Explicit generators in rectangular affine \( W \)-algebras of type \( A \)

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Abstract

We produce in an explicit form free generators of the affine \( W \)-algebra of type \( A \) associated with a nilpotent matrix whose Jordan blocks are of the same size. This includes the principal nilpotent case and we thus recover the quantum Miura transformation of Fateev and Lukyanov.

1 Main results

Let \( \mathfrak{g} \) be a reductive Lie algebra over \( \mathbb{C} \) equipped with a symmetric invariant bilinear form \( \kappa \) and let \( f \) be a nilpotent element of \( \mathfrak{g} \). The corresponding affine \( W \)-algebra \( W^{\kappa}(\mathfrak{g}, f) \) is defined by the generalized quantized Drinfeld–Sokolov reduction; see [5], [7] and [8].

In this note we take \( \mathfrak{g} = \mathfrak{gl}_N \). The Jordan type of a nilpotent element \( f \in \mathfrak{gl}_N \) is a partition of \( N \). We will work with the elements \( f \) corresponding to partitions of the form \((l^n)\) so that the associated Young diagram is the \( n \times l \) rectangle with \( nl = N \). Our main result is an explicit construction of free generators of the \( W \)-algebra \( W^{\kappa}(\mathfrak{g}, f) \). Moreover, we calculate the images of these generators with respect to the Miura transformation. In particular, if \( f \) is the principal nilpotent (i.e., \( n = 1 \)) we thus reproduce the description of the \( W \)-algebra due to Fateev and Lukyanov [4]. The results can be regarded as ‘affine analogues’ of the construction of the corresponding finite \( W \)-algebras originated in [2], [10] and extended to arbitrary nilpotent elements \( f \) in [3].

To describe the results in more detail, identify \( \mathfrak{g} \) with the tensor product of \( \mathfrak{gl}_l \) and \( \mathfrak{gl}_n \) via the isomorphism \( \mathfrak{gl}_l \otimes \mathfrak{gl}_n \to \mathfrak{g} \) defined by

\[
ed_{ij} \otimes e_{rs} \mapsto e_{(i-1)n+r,(j-1)n+s}, \tag{1.1}\]

where the \( e_{ij} \) denote the standard basis elements of the corresponding general linear Lie algebras. Set

\[
f_l = \sum_{i=1}^{l-1} e_{i+1,i} \in \mathfrak{gl}_l
\]
and
\[ f = f_1 \otimes I_n = \sum_{i=1}^{t-1} \sum_{j=1}^n e_{i(n+j)(i-1)n+j} \in \mathfrak{g}, \]

where \( I_n \in \mathfrak{gl}_n \) is the identity matrix. The matrix \( f \) is a nilpotent element of \( \mathfrak{g} \) of Jordan type \((l^n)\). Let
\[ \mathfrak{gl}_l = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p \]
be the standard principal grading of \( \mathfrak{gl}_l \), obtained by defining the degree of \( e_{ij} \) to be equal to \( j - i \). Set
\[ \mathfrak{gl}_{l,\leq 0} = \bigoplus_{p \leq 0} \mathfrak{g}_p \quad \text{and} \quad \mathfrak{gl}_{l,< 0} = \bigoplus_{p < 0} \mathfrak{g}_p. \]
The isomorphism (1.1) then induces the \( \mathbb{Z} \)-grading on \( \mathfrak{g} \),
\[ \mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p, \quad \mathfrak{g}_p = (\mathfrak{gl}_l)_p \otimes \mathfrak{gl}_n, \]
which is a good grading for \( f \) in the sense of [7]. We also set
\[ \mathfrak{b} = \bigoplus_{p \leq 0} \mathfrak{g}_p = \mathfrak{gl}_{l,\leq 0} \otimes \mathfrak{gl}_n \quad \text{and} \quad \mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p = \mathfrak{gl}_{l,< 0} \otimes \mathfrak{gl}_n. \tag{1.2} \]

For any \( k \in \mathbb{C} \), we let \( \kappa \) be any symmetric invariant bilinear form on \( \mathfrak{g} \) such that
\[ \kappa(x,y) = k \operatorname{tr}(xy) \quad \text{for} \quad x, y \in \mathfrak{sl}_N \subset \mathfrak{gl}_N. \tag{1.3} \]
For elements \( x, y \in \mathfrak{b} \) set
\[ \kappa_b(x,y) = \kappa(x,y) + \frac{1}{2} \operatorname{tr}_g(\operatorname{ad} x \operatorname{ad} y) - \frac{1}{2} \operatorname{tr}_{g_0} p_0(\operatorname{ad} x \operatorname{ad} y), \]
where \( p_0 \) denotes the restriction of the operator to \( \mathfrak{g}_0 \). Then \( \kappa_b \) defines a symmetric invariant bilinear form on \( \mathfrak{b} \).

