Numerical Methods for the Bogoliubov-Tolmachev-Shirkov model in superconductivity theory

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Abstract

In the work, the numerical methods are designed for the Bogoliubov-Tolmachev-Shirkov model in superconductivity theory. The numerical methods are novel and effective to determine the critical transition temperature and approximate to the energy gap function of the above model. Finally, a numerical example confirming the theoretical results is presented.

Keywords: Bogoliubov-Tolmachev-Shirkov model; Critical Temperature; Numerical Methods.

1. Introduction

In the Bardeen-Cooper-Schrieffer(BCS) quantum theory of superconductivity, the superconducting state is characterized by a positive gap function, \( \Delta(x) \), which is the solution of the BCS equation

\[
\Delta(x) = \int_\Omega dy K(x, y) \varphi_\beta(y, \Delta(y)),
\]

\[ (1.1) \]
where
\[ \varphi_\beta(y, \Delta(y)) = H_\beta(((y)^2 + \Delta^2(y))^{1/2})\Delta(y), \quad (1.2) \]
with
\[ H_\beta(t) = \frac{\tanh(1/2\beta t)}{t}. \quad (1.3) \]

Where \( \Omega \) is a bounded region, \( \beta \) is the inverse of the absolute temperature, \( T \geq 0 \), \( K(x, y) = -V_{xy} \) the negative matrix elements of the interaction potential of electrons with wave vectors \( x, y \in \mathbb{R}^3 \), and \( x^2 = |x|^2 \), where \( V_{xy} \) is generally the sum of two term: the first term, positive, arise from the repulsive coulomb force, while the second one, negative, from the attractive phonon force. As for the physical solution of BCS model, some researchers have studied, such as [6], [4], [9], [3], [12], [5] and so on.

For simplicity, one often consider the BCS gap equation in one dimension:
\[ \Delta(x) = \int_I K(x, y) \frac{\tanh((1/2T)\sqrt{y^2 + \Delta^2(y)})}{\sqrt{y^2 + \Delta^2(y)}} \Delta(y) dy, \quad (1.4) \]
where \( I = [-a, a] \) is a finite interval, \( T \geq 0 \) is the absolute temperature, \( \Delta(x) \) is the energy gap function so that \( \Delta(x) = 0 \) corresponds to the normal phase and \( \Delta(x) \neq 0 \) corresponds to the superconducting phase, the original BCS assumption was given that the interaction kernel \( K(x, y) \) is positive throughout the cut-off range from the Fermi surface up to a level \( a > 0 \), which implies that the attractive phonon interaction is everywhere dominant.

Recently, under the case of the interaction kernel \( K(x, y) \) that
\[ K(x, y) > 0, K(x, y) \leq \sigma(y), \quad \frac{\sigma(x)}{x^2 + 1} \in L(\mathbb{R}^3), \quad (1.5) \]
Du and Yang in [2] give some theoretical results: the BCS equation (1.1) has a positive gap solution \( \Delta(x) > 0 \), representing the occurrence of superconductivity, while for \( T = 1/\beta > 1/\beta_c = T_c \), the only solution of (1.1) is the trivial one, \( \Delta(x) \equiv 0 \), indicating the dominance of the normal phase; also give a numerical method by the Min-Min scheme and Max-Max scheme to determine a critical temperature \( T_c > 0 \).
However, this assumption (1.5) is only a simplified one. In order to make the model more realistic, Bogoliubov, Tolmachev, and Shirkov in [1] considered the model (1.4) in which the interaction kernel function \( K(x,y) \) is given by the form

\[
K(x,y) = K_{\text{phonon}}(x,y) + K_{\text{Coulomb}}(x,y),
\]

where

\[
K_{\text{phonon}}(x,y) \equiv \frac{K_1}{2} > 0, \quad |x|, |y| < a;
\]

\[
K_{\text{phonon}}(x,y) = 0 \quad \text{otherwise},
\]

\[
K_{\text{Coulomb}}(x,y) \equiv -\frac{K_2}{2} < 0, \quad |x|, |y| < b;
\]

\[
K_{\text{Coulomb}}(x,y) = 0 \quad \text{otherwise},
\]

and \( K_1, K_2 \) are constants, \( a > 0 \) is normally taken to be the Debye energy, \( a = \hbar \omega_D \), and \( b > a \) is a cut-off energy for the range of the screened Coulomb repulsion.

