Symbolic Control for Stochastic Systems
via Finite Parity Games

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Abstract

We consider the problem of computing the maximal probability of satisfying an \(\omega\)-regular specification for stochastic, continuous-state, nonlinear systems evolving in discrete time. The problem reduces, after automata-theoretic constructions, to finding the maximal probability of satisfying a parity condition on a (possibly hybrid) state space. While characterizing the exact satisfaction probability is open, we show that a lower bound on this probability can be obtained by (I) computing an under-approximation of the qualitative winning region, i.e., states from which the parity condition can be enforced almost surely, and (II) computing the maximal probability of reaching this qualitative winning region.

The heart of our approach is a technique to symbolically compute the under-approximation of the qualitative winning region in step (I) via a finite-state abstraction of the original system as a \(2^{1/2}\)-player parity game. Our abstraction procedure uses only the support of the probabilistic evolution; it does not use precise numerical transition probabilities. We prove that the winning set in the abstract \(2^{1/2}\)-player game induces an under-approximation of the qualitative winning region in the original synthesis problem, along with a policy to solve it. By combining these contributions with (a) a symbolic fixpoint algorithm

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Preprint submitted to Elsevier September 30, 2022
to solve 2\(1/2\)-player games and (b) existing techniques for reachability policy synthesis in stochastic nonlinear systems, we get an abstraction-based algorithm for finding a lower bound on the maximal satisfaction probability.

We have implemented the abstraction-based algorithm in Mascot-SDS (Majumdar et al., 2020), where we combined the outlined abstraction step with our recent tool FairSyn (Banerjee et al., 2021) that solves 2\(1/2\)-player parity games more efficiently than existing algorithms. We evaluated our implementation on the nonlinear model of a perturbed bistable switch from the literature. We show empirically that the lower bound on the winning region computed by our approach is precise, by comparing against an over-approximation of the qualitative winning region. Moreover, thanks to FairSyn, we outperform a recently proposed tool for solving this problem by a large margin. In fact, in many cases the existing tool crashed after consuming too much memory on a standard laptop, whereas our tool consumed small amount of memory and produced results within reasonable amount of time.

**Keywords:** Correct-by-design controller synthesis, controlled Markov processes, stochastic dynamical systems, \(\omega\)-regular specifications

### 1. Introduction

Controlled Markov processes (CMPs) over continuous state spaces and evolving in discrete time form a general model for temporal decision making under stochastic uncertainty. In recent years, the problem of finding or approximating optimal policies in CMPs for specifications given in temporal logics or automata has received a lot of attention. While there is a steady progression towards more expressive models and properties (Svoreňová et al., 2017; Haesaert & Soudjani, 2020; Dutreix & Coogan, 2020; Majumdar et al., 2020; Dutreix et al., 2020; Lavaei et al., 2021), a satisfactory solution that can handle nonlinear models for general \(\omega\)-regular specifications in a symbolic way is still open. In this paper, we make progress toward a solution to this general problem.

For finite-state Markov decision processes (MDP), one can find optimal poli-
cies for $\omega$-regular specifications by decomposing the problem into two parts (Courcoubetis & Yannakakis, 1990; Bianco & de Alfaro, 1995; de Alfaro, 1998; Baier & Katoen, 2008). (I) Using graph-theoretic techniques that ignore the actual transition probabilities, one can find the set of states that ensures the satisfaction of the specification almost surely. Further, for any state in this almost sure winning region, an optimal policy for almost sure satisfaction of the specification can be derived. (II) One finds an optimal policy to reach the almost sure winning region using linear programming or traditional dynamic programming approaches. Combining both policies returns an optimal policy for the overall synthesis problem.

Unfortunately, this two-step solution approach does not carry over to optimal policy synthesis for all $\omega$-regular specifications given a continuous-state CMP. First, we do not have characterizations of optimal policies for almost sure satisfaction in this case—such as whether randomization or memory is necessary. Second, in contrast to finite-state MDPs, it is possible that the almost sure winning region of a CMP is empty, even if there is a policy that satisfies the specification with positive probability (Majumdar et al., 2020).

However, we show in this paper, that the same decomposition can be used to compute an under-approximation for the optimal policy instead: that is, the resulting policy gives a lower bound on the probability of satisfying a given $\omega$-regular specification from every state. While existing techniques (Soudjani & Abate, 2013; Tkachev et al., 2017; Haesaert & Soudjani, 2020) can be used in step (II) to compute the reachability probability with any given precision, this paper provides a new technique to under-approximate the set of states of a CMP that almost surely satisfies a parity specification in step (I) of the decomposition. A parity specification is a canonical representation for all $\omega$-regular properties (Emerson & Jutla, 1991; Thomas, 1995); thus, our approach provides a way to under-approximate any $\omega$-regular specification.

The main contribution of our paper is to show that the approximate solution to step (I) can be computed by a symbolic algorithm over a finite state abstraction of the underlying CMP that is using only the support of the probabilistic
evolution of the system. This abstraction-based policy synthesis technique is inspired by abstraction-based controller design (ABCD) for non-stochastic systems (Reissig et al., 2017; Tabuada, 2009; Nilsson et al., 2017). In ABCD, a nonlinear dynamical system is abstracted into a discrete two-player game over a finite discrete state space obtained by partitioning the continuous state space into a finite set of cells. The resulting abstract two-player game is then used to synthesize a discrete controller which is then refined into a continuous controller for the original system.

In ABCD, the abstract two-player game models the interplay between the controller (Player 0) and the dynamics (Player 1) such that the resulting abstract controller (i.e., the winning strategy of Player 0) can be correctly refined to the original system. This requires a very powerful Player 1; in every instance of the play (corresponding to the system being in one particular abstract cell \( \hat{s} \)) and for any input \( u \) chosen by Player 0, Player 1 can adversarially choose both (a) the actual continuous state \( s \) within \( \hat{s} \) to which \( u \) is applied and (b) any continuous disturbance affecting the system at this state \( s \).

The key insight in our work is that the stochastic nature of the underlying CMP does not require a fully adversarial treatment of continuous disturbances by Player 1 in the abstract game to allow for controller refinement. Intuitively, disturbances need to be handled in a fair way: In the long run, all transitions with positive probability will eventually occur. We show, that the resulting fairness assumption on the behavior of Player 1 can be modeled by an additional random player (also called 1/2 player) resulting in a so called 2\(1/2\)-player game (Condon, 1992; Chatterjee et al., 2003; Chatterjee & Henzinger, 2012) as the abstraction.

This provides a conceptually very appealing result of our paper. Using 2\(1/2\)-player games as abstractions of CMPs allows to utilize the machinery of symbolic game solving, analogous to ABCD techniques for non-stochastic systems, while capturing the intuitive differences between the problem instances by the use of a random player in the abstract game. Most interestingly, the stochastic nature of the resulting abstract game cases the abstract synthesis problem compared
to standard ABCD, in the sense that controllers for a given specification are more likely to be found. The price to be paid, however, is that solving 2\textsuperscript{1/2}-player games is computationally harder than solving two-player games. To mitigate this problem we utilize the recent symbolic algorithm for solving 2\textsuperscript{1/2}-player games by Banerjee et al. (2022) which has almost the same worst-case complexity as the usual algorithm for (non-stochastic) two player games. With this, our approach becomes computationally tractable and vastly outperforms recent tools using different solution approaches.

In conclusion, we obtain a symbolc algorithm to compute an under-approximation of the almost sure winning region in a continuous-state CMP for all \(\omega\)-regular specifications. Moreover, similar to the results for finite-state MDPs, this shows that the (approximate) solution to step (I) does not need to handle the actual transition probabilities. They are only needed in step (II), where existing techniques can be used. In addition, we yield an efficient implementation of our approach by utilizing a recent symbolic game solving algorithm for 2\textsuperscript{1/2}-player parity games.

A preliminary version of our paper appeared in ADHS ’21 (Majumdar et al., 2021). In this manuscript we have made substantial improvement over the conference version; the main additional contributions are as follows:

1. Most importantly, we have completed the exposition of our solution to the stated synthesis problem by

   (a) additionally addressing the quantitative aspect of the optimal policy synthesis problem in Thm. 3.2 and Thm. 3.3 while the conference version of the paper only dealt with the qualitative aspect of the problem, and

   (b) including the new symbolic algorithm for solving the resulting 2\textsuperscript{1/2}-player games adapted from Banerjee et al. (2022), while the conference version utilized the existing algorithm by Chatterjee et al. (2003).

2. We have added the proof of Thm. 4.7 which is the core theorem of this paper and which was omitted in the conference version.
3. We have substantially improved the experimental evaluation (see Sec. [5]) of our approach by considering benchmark examples proposed by other authors, and comparing the performance of our tool against the available tool from the literature.

4. We have provided examples and extensive explanation of the required steps throughout the manuscript.

Related Work. Our paper extends the recent results of Majumdar et al. (2020) from Büchi specifications to parity specifications. Seen through the lens of 21/2-player games, the algorithm of Majumdar et al. (2020) can be seen as directly solving a Büchi game symbolically on a non-probabilistic abstraction, by implicitly reducing the 21/2-player games to two player games on graphs with extreme fairness assumptions (Hart et al., 1983). While it may be possible to present a similar “direct” symbolic algorithm for parity games, the details of handling fairness symbolically get difficult. Our exposition in this paper helps separate out the different combinatorial aspects: the representation of the abstraction and the solution of the game on the abstraction, leading to a clean proof of correctness.

21/2-player games have been used as abstractions of probabilistic systems, both in the finite case (Kwiatkowska et al., 2020) and for stochastic linear systems (Svoreňová et al., 2017). Our paper subsumes the result of both these cases by showing a computational procedure to abstract a general, nonlinear CMP into a finite-state 21/2-player game. Further the existing approach for stochastic linear systems (Svoreňová et al., 2017) only considers specifications in the GR(1) fragment of linear temporal logic, whereas we can handle any ω-regular specification. Another difference is that the linearity assumption enables the use of symbolic algorithms based on polyhedral manipulations, when the specification is given using polytopic predicates on the state space. Instead, our abstractions are based on gridding the state space, as in ABCD for non-stochastic nonlinear systems.

Stochastic nonlinear systems were abstracted to finite-state bounded-parameter
Markov decision processes (BMDP) by Dutreix et al. (2020) for the purpose of controller synthesis. By using algorithms for finding controllers on BMDPs against deterministic Rabin automata, they provide an alternative approach for approximating the optimal controller against \( \omega \)-regular specifications. Their method is conceptually very different. It explicitly computes lower and upper bounds of all involved transition probabilities for constructing the BMDP, and computes winning regions by an enumerative algorithm taking these probability bounds into account. On the other hand, our approach shows a clean separation between step (II), which requires knowing explicit transition probabilities but can be solved by existing techniques, and step (I), for which this knowledge is not needed. This allows us to provide a conceptually simpler and computationally superior \textit{symbolic algorithm} approximately solving (I) via abstract 2\( \frac{1}{2} \)-player games.

In principle, one can alternatively obtain a similar 2\( \frac{1}{2} \)-player game, like us, by first reducing a CMP to a BMDP using the method of Dutreix et al. (2020) and then reducing the BMDP to a 2\( \frac{1}{2} \)-player game using the results of Weininger et al. (2019). However, this would still require going through the expensive construction of the BMDP, whereas we present a direct and computationally efficient way to obtain a 2\( \frac{1}{2} \)-player game through reachable set computations for the CMP. The computational benefit of our approach over the BMDP based method is enormous: On a couple of benchmark examples taken from the paper of Dutreix et al. (2020) itself, our uniform-grid based implementation was up to around 150 times faster and used up to around 150 times smaller amount of memory than their adaptive refinement-based implementation. The details of the experiments can be found in Sec. 6.

2. Stochastic Nonlinear Systems

2.1. Preliminaries

For any set \( A \), a sigma-algebra on \( A \) comprises subsets of \( A \) as events that includes \( A \) itself and is closed under complement and countable unions. We
consider a probability space \((\Omega, \mathcal{F}_\Omega, P_\Omega)\), where \(\Omega\) is the sample space, \(\mathcal{F}_\Omega\) is a sigma-algebra on \(\Omega\), and \(P_\Omega\) is a probability measure that assigns probabilities to events. An \(((S, \mathcal{F}_S)\text{-valued})\) random variable \(X\) is a measurable function of the form \(X : (\Omega, \mathcal{F}_\Omega) \to (S, \mathcal{F}_S)\), where \(S\) is the codomain of \(X\) and \(\mathcal{F}_S\) is a sigma-algebra on \(S\). Any random variable \(X\) induces a probability measure on its space \((S, \mathcal{F}_S)\) as \(P(\{A\}) = P_\Omega(\{X^{-1}(A)\})\) for any \(A \in \mathcal{F}_S\). We often directly discuss the probability measure on \((S, \mathcal{F}_S)\) without explicitly mentioning the underlying probability space \((\Omega, \mathcal{F}_\Omega, P_\Omega)\) and the function \(X\) itself.

A topological space \(S\) is called a Borel space if it is homeomorphic to a Borel subset of a Polish space (i.e., a separable and completely metrizable space). Examples of a Borel space are the Euclidean spaces \(\mathbb{R}^n\), its Borel subsets endowed with a subspace topology, as well as hybrid spaces. Any Borel space \(S\) is assumed to be endowed with a Borel sigma-algebra (i.e., the one generated by the open sets in the topology), which is denoted by \(\mathcal{B}(S)\). We say that a map \(f : S \to Y\) is measurable whenever it is Borel measurable.

Given an alphabet \(A\), we use the notation \(A^*\) and \(A^\omega\) to denote respectively the set of all finite words, the set of all infinite words formed using the letters of the alphabet \(A\), and use \(A^\infty\) to denote the set \(A^* \cup A^\omega\). Let \(X\) be a set and \(R \subseteq X \times X\) be a relation. For simplicity, let us assume that \(\text{dom } R := \{x \in X \mid \exists y \in X . (x, y) \in R\} = X\). For any given \(x \in X\), we use the notation \(R(x)\) to denote the set \(\{y \in X \mid (x, y) \in R\}\). We extend this notation to sets: For any given \(Z \subseteq X\), we use the notation \(R(Z)\) to denote the set \(\bigcup_{z \in Z} R(z)\). Given a set \(A\), we use the notation \(\text{Dist}(A)\) to denote the set of all probability distributions over \(A\).

