Weakly Coretractable Modules

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Abstract: If R is a ring with identity and M is a unitary right R-module. Here we introduce the class of weakly coretractable module. Some basic properties are investigated and some relationships between these modules and other related one are introduced.

1. Introduction
Throughout this paper, R is a ring with unity and all modules are unitary right R-modules. The notion of coretractable module appeared in [3]. However, Amini [2], studied this concept as a dual of retractable modules, “M is said to be coretractable if ∀N of M, there exists a nonzero mapping f∈Homₐ(M/N, M)”[3]. In [6], we introduced the concept of strongly coretractable modules, where "an R-module M is called strongly coretractable module if for all a submodule N of M, there exists a nonzero R-homomorphism f:M/N→M such that Imf+N=M"[6]. Equivalently, "M is strongly coretractable if for each a proper submodule N of M, there exists a nonzero mapping f∈Endₐ(M) such that Imf+N=M" [6], [7]. "It is clear that every strongly coretractable module is coretractable, but not conversely for example the Z-module Z₄ is coretractable but not strongly coretractable module" and then some generalizations are introduced see [8], [9], [10] and [11]. We introduce the notion of weakly coretractable module, where an R-module M is called weakly coretractable if ∀K of M, there exists a nonzero mapping f∈Endₐ(M) such that f(K)=0. Clear that every coretractable is weakly coretractable, but not conversely. Section one of this work is devoted to recall some basic properties of coretractable modules. Also we added some new results (up to our knowledge). In section two we introduced and studied weakly coretractable modules. Also many connections between it and other classes of modules were given.

2. On Coretractable Modules
First recall the following definition:
"Definition(1.1) [3]: An R-module M is called coretractable if for each a proper submodule S of M, there exists a nonzero R-homomorphism f:M/S→M".

Examples and Remarks (1.2):
(1) A module M is coretractable iff ∀S<M, ∃ 0≠f∈Endₐ(M) such that f(S)=0; S⊆kerf.
Proof: \((\Rightarrow)\) Suppose that \(S\triangleleft M, \exists f: M \rightarrow M, f \neq 0 \) and \(f(S) = 0\). Define \(g: M/S \rightarrow M\) by \(g(m+S) = f(m)\), it is clear that \(g\) is well-defined \(R\)-homomorphism. Now, since \(f \neq 0\) and \(f(S) = 0\), then \(\exists m \in M, m \notin S\) such that \(f(m) \neq 0\). Thus \(g(m+S) = f(m) \neq 0\). Thus \(\text{Hom}_R(M/S, M) \neq 0\) \(\forall S \triangleleft M\).

\((\Leftarrow)\) Let \(S \triangleleft M\), since \(\exists f: M/S \rightarrow M, f \neq 0\). It follows that \(f \pi \in \text{End}_R(M)\) and \(f \pi(S) = f(\pi(S)) = f(0) = 0\), where \(\pi\) is the natural epimorphism from \(M\) into \(M/S\). Thus \(M\) is coretractable.

(2) "Clearly every semisimple module is coretractable, and hence every \(R\)-module over a semisimple ring is coretractable \([2]\)."

But it may be that coretractable module not semisimple as the \(Z\)-module \(Z_4\).

(3) "\(Z_{p^\infty}\) is coretractable \(Z\)-module, since \(Z_{p^\infty} \oplus Z_{p^\infty} (\forall S < Z_{p^\infty})\) \([2]\)."

(4) "\(Z_n\) is coretractable \(Z\)-module \((\forall n \in \mathbb{Z}^+)\)" \([2]\).

(5) See the \(Z\)-module \(Q\). Suppose \(Q\) is a coretractable module. Since \(Z \triangleleft Q\), \(\exists f \in \text{End}_R(Q)\), \(f \neq 0\) with \(f(Z) = 0\). Now, for any \(s/t \in Q\), \(f(s/t) = f(1/t) s\). But \(0 = f(1) = f(1/t) s\) and hence \(f(s/t) = 0\), \(f = 0\), which is a contradiction. Thus \(Q\) is not coretractable module.

