

**H∞ optimization of Maxwell dynamic vibration absorber with multiple negative stiffness springs**

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**Abstract**

The Maxwell model with viscoelastic material and multiple negative stiffness springs is introduced into dynamic vibration absorber system, and all the system parameters are optimized in detail. The analytical solution of the primary system is exhibited according to the established motion differential equation. The dimensionless system parameters, including the optimum natural frequency ratio, the optimum damping ratio and the first optimum negative stiffness ratio of dynamic vibration absorber, are obtained based on H∞ optimization principle and the fixed-point theory. Considering system stability, the other optimum negative stiffness ratio is also determined. Furthermore, by the comparisons of the presented dynamic vibration absorber with other traditional dynamic vibration absorbers, it is found that the dynamic vibration absorber in this paper has better vibration reduction effect in the case of both harmonic and random excitation.

**Keywords**

Negative stiffness spring, dynamic vibration absorber, Maxwell model, fixed-point theory, H∞ optimization

**Introduction**

In many engineering fields, viscoelastic material always behaves in stiffness and damping characteristics simultaneously. The Maxwell model could describe these mechanic characteristics very well,¹ and attracted many scholars. For example, Wu et al.² analyzed the nonlinear seismic response of two highly adjacent reinforced concrete structures connected with Maxwell model under large earthquake excitation, and proposed the optimum parameters for dampers under different seismic excitation. A simplified Maxwell model was proposed in He and Wang,³ in which the influence of anti-yaw damper series stiffness on critical speed was investigated in details. Recent years, fractional-order calculus was introduced into the equation of Maxwell viscoelasticity fluid, of which the theoretical result agreed well with experimental data.⁴ Zheng et al.⁵ designed a generalized Maxwell model to demonstrate the viscoelastic property of asphalt mixtures, and the rationality of the constitutive relation and numerical method was verified. The generalized Maxwell model of five elements⁶ was used to simulate the dynamical behavior of stress relaxation of the elastic pad in WJ-8 fastener of high speed railway, and the results verified that the fitting effect was well.

Dynamic vibration absorber (DVA) is one of the common devices for vibration control. Because it possesses many advantageous properties such as reliability, efficiency, and low cost, DVA is widely used in engineering practice related to many fields. The research on DVA has attracted a lot of scholars for more than 100 years, and

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many different DVA structures were proposed to solve many engineering problems.\textsuperscript{7–19} With the development and improvement of vibration control technology, negative stiffness (NS) device is gradually used in vibration system. Because it can make the system behave in many advantages such as low natural frequency, high bearing capacity and good control performance, more and more NS device was applied in vibration reduction in recent years. The earlier report about NS system is by Alabuzhev and Rivin,\textsuperscript{20} which introduced the realization forms and theoretical analysis of NS mechanism in case of vibration control. Antoniadis et al.\textsuperscript{21} proposed a novel vibration damping and isolation concept based on a stiff and stable linear oscillator with a NS element. Acar et al.\textsuperscript{22} investigated an adaptive passive DVA with a NS device and found that it could suppress the amplitude of the system under some parameter conditions. A new type of DVA with NS characteristics was designed in Peng and Shen,\textsuperscript{23} and a conclusion that DVA with NS had better control performance than traditional ones was obtained. The new three-element DVA was discussed in Shen et al.\textsuperscript{24} and Wang and Shen,\textsuperscript{25} and it had better control performances in the case of harmonic and random excitations. Cheung and Wong\textsuperscript{26} discussed the grounded DVA, and found that the results in Ren\textsuperscript{14} and Liu and Liu\textsuperscript{15} are local minimum. Tursun and Eskinat\textsuperscript{27} presented a method for calculating the optimal parameters of absorbers in any number on uniform beam according to the H\textsuperscript{2} optimization criteria. Based on three typical optimization criteria, Asami\textsuperscript{28} optimized the double-mass DVAs arranged in series or in parallel attached to an undamped SDOF system. Zang et al.\textsuperscript{29} introduced a lever mechanism to nonlinear DVA, and obtained some typical results. Shen et al.\textsuperscript{30,31} studied the effects of negative stiffness and lever mechanism on DVA, and presented some meaningful results.