We denote the set of nonnegative integers by \(\mathbb{N} := \{0, 1, 2, \ldots\}\) and the set of integers in an interval by \([a; b] := \{a + k \mid k \in \mathbb{N}, k \leq b - a\}\). We also use the symbols \(\in_{\text{even}}\) and \(\in_{\text{odd}}\) to denote memberships in the set of even and odd integers within a given set of integers: For example, for a given set of natural numbers \(M \subseteq \mathbb{N}\), the notation \(n \in_{\text{even}} M\) is equivalent to \(n \in M \cap \{0, 2, 4, \ldots\}\), and the notation \(n \in_{\text{odd}} M\) is equivalent to \(n \in M \cap \{1, 3, 5, \ldots\}\).
2.2. Controlled Markov Processes

A controlled Markov process (CMP) is a tuple \( \mathcal{G} = (\mathcal{S}, \mathcal{U}, T_s) \), where \( \mathcal{S} \) is a Borel space called the state space, \( \mathcal{U} \) is a finite set called the input space, and \( T_s \) is a conditional stochastic kernel \( T_s : \mathcal{B}((\mathcal{S})) \times \mathcal{S} \times \mathcal{U} \rightarrow [0, 1] \) with \( \mathcal{B}(\mathcal{S}) \) being the Borel sigma-algebra on the state space and \( (\mathcal{S}, \mathcal{B}(\mathcal{S})) \) being the corresponding measurable space. The kernel \( T_s \) assigns to any \( s \in \mathcal{S} \) and \( u \in \mathcal{U} \) a probability measure \( T_s(|s, u) \) on the measurable space \( (\mathcal{S}, \mathcal{B}(\mathcal{S})) \) so that for any set \( A \in \mathcal{B}(\mathcal{S}) \), \( P_{s,u}(A) = \int_A T_s(ds|s, u) \), where \( P_{s,u} \) denotes the conditional probability \( P(|s, u) \).

In general, the input space \( \mathcal{U} \) can be any Borel space and the set of valid inputs can be state dependent. While our results can be extended to this setting, for ease of exposition, we consider the special case where \( \mathcal{U} \) is a finite set and any input can be taken at any state. This choice is motivated by the digital implementation of control policies with a finite number of possible actuations.

The evolution of a CMP is as follows. For \( k \in \mathbb{N} \), let \( X^k \) denote the state at the \( k \)-th time step and \( A^k \) the input chosen at that time. If \( X^k = s \in \mathcal{S} \) and \( A^k = u \in \mathcal{U} \), then the system moves to the next state \( X^{k+1} \), according to the probability distribution \( P_{s,u} \). Once the transition into the next state has occurred, a new input is chosen, and the process is repeated.

Given a CMP \( \mathcal{G} \), a finite path of length \( n + 1 \) is a sequence

\[
  w^n = (s^0, s^1, \ldots, s^n), \quad n \in \mathbb{N},
\]

where \( s^i \in \mathcal{S} \) are state coordinates of the path. The space of all paths of length \( n + 1 \) is denoted \( \mathcal{S}^{n+1} \). An infinite path of the CMP \( \mathcal{G} \) is the sequence \( w = (s^0, s^1, s^2, \ldots) \), where \( s^i \in \mathcal{S} \) for all \( i \in \mathbb{N} \). The space of all infinite paths is denoted by \( \mathcal{S}^\omega \). The spaces \( \mathcal{S}^{n+1} \) and \( \mathcal{S}^\omega \) are endowed with their respective product topologies and are Borel spaces.

A stationary control policy is a universally measurable function \( \rho : \mathcal{S} \rightarrow \mathcal{U} \) such that at any time step \( n \in \mathbb{N} \), the input \( w^n \) is taken to be \( \rho(s^n) \in \mathcal{U} \). As we only deal with stationary control policies in this paper, we simply refer to
them as policies for short. We denote the class of all such policies by Π. The function ρ is also called state feedback controller in control theory.

For a CMP S, any policy ρ ∈ Π together with an initial probability measure α : B(S) → [0, 1] on the states induces a unique probability measure on the canonical sample space of paths (Hernández-Lerma & Lasserre, 1996), denoted by P_ρ^α with the expectation E_ρ^α. In the case when the initial probability measure is supported on a single state s ∈ S, i.e., α(s) = 1, we write P_ρ^s and E_ρ^s in place of P_ρ^α and E_ρ^α, respectively. We denote the set of probability measures on (S, B(S)) by D.

Given any ω-regular specification φ defined using a set of predicates over the state space S of S, we use the notation S |= φ to denote the set of all paths of S which satisfy φ. Thus, P_ρ^α(S |= φ) denotes the probability of satisfaction of φ by S under the effect of the control policy ρ, when the initial probability measure is given by α. Often we will use Linear Temporal Logic (LTL) notation to express ω-regular properties. The syntax and semantics of LTL can be found in standard literature (Baier & Katoen, 2008).

A stochastic dynamical system Σ is described by a state evolution

\[ s^{k+1} = f(s^k, u^k, \varsigma^k), \quad k \in \mathbb{N}, \]

where s^k ∈ S and u^k ∈ U are states and inputs for each k ∈ N, and (\varsigma^0, \varsigma^1, ... ) is assumed to be a sequence of independent and identically distributed (i.i.d.) random variables representing a stochastic disturbance. The map f gives the next state as a function of current state, current input, and the disturbance. One can construct a CMP over states S and inputs U from (1) by noticing that for any given state s^k and input u^k at time k, the next state is a random variable defined as a function of ς^k. Thus, T_s(\cdot|s^k, u^k) is exactly the distribution of the random variable f(s^k, u^k, ς^k) and can be computed based on the distribution of ς^k and the map f itself (Kallenberg, 2002).
2.3. Parity Specifications

Let $\mathfrak{S} = (S, \mathcal{U}, T)$ be a CMP and suppose $\mathcal{P} = \langle B_1, \ldots, B_\ell \rangle$ is a partition of $S$ with measurable sets $B_1, \ldots, B_\ell$; that is, $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^\ell B_i = S$. We allow some $B_i$’s to be empty. For each $B_i$, we call the integer $i$ its priority.

Intuitively, an infinite path $w \in S^\omega$ satisfies the parity specification with respect to $\mathcal{P}$ if the highest subset $B_i$ visited infinitely often has even priority. This specification is formalized in Linear Temporal Logic (LTL) notation 

\[
Parity(\mathcal{P}) := \bigwedge_{i \in \text{odd}[1:\ell]} \left( \square \Diamond B_i \implies \bigvee_{j \in \text{even}[i+1:\ell]} \square \Diamond B_j \right),
\]

which requires that infinitely many visits to an odd priority subset ($\square \Diamond B_i$) must imply infinitely many visits to a higher even priority subset ($\square \Diamond B_j$). We indicate the set of all infinite paths $w \in S^\omega$ of a CMP $\mathfrak{S}$ that satisfy the property $Parity(\mathcal{P})$ by $\mathfrak{S} \models Parity(\mathcal{P})$. The proof of measurability of the event $\mathfrak{S} \models Parity(\mathcal{P})$ goes back to the work by Vardi (1985) that provides the proof for probabilistic finite state programs. The proof for a CMP follows similar principles, using the observation that $\mathfrak{S} \models Parity(\mathcal{P})$ can be written as a Boolean combination of events $\mathfrak{S} \models \square \Diamond A$, where $A$ is a measurable set, and $\square \Diamond A$ is a canonical $G_\delta$ set in the Borel hierarchy.

It is well-known that every $\omega$-regular specification whose propositions range over measurable subsets of the state space of a CMP can be modeled as a deterministic parity automaton (Gradel & Thomas, 2002, Thm. 1.19). By taking a synchronized product of this parity automaton with the CMP, we can obtain a product CMP and a parity specification over the product state space such that every path satisfying the parity specification also satisfies the original $\omega$-regular specification and vice versa. Moreover, a stationary policy for the parity objective gives a (possibly history-dependent) policy for the original specification. Thus, without loss of generality, we assume that an $\omega$-regular objective is already given as a parity condition using a partition of the state space of the
3. Problem Definition

We are interested in computing the maximal probability that a given parity specification can be satisfied by paths of a CMP \( \mathcal{S} \) starting from a particular state \( s \in \mathcal{S} \) under stationary policies. Given a control policy \( \rho \in \Pi \) and an initial state \( s \in \mathcal{S} \), we define the satisfaction probability and the supremum satisfaction probability as

\[
f(s, \rho) := P^\rho_s(\mathcal{S} \models \text{Parity}(\mathcal{P})) \quad \text{and} \quad (3)
f^*(s) := \sup_{\rho \in \Pi} P^\rho_s(\mathcal{S} \models \text{Parity}(\mathcal{P})), \quad (4)
\]

respectively. An optimal control policy for the parity condition is a policy \( \rho^* \) such that \( f^*(s) = f(s, \rho^*) \) for all \( s \in \mathcal{S} \). Note that an optimal policy need not exist, since the supremum may not be achieved by any policy. Our goal is to study the following optimal policy synthesis problem.

**Problem 1** (Optimal Policy Synthesis). Given \( \mathcal{S} \) and a parity specification \( \text{Parity}(\mathcal{P}) \), find an optimal control policy \( \rho^* \), if it exists, together with \( f^*(s) \) such that \( P^\rho_s(\mathcal{S} \models \text{Parity}(\mathcal{P})) = f^*(s) \).

While the satisfaction probability (3) and the supremum satisfaction probability (4) are both well-defined, we are not aware of any work characterizing necessary or sufficient conditions for existence of optimal control policies on continuous-space CMPs for parity specifications. Additionally, we restrict attention to stationary policies. While it is possible to define more general classes of policies, that depend on the entire history and use randomization over \( \mathcal{U} \), we are again unaware of any work that characterizes the class of policies that are sufficient for optimal control of CMPs for parity specifications. For finite-state systems, stationary policies are sufficient and we restrict attention to them.

Since we cannot prove existence or computability of optimal policies, in this paper, we focus on providing an approximation procedure to compute a
possibly sub-optimal control policy and guaranteed lower bounds on the optimal satisfaction probability. Our procedure relies on first approximating almost sure winning regions (i.e., where the specification can be satisfied with probability one), and then solving a reachability problem, as formalized next.

**Definition 3.1 (Almost sure winning region).** Given a CMP $\mathcal{S}$, a policy $\rho$, and a parity specification $\text{Parity}(P)$, the state $s \in \mathcal{S}$ satisfies the specification almost surely if $f(s, \rho) = 1$. The almost sure winning region—or simply the winning region—of the policy $\rho$ is defined as

$$\text{WinDom}(\mathcal{S}, \rho) := \{s \in \mathcal{S} | f(s, \rho) = 1\}. \quad (5)$$

We also define the maximal almost sure winning region—or simply the maximal winning region—as

$$\text{WinDom}^*(\mathcal{S}) := \{s \in \mathcal{S} | f^*(s) = 1\}. \quad (6)$$

Note that $\text{WinDom}(\mathcal{S}, \rho) \subseteq \text{WinDom}^*(\mathcal{S})$ for any control policy $\rho \in \Pi$. It is clear by definition of the winning region that for any policy $\rho$, the satisfaction probability $P^\rho_s(\mathcal{S} \models \text{Parity}(P))$ is equal to 1 for any $s$ in the winning region $W := \text{WinDom}(\mathcal{S}, \rho)$. The next theorem states that this satisfaction probability is lower bounded by the probability of reaching the winning region $W$ for any $s \not\in W$. We denote such a reachability by $(\mathcal{S} \models \Diamond W) := \{w = (s^0, s^1, s^2, \ldots) | \exists n \in \mathbb{N}. s^n \in W\}$.

**Theorem 3.2.** For any control policy $\rho \in \Pi$ on CMP $\mathcal{S}$, and $W := \text{WinDom}(\mathcal{S}, \rho)$, we have

$$P^\rho_s(\mathcal{S} \models \text{Parity}(P)) = 1 \quad \text{if } s \in W \quad \text{and} \quad (7)$$

$$P^\rho_s(\mathcal{S} \models \text{Parity}(P)) \geq P^\rho_s(\mathcal{S} \models \Diamond W) \quad \text{if } s \not\in W.$$

The proof can be found in the appendix. The inequality in the second part of (7) is because the Parity($P$) specification may be satisfied with positive probability even though the path always stays outside of $W$. When the state space is finite (i.e., for finite Markov decision processes), equality holds. [Baier &]
Katoen, 2008). However, equality need not hold for general CMPs: Majumdar et al. (2020) showed an example where the maximal almost sure winning region is empty even though a Büchi specification is satisfied with positive probability.

The next theorem establishes that for any policy \( \rho \), the winning region is an absorbing set, i.e., paths starting from this set will stay in the set almost surely.

**Theorem 3.3.** For any control policy \( \rho \), the set \( W = \text{WinDom}(\mathcal{G}, \rho) \) is an absorbing set, i.e., \( T_s(W \mid s, \rho(s)) = 1 \) for all \( s \in W \). This implies \( P^s_\rho(\mathcal{G} \models \Diamond \mathcal{S} \setminus W) = 0 \) for all \( s \in W \).

The proof of this theorem utilizes the fact that Parity(\( \mathcal{P} \)) is a long-run property and its satisfaction is independent of the prefix of a path. The proof is provided in the appendix.

Notice that Thm. 3.2 and Thm. 3.3 are stated for any fixed control policy \( \rho \), but these theorems enable us to decompose the maximization of \( P^s_\rho(\mathcal{G} \models \text{Parity}(\mathcal{P})) \) with respect to \( \rho \) into two sub-problems. First, find a policy that gives the largest winning region \( W \) and employ that policy when the current state is in \( W \). Then, find a policy that maximizes the reachability probability \( P^s_\rho(\mathcal{G} \models \Diamond W) \) and employ that policy as long as the current state is not in \( W \).

Computation of the reachability probability has been studied extensively in the literature for both infinite horizon (Tkachev & Abate, 2012; Tkachev et al., 2017; Haesaert & Soudjani, 2020; Haesaert et al., 2021) and finite horizon (Soudjani & Abate, 2013; 2012; Lahijanian et al., 2015; Kariotoglou et al., 2017; Lesser & Oishi, 2017; Soudjani et al., 2017; Vinod & Oishi, 2018; Lavaei et al., 2019; Jagtap et al., 2020) using different abstract models and computational methods. Together with an algorithm that underapproximates the region of almost sure satisfaction, these approaches can be used to provide a lower bound on the probability of satisfaction of the parity condition. In the rest of the paper, we focus on the following problem (the first half of (7)).

**Problem 2 (Approximate Maximal Winning Region).** Given \( \mathcal{G} \) and a parity specification Parity(\( \mathcal{P} \)), find a (sub-optimal) control policy \( \rho \in \Pi \), its winning
region \( \text{WinDom}(\mathcal{G}, \rho) \neq \emptyset \), and a bound on the absolute volume of the set difference \( \text{WinDom}^*(\mathcal{G}) \backslash \text{WinDom}(\mathcal{G}, \rho) \), which we call the approximation error.

