THE WEAK LIMITING BEHAVIOR OF THE DE HAAN RESNICK ESTIMATOR OF THE EXPONENT OF A STABLE DISTRIBUTION

GANE SAMB LO

Abstract. The problem of estimating the exponent of a stable law received a considerable attention in the recent literature. Here, we deal with an estimate of such an exponent introduced by De Haan and Resnick when the corresponding distribution function belongs to the Gumbel’s domain of attraction. This study permits to construct new statistical tests. Examples and simulations are given. The limiting law are shown to be the Gumbel’s law and particular cases are given with norming constants expressed with iterated logarithms and exponentials.

Nota Bena. This paper is an unpublished of the author, part of his PhD thesis, UPMC, Paris VI, 1986.

1. INTRODUCTION AND RESULTS

Many biological phenomena seem to fit the Zipf’s form:

\[(1.1) \quad 1 - G(x) = C \cdot x^{-1/c}, \quad c > 0 \text{ and } C > 0.\]

For instance, we can cite the plot against r of the population of the r-th largest city (see e.g. Hill [9]. This motivated considerable works on the problem of estimating c. More generally, if \(X_1, X_2, \ldots, X_n\) are independent and identical copies of a random variable (rv) X such that \(F(x) = \mathbb{P}(X \leq x)\) satisfies

\[(1.2) \quad \forall t > 0, \quad \lim_{x \uparrow +\infty} \frac{1 - F(\log(tx))}{1 - F(\log(x))} = t^{-1/c}.\]

Several estimates (Hill [9], S. Csorgo-Deheuvels-Mason [2] have been proposed. De Haan and Resnick [6] introduced

\[K\text{e}\text{y words and phrases.} \quad \text{Regular and slowly varying functions, Domain of attraction, norming constants, Order Statistics, Limiting laws.}\]
\[ T_n = (X_{n,n} - X_{n-k,n}) / \log k, \]

where \( X_{1,n}, X_{2,n}, \ldots, X_{n,n} \) are the order statistics of \( X_1, X_2, \ldots, X_n \) and \( k \) is a sequence of integers satisfying

\[(K) \quad 0 < k < n, \quad k = k(n) \to +\infty, \quad k/n \to 0 \text{ as } n \to +\infty\]

De Haan and Resnick [6] have proved that (1.2) implies under some conditions that:

(1.3) \[ T_n \to^P c, \text{ in probability } (\to^P) \]

and

(1.4) \[ \frac{\log k}{c} (T_n - c) \to \Lambda, \text{ in distribution } (\to^d) \]

where \[ \Lambda(x) = \exp(e^{-x}) \]

is the Gumbel law.

In order to contribute to a complete asymptotic theory for the inference about the upper tail of a distribution (as specified in LO [10], Section 3), we deal with the asymptotic behavior of \( T_n \), here, in the case where (1.2) fails. Notice that (1.2) mean that \( F(\log(\cdot)) \) belongs to the Frechet’s domain of attraction. Here, we restrict ourselves to the case where \( F(\log(\cdot)) \) belongs to the Domain of attraction of the Gumbel law, \( D(\Lambda) \). These results are stated in this section [1] proved in section [2] and illustrated in Section [3].

Before the statements of the results, we need some further notation. Define

\[ A = \inf \{ x, \ F(x) = 1 \}, \quad B = \sup \{ x, \ F(x) = 0 \}, \]

\[ R(t) = (1 - F(t))^{-1} \int_t^A (1 - F(v)) \ dv, \quad B \leq t < A \]

and

\[ Q(u) = F^{-1}(u) = \inf \{ x, \ F(x) \geq u \} \]
is the quantile function of $X$. We shall assume, when appropriate, that

$$(H1) \quad F(\log(x)) \in D(\Lambda).$$

$$(H2) \quad F(x) \text{ is ultimately strictly increasing and continuous.}$$

We will prove that if $(H1)$ and $(H2)$ are satisfied, then $F \in D(\Lambda)$ (see Lemma 2). So, we can use the De Haan (see [5]) representation for the quantile function associated with a distribution function $F$ such that $F \in D(\Lambda)$ and $F(x)$ is ultimately strictly increasing as $x \uparrow A$ (see De Haan [5], Theorems 1.4.1 and 2.4.2):

$$Q(1-u) = c_0 + r(u) + \int_u^1 \frac{r(t)}{t} \, dt, \quad \text{as } u \downarrow 0.$$  

where $c_0$ is some constant anf $r(u)$ is a positive function slowly varying at zero (S.V.Z).

