Designing Energy Spectra in Finite Phase Plane

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Abstract. In recent years the finite phase plane has been used to accommodate spin 1/2 particles which play a decisive role as a tool in quantum computing [1]. For this, one often uses conjugate operators in the framework of the Wigner function approach. Conjugate variables are important both in classical and quantum mechanics. The best known example of such operators are the coordinate $x$ and the momentum $p$ [2]. In elementary quantum mechanics there are very few such pairs of conjugate operators. One of the striking results in a finite phase plane of dimension $M$ is that there exist more than one pair of conjugate operators in the $xp$-plane. In this talk we show how to build 2 conjugate $kq$-representations ($k$-quasimomentum, $q$-quasicoordinate) [3]. These conjugate representations are then applied to Harper-like Hamiltonians, and it's shown how to design physical systems with energy spectra with any desired number of discrete energy levels, and with prescribed degeneracy. A general theory of quantum mechanics in finite phase plane was developed by Schwinger [4].

1. Introduction
In recent years the finite phase plane has been used to accommodate spin 1/2-particles which play a decisive role as a tool in quantum computing [1]. For this, one often uses conjugate operators in the framework of the Wigner function approach. Conjugate variables are important both in classical and quantum physics. The best known example of such variables are the coordinate $x$ and the momentum $p$. They are the canonical coordinates in the elegant Hamilton-Jacobi theory in classical mechanics, and as operators they also form the $x$ and $p$-representations in quantum mechanics. They are what is called a pair of mutually conjugate operators [2]. This means that the absolute value of the scalar product of the eigenvector $|x\rangle$ of $x$ with the eigenvector $|p\rangle$ of $p$ is a constant, independent of $x$ and $p$

$$|\langle x|p\rangle| = \frac{1}{\sqrt{h}} = \text{constant (h is the Planck constant)}$$

The physics of this equation is basic and it says that in the state $|p\rangle$ all the values of the conjugate variable $x$ are equally probable, and, vice versa, in the state $|x\rangle$, all the values of $p$ are equally probable. In elementary quantum mechanics there are very few such pairs of conjugate operators. Thus, another well known example of such a pair is the $z$-component of the angular momentum and the angle. Things however, change when one goes to a finite phase plane of dimension $M$. It turns out that the results depend then on $M$, e.g., whether $M$ is prime or not, and what are the factors into which $M$ decomposes. One of the striking results is that for finite $M$, there exist more than one pair of mutually conjugate operators in the $xp$-plane.
In this talk we show that similar results exist for the $kq$-representation ($k$-quasimomentum, $q$-quasicoordinate) [3]. Strictly speaking, in an infinite phase plane there is no quantum mechanical representation, which is conjugate to the $kq$-representation.

We construct two pairs of quasicoordinates and quasimomenta in finite phase plane, which form sets of conjugate variables. In such a plane the coordinate $x$ is quantized with a step $c$, and the momentum $p$ with a step $\frac{2\pi}{Mc}$ where $Mc$ is the size of the phase plane in the $x$-direction. The construction depends crucially on the possibility of writing $M = M_1M_2$ with $M_1$ and $M_2$ relatively prime. The conjugate variables are applied to Harper-like Hamiltonians. It is shown how to design physical systems with energy spectra containing any desired number of discrete energy levels, say $M_1$, each of them having a prescribed degeneracy $M_2$.

A general theory of quantum mechanics in finite phase plane was developed by Schwinger [4].

2. Symmetric Coordinates in Solids - The $kq$ representation

In what follows, we give a short summary of the $kq$-representation [3]. It is closely related to symmetric coordinates in solids. The concept of symmetric coordinates can best be explained by using as an example a spherical potential in atomic physics, $V(r)$ which depends only on $r$, the absolute value of $\vec{r}$. It is convenient to choose the spherical coordinates $r, \vartheta, \varphi$ in solving Schrödinger’s equation for the atom. The reason for this is that the potential depends only on $r$ and it is therefore possible to separate the coordinates and look for the wave function $\psi$ in a product form $\psi(r, \vartheta, \varphi) = R(r)Y(\vartheta, \varphi)$, of the radial, $R(r)$, and angular $Y(\vartheta, \varphi)$ parts. As is well known, it is this separation of variables that makes the problems in atomic physics solvable. The separation here was achieved by choosing as one of the coordinates the same coordinate $r$ on which and on which only the potential $V(r)$ depends. The other two coordinates have to be chosen in such a way that together with $r$ they form what is called in quantum mechanics a complete set of commuting coordinates [2]. Thus, one way of choosing such a set is to add to $r$ the angles $\vartheta$ and $\varphi$. $r, \vartheta, \varphi$ are usually called the symmetric coordinates for the central field problem. In this case one can choose the symmetric coordinates also in another way. Namely, $r, \ell^2, \ell_z$ where $\ell^2$ and $\ell_z$ are the square of the angular momentum and its $z$-component, respectively, and they are, as is well known, constants of motion. The latter choice will correspond quite closely to the way of choosing symmetric coordinates in solids.