However, to the best of our knowledge, most of the reported works mainly involved single NS device, and the influence of multiple NS springs on vibration control is little considered. Therefore, we will investigate the Maxwell model with viscoelastic material, and multiple negative stiffness (NS) springs are introduced into DVA system to discuss the control effect. The paper is organized as follows. A Maxwell DVA with multiple NS coefficients is established in the next section. The subsequent section presents the study on the optimum parameters of the stiffness and damping of the absorber, and two optimum NS ratios are obtained in the condition of ensuring the system stability and optimizing the response. Under the harmonic and random excitations, the better control performances of the new DVA model are verified in the penultimate sections, and the primary conclusions made are given at last.

**Approximate solution for Maxwell DVA with multiple NS springs**

The Maxwell DVA coupling with multiple NS springs is exhibited in Figure 1, where $m_1$, $m_2$, $k_1$ and $k_2$ are the masses and linear stiffness coefficients of the primary system and DVA, respectively. $x_1$, $x_2$ and $x_3$ denote the
displacements of the primary system, DVA and the split point of spring and damping in Maxwell model. $F_0$ and $\omega$ describe the amplitude and frequency of the force excitation. $k_3$ and $c$ express the NS and damping coefficients of viscoelastic Maxwell model. The NS coefficient of the grounded spring is denoted by $F$. The displacements of the primary system, DVA and the split point of spring and damping in Maxwell model. Hao et al.\textsuperscript{23–25,30,31} system with negative stiffness spring may have better control performance, which had been illustrated in some DVAs.\textsuperscript{23–25,30,31} But the appropriate configuration and parameters selection of vibration system with negative stiffness spring may have better control performance, which had been illustrated in some DVAs.\textsuperscript{23–25,30,31}

According to Newton’s second law, the motion equation could be written as

$$\begin{align*}
\begin{cases}
m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = F_0\sin(\omega t) \\
m_2\ddot{x}_2 + k_2(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_3) + k_4x_2 = 0 \\
c(\dot{x}_3 - \dot{x}_2) + k_3x_3 = 0
\end{cases}
\end{align*}$$

(1)

By parametric transformations

$$\omega_1 = \sqrt{\frac{k_1}{m_1}}, \quad \omega_2 = \sqrt{\frac{k_2}{m_2}}, \quad \zeta = \frac{c}{2m_2\omega_2}, \quad \mu = \frac{m_2}{m_1}, \quad \alpha_1 = \frac{k_3}{k_2}, \quad \alpha_2 = \frac{k_4}{k_2}, \quad f = \frac{F_0}{m_1}$$

Equation (1) evolves into

$$\begin{align*}
\ddot{x}_1 + \omega_1^2x_1 + \mu\omega_2^2(x_1 - x_2) = f\sin(\omega t) \\
\ddot{x}_2 + \omega_2^2(x_2 - x_1) + 2\zeta\omega_2(x_2 - \dot{x}_3) + x_2\omega_2^2x_2 = 0 \\
2\zeta\omega_2(x_3 - \dot{x}_2) + \alpha_1\omega_2^2x_3 = 0
\end{align*}$$

(2)

Supposing $x_1 = X_1e^{j\omega t}$, $x_2 = X_2e^{j\omega t}$ and $x_3 = X_3e^{j\omega t}$, one can obtain

$$X_1 = \frac{f(jA_1 + B_1)}{FC_1 + D_1}$$

(3)

where $j$ is the imaginary unit and the other parameters are expressed as

$$\begin{align*}
A_1 &= -2\zeta\omega_2\omega_2^2 - (1 + \alpha_1 + \alpha_2)\omega_2^2 \\
B_1 &= \alpha_1(1 + \alpha_2)\omega_2^2 - \alpha_1\omega_2^2
\end{align*}$$

