ASPECTS OF T-DUALITY IN OPEN STRINGS

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Abstract

We study T-duality for open strings in various D-manifolds in the approach of canonical transformations. We show that this approach is particularly useful to study the mapping of the boundary conditions since it provides an explicit relation between initial and dual variables. We consider non-abelian duality transformations and show that under some restrictions the dual is a curved \((d - \text{dim}G - 1)\) D-brane, where \(d\) is the dimension of the space-time and \(G\) the non-abelian symmetry group. The generalization to \(N = 1\) supersymmetric sigma models with abelian and non-abelian isometries is also considered.
1 Introduction

Recently there has been renewed interest in the study of open string theories with the last developments in string dualities (see for instance [1] and references therein). In [2] Polchinski showed that open strings with certain exotic boundary conditions (D-branes) were the carriers of the RR charges required by string duality [3] being the quantum of the charge precisely that predicted by duality [4]. This identification allowed for many new tests of string duality. For instance the heterotic string soliton needed by the S-duality conjecture has been identified as a D-string in the type I superstring [5]. D-brane technology has also provided a statistical interpretation of Black Hole entropy. For very pedagogical reviews on D-branes and complete set of references see [6].

D-branes first arised as particular features under T-duality in the ories of open strings [7, 8]. The well-known duality mapping of toroidal compactifications:

\[
\begin{align*}
\partial^+ x &\rightarrow -\frac{1}{R^2} \partial^+ \tilde{x}, \\
\partial^- x &\rightarrow \frac{1}{R^2} \partial^- \tilde{x}
\end{align*}
\]

maps Neumann boundary conditions: \( \partial_n x = 0 \), to Dirichlet boundary conditions: \( \partial_\tilde{x} = 0 \), where \( \partial_n \) and \( \partial_t \) are the normal and tangent derivatives to the boundary. The ends of the strings are then confined to the \( \tilde{x} \) plane, which is itself dynamical. These particular objects with mixed Neumann and Dirichlet boundary conditions are the D-branes [7]. For type I superstrings crosscap boundary conditions for the unoriented topologies are mapped to orientifold conditions [7, 9] and the dual D-brane is hidden in the orientifold plane. These dual theories may seem rather exotic, but they are just a more suitable description at small distances of the same original open string theory.

The open string-D-brane dualities of toroidal compactifications have been extended recently to more general backgrounds. Namely, to backgrounds with abelian [10, 11, 12, 13] and non-abelian isometries [14]. In these references the gauging procedure to T-duality (for a review see [15]) was followed. Backgrounds without isometries have also been studied in [40] within the Poisson-Lie T-duality.

In this article we study T-duality for open strings in various backgrounds within the canonical transformation approach [16, 17]. This approach is particularly useful in obtaining information about the boundary conditions, since it provides an explicit mapping between initial and dual variables. In particular, it generalizes (1. 1) to the specific type of sigma model and duality symmetry under consideration. Once the duality mapping is identified we show how it modifies the boundary conditions. For backgrounds with abelian isometries we will reproduce in a very simple manner some of the results already presented in the references above. In the non-abelian case we will see that the canonical transformation gives results differing from the ones in [14]. We will give some arguments in favor of the canonical transformation description of non-abelian duality for D-manifolds, basically due to the existence of boundaries.

In section 2 we study abelian T-duality of bosonic open and closed strings. We reproduce in a very simple way the results in [10, 11]. If the initial \( d \) dimensional theory satisfies Neumann conditions then the generalization of (1. 1) obtained with the canonical transformation yields Dirichlet boundary conditions for one of the coordinates (for backgrounds

\footnote{In [11] there is also a brief study with canonical transformations.}
with just one abelian isometry) and Neumann for the rest, i.e. the dual theory is a \((d - 2)\) D-brane. For backgrounds of unoriented strings it is also easy to see that crosscap boundary conditions are mapped to orientifold conditions \([10]\).

In section 3 we generalize these results to the case of \(N = 1\) open superstring sigma models. With the canonical transformation description we are able to consider the most general case of classical boundary conditions (minimizing both the bulk and the boundary) and explore their transformation under abelian T-duality. We show that in order to get a dual super D-brane some restrictions over the original backgrounds must be imposed. For standard Neumann R-NS boundary conditions we recover the results in \([10]\).

In section 4 we consider backgrounds with non-abelian isometries. We argue that the canonical transformation approach is more adequate for open string world-sheets since the existence of boundaries makes unclear the validity of the more conventional gauging procedure. We focus in the particular class of backgrounds with non-abelian isometries for which the canonical transformation description is known, i.e. the ones where the isometry group acts without isotropy. We show that only in those cases in which the equivalence between the initial and dual theories can be established at the quantum level\(^4\), the dual of Neumann boundary conditions for the coordinates transforming under the non-abelian symmetry group are generalized Dirichlet boundary conditions, generalized in the sense that it is the momentum defined in a given curved background which is zero at the ends of the string. The inert coordinates under the non-abelian isometry still satisfy generalized Neumann conditions in the dual. Therefore the dual is a curved \((d - \text{dim} G - 1)\) D-brane, where \(G\) is the non-abelian symmetry group, static, as we will show. We will see that for unoriented strings crosscap boundary conditions are mapped to generalized (in the same sense as above) orientifold conditions. The D-branes are then hidden in the orientifold surfaces. These results mean that flat D-branes and orientifolds are just particular results under T-duality. We also present the \(N = 1\) supersymmetrization of some of the models considered.

2 Abelian Duality

2.1 Let us consider open and closed strings propagating in a \(d\) dimensional background of metric, antisymmetric tensor and abelian gauge field\(^5\). In the neutral case the action can be written:

\[
S = \int_{\Sigma} d\sigma_+ d\sigma_- (g_{ij} + b_{ij}) \partial_+ x^i \partial_- x^j + \int_{\partial \Sigma} V_i \partial_t x^i
\]

(2.1)

where \(V_i\) denotes the abelian background gauge field and \(\partial_t\) is the tangent derivative to the boundary\(^6\). The boundary term can be absorbed in the action by just considering:

\[
S = \int_{\Sigma} d\sigma_+ d\sigma_- (g_{ij} + B_{ij}) \partial_+ x^i \partial_- x^j
\]

(2.2)

with \(B_{ij} = b_{ij} + F_{ij} = b_{ij} + \partial_i V_j - \partial_j V_i\). The torsion term is absent for the unoriented topologies. Let us assume that there exists a Killing vector \(k^i\) such that \(\mathcal{L}_k g_{ij} = 0\) and

\(^4\)We will clearly specify what we mean by this.

\(^5\)We only consider throughout the paper abelian background gauge fields. For a non-abelian treatment see \([11]\).

\(^6\)We consider \(\sigma = \text{constant} \) boundaries throughout the paper but in certain, specified, cases.
\( \mathcal{L}_k b_{ij} = 0 \) (this means we can have: \( \mathcal{L}_k b_{ij} = \partial_i v_j - \partial_j v_i \) and \( \mathcal{L}_k V_i = -v_i + \partial_i \varphi \), for some \( v_i, \varphi \)). Going to the system of adapted coordinates to the isometry \( \{ \theta, x^\alpha \} \), \( \alpha = 1, \ldots, d-1 \), such that \( k = \partial_\theta \) the action reads:

\[
S = \frac{1}{2} \int_\Sigma d^2 \sigma (g_{00} \partial_+ \theta \partial_- \theta + (g_{0\alpha} + B_{0\alpha}) \partial_+ \theta \partial_- x^\alpha + (g_{0\alpha} - B_{0\alpha}) \partial_+ x^\alpha \partial_- \theta + (g_{\alpha\beta} + B_{\alpha\beta}) \partial_+ x^\alpha \partial_- x^\beta).
\]

(2.3)

Let us briefly discuss the gauging procedure to construct the abelian T-dual [18]. This method has been applied to open strings in [10, 11]. The global isometry of the model can be made local by replacing ordinary derivatives of the adapted coordinate by covariant derivatives, introducing a gauge field \( A_\pm \) with gauge variation \( \delta A_\pm = -\partial_\pm \epsilon \) (\( \epsilon \) being the gauge parameter). This gauge field is imposed to be flat by means of a Lagrange multiplier term. In non-trivial world-sheets (with boundaries or non-trivial genera) one has to be especially careful in imposing this condition. In [19] the higher genus case was studied in detail for closed strings. It was shown that in order to obtain \( A \) pure gauge in non-trivial world-sheets the Lagrange multiplier term must be introduced as

\[
\int_\Sigma \tilde{\theta} \wedge A
\]

(2.4)

where \( \tilde{\theta} \) must be multivalued to accomplish gauge invariance under large gauge transformations. This multivaluedness can be accounted for with a harmonic contribution to \( d \tilde{\theta} \). The integration on the exact component fixes the constraint that \( A \) is closed and, further, the integration on the harmonic component imposes that the harmonic contribution to \( A \) is cero.

For open strings we have to consider as well the harmonic contribution from the boundary, whose integration imposes the constraint

\[
\int_{\partial \Sigma} A = 0.
\]

(2.5)

In this way the pure gauge condition \( A_\pm = \partial_\pm \theta_0 \) is obtained. Doing the transformation \( \theta \to \theta + \theta_0 \) the original theory is recovered in gauge invariant variables (recall: \( \delta \theta = \epsilon, \delta A_\pm = -\partial_\pm \epsilon \Rightarrow \delta \theta_0 = -\epsilon \)). On the other hand integrating out \( A_\pm \) and fixing the gauge \( \theta = 0 \) the dual theory is obtained. The dual backgrounds are given by Buscher’s formulas [20]:

\[
\begin{align*}
\tilde{g}_{00} &= \frac{1}{g_{00}}, & \tilde{G}_{0\alpha} &= -\frac{B_{0\alpha}}{g_{00}}, & \tilde{B}_{0\alpha} &= \frac{-g_{0\alpha}}{g_{00}}, \\
\tilde{G}_{\alpha\beta} &= g_{\alpha\beta} - \frac{g_{\alpha\alpha} g_{\beta\beta} - B_{0\alpha} B_{0\beta}}{g_{00}}, & \tilde{B}_{\alpha\beta} &= B_{\alpha\beta} - \frac{g_{\alpha\alpha} B_{0\beta} - g_{\beta\beta} B_{0\alpha}}{g_{00}}
\end{align*}
\]

(2.6)

where now \( B \) involves the background gauge field and we have denoted the dual backgrounds also with capital letters to account for this dependence. \( \tilde{\theta} \) plays now the role of 0-coordinate. The regularization of the determinant coming from the integration over the gauge fields produces the shift of the dilaton required by conformal invariance [20, 21]: \( \tilde{\Phi} = \Phi - \log \det \tilde{g}_{00} \).

