Strong entanglement criterion involving momentum weak values

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In recent years weak values have been used to explore interesting quantum features in novel ways. In particular, the real part of the weak value of the momentum operator has been widely studied, mainly in connection with (nonlocal) Bohmian trajectories. Here we focus on the imaginary part and its role in relation with the entanglement of a bipartite system. We establish an entanglement criterion based on weak momentum correlations, that allows to discern whether the entanglement is encoded in the amplitude and/or in the phase of the wave function. Our results throw light on the physical role of the real and imaginary parts of the weak values, and stress the relevance of the latter in the multi-particle scenario.

I. INTRODUCTION

The usual operator algebra of quantum mechanics, when applied to the linear momentum operator, leads to a complex vector in configuration space composed of a flux velocity $v$ and an osmotic or diffusive velocity $u$. The former is widely known as the flow velocity associated with the probability current, and it is also recognized as the particle velocity field in Bohmian mechanics [1]-[3]. The diffusive velocity, by contrast, has received little attention despite its intimate connection to distinctive quantum properties [4], such as the existence of irreducible (quantum) fluctuations and the nonclassical features related to the so-called quantum (or Bohm) potential. In fact most of the studies of $u$ have been circumscribed to the realm of stochastic quantum mechanics or the hydrodynamic (or classical-like) formulation of (single-particle) quantum mechanics [5]-[9]. More specifically, within standard quantum mechanics the role of the diffusive velocity in systems composed of more than one particle has, to our knowledge, never been analysed.

Here we carry out such an analysis for a bipartite system, and show, first, that the diffusive velocity associated with each of the two particles plays a prominent role in expressions related to the quantumness of the system (as in the single-particle case), and more specifically in connection with entanglement. Notably, correlations involving the diffusive velocities are obtained that serve as entanglement indicators and allow us to discern whether the entanglement is encoded in the probability distribution ($A$-entanglement), and/or in the phase ($P$-entanglement) of the bipartite wave function. This discriminating property, together with the fact that such entanglement criterion involves only bilinear products of the velocities, differs from the separability criteria for continuous variables that typically rest on variances and covariance matrices [10]-[12], higher-order moments [13, 14] or entropic functions of global variables involving canonically conjugate variables [15] (for a recent account of entanglement criteria based on uncertainty relations see [16] and references therein).

We further find that the $A$- and $P$- entanglement signals via correlations involving the diffusive velocities can be certified in a natural way by resorting to the weak values [17, 18] associated with the momentum operator. The connection ensues from the fact that $v$ and $u$ coincide, respectively, with the real and imaginary parts of the weak value of $\hat{p}$ (with postselection state $|\vec{x}_1, \vec{x}_2\rangle$), an observation that has led a number of authors to explore interesting features of the quantum phenomenon in novel ways. However, most of the studies so far focus on the real ($v$) part [19, 20] and primarily on the (theoretical and experimental) study of nonlocal (Bohmian) trajectories [21]-[25]. In addition to contributing to the discussion of both $v$ and $u$ in the weak-value context (see for example [26]), here we take a close look at entanglement from such perspective. The result is an entanglement criterion based on weak momentum correlations, valid for any bipartite state of continuous variables; the criterion proposed is strong in the sense that it serves to distinguish between $A$- and $P$- entanglement. Along our derivations, we delve into the physical meaning of the real and imaginary parts of the weak value of an arbitrary Hermitian operator, and stress their role in the expression for the quantum correlation between a pair of observables.

The paper is organized as follows. In Section II A we introduce the flux and diffusive velocities in a bipartite state. Section II B is devoted to exhibiting the relevance of the diffusive velocities in the context of the quantum correlation between particle momenta, thereby bringing to the fore the importance of $u$ in connection with paradigmatic quantum features. The link between $u$ and quantumness is taken further in Section II C, where the $A$- and $P$- entanglement criteria based on correlations involving the diffusive velocities are presented. In Section III A we introduce the reader to the weak values, focusing on the role of their real and imaginary parts. In Section III B we proceed to construct the strong entanglement criterion based on weak values of momentum operators, and also propose a generalization of it. Finally, we present some conclusions in Section IV.
II. DIFFUSIVE VELOCITY AND QUANTUMNESS