In Sec. 4–5, we provide a solution for Prob. 2 via the paradigm of abstraction-based controller design. Not surprisingly, we get a tighter (i.e., larger) approximation of \( \text{WinDom}^*(\mathcal{G}) \) if we use a finer discretization of the state space during the abstraction step. We also provide an over-approximation of \( \text{WinDom}^*(\mathcal{G}) \), and show closeness of the under- and over-approximation of \( \text{WinDom}^*(\mathcal{G}) \) in the numerical example provided in Sec. 6.

4. Abstraction-Based Policy Synthesis

The main result of our paper is a solution to Prob. 2 via a symbolic algorithm over abstract \( 2^{1/2} \)-player games in the spirit of abstraction-based controller design (ABCD). ABCD is typically used to compute temporal-logic controllers for non-stochastic nonlinear dynamical systems (Reissig et al., 2017; Tabuada, 2009; Nilsson et al., 2017) in two steps. First, the system is abstracted into an abstract finite-state two-player game. This game is then used to synthesize a discrete controller which is then refined into a continuous controller for the original system. In standard ABCD techniques, the abstract game has two players: Player 0 simulating the controller and choosing the next control input \( u \) based on the currently observed abstract state \( \tilde{s} \), and Player 1 simulating the adversarial effect of (a) choosing any continuous state \( s \) in \( \tilde{s} \) to which \( u \) is applied and (b) choosing any continuous disturbance \( d \) that effects the resulting transition.

The key insight in our abstraction step is that the stochastic nature of the underlying CMP allows choosing disturbances in a fair random way instead of purely adversarially. We model this by introducing an additional random player (also called \( 1/2 \) player) resulting in a so called \( 2^{1/2} \)-player game (Condon, 1992; Chatterjee et al., 2003; Chatterjee & Henzinger, 2012). In the resulting abstract game, only the effect of the discretization is handled by Player 1 in an adversarial manner. The random player picks the applied disturbance uniformly at random.
After introducing necessary preliminaries on 2\slash 1-player games in Sec. 4.1, we show how a CMP can be abstracted into a 2\slash 1-player game in Sec. 4.2. We then recall in Sec. 4.3 a symbolic procedure to find winning regions in 2\slash 1-player games for parity specifications. Finally, we state in Sec. 4.4 how an almost-sure winning strategy in the abstract 2\slash 1-player game is refined, and that the resulting control policy is almost sure winning for the original CMP and its associated parity specification. This establishes soundness of our ABCD technique to solve Problem 2.

4.1. Preliminaries: 2\slash 1-Player Parity Games

The game graph. A 2\slash 1-player game graph is a tuple $G = \langle V, E, (V_0, V_1, V_r) \rangle$, where $V$ is a finite set of vertices, $E$ is a set of directed edges $E \subseteq V \times V$, and the sets $V_0, V_1, V_r$ form a partition of the set $V$. A Markov chain is a particular type of 2\slash 1-player game graph with $V = V_r$ (i.e. $V_0 = V_1 = \emptyset$).

A 2\slash 1-player parity game is a pair $\langle G, P \rangle$, where $G$ is a 2\slash 1-player game graph, and $P = \langle B_1, \ldots, B_\ell \rangle$ is a tuple of $\ell$ disjoint subsets of $V$, some of which can possibly be empty. The tuple $P$ induces the parity specification $\text{Parity}(P)$ over the set of vertices $V$ in the natural way. In order to ensure that $\text{Parity}(P)$ is well defined, we impose the restriction that every infinite run must have infinitely many occurrences of vertices from at least one of the nonempty sets in $P$. In other words, we require that every set of vertices $U \subseteq V$ for which there is no $i \in [1; \ell]$ with $U \cap B_i \neq \emptyset$ must be “transient” vertices.

The players and their strategies. We assume that there are two players $\text{Player} 0$ and $\text{Player} 1$, who are playing a game by moving a token along the edges of the game graph $G$. In every step, if the token is located in a vertex in $V_0$ or $V_1$, $\text{Player} 0$ or $\text{Player} 1$ respectively moves the token to one of the successors according to the edge relation $E$. On the other hand, if the token is located in a vertex $v \in V_r$, then in the next step the token moves to a vertex $v'$ which is chosen uniformly at random from the set $E(v)$. Strategies of $\text{Player} 0$ and $\text{Player} 1$ are respectively the functions $\pi_0: V^*V_0 \rightarrow \text{Dist}(V)$ and $\pi_1: V^*V_1 \rightarrow \text{Dist}(V)$ such that for all $w \in V^*$, $v_0 \in V_0$ and $v_1 \in V_1$, we have
\(\text{supp } \pi_0(wv_0) \subseteq E(v_0)\) and \(\text{supp } \pi_1(wv_1) \subseteq E(v_1)\). We use the notation \(\Pi_0\) and \(\Pi_1\) to denote the set of all strategies of Player 0 and Player 1 respectively. A strategy \(\pi_i\) of Player \(i\), for \(i \in \{0, 1\}\), is deterministic memoryless if for every \(w_1, w_2 \in V^*\) and for every \(v \in V_i\), \(\pi_i(w_1v) = \pi_i(w_2v)\) holds; we simply write \(\pi_i(v)\) in this case. We use the notation \(\Pi_i^{\text{DM}}\) to denote the set of all deterministic memoryless strategies of Player \(i\). Observe that \(\Pi_i^{\text{DM}} \subseteq \Pi_i\).

**Runs and winning conditions.** An infinite (finite) run of the game graph \(G\), compatible with the strategies \(\pi_0 \in \Pi_0\) and \(\pi_1 \in \Pi_1\), is an infinite (a finite) sequence of vertices \(r = v^0v^1v^2 \ldots (r = v^0 \ldots v^n\) for some \(n \in \mathbb{N}\)) such that for every \(k \in \mathbb{N}\), (a) \(v^k \in V_0\) implies \(v^{k+1} \in \text{supp } \pi_0(v^0 \ldots v^k)\), (b) \(v^k \in V_1\) implies \(v^{k+1} \in \text{supp } \pi_1(v^0 \ldots v^k)\), and (c) \(v^k \in V_r\) implies \(v^{k+1} \in E(v^k)\). Given an initial vertex \(v^0\) and a fixed pair of strategies \(\pi_0 \in \Pi_0\) and \(\pi_1 \in \Pi_1\), we obtain a probability distribution over the set of infinite runs of the system. For a measurable set of runs \(R \subseteq V^\omega\), we use the notation \(P_{v^0}^{\pi_0,\pi_1}(R)\) to denote the probability of obtaining the set of runs \(R\) when the initial vertex is \(v^0\) and the strategies of Player 0 and Player 1 are fixed to respectively \(\pi_0\) and \(\pi_1\). For an \(\omega\)-regular specification \(\varphi\), defined using a predicate over the set of vertices of \(G\), we write \((G \models \varphi)\) to denote the set of all infinite runs for all possible strategies of Player 0 and Player 1 which satisfy \(\varphi\). For example, \((G \models \text{Parity}(\mathcal{P}))\) denotes the set of all infinite runs for all possible strategies of Player 0 and Player 1 which satisfy the parity condition \(\text{Parity}(\mathcal{P})\). We say that Player 0 wins \(\text{Parity}(\mathcal{P})\) almost surely from a vertex \(v \in V\) (or \(v\) is almost sure winning for Player 0) if Player 0 has a strategy \(\pi_0 \in \Pi_0\) such that for all \(\pi_1 \in \Pi_1\) we have \(P_{v}^{\pi_0,\pi_1}(G \models \text{Parity}(\mathcal{P})) = 1\). We collect all vertices for which this is true in the almost sure winning region \(W(G \models \text{Parity}(\mathcal{P}))\).

### 4.2. Abstraction: CMPs to 21/2-Player Games

Given a CMP \(\mathcal{G} = (S, \mathcal{U}, T_s)\) and a parity specification \(\text{Parity}(\mathcal{P})\) for a partition \(\mathcal{P}\) of the state space \(S\) we construct an abstract 21/2-player game.

**State-space abstraction.** We introduce a finite partition \(\tilde{S} := \{\tilde{s}_i\}_{i \in I}\) such that \(\cup_{i \in I} \tilde{s}_i = S\), \(\tilde{s}_i \neq \emptyset\) and \(\tilde{s}_i \cap \tilde{s}_j = \emptyset\) for every \(\tilde{s}_i, \tilde{s}_j \in \tilde{S}\) with \(i \neq j\). Further-
more, we assume that the partition \( \hat{S} \) is consistent with the given priorities \( P \), i.e., for every partition element \( \hat{s} \in \hat{S} \), and for every \( x, y \in \hat{s} \), \( x \) and \( y \) belong to the same partition element in \( P \) (i.e., \( x \) and \( y \) are assigned the same priority). We call the set \( \hat{S} \) the abstract state space and each element \( \hat{s} \in \hat{S} \) an abstract state.

We introduce the abstraction function \( Q: S \to \hat{S} \) as a mapping from the continuous to the abstract states: For every \( s \in S \), \( Q: s \mapsto \hat{s} \) such that \( s \in \hat{s} \). We define the concretization function as the inverse of the abstraction function: \( Q^{-1}: \hat{S} \to 2^S \), \( Q^{-1}(\hat{s}) = \{ s \in S \mid s \in \hat{s} \} \). We generalize the use of \( Q \) and \( Q^{-1} \) to sets of states: For every \( U \subseteq S \), \( Q(U) = \bigcup_{s \in U} Q(s) \), and for every \( \hat{U} \subseteq \hat{S} \), \( Q^{-1}(\hat{U}) = \bigcup_{\hat{s} \in \hat{U}} Q^{-1}(\hat{s}) \).

**Transition abstraction.** We also introduce an over- and an under-approximation of the probabilistic transitions of the CMP \( \mathcal{S} \) using the non-deterministic abstract transition functions \( \hat{F}: \hat{S} \times U \to 2^{\hat{S}} \) and \( \hat{F}^{-1}: \hat{S} \times U \to 2^{\hat{S}} \) with the following properties:

\[
\begin{align*}
\hat{F}(\hat{s}, u) &\supseteq \{ \hat{s}' \in \hat{S} \mid \exists s \in \hat{s} . T_a(\hat{s}' \mid s, u) > 0 \}, \quad (8a) \\
\hat{F}^{-1}(\hat{s}, u) &\subseteq \{ \hat{s}' \in \hat{S} \mid \exists \varepsilon > 0 . \forall s \in \hat{s} . T_a(\hat{s}' \mid s, u) \geq \varepsilon \}. \quad (8b)
\end{align*}
\]

To understand the need for both \( \hat{F} \) and \( \hat{F}^{-1} \) and the way they are constructed, consider the following example. Intuitively, given an abstract state \( \hat{s} \) and an input \( u \), the set \( \hat{F} \) over-approximates the set of abstract states reachable by probabilistic transitions from \( \hat{s} \) on input \( u \). On the other hand, \( \hat{F}^{-1} \) under-approximates those abstract states which can be reached by every state in \( \hat{s} \) with probability bounded away from zero.

**Example 4.1.** Consider the two CMPs, \( \mathcal{S}_A \) and \( \mathcal{S}_B \):

\[
\mathcal{S}_A : \begin{array}{c}
\hat{s}_1 \quad \hat{s}_2 \quad \hat{s}_3 \\
\hat{s}_1 \quad \hat{s}_2 \quad \hat{s}_3
\end{array} \quad \mathcal{S}_B : \begin{array}{c}
\hat{s}_1 \quad \hat{s}_2 \quad \hat{s}_3 \\
\hat{s}_1 \quad \hat{s}_2 \quad \hat{s}_3
\end{array}
\]

The circles are concrete states \( s_i \), the dashed boxes denote abstract states \( \hat{s}_i \), and the edges denote transitions with positive probability between concrete
states $s_i$. Consider the left abstract state $\hat{s}_1$. Here, the adversary decides which concrete state (i.e., $s_1$ or $s_2$) the game is in. In both $\mathcal{S}_A$ and $\mathcal{S}_B$, $\mathcal{F}$ says that both $\hat{s}_1$ and $\hat{s}_2$ are reachable from $\hat{s}_1$. In $\mathcal{S}_A$, $\mathcal{F}$ contains both $\hat{s}_1$ and $\hat{s}_2$, in $\mathcal{S}_B$, $\mathcal{F}$ is empty. An adversary that plays according to $\mathcal{F}$ is too strong: it can keep playing the self loop in $s_2$, while the stochastic nature of the CMP ensures that eventually $s_2$ will transition to $s_3$. In order to follow the probabilistic semantics, we must ensure the adversary picks a distribution whose support contains both abstract states.

In $\mathcal{S}_B$, the probabilistic behavior of the two concrete states $s_1$ and $s_2$ are very different: $s_1$ stays in $\hat{s}_1$ with probability one and $s_2$ stays in $\hat{s}_1$ or moves probabilistically to $\hat{s}_2$. To ensure correct behavior, we look at possible supports of distributions induced by the dynamics: these are the possible subsets of abstract states between $\mathcal{F}$ and $\overline{\mathcal{F}}$. Here, the game either stays in $\hat{s}_1$ or (eventually) moves to $\hat{s}_2$ and, in our reduction, we force the adversary to commit to one of the two options.

The parameter $\varepsilon$ states that there is a uniform lower bound on transition probabilities for all states in an abstract state. This ensures that, provided $\hat{s}$ is visited infinitely often and $u$ is applied infinitely often from $\hat{s}$, then $\hat{s}'$ will be reached almost surely from $\hat{s}$. In the absence of a uniform lower bound, this property need not hold for infinite state systems; for example, if the probability converges to zero, the probability of escaping $\hat{s}$ can be strictly less than one.

**Algorithmic computation of $\overline{\mathcal{F}}$ and $\mathcal{F}$.** While it is difficult to compute $\overline{\mathcal{F}}$ and $\mathcal{F}$ in general, they can be approximated for the important subclass of stochastic nonlinear systems with affine disturbances

$$s^{k+1} = f(s^k, u^k) + \varsigma^k, \quad k \in \mathbb{N},$$

where $\varsigma^0, \varsigma^1, \ldots$ are independent and identically distributed random variables from a distribution with a bounded support $D$, and we assume we are only interested in a compact region $\hat{S}$ of the state space. In this case, for any abstract state $\hat{s}$ and any $u \in U$, one can compute an approximation $\text{ReachSet}(\hat{s}, u)$ with
$\text{ReachSet}(\hat{s}, u) \supseteq \{ s' \in S \mid \exists s \in \hat{s} : f(s, u) = s' \}$ using standard techniques (Reissig et al. 2017; Coogan & Arcak 2015; Althoff & Krogh 2013). Define $S_1, S_2 : 2^{\hat{S}} \times U \rightarrow 2^S$ such that

$$
S_1 : (\hat{s}, u) \mapsto D \oplus \text{ReachSet}(\hat{s}, u),
$$

$$
S_2 : (\hat{s}, u) \mapsto D \ominus (\hat{s}, u).
$$

where the minus sign ($-\text{ReachSet}(\hat{s}, u)$) is applied to each individual element of $\text{ReachSet}(\hat{s}, u)$ and $\oplus$ and $\ominus$ are Minkowski sum and difference, respectively.

