Defect and degree of the Alexander polynomial

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Abstract Defect characterizes the depth of factorization of terms in differential (cyclotomic) expansions of knot polynomials, i.e. of the non-perturbative Wilson averages in the Chern-Simons theory. We prove the conjecture that the defect can be alternatively described as the degree in $q^{\pm 2}$ of the fundamental Alexander polynomial, which formally corresponds to the case of no colors. We also pose a question if these Alexander polynomials can be arbitrary integer polynomials of a given degree. A first attempt to answer the latter question is a preliminary analysis of antiparallel descendants of the 2-strand torus knots, which provide a nice set of examples for all values of the defect. The answer turns out to be positive in the case of defect zero knots, what can be observed already in the case of twist knots. This proved conjecture also allows us to provide a complete set of $C$-polynomials for the symmetrically colored Alexander polynomials for defect zero. In this case, we achieve a complete separation of representation and knot variables.

1 Introduction

The HOMFLY-PT polynomials [1–10] are the main non-perturbative observables in the Chern-Simons theory [11, 12], and at the moment they are the most important source of information about such quantities – much worse understood than correlators in matrix models [13, 14] and intimately related supersymmetric (AGT-controlled) low-energy theories [15, 16]. Being originally defined as certain contractions of quantum $\mathcal{R}$-matrices [17–27], the HOMFLY polynomials possess a lot of additional structures, which are not immediately obvious from the Chern-Simons formulation and are instead the implications of generic representation theory. They range from the very polynomiality of Wilson loop averages in appropriate variables $q = \exp \left( \frac{2\pi i}{g_s N_g} \right)$ and $A = q^N$, to description in terms of Khovanov-Rozansky complexes [28–33]. Of special interest in this list is the differential expansion and its properties.

The differential expansion (DE) for the reduced HOMFLY polynomial in symmetric representations for the figure-eight knot $4_1$ was introduced in [34] and was further generalized to other knots [35–41]. Originally it was suggested in the form

$$\mathcal{H}_{(v)}^K \equiv \sum_{j=0}^{r} \frac{[r]!}{[j]!(r-j)!} F_{(U)}^K(A,q) \prod_{i=0}^{j-1} \{Aq^{r+i}\}{\{Aq^{r-i}\}}$$

(1.1)

with Laurent polynomials $F_{(U)}^K(A,q)$. Here, as usual, $\{x\} := x - x^{-1}$ and $[n] := \{q^n\}/\{q\}$. The DE is also known under the name of cyclotomic expansion [38,42–47]. The question mark in (1.1) stands because the relation is not quite true for all knots $K$, in generic case it is actually weaker [48]:

$$\mathcal{H}_{(v)}^K \equiv \sum_{j=0}^{r} \frac{[r]!}{[j]!(r-j)!} G_{(U)}^K(A,q) \{A/q\} \prod_{i=0}^{j-1} \{Aq^{r+i}\} .$$

(1.2)

This DE for symmetric representations actually follows from representation theory, and it involves polynomial $G_{(U)}^K$ rather than $F_{(U)}^K$. The defect [48] characterizes the degree of its enhancement towards $F_{(U)}^K$.

The variables $G_{(U)}^K$ can provide better coordinates in the space of knots than the HOMFLY polynomials themselves [49], still they are far from being free parameters. In particular, they satisfy the $C$-polynomial equations [41,50,51], which are still far from being well understood and classified.

In this paper, we clarify the situation with an alternative description of the defect $\delta$ [48] – as dictated by the
degree $\delta + 1$ of the fundamental Alexander polynomial in $q^{\pm 2}$. We explicitly prove that this is indeed a corollary of definition (1.2) and derive explicit expressions for the DE coefficients $G_{1}[r](1, q)$ at $A = 1$ in terms of the coefficients of the fundamental Alexander polynomial. This description provides an explicit restriction on $G_{1}[r](1, q)$ – they are not free. What still can be free are the fundamental coefficients $G_{1}[1](1, q) = F_{1}[1](1, q)^{1}$ at $r = 1$ – they could be arbitrary integer symmetric polynomials of degree $\delta$. We show that this is indeed the case for $\delta = 0$ and provide some ideas on how this problem can be investigated for higher $\delta$. Namely, we use the defect-preserving antiparallel evolution [52] of the 2-strand knots, which provides rich sets of examples for all values of the defect. This also allows us to highlight the properties of the antiparallel evolution and derive non-trivial examples, when the defect drops down at particular points (the conjecture of [52] does not allow it to raise but allows occasional drop downs).

The paper is organized as follows. In Sects. 2, 3 we remind the definitions of the DE and its defect for the case of symmetric representations. The central Sect. 4 proves that defect $\delta$ is unambiguously related to the degree $\delta + 1$ of the fundamental Alexander, and in Sect. 4.2.2 we express all the coefficients $G_{[r]}(1, q)$ of DE for the symmetric Alexander polynomials through $G_{[1]}(1, q)$. These are our main results.

The remaining part of the paper concerns open questions and is not yet conclusive. We discuss the following issues.

- First, if all $G_{[r]}(1, q)$ are expressed through $G_{1}[1](1, q)$, can the latter one be arbitrary (free)? It is for $\delta = 0$, but what about the higher defects? To address this question, in Sect. 5 we rewrite the expressions from Sect. 4.2.2 in a more explicit/detailed form, and consider a similarly rich family of antiparallel descendants of the two-strand torus knots. This is an interesting family by itself, but the corresponding sets $G_{1}[1](1, q)$ for $\delta \geq 1$ look unexpectedly restricted. At the moment, it is unclear if this is a property of this particular set (it is not as big as it seems) or $G_{1}[1](1, q)$ are indeed far from being free.
- Second, is the defect indeed as simple as described in Sect. 3, or is there a more sophisticated structure, associated with more complicated degeneracy diagrams? We do not go deep into this problem (raised already in [48] and [52]), we just provide some illustrations of “accidental” degeneracies in Sect. 6. This also helps to reformulate the statement about invariance of the defect under antiparallel evolution more carefully – defect can actually drop down at some evolution points, and there remains a question, if this should be treated as “accidents” or as a manifestation of additional structures, promoting defect from just a number to a slightly more involved characteristic of the DE.
- Third, can we express the fact that all $G_{[r]}(1, q)$ are made from $G_{[1]}(1, q)$ as some equations for $G_{[r]}(1, q)$? In Sect. 7, we show that such “Alexander $C$-polynomials” can indeed be defined and they are considerably simpler than generic $C$-polynomials for the HOMFLY polynomial and superpolynomials.
- Fourth, can we lift the description of $G_{[r]}(1, q)$ through $G_{1}[1](1, q)$ to non-symmetric representations? In Sect. 8, we make first steps in this direction.

Conclusion in Sect. 9 contains a brief summary.

2 On the origins of the differential expansion

In this paper, we mostly consider the differential expansion of the HOMFLY polynomials in the simplest case of symmetric representations. All the HOMFLY polynomials are reduced, i.e. they are Wilson averages in an irreducible representation $R$, divided by the dimension of the representation. The differential (cyclotomic) expansion is what follows from elementary representation theory and from the fact that it is respected by the HOMFLY polynomials. Namely, we get use of the properties below.

- In sl$_{N}$ algebra, $[R_{1}, \ldots, R_{N}] = [R_{1} + \delta R, \ldots, R_{N} + \delta R]$ for any integer $\delta R$.
- In sl$_{N}$ algebra, a representation $R$ and its conjugate $\bar{R}$ are equivalent.
- The HOMFLY polynomial possesses the symmetry under the transposition of a diagram [53]:

$$H_{R}^{K}(q, A) = H_{R}^{K}(q^{-1}, A) , \quad (2.1)$$

where $R^{\vee}$ denotes the transposition of a Young diagram $R$.