Since the kernel of the Bogoliubov-Tolmachev-Shirkov model is not positive but alternating, so the numerical methods in [2] do not work. And as we have known, there exist no effective numerical methods to handle this case. So, to overcome the above difficulties, we will develop a new numerical method to deal with the above model in this work.

The paper is organized as follows. In section 2, we design the Min-Mixed scheme and Max-Mixed scheme to Bogoliubov-Tolmachev-Shirkov model. And we show that these approximations lead to two numerical critical temperatures \( \tau_c^\prime \) and \( \tau_c^\prime \), and \( \tau_c^\prime \leq T_c \leq \tau_c^\prime \). And also, we prove that there exist \( (u,v)_m \) and \( (u,v)_M \) such that \( (u,v)_m \leq (u,v) \leq (u,v)_M \). In section 3, we give a numerical test confirming the theoretical numerical results, and we obtain some important and interesting physical phenomenon.

2. Numerical Methods

For the Bogoliubov-Tolmachev-Shirkov model, physicists expect the existence of a unique transition temperature \( T_c > 0 \) so that, when \( T < T_c \),
has a positive solution representing the superconducting phase, but when
$T > T_c$, the only solution is the trivial zero solution, representing the normal
phase. Besides, as $T \to T_c$, the positive solution goes to zero.

With this form of the interaction kernel reflecting the mixed interaction
of the phonon attraction and the Coulomb repulsion, one seeks(see [1][7][8])
a piecewise constant solution of the form

$$
\Delta(x) = \begin{cases} 
\Delta_1, & |x| < a; \\
\Delta_2, & a < |x| < b; \\
0, & \text{otherwise.}
\end{cases}
$$

Hence, using (1.4), (1.6), (1.7) and (2.1), we arrive at the coupled system

$$
\Delta_1 = (K_1 - K_2)A_\beta(\Delta_1) - K_2B_\beta(\Delta_2), \\
\Delta_2 = -K_2(A_\beta(\Delta_1) + B_\beta(\Delta_2)),
$$

where $A_\beta$ and $B_\beta$ are the nonlinear transformations defined by

$$
A_\beta(\Delta) = \Delta \int_0^a f_\beta(\sqrt{\Delta^2 + x^2}) dx = \Delta \int_0^a \frac{\tanh(1/2\beta\sqrt{\Delta^2 + x^2})}{\sqrt{\Delta^2 + x^2}} dx, \\
B_\beta(\Delta) = \Delta \int_a^b f_\beta(\sqrt{\Delta^2 + x^2}) dx = \Delta \int_a^b \frac{\tanh(1/2\beta\sqrt{\Delta^2 + x^2})}{\sqrt{\Delta^2 + x^2}} dx.
$$

The normal phase is characterized by the trivial solution of (2.2): $\Delta_1 = 0$, $\Delta_2 = 0$, and the superconducting phase is characterized by any nontrivial
solution of (2.2) of the form

$$
\Delta_1 > 0, \Delta_2 < 0.
$$

And a rigorously superconducting-normal phase transition theorem for
the phonon-Coulomb interaction model of Bogoliubov-Tolmachev-Shirkov
within the BCS theory has been established in [11]:

**Theorem 2.1.** There exists a unique and positive transition temperature,
$T_c = 1/\beta_c$, so that when $T < T_c$, the system (2.2) has a nontrivial
solution of the form (2.4), and, when $T > T_c$, the only solution of (2.2)
is the trivial
solution, $\Delta_1 = \Delta_2 = 0$. 

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Next, we design a numerical method to determine the critical temperature. For convenience, introducing the new variables $u = \Delta_1$ and $v = -\Delta_2$, and using (2.2), we have

\begin{align*}
  u &= (K_1 - K_2)A_\beta(u) + K_2B_\beta(v), \\
  v &= K_2A_\beta(u) - K_2B_\beta(v).
\end{align*}

It is seen that the superconducting phase is given by any positive solution of (2.5): $u > 0$, $v > 0$.

Observing the structure of $A_\beta$ and $B_\beta$ of (2.3), it is difficult to solve this equations directly. So, in next discussion, we introduce two discretized versions, called the min-mixed and max-mixed approximations.

Now, we first introduce a partition of the interval $I$ as follows. Let \( \{I_j\}_{1 \leq j \leq n} \) be a collection of open subsets of $I$ such that

\[ I_j \cap I_k = \emptyset \quad (j \neq k), \quad \bigcup_{j=1}^n \bar{I}_j \supset I. \]

To give the numerical methods for the model (2.5), we do with the problem by the following two cases.

