A CHARACTERIZATION OF THE $\hat{A}$-GENUS AS A LINEAR COMBINATION OF PONTRYAGIN NUMBERS

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ABSTRACT. We show in this short note that if a rational linear combination of Pontrjagin numbers vanishes on all simply-connected $4k$-dimensional closed connected and oriented spin manifolds admitting a Riemannian metric whose Ricci curvature is nonnegative and nonzero at any point, then this linear combination must be a multiple of the $\hat{A}$-genus, which improves on a result of Gromov and Lawson. Our proof combines an idea of Atiyah and Hirzebruch and the celebrated Calabi-Yau theorem.

1. INTRODUCTION AND MAIN OBSERVATION

Unless otherwise stated, all manifolds considered in this note are smooth, closed, connected and oriented. All metrics mentioned in this note refer to Riemannian metrics.

Atiyah and Hirzebruch proved in [AH70] that if a $4k$-dimensional spin manifold $M$ admits a non-trivial (smooth) circle action ($S^1$-action), then the $\hat{A}$-genus of $M$ must vanish. Indeed, they showed via the Atiyah-Bott-Singer fixed point formula that the equivariant index $\text{spin}(g, M) \in R(S^1)$ of the Dirac operator on $M$ vanishes identically. In particular, the $\hat{A}$-genus $\hat{A}(M) = \text{spin}(1, M)$ must vanish. Note that the $\hat{A}$-genus is a rational linear combination of Pontrjagin numbers and $\hat{A}(\cdot)$ can be viewed as a ring homomorphism $\hat{A}(\cdot) : \Omega^*_{SO} \otimes \mathbb{Q} \to \mathbb{Q}$, where $\Omega^*_{SO}$ is the oriented cobordism ring ([Hi66]). In addition to the above-mentioned main result, by suitably choosing generators of $\Omega^*_{SO} \otimes \mathbb{Q}$ and a clever manipulation for them, Atiyah and Hirzebruch also proved in [AH70] that the $\hat{A}$-genus can be characterized as the only linear combination of Pontrjagin numbers that vanishes on all $4k$-dimensional spin manifolds admitting a non-trivial circle action. To be more precise, they showed that ([AH70, §2.3]) a rational linear combination of Pontrjagin numbers vanishes on all spin manifolds equipped with a non-trivial smooth circle action if and only if it is a multiple of the $\hat{A}$-genus.

Recall that on $4k$-dimensional spin manifolds the vanishing of the $\hat{A}$-genus is also an obstruction to the existence of a positive scalar curvature metric, which is a classical result of Lichnerowicz ([Li63]) and whose argument is based on a Bochner-type formula related to the Dirac operator and the Atiyah-Singer index theorem (a detailed and excellent proof can be found in [Wu88, Ch.5]). Another breakthrough related to the existence of a positive scalar curvature metric on spin manifolds came from Gromov and Lawson. They proved in [GL80, Theorem B] that if a simply-connected spin manifold $X$ is spin cobordant to a manifold equipped with a positive scalar curvature metric, then $X$ itself also carries such a metric. As an application of this result, Gromov-Lawson obtained the same type result as that of

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Atiyah-Hirzebruch in the language of cobordism theory ([GL80, Corollary B]): a rational linear combination of Pontrjagin numbers vanishes on all 4k-dimensional spin manifolds carrying a positive scalar curvature metric if and only if it is a multiple of the $\hat{A}$-genus.

Another result which has a similar flavor to the above Atiyah-Hirzebruch and Gromov-Lawson’s results should also be mentioned. In [Ko10], Kotschick showed that a rational linear combination of Pontrjagin numbers which is bounded on all 4k-dimensional manifolds with a nonnegative sectional curvature metric if and only if it is a multiple of the signature. After [Ko10, Corollary 2], the author commented that his result bears some resemblance to Gromov-Lawson’s above result.

However, although Atiyah-Hirzebruch’s result is much earlier, clearly neither [GL80] nor [Ko10] was aware of it as they didn’t cite [AH70]. In spirit, both Gromov-Lawson and Kotschick’s proofs are similar to that of Atiyah-Hirzebruch (see [GL80, p. 432] and [Ko10, p. 139]): suitably choose generators for $\Omega^*_\text{SO} \otimes \mathbb{Q}$ satisfying certain restrictions and then apply a proportionality argument to lead to the desired proof. This idea was further taken up by Kotschick et al to solve a long-standing problem posed by Hirzebruch in 1954 ([Ko12], [KS13]).