In [10, 11] due account was taken of the boundary conditions with the result that the dual of (2.1) with Neumann conditions in the boundary is a Dirichlet \((d-2)\)-brane.

For closed strings it is also possible to construct the dual by the method of the canonical transformation [10, 17]. The mapping:

\[
\begin{align*}
p_\theta &= -\tilde{\theta}' \\
p_{\tilde{\theta}} &= -\theta'
\end{align*}
\]

(2.7)
from \{\theta, p_\theta\} to \{\tilde{\theta}, p_{\tilde{\theta}}\}$, where $p_\theta, p_{\tilde{\theta}}$ are the conjugate momenta to $\theta, \tilde{\theta}$, yields the dual theory defined by Buscher’s formulas. This relation provides as well the generalization of the non-local duality transformation $d \to \frac{1}{2\pi} * d$ of toroidal compactifications to arbitrary backgrounds with abelian isometries. We just need to substitute:

$$p_\theta = g_{00}\dot{\tilde{\theta}} + g_{0a}\dot{x}^a - b_{0a}x^a$$

$$p_{\tilde{\theta}} = \frac{1}{g_{00}}(\dot{\theta} - b_{0a}\dot{x}^a + g_{0a}x'^a)$$  \hspace{1cm} (2. 8)

in (2. 7) to obtain:

$$\partial_+ \theta = -\tilde{g}_{00}\partial_+ \tilde{\theta} - \tilde{k}_i\partial_+ \tilde{x}^i = -\tilde{k}_i\partial_+ \tilde{x}^i$$

$$\partial_- \theta = \tilde{g}_{00}\partial_- \tilde{\theta} + \tilde{k}_i\partial_- \tilde{x}^i = \tilde{k}_i\partial_- \tilde{x}^i$$  \hspace{1cm} (2. 9)

where we have defined $k_i^\pm = g_{0i} \pm b_{0i}$.

The generating functional is given by:

$$F = \frac{1}{2} \oint d\sigma (\tilde{\theta}\theta' - \tilde{\theta}'\theta).$$  \hspace{1cm} (2. 10)

$F$ being linear in $\theta$ and $\tilde{\theta}$ implies that the classical canonical transformation (2. 7) is also valid quantum mechanically and we can write the relation:

$$|\tilde{\theta}\rangle = \int D\theta(\sigma) e^{iF[\tilde{\theta}, \theta(\sigma)]} |\theta(\sigma)\rangle$$  \hspace{1cm} (2. 11)

up to some normalization factor, between the corresponding Hilbert spaces [22]. However renormalization effects still need to be considered and in fact there are some results showing that they give corrections to Buscher’s backgrounds [23]. Also to really establish the quantum equivalence between the initial and dual theories we need to reproduce the dilaton shift within the canonical transformations approach. Consider a constant toroidal background of radius $R$. The measure in configuration space is given by $D\theta \det R$. We can regularize this determinant as $R^{B_0}$, where $B_0$ is the dimension of the space of 0-forms in the two dimensional world-sheet (regularized in a lattice, for instance). With this prescription one realizes that the usual measure in phase space: $D\theta Dp_\theta$ gives upon integration in $p$: $D\theta R^{B_1}$, where $B_1$ is the dimension of the space of 1-forms in the world-sheet and emerges because the momenta are 1-forms. Therefore it differs from our definition of measure in configuration space. In order to reproduce the partition function in configuration space we need to include explicit factors on $R$ in the definition of the measure in phase space. One can check that considering these factors the correct shift of the dilaton is obtained after performing the canonical transformation. These arguments however are only rigorous for constant backgrounds. We believe that a similar reasoning could be applied to the general case.

The relation (2. 11) between the states of the initial and dual theories implies that the duality transformation is valid for arbitrary Riemann surfaces, because $|\theta(\sigma)\rangle$ can be

\footnote{This way of regularizing the determinants has been shown to reproduce the correct modular anomaly under S-duality in abelian gauge theories [41, 42].}
the result of integrating the original theory on an arbitrary Riemann surface with boundary. Then the previous results apply with no modification to open and closed strings backgrounds [11] with $b$ replaced by $B$ in order to absorb the background gauge field. In this case we also need to care about the boundary conditions. The canonical transformation approach is particularly adequate to deal with boundary conditions since it provides an explicit relation between the target space coordinates of the original and dual theories. From (2.9) we get:

\[ \dot{\tilde{\theta}} = -(g_{00}' \theta + g_{0\alpha} \theta'^\alpha - B_{0\alpha} \dot{x}^\alpha) \]  

(2.12)

and

\[ g_{00}' \theta + g_{\alpha\beta} x'^\beta + B_{0\alpha} \dot{\theta} - B_{\alpha\beta} \dot{x}^\beta = \tilde{G}_{00}' \tilde{\theta} + \tilde{G}_{\alpha\beta} x'^\beta + \tilde{B}_{0\alpha} \dot{\tilde{\theta}} - \tilde{B}_{\alpha\beta} \dot{\tilde{x}}^\beta. \]  

(2.13)

Then, Neumann boundary conditions for the original theory [24]:

\[ g_{ij} x'^j - B_{ij} \dot{x}^j = 0 \]  

(2.14)

imply:

\[ \dot{\tilde{\theta}} = 0 \]
\[ \tilde{G}_{00}' \tilde{\theta}' + \tilde{G}_{\alpha\beta} x'^\beta + \tilde{B}_{0\alpha} \dot{\tilde{\theta}} - \tilde{B}_{\alpha\beta} \dot{\tilde{x}}^\beta = 0. \]  

(2.15)

These mixed boundary conditions represent a flat Dirichlet $(d - 2)$-brane in the dual background. Also from (2.9) we can deduce the collective motion of the brane. Decomposing $B_{0\alpha} = b_{0\alpha} - \partial_\alpha V_0$ we realize that the usual Buscher’s backgrounds for closed strings (with the torsion $b$) are gotten provided we redefine $\tilde{\theta} \equiv \tilde{\theta} + V_0(x^\alpha)$. Therefore $V_0(x^\alpha)$ gives the transverse position of the brane in the dual theory. If we dualize $n$ commuting isometries it is straightforward to check that a Dirichlet $(d - n - 1)$-brane is obtained in the dual. It is perhaps worth mentioning that there are some particular backgrounds (those whose conserved currents associated to the isometry are chiral [18]) which are at the same time backgrounds of open strings and D-branes depending on the boundary conditions, which are in turn related by a T-duality transformation.

In [10] this mapping from Neumann to Dirichlet boundary conditions was studied within the gauging procedure. As an intermediate stage it was necessary to generalize Neumann boundary conditions for the gauged action and the choice they made was to impose (2.14) with covariant derivatives replacing ordinary derivatives of the adapted coordinate. The justification for this is that the integration over the Lagrange multiplier yields $A_\pm = \partial_\pm \theta_0$, pure gauge, and after substituting in the covariant derivatives and making the isometric transformation $\theta \rightarrow \tilde{\theta} + V_0$ the original Neumann boundary conditions in gauge invariant variables are recovered. On the other hand, substituting $A_\pm$ from their equations of motion and fixing the gauge choosing $\theta = 0$ Dirichlet boundary conditions are obtained for $\tilde{\theta}$ and Neumann for the rest of the coordinates. In the canonical transformation approach we have made use of the explicit relations between original and dual variables and further justified the choice made in [10].

A particular case is when $V_0$ is taken pure gauge locally breaking $U(N)$ to $U(1)^N$, i.e. when a Wilson line $V_0 = \text{diag}\{\theta_1, \ldots, \theta_N\}$ is included. In this case we get a maximum of $N$ D-branes in the dual theory with fixed positions at $\theta_i$, $i = 1, \ldots, N$ [3].
Let us finish this section by analyzing the unoriented topologies. Invariance under worldsheet parity implies that the antisymmetric tensor and the abelian gauge field are projected out of the spectrum. We can still have non-abelian gauge fields in $SO(N)$ and $USp(N)$ but they must be treated differently (see for instance [11]). Unoriented topologies can be obtained from oriented ones by identifications of points on the boundary [25]. For instance the projective plane is obtained from the disk by identifying opposite points. The topology thus obtained is a crosscap. Under abelian T-duality we should get the mapping from crosscap to orientifold conditions [10]. Crosscap boundary conditions for the coordinate adapted to the isometry:

$$\dot{\theta} (\sigma + \pi) = -\dot{\theta}(\sigma)$$
$$\theta'(\sigma + \pi) = \theta'(\sigma),$$

(2.16)

where we are parametrizing the boundary of the disk by $(0, 2\pi)$ and identifying opposite points: $\theta(\sigma + \pi) = \theta(\sigma)$, translate to:

$$p_\theta(\sigma + \pi) = -p_\theta(\sigma)$$
$$\theta'(\sigma + \pi) = \theta'(\sigma)$$

(2.17)

in phase space. Then (2.7) implies:

$$\ddot{\theta'} (\sigma + \pi) = -\ddot{\theta}'(\sigma)$$
$$p_{\tilde{\theta}}(\sigma + \pi) = p_{\tilde{\theta}}(\sigma),$$

(2.18)

which are orientifold conditions in phase space since $p_{\tilde{\theta}}(\sigma + \pi) = p_{\tilde{\theta}}(\sigma)$ implies $\dot{\tilde{\theta}}(\sigma + \pi) = \dot{\tilde{\theta}}(\sigma)^9$. The orientifold plane is at $\tilde{\theta} = 0$ and it’s non-dynamical, because the abelian gauge field is zero in the unoriented case. These conditions can also be easily worked out in configuration space from (2.9). The rest of the coordinates still satisfy crosscap boundary conditions. This implies that we cannot have $\tilde{g}_{0i}$ components in the dual metric, which it can be seen to be the case since $B_{0i}$ must be zero in the initial theory.