Using $S_1$ and $S_2$, the functions $F(\cdot, \cdot)$ and $\overline{F}(\cdot, \cdot)$ can be computed as (Majumdar et al. 2020, Thm. 6.1): (1) $\hat{s}' \in \overline{F}(\hat{s}, u)$ iff either $S_1(\hat{s}, u) \subseteq S'$ and $\hat{s}' \cap S_1(\hat{s}, u) \neq \emptyset$ or $S_1(\hat{s}, u) \not\subseteq S'$ and $\hat{s}'$ is a special sink state; and (2) $\hat{s}' \in F(\hat{s}, u)$ iff either $\lambda(\hat{s}' \cap S_2(\hat{s}, u)) > 0$ or $\lambda(S_2(\hat{s}, u) \setminus S') > 0$ and $\hat{s}'$ is a special sink state, and where $\lambda(\cdot)$ denotes the Lebesgue measure (generalized volume) of a set.

**Remark 1.** We remark that standard algorithms (Reissig et al. 2017; Coogan & Arcak 2015) for computing $\text{ReachSet}(\cdot, \cdot)$, including the one that is used in our implementation, have the following monotonicity property: For every $\hat{s}, \hat{s}' \in \hat{S}$ and for every $u \in U$, if $\hat{s} \subseteq \hat{s}'$ then

$$
\text{ReachSet}(\hat{s}, u) \subseteq \text{ReachSet}(\hat{s}', u).
$$

Monotonicity of $\text{ReachSet}(\cdot, \cdot)$ implies $\overline{F}(\cdot, \cdot)$ and $F(\cdot, \cdot)$ are monotone: For every $\hat{s}, \hat{s}' \in \hat{S}$ and for every $u \in U$, if $\hat{s} \subseteq \hat{s}'$ then

$$
\overline{F}(\hat{s}, u) \subseteq \overline{F}(\hat{s}', u) \quad \text{and} \quad F(\hat{s}, u) \supseteq F(\hat{s}', u). \quad (9)
$$

This property ensures that we get a better (i.e., larger) approximation of the optimal almost sure winning domain $\text{WinDom}^*(S)$ in problem 3 if we use a finer discretization of the state space during the abstraction step. This observation can be empirically confirmed from the experiments in Sec. 6.

**Abstract 21/2-player game graph.** Given the abstract state space $\hat{S}$ and the over and under-approximations of the transition functions $\overline{F}$ and $F$, we are ready to construct the abstract 21/2-player game graph induced by a CMP.
Definition 4.2. Let $\mathcal{G}$ be a given CMP. Then its induced abstract 2^{1/2}-player game graph is given by $G = \langle V, E, \langle V_0, V_1, V_r \rangle \rangle$ such that

- $V_0 = \hat{S}$ and $V_1 = \hat{S} \times \mathcal{U}$;
- $V_r = \bigcup_{v_1 \in V_1} V_r(v_1)$, where \( V_r(v_1) := \{ v_r \subseteq \hat{S} \mid F(v_1) \subseteq v_r \subseteq \mathcal{T}(v_1), 1 \leq |v_r| \leq |F(v_1)| + 1 \}; \)
- and it holds that
  - for all $v_0 \in V_0$, $E(v_0) = \{(v_0, u) \mid u \in \mathcal{U}\}$
  - for all $v_1 \in V_1$, $E(v_1) = V_r(v_1)$, and
  - for all $v_r \in V_r$, $E(v_r) = \{ v_0 \in V_0 \mid v_0 \in v_r \}$.

Note that $V_r(v_1)$ contains non-empty subsets of $\hat{S}$ that includes all the abstract states in $F(v_1)$ and possibly include only one additional element from $\mathcal{T}(v_1)$. The construction is illustrated in Fig. 1.

In the reduced game, Player 0 models the controller, Player 1 models the effect of discretization of the state space of $\mathcal{G}$, and the random edges from the states in $V_r$ model the stochastic nature of the transitions of $\mathcal{G}$. Intuitively,
the game graph in Def. 4.2 captures the following interplay which is illustrated in Fig. 1. At every time step, the control policy for $\mathcal{S}$ has to choose a control input $u \in U$ based on the current vertex $\hat{s}$ of $\mathcal{G}$. Since the control policy is oblivious to the precise continuous state $s \in \mathcal{S}$ of $\mathcal{S}$, hence $u$ is required to be an optimal choice for every continuous state $s \in \hat{s}$. This requirement is materialized by introducing a fictitious adversary (i.e. Player 1) who, given $\hat{s}$ and $u$, picks a continuous state $s \in \hat{s}$ from which the control input $u$ is to be applied. Now, we know that no matter what continuous $s$ is chosen by Player 1, $T_s(F(\hat{s}, u) \mid s, u) > \varepsilon$ holds for some $\varepsilon > 0$. This explains why every successor of the $(\hat{s}, u) \in V_1$ states contains the set of vertices $F(\hat{s}, u)$. Moreover, depending on which exact $s \in \hat{s}$ Player 1 chooses, with positive probability the system might go to some state in $F(\hat{s}, u) \setminus F(\hat{s}, u)$. This is materialized by adding every state in $F(\hat{s}, u) \setminus F(\hat{s}, u)$ at a time to the successors of the states in $V_1$ (see Def. 4.2). Finally, we assume that the successor from every state in $V_r$ is chosen uniformly at random (indicated by dotted edges in Def. 4.2). Later, it will be evident that the exact probability values are never used for obtaining the almost sure winning region, and so we could have chosen any other probability distribution.

Abstract parity specification. To conclude the abstraction of a given CMP $\mathcal{S}$ and its parity specification $\mathcal{P} = \{B_1, \ldots, B_\ell\}$, we have to formally translate the priority sets $B_i$ defined over subsets of states of the CMP into a partition of the vertices of the abstract 2$^1/2$-player game graph $\mathcal{G}$ induced by $\mathcal{S}$. To this end, recall that we have assumed that the state space abstraction $\hat{\mathcal{S}}$ respects the priority set $\mathcal{P}$.

**Definition 4.3.** Let $\mathcal{S}$ be a CMP with parity specification $\text{Parity}(\mathcal{P})$ and $\mathcal{G}$ the abstract 2$^1/2$-player game graph induced by $\mathcal{S}$. Then the induced abstract parity specification $\hat{\mathcal{P}} = \{\hat{B}_1, \ldots, \hat{B}_\ell\}$ is defined such that $\hat{B}_i = \{v_0 \in V_0 \mid Q^{-1}(v_0) \subseteq B_i\}$ for all $i \in [1; \ell]$. We denote the resulting 2$^1/2$-player parity game by the tuple $\langle \mathcal{G}, \hat{\mathcal{P}} \rangle$.

We note that the choice of the abstract parity set $\hat{\mathcal{P}}$ does not partition
the state space. Indeed, we implicitly assign an “undefined” color “−” to all nodes $V_1 \cup V_r$. Thereby, we only interpret the given parity specification over a projection of a run to its player 0 nodes. Formally, a run $r$ over the abstract game graph $G$ starting from a vertex $s^0 \in V_0$ is of the form:

$$r = s^0,(s^0,u^0),\{s^0,0,\ldots,s^0,i_0\},s^1,(s^1,u^1),\{s^1,0,\ldots,s^1,i_1\},\ldots$$

where $s^k \in \{s^{k,0},\ldots,s^{k,i_k}\}$ for all $k \in \mathbb{N}$. The projection of the run $r$ to the Player 0 states is defined as Proj$_{V_0}(r) = s^0, s^1, \ldots$. Let $\varphi$ be an $\omega$-regular specification defined using a set of predicates over $V_0$. We use the convention that $(G \models \varphi)$ will denote the set of every infinite run $r$ of $G$, for any arbitrary pair of strategies of Player 0 and Player 1, such that Proj$_{V_0}(r)$ satisfies $\varphi$. This convention is well-defined because every infinite run of $G$ will have infinitely many occurrences of vertices from $V_0$ in it: This follows from the strict alternation of the vertices in $V_0$, $V_1$, and $V_r$, as per Def. 4.2.

4.3. Abstract Controller Synthesis

Suppose $\langle G, \hat{P} \rangle$ is the 2$^{1/2}$-player parity game constructed from the CMP $\mathcal{S}$ according to Def. 4.2. Then solving the abstract controller synthesis problem amounts to computing the almost sure winning strategy $\pi_0$ of Player 0 over $\langle G, \hat{P} \rangle$. In this section, we present a symbolic algorithm for this computation on 2$^{1/2}$-player parity games. While the established algorithm (Chatterjee et al., 2003) solves 2$^{1/2}$-player games through a reduction to usually much larger 2-player games, our algorithm is direct and much faster—both in theory and in practice (Banerjee et al., 2021, 2022). Our symbolic algorithm for 2$^{1/2}$-player parity games has already appeared as a special case of fair adversarial games in our (not peer-reviewed) technical report (Banerjee et al., 2021), though the proof of correctness is via a reduction to fair adversarial Rabin games. In this section, as one of our main contributions of this paper, we present a direct and simpler proof for the same algorithm.

Let us review the symbolic algorithm for 2$^{1/2}$-player parity games from Banerjee et al. (2021), before we give our new proof. The main ingredients
of our algorithm are the following set transformers: For any given sets of vertices $S,T \subseteq V$ of $G$, define

$$C_{pre}(S) := \{ v \in V_0 \mid E(v) \cap S \neq \emptyset \} \cup \{ v \in V_1 \cup V_r \mid E(v) \subseteq S \},$$

$$A_{pre}(S,T) := C_{pre}(T) \cup \{ v \in V_r \mid E(v) \subseteq S \land E(v) \cap T \neq \emptyset \}.$$ 

Intuitively, the set $C_{pre}(S)$ is the set of vertices from which Player 0 can force the game to reach $S$ in one step. On the other hand, the set $A_{pre}(S,T)$ is a subset of $C_{pre}(S)$ (i.e. the game remains in $S$ after one step) from which additionally there is a positive probability transition to the set $T$.

Without loss of generality, we assume that the highest priority $\ell$ appearing in the abstract parity specification $\widehat{P} = \{ \widehat{B}_1, \ldots, \widehat{B}_\ell \}$ is even. (When this is not the case by default, we add a dummy set $\widehat{B}_\ell = \emptyset$ with the next higher even priority $\ell$.) With these, the fixpoint algorithm for solving parity games can be expressed using the following $\mu$-calculus expression (cf. (Banerjee et al., 2021, Eq.(21)):

$$\nu Y_\ell, \mu X_{\ell-1} \ldots \mu X_3, \nu Y_2, \mu X_1.
\quad (\widehat{B}_1 \cap A_{pre}(Y_1, X_1)) \cup (\widehat{B}_2 \cap C_{pre}(Y_2)) \cup
\quad (\widehat{B}_3 \cap A_{pre}(Y_3, X_3)) \cup (\widehat{B}_4 \cap C_{pre}(Y_4)) \cup
\quad \ldots
\quad (\widehat{B}_{\ell-1} \cap A_{pre}(Y_{\ell-1}, X_{\ell-1})) \cup (\widehat{B}_{\ell} \cap C_{pre}(Y_{\ell})). \quad (10)$$

In the rest of this section, we make the number of priorities explicit by using the notation $\widehat{P}_\ell$ to denote the set of priorities $\{ \widehat{B}_1, \ldots, \widehat{B}_\ell \}$ for some even $\ell$.

In order to obtain a direct proof of the above parity fixpoint for 2/2-player games in Thm. 4.5 we utilize an argument similar to the one used by Emerson & Jutla (1991) in the correctness proof of their symbolic algorithm for solving 2-player parity games. Like them, we use an induction over the number of priorities $\ell$, and inductively decompose the parity specification $\widehat{P}_\ell$ into two parts $R := \widehat{B}_\ell \cup \widehat{B}_{\ell-1}$ and $S := \cup_{i \in [1:\ell-2]} \widehat{B}_i$. Almost sure satisfaction of $\text{Parity}(\widehat{P}_\ell)$ involves: (I) when in $R$, either visit $\widehat{B}_\ell$ infinitely often or eventually reach the almost sure winning region for $\text{Parity}(\widehat{P}_{\ell-2})$, which is a subset of $S$, and (II) when
in $S$, either eventually reach the almost sure winning region for $Parity(\hat{P}_{\ell-2})$ or eventually reach the vertices in region $R$ from where (I) can be satisfied. Clearly, there is a two-way dependence between (I) and (II), which is handled in the nested fixpoint computation by iteratively solving (I) and (II) one after the other until convergence.

To easily reason about this nested dependence in our inductive proof of correctness, we “flatten out” the nested fixpoint computation by fixing the solution of (I) using a symbolic variable $T$, that itself gets updated subsequently based on the solution of (II). A visualization of this decomposition of the fixpoint for $Parity(\hat{P}_{\ell})$ (as in (10)) into (I) and (II) using the variable $T$ is as follows:

\[
\nu Y_\ell, \mu X_{\ell-1}, \nu Y_{\ell-2}, \mu X_{\ell-3} \cdots \mu X_3, \nu Y_2, \mu X_1.
\]

\[
(\tilde{B}_1 \cap \text{Apre}(Y_2, X_1)) \cup (\tilde{B}_2 \cap \text{Cpre}(Y_2)) \cup
(\tilde{B}_3 \cap \text{Apre}(Y_4, X_3)) \cup (\tilde{B}_4 \cap \text{Cpre}(Y_4)) \cup
\ldots
(\tilde{B}_{\ell-3} \cap \text{Apre}(Y_{\ell-2}, X_{\ell-3})) \cup (\tilde{B}_{\ell-2} \cap \text{Cpre}(Y_{\ell-2})) \cup
(\tilde{B}_{\ell-1} \cap \text{Apre}(Y_\ell, X_{\ell-1})) \cup (\tilde{B}_{\ell} \cap \text{Cpre}(Y_\ell))
\]

For fixed values of $Y_\ell$ and $X_{\ell-1}$, the value of $T$ (i.e., part (I)) is fixed, which is updated subsequently based on the outcome of the fixpoint for $Parity(\hat{P}_{\ell-2})$ (i.e., part (II)). Using induction over $\ell$, we show that the above fixpoint computes exactly the set of vertices from where (I) or (II) can be fulfilled almost surely.

