The consensus in the two-feature two-state one-dimensional Axelrod model revisited

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Abstract. The Axelrod model for the dissemination of culture exhibits a rich spatial distribution of cultural domains, which depends on the values of the two model parameters: $F$, the number of cultural features and $q$, the common number of states each feature can assume. In the one-dimensional model with $F = q = 2$, which is closely related to the constrained voter model, Monte Carlo simulations indicate the existence of multicultural absorbing configurations in which at least one macroscopic domain coexist with a multitude of microscopic ones in the thermodynamic limit. However, rigorous analytical results for the infinite system starting from the configuration where all cultures are equally likely show convergence to only monocultural or consensus configurations. Here we show that this disagreement is due simply to the order that the time-asymptotic limit and the thermodynamic limit are taken in the simulations. In addition, we show how the consensus-only result can be derived using Monte Carlo simulations of finite chains.

Keywords: critical phenomena of socio-economic systems, interacting agent models, stochastic processes
1. Introduction

The study of Axelrod’s model for the dissemination of culture [1] by the statistical physics community has revealed a rich dynamic behavior with a nonequilibrium phase transition separating stationary regimes characterized by distinct distributions of domain sizes [2–6]. Essentially, the different regimes are characterized by the presence or not of cultural domains of macroscopic size in the thermodynamic limit. In Axelrod’s model, the agents are represented by strings of cultural features of length $F$, where each feature can adopt $q$ distinct states or traits. Axelrod uses the term culture to indicate any set of individual attributes that are susceptible to social influence [1].

A feature that sets Axelrod’s model apart from most lattice models that exhibit nonequilibrium phase transitions [7] is that for finite systems all stationary states of the dynamics are absorbing configurations, i.e. the dynamics always freezes in one of those configurations. This contrasts with lattice models that exhibit an active state in addition to a macroscopic number of absorbing states [8] and the phase transition occurs between the active state and the (equivalent) absorbing states. Since according to the rules of Axelrod’s model the interaction between two neighboring agents occurs with a probability proportional to the number of cultural states they have in common, agents who do not have any cultural state in common cannot interact and the interaction between agents who share all their cultural states does not result in any change. Hence we can guarantee that at an absorbing configuration any pair of neighbors are either identical or completely different regarding their cultural states. In principle, Axelrod’s model can exhibit monocultural (consensus) absorbing configurations as well as multicultural absorbing configurations.

As Axelrod’s model can be seen as $F$ coupled voter models [9], most of the information we have on the behavior of the model in regular lattices was obtained using Monte Carlo simulations of lattices of finite linear size $L$ and then properly extrapolating the results.
to the thermodynamic limit $L \to \infty$ within a well-established framework in statistical physics (see, e.g. [10]). Hence our surprise with the recent claim by Lanchier [11] (see also [12]) that in the particular case $F = q = 2$ of the one-dimensional system, which is isomorphic to the constrained voter model [13, 14], the Monte Carlo simulations [15] yielded predictions that seemed to disagree with his analytical results, leading to the assertions that ‘spatial simulations are usually difficult to interpret’ and that ‘there is a need for rigorous analytical results’ [11]. In particular, whereas the Monte Carlo results indicate the presence of multicultural absorbing configurations in the thermodynamic limit, Lanchier’s analysis shows that only the consensus configurations exist in that limit. Actually, the convergence of the one-dimensional Axelrod’s model to a consensus for $q = 2$ can be shown rigorously regardless of the value of $F$ [12].

Here we argue that the reason for that discrepancy is the order in which the time-asymptotic limit $\tau \to \infty$ and the chain size limit $L \to \infty$ are taken in the simulations. In particular, in the Monte Carlo studies one usually takes the limit $\tau \to \infty$ first and then the limit $L \to \infty$. We show that in order to obtain the results of Lanchier’s approach, which considers a chain of infinite size at the very outset, we need to take the limit $L \to \infty$ before the time-asymptotic limit $\tau \to \infty$ in the Monte Carlo simulations. In doing so, we were able to reproduce numerically Lanchier’s finding that the $F = q = 2$ Axelrod model exhibits only a consensus phase in one dimension.

