On the Recovery of Galaxy Properties from SED Fitting Solutions

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ABSTRACT. We explore the ability of four different inverse population synthesis codes to recover the physical properties of galaxies from their spectra by SED fitting. Three codes, DynBaS, TGASPEX, and GASPEX, have been implemented by the authors and are described in detail in the paper. STARLIGHT, the fourth code, is publicly available. DynBaS selects dynamically a different spectral basis to expand the spectrum of each target galaxy, and TGASPEX uses an unconstrained age basis, whereas GASPEX and STARLIGHT use for all fits a fixed spectral basis selected a priori by the code developers. Variable and unconstrained basis reflect the peculiarities of the fitted spectrum and allow for simple and robust solutions to the problem of extracting galaxy parameters from spectral fits. We assemble a Synthetic Spectral Atlas of Galaxies (SSAG),3 comprising 100,000 galaxy spectra corresponding to an equal number of star formation histories based on the recipe of Chen et al. We select a subset of 120 galaxies from SSAG with a color distribution similar to that of local galaxies in the seventh data release (DR7) of the Sloan Digital Sky Survey (SDSS), and produce 30 random noise realizations for each of these spectra. For each spectrum, we recover the mass, mean age, metallicity, internal dust extinction, and velocity dispersion characterizing the dominant stellar population in the problem galaxy. All methods produce almost-perfect fits to the target spectrum, but the recovered physical parameters can differ significantly. Our tests provide a quantitative measure of the accuracy and precision with which these parameters are recovered by each method. From a statistical point of view, all methods yield similar precisions, whereas DynBaS produces solutions with minimal systematic biases in the distributions of residuals for all of these parameters. We caution the reader that the results obtained in our consistency tests represent lower limits to the uncertainties in parameter determination. Our tests compare theoretical galaxy spectra built from the same synthesis models used in the fits. Using different synthesis models and the lack of particular stellar types in the synthesis models but present in real galaxies will increase these errors considerably. Additional sources of error expected to be present in real galaxy spectra are not easy to emulate, and again will result in larger errors.

1. INTRODUCTION AND MOTIVATION

The stellar content of unresolved galaxies can be studied only through integrated photometry and/or spectroscopy. The advent of large surveys centred at several rest-frame spectral
ranges has encouraged the development of tools to characterize the stellar populations dominating the light in these galaxies. The literature of the last 10 years is rich in papers on inverse population synthesis, which use spectral energy distribution (SED)-fitting and parameter extraction algorithms as a natural application of population synthesis to derive the star formation history (SFH) of distant and nearby galaxies. See papers by Heavens et al. (2000), Tojeiro et al. (2007), Ocvirk et al. (2006), Cid Fernandes et al. (2005), and Koleva et al. (2009) for details on different methods.

SED fitting refers to the use of different algorithms to select from a set of spectral evolution models, e.g., Bruzual & Charlot (2003, hereafter BC03), a single age spectrum, or a linear combination of spectra of different age, to construct the synthetic spectrum $F^\text{syn}_\lambda$ that minimizes the goodness-of-fit, e.g.,

$$
\chi^2 = \sum_i \frac{(F^\text{obs}_\lambda - F^\text{syn}_\lambda)^2}{\sigma^2_\lambda}.
$$

Here, $\sigma^2_\lambda$ is the standard deviation of the Gaussian distribution characterizing the error in the flux of the target spectrum $F^\text{obs}_\lambda$ at wavelength $\lambda$. Since the quality of the spectral fit, measured by $\chi^2$ in equation (1), depends on the adequacy of the spectral evolution model to describe the problem spectrum, most SED-fitting algorithms will produce equally good fits to the target spectrum. However, flexible SED-fitting codes that use a relatively large and variable number of individual spectra to build $F^\text{syn}_\lambda$ will, in general, produce better fits to $F^\text{obs}_\lambda$ than the more rigid codes.

The ultimate goal of inverse modeling is not the spectral fit itself, but the interpretation of the components of the synthetic spectrum which minimize $\chi^2$ in terms of the SFH of the problem galaxy, i.e., the run of star formation with time over the galaxy lifetime. The derived spectral components can then be used together with the evolutionary synthesis models to recover basic physical parameters of the stellar population in this galaxy, e.g., stellar mass, mass-weighted age, flux-weighted age, average metallicity, mass-to-light ratio, and dust content. Parameter extraction is model-dependent, and different algorithms can recover different values for these parameters from equally good spectral fits.

As motivation for our study, we show in Figure 1 the results of fitting the SED of two SDSS DR7 galaxies (Abazajian et al. 2009) using two different algorithms: DynBaS3D and TGASPEX (these are introduced in § 2.2). The fitted SEDs are indistinguishable in the scale of these plots (main panel). This is illustrated also by the residual plots (two bottom panels) and evidenced by the minute differences in the values of reduced $\chi^2$. However, the stellar mass derived for the early-type galaxy (left panels) using TGASPEX is 30% larger than the mass obtained with DynBaS3D. For the late-type galaxy in the right-hand panels, the effect is even more striking. The difference in reduced $\chi^2$ is only 0.002, but the derived mass differs by 110%. In the top panels of Figure 1, we show for both solutions the fraction of the galaxy mass in each of the selected population components versus log age. This function can be interpreted as the SFH, as will be discussed in a forthcoming paper. For the early-type galaxy, both algorithms give approximately the same mass distribution, resulting in a small difference in the derived total mass. For the bluer galaxy, DynBaS3D gives a solution with a younger component than TGASPEX, and the inferred galaxy mass differs by a factor of 2. Several degeneracies present in spectral evolution models contribute to this ambiguity. The discrepant results do not depend in an obvious manner on the signal-to-noise ratio (SNR) of the target spectrum, as we discuss in § 3.

A quantitative measurement is then needed of the uncertainty in the derived galaxy physical parameters that takes into account these model limitations, whose origin has been discussed in the papers cited above, as well as in the review paper by Walcher et al. (2011).

In summary, star formation and evolution are complex processes, which overlap in galaxies over time scales ranging from a few million to billions of years. These processes are highly simplified in stellar population synthesis models. Additionally, by the nature of stellar evolution, at old ages the evolution is slow in almost all observable spectral features. As a consequence, the footprints of evolution are blurred in integrated observational properties, especially when multiple generations of stars are involved. Despite these limitations, population synthesis modeling and its complementary SED-fitting and parameter extraction algorithms, have been improved remarkably, in parallel with the quality of the large amount of observational material available today.