Example 1.1. Let
\[ \kappa(x,y) = \frac{k}{2N} \operatorname{tr}_g(\operatorname{ad} x \operatorname{ad} y) = k \left( \operatorname{tr}(xy) - \frac{1}{N} \operatorname{tr}(x) \operatorname{tr}(y) \right), \quad x, y \in \mathfrak{g}. \]
Then for \( i \geq i' \) and \( j \geq j' \) we have
\[ \kappa_b(e_{ii'}, e_{ppq}, e_{jj'}, e_{rs}) \]
\[ = \delta_{ii'} \delta_{jj'} \left( (k + nl) \left( \delta_{ij} \delta_{ps} \delta_{qr} - \frac{1}{nl} \delta_{pq} \delta_{rs} \right) - n \delta_{ij} \left( \delta_{ps} \delta_{qr} - \frac{1}{n} \delta_{pq} \delta_{rs} \right) \right) \]
with \( N = nl \), as before. \qed
Let \( \hat{b} = b[t, t^{-1}] \oplus \mathbb{C} 1 \) be the Kac–Moody affinization of \( b \) with respect to the cocycle \( \kappa_b \), and let \( V^{\kappa_b}(b) \) be the universal affine vertex algebra associated with \( b \) and \( \kappa_b \) [6]:

\[
V^{\kappa_b}(b) = U(\hat{b}) \otimes_{U(b[t] \oplus \mathbb{C} 1)} \mathbb{C},
\]

where \( \mathbb{C} \) is regarded as the one-dimensional representation of \( b[t] \oplus \mathbb{C} 1 \) on which \( b[t] \) acts trivially and \( 1 \) acts as 1. Note that by the Poincaré–Birkhoff–Witt theorem, \( V^{\kappa_b}(b) \) is isomorphic to \( U(b[1^{-1}, 1]) \) as a \( \mathbb{C} \)-vector space.

Due to [8, 9], the \( \mathcal{W} \)-algebra \( \mathcal{W}^\kappa(g, f) \) can be realized as a vertex subalgebra of \( V^{\kappa_b}(b) \). Our aim is to give explicit description of the generators of \( \mathcal{W}^\kappa(g, f) \) inside \( V^{\kappa_b}(b) \). We will use the identification

\[
\mathfrak{gl}_{t, \leq 0}[t^{-1}]t^{-1} \otimes \mathfrak{gl}_n \cong b[t^{-1}]t^{-1},
\]

defined by

\[
e_{ij}[-m] \otimes e_{pq} \mapsto e_{(j-1)n+p, (i-1)n+q}[-m], \quad m \geq 1,
\]

for \( 1 \leq i \leq j \leq l \) and \( 1 \leq p, q \leq n \), where we write \( x[r] = x t^r \) for any \( r \in \mathbb{Z} \).

By analogy with [3, Sec. 12], consider the tensor algebra \( T(\mathfrak{gl}_{t, \leq 0}[t^{-1}]t^{-1}) \) of the vector space \( \mathfrak{gl}_{t, \leq 0}[t^{-1}]t^{-1} \) and let \( M_n \) denote the matrix algebra with the basis formed by the matrix units \( e_{ij} \), \( 1 \leq i, j \leq n \). Define the algebra homomorphism

\[
\mathcal{T} : T(\mathfrak{gl}_{t, \leq 0}[t^{-1}]t^{-1}) \to M_n \otimes U(b[t^{-1}]t^{-1}), \quad x \mapsto \mathcal{T}(x) = \sum_{i,j=1}^n e_{ij} \otimes \mathcal{T}_{ij}(x)
\]

by setting

\[
\mathcal{T}_{ij}(x) = x \otimes e_{ij} \in \mathfrak{gl}_{t, \leq 0}[t^{-1}]t^{-1} \otimes \mathfrak{gl}_n = b[t^{-1}]t^{-1}
\]

for \( x \in \mathfrak{gl}_{t, \leq 0}[t^{-1}]t^{-1} \). By definition, for any \( x, y \in T(\mathfrak{gl}_{t, \leq 0}[t^{-1}]t^{-1}) \) we have

\[
\mathcal{T}_{ij}(xy) = \sum_{r=1}^n \mathcal{T}_{ir}(x) \mathcal{T}_{rj}(y) = \sum_{r=1}^n (x \otimes e_{ri})(y \otimes e_{jr}).
\]

