Abstract

The dispersionful analogue, by means of Lax formalism, of the zero-genus universal Whitham hierarchy together with its algebraic orbit finite-field reductions is considered. The theory is illustrated by several significant examples.

1 Introduction

In [1] Krichever introduced the so-called universal Whitham hierarchies by means of moduli spaces of Riemann surfaces of all genera. We are interested in the zero-genus case, which we will henceforth refer to as the Whitham hierarchy. This hierarchy contains such particular nonlinear integrable systems like dispersionless limits of KdV, Toda, AKNS (or equivalently nonlinear Schrödinger) soliton equations. However, the class of integrable systems is much wider. There are very important reductions to the Whitham hierarchy, related to the so-called algebraic orbits [2, 3], leading to the construction of (1+1)-dimensional integrable dispersionless systems that are generated by rational Lax functions. Thus, there arises a question of the construction of a theory analogous to the Whitham hierarchy leading to the soliton systems, i.e. systems with dispersion. The main reason for the lack of such a complete theory are problems with the notion of higher order "finite" poles in the case of algebra of pseudo-differential operators.

The part of the Whitham hierarchy related to the puncture at infinity is nothing but dispersionless limit of the well-known infinite-field KP hierarchy. In [4] Krichever considered reductions of KP hierarchy generated by rational (pseudo-differential) Lax operators being quotients of appropriate purely-differential operators. Besides, he has shown that there exist additional symmetries that are not themselves reductions of KP hierarchy. From the more general point of view 'quantization' of dispersionless systems from the Whitham hierarchy generated by rational Lax functions with "finite" poles of first order is solved in [5]. This solution is given by means of Lax operators of KdV type with the additional so-called source terms [6]. This class of integrable soliton systems contains, among others, full AKNS hierarchy. In the series of articles [7]-[9] Bonora and Xiong investigated integrable differential-difference hierarchies of the Toda type arising in the context of the multi-matrix models. They showed how this hierarchies, using one flow, can be turned into completely equivalent purely differential hierarchies that can be recognized as special rational reductions of the KP hierarchy. They also considered dispersionless limit of the above integrable systems.
Let us illustrate such a procedure on the simplest example. The Toda hierarchy, being a hierarchy of mutually commuting lattice soliton equations, has a Lax formulation expressed by shift operators. The pair of linear Lax equations have the form

\[ \Psi_{n+1} + u_n \Psi_n + v_n \Psi_{n-1} = \lambda \Psi \]  
(1.1)
\[ \partial_x \Psi_n = \Psi_{n+1} + u_n \Psi_n, \]  
(1.2)

where the second equation defines the evolution of the eigenfunction \( \Psi \). The compatibility between equations (1.1) and (1.2) leads to the Toda equation. It is known that the above auxiliary linear problem can be rewritten with the use of pseudo-differential operators as it follows from (1.2) that \( \Psi_{n+1} = (\partial_x + u_n) \Psi_n \) and \( \Psi_{n-1} = (\partial_x + u_{n-1})^{-1} \Psi_n \) [7], [10]-[12]. Then, the linear equation (1.1) takes the form

\[ (\partial_x + v_n(\partial_x - u_{n-1})^{-1}) \Psi = \lambda \Psi \]

of the eigenvalue problem related to the AKNS system. Consequently, the full Toda hierarchy in the representation of pseudo-differential operators takes the form of the well known AKNS hierarchy. Following the above idea in the opposite direction one is able to construct, with the aid of shift operators, a dispersionful analogue of the Whitham hierarchy.

Very recently Takasaki considered dispersionless limit of multi-component KP hierarchy with charges (lattice variables), that can be treated as a generalization of the Toda hierarchy [13]. In this sense the original Toda hierarchy is equivalent to the two-component KP hierarchy. Rewriting the multi-component KP hierarchy in the scalar Lax formalism Takasaki showed that the above dispersionless limit is the Whitham hierarchy [13]. Further in [14] Takasaki and Takebe analysed the above issue from the point of view of Hirota equations and Fay identities.

The theory of Whitham hierarchy and its dispersionful analogue is inseparably connected with problems of systematic constructions of integrable nonlinear dynamical systems. It is well known that a very powerful tool, called the classical \( R \)-matrix formalism, can be used for systematic construction of (1+1)-dimensional field and lattice integrable dispersive systems (soliton systems) [15]-[18] as well as dispersionless integrable field systems [19]-[22]. Moreover, the \( R \)-matrix approach allows a construction of Hamiltonian structures and conserved quantities.