These facts restrict the HOMFLY group structure drastically. We use them to obtain the differential expansion for the symmetric HOMFLY polynomials. To begin with, for $N = 1$ all representations are trivial and the HOMFLY polynomials are unities. In particular, for $R = [r]$ we have

$$H_{[r]}^{K}(q, A = q) = 1 \Rightarrow H_{[r]}^{K}(q, A) - 1 : \{A/q\} . \quad (2.2)$$

At the same time, for $N = r$ the antisymmetric representation $[1^{r}]$ is the same as singlet, $\emptyset$, so

$$H_{[1^{r}]}^{K}(q, A = q') = 1 \Rightarrow H_{[1^{r}]}^{K}(q, A) - 1 : \{A/q'\} , \quad (2.3)$$

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1 We sometimes omit the upper index $K$ to shorten expressions.
and utilizing the basic transposition property (2.1) and taking into account (2.2), we obtain:

\[ \mathcal{H}^K_{[1]}(q, A) - 1 \sim \{ Aq^r \} \{ A/q \}. \tag{2.4} \]

In fact, for \( r > 1 \) this is not the full story. We illustrate this by the next simplest example. For \( N = 3 \), we have an additional property: \( \mathcal{H}^K_{[1]}(q, A = q^3) = \mathcal{H}^K_{[1,1]}(q, A = q^3) \), because in this case the two representations \([1]\) and \([1, 1]\) are the same. This fact implies that

\[ \mathcal{H}_{[1,1]} - \mathcal{H}_{[1]} \sim \{ Aq^3 \} \quad \mathcal{H}_{[2]} \]

\[ -\mathcal{H}_{[1]} \sim \{ Aq^3 \} \quad \mathcal{H}_{[2]} - \mathcal{H}_{[1]} \sim \{ Aq^3 \} \{ A/q \}. \tag{2.5} \]

From (2.4), we get

\[ \mathcal{H}^K_{[1]}(q, A) = 1 + \{ Aq \} \{ A/q \} F^K_{[1]}(q, A). \tag{2.6} \]

with Laurent polynomial \( F_{[1]}(q, A) \). Then

\[ \mathcal{H}^K_{[2]}(q, A) = \mathcal{H}^K_{[1]}(q, A) + g^K_2(q, A) \{ Aq^3 \} \{ A/q \} \]

\[ = 1 + (\{ Aq \} F^K_{[1]}(q, A) + h^K_2(q, A) \{ Aq^3 \}) \{ A/q \}. \tag{2.7} \]

From (2.4), it follows that \( \mathcal{H}^K_{[2]}(q, A) - 1 \sim \{ Aq^2 \} \{ A/q \} \), so we set \( h_2 = F_{[1]} + \{ Aq^2 \} G_{[2]} \) with some polynomial \( G_{[2]}(q, A) \) and obtain

\[ \mathcal{H}_{[1]} = 1 + F_{[1]} \cdot \{ Aq \} \{ A/q \}, \]

\[ \mathcal{H}_{[2]} = 1 + [2] F_{[1]} \cdot \{ Aq^2 \} \{ A/q \} + G_{[2]} \cdot \{ Aq^3 \} \{ A/q \}. \tag{2.8} \]

In the same way, one can iteratively deduce that for an arbitrary knot

\[ \mathcal{H}^K_{[r]} = \sum_{k=0}^{r} \frac{[r]!}{[k]! [r-k]!} \cdot G^K_{[r]}(A, q) \cdot \left( \prod_{i=0}^{k-1} \{ Aq^{r+i} \} \right) \{ A/q \}. \tag{2.9} \]

with some polynomial factors \( G^K_{[r]}(A, q) \).

### 3 Defect of the differential expansion of the symmetric HOMFLY

Now, there comes something else – perhaps, a little closer to the mysterious nature of knots. The coefficients \( G_{[s]} \) turn out to factorize further. Their factorization property depends on the knot parameter \( \delta^K \) named defect:

\[ G^{(s)}_{[s]}(A, q) = G^{(s)}_{[s]}(A, q) \cdot \prod_{i=1}^{\text{floor}\left( \frac{s-1}{2} \right)} \{ Aq^{i-1} \}. \tag{3.1} \]

Factorization is maximal for \( \delta = 0 \), the corresponding coefficients\(^2\) \( G^{(0)}_{[s]} \) are usually denoted by \( F_{[s]} \). For \( s = 1 \) always \( G_{[1]} = F_{[1]} \), we use this fact to distinguish between the first and all other coefficients of the differential expansion.

The property (3.1) can be depicted by the following ladder diagrams:

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\(^2\) To shorten the notations, we sometimes use \( \delta \) instead of \( \mathcal{K}^{(\delta)} \) super-script.
It looks like a highly non-trivial structure, and neither its origins nor its universality are understood. Also unknown is identification of the defect with more conventional discrete topological numbers, which are associated with knots.

As we will see in Sect. 6, the list of diagrams above can be not exhaustive: sometimes some boxes can be eliminated – there are additional zeroes in coefficients of the DE. Moreover, there can be some regularity – new zeroes appear at the boundaries of our diagrams, and preserve their ladder structure. This can be a signal that defect is not just a number, but a more complicated structure, perhaps, like a Young diagram, as suggested in original [48]. These additional zeroes also affect precise formulation of antiparallel invariance of the defect, which is also discussed in Sect. 6.

In this paper, we address/prove still another – and simpler, characterization of the defect: the conjecture [48] that it is equal to the degree of the fundamental Alexander polynomial minus one. This is a relatively simple statement, it concerns only the bottom level of the above diagrams. Indeed, the Alexander polynomial arises at $A = 1$ and then (3.1) means that there are just a few non-vanishing

$$G_{[s]}^{K(\delta)}(1, q) = G_{[s]}^{K(\delta)}(1, q), \quad 1 < s \leq \delta + 1$$

shown by small black dots in pictures (3.2)–(3.5). Big dots stand for $F_{[1]}(1, q)$, which can vanish only occasionally (see Sect. 6 below). For higher $s$, the product at the r.h.s. of (3.1) contains $\{A\}$, which vanishes at $A = 1$. Despite the fact that the statement seems simple, it becomes highly informative if one assumes that (3.1) is known – then it allows to predict a lot about the colored HOMFLY polynomial (and, actually, superpolynomials as well) by making a nearly trivial calculation for the fundamental Alexander polynomial.

4 Defect defines the degree of the fundamental Alexander polynomial

In this section, we prove that the defect (in most cases, see Sect. 6) can be defined as the degree of the fundamental Alexander polynomial minus one. It turns out that it is a consequence of two facts:

- The reduced symmetric Alexander polynomial possesses the following property:

$$A_r(q) = A_{[1]}(q^r).$$

(4.1)

- The DE (2.9) for the symmetric Alexander polynomial ($A = 1$) becomes

$$A_{[r]} = \sum_{k=0}^{\min(r, \delta+1)} \frac{[r]!}{[k]!\{r-k\}!} \cdot G_{[k]}^{K}(1, q)$$

$$\cdot \left( \prod_{i=0}^{k-1} \{q^{r+i}\} \right) \{q^{-1}\}$$

(4.2)

3 In [52], it was conjectured that the defect does not change under substitution of any knot crossing with an antiparallel braid, except for a few knots for which the defect occasionally drops down.

4 This symmetry actually holds for all single-hook representations.
with \(F_{[1]}(1, q) = G_{[1]}(1, q)\). In particular
\[
A_{[1]} = 1 - F_{[1]}(1, q)\langle q^r \rangle, \tag{4.3}
\]
where recall that \(\{x\} := x - x^{-1}\), so that the quantum numbers are \([n] := \frac{[q^n]}{[q]}\).

### 4.1 Small defect examples

Before providing a proof for the general defect \(\delta\), we parse examples for small values of the defect in order to clarify the arguments.

- **For the defect** \(\delta = 0\) the DE (4.2) contains two terms for all symmetric representations \([r]\):
  \[
  A_{[r]} = 1 - [r]F_{[1]}(1, q)\langle q^r \rangle, \tag{4.4}
  \]
  Then (4.1), (4.3), (4.4) imply
  \[
  [r]\langle q^r \rangle F_{[1]}(1, q) = \langle q^r \rangle^2 F_{[1]}(1, q^r) \implies F_{[1]}(1, q^r) = F_{[1]}(1, q) = \text{const} = a. \tag{4.5}
  \]
  The trick is to look at the \(r\) dependence, which at the l.h.s. is just quadratic in \(q^{-r}\) because \(F_{[1]}(1, q^r)\) is independent of \(r\). This restricts the \(q\)-dependence of \(F_{[1]}(1, q^r)\) at the r.h.s. – actually to nothing.

  Note that from (4.5), it follows that the degree of the fundamental Alexander polynomial is \(\delta + 1 = 1\). However, sometimes it happens that \(F_{[1]}(1, q) = 0\) and the degree of the fundamental Alexander polynomial drops to 0. Examples are provided in Sect. 6.1.

- **For the defect** \(\delta = 1\) it is released, but just a little. For \(r > 1\) there are three terms:
  \[
  A_{[r]} = 1 - [r]F_{[1]}(1, q)\langle q^r \rangle, \tag{4.6}
  \]
  and therefore
  \[
  F_{[1]}(1, q) = b \cdot (q^2 + q^{-2}) + a, \tag{4.7}
  \]
  with some constants \(a\) and \(b\). At the l.h.s., we have a quartic polynomial of \(q^{2r}\), therefore the same must happen to the r.h.s., thus \(F_{[1]}\) is at most quadratic. Note that this does not mean, that \(G_{[2]}(1, q)\) is independent of \(q\) – the dependence exists, but it is fully fixed in terms of the \(q\)-dependence of \(F_{[1]}(1, q)\).