Let us equip the tensor product space \( T(\mathfrak{gl}_{t, \leq 0}[t^{-1}]t^{-1}) \otimes \mathbb{C} [\tau] \) with an associative algebra structure in such a way that the natural embeddings

\[
T(\mathfrak{gl}_{t, \leq 0}[t^{-1}]t^{-1}) \hookrightarrow T(\mathfrak{gl}_{t, \leq 0}[t^{-1}]t^{-1}) \otimes \mathbb{C} [\tau] \quad \text{and} \quad \mathbb{C} [\tau] \hookrightarrow T(\mathfrak{gl}_{t, \leq 0}[t^{-1}]t^{-1}) \otimes \mathbb{C} [\tau]
\]

are algebra homomorphisms and the generator \( \tau \) satisfies the relations

\[
[\tau, x[-m]] = mx[-m-1] \quad \text{for} \quad x \in \mathfrak{gl}_{t, \leq 0} \quad \text{and} \quad m \in \mathbb{Z}.
\]
Furthermore, the tensor product space \( U(b[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau] \) will also be considered as an associative algebra in a similar way. We will extend \( \mathcal{T} \) to the algebra homomorphism

\[
\mathcal{T} : T(g_{t,\leq 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau] \to M_n \otimes U(b[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]
\]

by setting \( \mathcal{T}_{ij}(uS) = \mathcal{T}_{ij}(u) \) for \( u \in T(g_{t,\leq 0}[t^{-1}]t^{-1}) \) and any polynomial \( S \in \mathbb{C}[\tau] \).

Set \( \alpha = k + n(l - 1) \) and consider the matrix

\[
B = \begin{bmatrix}
\alpha \tau + e_{11}[-1] & -1 & 0 & \ldots & 0 \\
e_{21}[-1] & \alpha \tau + e_{22}[-1] & -1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
e_{l-11}[-1] & e_{l-22}[-1] & \ldots & \alpha \tau + e_{l-1l-1}[-1] & -1 \\
e_{l1}[-1] & e_{l2}[-1] & \ldots & \ldots & \alpha \tau + e_{ll}[-1]
\end{bmatrix}
\]

with entries in \( T(g_{t,\leq 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau] \). Its column-determinant \( \text{cdet} B \) is defined as the usual alternating sum of the products of the entries taken in the order determined by the column numbers of the entries\(^1\). So \( \text{cdet} B \) is an element of \( T(g_{t,\leq 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau] \) and we can write

\[
\mathcal{T}_{ij}(\text{cdet} B) = \sum_{r=0}^{i} W_{ij}^{(r)}(\alpha \tau)^{i-r}
\]

for certain coefficients \( W_{ij}^{(r)} \) which are elements of \( U(b[t^{-1}]t^{-1}) \), and we can also regard them as elements of \( V^\kappa b(b) \). The following is our main result.

**Theorem 1.2.** All coefficients \( W_{ij}^{(r)} \) belong to the \( \mathcal{W} \)-algebra \( \mathcal{W}^\kappa(g, f) \). Moreover, the \( \mathcal{W} \)-algebra \( \mathcal{W}^\kappa(g, f) \subset V^\kappa b(b) \) is freely generated by the elements \( W_{ij}^{(r)} \) with \( 1 \leq i, j \leq n \) and \( r = 1, 2, \ldots, l \).

Set \( \mathfrak{l} = (gl_i)_0 \otimes gl_n \subset gl_N \). Then the projection \( b \to \mathfrak{l} \) induces the vertex algebra homomorphism \( V^\kappa b(b) \to V^\kappa(b)(\mathfrak{l}) \), which restricts to the map

\[
\nu : \mathcal{W}^\kappa(g, f) \to V^\kappa(b)(\mathfrak{l}),
\]

called the (quantum) Miura transformation. This is an injective vertex algebra homomorphism. The following formula for the images of the elements \( W_{ij}^{(r)} \) under the Miura transformation is an immediate consequence of Theorem 1.2.

