COLLAPSING AND DIRAC-TYPE OPERATORS

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Abstract. We analyze the limit of the spectrum of a geometric Dirac-type operator under a collapse with bounded diameter and bounded sectional curvature. In the case of a smooth limit space $B$, we show that the limit of the spectrum is given by the spectrum of a certain first-order differential operator on $B$, which can be constructed using superconnections. In the case of a general limit space $X$, we express the limit operator in terms of a transversally elliptic operator on a $G$-manifold $\tilde{X}$ with $X = \tilde{X}/G$. As an application, we give a characterization of manifolds which do not admit uniform upper bounds, in terms of diameter and sectional curvature, on the $k$-th eigenvalue of the square of a Dirac-type operator. We also give a formula for the essential spectrum of a Dirac-type operator on a finite-volume manifold with pinched negative sectional curvature.

1. Introduction

In a previous paper we analyzed the limit of the spectrum of the differential form Laplacian on a manifold, under a collapse with bounded diameter and bounded sectional curvature [18]. In the present paper, we extend the analysis of [18] to geometric Dirac-type operators. As the present paper is a sequel to [18], we refer to the introduction of [18] for background information about collapsing with bounded curvature and its relation to analytic questions.

Let $M$ be a connected closed oriented Riemannian manifold of dimension $n > 0$. If $M$ is spin then we put $G = \text{Spin}(n)$ and if $M$ is not spin then we put $G = \text{SO}(n)$. The spinor-type fields that we consider are sections of a vector bundle $E^M$ associated to a $G$-Clifford module $V$, the latter being in the sense of Definition 2 of Section 2. The ensuing Dirac-type operator $D^M$ acts on sections of $E^M$. We will think of the spectrum $\sigma(D^M)$ of $D^M$ as a set of real numbers with multiplicities, corresponding to possible multiple eigenvalues. For simplicity, in this introduction we will sometimes refer to the Dirac-type operators as acting on spinors, even though the results are more general.

We first consider a collapse in which the limit space is a smooth Riemannian manifold. The model case is that of a Riemannian affine fiber bundle.

Definition 1. A Riemannian affine fiber bundle is a smooth fiber bundle $\pi : M \to B$ whose fiber $Z$ is an infranilmanifold and whose structure group is reduced from $\text{Diff}(Z)$ to $\text{Aff}(Z)$, along with

- A horizontal distribution $T^H M$ whose holonomy lies in $\text{Aff}(Z)$,
- A family of vertical Riemannian metrics $g^{T^Z}$ which are parallel with respect to the flat affine connections on the fibers $Z_b$ and
- A Riemannian metric $g^{TB}$ on $B$.

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Given a Riemannian affine fiber bundle $\pi : M \to B$, there is a Riemannian metric $g^{TM}$ on $M$ constructed from $T^H M$, $g^{TZ}$ and $g^{TB}$. Let $R^M$ denote the Riemann curvature tensor of $(M, g^{TM})$, let $\Pi$ denote the second fundamental forms of the fibers $\{Z_b\}_{b \in B}$ and let $T \in \Omega^2(M; TZ)$ be the curvature of $T^H M$. Given $b \in B$, there is a natural flat connection on $E^M|_{Z_b}$ which is constructed using the affine structure of $Z_b$. We define a Clifford bundle $E^B$ on $B$ whose fiber over $b \in B$ consists of the parallel sections of $E^M|_{Z_b}$. The operator $D^M$ restricts to a first-order differential operator $D^B$ on $C^\infty(B; E^B)$. If $V$ happens to be the spinor module then we show that $D^B$ is the “quantization” of a certain superconnection on $B$. For general $V$, there is an additional zeroth-order term in $D^B$ which depends on $\Pi$ and $T$.

We show that the spectrum of $D^M$ coincides with that of $D^B$ up to a high level, which depends on the maximum diameter $\text{diam}(Z)$ of the fibers $\{Z_b\}_{b \in B}$.

**Theorem 1.** There are positive constants $A$, $A'$ and $C$ which only depend on $n$ and $V$ such that if $\| R^2 \|_\infty \text{diam}(Z)^2 \leq A'$ then the intersection of $\sigma(D^M)$ with the interval

$$[-(A \text{diam}(Z)^2 - C (\| R^M \|_\infty + \| \Pi \|_\infty^2 + \| T \|_\infty^2))^{1/2},$$

$$(A \text{diam}(Z)^2 - C (\| R^M \|_\infty + \| \Pi \|_\infty^2 + \| T \|_\infty^2))^{1/2}]$$

(1.1)

equals the intersection of $\sigma(D^B)$ with (1.1).

If $Z = S^1$, $\Pi = 0$ and $V$ is the spinor module then we recover some results of [1]; see also [12, Theorem 1.5]. The proof of Theorem 1 follows the same strategy as the proof of the analogous [18, Theorem 1]. Consequently, in the proof of Theorem 1, we only indicate the changes that need to be made in the proof of [18, Theorem 1] and refer to [18] for details.

Given $B$, Cheeger, Fukaya and Gromov showed that under some curvature bounds, any Riemannian manifold $M$ which is sufficiently Gromov-Hausdorff close to $B$ can be well approximated by a Riemannian affine fiber bundle [11]. Using this fact, we show that the spectrum of $D^M$ can be uniformly approximated by that of a certain first-order differential operator $D^B$ on $B$, at least up to a high level which depends on the Gromov-Hausdorff distance between $M$ and $B$.

Given $\epsilon > 0$ and two collections of real numbers $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$, we say that $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ are $\epsilon$-close if there is a bijection $\alpha : I \to J$ such that for all $i \in I$, $|b_{\alpha(i)} - a_i| \leq \epsilon$.

**Theorem 2.** Let $B$ be a fixed smooth connected closed Riemannian manifold. Given $n \in \mathbb{Z}^+$, take $G \in \{\text{SO}(n), \text{Spin}(n)\}$ and let $V$ be a $G$-Clifford module. Then for any $\epsilon > 0$ and $K > 0$, there are positive constants $A(B, n, V, \epsilon, K)$, $A'(B, n, V, \epsilon, K)$, and $C(B, n, V, \epsilon, K)$ so that the following holds. Let $M$ be an $n$-dimensional connected closed oriented Riemannian manifold with a $G$-structure such that $\| R^M \|_\infty \leq K$ and $d_{GH}(M, B) \leq A'$. Then there are a Clifford module $E^B$ on $B$ and a certain first-order differential operator $D^B$ on $C^\infty(B; E^B)$ such that

1. $\left\{ \sin^{-1}\left(\frac{\lambda}{\sqrt{2K}}\right) : \lambda \in \sigma(D^M) \text{ and } \lambda^2 \leq A d_{GH}(M, B)^{-2} - C \right\}$ is $\epsilon$-close to a subset of

   $\left\{ \sin^{-1}\left(\frac{\lambda}{\sqrt{2K}}\right) : \lambda \in \sigma(D^B) \right\}$, and
2. \( \sinh^{-1} \left( \frac{\lambda}{\sqrt{2k}} \right) : \lambda \in \sigma(DB) \) and \( \lambda^2 \leq A d_{GH}(M, B)^{-2} - C \) is \( \epsilon \)-close to a subset of \( \left\{ \sinh^{-1} \left( \frac{\lambda}{\sqrt{2k}} \right) : \lambda \in \sigma(DM) \right\} \).

The other results in this paper concern collapsing to a possibly-singular space. Let \( X \) be a limit space of a sequence \( \{M_i\}_{i=1}^{\infty} \) of \( n \)-dimensional connected closed oriented Riemannian manifolds with uniformly bounded diameter and uniformly bounded sectional curvature. In general, \( X \) is not homeomorphic to a manifold. However, Fukaya showed that \( X \) is homeomorphic to \( \tilde{X} / G \), where \( \tilde{X} \) is a manifold and \( G \) is a compact Lie group which acts on \( \tilde{X} \). This comes from writing \( M_i = P_i / G \), where \( G = \text{SO}(n) \) and \( P_i \) is the oriented orthonormal frame bundle of \( M_i \). There is a canonical Riemannian metric on \( P_i \). Then \( \{P_i\}_{i=1}^{\infty} \) has a subsequence which Gromov-Hausdorff converges to a manifold \( \tilde{X} \). As the convergence argument can be done \( G \)-equivariantly, the corresponding subsequence of \( \{M_i\}_{i=1}^{\infty} \) converges to \( X = \tilde{X} / G \). In general, \( \tilde{X} \) is a smooth manifold with a metric which is \( C^{1, \alpha} \) regular for all \( \alpha \in (0, 1) \).

In [13] we dealt with the limit of the spectra of the differential form Laplacians \( \{\Delta_{M_i}\}_{i=1}^{\infty} \) on the manifolds \( \{M_i\}_{i=1}^{\infty} \). We defined a limit operator \( \Delta^X \) which acts on the “differential forms” on \( X \), coupled to a superconnection. In order to make this precise, we defined the “differential forms” on \( X \) to be the \( G \)-basic differential forms on \( \tilde{X} \). We constructed the corresponding differential form Laplacian \( \Delta^X \) and showed that its spectrum described the limit of the spectra of \( \{\Delta_{M_i}\}_{i=1}^{\infty} \). We refer to [13] for the precise statements.

In the case of geometric Dirac-type operators \( D_{M_i} \), there is a fundamental problem in extending this approach. Namely, if \( \tilde{X} \) is a spin manifold on which a compact Lie group \( G \) acts isometrically and preserving the spin structure then there does not seem to be a notion of \( G \)-basic spinors on \( \tilde{X} \). In order to get around this problem, we take a different approach. For a given \( n \)-dimensional Riemannian spin manifold \( M \), put \( G = \text{Spin}(n) \), let \( P \) be the principal \( \text{Spin}(n) \)-bundle of \( M \) and let \( V \) be the spinor module. One can identify the spinor fields on \( M \) with \( (C^\infty(P) \otimes V)^G \), the \( G \)-invariant subspace of \( C^\infty(P) \otimes V \). There are canonical horizontal vector fields \( \{\mathfrak{G}_j\}_{j=1}^{n} \) on \( P \) and the Dirac operator takes the form \( D^M = -i \sum_{j=1}^{n} \gamma^j \mathfrak{G}_j \). Furthermore, \( (D^M)^2 \) can be written in a particularly simple form. As in equation (4.2) below, when acting on \( (C^\infty(P) \otimes V)^G \), \( (D^M)^2 \) becomes the scalar Laplacian on \( P \) (acting on \( V \)-valued functions) plus a zeroth-order term.