The aim of this article is to consider the dispersionful analogue of the Whitham hierarchy and its algebraic orbit reductions on the level of the explicit Lax representations. The article is divided onto two parallel parts concerning dispersionless and dispersive cases, respectively. First the basic facts about the Whitham hierarchy together with its algebraic orbits, given by some meromorphic functions, are presented. Next an alternative formulation of the original Whitham hierarchy is constructed. This formulation is the one that in the formal quantization procedure yields the dispersionful Whitham hierarchy. Initially the dispersionful Whitham hierarchy is presented by means of pseudo-differential operators, but the one-to-one analogue of the original Whitham hierarchy is given with the simultaneous use of the shift operators following reference [13]. This is a multi-component Toda-like formulation of the dispersionful Whitham hierarchy. The finite-field reductions being counterparts of algebraic orbits are also considered. The whole theory is illustrated by the rich set of significant examples. Finally some conclusions are given.
2 Whitham hierarchy

2.1 Basic definitions and Hamilton-Jacobi problems

Consider a set of Lax functions \( z_\alpha(p,t) \) \((\alpha \in \{\infty, 1, ..., N\})\) in a complex variable \( p \) being formal Laurent series at \( \infty \) and 'finite' punctures \( q_i \) on the Riemann sphere

\[
\begin{align*}
z_\infty &= p + \sum_{l=1}^{\infty} u_{\infty,l}(t)p^{-l} \\
z_i &= u_{i,-1}(t)(p-q_i(t))^{-1} + \sum_{l=0}^{\infty} u_{i,l}(t)(p-q_i(t))^l & i \in 1, ..., N,
\end{align*}
\]  

respectively. In (2.1-2.2) the coefficients \( u_{\alpha,l} \) and the poles \( q_i \) are smooth dynamical fields depending on the infinite set of evolution parameters ('times') \( t = \{t_\alpha\} \). One of these times is distinguished as a spatial variable: \( t_\infty_1 =: x \). Times \( t_\alpha \) are coupled with the generating functions \( \Omega_\alpha \) defined in the following way

\[
\Omega_\alpha(p,t) := \begin{cases} 
  (z^n_\infty)_{(\alpha,+)} & \text{for } \alpha = \infty, \ n = 1, 2, 3, ... , \\
  -(z^n_\alpha)_{(\alpha,-)} & \text{for } \alpha = i, \ n = 1, 2, 3, ... , \\
  \ln(p-q_i) & \text{for } \alpha = i, \ n = 0,
\end{cases}
\]

where \((\cdot)_{(\alpha,+)}\) are the projections onto the principal parts of Laurent expansions at poles \( \infty \) and \( q_i \) such that \((\sum_k a_k p^k)_{(\infty,+)} := \sum_{k \geq 0} a_k p^k \) and \((\sum_k a_k (p-q_i)^k)_{(i,-)} := \sum_{k < 0} a_k (p-q_i)^k \), respectively. Then, the Whitham hierarchy is the set of zero-curvature equations

\[
\frac{\partial \Omega_\alpha}{\partial t_\beta} - \frac{\partial \Omega_\beta}{\partial t_\alpha} + \{\Omega_\alpha, \Omega_\beta\} = 0 \quad (2.4)
\]

for the following hierarchies of mutually commuting evolution systems given in the Lax form

\[
\frac{\partial z_\alpha}{\partial t_\beta_m} = \{\Omega_\beta_m, z_\alpha\} \quad (2.5)
\]

where the Poisson bracket is the canonical one

\[
\{f,g\} := \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} \quad (2.6)
\]

Let us use the short notation \( t_\alpha \leftrightarrow t_\beta \) for the equation, related to fixed times \( t_\alpha \) and \( t_\beta \), from the Whitham hierarchy (2.4). Notice, that \( \Omega_\infty_1 = p \) thus \( t_\infty_1 \leftrightarrow t_1 \) is trivially satisfied and the equations from Lax hierarchies (2.5) for \( \beta = \infty, m = 1 \) are the translation symmetries \( (z_\alpha)_{t_\infty_1} = (z_\alpha)_x \) in the variable \( x \) chosen as a spatial one.

The above theory of Whitham hierarchy is well defined due to the standard approach following from the classical \( R \)-matrix formalism \[2, 15, 22\]. The Lax hierarchy (2.5) for \( \alpha = \beta = \infty \) is the well known infinite-field Benney chain, containing many well known finite-field reductions being dispersionless limits (see for example [21]) of soliton systems [16] from the KP hierarchy.