The remainder of the paper is organized as follows. In section 2 we present a brief account of Axelrod’s model and point out its connection with the constrained voter model in the case $F = q = 2$. The usual order of limits $\tau \to \infty$ first and then $L \to \infty$ is considered in section 3 and the reverse order in section 4. Finally, section 5 offers our concluding remarks.

2. Model

In the one-dimensional two-feature two-state Axelrod’s model each agent is characterized by a set of $F = 2$ cultural features and each feature can take on $q = 2$ different states, which we label by 0 and 1. Hence there are four distinct cultures $(0,0), (0,1), (1,0)$ and $(1,1)$ in total. In the initial configuration each agent is assigned one of these cultures with equal probability. The agents are fixed in the sites of a chain of length $L$ with periodic boundary conditions (i.e. a ring). According to the dynamics of the original model [1], at each time $\tau$ we pick an agent at random—the target agent—as well as one of its neighbors. As usual in such asynchronous update scheme we choose the time unit as $\Delta \tau = 1/L$. These two agents interact with probability equal to their cultural similarity, defined as the fraction of common cultural features. This rule models homophily, which is the tendency of individuals to interact preferentially with similar others. An interaction consists of selecting at random one of the distinct features, and making the selected feature of the target agent equal to its neighbor’s corresponding state. This rule models social influence since the agents become more similar after they interact. Hence two neighboring agents with antagonistic cultures $(0,0)$ and $(1,1)$ or $(0,1)$ and $(1,0)$ do not interact, whereas agents with, say, cultures $(0,0)$ and $(0,1)$ can interact with probability 1/2. In the case the two agents are identical, the interaction produces no changes. This procedure is repeated until the system is frozen into an absorbing configuration. Clearly, there are
four different types of monocultural absorbing configurations corresponding to each of the four possible cultures, whereas a multicultural absorbing configuration must either be a concatenation of the cultures $(0, 0)$ and $(1, 1)$ or of the cultures $(0, 1)$ and $(1, 0)$.

The three-opinion constrained voter model identifies the cultures $(0, 1)$ and $(1, 0)$ with a single centrist opinion labeled by 0 and the other two cultures $(0, 0)$ and $(1, 1)$ with a leftist and a rightist opinion labeled by $-$ and $+$, respectively [13,14]. Leftists and rightists are considered too incompatible to interact so the interactions are between centrists $(0)$ and leftists $(−)$ or between centrists $(0)$ and rightists $(+)$ only and follow the usual rules of the voter model [9]. The fact that in the constrained voter model the interaction between, say, 0 and $−$ takes place with probability 1 whereas in Axelrod’s model the interaction between $(0, 1)$ and $(0, 0)$ occurs with probability $1/2$ implies only a rescale of time, so that the relaxation in Axelrod’s model takes twice as long as in the constrained voter model. In addition, the centrist consensus (i.e. the extinction of both leftists and rightists) should be interpreted either as the consensus of one of the cultures $(0, 1)$ and $(1, 0)$ or as the multicultural coexistence of those two cultures. As expected, Monte Carlo simulations of the constrained voter model yielded multicultural coexistence between the two extremist opinions as well as consensus of one of the three opinions [13], as in Axelrod’s model [15].

3. Monte Carlo study of the absorbing configurations

As pointed out already, the main appeal of Axelrod’s model to the statistical physics community is probably the existence of a phase transition that separates absorbing configurations, which differ in the statistical organization of their cultural domains, in the space of parameters $(F, q)$ [2,15,16]. Hence the typical statistical mechanics analysis of Axelrod’s model consists of taking first the limit $\tau \to \infty$ for finite $L$, so one is guaranteed to reach the absorbing configurations, and then extrapolating the results of finite $L$ to the thermodynamic limit. For the sake of completeness, in this section we present the results of the analysis of the absorbing configurations of Axelrod’s model for $F = q = 2$, expanding upon the study of Vilone et al [15].