**Theorem 1.3.** We have

\[
\sum_{r=0}^{l} \nu(W_{ij}^{(r)})(\alpha \tau)^{l-r} = \mathcal{T}_{ij} \left( (\alpha \tau + e_{11}[-1]) \ldots (\alpha \tau + e_{ll}[-1]) \right).
\]

\(^1\)It is easy to verify that \( \text{cdet} B \) coincides with the row-determinant of \( B \) defined in a similar way.
Note that the principal \( W \)-algebra of type \( A \) corresponds to the case \( n = 1 \) (and \( N = l \)). The elements \( W^{(r)} \) are defined via the expansion of \( \text{cdet} \, B \),

\[
\text{cdet} \, B = \sum_{r=0}^{l} W^{(r)} (\alpha \tau)^{l-r}.
\]

By applying the Miura transformation we recover the formula of Fateev and Lukyanov \[4\].

**Corollary 1.4.** The principal \( W \)-algebra \( W^{(g,f)} \) is freely generated by the elements \( W^{(1)}, \ldots, W^{(l)} \). Moreover, we have

\[
\sum_{r=0}^{l} \nu(W^{(r)}) (\alpha \tau)^{l-r} = (\alpha \tau + e_{11}[-1]) \ldots (\alpha \tau + e_{ll}[-1]).
\]

**Example 1.5.** Take \( n = l = 2 \) so that \( N = 4 \). We have

\[
\text{cdet} \, B = (\alpha \tau)^2 + (e_{11}[-1] + e_{22}[-1])(\alpha \tau) + e_{11}[-1]e_{22}[-1] + e_{21}[-1] + \alpha e_{22}[-2].
\]

Hence

\[
\begin{align*}
W^{(1)}_{11} &= e_{11}[-1] + e_{33}[-1], & W^{(1)}_{22} &= e_{22}[-1] + e_{44}[-1], \\
W^{(1)}_{21} &= e_{12}[-1] + e_{34}[-1], & W^{(1)}_{12} &= e_{21}[-1] + e_{43}[-1], \\
W^{(2)}_{11} &= e_{11}[-1]e_{33}[-1] + e_{21}[-1]e_{34}[-1] + e_{31}[-1] + \alpha e_{33}[-2], \\
W^{(2)}_{22} &= e_{12}[-1]e_{34}[-1] + e_{22}[-1]e_{44}[-1] + e_{42}[-1] + \alpha e_{44}[-2], \\
W^{(2)}_{21} &= e_{12}[-1]e_{33}[-1] + e_{22}[-1]e_{34}[-1] + e_{32}[-1] + \alpha e_{34}[-2], \\
W^{(2)}_{12} &= e_{11}[-1]e_{43}[-1] + e_{21}[-1]e_{44}[-1] + e_{41}[-1] + \alpha e_{43}[-2].
\end{align*}
\]

For the images under the Miura transformation we have

\[
\begin{align*}
\nu(W^{(1)}_{11}) &= e_{11}[-1] + e_{33}[-1], & \nu(W^{(1)}_{22}) &= e_{22}[-1] + e_{44}[-1], \\
\nu(W^{(1)}_{21}) &= e_{12}[-1] + e_{34}[-1], & \nu(W^{(1)}_{12}) &= e_{21}[-1] + e_{43}[-1], \\
\nu(W^{(2)}_{11}) &= e_{11}[-1]e_{33}[-1] + e_{21}[-1]e_{34}[-1] + \alpha e_{33}[-2], \\
\nu(W^{(2)}_{22}) &= e_{12}[-1]e_{34}[-1] + e_{22}[-1]e_{44}[-1] + \alpha e_{44}[-2], \\
\nu(W^{(2)}_{21}) &= e_{12}[-1]e_{33}[-1] + e_{22}[-1]e_{34}[-1] + \alpha e_{34}[-2], \\
\nu(W^{(2)}_{12}) &= e_{11}[-1]e_{43}[-1] + e_{21}[-1]e_{44}[-1] + \alpha e_{43}[-2].
\end{align*}
\]

