COMBINATORIAL ASPECTS OF Exceptional Sequences ON (RATIONAL) SURFACES

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Abstract. We investigate combinatorial aspects of exceptional sequences in the derived category of coherent sheaves on certain smooth and complete algebraic surfaces. We show that to any such sequence there is canonically associated a complete toric surface whose torus fixpoints are either smooth or cyclic T-singularities (in the sense of Wahl) of type $\frac{1}{kr}(1, kr - 1)$. We also show that any exceptional sequence can be transformed by mutation into an exceptional sequence which consists only of objects of rank one.

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1. Introduction

In this article we want to work out certain combinatorial aspects associated with exceptional sequences in the derived category $D^b(X)$ of coherent sheaves on rational surfaces. In earlier work [HP11] it was found, somewhat surprisingly, that to any exceptional sequence of invertible sheaves on a rational surface there is associated in a canonical way the combinatorial data of a smooth complete toric surface. This finding suggests that in many interesting cases there could be a link between semi-orthogonal decompositions of derived categories and toric geometry. However, so far this is not very well understood, even for the case of line bundles. An important development in this direction is work by Hacking and Prokhorov [HP10] and Hacking [Hac13]. In [HP10], singular surfaces with ample anticanonical divisor and Picard number one which admit $\mathbb{Q}$-Gorenstein smoothings are classified. These surfaces necessarily have $T$-singularities in the sense of Wahl [Wah81]. Among such surfaces, there is one family of weighted projective planes $\mathbb{P}(e^2, f^2, g^2)$ such that $e, f, g$ satisfy the Markov equation $e^2 + f^2 + g^2 = 3efg$.

This classification resembles Rudakov’s interpretation [Rud89] of the classification of exceptional bundles on $\mathbb{P}^2$ by Drezet and Le Potier [DL85]. Rudakov showed that any exceptional sequence $\mathcal{E}, \mathcal{F}, \mathcal{G}$ on $\mathbb{P}^2$ is essentially uniquely determined by the ranks $(e, f, g)$ and the possible ranks correspond to solutions of the Markov equation $e^2 + f^2 + g^2 = 3efg$. In [Hac13], Hacking shows that indeed there exists a

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natural bijective correspondence between degenerations of $\mathbb{P}^2$ and exceptional bundles on $\mathbb{P}^2$. More generally, for a rather large class of surfaces (which includes rational surfaces and certain surfaces of general type) Hacking constructs a correspondence between exceptional vector bundles and Q-Gorenstein degenerations.

In this article, we want to follow some ideas of both [HP11] and [Hae13] and show that the relation between exceptional sequences and toric surfaces with $T$-singularities is a quite general phenomenon. Our main result is the following:

**Theorem (11.3):** Let $X$ be a numerically rational surface and let $E = E_1, \ldots, E_n$ be a numerically exceptional sequence whose length equals $\text{rk} K_{0}^{\text{num}}(X)$ such that $\text{rk} E_i = c_i \neq 0$ for every $i$. Then to this sequence there is associated in a canonical way a complete toric surface $Y(E)$ with $n$ torus fixed points which are either smooth (if $c_i^2 = 1$) or $T$-singularities of type $\frac{1}{k_i}(1, k_i c_i - 1)$, where $\gcd(k_i, c_i) = 1$. Moreover, this correspondence induces a natural isomorphism of Chow rings $\text{CH}^*(Y(E))_{\mathbb{Q}} \rightarrow \text{CH}_{\text{num}}^*(X)_{\mathbb{Q}}$ which maps $K_Y(E)$ to $K_X$.

Here, $K_{0}^{\text{num}}(X)$ denotes the numerical Grothendieck group of $X$ which is defined as the quotient of the Grothendieck group $K_0(X)$ by the null space of the Euler form, and $\text{CH}_{\text{num}}^*(X)$ denotes the $i$-th Chow group modulo numerical equivalence. In particular, $\text{CH}_{\text{num}}^1(X)$ is the Néron-Severi lattice of $X$. The ring structure of $\text{CH}_{\text{num}}^*(X)$ is induced from $\text{CH}^*(X)$.

The term **numerically exceptional** refers to a weaker version of exceptionality and semi-orthogonality which only requires the vanishing of Euler characteristics rather than Hom-vanishing (see Definition 2.2). By $X$ being a numerically rational surface, we mean that $X$ is an algebraic surface whose effective numerical properties are that of a rational surface. More precisely, we define:

**Definition:** Let $X$ be a smooth complete surface defined over a ground field. We call $X$ **numerically rational** if the following hold:

1) $\chi(\mathcal{O}_X) = 1$.

2) $K_X^2 = 12 - \text{rk} K_{0}^{\text{num}}(X)$.

By construction, both $K_{0}^{\text{num}}$ and $\text{CH}_{\text{num}}^*(X)$ are torsion free abelian groups, and the Chern isomorphism $K_0(X)_{\mathbb{Q}} \rightarrow \text{CH}^*(X)_{\mathbb{Q}}$ descends to a ring isomorphism $K_{0}^{\text{num}}(X)_{\mathbb{Q}} \rightarrow \text{CH}_{\text{num}}^*(X)_{\mathbb{Q}}$. So $K_{0}^{\text{num}}(X)$ and $\text{CH}_{\text{num}}^1(X)$ both are finitely generated with $\text{rk} K_{0}^{\text{num}}(X) = \text{rk} \text{CH}_{\text{num}}^1(X) + 2$. The class of numerically rational surfaces in particular includes surfaces of general type with $p_g = q = 0$.

We will always assume that our numerically rational surface admits a **numerically exceptional sequence of maximal length**, i.e. a semi-orthonormal basis of $K_{0}^{\text{num}}(X)$. Note that condition 2) can be dropped in many cases (see Remark 10.12), though it is an open question whether it can be removed entirely.

Another interesting result which will be part of our analysis leading to Theorem 11.3 is the following:

**Theorem (10.14):** Let $X$ be a numerically rational surface. Then any numerically exceptional sequence of maximal length on $X$ can be transformed by mutation into a numerically exceptional sequence consisting only of objects of rank one.

Both theorems are a fortiori apply as well to proper exceptional sequences. The main reason why we formulated them for numerically exceptional sequences is that indeed they are purely a result of the surprisingly rich Riemann-Roch arithmetic. The correspondence between exceptional sequences and toric surfaces therefore does not depend on any geometric construction or any refined geometric properties of $X$. This is of particular interest in light of recent work (e.g. [BGKS15]) where exceptional sequences have been constructed on certain complex surfaces of general type with $p_g = q = 0$. These sequences are almost full — their complements in the derived category are among the first examples of so-called phantom categories. By [Via15] Theorem 3.1 our results are applicable to all surfaces of general type with $p_g = q = 0$ including those of [BGKS15]. Indeed it is an open question whether the existence of a full exceptional sequence on a variety implies that this variety is rational. So far, no example of a non-rational variety which admits such a sequence is known. We believe that our results in conjunction with Hacking’s provide tentative evidence that indeed the existence of a full exceptional sequence implies rationality.

One important aspect of our results is the connection between mutation of exceptional sequences and what we propose to call minimal model program for a class of toric surfaces which includes, but is
strictly bigger than, the class of smooth toric surfaces. We will spend the remainder of this introduction to explain this connection and to relate it to some of the technical results in the paper. Assume that \( E_1, \ldots, E_n \) is an exceptional sequence, where \( n = \text{rk} \ A_\text{num}^1(X) \) and for simplicity, \( 0 \neq e_i : = \text{rk} \ E_i \) for every \( i \). Generalizing an idea of [HPT1], we denote \( A_1 : = c_1(E_{i+1})/c_{i+1} - c_1(E_i)/e_i \in \text{CH}_\text{num}^1(X)_\mathbb{Q} \) for \( 1 \leq i < n \), \( A_n : = c_1(E_1)/e_1 - c_1(E_n)/e_n - K_X \), and \( A \) the \( \mathbb{Z} \)-linear span of \( A_1, \ldots, A_n \) in \( \text{CH}_\text{num}^1(X)_\mathbb{Q} \). Then \( A \) is a free \( \mathbb{Z} \)-module of rank \( n - 2 \) and we have a short exact sequence:

\[
0 \rightarrow \mathbb{Z}^2 \stackrel{L}{\rightarrow} \mathbb{Z}^n \stackrel{c}{\rightarrow} A \rightarrow 0,
\]

where \( c \) is the map which sends the \( i \)-th standard basis vector of \( \mathbb{Z}^n \) to \( A_i \). We can now choose to associate the rows \( l_1, \ldots, l_n \) of \( L \) with the columns of \( c \) (the \( A_i \), disregarding a choice of basis for \( A \)), i.e. we consider \( l_i \in \mathbb{Z}^2 \) as associated to \( A_i \in A \). This association is often called Gale duality. It is elementary to see that Gale duality is complementary with respect to linear dependency, e.g. if some of the \( l_i \) form a basis of \( \mathbb{Z}^2 \) then the complementary \( A_i \) form a basis of \( A \); if some of the \( l_i \) form a minimal linearly dependent set, then the complementary \( A_j \) generate a hyperplane in \( A \), and so on. A substantial part of this paper is devoted to determine from the \( A_i \) and their respective intersection products that the \( l_i \) form a circularly ordered set of primitive lattice vectors which generate the fan of a complete toric surface (Proposition [10.7]) which has at most \( T \)-singularities (Theorem [11.3]). The corresponding fans are characterized by their collection of primitive vectors \( l_1, \ldots, l_n \) and nonzero integers \( e_1, \ldots, e_n \) such that:

1) the determinants of two adjacent lattice vectors are squares: \( \text{det}(l_{i-1}, l_i) = c_i^2 \),
2) the differences \( l_i - l_{i-1} \) have lattice length \( |e_i| \). We think of these difference as segments of the “circumference” of the fan which connects the \( l_i \). In particular, as in the depictions below, one often does not care about the orientation of these segments (see Section 9 however).

Now recall that a complete toric surface \( Y(E) \) has \( n \) torus invariant prime divisors \( D_1, \ldots, D_n \), which form a cycle, i.e. \( D_i \approx \mathbb{P}^1 \) for every \( i \) and \( D_i \cdot D_j = 0 \) whenever \( |i - j| > 1 \) and \( D_i \cdot D_{i+1} = 1/e_{i+1}^2 \in \mathbb{Q} = \text{CH}^2(Y)_{\mathbb{Q}} \) for every \( i \), where the product on \( \text{CH}^*(Y(E))_{\mathbb{Q}} \) is the orbifold intersection product. For our constructions, we have the correspondence \( A_i \leftrightarrow l_i \) via Gale duality, and toric geometry relates \( l_i \leftrightarrow D_i \). The ring isomorphism \( \text{CH}^*(Y(E))_{\mathbb{Q}} \rightarrow \text{CH}^\text{num}_\text{num}^*(X)_\mathbb{Q} \) of Theorem [11.3] is determined by mapping \( D_i \) to \( A_i \) (see Remark [11.4]).

To give an example, consider some Hirzebruch surface \( \mathbb{F}_a \) for some \( a \geq 0 \) and denote \( P, Q \) the primitive integral generators of its nef cone, where \( P^2 = 0 \) is the class of the fiber and \( Q^2 = a \) is the class of the relative ample divisor of the fibration \( \mathbb{F}_a \rightarrow \mathbb{P}^1 \). Then for every \( s \in \mathbb{Z} \),

\[
\mathcal{O}, \mathcal{O}(P), \mathcal{O}((s + 1)P + Q), \mathcal{O}((s + 2)P + Q)
\]

is a full exceptional sequence on \( \mathbb{F}_a \) (see e.g. [HPT1 Proposition 5.2]). According to [HPT1 Theorem 3.5] the toric fan associated to this sequence is the Hirzebruch surface \( \mathbb{F}_b \) where \( b = |a + 2s| \). The associated fan is specified by lattice vectors \( l_1, l_2, l_3, l_4 \), where, if we choose, say, \( l_1, l_2 \) as a basis of \( \mathbb{Z}^2 \) then we have \( l_3 = -l_1 - (a + 2s)l_2 \) and \( l_4 = -l_2 \). Figure 1 shows the fan associated to this toric surface for the case \( a = 3 \) and \( s = 1 \). The numbers, which in this case are always 1, represent the lattice volumes \( \text{det}(l_i, l_{i+1}) \). The circumference is indicated by the dashed line. We can see that the lattice lengths of its segments are always 1. Applying a right mutation to the pair \( \mathcal{O}(P), \mathcal{O}((s + 1)P + Q) \) yields another exceptional sequence

\[
\mathcal{O}, \mathcal{O}((s + 1)P + Q), \mathcal{R}, \mathcal{O}((s + 2)P + Q)
\]

with \( \mathcal{R} = RL((s + 1)P + Q)\mathcal{O}(P) \) and \( r := \text{rk} \mathcal{R} = a + 1 + 2s \). The effect of the mutation to the combinatorial picture is the mutation of \( l_2 \) into \( l'_2 = l_2 - (a + 2 + 2s)l_1 \) (see Proposition 5.3 and Lemma 9.3) and the corresponding toric surface has a cyclic singularity of type \( \frac{1}{2r+1}(1, r(r - 1) - 1) \). Figure 2 shows the case \( a = 3, s = 1 \) with \( r = 6 \), so that in the fan we have created a cone of lattice volume 36 and the corresponding circumference segment of length 6.

In [Orl93], Orlov described how exceptional sequences (and more general semi-orthogonal decompositions) can be completed along blow-ups. Consider for example the exceptional sequence \( \mathcal{T}, \mathcal{O}(2), \mathcal{O}(4) \) on \( \mathbb{P}^2 \), where \( \mathcal{T} \) denotes the tangent bundle, and a blow-up \( b : X \rightarrow \mathbb{P}^2 \) in one point; we denote \( E \) the exceptional curve with \( E^2 = -1 \). Figure 3 shows the fan corresponding to this sequence which describes the weighted projective plane \( \mathbb{P}(1, 1, 4) \). The pull-back of this sequence \( b^*\mathcal{T}, b^*\mathcal{O}(2), b^*\mathcal{O}(4) \) is an exceptional sequence on \( X \) which can be completed to the full sequence \( \mathcal{O}_E(E), b^*\mathcal{T}, b^*\mathcal{O}(2), b^*\mathcal{O}(4) \). The object \( \mathcal{O}_E(E) \) has rank zero which leads in the combinatorial picture to a doubling of a primitive
Figure 1. The fan associated to $\mathcal{O}, \mathcal{O}(P), \mathcal{O}(2P + Q), \mathcal{O}(3P + Q)$ on $\mathbb{P}_3$.

Figure 2. The fan associated to $\mathcal{O}, \mathcal{O}(2P + Q), \mathcal{R}, \mathcal{O}(3P + Q)$ on $\mathbb{P}_3$.

Figure 3. The fan associated to $\mathcal{T}, \mathcal{O}(2), \mathcal{O}(4)$ on $\mathbb{P}_2$.

Figure 4. The fan associated to $\mathcal{O}_E(E), b^*\mathcal{T}, b^*\mathcal{O}(2), b^*\mathcal{O}(4)$ on $X$.

Figure 5. The fan for $b^*\mathcal{T}, R', b^*\mathcal{O}(2), b^*\mathcal{O}(4)$.

Figure 6. The fan for $\mathcal{O}_E(E), b^*\mathcal{O}(2), b^*\mathcal{O}(3), b^*\mathcal{O}(4)$.

vector $(l_3$ in this case, see Figure 4) or, if we like, to the creation of a new cone of volume zero (see Proposition 6.4 and Theorem 4.11). Applying right mutation to the pair $\mathcal{O}_E(E), b^*\mathcal{T}$ yields a sequence $(b^*\mathcal{T}, R', b^*\mathcal{O}(2), b^*\mathcal{O}(4))$ where $\text{rk} R' = -4$. Figure 5 shows the effect of this mutation: one of the multiple lattice vectors "jumps" into the neighbouring cone, thereby subdividing it into two cones of lattice volumes 4 and 16, respectively. The corresponding toric surface therefore has two cyclic $T$-singularities of orders 4 and 16. If instead we apply a right mutation to the pair $b^*\mathcal{T}, b^*\mathcal{O}(2)$, we obtain a sequence isomorphic to $\mathcal{O}_E(E), b^*\mathcal{O}(2), b^*\mathcal{O}(3), b^*\mathcal{O}(4)$. Figure 6 shows that we end up with the fan for $\mathbb{P}_2$, again with $l_3$ doubled. The reader may have noticed that the cyclic enumerations of the $l_i$ in Figure 5 are now
shifted by one position as compared to the characterisation given earlier. For a complete description of the correspondence between an exceptional sequence \( E_1, \ldots, E_n \) that may contain objects of rank zero and lattice vectors \( l_1, \ldots, l_n \) we refer to Sections 5 and 7 in particular compare Example 6.3.

The transition from \( b^* T, R', b^* O(2), b^* O(4) \) to \( O_E(E), b^* O(2), b^* O(3), b^* O(4) \) illustrates for a simple case how the minimal model program works for toric surfaces which are associated to an exceptional sequence on some numerically rational surface (see Example 5.4 for an explicit representation of the associated toric systems). In general, assume we have a numerically exceptional sequence of maximal length \( E = E_1, \ldots, E_n \) on such a surface \( X \) and the corresponding toric surface \( Y = Y(E) \).

1. For every pair \( E_i, E_{i+1} \), the ranks \( r_i, r_{i+1} \) and Euler characteristic \( \chi(E_i, E_{i+1}) \) translate into certain local convexity properties of the fan of \( Y \). These properties and their behaviour under mutation are studied in Section 9.
2. By Lemma 9.1 there is a criterion when we can use these properties in order to reduce the ranks of objects in \( E \) by mutation.
3. Taking global properties into account, we will show that we can use this criterion to transform our sequence by mutation into a sequence \( E' = E_1, \ldots, E_t, F_1, \ldots, F_{n-t} \) where either \( t = n - 3 \) or \( t = n - 4 \), \( r_k Z_i = 0 \) for every \( i \), and \( r_k F_j \neq 0 \) for every \( j \) (Corollaries 9.12 & 10.8). As in above examples, this means that the associated fan \( Y(E') \) is generated by either 3 or 4 lattice vectors, one of which appears with multiplicity \( n - t + 1 \). Note however that in Section 7 we will see that the distribution of multiplicities of these lattice vectors is essentially arbitrary and in Section 9 we decide to accumulate them onto one lattice vector purely for convenience.
4. The cases \( t = n - 3 \) and \( t = n - 4 \) are easily analyzed. In the first case (see 10.2), the triple \( r_k F_1, r_k F_2, r_k F_3 \) is a solution of the Markov equation and, as has already been explored by Rudakov [Rud89], by further mutation we can transform the \( F_j \) into objects of rank \( \pm 1 \). The corresponding toric surface then is \( F^2 \). In the case \( t = n - 4 \) we will have directly \( r_k F_j = \pm 1 \) for every \( j \), in which case the corresponding fan will be that of a Hirzebruch surface (see 10.4).

In summary, the minimal model program for toric surfaces associated to exceptional sequences consists of minimizing lattice volumes via mutation. Occasionally, we may create a cone of volume zero, which then will live on as the multiplicity of some primitive vector (this effect corresponds to the blow-down of a smooth toric surface at a \((-1)\)-divisor). Analogous to blowing down smooth toric surfaces, the process ends when we arrive at a Hirzebruch surface or \( F^2 \). If we neglect multiple lattice vectors, we get the following result.

**Theorem 1.1** (see Corollaries 9.12 & 10.8): Let \( Y \) be a toric surface associated to a numerically exceptional sequence of maximal length on a numerically rational surface. Then it can be transformed via mutation into a Hirzebruch surface or a projective plane.

Note that we have not explicitly formulated the theorem in the body of this article and state it here in order to give a summary of some of the technical statements of this paper and for the reader’s guidance. Also note that, if we translate the usual braid group action on exceptional sequences to a braid group action on toric surfaces with \( T \)-singularities, then the theorem could further be strengthened to the effect that the minimal model program for smooth toric surfaces embeds into this braid group action. However, the classification of toric surfaces with \( T \)-singularities is of broader interest (see e.g. [KNP15] for recent results) and we leave a more detailed treatment to future work.

**Overview.** In Section 2 we introduce some basic notions. The reader will probably avoid some confusion by paying attention to standing conventions as stated in paragraphs 2.1 and 2.9. Section 3 is devoted to the exploitation of the Riemann-Roch formula for exceptional objects. Sections 4 and 7 deal with some crucial aspects which arise if exceptional objects of rank zero are involved. In Sections 5 and 6 toric systems and their Gale dual are introduced. The latter will be analyzed locally in Sections 8 and 9. The global analysis and the main theorems are contained in Sections 10 and 11. Section 12 then concludes with some observations related to \( T \)-singularities. For easier reference, we collect some facts on toric surfaces in an appendix.

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2. SOME GENERALITIES

2.1 (Standing conventions throughout the rest of this paper). We assume that \( X \) is a numerically rational surface as defined in the introduction over some ground field \( K \). We denote \( D^b(\mathcal{X}) \) the bounded derived category of coherent sheaves on \( X \). We will always write objects of \( D^b(\mathcal{X}) \) in calligraphic style, \( \mathcal{E}, \mathcal{F}, \ldots, \mathcal{Z} \). Then their ranks will be denoted in the corresponding lower case letters \( e, f, \ldots, z \). By \( n \) we will always denote the rank of \( K_0^{\text{num}}(\mathcal{X}) \).

For any two objects \( \mathcal{E}, \mathcal{F} \) of \( D^b(\mathcal{X}) \) the Euler characteristic is defined as:

\[
\chi(\mathcal{E}, \mathcal{F}) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \text{Hom}_{D^b(\mathcal{X})}(\mathcal{E}, \mathcal{F}[k]) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \text{Ext}^k_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}).
\]

**Definition 2.2:**

(i) We call an object \( \mathcal{E} \) of \( D^b(\mathcal{X}) \) exceptional if \( \text{End}(\mathcal{E}) \simeq K \) and \( \text{Hom}_{D^b(\mathcal{X})}(\mathcal{E}, \mathcal{E}[k]) = 0 \) for all \( k \neq 0 \). We call \( \mathcal{E} \) numerically exceptional if \( \chi(\mathcal{E}, \mathcal{E}) = 1 \).