Qualitatively, one can obtain the symmetric coordinates for crystalline solids in the following way. Let us restrict ourselves to a one-dimensional case, and consider the operator ($a$ is the lattice constant):

$$\tau\left(\frac{2\pi}{a}\right) = \exp\left(i\frac{2\pi \cdot x}{a}\right). \quad (1)$$

Its eigenvalues can be given by $\exp(i\frac{2\pi \cdot q}{a})$ where $q$ varies between 0 and $a$ and is called the quasicoordinate. While $x$ defines the position of a particle and no information can be added about the momentum when $x$ is precisely known, this is not so with the operator $\tau\left(\frac{2\pi}{a}\right)$. The latter gives only partial information about the position which is expressed by the quasicoordinate $q$. When $q$ is precisely known one can still give some information about the momentum. This follows from the simple fact that the well known translation operator, $T(a)$,

$$T(a) = \exp\left(\frac{i}{\hbar}pa\right), \quad (2)$$

commutes with the operator (1). This is easy to check

$$\left[T(a), \tau\left(\frac{2\pi}{a}\right)\right] = 0. \quad (3)$$
The eigenvalues of $T(a)$ are $\exp(ika)$ and they are known to define the quasimomentum $k$ in the theory of solids. $k$ varies from 0 to $\frac{2\pi}{a}$. In what follows we shall show that $k$ and $q$ are the symmetric coordinates in solids and they form what is called the $kq$-representation. It turns out that $T(a)$ in (2) is the additional function that can be added to $\tau(\frac{2\pi}{a})$ in (1) in order to replace fully the coordinate $x$. For deriving the $kq$-representation we start with a short description of the concept of a representation by using the well known examples of an $x$ and $p$-representation. These two representations are the most often used ones in quantum mechanics. In general, a representation is defined by giving a Hilbert space of states $|\psi\rangle$ and by defining the operators $x$ and $p$ (the coordinate and momentum operators correspondingly) on this space in such a way as to make them satisfy the canonical commutation relation

$$[x, p] = i\hbar. \tag{4}$$

Here $x$ and $p$ are Hermitian operators. Thus the $x$-representation is given by the space of functions $\langle x|\psi\rangle = \psi(x)$ and the operators $x$ (a multiplication operator) and $p = -i\hbar \frac{\partial}{\partial x}$. On the other hand, the $p$-representation is given by the space of functions $\langle p|F\rangle = F(p)$ and the operators $x = i\hbar \frac{\partial}{\partial p}$ and $p$ (a multiplication operator). These are two realizations of the abstract representation of $x$ and $p$ satisfying Rel. (4) in the space of $|\psi\rangle$.

It can directly be checked that the eigenfunctions of $T(a)$ and $\tau(\frac{2\pi}{a})$ assume in the $x$-representation (and correspondingly in the $p$-representation) the following form [3]:

$$\langle x|kq\rangle = \left(\frac{a}{2\pi}\right)^{\frac{1}{2}} \sum_n \exp(ikan) \delta(x - q - na), \tag{5}$$

$$\langle p|kq\rangle = \left(\frac{\hbar}{a}\right)^{\frac{1}{2}} \exp(-ikq) \sum_m \exp \left(\frac{iq^2}{a}m\right) \delta(p - \hbar k - \frac{2\pi}{a}hm). \tag{6}$$

As was said above, in atomic physics $r\psi_\varphi$ are the symmetric coordinates: the potential depends on $r$ only, while the angular momentum which is a constant of motion, is a function of $\vartheta$ and $\varphi$. In crystalline solids the periodic potential depends on the structure in Eq. (1) only (it does not depend explicitly on the coordinate $x$), e.g., on the quasicordinate $q$. And, as is well known, the quasimomentum $k$ is a constant of motion. There is therefore an analogy between the symmetric spherical coordinates $r\psi_\varphi$ for a central potential and the symmetric coordinates $k$ and $q$ in a periodic potential.

We mentioned 3 basic representations in elementary quantum mechanics: the $x$-representation, the $p$-representation and the $kq$-representation. In the latter, the $x$ and $p$-operators are given as follows:

$$x = i\frac{\partial}{\partial k} + q, \quad p = -i\hbar \frac{\partial}{\partial q}. \tag{7}$$

To complete the section on the $kq$-representation we give below the boundary conditions on the wave function $C(k, q)$ and its relation to $\psi(x)$ [3]. $C(k, q)$ satisfies the following boundary conditions

$$C(k, q) = C(k + \frac{2\pi}{a}, q) = \exp(-ika)C(k, q + a). \tag{8}$$

Its connection to $\psi(x)$ is

$$C(k, q) = \left(\frac{a}{2\pi}\right)^{\frac{1}{2}} \sum_n \exp(ikan)\psi(q - na). \tag{9}$$
Vice versa
\[ \psi(x) = \left( \frac{a}{2\pi} \right)^{\frac{3}{2}} \int C(k, x) dk. \] (10)

Being a quantum mechanical representation, the \( kq \)-representation can be used to replace either the \( x \) or the \( p \)-representation. As was pointed out above, \( x \) and \( p \) form conjugate representations. In the infinite phase plane there does not seem to exist, in the strict sense, a representation which is conjugate to the \( kq \)-representation. In the next sections we show that this is not the case in the finite phase plane, where one can construct a representation which is conjugate to the \( kq \)-representation.

