Renormalization of NN-Scattering with One Pion Exchange and Boundary Conditions

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A non-perturbative renormalization scheme for Nucleon-Nucleon interaction based on boundary conditions at short distances is presented and applied to the One Pion Exchange Potential. It is free of off-shell ambiguities and ultraviolet divergences, provides finite results at any step of the calculation and allows to remove the short distance cut-off in a suitable way. Low energy constants and their non-perturbative evolution can directly be obtained from experimental threshold parameters in a completely unique and model independent way when the long range explicit pion effects are eliminated. This allows to compute scattering phase shifts which are, by construction consistent with the effective range expansion to a given order in the C.M. momentum $p$. In the singlet $^1S_0$ and triplet $^3S_1 - ^3D_1$ channels ultraviolet fixed points and limit cycles are obtained respectively for the threshold parameters. Data are described satisfactorily up to CM momenta of about $p \sim m_\pi$.

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I. INTRODUCTION

Effective field theories (EFT) are a powerful tool to deal with non-perturbative low energy physics. Over the last years, they have provided promising results as regards a systematic and model independent understanding of hadronic and nuclear physics. The scale separation between long and short distance physics makes the development of a systematic power counting possible. After the original proposal of Weinberg’s 1 to design a power counting based on applying ChPT to the potential many works have followed implementing such a counting, 2, 3, 4 with finite cut-offs or proposing a counting in the renormalized S-matrix 5, 6 which has also been pursued to NNLO 7. The relation of both the Weinberg (W) and Kaplan-Savage-Wise (KSW) counting has been understood as perturbative expansions about infrared fixed points in the limit of small and large scattering lengths 8 respectively (see also Ref. 9 for a discussion on long range forces in that context). For systems with a large scattering length as it turns out to be the case in low energy NN scattering, the Weinberg counting may be modified to iterate the scattering length to all orders, but then the connection to ChPT must be given up 10. On the other hand the KSW counting, although systematic, does not converge at NNLO 8. In Ref. 11 a new counting (BBSvK) involving also the chiral limit should be invoked. According to these authors, one should treat non-perturbatively the NN potential in the chiral limit and consider finite pion mass corrections perturbatively on top of that. For a recent and more complete review on these and related issues see e.g. Ref. 12 and references therein.

In order to properly define a truly EFT three essential requirements must be met. Firstly, one needs a power counting scheme, i.e. a dimensionless expansion parameter, which controls a priori the accuracy of a calculation and provides an error estimate of the neglected terms. The second requirement is the mathematical need for a regularization method and subsequent renormalization scheme which is consistent with the physical power counting, independent on any ultraviolet cut-off. Finally, there is the question of practical convergence, which can only be decided a posteriori on the light of practical calculations.

The issue of regularization and renormalization in the present context is not at all trivial, particularly if the power-counting scheme involves summing up some infinite set of diagrams. There are well known examples in the literature that shows that not all regularization methods comply to the physical power counting, like e.g. πN scattering for relativistic baryons in dimensional regularization 13. Such an approach is not consistent with the non-relativistic limit in the case of heavy baryons.

One has to either device a specific scheme, the so-called infrared regularization 14, or to make a non-relativistic limit first and introduce dimensional regularization afterwards 15. In the context of NN scattering it has also been shown 16 that the renormalized scattering amplitude for a theory without pions with a contact and a derivative term, i.e. a truncated potential, depends on whether one uses a cut-off regularization or dimensional regularization as an intermediate step of the calculation. The problem was latter understood by using other subtraction schemes than the MS in dimensional regularization, the so-called power divergence scheme (PDS) 2 (see also Refs. 17, 18 for off-shell (OS) subtraction schemes.)

As usual in field theory, one essential ingredient in the
whole construction is the choice of an appropriate regulator, which may eventually be removed. In perturbation theory, there are many such regulators, like e.g. dimensional regularization. Beyond perturbation theory, like in the two body scattering problem one can use those regulators within an order by order analysis aiming at finite renormalized scattering equation. For potentials which are purely short range, i.e., contact terms and derivative corrections there-off, this problem has successfully been tackled. The problem becomes subtle when, in addition, long range forces are added. Typically, these long range potentials develop some strong singularity at short distances. The paradigm of the problem is probably best exemplified by the $^1S_0$ channel with One Pion Exchange (OPE) potential. On the one hand, the regular OPE alone produces a finite scattering amplitude. On the other hand the short range potential is infinite but renormalizable, and hence a finite scattering amplitude can be built according to the standard principles of renormalization theory. This suggests, as it turns out to be the case, that the problem may still be renormalizable within a scheme where the long range potential is treated perturbatively. Whether or not this mathematical requirement is physically justified has been the subject of much debate in the recent past, with a negative answer after the findings of Ref. [16, 18, 20].

The mathematical problem actually arises when both short and long range potentials are added and none of them can be considered small; there seems no obvious way to renormalize the scattering equation non-perturbatively. By this we mean regularizing the equations first and removing the regulators afterwards, or at least make the mass scale of the regulator much larger than any other mass scale in the problem. The main interest in making such a non-perturbative renormalization is that any perturbative scheme can be thought of as an approximation to renormalized equations. The non-perturbative renormalization of NN interaction in the singlet $^1S_0$ and triplet $^3S_1 - ^3D_1$ channels has been studied several times in the literature by different regularization methods. In Ref. [15] a subtraction method was developed for the Lippmann-Schwinger equation to construct a finite $T$ matrix for contact interactions added to OPE. Renormalization is indeed achieved by taking the subtraction scale to be much larger than any other mass scale and checking for independence of results in this limit. The resulting description of the $^1S_0$ phase shift is only valid to very low energies, requiring for inclusion of derivative terms. Unfortunately, the method has not been extended to that case. Derivative interactions can be included within a tree-dimensional cut-off regularization [20, 21]. Actually, in the cut-off regularization analysis of Ref. [21] a strong breaking of angular momentum is observed unless either a vanishing bare mixing coupling is chosen or a (according to the authors) unlikely fine tuning sets in for a non-trivial fixed point. We will show below that indeed the bare mixing vanishes but not in an uniform way, but rather following a limit cycle pattern. Inspired by the N/D method, the work of Ref. [22] makes a re-summation of the KSW amplitudes introducing a on-shell potential to which chiral counting is applied. A new parameter, which is considered to be of order zero in the chiral counting is introduced. The dynamical origin of this parameter is unclear. The non-perturbative dimensional regularization of the OPE potential with derivative interactions has been studied in coordinate space [23] as well as in momentum space [24] in the $^1S_0$ channel. In both cases a three-parameter fit can be achieved with no explicit two pion exchange contribution up to CM momentum as large as $p \sim 400$ MeV. Ref. [24] opens up some hope as how the triplet $^3S_1 - ^3D_1$ channel might also be renormalized non-perturbatively within dimensional regularization, but so far there are no practical calculations. The early finite sharp cut-off momentum space treatments of Ref. [4] have been improved by implementing a better regularization scheme [25] which allows to make take larger cut-off values, yet finite. More recent works supporting the W-counting have also appeared [26, 27].

From a diagrammatic point of view momentum space treatments based on the Lippmann-Schwinger equation are more natural within a Lagrangian framework and allow explicit consideration of nonlocal potentials. On the other hand, the long range NN potentials making use of chiral symmetry constraints are local, and for those the analysis of non-perturbative renormalization in coordinate space becomes much simpler, as will be shown along this work. In addition, the Schrödinger equation is a second order operator and mixed boundary conditions define a complete and unique solution of the scattering problem in the whole space at both sides of the boundary. This sharp boundary separation of the space is naturally formulated in coordinate space for a local potential. Boundary conditions for NN scattering were used many years ago (see e.g. Ref. [28] and references therein), and there has been renewed interest motivated by the developments within EFT [29, 30, 31]. Actually, the thorough analysis of Ref. [31] shows that in the absence of long range forces a low momentum expansion of the potential within EFT framework for the Lippmann-Schwinger equation is completely equivalent to an effective range expansion (ERE) and also to an energy expansion of a generic boundary condition at the origin in coordinate space for the Schrödinger equation. Moreover, the reference partial wave analysis of the Nijmegen group [32] uses this method to successfully describe the a large NN scattering data base, when long range potentials are used. While in the first works phenomenological potentials where used, more recent studies consider potentials deduced from ChPT theory with a rather satisfactory description of the experimental scattering data [33]. The minimal boundary radius which can still provide an ac-
acceptable $\chi^2/DOF$ is about $R_S = 1.4 - 1.8$fm. Obviously, if the radius cannot be lowered without spoiling the quality of the fit, the short distance cut-off becomes an indispensable parameter of the theory, which cannot be removed. The corresponding momentum space cut-off $\Lambda = 2\pi / R_S \sim 600$MeV is comparable to the one needed in early momentum space treatments \cite{24}. Within the spirit of an EFT it would actually be more appropriate to take instead larger $\Lambda$’s or equivalently shorter $R_S$’s and to check for insensitivity of results in the low energy regime. Thus, there arises the natural question whether in fact this EFT procedure can be implemented.

In our previous work \cite{34} we showed that for the $^1S_0$ singlet channel with OPE the boundary radius can be effectively removed without spoiling a good description of the corresponding phase shift up to the a priori expected CM momentum of $k \sim m_\pi$, where the Two Pion Exchange (TPE) effects should start playing a role. The first order differential equation satisfied by the boundary condition of the problem defined in the interval $R < r < \infty$ as a function of the boundary radius was very helpful, since the whole problem could be mapped into a variable phase equation \cite{35} of a truncated potential in the region $0 < r \leq R$ with a non-trivial initial condition at the origin, encoding the short distance physics. In this way, the long range pions could be eliminated and the evolution of the threshold parameters as a function of the boundary radius could be determined non-perturbatively. Actually, a trivial ultraviolet fixed point limit for the scattering length was found non-perturbatively. Remarkably, this behaviour coincides with the one found in Ref. \cite{10} within a perturbative treatment. This trivial fixed point at the origin implies a fine tuning of the short distance physics in order to reproduce the physical scattering length. In this paper we want to extend our results for the interesting case of the triplet $^3S_1 - ^3D_1$ channel. The solution of the boundary condition problem requires solving a coupled set of Schrödinger equations. Instead of doing so, we prefer to directly compute the change of the boundary condition by an equivalent variable phase approach \cite{35} with non-trivial initial conditions which encode the short distance physics \cite{34}. This provides, in addition, a direct and quite transparent connection to renormalization group ideas \cite{34}.

