A small universe after all?

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The cosmic microwave background radiation allows us to measure both the geometry and topology of the universe. It has been argued that the COBE-DMR data already rule out models that are multiply connected on scales smaller than the particle horizon. Here we show the opposite is true: compact (small) hyperbolic universes are favoured over their infinite counterparts. For a density parameter of $\Omega_c = 0.3$, the compact models are a better fit to COBE-DMR (relative likelihood $\sim 20$) and the large-scale structure data ($\sigma_8$ increases by $\sim 25\%$).

Measurements of the cosmic microwave background radiation (CMBR) provide a powerful probe of the geometry and topology of the universe. The geometry of the universe is reflected in the height, position and spacing of acoustic peaks in the CMBR angular power spectrum [1], while the topology of the universe is betrayed by matched circles in the microwave sky [2,3]. The topology of the universe also influences the angular power spectrum by “quantising” the spectrum of density fluctuations, and in many cases, by imposing a long wavelength cut-off. An infrared cut-off in the spectrum of density fluctuations translates into a suppression of large angle CMBR fluctuations on the surface of last scatter. This effect has been used to rule out flat [4] and hyperbolic [5] models with toroidal topologies.

Earlier we conjectured [6] that similar negative conclusions would not apply to generic compact hyperbolic models as the majority of the large angle CMBR power in a hyperbolic universe comes not from the surface of last scatter, but from the decay of curvature perturbations along the line of sight. To a lesser extent, the same will be true in the newly popular flat models with a cosmological constant, or in flat models with other forms of exotic dark matter such as tangled string networks [7]. In what follows we shall not only verify our conjecture, but show that finite hyperbolic models actually provide a significantly better fit to the COBE-DMR data than their infinite counterparts.

Our main focus will be on universes with compact hyperbolic spatial sections as these are the most appealing from a theoretical standpoint. However, the growing body of observational evidence favouring a flat universe with a cosmological constant prompts us to reconsider models with three-torus topology. We refer to a model as being “small” if its comoving spatial volume is less than the comoving volume enclosed by the particle horizon (measured in the covering space). The ratio of the horizon volume to the volume of the space exceeds 500 for several of the models we looked at. Small universe models are obtained from the usual FRW models by making identifications between different points in space. These identifications break global isotropy and homogeneity, but do not alter the evolution history. There is one caveat to the last statement: by altering the mode spectrum, the topological identifications will alter the vacuum structure, leading to a Casimir-like vacuum energy that could alter the dynamics. If an effective cosmological constant could be linked to the universe having non-trivial topology, we would have a strong motivation for reconsidering flat models.

On large angular scales, temperature fluctuations in the CMBR are related to the fluctuations in the gauge-invariant gravitational potential $\Phi$ by the Sachs-Wolfe equation

$$
\frac{\delta T(\theta, \phi)}{T} = \frac{1}{3} \Phi(\eta_{re}, r_{als}, \theta, \phi) + 2 \int_{\eta_{als}}^{\eta_0} \Phi(\eta, r, \theta, \phi) \, d\eta.
$$

(1)

Here $\eta$ denotes the conformal time, $\Phi' = \partial_\eta \Phi$, and $\eta_{als}$ and $\eta_0$ denote recombination and the present day respectively. The evolution of the gauge-invariant potential from last scatter until today is described by

$$
\Phi'' + 3H(1 + c_s^2)\Phi' - c_s^2 \nabla^2 \Phi + (2H' + (1 + c_s^2)H)\Phi = 0,
$$

(2)

where $H$ is the conformal Hubble factor and $c_s$ is the sound speed in the cosmological fluid. In a flat matter dominated universe we have $c_s = 0$, $H = 2/\eta$ and (2) tells us that $\Phi' = 0$. Consequently, the second term in (1) vanishes and the temperature fluctuations are all imprinted on large scales when matter and radiation decouple. The presence of either curvature or a cosmological constant alters the time evolution of the expansion rate $H$, leading to a decay of the potential $\Phi$. In these models, the line-of-sight integral (Integrated Sachs-Wolfe, or ISW effect) in (1) can be the dominant source of large scale temperature fluctuations.