Finally, we define this assumption of $k$. We say that $k$ satisfies $(Kr(\lambda))$ and its satisfies $(K)$ and this extra-condition:

$$r(1/n) / r(k/n) \to \lambda, \ 0 \leq \lambda < +\infty$$

Our main results are

**Theorem 1.** Let $(H1)$ and $(H2)$ be satisfied, then

(i) for any sequence $k$ satisfying $(K)$, we have

$$(X_{n,n} - X_{n-k,n}) / \log k \to^p 0$$

(ii) for any sequence $k$ satisfying $(Kr(\lambda))$, we have

$$(1.7) \quad \frac{\log k}{r(k/n)} \left\{ \frac{X_{n,n} - X_{n-k,n}}{\log k} \right\} - \frac{Q(1 - 1/n) - Q(1 - k/n)}{r(k/n)} \to^d \lambda \times \Lambda$$

with $a_N = r(k/n)$ and $b_n = \{Q(1 - 1/n) - Q(1 - k/n)\} / r(k/n)$.

**Corollary 1.** Let $A > 0$. Then, if $(H1)$ and $(H2)$ hold, $(1.6)$ and $(1.7)$ remain true if we replace

$X_{n,n}, \ X_{n-k,n}$ by $\log X_{n,n}, \ \log X_{n-k,n}$

$Q(1 - s)$ by $\log Q(1 - as), \ a = P(X > 0)$
and if \( k \) satisfies \( Ks (\lambda) \) at the place of \( Kr (\lambda) \). Moreover, if \( \log A > 0 \), we may repeat the operation. Conversely

**Corollary 2.** Let \( r (u) \to 0 \) as \( u \downarrow 0 \). Then, if \((H1)\) and \((H2)\) are satisfied, \((1.6)\) and \((1.7)\) hold if we replace

\[
X_{n,n}, \ X_{n-k,n} \ \text{by} \ \exp (X_{n,n}), \ \exp (X_{n-k,n})
\]

\[
Q (1 - u) \ \text{by} \ \exp (Q (1 - u))
\]

\[
r (u) \ \text{by} \ t (u) = \exp (Q (1 - u)) \ R (Q (1 - u))
\]

and if \( k \) satisfies \( Kt (\lambda) \) at the place of \( Kr (\lambda) \).

**Corollary 3.** (Particular cases). Here, we restrict ourselves to the case where \( k = \lfloor (\log n) \ell \rfloor \), where \( \lfloor \cdot \rfloor \) denotes the integer part, and \( \ell \) is any positive number.

1) **Normal case**: \( X \sim \mathcal{N} (0,1) \)

\[(i)\]

\[
(\ell (2 \log n)^{\frac{1}{2}} (\log \log n)) \cdot T_n \cdot (1 + o (1)) - \ell (\log \log n) (1 + o (1)) \to^d \Lambda
\]

\[(ii)\]

\[
(2 \log n)^{\frac{1}{2}} T_n \to^P 1
\]

2) **Exponential case**: \( \exp (X) \sim \mathcal{E} (1) \). (or general gamma case, see the proof).

\[(i)\]

\[
(1 + o (1)) \cdot \ell (\log n) (\log \log n) \cdot T_n - \ell (\log \log n) (1 + o (1)) \to^d \Lambda
\]

\[(ii)\]

\[
(\log n) \cdot T_n \to^P 1
\]

3) Suppose that \( X = \log_p \sup (e_{p-1} (1), Z) \), a.s., where \( Z \sim \mathcal{N} (0,1) \), \( p \geq 1 \), \( \log_p \) (resp. \( e_p \)) denotes the \( p \)-th logarithm (resp. exponential), with by convention \( \log_0 x = 1 \) for all \( x > 0 \). Let
Then we have:

(i) First,
\[ \ell (\log \log n) C_n T_n - \ell (\log \log n) (1 + o(1)) \to^d \Lambda; \]

(ii) Next,
\[ C_n T_n \to^P 1. \]