The conjecture about the defect still holds: the **degree of the Alexander polynomial** turns out to be \(\delta + 1 = 2\).

We return to defect 0 if \(b = 0\). To understand if the conjecture about defect-Alexander relation still holds at such special points, it is necessary to check if factorization of entire HOMFLY also increases. Some examples are provided in Sect. 6.2.

- **For the defect** \(\delta = 2\) and \(r > 2\) there are four terms:
  \[
  A_{[r]}(q) = 1 - [r]F_{[1]}(1, q)\langle q^r \rangle, \tag{4.8}
  \]
  that implies
  \[
  F_{[1]}(1, q) = G_{[1]}(1, q)\langle q^r \rangle = \Bigg\{q^{r+1}\Bigg\}^2 F_{[2]}(1, q) G_{[3]}(1, q) \frac{[q^2]}{[q^3]}, \tag{4.9}
  \]
  and
  \[
  F_{[1]}(1, q) = c \cdot (q^4 + q^{-4}) + b \cdot (q^2 + q^{-2}) + a, \tag{4.10}
  \]
  with some knot-dependent integers \(a, b, c\). The degree of the Alexander polynomial is still \(\delta + 1 = 3\), except some points, where \(a, b, c, a\) turn to zero.

For example, for the defect-2 torus knot \(7_1\)
\[
F_{[1]}(1, q) = -(q^4 + q^{-4}) - (q^2 + q^{-2}) - 2, \tag{4.11}
\]
and
\[
G_{[3]}(1, q) = -(q^4 + q^2 + q^{-2} + q^{-4}) - 1, \tag{4.11}
\]
i.e. \(a = b = c = -1\).

---

5 Of course, the \(q\)-dependence of entire \(F_{[1]}(A, q)\) and \(G_{[2]}(A, q)\) with \(A \neq 1\) can be far more involved. For example, for the defect-1 knot [7, 5]:

\[
F_{[1]}(1, q) = -2(q^2 + q^{-2}), \tag{4.12}
\]

\[
G_{[3]}(1, q) = -2(q^2 - q^{-2}) = -2(q^2). \tag{4.13}
\]
4.2 The case of generic defect

In this subsection, we raise the examples from the previous subsection to generic \( \delta \). Again, our goal is to find the restriction on \( F_{11}(1, q) \) -- and then it allows to express all non-vanishing \( G_{[r]}(1, q) \) through this \( F_{11}(1, q) \). Expectedly or not, this expression is rather involved -- it is the usual amusement in knot theory to observe how fast the complicated structures emerge from the trivial inputs. The lessons for a generic quantum field theory are still to be drawn.

4.2.1 Degree of the Alexander polynomial

Substituting the DE at \( A = 1 \) (4.2) into (4.1), we get:

\[
1 - [r] F_{11}(1, q)(q^r)[q] \sum_{j=2}^{\min(r, \delta+1)} \frac{[r]!}{[j]! [r-j]!} \Delta_{jj}(1, q) \prod_{i=1}^{j} \{q^{r+i-1}\} = 1 - F_{11}(1, q)[q^{r}]^2.
\] (4.12)

This equality must be true for all \( r \). The l.h.s. depends on \( q^r \) only through binomial coefficients and differentials, and it is clear that it is a polynomial of degree 2 \( j_{\text{max}} = 2\delta + 2 \) in \( q^{2r} \).

The r.h.s. depends on \( q^{2r} \) through the factor \( [q^r]^2 \) of degree 2 and \( F_{11}(1, q) \). Since all the dependencies are actually on even powers of \( q \), this implies that \( F_{11}(1, q) \) is actually a polynomial of \( q^{\delta+2} \) of degree \( \delta \), and the whole fundamental Alexander polynomial has degree \( \delta + 1 \). Moreover, since \( \Delta_{11}(q) = \Delta_{11}(q^{-1}) \) (2.1), \( F_{11}(1, q) \) must be symmetric under the change \( q \rightarrow q^{-1} \).

In other words,

\[
F_{11}(1, q) = a_{0}^{(\delta)} + \sum_{j=1}^{\delta} a_{j}^{(\delta)} (q^{2j} + q^{-2j})
\] (4.13)

with \( q \)-independent integers \( a_{j}^{(\delta)} \). We also put the label \( (\delta) \) on \( F_{11}(1, q) \) and in what follows -- on \( G_{[k]}(1, q) \), to emphasize that at \( A = 1 \) they depend on the knot in a huge variety of objects) only through its defect and just a few additional parameters \( a_{i}^{(\delta)} \).

Now, the conjecture about the degree of the fundamental Alexander polynomial is proved, and in the next subsection we present explicit expressions of \( G_{[k]}(1, q) \) through the coefficients \( a_{i}^{(\delta)} \) of \( F_{11}(1, q) \).

4.2.2 Expressions for non-vanishing \( G_{[k]}(1, q) \)

Imposing (4.13), we can rewrite (4.12) as

\[
\sum_{j=1}^{\delta} a_{j}^{(\delta)} \cdot (q^{2jr} - q^{2j} - q^{-2j} + q^{-2jr}) = \sum_{j=2}^{\min(r, \delta+1)} g_{j}^{(\delta)}(q) \cdot \prod_{i=1}^{j-1} \{q^{r+i-j}\}(q^{r-i})
\] (4.14)

where we denote

\[
g_{j}^{(\delta)}(q) := \frac{G_{[j]}(1, q)}{\prod_{i=2}^{j} \{q^{r}\}}, \quad 2 \leq j \leq \delta + 1.
\] (4.15)

Equation (4.14) must hold for any \( r \), so we can extract coefficients at certain powers of \( q^{r} \) and obtain the following system of equations:

\[
\begin{align*}
a_{\delta}^{(\delta)} & = \frac{G_{[\delta]}(1, q)}{\prod_{i=2}^{\delta} \{q^{r}\}}, \\
a_{\delta-1}^{(\delta)} & = \frac{G_{[\delta-1]}(1, q)}{\prod_{i=2}^{\delta-1} \{q^{r}\}}, \\
a_{\delta-2}^{(\delta)} & = \frac{G_{[\delta-2]}(1, q)}{\prod_{i=2}^{\delta-2} \{q^{r}\}} \\
& \vdots
\end{align*}
\] (4.16)

In general

\[
a_{j}^{(\delta)} = \sum_{k=0}^{\delta-j} g_{j+k+1}^{(\delta)} \sigma_{j,k},
\] (4.17)

where

\[
\sigma_{j,k} := \sum_{i=0}^{k} (-1)^{i} q^{i(\delta+1-k)} k^{i} \frac{(q^{-2}; q^{-2})_{j+k}}{(q^{-2}; q^{-2})_{j+k-i} (q^{2}; q^{2})_{k-i} (q^{2}; q^{2})_{j+i}}.
\] (4.18)

Note that relations (4.17) connect knot-factors \( F_{11}(1, q) \) and \( G_{[k]}(1, q) \), so that actually they are the symmetric Alexander C-polynomials (see Sect. 7).

One can revert relations (4.17) and express all \( g_{j} \) through \( a_{j} \):

\[
\begin{align*}
& g_{\delta+1}^{(\delta)} = a_{\delta}^{(\delta)}, \\
& g_{\delta}^{(\delta)} = a_{\delta-1}^{(\delta)} + a_{\delta}^{(\delta)} \cdot (q^{\delta+1} + q^{-\delta-1})[\delta], \\
& g_{\delta-1}^{(\delta)} = a_{\delta-2}^{(\delta)} + a_{\delta-1}^{(\delta)} \cdot (q^{\delta} + q^{-\delta})[\delta] + a_{\delta}^{(\delta)}, \\
& \vdots
\end{align*}
\] (4.19)

In general

\[
\begin{align*}
& g_{i+1}^{(\delta)} = \sum_{m=0}^{\delta-i} a_{i+m}^{(\delta)} \cdot \xi_{m,i+m}, \quad i = 1, \ldots, \delta
\end{align*}
\] (4.20)

and

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Expressions (4.20) are another form of C-polynomials (4.17).

