CHRISTOPHER HOOLEY
7 August 1928 — 13 December 2018
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Christopher Hooley was one of the leading analytic number theorists of his day, world-wide. His early work on Artin’s conjecture for primitive roots remains the definitive investigation in the area. His greatest contribution, however, was the introduction of exponential sums into every corner of analytic number theory, bringing the power of Deligne’s ‘Riemann hypothesis’ for varieties over finite fields to bear throughout the subject. For many he was a figure who bridged the classical period of Hardy and Littlewood with the modern era. This biographical sketch describes how he succeeded in applying the latest tools to famous old problems.

LIFE AND CAREER

Childhood to student days

Christopher Hooley was born on 7 August 1928 in Edinburgh, the only child of Leonard and Barbara Hooley. He attended Wilmslow Preparatory School and then Abbotsholme School. After finishing school, Hooley served in Egypt as a subaltern with the Royal Engineers, and later became a captain in the Royal Army Educational Corps, based for a time at Eltham Palace in South East London. He went on to study mathematics at Cambridge.

Hooley gave his own description of his time at Cambridge in a talk at a conference in Bristol celebrating 50 years since his seminal work on Artin’s conjecture (Biggs 2017):

Having been ill-prepared for Part II of the Maths Tripos on account of my army service, I took this at the beginning of my second year. Also on account of my army service I was eligible for a F.E.T. [further education and training] Grant for a period of up to six years. So I took Part III

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in my fourth year. By this time I had acquired a great interest in matrices and I judge now had by that time some material for publication. I therefore offered as one of my subjects in Part III a book by Wedderburn on matrices. As a result among my questions in the Part III exam was one on elementary divisors. But what I was asked to prove was incorrect. So I responded in two ways (i) by providing a counter example and then (ii) by proving what I thought the examiner meant me to establish. Nevertheless I did not succeed in securing a ‘Distinction’ as opposed to a pass. In consequence the lecturer I thought might supervise my research refused to take me on. There was then a lot of discussion between myself and Corpus Christi College. Also between myself and my Mother and Father, who thought that I shouldn’t pursue my mathematical ambitions further. I owe a great debt of gratitude to Birgitta, my fiancée who encouraged me to persevere in my ambitions.

What followed on was rather traumatic. I tried the celebrated Prof. Hall, the group theorist, who was very polite but said he could not find any suitable topic for me to study. I then approached J.A. Todd because his sort of projective geometry involved a lot of matrix theory. He was not very encouraging saying that before the war only the very best went on to mathematical research. I confided my difficulties to Dr Michael Drazin (the elder brother of Philip who was later a Professor in Bristol). He like me went to the tea and morning dances at the Dorothy Café (Hawkins) and was a Prize Fellow of Trinity, later a Professor at Duke University in the south of the U.S.A. He said, ‘Why don’t you try Mr Ingham?’; he said his pupils usually did very well. So I took his advice and made an appointment to see Ingham at King’s. He was also very polite but said he would have to see my answers in Part III before taking matters further. A bit later he asked to see me again and asked me to look at a question related to some work of Mirsky. At the end of the long vacation I had something to shew1 him as a result of which he agreed to supervise my research. Then, after further negotiation between the War Office and my College, it was agreed my work would be supported for a further two years—this was actually most generous of the War Office.

The problem set by Ingham was to find an asymptotic formula for the sum

\[ \sum_{n \leq x} d(n)d(n + a)d(n + b), \quad a \neq b \neq 0 \]

extending his formula for

\[ \sum_{n \leq x} d(n)d(n + a). \]

One of the objects of this would be then that one should replace \( d(n) \) by \( r(n) \), the number of representations of \( n \) as the sum of two squares. One would thus establish the infinitude of triples \( n, n + a, n + b \) that were sums of two squares. Having told Ingham that I had hardly studied any number theory before, he merely recommended that I read a specific four chapters of Hardy and Wright—you can imagine how unlikely such advice would be today. I found the problem set impossibly difficult. Indeed it remains unsolved so far as I know up to this day. Even Iwaniec with all his perspicuity has I think failed to solve it.2 However, I have solved the problem of triplets of sums of two squares that it was Ingham’s intention for us to solve. [See (6).]

However, I came across the problem of asymptotic formulae for

\[ \sum_{n \leq x} d(n)d_3(n + a), \quad \sum_{n \leq x} d_3(n)d_3(n + a) \]

1 Here we see some of Hooley’s well-known love of archaic wording.
2 This triple correlation problem for the divisor function remains an important open problem.
Christopher Hooley

that were discussed by Titchmarsh in the Quarterly Journal.\textsuperscript{3} The methods I had developed in connection with Ingham’s problem, I used to solve the first problem with some complexity. This with some work on the latter problem gave me a Prize Fellowship at Corpus.

Christopher and Birgitta had married in 1954, and their first child, Thomas, was born early in 1956. The fellowship at Corpus lasted from 1955 to 1958, in which year Hooley completed his PhD with a thesis entitled ‘Some theorems in the additive theory of numbers’. At the end of the fellowship he took up a post in Bristol, where Heilbronn was head of department. After the move, the family bought a house, Rushmoor Grange, some distance outside the city, in Backwell. This seventeenth-century listed building was to remain the family home for the next 50 years. After seven years in Bristol, and the birth of a second son, Adam, Hooley was appointed as professor of pure mathematics in Durham. However, he only stayed for a couple of years before returning south, becoming head of the pure mathematics department of the University College of South Wales and Monmouthshire, a constituent college of the University of Wales.

The Cardiff years

Hooley’s predecessor in Cardiff, Aubrey Ingleton, had resigned for family reasons after only a year, and it was a difficult time for University College to find a suitable replacement. Hooley’s arrival was greeted with much relief, and naturally with delight. Having retained the family home in Somerset during his time in Durham, Hooley somehow managed to persuade the College to overlook the rule that all staff had to live within a radius of 25 miles, and regularly drove between Bristol and Cardiff. Hooley rapidly created a powerful group of analytic number theorists and added to the existing strengths of analysis and group theory.

