GROUP GEOMETRICAL AXIOMS FOR MAGIC STATES OF QUANTUM COMPUTING

MICHEL PLANAT†, RAYMOND ASCHHEIM‡, MARCELO M. AMARAL‡ AND KLEE IRWIN‡

ABSTRACT. Let $H$ be a non trivial subgroup of index $d$ of a free group $G$ and $N$ the normal closure of $H$ in $G$. The coset organization in a subgroup $H$ of $G$ provides a group $P$ of permutation gates whose common eigenstates are either stabilizer states of the Pauli group or magic states for universal quantum computing. A subset of magic states consists of MIC states associated to minimal informationally complete measurements. It is shown that, in most cases, the existence of a MIC state entails that the two conditions (i) $N = G$ and (ii) no geometry (a triple of cosets cannot produce equal pairwise stabilizer subgroups), or that these conditions are both not satisfied. Our claim is verified by defining the low dimensional MIC states from subgroups of the fundamental group $G = \pi_1(M)$ of some manifolds encountered in our recent papers, e.g. the 3-manifolds attached to the trefoil knot and the figure-eight knot, and the 4-manifolds defined by 0-surgery of them. Exceptions to the aforementioned rule are classified in terms of geometric contextuality (which occurs when cosets on a line of the geometry do not all mutually commute).

MSC codes: 81P68, 51E12, 57M05, 81P50, 57M25, 57R65, 14H30

1. Introduction

Interpreting quantum theory is a long standing effort and not a single approach can exhaust all facets of this fascinating subject. Quantum information owes much to the concept of a (generalized) Pauli group for understanding quantum observables, their commutation, entanglement, contextuality and many other aspects, e.g. quantum computing. Quite recently, it has been shown that quantum commutation relies on some finite geometries such as generalized polygons and polar spaces [1]. Finite geometries connect to the classification of simple groups as understood by prominent researchers as Jacques Tits, Cohen Thas and many others [2, 3].

In the Atlas of finite group representations [4], one starts with a free group $G$ with relations, then the finite group under investigation $P$ is the permutation representation of the cosets of a subgroup of finite index $d$ of $G$ (obtained thanks to the Todd-Coxeter algorithm). As a way of illustrating this topic, one can refer to [3, Table 3] to observe that a certain subgroup of index 15 of the symplectic group $S'_4(2)$ corresponds to the $2QB$ (two-qubit) commutation of the 15 observables in terms of the generalized quadrangle of order two, denoted $GQ(2, 2)$ (alias the doily). For $3QB$, a subgroup of index 63 in the symplectic group $S_6(2)$ does the job and the commutation relies on the symplectic polar space $W_5(2)$ [3, Table 7]. An alternative way to approach $3QB$ commutation is in terms of the generalized hexagon $GH(2, 2)$
(or its dual) which occurs from a subgroup of index 63 in the unitary group $U_3(3)$ [3, Table 8]. Similar geometries can be defined for multiple qudits (instead of qubits).

The straightforward relationship of quantum commutation to the appropriate symmetries and finite groups was made possible thanks to techniques developed by the first author (and coauthors) that we briefly summarize. This will be also useful at a later stage of the paper with the topic of magic state quantum computing.

The rest of this introduction recalls how the permutation group organizing the cosets leads to the finite geometries of quantum commutation (in Sec. 1.1) and how it allows the computation of magic states of universal quantum commutation (in Sec. 1.2). In this paper, it is shown that magic states themselves may be classified according to their coset geometry with two simple axioms (in Sec. 1.3).