Examining the bi-Hamiltonian structure of equations (2.5) in the spirit of the classical \( R \)-matrix theory [22], one observes that hierarchies (2.5) for different fixed \( \alpha \) and \( n \geq 1 \) form
different bi-Hamiltonian hierarchies, mutually commuting, with respect to the same Poisson tensors. The symmetries with \( n = 1 \) are the starting symmetries for the recurrence procedure. In the same way equations for \( \alpha = i \) and \( n = 0 \), constructed for \( \Omega = \ln(p - q_i) \), generate other bi-Hamiltonian hierarchies, which except the first symmetries are not constructed within the above scheme of the Whitham hierarchy with generating functions restricted to (2.3). In this sense the above construction is still incomplete. The appearance of the logarithmic function in (2.3) could be explained in several ways. For example in [23] all terms (2.3), including the logarithmic ones, are constructed by the Lie algebraic splitting of Hamiltonian vector fields. How to generalize the Whitham hierarchy so that it contains the whole bi-Hamiltonian hierarchies generated by functions containing logarithmic terms is not clear yet. Restricting to the particular reductions of the Whitham hierarchy given by meromorphic Lax functions with only two poles, first \( \infty \) and second 'finite' one, one can manage with this problem proceeding like in [24], where logarithmic terms were represented by a sum of two infinite series convergent at these poles. For the same class of Lax functions the more correct approach is given in [25], where the logarithmic terms are constructed using contour integrals. However, the generalization of such approaches to the whole Whitham hierarchy makes problems.

**Example 2.1** Consider the part of the Whitham hierarchy involving times: \( t_{\infty 1} =: x \), \( t_{\infty 2} =: t \), \( t_{i1} =: y_i \) and \( t_{i0} =: s_i \). The above notation will be used in the whole article. The functions (2.3) related to these times have the form

\[
\Omega_{\infty 1} = p \quad \Omega_{\infty 2} = p^2 + 2w \quad \Omega_{i1} = -a_i(p - q_i)^{-1} \quad \Omega_{i0} = \ln(p - q_i),
\]

where \( w := u_{\infty,1}, a_i := u_{i,-1} \). Then, for \( t \leftrightarrow y_i \) one finds

\[
(q_i)_t = (2w + q_i^2)_x \quad (a_i)_t = (2a_i q_i)_x \quad w_{y_i} = (a_i)_x \tag{2.7}
\]

the generalized Benney gas system. For \( y_i \leftrightarrow y_j \) and \( s_i \leftrightarrow y_j \) we have

\[
(q_j)_{y_i} = \left( \frac{a_i}{q_i - q_j} \right)_x \quad (a_j)_{y_i} = (a_i)_{y_j} = \left( \frac{a_i a_j}{(q_i - q_j)^2} \right)_x \quad i \neq j \tag{2.8}
\]

and

\[
(q_i)_{y_i} = (a_i)_{s_i} \quad (q_j)_{s_i} = \left( \frac{(q_i - q_j)}{a_i} \right)_x \tag{2.9}
\]

\[
(q_j)_{s_i} = (q_i)_{s_j} = \left( \frac{(q_i - q_j)}{q_i - q_j} \right)_x \quad (a_i)_{s_j} = (q_j)_{y_i} \quad i \neq j, \tag{2.10}
\]

respectively. Eliminating \( q_i \) from (2.9) one obtains the dispersionless multi-dimensional Toda equation or equivalently the Boyer-Finley equation, which appears in general relativity theory. For \( s_i \leftrightarrow t \) one finds

\[
w_{s_i} = (q_i)_x
\]

and the first equation from (2.7). For \( s_i \leftrightarrow s_j \) we have the first equation from (2.10). The set of equations from this example has been considered in [23].
Furthermore, there exists a set of pseudo-potential functions $S_\alpha(z_\alpha, t)$ satisfying a set of the Hamilton-Jacobi problems

$$z_\alpha = z_\alpha \left( \partial_x S_\alpha, t \right)$$  \hspace{1cm} (2.11)

$$\frac{\partial S_\alpha}{\partial t_{\beta n}} = \Omega_{\beta n} \left( \partial_x S_\alpha, t \right)$$  \hspace{1cm} (2.12)

and compatibility conditions

$$\frac{\partial^2 S_\gamma}{\partial t_{\alpha m} \partial t_{\beta n}} = \frac{\partial^2 S_\gamma}{\partial t_{\beta n} \partial t_{\alpha m}},$$  \hspace{1cm} (2.13)

since $p = \partial_x S_\alpha$. The above equations must be correctly understood, i.e. (2.11) are Hamilton-Jacobi equations with fixed 'energy' levels and therefore $S_\alpha$ depend on $z_\alpha$ treated as parameters. Eliminating $\partial_x S_\alpha$ from (2.11-2.12) the compatibility conditions (2.13) lead to the Whitham hierarchy (2.4) and simultaneously to Lax hierarchies (2.5).