Abelian T-duality has also been studied at the level of the effective world volume actions in [26, 10, 12, 13].

# 3 Superstrings

In this section we analyze abelian T-duality for $N = 1$ open superstring sigma models in the approach of canonical transformations. This approach has been applied to closed superstrings in [27]. The action in $(1, 1)$ superspace reads:

$$S = \int_{\Sigma} d\sigma_+ d\sigma_- d^2 \theta (g_{ij} + B_{ij}) D_+ X^i D_- X^j$$

(3.1)

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9We have to make first a Wick rotation to imaginary time.
10Integration on the first equation implies $\dot{\theta}(\sigma + \pi) = -\dot{\theta}(\sigma)$ so that the worldsheet parity reversal is accompanied by a $Z_2$ transformation in space-time. These kinds of constructions are the orientifolds [7, 9].
where the superfields \( X^i(\theta, \sigma^\pm) = x^i + \theta_+ \psi^i_+ + \theta_- \psi^i_- + \theta_+ \theta_- F^i \), the superderivatives \( D_+ = \partial_{\theta_-} - i \theta_- \partial_{\sigma_+} \), \( D_- = - \partial_{\theta_+} + i \theta_+ \partial_{\sigma_-} \) and \( B_{ij} = g_{ij} + F_{ij} \), as in the previous section. Once the superspace integration \( \int d^2 \theta = D_+ D_- |_{\theta_+ = 0} \) and the on-shell substitution of the auxiliary field \( F^i \) are made, we get the usual target-space action’s covariant form in components in addition to a boundary contribution which includes the supersymmetric circulation of the abelian electromagnetic field:

\[
S = \int_{\Sigma} d\sigma_+ d\sigma_- \{(g_{ij} + B_{ij})\partial_+ x^i \partial_- x^j - ig_{ij} \psi_+^i \nabla_- (\psi_+^j) - ig_{ij} \psi_-^i \nabla_+ (\psi_-^j) + \frac{1}{2} R_{ijkl}(\Gamma_\pm)^{(\pm)} \psi_+^i \psi_+^j \psi_-^k \psi_-^l \} + \int_{\partial \Sigma} \{V_i \partial_+ x^i + \frac{i}{4} B_{ij}(\psi_+^i \psi_+^j + \psi_-^i \psi_-^j)\} \quad (3.2)
\]

where

\[
\nabla_\pm \psi^i = \partial_+ \psi^i + \partial_- x^i \Gamma_\pm^{ij} \psi^j \\
\Gamma_\pm^{ij} = \Gamma_\pm^{ij} \pm H_{jk} \\
H_{ijk} = \frac{1}{2}(b_{ij,k} + b_{jk,i} - b_{ik,j}) \quad (3.3)
\]

and \( \Gamma_\pm^{ij} \) is the Levi-Civita connection.

For our purposes it is preferable to absorb the boundary term in the bulk and work with an action:

\[
S = \int_{\Sigma} d\sigma_+ d\sigma_- \{(g_{ij} + B_{ij})\partial_+ x^i \partial_- x^j - ig_{ij} \psi_+^i \nabla_- (\psi_+^j) - ig_{ij} \psi_-^i \nabla_+ (\psi_-^j) - i\partial_-(g_{ii} - B_{ii})\psi_+^i \partial_- x^j \psi_+^j - i\partial_+(g_{ii} + B_{ii})\psi_-^i \partial_+ x^j \psi_-^j + \frac{1}{2} R_{ijkl}(\Gamma_\pm)^{(\pm)} \psi_+^i \psi_+^j \psi_-^k \psi_-^l \}. \quad (3.4)
\]

Let us assume that there exists an abelian isometry \( \delta x^i = \epsilon k_i \) (the conditions being the same than in the pure bosonic case). In adapted coordinates to the isometry \( \{\theta, \psi_0^0, x^\alpha, \psi_0^\alpha\} \), \( \alpha = 1, \ldots, d-1 \), and retaining only the world-sheet derivative terms of the zero components, the action reduces to:

\[
S = \frac{1}{2} \int_{\Sigma} d^2 \sigma \{g_{00}(\dot{\theta}^2 - \theta'^2) + 2j_- (\dot{\theta} + \theta') + 2j_+(\dot{\theta} - \theta') - i(g_{00}\psi_0^0 + k_0^+ \psi_0^\alpha)(\psi_0^- - \psi_0^\alpha) - i(g_{00}\psi_0^0 + k_0^+ \psi_0^\alpha)(\psi_0^- + \psi_0^\alpha') + V\} \quad (3.5)
\]

where we have defined

\[
k_0^\pm = g_{0i} \pm B_{0i} \\
j_\pm = \frac{1}{2}(k_0^+ \partial_+ x^\alpha + ik_0^+ \partial_- x^\alpha) \\
V = (g_{\alpha\beta} + B_{\alpha\beta})\partial_+ x^\alpha \partial_- x^\beta - i\psi_+^\alpha g_{i\alpha} + B_{i\alpha})\partial_- \psi_+^\alpha - i\psi_-^\alpha (g_{i\alpha} - B_{i\alpha})\partial_+ \psi_+^\alpha - i\partial_-(g_{\alpha\alpha} - B_{\alpha\alpha})\psi_+^\alpha \partial_- x^\alpha \psi_+^\alpha - i\partial_+(g_{\alpha\alpha} + B_{\alpha\alpha})\psi_-^\alpha \partial_+ x^\alpha \psi_-^\alpha + \frac{1}{2} R_{ijkl}(\Gamma_\pm)^{(\pm)} \psi_+^i \psi_+^j \psi_-^k \psi_-^l. \quad (3.6)
\]
The canonical momenta associated to the zero coordinates are

\[
\Pi_\pm = \frac{\delta S}{\delta \dot{\psi}_\pm} = \frac{i}{2} (g_{00} \dot{\psi}_\pm^0 + \tilde{k}_\alpha^\pm \dot{\psi}_\pm^\alpha)
\]  
(3. 7)

\[
p_\theta = \frac{\delta S}{\delta \theta} = g_{00} \dot{\theta} + j_+ + j_-
\]  
(3. 8)

where (3. 7) are two first class constraints.

The following generating functional:

\[
\mathcal{F} = \frac{1}{2} \int d\sigma \{ \theta' \dot{\theta} - \theta \ddot{\theta} - i \psi_+^0 \dot{\psi}_+^0 + i \psi_-^0 \dot{\psi}_-^0 \}
\]  
(3. 9)

yields the abelian T-dual, with backgrounds given by Buscher’s formulas\textsuperscript{11}. It induces the change of variables in phase space \textsuperscript{27}:

\[
\Pi_\pm = -\frac{\delta F}{\delta \dot{\psi}_\pm^0} = \mp \frac{i}{2} \psi_\pm^0, \quad \Pi_\pm = \frac{\delta F}{\delta \psi_\pm^0} = \mp \frac{i}{2} \dot{\psi}_\pm^0
\]
\[
p_\theta = -\frac{\delta F}{\delta \theta} = -\theta', \quad p_\theta = \frac{\delta F}{\delta \theta} = -\bar{\theta'}
\]  
(3. 10)

\(\mathcal{F}\) being linear in the original and dual variables implies that the original and dual theories are also equivalent quantum mechanically, as in the bosonic case.

(3. 10) reads, in configuration space variables:

\[
\psi_\pm^0 = \mp (\tilde{g}_{00} \tilde{\psi}_\pm^0 + \tilde{k}_\alpha^\pm \tilde{\psi}_\pm^\alpha)
\]
\[
\psi_\pm^\alpha = \tilde{\psi}_\pm^\alpha
\]  
(3. 11)

for the fermions, and:

\[
\partial_+ \theta = -\tilde{g}_{00} \partial_+ \tilde{\theta} - \tilde{k}_\alpha^\pm \partial_+ \tilde{x}^\alpha - i \tilde{k}_{[i,j]} \tilde{\psi}_+^i \tilde{\psi}_+^j = -\tilde{k}_i^\pm \partial_+ x^i - i \tilde{k}_{[i,j]} \tilde{\psi}_+^i \tilde{\psi}_+^j
\]
\[
\partial_- \theta = \tilde{g}_{00} \partial_- \tilde{\theta} + \tilde{k}_\alpha^\pm \partial_- \tilde{x}^\alpha + i \tilde{k}_{[i,j]} \tilde{\psi}_-^i \tilde{\psi}_-^j = \tilde{k}_i^\pm \partial_- x^i + i \tilde{k}_{[i,j]} \tilde{\psi}_-^i \tilde{\psi}_-^j;
\]  
(3. 12)

for the bosons, where \(\tilde{k}_{\alpha}^\pm = \tilde{G}_{0i} \mp \tilde{B}_0i\). This is the abelian duality mapping in the \(N = 1\) case and it can be obtained as well from (2. 9) replacing bosonic fields by superfields and derivatives by superderivatives.