Remark 1. Mason [12] has proved that the Hill (9) estimate

\[ H_n = k^{-1} \sum_{i=1}^{i=k} (X_{n-i+1,n} - X_{n-k,n}) \]

is characteristic of a distribution satisfying (1.2) in the following sense: suppose that \( A = +\infty \), then for any real number \( c, 0 < c < +\infty \), one has

\[ T_n \to^P c, \]

for any sequence \( k \) satisfying (K) if and only if \( F \) satisfies (1.2). In order to compare \( H_n \) and \( T_n \), we remark that this property is not obtained for \( T_n \). Indeed, the Mason distribution defined by

\[ F^{-1} (1 - 2^{-m}) = m, m = 0, 1, 2, \ldots \]

\[ = m + (2^{-m} - u) 2^{m+1}, \text{ if } 2^{-m-1} < u < 2^{-m} \]

satisfies the following property

\[ \frac{F^{-1} (1 - s) - F^{-1} (1 - bs)}{\log b} + (\log 2)^{-1} \text{ as } s \to 0, b \to +\infty \text{ and } bs \to 0. \]

Thus, by letting \( b = U_{k,n} U_{1,n}^{-1}, s = U_{1,n} \), we get \( T_n \to^P (\log 2)^{-1} \), where we have used the well known representation \( X_{n-i+1,n} =^d F^{-1} (U_{n-i+1,n}) \)

\[ d = F^{-1} (1 - U_{i,n}) \text{ and } U_{1,n} \leq U_{2,n} \leq \ldots \leq U_{n,n} \]

are the order statistics of a sequence of independent rv’s uniformly distributed on (0, 1). However, \( 1 - F (\log (x)) \) does not vary regularly at infinity, in other words, does not satisfy (1.2), \( c = (\log 2)^{-1} \). (see e.g. Mason (1982), Appendix).
2. PROOFS OF THE RESULTS

First, we show how to derive corollaries 1 and 2 from the Theorems. To begin with, we need four lemmas and we define $L$ as the set of distribution functions $F$ satisfying $(H1)$ and $(H2)$

**Lemma 1.** If $F \in L$, then (1.5) holds and $\lim_{u \downarrow 0} \frac{r(u)}{R(Q(1-u))} = 1$. It follows that $R(Q(1-u))$ is S.V.Z.

**Lemma 2.** Let $A > 0$. Then, if $X$ has a distribution function $F \in L$, $\log \sup (0, X)$ is defined a.s., and has a distribution function $G \in L$ and

$$S(G^{-1}(1-u)) \sim \frac{R(Q(1-u))}{Q(1-u)}$$

as $u \downarrow 0$, where $S(t) = \int_{e^{-A}}^{e^{B}} \log \frac{1}{1-G(v)} dv$, $-\infty < t < \log A$.

**Lemma 3.** Let $R(t) \to 0$ as $t \uparrow 0$. Then, if $X$ has a distribution function $F \in L$, $\exp (X)$ has a distribution function $H$ such that $H \in L$ and $R(Q(1-u)) \sim \frac{T(H^{-1}(1-u))}{H^{-1}(1-u)}$ as $u \downarrow 0$, where $T(t) = (1 - H(t))^{-1} \int_{e^{B}}^{e^{A}} (1 - H(v)) dv$, $e^{B} < t < e^{A}$

**Lemma 4.** If $E \in L$, then $Q(1-u)$ is S.V.Z.

**Proof.** Proofs of Lemmas 1, 2, 3 and 4.

Lemmas 1, 2 and 3 are proved in Lo [11] via lemma 3.2, 3.3 and 3.4. Lemma 4 is proved in Lo [10] via its lemma 2.

**Proof of corollaries 1 and 2**

Let $G$ be the distribution function of $\log \sup (0, X)$. Lemma 2 says that $(H1)$ and $(H2)$ imply that $F \in D(\Lambda)$. But $g(\log x) = F_{1}(x)$, where $F_{1}$ is the distribution function if $\sup(0, X)$. And it is obvious that $F_{1} \in D(\Lambda)$ if $A > 0$. If follows that $(H1)$ and $(H2)$ are true for $G$. So, we may write (1.6) and (1.7) for $G$. Furthermore, we have

$$G^{-1}(1-u) = \log Q(1-\alpha s), \quad a = P(X > 0)$$

and lemmas 1.2 say that we may replace $r(u) \sim R(Q(1-u))$ by $S(G^{-1}(1-u)) \sim \frac{R(Q(1-u))}{Q(1-u)}$.