One can straightforward express $g_j$ through $a_j$ using (4.17) and obtain another form of (4.20):

$$g_{2k+1}^{(\delta)} = \sum_{j=0}^{k} a_{\delta-k+j}^{(\delta)} \eta_{j,k}$$

$$\sum_{j=0}^{k} a_{\delta-k+j}^{(\delta)} \left\{ \sum_{\Delta \leq j} (-1)^{l(\Delta)} \sum_{l+j+\Delta_i \leq \delta-k+j} \prod_{l=1}^{j} \sigma_{l,j,\Delta_i} \right\}$$

(4.22)

where at the r.h.s. there stands the sum over all Young diagrams $\Delta$ of the size $j$. For example:

$$\eta_{0,k} = 1,$$

$$\eta_{1,k} = -\sigma_{k-1},$$

$$\eta_{2,k} = \sigma_{k-1} \sigma_{k-1} \sigma_{k-1} \sigma_{k-1,0}. $$

(4.23)

4.2.3 Examples

We now list the implications of general formulas (4.13) and (4.20) for particular small values of $\delta$. These expressions can be simpler to deal with in practical implications.

- In the case of the defect $\delta = 1$

$$F^{(1)}_{[1]}(1, q) = a_0^{(1)} + a_1^{(1)} \cdot (q^2 + q^{-2})$$

(4.24)

and

$$g_2^{(1)}(q) = a_1^{(1)} \cdot \xi_0, 1 = a_1^{(1)} \cdot \xi_0, 1$$

(4.25)

We see that $F^{(1)}_{[1]}$ and $g_2^{(1)} \sim G_2^{[1]}$ are not independent. They are related by the Alexander C-polynomial (see Sect. 7).

The obvious question is (see Sect. 5): Are all integer pairs $(a_1^{(1)}, a_0^{(1)})$ allowed for knots? In other words, can one get an arbitrary integer for $g_2^{(1)}$ and an arbitrary integer symmetric Laurent polynomial of degree 1 in $q^\pm 2$ for $F^{(1)}_{[1]}$?

- For the defect $\delta = 2$

$$F^{(2)}_{[1]}(1, q) = a_0^{(2)} + a_1^{(2)} \cdot (q^2 + q^{-2}) + a_2^{(2)} \cdot (q^4 + q^{-4})$$

(4.26)

and

$$g_2^{(2)}(q) = \sum_{m=0}^{0} a_{m+1}^{(2)} \xi_{m+1} = a_1^{(2)} \cdot \xi_{0,1} + a_2^{(2)} \cdot \xi_{1,2} = a_1^{(2)} + a_2^{(2)} \cdot [2](q^3 + q^{-3})$$

(4.27)

This time we can again ask if all integer triples $(a_2^{(2)}, a_1^{(2)}, a_0^{(2)})$ are allowed (see Sect. 5). From the very beginning we see that $g_2^{(2)}(q)$ has degree 2, not 1 (in $q^\pm 2$) but only two parameters $a_1^{(2)}, a_2^{(2)}$, so that a generic integer polynomial of degree 2 cannot appear in the role of $g_2^{(2)}(q)$.

- For the defect $\delta = 3$

$$F^{(3)}_{[1]}(1, q) = a_0^{(3)} + a_1^{(3)} \cdot (q^2 + q^{-2})$$

$$+ a_2^{(3)} \cdot (q^4 + q^{-4}) + a_3^{(3)} \cdot (q^6 + q^{-6})$$

(4.28)

and

$$g_2^{(3)}(q) = \sum_{m=0}^{2} a_{m+1}^{(3)} \xi_{m+1} = a_1^{(3)} + a_2^{(3)} \cdot [2](q^3 + q^{-3})$$

$$+ a_3^{(3)} \cdot [3](q^6 + 1 + q^{-6}),$$

$$g_3^{(3)}(q) = \sum_{m=0}^{1} a_{m+2}^{(3)} \xi_{m+2} = a_2^{(3)} \cdot \xi_{0,2} + a_3^{(3)} \cdot \xi_{1,3} = a_2^{(3)}$$

$$+ a_3^{(3)} \cdot [3](q^4 + q^{-4}),$$

$$g_4^{(3)}(q) = \sum_{m=0}^{0} a_{m+3}^{(3)} \xi_{m+3} = a_3^{(3)} \cdot \xi_{0,3} = a_3^{(3)}$$

(4.29)

Again, $g_2^{(3)}$ and $g_3^{(3)}$ are not arbitrary integer Laurent polynomials. In the next section, we will discuss whether all integer $a_j$ can be met in the symmetric Alexander DE.

4.3 Intermediate summary

We have proved that the defect $\delta$ defines the degree of the fundamental Alexander polynomial. Moreover, all the dependence on a knot $K$ for all symmetric Alexander polynomials is concentrated in $\delta$ parameters (numbers) $a_{j}^{(\delta)}, j = 1, \ldots, \delta$. Note that any given $a_{j}^{(\delta)}$ enters only $F^{(1)}_{[1]}(1, q)$ and the first few $G_{[j]}(1, q)$ with $i \leq j$. This triangularity implies that the degree of $F^{(1)}_{[1]}$ defines the number of non-vanishing $G_{[j]}$. Only $a_{j}^{(\delta)}$ appears in all $\delta$ DE coefficients $G_{[j]}(1, q)$, which contribute to the Alexander polynomial (note that the other $G_{[j]}(1, q)$ with $i > \delta + 1$ need not vanish, but they do not contribute to the DE at $A = 1$).

Relations (4.20), (4.22) are somewhat sophisticated. A useful way to represent them is through recurrence conditions (equations), which are also known as C-polynomials [51] for the Alexander polynomials, see Sect. 7 below. Still, another question to address is if the same parameters $a_{j}^{(\delta)}$ are sufficient to describe the reduced Alexander polynomials in all other representations. We remind that the reduced Alexander polynomials are non-vanishing for all representations. The simplifying reduction condition, similar to (4.1), remains
true for all single-hook representations, but it breaks violently for multiple hooks. The last direction will be addressed elsewhere, while one-hook representations will be considered in Sect. 8. In the next section of the present paper we will restrict to a kind of inverse question – if everything about the Alexander polynomials is fully controlled by $a_i^{(\delta)}$, then are they free parameters or somehow constrained as well?

5 Integrality

A remarkable property of knot polynomials is that all their coefficients are integer. For the HOMFLY polynomials, this follows technically from the $R$-matrix formalism. In general, this is the ground for cohomological descriptions in terms of various complexes, like in Khovanov-Rozansky approach [31].

In other words, all the coefficients $a_j$ in (4.13) are integers and all $g_j$ in (4.15) are Laurent polynomials in $q$ with integer coefficients. From (4.20), it is not obvious why $g_j^{(\delta)}$ are integer Laurent polynomials in $q$, because of the monstrous coefficients $\delta_{m,n}$ (4.21). Still, there is a bit simpler formula (4.22). From this expansion, it is clear that $g_j^{(\delta)}$ are integer Laurent polynomials in $q$. Indeed, one rewrites $\sigma_{j,k}$ in the form

$$\sigma_{j,k} = \sum_{i=0}^{k} (-1)^i q^{k(k-1)/2} \left[ \frac{j+k}{i} \right]_{q^2} \left( \frac{j+i}{q^2} \right),$$

(5.1)

so that $\sigma_{j,k}$ is constructed with quantum binomial coefficients which are integer Laurent polynomials in $q$.

Now, there is a question whether one can get any integers as the values of $a$ and $g$ any Laurent polynomials with integer coefficients. In the following subsections we make a first attempt to understand the abundance of values of $a_i^{(\delta)}$ in the space of knots. We look at the simplest possibility – at the families of antiparallel descendants of 2-strand torus knots with the hope that they represent big enough sets for each particular defect. The result turns out to be modest if not discouraging – the values of $a_i^{(\delta)}$ appearing in these families look rather poor, at least for $\delta > 0$. Still, the exercise is interesting and it also provides a more accurate formulation for the defect-preservation conjecture of [41] (see Sect. 6), which makes it fully consistent with the defect-degree correspondence studied in the present paper.

We would like to mention a related result in [54], which states that the scalar equation satisfied by colored Alexander polynomials implies certain integrality property of Labastida-Marino-Ooguri-Vafa conjecture. It is interesting to search possible connections with our studies.