A remarkable result about the case $F = q = 2$ is that when the four cultures are present in the same proportion in the initial configuration, a fraction $\xi$ of runs freezes in monocultural absorbing configurations whereas the remaining fraction $1 - \xi$ freezes in multicultural configurations. The dependence of $\xi$ on $L$, which is shown in figure 1, is very weak and we found that $\xi \to 0.33$ exponentially fast with increasing $L$.

In order to understand the nature of the multicultural absorbing configurations we present in figure 2 the mean density of domains $N_d/L$ and the mean fraction of sites that belong to the largest domain $s_m$. These results show that $N_d$ increases with $L$ according to the power law $L^{1−2\psi}$ with $\psi = 0.36$ and that $s_m \to 0.576$ in the thermodynamic limit, which indicates a rich multicultural equilibrium regime characterized by the coexistence of at least one macroscopic domain with a large number of microscopic ones. This conclusion is in stark contrast to the claim made by Lanchier [11] that Axelrod’s model reaches consensus (i.e. converges to a monocultural equilibrium) for $F = q = 2$ in the thermodynamic limit. We stress that only runs that have frozen in multicultural configurations were considered in the calculation of the averages exhibited in figure 2.
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**Figure 1.** Fraction of runs trapped in monocultural absorbing configurations $\xi$ as function of the chain size $L$. The total number of runs is $10^6$ for each $L$. The solid line is the fitting $\xi = 0.33 + 0.017 \exp(-L/387)$. The asymptotic value of $\xi$ is very robust whereas the other two adjustable parameters can vary considerably with changes in the range of the fitting.

**Figure 2.** Left panel: Mean density of domains $N_d/L$ as function of the chain size $L$ for the multicultural absorbing configurations. The solid line is the fitting $N_d/L = 1.41L^{-2\psi}$ with $\psi = 0.36$ and the dashed line is the fitting when the exponent is held fixed at $\psi = 1/3$ (see section 5). Right panel: Mean fraction of sites $s_m$ that belong to the largest cultural domain as function of the chain size $L$ for the multicultural absorbing configurations. The solid line is the fitting $s_m = 0.576 + 0.33/L^{0.72}$. The asymptotic value of $s_m$ is very robust to changes in the fitting function or fitting range.

This is the reason why our estimate for $s_m$ differs from that of Vilone et al. [15], which represent averages over all runs. Of course, their results are easily derived from ours. For instance, averaging over all runs yields $s_m = \xi + (1 - \xi) s_m \to 0.717$ for $L \to \infty$.

The reason for the stern discrepancy between the results of the Monte Carlo simulation and Lanchier’s analysis becomes apparent when we study how the relaxation time $\tau^*$ scales with the chain size $L$. Figure 3 shows that regardless of the nature of the absorbing
configuration we find $\tau^* \sim L^2$. In addition, these results show that the dynamics takes about 3.4 times as long to freeze in a monocultural absorbing configuration than in a multicultural one.