In 1988, University College merged with the University of Wales Institute of Science and Technology (UWIST), the new institution later becoming Cardiff University. Hooley became head of the combined mathematics departments and stayed in that role until his retirement in 1995. He was Distinguished Research Professor at Cardiff until 2008.

Those who knew him at Cardiff say that he was usually happy to keep himself to himself, and as a result of this he was not always easily approachable. Indeed, he was well known for having had no jointly authored research papers, something quite remarkable to twenty-first-century mathematicians. Moreover, he had just one research student, George Greaves, who later joined the staff with him in Cardiff.

Back to Bristol

After retiring at Cardiff, Hooley was given a succession of visiting positions at the University of Bristol. A key component of these appointments was that he gave an annual series of six to eight lectures for postgraduate students, Heilbronn Fellows, and other researchers. These became known locally as the Hooley Lectures. He gave the first course in 2009, and they continued until 2016, after which ill-health prevented him continuing. He prepared these lectures very carefully and took great pride in them. The subject changed each year, covering everything from sieve methods, to Dwork’s work on the rationality of the Zeta-function and Gauss’ theory of composition. Students found them insightful and interesting, with great attention to the historical development of the subject. They were also greatly amused by the

\textsuperscript{3} See Titchmarsh (1942).
wide-ranging and arcane vocabulary. Everyone enjoyed hearing about ‘the tyro’s approach to the sinister side of the equation’. The classical style in which they were delivered seemed unchanged from when Hooley himself was a student and often involved archaic terminology. It was clear that he did not welcome questions during the lectures, as these interrupted his flow, but afterwards he enjoyed discussing the content at some length informally.

Hooley was sceptical of some aspects of modern technology. Tim Browning recalls him looking for a blackboard eraser and accidentally picking up the remote control for the projector, attempting to clean the board with it and then mislabelling it as a ‘mobile telephone’ in a scathing tone. He would write his papers by hand and have a secretary type them. In his Cardiff days he would insist on using foolscap paper size, when stationers had almost universally switched to A4. Later he would have graduate student assistants to turn his manuscripts into LaTeX. These manuscripts were extremely accurately prepared, so that the LaTeX files needed no further correction—if the student did their job properly!

He was a regular participant at Bristol’s weekly Number Theory Seminar, and often went out afterwards for a drink (in his case a whiskey, he preferred Irish) with the speaker—something he had done less frequently in Cardiff. Around the Bristol School of Mathematics his style was formal and he liked to refer to everyone by title and surname. He rarely used first names. He took pains to enquire whether female administrators were Mrs or Miss and referred to them in this way. He seemed never to understand the concept of Ms. He developed a particular friendship with two members of the Heilbronn Institute for Mathematical Research (HIMR) administration team—Chrystal Cherniwchan and Fran Blake—and indeed they were invited to the celebration his family organized for his ninetieth birthday. It seems that they had a much easier relationship than any of his mathematical colleagues had managed. In particular, Hooley shared with Chrystal Cherniwchan an interest in dogs, and this was a subject of many lengthy conversations. Hooley seemed intimidated by the idea of pronouncing Chrystal Cherniwchan’s surname and embarrassed by the possibility of getting it wrong, so she was one of the very few people whom he referred to by her first name only. In fact, having reached that point, he went on to use first names for the other HIMR administrators. However, they never felt able to reciprocate.

**Home life**

Christopher Hooley had many interests, most of which were pursued with vigour. From early on as a pupil at Abbotsholme School during the Second World War he developed a love not only of mathematics, but also of classical, military and naval history. In years to come he would delight in discussing the history of western and eastern Roman empires with his colleagues from the history department at Cardiff, who would ruefully admit they were out of their comfort zone, using that standard excuse ‘not my period’—Hooley counted all European history of the two millennia AD as his period. This interest went along with a love of antiques and especially of collecting West Country friendly society brasses. Many days were spent scouring antique shops and following up leads to remote places in Somerset during the 1970s and 80s. Birgitta, denied pets as a child in Sweden, made sure that the family always had plenty of dogs—at one time six—and sometimes cats as well; this interest led to another hobby, that of travelling to terrier shows in the summer months, from which Birgitta, Christopher and their two sons, Thomas and Adam, often returned proudly bearing coloured rosettes for their dogs’

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4 This section is based on material kindly supplied by Thomas and Adam Hooley.
Christopher Hooley

successes. Supportive as a father, he was nonetheless quick to spot and point out inaccuracies of memory in the reminiscences of his sons at his ninetieth birthday party, held in Bristol just four months before he died in December 2018.

Hooley’s grandfather had taught carpentry in Macclesfield and, on inheriting his tools, Hooley set about using them to restore or completely remake a large number of period sash windows in the family home in Somerset, often at some considerable height. Once done, he turned his hand to panelling, lath and plaster, and similar skills with great success (figure 1).

The practical side was also present when it came to motoring. The first family car, a 1950s Wolseley 6/80, was kept and later given a thorough restoration in the 1980s; he showed no fear in creating an entire wiring loom from scratch. These interests and his family life provided the counterpoint to Hooley’s academic career. At times unconventional, for instance researching deep into the night with his beloved whiskey and cigarette alongside an eclectic mixture of steam railway magazines, Dante, Milton and paperback thrillers, he had a ready wit and excellent sense of humour.

Hooley was devastated by the death of Birgitta, from cancer, in 2013. Although he managed to form new friendships with generous and caring neighbours, it was clear that he missed his wife terribly in his last five years. He had been interested in religion since his student days, but only became a regular church-goer after Birgitta’s death, and was confirmed in 2014.