Inverse population synthesis algorithms can be classified as either parametric or nonparametric. In the parametric case, the galaxy SFH is assumed to follow an analytic expression that depends on one or more parameters (see, e.g., da Cunha et al. 2008; Acquaviva et al. 2011; Lee et al. 2010; Maraston et al. 2010; Guaita et al. 2011). It is common practice to assume that galaxies begin to form stars at time $t_\text{form}$ according to an exponentially declining or increasing star formation rate (SFR) with e-folding time $\tau$ (which we will refer to as $\tau$-models). The goal in this case is to determine from the spectral fit the best values of the parameters $t_\text{form}$ and $\tau$. In the nonparametric case, no functional form is assumed for the SFR. The galaxy SFH is then recovered from the stellar mass formed at each age bin in a previously defined grid of time steps, e.g., MOPED, VESPA, STECMAP, STARLIGHT, ULYSS (Heavens et al. 2000; Tojeiro et al. 2007; Ocvirk et al. 2006; Cid Fernandes et al. 2005; Koleva et al. 2009).

In general, the SFH of a galaxy cannot be inferred from the same time resolution available in population synthesis models. Testing their algorithm on spectra similar to the SDSS galaxy spectra, Cid Fernandes et al. (2005) concluded that a robust description of the SFH is only achievable in terms of three populations: young, intermediate, and old. A discussion on this
issue can also be found in Cid Fernandes et al. (2014). Similarly, Tojeiro et al. (2007), using their code VESPA, report that it is possible to recover a maximum of between two and five independent stellar population components from a preselected set of 16 age bins. Both these algorithms use a fixed-age basis of model galaxy spectra to decompose the target spectrum in equation (1).

Our goal in this article is to characterize the uncertainties in the physical parameters describing the SFH of a galaxy derived from nonparametric SED-fitting algorithms and to provide the user with the typical error and bias associated to these parameters. The paper is structured as follows. In § 2, we present the details of GASPEX, TGASPEX, and DynBaS, our nonparametric SED-fitting algorithms. In § 3, we contrast the performance of these methods on target spectra from a library of theoretical spectra (SSAG) based on the SFH recipes of Chen et al. (2012) to simulate SDSS spectra of galaxies. The details of this physically motivated spectral library are described in Appendix B. We compare our results with those obtained with the well-known public code STARLIGHT (Cid Fernandes et al. 2005, 2014). In § 4, we discuss the advantage of using flexible over fixed-age basis algorithms and present the conclusions of our study.

2. THE STELLAR POPULATION CONTENT OF GALAXIES

Following BC03, the SED at time $t$ of a composite stellar population in which stars are formed with a constant initial mass function (IMF), can be synthesized from

$$F_{\lambda}(t) = \int_{0}^{t} \Psi(t - t') f_{\lambda}(t', \zeta(t - t')) dt',$$

(2)

where $\Psi(t)$ and $\zeta(t)$ are the SFR and the metal-enrichment law, respectively, and $f_{\lambda}(t, Z)$ is the SED at age $t$ of a simple stellar population (SSP) of metallicity $Z$ for the indicated IMF. Several sets of spectra for SSPs are available in the literature, e.g., BC03 models. These models provide a set of evolving spectra.
computed at a discrete number of time steps. Thus, it is more appropriate to write equation (2) as a linear combination of spectra:

\[ F_\lambda(t) = \sum_{i,j} a_{i,j} f_\lambda(t_i, Z_j), \tag{3} \]

where the double sum is formally performed over the \( N_t \) time steps \( t_i \) available in the SSP model for metallicity \( Z_j \) of a set of \( K_Z \) metallicities, i.e., \( j = 1, \ldots, K_Z \). We can think of equation (3) as a sum in vector space. The coefficient \( a_{i,j} \) represents the weight assigned to the basis vector \( f_\lambda(t_i, Z_j) \) in this space. The SFR at time \( (t - t_i) \) can then be derived directly from \( \Psi(t - t_i) = a_i/\Delta t_i \), where \( a_i = \sum_j a_{i,j} \), and \( \Delta t_i = (t_{i+1} - t_{i-1})/2 \).

2.1. SED Fitting

Spectral fitting techniques seek to determine the SFH of a galaxy, i.e., \( \Psi(t) \) and \( \zeta(t) \), by minimizing the difference between the target spectrum \( F^{\text{obs}}_\lambda \) and a model spectrum \( F^{\text{mod}}_\lambda \). For these fits we can either use models that follow a parametric SFR, e.g., \( \tau \)-models with \( \Psi(t) \propto \exp^{-t/\tau} \), or make no a priori assumption about the distribution of \( a_{i,j} \) in equation (3). In the latter case, the SFH can be estimated by computing the non-negative values of the coefficients \( a_{i,j} \) that minimize the merit function

\[ \chi^2 = \sum_l \left[ \frac{F^{\text{obs}}_\lambda - \sum_{i,j} a_{i,j} f_\lambda(t_i, Z_j)}{\sigma_l^2} \right]^2, \tag{4} \]

used to measure the goodness-of-fit. The sum in equation (4) extends over all the wavelength points \( \lambda_l \) in common in both spectra, and \( \sigma_l \) is the standard deviation characterizing the error in the flux \( F^{\text{obs}}_\lambda \), assumed to follow a Gaussian distribution.

In this article, we follow the nonparametric approach, in which the SFH is not expressed as a function of one or more parameters. For the \( f_\lambda(t_i, Z_j) \) in equation (4), we use the BC03 SSP models computed for the Padova 1994 stellar evolutionary tracks, the STELIB stellar atlas, and the Chabrier (2003) IMF (see BC03 for details and references). To model the galaxy spectrum, we can use in equations (3) and (4) the full age and metallicity resolution available in the BC03 models, i.e., \( N_t = 221 \) time-steps and \( K_Z = 7 \) metallicities. However, the problem of retrieving physical information from \( N_t \times K_Z = 1547 \) coefficients presents difficulties. Some of these difficulties are related to well-known degeneracies characteristic of stellar populations (see, e.g., Conroy 2013), with the consequence that more than one SFH will produce the same SED, but the implied galaxy properties will be different for each of these SFHs (cf., Fig. 1). For this reason, solving the problem of SED fitting goes beyond the solution of a mathematical equation. The chosen solution must warrant that the physical parameters characterizing the problem galaxy (e.g., galaxy mass, mean stellar age, dust content, and mean metallicity) are well recovered, or, at least, the fitting algorithm must provide us with a reasonable estimate of the error on each of these quantities.