4. Monte Carlo study of the finite time dynamics

The previous section summarized the Monte Carlo findings for the stationary regime of Axelrod’s model with $F = q = 2$, where the limit $\tau \to \infty$ is taken keeping the chain size $L$ fixed. To obtain the results of Lanchier [11] we have to take the limit $L \to \infty$ for a fixed time $\tau$. Here we focus on two critical measures of the dynamics. The first measure is the density of bonds (links or edges) that connect two sites that have no features in common. Since those sites do not interact we refer to the bonds connecting them as walls and denote their density by $n_F$. The second measure is the density of bonds that connect sites with exactly one feature in common. Since those sites can interact we refer to those bonds as active bonds and denote their density by $n_A$. In doing so we conform to the terminology of Vilone et al [15], in which $n_F$ stands for the density of bonds connecting sites that differ by all $F$ features. In the terminology of Lanchier [11], the measures $n_F$ and $n_A$ are the densities of 0-edges and 1-edges, respectively, which were shown to vanish in the asymptotic limit $\tau \to \infty$ for a chain of infinite size. We note that for the one-dimensional lattice with periodic boundary conditions (i.e. a ring) the density of walls $n_F$ becomes identical to the density of domains $N_d / L$ when the dynamics freezes in the absorbing configurations.

Figures 4 and 5 show the time evolution of the density of walls and active bonds, respectively, for several values of chain size $L$. For each $L$ there are two sets of data, according to whether the dynamics freezes in a monocultural absorbing configuration (empty symbols) or to whether it freezes in a multicultural configuration (filled symbols).
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Figure 4. Density of walls $n_F$ as function of the time (Monte Carlo steps) $\tau$ for chains of size $L = 5 \times 10^3 (\square), 1 \times 10^4 (\triangle), 2 \times 10^4 (\triangle)$ and $5 \times 10^4 (\circ)$. The open symbols represent averages taken over runs that converged to monocultural (consensus) absorbing configurations, whereas the filled symbols represent the statistics over the runs that converged to multicultural configurations. The solid line is the fitting $n_F = 0.52 \tau^{-\psi}$, with $\psi = 0.36$, of the data for $L = 5 \times 10^4$ in the range $\tau \in [10^3, 10^7]$. The dashed line is the fitting with the exponent held fixed at $\psi = 1/3$ (see section 5).

Figure 5. Density of active bonds $n_A$ as function of the time $\tau$. The symbol convention is the same as in figure 4. The solid straight line is the fitting $n_A = 0.38 \tau^{-1/2}$ in the region where $n_A$ is insensitive to the chain size.

In agreement with the results of the previous section, for any finite $L$ the statistics over the runs leading to multicultural absorbing configurations results in a nonzero value for $n_F$, even in the limit $\tau \to \infty$. Of course, for the runs leading to monocultural absorbing configurations we find that $n_F \to 0$ even for finite $L$. The point here is that unless $\tau$ is on the order of $L^2$ it is not possible to distinguish between those two cases. More importantly, we can immediately realize what happens in the case that $L \to \infty$ with $\tau$ finite: when
the limit $L \to \infty$ is taken before the limit $\tau \to \infty$, all data fall on the power law function $n_F \sim \tau^{-\psi}$, with $\psi = 0.36$ for large $\tau$ (solid line in figure 4), from where we conclude that $n_F \to 0$ as $\tau \to \infty$ in agreement with Lanchier [11].

Regarding the density of active bonds $n_A$ shown in figure 5, this quantity tends to zero for finite $L$ and large $\tau$ irrespective of the nature of the absorbing configuration. In particular, in the limit $L \to \infty$ we find $n_A \sim \tau^{-1/2}$ for large $\tau$. This figure reveals also that the relaxation towards monocultural absorbing configurations is much slower than towards multicultural configurations, in agreement with the results of figure 3. However, the additional information figure 5 offers is that the slowing down takes place near the end of the runs when the density of active bonds levels off before resuming its decrease towards zero.

It is important to note that Vazquez et al [13] have calculated the relaxation of $n_A$ exactly by mapping the constrained voter model on a spin-1/2 ferromagnetic Ising chain with zero-temperature Glauber dynamics [17]. Most interestingly, those authors used the asymptotic decay of $n_A \sim \tau^{-1/2}$, which is due to the underlying diffusive dynamics, to obtain the scaling of the mean relaxation time $\tau^* \sim L^2$ since this is the typical time needed for the active bonds to diffuse throughout the chain and be eliminated [13]. However, since that mapping, applies to the dynamics of $n_A$ only the relaxation of $n_F$ has to be studied through Monte Carlo simulations.