Finally, remark that if $A > 0$, $k$ satisfies $(K)$, we have for large values of $n$, $\sup(0, X_{n-k,n}) = X_{n-k,n}$ and $\sup(0, X_{n,n}) = X_{n,n}$, a.s.. With the above remarks, we can see that Corollary 1 is proved.

**Corollary 2** is proved a similar way with Lemma 3.
Proof of the part (i) of the theorem. Let \( G(x) = F(\log x) \). Since \( G \in L \), lemma 1 implies that \( G^{-1}(1-u) \) is S.V.Z. At this step, we need the Karamata’s representation for functions S.V.Z.

\[
(2.1) \quad G^{-1}(1-u) = c(u) \exp \left( \int_u^1 (\varepsilon(s)/s) \, ds \right),
\]

where \( c(s) \to c \), \( 0 < c < +\infty \), \( \varepsilon(s) \to 0 \) as \( s \downarrow 0 \). So,

\[
(2.2) \quad Q(1-u) = \log G^{-1}(1-u) = \log c(u) + \int_u^1 (\varepsilon(s)/s) \, ds.
\]

We recall that

\[
(2.3) \quad \{X_{i,n}, \ 1 \leq i \leq n\} \overset{d}{=} \{Q(U_{i,n}), \ 1 \leq i \leq 1\},\ U_{i,n} \overset{d}{=} 1 - U_{n-i+1,n},
\]

where \( 0 = U_{0,n} \leq U_{1,n} \leq U_{2,n} \leq \ldots \leq U_{n,n} \leq U_{n+1,n} = 1 \) are the order statistics of a sequence of independent rv’s uniformly distributed on \((0,1)\). Therefore, (2.3) implies

\[
(2.4) \quad (\log k) T_n = X_{n,n} - X_{n-k,n} \overset{d}{=} Q(1-U_{1,n}) - Q(1-U_{k-1,n}) = (\log k) T_n^*.
\]

Let us apply (2.3). We obtain that

\[
0 \leq T_n^* \leq \frac{|\log c(U_{1,n}) / c(U_{k-1,n})|}{\log k} + \frac{|\log n(U_{k+1,n})|}{\log k} \sup_{0 \leq s \leq U_{k+1,n}} |\varepsilon(s)|
\]

\[
(2.5) \quad \leq: A_{n1} + A_{n2}.
\]

Obviously, we have

\[
(2.6) \quad A_{n1} \to^P, \text{ since } U_{1,n} \to^P 0 \text{ and } U_{k+1,n} \to^P 0 \text{ if } k \text{ satisfies (K)}.
\]

By (2.1), we also have that

\[
(2.7) \quad \sup_{0 \leq s \leq U_{k+1,n}} |\varepsilon(s)| = o_p(1)
\]

Therefore, we can see that (2.5), (2.6) and (2.7) will imply the part (i) of theorem 2 if we prove that \( (\log nU_{k+1,n}) / \log k = o_p(1) \). But (see e.g. De Haan and Balkema [1]),
\[ n^{-\frac{1}{2}} \left( U_{k,n} - \frac{k}{n} \right) \overset{d}{\to} N(0,1), \text{ which implies} \]

\[ n (k + 1)^{-1} U_{k+1,n} \to P 1. \]

We deduce from (2.9) that

\[ (nk^{-1} U_{k+1,n}) \to P 1. \]

So,

\[ (\log n U_{k+1,n}) / \log k = 0_p (1), \]

which completes the proof of the part (i) of the theorem.

**Proof of the part (ii) of the theorem.** Let suppose that \((H1)\) and \((H2)\) hold. Then, \((1.5)\) holds.

\[ Q (1 - u) = c_o + r (u) + \int_u^1 (r (s) / s) \, ds, \text{ as } u \downarrow 0. \]

We have

\[ T_n^* = \left\{ Q(1 - U_{1,n}) - Q(1 - 1/n) \right\} / \log k + \left\{ Q(1 - k/n) - Q(1 - U_{k+1,n}) \right\} / \log k + a_n b_n / \log k \]

\[ =: A_3 / \log k + A_{n4} / \log k + a_n b_n / \log k, \]

where \(a_n\) and \(b_n\) are defined in the statement of the theorem. First, we prove that

\[ A_{n4} / a_n \to P 0. \]