Let the form \( \kappa_b \) be as in Example 1.1. The values \( \kappa_b(x, y) \) are then given in the following table, where the columns and rows correspond to the \( x \) and \( y \) variables, respectively:
These values can be used to calculate the operator product expansion formulas for the generators of \( W^\kappa(g, f) \). In particular, set
\[
L = \frac{1}{2(k+4)} \left( -2(W_{11}^{(2)} + W_{22}^{(2)}) + W_{12}^{(1)}W_{21}^{(1)} + \frac{3}{4}(W_{11}^{(1)}W_{11}^{(1)} + W_{22}^{(1)}W_{22}^{(1)}) - \frac{1}{2}W_{11}^{(1)}W_{22}^{(1)} - (k+2)(W_{11}^{(1)} + W_{22}^{(1)}) - (W_{11}^{(1)} - W_{22}^{(1)})' \right),
\]
where the primes indicate the action of \( \text{ad} \tau \) taking \( e_{ij}[-1] \) to \( e_{ij}[-2] \). Then \( L \) is the conformal vector of \( W^\kappa(g, f) \):
\[
L(z)L(w) \sim -\frac{12k^2 + 41k + 32}{2(k+4)^2(z-w)^4} + \frac{2}{(z-w)^2}L(w) + \frac{1}{z-w}\partial L(w).
\]

2 Proof of Theorem 1.2

Recall the notation (1.2) and let \( \widehat{a} = \widehat{a}_0 \oplus \widehat{a}_1 \) be the Lie superalgebra such that \( \widehat{a}_0 = \widehat{b} \) and \( \widehat{a}_1 = m[t, t^{-1}] \), where \( m[t, t^{-1}] \) is regarded as the supercommutative Lie superalgebra, while
\[
[x, y] = \text{ad} x(y) \quad \text{for} \quad x \in \widehat{a}_0 \quad \text{and} \quad y \in \widehat{a}_1.
\]
We will write \( \psi_{ij}[-m] \otimes e_{pq} \) for the element
\[
e_{ij}[-m] \otimes e_{pq} \in \mathfrak{gl}_{t, <0}[t^{-1}]t^{-1} \otimes \mathfrak{gl}_{t} = m[t^{-1}]t^{-1}
\]
with \( m \geq 1 \), when it is considered as an element of \( \widehat{a}_1 \).

Let \( V^\kappa(b) \) be the representation of \( \widehat{a} \) induced from the one-dimensional representation of \( (b[t] \oplus \mathbb{C}1) \oplus m[t]t \) on which \( b[t] \subset \widehat{a}_0 \) and \( m[t]t \subset \widehat{a}_1 \) act trivially and \( 1 \) acts as \( 1 \). Then \( V^\kappa(b) \) is naturally a vertex algebra which contains \( V^\kappa(b) \) as its vertex subalgebra. We will regard \( V^\kappa(b) \) as a (non-associative) algebra with respect to the \((-1)-product
\[
V^\kappa(b) \otimes V^\kappa(b) \to V^\kappa(b), \quad a \otimes b \mapsto a_{(-1)}b,
\]
\[
\begin{array}{ccccccccc}
| e_{11} & e_{22} & e_{33} & e_{44} & e_{12} & e_{21} & e_{34} & e_{43} \\
| \frac{3k+2}{4} & -\frac{k}{4} & -\frac{k+1}{4} & \frac{3k+1}{4} & 0 & 0 & 0 & 0 \\
| -\frac{k}{4} & \frac{3k+8}{4} & -\frac{k+4}{4} & \frac{k+4}{4} & 0 & 0 & 0 & 0 \\
| -\frac{k+4}{4} & -\frac{k+4}{4} & \frac{3k+8}{4} & -\frac{k}{4} & 0 & 0 & 0 & 0 \\
| -\frac{k+4}{4} & -\frac{k+4}{4} & -\frac{k}{4} & \frac{3k+8}{4} & 0 & 0 & 0 & 0 \\
| e_{12} & 0 & 0 & 0 & 0 & k+2 & 0 & 0 \\
| e_{21} & 0 & 0 & 0 & 0 & k+2 & 0 & 0 \\
| e_{34} & 0 & 0 & 0 & 0 & 0 & k+2 & 0 \\
e_{34} & 0 & 0 & 0 & 0 & 0 & k+2 & 0
\end{array}
\]
where the Fourier coefficients $a_n$ are defined in the usual way from the state-field correspondence map,