### 2.2 Algebraic orbit reductions

There is important class of finite-field reductions of the Whitham hierarchy, being the so-called algebraic orbits given by meromorphic functions $E$ satisfying

$$E = z_\alpha^{n_\alpha} = z_i^{n_i} \text{ for } n_\alpha \in \mathbb{Z}_.$$

Then, the most general form of $E$ is given by

$$E = p^{n_\infty} + \sum_{l=0}^{n_\infty-2} a_{\infty,l} p^l + \sum_{i=1}^N \sum_{l_i=1}^{n_i} a_{i,l_i} (p - q_i)^{-l_i}$$  \hspace{1cm} (2.14)

and then the dynamical fields from functions $z_\alpha$ are given by polynomials of fields from (2.14), i.e. expanding $E$ at $\infty$ $E = z_\alpha^{n_\infty}$ and expanding $E$ at $q_i$ $E = z_i^{n_i}$. The goal of such reductions of the Whitham hierarchy is the construction of (1+1)-dimensional integrable finite-field dispersionless systems. By integrable systems we understand those which have infinite hierarchy of commuting symmetries. In this case pseudo-potential functions reduce to one $S_\alpha(z_\alpha, t) := S(E, t)$ and the set of Lax hierarchies (2.5) reduces also to one Lax hierarchy in the form

$$\frac{\partial E}{\partial t_{\beta m}} = \{ \Omega_{\beta m}, E \},$$

where the functions $\Omega_{\beta m}$ are generated around poles of $E$.

A more general theory of meromorphic Lax representations, not only for the canonical Poisson bracket (2.6), including the case (2.14) allowing a construction of integrable dispersionless systems together with multi-Hamiltonian structures is presented in [22]. In the same paper, it is also shown that one can construct Lax hierarchies not only at poles of meromorphic Lax function $E$ but also at its zeros. Another class of reductions of the Whitham hierarchy related to algebraic orbits is presented in [3].

In all examples in this article we will in general present only the first nontrivial evolution equations from related hierarchies.
Example 2.2 The dispersionless AKNS hierarchy.

We will consider the meromorphic Lax function in the form

\[ E = p + \sum_i a_i (p - q_i)^{-1} = z_\infty = z_1, \]

(2.15)
i.e. the case of (2.14) for \( n_\infty = n_i = 1 \). Then the system generated by \( \Omega_{\infty 2} = p^2 + 2 \sum_i a_i \) is

\[
\begin{align*}
(a_i)_t &= 2 (a_i q_i)_x, \\
(q_i)_t &= 2 \left( \sum_j a_j + q_i^2 \right)_x. 
\end{align*}
\]

(2.16)

This is the dispersionless limit of multi-component AKNS system. For \( \Omega_{k0} = \ln(p - q_k) \) we have the following equations

\[
\begin{align*}
(a_i)_{s_k} &= \left( \frac{a_i}{q_i - q_k} \right)_x, \\
(q_i)_{s_k} &= (\ln |q_i - q_k|)_x, \\
(a_k)_{s_k} &= (q_k)_x - \sum_{k \neq k} \left( \frac{a_k}{q_k - q_k} \right)_x, \\
(q_k)_{s_k} &= (\ln a_k)_x, 
\end{align*}
\]

where \( i \neq k \). For \( \Omega_{k1} = -a_k(p - q_k)^{-1} \) one finds

\[
\begin{align*}
(a_i)_{y_k} &= \left( \frac{a_i a_k}{(q_i - q_k)^2} \right)_x, \\
(q_i)_{y_k} &= - \left( \frac{a_k}{q_i - q_k} \right)_x, \\
(a_k)_{y_k} &= (a_k)_x - \sum_{i \neq k} \left( \frac{a_i a_k}{q_i - q_k} \right)_x, \\
(q_k)_{y_k} &= (q_k)_x - \sum_{i \neq k} \left( \frac{a_i}{q_i - q_k} \right)_x, 
\end{align*}
\]

(2.17)

where \( i \neq k \). The equations from this example are of course compatible with the equation from Example 2.1 as in this case \( w = \sum_i a_i \).