(6) See \(M = \mathbb{Z} \oplus \mathbb{Z}^2\) as \(Z\)-module. Let \(N = 3\mathbb{Z} \oplus \mathbb{Z}^2\), \(M/N \cong \mathbb{Z}^2\). But there is no nonzero mapping \(f: \mathbb{Z}^2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}_2\), since if we assume there exists \(x \in \mathbb{Z}^2\) and \(f(x) \neq (0, 0)\). So if \(f(x) = (n, m)\), \(n \neq 0\), then \(f(x, 3) = f(0, 0)\). On the other hand, \(f(x, 3) = f(\bar{x}) = 3 = (m, n)\). So if \(f(x) \neq (0, 0)\) which is a contradiction. Similarly if \(f(x) = (0, 1)\), then \(f(x, 3) = f(0, 0) \neq f(0, 0)\), but \(f(0, 0) = f(0, 0)\). Thus \(\mathbb{Z}^2\) is not coretractable module.

(7) See \(M = \mathbb{Z} \oplus \mathbb{Z}^2\) as \(Z\)-module. Let \(N = \mathbb{Z} \oplus \mathbb{Z}_2\) suppose \(\exists g \in \text{End}_R(M)\), \(g \neq 0\), \(g(N) = 0\) then \(g(a, b) = 0 \forall a \in \mathbb{Z}, b \in \mathbb{Z}_2\) Now, for all odd integers \(x\) and \(y\), \(x = 2m + 1\) and \(y = 2n + 1\) for some \(m, n \in \mathbb{Z}\), \(g(x, y) = g(2m + 1, 2n + 1) = g(2m + 1, 2n) + g(0, 1) = 0\) \(g(0, 1) = g(0, 1)\), but \(g(0, 2) = g(0, 1) 2\), then \(g(0, 1) = 0\), hence \(g(x, y) = 0\). Now for all even integer 2\(m\) and odd integer 2\(n+1\); \(g(2m, 2n+1) = g(2m, 2n) + g(0, 1) = 0\). Thus \(g(M) = 0\) that is \(g\) is a zero mapping which is a contradiction. Thus \(M\) is not coretractable module.

(8) Coretractability is preserved by an isomorphism.

Recall that "a submodule \(B\) of \(M\) is relative complement of \(A\) in \(M\) if \(B\) is maximal with respect to the property \(B \cap A = 0\) \([4]\). "A submodule \(E\) of \(M\) is essential if \(\forall W \leq M\) if \(E \cap W = 0\), then \(W = 0\) (denoted \(E \subseteq M\))\) \([4]\). The following Proposition appears in \([2]\) without proof.

**Proposition(1.3):** An \(R\)-module \(M\) is coretractable iff \(\text{Hom}_R(M/E, M) \neq 0\ \forall E \subseteq M\).

**Proof:** It is clear.

**Proposition(1.4):** An \(R\)-module \(M\) is coretractable iff \(M\) is coretractable \(\bar{R}\)-module (where \(\bar{R} = R/\text{ann}M\) ).

**Proof:** Since every \(R\)-submodule of \(M\) is \(\bar{R}\)-submodule of \(M\) and conversely, also every \(R\)-homomorphism is an \(\bar{R}\)-homomorphism and conversely. Hence the result follows directly.

"A module \(M\) is cogenerator if every nonzero homomorphism \(f: M_1 \rightarrow M_2\) where \(M_1\) and \(M_2\) are modules, there is \(g: M_2 \rightarrow M\) such that \(g \circ f \neq 0\ \([5]\, \text{P.} \, 507\) and \([6], \text{P.} \, 53\). Equivalently an \(R\)-module \(M\) is called a cogenerator if for any \(R\)-module \(N\) and \(0 \neq x \in \text{Ker} N\), there exists \(g: N \rightarrow M\) such that \(g(x) \neq 0\)" \([5], \text{P.} \, 507\).

**Proposition(1.5):** Every cogenerator \(R\)-module is coretractable module.

**Proof:** Let \(M\) be a cogenerator module and let \(S \triangleleft M\), then \(M/S \neq 0\). Let \(x = m + S \neq 0\), \(\exists g: M/S \rightarrow M\) such that \(g(x) \neq 0\), that is \(g \neq 0\) and hence \(M\) is coretractable module.

The converse of Proposition (1.5) may not be true. Consider \(Z_2\) is coretractable (by part (4)), but it is not cogenerator module. Since the only nonzero mapping from \(Z\) into \(Z_2\) is given by:

\[
g(x) = \begin{cases} 
0 & \text{if } x \text{ is even integer} \\
1 & \text{if } x \text{ is odd integer}
\end{cases}
\]

Thus for each nonzero even integer \(x\), there is no \(g: Z \rightarrow Z_2\) such that \(g(x) \neq 0\).