(ii) A sequence of objects \( \mathcal{E}_1, \ldots, \mathcal{E}_t \) is called an exceptional sequence if all \( \mathcal{E}_i \) are exceptional and \( \text{Hom}_{D^b(\mathcal{X})}(\mathcal{E}_i, \mathcal{E}_j[k]) = 0 \) for all \( i > j \) and all \( k \in \mathbb{Z} \). Similarly, we call it a numerically exceptional sequence if all \( \mathcal{E}_i \) are numerically exceptional and \( \chi(\mathcal{E}_i, \mathcal{E}_j) = 0 \) for all \( i > j \).

(iii) Denote \( \omega_X = \mathcal{O}(K_X) \) the canonical sheaf on \( X \). Then we can extend any exceptional sequence \( \mathcal{E}_1, \ldots, \mathcal{E}_t \) to an infinite sequence \( \ldots, \mathcal{E}_i, \mathcal{E}_{i+1}, \ldots \) such that \( \mathcal{E}_{i+t} = \mathcal{E}_i \otimes \omega_X^{-1} \) holds for any \( i \in \mathbb{Z} \). We call such a sequence a cyclic exceptional sequence. If the sequence is only numerically exceptional, then we call it cyclic numerically exceptional. For any \( i \in \mathbb{Z} \) we call the subsequence \( \mathcal{E}_{i+1}, \ldots, \mathcal{E}_{i+t} \) a winding.

(iv) An exceptional sequence is called strongly exceptional if \( \text{Hom}_{D^b(\mathcal{X})}(\mathcal{E}_i, \mathcal{E}_j[k]) = 0 \) for all \( i, j \) and all \( k \neq 0 \). A cyclic exceptional sequence is called cyclic strongly exceptional if every winding is strongly exceptional.

(v) Any collection of objects in \( D^b(\mathcal{X}) \) is called full if it generates \( D^b(\mathcal{X}) \).

As general references for exceptional sequences we refer to [Bon90] and [Rud90].

2.3. Note that any sub-interval of length at most \( t \) of a cyclic (numerically) exceptional sequence is a (numerically) exceptional sequence (see [HP11] Proposition 5.1). By convention, if we are given a fixed exceptional sequence \( \mathcal{E}_1, \ldots, \mathcal{E}_t \), then we will always implicitly assume that it is extended cyclically, i.e. we consider \( \mathcal{E}_i \) for any \( i \in \mathbb{Z} \), denoting any element of the original sequence twisted by an appropriate power of \( \omega_X \) as in 2.2 (iii).

2.4. If \( \mathcal{E}_1, \ldots, \mathcal{E}_t \) is a (numerically) exceptional sequence, then so is \( \mathcal{E}_1, \ldots, \mathcal{E}_{i-1}, \mathcal{E}_i[j], \mathcal{E}_{i+1}, \ldots, \mathcal{E}_t \) for any \( i \) and any \( j \in \mathbb{Z} \). So, as long as we are not interested in concrete representations for the \( \mathcal{E}_i \), we have some flexibility in considering exceptional sequences up to shift. For instance, there usually is no loss of generality to assume \( c_i \geq 0 \) for all \( i \). Note that for any object \( \mathcal{E} \) in \( D^b(\mathcal{X}) \) we have \( \chi(\mathcal{E}) = -\chi(\mathcal{E}[1]) \) which implies \( c_1(\mathcal{E}) = -c_1(\mathcal{E}[1]) \) and \( c_2(\mathcal{E}) + c_2(\mathcal{E}[1]) = c_1(\mathcal{E})^2 \).

2.5. For any pair of objects \( \mathcal{E}, \mathcal{F} \) there exist the following two distinguished triangles:

\[
\begin{align*}
L_\mathcal{E} \mathcal{F} & \longrightarrow \text{RHom}(\mathcal{E}, \mathcal{F}) \otimes_K \mathcal{E} \longrightarrow \mathcal{F}, \\
\mathcal{E} & \longrightarrow \text{RHom}(\mathcal{E}, \mathcal{F}^*) \otimes_K \mathcal{F} \longrightarrow R_\mathcal{F} \mathcal{E},
\end{align*}
\]

where \( \text{can} \) in both cases denotes the canonical evaluation map. If \( \mathcal{E}, \mathcal{F} \) form an exceptional pair, then it follows that both \( \mathcal{F}, R_\mathcal{F} \mathcal{E} \) and \( L_\mathcal{E} \mathcal{F}, \mathcal{E} \) form exceptional pairs as well.

**Definition 2.6:** For an exceptional or numerically exceptional pair \( \mathcal{E}, \mathcal{F} \), we call the pairs \( \mathcal{F}, R_\mathcal{F} \mathcal{E} \) and \( L_\mathcal{E} \mathcal{F}, \mathcal{E} \) its right- and left-mutation, respectively.
3.1. For any two objects \( \mathcal{E}, \mathcal{F} \) the following formula holds:
\[
c_1(\mathcal{E}, \mathcal{F}) = ec_1(\mathcal{F}) - fc_1(\mathcal{E}).
\]
This is immediately clear for vector bundles, because in this case \( R\text{Hom}(\mathcal{E}, \mathcal{F}) \simeq \mathcal{E}^* \otimes \mathcal{F} \). The extension to the general case follows from the fact that, because \( X \) is smooth, any object in \( D^b(X) \) is quasi-isomorphic to a finite complex of vector bundles. In the case \( \mathcal{E} \) and \( \mathcal{F} \) have nonzero rank it follows that
\[
s(\mathcal{E}, \mathcal{F}) = s(R\text{Hom}(\mathcal{E}, \mathcal{F})).
\]
3.2 (Riemann-Roch formula). For any $\mathcal{E}, \mathcal{F}$ in $D^b(X)$, the Riemann-Roch formula is:

$$\chi(\mathcal{E}, \mathcal{F}) = ef - \frac{1}{2} K_X c_1(\mathcal{E}, \mathcal{F}) + \frac{1}{2} (f c_1(\mathcal{E})^2 + ec_1(\mathcal{F})^2 - 2c_1(\mathcal{E})c_1(\mathcal{F})) - (fc_2(\mathcal{E}) + ec_2(\mathcal{F})).$$

We now collect some identities which we obtain from simple inspection of the Riemann-Roch formula.

3.3. Let $\mathcal{E}$ be any object in $D^b(X)$.

(i) If $e = 0$ then $\mathcal{E}$ is numerically exceptional iff $c_1(\mathcal{E})^2 = -1$.

(ii) If $e \neq 0$ then $\mathcal{E}$ is numerically exceptional iff

$$c_2(\mathcal{E}) = \frac{1}{2e}(e^2 + (e-1)c_1(\mathcal{E})^2 - 1).$$

3.4. Now for objects $\mathcal{E}, \mathcal{F}$ with $\chi(\mathcal{E}, \mathcal{F}) = \chi(\mathcal{F}, \mathcal{F}) = 1$, we can use $\boxtimes$ to simplify the Riemann-Roch formula:

(i) If $e, f \neq 0$, then:

$$\chi(\mathcal{E}, \mathcal{F}) = -\frac{1}{2} K_X c_1(\mathcal{E}, \mathcal{F}) + \frac{1}{2ef} (c_1(\mathcal{E}, \mathcal{F})^2 + e^2 + f^2).$$

(ii) If $e = 0$ and $f \neq 0$, then:

$$\chi(\mathcal{E}, \mathcal{F}) = \frac{f}{2} K_X c_1(\mathcal{E}) - \left(\frac{f}{2} + c_1(\mathcal{E})c_1(\mathcal{F}) + fc_2(\mathcal{E})\right).$$

(iii) If $e \neq 0$ and $f = 0$, then:

$$\chi(\mathcal{E}, \mathcal{F}) = -\frac{e}{2} K_X c_1(\mathcal{F}) - \left(\frac{e}{2} + c_1(\mathcal{E})c_1(\mathcal{F}) + ec_2(\mathcal{F})\right).$$

(iv) If $e = f = 0$, then:

$$\chi(\mathcal{E}, \mathcal{F}) = \chi(\mathcal{F}, \mathcal{E}) = -c_1(\mathcal{E})c_1(\mathcal{F}).$$

3.5. Anti-symmetrizing of the Euler form yields for any two objects $\mathcal{E}, \mathcal{F}$:

(i) $\chi(\mathcal{E}, \mathcal{F}) - \chi(\mathcal{F}, \mathcal{E}) = -K_X c_1(\mathcal{E}, \mathcal{F})$.

If $\mathcal{E}$ and $\mathcal{F}$ are numerically exceptional, we moreover get by symmetrizing the Euler form:

(ii) If $e, f \neq 0$, then $\chi(\mathcal{E}, \mathcal{F}) + \chi(\mathcal{F}, \mathcal{E}) = -\frac{1}{2}(c_1(\mathcal{E}, \mathcal{F})^2 + e^2 + f^2)$.

(iii) If $e = 0$ and $f \neq 0$, then $\chi(\mathcal{E}, \mathcal{F}) + \chi(\mathcal{F}, \mathcal{E}) = -(f + 2c_1(\mathcal{E})c_1(\mathcal{F}) + 2fc_2(\mathcal{E}))$.

(iv) If $e \neq 0$ and $f = 0$, then $\chi(\mathcal{E}, \mathcal{F}) + \chi(\mathcal{F}, \mathcal{E}) = -(e + 2c_1(\mathcal{E})c_1(\mathcal{F}) + 2ec_2(\mathcal{F}))$.

(v) If $e = f = 0$, then $\chi(\mathcal{E}, \mathcal{F}) + \chi(\mathcal{F}, \mathcal{E}) = -2c_1(\mathcal{E})c_1(\mathcal{F})$.

In the case that $\mathcal{E}$ and $\mathcal{F}$ are numerically exceptional and $\chi(\mathcal{F}, \mathcal{E}) = 0$, the formulas $\boxtimes$ yield particularly nice identities for the Euler characteristic $\chi(\mathcal{E}, \mathcal{F})$:

(vi)

$$\chi(\mathcal{E}, \mathcal{F}) = -K_X c_1(\mathcal{E}, \mathcal{F}) = \begin{cases} \frac{1}{2}(c_1(\mathcal{E}, \mathcal{F})^2 + e^2 + f^2) & \text{if } e, f \neq 0, \\ -(f + 2c_1(\mathcal{E})c_1(\mathcal{F}) + 2fc_2(\mathcal{E})) & \text{if } e = 0, f \neq 0, \\ -(e + 2c_1(\mathcal{E})c_1(\mathcal{F}) + 2ec_2(\mathcal{F})) & \text{if } e \neq 0, f = 0, \\ -2c_1(\mathcal{E})c_1(\mathcal{F}) & \text{if } e = f = 0. \end{cases}$$

Lemma 3.6: The pair $\mathcal{E}, \mathcal{F}$ is numerically exceptional iff $\mathcal{F}, \mathcal{E} \otimes \omega_X^{-1}$ is and properly exceptional iff $\mathcal{F}, \mathcal{E} \otimes \omega_X^{-1}$ is. Moreover, the following equality holds:

$$\chi(\mathcal{E}, \mathcal{F}) + \chi(\mathcal{F}, \mathcal{E} \otimes \omega_X^{-1}) = ef K_X^2$$

Proof. The first two assertions follow from Serre duality. For the last assertion, we use $\boxtimes$ and $c_1(\mathcal{E}, \mathcal{F} \otimes \omega_X^{-1}) = -c_1(\mathcal{E}, \mathcal{F}) - ef K_X$. \hfill $\square$

3.7. From $\boxtimes$ for any three objects $\mathcal{E}, \mathcal{F}, \mathcal{G}$ we get:

$$fc_1(\mathcal{E}, \mathcal{G}) = gc_1(\mathcal{E}, \mathcal{F}) + ec_1(\mathcal{F}, \mathcal{G}).$$
If moreover these objects form a numerically exceptional triple, we can multiply both sides of this equation with \(-K_X\) and then with \(\mathbb{E}, \mathbb{F}, \mathbb{G}\) the equality extends to the Euler characteristic:

\[
f \chi(\mathbb{E}, \mathbb{G}) = g \chi(\mathbb{E}, \mathbb{F}) + e \chi(\mathbb{F}, \mathbb{G}).
\]

**Proposition 3.8:** Let \(\mathbb{E}, \mathbb{F}, \mathbb{G}\) in \(D^b(X)\) with \(e, f, g \neq 0\) such that \(\mathbb{E}, \mathbb{F}\) and \(\mathbb{F}, \mathbb{G}\) form numerically exceptional pairs. Then \(\mathbb{E}, \mathbb{F}, \mathbb{G}\) forms a numerically exceptional triple (i.e. \(\chi(\mathbb{G}, \mathbb{E}) = 0\)) if and only if \(c_1(\mathbb{E}, \mathbb{F}) \cdot c_1(\mathbb{F}, \mathbb{G}) = eg\).

**Proof.** By equation \(\mathbb{E}, \mathbb{F}, \mathbb{G}\) we have

\[
\chi(\mathbb{G}, \mathbb{E}) = \frac{-K_X}{2} c_1(\mathbb{G}, \mathbb{E}) + \frac{1}{2eg} (c_1(\mathbb{G}, \mathbb{E})^2 + e^2 + g^2).
\]

Using \(\mathbb{E}, \mathbb{F}, \mathbb{G}\) and \(\mathbb{F}, \mathbb{G}\) we get:

\[
\chi(\mathbb{G}, \mathbb{E}) = \frac{K_X}{2f} (c_1(\mathbb{G}, \mathbb{F}) + gc_1(\mathbb{F}, \mathbb{E})) + \frac{1}{2efg} (e^2c_1(\mathbb{G}, \mathbb{F})^2 + g^2c_1(\mathbb{F}, \mathbb{E})^2 + 2egc_1(\mathbb{G}, \mathbb{F}) \cdot c_1(\mathbb{F}, \mathbb{E})) + \frac{e^2 + g^2}{2eg} =
\]

\[
\frac{e}{f} (\chi(\mathbb{G}, \mathbb{F}) - \frac{f^2}{2fg}g) + \frac{g}{f} (\chi(\mathbb{F}, \mathbb{E}) - \frac{e^2 + f^2}{2ef}) + \frac{1}{f} c_1(\mathbb{G}, \mathbb{F}) \cdot c_1(\mathbb{F}, \mathbb{E}) + \frac{e^2 + g^2}{2eg} =
\]

\[
\frac{1}{f} (-eg + c_1(\mathbb{G}, \mathbb{F}) \cdot c_1(\mathbb{F}, \mathbb{E})).
\]

Hence, we get \(\chi(\mathbb{G}, \mathbb{E}) = 0\) iff \(c_1(\mathbb{G}, \mathbb{F}) \cdot c_1(\mathbb{F}, \mathbb{E}) = c_1(\mathbb{E}, \mathbb{F}) \cdot c_1(\mathbb{G}, \mathbb{G}) = eg\). \(\square\)

**Remark:** We point out that for the case that \(\mathbb{E}, \mathbb{F}\) and \(\mathbb{F}, \mathbb{G}\) are proper exceptional pairs, Proposition 3.8 yields only a necessary, but not sufficient criterion for \(\mathbb{E}, \mathbb{F}, \mathbb{G}\) to form a proper exceptional triple.

**3.9.** For a numerically exceptional triple \(\mathbb{E}, \mathbb{F}, \mathbb{G}\), the Chern classes and Euler characteristic transform for right mutation as follows:

\[
c_1(\mathbb{F}, R_\mathbb{F}\mathbb{E}) = c_1(\mathbb{E}, \mathbb{F}), \quad c_1(\mathbb{R}_\mathbb{F}\mathbb{E}, \mathbb{G}) = \chi(\mathbb{E}, \mathbb{F})c_1(\mathbb{F}, \mathbb{G}) - c_1(\mathbb{E}, \mathbb{G})
\]

\[
\chi(\mathbb{F}, R_\mathbb{F}\mathbb{E}) = \chi(\mathbb{E}, \mathbb{F}), \quad \chi(\mathbb{R}_\mathbb{F}\mathbb{E}, \mathbb{G}) = \chi(\mathbb{E}, \mathbb{F})\chi(\mathbb{F}, \mathbb{G}) - \chi(\mathbb{E}, \mathbb{G}).
\]

Similarly, for left mutation, we get:

\[
c_1(\mathbb{L}_\mathbb{F}\mathbb{G}, \mathbb{F}) = c_1(\mathbb{F}, \mathbb{G}), \quad c_1(\mathbb{E}, L_\mathbb{F}\mathbb{G}) = \chi(\mathbb{F}, \mathbb{G})c_1(\mathbb{E}, \mathbb{F}) - c_1(\mathbb{E}, \mathbb{G})
\]

\[
\chi(\mathbb{L}_\mathbb{F}\mathbb{G}, \mathbb{F}) = \chi(\mathbb{F}, \mathbb{G}), \quad \chi(\mathbb{E}, L_\mathbb{F}\mathbb{G}) = \chi(\mathbb{F}, \mathbb{G})\chi(\mathbb{E}, \mathbb{F}) - \chi(\mathbb{E}, \mathbb{G}).
\]

If \(e, f \neq 0\), we have by \(\mathbb{E}, \mathbb{F}, \mathbb{G}\) that \(\chi(\mathbb{E}, \mathbb{F}) = \frac{1}{ef}(a + e^2 + f^2)\) where \(a = c_1(\mathbb{E}, \mathbb{F})^2\), and the rank formulas of \(\mathbb{E}, \mathbb{F}\) specialize as follows:

\[
\text{rk } L_\mathbb{F}\mathbb{E} = \frac{a + e^2}{f} \quad \text{and} \quad \text{rk } R_\mathbb{F}\mathbb{E} = \frac{a + f^2}{e}.
\]

**4. Exceptional Sequences Containing Objects of Rank Zero**

As we have seen in paragraphs \(\mathbb{E}, \mathbb{F}, \mathbb{G}\) the Riemann-Roch formula does not lead to a uniform treatment of exceptional objects of rank zero as it does for objects of nonzero rank. Indeed, one should think of such objects as associated to exceptional divisors of blow-ups, as indicated by a classical construction due to Orlov.

**Example 4.1:** Let \(b: \tilde{X} \to X\) be a blow-up of a point with exceptional divisor \(E\) and let \(\mathbb{E}_1, \ldots, \mathbb{E}_n\) be a full exceptional sequence on \(X\). Then by \(\mathbb{E}, \mathbb{E}_1, \ldots, \mathbb{E}_n\) is a full exceptional sequence on \(\tilde{X}\). Clearly, \(c_2(\mathbb{O}_E(E)) = 0\) and \(c_1(\mathbb{O}_E(E))c_1(\mathbb{E}_i) = 0\) for all \(i\). It follows from \(\mathbb{E}, \mathbb{F}\) that \(\chi(\mathbb{O}_E(E), \mathbb{L}^b\mathbb{E}_i) = -c_i\) for every \(i\).

**4.2.** A distinctive feature of this example is that the first Chern class of \(\mathbb{O}_E(E)\) is orthogonal to the first Chern classes of the rest of the sequence and that its second Chern class is zero. We will see (Theorem 5.11) that, possibly after twisting with a line bundle, this is always true for rank zero objects in an exceptional sequence of maximal length. However, as we can see by formulas \(\mathbb{E}, \mathbb{F}\) this is not an
immediate consequence of the Riemann-Roch formula and we have not yet developed enough machinery to prove this fact. In this section we will describe some general features of exceptional objects of rank zero and their semi-orthogonal complements sufficient to motivate and state Theorem 4.11 but we will only be able to prove the theorem in Section 10.

Let $E_1, \ldots, E_n$ be an exceptional sequence and denote $Z_1, \ldots, Z_t$ the maximal sub-sequence consisting of objects of rank zero. The following lemma shows that the $Z_i$ can never represent a semi-orthogonal basis of $K_0^\num(X)$.

**Lemma 4.3:** Under above assumptions we have $t \leq n - 3$.

**Proof.** By Lemmas 3.3 [vii] and 3.5 [vii], the Chern classes $c_1(Z_i)$ form an orthogonal system of vectors of length $-1$ in $\CH_1^\num(X)$. Then $t \leq n - 3 = \rk \CH_1^\num(X) - 1$ by the Hodge index theorem. □

Given a divisor $D$, we can consider the twisted sequence $E_1(D), \ldots, E_n(D)$. For the sub-sequence of the $Z_i$, we observe the following:

**Lemma 4.4:** Let $m_1, \ldots, m_t$ be any integers. Then with above notation, there exists a divisor $D$ such that $c_2(Z_i(D)) = m_i$ for every $1 \leq i \leq t$.

**Proof.** By the multiplicative property of the Chern character we have $ch(Z_i(D)) = ch(Z_i) \cdot ch(O(D))$ for every $i$. From this we compute that $c_2(Z_i(D)) = c_2(Z_i) - c_1(Z_i) D$. As we have observed in the proof of Lemma 4.3, the $c_1(Z_i)$ form an orthogonal set of vectors of length $-1$ with respect to the intersection form. Hence, the divisor $D = - \sum_{i=1}^t (c_2(Z_i) - m_i)c_1(Z_i)$ satisfies $c_1(Z_i) D = c_2(Z_i) - m_i$ for all $i$. □

Now consider any exceptional object $Z$ of rank zero. We want to describe the relative configurations of the left- and right-orthogonal complements in $K_0^\num(X)$.

4.5. Both $\chi(-, Z)$ and $\chi(Z, -)$ induce linear forms on $K_0^\num(X)$ and with respect to these forms we denote $L$ and $R$ the left- and right-orthogonal complements of $Z$ in $K_0^\num(X)$, respectively. By the integrality of the Euler form and the fact that $\chi(Z, Z) = 1$ it follows that $L$ and $Z$ (respectively, $R$ and $Z$) generate $K_0^\num(X)$. Furthermore, because the first Chern class is additive on complexes, we obtain another linear form on $K_0^\num(X)$ which is given as

$$\phi_Z : K_0^\num(X) \rightarrow \mathbb{Z}, \quad E \mapsto c_1(Z)c_1(E).$$

We denote its orthogonal complement in $K_0(X)$ by $C$. Note that because $Z$ is primitive in $K_0^\num(X)$ and $\chi(Z, Z) = 1 = -c_1(Z)^2$ it follows that all these forms are integral and primitive in $K_0^\num(X)^*$. By a result of Thomason [Tho97, Theorem 2.1], the subgroups of $K_0(X)$ correspond precisely to the full dense triangulated subcategories of $D^b(X)$. In particular, we denote by $\mathcal{L}$ and $\mathcal{R}$ those subcategories which correspond to the preimages of $L$ and $R$ in $K_0(X)$, respectively. For any object $E$ of $\mathcal{L}$ we have by 3.5 [ii], [v]:

$$\chi(Z, E) = \begin{cases} 0 & \text{if } e = 0, \\ -e(1 + 2c_1(Z)s(E) + 2c_2(Z)) & \text{otherwise.} \end{cases}$$

Similarly, for any object $F$ of $\mathcal{R}$ we have:

$$\chi(F, Z) = \begin{cases} 0 & \text{if } f = 0, \\ -f(1 + 2c_1(Z)s(F) + 2c_2(Z)) & \text{otherwise.} \end{cases}$$

**Lemma 4.6:** Let $E, E' \in \mathcal{L}$ and $F, F' \in \mathcal{R}$. Then with above notation, we have $c_1(Z)s(E) = c_1(Z)s(E')$ and $c_1(Z)s(F) = c_1(Z)s(F')$ whenever $e, e', f, f' \neq 0$.