Example 2.3 Consider Lax function (2.14) for \( N = 1 \) with the pole of first order at \( \infty \) and second order pole at \( q := q_1 \):

\[ E = p + u(p - q)^{-1} + v(p - q)^{-2} = z_\infty = z_1^2. \]

(2.18)

Let \( y := y_1, s := s_1 \). Then, one finds the following equations

\[
\begin{align*}
\Omega_{\infty 2} = p^2 + 2 u & \implies u_t = 2 (ug)_x + 2v_x, \\
v_t &= 2v_x q + 4vq_x, \\
q_t &= 2u_x + 2q q_x, 
\end{align*}
\]

(2.19)
\[ \Omega_{10} = \ln(p - q) \implies u_s = q_x \]
\[ v_s = u_x - \frac{uv_x}{2v} \]
\[ q_s = \frac{v_x}{2v}, \quad (2.20) \]

\[ \Omega_{11} = -\sqrt{v(p - q)^{-1}} \implies u_y = (\sqrt{v})_x \]
\[ v_y = \sqrt{v} \left( q - \frac{1}{4} u^2 v \right)_x \]
\[ q_y = \left( \frac{u}{2\sqrt{v}} \right)_x. \quad (2.21) \]

Now \( w = u \) from Example 2.1.

**Example 2.4** Let us take now (2.14) for \( N = 1 \) with \( n_\infty = 2 \) and \( n_1 = 1 \), i.e.
\[ E = p^2 + u + \frac{a}{p - q} = z_\infty^2 = z_1. \]

In this case \( E = \Omega_{\infty2} - \Omega_{11} \), thus we have that \( E_t = \{ \Omega_{\infty2}, E \} = \{ \Omega_{11}, E \} = E_y \) (\( y := y_1 \)) and consequently \( y \) and \( t \) can be identified, i.e. \( t \equiv y \). The system for these times is
\[ \Omega_{\infty2} = p^2 + u \implies u_t = 2a_x \]
\[ a_t = (2aq)_x \]
\[ q_t = (q^2 + u)_x. \quad (2.22) \]

For the times \( \tau := t_\infty3 \) and \( s := s_1 \) one finds the following systems
\[ \Omega_{\infty3} = p^3 + \frac{3}{2} up + \frac{3}{2} a \implies u_\tau = \left( \frac{3}{4} u^2 + 3aq \right)_x \]
\[ a_\tau = \left( 3aq^2 + \frac{3}{2} ua \right)_x \]
\[ q_\tau = \left( q^3 + \frac{3}{2} uq + \frac{3}{2} a \right)_x \quad (2.23) \]

and
\[ \Omega_{10} = \ln(p - q) \implies u_s = 2q_x \]
\[ a_s = (q^2 + u)_x \]
\[ q_s = (\ln a)_x. \]

### 2.3 Alternative formulation

The choice of the evolution parameter \( t_{\infty1} \) to be considered as a spatial parameter \( x \) is ambiguous. Since \( \Omega_{k0} = \ln(p - q_k) \) by (2.12) the following relation is valid
\[ \partial_x S_\alpha = \exp(\partial_{s_k} S_\alpha) + q_k, \quad (2.24) \]
where $s_k := t_{k0}$. Now fixing $k$, we can eliminate $\partial_{s_k} S_\alpha$ from \eqref{2.12} instead of $\partial_x S_\alpha$ and consider $s_k$ as a spatial variable. Then, for $\lambda_k = \exp(\partial_{s_k} S_\alpha)$ the Whitham hierarchy \eqref{2.4} and related Lax hierarchies \eqref{2.5} have the same form, but the Poisson bracket is given by

$$\{f, g\} = \lambda_k \left( \frac{\partial f}{\partial \lambda_k} \frac{\partial g}{\partial s_k} - \frac{\partial f}{\partial s_k} \frac{\partial g}{\partial \lambda_k} \right).$$ \hfill (2.25)

In such a situation the variables $p$ and $\lambda_k$ should be considered rather only as auxiliary parameters. On the level of zero-curvature equations \eqref{2.4} and Lax hierarchies \eqref{2.5} the above procedure is little bit more complicated. From \eqref{2.24} it follows that at first, one has to perform the transformation $\lambda_k = p - q_k$. Hence, $\partial_p \leftrightarrow \partial_{\lambda_k}$ and $\partial_{t_{an}} \leftrightarrow \partial_{t_{an}} - (q_k)_{kan} \partial_{\lambda_k}$. Next, all terms in \eqref{2.4} and \eqref{2.5} containing derivatives with respect to $x$ have to be replaced using equations $t_{an} \leftrightarrow s_k$ for $\Omega_{k0} = \ln \lambda_k$ and \eqref{2.5} for $\beta = k, m = 0$. As a result, one obtains, in a preserved form, the Whitham hierarchy \eqref{2.4} and related Lax hierarchies \eqref{2.5} for the Poisson bracket given by \eqref{2.25}. On the level of explicit equations both formulations give a compatible set of equations. In the case of explicit equations from Lax hierarchies \eqref{2.5} the above passage between both formulations relies on the change between spatial variable $x$ and $s_k$ through use of explicit equations \eqref{2.5} for $\beta = k, m = 0$ or $\beta = \infty, m = 1$, respectively.

