Detecting topological exceptional points in a parity-time symmetric system with cold atoms

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We reveal a novel topological property of the exceptional points in a two-level parity-time symmetric system and then propose a scheme to detect the topological exceptional points in the system, which is embedded in a larger Hilbert space constructed by a four-level cold atomic system. We show that a tunable parameter in the presented system for simulating the non-Hermitian Hamiltonian can be tuned to sweep the eigenstates through the exceptional points in parameter space. The non-trivial Berry phases of the eigenstates obtained in this loop from the exceptional points can be measured by the atomic interferometry. Since the proposed operations and detection are experimentally feasible, our scheme may pave a promising way to explore the novel properties of non-Hermitian systems.

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I. INTRODUCTION

Non-Hermitian Hamiltonian used to describe open or dissipative systems usually have complex eigenvalues. However, it is recently found that a series of non-Hermitian Hamiltonians have real eigenvalues if they are invariant under the parity (P) and time-reversal (T) union operation [1–3]. Meanwhile, the eigenstates of these Hamiltonians are also commuted with the PT-symmetry operation [4]. Due to various intriguing properties, the PT-symmetric systems have raised broad attentions. Some PT-symmetric models have been experimentally realized in physical systems, such as active LRC circuits [5, 6], two coupled waveguides [7], photonic lattices [8, 9], microwave billiard [10], transmission line [11], whispering-gallery microcavities [12, 13] and single-mode lasing [14, 15]. Recently, the schemes for simulating the PT-symmetry potentials with cold atoms in optical systems have been theoretically proposed [16–18] and experimentally realized [19].

Exceptional point (EP) is a special point in parameter space of non-Hermitian systems where both eigenvalues and eigenstates coalesce into only one value and state [20, 21]. One of the important properties of EPs is that the states of EPs are chiral [22], which has been experimentally observed [23, 24]. On the other hand, the chirality leads to an unique effect that the eigenstates exchange themselves but only one of them obtains a π Berry phase in a cyclic evolution [25–26]. This chiral phenomenon of EPs has been experimentally demonstrated in microwave cavity for the first time [27, 28]. Very recently, a full dynamical encircling of EP has been realized [29] and a non-reciprocal topological energy transfer due to dynamical encircling of such point has been measurement [30]. In contrast, other theoretical and numerical results suggest in this case the eigenstates change to the other one but both of them obtain a π/2 Berry phase due to the linear dependence of eigenstates [31, 32], which has been verified in a recent experiment [33].

In this paper, we demonstrate that there are two different chiral EPs in parameter space and the non-trivial Berry phase of EPs emerge different due to the chirality breaking when the eigenstates exactly to sweep through EPs in a cyclic evolution. Then we propose a scheme to realize the PT-symmetric Hamiltonian in cold atomic systems that the parameters can be exactly controlled in time. Based on the idea of the Naimark-Dilated operation [34] and the embedding quantum simulator [35], we show that a two-level PT-symmetric Hamiltonian can be constructed through a four-level Hermitian Hamiltonian in an embedding cold atomic system. Finally, we propose to detect this Berry phase through the observation of the phase difference between atomic levels, which can be measured through atomic interferometry. The proposed scheme provides a promising approach to realize the PT-symmetric Hamiltonian in cold atomic systems and to further explore the exotic properties of the EPs.

The paper is organized as follows. Section II describes our two-level PT-Hamiltonian and the topological properties of the intrinsic EPs. In Sec. III, we propose an experimentally feasible scheme for emulating the non-Hermitian two-level Hamiltonian in a Hermitian four-level cold atomic system. In Sec. IV, we show that the topological EPs can be measured by the atomic interferometry in the cold atom system. Finally, a brief discussion and a short conclusion are given in Sec. V.

