A new counting methods, including the issue of counting labelled self-complementary graphs

Shinsei Tazawa
Department of Mathematics, Faculty of Science and Engineering, Kinki University, Higashi-Osaka 577-8502, Japan

abstract

Harary and Palmer announced an enumeration problem of labelled self-complementary graphs at the end of their book (Graphical Enumeration, Academic Press, New York and London, 1973). This paper resolves this problem. A method for solving this problem leads to the derivation of following formulas:
(a) A formula on the number of labelled graphs with the given order of automorphism groups of those graphs. (b) A formula on the number of unlabelled graphs with the given order of automorphism groups of those graphs. (c) A formula on the number of labelled self-complementary graphs with the given order of automorphism groups of those graphs. (d) A formula on the number of unlabelled self-complementary graphs with the given order of automorphism groups of those graphs.

Keywords: Graphical enumeration, Graph, Self-complementary graph, Automorphism group, Generating function

1 Introduction

This paper considers finite graphs. They do not have multiple edges nor loops on vertices. Harary and Palmer[5] announced many graphical enumeration problems at the end of their book, including an enumeration problem of labelled self-complementary graphs. This paper resolves the latter problem. A method for solving this problem leads to the derivation of following formulas:
(a) A formula on the number of labelled graphs with the given order of automorphism groups of those graphs. (b) A formula on the number of unlabelled graphs with the given order of automorphism groups of those graphs. (c) A formula on the number of labelled self-complementary graphs with the given order of automorphism groups of those graphs. (d) A formula on the number of unlabelled self-complementary graphs with the given order of automorphism groups of those graphs.

A formula on the number of unlabelled graphs was given by Harary[3] by using Pólya theorem[6]. This number is found by taking the sum of the numbers in (b) over the orders of the automorphism groups of those graphs. The enumeration of unlabelled self-complementary graphs dates back to the Read’s investigation[7] in 1963 and Read[7] gave a formula on the number of unlabelled self-complementary graphs, by using De Bruijn’s theorem[1,2]. This number is found by taking the sum of the numbers in (d) over the orders of the automorphism groups of those graphs.

In the section 2 we consider to assign a number to each labelled graph. This assignment is very useful in the sense that from the number having been assigned to a graph, the graph can be uniquely determined. The section 3 discusses about the automorphism groups of graphs. The section 4 treats the
enumerations of labelled graphs and unlabelled graphs with the orders of automorphism groups of those graphs. In the section 6 we treat the enumerations of labelled self-complementary graphs, which is a main purpose in this paper, and the unlabelled self-complementary graphs.

2 The indexes of graphs

For a positive integer \( n \) let \( V \) be the set \( \{1, 2, ..., n\} \) and let \( \binom{V}{2} = \{\{i, j\} \subset V | i \neq j\} \). We here denote each element \( \{i, j\} \) of \( \binom{V}{2} \) by \( ij \) when \( i < j \) and by \( ji \) when \( i > j \). For a subset \( E \) of \( \binom{V}{2} \) we call the pair \( (V, E) \) a labelled graph or a graph on \( V \). \(|V|\) is called the order of the graph and \( E \) is called the edge-set of the graph. We define a bijection \( w \) from \( \binom{V}{2} \) to \( \{2^0 = 1, 2^1, 2^2, \cdots, 2^\lambda - 1\} \) by

\[
    w(ij) = 2^{\lambda - p_{ij}} \quad ij \in \binom{V}{2},
\]

where \( \lambda = \binom{n}{2} \) and \( p_{ij} = \frac{(2n - i)(i - 1)}{2} + (j - i) \) for \( ij \in \binom{V}{2} \). Let \( \mathcal{G}_n \) be the set of all labelled graphs of order \( n \) and let \( \mathcal{N}_n = \{0, 1, 2, 3, \cdots, 2^\lambda - 1\} \). We define a map \( N \) from \( \mathcal{G}_n \) to \( \mathcal{N}_n \) by

\[
    N(G) = \begin{cases} 
    \sum_{ij \in E} w(ij) = \sum_{ij \in E} 2^{\lambda - p_{ij}} & \text{if } E \text{ is not empty} \\
    0 & \text{if } E \text{ is empty.}
\end{cases}
\]

\( N(G) \) is called the index of \( G \). We consider the binary number representation of an integer \( L \) satisfying \( 0 \leq L \leq 2^\lambda - 1 \) and the binary number of \( L \) is denoted by \( L^{(2)} \), where \( L^{(2)} \) is a number of \( \binom{n}{2} \) figures. For example a decimal number 29 is represented as the binary number 011101, whose leading digit "0" is not omitted. In the binary number \( L^{(2)} \) of \( L \), the \( p_{ij} \)-th digit is written by \( L^{(2)}_{ij} \) for each element \( ij \in \binom{V}{2} \). The binary number of the index \( N(G) \) of a labelled graph \( G \) is denoted by \( N^{(2)}(G) \) and its \( p_{ij} \)-th digit is written by \( N^{(2)}_{ij}(G) \). The index \( N(G) \) of a labelled graph \( G = (V, E) \) has the following property:

\[
    N^{(2)}_{ij}(G) = 1 \quad \text{if and only if} \quad ij \in E.
\]

We first give the following result.