In the following, we discuss three different nonparametric approaches to solving the SED-fitting problem: GASPEX, a fixed-basis method; TGASPEX, an unconstrained basis method; and DynBaS, a dynamical basis selection method.

2.2. Three Nonparametric Methods

2.2.1. GASPEX: A Fixed Basis Method

The Galaxy Spectrum Parameter Extraction (GASPEX) code is suitable for deriving the SFH of a galaxy using a rather simple SED-fitting procedure. GASPEX minimizes \( \chi^2 \) using a fixed set of \( n \) model spectra (the same for all problem galaxies) as a basis in equation (4). If \( m \) is the number of wavelength points in the problem spectrum, we can think of \( F^{\text{obs}}_\lambda/\sigma_\lambda \) in equation (4) as the \( m \) components of an \( 1 \times m \) vector \( O \). The subset of \( n \) model spectra, \( f_\lambda(t_i, Z_j)/\sigma_\lambda \), can then be arranged as an \( m \times n \) matrix \( A \), where the number of time steps \( t_i \) and metallicities \( Z_j \) used in the fit adds to \( n \). The coefficients \( a_{i,j} \) in equation (4) will then define the \( n \times 1 \) solution vector \( A \). Minimizing \( \chi^2 \) in equation (4) is equivalent to solving for \( A \) the following matrix equation

\[ O = M \times A. \tag{5} \]

From physical grounds, all components of the solution vector \( A \) must be nonnegative, and the sum of these components is proportional to the problem galaxy mass. The nonnegative least-squares algorithm (NNLS) developed by Lawson & Hanson (1974) is especially suited for this purpose. GASPEX uses the NNLS routine to solve equation (5) for \( A \) by least-squares techniques, minimizing the squared residuals of the \( O = M \times A \) spectral fit, subject to the conditions \( a_i \geq 0 \) for all \( i \), and returns the \( n \) components of the vector \( A \) that minimize \( \chi^2 \) defined in equation (4).

An important issue in SED fitting is the degeneracy introduced in the results by using a large number of template SSP spectra as a basis in equation (4). Therefore, one must carefully choose a subset of templates that minimize this degeneracy. In Appendix A, we explain the procedure used to select an \( n = 12 \) element optimal basis for GASPEX.

2.2.2. TGASPEX: An Unconstrained Basis Method

TGASPEX is the extension of the GASPEX algorithm using as a basis the full set of 221 spectra per metallicity in the BC03 models. In this case, equation (5) is solved using the NNLS routine with \( n \leq N_t \times K_Z = 221 \times 7 \). The value of \( n \) can be

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4 Preliminary results obtained with this code were presented briefly in Mateu et al. (2001) and Magris et al. (2007), but without a full introduction to the method.
reduced if we have some knowledge about the target spectrum.
For instance, for single metallicity fits \( n = N_t = 221 \), in con-
trast with GASPEX which uses only \( n = 12 \).

2.2.3. DynBaS: A Dynamical Basis Selection Method

Mateu (2009) briefly introduced the basic idea of the non-
parametric dynamical basis-selection algorithm (DynBaS) for
SED-fitting and parameter extraction. In this method, \( F^\text{obs}_\lambda \) in
equation (1) is fitted using \( N \) model spectra selected dynami-
cally from the full set available in the stellar population synthe-
sis models (e.g., \( N \) out of 221 in the case of the BC03 models
for a single metallicity). The specific \( N \) spectra will vary with
the problem galaxy. The galaxy physical parameters are then
recovered from the weight assigned to each of the \( N \) com-
nents in the spectral fit. This method guarantees an optimal so-
olution for a given dimension \( N \) of the spectral basis. This
algorithm is quite flexible and easy to implement, but its use-
fulness remains to be established through internal consistency
tests using a large sample of spectra of galaxies of different
types. We denote this code as DynBaSND to indicate the num-
ber of \( N \) spectra selected out of 221 in the case of the BC03 models
for a single metallicity.

2.2.4. SSAG: A Statistical Spectral Attenuation

SSAG (2015) provides the same basic idea as DynBaSND,
but it is focused on parameter extraction of stellar popula-
tions using a large set of templates. SSAG uses a statistical
approach to select the best set of templates for each galaxy
based on the observed spectra. This method is quite flexible
and easy to implement, and its usefulness remains to be estab-
lished through internal consistency tests using a large sample of
spectra of galaxies of different types. We denote this code as
SSAG to indicate the number of \( N \) spectra selected out of 221
in the case of the BC03 models for a single metallicity.

2.3. Treatment of Dust Extinction and Stellar Velocity Dispersion

We incorporate the dust extinction and line-of-sight velocity
dispersion as model parameters, which we treat in a separate
manner given that the model dependency on them is nonlinear.

2.3.1. Extinction by Dust

As indicated in Appendix B, in the SSAG we use the Charlot
& Fall (2000) model to simulate the absorption of starlight by
dust in galaxies. This model treats the effects of dust in star
forming galaxies as a two-phase process. In the first phase,
young stars are embedded in their native clouds for a short time
(10^7 yr), and the effects of dust are described by an optical
depth parameter \( \tau^\text{BC}_V \). The second phase corresponds to the
absorption in the ambient interstellar medium, and is characterized
by an optical depth \( \tau^\text{ISM}_V \).

It is too ambitious to attempt to recover separately the values
of \( \tau^\text{BC}_V \) and \( \tau^\text{ISM}_V \) using SED-fitting algorithms, especially for
massive data processing. Tojeiro et al. (2007) tested VESPA
on model spectra which were simulated using the two-phase
dust model. They report a good recovery of \( \tau^\text{ISM}_V \) and a poorer
one of \( \tau^\text{BC}_V \). They also analyzed their sample using a single dust
component model, and concluded that the one-component
model gives acceptable results that are only slightly improved
by the two-parameter fitting. Chen et al. (2012), using principal
component analysis, also estimate dust absorption using just
one parameter, although their model spectral library was
generated using the two-component Charlot & Fall (2000)
prescription.