3. Finite Phase Plane

Quantum mechanics in a finite phase plane can be achieved by applying boundary conditions on the coordinate \( x \) for the wavefunction \( \psi(x) \) and on its Fourier transform \( F(p) \)

\[ \psi(x + M c) = \psi(x); \quad F \left( p + \frac{2\pi}{c} \right) = F(p), \] (11)

where \( M \) is an integer and \( c \) is a constant. As a consequence of these boundary conditions, the coordinate \( x \) and the momentum \( p \) are quantized and they assume the following discrete values:

\[ x = sc, \quad s = 1, \ldots, M; \quad p = \frac{2\pi}{Mc}t, \quad t = 1, \ldots, M. \] (12)

In this framework the operators \( x \) and \( p \) are replaced by the exponential operators \( \tau \left( \frac{2\pi}{Mc} \right) \) and \( T(c) \)

\[ \tau \left( \frac{2\pi}{Mc} \right) = \exp \left( i \frac{2\pi}{Mc} x \right), \quad T(c) = \exp \left( \frac{i}{\hbar} pc \right). \] (13)

From equations (12) and (13), it follows that

\[ \left[ \tau \left( \frac{2\pi}{Mc} \right) \right]^M = [T(c)]^M = 1. \] (14)

Eqs. (11)-(14) define an \( M \)-dimensional vector space in finite phase plane. Let us show that in such a space one can define conjugate \( kq \)-representations. For this we assume that

\[ M = M_1 M_2 \] (15)

with \( M_1 \) and \( M_2 \) relatively prime numbers. This assumption is crucial for what follows. Following Eq. (15), we introduce two constants

\[ a = M_1 c \quad \text{and} \quad b = M_2 c, \] (16)

and define two \( kq \)-representations based on the two complete sets of commuting operators [3]

\[ \tau \left( \frac{2\pi}{a} \right) = e^{ix \frac{2\pi}{a}}, T(a) = e^{i\frac{2\pi}{a}x}; \quad \tau \left( \frac{2\pi}{b} \right) = e^{ix \frac{2\pi}{b}}, T(b) = e^{i\frac{2\pi}{b}x}. \] (17)

We have

\[ \left[ \tau \left( \frac{2\pi}{a} \right), T(a) \right] = \left[ \tau \left( \frac{2\pi}{b} \right), T(b) \right] = 0 \] (18)
but
\[ T(a)\tau(\frac{2\pi i}{a}) = \tau(\frac{2\pi i}{a}) T(a) \exp(2\pi i \frac{M_1}{M_2}) \]
and
\[ T(b)\tau(\frac{2\pi i}{a}) = \tau(\frac{2\pi i}{a}) T(b) \exp(2\pi i \frac{M_1}{M_2}). \]

(19)

It therefore follows that the operators \( T(a) \) and \( \tau(\frac{2\pi i}{a}) \) and their powers form a set of \( M \) commuting operators. The same can be said about the operators \( T(b) \) and \( \tau(\frac{2\pi i}{a}) \). This means that these operators together with all their products lead to \( M^2 \) distinct operators which replace the \( M^2 \) operators in Eq. (13).

Let us now define the eigenvectors and eigenvalues of each of the two commuting sets of operators in Eq. (17):

\[ T(\frac{2\pi i}{a})|k, q\rangle = e^{iax} |k, q\rangle; \ T(a)|k, q\rangle = e^{ika}|k, q\rangle \]

(20)

\[ \tau(\frac{2\pi i}{b})|K, Q\rangle = e^{i\frac{\pi}{b}Q} |K, Q\rangle; \ T(b)|K, Q\rangle = e^{iKb}|K, Q\rangle \]

(21)

where \(|k, q\rangle\) and \(|K, Q\rangle\) are, respectively, the eigenvectors of the pairs of commuting operators \( \tau(\frac{2\pi i}{a}) \), \( T(a) \) and \( \tau(\frac{2\pi i}{b}) \), \( T(b) \) in Eq. (17). In the \( x \)-representation the eigenvectors in Eqs. (20) and (21) are (see Eq. (5) above and Ref. [5])

\[ \langle x | k, q \rangle = \frac{1}{\sqrt{M_2}} \sum_{s=1}^{M_2} \exp(iks) \Delta(x - q - sa), \]

(22)

\[ \langle x | K, Q \rangle = \frac{1}{\sqrt{M_1}} \sum_{t=1}^{M_1} \exp(iKtb) \Delta(x - Q - tb), \]

(23)

where \( \Delta(x) \) is 1 when \( x \) is a multiple of \( Mc \), and is zero otherwise. In Eqs. (22) and (23) the variables \( k, q, K \) and \( Q \) assume the following values

\[ k = \frac{2\pi}{Mc}f, \ f = 1, \ldots, M_2, \ q = gc, \ g = 1, \ldots, M_1, \]

(24)

\[ K = \frac{2\pi}{Mc}f', \ f' = 1, \ldots, M_1, \ Q = g'c, \ g' = 1, \ldots, M_2. \]

(25)

The Eqs. (22) and (23) in finite phase plane were written following the corresponding Eq. (5) in the infinite phase plane [5].

