A CONSTRUCTION OF SWAP OR SWITCH POLYNOMIALS

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Abstract. We discuss several constructions of swap polynomials that is non commutative polynomials in matrix variables $x_i \in M_d(\mathbb{Q})$ with values in $M_d(\mathbb{Q}) \otimes^2$ which are multiples of the transposition operator $(1, 2)$.

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1. INTRODUCTION

Given a positive integer $d$ and a field $F$ denote by $M_d(F)$ the algebra of $d \times d$ matrices with entries in $F$. We will assume that $F$ has characteristic 0 and in fact work with $F = \mathbb{Q}$ and then denote $M_d := M_d(\mathbb{Q})$.

Denote by $F\langle X \rangle = F\langle x_1, \ldots, x_i, \ldots \rangle$ the free algebra in the variables $x_i$. The elements of $F\langle X \rangle$ are usually called non commutative polynomials. Given an associative algebra $A$ over $F$ by definition the homomorphisms of $F\langle X \rangle$ to $A$ correspond to maps $X \rightarrow A$ and can be thought of as evaluations of the variables $X$ in $A$. Given any positive integer $k$ we may consider the algebra $A^{\otimes k}$ and then for every evaluation $\pi : F\langle X \rangle \rightarrow A$ of $X$ in $A$ we have a corresponding evaluation $\pi^{\otimes k} : F\langle X \rangle^{\otimes k} \rightarrow A^{\otimes k}$.

Definition 1.1. The elements of $F\langle X \rangle^{\otimes k}$ will be called tensor polynomials. They can be thought of as (non commutative) polynomials in the tensor variables $x_j^{(i)} := 1^{\otimes i-1} \otimes x_j \otimes 1^{\otimes k-i}$. An element $f \in F\langle X \rangle^{\otimes k}$ is called
a tensor polynomial identity for $A$ (short a TPI) if it vanishes under all evaluations of $X$ in $A$.

We then devote our study to permutation valued polynomials and in particular to swap polynomials.

**Definition 1.2.** Consider a non commutative 2–tensor polynomial $f \in F\langle X \rangle \otimes 2$ in variables $x_i$. Then $f$ is called a swap polynomial for $d \times d$ matrices if, when evaluated in the algebra of $d \times d$ matrices it is not a TPI and takes values only multiples of the exchange (or swap) operator $(1, 2) : a \otimes b \mapsto b \otimes a$, $(1, 2) \in M_d^{\otimes 2}$.

A 2–tensor polynomial $f = \sum_i A_i \otimes B_i$ is balanced if all $A_i, B_i$ are homogeneous of the same degrees.

This notion appears in a recent preprint Translating Uncontrolled Systems in Time, by David Trillo, Benjamin Dive, and Miguel Navascués, arXiv:1903.10568v2 [quant–ph] 28 May 2020 [19]. The authors introduce some special tensor valued polynomials. In particular they prove the existence of such operators using the classical theory of central polynomials developed independently by Formanek and Razmyslov. The resulting swap polynomials have usually very large degrees, although they exhibit such a polynomial for 2 tensor $2 \times 2$ matrix variables of degree 10. In their notations the degree is 5, or $5 + 5$, which according to their computations is the least degree for a balanced swap polynomial. They consider a tensor variable $x_i \otimes x_j$ of degree 1 while in this paper we consider it of degree 2 in the matrix variables, so their polynomial is of degree 5 in the tensor variables.

Swap polynomials belong to the interesting class of permutation valued polynomials that is polynomials whose values are a scalar times a constant linear combination of permutation operators.

The existence of such polynomials is a consequence of the existence of central polynomials and follows easily from the so called Goldman elements as in Saltman, [18], see Theorem 2.3.

We discuss first the case of $2 \times 2$ matrices and two variables, which can be described almost completely. Then, for general $d \times d$ matrices we propose a canonical construction of a swap polynomial of degree $2d^2$ in $2d^2$ variables.

In particular we exhibit a swap polynomial of degree 8 (or 4 in their language) but in 8 variables for $2 \times 2$ matrices.

The construction is based on a canonical central polynomial proposed by Regev and proved to be non zero by Formanek. The advantage of these polynomials is that they are alternating in two sets of $d^2$ variables which implies that we know an explicit formula for their values.

The paper [19] was pointed out to me by Felix Huber while discussing his recent work Positive maps and trace polynomials from the Symmetric Group arXiv:2002.12887 28–2–2020.

2. Tensor Polynomials

2.1. Azumaya algebras, the Goldman element. The following Definition and Theorem is attributed, with no reference, to Oscar Goldman in
the book of M. A. Knus, M. Ojanguren, *Théorie de la descente et algèbres d’Azumaya* page 112 [10].

Let $R$ be a rank $n^2$ Azumaya algebra over its center $A$. By definition of Azumaya algebra the map

$$\pi : R \otimes_A R^{op} \to \text{End}_A(R), \quad \pi(\sum_i a_i \otimes b_i)(x) = \sum_i a_i x b_i$$

is an isomorphism, moreover there is a faithfully flat extension $A \to B$ so that $B \otimes_A R \simeq M_n(B)$ and the trace of $M_n(B)$ restricted to $R$ takes values in $A$, so we have an $A$–linear map $tr : R \to A \subset R$.

**Definition 2.2.** We define the Goldman element $t \in R \otimes_A R$ by $\pi(t)(x) := tr(x)$.

**Theorem 2.3.** The Goldman element satisfies

1. $t^2 = 1,$
2. $(ta \otimes b)t^{-1} = b \otimes a.$

**Proof.** Under the faithfully flat extension $A \to B$ the element $t$ must map to $\sum_{i,j=1}^n e_{i,j} \otimes e_{j,i}$, by uniqueness since this element satisfies the same property:

$$\sum_{i,j=1}^n e_{i,j} e_{h,k} e_{j,i} = \begin{cases} 0 & \text{if } h \neq k \\ 1 & \text{if } h = k \end{cases}.$$

Then the properties of Formula (1) can be verified directly:

$$t^2 = \sum_{i,j=1}^n e_{i,j} \otimes e_{j,i} \sum_{h,k=1}^n e_{h,k} \otimes e_{k,h} = \sum_{i,j=1}^n e_{i,i} \otimes e_{j,j} = 1.$$

The second property can again be verified in the split algebra and it is

$$(\sum_{i,j=1}^n e_{i,j} \otimes e_{j,i}) e_{a,b} \otimes e_{c,d}(\sum_{h,k=1}^n e_{h,k} \otimes e_{k,h}) = e_{c,d} \otimes e_{a,b}.$$

By Corollary 10.4.3 of [1] we can take as $B = A_n(R)$ the commutative ring giving the universal map into matrices that is for any map $j : R \to M_n(C)$ one has a map $\bar{j} : A_n(R) \to C$ making commutative the diagram.

$$(2) \quad R \xrightarrow{i} M_n(A_n(R)) \xrightarrow{j} M_n(C) \xrightarrow{\bar{j}} M_n(j)$$

It follows that if $\rho : R \to S$ is a ring homomorphism of rank $n^2$ Azumaya algebras we have

$$(3) \quad R \xrightarrow{i_R} M_n(A_n(R)) \xrightarrow{\rho} M_n(i_S \circ \rho) \xrightarrow{\bar{\rho}} M_n(i_S) \xrightarrow{i_S} M_n(A_n(S))$$

and $t_R, t_S$ the respective Goldman elements we have $\rho(t_R) = t_S$, hence:
Corollary 2.4. Under any splitting $C \otimes_A R \simeq M_n(C)$, the element $t$ maps to the switch operator on $C^n \otimes C^n$.

Proof. The element $t$ maps to $\sum_{i,j=1}^n e_{i,j} \otimes e_{j,i}$ which is the switch operator on $C^n \otimes C^n$ since
\[
\sum_{i,j=1}^n e_{i,j} \otimes e_{j,i}(e_a \otimes e_b) = e_b \otimes e_a.
\]

\[\square\]

Remark 2.5. The properties of Formula (1) determine $t$ only up to a multiplicative scalar $\alpha$ with $\alpha^2 = 1$.

In particular if $A$ is a domain $\alpha = \pm 1$.

If $R$ is a free rank $n^2$ module over $A$ with basis $a_1, a_2, \ldots, a_{n^2}$ then there is a unique dual basis for the trace form $tr(xy)$. That is, there are unique elements $a_1^*, a_2^*, \ldots, a_{n^2}^*$ with $tr(a_i a_j^*) = \delta_i^j$. Then we have
\[
t = \sum_{i=1}^{n^2} a_i \otimes a_i^*.
\]

This depends upon the fact that the element $\sum_{i=1}^{n^2} a_i \otimes a_i^*$ is independent of the basis chosen. For $R = M_n(A)$ the dual basis of elementary matrices is $e_{i,j}^* = e_{j,i}$. So under the faithfully flat splitting the element $\sum_{i=1}^{n^2} a_i \otimes a_i^*$ coincides with $\sum_{i,j=1}^n e_{i,j} \otimes e_{j,i}$.