The base case of the inductive proof is stated in the following lemma. Here we consider the case $\ell = 2$ along with the set $T$ that we informally talked about earlier. This case can be seen as a Büchi-or-reachability specification.

**Lemma 4.4.** Let $B, T \subseteq V$ be two sets of vertices of a $2^{1/2}$-player game graph $G$, and let $B^c$ denote the complement of $B$. Then the fixed point of the following $\mu$-calculus expression is Player 0’s almost sure winning region for the winning
condition □♦ B ∨ ◇ T:

\[ ∀ Y. μX. (B^c \cap \text{Apre}(Y, X)) \cup (B \cap \text{Cpre}(Y)) \cup T. \]  \hspace{1cm} (11)

For the proof, we will use some concepts which need to be defined first. Let \( π_0 \in Π_0^{DM} \) and \( π_1 \in Π_1^{DM} \) be a pair of arbitrary deterministic memoryless strategies of the two players. Fixing \( π_0 \) and \( π_1 \) in the game graph \( G \) induces a Markov chain \( M = \langle V', E', \langle \emptyset, \emptyset, V_r' \rangle \rangle \) with \( V' = V'_r = V \) and \( E' \subseteq E \) where every \( v \in V_0 \cup V_1 \) has a single successor in \( E' \) governed by the strategies \( π_0 \) and \( π_1 \). Concretely, for every \( v \in V_0 \), \((v, v') \in E' \) iff \( π_0(v) = v' \) and for every \( v \in V_1 \), \((v, v') \in E' \) iff \( π_1(v) = v' \). We call \( M \) the Markov chain induced on \( G \) by \( π_0 \), \( π_1 \). We will also use the concept of Bottom Strongly Connected Components (BSCC) of a given Markov chain \( M = \langle V, E, \langle \emptyset, \emptyset, V_r \rangle \rangle \): A set of vertices \( C \subseteq V \) of \( M \) is called a BSCC if (1) the subgraph \( \langle C, E \cap (C \times C) \rangle \) is a strongly connected component of the graph \( \langle V, E \rangle \), and (2) no vertex in \( C \) has an outgoing edge to a vertex outside \( C \), i.e. \( E(C) \subseteq C \).

**Proof of Lem. 4.4.** We prove Lem. 4.4 in two parts:

**Soundness:** Suppose \( Y^* \) is the fixed point of the expression (11). Consider the computation of the inner fixpoint over \( X \) when the outer fixpoint has reached the last iteration \( Y^* \). We get a sequence of \( X^i \)-s as in the following:

\[
\begin{align*}
X^0 &= \emptyset \\
X^1 &= (B \cap \text{Cpre}(Y^*)) \cup T \\
X^2 &= (B^c \cap \text{Apre}(Y^*, X^1)) \cup (B \cap \text{Cpre}(Y^*)) \cup T \\
X^3 &= (B^c \cap \text{Apre}(Y^*, X^2)) \cup (B \cap \text{Cpre}(Y^*)) \cup T \\
&\vdots \\
Y^* &= X^p = (B^c \cap \text{Apre}(Y^*, X^p)) \cup (B \cap \text{Cpre}(Y^*)) \cup T, \hspace{1cm} (12)
\end{align*}
\]

where \( X^p \) is the fixed point of the inner fixpoint equation. Observe that \( X^0 \subseteq X^1 \subseteq X^2 \subseteq \ldots \subseteq X^p \). Since we do not care what happens to the
runs after they reach $T$, let us assume that every vertex in $T$ is a sink vertex with a self loop on it; clearly every vertex in $T$ forms a BSCC on its own. Let $v$ be any arbitrary vertex in $Y^*$. From \cite{12} it follows that there exists a strategy $\pi_0$ of Player 0, such that for every strategy $\pi_1$ of Player 1, there is a run of positive probability of length $\leq p$ that reaches either $B$ or $T$.

Moreover, $\pi_0$ ensures that no run from $v$ goes outside $Y^*$ even after reaching $B$, so that with positive probability either $B$ can be reached again or $T$ can be eventually reached. These imply that every path in the Markov chain, induced on $\mathcal{G}$ by $\pi_0$ and an arbitrary Player 1 strategy $\pi_1$, will eventually reach a BSCC that either intersects with $B$ or is in $T$, almost surely. From the ergodicity property of Markov chains (see Kemeny et al. \cite{1976} Prop. 4.27, 4.28) it follows that $P_{\pi_0;\pi_1}(\mathcal{G} \models \square B \lor \square T) = 1.$

Completeness: Let $Y^0 = V \supseteq Y^1 \supseteq Y^2 \supseteq \ldots \supseteq Y^q$ be the sequence of sets of vertices represented by the $Y$ fixpoint variable, where $q$ is the iteration index where the fixpoint converges (i.e. the minimum index $q$ such that $Y^q = Y^{q+1}$). We show, by induction, that for every iteration index $i \in [0; q]$, the almost sure winning region for the parity specification is contained in $Y^i$.

The base case is when $i = 0$, in which case the claim follows trivially by observing that $Y^0 = V$. Suppose the claim holds for iteration $i$, i.e. $Y^i$ contains the almost sure winning region and every vertex outside $Y^i$ is losing with positive probability. When the iterations over $Y$ reaches $i + 1$, for every vertex $v$ in $Y^{i+1} \setminus Y^i$, for every Player 0 strategy there is a Player 1 strategy such that either (a) the plays exit $Y^i$ in one step, surely or with positive probability, or (b) no positive probability path exists from $v$ that reaches $B \cap Cpre(Y^i)$ or $T$. (Both follow from the definition of $Cpre$ and $Apre$ and the fixpoint expression.)

When (a) is true, $v$ is not almost surely winning because every vertex outside $Y^i$ is not almost surely winning by induction hypothesis. When
(b) is true, \( v \) is not almost surely winning because it is no longer possible to reach a vertex in \( B \) from which the play surely does not leave \( Y^i \). Thus \( v \) cannot belong to the almost sure winning region, implying our induction claim.

Now we establish the correctness of our proposed parity fixpoint given in (10).

**Theorem 4.5.** \textit{The fixpoint (10) is a sound and complete algorithm for computing the almost sure winning region for the parity winning condition \( \hat{P} = \{\hat{B}_1, \ldots, \hat{B}_\ell\} \).}

The proof of Thm. 4.5 uses an induction over the number of priorities \( \ell \). The soundness claim uses the same ergodicity property of Markov chains, as in the proof of Lem. 4.4 to establish that either the \( \hat{B}_\ell \) set is visited infinitely often or eventually the almost sure winning region for \( \hat{P}_{\ell-2} \) is reached almost surely. The completeness claim uses an induction over the indices of the outermost fixpoint variable \( Y_\ell \) as in the proof of Lem. 4.4 and shows that every almost sure winning vertex will remain included in every iteration over \( Y_\ell \). The complete proof is included in the appendix.

### 4.4. Controller Refinement

Now consider that the abstract 2\(^{1/2}\)-player parity game \( \langle G, \hat{P} \rangle \) constructed from the CMP \( S \) via Def. 4.2 and Def. 4.3 has been solved as discussed in Sec. 4.3. Hence, we know the almost-sure Player 0 winning region \( W(G | \text{Parity}(\hat{P})) \) and the associated deterministic memoryless almost-sure Player 0 winning policy \( \pi_0 \in \Pi^{DM} \). Then we refine \( \pi_0 \) to a control policy \( \rho \in \Pi \) for the CMP \( S \) under parity condition \( \text{Parity}(\hat{P}) \) as follows.

**Definition 4.6.** Let \( S \) be a CMP with parity specification \( \text{Parity}(P) \) and \( \langle G, \hat{P} \rangle \) its induced finite 2\(^{1/2}\)-player parity game with deterministic memoryless almost sure Player 0 winning strategy \( \pi_0 \in \Pi^{DM} \). Then the control policy \( \rho \in \Pi \) is
called the refinement of $\pi_0$ if and only if for every $s \in S$, if $s \in \hat{s}$ for some $\hat{s} \in \hat{S}$, and if $\pi_0(\hat{s}) = (\hat{s}, u) \in V_1$ for some $u \in U$, then $\rho(s) := u$.

With the completion of this last step of our ABCD method for stochastic nonlinear systems we can finally state our main theorem providing a solution to Problem 2, which we prove in Sec. 5.

**Theorem 4.7 (Solution of Problem 2).** Let $\mathcal{S}$ be a CMP and $\text{Parity}(P)$ be a given parity specification. Let $\langle G, \hat{P} \rangle$ be the abstract 2-player game defined in Def. 4.2. Suppose, a vertex $\hat{s} \in V_0$ is almost sure winning for Player 0 in the game $\langle G, \hat{P} \rangle$, and $\pi_0 \in \Pi^{\text{DM}}$ is the corresponding Player 0 winning strategy. Then the refinement $\rho$ of $\pi_0$ ensures that $\hat{s} \subseteq \text{WinDom}(\mathcal{S}, \rho)$.

**Remark 2.** An over-approximation of the optimal almost sure winning domain $\text{WinDom}^*(\mathcal{S})$ of $\mathcal{S}$ w.r.t. $\text{Parity}(P)$ can be computed via $\langle G, \hat{P} \rangle$ as well. To obtain an over-approximation, we solve this abstract game cooperatively. That is, we let player Player 0 choose both its own moves and the moves of player $p_1$ to win almost surely with respect to $\text{Parity}(\hat{P})$. Then the approximation error of Problem 2 can be upper-bounded as the Lebesgue measure of the set difference of the over- and the under-approximation, as will be done for the experiments in Sec. 6.

5. **Proof of Theorem 4.7**

**Proof outline.** To prove Thm. 4.7, we first decompose both the original and the abstract parity specifications $\text{Parity}(P)$ and $\text{Parity}(\hat{P})$ into a combination of more manageable safety and reachability sub-parts. That is, for every state reachable by a finite run in $\mathcal{S}$, we consider a local safety specification $\psi_S$ and a local reachability specification $\psi_R$ defined by

$$\psi_S := \Box \neg B_i \quad \text{and} \quad \psi_R := \Diamond \left( \psi_S \lor \bigvee_{j \in \text{odd}[i+1:k]} B_j \right).$$

Intuitively, $\psi_R$ requires that every time an odd priority—say $B_i$—is visited in $\mathcal{S}$, eventually either $B_i$ should never occur or an even priority $B_j$ with $j > i$ should
occur, almost surely. Similarly, for the abstract 2\textsuperscript{1/2}-player parity game \langle G, \hat{\mathcal{P}} \rangle we consider the local safety specification \hat{\psi}_S and a local reachability specification \hat{\psi}_R defined by

\[
\hat{\psi}_S := \Box \neg \hat{B}_i \quad \text{and} \quad \hat{\psi}_R := \Diamond \left( \hat{\psi}_S \lor \bigvee_{j \in \text{even}[i+1;k]} \hat{B}_j \right).
\]  

While the above decomposition needs to be established both for \(G\) and for \(S\), the directions of the respective proof differ. For \(S\) we show that if \(\psi_R\) holds for a state reachable by a finite run over \(S\), then the original specification \textit{Parity}(\(P\)) is satisfied by a continuation of the run using the refined policy \(\rho\) (\textbf{Step 1}). For \(G\) we show that if \textit{Parity}(\(\hat{P}\)) is satisfied, then \(\hat{\psi}_R\) holds for every state visited by a run compatible with the almost-sure winning strategy \(\pi_0\) in \langle G, \hat{\mathcal{P}} \rangle (\textbf{Step 2}). Further, we show that satisfaction of \(\hat{\psi}_S\) (resp. \(\hat{\psi}_R\)) in \(G\) implies satisfaction of \(\psi_S\) (resp. \(\psi_R\)) in \(S\) (\textbf{Step 3-5}). With this, we have all ingredients to prove Thm. 4.7 (\textbf{Step 6}).

\textbf{Step 1: Decomposition of Parity(\(P\)).} We prove a \textbf{sufficient} condition for satisfaction of \textit{Parity}(\(P\)) in \(S\) if \(\psi_R\) holds.

\textbf{Lemma 5.1.} Let \(S\) be a CMP, \textit{Parity}(\(P\)) be a parity specification, \(s^0 \in S\) be a given initial state, and \(\rho\) be a control policy. Suppose the following holds for every finite path \((s^0, \ldots, s^n) \in S^{n+1}\) of \(S\) and every \(i \in \text{odd}[1;k]\):

\[
s^n \in B_i \implies P_{s^0}^\rho (S \models \psi_R) = 1.
\]  

Then \(P_{s^0}^\rho (\emptyset \models \textit{Parity}(P)) = 1\).

\textbf{Proof of Lem. 5.1.} Define for any arbitrary \(i \in \text{odd}[1;k]\) the event \(E_i := (S \models \psi_i)\) with the specification \(\psi_i := \left( \Box \Diamond B_i \implies \bigvee_{j \in \text{even}[i+1;k]} \Diamond \Box B_j \right)\). We want to show that \(P_{s^0}^\rho (S \models \textit{Parity}(P)) = P_{s^0}^\rho (\bigcap E_i) = 1\) where \(i \in \text{odd}[1;k]\). We prove this by showing \(P_{s^0}^\rho (\overline{E_i}) = 0\) for every \(i \in \text{odd}[1;k]\). Once we show this, the result follows according to the standard inequalities:

\[
P_{s^0}^\rho (\bigcap E_i) = 1 - P_{s^0}^\rho (\bigcup \overline{E_i}) \geq 1 - \sum_{i} P_{s^0}^\rho (\overline{E_i}) = 1
\]

where \(P_{s^0}^\rho (\overline{E_i}) = P_{s^0}^\rho ((S \models \Box \Diamond B_i) \cap (S \models \Box \Diamond \neg \hat{B}_j))\),
with $i \in_{\text{odd}} [1; k]$ and $j \in_{\text{even}} [i + 1; k]$. Define the random variable $\tau$ to be the largest time instance when the trajectory visits one of the sets $B_j$. Also define $\tau' > \tau$ to be the first time instance after $\tau$ when the trajectory visits $B_i$ again. Note that for any trajectory satisfying $\square \Diamond B_i$ and $\wedge_j \Diamond \Box \neg B_j$, both $\tau$ and $\tau'$ are well-defined and bounded. According to the assumption (15), we have

$$P_{v^0}^0(\overline{E}_i \mid \tau' = n, s^0, s^1, \ldots, s^n) = P_{v^0}^0(([\mathcal{G} \models \square \Diamond B_i) \cap ([\mathcal{G} \models \wedge_j \Diamond \Box \neg B_j])) = 0.$$  

By taking the expectation with respect to the condition $(\tau', s^0, \ldots, s^n)$, we conclude that

$$P_{v^0}^0(\mathbb{E}_i) = \mathbb{E}_{v^0}^0[ P_{v^0}^0(\overline{E}_i \mid \tau' = n, s^0, s^1, \ldots, s^n)] = \mathbb{E}_{v^0}^0[0] = 0,$$

i.e., $\overline{E}_i$ has a zero probability. \hfill \Box

**Step 2: Decomposition of Parity($\widehat{P}$).** We present a necessary condition for satisfaction of Parity($\widehat{P}$) in $\mathcal{G}$ if $\widehat{\psi}_R$ holds.