As in the pure bosonic case (3. 9) can be derived from the total time derivative term that is induced in the usual gauging procedure. Starting with the superspace action (2. 4) we replace the superderivatives of the adapted coordinate to the isometry by covariant superderivatives and add the following super-Lagrange multipliers term \textsuperscript{28}:

\[
\int \frac{\partial}{\partial \Sigma} (D_+ \tilde{X}_0^0 A_- + D_- \tilde{X}_0^0 A_+)
\]  
(3. 13)

where \(\tilde{X}_0^0\) is the superfield: \(\tilde{X}_0^0 = \tilde{\theta} + \theta \tilde{\psi}_-^0 + \theta \tilde{\psi}_+^0 + \theta_+ \tilde{\psi}_-^0 + \theta_- \tilde{\psi}_+^0\) and \(A_\pm\) is the gauge superfield \(A_\pm = f_\pm + \theta_+ a_\pm^+ + \theta_-' a_\pm^- + \theta_+ \theta_-' F_\pm\). The integration on the Lagrange multiplier imposes

\[\text{11}\ H \text{ and } \tilde{H} \text{ are equal up to a spatial derivative } \frac{\partial}{\partial \Sigma} \left( \frac{1}{2} \delta_{\alpha} (\tilde{\psi}_+^0 \tilde{\psi}_+^0 + \tilde{\psi}_-^0 \tilde{\psi}_-^0) \right). \]  
We will see how this term emerges in the derivation of the generating functional via the gauging procedure.
the constraint that \( A_\pm = D_\pm X^0 \), whose substitution in the gauged action yields the original theory (after fixing the gauge) plus a total time derivative term. The integration over the gauge fields \( A_\pm \) yields the dual theory. We then have:

\[
\tilde{S} = S + \int (D_+ X^0 D_- X^0 + D_- X^0 D_+ X^0)
\]  

(3.14)

where \( X^0 \) is the superfield containing the bosonic coordinate adapted to the isometry: \( X^0 = \theta + \theta_+ \psi^0_\theta + \theta_- \psi^0_- + \theta_+ \theta_- f \). Under a canonical transformation \( \{ q^i, p_i \} \rightarrow \{ Q^i, P_i \} \) the generating functional is such that:

\[
\dot{q}^i p_i - H = \dot{Q}^i P_i - \tilde{H} + \frac{dF}{dt}
\]  

(3.15)

where \( H \) and \( \tilde{H} \) denote the original and canonically transformed Hamiltonians respectively. Hence, from (3.14) we can read (3.9) after performing the superspace integration. In this calculation one also obtains the spatial derivative term that was needed to prove the equivalence between \( H \) and \( \tilde{H} \).

We now focus on the study of the boundary conditions. We are going to look at the most general class of classical boundary conditions, i.e. those for which the variation of the whole action vanishes. Let us consider the general conditions:

\[
\psi_+^i = \bar{R}^i(x^k; \psi_-^j)
\]  

(3.16)

for the fermions, where by \( \bar{R}^i(x^k; \psi_-^j) \) we mean:

\[
\bar{R}^i(x^k; \psi_-^j) \equiv \sum_{l=odd}^d \psi_-^{j_1} \ldots \psi_-^{j_l} R_{j_1 \ldots j_l}^i(x^k),
\]  

(3.17)

i.e. we denote with a bar these kinds of expansions to distinguish them from ordinary target space tensors\(^{12}\). The variation of the action in the boundary:

\[
\delta S_{\text{boundary}} = \int dt \{ \delta x^i ( (g_{ij} + B_{ij}) \partial_- x^j - (g_{ij} - B_{ij}) \partial_+ x^j + i(g_{ij} + B_{ij})_k \psi_-^k \psi_-^j - \\
i(g_{ij} - B_{ij})_k \psi_-^k \psi_+^j + i(g_{ij} + B_{ij}) (\delta \psi_-^i \psi_-^j - \delta \psi_+^i \psi_+^j) \}
\]  

(3.19)

vanishes if the following conditions are satisfied:

\[
(g_{ij} + B_{ij}) \psi_-^j = (g_{kj} + B_{kj}) \frac{\delta \bar{R}^j}{\delta \psi_-^k} \bar{R}^k
\]  

(3.20)

\(^{12}\)Although it is clear that usual tensors can be viewed as barred ones with vanishing expansion except for the zeroth term. With this notation an arbitrary barred tensor is, in components:

\[
\bar{V}_{k_1 \ldots k_m}^{i_1 \ldots i_n} = \bar{V}_{k_1 \ldots k_m}^{i_1 \ldots i_n}(x; \psi_-) \equiv \sum_{l=0}^d \psi_-^{i_1} \ldots \psi_-^{i_l} V_{k_1 \ldots k_m j_1 \ldots j_l}^{i_1 \ldots i_n}(x).
\]  

(3.18)
\[ g_{ij} \dot{x}^j - B_{ij} \dot{x}^i - \frac{i}{2} (g_{ij} + B_{ij}) \psi_-^j \psi_-^j + \frac{i}{2} (g_{ij} - B_{ij}) \bar{R}^k \bar{R}^i + \frac{i}{2} (g_{kj} + B_{kj}) \frac{\partial \bar{R}^i}{\partial x^j} \bar{R}^k = 0 \] (3. 21)

at the ends of the string. (3. 20) imposes a constraint over the possible tensors \( R^{i_1 \ldots i_l} \) allowed as classical boundary conditions. For instance the linear case: \( \psi_+^j = J^i \cdot (x) \psi_-^j \) considered in [24] is a solution provided \( J \) satisfies \( J^i (g + B) J = g - B \). \( J \) can be chosen such that \( J^2 = J \), \( J^2 = 1 \) and \( BJ = -JB \).

Clearly (3. 21) is not manifestly covariant since \( \partial_i \bar{R}^j \) is not a tensor. It can be seen that the covariant derivative of a general barred tensor (3. 18) is given by:

\[ \nabla_j \bar{V}^{i_1 \ldots i_n}_{k_1 \ldots k_m} = \nabla_j \bar{V}^{i_1 \ldots i_n}_{k_1 \ldots k_m} - \psi_-^j \Gamma^p_{ji} \frac{\partial \bar{V}^{i_1 \ldots i_n}_{k_1 \ldots k_m}}{\partial \psi_-^p} \] (3. 22)

where

\[ \nabla_j \bar{V}^{i_1 \ldots i_n}_{k_1 \ldots k_m} = \partial_j \bar{V}^{i_1 \ldots i_n}_{k_1 \ldots k_m} + \sum_{r=1}^n \Gamma^r_{ji} \bar{V}^{i_1 \ldots \hat{i}_r \ldots i_n}_{k_1 \ldots k_m} - \sum_{s=1}^m \Gamma^k_{jk} \bar{V}^{i_1 \ldots i_n}_{k_1 \ldots \hat{k} \ldots k_m}, \] (3. 23)

since in this way also the target space indices contracted with the fermions are covariantized. Then the manifestly covariant bosonic boundary condition is given by:

\[ g_{ij} \dot{x}^j - B_{ij} \dot{x}^i - \frac{i}{2} (g_{kj} + B_{kj}) \bar{R}^k \partial_i \bar{R}^i - \frac{i}{2} \nabla_k B_{ij} (\psi_-^i \psi_-^j + \bar{R}^k \bar{R}^i) = 0 \] (3. 24)

where use have been made of the constraints (3. 20). This expression is the generalization to the \( N = 1 \) supersymmetric case of Neumann boundary conditions.

The linear classical boundary conditions \( \psi_+^j = J^i \cdot (x) \psi_-^j \) are such that the fermionic boundary contribution to the action: \( B_{ij} (\psi_-^i \psi_-^j + \psi_-^j \psi_-^j) \) vanishes. This means that in some sense they are too restrictive since they do not allow for any fermionic dynamics at the boundary. We want to investigate if this feature is general to all classical boundary conditions. The variation of \( B_{ij} (\psi_-^i \psi_-^j + \psi_-^j \psi_-^j) \) is given by:

\[ \frac{\delta}{\delta \psi_-^i} (B_{ij} (\psi_-^i \psi_-^j + \bar{R}^i \bar{R}^j)) = 2 (\frac{\delta \bar{R}^i}{\delta \psi_-^i} \bar{R}^j g_{ij} - g_{kj} \psi_-^j) \] (3. 25)

after substitution of the constraints (3. 20). If we denote \( \bar{E}_k \equiv \frac{\delta}{\delta \psi_-^k} (B_{ij} (\psi_-^i \psi_-^j + \bar{R}^i \bar{R}^j)) \), the integrability conditions \( \frac{\delta \bar{E}_k}{\delta \psi_-^j} + \frac{\delta \bar{E}_j}{\delta \psi_-^k} = 0 \) imply:

\[ g_{ik} = \frac{\delta \bar{R}^i}{\delta \psi_-^k} \delta \bar{R}^j \frac{\delta g_{ij}}{\delta \psi_-^k} \] (3. 26)

Substituting in (3. 23) we finally get:

\[ \frac{1}{2} B_{ij} (\psi_-^i \psi_-^j + \bar{R}^i \bar{R}^j) = -g_{ij} \bar{R}^i \frac{\delta \bar{R}^j}{\delta \psi_-^j} \psi_-^j. \] (3. 27)

In the linear case the right-hand side is zero and therefore the fermionic contribution to the boundary action vanishes. However this is not so in the general case. The usual R-NS boundary conditions: \( \psi_+^i = \eta \psi_-^i \), with \( \eta = \pm 1 \), also produce non-trivial dynamics in the
boundary since they are chosen as minima of the bulk in (3.2). They are classical boundary conditions when \( B_{ij} = 0 \).

Let us now study how the general classical boundary conditions (3.10) and (3.24) transform under abelian duality.

The conditions for the fermions: \( \psi^i = \tilde{R}^i(x^k; \psi^-_j) \), where \( \tilde{R}^i \) must have vanishing Lie derivative in the Killing direction \(^{13}\), transform under (3.11) as:

\[
\tilde{\psi}^\alpha = \tilde{R}^\alpha(\tilde{x}; \frac{\psi^0 - k^+_\alpha \psi^-_\beta}{g_{00}}, \psi^-_\beta) = \tilde{R}^\alpha \\
\tilde{\psi}^0 = -\frac{1}{g_{00}}(\tilde{R}^0 + \tilde{k}^-_\alpha \tilde{R}^\alpha),
\]

where our notation is such that given a function \( \tilde{\xi} = \tilde{\xi}(\tilde{x}^k; \psi^0, \psi^-_\beta) \), \( \tilde{\xi} \) is defined by: \( \tilde{\xi} = \tilde{\xi}(\tilde{x}^k; \psi^0, \psi^-_\beta) \).