Remark 2.6. In the Physics literature for $n = 2$ and $A = \mathbb{C}$ one has the Pauli matrices which form, up to the normalizing factor $\frac{1}{\sqrt{2}}$, an orthonormal basis for the trace form restricted to Hermitian matrices where it is positive.

2.7. The case of generic matrices. In the Theory of algebras with polynomial identities, see [1], one has the basic algebra $R_{k,d} := F[\xi_1, \ldots, \xi_k]$ of polynomials in $k$ generic $d \times d$ matrices $\xi_i = (\xi_{h,k}^{(i)})$ which one identifies with the non commutative polynomial functions in $k$ matrix variables. Here the variables $\xi_{h,k}^{(i)}$ are the coordinates of the $kd^2$ vector space $M_d(F)^k$. See [1] for details. This algebra is a domain with a center $Z$. If $G$ is the field of fractions of $Z$ we have, as soon as $k > 1$, that $R_{k,d} \otimes_Z G := D_{k,d}$ is a division algebra of dimension $d^2$ over its center $G$. Moreover $G$ is the field of rational functions on $M_d^k$ invariant under conjugation action by $GL(d, F)$ the group of invertible matrices. Finally the polynomial functions on $M_d^k$ invariant under conjugation are generated by the traces of the monomials in the $\xi_i$. Thus we have the Goldman element $t \in D_{k,d} \otimes_G D_{k,d}$.

Notice that this element is independent of the variables $\xi_i$ in particular we may find it in the algebra of just two variables.

In general a rational function $f \in D_{k,d}$ can be evaluated on an open set of matrices and on this set the evaluation of $t$ is the switch operator $(1, 2)$. To find a swap polynomial is equivalent to find a common denominator $c$ for
\( t \) which is a scalar polynomial, that is a central polynomial. In fact this can be done in several ways.

The algebra \( R_{k,d} := F[\xi_1, \ldots, \xi_k], \) \( k \geq 2 \) has a non trivial center due to Formanek [4] and Razmyslov [17]. If \( c \in F[\xi_1, \ldots, \xi_k] \) is an element of the center with no constant term then inverting \( c \) one has that the algebra \( F[\xi_1, \ldots, \xi_k][c^{-1}] \) is Azumaya of rank \( n^2 \), Corollary 10.3.5 of [1]. Thus it has a Goldman element which in fact coincides with that \( t \) of \( D_{k,d} \).

Thus for each such \( c \) there exist \( a_i, b_i \in F[\xi_1, \ldots, \xi_k] \) and \( h > 0 \) so that we have \( t = c^{-h} \sum a_i \otimes b_i, a_i, b_i \in F[\xi_1, \ldots, \xi_k] \). In other words, by adding an extra variable \( \zeta \) we have the identity

\[
(5) \quad c^h t = \sum a_i \otimes b_i, \quad \sum a_i \zeta b_i = c^h \text{tr}(\zeta).
\]

The element \( \sum a_i \otimes b_i \) is thus a swap polynomial.

**Theorem 2.8.** A tensor polynomial \( \sum a_i \otimes b_i \) is a swap polynomial if and only if adding a variable \( \zeta \) we have that \( \sum a_i \zeta b_i \) is a central polynomial which vanishes for \( \zeta \) with \( \text{tr}(\zeta) = 0 \).

**Proof.** If \( \sum a_i \otimes b_i \) is a swap polynomial then it is equal, as function on matrices, to \( \alpha(1,2) = \alpha \sum_{i,j=1}^{n} e_{i,j} \otimes e_{j,i} \) with \( \alpha \) an invariant scalar function. Then

\[
\sum a_i \zeta b_i = \alpha \sum_{i,j=1}^{n} e_{i,j} \zeta e_{j,i} = \alpha \cdot \text{tr}(\zeta) 1.
\]

Conversely if \( \sum a_i \zeta b_i = \beta 1 \) is a central polynomial, \( \beta \) some polynomial invariant, which vanishes with \( \zeta \) with \( \text{tr}(\zeta) = 0 \) then \( \beta = \text{tr}(\zeta) \alpha \) is divisible by \( \text{tr}(\zeta) \). Since \( \beta \) is linear in \( \zeta \) we have \( \alpha \) is independent of \( \zeta \) and \( \sum a_i \zeta b_i = \text{tr}(\zeta) \alpha 1 \) implies

\[
\sum a_i \otimes b_i = \alpha t.
\]

\( \Box \)

**Remark 2.9.** For a swap polynomial \( \sum a_i \otimes b_i = \alpha t \) we have \( n \alpha = \sum a_i b_i \) so as soon as the characteristic does not divide \( n \) we also have that \( n \alpha = \sum a_i b_i \) is a central polynomial.

3. **2 \times 2 matrices**

3.1. **Swap polynomials in two variables.** In [19] the authors explain a method to construct balanced swap polynomials, Definition 1.2. The condition of being balanced is necessary for their applications. They exhibit one, let us denote it by \( Q(x,y) \), in two variables \( x,y \) for \( 2 \times 2 \) matrices, involving 40 terms and of degree 10 (or 5 +5 in their terminology).

\[
(6) \quad Q(x,y) :=
\]

\[
xy^2xy \otimes xy^2xy - xy^2xy \otimes y^2x^2y - xy^3x \otimes xy^2xy + xy^3x \otimes xy^3x + xy^3x \otimes xyxyy - xy^3x \otimes y^2xy - xy^4 \otimes xyryxy + xy^4 \otimes yx^2yx - yxy^2x \otimes xy^2xy - yxy^2x \otimes xy^3x + yxy^2x \otimes yxyryxy + yxy^2x \otimes yx^3x^2 + yxy^3 \otimes yxy^2x - yxy^3 \otimes y^2x^2 + y^2xy \otimes yx^2xy - y^2xy \otimes xy^2xy - y^2xy \otimes xy^3x - y^2xy \otimes xyxyy + y^2xy \otimes xy^2x
\]
The multiplication table is given by Cayley–Hamilton:

\[-y^2xy^2 \otimes xyx^2y + y^2xy^2 \otimes xyxx + y^2x^2 \otimes yx^2yx + y^3x^2 \otimes x^2xy + y^3xy \otimes x^2y - y^3x^2 \otimes y^3x^2y - y^3xy \otimes x^2y^2x - y^3xy \otimes y^2xyx + y^4x \otimes x^2y^2x + y^4x \otimes xyxyx \]

\[-y^4x \otimes x^2y - y^4x \otimes xyx^2 + y^4x \otimes y^2x^3 + y^5 \otimes x^2y^2x - y^5 \otimes xyx^3 \]

This has been built by a computer program, and I have verified it, and now I want to give a theoretical explanation for its existence and that of many other swap polynomials, Theorem 3.11. Its value, checked by Computer, is

\[(7) \quad \text{tr}(y)^2 \det([x, y])^2 t = \text{tr}(y)^2[x, y]^4 t.\]

Formulas. This topic is treated in detail in Chapter 9 of [1]. We want to recall some formulas which will be useful to understand swap polynomials.

The case of two $2 \times 2$ matrices $x, y$ is fully treated in the 1981 paper of Formanek, Halpin [6] and will be quickly reviewed here. Start with Exercise 9.1.1 of [1].

**Proposition 3.2.** The ring $T$ of invariants of 2, $2 \times 2$ matrices $x, y$ is the polynomial ring in 5 generators

\[(8) \quad \text{tr}(x), \text{det}(x), \text{tr}(y), \text{det}(y), \text{tr}(xy).\]

The ring $S$ of equivariant maps of 2, $2 \times 2$ matrices $x, y$ to $2 \times 2$ matrices (also called the trace algebra) is a free module $S = T + Tx + Ty + Txy$ over the ring of invariants with basis

\[(9) \quad 1, x, y, xy.\]

The multiplication table is given by Cayley–Hamilton:

\[x^2 = \text{tr}(x)x - \text{det}(x), \quad y^2 = \text{tr}(y)y - \text{det}(y),\]

\[yx = -xy + \text{tr}(x)y + \text{tr}(y)x + \text{tr}(xy) - \text{tr}(x)\text{tr}(y).\]

Of course in characteristic 0 we may replace the generators of Formula (8) with

\[(11) \quad \text{tr}(x), \text{tr}(x^2), \text{tr}(y), \text{tr}(y^2), \text{tr}(xy).\]

Let $R := F(x, y) \subset S$ be the subalgebra generated by the two generic matrices $x, y$. This is also the free algebra in two variables modulo the ideal of polynomial identities of $2 \times 2$ matrices.

