Smooth phase transition of energy equilibration in a springy Sinai billiard

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Statistical equilibration of energies in a slow-fast system is a fundamental open problem in physics. In a recent paper, it was shown that the equilibration rate in a springy billiard can remain strictly positive in the limit of vanishing mass ratio (of the particle and billiard wall) when the frozen billiard has more than one ergodic component [Proc. Natl. Acad. Sci. USA 114, E10514 (2017)]. In this paper, using the model of a springy Sinai billiard, it is shown that this can happen even in the case where the frozen billiard has a single ergodic component, but when the time of ergodization in the frozen system is much longer than the time of equilibration. It is also shown that as the size of the disc in the Sinai billiard is increased from zero, thereby leading to a decrease in the time required for ergodization in the frozen system, the system undergoes a smooth phase transition in the equilibration rate dependence on mass ratio.

Since the seminal work done by James C. Maxwell, Ludwig Boltzmann, J. William Gibbs and few others in the 1800s, statistical physics has made immense progress for equilibrium systems and many of the laws proposed by statistical physicists have found applications in diverse fields like information theory [1], economics [2] and now even in machine learning [3]. The most fundamental connection between statistical mechanics and information theory is that of entropy. Boltzmann and Gibbs gave the first probabilistic description of entropy in the late 1800s, but it was Claude E. Shannon who came up with a mathematically rigorous derivation for the entropy formula in 1948 while studying communication systems [4]. Since then, several other statistical mechanics concepts like the maximum entropy principle, random matrix theory and even spin glasses have found many applications in information theory and communication systems [1]. Applications of statistical physics and nonlinear dynamics concepts in economics have given birth to a whole field of econophysics, which attempts at coming up with various mathematical models to explain economic phenomenon and predict its future course [2]. Machine learning is not a new concept, but recent advances in processing power of computer chips has made it possible for humans to harness its power on reasonable time scales, thereby leading to a complete paradigm shift in the way we think of computations and algorithms. And perhaps not surprisingly, several statistical physics concepts like the Boltzmann-Gibbs distribution and stochastic dynamics have found to be very useful in building and explaining models of machine learning [3].

Although lot of progress has been made in the field of non-equilibrium statistical mechanics, most of the well established laws of statistical mechanics mainly pertain to systems under equilibrium. One of the fundamental open problems in statistical physics is about how dynamical systems actually reach this state of statistical equilibrium! A partial answer is provided by the ergodic hypothesis which states that all accessible microstates of a given system are equiprobable over sufficiently long periods of time [1, 5]. However, there are very few dynamical systems which have been actually proven to be ergodic [6, 7, 9–12]. And even for ergodic systems, the time required for ergodization may be so long that it may be practically irrelevant. In the case of slow-fast ergodic systems, it has also been shown that there are adiabatic invariants which can prevent equilibration of the full system over very long periods of time [13, 14]. Hence, there is a theoretical as well as practical need to understand the equilibration properties of systems from the dynamical perspective.

In a recent paper, it has been shown that equilibration of energies can be achieved in slow-fast systems on reasonable time scales if the frozen system has more than one ergodic components [15]. This was numerically demonstrated by studying the dynamics of a point-like particle of small finite mass, \( m \), in a springy billiard where one of the walls is massive, \( M \gg m \), and is connected to a linear spring. Three different springy billiards were studied in that paper : springy barred rectangle, springy mushroom and springy stadium. The total energy of the springy billiard system is conserved in each case since its an autonomous system. It was found that the partial energies of the particle, \( E_p \), and the massive billiard wall, \( E_b \), reached a state of equipartition and equilibration asymptotically with time for all the three springy billiards when the mass ratio \( m/M \) was non-zero. However, in the limit of vanishing mass ratio, only the springy barred rectangle and springy mushroom retained a non-zero equilibration rate, whereas the equilibration rate for the springy stadium went to zero. It was shown that this difference in behavior can be explained through a mathematical model by taking into account the fact that the springy barred rectangle and springy mushroom have more than one ergodic components in the frozen state (called VFS systems, variable partition of the fast subspace), and the springy stadium has only one ergodic component (called EFS systems, ergodic fast subsystem for almost all values of the slow variables). However, one similarity between the springy mushroom and the springy stadium was that...
Figure 1: Springy Sinai billiard, which consists of a particle of mass, \( m \ll 1 \), moving within the billiard boundaries undergoing elastic reflections at each collision with the boundaries (including the disc in between). The bottom wall of the billiard has a mass, \( M = 1 \), and is attached to a spring such that its natural frequency of oscillations is \( \omega = 1 \). When the disc radius is zero, this becomes a polygon (which is non-ergodic if all its angles are rational multiples of \( \pi \)) and for a non-zero disc radius, this becomes a Sinai billiard which is known to be hyperbolic. It is this transition in the billiard properties as we change the disc radius that leads to the phase transition that is demonstrated in this paper. The parameter values chosen are \( L = 2 \), \( \theta_0 = \pi/18 = \theta_\ell \), \( 0 \leq r \leq 1 \), \( M = 1 \) and \( 4 \times 10^{-6} \leq m \leq 12 \times 10^{-6} \).