**Case I:** $K_1 > K_2$.

In order to design the numerical scheme, we firstly introduce a definition.

**Definition 2.1.** We say that the pair $(u, v)$ is positive (nonnegative), if $u > 0$, $v > 0$ ($u \geq 0$, $v \geq 0$). Besides, we say $(u, v) > (u', v')$ if $u > u'$, $v > v'$ if $(u - u' > 0, v - v' > 0)$ if $u \geq 0$, $v \geq 0$). We use the notation

\[ \chi = \{(u, v) \in \mathbb{R} \times \mathbb{R} | u \geq 0, v \geq 0\}. \]
2.1. Min-Mixed and Max-Mixed schemes

The discrete scheme of the Bogoliubov-Tolmachev-Shirkov model below is

\[
\begin{align*}
    u &= (K_1 - K_2) u \sum_{k}^{N} \min_{x \in I_k} f_{\beta}(\sqrt{u^2 + x^2}) |I_k| \\
    &\quad + K_2 v \sum_{k}^{N} \min_{x \in I'_k} f_{\beta}(\sqrt{v^2 + x^2}) |I'_k|, \\
    v + K_2 v \sum_{k}^{N} \max_{x \in I'_k} f_{\beta}(\sqrt{v^2 + x^2}) |I'_k| \\
    &= K_2 u \sum_{k}^{N} \min_{x \in I_k} f_{\beta}(\sqrt{u^2 + x^2}) |I_k|.
\end{align*}
\]

(2.6)

We next will show that (2.6) has a positive solution if and only if it has a subsolution \((u_0, v_0)\) satisfying \(u_0 > 0\), \(v_0 \geq 0\) and

\[
\begin{align*}
    u_0 &\leq (K_1 - K_2) u_0 \sum_{k}^{N} \min_{x \in I_k} f_{\beta}(\sqrt{u_0^2 + x^2}) |I_k| \\
    &\quad + K_2 v_0 \sum_{k}^{N} \min_{x \in I'_k} f_{\beta}(\sqrt{v_0^2 + x^2}) |I'_k|, \\
    v_0 + K_2 v_0 \sum_{k}^{N} \max_{x \in I'_k} f_{\beta}(\sqrt{v_0^2 + x^2}) |I'_k| \\
    &\leq K_2 u_0 \sum_{k}^{N} \min_{x \in I_k} f_{\beta}(\sqrt{u_0^2 + x^2}) |I_k|.
\end{align*}
\]

(2.7)
To this end, we first define the iterative scheme

\[ u_{n+1} = (K_1 - K_2)u_{n+1} \sum_{k}^{N} \min_{x \in I_k} f_{\beta}(\sqrt{u_{n+1}^2 + x^2})|I_k| \]

\[ + K_2 v_n \sum_{k}^{N} \min_{x \in I'_k} f_{\beta}(\sqrt{v_n^2 + x^2})|I'_k|, \]

\[ v_{n+1} + K_2 v_{n+1} \sum_{k}^{N} \max_{x \in I'_k} f_{\beta}(\sqrt{v_{n+1}^2 + x^2})|I'_k| \]

\[ = K_2 u_{n+1} \sum_{k}^{N} \min_{x \in I_k} f_{\beta}(\sqrt{u_{n+1}^2 + x^2})|I_k|, \]

\[ n = 1, 2, \ldots; \quad v_1 = v_0. \] (2.8)

The solution of (2.6) is denoted by \((u, v)_m\), and (2.6) is called by the Min-Mixed scheme.

The discrete scheme of the Bogoliubov-Tolmachev-Shirkov model up is

\[ u = (K_1 - K_2)u \sum_{k}^{N} \max_{x \in I_k} f_{\beta}(\sqrt{u^2 + x^2})|I_k| \]

\[ + K_2 v \sum_{k}^{N} \max_{x \in I'_k} f_{\beta}(\sqrt{v^2 + x^2})|I'_k|, \] (2.9)

\[ v + K_2 v \sum_{k}^{N} \min_{x \in I'_k} f_{\beta}(\sqrt{v^2 + x^2})|I'_k| \]

\[ = K_2 u \sum_{k}^{N} \max_{x \in I_k} f_{\beta}(\sqrt{u^2 + x^2})|I_k|. \]
In fact, \((u_0, v_0)\) is also a subsolution of (2.9), namely,