The structure of $R$ is deduced in [6] from the following identities:

\[ [x, y]x = \text{tr}(x)[x, y] - x[x, y], \quad [x, y]y = \text{tr}(y)[x, y] - y[x, y] \]

\[ \text{det}(x)[x, y] = x[x, y]x, \quad \text{det}(y)[x, y] = y[x, y]y \]

\[ \text{tr}(xy)[x, y] = xyxy - xyxy. \]

From which it follows (Lemma (2) of [6]):

**Proposition 3.3.** The commutator ideal $R[x, y]R$ equals $S[x, y] = S[x, y]S$.

From this Theorem (3) of [6] follows.
Theorem 3.4. We have the decomposition of $R$ as vector space:

\[(13)\quad R = \oplus_{i,j} F x^i y^j \oplus S[x, y].\]

Recall that, given an inclusion $A \subset B$ of rings, the conductor (of $A$ in $B$) is the maximal ideal $I$ of $A$ which is also an ideal of $B$. In other words it is the set of $a \in A$ with $Ba + aB \subset A$.

Proposition 3.5. The commutator ideal $R[x, y]R$ is the conductor of $R$ in $S$.

Proof. By the previous proposition $R[x, y]R$ is an ideal in $S$ so it is contained in the conductor. Now the conductor is a bigraded ideal and $R/R[x, y]R$ has a basis of the ordered monomials $x^i y^j$. So if we had an element in the conductor not in $R[x, y]R$ we would have that one of those monomials is in the conductor. If $x^i y^j$ is in the conductor we have $x^i y^j tr(x) = f(x, y)$ for some non commutative polynomial. Setting $y = 1$ we obtain some identity $x^i tr(x) = \alpha x^{i+1}$, $\alpha \in F$. This implies $tr(x) = \alpha x$ a contradiction. \qed

The aim of the paper [6] was in particular to compute the Poincaré series $P(R) = \sum_{i,j=1}^{\infty} \dim(R_{i,j}) t^i s^j$ where $\dim(R_{i,j})$ is the dimension of $R$ in bidegree $i,j$ with respect to $x,y$. Now we clearly have

\[
\begin{align*}
P(T) &= \frac{1}{(1-t)(1-s)(1-t^2)(1-s^2)(1-ts)}, \\
P(S) &= (1+t+s+ts)P(T).
\end{align*}
\]

(14) \quad $P(R) = \frac{1}{(1-t)(1-s)} + tsP(S)$, \quad $P(Z) = 1 + (s^2 t^2)P(T)$.

Here $Z$ is the center of $R$ and the last Formula follows from Theorem 3.9.

Finally the Poincaré series of the free algebra is $(1-s-t)^{-1}$ so the Poincaré series of the polynomial identities in 2 variables for $2 \times 2$ matrices is, Theorem (10) of [6]:

\[
s^2 t^2 (s+t-st)(1-s)^{-2}(1-t)^{-2}(1-st)^{-1}(1-s-t)^{-1}.
\]

In fact it is important to describe some basic elements in the conductor of the ring $R_n$ of generic $2 \times 2$ matrices in $n \geq 3$ variables inside the corresponding trace ring $S_n$.

Lemma 3.6. An element $f(x_1, \ldots, x_n) \in R_n$ is in the conductor of $S_m$ for all $m \geq n$ if and only if, adding an extra variable $x_{n+1}$, we have that $tr(x_{n+1})f(x_1, \ldots, x_n) \in R_{n+1}$.

Proof. Write the polarized form of the Cayley–Hamilton identity in the form

\[
tr(z)tr(w) = -zw - wz + tr(z)w + tr(w)z + tr(zw).
\]

One has recursively that $\prod_{i=1}^{m} tr(z_i) = \sum_{j} A_j tr(B_j)$ with $A_j, B_j$ explicit non commutative polynomials.

The algebra $S_m$ is generated over $R_m$ by the elements $tr(M)$ with $M$ a monomial in the generic matrices (of degree $\leq 3$). Therefore $f(x_1, \ldots, x_n) \in R_n$ is in the conductor of $S_m$ if it absorbs the elements $\prod_{i=1}^{m} tr(z_i)$. But by the previous identity one is reduced to a single trace. \qed
Remark 3.7. In Theorem 10.4.8 of [1] we generalize to generic matrices of any size proving that $f(x_1, \ldots, x_n) \in R_n$ is in the conductor of $S_n$ if and only if for an extra variable $x_{n+1}$ we have $\det(x_{n+1})f(x_1, \ldots, x_n) \in R_{n+1}$.

Corollary 3.8. 1) The polynomial $[[x_1, x_2], x_3]$ is in the conductor of the generic matrices inside the trace algebra.

2) The central polynomial $[x, y]^2$ is in the conductor of the center of the generic matrices inside the invariant ring, that is for all $m$ we have $\prod_{i=1}^m \text{tr}(z_i)[x, y]^2$ is also a non commutative central polynomial.

Proof. Recall the basic Formula 9.50 of [1]:

\begin{equation}
[z[x_1, x_2] + [x_1, x_2]z, x_3] = \text{tr}([x_1, x_2], x_3).
\end{equation}

With some elementary manipulations one obtains from this

\begin{equation}
\text{tr}(z)[x, y]^2 = [zx[x, y] + x[x, y]z, y] - x[z[x, y] + [x, y]z, y] =
zx[x, y]y - yzx[x, y] - xz[x, y]y + xyz[x, y] + [x, y]^2z.
\end{equation}

\hfill □

Recall also that $[x, y]^2 = -\det([x, y])$ is an irreducible polynomial vanishing exactly on the subvariety $V$ of pairs of matrices $x, y$ which are NOT irreducible. If $c(x, y)$ is any central polynomial in these two variables (with no constant term) then it vanishes on $V$ (Proposition 10.2.2 of [1]). Therefore we have $c(x, y) = [x, y]^2\alpha$, $\alpha \in T$. Conversely we have Theorem (5) of [6]

Theorem 3.9. The center $Z$ of $R$ equals $F + [x, y]^2T$.

Proof. Every element of the form $c(x, y) = [x, y]^2\alpha$, $\alpha \in T$ can be expressed as a central polynomial by Corollary 3.8 2).

Formula (16) gives, by Theorem 2.8 §2.1, the swap polynomial of degree 4 in two variables:

\begin{equation}
P(x, y) := 1 \otimes ([x, y]^2 + x[x, y]) - y \otimes x[x, y] - x \otimes [x, y]y + xy \otimes [x, y]
\end{equation}

with value $[x, y]^2t$ but quite unbalanced.

Notice that also $P(y, x) = [x, y]^2t$ and

\begin{equation}
P(y, x) := 1 \otimes ([x, y]^2 - y[x, y]x + x \otimes [x, y] + y \otimes [x, y]x - xy \otimes [x, y]
\end{equation}

But now from this we can build a balanced swap polynomial taking the same value as $Q(x, y)$ of Formula (6).

In fact $Q(x, y) = \text{tr}(y)[x, y]^22P(x, y) = \text{tr}(y)[x, y]^2P(y, x)$. The first equals

\begin{equation}
\text{tr}(y)[x, y]^2 \otimes ([x, y]^2 + x[x, y]) - [x, y]^2y \otimes \text{tr}(y)[x, y] - [x, y]^2x \otimes \text{tr}(y)[x, y] + \text{tr}(y)[x, y]^2xy \otimes \text{tr}(y)[x, y]
\end{equation}

similar for the second. All terms of these trace tensor polynomials are balanced except the last ones

\begin{equation}
\text{tr}(y)[x, y]^2xy \otimes \text{tr}(y)[x, y], -\text{tr}(y)[x, y]^2yx \otimes \text{tr}(y)[x, y]^2
\end{equation}
Each of the balanced terms containing traces, both on the left and on the right of $\otimes$ are in the ideal generated by $[x, y]$ so can be written as non commutative polynomials in $x, y$. As for the last term remark that, developing $[x, y]^2$ as polynomial in the 5 basic generators of the invariants, one has a polynomial of degree 4 in $x, y$ sum of monomials, each of which can be split as the product of two scalar terms of degree 2.

Then we replace $P(x, y)$ with the sum $\frac{1}{2}(P(x, y) + P(y, x))$ so that the two unbalanced terms give
\[ \frac{1}{2} tr(y)[x, y]^3 \otimes tr(y)[x, y]. \]

In the left term we first replace $[x, y]^2$ by the polynomial in the basic generators and then move to the right for each monomial one of the two scalar factors of degree 2. We obtain a balanced trace tensor polynomial of the same type as before which can be written as a balanced tensor polynomial.

**Remark 3.10.** I have not verified if, by applying these formulas, one may obtain exactly formula (6) or another formula giving the same swap polynomial up to a tensor polynomial identity (this will depend on which of the monomials we move to the right and in which order to apply Formulas (12)). Notice that
\[ tr(x)tr(y)[x, y] \otimes [x, y] = (x[x, y] + [x, y]x) \otimes (y[x, y] + [x, y]y) = (y[x, y] + [x, y]y) \otimes (x[x, y] + [x, y]x). \]

Therefore the way to express a balanced polynomial is not unique.

