Time-dependent Schrödinger equations having isomorphic symmetry algebras.

I. Classes of interrelated equations.

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ABSTRACT

In this paper, we focus on a general class of Schrödinger equations that are time-dependent and quadratic in $X$ and $P$. We transform Schrödinger equations in this class, via a class of time-dependent mass equations, to a class of solvable time-dependent oscillator equations. This transformation consists of a unitary transformation and a change in the “time” variable. We derive mathematical constraints for the transformation and introduce two examples.

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1 Introduction

Finding analytical solutions to time-dependent Schrödinger equations has been a mathematical problem of considerable interest. Such equations are relevant to the study of dissipative systems in quantum theory. Solutions to the time-dependent Schrödinger equation (setting \( \hbar = m = 1 \))

\[
\{ H - i\partial_t \} \Psi(x, t) = 0, \tag{1}
\]

where the Hamiltonian \( H \) is time-dependent, describe the evolution of such systems.

Several calculations for a general class of Hamiltonians that are quadratic in \( x \) and \( p \) have been studied \([1]-[10]\). We write the following general form for these Hamiltonians

\[
H_1 = \left[ 1 + k(t) \right] \frac{p^2}{2} + \frac{1}{2} h(t) (xp + px) + g(t)p + h^{(2)}(t)x^2 + h^{(1)}(t)x + h^{(0)}(t), \tag{2}
\]

where \( k, h, g, \) and \( h^{(j)}, j = 0, 1, 2, \) are suitably well-behaved real functions of time. We designate such time-dependent Hamiltonians by \( TQ \).

A subclass of time-dependent Hamiltonians, \( TM \), with “time-dependent masses” is \([11]-[24]\),

\[
H_2 = f(t) \frac{p^2}{2} + f^{(2)}(t)x^2, \tag{3}
\]

where \( f \) and \( f^{(2)} \) are suitably well-behaved real functions of time. Using Lewis invariants \([25, 26]\), analytical solutions have been obtained for some Schrödinger equations with this type of Hamiltonian \([20, 22]\). Often, the function \( f(t) \) has the form \( \exp(\pm \Upsilon t) \), \( \Upsilon \) a real constant.

A second subclass of time-dependent Hamiltonians, \( TO \), is the time-dependent harmonic oscillator in one dimension. This has been studied extensively \([23]-[37]\). Its Hamiltonian is

\[
H_3 = \frac{p^2}{2} + g^{(2)}(t)x^2 + g^{(1)}(t)x + g^{(0)}(t), \tag{4}
\]

where the coefficients \( g^{(j)}(t), j = 1, 2, 3, \) are suitably well-behaved real functions of time. One of the earliest symmetry techniques used to solve this equation was the method of Lewis invariants \([23, 26]\). Its Lie space-time symmetry algebra has been identified as \( sl(2, \mathbb{R}) \square w_1 \) by one of the authors \([30]\) and its complexification, \( su(1,1) \square w_1^c \), has been used to construct solution spaces.
for Eq. (1) \cite{31, 35-37}. The subalgebra \( w_1 \) is a Heisenberg-Weyl algebra in one dimension and \( w_1^c \) is its complexification.

In this paper we show that the three classes of Schrödinger equations mentioned above can be interrelated by transformations. Generally speaking, these transformations can be given by two actions: a unitary transformation \((TQ \rightarrow TM)\) and a change in “time” variable \((TM \rightarrow TO)\). After giving notation for the three classes of Schrödinger equations in Section 2, we describe the unitary transformation and the change in “time” variable in Section 3. In Section 4, we apply our analysis to two TM systems commonly found in the literature. (For example, see Refs. \cite{19, 24}.) We close with a short summary.

In paper II \cite{38} we continue by studying an algebraic approach to solving Schrödinger equations for all three classes of systems, TQ, TM, and TO. We shall show that these three systems have isomorphic Schrödinger algebras, 6-dimensional Lie algebras of space-time symmetries. A subalgebra, having the structure of an oscillator algebra, will be used to derive expressions for number-state, coherent-state, and squeezed-state wave functions for each of the three classes of systems. Expectation values and uncertainty products will also be obtained and the classical equations of motion determined.

Elsewhere \cite{39} we will concentrate on the TM systems discussed in Section 4 of this paper. We will apply the general procedures worked out here and in paper II to obtain number states, coherent states, and squeezed states for the TM systems and examples of symmetry-related TO and TQ systems. Our treatments of TM systems will be detailed and new results will emerge.