### 1.1. Finite geometries from cosets [3, 6, 7]

One needs to define the rank $r$ of a permutation group $P$. First it is expected that $P$ acts faithfully and transitively on the set $\Omega=\{1,2,\cdots,n\}$ as a subgroup of the symmetric group $S_n$. The action of $P$ on a pair of distinct elements of $\Omega$ is defined as $(\alpha,\beta)^p=(\alpha^p,\beta^p)$, $p\in P$, $\alpha\neq\beta$. The orbits of $P$ on $\Omega\times\Omega$ are called orbitals, and the number of orbits is called the rank $r$ of $P$ on $\Omega$. The rank of $P$ is at least two, and the two-transitive groups identify to the rank two permutation groups. Next, selecting a pair $(\alpha,\beta)\in\Omega\times\Omega$, $\alpha\neq\beta$, one introduces the two-point stabilizer subgroup $P_{(\alpha,\beta)}=\{p\in P|(\alpha,\beta)^p=(\alpha,\beta)\}$. There exist $1<m\leq r$ such non-isomorphic (two-point stabilizer) subgroups $S_m$ of $P$. Selecting one with $\alpha\neq\beta$, one defines a point/line incidence geometry $G$ whose points are the elements of $\Omega$ and whose lines are defined by the subsets of $\Omega$ sharing the same two-point stabilizer subgroup. Thus, two lines of $G$ are distinguished by their (isomorphic) stabilizers acting on distinct subsets of $\Omega$. A non-trivial geometry arises from $P$ as soon as the rank of the representation $\mathcal{P}$ of $P$ is $r>2$, and simultaneously, the number of non-isomorphic two-point stabilizers of $\mathcal{P}$ is $m>2$. Further, $G$ is said to be contextual (shows geometrical contextuality) if at least one of its lines/edges corresponds to a set/pair of vertices encoded by non-commuting cosets [7].

Figure 1 illustrates the application of the two-point stabilizer subgroup approach just described for the index 15 subgroup of the symplectic group is $S_4'(2)=A_6$ whose finite representation is $H=\langle a,b|a^2=b^4=(ab)^5=1\rangle$. The finite geometry organizing the coset representatives is the generalized quadrangle $GQ(2,2)$. The other representation is in terms of the two-qubit Pauli operators, as first found in [1, 8]. It is easy to check that all lines not passing through the coset $e$ contains some mutually not commuting cosets so that the $GQ(2,2)$ geometry is contextual. The embedded $(3\times3)$-grid shown in bold (the so-called Mermin square) allows a $2QB$ proof of Kochen-Specker theorem [5].

### 1.2. Magic states in quantum computing

Now we recall our recent work about the relation of coset theory to the magic states of universal quantum computing. Bravyi & Kitaev introduced the principle of ‘magic state distillation’ [9]: universal quantum computation, the possibility of
getting an arbitrary quantum gate, may be realized thanks to stabilizer operations (Clifford group unitaries, preparations and measurements) and an appropriate single qubit non-stabilizer state, called a ‘magic state’. Then, irrespective of the dimension of the Hilbert space where the quantum states live, a non-stabilizer pure state was called a magic state [10]. An improvement of this concept was carried out in [11] showing that a magic state could be at the same time a fiducial state for the construction of a minimal informationally complete positive operator-valued measure, or MIC, under the action on it of the Pauli group of the corresponding dimension. Thus UQC in this view happens to be relevant both to magic states and to MICs. In [11], a $d$-dimensional magic state is obtained from the permutation group that organizes the cosets of a subgroup $H$ of index $d$ of a two-generator free group $G$. This is due to the fact that a permutation may be realized as a permutation matrix/gate and that mutually commuting matrices share eigenstates - they are either of the stabilizer type (as elements of the Pauli group) or of the magic type. It is enough to keep magic states that are...
simultaneously fiducial states for a MIC because the other magic states may
lose the information carried during the computation. A catalog of the
magic states relevant to UQC and MICs can be obtained by selecting $G$
as the two-letter representation of the modular group $\Gamma = PSL(2, \mathbb{Z})$ [12].
The next step, developed in [13, 14], is to relate the choice of the starting group
$G$ to three-dimensional topology. More precisely, $G$ is taken as the fundamental
group $\pi_1(\Delta)$ of a 3-manifold $\Delta$ defined as the complement of a
knot or link $L$ in the 3-sphere $S^3$. A branched covering of degree $d$ over the
selected $\Delta$ has a fundamental group corresponding to a subgroup of index
$d$ of $\pi_1(\Delta)$ and may be identified as a sub-manifold of $\Delta$, the one leading
to a MIC is a model of UQC. In the specific case of $\Gamma$, the knot involved is
the left-handed trefoil knot $T_1 = 3^1$, as shown in [12] and [13, Sec. 2].

1.3. Coset geometry of magic states. The goal of the paper is to classify
the magic states according to the coset geometry where they arise. We start
from a non trivial subgroup $H$ of index $d$ of a free group $G$ and we denote
$N$ the normal closure of $H$ in $G$. As above, the coset organization in a
subgroup $H$ of $G$ provides a group $P$ of permutation gates whose common
eigenstates are either stabilizer states of the Pauli group or magic states for
universal quantum computing. A subset of magic states consists of MIC
states associated to minimal informationally complete measurements.