**Proof.** We have $c_1(Z, E) = -ec_1(Z)$ for any $E$ in $\mathcal{L}$ and hence by Lemma 3.5 [ii], we get $e\chi(Z, E') = e'\chi(Z, E)$. Then the assertion follows from 3.5 [ii]. The statement for $F$ and $F'$ follows analogously. □

**Definition 4.7:** With above notation, for some $E \in \mathcal{L}$, $F \in \mathcal{R}$ with $e, f \neq 0$, we denote

$$\delta_Z := c_1(Z)s(E) + c_2(Z),$$

$$\varepsilon_Z := c_1(Z)s(F) + c_2(Z).$$

By Lemma 4.6 $\delta_Z$ and $\varepsilon_Z$ are independent of the choice of $E$ and $F$. 


Lemma 4.8:  (i) We have $K_Xc_1(Z) = -(1 + 2\delta_Z) = 1 + 2\varepsilon_Z$ and therefore $\delta_Z + \varepsilon_Z + 1 = 0$.
(ii) The restriction of $\chi(Z, -)$ to $L$ coincides with $-(1 + 2\delta_Z)$ times the rank function.
(iii) The restriction of $\chi(-, Z)$ to $R$ coincides with $-(1 + 2\varepsilon_Z)$ times the rank function.
(iv) The following formulas hold for any object $G$ of $D^b(X)$:
$$
\chi(Z, G) + c_1(G)c_1(Z) + gc_2(Z) = -g(1 + \delta_Z) = gc_Z, \\
\chi(G, Z) + c_1(G)c_1(Z) + gc_2(Z) = g\delta_Z = -g(1 + \varepsilon_Z).
$$

In particular, by taking a rank one object for $G$ we see that both $\delta_Z$ and $\varepsilon_Z$ are integers.

Proof. [ii] For any $E \in \mathcal{E}$, $F \in \mathcal{R}$ with $e, f \neq 0$ follows from [iv] that $\chi(Z, E) = eK_Xc_1(Z) = -e(1 + 2\delta_Z)$ and $\chi(F, Z) = fK_Xc_1(Z) = -f(1 + 2\varepsilon_Z)$.

[iii] For any $E \in \mathcal{E}$, we have $\chi(Z, E) = -e(1 + 2\delta_Z)$ by [iv]. We conclude as in (ii).

(iv) From (3.2) and (i) we get immediately
$$
\chi(Z, G) + c_1(G)c_1(Z) + gc_2(G) = \frac{1}{2}K_Xc_1(Z) - \frac{1}{2} = -g(1 + \delta_Z)
$$
and
$$
\chi(G, Z) + c_1(G)c_1(Z) + gc_2(G) = -\frac{1}{2}K_Xc_1(Z) - \frac{1}{2} = g\delta_Z = -g(1 + \varepsilon_Z).$$

4.9. Let $G$ be an object in $\mathcal{E} \cap \mathcal{R}$, then the left hand sides of both equations in Lemma 4.8 coincide and, by the integrality of $\delta_Z$, the right hand sides can only be equal if $g = 0$. Then from the existence of objects of nonzero rank it follows that $L \cap \mathcal{R}$ and $[Z]$ cannot generate $K_{\mathcal{mum}}^m(X)$. Moreover, both $L$ and $R$ are saturated sublattices of corank 1 in $K_{\mathcal{mum}}^m(X)$. It follows that $L \neq R$ and $L \cap \mathcal{R}$ is a saturated sublattice of corank two in $K_{\mathcal{mum}}^m(X)$ consisting of objects of rank zero. Furthermore, it follows that the linear forms $\chi(-, Z)$ and $\chi(Z, -)$ are linearly independent.

If we denote $L := K_{\mathcal{mum}}^m(X)/L \cap \mathcal{R} \cong \mathbb{Z}^2$ it follows immediately that $\chi(-, Z)$ and $\chi(Z, -)$ descend to linearly independent linear forms on $L$. Moreover, as $\text{rk}(L \cap \mathcal{R}) = 0$, the rank function descends as well, as does $\phi_Z$ by Lemma 4.3 [iv].

Another consequence of Lemma 4.8 [iv] is that the intersection products $c_1(Z)s(E)$ and $c_1(Z)s(F)$ are integral for $E \in \mathcal{E}$ and $F \in \mathcal{R}$, respectively. So, noting that $c_1(Z(D)) = c_1(Z)$, $\delta_{Z(D)} = \delta_Z$, and $\varepsilon_{Z(D)} = \varepsilon_Z$ for any divisor $D$, we can choose by Lemma 4.4 a divisor $D$ such that either $L = \ker \phi_{Z(D)}$ or $R = \ker \phi_{Z(D)}$. So, up to twist by an invertible sheaf we may assume without loss of generality that, say, $\chi(-, Z) = -\phi_Z$ and, in particular, $O \in \mathcal{E}$, $\omega \in \mathcal{R}$. Figure 7 shows the configurations of $L$ and $R$ in $L$ for the case $\delta_Z = c_2(Z) = 2$.

Figure 7. The configuration of $L$ and $R$ in $L$ for $\delta_Z = 2$.

Here, $Z$ and $O$ represent a basis of $L$ whose dual is naturally given by $\chi(-, Z) = -\phi_Z$ and $\text{rk}$. In $L$, the classes of rank one objects have coordinates $O + kZ$ which can be represented by line bundles $O(kc_1(Z))$. In particular, $\omega \sim O((1 + 2\delta_Z)c_1(Z))$. With the relation $\delta_Z + \varepsilon_Z + 1 = 0$ and $\delta_Z = c_2(Z)$ we moreover observe with [iv]
$$
\varepsilon_Z = \delta_{Z[1]} \text{ and } \delta_Z = \varepsilon_{Z[1]}.
$$

4.10. We are interested in the particular situation, where $Z$ is part of an exceptional sequence of the form, say, $Z, E_2, \ldots, E_n$ where we can assume without loss of generality that $c_1(Z)c_1(E_i) = 0$ for any $i$. Then the classes of $E_2, \ldots, E_n$ in $K_{\mathcal{mum}}^m(X)$ form a semiorthogonal basis of $L$ and $E_2 \otimes \omega, \ldots, E_n \otimes \omega$ is a semiorthogonal basis of $R$. Alternatively, by $n - 1$ left mutations we can move $Z$ to the rightmost end of a sequence $F_1, \ldots, F_{n-1}, Z$ and obtain another semiorthogonal basis $F_1, \ldots, F_{n-1}$ of $R$. 
In any case, \( \delta_Z = c_2(Z) \) represents the “spread” of \( L \) and \( R \) in \( L \). In Example 4.11 we have seen that \( \delta_{\pi(E)} = 0 \) holds and therefore \( L \) and \( R \) generate \( K_{\text{num}}^0(X) \). It turns out that this indeed is a general feature of exceptional sequences, as the following theorem shows, which, however, we cannot yet prove.

**Theorem 4.11:** Let \( Z \) be an exceptional object of rank zero which can be included in an exceptional sequence \( Z, E_2, \ldots, E_n \). Then \( \delta_Z \in \{0, -1\} \).

The proof will be postponed until Section 10. Until then we will have to take a defect \( \delta_Z \) into account whenever an exceptional rank zero object \( Z \) is part of our exceptional sequence. However, once Theorem 4.11 is established, the following corollary shows that we can essentially forget about them.

**Corollary 4.12:**
(i) If \( \delta_Z = 0 \) then \( \chi(Z[1], -) \) coincides with the rank function on \( L \). If \( \delta_Z = -1 \) then \( \chi(\cdot, Z[1]) \) coincides with the rank function on \( R \).
(ii) If \( Z, E_2, \ldots, E_n \) is an exceptional sequence then so is \( Z[e](D), E_2(D), \ldots, E_n(D) \) for any divisor \( D \) and \( e \in \mathbb{Z} \). In particular, we can choose \( D \) and \( e \in \{0, 1\} \) such that

\[
\delta_Z[e](D) = 0 \quad \text{and} \quad -c_1(Z[e](D))K_X = 1.
\]

5. Toric Systems

In [HP11], exceptional sequences of invertible sheaves \( \mathcal{O}(D_1), \ldots, \mathcal{O}(D_n) \) have been considered. For such sequences, so-called toric systems have been introduced, which represent a normal form for such sequences. More precisely, a toric system is simply given by forming the differences \( A_i := D_{i+1} - D_i \) for all \( 1 \leq i < n \) and \( A_n := D_1 - D_n - K_X \). Such a toric system satisfies the following equations:

(i) \( A_i \cdot A_{i+1} = 1 \) for all \( i \),
(ii) \( A_i \cdot A_j = 0 \) if \( i \neq j \) and \( \{i, j\} \neq \{k, k+1\} \) for any \( 1 \leq k \leq n \),
(iii) \( \sum_{i=1}^n A_i = -K_X \).

In [HP11] the peculiar fact was observed that a toric system is equivalent to the data of a smooth complete toric surface, which this way becomes a combinatorial invariant of an exceptional sequence of invertible sheaves.

In this section we will extend the notion of toric systems to the case of general exceptional sequences. This generalization will be straightforward for the most part, with two notable differences:

1. It is necessary to pass to rational Chern classes, i.e. to an exceptional sequence we will associate elements \( A_i \) in a similar fashion, but they are now constructed as elements of \( \text{CH}^2_{\text{num}}(X)_\mathbb{Q} \).
2. Objects of rank zero cannot be treated uniformly together with objects of nonzero rank.

5.1. We start with an exceptional sequence \( E_1, \ldots, E_t \), where \( e_i \neq 0 \) for all \( i \), which we assume extended to a cyclic exceptional sequence. This in particular implies that \( c_1(E_i, E_{i+1}) = c_1(E_{i+1}, E_{i+1+t}) \) for all \( i \). For \( t > 2 \) the following are straightforward consequences of [5.1] Proposition 5.8 Serre duality, and the Riemann-Roch formula:

(i) \( c_1(E_{i-1}, E_i) \cdot c_1(E_i, E_{i+1}) = e_{i-1}e_{i+1} \) for every \( i \in \mathbb{Z} \).
(ii) \( c_1(E_{i-1}, E_i) = c_1(E_{i-1}, E_{i-1}) = 0 \) for \( 1 < |i-j| < t-1 \).

5.2. Observe that for any three objects \( \mathcal{E}, F, G \) we have \( s(\mathcal{E}, F) + s(\mathcal{F}, \mathcal{G}) = s(\mathcal{E}, \mathcal{G}) \) by [5.7] The intersection product extends in a natural way to a \( \mathbb{Q} \)-valued bilinear form on \( \text{CH}_{\text{num}}^2(X)_\mathbb{Q} \), so that we can reformulate the equalities of 5.1 as follows:

(i) \( s(E_{i-1}, E_i) \cdot s(E_i, E_{i+1}) = 1/e_i^2 \),
(ii) \( s(E_{i-1}, E_i) \cdot s(E_{i-1}, E_{i-1}) = 0 \) for \( 1 < |i-j| < t-1 \).

Moreover, by \( s(E_i, E_{i+1}) = s(E_i, E_1 \otimes \omega^{-1}) = s(E_i, E_1) - K_X \), we have:

(iii) \( \sum_{i=1}^t s(E_i, E_{i+1}) = -K_X \).

5.3. Assume that we have an exceptional sequence \( E_1, \ldots, E_t \) and assume that one of the \( E_i \) has rank zero. By choosing the appropriate winding in the cyclic sequence, we can always assume without loss of generality that we have \( E_i \simeq \mathcal{Z} \) with \( z = 0 \). Then for any pair \( E_i, E_j \) with \( e_i, e_j \neq 0 \) we have by Lemma 4.6 that \( c_1(\mathcal{Z})s(E_i, E_j) = 0 \) if \( 1 < i < j \leq t \). If \( 1 < j < i \leq t \) and \( i-j < t \), then \( E_i = E_{i+t} \otimes \omega \) and therefore \( s(E_i, E_1) = s(E_{i+t}, E_1) - K_X \), hence \( c_1(\mathcal{Z})s(E_i, E_1) = -c_1(\mathcal{Z})K_X = -(1 + 2\delta_Z) \) for some undetermined integer \( \delta_Z \) by Lemma 4.8 [6].
5.4. Now consider an arbitrary exceptional sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$. Then we can partition $\{1, \ldots, n\} = I \sqcup J$, where $I = \{i_1 < \cdots < i_t\}$, $J = \{j_1 < \cdots < j_{n-t}\}$, and such that $e_i = 0$ iff $i \in I$. Then we set:

$$E_k := c_j(\mathcal{E}_{i_k}) \quad \text{for} \quad 1 \leq k \leq t,$$
$$A_k := s(\mathcal{E}_{j_k}, \mathcal{E}_{j_{k+1}}) \quad \text{for} \quad 1 \leq k < n-t,$$
$$A_{n-t} := s(\mathcal{E}_{j_{n-t}}, \mathcal{E}_{j_1} \otimes \omega^{-1})$$

Clearly, the $A_k$ satisfy the conditions listed in [5.2] and it follows from from [3.3], [3.5] and [5.3] that $E_2^2 = -1$ and there exist integers $\delta_1, \ldots, \delta_t$ such that $-K_X E_i = 1 + 2\delta_i$ for all $i \in I$, and $E_i \cdot E_k = 0$ for all $i \neq k$. Moreover, by [5.3] there exists for every $i \in I$ precisely one $j \in J$ such that $E_i \cdot A_j \neq 0$.

For easier notation, we give a formal definition for above data.

**Definition 5.5:** An abstract toric system on $X$ is given by the following data:

1. For $0 \leq t \leq n-3$ a collection of integral divisor classes $E_1, \ldots, E_t$ and integers $\delta_1, \ldots, \delta_t$ with $E_i^2 = 1$ and $-K_X E_i = 1 + 2\delta_i$ for all $i \in I$, and $E_i \cdot E_k = 0$ for all $i \neq k$.
2. A sequence of ranks $r_1, \ldots, r_{n-t} \in \mathbb{Z} \setminus \{0\}$ with $\gcd\{r_1, \ldots, r_{n-1}\} = 1$.
3. A sequence of $\mathbb{Q}$-divisor classes $A_1, \ldots, A_{n-t} \in CH^1_{num}(X)_{\mathbb{Q}}$ such that
   - $r_i \cdot A_{k+1}(A_1 + \cdots + A_{k+1})$ is integral for every $i$ and every $0 \leq k < n-t$,
   - $A_1 \cdot A_{i+1} = \frac{1}{r_{i+1}}$ for every $i$,
   - $A_i \cdot A_j = 0$ if $i \neq j$ and $\{i, j\} \neq \{k, k+1\}$ for any $1 \leq k \leq t$,
   - $\sum_{i=1}^{n-t} A_i = -K_X$.
4. A function $\phi : \{1, \ldots, t\} \to \{1, \ldots, n-t\}$ such that $E_i \cdot A_j \neq 0$ if and only if $j = \phi(i)$ (and thus $E_i \cdot A_{\phi(i)} = 1 + 2\delta_i$ by (1) and (5.3)).

Note that the indices of the $r_i$ and $A_i$ are to be read cyclically; in particular, we have $A_{n-t} \cdot A_1 = 1/r_1^2$. Also note that by Lemma 4.3 we can make the implicit assumption that $n-t \geq 3$. Moreover, $\phi$ and the $r_i$ are completely determined by the divisors $A_j$, $E_k$, so, usually we will specify an abstract toric system only by the data $E_1, \ldots, E_t, A_1, \ldots, A_{n-t}$.

A toric system is an abstract toric system which can be constructed from an exceptional sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$ by the procedure described in 5.4.

Once Theorem 4.11 is proven, by Corollary 4.12 it will in many situations be harmless to require that the $\delta_i$ are zero.

**Remark 5.6:** In the following we will exclusively consider actual (i.e. non-abstract) toric systems. In [HP11], §2, some effort has been devoted to the inverse problem, i.e. the question whether for a given toric system we can check implications such as vanishing of the $\chi(O(-A_1))$. However, contrary to the case of line bundles, the association of toric systems to exceptional objects is not as straightforward at this stage. Instead, our strategy in the subsequent sections will be to reduce such questions to the case of sequences of rank one objects (see Remark 10.10 below).

**Example 5.7:** Consider the strongly exceptional sequence $\mathcal{T}, O(2), O(4)$ on $\mathbb{P}^2$, where $\mathcal{T}$ denotes the tangent sheaf. If we denote $H$ the class of a line in $CH^1(\mathbb{P}^2)$, then the toric system associated to this sequence is given by $A_1, A_2, A_3 = \frac{1}{2}H, 2H, \frac{1}{2}H$. Now we take any point $x \in \mathbb{P}^2$ and denote $b : \mathbb{F}_1 \simeq \mathbb{P}^2 \to \mathbb{P}^2$ the blow-up at $x$ with exceptional curve $E$. For ease of notation we identify $E$ with its class in $CH^1(\mathbb{F}_1)$. We also identify $H$ with its pull-back in $CH^1(\mathbb{F}_1)$. Completing to a full exceptional sequence by adding $O_E(E)$ we get $O_E(E), b^*T, b^*O(2), b^*O(4)$. Then the toric system associated to this sequence is given by the $(-1)$-divisor $E_1 = E$, the rational classes $A_1, A_2, A_3 = \frac{1}{2}H, 2H, \frac{1}{2}H - E$, and $\phi : \{1\} \to \{1, 2, 3\}$ with $\phi(1) = 3$. Now, by right mutating the pair $O_E(E), b^*T$, we obtain $b^*T, R, b^*O(2), b^*O(4)$. We have $\chi(O_E(E), b^*T) = -2$, hence $rk R = -4$. Moreover, we get $s(b^*T, R) = \frac{1}{4}E$ and consequently the new toric system consists of four rational divisor classes which are given by

$$\frac{1}{4}E, \frac{1}{2}H, \frac{1}{4}E, 2H, \frac{1}{2}H - E.$$
6. Toric systems and their Gale dual

Let $A = E_1, \ldots, E_t$, $A_1, \ldots, A_{n-t}$ be an abstract toric system. It will be an important technical aspect to consider the projection of the $A_j$ onto the orthogonal complement of the $E_i$ in $\text{CH}^1_{\text{num}}(X)_\mathbb{Q}$. Recall that $E_i \cdot A_{\phi(i)} = -(1 + 2\delta_i)$ and $E_i \cdot A_j = 0$ for $j \neq \phi(i)$.

**Definition 6.1:** The contraction $\tilde{A}$ of $A$ is given by $\tilde{A}_1, \ldots, \tilde{A}_{n-t}$, where

$$\tilde{A}_i = A_i + \sum_{j : \phi(j) = i} (1 + 2\delta_j)E_j$$

Both $A$ and $\tilde{A}$ give rise to subgroups of $\text{CH}^1_{\text{num}}(X)_\mathbb{Q} \cong \mathbb{Q}^{n-2}$ given by $A := \{E_1, \ldots, E_t, A_1, \ldots, A_{n-t}\}_\mathbb{Z}$ and $\tilde{A} := (A_1, \ldots, A_{n-t})_\mathbb{Z}$. Clearly, both $A$ and $\tilde{A}$ are finitely generated and torsion free $\mathbb{Z}$-modules of rank at most $n - 2$. It is easy to see that the $A_i$ still satisfy conditions (3i), (3ii), (3iii) of Definition 5.5, but $\sum_{i=1}^{n-t} \tilde{A}_i = -K_X + \sum_{j=1}^{t}(1 + 2\delta_j)E_j$.

**Proposition 6.2:** $\text{rk} \ A = n - 2$ and $\text{rk} \ \tilde{A} = n - t - 2$.

**Proof.** As the $E_i$ form an orthogonal system of divisors which by construction contain $\tilde{A}$ in their orthogonal complement, it suffices to show that $\text{rk} \ \tilde{A} = n - t - 2$. Starting with the observation that by Definition 5.5 the $A_i$, and in particular $A_1$, are all nonzero we will show by induction that $A_1, \ldots, A_i$ are $\mathbb{Q}$-linearly independent for $1 \leq i \leq n - t - 2$. So, for $1 < i < n - t - 2$ we assume that $A_1, \ldots, A_{i-1}$ are linearly independent. Then, for any $\mathbb{Q}$-linear combination $B := \sum_{j=1}^{i-1} \alpha_j A_j$, we have $B : A_{i+1} = 0$. However, we have $\tilde{A}_1, \tilde{A}_{i+1} = \frac{1}{t_i} \neq 0$, hence $\tilde{A}_i$ cannot be contained in the linear span of $\tilde{A}_1, \ldots, \tilde{A}_{i-1}$, hence $\tilde{A}_1, \ldots, \tilde{A}_i$ are linearly independent for all $1 \leq i < n - t - 2$ and the assertion follows. □

**6.3.** Consider the structural linear maps $c : \mathbb{Z}^n \to A$ and $\tilde{c} : \mathbb{Z}^{n-t} \to \tilde{A}$. That is, if we denote $b_1, \ldots, b_n$ the standard basis of $\mathbb{Z}^n$, then we have $c(b_i) = E_i$ for $1 \leq i \leq t$ and $c(b_i) = A_{i-t}$ for $t < i \leq n$. For $\mathbb{Z}^{n-t}$ with standard basis $b'_1, \ldots, b'_{n-t}$, we have $\tilde{c}(b'_i) = \tilde{A}_i$ for every $i$. We define a linear map $\Phi : \mathbb{Z}^{n-t} \to \mathbb{Z}^n$ by setting $\Phi(b'_i) = b_{i+t} + \sum_{j : \phi(j) = i} (1 + 2\delta_j)b_j$ for every $1 \leq i \leq n - t$. Then $\Phi$ induces a linear map $\tilde{\Phi} : \tilde{A} \to A$ with $\tilde{\Phi}(\tilde{A}_i) = A_i + \sum_{j : \phi(j) = i} (1 + 2\delta_j)E_j$ for $1 \leq i \leq n - t$. Clearly, both $\Phi$ and $\tilde{\Phi}$ are injective and their image is saturated in $\mathbb{Z}^n$ and $\tilde{A}$, respectively. We obtain the following commutative diagram:

$$
\begin{array}{ccc}
0 & \xrightarrow{M} & \mathbb{Z}^{n-t} \\
\downarrow & & \downarrow \phi \\
0 & \xrightarrow{M'} & \mathbb{Z}^n \\
\end{array}
$$

where we set $M = \ker \tilde{c}$ and $M' = \ker(c)$. We can represent $L$ and $\tilde{L}$ as row matrices with rows $l_1, \ldots, l_n \in (M')^*$ and $\tilde{l}_1, \ldots, \tilde{l}_{n-t} \in M^*$, respectively. We have $M, M' \simeq \mathbb{Z}^2$ by Proposition 6.2. Clearly, $\Psi$ is injective and it follows from the saturatedness of $\Phi(\mathbb{Z}^{n-t})$ in $\mathbb{Z}^n$ that its cokernel is trivial, hence $\Psi$ is an isomorphism. Moreover, by dualizing the left part of the diagram and the construction of $\Psi$, we immediately obtain following statement.