II. TOPOLOGICAL EXCEPTIONAL POINTS IN A PT-SYMMETRIC SYSTEM

If a Hamiltonian \( H \) is non-Hermitian for describing an open or dissipative system, there are gain or loss effects in this system and the eigenvalues are generally complex values. However, if gain and loss of this system are balanced, this system remains stable and all eigenvalues are real numbers. This phenomenon is described by the PT-symmetric theory. Supposing that in this case \( \sigma_i (i = x, y, z) \) are Pauli matrices, the parity operator \( P \) is \( \sigma_x \) and the time-reversal operator \( T \) is the complex conjugation operator, which is an antilinear operator. A
simple PT-symmetric Hamiltonian can be constructed as

\[ \hat{H}_{PT} = S \left( i \sin(\alpha) - i \sin(\alpha) \right) , \]

where \( S \) is a general scaling factor of the matrix. The angle \( \alpha \) characterizes the non-Hermiticity of the Hamiltonian. When \( \alpha = 0 \), the Hamiltonian \( \hat{H} \) is a Hermitian operator, when \( \alpha \neq 0 \) the Hamiltonian \( \hat{H} \) becomes a non-Hermitian operator. In the case of \( \alpha = \pm \pi/2 \), the eigenvalues and the eigenstates coalesce into a single value and state, respectively.

The eigenvalues of Eq. (1) are \( E_{\pm}(\alpha) = \mp \chi = \mp S \cos(\alpha) \) and the corresponding eigenstates are given by

\[ |E_{+}(\alpha)\rangle = \frac{e^{i\alpha/2}}{\sqrt{2 \cos(\alpha)}} \left( \begin{array}{c} 1 \\ e^{-i\alpha} \end{array} \right) , \]
\[ |E_{-}(\alpha)\rangle = \frac{ie^{-i\alpha/2}}{\sqrt{2 \cos(\alpha)}} \left( \begin{array}{c} 1 \\ -e^{i\alpha} \end{array} \right) . \]

In addition, the non-Hermitian Hamiltonian \( \hat{H}_{PT} \) has a bi-orthogonal basis \( |E_{\pm}(\alpha)\rangle, |\Lambda_{\pm}(\alpha)\rangle \) [37]:

\[ \hat{H}_{PT}|\Lambda_{\pm}(\alpha)\rangle = E_{\pm}(\alpha)|\Lambda_{\pm}(\alpha)\rangle , \]
\[ |\Lambda_{\pm}(\alpha)\rangle = -|E_{\mp}(\alpha)\rangle . \]

When \( \alpha = \pm \pi/2 \), the two eigenvalues become \( E_{\pm} = 0 \) and the corresponding eigenstates coalesce at the same time:

\[ |E(\pi/2)\rangle \propto \left( \begin{array}{c} 1 \\ -i \end{array} \right) , \]
\[ |E(-\pi/2)\rangle \propto \left( \begin{array}{c} 1 \\ i \end{array} \right) . \]

The signs before \( i \) in Eq. (4) depend on the system and give the chirality of these specific degeneracy points dubbed as EPs [26, 22, 25, 24], such that the two EPs are different from each other. The Hamiltonian thus is restricted to purely real eigenvalues, and the time evolution operator \( \hat{U}_{PT}(t) = e^{-i\hat{H}_{PT}t} \) is unitary with the explicit form

\[ \hat{U}_{PT}(t) = \frac{1}{\cos(\alpha)} \left( \begin{array}{cc} \cos(\chi t - \alpha) & -i \sin(\chi t) \\ -i \sin(\chi t) & \cos(\chi t + \alpha) \end{array} \right) . \]

With the analytical solution of the PT-symmetric Hamiltonian, we can study the topological properties of the EPs. Considering the eigenvalues \( E_{\pm}(\alpha) \), we can find that the eigenstates \( |E_{+}(\alpha)\rangle \) and \( |E_{-}(\alpha)\rangle \) respectively represent the higher and lower levels when \( \alpha \in [-\pi/2 + 2N\pi, \pi/2 + 2N\pi] \), but respectively represent the lower and higher levels when \( \alpha \in [\pi/2 + 2N\pi, 3\pi/2 + 2N\pi] \), with \( N \) being a positive integer. This means that the definition of the domain in the system is \( [-\pi/2, \pi/2] \) and the eigenstates exchange themselves when \( \alpha \) sweeps through the EPs every time. With the eigenvalues \( E_{\pm}(\alpha) = \pm S \cos(\alpha) \), one can also find that the corresponding point of \( \alpha \) in the other Riemann surface is the point of \( \pm \pi - \alpha \). Due to the degeneracy of the EPs, the eigenstates obtain non-Abelian geometric phases through passing the EPs.