Related works in spirit to the present are those of Refs. \cite{34, 36} and \cite{4}. In Refs. \cite{34, 36} a square well potential is used to regulate the short distance behaviour simulating a smeared delta function. The renormalization group flow for the potential strength is not uniquely defined. This phenomenon is also found in the theory of self-adjoint extensions of the Schrödinger operators \cite{37}. In Ref. \cite{4} a delta shell regulator located at a finite distance is assumed as the short distance potential whereas the long distance piece is solved exactly using a distorted wave basis. This formalism has been so far used to the study of renormalization of repulsive singular potentials (like $1/r^2$). The reason may have to do with the need for a well defined renormalization at the origin. A common feature of both regularization schemes is that the wave function at the origin is uniquely determined by the regularity condition, $u(0) = 0$. The boundary condition regularization that we use in this paper provides a uniquely defined renormalization group flow \cite{34}, to treat both repulsive and attractive singular potentials \cite{35}. In addition, the boundary condition admits a simple physical interpretation: it can be mapped into a variable phase shift problem \cite{52} with a truncated potential. This interpretation directly provides the non-perturbative renormalization flow of low energy parameters and a quite transparent analysis of both infrared as well as ultraviolet fixed points and limit cycles \cite{35}.

In this paper we analyze precisely how the energy dependent boundary condition must change as we move the boundary radius for fixed energy to achieve independence of physical observables such as scattering phase shifts. By doing so we are effectively changing the Hilbert space since the wave function in the outer region is defined only from the boundary to infinity. An advantage of this procedure is that we never need to invoke off-shellness explicitly; at any step we are dealing with an on-shell problem. In addition, we work directly with finite quantities and no divergences appear at any step of the calculation when the boundary radius is taken to zero from above.

Another advantage of our construction, as it will become clear along the paper, is that we only need the potentials and physical threshold parameters as input of the calculation (the cut-off dependence is removed completely, so this is not a parameter). This implies, in particular, that given this information we never have to make a fit (except perhaps for the determination of the threshold parameters); our calculations are predictions for the phase shifts that are consistent, by construction, with a low energy expansion up to a given order. Thus, the potential danger of compromising the low energy fit due to a global fit up to 300MeV may be precluded from the start. This is at difference with the standard way of proceeding where the low energy parameters are fitted to the phase shifts and the threshold parameters are then recomputed. Actually, our analysis is equivalent to making a fit only in the low energy region, where explicit pions do not contribute, and predicting the intermediate energy region. We believe this is a possible and practical way of learning about the role of explicit pions in the NN interaction. Actually, our motivation was partly to see whether OPE can actually be seen in low partial waves in the intermediate energy range $m_\pi/2 \leq k \leq m_\pi$.

In the present paper we analyze the OPE potential \cite{5} 
\begin{equation}
(U = 2\mu V \pm \mu = M_N/2)
\end{equation}

which reads
\begin{align}
U(\vec{x}) &= U_C(r) + S_1 U_T(r), \\
\text{with } \hat{\sigma} &= \vec{x}/r \\
S_{12} &= 3\vec{\sigma}_1 \cdot \vec{\sigma}_2 \cdot \hat{x} - \vec{\sigma}_1 \cdot \hat{x} \cdot \vec{\sigma}_2, \\
U_C &= -\frac{m_\pi^2 M_N g_A^2}{16\pi f^2}\frac{e^{-m_\pi r}}{r}
\end{align}
\[ U_T = -\frac{m_N^2 M_N g_A^2}{16\pi f^2_\pi} e^{-m_\pi r} \left( 1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2} \right) \]  

(4)

Where \( M_N \) is the nucleon mass, \( m_\pi \) the pion mass, \( f_\pi \) the pion weak decay constant and \( g_A \) the nucleon axial coupling constant. In the numerical calculations below we take \( M_N = 938.92 \text{ MeV} \), \( f_\pi = 93 \text{ MeV} \), \( m_\pi = 138 \text{ MeV} \) and \( g_A = 1.25 \). Note that the singularity at the origin of the tensor potential

\[ U_T \rightarrow \frac{3M_N g_A^2}{16\pi f^2_\pi r^3} \quad r \rightarrow 0 \]  

(5)

is independent on the pion mass \( m_\pi \).

The plan of the paper is as follows. In Sect. II we present the basic object of our analysis, the variable S-matrix which we supplement with general mixed boundary conditions in the general case of coupled channel scattering. We also discuss the role played by the irregular solutions for singular potentials in the spirit of an effective field theory. After that we rewrite in Sect. III the variable S-matrix equation for the variable K-matrix and R-matrix in a way that the low energy limit may be taken. As a result, we find the boundary radius evolution of threshold parameters. As a first application we apply in Sect. IV the obtained equations to determine the low energy threshold parameters from well established NN potentials. In Sect. V we study the short distance behaviour of the threshold parameters. There we show that one has for the \( ^1S_0 \) and \( ^3S_1 - ^3D_1 \) channels an UV fixed point and a UV limit cycle for the scattering lengths. In Sect. VI we present our numerical results, both for the threshold parameters as well as for the \( ^1S_0 \) and \( ^3S_1 - ^3D_1 \) phase shifts. Finally in Sect. VII we present some final remarks, conclusions and perspectives for future work.

II. VARIABLE S-MATRIX WITH BOUNDARY CONDITIONS

In order to generalize to triplet states the results of Ref. 34 for the singlet channel case, we introduce the variable S-matrix formalism for the general coupled channel case. For potentials which are either regular or singular repulsive at the origin the procedure is standard 35 and it has many variants. For completeness and to make the exposition more self contained we present here our particular derivation which also applies to singular attractive potentials and at the same time introduce our basic notation for the rest of the paper. Although in the case under study we are interested in, at most, two coupled channels, the formalism can be developed for the general case with almost no additional effort.

The scattering amplitude for NN scattering can be written as a partial wave expansion

\[ f = \frac{1}{2i^k} \sum_{l=0}^{\infty} (S - 1) P_l(\cos \theta) , \]  

(6)

where \( S \) is the S-matrix for coupled channels. The coupled channel Schrödinger equation for the relative motion reads

\[ -u''(r) + \left[ U(r) + \frac{l^2}{r^2} \right] u(r) = k^2 u(r) , \]  

(7)

where \( U(r) \) is the coupled channel matrix potential, \( l^2 = \text{diag}(l_1(l_1 + 1), \ldots, l_N(l_N + 1)) \) is the angular momentum, \( u(r) \) is the reduced matrix wave function and \( k \) the C.M. momentum. We assume for \( u(r) \) the mixed boundary condition

\[ u'(R) + L_k(R)u(R) = 0 , \]  

(8)

where \( L_k(R) \) is a real hermitean matrix in coupled channel space, which in our framework encodes the unknown physics at distances \( r \) below the boundary radius \( R \). In addition, we assume the asymptotic normalization condition

\[ u(r) \rightarrow u_{\text{in}}(r) - u_{\text{out}}(r)S , \]  

(9)

with \( S \) the standard coupled channel S-matrix. The corresponding out-going and in-going free spherical waves are given by

\[ u_{\text{out}}(r) = \text{diag}(\hat{h}_1^+(kr), \ldots, \hat{h}_N^+(kr)) , \]  

(10)

\[ u_{\text{in}}(r) = \text{diag}(\hat{h}_1^-(kr), \ldots, \hat{h}_N^-(kr)) , \]  

(11)

with \( \hat{h}_l^\pm(x) \) the reduced Hankel functions of order \( l \), \( \hat{h}_l^+(x) = xH_{l+1/2}^{\pm}(x) \) ( \( \hat{h}_0^+ = e^{\pm ix} \) ), and satisfy the free Schrödinger’s equation for a free particle,

\[ -u''_{\text{out}}(r) + \frac{l^2}{r^2} u_{\text{out}}(r) = k^2 u_{\text{out}}(r) , \]  

(12)

\[ -u''_{\text{in}}(r) + \frac{l^2}{r^2} u_{\text{in}}(r) = k^2 u_{\text{in}}(r) , \]  

(13)

The boundary condition, Eq. (8), for the outer boundary values problem, Eq. (10) and Eq. (11), can be interpreted in simple physical terms of a complementary inner problem where the potential \( U(r) \) acts in the interval \( R \leq r < \infty \) and \( U(r) \) acts in the interval \( 0 < r \leq R \), which inherits the dependence on the chosen boundary radius \( R \). The equation satisfied by the variable S-matrix can be obtained

\[ \text{1 This is the most general boundary condition that makes the coupled channel Hamiltonian self-adjoint in the interval } R \leq r < \infty . \]
from Schrödinger’s equation applied to the matrix \( L(R) \) yielding

\[
L_k(R)' + L_k(R)^2 = U(R) + \frac{l^2}{r^2} - k^2 .
\]  

(15)

From here \(^2\) it is straightforward to obtain the equation for the variable S-matrix,

\[
2k \frac{dS(R)}{dR} = \left[ \left( S(R) \hat{h}^+(R) - \hat{h}^-(R) \right) U(R) \right] \times \left[ \left( \hat{h}^-(R) - \hat{h}^+(R) S(R) \right) \right].
\]  

(16)

This is a first order non-linear matrix differential equation which can be solved by standard means, provided the S-matrix is known at one given scale. One of the interesting aspects of this equation is that there is no need to invoke any off-shellness; for any value of the boundary radius we have a different on-shell scattering problem. In appendix \(^A\) we show an alternative derivation based on continuous deformations of the potential with a fixed boundary condition. As we will discuss below, Eq. (16) describes the renormalization group flow of the S-matrix as a function of the distance scale \( R \) where the long range potential is cut-off.