By \((1.5)\), we have \(A_{n4} = r(k/n) - r(U_{k+1,n}) + \int_{U_{k+1,n}}^n (r(s) / s) \, ds. \)

Remark that since \(r(u)\) is slowly varying at 0, we have on account of \((2.10)\) that

\[ \frac{r(k/n)}{r(U_{k+1,n})} \to 1, \text{ in probability}. \]
Furthermore,
\[(2.15) \quad a_n^{-1} \left| \int_{k/n}^{U_{k+1,n}} \left( r(s)/s \right) \, ds \right| \leq \left| \log \left( nk^{-1}U_{k+1,n} \right) \sup_{s \in I_n} \frac{r(s)}{r(k/n)} \right|,
\]
where
\[I_n = \left\{ \min \left( \frac{k}{n}, U_{k+1,n} \right), \max \left( \frac{k}{n}, U_{k+1,n} \right) \right\}
\]
is a random interval. At this step, we need this

**Lemma 5.** Let \( r(u) \) be a positive function S.V.Z. Let \((u_n)\) be the sequence of rv’s and \((d_n)\) be a sequences of real numbers such that
\[d_n \to +\infty, u_n \to P 0, (d_n, u_n) = 0_p (1), \text{ and } (d_n, u_n)^{-1} = 0_p (1) \text{ as } n \to +\infty
\]
then
\[1 + o_p (1) = \inf_{s \in J_n} \frac{r(s)}{r(1/d_n)} \leq \sup_{s \in J_n} \frac{r(s)}{r(1/d_n)} = 1 + o_p (1), \text{ as } n \to +\infty
\]
where
\[J_n = \left( \min \left( \frac{1}{d_n}, u_n \right), \max \left( \frac{1}{d_n}, u_n \right) \right).
\]

**Proof of Lemma 5** The proof is the same as that of Lemma 3.5 in Lo (1985b).

**Proof of the part (ii) of the theorem (continued).**

By \[(2.11), \quad \frac{k}{n}U_{k+1,n} \to P 1.\] So, we may apply lemma 5 to (2.15) and get
\[(2.16) \quad a_n^{-1} \left| \int_{k/n}^{U_{k+1,n}} \left( r(s)/s \right) \, ds \right| \leq o_p (1). (1 + o_p (1))
\]
Combining (2.14) and (2.16), we get (2.14). We now concentrate on \(A_{n3}\) and show that
\[(2.17) \quad A_{n3}/a_n \to^d \lambda \times \Lambda,
\]
whenever if \(k\) satisfies (\(Kr(\lambda)\)). We have
\[A_{n3} = Q \left( 1 - U_{1,n} \right) - Q \left( 1 - 1/n \right)
\]
\begin{align*}
(2.18) 
&= r(U_{1,n}) - r(1/n) - \int_{U_{1,n}}^{1/n} (r(s)/s) \, ds.
\end{align*}

Recall that

\begin{align*}
(2.19) 
P(n U_{1,n} \geq x) &\to e^{-x}, \quad \text{as } n \to +\infty.
\end{align*}

This means that \( nU_{1,n} = 0_p(1) \) and \( (nU_{1,n})^{-1} = 0_p(1) \). Thus, we may apply Lemma 5 and get

\begin{align*}
(2.20) 
r(1/n)/r(U_{1,n}) &\to^P 1 \text{ as } n \to +\infty.
\end{align*}

Then, if \( k \) satisfies \((Kr(\lambda))\), we have

\begin{align*}
(2.21) 
(r(1/n) - r(U_{1,n})) / a_n &\to^P 0, \quad \text{as } n \to +\infty.
\end{align*}

Now, let

\[ B_n = \int_{U_{1,n}}^{1/n} (r(s)/s) \, ds. \]

We have

\[ B_n / a_n = \{r(1/n)/r(k/n)\} \cdot \int_{U_{1,n}}^{1/n} \left\{ \frac{r(s)}{r(1/n)} \right\} \frac{ds}{s}. \]

Then, it follows, from Lemma 5 and the fact that \( r(u) \) is positive, that if \( k \) satisfies \((Kr(\lambda))\), we have

\begin{align*}
(2.22) 
\{B_n/a_n\} + \lambda \log nU_{1,n} &= \log nU_{1,n} \cdot o_p(1).
\end{align*}

By \((2.19)\), we see that

\[ \lim_{n \to +\infty} P(- \log nU_{1,n} \leq x) = e^{-e^{-x}} = \Lambda(x). \]

We get finally that if \( k \) satisfies \((Kr(\lambda))\), one has

\begin{align*}
(2.23) 
B_n / a_n &\to^d \lambda \Lambda.
\end{align*}

\((2.17)\) and \((2.14)\) together imply the theorem.