in both the cases, the equilibration rate varied with mass as \( \sqrt{m/M} \), which is same as what was predicted earlier for the case of uniformly hyperbolic systems \([13, 14]\). For the case of the springy barred rectangle, the equilibration rate was found to be independent of the mass ratio as also predicted by the mathematical model \([15]\). Though the above result is expected to hold for systems in general which can clearly be classified as being VFS or EFS, the behavior of systems which are in between can be lot more interesting. One example of such a system is the springy Sinai billiard as shown in Fig. 1 which essentially consists of a circular disc within a trapezium.

In this paper, the springy Sinai billiard has been studied and found to have several very interesting properties so far not reported in any other slow-fast system. Most importantly, it has been found that this billiard can have a non-zero value of the equilibration rate in the limit of vanishing mass ratio even when the frozen system has a single ergodic component. When the disc radius is increased, the equilibration rate tends to zero in the limit of vanishing mass ratio as is typical for EFS systems. Hence, this billiard is found to undergo a smooth phase transition from a VFS-like system to an EFS-like system as we increase the disc radius. Interestingly, for low values of the disc radius, the equilibration rate dependence on the mass ratio is also found to be non-monotonic. This hints at the possibility that this dynamical system can be a very good candidate for the discovery of interesting dynamical properties not commonly found in other springy billiards.

The conventional Sinai billiard consists of a circular disc at the center of a rectangle and in fact, was one of the first dynamical billiards to be shown to be hyperbolic \([6, 8]\). If one of the walls of this billiard is attached to a linear spring, it is expected to show similar equilibration properties as that of the springy Stadium \([15]\). In the limit of vanishing disc radius, the springy Sinai billiard is reduced to a springy rectangular billiard which is known to be integrable \([10]\), and hence does not have any equilibration of energies even for a non-zero mass ratio \([15]\). In an integrable billiard attached to a spring, the partial energies of the particle and the oscillating bar keep varying periodically about a certain average without reaching equilibration \([13, 15]\).

As shown in Fig. 1, the walls of the springy Sinai billiard have been made slanted in this work to create a trapezium, which is known to be non-integrable \([10, 21, 23]\) and hence, is expected to have equilibration of energies asymptotically even in the limit of zero mass ratio. In fact, among all possible varieties of polygons, it is known that only four are integrable: rectangle, equilateral triangle, right angled triangle with other two angles \( \pi/4 \), and right angled triangle with other two angles \( \pi/3 \) and \( \pi/6 \). If the angles of the non-integrable polygon are rational multiples of \( \pi \), then the billiard is also known to be non-ergodic and hence, is expected to show a behavior similar to that of VFS systems \([15]\). Very little is known about the ergodic properties of polygons with angle(s)
For low values of $r/L$, $\gamma$ is non-monotonic in $m/M$ and has a non-zero value in the limit of vanishing mass ratio. However, as the value of $r/L$ crosses a certain critical value, $\gamma$ shows is monotonic in $m/M$ and tends towards a zero limiting value for large enough radius. For intermediate values of $r/L \approx 0.25$, the value of $\gamma$ becomes independent of the mass ratio beyond a certain threshold, which is similar to the behavior observed in the springy barred rectangle [15].