2 Notation

For computational purposes, it is more convenient to rearrange the Schrödinger equation (1), with quadratic Hamiltonian (2), into the form

\[
S_1 \Phi(x,t) = \left\{ \left[ 1 + k(t) \right] \partial_{xx} + 2i \partial_t + h(t) \left( -ix \partial_x - i/2 \right) + g(t) \left( -i \partial_x \right) \right. \\
\left. -2h^{(2)}(t)x^2 - 2h^{(1)}(t)x - 2h^{(0)}(t) \right\} \Phi(x,t) = 0, \tag{5}
\]

where \( k, h, g, \) and \( h^{(j)}, j = 0, 1, 2 \) are suitably well-behaved, real functions of \( t \).
Next, we introduce the following operator algebra

$$T = i \partial_t, \quad P = -i \partial_x, \quad X = x, \quad I = 1,$$  \hspace{1cm} (6)

$$P^2 = -\partial_{xx}, \quad X^2 = x^2, \quad D = \frac{1}{2} (XP + PX) = -ix \partial_x - i/2.$$  \hspace{1cm} (7)

These operators have the following nonzero commutation relations

$$[X, P] = iI,$$  \hspace{1cm} (8)

$$[X^2, P^2] = 4iD, \quad [D, X^2] = -2iX^2, \quad [D, P^2] = 2iP^2,$$  \hspace{1cm} (9)

$$[P^2, X] = -2iP, \quad [X^2, P] = 2iX, \quad [D, X] = -iX, \quad [D, P] = iP.$$  \hspace{1cm} (10)

This Lie algebra of operators has a structure isomorphic to $sl(2, \mathbb{R}) \boxtimes w_1$, a Schrödinger algebra. The operators $\{X, P, I\}$ form a basis for a Heisenberg-Weyl algebra, $w_1$ (see Eq. (8)), and the operators $\{X^2, P^2, D\}$ form a basis for the special linear algebra $sl(2, \mathbb{R})$ (see Eq. (9)).

When we express the Schrödinger equation (5) in terms of these operators, we obtain

$$S_1 \Phi(x, t) = \left\{ -[1 + k(t)]P^2 + 2T + h(t)D + g(t)P \
-2h^{(2)}(t)X^2 - 2h^{(1)}(t)X - 2h^{(0)}(t)I \right\} \Phi(x, t) = 0.$$  \hspace{1cm} (11)

Equations of the type (11) are time-dependent quadratic Schrödinger equations, the class $TQ$.

Next, we turn to those Schrödinger equations (1) which have Hamiltonians (3). With the operator notation, (6) and (7), we write the Schrödinger equation as

$$S_2 \Theta(x, t) = \left\{ -f(t)P^2 + 2T - 2f^{(2)}(t)X^2 - 2f^{(1)}(t)X - 2f^{(0)}(t)I \right\} \Theta(x, t) = 0.$$  \hspace{1cm} (12)

Eq. (12) is representative of the so-called “time-dependent mass” equations. We have denoted this class of by $T.M$. [The term “time-dependent mass” comes from the fact that $f(t)$ multiplies $P^2$, just as $1/m$ would if we did not have units $m = 1$.]

The second class of Schrödinger equations, the time-dependent oscillator equations, denoted by $TO$, can be written as (here the “time” variable is indicated with a prime)

$$S_3 \Psi(x, t') = \left\{ -P^2 + 2T' - 2g^{(2)}(t')X^2 - 2g^{(1)}(t')X - 2g^{(0)}(t')I \right\} \Psi(x, t') = 0.$$  \hspace{1cm} (13)
In the next section, we derive a connection between \( t \) and \( t' \).

We emphasize again that the two classes of equations, \( TM \) and \( TO \), are subclasses of the class \( TQ \). Here, our main focus will be on answering the question: Do transformations exist that relate a given Schrödinger equation in one class to a Schrödinger equation in another class? In the next section, we will find a unitary transformation that interrelates all \( TQ \) Schrödinger equations (11). Then, we identify the conditions that allow a \( TQ \) equation to be transformed into a \( TM \) equation (12). That equation can in turn be mapped into a solvable Schrödinger equation (13) in the class \( TO \). Also, we derive conditions for the inverse transformations. That such transformations exist is due to the classes \( TM \) and \( TO \) being subclasses of \( TQ \).

3 The Transformation

3.1 The Form of the Unitary Transformation

Let us consider the following unitary transformation

\[
R(\mu, \nu, \kappa) = \exp\{i\mu P\} \exp\{i\nu D\} \exp\{i\kappa P^2\},
\]

where \( \kappa, \mu, \) and \( \nu \) depend upon \( t \). (We shall normally not indicate this time dependence.) When we apply this transformation to the \( TQ \) Schrödinger equation (11), we have

\[
R(\mu, \nu, \kappa)S_1 R^{-1}(\mu, \nu, \kappa)R(\mu, \nu, \kappa)\Phi(x, t) = 0,
\]

\[
\tilde{S}_1 \tilde{\Phi}(x, t) = 0,
\]

where Eq. (16) follows from Eq. (15) by the definitions

\[
\tilde{S}_1 = \exp\{i\mu P\} \exp\{i\nu D\} \exp\{i\kappa P^2\} S_1 \exp\{-i\kappa P^2\} \exp\{-i\nu D\} \exp\{-i\mu P\},
\]

\[
\tilde{\Phi}(x, t) = \exp\{i\mu P\} \exp\{i\nu D\} \exp\{i\kappa P^2\} \Phi(x, t) = R(\mu, \nu, \kappa)\Phi(x, t).
\]