**Corollary(1.6):** For an \(R\)-module \(M\), if \(\prod_{i \in \Lambda} M\) is a cogenerator module, then \(M\) is coretractable module.
Proof: By [5, Corollary19.7, P. 508], M is cogenerator and hence by Proposition(1.5), M is coretractable.

Corollary(1.7): If R is cogenerator. Then any faithful R-module is coretractable.

Proof: Take M is faithful R-module. Then by [5, Proposition(19.19), P. 512], M is cogenerator and hence by Proposition(1.5), M is a coretractable module.

Note that "for any module M and a cogenerator C, C⨁M is a cogenerator and so is a coretractable module, but M need not be coretractable module. So that Coretractability is not preserved by taking submodules, factor modules and direct summands "[2]. "However there are some special cases, but first recall that: a submodule S of an R-module M is called fully invariant if f(S) is contained in S for every R-endomorphism f of M" [4].

Proposition(1.8): [2] "Let F=⨁I F be a coretractable R-module. If F is a fully invariant submodule of M or F cogenerates M, then F is also coretractable. In particular, if ⨁I F is coretractable for some index set I, then so is F".

Consider M=Z⨁Q, Endz(M)≅HomZ(Z,Q), EndQ(M)≅Z, so for any φ∈Endz(M).

(f(n 0)
m x), for some n, m∈Z and x∈Q. Then φ(Q)={0 0}(n 0)(m x) y∈Q\{0 0 0 y}, x, y∈Q\{0 0 0 y}. So φ(Q)⊆Q. Thus Q is fully invariant in M, but Q isn't coretractable.

Proposition(1.9): [2] "Let M1, M2, ..., Mn be coretractable modules, then so is ⨁n i=1 Mi".

As application of Proposition(1.9), each of the Z-module Z, Z⨁Z, Z⨁Z, Zp, Z2 is any prime number, n∈Z, is coretractable.

Proposition(1.10): If M is module over commutative ring R such that [N:M]=annM≠annN ∀N of M. Then every factor module of M is a coretractable module, where [N:M]={r∈R: Mr⊆N} and annN={r∈R: Nr=0}.

Proof: Suppose L/N<M/N, so N<L<M. By 3rd fundamental isomorphism theorem (M/N)/(L/N)≅M/L. Hence there exists an isomorphism g:(M/N)/(L/N)→M/L. Since annL≠annM, then there exists t∉annL and t∉annM. Define f:M/L→M by f(m+L)=mt. Clear f is well-defined and R-homomorphism.

Consider the sequence (M/N)/(L/N)→M/L→M→M/N; that is πₖfₖgₖ(M/N)/(L/N)→M/N and πₖfₖgₖ(M/N)/(L/N)={πₖfₖ(M/L)={πₖ(Mt)}}={πₖ(Mt)}. Since M is coretractable, then M/N=annM which is a contradiction.

Proposition(1.11): If M is a module such that ∀S<M, ∃D⊆⨁M such that S⊆D⊂M. Then M is coretractable.

Proof: Let S<M. By hypothesis S⊆D⊂M for some direct summand D of M, there is f:M→M, f≠0. Define h: M/S→M/D by h(x+S)=x+D ∀x∈M, then h is well-defined. Since f≠0, there exists x +D∈M/D, x+D≠0 and f(x+D)=0, x∉D and hence x∉S. Thus x+S≠0 and h(x+S)=x+D≠0. Now, fₖh: M/S→M and fₖh ≠0 since fₖh(x+S)=f(h(x+S))=f(x+D)≠0. Therefore M is a coretractable module.

Recall that Schur's Lemma stated "If M is simple module, then S=End(M) is division ring "[4].

Proposition(1.12): An R-module M is simple iff M is coretractable and EndR(M) is a division ring.

Proof: (⇒) Since M is simple, clearly M is coretractable and so EndR(M) is division ring by Schur’s Lemma.

(⇐) Let S<M and S≠0. As M is coretractable, ∃g∈EndR(M), g≠0 such that g(S)=0, hence g is not one-one and that contradiction with EndR(M) is a division ring. Thus M is a simple module.

