Quasi-exactly-solvable confining solutions for spin-1 and spin-0 bosons in (1+1)-dimensions with a scalar linear potential

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Abstract

We point out a misleading treatment in the recent literature regarding confining solutions for a scalar potential in the context of the Duffin-Kemmer-Petiau theory. We further present the proper bound-state solutions in terms of the generalized Laguerre polynomials and show that the eigenvalues and eigenfunctions depend on the solutions of algebraic equations involving the potential parameter and the quantum number.

Keywords: Duffin-Kemmer-Petiau theory, scalar potential, bound state
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1. Introduction

The Duffin-Kemmer-Petiau (DKP) formalism \cite{1, 2, 3, 4} describes spin-0 and spin-1 bosons and has been used to analyze relativistic interactions of spin-0 and spin-1 hadrons with nuclei as an alternative to their conventional second-order Klein-Gordon (KG) and Proca counterparts (see, e.g. \cite{5} for a comprehensive list of references). The DKP formalism enjoys a richness of couplings not capable of being expressed in the KG and Proca theories \cite{6, 7}. Although the formalisms are equivalent in the case of minimally coupled vector interactions \cite{8, 9, 10}, the DKP formalism opens new horizons as far as it allows other kinds of couplings which are not possible in the KG and Proca theories.

The scalar interaction refers to a kind of coupling that behaves like a scalar (invariant) under a Lorentz transformation. Though the scalar interaction finds many of their applications in nuclear and particle physics, it could also simulate an effective mass in solid state physics. Due to weak potentials, relativistic effects are considered to be small in solid state physics, but the relativistic wave equations can give relativistic corrections to the results obtained from the nonrelativistic wave equation, therefore the relativistic extension of this problem is also of interest and remains unexplored. The scalar interaction in the context of the DKP theory has been reported in the literature for a smooth step potential \cite{11}, step potential \cite{12}, Coulomb potential \cite{13} and linear potential \cite{14}. In Ref. \cite{14}, the authors examined the time-independent DKP equation in a (1+1)-dimension with the scalar linear potential and claimed to have obtained exact conditions selecting the eigenvalues. In that paper, the authors misidentified the correct asymptotic behavior of Kemmer’s function, kept out Kummer’s function and used Tricomi’s function as a particular solution.

The purpose of this paper is to review the DKP equation in the presence of a scalar linear potential for spin-1 and spin-0 bosons in (1+1)-dimensions. Following the appropriate \textit{modus operandi}, we show that the problem is exactly solvable on the whole line for a restrict class of potential parameters and quantum numbers. In this circumstance, the eigenenergies are solutions of algebraic equations and the eigenfunctions are expressed in terms of the generalized Laguerre polynomials.

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2. The Duffin-Kemmer-Petiau equation

The DKP equation for a free boson is given by \( (i\beta^\mu \partial_\mu - m) \Psi = 0 \) (1)

where the matrices \( \beta^\mu \) satisfy the algebra

\[
\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu \nu} \beta^\lambda + g^{\lambda \nu} \beta^\mu
\] (2)

and the metric tensor is \( g^{\mu \nu} = \text{diag}(1, -1, -1, -1) \). The algebra expressed by (2) generates a set of 126 independent matrices whose irreducible representations are a trivial representation, a five-dimensional representation describing the spin-0 particles and a ten-dimensional representation associated to spin-1 particles.

The second-order KG and Proca equations are obtained when one selects the spin-0 and spin-1 sectors of the DKP theory. A well-known conserved four-current is given by

\( J^\mu = \frac{1}{2} \bar{\Psi} \beta^\mu \Psi \) (3)

where the adjoint spinor \( \bar{\Psi} = \Psi^\dagger \eta_0 \), with \( \eta_0 = 2 \beta^0 \beta^0 - 1 \) in such a way that \( (\eta^0 \beta^\mu)^\dagger = \eta^0 \beta^\mu \) (the matrices \( \beta^\mu \) are Hermitian with respect to \( \eta^0 \)). Despite the similarity to the Dirac equation, the DKP equation involves singular matrices, the time component of \( J^\mu \) is not positive definite but it may be interpreted as a charge density. The factor \( 1/2 \) multiplying \( \bar{\Psi} \beta^\mu \Psi \), of no importance regarding the conservation law, is in order to hand over a charge density conformable to that one used in the KG theory and its nonrelativistic limit [5]. Then the normalization condition

\[
\int d\tau \ J^0 = \pm 1
\] (4)

can be expressed as

\[
\int d\tau \bar{\Psi} \beta^0 \Psi = \pm 2
\] (5)

where the plus (minus) sign must be used for a positive (negative) charge.