By using the theorem [40]

\[
\exp\{B\}A \exp\{-B\} = A + [B, A] + \frac{1}{2!}[B, [B, A]] + \cdots
\]
and the commutation relations in Eqs. (8) through (10), the transformation of the operators can be carried out analytically. (See Ref. [41].)

\[
RXR^{-1} = e^{\nu} X + 2\kappa e^{-\nu} P + e^{\nu} \mu I, \tag{20}
\]

\[
RX^2R^{-1} = e^{2\nu} X^2 + 4\kappa D + 4\kappa^2 e^{-2\nu} P^2 + 2e^{2\nu} \mu X + 4\kappa \mu P + e^{2\nu} \mu^2 I, \tag{21}
\]

\[
RPR^{-1} = e^{-\nu} P, \tag{22}
\]

\[
RP^2R^{-1} = e^{-2\nu} P^2, \tag{23}
\]

\[
RDR^{-1} = D + \mu P + 2\kappa e^{-2\nu} P^2, \tag{24}
\]

\[
RT R^{-1} = T + \left( \frac{d\mu}{dt} + \mu \frac{d\nu}{dt} \right) P + \frac{d\nu}{dt} D + \frac{d\kappa}{dt} e^{-2\nu} P^2. \tag{25}
\]

Keeping in mind that \(\kappa, \mu, \) and \(\nu\) are time-dependent, the Schrödinger operator \(\tilde{S}_1\) is

\[
\tilde{S}_1 = -[1 + \tilde{k}(t)] P^2 + 2T + \tilde{h}(t) D + \tilde{g}(t) P - 2\tilde{h}^{(2)}(t) X^2 - 2\tilde{h}^{(1)}(t) X - 2\tilde{h}^{(0)}(t) I. \tag{26}
\]

In Eq. (26), the coefficients of the operators \(P, D, \) and \(P^2\) are, respectively,

\[
\tilde{g}(t) = 2 \frac{d\mu}{dt} + e^{-\nu} \left[ g(t) - 4h^{(1)}(t) \kappa \right] + \mu \left[ 2 \frac{d\nu}{dt} + h(t) - 8h^{(2)}(t) \kappa \right], \tag{27}
\]

\[
\tilde{h}(t) = 2 \frac{d\nu}{dt} + h(t) - 8h^{(2)}(t) \kappa, \tag{28}
\]

\[
1 + \tilde{k}(t) = \left[ -2 \frac{d\kappa}{dt} - 2h(t) \kappa + 8h^{(2)}(t) \kappa^2 + k(t) + 1 \right] e^{-2\nu}. \tag{29}
\]

The coefficients of \(X^2, X, \) and \(I\) are, respectively,

\[
\tilde{h}^{(2)}(t) = h^{(2)}(t)e^{2\nu}, \quad \tilde{h}^{(1)}(t) = h^{(1)}(t)e^{\nu} + 2h^{(2)}(t)e^{2\nu} \mu,
\]

\[
\tilde{h}^{(0)}(t) = h^{(0)}(t) + h^{(1)}(t)e^{\nu} \mu + h^{(2)}(t)e^{2\nu} \mu^2. \tag{30}
\]

Since the mapping \(R(\mu, \nu, \kappa)\) is unitary, Eq. (16), with \(\tilde{S}_1\) given by Eq. (26), has the same form as Eq. (11). Eqs. (27) to (30) give the conditions connecting the two \(TQ\) equations, \(S_1\) and \(\tilde{S}_1\).

### 3.2 The Transformation \(TQ \rightarrow TM\)

To transform Eq. (11), \(S_1 \Phi = 0\), into a \(TM\)-type equation (12), \(S_2 \Theta = 0\), we require that the coefficients \(\tilde{h}, \tilde{g}, \) and \(\tilde{k}\) in Eqs. (27) through (29) satisfy the conditions:

\[
\tilde{g}(t) = \tilde{h}(t) = 0, \quad 1 + \tilde{k}(t) = f(t). \tag{31}
\]
Under these circumstances, the operator $\tilde{S}_1$ in Eq. (26) reduces to an $S_2$ operator such as in Eq. (12). Then Eqs. (27), (28), and the first equality in (31) imply that $\mu$ and $\nu$ satisfy

$$\frac{2}{\mu} \frac{d\nu}{dt} + h(t) - 8 h^{(2)}(t) \kappa = 0, \quad (32)$$

$$2 \frac{d\mu}{dt} + e^{-\nu} \left(g(t) - 4 h^{(1)}(t) \kappa\right) = 0. \quad (33)$$