(4)

$$\begin{align*}
C_1 &= 2\zeta\omega_2\omega_2^2 + (1 + \alpha_1 + \alpha_2)\omega_2^4 + \mu(\alpha_1 + \alpha_2)\omega_2^4 + \omega_2^2(-\omega_2^2 - (1 + \alpha_1 + \alpha_2 + \mu)\omega_2^2) \\
D_1 &= \alpha_1\omega_2(\omega_2^4 + (1 + \alpha_2)\omega_2^2 + \mu\alpha_2\omega_2^4 - \omega_2^2(\omega_2^2 + (1 + \alpha_2 + \mu)\omega_2^2))
\end{align*}$$

Introducing the parameters $\lambda = \frac{\omega_2}{\omega_1}$, $\nu = \frac{\omega_2}{\omega_1}$ and $\delta = \frac{F_0}{m_1}$ and supposing $A$ is the normalized amplitude amplification factor of the primary system, one could obtain

$$A^2 = \left|\frac{X_1}{\delta}\right|^2 = \frac{\zeta^2A_1^2 + B_1^2}{\zeta^2C_1^2 + D_1^2}$$

(5)

where

$$\begin{align*}
A_2 &= 2\zeta[(1 + \alpha_1 + \alpha_2)\nu^2 - \lambda^2] \\
B_2 &= \alpha_1(1 + \alpha_2)\nu^3 - \alpha_1\lambda^2\nu \\
C_2 &= 2\zeta[\lambda^4 + (1 + \alpha_1 + \alpha_2)\nu^4 + \mu(\alpha_1 + \alpha_2)\nu^4 + \lambda^2[1 - (1 + \alpha_1 + \alpha_2 + \mu)\nu^2]] \\
D_2 &= \alpha_1\nu(\lambda^4 + (1 + \alpha_2)\nu^2 + \mu\alpha_2\nu^4 - \lambda^2[1 + (1 + \alpha_2 + \mu)\nu^2])
\end{align*}$$

(6)
After simple deduction for equation (5), it could be found that the different normalized amplitude–frequency curves with different damping ratios will always intersect at three fixed points, in which the other parameters are fixed. The conclusion has been proven by Den Hartog, and can be observed in Figure 2, where the different damping ratios are taken as 0.2, 0.5 and 0.9, respectively, and other parameters are fixed as \( l = 0.1 \), \( a_1 = -0.5 \), \( a_2 = -0.1 \) and \( \nu = 1.5 \). Here the three fixed points are denoted by P, Q and R, respectively. In order to solve the analytical expression of the fixed points theoretically, the following equation should be met

\[
\frac{A_2}{C_2} = \frac{B_2}{D_2}
\]

(7)

After replacing its numerator and denominator with equation (6), the equation can be obtained

\[
\lambda^6 + a_1 \lambda^4 + a_2 \lambda^2 + a_3 = 0
\]

(8)

where

\[
\begin{align*}
a_1 &= -[1 + (2 + x_1 + 2x_2 + \mu) \nu^2] \\
a_2 &= (2 + x_1 + 2x_2) \nu^2 + [1 + \mu + x_1(1 + x_2 + \mu) + x_2(2 + x_2 + 2\mu) \nu^4] \\
a_3 &= -\{2(1 + x_2)(1 + x_1 + x_2) \nu^4 + [x_1 + 2x_1x_2 + 2x_2(1 + x_2)]\mu \nu^6\}/2
\end{align*}
\]

(9)

When \( \zeta = 0 \), one can get

\[
|A| = \left| \frac{X_1}{\delta} \right| = \left| \frac{(1 + x_2) \nu^2 - \lambda^2}{\lambda^4 + (1 + x_2) \nu^2 + \mu x_2 \nu^4 - \lambda^2 [1 + (1 + x_2 + \mu) \nu^2]} \right|
\]

(10)

If \( \zeta = \infty \), the following result holds

\[
|A| = \left| \frac{X_1}{\delta} \right| = \left| \frac{(1 + x_1 + x_2) \nu^2 - \lambda^2}{\lambda^4 + (1 + x_1 + x_2) \nu^2 + \mu (x_1 + x_2) \nu^4 + \lambda^2 [1 - (1 + x_1 + x_2 + \mu) \nu^2]} \right|
\]

(11)
Because the amplitude–frequency curves with three fixed points are not related to the damping ratio, equations (10) and (11) are equal, i.e.