To conclude this section, we note that $L = 2n_F + n_A$ is a Lyapunov function for the one-dimensional Axelrod model with $F = q = 2$ and periodic boundary conditions [16]. Since the dynamics never increases the value of $\mathcal{L}$, the consensus configurations are global minima of this function ($\mathcal{L} = 0$), whereas the multicultural configurations are local minima ($\mathcal{L} = 2n_F > 0$) for finite systems. As a result, the multicultural absorbing configurations are unstable to small local perturbations (cultural drift), which then drive the system towards one of the consensus configurations.

5. Conclusion

We show that the reason for the discordance between the Monte Carlo simulations of the two-feature two-state one-dimensional Axelrod model [15] (or, equivalently, the constrained voter model [13]) and the rigorous analytical results of Lanchier for infinite chain sizes [11] is that the limits $\tau \to \infty$ and $L \to \infty$ do not commute in the Monte Carlo simulations. This difficulty should be expected somehow: since the mean relaxation time scales with $L^2$, taking the limit $L \to \infty$ with finite $\tau$, as we have done here in order to obtain Lanchier’s results, will keep the system away from the stationary regime no matter how large one chooses the value of $\tau$.

Actually, the fact that the order in which the limits $L \to \infty$ and $\tau \to \infty$ are taken influences the results was already explicit in the scaling form proposed by Vilone et al for the mean density of walls,

$$n_F (\tau, L) = \tau^{-\psi} g_F \left( \tau / L^2 \right)$$

(1)

where the scaling function is such that $g_F (x) = \text{const}$ for $x \ll 1$ and $g_F (x) \sim x^\psi$ for $x \gg 1$ [15]. Here $n_F$ represents an average over all runs or, equivalently for our purposes, over the runs trapped in multicultural absorbing configurations. For those runs we recall
from figure 2 that \( n_F \sim N_d/L \sim L^{-2\psi} \) for \( \tau \gg L^2 \). Vilone et al estimated \( \psi = 1/3 \) using chains of size up to \( L = 5 \times 10^3 \). While this estimate is practically indistinguishable from ours (i.e. \( \psi = 0.36 \)) in the scale of figure 4, it is clearly inaccurate when we consider the equilibrium regime \( n_F \sim N_d/L \) shown in figure 2. We point out, however, that the value of the exponent \( \psi \) does not seem to reflect any meaningful property of the model since it varies with the initial frequency of cultures, as shown in the context of the constrained voter model [13].

A word is in order about the behavior of the one-dimensional Axelrod model for more general values of the parameters \( F \) and \( q \). For \( F = 2 \) and \( q > 2 \) the dynamics of the infinite system converges to highly fragmented multicultural configurations [11,15] and the mean-field expression \( n_F (\tau \to \infty) = 1 - 2/q \) fits perfectly the numerical results [15]. For \( q = 2 \) and \( F > 2 \) the dynamics of the infinite system converges to a consensus [12], in agreement with the simulations [15]. For \( F > 2 \) and \( q > 2 \), the Monte Carlo simulations indicate that the infinite system exhibits a discontinuous transition between a consensus phase that exists for small \( q \) and a multicultural phase that exists for large \( q \) [15]. The discordance between the full mathematical analysis of the infinite system and the Monte Carlo simulations discussed at length in this paper occurs for \( F = q = 2 \) only, and provides a simple example of a situation where a rigorous solution of a model in the thermodynamic limit can miss important facts that necessarily would appear in applications to finite systems [18].

In sum, the scaling form (1), which is validated by the Monte Carlo simulations, explains how the order of the limits \( \tau \to \infty \) and \( L \to \infty \) determines the nature of the absorbing configurations in the two-feature two-state one-dimensional Axelrod model and shows that there is really no contradiction between the Monte Carlo and Lanchier’s results.

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