**Theorem 1.** The map \( N \) is a bijection.

**Proof.** Consider \( L \in \mathcal{N}_n \). If \( L = 0 \), then for the empty graph \( G \) whose edge-set is empty, we have \( N(G) = L \). Suppose \( L \neq 0 \). Then the graph \( G \) on \( V \) having the set \( E = \{ij \in \binom{V}{2} | L^{(2)}_{ij} = 1\} \) as its edge-set is an element of \( \mathcal{G}_n \) and it is obvious that \( N(G) = L \) holds. For \( G_1, G_2 \in \mathcal{G}_n \), if \( G_1 \) and \( G_2 \) are different, then \( N^{(2)}(G_1) \) is not equal to \( N^{(2)}(G_2) \). This fact implies that \( N(G_1) \) is not equal to \( N(G_2) \). Hence \( N \) is bijection. \( \square \)

As an illustration, consider a graph \( G = (V, E = \{13, 14, 23, 34\}) \) on \( V = \{1, 2, 3, 4\} \). Since \( p_{13} = 2, p_{14} = 3, p_{23} = 4 \) and \( p_{34} = 6 \), it is easy to check that \( N(G) = 29 \) and \( N^{(2)}(G) = 011101 \). Conversely, a binary number \( L^{(2)} = 011101 \) yields the labelled graph with the index \( L \), since \( L^{(2)}_{13} = L^{(2)}_{14} = L^{(2)}_{23} = L^{(2)}_{34} = 1 \).
and \(L_{12}^{(2)} = L_{24}^{(2)} = 0\). This graph is what was given in the beginning of this illustration.

The complement \(\overline{G}\) of a graph \(G = (V, E)\) is the graph on \(V\) with edge-set \(\left(\binom{V}{2}\right) - E\). The index \(N(\overline{G})\) of \(\overline{G}\) is \(2^\lambda - 1 - N(G)\). It is easy to see that \(N^{(2)}(\overline{G})\) is obtained by interchanging 0 and 1 in \(N^{(2)}(G)\).

## 3 Graph and group

Let \(A\) be the symmetric group on \(V\). In this paper we assume \(n \geq 3\). Each permutation \(\alpha \in A\) induces a permutation \(\alpha'\) which acts on \(\binom{V}{2}\) such that for every element \(ij \in \binom{V}{2}\),

\[
\alpha'\{i, j\} = \{\alpha i, \alpha j\}.
\]

The permutation group \(A' = \{\alpha' | \alpha \in A\}\) is called the pair group of \(A\). The degree of \(A'\) is \(\lambda\) and \(A \cong A'\), since \(n \geq 3\). For a graph \(G = (V, E)\) and \(\alpha \in A\), the graph which has \(\{(\alpha i, \alpha j) | ij \in E\}\) as its edge-set is denoted by \(\alpha G\).

For a graph \(G\), an element \(\alpha \in A\) satisfying \(\alpha G = G\) is called the automorphism of \(G\). The automorphism group \(\Gamma(G) = \{\alpha | \alpha G = G\}\) of \(G\) is called the group of \(G\), and the number of elements which belong to the group is called the group-order.

A graph \(G\) is said to be self-complementary if \(G\) and \(\overline{G}\) are isomorphic, that is, if there exists \(\alpha \in A\) such that \(\alpha G = \overline{G}\) holds. It is well-known that every self-complementary graph \(G\) has \(n \equiv 0, 1\) (mod 4) vertices. Note that while a self-complementary graph \(G\) is isomorphic to \(\overline{G}\), \(G\) and \(\overline{G}\) do not have the same index. Let \(S_n\) be the set of all labelled self-complementary graphs on \(V\). Then we define a set \(A_S = \{\alpha \in A | \exists G \in S_n, \alpha G = \overline{G}\}\) and we also define a set \(A_S(G) = \{\alpha \in A_S | \alpha G = \overline{G}\}\) for \(G \in S_n\). For \(\alpha' \in A'\) let \(Z(\alpha')\) be the set of all cycles in the disjoint cycle decomposition of \(\alpha'\). Then the following lemma can easily be seen from the definitions of \(\Gamma(G)\) and \(A_S(G)\).

### Lemma 1. @

(i) For \(\alpha \in A\) and \(G = (V, E) \in S_n\), \(\alpha\) is an element of \(\Gamma(G)\) if and only if for each \(z \in Z(\alpha')\),

\[
ij \in E \text{ if and only if } z(ij) \in E, \quad \forall ij \in z
\]

holds.

(ii) For \(\alpha \in A\) and \(G = (V, E) \in S_n\), \(G\) is a self-complementary graph satisfying \(\alpha G = \overline{G}\), that is, \(\alpha \in A_S(G)\) if and only if for each \(z \in Z(\alpha')\),

\[
ij \in E \text{ if and only if } z(ij) \notin E, \quad \forall ij \in z
\]

holds.

