Phase separation and vortex states in binary mixture of Bose-Einstein condensates

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Abstract

The phase separation and vortex states in two-component Bose-Einstein condensate consisting of $|F = 1, m_f = -1 >$ and $|2, 1 >$ internal spin states of $^{87}$Rb atoms are considered in the framework of Thomas-Fermi approximation. It is shown that in nonrotating system the atoms in the state $|1, -1 >$ form a shell about the atoms in the state $|2, 1 >$. The critical angular velocity for each state is calculated. These velocities depend drastically on the relative concentrations of the components, the critical angular velocity of the outer component being less than the angular velocity of the inner one. It is shown that the atoms in the $|1, -1 >$ state can form a rotating ring about the resting core of the atoms in the state $|2, 1 >$.

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The realization of Bose-Einstein Condensation (BEC) in dilute atomic gases offers new opportunities for studying quantum degenerate fluids\cite{1}. These condensates, which contain thousands of atoms confined to microscale clouds, have similarities to superfluidity and laser, and provide new testing ground for many body physics.

The modern theoretical description of dilute BEC originates from Bogoliubov’s seminal 1947 paper where he showed that weak repulsive interaction qualitatively change the
excitation spectra from quadratic free particle form to a linear phonon-like structure. To describe the trapped condensates at $T = 0$ one can use the Gross-Pitaevskii (GP) (nonlinear Schrödinger) equation for the condensate wave function [2]. This equation appears as the generalization of the Bogoliubov theory for the inhomogeneous phase. It was widely used to discuss the ground state properties and collective excitations in BEC.

Bulk superfluids are distinguished from normal fluids by their ability to support dissipationless flow. This ability is closely related to the existence of stable quantized vortices. Such vortices have been widely studied in superfluid $^4$He. Recently clear experimental evidences of the existence of a vortex in trapped BEC [3–5] was reported. Unlike superfluid helium, the trapping potential makes alkali BEC nonuniform. Theoretical work has been concentrated on the critical angular velocity of the vortex creation, the collective excitations of BEC in the presence of the vortex and considerations of stability of the vortex [6–13]. It was shown that unlike superfluid helium where the vortex is locally stable, for weakly interacting gases stable quantized vortices exist only in a driven system, and become unstable without imposed rotation. So this system cannot be considered a superfluid [6,10,11].

In the study of quantum fluids, the most interesting behavior has been found in the behavior of fluid mixtures. At the present time, two experimental groups have observed trapped multiple condensates as realized in a magnetic trap in rubidium [14] and in an optical trap in sodium [15]. In these experiments, the spatial separation of condensates has been observed. One can distinguish two types of spatial separation: (a) potential separation, caused by external trapping potentials; (b) phase separation, which can occur in the absence of external potentials due to the interaction between two condensates. It is the latter type of phase separation that we consider in this paper. This type of phase separation has been observed in the experiments on the simultaneously trapped condensates, consisting of the $^{87}$Rb atoms in the $|2,1 >$ and $|1,−1 >$ spin states (states 2 and 1 correspondingly) [14]. In this case the intraspecies and interspecies scattering lengths denoted correspondingly as $a_{11}, a_{22}, a_{12}$ are in the proportion $a_{11} : a_{12} : a_{22} = 1.03 : 1 : 0.97$ with the average of the three being $55(3)\text{Å}$ [14,16].
In this paper we consider the behavior of binary mixture of Bose-Einstein condensates of alkali atoms. We calculate the critical angular velocity needed to create stable vorticies in either component in the rotating frame. This quantity is of crucial importance in view of the experimental possibility of creating vortices by rotation of the confining trap [5].

The physics of interpenetrating Bose fluids is very rich and far from a complete understanding. While properties of a rotating single component Bose fluids are much studied [6]-[13], the rotating mixture of alkali atoms presents a new frontier, that is essentially virgin territory.

In order to derive analytic results, some approximations must be used. A commonly used one is the Thomas-Fermi Approximation (TFA), which ignores the kinetic energy terms. It has been shown that in the case of one component condensates the TFA results agree well with the numerical calculations for large particle numbers, except for a small region near the boundary of the condensate [4,8]. In fact, even for a small number of particles the TFA still usually gives qualitatively correct results. The TFA provides an excellent starting point of study. However, the TFA should not be relied upon when a quantitative comparison of experiment and theory is important. In this case a numerical approach based on the Monte-Carlo simulation becomes necessary.

Let us first consider the phase separation in binary mixture without rotation.