5.1 2-strand torus knots

The $(2m+1)$ torus knot (a closure of 2 strands with $2m+1$ crossings) has maximal possible (for given number of intersections) defect $\delta = m - 1$ and

$$a_i^{(m-1)} = -\text{floor} \left( \frac{m+1-i}{2} \right), \quad i = 0, \ldots, m-1. \quad (5.2)$$

For example, for $9_1$ parameter $m = 4$ and

$$a_0^{(3)} = -\text{floor}(5/2) = -2, \quad a_1^{(3)} = -\text{floor}(4/2) = -2, \quad a_2^{(3)} = -\text{floor}(3/2) = -1, \quad a_3^{(3)} = -\text{floor}(2/2) = -1.$$

Substituting (5.2) into (4.13) and (4.20), we get

$$F_{(m+1)}^{(m+1)}(1, q^{(4.13)}) a_0^{(m-1)},$$

$$+ \sum_{i=1}^{m-1} a_i^{(m-1)} (q^{2i} + q^{-2i}) (5.2) = -[m+1][m],$$

$$G_{(2)}^{(m+1)}(1, q^{(4.13)}) = -\frac{[m+2]!}{[3][4] m!} q^m,$$

$$G_{(3)}^{(m+1)}(1, q^{(4.13)}) = -\frac{[m+3]!}{[4][5] m!} q^m,$$

$$G_{(4)}^{(m+1)}(1, q^{(4.13)}) = -\frac{[m+4]!}{[5][6] m!} q^m,$$

$$G_{(m-1)}^{(m+1)}(1, q^{(4.13)}) = -\frac{[m-1]!}{[m-2][3]! m!} q^m,$$

$$G_{(m-j)}^{(m-1)}(1, q^{(4.13)}) = -\frac{[m-j]!}{[m-j][2]! m!} q^m,$$

$$G_{(m-j)}^{(m-1)}(1, q^{(4.13)}) = -\frac{[m-j]!}{[m-j][2]! m!} q^m,$$

$$G_{(m-j)}^{(m-1)}(1, q^{(4.13)}) = -\frac{[m-j]!}{[m-j][2]! m!} q^m.$$ 

All these DE coefficients are nicely factorized.

5.2 Antiparallel descendants of torus knots

The set (5.2) provides just one point per defect in $m = \delta + 1$-dimensional space of parameters $a_i^{m-1}$. However, this may be not a big problem: we can now use the defect-preserving antiparallel evolution [52] in each intersection, which gives rise to a $2m+1$-dimensional family of antiparallel pretzels with the same defect $\delta = m - 1$. This dimension is more than enough, the question is only if arbitrary integer vectors $a_i^{m-1}$ appear in this family. Regarding $G_{(k)}$, as we have first seen in Sect. 4.2.3, they are not generic integer symmetric Laurent polynomials of a given degree – the degree is typically higher than the number of contributing parameters $a_i^{(\delta)}$. 
According to [25], the HOMFLY polynomial for the odd antiparallel pretzel of genus 2m is equal to

\[ H_{[r]}^{(n_1, \ldots, n_{2m+1})} = \sum_{j=1}^{r+1} \prod_{a=1}^{2m+1} \frac{d_{[r]}^{2m}}{a^m - 1/2} \frac{\left(S_{r}^{2n_a - 1} s_{1,j}\right)}{(A)} . \]  

(5.4)

We restrict it to a symmetric representation \( R = [r] \) and refer to Sect. 3 of [41] for detailed notation in this case. We also do not put bars over \( n_a \), what is usually done to distinguish antiparallel evolution, since this is the only one which is matter for the formulas. To further simplify them, we write \( n_a \) in the labels and in the denotations of pretzel knots instead of 2n a - 1.

For fundamental representation, we obtain:

\[ H_{[1]}^{(n_1, \ldots, n_{2m+1})} = \frac{Aq^{-1}}{[q^2]} \frac{2n_{2m}}{1} \frac{A^{2n_{2m} - 2} q^{-1} - q A^{2n_a} + q - q^{-1}}{(A)} \]

\[ + \frac{Aq}{[q^2]} \prod_{a=1}^{2m+1} -q A^{2n_a - 2} + A^{2n_a} - q^{-1} + q^{-1} \]  

(5.5)

Taking limit \( A \to 1 \), we get that the degree of \( q^{\pm 2} \) in \( F_{[1]} \) is equal to \( m \), and the corresponding coefficients of \( F_{[1]} \) are given by formulas (5.8), (5.10) and (5.13)–(5.22).

Strictly speaking, the defect is not fully preserved by antiparallel evolution. For example, the trefoil 31 is not only the parent-knot in defect-zero family, but also a descendant of all other knots: \( 31 = (1, 1, 1, 1, 1, 1, 0) = (1, 1, 1, 1, 1, 0, 0) = \ldots \) As we will see (Sect. 6), there are also other defect-zero points in higher-defect evolution families. What is important for this paper, these dropdowns are easily captured by the fundamental Alexander – its power also drops.

5.3 Defect \( \delta = m - 1 = 1 \): descendants of 31

In this case there is just a single coefficient \( a_0^{(0)} \) and all \( a_j^{(0)} = 0 \). Thus, the question is just if all integer \( a_0^{(0)} \) are possible. The answer is positive, and an example is provided just by the family of twist knots [55], which form a class of descendants of 31:

\[ a_0^{(0)} \text{ (twist}_a) = F_{[1]}^{\text{twist}} (q, A = 1) = n . \]  

In particular,

\[ F_{[1]}^{31} (q, A = 1) = -1 , \]

\[ F_{[1]}^{41} (q, A = 1) = 1 , \]

\[ F_{[1]}^{51} (q, A = 1) = 2 , \]

\[ F_{[1]}^{51} (q, A = 1) = 3 , \]

\[ \ldots \]  

(5.7)

More examples are given by the larger families of double braids [35] and 3-prezels [41], also obtained by the defect-preserving antiparallel evolution from the trefoil 31 [52] and thus all having the defect zero:

\[ a_0^{(0)} \text{ (pretzel}_{31-1,2m-1,2n-1}) = l m n - (l - 1)(m - 1)(n - 1) . \]  

(5.8)

They provide more delicate information about the abundance/frequency of particular values of \( a_0^{(0)} \) in the population of knots – which is a question at the next level of complexity.

5.4 Defect \( \delta = m - 1 = 1 \): descendants of 51

Using (5.5) for defect \( \delta = m - 1 = 1 \) (descendants of 51) and the notation

\[ F_{[1]}^{(n_1, \ldots, n_5)} (1, q) = a_0^{(n_1, \ldots, n_5)} + a_1^{(n_1, \ldots, n_5)} (q^2 + q^{-2}) , \]

we get

\[ a_0^{(n_1, \ldots, n_5)} = 2 \sum_{1 \leq a<b<c<d \leq} n_a n_b n_c n_d \]

\[ -2 \sum_{1 \leq a < b < c , d \leq} n_a n_b n_c + \sum_{1 \leq a < b \leq} n_a n_b - 1 , \]

\[ a_1^{(n_1, \ldots, n_5)} = \sum_{1 \leq a < b < c \leq} n_a n_b n_c - \sum_{1 \leq a < b \leq} n_a n_b + \sum_{1 \leq a < b \leq} n_a - 1 . \]  

(5.10)

In particular,

\[ a_0^{(n_1, 2, 1, 1, 1)} = n_1 - 2 , \]

\[ a_1^{(n_1, 2, 1, 1, 1)} = -2n_1 , \]  

(5.11)

what provides us with a full-fledged 1-dimensional subspace in the 2-dimensional space of integer \((a_0,a_1)\).

For our analysis of the Alexander polynomial for this family, Eq. (5.10) could be practical. Still, it is rather difficult to extract conclusive statements. Our preliminary impression is that the 2-dimensional set of \((a_0,a_1)\) for descendants of 51 is severely restricted and does not cover the full integer lattice. For example, for \( a_1 = \pm 2 \) there are only odd \( a_0 \) in this family, moreover, they seem rather rare: we have found \( a_0 = 43, 17, 7, 5, -1, -11 \) for \( a_1 = -2 \) and \( a_0 = -1, -3, -9 \) for \( a_1 = 2 \) in the range \(-10 \leq n_1, \ldots, n_5 \leq 10\).

