EIGENVALUES, ABSOLUTE CONTINUITY AND LOCALIZATIONS
FOR PERIODIC UNITARY TRANSITION OPERATORS

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ABSTRACT. The localization phenomenon for periodic unitary transition operators on a Hilbert space consisting of square summable functions on an integer lattice with values in a complex vector space, which is a generalization of the discrete-time quantum walks with constant coin matrices, are discussed. It is proved that a periodic unitary transition operator has an eigenvalue if and only if the corresponding unitary matrix-valued function on a torus has an eigenvalue which does not depend on the points on the torus. It is also proved that the continuous spectrum of the periodic unitary transition operators is absolutely continuous. As a result, it is shown that the localization happens if and only if there exists an eigenvalue, and the long time average of the transition probabilities coincides with the point-wise norm of the projection of the initial state to the direct sum of eigenspaces. Some parts in the results explained above are deduced from a general discussion for unitary operators over any locally finite connected graph, and an analytic perturbation theory for matrices in several complex variables is employed to show the results mentioned above for periodic unitary transition operators.

1. Introduction

The notion of discrete-time quantum walks are defined as a unitary operator which is a quantum analogue of the usual random walks, and it was discovered in quantum physics (2) and computer science (1, 3, 15). As in the theory of usual random walks, the transition probability is naturally defined, and it is one of the central issues to clarify various asymptotic behavior, as time goes to infinity, of the transition probability, such as to determine weak-limit distributions, to obtain local asymptotic formulas and so on. It is well-known that the asymptotic behavior of the transition probability for quantum walks is quite different from that of usual random walks. The weak-limit distributions for the quantum walks are usually ballistic (13, 18), and the pointwise asymptotics of the transition probability is quite different from that of the classical random walks (17). One of typical properties of transition probability for quantum walks is a localization, which is a phenomenon that the probability of the walker to be found at a point does not tend to zero as time goes to infinity. Interesting is that the localization phenomenon happens for some simple classes of quantum walks, even on the one-dimensional integer lattice, as discussed in 8, 9, 10. There are numerous literatures on the localization phenomenon for quantum walks and it is difficult to refer all of them, and thus we just refer 14 where the localization properties of a quantum walk on certain infinite graphs called spidernets, not on the usual integer lattices, is discussed.

To explain the setting we consider in this paper more concretely, let us prepare some notation. Let $G = (V, E)$ be a locally finite connected graph with the set $V$ of vertices and the set $E$ of oriented edges. Let $D$ be a positive integer. Let $U$ be a unitary operator on the Hilbert space $\ell^2(V, \mathbb{C}^D)$ consisting of all square summable functions on $V$ with values in the $D$-dimensional complex vector space $\mathbb{C}^D$. For $f, g \in \ell^2(V, \mathbb{C}^D)$, the inner product on $\ell^2(V, \mathbb{C}^D)$ is defined by

$$\langle f, g \rangle = \sum_{x \in V} \langle f(x), g(x) \rangle_{\mathbb{C}^D},$$

(1.1)
where $(\cdot, \cdot)_{\mathbb{C}^D}$ denotes the standard Hermitian inner product on $\mathbb{C}^D$. For a non-negative integer $n$, a non-zero vector $w \in l^2(V, \mathbb{C}^D)$ and a vertex $x \in V$, we define the quantity $p_n(w; x)$ by

$$p_n(w; x) = \|U^n w(x)\|_{\mathbb{C}^D},$$

where $\| \cdot \|_{\mathbb{C}^D}$ denotes the standard norm on $\mathbb{C}^D$. It is straightforward to see that the sum of $p_n(w; x)$ over all $x \in V$ equals $\|w\|^2$, the norm square of $w$ in $l^2(V, \mathbb{C}^D)$, and hence $p_n(w; x)$ defines a probability distribution on $V$ when $\|w\| = 1$. For this reason, we call, in this paper, $p_n(w; x)$ the transition probability for the unitary operator $U$ with the initial state $w$ even when $w$ is not a unit vector.

**Definition 1.1.** The transition probability for $U$ with initial state $w$ is said to be localized at a vertex $x \in V$ if $\limsup_{n \to \infty} p_n(w; x) > 0$.

As in Section 2, a relationship between the spectral structures of $U$ and the localization of the transition probability on general graphs can be discussed to a certain extent. However, the above situation is too general to obtain further results. In this paper, we mainly consider the integer lattice $\mathbb{Z}$ transition probability on general graphs can be discussed to a certain extent. However, the above situation is too general to obtain further results. In this paper, we mainly consider the integer lattice $\mathbb{Z}$ transition operators, It is natural to use the Fourier transform because of their invariance under the finite number of such operators are periodic unitary transition operators. To analyze periodic unitary transition operators, it is natural to use the Fourier transform because of their invariance under the action of $\mathbb{Z}^d$. Let $T^d$ be the $d$-dimensional torus in $\mathbb{C}^d$ defined by

$$T^d = \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d; |z_j| = 1 \ (j = 1, \ldots, d)\}.$$ 

Let $\mathcal{H} = L^2(T^d, \mathbb{C}^D)$ be the Hilbert space consisting of all square integrable functions on $T^d$ (with respect to the Lebesgue measure on $T^d$) with values in $\mathbb{C}^D$. The normalized Lebesgue measure on $T^d$ is denoted by $\nu_d$. The inner product on $\mathcal{H}$ is defined in a usual manner with the standard Hermitian inner product on $\mathbb{C}^D$. Let $\mathcal{F} : \mathcal{H} \to H$ be the Fourier transform defined by

$$\mathcal{F}(f)(x) = \int_{T^d} \bar{z}^x f(z) \, d\nu_d(z) \quad (f \in \mathcal{H}),$$

where we write $z^x = z_1^{x_1} \cdots z_d^{x_d}$ for a point $z = (z_1, \ldots, z_d)$ in the complex torus $T^d_d = (\mathbb{C} \setminus \{0\})^d$ and a lattice point $x = (x_1, \ldots, x_d)$ in $\mathbb{Z}^d$. The Fourier transform $\mathcal{F}$ is a unitary operator with the inverse $\mathcal{F}^* = \mathcal{F}^{-1} : H \to \mathcal{H}$ given by

$$(\mathcal{F}^* g)(z) = \sum_{x \in \mathbb{Z}^d} g(x) z^x \quad (z \in T^d).$$

For a periodic unitary transition operator $U$, we define a unitary operator $\mathcal{U}$ on $\mathcal{H}$ by $\mathcal{U} = \mathcal{F}^* U \mathcal{F}$. Then the unitary operator $\mathcal{U}$ can be written in the form

$$(\mathcal{U} f)(z) = \hat{U}(z) f(z) \quad (f \in \mathcal{H}, \ z \in T^d)$$

where $\hat{U}(z)$ is denoted by $\hat{U}$. The Fourier transform on $\mathcal{H}$ is defined in a usual manner with the standard Hermitian inner product on $\mathbb{C}^D$. It is straightforward to see that the sum of $p_n(w; x)$ over all $x \in V$ equals $\|w\|^2$, the norm square of $w$ in $l^2(V, \mathbb{C}^D)$, and hence $p_n(w; x)$ defines a probability distribution on $V$ when $\|w\| = 1$. For this reason, we call, in this paper, $p_n(w; x)$ the transition probability for the unitary operator $U$ with the initial state $w$ even when $w$ is not a unit vector.

**Definition 1.2.** A unitary operator $U$ on $H = l^2(\mathbb{Z}^d, \mathbb{C}^D)$ is said to be a periodic unitary transition operator if the following two conditions are satisfied.

- There exists a finite set $S \subset \mathbb{Z}^d$, called the set of steps, such that for any $x \in \mathbb{Z}^d$, $y \in \mathbb{Z}^d \setminus (x + S)$ and any $\phi \in \mathbb{C}^D$, we have $U(\delta_x \otimes \phi)(y) = 0$.
- The unitary operator $U$ commutes with the natural action of the abelian group $\mathbb{Z}^d$ on $H$.