Of course one may also exchange $x$ and $y$. A more symmetric swap polynomial of degree 5 both in $x$ and $y$ is, by the same argument, $tr(x)tr(y)[x, y]^4\mathfrak{t}$. We will refer to these 3 polynomials as $Q_i(x, y)$, $i = 1, 2, 3$.

The authors of [19] have in fact verified that there are no balanced swap polynomials in two variables of degree $< 10$ and only these 3 in degree 10. Later we will see that there is a balanced swap polynomial of degree 8 but in 8 variables, Theorem 4.14.

**Theorem 3.11.** For every invariant $A = A_1A_2$ product of $2h > 0$ factors of degree 1 giving $A_1$ and $k$ factors of degree 2 giving $A_2$. If either $k = 2\ell$ is even, or $k = 2\ell + 1$ and $h \geq 2$ we have that $A[x, y]^{4\mathfrak{t}}$ is the value of a balanced swap polynomial.

**Proof.** First if $k = 2\ell$ we split $A_2 = B_1B_2$ each with $\ell$ elements and $A_1 = C_1C_2C_3$ with $C_1 = tr(a)tr(b)$, $a, b \in \{x, y\}$ a product of 2 traces and $C_2, C_3$ of the same degree.

Then $C_1[x, y]^2P(x, y) = Q_i(x, y)$, depending on the variables appearing in the traces. Then we multiply by $B_1B_2C_2C_3$ and we distribute $B_1C_2$ on the first factor and $B_2C_3$ on the second. We obtain a balanced polynomial involving traces but again by the same argument all terms can be expressed as non commutative polynomials. The second case is similar. $\square$

It remains open the question if one can have a balanced swap polynomial which does not satisfy the previous conditions or even just whose value is
not divisible by \([x, y]^4\). My guess is NO. This may be related to the fact that
\(-[x, y]^4\) is the discriminant of the basis (9).

The algebra \(S\) becomes an Azumaya algebra after inverting the element
\([x, y]^2\) and in fact

**Proposition 3.12.** The polynomial \(P(x, y)\) is also an expression of \([x, y]^2t\)
by using the dual basis to \(x, y\).

**Proof.** In fact, recall from page 374 of [1] the matrix of the trace form, of the
basis (9), is:

\[
D = \begin{bmatrix}
tr(1) & tr(x) & tr(y) & tr(xy) \\
tr(x) & tr(x^2) & tr(xy) & tr(x^2y) \\
tr(y) & tr(yx) & tr(y^2) & tr(yxy) \\
tr(xy) & tr(x^2y) & tr(xy^2) & tr((xy)^2)
\end{bmatrix}, \quad \det(D) = -[x, y]^4
\]

One can compute that the cofactor matrix \(\Lambda\) of \(D\) is divisible by \([x, y]^2\)
so is \(\bar{\Lambda} = [x, y]^2\Lambda\). Setting

\[
\Lambda = \begin{bmatrix}
2A & -tr(x) \det(y) & -\det(x)tr(y) & tr(x)tr(y) - tr(xy) \\
-tr(x) \det(y) & 2 \det(y) & tr(xy) & -tr(y) \\
-\det(x)tr(y) & tr(xy) & 2 \det(x) & -tr(x) \\
tr(x)tr(y) - tr(xy) & -tr(y) & -tr(x) & 2
\end{bmatrix}
\]

From this one has that the dual basis, for the trace form of the basis (9), up
to the scalar \([x, y]^2\) is

\([x, y]^2 + x[x, y]y, -[x, y]y, -x[x, y], [x, y]\)

Then \([x, y]^2t\) is given by the dual bases expression (4)

\[
= 1 \otimes ([x, y]^2 + x[x, y]y) - x \otimes [x, y]y - y \otimes [x, y] + xy \otimes [x, y]
\]

is again the polynomial \(P(x, y)\) of Formula (17). \(\square\)

It remains to exhibit with an explicit formula a balanced swap polynomial
\(g(\xi) = f(\xi)(1, 2)\) for all \(d\). An explicit construction is performed in §4.1 and
§4.13.1.

4. Swap polynomials

A general approach to balanced swap polynomials is the following. Start
from any swap trace polynomial

\[
G := \sum_i A_i \otimes B_i = f(x_1, \ldots, x_n)t
\]

with \(A_i, B_i\) trace polynomials on \(d \times d\) matrices, \(f(x_1, \ldots, x_n)\) a scalar invariant function. This can be for instance constructed, for any \(n \geq 2\), by taking
\(n^2\) monomials \(A_i\) in the generic matrices with \(\Delta := \det(tr(A_iA_j)) \neq 0.\)
Solving for the dual basis (by Cramer’s rule) \(B_i = \sum_{i=1}^{d^2} x_{i,j} A_i\) the equations

\[
\Delta \delta_j = tr(A_j B_i) = \sum_{i=1}^{d^2} x_{i,j} tr(A_j A_i) \overset{(4)}{=} \sum_{i=1}^{n^2} A_i \otimes B_i = \Delta t.
\]
One sees first that the homogeneous components of \( G \) of degree \( h \) (i.e. \( \deg A_i + \deg B_i = h \)) are still swap trace polynomials. It is easy to construct from this a balanced swap polynomial. First by multiplying by a suitable product \( \prod_i \text{tr}(z_i) \). This can be distributed in each of the two factors to make them balanced.

Next one has special central polynomials \( u(x) \) which are in the conductor of the inclusion of the ring of central polynomial inside the ring of invariants. That is if \( f(x) \) is any invariant, i.e. a polynomial in the traces we have \( u(x)f(x) \) is a central polynomial (traces disappear). So taking two such elements of the same degree one has

\[
F(x) := \sum_i u_1(x) A_i \otimes u_2(x) B_i = u_1(x) u_2(x) f(x)(1,2)
\]

and \( F(x) \) is a balanced swap polynomial. Moreover \( u_1(x) u_2(x) f(x) \) is a central polynomial.

4.1. Dual bases and Capelli polynomials. The previous general procedure is more effective using the approach to central polynomials of Razmyslov, [17] i.e. antisymmetry as follows.

Given a noncommutative polynomial \( f \) in several variables which is linear in a given variable \( x_i \), write it in the form \( f = \sum_k a_k x_i b_k \). Consider it as a function on matrices, set \( f_i := \sum_k b_k a_k \) we then have:

\[
\text{tr}(f) = \text{tr}(x_i f_i).
\]

If \( f = \sum_k a_k x_i b_k \) depends linearly also upon another variable \( x_j \) and it changes sign by exchange of \( x_i, x_j \), then when we substitute in \( f, x_i \) with \( x_j \) we get \( \sum_k a_k x_j b_k = 0 \) and we also deduce:

\[
\text{tr}(x_j f_i) = \text{tr}(\sum_k a_k x_j b_k) = 0.
\]

Therefore if \( f(x_1, \ldots, x_n, y) \) is a noncommutative polynomial which is linear in each variable \( x_i \) and alternating in \( x_1, \ldots, x_n \) and \( \text{tr}(f(x_1, \ldots, x_n, y)) \) is different from 0 we have \( \text{tr}(x_j f_i) = \delta_j^i \text{tr}(f(x_1, \ldots, x_n, y)) \). Thus we have, up to the scalar function \( \text{tr}(f(x_1, \ldots, x_n, y)) \) the polynomials \( f_i \) form a dual basis of the \( n^2 \) variables \( x_j \). From Formula (4) we also have that

\[
\tau = \text{tr}(f(x_1, \ldots, x_n, y))^{-1} \sum_{i=1}^{n^2} x_i \otimes f_i, \quad \text{the Goldman element}
\]

\[
(20) \quad \Rightarrow \quad h(x, y) := \sum_{i=1}^{n^2} x_i y_0 f_i(x_1, \ldots, \bar{x}_i, \ldots, x_n, y) = \text{tr}(y_0) \text{tr}(f(x, y)).
\]

Thus we have a central polynomial \( h(x, y) \) of degree \( \deg(f) + 1 \) and, by Formula (4), the swap polynomial:

\[
(21) \quad H := \sum_{i=1}^{n^2} x_i \otimes f_i(x_1, \ldots, \bar{x}_i, \ldots, x_n, y) = \text{tr}(f(x_1, \ldots, x_n, y)) \tau.
\]
This $H$ is of course unbalanced, but if $m$ is the degree of $f$ and we choose any central polynomial $c$ of degree $m - 2$ one has that

$$c \cdot H = \sum_{i=1}^{n^2} c \cdot x_i \otimes f_i(x_1, \ldots, \hat{x}_i, \ldots, x_{n^2}, y) = c \cdot tr(f(x_1, \ldots, x_{n^2}, y))t$$

is balanced. At worst, if we cannot find such a central polynomial, replacing $f$ with $\bar{f} = fuzw$ of degree $m + 3$ we may take as $c = h(x, y)$ of degree $m + 1 = (m + 3) - 2$.