Following this viewpoint, it makes sense to define the limiting “spinor fields” on \( X \) to be the elements of \( (C^\infty(\tilde{X}) \otimes V)^G \). We can then extend Theorem 1 to the setting of \( G \)-equivariant Riemannian affine fiber bundles. Namely, the limit operator \( D^X \) turns out to be a \( G \)-invariant first-order differential operator on \( C^\infty(\tilde{X}) \otimes V \), transversally elliptic in the sense of Atiyah [2], which one then restricts to the \( G \)-invariant subspace \( (C^\infty(\tilde{X}) \otimes V)^G \).

In Theorem 3 below, we show that the analog of Theorem 1 holds, in which \( D^B \) is replaced by \( D^X \).

Theorem 3 refers to a given \( G \)-equivariant Riemannian affine fiber bundle. In order to deal with arbitrary collapsing sequences, we use the aforementioned representation of \( (D^M)^2 \) as a Laplace-type operator on \( P \). If \( \{M_i\}_{i=1}^{\infty} \) is a sequence of \( n \)-dimensional Riemannian manifolds with uniformly bounded diameter and uniformly bounded sectional curvature then we show that after taking a subsequence, the spectra of \( \{(D^M_i)^2\}_{i=1}^{\infty} \) converge to
the spectrum of a Laplace-type operator on a limit space. Let \( \{\lambda_k(\|D^M_i\|)\}_{k=1}^\infty \) denote the eigenvalues of \( |D^M| \), counted with multiplicity.

**Theorem 3.** Given \( n \in \mathbb{Z}^+ \) and \( G \in \{\text{SO}(n), \text{Spin}(n)\} \), let \( \{M_i\}_{i=1}^\infty \) be a sequence of connected closed oriented \( n \)-dimensional Riemannian manifolds with a \( G \)-structure. Let \( V \) be a \( G \)-Clifford module. Suppose that for some \( D, K > 0 \) and for each \( i \in \mathbb{Z}^+ \), we have \( \text{diam}(M_i) \leq D \) and \( \|R^M_i\|_\infty \leq K \). Then there are

1. A subsequence of \( \{M_i\}_{i=1}^\infty \), which we relabel as \( \{M_i\}_{i=1}^\infty \),
2. A smooth closed \( G \)-manifold \( X \) with a \( G \)-invariant Riemannian metric \( g^TX \) which is \( C^{1,\alpha} \)-regular for all \( \alpha \in (0, 1) \),
3. A positive \( G \)-invariant function \( \chi \in C(X) \) with \( \int_X \chi \, d\text{vol} = 1 \) and
4. A \( G \)-invariant function \( V \in L^\infty(X) \otimes \text{End}(V) \)

such that if \( \Delta^X \) denotes the Laplacian on \( L^2(X, \chi \, d\text{vol}) \otimes V \) \([14, (0.8)]\) and \( |D^X| \) denotes the operator \( \sqrt{\Delta^X + V} \) acting on \( (L^2(X, \chi \, d\text{vol}) \otimes V)^G \) then for all \( k \in \mathbb{Z}^+ \),

\[
\lim_{i \to \infty} \lambda_k(\|D^M_i\|) = \lambda_k(\|D^X\|). \tag{1.2}
\]

In the special case of the signature operator, the proof of Theorem 3 is somewhat simpler than that of the analogous \([18, \text{Proposition 11}]\), in that we essentially only have to deal with scalar Laplacians. However, \([18, \text{Proposition 11}]\) gives more detailed information. In particular, it expresses the limit operator in terms of a basic flat degree-1 superconnection on \( X \). This seems to be necessary in order to prove the results of \([18]\) concerning small eigenvalues. Of course, one does not expect to have analogous results concerning the small eigenvalues of general geometric Dirac-type operators, as their zero-eigenvalues have no topological meaning.

As an application of Theorem 3, we give a characterization of manifolds which do not have a uniform upper bound on the \( k \)-th eigenvalue of \( |D^M| \), in terms of diameter and sectional curvature.

**Theorem 4.** Let \( M \) be a connected closed oriented manifold with a \( G \)-structure. Let \( V \) be a \( G \)-Clifford module. Suppose that for some \( K > 0 \) and \( k \in \mathbb{Z}^+ \), there is no uniform upper bound on \( \lambda_k(\|D^M\|) \) among Riemannian metrics on \( M \) with \( \text{diam}(M) = 1 \) and \( \|R^M\|_\infty \leq K \). Then \( M \) admits a possibly-singular fibration \( M \to X \) whose generic fiber is an infranilmanifold \( Z \) such that the restriction of \( E^M \) to \( Z \) does not have any nonzero affine-parallel sections.

More precisely, the possibly-singular fibration \( M \to X \) of Theorem 4 is the \( G \)-quotient of a \( G \)-equivariant Riemannian affine fiber bundle \( P \to X \). Theorem 4 is an analog of \([18, \text{Theorem 5.2}]\). A simple example of Theorem 4 comes from considering spinors on \( M = S^1 \times N \), where \( N \) is a spin manifold and the spin structure on \( S^1 \) is the one that does not admit a harmonic spinor. Upon shrinking the \( S^1 \)-fiber, the eigenvalues of \( D_M \) go off to \( \pm \infty \). More generally, let \( \pi : M \to B \) be an affine fiber bundle. Theorem 4 implies that if \( E^M|_Z \) does not have any nonzero affine-parallel sections then upon collapsing \( M \) to \( B \) as in \([10, \text{Section 6}]\), the eigenvalues of \( D_M \) go off to \( \pm \infty \).

Finally, we give a result about the essential spectrum of a geometric Dirac-type operator on a finite-volume manifold of pinched negative curvature, which is an analog of \([19, \text{Theorem 2}]\). Let \( M \) be a complete connected oriented \( n \)-dimensional Riemannian manifold
with a $G$-structure. Suppose that $M$ has finite volume and its sectional curvatures satisfy $-b^2 \leq K \leq -a^2$, with $0 < a \leq b$. Let $V$ be a $G$-Clifford module. Label the ends of $M$ by $I \in \{1, \ldots, N\}$. An end of $M$ has a neighborhood $U_I$ whose closure is homeomorphic to $[0, \infty) \times Z_I$, where the first coordinate is the Busemann function corresponding to a ray exiting the end, and $Z_I$ is an infranilmanifold. Let $E^M$ be the vector bundle on $M$ associated to the pair $(G, V)$ and let $D^M$ be the corresponding Dirac-type operator. If $U_I$ lies far enough out the end then for each $s \in [0, \infty)$, $C^\infty\left(\{s\} \times Z_I; \left(E^M\right)_{\{s\} \times Z_I}\right)$ decomposes as the direct sum of a finite-dimensional space $E^B_{I,s}$, consisting of “bounded energy” sections, and its orthogonal complement, consisting of “high energy” sections. The vector spaces $\{E^B_{I,s}\}_{s \in [0, \infty)}$ fit together to form a vector bundle $E^B_I$ on $[0, \infty)$. Let $P_0$ be orthogonal projection from $\bigoplus_{I=1}^N C^\infty\left(U_I; E^M|_{U_I}\right)$ to $\bigoplus_{I=1}^N C^\infty([0, \infty); E^B_I)$. Let $D^M_{\text{end}}$ be the restriction of $D^M$ to $\bigoplus_{I=1}^N C^\infty\left(U_I; E^M|_{U_I}\right)$, say with Atiyah-Patodi-Singer boundary conditions. Then $P_0 D^M_{\text{end}} P_0$ is a first-order ordinary differential operator on $\bigoplus_{I=1}^N C^\infty([0, \infty); E^B_I)$.

**Theorem 5.** The essential spectrum of $D^M$ is the same as that of $P_0 D^M_{\text{end}} P_0$.

There is some intersection between Theorem 5 and the results of [4, Theorem 0.1], concerning the essential spectrum of $D^M$ when $n = 2$ and under an additional curvature assumption, and [3, Theorem 1], concerning the essential spectrum of $D^M$ when $M$ is hyperbolic and $V$ is the spinor module.

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### 2. Dirac-type Operators and Infranilmanifolds

Given $n \in \mathbb{Z}^+$, let $G$ be either $\text{SO}(n)$ or $\text{Spin}(n)$.

**Definition 2.** A $G$-Clifford module consists of a finite-dimensional Hermitian $G$-vector space $V$ and a $G$-equivariant linear map $\gamma : \mathbb{R}^n \to \text{End}(V)$ such that $\gamma(v)^2 = |v|^2 \text{Id}$ and $\gamma(v)^* = \gamma(v)$.

Let $M$ be a connected closed oriented smooth $n$-dimensional Riemannian manifold. Put $G = \text{Spin}(n)$ or $G = \text{SO}(n)$, according as to whether or not $M$ is spin. If $M$ is spin, fix a spin structure. Let $P$ be the corresponding principal $G$-bundle, covering the oriented orthonormal frame bundle. Its topological isomorphism class is independent of the choice of Riemannian metric. Given the Riemannian metric, there is a canonical $\mathbb{R}^n$-valued 1-form $\theta$ on $P$, the soldering form.