A. Flow and diffusive velocities

Consider a two-particle quantum system in a state described by the wave function

\[ \psi(x_1, x_2, t) = \sqrt{\rho(x_1, x_2, t)} e^{iS(x_1, x_2, t)}, \]

with \( S \) a real function and \( \rho = \psi^* \psi \). In what follows we assume that the system is bounded so that \( \psi \) vanishes at infinity. Let \( \hat{p}_i = -i\hbar \nabla_i \) be the momentum operator of the \( i \)-th particle (\( i = 1, 2 \)) with mass \( m_i \). Direct calculation gives

\[ \hat{p}_i \psi = m_i (v_i - iu_i) \psi, \]

where the (real) velocity vectors \( v_i \) and \( u_i \) are given, respectively, by

\[ v_i = \frac{\hbar}{m_i} \nabla_i S, \quad u_i = \frac{\hbar}{2m_i} \nabla_i \rho. \]

The (quantum) expectation value of \( \hat{p}_i \), here denoted by \( \langle \hat{p}_i \rangle_q \), is thus (in what follows all integrations are performed over the entire configuration space)

\[ \langle \hat{p}_i \rangle_q = \int \psi^* \hat{p}_i \psi \, dx_1 dx_2 \]

\[ = m_i \langle v_i - iu_i \rangle \rho \, dx_1 dx_2 = m_i \langle v_i \rangle, \]

where \( \langle \cdot \rangle \) (without a subindex) stands for the mean value of c-numbers (instead of q-numbers), defined as \( \langle \cdot \rangle = \int \rho \, dx_1 dx_2 \). Notice that in the last equality we took into account that \( \langle u_i \rangle = 0 \), since \( \rho \) vanishes at infinity. The expectation value of \( \hat{p}_i \), coincides therefore with the mean value of the momentum \( m_i v_i \), defined in terms of the flow velocity \( v_i \). This is the velocity related to the probability current \( j_i = \rho v_i \), that appears in the continuity equation \( \frac{d\rho}{dt} + \sum_j \nabla_i j_i = 0 \). It is also the actual velocity \( dx_i / dt \) of the \( i \)-th particle in the context of Bohmian Mechanics [1, 20]. The diffusive velocity \( u_i \), by contrast, does not contribute to \( \langle \hat{p}_i \rangle_q \), and although it appears on an equal footing with \( v_i \) in Eq. (2), it is normally absent in the usual quantum mechanics parlance. However, it certainly acquires importance when dealing with bilinear products of the form \( \langle \hat{p}_i \hat{p}_j \rangle_q \), and particularly in relation with the entanglement between the two parties. The results below will show that \( u \) has a role of its own, one that allows us to identify this velocity as a carrier of the quantumness of the system.

B. Momentum correlations involving \( u_i \)

In order to exhibit the presence of \( u_i \) in the quantum features of the bipartite system, we start by resorting to Eq. (2) and write

\[ \hat{p}_i \cdot \hat{p}_j \psi = \hat{p}_i \cdot [m_j (v_j - iu_j) \psi] \]

\[ = m_i m_j v_i \cdot v_j \psi + \langle \pi^{uv}_{ij} \rangle \psi, \]

where we have defined

\[ \pi^{uu}_{ij} = -m_i m_j u_i \cdot u_j - \hbar m_i \nabla_i \cdot u_j, \]

\[ \pi^{uv}_{ij} = -m_i m_j (v_i \cdot u_j + v_j \cdot u_i) - \hbar m_i \nabla_i \cdot v_j. \]

Notice that since \( m_i \nabla_i \cdot u_j = m_i \nabla_i \cdot u_i \), and \( m_i \nabla_i \cdot v_j = m_i \nabla_i \cdot v_i \), both \( \pi^{uu}_{ij} \) and \( \pi^{uv}_{ij} \) are symmetric under the exchange \( i \leftrightarrow j \). We thus obtain

\[ \langle \hat{p}_i \hat{p}_j \rangle_q = m_i m_j \langle v_i \cdot v_j \rangle + \langle \pi^{uu}_{ij} + i\pi^{uv}_{ij} \rangle. \]