### Lemma 2. @

(i) For \(\alpha \in A\) and \(G \in S_n\), \(\alpha\) is an element of \(\Gamma(G)\) if and only if for each \(z \in Z(\alpha')\),

\[
N_{ij}^{(2)}(G) = 1 \text{ for } \forall ij \in z \quad \text{or} \quad N_{ij}^{(2)}(G) = 0 \text{ for } \forall ij \in z
\]

holds.
(ii) For \( \alpha \in A \) and \( G \in G_n \), \( G \) is a self-complementary graph satisfying \( \alpha G = \bar{G} \), that is, \( \alpha \in A_{S}(G) \) if and only if for each \( z \in Z(\alpha') \),

\[
N^{(2)}_{ij}(G) + N^{(2)}_{z(ij)}(G) = 1 \quad \text{for} \quad \forall ij \in z
\]

holds.

Proof. (i) is immediate from the combination of (3) and (i) in Lemma 1. We next prove (ii). Let \( E \) be the edge-set of \( G \). Then (3) can be written as follows:

\[
z(ij) \not\in E \quad \text{if and only if} \quad N^{(2)}_{z(ij)}(G) = 0.
\]

Therefore, it is easy to see from (3), (8) and (ii) in Lemma 1 that the desired result is obtained.

Theorem 2. Let \( K(P) \) be the set of all labelled graphs of order \( n \) all of which have a property \( P \) and let \( \xi \) be a positive integer. If the group-order of the group of every graph in \( K(P) \) is \( \xi \), then the number of graphs which belong to \( K(P) \) and which are not mutually isomorphic is, that is, the number of unlabelled graphs in \( K(P) \) is given by

\[
\frac{\xi|K(P)|}{n!}.
\]

Proof. Let \( l(G) \) be the number of ways of labelling a graph \( G \). Then using a well-known \( l(G) = \frac{n^n}{|\Gamma(G)|} \) [5; p4], we get the equality (9), since \( |\Gamma(G)| = \xi \) for \( G \in K(P) \). This complete the proof.

We have the following results with respect to self-complementary graphs.

Theorem 3. Suppose that \( G \) is an element of \( S_n \). Then for any \( \alpha \in A_{S}(G) \), \( \alpha^{-1}A_{S}(G) = \Gamma(G) \) holds.

Proof. It follows that for any \( \beta \in A_{S}(G) \), \( (\alpha^{-1}\beta)G = \alpha^{-1}(\beta G) = \alpha^{-1}\bar{G} = G \) holds. Therefore we have \( \alpha^{-1}\beta \in \Gamma(G) \), which implies \( \alpha^{-1}A_{S}(G) \subseteq \Gamma(G) \). Conversely, Any element \( \gamma \) of \( \Gamma(G) \) can be written as \( \gamma = \alpha^{-1}\alpha \gamma \). Thus \( (\alpha\gamma)G = \alpha(\gamma G) = \alpha G = \bar{G} \) is obtained. Therefore, we have \( \alpha\gamma \in A_{S}(G) \), that is, \( \gamma \in \alpha^{-1}A_{S}(G) \), which implies \( \Gamma(G) \subseteq \alpha^{-1}A_{S}(G) \). Hence \( \alpha^{-1}A_{S}(G) = \Gamma(G) \) holds.

Theorem 3 gives the following.

Corollary 1. For \( G \in S_n \), \( |A_{S}(G)| = |\Gamma(G)| \) holds.

For \( \alpha \in A \) let \( m_k(\alpha) \) be the number of cycles of length \( k \) in the disjoint cycle decomposition of \( \alpha \). Then we prove

Theorem 4. Let \( n \) be a positive integer satisfying \( n \equiv 0, 1 \pmod{4} \). Then \( \alpha \) is an element of \( A_{S} \) if and only if the following condition holds:

\[
m_1(\alpha) \leq 1 \quad \text{and} \quad m_k(\alpha) \begin{cases} \geq 1 & \text{if } k \equiv 0 \pmod{4} \\ = 0 & \text{if } k \not\equiv 0 \pmod{4} \end{cases}
\]

for \( k = 2, 3, \ldots, n \).
Proof. Suppose $\alpha \in A_S$. Then there exists $G \in S_n$ satisfying $\alpha G = \overline{G}$. Accordingly, it is easy to see that the length of each cycle in $Z(\alpha')$ must be even. This fact implies $m_1(\alpha) \leq 1$ and $m_k(\alpha) = 0$ for odd $k(\geq 3)$. If $\alpha$ has a cycle of length $2k$ providing $k$ is odd, then it generates a cycle of odd length $k$ in $\alpha'$ which contradicts the fact that the length of each cycle in $Z(\alpha')$ is even.