With respect to the standard basis $\{e_j\}_{j=1}^n$ of $\mathbb{R}^n$, we write $\gamma^j = \gamma(e_j)$. We also take a basis $\{\sigma_{ab}\}_{a,b=1}^n$ for the representation of the Lie algebra $\mathfrak{g}$ on $V$, so that $(\sigma_{ab})^* = -\sigma_{ab}$ and

$$[\sigma_{ab}, \sigma_{cd}] = \delta_{ad} \sigma_{bc} - \delta_{ac} \sigma_{bd} + \delta_{bc} \sigma_{ad} - \delta_{bd} \sigma_{ac}. \quad (2.1)$$

The $G$-equivariance of $\gamma$ implies

$$[\gamma^a, \sigma_{bc}] = \delta_{ab} \gamma^c - \delta_{ac} \gamma^b. \quad (2.2)$$

**Examples:**

1. If $G = \text{Spin}(n)$ and $V$ is the spinor representation of $G$ then $\sigma_{ab} = \frac{1}{4} [\gamma^a, \gamma^b]$. 

2. If $G = \text{SO}(n)$ and $V = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$, let $E^j$ and $I^j$ denote exterior and interior multiplication by $e^j$, respectively. Put $\gamma^j = i (E^j - I^j)$ and $\bar{\gamma}^j = E^j + I^j$. Then $\sigma^{ab} = \frac{1}{i} \left( [\gamma^a, \gamma^b] + [\bar{\gamma}^a, \bar{\gamma}^b] \right)$.

Put $E^M = P \times_G V$. The Dirac-type operator $D^M$ acts on the space $C^\infty(M; E^M)$. As the topological vector space $C^\infty(M; E^M)$ is independent of any choice of Riemannian metric on $M$, it makes sense to compare Dirac-type operators for different Riemannian metrics on $M$; see [17, Section 2] for further discussion.

Let $g^{TM}$ be the Riemannian metric on $M$. Let $\omega$ be the Levi-Civita connection on $P$. Let \{\tau_j\}_{j=1}^n be a local oriented orthonormal basis of $TM$, with dual basis \{\tau^j\}_{j=1}^n. Then we can write $\omega$ locally as a matrix-valued 1-form $\omega^a_b = \sum_{j=1}^n \omega^a_{bj} \tau^j$, and

$$D^M = -i \sum_{j=1}^n \gamma^j \nabla_{e^j} = -i \sum_{j=1}^n \gamma^j \left( e_j + \frac{1}{2} \sum_{a,b=1}^n \omega^a_{bj} \sigma^{ab} \right). \quad (2.3)$$

We have the Bochner-type equation

$$(D^M)^2 = \nabla^* \nabla - \frac{1}{8} \sum_{a,b,i,j=1}^n R^M_{abij} (\gamma^i \gamma^j - \gamma^j \gamma^i) \sigma^{ab}. \quad (2.4)$$

As the set of Riemannian metrics on $M$ is an affine space modeled on a Fréchet space, it makes sense to talk about an analytic 1-parameter family $\{c(t)\}_{t \in [0,1]}$ of metrics. Then for $t \in [0,1]$, $\dot{c}(t)$ is a symmetric 2-tensor on $M$. Let $\|\dot{c}(t)\|_{c(t)}$ denote the norm of $\dot{c}(t)$ with respect to $c(t)$, i.e.

$$\|\dot{c}(t)\|_{c(t)} = \sup_{v \in TM_{c(t)}} \frac{|\dot{c}(t)(v,v)|}{c(t)(v,v)}. \quad (2.5)$$

Put $l(c) = \int_0^1 \|\dot{c}(t)\|_{c(t)} \, dt$. We extend the definition of $l(c)$ to piecewise-analytic families of metrics in the obvious way. Given $K > 0$, let $\mathcal{M}(M, K)$ be the set of Riemannian metrics on $M$ with $\|R^M\|_\infty \leq K$. Let $d$ be the corresponding length metric on $\mathcal{M}(M, K)$, computed using piecewise-analytic paths in $\mathcal{M}(M, K)$. Let $\sigma(D^M, g^{TM})$ denote the spectrum of $D^M$ as computed with $g^{TM}$, a discrete subset of $\mathbb{R}$ which is counted with multiplicity.

**Proposition 1.** There is a constant $C = C(n, V) > 0$ such that for all $K > 0$ and $g_1^{TM}, g_2^{TM} \in \mathcal{M}(M, K)$,

$$\left\{ \sinh^{-1} \left( \frac{\lambda}{\sqrt{K}} \right) : \lambda \in \sigma(D^M, g_1^{TM}) \right\} \quad (2.6)$$

and

$$\left\{ \sinh^{-1} \left( \frac{\lambda}{\sqrt{K}} \right) : \lambda \in \sigma(D^M, g_2^{TM}) \right\} \quad (2.7)$$

are $C \, d(g_1^{TM}, g_2^{TM})$-close.

**Proof.** It is enough to show that there is a number $C$ such that if $\{c(t)\}_{t \in [0,1]}$ is an analytic 1-parameter family of metrics contained in $\mathcal{M}(M, K)$ then $\left\{ \sinh^{-1} \left( \frac{\lambda}{\sqrt{K}} \right) : \lambda \in \sigma(D^M, c(0)) \right\}$ and $\left\{ \sinh^{-1} \left( \frac{\lambda}{\sqrt{K}} \right) : \lambda \in \sigma(D^M, c(1)) \right\}$ are $Cd(c(0), c(1))$-close. By eigenvalue perturbation
theory [24, Chapter XII], the subset $\bigcup_{t \in [0,1]} \{ t \} \times \sigma(D^M, c(t))$ of $\mathbb{R}^2$ is the union of the graphs of functions $\{ \lambda_j(t) \}_{j \in \mathbb{Z}}$ which are analytic in $t$. Thus it is enough to show that for each $j \in \mathbb{Z}$,

$$\left| \sinh^{-1} \left( \frac{\lambda_j(1)}{\sqrt{K}} \right) - \sinh^{-1} \left( \frac{\lambda_j(0)}{\sqrt{K}} \right) \right| \leq C l(c). \quad (2.8)$$

Let $D(t)$ denote the Dirac-type operator constructed with the metric $c(t)$. It is self-adjoint when acting on $L^2(E^M, d\text{vol}(t))$. In order to have all of the operators $\{ D(t) \}_{t \in [0,1]}$ acting on the same Hilbert space, define $f(t) \in C^\infty(M)$ by $f(t) = \frac{d\text{vol}(t)}{d\text{vol}(0)}$. Then the spectrum of $D(t)$, acting on $L^2(E^M, d\text{vol}(t))$, is the same as the spectrum of the self-adjoint operator $f(t)^{1/2} D(t) f(t)^{-1/2}$ acting on $L^2(E^M, d\text{vol}(0))$. One can now compute $\frac{d\lambda_j}{dt}$ using eigenvalue perturbation theory, as in [20, Chapter XII]. Let $\psi_j(t)$ be a smoothly-varying unit eigenvector whose eigenvalue is $\lambda_j(t)$. Define a quadratic form $T(t)$ on $TM$ by

$$T(t)(X,Y) = \langle \psi_j, -i (\gamma(X)\nabla_Y \psi_j + \gamma(Y)\nabla_X \psi_j) \rangle + \langle -i (\gamma(X)\nabla_Y \psi_j + \gamma(Y)\nabla_X \psi_j), \psi_j \rangle. \quad (2.9)$$

Using the metric $c(t)$ to convert the symmetric tensors $\dot{c}(t)$ and $T(t)$ to self-adjoint sections of $\text{End}(TM)$, one finds

$$\frac{d\lambda_j}{dt} = -\frac{1}{8} \int_M \text{Tr}(\dot{c}(t) T(t)) \, d\text{vol}(t). \quad (2.10)$$

(This equation was shown for the pure Dirac operator, by different means, in [11].) Then

$$\left| \frac{d\lambda_j}{dt} \right| \leq \text{const.} \, \| \dot{c}(t) \|_{c(t)} \int_M \text{Tr}(|T(t)|) \, d\text{vol}(t). \quad (2.11)$$

Letting $\{ x_i \}_{i=1}^n$ be an orthonormal basis of eigenvectors of $T(t)$ at a point $m \in M$, we have

$$\text{Tr}(|T(t)|) = \sum_{i=1}^n |T(t)(x_i, x_i)|. \quad \text{Then from (2.11), we obtain}$$

$$\int_M \text{Tr}(|T(t)|) \, d\text{vol}(t) \leq \text{const.} \left( \int_M |\nabla \psi_j|^2 \, d\text{vol}(t) \right)^{1/2}. \quad (2.12)$$

From (2.4),

$$\int_M |\nabla \psi_j|^2 \, d\text{vol}(t) \leq \lambda_j^2 + \text{const.} \, K. \quad (2.13)$$

In summary, from (2.11), (2.12) and (2.13), there is a positive constant $C$ such that

$$\left| \frac{d\lambda_j}{dt} \right| \leq C \, \| \dot{c}(t) \|_{c(t)} \left( \lambda_j^2 + K \right)^{1/2}. \quad (2.14)$$

Integration gives equation (2.8). The proposition follows. \qed

For some basic facts about infranilmanifolds, we refer to [18, Section 3]. Let $N$ be a simply-connected connected nilpotent Lie group. Let $\Gamma$ be a discrete subgroup of $\text{Aff}(N)$ which acts freely and cocompactly on $N$. Put $Z = \Gamma \backslash N$, an infranilmanifold. There is a canonical flat linear connection $\nabla^{aff}$ on $TZ$. Put $\widehat{\Gamma} = \Gamma \cap N$, a cocompact subgroup of $N$. There is a short exact sequence

$$1 \rightarrow \widehat{\Gamma} \rightarrow \Gamma \rightarrow F \rightarrow 1, \quad (2.15)$$
with $F$ a finite group. Put $\hat{Z} = \hat{\Gamma}\setminus N$, a nilmanifold which finitely covers $Z$ with covering group $F$.

Let $g^{TZ}$ be a Riemannian metric on $Z$ which is parallel with respect to $\nabla^{aff}$. Let us discuss the condition for $Z$ to be spin. Suppose first that $Z$ is spin. Choose a spin structure on $Z$. Fix the basepoint $z_0 = \Gamma e \in Z$. As $\nabla^{aff}$ preserves $g^{TZ}$, its holonomy lies in $SO(n)$. Hence $\nabla^{aff}$ lifts to a flat connection on the principal $Spin(n)$-bundle, which we also denote by $\nabla^{aff}$. There is a corresponding holonomy representation $\Gamma \to Spin(n)$.