Of course in the new auxiliary variable $\lambda_k$ the Lax functions \eqref{2.1,2.2} take appropriate form of Laurent series. The "finite" poles $q_i$ are then moved to $q_i - q_k$. Particularly $z_i$ takes the form of a Laurent series at 0

$$z_i = a_{i,-1} \lambda_i^{-1} + \sum_{l=0}^{\infty} a_{i,l} \lambda_i^l,$$

so the dynamical field $q_i$ in $z_i$ is missed. Then, the evolution of $q_i$ can be calculated from the representation of $z_\infty$ as

$$z_\infty = \lambda_k + q_k + \sum_{l=1}^{\infty} a_{\infty,l} \lambda_k^{-l}.$$

In fact, we can use all auxiliary variables $p$ and $\lambda_i$ (for all $i$) simultaneously. This formulation of the Whitham hierarchy is correct if all equations (relations) are understood on the level of the Hamiltonian-Jacobi problems, i.e. $p \equiv \partial_x S_\alpha, \lambda_i \equiv \exp(\partial_{s_i} S_\alpha)$ are not considered as independent variables but as shortened notation. Then, the Poisson bracket is given in the following general form

$$\{f, g\} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} + \sum_i \lambda_i \left( \frac{\partial f}{\partial \lambda_i} \frac{\partial g}{\partial s_i} - \frac{\partial f}{\partial s_i} \frac{\partial g}{\partial \lambda_i} \right).$$

In this case one has to first evaluate the Poisson bracket and then use the relation $p \equiv \lambda_i + q_i \equiv \lambda_j + q_j$ to draw consistent equations from \eqref{2.4} or \eqref{2.5}. However, one has to proceed carefully in order to not lose the evolution of the fields $q_i$.

**Example 2.5** With the use of $\lambda_i$, the functions \eqref{2.3} related to the set of evolution parameters $x, t, s_i$ and $y_i$ are given in the form

$$\Omega_{\infty 1} = \lambda_k + q_k \quad \Omega_{\infty 2} = \lambda_k^2 + 2q_k \lambda_k + q_k^2 + 2w \quad \Omega_{i1} = -a_i^{-1} \quad \Omega_{i0} = \ln \lambda_i,$$

where $k$ is arbitrary from allowed values. Then, one obtains a set of equations being consistent with the set of equations from Example 2.1.
2.4 Finite-field reductions

The meromorphic reductions (2.14) can be represented using variables \( \lambda_i = p - q_i \) simultaneously in the form

\[
E = \lambda_k^n + u \lambda_k^{n-1} + \sum_{l=0}^{n-2} u_{\infty,l} \lambda_k^l + \sum_{i=1}^{N} \sum_{i_l=1}^{n_i} a_{i,l} \lambda_i^{-l_i}, \tag{2.26}
\]

where \( k \) is arbitrary and \( u = n_{\infty} q_k \). However, as we are interested in the construction of \((1+1)\)-dimensional equations we should use the form of \( E \) only with one auxiliary parameter \( \lambda_k \) for a fixed \( k \). In this case, \( \Omega_k = \ln \lambda_k \) leads to the trivially satisfied equations related to the translational symmetry, this time, in the variable \( s_k \). Therefore, the equation from the hierarchy (2.5) for \( \alpha = k \) which was earlier calculated for \( \Omega_k = \ln(p - q_k) \) can now be calculated for \( \Omega_{\infty,1} = \lambda_k + q_k \) but in the reversed form, i.e. with \( x \) as an evolution parameter.

**Example 2.6** Multi-component dispersionless Toda hierarchy.

In the auxiliary parameter \( \lambda_k = p - q_k \) the Lax function (2.15) takes the form

\[
E = \lambda_k + q_k + a_k \lambda_k + \sum_{i \neq k} a_i (\lambda_k + q_k - q_i)^{-1}.
\]

The first nontrivial system is a counterpart of translational symmetry in \( x \) for (2.15)

\[
\Omega_{\infty,1} = \lambda_k + q_k \quad \implies \quad (a_i)_x = (q_i - q_k)(a_i)_{s_k} + a_i(q_k)_{s_k}
\]

This is the multi-component dispersionless Toda equation. The counterpart of dispersionless AKNS system (2.16) is only the next system from the hierarchy calculated at \( \infty \). For \( \Omega_{k1} = -a_k \lambda_k^{-1} \) one obtains a counterpart of the system (2.17), which can be also calculated directly from (2.17) with the use of (2.27),

\[
(a_i)_y = \frac{(a_i)_{s_k} a_k - a_i(a_k)_{s_k}}{q_i - q_k} \quad i \neq k \\
(a_k)_y = a_k(q_k)_{s_k} - \sum_{j \neq k} (a_j)_y \\
(q_i)_y = (a_k)_{s_k}.
\]

As a result, the dispersionless AKNS hierarchy from Example 2.2, with \( x \) as a spatial variable, becomes a dispersionless multi-component Toda hierarchy with the spatial variable \( s_k \).