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \quad \text{for} \quad a \in V^\kappa(b).$$

By \cite{9} the $\mathcal{W}$-algebra is given by

$$\mathcal{W}^\kappa(g, f) = \{ v \in V^\kappa(b) \mid Q v = 0 \},$$

where $Q : V^\kappa(a) \to V^\kappa(a)$ is the derivation of the non-associative algebra $V^\kappa(b)$ defined by the following properties. First, $Q$ commutes with the translation operator $D$ of the vertex algebra $V^\kappa(b)$, that is, $[Q, D] = 0$. Moreover, we have the commutation relations

$$[Q, e_{ji} \otimes e_{pq}] = \sum_{a=i}^{j-1} \sum_{r=1}^{n} (e_{ai} \otimes e_{rq})(\psi_{ja} \otimes e_{pr}) - \sum_{a=i+1}^{j} \sum_{r=1}^{n} (\psi_{ai} \otimes e_{rq})(e_{ja} \otimes e_{pr})$$

$$+ \alpha \psi'_{ji} \otimes e_{pq} + e_{i+1} \otimes e_{pq} - e_{i} \otimes e_{pq}$$

and

$$[Q, \psi_{ji} \otimes e_{pq}] = \frac{1}{2} \sum_{i<r<j, 1<s<n} (\psi_{jr} \otimes e_{ps})(\psi_{ri} \otimes e_{qs}) - \frac{1}{2} \sum_{i<r<j, 1<s<n} (\psi_{ri} \otimes e_{sp})(\psi_{jr} \otimes e_{qs}),$$

where we used the abbreviations

$$e_{ij} \otimes e_{pq} = (e_{ij} \otimes e_{pq})[-1] 1, \quad \psi_{ij} \otimes e_{pq} = (\psi_{ij} \otimes e_{pq})[-1] 1,$$

$$\psi'_{ij} \otimes e_{pq} = D(\psi_{ij} \otimes e_{pq})[-1] 1 = (\psi_{ij} \otimes e_{pq})[-2] 1,$$

and set $\psi'_{ii} = 0$. Also, we used the fact that

$$\text{tr}_m p_+ (\text{ad} (e_{ij} \otimes e_{pq}) \text{ad} (e_{ij} \otimes e_{pq})) = n(l + i - j - 1)$$

for $1 \leq i < j \leq l$ and $1 \leq p, q \leq n$, where $p_+$ denotes the restriction of the operator to $m$.

Our goal now is to reduce the calculations to the principal nilpotent case. To this end, when $n = 1$, we will write $\bar{a}$ and $\bar{b}$ respectively, instead of $a$ and $b$, and replace $k$ with $k + (n-1)(l-1)$ in \cite{13}. Consequently, $V^{\kappa\bar{b}}(\bar{a})$ will denote the vertex algebra $V^{\kappa\bar{b}}(\bar{a})$ with $n = 1$ (and $k$ replaced by $k + (n-1)(l-1)$). We let $\overline{Q}$ denote the operator $Q$ for $V^{\kappa\bar{b}}(\bar{a})$. We have

$$\overline{Q, e_{ji}} = \sum_{a=i}^{j-1} e_{ai} \psi_{ja} - \sum_{a=i+1}^{j} \psi_{ai} e_{ja} + \alpha \psi'_{ji} + \psi_{j+1} - \psi_{ji-1},$$

$$\overline{Q, \psi_{ji}} = \frac{1}{2} \sum_{i<r<j} (\psi_{jr} \psi_{ri} - \psi_{ri} \psi_{jr}),$$

and set

$$\overline{Q, \psi_{ii}} = 0.$$
where we used the notation $e_{ji} = e_{ji}[-1]$, $\psi_{ij} = \psi_{ij}[-1]$, $\psi'_{ij} = \psi_{ij}[-2]$, and we set $\psi'_{ii} = 0$.