In the case of a regular potential, Eq. (16) has to be supplemented with an initial condition at the origin, namely the trivial one (corresponding to the absence of a potential), and its asymptotic value yields the full \( S \)-matrix;

\[
S(0) = \mathbf{1}, \quad S = S(\infty) \quad \text{(regular)}
\]  

(17)

In this paper we are concerned with the OPE potential which has a singular \( 1/r \) behaviour at the origin in the \( 1S_0 \) singlet channel and singular \( 1/r^3 \) behaviour at the origin due to the tensor force in the \( 3S_1 - 3D_1 \) triplet channel. While in the single channel the singularity is a mild one in the sense that there still exists a unique regular solution at the origin, \( u(0) = 0 \) (like in the Coulomb potential), in the triplet channel both linearly independent solutions to Schrödinger’s equation vanish at the origin, and the regularity condition \( u(0) = 0 \) does not uniquely specify the solution.

The point of view we take in the present work is that of an EFT; low energy physics should not depend on the detailed knowledge of the interaction at short distances. This applies, in particular, to the case of a singular potential as will also become clear below. Following the lines already sketched in our previous work \(^3\), we take instead the value at infinity as the initial value for the variable S-matrix. Of course, for a short range potential, this procedure corresponds to start integrating at sufficiently large distances (where the potential may be neglected). An advantage of this procedure is that by construction a unique solution \( S(R) \) is obtained. Even for a regular potential, it is clear that a generic choice of \( S(\infty) \) cannot yield by integrating towards the origin the result \( S(0) = \mathbf{1} \) besides the very exceptional cases which accidentally correspond to the regular solution at the origin. Thus, we expect in general an admixture of both the regular and irregular solutions, which corresponds to a mixed boundary condition close to the origin,

\[
\lim_{R \to 0^+} \{ u'(R) + L(R)u(R) \} = 0
\]  

(18)

In the case of a singular potential both solutions vanish and we equally have a unique mixed boundary condition as in Eq. (18). Thus, we may define the short distance S-matrix as the extrapolation to the origin of a given solution at infinity,

\[
S_s \equiv \lim_{R \to 0^+} S(R) , \quad S = S(\infty) \quad \text{(general)}
\]  

(19)

Actually, the precise meaning of the previous limit will be the main topic of the present work. We anticipate already that we will find ultraviolet fixed points for the singlet \( 1S_0 \) channel and limit cycles for the \( 3S_1 - 3D_1 \) triplet channel. Eqs. (16) and (20) are well known in potential scattering (for a review see e.g. Ref. \(^3\)), but they have always been used assuming the trivial initial conditions \( S(0) = \mathbf{1} \).

Obviously, if one would literally use the full \( S \)-matrix and integrate downward, nothing could be achieved, since that would correspond to eliminating the full potential. A more interesting perspective, already pursued in Ref. \(^3\) for the singlet \( 1S_0 \) channel, consists of regarding the low energy limit of the previous equations, extracting the threshold parameters at short distances by integrating downward from their experimental values and integrate back upward the variable \( S \)-matrix equation to infinity. Physically, this procedure corresponds to explicitly separate the OPE contributions on top of any low energy approximation, like e.g. the effective range expansion.

In the case of one channel, like the \( 1S_0 \), the S-matrix can be parameterized as \( S_t(k, R) = \exp(2i\delta_t(k, R)) \) with \( \delta_t(k, R) \) the variable phase. Eq. (16) becomes rather simple \(^5\) for \( s \)-waves, yielding

\[
\frac{d\delta_0(k, R)}{dR} = \frac{1}{k} U(R) \sin^2(kR + \delta_0(k, R)) .
\]  

(20)

and the obvious conditions both at the origin and at infinity must be satisfied

\[
\lim_{R \to 0} \delta_0(k, R) = \delta_0^0(k) \quad \lim_{R \to \infty} \delta_0(k, R) = \delta_0(k) .
\]  

(21)

The OPE potential in the coupled \( 3S_1 - 3D_1 \) in the triplet channel space is given by

\[
U(r) = \begin{pmatrix} U_s(r) & U_{sd}(r) \\ U_{sd}(r) & U_d(r) \end{pmatrix} ,
\]  

(22)
where
\[ U_* = U_C \quad U_{sd} = 2\sqrt{2}U_T \quad U_d = U_C - 2U_T. \] (23)

The two coupled channels S-matrix can be represented in the Blatt-Biedenharn (BB) or Eigen phase parameterization
\[ S = \begin{pmatrix} \cos \epsilon & -\sin \epsilon \\ \sin \epsilon & \cos \epsilon \end{pmatrix} \begin{pmatrix} e^{2i\delta_i} & 0 \\ 0 & e^{2i\delta_2} \end{pmatrix} \begin{pmatrix} \cos \epsilon & \sin \epsilon \\ -\sin \epsilon & \cos \epsilon \end{pmatrix} \] (24)
which will be used along this paper. The relation to the standard coupled channel K-matrix is given by
\[ S = (K + ik)(K - ik)^{-1}, \] (25)
where
\[ \tan(2\epsilon) = \frac{2K_{12}}{K_{11} - K_{22}}, \] (26)
\[ -\tan \delta_- = \frac{K_{11} + K_{22} + \frac{K_{11} - K_{22}}{\cos 2\epsilon}}{2}, \] (27)
\[ -\tan \delta_+ = \frac{K_{11} + K_{22} - \frac{K_{11} - K_{22}}{\cos 2\epsilon}}{2}. \] (28)

Due to unitarity of the S-matrix in the low energy limit, \( k \to 0 \) we have
\[ (S - 1)_{ij} = -2i\alpha_{ij}k^{i+j+1} + \ldots, \] (29)
with \( \alpha_{ij} \) the (hermitean) scattering length matrix. The low energy limit acquires its simplest form in the Stapp-Ypsilantis-Metropolis (SYM or Nuclear bar) parameterization
\[ S = \begin{pmatrix} e^{2i\delta_1} \cos 2\epsilon & ie^{i(\delta_1 + \delta_2)} \sin 2\epsilon \\ ie^{i(\delta_1 + \delta_2)} \sin 2\epsilon & e^{2i\delta_2} \cos 2\epsilon \end{pmatrix} \] (30)
which is related to the BB phase shifts by
\[ \delta_1 + \delta_2 = \delta_+ + \delta_-, \] (31)
\[ \sin(\delta_1 - \delta_2) = \frac{\tan(2\epsilon)}{\tan(2\epsilon)} \] (32)
The low energy limit in the SYM representation becomes
\[ \delta_1 \to -\alpha_0k, \quad \delta_2 \to -\alpha_2k^5, \quad \epsilon \to -\alpha_0k^3. \] (33)
The scaled K-matrix, \( \hat{K} \), has a good low energy behaviour and is defined by making an energy dependent transformation
\[ \hat{K} = kDKD, \] (34)
with \( D = \text{diag}(k^{l_1}, \ldots, k^{l_N}) \). The scaled K-matrix admits the coupled channel analog of the effective range expansion
\[ \hat{K} = -a^{-1} + \frac{1}{2}r k^2 + v k^4 + \ldots, \] (35)
where \( a, r \) and \( v \) are the scattering length matrix, effective range and curvature parameters respectively.

## III. EVOLUTION OF LOW ENERGY PARAMETERS

In order to take this low energy limit and corrections there-off, we introduce the variable or running \( K \)-matrix
\[ S(R) = (K(R) + ik)(K(R) - ik)^{-1}, \] (36)
as well as the reduced Bessel functions
\[ \hat{y}(x) = x\hat{y}(x), \quad \hat{y}(x) = x\hat{y}(x), \] (37)
i.e. \( \hat{y}(x) = x\hat{y}(x) \). Thus,
\[ \hat{y} = \frac{1}{2i} \left( \hat{y}^{(+)} - \hat{y}^{(-)} \right), \] (38)
\[ -\hat{y} = \frac{1}{2} \left( \hat{y}^{(+)} + \hat{y}^{(-)} \right). \] (39)

Then, we get
\[ K'(k, R) = \left( \frac{1}{k} K(k, R) \hat{j}(kR) - \hat{y}(kR) \right) U(R), \] \[ \times \left( \frac{1}{k} j(kR)K(k, R) - \hat{y}(kR) \right). \] (40)
The scaled K-matrix, \( \hat{K}(R) \), has a better low energy behaviour and is defined by making an energy dependent transformation
\[ \hat{K}(R) = kDK(R)D, \] (41)
with \( D = \text{diag}(k^{l_1}, \ldots, k^{l_N}) \). We get
\[ \hat{K}'(k, R) = \left( \frac{1}{k} j'j(kR)D^{-1} - \hat{y}(kR)D \right) U(R), \] \[ \times \left( \frac{1}{k} j(kR)D^{-1} \hat{K}(k, R) - \hat{y}(kR)D \right). \] (42)
The scaled K-matrix admits the analog of the effective range expansion
\[ \hat{K}(R) = -a(R)^{-1} + \frac{1}{2}r(R)k^2 + v(R)k^4 + \ldots, \] (43)
where \( a(R), r(R) \) and \( v(R) \) are the corresponding running scattering length matrix, effective range and curvature parameters respectively. In this form the low energy limit can be easily taken. Defining the matrix functions and their low energy expansion
\[ A_k(R) = \frac{j(kR)}{k} D^{-1} = A_0 + k^2A_2 + k^4A_4 + \ldots, \]
\[ B_k(R) = y(kR)D = B_0 + k^2B_2 + k^4B_4 + \ldots, \] (44)
we get the system of coupled equations \(^3\)
\[ \frac{d}{dR} [a(R)]^{-1} = -[(a(R))^{-1} A_0 + B_0] U(R) \]
\[^3\] The Equation for \( v_2 \) in the coupled channel case is too long to be reproduced here, but can be obtained in a straightforward way.
\[
\frac{d}{dR} r(R) = \left[ \text{a}(R)^{-1} A_0 + B_0 \right] U(R) \times \left[ (A_0)^{-1} A + B_0 \right] \quad (45)
\]