**Example 2.7** In \( \lambda := \lambda_1 = p - q \) the Lax operator (2.18) has the form

\[
E = \lambda + q + u \lambda^{-1} + v \lambda^{-2} \tag{2.28}
\]

and the first nontrivial equation becomes

\[
\Omega_{\infty,1} = \lambda + q \quad \implies \quad u_x = u q_s + v_s \\
v_x = 2 v q_s \\
q_x = u_s. \tag{2.29}
\]
The counterpart of (2.20) is only the next system from the hierarchy. Then,

\[ \Omega_{11} = -\sqrt{v} \lambda^{-1} \quad \Rightarrow \quad u_y = \sqrt{v} q_s, \]
\[ v_y = v \left( \frac{u}{\sqrt{v}} \right)_s, \]
\[ q_y = (\sqrt{v})_s \]

is the counterpart of (2.21).

**Example 2.8** In terms of \( \lambda := p - q \) we can also rewrite the algebraic orbit of Example 2.4 as:

\[ E = \lambda^2 + 2q\lambda + q^2 + u + a\lambda^{-1}. \]

So, we have

\[ \Omega_{\infty 1} = \lambda + q \quad \Rightarrow \quad u_x = a_s - qu_s, \]
\[ a_x = a q_s, \quad q_x = \frac{u_s}{2}. \] (2.30)

As before \( E = \Omega_{\infty 2} - \Omega_{11} \). The counterpart of (2.22) is

\[ \Omega_{\infty 2} = \lambda^2 + 2q\lambda + q^2 + u \quad \Rightarrow \quad u_t = 2aq_s, \]
\[ a_t = a(q^2 + u)_s, \quad q_t = a_s. \] (2.31)

### 3 Dispersionful Whitham hierarchy

#### 3.1 Pseudo-differential operators and auxiliary linear problems

In this section we will use the same set of evolution parameters \( t_\alpha \) and the same notation \( x := t_\infty, s_i := t_0 \) and \( y_i := t_1 \) as in the previous section.

It is well known that the Benney momentum chain can be obtained from the infinite-field KP hierarchy in the quasi-classical limit, which suggests that the dispersive counterpart of the Whitham hierarchy can be constructed by means of pseudo-differentials operators \( \partial := \partial_x \) satisfying the generalized Leibniz rule

\[ \partial^m u(x) = \sum_{n \geq 0} \binom{m}{n} u(x, nx) \partial^{m-n}, \] (3.1)

where \( \binom{m}{n} = (-1)^n \binom{-m+n-1}{n} \) for \( m < 0 \). From (3.1) it follows that

\[ (\partial - \nu) u = u(\partial - \nu) + u_x \]
\[ (\partial - \nu)^{-1} u = u(\partial - \nu)^{-1} - (\partial - \nu)^{-1} u_x (\partial - \nu)^{-1} \]
\[ = u(\partial - \nu)^{-1} - u_x (\partial - \nu)^{-2} + u_{2x} (\partial - \nu)^{-3} - \ldots, \]
where \( v \) can be equal zero. Some other useful formulae, needed in further calculations, are
\[
\begin{align*}
\partial (\partial - v)^{-1} &= 1 + v (\partial - v)^{-1} \\
(\partial - v)^{-1} \partial &= 1 + (\partial - v)^{-1} v \\
(\partial - v)^{-1} u &= u (\partial - v + \frac{u}{u})^{-1} \\
[(\partial - v)^{-1}]_t &= (\partial - v)^{-1} v (\partial - v)^{-1}.
\end{align*}
\]