The corresponding bosonic covariant boundary conditions:

\[
g_{ij}x^j - B_{ij}\dot{x}^j + \tilde{W}_i(x^\alpha; \psi^-_\beta) = 0
\]

with

\[
\tilde{W}_i = -\frac{i}{2}(g_{kj} + B_{kj})\tilde{R}^k \tilde{\nabla}_j \tilde{R}^j - \frac{i}{2} \nabla_k B_{ij}(\psi^k \psi^-_j + \tilde{R}^k \tilde{R}^j)
\]

are a little bit more involved. In phase space variables \( (\Pi_\pm, \psi^0_\pm, p_\theta, \theta) \) these conditions read:

\[
g_{i0} \dot{\theta} + g_{i\beta} x^\beta - B_{i0}(\frac{p_\beta - j^+ - j^-}{g_{00}}) - B_{i\beta} \dot{x}^\beta + \tilde{W}_i = 0.
\]

In dual variables and splitting the currents into their bosonic and fermionic parts \( j^f_\pm \equiv j^f_\pm - \frac{1}{2} k^\pm_\sigma \partial_\pm x^\sigma \) we get for the zero and \( \alpha \) components:

\[
\dot{\theta} + j^+_\alpha - j^-_\alpha - \tilde{W}_\alpha = 0
\]

\[
\tilde{G}_{\alpha i} \dot{x}^i - \tilde{B}_{\alpha i} \dot{x}^i + \tilde{G}_{0\alpha}(j^f_+ + j^f_-) + \tilde{B}_{0\alpha}(j^f_+ - j^f_-) + \tilde{W}_\alpha = 0
\]

where

\[
j^f_+ = -\frac{i}{2}(\partial_\alpha \frac{1}{k^2} \tilde{R}^\alpha \tilde{R}^\alpha - \partial_\alpha \frac{\tilde{k}^-_\beta \tilde{R}^\beta}{k^2})
\]

\[
j^f_- = -\frac{i}{2} \frac{1}{k^2}(\partial_\alpha \tilde{k}^2 \psi^-_\alpha \psi^-_\beta + \partial_\alpha \tilde{k}^+ \psi^-_\alpha \psi^-_\beta).
\]

\( \tilde{W}_0, \tilde{W}_\alpha \) have complicated expressions in terms of the dual backgrounds and since they will be irrelevant in what follows we shall omit them. It is clear that for arbitrary backgrounds (3.32) cannot be interpreted as a Dirichlet boundary condition. To discuss this further let

\(^{13}\)This is implied by the constraints (3.20).
us restrict ourselves to the more familiar case of Neumann R-NS boundary conditions for the initial theory:

\[ g_{ij} x'^j - B_{ij} x^j = 0 \]
\[ \bar{\psi}_+ = \eta \psi^\alpha; \quad \eta = \pm 1 \] (3.35)

The dual boundary conditions are then:

\[ \bar{\psi}_+ = \eta \bar{\psi}_\alpha \]
\[ \bar{\psi}_0 + \eta \bar{\psi}_0 = 2 \eta k_0^\alpha \bar{\psi}_\alpha \]
\[ \dot{\bar{\theta}} = i \partial_\alpha k^\beta \bar{\psi}_- \bar{\psi}_\beta \]
\[ \bar{G}_{\alpha i} \dot{x}_i - \bar{B}_{\alpha i} \dot{x}_i = -i k_0^\alpha \partial_\beta k^2 \bar{\psi}_- \bar{\psi}_0 + i (k^\alpha_\beta \partial_\beta k^\sigma_\alpha - k^\sigma_\alpha \partial_\beta k^\beta_\alpha) \bar{\psi}_- \bar{\psi}_\sigma \] (3.36)

where \( k_0^\alpha = B_{0\alpha} \). These results are in agreement with (3.12) in [10]. The non-trivial terms (those that spoil Dirichlet NS-R boundary conditions in the dual) are all proportional to \( B_{0\alpha} \).

Therefore a super D-brane is obtained in the dual only if the original background is such that \( B_{0\alpha} = 0 \). If this occurs (3.36) turns into: \( \bar{\psi}_0 = -\eta \bar{\psi}_0 \), accounting for the reversal of space-time chirality under T-duality [29], \( \dot{\bar{\theta}} = 0 \), and Neumann R-NS boundary conditions for the rest of the coordinates. This is the case, in particular, for the type I superstring where the D-brane is actually an orientifold. In this theory consistency conditions restrict the possible D-manifolds to one, five and nine-branes [4]. Since the only consistent open superstring theory is the type I superstring, which contains unoriented topologies, it is interesting to analyze in a little bit more detail the unoriented world-sheets. As in the previous section we consider the projective plane, obtained from the disk by identifying opposite points. Crosscap boundary conditions for the fermions contain an \( i \) factor due to the fact that we are taking a constant time boundary [24]:

\[ \psi^i_+ (\sigma + \pi) = i \eta \psi^i_- (\sigma), \quad \eta = \pm 1 \] (3.37)
\[ x'^i (\sigma + \pi) = x'^i (\sigma) \]
\[ \dot{x}^i (\sigma + \pi) = -\dot{x}^i (\sigma). \] (3.38)

These conditions are mapped under (3.11) and (3.12) to:

\[ \bar{\psi}_\alpha^+ (\sigma + \pi) = i \eta \bar{\psi}_\alpha^- (\sigma) \]
\[ \bar{\psi}_0^+ (\sigma + \pi) = -i \eta \bar{\psi}_0^- (\sigma) \] (3.39)

for the fermions, giving the usual change of sector for the 0-component, and to:

\[ \dot{\bar{\theta}} (\sigma + \pi) = \dot{\bar{\theta}} (\sigma) \]
\[ \dot{\bar{\theta}} (\sigma + \pi) = -\dot{\bar{\theta}} (\sigma) \]
\[ \dot{x}^\alpha_\sigma (\sigma + \pi) = -\dot{x}^\alpha_\sigma (\sigma) \]
\[ \dot{x}^\alpha_\sigma (\sigma + \pi) = \dot{x}^\alpha_\sigma (\sigma) \] (3.40)

for the bosons, i.e. orientifold conditions for the \( \bar{\theta} \) coordinate and crosscap for the rest. Therefore the dual theory is an orientifold, static since the abelian electromagnetic field is absent for unoriented strings.
4  Non-abelian duality

In this section we analyze non-abelian duality transformations in backgrounds of open and closed strings. We follow the canonical transformations approach. We argue that this approach is more suitable to discuss non-abelian duality in world-sheets with non-trivial genera or boundaries, therefore for open strings, since it doesn’t convey some subtleties involved in the more usual gauging procedure [30]. The gauging approach has been applied to backgrounds of open and closed strings in [14]. Also, particular subclasses of the backgrounds we will be considering have been studied in [40] within the Poisson-Lie T-duality.

Let us consider a sigma model:

\[ S = \int_{\Sigma} d\sigma d\sigma_- (g_{ij} + B_{ij}) \partial_+ x^i \partial_- x^j \]  

(4.1)

where \( B_{ij} = b_{ij} + F_{ij} \), with a non-abelian isometry group \( G \) generated by the Killing vectors \( k^i_a, a = 1, \ldots, \dim G \). In order to construct the non-abelian dual with respect to this set of isometries following the gauging procedure [30] we need to have invariance under the local gauge transformations \( \delta x^i = \epsilon^a(\sigma) k^i_a(x) \). This is accomplished by introducing gauge fields in the Lie algebra \( A^a_\pm \), with gauge variation \( \delta A^a_\pm = -\partial_\pm \epsilon^a + f^a_{bc} A^b_\pm \epsilon^c \) (\( f_{abc} \) are the structure constants of the Lie algebra), and replacing ordinary derivatives by covariant derivatives:

\[ \partial_\pm x^i \to D_\pm x^i = \partial_\pm x^i + k^i_a A^a_\pm. \]  

(4.2)

The gauged action reads:

\[ S_{\text{gauged}} = S + \int d\sigma d\sigma_- (g_{ij} + B_{ij})(k^i_a k^j_b A^a_+ A^b_- + A^a_- k^j_o \partial_+ x^i + A^a_+ k^i_o \partial_- x^j). \]  

(4.3)

In homologically trivial world-sheets the Lagrange multipliers term

\[ -\int \Sigma Tr(\chi (dA + A \wedge A)) \]  

(4.4)

is added to impose the pure gauge condition \( A = -dgg^{-1}, g \in G \), upon integration on \( \chi \) and thus recover the original theory after fixing the gauge. For higher genus or in the presence of boundaries it is not known how to impose the pure gauge constraint in a gauge invariant way. This problem already appeared in closed strings when trying to generalize [30] to arbitrary genera [19, 31]. In particular, when homologically non-trivial cycles exist in the world-sheet the integration on \( \chi \) in [4.4] implies:

\[ A = -d(gg_c) (gg_c)^{-1} \]  

(4.5)

where we have the contribution of the non-trivial flat connection along the boundaries and the \( 2h \) cycles (\( h \) is the genus of the world-sheet):

\[ g_c = Pe^{\int A} \]  

(4.6)

(here \( P \) stands for path ordering). Imposing \( g_c = 1 \) in a gauge invariant way introduces non-local terms in the action, spoiling the non-abelian dual construction. In closed strings it
is possible to deal with higher genera using the canonical transformation approach \[32, 33, 34, 35\]. The idea is that once we have a relation like (2.11) between the Hilbert spaces of the original and dual theories, \(|\theta(\sigma)\rangle\) can be the result of integrating the original theory on an arbitrary Riemann surface with boundary. This means in particular that we can apply the canonical description to open string world-sheets. Non-abelian duality as an explicit canonical transformation is only known in those cases in which the isometry acts without isotropy, i.e. without fixed points, therefore restricting the backgrounds where non-abelian duality can be studied. The most general sigma model of this kind is \[31\]:

\[
S[g, x] = \int d\sigma_+ d\sigma_- [E_{ab}(x)(\partial_+ gg^{-1})^a(\partial_- gg^{-1})^b + F_{aa}^R(x)(\partial_+ gg^{-1})^a \partial_- x^a + F_{a\alpha}(x)\partial_+ x^a \partial_- x^\alpha + F_{aa}^L(x)\partial_+ x^a (\partial_- gg^{-1})^a + F_{a\beta}(x)\partial_+ x^a \partial_- x^\beta],
\]

where \(g \in G\), a Lie group (which we take to be compact), and \(\partial_\pm gg^{-1} = (\partial_\pm gg^{-1})^a T_a\) with \(T_a\) the generators of the corresponding Lie algebra.\[14\] This model is invariant under \(g \to gh\), with \(h \in G\). We can regard (4.7) as a background of open and closed strings (neutral case) with the abelian gauge field absorbed in the corresponding torsion terms.