These equations generalize to the coupled channel case those already found in Ref. [34] and have to be supplemented with some initial conditions, at e.g. infinity,

\[
a(\infty) = a, \quad r(\infty) = r, \quad \ldots \quad (47)
\]

For the case of \(s\)-wave one channel scattering Eq. (48) becomes

\[
\frac{dK(k, R)}{dR} = U(R) \left[ K(k, R) \frac{\sin kR}{k} + \cos kR \right]^2. \quad (48)
\]

where

\[
K(k, R) = k \cot \delta(k, R), \quad (49)
\]
yielding at low energies an effective range expansion,

\[
k \cot \delta(k, R) = -\frac{1}{\alpha_0(R)} + \frac{1}{2} r_0(R) k^2 + v_2(R) k^3 \ldots, \quad (50)
\]

where

\[
\frac{d\alpha_0}{dR} = U(R) (\alpha_0 - R)^2 \quad (51)
\]

\[
R^4 \alpha_0 = 9 U \alpha_0^2 + (\alpha_0 - R) R^2 \left[ (\alpha_0 - R) U_s + 6 \alpha_0 U_{sd} \right],
\]

\[
15 R^2 \alpha_0^2 = -15 \alpha_0 R^2 \left[ (\alpha_0 - R) U_s + 2 \left( 45 \alpha_0^2 - (\alpha_0 - R) (-45 \alpha_2 + R^3) \right) U_{sd} - 3 \alpha_0 \left( 45 \alpha_2 + R^3 \right) U_d \right],
\]

\[
225 R^4 \alpha_0^2 = 225 \alpha_0^2 R^4 U_s - 30 \alpha_0 R^2 \left( -45 \alpha_2 + R^3 \right) U_{sd} + \left( -45 \alpha_2 + R^3 \right)^2 U_d. \quad (58)
\]

Note that all three running low energy parameters \(\alpha_0\), \(\alpha_0^2\) and \(\alpha_2\) (the explicit \(R\)-dependence has been suppressed for simplicity) are coupled due to the mixing potential \(U_{sd}\). Thus, it would be inconsistent to take any of them as a constant; exact renormalization group invariance requires mixing between the \(S\) and \(D\) channels. As we see the mixing is related both to a non-vanishing of the mixing potential \(U_{sd}\) and a non vanishing value of \(\alpha_{sd}\) at a given point. If by some accident both vanish at a given point, the mixing will vanish.

The evolution of the low energy parameters can be translated into the corresponding evolution of the short distance boundary condition as a function of the boundary radius. Defining the dimensionless quantity

\[
C_k(R) = 1 - RL_k(R) = \left[ 1 - Ru'_k(R)u_k(R)^{-1} \right], \quad (59)
\]

\[
\frac{dr_0}{dR} = 2U(R) R^2 \left( 1 - \frac{R}{\alpha_0} \right) \left( \frac{r_0}{R} + \frac{R}{3\alpha_0} - 1 \right) \quad (52)
\]

\[
\frac{dr_2}{dR} = \frac{U(R)}{R} \left\{ \frac{1}{4} \left( \frac{r_0}{R} + \frac{R}{3\alpha_0} - 1 \right)^2 + 2 \left( 1 - \frac{R}{\alpha_0} \right) \left( -\frac{1}{12} \frac{r_0}{R} + \frac{v_2}{R^3} - \frac{1}{120} \frac{R}{\alpha_0} + \frac{1}{24} \right) \right\} \quad (54)
\]

These equations have been studied by us in Ref. [34] for analyzing the OPE in the singlet \(^1S_0\) channel.

In the \(^3S_1 - ^3D_1\) coupled channel case the threshold parameters matrices are

\[
a = \begin{pmatrix} \alpha_0 & \alpha_0^2 \\ \alpha_0^2 & \alpha_2 \end{pmatrix}, \quad \alpha_0 \quad (55)
\]

\[
r = \begin{pmatrix} r_0 & r_{02} \\ r_{02} & r_2 \end{pmatrix}, \quad \alpha_0 \quad (56)
\]

\[
v = \begin{pmatrix} v_0 & v_{02} \\ v_{02} & v_2 \end{pmatrix}, \quad \alpha_0 \quad (57)
\]

The explicit form of the equations for the \(^3S_1 - ^3D_1\) running scattering lengths reads

\[
RC_k'(R) = C_k(1 - C_k) + U(R) R^2 + 1^2 - k^2 R^2. \quad (60)
\]

Expanding into powers of the momentum \(k\) one gets

\[
C_k(R) = C_0(R) + k^2 R^2 C_2(R) + \ldots \quad (61)
\]

For the singlet \(^1S_0\) channel we have, in particular, the following relation

\[
C_0 = \frac{\alpha_0(R)}{R - \alpha_0(R)}. \quad (62)
\]

Note that for \(R \to \infty\) we have a fixed point behaviour \(C_0 \to 0\) unless \(\alpha = \infty\) in which case \(C_0 \to 1\). The evolution of the boundary condition with the short distance
boundary radius for the $^3S_1-^3D_1$ in terms of the running scattering lengths is given by

$$C_s^0 = 1 + \frac{R(R^5 - 45\alpha_{22})}{45\alpha_{22}^2 + (\alpha_{00} - R)(R^5 - 45\alpha_{22})}, \quad (63)$$

$$C_{sd}^0 = \frac{15\alpha_{02}R^3}{45\alpha_{22}^2 + (\alpha_{00} - R)(R^5 - 45\alpha_{22})}, \quad (64)$$

$$C_d^0 = 3 - \frac{5R - (\alpha_{00} - R)R^5}{45\alpha_{22}^2 + (\alpha_{00} - R)(R^5 - 45\alpha_{22})}. \quad (65)$$

(R-dependence has been suppressed for simplicity.) Again, for $R \to \infty$ we have for non exceptional values of the parameters $C_s^0 \to 0$, $C_{sd}^0 \to 0$ and $C_d^0 \to -2$. In Ref. 38 a more detailed study on these issues will be carried out.

IV. DETERMINATION OF LOW ENERGY PARAMETERS AND THE THEORY WITHOUT EXPLICIT PIONS

An essential ingredient of our formalism is to parameterize the scattering data directly in terms of low energy threshold parameters, like $\alpha$, $r$ and $v$, defined through Eq. (35). Unfortunately, besides $\alpha$ and $r_0$ in the singlet and triplet channels, the PWA data base does not provide values for them. They could be obtained from a fit to the NN data base in the pertinent channels, at sufficiently low energies. Such a procedure turns out to be numerically unstable, particularly for the $v$ parameter, because it depends very strongly on the energy window chosen for the fit (see Appendix B). On the other hand, the NN data base provides explicit potentials, some of them local like the NijmII and Reid93 potentials, for which the variable phase approach may directly be applied. In such a way we can uniquely and accurately determine all the needed low energy threshold parameters by integrating Eqs. (10) upwards from the origin to infinity with trivial boundary conditions. For illustration purposes the evolution for the NijmII potential for the threshold parameters is depicted in Fig. 1. We remind that these curves represent the low energy threshold parameters corresponding to a potential truncated at a given distance, $R$. The behaviour at the origin has to do with the strong repulsive core of the potential in the $s$-wave channel. Also, the divergence of the scattering lengths at about $R \sim 2\text{fm}$ signals that a bound state has appeared. The values at infinity correspond to the physical values. Our results, in appropriate powers of $\hbar$, can be summarized as follows for the NijmII and the Reid93 (in brackets) potential

- Singlet $^1S_0$ NijmII (Reid93)
  $$\alpha_0 = -23.74(3), \quad r_0 = 2.67(75), \quad v_2 = -0.48(9) \quad (66)$$

- Singlet $^1S_0$ Reid93
  $$\alpha_0 = 5.001(3), \quad r_0 = 1.833, \quad v_2 = 0.131(41) \quad (67)$$

- Triplet $^3S_1-^3D_1$ without mixing NijmII (Reid93)
  $$\alpha_0 = 5.001(3), \quad r_0 = 1.833, \quad v_2 = 0.131(41) \quad (67)$$

- Triplet $^3S_1-^3D_1$ with mixing NijmII (Reid93)
  $$a = \begin{pmatrix} 5.419(22) & 1.647(6) \\ -6.504(453) \end{pmatrix} \quad (68)$$
  $$r = \begin{pmatrix} 1.833 & 0.404(12) \\ -3.522(66) \end{pmatrix} \quad (70)$$
  $$v = \begin{pmatrix} -0.131(41) & -0.274(64) \\ -3.70(80) \end{pmatrix} \quad (71)$$

The $^3S_1$ channel without mixing parameters have been obtained from the $^3S_1-^3D_1$ channel for the $^3S_1$ component, and $\alpha_0 = 1/(\alpha_{-1})_{00}$, complying to the low energy expansion of the scaled $K$ matrix, Eq. (35). Although we will be using the NijmII parameters, we have also presented the ones corresponding to the Reid93 case to provide an idea on the size of errors.

Once the threshold parameters have been determined we can use the coupled channel effective range expansion, Eq. (35) to find out to what extent does this expansion apply. On theoretical grounds we expect this expansion to converge within the region of analyticity of the $S$-matrix, which presents a left cut at $k = \pm im_{\pi}/2$. In Fig. 2 we compare the quality of the ERE including LO, NLO and NNLO contributions to the original data of Ref. 32. As we see, to describe the data within the ERE approach up to the convergence radius $m_{\pi}/2$ one has to go at least to NLO. The description of the data below $m_{\pi}/2$ is improved, as expected, with higher orders in the ERE. Above this region, where OPE should play a role, this is not necessarily so. Actually, we see that in the $^3S_1$ end $E_1$ channels the NNLO is worse than the NLO approximation. We emphasize that these curves are not obtained from a fit to the data.