\[
\begin{align*}
    u_0 & \leq (K_1 - K_2)u_0 \sum_k^N \max_{x \in \mathcal{I}_k} f_\beta(\sqrt{u_0^2 + x^2})|\mathcal{T}_k| \\
    & + K_2v_0 \sum_k^N \max_{x \in \mathcal{T}_k} f_\beta(\sqrt{v_0^2 + x^2})|\mathcal{T}_k|, \\
    v_0 & + K_2v_0 \sum_k^N \min_{x \in \mathcal{T}_k} f_\beta(\sqrt{v_0^2 + x^2})|\mathcal{T}_k| \\
    & \leq K_2u_0 \sum_k^N \max_{x \in \mathcal{T}_k} f_\beta(\sqrt{u_0^2 + x^2})|\mathcal{T}_k|.
\end{align*}
\]  

(2.10)

And the iterative scheme is defined by

\[
\begin{align*}
    u_{n+1} & = (K_1 - K_2)u_n + K_2v_n \sum_k^N \max_{x \in \mathcal{T}_k} f_\beta(\sqrt{u_n^2 + x^2})|\mathcal{T}_k| \\
    v_{n+1} & = v_n + K_2v_n \sum_k^N \min_{x \in \mathcal{T}_k} f_\beta(\sqrt{v_n^2 + x^2})|\mathcal{T}_k| \\
    & = K_2u_n \sum_k^N \max_{x \in \mathcal{T}_k} f_\beta(\sqrt{u_n^2 + x^2})|\mathcal{T}_k|, \\
    n & = 1, 2, \ldots; \quad v_1 = v_0.
\end{align*}
\]  

(2.11)

\((u, v)_M\) is used to denote the solution of (2.9), and (2.9) is called by the Max-Mixed scheme.

**Remarks 2.1.** The Min-Mixed scheme and Max-Mixed scheme are different from the Min-Min scheme and Max-Max scheme of [2]: the problem in the work is a system, while the problem of [2] is a single equation; the discrete schemes are very different.

In order to prove the numerical solutions of the discrete system (2.6) and
[2.9], we need to give the following lemmas. Denote

\[
A_h(u) = u \sum_{k}^{N} \min_{x \in I_k} f_{\beta}(\sqrt{u^2 + x^2}) |I_k|,
\]

\[
B_h(v) = v \sum_{k}^{N} \min_{x \in I_k'} f_{\beta}(\sqrt{v^2 + x^2}) |I_k'|.
\]

(2.12)

Lemma 2.1.

\[H_h(u) = u - (K_1 - K_2)A_h(u)\]

is monotone about \(u\).

Proof. The proof is similar to that for the continuous case in [11] and is skipped here. \(\Box\)

Lemma 2.2. When \(\beta > 0\) is small, the only solution of (2.6) is the zero solution.

Proof. This is because

\[A_h(u) \leq \frac{1}{2} \beta au,\]

and

\[B_h(v) \leq \frac{1}{2} \beta (b - a)v.\]

Therefore, when \(\beta\) is small, the only non-negative solution of (2.6) is the trivial solution \(u = 0, \ v = 0\). \(\Box\)

Lemma 2.3. When \(\beta > 0\) is sufficiently large, (2.6) has a subsolution \((u_0, v_0)\) as it is defined in (2.7).

Proof. Indeed, we may start from the simple BCS discrete equation

\[u = (K_1 - K_2)A_h(u),\]

(2.13)

which may be obtained by setting \(v = 0\) in the first equation in (2.6). When \(\beta\) is large, (2.13) has a positive solution, say \(u_0\) (see [2]). Let \(v_0 = 0\). Then the pair \((u_0, v_0)\) satisfying (2.7) is a subsolution. \(\Box\)
Lemma 2.4. There is a \( \delta_0 > 0 \), so that for any \( u^0 \geq \delta_0 \), \( u^0 \) is a supersolution of the first equation of (2.6) for any \( v \), in the sense that:

\[
u^0 - (K_1 - K_2)A_h(u^0) \geq K_2B_h(v), \quad \forall v. \tag{2.14}\]

Proof. Since the function \( A_h(u) \), \( B_h(v) \) are bounded uniformly with respect to the parameter \( \beta \), so we have

\[
A_h(u) \leq C, \\
B_h(v) \leq C.
\]

for some absolute constant \( C > 0 \), there is an absolute constant \( \delta_0 > 0 \) so that

\[
\delta_0 - (K_1 - K_2)A_h(\delta_0) \geq K_2B_h(v), \quad \forall v. \tag{2.15}\]

then using Lemma 2.1, we can obtain if \( u^0 \geq \delta_0 \), \( u^0 \) is a supersolution which satisfies (2.14). \( \square \)

Lemma 2.5. The Min-Mixed interaction scheme (2.6) has a positive solution if and only if there is a nontrivial subsolution \((u_0, v_0)\).