Conversely, suppose that we do not know \textit{a priori} if $Z$ is spin. Suppose that the affine holonomy $\Gamma \to F \to SO(n)$ lifts to a homomorphism $\Gamma \to Spin(n)$. Naturally, the existence of this lifting is independent of the particular choice of $g^{TZ}$. Then there is a corresponding spin structure on $Z$ with principal bundle $\Gamma \setminus (N \times Spin(n))$. The different spin structures on $Z$ correspond to different lifts of $\Gamma \to SO(n)$ to $\Gamma \to Spin(n)$. These are labelled by $H^1(\Gamma; \mathbb{Z}_2) \cong H^1(Z; \mathbb{Z}_2)$. Note that there are examples of nonspin flat manifolds [8]. Also, even if $Z$ is spin and has a fixed spin structure, the action of $Aff(Z)$ on $Z$ generally does not lift to the principal $Spin(n)$-bundle, as can be seen for the $SL(n,Z)$-action on $Z = T^n$.

Now let $G$ be either $SO(n)$ or $Spin(n)$. Let $V$ be a $G$-Clifford module. Suppose that $Z$ has a $G$-structure. If $G = SO(n)$ then we have the affine holonomy homomorphism $\rho : \Gamma \to SO(n)$. If $G = Spin(n)$ then we have a given lift of it to $\rho : \Gamma \to Spin(n)$. In either case, there is an action of $G$ on $V$ coming from $\Gamma \to G \to Aut(V)$. The vector bundle $E^Z$ can now be written as $E^Z = \Gamma \setminus (N \times V)$. We see that the vector space of sections of $E^Z$ which are parallel with respect to $\nabla^{aff}$ is isomorphic to $V^T$, the subspace of $V$ which is fixed by the action of $\Gamma$.

If $V$ is the spinor representation of $G = Spin(n)$ then let us consider the conditions for $V^T$ to be nonzero. First, as the restriction of $\rho : \Gamma \to Spin(n)$ to $\hat{\Gamma}$ maps $\hat{\Gamma}$ to $\pm 1$, we must have $\rho|_{\hat{\Gamma}} = 1$. Given this, the homomorphism $\rho$ factors through a homomorphism $F \to Spin(n)$. Then we have $V^T = V^F$. This may be nonzero even if the homomorphism $F \to Spin(n)$ is nontrivial.

Returning to the case of general $V$, as $g^{TZ}$ is parallel with respect to $\nabla^{aff}$, the operator $D^Z$ preserves the space $V^T$ of affine-parallel sections of $E^Z$. Let $D^{inv}$ be the restriction of $D^Z$ to $V^T$.

\textbf{Proposition 2.} There are positive constants $A$, $A'$ and $C$ depending only on $\dim(Z)$ and $V$ such that if $\| R^Z \|_\infty \ diam(Z)^2 \leq A'$ then the spectrum $\sigma(D^Z)$ of $D^Z$ satisfies

\[
\begin{align*}
\sigma(D^Z) \cap & \left[ - (A \ diam(Z)^{-2} - C \| R^Z \|^2_\infty)^{1/2}, (A \ diam(Z)^{-2} - C \| R^Z \|^2_\infty)^{1/2} \right] = \\
\sigma(D^{inv}) \cap & \left[ - (A \ diam(Z)^{-2} - C \| R^Z \|^2_\infty)^{1/2}, (A \ diam(Z)^{-2} - C \| R^Z \|^2_\infty)^{1/2} \right].
\end{align*}
\] (2.16)

\textbf{Proof.} As $D^Z$ is diagonal with respect to the orthogonal decomposition

\[
C^\infty(Z; E^Z) = V^T \oplus (V^T)^{\perp},
\] (2.17)

it is enough to show that there are constants $A$, $A'$ and $C$ as in the statement of the proposition such that the eigenvalues of $(D^Z)^2|_{(V^T)^{\perp}}$ are greater than $A \ diam(Z)^{-2} - C \| R^Z \|_\infty$. As in the proof of [8, Proposition 2], we can reduce to the case when $F = \{e\},$
i.e. $Z$ is a nilmanifold $\Gamma \backslash N$. Then
\[ C^\infty(Z; E^Z) \cong (C^\infty(N) \otimes V)^\Gamma. \] (2.18)
Using an orthonormal frame $\{e_i\}_{i=1}^{\dim(Z)}$ for the Lie algebra $\mathfrak{n}$ as in the proof of [18, Proposition 2], we can write
\[ \nabla_\text{aff}^{e_i} = e_i \otimes \text{Id}. \] (2.19)
and
\[ \nabla_\text{e_i} = (e_i \otimes \text{Id}) + \left( \text{Id} \otimes \frac{1}{2} \sum_{a,b=1}^{\dim(Z)} \omega_{ab} \sigma^{ab} \right). \] (2.20)
The rest of the proof now proceeds as in that of [18, Proposition 2], to which we refer for details. \(\square\)

3. Collapsing to a Smooth Base

For background information about superconnections and their applications, we refer to [2]. Let $M$ be a connected closed oriented Riemannian manifold which is the total space of a Riemannian submersion $\pi: M \to B$. Suppose that $M$ has a $G^M$-structure and that $V^M$ is a $G^M$-Clifford module as in Section 2. If $G^M = \text{SO}(n)$, put $G^Z = \text{SO}(\dim(Z))$ and $G^B = \text{SO}(\dim(B))$. If $G^M = \text{Spin}(n)$, put $G^Z = \text{Spin}(\dim(Z))$ and $G^B = \text{Spin}(\dim(B))$. As a fiber $Z_b$ has a trivial normal bundle in $M$, it admits a $G^Z$-structure. Fixing an orientation of $T_bB$ fixes the $G^Z$-structure of $Z_b$. Note, however, that $B$ does not necessarily have a $G^B$-structure. For example, if $M$ is oriented then $B$ is not necessarily oriented, as is shown in the example of $S^1 \times_{Z_2} S^2 \to \mathbb{R}P^2$, where the generator of $Z_2$ acts on $S^1$ by complex conjugation and on $S^2$ by the antipodal map. And if $M$ is spin then $B$ is not necessarily spin, as is shown in the example of $S^5 \to \mathbb{C}P^2$. What is true is that if the vertical tangent bundle $TZ$ has a $G^Z$-structure, then $B$ has a $G^B$-structure.

Put $E^M = P \times_{G^M} V^M$. There is a Clifford bundle $C$ on $B$ with the property that $C^\infty(B; C) \cong C^\infty(M; E^M)$ [2, Section 9.2]. If $\dim(Z) > 0$ then $\dim(C) = \infty$. To describe $C$ more explicitly, let $V^M = \bigoplus_{i \in L} V^B_i \otimes V^Z_i$ be the decomposition of $V^M$ into irreducible representations of $G^B \times G^Z \subset G^M$.

Examples:
1. If $G^M = \text{Spin}(n)$ and $V^M$ is the spinor representation then $V^B$ and $V^Z$ are spinor representations.
2. If $G^M = \text{SO}(n)$ and $V^M = \Lambda^* (\mathbb{R}^n) \otimes \mathbb{R} \mathbb{C}$ then $V^B = \Lambda^* (\mathbb{R}^{\dim(B)}) \otimes \mathbb{R} \mathbb{C}$ and $V^Z = \Lambda^* (\mathbb{R}^{\dim(Z)}) \otimes \mathbb{R} \mathbb{C}$.

Let $U$ be a contractible open subset of $B$. Choose an orientation on $U$. For $b \in U$, let $E^Z_{b,l}$ be the vector bundle on $Z_b$ associated to the pair $(G^Z, V^Z_l)$. Then $E^M |_{Z_b} \cong \bigoplus_{i \in L} V^B_i \otimes E^Z_{b,i}$. The vector bundles $\{E^Z_{b,l}\}_{b \in U}$ are the fiberwise restrictions of a vector bundle $E^Z_l$ on $\pi^{-1}(U)$, a vertical “spinor” bundle. There is a pushforward vector bundle $W_l | U$ whose fiber $W_{i,b}$ over $b \in U$ is $C^\infty(Z_b; E^Z_{b,l})$. If $\dim(Z) > 0$ then $\dim(W_l) = \infty$. There are Hermitian inner products $\{h^{W_l}\}_{l \in L}$ on $\{W_l\}_{l \in L}$ induced from the vertical Riemannian metric $g^T_Z$. 


Furthermore, there are Clifford bundles $\{C_l\}_{l \in L}$ on $U$ for which the fiber $C_{l,b}$ of $C_l$ over $b \in U$ is isomorphic to $V^B_l \otimes W_{l,b}$. By construction, $C^\infty \left( Z_b; E^M \big|_{Z_b} \right) \cong \bigoplus_{l \in L} C_{l,b}$. The Clifford bundles $\{C_l\}_{l \in L}$ exist globally on $B$ and $C = \bigoplus_{l \in L} C_l$. The Dirac-type operator $D^M$ decomposes as $D^M = \bigoplus_{l \in L} D^M_l$, where $D^M_l$ acts on $C^\infty (B; C_l)$.