V. SHORT DISTANCE BEHAVIOUR FOR OPE: FIXED POINTS AND LIMIT CYCLES

In this section we analyze the short distance behaviour of the equations for the scattering lengths for the singlet $^1S_0$, Eq. (21), and the triplet $^3S_1$, Eq. (64), channels in the short distance limit. According to Eq. (12) this is equivalent to study the mixed boundary condition at short distances.

We study first the case of OPE in the singlet $^1S_0$ channel. At short distances $R << 1/m_{\pi}$ the OPE potential behaves like the Coulomb potential. Eq. (51) can be easily solved in two extreme cases, $\alpha_0 << R$ and $\alpha_0 >> R$. While in the first case we get

$$\alpha_0(R) \rightarrow -\frac{g_A^2 M_N^2}{16\pi f_{\pi}^2} \frac{1}{g_{\pi}^2(R)/R_0}, \quad \alpha_0 << R \quad (72)$$

in the second case one solution behaves as

$$\alpha_0(R) \rightarrow -\frac{g_A^2 M_N^2}{16\pi f_{\pi}^2} \frac{1}{g_{\pi}^2(R)/R_0}, \quad \alpha_0 << R \quad (73)$$
FIG. 1: Evolution of the $^3S_1$, $^3D_1$, and $E_1$ NN-threshold parameters from the trivial values at the origin using the NijmII potential to infinity. Top panel: scattering lengths $\alpha_0(R)$ (in fm), $\alpha_{02}(R)$ (in fm$^3$) and $\alpha_2(R)$ (in fm$^5$). Bottom panel: effective ranges $v_0(R)$ (in fm), $v_{02}(R)$ (in fm$^3$) and $v_2(R)$ (in fm$^5$). In the left panel we represent a global picture from the 1fm to 40 fm. The lower panel is a detailed picture in the short distance region below 1fm.

where $R_0$ is a reference scale fulfilling $R < R_0 \ll 1/m_\pi$. As we see, $\alpha_0(R)$ goes to zero in both case but, while Eq. (72) goes rapidly $\alpha'_0(R) \to 0$, Eq. (73) goes very slowly and with $\alpha'_0(R) \to -\infty$ at short distances. In momentum space the $R \to 0$ limit corresponds to the ultraviolet limit. Eq. (73) resembles a sort of asymptotic freedom and, hence we have an ultraviolet fixed point. One can see that the first case, Eq. (72), corresponds
FIG. 2: Theory without explicit pions. $^3S_1$, $^3D_1$ and $E_1$ at LO (contact terms), NLO ($k^2$ terms) and NNLO ($k^4$ terms) predicted phase shifts for the triplet channel within a pure effective range expansion approximation, with the low energy threshold parameters obtained from solving the evolution equations for the threshold parameters, Eq. (46) with the Nijm II potential and using the reduced K-matrix, Eq. (35). Data are the PWA from Ref. [32].

to selecting the regular solution at the origin, whereas Eq. (73) corresponds to a generic case, which always contains an admixture of the irregular solution. The regular case at the origin corresponds to integrate from the origin starting with the trivial initial condition $\delta(k, 0) = 0$ up to infinity. As we have discussed in Ref. [34] the result corresponds to a pure OPE interaction, with no short-distance interactions. The important thing to realize is that regardless of the value of $\alpha_0$ at infinity, removing one-pion exchange goes into the same value at the origin, as implied by Eq. (73). This also implies that any tiny deviation of the $\alpha_0(R)$ at small distances results in huge variations at infinity. Thus, removing OPE in the $^3S_0$ channel implies an extreme fine tuning of the scattering length at short distances, and hence of the boundary condition at the origin.

We turn now to the case of the $^3S_1$ $^3D_1$ channel, where the tensor force plays a role. In the region close to the origin the wave function oscillates wildly and hence a WKB approximation may be used. The calculation is simplified by taking into account that for the OPE interaction the potential matrix is diagonalized by an $r$-
independent unitary transformation, i.e.

$$\mathbf{M} \mathbf{U}(r) \mathbf{M}^{-1} = \begin{pmatrix} U_C(r) - 4U_T(r) & 0 \\ 0 & U_C(r) + 2U_T(r) \end{pmatrix},$$  \hspace{1cm} (74)

with

$$\mathbf{M} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1 \\ -\sqrt{2} & 1 \end{pmatrix}. \hspace{1cm} (75)$$

Note that this transformation does not diagonalize the full potential \( U + \frac{1}{2}/r^2 \) including the centrifugal barrier, which for \( r \to 0 \) may be neglected. Thus, in the short distance limit we may decouple all our equations into pairs, and in particular we can apply the transformation to the boundary condition, Eq. (8) at zero energy

$$\mathbf{M} \mathbf{l}_0(R) \mathbf{M}^{-1} = \text{diag}(l_1(R), l_2(R)), \hspace{1cm} (76)$$

where \( l_1(R) \) and \( l_2(R) \) are the logarithmic derivatives at zero energy of the decoupled problem with potentials \( U_1 = U_C - 4U_T \) and \( U_2 = U_C + 2U_T \) respectively. After straightforward algebra we get

$$\alpha_0(R) = \frac{3Rl_2(R)(Rl_1(R) + 1) - 2}{4l_2(R) + l_1(R)(3Rl_2(R) + 2)}, \hspace{1cm} (77)$$

$$\alpha_{02}(R) = -\frac{\sqrt{2}R^3}{3} \frac{l_1(R) - l_2(R)}{4l_2(R) + l_1(R)(3Rl_2(R) + 2)}, \hspace{1cm} (78)$$

$$\alpha_2(R) = \frac{R^5}{15} \frac{l_1(R)(Rl_2(R) - 1) - 2l_2(R)}{4l_2(R) + l_1(R)(3Rl_2(R) + 2)}. \hspace{1cm} (79)$$

Now, as we approach the origin the tensor potential dominates, and the potential \( U_1 \) and \( U_2 \) behave as repulsive and attractive \( 1/r^3 \) potentials respectively, corresponding to take \( l_1 \to \infty \) and \( l_2(R) \) by the zero energy limit of the logarithmic derivative of a WKB function,

$$\alpha_0(R) \to R \frac{3Rl_2(R)}{3Rl_2(R) + 2}, \hspace{1cm} (80)$$

$$\alpha_{02}(R) \to -\frac{\sqrt{2}R^3}{3} \frac{1}{3Rl_2(R) + 2}, \hspace{1cm} (81)$$

$$\alpha_2(R) \to \frac{R^5}{15} \frac{Rl_2(R)}{3Rl_2(R) + 2}, \hspace{1cm} (82)$$

with

$$Rl_{WKB}(R) = \frac{3}{4} + \frac{1}{2} \sqrt{\frac{R_M}{R}} \cot \left( \Delta + \sqrt{\frac{R_M}{R}} \cot \left( \Delta \right) \right). \hspace{1cm} (83)$$

Here \( \Delta \) is an energy independent phase, and \( R_0 \) a reference point, given by

$$R_0\ell_{WKB}(R_0) = \frac{3}{4} + \frac{1}{2} \sqrt{\frac{R_M}{R_0}} \cot \left( \Delta \right), \hspace{1cm} (84)$$

and

$$R_M = 3g^2M \frac{2}{f^2\pi} = 16\text{fm}. \hspace{1cm} (85)$$

As we see the scattering lengths \( \alpha_0, \alpha_{02} \) and \( \alpha_2 \) present on oscillatory behaviour as we approach the origin, so they do not converge to a well defined value; as we approach to the origin the \( \alpha \)'s take all possible values. This situation corresponds to a limit cycle at short distances.

A way of avoiding the unbound variation of the scattering lengths consists of going to the origin stepwise through some envelope subsequence defined by a fixed condition for \( l_{WKB}(R) \). For instance, if we define a cycle by the condition \( \alpha(R_n) = 0 \), we have \( R_n l_{WKB}(R_n) = 0 \), yielding

$$\alpha_0(R_n) = 0$$

$$\alpha_{02}(R_n) = -\frac{\sqrt{2}R^3}{6}$$

$$\alpha_2(R_n) = -\frac{R_n^5}{30}$$

Another possibility would to take \( l_{WKB}(R_n) = \infty \) in which case one has

$$\alpha_0(R_n) = R_n$$

$$\alpha_{02}(R_n) = 0$$

$$\alpha_2(R_n) = \frac{R_n^5}{30}$$

As we see, there are infinitely many such possibilities, although all of them go towards the trivial values, \( \alpha_0(0^+) = \alpha_{02}(0^+) = \alpha_2(0^+) = 0 \). Actually, any of the choices corresponds to a different starting condition at infinity, modulo a cycle. Conversely, if we go to very short distances, where the scattering lengths vary wildly; any tiny perturbation there results in a completely different value at infinity. So, we see again that an extreme fine tuning of the threshold parameters at short distances is required. Finally, let us mention that close to the origin the sequence of cycles can be determined by the solution of the equation

$$\frac{3}{2} + x_n \cot(\Delta + x_n) = 0 \hspace{1cm} x_n = \sqrt{\frac{R_M}{R_n}} \hspace{1cm} (86)$$

in the limit \( x_n \to 0 \).

In practical numerical calculations the finite integration step \( \Delta R \) provides a given resolution scale, and these infinite limit cycles may not observed due to the rapid oscillations. Instead, one sees the envelope corresponding to the stationary points of the scattering lengths. This point will become clear below, Sect. [VI]

VI. NON-PERTURBATIVE SOLUTIONS

A. Evolution of the Low energy parameters

The exact mathematical analysis of the general set of Equations is rather complicated since we are dealing with a non-linear system of equations. In Ref. [55] simple cases
are analyzed analytically and the general features which can be deduced there are consistent with the numerical results we have obtained in the present work.