2.1. Interaction in the DKP equation

With the introduction of interactions, the DKP equation can be written as

\( (i\beta^\mu \partial_\mu - m - U) \Psi = 0 \) (6)

where the more general potential matrix \( U \) is written in terms of 25 (100) linearly independent matrices pertinent to five (ten)-dimensional irreducible representation associated to the scalar (vector) sector. In the presence of interaction, \( J^\mu \) satisfies the equation

\[
\partial_\mu J^\mu + \frac{i}{2} \bar{\Psi} (U - \eta^0 U^\dagger \eta^0) \Psi = 0
\] (7)

Thus, if \( U \) is Hermitian with respect to \( \eta^0 \) then four-current will be conserved. The potential matrix \( U \) can be written in terms of well-defined Lorentz structures. For the spin-zero sector there are two scalar, two vector and two tensor terms [6], whereas for the spin-one sector there are two scalar, two vector, a pseudoscalar, two pseudovector and eight tensor terms [7]. The tensor terms have been avoided in applications because they furnish noncausal effects [6,7]. The condition (7) has been used to point out a misleading treatment in the recent literature regarding analytical solutions for nonminimal vector interactions [15,16,17,18].
2.2. Scalar coupling in the DKP equation

Considering only scalar interaction, the DKP equation can be written as

\[(i\beta^\mu \partial_\mu - m - S) \Psi = 0\] (8)

with \(S\) denoting the scalar potential function.

For the case of spin-0, we use the representation for the \(\beta^\mu\) matrices given by [19]

\[\beta^0 = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & \rho^i \\ -\rho^i & 0 \end{pmatrix}, \quad i = 1, 2, 3\] (9)

where

\[\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\]

\[\rho_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\] (10)

\(\bar{0}, \tilde{0}\) and \(0\) are 2×3, 2×2 and 3×3 zero matrices, respectively, while the superscript \(T\) designates matrix transposition. The five-component spinor can be written as \(\Psi^T = (\Psi_1, ..., \Psi_5)\) in such a way that the DKP equation for a boson constrained to move along the \(x\)-axis decomposes into

\[-i (m + S) \Psi_2, \quad \partial_1 \Psi_1 = -i (m + S) \Psi_3\]

\[\partial_0 \Psi_2 - \partial_1 \Psi_3 = -i (m + S) \Psi_1\]

\[\Psi_4 = \Psi_5 = 0\] (11)

and \(J^\mu\) can be written as

\[J^0 = \text{Re}(\Psi_2^* \Psi_1), \quad J^1 = -\text{Re}(\Psi_3^* \Psi_1), \quad J^2 = J^3 = 0\] (12)

For the case of spin-1, the \(\beta^\mu\) matrices are [20]

\[\beta^0 = \begin{pmatrix} 0 & \bar{0} & \bar{0} & \bar{0} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & \bar{0} & e_i & \bar{0} \\ 0 & 0 & 0 & -is_i \\ -e_i & 0 & 0 & 0 \\ -is_i & 0 & 0 & 0 \end{pmatrix}\] (13)

where \(s_i\) are the 3×3 spin-1 matrices \((s_i)_{jk} = -i\delta_{ij}\), \(e_i\) are the 1×3 matrices \((e_i)_{1j} = \delta_{ij}\) and \(\bar{0} = (0 \ 0 \ 0)\),

while \(1\) and \(0\) designate the 3×3 unit and zero matrices, respectively. The spinor \(\Psi^T = (\Psi_1, ..., \Psi_{10})\) can be partitioned as

\[\Psi_7^T = (\Psi_3, \Psi_4, \Psi_5), \quad \Psi_{II}^T = (\Psi_6, \Psi_7, \Psi_2), \quad \Psi_{III}^T = (\Psi_{10}, -\Psi_9, \Psi_1)\] (14)

so that the one-dimensional DKP equation can be expressed in the form

\[-i (m + S) \Psi_{II}, \quad \partial_1 \Psi_{II} = -i (m + S) \Psi_{III}\]

\[\partial_0 \Psi_{II} - \partial_1 \Psi_{III} = -i (m + S) \Psi_I\]

\[\Psi_8 = 0\] (15)
In addition, expressed in terms of (14) the current can be written as
\[ J^0 = \text{Re} \left( \Psi_I^\dagger \Psi_I \right), \quad J^1 = -\text{Re} \left( \Psi_{II}^\dagger \Psi_I \right), \quad J^2 = J^3 = 0 \] (16)