We can now find the probability of measuring the variables \( k \) and \( q \), when the particle is in the state \(|K, Q\rangle\). For this we first find \( \langle k, q | K, Q \rangle \) (we use Eqs. (22) and (23) and follow closely the publication in Ref. [6])

\[ \langle k, q | K, Q \rangle = \sum_{x=c}^{M_c} \langle k, q | x \rangle \langle x | K, Q \rangle = \frac{1}{\sqrt{M_1M_2}} \sum_{s=1}^{M_2} \sum_{t=1}^{M_1} \exp(-iks + iKtb) \Delta(Q + tb - q - sa). \]

(26)

Because of the \( \Delta \)-function, \( \langle k, q | K, Q \rangle \) in Eq. (26) does not vanish only when \( Q - q + tb - sa = 0 \), modulo \( M_c \). Let us assume that \( Q = mc, q = nc \). Having in mind that \( a = M_1c \) and \( b = M_2c \) [Eq. (16)], the equation

\[ Q - q + tb - sa = 0, \text{ modulo } M_c \]

(27)

becomes

\[ tM_2 - sM_1 = r, \text{ modulo } M, \]

(28)
shown that for each value of \( r \) between 1 and \( M \) there is a single pair \((s,t)\) that solves Eq. (28). This means that the Diophantine Eq. (28) has exactly \( M \) different triples \((r,s,t)\) that solve it with \( r \) running from 1 to \( M \), \( s \) from 1 to \( M_2 \) and \( t \) from 1 to \( M_1 \). An example of such triples is given in Table 1 for \( M = 10, M_1 = 2, M_2 = 5 \).

| \( s \) | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 |
|-----|---|---|---|---|---|---|---|---|---|---|
| \( t \) | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |
| \( r \) | 3 | 8 | 1 | 6 | 9 | 4 | 7 | 2 | 5 | 10 |

Table 1. Solutions of Eq. (28) for \( M = 10, M_1 = 2, M_2 = 5 \)

where \( r = n - m \), and \( M_1 \) and \( M_2 \) are relatively prime as was pointed out before. It can be shown that for each value of \( r \) between 1 and \( M \) there is a single pair \((s,t)\) that solves Eq. (28).

Let us now come back to Eq. (26) for \( \langle k,q|K,Q \rangle \). For any two coordinates \( q \) and \( Q \) the \( \Delta \)-function in Eq. (26) will be 1, when Eq. (27) [or Eq. (28)] is satisfied, and zero otherwise. From here it follows that the double sum in Eq. (26) will reduce to one term, and we find for \( \langle k,q|K,Q \rangle \)

\[
\langle k,q|K,Q \rangle = \frac{1}{\sqrt{M_1 M_2}} \exp(-iksa + iKtb),
\]

(29)

where \( q - Q = rc \), and \( r,s,t \) satisfy Eq. (28). This actually means that \( \langle k,q|K,Q \rangle \) does not depend on \( q \) and \( Q \) explicitly, and the dependence on \( k \) and \( K \) is given by just the phase in Eq. (29). One has to keep in mind however, that \( s \) and \( t \) in the phase in Eq. (29) are determined by \( r \) in Eq. (28). From Eq. (29) it follows

\[
|\langle k,q|K,Q \rangle|^2 = \frac{1}{M_1 M_2} = \frac{1}{M},
\]

(30)

The result in Eq. (30) shows that when the particle is in the eigenstate \(|K,Q\rangle\) of the commuting operators \( T(b) \) and \( \tau(\frac{2\pi}{b}) \) [see Eq. (17)], the probability of measuring \( k \) and \( q \) does not depend on \( q \) and \( Q \). The same can be said about measuring \( K \) and \( Q \) in the eigenstate \(|k,q\rangle\). We can therefore claim that the two sets of commuting operators in Eq. (17) are conjugate (like the operators \( x \) and \( p \)).