§2: Weakly coretractable Modules

In this section, we introduce the concept of weakly coretractable modules which is a generalization of coretractable modules. Several examples and many properties related with this concept are given.
Definition(2.1): An R-module $M$ is called weakly coretractable module if $\forall N$ of $M$, there exists $f \in \text{End}_R(M)$, $f \neq 0$ and $I^2(N) = 0$; $(N \subseteq \ker f)$. A ring $R$ is called weakly coretractable if $R$ is a weakly coretractable $R$-module.

One can easy see that a module $M$ is weakly coretractable if any proper submodule $S$ of $M$, there is a nonzero homomorphism $f \in \text{End}_R(M)$ and there is $n \in \mathbb{Z}$ with $I^2(N) = 0$.

Examples and Remarks(2.2):

(1) Every coretractable module is weakly coretractable. But conversely;

Example(1): Consider $M = \mathbb{Z} \oplus \mathbb{Z}$ is not coretractable by Examples and Remarks(1.2(6)).

$$\text{End}_2(M) \cong \begin{pmatrix} \text{End}(\mathbb{Z}) & \text{Hom}(\mathbb{Z}, \mathbb{Z}) \\ \text{Hom}(\mathbb{Z}, \mathbb{Z}) & \text{End}(\mathbb{Z}) \end{pmatrix},$$

where $\text{End}(\mathbb{Z}) \cong \mathbb{Z}$, $\text{Hom}(\mathbb{Z}, \mathbb{Z}) = 0$ and $\text{End}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$.

Let $f_1$, zero map $\cong \mathbb{Z}$ where $f_1(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$ and $\text{End}(\mathbb{Z}) \cong \mathbb{Z}$. Any $f \in \text{End}_R(M)$ has one of the following form:

$\{ (n \begin{pmatrix} 0 \\ 1 \end{pmatrix}), (n \begin{pmatrix} 0 \\ 1 \end{pmatrix}), (n \begin{pmatrix} 0 \\ 1 \end{pmatrix}), (0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}), (0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}), (0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}), (n \begin{pmatrix} 0 \\ 1 \end{pmatrix}), (n \begin{pmatrix} 0 \\ 1 \end{pmatrix}) : \text{with } n \neq 0 \}$.

Let $N < M$. If $N$ contains $(x, y)$ with $x \neq 0$, then there exists $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f(\begin{pmatrix} 0 \\ 1 \end{pmatrix})$. Hence $f(N) = 0$. Otherwise, $f(N) = 0$ and hence $I^2(N) = 0$, so $M$ is weakly coretractable.

Example(2): Consider $M = \mathbb{Z} \oplus \mathbb{Z}$ isn't coretractable module see Examples and Remarks(1.2(7)).

$$\text{End}_2(M) \cong \begin{pmatrix} Z & Z \\ Z & Z \end{pmatrix}.$$  

Let $N < M$. If $N$ has an element $(x, y)$ such that $x \neq y$, so there is $f = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \text{End}_2(M)$ and $f(x) = (x - y, x - y) = (0, 0)$. So $f(N) \neq 0$ but it is easy to check that $I^2(M) = 0$, so $f(N) = 0$. If all elements of $N$ has the form $(x, x), x \in \mathbb{Z}$. Then there exists also $f = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \text{End}_2(M)$, $f(x) = (-1, -1) = (-1, -1)$.

Therefore $M$ is weakly coretractable.

(2) A direct summand of weakly coretractable module may be not weakly coretractable.

Consider $M = \mathbb{Z} \oplus \mathbb{Z}$ is weakly coretractable $\mathbb{Z}$-module and $Z$ is direct summand of $M$ and it isn’t weakly coretractable because for any proper submodule $N$ of $Z$, $N = \mathbb{Z} n > 0$ for some positive integer $n$ and for any $f \in \text{End}_2(Z), f \neq 0, f(x) = zm$ for some $m \in \mathbb{Z}, m \neq 0$, then $f \neq 0$ and so $I^2(N) \neq 0$.

(3) The epimorphism image of weakly coretractable module may be not necessary weakly coretractable.

Consider $f : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}$ be a natural projection and $\mathbb{Z} \oplus \mathbb{Z}$ weakly coretractable, but $\mathbb{Z}$ is not weakly coretractable.