Also, Eqs. (29) and the third equality in (31) yield the equation

$$\frac{d\kappa}{dt} + h(t) \kappa - 4 h^{(2)}(t) \kappa^2 = \frac{1}{2} (1 + k(t)) - \frac{1}{2} e^{2\nu}. \quad (34)$$

Eqs. (32) to (34) are a set of coupled, first-order, nonlinear, ordinary differential equations for the functions $\kappa$, $\mu$, and $\nu$. When solutions to these equations are obtained, one can calculate the functions $f^{(j)}$, $j = 0, 1, 2$ in Eq. (12) from

$$f^{(j)} = \tilde{h}^{(j)}, \quad j = 0, 1, 2. \quad (35)$$

Under these conditions, with $\Phi(x, t)$ given in Eq. (18),

$$\Theta(x, t) = \tilde{\Phi}(x, t) = R(\mu, \nu, \kappa) \Phi(x, t). \quad (36)$$

We refer to Eqs. (31) through (36) as the $(TQ \rightarrow TM)$-connecting equations.

### 3.3 The Transformation $TM \rightarrow TO$

As an aside, we note that, with the conditions

$$\tilde{g} = \tilde{h} = \tilde{k} = 0, \quad (37)$$

a $TQ$ Schrödinger equation (11) could be directly transformed into a $TO$ Schrödinger equation (13), if the set of coupled nonlinear equations (32), (33), and (38) below had a solution.

$$\frac{d\kappa}{dt} + h(t) \kappa - 4 h^{(2)}(t) \kappa^2 = \frac{1}{2} (1 + k(t)) - \frac{1}{2} e^{2\nu}. \quad (38)$$

Instead of using the above approach, we shall employ an alternative method involving a change in “time” variable to go from the $TM$ equation to the $TO$ equation. Since we already have $TQ \rightarrow TM$, the time transformation will complete the $TQ \rightarrow TM \rightarrow TO$ path.
We start by multiplying both sides of Eq. \ref{eq:12} by $1/f(t)$, where we assume that $f(t) \neq 0$ for all $t$. This yields
\[ S_2 \Theta(x, t) = \left\{ -P^2 + \frac{2}{f(t)} T - 2q^{(2)}(t)X^2 - 2q^{(1)}(t)X - 2q^{(0)}(t)I \right\} \Theta(x, t) = 0, \tag{39} \]
where
\[ S_2 = \frac{1}{f(t)} S_2, \tag{40} \]
\[ q^{(j)}(t) = \frac{f^{(j)}(t)}{f(t)}. \tag{41} \]
Now change to a new “time” variable $t' = t'(t)$. Focusing on $(1/f(t))T$ in Eq. \ref{eq:39} gives
\[ \frac{1}{f(t)} T = -i \frac{1}{f(t)} \frac{\partial}{\partial t} = -i \frac{1}{f(t)} \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'}. \tag{42} \]
Setting the product
\[ \frac{1}{f(t)} \frac{\partial t'}{\partial t} = 1 \tag{43} \]
and solving for $t'(t)$ we find that
\[ t' - t'_o = \int_{t'_o}^{t'} ds f(s). \tag{44} \]
(Some time transformations may not be defined for all $t$ or $t'$. See the examples below.)

Suppose that $t'(t)$ has an inverse. Then, writing $t = t(t')$, we define the functions
\[ \tilde{f}(t') = (f \circ t)(t'), \]
\[ \tilde{q}^{(j)}(t') = (\tilde{f}^{(j)} \circ t)(t'), \quad \tilde{h}^{(j)}(t') = (\tilde{h}^{(j)} \circ t)(t'), \quad j = 0, 1, 2, \tag{45} \]
\[ \tilde{\kappa}(t') = (\kappa \circ t)(t'), \quad \tilde{\nu}(t') = (\nu \circ t)(t'), \quad \tilde{\mu}(t') = (\mu \circ t)(t'). \tag{46} \]
With the aid of the identities in Eq. \ref{eq:30} and Eq. \ref{eq:35}, we express $g^{(j)}$, $j = 0, 1, 2$, as
\[ g^{(2)}(t') = \frac{\tilde{g}^{(2)}(t')}{f(t')} = \tilde{h}^{(2)}(t') e^{2\tilde{\nu}(t')} \frac{1}{f(t')}, \tag{47} \]
\[ g^{(1)}(t') = \frac{\tilde{g}^{(1)}(t')}{f(t')} = \left[ 2\tilde{\mu}(t') \tilde{h}^{(2)}(t') e^{2\tilde{\nu}(t')} + \tilde{h}^{(1)}(t') e^{\tilde{\kappa}(t')} \right] \frac{1}{f(t')}. \tag{48} \]
Using Eqs. (43) and (47) through (49), we have mapped Eq. (39) [that is, Eq. (12)] into a new Schrödinger equation having the form of a TO-equation (13). We refer to Eqs. (32) to (34), (44) and its inverse, and (47) through (49) as the (TM → TO)-connecting equations. Also, we write the wave function, Ψ(x, t′), as the composition

$$\Psi(x, t') = (\Theta \circ t)(x, t').$$

(50)

This completes the transformation of Eq. (12) into Eq. (13). Now, let us look at the reverse transformations, TO → TM and TM → TQ.