**Lemma 5.2.** Let $\langle \mathcal{G}, \widehat{P} \rangle$ be a $2^1/2$-player parity game, and $v^0$ be a given vertex of $\mathcal{G}$. Suppose $\pi_0^* \in \Pi_0^{\mathcal{G}}$ is a Player 0 strategy such that $\inf_{\pi_1 \in \Pi_1} P_{v^0}^{\pi_0^*, \pi_1}(\mathcal{G} \models \text{Parity}(\widehat{P})) = 1$. Then given every finite path $v^0 \ldots v^n \in V^*$ such that there exists a Player 1 strategy $\pi_1 \in \Pi_1$ with $P_{v^0}^{\pi_0^*, \pi_1}(\mathcal{G} \models v^0 \ldots v^n) > 0$, the following holds for every $i \in_{\text{odd}} [1; k]$:

$$v^n \in \widehat{B}_i \implies \inf_{\pi_1 \in \Pi_1} P_{v^n}^{\pi_0^*, \pi_1}(\mathcal{G} \models \Diamond \widehat{\psi}_R) = 1. \quad (16)$$

The only new factor in Eq. (16) is the presence of the adversarial effect of the Player 1 strategies.

**Proof.** It follows from the definition of the parity specification in (2) that a vertex $v^0 \in V$ is almost sure winning using the strategy $\pi_0^*$ if the following condition is fulfilled:

$$\inf_{\pi_1 \in \Pi_1} P_{v^0}^{\pi_0^*, \pi_1}(\mathcal{G} \models \wedge_{i \in_{\text{odd}} [1; k]} \square \left( \overline{B}_i \implies \Diamond \psi_R \right)) = 1. \quad (17)$$
From the semantics of LTL, (17) implies:
\[
\inf_{\pi_1 \in \Pi_1} P^{\pi_0, \pi_1}_{v_0} (G|\models \bigwedge_i \forall m \in \mathbb{N}. \ (v^m \in \widehat{B}_i) \implies \Diamond \widehat{\psi}_R) = 1, \tag{18}
\]
with \(i \in \text{odd} \ [1; k]\). We show that (18) implies for every finite run \(v^0 \ldots v^n \in V^*\), occurring with a positive probability \(p_1 > 0\) for some strategy of Player 1, (16) holds. Suppose, for contradiction’s sake, there exists some \(i \in \text{odd} \ [1; k]\) such that \(v^n \in \widehat{B}_i\) and (16) does not hold, i.e., \(\inf_{\pi_1 \in \Pi_1} P^{\pi_0, \pi_1}_{v_0} (G|\models \Diamond \widehat{\psi}_R) < 1\), implying existence of some \(0 < p_2 \leq 1\) with \(\sup_{\pi_1 \in \Pi_1} P^{\pi_0, \pi_1}_{v_0} (G \not\models \Diamond \widehat{\psi}_R) = p_2\). This results in satisfaction of the parity specification with a probability of at most \((1 - p_1 \cdot p_2) < 1\), contradicting (18).

\[\square\]

Step 3: Refinement of \(\widehat{\psi}_S\) to \(\psi_S\). We show that almost sure safety with respect to a given set \(U\) in \(G\) implies the same with respect to the set \(Q^{-1}(U)\) in \(S\); this will later be used to infer \(\widehat{\psi}_S \implies \psi_S\).

Proposition 5.3. Let \(S\) be a CMP and \(G\) be a finite 2/2-player game graph as defined in Def. 4.2. Suppose \(U \subseteq V_0\) is a given set of vertices of \(G\), and assume that there is a Player 0 vertex \(v \in U\) for which there is a strategy \(\pi_0 \in \Pi_0^{\text{DM}}\) of Player 0 such that \(\inf_{\pi_1 \in \Pi_1} P^{\pi_0, \pi_1}_{v} (G|\models \Box U) = 1\). Then the refinement \(\rho \in \Pi\) of \(\pi_0\) ensures that for every state \(s \in v\), \(P^{\rho}_{s}(S|\models \Box Q^{-1}(U)) = 1\).

Proof. It is known that for safety properties, almost sure satisfaction coincides with sure satisfaction, i.e., \(\inf_{\pi_1 \in \Pi_1} P^{\pi_0, \pi_1}_{v} (G|\models \Box U) = 1\) if and only if for every strategy \(\pi_1 \in \Pi_1\), every infinite run of \(G\) stays inside \(U\) at all time (de Alfaro & Henzinger, 2000). In other words, there must be a controlled invariant set \(W\) inside \(U\) for the strategy \(\pi_0\), and \(v \in W\). This controlled invariant set can be obtained by considering the 2-player game, obtained from \(G\) by removing all the random vertices, and redirecting the outgoing transitions of a given Player 1 vertex \(v' \in V_1\) to the Player 0 vertices within the set \(\mathcal{F}(v', \pi_0(v')) \subseteq V_0\). Since \(\mathcal{F}(v', \pi_0(v'))\) overapproximates the set of all the continuous states reachable from \(v'\) using the input \(\pi_0(v')\), hence if Player 0 can fulfill \(\Box U\) using the strategy \(\pi_0(v')\), then \(\rho\) can fulfill \(\Box_Q^{-1}(U)\) from every state \(s \in v'\) in \(S\).

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(This follows from the standard arguments in abstraction-based control using over-approximation based abstractions [Reissig et al., 2017].)

**Step 4: Refinement of \( \hat{\psi_R} \) to \( \psi_R \).** We show that almost sure reachability with respect to a given set \( U \) in \( \mathcal{G} \) implies the same with respect to the set \( Q^{-1}(U) \) in \( \mathcal{G} \); this will be used to infer \( \hat{\psi_R} \implies \psi_R \).

Let \( \mathcal{G} \) be a CMP. Let us consider a reachability specification \( \lozenge U \), for a set \( U \subseteq V_0 \), in the game \( \mathcal{G} \) defined in Def. 4.2. Suppose \( \pi_0 \in \Pi_0^{DM} \) is some strategy of Player 0.

We introduce a ranking function \( r : V_0 \to \mathbb{N} \cup \{\infty\} \) as a certificate for almost sure satisfaction of the specification \( \lozenge U \). The ranking function \( r \) is defined inductively as follows:

\[
\begin{align*}
    r(v) = \begin{cases} 
        0 & v \in U, \\
        \infty & \inf_{\pi_1 \in \Pi_1} P_{v,\pi_0}^{\pi_0,\pi_1}(G \models \lozenge U) < 1, \\
        i + 1 & \min \{n \in \mathbb{N} \mid \inf_{\pi_1 \in \Pi_1} P_{v,\pi_0}^{\pi_0,\pi_1}(G \models \lozenge r^{-1}(n)) > 0 \} \\
        & \quad = i \land \inf_{\pi_1 \in \Pi_1} P_{v,\pi_0}^{\pi_0,\pi_1}(G \not\models \lozenge r^{-1}(\infty)) = 1.
    \end{cases}
\end{align*}
\]

(19)

Note that every vertex \( v \in V \) gets a rank: If \( r(v) \neq \infty \), then \( \inf_{\pi_1 \in \Pi_1} P_{v,\pi_0}^{\pi_0,\pi_1}(G \models \lozenge U) = 1 \) by definition of \( r \). In this case, there must exist some path to \( U \), i.e., \( \inf_{\pi_1 \in \Pi_1} P_{v,\pi_0}^{\pi_0,\pi_1}(G \not\models \lozenge r^{-1}(\infty)) = 1 \) must be true, and moreover \( \inf_{\pi_1 \in \Pi_1} P_{v,\pi_0}^{\pi_0,\pi_1}(G \models \lozenge r^{-1}(n)) > 0 \) will be true for some \( n \). Thus, \( r(v) = n + 1 \).

From the ranking function \( r(\cdot) \) defined in (19), it is clear that \( \inf_{\pi_1 \in \Pi_1} P_{v,\pi_0}^{\pi_0,\pi_1}(G \models \lozenge U) = 1 \) implies \( r(v) \neq \infty \). We first identify some local structural properties of the abstract transition functions \( \overline{F} \) and \( \overline{F} \) evaluated on some abstract states with finite ranking.

**Lemma 5.4.** Suppose \( \pi_0 \in \Pi_0^{DM} \) is some strategy of Player 0. For every \( v \in V_0 \) with \( r(v) = i \neq \infty, i > 0 \), both \( \overline{F}(v, \pi_0(v)) \cap r^{-1}(\infty) = \emptyset \) and either of the following holds:

1. \( \overline{F}(v, \pi_0(v)) \cap r^{-1}(i - 1) \neq \emptyset \), or
2. \( \overline{F}(v, \pi_0(v)) = \emptyset \) and \( \overline{F}(v, \pi_0(v)) \subseteq r^{-1}(i - 1) \).
**Proof.** Firstly, $F(v, \pi_0(v)) \cap r^{-1}(\infty) = \emptyset$ should always hold as otherwise Player 1 would have a strategy to reach a state in $r^{-1}(\infty)$ with nonzero probability in the next step.

Suppose (2) does not hold, implying either (a) $F(v, \pi_0(v)) \neq \emptyset$, or (b) the existence of a vertex $v' \in F(v, \pi_0(v))$ with $r(v') \neq i - 1$. Then $F(v, \pi_0(v)) \cap r^{-1}(i - 1) \neq \emptyset$ must hold, as otherwise, for case (a) and (b) Player 1 would have strategies $\pi_1$ with $\pi_1(v, \pi_0(v)) = (F(v, \pi_0(v)))$ and $\pi_1(v, \pi_0(v)) = (F(v, \pi_0(v)) \cup \{v'\})$ respectively, such that $P^v_{v_0, \pi_1}(G \models \bigcirc r^{-1}(i - 1)) = 0$.

On the other hand, suppose (1) does not hold. Then $F(v, \pi_0(v)) = \emptyset$ must be true, as otherwise Player 1 would have a strategy $\pi_1$ with $\pi_1(v, \pi_0(v)) = (F(v, \pi_0(v)))$ such that $P^v_{v_0, \pi_1}(G \models \bigcirc r^{-1}(i - 1)) = 0$. Moreover, $F(v, \pi_0(v)) \subseteq r^{-1}(i - 1)$ must also be true, as otherwise there would exist a vertex $v' \in F(v, \pi_0(v))$ with $r(v') \neq i - 1$, and Player 1 would have a strategy $\pi_1$ with $\pi_1(v, \pi_0(v)) = (F(v, \pi_0(v)) \cup \{v'\}) = (\{v'\})$ such that $P^v_{v_0, \pi_1}(G \models \bigcirc r^{-1}(i - 1)) = 0$. 

The following lemma establishes soundness of the reduction with respect to reachability specifications.

**Proposition 5.5.** Let $\mathcal{G}$ be a CMP and $\mathcal{G}$ be a finite 21/2-player game graph as defined in Def. 5.4. Suppose there is a Player 0 vertex $v \in V_0$ in $\mathcal{G}$ and a set of vertices $U \subseteq V_0$ for which there is a strategy $\pi_0 \in \Pi_0^{DM}$ of Player 0 such that $\inf_{\pi_1 \in \Pi_1} P^v_{v_0, \pi_1}(G \models \bigcirc U) = 1$. Then the refinement $\rho \in \Pi$ of $\pi_0$ ensures that for every state $s \in v$, $P^v_s(\mathcal{G} \models \bigcirc Q^{-1}(U)) = 1$.

**Proof.** It follows from the definition of the ranking function in (19) that the set of almost sure winning vertices for the specification $\bigcirc U$ is given by all the vertices with finite rank. We show that for every vertex $v$ with a finite rank, the refinement $\rho \in \Pi$ of $\pi_0$ ensures that from every state $s \in v$, $P^v_s(\mathcal{G} \models \bigcirc Q^{-1}(U)) = 1$.

First, trajectories starting from any state $s \in v$ with $r(v) \neq \infty$ never go to the region $Q^{-1}(r^{-1}(\infty))$. This follows from the identity $F(v, \pi_0(v)) \cap r^{-1}(\infty) = \emptyset$ in Lem. 5.4 and because $F(v, \pi_0(v))$ is an overapproximation of the one step
reachable set from the states within vertex \( v \). Hence, every infinite trajectory of \( \mathcal{G} \) starting from \( s \) will visit the states in \( S \setminus Q^{-1}(r^{-1}(\infty)) \) infinitely often.