Hooley had been a life-long smoker, and appeared not to suffer any ill-effects. He would enjoy long walks in the Cotswolds and Mendips with no effort. However, quite suddenly towards the end of 2016 he lost the circulation in one leg, and was left with little option beyond amputation. He found the prosthetic leg very uncomfortable and eventually gave up on it, using a wheelchair instead (figure 2). He was fortunate enough to have the kind assistance of a neighbour who drove him to mathematical meetings and would accompany him at associated social events.

Professor Christopher Hooley FRS died quietly, after being unconscious for 24 hours, on the afternoon of Thursday 13 December 2018.

Research work

Artin’s conjecture on primitive roots

Hooley’s most widely cited paper was his work ‘On Artin’s conjecture’ (4). This concerned the number \(N_a(x)\) of primes \(p \leq x\) for which a given integer \(a\) is a primitive root. It was conjectured by Emil Artin that there should be infinitely many such primes \(p\) whenever \(a\) is not 0, \(\pm 1\) or a square. Indeed when \(a\) is not excluded he conjectured that

\[N_a(x) \sim A(a) \pi(x) \quad (x \to \infty),\]

with an explicitly given positive constant \(A(a)\). Calculations by D. H. Lehmer suggested that Artin’s prediction for this constant was not always correct, and Heilbronn suggested a revised formula for \(A(a)\). However, no-one had come close to a proof. Thus it must have been something of a surprise when Hooley’s paper appeared, establishing Artin’s conjectured asymptotic formula for all relevant \(a\) (with the constant \(A(a)\) as revised by Heilbronn), subject to the Riemann hypothesis for the Dedekind Zeta-functions of an appropriate family of number-fields (the ‘extended Riemann hypothesis’, ERH).
Figure 1. Christopher Hooley at work on his house, Rushmoor Grange in Backwell, Somerset, where he lived for nearly 60 years. (Online version in colour.)
Hooley had been familiar with Artin’s conjecture since his Cambridge days, and indeed Ingham had suggested he look into it. However, a remark by Heilbronn, while on a visit to Cambridge, suggested that he had already cracked it. As a result, Hooley paid the problem no further attention until after his arrival in Durham, where Vernon Armitage mentioned it in terms suggesting it was still open. Hooley took the train back to Bristol that evening, where his wife and children were still living. All the way back he worked on the problem, ensconced in the dining car with a cigarette and a whiskey, and by the time he arrived he had the proof worked out.

In 1967 those working in elementary and analytic number theory—the area to which Artin’s statement apparently belongs—would have thought Dedekind Zeta-functions, as used by Hooley, to be obscure and erudite constructions, with no conceivable connection to such down-to-earth objects as primitive roots. In fact, as Hooley notes in his paper, the function-field analogue of his theorem had been proved by Bilharz back in 1937. (Indeed, Bilharz’s work became unconditional after Weil’s later proof of the Riemann hypothesis for curves over finite fields.) It is not clear whether Hooley studied the Bilharz paper, which would probably have given him a strong hint as to how to tackle Artin’s conjecture.

The majority of Hooley’s oeuvre uses rather traditional analytic tools, and it is not until his 1986 paper (17), using Hasse–Weil $L$-functions, that higher degree $L$-functions occur again. Hooley’s work on the Artin conjecture was published soon after he left Bristol, where he had been a colleague of Heilbronn’s. One could speculate that Hooley’s use of Dedekind Zeta-functions as tools in analytic number theory had been influenced by Heilbronn’s expertise in the area.
While there has been no unconditional treatment of Artin’s conjecture, it was shown unconditionally by Gupta and Murty (1984) that there is at least one integer $a$ for which $N_a(x)$ tends to infinity. This was refined by Heath-Brown (1986) to show that if $a, b, c$ are multiplicatively independent, and if none of $a, b, c, -3ab, -3bc - 3ca$, or $abc$ is a square, then at least one of $N_a(x), N_b(x), N_c(x)$ tends to infinity. In particular it follows that there are at most two prime values of $a$ for which $N_a(x)$ could remain bounded.

The ideas from Hooley’s paper have, somewhat surprisingly, been used to handle the Euclidean algorithm in number-fields. Thus it was shown by Weinberger (1973), under the assumption that the Riemann hypothesis holds for all Dedekind Zeta-functions, that the ring of integers of a number field of class number 1 is a Euclidean Domain, provided only that the
unit group is infinite. (For \( \mathbb{Q}\left(\sqrt{-163}\right) \), for example, the unit group is finite, and there is no Euclidean algorithm.)

Hooley’s argument begins with the observation that \( a \) is a primitive root of a prime \( p \) if and only if it is not a \( q \)-th power modulo \( p \) for every prime \( q \mid p - 1 \). This allows him to set up sieve inequalities

\[
N(x, \alpha) - M(x; \alpha, x) \leq N_a(x) \leq N(x, \alpha),
\]

in which \( N(x, \alpha) \) counts primes \( p \leq x \) for which \( a \) is not a \( q \)-th power for primes \( q \leq \alpha \) dividing \( p - 1 \), while \( M(x; u, v) \) counts primes \( p \leq x \) for which \( a \) is a \( q \)-th power modulo \( p \) for some prime \( q \in (u, v] \) dividing \( p - 1 \).

When \( p \) does not divide \( aq \), the primes \( p \equiv 1 \pmod{q} \), for which \( a \) is a \( q \)-th power modulo \( p \), are precisely those that split completely over \( K = \mathbb{Q}\left(\sqrt[3]{a}, \sqrt[9]{a}\right) \), so that they can be counted by the prime ideal theorem for \( K \). The standard zero-free region for the Dedekind Zeta-function \( \zeta_K(s) \) is then enough to deduce unconditionally that

\[
N(x, \alpha) \sim A(a)\pi(x), \quad (x \to \infty),
\]

if \( \alpha = \alpha(x) \) tends to zero suitably slowly. (Hooley uses the ERH here, though this is unnecessary, and takes \( \alpha = \frac{1}{6} \log x \).) In particular one may obtain the correct asymptotic upper bound for \( N_a(x) \) unconditionally.