In order to write $D^M_l$ explicitly, let us recall the Bismut superconnection on $W_l$. We will deal with each $l \in L$ separately and so we drop the subscript $l$ for the moment. We use the notation of [9, Section III(c)] to describe the local geometry of the fiber bundle $M \to B$, and the Einstein summation convention. Let $\nabla^{TZ}$ denote the Bismut connection on $Z_l$, which we extend to a connection on $E^Z_l$. The Bismut superconnection on $W_l$ [7, Proposition 10.15] is of the form

$$A = D^W + \nabla^W - \frac{1}{4} c(T). \quad (3.1)$$

Here $D^W$ is the fiberwise Dirac-type operator and has the form

$$D^W = -i \gamma^j \nabla_{e_j}^{TZ} = -i \gamma^j \left( e_j + \frac{1}{2} \omega_{pqj} \sigma^{pq} \right). \quad (3.2)$$

Next, $\nabla^W$ is a Hermitian connection on $W$ given by

$$\nabla^W = \tau^\alpha \left( \nabla_{e_\alpha}^{TZ} - \frac{1}{2} \omega_{\alpha jj} \right) = \tau^\alpha \left( e_\alpha + \frac{1}{2} \omega_{jka} \sigma^{jk} - \frac{1}{2} \omega_{\alpha jj} \right). \quad (3.3)$$

Finally,

$$c(T) = i \omega_{\alpha \beta j} \gamma^j \tau^\alpha \tau^\beta. \quad (3.4)$$

The superconnection $A$ can be "quantized" into an operator $D^A$ on $C^\infty (B; V^B \otimes W)$. Explicitly,

$$D^A = -i \gamma^j \left( e_j + \frac{1}{2} \omega_{pqj} \sigma^{pq} \right)
- i \gamma^\alpha \left( e_\alpha + \frac{1}{2} \omega_{\beta \gamma \alpha} \sigma^{\beta \gamma} + \frac{1}{2} \omega_{jka} \sigma^{jk} - \frac{1}{2} \omega_{\alpha jj} \right)
+ i \frac{1}{2} \omega_{\alpha \beta j} \gamma^j \sigma^{\alpha \beta}. \quad (3.5)$$

Let $\mathcal{V} \in \text{End}(C_l)$ be the self-adjoint operator given by

$$\mathcal{V} = -i \left( \omega_{\alpha \beta j} \gamma^j \sigma^{\alpha \beta} + \frac{1}{2} \omega_{\alpha jj} \gamma^\alpha + \omega_{\alpha \beta j} (\gamma^j \sigma^{\alpha \beta} + \gamma^\alpha \sigma^{j \beta}) \right). \quad (3.6)$$

Then restoring the index $l$ everywhere,

$$D^M_l = D^A_l + \mathcal{V}_l. \quad (3.7)$$

Examples:
1. If $G^M = \text{Spin}(n)$ and $V^M$ is the spinor representation then $\mathcal{V} = 0$.
2. If $G^M = \text{SO}(n)$ and $V^M = \Lambda^*(\mathbb{R}^n) \otimes \mathbb{C}$ then

$$\mathcal{V} = -\frac{1}{4} i \left( \omega_{\alpha \beta j} \gamma^j [\tilde{\gamma}^{\alpha}, \tilde{\gamma}^j] + \omega_{\alpha \beta j} (\gamma^j [\tilde{\gamma}^{\alpha}, \tilde{\gamma}^j] + \gamma^\alpha [\tilde{\gamma}^j, \tilde{\gamma}^j]) \right). \quad (3.8)$$
Now suppose that $\pi : M \to B$ is a Riemannian affine fiber bundle. Then $E^M|_{Z_b}$ inherits a flat connection from the flat affine connections on $\{E^Z_{b,l}\}_{l \in L}$. Let $E^B$ be the Clifford bundle on $B$ whose fiber over $b \in B$ is the space of parallel sections of $E^M|_{Z_b}$. Then $D^M$ restricts to a first-order differential operator $D^B$ on $C^\infty(B; E^B)$.

Given $b \in U$ and $l \in L$, let $W^\text{inv}_{l,b}$ be the finite-dimensional subspace of $W_{l,b}$ consisting of affine-parallel elements of $C^\infty(Z_b; E^Z_{b,l})$. From the discussion in Section 2, $W^\text{inv}_{l,b}$ is isomorphic to $(V^l_Z)^\Gamma$. The vector spaces $W^\text{inv}_{l,b}$ fit together to form a finite-dimensional subbundle $W^\text{inv}_l$ of $W_l$. There is a corresponding finite-dimensional Clifford subbundle $C^\text{inv}_l$ of $C_l$ whose fiber over $b \in U$ is isomorphic to $V^B_l \otimes W^\text{inv}_{l,b}$. Again, $C^\text{inv}_l$ exists globally on $B$. Then $E^B = \bigoplus_{l \in L} C^\text{inv}_l$. Let $D^B_l$ be the restriction of $D^M_l$ to $C^\infty(B; C^\text{inv}_l)$. Then

$$D^B = \bigoplus_{l \in L} D^B_l.$$  \hfill (3.9)

The superconnection $A_l$ restricts to an superconnection $A^\text{inv}_l$ on $W^\text{inv}_l$, the endomorphism $\mathcal{V}_l$ restricts to an endomorphism of $C^\text{inv}_l$ and $D^M_l$ restricts to the first-order differential operator

$$D^B_l = D^A_l + \mathcal{V}_l^\text{inv}$$  \hfill (3.10)

on $C^\infty(B; C^\text{inv}_l)$.

**Proof of Theorem 1:**

The operator $D^M_l$ is diagonal with respect to the orthogonal decomposition

$$C_l = C^\text{inv}_l \oplus \left( C^\text{inv}_l \right)^\perp.$$  \hfill (3.11)

Thus it suffices to show that there are constants $A$, $A'$ and $C$ such that the spectrum of $\sigma(D^M_l)$, when restricted to $(C^\text{inv}_l)^\perp$, is disjoint from (1.1).

For simplicity, we drop the subscript $l$. Given $\eta \in C^\infty \left( B; (C^\text{inv})^\perp \right) \subset C^\infty(M; E^M)$, it is enough to show that for suitable constants,

$$\langle D^M\eta, D^M\eta \rangle \geq \left( \text{const. diam}(Z)^{-2} - \text{const.} \left( \| R^M \|_\infty + \| \Pi \|_\infty^2 + \| T \|_\infty^2 \right) \right) \langle \eta, \eta \rangle.$$  \hfill (3.12)

Using (2.4), it is enough to show that

$$\langle \nabla^M\eta, \nabla^M\eta \rangle \geq \left( \text{const. diam}(Z)^{-2} - \text{const.} \left( \| R^M \|_\infty + \| \Pi \|_\infty^2 + \| T \|_\infty^2 \right) \right) \langle \eta, \eta \rangle.$$  \hfill (3.13)

We can write $\nabla^M = \nabla^V + \nabla^H$, where

$$\nabla^V : C^\infty(M; E^M) \to C^\infty(M; T^*Z \otimes E^M)$$  \hfill (3.14)

denotes covariant differentiation in the vertical direction and

$$\nabla^H : C^\infty(M; E^M) \to C^\infty(M; \pi^*T^*B \otimes E^M)$$  \hfill (3.15)
denotes covariant differentiation in the horizontal direction. Then
\[
\langle \nabla^M \eta, \nabla^M \eta \rangle = \langle \nabla^V \eta, \nabla^V \eta \rangle + \langle \nabla^H \eta, \nabla^H \eta \rangle
\geq \langle \nabla^V \eta, \nabla^V \eta \rangle
= \int_B \int_{Z_b} |\nabla^V \eta|^2(z) \ d\text{vol}_{Z_b} \ d\text{vol}_B.
\]
(3.16)

On a given fiber \(Z_b\), we have
\[
E^M \big|_{Z_b} \cong V^B \otimes E^Z_b.
\]
(3.17)

Hence we can also use the Bismut connection \(\nabla^{TZ}\) to vertically differentiate sections of \(E^M\).
That is, we can define
\[
\nabla^{TZ} : C^\infty(M; E^M) \to C^\infty(M; T^*Z \otimes E^M).
\]
(3.18)

Explicitly, with respect to a local framing,
\[
\nabla^{TZ}_{e_j} = e_j \eta + \frac{1}{2} \omega_{pqj} \sigma^{pq} \eta
\]
(3.19)

and
\[
\nabla^V_{e_j} = e_j \eta + \frac{1}{2} \omega_{pqj} \sigma^{pq} \eta + \omega_{\alpha kj} \sigma^{\alpha k} \eta + \frac{1}{2} \omega_{\alpha \beta j} \sigma^{\alpha \beta} \eta.
\]
(3.20)

Then from (3.16), (3.19) and (3.20),
\[
\langle \nabla^M \eta, \nabla^M \eta \rangle \geq \int_B \left[ \int_{Z_b} |\nabla^{TZ} \eta|^2(z) - \text{const.} (\| T_b \|^2 + \| \Pi_b \|^2) |\eta(z)|^2 \right] \ d\text{vol}_{Z_b} \ d\text{vol}_B.
\]
(3.21)

Thus it suffices to bound \(\int_{Z_b} |\nabla^{TZ} \eta|^2(z) \ d\text{vol}_{Z_b}\) from below on a given fiber \(Z_b\) in terms of \(\langle \eta, \eta \rangle_{Z_b}\), under the assumption that \(\eta \in (W_0^{inv})^\perp\). Using the Gauss-Codazzi equation, we can estimate \(\| R^{Z_b} \|_{\infty}\) in terms of \(\| R^{M} \|_{\infty}\) and \(\| \Pi \|_{\infty}^2\). Then the desired bound on \(\int_{Z_b} |\nabla^{TZ} \eta|^2(z) \ d\text{vol}_{Z_b}\) follows from Proposition 4. \(\square\)

**Proof of Theorem 4:**
Let \(g_0^{TM}\) denote the Riemannian metric on \(M\). From Proposition 4 if a Riemannian metric \(g_1^{TM}\) on \(M\) is close to \(g_0^{TM}\) in \((\mathcal{M}(M, 2K), d)\) then applying the function \(x \to \sinh^{-1} \left( \frac{x}{\sqrt{2R}} \right)\) to \(\sigma(D^M, g_0^{TM})\) gives a collection of numbers which is close to that obtained by applying \(x \to \sinh^{-1} \left( \frac{x}{\sqrt{2R}} \right)\) to \(\sigma(D^M, g_1^{TM})\). We will use the geometric results of [11] to find a metric \(g_2^{TM}\) on \(M\) which is close to \(g_0^{TM}\) and to which we can apply Theorem 4.