The gravitational potential can be expanded in terms of the eigenmodes, $\Psi_q$, of the Laplacian:

$$
\Phi = \sum_q \sum_n \delta_q^n \Psi_q.
$$

(3)

In the above equation $n$ denotes the multiplicity of each eigenmode and $\delta$ denotes the amplitude. The eigenmodes are found as solutions of the equation $\nabla\Psi_q = -q^2 \Psi_q$. The geometry of the space determines the form of the Laplacian operator $\nabla$, while the topology determines the boundary conditions. Though it is a simple exercise to write down the eigenmodes for any of the 10 flat topologies, hyperbolic manifolds have defied description.
The first breakthrough came last year when Inoue found the 14 lowest eigenmodes of the Thurston space [10] using a numerical method [11] originally developed for 2-dimensional manifolds. By refining the method and using a more powerful computer, Inoue extended the count to include the first 36 eigenmodes. These modes have since been used to study the CMBR [12] in a universe with Thurston topology. Using the same numerical method, Aurich [13] has studied the CMBR in a small hyperbolic orbifold with tetrahedral topology. Recently a new, fully automated, algorithm for finding the eigenmodes [14] was discovered and implemented. The list of solutions has grown from 36 to several thousand in the past two weeks. Our cosmological simulations use the first 100+ modes for each of 4 spaces selected from the SnapPea [15] census of compact hyperbolic manifolds. Our selections are: the Weeks space, m003(-3,1), as it is smallest known; the Thurston space, m003(-2,3), in order to compare our results to Inoue’s; and two larger examples, s718(1,1) and v3509(4,3), to see how the size of the space influences the CMBR. In units of the curvature radius cubed, the volumes of our selections are 0.9427, 0.9814, 2.2726 and 6.2392 respectively. To get a feel for how the eigenmodes look, Figure 1 shows the lowest eigenmode of the Weeks space extended across the entire Poincaré ball.

The cosmological simulations are readily performed in spherical coordinates, where the eigenmodes can be expanded:

$$\Psi_q = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{q\ell m} R_{q\ell}(r) Y_{\ell m}(\theta, \phi).$$  \hspace{1cm} (4)

Here the $Y_{\ell m}$’s are spherical harmonics and the radial function $R_{q\ell}$ are either spherical (flat space) or hyperspherical (hyperbolic space) Bessel functions. It is worth emphasising how dramatically different the eigenmodes look in compact flat spaces and compact hyperbolic spaces. In flat space the expansion coefficients $A_{q\ell m}$ are peaked around certain values of $\ell$, and we can always rotate the coordinate system so that $A_{q\ell m} = 0$ for $m \neq 0$. In other words, global isotropy is badly broken. In contrast, the $A_{q\ell m}$’s for hyperbolic eigenmodes are statistically independent of $\ell$ and $m$. The $A_{q\ell m}$’s are gaussian distributed pseudo-random numbers with variance proportional to $1/k^2$ [14]. While it is possible to assign a wavenumber $k = \sqrt{q^2 - 1}$ to hyperbolic eigenmodes, it is impossible to assign a wavevector. They are essentially omni-directional. You do not need inflation to explain why the microwave sky is nearly isotropic in a small hyperbolic universe [2].

Once the eigenmodes have been found, it is a simple though laborious task to evaluate equation (1) using equations (2) and (3). We took the mode amplitudes to be gaussian random numbers with variance $\sqrt{k}/(k^2 + 4)$ (hyperbolic space) or $1/k^{3/2}$ (flat space). These are the standard inflationary power spectra.

Figure 2 shows the angular power spectrum for a universe with density parameter $\Omega_0 = 0.3$ and Weeks topology. Also shown is the angular power spectrum for the corresponding infinite model, along with the COBE-DMR data points [17]. The effect of varying the density parameter is shown in Figure 3, while the effect of varying the volume of the space is shown in Figure 4. Notice that the large volume space v3509(4,3) produces an angular power spectrum very similar to the infinite model shown in Figure 1.

