Thermodynamical Properties of Hall Systems

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Abstract

We study quantum Hall effect within the framework of a newly proposed approach, which captures the principal results of some proposals. This can be established by considering a system of particles living on the non-commutative plane in the presence of an electromagnetic field and quantum statistical mechanically investigate its basic features. Solving the eigenvalue equation, we analytically derive the energy levels and the corresponding wavefunctions. These will be used, at low temperature and weak electric field, to determine the thermodynamical potential $\Omega_{nc}$ and related physical quantities. Varying $\Omega_{nc}$ with respect to the non-commutativity parameter $\theta$, we define a new function that can be interpreted as a $\Omega_{nc}$ density. Evaluating the particle number, we show that the Hall conductivity of the system is $\theta$-dependent. This allows us to make contact with quantum Hall effect by offering different interpretations. We study the high temperature regime and discuss the magnetism of the system. We finally show that at $\theta = 2l_B^2$, the system is sharing some common features with the Laughlin theory.

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1 Introduction

The quantum Hall effect (QHE) \([1]\) is a physical phenomenon that appears in two-dimensional electrons at low temperature and in the presence of a strong uniform magnetic field. It is an interesting subject, not only because of its precise quantized plateaus of the Hall conductivity \([2]\), but also of its relationship to different theories and mathematical formalism. Non-commutative (NC) geometry \([3]\) is one of the most successful tools that has been used to deal with the basic features of QHE. This is not surprising because this phenomena itself is of the non-commutative nature \([4, 5]\). In fact, to glue the system in the lowest Landau level, the confining potential energy should be strong enough. Therefore particles can not jump to the next level because of the large gap energy. Mathematically this effect can be traduced by a non-commuting spatial variables.

There are several works have been reported on the subject by quantum mechanically studying spinless particles. Among them, we quote the authors in \([8]\) who analyzed electrons in uniform external magnetic and electric fields, which move in a plane whose coordinates are non-commuting. They tuned on the non-commutativity parameter \(\theta\) such that electrons moving in the non-commutative coordinates are interpreted as either leading to the fractional QHE or composite fermions in the usual coordinates. In fact, these results will be discussed in terms of our main idea that will be analyzed in the present paper.

Motivated by the intrinsic relation between the NC geometry and QHE, we develop an approach to investigate the basic features of QHE. Based on the statistical mechanics technical, we analyze the thermodynamical properties of particles on the NC plane \(\mathbb{R}^2_\theta\). In this analysis, we involve the spin as an additional degree of freedom. More precisely, we consider particles of spin \(\frac{1}{2}\) living on \(\mathbb{R}^2_\theta\) in the presence of an uniform electromagnetic field and study its thermodynamical properties. Indeed, to characterize the system behaviour at low temperature as well as weak electric field, we determine the thermodynamic potential (TPO). This can be done by evaluating the corresponding physical quantities. Getting the particle number, we derive the Hall conductivity \(\sigma_{\text{nc}}^H\) in terms of \(\theta\). It allows us to define the widths of different plateaus entering in the game as \(\theta\)-function. Moreover by using some data, we show that \(\theta\) can experimentally be fixed. Giving different values to \(\theta\), we differently interpret \(\sigma_{\text{nc}}^H\). Taking advantage of the non-commutativity, we obtain a general results those can be used to explain fractional QHE, composites fermions and multilayer Hall systems. We analyze with the non-commutativity formalism the high temperature regime. In fact, the magnetism of our system will be discussed by writing down the critical point. In the end, we show that at the point \(\theta = 2l_B^2\) our system behaves like Laughlin one for the filling factor \(\nu = 1\).

The present paper is organized as follows. In section 2, the Hamiltonian describing the system under consideration on the ordinary plane \(\mathbb{R}^2\) will be given as well as its spectrum. This will be generalized by adopting the star product and therefore will offer for us a mathematical tools to deal with our proposal. In section 3, basing on the Fermi–Dirac statistics, we determine the corresponding partition function that leads to TPO. This will be used to define particle number, magnetization and
a TPO density. In section 4, using the standard definition, we obtain a general form for the Hall conductivity. After getting the filling factor, we offer different interpretations and establish some links with other results related to QHE, in section 5. The high temperature analysis and the magnetism of the system will be considered in section 6. Section 7 will devoted to discuss the critical point, i.e. $\theta = 2l_B^2$. We conclude and give some perspectives in the last section.

2 Hamiltonian and spectrum

To deal with our task, we need to fix some mathematical tools. Firstly, we define the Hamiltonian system on $\mathbb{R}^2$ and give its spectrum. Secondly, by adopting the NC geometry we show how the obtained results on $\mathbb{R}^2$ can be generalized to those on $\mathbb{R}^2_{\theta}$.

2.1 Ordinary plane $\mathbb{R}^2$

We consider one particle of spin $\frac{1}{2}$ and mass $m_0$ living on $\mathbb{R}^2$ in the presence of a magnetic $\vec{B} = B\vec{e}_z$ and electric $\vec{E} = E\vec{e}_y$ fields. Assuming that this system is confined in a finite surface such that $S = L_x L_y$. In the Landau gauge

$$\vec{A} = (-yB, 0, 0)$$

the system is described by the Hamiltonian [6]

$$H = \frac{1}{2m} \left( p_x - \frac{eB}{c} y \right)^2 + \frac{p_y^2}{2m} + g\mu_B \hat{S}B + eEy$$

where $m$ is the effective particle mass in a crystal lattice. In (2), the third term is resulting from the interaction between spin and $\vec{B}$, the last is reflecting the dipolar momentum and $\vec{E}$ interaction, $g$ is the Landé factor and $\mu_B = \frac{e\hbar}{2mc}$ is the Bohr magneton.

The spectrum of $H$ can analytically be obtained by solving the eigenvalue equation

$$H\Psi = E\Psi.$$  \hspace{1cm} (3)

Consequently, the energy levels are

$$E_{n, m_s}^\theta = \hbar \omega_c \left( n + \frac{1}{2} \right) + g\mu_B m_s B - \frac{E}{B} \hat{p}, \quad n = 1, 2, 3 \cdots, \ m_s = \pm \frac{1}{2}$$

where $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency and $\hat{p}$ is a new momentum defined as

$$\hat{p} = -p_x + \frac{mcE}{2B}.$$  \hspace{1cm} (4)

It is submitted to the constraint [6]

$$|\hat{p}| \leq \frac{eBL_y}{2c}$$

for a weak electric field. The wavefunctions corresponding to (4) are given by

$$\Psi_n(x, y) = e^{\frac{i}{\hbar}px_y} \exp \left[ -\frac{1}{2l_B^2} (y - y_0)^2 \right] H_n \left[ \frac{1}{l_B} (y - y_0) \right]$$
where $H_n$ are the Hermite polynomials and $l_B = \sqrt{\frac{\hbar c}{eB}}$ is the magnetic length. This latter plays an important role in the QHE world. In fact, it defines the area occupied by the quantum Hall droplet for particles in the lowest Landau level.