The rational trapezium with a circular disc in between is a very interesting system. When the radius of the disc is non-zero, the system is known to be ergodic, and the time-scale of ergodization decreases as the disc radius increases [7]. Hence, when the disc radius of this springy Sinai billiard is large enough, it is expected to behave like an EFS system, but it is apriori not clear what may happen when the disc radius is small enough. This is because for small disc radius, the dynamical properties of the non-ergodic non-integrable trapezium may become more dominant. Hence, it might behave like an EFS system for all non-zero values of the disc radius, or might undergo a phase transition to a VFS-like system when the disc radius is small enough. This question is numerically studied in this paper and the latter possibility is found to be true. The springy Sinai billiard indeed undergoes a smooth phase transition from an EFS system to a VFS-like system as the disc radius is decreased.

The springy Sinai billiard shown in Fig. 1 consists of a circular disc of radius, $r$, contained within a trapezium. In this paper, the values taken are $L = 2$, $\theta_b = \pi/18 = \theta_i$, $0 \leq r \leq 1$, $M = 1$ and $4 \times 10^{-7} \lesssim m \lesssim 9 \times 10^{-5}$. Simulations were performed for other values of $\theta_i, \theta_b$ and qualitatively similar results were found as those reported in this paper. The spring attached to the massive billiard boundary has a spring constant such that its angular frequency is $\omega = 2\pi$. Numerical simulations are performed for an ensemble of 10000 particles using the same algorithm described in [15]. The particle moves inside the billiard in straight lines and undergoes elastic collisions at the boundaries. The particle velocity after collision with a static wall is simply given by the law of reflection. In this case, the particle velocity after elastic collisions, where the angle of incidence is equal to the angle of reflection. In this case, the particle velocity only undergoes a change in direction and its speed remains the same. When the particle undergoes elastic collisions with the oscillating bar, the time and position of collision is calculated using a combination of bisection and Newton method. The equilibration rate is estimated using a linear least squares fit of $\log |E_b - 0.5|$ over a time-interval in which the bar energy, $E_b$, changes by a factor of $e$. In each simulation, there is only a single particle in the springy billiard and then an average is taken over 10000 different randomly chosen initial conditions. The total energy of the system stays constant at $E = 1$, which is the sum of the particle kinetic energy, $E_p$, and the energy of the oscillating bar, $E_b$. The energy $E_b$ is a sum of the kinetic energy of the bar and the poten-
tial energy of the attached spring. There is an exchange of energy between $E_p$ and $E_b$ each time there is a collision between the oscillating bar and the particle. In between such collisions, the values of $E_p$ and $E_b$ remain unchanged.

Figure 2 shows the plot of the bar energy, $E_b$, with time for few values of $r/L$ and $m/M$ each. As can be clearly seen, the bar energy reaches its equilibrium value of 0.5 in all cases, which is the expected behavior for billiards which are non-integrable [15]. The equilibration proceeds approximately as an exponential function, and so $E_b$ can be written as

$$E_b (t) \approx 0.5 + [E_b (0) - 0.5] e^{-\gamma t} \quad (1)$$

where $\gamma$ is the equilibration rate and depends on $E_b (0)$, $m/M$ as well as the billiard parameters. In this paper, we have kept all other billiard parameters fixed, except for the disc radius, $r$.