The rest of the proof shows that if a trajectory visits the states \( S \setminus Q^{-1}(r^{-1}(\infty)) \cup r^{-1}(0) \) infinitely often, then the trajectory will almost surely satisfy \( Q^{-1}(r^{-1}(0)) = Q^{-1}(U) \). The is by induction over the largest rank assigned by \( r \). For the base case, let the largest rank assigned by \( r \) be 2. We show that every state \( s \in S \) starting from inside a vertex \( v \) with \( r(v) = 1 \) or \( r(v) = 2 \) will almost surely reach \( Q^{-1}(U) \), i.e., \( P_s^p(\mathcal{G} \models □ Q^{-1}(r^{-1}(1) \cup r^{-1}(2))) = 0 \). Note that the events \( \{ Q^{-1}(r^{-1}(2)) \} \) and \( \{ □ Q^{-1}(r^{-1}(1)) \} \) form a partition of the event of \( \{ Q^{-1}(r^{-1}(1) \cup r^{-1}(2)) \} \). Therefore,

\[
P_s^p(\mathcal{G} \models □ Q^{-1}(r^{-1}(1) \cup r^{-1}(2))) = P_s^p(\mathcal{G} \models □ Q^{-1}(r^{-1}(1) \cup r^{-1}(2)) \land □ Q^{-1}(r^{-1}(2))) + P_s^p(\mathcal{G} \models □ Q^{-1}(r^{-1}(1) \cup r^{-1}(2)) \land □ Q^{-1}(r^{-1}(1))).
\]

The first term is upper bounded by \( P_s^p(\mathcal{G} \models □ Q^{-1}(r^{-1}(2))) \) which is zero, because \( P_s^p(\mathcal{G} \models □ Q^{-1}(r^{-1}(2))) = \prod_{n=1}^{\infty} (1 - \varepsilon)^n = 0 \). The second term is also zero because the event requires the number of transitions from \( Q^{-1}(r^{-1}(1)) \) to be infinite. To see this, let \( i_n = (i_{n0}, i_{n1}, \ldots, i_{nn}) \) be the first \( (n+1) \) time instances that a trajectory visits \( Q^{-1}(r^{-1}(1)) \). Then,

\[
P_s^p(\mathcal{G} \models □ Q^{-1}(r^{-1}(1) \cup r^{-1}(2)) \land □ Q^{-1}(r^{-1}(1))) = \sum_{i_n} P_s^p(\mathcal{G} \models □ Q^{-1}(r^{-1}(1) \cup r^{-1}(2)) \land □ Q^{-1}(r^{-1}(1)) | i_n) P_s^p(i_n) \leq \sum_{i_n} (1 - \varepsilon)^n P_s^p(i_n) = (1 - \varepsilon)^n.
\]

The last inequality is due to either Cond. (1) or Cond. (2) of Lem. 5.4 applied to the vertices in \( v \in r^{-1}(1) \). Note that this inequality holds for any \( n \). By taking the limit when \( n \) goes to infinity, we have that this second term is also zero. Hence the base case is established.

For the induction hypothesis, assume that the claim holds when the maximum rank assigned by the function \( r \) is \( i \). Then for the induction step, i.e.,
when the maximum rank is $i+1$, we can follow same argument, as we did for the states with rank 2 in the base case, to show that every infinite trajectory inside $S \setminus Q^{-1}(r^{-1}(\infty) \cup r^{-1}(0))$ will never get trapped inside $Q^{-1}(r^{-1}(i + 1))$, which will mean that the trajectory will visit the states in $S \setminus Q^{-1}(r^{-1}(\infty) \cup r^{-1}(0) \cup r^{-1}(i + 1))$ infinitely often. Then it follows from the induction hypothesis that the trajectories will reach $Q^{-1}(U)$ almost surely.

\[\square\]

**Step 5: Refinement of runs.** We show that every finite path in $S$ can be mapped to a positive probability finite run in $G$; this will be used to establish a bridge from the universal quantification over finite paths in $S$ to the universal quantification over finite runs in $G$.

**Lemma 5.6.** Let $S$ be a CMP, $G$ be the abstract game graph as defined in Def. 4.2, $\pi_0 \in \Pi_0^{DM}$ be an arbitrary Player 0 strategy in the game $G$, and $s \in S$ be a state of $S$. Suppose $\rho \in \Pi$ is the refinement of $\pi_0$. Then for every finite trajectory $s^0 \ldots s^n \in S^*$ of $S$ in the support of the distribution $P_\rho s^0$, there exists a Player 1 strategy $\pi_1 \in \Pi_1$ such that $P_{\rho s^0}(G | s^0 \ldots s^n) > 0$, where $s^i = Q(s^i)$ for every $i \in [0; n]$.

**Proof.** The initial state $s^0 \in S^0$, and for every $0 \leq i < n$, from the definition of $T$ it follows that $s^{i+1} \in F(s^i, \pi_0(s^i))$. Thus, from every Player 1 vertex $(s^i, \pi_0(s^i))$, there is a successor vertex in $V_r$ whose successor is $s^{i+1}$. Hence, for every $0 \leq i < n$, there is some move of Player 1 which causes a transition to $s^{i+1}$ with some positive probability $p^i$. Then $P_{\rho s^0}(G | s^0 \ldots s^n) = \prod_{i=0}^{n-1} p^i > 0$. \[\square\]

**Step 6: The final assembly of the proof.** Finally, we finish the proof of Thm. 4.7 by stitching everything together. It is known that memoryless strategies suffice for winning almost surely in $2^1/2$-player parity games [Zielonka 2004]. Let $\pi_0^* \in \Pi_0^{DM}$ be the witness strategy of Player 0 to almost surely win from the vertex $\tilde{s}^*$ in the game $(G, \tilde{P})$, and $\rho^*$ be the refinement of $\pi_0^*$. We claim that $\tilde{s}^* \subseteq \text{WinDom}(G, \rho^*)$.

We will show that for every finite path of $S$ starting within $\tilde{s}^*$ and ending in some odd priority state $B_i$, eventually either $B_i$ will not be visited any more,
or a state of higher even color will be visited. Then the claim will follow from Lem. 5.1. We know from Lem. 5.6 that existence of a finite path \( s^0 \ldots s^n \) of \( \mathcal{G} \) implies existence of an abstract run \( \hat{s}^0 \ldots \hat{s}^n \) such that \( \sup_{\pi_1 \in \Pi_1} P_{\hat{s}^0; \pi_1}^\mathcal{G} (\mathcal{G} \models \hat{s}^0 \ldots \hat{s}^n) > 0 \). Moreover, due to the priority preserving partitions we have \( \hat{s}^n \in \hat{B}_i \), where \( i \) is odd. Since \( \pi_0^* \) is an almost sure winning strategy, hence, by using Lem. 5.2 we know that the following holds:

\[
\inf_{\pi_1 \in \Pi_1} P_{\hat{s}^0; \pi_1}^\mathcal{G} (\mathcal{G} \models \Diamond \neg \hat{B}_i \lor \bigcup_{j \in \text{even}[i+1; k]} \hat{B}_j) = 1. \tag{20}
\]

From Prop. 5.3 we know that the set of abstract states from which the specification \( \Box \neg \hat{B}_i \) is satisfied (almost) surely using the strategy \( \pi_0^* \) are also the set of continuous states from which the specification \( \Box \neg B_i \) is satisfied almost surely using the controller \( \rho^* \). Together with Prop. 5.5 and Lem. 5.1 we can infer Thm. 4.7 from (20).

6. Numerical Example

We consider the controller synthesis problems for a two-dimensional stochastic bistable switch with a couple of different parity specifications; the examples have been adopted, mutatis mutandis, from the work of Dutreix et al. (2020).

The dynamics of the system is modeled by the following difference equations:

\[
s^{k+1}_1 = s^k_1 + (-a \cdot s^k_1 + s^k_2) \cdot \tau + u^k_1 + \zeta^k_1 \tag{21}
\]

\[
s^{k+1}_2 = s^k_2 + \left( \frac{(s^k_1)^2}{(s^k_2)^2 + 1} - b \cdot s^k_2 \right) \cdot \tau + u^k_2 + \zeta^k_2, \tag{22}
\]

where \( s_1, s_2 \) are the state variables, \( u_1, u_2 \) are the control inputs, \( a, b \) are constant parameters, \( \tau \) is the sampling time, and \( \zeta_1, \zeta_2 \) are stochastic noises. We assume that the domain of the state variables is \([0,0,4.0] \times [0,0,4.0]\), and we saturate the state trajectories at the boundary of the domain. We consider a finite set of values for both of the control inputs: for every \( k \), \( u^k_1, u^k_2 \in \{-0.05, 0.0, 0.05\} \).

The values of the constants are given by: \( a = 1.3, b = 0.25, \) and \( \tau = 0.05 \).

Finally, we assume that the stochastic noise samples \((\zeta^k_1, \zeta^k_2)\) are drawn from
a piecewise continuous density function with the support $D = [-0.4, -0.2] \times [-0.4, -0.2]$.

Let $A, B, C, D$ be sets of states, as shown in Fig. 2a. We are interested in synthesizing the almost sure winning controllers for the above system for the following two LTL specifications:

$$\varphi_1 := \Box((\neg A \land \Diamond A) \rightarrow (\Diamond \Diamond A \land \Diamond \Diamond \Diamond A)),$$
$$\varphi_2 := (\Box \Diamond B \rightarrow \Diamond C) \land (\Diamond D \rightarrow \Box \neg C).$$

The specifications $\varphi_1$ and $\varphi_2$ can be represented using the parity automata shown in Fig. 2b and 2c. Firstly, for each of the two specifications, we compute a product with the system model. Secondly, we apply the algorithm from Sec. 4 on the product system to solve the synthesis problem. Both of these steps have been implemented on the open-source tool Mascot-SDS (Majumdar et al., 2020). By symbolically encoding the abstract 2/1/2-player game using BDD-s, and by using sophisticated acceleration techniques for solving symbolic fixpoint algorithms from the literature (Long et al., 1994; Piterman & Pnueli, 2006), we achieve significant improvement in performance, in comparison with the implementation of the enumerative algorithm of Dutreix et al. (2020) in the tool called StochasticSynthesis.

The tool StochasticSynthesis has a couple of additional features compared to our implementation in Mascot-SDS. Firstly, it performs an adaptive abstraction refinement procedure for achieving better computational efficiency over uniform abstractions. This is an orthogonal optimization tool that is known to be effective in discretization-based approaches for controller synthesis (Soudjani &

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[1] Dutreix et al. (2020) considered the density function to be given by truncated Gaussian distribution with support $D$. In our work, we disregard the shape of the distribution because we restrict the focus to only the qualitative satisfaction of the specification.

[2] Dutreix et al. (2020) modeled $\varphi_1$ and $\varphi_2$ using Rabin automata, and we transformed them into (language-) equivalent parity automata to match our setup.

[3] Repository: https://github.com/gtfactslab/StochasticSynthesis, commit nr.: 888b9dcf67369a732b8c225d790b3343e4442ea5
Abate, 2013; Girard et al., 2016; Nilsson et al., 2017; Hsu et al., 2018), and
we expect that our uniform abstraction-based synthesis procedure will benefit
further from this in the future. Secondly, StochasticSynthesis also addresses the
quantitative aspect of the synthesis problem, which is to maximize the proba-
bility of satisfying the given specification. In all our experiments, we disabled
this second feature of StochasticSynthesis, since the quantitative part is not the
main algorithmic contribution of our paper.

We performed all the experiments on a Macbook Pro (2015) laptop equipped
with a 2.7 GHz Intel Core i5 processor and a 16 GB RAM.

We performed two sets of experiments to compare the performance of Mascot-
SDS and StochasticSynthesis, one with the adaptive refinement feature of Stochas-
ticSynthesis disabled and one with the same enabled. The results have been
summarized in Tab. 1, Fig. 3 and Fig. 4. All the experiments empirically show
that the approximation error reduces with finer discretization, as already men-
tioned in Rem. 2. We highlight the other main findings in the following: (A)
When the abstraction-refinement feature of StochasticSynthesis was disabled,
Mascot-SDS outperformed StochasticSynthesis by a large margin. In fact, for
some levels of discretization, StochasticSynthesis crashed due to memory limita-
tion, whereas Mascot-SDS consumed quite manageable amount of memory and
synthesized controllers within reasonable amount of time. (B) Even when the
abstraction-refinement feature of StochasticSynthesis was enabled, for achieving
the same level of approximation error, Mascot-SDS was significantly faster for
\( \varphi_1 \) and was competitive for \( \varphi_2 \), and consumed much less memory for both \( \varphi_1 \)
and \( \varphi_2 \): For \( \varphi_1 \), at one point Mascot-SDS was more than 150 times faster and
consumed around 150 times lesser memory. These findings demonstrate the su-
perior capabilities of our symbolic solution approach, which can be potentially
further improved by using adaptive refinement techniques.
(a) The state predicates.

(b) Parity automaton for $\varphi_1$; the prioritized partition is $P = \langle \emptyset, \{q_2, q_3, q_4\}, \{q_0, q_1\}\rangle$.

$\neg A$ stands for any element in $\{\emptyset, B, C, D\}$.

(c) Parity automaton for $\varphi_2$; the prioritized partition is $P = \langle \{q_0, q_2, q_4\}, \{q_1, q_3, q_5, q_6\}\rangle$.

Figure 2: The state predicates and the parity specifications.
| Specification | Size of abstract states | Approximation error | Abstraction time | Total synthesis time | Peak memory footprint |
|---------------|-------------------------|---------------------|-----------------|---------------------|-----------------------|
|               |                         | Mascot-SDS          | SS              | Mascot-SDS          | SS                    |
| \( \varphi_1 \) | \( \frac{1}{2} \times \frac{1}{2} \) | 7.0                 | 9.0             | 0.003 s             | 0.4 s                 | 0.02 s               | 8 s                  | 21 MiB               | 125 MiB              |
|               | \( \frac{1}{4} \times \frac{1}{4} \) | 6.6                 | 5.9             | 0.04 s             | 13 s                  | 2 s                  | 18 s                 | 23 MiB               | 1 GiB                |
|               | \( \frac{1}{8} \times \frac{1}{8} \) | 4.0                 | 3.0             | 0.2 s              | 35 min 5 s            | 0.7 s                | 9 min 18 s           | 26 MiB               | 81 GiB               |
|               | \( \frac{1}{16} \times \frac{1}{16} \) | 1.7                 | OoM             | 0.9 s              | OoM                   | 5 s                  | OoM                  | 50 MiB               | 127 GiB              |
|               | \( \frac{1}{32} \times \frac{1}{32} \) | 0.8                 | OoM             | 5 s                | OoM                   | 37 s                 | OoM                  | 171 MiB              | 127 GiB              |
| \( \varphi_2 \) | \( \frac{1}{2} \times \frac{1}{2} \) | 11.0                | 11.8            | 0.004 s            | 0.6 s                 | 5 s                  | 30 s                 | 864 MiB              | 156 MiB              |
|               | \( \frac{1}{4} \times \frac{1}{4} \) | 6.8                 | 3.4             | 0.02 s             | 15 s                  | 13 s                 | 55 s                 | 866 MiB              | 1 GiB                |
|               | \( \frac{1}{8} \times \frac{1}{8} \) | 2.0                 | 1.8             | 0.2 s              | 34 min 20 s           | 1 min 10 s           | 16 min 1 s           | 890 MiB              | 81 GiB               |
|               | \( \frac{1}{16} \times \frac{1}{16} \) | 1.1                 | OoM             | 0.9 s              | OoM                   | 4 min 37 s           | OoM                  | 994 MiB              | 126 GiB              |
|               | \( \frac{1}{32} \times \frac{1}{32} \) | 0.6                 | OoM             | 5 s                | OoM                   | 45 min 39 s          | OoM                  | 1 GiB                | 127 GiB              |

Table 1: Performance comparison between Mascot-SDS and StochasticSynthesis (abbreviated as SS) [Dutreix et al., 2020] when both tools are restricted to use uniform grid-based abstraction only. Col. 1 shows the specification considered, Col. 2 shows the size of each individual abstract state, Col. 3 and 4 compare the approximation errors, Col. 5 and 6 compare the abstraction computation times, Col. 7 and 8 compare the total synthesis times (combined time for computing the over- and the under-approximations of the almost sure winning regions), and Col. 9 and 10 compare the peak memory footprint (as measured using the “time” command) for both tools. “OoM” stands for out-of-memory. The length of the sides of the abstract states is measured in units, and the approximation error is measured in sq. units (the area covered by the abstract states which are in the over-approximation but not in the under-approximation).
Figure 3: Performance comparison between Mascot-SDS and StochasticSynthesis (abbreviated as SS) (Coogan & Arcak, 2015) when the latter was allowed to use its inbuilt abstraction refinement process for better performance. The different points on the plots were obtained by running Mascot-SDS using different sizes of abstract states, and running SS using different numbers of refinement stages.
Figure 4: Visualization of the under-approximations—in green—and the over-approximations—in yellow and green, combined—of the almost sure winning regions for the specifications $\varphi_1$ and $\varphi_2$ as computed using the tools Mascot-SDS and StochasticSynthesis; in red are the complements of the over-approximations. For Mascot-SDS, the abstract states were chosen of the size $\frac{1}{32} \times \frac{1}{32}$. For StochasticSynthesis, initially the abstract states were chosen of the size $1 \times 1$, and then the tool was made to execute 14 refinement steps to improve the approximation of the solution.
7. Acknowledgements

R. Majumdar and K. Mallik are funded through the DFG project 389792660 TRR 248–CPEC, A.-K. Schmuck is funded through the DFG project (SCHM 3541/1-1), and S. Soudjani is funded through the EPSRC New Investigator Award CodeCPS (EP/V043676/1).