In the case of a two-species condensate, letting \( \psi_i(r) \) \((i = 1, 2)\) be the wave function of species \( i \) with particle number \( N_i \), we can write the two coupled nonlinear Schrödinger (Gross-Pitaevskii) equations as:

\[
-\frac{\hbar^2}{2m_1} \nabla^2 \psi_1(r) + \frac{1}{2} m_1 \omega_1^2(x^2 + y^2 + \lambda^2 z^2) \psi_1(r) - \mu_1 \psi_1(r) + G_{11} |\psi_1(r)|^2 \psi_1(r) + G_{12} |\psi_2(r)|^2 \psi_1(r) = 0; \tag{1}
\]

\[
-\frac{\hbar^2}{2m_2} \nabla^2 \psi_2(r) + \frac{1}{2} m_2 \omega_2^2(x^2 + y^2 + \lambda^2 z^2) \psi_2(r) - \mu_2 \psi_2(r) + G_{22} |\psi_2(r)|^2 \psi_2(r) + G_{12} |\psi_1(r)|^2 \psi_2(r) = 0. \tag{2}
\]

Equations (1) and (2) were obtained by minimization of the energy functional of the trapped bosons of masses \( m_1 \) and \( m_2 \) given by:
\begin{equation}
E(\psi_1, \psi_2) = \int d^3r \left[ \frac{\hbar^2}{2m_1} |\nabla \psi_1(r)|^2 + \frac{1}{2} m_1 \omega_1^2 (x^2 + y^2 + \lambda^2 z^2) |\psi_1(r)|^2 + \frac{\hbar^2}{2m_2} |\nabla \psi_2(r)|^2 + \frac{1}{2} m_2 \omega_2^2 (x^2 + y^2 + \lambda^2 z^2) |\psi_2(r)|^2 + \frac{G_{11}}{2} |\psi_1(r)|^4 + \frac{G_{22}}{2} |\psi_2(r)|^4 + G_{12} |\psi_1(r)|^2 |\psi_2(r)|^2 \right].
\end{equation}

(3)

The chemical potentials \(\mu_1\) and \(\mu_2\) are determined by the relations \(\int d^3r |\psi_i|^2 = N_i\).

The trap potential is approximated by an effective three-dimensional harmonic-oscillator potential well, which is cylindrically symmetric about \(z\) axis, \(\lambda\) being the ratio of angular frequencies in the axial direction \(\omega_{zi}\) to that in the transverse direction \(\lambda = \omega_{zi}/\omega_i\). The experimental value of \(\lambda\) is \(\lambda = \sqrt{8}\). The interaction strengths, \(G_{11}, G_{22}, G_{12}\) are determined by the \(s\)-wave scattering lengths for binary collisions of like and unlike bosons: \(G_{ii} = 4\pi \hbar^2 a_{ii}/m_i; G_{12} = 2\pi \hbar^2 a_{12}/m\), where \(m^{-1} = m_1^{-1} + m_2^{-1}\).

Let us consider now the phase separation due to interaction between the two condensates. In this case

\begin{equation}
\frac{1}{2} m_1 \omega_1^2 = \frac{1}{2} m_2 \omega_2^2.
\end{equation}

(4)

Let us simplify the equations by using dimensionless variables. We define the length scale

\begin{equation}
a_\perp = \left( \frac{\hbar}{m_1 \omega_1} \right)^{1/2},
\end{equation}

(5)

and the dimensionless variables

\begin{equation}
\mathbf{r} = a_\perp \mathbf{r}',
\end{equation}

(6)

\begin{equation}
E = \hbar \omega_1 E',
\end{equation}

(7)

\begin{equation}
\psi_i(\mathbf{r}) = \sqrt{N_i/a_\perp^3} \psi'_i(\mathbf{r}).
\end{equation}

(8)

The wave function \(\psi'_i(\mathbf{r}')\) is normalized to 1. In terms of these variables the Gross-Pitaevskii energy functional takes the form:

\begin{equation}
E' = \frac{1}{2} \int d^3r' \left[ N_1 |\nabla' \psi'_1|^2 + N_1 (x'^2 + y'^2 + \lambda^2 z'^2) |\psi'_1|^2 + N_2 \beta^2 |\nabla' \psi'_2|^2 + N_2 (x'^2 + y'^2 + \lambda^2 z'^2) |\psi'_2|^2 + \frac{1}{2} N_1 u_1 |\psi'_1|^4 + \frac{1}{2} N_2 u_2 \beta^2 |\psi'_2|^4 + \frac{2\pi a_{12} m_1}{a_\perp m} N_1 N_2 |\psi'_1|^2 |\psi'_2|^2 \right].
\end{equation}