On the other hand, suppose that the permutation $\alpha$ of $A$ satisfies the conditions (10) and (11). Then the length of each cycle in $Z(\alpha')$ for the permutation $\alpha' \in A'$ induced by $\alpha$ is always even. Accordingly, we can construct the binary number $L^{(2)}$ of $\binom{n}{2}$ figures satisfying

$$L_{ij}^{(2)} + L_{z(ij)}^{(2)} = 1, \quad \forall ij \in z$$

for $z \in Z(\alpha')$. If we consider the labelled graph $G$ on $V$ which has $E = \{ij \in \binom{V}{2} | L_{ij}^{(2)} = 1\}$ as its edge-set, (ii) in Lemma 2 tells us that $G$ is a self-complementary graph of order $n$ satisfying $\alpha G = \overline{G}$. Hence we have $\alpha \in A_S$. \qed

4 Counting of graphs

For each $z \in Z(\alpha')$ in $\alpha \in A$, we consider

$$W(z) = \sum_{ij \in z} w(ij). \quad (12)$$

Then to $\alpha \in A$ there corresponds the following polynomial of $x$:

$$F^{(n)}(x) = \prod_{z \in Z(\alpha')} f_z^{(n)}(x), \quad (13)$$

where

$$f_z^{(n)}(x) = 1 + x^{W(z)}. \quad (14)$$

Let us consider the generating function for labelled graphs of order $n$

$$F(n)(x) = \sum_{\alpha \in A} F^{(n)}(x). \quad (15)$$

Let $c_L^{(\alpha)}$ be the coefficient of $x^L$ in the power series expansion of $F^{(n)}(x)$ for $\alpha \in A$, and let $c_L$ be the coefficient of $x^L$ in the power series expansion of $F(n)(x)$. Then the following equality is an immediate consequence of (15):

$$c_L = \sum_{\alpha \in A} c_L^{(\alpha)}. \quad (16)$$

Let $f(x)$ be a polynomial of $x$. Then by the symbol $a_n x^n \vdash f(x)$, $a_n x^n$ will mean the term in the polynomial $f(x)$. On the other hand, If $a_n x^n$ is not any term of $f(x)$, that is, if $a_n x^n \not\vdash f(x)$, we use the convention $a_n = 0$.

Let $lq(n) = |G_n|$. Then $lq(n)$ is equal to $2^\lambda$, since $|\mathcal{G}_n| = 2^\lambda$, as well-known. Consider an $n!$ by $lq(n)$ matrix $M(\mathcal{G}_n) = ||c_L^{(\alpha)}||$, which has a row for each $\alpha \in A$ and a column for each $G \in \mathcal{G}_n$, provided that $c_L^{(\alpha)} = 0$ when $c_L^{(\alpha)} x^L \not\vdash F(n)(x)$.

Since by Theorem 1 there is a one-to-one correspondence between $\mathcal{G}_n$ and $\mathfrak{G}_n$, the column corresponding to $G \in \mathcal{G}_n$ is referred to as the $N(G)$-column.
Property 1. In the matrix $M(G_n)$, if $c_L^{(a)} \neq 0$, then $c_L^{(a)} = 1$ holds.

Proof. Consider $\alpha \in A$. If $L = 0$, then since the constant term in the power series expansion of $F_\alpha^{(n)}(x)$ is 1, we have $c_L^{(a)} = 1$. Suppose that $L \neq 0$. Then note first that $c_L^{(a)} x^L = F_\alpha^{(n)}(x)$, since $c_L^{(a)} \neq 0$. From the observation of [13], there exists a subset $K = \{z_1, z_2, \cdots, z_i\}$ of $\mathbb{Z}(\alpha')$ such that $L = W(z_1) + W(z_2) + \cdots + W(z_i)$. It can easily seen that the binary number $L^{(2)}$ satisfies

$$L^{(2)}_{ij} = 1, \forall ij \in z, \forall z \in K \quad \text{and} \quad L^{(2)}_{ij} = 0, \forall ij \in z, \forall z \in Z(\alpha') - K.$$

Thus if we consider the labelled graph $G \in G_n$ whose edge-set is $\{ij \in (V)|L^{(2)}_{ij} = 1\}$, then it follows that its index $N(G)$ is $L$ and that $\alpha G = G$ holds by (i) in Lemma 2. Since $L^{(2)}$ is determined by $L$, vice versa, $L$ is determined by $L^{(2)}$, the subset $K$ being discussed just above can be uniquely determined by $L$. This implies that $c_L^{(a)} = 1$. □

This property states that for $\alpha \in A$, every coefficient of the power series expansion of $F_\alpha^{(n)}(x)$ is 1.

Property 2. The coefficient $c_L$ of $x^L$ in $F^{(n)}(x)$ is equal to the group-order of the group of labelled graph having the index $L$.