As we have said the set of equations, Eq. (51) and Eq. (52) can be numerically solved. Given the fact that as we approach the origin the tensor part of the potential develops a singularity it is important to carefully check for numerical accuracy at short distances. A crucial property which must be fulfilled by any algorithm is that of exact reversibility; i.e. evolving upwards or downwards should be inverse operations of each other. This is an stringent test and, moreover, the only way to make sure that when the long range piece of the potential is switched on for the K-matrix integration we have consistency with the effective range expansion up to the relevant order (see also below). We prefer to impose this reversibility exactly, independently on the number of mesh points used in the integration, so that any numerical irreversibility is merely attributable to computer arithmetic round-off errors. This feature will prove extremely relevant when computing the phase shifts below since our calculation requires upwards integration from lower distances. In all calculations presented in this paper we have checked that the correct threshold behaviour is obtained.

Quite generally, we find stable results when we take the long distance cut-off to be $R_\infty = 20\text{fm}$. On the other hand the lowest radius we can achieve numerically and preserving reversibility is $R_S = 0.1\text{fm}$, mainly due to computer arithmetic round-off errors triggered by the singularity of the potential. One could further lower the radius by a semi-classical approximation as outlined in Sect. V since as the origin is approached the wave function undergoes an increasing number of oscillations and WKB methods can be applied. Nevertheless, as we will see below, for our short distance cut-off the phase shifts for CM momenta up to $k = 250\text{MeV}$ are rather stable numerically.

The strong dependence of the low energy threshold parameters on the short distance cut-off provides a clue to the fact that there seems to be a lower finite limit for the boundary radius $R_S = R_{\text{min}} \sim 1.4\text{fm}$ with still an acceptable fit; if the boundary radius is lowered, the parameters encoding the short distance boundary condition which are used as fitting parameters depend in a non-smooth way on $R_S$. In addition, the strong singularity at the origin triggers a fine tuning in those parameters. According to our previous discussion, this short distance fine tuning of low energy parameters is absolutely necessary to comply with the independence of the scattering amplitude on the short distance boundary radius. For such a situation, a fit based on successive adiabatic changes of $R_S$ becomes impractical since the fitting parameters do not change adiabatically and also because these parameters should have to be determined to extraordinary high precision. In addition, the way how the limit $R_S \to 0$ should be taken differs from channel to channel. Our method provides a practical way to overcome the difficulty, given the fact that the boundary radius is taken exactly to zero along the renormalization trajectories while keeping the low energy threshold parameters at fixed values.

1. $^1S_0$ and $^3S_1$-without mixing channels

In Fig. 3 we show our results for the evolution of the threshold parameters $\alpha_0$, $r_0$ and $v_2$ in the singlet $^1S_0$ and triplet $^3S_1$-without mixing (i.e. neglecting the tensor force) channels. The main difference one can appreciate from the comparison of both channels is that while the scattering length for the $^1S_0$ channel exhibits a monotonic trend towards the origin, the scattering length in the $^3S_1$ channel diverges at a distance of about $0.7\text{fm}$. The interpretation of this fact in our framework is clear: the central part of the OPE potential is purely attractive. Thus, by eliminating the pions down to a certain distance, we are effectively building some repulsion, until we lose a bound state. An alternative interpretation is that as we switch on the OPE potential from the origin up to certain distance we can accommodate a bound state above $0.7\text{fm}$. With this interpretation in mind, we should add $180^\circ$ to the $^3S_1$ phase shift to comply with Levinson’s theorem.

2. $^3S_1-^3D_1$ channel

We finally analyze the triplet $^3S_1-^3D_1$-channel taking into account the tensor force. In Fig. 3 we show our numerical solutions of the set of Eqs. (53). Starting at sufficiently long distances (in practice $R_\infty = 20\text{fm}$ turns out to be adequate) and evolve downwards to the origin. Operationally this corresponds to eliminate OPE in the triplet channel. As can be clearly seen for distances above $R \sim 3\text{fm}$ nothing dramatic happens and a monotonic trend is observed. At smaller distances $\sim 2\text{fm}$, however, we note a rapid change in the running scattering lengths. Again, a rather flat evolution follows until the region below $1\text{fm}$. A zoomed picture is plotted in the lower panel for distances shorter than $1\text{fm}$ where the cyclic structure of evolution becomes evident, as expected from our analysis in Sect. V. The number of cycles increases without any bound as the origin is approached. This situation is dramatically different from that found in the case without tensor mixing, since there OPE produced an ultraviolet fixed point. The situation we encounter here is not new and has already been described in the context of non-coupled channels. The limit cycle structure naturally raises the problem of undefined values of the short distance parameters as we take the limit $R \to 0$. The point is that there is a way of taking the limit through equivalent points defined by the property $\alpha(R_n) = \alpha(R_{n+1})$; any two such points produce identical low energy parameters at infinity. Thus, the limit $R_n \to \infty$ through equivalent points produces the same parameters at long distances. The cycles in $\alpha_0$ and $\alpha_2$...
are hardly seen in the plot due to a low resolution $\Delta R$ compared with the typical cycle spacing.

B. Phase-Shifts

The standard way of proceeding would be to determine the low energy constants or, equivalently, the short distance parameters directly from a fit to the data in a large energy range (say up to $k \sim m_\pi$ where the two pion exchange left cut opens up should start contributing) for the theory with OPE. The low energy parameters would have to be recomputed, and the description at lower energies ($k < m_\pi$) might become even worse than a pure effective range expansion (see e.g. Refs. [39, 40]). Obviously, this is an undesirable situation. The effective range expansion is convergent up to the OPE left cut, located at $k = \pm im_\pi/2$ and should be applied only there. Our formalism can be specifically constructed to avoid such a situation. Once the threshold parameters are determined in the short distance limit $R_S \to 0$, our phase shifts become pure predictions without any additional parameter fitting obtained to a given order $k^2$ expansion of the initial condition by integrating Eq. (48) using the effective range type of initial condition,

$$\mathbf{K}_S = \mathbf{K}(R_S) = -a_S^{-1} + \frac{1}{2} r_S k^2 + v_S k^4 + \ldots \quad (87)$$

with $R_S \to 0$. The solution of Eq. (48) at $R \to \infty$ gives a solution which when expanded in powers of $k^2$ exactly reproduces ERE to the order imposed by the initial condition, Eq. (45). Thus, the difference beyond the displayed terms is merely attributable to the OPE potential.

In what follows we use LO, NLO, NNLO, etc. to denote keeping up to the first, second, third order terms in Eq. (87) respectively.

1. $^1S_0$ and $^3S_1$-without mixing channels

In Fig. 5 we show the results for the phase shifts for both $^1S_0$ and $^3S_1$-without mixing channels depending on the number of terms kept in the low energy expansion at short distances. Our results exhibit a good convergence rate. For comparison we also depict the effective range expansion results without explicit pions, which is expected to work at low energies only. As we see, the effect of introducing pions always improves the results. This can be fully appreciated at NNLO, where ERE does a poor job above CM momenta $\sim 100\text{MeV}$, but explicit OPE effects enlarge the energy range up to about $\sim 140\text{MeV} \sim m_\pi$, where we expect explicit two pion exchange contributions to start playing a role.

2. $^3S_1 - 3D_1$ channel

Once the short distance evolution of the low energy parameters are known one may compute the phase shifts to any order of the approximation in a $k^2$ expansion of the initial condition without any additional parameter fitting by integrating Eq. (48) upwards with a suitable initial condition at a short distance radius. As a matter of fact the practical choice of the radius in the numerical calculation is far from obvious, particularly in the triplet channel case where the low energy parameters take unbounded values in an increasingly finer scale at short distances (see e.g. Fig. 4). It is most practical to use the WKB approximation to match the numerical solution at a radius $R_{WKB}$ which can safely be taken in the range $\sim 0.5\text{fm}$. The results for LO (contact terms), NLO ($k^2$ terms) and NNLO ($k^4$ terms) are presented in Fig. 6 and compared to the partial wave analysis of Ref. [32]. As we see the best scheme to take into account the OPE potential corresponds to use the NLO initial condition. This means on the one hand that while the scattering lengths may be considered large and comparable to the effective ranges the curvature parameters $v_2$ can be considered to be small.

C. Finite cut-off effects

Finite short distance cut-off effects in the scattering phase shifts can be seen in Fig. 8 for finite radii $R_S = 1.4\text{fm}$ and $R_S = 1.8\text{fm}$ as compared to the renormalized $R_S = 0$ case, for the OPE-LO, OPE-NLO and OPE-NNLO approximations. As one naively expects these finite effects increase for larger energies, since they probe smaller wavelengths. A very important feature which can be deduced from the plots is that these effects are sizable for momenta where TPE effects should not play a decisive role $m_\pi/2 < k < m_\pi$ role. Thus, letting a finite short distance boundary radius $R_S \sim 1.4\text{fm}$ provides a large systematic error, already in the region where OPE dominates. Thus, it is not clear whether TPE can be seen in the central NN waves with a finite cut-off distance of about $R_c = 1.4\text{fm}$. Of course, one should include TPE contributions in order to make a definite statement.

D. Are pions perturbative?

The discussion of which power counting is the appropriate one for the NN interaction corresponds physically
to the question whether or not the pion cloud can be considered to be perturbative. It is important to realize that within our framework we are considering OPE departures from the effective range expansion to a given order. Thus, at sufficiently low $k$ explicit pion effects can always be considered perturbative. This is so regardless of the number of $k^2$ terms included in the initial condition. Actually, the point is rather if the low energy threshold parameters can be considered large or small. According to our results in Fig. (6) it seems that best possible agreement can be obtained when both the scattering lengths and the effective ranges are taken to be large, while other low energy parameters can be taken to be small. To properly emphasize this point we plot in Fig. (7) the scaled K matrix computed including OPE and compared to the ERE to LO, NLO and NNLO. Given this fact we expect a kind of consistent long distance perturbation theory to work. The details of such an expansion will be presented elsewhere.