5.5 Defect \( \delta = m - 1 > 1 \): descendants of \((2m + 1)_1\)

Using (5.5), we can calculate
\[ F_{[1]}^{(n_1, \ldots, n_{2m+1})}(1, q) = a_0^{(n_1, \ldots, n_{2m+1})} + \sum_{i=1}^{m-1} a_i^{(n_1, \ldots, n_{2m+1})}(q^{2i} + q^{-2i}) \]  

(5.12)

for any \( m \). It appears that the highest coefficient is always very simple:

\[ a_{m-1}^{(n_1, \ldots, n_{2m+1})} = (-)^{m+1}\left( \prod_{a=1}^{2m+1} n_a - \prod_{i=1}^{2m+1} (n_a - 1) \right), \]

(5.13)

For other coefficients it is convenient to use the following parametrization:

\[ a_i^{(n_1, \ldots, n_{2m+1})} = \sum_{k=1}^{2m} u_k^{(i)} \left( \sum_{1 \leq a_1 < \ldots < a_k \leq 2m+1} n_{a_1} \cdots n_{a_k} \right). \]

(5.14)

Then

(5.13) : \( u_k^{(m-1)} = (-)^{k+1} \)

\( u_k^{(m-2)} = (-)^k (k - 1 - \delta_{k,2m}) \)

\( u_k^{(m-3)} = (-)^{k+1} \left( \frac{k^2 - 3k + 4}{2} - \delta_{k,2m-1} + (2m - 1)\delta_{k,2m} \right) \)

\( u_k^{(m-4)} = (-)^k \left( \frac{k^3 - 6k^2 + 17k - 12}{6} - \delta_{k,2m-2} - (2m - 2)\delta_{k,2m-1} - (2m^2 - 3m + 2)\delta_{k,2m} \right) \)

\( u_k^{(m-5)} = (-)^{k+1} \left( \frac{k^4 - 10k^3 + 47k^2 - 86k + 72}{24} - \delta_{k,2m-3} - (2m - 3)\delta_{k,2m-2} - (2m^2 - 5m + 4)\delta_{k,2m-1} - \right. \)

\left. \frac{(2m-1)(2m^2 - 5m + 6)}{3} \delta_{k,2m} \right) \)

\( \ldots \)

(5.15)

- For example, (5.10) gives for \( m = 2 \) (descendants of \( 5_1 \)):

\[
\begin{array}{ccccccc}
 k & 0 & 1 & 2 & 3 & 4 & \\
 u_k^{(1)} & -1 & 1 & -1 & 1 & 1 & \\
 u_k^{(0)} & 1 & 0 & 1 & -2 & 2 & \\
\end{array}
\]

(5.16)

- while for \( m = 3 \) (descendants of \( 7_1 \)):

\[
\begin{array}{ccccccccc}
 k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \\
 u_k^{(2)} & -1 & 1 & -1 & 1 & -1 & 1 & 1 & \\
 u_k^{(1)} & 1 & 0 & 1 & -2 & 3 & -4 & 4 & \\
 u_k^{(0)} & -2 & 1 & -1 & 2 & -4 & 6 & -6 & \\
\end{array}
\]

(5.17)

The general case can be described as follows:

\[ u_k^{(m-j)} = (-)^{k+j} \left( U_j(k) - \sum_{i=0}^{j-2} V_{j,i}(m)\delta_{k,2m-(j-2-i)} \right), \]

(5.20)

where the coefficients \( U_j \) and \( V_{j,i} \) are polynomials of degree \( j-1 \) in \( k \) and degree \( i \) in \( m \) respectively, which are defined recursively:

\[ U_{j+1}(k+1) - U_{j+1}(k) = U_j(k), \]

\[ U_j(k) = \begin{cases} 0 & \text{for even } j \\ 1 & \text{for odd } j \end{cases}, \]

or \( U_j(k) = (-)^{j+1} \frac{j+1}{2} \).
\[ V_{j+1,i}(m) = V_{j,i} \left( m - \frac{1}{2} \right), \]
\[ U_j(2m-1) - V_{j,j-3}(m) = U_j(2m) - V_{j,j-2}(m), \]
(5.21)

where the forth constraint says that the last two coefficients in front of \( \delta k_{2m-1} \) and \( \delta k_{2m} \) are the same. In particular

\[ V_{j,0} = 1 \]
\[ V_{j,1} = 2m - (j - 2) \]
\[ V_{j,2} = 2m^2 - (2j - 5)m + \frac{j^2 - 5j + 8}{2}, \]
\[ \ldots \]
(5.22)
in accordance with the examples in (5.15).

Expressions (5.14) restrict integer sets \( \{a_0, \ldots, a_8\} \), so that for \( \delta > 0 \) they are not arbitrary, at least for the family of antiparallel descendants of torus knots. In the next subsection we will provide a kind of probable explanation for sets of \( a_j \) to be so restricted.

5.6 Example of 9_1

To emphasize the peculiarity of torus knots, we add an example of

defect 3: knot \( 9_1 \)
\[ a_0^{(3)} = -2, \quad a_1^{(3)} = -2, \quad a_2^{(3)} = -1, \quad a_3^{(3)} = -1. \]
(5.23)
The point is that all the four non-vanishing combinations are nicely factorized expressions (5.3):

\[ F_{[1]}^{9_1}(1, q) = -\frac{[4][5]}{2}, \]
\[ G_{[2]}^{9_1}(1, q) = -q[5][6], \]
\[ G_{[3]}^{9_1}(1, q) = -q^2[2][3][7], \]
\[ G_{[4]}^{9_1}(1, q) = -q^3[2][3][4]. \]
(5.24)

In this case, this factorization is a result of fine tuning of the coefficients \( a_i^{(3)} \), which are very special. It can happen that the antiparallel descendants continue to carry some weakened traces of this peculiarities and this is the reason why they do not describe the general situation with a given defect. In any case, the question, if the coefficients \( a_i^{(3)} \) are all independent or not, remains open.

6 Consequences for the defect-preservation conjecture

In this section, we discuss the defect drop downs and additional peculiarities to the standard defect-diagrams (3.2)-(3.5). The analysis is simpler to conduct by the use of the connection with the degree of the fundamental Alexander polynomial. Namely, the mentioned deviations occur if several highest coefficients \( a_j^{(3)} \) (4.13) vanish. In this case, there are two options:

1. The defect drops down.
2. The defect stays the same, but \( F_{[1]} \) and some of \( G_{[a]} \) turn out to additionally factorize.

Below, we will consider concrete examples and discuss these properties in more detail. It turns out that some mentioned peculiarities and drop downs can be classified, so that such ones are not occasional.

6.1 Descendants of 3_1

In this case, just a \( 1 \)-parametric family of twist knots within the \( 3 \)-dimensional set of trefoil descendants is sufficient to provide any value of \( a_0 \) (see Sect. 5.3). Still, triple pretzels provide additional curious examples. As already mentioned in [52], there are knots with the unit Alexander polynomial, \( a_0 = 0 \), in this family, which are not unknots, for example

\[ F_{[1]}^{(-1,3,4)}(A, q) = [A]A^3(A^2 + 1)(A^4 + A^2 + 1) \]
(6.1)

and vanishes at \( A = 1 \). Actually, this knot is a point of a whole family \((-k, 2k+1, 2k+2)\). Moreover, there are many other evolution points, where the Alexander polynomial turn to 1. Formally, we could associate to these cases a new sort

\[ F_{[1]}^{a_n} = \cdots = 0 \]

6 Recall that for pretzel knots \((2n_1 - 1, \ldots, 2n_{2m+1} - 1)\), we denote \( F \)- and \( G \)-factors superscripts as \((n_1, \ldots, n_{2m+1})\).

7 Actually, for triple pretzels, all the knots have \( F_{[1]}(A, q) \) independent of \( q \) for arbitrary \( A \) – not only their Alexander limit \( F_{[1]}(1, q) \) at \( A = 1 \). However, this is not a general property of the defect-zero knots – already in the next Sect. 5.4 we will mention the members of \( 5_1 \) family, where \( F_{[1]}(A, q) \) are linear in \( q^5 \), as all the members of that family. Still, the defect is \( \delta = 0 \), because this \( q \)-dependence disappears at \( A = 1 \). Just a couple of non-trivial examples is provided by

\[ \delta^{(3,2,-1,\ldots,-8)} = 0 \quad \text{and} \quad F_{[1]}^{(3,2,-1,\ldots,-8)} = \cdots \]
\[ = 2 + [A]\left(U_0(A) + U_1(A)(q^2 + q^{-2})\right), \]
\[ \delta^{(3,2,-1,\ldots,-8)} = 0 \quad \text{and} \quad F_{[1]}^{(3,2,-1,\ldots,-8)} = \cdots \]
\[ = 2 + [A]\left(U_0(A) + V_1(A)(q^2 + q^{-2})\right) \]
(6.2)

with somewhat complicated polynomials \( U \) and \( V \). Then, there are defect-0 knots among the antiparallel descendants of \( 7_1 \), with \( F_{[1]} \) depending on the squares of \( q^{\pm 2} \) at \( A \neq 1 \) and so on.
of a diagram:

\[ \text{sometimes for} \]
\[ \text{defect } \delta^K = 0: \]

In this way, one can actually reveal a more sophisticated structure in the defect (like a Young diagram, not just a single number).

6.2 Descendants of 5_1

With the help of (5.10), one immediately observes numerous zeroes of \( a_1 \) within the 5-fold antiparallel family – does this signal about a deviation from the defect-preservation theorem? Not quite, this rather calls for a more accurate formulation. Antiparallel evolution can, of course, convert \( 5_1 = (1, 1, 1, 1, 1) \) into \( 3_1 = (1, 1, 1, 1, 0) \) and likewise the other knots from the defect-one family into those from defect-zero. Thus, the defect-preservation means that the defect is not increased, but it can drop down at particular values of evolution parameters. This never happens when they all have the same signs, otherwise one should be careful with this kind of the defect drop-downs.