Proof. In the matrix $M(G_n)$, [$\mathbb{I}\mathbb{I}$] denotes the $L$-th column sum. The $L$-th column is associated with the labelled graph $G$ having the index $N(G) = L$, and [$\mathbb{I}\mathbb{I}$] becomes $c_L = \sum_{\alpha \in \Gamma(G)} c_L^{(a)}$. Applying Property 1, we get $c_L = |\Gamma(G)|$. □

If we consider two sets: $O(G_n) = \{|\Gamma(G)||G \in G_n\}$ and $C(G_n) = \{c_L|c_L x^L = F^{(n)}\}$, then Property 2 gives the following theorem, which states that the list of the group-orders of the groups of all graphs of order $n$ can be obtained by expanding $F^{(n)}(x)$.

Theorem 5. $O(G_n) = C(G_n)$ holds.

For $\xi \in O(G_n)$, let $lg(n; \xi)$ be the number of labelled graphs $G$ of order $n$ such that $|\Gamma(G)| = \xi$, and let $g_\xi^{(n)}$ be the number of unlabelled graphs $G$ of order $n$ such that $|\Gamma(G)| = \xi$. For a positive integer $\xi$, put $L_\xi(G_n) = \{L|c_L = \xi, c_L x^L \models F^{(n)}\}$. Then we have the following three theorems.

Theorem 6. The equality

$$lg(n; \xi) = |L_\xi(G_n)|, \quad \xi \in C(G_n)$$

(17)

holds.

Proof. As seen from the statement of Property 2 and its proof, each labelled graph $G$ in $G_n$ having $|\Gamma(G)| = \xi$ is associated with the $N(G)$-th column of $M(G_n)$ whose column sum is $\xi$, and each element of $L_\xi(G_n)$ is also associated with a column of $M(G_n)$ whose column sum is $\xi$. Accordingly, $lg(n; \xi)$ is the number of columns of $M(G_n)$ whose column sum is $\xi$ and also $|L_\xi(G_n)|$ is the number of columns of $M(G_n)$ whose column sum is $\xi$. Hence we obtain the equality (17), noting Theorem 5. □
Theorem 7. The equality

\[ g^{(n)}_\xi = \frac{\xi}{n!} |L_\xi(G_n)|, \quad \xi \in C(G_n) \]  

holds.

Proof. In Theorem 2, let \( P \) denote the property that the group-order of the group of a labelled graph in \( G_n \) is \( \xi \). Then \( \mathcal{K}(P) \) is the set of all labelled graphs \( G \) in \( G_n \) satisfying \( |\Gamma(G)| = \xi \), that is, \( |\mathcal{K}(P)| = lg(n; \xi) \). Hence using Theorem 2, Theorem 5 and Theorem 6, the equality (18) is obtained.

Let \( g(n) \) be the number of unlabelled graphs of order \( n \). A formula on \( g(n) \) has been given by Harary[3]. But an enumeration method having been here presented will derive a formula on \( g(n) \), which is stated in the following.

Theorem 8. The number \( g(n) \) of unlabelled graphs of order \( n \) is given by

\[ g(n) = \sum_{\xi \in C(G_n)} \frac{\xi}{n!} |L_\xi(G_n)|. \]  

Proof. Since \( g(n) = \sum_{\xi \in C(G_n)} g^{(n)}_\xi \), using Theorem 5 and Theorem 7 we get the equality (19).

5 Numerical results for graphs

We have the numerical examples for \( n \geq 3 \), since \( n \geq 3 \) in this paper. Table 1 gives the numerical examples for labelled graphs. Each pair \( (\xi, lg(n; \xi)) \) in the Table 1 shows the number \( lg(n; \xi) \) of labelled graphs of order \( n \) with the group-order \( \xi \) for a few small values of \( n \geq 3 \). The last column of Table 1 shows the number \( lg(n) \) of labelled graphs of order \( n \), which is well-known. On the other hand, Table 2 gives the numerical examples for unlabelled graphs. Each pair \( (\xi, g^{(n)}_\xi) \) in the Table 2 shows the number \( g^{(n)}_\xi \) of unlabelled graphs of order \( n \) with the group-order \( \xi \). The last column of Table 2 shows the number \( g(n) \) of unlabelled graphs of order \( n \).
Table 1: Labelled graphs

| n | (ξ, lg(n; ξ)) | lg(n) |
|---|----------------|-------|
| 3 | (2, 6), (6, 2) | $2^3$ |
| 4 | (2, 36), (4, 12), (6, 8), (8, 6), (24, 2) | $2^6$ |
| 5 | (2,660), (4,180), (6,40), (8,60), (10,12), (12,60), (24,10), (120,2) | $2^{10}$ |
| 6 | (1,5760), (2,16560), (4,6480), (6,960), (8,1260), (10,144), (12,1080), (16,270), (24,60), (36,40), (48,120), (72,20), (120,12), (720,2) | $2^{15}$ |
| 7 | (1,766080), (2,892080), (4,312480), (6,31920), (8,46620), (10,1008), (12,29400), (14,720), (16,6300), (20,270), (24,60), (36,840), (48,2940), (72,280), (120,84), (144,210), (240,126), (720,14), (5040,2) | $2^{21}$ |