The real difficulty in the proof is then to show that \( M(x; \alpha, x) = o(\pi(x)) \). This is achieved by considering separately \( M(x; \alpha, \beta) \), \( M(x; \beta, \gamma) \) and \( M(x; \gamma, x) \), where \( \beta = x^{1/2}(\log x)^{-2} \) and \( \gamma = x^{1/2}(\log x)^{2} \). It is only the range \( \alpha < q \leq \beta \) where ERH is used. If we write \( \pi_{a,q}(x) \) for the number of primes \( p \equiv 1 \pmod{q} \) up to \( x \), such that \( a \) is a \( q \)-th power modulo \( p \), then ERH yields

\[
\pi_{a,q}(x) = \frac{\pi(x)}{q(q - 1)} + O(x^{1/2} \log x). \tag{1}
\]

Here the exponent \( 1/2 \) appearing in the error term is a consequence of the zeros of the relevant Dedekind Zeta-functions being on the \( \frac{1}{2} \)-line. This estimate gives a satisfactory bound for \( M(x; \alpha, \beta) \) provided \( \alpha \to \infty \) and \( \beta = o(x^{1/2}(\log x)^{-2}) \).

For the intermediate range \( \beta < q \leq \gamma \) Hooley uses the Brun–Titchmarsh bound

\[
\pi_{a,q}(x) \leq \pi(x; q, 1) \leq \frac{x}{(q - 1)\log(x/q)}, \tag{2}
\]

which gives a satisfactory estimate since \( \log \gamma - \log \log \beta = o(1) \). Finally, large values of \( q \) are handled using the observation that \( p \) must divide \( a^m - 1 = a^{(p - 1)/q} - 1 \) for some \( m \leq x_{\gamma^{-1}} \). Each \( m \) can correspond to at most \( O(m) \) primes \( p \), again giving a satisfactory bound.

Hooley remarks that one can show that \( N_a(x) \to \infty \) under a weaker hypothesis than ERH. Although he provides no details, one can guess what he had in mind. One can estimate \( M(x; x^\mu, \beta) \) using [2], leading to a bound

\[
M(x; x^\mu, \beta) \ll \left(\frac{1}{2} - \mu\right) \pi(x).
\]

One can then deduce that \( N_a(x) \gg \pi(x) \), provided that \( \mu \) is taken to be a constant sufficiently close to \( \frac{1}{2} \). For the range \( \alpha < q \leq x^\mu \) it would then suffice to have an analogue of [1] with an error term \( O(x^\phi \log x) \) in which \( \phi < 1 - \mu \). One therefore sees that a quasi Riemann hypothesis would suffice, in which zeros of the relevant Dedekind Zeta-functions had real part at most \( \phi \). It seems likely that one could produce a variant of this argument in which
\[ \phi = 1 - (2e)^{-1} + \varepsilon. \]

This would use the Bombieri–Vinogradov Theorem in place of the Brun–Titchmarsh Theorem, and replace \( \pi_{a,q}(x) \) by the count for primes \( p \) as before, but with the additional condition that \( a \) is not an \( r \)-th power modulo \( p \) for any prime \( r \leq \alpha \).

The idea of proving a lower bound rather than an asymptotic formula, by using a relatively trivial estimate (the Brun–Titchmarsh Theorem in this instance) over part of the range, is something that can be employed in other quite different circumstances. Thus in Hooley’s work of 2009 (31), for example, it is used in the context of power-free values of polynomials.

**The use of exponential sums**

More than anything, Hooley is credited with promoting the use of ‘arithmetic’ exponential sums—sometimes called complete exponential sums. He was certainly not the first to use these. Indeed, Kloosterman’s classic paper (Kloosterman 1926) introduced the sum

\[ \sum_{n=1}^{p-1} \exp \left( \frac{2\pi i}{p} (an + bn) \right), \]

where \( \bar{n} \) is the inverse modulo \( p \). In his paper, Kloosterman gave a non-trivial bound for the sum, and used it in an arithmetic context. However, Hooley employed exponential sums of very many kinds, in all sorts of problems, and contributed to the literature on estimates for such sums as well.

When the sum has essentially one variable, one gets square-root cancellation via Weil’s ‘Riemann hypothesis’ (Weil 1941) for curves over finite fields. For higher dimensional sums, Deligne’s work (Deligne 1974) gives square-root cancellation unless there is a geometric reason why one should not—but rigorously identifying such reasons can be very difficult.

Some of Hooley’s work uses the strength of Weil’s bounds in a quite straightforward way. Thus, in an early work (1), he shows, for example, that

\[ \sum_{n \leq x} d(n^2 + 1) = C_1 x \log x + C_2 x + O(x^{8/9} \log^3 x). \]

An asymptotic formula without a power saving would be relatively easy. However, Hooley shows quite ingeniously how to convert the question into one involving Kloosterman’s sum. Once this is achieved, the required result follows using Weil’s estimate. These ideas were applied again (3) in showing that \( n^2 + 1 \) has a prime factor larger than \( n^{11/10} \) for infinitely many integers \( n \). Weil’s bound for the Kloosterman sum is also applied crucially in another paper (5) on the Brun–Titchmarsh Theorem, but here it is used in a more analytic setting, in conjunction with Perron’s formula.

Other papers use bounds for exponential sums in a much more subtle way, in situations where a simple order-estimate for the sum is not sufficient. Perhaps the first example of this is in Hooley’s paper (2) on the distribution of roots of congruences \( F(x) \equiv 0(\text{mod } k) \), for a fixed irreducible polynomial \( F \) and varying \( k \). Here one considers

\[ S(k) = \sum_{j=1}^{J} \exp \left( \frac{2\pi i m_j}{k} \right), \]

where \( m_1, \ldots, m_J \) are the roots of the polynomial congruence, with \( J = J(k) \) depending on \( k \). On average \( J(k) \) has order 1, but the average is dominated by large values of \( J(k) \), with most
Having $J(k) = 0$. This enables Hooley to show that
\[ \sum_{k \leq x} |S(k)| \ll x (\log x)^{-\delta}, \]
(for some small constant $\delta > 0$) even though one cannot bound individual values of $S(k)$ non-trivially. Thus, one makes only a log-power saving over the trivial bound.