Let us parametrize the Lie group using the Maurer-Cartan forms \(\Omega^a_x\), such that \n
\[
(\partial_\pm gg^{-1})^a = \Omega^a_x(\theta)\partial_\pm \theta^k.
\]

The abelian background gauge fields that are compatible with the non-abelian isometry \(g \to gh\) have the form \(V_i = \frac{1}{2} \Omega^a_i C^a(x)\), with \(C^a\) arbitrary, and \(V_a \theta^a\)-independent. Then \(E_{ab} = b_{ab} + f_{abc} C^c(x)\), where \(b_{ab}\) is the closed strings antisymmetric tensor and \(F_{aa}^R = f_{aa}^R - \frac{1}{2} \partial_a C^a(x)\), \(F_{aa}^L = f_{aa}^L + \frac{1}{2} \partial_a C^a(x)\) (with \(f^R, f^L\) the corresponding closed strings backgrounds).

The following canonical transformation from \{\(\theta^a, \Pi_i\)\} to \{\(\chi^a, \tilde{\Pi}_a\)\}:

\[
\Pi_i = -\Omega^a_i \chi^a + f_{abc} \chi^a \Omega^b_j \Omega^c_k \theta^{jk} \\
\tilde{\Pi}_a = -\Omega^a_i \theta^i
\]

produces the non-abelian dual of (4.7) with respect to its isometry \(g \to gh\):

\[
\tilde{S} = \int d\sigma_+ d\sigma_- [(E + ad\chi)^{-1}_i(\partial_+ \chi^a + F_{aa}^L(x)\partial_+ x^a)(\partial_- \chi^b - F_{b\beta}^R(x)\partial_- x^\beta) + F_{a\beta} \partial_+ x^a \partial_- x^\beta] \\
\]

(4.10)

This was first realized in \[32\] for the case of \(SU(2)\) principal chiral models (where \(E_{ab} = \delta_{ab}, F_{aa}^R = F_{aa}^L = F_{a\beta} = 0\)), generalized in \[33, 34\] to arbitrary group, and shown to apply also to this more general case in \[37\]. The dual backgrounds are given by:

\[
\tilde{G}_{ab} = \frac{1}{2} M_{(ab)}, \quad \tilde{B}_{ab} = \frac{1}{2} M_{[ab]}, \quad \tilde{G}_{aa} = \frac{1}{2} (-M_{ab} F_{ba}^R + M_{ba} F_{ab}^L) \\
\tilde{B}_{aa} = -\frac{1}{2} (M_{ab} F_{ba}^R + M_{ba} F_{ab}^L), \quad \tilde{G}_{a\beta} = -\frac{1}{2} M_{ab} (F_{aa}^R F_{b\beta}^R + F_{b\beta}^L F_{aa}^R) + \frac{1}{2} F_{a\beta} \\
\tilde{B}_{a\beta} = -\frac{1}{2} M_{ab} (F_{aa}^L F_{b\beta}^R - F_{b\beta}^L F_{aa}^R) + \frac{1}{2} F_{a\beta}^{14}
\]

\([T_a]\) are normalized such that \(Tr(T_a T_b) = \delta_{ab}\).
where we use capital letters for the metric and torsion to account for the fact that the initial abelian background gauge field is absorbed in $M$ and $F^{L}, F^{R}, F$. $M \equiv (E + \text{ad}\chi)^{-1}$, $(\text{ad}\chi)_{ab} = f_{abc}\chi^{c}$ and $M_{(ab)} = M_{ab} + M_{ba}$, $M_{[ab]} = M_{ab} - M_{ba}$.

(1.9) reads, in configuration space variables:

\[
\Omega_{i}^{a}\partial_{+}\theta^{i} = -M_{ba}(\partial_{+}\chi^{b} + F_{ab}^{L}\partial_{+}x^{a}) = -(\tilde{G}_{ab} - \tilde{B}_{ab})\partial_{+}\chi^{b} - (\tilde{G}_{aa} - \tilde{B}_{aa})\partial_{+}x^{a}
\]

\[
\Omega_{i}^{a}\partial_{-}\theta^{i} = M_{ab}(\partial_{-}\chi^{b} - F_{ba}^{R}\partial_{-}x^{a}) = (\tilde{G}_{ab} + \tilde{B}_{ab})\partial_{-}\chi^{b} + (\tilde{G}_{aa} + \tilde{B}_{aa})\partial_{-}x^{a}.
\] (4.12)

These relations generalize (2.9) for non-abelian duality transformations, the main difference being that the components of the torsion in the Lie algebra variables appear explicitly. They identify the non-local transformation responsible for non-abelian duality.

Up to now only the transformations in the bulk have been accounted for. In order to study the open strings sector we have to analyze as well the boundary conditions. Using (4.12) it is easy to see that Neumann boundary conditions in the initial theory:

\[
E_{(ab)}\Omega_{j}^{a}\theta^{j} - E_{[ab]}\Omega_{j}^{b}\dot{\theta}^{j} - F_{aa}^{R}\partial_{-}x^{a} + F_{aa}^{L}\partial_{+}x^{a} = 0
\] (4.13)

imply for the dual:

\[
(2\delta_{ab} - (\text{ad}\chi)_{ac}M_{[bc]}\chi^{c} + (\text{ad}\chi)_{ab}(M_{bc}\chi^{c} + M_{bc}F_{ca}^{R}\partial_{-}x^{a} + M_{cb}F_{ac}^{L}\partial_{+}x^{a})) = 0
\] (4.14)

or

\[
E_{(ab)}(\tilde{G}_{bc}\chi^{c} - \tilde{B}_{bc}\chi^{c} + \tilde{G}_{ba}\dot{x}^{a} - \tilde{B}_{ba}x^{a}) - E_{[ab]}(\tilde{G}_{bc}\chi^{c} + \tilde{G}_{ba}\dot{x}^{a} - \tilde{B}_{ba}x^{a} - (F_{aa}^{L} - F_{aa}^{R})\dot{x}^{a} - (F_{aa}^{L} + F_{aa}^{R})x^{a} = 0
\] (4.15)

in terms of the dual backgrounds. Therefore the general dual boundary conditions are a combination of Dirichlet and Neumann boundary conditions with some additional terms depending on the inert coordinates. For the $x^{a}$ coordinates it can be seen that

\[
g_{\alpha i}\theta^{i} + g_{\alpha \beta}x^{\beta} - B_{\alpha i}\dot{\theta}^{i} - B_{\alpha \beta}\dot{x}^{\beta} = 0
\] (4.16)

implies:

\[
\tilde{G}_{aa}\chi^{a} + \tilde{G}_{\alpha \beta}x^{\beta} - \tilde{B}_{aa}\dot{x}^{a} - \tilde{B}_{\alpha \beta}\dot{x}^{\beta} = 0
\] (4.17)

i.e. they still verify Neumann boundary conditions.

Let us now study those cases in which (4.15) reduces to generalized Dirichlet conditions. First let us make the change of variables:

\[
\chi^{a} \equiv \chi^{a} + \frac{1}{2}C^{a}(x)
\] (4.18)

in order to eliminate the dependence of the matrix $M$ on the background gauge field. Dirichlet conditions are obtained when the initial torsion $E_{[ab]} = b_{ab} + f_{abc}C^{c}$ is zero and also $F_{aa}^{R} = F_{aa}^{L} = 0$. Then (4.15) reduces to:

\[
\tilde{g}_{ab}\dot{x}^{b} - \tilde{b}_{ab}\dot{x}^{b} = 0
\] (4.19)
where $\tilde{g}, \tilde{b}$ are the closed strings backgrounds (given by (4.11) without the contribution of the background gauge field). We can then conclude that for certain kinds of sigma-models with non-abelian isometries a curved $(d - \dim G - 1)$ D-brane is obtained in the dual with metric $\tilde{g}_{ab}$ and torsion $\tilde{b}_{ab}$. In this more general case it is not the velocity that vanishes at the ends of the string but the whole momentum associated to the non-flat background. The transverse dynamics of the brane would be given by (4.18), therefore induced by the background gauge fields: $\chi^a \equiv \chi^a + \frac{1}{2} C^a(x) = \chi^a + \omega^{ai} V_i$, with $\omega^{ai}$ such that $\omega^{ai} \Omega^b_i = \delta^{ab}$, but the condition $E_{[ab]} = 0$ implies that the $i$-components of the antisymmetric tensor and the background gauge field must be zero and therefore the brane is static. The backgrounds of unoriented strings are among the ones for which we get Dirichlet boundary conditions in the dual. We will analyze in the next subsection their $N = 1$ supersymmetrization.

It is important to point out that the conditions that must be satisfied in order to obtain a Dirichlet brane in the dual ($E_{[ab]} = F^L_{\alpha a} = F^R_{\alpha a} = 0$) are precisely those required to have a symmetry under $g \rightarrow hg$, $h \in G$, in the initial sigma-model\textsuperscript{15}. In these cases we also find a symmetry in the dual theory under $\chi$ transforming in the adjoint representation\textsuperscript{16} and it is easy to see that the canonical transformation couples the conserved currents associated to the left symmetry of the initial theory:

$$J^a_+ = \frac{1}{2} E_{(ab)} \Omega^b_i \partial_+ \theta^i$$

$$J^a_- = \frac{1}{2} E_{(ab)} \Omega^b_i \partial_- \theta^i$$

(4.20)

and the ones associated to $\chi \rightarrow h \chi h^{-1}$ in the dual\textsuperscript{17}:

$$\tilde{J}^a_+ = \partial_+ \chi^a - \frac{1}{2} E_{(ab)} M_{cb} \partial_+ \chi^c$$

$$\tilde{J}^a_- = - \partial_- \chi^a + \frac{1}{2} E_{(ab)} M_{bc} \partial_- \chi^c.$$

(4.21)

The existence of a symmetry in the dual theory allows to establish the equivalence between the original and dual theories at the level of the Hilbert spaces. The generating functional responsible for the canonical transformation (4.9) is:

$$\mathcal{F}[\chi, \theta] = \oint d\sigma Tr(\chi \partial_\sigma gg^{-1})$$

(4.22)

i.e. it is linear in the dual variables but non-linear in the original ones. This means that in general it will receive quantum corrections when implemented at the level of the Hilbert spaces\textsuperscript{22}, the reason being that we cannot prove a relation like

$$|\chi^a \rangle = \int \mathcal{D}\theta^i(\sigma) e^{i\mathcal{F}[\chi^a, \theta^i(\sigma)\theta^i(\sigma)}} \langle \theta^i(\sigma)\rangle$$

(4.23)

\textsuperscript{15}We need to have as well $E_{(ab)} = f(x^\alpha) \delta_{ab}$ but this can always be achieved by choosing a proper normalization for the generators of the Lie algebra.