In this article, we use a one-parameter, single-screen model
to describe the effects of dust attenuation in our sample of prob-
lem galaxies. We adopt the extinction curve from Charlot & Fall
(2000): \( \tau_\lambda = \tau_V (\lambda/5500 \text{ Å})^{-0.7} \), and look for the minimum
value of \( \chi^2 \) in equation (4), when the model SED is replaced
by the respective reddened SED: \( f^\text{redd}_\lambda = f_\lambda \exp(-\tau_\lambda) \). In prac-
tice, for each galaxy in the sample and for each algorithm de-
scribed above, we repeat the fits using a fixed set of values of
\( \tau_V \), including \( \tau_V = 0 \). The value of \( \tau_V \) for which \( \chi^2 \) attains an
absolute minimum for the specific algorithm is then adopted as
its best estimate of \( \tau_V \), and is reported in terms of the \( V \)-band
extinction, \( A_V = 1.086 \tau_V \).

2.3.2. Line-of-Sight Velocity Dispersion

The same brute-force procedure used to find \( A_V \) is followed
to determine the value of \( \sigma_v \), the velocity dispersion charac-
terizing the line-of-sight stellar motions in the problem galaxy. For
each galaxy and for each algorithm, we repeat the fits convolv-
ing the model SEDs with a Gaussian kernel, increasing \( \sigma_v \) from
0 to 300 km s\(^{-1}\) in steps of 10 km s\(^{-1}\). Our estimate of \( \sigma_v \) is
taken as the value that produces the minimum \( \chi^2 \). Given the
quality of the spectral fits achieved by the different codes, the
adopted value of \( \sigma_v \) is practically independent of the algorithm.
A quantitative assessment of the overall performance of the different fitting methods on our sample is summarized in Figure 3. We use the median of the residual distribution as a measure of the bias, and its width, defined here as \((P84−P16)/2\), as an indicator of the precision in the determination of the corresponding parameter, both measured globally over the whole galaxy sample. Below we discuss in detail the implications of Figures 2 and 3 and Table 1 for the case \(\text{SNR}_r = 20\) and 30.

### 3.3.1. Mass

For spectra with \(\text{SNR}_r = 30\), all methods recover the stellar mass with essentially the same precision, \(\lesssim 17\%\), but differ in the bias (see Table 1). \text{DynBaS3D} tends to underestimate \(M\) by \(\sim 6\%\), and \text{GASPEX} and \text{TGASPEX}, on average, tend to overestimate \(M\) by a similar amount (bias \(\sim 7\%\)). \text{DynBaS2D} shows the most biased estimation of \(M\) (bias \(\sim 13\%\)) and an extended asymmetrical distribution of residuals. For lower-quality spectra with \(\text{SNR}_r = 20\), we find larger differences in the bias reported for each method. In this case, \text{DynBaS3D} provides the less biased estimate of the stellar mass \(M\) (negative bias of 5\%), while \text{GASPEX} and \text{TGASPEX} recover \(M\) with a bias of \(\sim 10\%\). Results in Table 1 show that the residual in the determination of \(M\) for \text{GASPEX} and \text{TGASPEX} are less biased for higher SNR spectra, while \text{DynBaS3D} shows the inverse trend. The \text{STARLIGHT} residuals are very similar to the \text{GASPEX} and \text{TGASPEX} residuals, with a distribution slightly more biased but also slightly more precise.

### 3.3.2. Mass-Weighted Log-Age

Even though the precision for all methods is comparable \((0.15−0.17\ \text{dex} \text{ and } 0.12−0.15\ \text{dex}, \text{ for } \text{SNR}_r = 20 \text{ and 30, respectively})\), the bias in the residuals of \(\text{MWlogAge}\) shows the same trend as the stellar mass \(M\) residuals: \text{DynBaS2D} and \text{DynBaS3D} tend to underestimate \(\text{MWlogAge}\) (bias \(\sim −0.11\) and \(\lesssim −0.05\) dex, respectively) with a weak dependence on the quality of the spectra, while \text{GASPEX} and \text{TGASPEX} overestimate \(\text{MWlogAge}\) with bias \(\lesssim 0.11\) and \(\lesssim 0.09\), for \(\text{SNR}_r = 20\) and 30, respectively. As for the mass \(M\), the \(\text{MWlogAge}\) determination improves with \(\text{SNR}\) for \text{GASPEX} and \text{TGASPEX}, while it worsens for \text{DynBaS2D} and \text{DynBaS3D}. The results obtained with \text{STARLIGHT} are practically identical to those of \text{GASPEX} and \text{TGASPEX}.

### 3.3.3. Flux-Weighted Log-Age

\text{DynBaS3D} and \text{TGASPEX} produce almost unbiased estimates of \(\text{FWlogAge}\), with a bias \(\lesssim −0.01\) and \(\lesssim 0.03\) dex, respectively. All methods, \text{DynBaS2D}, \text{DynBaS3D}, \text{GASPEX}, and \text{TGASPEX} provide equally precise estimates of \(\text{FWlogAge}\) (precision \(\lesssim 0.1\) dex), practically independent of the \(\text{SNR}\). \text{STARLIGHT} gives the same statistical estimation of \(\text{FWlogAge}\) as \text{TGASPEX}.

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\^\text{6}The \(r\)-band flux is used as weight to compute the flux-weighted log-age.

\^\text{7}Residuals are defined as \(\Delta x = (x−x_{\text{true}})\), where \(x\) is the recovered value of a galaxy property, and \(x_{\text{true}}\) its true value. The true values are determined from the galaxy SFH and the population synthesis models for all the properties except for \(A_V\). For the latter we use \(\Delta A_V = A^\text{red} − A^\text{blue}\), where \(A^\text{red}\) and \(A^\text{blue}\) represent the \(V\)-band magnitude of the reddened and unreddened model galaxy computed for the same SFH, respectively. For the total mass and velocity dispersion, we use the fractional residuals \(\Delta M/M_{\text{true}}\) and \(\Delta \sigma/\sigma_{\text{true}}\), respectively.

\^\text{8}The 16th and 84th percentile are the values below which 16\% and 84\% of the residuals may be found. For a normal distribution, they correspond approximately to \(−1\sigma\) and \(+1\sigma\), respectively.
3.1.4. Mass-Weighted Log-Metallicity

The typical precision achieved in the recovery of $\text{MWlogZ}$ is quite good, and the distributions of residuals are almost independent of the SNR of the target spectrum. The DynBaS3D, GASPEX, TGASPEX, and STARLIGHT residuals show very little bias, $0.01$, $-0.04$, $-0.03$, and $-0.04$ dex, respectively, with a precise estimate of $\text{MWlogZ} \leq 0.09$ dex. For DynBaS2D the $\text{MWlogZ}$ residual distribution is bimodal, indicating a limitation of this method to determine a galaxy metallicity from its spectrum.