The simplest $f$ satisfying the previous conditions is the Capelli polynomial $C_n^2$, of degree $2n^2 - 1$. Following Razmyslov for each $m$ one sets

$$C_m(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_{m-1})$$

(22)

$$= \sum_{\sigma \in S_m} \epsilon_\sigma x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots x_{\sigma(m-1)} y_{m-1} x_{\sigma(m)}.$$

In fact this is only an analogy of the classical Capelli identity, which is instead an identity of differential operators (see [15]).

Thus we have an explicit central polynomial of degree $2n^2$ and an explicit balanced swap polynomial of degree $4n^2 + 2$.

We will see later, in §4.13.1, how to lower this degree and at the same time constructing more canonical swap polynomials.

4.2. Antisymmetry. Let us recall some basic facts. Denote for simplicity by $G = GL(n, F)$ the group of invertible matrices which acts on $n \times n$ matrices by conjugation.

**Theorem 4.3.** The invariants of $n \times n$ matrices are generated by elements $tr(M)$ where $M$ are monomials (of degree $\leq n^2$ by Razmyslov).

Among these invariants the ones that are multilinear and alternating have a very special structure.

In fact these invariants have an exterior multiplication. The algebra of these invariants, under exterior multiplication, is the algebra of invariant multilinear alternating functions $(\wedge M_d(F^*)^G$).

In turn this algebra can be identified to the cohomology of the unitary group. As all such cohomology algebras it is a Hopf algebra and by Hopf’s Theorem it is the exterior algebra generated by the primitive elements.

The primitive elements of $(\wedge M_d(F^*)^G$ are [11]:

(23) $T_{2i-1} = T_{2i-1}(x_1, \ldots, x_{2i-1}) : = tr(St_{2i-1}(x_1, \ldots, x_{2i-1}))$.

Recall the standard polynomial in $k$ variables is defined as

(24) $St_k(X) := \sum_{\sigma \in S_k} \epsilon_\sigma x_{\sigma(1)} \cdots x_{\sigma(k)} = AltX x_1 x_2 \cdots x_k$.

$S_k$ denotes the symmetric group and $\epsilon_\sigma$ the sign of the permutation $\sigma$.

In particular, since these elements generate an exterior algebra, a product of elements $T_i$ is non zero if and only if the $T_i$ involved are all distinct. Given $k$ distinct $T_i$ their product depends on the order only up to sign.
The $2^n$ different products form a basis of $(\bigwedge M_d(F)^*)^G$. In dimension $d^2$ the only non-zero product of these elements containing $d^2$ variables is

$$T_d(x_1, x_2, \ldots, x_{d^2}) = T_1 \wedge T_3 \wedge T_5 \wedge \cdots \wedge T_{2d-1}.$$  

Notice that $\text{tr}(St_2(x_1, \ldots, x_{2i})) = 0$, $\forall i$.

As a consequence, we have:

**Proposition 4.4.** Any multilinear antisymmetric function of $x_1, \ldots, x_{d^2}$ is a multiple of $T_1 \wedge T_3 \wedge T_5 \wedge \cdots \wedge T_{2d-1}$.

**Remark 4.5.** The function $\det(x_1, \ldots, x_{d^2})$ is an alternating invariant of matrices, so it must have an expression as in Formula (25). In fact the computable integer constant is known up to a sign [5]:

$$T_d(x_1, \ldots, x_{d^2}) = C_d \det(x_1, \ldots, x_{d^2}), \quad C_d := \pm \frac{1!3!5! \cdots (2d-1)!}{1!2! \cdots (d-1)!} \in \mathbb{Z}.$$  

(27) $\pm \{1, 6, 360, 302400, 4572288000, 152092588032000, \ldots\}$

**Proposition 4.6.**

1. There is an element $A_G \in M_d(F)^\otimes n$ invariant under the diagonal action of the linear group $G = GL_d$ such that

$$G(X_1, \ldots, X_k) = \prod_{i=1}^k T_d(X_i) A_G, \quad A_G \in \Sigma_n(F^d) \subset M_d(F)^\otimes n.$$  

2. We have

$$\text{tr}(\sigma \circ G(X_1, \ldots, X_k)) = \prod_{i=1}^k T_d(X_i) \text{tr}(\sigma \circ A_G), \quad \forall \sigma \in S_n.$$  

3. If $n = 2$ then $G(X_1, \ldots, X_k) \neq 0$ is a swap polynomial if and only if:

$$d \cdot \text{tr}(G(X_1, \ldots, X_k)) = \text{tr}((1, 2)G(X_1, \ldots, X_k)) \iff d \cdot \text{tr}(A_G) = \text{tr}((1, 2)A_G).$$

**Proof.** The first two parts are clear. As for 3. we have $A_G = a \cdot Id + b(1, 2)$ and $G(X_1, \ldots, X_k)$ is a swap polynomial if and only if $a = 0$ (and $b \neq 0)$:

$$\text{tr}(A_G) = \text{tr}(a \cdot Id + b(1, 2)) = a \cdot d^2 + b \cdot d$$

$$\text{tr}((1, 2)A_G) = \text{tr}(a(1, 2) + b \cdot Id) = a \cdot d + b \cdot d^2$$

so

$$a \cdot d + b \cdot d^2 = d(a \cdot d^2 + b \cdot d) \iff a = 0.$$  

Let us also remark:

**Remark 4.7.**

$$\text{tr}(a \cdot Id + b(1, 2)) = 0 \iff a = -d \cdot b, \quad \text{tr}((1, 2)(a \cdot Id + b(1, 2))) = 0 \iff b = -d \cdot a.$$
Remark 4.8. One has that the element $\Phi_d(1)$ is invertible in the center of $\Sigma_n(F^d)$. Hence $Wg(n, d) = \sum_{\mu \vdash d} a_\mu c_\mu \in \Sigma_n(F^d)$ is the image of a class function ($c_\mu$ denotes the sum over the conjugacy class relative to the partition $\mu$). Of course the expression is unique only if $d \geq n$, in this case one may interpret the Weingarten function as a function of $\sigma \in S_n$ depending on $d$.

An explicit formula through characters is, see [2], or [14]

$$a_\mu = \sum_{\lambda \vdash n,\ h(\lambda) \leq d} \frac{\chi(\lambda)\lambda^2\chi(\mu)}{s_{\lambda,d}(1)}.$$ 

Where $\mu$ is the cycle partition of $\sigma$, $\chi(\lambda)\lambda^2\chi(\mu)$ the character of $\sigma$ in the irreducible representation of $S_n$ corresponding to $\lambda$ and $s_{\lambda,d}(1)$ is the dimension of the corresponding irreducible representation of $GL(d,F)$.

Remark 4.9. It is known [13] (see also [14] Theorem 1.29) that the function $a_\mu$ is always nonzero and positive if $\mu$ is the cycle partition of an even permutation and negative for odd permutations.

For a computation using Mathematica of the list $d^2 \sum_{\mu \vdash d} a_\mu c_\mu$ and $d \leq 8$, for $n = d$ see [14].

Remark 4.10. In Proposition 16 of [9] we have shown that, if we distribute the $d^2$ variables $Y$ in $k$ monomials $n_i(Y)$ each of degree $h_i$ (with $\sum_i h_i^k = d^2$) then $Alt_Y n_1(Y) \otimes \cdots \otimes n_k(Y)$ is 0 unless the $h_i$ are a permutation of a refinement of the sequence $\delta_d := 1, 3, \ldots, 2d - 1$. In this case we have $tr(\sigma^{-1}G_d(Y_1, \ldots, Y_{d^2})) = 0$ unless $\sigma$ glues together the monomials so to recover the partition $\delta_d$. In this case $tr(\sigma^{-1}G_d(Y_1, \ldots, Y_{d^2})) = \pm T_d(Y)$. The sign is that of the permutation that $\sigma$ induces on the subset of the indices $i$ for which $h_i$ is odd.