Comparison of (11) with (15) evidences that the spinors \( \Psi_I, \Psi_{II} \) and \( \Psi_{III} \) behave like the spinor components \( \Psi_1, \Psi_2 \) and \( \Psi_3 \), respectively, from the spin-0 sector of the DKP theory. More than this, comparison of (12) with (16) places on view that the spin-1 sector of the DKP theory looks formally like the spin-0 sector [12].

For a time-independent scalar potential, one can write \( \Psi(x,t) = \psi(x) \exp(-iEt) \). With the abbreviations
\[ \phi_1 = \psi_1(\psi_I), \quad \phi_2 = \psi_2(\psi_{II}) \] and \( \phi_3 = \psi_3(\psi_{III}) \) the time-independent DKP equation for the spin-0 (spin-1) sector of the DKP theory splits into
\[ \frac{d}{dx} \left( \frac{1}{m+S} \frac{d\phi}{dx} \right) + \frac{E^2 - (m+S)^2}{m+S} \phi = 0 \] (17)
\[ \phi_2 = \frac{E}{m+S} \phi \] (18)
\[ \phi_3 = \frac{i}{m+S} \frac{d\phi}{dx} \] (19)

with
\[ J^0 = \frac{E}{m+S} |\phi|^2, \quad J^1 = \frac{1}{m+S} \text{Im} \left( \phi_1^d \frac{d\phi}{dx} \right) \] (20)

2.3. Linear potential

In Ref. [14], the authors used
\[ S(x) = \lambda |x|, \quad \lambda > 0 \] (21)
and the changes
\[ z = 1 + \frac{\lambda}{m} |x|, \quad \phi(z) = z^{1+\beta} e^{-z^2/2g} f(z) \] (22)
with the definitions
\[ g = \frac{\lambda}{m^2}, \quad \varepsilon = \frac{E}{gm}, \quad \beta^2 = 1 \] (23)

With the additional change of variable
\[ t = \frac{z^2}{g} \] (24)
they finally arrived at
\[ \frac{d^2 f(t)}{dt^2} + (\beta + 1 - t) \frac{df(t)}{dt} + \left( \frac{g\varepsilon^2}{4} - \frac{\beta + 1}{2} \right) f(t) = 0 \] (25)

and presented the solution
\[ f(t) = AM(a,b,t) + BU(a,b,t) \] (26)
where \( A \) and \( B \) are arbitrary constants, \( M(a,b,t) \) and \( U(a,b,t) \) are Kummer’s and Tricomi’s functions respectively, and
\[ a = \frac{\beta + 1}{2} - \frac{g\varepsilon^2}{4}, \quad b = \beta + 1 \] (27)

All of their remaining analysis rested on identifying Eq. (26) with the general solution of the confluent hypergeometric equation. Furthermore, they assumed that the asymptotic behaviour of Kummer’s function is given by \( e^t t^{a-b} \), kept out Kummer’s function and used Tricomi’s function as a particular solution.
3. Solutions on the half line

For solutions of the confluent hypergeometric equation and their properties we refer to Abramowitz and Stegun [21]. Kummer’s function is expressed as

\[ M(a, b, w) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)}{\Gamma(b+j)} \frac{w^j}{j!} = 1 + \frac{aw}{b} + \frac{a(a+1)w^2}{b(b+1)2!} + \ldots \] (28)

where \( \Gamma(w) \) is the gamma function with simple poles at \( w = 0, -1, -2, -3, \ldots \) and the defining property \( \Gamma(w+1) = w\Gamma(w) \). On the other hand, Tricomi’s function is expressed in terms of Kummer’s function as

\[ U(a, b, w) = \frac{\pi}{\sin \pi b} \left( \frac{M(a, b, w)}{\Gamma(1+a-b)} - w^{1-b}M(1+a-b, 2-b, w) \right) \] (29)