3.1.5. Extinction

We have used DynBaS2D, GASPEX, and TGASPEX to estimate $A_v$ by brute force as indicated in § 2.3.1. DynBaS2D, GASPEX, TGASPEX, and STARLIGHT provide practically unbiased estimates of $A_v$, bias between $-0.01$ and $-0.04$, with
precision \leq 0.08. We did not perform the brute-force determination of \( A_V \) for DynBaS3D, since for a reduced subsample of galaxies we recovered the same value of \( A_V \) as for DynBaS2D. The value of \( A_V \) does not change significantly with SNR.

### 3.1.6. Velocity Dispersion

Figure 4 shows as histograms the distributions of fractional residuals in the derivation of \( \sigma_v \) for GASPEX and \textsc{STARLIGHT}. The velocity dispersion recovered by GASPEX is almost unbiased (bias 0.02), with a precision of 18 and 13\% for SNR\(_r\) = 20 and 30, respectively. \textsc{STARLIGHT} produces a narrower distribution of fractional residuals with precision = 0.07 and 0.05 for SNR\(_r\) = 20 and 30, respectively, but underestimates \( \sigma_v \) slightly (bias = -0.03).

### 3.1.7. Residuals as a Function of Galaxy Color

The \( M, MW\text{log}Age, FW\text{log}Age, MW\text{log}Z, \) and \( A_V \) residuals as a function of galaxy \((u - r)\) color are shown in Figure 5 for the 3600 spectra in our sample with SNR\(_r\) = 20. Figure 6 provides a graphical summary of the residual distributions of Figure 5, showing the bias and precision obtained with all methods for galaxies in three \((u - r)\) color bins (bluer galaxies on top panels). This figure clearly shows that for all methods the bias decreases (the distribution of points shrinks toward zero in the horizontal direction), and the precision improves (points move toward zero in the vertical direction), when going from the bluer to the redder bins. Overall, DynBaS3D provides the least biased estimate of all these properties for galaxies of any \((u - r)\) color. The values of \( M \) and \( MW\text{log}Age \) recovered by GASPEX, TGASPEX, and \textsc{STARLIGHT} show a systematic bias, being larger than the respective true value for galaxies of any color. Precision does not vary significantly between the different methods as a function of galaxy color. The same trend is observed for mock spectra with SNR\(_r\) = 30.

### 3.1.8. Residuals and Chemical Evolution in Galaxies

To assess the effect of the metallicity evolution in the recovered physical parameters, we show in Figure 7 the residuals in the \( M \) and \( MW\text{log}Age \) determination using DynBaS3D and TGASPEX as representative of a dynamical basis-selection and an unconstrained basis method, for four different synthetic galaxies with SNR\(_r\) = 20. The metallicity in these galaxies

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**Table 1**

| Parameter | Method  | Median | P16 | P84 | Precision |
|-----------|---------|--------|-----|-----|-----------|
| \( \Delta M/M_{\text{true}} \) | GASPEX | 0.108  | -0.061 | 0.373 | 0.217 | 0.077  | -0.051 | 0.283 | 0.167 |
|           | DynBaS2D | -0.127 | -0.315 | 0.029 | 0.172 | -0.128 | -0.315 | 0.017 | 0.166 |
|           | DynBaS3D | -0.045 | -0.209 | 0.152 | 0.181 | -0.060 | -0.202 | 0.090 | 0.146 |
|           | TGASPEX | 0.101  | -0.065 | 0.382 | 0.224 | 0.067  | -0.056 | 0.275 | 0.165 |
|           | \textsc{STARLIGHT} | 0.119 | -0.038 | 0.357 | 0.197 | 0.104 | -0.021 | 0.293 | 0.157 |

| \( \Delta MW\text{log}Age \) | GASPEX | 0.112 | -0.020 | 0.301 | 0.161 | 0.090 | -0.010 | 0.245 | 0.128 |
|           | DynBaS2D | -0.105 | -0.282 | 0.025 | 0.153 | -0.107 | -0.282 | 0.007 | 0.145 |
|           | DynBaS3D | -0.035 | -0.184 | 0.110 | 0.147 | -0.050 | -0.182 | 0.063 | 0.123 |
|           | TGASPEX | 0.095  | -0.041 | 0.303 | 0.172 | 0.072  | -0.033 | 0.238 | 0.136 |
|           | \textsc{STARLIGHT} | 0.124 | -0.012 | 0.304 | 0.158 | 0.113 | 0.001 | 0.269 | 0.134 |

| \( \Delta FW\text{log}Age \) | GASPEX | 0.050 | -0.021 | 0.121 | 0.071 | 0.041 | -0.013 | 0.096 | 0.055 |
|           | DynBaS2D | -0.024 | -0.127 | 0.071 | 0.099 | -0.026 | -0.121 | 0.061 | 0.091 |
|           | DynBaS3D | -0.006 | -0.089 | 0.071 | 0.080 | -0.012 | -0.083 | 0.055 | 0.069 |
|           | TGASPEX | 0.028  | -0.043 | 0.104 | 0.074 | 0.020  | -0.036 | 0.081 | 0.058 |
|           | \textsc{STARLIGHT} | 0.030 | -0.043 | 0.105 | 0.074 | 0.025 | -0.036 | 0.084 | 0.060 |

| \( \Delta MW\text{log}(Z) \) | GASPEX | -0.035 | -0.134 | 0.044 | 0.089 | -0.027 | -0.115 | 0.033 | 0.074 |
|           | DynBaS2D | 0.021 | -0.120 | 0.115 | 0.118 | 0.022 | -0.105 | 0.113 | 0.109 |
|           | DynBaS3D | 0.012 | -0.094 | 0.092 | 0.093 | 0.013 | -0.077 | 0.080 | 0.078 |
|           | TGASPEX | -0.025 | -0.113 | 0.062 | 0.088 | -0.020 | -0.093 | 0.047 | 0.070 |
|           | \textsc{STARLIGHT} | -0.036 | -0.121 | 0.040 | 0.080 | -0.033 | -0.102 | 0.027 | 0.065 |

| \( \Delta A_V \) | GASPEX | -0.038 | -0.121 | 0.026 | 0.073 | -0.035 | -0.111 | 0.012 | 0.061 |
|           | DynBaS2D | -0.008 | -0.088 | 0.065 | 0.077 | -0.011 | -0.085 | 0.055 | 0.070 |
|           | TGASPEX | -0.023 | -0.098 | 0.042 | 0.070 | -0.021 | -0.086 | 0.030 | 0.058 |
|           | \textsc{STARLIGHT} | -0.028 | -0.097 | 0.031 | 0.064 | -0.026 | -0.082 | 0.025 | 0.054 |

| \( \Delta \sigma_v/\sigma_{v,\text{true}} \) | GASPEX | 0.020 | -0.148 | 0.211 | 0.180 | 0.018 | -0.099 | 0.161 | 0.130 |
|           | \textsc{STARLIGHT} | -0.033 | -0.099 | 0.036 | 0.067 | -0.027 | -0.074 | 0.020 | 0.047 |

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*a* We use the median as a measure of the bias of the distribution.