More interesting are the cases like \( (3, 2, -1, -4, -5) \) and \( (3, 2, -1, -3, -8) \) with \( F_{[1]}(1, q) = 2 \) – they have defect zero, but do not belong to the triple-pretzel family. Unlike triple-pretzels, they have \( q \)-dependent \( F_{[1]}(A, q) \), just \( q \)-dependence vanishes at \( A = 1 \).

Surprisingly or not, but we have not found any examples with the anomalous diagram

\[ \text{sometimes for} \]
\[ \text{defect } \delta^K = 1 \]

among the descendants of \( 5_1 \). The DE coefficients \( F_{[1]} \sim [A] \) and vanishes for the Alexander polynomial only for the knots of the type \((-k, 2k+1, 2k+2, 1, 0)\) which have defect 0 and anomalous diagram (6.3), or for \((k, 0, 1, 0, 1)\) which are just unknots. It is currently unclear if other kinds of anomalous diagrams, e.g.

\[ \text{HYPOTETICAL:} \]
\[ \text{sometimes for} \]
\[ \text{defect } \delta^K = 1 \]

also appear within the family generated by \( 5_1 \). Clearly, one needs to consider richer family of defect-one knots, which is beyond the scope of the present paper.

7 Recurrence relations (C-polynomials) for the colored Alexanders

For a given knot, the HOMFLY polynomial for different representations are not independent – infinite set of relations between them is named "quantum \( A \)-polynomial", because in the quasiclassical limit it reproduces the well-known topological invariant. However, from the point of view of the knot moduli space, the knot polynomials are in any way superficial, and it makes more sense to look at the more fundamental variables. As explained in [49], the DE coefficients are better suited for this purpose. Still, they are not free, and relations between them are named \( C \)-polynomials [41,50,51] (the word “quantum” is omitted because classical topology does not seem to tell anything interesting for description of their quasiclassical limit).

Relations survive even in the case \( A = 1 \), i.e. the \( C \)-polynomials exist even for the Alexander polynomials. However, in this case they can be studied exhaustively – and actually this was done in Sect. 4.2.2 of the present paper. As we show there, at least for symmetric representations, the Alexander \( C \)-polynomials – relations between \( G_{[r]}(1, q) \) with different \( r \), depend only on the defect, but not on the other details of knot topology. This is exactly what we want from the future ideal relations between the colored HOMFLY polynomials – to separate the dependence on the representation from that one of the knot, and in the case of the Alexander polynomial this is partly done by separation the defect \( \delta \) from the variables \( a_j^{(\delta)} \) at given \( \delta \). It is yet unclear whether this completes the story, because we did not yet manage to understand if \( a_j^{(\delta)} \) are free integers or there are further relations between them. Actually, we have shown that they are free for \( \delta = 0 \), and this is the case where we can claim that we know a complete set of the Alexander \( C \)-polynomials. The situation with other \( \delta \) remains to be clarified.

Another question is to find the relation between Alexander and HOMFLY \( C \)-polynomials – whether the former ones can be lifted to the latter ones and whether the latter ones can be
restricted to the former ones. It is even unclear which set is bigger. In fact, the theory of C-polynomials is just at its very beginning, they are very hard to find, the known examples look ugly and are very difficult to study. Moreover, in the HOMFLY case they split into different classes, say, for given A, for given q, for certain combinations of those. In this sense, our result for the Alexander case is very encouraging – it proves that this kind of problems can have an exhaustive and elegant solution. Therefore, we hope for a new interest and progress with C-polynomials in foreseeable future.

8 Other one-hook representations

In this section, we consider relations between coefficients of the DE for other one-hook representations \([r, 1^L]\).

8.1 Generalities

As we have already mentioned near (4.1), this scaling relation is actually true not only for symmetric representations, but for arbitrary 1-hook diagrams [34,56]:

\[
\mathcal{A}_{[r, 1^{r-1}]}(q) = R_{[1]}(q^{r+s-1})
\]

\[
\equiv 1 - F_{[1]}(1, q^{r+s-1})[q^{r+s-1}]^2
\]

\[
\equiv 1 - [r + s - 1]^2 [q]^2 \cdot \left( a_{0}^{(s)} + \sum_{j=1}^{\delta} a_{j}^{(s)} \cdot (q^{2 \cdot j (r+s-1)} + q^{-2 \cdot j (r+s-1)}) \right). \tag{8.1}
\]

This allows us to impose restrictions on the DE coefficients in the case of such representations. The main problem here is that these representations are not rectangular, there is a complication with multiplicities, and not so much is known about the DE for them.

In fact, the matching is not quite trivial, because [39]

(i) there are multiplicities, i.e. \( Q \) in the sum

\[
H_R = \sum_{Q < R + X} F_Q Z_R^Q \tag{8.2}
\]

is not in one-to-one correspondence with sub-diagrams of \( R \),

(ii) since now the number of \( Q \) exceeds that one of \( R \) there is an ambiguity in the coefficients \( F_Q \)

(iii) the Z-factors do not vanish automatically at \( A = 1 \) even for defect \( \delta = 0 \).

See [40,57] for further available details.

8.2 Representation \( R = [2, 1] \) for the defect \( \delta = 0 \)

For the simplest representation [21]:

\[
\begin{align*}
Z_{[2, 1]}^{[1]} &= 1, \\
Z_{[2, 1]}^{[2]} &= \frac{[3][A^2] + [3][2][Aq^2][A/q^2]}{[2]^2}, \\
Z_{[2, 1]}^{[3]} &= \frac{[3][2][Aq^3][Aq^2][A]/[A/q^2]}{[2]^2}, \\
Z_{[2, 1]}^{[4]} &= \frac{[3][2][Aq^2][A]/[A/q^2][A/q^3]}{[2]^2}, \\
Z_{[2, 1]}^{[5]} &= \frac{[3][2][Aq^2][A]/[A/q^2][A/q^3]}{[2]^2}, \\
Z_{[2, 1]}^{[6]} &= \frac{[3][2][q]^4[Aq^2]/[A/q^2]}{[2]^2}.
\end{align*}
\tag{8.3}
\]

and only the Z-factors in the second line vanish at \( A = 0 \) (this matters when the defect is zero and all \( F_Q \) are not singular at \( A = 1 \)). Thus, for the Alexander polynomial we have:

\[
\begin{align*}
\mathcal{A}_{[2, 1]}^{(\delta=0)} &= 1 - [3][2]^2 [q]^2 F_{[1]}^{(\delta=0)}(1, q) \\
&\quad - [3][2]^2 [q]^6 F_{[2]}^{(\delta=0)}(1, q) - F_{[2]}^{(\delta=0)}(1, q).
\end{align*}
\tag{8.4}
\]

The very different Z-factors \( Z_{[2, 1]}^{[2]} \) and \( Z_{[2, 1]}^{[6]} \) almost coincide at \( A = 1 \) (just differ in sign), the ambiguity in the F-coefficients reduces to coinciding shifts of \( F_{[2, 1]}^{(\delta=0)}(1, q) \) and \( F_{[2]}^{(\delta=0)}(1, q) \) (and certain change of \( F_{[2, 2]}^{(\delta=0)}(1, q) \), which is beyond our consideration here) – in the result, what is well defined is just the difference \( F_{[2, 1]}^{(\delta=0)}(1, q) - F_{[2]}^{(\delta=0)}(1, q) \).

Since from (4.5) we know that for defect \( \delta = 0 \) the DE coefficient \( F_{[1]}^{(\delta=0)}(1, q) \) is actually independent of \( q \), comparison with (8.1) implies that

\[
F_{[2, 1]}^{(\delta=0)}(1, q) = F_{[2]}^{(\delta=0)}(1, q). \tag{8.5}
\]

Indeed, for defect-zero twist knots \( T_{m} \), the DE coefficients \( F_{[2, 1]} \) and \( F_{[2]} \) are the sums of elements of respectively the fifth and the sixth lines of the KNTZ triangular matrix \( B_{[2, 1]}^{m} \), see (13) of [39],

\[
B_{[2, 1]} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-A^2 & A^2 & 0 & 0 & 0 & 0 \\
A^4 & -2[A^4] & A^4 & 0 & 0 & 0 \\
-A^6 &[3]A^6 & -[3][A^6] & -[3][A^6] & A^6 & 0 \\
-A^6 & [3]A^6 & (A^2 q^6 - A^4) & (A^2 q^6 - A^4) & A^4 & 0 \\
\end{pmatrix}, \tag{8.6}
\]

and it is easy to check that for all integer \( m \) the difference

\[
F_{[2, 1]}^{T_{m}} - F_{[2]}^{T_{m}} \sim \{ A \}, \tag{8.7}
\]
i.e. it vanishes at $A = 1$. The proportionality coefficient is quite a complicated polynomial of $A^{\pm 1}$ and $q^{\pm 1}$, thus, the statement does not look trivial.