We will regard $V^{\kappa b}(a) \otimes \mathbb{C}[\tau]$ as a non-associative algebra with the natural subalgebras $V^{\kappa b}(a)$ and $\mathbb{C}[\tau]$ together with the relation $[\tau, u] = D u$ for $u \in V^{\kappa b}(a)$. Similarly, the tensor product $V^{\kappa b}(\tilde{a}) \otimes \mathbb{C}[\tau]$ will be regarded as a non-associative algebra with the relation $[\tau, u] = \tilde{D} u$ for $u \in V^{\kappa b}(\tilde{a})$, where $\tilde{D}$ denotes the translation operator of the vertex algebra $V^{\kappa b}(\tilde{a})$. Define the non-associative algebra homomorphism

$$\tilde{T} : V^{\kappa b}(\tilde{a}) \otimes \mathbb{C}[\tau] \rightarrow M_n \otimes V^{\kappa b}(a) \otimes \mathbb{C}[\tau], \quad x \mapsto \tilde{T}(x) = \sum_{p,q=1}^{n} e_{pq} \otimes \tilde{T}_{pq}(x)$$

by

$$\tilde{T}_{pq}(e_{ji}[-m]) = e_{ji}[-m] \otimes e_{qp}, \quad \tilde{T}_{pq}(\psi_{ji}[-m]) = \psi_{ji}[-m] \otimes e_{qp} \quad \text{and} \quad \tilde{T}_{pq}(\tau) = \tau.$$

We extend the definition of column-determinant to matrices $A = [a_{ij}]$ with entries in a non-associative algebra by using right-normalized products,

$$\widetilde{\text{cdet}} A = \sum_{\sigma \in \mathcal{S}_l} \text{sgn} \sigma \cdot a_{\sigma(1)}(a_{\sigma(2)}(a_{\sigma(3)}(\ldots (a_{\sigma(l-1)} a_{\sigma(l)})\ldots))).$$

(2.4)

Note the relation

$$\tilde{T}(\widetilde{\text{cdet}} B) = T(\text{cdet} B),$$

where $\widetilde{\text{cdet}} B$ is regarded as an element of $V^{\kappa b}(\tilde{a}) \otimes \mathbb{C}[\tau]$. The first part of Theorem 2.2 will now be implied by the following two propositions.

**Proposition 2.1.** For any $a \in V^{\kappa b}(\tilde{a}) \otimes \mathbb{C}[\tau]$ and $1 \leq p, q \leq n$ we have the relations

$$[Q, \tilde{T}_{pq}(a)] = \tilde{T}_{pq}([Q, a]).$$

**Proof.** This follows immediately from the definitions of the operators $Q$ and $\tilde{Q}$. \hfill \square

**Proposition 2.2.** We have the relation

$$[Q, \widetilde{\text{cdet}} B] = 0.$$

**Proof.** We use induction on $l$. For any $0 \leq s \leq l$ consider the submatrix $B^{(s)}$ of $B$ corresponding to its last $s$ rows and columns, which is given by

$$\begin{bmatrix}
\alpha \tau + e_{l-s+1l-s+1}[-1] & -1 & 0 & \ldots & 0 \\
e_{l-s+2l-s+1}[-1] & \alpha \tau + e_{l-s+2l-s+2}[-1] & -1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
e_{l-1l-s+1}[-1] & e_{l-1l-s+2}[-1] & \ldots & \alpha \tau + e_{l-1l-1}[-1] & -1 \\
e_{ll-s+1}[-1] & e_{ll-s+2}[-1] & \ldots & \ldots & \alpha \tau + e_{ll}[-1]
\end{bmatrix}$$

Using the definition (2.4), set $D^{(s)} = \widetilde{\text{cdet}} B^{(s)}$. 

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Lemma 2.3. We have the column expansion formula

\[ D^{(s)} = \sum_{i=1}^{s} B_{ij}^{(s)} D^{(s-i)}, \]

where \( B_{ij}^{(s)} \) denotes the \((i, j)\) entry of \( B^{(s)} \).

Suppose that \( s < l \). By the induction hypothesis, the commutator \([\hat{Q}, D^{(s)}]\) equals

\[
\sum_{i=1}^{s} \det \begin{bmatrix}
0 & -1 & 0 & \ldots & 0 \\
0 & \alpha \tau + e_{l-s+2l-s+2}[-1] & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & e_{l-1l-s+2}[-1] & \ldots & \alpha \tau + e_{l-1l-1}[-1] & -1 \\
0 & e_{l+1l}[-1] & \ldots & \alpha \tau + e_{l+1}[l] & \alpha \tau + e_{l+1}[-1]
\end{bmatrix}
\]

so that

\[
[\hat{Q}, D^{(s)}] = -\sum_{i=1}^{s} \psi_{l-i+1l-s}[-1] D^{(i-1)}. \tag{2.5}
\]