\textsuperscript{16}This symmetry was expected since the left and right symmetries of the original theory commute and we are dualizing with respect to the right action only, so we should get a reminiscent of the left symmetry in the dual.

\textsuperscript{17}Up to a total derivative term which for principal chiral models ($E_{ab} = \delta_{ab}$) is the responsible for having curvature free currents in the dual, that are coupled to the curvature free currents of the principal chiral model\textsuperscript{32, 33}.
using the eigenfunctions of the respective Hamiltonians. However, it was shown in \[32, 33\] that such a relation can in fact be proven using the eigenfunctions of the respective conserved currents in the initial and dual theories. Of course for this to be true we need to have a symmetry in the dual theory, which is not the case for arbitrary backgrounds. As in the abelian case there can still be renormalization effects modifying the classical backgrounds.

We should mention here that a dilaton shift is needed in order to preserve conformal invariance \[30\], exactly as in the abelian case. This remains an open question within the canonical transformation description whose resolution we believe should be along the lines previously mentioned in the abelian case.

Therefore we can conclude that the non-abelian dual of the subclass of sigma-models (4.7) for which the canonical transformation description is valid quantum mechanically (up to the dilaton problem and the renormalization effects we have just mentioned) is a curved \((d - \dim G - 1)\) D-brane. More general cases have duals that cannot be interpreted as Dirichlet branes.

Let us finish this section with a few comments on unoriented strings. The previous backgrounds \((E_{ab} = F_{aa}^L = F_{aa}^R = 0)\) are further restricted by \(F_{\alpha\beta} = \frac{1}{2} F_{(\alpha\beta)}\). Crosscap conditions for the \(\theta^i\)-coordinates:

\[
\dot{\theta}^i(\sigma + \pi) = -\dot{\theta}^i(\sigma) \quad \theta^i(\sigma + \pi) = \theta^i(\sigma)
\]  

are mapped under (4.12) to

\[
\tilde{g}_{ab} \chi^b(\sigma + \pi) - \tilde{b}_{ab} \chi^b(\sigma + \pi) = -(\tilde{g}_{ab} \chi^b(\sigma) - \tilde{b}_{ab} \chi^b(\sigma)),
\]

\[
\tilde{g}_{ab} \chi^b(\sigma + \pi) - \tilde{b}_{ab} \chi^b(\sigma + \pi) = \tilde{g}_{ab} \chi^b(\sigma) - \tilde{b}_{ab} \chi^b(\sigma).
\]

Thus we find a curved orientifold in the dual with metric and torsion \(\tilde{g}_{ab}, \tilde{b}_{ab}\) and generalized orientifold conditions given by equal momenta at the identification points (the second equation in (4.25)) and opposite sign momentum flows out of them. This is the generalization to curved orientifolds of the usual conditions:

\[
\dot{x}^i(\sigma + \pi) = \dot{x}^i(\sigma) \quad x^i(\sigma + \pi) = -x^i(\sigma).
\]

4.1 Superstrings

Let us now study the \(N = 1\) supersymmetrization of the non-abelian models considered. Non-abelian duality in these theories has been studied in \[38, 39\] and more extensively in \[38, 39\] for principal chiral models. Our aim here is to focus on the mapping of the boundary conditions. For simplicity we are going to restrict ourselves to the case of principal chiral models. Their \(N = 1\) supersymmetrization is given by (3.2) with \(g_{ij} = \Omega_i^a \Omega_j^a\) and \(b_{ij} = 0\). We take as well zero background gauge fields, so that the examples we will be considering are suitable as backgrounds of unoriented strings, which are the ones interesting to us since the only consistent open superstring theory, the type I superstring, contains unoriented strings.
Following [33] we can use tangent space variables for the fermions \( \phi^a_{\pm} = \Omega_{a}^{i} \psi_{i}^{\pm} \), in which case the action reads:

\[
S = \int_{\Sigma} d\sigma_{+} d\sigma_{-} \left[ (\partial_{+} gg^{-1})^{a}(\partial_{-} gg^{-1})^{a} - i\phi_{+}^{a} \partial_{-} \phi_{-}^{a} - i\phi_{-}^{a} \partial_{+} \phi_{+}^{a} + \frac{i}{2} f_{abc} \phi_{+}^{a} (\partial_{-} gg^{-1})^{b} \phi_{+}^{c} + \right.
\]

\[
\left. \frac{i}{2} f_{abc} \phi_{-}^{a} (\partial_{+} gg^{-1})^{b} \phi_{-}^{c} + \frac{1}{8} f_{adbc} f_{be} \phi_{+}^{d} \phi_{+}^{e} \phi_{-}^{f} \phi_{-}^{g} \right].
\]

(4. 27)

Working in phase space variables \( \{ (\theta^{i}, \Pi_{i}), (\phi_{a}^{i}, \Pi_{a}^{i}) \} \):

\[
\Pi_{i} = \Omega_{a}^{i} \phi_{a}^{i} \dot{\theta}^{i} + \frac{i}{4} f_{abc} \Omega_{a}^{i} (\phi_{+}^{a} \phi_{+}^{c} + \phi_{-}^{a} \phi_{-}^{c})
\]

(4. 28)

\[
\Pi_{a}^{\phi} = \frac{i}{2} \phi_{a}^{\pm},
\]

(4. 29)

where (4. 27) are a set of first class constraints, the non-abelian dual of (4. 27) with respect to the right action of the whole symmetry group \( G \) can be obtained through a canonical transformation from \( \{ (\theta^{i}, \Pi_{i}), (\phi_{a}^{i}, \Pi_{a}^{i}) \} \) to \( \{ (\chi^{a}, \Pi_{a}), (\tilde{\phi}_{a}^{\pm}, \Pi_{a}^{\phi}) \} \). Namely:

\[
\Pi_{i} = - (\Omega_{a}^{i} \chi^{a} + f_{abc} \chi^{a} \Omega_{a}^{b} \Pi_{a}^{c} \dot{\theta}^{i})
\]

\[
\tilde{\Pi}_{a} = - (\Omega_{a}^{i} \theta^{a} + \frac{i}{4} f_{abc} (\phi_{+}^{a} \phi_{+}^{c} - \phi_{-}^{a} \phi_{-}^{c}))
\]

(4. 30)

for the bosonic momenta, and:

\[
\Pi_{a}^{\phi} = \mp \frac{i}{2} (\tilde{\phi}_{a}^{\pm} + f_{abc} \chi^{b} \phi_{a}^{c})
\]

\[
\tilde{\Pi}_{a}^{\phi} = \mp \frac{i}{2} \tilde{\phi}_{a}^{\pm}
\]

(4. 31)

for the fermionic ones. Its generating functional is:

\[
\mathcal{F} = \oint d\sigma [\chi^{a} \Omega_{a}^{i} \theta^{i} + \frac{i}{4} f_{abc} \chi^{a} (\phi_{+}^{b} \phi_{+}^{c} - \phi_{-}^{b} \phi_{-}^{c}) - \frac{i}{2} (\phi_{+}^{a} \phi_{+}^{a} - \phi_{-}^{a} \phi_{-}^{a})].
\]

(4. 32)

As in the abelian case \( H \) and \( \tilde{H} \) are related up to a spatial derivative, which in this case is:

\[
\frac{i}{2} \partial_{\sigma} (\tilde{\phi}_{a}^{a} \phi_{a}^{a} + \phi_{a}^{a} \phi_{a}^{a} + \frac{1}{2} (\lambda_{ab} \phi_{a}^{b} + \phi_{a}^{a} \phi_{a}^{b})).
\]

(4. 33)

The dual action is given by:

\[
\tilde{S} = \int_{\Sigma_{+}} d\sigma_{+} d\sigma_{-} [M_{ab} (\partial_{+} \chi^{a} \partial_{-} \chi^{b} - i\tilde{\phi}_{+}^{a} \partial_{-} \phi_{-}^{b} + i\partial_{+} \phi_{+}^{a} \tilde{\phi}_{-}^{b}) + i M_{bc} f_{cde} M_{da} \phi_{+}^{a} \tilde{\phi}_{+}^{c} \partial_{+} \chi^{b} + \]

\[
i M_{ac} f_{cde} M_{db} \phi_{+}^{a} \tilde{\phi}_{+}^{d} \partial_{+} \chi^{b} + L_{abcd} \phi_{+}^{a} \tilde{\phi}_{+}^{b} \phi_{+}^{c} \phi_{+}^{d}]
\]

(4. 34)

with \( M = (1 + \text{ad} \chi)^{-1} \) and \( L_{abcd} = -(f_{agf} f_{ieb} + f_{aie} f_{gfb}) M_{ci} M_{eg} M_{fd} \). (4. 34) is manifestly \( N = 1 \) supersymmetric having the form of (3. 4) with variables living in the Lie algebra and
metric and antisymmetric tensor $\frac{1}{2}M_{(ab)}$, $\frac{1}{2}M_{[ab]}$ respectively. The dual momenta are:

$$\bar{\Pi}_a = \frac{1}{2} (M_{(ab)} \chi^b - M_{[ab]} \chi^b - i(M \dot{\phi}_- M)_a \tilde{\phi}_-^b - i(M \dot{\phi}_+ M)_{ba} \tilde{\phi}_+^b)$$  \hspace{1cm} (4. 35)

$$\bar{\Pi}_{\tilde{\phi}_+} = \frac{i}{2} M_{ba} \tilde{\phi}_+^b$$

$$\bar{\Pi}_{\tilde{\phi}_-} = \frac{i}{2} M_{ab} \tilde{\phi}_-^b$$  \hspace{1cm} (4. 36)

where $\text{ad} \tilde{\phi}_\pm = f_{abc} \tilde{\phi}_c^a$. From (4. 36) and (4. 31) we see that the fermions simply transform with the change of scale:

$$\phi_+^a = -M_{ba} \tilde{\phi}_+^b$$

$$\phi_-^a = M_{ab} \tilde{\phi}_-^b.$$  \hspace{1cm} (4. 37)