Another example in which delicate methods allow a log-power saving is given by his work that proves an asymptotic formula for the number of representations of a large integer $N$ as a sum of two squares and three cubes. The exponential sum that occurs here is
\[ F(k) := \sum_{a,b,c} \left| \sum_{x^3 + y^3 + z^3 \equiv N(k) \pmod{k}} \exp \left( \frac{2\pi i}{k}(ax + by + cz) \right) \right|^4, \]
where $a, b, c$ and $x, y, z$ run modulo $k$. This is multiplicative in $k$ and is easily seen, via a second moment estimate, to satisfy $F(k) \leq k^3 \rho(N, k)^{1/2}$, where $\rho(N, k)$ is the number of solutions to the congruence $x^3 + y^3 + z^3 \equiv N \pmod{k}$. Such an estimate is insufficient, by a log log factor. However, using a deep result of Milne, Hooley deduces that one has a lower bound
\[ \sum_{a,b,c} \left| \sum_{x^3 + y^3 + z^3 \equiv N \pmod{p}} \exp \left( \frac{2\pi i}{p}(ax + by + cz) \right) \right|^4 \geq 2p^7 + O(p^{13/2}), \]
when $k$ is a prime $p \equiv 1(\text{mod } 3)$. This, together with the second moment bound, is shown to imply that $F(p) \leq Ap$ with an absolute constant $A < 1$, for all sufficiently large $p \equiv 1(\text{mod } 3)$. As a result, Hooley obtains a small log-power saving for $F(k)$, on average, and this suffices for his asymptotic formula.

In addition to promoting the use of exponential sums in applications, Hooley contributed to the theory of the estimation of such sums. In general, analytic number theorists had left the theory to those with expertise in algebraic geometry, so that it is quite remarkable that Hooley was able to say something of value. He examined exponential sums of the shape
\[ \sum_{g(x,y,z) \equiv 0 \pmod{p}} \exp \left( \frac{2\pi i}{p} f(x, y, z) \right) \]
where $f$ and $g$ are integral polynomials, and $x, y, z$ run over integers modulo $p$ (16). He shows, under mild conditions, that the sum is $O(p)$, with an implied constant depending only on the degrees of $f$ and $g$. While his argument uses some basic properties of $L$-functions of varieties over finite fields, including Deligne’s ‘Riemann hypothesis’, it avoids any of the more advanced theory appearing in other approaches to such problems.

A second paper that should be mentioned here is the work counting points on a projective complete intersection $V$ defined over a finite field $\mathbb{F}_q$ (20). If $V$ has dimension $n$, it is shown that the number of points over the finite field is
\[ (q^{n+1} - 1)/(q - 1) + O(q^{(n+d-1)/2}), \]
where $d$ is the dimension of the singular locus. For a smooth variety one has $d = -1$ by convention, so for this case one just has the Deligne estimate. For singular varieties Hooley gives two proofs, both of which use induction via hyperplane slices.
It was shown by Davenport (Davenport 1963) that any cubic form
\[ C(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n] \]
in \( n \geq 16 \) variables has a non-trivial integer zero. This result was improved by Heath-Brown (Heath-Brown 1983) to allow \( n \geq 10 \), provided that the form \( C \) is non-singular. The two methods differ substantially, and in particular Heath-Brown used Kloosterman’s version of the circle method, coupled with Deligne’s estimates for exponential sums. In order to go below \( n = 10 \) one would have to assume that the projective variety \( C = 0 \) has points everywhere locally.

In the first of his papers on the subject (18), Hooley made significant refinements to Heath-Brown’s methods, and showed that any non-singular cubic in \( n \geq 9 \) variables has non-trivial integer zeros, assuming of course that the local condition is met. Hooley’s most important advance lay in the treatment of the exponential sums. Heath-Brown’s analysis only just falls short of allowing \( n \geq 9 \). In essence, it involves an asymptotic formula with main term of order \( B^{n-3} \) and error term of order \( B^{(3n-3)/4+\varepsilon} \), for any fixed \( \varepsilon > 0 \). With more work, Hooley was able to replace \( B^n \) by a log log factor, reaching a situation somewhat similar to that he had encountered with his work (14) on sums of two squares and three cubes; and, indeed, a rather similar solution was developed. One wants to estimate

\[ \sum_{m_1, \ldots, m_n} \left| \sum_h \sum_{r_1, \ldots, r_n} \exp \left\{ \frac{2\pi i}{p} (hC(r_1, \ldots, r_n) + m_1r_1 + \cdots + m_nr_n) \right\} \right|, \]

where all the variables run modulo a prime \( p \). A second moment bound easily produces an estimate \( p^{(3n+1)/2} + O(p^{(3n-1)/2}) \) for the first moment sum above. Then, using a lower bound for the fourth moment, due to Katz, Hooley deduces that the first moment above is at most \( A p^{(3n+1)/2} \), for some absolute constant \( A < 1 \), on a set of primes \( p \) of positive Dirichlet density. This ultimately leads to a saving of a small power of a logarithm, which suffices for the theorem.

His paper (18) involves two further technical points of interest. First, one uses Kloosterman’s form of the circle method, in which one averages non-trivially over the numerators \( a \) of rational numbers \( a/q \) approximating points in the range of integration. In order not to waste unnecessary logarithm powers, this operation has to be performed particularly carefully, using smoothed Farey arcs, but also using smooth weights in the sum over denominators \( q \). In this last step one sees a precursor to Hooley’s ‘second Kloosterman refinement’, of which we will say more later. The second technical point relates to rational points on the dual variety. Such points were a cause of difficulty in Heath-Brown’s paper, but Hooley observes that a non-singular rational point on the dual variety automatically produces a rational zero of \( C \), so that one need only worry about singular points on the dual, which cannot be very frequent.