Proof. Using Lemma 2.4, there is an absolute constant \( u^0 > 0 \) so that

\[
u^0 - (K_1 - K_2)A_h(u^0) \geq K_2B_h(v), \quad \forall v. \tag{2.16}\]

In the iterative scheme (2.8), if \( v_1 = v_0 \geq 0 \), then \( u_2 > 0 \) and \( u_0 \leq u_2 \leq u^0 \) by

\[
u_0 - (K_1 - K_2)A_h(u_0) \leq K_2B_h(v_0) \tag{2.17}\]

and (2.16). Since the function

\[
J_h(v) = v + K_2v \sum_k N \max_{x \in \bar{T}_k} f_{\beta}(\sqrt{v^2 + x^2})|\bar{T}_k|, \tag{2.18}\]

strictly increases with \( J_h(0) = 0 \) and \( J_h(\infty) = \infty \), the equation

\[
J_h(v) = s, \tag{2.19}\]

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has a unique solution, say $v$, in $[0, \infty]$ for each $s \in [0, \infty]$ and $v$ increases as $s$ increases. Hence, in (2.8), $v_2 > 0$ is well defined and $v_2 \geq v_1 = v_0$. Assume that the inequalities

$$0 < u_0 = u_1 \leq u_2 \leq \ldots \leq u_l \leq u^0,$$  \hspace{1cm} (2.20)

$$0 \leq v_0 = v_1 \leq v_2 \leq \ldots \leq v_l,$$  \hspace{1cm} (2.21)

hold at some step $l$. Then, in view of (2.20) and (2.21), $u_l$ and $v_l$ satisfy

$$K_2 \mathcal{B}_h(v_{l-1}) \leq K_2 \mathcal{B}_h(v_l).$$ \hspace{1cm} (2.22)

Hence we arrive at $u_{l+1} \geq u_l$ after comparing (2.22) with (2.21) and reviewing the definition of $u_{l+1}$. Thus

$$K_2 \mathcal{A}_h(u_l) \leq K_2 \mathcal{A}_h(u_{l+1}).$$ \hspace{1cm} (2.23)

Obviously, $v_l \leq v_{l+1}$ in view of (2.18). Of course, $u_{l+1} \leq u^0$ because $u^0$ has been chosen to be a (universal) supersolution (see (2.16)). Therefore, we have shown that (2.20) and (2.21) are valid in general.

The boundedness of the sequence $\{v_n\}$ follows from the boundedness of the sequence $\{u_n\}$ and the second equation in (2.8). In fact,

$$v_n \leq K_2 \mathcal{A}_h(u^0), \quad n = 1, 2, \ldots$$

Taking $n \to \infty$ in the scheme (2.8), we obtain a numerical solution pair $(u, v)_m$ of the Bogoliubov-Tolmachev-Shirkov model.

\[\square\]

**Lemma 2.6.** Let

$$\Lambda = \{\beta > 0 \mid \text{When } N \text{ is sufficiently large, (2.6) has a positive solution pair}\},$$

and

$$\beta'_c = \inf \{\beta \mid \beta \in \Lambda\},$$

then $\Lambda$ is connected and $\beta'_c > 0$. Moreover, we have the relations $(\beta'_c, \infty) \subset \Lambda$ and $[0, \beta'_c) \cap \Lambda = \emptyset$.
Proof. To see this, we show that, if $\beta \in \Lambda$, then $\beta + \varepsilon \in \Lambda$ for any $\varepsilon > 0$.