(4) Every strongly coretractable $R$-module is coretractable and so it is weakly coretractable.

(5) If $M$ is an $R$-module with $\text{End}_R(M)$ is semiprime, $M$ is weakly coretractable iff $M$ is coretractable.

Proof: ($\Rightarrow$) Let $S < M$. As $M$ is weakly coretractable, so $\exists g \in \text{End}_R(M), g \neq 0$ and $g^2(S) = 0$.

Since $\text{End}_R(M)$ is semiprime, $g^2 \neq 0$ implies $M$ is coretractable.

($\Leftarrow$) It is clear.

"A module $M$ is strongly Rickart iff ker$f_1 = r_M(f_1)$ is a fully invariant direct summand in $M$ for all $f \in \text{End}_R(M)$" [1]. "If $M$ is strongly Rickart module then for all $f \in \text{End}_R(M), r_M(f) = r_M(f^2)$ (That is ker$f = \ker(f^2)$" [1, corollary(1.9)].

**Proposition(2.3):** If $M$ is strongly Rickart module. $M$ is weakly coretractable iff $M$ is coretractable.
Proof: ($\Rightarrow$) Let $S$ be a submodule of $M$. If $S$ is weakly cocontractable, then $g \in \text{End}_R(M)$, $g \neq 0$ and $S \subseteq \text{ker} g^2$. Thus $M$ is weakly cocontractable. 

($\Leftarrow$) It is clear.

As an application of this corollary, the following examples are introduced.

**Examples (2.4):**

1. $Q$ and $Z$ as $Z$-module are strongly Rickart modules. But each of $Z$ and $Q$ is not coretractable module, so they are not weakly cocontractable modules.

2. $Q \oplus Z_2$ is strongly Rickart, but it’s not cocontractable and hence it’s not weakly cocontractable.

3. $M = Z \oplus Z$ is weakly cocontractable $Z$-module but not cocontractable (see Ex. and Rem. 2.2(1)).

   However, $M$ is not strongly Rickart since there exists $f \in \text{End}_R(M)$ defined by $f(a, b) = (a, 0)$, $\text{ker} f = 0 \oplus Z$, but $\text{ker} f$ is not fully invariant submodule since there exists $g \in \text{End}_Z(M)$, $g(a, b) = (b, 0)$, so $g(\text{ker} f) = g(0 \oplus Z) = Z \oplus 0 \subseteq \text{ker} f$.

   "A submodule $N$ of $M$ is stable if each $f \in \text{Hom}(N, M)$, $f(N) \subseteq N$ and $M$ is fully stable if every submodule of $M$ is stable. Every stable submodule is fully invariant but not conversely." 

   **Proposition (2.5):** Let $M$ be an $R$-module. If $M$ is a fully stable weakly cocontractable $R$-module, then every submodule $N$ of $M$ is weakly cocontractable.

   Proof: Let $W < N$. As $M$ is weakly cocontractable, then $g \in \text{End}_R(M)$, $g \neq 0$ and $f(N) = 0$. Let $g = f \in N: N \rightarrow M$. As $M$ is fully stable, $g(N) \subseteq N$, thus $g \in \text{End}(N)$, and $g^2(W) = g(g(W))$. But $g^2(W) \subseteq g^2(N)$ since $W \subseteq N$. But $g^2(N) = f^2(N) = 0$. Thus $g^2(W) = 0$ and hence $N$ is weakly cocontractable.

   **Proposition (2.6):** Let $M = M_1 \oplus M_2$ where $M_1$ and $M_2$ are weakly cocontractable modules. If $M$ is a duo module (or $\text{ann} M_1 \cap \text{ann} M_2 = R$ or distributive module), then $M$ is a weakly cocontractable module.

   **Proof:** Let $N$ be a proper submodule of $M$. Since $M$ is duo module (or $\text{ann} M_1 \cap \text{ann} M_2 = R$ or distributive module), then $N = (N \cap M_1) \oplus (N \cap M_2) = N_1 \oplus N_2$, where $N_1 = N \cap M_1$ and $N_2 = N \cap M_2$.

   **Case (1):** $N_1 = M_1$ and $N_2 = M_2$.

   There exists $g \in \text{End}_R(M_2)$, $g \neq 0$, $g^2(N_2) = 0$. Define $h \in \text{End}_R(M)$ by $h(a, b) = (0, g(b)) \neq 0 \forall (a, b) \in M$. But $h(a, b) = (0, g(b)) \neq 0 \forall (a, b) \in M$, and so $h^2(N_2) = 0$.