**Proposition 6.4:** Denote $N := M^*$, $N' := (M')^*$ and consider the dual maps

$$
\begin{array}{ccc}
\mathbb{Z}^n & \xrightarrow{L^T} & N' \\
\downarrow \phi^T & & \downarrow \Psi^T \\
\mathbb{Z}^{n-t} & \xrightarrow{\tilde{L}^T} & N \\
\end{array}
$$

where we identify the column vectors $l_i^T$ and $\tilde{l}_i^T$ with the images of the $i$-th standard basis vector of $\mathbb{Z}^n$ and $\mathbb{Z}^{n-t}$, respectively, in $N$. Then $\Psi^T$ is an isomorphism which maps $l_i$ to $\tilde{l}_{i-t}$ for $t < i \leq n$ and $l_i$ to $(1 + 2\delta_i)\tilde{l}_{\phi(i)}$ for $1 \leq i \leq t$.

So we can naturally identify $M = M'$ and $N = N'$, respectively, and consider $\Psi^T$ as the identity map. Then both $L^T$ and $\tilde{L}^T$ give rise to almost the same set of column vectors: the column vectors of $\tilde{L}^T$
coincide with the last \( n - t \) column vectors of \( L^T \) and the first \( t \) column vectors of \( L^T \) are multiples of column vectors of \( L^T \) by some factors \( (1 + 2\delta_k) \). We will see later that the columns of \( L^T \) appear with multiplicity 1 and, once Theorem 1.1 is established, we can assume that every column vector \( l_{t+j} \) of \( L^T \) occurs (up to sign) with multiplicity 1 + \(|\{i \mid \phi(i) = j\}|\).

**Example 6.5:** In Example 5.7, a toric system was given with \(-1\)-divisor \( E \) and \( A_1, A_2, A_3 = \frac{1}{2}H, 2H, \frac{1}{2}H - E \) such that \( E \cdot (\frac{1}{2}H - E) = 1 \). For the Gale dual, we obtain vectors \( l_1, l_2, l_3, l_4 \) which for a suitable choice of basis can be represented as \( l_1 = l_4 = (1, 0), l_2 = (0, -1), l_3 = (0, -1), i.e. up to latter multiplicity, the \( l_i \) generate fan of \( \mathbb{P}(1, 1, 4) \). The mutated toric system \( A'_1, A'_2, A'_3, A'_4 = \frac{1}{2}E, \frac{1}{2}H - E, 2H, \frac{1}{2}H - E \) has Gale duals \( \tilde{l}_1, \tilde{l}_2, \tilde{l}_3, \tilde{l}'_4 \) with \( \tilde{l}'_4 = (3, 4) \) which can be interpreted to generate the fan of a weighted blow-up of \( \mathbb{P}(1, 1, 4) \) with two singular points of order 4 and 16, respectively. The corresponding fans are shown in figures 4 and 5 in the introduction. Note that with our current terminology, the enumeration of the \( \tilde{l}_i \) in figure 4 is that of the \( l_i \) rather than the \( l_i \).

In both cases the \( l_i \) are primitive lattice vectors and generate the fan of a complete toric surface. It is easy to see that the singularities are \( T \)-singularities. We will show that this and the observation that the multiplicity \( l_1 = l_4 \) translates into a weighted blow-up via mutation are general properties of toric systems.

### 7. Moving around objects of rank zero

Let \( E = E_1, \ldots, E_n \) be an exceptional sequence and denote \( E_1, \ldots, E_t, A_1, \ldots, A_n-t \) its associated toric system and \( A_1, \ldots, A_{n-t} \) its contraction. Via Gale duality, we have extracted certain collections of integer vectors from both data in terms of rows of certain matrices \( E \) and \( L \), respectively. By Proposition 6.3 the rows of \( L \) coincide with the last \( n - t \) rows of \( L \) and for \( 1 \leq i \leq t \), the \( i \)-th row \( l_i \) coincides with \((1 + 2\delta_i)\tilde{l}_{t+i}(\phi(i)) \). It will be the subject of the subsequent sections to show that the vectors \( \tilde{l}_1, \ldots, \tilde{l}_{n-t} \) are cyclically ordered and generate the fan associated to a complete toric surface. This section is devoted to the first \( t \) columns of \( L \) and their behaviour under mutation.

**7.1.** Consider an exceptional triple \( E, Z, F \) with \( e, f \neq 0 \) and \( z = 0 \). By moving \( Z \) to the left or right via mutating \( E \) or \( F \), respectively, we obtain exceptional triples \( Z, R_Z E, F \) and \( E, L_Z F, Z \), respectively. With \( 2.3 \) and Lemma 1.3 we see that \( s(R_Z E) = s(E) + (1 + 2\delta_Z)E_1(Z) \) and \( s(L_Z F) = s(F) - (1 + 2\delta_Z)c_1(Z) \). Thus we get \( s(R_Z E, F) = s(E, L_Z F) = s(E, F) - (1 + 2\delta_Z)c_1(Z) \). If we can extend our exceptional triple, say, to the left, i.e. we have an exceptional sequence \( D, E, Z, F \) with \( d \neq 0 \), then we get furthermore that \( s(D, L_Z F) = s(D, F) + (1 + 2\delta_Z)c_1(Z) \). In the following proposition we apply this simple modification of Chern classes to toric systems.

**Proposition 7.2:** Let \( E = E_1, \ldots, E_n \) be an exceptional sequence with associated toric system \( E_1, \ldots, E_t, A_1, \ldots, A_{n-t}, \phi \). Assume that \( E_j \) has rank zero for some \( 1 \leq k \leq n \) and let \( 1 \leq i \leq t \) such that \( E_i = e_1(\phi) \). Consider the mutations \( L_k E \) and \( R_k E \). Then the corresponding toric systems are given by \( E_1', \ldots, E_t', A_1', \ldots, A_n', \phi' \) (for \( L_k E \)) and \( E_1'', \ldots, E_t'', A_1'', \ldots, A''_n, \phi'' \) (for \( R_k E \)), where 

(i) If \( e_{k+1} = 0 \) (resp. \( e_{k+1} = 0 \)), then \( E_i' = -E_{i+1}, E_i'' = E_i \) and \( E_j' = E_j \), otherwise, \( A_j = A_j \) for all \( j \) and \( \phi' = \phi \) (resp. \( E_i'' = -E_i, E_j'' = E_j \), otherwise, \( A_j = A_j \) for all \( j \) and \( \phi'' = \phi \)).

(ii) If \( e_{k+1} = 0 \) then \( E_j' = E_j \) for all \( j \), \( A_{\phi(j)}' = A_{\phi(j)} - (1 + 2\delta_i)E_i \), \( A_{\phi(j)+1}' = A_{\phi(j)+1} + (1 + 2\delta_i)E_i \), and \( A_j' = A_j \) otherwise. Moreover, \( \phi'(j) = \phi(j) \) for \( j \neq i \) and \( \phi'(i) = \phi(i) + 1 \).

(iii) If \( e_{k+1} = 0 \) then \( E_j'' = E_j \) for all \( j \), \( A_{\phi(j)+1}'' = A_{\phi(j)+1} + (1 + 2\delta_i)E_i \), \( A_{\phi(j)}'' = A_{\phi(j)} - (1 + 2\delta_i)E_i \), and \( A_j'' = A_j \). Moreover, \( \phi''(j) = \phi(j) \) for \( j \neq i \) and \( \phi''(i) = \phi(i) - 1 \).

**Proof.** By 5.3, 6.1, we have for any exceptional pair \( E, F \) with \( e = f = 0 \) that \( [L_Z E] = [F] \) and \( [R_Z F] = [-E] \) in \( K^{num}(X) \), so that on the level of toric systems \( E_i \) and \( E_{i+1} \) (respectively \( E_i \) and \( E_{i+1} \)) get replaced by \(-E_i\) and \( E_i \) (respectively \( E_i \) and \(-E_i \)). The assertion correspondingly just reflects the reshuffling of data.

We have already seen in 7.1 that the \( A_j \) behave in the described way (in particular, the \( A_j \) for \( j \neq \phi(i), \phi(i+1) \) remain the same). Also, as the sequence of \( E_j \)’s remains constant, the function \( \phi(i) \) changes as described.

\( \square \)
By Proposition 6.3 the translation of above modifications into the Gale dual picture can be described as follows.

**Corollary 7.3:** With the notation of Proposition 7.4 we denote $L'$ and $L''$ (respectively $\tilde{L}'$ and $\tilde{L}''$) the Gale dual matrices corresponding to $L_kE$ and $R_{k-1}E$, respectively. Then:

(i) $\tilde{L}' = \tilde{L}'' = L$.

(ii) If $e_{k+1} = 0$ (resp. $e_{k-1} = 0$) then $l'_i = -l_{k+1} = -(1 + 2\delta_i l_{i+\phi(i+1)}), l'_{k+1} = l_i$ (resp. $l''_i = l_i$ and $l''_{k+1} = l_i$ otherwise.

(iii) If $e_{k+1} \neq 0$ then $l'_i = -(1 + 2\delta_i) l_{i+\phi(i)}$ and $l'_{j} = l_j$ for all $j \neq i$.

(iv) If $e_{k-1} \neq 0$ then $l''_i = -(1 + 2\delta_i) l_{i+\phi(i)}$ and $l''_{j} = l_j$ for all $j \neq i$.

**7.4.** Whenever we have an exceptional sequence of maximal length which may contain objects of rank zero, it follows from our discussion in paragraph 4.9 that it as well contains objects of nonzero rank which can “move around” rank zero objects via mutation without essentially changing the combinatorial data such objects flexibly. The statements in this section imply that we have such a flexibility where we can “move around” rank zero objects via mutation without essentially changing the combinatorial data associated to it. To exemplify this, consider the exceptional sequence $O_E(E), b^*O(2), b^*O(3), b^*O(4)$ from the introduction. In suitable coordinates, one computes the matrices as:

$$L = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{L} = \begin{pmatrix} \tilde{l}_1 \\ \tilde{l}_2 \\ \tilde{l}_3 \\ \tilde{l}_4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding toric system is given by data $E = E, A_1, A_2, A_3, \phi$, with $\phi(1) = 3$. In anticipation of Theorem 11.3 Figure 8 depicts the $\tilde{i}_8$ as primitive generators for the fan of $\mathbb{P}^2$, where the presence of a second copy of $\tilde{l}_3$ in $L$ is indicated by a double arrow. Here we express the fact that the exceptional sequence contains an object of rank zero at its leftmost position by attaching the multiplicity 2 to $\tilde{i}_5$. Now, if we move $O_E(E)$ to the right by mutation, we obtain the sequence $L_{O_E(E)}b^*O(2), O_E(E), b^*O(3), b^*O(4)$ and, by applying a shift to $O_E(E)$, we get $L_{O_E(E)}b^*O(2), O_E(E)[1], b^*O(3), b^*O(4)$. If we denote the corresponding matrices by $L'$ and $\tilde{L}'$, respectively, we get:

$$(L')^T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{L}' = \begin{pmatrix} \tilde{l}'_1 \\ \tilde{l}'_2 \\ \tilde{l}'_3 \\ \tilde{l}'_4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where the first equality follows from Corollary 7.3 and the application of the shift functor translates to the second equality. Moreover, we have $L' = \tilde{L} = L$, so, up to multiplicities, the fan we associate to each of the three cases is the same. Figure 9 shows the effect that the multiplicity 2 shifts counterclockwise from $\tilde{l}_3$ to $\tilde{l}_1$. We may refer to this effect colloquially as “hopping” of multiplicities.

In general, if we apply, say, $L_kE$ to $E = \ldots, E_{k-1}, E_k, E_{k+1}, \ldots$ with $e_k = 0$ and $e_{k+1} \neq 0$, and we get $\ldots, E_{k-1}, L_kE, E_{k+1}, E_k, \ldots$, then by Corollary 7.3 the matrices $\tilde{L}$ and $L'$ (equivalently, the respective
last \( n - t \) rows of \( L \) and \( L' \) coincide. The only difference between \( L \) and \( L' \) then is that the \( i \)-th row (where \( 1 \leq i \leq t \) such that \( E_i = c_1(\mathcal{E}_k) \)) flips from \((1 + 2\delta_i)\tilde{L}_{\phi(i)} \) to \(-(1 + 2\delta_i)\tilde{L}_{\phi(i)+1} \). In other terms, one could think of the \( \tilde{L}_i \) to be endowed with “multiplicities”, i.e. if \( A_j = s(\mathcal{E}_k, \mathcal{E}_{k'}) \) for \( k < k' \) and \( c_k, c_{k'} \neq 0 \) then \(|\phi^{-1}(j)|\) is the number of rank zero objects in the exceptional sequence between \( \mathcal{E}_k \) and \( \mathcal{E}_{k'} \) and correspondingly, \( L \) contains as many many additional rows which are collinear to \( \tilde{L}_j \) for \( l_{i+j} \). If we disregard the factors \((1 + 2\delta_i) \), we should think of these additional rows as copies of \( l_{i+j} \). The latter will be fully justified once we have proven Theorem 4.4 which implies \( 1 + 2\delta_i = \pm 1 \). Then by Corollary 4.12 there will be no loss of generality to assume that \( 1 + 2\delta_i = 1 \) for every \( i \). Ultimately, we can view the vectors \( \tilde{L}_1, \ldots, \tilde{L}_{n-t} \) as the essential data associated to a toric system of an exceptional sequence, where every \( \tilde{L}_i \) comes with a multiplicity for bookkeeping of the rank zero objects in the sequence. In particular, by Corollary 7.4, moving rank zero objects around via mutations then is reflected simply by a “hopping” of the corresponding multiplicities. So, as far as the \( \tilde{L}_i \) are concerned, there will in many situations be no harm to pretend that they have multiplicity one. E.g. we can without loss of generality assume that \( \phi \) is constant so that the rank zero objects form an uninterrupted sub-sequence anywhere in the sequence.

8. Local Constellations

In order to simplify notation and to reduce the number of trivial caveats, in this section we will make the assumption that our exceptional sequence \( \mathcal{E} = \mathcal{E}_1, \ldots, \mathcal{E}_n \) contains no objects of rank zero (and hence \( L = \tilde{L} \)). In the spirit of 7.3 “up to multiplicities” the results extend trivially to the general case.

8.1. We denote \( a_i := c_2^i e_{i+1}^2 A_i^2 \in \mathbb{Z} \). The intersection product in \( CH_{\text{num}}(X) \) induces a bilinear form on \( A \). For any \( A_i \), we denote \( A_i^+ \) the orthogonal complement of \( A_i \) in \( A \) with respect to this form. Clearly, \( A_i^+ \) contains all \( A_j \) for \( j \not\in \{i-1, i, i+1\} \) and therefore the quotient \( A_i/A_i^+ \) is isomorphic to \( \mathbb{Z} \) and \( A/(A_j \in A_i^+)_\mathbb{Z} \cong \mathbb{Z} \oplus F_i \) where \( F_i \cong A_i^+/(A_j \in A_i^+) \mathbb{Z} \). As we have seen in Section 6 (see in particular the proof of Proposition 6.2), the set \( \{ A_j \mid j \neq i, i+1 \} \) is linearly independent in \( A \), hence the set \( \{ A_j \mid j \neq i-1, i, i+1 \} \subseteq A_i^+ \) has finite index in \( A_i^+ \) and thus \( F_i \) is finite. Moreover, by the general properties of Gale duality, this implies that \( l_i \) and \( l_{i+1} \) are linearly independent in \( N \). So we have the following exact commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & M & \overset{L}{\to} & \mathbb{Z}^n & \overset{c}{\to} & A & \to & 0 \\
0 & \to & M & \overset{L_{i-1,i+1}}{\to} & \mathbb{Z}^n & \overset{c_i}{\to} & \mathbb{Z} \oplus F_i & \to & 0.
\end{array}
\]

Dualizing the lower row, we get the exact sequence:

\[
0 \to \mathbb{Z} \overset{\epsilon_i^T}{\to} \mathbb{Z}^n \overset{L_{i-1,i+1}^T}{\to} N \overset{F_i}{\to} 0
\]

where by slight abuse of notation we denote \( \epsilon_i^T \) the dual of \( \epsilon_i \) and we identify \( \text{Ext}^1(\mathbb{Z} \oplus F_i, \mathbb{Z}) \cong F_i \). An elementary computation shows

\[
d_i := (\det(l_i, l_{i+1}), \det(l_{i+1}, l_{i-1}), \det(l_{i-1}, l_i))^T \in \ker(L_{i-1,i+1}^T).
\]

and \( \ker(L_{i-1,i+1}^T) \) is generated by \( \frac{1}{g_i}d_i \), where \( g_i = \gcd(\det(l_i, l_{i+1}), \det(l_{i+1}, l_{i-1}), \det(l_{i-1}, l_i)) \). We leave it as an exercise for the reader to show that \( L_{i-1,i+1}^T \) is surjective iff \( g_i = 1 \). Applying this exercise to the exact sequence

\[
0 \to \mathbb{Z} \overset{\epsilon_i^T}{\to} \mathbb{Z}^n \overset{L_{i-1,i+1}^T}{\to} \text{im} L_{i-1,i+1}^T \to 0
\]

we obtain by composition with the inclusion \( \text{im} L_{i-1,i+1} \hookrightarrow N \) that

\[
g_i = |F_i|.
\]

Now we want to determine \( d_i \).

**Proposition 8.2:** Up to a choice of orientation in \( N \), we have \( d_i = h(c_{i+1}^2, a_i, c_i^2) \) for every \( i \) and some \( h \in \mathbb{Z} \) with \( h > 0 \). In particular,

\[
\det(l_{i-1}, l_i) = h e_i^2 \quad \text{and} \quad \det(l_{i+1}, l_{i-1}) = h a_i.
\]
Proof. We first fix some $i$. The projection $A \to \mathbb{Z} \oplus F_i$ induces a $\mathbb{Q}$-valued linear form on $\mathbb{Z} \oplus F_i$ which is given by $(A_i, \ldots)$ and which vanishes on $F_i$. We project further and obtain a $\mathbb{Q}$-valued linear form on $\mathbb{Z}$ which is completely determined by a number $q \in \mathbb{Q}$ such that $A_i \cdot A_j = q A_j$ for $j = i - 1, i, i + 1$, where we denote $\tilde{A}_j$ the image of $A_j$ in $\mathbb{Z}$. In particular, we have $q A_{i-1} = 1/e_i^2, q A_i = a_i/e_i^2 e_{i+1}^2$, and $q A_{i+1} = 1/e_i^2 e_{i+1}^2$ that we can simply interpret as equations of rational numbers.

Now we set $q := x/e_i^2 e_{i+1}^2$ for some $x \in \mathbb{Q}$ and obtain by rearranging the equations:

$$\bar{A}_{i-1} = e_i^2 e_{i+1}^2 / x, \quad \bar{A}_i = a_i / x, \quad \bar{A}_{i+1} = e_i^2 / x.$$ 

As the $\tilde{A}_j$’s are integral and generate $\mathbb{Z}$, we get $x = \pm g_i^2$ where $g_i^2 := \gcd\{e_{i+1}^2, a_i, e_i^2\}$. Up to the choice of a generator of $A/A_i^2 \cong \mathbb{Z}$, we can assume $x = g_i^2$. By construction, the short exact sequence corresponding to the surjection $\mathbb{Z}^3 \to \mathbb{Z} = \langle \bar{A}_{i-1}, \bar{A}_i, \bar{A}_{i+1} \rangle$ is dual to the short exact sequence $\mathbb{Z} \to \mathbb{Z}^3 \to \text{im} \bar{L}_{i-1,i+1}^1$ from \[\text{Cor} \] hence from the discussion there it follows that $\bar{e}_i^2 = \frac{1}{g_i^2}(e_{i+1}^2 + a_i, e_i^2)^T$. Moreover, it follows that $d_i = h_i(e_{i+1}^2, a_i, e_i^2)$, where we denote $h_i := g_i / g_i^2 = \det(l_{i-1}, l_i) / e_i^2 = \det(l_i, l_{i+1}) / e_{i+1}^2 \in \mathbb{Q}$.

This implies $h_i = h_{i+1}$ for every $i$, hence by induction there exists $h \in \mathbb{Q}$ such that $h_i = h$ for every $i$. Moreover, because for an exceptional sequence of maximal length the gcd of the $e_i$ is 1, it follows that $h \in \mathbb{Z}$.

\[\text{Corollary 8.3:} \]

(i) We can choose an orientation of $N$ such that for every $i$ the pair $l_i, l_{i+1}$ is positively oriented, i.e. $\det(l_i, l_{i+1}) > 0$.

(ii) For every $i$, the pair $l_i, l_{i+1}$ generates a strictly convex polyhedral cone in $N_Q$.

(iii) Every triple $l_{i-1}, l_i, l_{i+1}$ satisfies the relation

$$e_i^2 l_{i-1} + a_i l_{i} + e_{i+1}^2 l_{i+1} = 0$$

and generates a fan which contains two 2-dimensional cones which intersect in the common facet $Q_{\geq 0} l_i$.