We can now generalize it in two directions: to other 1-hook representations (see Sects. 8.4 and 8.5) and to non-vanishing defects (see Sect. 8.3).

8.3 Representation $R = [2, 1]$ for the defect $\delta \geq 1$

For non-vanishing defect, some DE coefficients $F^Q_8$ become singular at $A = 1$, and there are more terms of the DE contributing to the Alexander polynomial. This can be illustrated with the example of two-strand torus knots, which have maximal defect per intersection.

In the case of $R = [2, 1]$, we expect that for $\delta \geq 1$ the coefficients $F_{[2]}$ and $F_{[1, 1]}(q) = F_{[2]}(q^{-1})$ are proportional to $\{A\}^1$, so that there are two more contributions to the generalization of (8.4):

$$A_{[2, 1]}^{(\delta > 0)} = 1 - [3]_2^2 [q]^2 F_{[1]}^{(\delta > 0)}(1, q)$$

$$- [3]_2^2 [2]_2^2 [q]^3 \left( G_{[2]}^{(\delta > 0)}(1, q) - G_{[2]}^{(\delta > 0)}(1, q^{-1}) \right)$$

$$- [3]_2^2 [2]_2^2 [q]^6 \left( F_{[2, 1]}^{(\delta > 0)}(1, q) - F_{X_2}^{(\delta > 0)}(1, q) \right).$$

This time, (8.1) means that

$$F_{[1]}^{(\delta > 0)}(1, q^3) = F_{[1]}^{(\delta > 0)}(1, q) + [2]_2 [q] \left( G_{[2]}^{(\delta > 0)}(1, q)$$

$$- G_{[2]}^{(\delta > 0)}(1, q^{-1}) \right) + [2]_2^2 [q]^4 \left( F_{[2, 1]}^{(\delta > 0)}(1, q) - F_{X_2}^{(\delta > 0)}(1, q) \right).$$

$$= \sum_{j=1}^{\delta} a_j (q) \left( q^j - q^{-j} \right) = [2]_2 [q]$$

$$\left( G_{[2]}^{(\delta > 0)}(1, q) - G_{[2]}^{(\delta > 0)}(1, q^{-1}) \right) + [2]_2^2 [q]^4 \left( F_{[2, 1]}^{(\delta > 0)}(1, q) - F_{X_2}^{(\delta > 0)}(1, q) \right).$$

In particular, for $\delta = 1$ we get from the identity $q^6 + q^{-6} - q^2 - q^{-2} = (q^2 - q^{-2})^2 (2 + [q]^2)$:

$$G_{[2]}^{(\delta = 1)}(1, q) = a_1^{(1)} (q^2 - q^{-2})$$

$$F_{[2, 1]}^{(\delta = 1)}(1, q) - F_{X_2}^{(\delta = 1)}(1, q) = a_1^{(1)}.$$  

Indeed, for the simplest defect 1 torus knot $S^1$, we have $a_1^{(1)} = -1$, and (8.9) is satisfied. The first statement in (8.9) is the same as (4.25).

Likewise, for $\delta = 2$ we get

$$a_1^{(2)} (q^6 + q^{-6} - q^2 - q^{-2})$$

$$+ a_2^{(2)} (q^{12} + q^{-12} - q^4 - q^{-4})$$

$$= [2]_2 [q] \left( G_{[2]}^{(\delta > 0)}(1, q) - G_{[2]}^{(\delta > 0)}(1, q^{-1}) \right)$$

$$+ [2]_2^2 [q]^4 \left( F_{[2, 1]}^{(\delta > 0)}(1, q) - F_{X_2}^{(\delta > 0)}(1, q) \right).$$

This relation per se is not sufficient to find both terms in the second line, but we can substitute (4.27) for $G_{[2]}^{(2)}(1, q) = \left( a_1^{(2)} + a_2^{(2)} \right) [2]_2 (q^3 + q^{-3}) [q^2]$ to get

$$F_{[2, 1]}^{(2)}(1, q) - F_{X_2}^{(2)}(1, q) = a_1^{(2)} + a_2^{(2)}$$

$$\cdot \left( q^6 + 2q^4 + 3q^2 + 2 + 3q^{-2} + 2q^{-4} + q^{-6} \right).$$

Indeed, for defect-2 torus knot $T^2_2$ we have $a_1^{(1)} = a_2^{(1)} = 1$, and

$$G_{[2]}^{(2)}(1, q) = [5]_2 (q^2 - q^{-2})$$

$$F_{[2, 1]}^{(2)}(1, q) - F_{X_2}^{(2)}(1, q) = [3] [5].$$

8.4 Representations $R = [r, 1]$ for the defect $\delta = 0$

For the defect $\delta = 0$ and any $r$, the only non-vanishing at $A = 1$ is the quadruple $Z_{[r, 1]}^{[0]} = 1$, $Z_{[r, 1]}^{[1]}$, $Z_{[r, 1]}^{[2]}$, and $Z_{[r, 1]}^{[2]}$, so that

$$A_{[r, 1]}^{(\delta = 0)} = 1 - [r + 1]_2 [q]^2 F_{[1]}^{(\delta = 0)}(1, q)$$

$$- [2]_2 [r + 1]_2 [q] [r - 1]_2 [q]^6$$

$$\times \left( F_{[2, 1]}^{(\delta = 0)}(1, q) - F_{X_2}^{(\delta = 0)}(1, q) \right).$$

Comparison with (8.1) gives:

$$F_{[2, 1]}^{(\delta = 0)}(1, q) = F_{X_2}^{(\delta = 0)}(1, q),$$

which is just the same (8.5) for all $r$.

8.5 Representations $R = [r, 1^{r-1}]$ for the defect $\delta = 0$

First, let us obtain the DE for the Alexander polynomial for $R = [r, 1^{r-1}]$ and the defect $\delta = 0$. For this representation:

$$\mathcal{H}_{[r, 1-r]}^K(q, A = q^{L+1}) = \mathcal{H}_{[r, 1-r]}^K(q, A = q^3)$$

$$\Rightarrow \mathcal{H}_{[r, 1^{r-1}]}^K(q, A)$$

$$\sim [A q^3] .$$

Looking at lots of examples, we induce that

$$A_{[r, 1^{r-1}]}^K = A_{[r, 1]}^K$$

$$+ G_{[1]}(1, q) [q^{-3}] [q^{2r+s-2}]$$

$$= 1 - F_{[1]}(1, q) [q^{-3}] [q^{2r+s-2}]$$

$$+ G_{[1]}(1, q) [q^{-3}].$$

(8.17)

For example, we have:

- for $3_1$: $G_{[1]}(1, q) = -1$
- for $4_1$: $G_{[1]}(1, q) = 1$
- for $5_2$: $G_{[1]}(1, q) = -2$
Note that result (8.17) are in a straightforward correspondence with the results of Sects. 8.2 and 8.4.
Comparing (8.17) and (8.1), we obtain
\[ G_{1}\!(1, q) = F_{t}\!(1, q) = a_{0}^{(\delta)}. \] (8.18)

9 Conclusion

The purpose of this paper is to study the implications of (1.2) and (4.1). In other words, we studied the combined implications of (1.2) and (4.1).

The main result is the proof of the conjecture [48] that the defect \( \delta \), which was originally defined as characteristic of factorization depth of the coefficients of differential expansion for the HOMFLY polynomials in symmetric representations, can be alternatively related to the degree \( \delta + 1 \) of the fundamental Alexander polynomial (see Sect. 4.2.1). We have provided explicit formula for all other DE coefficients at \( A = 1 \) in terms of the fundamental Alexander polynomial and explain that they are not free (see Sect. 4.2.2).

We address the question, if the fundamental Alexander coefficients are free, and as an example demonstrate that for \( \delta = 0 \) they are (see Sect. 5.3). We use for this purpose the family of antiparallel descendants of the trefoil. Our results in this case can be interpreted as the full separation of representation and knot variables, i.e. a complete set of \( C \)-polynomials (see Sect. 7).

Looking at different descendants of the trefoil we also observe the defect peculiarities, which make the statement of the defect-preserving theorem more precise (see Sect. 6).

Similar calculations for other defects (not only \( \delta = 0 \)), other representations (not only symmetric) and other knot polynomials (not only Alexanders) remain to be done. We describe just the simplest steps in these directions (see Sects. 5.4, 5.5, 5.6 and 8), which prove that they are indeed interesting and doable.

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