(9)
Here \( \beta^2 = m_1/m_2 = \omega_2^2/\omega_1^2 \) and \( u_i = 8\pi a_i N_i/a_\perp \). In deriving Eq. (9) we used Eq. (4). Eqs. (4) and (2) can be rewritten as:

\[
- \nabla'^2 \psi_1' + (x'^2 + y'^2 + \lambda^2 z'^2) \psi_1' - \mu_1' \psi_1' + u_1 |\psi_1'|^2 \psi_1' + \frac{4\pi a_{12} N_2 m_1}{m} |\psi_2'|^2 \psi_1 = 0; \tag{10}
\]

\[
- \beta^2 \nabla'^2 \psi_2' + (x'^2 + y'^2 + \lambda^2 z'^2) \psi_2' - \mu_2' \psi_2' + u_2 |\psi_2'|^2 \psi_2 + \frac{4\pi a_{12} N_1 m_1}{m} |\psi_1'|^2 \psi_2 = 0; \tag{11}
\]

where \( \mu_i' = 2\mu_i/h\omega_1 \).

In the TFA, Eqs. (4), (10) and (11) can be further simplified by omitting the kinetic energy. The phase segregated condensates do not overlap, so we can neglect the last terms in Eqs. (4), (10) and (11), obtaining from (10) and (11), in separate regions that they do not overlap, simple algebraic equations:

\[
|\psi_1'(r')|^2 = \frac{1}{u_1} \left( \mu_1' - \left( \rho'^2 + \lambda^2 z'^2 \right) \right); \tag{12}
\]

\[
|\psi_2'(r')|^2 = \frac{1}{u_2\beta^2} \left( \mu_2' - \left( \rho'^2 + \lambda^2 z'^2 \right) \right). \tag{13}
\]

Here \( \rho'^2 = x'^2 + y'^2 \). From Eqs. (12) and (13) one can see that the condensate density has the ellipsoidal form.

In the case of phase separation, the energy of the system can be written in the form

\[
E = E_1 + E_2, \tag{14}
\]

where

\[
E_1 = \frac{1}{2} \hbar \omega_1 N_1 \left[ \mu_1' - \frac{1}{2} u_1 \int d^3 r' |\psi_1'|^4 \right], \tag{15}
\]

\[
E_2 = \frac{1}{2} \hbar \omega_1 N_2 \left[ \mu_2' - \frac{1}{2} u_2 \beta^2 \int d^3 r' |\psi_2'|^4 \right]. \tag{16}
\]

In order to obtain Eqs. (15)–(16), Eqs. (12)–(13) have been used.

To investigate the phase separation in the mixture we first suppose that the condensate 1 atoms form an ellipsoidal shell about the condensate 2 atoms (we will call this configuration as the configuration ”a”). To determine the position of the boundary between the...
condensates, we use the condition of thermodynamic equilibrium \[19\]: the pressures exerted by both condensates must be equal:

\[ P_1 = P_2. \] (17)

The pressure is given by \[20\]:

\[ P_i = \frac{G_{ii}}{2} |\psi_i|^4. \] (18)

The condensate 2 has the form of the ellipsoid with long semiaxis \( q \):

\[ \rho^2 + \lambda^2 z^2 = q^2. \] (19)

From Eqs. \((12)-(13)\) and \((17)-(19)\) one has the equation for \( q \):

\[ \mu'_1 - q^2 = \kappa \mu'_2 - \kappa q^2, \] (20)

where \( \kappa = \sqrt{(a_{11}m_2)/(a_{22}m_1)}. \)

Chemical potentials \( \mu'_1 \) and \( \mu'_2 \) can be obtained using the normalization conditions \( \int d^3 r' |\psi'_1|^2 = \int d^3 r' |\psi'_2|^2 = 1 \) and are given by:

\[ \mu'_1 = \frac{\mu'^0_1}{(1 - \frac{\mu'_1}{\mu'_2} + \frac{\mu'_1}{\mu'_2})^{2/5}}, \] (21)

\[ \mu'_2 = \frac{3}{(\mu'_1)^{3/2}q^3} \left( \frac{2\beta^2(\mu'^0_2)^{5/2}}{15} + \frac{(\mu'_1)^{5/2}q^5}{5} \right), \] (22)

where \( q = \sqrt{\mu'_1 q'} \) and

\[ \mu'^0_1 = \left( \frac{15\lambda u_i}{8\pi} \right)^{2/5}. \] (23)