It is shown in the paper that, in many cases, the existence of a MIC
state entails that the two conditions (i) $N = G$ and (ii) no geometry (a
triple of cosets cannot produce equal pairwise stabilizer subgroups), or that
these conditions are both not satisfied. Our claim is verified by defining
the low dimensional MIC states from subgroups of the fundamental group
$G = \pi_1(\Delta)$ of manifolds encountered in our recent papers, e.g. the 3-
manifolds attached to the trefoil knot and the figure-eight knot, and the
4-manifolds defined by 0-surgery of them.

Exceptions to the aforementioned rule are classified in terms of geometric
contextuality (which occurs when cosets on a line of the geometry do not
all mutually commute).

In section 2, one deals with the case of MIC states obtained from the sub-
groups of the fundamental group of Figure-of-Eight knot hyperbolic manifold
and its 0-surgery. In section 3, the MIC states produced with the trefoil knot
manifold and its 0-surgery are investigated.

2. MIC states pertaining to the Figure-of-Eight knot and its
0-surgery

We first investigate the relation of MIC states to the group geometrical
axioms (i)-(ii) (or their negation) in the context of the Figure-of-Eight knot
$K_{4a1}$ (in Sec. 2.2) and its 0-surgery (in Sec. 2.1). The fundamental group
of the complement of $K_{4a1}$ in the 3-sphere $G = \pi_1(S^3 \setminus K_{4a1})$, and its
connection to MICs, is first studied in [13, Table 2], below are new results
and corrections.

2.1. Group geometrical axioms applied to the fundamental group
$\pi_1(Y)$. The manifold $Y$ defined by 0-surgery on the knot $K_{4a1}$ is of special
interest as shown in [14, Sec 2] and references therein. The number of subgroups of index $d$ of the fundamental group $\pi_1(Y)$ is as follows

$$\eta_d[\pi_1(Y)] = [1, 1, 1, 2, 2, 5, 1, 2, 2, 4, 3, 17, 1, 1, 2, 3, 3, 6, 1, 3, 1, 43, \ldots],$$

where a bold number means that a MIC exists at the corresponding index.

In Table 1, one summarizes the check of our axioms (i) and (ii) applied to $\pi_1(Y)$. A triangle $\Delta$ means that a geometry does exist (corresponding to at least a triple of cosets with equal pairwise stabilizer subgroups), thus with (ii) is violated. According to our theory, for a MIC to exist, we should have (i) and (ii) satisfied, or both of them violated. The former case occurs for $d = 9, 11$ and 19. The latter case occurs for $d = 6$ where the geometry is that of the octahedron (with the 3-partite graph $K(2, 2, 2)$) and $d = 20$, where the geometry is encoded by the complement of the line graph of the bipartite $K(4, 5)$. In all of these five cases, a $pp$-valued MIC does exist.

For dimension 4, the bold triangle points out a violation since (i) is true and (ii) is false while the 2QB-MIC exists. In this case the geometry is the tetrahedron (with complete graph $K_4$) but not all cosets on a line/triangle are mutually commuting, a symptom of geometric contextuality, as shown in Fig. 2.

2.2. Group geometrical axioms applied to the fundamental group $\pi_1(S^3 \setminus K4a1)$. The submanifolds obtained from the subgroups of index $d$ of the fundamental group $\pi_1(S^3 \setminus K4a1)$ for the Figure-of-Eight knot complement are given in Table 2 (column 3), as identified in SnapPy [15] (this corrects a few mistakes of [13, Table 2]).

As for the subsection above, when axioms (i) and (ii) are simultaneously satisfied (or both are not satisfied), a MIC is created. Otherwise no MIC exist in the corresponding dimension, as expected.

There are three exceptions where (i) is true and a geometry does exist (when (ii) fails to be satisfied). This first occurs in dimension 4 with a 2QB MIC arising from the 3-manifold otet0800002, in this case geometric contextuality occurs.