6 Counting of self-complementary graphs

In this section we make a consideration about the enumeration of self-complementary graphs. This enumeration is made along the argument in the preceding section. Let $n$, throughout this section, be an integer satisfying $n \equiv 0, 1 \pmod{4}$ and $n \geq 4$. For $\alpha \in A_S$, the length of each cycle in $Z(\alpha')$ is always even. We denote a cycle $z \in Z(\alpha')$ by $z = (i_1j_1, i_2j_2, \cdots, i_{k-1}j_{k-1}, i_kj_k)$. The cycle can be represented in some forms in the usual cyclic representation, as well-known. For example, the cycle(12, 23, 34, 14) can also be written (23, 34, 14, 12), (34, 14, 12, 23) or (14, 12, 23, 34). We assume that the leading position of $z$ is occupied by the element at which the value of $w$ is the maximum among those elements of $z$. Let $P_1(z) = \{i_1j_1, i_3j_3, \cdots, i_{k-1}j_{k-1}\}$ be the set of elements in the odd position in the cycle $z$ and let $P_2(z) = \{i_2j_2, i_4j_4, \cdots, i_kj_k\}$ be the set of elements in the even position in $z$. Then, note that for $ij \in z$, $ij \in P_1(z)$ if and only if $z(ij) \in P_2(z)$. For each $z \in Z(\alpha')$ in $\alpha \in A_S$, we consider

$$W_1(z) = \sum_{ij \in P_1(z)} w(ij) \quad \text{and} \quad W_2(z) = \sum_{ij \in P_2(z)} w(ij). \quad (20)$$

Then to $\alpha \in A_S$ there corresponds the following polynomial of $x$:

$$U^{(n)}_\alpha(x) = \prod_{z \in Z(\alpha')} u^{(n)}_z(x), \quad (21)$$

where

$$u^{(n)}_z(x) = x^{W_1(z)} + x^{W_2(z)}, \quad z \in Z(\alpha'). \quad (22)$$
Table 2: Unlabelled graphs

| n  | (ξ, g_ξ(n))                    | g(n) |
|-----|--------------------------------|------|
| 3   | (2,2), (6,2)                   | 4    |
| 4   | (2,3), (4,2), (6,2), (8,2), (24,2) | 11   |
| 5   | (2,11), (4,6), (6,2), (8,4), (10,1), (12,6), (24,2), (120,2) | 34   |
| 6   | (1.8), (2.46), (4.36), (6.8), (8.14), (10.2), (12.18), (16.6), (24.2), (36.2), (48.8), (72.2), (120.2), (720.2) | 156  |
| 7   | (1.152), (2.354), (4.248), (6.38), (8.74), (10.2), (12.70), (14.2), (16.20), (20.4), (24.24), (36.6), (48.28), (72.4), (120.2), (144.6), (240.6), (720.2), (5040.2) | 1044 |

Let us consider the generating function for labelled self-complementary graphs of order $n$

$$U(n)(x) = \sum_{\alpha \in A_S} U_\alpha(n)(x).$$  \hfill (23)

Let $lsc(n) = |S_n|$ be the number of labelled self-complementary graphs of order $n$. Then we shall show that $lsc(n)$ is equal to the number of terms in the power series expansion of $U(n)(x)$. For $G \in S_n$, there exists $\alpha \in A_S$ satisfying $\alpha G = G$.

It is seen from (ii) in Lemma 2 that the index $N(G)$ of $G$ satisfies

$$N_{ij}(G) + N_{z(ij)}(G) = 1, \forall ij \in z$$ \hfill (24)

for each $z \in Z(\alpha')$. For $z \in Z(\alpha')$, if we put $P(z) = \{ij \in z|N_{ij}^{(2)}(G) = 1\}$, then it follows from (24) that $P(z)$ agrees with either $P_1(z)$ or $P_2(z)$. Furthermore, if we put $W(z) = \sum_{ij \in P(z)} w(ij)$ for each $z \in Z(\alpha')$, then it is easy to check that $N(G) = \sum_{z \in Z(\alpha')} W(z)$. Thus $\prod_{z \in Z(\alpha')} x^{W(z)} = x^{N(G)}$ is a term in the power series expansion of $U_\alpha(n)(x)$, which implies that this term is appeared in the power series expansion of $U(n)(x)$. Since $N$ is a bijection from $G_n$ to $\mathfrak{H}_n$ by Theorem 1, we have the following theorem.

**Theorem 9.** Let $\mathcal{L}(S_n) = \{L|d_Lx^L \vdash U(n)(x)\}$. Then the equality $lsc(n) = |\mathcal{L}(S_n)|$ holds.