Figure 3 shows a plot of the equilibration rate, $\gamma$, versus the mass ratio, $m/M$, for a few values of $r/L$ when $E_b (0) = 0.9$. As can be seen, for larger values of $r/L$, the equilibration rate is increases with an increase in $m/M$ as a power-law and tends to zero in the limit of vanishing mass ratio as is expected of EFS systems. However, for lower values of $r/L$, $\gamma$ becomes non-monotonic and has a non-zero value in the limit of vanishing mass ratio, which is typical of VFS systems. For some values of the disc radius around $r/L \sim 0.12$, we also see that the equilibration rate is independent of the mass ratio, which is similar to the behavior found for the springy barred rectangle [15]. A qualitatively similar behavior is observed when $E_b (0) = 0.1$ as shown in Fig. 1. Hence, it is reasonable to conclude that this is typical behavior of the springy Sinai billiard. This result is significant since it is usually believed that ergodic systems should not have equilibration of energies in the limit of vanishing mass ratio [13, 14]. But this result shows that if the ergodicity is weak, then even ergodic systems can have equilibration of energies in this limit.

These results are particularly relevant to practical systems, since most of them are neither strictly VFS or EFS and actually fall somewhere in between, in the same sense as most real systems have a mixed phase space. Hence, most real systems are expected to show this kind of a smooth phase transition as the relevant parameters are varied. The criteria for observing a similar behavior is that for some set of parameters, the system should become strongly ergodic and for some other set of parameters, the system should become strongly non-ergodic, while remaining non-integrable for all parameter values.

The simultaneous requirement of non-ergodicity and non-integrability for some parameter values is important since in the springy Sinai billiard shown in Fig. 1 if the bounding polygon is a rectangle (non-ergodic, but integrable) instead of a trapezium, then the phase transition will not be observed. This is because there is no equilibration of energies in an integrable system with a linear spring and the bar energy keeps oscillating about a certain mean value for all time. However, there might be interesting equilibration effects even in integrable billiards when the spring becomes nonlinear [20].

These results can also have very interesting implications in the study of Fermi acceleration in dynamical billiards [21-24]. Fermi acceleration is the study of particle dynamics within billiards in a similar manner as studied in this work, with the only difference that there the billiard wall is infinitely massive and, hence, the total energy of the system is not a conserved quantity. So, instead of equilibration, what is observed is an unbounded increase of energy of the particle ensemble with time if the underlying billiard is non-integrable. In most such systems studied so far, this energy growth has been found to be either exponential or quadratic in time. The prevailing understanding is that the energy growth rate is quadratic-in-time if the underlying frozen billiard is ergodic [21] and exponential-in-time if it is non-ergodic [22, 24]. One of the open questions in this area is whether polygons, which are known to be pseudo-integrable [25], in general have an exponential growth of energy or not. Some indirect evidence has been found which indicates an exponential-in-time growth of energy in polygons [22], but it is not yet well established. And as shown in [15], there is a strong connection between equilibration rates in springy billiards in the limit of vanishing mass ratio and exponential acceleration when the same system is studied in the context of Fermi acceleration. The results reported in this paper provide more evidence to support the case of exponential energy growth in polygons in general. A more careful study of these models might also be helpful in figuring out whether irrational polygons are ergodic or not!

This work primarily has three limitations, which can serve as fruitful directions for future work. Firstly, no mathematical model could be proposed in this paper to predict the equilibration rate in the limit of vanishing mass ratio as was done for the springy barred rectangle and the springy mushroom in [15]. Proposing such a mathematical model requires clear knowledge of the ergodic partitions of the frozen system and the probability of jumping across these partitions. Both these pieces of information were available for the springy billiards studied in [15], but as of now, are not available for the springy polygonal billiard mainly because, in this case, the ergodic components are not that well separated as compared to the springy barred rectangle or the springy mushroom. In the case of the springy polygon, the ergodic components are intricately intertwined with each other, and hence predicting the limiting equilibration rate will perhaps require a completely new approach compared to that adopted in [15]. Secondly, it is not clear why the equilibration rate, $\gamma$, is non-monotonic with respect to the mass ratio for low values of $r/L$. Perhaps there is some kind of resonance phenomenon taking place for certain values of $m/M$ for lower values of the disc radius, which leads to a maximization of the equilibration rate. Thirdly, the functional dependence of the equili-
bration rate on \( m/M \) is unclear for lower values of the disc radius. This information is required so as to be able to predict the value of the equilibration rate in the limit of vanishing mass ratio. We can graphically see that this limiting value is most likely non-zero, but a proper empirical estimation is needed in order to be sure.

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