Figure 2. The contextual geometry associated to the 2QB-MIC and permutation group $P = A_4$ in Table 1. The line/triangle $\{a, ab, ab^{-1}\}$ is not made of mutually commuting cosets, thus geometric contextuality occurs.
Table 1. Table of subgroups of the fundamental group
\( \pi_1[S^3 \setminus K4a1(0,1)] \) [with \( K4a1(0,1) \) the 0-surgery over the Figure-of-Eight knot]. The permutation group \( P \) organizing the cosets in column 2. If (i) is true, unless otherwise specified, the graph of cosets leading to a MIC is that of the \( d \)-simplex [and/or the condition (ii) is true: no geometry]. The symbol \( \Delta \) means that (ii) fails to be satisfied. When there exists a MIC with (i) true and (ii) false, the geometry is shown in bold characters (here this occurs in dimension 4, see Fig. 2). If it exists, the MIC is \( pp \)-valued as given in column 4. In addition, \( K(2,2,2) \) is the binary tripartite graph (alias the octahedron) and \( \overline{L}(K(4,5)) \) means the complement of the line graph of the bipartite graph \( K(4,5) \).

| \( d \) | \( P \)         | (i) | pp | geometry                  |
|-------|----------------|-----|----|----------------------------|
| 4     | \( A_4 \)      | yes | 2  | 2QB MIC, \( \Delta \)     |
| 5     | yes            |     |    | \( \Delta \)               |
| 6     | \( A_4 \)      | no  | 2  | 6-dit MIC, \( K(2,2,2) \) |
| 9     | \( (36, 9) \cong 3^2 \times 4 \) | yes | 2  | 2QT MIC                    |
| 11    | \( (55,1) = 11 \times 5, \times 2 \) | yes | 3  | 11-dit MIC                 |
| 16    | \( (48,3) \cong 4 \times A_4 \) | yes |    | \( \Delta \)               |
| 19    | \( (171,3) \cong 19 \times 9 \) | yes | 3  | 19-dit MIC                 |
| 20    | \( (120,39) \cong 4 \times (5 \times (6,2)) \) | yes |    | \( \overline{L}(K(4,5)) \) |

contextuality occurs in the cosets as in Fig. 2 of the previous subsection. Then, it occurs in dimension 7 (corresponding to 3-manifolds \( \text{otet1400002} \) and \( \text{otet14000035} \)) when the geometry of cosets is that of the Fano plane shown in Fig. 3a. Finally it occurs in dimension 10 when the geometry of cosets is that of a \([10_3] \) configuration shown in Fig. 3b [16, p 74].

In addition to the latter cases, false detection of a MIC may occur (this is denoted ‘fd’) in dimension 8 as shown in Table 2.
Figure 3. Contextual geometries associated to ‘(i) true and (ii) false’ for the MICs of the Figure-of-Eight knot $K_{4a1}$ listed in Table 1: (a) the Fano plane related to the manifold $\text{otet1400002}$ at index 7, (b) the configuration $[10_5]$ at index 10. The bold lines are for cosets that are not all mutually commuting. Each line corresponds to pair of cosets with the same stabilizer subgroup isomorphic to $\mathbb{Z}_2^2$.

3. MIC states pertaining to the trefoil knot and its 0-surgery

We now investigate the relation of MIC states to the group geometrical axioms (i)-(ii) (or their negation) in the context of the trefoil knot $3_1$ (in Sec. 3.2) and its 0-surgery (in Sec. 3.1). The fundamental group of the complement of $3_1$ in the 3-sphere $G = \pi_1(S^3 \setminus 3_1)$, and its connection to MICs, is studied in [13, Table 1] and below.

3.1. Group geometrical axioms applied to the fundamental group $\pi_1(\tilde{E}_8)$.

The manifold $\tilde{E}_8$ is defined by 0-surgery on the trefoil knot $3_1$ and is of special interest as shown in [14, Sec 3] and references therein. The number of subgroups of index $d$ of the fundamental group $\pi_1(Y)$ is as follows

$$\eta_d[\tilde{E}_8] = [1, 1, 2, 2, 1, 5, 3, 2, 4, 1, 1, 12, 3, 3, 4, 3, 1, 17, 3, 2, 8, 1, 1, 27, 2, \ldots]$$

where a bold number means that a MIC exists at the corresponding index.

Such cases are summarized in Table 3. As expected this occurs when the axioms (i) and (ii) are both true, or are both false. The latter case occurs at index 6 with geometry of the octahedron [and graph $K(2,2,2)$] and at index 15 with a geometry of graph $K(5,5,5)$.