Proof. (i) Obvious. We will assume the choice of this orientation for the rest of the proof.

(ii) Follows from $\det(l_i, l_{i+1}) = h e_i^2 > 0$ for every $i$.

Because $\det(l_i, l_{i+1}) > 0$ and $\det(l_{i-1}, l_i, l_{i+1}) > 0$, the lattice vectors $l_{i-1}$ and $l_{i+1}$ lie in the opposite interiors of the half spaces which are bounded by the line $Q l_i$. Hence the cones generated by $l_{i-1}, l_i$ and $l_i, l_{i+1}$, respectively, intersect at the common facet $Q_{\geq 0} l_i$ and therefore form a fan.

Note that for the rest of the paper we will always implicitly assume that we have chosen an orientation of $N$ which conforms to Corollary 8.3 (i) and will mention it no further.

The following lemma shows that the case $a_i = 0$ corresponds to a very special configuration.

\[\text{Lemma 8.4:} \]

If $a_i = 0$ then $e_i^2 = e_{i+1}^2$ and $l_{i-1} = -l_{i+1}$.

Proof. By \[\text{Lemma 8.4 (i)}\] we have $\chi(E_i, E_{i+1}) = e_{i+1}/e_i$. If we denote $g := \gcd\{e_i, e_{i+1}\}$ and $e_i' := e_i / g$, $e_{i+1}' := e_{i+1} / g$, then we get immediately $\chi(E_i, E_{i+1}) = (e_i')^2 + (e_{i+1}')^2$. But then $e_i'$ divides $(e_{i+1}')^2$ and $e_{i+1}'$ divides $(e_i')^2$. Because $\gcd\{e_i', e_{i+1}'\} = 1$, this implies $(e_i')^2 = (e_{i+1}')^2 = 1$, hence $e_i^2 = e_{i+1}^2 = g^2$. For the second assertion, observe that the relation $e_i^2 l_{i-1} + e_{i+1}^2 l_{i+1} = 0$ holds.

We want now describe what happens to the vectors $l_i$ if we perform a mutation of the sequence $E$. If we apply a mutation $L_i E$ or $R_i E$, then we can construct by above procedure a sequence of vectors $l_i', \ldots, l_{i+1}' \in N'$, where $N'$ is the dual of the kernel of the structural morphism $e'$ corresponding to the new toric system. The following lemma shows that in terms of the $l_i$, the effect of mutation is local.

\[\text{Proposition 8.5:} \]

With above notation, we can naturally identify $N$ with $N'$ such that $l_j' = l_j$ for all $j \neq i$.

Proof. Without loss of generality, we only consider left mutations $L_i E$; the case of right mutations then follows analogously. On the level of toric systems, the effects of such a mutation can be described by the formulas of 8.4 and 8.9. For this, we distinguish two cases, depending on whether $R := \ker L_i E_{i+1}$ is nonzero or not.
In the first case, we obtain a toric system $A'_1, \ldots, A'_n$, where $A'_i = A_i - \frac{c_{i+1}}{h} A_i$, $A'_i = A_i + A'_{i+1}$, and $A'_1 = A_1$. In particular, $(A'_1, \ldots, A'_n)$ contains all $A_j$ with $j \neq i, i+1$. By Proposition 6.2 these $A_j$ are linearly independent and therefore form a $Q$-basis of $CH^1(X)$. By rescaling these basis vectors by a factor $\frac{1}{c_{i+1}}$, we can represent the $Q$-linear extension of the maps $c$ and $c'$ in 6.3 by the following matrices:

$$
c = \begin{pmatrix}
e_2 & a_2 & e_2 & 0 & \cdots & 0 \\
x_2 & y_2 & e_3 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_{n-2} & y_{n-2} & 0 & \cdots & e_3 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
$$

and

$$
c' = \begin{pmatrix}
e_2 - \frac{c_{i+1}}{h} a_2 & \frac{c_{i+1}}{h} a_2 & e_2 & 0 & \cdots & 0 \\
x_2 & \frac{c_{i+1}}{h} x_2 & x_2 + y_2 & e_3 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_{n-2} & \frac{c_{i+1}}{h} x_{n-2} & y_{n-2} + x_{n-2} & 0 & \cdots & e_3 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
$$

where by cyclic renumbering we assume without loss of generality that $i = 2$. The $x_j, y_j$ are determined by the relations $x_j l_2 + y_j l_3 + e_3 l_2 = 0$, in particular we have $hx_j = \det(l_3, l_{j+2})$ and $hy_j = \det(l_{j+1}, l_2)$ for every $j$. Consider first the case $a_2 \neq 0$. Then $l'_2 = \frac{1}{a_2}(e_2 l'_j + R^2 l'_j)$. Now it follows from a direct calculation that we can represent the Gale transforms $l'_1, \ldots, l'_n$ by $l_1, l_2, l_3, \ldots, l_n$, in particular, we have $l'_2 = \frac{1}{a_2}(e_2 l_1 + R^2 l_1)$. In the case $a_2 = 0$, we use any row with $x_j \neq 0$ in order to find the representation $l'_2 = l_2 + 2l_1$. As before, we check that we can represent $l'_1, \ldots, l'_n$ by $l_1, l_2, l_3, \ldots, l_n$.

In the second case, our toric system is given by $E_1, A'_1, \ldots, A'_{n-1}, 0$, where $A'_j = A_j$ for $j < i$, $A'_i = A_i + A'_{i+1}$, $A'_j = A_{j-1}$ for $j > i$, and $\phi(1) = i - 1$. By similar arguments as in the proof of Proposition 6.3, we can conclude that the effect of the mutation is that the vector $l_i$ “hops” onto $l_{i-1}$.

The following statement shows that we can get rid of the factor $h$ in Proposition 8.2.

**Proposition 8.6:** (i) Let $h$ be as in Proposition 8.2. Then $h = 1$ and thus

$$
det(l_{i-1}, l_i) = e_i^2 
$$

and

$$
det(l_{i+1}, l_{i-1}) = a_i
$$

for every $i$.

(ii) The $l_i$ are primitive lattice vectors.

**Proof.** If we have shown that $h$ divides det$(l_i, l_{i+1})$ and det$(l_{i-1}, l_{i+1})$ for every $i$. We will show that $h \neq 1$ implies that the $l_i$ generate a proper sublattice of $N$, which is a contradiction. Obviously, for $n \leq 5$ we immediately have that $h$ divides det$(l_i, l_j)$ for every $i \neq j$, hence there is nothing to show. Soo without loss of generality, we can assume $n > 5$. We start with the following claim: for every $i < j$ with $2 < j - i < n$, $h$ divides det$(l_i, l_j) \cdot e_i^2 e_{i+2} \cdots e_{j-1}$. To see this, we start with the equality

$$
e_i^2 l_{j-2} + a_{j-1} l_{j-1} + e_{j-1} l_j = 0
$$

Applying det$(l_i, -)$ to this equation, we get:

$$
e_i^2 \det(l_i, l_{j-2}) + a_{j-1} \det(l_i, l_{j-1}) + e_{j-1} \det(l_i, l_j) = 0.
$$

Now the claim follows by induction starting with $j = i + 3$. As a corollary of this claim we conclude that for any $i, j$ with $|i-j| > 2$, $h$ divides

$$
det(l_i, l_j) \cdot \gcd(e_{i+1}^2, e_{i+2}^2, \ldots, e_{j-1}^2, e_{j+2}^2, \ldots, e_{i-1}^2).
$$

Now assume $p$ is a prime factor of $h$, which is not a prime factor of any of the $e_i$. Then above equation shows that $p$ divides det$(l_i, l_j)$ for every $i \neq j$, hence the sublattice generated by the $l_i$ in $N$ has an index which is a multiple of $p$, which is absurd, as the $l_i$ generate $N$. Hence, such a prime factor cannot exist and thus every prime factor of $h$ must show up as a prime factor of at least one of the $e_i$. To complete our argument, it suffices to show that for a prime factor $p$ of $h$, $p$ divides det$(l_i, l_j)$ for any pair $i \neq j$. As before, this is absurd, hence such a prime factor cannot exist.

We now observe that by Proposition 8.3 $h$ remains invariant under mutation. There exists always at least one $e_i$ which is not divisible by $p$ and, by renumbering we can always arrange that $p$ does not divide $e_3, \ldots, e_k$ for some $k > 2$. Then it follows from our claim above that $p$ divides det$(l_i, l_j)$ for all $1 \leq i < j \leq k + 1$. If $k > n - 3$, we are done. Otherwise, we will show that, possibly after mutation, $p$ divides det$(l_i, l_j)$ for all $1 \leq i < j \leq k + 2$. The statement is trivial if $p$ does not divide $e_{k+1}$. If $p$ is a divisor of $e_{k+1}$, we perform a right mutation of the pair $E_k, E_{k+1}$, resulting in the pair $E_{k+1}, R_{E_{k+1}} E_k$ and moving $l_k$ to $l'_k$ by Proposition 8.3. Using our claim on $l_1, \ldots, l_{k-1}, l'_k$ and the properties stated in the beginning
of the proof, we see that $p$ divides $\det(l_i, l_i')$ for every $1 \leq i \leq k + 1$, $i \neq k$, hence $l_1, \ldots, l_{k-1}, l_k', l_{k+1}$ still generate a sublattice of index divisible by $p$ in $\mathcal{N}$. Now, because $\text{rk} R e_{k+1} E_k = \chi(E_k, E_{k+1}| e_k - e_{k+1}$, where $p$ divides $e_{k+1}$ and does not divide $e_k$, it follows that $p$ does not divide $\text{rk} R e_{k+1} E_k$. Hence, as we did in the induction step for above claim, we conclude that $p$ divides $\det(l_i, l_{k+2})$. Similarly, we conclude that $p$ divides $\det(l_j, l_{k+2})$ for $1 < j < k + 2$. By induction it follows that $p$ divides $\det(l_i, l_j)$ for every $i \neq j$, which implies that the $l_i$ cannot generate $N$, which is a contradiction. Therefore $h$ has no prime factors, i.e. $h = 1$, which completes our proof.

Assume there is one $l_i$, which is not primitive. Without loss of generality, we cyclically renumber the sequence such that $i = 1$. Then $l_1 = p l_1$ for some $p > 1$ and $l_1$ is a primitive lattice vector. Then $p$ divides both $e_1^2 = \det(l_3, l_1)$ and $e_1^2 = \det(l_3, l_2)$. Now for every $3 < i < n$ with $e_i \neq 0$, we can perform right-mutations in order to move $E_i$ to the left:

$E_1, E_1, R e_1 E_2, \ldots, R e_1 E_{i-1}, E_{i+1}, \ldots, E_n.$

As these mutations do not alter $l_n$ and $l_1$, the exceptional pair $E_1, E_i$ corresponds to rays $l_n, l_1, l_2'$ and $e_i^2 = \det(l_1, l_2')$. Therefore $p$ divides $e_i^2$ as well, and hence we get $\gcd\{e_1^2, \ldots, e_n^2\} \neq 1$ and thus $\gcd\{e_1, \ldots, e_n\} \neq 1$. But this contradicts the fact that the rank morphism from $K_0^{\text{num}}(X)$ to $\mathbb{Z}$ is surjective.

Another special configuration arises if $L_{E_i} E_{i+1}$ or $R_{E_{i+1}} E_i$ has rank zero. Recall that in each case we have defect terms $\delta$ and $\delta'$, respectively, in the sense of Definition 4.7.

Lemma 8.7: If $\text{rk} L_{E_i} E_{i+1} = 0$ then $e_i^2 (1 + 2\delta) l_i - e_i^2 l_i - e_{i+1} = 0$. If $\text{rk} R_{E_{i+1}} E_i = 0$ then $e_i^2 (1 + 2\delta') l_i - e_i^2 l_i + e_{i+1} = 0$.

Note that once Theorem 4.11 is proved we can assume $(1 + 2\delta) = 1$ and $(1 + 2\delta') = 1$, respectively.

Proof. We only prove the first equation. Observe that $0 = \text{rk} L_{E_i} E_{i+1} = \chi(E_i, E_{i+1}) e_i - e_{i+1}$ by 2.8 and $\chi(E_i, E_{i+1}) = \chi(L_{E_i} E_{i+1}, E_i) = -e_i (1 + 2\delta)$ by Proposition 4.8 (ii), hence $e_{i+1} = -e_i^2 (1 + 2\delta)$ and with $a_i = \det(l_{i+1}, l_i) = -e_i^2$ the statement follows.

Remark 8.8: As remarked at the beginning of this section, in order to simplify the presentation we considered only the case where our exceptional sequence contains only objects of nonzero rank. Given an arbitrary sequence of maximal length, we can always produce such a sequence by mutation. More precisely, by 7.4 we can assume that $e_i = 0$ for $1 \leq i \leq t$ for some $0 \leq t \leq n - 3$ and $e_i \neq 0$ for $t < i \leq n$. Then by right-mutation, we can produce a sequence

$E_{t+1}, R e_{t+1} E_{t+1}, \ldots, R e_{t+1} E_1, E_{t+1}, \ldots, E_n$

with $\text{rk} R e_{t+1} E_1 = -e_i^2 (1 + \delta) \neq 0$ for $1 \leq i \leq t$. The local configuration of the new $l_i$ then arises iteratively from Lemma 8.7. With Theorem 4.11 in the case $e_i = \pm 1$ this corresponds (at least locally, so far) to a series of smooth toric blow-ups.

For any exceptional pair $E_0, E_1$, we define inductively $E_{i+2} = R e_{i+2} E_i$ for $i \geq 0$. The Chern classes $s(E_i, E_{i+1})$ are all collinear to $c_1(E_0, E_1)$ in $CH^1(X)_{\mathbb{Q}}$, where the proportionality is successively given by the quotients of ranks $e_i/e_{i+1}$. These ranks are determined in the following proposition which generalizes a similar statement by Rudakov [Rud90] §4. It is not needed in the remainder of this paper but it might be of some general interest.

Proposition 8.9: Let $E_0, E_1$ be an exceptional pair with $\chi(E_0, E_1)^2 \neq 4$ and for $i \geq 0$ define inductively $E_{i+2} = R e_{i+2} E_i$. Moreover, denote $\alpha_\pm = \frac{1}{4} (\chi(E_0, E_1) \pm \sqrt{(\chi(E_0, E_1)^2 - 4})$ the roots of the polynomial $x^2 - \chi(E_0, E_1)x + 1$. Then for $i \geq 2$ we get:

$e_i = \frac{\alpha_{i+1} - \alpha_i}{\alpha_+ - \alpha_-} e_0 - \frac{\alpha_{i+1} - \alpha_i}{\alpha_+ - \alpha_-} (\chi(E_0, E_1)e_0 - e_1)$.

Proof. Follows from standard arguments for solving the recurrence relation

$e_{i+2} = \chi(E_0, E_1)e_{i+1} - e_i$.

for $i \geq 0$. □
Remark 8.10: For the two cases with \( \chi(\mathcal{E}_0, \mathcal{E}_1)^2 = 4 \) we can directly use the first equation in the proof of Proposition 8.9 and get by induction:
\[
e_i = i e_1 + e_0(1 - i) \quad \text{if } \chi(\mathcal{E}_0, \mathcal{E}_1) = 2,
\]
\[
e_i = (-1)^{i+1}(i e_1 - e_0(1 - i)) \quad \text{if } \chi(\mathcal{E}_0, \mathcal{E}_1) = -2.
\]
Moreover, note the periodic behaviour for the cases \( \chi(\mathcal{E}_0, \mathcal{E}_1)^2 \leq 1 \):
(i) If \( \chi(\mathcal{E}_0, \mathcal{E}_1) = 0 \) then \( e_i = \begin{cases} (-1)^{i/2} e_0 & \text{for } i \text{ even,} \\ (-1)^{(i-1)/2} e_1 & \text{for } i \text{ odd.} \end{cases} \)
(ii) If \( \chi(\mathcal{E}_0, \mathcal{E}_1)^2 = 1 \) then \( e_i = \begin{cases} (-\chi(\mathcal{E}_0, \mathcal{E}_1))^{i/3} e_0 & \text{for } i \equiv 0(3), \\ (-\chi(\mathcal{E}_0, \mathcal{E}_1))^{(i-1)/3} e_1 & \text{for } i \equiv 1(3), \\ (-\chi(\mathcal{E}_0, \mathcal{E}_1))^{(i-2)/3} (\chi(\mathcal{E}_0, \mathcal{E}_1) e_1 - e_0) & \text{for } i \equiv 2(3). \end{cases} \)

9. Mutations

In this section we consider an exceptional pair \( \mathcal{E}, \mathcal{F} \) with \( e, f \not= 0 \) and \( a := c_1(\mathcal{E}, \mathcal{F})^2 \). We assume that this pair can be extended to an exceptional sequence of length \( n, \mathcal{E}, \mathcal{F}, \mathcal{E}_3, \ldots, \mathcal{E}_n \). Then the set of Gale duals of the associated toric system contains primitive lattice vectors \( e, f \in \mathbb{N} \) such that the following relation holds:
\[
f^2 l_e + af + e^2 l_f = 0.
\]
Note that by Proposition 8.6 and Lemma A.5 \( e, f \) are essentially uniquely determined by the integers \( e^2, a, f^2 \).

9.1. As in Definition A.2 we have circumference segments \( p_e := l - e, p_f := f - l \) and it is convenient to define
\[
w_e := \frac{1}{e} p_e, \quad \text{and} \quad w_f := \frac{1}{f} p_f.
\]
Using 3.5 and 3.9 we immediately get the following formulas:
\[
\det(w_e, w_f) = \frac{1}{ef}(a + e^2 + f^2) = \chi(\mathcal{E}, \mathcal{F}),
\]
\[
\det(w_f, l_e) = \frac{a + e^2}{f} = \text{rk} L\mathcal{E}\mathcal{F},
\]
\[
\det(w_e, l_f) = \frac{a + f^2}{e} = \text{rk} R\mathcal{F}\mathcal{E}.
\]
Note that mutation can change the orientation of the \( w \)'s with respect to the \( p \)'s.

In our subsequent analysis, the circumference segments will play a crucial role. We start with the following observation.

Lemma 9.2: Both \( w_e \) and \( w_f \) are integral.

Proof. After a choice of coordinates we can assume without loss of generality that \( l_e = (1, 0), l = (x, e^2), l_f = (y, -a) \) for some integers \( x, y \). Then \( p_e = (x - 1, e^2) \) and \( p_f = (y - x, -a - e^2) \) and \( ef\chi(\mathcal{E}, \mathcal{F}) = \det(p_e, p_f) = e^2(1 - y) - a(x - 1) \). So, in particular, \( e \) divides \( a(x - 1) \) and therefore \( e \) divides \( a \cdot p_e \). Moreover, as \( e \) divides \( a + f^2 \), it also divides \( (a + f^2)p_e \) and therefore also \( f^2 \cdot p_e \). Hence, \( e \) divides \( \gcd(a, e^2, f^2) \cdot p_e \). Now, via mutation, we can replace \( \mathcal{F} \) by any \( \mathcal{E}_i \) with \( e_i \neq 0 \). In the fan, this leaves \( l_e, l \) and \( p_e \) unchanged, and we obtain analogously that \( e \) divides \( \gcd(e_1(\mathcal{E}, \mathcal{E}_1)^2, e^2, e_f^2) \cdot p_e \). So we get that \( e \) divides \( \gcd(e^2, f^2, e_f^2 | e_i \neq 0 \) \cdot \( p_e \). But \( \gcd(e^2, f^2, e_f^2 | e_i \neq 0 \} = 1 \) and the assertion follows for \( w_e \) and, by exchanging the roles of \( \mathcal{E} \) and \( \mathcal{F} \), also for \( w_f \). \( \square \)

By Proposition 8.5 the effect of mutation is local in the sense that if we apply a mutation to the pair \( \mathcal{E}, \mathcal{F} \), say, then the triple \( l_e, l, l_f \) gets transformed to a triple \( l_e, l', l_f \) and the other \( l \) remain constant. The transformation of \( l \) to \( l' \) can be described nicely with help of the circumference segments. We consider
We can consider the value \( f \) case we get the convexity of the configuration of lattice vectors \( l \) between Figure 10 schematically depicts these possibilities for the case \( a < \). The second equality follows from a simple rearrangement of terms. We only prove the formulas for \( l \); the case \( l'' \) then follows analogously. By A.5 \( l' \) is completely determined by \( a \) and the volumes \( f^2 \) and \( (a+f^2) \) relative to \( l_f \) and \( l_e \), respectively. Then the first equality follows from

\[
\det(l_f + \frac{a+f^2}{e} w_e, l_f) = \left( \frac{a+e^2}{f} \right)^2 \quad \text{and} \quad \det(l_e, l_f + \frac{a+f^2}{e} w_e) = f^2,
\]

which both are immediate consequences of formulas 9.1. The second equality follows from a simple rearrangement of terms. \( \square \)

For \( \chi(\mathcal{E}, \mathcal{F})^2 \leq 1 \) we see that the transformations of \( w_e \) and \( w_f \) reflect the periodic behaviour under subsequent permutations which we have observed in Remark 8.10. For \( \chi(\mathcal{E}, \mathcal{F})^2 > 0 \) we observe the following.

**Corollary 9.4:** With above notation, consider the pair \( \mathcal{F}, R_{\mathcal{F}} \mathcal{E} \) and assume that \( \chi(\mathcal{E}, \mathcal{F}) \neq 0 \). If \( \text{rk} R_{\mathcal{F}} \mathcal{E} \neq 0 \), then \( w_e \) and \( w_f \) transform in the sublattice which they generate by the matrix \( \begin{pmatrix} \chi(\mathcal{E}, \mathcal{F})^{-2} & 0 \\ 0 & 1 \end{pmatrix} \). If \( \text{rk} L_{\mathcal{E}} \mathcal{F} \neq 0 \), \( w_e \) and \( w_f \) transform by \( \begin{pmatrix} 0 & \chi(\mathcal{E}, \mathcal{F})^{-1} \\ 1 & 0 \end{pmatrix} \)

**Remark 9.5:** For \( \chi(\mathcal{E}, \mathcal{F})^2 > 4 \), Corollary 9.4 gives us another method for computing the terms \( \frac{a_i' - a_i}{\alpha_i - \alpha} \) of Corollary 9.3 for \( \mathcal{E}_0 = \mathcal{E} \) and \( \mathcal{E}_1 = \mathcal{F} \). It follows by induction that for \( i \geq 1 \), \( \frac{a_i - a_i}{\alpha_i - \alpha} \) coincides with the upper left entry of the \((i-1)\)-st power of the matrix \( \begin{pmatrix} \chi(\mathcal{E}, \mathcal{F})^{-1} & 0 \\ 0 & 1 \end{pmatrix} \)

The following statement is less nice but stronger, as it in particular implies that \( w_e \) and \( w_f \) do not change their lattice length in \( N \) under mutation.