First, as in [11] (2.4.1), by the smoothing results of Abresch and others [11, Theorem 1.12], for any \(\epsilon > 0\) we can find metrics on \(M\) and \(B\) which are \(\epsilon\)-close in the \(C^1\)-topology to the original metrics such that the new metrics satisfy \(\| \nabla^i R \|_{\infty} \leq A_i(n, \epsilon)\) for some appropriate sequence \(\{A_i(n, \epsilon)\}_{i=0}^\infty\). Let \(g_i^{TM}\) denote the new metric on \(M\). In the proof of the smoothing result, such as using the Ricci flow [23, Proposition 2.5], one obtains an explicit smooth 1-parameter family of metrics on \(M\) in \(\mathcal{M}(M, K')\), for some \(K' > K\), going from \(g_0^{TM}\) to \(g_1^{TM}\). We can approximate this family by a piecewise-analytic family. Hence one obtains an upper bound on \(d(g_0^{TM}, g_1^{TM})\) in \(\mathcal{M}(M, K')\), for some \(K' > K\), which depends
on $K$ and is proportionate to $\epsilon$. (Note that $d$ is essentially the same as the $C^0$-metric on $\mathcal{M}(M, K)$.) By rescaling, we may assume that $\| \mathcal{R}^M \|_\infty \leq 1$, $\| \mathcal{R}^B \|_\infty \leq 1$ and $\text{inj}(B) \geq 1$. We now apply [11] Theorem 2.6, with $B$ fixed. It implies that there are positive constants $\lambda(n)$ and $c(n, \epsilon)$ so that if $d_{GH}(M, B) \leq \lambda(n)$ then there is a fibration $f : M \to B$ such that

1. $\text{diam} (f^{-1}(b)) \leq c(n, \epsilon) d_{GH}(M, B)$.
2. $f$ is a $c(n, \epsilon)$-almost Riemannian submersion.
3. $\| \Pi_{f^{-1}(b)} \|_\infty \leq c(n, \epsilon)$.

As in [11], the Gauss-Codazzi equation, the curvature bound on $M$ and the second fundamental form bound on $f^{-1}(b)$ imply a uniform bound on $\{ \| R^{f^{-1}(b)} \|_\infty \}_{b \in B}$, along with the diameter bound on $f^{-1}(b)$, this implies that if $d_{GH}(M, B)$ is sufficiently small then $f^{-1}(b)$ is almost flat.

From [11], Propositions 3.6 and 4.9, we can find another metric $g_2^{TM}$ on $M$ which is $\epsilon$-close to $g_1^{TM}$ in the $C^1$-topology so that the fibration $f : M \to B$ gives $M$ the structure of a Riemannian affine fiber bundle. Furthermore, by [11], Proposition 4.9, there is a sequence $\{ A_i(n, \epsilon) \}_{i=0}$ so that we may assume that $g_1^{TM}$ and $g_2^{TM}$ are close in the sense that

$$\| \nabla^i (g_1^{TM} - g_2^{TM}) \|_\infty \leq A_i(n, \epsilon) d_{GH}(M, B),$$

where the covariant derivative in (3.22) is that of the Levi-Civita connection of $g_2^{TM}$. Then we can interpolate linearly between $g_1^{TM}$ and $g_2^{TM}$ within $\mathcal{M}(M, K''')$ for some $K''' > K'$, and obtain an upper bound on $d_{GH}(g_1^{TM}, g_2^{TM})$ in $\mathcal{M}(M, K'')$ which is proportionate to $\epsilon$. From [21], Theorem 2.1, we can take $K'' = 2K$ (or any number greater than $K$).

We now apply Theorem 1 to the Riemannian affine fiber bundle with metric $g_2^{TM}$. It remains to estimate the geometric terms appearing in (1.1). We have an estimate on $\| \Pi \|_\infty$ as above. Applying O’Neill’s formula [3] (9.29) to the Riemannian affine fiber bundle, we can estimate $\| T \|_\infty^2$ in terms of $\| \mathcal{R}^M \|_\infty$ and $\| \mathcal{R}^B \|_\infty$. Putting this together, the theorem follows. □

4. Collapsing to a Singular Base

Let $p : P \to M$ be the principal $G$-bundle of Section 2. Let $\{ \mathfrak{g}_j \}_{j=1}^n$ be the horizontal vector fields on $P$ such that $\theta(\mathfrak{g}_j) = e_j$. Put $\mathcal{D}^P = -i \sum_{j=1}^n \gamma^j \mathfrak{g}_j$, acting on $C^\infty(P) \otimes V$.

There is an isomorphism $C^\infty(M; E^M) \cong (C^\infty(P) \otimes V)^G$. Under this isomorphism, $\mathcal{D}^M \cong \mathcal{D}^P |_{(C^\infty(P) \otimes V)^G}$. The Bochner-type equation (2.4) becomes

$$(\mathcal{D}^M)^2 \cong - \sum_{j=1}^n \mathfrak{g}_j^2 - \frac{1}{8} \sum_{a, b, i, j=1}^n (p^* \mathcal{R}^M)_{abij} (\gamma^i \gamma^j - \gamma^j \gamma^i) \sigma^{ab}$$

when acting on $(C^\infty(P) \otimes V)^G$.

Let $\{ x_a \}_{a=1}^{\dim(G)}$ be a basis for the Lie algebra $\mathfrak{g}$ which is orthonormal with respect to the negative of the Killing form. Let $\{ \mathfrak{g}_a \}_{a=1}^{\dim(G)}$ be the corresponding vector fields on $P$. Then $-\sum_{a=1}^{\dim(G)} \mathfrak{g}_a^2$ acts on $(C^\infty(P) \otimes V)^G$ as $cV \in \mathbb{R}$, the Casimir of the $G$-module $V$. Give $P$ the Riemannian metric $g^{TP}$ with the property that $\{ \mathfrak{g}_j, \mathfrak{g}_a \}$ forms an orthonormal basis of vector fields. Let $\Delta^P$ denote the corresponding (nonnegative) scalar Laplacian on $P$, extended to act on $C^\infty(P) \otimes V$. Then when acting on $(C^\infty(P) \otimes V)^G$, equation (4.1) is
equivalent to
\[
(D^M)^2 \cong \Delta^P - \frac{1}{8} \sum_{a,b,i,j=1}^n (p^* R^M)_{abij} (\gamma^i \gamma^j - \gamma^j \gamma^i) \sigma^{ab} - c_v \text{Id.} \tag{4.2}
\]

**Definition 3.** A $G$-equivariant Riemannian affine fiber bundle structure on $P$ consists of a Riemannian affine fiber bundle structure $\tilde{\pi} : P \to \tilde{X}$ which is $G$-equivariant.

Given a $G$-equivariant Riemannian affine fiber bundle, let $\tilde{Z}$ be the fiber of $\tilde{\pi} : P \to \tilde{X}$, an infranilmanifold. For collapsing purposes it suffices to take $\tilde{Z}$ to be a nilmanifold $\Gamma \backslash N$ \cite[(7.2)]{11}. We assume hereafter that this is the case. Put $X = \tilde{X}/G$, a possibly singular space. As $N$ acts isometrically in a neighborhood of a given fiber $\tilde{Z}$ and preserves the horizontal subspaces of $P \to M$, it follows that the vector fields $\{X_j\}_{j=1}^n$ are projectable with respect to $\tilde{\pi}$ and push forward to vector fields $\{\chi_j\}_{j=1}^n$ on $\tilde{X}$. Put $D^X = -i \sum_{j=1}^n \gamma^j \chi_j$, acting on $C^\infty(\tilde{X}) \otimes V$. Let $v \in C^\infty(\tilde{X})$ be given by $v(\tilde{x}) = \text{vol}(\tilde{Z}_x)$. We give $C^\infty(\tilde{X}) \otimes V$ the weighted $L^2$-inner product with respect to the weight function $v$.

We recall that there is a notion of a pseudodifferential operator being transversally elliptic with respect to the action of a Lie group $G$ \cite[Definition 1.3]{3}.

**Lemma 1.** $D^X$ is transversally elliptic on $\tilde{X}$.

**Proof.** Let $s(D^X) \in C^\infty(T^* \tilde{X}) \otimes \text{End}(V)$ denote the symbol of $D^X$. Suppose that $\xi \in T_{\tilde{x}}^* \tilde{X}$ satisfies $\xi(\tilde{v}) = 0$ for all $\tilde{v} \in T_{\tilde{x}} \tilde{X}$ which lie in the image of the representation of $\mathfrak{g}$ by vector fields on $\tilde{X}$. Then if $p \in \tilde{\pi}^{-1}(\tilde{x})$, we have that $(\tilde{\pi}^* \xi)(r) = 0$ for all $r \in T_pP$ which lie in the image of the representation of $\mathfrak{g}$ by vector fields on $P$. In other words, $\tilde{\pi}^* \xi$ is horizontal. Suppose in addition that $s(D^X)(\xi) = 0$. Then $s(D^P)(\tilde{\pi}^* \xi) = 0$. As $D^P$ is horizontally elliptic, it follows that $\tilde{\pi}^* \xi = 0$. Thus $\xi = 0$, which proves the lemma. \qed

**Definition 4.** For notation, write $C^\infty(X;E^X) = (C^\infty(\tilde{X}) \otimes V)^G$. Let $D^X$ be the restriction of $D^\tilde{X}$ to $C^\infty(X;E^X)$.

It will follow from the proof of the next theorem that $D^X$ is self-adjoint on the Hilbert space completion of $C^\infty(X;E^X)$ with respect to the (weighted) inner product. As $D^X$ is transversally elliptic, it follows that $D^X$ has a discrete spectrum \cite[Proof of Theorem 2.2]{2}.

Let $\tilde{\Pi}$ denote the second fundamental forms of the fibers $\{\tilde{Z}_x\}_{x \in \tilde{X}}$. Let $\tilde{T} \in \Omega^2(P; T\tilde{Z})$ be the curvature of the horizontal distribution on the affine fiber bundle $P \to \tilde{X}$.