In fact, for $\beta \in \Lambda$, let $(u, v)_m$ be a positive solution pair of (2.6). We rewrite (2.6) as

\[
\begin{align*}
\alpha &= (K_1 - K_2)u \sum_{k} \min_{x \in I_k} f_\beta(\sqrt{u^2 + x^2})|I_k| \\
&\quad + K_2 v \sum_{k} \min_{x \in \bar{I}_k} f_\beta(\sqrt{v^2 + x^2})|\bar{I}_k|, \\
\beta &= K_2 u \sum_{k} \min_{x \in I_k} f_\beta(\sqrt{u^2 + x^2})|I_k|. 
\end{align*}
\] (2.24)

Since $v > 0$, we may choose $r \in (0, 1)$ so that

\[
B_{\beta + \varepsilon}(rv) = B_\beta(v). \tag{2.25}
\]

However, from (2.24), we have

\[
\begin{align*}
u < (K_1 - K_2)u \sum_{k} \min_{x \in I_k} f_{\beta + \varepsilon}(\sqrt{u^2 + x^2})|I_k| \\
&\quad + K_2 v \sum_{k} \min_{x \in \bar{I}_k} f_\beta(\sqrt{v^2 + x^2})|\bar{I}_k|, \\
v + K_2 v \sum_{k} \max_{x \in \bar{I}_k} f_\beta(\sqrt{v^2 + x^2})|\bar{I}_k| \\
&< K_2 u \sum_{k} \min_{x \in I_k} f_{\beta + \varepsilon}(\sqrt{u^2 + x^2})|I_k|. 
\end{align*}
\] (2.26)
Combining (2.25) and (2.26), we obtain

\[ u \leq (K_1 - K_2)u \sum_{k} \min_{x \in I_k} f_{\beta + \epsilon}(\sqrt{u^2 + x^2})|I_k| \]

\[ + K_2 ru \sum_{k} \min_{x \in I_k'} f_{\beta + \epsilon}(\sqrt{(rv)^2 + x^2})|I_k'|, \]

\[ rv + K_2 rv \sum_{k} \max_{x \in I_k'} f_{\beta + \epsilon}(\sqrt{(rv)^2 + x^2})|I_k'| \]

\[ \leq K_2 u \sum_{k} \min_{x \in I_k} f_{\beta + \epsilon}(\sqrt{u^2 + x^2})|I_k|. \]

In other words, we have recovered (2.7) with \( u_0 = u, v_0 = rv \), and \( \beta \) being replaced by \( \beta + \epsilon \). Consequently, \( \beta + \epsilon \in \Lambda \).

Using Lemma 2.1-Lemma 2.6, we obtain the following important result:

**Theorem 2.2.** There exists a number \( \beta'_c > 0 \) so that (2.6) has a nontrivial solution: \( u > 0, v > 0 \), for any \( \beta : \beta'_c < \beta \leq \infty \), while for \( \beta < \beta'_c \), the only solution of (2.6) is the trivial one, \( u = 0, v = 0 \).

**Remarks 2.2.** From Theorem 2.2, we do not know if the only solution of (2.6) is the zero solution when \( \beta = \beta'_c \). We guess that it is true (one can see Fig. 1 and Fig. 4), but we are not able to prove it.

In fact, we can obtain another important theorem.

**Theorem 2.3.** There exists a number \( \beta''_c > 0 \) so that (2.9) has a nontrivial solution \( u > 0, v > 0 \) for any \( \beta : \beta''_c < \beta \leq \infty \), while for \( \beta < \beta''_c \), the only solution of (2.9) is the trivial one, \( u = 0, v = 0 \).

**Proof.** Similar to Theorem 2.2, the proof of this theorem can be carried out.

Additionally, let us see an interesting comparison theorem.
Theorem 2.4. Let $\beta_c$, $\beta'_c$ and $\beta''_c$ are the corresponding critical numbers of (2.5), (2.6) and (2.9), respectively. Then

$$\beta'_c \geq \beta_c \geq \beta''_c.$$ 

Besides, if $(u,v)_m$, $(u,v)_M$ are solutions of (2.6) and (2.9) respectively, $(u,v)$ is the solution of (2.5), then

$$(u,v)_m \leq (u,v) \leq (u,v)_M.$$ 

Proof. For $\beta > \beta'_c$, let $(u,v)_m$ be a nontrivial solution of (2.6) in $\chi$. Then $(u,v)_m$ is a subsolution of (2.5). Thus $(u,v)_m \leq (u,v)$ which can be obtained by interating from $(u,v)_m$. Consequently, $\beta > \beta_c$. Clearly, $\beta'_c \geq \beta_c$ and $(u,v)_m \leq (u,v)$.