Notice that \( M(a, b, w) \) corresponds to \( F(a, b, w) \) in Ref. [14]. Notice also that the identity \( \Gamma(w)\Gamma(1-w) = \pi/\sin \pi w \) carries Eq. (29) on exactly the same form as Eq. (28) in Ref. [14]. The Wronskian of \( M(a, b, w) \) and \( U(a, b, w) \) is given by

\[ W(M, U) = -\frac{\Gamma(b)}{\Gamma(a)} w^{-b} e^w \] (30)

so that these functions are linearly independent solutions of the confluent hypergeometric equation only if \( a \neq -n \), where \( n \) is a nonnegative integer, and \( b \neq -1, -2, -3, \ldots \). Furthermore, these functions present asymptotic behaviours as \( |w| \to \infty \) dictated by

\[ M(a, b, w) \sim \frac{\Gamma(b)}{\Gamma(b-a)} e^{-i\pi a} w^{-a} + \frac{\Gamma(b)}{\Gamma(a)} e^w w^{a-b} \] (31)

\[ U(a, b, w) \sim w^{-a} \]

It is true that the presence of \( e^w \) in the asymptotic behaviour of \( M(a, b, w) \) perverts the normalizability of \( \phi(w) \). Nevertheless, this unfavorable behaviour can be remedied by demanding \( a = -n \) and \( b \neq -\tilde{n} \), where \( \tilde{n} \) is also a nonnegative integer. In this case, one has to consider \( b = 2 (\beta = +1) \). Therefore, the asymptotic behaviour of Kummer’s function is proportional to \( w^a \) and Tricomi’s function becomes proportional to Kummer’s function.

As a matter of fact, \( M(-n, b, w) \) with \( b > 0 \) and \( w \in [0, \infty) \) is proportional to the generalized Laguerre polynomial \( L_n^{(b-1)}(w) \), a polynomial of degree \( n \) with \( n \) distinct positive zeros in the range \([0, \infty)\). The requirement \( a = -n \) and \( b = 2 \) implies into the solution on the half line

\[ E_n = \pm \frac{2m}{\sqrt{s}} \sqrt{n+1} \] (32)

\[ \phi_n (|x|) = N_n t e^{-t/2} L_n^{(1)} (t) \]

where \( t \) can be written as

\[ t = \zeta \left( 1 + \frac{|x|}{\lambda_C} \right)^2 \] (33)

\( N_n \) is a normalization constant, \( \zeta = 1/g > 0 \) and \( \lambda_C = 1/m \). It is useful to list the first few generalized Laguerre polynomials \( L_n^{(1)}(w) \) as standardized in Ref. [21]:

\[
\begin{align*}
L_0^{(1)}(w) &= 1 \\
L_1^{(1)}(w) &= -w + 2 \\
L_2^{(1)}(w) &= w^2/2 - 3w + 3
\end{align*}
\]
and in general we have
\[ L_n^{(1)}(w) = \sum_{j=0}^{n} \frac{\Gamma(n+2)(-w)^j}{\Gamma(j+2)j!(n-j)!} \]  
(35)

The following differential property
\[ \frac{dL_n^{(1)}(w)}{dw} = nL_n^{(1)}(w) - (n+1)L_{n-1}^{(1)}(w) \]  
(36)

allows us to expand the derivative of a generalized Laguerre polynomial in terms of other generalized Laguerre polynomials with the same superscripts. \( L_n^{(1)}(w) \) is to be interpreted as zero.

4. Solutions on the whole line

Following the ideas of the preceding section, we proceed now to find the eigenfunctions on the whole line. Because jump discontinuities of \( \phi_n(x) \) and \( d\phi_n(x)/dx \) would imply in the presence of Dirac delta functions and their first derivatives in Eq. (17), respectively, lawful symmetric and antisymmetric extensions of \( \phi(|x|) \) given on the half line to the whole line are possible only if \( \phi_n(x) \) and \( d\phi_n(x)/dx \) are continuous at the origin. The continuity requirement implies that \( d\phi_n(|x|)/dx|_{x=0^+} \) vanishes if \( \phi_n(x) \) is an even function, and \( \phi_n(|x|)|_{x=0^+} \) vanishes for an odd function. One has
\[ \phi_n(|x|)|_{x=0^+} = N_n \zeta e^{-\zeta/2} L_n^{(1)}(\zeta) \]  
(37)

and, with the aid of Eq. (36), one finds
\[ \left. \frac{d\phi_n(|x|)}{dx} \right|_{x=0^+} = \frac{2N_n e^{-\zeta/2}(n+1)}{\lambda_C} \left\{ \left[ 1 - \frac{\zeta}{2(n+1)} \right] L_n^{(1)}(\zeta) - L_{n-1}^{(1)}(\zeta) \right\} \]  
(38)