It is obvious that the discrete-time quantum walks with constant coin matrices and products of finite number of such operators are periodic unitary transition operators. To analyze periodic unitary transition operators, it is natural to use the Fourier transform because of their invariance under the action of $\mathbb{Z}^d$. Let $T^d$ be the $d$-dimensional torus in $\mathbb{C}^d$ defined by

$$T^d = \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d; |z_j| = 1 \ (j = 1, \ldots, d)\}.$$
Theorem
Our main theorem is then described as follows.

2.1. Theorem

E has an eigenvalue \( \omega \) if and only if the matrix-valued function \( \hat{U}(z) \) on \( T^d \) has an eigenvalue \( \omega \) for all points \( z \) in \( T^d \). Hence the number of eigenvalues of \( E \) is finite.

(2) The continuous spectrum of \( E \) is absolutely continuous with respect to the Lebesgue measure on the unit circle \( S^1 \) in \( \mathbb{C} \).

(3) If \( E \) has no eigenvalues, then we have \( \lim_{n \to \infty} p_n(w; x) = 0 \) for any \( w \in H \) and \( x \in \mathbb{Z}^d \).

(4) Let \( \{\omega_1, \ldots, \omega_K\} \) be the set of eigenvalues of \( E \), and let \( \pi_j \) \( (j = 1, \ldots, K) \) be the orthogonal projection onto the eigenspace corresponding to \( \omega_j \). We set \( \pi_p = \pi_1 + \cdots + \pi_K \). Then, for any \( w \in H \) and \( x \in \mathbb{Z}^d \), the long-time average

\[
\overline{p}(w; x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} p_n(w; x)
\]

exists and satisfies the following.

\[
\overline{p}(w; x) = \sum_{j=1}^{K} \| \pi_j w(x) \|_{\ell^2}^2, \quad \sum_{x \in \mathbb{Z}^d} \overline{p}(w; x) = \| p_p w \|_2^2.
\]

Organization of the present paper is as follows. First, in Section 2 a relationship between localization and spectral structures of unitary operators defined over general graphs is discussed. In Section 3 some simple but important properties of periodic unitary transition operators are summarized. The item (1) in Theorem 1.3 is an easy consequence of the formula (1.5), but its proof is written, for completeness of the presentation, in Section 4. In Section 5, absolute continuity of the continuous spectrum is discussed.

2. Localization for general unitary operators

Let \( G = (V, E) \) be, as in Section 1, a locally finite connected graph. Let \( U \) be a unitary operator on the Hilbert space \( H = \ell^2(V, \mathbb{C}^D) \) with the inner product given by (1.3). In this section, we discuss some relationships between the localization for the transition probability \( p_n(w; x) \) defined in (1.2) and the spectral structures for the unitary operator \( U \). Let \( \text{spec}(U) \) and \( \text{spec}(U)_p \) denote the spectrum and the set of eigenvalues of \( U \), respectively. Since \( U \) is unitary, \( \text{spec}(U) \) is contained in the unit circle \( S^1 \) in the complex plane, and the spectral resolution of \( U \) takes the form

\[
U = \int_{S^1} \lambda \, dE(\lambda),
\]

where \( E \) is a projection-valued measure on \( S^1 \). If \( \omega \in \text{spec}(U)_p \), then \( \pi_\omega = E(\{\omega\}) \) is the projection onto the eigenspace corresponding to \( \omega \). When \( \text{spec}(U)_p = \{\omega_1, \ldots, \omega_K\} \), we write \( \pi_j = \pi_\omega_j \) and in this case we set \( \pi_p = \pi_1 + \cdots + \pi_K \) and \( \pi_c = I - \pi_p \). We show the following.

Theorem 2.1. Suppose that \( \text{spec}(U)_p \) is a non-empty finite set and we set \( \text{spec}(U)_p = \{\omega_1, \ldots, \omega_K\} \) with \( 1 \leq K < +\infty \). Then, for each \( w \in H \) and \( x \in V \), we have

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} p_n(w; x) \geq \sum_{j=1}^{K} \| \pi_j w(x) \|_{\ell^2}^2.
\]
Furthermore, if $\pi_c w$ has absolutely continuous spectral measure, then the long-time average $\mathcal{P}(w; x)$ defined in (2.7) exists for any $x \in V$ and satisfies the following.

$$\mathcal{P}(w; x) = \sum_{j=1}^{K} \|\pi_j w(x)\|_{C^D}^2, \quad \sum_{x \in \mathbb{Z}^d} \mathcal{P}(w; x) = \|\pi_p w\|^2.$$  

To prove Theorem 2.1 we decompose $p_n(w; x)$ as follows.

$$p_n(w; x) = p_n(\pi_p w; x) + p_n(\pi_c w; x) + r_n(w; x),$$  

where $r_n(w; x)$ is defined by

$$r_n(w; x) = 2\Re \left( \langle U^n \pi_p w(x), U^n \pi_c w(x) \rangle_{C^D} \right).$$  

**Lemma 2.2.** For any $w \in H$ and $x \in V$, we have

$$\mathcal{P}(\pi_p w; x) = \sum_{j=1}^{K} \|\pi_j w(x)\|_{C^D}^2.$$  

**Proof.** We can write

$$\frac{1}{N} \sum_{n=0}^{N-1} p_n(\pi_p w; x) = \frac{1}{N} \sum_{j,k=1}^{K} \left( \omega_j \omega_k^{-1} \right)^n \langle \pi_j w(x), \pi_k w(x) \rangle_{C^D}.$$  

We observe that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left( \omega_j \omega_k^{-1} \right)^n = \begin{cases} 0 & \text{when } j \neq k, \\ 1 & \text{when } j = k, \end{cases}$$  

from which the lemma follows. \hfill \Box

**Lemma 2.3.** For any $w \in H$, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{y \in V} \left| \sum_{n=0}^{N-1} r_n(w; y) \right| = 0.$$  

**Proof.** By (2.6), we have

$$\frac{1}{N} \sum_{n=0}^{N-1} r_n(w; y) = 2 \sum_{j=1}^{K} \Re \left( \langle \pi_j w(y), (T_{N,j} \pi_c w(y)) \rangle_{C^D} \right),$$  

where the operators $T_{N,j}$ are defined by

$$T_{N,j} = \frac{1}{N} \sum_{n=0}^{N-1} (\omega_j^{-1} U)^n.$$  

Taking the absolute value of the both side of (2.4), summing them over all $y \in V$ and applying Cauchy-Schwarz inequality for the sum in $y$, we obtain

$$\frac{1}{N} \sum_{y \in \mathbb{Z}^d} \left| \sum_{n=0}^{N-1} r_n(w; y) \right| \leq 2 \sum_{j=1}^{K} \|\pi_j w\| \|T_{N,j} \pi_c w\|. \tag{2.6}$$  

By von Neumann’s mean ergodic theorem ([16, Theorem II.11]), $T_{N,j}$ tends to the orthogonal projection $\pi_j$ onto the eigenspace corresponding to the eigenvalue $\omega_j$ in the strong operator topology. In particular, we have $\lim_{N \to \infty} \|T_{N,j} \pi_c w\| = \|\pi_j \pi_c w\| = 0$ since $\pi_j \pi_c = 0$. Therefore taking the limit $N \to \infty$ in (2.6) shows the assertion. \hfill \Box
Lemma 2.4. Let \( w \in H \). Suppose that \( w \) has absolutely continuous spectral measure. Then, for any \( x \in V \), we have \( \lim_{n \to \infty} p_n(w; x) = 0 \).