For this degenerate non-Hermitian system, the Berry phase in the cyclic evolution is [38]

\[ \gamma = \oint C Ad\alpha , \]

where \( A \) is the non-Abelian Berry connection:

\[ A = i \left( (\Lambda_{+}|d\alpha|E_{+}) \langle \Lambda_{+}|d\alpha|E_{-} \rangle \right) , \]

with \( d\alpha \) being the \( \alpha \) derivative. For the PT-symmetric Hamiltonian \( \hat{H}_{PT} \) here, we can obtain the non-Abelian geometric phases for two different loops from \( \alpha \) to \( \pm \pi - \alpha \) (which pass through two EPs of different chiralities) as

\[ \gamma_{\alpha \rightarrow \pm \pi - \alpha} = \left( \begin{array}{cc} 0 & \pm \frac{i\pi}{2} \\ \pm \frac{i\pi}{2} & 0 \end{array} \right) . \]

Consequently, when \( \alpha \) successively sweeps through two different EPs in the same evolutionary direction, the eigenstates become original with an additional \( \pi \) Berry phase. In this case, the eigenstates under the whole evolution can be written as

\[ |E_{\pm}(\alpha)\rangle \rightarrow e^{i\pi/2}|E_{\mp}(\pi - \alpha)\rangle \rightarrow -|E_{\pm}(\alpha)\rangle , \]
\[ |E_{\pm}(\alpha)\rangle \rightarrow e^{-i\pi/2}|E_{\mp}(\pi - \alpha)\rangle \rightarrow -|E_{\pm}(\alpha)\rangle . \]

Unlike \( |E_{\pm}(\alpha)\rangle \rightarrow |E_{\mp}(\pi)\rangle \rightarrow -|E_{\pm}(\alpha)\rangle \) or \( |E_{\pm}(\alpha)\rangle \rightarrow -|E_{\pm}(\alpha)\rangle \) in the general case, here Eq. (7) shows that there is an obviously different behavior in the intermediate process when one passes through different EPs successively.

The above results show that in non-Hermitian systems, the eigenvalue surfaces exhibit a complex-square-root topology with a branch point named by EP, which can also be considered as a critical point where a transition from PT-symmetric phase to broken-PT-symmetric phase. A consequence of this topology is that encircling an EP once in the parameter space results in the exchange of both eigenvalues and eigenstates. This means that one has to encircle an EP twice to recover the original eigenvalues and eigenstates. On the other hand, one of the eigenstates acquires a Berry phase of \( \pm \pi \) when encircling an EP once and the other one acquires the same phase in the second loop. However, because one not longer distinguishes the clockwise or anticlockwise direction of the state evolution, the result of passing through the EPs once will be different. In this system, one acquires this phase in each loop which is determined by the chirality of the EP and the evolutionary direction of the eigenstates in the parameter space.

### III. REALIZATION OF THE TWO-LEVEL PT-SYMMETRIC HAMILTONIAN IN A FOUR-LEVEL COLD ATOMIC SYSTEM

The PT-symmetric Hamiltonian may be difficult to achieve in a practical non-Hermitian two-level system. In this section, we propose to use a four-level Hermitian system to simulate the two-level PT-symmetric Hamiltonian (1). In the basis \( \{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}^2 \), the four-level Hermitian Hamiltonian
the evolution of the states in Eq. (3) in the two-level PT-symmetric Hamiltonian.