**Lemma 9.6:** We have \( \det (w_e, w_f) w_e + w_f = G w_f \) and \( \det (w_e, w_f) w_f + w_e = G' w_e \) for \( G, G' \) in \( \text{GL}_2(N) \).

**Proof.** We consider only the first equality; the second equality follows analogously. By construction, \( \det (w_e, w_f) w_e + w_f = \frac{1}{2} (l'' - l_e) \) which is a lattice vector by Lemma 9.2. We show that its lattice length coincides with that of \( w_f \), which then implies the assertion. For this, we consider the exceptional triples \( \mathcal{E}_0, \mathcal{F}, R_{\mathcal{F}} \mathcal{E} \) and \( L_{\mathcal{E}_0} \mathcal{E}, \mathcal{E}_0, \mathcal{F} \) (for simplicity, we assume that \( c_0 \neq 0 \), which can always be arranged). In the first case, the exceptional pair \( \mathcal{E}_0, \mathcal{F} \) gives rise to a relation \( f^2 l_{n-1} + b l_e + c^2 d l_f = 0 \), and in the second case we get \( f^2 l'_e + b l_e + c^2 d l_f = 0 \), where \( l'_e \) is the mutation of \( l_e \) corresponding to \( L_{\mathcal{E}_0} \mathcal{E} \). As the triples \( l_{n-1}, l_e, l' \) and \( l'_e, l_f, l_l \) correspond to the same triple of volumes \( f^2, b, c^2 \), it follows by Lemma A.5 that \( l' - l_e = G \cdot (l_f - l) \) for some \( G \in \text{GL}_2(N) \), which implies the assertion. \( \square \)

**9.7.** We can consider the value \( e f \chi(\mathcal{E}, \mathcal{F}) = \det (p_e, p_f) = a + e^2 + f^2 \) as a lattice-geometric measure of the convexity of the configuration of lattice vectors \( l_e, l_f \). More precisely, if we consider the angle between \( p_e \) and \( p_f \) at \( l \), then we have three possibilities:

\[
\begin{align*}
a + e^2 + f^2 &> 0 \\ a + e^2 + f^2 &< 0 \\ a + e^2 + f^2 &= 0
\end{align*}
\]

(\text{convex}), (\text{concave}), (\text{flat}).

Figure 10 schematically depicts these possibilities for the case \( a < 0 \).
**Definition 9.8:** Corresponding to above inequalities, we call an exceptional pair \( \mathcal{E}, \mathcal{F} \) with \( e, f \neq 0 \) either convex, concave, or flat. We call the value \( a + e^2 + f^2 \) the convexity of the pair \( \mathcal{E}, \mathcal{F} \).

The following lemma shows that very often we can decrease the convexity of an exceptional pair by mutation.

**Lemma 9.9:** Let \( \mathcal{E}, \mathcal{F} \) be a convex exceptional pair and assume that \( a < 0 \). Then either \( \max\{\text{rk } L_{e}\mathcal{F}\}^2, e^2\} < \max\{e^2, f^2\} \) or \( \max\{\text{rk } R_{e}\mathcal{E}\}^2, f^2\} < \max\{e^2, f^2\} \).

**Proof.** We can assume without loss of generality that \( f < 0 \). As we minimize the absolute value of ranks it is guaranteed that this iteration will terminate. We then finalize this procedure by moving all rank zero objects by right mutations to the leftmost range. □

**9.10.** Observe that flatness implies \( a < 0 \) and that mutation preserves flatness. In particular, for a flat pair we have \( \text{rk } L_{e}\mathcal{F} = -f \) and \( \text{rk } R_{e}\mathcal{E} = -e \) and it can never occur that iterative mutations of a non-flat pair can result in a flat pair.

**9.11.** So, the lemma implies that for any convex pair with \( a < 0 \), we can obtain by mutation a sequence of exceptional pairs where both the maximal rank as well as the convexity strictly decrease while \( a \) remains constant, until either we arrive at a concave pair or one of the exceptional objects acquires ranks zero.

**Corollary 9.12:** Let \( \mathcal{E}_1, \ldots, \mathcal{E}_n \) be an exceptional sequence, then we can produce by mutation an exceptional sequence \( Z_1, \ldots, Z_t, F_1, \ldots, F_n-t \) such that the \( Z_i \) have rank zero and for any \( 1 \leq i \leq n-t \), if \( F_i, F_{i+1} \) is not a flat or concave exceptional pair then \( \bar{A}_i^2 \geq 0 \), where \( \bar{A}_i \) is the corresponding element of the contracted toric system.

**Proof.** We iterate the following procedure.

1) Choose any pair \( \mathcal{E}_i, \mathcal{E}_j \) with \( 1 \leq i < j \leq n+1 \) (recall that we are working with the cyclically extended sequence) and \( e_i, e_j \neq 0 \) such that \( e_k = 0 \) for all \( i < k < j \). If \( \mathcal{E}_i, \mathcal{E}_j \) is a convex pair and \( c_i(\mathcal{E}_i, \mathcal{E}_{i+1})^2 < 0 \) then continue with steps 2) and 3).

2) Move all \( \mathcal{E}_k \) for \( i < k < j \), to the position left of \( \mathcal{E}_i \) via right mutation. We denote \( R_{\mathcal{E}_{i-1}} \cdots R_{\mathcal{E}_{i+1}} \mathcal{E}_i =: \mathcal{E}_i' \). The resulting pair \( \mathcal{E}_i', \mathcal{E}_j \) is still convex and \( (\text{rk } \mathcal{E}_i')^2 = e_i^2 \).

3) We iterate mutations as in 9.11 and successively minimize the maximal rank of the resulting mutated pairs until either the pair becomes concave or a mutation results in an object of rank zero. We repeat these steps until the resulting sequence does not contain any convex pair \( \mathcal{E}_i, \mathcal{E}_j \) with \( c_1(\mathcal{E}_i, \mathcal{E}_j)^2 < 0 \). As we minimize the absolute value of ranks it is guaranteed that this iteration will terminate. We then finalize this procedure by moving all rank zero objects by right mutations to the leftmost range. □

**10. The Global Picture**

So far, we have established that the Gale transforms of a toric system at least locally represent the data of a complete toric surface. That is, by Corollary 8.3 and Proposition 8.8 every triple \( l_{i-1}, l_i, l_{i+1} \) generates a fan with two maximal cones which intersect in a facet such that \( l_{i-1}, l_i, l_{i+1} \) are the primitive vectors which generate the 1-dimensional cones. In this section we want to show that all the \( l_i \) indeed form the set of primitive vectors which generate the fan of a complete toric surface. For this, we can...
start with the two cones $\sigma_1, \sigma_2$, generated by, say, $l_1, l_2, l_3$. Then clearly, we can try to add a third cone $\sigma_3$ generated by $l_1$ and $l_4$. By construction, this cone lies in counterclockwise direction from the first two cones, and we know that $\sigma_2$ and $\sigma_3$ again form a fan with two maximal cones. However, so far we do not have any information on whether $\sigma_1, \sigma_2, \sigma_3$ fit together to form a proper fan. For this, we would have to prove that either $\sigma_1 \cap \sigma_3 = \{0\}$ or $\sigma_1 \cap \sigma_3 = \mathbb{Q}_{\geq 0} l_1$ (which implies $l_1 = l_4$). Similarly, if we successively add $\sigma_i = (l_i, l_{i+1})\mathbb{Q}_{\geq 0}$ in counterclockwise fashion, we have to show that $\sigma_i$ obeys the correct intersection properties with the previously added cones.

The only possibility that this construction can violate these intersection properties is that for some $3 \leq i < n$, the intersection $\sigma_1 \cap \sigma_i$ is a two-dimensional cone by itself. Then $\sigma_1, \ldots, \sigma_i$ cover all of $\mathbb{N}_2$ for some $i < n$ without closing up to a proper fan. Continuing this way, we would end up with a sequence of cones which in counterclockwise order cover $\mathbb{N}_2$ several times by “winding” around the origin until finally $\sigma_n$ and $\sigma_1$ close up these windings via the triple $l_n, l_1, l_2$. We are going to show that there indeed can only be one winding.

10.1. Recall that in Section 8 we used the simplified assumption that $e_i \neq 0$ for all $i$. We cannot make this simplification in this section, but instead we will make use of the fact that the number of windings does not depend on the existence of objects of rank zero among the $\tilde{\mathcal{E}}_i$. Indeed, as already remarked in §7.4, if $e_i = 0$ for some $i$ then we can use mutation to produce a sequence which only contains objects of nonzero rank. By Lemmas 8.7 and 9.3 this is a completely local operation which cannot change the number of windings. This is also true for the converse process, where rank zero objects are created by mutation. Our strategy will be to use Corollary 9.12 in order to create a sequence of the form $\mathcal{Z}_1, \ldots, \mathcal{Z}_{n-3}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ with $z_i = 0$ and $e_i \neq 0$. Without loss of generality we can assume that $e_i > 0$ for all $i$. Then with the notation of the previous section and Section 8 we have one single relation $e_2^2 l_1 + e_3^2 l_2 + e_3^2 l_3 = 0$. The problem of additional windings does not occur in this case and we obtain a fan of a weighted projective space $\mathbb{P}(c_1^2, c_2^2, c_3^2)$ and we would like to determine the possible values for the $e_i$.

The $E_i = c_1(\mathcal{Z}_i)$ form the basis of a maximal negative definite subspace of $\text{CH}_{\text{num}}^1(X)$. As we have seen in Section 8 possibly after twisting the sequence with an appropriate line bundle we can assume that $E_i \cdot c_1(\mathcal{E}_j) = 0$ for all $i, j$. Moreover, because the exceptional sequence generates $\text{K}_{\text{num}}^1(X)$, the $E_i$ together with the $c_1(\mathcal{E}_i)$ form a generating set of $\text{CH}_{\text{num}}^1(X)$. Now, as $E_1, \ldots, E_{n-3}$ together with $A_1, A_2, A_3$ generate a $\mathbb{Q}$-span of $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ is isomorphic to $\mathbb{Q}$ and contained in the orthogonal complement of the $E_i$. If we denote $H \in \langle \tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \rangle \mathbb{Q} \cap \text{CH}_{\text{num}}^1(X)$ a minimal integral element, then by the unimodularity of the intersection pairing, $E_1, \ldots, E_n$ and $H$ necessarily form an orthogonal basis of $\text{CH}_{\text{num}}^1(X)$, in particular we have $H^2 = 1$ and $\tilde{A}_i = \alpha_i H$ with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}$. Now we denote $J := -K_X + \sum_{i=1}^n (1 + 2\delta_i) E_i = \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3 = \gamma H$ for some $\gamma \in \mathbb{Z}$. Then from

\[ \tilde{A}_{i-1} \cdot \tilde{A}_i = \frac{1}{e_i^2} \]

for $i = 1, 2, 3$ we compute

\[ \alpha_i^2 = \frac{e_i^2}{e_{i+2}^2 + e_{i+1}^2} . \]

If $\alpha_i = -e_{i+1}/(e_i e_{i+1})$ for one $i$, it follows that $\alpha_i = -e_{i+2}/(e_i e_{i+1})$ for all $i$, hence, after possibly exchanging $H$ with $-H$, we can assume without loss of generality that $\alpha_i = e_{i+2}/(e_i e_{i+1})$ for $i = 1, 2, 3$. Now with the identities $[3, 5]$, $[4]$ & $[15]$: $\chi(\mathcal{E}_1, \mathcal{E}_2) = \frac{1}{e_1 e_2} (c_1(\mathcal{E}_1, \mathcal{E}_2)^2 + e_2^2 + e_2^2) = \frac{1}{e_1 e_2} (e_3^2 + e_3^2 + e_3^2) = -K_X \cdot c_1(\mathcal{E}_1, \mathcal{E}_2) = J \cdot c_1(\mathcal{E}_1, \mathcal{E}_2) = \gamma e_3$. So, the $e_i$ must satisfy the following equation:

\[ e_1^2 + e_2^2 + e_3^2 = \gamma e_1 e_2 e_3 . \]

It is well known that this equation admits integer solutions if and only if $\gamma \in \{1, 3\}$ (see e.g. [Aig13 §2.1]) and $(e_1, e_2, e_3)$ is a solution for the case $\gamma = 1$ iff $(e_1/3, e_2/3, e_3/3)$ is a solution for the case $\gamma = 3$. 


As necessarily \( \gcd\{e_1, e_2, e_3\} = 1 \), the result is that \( \gamma = 3 \) and the \( e_i \) satisfy the Markov equation:

\[
e_i^2 + e_j^2 + e_k^2 = 3e_ie_je_k.
\]

For the case \( n = 3 \) this reproduces a well-known result of Rudakov [Rud80] for \( \mathbb{P}^2 \) from a purely combinatorial perspective. Also, as in the case for \( \mathbb{P}^2 \) we can use mutations such that after finitely many steps we obtain \( e_1^2 = e_2^2 = e_3^2 = 1 \). Above observation is also a combinatorial variant of results of Hacking [HP10, Hac13] which gives a correspondence of the construction of exceptional sequences on \( \mathbb{P}^2 \) to \( \mathbb{Q} \)-Gorenstein degenerations of \( \mathbb{P}^2 \) whose exceptional fiber is a weighted projective spaces \( \mathbb{P}(e_1^2, e_2^2, e_3^3) \), where the \( e_i \) satisfy the Markov equation.

We are now finally able to deal with the second Chern classes of rank zero objects. The following result is an important step towards proving Theorem 1.11.

**Proposition 10.3:** As in 10.2, let \( Z_1, \ldots, Z_{n-3}, E_1, E_2, E_3 \) be an exceptional sequence with \( z_i = 0 \) and \( e_j \neq 0 \). Then \( \delta_i \in \{0, 1\} \) for every \( i \).

**Proof.** With the notation of 10.2 and our general assumption that \( K_X^2 = 12 - n \), we have

\[
12 - n = K_X^2 = (\sum_{i=1}^{n-3} (1 + 2\delta_i) E_i)^2 - 4\sum_{i=1}^{n-3} \delta_i(\delta_i + 1)
\]

therefore \( \sum_{i=1}^{n-3} \delta_i(\delta_i + 1) = 0 \) which implies \( \delta_i \in \{0, -1\} \) for every \( i \). \( \square \)

### 10.4. Another special case which we have to settle is \( n - t = 4 \). This case exhibits a nice symmetry with \( a_{i+1} = \det(\tilde{l}_{i+2}, \tilde{l}_i) = -\det(\tilde{l}_i, \tilde{l}_{i+2}) = -a_{i-1} \) for every \( i \).

If no opposing pair, i.e. an \( \tilde{l}_i \) such that \( \tilde{l}_{i+2} = -\tilde{l}_i \) exists then it is elementary to see that the \( \tilde{l}_i \) can produce only one winding and for some \( i \) we have \( a_{i-2}, a_{i-1} > 0 \) and \( a_i, a_{i+1} < 0 \). If we move the rank zero objects such that our exceptional sequence is of the form \( Z_1, \ldots, Z_{n-4}, E_1, E_{i+1}, E_{i+2}, E_{i+3} \), then the two pairs \( E_i, E_{i+1} \) and \( E_{i+1}, E_{i+2} \) cannot be both concave, because otherwise the \( \mathbb{Q}_{\geq 0} \)-span of the \( \tilde{l}_i \) could not generate \( N_Q \). This means that whenever there is no opposing pair \( \tilde{l}_{i+2} = -\tilde{l}_i \), we can use Lemma 9.9 in order to decrease the maximal rank of a concave pair. By iteration we will eventually end up in one of two cases:

1) We produce a pair \( \tilde{l}_{i+2} = -\tilde{l}_i \).
2) We produce a sequence of the form \( Z_1, \ldots, Z_4, Z, F_1, F_2, F_3 \) with \( z = 0 \) and \( f_{i+2}^2 > 0 \) for \( i = 1, 2, 3 \).

Then it follows from 10.2 that the \( f_i \) must satisfy the Markov equation \( f_1^2 + f_2^2 + f_3^2 = 3f_1f_2f_3 \).

**Proposition 10.5:** Let \( Z_1, \ldots, Z_{n-4}, E_1, E_2, E_3, E_4 \) be an exceptional sequence with \( z_i = 0 \) and \( e_j \neq 0 \) and assume that \( \tilde{l}_j = -\tilde{l}_{j+2} \) for some \( 1 \leq j \leq 4 \). Then \( \delta_i \in \{0, -1\} \) for every \( i \) and \( e_j^2 = 1 \) for every \( j \).

**Proof.** If \( \tilde{l}_j = -\tilde{l}_{j+2} \), then \( a_{j-1} = a_{j+1} = \det(\tilde{l}_j, \tilde{l}_{j+2}) = 0 \). So, by Lemma 8.4 we get that \( e_{j-1}^2 = e_j^2 \) and \( e_{j+1}^2 = e_{j+2}^2 \). As in 10.2 we have an orthogonal decomposition \( CH^1_{num}(X)_Q \supset A = \langle E_1, \ldots, E_{n-4} \rangle \subset \langle \tilde{A}_1, \ldots, \tilde{A}_4 \rangle \mathbb{Z} \) with \( \sum_{k=1}^{4} \tilde{A}_j = J = -K_X - \sum_{i=1}^{n-4} (1 + 2\delta_i) E_i \) and get:

\[
K_X^2 = 12 - n = J^2 - (n - 4) - 4\sum_{i=1}^{n-4} \delta_i(\delta_i + 1)
\]

Moreover, with \( a_{j-1} = a_{j+1} = 0 \), \( a_j = -a_{j+2} \), \( e_{j-1} = e_{j+1} \), and \( e_j = e_{j+2} \), we get \( \tilde{A}_{j-1}^2 = \tilde{A}_{j+1}^2 = 0 \), \( \tilde{A}_j^2 = -\tilde{A}_{j+2}^2 \), and:

\[
J^2 = \left( \sum_{k=1}^{4} \tilde{A}_k \right)^2 = 4(\frac{1}{e_j^2} + \frac{1}{e_{j+1}^2}).
\]
Substituting this into the previous formula, we get:
\[
\sum_{i=1}^{n-4} \delta_i (\alpha_i + 1) = \frac{1}{c_i} + \frac{1}{c_{i+1}} - 2,
\]
where the right hand side is integral iff $c_i = c_{i+1} = 1$. So,
\[
\sum_{i=1}^{n-4} \delta_i (\alpha_i + 1) = 0,
\]

hence $\delta_i \in \{0, -1\}$ for every $i$. \quad \square

By Proposition 10.5 the $\bar{l}_i$ generate the fan of a Hirzebruch surface and we recover a special case of the corresponding result for exceptional sequences of line bundles as was proven in [HP11] Theorem 3.5
We also have reproduced a result of Nogin [Nog01 §3] which states that every exceptional sequence on a Hirzebruch surface can be mutated to a sequence consisting of objects of rank one (see also Example 10.11).

The following Lemma helps to separate the cases $n - t \leq 4$ and $n - t > 4$.

Lemma 10.6: Let $n - t \geq 4$.

(i) Assume that $\tilde{A}_i^2 \geq 4$ for some $i$, then there exists at most one other $\tilde{A}_j$ such that $\tilde{A}_j^2 > 0$. If so, then $j$ is either $i - 1$ or $i + 1$.

(ii) If $i \neq j$, $\tilde{A}_i^2 = 0$, $\tilde{A}_j^2 = 0$, and $\tilde{A}_i \cdot \tilde{A}_j = 0$ then $j = i + 2$ and $n - t = 4$.

(iii) If $\tilde{A}_i^2 = \tilde{A}_{i+1}^2 = 0$ for some $i$, then either $n - t = 4$ and $\tilde{A}_j^2 = 0$ for all $j$, or $n - t > 4$ and $\tilde{A}_i^2 < 0$ for all $j \neq i, i + 1$.

Proof. $CH_{\text{num}}^1(X)_Q$ is a metric space with respect to the intersection product which has an orthogonal decomposition $CH_{\text{num}}^1(X)_Q \simeq C^+ \perp C^-$ into strictly positive and strictly negative definitive part. This induces a decomposition of $\tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_{n-i}) = \tilde{A}^+ \perp \tilde{A}^-$ where $\tilde{A}^- = \tilde{A} \cap C^-$. By the Hodge Index Theorem we have $\dim_Q \tilde{A}^+ \leq 1$. Up to cyclic renumbering let us now assume that $\tilde{A}_1^2 \geq 0$. Then clearly $\tilde{A}_1$ and $\tilde{A}^-$ generate $\tilde{A}$ as a $Q$-vector space and for every $i$ we can write $\tilde{A}_i = \alpha_i \tilde{A}_1 + n_i$ for some $\alpha_i \in Q$ and $n_i \in \tilde{A}^-$. Now, for any $2 < i < n$ we have $0 = \tilde{A}_1 \cdot \tilde{A}_i = \tilde{A}_1 \cdot (\alpha_i \tilde{A}_1 + n_i) = \alpha_i \tilde{A}_1^2 + \tilde{A}_1 n_i$, hence $\alpha_i \tilde{A}_1^2 = -\tilde{A}_1 n_i$. Now, for $2 < i < n$ we compute $\tilde{A}_i^2 = (\alpha_i \tilde{A}_1 + n_i)^2 = \alpha_i \tilde{A}_1 \cdot (\alpha_i \tilde{A}_1 + 2n_i) + n_i^2 = \alpha_i \tilde{A}_1 + \alpha_i n_i + n_i^2 = \alpha_i \tilde{A}_1 n_i + n_i^2 = -\alpha_i^2 \tilde{A}_1^2 + n_i^2 \leq 0$, as both $-\alpha_i^2 \tilde{A}_1^2 \leq 0$ and $n_i^2 \leq 0$. If $\tilde{A}_i^2 > 0$ then this inequality is strict for every $i$. This leaves only $\tilde{A}_2$ and $\tilde{A}_n$ which can have positive self-intersection number. But because $n \geq 4$, we also have $\tilde{A}_2 \cdot \tilde{A}_n = 0$. So if one of $\tilde{A}_2, \tilde{A}_n$ has positive self-intersection, then we can argue as for $\tilde{A}_1$ and conclude that the other must have negative self-intersection and $i$ follows.