### 3.4 The Transformations TO → TM and TM → TQ

In the TO Schrödinger equation (13), suppose that the t'-dependent functions, $$g^{(j)}$$, $$j = 0, 1, 2$$, are suitably well-behaved, real functions but otherwise unspecified. Furthermore, assume that $$t' = t'(t)$$ is any suitable, invertible function of t, and denote its inverse by $$t = t(t')$$. With foresight, define the function $$f(t)$$ by

$$f(t) = \frac{\partial t'}{\partial t}. \quad (51)$$

Then, changing the “time” variable in Eq. (13) and multiplying the result by $$f(t)$$, we obtain

$$S_2 \Theta(x, t) = \left\{-f(t)P^2 + 2T - 2f^{(2)}(t)X^2 - 2f^{(1)}(t)X - 2f^{(0)}(t)I \right\} \Theta(x, t) = 0,$$

(52)

where we have set

$$f^{(j)}(t) = f(t)(g^{(j)} \circ t')(t), \quad (53)$$

for $$j = 0, 1, 2$$, and

$$\Theta(x, t) = (\Psi \circ t')(x, t). \quad (54)$$

Eq. (52) is of the same form as the TM Schrödinger equation (12).

We refer to Eqs. (51), (54), and (53) as the (TO → TM)-connecting equations.
In the TM Schrödinger equation (12), assume that \( f \) and \( f^{(j)}, j = 0, 1, 2 \), are suitably well-behaved, real functions of \( t \). We shall now determine a transformation \( R(\mu, \nu, \kappa) \) of the type (14) that will transform Eq. (12) of the TM class into a TQ Schrödinger equation (11).

We choose \( R(\mu, \nu, \kappa) \) such that

\[
R^{-1}(\mu, \nu, \kappa)S_2R(\mu, \nu, \kappa)R^{-1}(\mu, \nu, \kappa)\Theta(x, t) = 0. \tag{55}
\]

Substitute Eq. (14) and the Schrödinger operator of Eq. (12) into Eq. (55). Then, using Eq. (19) and the commutation relations (8)-(10), we obtain the TQ equation (11), where

\[
1 + k(t) = 2 \left( \frac{d\kappa}{dt} - 2\kappa \frac{d\nu}{dt} \right) + 8f^{(2)}(t)\kappa e^{-2\nu} + f(t) e^{2\nu}, \tag{56}
\]

\[
h(t) = -2 \frac{d\nu}{dt} + 8f^{(2)}(t)\kappa e^{-2\nu}, \tag{57}
\]

\[
g(t) = -2 \frac{d\mu}{dt} e^{-\nu} - 8f^{(2)}(t)\kappa \mu e^{-\nu} + f^{(1)}(t)\kappa e^{-\nu}, \tag{58}
\]

\[
h^{(2)}(t) = f^{(2)}(t)e^{-2\nu}, \tag{59}
\]

\[
h^{(1)}(t) = f^{(1)}(t)e^{-\nu} - 2f^{(2)}(t)\mu e^{-\nu}, \tag{60}
\]

\[
h^{(0)}(t) = f^{(0)}(t) - f^{(1)}(t)\mu + f^{(2)}(t)\mu^2. \tag{61}
\]

We refer to Eqs. (56) to (61) as the \((TM \rightarrow TQ)\)-connecting equations. As we might expect, Eqs. (56)-(58) are equivalent to Eqs. (32)-(34). This can be seen by solving for the derivatives \( d\nu/dt \) and \( d\mu/dt \) and inverting Eqs. (59) to (61) for the functions \( f^{(j)}, j = 0, 1, 2 \).

4 Examples with \( f(t) = e^{-2\nu(t)} \)

4.1 Form of Examples

Now that the analysis of the transformations connecting the three Schrödinger equations (11), (12), and (13) has been completed, we illustrate how the mapping works with two examples of the \( TQ \rightarrow TM \rightarrow TO \) transformations.

But here, and in later work, we shall restrict \( f(t) \) in Eq. (12) to one particular form:

\[
f(t) = e^{-2\nu(t)}. \tag{62}
\]
In this case, the Schrödinger operator in Eq. (12) becomes

\[ \hat{S}_2 = -e^{-2\nu} P^2 + 2T - 2f^{(2)}(t)X^2 - 2f^{(1)}(t)X - 2f^{(0)}(t)I. \]  

(63)

The ‘hat’ indicates the restriction (62). The corresponding Schrödinger equation is

\[ \hat{S}_2 \hat{\Theta}(x, t) = \left\{ -e^{-2\nu} P^2 + 2T - 2f^{(2)}(t)X^2 - 2f^{(1)}(t)X - 2f^{(0)}(t)I \right\} \hat{\Theta}(x, t) = 0, \]  

(64)

\[ \hat{\Theta}(x, t) = \Theta(x, t, f(t) = e^{-2\nu}). \]  

(65)

This equation is a special case of the class TM equations (12), where \( f(t) \) is identified with \( \exp(-2\nu) \). The two examples are actually of this form.