Proof. Let \( H_{ac} \) denote the space of vectors \( h \in H \) such that the corresponding spectral measure \( \| E(\Lambda) h \|^2 \) (where \( \Lambda \subset S^1 \) is a Borel subset) on the circle \( S^1 \) is absolutely continuous with respect to the Lebesgue measure on \( S^1 \). Then, as is well-known, \( H_{ac} \) is a closed subspace (see [12, Chapter X] for details on the absolute continuity). Denoting the standard basis in \( \mathbb{C}^D \) by \( \{ e_1, \ldots, e_D \} \), we have

\[
p_n(w, x) = \sum_{i=1}^{D} |\langle U^n w, \delta_x \otimes e_i \rangle|^2.
\]

Thus, it is enough to prove that, for any \( v \in H \), \( \lim_{n \to \infty} \langle U^n w, v \rangle = 0 \) when \( w \in H_{ac} \). Let \( \pi_{ac} \) be the orthogonal projection onto \( H_{ac} \). Since \( U \) and \( \pi_{ac} \) commutes, the spectral projection \( E(\Lambda) \) for \( U \) also commutes with \( \pi_{ac} \), and hence we see \( \langle E(\Lambda) w, v \rangle = \langle E(\Lambda) w, E(\Lambda) \pi_{ac} v \rangle \). This shows that

\[
\|\langle E(\Lambda) w, v \rangle\|^2 \leq \|E(\Lambda) w\|^2 \|E(\Lambda) \pi_{ac} v\|^2.
\]

Therefore, the complex valued measure \( d\langle E(\lambda) w, v \rangle \) is absolutely continuous with respect to the Lebesgue measure on \( S^1 \). Let \( \rho \) be the Radon-Nykodym derivative of the measure \( \langle E(\Lambda) w, v \rangle \). Since \( \rho \) is an \( L^1 \)-function on \( S^1 \), the Riemann-Lebesgue lemma shows

\[
\langle U^n w, v \rangle = \int_{-\pi}^{\pi} e^{in\theta} \rho(e^{i\theta}) d\theta \to 0 \quad (n \to \infty),
\]

which shows the lemma. \qed

Proof of Theorem 2.1 By (2.2) and Lemmas 2.2 and 2.3 we see that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} p_n(w; x) = \sum_{j=1}^{K} \|\pi_j w(x)\|_{\mathbb{C}^D}^2 + \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} p_n(\pi_c w, x),
\]

from which the first part of Theorem 2.1 follows. The second part of Theorem 2.1 follows from Lemmas 2.2, 2.3 and 2.4. \qed

According to Theorem 2.1 the items (3) and (4) in Theorem 1.3 follow from the items (1) and (2). In the following sections we prove the items (1) and (2) in Theorem 1.3.

3. Simple properties of periodic unitary transition operators

For \( i = 1, \ldots, d \), let \( \tau_i \) be a shift operator on \( H = \ell^2(\mathbb{Z}^d, \mathbb{C}^D) \) defined by

\[
(\tau_i g)(x) = g(x - e_i) \quad (g \in H, \ x \in \mathbb{Z}^d),
\]

where \( \{ e_1, \ldots, e_d \} \) denotes the standard basis of \( \mathbb{Z}^d \) over \( \mathbb{Z} \). Then, \( \tau_i \)'s are unitary operators on \( H \). We note that \( \tau_i \) and \( \tau_j \) commutes for any \( i, j \), and hence, for \( \alpha = \sum_{i=1}^{d} \alpha_i e_i \), we may write

\[
\tau^\alpha = \tau_1^{\alpha_1} \cdots \tau_d^{\alpha_d}.
\]

The action of \( \mathbb{Z}^d \) on \( H \) is given by the operators \( \tau^\alpha \).

Lemma 3.1. Let \( U \) be a periodic unitary transition operator on \( H \) with the set of steps \( S \subset \mathbb{Z}^d \). For each \( \alpha \in S \), define a linear map \( C(\alpha) \) on \( \mathbb{C}^D \) by

\[
C(\alpha) \phi = U(\delta_0 \otimes \phi)(\alpha) \quad (\phi \in \mathbb{C}^D).
\]
Then, $U$ can be written in the following form.

$$U = \sum_{\alpha \in S} \tau^\alpha C(\alpha).$$

Proof. It is straightforward to see that, for any $\phi \in \mathbb{C}^D$ and $x \in \mathbb{Z}^d$,

$$U(\delta_x \otimes \phi) = \sum_{\alpha \in S} \tau^\alpha \delta_x \otimes C(\alpha) \phi = \left( \sum_{\alpha \in S} \tau^\alpha C(\alpha) \right) (\delta_x \otimes \phi).$$

Since the set $\{\delta_x \otimes \phi\}_{x \in \mathbb{Z}^d, \phi \in \mathbb{C}^D}$ spans $H$, we conclude the assertion.

The operator $F^* \tau_i F$, where $F : \mathcal{H} \to H$ is the Fourier transform defined in (1.3) and $F^*$ is its inverse, is the multiplication operator defined by the coordinate function $z_i$, where $z = (z_1, \ldots, z_d) \in T^d$. Hence, the operator $U = F^* UF$ is written in the form (1.5) with the matrix $\hat{U}(z)$ given by the formula

$$\hat{U}(z)\phi = \sum_{\alpha \in S} z^\alpha C(\alpha) \phi = U(1 \otimes \phi)(z) \quad (\phi \in \mathbb{C}^D).$$

Therefore, by Lemma 3.1 and the assumption that $U$ is a unitary operator, we see

$$\sum_{\alpha, \beta \in S, \alpha - \beta = \gamma} C(\beta)^* C(\alpha) = \delta_{\gamma, 0} I$$

for any $\gamma \in \mathbb{Z}^d$. From this and (3.1), it follows that the matrix $\hat{U}(z)$ is unitary for any $z \in T^d$. We note that $\hat{U}(z)$ is a Laurent polynomial in $z \in T^d$ and hence it is naturally extended to the points $z$ in the complex torus $T_d^\mathbb{C} = (\mathbb{C} \setminus \{0\})^d$. However, it should be remarked that, in general, $\hat{U}(z)$ is not normal when $z$ is not in the real torus $T_d^\mathbb{R}$. We remark also that, for a bounded operator $A$ on $H = \ell^2(\mathbb{Z}^d, \mathbb{C}^D)$ commuting with the action of $\mathbb{Z}^d$, the bounded operator $F^* A F$ on $\mathcal{H} = L^2(T^d, \mathbb{C}^D)$ commutes with multiplication operators by continuous functions on $T^d$. In what follows, we write $\nu_d$ the Lebesgue measure on the $d$-dimensional torus $T^d$ normalized so that $\nu_d(T^d) = 1$. The characteristic polynomial

$$\chi(\zeta, z) = \det(\zeta - \hat{U}(z)) \quad (z \in T^d_d, \zeta \in \mathbb{C})$$

of $\hat{U}(z)$ is a polynomial in $\zeta$ of degree $D$ with coefficients in the ring of Laurent polynomials in $z \in T^d_d$.