Now, taking into account that for any bounded vector \( \rho g(x_1, x_2) \)

\[ \langle \nabla_i \cdot g \rangle = \int (\nabla_i \cdot g) \rho \, dx_1 dx_2 \]

\[ = - \int g \cdot (\nabla_i \rho) \, dx_1 dx_2 \]

\[ = - \frac{2m_i}{\hbar} \langle g \cdot u_i \rangle, \]

we get (for bounded \( \nabla_i \rho \) and \( j_i \), respectively)

\[ \langle \nabla_i \cdot u_j \rangle = - \frac{2m_i}{\hbar} \langle u_i \cdot u_j \rangle, \]

\[ \langle \nabla_i \cdot v_j \rangle = - \frac{2m_i}{\hbar} \langle u_i \cdot v_j \rangle. \]

This implies \( \langle \pi^{uu}_{ij} \rangle = m_i m_j \langle u_i \cdot u_j \rangle \) and \( \langle \pi^{uv}_{ij} \rangle = 0 \), and consequently from Eqs. (4) and (8),

\[ C_{\hat{p}_i, \hat{p}_j} = m_i m_j C_{v_i, v_j} + m_i m_j \langle u_i \cdot u_j \rangle, \]

with \( C_{y,z} \) the correlation \( \langle y \cdot z \rangle - \langle y \rangle \langle z \rangle \).

Equation (12) shows that the correlation between the diffusive velocities plays a central role in deviating the quantum correlation \( C_{\hat{p}_i, \hat{p}_j} \) from the correlation between the flux momenta (or in Bohmian terms, from the correlation between the actual momenta of the particles). That such deviation reflects nonclassical features will become clearer below (see Eq. (17)). At this point it can be verified by putting \( i = j \) in the above equations; in particular, Eq. (12) gives for the quantum momentum dispersion [4]

\[ \sigma_{\hat{p}_i}^2 = m_i \sigma_{v_i}^2 + m_i \langle u_i^2 \rangle, \]

which shows that whilst \( u_i \) does not contribute to the expectation value of \( \hat{p}_i \), it does contribute to its fluctuations. Moreover, whereas \( \sigma_{v_i}^2 \) may vanish, \( \langle u_i^2 \rangle \) is always strictly greater than zero (we are considering bounded states, so the case \( \rho = \text{constant} \), or rather \( u_i = 0 \), is ruled out from our analysis). In other words, the presence of \( u_i \) in Eq. (13) reflects the irreducible dispersive nature
of the system characteristic of the quantum phenomenon. 

Now for \( i = j \), the term \( \pi_{ij}^{uu} \) entering in Eq. (8) becomes

\[
\pi_{ij}^{uu} = 2m_i V_{Q_i},
\]

with

\[
V_{Q_i} = -\frac{1}{2}(m_i u_i^2 + \hbar \nabla_i \cdot u_i).
\]

Remarkably, \( V_{Q_i} \) is closely related to the so-called quantum potential, which lies at the core of Bohmian mechanics. Indeed, the quantum potential is defined as

\[
V_Q = \sum_k \frac{-\hbar^2 \nabla_k^2 U}{\sqrt{\rho}}
\]

(summed over all the particles of the system), which can be rewritten, using Eqs. (3) and (15), as

\[
V_Q(\rho) = \sum_k V_{Q_k}(u_k).
\]

It is well known that the quantum potential endows the system with nonclassical attributes, particularly with its characteristic nonlocalities [3]. However, little attention has been paid to the fact that \( V_Q \) is directly linked to the diffusive velocities, as shown in Eq. (15). Notice also that, in line with the above results, Eq. (13) can alternatively be expressed as

\[
\sigma_{\hat{p}_i}^2 = m_i^2 \sigma_{u_i}^2 + 2m_i V_{Q_i},
\]

which relates the momentum dispersion with the \( i \)-th particle’s quantum potential.