The corresponding non-local transformation for the bosonic part is given by:

$$\Omega^a_i \partial_+ \theta^i = -M_{ba} \partial_+ \chi^b + \frac{i}{2} (f_{dab} M_{pf} M_{ed} + 2 f_{cde} M_{da} M_{fc} \phi^e_+ \phi^f_+)$$

$$\Omega^a_i \partial_- \theta^i = M_{ab} \partial_- \chi^b + \frac{i}{2} (f_{dab} M_{de} M_{bf} + 2 f_{cde} M_{ad} M_{cf} \phi^e_- \phi^f_-).$$  \hspace{1cm} (4. 38)

In terms of the dual backgrounds this corresponds to:

$$(\partial_+ gg^{-1})^a = -(g_{ab} - \tilde{b}_{ab}) \partial_+ \chi^b - i \partial_c (g_{ab} - \tilde{b}_{ab}) \tilde{\phi}_+^c \tilde{\phi}_+^b - i (\phi_+^2)^a$$

$$(\partial_- gg^{-1})^a = (g_{ab} + \tilde{b}_{ab}) \partial_- \chi^b + i \partial_c (g_{ab} + \tilde{b}_{ab}) \tilde{\phi}_-^c \tilde{\phi}_-^b - i (\phi_-^2)^a,$$  \hspace{1cm} (4. 39)

where the last terms need still be written in terms of the dual fermions. In this form we see that they generalize (3. 12) by means of the last quadratic terms in the fermions, which are zero in the abelian case. (4. 37) and (4. 38) can also be obtained from the corresponding (4. 12) in superspace [37] by introducing chiral superfields [38]. As in the pure bosonic non-abelian case the canonical transformation couples the conserved currents associated to the left symmetry $g \rightarrow hg$ of the initial theory and the ones associated to transformations in the adjoint in the dual. Namely:

$$J^{(L)}_+^a = (\partial_+ gg^{-1})^a + i (\phi_+^2)^a = \Omega^a_i \partial_+ \theta^i + \frac{i}{2} f_{abc} \phi_+^b \phi_+^c$$

$$J^{(L)}_-^a = (\partial_- gg^{-1})^a + i (\phi_-^2)^a = \Omega^a_i \partial_- \theta^i + \frac{i}{2} f_{abc} \phi_-^b \phi_-^c$$  \hspace{1cm} (4. 40)

with

$$\tilde{J}_+^a = \partial_+ \chi^a - M_{ba} \partial_+ \chi^b + i (M \dot{\phi}_+ M)_{ba} \tilde{\phi}_+^b$$

$$\tilde{J}_-^a = -\partial_- \chi^a + M_{ab} \partial_- \chi^b - i (M \dot{\phi}_- M)_{ab} \tilde{\phi}_-^b.$$  \hspace{1cm} (4. 41)

Then we can also establish the quantum equivalence between the two theories, since it is easy to check that the following relation holds:

$$\tilde{J}_+^a e^{iF} = J^{(L)}_+^a e^{iF}.$$  \hspace{1cm} (4. 42)

\footnote{Up to the same total derivative of the bosonic case.}
We can then prove
\[
\tilde{\psi}_k[\chi, \tilde{\phi}] = N(k) \int \prod_{i=1}^{\dim G} D\theta^i(\sigma) e^{iF[\chi, \tilde{\phi}, \theta(\sigma), \phi]} \psi_k[\theta(\sigma), \phi] \tag{4.43}
\]
with $F$ given by the classical expression (4.32) and $\psi_k$ and $\tilde{\psi}_k$ eigenfunctions of the respective conserved currents with the same eigenvalue.

Let us now study the mapping of the boundary conditions. For simplicity we are going to restrict ourselves to R-NS boundary conditions for the fermions:
\[
\phi^a_+ = \eta \phi^a_-, \quad \eta = \pm 1. \tag{4.44}
\]
From (4.37) we get in the dual:
\[
\tilde{\phi}^a_+ = -\eta M^{-1}_{ba} M_{bc} \tilde{\phi}^c_-, \tag{4.45}
\]
which are not NS-R boundary conditions. We can just point out that they could be interpreted as NS-R plus corrections in $ad\chi$.

Concerning the bosons, if we start with Neumann boundary conditions: $\Omega^a_i \theta^i = 0$, we get in the dual:
\[
M_{(ab)} \chi^b - M_{[ab]} \chi^b - i(M_{ac} f_{cde} M_{db} - \frac{1}{2} f_{ade} M_{db} M_{ce}) \tilde{\phi}^e_+ \tilde{\phi}^b_+ - i(M_{bc} f_{cde} M_{da} + \frac{1}{2} f_{adc} M_{bd} M_{ec}) \tilde{\phi}^e_+ \tilde{\phi}^b_+ = 0, \tag{4.46}
\]
which in terms of the dual backgrounds can be written as:
\[
\tilde{g}_{ab} \chi^b - \tilde{b}_{ab} \chi^b + \frac{i}{2} \partial_e (\tilde{g}_{ab} + \tilde{b}_{ab}) \tilde{\phi}^e_+ \tilde{\phi}^b_+ + \frac{i}{2} \partial_e (\tilde{g}_{ab} - \tilde{b}_{ab}) \tilde{\phi}^e_+ \tilde{\phi}^b_+ = 0. \tag{4.47}
\]
This equation represents the vanishing of $\tilde{\Pi}_a$ (given by (4.35)) at the ends of the string. Therefore we find a curved $N = 1$ supersymmetric D-brane (in this particular example (-1)-brane, since we haven’t allowed for inert coordinates) with metric and torsion given by $\tilde{g}_{ab}$ and $\tilde{b}_{ab}$. However since the only consistent open superstring theory contains unoriented topologies the D-brane is an orientifold, as happened in the abelian case. In particular, one can see that crosscap boundary conditions are mapped to:
\[
\tilde{\Pi}_a(\sigma + \pi) = \tilde{\Pi}_a(\sigma)
\]
\[
(\tilde{g}_{ab} \chi^b - \tilde{b}_{ab} \chi^b + \frac{i}{2} \partial_e (\tilde{g}_{ab} + \tilde{b}_{ab}) \tilde{\phi}^e_+ \tilde{\phi}^b_+ - \frac{i}{2} \partial_e (\tilde{g}_{ab} + \tilde{b}_{ab}) \tilde{\phi}^e_+ \tilde{\phi}^b_+)|_{\sigma + \pi} =
\]
\[
-(\tilde{g}_{ab} \chi^b - \tilde{b}_{ab} \chi^b + \frac{i}{2} \partial_e (\tilde{g}_{ab} - \tilde{b}_{ab}) \tilde{\phi}^e_+ \tilde{\phi}^b_+ - \frac{i}{2} \partial_e (\tilde{g}_{ab} - \tilde{b}_{ab}) \tilde{\phi}^e_+ \tilde{\phi}^b_+)|_{\sigma}, \tag{4.48}
\]
where the second equation represents that the momenta flowing out of the identification points must have opposite signs, as in the bosonic non-abelian case. The dual fermions satisfy:
\[
\tilde{\phi}^a_+ (\sigma + \pi) = -i \eta M^{-1}_{ba} M_{bc} \tilde{\phi}^e_-(\sigma). \tag{4.49}
\]
\footnote{In these models they are classical boundary conditions since we have null torsion.}
5 Conclusions

We have shown that the canonical transformation approach is particularly useful to study T-duality in open string theories. It provides the explicit non-local mapping between the target space coordinates of the original and dual theories. From it the derivation of the dual boundary conditions is straightforward. In backgrounds with abelian isometries we have reproduced the well-known results of open strings - D-branes duality [10, 11] in a very simple manner. In the $N = 1$ supersymmetric case we have been able to address arbitrary classical boundary conditions in the initial theory and study their transformation under abelian T-duality, showing that in order to obtain a dual super-D-brane some conditions over the original backgrounds must be satisfied. For type I superstrings we have found an orientifold in the dual [10].

We have also applied the canonical transformation description of non-abelian duality to backgrounds of open and closed strings with non-abelian symmetry groups. In these cases the equivalence between initial and dual theories at the quantum level is necessary to get generalized Dirichlet boundary conditions in the dual, these conditions representing that the momenta at the ends of the string must vanish in a certain curved manifold. The dual is then a curved $(d - \dim G - 1)$ D-brane, static since the previous conditions of quantum equivalence force null background gauge field components on the Dirichlet directions. For unoriented topologies we obtain curved orientifolds in which the D-branes are hidden. The $N = 1$ supersymmetrization of these theories shows that the dual is a $N = 1$ supersymmetric curved D-brane (or orientifold if we consider unoriented strings). All these results show that flat D-branes and orientifolds are just particular features under T-duality.

The canonical transformation description of T-duality has been very powerful in obtaining information about the transformation of boundary conditions. However we have to remark that some problems are still open within this approach. An explicit proof of the transformation of the dilaton is lacking, although we believe it should be along the lines of how the modular anomaly appears in abelian gauge field theories [41, 42]. Also it remains an open problem the study of more general examples of non-abelian backgrounds, namely those with isotropic isometries, like WZW models, for which an explicit canonical transformation description is still not available. It could be interesting to study the kind of structures that emerge in the dual.

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Note added

After this work was completed we learned that in [10] curved D-branes were also obtained from Poisson-Lie T-duality transformations. We would like to thank C. Klimcik for bringing this information to our attention and for very helpful discussions.
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