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8. Appendix

8.1. Proof of Statements

Proof of Thm. 8.2. By the definition of the winning set, we already know that $P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P})) = 1$ for all $s \in \text{WinDom}(\mathcal{G}, \rho)$. Take any $s \notin W := \text{WinDom}(\mathcal{G}, \rho)$. Define $\tau$ to be the first time step when the path visits the set $W$. Note that $\tau$ is a random variable taking values in $\mathbb{N} \cup \{\infty\}$. We use the law of total probability by making the event $(\mathcal{G} \models \text{Parity}(\mathcal{P}))$ conditional on $\tau$.

Then we have

$$P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P})) = \sum_{n=0}^{\infty} P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P}) | \tau = n)P_\rho^s(\tau = n)$$

$$+ P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P}) | \tau = \infty)P_\rho^s(\tau = \infty)$$

$$= \mathbb{E}_s^\rho \left[ P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P}) \mid s^1, s^2, \ldots, s^n, \tau = n) \right]$$

$$+ P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P}) \land \tau = \infty)$$

$$= \sum_{n=0}^{\infty} P_\rho^s(s^1, s^2, \ldots, s^{n-1} \in S \setminus W, s^n \in W)$$

$$+ P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P}) \land \mathcal{G} \models \Box \setminus W)$$

$$\geq P_\rho^s(\mathcal{G} \models \Diamond W).$$

The equality (*) holds due to $s^n \in W$ and $P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P})) = 1$. $\square$

Proof of Thm. 8.3. For any $s \in W$, we have

$$P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P})) = \int_S P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P})) T_s(ds_1 | s, \rho(s))$$

$$= \int_W T_s(ds_1 | s, \rho(s)) + \int_{S \setminus W} P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P})) T_s(ds_1 | s, \rho(s)).$$

This means

$$\int_{S \setminus W} (1 - P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P}))) T_s(ds_1 | s, \rho(s)) = 0 \Rightarrow$$

$$\forall \epsilon > 0, P_\rho^s (1 - P_\rho^s(\mathcal{G} \models \text{Parity}(\mathcal{P}))) 1_{S \setminus W}(s_1) \geq \epsilon \leq \frac{0}{\epsilon} = 0.$$
where the last inequality is a consequence of Markov’s inequality for non-negative random variables. By taking the union over a monotone positive sequence \( \{ \epsilon_n \to 0 \} \), we get

\[
\begin{align*}
\mathbb{P}_\rho [ (1 - \mathbb{P}_\rho (\Theta \models \text{Parity}(\mathcal{P}))) 1_{S \setminus W}(s_1) > 0 ] &= 0, \\
\mathbb{P}_\rho [ s_1 \in S \setminus W \text{ and } \mathbb{P}_\rho (\Theta \models \text{Parity}(\mathcal{P})) < 1 ] &= 0, \\
\mathbb{P}_\rho [ s_1 \in S \setminus W ] &= 0.
\end{align*}
\]

\[\square\]

**Proof of Thm. 4.5.** Consider the following expression:

\[
I_\ell(T) := \nu Y_\ell, \mu X_{\ell-1} \ldots \mu X_3, \nu Y_2, \mu X_1.
\]

\[
(\hat{B}_1 \cap \text{Apre}(Y_2, X_1)) \cup (\hat{B}_2 \cap \text{Cpre}(Y_2)) \cup
\]

\[
(\hat{B}_3 \cap \text{Apre}(Y_4, X_3)) \cup (\hat{B}_4 \cap \text{Cpre}(Y_4)) \cup
\]

\[
\ldots
\]

\[
(\hat{B}_{\ell-1} \cap \text{Apre}(Y_\ell, X_{\ell-1})) \cup (\hat{B}_\ell \cap \text{Cpre}(Y_\ell)) \cup T,
\]

(23)

We prove the stronger claim that the fixed point of \( I_\ell(T) \) is the set of vertices from which Player 0 can almost surely enforce the specification \( \text{Parity}(\bar{\mathcal{P}}) \lor \Diamond T \). Then the claim will follow, by observing that (10) can be rewritten as \( I_0(\emptyset) \).

We prove the claim by induction over the number of priorities \( \ell \). The base case of our inductive proof is the case \( \bar{\mathcal{P}}_2 = \{ \hat{B}_1, \hat{B}_2 \} \), where

\[
I_2(T) = \nu Y_2, \mu X_1. \ (\hat{B}_1 \cap \text{Apre}(Y_2, X_1)) \cup (\hat{B}_2 \cap \text{Cpre}(Y_2)) \cup T,
\]

(24)

which is exactly equal to the almost sure winning region for the winning condition \( \varphi := \Box \Diamond \hat{B}_2 \lor \Diamond T \) (by Lem. 4.4 with the observation that \( \hat{B}_1 = \hat{B}_2 \) since the priorities partition the set of vertices). The base case is established by observing that \( \varphi = \text{Parity}(\bar{\mathcal{P}}_2) \lor \Diamond T \).

As the induction hypothesis, suppose when the set of priorities is \( \bar{\mathcal{P}}_{\ell-2} \), then the fixed point of \( I_{\ell-2}(T) \) is exactly equal to the almost sure winning region for

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the specification $\text{Parity}(\hat{P}_{\ell-2}) \lor \Box T$. We show that the claim will also hold when
the set of priorities is $\hat{P}_{\ell}$. Our proof of the induction claim uses the observation
that almost sure satisfaction of $\text{Parity}(\hat{P}_{\ell}) \lor \Box T$ is achieved by either almost
surely reaching the almost sure winning region for $\text{Parity}(P_{\ell-2})$ or almost surely
reaching a BSCC where $\hat{B}_{\ell}$ or $T$ can be repeatedly visited.

Suppose $G = \langle V, E, \langle V_0, V_1, V_r \rangle \rangle$ is the game graph and $\hat{P}_{\ell}$ is the set of
priorities. Observe that in $G$, the almost sure winning region for $\text{Parity}(\hat{P}_{\ell-2})$
is not well-defined, because the priorities $\hat{P}_{\ell-2}$ no longer create a partition
of the set of vertices as required by the parity condition (assuming $\hat{B}_{\ell-1}$ and
$\hat{B}_{\ell}$ to be nonempty). For this subtle technical reason, we need to consider a
subgame constructed by removing the $\hat{B}_{\ell-1}$ and $\hat{B}_{\ell}$ vertices from the vertices
in $G$. Formally, we will consider the subgame $\tilde{G} = \langle \tilde{V}, \tilde{E}, \langle \tilde{V}_0, \tilde{V}_1, \tilde{V}_r \rangle \rangle$, where
$\tilde{V} \subset V$ is obtained by replacing all the $\hat{B}_{\ell}$ and $\hat{B}_{\ell-1}$ vertices in $V$ by a special
sink vertex $v_{sink}$ in $\tilde{V}$, and by setting $\tilde{E} := (E \cap \tilde{V} \times \tilde{V}) \cup \{ (v, v_{sink}) \mid \exists v' \in
\hat{B}_{\ell-1} \cup \hat{B}_{\ell} . (v, v') \in E \} \cup \{ (v_{sink, \text{sink}}) \}, \tilde{V}_0 := (V_0 \cap \tilde{V}) \cup v_{sink}, \tilde{V}_1 := V_1 \cap \tilde{V},$
and $\tilde{V}_r := V_r \cap \tilde{V}$. Let $v_{sink} \in \hat{B}_{\ell-3}$ (any odd priority will work). We define the
almost sure winning region for the specification $\text{Parity}(\hat{P}_{\ell-2})$ in the game graph
$G$ as the almost sure winning region for the same specification in the subgame graph $\tilde{G}$.

Now we rewrite $I_{\ell}(T)$ as in the following:

$$I_{\ell}(T) =$$
$$\nu Y_\ell \mu X_{\ell-1} \cdot$$

$$I_{\ell-2} \left( (\hat{B}_{\ell-1} \cap \text{Apre}(Y_\ell, X_{\ell-1})) \cup (\hat{B}_{\ell} \cap \text{Cpre}(Y_\ell)) \right) \cup T. \quad (25)$$

We establish the soundness and the completeness claims separately:

**Soundness:** Let $Y_\ell^*$ be the fixed point of the outer $\mu$-calculus iterations over
$Y_\ell$ in the expression $(25)$. For the last outer iteration with $Y_\ell = Y_\ell^*$,
we obtain a growing sequence of $X_{\ell-1}$s which represent the almost sure
winning regions of the following specifications (applying the definition of

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\( I_{\ell-2}(\cdot) \):

\[
X^0_{\ell-1} : \emptyset
\]

\[
X^1_{\ell-1} : \text{Parity}(\widehat{P}_{\ell-2}) \lor \Diamond \left( \widehat{B}_\ell \cap C_{\text{pre}}(Y^*_\ell) \right) \lor T
\]

\[
X^2_{\ell-1} : \text{Parity}(\widehat{P}_{\ell-2}) \lor \Diamond \left( (\widehat{B}_{\ell-1} \cap \text{Apre}(Y^*_\ell, X^1_{\ell-1})) \cup (\widehat{B}_\ell \cap C_{\text{pre}}(Y^*_\ell)) \right) \lor T
\]

\[
\vdots
\]

\[
X^p_{\ell-1} : \text{Parity}(\widehat{P}_{\ell-2}) \lor \Diamond \left( (\widehat{B}_{\ell-1} \cap \text{Apre}(Y^*_\ell, X^{p-1}_{\ell-1})) \cup (\widehat{B}_\ell \cap C_{\text{pre}}(Y^*_\ell)) \right) \lor T,
\]

where \( p \geq 1 \) is the index at which the fixpoint over \( X_{\ell-1} \) converges. We use a similar argument as in Lem. 4.4. First, let us assume that the all the vertices in the almost sure winning region of \( \text{Parity}(\widehat{P}_{\ell-2}) \) and the set \( T \) are transformed into sink states with self-loops. (We do not care anymore once the runs reach these sets.) Consider any vertex \( v \in Y^*_\ell \). From the sequence (26), we can infer that there is a Player 0 strategy \( \pi_0 \) such that for every strategy \( \pi_1 \) of Player 1, there is a run of positive probability of length \( \leq p \) that either (a) reaches \( \widehat{B}_\ell \cap C_{\text{pre}}(Y^*_\ell) \) or (b) reaches the almost sure winning region for \( \text{Parity}(\widehat{P}_{\ell-2}) \) or (c) reaches \( T \). Moreover, \( \pi_0 \) ensures that no run from \( v \) goes outside \( Y^*_\ell \) even after reaching \( \widehat{B}_\ell \), so that (a) can be repeated or (b) or (c) can be eventually executed with positive probability. These imply that every path in the Markov chain, induced on \( G \) by \( \pi_0 \) and any arbitrary Player 1 strategy \( \pi_1 \), will eventually reach a BSCC that either (a) intersects with \( \widehat{B}_\ell \) or (b) is in the almost sure winning region for \( \text{Parity}(\widehat{P}_{\ell-2}) \) or (c) is in \( T \). Then the induction claim follows from the ergodicity property of Markov chains (see Kemeny et al. [1976, Prop. 4.27, 4.28]).

**Completeness:** Let \( Y^0_\ell = V \supseteq Y^1_\ell \supseteq Y^2_\ell \supseteq \ldots \supseteq Y^q_\ell \) be the sequence of sets of vertices represented by the \( Y_\ell \) fixpoint variable, where \( q \) is the iteration index where the fixpoint converges (i.e. the minimum index \( q \) such that \( Y^q_\ell = Y^{q+1}_\ell \)). We show, by induction, that for every iteration index \( i \in [0; q] \), the almost sure winning region for the parity specification
is contained in \( Y^i \).

The base case is when \( i = 0 \), in which case the claim follows trivially by observing that \( Y^0 \) = \( V \). Suppose the claim holds for iteration \( i \), i.e. \( Y^i \) contains the almost sure winning region and every vertex outside \( Y^i \) is losing with positive probability. When the iterations over \( Y \) reaches \( i + 1 \), for every vertex \( v \) in \( Y^{i+1} \setminus Y^i \), for every Player 0 strategy there is a Player 1 strategy such that either (a) the plays eventually exit \( Y^i \), surely or with positive probability, or (b) no positive probability path exists from \( v \) that eventually reaches \( \hat{B} \cap Cpre(Y^i) \) or \( T \) or the almost sure winning region for \( Parity(\hat{P} \ell) \). (Both follow from the definition of \( Cpre \) and \( Apre \) and the fixpoint expression.) Both (a) and (b) result in failure to satisfy the parity winning condition \( Parity(\hat{P} \ell) \), which imply that every vertex \( v \) in \( Y^{i+1} \setminus Y^i \) does not belong to the almost sure winning region of \( Parity(\hat{P} \ell) \). This proves our induction claim.

\[ \square \]