To close this part, we mention that the above results have been used to deal with the thermodynamical properties for the Hall particles of spin $\frac{1}{2}$ on $\mathbb{R}^2$. More precisely, integer QHE as well as multilayer Hall systems have been discussed. Later, we propose a newly approach based on the NC geometry that captures the basic features of [7] and leads to different results.

### 2.2 Non-commutative plane $\mathbb{R}^2_\theta$

As we claimed before, to deal with our task we may proceed by using the NC geometry [3]. This can be done by introducing the commutator

$$[x_j, x_k] = i\theta_{jk}$$

where $\theta_{jk} = \epsilon_{jk}\theta$ is the non-commutativity parameter, $\theta$ is a real and free quantity. This relation can be realized by considering the star product definition

$$f(x) \ast g(x) = \exp \left\{ i\frac{\theta_{jk}}{2} \partial_j \partial_k \right\} f(y)g(z) \bigg|_{x=y=z}$$

where $f$ and $g$ are two arbitrary functions, supposed to be infinitely differentiables. In what follows, we will use the standard commutation relations

$$[p_j, x_k] = -i\hbar\delta_{jk}, \quad [p_j, p_k] = 0$$

supplemented by the relation (8), which together define a generalized quantum mechanics and leads to the standard case once $\theta$ is switched off.

Now we show how the above tools can be employed to define the non-commutative version of the Hamiltonian (2). In doing so, we start by noting that $H$ acts on an arbitrary function $\Psi(\vec{r}, t)$ as

$$H \ast \Psi(\vec{r}, t) = H_{nc} \Psi(\vec{r}, t).$$

Applying the definition (9), we obtain the required Hamiltonian

$$H_{nc} = \frac{1}{2m} \left( \tilde{p}_x - \frac{eB}{\alpha c} \tilde{y} \right)^2 + \frac{p_y^2}{2m} + g\mu_B \tilde{S}B + \frac{eE}{\alpha^2} \tilde{y} - \frac{mc^2E^2}{2B^2} \left( 1 - \frac{1}{\alpha^2} \right)$$

where the quantities $\tilde{p}_x$ and $\tilde{y}$ are given by

$$\tilde{p}_x = \alpha p_x, \quad \tilde{y} = \alpha y + \frac{mc^2E}{eB^2} (\alpha - 1).$$

The important parameter of our theory is

$$\alpha = 1 - \frac{\theta}{2l_B^2}.$$
We have some remarks in order. First, comparing (2) with (12), one can see that (12) is including extra terms. Otherwise, it is clear that $H_{nc}$ is a generalized version of (2). Therefore, it is a good task to deeply investigate the consequences of $H_{nc}$ on the Hall effect. Second, according to (14) one can notice that there is a singularity at $\theta = 2l_B^2$ that should be taking into account. For this, we offer a section to discuss how, at this point, the present system can be linked to the Laughlin theory. Third, without loss of generality, we assume that $\alpha > 0$ in the forthcoming study.

To fully establish our mathematical tools, we need to derive the spectrum of the non-commutative Hamiltonian (12). As usual, this can be achieved by solving the eigenvalue equation

$$H_{nc}\Psi = E_{nc}\Psi$$

(15)

to get the energy levels

$$E_{n,m,s}^{nc} = \hbar \tilde{\omega}_c \left( n + \frac{1}{2} \right) - e E_B \tilde{p}_\theta + m_s g \frac{\hbar e B}{2m_0 c}, \quad n = 0, 1, 2 \ldots, m_s = \pm \frac{1}{2}$$

(16)

where $\tilde{\omega}_c$ and $\tilde{p}_\theta$ are given by

$$\tilde{\omega}_c = \frac{\omega_c}{\alpha}, \quad \tilde{p}_\theta = -\alpha p_x + \frac{mc E_B}{2}$$

(17)

In similar way to (6), for a weak electric field $\tilde{p}_\theta$ must fulfill the condition

$$\left| \tilde{p}_\theta - (1 - \alpha) \frac{mc E_B}{2} \right| \leq \alpha \frac{e B L_y}{2c}.$$  

(18)

As we will see later, this will be helpful in order to approximatively evaluate TPO on $\mathbb{R}_0^2$. The wavefunctions can be obtained as

$$\Psi_n(x, \tilde{y}, \theta) = e^{\frac{i}{\hbar}\tilde{p}_x x} \exp \left[ -\frac{1}{2l_B^2} (\tilde{y} - \tilde{y}_0)^2 \right] H_n \left[ \frac{1}{l_B} (\tilde{y} - \tilde{y}_0) \right]$$

(19)

where the new quantity

$$\tilde{l}_B = \sqrt{\frac{\alpha\hbar c}{\epsilon B}}$$

(20)

can be interpreted as an effective magnetic length. Moreover by analogy to $l_B^2$, (20) can be seen as an effective area of the quantum Hall droplet on $\mathbb{R}_0^2$. For $\alpha = 1$, we recover the spectrum of the system on $\mathbb{R}_0^2$ given before.

At this point, we have settled all formalism needed to do our task. In fact, we will show it can be used to derive more general results those can be related to the real physical systems. In doing so, let us begin by evaluating different thermodynamical quantities.

3 Thermodynamical properties

In dealing with QHE, we start by applying quantum statistical mechanics to derive different physical quantities. Indeed, we evaluate the partition function of our system in order to get the corresponding
TPO. Subsequently, we introduce the Mellin transformation to obtain a simplified form of TPO. Consequently, we end up with two different residues, which showing that TPO can be separated into two parts: monotonic and oscillating PTO’s. Combing all to derive the particle number and related matters as well as discussing the magnetization.