The same computation leads to $\tilde{A}_i^2 = n_i^2 = 0$, hence $n_i = 0$ and thus $\tilde{A}_i = \alpha_i \tilde{A}_1$. Then $\tilde{A}_i \cdot \tilde{A}_j \neq 0$ and $\tilde{A}_i \cdot \tilde{A}_j \neq 0$, which is only possible if $n = 4$ and $i = 3$.

We know from (i) that there cannot be any $\tilde{A}_j$ with $\tilde{A}_j^2 > 0$. If there exists a third $\tilde{A}_j$ with $\tilde{A}_j^2 = 0$, then $\tilde{A}_j \cdot \tilde{A}_i = 0$ or $\tilde{A}_j \cdot \tilde{A}_{i+1} = 0$ and by (iii) this implies $n - t = 4$ and $l_1 = -l_3$, $l_2 = -l_4$ by Lemma 8.2. Note that $\det(l_{i-1}, l_{i+1}) = 0$ for any $i$ implies $\alpha_i = 0$ and thus $\tilde{A}_i^2 = 0$. \quad \square

Proposition 10.7: Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be a numerically exceptional sequence with contracted toric system $\tilde{A}_1, \ldots, \tilde{A}_{n-t}$. Then the Gale duals $\bar{l}_1, \ldots, \bar{l}_{n-t}$ generate the fan corresponding to a complete toric surface such that the $\bar{l}_i$ are primitive vectors of the rays in this fan and the maximal cones are generated by $\bar{l}_i, \bar{l}_{i+1}$ for $1 \leq i < n - t$ and $\bar{l}_{n-t}, \bar{l}_1$.

Proof. By Corollary 9.12 we can produce by mutation an exceptional sequence $Z_1, \ldots, Z_s, F_1, \ldots, F_{n-s}$ with $s \geq t$ and contracted toric system $B_1, \ldots, B_{n-s}$, such that any exceptional pair $F_i, F_{i+1}$ (where we read the indices cyclically modulo $n - s$) with $B_i^2 < 0$ is concave or flat. We denote $k_1, \ldots, k_{n-s}$ the Gale duals of the $B_i$. The cases $n - s \leq 4$ have been covered in [HP12] and [HP13] so we assume $n - s \geq 5$. It follows from Lemma 10.6 that there exist at most two $B_i$ with $B_i^2 > 0$ which then must be adjacent.

Now consider any subsequent $B_j, B_{j+1}, \ldots, B_{j+r}$ such that $B_{j+r}^2 < 0$ for all $0 \leq j' \leq r$. Then the sequence of circumference segments $p_{j}, \ldots, p_{j+r}$ is non-convex with respect to the origin, i.e. for any $0 \leq j' < r$, the vector $k_{j'+1}$ is contained in the convex hull of $0, k_{j'+1}, k_{j'+1}$. But this implies that
all the $k_{j+j'}$ for $-1 \leq j' \leq r$ are contained in the same half space whose boundary is given by $Qk_{j-1}$. In particular, $k_{j-1}$ is the only one which is contained in the boundary.

With this observation it follows that there must be at least one $B_i$ with $B_i^2 > 0$, say $B_j$, and $B_j,\ldots,B_n$ is the maximal sequence with $B_j^2 < 0$, where $j = 2$ or $j = 3$. This implies that all cones except for possibly two (if $j = 2$) or three (if $j = 3$) are contained in a half space and it follows from elementary geometric arguments that we cannot produce more than one winding this way.

Now, as we have argued we can conclude that also the original contracted toric system does not produce more than one winding and the assertion follows.

Corollary 10.8: Let $E_1,\ldots,E_n$ and $Z_1,\ldots,Z_t, F_1,\ldots,F_{n-t}$ be numerically exceptional sequences as in Corollary 9.12. Then either $t = n - 3$ and $f_1, f_2, f_3$ satisfy the Markov equation of (10.2) or $t = n - 4$ and the $F_i$ are objects of ranks $\pm 1$. In particular, any exceptional sequence can by mutation be transformed into an exceptional sequence consisting only of objects of ranks $\pm 1$ and 0.

Proof. The assertions follow from Proposition 10.7 and the elementary observation that a fan of a complete toric surface with at least 5 rays always contains a configuration of adjacent primitive vectors, $k_1,k_2,k_3$, say, with $\det(k_3,k_1) < 0$ and which are convex in the sense of definition 9.7. Hence, the procedure of Corollary 9.12 always results in an exceptional sequence with $n - t \leq 4$. Then as in the last part of 10.3, if $n - t = 4$ then by Proposition 10.8 we know that the $F_i$ have ranks $\pm 1$. If $n - t = 3$, the $F_i$ might have ranks different from $\pm 1$, but as in the case of $\mathbb{P}^2$ we can conclude that by further mutation we can transform the sequence to $Z_1,\ldots,Z_{n-3}, F'_1, F'_2, F'_3$ with $\text{rk} F'_i = \pm 1$ for $i = 1,2,3$.

We are now ready to prove Theorem 10.11

Proof of Theorem 10.11: Let $Z,E_2,\ldots,E_h$ be an exceptional sequence with $z = 0$. If we follow the procedure described in the proof of Corollary 9.12 we can produce an exceptional sequence of the form $Z_1 = Z, Z_2,\ldots,Z_t,E_1,\ldots,E_{n-t}$, where $z_i = 0$ and $e_j^2 \neq 0$. By Corollary 10.8 we can even assume that $n - t = 3$ or $n - t = 4$ and $e_j^2 = 1$ for all $j$. Then we can apply either Proposition 10.3 or Proposition 10.5 to show that $\delta_i \in \{0, -1\}$ for every $i$ and in particular $\delta_1 = \delta_2 \in \{0, -1\}$.

The following theorem can also be considered as a corollary of Proposition 10.7 and Corollary 10.8.

Theorem 10.9: Let $X$ be a numerically rational surface. Then any numerically exceptional sequence on $X$ can be transformed by mutation into a numerically exceptional sequence consisting only of objects of rank one.

Proof. By Corollary 10.8 every sequence can be transformed into an exceptional sequence of objects of rank one and zero. To this sequence is associated the fan of a toric surface which is given by the Gale duals $l_1,\ldots,l_{n-t}$ of the contracted toric system. By assumption, we have $\det(l_{i+1}) = 1$ for all $i$, hence the toric surface is smooth. If we instead consider the Gale duals of the uncontracted toric system, we obtain by Proposition 6.3 the same fan, but where the rays come with possible multiplicities, i.e. if $E_i,E_{i+1},\ldots,E_j$ is a sub-sequence for $i < j$ such that $e_k = 0$ for all $i < k < j$, then the multiplicity of the Gale dual of the contract element $s(E_i) - s(E_i) + \sum_{k=i+1}^{j-1} e_i(E_k)$ is $j - i$. For any pair $E_i,E_{i+1}$ with $e_i = 0$, we have $\text{rk} L_E E_{i+1} = -e_i$ and $\text{rk} R_{E_{i+1}} E_i = -e_{i+1}$ (and similarly if $e_{i+1} = 0$). We have seen in Section 7 the mutation from $E_i,E_{i+1}$ to $L_E E_{i+1}, E_i$ leaves the associated fan unchanged, except for a possible exchange of multiplicities. On the other hand, if $e_{i+1} \neq 0$, then by Theorem 4.11 and Lemma 8.7 the mutation to $E_{i+1}, R_{E_{i+1}} E_i$ yields a new object of rank $\pm 1$, and the associated fan obtains a new primitive vector $l$, whose position is given by the relation $e_{i+1}^2 l_k - e_{i+1}^2 l_i + e_{i+1} e_{i+1} l_{k+1} = 0$ for the corresponding $1 \leq k \leq n - t$, and with $e_{i+1}^2 = 1$ we get $l' = l_k + l_{k+1}$. That is, the right mutation yields a bigger fan which corresponds to a toric blow-up of the original fan. Similarly, if $e_i \neq 0$ and $e_{i+1} = 0$, then left mutation results in a blow-up as well. Now we can iterate the following two steps of mutations $E_i,E_{i+1}$ to $L_E E_{i+1}, E_i$ (respectively $E_{i+1}, R_{E_{i+1}} E_i$):

1) If $e_i = 0$ (respectively $e_{i+1} = 0$), which only affect multiplicities of the primitive vectors.
2) If $(e_i^2, e_{i+1}^2) = (1,0)$ (respectively $(e_i^2, e_{i+1}^2) = (0,1)$), which correspond to smooth blow-ups.

Iterating these steps at will we can realize every smooth toric surface which can be obtained from the original $l_1,\ldots,l_{n-t}$ by at most $t$ blow-ups.
Remark 10.10: By Theorem [10.9] any toric system coming from a numerically exceptional sequence indeed can be constructed by mutation from a toric system associated to rank one objects as were considered in [HP11] (see also Remark 5.6). Note that the relevant analysis of [HP11, §2] is strictly on the numerical level and therefore also applies to any numerical exceptional sequence whose elements’ classes in $K_0^{num}(X)$ coincide with that of invertible sheaves. Moreover, note that any numerically exceptional object $L$ of rank one has the same class in $K_0^{num}(X)$ as an invertible sheaf $O(D)$, where the class of $D$ in $\text{CH}^1(X)$ coincides with $c_1(L)$. This is easy to see from $c_2(L) = 0$ by [3.3] and by the injectivity of the Chern-isomorphism with $\text{ch}(L) = \text{ch}(O(D)) \in \text{CH}^1(X)_Q$.

Example 10.11: We construct examples of exceptional objects of rank one which are not isomorphic to invertible sheaves. Consider an even Hirzebruch surface $\mathbb{F}_a$, where $a = 2b$ for $b > 0$ and denote $P, Q$ the generators of its nef cone, where $P^2 = 0, Q^2 = a$ and $P \cdot Q = 1$. In [HP11, Proposition 5.2] it was shown that $\mathbb{F}_a$ admits two families of toric systems for exceptional sequences consisting of objects of rank one, which are of the following form:

$$F_1(s) : (-s-b)P + Q, P + s(-bP + Q), -bP + Q, P + s(-bP + Q)$$

for $s \in \mathbb{Z}$.

Note that both families meet at $s = 0$. Any toric system of type $F_1(s)$ corresponds to an exceptional sequence of invertible sheaves $L = L_1, \ldots, L_4$. Assume that, say, $c_1(L_2, L_3) = P$, then $rk L_{L_2, L_3} = rk R_{L_2, L_3} = 1$ and both mutations are isomorphic to invertible sheaves, and $c_1(L_{L_2, L_3, L_2}) = c_1(L_3, R_{L_2, L_3}) = P$. Moreover, $c_1(L_1, L_{L_2, L_3}) = c_1(L_1, L_2) - 2P$ and $c_1(R_{L_2, L_3}) = c_1(L_3, L_4) - 2P$. That is, from a sequence of type $F_1(s)$ we obtain by left mutation of the pair $L_2, L_3$ a sequence of type $F_1(s-1)$ and by right mutation a sequence of type $F_1(s+1)$ and we can conclude that the exceptional sequences of invertible sheaves of type $F_1(s)$ can be transformed into each other via mutations (note 2 facts: 1. that this is strictly true only up to overall twist by an invertible sheaf; 2. $Y(L) \simeq F_2$, which due to the type of the intersection form is the only allowed case). We can argue similarly that exceptional sequences of type $F_2(s)$ can be transformed into each other via mutations.

Both families intersect in the particular case $s = 0$, where $Y(L) \simeq P^1 \times P^1$. Then, starting with an exceptional sequence of invertible sheaves representing $F_1(0) = F_2(0)$, we can produce both families by either mutating the pair $L_2, L_3$ (to obtain $F_1(s)$ or $L_1, L_2$ for $F_2(s)$). However, by [HP11, Proposition 5.2], for $b > 0$ and $s \neq 0$, the resulting sequences of type $F_2(s)$ cannot consist entirely of invertible sheaves.

Remark 10.12: Recent work by Vial [Via15] suggests that it should be possible to relax the conditions on our surface $X$ by dropping the requirement that $K_X^2 = 12 - n$. We will give a brief outline on how this condition can be dropped in many cases, though we are not able to remove it completely.

First, we point out that Corollary [10.3] can indeed be proved without $K_X^2 = 12 - n$. An analysis of the proof shows that this condition enters only in Proposition [10.3] in order to derive the integrality of $1/c_j + 1/c_{j+1}$ (and thus $c_j^2 = c_{j+1}^2 = 1$). Without the condition, we get only that $4 \left( 1/c_j + 1/c_{j+1} \right)$ is integral, so that we have, up to order, the additional possibilities $c_j^2 = c_{j+1}^2 = 4$ and $c_j^2 = 1, c_{j+1}^2 = 4$. We can exclude the first case right away, because necessarily gcd$(e_1, e_2, e_3, e_4) = 1$. For the second case, we can assume without loss of generality that $c_1^2 = c_2^2 = 1, c_3^2 = c_4^2 = 4$ and $\Delta_t = -\tilde{\Delta}_t$. Moreover, $\tilde{\Delta}_t$ form a basis and we can assume without loss of generality that $\Delta_t = -4\tilde{\Delta}_t + (4k + 1)\tilde{\Delta}_t$ for some integer $k$. By successive mutation of the pair $E_3, E_4$, we can arrange that $k = 0$ (see Section 9). Then the vectors $\hat{\Delta}_t, \tilde{\Delta}_t, \tilde{\Delta}_t$ are in a convex configuration with $a_2 < 0$ in the sense of Lemma 10.9. Then by mutating the pair $E_3, E_4$, we obtain $Z_1, \ldots, Z_{n-4}, E_1, L_{E_2, E_3, E_2, E_4}$, where we compute $rk L_{E_2, E_3} = 0$, which puts us into the position of Proposition 10.3. By the first displayed formula in the proof of Proposition 10.3, we have:

$$K_X^2 = 9 - n - 4 \sum_{i=1}^{n-4} \delta_i(\delta_i + 1) \equiv 1 - n \quad (8).$$

However, for the case $n - t = 3$ we have by the formula in the proof of Proposition 10.3:

$$K_X^2 = 12 - n - 4 \sum_{i=1}^{n-3} \delta_i(\delta_i + 1) \equiv 4 - n \quad (8).$$
With this contradiction, we can exclude the case $e_j^2 = 1, e_{j+1}^2 = 4$ and conclude that Corollary 10.8 indeed holds without the assumption $K_X^2 = 12 - n$.

Second, with the previous remarks we have established that

$$4 \sum_i \delta_i(\delta_i + 1) = (10 - K_X^2) - \text{rk} \, \text{CH}^1_{\text{num}}(X),$$

in particular both sides of this equation are divisible by 8 (this was pointed out to me by C. Vial). Hence, unless $(10 - K_X^2) - \text{rk} \, \text{CH}^1_{\text{num}}(X)$ is nonzero and divisible by 8, the condition $\chi(O_X) = 1$ and the existence of an exceptional sequence of maximal length already imply $K_X^2 = 12 - n$. All our main results then remain true under these weaker assumptions.

11. THE MAIN THEOREM

We are now in possession of everything we need in order to show our main theorem.

Theorem 11.1: Let $X$ be a numerically rational surface with $\text{rk} \, K_0^\text{num}(X) = n$ and $E = E_1, \ldots, E_n$ a numerically exceptional sequence. Then to the maximal sub-sequence of objects of nonzero rank $E_{i_1}, \ldots, E_{i_{n-1}}$ there is associated in a canonical way a complete toric surface $Y(E)$ with $n$ torus fixpoints. These fixpoints are either smooth (if $e_{i_j} = 1$) or $T$-singularities of type $\frac{1}{e_{i_j}}(1, k_{i_j} e_{i_j} - 1)$ with $\gcd(k_{i_j}, e_{i_j}) = 1$.

Remark 11.2: To justify “canonical”, we should, strictly speaking, also incorporate the multiplicities of the rays of $Y(E)$ coming from the rank zero objects. However, in light of our discussion in Section 7 at least on the combinatorial level there seems not much to be gained from this.

In absence of rank zero objects, we can state Theorem 11.1 in a more convenient form.

Theorem 11.3: Let $X$ be a numerically rational surface and let $E = E_1, \ldots, E_n$ be a numerically exceptional sequence of maximal length such that $(\text{rk} \, E_i)^2 = e_i^2 > 0$ for every $i$. Then to this sequence there is associated in a canonical way a complete toric surface $Y(E)$ with $n$ torus fixpoints which are either smooth (if $e_i^2 = 1$) or $T$-singularities of type $\frac{1}{e_i^2}(1, k_i e_i - 1)$ with $\gcd(k_i, e_i) = 1$. Moreover, this correspondence induces a natural isomorphism of Chow rings $\text{CH}^*(Y(E))_\mathbb{Q} \to \text{CH}^*_{\text{num}}(X)_\mathbb{Q}$ which maps $K_{Y(E)}$ to $K_X$.

Remark 11.4: Note that the ring isomorphism exists by construction and is given by mapping the class of the $t$-th torus invariant prime divisor $D_t$ of $Y(E)$ to $A_t$ of the associated toric system (see A.6). It follows that $K_{Y(E)} = - \sum_{i=1}^n D_i$ maps to $K_X = - \sum_{i=1}^n A_i$.

Proof of Theorem 11.1: By Proposition 10.4 we have constructed our toric variety as the Gale duals of the contracted toric system. It only remains to show that this surface indeed can only have at most $T$-singularities. Consider the vectors $l_1, \ldots, l_n$ generating the fan of $Y(E)$ and their corresponding $n - t$ circumference segments $p_i = l_i - l_{i-1}$. For any given $i$ with $e_i^2 > 1$, with a convenient choice of coordinates we can represent the vectors $l_{i-1}, l_i$ as $l_{i-1} = (1, 0)$ and $l_i = (-x, e_i^2)$ for some $0 < x < e_i^2$. We have seen in Lemma 9.2 that the vector $w_i = \frac{1}{e_i^2} p_i$ is integral, hence we get $x = k_i e_i - 1$ for some $1 \leq k_i \leq e_i$. So, by Lemma A.3 in order to show that $\gcd(k_i, e_i) = 1$ it suffices to show that $w_i$ is primitive. By Lemmas 9.2 and 9.6 we know that any left or right mutation of a pair $E_i, E_{i+1}$ with $e_i, e_{i+1} \neq 0$ which results in a pair $E'_i, E'_{i+1}$ such that $\epsilon_i, \epsilon_{i+1} \neq 0$ leaves the lattice lengths of the involved $w_i$ invariant. By Theorem 10.9 we can transform any sequence by mutation into a sequence of length $n$ of objects of rank one, where obviously the $w_i$ are primitive for every $1 \leq j \leq n$. However, in this process we might destroy or create new circumference segments by creating or destroying objects of rank zero, and it remains to check whether we obtain non-primitive $w_j$ this way. By Theorem 10.11 and Lemma A.7 for every pair $E_i, E_{i+1}$ with $\text{rk} \, E_i E_{i+1} = 0$ we have the relation $e_i^2 l_{i-1} - e_{i+1}^2 l_i + e_{i+1}^2 l_{i+1} = 0$, respectively we have $l_i = e_i^2 l_{i-1} - l_{i+1}$. Then mapping $l_i$ to $l_{i-1}$ extends to a linear map on $N$ which leaves $l_{i-1}$ invariant and thus the pairs $l_{i-1}, l_i$ and $l_i, l_{i+1}$ both correspond to cones whose associated affine toric surfaces are isomorphic, hence $l_{i+1} - l_{i-1}$ has lattice length $e_i$ iff $l_i - l_{i-1}$ has. The difference $l_{i+1} - l_i = e_i^2 l_{i-1}$ clearly corresponds to the circumference segment of a $T$-singularity of order $e_i^2$. This concludes the proof of the theorem. □
11.5. We conclude this section with an observation on cyclic strongly exceptional sequences. In [HP11] Theorems 5.13, 5.14 it was shown that any rational surface which admits a cyclic strongly exceptional sequence necessarily has Picard-rank at most 7 and that every del Pezzo surface meeting this conditions indeed does admit a cyclic strongly exceptional sequence of invertible sheaves. More generally, in [van09] van den Bergh constructed cyclic strongly exceptional sequences for all del Pezzo surfaces, which for the Picard-ranks 8 and 9 cannot consist of line bundles only. The crucial observation in [HP11] was the fact that for the rank one case the associated toric surface must be a weak del Pezzo surface or, equivalently, that the fan must be convex. This is easily seen to be true also in the general case, as by [HP11] \( \chi(E_i, E_{i+1}) \geq 0 \) implies the convexity of the triple \( l_{i-1}, l_i, l_{i+1} \). The following implication for the Picard-rank then is quite straightforward. Note that \( K_X^2 > 0 \) implies that \( \text{rk} \text{CH}^1_{\text{num}}(X) < 10 \).

**Theorem 11.6:** Let \( X \) be a numerically rational surface. Then the length of a cyclic strongly exceptional sequence on \( X \) is at most 11. In particular, if \( X \) admits a cyclic strongly exceptional sequence of maximal length, then \( K_X^2 > 0 \).

**Proof.** By Proposition 4.3 we can choose \( E_i, E_j \) with \( e_i e_j > 0 \) and \( i < j \). From strong cyclicity it follows that \( \chi(E_i, E_j), \chi(E_j, E_i) \geq 0 \). From the fact that the fan for \( Y(E) \) necessarily has a convex configuration of primitive vectors it follows that at least one inequality is strict, where up to cyclic renumbering we can assume without loss of generality that \( \chi(E_i, E_j) > 0 \). By Lemma 3.6 we have \( \chi(E_j, E_i) = e_i e_j K_X^2 - \chi(E_i, E_j) > 0 \), hence

\[
e_i e_j K_X^2 \geq \chi(E_i, E_j) > 0.
\]

from which our assertion follows. \( \Box \)

12. Some remarks on the singularities

Consider a numerically exceptional sequence \( E = E_1, \ldots, E_n \), where for simplicity we will assume that \( e_i > 0 \) for all \( i \). Then by the classification of Section 11, \( Y(E) \) belongs to a very nice class of toric surfaces with the following properties:

1) \( \text{det}(l_i, l_{i+1}) = e_i^2 \) for every \( i \).
2) The circumference segments \( p_i = l_i - l_{i-1} \) have lattice length \( e_i \) for every \( i \).
3) \( K_Y^2(E) = K_X^2 = 12 - n \).