Exceptions to the rules are when a MIC exists with (i) true but not (ii). This occurs in dimension 3 (for the Hesse SIC) since the free group has a single generator (a trivial case), at index 4 (for the $2QB$ MIC) with a contextual geometry as in Fig. 2 and at index 21 with a contextual geometry (not shown) of graph $K(3,3,3,3,3,3)$.

3.2. Group geometrical axioms applied to the fundamental group $\pi_1(S^3 \setminus 3_1)$. The characteristics of submanifolds obtained from the subgroups of index $d$ of the fundamental group $\pi_1(S^3 \setminus 3_1)$ for the trefoil knot complement are given
Table 2. Table of 3-manifolds $M^3$ found from subgroups of finite index $d$ of the fundamental group $\pi_1(S^3 \setminus K_{4a1})$ (alias the $d$-fold coverings over the Figure-of-Eight knot 3-manifold). The covering type ‘ty’ in column 2, the manifold identification ‘$M^3$’ in column 3 and the number of cusps ‘cp’ in column 4 are from SnapPy [15]. For $d = 9$ and 10, SnapPy does not provide results so that we only identify the permutation group $P = \text{SmallGroup}(o, k)$ (abbreviated as $(o, k)$), where $o$ is the order and $k$ is the $k$-th group of order $o$ in the standard notation (that is used in Magma). If it exists, the MIC is ‘pp’-valued. If (i) is true, unless otherwise specified, the graph of cosets leading to a MIC is that of the $d$-simplex [and/or the condition (ii) is true: no geometry]. The symbol $\Delta$ means that (ii) fails to be satisfied. When there exists a MIC with (i) true and (ii) false, the geometry is shown in bold characters. The symbol ‘fd’ means a false detection of a MIC when (i) and (ii) are satisfied simultaneously while a MIC does not exist. The abbreviations ‘Fano’, ‘$d$-ortho’ and ‘$[10_3]$’ are for the Fano plane, the $d$-orthoplex and the corresponding geometric configuration.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$d$ & ty & $M^3$ (or $P$) & cp & (i) & pp & geometry \\
\hline
2 & cyc & otet\text{04}00002, \text{m}206 & 1 & yes & & 2QB MIC, \Delta \\
3 & cyc & otet\text{06}00003, \text{s}961 & 1 & no & & \\
\hline
4 & irr & otet\text{08}00002, L10g46, t_{12840} & 2 & yes & yes & 2QB MIC, \Delta \\
& cyc & otet\text{08}00007, t_{12839} & 1 & no & & \\
\hline
5 & cyc & otet\text{10}00009 & 1 & no & & \\
& irr & otet\text{10}00006, L8o20 & 3 & yes & yes & 5-dit MIC \\
& irr (\times 2) & otet\text{10}00026 & 2 & yes & 5-dit MIC & \\
\hline
6 & cyc & otet\text{12}00013 & 1 & no & & \\
& irr & otet\text{12}00009 & 1 & no & & \\
& irr & otet\text{12}00038 & 1 & yes & 10 & 6-dit MIC \\
& irr & otet\text{12}00041 & 2 & no & & \\
& irr (\times 2) & otet\text{12}00017 & 2 & no & & \\
& irr (\times 4) & otet\text{12}00090 & 2 & yes & 2 & 6-dit MIC \\
\hline
7 & cyc & otet\text{14}00019 & 1 & no & & \\
& irr (\times 4) & otet\text{14}00002, L14n55217 & 3 & yes & yes & 7-dit MIC, $\Delta$ : Fano \\
& irr (\times 4) & otet\text{14}00035 & 1 & yes & yes & 7-dit MIC, $\Delta$ : Fano \\
\hline
8 & cyc & otet\text{16}00026 & 1 & no & & \\
& irr (\times 2) & otet\text{16}00035 & 1 & no & & \\
& irr & otet\text{16}00079 & 2 & yes & yes & 16-cell \\
& irr (\times 2) & otet\text{16}00016 & 2 & yes & yes & 16-cell \\
& irr & otet\text{16}00092 & 2 & yes & & \\
& irr & otet\text{16}00091 & 2 & yes & & \\
& irr & otet\text{16}00013, L14n17678 & 2 & no & & \\
\hline
9 & (\times 2) & (36, 9) \cong 3^2 \rtimes 4 & yes & 2 & & 2QT MIC \\
& (\times 2) & (504, 156) = PSL(2, 8) & yes & 3 & & 2QT MIC \\
& (\times 2) & (216, 153) \cong 3^2 \rtimes (24, 3) & yes & 2 & & 2QT MIC \\
\hline
10 & (\times 6) & (160, 234) \cong 2^4 \rtimes 10 & yes & 5 & & 10-dit MIC \\
& (\times 2) & (120, 34) = S_5 & yes & 4 & & 10-dit MIC, $\Delta$ : $[10_3]$ \\
& (\times 2) & (120, 34) = S_5 & no & 7 & 10-dit MIC, 5-ortho & \\
& (\times 2) & (360, 118) = A_6 & yes & 5 & & 10-dit MIC \\
\hline
\end{tabular}
\end{table}