3.1 Thermodynamic potential

Normally in statistical mechanics, TPO can be obtained by determining the partition function describing the system. In our case, it is defined by

\[ Z_{nc} = \text{Tr} \left[ \exp \left( \frac{\tilde{\mu} - H_{nc}}{k_B T} \right) \right] \]  

(21)

where \( \tilde{\mu} \) is the chemical potential and trace is over all states given by (19). Since we are considering a system of fermions, the Fermi-Dirac statistics enforces to get

\[ Z_{nc} = \prod_{n,m,s,\tilde{p}_\theta} \left[ 1 + \exp \left( \frac{\tilde{\mu} - E_{n,m,s,\tilde{p}_\theta}}{k_B T} \right) \right] . \]  

(22)

This showing that TPO is

\[ \Omega_{nc} = -k_B T \sum_{n,m,s,\tilde{p}_\theta} \log \left[ 1 + \exp \left( \frac{\tilde{\mu} - E_{n,m,s,\tilde{p}_\theta}}{k_B T} \right) \right] . \]  

(23)

Keep in mind that \( \tilde{p}_\theta \) is a continue variable. By introducing a set of the dimensionless quantities

\[ p_\theta = \frac{\tilde{p}_\theta}{mc}, \quad \mu = \frac{\tilde{\mu}}{mc^2}, \quad \lambda = \frac{k_B T}{mc^2}, \quad \epsilon_{nc} = \frac{E_{nc}}{mc^2} \]  

(24)

we write \( \Omega_{nc} \) as

\[ \Omega_{nc} = -mc^2 \lambda \alpha N_\phi \int_{\frac{a+b}{2}}^{a+b} dp_\theta \frac{1}{b} \sum_{n,m,s} \ln \left[ 1 + \exp \left( \frac{\mu - \epsilon_{nc}}{\lambda} \right) \right] \]  

(25)

where \( a \) and \( b \) are given by

\[ a = (1 - \alpha) \frac{E}{B}, \quad b = \alpha \frac{eBL_y}{mc^2}. \]  

(26)

The number \( N_\phi \) is the quantized flux:

\[ N_\phi = \frac{\phi}{\phi_0} = \frac{eBS}{hc}. \]  

(27)

In the QHE world, it is well-known that \( N_\phi \) is an important quantity that should be determined. This comes from the fact that \( N_\phi \) is exactly the degeneracy of Landau levels, which defines the filling factor, see subsection (4.1).

To calculate (25), we need to introduce a relevant method. This can be realized by making use of the Mellin transformation. In general, it is defined by

\[ f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(t)x^{-t} dt \quad \iff \quad \tilde{f}(t) = \int_0^{\infty} f(x)x^{t-1} dx \]  

(28)
for an arbitrary function $f(x)$. By applying (28) to our case with respect to the variable $e^{\mu/\lambda}$ and after a straightforward calculation, we find

$$\Omega^{nc} = \pm mc^2 \lambda \alpha N_\phi \sum_s \text{Res} \frac{s \exp \left( \frac{\mu}{\lambda} \right)}{s \sin \pi s} \int_{-\infty}^{\infty} \frac{dp}{b} \sum_{n,m} \exp \left( -\frac{s E_n^{nc}}{mc^2 \lambda} \right)$$

(29)

where the upper and lower sign refer, respectively, to closing the contour to the left and right of the imaginary axis for $\bar{\mu} > 0$ and $\bar{\mu} < 0$. The integral gives

$$\Omega^{nc} = \pm mc^2 \lambda \alpha N_\phi \sum_s \text{Res} \left[ \frac{\pi \exp \frac{s \bar{\mu}}{\lambda}}{s \sin(\pi s)} \frac{\cosh(s \frac{g^* \kappa}{2 \lambda})}{\sinh(s \frac{g^* \kappa}{2 \lambda})} \right]$$

(30)

and the new involved parameters

$$g^* = \frac{gm}{2m_0}, \quad \kappa = \frac{B}{B'}, \quad B' = \frac{m^2 c^3}{\epsilon \hbar}, \quad \bar{\mu} = \mu + \frac{a E}{2 B}.$$  

(31)

In conclusion, one has to distinguish two different cases. Firstly $\bar{\mu} > 0$, the sum of residues can be calculated at the negative poles in the real semiaxis $s = 0, -1, -2, -3, \cdots$ and in the imaginary semiaxis at the poles $s_l = 2\pi i l \frac{\alpha \lambda}{\kappa}$. Secondly $\bar{\mu} < 0$, the sum of residues can be calculated by closing the integration contour to the right, i.e. considering only the positive real semiaxis, however the residues can be evaluated at the poles $s = 1, 2, 3, \cdots$.

It is not easy to directly evaluate $\Omega^{nc}$ in (30), but it can be achieved by introducing some approximations. For this, we consider the high and low temperature regimes as well as assume that the electric field is weak. In doing so, we separate TPO into two parts: monotonic and oscillating.

### 3.2 Low temperature regime

To completely determine TPO, we consider two different cases those are low and high temperature regimes. Note that, the first case is governed by the constraint $\lambda \ll \bar{\mu}$. Since we have two poles, one can separate TPO into two parts, which are the monotonic $\Omega^{nc}_{mo}$ and oscillating $\Omega^{nc}_{os}$ terms corresponding, respectively, to two poles $s = 0$ and $s_l = 2\pi i l \frac{\alpha \lambda}{\kappa}$.

Now we evaluate each part separately. Indeed, using the residue methods, one can show that the monotonic part is

$$\Omega^{nc}_{mo} = -mc^2 N_\phi \kappa \alpha \left[ \frac{1}{4\pi^2} \alpha \xi^2 + \frac{1}{4} \left( \alpha g^* \kappa - \frac{1}{3} \right) + \frac{\pi^2}{3} \alpha \left( \frac{\lambda}{\kappa} \right)^2 + \alpha^3 \eta^2 \frac{1}{12} \right]$$

(32)

where $\xi$ and $\eta$ are

$$\xi = \frac{2\pi \bar{\mu}}{\kappa}, \quad \eta = \frac{e E \lambda y}{\kappa mc^2}.$$  

(33)

Note that, in deriving $\Omega^{nc}_{os}$ we have dropped the contributions resulting from other residues $s \neq 0$, because they are exponentially small. On the other hand, the oscillating TPO can be evaluated by assuming that the condition $\lambda \ll \frac{\kappa}{\alpha}$ is holding. This implies that $\Omega^{nc}_{os}$ can be written as

$$\Omega^{nc}_{os} = -mc^2 N_\phi \kappa \sum_{l=1}^{\infty} (-1)^{l+1} \frac{\cos(\alpha \xi l) \cos(\alpha g^* \pi l)}{\pi^2 l^2} \frac{\sin(\pi \eta l)}{\pi \eta l}.$$  

(34)
Therefore, combining all we get
\[ \Omega_{\text{lo}}^{\text{nc}} = \Omega_{\text{mo}}^{\text{nc}} + \Omega_{\text{os}}^{\text{nc}}. \] (35)

As claimed before, we are considering a very weak electric field. This assumption is equivalent to the constraint
\[ E \ll \frac{\kappa mc^2}{eL_y} \iff \eta \ll 1. \] (36)

In this limit, we show
\[ \Omega_{\text{lo}}^{\text{nc}} = -mc^2N_\phi k\alpha \left[ \frac{1}{4\pi^2}\alpha^2 + \frac{1}{4}\left(\alpha g^* - \frac{1}{3}\alpha\right) + \frac{\pi^2}{3}\alpha \left(\frac{\lambda}{\kappa}\right)^2 + \frac{1}{\alpha\pi^2} F^{\text{nc}}(\xi) \right] \] (37)

where \( F^{\text{nc}}(\xi) \) is given by
\[ F^{\text{nc}}(\xi) = \sum_{l=1}^{\infty} (-1)^{l+1} \frac{1}{l^2} \cos(\alpha \xi l) \cos(\pi \alpha g^* l). \] (38)

It can be rearranged as
\[ F^{\text{nc}}(\xi) = \frac{1}{2} \sum_{l=1}^{\infty} (-1)^{l+1} \frac{1}{l^2} \left[ \cos \left\{ \alpha (\xi + g^* \pi) l \right\} + \cos \left\{ \alpha (\xi - g^* \pi) l \right\} \right]. \] (39)

This function will play an important role in discussing QHE on \( \mathbb{R}^2_\theta \). In fact, we will see how it can be linked to the filling factor.