**Proof of the corollary**

\[ \]
Previously in the occasion of our study of the same particular cases for the Hill’s estimator (see Lo [10]), Lemma 5 and Corollary 5) we have proved that (2.24)

\[
\lim_{u \downarrow 0} R \left( Q \left( 1 - u \right) \right) / \rho \left( u \right) \to 1,
\]

where \( \rho \left( u \right) = u, Q' \left( 1 - u \right), Q' \left( u \right) = \frac{dQ(u)}{du} \) for values of \( u \) near 1.

With (2.24), we may handle the different points of Corollary 3. Here, we concentrate on the case where \( k = \left( (\log n)^{\ell} \right), \ell > 0. \)

1) **Normal case** : \( F \left( x \right) = \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} e^{-t^2/2} dt. \)

Remark that \( F \left( \log x \right) \) is the distribution function of the log-normal law. It follows that \( F \left( \log \left( \cdot \right) \right) \in D(\Lambda). \) \( \left( H2 \right) \) is obviously true. On the other hand, it is well known that

\[
Q \left( 1 - s \right) = \left( 2 \log (1/s) \right)^{\frac{1}{2}} + \frac{\log \log (1/s) + 4\pi + o \left( 1 \right)}{2 \left( 2 \log (1/s) \right)}, \text{ as } s \downarrow 0
\]

(2.25)

\[
\rho \left( s \right) = s Q' \left( 1 - s \right) = \left( 2 \log (1/s) \right)^{-\frac{1}{2}} \left( 1 + o \left( 1 \right) \right), \text{ as } s \downarrow 0.
\]

Notice that we might have used (see Galambos [7], p. 66)

\[
R \left( t \right) = t^{-1} \left( 1 + o \left( t^{-3} \right) \right) \text{ and } \rho \left( s \right) \sim R \left( Q \left( 1 - s \right) \right).
\]

Then

\[
a_n^{-1} = \rho^{-1} \left( k/n \right) \left( 1 + o \left( 1 \right) \right) = \left( 2 \log n \right)^{\frac{1}{2}} \left( 1 + o \left( 1 \right) \right)
\]

and

\[
b_n = Q \left( 1 - 1/n \right) - Q \left( 1 - k/n \right) = \frac{\ell \log \log n}{\left( 2 \log n \right)^{\frac{1}{2}}} \left( 1 + o \left( \frac{\log n}{\log \log n} \right) \right)
\]

Therefore,

\[
b_n = \left\{ \ell \log \log n \right\} \left( 1 + o \left( 1 \right) \right)
\]

2) **Exponential case** : \( F \left( \log x \right) = 1 - e^{-ax}, \alpha > 0. \)

More generally, since the tail of the quantile function associated with a general gamma law \( \gamma \left( r, \alpha \right), r > 0, \alpha > 0, \) admits the expansion

\[
H^{-1} \left( 1 - u \right) = \left( \log \frac{1}{u} \right) \left( 1 + o \left( 1 \right) \right)
\]

(2.27)
the behavior of $T_n$ is same for all Gamma laws because (2.27) doesn’t depend neither on $r$, nor on $\alpha$. That is why we only consider

$$F (\log x) = 1 - e^{-x}.$$  

Therefore

$$Q (1 - u) = \log \log (1/s), \quad \rho (u) = (\log (1/s))^{-1}.$$  

Then

$$a_n = (\log n) (1 + o(1)), \quad b_n = (\ell \log \log n) (1 + o(1))$$  

At this step, we apply the Theorem to conclude.

3) In this case, we have

$$T_n = \frac{\{\log_p Z_{n,n} - \log_p Z_{n-k,n}\}}{\log k},$$  

for large values of $n$, where $Z_{1,n}, Z_{2,n}, ..., Z_{n,n}$ are the order statistics of a sequence of independent and standard Gaussian rv’s.

We also have

$$(2.28) \quad 1 - G (x) = m^{-1} \left( \int_{x}^{+\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{t^2}{2}} dt \right),$$  

where $G$ is the distribution function associated to sup $(e_{p-1}^1, Z)$, and $m = p (Z > e_{p-1}^1)$.