Let $d_L^{(n)}$ be the coefficient of $x^L$ in the power series expansion of $U_\alpha(n)(x)$ for $\alpha \in A_S$, and let $d_L$ be the coefficient of $x^L$ in the power series expansion of
Consider an \(|A_S|\) by \(lsc(n)\) matrix \(M(S_n) = ||d_L^{(\alpha)}||\), which has a row for each \(\alpha \in A_S\) and a column for each \(G \in S_n\), provided that \(d_L^{(\alpha)} = 0\) when \(d_L^{(\alpha)} x_L \neq U^{(n)}_\alpha(x)\). \(M(S_n)\) may be regarded as a submatrix of \(M(G_n)\), if \(c_L^{(\alpha)}\) in the latter is replaced by \(d_L^{(\alpha)}\) in the former. Thus the column corresponding to \(G \in S_n\) is referred to as the \(N(G)\)-column.

**Property 3.** In the matrix \(M(S_n)\), if \(d_L^{(\alpha)} \neq 0\), then \(d_L^{(\alpha)} = 1\) holds.

Proof. Consider \(\alpha \in A_S\). Note first that \(d_L^{(\alpha)} x_L = U^{(n)}_\alpha(x)\), since \(d_L^{(\alpha)} \neq 0\). Also, note that \(L \neq 0\), since we assume that \(n \geq 4\). Put \(Z(\alpha') = \{z_1, z_2, \ldots, z_k\}\) and put \(I = \{1, 2\}\). Then from the observation of \((24)\) there exists a \(k\)-tuple \((l_1, l_2, \ldots, l_k) \in I \times I \times \cdots \times I\) such that \(L = W_{l_1}(z_1) + W_{l_2}(z_2) + \cdots + W_{l_k}(z_k)\).

It can easily seen that the binary number \(L^{(2)}\) satisfies

\[
L_{ij}^{(2)} = 1, \quad \forall i, j \in P_{l_1}(z_1) \cup P_{l_2}(z_2) \cup \cdots \cup P_{l_k}(z_k)
\]

\[
L_{ij}^{(2)} = 0, \quad \forall i, j \in P_{l_1'}(z_1) \cup P_{l_2'}(z_2) \cup \cdots \cup P_{l_k'}(z_k),
\]

where \(l_h' \neq l_h, l_k' \in I (h = 1, 2, \ldots, k)\). Thus if we consider the labelled graph \(G \in G_n\) whose edge-set is \(\{ij \in (\cup)^2 | L_{ij}^{(2)} = 1\}\), then it is obvious that its index \(N(G)\) is \(L\), and (ii) in Lemma \((2)\) tells us that \(G\) is a self-complementary graph satisfying \(\alpha \in A_S(G)\). As seen at the end of the proof of Property \((4)\) since \(L^{(2)}\) is determined by \(L\), vice versa, \(L\) is determined by \(L^{(2)}\), the \(k\)-tuple \((l_1, l_2, \ldots, l_k)\) being discussed just above can be uniquely determined by \(L\). This implies that \(d_L^{(\alpha)} = 1\).

This property states that for \(\alpha \in A_S\), every coefficient of the power series expansion of \(U^{(n)}_S(x)\) is 1.

**Property 4.** The coefficient \(d_L\) of \(x_L\) in \(U^{(n)}(x)\) is equal to the group-order of the group of labelled self-complementary graph having the index \(L\).

Proof. In the matrix \(M(S_n)\), \((25)\) denotes the \(L\)-th column sum. The \(L\)-th column is associated with the labelled self-complementary graph \(G\) having the index \(N(G) = L\), and \((25)\) becomes \(d_L = \sum_{\alpha \in A_S(G)} d_L^{(\alpha)}\). Applying Property \((3)\) we get \(d_L = |A_S(G)|\). Hence we have \(d_L = |\Gamma(G)|\) from Corollary \((4)\).

If we consider two sets: \(O(S_n) = \{||\Gamma(G)|| G \in S_n\}\) and \(D(S_n) = \{d_L | d_L x_L = U^{(n)}\}\), then Property \((3)\) gives the following theorem, which states that the list of the group-orders of the groups of all self-complementary graphs of order \(n\) can be obtained by expanding \(U^{(n)}(x)\).

**Theorem 10.** \(O(S_n) = D(S_n)\) holds.
For $\xi \in \mathcal{O}(S_n)$, let $lsc(n; \xi)$ be the number of \textit{labelled} self-complementary graphs $G$ of order $n$ such that $|\Gamma(G)| = \xi$, and let $s_\xi^{(n)}$ be the number of \textit{unlabelled} self-complementary graphs $G$ of order $n$ such that $|\Gamma(G)| = \xi$. For a positive integer $\xi$, put $L_\xi(S_n) = \{ L | d_L = \xi, d_L x^L \models U^{(n)}(x) \}$. Then we have the following three theorems.

**Theorem 11.** The equality

$$lsc(n; \xi) = |L_\xi(S_n)|, \quad \xi \in \mathcal{D}(S_n)$$

(26)

holds.