Now we present an experimentally realizable scheme implementing with cold atoms. We consider an atomic cloud of $^{87}$Rb with five internal states in the ground-state manifold, noted by $|1 + m_F\rangle = |F = 1, m_F = -1, 0\rangle$ and $|4 + m_F\rangle = |F = 2, m_F = -1, 0\rangle$, which are separated by the hyperfine splitting $\omega HF$ and the Zeeman splitting $\omega Z$ caused by a uniform static magnetic field. One can apply four microwave radios to couple these atomic levels through the anulus configuration as shown in Fig. 1. Here $\Omega_1$ and $\omega_1$ denote the Rabi frequencies and the frequencies of the corresponding microwave radios, respectively. In particular, we use resonant microwaves to couple $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |4\rangle$ and use radio-frequency fields to couple $|1\rangle \leftrightarrow |2\rangle$ and $|3\rangle \leftrightarrow |4\rangle$, respectively. Supposing that $\omega_{21}$ are the energies of $|j\rangle$ and the energy of $|1\rangle$ is the zero of energy, the total Hamiltonian can be written as $H = H_0 + H_{int}$ with

$$H_0 = \sum_j \omega_{j} |j\rangle\langle j|,$$

$$H_{int} = \Omega_1 e^{i\omega_{1} t} |3\rangle\langle 1| + \Omega_2 e^{i\omega_{2} t} |3\rangle\langle 4| + \Omega_3 e^{i\omega_{3} t} |4\rangle\langle 3| + \Omega_4 e^{i\omega_{4} t} |2\rangle\langle 1| + H.c.$$

In the bare-state basis $(|1\rangle, |2\rangle, |3\rangle, |4\rangle)^T$, one has

$$V = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{-\omega_{1} t} & 0 & 0 \\
0 & 0 & e^{-\omega_{2} t} & 0 \\
0 & 0 & 0 & e^{-\omega_{3} t}
\end{pmatrix}$$

and the Hamiltonian in the rotating frame becomes

$$\hat{H} = i \frac{dV}{dt} V + V^\dagger H V = \begin{pmatrix}
0 & \Omega_4 & \Omega_1 & 0 \\
\Omega_4^* & -\omega_4 - \omega_{e1} + \omega_2 & 0 & 0 \\
\Omega_1^* & 0 & -\omega_1 - \omega_{e1} + \omega_3 & 0 \\
0 & e^{i(\omega_1 + \omega_2 - \omega_3 - \omega_4)t} & 0 & -\omega_1 - \omega_3 - \omega_{e1} + \omega_4
\end{pmatrix}$$

To ensure the Hamiltonian being time-independent, we choose $\omega_1 + \omega_2 = \omega_3 + \omega_4$. On the other hand, considering the resonance condition $\Delta_1 = \omega_{e3} - \omega_{e1} - \omega_1 = 0$, $\Delta_2 = \omega_{e4} - \omega_{e3} - \omega_2 = 0$, $\Delta_3 = \omega_{e4} - \omega_{e2} - \omega_3 = 0$, $\Delta_4 = \omega_{e2} - \omega_{e1} - \omega_4 = 0$, with $\omega_1 = \omega_2 = \omega_4 = \omega_2$.

In particular, we choose the corresponding Rabi frequencies $\Omega_1 = -\Omega_2 = i S \sin(\alpha) \cos(\alpha)$ and $\Omega_2 = \Omega_4 = S \cos^2(\alpha)$. Under these conditions, the Hamiltonian $\hat{H}$ can be represented as $H_F$ in Eq. (10):

$$\hat{H} = \begin{pmatrix}
0 & \Omega_4 & \Omega_1 & 0 \\
\Omega_4^* & 0 & 0 & 0 \\
\Omega_1^* & 0 & 0 & \Omega_3 \\
0 & \Omega_3^* & \Omega_2 & 0
\end{pmatrix} = H_F.$$