Now, direct calculation of \( \langle \hat{x}_i \cdot \hat{p}_i - \hat{p}_1 \cdot \hat{x}_i \rangle_q \) using Eq. (9), gives

\[
\langle \hat{x}_i \cdot \hat{p}_i - \hat{p}_1 \cdot \hat{x}_i \rangle_q = i\hbar \langle \nabla_i \cdot x_i \rangle = -2im_i \langle x_i \cdot u_i \rangle.
\]

This result displays the equivalence (in terms of mean values) between the fundamental commutator \( [\hat{x}_i, \hat{p}_i] \neq 0 \) and the (nonzero) correlation \( \langle x_i \cdot u_i \rangle \), thus revealing an intimate connection between the presence of \( u_i \) and the far-reaching consequences (as, e.g., the existence of irreducible fluctuations) of a nonzero fundamental commutator.

The results of this subsection serve to sustain the statement that \( u_i \) can be thought of as a kinematic term that bears the quantumness of the system. In the next section we take this statement further, by establishing a relation between the diffusive velocities and the presence of entanglement in state \( \psi \).

### C. Role of \( u \) in entanglement

The state \( \psi \) is non-entangled, that is, \( \psi(x_1, x_2, t) = \psi_1(x_1, t) \psi_2(x_2, t) \) with \( \psi_i = \sqrt{\rho_i} \exp(iS_i) \) representing the wave function of subsystem \( i \), if and only if:

1. \( \rho \) factorizes as \( \rho(x_1, x_2, t) = \rho_1(x_1, t) \rho_2(x_2, t) \), and
2. \( S \) decomposes as \( S(x_1, x_2, t) = S_1(x_1, t) + S_2(x_2, t) \).

For \( i \neq j \), we see that condition 1 implies \( \nabla_i \cdot u_j = 0 \), whence (using Eq. (10)) \( \langle u_1 \cdot u_2 \rangle = 0 \). Analogously, condition 2 implies \( \nabla_i \cdot v_j = 0 \), whence (using Eq. (11)) \( \langle u_1(2) \cdot v_2(1) \rangle \neq 0 \). This leads to the following entanglement criteria (with \( i \neq j \)):

\[
\langle u_1 \cdot u_2 \rangle \neq 0 \Rightarrow \nabla_i \cdot u_j \neq 0 \Rightarrow \psi \text{ is } A\text{-entangled},
\]

\[
\langle u_1 \cdot v_j \rangle \neq 0 \Rightarrow \nabla_i \cdot v_j \neq 0 \Rightarrow \psi \text{ is } P\text{-entangled},
\]

where the term ‘A-entangled’ indicates that the entanglement is encoded in the non-factorizability of the amplitude \( \sqrt{\rho} \), whereas ‘P-entangled’ means it is encoded in the non-additivity of the phase \( S \). With Eqs. (17), the previous observation that the diffusive velocities typically come up in expressions that bring to the fore the quantum properties of the system is reinforced, now in the context of entanglement—considered the most distinctive quantum feature of composite systems.

Now, returning to Eq. (12), we see that the correlation between the diffusive velocities contributes to the quantum momentum correlations, hence any deviation of \( C_{\hat{p}_i, \hat{p}_j} \) from the Bohmian momenta correlation constitutes a trace of \( A \)-entanglement. However, correlations of the form \( \langle u_i \cdot v_j \rangle \) do not contribute to \( C_{\hat{p}_i, \hat{p}_j} \). This does not mean that the correlation \( C_{\hat{p}_i, \hat{p}_j} \) is insensitive to any \( P \)-entanglement present in the correlations \( C_{u_i, v_j} \). Yet this \( P \)-entanglement does not modify the quantum correlations with respect to the (classically expected) correlations between the flux momenta.

The above considerations invite us to explore whether the two conditions (17) can be brought together into a single quantity endowed with physical meaning, that serves to establish a strong entanglement criterion in the sense that it would not only be useful in attesting entanglement, but also in discerning whether it is encoded in the amplitude and/or in the phase of the wave function. In the following Section we tackle this problem.