**Proof.** As seen from the statement of Property 4 and its proof, each labelled graph $G$ in $S_n$ having $|\Gamma(G)| = \xi$ is associated with the $N(G)$-th column of $M(S_n)$ whose column sum is $\xi$, and each element of $L_\xi(S_n)$ is also associated with a column of $M(S_n)$ whose column sum is $\xi$. Accordingly, $lsc(n; \xi)$ is the number of columns of $M(S_n)$ whose column sum is $\xi$, and also $|L_\xi(S_n)|$ is the number of columns of $M(S_n)$ whose column sum is $\xi$. Hence we obtain the equality (26), noting Theorem 10.

**Theorem 12.** The equality

$$s_\xi^{(n)} = \frac{\xi}{n!} |L_\xi(S_n)|, \quad \xi \in \mathcal{D}(S_n)$$

(27)

holds.

**Proof.** In Theorem 2, let $P$ denote the property that the group-order of the group of a labelled self-complementary graph in $S_n$ is $\xi$. Then $K(P)$ is the set of all labelled self-complementary graphs $G$ in $S_n$ satisfying $|\Gamma(G)| = \xi$, that is, $|K(P)| = lsc(n; \xi)$. Hence using Theorem 2, Theorem 10 and Theorem 11, the equality (27) is obtained.

Let $sc(n)$ be the number of unlabelled self-complementary graphs of order $n$. A formula on $sc(n)$ has been given by Read[7]. But an enumeration method having been here presented will derive a formula on $sc(n)$, which is stated in the following.

**Theorem 13.** The number $sc(n)$ of unlabelled self-complementary graphs of order $n$ is given by

$$sc(n) = \sum_{\xi \in \mathcal{D}(S_n)} \frac{\xi}{n!} |L_\xi(S_n)|.$$  

(28)

**Proof.** Since $sc(n) = \sum_{\xi \in \mathcal{O}(S_n)} s_\xi^{(n)}$, using Theorem 10 and Theorem 12 we get the equality (28).

7 Numerical results for self-complementary graphs

This section gives the numerical results for self-complementary graphs of a few small orders $n$ satisfying $n \equiv 0, 1(\mod 4)$. Table 3 gives the numerical examples for labelled self-complementary graphs. Each pair $(\xi, lsc(n; \xi))$ in the
Table 3 shows the number \( lsc(n; \xi) \) of labelled self-complementary graphs of order \( n \) with the group-order \( \xi \). The last column of Table 3 shows the number \( lsc(n) \) of labelled self-complementary graphs of order \( n \). On the other hand, Table 4 gives the numerical examples for unlabelled self-complementary graphs. Each pair \( (\xi, s^{(n)}_\xi) \) in the Table 4 shows the number \( s^{(n)}_\xi \) of unlabelled self-complementary graphs of order \( n \) with the group-order \( \xi \). The last column of Table 4 shows the number \( sc(n) \) of unlabelled self-complementary graphs of order \( n \).

Table 3: Labelled self-complementary graphs

| \( n \) | \( (\xi, lsc(n; \xi)) \) | \( lsc(n) \) |
|---|---|---|
| 4 | (2, 12) | 12 |
| 5 | (2, 60), (10, 12) | 72 |
| 8 | (2, 60480), (4, 20160), (8, 15120), (32, 2520) | 98280 |
| 9 | (2,3265920), (4,544320), (8,226800), (20,36288), (32,45360), (72,5040) | 4123728 |

Table 4: Unlabelled self-complementary graphs

| \( n \) | \( (\xi, s^{(n)}_\xi) \) | \( sc(n) \) |
|---|---|---|
| 4 | (2, 1) | 1 |
| 5 | (2, 1), (10, 1) | 2 |
| 8 | (2, 3), (4, 2), (8, 3), (32, 2) | 10 |
| 9 | (2, 18), (4, 6), (8, 5), (20, 2), (32, 4), (72, 1) | 36 |

Acknowledgements

The author wishes to express his deepest gratitude to Professor Fumikazu Tamari, who is in Fukuoka University of Education, for frequent, stimulating, and helpful discussions, and for his kind suggestions with respect to the format of this paper. The author also wishes to thank Professors Yasuo Ohno and Tsunenobu Asai, who are in Department of Mathematics, Kinki University, for their helpful advice.
References

[1] N.G. de Bruijn, Generalization of Pólya’s fundamental theorem in enumeration combinatorial analysis, Indagationes Math. 21(1959), 59-69.

[2] N.G. de Bruijn, Pólya’s theory of counting, in: E. F. Beckenbach (Ed.), Applied Combinatorial Mathematics: Wiley, New York, 1964, pp. 144-184.

[3] F. Harary, The number of linear, directed, rooted, and connected graphs, Trans. Amer. Math. Soc. 78(1955), 445-463.

[4] F. Harary, Graph Theory, Addison-Wesley Pub., Massachusetts, 1972.

[5] F. Harary and E.M. Palmer, Graphical Enumeration, Academic Press, New York and London, 1973.

[6] G. Polya, Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen, Acta Math. 68(1937), 145-254.

[7] R.C. Read, On the number of self-complementary graphs and digraphs, Journal London Math. Soc. 38(1963), 99-104.