in Table 4 by using SnapPy [15] and Sage [17] for identifying the corresponding subgroup of the modular group $\Gamma$ [12] (this improves [13, Table 1]).
As for the above sections, when axioms (i) and (ii) are simultaneously satisfied (or both are not satisfied), a MIC is created. Otherwise no MIC exist in the corresponding dimension, as one should expect.

There are a few exceptions where (i) is true and a geometry does exist (when (ii) fails to be satisfied). This first occurs in dimension 3 for the Hesse SIC where the free subgroup is trivial with a single generator. The next exceptions are for the 6-dit MIC related to the permutation group $S_4$ with the contextual geometry of the octahedron shown in Fig. 4a, in dimension 9 for the 2QT MIC related to the permutation group $3_3 \rtimes S_4$ with a contextual geometry consisting of three disjoint lines, and in dimension 10 for a 10-dit MIC related to the permutation group $A_5$ and the contextual geometry of the so-called Mermin pentagram. The latter geometry is known to allow a $3QB$ proof of the Kochen-Specker theorem [5].

4. Conclusion

Previous work about the relationship between quantum commutation and coset-generated finite geometries has been expanded here by establishing a connection between coset-generated magic states and coset-generated finite geometries. The magic states under question are those leading to MICs (with minimal complete quantum information in them). We found that, given an appropriate free group $G$, two axioms (i): the normal closure $N$ of the subgroup of $G$ generating the MIC is $G$ itself and (ii): no coset-geometry should exist, or the negation of both axioms (i) and (ii), are almost enough to classify the MIC states. The few exceptions rely on configurations that admit geometric contextuality. We restricted the application of the theory to the fundamental group of the 3-manifolds defined from the Figure-of-Eight knot (an hyperbolic manifold) and from the trefoil knot, and to 4-manifolds $Y$ and $E_8$ obtained by 0-surgery on them. It is of importance to improve of knowledge of the magic states due to their application to quantum computing and we intend to pursue this research in future work.
Figure 4. Contextual geometries associated to ‘(i) true and (ii) false’ for the MICs of the trefoil knot $3_1$ listed in Table 2: (a) the octahedron, related to the subgroup $\Gamma_0(4)$ of $\Gamma$ at index 6, (b) three disjoint lines $K_3^3$ at index 9, (c) The Mermin’s pentagram at index 10. The bold lines are for cosets that are not all mutually commuting.