From equations \((21)-(23)\) one can determine the chemical potentials \( \mu'_1 \) and \( \mu'_2 \) and the semiaxis of the phase boundary ellipsoid \( q \) as functions of \( N_1 \) and \( N_2 \). The energy of the configuration ”a” \( E_a = E_{a1} + E_{a2} \) is given by:

\[ E_{a1} = \frac{1}{2} \hbar \omega_1 N_1 \left\{ \mu'_1 - \frac{15}{4} \left( \frac{\mu'^0_1}{\mu'^0_2} \right)^{7/2} \left( 8 \frac{q^3}{105} - \frac{2}{5} q^5 + \frac{q^7}{7} \right) \right\}, \] (24)

\[ E_{a2} = \frac{1}{2} \hbar \omega_1 N_2 \left\{ \mu'_2 - \frac{15}{4} \beta^2 \left( \frac{\mu'^0_2}{\mu'^0_2} \right)^{3/2} \left( \frac{\mu'_2 q^3}{3} - \frac{2\mu'_2 \mu'_1 q^5}{5} + \frac{\mu'_2 q^7}{7} \right) \right\}. \] (25)
Let us now consider the opposite case when the condensate 2 atoms form an ellipsoidal shell about the condensate 1 atoms (configuration "b"). In this case Eqs. (20)-(25) can be rewritten in the form:

\[ \mu''_1 - q_1'^2 = \kappa (\mu''_2 - q_1'^2) , \]

(26)

\[ (\mu''_2)^{5/2} = \frac{\beta^2(\mu''_2)^{5/2}}{1 - \frac{5}{2}q_1'^3 + \frac{3}{2}q_1'^5} , \]

(27)

\[ \frac{15 (\mu''_2)^{3/2}}{2 (\mu''_1)^{5/2}} \left( \frac{\mu''_1 q_1'^3}{3} - \frac{\mu''_2 q_1'^5}{5} \right) = 1 , \]

(28)

\[ E_b = E_{b_1} + E_{b_2} , \]

\[ E_{b_1} = \frac{1}{2} \hbar \omega_1 N_1 \left\{ \mu''_1 - \frac{15 (\mu''_2)^{3/2}}{4 (\mu''_1)^{5/2}} \left( \mu''_1 q_1'^3 \frac{3}{3} - 2 \mu''_1 q_1'^5 \frac{5}{5} + \mu''_2 q_1'^7 \frac{7}{7} \right) \right\} , \]

(29)

\[ E_{b_2} = \frac{1}{2} \hbar \omega_1 N_2 \left\{ \mu''_2 - \frac{15 (\mu''_1)^{7/2}}{4 \beta^2(\mu''_2)^{5/2}} \left[ \frac{8}{105} - \left( \frac{q_1'^3}{3} - \frac{2}{5} q_1'^5 + \frac{q_1'^7}{7} \right) \right] \right\} . \]

(30)

Here \( \mu''_1 \) and \( \mu''_2 \) are the chemical potentials in the configuration "b", \( q_1 = \sqrt{\mu''_2 q_1'} \) is the long semiaxis of the boundary ellipsoid, \( E_b \) is the energy of the configuration "b".

To estimate which configuration is stable, one has to compare \( E_a \) and \( E_b \). Let us first consider the limiting cases \( n_2 = N_2/N_1 \ll 1, \) and \( n_1 = N_1/N_2 \ll 1. \)

In the former case \( n_2 \ll 1 \) the approximate solution of Eqs. (20)- (22) has the form:

\[ q' = q_0 \left( 1 + \frac{1}{3} \left( 1 - \frac{2}{5} \kappa \right) q_0^2 - \frac{5}{6} q_0^3 \right) , \]

(31)

\[ \mu'_1 = \mu''_1 (1 + q_0^2) , \]

(32)

\[ \mu'_2 = \frac{\mu''_1}{\kappa} \left( 1 + (\kappa - 1)q_0^2 + q_0^3 \right) . \]

(33)

where

\[ q_0 = \left( \frac{2n_2}{5\kappa} \right)^{1/3} . \]

(34)

From Eqs. (20)- (28) one has:

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\[ q'_1 = 1 - p_0 - \frac{5}{2} \left( \kappa - \frac{2}{3} \right) p_0^2 - \frac{25}{8} \left( \kappa^2 - \frac{14}{5} \kappa + \frac{224}{225} \right) p_0^3, \quad (35) \]

\[ \mu''_{1} = \mu_{1}^{0} (1 + 3 \kappa^2 p_0^2), \quad (36) \]

\[ \mu''_{2} = \mu_{1}^{0} \left( 1 - 2 (\kappa - 1) p_0 + \frac{1}{3} (6 \kappa^2 + 4 \kappa - 1) p_0^2 \right), \quad (37) \]

where

\[ p_0 = \left( \frac{2 n_2}{15 \kappa^2} \right)^{1/2}. \quad (38) \]

Using Eqs. (31)-(38), it may be easily shown that

\[ \Delta E = E_a - E_b = \frac{1}{2} \hbar \omega N_1 \mu_1^{0} \frac{1 - \kappa}{\kappa} n_2. \quad (39) \]

From Eq. (39) one can see that for \( \kappa > 1 \), \( \Delta E < 0 \), so the configuration "a" is stable.

Let us now consider the case \( n_1 \ll 1 \). Approximate solution for the configuration "a" is given by:

\[ q'_1 = 1 - x - \frac{5 (3 - 2 \kappa)}{6 \kappa} x^2 - \frac{224 \kappa^2 - 630 \kappa + 225}{72 \kappa^2} x^3, \quad (40) \]

\[ \mu'_1 = \mu_{2}^{0} \beta^{4/5} \left( 1 + 2 \frac{(\kappa - 1)}{\kappa} x - \frac{(\kappa^2 - 4 \kappa - 6)}{3 \kappa} x^2 \right), \quad (41) \]

\[ \mu'_2 = \mu_{2}^{0} \beta^{4/5} \left( 1 + \frac{3}{\kappa^2} x^2 \right). \quad (42) \]

Here

\[ x = \left( \frac{2 \kappa^2 n_1}{15} \right)^{1/2}. \quad (43) \]

In the configuration "b" solution has the form;

\[ q'_1 = y + \frac{(\kappa - 2/5)}{3 \kappa} y^3 - \frac{5}{6} y^4, \quad (44) \]

\[ \mu''_{1} = \beta^{4/5} \mu_2^{0} \left[ \kappa - (\kappa - 1) y^2 + \kappa y^3 - \frac{2 (\kappa - 1) (5 \kappa - 2)}{15 \kappa} y^4 \right], \quad (45) \]

\[ \mu''_{2} = \beta^{4/5} \mu_2^{0} (1 + y^3), \quad (46) \]

where

\[ y = \left( \frac{2 \kappa n_1}{5} \right)^{1/3}. \quad (47) \]
The energy difference is:

$$\Delta E = \frac{1}{2} \hbar \omega_1 N_2 \mu_2^0 n_1 (1 - \kappa).$$  \hspace{1cm} (48)$$

From Eqs. (39) and (48) it is seen that the configuration "a" has lower energy if \( \kappa = \sqrt{(a_{11} m_2)/(a_{22} m_1)} > 1 \). For \( m_1 = m_2 \) this is consistent with the qualitative assertion and experimental observation that it is energetically favorable for the atoms with the larger scattering length to form a lower-density shell about the atoms with the smaller scattering length [14,21].

To evaluate \( \Delta E \) in general case it is worth first to estimate the energy of the phase boundary which arises due to the gradient terms omitted in the TFA. The surface energy per unit area, the surface tension, is defined as \( \sigma = E_s/S \), where \( E_s \) is the surface energy, and \( S \) is the interface area. \( \sigma \) may be written in the form [17,18]:

$$\sigma = \frac{\hbar \omega_1}{2 \sqrt{2} a_1^2} \left( \frac{a_{12}}{\sqrt{a_{11} a_{22}}} - 1 \right)^{1/2} (u_1 u_2 N_1 N_2)^{1/4} |\psi'_1||\psi'_2| (N_1 |\psi'_1|^2 + N_2 |\psi'_2|^2)^{1/2}. \hspace{1cm} (49)$$

Taking into account that the surface area of the ellipsoid with the semiaxis \( a_\perp q \) has the form:

$$S = 2\pi a_\perp^2 q^2 \left( 1 + \frac{1}{\lambda \sqrt{x^2 - 1}} \log \frac{1}{\lambda - \sqrt{x^2 - 1}} \right), \hspace{1cm} (50)$$

one can estimate the contribution of the surface energy \( E_s = \sigma S \) to the total energy of each configuration. To be specific, we will use the parameters corresponding to the experiments on \(^{87}\text{Rb} \) atoms. In this case \( m_1 = m_2, a_\perp = 2.4 \times 10^{-4} cm, N = N_1 + N_2 = 0.5 \times 10^6 \) atoms.