Statistical mechanics teaches us that the thermodynamical potential is the first step that should be determined in order to study the system behaviour. This is because of its relationships to other physical quantities. To clarify this point, we start by evaluating the particles number:
\[ N_{\text{lo}}^{\text{nc}} = -\frac{1}{mc^2} \frac{\partial \Omega_{\text{lo}}^{\text{nc}}}{\partial \mu}. \] (40)

After a straightforward calculation, we find
\[ N_{\text{lo}}^{\text{nc}} = N_\phi \frac{\alpha}{\pi} \left[ \alpha \xi + \frac{2}{\alpha} \frac{\partial F^{\text{nc}}(\xi)}{\partial \xi} \right] \] (41)

which is depending to \( N_\phi \) and \( \alpha \). In fact, this will be helpful in defining the Hall conductivity on \( \mathbb{R}^2_\theta \) and thus getting its final form.

It might be a good task to see how the magnetization of the present system does look like. In doing so, we start by noticing that the magnetic field is along \( z \)-direction, then there is only one component following the \( z \) axis of the magnetic moment of particles on \( \mathbb{R}^2_\theta \). It is given by
\[ M^{\text{nc}} = -\frac{\partial \Omega_{\text{lo}}^{\text{nc}}}{\partial B}. \] (42)

It can be calculated to get
\[ M_{\text{lo}}^{\text{nc}} = mc\mu \frac{eS}{\hbar} \left[ \frac{\xi}{2\pi^2} \left( \alpha^2 - \alpha \right) + \frac{1}{2\xi} \left( \alpha (2\alpha - 1) g^* - \frac{1}{3} \right) + \frac{2F^{\text{nc}}(\xi)}{\pi^2\xi} \right. \\
+ \frac{2\pi^2}{3\xi} \left( \alpha^2 - \alpha \right) \left( \frac{\lambda}{\kappa} \right)^2 - \frac{1}{\pi^2} \frac{\partial F^{\text{nc}}(\xi)}{\partial \xi} - \frac{a}{\pi \kappa B} \left( \frac{\alpha^2}{2} + \frac{1}{\xi} \frac{\partial F^{\text{nc}}(\xi)}{\partial \xi} \right) \left. \right]. \] (43)
The obtained form of $M_{\text{lo}}^{\text{nc}}$ can deeply be investigated to make contact with early works on the magnetization [9, 10, 11]. With this, one may have a full picture on the magnetism of the system at low temperature.

We are not going to stop at this level in deriving physical quantities. In fact, since TPO is $\theta$-dependent, one may define a new function by making variation with respect to $\theta$. This is equivalent to write

$$\chi_{\text{lo}}^{\text{nc}} = -\frac{\partial \Omega^{\text{nc}}_{\text{lo}}}{\partial \theta}$$

which leads to

$$\chi_{\text{lo}}^{\text{nc}} = mc\bar{\mu}_e S (\alpha - 1) \left[ \frac{\alpha}{2\pi^2} \xi - \frac{\alpha^2}{2\pi\kappa} \left( \frac{E}{B} \right)^2 + \frac{\alpha g^*}{2\xi} + \frac{2\pi^2}{3} \frac{\alpha}{\xi} \left( \frac{\lambda}{\kappa} \right)^2 + \frac{1}{\pi^2} \frac{\partial F^{\text{nc}}(\xi)}{\partial \alpha} \right].$$

Because of $\theta$ has length square of dimension, thus $\chi_{\text{lo}}^{\text{nc}}$ can be interpreted as a TPO density of a system possessing $\theta$ as surface. This is the case for instance for a quantum Hall droplet where its area is $l_B^2 = \frac{\hbar c}{eB}$, which corresponds to $\alpha = \frac{1}{2}$ in our case. Of course (45) goes to zero once we set $\alpha = 1$.

4 Hall conductivity

To make contact with the Hall effect, we should determine the Hall conductivity in terms of our language. It can be done by adopting the usual definition of the Hall current as well as using the Heisenberg equation for the particle velocity on $\mathbb{R}^2$. As we will see soon, these will offer for us a general form of the filling factor that can be used to show that our system is describing a real physics.

4.1 Filling factor on $\mathbb{R}^2$

To establish different definitions to deal with our issues on the generalized plane, let us sketch some discussion about the Hall conductivity on $\mathbb{R}^2$. We start from classical to quantum mechanics in order to emphasis the differences.

Let us consider system of $N$ particles living on the plane in the presence of magnetic $\vec{B} = B\vec{e}_z$ and electric $\vec{E} = B\vec{e}_y$ fields. Experimentally, the classical result is showing that the inverse of $\sigma_H$ is behaving like a straightforward line in terms of $B$, which has given birth to the Hall effect. Theoretically, this effect can be explained by studying Newton mechanically the motion of a Hall system. In the end, one can obtain the Hall conductivity

$$\sigma_H = \frac{-\rho e c}{B}$$

where the particle density $\rho = \frac{N}{S}$ is resulting from the Hall current definition. This is showing how the conductivity is linked to the magnetic field. Moreover, it is easily seen that (46) is actually reflecting the experiment result.

Now let us see how $\sigma_H$ does look like in quantum mechanics. Resorting the spectrum of the Hamiltonian describing the above system, without interactions, we can derive $\sigma_H$ in terms of the
fundamental constants. This is

\[ \sigma_H = -\nu \frac{e^2}{h} \]  (47)

where the filling factor \( \nu \) is given by

\[ \nu = \frac{\rho ch}{eB} = 2\pi l_B^2 \rho. \]  (48)

Clearly, \( \nu \) is actually measuring different values of \( B \) for a fixed density \( \rho \). Furthermore, to talk about QHE \( \nu \) should be quantized over a large range of a strong magnetic. This can be traduced by defining \( \nu \) as

\[ \nu = \frac{N}{N_\phi} \]  (49)

where \( N_\phi \) represents also the degree of degeneracy of each Landau level. Otherwise, \( \nu \) is the ratio between particle number and number of the accessible states per Landau level. To get integer QHE or fractional QHE, \( \nu \) should be, respectively, integrally or fractionally quantized.