Lemma 3.2. We have the following.

$$\text{spec}(U) = \{ \lambda \in S^1 \mid \text{There exists a point } z \in T^d \text{ such that } \lambda \text{ is an eigenvalue of } \hat{U}(z) \}. $$

Proof. It is easy to show that the spectrum $\text{spec}(U)$ is contained in the set in the right-hand side above. Suppose that $\lambda$ is in the resolvent set of $U$. Then, $R = (\lambda - U)^{-1}$ is a bounded operator on $H$. Let $W \subset T^d$ be a Borel set with $\nu_d(W) = 1$ such that $\hat{R}(z)\phi := (\lambda - U)^{-1}(1 \otimes \phi)(z)$ is defined for $z \in W$ and $\phi \in \mathbb{C}^D$. For any $z \in W$ and $\phi \in \mathbb{C}^D$, we see

$$\phi = (\lambda - U)(\lambda - U)^{-1}(1 \otimes \phi)(z) = (\lambda - \hat{U}(z))\hat{R}(z)\phi,$$

which shows that $\hat{R}(z) = (\lambda - \hat{U}(z))^{-1}$ for $z \in W$. Thus the open set $V = \{ z \in T^d \mid \chi(\lambda, z) \neq 0 \}$ contains $W$. Hence $\nu_d(V) = 1$, $V$ is open dense in $T^d$ and $T^d \setminus V$ does not contain any non-empty open set. We set $R(z) = (\lambda - \hat{U}(z))^{-1}$. Note that $R(z)$ is continuous in $z \in V$. We claim that $V = T^d$. To show this, suppose contrary that $T^d \setminus V \neq \emptyset$ and we take $z_0 \in T^d \setminus V$. We take $\phi_o \in \mathbb{C}^D$ such that $\| \phi_o \|_{\mathbb{C}^D} = 1$ and $(\lambda - \hat{U}(z_0))\phi_o = 0$. For any positive integer $n$, there exists a neighborhood $U_n$ of $z_o$ in $T^d$ such that $\| (\lambda - \hat{U}(z))\phi_o \|_{\mathbb{C}^D} < 1/(n + 1)$ for any $z \in U_n$. Then, for any $z \in U_n \cap V$, we have

$$1 = \| \hat{R}(z)(\lambda - \hat{U}(z))\phi_o \|_{\mathbb{C}^D} \leq \| \hat{R}(z)\phi_o \|_{\mathbb{C}^D} \| (\lambda - \hat{U}(z))\phi_o \|_{\mathbb{C}^D} \leq \| \hat{R}(z)\phi_o \|_{\mathbb{C}^D} / (n + 1), $$

where $\| \cdot \|_{\mathbb{C}^D}$ denotes the operator norm. For each $n$, we fix $z_n \in U_n \cap V$. By the definition of the operator norm $\| \hat{R}(z)\phi_o \|_{\mathbb{C}^D}$, one can find $\phi_n \in \mathbb{C}^D$ with $\| \phi_n \|_{\mathbb{C}^D} = 1$ such that $n < \| \hat{R}(z_n)\phi_n \|_{\mathbb{C}^D}$. There
exists a neighborhood $V_n$ of $z_n$ such that $V_n \subset U_n \cap V$ and $n < \|\hat{R}(z)\phi_n\|_{C_D}$ for any $z \in V_n$. We take a continuous function $f_n$ on $T^d$ such that the support of $f_n$ is contained in $V_n$ and $\|f_n\|_{L^2(T^d)} = 1$. We set $v_n = f_n \otimes \phi_n = f_n(1 \otimes \phi_n) \in \mathcal{H}$ which satisfies $\|v_n\| = 1$. Since $\mathcal{R} = (\lambda - \mathcal{U})^{-1}$ commutes with the multiplication operator by $f_n$, we see $(\mathcal{R}v_n)(z) = f_n(z)\hat{R}(z)\phi_n$ for any $z \in W$. Then,

$$\|\mathcal{R}\|_{\text{op}}^2 \geq \|\mathcal{R}v_n\|^2 = \int_{V_n} |f_n(z)|^2 \|\hat{R}(z)\phi_n\|^2_{C_D} d\nu_d(z) \geq n^2,$$

which contradicts the fact that $\mathcal{R}$ is a bounded operator. This shows that $V = T^d$, and hence $\lambda$ is not an eigenvalue of $\hat{U}(z)$ for all $z \in T^d$.

\section{Eigenvalues of periodic unitary transition operators}

In this section and the next section, we frequently use the following

\textbf{Lemma 4.1.} Let $\Delta \subset \mathbb{C}^d$ be a domain. Let $f$ be a holomorphic function on $\Delta$. Suppose that $f^{-1}(0) \cap T^d$ has positive Lebesgue measure on $T^d$. Then $f$ is identically zero on $\Delta$.

\textbf{Proof.} First, let us assume that $\Delta$ is a polydisk in $\mathbb{C}^d$, that is, $\Delta$ is a product of $d$ disks in $\mathbb{C}$. We use an induction on the dimension $d$. When $d = 1$, the assertion is trivial. Suppose that, for $d \geq 2$, the assertion is true in the dimension $d - 1$. Let $\Delta \subset \mathbb{C}^d$ be a polydisk and let $f$ be a holomorphic function on $\Delta$. Suppose that $f$ is not identically zero. We would like to show that $\nu_d(f^{-1}(0) \cap T^d) = 0$. We may assume that $\Delta \cap T^d \neq \emptyset$. In the following, a point in $\mathbb{C}^d$ is written as $(z, w)$ with $z \in \mathbb{C}^{d-1}$ and $w \in \mathbb{C}$. We set $\Delta = V_{d-1} \times V_1$ where $V_{d-1} \subset \mathbb{C}^{d-1}$ is a polydisk and $V_1 \subset \mathbb{C}$ is a disk. For any $(z, w) \in \Delta$, we set $g_w(z) = h_z(w) = f(z, w)$. The functions $g_w$ and $h_z$ are holomorphic on $V_{d-1}$ and $V_1$, respectively. Fubini’s theorem shows

$$\nu_d(f^{-1}(0) \cap T^d) = \int_{\mathbb{C}^d} \nu_d-1(g_w^{-1}(0) \cap T^{d-1}) \, d\nu_1(w).$$

We set

$$M = \{w \in S^1; \nu_d-1(g_w^{-1}(0) \cap T^{d-1}) > 0\}.$$

We show that $\nu_1(M) = 0$. By inductive hypothesis, $w \in M$ if and only if $g_w$ is identically zero on $V_{d-1}$. Suppose that $M \neq \emptyset$ and we take $w_0 \in M$. Since $g_{w_0} = 0$ on $V_{d-1}$, $h_z(w_0) = 0$ for any $z \in V_{d-1}$. Since $f$ is not identically zero, one can take a point $z_0 \in V_{d-1}$ such that $h_{z_0}$ is not identically zero on $V_1$. For such a point, one can take $z_0$ to be in $V_{d-1} \cap T^{d-1}$. Indeed, suppose that $h_z$ is identically zero on $V_1$ for any $z \in V_{d-1} \cap T^{d-1}$, which means that $f(z, w) = 0$ for any $z \in V_{d-1} \cap T^{d-1}$ and $w \in V_1$. Thus, for any $w \in V_1$, $g_w = 0$ on $V_{d-1} \cap T^{d-1}$. We note that $V_{d-1} \cap T^{d-1} \neq \emptyset$ because we have assumed that $\Delta \cap T^d \neq \emptyset$. Since $V_{d-1} \cap T^{d-1}$ is a non-empty open set in $T^{d-1}$, we have $\nu_{d-1}(V_{d-1} \cap T^{d-1}) > 0$. Then, by inductive hypothesis, $g_w = 0$ identically on $V_{d-1}$. Thus $f = 0$ identically on $\Delta$, which is a contradiction. Now, we take $z_0 \in V_{d-1} \cap T^{d-1}$ such that $h_{z_0}$ is not identically zero on $V_1$. Since $h_{z_0}$ is holomorphic on the disk $V_1$ in $\mathbb{C}$, $h_{z_0}^{-1}(0)$ is a discrete set containing $w_0$. There exists an open neighborhood $I$ of $w_0$ in $V_1$ such that $h_{z_0}^{-1}(0) \cap I = \{w_0\}$. We have $M \cap I = \{w_0\}$. Indeed, suppose that we have a point $w_1 \in M \cap I$ different from $w_0$. Then, $g_{w_1}$ is identically zero on $V_{d-1}$. Hence $h_{z_0}(w_1) = 0$ which is a contradiction. Therefore, we have $M \cap I = \{w_0\}$. This means that $M$ is a discrete set on the compact set $S^1$. Hence $M$ is a finite set and its Lebesgue measure is zero.