Two later papers in the series (19) and (32) weaken the requirement that \( C \) should be non-singular. In particular, it is shown (32) that \( C(x_1, \ldots, x_n) \) has a non-trivial integer zero whenever there are zeros everywhere locally, provided that \( n \geq 9 \), and provided that any singularities of the corresponding projective variety are isolated (ordinary) double points.

A further question of interest concerns the ‘weak approximation’ property. Thus one may ask, for example, whether one can require the integer zeros of \( C \) to satisfy appropriate
congruence restraints, and to lie in a given region of $\mathbb{R}^n$. In general, the circle method handles such questions almost automatically. However, Hooley’s first paper on the subject (18) sometimes produces zeros of $C$ out of points on the dual variety, and one cannot control these locally. This deficiency was rectified later (21), where satisfactory exponential sum estimates were established, even when the corresponding parameter lies on the dual variety. In consequence, a smooth projective variety $C(x_1, \ldots, x_n) = 0$ satisfies the weak approximation property as soon as $n \geq 9$.

One of Hooley’s most interesting contributions to the analytic theory of cubic forms has its roots in his paper on Waring’s problem for cubes (17). The paper gives a conditional proof of the Hardy–Littlewood asymptotic formula for the number of representation of a positive integer $N$ as a sum of 7 or more cubes, with an error term which saves a power of $N$. This is an easy deduction from a powerful new conditional upper bound

$$\sum_{m \leq x} R_3(m)^2 \ll x^{20/19+\varepsilon},$$

for any fixed $\varepsilon > 0$, in which $R_3(m)$ counts representations of $m$ as a sum of three cubes. In a subsequent paper (23), the exponent was improved to $1 + \varepsilon$, which is optimal, apart from the $\varepsilon$. The upper bound [3] is proved via the circle method. If one denotes the contribution from a Farey arc around $a/q$ by $f(a,q)$, say, then the Kloosterman refinement can be viewed as a procedure for estimating

$$F(q) = \sum_{a=1 \atop (a,q)=1}^{q-1} f(a,q)$$

non-trivially, giving a saving relative to the sum $\sum_a |f(a,q)|$. Prior to Hooley’s work, one would then estimate $\sum_q |F(q)|$. However, Hooley was able to show how one could estimate $\sum_q F(q)$ non-trivially, if one assumed standard facts about certain Hasse–Weil $L$-functions. One may express $F(q)$ in terms of the counting functions

$$N(p; h_1, \ldots, h_6) = \# \left\{ (n_1, \ldots, n_6) \pmod{p} : \sum_{i=1}^6 n_i^3 \equiv \sum_{i=1}^6 n_i h_i \equiv 0 \pmod{p} \right\}$$

for primes $p | q$, for various 6-tuples $(h_1, \ldots, h_6)$. These counting functions relate to the projective cubic 3-fold

$$X(h_1, \ldots, h_6) : \sum_{i=1}^6 x_i^3 = \sum_{i=1}^6 x_i h_i = 0,$$

and the Hasse–Weil $L$-functions of these varieties are the ones that arise in Hooley’s analysis. What is required is that the completed $L$-function has a meromorphic continuation to the whole complex plane, being regular except possibly at $s = 5/2$ and $3/2$; that there is a functional equation of the standard type; and that all zeros of the completed function lie on the critical line $\sigma = 2$.

These assumptions allow one to obtain cancellation in $\sum_q F(q)$, leading to the mean value bound [3].

This double Kloosterman refinement opens up new possibilities for the classical circle method. Heath-Brown applied it respectively to quadratic forms (Heath-Brown 1996) and to diagonal cubics in four variables (Heath-Brown 1998). In the latter case, the Hasse–Weil
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\(L\)-functions reduce to \(L\)-functions of elliptic curves, for which the analytic continuation and functional equation are known.

Hooley went on to explore the application of the double Kloosterman refinement to general (non-diagonal) cubic forms, and in two of his last papers (33, 34), he considered integer zeros of non-singular cubic forms in 8 variables. Subject to assumptions about the analytic continuation and Riemann hypothesis for the Hasse–Weil \(L\)-functions of hyperplane slices of the corresponding cubic, he was able to establish the Hasse principle and weak approximation for zeros of such forms. The technical difficulties involved are formidable, particularly in relation to exponential sums to prime power modulus. Indeed if it were not for such issues, Hooley’s methods would be able to handle cubics in only 7 variables.

The Barban–Davenport–Halberstam Theorem

Hooley published a series of no less than 19 papers on the Barban–Davenport–Halberstam Theorem, thereby eclipsing Liouville’s well-known ‘eighteen articles’. Indeed, in addition to the main series there was a survey (27) as well as various other publications that touched on the subject.

The original theorem, proved by Barban (Barban 1963), defines

\[
E(X; q, a) = \theta(x; q, a) - \frac{x}{\phi(q)}, \quad \text{where } \theta(x; q, a) = \sum_{p \leq x, p \equiv a (\text{mod} q)} \log p,
\]

the sum being over primes, and states that for any \(A > 0\) there is a corresponding \(B = B(A)\) such that

\[
\sum_{q \leq Q} \sum_{1 \leq a \leq q, (a, q) = 1} E(x; q, a)^2 \ll A x^2 (\log x)^{-A},
\]

for \(Q \leq x (\log x)^{-B}\). The result was discovered independently by Davenport and Halberstam (Davenport & Halberstam 1966) a few years later. This work was refined by Montgomery (Montgomery 1970), who showed that

\[
\sum_{q \leq Q} \sum_{1 \leq a \leq q, (a, q) = 1} E(x; q, a)^2 = Q x \log x + O_A(Q x (\log(2x/Q))) + O_A(x^2 (\log x)^{-A}),
\]

for \(1 \leq Q \leq x\), for any fixed \(A > 0\), thereby giving an asymptotic formula when \(x/Q\) is at most a power of \(\log x\).