In particular if $k = d$ this means that the $h_i$ are a permutation of the sequence $\delta_d$. For the sequence $\delta_d$ it follows

$$n_i(Y) = y_{(i-1)^2+1} \cdots y_{i^2}, \quad G_d(Y_1, \ldots, Y_{d^2}) := Alt_Y(n_1(Y) \otimes \cdots \otimes n_d(Y))$$

$$tr(\sigma \circ G_d(Y_1, \ldots, Y_{d^2})) = \begin{cases} T_d(Y) \quad & \text{if } \sigma = 1 \\ 0 \quad & \text{otherwise.} \end{cases}$$

That is $\Phi_d(G_d(Y_1, \ldots, Y_{d^2})) = T_d(Y) 1$ and, Proposition 26 of [9] gives:

$$G_d(Y_1, \ldots, Y_{d^2}) := Alt_Y(n_1(Y) \otimes \cdots \otimes n_d(Y)) = T_d(Y) Wg(d,d).$$
4.11. A construction of swap polynomials. Our final construction of swap polynomials is based on Proposition 4.6. Suppose we have two tensor polynomials \( G_i(X_1, \ldots, X_k) \in F(X)^{\otimes 2}, \ i = 1, 2 \) as in Formula (28)

\[
G_i(X_1, \ldots, X_k) = \prod_{j=1}^{k} T_d(X_j)A_i, \quad A_i \in \Sigma_2(F^d) \subset M_d(F)^{\otimes 2}, \ i = 1, 2.
\]

We then have \( A_i = a_i Id + b_i(1, 2), \ i = 1, 2. \)

Then we may assume \( a_i \neq 0, \ i = 1, 2 \) otherwise one is already a swap polynomial. We clearly have:

**Theorem 4.12.**

\[-a_2G_1 + a_1G_2 = T_d(X)T_d(Y)(-a_2b_1 + a_1b_2)(1, 2)\]

is a swap polynomial (provided that \((-a_2b_1 + a_1b_2) \neq 0\)) that is the two polynomials are not proportional.

So the issue is to find a pair of such polynomials. First it is easy to see, cf. [9], that if \( k = 1 \) a multilinear balanced antisymmetric tensor polynomial \( G(x_1, \ldots, x_d) \in M_d(F)^{\otimes 2}, \ d \geq 2 \) vanishes when computed in \( d \times d \) matrices, so the minimum number of sets \( X_i \) to consider is 2. The approach to find two such polynomials (balanced) for \( k = 2 \), rests on the Theory of Formanek developed to prove Theorem 4.13.

In principle a simple way of finding a polynomial multilinear and alternating in each of two sets of \( d^2 \) variables \( X := (x_1, \ldots, x_{d^2}); Y = (y_1, \ldots, y_{d^2}) \) is the following. Take any monomial \( M(X, Y) \) product in some order of the \( 2d^2 \) variables \( X, Y \) and alternate \( F_M(X, Y) := Alt_X Alt_Y M(X, Y). \)

Also split \( M(X, Y) := AB \) with \( A, B \) each of length \( d^2 \), and alternate \( G_M(X, Y) := Alt_X Alt_Y A \otimes B. \) The real issue is to choose \( M(X, Y) \) so that \( G_M(X, Y) \) is not a tensor polynomial identity. Of course if \( F_M(X, Y) \) is not a polynomial identity then \( G_M(X, Y) \) is not a tensor polynomial identity [9].

A theorem of Formanek relative to a conjecture of Regev, see [5], states that a certain explicit polynomial \( F(X, Y) \) in \( d^2 \) variables \( X = \{x_1, \ldots, x_{d^2}\} \) and \( d^2 \) variables \( Y = \{y_1, \ldots, y_{d^2}\} \) is non zero. A general discussion can be found in theorems of Giambruno Valenti [7].

From the Regev polynomial we shall construct two tensor polynomials with the required properties.

The definition of \( F(X, Y) \) is this. Decompose \( d^2 = 1 + 3 + 5 + \ldots + (2d - 1) \) and accordingly decompose the \( d^2 \) variables \( X \) and the \( d^2 \) variables \( Y \) in the list and construct the monomials \( m_i(X), i = 1, \ldots, d \) and similarly \( m_i(Y) \) as

\[
m_i(X) = x_{(i-1)^2+1} \cdots x_{i^2}, \quad m_i(Y) = y_{(i-1)^2+1} \cdots y_{i^2}.
\]

(35)

\[
m_1(X) = x_1, m_2(X) = x_2x_3x_4, m_3(X) = x_5x_6x_7x_8x_9, \ldots.
\]

We finally define

\[
F(X, Y) := Alt_X Alt_Y (m_1(X)m_1(Y)m_2(X)m_2(Y) \ldots m_d(X)m_d(Y)),
\]

where \( Alt_X \) (resp. \( Alt_Y \)) is the operator of alternation in the variables \( X \) (resp. \( Y \)).
Theorem 4.13. [see [5] or [14]]

\[ F(X, Y) = (-1)^{d-1} \frac{1}{(d!)^2(2d-1)} T_d(X)T_d(Y)Id_d \]

\[ (26) \quad (-1)^{d-1} \frac{C_d^2}{(d!)^2(2d-1)} \Delta(X)\Delta(Y)Id_d; \quad \Delta(X) = \det(x_1, \ldots, x_d). \]

In fact this follows from the special value for \( \sigma = (1, 2, \ldots, d) \), the full cycle, of the Weingarten function:

\[ a_\sigma = (-1)^{d-1} \frac{d}{(d!)^2(2d-1)} \]

Thus \( F(X, Y) \) is a central polynomial. In fact it has also the property of being in the conductor of the ring of polynomials in generic matrices inside the trace ring. In other words by multiplying \( F(X, Y) \) by any invariant we still can write this as a non commutative polynomial. This follows by polarizing in \( z \) the identity, cf Propositions 10.2.10 and 10.215 or Theorem 10.4.8 of [1]

\[ \det(z)^d F(X, Y) = F(zX, Y) = F(X, zY). \]

In order to use this for tensor polynomials we start from a general fact.

Let us consider two decompositions of \( d^2 \) as a sum of \( c \leq d \) positive integers:

\[ \underline{h} := (h_1, \ldots, h_c); \quad \underline{k} := (k_1, \ldots, k_c) \mid \sum_i h_i = \sum_i k_i = d^2. \]

Decompose accordingly the list \( X \) (resp \( Y \)) in \( c \leq d \) lists with the \( i^{th} \) list formed by the \( h_i \) (resp \( k_i \)) variables successive to the ones of the previous lists. Define \( m_i(X) \) the ordered product of the variables in the \( i^{th} \) list relative to \( X \) similarly \( n_i(Y) \) for \( Y \). We have \( m_i(X) \) has degree \( h_i \) and \( n_j(Y) \) has degree \( k_j \). Define next \( N_i = N_i(X, Y) := m_i(X)n_i(Y), \; i = 1, \ldots, c. \)

\[ N_1 \otimes N_2 \otimes \ldots \otimes N_d = m_1(X) \otimes m_2(X) \otimes \ldots \otimes m_d(X) \otimes n_1(Y) \otimes n_2(Y) \otimes \ldots \otimes n_d(Y). \]

From Remark 4.10 it follows that the element

\[ \text{Alt}_X \text{Alt}_Y(N_1 \otimes N_2 \otimes \ldots \otimes N_d) \]

is 0 unless both \( \underline{h} \) and \( \underline{k} \) are permutations \( \eta, \zeta \) of the sequence 1, 3, 5, \ldots, \( 2d - 1 \).

In this last case always by the same remark since

\[ \text{Alt}_Y(n_1(Y) \otimes n_2(Y) \otimes \ldots \otimes n_d(Y)) = \pm T_d(Y)Wg(d, d) = \pm T_d(Y) \sum_{\tau \in S_d} a_{\tau \tau} \]

we have by Formula (32), for \( \sigma \in S_d \):

\[ \pm \text{Alt}_X \text{Alt}_Y \text{tr}(\sigma^{-1} \circ N_1 \otimes N_2 \otimes \ldots \otimes N_d) \]

\[ = T_d(Y) \sum_{\tau} a_{\tau \tau} \text{tr}(\sigma^{-1} \circ \text{Alt}_X m_1(X) \otimes m_2(X) \otimes \ldots \otimes m_d(X) \circ \tau) \]

\[ \stackrel{(32)}{=} \pm T_d(Y)T_d(X)a_\sigma. \]
The sign is $\epsilon_{\eta,\zeta}$ if $\eta$ and $\zeta$ are permutations $\eta, \zeta$ of $1, 3, 5, \ldots, 2d - 1$.

Take a monomial $M(X, Y)$ product in some order of the $2d^2$ variables $X, Y$. If we split $M(X, Y) = AB$ as a product of two factors each of length $d^2$ we may construct the 2–tensor valued polynomial

$$ F_M^{\otimes 2}(X, Y) := \text{Alt}_X \text{Alt}_Y A \otimes B $$  

(41)

If $F_M(X, Y)$ is not a polynomial identity we can take $G_1(X, Y) = F_M^{\otimes 2}(X, Y)$ from the previous Formula.

Instead, if $F_M(X, Y)$ is a polynomial identity for $d \times d$ matrices, then $\text{tr}((1,2)F_M^{\otimes 2}(X, Y)) = \text{tr}(F_M(X, Y)) = 0$ so $F_M^{\otimes 2}(X, Y)$ can be taken as $G_2(X, Y)$ provided $\text{tr}(F_M^{\otimes 2}(X, Y)) = \text{Alt}_X \text{Alt}_Y \text{tr}(A) \text{tr}(B) \not= 0$.