$$|A| = \left| \frac{X_1}{\delta} \right| = \frac{(2 + 2x_2 + x_1)\nu^2 - 2\mu^2}{x_1\nu^2 + x_1\mu\nu^4 - x_1\nu^2\lambda^2}$$

(12)

If $\lambda_p^2$, $\lambda_Q^2$, and $\lambda_R^2$ are supposed as the three real roots of equation (12), the ordinates of $P$, $Q$, and $R$ could be expressed as

$$\left| \frac{X_1}{\delta} \right|_P = \frac{(2 + 2x_2 + x_1)\nu^2 - 2\mu^2}{x_1\nu^2 + x_1\mu\nu^4 - x_1\nu^2\lambda_P^2},$$

$$\left| \frac{X_1}{\delta} \right|_Q = \frac{(2 + 2x_2 + x_1)\nu^2 - 2\mu^2}{x_1\nu^2 + x_1\mu\nu^4 - x_1\nu^2\lambda_Q^2},$$

$$\left| \frac{X_1}{\delta} \right|_R = \frac{(2 + 2x_2 + x_1)\nu^2 - 2\mu^2}{x_1\nu^2 + x_1\mu\nu^4 - x_1\nu^2\lambda_R^2}$$

(13)

Obviously, the expectancy of vibration reduction is as far as possible for minimizing the maximum amplitude of the amplitude–frequency curve. Based on the fixed-point theory, we expect that the three fixed points will nearly locate at the peaks of the amplitude–frequency curves. In order to realize the optimum control effect, the optimum parameters should be searched, and the process of regulating parameters is divided into two steps.

The first step is adjusting the fixed points $P$ and $R$ to equal height, i.e. $X_{\delta P} = X_{\delta R}$. One could obtain

$$x_1 = 2(-1 - x_2 + \mu + \frac{1}{\nu^2})$$

(14)

Equation (8) involves into

$$(1 + \mu\nu^2 - \lambda^2)(2(1 + x_2)\nu^2 + [(1 + x_2)^2 + \mu + 2\mu\nu^4 - 2(1 + \mu\nu^2)\lambda^2 + \lambda^4]) = 0$$

(15)

Based on equation (15), we can get

$$\lambda_P^2 = 1 + \mu\nu^2 - \sqrt{1 - 2(1 + x_2 - \mu)\nu^2 + [(1 + x_2)^2 - \mu(1 + x_2) + \mu^2]\nu^4}$$

$$\lambda_Q^2 = 1 + \mu\nu^2$$

$$\lambda_R^2 = 1 + \mu\nu^2 + \sqrt{1 - 2(1 + x_2 - \mu)\nu^2 + [(1 + x_2)^2 - \mu(1 + x_2) + \mu^2]\nu^4}$$

(16)

Then equation (13) becomes into

$$\left| \frac{X_1}{\delta} \right|_{P,R} = \frac{1}{1 - (1 + x_2 - \mu)\nu^2},$$

$$\left| \frac{X_1}{\delta} \right|_Q = \frac{1 + (-1 - x_2 + \mu)\nu^2}{\mu\nu^4}$$

(17)

Letting the heights of $P$ (or $R$) and $Q$ are the same, the optimum natural frequency ratio $\nu_{opt}$ can be expressed as

$$\nu_{opt} = \frac{1}{1 + x_2 - \sqrt{\mu} - \mu}$$

(18)
Substituting equation (18) into equation (14), the first optimum value of \( a_1 \) could be obtained

\[
a_{1_{\text{opt}}} = -2\sqrt{\mu}
\]  

By means of the optimum values, the response at the fixed points will be

\[
\left| \frac{X_1}{\delta} \right|_{P,Q,R}^2 = \frac{(-1 - x_2 + \sqrt{\mu} + \mu)^2}{\mu}
\]