Since $X = \log \sup (e_{p-1}^1, Z)$, one has

$$Q (1 - u) = \log_p G^{-1} (1 - \mu u) = \log_p \left\{ \frac{(2 \log (1/ms))^\frac{1}{2} + \frac{\log \log (1/ms) + o (1)}{2 (2 \log (1/ms))^{\frac{1}{2}}} \right\}. $$  

It follows from (2.26) and (2.28) that

$$\rho^{-1} (s) = (2 \log (1/s))^{\frac{1}{2}} \prod_{j=1}^{j=p-1} \log_j (2 \log (1/s))^{\frac{1}{2}} (1 + o (1))$$  

Then

$$a_n \sim 2 \log n \prod_{j=p}^{j=p-1} \log_j n$$
and from (2.28), we deduce after some calculations that

\[ b_n = \ell \log \log n (1 + o(1)). \]

Remark that the part 3 of the corollary might have been derived from the part 1 of the same corollary after p applications of Corollary 11. We have given the normal case as example but such an operation is possible whenever \( Z_i \) has a distribution function \( F \) such that \( F(\log(.)) \in D(\Lambda) \) and \( \log p^{-1} A > 0 \). Even when \( F(\log(.)) \in D(\Psi) \), where \( \Psi(x) = e^{-1/x} \) is the Frechet law, we can have the part 3 since \( F(\log(.)) \in D(\Psi) \) implies that \( F(.) \in D(\Lambda) \). □
3. Simulations

Here, we will illustrate the behavior of $T_n$ in the three cases.

(i) $\exp(X) \sim E(1)$, (ii) $\exp(X) = \sup(0, Z), Z \sim N(0, 1)$, (iii) $F(\log x) = 1 - 1/x$.

For making our simulations, we have generated an ordered sample $u_i$, $1 \leq i \leq 4000$ from a uniform rv. Therefore, we have constructed:

(i) an order sample of the standard exponential law

$$y_i = -\log(1-u_i)$$

and defined $T_{n1} = (\log y_n - \log y_{n-k}) / \log k$, $k = [n^{1/4}]$, $T_{n1} = C_n(y_i)$;

(ii) an ordered sample of the standard Normal law for $u_i \uparrow 1$

$$x_i = (−2 \log (1 - u_i))^{1/2}$$ (see e.g. 3.2)

and defined

$$T_{n2} = C_n(x_i);$$

(iii) an ordered sample of the Pareto law

$$z_i = (1-u_i)^{-1}$$ and defined $T_{n3} = C_n(z_i)$

Before we proceed any further, we remark that with $x_i = (−2 \log (1 - u_i))$, we get:

$T_{n2} = \frac{1}{2} T_{n1}$. In fact, we have $T_{n2}^* = \frac{1}{2} T_{n1}$

$$+ 0 \left( \frac{\log \log (1/u_{n-k,n})}{4 \log k \log (1/1 - u_{n-k,n})} \right)$$

where $T_{n2}^*$ is the exact value of $T_{n2}$ if we use the true quantile function. With our data, we get $T_{n2}^* = \frac{1}{2} T_{n1}^{-1} \pm 0.0179$ and $(\log n) T_{n2}^* = (\frac{1}{2} \log n) T_{n1} \pm 0.14$.

The simulations are given as follows:

3.1. The data. Here are the simulation outcomes.
BEHAVIOR OF DE HAAN-RENICK ESTIMATOR

| \(n\) | \(\frac{1}{2} \log n T_{n1}\) | \((\log n) T_{n2}\) | \(T_{n3}\) | \(u_{4000-i+1}\) |
|-------|----------------|----------------|-------|----------------|
| 3991  | 0.3294         | 0.3294         | 0.3302| 0.002435       |
| 3992  | 0.3391         | 0.3391         | 0.4042| 0.001631       |
| 3993  | 0.3625         | 0.3625         | 0.4334| 0.002620       |
| 3994  | 0.3550         | 0.3550         | 0.4324| 0.001337       |
| 3995  | 0.4598         | 0.4598         | 0.5954| 0.000988       |
| 3996  | 0.4693         | 0.4593         | 0.6124| 0.000437       |
| 3997  | 0.4689         | 0.4689         | 0.6130| 0.000418       |
| 3998  | 0.4977         | 0.4977         | 0.6625| 0.000308       |
| 3999  | 0.5116         | 0.5116         | 0.6963| 0.000297       |
| 4000  | 0.6942         | 0.9942         | 1.0033| 0.000095       |