REFERENCES

[1] M. Planat, Pauli graphs when the Hilbert space dimension contains a square: Why the Dedekind psi function? J. Phys. A Math. Theor. 2011, 44, 045301.
[2] Thas, J.; van Maldeghem, H. Generalized polygons in finite projective spaces. In Distance-Regular Graphs and Finite Geometry, in Special Issue: Conference on Association Schemes, Codes and Designs, Proceedings of the 2004 Workshop on Distance-Regular Graphs and Finite Geometry (Com 2 MaC 2004), Busan, Korea, 19–23 July 2004.
[3] M. Planat and H. Zainuddin, Zoology of Atlas-Groups: Dessins D’enfants, Finite Geometries and Quantum Commutation, Mathematics (MDPI) 2017, 5, 6.
[4] Wilson, R.; Walsh, P.; Tripp, J.; Suleiman, I.; Parker, R.; Norton, S.; Nickerson, S.; Linton, S.; Bray, J.; Abbott, R. ATLAS of Finite Group Representations, Version 3. Available online: http://brauer.maths.qmul.ac.uk/Atlas/v3/exc/TF42/ (accessed on June 2015).
[5] Planat, M. On small proofs of the Bell-Kochen-Specker theorem for two, three and four qubits. Eur. Phys. J. Plus 2012, 127, 86.
[6] M. Planat, A. Giorgetti, F. Holweck and M. Saniga, Quantum contextual finite geometries from dessins d’enfants, Int. J. Geom. Meth. in Mod. Phys. 12 1550067 (2015).
[7] Planat, M. Geometry of contextuality from Grothendieck’s coset space. Quantum Inf. Process. 2015, 14, 2563–2575.
[8] M. Saniga and M. Planat, Multiple qubits as symplectic polar spaces of order two, Adv. Stud. Theor. Phys. 1, 1 (2007).
[9] Sergey Bravyi and Alexei Kitaev, Universal quantum computation with ideal Clifford gates and noisy ancillas, Phys. Rev. A 71 022316 (2005).
[10] V. Veitch, S. A. Mousavian, D. Gottesman and J. Emerson, New J. of Phys. 16, Article ID 013009 (2014).
[11] M. Planat and Z. Gedik, Magic informationally complete POVMs with permutations, R. Soc. open sci. 4 170387 (2017).
[12] M. Planat, The Poincaré half-plane for informationally complete POVMs, *Entropy* **20**, 16 (2018).
[13] M. Planat, R. Aschheim, M. M. Amaral and K. Irwin, Universal quantum computing and three-manifolds, *Universal quantum computing and three-manifolds Symmetry* **10**, 773 (2018).
[14] M. Planat, R. Aschheim, M. M. Amaral and K. Irwin, Quantum computing, Seifert surfaces and singular fibers, *Quantum Reports* **1**, 12-22 (2019).
[15] M. Culler, N. M. Dunfield, M. Goerner, and J. R. Weeks, SnapPy, a computer program for studying the geometry and topology of 3-manifolds, http://snappy.computop.org.
[16] B. Grünbaum, *Configurations of points and lines* (Graduate studies in mathematics, vol. 103, AMS, Providence, Rhode Island, 2009).
[17] William A. Stein et al. Sage Mathematics Software (Version 8.6), The Sage Development Team, 2019, http://www.sagemath.org.

† Université de Bourgogne/Franche-Comté, Institut FEMTO-ST CNRS UMR 6174, 15 B Avenue des Montboucons, F-25044 Besançon, France.
E-mail address: michel.planat@femto-st.fr