Hence, by Lemma 2.3, we have

\[
[\hat{Q}, D^{(l)}] = \sum_{i=1}^{l} [\hat{Q}, B_{i1}^{(l)}] D^{(l-i)} + \sum_{i=1}^{l} B_{i1}^{(l)} [\hat{Q}, D^{(l-i)}].
\]

Now we use the definition of \( \hat{Q} \) and relation (2.5) to write this expression as

\[
\sum_{i=1}^{l} \left( \sum_{a=1}^{l+1} e_{a1} \psi_{ia} - \sum_{a=2}^{i} \psi_{a1} e_{ia} + \alpha \psi_{i1}^{[1]} + \psi_{i11} \right) D^{(l-i)} - \sum_{i=1}^{l} B_{i1}^{(l)} \sum_{a=1}^{l-i} (\psi_{l-a+1i} D^{(a-1)}),
\]

where, as before, we write \( e_{ij} = e_{ij}[-1], \psi_{ij} = \psi_{ij}[-1] \) and \( \psi_{ij}^{[1]} = \psi_{ij}[-2] \) for brevity. Thus,

\[
[\hat{Q}, D^{(l)}] = \sum_{i=1}^{l} \sum_{a=1}^{l-1} e_{a1} (\psi_{ia} D^{(l-i)}) - \sum_{i=1}^{l} \sum_{a=2}^{i} \psi_{a1} (e_{ia} D^{(l-i)}) + \sum_{i=1}^{l} (\alpha \psi_{i1}^{[1]} + \psi_{i11}) D^{(l-i)}
\]

- \[ \sum_{i=1}^{l} B_{i1}^{(l)} \sum_{a=1}^{l-i} (\psi_{l-a+1i} D^{(a-1)}). \]
which equals
\[
- \alpha \sum_{a=2}^{l} \psi_{a1} \tau D^{(l-a)} - \sum_{a=2}^{l} \sum_{i=a}^{l} \psi_{a,i} (e_{i,a}D^{(l-i)}) + \sum_{i=1}^{l} \psi_{i+1} D^{(l-i)}
\]
\[
= - \sum_{a=1}^{l} \psi_{a1} \left( \sum_{a=i}^{l} (\delta_{i,a} \alpha \tau + e_{i,a}) D^{(l-i)} - D^{(l-a+1)} \right)
\]
\[
= \sum_{a=1}^{l} \left( D^{(l-a+1)} - \sum_{i=1}^{l-a+1} B_{i,a} D^{(l-a+1-i)} \right) = 0,
\]
where the last equality holds by Lemma 2.3. Here we used the relations
\[
(e_{ji}[-m] \psi_{pq}[-n]) u = e_{ji}[-m] (\psi_{pq}[-n] u),
\]
\[
(\psi_{pq}[-n] e_{ji}[-m]) u = \psi_{pq}[-n] (e_{ji}[-m] u),
\]
which hold under the assumption
\[
u \in \text{span of } \{e_{i'j'}[-m'], \psi_{p'q'}[-n'] \mid j' > j \text{ and } q' > p\}.
\]
This completes the proof of the proposition. \(\square\)

To see the second part of Theorem 1.2, consider the grading of \(V^{\kappa \mathfrak{b}}(\mathfrak{b})\) induced by the grading of \(\mathfrak{b}\). One has
\[
W_{ij}^{(r)} = T_{ij} \left( \sum_{s=1}^{l-r+1} e_{r+s-1} s[-1] \right) + \text{(terms of higher degree)}.
\]
The elements \(\sum_{s=1}^{l-r+1} e_{r+s-1} s\) with \(r = 1, \ldots, l\) form a basis of \(\mathfrak{g}_{l}^{l}\) and the elements
\[
\sum_{s=1}^{l-r+1} e_{r+s-1} s \otimes e_{ji}, \quad r = 1, \ldots, l \text{ and } i, j = 1, \ldots, n,
\]
form a basis of \(\mathfrak{g}_{l}\). Hence the claim follows from \(|S\text{ Theorem }4.1|\) (cf. \(|P\text{ Theorem }5.5.1|\)) thus completing the argument.

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