IV. DETECTING THE BERRY PHASE IN THE PT-SYMMETRIC SYSTEM

In the section, we show how to detect the Berry phases of the mimicked EPs in the four-level cold atomic system. First, we need an additional atomic level for this measurement, which is denoted by $|0\rangle$ in the cold atomic system as shown in Fig. 1. We assume the atoms are initially pumped to $|0\rangle$ and the transitions $|0\rangle \rightarrow |i\rangle (i = 1, 2, 3, 4)$ can be re-
alized successively through the stimulated-Raman-adiabatic-passage (STIRAP) [41, 42]. It is noted that the microwave radiations must be phase-locking between each STIRAP for keeping the coherence between the states. On the other hands, only the phase difference between $|0\rangle$ and $|i\rangle$ is needed to be concerned, so it is nonsignificant what the population differences between $|0\rangle$ and other levels are. Under this condition, we make $|0\rangle \rightarrow |0\rangle + \eta |\psi_\pm(\alpha_0)\rangle^T$ from the very beginning, where $\eta$ is an arbitrary real number, for the preparation of the PT-symmetric initial state. The phase difference between $|1\rangle$ and $|2\rangle$ is the only distinction between $|E_+(\alpha)\rangle$ for a given $\alpha$, and the phase differences between $|0\rangle$ and $|1\rangle$ ($|2\rangle$) can be used to detect the Berry phases of $|E_\pm(\alpha)\rangle$. Thus we can detect the Berry phases by measuring the phase differences between the corresponding atomic levels.

The phase difference between two atomic levels can be measured through the atomic interferometry. For an arbitrary state denoted by $|a\rangle + e^{-i\varphi}|b\rangle$, a $\pi/2$-pulse operation takes the form

$$U_{\pi/2, \phi} = \begin{pmatrix} 1 & -ie^{-i\varphi} \\ ie^{i\varphi} & 1 \end{pmatrix},$$

with $\phi$ being a controllable phase of the $\pi/2$-pulse. After applying a $\pi/2$-pulse to the state $|a\rangle + e^{-i\varphi}|b\rangle$, one can find the relationship between the atomic populations and the phase differences as

$$N_{\alpha, b} = \frac{1}{2} \left[ 1 \mp \sin(\phi + \varphi) \right].$$

Considering Eq. (2) and Eq. (17), the atomic populations of the different levels are given by

$$N_{1, 2}(\alpha) = \begin{cases} \left[ 1 \mp \sin(\phi + \alpha) \right]/2 & \text{for } |E_+(\alpha)\rangle \\ \left[ 1 \mp \sin(\phi + \pi - \alpha) \right]/2 & \text{for } |E_-(\alpha)\rangle \end{cases}.$$

For a given $\alpha$, the function of $\phi$ in Eq. (18) can be used to distinguish $|E_+(\alpha)\rangle$ and $|E_-(\alpha)\rangle$, so we can verify whether the eigenstates exchange themselves through measuring the function of $\phi$ in Eq. (13) when sweeping an EP.

After confirming the two states $|E_\pm(\alpha)\rangle$, we can measure the phase difference between $|0\rangle$ and $|1\rangle$ to determine the Berry phases $\gamma_d$ for $|E_\pm(\alpha)\rangle$, which is related to the value of $\alpha$ and the evolution loop. For $|0\rangle$ and $|1\rangle$, we can also find the relationship between atomic populations and total phase differences:

$$N_{0,1}(\alpha) = \begin{cases} (1 \mp \sin \varphi_+) / 2 & \text{for } |E_+(\alpha)\rangle \\ (1 \mp \sin \varphi_-) / 2 & \text{for } |E_-(\alpha)\rangle \end{cases},$$