The main purpose of this short note is to show that, by using the generators of $\Omega^*_\text{SO} \otimes \mathbb{Q}$ chosen in [AH70], together with the celebrated Calabi-Yau theorem, we can give a direct proof of Gromov-Lawson’s this result, which avoids using their main result [GL80, Theorem]. Moreover, because of the use of the Calabi-Yau theorem, our statement is indeed more sharper and thus improves on their original result. We now state our main observation in this note, whose proof will be given in Section 2.

Note that the Ricci curvature of a metric on a manifold is an assignment of a quadratic form at the tangent space of any point of this manifold. The Ricci curvature is said to be nonnegative and nonzero at any point if, at any point of this manifold, this quadratic form is nonnegative definite but nonzero. With this notion understood, our main observation is the following

**Theorem 1.1.** If a rational linear combination of Pontrjagin numbers vanishes on all 4k-dimensional simply-connected spin manifolds equipped with a metric whose Ricci curvature is nonnegative and nonzero at any point, then it must be a multiple of the $\hat{A}$-genus.

Clearly the conclusion in Theorem 1.1 remains true if we weaken the restrictions imposed on the manifolds in Theorem 1.1. Recall that the scalar curvature is nothing but the trace of the real symmetric matrix which represents the quadratic form of the Ricci curvature under some orthonormal basis. So the condition that the Ricci curvature be nonnegative and nonzero at any point means that the eigenvalues of this matrix are all nonnegative and at least one of them is positive. In particular the trace is positive. This means the condition of the Ricci curvature required in Theorem 1.1 implies that the scalar curvature of this metric is positive. Therefore, Theorem 1.1, together with the Lichnerowicz vanishing theorem, yields the following corollary.

**Corollary 1.2.** (Gromov-Lawson, [AH70, Corollary B]) A rational linear combination of Pontrjagin numbers vanishes on all 4k-dimensional spin manifolds equipped with a positive scalar curvature metric if and only if it is a multiple of the $\hat{A}$-genus.
2. Proof of Theorem 1.1

Let $\mathbb{HP}^k$ denote the quaternionic projective space whose real dimension is $4k$. It is well-known that $\mathbb{HP}^k$ admits a metric with positive sectional curvature and thus it has positive Ricci curvature. Let $V_2(4)$ be the smooth hypersurface of degree 4 in the 3-dimensional complex projective space $\mathbb{CP}^3$, which is a compact Kähler surface and whose first Chern class $c_1(V_2(4)) = 0$ in $H^2(V_2(4); \mathbb{Z})$ (Kummer surface). For simplicity we denote by $N^1 := V_2(4)$ and $N^k := \mathbb{HP}^k$ $(k \geq 2)$.

For our later use, we record some facts related to $N^k$ in the following lemma. Although some of them have been sketchily explained in [AH70, §2.3], for the reader’s convenience, we would like to either point out a reference or indicate its proof if some non-standard fact is stated/recorded.

**Lemma 2.1.**

1. \[ \Omega^*_V \otimes \mathbb{Q} = \mathbb{Q}[N^1, N^2, N^3, \ldots]. \]
   
   In other words, \{$(N^k)_{k=1}^\infty$\} is a basis sequence for the oriented cobordism graded ring tensored with $\mathbb{Q}$. This means, if we denote by $\Omega^k$ the restriction of $\Omega^*_V$ to the 4k-dimensional manifolds, then the vector space $\Omega^k \otimes \mathbb{Q}$ has a basis \{$(\prod_{i=1}^t N^{k_i})_{k_i}$\}, where $(k_1, \ldots, k_t)$ runs over all the partitions of weight $k$ $(= k_1 + k_2 + \cdots + k_t)$.

2. All $N^k$ and any of their finitely many product $\prod_{i=1}^t N^{k_i}$ are simply-connected and spin.

3. $\hat{A}(N^1) = 2$ and $\hat{A}(N^k) = 0$ with $k \geq 2$.

4. Suppose we have a sequence of $t$ positive-dimensional Riemannian manifolds $(M_i, g_i)$ $(1 \leq i \leq t)$ such that any Ricci curvature $\text{Ric}(g_i)$ is either positive or identically zero and at least one of these $\text{Ric}(g_i)$ is positive. Then the Ricci curvature $\text{Ric}(\prod_{i=1}^t M_i, \prod_{i=1}^t g_i)$ of the Riemannian product manifold $(\prod_{i=1}^t M_i, \prod_{i=1}^t g_i)$ is nonnegative and nonzero at any point.