With condition (62), Eqs. (32) and (33) remain the same. But, the right hand side of Eq. (34) is simplified to \( \frac{1}{2} k(t) \), that is

\[ \frac{dk}{dt} + h(t)\kappa - 4h^{(2)}(t)\kappa^2 = \frac{1}{2} k(t), \]  

(66)

which is a Riccati equation for \( \kappa \). The \( f^{(j)}, \) \( j = 0, 1, 2, \) in Eqs. (33) and (34) are still given by Eqs. (31) and (36), but \( \kappa \) is now a solution of the Riccati equation (34).

Since Eq. (66) is a Riccati equation, we can proceed analytically in the examples below. Furthermore, previously studied systems [11]-[24] are encompassed within the restriction (62).

Before continuing with the examples, we need to specifically incorporate (62) into Eq. (44) for the TM \( \rightarrow \) TO transformation. This becomes

\[ t' - t_o = \int_{t_o}^{t} ds e^{-2\nu(s)}. \]  

(67)

In addition, Eqs. (17) through (19) yield

\[ g^{(2)}(t') = \tilde{q}^{(2)}(t')e^{2\nu(t')} = \tilde{h}^{(2)}(t')e^{4\nu(t')}, \]  

(68)

\[ g^{(1)}(t') = \tilde{q}^{(1)}(t')e^{2\nu(t')} = 2\tilde{\mu}(t')\tilde{h}^{(2)}(t')e^{4\nu(t')} + \tilde{h}^{(1)}(t')e^{3\nu(t')}, \]  

(69)

\[ g^{(0)}(t') = \tilde{q}^{(0)}(t')e^{2\nu(t')} = \tilde{h}^{(0)}(t')e^{2\nu(t')} + \tilde{h}^{(1)}(t')e^{3\nu(t')}\tilde{\mu}(t') + \tilde{h}^{(2)}(t')e^{4\nu(t')}\tilde{\mu}^2(t'). \]  

(70)
4.2 Example 1

The first example is a frequently studied TM equation [11]-[23] of the form

\[ \hat{S}_2 \hat{\Theta}(x, t) = \left\{ -e^{\Upsilon(t-t_o)} P^2 + 2T - \omega^2 e^{-\Upsilon(t-t_o)} X^2 \right\} \hat{\Theta}(x, t) = 0, \tag{71} \]

with associated TM Hamiltonian

\[ \hat{H}_2 = -\frac{1}{2} e^{\Upsilon(t-t_o)} \partial_{xx} + \frac{1}{2} \omega^2 e^{-\Upsilon(t-t_o)} x^2. \tag{72} \]

Both \( \Upsilon \) and \( \omega^2 \) are real constants and we also take \( \omega^2 \) to be positive.

First, we shall find a TQ equation related to the TM equation (71) via the mapping (55).

Comparing the Schrödinger equations (64) and (71), we observe that,

\[ \nu = -\frac{1}{2} \Upsilon(t-t_o), \quad f^{(2)}(t) = \frac{1}{2} \omega^2 e^{-\Upsilon(t-t_o)}, \quad f^{(1)}(t) = f^{(0)}(t) = 0. \tag{73} \]

Therefore, Eqs. (59) to (61), yield

\[ h^{(2)}(t) = \frac{1}{2} \omega^2, \quad h^{(1)}(t) = -\omega^2 e^{-\frac{1}{2} \Upsilon(t-t_o)} \mu, \quad h^{(0)}(t) = \frac{1}{2} \omega^2 e^{-\Upsilon(t-t_o)} \mu^2. \tag{74} \]

Conditions (74) and Eqs. (32), (33), and (66) together imply that

\[ h(t) = \Upsilon + 4\omega^2 \kappa, \tag{75} \]

\[ 2 \frac{d\mu}{dt} + e^{\Upsilon(t-t_o)} g(t) + 4\omega^2 \mu \kappa = 0, \tag{76} \]

\[ \frac{d\kappa}{dt} + \Upsilon \kappa + 2\omega^2 \kappa^2 = \frac{1}{2} k(t). \tag{77} \]

Since there are a nondenumerable number of choices for \( k(t) \) and \( g(t) \), there are an nondenumerable number of TQ equations that can be mapped into the TM equation (71), with Eqs. (73) through (77) as the connecting equations.