The second step is regulating the optimum damping ratio \( \xi \) to make the fixed points nearly locate at the two resonance peaks. Figure 3 presents the amplitude–frequency curves under different damping ratios, in which the case we expect can be realized. When the two resonance peaks are almost at the same height level, the tangent line at the point Q on the amplitude–frequency curve is nearly level. The abscissa value of point Q in equation (16) has been obtained, so that the approximate optimum damping ratio can be solved according to the several above-mentioned optimum conditions and presented as

\[
\begin{align*}
\frac{\partial A^2}{\partial \lambda^2} &= 0 \\
\lambda_1 &= -2\sqrt{\mu} \\
\nu &= \sqrt{\frac{1}{1 + x_2 - \sqrt{\mu} - \mu}} \\
\lambda_Q^2 &= 1 + \mu
\end{align*}
\]

Solving equation (21), one can obtain

\[
\frac{\sqrt{\mu} - \xi^2(1 + x_2 - \sqrt{\mu})^2 + \mu^2}{-1 - x_2 + \sqrt{\mu} + \mu} = 0
\]

and the optimum damping ratio should be

\[
\xi_{\text{opt}} \approx \sqrt{\frac{\mu}{1 + x_2 - \sqrt{\mu}}}
\]
where $a_2$ is a parameter related to NS, and it can regulate the vibration control performance. According to the characteristic of NS, preload generally exists in the NS devices, which causes a pre-displacement of the primary system. We can adopt the pre-displacement as the amplitudes at the fixed points. It means the response about zero-frequency excitation is equal to the one at the fixed points, and the equation is presented as

$$|X_1| \delta_{i=} = |X_1| \delta_{P,Q,R}$$

(24)

i.e.

$$\sqrt{(1 + a^2)^2 / \left(1 + a^2 \mu \alpha^2 / (1 + a^2 + \sqrt{\mu} - \mu)\right)} = \sqrt{(-1 - a^2 + \sqrt{\mu + \mu})^2 / \mu}$$

(25)

Based on equation (25), the NS ratios are shown as

$$a_{2a} = -1 - \sqrt{\mu}$$
$$a_{2b} = -1 + \sqrt{\mu}$$
$$a_{2c} = -1 - (1 + \sqrt{2}) \sqrt{\mu}$$
$$a_{2d} = -1 + (1 + \sqrt{2}) \sqrt{\mu}$$
$$a_{2e} = -1 + \sqrt{\mu} + \mu$$

(26)

Considering equations (18) and (23), one can find that all the roots except $a_{2d}$ will make the optimum natural frequency ratio as imaginary number and the optimum damping ratio infinite. So we only select $a_{2d}$ as the optimum NS ratio, i.e.

$$a_{2opt} = a_{2d} = -1 + (1 + \sqrt{2}) \sqrt{\mu}$$

(27)

Then the adjustable parameters of the DVA are analytically obtained as

$$v_{opt} = \sqrt{1 / 2(\sqrt{\mu} - \mu)}$$
$$a_{1opt} = -2\sqrt{\mu}$$
$$\xi_{opt} = (\frac{1}{4})^\frac{1}{3}$$

(28)

The amplitude–frequency curve under the above optimum parameters is plotted in Figure 4, and the optimization purposes are basically achieved. Here we would like to point out that the two resonance peaks of the curve are not accurately equal, and the reason lies in two aspects. On one hand, it can be seen from Figure 3 that when the two resonance peaks are at the same height, the point with zero slope is near the point Q, that means the zero slope at point Q in the curve is approximate but not precise. On the other hand, the selection principle for choosing the optimal NS ratio is also approximate.

**Comparisons between the presented DVA and other DVAs**

In this section, we will test the control performance of the presented DVA with multiple negative stiffness springs, by comparing it with some classical DVAs proposed by Den Hartog,\textsuperscript{9} Ren,\textsuperscript{14} and Asami\textsuperscript{16} shown in Figure 5(a) to (c). In addition, the model with NS of removing the ground spring is also shown in Figure 5(d). Here all the original mass ratios of the five models are selected as $\mu = 0.1$, and the other optimal system parameters are determined by the results in literature\textsuperscript{9,14,16} and this paper. Based on the original mass ratios and the other
system parameters, one could obtain the normalized amplitude amplification factors of all the five primary systems shown in Figure 6. Here the optimized result of the DVA with single NS (Figure 5(d)) can also be obtained and shown in Figure 6 denoted by N. By comparison, it can be concluded that the DVA in this paper can not only suppress the amplitude of the primary system in resonance region significantly, but also extend the efficient frequency range of vibration absorption.