E. Evolution of the short distance boundary condition

As we have said, the short distance singularity of the OPE potential enforces a very precise determination of the running low energy threshold parameters at short distances, and hence of the boundary condition. We can directly determine this dependence by using Eq. (62) and Eq. (65). For simplicity and to illustrate the point we just display in Fig. (9) the behaviour of the boundary condition parameters as a function of the short distances boundary radius in the zero energy limit, both for the singlet $^1S_0$ and triplet $^3S_1$ channel without mixing and for the triplet $^3S_1 - ^3D_1$ channel. The fixed point and limit cycle behaviour obtained for the running of the low energy threshold parameters maps into a similar behaviour for the short distance boundary condition. From the picture it is clear that the standard procedure of integrating the Schrödinger equation upwards from a given short distance boundary radius to infinity in order to fit the low energy parameters would require a very high precision determination of a rapidly varying boundary condition in the case of the triplet $^3S_1 - ^3D_1$ channel. It is clear that a determination of the $C_0$ constants from a fit to the phase shifts in the low energy region would be extremely delicate in the limit $R_c \to 0$ in practice. Instead, the present approach computes directly the boundary condition in a power expansion of the energy at any given radius from the physical values of the low energy parameters. Actually, our method is equivalent to integrate the Schrödinger equation from that short distance boundary radius to infinity. In adittion, the singular and attractive nature of the OPE potential allows a WKB treatment of the short distance singularity, and allows to eliminate the finite cut-off radius taking the limit $R_c \to 0$.

Obviously, the present framework can be extended to reanalyze the role of TPE potentials in a non perturbative way and completely free of finite cut-off artifacts, where
FIG. 4: Evolution of the $^3S_1$, $^3D_1$ and $E_1$ NN-threshold parameters from the physical values at infinity down to the origin using the OPE potential. Top panel: scattering lengths $\alpha_0(R)$ (in fm), $\alpha_{02}(R)$ (in fm$^3$) and $\alpha_2(R)$ (in fm$^5$). Bottom panel: effective ranges $v_0(R)$ (in fm), $v_{02}(R)$ (in fm$^3$) and $v_2(R)$ (in fm$^5$). In the left panel we represent a global picture from the 1fm to 20 fm. The lower panel is a detailed picture in the short distance region below 1fm. Limit cycles are clearly visible in the s-wave scattering length $\alpha_0$ and effective ranges. The $E_1$ and $^3D_1$ scattering lengths $\alpha_{02}$ and $\alpha_2$ go quickly to zero below 0.25 fm.
FIG. 5: Predicted phase shifts in the $^1S_0$ (left panel) and the $^3S_1$-without mixing (right panel) channels for NN scattering as a function of the CM momentum in MeV. In the $^3S_1$ channel we assume no mixing according to Eq. (48) when OPE potential is switched on and the initial condition is a low energy expansion of the $K$-matrix at short distances, (See Eq. (87) in the main text). LO means keeping $\alpha_{S,0}$ only, NLO keeping $\alpha_{S,0}$ and $r_{0,S}$ and NNLO keeping $\alpha_{S,0}, r_{0,S}$ and $v_{2,S}$. The short range parameters are directly determined by evolving the low energy parameters from their experimental values, Eq. 66) and Eq . 67) ER-LO, ER-NLO and ER-NNLO corresponds to a pure effective range expansion keeping $\alpha_{0}$ only, $\alpha_{S,0}$ and $r_{0}$, $\alpha_{0}$, $r_{0}$ and $v_{2}$ respectively. No further fit is involved. Data are the PWA from Ref. [32].

VII. CONCLUSIONS

In the present paper we have analyzed the renormalization of the OPE interaction in the presence of contact and derivative interactions of any order for NN scattering both for the singlet and triplet channel states. The basic point of our approach is to regularize the unknown short distance physics by means of a boundary condition at a certain boundary radius, above which the OPE potential is assumed to work, i.e. where pions are treated explicitly. Below that scale pions contribute implicitly to the scattering properties although always in combination with other effects which cannot be disentangled unless a given distance scale is specified. Actually, when the boundary radius goes to infinity, above the pion Compton wavelength, we have a low energy theory of contact interactions and derivatives there-off. As the boundary radius goes below the OPE range, we have a theory where pions are eliminated above the scale set by the boundary.

This allows to remove explicitly pion effects in the threshold parameters for the OPE potential in an unambiguous and model independent way. The renormalization group flow implied by our non-perturbative equations is unique provided the OPE potential is assumed to be valid all the way down to the origin. This is obviously not a realistic assumption but it is absolutely necessary to go to these small distances in order to get rid of any finite short-distance cut-off effect and properly define the OPE contributions to the scattering observables. This result fully complies to the spirit of an effective field theory, the terms in a low momentum expansion of the amplitude are shape independent while the remaining powers depend both on the long distance OPE details (like the left branch cut) and the shape independent low energy parameters themselves.

The short distance behaviour of threshold parameters present either a ultraviolet fixed point structure in the $^1S_0$ and $^3S_1$-without-mixing channels whereas we find limit cycles for the $^3S_1-^3D_1$ channel due to the singular and attractive behaviour of the OPE contribution to the tensor potential. This means that in the latter case there is not a monotonous trend at short distances. A direct consequence of having both ultraviolet fixed points and limit cycles for the threshold parameters is that a delicate fine tuning of the short distance physics is implied. In addition, for the experimental values of the threshold parameters one obtains huge changes for distances below

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5 Unlike the OPE where one has both an attractive and repulsive $1/r^3$ singularity (see Sect. [X]), in the TPE case one encounters attractive $1/r^6$ singularities for coupled channels.
FIG. 6: $^3S_1$, $^3D_1$ and $E_1$ at LO (contact terms), NLO ($k^2$ terms) and NNLO ($k^4$ terms) predicted phase shifts for the triplet channel. The initial condition is a low energy expansion of the $K$-matrix at short distances. By construction the low energy parameters $\alpha$, $r$ and $v$ coincide with those extracted from the NijmII potential. Data are the PWA from Ref. [32].

2fm when OPE effected are removed. Nevertheless, we find moderate changes in the phase shifts due to explicit pion effects. Actually, in the $^1S_0$ and $^3S_1$-without mixing channels the effect is found to be compatible with a perturbative treatment. In the $^3S_1-^3D_1$ channel the effect is a bit more complicated due to the presence of ultraviolet limit cycles triggered by the singular character of the tensor potential; the coupled channel amplitudes are non-perturbatively renormalizable while they become perturbatively non-renormalizable. This makes a naive perturbative treatment slightly more subtle. One of the advantages of having a renormalizable theory is that non-perturbative equations make sense, and any perturbative treatment should arise as a controllable approximation to the full equations. As we have pointed out along the paper, this is probably an advantage of using coordinate space methods and a boundary condition renormalization versus momentum space methods.

Taking into account all the nice features of the present calculation, in particular, getting a handle on the finite cut-off corrections, the results presented in this paper are very satisfactory suggesting several improvements. Explicit Two Pion Exchange contributions are expected to contribute significantly at about $1.5-2$ fm at the level
of the potential, so our results for the evolution of the threshold parameters should not be considered realistic below that scale, or equivalently above CM momenta of about $100 - 150$ MeV, as it seems to be the case. In addition, our description should be enlarged to include higher partial waves. For peripheral waves one expects perturbative methods to work since there is a strong centrifugal suppression of the wave function at the origin, and perturbative renormalization methods can be applied. For those the present approach does not have much to say. Low partial waves, however, are particularly interesting since a re-summation of pion exchanges seems crucial to understand the data. Work along these lines will be presented elsewhere [38].
FIG. 8: The effect of having a finite short distance boundary radius for the OPE potential on top of the effective range expansion (ERE). We compare the theory with finite radii $R = 1.4\text{fm}$ and $R = 1.8\text{fm}$ with the renormalized theory $R = 0$. Top panel: $^3S_1$, $^3D_1$ and $E_1$ OPE-LO. Middle panel: same but for NLO. Bottom panel: same but NNLO. In all cases the low energy threshold parameters coincide with those extracted from the NijmII potential. Data are the PWA from Ref. 32.

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APPENDIX A: ANOTHER DERIVATION OF THE VARIABLE $S$-MATRIX

In order to deduce a variable $S$-matrix equation, we determine first the infinitesimal change of the $S$ matrix under a general deformation of the potential $U(r) \rightarrow U(r) + \Delta U(r)$. Using Schrödinger’s equation and the standard Lagrange’s identity adapted to this particular
FIG. 9: Evolution of the dimensionless short distance boundary conditions at zero energy $C_0(R) = 1 - R L_0(R) = 1 - R u_0(R) - u(R)^{-1}$ with the boundary radius $R$ due to OPE potential. Top: Singlet $^1S_0$ channel (left) and Triplet $^3S_1$ channel (right) without mixing using Eq. (62). Bottom: The triplet $^3S_1 - ^3D_1$ channel. The coefficients $C^0_{ss}$, $C^0_{sd}$, and $C^0_{dd}$ are related to the running scattering lengths $\alpha_{00}$, $\alpha_{02}$, and $\alpha_{22}$ through Eq. (65). Large scale (left) and zoomed (right) picture.

In particular, for the parametric family of potentials $\bar{U}(r, R) = 2i k S(R) S'(R) = \theta(R - r) U(r)$ we get

$$2i k S(R) S'(R) = u(R)^{-1} U(R) u(R)$$  \hspace{1cm} (A3)

and using the value of the wave function at the outer boundary

$$u(R) = h^-(R) - h^+(R) S(R)$$  \hspace{1cm} (A4)

we finally get the variable S-matrix equation, Eq. (10).