For the first we can take the polynomial of Formula (37), we need to find the second.

4.13.1. A construction for $d = 2h$ even. Let us use Formula 40 to construct two polynomials $G_1, G_2$ satisfying the hypotheses of Theorem 4.12 when $d = 2h$ is even.

Consider the three monomials products of $d$ factors in the variables $X, Y$

$$ A = m_1(X)m_2(Y)m_3(X) \ldots m_{d-1}(X)m_d(Y), $$
$$ B = m_d(X)m_{d-1}(Y) \ldots m_3(Y)m_2(X)m_1(Y), $$
$$ C = m_1(Y)m_d(X)m_{d-1}(Y) \ldots m_3(Y)m_2(X). $$

All of degree $d^2$ and $A, B$ or $A, C$ involving disjoint sets of variables. Set

$$ G_1(X, Y) = \text{Alt}_X \text{Alt}_Y A \otimes B, \quad G_2(X, Y) = \text{Alt}_X \text{Alt}_Y A \otimes C. $$  

(42)

Both are multilinear balanced tensor polynomials in the variables $X, Y$.

We have $\text{tr}(B) = \text{tr}(C)$ and $AC = A'm_d(Y)m_1(Y)C'$ so,

$$ \text{Alt}_X \text{Alt}_Y AB = F(X, Y) = 0, \text{Alt}_Y m_d(Y)m_1(Y) = \text{St}_{2d}(Y) = 0 \implies \text{Alt}_X \text{Alt}_Y AC = 0. $$

Apply Formula 40 to the $N_i$ decomposing $AB$ and $AC$;

$$ \text{tr}(G_2(X, Y)) = \text{tr}(G_1(X, Y)) = \text{Alt}_X \text{Alt}_Y \text{tr}(A) \text{tr}(B) \overset{\text{(40)}}{=} T_d(X)T_d(Y)a_{h,h}. $$  

(43)

$$ \text{tr}((1, 2) \circ G_1(X, Y)) = \text{Alt}_X \text{Alt}_Y \text{tr}(AB) = \overset{\text{(40)}}{=} T_d(X)T_d(Y)a_d. $$

(44)

\[\text{tr}((1, 2) \circ G_2(X, Y)) = 0.\]

Set $A_i := A_{G_i} = a_i I_d + b_i (1, 2)$, $i = 1, 2$.

$$ \text{tr}(A_i) = a_i \cdot d^2 + b_i \cdot d, \quad \text{tr}((1, 2) A_i) = a_i \cdot d + b_i \cdot d^2. $$

From Formulas (43) and (44)

$$ a_2 d + b_2 d^2 = 0, \quad a_1 d + b_1 d^2 = a_d, \quad a_2 \cdot d^2 + b_2 \cdot d = a_1 \cdot d^2 + b_1 \cdot d = a_{h,h}. $$

Therefore solving the system of linear equations we have:

$$ a_1 = \frac{a_d - d \cdot a_{h,h}}{d(1 - d^2)}, \quad b_1 = \frac{d \cdot a_d - a_{h,h}}{d(1 - d^2)}, \quad b_2 = \frac{a_{h,h}}{d(1 - d^2)}, \quad a_2 = -\frac{a_{h,h}}{(1 - d^2)}. $$

We want to use Formula (29) and hence
Theorem 4.14. $A_G = -a_2A_1 + a_1A_2$ is a balanced swap polynomial with value $\mathcal{T}_d(X)\mathcal{T}_d(Y)(-a_2b_1 + a_1b_2)(1,2)$.

\[ d-a_{h,h}G_1(X,Y) + (d-a_{h,h})G_2(X,Y) = \frac{a_{h,h}(d \cdot a_d - a_{h,h})}{(1-d)(1-d^2)d^2}\mathcal{T}_d(X)\mathcal{T}_d(Y)(1,2). \]

For $d = 2, 4, 6$ we have, after multiplying by $d!^2$ that

\[ a_2 = \frac{2}{3}, \quad a_4 = -\frac{4}{7}, \quad a_6 = -\frac{6}{11}, \quad a_{1,1} = \frac{4}{3}, \quad a_{2,2} = \frac{22}{35}, \quad a_{3,3} = \frac{300}{539}. \]

\[ d = 2, \quad \frac{8}{3}G_1 - \frac{10}{3}G_2 = -\frac{8}{27}\mathcal{T}_2(X)\mathcal{T}_2(Y)(1,2) = -\frac{32}{3}D(X)D(Y)(1,2) \]

\[ 4G_1 - 5G_2 = -16D(X)D(Y)(1,2); \quad d = 4, \quad 22G_1 - 27G_2 = -\frac{561}{7}D(X)D(Y)(1,2); \]

\[ d = 6, \quad 75G_1 - 389\frac{3}{4}G_2 = -\frac{310564800}{6523}D(X)D(Y)(1,2). \]

Remark 4.15. By Remark 4.9 have that $a_{h,h} > 0$ while $a_d < 0$, so the coefficient $\frac{a_{h,h}(d \cdot a_d - a_{h,h})}{(1-d)(1-d^2)d^2}$ is $< 0$ and the tensor polynomial is thus nonzero.

4.15.1. The case $d$ odd. The previous construction does not apply to the case $d$ odd. In this case

\[ A = m_1(X)m_2(Y)m_3(X) \ldots m_{d-1}(Y)m_d(X), \]

\[ B = m_d(Y)m_{d-1}(X) \ldots m_3(Y)m_2(X)m_1(Y). \]

\[ tr(Alt_X A) = tr(Alt_X m_1(X)m_d(Y)m_2(Y)m_3(X) \ldots m_{d-1}(Y)) = 0 \]

since $Alt_X m_1(X)m_d(X) = St_{2d}(X) = 0$.

We construct $G_1(X,Y) = Alt_X Alt_Y A \otimes B$ and have $tr(G_1(X,Y)) = 0$, $tr((1,2) \circ G_1(X,Y)) = \mathcal{T}_d(X)\mathcal{T}_d(Y)a_d$.

\[ G_1(X,Y) = \mathcal{T}_d(X)\mathcal{T}_d(Y)(a + (1,2)b, \quad ad + bd = 0, \quad ad + bd^2 = a_d \]

\[ \Rightarrow b = -\frac{a_d}{(1-d)}, \quad a = \frac{a_d}{d(1-d^2)}. \]

We need another tensor polynomial $G_2(X,Y)$ with $tr(G_2(X,Y)) \neq 0$. In fact we will give one with $tr((1,2)G_2(X,Y)) = 0$.

Let us do it for $d = 3$ and the general case is similar. We start from the monomials of Formula (35)

\[ m_1(X) = x_1, m_2(X) = x_2x_3x_4, m_3(X) = x_5x_6x_7x_8x_9. \]

We split the monomials $m_2$ and consider

\[ A = x_1y_1x_3x_4m_3(Y), \quad B = y_2x_2y_3y_4m_3(X). \]

\[ G(X,Y) := Alt_X Alt_Y A \otimes B, \quad tr((1,2)G(X,Y)) = tr(Alt_X Alt_Y AB) = 0 \]

since in $AB$ appears the factor $m_3(Y)y_2$ of degree 6 whose alternation is 0. We need to show that $tr(G(X,Y)) = Alt_X Alt_Y tr(A)tr(B)$ is different from 0. Next $tr(B) = tr(m_3(X)y_2x_2y_3y_4)$ and consider

\[ (46) \quad M_1 := x_1 \otimes x_3x_4 \otimes m_3(X) \otimes x_2, \quad M_2 := y_1 \otimes m_3(Y) \otimes y_2 \otimes y_3y_4 \]

\[ P = M_1M_2, \quad tr((1,2)(3,4)P) = tr(A)tr(B) \]
Lemma 4.16. \( Alt_X tr(\sigma^{-1}M_1) = Alt_Y tr(\sigma^{-1}M_2) = 0 \) except for the following cases:

\[
\begin{align*}
tr((1,2)x_1 \otimes x_3 x_4 & \otimes m_3(X) \otimes x_2) = tr(x_2)tr(x_1 x_3 x_4)tr(m_3(X)) \\
tr((2,4)x_1 \otimes x_3 x_4 & \otimes m_3(X) \otimes x_2) = tr(x_1)tr(x_2 x_3 x_4)tr(m_3(X)) \\
tr((1,4)y_1 \otimes m_3(Y) & \otimes y_2 \otimes y_3 y_4) = tr(y_2)tr(y_1 y_3 y_4)tr(m_3(Y)) \\
tr((3,4)y_1 \otimes m_3(Y) & \otimes y_2 \otimes y_3 y_4) = tr(y_1)tr(y_2 y_3 y_4)tr(m_3(Y))
\end{align*}
\]