Now, let $\Delta \subset \mathbb{C}^d$ be a general domain and let $f$ be a holomorphic function on $\Delta$. Since $\Delta$ is second countable, one can find a countable open covering $\{\Delta_n\}_{n=1}^{\infty}$ of $\Delta$ such that each $\Delta_n$ is a polydisk. Suppose that $\nu_d(f^{-1}(0) \cap T^d) > 0$. Then, we have

$$0 < \nu_d(f^{-1}(0) \cap T^d) \leq \sum_n \nu_d(\Delta_n \cap f^{-1}(0) \cap T^d),$$

which contradicts the fact that $\Delta$ is second countable.
and hence there exists an integer $n$ such that \( \nu_d(\Delta_n \cap f^{-1}(0) \cap T^d) > 0 \). Thus, from what we have just proved above, $f$ is identically zero on $\Delta_n$. Now the identity theorem for holomorphic functions shows that $f$ is identically zero on $\Delta$. \hfill \Box

**Lemma 4.2.** Suppose that $U$ has an eigenvalue $\lambda$. Then $\lambda$ is an eigenvalue of $\hat{U}(z)$ for any $z \in T_C^d$.

*Proof.* Let $0 \neq w \in \mathcal{H}$ be an eigenvector of $U$ with the eigenvalue $\lambda$. Then, there exists a Borel set $W \subset T^d$ such that $\nu_d(W) = 1$ and $\hat{U}(z)w(z) = \lambda w(z)$ for all $z \in W$. We may set $w(z) = 0$ for $z \in T^d \setminus W$. We have $\chi_\lambda(z) = 0$ for any $z \in T^d \setminus w^{-1}(0)$, where $\chi_\lambda(z) := \chi(\lambda, z)$ is a Laurent polynomial in $z \in T_C^d$. Since $w \neq 0$, we have $\nu_d(T^d \setminus w^{-1}(0)) > 0$. Therefore, Lemma 4.1 shows that the function $\chi_\lambda$ is identically zero on $T^d_C$. Thus, $\lambda$ is an eigenvalue of $\hat{U}(z)$ for any $z \in T^d_C$. \hfill \Box

We have to show the converse to Lemma 4.2. Suppose that $\lambda \in S^1$ is an eigenvalue of the unitary matrix $\hat{U}(z)$ for any $z \in T^d$. By Lemma 4.1 we have $\chi(\lambda, z) = 0$ for any $z \in T_C^d$. Therefore, for each fixed $z \in T_C^d$, the polynomial $\chi(\zeta, z)$ in $\zeta \in \mathbb{C}$ is divisible by $\zeta - \lambda$. However, the multiplicity of $\zeta - \lambda$ in $\chi(\zeta, z)$ depends on $z \in T_C^d$. To handle this situation, we use a method described in [4, Supplement]. Let $R_d$ denote the ring of Laurent polynomials in $z = (z_1, \ldots, z_d)$ with complex coefficients. The quotient field of $R_d$, denoted by $k_d$, is the field of rational functions on $\mathbb{C}^d$. Let $R_d[\zeta]$, resp. $k_d[\zeta]$, be the ring of polynomials in one variable $\zeta$ with coefficients in $R_d$, resp. in $k_d$. We have $\chi \in R_d[\zeta] \subset k_d[\zeta]$. Since $\chi(\lambda, z) = 0$ for any $z \in T_C^d$, the polynomial $\zeta - \lambda \in k_d[\zeta]$ divides $\chi$ in $k_d[\zeta]$.

**Lemma 4.3.** Let $f \in k_d[\zeta]$ and $\lambda \in S^1$. Suppose that $f$ is divisible by $\zeta - \lambda$ in $k_d[\zeta]$. We write $f = (\zeta - \lambda)q$ with $q \in k_d[\zeta]$. If $f$ is in $R_d[\zeta]$, then $q$ is also in $R_d[\zeta]$.

*Proof.* We write $f$ and $q$ as

$$
\chi(\zeta, z) = \sum_{j=0}^{D} D^{-j} \zeta D^{-j}(z), \quad q(\zeta, z) = \sum_{j=0}^{D-1} q_{D^{-1}-j} \zeta^j.
$$

By assumption, $p_j(z) \in R_d (0 \leq j \leq D)$. Since $f = (\zeta - \lambda)q$, we have

$$
p_0(z) = q_0(z), \quad p_k(z) = q_k(z) - \lambda q_{k-1}(z) \quad (1 \leq k \leq D - 1), \quad p_D(z) = -\lambda q_{D-1}(z).
$$

It is clear from the above expression that $q_j(z) \in R_d (0 \leq j \leq D - 1)$. \hfill \Box

*Proof of Theorem 4.3 (1).* Suppose, as above, that $\lambda \in S^1$ is an eigenvalue of $\hat{U}(z)$ for any $z \in T^d$ and hence for any $z \in T_C^d$. We divide $\chi$ by $\zeta - \lambda$ in $k_d[\zeta]$ and write $\chi(\zeta, z) = (\zeta - \lambda)^m q(\zeta, z)$, where $q$ is not divisible by $\zeta - \lambda$ in $k_d[\zeta]$. By Lemma 4.3, $q$ is in $R_d[\zeta]$. By the Hamilton-Cayley theorem, we have

$$
0 = \chi(\hat{U}(z), z) = (\hat{U}(z) - \lambda)^m q(\hat{U}(z), z) \quad (4.1)
$$

as a matrix for any $z \in T_C^d$. We note that $q(\hat{U}(z), z)$ is not identically zero on $T^d$. Indeed, suppose contrary that $q(\hat{U}(z), z) = 0$ for any $z \in T^d$. Then, for any fixed $z \in T^d$, $q(\zeta, z)$ is divisible by the minimal polynomial of the matrix $\hat{U}(z)$. Since $\hat{U}(z)$ has $\lambda$ as an eigenvalue, $q(\zeta, z)$ is divisible by $\zeta - \lambda$ for any $z \in T^d$. Hence $q(\lambda, z) = 0$ for any $z \in T_C^d$ by Lemma 4.1. Thus, $q$ is divisible by $\zeta - \lambda$ in $k_d[\zeta]$ which is a contradiction. Therefore, $q(\hat{U}(z), z)$ is not identically zero. By the Fourier transform, the bounded operator $q(U, \tau)$ on $H$ is not identically zero. From (4.1), it follows that $0 = \chi(U, \tau) = (U - \lambda)^m q(U, \tau)$. We take $w \in H$ such that $h := q(U, \tau)w$ is not zero. Since $(U - \lambda)^m h = 0$, there exists a number $j (1 \leq j \leq m)$ such that $(U - \lambda)^j h \neq 0$ and $(U - \lambda)^{j+1} h = 0$. Then $v = (U - \lambda)^j h$ is an eigenvector of $U$ with the eigenvalue $\lambda$. \hfill \Box
5. Absolute continuity of the continuous spectrum