*b* We define precision as \((P84 - P16)/2\).

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evolves in time as indicated in the respective right hand side panel of Figure 7. For clarity, we show only the residuals of the mass-related physical quantities, the most sensitive ones to the fitting method, as shown in the previous sections. For DynBaS3D and TGASPEX, the residual distributions are very similar for the four galaxies, especially for galaxies in rows (a) and (b). For galaxy (c), \(M\) is clearly underestimated by DynBaS3D, and the TGASPEX residual distribution is nearly unbiased, while \(MW\log\text{Age}\) is underestimated by DynBaS3D and overestimated by TGASPEX. Finally, for galaxy (d), \(M\) and \(MW\log\text{Age}\) are underestimated by both DynBaS3D and TGASPEX. The mean metallicity \(MW\log\text{Z}\) is not well-recovered by any of the methods and is not shown in Figure 7. The conclusions do not depend on the explored values of SNR. These examples illustrate that the residual distributions are not dramatically different for the two methods. An analysis of their behavior as a function of the chemical evolution of each galaxy requires a much more detailed analysis, which is out of the scope of this work.

3.1.9. Summary

DynBaS3D shows a slightly better overall performance in the recovery of physical parameters of galaxies with SNR\(r = 20\). The differences between methods are more notorious in the determination of \(M\) and \(MW\log\text{Age}\), and decrease when SNR\(r\) increases from 20 to 30. Nevertheless, it should be pointed out that the precision obtained with GASPEX, DynBaS2D, DynBaS3D, TGASPEX, and \textsc{Starlight} are comparable for the six parameters explored above. The results obtained for

![Fig. 3.—Precision vs. bias obtained with GASPEX, DynBaS2D, DynBaS3D, TGASPEX, and \textsc{Starlight} for the recovery of the total mass \(M\), mass-weighted log-age \(MW\log\text{Age}\), flux-weighted log-age \(FW\log\text{Age}\), mass-weighted log-metallicity \(MW\log\text{Z}\), and V-band extinction \(A_V\) (panels from left to right). The bias and precision have been defined, respectively, as the median of the residuals and the half distance between the 16th and 84th percentile of residuals (see text for discussion). The top panels correspond to SNR\(r = 20\) mock galaxy spectra, and the bottom ones to SNR\(r = 30\).](image-url)

![Fig. 4.—Distributions of residuals in the determination of the stellar velocity dispersion \(\sigma_v\) using GASPEX and \textsc{Starlight} on mock spectra with SNR\(r = 20\). The vertical dashed lines indicate the median of residuals for each model. The distributions are normalized to unit area.](image-url)
$M$, $MWlogAge$, $FWlogAge$, $MWlogZ$, $A_V$, and $\sigma_v$ show very minor differences for the different algorithms and can be considered equivalent. For a reduced set of mock galaxies with time-evolving metallicity, DynBaS3D and TGASPEX provide an analogous determination of $M$ and $MWlogAge$.

4. DISCUSSION AND CONCLUSIONS

The goal of SED-fitting is to determine from galaxy spectra the physical parameters characterizing stellar populations in galaxies. Some of these parameters dictate the past and present evolution of galaxies. We have discussed the results of blind tests on SED-fitting using the GASPEX, DynBaS2D, DynBaS3D, and TGASPEX algorithms developed by the authors, and the public code STARLIGHT (Cid Fernandes et al. 2005, 2014), applied to a sample of 120 SSAG spectra selected to cover the range of galaxy spectra seen in the Universe. All these methods provide roughly equally good spectral fits, but show differences when recovering physical parameters. Our blind tests allow us to report a quantitative assessment of the precision (error) and bias in the

Fig. 5.—Residuals as a function of $(u-r)$ color for SSAG spectra with SNR$_r = 20$. The median and the 16th–84th percentile of the residual distributions are indicated with solid and dashed lines, respectively.
determination of $M$, $MW\log Age$, $FW\log Age$, $MW\log Z$, $A_V$, and $\sigma_v$, using the GASPEX, DynBaS2D, DynBaS3D, TGASPEX, and STARLIGHT codes. The values reported in Table 1 should be used to assign lower limits to the errors when spectra of observed galaxies are fitted with any of these algorithms.

DynBaS3D (dynamically selected spectral basis with three components) shows the best performance, producing, in general, the least biased determination of stellar mass, age, and metallicity, for mock galaxies with SNR$_r$ = 20, with the same precision as DynBaS2D (dynamically selected spectral basis with two components), GASPEX (fixed 12-dimensional basis per metallicity), and TGASPEX (unconstrained basis). The visual extinction $A_V$ was recovered only with DynBaS2D, GASPEX, and TGASPEX, with no significant differences in the results. The determination of the physical parameters provided by all methods is more precise, in general, for SNR$_r$ = 30 than for SNR$_r$ = 20. GASPEX, TGASPEX and STARLIGHT produce less-biased determinations of $M$, $MW\log Age$, and $FW\log Age$ when SNR$_r$ is increased from 20 to 30, while DynBaS2D and DynBaS3D tend to give more-biased results.

STARLIGHT, on average, overestimates mass and mass-weighted log-age, and shows roughly the same dispersion in the residuals as the other codes. The performance of GASPEX, TGASPEX, and STARLIGHT is statistically comparable.

DynBaS3D is a powerful and easy to implement algorithm to fit spectra and recover galaxy parameters. The three stellar populations obtained as the DynBaS3D solution represent the relevant events in the SFH of a galaxy, and can be used for a statistical estimation of the SFH over the galaxy lifetime. This will be addressed in an upcoming paper (Mateu, Magris & Bruzual, in preparation). In particular, DynBaS provides a good description of the SFH in stellar systems with isolated episodes of star formation, which are usually blurred when using other SED-fitting methods (Cabrera-Ziri et al. 2014). If a more detailed estimate of the SFH is desired, TGASPEX is a more suitable choice.