### III. WEAK VALUES OF THE MOMENTUM

#### A. Weak values and local mean averages

Let us consider an operator \( \hat{A} \), a preselection state \( |\psi\rangle \), and a postselection state \( |\phi\rangle \). Formally, the corresponding weak value of \( \hat{A} \) is a complex number defined as [17, 18]

\[
\langle \hat{A} \rangle_{\psi, |\phi\rangle} = \frac{\langle \phi | \hat{A} | \psi \rangle}{\langle \phi | \psi \rangle},
\]

Operationally, the weak values of (every power of) an Hermitian operator \( \hat{A} \) characterize the relative correction to the detection probability \( P_0 = |\langle \phi | \psi \rangle|^2 \) due to an intermediate perturbation \( \hat{U}_\alpha = e^{-i\alpha \hat{A}} \). Specifically, if the state \( |\psi\rangle \) is affected by the unitary operation \( \hat{U}_\alpha \),
the detection probability of the postselection state $|\phi\rangle$ is
\[ P_\alpha = |\langle \phi | e^{-iA} |\psi\rangle|^2, \]
whence
\[ \frac{P_\alpha}{P_0} = \left| \sum_{n=0}^{\infty} \frac{(-i\alpha)^n}{n!} \langle \phi | A^n |\psi\rangle \right|^2. \tag{19} \]

To first order in $\alpha$ (or equivalently for a ‘weak’ perturbation) the quotient $P_\alpha/P_0$ goes as $|1 - i\alpha \langle \hat{A} |\psi\rangle|^2$, and the weak value (18) completely determines the relative correction to $P_0$ [27].

Physically, the real part of $A\langle \psi |\phi\rangle$, with $\hat{A}$ an Hermitian operator, can be understood as the ‘$\phi$-local’ value of the corresponding dynamical variable $A$ in the state $\psi$, when the description is made in the $\phi$-representation. This can be seen as follows. Given the state $|\psi(t)\rangle$ and an element $|\phi\rangle$ of an orthonormal basis of the corresponding Hilbert space (in what follows a continuous one is assumed), the function $\psi(\phi, t) = \langle \phi |\psi\rangle$ gives the state $\psi$ in the $\phi$-representation. Moreover, the operator $\hat{A}$ in that same representation, $\hat{A}_\phi$, is defined in such a way that $\hat{A}_\phi \psi(\phi) = \langle \phi |\hat{A} |\phi\rangle$, whence
\[ \langle \hat{A}_\phi(\psi, \phi) \rangle = \langle \hat{A}_\phi \psi(\phi, t) \rangle, \tag{20} \]
and the expectation value of $\hat{A}$ in the state $|\psi\rangle$ can be expressed as:
\begin{align*}
\langle \hat{A} \rangle &= \langle \psi |\hat{A} |\psi\rangle = \int \psi^*(\phi, t) \hat{A}_\phi \psi(\phi, t) d\phi \\
&= \int \rho(\phi, t) \langle \hat{A} \rangle^{(\psi, \phi)} d\phi \\
&= \langle \text{Re} \langle \hat{A} \rangle^{(\psi, \phi)} \rangle + i \langle \text{Im} \langle \hat{A} \rangle^{(\psi, \phi)} \rangle, \tag{21}
\end{align*}
where $\rho(\phi, t) = |\psi(\phi, t)|^2$ stands for the probability density function in $\phi$-space. For $\hat{A}$ Hermitian, the last line implies that
\[ \langle \text{Im} \langle \hat{A} \rangle^{(\psi, \phi)} \rangle = 0, \tag{22} \]
hence Eq. (21) states that the expectation value of $\hat{A} = \hat{A}^\dagger$ in the state $|\psi\rangle$ is just the average of $\text{Re} \langle \hat{A} \rangle^{(\psi, \phi)}$ weighted with the probability distribution $\rho(\phi, t)$. In this sense, $\text{Re} \langle \hat{A} \rangle^{(\psi, \phi)}$ plays the role of the $\phi$-local (i.e., defined at each point $\phi$) value of $A$ in the $\phi$-space. For example, if the postselection state is chosen as $|\phi\rangle = |x\rangle$, we have
\begin{align*}
\langle \hat{A} \rangle &= \int \rho(x, t) \langle \hat{A} \rangle^{(\psi, x)} dx \\
&= \int Q(x, p, t) A(x, p) dx dp, \tag{23}
\end{align*}
with $Q$ an appropriate (pseudo)-probability density function in phase space, such that $\rho(x, t) = \int Q dp$. From Eqs. (21) and (23) we get that, up to a term with vanishing mean value, $\text{Re} \langle \hat{A} \rangle^{(\psi, x)}$ coincides with
\[ \langle A \rangle_{\psi}(x) = \frac{1}{\rho} \int Q(x, p, t) A(x, p) dp, \tag{24} \]
which is no other than the $(x)$-local average of the variable $A$, obtained when we partially average $A$ over the momentum space.