In conclusion, we need to determine the analogue of \( \nu \) on \( \mathbb{R}_\theta^2 \) and see how can be used to offer different interpretations and establish some links. To do this end, we assume that all definitions above will be also valid on the non-commutative plane.

### 4.2 Effective filling factor

According to the above analysis, it follows that the filling factor is a relevant quantity, which has to be determined in having the Hall effect. In doing so, we use (47) to write the non-commutative Hall conductivity as

\[ \sigma_{nc}^H = -\nu_{nc} \frac{e^2}{h} \]  (50)

where \( \nu_{nc} \) is its corresponding effective filling factor. Using (49), \( \nu_{nc} \) can be seen as ratio between the particles number \( N_{lo}^{nc}(\xi) \) and the degeneracy of the Landau levels \( N_\phi \), i.e. quantum flux. Otherwise, one may write \( \nu_{nc} \) as

\[ \nu_{nc}(\xi) = \frac{N_{lo}^{nc}(\xi)}{N_\phi}. \]  (51)

Since we have all ingredients, it is easy to see

\[ \nu_{nc}(\xi) = \frac{\alpha}{\pi} \left[ \alpha \xi + \frac{2}{\alpha} \frac{\partial F^{nc}(\xi)}{\partial \xi} \right]. \]  (52)

Note that, (52) is showing that there is a relation between the non-commutativity of space and the filling factor of the present system. To clarify this point, we simplify \( \nu_{nc}(\xi) \) to another form. Indeed, using the identity

\[ \sum_{i=1}^{\infty} (-1)^{i+1} \frac{\cos(IX)}{I^2} = \frac{\pi^2}{12} - \frac{x^2}{4} \]  (53)

we show that \( F^{nc}(\xi) \) can be written as

\[ F^{nc}(\xi) = \begin{cases} \frac{\pi^2}{12} - \frac{\alpha^2}{4} \xi^2 - \frac{1}{3} \alpha^2 g^* \pi^2 + \alpha \pi \xi - \frac{1}{2} (1 - \alpha g^*) \pi^2 & \text{if } -u_1 \leq \xi \leq u_1 \\ \frac{\pi^2}{12} - \frac{\alpha^2}{4} \xi^2 - \frac{1}{3} \alpha^2 g^* \pi^2 + \alpha \pi \xi - \frac{1}{2} (1 - \alpha g^*) \pi^2 + \frac{\alpha \pi \xi - \frac{1}{2} (1 - \alpha g^*) \pi^2}{2} & \text{if } u_1 \leq \xi \leq u_2 \end{cases} \]  (54)
where $u_1$ and $u_2$ are given by

\[ u_1 = \left( \frac{1}{\alpha} - g^* \right) \pi, \quad u_2 = \left( \frac{1}{\alpha} + g^* \right) \pi. \quad (55) \]

Moreover, (54) can be generalized to any odd integer value $i$, such as

\[ F_{nc}(\xi) = \begin{cases} \frac{i^2}{12} - \frac{1}{4} \alpha^2 \xi^2 - \frac{1}{4} g^* \alpha^2 \pi^2 - \frac{1}{4} i^2 \pi^2 + \frac{1}{2} i \alpha \xi \pi + \frac{1}{2} \alpha g^* \pi^2 & \text{if } u_1' \leq \xi \leq u_2' \\frac{i^2}{12} - \frac{1}{4} \alpha^2 \xi^2 - \frac{1}{4} g^* \alpha^2 \pi^2 - \frac{1}{4} i^2 \pi^2 - \frac{1}{4} \frac{1}{2} \alpha \xi (i + 1) \pi & \text{if } u_2' \leq \xi \leq u_3' \end{cases} \quad (56) \]

where $u_1'$, $u_2'$ and $u_3'$ are

\[ u_1' = \left( \frac{i}{\alpha} - g^* \right) \pi, \quad u_2' = \left( \frac{i}{\alpha} + g^* \right) \pi, \quad u_3' = \left( \frac{i + 2}{\alpha} - g^* \right) \pi. \quad (57) \]

Therefore, the final form of the filling factor is given by

\[ \nu_{nc}(\xi) = \begin{cases} 0 & \text{if } 0 \leq \xi \leq u_1' \\ i \alpha & \text{if } u_1' \leq \xi \leq u_2' \\ (i + 1) \alpha & \text{if } u_2' \leq \xi \leq u_3' \end{cases} \quad (58) \]

This is more suggestive and thus one needs to look deeply for its interpretation. Moreover, we will see that the different intervals can be used to discuss a possible measurement of $\theta$. These materials will be the subject of the next section.

We still need different definitions settled above for the Hall effect on $\mathbb{R}^2$. Indeed, by comparing (48) with (58), it follows that one can define an effective magnetic field as

\[ B_{\text{eff}} = \frac{B}{\alpha} \equiv \frac{B}{1 - \frac{g}{2\gamma}}. \quad (59) \]

This implies that its effective filling factor is

\[ \nu_{\text{eff}} \equiv \nu_{nc}(\xi) = \frac{\rho_{\text{ch}}}{e B_{\text{eff}}} \quad (60) \]

where the quantity $\frac{\rho_{\text{ch}}}{e B_{\text{eff}}}$ is integrally quantized and more specifically is corresponding to the case when $\alpha = 1$ in (58), namely $\nu_{nc}(\xi)|_{\alpha=1}$.

Equations (59) and (60) are interesting results in sense that they will be employed to clarify our motivation behind the present work. In fact, we use $B_{\text{eff}}$ and $\nu_{\text{eff}}$ to offer some interpretations and also show different bridges between our proposal and some results related to QHE.

## 5 Interpretation

In the beginning, let us notice that particles of spin $\frac{1}{2}$ living on $\mathbb{R}^2$ when uniform external magnetic as well as electric fields are applied can be envisaged as the usual motion of particles experiencing an effective magnetic field (59). This can be regarded as standard results, but let us go deeply to see what we can do with different values of $\theta$. 

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5.1 Measuring \( \theta \)

It is a good job to look for an experiment realization of the non-commutativity parameter \( \theta \). For this, we offer one possibility that can be used to identify our system to GaAs. In doing so, let us return to comment the filling factor (58).