A TQ Schrödinger equation with a time-independent Hamiltonian can be obtained by setting \( k(t) = g(t) = 0 \) for all \( t \). With the initial condition, \( R(t = t_o) = I \), the Riccati equation (77) with \( k(t) = 0 \) yields the trivial solution \( \kappa = 0 \), for all \( t \), as the only solution. Furthermore, Eq. (76) with \( g(t) = 0 \) and \( \kappa = 0 \) implies that \( \mu = 0 \), for all \( t \), [subject to the initial condition}
\[ R(t = t_0) = I \]. Hence, \( h(t) = \Upsilon \) and the mapping has the general form

\[ R(0, \nu, 0) = \exp \left[ i\nu D \right], \tag{78} \]

\( \nu \) given in Eq. (73). Under these conditions, the \( TQ \) equation (11) and Hamiltonian become

\[ S_1 \Phi(x, t) = \left\{ -P^2 + 2T + \Upsilon D - \omega^2 X^2 \right\} \Phi(x, t) = 0, \tag{79} \]

\[ H_1 = -\frac{1}{2} \partial_{xx} - \frac{1}{4} \Upsilon \left( -2ix\partial_x - i \right) + \frac{1}{2} \omega^2 x^2. \tag{80} \]

To find the equivalent \( TO \) equation, we change the “time” variable. From Eq. (67)

\[ t' - t'_{0} = \frac{1}{\Upsilon} \{ \exp [\Upsilon(t - t_0)] - 1 \}. \tag{81} \]

The inverse mapping is

\[ t - t_0 = \frac{1}{\Upsilon} \ln \left[ 1 + \Upsilon(t' - t'_0) \right], \tag{82} \]

where certain restrictions on \( t' \) apply. That is, if \( (t - t_0) \in [0, \infty) \), then when \( \Upsilon > 0 \), \( (t' - t'_0) \) lies in the interval \( [0, \infty) \) and when \( \Upsilon < 0 \), \( (t' - t'_0) \) lies in the interval \( [0, 1/|\Upsilon|) \).

From the (\( TM \rightarrow TO \))-connecting equations (68) to (70), we see that

\[ g^{(2)}(t') = \frac{1}{2} \frac{\omega^2}{[1 + \Upsilon(t' - t'_0)]^2}, \quad g^{(1)}(t) = g^{(0)}(t) = 0, \tag{83} \]

and the \( TO \) Schrödinger equation (13) and Hamiltonian are

\[ S_3 \Psi(x, t') = \left\{ -P^2 + 2T' - \frac{\omega^2}{[1 + \Upsilon(t' - t'_0)]^2} X^2 \right\} \Psi(x, t') = 0, \tag{84} \]

\[ H_3 = -\frac{1}{2} \partial_{xx} + \frac{\omega^2/2}{[1 + \Upsilon(t' - t'_0)]^2} x^2. \tag{85} \]

We shall discuss this example in detail elsewhere (39).

4.3 Example 2

The second example is also a \( TM \) Schrödinger equation. This time it is of the form

\[ \hat{S}_2 \hat{\Theta}(x, t) = \left\{ -\left( \frac{t_0}{t} \right)^a P^2 + 2T - \left( \frac{t}{t_0} \right)^b \omega^2 X^2 \right\} \hat{\Theta}(x, t) = 0, \tag{86} \]

where \( a \) and \( b \) are real numbers. The associated \( TM \) Hamiltonian is

\[ \hat{H}_2 = -\frac{1}{2} \left( \frac{t_0}{t} \right)^a \partial_{xx} + \frac{1}{2} \omega^2 \left( \frac{t}{t_0} \right)^b x^2, \tag{87} \]
for real values of $a$ and $b$. For positive values of $a$ and $b$, this is the Hamiltonian system studied by Kim [24]. We shall not consider the $a = 0$ case for which Eq. (86) already has TO form.

As in Example 1, we shall find a TQ equation related to the TM equation (86) via the mapping (55). Comparing the Schrödinger equations (64) and (86), we observe that,

$$\nu = \frac{a}{2} \ln \left(\frac{t}{t_o}\right), \quad f^{(2)}(t) = \frac{1}{2} \omega^2 \left(\frac{t}{t_o}\right)^b, \quad f^{(1)}(t) = f^{(0)}(t) = 0.$$  \hfill (88)

Therefore, Eqs. (59) to (61), yield

$$h^{(2)}(t) = \frac{1}{2} \omega^2 \left(\frac{t}{t_o}\right)^{b-a}, \quad h^{(1)}(t) = -\omega^2 \left(\frac{t}{t_o}\right)^{b-a/2} \mu, \quad h^{(0)}(t) = \frac{1}{2} \omega^2 \left(\frac{t}{t_o}\right)^b \mu^2.$$  \hfill (89)