The imaginary part of $\langle \hat{A} \rangle^{(\psi, \phi)}$, in its turn, becomes relevant when bilinear expressions, specifically correlations, are considered. Let $\hat{A}$ and $\hat{B}$ denote two Hermitian and commuting operators (so that $\hat{A}\hat{B}$ is also Hermitian). With the aid of Eqs. (21) and (22) it can be shown that the quantum correlation between $\hat{A}$ and $\hat{B}$ reads (we omit the superindex $(\psi, \phi)$ in the expression for the weak values)
\begin{align*}
C_{\hat{A}, \hat{B}} &= \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle \\
&= \langle \text{Re} \langle \hat{A}\hat{B} \rangle \rangle - \langle \text{Re} \langle \hat{A} \rangle \rangle \langle \text{Re} \langle \hat{B} \rangle \rangle \\
&= C_{\text{Re}(\hat{A}), \text{Re}(\hat{B})} - C_{\text{Im}(\hat{A}), \text{Im}(\hat{B})} + \text{Re} \langle C_{\hat{A}\hat{B}} \rangle, \tag{25}
\end{align*}
where we have defined $C_{\hat{A}, \hat{B}} = \langle \hat{A} \hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle$. In what follows we will refer to this quantity as the weak correlation (between $\hat{A}$ and $\hat{B}$). According to the discussion following Eq. (22), the first term in Eq. (25) can be interpreted as the correlation between the local values of $A$ and $B$. The difference between the latter and the (standard) quantum correlation $C_{\hat{A}, \hat{B}}$ is thus determined by the (correlation between the) imaginary parts of $\langle \hat{A} \rangle$ and $\langle \hat{B} \rangle$, and the (real part of the) weak correlation $C_{\hat{A}, \hat{B}}$.

### B. Strong entanglement criteria with momentum weak values

Weak values acquire relevance in our analysis since, according to Eqs. (2) and (20), $m_i (\mathbf{v}_i - i\mathbf{u}_i)$ is precisely the weak value of the momentum operator of the $i$-th particle, with preselection state $|\psi(t)\rangle$ and postselection state $|\phi\rangle = |x\rangle = |x_1 x_2\rangle$ [28]. The recognition that the flux velocity is the real part of $\langle \hat{p}_1 \rangle^{(\psi, x)}$ [19, 20] has led to the experimental observation of Bohmian nonlocal trajectories using entangled photons [21–23, 25], and further proposals of experimental monitoring of such trajectories [24, 28]. In its turn, as we have seen above, consideration of the imaginary part of $\langle \hat{p}_1 \rangle^{(\psi, x)}$ allows (among other things) for the determination of the quantum potential, and thus for further studies of Bohmian Mechanics (in the single-particle problem, $\mathbf{u}$ suffices to determine $V_Q$; in the multi-particle problem, the entire set $\{\mathbf{u}_k\}$ is required). Another way of monitoring the quantum potential is via
the real part of the weak value of the total kinetic energy,

\[ \text{Re} \left\langle \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} \right\rangle_w^{(\psi, x)} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + V_Q. \] (26)