In Fig. 1(a) we show the energies of configurations "a" and "b" (including the surface energy) \( E_a/(\hbar \omega_1 N) \) (solid line) and \( E_b/(\hbar \omega_1 N) \) (dashed line) as functions of \( \log_{10}(n_2) \). One can see that \( E_a \) is always lower than \( E_b \). Fig. 1 (b) represents the difference \( \Delta E = (E_a - E_b)/(\hbar \omega_1 N) \). The behavior of \( \Delta E \) for small and large values of \( n_2 \) is well described by Eqs. (39) and (48). Fig. 1 (c) illustrates the behavior of the surface energy as a function of \( n_2 \). It should be noted that the surface energy is much smaller than the interaction energy because the scattering lengths \( a_{ij} \) have very close values (see Eq. (49)).
Let us now consider a trap rotating with frequency $\Omega$ along the $z$-axis.

For vortex excitation with angular momentum $\hbar l$, the condensate wave function is given by

$$\psi_l(r) = |\psi_l(r)| e^{il\phi}. \quad (51)$$

In a rotating frame the energy functional of the system is

$$E_{\text{rot}}(l_1, l_2) = E(\psi_{l_1}, \psi_{l_2}) + \int d^3r (\psi_{l_1}^* + \psi_{l_2}^*) i\hbar \Omega \partial_\phi (\psi_{l_1} + \psi_{l_2}). \quad (52)$$

After substituting the wave function for the vortex excitation (51) in Eq. (52), the effective confinement potential for the bosons becomes

$$l_1^2 \hbar^2/2m_1 \rho^2 + l_2^2 \hbar^2/2m_2 \rho^2 + V_1 + V_2,$$

where $V_i = m_i \omega_i (\rho^2 + \lambda^2 z^2)/2$ and $\rho^2 = x^2 + y^2$. So within the TFA the density of the vortex state, in separate regions that they do not overlap, has the form:

$$|\psi'_1(r')|^2 = \frac{1}{u_1} \left( \mu'_1(l_1) - (\rho^2 + \lambda^2 z^2) - \frac{l_1^2}{\rho^2} \right); \quad (53)$$

$$|\psi'_2(r')|^2 = \frac{1}{u_2 \beta^2} \left( \mu'_2(l_2) - (\rho^2 + \lambda^2 z^2) - \frac{\beta^2 l_2^2}{\rho^2} \right). \quad (54)$$

The important new qualitative feature of a vortex in the TFA is the appearance of a small hole of radius $\xi_i$, $\xi_i^2 \propto l_i^2/\mu_i(l_i)$, but the remainder of the condensate density is essentially unchanged. The fractional change in the chemical potentials caused by the vortex $(\mu'_i(l_i) - \mu'_i)/\mu'_i$ can be shown to be small \[6,9\], of the order of $1/N^{4/5}$. In the calculation of physical quantities involving the condensate density it is sufficient to retain the no-vortex density and simply cut off any divergent radial integrals at the appropriate core sizes $\xi_1^2 = l_1^2/\mu'_1$ or $\xi_2^2 = \beta^2 l_2^2/\mu'_2$. Note that using the unperturbed density for calculation of the vortex properties corresponds to the hydrodynamic limit.

In the case of the phase segregated condensate, one finds from Eqs. (51-52) and (15-16) that the energy change due to the presence of the vortices $\Delta E = E_{\text{rot}}(l_1, l_2) - E_{\text{rot}}(0, 0)$ has the form:

$$\Delta E = \Delta E_{N_1} + \Delta E_{N_2} = \ldots$$
\[
\frac{1}{2} \hbar \omega_1 N_1 \int d\mathbf{r} \left( \frac{l_1^2}{\rho^2} |\psi'_1|^2 - \frac{2\Omega l_1}{\omega_1} |\psi'_1|^2 \right) + \\
\frac{1}{2} \hbar \omega_1 N_2 \int d\mathbf{r} \left( \frac{l_2^2 \beta^2}{\rho^2} |\psi'_2|^2 - \frac{2\Omega l_2}{\omega_1} |\psi'_2|^2 \right).
\] 

(55)

In the hydrodynamic limit \( \psi'_i \) is given by Eqs. (12) and (13).