It is clear that the widths of the plateaus coincide with different intervals appearing in different values of (58). For the second and third equations in (58), the widths, respectively, are given by

\[
W_{21} = 2\pi g^*, \quad W_{32} = 2\pi \left( \frac{1}{\alpha} - g^* \right). \tag{61}
\]

Consequently, there are two critical points those should be discussed in order to get more information about our system. These are

\[
g_{21}^* = 0, \quad g_{32}^* = \frac{1}{\alpha}. \tag{62}
\]

Let us discuss each case separately. Indeed, \( g_{21}^* \) is actually corresponding to the filling factor

\[
\nu^{nc}(\xi) = (i + 1)\alpha. \tag{63}
\]

It follows that, there is no splitting of spin degree of freedom in each Landau level. However, in the second case all Landau levels except the ground state are spin degenerate and we have

\[
\nu^{nc}(\xi) = i\alpha. \tag{64}
\]

It turns out that \( g_{32}^* \) is important value because it is linked to the non-commutativity of the system. In fact, since \( \theta \) is a free parameter one can get different \( g_{32}^* \) in order to link our system to different real physics systems. In what follows, we offer an experiment measurement of \( \theta \) by using some data. Indeed, according to [12] for the Hall system GaAs, we have

\[
m \approx 0.07 m_0, \quad g \approx 0.8, \quad g^* = 0.03. \tag{65}
\]

This implies that at the second point, one may fix \( \theta \) as

\[
\alpha \approx 33.33 \iff \theta \approx -64.66 l_B^2. \tag{66}
\]

Thus, when \( \theta \approx -64.66 l_B^2 \) our system can be regarded as GaAs. In conclusion, this results is interesting in sense that one may use different experiment data to see how the non-commutativity parameter can experimentally be detected. Note that, one possibility has been realized by one of the present authors in [16].

5.2 Dayi and Jellal approach

Dayi and Jellal proposed an approach based on the NC geometry tools to describe the Hall effect of a system of electrons [8]. In fact, their corresponding filling factor is found to be

\[
\nu_{dj} = \frac{\pi}{2} \rho(l_B^2 - \theta^*). \tag{67}
\]
which is identified to the observed fractional values for FQHE. This approach also allowed to make a link with the composite fermion approach \[14\] by setting an effective magnetic field

\[ B_{dj} = \frac{B}{1 - \theta l_B^2} \]  

(68)

similar to that felt by the composite fermions, with \( l_B' = 2l_B \) and \( \theta' \) is a non-commutativity parameter.

Now let see if the above approach has some overlapping with our analysis. We start by noting that the relations (67) and (68) have been obtained for a spinless electrons. The fact that of taking into account of spin as an additional degree of freedom plus the system confinement in rectangular area offered for us a possibility to have plateaus and therefore an experiment measurement for \( \theta \). On the other hand, for zero spin, one may have the identification

\[ \nu_{\text{eff}} |_{\theta = \theta_{dj}} = \nu_{dj} \]

(69)
at the critical point \( g^* = 0 \) and thus the comparison should be done with respect to (63). It follows that, one can solve the equation

\[ (i + 1)\alpha_{dj} = \frac{\pi}{2} l_B^2 \theta' \alpha' \]

(70)
to establish different relations between \( \theta \) and \( \theta' \), or let say between \( \alpha \) and \( \alpha' \) with

\[ \alpha' = 1 - \frac{\theta'}{l_B^2}. \]

(71)
It is easily seen that one may have

\[ \alpha_{dj} = \frac{\nu}{2(i + 1)} \alpha' \]

(72)
where the filling factor \( \nu \) is given by (48). In terms of \( \theta \), we obtain

\[ \theta_{dj} = 2l_B^2 - \frac{\nu}{2(i + 1)} (2l_B^2 - \theta'). \]

(73)
It reflects that the results derived in [8] can be linked to the present approach if our parameter \( \theta \) is fixed to be \( \theta_{dj} \). It is obvious that, whenever we have \( \theta = \theta' \), \( \nu \) should be quantized as even integral values, such as

\[ \nu = 2(i + 1). \]

(74)
This is showing that there is missing part of the filling factor, which is the odd values. Therefore, we have a partially integer QHE. To overcome this situation, we will show how to get a full picture of the integer QHE.

5.3 Other approaches

We are looking for other explanations of our results. Indeed, by tuning on \( \theta \) we offer four different interpretations of \( \nu_{\text{eff}} \). These are related to integer QHE, fractional QHE, composite fermions and multilayer Hall systems.
• **Integer quantum Hall effect**: it is corresponding to an integral quantized Hall conductivity, i.e. the filling factor is integer. To reproduce this effect from our consideration, we simply set
\[ \alpha = 1 \]  
which basically corresponds to \( \theta \) is switched off. In this case our analysis coincide with that has been reported on the subject in [7]. Consequently, this result is showing that our approach is relevant for QHE and is general in sense that one can get other links.

• **Fractional quantum Hall effect**: is one of the most interesting features of low-dimensional systems [13]. It is characterized by the observed Hall conductivity
\[ \sigma_H = f \frac{e^2}{h} \]  
where \( f \) is denoting the fractional quantized values of the filling factor \( \nu \). This effect can be interpreted in terms of the Hall effect on \( \mathbb{R}^2_\theta \). More precisely, we identify the effective filling factor \( \nu_{\text{eff}} \) with the observed value \( f \) by requiring that
\[ \nu_{\text{eff}}|_{\theta=\theta_f} = f. \]  
This can be solved by considering different plateau widths. According to [61], for \( W_{21} \) we get the first critical value of \( \theta \)
\[ \theta_{t_1} = 2l_B^2 \left( 1 - \frac{f}{t} \right). \]  
By considering \( W_{32} \), we obtain the second
\[ \theta_{t_2} = 2l_B^2 \left( 1 - \frac{f}{t + 1} \right). \]  
Therefore, when \( \theta \) is fixed to be \( \theta_f \), with \( i = 1, 2 \), one can envisage the Hall effect on \( \mathbb{R}^2_\theta \) as the usual fractional QHE on the plane.

• **Composite fermions**: are new kind of particles appeared in condensed matter physics to provide an explanation of the behavior of particles moving in the plane when a strong magnetic field \( B \) is present [14]. Particles possessing \( 2p \), with \( p \in \mathbb{N}^* \), flux quanta (vortices) can be thought of being composite fermions. One of the most important features of them is they feel effectively a magnetic field given by
\[ B^* = B - 2p\Phi_0\rho. \]  
To interpret particles of spin \( \frac{1}{2} \) living on \( \mathbb{R}^2_\theta \) in the magnetic field \( B \) and electric field \( E \) as the usual composite fermions we should tune on \( \theta \), such as
\[ B_{\text{eff}}|_{\theta=\theta_{\text{cf}}} = B^* \]  
where \( B_{\text{eff}} \) is given by [59]. We solve [81] to obtain
\[ \theta_{\text{cf}} = 2l_B^2 \left[ 1 - \left( 1 - \frac{2p\Phi_0\rho}{B} \right)^{-1} \right]. \]
In the limit of a strong magnetic field, it leads
\[ \theta_{cf} \approx -4\pi p l_B^4. \] (83)

Therefore, composite fermions can be envisaged as particles living on \( \mathbb{R}^2_\theta \) in the magnetic field \( B \) and the electric field \( E \) when we set \( \theta = \theta_{cf} \).