The questions addressed by Hooley include: Can one improve the error terms? What error terms can one prove under the Generalized Riemann Hypothesis (GRH)? What can one say about

\[
\sum_{1 \leq a \leq q, (a, q) = 1} E(x; q, a)^2
\]

for individual moduli \(q\)? What lower bounds can be established for ranges of \(x\) and \(Q\) where one cannot prove an asymptotic formula? What happens when one replaces the sum over all primitive residue classes \(a\) by a sum over a restricted range? To what extent can one replace the primes by other well-distributed sequences?

In the first two papers of the series (7, 8), Hooley replaces Montgomery’s error terms by \(C Q x + O_A(x^{3/4} Q^{1/4}) + O_A(x^2 (\log x)^{-A})\) for a certain absolute constant \(C\). Moreover, under
GRH he obtains $CQx + O_\varepsilon(x^{3/4}Q^{5/4}) + O_\varepsilon(x^{3/2+\varepsilon})$ for any fixed $\varepsilon > 0$, giving an asymptotic formula as soon as $Q$ is larger than $x^{1/2+\varepsilon}$. This theme is extended in the fourth article (10) where similar but weaker results are proved for

$$\sum_{q \leq Q} \max_{y \leq x} \sum_{1 \leq a \leq q \over (a,q)=1} E(y; q, a)^2.$$

The third paper (9) begins the investigation of generalizations to sequences other than primes. Let $s$ run over any strictly increasing sequence of positive integers, and denote by $S(x; q, a)$ the number of $s \leq x$ in the congruence class $a \pmod{q}$. It is assumed that the sequence satisfies a Siegel–Walfisz condition in the form

$$S(x; q, a) = g(q, (q, a))x + O_A(x(\log x)^{-A})$$

uniformly over all $q$ and $a$, for any given $A > 0$, where $g$ is a suitable density function. For the primes, one can deduce a version of the Siegel–Walfisz condition from the original Barban theorem, so some such assumption is clearly necessary if progress is to be made. Although Hooley assumes that the sequence of integers $s$ has constant density, one can adapt the arguments to apply to weighted sequences and hence to include the original Barban formulation. The conclusion (9) is then that

$$\sum_{q \leq Q} \sum_{1 \leq a \leq q} \{S(x; q, a) - g(q, (q, a))x\}^2 \ll_A Q^x + x^2(\log x)^{-A}, \quad [4]$$

for $1 \leq Q \leq x$. The ninth article (24) takes this further, under the assumption that $g(q, q)$ is multiplicative in $q$, proving that the sum above is

$$(C_1 + o(1))Qx + O_A(x^2(\log x)^{-A})$$

when $x/Q \to \infty$, and is $C_2x + O_A(x^2(\log x)^{-A})$ when $Q = x$. Hooley went on to weaken the Siegel–Walfisz condition in the next paper (25), replacing $g(q, (q, a))$ by $f(q, a)$ (which could, for example, be different for different $a$ coprime to $q$). In particular, he showed that one could produce the previous asymptotic formulae without any multiplicativity condition. Continuing the investigation of general sequences, the fourteenth article (28) strengthens the Siegel–Walfisz assumption, now requiring that

$$S(x; q, a) = f(q, a)x + O_A(x^a q^{-a})$$

for $q \leq x^{1/2}$, for some positive constant $\alpha < 1/2$. Under this assumption, the bound [4] can be sharpened, giving the upper bound $O_A(Q^{2-2\alpha}x^{2\alpha}\log 2(2x/Q))$. Still pursuing the theme of general sequences, the last two papers in the series, (29) and (30), consider lower bounds for

$$G_\lambda(x, Q) := \sum_{q \leq Q} \sum_{1 \leq a \leq \lambda q} \{S(x; q, a) - f(q, a)x\}^2,$$

both when $\lambda = 1$ and when $0 < \lambda < 1$. In particular when $x/Q \to \infty$ one has

$$G_1(x, Q) \geq (\theta(C) + o(1))Q^x + O_A(x^2(\log x)^{-A})$$

for any fixed $A > 0$, where $C = f(1, 0)$, and $\theta$ is an explicitly given function. The values $\theta(1) = 1/12$ and $\theta(1/2) = 1/8$ are shown to be optimal. Moreover, one has $G_\lambda(x, Q) \sim \lambda G(x, Q)$ whenever $G(x, Q)/Q^x \to \infty$, but otherwise $G_\lambda(x, Q)$ can behave differently.
The square sieve

The so-called ‘square sieve’ originates from Hooley’s paper (12) in which it is shown that a positive integer \( N \) has \( O(N^{11/18+\varepsilon}) \) representations as a sum of four positive cubes, for any fixed \( \varepsilon > 0 \). The exponent \( 2/3 + \varepsilon \) is trivial, but to get a power saving required substantial new ideas. If \( N = a^3 + b^3 + c^3 + d^3 \) then, on writing \( \sigma = c + d \), \( \delta = c - d \) and \( M = 4N - 4a^3 - 4b^3 - \sigma^3 \), we find that \( M = 3\sigma \delta^2 \). For each possible value of \( \sigma \) one then wants to count the number of possible pairs \( a, b \) such that \( 3\sigma | M \), and moreover such that \( M/3\sigma \) is a square. An argument based on the first of these conditions would recover a bound \( O(N^{2/3+\varepsilon}) \) for the number of representations, so that the key is to extract a further saving from the condition that \( M/3\sigma \) is a square.