‡ Quantum Gravity Research, Los Angeles, CA 90290, USA
E-mail address: raymond@QuantumGravityResearch.org
E-mail address: Klee@QuantumGravityResearch.org
E-mail address: Marcelo@QuantumGravityResearch.org
| d | ty | cp | P          | (i) | pp | type in Γ | geometry          |
|---|----|----|------------|-----|----|-----------|-------------------|
| 2 | cyc | 1  | (2,1) ≡ 2 | no  | 1  | Γ₀(2)    | Hesse SIC, Δ, L7a1 |
| 3 | cyc | 1  | (3,1) ≡ 3 | no  | 1  | Γ₀(3)    | 2QB SIC, Δ, L6a3  |
|   | irr | 2  | (6,1) ≡ 6 | yes | 2  | 4A₀      | 2QB SIC            |
| 4 | cyc | 1  | (4,1) ≡ 4 | no  | 1  | Γ₀(3)    | 2QB SIC, Δ, L6a3  |
|   | irr | 2  | (12,3) = A₄ | yes | 2  | 4A₀      | 2QB SIC            |
|   | irr | 1  | (24,12) = S₄ | yes | 2  | 4A₀      | 2QB SIC            |
| 5 | cyc | 2  | (5,1) ≡ 5 | no  | 1  | 5A₀      | 5-dit MIC          |
|   | irr | 3  | (60,5) = A₅ | yes | 1  | 5A₀      | 5-dit MIC          |
| 6 | reg | 3  | (6,1) ≡ 6 | no  | 2  | Γ₀(2)    | 6-dit MIC, 6₃ [14] |
|   | cyc | 3  | (6,2) = 3×2 | no  | 2  | Γ₀(2)    | 6-dit MIC, K(2,2,2) |
|   | irr | 2  | A₄        | yes | 2  | 3C₀      | 6-dit MIC          |
|   | irr | 1  | (24,13) = 3×8 | no | 2  | 6B₀      | 6-dit MIC          |
|   | irr | 1  | (18,3) ≡ 3²×2 | no | 2  | 6A₀      | 6-dit MIC          |
|   | irr | 3  | S₄        | yes | 2  | Γ₀(4)    | 6-dit MIC, Δ : octa |
|   | irr | 2  | A₅        | yes | 2  | Γ₀(5)    | 6-dit MIC          |
|   | irr | 2  | S₄        | yes | 2  | 4C₀      | 6-dit MIC, Δ : octa |
| 7 | cyc | 1  | (7,1) ≡ 7 | no  | 2  | NC(0,6,6,1,1,[i7]) | 7-dit MIC          |
|   | irr | 2  | (42,1) ≡ 7×(6,2) | yes | 2  | NC(0,6,6,1,1,[i7]) | 7-dit MIC          |
|   | irr | 1  | (168,42) = PSL(2,7) | yes | 2  | NC(0,10,1,1,[i7]) | 7-dit MIC          |
|   | irr | 2  | SL(2,7)   | yes | 2  | NC(0,10,1,1,[i7]) | 7-dit MIC          |
| 8 | cyc | 1  | (8,1) ≡ 8 | no  | 2  | 6C₀      | 6-dit MIC          |
|   | irr | 2  | (24,13) = A₄ | yes | 2  | 4D₀      | 6-dit MIC          |
|   | irr | 2  | (24,3) ≡ 2.A₄ | yes | 2  | 4D₀      | 6-dit MIC          |
|   | irr | 1  | PSL(2,7)  | yes | 2  | Γ₀(7)    | 6-dit MIC          |
|   | irr | 2  | SL(2,7)   | yes | 2  | NC(0,8,2,2, [s²]) | 6-dit MIC          |
|   | irr | 2  | (48,29) ≡ 2×(24,3) | yes | 2  | 8A₀      | 16-cell            |
| 9 | cyc | 1  | (9,1) ≡ 9 | no  | 7  | 6D₀      | 9-dit MIC, K(3,3,3) |
|   | irr | 2  | (18,3) = 3²×(18,3) | no | 7  | 6D₀      | 9-dit MIC, K(3,3,3) |
|   | irr | 1  | (324,160) ≡ 3³×A₄ | no | 7  | 9A₀      | K(3,3,3), K₃²    |
|   | irr | 3  | (54,5)    | yes | 7  | NC(0,6,1,0,[i7]) | 9-dit MIC          |
|   | irr | 3  | (162,10) = 3²×6 | yes | 7  | NC(0,6,1,0,[i7]) | 9-dit MIC          |
|   | irr | 1  | (504,156) = PSL(2,8) | yes | 3  | NC(1,9,1,0,[i7]) | 2QT SIC            |
|   | irr | 2  | (432,734) ≡ 3²×(48,29) | yes | 2  | NC(0,8,3,0,[i7]) | 2QT SIC            |
|   | irr | 3  | (648,703) ≡ 3³×S₄ | yes | 2  | NC(0,12,1,0,[i7]) | 2QT SIC, Δ : K₃²   |
| 10| cyc | 1  | (120,35) ≡ 2×A₅ | no  | 5  | 10A₀     | 10-dit MIC, Δ : MP |
|   | irr | 2  | A₅        | yes | 5  | 5C₀      | 10-dit MIC, Δ : MP |
|   | irr | 2  | (720,764) ≡ A₆×2 | yes | 9  | NC(0,10,0,4,[i7]) | 10-dit MIC         |

**Table 4.** Subgroups of index d of the fundamental group π₁(S³ \ {z})
(also the d-fold coverings over the trefoil knot 3-manifold. The meaning of symbols is as in Table 2. When the subgroup in question is a subgroup of the modular group Γ, it is identified as a congruence subgroup or by its signature NC(0, N, ν₀, ν₁, [iν]) (see [12] for the meaning of entries). The permutation group P = SmallGroup(o, k) is abbreviated as (o, k). As in Table 1, if (i) is true, unless otherwise specified, the graph of cosets leading to a MIC is the d-simplex [and/or the condition (ii) is true]. Exceptions (with geometry identified in bold characters) are for a MIC with (i) true and (ii) false. For index 9 and 10, some subgroups of large order could not be checked as leading to a MIC or not, they are not shown in the table. The abbreviation ‘octa’ is for the octahedron, ‘MP’ is for the Mermin pentagram and K₃² means three disjoint triangles.)