1) The right column gives values of \((u_i)\), first. With the symmetry of the uniform maw, we have

\[\{1 - u_i, 1 \leq i \leq 4000\} = \{u_{4000-i+1}, 1 \leq i \leq 4000\}\]

2) If \(k = \lfloor n^{\frac{3}{5}} \rfloor\), similar calculations as in the proof of the part 2 of corollary 3 show that \(a_n \sim \log n\) and \(a_nb_n \sim \log 2\). Therefore, if \(T_{n1}\) denotes the De Haan/Resnick estimate for \(\exp(X) \sim E(1)\), we get

\[(3.1) \quad \left(\frac{1}{2} \log n\right) T_{n1} \rightarrow^P \log 2.\]

The same considerations from part 3 of corollary 3 \((p = 1)\) yield

\[(3.2) \quad (\log n) T_{n2} \rightarrow^P \log 2,\]

where \(\bar{T}_{n2}\) denotes the De Haan/Resnick estimate for \(X = \log \sup(0, Z)\), \(Z \sim (0,1)\). Notice that (3.1) and (3.2) are well illustrated by our simulations since \(\log 2 \sim 0.69314\).

3) The column 4 illustrates the result of De Haan/Resnick (1980)

\[T_{n3} \rightarrow^P 1.\]

4. Conclusions

We have proved that a suitable choice of \(k\) (for instance \(k \sim \{\log n\}^\ell\)), we can find the norming constants \(d_n\) such that
\[ d_n T_n \to^P 1 \]

In addition, we have given the limit law as the Gumbel distribution. The same work has been already done by De Haan and Resnick (1980) under the hypothesis (1.2). So, for a wide range of distributions belonging in \( D (\Lambda) \), we can provide statistical tests. For instance, we may obtain tests for a Normal model against an Exponential one based on (3.1) and (3.2). Similar tests are specified in LO (1985a)
REFERENCES

[1] Balkema., A.A and De Haan, L. (1974) : Limit for order statistics. In : Colloquia. Math. Soc. Boylai. Limit theorems of Probability, Keszthely, 17-20.

[2] Csorgo, S., Deheuvels, P and Mason, D.M. (1983) : Kernel estimates for the tail index of a distribution. Rapport Technique, Université Paris 6. To appear in Annals of Statistics.

[3] Csorgo, S. and Mason, D.M. (1984) : Central limit theorems for sums of extreme values. Proc. Cambr. Phil. Math. Soc.

[4] Davis, R. and Resnick, S.I. (1984) : Tail estimates motivated by extreme value theory. Ann. Statist., Vol 12, (4), 1467-1487

[5] De Haan, L. (1970) : On regular variation and its application to the weak convergence of sample extreme. Mathematical Centre, Amsterdam.

[6] De Haan, L. and Resnick, S.I. (1980) : A simple asymptotic estimate for the tail index of a stable distribution. J. Roy. Statist. Soc. B 42, 83-87

[7] Galambos, J. (1978) : The asymptotic theory of extreme order Statistics. Wiley, New-York.

[8] Hill, B.M. (1974) : The rank-frequency from of Zipf’s law. Journ. Amer. Assoc. Dec., Vol 69, n° 348. Theory and Methods Section.

[9] Hill, B.M. (1975) : A simple general approach to the inference about the tail index of a distribution. Ann. Statist. 3. 1163-1174

[10] Lo, G.S. (1985a). Asymptotic behaviour of the Hill estimator and application,(1986), Journal of Applied Probability, 922-936, (23).

[11] - Lo, G.S. (1985b). On the asymptotic normality of extreme values, (1989), Journal of Statistical Planning and Inference, 89-94, (22).

[12] Mason, D.M. (1982) Laws of large numbers for sums of extreme values. Ann. Probab. Vol 10, (3), pp. 754-764.

[13] Teugels, J.L. (1981). Limit theorems on order Statistics. Ann. Probab. 9, 868-880.

Lo Gane Samb, Université Paris 6. L.S.T.A. Tour 45-46, E.3., 4, Place Jussieu. 75230, Paris Cedex 05.