Let us now consider the weak value of the operator \( \hat{p}_1 \cdot \hat{p}_2 \). According to Eqs. (5) and (20), it is

\[ \langle \hat{p}_i \cdot \hat{p}_j \rangle_w = \langle \hat{p}_i \cdot \hat{p}_j \rangle \psi_w \]

\[ = \int \left[ m_1 (v_i \cdot v_j - \mathbf{u} \cdot \mathbf{u}_j) - im_1 (v_i + \mathbf{u}_i \cdot \mathbf{u}_j) - ihm_1 \nabla_i \cdot \mathbf{v}_j \right] \frac{d^3 \mathbf{v}_j}{(2\pi)^3} \]

\[ = \langle \hat{p}_i \rangle_w \cdot \langle \hat{p}_j \rangle_w - hm_1 (\nabla_i \cdot \mathbf{u}_j + i\nabla_i \cdot \mathbf{v}_j), \]

so the weak correlation between the momenta is given by

\[ C_{\hat{p}_1, \hat{p}_2}^w = \langle \hat{p}_1 \rangle_w \cdot \langle \hat{p}_2 \rangle_w \]

(27)

Substitution of the real part of this expression into Eq. (25) gives, using Eq. (10), the correlation (12) as expected. But beyond contributing to \( C_{\hat{p}_1, \hat{p}_2} \), the weak correlation \( C_{\hat{p}_1, \hat{p}_2}^w \) provides additional information regarding the entanglement. Indeed, in line with Eqs. (17), a strong entanglement criterion can now be stated as:

\[ \text{Re} \ C_{\hat{p}_1, \hat{p}_2}^w \neq 0 \Rightarrow \psi(x_1, x_2, t) \text{ is A-entangled.} \] (29a)

\[ \text{Im} \ C_{\hat{p}_1, \hat{p}_2}^w \neq 0 \Rightarrow \psi(x_1, x_2, t) \text{ is P-entangled.} \] (29b)

Thus, \( C_{\hat{p}_1, \hat{p}_2}^w \) suffices to determine not only whether \( \psi \) is entangled, but also the type of entanglement involved. According to Eqs. (29) and the last paragraphs in Section II C, we see that it is this weak correlation, and not \( C_{\hat{p}_1, \hat{p}_2} \), what provides information of both types of entanglement on an equal footing. A proposal to quantify the amount of each kind of entanglement can be seen in [29].

In the one-dimensional case, the conditions Re \( C_{\hat{p}_1, \hat{p}_2}^w \neq 0 \) and Im \( C_{\hat{p}_1, \hat{p}_2}^w \neq 0 \) are not only sufficient but also necessary to guarantee the corresponding type of entanglement. This follows from the fact that in 1D the conditions \( \nabla_i \cdot \mathbf{u}_j = 0 \) and \( \nabla_i \cdot \mathbf{v}_j = 0 \) become, respectively, \( du_j/dx_j = 0 \) and \( dv_j/dx_j = 0 \), which according to Eq. (3) lead to \( \rho = \rho_1(x_1)\rho_2(x_2) \) and \( S = S_1(x_1) + S_2(x_2) \). Consequently, A-entanglement implies \( du_j/dx_j \neq 0 \), whereas P-entanglement implies \( dv_j/dx_j \neq 0 \), and we are finally led to

\[ \text{Re} \ C_{\hat{p}_1, \hat{p}_2}^w \neq 0 \Leftrightarrow \psi(x_1, x_2, t) \text{ is A-entangled.} \]

(30a)

\[ \text{Im} \ C_{\hat{p}_1, \hat{p}_2}^w \neq 0 \Leftrightarrow \psi(x_1, x_2, t) \text{ is P-entangled.} \]

(30b)

The structure of the entanglement criteria (29) holds also for other representations and operators, under certain conditions. Specifically, we can consider operators \( \hat{A} \) and \( \hat{B} \) representing, respectively, a dynamical variable of particle 1 and 2, and an orthonormal basis \( \{ |\psi \rangle = |\alpha\beta \rangle = |\alpha \rangle_1 \otimes |\beta \rangle_2 \} \) of the bipartite Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). In the \( \phi \)-representation, the state \( |\psi(t)\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \) is thus described by the wave function \( \psi(\alpha, \beta, t) = \sqrt{\rho(\alpha, \beta, t)} e^{iS(\alpha, \beta, t)} \), and \( \hat{A} \) and \( \hat{B} \) become represented by local operators \( \hat{A}_\phi = \hat{A}_\alpha \) and \( \hat{B}_\phi = \hat{B}_\beta \). If the representation is such that \( \hat{\alpha} |\alpha \rangle = |\alpha \rangle \) with \( [\hat{\alpha}, \hat{A}] = \pm ih \), and \( [\hat{\beta} |\beta \rangle = |\beta \rangle \) with \( [\hat{\beta}, \hat{B}] = \pm ih \), then \( \hat{A}_\alpha = \mp ih \partial/\partial \alpha \), and \( \hat{B}_\beta = \mp ih \partial/\partial \beta \). All this gives