When \( L_n^{(1)}(\zeta) \neq 0 \) one finds \( \phi_n(|x|)|_{x=0^+} \neq 0 \). In this case, one obtains from Eq. (38) that \( d\phi_n(|x|)/dx|_{x=0^+} \) vanishes if \( \zeta \) and \( n \) satisfy the following \( (n+1) \)-degree algebraic equation in \( \zeta \):
\[ 1 - \frac{\zeta}{2(n+1)} = \frac{L_{n-1}^{(1)}(\zeta)}{L_n^{(1)}(\zeta)} \]  
(39)

Thus, the solution given by Eq. (32) is acceptable on the whole line as an even-parity function only if the potential parameter satisfies the constraint relation expressed by Eq. (39). In particular \( \zeta = 2 \) with \( n = 0 \), and \( \zeta = 3 \pm \sqrt{5} \) with \( n = 1 \).

On the other hand, the odd-parity solutions are related to the zeros of the generalized Laguerre polynomial: \( L_n^{(1)}(\zeta) = 0 \). This is a \( n \)-degree algebraic equation in \( \zeta \) depending on \( n \). There are \( n \) positive roots for a given \( n \). One finds no solution with \( n = 0 \). Nevertheless, one finds \( \zeta = 2 \) with \( n = 1 \), and \( \zeta = 3 \pm \sqrt{5} \) with \( n = 2 \), for example.

Using (4) and (20), the normalization constant can be written as
\[ N_n = \sqrt{\frac{\lambda}{\delta|x_n|}}. \]  
(40)

where \( \delta = \int_{1/g}^{\infty} t e^{-t} |L_n^{(1)}(t)|^2 dt \). In Figure 1, we illustrate the results for \( \zeta = 2 \). Although \( n \) is equal to the number of nodes of \( \phi_n \) in Figure 1, one should not expect this relation in a systematic way because \( t \) in the second line of Eq. (32) is restricted to the interval \([1/g, \infty)\). By way of addition, one may expect a two-fold degeneracy, with even and odd eigenfunctions for the same potential parameter and quantum number when both \( \phi_n(x) \) and \( d\phi_n(x)/dx \) vanish at the origin in such a way that \( \zeta \) and \( n \) satisfy the following set of equations: \( L_n^{(1)}(\zeta) = 0 \) and \( L_{n-1}^{(1)}(\zeta) = 0 \). Nevertheless, no numerical solution is found for this system of equations for \( n \leq 100 \), showing that the zeros of \( L_n^{(1)}(w) \) are different from the zeros of \( L_{n-1}^{(1)}(w) \). After all, this absence of degeneracy is plausible in view of the so-called nondegeneracy theorem for bound states in one-dimensional nonsingular potentials [22].
5. Final remarks

As commented in the Introduction, the authors of Ref. [14] misidentified the correct asymptotic behaviour of Kummer’s function, kept out Kummer’s function and used Tricomi’s function as a particular solution. These facts should be enough to nullify the candidature of the set of solutions presented in Ref. [14] as bona fide solutions. Surprisingly, that set of solutions is licit because $U(a, 0, t)$ presents a good behaviour at the neighbourhood of $1/g$ as well as a good asymptotic behaviour. Nevertheless, the solutions have to be found by numerical methods.

We analyzed in detail the solutions of Eq. (17) with the linear potential by given careful consideration to asymptotic behaviour of Kummer’s function. In that process, we have shown that the Sturm-Liouville problem has been transmuted in a simpler problem of solving algebraic equations for the eigenvalues and that the eigenfunctions are expressed in terms of the generalized Laguerre polynomials. The quantization condition comes into sight already for the problem defined on the half line and the extensions for the whole line imply into extra algebraic equations constraining the potential parameter and the quantum number. In general, one finds different potential parameters for different quantum numbers.

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