Proof. The other \( \sigma \) give a product of trace monomials of lengths different from the partition 1, 3, 5 hence give 0. \( \square \)

We deduce for \( Alt_Y M_2 \) Formula (49)

\[
(48) \quad Alt_Y tr((3,4)M_2) = \mathcal{T}_5(Y), \quad Alt_Y tr((1,4)M_2) = -\mathcal{T}_3(Y);
\]

\[
\Phi_d(Alt_Y M_2) = \mathcal{T}_3(Y)[(3,4) - (1,4)]
\]

\[
(49) \quad Alt_Y M_2 = \mathcal{T}_5(Y)|(3,4) - (1,4)|Wg(4,3).
\]

We deduce for \( Alt_X M_1 \) Formula (51)

\[
(50) \quad Alt_X tr((2,4)M_1) = \mathcal{T}_3(X), \quad Alt_X tr((1,2)M_2) = -\mathcal{T}_3(X);
\]

\[
\Phi_d(Alt_X M_1) = \mathcal{T}_3(X)[(2,4) - (1,2)]
\]

\[
(51) \quad Alt_X M_1 = \mathcal{T}_3(X)|(2,4) - (1,2)|Wg(4,3).
\]

Thus \( tr(G(X,Y)) = Alt_X Alt_Y tr(A)tr(B) \) is given, (47), by \( \mathcal{T}_3(Y) \) times:

\[
Alt_X tr ((1,2)(3,4)x_1 \otimes x_3 x_4 \otimes m_3(X) \otimes x_2 \cdot [(3,4) - (1,4)]Wg(4,3))
\]

\[
(52) = \sum_{\sigma \in S_4} b_\sigma Alt_X tr ((1,2)(3,4)x_1 \otimes x_3 x_4 \otimes m_3(X) \otimes x_2 \cdot [(3,4) - (1,4)]\sigma).
\]

By Lemma 4.16 we know that we have non zero contributions \( \pm b_\sigma \mathcal{T}_5(X) \), to Formula (52), from \( Wg(4,3) = \sum_\sigma b_\sigma \sigma \) only when

\[
\begin{align*}
- b_\sigma \text{ if } (3,4)\sigma(1,2)(3,4) = (1,2), \quad b_\sigma \text{ if } (3,4)\sigma(1,2)(3,4) = (2,4) \\
b_\sigma \text{ if } (1,4)\sigma(1,2)(3,4) = (1,2), \quad - b_\sigma \text{ if } (1,4)\sigma(1,2)(3,4) = (2,4)
\end{align*}
\]

The corresponding 4 values of \( \sigma \) are:

\[
\begin{align*}
\sigma = (3,4)(1,2)(3,4)(1,2) = 1, \quad & \text{sign} - \\
\sigma = (3,4)(2,4)(1,2)(3,4) = (1,3,2), \quad & \text{sign} + \\
\sigma = (1,4)(1,2)(1,2)(3,4) = (1,4,3), \quad & \text{sign} - \\
\sigma = (1,4)(2,4)(1,2)(3,4) = (2,4,3), \quad & \text{sign} +
\end{align*}
\]

But \( b_\sigma \) is a class function so the contribution is \(-b_{14} + b_{3,1}\) and

\[
tr(G(X,Y)) = Alt_X Alt_Y tr(A)tr(B) = \mathcal{T}_5(X)\mathcal{T}_3(Y)[-b_{14} + b_{3,1}]
\]

\[
(4!)^2\left[ b_{2,2} - b_{3,1} \right] = -\frac{61}{5} - \frac{3}{5} = -\frac{64}{5}.
\]
Now the expression of $W_g(4,3) = \sum_\sigma b_\sigma \sigma$ is not unique (cf. Remark 4.8) since $\sum_\sigma e_\sigma \sigma = 0$ but the difference $b_\sigma - b_\tau$ for two permutations of the same parity or the sum $b_\sigma + b_\tau$ for two permutations of opposite parity is well defined.

Examples of $b_\lambda \cdot (d + 1)!^2$ for $d = 2, 3, 4, 5$:

$$
\begin{align*}
\frac{7}{4} c_3 + \frac{17}{4} c_{2,1} + \frac{37}{15} c_4 &- \frac{11}{5} c_{3,1} - \frac{7}{3} c_{2,1,1} + \frac{61}{5} c_{1,1,1,1} \\
- \frac{533}{168} c_5 + \frac{143}{168} c_{4,1} + \frac{503}{168} c_{3,2} + \frac{61}{24} c_{3,1,1} - \frac{53}{168} c_{2,2,1} - \frac{1417}{168} c_{2,1,1,1} + \frac{5227}{168} c_{1,1,1,1,1} \\
\frac{1627}{420} c_6 - \frac{451}{105} c_{5,1} - \frac{389}{105} c_{4,2} - \frac{104}{35} c_{4,1,1} - \frac{1601}{420} c_{3,3} + \frac{151}{210} c_{3,2,1} + \frac{991}{105} c_{3,1,1,1} \\
+ \frac{701}{210} c_{2,2,2} + \frac{289}{70} c_{2,2,1,1} - \frac{4649}{210} c_{2,1,1,1,1} + \frac{5227}{168} c_{1,1,1,1,1,1}
\end{align*}
$$

The general case $d + 1 = 2h$ reduces to this, see Formula (53), by considering the two monomials

$$
C = m_4(Y)m_5(X) \ldots m_{d-1}(Y)m_d(X); D = m_4(X)m_5(Y) \ldots m_{d-1}(X)m_d(Y)
$$

substituting

$$
Alt_Y Alt_X AC \otimes BD, \quad Alt_Y Alt_X ACBD = 0
$$

$tr(AC)(BD) = tr(\tau_{h}^{-1} Q)$, $\tau_h = (1, 2, \ldots , h)(h + 1, \ldots , 2h)$, $Q = Q_1 \cdot Q_2$

$Q_1 = x_1 \otimes x_3 x_4 \otimes m_3(X) \otimes x_2 \otimes m_4(X) \otimes m_5(X) \otimes \ldots \otimes m_d(X)$

$Q_2 = y_1 \otimes m_3(Y) \otimes y_4 y_3 \otimes y_2 \otimes m_4(Y) \otimes m_5(Y) \otimes \ldots \otimes m_d(Y)$

continuing in the same way we need the non zero contributions $\pm b_\sigma$ from $W_g(d + 1, d) = \sum_\sigma b_\sigma \sigma$ only when

$-b_\sigma$ if $(3, 4) \sigma \tau_h^{-1} = (1, 2)$, $b_\sigma$ if $(3, 4) \sigma \tau_h^{-1} = (2, 4)$

$b_\sigma$ if $(1, 4) \sigma \tau_h^{-1} = (1, 2)$, $-b_\sigma$ if $(1, 4) \sigma \tau_h^{-1} = (2, 4)$.

The corresponding 4 values of $\sigma$ are:

$\sigma = (1, 2)(3, 4) \tau_h = (2, 4, \ldots , h)(h + 1, \ldots , 2h)$

$\sigma = (3, 4)(2, 4) \tau_h = (2, 3, 4) \tau_h = (1, 3, 2, 4, \ldots , h)(h + 1, \ldots , 2h)$

$\sigma = (1, 4)(1, 2) \tau_h = (1, 2, 4) \tau_h = (2, 3, 1, 4, \ldots , h)(h + 1, \ldots , 2h)$

$\sigma = (1, 4)(2, 4) \tau_h = (1, 4, 2) \tau_h = (2, 3)(4, \ldots , h)(h + 1, \ldots , 2h)$.

$$
(53) \quad tr(G(X,Y)) = T_d(X)T_d(Y)[-b_{1,2,h-2,h} + 2b_{h,h} - b_{1,2,h-3,h}]
$$

$d = 5$, $h = 3$, $(5)!^2 [-b_{1,3,3} + 2b_{3,3} - b_{1,2,3}] = -\frac{991}{105} - 2\frac{1601}{420} - \frac{151}{210} = -\frac{1867}{105}$.

It remains to prove that for all $h$ we have $-b_{1,2,h-2,h} + 2b_{h,h} - b_{1,2,h-3,h} \neq 0$. Unfortunately we cannot use directly Novak’s argument, so we leave this open.

In fact we have seen that the element $\Phi_d(1)$ is invertible in the center of $\Sigma_n(F^d)$ (cf. Remark 4.8). Novak’s result follows from writing the inverse as a Neumann series depending on the parameter $d$ with terms in $\mathbb{C}[S_n]$. Then the fact that $\Sigma_n(F^d)$ equals the group algebra $\mathbb{C}[S_n]$ only for $d \geq n$ reflects into the fact that this series converges only for $d \geq n$. Maybe a closer look
at Novak’s proof and the spectrum of Jucy–Murphy elements can solve this problem (see [14]).

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