For \( K_Y^2(E) \), we have by [4, 3, 3] and [4] the formulas (where \( q_i = p_i / e_i^2 \) in the notation of [4, 3]):

\[
K_Y^2(E) = \sum_{i=1}^n \text{det}(q_i q_{i+1}) = \sum_{i=1}^n \frac{1}{e_i e_{i+1}} \chi(E_i, E_{i+1}) = \sum_{i=1}^n c_i (E_i, E_{i+1})^2 + e_i^2 + e_{i+1}^2.
\]

As we have already stated in the introduction, this kind of surface can be classified in terms of mutations, for instance by starting from the fan of a smooth toric surface. However, a classification of such surfaces independent of exceptional sequences might be of interest as well (e.g. one could relax property 3) above and just require that \( K_Y^2(E) \) be integral). This is beyond the scope of this article, but with this perspective we want conclude with some general remarks on the singularities which occur in our setting.

Cyclic \( T \)-singularities of the form \( \frac{d e^2}{\text{det}} \), where \( d, e, k, d > 0 \) and \( \gcd\{k, e\} = 1 \), have been classified Kollar and Shepherd-Barron. Their statement is:

**Proposition 12.1:** [KSSS] Proposition 3.11] A cyclic singularity is of class \( T \) if and only if its continued fraction expansion \([b_1, \ldots, b_r]\) is of one of the following forms:

(i) \([4] \) and \([3, 2, \ldots, 2, 3]\) are of class \( T \),
(ii) If \([b_1, \ldots, b_r]\) is of class \( T \) then so are \([2, b_1, \ldots, b_r + 1]\) and \([b_1 + 1, \ldots, b_r, 2]\).
(iii) Every singularity of class \( T \) that is not a rational double point can be obtained by starting with one of the singularities described in (i) and iterating the steps described in (ii).

12.2. It follows from the analysis in the proof of [KSSS] Proposition 3.11] that \( d = 1 \) implies that the \( T \)-singularities of type \( \frac{d e^2}{\text{det}} \) are precisely those obtained from [4] and iterating step (i). Consider the minimal desingularization of \( Z \to Y = Y(E) \). Denote \( I := \{1 \leq k \leq n \mid c_k^2 \neq 1\} \), then the fan of
Z is obtained from Y by inserting $t_k$ new primitive vectors corresponding to the continued fractions $[b_1^k, \ldots, b_n^k]$. We have

$$K^2_Z = \sum_{i=1}^{n} a_i + 2n - \sum_{k \in I} \left( \sum_{j=1}^{t_k} b_j^k - 2l_i \right) = 12 - n - \sum_{k \in I} t_k,$$

where the $a_i$ are the self-intersection of the pullbacks of the original $n$ divisors to the surface $Z$. A simple induction using Proposition 12.1 shows that $\sum_{j=1}^{t_k} b_j^k = 1 + 3l_k$ for every $k \in I$, hence

$$\sum_{i=1}^{n} a_i = 12 - 3n + |I|.$$

This is almost the same formula as for the sum of self-intersection numbers of the toric prime divisors on a smooth toric surface with $n$ rays, except that we obtain the number of singularities as an extra term.

12.3. Consider $e, f > 1$ and a triple of primitive lattice vectors $l_e, l_f, l_l$ such that the cones generated by $l_e, l_f$ and $l_l$, respectively, correspond to $T$-singularities of types \(12(1, \alpha e - 1)\) and \(12(1, \beta f - 1)\), respectively. We can choose coordinates such that $l = (0, 1)$ and $l_e = (e^2, -\alpha e + 1 + \lambda_1 e^2)$, $l_f = (-f^2, -\beta f + 1 + \lambda_2 f^2)$. It is easy to verify that the term $\lambda \equiv \lambda_1 + \lambda_2$ does not depend on any choice of coordinates with $l = (0, 1)$. Using \(9.1\) we compute:

$$\det(l_e, l_e) = e^2f^2(\frac{\alpha}{e} + \frac{\beta}{f} + \lambda) - e^2 - f^2 = e^2f^2(\chi(E, F)) - e^2 - f^2,$$

hence \(\frac{1}{ef}\chi(E, F) = \frac{\alpha}{e} + \frac{\beta}{f} + \lambda\). If $e = 1, f > 1$ (or $e > 1, f = 1$ or $e = f = 1$, respectively), then the same calculation yields \(\frac{1}{e}\chi(E, F) = 1 + \frac{\alpha}{e} + \lambda\) (respectively \(\frac{1}{f}\chi(E, F) = 1 + \frac{\beta}{f} + \lambda\) and \(\chi(E, F) = 2 + \lambda\)). In the case $e = f = 1, \lambda$ coincides with the self-intersection number of the toric divisor associated to $l$.

12.4. Now, if we transfer above considerations to $Y(E)$, we have in terms of reduced circumference segments for every $1 \leq i \leq n$ that \(\frac{1}{c_{e_i+1}}\chi(E_i, E_{i+1}) = \det(q_i, q_{i+1})\), where

$$\det(q_i, q_{i+1}) = \begin{cases} \frac{\alpha_i}{e_i} + \frac{\beta_i}{c_{i+1}} + \lambda_i & \text{if } e_i, c_{i+1} > 1 \\ 1 + \frac{\beta_i}{c_{i+1}} + \lambda_i & \text{if } e_i = 1, c_{i+1} > 1 \\ \frac{\alpha_i}{e_i} + 1 + \lambda_i & \text{if } e_i > 1, c_{i+1} = 1 \\ 2 + \lambda_i & \text{if } e_i > 1, c_{i+1} = 1. \end{cases}$$

It is a consequence of local change of coordinates that $\alpha_i = e_i - \beta_{i-1}$ for every $i$ and in particular, for any $i$ with $e_i > 1$ we have $\frac{\alpha_i}{e_i} + \frac{\beta_{i+1}}{e_i} = 1$. Thus we get:

$$K^2_Y(E) = 12 - n = \sum_{i=1}^{n} \det(q_i, q_{i+1}) = \sum_{i=1}^{n} \lambda_i + 2n - |I|,$$

where $I = \{1 \leq i \leq n \mid e_i > 1\}$ as in 12.2 hence

$$\sum_{i=1}^{n} \lambda_i = 12 - 3n + |I|.$$

That is, the sum of the $\lambda_i$ coincides with the sum of the $a_i$ from 12.2. Using mutation, one can trace the $a_i$ and the $\lambda_i$ to see that indeed $a_i = \lambda_i$ for every $i$, though we will leave the verification to the reader.

**Appendix: Toric surfaces**

A toric surface is a normal algebraic surface $X$ on which an algebraic torus $T \simeq \mathbb{G}_m(\mathbb{K})^2$ acts such that $T$ embeds into $X$ as an open dense orbit and the group action extends the multiplication on $T$. In this appendix we want to remind the reader on standard material on toric surfaces as it can be found in standard textbooks such as [CLS11]. However, we will need to paraphrase some of the material in order to suit its use in the main text.

Let us denote $M \simeq \mathbb{Z}^2$ the character group of $X$ and $N$ its dual module. We denote $M \otimes \mathbb{Q} := M \otimes \mathbb{Z} \otimes \mathbb{Q}$ and $N \otimes \mathbb{Q} := N \otimes \mathbb{Z} \otimes \mathbb{Q}$. The complement of $T$ in $X$ (if nonempty) is given by a union of divisors $D_1 \cup \cdots \cup D_n$.

We are interested in three cases:
(1) $X$ is affine and has a fixpoint with respect to the torus action.
(2) $X$ can be covered by two affine toric varieties and has two fixpoints.
(3) $X$ is complete.

Each of these cases is completely determined by a collection of primitive lattice vectors $l_1, \ldots, l_n$ in $N$. In every case, we will write $L$ for the matrix whose rows are given by the $l_i$, which we will interpret as $\mathbb{Z}$-linear map from $M$ to $\mathbb{Z}^n$.

**A.1 (The affine case).** In the affine case, we have $n = 2$ and the $T$-fixpoint is $D_1 \cap D_2$, where $D_1$ and $D_2$ both are isomorphic to $\mathbb{A}_K^2$. The vectors $l_1$ and $l_2$ generate over $\mathbb{Q}_{\geq 0}$ a strictly convex polyhedral cone in $N_\mathbb{Q}$. In general, the fixpoint of $X$ is a quotient singularity, i.e. $X \simeq \mathbb{A}_k^2/\mu_v$, where $\mu_v = \text{spec} K[G]$ the abelian group scheme over $K$ corresponding to the cyclic group $G \simeq N/(\mathbb{Z}l_1 + \mathbb{Z}l_2) \simeq \mathbb{Z}/v\mathbb{Z}$.

As $l_1$ and $l_2$ are primitive, one can choose a suitable basis for $N$ such that $l_1 = (1,0)$ and $l_2 = (-k,v)$, where $\gcd(k,v) = 1$ and $0 < k < v$. In the case that $K$ is algebraically closed, and char $K$ and $v$ are coprime, this yields a customary representation for the action of $\mu_v$ on $\mathbb{A}_k^2$ as $\text{diag}(\eta, \eta^k)$, where $\eta \in K$ is a $v$-th root of unity. We also use the notation $\frac{1}{v}(1,k)$ to denote a toric singularity which up to choice of coordinates can be represented in this way. Note that $v = \det(l_1,l_2)$, which equals the lattice volume of the parallelogram spanned by $l_1$ and $l_2$. We will therefore very often refer to $\det(l_1,l_2)$ as the volume of $l_1$ and $l_2$. A useful invariant for us will be the lattice vector $l_2 - l_1$:

**Definition A.2:** We call $l_2 - l_1$ the circumference segment of the pair $l_1,l_2$. We call $\frac{1}{\det(l_1,l_2)}(l_2 - l_1)$ the reduced circumference segment.

Note that the circumference segment $l_2 - l_1$ in general is not primitive and the reduced segment in general is not integral.

**Lemma A.3:** Assume that $\mu_v$ acts on $\mathbb{A}_k^2$ with weights $\frac{1}{v}(1,k)$. Then the lattice length of $l_2 - l_1$ is $\gcd(v,k+1)$.

**Proof.** With above coordinate representation, we get $l_2 - l_1 = (-k - 1,v) = g((-k - 1)/g, v/v) =: gP$, where $g = \gcd(k + 1, v)$. Therefore $l_2 - l_1$ is a $(v,k+1)$-primitive lattice vector. \hfill $\Box$

**A.4 (The minimal linearly dependent case).** Let $l_1, l_2, l_3$ be three primitive lattice vectors such that $l_1$ and $l_2$ lie in opposite half spaces with respect to the line $Ql$. Then $l_1, l_2, l_3$ generate strictly convex polyhedral cones in $N_\mathbb{Q}$ which intersect in the common facet $Ql$ and therefore generate a fan corresponding to a two-dimensional toric surface which is covered by two affine toric surfaces each of which contains a single torus fixpoint. We choose an orientation on $N_\mathbb{Q}$ such that $l_1, l_2, l_3$ are ordered counter-clockwise. Then they satisfy a relation

$$w_1 + al + vl_2 = 0,$$

where $v = \det(l_1,l), w = \det(l,l_2), a = \det(l_2,l_1)$, and $v, w > 0$. This relation is unique up to a common scalar multiple of the coefficients. If $a = 0$, then $v = w$ and $l_1 = -l_2$. We have the two circumference segments $p_1 = l - l_1$ and $p_2 = l_2 - l$ and the corresponding reduced circumference segments $q_1 = \frac{1}{v}p_1$ and $q_2 = \frac{1}{w}p_2$. We observe:

$$\det(p_1,p_2) = \det(l - l_1, l - l_2) = a + v + w, \quad \det(q_1,q_2) = \frac{1}{vw}(a + v + w).$$

It follows that the sum $a + v + w$ determines the convexity of the configuration of lattice vectors. That is, $l$ is not contained in the convex hull of $l_1, l_2$ and the origin if and only if $a + v + w > 0$. The vectors $l_1, l_2, l_3$ lie on a line in $N_\mathbb{Q}$ if and only if $a + v + w = 0$.

**Lemma A.5:** Let $a_1, a_3 > 0$ and $a_2$ be integers and $l_1, l_2, l_3 \in N$ primitive such that $a_{\pi(1)} = (\text{sgn} \pi) \cdot \det(l_{\pi(2)}, l_{\pi(3)})$ for any permutation $\pi \in S_3$. Then

(i) $a_1l_1 + a_2l_2 + a_3l_3 = 0$,

(ii) $\gcd(a_i, a_j) = \gcd(a_i, a_2, a_3)$ for any $1 \leq i \neq j \leq 3$,

(iii) The $l_i$ are uniquely determined up to a transformation by $\text{GL}_2(N)$.

**Proof.** We have seen (i) already in A.4.
Without loss of generality we restrict to the case $i = 1, j = 2$. Clearly, \( \gcd\{a_1, a_2, a_3\} \) divides \( \gcd\{a_1, a_2\} =: g \). Now we write \( a_1 l_1 + a_2 l_2 = -a_3 l_3 \). Clearly, \( g \) divides the left hand side and hence the right hand side. But \( l_3 \) is primitive, so \( g \) cannot divide \( l_3 \), hence \( g \) divides \( a_3 \) and thus \( g \) divides \( \gcd\{a_1, a_2, a_3\} \).

Up to a choice of basis for \( N \), we can write \( l_1 = (1, 0), l_2 = (x, a_3), l_3 = (y, a_2) \), where \(-a_3 < x < 0 \) and \( y \) is determined by the relation \( a_1 + a_2 x + a_3 y = 0 \). We set \( a_i := a_i' g \), where \( g := \gcd\{a_1, a_2, a_3\} \). Then the \( a_i' \) are pairwise coprime by (iii) and equation \( a_1' + a_2' x + a_3' y = 0 \) holds as well. Then the set of solutions \( (x, y) \) is given by the set \( (x_0, y_0) + k(-a_3, a_2) \), where \( k \in \mathbb{Z} \) and \((x_0, y_0)\) is a special solution. Multiplying by \( g \), the solutions are given by the set \((x_0, y_0) + k(-a_3, a_2)\) or \( k \in \mathbb{Z} \) and \((x_0, y_0)\) some special solution. It follows that the condition \(-a_3 < x < 0\) determines \( x \) (and therefore \( y \)) uniquely. \( \square \)

A.6 (The complete case). In this case the \( D_i \) form a cycle of \( \mathbb{P}^1 \)'s and there are \( n \) torus fixpoints which are given by the intersections \( D_i \cap D_{i+1} \). Here it is customary to consider in this case the integers \( 1, \ldots, n \) as a system of representatives for \( \mathbb{Z} / n \mathbb{Z} \), i.e. we read the indices modulo \( n \). The Chow group of \( X \) is determined by the following standard short exact sequence

\[
0 \rightarrow M \rightarrow \mathbb{Z}^n \rightarrow \text{CH}^1(X) \rightarrow 0,
\]

such that the \( i \)-th standard basis vector of \( \mathbb{Z}^n \) maps to the class of \( D_i \) in \( 
\text{CH}^1(X) \). The canonical divisor on \( X \) can be represented by \( K_X = -\sum_{i=1}^{n} D_i \). On \( \text{CH}^1(X) \), there exists a \( \mathbb{Q} \)-valued intersection product which is completely determined by triple relations as in \( \text{A.4} \). That is, if for every \( i \) we denote \( v_i := \det(l_{i-1}, l_i) \) and \( a_i := \det(l_{i+1}, l_{i-1}) \), then we have for every \( i \) the following relation:

\[
v_{i+1} l_{i+1} - a_i l_i + v_i l_{i-1} = 0, \quad \text{respectively} \quad \frac{1}{v_i} l_{i-1} + \frac{a_i}{v_i v_{i+1}} l_i + \frac{1}{v_{i+1}} l_{i+1} = 0,
\]

which translates to intersection products:

\[
D_{i-1} D_i = \frac{1}{v_i}, \quad D_i^2 = \frac{a_i}{v_i v_{i+1}}, \quad D_i D_{i+1} = \frac{1}{v_{i+1}} \quad \text{for every} \ i
\]

and \( D_i D_j = 0 \) else. Note that we choose an orientation on \( N_\mathbb{Q} \) such that \( v_i > 0 \) for every \( i \).

For any \( i \), we have a circumference segment \( p_i := l_i - l_{i-1} \) and its reduction \( q_i := \frac{p_i}{v_i} \). As in \( \text{A.4} \) we have equations

\[
\det(p_i, p_{i+1}) = a_i + v_i + v_{i+1} \quad \text{and} \quad \det(q_i, q_{i+1}) = \frac{1}{v_i v_{i+1}} (a_i + v_i + v_{i+1}) = D_i (D_{i-1} + D_i + D_{i+1})
\]

for every \( i \).

Lemma A.7: We have \( K_X^2 = \sum_{i=1}^{n} \det(q_i, q_{i+1}) = \sum_{i=1}^{n} \frac{a_i + v_i + v_{i+1}}{v_i v_{i+1}} \).

\[
\text{Proof.} \quad \text{By above discussion and} \quad K_X = -\sum_{i=1}^{n} D_i. \quad \square
\]

Note that in the case that \( X \) is smooth, we have \( v_i = 1 \) for every \( i \) and this formula specializes to \( K_X^2 = 12 - n = 2n + \sum_{i=1}^{n} a_i \) or equivalently, \( \sum_{i=1}^{n} a_i = 12 - 3n \).

A.8. The minimal resolution of a toric surface singularity can be described with help of Hirzebruch-Jung continued fractions. That is, in the situation of \( \text{A.4} \) we have \( l_1 = (1, 0) \) and \( l_2 = (-k, v) \) and we write the quotient \( \mathbb{Z} \) by \( [b_1, \ldots, b_r] := b_1 - 1(b_2 - 1(b_3 - 1(\cdots (b_{r-1} - 1/b_r)\cdots)) \) and the \( b_i \) are integers \( \geq 2 \). Now, in the first step of the resolution, we introduce a new primitive vector \( (0, 1) \) which subdivides the cone into smaller cones of volumes \( 1 \) and \( k \), respectively. Ultimately, this new vector will correspond to a component with self-intersection number \( -b_1 \) of the the exceptional divisor in the minimal resolution. After this first step we see that the fraction \( \frac{b}{v} \) can be interpreted as the ratio of the volume of the original cone and the volume of the (possibly still) singular cone after the first resolution step. Iterating this, we see that the continued fraction \( [b_1, \ldots, b_r] \) equals the ratio of volumes \( V_{i-1}/V_i \), where \( V_i \) is the volume of the remaining singular cone after the \( i \)-th resolution step (we set \( V_0 := v \) and \( V_1 := k \), and it follows that \( V_r = 1 \)).

We now are interested in the behaviour of the canonical self-intersection number under minimal resolutions. That is, we consider a complete toric surface \( X \) which has a singular point and \( Y \rightarrow X \) a
toric morphism which is the result of $s \leq r$ steps of the minimal desingularization procedure for this point as described in [A8].

**Proposition A.9:** With notation of [A8] we have

$$K_X^2 - K_Y^2 = \sum_{i=1}^{s} \frac{(V_i - V_{i+1})^2}{V_i - V_{i+1}}.$$ 

**Proof.** We consider the very first step of of the resolution, i.e. we factor $Y = Z_s \rightarrow \cdots \rightarrow Z_1 \rightarrow Z_0 = X$, where every map $Z_i \rightarrow Z_{i-1}$ is a partial resolution step which precisely adds one more toric divisor according to the procedure of [A8]. For the first step, we use the presentation of $K_X^2$ and $K_Y^2$ of Lemma A.7. Up to cyclic renumbering and a choice of basis of $N$, we can assume that the singular point on $X$ being resolved is represented by the cone generated by $l_1 = (1,0)$ and $l_2 = (-k, v_2)$ and one primitive vector $l = (0,1)$ is added as described in [A8]. Using the relations $v_{i+1} l_{i-1} + a_i l_i + v_i l_{i+1} = 0$, we compute

$$l_1 + a_1 l_2 + v_2 l = 0 \text{ with } a = \frac{a_2 - v_2 k}{v_3} \text{ and } v_4 l + b_3 + k l_4 = 0 \text{ with } b = \frac{a_3 k - v_4}{v_3}.$$ 

Now, the effect of the canonical self-intersection number is local in the sense that, if we write $K_X^2 =: R + \frac{a_2 + v_2 k}{v_3} + \frac{a_3 + v_3 k}{v_4}$, then

$$K_X^2 - R = \frac{a + v^2 + 1}{v_2} + \frac{-v^2 + 1}{k} + \frac{b + k + v_4}{k v_4} = \frac{a_2 + v_2 k}{v_2 v_3} + \frac{a_3 + v_3 k}{v_3 v_4} + 2 + \frac{2}{k} - \frac{v_3}{k} - \frac{1}{v_3} - \frac{2}{v_3}.$$ 

Iterating this procedure implies the assertion.

It is useful to understand also how the terms $\det(q_i, q_{i+1})$ behave under resolution of singularities.

**Corollary A.10:** Let $l_1, l_2, l_3$ and $w, a, v$ as in [A7] such that $l_1$ and $l_3$ generate a singular cone and assume that we have a minimal resolution corresponding to a continued fraction $[a_1, \ldots, b_r]$. If we denote $v := V_0, V_1, \ldots, V_s$ for $s \leq r$ the associated volumes and $\xi_1, \ldots, \xi_s$ the successively added primitive vectors, then we have a relation

$$w \xi_s + b_2 + V_3 l_3 = 0 \text{ such that } \frac{a + v + w}{w v} - \frac{b + V_3 + w}{V_3 w} = \sum_{i=1}^{s} \frac{V_i - V_{i-1} + 1}{V_i V_{i-1}}.$$ 

**Proof.** As in the proof of [A9] it is enough to consider one iteration step. We have seen that we can write $l_1 = (1,0)$, $l_2 = (-V_1, v)$, $l_3 = (y, -a)$ and thus we obtain the equation $y = \frac{1}{2}(a V_1 - w)$. It follows that for $\xi_1 = (0,1)$ we have a relation $w \xi_1 + b_2 + V_1 l_3$, where $b := y$. By plugging in $b$, we get

$$\frac{a + V_0 + w}{V_0 w} - \frac{b + V_1 + w}{V_1 w} = \frac{V_0 - V_0 + 1}{V_0 V_1}.$$ 



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