Finally we discuss the absolute continuity of the continuous spectrum of a periodic unitary transition operator $U$. As before, let $\text{spec}(U)_p = \{\omega_1, \ldots, \omega_K\}$ be the set of eigenvalues of $U$ and let $\pi_j$ be the orthogonal projection onto the eigenspace of $U$ corresponding to $\omega_j$. We set $\pi_p = \pi_1 + \cdots + \pi_K$ and $\pi_0 = I - \pi_p$. If $\text{spec}(U)_p = \emptyset$, then we set $\pi_p = 0$. As in a previous section, the characteristic polynomial $\chi$ defined in (3.2) is divisible by $\zeta - \omega_j$ in the ring of polynomials $k_d[\zeta]$ with coefficients in the field $k_d$ of rational functions in $z = (z_1, \ldots, z_d)$. The characteristic polynomial $\chi$ is monic in the sense that the coefficient of the leading term of $\chi$ is the identity in $k_d$. The ring $k_d[\zeta]$ is a unique factorization domain (UFD for short), and hence $\chi \in k_d[\zeta]$ can be decomposed into powers of monic irreducible elements in $k_d[\zeta]$. It should be remarked that the ring $R_d$ of Laurent polynomials in $z = (z_1, \ldots, z_d)$ is a UFD because it is a localization of the ring $\mathbb{C}[z_1, \ldots, z_d]$ of polynomials in $z$. Thus, $R_d[\zeta]$ is also a UFD, and hence $\chi$ can be decomposed into irreducibles in $R_d[\zeta]$. However, we need to use some properties of the notion of discriminants which are valid for polynomials with coefficients in a field. Since the polynomials $\zeta - \omega_j$ which divide $\chi$ are irreducible, the decomposition of $\chi$ into monic irreducible elements in $k_d[\zeta]$ can be written as

$$
\chi(\zeta, z) = \prod_{j=1}^{K} (\zeta - \omega_j)^{m_j} \times \prod_{\rho=1}^{r} \pi_\rho(\zeta, z)^{n_\rho},
$$

(5.1)

where $\pi_\rho \in k_d[\zeta]$ are monic irreducible polynomials in $k_d[\zeta]$, and $\zeta - \omega_j$ ($j = 1, \ldots, K$) and $\pi_\rho$ ($\rho = 1, \ldots, r$) are mutually different. We set

$$
\pi(\zeta, z) = \prod_{j=1}^{K} (\zeta - \omega_j) \times \prod_{\rho=1}^{r} \pi_\rho(\zeta, z)
$$

(5.2)

so that $\chi^{-1}(0) = \pi^{-1}(0) \subset T^{d+1}_d$. If $\pi_\rho$ ($1 \leq \rho \leq r$) is of the form $\zeta - \lambda$ with a constant $\lambda \in \mathbb{C}$, then $\chi(\lambda, z) = 0$ for any $z \in T^{d}_d$ and hence by Theorem 1.3 (1), $\lambda$ is an eigenvalue of the operator $U$. Thus, $\lambda = \omega_j$ for some $j = 1, \ldots, K$. However, $\pi_\rho$ is different from $\zeta - \omega_j$, and hence it is a contradiction. Thus, $\pi_\rho$ is not of the form $\zeta - \lambda$ with a constant $\lambda$. The fact that the polynomials $\pi_\rho$ are indeed in $R_d[\zeta]$ follows from the following lemma, which is a generalization of Lemma 4.3.

**Lemma 5.1.** Let $p, q \in k_d[\zeta]$ be monic polynomials and let $r \in R_d[\zeta]$. If $r = pq$, then $p, q \in R_d[\zeta]$.

**Proof.** It follows from Lemma 2 in [4, Supplement] that each coefficient of $p$ and $q$ is holomorphic on $T^{d+1}_d$. Thus, what we need is to prove that a rational function $m(z)$ in $\mathbb{C}^d$ which is holomorphic in $z \in T^{d+1}_d$ is actually a Laurent polynomial in $z$. We use an induction on $d$. Let us consider the case where $d = 1$. Let $g/f$ (where $f, g \in \mathbb{C}[z]$) be a rational function holomorphic on $T^{d}_d$.

Suppose that the assertion holds in the dimension $d - 1$. For $z \in \mathbb{C}^d$, we write, as before, $z = (\xi, w)$ with $\xi \in \mathbb{C}$ and $w \in \mathbb{C}^{d-1}$. Let $m(\xi, w)$ be a rational function in $(\xi, w) \in \mathbb{C}^{d}$ holomorphic in $T^{d}_d$. We write $m(\xi, w) = g(\xi, w)/f(\xi, w)$ with polynomials $f$ and $g$ having no common irreducible factors in $\mathbb{C}[\xi, w]$. The assertion is trivial if $f$ is constant. Thus, suppose that $f$ is non constant. First, suppose that $g$ is a non-zero constant. Since $m = g/f$ is holomorphic on $T^{d}_d$, $f$ can not be zero on $T^{d}_d$. Let $s = \xi w_1 \cdots w_{d-1}$. By the Hilbert’s Nullstelensatz (see, for instance, [4, Chapter 4]), each irreducible factor of $f$ divides $s$, and hence $f$ is a monomial. Next, suppose that $g$ is not constant. We may suppose that $g$ has a positive degree in $\xi$, and we write $g = b_0 \xi^m + \cdots + b_m$ with $m > 0$, $b_j \in \mathbb{C}[w]$ and $b_0 \neq 0$. If $f$ is constant in $\xi$, then $f \in \mathbb{C}[w]$, and $m = (b_0/f) \xi^m + \cdots + (b_m/f)$. Since $b_j/f$’s are rational functions on $\mathbb{C}^{d-1}$ holomorphic in $T^{d-1}_d$, the inductive hypothesis shows that these are Laurent polynomials in $w$. Thus, we may suppose that $f$ and $g$ have positive degrees in $\xi$. Let $f$ be a polynomial in $\xi$ of degree $l > 0$, and we write $f = a_0 \xi^l + \cdots + a_l$ with $a_j \in \mathbb{C}[w]$ and $a_0 \neq 0$. Let $r \in \mathbb{C}[w]$ be the resultant of the
polynomials $f$, $g$ with respect to $\xi$. (See, for instance, [8, Chapter 3] for the properties of resultants.) Since $f$ and $g$ have no common factors, $r$ is not the zero polynomial. Suppose that $a_l \neq 0$. We set $a = a_lr \in \mathbb{C}[w]$. Then the polynomial $a$ is not the zero polynomial and $U = \{w \in \mathbb{C}^{d-1}; a[w] \neq 0\}$ is an open dense subset in $\mathbb{C}^{d-1}$. Fix $w_o \in U \cap T^d_{\mathbb{C}}$. Since $r(w_o) \neq 0$, $f(\xi, w_o)$ and $g(\xi, w_o)$ have no common zeros as polynomials in $\xi \in \mathbb{C}$. Since $m = g/f$ is holomorphic on $T^d_{\mathbb{C}}$, $f(\xi, w_o) = 0$ only at $\xi = 0$. However, since $a_l(w_o) \neq 0$, we have $f(\xi, w_o) \neq 0$ for any $\xi \in \mathbb{C}$, which is a contradiction to the assumption that $f$ has a positive degree in $\xi$. Thus, $a_l$ is zero. This shows that $f$ is divisible by $\xi$. We write $f = \xi^kh$ where $k > 0$ and $h$ is a polynomial in $\mathbb{C}[\xi, w]$ which is not divisible by $\xi$. Then $\xi^km = g/h$ is again a rational function holomorphic on $T^d_{\mathbb{C}}$. Applying the above discussion shows that $h$ must be a constant in $\xi$. Thus, $h \in \mathbb{C}[w]$. Therefore, the previous discussion shows that $\xi^km = g(\xi, w)/h(w)$ is a Laurent polynomial in $(\xi, w)$.

By Lemma 5.1 each polynomial $\pi_o$ in (5.2) is in $R_d[\zeta]$. Let $d_\pi$ denote the discriminant of $\pi$, namely, $d_\pi = \pm$ resultant of $\pi$ and $\partial_\zeta \pi$. $d_\pi$ is a Laurent polynomial in $z \in T^d_{\mathbb{C}}$. Note that $d_\pi$ is identically zero if and only if $\pi$ and $\partial_\zeta \pi$ has a common irreducible factor in $k_d[\zeta]$. Since $\pi$ is a product of mutually different irreducible elements in $k_d[\zeta]$, $d_\pi$ is not identically zero. Hence the discriminant set $D = d_\pi^{-1}(0) \subset T^d_{\mathbb{C}}$ is an algebraic variety such that $T^d_{\mathbb{C}} \setminus D$ is a connected open dense set (7, Chapter 1). By Lemma 4.1 $T^d \cap D$ is a closed set in $T^d$ having Lebesgue measure zero. For any $z \in T^d_{\mathbb{C}} \setminus D$, the equation $\pi(\zeta, z) = 0$ has $K + L = \deg \pi$ roots because $\pi(\zeta, z)$ does not have multiple roots for $z \notin D$. (See [8] for the properties of discriminants.)