We quantified how well each model fit the COBE-DMR data.
data by modeling the spread in the $\Delta T$'s by smooth probability distributions (the statistics are not gaussian) and performing a standard likelihood analysis. Our results are based on 1000 realizations of each topology for five values of the density parameter. The likelihoods relative to the corresponding infinite model are listed in Table 1, along with the likelihoods of the infinite hyperbolic models relative to a fiducial flat matter dominated universe.

The lower the density, the better the compact models fare relative to their infinite counterparts. The mechanism behind this result can be seen at work in Figure 1. As we decrease the density, the infrared cut-off in the mode spectrum reduces the contribution from the first term in equation (3), while the line of sight contribution from the second term becomes increasingly important. Rather miraculously, the two effects almost precisely cancel out for the low volume models, resulting in a flat or mildly tilted spectrum. On small scales there is no difference in the shape of the angular power spectra for compact and infinite models, but the compact models have a higher COBE-DMR normalization. This helps raise the predicted size of density fluctuations on $8 h^{-1}$ Mpc scales from the low value of $\sigma_8 = 0.6$ for an infinite model, to the larger value of $\sigma_8 = 0.75$ for the Weeks model (both with $\Omega_m = 0.3$). Measurements of the present day cluster abundance [13] favour $\sigma_8 = 0.9 \pm 0.1$ if $\Omega_m = 0.3$. At this stage we should stress that our results primarily affect the fit to the COBE-DMR data. On small scales there is a growing body of observational evidence [14] for an acoustic peak at $\ell \sim 220$, which is consistent with the universe being flat, not hyperbolic. We should also mention that our results disagree with those of Bond et al [20] based on the method of images, and agree with those of Inoue et al [12] based on the finite element method.

| $\Omega_\odot$ | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
|---------------|-----|-----|-----|-----|-----|
| m003(-3,1)    | 20.1| 15.5| 4.2 | 2.9 | 2.7 |
| m003(-2,3)    | 30.3| 25.4| 7.5 | 4.1 | 5.0 |
| s718(1,1)     | 21.5| 8.2 | 3.7 | 2.1 | 1.5 |
| v3509(4,3)    | 12.2| 2.2 | 1.5 | 2.5 | 1.1 |
| infinite      | 0.14| 0.26| 0.52| 0.92| 1.2 |
cal three-torus with side length equal to half the horizon distance has a relative likelihood of 0.26 when $\Omega_\Lambda = 0$ and 0.25 when $\Omega_\Lambda = 0.7$. The real difference between these two models can be seen in Figure 5. Because the ISW effect mixes together different modes sampled at different points, it helps to hide the egregious breaking of global isotropy that lead to the matter dominated versions of these models being ruled out [4].

Returning to the compact hyperbolic models, we want to see if the ISW effect ruins the matched circle test [2] for non-trivial topology. The matched "circles in the sky" occur wherever the surface of last scatter self-intersects. Since the surface of last scatter is a 2-sphere, the intersections occur along circles. We see two copies of each circle of intersection, centered at different points on the sky. The portion of the microwave temperature coming from the surface of last scatter will be identical around each circle. However, the ISW contribution will be uncorrelated.

Taking one realization of the Weeks universe (Figure 6), we find the temperatures around a pair of matched circles (Figure 7a) and see that the match is poor. However, the ISW effect only operates on large angular scales, so we filter Figure 6 to remove all power below $\ell = 21$. The temperature match for the filtered sky is shown in Figure 7b. The correlation coefficient increases from 0.29 to 0.92 after filtering out modes with $\ell \leq 20$. The matched circle pairs will persist until the visible universe is simply connected, which occurs at around $\Omega_\Lambda = 0.95$ for most models in the SnapPea census. If we do live in a small universe, the Microwave Anisotropy Probe will find matched circles in the sky when it starts collecting data in 2001.

This work grew out of earlier collaborations and extensive discussion with Glenn Starkman and Jeff Weeks. Financial support was provided by NASA through their funding of the Microwave Anisotropy Probe satellite mission \url{http://map.gsfc.nasa.gov}.

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