Hooley achieves this by a remarkable procedure based on Selberg’s upper bound sieve. In his paper (12) he introduces sieve weights in the spirit of Selberg’s original work, and optimizes them. However, most modern applications of the square sieve use the following less delicate approach. Suppose one wants to count squares in a certain set \( S \). One takes the primes \( p \) in some dyadic range \( P < p \leq 2P \) and considers the sum

\[
\Sigma = \sum_{s \in S} \left( \sum_{P < p \leq 2P} \left( \frac{s}{p} \right) \right)^2,
\]

where \( (\cdot/p) \) is the Legendre symbol. When \( s \) is a square, the Legendre symbols are nearly all 1, so that the inner sum is about \( P/\log P \). It follows that, roughly speaking, the number of squares in \( S \) is bounded above by \( P^{-2}(\log P)^2 \Sigma \). However, if one expands the expression for \( \Sigma \) one sees that one is able to take advantage of any cancellation in sums of the shape

\[
\sum_{s \in S} \left( \frac{s}{pp'} \right)
\]

when \( p \neq p' \). In particular, if \( S \) is sufficiently well distributed to moduli of a suitable size, one expects to show that \( S \) contains relatively few squares.

In his paper (12) the set \( S \) above would consist of the numbers \( M/3\sigma \) for which \( 3\sigma | M \). Hooley was indeed able to obtain the necessary distribution results for these, but only though an application of estimates for exponential sums coming from Deligne’s Riemann hypothesis for varieties over finite fields.

Hooley went on to develop these ideas (15, 22, 26), applying them to count non-trivial integer solutions of \( F(u, v) = F(x, y) \) of size at most \( B \), for various binary forms \( F \). In particular he showed that the method could be adapted to count not just squares, but also integers represented by other polynomials. Hooley’s results for the application above have all now been overtaken by those coming from the determinant method (see Heath-Brown 2002, for example). In other situations, however, the square sieve and its variants remain important tools.

**Hooley’s \( \Delta \)-Function**

Hooley’s name is firmly attached to the function

\[
\Delta(n) = \max_u \sum_{d | n \atop u < d \leq \varepsilon u} 1,
\]
which he studied in his paper (13). In the paper he shows that

$$\sum_{n \leq x} \Delta(n) \ll x (\log x)^{4/\pi - 1}.$$  

This should be compared with the trivial bound $O(x \log x)$, coming from the observation that $\Delta(n) \leq d(n)$. In the paper Hooley applied this estimate, and other related bounds, to a number of diverse problems in analytic number theory. For these problems, previous workers had, in effect, worked with $d(n)$ rather than $\Delta(n)$. As an example, he showed that for any real irrational $\theta$, and any real number $\gamma$, there are infinitely many positive integers $n$ for which

$$\|n^2 \theta + \gamma\| \leq n^{-1/2} (\log n)^\kappa,$$

provided the constant $\kappa$ is greater than $2/\pi - 1/2 = 0.1366 \ldots$ (Here the notation $||\phi||$ is the distance from $\phi$ to the nearest integer.) For the problem above, the best previous result had only had $n^{-1/2+\varepsilon}$ on the right.

Subsequent work (Tenenbaum 1985) has shown that

$$\sum_{n \leq x} \Delta(n) \ll x (\log x)^\varepsilon$$

for any fixed $\varepsilon > 0$. Indeed, Tenenbaum’s result gives a sharper, more explicit, estimate.

As will be clear from our survey of Hooley’s oeuvre, quite a number of his papers achieved success through the saving of a small power of a logarithm. Thus the improvement from an exponent of 1 down to $4/\pi - 1$ had the potential for quite a few applications. Perhaps the most dramatic of these came in Vaughan’s work (Vaughan 1986) on Waring’s problem for cubes. Let $R_s(N)$ be the number of ways of writing an integer $N$ as a sum of $s$ cubes of non-negative integers. It had been shown by Hardy and Littlewood in 1922 that

$$R_s(N) \sim \frac{\Gamma(4/3)^s}{\Gamma(s/3)} \Xi_s(N) N^{s/3 - 1}$$

as $N \to \infty$, provided that $s \geq 9$. Here $\Xi_s$ is the Hardy–Littlewood singular series. As mentioned above, Hooley (17) had given a conditional proof of the asymptotic formula for $s = 7$ and 8, but no unconditional treatment was available. However, by combining Hooley’s $\Delta$-function techniques with various other ideas, Vaughan was able to prove the Hardy–Littlewood asymptotic formula for the case $s = 8$, which is where things currently stand. Vaughan achieves a saving of order $(\log N)^{4/\pi - 2 + \varepsilon}$, whereas the Hardy–Littlewood analysis saves a power of $N$, as does Hooley’s conditional approach.

**Conclusion**

Hooley was the author of nearly 100 papers. In addition to some remarkable one-off successes, he pioneered the use of exponential sums in arithmetic problems. Another recurrent theme was the introduction of novel sieve methods, and his Cambridge tract ‘Applications of sieve methods to the theory of numbers’ (11) showcases the impressive variety of sieve tools he developed. Modern analytic number theory owes a great deal to these ideas.

Hooley’s work was appreciated world-wide, as was recognized in his selection as a plenary speaker at the 1982 ICM in Warsaw—actually held in 1983, because of political events in
Poland. He made the UK a natural destination for anyone wanting to study analytic number theory, and was a key link in the tradition running from Hardy and Littlewood through to the present day.

Hooley’s writing was well-known for his love of such arcane phrases as ‘the dexter side of the ante-penultimate equation’. Equally, however, his papers were a model of clarity and accuracy. They were often couched in seemingly old-fashioned terms, both linguistically and mathematically, but they never failed to enlighten.

He will be remembered with fondness by colleagues and family alike (figure 3).

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Roger Heath-Brown studied mathematics at Cambridge, and completed his PhD there in 1977, working under Alan Baker in analytic number theory. He moved to Oxford in 1979, was elected as a Fellow of the Royal Society in 1993, and became Professor of Pure Mathematics in 1999. He retired in 2016.

He first met Christopher Hooley, as his PhD examiner, in 1977. His subject interests align closely with Hooley’s, and include prime number theory, the theory of the Riemann Zeta-function and Dirichlet L-functions, sieve methods, and the analytic theory of Diophantine equations.

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