5. The product manifold $\prod_{i=1}^t N^{k_i}$ admits a Riemannian metric whose Ricci curvature is nonnegative and nonzero at any point if at least one of these $k_i$ is larger than 1.

**Proof.**

1. This was observed in [AH70, §2.3]. Indeed, there exists a simple procedure, which is due to Thom, to detect whether or not a given sequence of $4k$-dimensional manifolds $(k = 1, 2, \ldots)$ is a basis sequence (cf. [Hi66, p. 79] or [HBJ92, p. 43]).

2. Simply-connectedness is clear. A manifold is spin if and only if its second Stiefel-Whitney class $\omega_2 = 0$. $N^k$ $(k \geq 2)$ are spin as $H^2(N^k; \mathbb{Z}_2) = 0$. $N^1$ is spin as $c_1(N^1) = 0$, whose module 2 reduction is exactly $\omega_2(N^1)$. The fact that the product $M_1 \times M_2$ of two (orientable) spin manifolds $M_1$ and $M_2$ is still spin follows from the Whitney direct sum formula for $\omega_2(M_1 \times M_2)$ and the fact that the first Stiefel-Whitney class is zero if and only if the manifold is orientable ([MS74]).

3. $\hat{A}(N^k) = 0$ for $k \geq 2$ follow form the main result in [AH70] as these $N^k$ admit nontrivial circle actions. $\hat{A}(N^1) = 2$ is quite well-known. In fact there is a closed formula to calculate the $\hat{A}$-genus of general complete intersections in the complex projective spaces ([Br83]).

4. This statement follows from the simple fact that the quadric form of the Ricci curvature $\text{Ric}(g_1 \times g_2)$ of the product metric $g_1 \times g_2$ is exactly the direct sum of those of $\text{Ric}(g_1)$ and $\text{Ric}(g_2)$. 
(5) As we have mentioned that on each $N^k$ ($k \geq 2$) one has a positive Ricci curvature metric. Since $c_1(N^1) = 0$, the celebrated Calabi-Yau theorem, which refers to S.-T. Yau’s solution to the famous Calabi conjecture ([Yau77]), tells us that $N^1$ carries a Kähler (hence Riemannian) metric whose Ricci curvature is identically zero. Hence this statement follows from (4). Note that it is this place where we apply the remarkable Calabi-Yau theorem! 

□

Now we are in a position to prove Theorem 1.1.

Proof. Fix a positive integer $k$.

Let $\hat{A}_k$ be the restriction of $\hat{A}$ to $\Omega^k \otimes \mathbb{Q}$. Namely, $\hat{A}_k$ is a linear homomorphism:

$$\hat{A}_k(\cdot) : \Omega^k \otimes \mathbb{Q} \to \mathbb{Q}.$$ 

Let $q$ be a rational linear combination of Pontrjagin numbers on $4k$-dimensional manifolds. Since Pontrjagin numbers are cobordism invariants, $q$ can also be viewed as a linear homomorphism

$$q : \Omega^k \otimes \mathbb{Q} \to \mathbb{Q}.$$ 

In view of these, it suffices to show that, if $q$ vanishes on all spin manifolds which admits a metric whose Ricci curvature is nonnegative and nonzero at any point, then $q$ is a multiple of $\hat{A}_k$.

If $k = 1$, the conclusion is clear as there has only one Pontrjagin number $p_1$ and $\hat{A}_1 = \frac{1}{24}p_1$. We now suppose that $k \geq 2$. By (1) of Lemma 2.1 we know that the vector space $\Omega^k \otimes \mathbb{Q}$ has a basis $\{\prod_{i=1}^t N^{k_i}\}$, where $(k_1, \ldots, k_t)$ runs over all the partitions of weight $k$. (5) of Lemma 2.1 tells us that $q$ vanishes on all these basis elements with only one possible exception $(N^1)^k$. The same holds for $\hat{A}_k$ by (3) of Lemma 2.1. This implies

$$q - q((N^1)^k) \frac{2^k}{2^k} \hat{A}_k$$

vanishes on all these basis elements and thus

$$q = q((N^1)^k) \frac{2^k}{2^k} \hat{A}_k \text{ on } \Omega^k \otimes \mathbb{Q},$$

which gives the desired proof. 

□

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