A very interesting property of conjugate operators is as follows. We call the commuting sets of pairs of operators in Eq. (17) the \( a \)-set and the \( b \)-sets, respectively. We can show that when the \( a \)-set operators in Eq. (17) operate on the eigenvectors of the \( b \)-set, the eigenvalues of these eigenvectors are shifted. Let us first find the eigenvalues of the vectors \( T(a)|K,Q \rangle \). We have, using the first equation in Eq. (19), and Eq. (21)

\[
\tau(\frac{2\pi}{b})T(a)|K,Q \rangle = e^{i(Q-a\frac{2\pi}{b})}T(a)|K,Q \rangle,
\]

(31)

where \( a = M_1 c \) [see Eq. (16)]. Eq. (31) shows that applying \( T(a) \) to the eigenvector \(|K,Q \rangle\) results in an eigenvector corresponding to the eigenvalue \( Q - a \). Similarly, if we apply \( T^2(a) \) to \(|K,Q \rangle\) we will get an eigenvector corresponding to \( Q - 2a \), and so on. Applying all the operators \( T^\ell(a) \) with \( \ell = 1, \ldots, M_2 \) to \(|K,Q \rangle\), we will obtain \( M_2 \) eigenvectors \( T^\ell(a)|K,Q \rangle \) corresponding to the eigenvalues \( Q - \ell a \). This means that starting with a fixed \( Q \), and \( \ell \) assuming the values \( \ell = 1, \ldots, M_2, Q - \ell a \) will assume all possible eigenvalues of \( \tau(\frac{2\pi}{b}) \) in Eq. (25) [or Eq. (31)]. This is easy to see because assuming \( Q = g'c \) [see Eq. (25)], \( Q - \ell a = g'c - \ell M_1 c \). Bearing in mind that \( M_1 \) and \( M_2 \) are relatively prime, the latter equation modulo \( M \) will lead to all possible values of...
Figure 1. (a) Eigenvalues of the $a$-set operators; (b) Eigenvalues of the $b$-set operators.

$Q$ in Eq. (25). In a similar way one can show that applying $\tau_m(\frac{2\pi}{a})$ with $m = 1, \ldots, M_1$, to the eigenvector $|K,Q\rangle$, one obtains all $M_1$ eigenvectors with eigenvalues $K = \frac{2\pi}{Mc}\ell^1$, as given in Eq. (25). Therefore we reach the following expected result for conjugate operators: when applying the $M$ operators of the $a$-set $[T(a)$ and $\tau(\frac{2\pi}{a})]$ to a fixed eigenvector $|K,Q\rangle$ of the $b$-set $[T(b)$ and $\tau(\frac{2\pi}{b})]$, we obtain all the eigenvectors of the $b$-set. And vice versa, when applying the $M$ operators of the $b$-set to a fixed eigenvector $|k,q\rangle$ of the $a$-set, we restore all the eigenvectors of the $a$-set. In Figs. 1a and 1b we plot the eigenvalues of the $a$-set and the $b$-set, respectively, for the example considered above, $M = 10$, $M_1 = 2$, $M_2 = 5$.

Alternatively, one may say that when all the $M = 10$ $a$-set operators are applied to a single eigenvector of the $b$-set this will lead to 10 different eigenvectors of the $b$-set, corresponding to the 10 eigenvalues in Fig. 1b. And, vice versa, when the $M = 10$ $b$-set operators are applied to a single eigenvector of the $a$-set, we will obtain the 10 different eigenvectors of the $a$-set corresponding to the 10 eigenvalues in Fig. 1a. What is important to stress here is that all the eigenvalues are included in this process, which is a consequence of the $a$-set and $b$-set being conjugate. For either of these sets, their eigenvalues lie in a unit cell of area $h$, the Planck constant. The covering of all the eigenvalues in the unit cells of the phase plane is reminiscent of ergodicity in statistical physics.

In the finite phase plane [see Eq. (11)] the Fourier transform $F(p)$ of $\psi(x)$ is

$$F(p) = \langle p|\psi \rangle = \sum_x \langle p|x\rangle \langle x|\psi \rangle = \frac{1}{\sqrt{M}} \sum_x \exp(-\frac{i}{\hbar}px)\psi(x),$$

(32)

where $x$ and $p$ are given in Eq. (12), and the result was used that the eigenstate $|p\rangle$ of $T(c)$ [Eq. (3)] in the $x$-representation is [4]

$$\langle x|p \rangle = \frac{1}{\sqrt{M}} \exp\left(\frac{i}{\hbar}px\right).$$

(33)

The absolute value of the square of $\langle x|p \rangle$ is a constant, $|\langle x|p \rangle|^2 = \frac{1}{M}$. This is a consequence of the fact that the operators in Eq. (13), $\tau(\frac{2\pi}{Mc})$ and $T(c)$, are conjugate. The relation between $C^{(a)}(k,q)$ and $\psi(x)$ [see Eq. (9) and Ref. [5]] is

$$C^{(a)}(k,q) = \langle k,q|\psi \rangle = \sum_x \langle k,q|x\rangle \psi(x) = \frac{1}{\sqrt{M_2}} \sum_{s=1}^{M_2} \exp(iks)\psi(q - sa).$$