Let us consider the stable configuration "a". In the hydrodynamic limit the location of the phase boundary is given by Eq. (19). From (55) one has:

\[
\Delta E_{N_1} = \frac{5l_1^2 (\mu_1')^{3/2}}{(\mu_1')^{5/2}} \left\{ \ln \left( \frac{2\mu_1'}{l_1} - \frac{4}{3} \right) - \frac{3}{2} q' \left( 1 - \frac{q'^2}{3} \right) \ln \left( \frac{2\mu_1' q'}{l_1} - \left( 1 - \frac{q'^2}{9} \right) \right) \right\} - \frac{2\Omega l_1}{\omega_1}. \tag{56}
\]

\[
\Delta E_{N_2} = \frac{15l_2^2 (\mu_1')^{1/2} q'}{2(\mu_2')^{5/2}} \left( \mu_2' - \frac{1}{3} \mu_1' q'^2 \right) \ln \left( \frac{2\sqrt{\mu_1' \mu_2' q'}}{l_2 \beta} - \left( \mu_2' - \frac{\mu_1' q'^2}{9} \right) \right) - \frac{2\Omega l_2}{\omega_1}. \tag{57}
\]

The critical angular velocities required to produce the vortex states in each condensate can be determined from the conditions \( \Delta E_{N_1} < 0, \Delta E_{N_2} < 0 \) and have the form:

\[
\frac{\Omega_{N_1}}{\omega_1} = \frac{5l_1}{2(\mu_1')^{5/2}} \left\{ \ln \left( \frac{2\mu_1'}{l_1} - \frac{4}{3} \right) - \frac{3}{2} q' \left( 1 - \frac{q'^2}{3} \right) \ln \left( \frac{2\mu_1' q'}{l_1} - \left( 1 - \frac{q'^2}{9} \right) \right) \right\}, \tag{58}
\]

\[
\frac{\Omega_{N_2}}{\omega_1} = \frac{15l_2^2 (\mu_1')^{1/2} q'}{4(\mu_2')^{5/2}} \left( \mu_2' - \frac{1}{3} \mu_1' q'^2 \right) \ln \left( \frac{2\sqrt{\mu_1' \mu_2' q'}}{l_2 \beta} - \left( \mu_2' - \frac{\mu_1' q'^2}{9} \right) \right). \tag{59}
\]

Let us consider the asymptotic behavior of the critical angular velocities (58) and (59) for \( N_2 \ll N_1 \) and \( N_1 \ll N_2 \). Using the approximate solutions (31)-(34), we obtain:

\[
\frac{\Omega_{N_1}}{\omega_1} = \frac{5l_1}{2\mu_1'} \left( \ln \left( \frac{2\mu_1^0}{l_1} - \frac{4}{3} \right) - \frac{15l_1}{4\mu_1'} \left( \ln \left( \frac{2\mu_1^0 q_0}{l_1} \right) - 1 \right) q_0, \right. \tag{60}
\]

\[
\left. \frac{\Omega_{N_2}}{\omega_1} = \frac{3l_2^2 \beta^2}{2\mu_1'^0 \kappa^{1/2} l_2 \beta} \left( \ln \left( \frac{2\mu_1^0 q_0}{\kappa^{1/2} l_2 \beta} \right) - 1 \right) - \frac{l_2^2 \beta^2}{\mu_1^0} \left( \ln \left( \frac{2\mu_1^0 q_0}{\kappa^{1/2} l_2 \beta} \right) \left( 1 - \frac{4}{5} \kappa \right) + \frac{7}{12} \kappa - \frac{3}{4} \right). \tag{61}
\]

From (58) and (59) one can see that if \( n_2 \to 0 \) the critical angular velocity of the external condensate \( \Omega_{N_1} \) tends to that of the pure condensate with the scattering length \( a_{11} \) (see Eq. (26) in Ref. [3]). The critical angular velocity of the inner condensate \( \Omega_{N_2} \) tends to infinity as \( n_2 \to 0 \). However, this consideration can not be applied to rapidly rotating gases with \( \Omega \) comparable to \( \omega_1 \) where the form of the condensate depends on \( \Omega \) [13].