- **Multilayers**: structure formed by several identical Hall systems and its integral quantization of the total Hall conductivity is given by [15]

\[ \sigma_{\text{tot}}^i = J \sigma_{\text{H}}^i \] (84)

where \( J \) is the number of the layers including in such structure and \( \sigma_{\text{H}}^i \) corresponding the Hall conductivity in a single layer.

To describe the integer quantization of multilayers from the motion of particles of spin \( \frac{1}{2} \) on \( \mathbb{R}^2_\theta \) in the presence of \( B \) and \( E \), one may quantize \( \alpha \) as
\[ \alpha_J = J. \] (85)

This is reflecting that how this kind of Hall structures can be identified to our system living on the non-commutative plane.

In conclusion, by tuning \( \theta \) on we got different interpretations. These concern integer QHE, fractional QHE, CF as well as multilayer Hall systems. Therefore, the above results are proving that our analysis is relevant for the Hall effect.

### 6 High temperature analysis

To complete our analysis, we need to discuss the high temperature regime of the present system on the non-commutative plane. Mathematically, this regime is equivalent to have the limit \( \lambda \gg \kappa \).

Moreover, our discussion will be achieved by requiring that the conditions \( \eta \ll 1 \) and \( \bar{\mu} < 0 \) are satisfied. These will be used to get the corresponding TPO and related matters. In particular, we analyze the magnetism by showing that there is a phase transition.

In getting TPO at high temperature, we evaluate (80) by closing the contour to the right of the imaginary axis and taking the lower sign. This is

\[ \Omega_{hi}^{\text{nc}} = mc^2 \lambda N_\phi \sum_s \text{Re} s \frac{\pi e^{\frac{s}{2}}}{s \sin \pi s} \frac{\cosh \left( \frac{s g \kappa}{2 \lambda} \right)}{\sinh \left( \frac{s \kappa}{2 \alpha \lambda} \right)}. \] (86)

Under the previous conditions, we find

\[ \Omega_{hi}^{\text{nc}} \approx mc^2 \lambda N_\phi \left[ 2\alpha^2 \frac{\lambda}{\kappa} L_2 \left( -e^{\bar{\mu}/\lambda} \right) - \left( \alpha^2 g^2 - \frac{1}{3} \right) \frac{\kappa}{4\lambda \left( 1 + e^{-\bar{\mu}/\lambda} \right)} \right]. \] (87)

up to the first order of \( \frac{\kappa}{\lambda} \). This is describing the system behaviour at high temperature, which goes to its analogue on \( \mathbb{R}^2 \) by choosing \( \alpha = 1 \).
Similarly to low temperature case, we use \((87)\) to derive some physical quantities. Indeed, from the standard definition, we show that the particle number is
\[
N^{nc}_{hi} = -N_\phi \left[ 2\alpha^2 \frac{\lambda}{\kappa} Li_1 \left( -e^{\bar{\mu}/\lambda} \right) - \left( \alpha^2 g^* 2 - \frac{1}{3} \right) \frac{\kappa}{16\lambda \cosh^2 \left( \frac{\mu}{4\lambda} \right)} \right]
\]  
(88)
where in general for any integer \(n\), the logarithmic function is given by
\[
Li_n \left( -e^{\bar{\mu}/\lambda} \right) = \sum_{l=1}^{\infty} \frac{1}{l^n} \left( -e^{\bar{\mu}/\lambda} \right)^l.
\]  
(89)
According to (51), the corresponding Hall conductivity can be written as
\[
\sigma^{nc}_{hi} = \frac{e^2}{\hbar} \left[ 2\alpha^2 \frac{\lambda}{\kappa} Li_1 \left( -e^{\bar{\mu}/\lambda} \right) - \left( \alpha^2 g^* 2 - \frac{1}{3} \right) \frac{\kappa}{16\lambda \cosh^2 \left( \frac{\mu}{4\lambda} \right)} \right]
\]  
(90)
which is the general form that can be derived for particles of spin \(\frac{1}{2}\) living on \(\mathbb{R}^2_\theta\). One may study its behaviour at different values of \(\theta\) to see what make difference with respect to low temperature case.

At this level, let us discuss the magnetism that can be exhibited by our system at high temperature. Indeed, we start by determining its magnetization, which is
\[
M^{nc}_{hi} = -mc^2 \lambda \frac{N_\phi}{B\kappa} \left[ 4\lambda \alpha (\alpha - 1) Li_2 \left( -e^{\bar{\mu}/\lambda} \right) - \left( \alpha g^* 2 (2\alpha - 1) - \frac{1}{3} \right) \frac{\kappa^2}{2\lambda \left( 1 + e^{-\bar{\mu}/\lambda} \right)} \right]
\]  
\[
+ \frac{aE}{2B} \left[ 2\alpha^2 Li_1 \left( -e^{\bar{\mu}/\lambda} \right) + \left( \alpha^2 g^* 2 - \frac{1}{3} \right) \left( \frac{\kappa}{4\lambda} \right)^2 \frac{1}{\cosh^2 \left( \frac{\mu}{4\lambda} \right)} \right].
\]  
(91)
Therefore, we end up with the following consequences. From (87) and (91), it follows that there is a phase transition, which is governed by the critical point
\[
g^* = \frac{1}{\alpha \sqrt{3}}.
\]  
(92)
More precisely, this is showing that the system is behaving like a diamagnetic for \(g^* < \frac{1}{\alpha \sqrt{3}}\). However, it has a paramagnetic nature if \(g^* > \frac{1}{\alpha \sqrt{3}}\). This is interesting in sense that one may differently fix \(\alpha\) to get different critical values and therefore different transitions in real physical systems.

On the other hand, (92) can be employed to experimentally measure \(\theta\). In doing so, let us solve (92) to find
\[
\theta = 2l_B^2 \left( 1 - \frac{1}{g^* \sqrt{3}} \right).
\]  
(93)
Using the experiment results given by (65), we obtain
\[
\theta = -36, 53 \ l_B^2
\]  
(94)
which is showing that the phase of our system is changing at this point. Therefore, it gives another way of making use of the evidently experiment of the non-commutativity parameter.
One can also define a TPO density at high temperature limit. After calculation, we end up with a new function as

$$\chi^{nc}_{hi} = (1 - \alpha) mc^2 \lambda \frac{N_\phi}{B\kappa} \left[ 4\lambda \alpha L_i \left( -e \tilde{\mu} \right) - \frac{\alpha g^2 \kappa^2}{2\lambda (1 + e^{-\frac{\mu}{\lambda}})} \right] - \frac{\alpha}{2} \left( 2\alpha^2 L_i \left( -e \tilde{\mu} \right) - (\alpha^2 g^2 - \frac{1}{3}) \left( \frac{\kappa}{4\lambda} \right)^2 \frac{1}{ \cosh^2 \left( \frac{\mu}{\lambda} \right)} \right) \left( \frac{E}{B} \right)^2. \quad (95)$$

With this we complete our analysis as concerning the high temperature regime. Of course, some questions related to the Hall conductivity that should be investigated deeply.