Note, that the variation of the potential is done with a
fixed boundary condition.

**APPENDIX B: DIFFICULTIES IN EXTRACTING THE LOW ENERGY PARAMETERS**

In this appendix we want to elaborate on the problems we have encountered while fitting the NN data base of Ref. [32] within a generalized coupled channel effective range expansion, Eq. (35). Unfortunately, this data base does not provide error estimates for their phase shifts (although 8 significant digits), nor the typical energy resolution where these data should be trusted, so some compromise must be made.

We use the NN-data and define the $\chi^2$ as

$$\chi^2 = \sum_{i=1}^{N} \left( \frac{\hat{K}_{\text{ER}} - \hat{K}_{\text{NN}}}{\Delta K_{\text{NN}}/2} \right)^2 \frac{M}{4p}$$

where we take $\Delta E_{\text{LAB}} = 0.01\text{MeV}$, and $\hat{K}_{\text{NN}}$ and $\Delta K_{\text{NN}}$ are the mean value and the standard deviation of the six potentials listed in the NN-data base [32], which can be taken as independent uncorrelated primary data. The factor $M/(4p)$ is the Jacobian of the transformation between the Lab-energy and the C.M. momentum, $E_{\text{LAB}} = 2p^2/M$, and would correspond to make an equidistant sampling in $p$, in the limit $\Delta E_{\text{LAB}} \rightarrow 0$ (this is why we take a small energy spacing). This weight factor is introduced in order to enhance the region at low momenta. On the other hand, very low momenta, must be excluded since the resulting mean value $K$-matrix is incompatible within the attributed errors with the expected theoretical behaviour, Eq. (35). Thus we take $E_{\text{LAB}} \geq 0.5\text{MeV}$, and the fit goes up to $E_{\text{LAB}} \leq 10\text{MeV}$, which corresponds to a C.M. momentum about $p = m_{\pi}/2$ where we expect the finite polynomial of the scaled $K$-matrix to truly represent an analytical function within the convergence radius up to the branch cut singularity located at $p = \pm im_{\pi}/2$.

The form of the fitting function is

$$\hat{K}_{\text{ER}} = -\beta + r_0 p^2 + v_2 p^4 + v_3 p^6 + v_4 p^8 + \ldots$$  \hspace{1cm} (B2)

In Fig. 11 we show as an illustration the $v_2$ parameter determined from a fit to the low energy region of the NN data base [32] as a function of the maximal LAB-energy considered in the fit. As we see, instead of a plateau within some energy window, we observe an ever changing value. We observe no stability depending on the number of terms considered in Eq. (32) either. For comparison we also plot the values we obtained by integrating the Eqs. (31), (32) and (34) with the NijmII and Reid93 potentials in Sect. IV which were quite stable numerically. As we see, the values obtained from the fit, in the chosen energy window are hardly compatible. The deceptive features extend to other channels, and non diagonal low energy threshold parameters such as the matrix elements of $a$ and $r$.

Finally, we have also tried, with no success, other methods for the determination of the low energy threshold parameters, like evaluation of derivatives within several algorithms. The reason for the failure has to do with round-off errors generated by the relatively small number of digits provided in the NN database.

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[1] S. Weinberg, Phys. Lett. B 251 (1990) 288. Nucl. Phys. B 363 (1991) 3.
[2] C. Ordonez, L. Ray and U. van Kolck, Phys. Rev. C 53, 2086 (1996) Phys. Rev. Lett. 72, 1982 (1994). C. Ordonez and U. van Kolck, Phys. Lett. B 291, 459 (1992).
[3] T. S. Park, K. Kubodera, D. P. Min and M. Rho, Phys. Rev. C 58 (1998) 637.
[4] E. Epelbaum, W. Glockle and U. G. Meissner, Nucl. Phys. A 637 (1998) 107 E. Epelbaum, W. Glockle, A. Kruger and U. G. Meissner, Nucl. Phys. A 645 (1999) 413 E. Epelbaum, W. Glockle and U. G. Meissner, Nucl. Phys. A 671, 295 (2000).
[5] D. B. Kaplan, M. J. Savage and M. B. Wise, Nucl. Phys. B 534, 329 (1998) Phys. Lett. B 424, 390 (1998). Nucl. Phys. B 478, 629 (1996).
[6] J. Gegelia, Contributed to Workshop on Methods of Nonperturbative Quantum Field Theory, Adelaide, Australia, 2-13 Feb. 1998. In *Adelaide 1998, Nonperturbative methods in quantum field theory* 30-35. Phys. Lett. B 429, 227 (1998). J. Gegelia, Phys. Lett. B 463 (1999) 133.
[7] S. Fleming, T. Mehen and I. W. Stewart, Nucl. Phys. A 677, 313 (2000) S. Fleming, T. Mehen and I. W. Stewart, Phys. Rev. C 61, 044005 (2000).
[8] M. C. Birse, J. A. McGovern and K. G. Richardson, Phys. Lett. B 464, 169 (1999).
[9] T. Barford and M. C. Birse, AIP Conf. Proc. 603, 229 (2001), [hep-ph/0206146](https://arxiv.org/abs/hep-ph/0206146).
[10] D. B. Kaplan, M. J. Savage and M. B. Wise, Nucl. Phys. B 478, 629 (1996).
[11] S. R. Beane, P. F. Bedaque, M. J. Savage and U. van Kolck, Nucl. Phys. A 700 (2002) 377.
[12] P. F. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. 52, 339 (2002).
[13] J. Gasser, M. E. Sainio and A. Svarc, Nucl. Phys. B 307, 779 (1988).
[14] T. Becher and H. Leutwyler, Eur. Phys. J. C 9, 643 (1999), [arXiv:hep-ph/9901354](https://arxiv.org/abs/hep-ph/9901354).
[15] E. Jenkins and A. V. Manohar, Phys. Lett. B 255, 558 (1991).
[16] D. R. Phillips, S. R. Beane and T. D. Cohen, Nucl. Phys. A 631 (1998) 447C.
[17] T. Mehen and I. W. Stewart, Phys. Rev. C 59, 2365 (1999)
[18] T. Mehen and I. W. Stewart, Phys. Lett. B 445, 378 (1999).
[19] T. Frederico, V. S. Timoteo and L. Tomio, Nucl. Phys. A 653 (1999) 200.
FIG. 10: The $v_2$ parameter for the $^1S_0$ (left) and the $^3S_1$ (right) channels determined from a fit to the low energy data of the NN database \cite{32} (see Eq. (B1) and main text, as a function of the maximal LAB-energy considered in the fit. $p^n$ means a fit including up to $p^n$ terms in the effective range expansion Eq. \bibitem{12}. “Database” means a fit to the average value of the corresponding scaled $K$-matrix. “Reid93” and “NijmII” means a fit to only this data. The values we obtained by integrating the Eqs. \bibitem{13}, \bibitem{14} and \bibitem{15} with the NijmII and Reid93 potentials in Sect. \ref{sec:potential_scattering}.\bibitem{20} J. Gegelia and G. Japaridze, Phys. Lett. B 517 (2001) 476\bibitem{21} D. Eiras and J. Soto, Eur. Phys. J. A 17, 89 (2003)\bibitem{22} J. A. Oller,\bibitem{23} D. B. Kaplan, Nucl. Phys. B 494, 471 (1997)\bibitem{24} J. Nieves, Phys. Lett. B 568, 109 (2003)\bibitem{25} E. Epelbaum, W. Gloeckle and U. G. Meissner, Eur. Phys. J. A 19, 401 (2004)\bibitem{26} J. F. Yang, arXiv:nucl-th/0310048\bibitem{27} J. Gegelia and S. Scherer, arXiv:nucl-th/0403052\bibitem{28} E. Lomon and H. Fesbach Rev. Mod. Phys. 39, 611 (1967)\bibitem{29} D. R. Phillips and T. D. Cohen, Phys. Lett. B 390 (1997) 7\bibitem{30} T. D. Cohen and J. M. Hansen, Phys. Lett. B 440, 233 (1998)\bibitem{31} U. van Kolck, Nucl. Phys. A 645, 273 (1999)\bibitem{32} V. G. Stoks, R. A. Kompl, M. C. Rentmeester and J. J. de Swart, Phys. Rev. C 48, 792 (1993). \url{http://www.nn-online.sci.kun.nl}\bibitem{33} M. C. M. Rentmeester, R. G. E. Timmermans, J. L. Friar and J. J. de Swart, Phys. Rev. Lett. 82 (1999) 4992\bibitem{34} M. Pavon Valderrama and E. Ruiz Arriola, Phys. Lett. B 580 (2004) 149\bibitem{35} F. Calogero “Variable Phase Space Approach to Potential Scattering” Academic Press, New York (1967).\bibitem{36} S. R. Beane, P. F. Bedaque, L. Childress, A. Kryjevski, J. McGuire and U. v. Kolck, Phys. Rev. A 64 (2001) 042103\bibitem{37} S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and H. Holden, \textit{Solvable Models in Quantum Mechanics} Texts and Monographs in Physics; Springer 1988.\bibitem{38} M. Pavón Valderrama and E. Ruiz Arriola (in preparation).\bibitem{39} T. D. Cohen and J. M. Hansen, Phys. Rev. C 59, 13 (1999)\bibitem{40} T. D. Cohen and J. M. Hansen, Phys. Rev. C 59, 3047 (1999)\bibitem{41} D. R. Entem and R. Machleidt, Phys. Lett. B 524, 93 (2002)\bibitem{42} D. B. Kaplan, M. J. Savage and M. B. Wise, Phys. Rev. C 59, 617 (1999)\bibitem{43} D. R. Phillips, G. Rupak and M. J. Savage, Phys. Lett. B 473, 209 (2000)\bibitem{44} J. V. Steele and R. J. Furnstahl, Nucl. Phys. A 645, 439 (1999)\bibitem{45} T. Mehen and I. W. Stewart, Nucl. Phys. A 665, 164 (2000)