A popular choice in SED-fitting is to assume that the SFR declines or increases exponentially with time (Maraston et al. 2010; Hansson et al. 2012; Lee et al. 2009; Pforr et al. 2012; Mitchell et al. 2013). This may introduce strong biases in the determination of the galaxy stellar mass and age when the true SFR is not well described by the chosen parametrization, as indicated by Lee et al. (2009), Pforr et al. (2012), and Mitchell et al. (2013). Such biases are naturally diminished by using nonparametric algorithms like DynBaS2D, DynBaS3D, GASPEX, and TGASPEX, improving the accuracy of parameter determination.

As a last caveat, we must mention that the uncertainties in the physical parameters derived in this article should be considered as lower limits to the errors in parameter estimation. Our tests
were conducted on synthetic galaxy spectra assembled from the same evolutionary models used to fit the target galaxies. Larger errors are expected when real galaxy spectra are fitted or when an independent set of models is used to fit our SSAG sample of mock galaxies.

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Fig. 7.—Distribution of residuals obtained with DynBaS3D and TGASPEX for the total mass \(M\) and the mass-weighted log-age \(MW\log\text{Age}\) (left and middle) for four synthetic galaxies with metallicity \(Z\) evolving in time (right) assuming SNR\(_r\) = 20.
APPENDIX A.
SELECTING THE OPTIMAL BASIS

We can think of the matrix $M$ in equation (A1) as a basis of an $n$-dimensional vector space. We are interested in finding in this space an optimal basis of dimension $<n$, which is as orthogonal as possible. Although conceptually simple, in practice most of the spectra in SSP models are far from orthogonal. To illustrate this point, we assume that the $N_t$ spectra $f_i \equiv f_\lambda(t_i) \equiv f_\lambda(\lambda, t_i, Z)$ available in a BC03 model (cf., eq. [3]) define a $N_t$-dimensional vector space. The “angle” between vectors $f_i$ and $f_j$ is obtained from the dot product $\theta_{ij}$ given by

$$\theta_{ij} = \cos^{-1}\left( \frac{f_i \cdot f_j}{\|f_i\| \|f_j\|} \right),$$

where

$$f_i \cdot f_j = \sum_{\lambda=\lambda_i} f_\lambda(t_i) f_\lambda(t_j).$$

The sum in equation (A2) extends over all wavelength points in the model spectra. We show in Figure 8 the angle between all possible pairs $(i, j)$ of model spectra in the $Z_\odot$ BC03 model. From this figure, it is clear that only vectors with large differences in age are strictly orthogonal.

To select an optimal basis that excludes nearly parallel spectra, we start by defining a two-dimensional basis choosing a young ($1.78 \times 10^7$ yr) and an old (13.75 Gyr) spectrum. We then fit the remaining $N_t - 2$ spectra in the model with a linear combination of this two-element basis. Using $S$, the sum of quadratic residuals, as a goodness-of-fit indicator, we select the spectrum with the highest $S$ (worse fit) and include it in the basis. We repeat this procedure, adding at each step a new spectrum to the previous basis.

In Figure 9, we show the median of the $S$ distribution as a function of $NB$, the number of spectra in the basis. We can see that the fits are improving (lower median) as we increase the number of elements in the basis until $NB \sim 10–12$ where $S$ stabilizes reaching a plateau, which indicates that adding more
spectra to the basis will not significantly improve the ability to reproduce the remaining ones and will only add degeneracies to the problem being solved. Thus we select as an optimal basis the one with $NB = 12$, the basis with the minimum number of spectra for which $S$ has reached this plateau for all model metallicities.

The selected ages in our optimal 12-dimensional basis are $1.78 \times 10^5$, $4.17 \times 10^6$, $8.71 \times 10^6$, $1.91 \times 10^7$, $3.60 \times 10^7$, $9.05 \times 10^7$, $3.60 \times 10^8$, $1.28 \times 10^9$, $2.75 \times 10^9$, $4.75 \times 10^9$, $9.25 \times 10^9$, and $13.75 \times 10^9$ yr.

A point worth noting is that a hand-picked basis can lead to systematic biases in the recovered SFH. We strongly encourage users of this sort of methods to try to find some criteria that allow them to reduce the degeneracy and minimize biases in the results.

### APPENDIX B.

**SYNTHETIC SPECTRAL ATLAS OF GALAXIES (SSAG)**

As part of this investigation, we assembled an atlas of synthetic spectra of galaxies (SSAG),\(^9\) containing a comprehensive set of 100,000 spectra. These spectra were built from BC03 models using physically motivated SFHs that follow the prescription of Chen et al. (2012), adapted as follows:

1. **SFR**: In their recipe, Chen et al. (2012) assume that a galaxy forms stars according to an exponentially declining SFR and that at some time in its life experiences a burst of star formation. A number of galaxies may undergo another event, after which the SFR starts to decrease at a faster rate than before. We refer to this last event as “truncation.” The following parameters, based on the Chen et al. (2012) recipe, define the SFRs used to build SSAG (see Fig. 10).

   a) $T_{\text{form}}$: Age at which the galaxy starts forming stars. $T_{\text{form}}$ is uniformly distributed between the Big Bang (13.7 Gyr ago) and 1.5 Gyr.

   b) $\gamma$: Characterizes the decline of the exponential SFR ($\propto \exp^{-\gamma t}$); $\gamma$ is uniformly distributed between 0 and 1 Gyr$^{-1}$.

   c) $M_{\text{cont}}$: Mass formed by the continuous SFR (i.e., without the burst contribution), obtained integrating the SFR from $T_{\text{form}}$ to the present.

   d) $B_{\text{age}}$: Starting age of the star formation burst. The burst can start at any time in the past, with the constraint that 15% of the galaxies experience the burst in the last 2 Gyr.

   e) $B_{\text{ext}}$: Duration of the burst episode, uniformly distributed in age between $3 \times 10^7$ and $3 \times 10^8$ yr. During $B_{\text{ext}}$, a constant SFR is superimposed on the underlying exponential SFR. $M_{\text{burst}}$ is the stellar mass formed during the burst event (excluding the mass formed by the underlying exponential SFR). The value of $M_{\text{burst}}$ is determined by the parameter $A = M_{\text{burst}}/M_{\text{cont}}$; $\log A$ is distributed uniformly between 0.03 and 4.

   f) $T_{\text{cut}}$: Starting age of the truncation event. Following Chen et al. (2012), we assume that 30% of the galaxies experience a truncation event. $T_{\text{cut}}$ is distributed in such a way that 10% of the truncation events occur in the last 2 Gyr. After $T_{\text{cut}}$ the exponential SFR declines faster, with $\tau$-folding time $\tau_{\text{cut}}$: $\log \tau_{\text{cut}}$ is distributed uniformly between 7 and 9.