   **Case (2):** $N_1 = M_1$ and $N_2 = M_2$.

   The proof is similar to case (1).

   **Case (3):** $N_1 = M_1$ and $N_2 < M_2$.

   Since $M_1$ and $M_2$ are weakly cocontractable modules, there exist $f \in \text{End}_R(M_1)$, $f \neq 0$ and $f_2^2(N_1) = 0$ and $g \in \text{End}_R(M_2)$, $g \neq 0$ and $g^2(N_2) = 0$. Define $h \in \text{End}_R(M)$ by $h(x, y) = (f(x), g(y))$, $h \neq 0$ and $h^2(N) = 0$. Therefore $M$ is weakly cocontractable.

   **Proposition (2.7):** Let $M = M_1 \oplus M_2$, where $M_1$ and $M_2$ are fully invariant in $M$ and $M$ is weakly cocontractable, then either $M_1$ or $M_2$ is weakly cocontractable.

   **Proof:** Let $N_1$ and $N_2$ be proper submodules of $M_1$ and $M_2$ respectively. Then $N = N_1 \oplus N_2$ is a proper submodule of $M$. As $M_1$ and $M_2$ are weakly cocontractable, $f \in \text{End}_R(M)$, $f \neq 0$, $N \subseteq \text{ker} f^2$.

   As $\text{End}_R(M) \cong \left( \begin{array}{c} \text{End}(M_1) \\ \text{Hom}(M_1, M_2) \\ \text{Hom}(M_2, M_1) \\ \text{End}(M_2) \end{array} \right)$. But $\text{Hom}(M_2, M_1) = 0$, $\text{Hom}(M_1, M_2) = 0$, (since $M_1$ and $M_2$ are fully invariant in $M$). Thus $\text{End}_R(M) \cong \left( \begin{array}{c} \text{End}(M_1) \\ 0 \\ 0 \\ \text{End}(M_2) \end{array} \right)$ and so $f = \left( \begin{array}{c} f_1 \\ 0 \\ 0 \\ f_2 \end{array} \right)$ for some $f_1 \in \text{End}(M_1), f_2 \in \text{End}(M_2)$. As $f \neq 0$, either $f_1 \neq 0$ or $f_2 \neq 0$. On other hand $0 = f_1^2(N_1) = f_2^2(N_2)$ or $f_1^2(N_2) = 0$, that is $f_1^2(N_1) = 0$ and $f_2^2(N_2) = 0$. Thus either $M_1$ or $M_2$ is weakly cocontractable.

   "A submodule $N$ of $M$ is coquasi-invertible if $\text{Hom}_R(M, N) = 0$ [12, P. 8] and a nonzero $R$-module $M$ is coquasi-Dedekind if every proper submodule of $M$ is coquasi-invertible" [12, P. 32].

   Also, $M$ is coquasi-Dedekind if for each $f \in \text{End}_R(M)$, $f \neq 0$, $f$ is an epimorphism [12, Theorem (2.1.4)].

   **Proposition (2.8):** Let $M$ be a weakly cocontractable and coquasi-Dedekind module. If $N$ is a proper fully invariant submodule of $M$, then $M/N$ is weakly cocontractable.
Proof: Let $U/N < M/N$. Then $U < M$, but $M$ is weakly coretractable, so $\exists f \in \text{End}R(M)$, $f \neq 0$ and $f(2U)=0$. Define $g:M/N\rightarrow M/N$ by $g(m+N)=f(m)+N \text{ for all } m \in M$. Then $g$ is well-defined since $N$ is fully invariant, also $g$ is an $R$-homomorphism. Beside this $g \neq 0$ because if $g=0$, $f(M) \subseteq N$ and so $f \in \text{Hom}(M,\ N)$ which is contradiction since $M$ is coquasi-Dedekind. Now $g^2(U/N)=g(g(U/N)= g(f(U)+N)= f(U)+N= N=0\text{MN}$. Thus $M/N$ is a weakly coretractable module.

The following gave a characterization for weakly coretractable ring $R$, when $R$ is a commutative semiprime.