In the opposite limit \( n_1 = N_1/N_2 \ll 1 \) the critical angular velocities can be written as
\[ \frac{\Omega_{N_1}}{\omega_1} = \frac{l_1}{2\mu_2^{\theta_{34/5}}} \ln \left( \frac{2\mu_2^{\theta_{34/5}}}{l_1} \right) + \frac{l_1}{3\kappa \mu_2^{\theta_{34/5}}} \left[ (3 - \kappa) \ln \left( \frac{2\mu_2^{\theta_{34/5}}}{l_1} \right) + 3 - 2\kappa \right] x, \quad (62) \]

\[ \frac{\Omega_{N_2}}{\omega_1} = \frac{5l_2^{\theta_{6/5}}}{2\mu_2^0} \left( \ln \left( \frac{2\mu_2^0}{l_2^{\theta_{1/5}}} \right) - \frac{4}{3} \right) + \frac{15l_2^{\theta_{6/5}}}{2\mu_2^0 \kappa^2} \left( \ln \left( \frac{2\mu_2^0}{l_2^{\theta_{1/5}}} \right) - 1 \right) x^2. \quad (63) \]

In deriving (62) we used the approximate solutions (40-43). Note that when \( n_1 \to 0 \), the critical angular velocity (63) has the same form as the critical velocity for the pure condensate with scattering length \( a_{22} \), \( \mu_0^{\theta_{34/5}} \) being the chemical potential.

Fig. 2 shows the critical angular velocities for the external (\( \Omega_{N_1} \)) and the inner (\( \Omega_{N_2} \)) condensates as functions of \( n_2 = N_2/N_1 \) for \( l_1 = l_2 = 1 \).

Using Eqs. (52) and (55), one can find the vortex configurations which correspond to the energy minimum for a given angular velocity \( \Omega/\omega_1 \). In Table 1 we represent angular momenta of condensates which correspond to the minimum of the total energy \( E_{\text{tot}}/\hbar \omega_1 \) of the system for different values of the angular velocity and \( n_2 \). In calculation we use the parameters for the \( ^{87}\text{Rb} \). \( E_{\text{tot}} \) is calculated as a sum of \( E \) (Eq. (14), the surface energy \( E_s \) (Eqs. (49-50)), \( \Delta E_{N_1} \) (Eq. (56)) and \( \Delta E_{N_2} \) (Eq. (57)).

To summarize, we have shown that in the case of \(^{87}\text{Rb} \) condensate the atoms in the state 1 form a shell about the atoms in the state 2, the critical angular velocity for each state being drastically dependent on the mutual concentrations. The critical angular velocity of the outer component is less than the angular velocity of the inner one. When the ratio of the number of state 2 atoms to the number of state 1 atoms is small enough the critical angular velocity of the inner state becomes very large: in the framework of the hydrodynamic approximation it turns out to be larger than the oscillator frequency characterizing the confining potential and to decrease smoothly with increasing number of atoms in the state 2. From the Table 1 one can see that atoms in the state 1 can form a rotating ring around the resting core of the atoms in the state 2.

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### TABLES

#### TABLE I.

| $\Omega/\omega_1$ | $n_2$ | $l_1$ | $l_2$ | $E_{tot}$       |
|------------------|-------|-------|-------|-----------------|
| 0.1              | 1.0   | 1     | 0     | $1.296863 \times 10^7$ |
| 0.1              | 10.0  | 2     | 0     | $1.287497 \times 10^7$ |
| 0.15             | 1.0   | 2     | 0     | $1.295151 \times 10^7$ |
| 0.2              | 1.0   | 3     | 1     | $1.292344 \times 10^7$ |
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FIGURES

FIG. 1. (a) The total energies of configurations "a" and "b" $E_a/(\hbar \omega_1 N)$ (solid line) and $E_b/(\hbar \omega_1 N)$ (dashed line) as functions of $\log_{10}(n_2)$. (b) The difference $\Delta E = (E_a - E_b)/(\hbar \omega_1 N)$ as a function of $\log_{10}(n_2)$. (c) The surface energies as functions of $\log_{10}(n_2)$. Solid line corresponds to the surface energy $E_{sa}$ for the configuration "a", dashed line - to the surface energy $E_{sb}$ for the configuration "b".

FIG. 2. Critical angular velocities $\Omega_{N_1}/\omega_1$ and $\Omega_{N_2}/\omega_2$ as functions of $\log_{10}(n_2)$ for the configuration "a". Solid line corresponds to the outer condensate 1, dashed line - to the inner condensate 2.
\[ \log_{10}(N_2 / N_1) \]
\[ \log_{10}(N_2 / N_1) \]

\[ \frac{\Omega_{N_1}}{\omega_1} \]

\[ \frac{\Omega_{N_2}}{\omega_1} \]