\[ \langle \hat{A}\hat{B} \rangle_w^{(\psi, \phi)} = \frac{1}{\psi} \langle \hat{B} \rangle_w^{(\psi, \phi)} \frac{1}{\psi} \langle \hat{A}_\alpha \rangle_w^{(\psi, \phi)} \]

(31)

and consequently

\[ C_{\hat{A}, \hat{B}}^w = \pm h \frac{\partial}{\partial \alpha} \text{Im} \langle \hat{B} \rangle_w^{(\psi, \phi)} + \mp h \frac{\partial}{\partial \alpha} \text{Re} \langle \hat{B} \rangle_w^{(\psi, \phi)}. \]

(32)

Now, direct calculation shows that whenever \( \rho(\alpha, \beta, t) = \rho_1(\alpha, t)\rho_2(\beta, t) \), then \( \partial [\text{Im} \langle \hat{B} \rangle_w^{(\psi, \phi)}]/\partial \alpha = 0 \), whereas if \( S(\alpha, \beta, t) = S_1(\alpha, t) + S_2(\beta, t) \), then \( \partial [\text{Re} \langle \hat{B} \rangle_w^{(\psi, \phi)}]/\partial \alpha = 0 \). Gathering results we arrive at Eqs. (29), with \( \hat{A} \) and \( \hat{B} \) instead of \( \hat{p}_1 \) and \( \hat{p}_2 \), and \( \alpha \) and \( \beta \) instead of \( x_1 \) and \( x_2 \).

IV. CONCLUDING REMARKS

Weak values of the momentum offer a highly interesting subject of research that is particularly suitable for the analysis of paradigmatic quantum features of the system. On one side, the real part of \( \langle \hat{p}_i \rangle_w^{(\psi, x)} \) allows for the study of quantum (Bohmian) trajectories and their concomitant nonlocality. On the other hand, as we have emphasised here, the imaginary part of \( \langle \hat{p}_i \rangle_w^{(\psi, x)} \) is intimately related to characteristic traits of quantumness, and in particular to entanglement detection. Indeed, the two-velocity correlations \( \langle \mathbf{u}_i \cdot \mathbf{u}_j \rangle \) and \( \langle \mathbf{v}_i \cdot \mathbf{v}_j \rangle \), both involving the diffusive velocity of one of the parties and referred to mean values of c-numbers, attest to the (A- or P-) entanglement of the bipartite state, as stated in Eqs. (17).

Interestingly, the usual quantum correlation \( C_{\hat{p}_1, \hat{p}_2} \) differs from the Bohmian correlation between the particle momenta precisely due to the A-entanglement, yet it does not explicitly include the companion term related to P-entanglement. This asymmetry is overcome by resorting to the weak-value formalism. Specifically, both types of entanglement become manifest and can be detected on an equal footing in the expression for the weak correlation \( C_{\hat{p}_1, \hat{p}_2}^w \). More generally, by appeal to pairs of canonically conjugate operators, both the real and the imaginary parts of the weak values prove to be useful in certifying the entanglement of the state of the system and to determine whether it is encoded in the wave function’s amplitude or phase.

Besides providing a physical meaning for both, the real and the imaginary part of the weak value of an Hermitian operator, our results point towards the convenience
of delving more deeply into the subject of the imaginary contributions and their role in the bipartite (and even in the multipartite) case, where novel entanglement criteria and ways of exploring quantum nonlocality may be found.

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[30] The dispersive nature exhibited by $u_i$ can be interpreted as a causal manifestation of a random behavior of the particle. Further discussions on $u$ in connection with stochastic quantum mechanics can be seen in [5, 6, 8], and within stochastic electrodynamics in [4].