Lemma 5.2. Let $z_o \in T^d_{\mathbb{C}} \setminus D$. Then, there exists a neighborhood $V \subset T^d_{\mathbb{C}} \setminus D$ of $z_o$ and holomorphic functions $\lambda_1(z), \ldots, \lambda_L(z)$ on $V$ such that, for each $z \in V$, the roots of the equation $\pi(\zeta, z) = 0$ is given by $\omega_1, \ldots, \omega_K, \lambda_1(z), \ldots, \lambda_L(z)$.

Proof. Let $\omega_1, \ldots, \omega_K, \lambda_1, \ldots, \lambda_L$ be the roots of the equation $\pi(\zeta, z_o) = 0$, each of which is different from others. Let $\lambda_o$ denote one of $\lambda_j$’s. Since $z_o \notin D$ and since $\pi(\lambda_o, z_o) = 0$, we have $\partial_\zeta \pi(\lambda_o, z_o) \neq 0$. Applying the implicit function theorem (7, Chapter 1), there exists a neighborhood $V$ of $z_o$ and a holomorphic function $\lambda(z)$ on $V$ such that $\pi(\lambda(z), z) = 0$ for any $z \in V$ and $\lambda(z_o) = \lambda_o$. This shows the lemma.

The projections onto algebraic (generalized) eigenspaces (called the eigenprojection in [12]) are also holomorphic near $z_o \in T^d_{\mathbb{C}} \setminus D$. Namely, we have the following.

Lemma 5.3. Let $z_o \in T^d_{\mathbb{C}} \setminus D$. Let $V_o$ be an open neighborhood of $z_o$ in $T^d_{\mathbb{C}} \setminus D$ as in Lemma 5.2 Then there is an open neighborhood $V_1$ of $z_o$ in $V_o$ and holomorphic projection-valued functions $R_j(z)$ ($1 \leq j \leq K$), $P_k(z)$ ($1 \leq k \leq L$) such that $R_j(z), P_k(z)$ are the projections onto the algebraic eigenspaces corresponding to $\omega_j, \lambda_k(z)$, respectively, for each $z \in V_1$.

Proof. As before, let $\omega_1, \ldots, \omega_K, \lambda_1, \ldots, \lambda_L$ be mutually different eigenvalues of the matrix $\hat{U}(z_o)$. Let $\lambda_o$ be one of them. We take a small circle $\Gamma$ around $\lambda_o$ such that $\Gamma$ and the disk bounded by $\Gamma$ do not contain any other eigenvalues. Let $\lambda_o(z)$ denote the eigenvalue of $\hat{U}(z)$ in $V_o$ such that $\lambda_o(z_o) = \lambda_o$ which is holomorphic on $V_o$, and let $\lambda(z)$ be any other holomorphic function which is an eigenvalue of $\hat{U}(z)$ in $V_o$. Since $\lambda_o(z)$ and $\lambda(z)$ are continuous, there exists a neighborhood $V_1$ of $z_o$ in $V_o$ such that, for each $z \in V_1$, $\lambda(z)$ is not contained in $\Gamma$ and the disk bounded by $\Gamma$ and $\lambda(z)$ is inside $\Gamma$. Then, the eigenprojection $P(z)$ onto the algebraic eigenspace corresponding to $\lambda_o(z)$ along other algebraic eigenspaces is given by the contour integral ([12, Chapter 1])

$$P(z) = \frac{1}{2\pi i} \int_{\Gamma} (\zeta - \hat{U}(z))^{-1} d\zeta.$$ (5.3)

From this expression, $P(z)$ is holomorphic in $z \in V_1$. □
We remark that, when \( z_0 \in T^d \setminus D \), \( R_j(z) \) and \( P_k(z) \) in Lemma 5.3 are smooth functions on \( T^d \cap V_1 \) which are the orthogonal projections onto the geometric eigenspaces because \( \hat{U}(z) \) is a unitary matrix for \( z \in T^d \). For \( R_j(z) \), we have the following.

**Lemma 5.4.** In the above, \( R_j(z) \) \((1 \leq j \leq K)\) is extended to \( T^d \setminus D \) as a smooth orthogonal projection-valued bounded function, and it is the orthogonal projection onto the eigenspace corresponding to the eigenvalue \( \omega_j \) of \( \hat{U}(z) \) for each \( z \in T^d \setminus D \).

**Proof.** For each fixed \( z \in T^d \), let \( R_j(z) \) be the eigenprojection onto the algebraic eigenspace corresponding to the eigenvalue \( \omega_j \) of \( \hat{U}(z) \) defined by a contour integral as in (5.3). \( R_j(z) \) is well-defined as a single-valued function on \( T^d \). By Lemma 5.3, it is holomorphic on \( T^d \setminus D \), and hence its restriction to \( T^d \setminus D \) is smooth. For \( z \in T^d \), \( \hat{U}(z) \) is a unitary matrix and hence \( R_j(z) \) is the orthogonal projection onto the (geometric) eigenspaces corresponding to \( \omega_j \). Thus, the operator norm of \( R_j(z) \) for \( z \in T^d \) is bounded.

**Lemma 5.5.** Let \( R_j(z) \) be the orthogonal projection-valued functions on \( T^d \setminus D \) in Lemma 5.4. Then, the orthogonal projection \( \pi_j \) onto the eigenspace of \( \hat{U} \) corresponding to the eigenvalue \( \omega_j \) is given by the following formula.

\[
(\pi_j w)(z) = R_j(z)w(z) \quad (w \in L^2(T^d, \mathbb{C}^d), z \in T^d \setminus D).
\]

**Proof.** Since \( T^d \cap D \) has Lebesgue measure zero and since \( R_j(z) \) is bounded on \( T^d \setminus D \), the right-hand side of the above expression defines a bounded operator \( R_j \) on \( L^2(T^d, \mathbb{C}^d) \) which commutes with \( \hat{U} \). By Lemma 5.4 the image of \( R_j \) is contained in that of \( \pi_j \). But, it is obvious that, for \( w \in \text{Im}(\pi_j) \), \( w(z) \) is an eigenvector of \( \hat{U}(z) \) with eigenvalue \( \omega_j \) when \( w(z) \neq 0 \). Thus, we have \( R_j w = w \), which completes the proof.