Next, take $\beta > \beta_c$ and assume that $(u,v)$ is a nontrivial solution of (2.5) in $\chi$. Then $(u,v)$ is a subsolution of (2.9). Thus $(u,v) \leq (u,v)_M (u,v)_M$ is a nontrivial solution of (2.9). Consequently, $\beta > \beta''_c$. So $\beta_c \geq \beta''_c$.

The proof of Theorem 2.4 is completed.

Case II: $K_1 \leq K_2$. In this case, (2.5) has been rewritted as

$$u + (K_2 - K_1)A_\beta(u) = K_2 B_\beta(v),$$
$$v + K_2 B_\beta(v) = K_2 A_\beta(u).$$

(2.28)

The Min-Mixed scheme below is

$$u + (K_2 - K_1)u \sum_{k}^{N} \max_{x \in T_k} f_\beta(\sqrt{u^2 + x^2})|T_k|$$

$$= K_2 v \sum_{k}^{N} \min_{x \in \bar{T}_k} f_\beta(\sqrt{v^2 + x^2})|\bar{T}_k|,$$

(2.29)

$$v + K_2 v \sum_{k}^{N} \max_{x \in \bar{T}_k} f_\beta(\sqrt{v^2 + x^2})|\bar{T}_k|$$

$$= K_2 u \sum_{k}^{N} \min_{x \in T_k} f_\beta(\sqrt{u^2 + x^2})|T_k|.$$
As before, we can show that the system (2.29) has a positive solution pair if and only if there exists a nontrivial subsolution, \((u_0, v_0)\), satisfying
\[
\begin{align*}
    u_0 + (K_2 - K_1)u_0 \sum_{k}^{N} \max_{x \in I_k} f_{\beta}(\sqrt{u_0^2 + x^2})|I_k| & \\
    \leq K_2v_0 \sum_{k}^{N} \min_{x \in I_k} f_{\beta}(\sqrt{v_0^2 + x^2})|I_k|,
\end{align*}
\]
where \(u_0, v_0\) are positive.

In fact, define the Min-Mixed iteration scheme below as
\[
\begin{align*}
    u_{n+1} + (K_2 - K_1)u_{n+1} \sum_{k}^{N} \max_{x \in I_k} f_{\beta}(\sqrt{u_{n+1}^2 + x^2})|I_k| & \\
    = K_2v_n \sum_{k}^{N} \min_{x \in I_k} f_{\beta}(\sqrt{v_n^2 + x^2})|I_k|,
\end{align*}
\]
\[
\begin{align*}
    v_{n+1} + K_2v_{n+1} \sum_{k}^{N} \max_{x \in I_k} f_{\beta}(\sqrt{v_{n+1}^2 + x^2})|I_k| & \\
    = K_2u_{n+1} \sum_{k}^{N} \min_{x \in I_k} f_{\beta}(\sqrt{u_{n+1}^2 + x^2})|I_k|,
\end{align*}
\]
where \(n = 1, 2, \ldots; \ v = v_0\).

Using the monotonicity of the function
\[
\begin{align*}
    P_h(u) = u + (K_2 - K_1)u \sum_{k}^{N} \max_{x \in I_k} f_{\beta}(\sqrt{u^2 + x^2})|I_k|,
\end{align*}
\]
and
\[
\begin{align*}
    Q_h(v) = v + K_2v \sum_{k}^{N} \max_{x \in I_k} f_{\beta}(\sqrt{v^2 + x^2})|I_k|,
\end{align*}
\]
we see that the sequences \( u_n \) and \( v_n \) are well defined and that
\[
\begin{align*}
u_0 &= u_1 \leq u_2 \leq \ldots \leq u_n \leq \ldots, \\
v_0 &= v_1 \leq v_2 \leq \ldots \leq v_n \leq \ldots,
\end{align*}
\] (2.32)