(34)
Table 2. $\sqrt{10}C^{(b)}(K,Q)$ for $M = 10$, $M_1 = 2$, $M_2 = 5$, when $C^{(a)}(k,q)$ is given in Eq. (37)

| $K$ | $Q$ | $c$ | $2c$ | $3c$ | $4c$ | $5c$ |
|-----|-----|-----|------|------|------|------|
| $\frac{2\pi}{10c}$ | $\frac{4\pi}{10c}$ | 1 | $\exp(i\frac{2\pi}{5})$ | $\exp(i\frac{3\pi}{5})$ | $\exp(i\frac{4\pi}{5})$ | $\exp(i\frac{5\pi}{5})$ |

It is of interest to find also an equation connecting the functions $C^{(b)}(K,Q)$ and $C^{(a)}(k,q)$. We have, by using Eq. (26) and the periodicity condition (8)

$$C^{(b)}(K,Q) = \sum_{k,q} \langle K,Q|k,q\rangle \langle k,q|\psi \rangle = \frac{1}{\sqrt{M}} \sum_{k} \sum_{t=1}^{M_1} \exp(-iKtb)C^{(a)}(k,Q+tb). \quad (35)$$

In deriving Eq. (35), we used the fact that for each fixed difference $q - Q = rc$, the Diophantine Eq. (28) determines uniquely $s$ and $t$. Similarly, one finds

$$C^{(a)}(k,q) = \frac{1}{\sqrt{M}} \sum_{K} \sum_{s=1}^{M_2} \exp(-iks\alpha)C^{(b)}(K,q+sa). \quad (36)$$

Eqs. (35) and (36) can be looked upon as analogs of the Fourier transform in Eq. (32). As an example of using Eq. (35), let us consider the case $M = 10$, $M_1 = 2$, $M_2 = 5$, and

$$C^{(a)}(k,q) = \begin{cases} 1, & \text{for } k = \frac{2\pi}{10c} \text{ and } q = c; \\ \text{and zero otherwise.} & \end{cases} \quad (37)$$

Eq. (35) will give us $C^{(b)}(K,Q)$ at all the points in Fig. 1b. Since $C^{(a)}(k,q)$ is maximally localized in Eq. (37) (1 at $k = \frac{2\pi}{10c}$, $q = c$, and zero otherwise), we should expect that the function $C^{(b)}(K,Q)$ in the conjugate space will be maximally extended, with $|C^{(b)}(K,Q)| = \frac{1}{\sqrt{10}}$ at all points $K$ and $Q$ in Fig. 1b. This is actually the case, as it follows from Eq. (35). Calculating $C^{(b)}(K,Q)$ from $C^{(a)}(k,q)$ by using Eq. (35) is entirely elementary. Let us find $C^{(b)}(\frac{2\pi}{10c},c)$. We have (the sum over $k$ leads to one term only at $k = \frac{2\pi}{10c}$)

$$C^{(b)}\left(\frac{2\pi}{10c},c\right) = \frac{1}{\sqrt{10}} \left[ e^{-\frac{i\pi}{10}} C^{(a)}\left(\frac{2\pi}{10c},c+5c\right) + e^{-\frac{i\pi}{10}} C^{(a)}\left(\frac{2\pi}{10c},c+10c\right) \right] = \frac{1}{\sqrt{10}} \quad (38)$$

In obtaining the result in Eq. (38) we used the fact that $C^{(a)}(\frac{2\pi}{10c},c+5c) = 0$ and $C^{(a)}(\frac{2\pi}{10c},c+10c) = 1$, as it follows from the boundary conditions on the $kq$-function in Eq. (8) and the conditions on $C^{(a)}(k,q)$ in Eq. (37). The results of $C^{(b)}(K,Q)$ for all $K$ and $Q$ are summarized in Table 2. As we see $|C^{(b)}(K,Q)| = \frac{1}{\sqrt{10}}$. Since the norm of $C^{(a)}(k,q)$ in Eq. (37) is 1, one has to expect that also the norm of $C^{(b)}(K,Q)$ will be 1, which is confirmed by the results of in Table 2. A graphical description of these results is given in Figs. 2a and 2b. These figures show that the function $C^{(a)}(k,q)$, which is localized in the $kq$-representation ($C^{(a)}(\frac{2\pi}{10c},c) = 1$ and zero otherwise), is delocalized in the $KQ$-representation. This feature is not surprising in view of the fact that the $kq$- and $KQ$-representations are conjugate. $C^{(a)}(k,q)$ and $C^{(b)}(K,Q)$ can be considered as Fourier transforms of each other.
Figure 2. (a) Plot of $|C(a)(k, q)|^2$ in Eq. (37). It is 1 (open big circle) at $k = \frac{2\pi}{10}$, $q = c$, and zero otherwise (black dots). (b) Plot of $|C(b)(K, Q)|^2$. It is $\frac{1}{10}$ at all points $K$ and $Q$ (open small circles).