**Theorem 6.** There are positive constants $A$, $A'$ and $C$ which only depend on $n$ and $V$ such that if $\|R^2\|_\infty \text{ diam}(\tilde{Z})^2 \leq A'$ then the intersection of $\sigma(D^M)$ with
\[
[-(A \text{ diam}(\tilde{Z})^{-2} - C \left(1 + \|R^M\|_\infty^2 + \|\tilde{\Pi}\|_\infty^2 + \|\tilde{T}\|_\infty^2\right))^{1/2},
\]
\[
\left(A \text{ diam}(\tilde{Z})^{-2} - C \left(1 + \|R^M\|_\infty^2 + \|\tilde{\Pi}\|_\infty^2 + \|\tilde{T}\|_\infty^2\right)\right)^{1/2}] \tag{4.3}
\]
equals the intersection of $\sigma(D^X)$ with (4.3).
Proof. Let us write
\[ C^\infty(P) \otimes V = (C^\infty(X) \otimes V) \oplus (C^\infty(X) \otimes V)^\perp, \] (4.4)
where we think of \( C^\infty(X) \otimes V \) as the elements of \( C^\infty(P) \otimes V \) which are constant along the fibers of the fiber bundle \( \tilde{\pi} : P \to \tilde{X} \). Taking \( G \)-invariant subspaces, we have an orthogonal decomposition
\[ C^\infty(M; E^M) = C^\infty(X; E^X) \oplus (C^\infty(X; E^X))^\perp, \] (4.5)
with respect to which \( D^M \) decomposes as
\[ D^M = D^X \oplus D^M|_{(C^\infty(X; E^X))^\perp}. \] (4.6)
As in the proof of Theorem 1, it suffices to obtain a lower bound on the spectrum of \( (D^M)^2|_{(C^\infty(X; E^X))^\perp} \). As \( (C^\infty(X; E^X))^\perp \subset (C^\infty(\tilde{X}) \otimes V)^\perp \), using (4.2) it suffices to obtain a lower bound on the spectrum of \( \Delta^P|_{(C^\infty(X) \otimes V)^\perp} \). This follows from the arguments of the proof of Theorem 1, using the fact that \( \| R^P \|_\infty \leq \text{const.} \). We omit the details. In fact, it is somewhat easier than the proof of Theorem 1, since we are now only dealing with the scalar Laplacian and so can replace Proposition 2 by standard eigenvalue estimates (which just involve a lower Ricci curvature bound); see [3] and references therein.

Proof of Theorem 3:
Everything in the proof will be done in a \( G \)-equivariant way, so we may omit to mention this explicitly. Let \( P_i \) be the principal \( G \)-bundle of \( M_i \), equipped with a Riemannian metric as in the beginning of the section. From the \( G \)-equivariant version of Gromov’s compactness theorem, we obtain a subsequence \( \{P_i\}_{i=1}^\infty \) which converges in the equivariant Gromov-Hausdorff topology to a \( G \)-Riemannian manifold \( (\tilde{X}, g^{TX}) \) with a \( C^{1,\alpha} \)-regular metric. As in [13, Section 3], the measure \( \chi d\text{vol}_{\tilde{X}} \) is a weak-* limit point of the pushforwards of the normalized Riemannian measures on \( \{P_i\}_{i=1}^\infty \). As in [13, p. 535], after smoothing we may assume that we have \( G \)-equivariant Riemannian affine fiber bundles \( \tilde{\pi}_i : P_i^G \to \tilde{X}_i \), with \( G \) acting freely on \( P_i^G \), along with \( G \)-diffeomorphisms \( \tilde{\phi}_i : P_i \to P_i^G \) and \( \Phi_i : \tilde{X} \to \tilde{X}_i \). Put \( M'_i = P_i^G/G \). Then \( \tilde{\phi}_i \) descends to a diffeomorphism \( \phi_i : M_i \to M'_i \) and we may also assume, as in the proof of Theorem 2, that
1. \( \phi_i^* g^{TM_i} \in M(M_i, \text{const. } K) \),
2. \( d(\phi_i^* g^{TM_i}, g^{TM_i}) \leq 2^{-i} \) in \( M(M_i, \text{const. } K) \) and
3. \( \lim_{i \to \infty} \Phi_i^* g^{TX_i} = g^{TX} \) in the \( C^{1,\alpha} \)-topology.
Using Proposition 4, we can effectively replace \( M_i \) by \( M'_i \) for the purposes of the argument. For simplicity, we relabel \( M'_i \) as \( M_i \) and \( P_i^G \) as \( P_i \). For the purposes of the limiting argument, using Theorem 6 and (4.2), we may replace the spectrum of \( |D^{X_i}| \) by the spectrum of the operator \( |D^{M_i}| \equiv \sqrt{\Delta X_i + \mathcal{V}_i} \) acting on \( C^\infty(X_i; E^{X_i}) = (C^\infty(\tilde{X}_i) \otimes V)^G \), where \( \mathcal{V}_i \) is the restriction of
\[ -\frac{1}{8} \sum_{a,b,i,j=1}^n (p^* R^{M_i})_{abij} \gamma^i \gamma^j - \gamma^i \gamma^j) \sigma^{ab} - c_V \text{Id}. \] (4.7)
to the elements of \((C^\infty(P_i) \otimes V)^G\) which are constant along the fibers of \(\tilde{\pi}_i : P_i \to \tilde{X}_i\), i.e. to \(C^\infty(X_i, E^{X_i})\).

From the curvature bound, we have a uniform bound on \(\{\|\mathcal{V}_i\|_\infty\}_{i=1}^\infty\). Using the weak-* compactness of the unit ball, let \(\mathcal{V}\) be a weak-* limit point of \(\{\mathcal{V}_i\}_{i=1}^\infty\) in \(L^\infty(\tilde{X}) \otimes \text{End}(V) = (L^1(\tilde{X}) \otimes \text{End}(V))^*\). We claim that with this choice of \(\tilde{X}, \chi\) and \(\mathcal{V}\), equation (1.2) holds.

To see this, we use the minimax characterization of eigenvalues as in [14, Section 5]. Using the diffeomorphisms \(\{\Phi_i\}_{i=1}^\infty\), we identify each \(\tilde{X}_i\) with \(\tilde{X}\). We denote by \(\langle \cdot, \cdot \rangle_{\chi, i}\) an \(L^2\)-inner product constructed using \(\Phi_i^* g^{TX_i}\) and the weight function \((\tilde{\pi}_i)_* (d\text{vol}_{P_i}))/\int_{\tilde{X}_i} (\tilde{\pi}_i)_* (d\text{vol}_{P_i})\). We denote by \(\langle \cdot, \cdot \rangle_{\chi}\) an \(L^2\)-inner product constructed using \(g^{TX}\) and the weight function \(\chi \, d\text{vol}_X\). As \(\Delta^{\tilde{X}}\) has a compact resolvent, it follows that \(|D^{\tilde{X}}|^2\) has a compact resolvent. Then

\[
\lambda_k(|D^{\tilde{X}}|)^2 = \inf_W \sup_{\psi \in W^{-0}} \frac{\langle d\psi, d\psi \rangle_{\chi_i} + \langle \psi, \mathcal{V}_i \psi \rangle_{\chi_i}}{\langle \psi, \psi \rangle_{\chi_i}},
\]

(4.8)

where \(W\) ranges over the \(k\)-dimensional subspaces of the Sobolev space \(H^1(X; E^{\tilde{X}})\). Given \(\epsilon > 0\), let \(W_\infty\) be a \(k\)-dimensional subspace such that

\[
\sup_{\psi \in W_\infty^{-0}} \frac{\langle d\psi, d\psi \rangle_{\chi_i} + \langle \psi, \mathcal{V}_i \psi \rangle_{\chi_i}}{\langle \psi, \psi \rangle_{\chi_i}} \leq \lambda_k(|D^{\tilde{X}}|)^2 + \epsilon.
\]

(4.9)

As \(\psi \otimes \psi\) lies in the finite-dimensional subspace \(W_\infty \otimes W_\infty^*\) of \(L^1(\tilde{X}) \otimes \text{End}(V)\), it follows that

\[
\lim_{i \to \infty} \langle \psi, \mathcal{V}_i \psi \rangle_{\chi_i} = \langle \psi, \mathcal{V} \psi \rangle_{\chi}
\]

(4.10)

uniformly on \(\{\psi \in W_\infty : \langle \psi, \mathcal{V}_i \psi \rangle_{\chi_i} = 1\}\). Then

\[
\lim_{i \to \infty} \sup_{\psi \in W_\infty^{-0}} \frac{\langle d\psi, d\psi \rangle_{\chi_i} + \langle \psi, \mathcal{V}_i \psi \rangle_{\chi_i}}{\langle \psi, \psi \rangle_{\chi_i}} = \sup_{\psi \in W_\infty^{-0}} \frac{\langle d\psi, d\psi \rangle_{\chi} + \langle \psi, \mathcal{V} \psi \rangle_{\chi}}{\langle \psi, \psi \rangle_{\chi}}.
\]

(4.11)

As

\[
\lambda_k(|D^{X_i}|)^2 = \inf_W \sup_{\psi \in W^{-0}} \frac{\langle d\psi, d\psi \rangle_{\chi_i} + \langle \psi, \mathcal{V}_i \psi \rangle_{\chi_i}}{\langle \psi, \psi \rangle_{\chi_i}},
\]

(4.12)

it follows that

\[
\limsup_{i \to \infty} \lambda_k(|D^{X_i}|) \leq \lambda_k(|D^{\tilde{X}}|).
\]

(4.13)

We now show that

\[
\liminf_{i \to \infty} \lambda_k(|D^{X_i}|) \geq \lambda_k(|D^{\tilde{X}}|).
\]

(4.14)

Along with (4.13), this will prove the theorem. Suppose that (4.14) is not true. Then there is some \(\epsilon > 0\) and some infinite subsequence of \(\{M_i\}_{i=1}^\infty\), which we relabel as \(\{M_i\}_{i=1}^\infty\), such that for all \(i \in \mathbb{Z}^+\),

\[
\lambda_k(|D^{X_i}|)^2 \leq \lambda_k(|D^{\tilde{X}}|)^2 - 2 \epsilon.
\]

(4.15)