Conditions (89) and Eqs. (32), (33), and (66) together imply that

$$h(t) = -\frac{a}{t} + 4\omega^2 \left(\frac{t}{t_o}\right)^{b-a} \kappa,$$  \hfill (90)

$$\frac{2 \, d\mu}{dt} + \left(\frac{4}{t}\right)^{a/2} g(t) + 4\omega^2 \left(\frac{t}{t_o}\right)^{b-a} \mu \kappa = 0,$$  \hfill (91)

$$\frac{d\kappa}{dt} - \frac{a}{t} \kappa + 2\omega^2 \left(\frac{t}{t_o}\right)^{b-a} \kappa^2 = \frac{1}{2} k(t).$$  \hfill (92)

There are a nondenumerable number of TQ equations, depending upon $g(t)$ and $k(t)$, that can be mapped into the TM equation (86), with Eqs. (88) through (92) as the connecting equations. To cite a specific example, if $g(t) = k(t) = 0$, then the only solutions to Eqs. (91) and (92) consistent with the initial condition $R(t = t_o) = I$, is $\kappa = \mu = 0$, for all $t$, and the transformation is of the form (78) with $\nu$ given in Eq. (88).

Since $h(t) = -a/t$, we see that the TQ Schrödinger equation (3) is

$$S_1 \Phi(x,t) = \left\{ -P^2 + 2T - \frac{a}{t} D - \omega^2 \left(\frac{t}{t_o}\right)^{b-a} X^2 \right\} \Phi(x,t) = 0.$$  \hfill (93)

The corresponding time-dependent TQ Hamiltonian is

$$H_1 = -\frac{1}{2} \partial_{xx} + \frac{a}{2t} \left( -2i x \partial_x - i \right) + \frac{1}{2} \omega^2 \left(\frac{t}{t_o}\right)^{b-a} x^2.$$  \hfill (94)

In the ($TM \rightarrow TO$) transformation, we compute a new “time” variable by substituting Eq. (88) for $\nu$ in (77) and performing the integration. We recognize two separate cases: $a = 1$ and $a \neq 0, 1$. Furthermore, we assume that $(t - t_o) \in [0, \infty)$. 

When $a = 1$, we find that
\[ t' - t'_o = t_o \ln \left( \frac{t}{t_o} \right), \] (95)
where $t' - t'_o \in [0, \infty)$. However, for $a \neq 0, 1$, we have
\[ t' - t'_o = \frac{t_o}{1 - a} \left[ \left( \frac{t}{t_o} \right)^{1-a} - 1 \right], \] (96)
where $t' - t'_o \in [0, \frac{1}{a}]$ if $a \in (1, \infty)$ and $(t' - t'_o) \in [0, \infty)$ if $a \in (-\infty, 0) \cup (0, 1)$.

From Eqs. (48) and (49), the functions $g^{(1)}(t') = g^{(0)}(t') = 0$ in Eq. (5). The TO Schrödinger equation (5) becomes
\[ S_3 \Psi(x, t') = \left\{ -P^2 + 2T' - 2g^{(2)}(t')X^2 \right\} \Psi(x, t') = 0, \] (97)
where Eq. (47) yields
\[ g^{(2)}(t') = \begin{cases} \frac{1}{2} \omega^2 \exp \left[ \left( \frac{1+b}{t_o} \right) (t' - t'_o) \right], & \text{for } a = 1, \\ \frac{1}{2} \omega^2 \left[ 1 + \left( \frac{1-a}{t_o} \right) (t' - t'_o) \right]^{\frac{1+b}{1-a}}, & \text{for } a \neq 0, 1. \end{cases} \] (98)

The Hamiltonian then has the form
\[ H_3 = -\frac{1}{2} \partial_{xx} + g^{(2)}(t') x^2. \] (99)

It is also possible to map many TM Schrödinger equations into a single TO equation. Example 2 provides an illustration of this. Look at the second equality in Eq. (98) ($a \neq 0, 1$). For all $b = -a$, we have $g^{(2)}(t') = \frac{1}{2} \omega^2$. Thus, we have a nondenumerable number of distinct TM equations, determined by each $a$ and $b = -a$, that are mapped into a single TO equation. This TO equation is independent of $a$ and $b$. This is a common phenomenon in Example 2.

We shall also discuss this example in detail elsewhere \[39\].

5 Summary

We have developed a general method for transforming Schrödinger equations of class $TQ$ into the subclasses of $TM$ and $TO$ equations. The transformation involves (i) a unitary mapping or (ii) a unitary mapping and a change in ‘time’ variable. This permits us to map (i) a $TQ$
Schrödinger equation into a $TM$ Schrödinger equation or (ii) into a $TO$ Schrödinger equation. In paper II, we shall use these transformations to show that all these equations have isomorphic space-time symmetry algebras. We shall exploit this isomorphism and the known generators for $TO$ equations to obtain solutions for each of the Schrödinger equations in the class $TQ$ and the subclasses $TM$ and $TO$. Then, we will compute displacement-operator coherent and squeezed states for each class and subclass.

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