5. Designing Energy Spectra

A natural application of the conjugate operators developed above is the active area of Harper-like Hamiltonians [7]. These are Hamiltonians which are periodic both in the momentum $p$ and coordinate $x$. As an elementary example we consider a Hamiltonian which is a function of the operators $T(a)$ and $\tau(\frac{2\pi}{a})$ (see Eqs.(17)-(19))

$$H = H \left[ T(a), \tau \left( \frac{2\pi}{a} \right) \right]$$

(39)

This Hamiltonian is periodic in momentum $p$ with period $\hbar \frac{2\pi}{a}$, and in coordinate $x$ with period $b$, which means that the operators $T(b)$ and $\tau(\frac{2\pi}{a})$ commute with $H$ in Eq. (39). Bearing in mind that $T(b)$ and $\tau(\frac{2\pi}{a})$ do not commute, we can label the eigenstates of $H$ by either $|\epsilon, K\rangle$ or $|\epsilon, q\rangle$ where $\epsilon$ is the eigenvalue of $H$, $K$ the eigenvalue of $T(b)$ and $q$ of $\tau(\frac{2\pi}{a})$. Let us choose the labeling $|\epsilon, K\rangle$ and write (see Eq. (21):

$$H|\epsilon, K\rangle = \epsilon|\epsilon, K\rangle, \quad T(b)|\epsilon, K\rangle = \exp(iKb)|\epsilon, K\rangle.$$  

(40)

Since $\tau(\frac{2\pi}{a})$ commutes with $H$, this means that also $\tau(\frac{2\pi}{a})|\epsilon, K\rangle$ is an eigenvector of $H$ with the same eigenvalue $\epsilon$. However, as was shown above (see text following Eq. (31)), when applying $\tau^m(\frac{2\pi}{a})$ with $m = 1, \ldots, M_1$ to the eigenvector of $T(b)$, one obtains $M_1$ eigenvectors of $T(b)$ with eigenvalues $K = \frac{2\pi}{M_1}j'$, as given in Eq. (25). This proves that the eigenvalue $\epsilon$ of the Hamiltonian $H$ in equation (31) is $M_1$-fold degenerate.

As a simple example let us consider the Harper-like Hamiltonian (we shall pay attention, that, unlike in Eq. (39), here we choose the operators $\cos(\frac{p}{\hbar}b)$ and $\cos(\frac{2\pi}{a}x)$)

$$H = V_1 \cos(\frac{p}{\hbar}b) + V_2 \cos(\frac{2\pi}{a}x),$$

(41)

where $V_1$ and $V_2$ are constants. In the $kq$-representation (the $a$-set) the eigenvalue equation for the Hamiltonian in Eq. (41) is

$$\frac{V_1}{2} [C(k, q + b) + C(k, q - b)] + V_2 \cos \left( \frac{2\pi}{a}q \right) C(k, q) = \epsilon C(k, q).$$  

(42)
From what was said above, it follows that this equation will have $M_1$ eigenvalues $\epsilon$, each of them $M_2$-fold degenerate. It is easy to solve it for our example of $M = 10$, $M_1 = 2$ and $M_2 = 5$. In this case we have 10 points in the finite phase plane (see Fig. 1), and respectively we have to solve 10 linear homogeneous equations for the 10 unknowns $C(k, q)$. However, because of the 5-fold degeneracy of each energy level, it is sufficient to solve 2 equations with 2 unknowns, which we choose as follows (see Eq. (42))

$$\frac{V_3}{2} [e^{i k_1 2 \alpha} C(k_1, q_2) + e^{i k_1 2 \alpha} C(k_1, q_2)] + V_2 \cos\left(\frac{\pi}{\alpha} q_1\right) C(k_1, q_1) = \epsilon C(k_1, q_1),$$

$$\frac{V_3}{2} [e^{i k_2 3 \alpha} C(k_1, q_1) + e^{i k_2 3 \alpha} C(k_1, q_1)] + V_2 \cos\left(\frac{\beta}{\alpha} q_2\right) C(k_1, q_2) = \epsilon C(k_1, q_2), \quad (43)$$

where we used the periodicity conditions of the $kq$-function $C(k, q)$ in Eq. (8). The notations in Eq. (43) are (see Eq. (24)): $k_1 = \frac{2\pi}{\alpha}$, $q_1 = c$, $q_2 = 2c$; $a = 2c$, and we have used the fact that in our example $b = 5c$. By using these values, Eqs. (43) become

$$V_1 e^{ik_1 \frac{2\pi}{\alpha}} C_2 - V_2 C_1 = \epsilon C_1,$$

$$V_1 e^{-ik_1 \frac{2\pi}{\alpha}} C_1 + V_2 C_2 = \epsilon C_2, \quad (44)$$

where $C_1 \equiv C(k_1, q_1)$ and $C_2 \equiv C(k_1, q_2)$ (see Eq. (24)). Solving Eq. (44) we find the eigenvalues $\epsilon$ and the ratio of $C_1$ and $C_2$

$$\epsilon_{1,2} = \pm \sqrt{V_1^2 + V_2^2}, \quad \frac{C_{1,2}}{C_1} = \frac{\epsilon_{1,2} + V_2}{V_1 \exp(i \frac{2\pi}{\alpha})} C_1^{(1,2)}, \quad (45)$$