where the total phases between the corresponding atomic levels in the case of different eigenstates $\varphi_+ = \gamma_d - \alpha/2 - \theta_\pm(\alpha)\pi/2$ and $\varphi_- = \gamma_d - (\pi - \alpha)/2 - \theta_\pm(\alpha)\pi/2$ with $\gamma_d = \mp S \cos(\alpha)t + \phi + \omega_2 t$ being the dynamic phase due to the evolution $\hat{U}_p$, the $\pi/2$-pulse and the Zeeman energy difference, respectively, and $\theta_\pm(\alpha) = \theta[\mp \cos(\alpha)]$ being a Heaviside unit step function whose value is 0 for $-\cos(\alpha) < 0$ and 1 for $-\cos(\alpha) > 0$. It is clear that the topology of the EP is the source of the Heaviside unit step function. Due to the symmetry of the trigonometric functions, confirming $|E_+(\alpha)\rangle$ and $|E_-(\alpha)\rangle$ in Eq. (19) needs two values of $\gamma_d$.

For simplicity, here we choose the dynamic phase as 0 and $\pi/2$, and the phases $\varphi_\pm$ and the atomic populations $N_{0,1}$ in the different eigenstates are shown in Fig. (2). For a given eigenstate, one can measure the phases $\varphi_\pm$ to obtain the Berry phases of the eigenstates from Eq. (19). In particular, for a given eigenstate and the parameter $\alpha$, we first measure $N_{1,2}$ in Eq. (13) to determine whether the eigenstate is through the function of $\phi$. Then one can detect the total phase $\varphi_\pm$ by measuring $N_{0,1}$ in Eq. (19), as shown in Fig. 2. After that, one can control the systemic parameter $\alpha$ to sweep an EP in parameter space that the eigenstate is predicted to obtain a non-Abelian phase in Eq. (8). Again one can successively measure $N_{1,2}$ and $N_{0,1}$ to determine whether the eigenstate has been changed and the variation of total phase. Here we can find that: i) The eigenstates obtain a phase of $+\pi/2$ or $-\pi/2$ alternately when $\alpha$ sweeps through the different EPs successively. ii) The eigenstates exchange themselves when $\alpha$ sweeps through an EP and the phase differences from $|E_+(\alpha)\rangle$ to $|E_\mp(\pi - \alpha)\rangle$ are always $\pm \pi/2$. iii) The eigenstates obtain a Berry phase of $\pm \pi$ when $\alpha$ sweeps $\pm 2\pi$. In short, the measurement of the phase differences between $|i\rangle$ ($i = 0, 1, 2$) provides a simple way to experimentally verify Eq. (9) and demonstrate the Riemann sheet structure and the intrinsic properties of the EPs.

**V. DISCUSSION AND SUMMARY**

In the above case, there is no dynamical phase contribution between the four states $|i\rangle$ ($i = 1, 2, 3, 4$), but the evolution of $|i\rangle$ must be still adiabatic in order to avoid non-adiabatic transitions. To be specific, the four Rabi frequencies should keep the adiabatic cyclic evolutions in the parameter space.
with the change rate of $\alpha$ being significantly smaller than the typical Rabi frequencies. This means that the evolution period of the system is much larger than the inverses of the energy gaps between the atomic levels. On the other hand, with the non-adiabatic transition between $|E_{\pm}(\alpha)\rangle$ being avoided, the change rate of $\alpha$ is also smaller than $2\chi$, which is the difference of eigenvalue. Thus, the adiabatic condition takes the form

$$\frac{d\alpha}{dt} \ll \Omega_i, 2\chi.$$  \hspace{1cm} (20)

To fulfill this condition, one should assume the system remains in the eigenstate of the Hamiltonian $\tilde{H}$ all the time.

In summary, we have proposed an experimental scheme to realize a two-level PT-symmetric system with parameter being controllable through an embedding four-level cold atomic system. We further clarify the different properties between encircling and passing through EPs. And then we demonstrate that the change of the eigenstates and the relevant topological phase in the case of passing through EPs can be confirmed by measuring the phases of atomic levels and this novel phenomenon can be probed through the standard phase measurement. Our work proposes a method to realize PT-Symmetric Hamiltonian in cold atomic systems and therefore provides a powerful tool to explore the properties of PT-symmetry and PEs further.

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