**Proposition (2.9):** If $R$ is a commutative semiprime. $R$ is weakly coretractable iff for each proper ideal $I$ of $R$, there exists $f:R/I^2 \rightarrow R$, $f \neq 0$.

**Proof:** (⇒) Let $I < R$. Hence $I^2 < R$ and so $\exists f \in \text{End}(R)$, $f \neq 0$ such that $f^2(I)=0$, $f \neq 0$. As $f \in \text{End}(R)$ there exists $r \neq 0$, $r \in R$, $f(x)=rx$ for all $x \in R$, hence $f^2(x)=r^2x$. Define $g:R/I^2 \rightarrow R$, by $g(x+I^2)=f^2(x)$ for each $x \in R$. It easy to check that $g$ is well-defined. As $g(1+I^2)=f^2(1)=r^2 \neq 0$, then $g \neq 0$.

(⇐) Let $I < R$. Then $I^2 < R$, and hence $\exists f:R/I^2 \rightarrow R$, $f \neq 0$. Let $g=f^2$, so that $g \in \text{End}(R)$ and $g \neq 0$, $g(f^2)=0$. Let $g(I)=r$. $g^2(I)=g(g(I))=g(rI)=r^2I$. But $g(I^2)=I^2 \neq 0$, then $I^2=r^2=1=0=(1 \ r)^2$, and so $I=0$ since $R$ is semiprime. Thus $g(I)=0$. Therefore $R$ is weakly coretractable.

In the class of finitely generated faithful multiplication $R$-modules, weakly coretractability transferred from a module $M$ to a ring $R$ and conversely.

**Theorem (2.10):** If $M$ is finitely generated faithful multiplication module. $M$ is weakly coretractable iff $R$ is weakly coretractable, where $R$ is commutative ring.

**Proof:** (⇒) Let $I < R$. Then $N = M/I < M$, hence $\exists f \in \text{End}_a(M)$, $f \neq 0$ and $f^2(N)=0$. But $f^2(N)=f^2(M)=I=0$. As $M$ is f. g. multiplication, $M$ is scalar module, so $\exists r \in R$, $r \neq 0$ and $f(m)=mr \forall m \in M$, hence $f(M)=Mr$. Thus $f^2(M)=f(M)r=I$. Now, define $g:R \rightarrow R$ by $g(a)=ra$, hence $g \neq 0$ and $g(I)=rI$. It follows that $g^2(I)=g(g(I))=g(rI)=r^2I$. But $Mr^2=0$ implies $r^2 \subseteq \text{ann}M=0$; that is $r^2=0$. Therefore $g^2(I)=0$ and $R$ is weakly coretractable.

(⇐) Let $N=M$ be a proper submodule of $M$. Then $I$ is a proper ideal in $R$, so there exists $f \in \text{End}_a(R)$ such that $f^2(I)=0$ and $f \neq 0$. Assume $f(1)=r$, then $f(I)=rI$. Define $h:M \rightarrow M$ by $h(x)=rx$ for each $x \in M$, $h$ is well-defined and $h^2(m)=h(h^2(\sum_{i=1}^n x_i a_i))$, $x_i \in M$ and $a_i \in I$, $i=1, 2, 3, ...$, $n$ and $m \in N$. Hence $h^2(m)=\sum_{i=1}^n x_i r^2 a_i = 0$ for each $m \in N$ and so $h^2(N)=0$. $M$ is weakly coretractable.

**Proposition (2.11):** Let $M$ be an $R$-module, then the following statements are equivalent:

1. $M$ is simple;
2. $M$ is a module and $\text{End}_a(M)$ is a division ring;
3. $M$ is weakly coretractable, and $\text{End}_a(M)$ is a division ring.

**Proof:** (1)⇒(2) It follows by Proposition (1.12).

(2)⇒(3) It is clear.

(3)⇒(1) Suppose $M$ is not simple; $N < M$ and $N \neq 0$. As $M$ is weakly coretractable, so $\exists g \in \text{End}_a(M)$, $g \neq 0$ and $g^2(N)=0$. Thus $g^2$ is not monomorphism. Hence $g^2 \neq 0$ (since $\text{End}_a(M)$ is a division ring).

But $g \neq 0$, so $g^{-1}$ exists and hence $g^{-1} g^2=g=0$ which is a contradiction. Thus $M$ is simple.

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