**Figure 4.** The amplitude–frequency curve under the optimum parameters $\mu = 0.1, x_1 = -0.632, x_2 = -0.237, \nu = 1.697$ and $\zeta = 0.473$.

**Figure 5.** The different models of DVAs. (a) The Voigt type DVA. (b) The DVA by Ren. (c) The DVA by Asmi. (d) The DVA with single negative stiffness.
The comparison with DVAs under the random excitation

In order to further verify the performance of the presented DVA and considering that the random excitations are common in the engineering practice, the response of the primary system to random excitation is investigated in this section. When the primary system is subjected to random excitation with zero mean, the power spectral density is denoted by \( S(\omega) = S_0 \). The power spectral density function of the absolute displacement response of different models are presented as

\[
\begin{align*}
S_O(\omega) &= |X_{1O}|^2 S_0, \\
S_V(\omega) &= |X_{1V}|^2 S_0, \\
S_R(\omega) &= |X_{1R}|^2 S_0, \\
S_A(\omega) &= |X_{1A}|^2 S_0, \\
S_N(\omega) &= |X_{1N}|^2 S_0
\end{align*}
\]

(29)

where the subscript \( O, V, R, A, \) and \( N \) represent for the Maxwell model with multiple negative stiffness springs, the Voigt type DVA by Den Hartog, the model by Ren, Asami, and the single NS model, respectively. The mean square responses of the primary systems about the abovementioned different DVAs are presented as

\[
\begin{align*}
\sigma_O^2 &= \int_{-\infty}^{\infty} S_O(\omega) d\omega = S_0 \int_{-\infty}^{\infty} |X_{1O}|^2 d\omega = \frac{\pi S_0 Y_1}{2 \omega_0^4 \mu^2 v^2 x_1^2 (1 + x_2 + \mu x_2 v^2)} \\
\sigma_V^2 &= \int_{-\infty}^{\infty} S_V(\omega) d\omega = S_0 \int_{-\infty}^{\infty} |X_{1V}|^2 d\omega = \frac{\pi S_0 Y_2}{2 \omega_0^4 \mu^2 v^2} \\
\sigma_R^2 &= \int_{-\infty}^{\infty} S_R(\omega) d\omega = S_0 \int_{-\infty}^{\infty} |X_{1R}|^2 d\omega = \frac{\pi S_0 Y_3}{2 \omega_0^4 \mu^2 v^2} \\
\sigma_A^2 &= \int_{-\infty}^{\infty} S_A(\omega) d\omega = S_0 \int_{-\infty}^{\infty} |X_{1A}|^2 d\omega = \frac{\pi S_0 Y_4}{2 \omega_0^4 \mu^2 v^2} \\
\sigma_N^2 &= \int_{-\infty}^{\infty} S_N(\omega) d\omega = S_0 \int_{-\infty}^{\infty} |X_{1N}|^2 d\omega = \frac{\pi S_0 Y_5}{2 \omega_0^4 \mu^2 v^2} \\
\end{align*}
\]

(30)

where

\[
\begin{align*}
Y_1 &= 4 \xi^2 (1 + x_2 + \mu x_2 v^2) \left\{ 1 - 2 (1 + x_1 + x_2 - \mu) v^2 + \left[ (1 + x_1 + x_2)^2 - (1 + 2 x_1 + 2 x_2) \mu + \mu^2 \right] v^4 \right\} \\
&\quad + \xi_1^2 v^2 \left\{ 1 + x_2 + \left[ -2 (1 + x_2)^2 + \mu + 2 \mu x_2 \right] v^2 + \left[ (1 + x_2)^3 - 2 x_2 (1 + x_2) \mu + x_2 \mu^2 \right] v^4 \right\} \\
Y_2 &= [1 + v^4 (1 + \mu)^2 + v^2 (4 \mu^2 + 4 \xi^2 - \mu - 2)] \\
Y_3 &= [1 + v^4 + (\mu + 4 \xi^2 + 2)] \\
Y_4 &= k^2 v^2 \left\{ 1 - (2 + \mu) v^2 + (1 + \mu) v^2 \right\} + 4 \xi^2 \left\{ 1 + (1 + k) v^2 \left[-2 + (1 + k) (1 + \mu) v^2 \right]\right\} \\
Y_5 &= 4 \xi^2 \left\{ 1 - 2 v^2 (1 + x - \mu) + v^4 \left[ (1 + x)^2 - \mu (1 + 2 x) + \mu^2 \right]\right\} + x^2 v^2 \left[ 1 + (\mu - 2) v^2 + v^4 \right]
\end{align*}
\]