For each \(i \in \mathbb{Z}^+\), let \(W_i\) be a \(k\)-dimensional subspace of \(H^1(X; E^{X_i})\) such that

\[
\sup_{\psi \in W_i^{-0}} \frac{\langle d\psi, d\psi \rangle_{\chi_i} + \langle \psi, \mathcal{V}_i \psi \rangle_{\chi_i}}{\langle \psi, \psi \rangle_{\chi_i}} \leq \lambda_k(|D^{X_i}|)^2 + \epsilon.
\]

(4.16)
Let \( \{f_{i,j}\}_{j=1}^k \) be a basis for \( W_i \) which is orthonormal with respect to \( \langle \cdot, \cdot \rangle_X \). Then for a given \( j \), the sequence \( \{f_{i,j}\}_{i=1}^\infty \) is bounded in \( H^1(X; E^X) \). After taking a subsequence, which we relabel as \( \{f_{i,j}\}_{j=1}^\infty \), we can assume that \( \{f_{i,j}\}_{i=1}^\infty \) converges weakly in \( H^1(X; E^X) \) to some \( f_{\infty,j} \). Doing this successively for \( j \in \{1, \ldots, k\} \), we can assume that for each \( j \), \( \lim_{i \to \infty} f_{i,j} = f_{\infty,j} \) weakly in \( H^1(X; E^X) \). Then from the compactness of the embedding \( H^1(X; E^X) \to L^2(X; E^X) \), we have strong convergence in \( L^2(X; E^X) \). In particular, \( \{f_{\infty,j}\}_{j=1}^\infty \) are orthonormal. Put \( W_\infty = \text{span}(f_{\infty,1}, \ldots, f_{\infty,k}) \).

If \( w_\infty = \sum_{j=1}^k c_j f_{\infty,j} \) is a nonzero element of \( W_\infty \), put \( w_i = \sum_{j=1}^k c_j f_{i,j} \). Then \( \{w_i\}_{i=1}^\infty \) converges weakly to \( w_\infty \) in \( H^1(X; E^X) \) and hence converges strongly to \( w_\infty \) in \( L^2(X; E^X) \). From a general result about weak limits, we have

\[
\langle w_\infty, w_\infty \rangle_{H^1} \leq \limsup_{i \to \infty} \langle w_i, w_i \rangle_{H^1}.
\]

(4.17)

Along with the \( L^2 \)-convergence of \( \{w_i\}_{i=1}^\infty \) to \( w_\infty \), this implies that

\[
\langle dw_\infty, dw_\infty \rangle_X \leq \limsup_{i \to \infty} \langle dw_i, dw_i \rangle_{X_i}.
\]

(4.18)

As \( w_i \otimes w_i^* \) converges in \( L^1(\tilde{X}) \otimes \text{End}(E) \) to \( w_\infty \otimes w_\infty^* \), we have

\[
\lim_{i \to \infty} \langle w_i, \mathcal{V}_i w_i \rangle_X = \lim_{i \to \infty} \left( \langle w_\infty, \mathcal{V}_i w_\infty \rangle_X + \langle w_i, \mathcal{V}_i w_i \rangle_X - \langle w_\infty, \mathcal{V}_i w_\infty \rangle_X \right)
\]

\[= \langle w_\infty, \mathcal{V} w_\infty \rangle_X. \]

(4.19)

Then

\[
\sup_{\psi \in W_\infty} \frac{\langle d\psi, d\psi \rangle_X + \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X} \leq \limsup_{i \to \infty} \sup_{\psi \in W_i} \frac{\langle d\psi, d\psi \rangle_{X_i} + \langle \psi, \mathcal{V}_i \psi \rangle_{X_i}}{\langle \psi, \psi \rangle_{X_i}}.
\]

(4.20)

Thus from (4.15), (4.16) and (4.20),

\[
\inf_{W} \sup_{\psi \in W_0} \frac{\langle d\psi, d\psi \rangle_X + \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X} \leq \lambda_k(|D^X|^2 - \epsilon),
\]

(4.21)

which is a contradiction. This proves the theorem. \( \square \)

**Proof of Theorem 4:**

Let \( \{g_t^{TM}\}_{t=1}^\infty \) be a sequence of Riemannian metrics on \( M \) as in the statement of the theorem, with respect to which \( \lambda_k(|D^M|) \) goes to infinity. Let \( P \) be the principal \( G \)-bundle of \( M \) and let \( \tilde{X} \) be the limit space of Theorem 3, a smooth manifold with a \( C^{1,\alpha} \)-regular metric. As the limit space \( X = \tilde{X}/G \) has diameter 1, it has positive dimension. As in the proof of Theorem 4, after slightly smoothing the metric on \( \tilde{X} \), there is a \( G \)-equivariant Riemannian affine fiber bundle \( \tilde{\pi} : P \to \tilde{X} \) whose fiber is a nilmanifold \( \tilde{Z} \). Let \( \tilde{x} \) be a point in a principal orbit for the \( G \)-action on \( \tilde{X} \), with isotropy group \( H \subset G \). Then \( H \) acts affinely on the nilmanifold fiber \( \tilde{Z}_{\tilde{x}} \). In particular, \( H \) is virtually abelian. The quotient \( Z = \tilde{Z}_{\tilde{x}}/H \) is the generic fiber of the possibly-singular fiber bundle \( \pi : M \to X \), the \( G \)-quotient of \( \tilde{\pi} : P \to \tilde{X} \). Then \( E^M|_Z = \tilde{Z}_{\tilde{x}} \times H V \). In particular, the vector space of affine-parallel sections of \( E^M|_Z \) is isomorphic to \( V^H \). On the other hand, if \( C^\infty(X; E^X) \neq 0 \) then \( |D^X| \) has an infinite discrete spectrum. Theorem 3 now implies that \( C^\infty(X; E^X) \cong \left( C^\infty(\tilde{X}) \otimes V \right)^G \) must be the zero space. As the orbit \( \tilde{x} \cdot G \) has a neighborhood consisting of principal orbits, the restriction
map from \((C^\infty (\tilde{X}) \otimes V)^G\) to \((C^\infty (\tilde{x} \cdot G) \otimes V)^G\) is surjective. However, \((C^\infty (\tilde{x} \cdot G) \otimes V)^G\) is isomorphic to \(V^H\). Thus \(V^H = 0\). This proves the theorem. \(\square\)

5. Proof of Theorem 5

As the proof of Theorem 5 is similar to \cite[PF. of Theorem 2]{19}, we only indicate the structure of the proof and the necessary modifications to \cite[PF. of Theorem 2]{13}.

The closure \(\overline{U}_I\) of an appropriate neighborhood of an end has the (affine) structure of an affine fiber bundle over \([0, \infty)\) with fiber \(Z_I\). The vector bundle \(E^B_I\) is the trivial vector bundle over \([0, \infty)\) whose fiber over \(s \in [0, \infty)\) consists of the affine-parallel sections of \(E^M_{\{s\} \times Z_I}\). As in \cite[Section 4]{13}, if \(U_I\) is sufficiently far out the end then we can use Propositions 4 and 2 of the present paper to construct an embedding of \(C^\infty([0, \infty); E^B_I)\) into \(C^\infty(\overline{U}_I; E^M_{|U_I})\) whose image consists of elements with “bounded energy” fiberwise restrictions. Let \(P_0\) be the Hilbert space extension of orthogonal projection from \(\bigoplus_{I=1}^N C^\infty(\overline{U}_I; E^M_{|U_I})\) to \(\bigoplus_{I=1}^N C^\infty([0, \infty); E^B_I)\). By standard arguments as in \cite[PF. of Proposition 2.1]{13}, the essential spectrum of \(D^M\) equals that of \(D^M_{\text{end}}\). With respect to the decomposition of the Hilbert space into \(\text{Im}(P_0) \oplus \text{Im}(I - P_0)\), we write

\[
D^M_{\text{end}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

(5.1)

The operators \(B\) and \(C\) are bounded, as can be seen by the method of proof of \cite[Proposition 2]{19}, replacing the operator \(\hat{d} + \hat{d}^*\) of \cite[PF. of Proposition 2]{19} by \(D^{Z_I}\). As in \cite[Proposition 3]{19}, the operator \(D\) has vanishing essential spectrum. Put \(L = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\). To prove the theorem, it suffices to show that \(D^M_{\text{end}}\) and \(L\) have the same essential spectrum. For this, it suffices to show that \((D^M_{\text{end}} + k i)^{-1} - (L + k i)^{-1}\) is compact for some \(k > 0\) \cite[Vol. IV, Chapter XIII.4, Corollary 1]{20}.

We use the general identity that

\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^{-1} + \alpha^{-1} \beta (\delta - \gamma \alpha^{-1} \beta)^{-1} \gamma \alpha^{-1} & - \alpha^{-1} \beta (\delta - \gamma \alpha^{-1} \beta)^{-1} \\ - (\delta - \gamma \alpha^{-1} \beta)^{-1} \gamma \alpha^{-1} & (\delta - \gamma \alpha^{-1} \beta)^{-1} \end{pmatrix}
\]

(5.2)

provided that \(\alpha\) and \(\delta - \gamma \alpha^{-1} \beta\) are invertible. Put

\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = D^M_{\text{end}} + k i = \begin{pmatrix} A + k i & B \\ C & D + k i \end{pmatrix}.
\]

(5.3)

If \(k\) is positive then \(\alpha\) and \(\delta\) are invertible, with \(\delta^{-1}\) being compact. If \(k\) is large enough then \(|\delta^{-1/2} \gamma \alpha^{-1} \beta \delta^{-1/2}| < 1\). Writing

\[
\delta - \gamma \alpha^{-1} \beta = \delta^{1/2} (I - \delta^{-1/2} \gamma \alpha^{-1} \beta \delta^{-1/2}) \delta^{1/2},
\]

(5.4)

we now see that \(\delta - \gamma \alpha^{-1} \beta\) is invertible. It also follows from (5.4) that \((\delta - \gamma \alpha^{-1} \beta)^{-1}\) is compact. Using (5.2), the theorem follows.
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