where the superscripts in the wave function denote the 2 solutions corresponding to $\epsilon_1$ and $\epsilon_2$. The other 8 solutions of Eqs. (44) will have the same energies $\epsilon_{1,2}$, while the wave functions will have the same expression as in Eq. (45), but with the exponents in the denominator $\exp(i \frac{2\pi m}{\alpha})$ with $m = 2, 3, 4, 5$. This corresponds to our general result that the solutions for each $\epsilon$ are 5-fold degenerate. It is interesting to point out that the solution in Eq. (44) also solves Eq. (42) for $M_1 = 2$ and any large odd $M_2$. The energies are identical with the ones in Eq. (45) while for the wave functions we will have $\exp[i \frac{2\pi}{M_2} (M_2 - 1)m]$ with $m = 1, 2, \ldots M_2$. This corresponds to a $M_2$-degeneracy of each energy level as predicted by symmetry. Similarly, one can show that for $M = 2M_2$, where $M_2$ is odd ($a = 2c$, $b = M_2 c$), the spectrum of the Harper-like Hamiltonian

$$H = V_1 \cos\left(\frac{p}{\hbar} a\right) + V_2 \cos\left(\frac{2\pi}{b}\right) \quad (46)$$

will consist of $M_2$ discrete energy levels, each of them being doubly degenerate.

From the symmetry analysis and the explicit examples above, it follows that the Hamiltonians in Eqs. (41) and (46) can be used in the design of physical systems with desired energy spectra, both from the point of view of the number of energy levels, and of their degeneracy. This can be achieved by the appropriate choice of $M_1$ and $M_2$ in Eq. (15). For example, when $M = 2 \times 3 \times 5 = 30$, there are 3 different choices for $M_1$ and $M_2$, $(M_1, M_2)$: $(2,15)$, $(3,10)$ and $(5,6)$. The number of energy levels and their degeneracies for these 3 choices are summarized in Table 3. We call the Hamiltonian in Eq. (41) $H = H_1$ and in Eq. (46) $H = H_2$.

The results in Table 3 are easy to understand. The Hamiltonian $H_1$ in Eq. (41) is built from the operators $T(b)$ and $\tau(\frac{2\pi}{\alpha})$ in Eq. (17), whose commutation relation is given in Eq. (19). One can check that

$$T^{M_1} (b) = \tau^{M_1} \left(\frac{2\pi}{\alpha}\right) = 1. \quad (47)$$

Also, these operators commute with the operators $T(a)$ and $\tau(\frac{2\pi}{b})$ in Eq. (17). Similarly,

$$T^{M_2} (a) = \tau^{M_2} \left(\frac{2\pi}{b}\right) = 1. \quad (48)$$
Table 3. Number of energy levels and their degeneracies for the Hamiltonian $H_1$ in Eq. (41) and $H_2$ in Eq. (46). For $M = 30$ there are 3 possible factorizations into relatively prime $M_1$ and $M_2$.

| $M = M_1 \times M_2$ | Number of Energy levels | Degeneracies of Energy levels |
|----------------------|-------------------------|-------------------------------|
| $(M_1, M_2) \equiv (2, 15)$ | 2 | 15 | 15 | 2 |
| $(M_1, M_2) \equiv (3, 10)$ | 3 | 10 | 10 | 3 |
| $(M_1, M_2) \equiv (5, 6)$ | 5 | 6 | 6 | 5 |

This actually means that the pair of operators $T(b)$ and $\tau\left(\frac{2\pi}{b}\right)$ and the pair $T(a)$ and $\tau\left(\frac{2\pi}{a}\right)$ split the degree of freedom in the $M$-dimensional space of operators in Eq. (13), into two degrees of freedom of $M_1$ and $M_2$ dimensional spaces, respectively. With the space of the operators of the Hamiltonian $H_1$ in Eq. (41) being $M_1$-dimensional, one should expect that, in general, $H_1$ will have $M_1$ energy levels, as in Table 3. The degeneracy of each of these levels will, in general, be $M_2$, as in Table 3. This can be deduced from the following basic symmetry principle in elementary quantum mechanics. The operators $T(a)$ and $\tau\left(\frac{2\pi}{a}\right)$ commute with the Hamiltonian $H_1$ in Eq. (41), but do not commute with one another and form an $M_2$-dimensional degree of freedom. This will, in general, lead to each level of the Hamiltonian $H_1$ in Eq. (41) to be $M_2$-fold degenerate. A similar analysis of the Hamiltonian in Eq. (46) will complete Table 3. We have shown here that the conjugate representations $kq$ and $KQ$ in combination with Harper’s equation enable us to design physical systems which have energy spectra with practically any desired number of discrete energy levels, each of them having a in advance prescribed degeneracy.

6. Summary
In summary, we would like to point out that finite phase plane has recently been used in a great variety of physical problems, such as quantum computing [1], quantum Hall effect [8], quantum maps [9], mutually unbiased bases, the Wigner function [10], Landau levels in a magnetic field and von Neumann lattices [5]. For a review article see Ref. [11].

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