(31)

**Figure 6.** The comparison of the presented DVA with other DVAs.
Figure 7. Time histories of the force excitation.

Figure 8. Time histories of the displacement without DVA.

Table 1. The variances and reduction ratios of the primary systems.

| Models                   | Displacement variances | Reduction ratio/% |
|--------------------------|------------------------|-------------------|
| Without DVA              | 3.114e-04              |                   |
| DVA by Den Hartog        | 2.5889e-05             | 91.69             |
| DVA by Ren               | 2.1160e-05             | 93.20             |
| DVA by Asami             | 2.3384e-05             | 92.49             |
| DVA by single NS model   | 1.3234e-05             | 95.75             |
| DVA in this paper        | 1.0324e-05             | 96.68             |

Figure 9. Time histories of the displacement with DVA by Den Hartog.
According to the literature\textsuperscript{9,14,16,26} and the results in this paper, the mean square responses of the primary systems under the optimum parameters can be calculated respectively when $\mu = 0.1$

$$
\begin{align*}
\sigma_D^2 &= \frac{2.619\pi S_0}{\omega_1^2}, & \sigma_V^2 &= \frac{6.401\pi S_0}{\omega_1^2}, & \sigma_R^2 &= \frac{5.780\pi S_0}{\omega_1^2} \\
\sigma_A^2 &= \frac{6.030\pi S_0}{\omega_1^2}, & \sigma_N^2 &= \frac{4.020\pi S_0}{\omega_1^2}
\end{align*}
$$

(32)

**Figure 10.** Time histories of the displacement with DVA by Ren.

**Figure 11.** Time histories of the displacement with DVA by Asami.

**Figure 12.** Time histories of the displacement with DVA by single NS model.
In order to get more meaningful results, we construct 50 s random excitation, that is composed of 5000 normalized random numbers with zero mean value and unit variance. Figure 7 gives the time history of the random excitation, and the time history of the primary system without DVA is presented in Figure 8. Here the primary mass \( m_1 = 1 \text{ kg} \) and the stiffness of the primary system \( k_1 = 100 \text{ N/m} \). For comparison, the random responses of all the five DVAs are also presented with their corresponding optimal values. Some statistical properties, including the response variances and decrease ratios of the primary systems for the different systems are summarized in Table 1.

From Figures 8 to 13 it can be found that all the five DVAs could suppress the vibration of the primary system to random excitation, especially the DVA in this paper. The results show that the Maxwell DVA with multiple negative stiffness springs possesses the minimum mean square response. This means the presented model can achieve better control performance than other DVAs under the random excitation. Moreover, the better results can exist in wide range about parameter mass ratios.

**Conclusions**

A Maxwell dynamic vibration absorber with multiple negative stiffness springs is proposed and investigated in detail. The optimum parameters of the stiffness and damping of the absorber are designed according to the fixed-point theory and \( H_\infty \) optimization principle. Especially, the two NS ratios are obtained in the case of ensuring the system stability and optimizing response. It is proved that the model of DVA with multiple negative stiffness springs can not only suppress the amplitude of the primary system in resonance region significantly, but also widen the efficient frequency range of vibration absorption with harmonic excitation. Furthermore, the Maxwell DVA in this paper behaves in better vibration reduction effect than other DVAs through comparison even if in the random excitation. The results should provide some theory references to design more effective vibration control devices in the engineering practice.

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