2. **Metallicity**: Each galaxy in SSAG is built using a single metallicity. The metallicity is distributed uniformly in $Z/Z_\odot$ in such a way that 5% of the galaxies have $0.05 < Z/Z_\odot < 0.2$, and the remaining 95% have $0.2 < Z/Z_\odot < 2.5$.

3. **Dust extinction**: The two-parameter Charlot & Fall (2000) prescription is used to model dust extinction in the SSAG galaxies. $\tau_V$, the optical depth in the $V$ band for stars younger than $10^7$ yr, takes values between 0 and 6. $\mu$, the fraction of the optical depth resulting from the ISM, ranges from 0 to 2. Inside these ranges the parameters follow Gaussian probability distribution functions of (mean, sigma) = (1.2, 0.985) and (0.3, 0.365), respectively.

4. **Velocity dispersion**: We convolve a Gaussian filter to the SSAG spectra to simulate the effects of the stellar velocity dispersion, characterized by $\sigma_v$, which take values distributed uniformly between 50 and 400 km s$^{-1}$.

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\(^9\) Available at http://www.astro.ljmu.ac.uk/~asticabr/SSAG.html.
Once a set of input parameters is specified, we use the csp_galaxev code distributed by Bruzual & Charlot to compute the corresponding SSAG spectrum. As input to this code, we use the BC03 SSP models for the Padova 1994 stellar evolutionary tracks, the STELIB stellar atlas, and the Chabrier (2003) IMF (see BC03 for details and references). When necessary, the BC03 models are interpolated in log Z to create an SSP model of the required Z/Z⊙. Note that in SSAG, every SFH has a single value of metallicity contrary to Chen et al. (2012). Also, for the synthesis of SSAG spectra, we used SSP models corresponding to Chabrier (2003) IMF instead of the Kroupa (2001) IMF used in Chen et al. (2012).

In Figure 11, the solid line encloses the region occupied by all the SSAG galaxies in the (u/C0 - g/C0, g/C0 - r/C0) color–color plane. The shaded area shows the distribution of z < 0.03 SDSS DR7 galaxies (Abazajian et al. 2009). The black dots in this figure represent the 120 SSAG galaxies used in this investigation.

The SSAG will be made publicly available in electronic form after paper acceptance, together with a detailed description of the software used to build and read it, its file structure, and the physical parameter distributions.

REFERENCES

Abazajian, K. N., Adelman-McCarthy, J. K., Agüeros, M. A., et al. 2009, ApJS, 182, 543
Acquaviva, V., Gawiser, E., & Guaita, L. 2011, ApJ, 737, 47
Bruzual, G., & Charlot, S. 2003, MNRAS, 344, 1000
Cabrera-Ziri, I., Bastian, N., Davies, B., et al. 2014, MNRAS, 441, 2754
Chabrier, G. 2003, PASP, 115, 763
Charlot, S., & Fall, S. M. 2000, ApJ, 539, 718
Chen, Y.-M., Kauffmann, G., Tremonti, C. A., et al. 2012, MNRAS, 421, 314
Cid Fernandes, R., Mateu, A., Sodré, L., Stasińska, G., & Gomes, J. M. 2005, MNRAS, 358, 363
Cid Fernandes, R., González Delgado, R. M., García Benito, R., et al. 2014, A&A, 561, A 130
Conroy, C. 2013, ARA&A, 51, 393
da Cunha, E., Charlot, S., & Elbaz, D. 2008, MNRAS, 388, 1595
Guaita, L., Acquaviva, V., Padilla, N., et al. 2011, ApJ, 733, 114
Hansson, K. S. A., Lisker, T., & Grebel, E. K. 2012, MNRAS, 427, 2376
Heavens, A. F., Jimenez, R., & Lahav, O. 2000, MNRAS, 317, 965
Koleva, M., Prugniel, P., Bouchard, A., & Wu, Y. 2009, A&A, 501, 1269
Kroupa, P. 2001, MNRAS, 322, 231
Lawson, C. L., & Hanson, R. J. 1974, Solving least squares problems (Upper Saddle River: Prentice-Hall)

Lee, S.-K., Ferguson, H. C., Somerville, R. S., Wiklind, T., & Giavalisco, M. 2010, ApJ, 725, 1644
Lee, S.-K., Idzi, R., Ferguson, H. C., et al. 2009, ApJS, 184, 100
Magris, G., Bruzual, G., & Mateu, J. 2007, in ASP Conf. Ser. 374, From Stars to Galaxies: Building the Pieces to Build Up the Universe, ed. A. Vallenari, R. Tantalo, L. Portinari, & A. Moretti (San Francisco: ASP), 411
Maraston, C., Pförj, J., Renzini, A., et al. 2010, MNRAS, 407, 830
Mateu, J. 2009, in Revista Mexicana Astron. Astrofis. Ser. Conf. 35, XII Latin American IAU Regional Meeting, G. Magris, G. Bruzual, & L. Carigi, México: Instituto de Astronomía, UNAM, 164–165
Mateu, J., Magris, G., & Bruzual, G. 2001, in ASP Conf. Ser. 230, Galaxy Disks and Disk Galaxies, ed. J. G. Funes, & E. M. Corsini (San Francisco: ASP), 323–324
Mitchell, P. D., Lacey, C. G., Baugh, C. M., & Cole, S. 2013, MNRAS, 435, 87
Ocvirk, P., Pichon, C., Laçon, A., & Thébaut, E. 2006, MNRAS, 365, 46
Pfőrj, J., Maraston, C., & Tonini, C. 2012, MNRAS, 422, 3285
Strateva, I., Ivezić, Ž., Knapp, G. R., et al. 2001, AJ, 122, 1861
ToFiejo, R., Heavens, A. F., Jimenez, R., & Panter, B. 2007, MNRAS, 381, 1252
Walcher, J., Groves, B., Budavári, T., & Dale, D. 2011, Ap&SS, 331, 1