### 7 Critical point

Finally, let us consider the critical point corresponding to the Hamiltonian describing the present system on the plane $\mathbb{R}^2$. In fact, we show that the Laughlin theory can be recovered at this point. In doing so, let us return to (14) in order to notice

$$\theta_c = 2l_B^2. \quad (96)$$

Inserting this in the non-commutative version of (2), we find a new Hamiltonian

$$H^{nc}|_{\theta_c} = \frac{p_y^2}{2m} + \frac{m\omega^2}{2} y^2 + eE \left( y + \frac{l_B^2}{\hbar} p_x \right). \quad (97)$$

It can be simplified by returning to (8) in order to write down a new commutator as

$$[x, y] = 2il_B^2. \quad (98)$$

Remember that this can be realized by dealing with the variable $y$ as a canonical momentum of $x$ and vis versa. This processing leads to define

$$y = 2\frac{l_B^2}{\hbar} p_x, \quad x = -2\frac{l_B^2}{\hbar} p_y. \quad (99)$$

They allow us to write (97) as

$$H^{nc}|_{\theta_c} = \frac{p_y^2}{2m} + \frac{m\omega^2}{2} y^2 + \frac{3}{2} eE_y. \quad (100)$$

To recover the well-know results on QHE, one may handle (100). This can be done by rearranging it to get one-dimensional harmonic oscillator. Indeed, we simply make the following change

$$Y = y + \frac{3eE}{m\omega^2}; \quad (101)$$

to end up with the Hamiltonian

$$H^{nc}|_{\theta_c} = \frac{p_Y^2}{2m} + \frac{m\omega^2}{2} Y^2 - \frac{9}{8} mc^2 \left( \frac{E}{B} \right)^2. \quad (102)$$
This is actually describing the lowest Landau level (LLL) of our system. Therefore, another theory of QHE can be reformulated by analyzing (102). Indeed, it gives the Landau levels for the system confined in LLL, since the term $\frac{9}{8}mc^2 (E_B)^2$ can be dropped for a strong magnetic field. Thus, one may write $H_{nc}|\theta_c$ as

$$H_{nc}|\theta_c = \frac{m\omega_c^2}{2}(X^2 + Y^2).$$ \hspace{1cm} (103)

This is just a confining potential and therefore can be used to make contact with the Laughlin theory [17] for FQHE. Indeed, let us define the creation and annihilation operators as

$$Z = \frac{1}{2} (X - iY), \quad \bar{Z} = \frac{1}{2} (X + iY)$$ \hspace{1cm} (104)

which satisfy the commutation relation

$$[Z, \bar{Z}] = I.$$

It is easily seen that $H_{nc}|\theta_c$ can be mapped in terms of $Z$ and $\bar{Z}$ to get

$$H_{nc}|\theta_c = \frac{\hbar \omega_c}{2} (Z\bar{Z} + \bar{Z}Z).$$ \hspace{1cm} (106)

Therefore, the corresponding spectrum is

$$E_n = \frac{\hbar \omega_c}{2} (2n + 1), \quad |n\rangle = \frac{Z^n}{\sqrt{n!}} |0\rangle, \quad n \in \mathbb{N}.$$ \hspace{1cm} (107)

Now let us consider the picture of $N$-electrons in LLL. Its total Hamiltonian is just sum of a single particle (106), namely

$$H_{tot} = \frac{m\omega_c^2}{2} \sum_{i=1}^{N} (X_i^2 + Y_i^2).$$ \hspace{1cm} (108)

It is obvious that its eigenvalues are just $N$ copies of $E_n$. The corresponding many-body state can be written in terms of the Slater determinant, such as

$$|1\rangle = \{ e^{i_1 \cdots i_N} Z_{i_1}^{n_1} \cdots Z_{i_N}^{n_N} \} |0\rangle$$ \hspace{1cm} (109)

which is nothing but the first Laughlin state that corresponds to $\nu = 1$ [17]. Other similar Laughlin states can also be constructed

$$|m\rangle = \{ e^{i_1 \cdots i_N} Z_{i_1}^{n_1} \cdots Z_{i_N}^{n_N} \}^m |0\rangle$$ \hspace{1cm} (110)

corresponding to the filling factor $\nu = \frac{1}{m}$, with $m$ odd integer. These shown that the critical point leads to an interesting results and therefore needed to be investigated.

8 Conclusion

By considering a system of particles of spin $\frac{1}{2}$ living on the non-commutative plane in the presence of a magnetic and a weak electric fields, we have investigated its basic features from quantum statistical mechanical point of view. After writing down the corresponding Hamiltonian, we have derived its
energy levels as well as the wavefunctions. They have been used to analyze the thermodynamical properties of our system. More precisely, we have given the partition function that allowed us to get the associated thermodynamical potential (TPO). Since a general form of TPO have been derived, we have introduced the Mellin transformation to get a simplified TPO. To completely evaluate this latter, we have introduced some condition supplemented by separating two different cases: low and high temperatures.

In the first case, we have derived a general form of the Hall conductivity that lead to different results. Indeed, from our consideration and making using of the experiment data, we have given a way to measure the non-commutativity parameter $\theta$. Subsequently, this has been tuned on to make contact with different results for QHE. In particular, we have obtained integer QHE simply by switching off $\theta$. On the other hand, the observed fraction values, characterizing fractional QHE, have been recovered by fixing $\theta$. Moreover, we have shown that $\theta$ can be chosen to interpret our system as a collection of composite fermions or multilayer Hall systems on the ordinary plane.

In the second case, we have derived the particle number as well as the corresponding Hall conductivity. To analyze the magnetism of our system, we have evaluated the magnetization. This allowed us to get a critical value in terms of $\theta$. This has been used to discuss the phase transition between two sectors. In fact, they are obtained to be diamagnetic and paramagnetic of nature. Furthermore, at the critical point, we have used the experiment data to fix $\theta$.

Finally we have analyzed the critical point of our non-commutative Hamiltonian, i.e. $\theta = 2l_B^2$. By making use of some rearrangement, we have ended up with one-dimensional harmonic oscillator. This latter is sharing many features with that corresponding to the lowest Landau level. Moreover, it has been used to make contact with the Laughlin theory.

Still some questions to be addressed. At the critical point, one may think about making use the matrix model theory to investigate thermodynamical properties of the system in LLL. On the other hand, a link to the spin Hall effect deserves an attention. We hope to return to these issues and related matter in forthcoming works.

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