For each \( z_0 \in T^d \setminus D \), we take a small neighborhood \( V_{z_0} \) such that \( \overline{V_{z_0}} \) is a compact subset contained in a neighborhood of \( z_0 \) as in Lemmas 5.2, 5.3. We set \( W_{z_0} = V_{z_0} \cap T^d \). We may assume that \( W_{z_0} \) is connected. Let \( \mathcal{H}(W_{z_0}) \) be the closed subspace in \( \mathcal{H} \) consisting of all \( w \in L^2(T^d, \mathbb{C}^d) \) whose essential support is contained in \( W_{z_0} \). It is obvious that the orthogonal projection onto \( \mathcal{H}(W_{z_0}) \) is given by the multiplication of the characteristic function of \( W_{z_0} \), and hence it commutes with \( \hat{U} \). Thus, the spectral projection of \( \hat{U} \) also commutes with the orthogonal projection onto \( \mathcal{H}(W_{z_0}) \). Let \( R_j(z) \), \( P_i(z) \) \((1 \leq j \leq K, 1 \leq i \leq L)\) denote the projection-valued holomorphic function on \( V_{z_0} \) given in Lemma 5.3. As in [12], we have

\[
\sum_{j=1}^K R_j(z) + \sum_{i=1}^L P_i(z) = I \quad (z \in V_{z_0}).
\]

By Lemma 5.5 for each \( w \in \mathcal{H}(W_{z_0}) \), we see

\[
w(z) = (\pi_p w)(z) + \sum_{i=1}^L P_i(z)w(z)
\]

for almost all \( z \in W_{z_0} \). Since \( w = \pi_p w + \pi_c w \), we have

\[
(\pi_c w)(z) = \sum_{i=1}^L P_i(z)w(z) \quad (w \in \mathcal{H}(W_{z_0})).
\]

From (5.4), we obtain

\[
\hat{U}^n \pi_c w(z) = \sum_{i=1}^L \lambda_i(z)^n P_i(z)w(z) \quad (w \in \mathcal{H}(W_{z_0}))
\]
for any integer \( n \). Since the ring of Laurent polynomials on \( S^1 \) is dense in the space \( C(S^1) \) of continuous functions on \( S^1 \) with respect to the supremum norm, we see

\[
    f(U)\pi_c w(z) = \sum_{i=1}^{L} f(\lambda_i(z))P_i(z)w(z) \quad (w \in \mathcal{H}(W_{z_o}))
\]

for any \( f \in C(S^1) \).

**Proposition 5.6.** Let \( V_{z_o} \subset T^d_{\mathbb{C}} \setminus D \) and \( W_{z_o} \subset T^d \setminus D \) be as above. Let \( w \in \mathcal{H}(W_{z_o}) \). Then the spectral measure \( ||E(\cdot)\pi_c w||^2 \) of \( \pi_c w \) is absolutely continuous with respect to the Lebesgue measure.

**Proof.** We first show that the gradient \( \nabla \lambda_k \) of the eigenvalues \( \lambda_k(z) \) \((1 \leq k \leq L)\), considered as a function on \( W_{z_o} \), does not vanish almost everywhere on \( W_{z_o} \). To show this, let \( (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d \) be a coordinates on \( W_{z_o} \), defined so that \( z = (e^{i\theta_1}, \ldots, e^{i\theta_d}) \in W_{z_o} \). Let \( \partial_{\theta_j} \) be the partial derivative with respect to \( \theta_j \). Then \( \partial_{\theta_j} \lambda_k = \frac{\partial}{\partial \theta_j} \lambda_k \) on \( W_{z_o} \), where \( \partial_{\theta_j} \lambda_k \) denote the derivative in \( \theta_j \) of the holomorphic function \( \lambda_k \). Let \( N_k \subset W_{z_o} \) be the set of points \( z \in W_{z_o} \) where \( \nabla \lambda_k(z) = 0 \). Thus, \( N_k \) is the intersection of the sets \( (\partial_{\theta_j} \lambda_k)^{-1}(0) \cap T^d \) for \( j = 1, \ldots, d \). If \( N_k \) has positive Lebesgue measure on \( T^d \), then by Lemma 4.12 \( \partial_{\theta_j} \lambda_k \) vanishes identically on \( V_{z_o} \) for any \( j = 1, \ldots, d \). Therefore, \( \lambda_k \) is constant on \( V_{z_o} \).

Now, we have \( \pi(\lambda_k, z) = 0 \) in \( z \in V_{z_o} \). By the identity theorem for holomorphic functions, \( \pi(\lambda_k, z) = 0 \) for any \( z \in T^d_{\mathbb{C}} \). This means that the irreducible monic polynomial \( \zeta - \lambda_k \in k[\zeta] \) divides \( \pi \), which is a contradiction to the fact that each \( \pi_k \) in [5.2] is not of the form \( \zeta - \lambda \). Therefore, \( \nabla \lambda_k \) does not vanish almost everywhere on \( W_{z_o} \).

Now, let \( f \) be a continuous function on \( S^1 \). Since \( P_i(z) \) and \( P_j(z) \) \((i \neq j)\) are orthogonal to each other for \( z \in W_{z_o} \), (5.6) shows

\[
    ||f(U)\pi_c w||^2 = \sum_{i=1}^{L} \int_{W_{z_o}} |f(\lambda_i(z))|^2 ||P_i(z)w(z)||^2_{\mathcal{H}} d\nu_{f}(z).
\]

As is shown in the above, we have \( \nu_d(W_{z_o}) \leq \nu_d(W_{z_o} \cap N_i^c) \) where \( N_i^c \) is the complement of \( N_i \). The function \( \lambda_i \) restricted to \( W_{z_o} \cap N_i^c \) has no critical points, and hence for each \( t \in S^1 \), the set \( \lambda_i^{-1}(t) \cap W_{z_o} \cap N_i^c \) is a smooth hyper-surface. For each \( t \in S^1 \) and \( i = 1, \ldots, K \), we set

\[
    \Gamma_i(t) = \int_{\lambda_i^{-1}(t) \cap W_{z_o} \cap N_i^c} ||P_i(z)w(z)||^2_{\mathcal{H}} dS_t(z) \quad \frac{dS_t(z)}{|\nabla \lambda_i(z)|}
\]

if \( \lambda_i^{-1}(t) \cap W_{z_o} \cap N_i^c \neq \emptyset \), where \( dS_t \) denotes the volume element of the hyper-surface \( \lambda_i^{-1}(t) \cap W_{z_o} \cap N_i^c \). Then the coarea formula (see [3] or [11] Appendix A) shows that \( \Gamma_i(t) \) is an \( L^1 \)-function on \( S^1 \) and

\[
    \int_{S^1} |f(t)|^2 d||E(t)\pi_c w||^2 = \int_{S^1} |f(t)|^2 \Gamma(t) d\nu_1(t), \quad \Gamma(t) = \sum_{i=1}^{L} \Gamma_i(t),
\]

which proves that the measure \( d||E(t)\pi_c w||^2 \) is absolutely continuous with respect to \( d\nu_1(t) \). \( \Box \)

**Proof of Theorem 1.3** (2). For each \( z_o \in T^d \setminus D \), we take a small neighborhood \( V_{z_o} \) such that \( \overline{V_{z_o}} \) is a compact set contained in a neighborhood of \( z_o \) as in Lemmas 5.2, 5.3. We set, as before, \( W_{z_o} = T^d \setminus V_{z_o} \). Then \( \{W_{z_o} \}_{z_o \in T^d \setminus D} \) is an open covering of \( T^d \setminus D \). We can choose a locally finite countable open covering \( \{W_i \}_{i=1}^{\infty} \) which is a refinement of a countable sub-covering of the covering \( \{W_{z_o} \}_{z_o \in T^d \setminus D} \). We take a partition of unity \( \{p_i \}_{i=1}^{\infty} \) subordinated to the covering \( \{W_i \}_{i=1}^{\infty} \). Then, the sum \( \sum_{i=1}^{\infty} M_i \) converges in the strong operator topology and equals the identity operator on \( \mathcal{H} \). Let \( w \in \mathcal{H} \). Let \( \Lambda \subset S^1 \) be a Borel subset having Lebesgue measure zero. Since \( U \) commutes with the operator \( M_i \) defined by the
multiplication by $\rho_i$, the spectral measure $E(\Lambda)$ and the projection $\pi_c$ also commute with $M_i$. Therefore, we obtain

$$\langle E(\Lambda)\pi_c w, \pi_c w \rangle = \infty \sum_{i=1}^{\infty} \int_{T^d \setminus D} \|E(\Lambda)\pi_c (\rho_i^{1/2}w)(z)\|^2_C d\nu_d(z),$$

Now Proposition 5.6 shows that $E(\Lambda)\pi_c (\rho_i^{1/2}w) = 0$ in $\mathcal{H}$. Therefore, we have $E(\Lambda)\pi_c w = 0$, which completes the proof of Theorem 1.3 (2). \qed

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