Since the function
\[
B_h(v) = v \sum_{k}^N \min_{x \in I_k} f_\beta(\sqrt{v^2 + x^2})|I_k|,
\]
and
\[
A_h(u) = u \sum_{k}^N \min_{x \in I_k} f_\beta(\sqrt{u^2 + x^2})|I_k|,
\]
are bounded, it follows from (2.31) that \( u_n \) and \( v_n \) are bounded sequences, Taking the limit \( n \to \infty \) in (2.31), we see that \( u = \lim_{n \to \infty} u_n \) and \( v = \lim_{n \to \infty} v_n \) make a solution pair to the system (2.29).
The Max-Mixed scheme up is
\[
\begin{align*}
u + (K_2 - K_1)u \sum_{k}^N \min_{x \in I_k} f_\beta(\sqrt{u^2 + x^2})|I_k| \\
&= K_2v \sum_{k}^N \max_{x \in I_k} f_\beta(\sqrt{v^2 + x^2})|I_k|,
\end{align*}
\] (2.33)
\[
\begin{align*}
u + K_2v \sum_{k}^N \min_{x \in I_k} f_\beta(\sqrt{v_{n+1}^2 + x^2})|I_k| \\
&= K_2u \sum_{k}^N \max_{x \in I_k} f_\beta(\sqrt{u^2 + x^2})|I_k|,
\end{align*}
\]
Obviously, \((u_0, v_0)\) defined in (2.30) is also a subsolution of (2.33), and the
Max-Mixed interation scheme up is

\[ u_{n+1} + (K_2 - K_1)u_{n+1} = K_2u_n + \sum_{k}^{N} \min_{x \in I_k} f_{\beta}(\sqrt{u_{n+1}^2 + x^2})|I_k| \]

\[ = K_2v_n + \sum_{k}^{N} \max_{x \in \bar{I}_k} f_{\beta}(\sqrt{v_{n+1}^2 + x^2})|\bar{I}_k|, \]

\[ v_{n+1} + K_2v_{n+1} = K_2u_{n+1} + \sum_{k}^{N} \min_{x \in \bar{I}_k} f_{\beta}(\sqrt{v_{n+1}^2 + x^2})|\bar{I}_k| \]

\[ = K_2u_{n+1} + \sum_{k}^{N} \max_{x \in I_k} f_{\beta}(\sqrt{u_{n+1}^2 + x^2})|I_k|, \]

\[ n = 1, 2, \ldots; \quad v = v_0. \]

The convergence of (2.34) which similar to (2.31) will no longer be proved here. And the choice of subsolution can reference [11]. In section 3, we shall present the numerical results.

3. Numerical Test

In this section, we shall calculate specifically a example which corresponding to the above section.

Case 1: \( K_1 > K_2 \). Taking

\[ a = 1, \quad b = 1.5, \]

\[ K_1 = 2, \quad K_2 = 0.1. \]
Figure 1: Min-Mixed iteration ($a = 1, \beta_c \approx 0.1$), $N = 50$, $(u(up), v(down))$ vs $\beta$

Figure 2: Max-Mixed iteration (from up to down), Min-Mixed iteration (from bottom to up), $u(left), v(right), \beta = 6, N = 50, 100, 200$
Next, we only increase the value of $a$ so that we observe the change of $\beta_c$. To do that, we choose $a = 1.1$. Comparing Fig 1 with Fig 4, we see that $\beta_c$ decreases as $a$ becomes bigger.
In fact, we can show the above fact is right numerically from Fig. 5 which fits the physical phenomenon very well.

Figure 5: Min-Mixed interaction $N = 50$, $\beta_c$ vs $a$

Now, we only change the value of $K_2$ to observe the change of $\beta_c$.

Figure 6: Min-Mixed interaction $N = 50$, $\beta_c$ vs $K_2$
From Fig. 6 we find that $\beta_c$ increases as $K_2$ increases.

**Case 2:** $K_1 \leq K_2$. Taking

$$a = 0.5, \ b = 1.5,$$

$$K_1 = 0.01, \ K_2 = 0.1.$$  \hfill (3.2)

Figure 7: Min-Mixed interation($N = 50$, $(u(up), v(down))_m$ vs $\beta$
Figure 8: Min-Mixed iteration (from bottom to up), $u(middle), v(left, right), \beta = 5, N = 50, 100, 200$

Figure 9: Max-Mixed iteration (from up to down), $u(middle), v(left, right), \beta = 5, N = 50, 100, 200$
Next, we only increase the value of $a$ so that we observe the change of $(u, v)_m$.

From Fig.11 we find that $u$ and $v$ of the Min-Mixed iterative scheme
decrease with $a$ closing to $b$. Thus, $\beta_c$ will increase as $a$ close to $b$.

We now just change $K_2$ to observe the change of $(u,v)_m$.

![Figure 12: Min-Mixed interation $N = 50$, $(u,v)_m$ vs $K_2$](image)

From Fig. 12 we can see that $(u,v)_m$ increases as $K_2$ increases. Namely, $\beta_c$ will decrease as $K_2$ increases.

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