Let us consider the following algebras, generated respectively by elements \( \partial, \partial^{-1} \) and \( (\partial - q_i), (\partial - q_i)^{-1} \),
\[
\begin{align*}
g_{\infty} &= g_{(\infty,+)} \oplus g_{(\infty,-)} = \left\{ \sum_{l \geq 0} a_l \partial^l \right\} \oplus \left\{ \sum_{l < 0} a_l \partial^l \right\} \\
g_i &= g_{(i,+)} \oplus g_{(i,-)} = \left\{ \sum_{l \geq 0} a_l (\partial - q_i)^l \right\} \oplus \left\{ \sum_{l < 0} a_l (\partial - q_i)^l \right\},
\end{align*}
\]
in which the operation, with respect to the above rules, is associative. The Lie algebra structure is defined through the commutator \([A, B] = AB - BA\), where \( A, B \in g_\alpha \), and these algebras are decomposed into Lie subalgebras. We would like to have on the ‘quantum’ level of pseudo-differential operators a theory parallel to the complex calculus of meromorphic functions, but after quantization we do not have to our disposal such a crucial tool as expansion into Taylor series and the clear-cut notion of poles for pseudo-differential operators. However, it is possible to construct formal rules of expansions which are similar to the expansions for meromorphic functions, i.e. \( g_{\infty}, g_j \subset g_{(i,+)} \) for \( j \neq i \) and \( g_{(i,\pm)} \subset g_{(\infty,\pm)} \), where \( a \subset b \) means that all elements from \( a \) can be expanded as a series from \( b \). Hence, the reasoning is analogous to the one on the Whitham hierarchy level. This is a crucial fact for the consistency of the equations (3.7) and (3.8). The above scheme will be considered in detail in the forthcoming article [26].

We define the infinite-field Lax operators in the form
\[
\begin{align*}
L_{\infty} &= \partial + \sum_{l=1}^{\infty} u_{\infty,l}(t) \partial^{-l} \\
L_i &= u_{i,-1}(t)(\partial - q_i(t))^{-1} + \sum_{l=0}^{\infty} u_{i,l}(t)(\partial - q_i(t))^l \\
&\quad i \in 1, ..., N,
\end{align*}
\]
where all coefficients depend on the same set of evolution parameters as before. The first one \( L_{\infty} \) defines the well-known KP hierarchy. Then
\[
\Omega_{\alpha n}(t) := \begin{cases} 
(L_{\infty}^n)_{(\infty,+)} & \text{for } \alpha = \infty \\
(L_i^n)_{(i,-)} & \text{for } \alpha = i,
\end{cases}
\]
where \( n = 1, 2, 3, ... \). Due to the lack of a good counterpart of logarithmic function, formulated by means of pseudo-differential operators, we omitted the case related to \( \Omega_{i0} \) in (2.3) with the natural logarithm. It will be considered separately.

Now we can define a set of auxiliary linear equations
\[
\begin{align*}
L_\alpha \Psi_\alpha &= z_\alpha \Psi_\alpha \\
\frac{\partial \Psi_\alpha}{\partial t_{\beta n}} &= \Omega_{\beta n} \Psi_\alpha.
\end{align*}
\]
for some eigenfunctions $\Psi_\alpha(t)$ and eigenvalues $z_\alpha$. Then, the compatibility conditions between (3.5) and (3.6) lead to the Lax hierarchies

$$\left[ \partial_{t_\beta n} - \Omega_{\beta n}, L_\alpha \right] \Psi_\alpha = 0. \quad (3.7)$$

The compatibility conditions among equations (3.6) lead to the zero-curvature equations

$$\left[ \partial_{t_{\alpha m}} - \Omega_{\alpha m}, \partial_{t_{\beta n}} - \Omega_{\beta n} \right] \Psi_\gamma = 0 \quad (3.8)$$

of Lax hierarchies (3.7). They are written as operation on eigenfunctions as this form guarantees that they are invariant with respect to multi-component Toda-like formulation presented in Section 3.3. Still we have to argue that they are well defined. One can show that equations (3.7) are self-consistent, i.e. Lax operators (3.2-3.3) are well normalized and the right- and left-hand sides of Lax hierarchies can be written in the same form. The related $R$-matrices are defined through decomposition of Lie algebras $g_\alpha$ and importantly they are mutually consistent due to the above rules of expansion. The mutual commutativity between symmetries from fixed Lax hierarchy (3.7) follows now from the scheme of $R$-matrix theory [15, 18].

We will use the notation $t_{\alpha m} \leftrightarrow t_{\beta n}$ for the related equation from the zero-curvature hierarchy (3.8). Notice, that $\Omega_{\infty,1} = \partial$. Hence, $t_{\alpha n} \leftrightarrow t_{\infty 1}$ is trivially satisfied and in the representation of pseudo-differential operators $(L_\alpha)_{t_{\infty 1}} = (L_\alpha)_{x}$.

The so-called quasi-classical limit of linear equations (3.5-3.6) leads to the Hamiltonian-Jacobi equations (2.11-2.12) for $n \neq 0$. To see that, at first one has to perform the transformation

$$t_{\alpha n} \mapsto \frac{1}{\hbar} t_{\alpha n} \implies \partial_{t_{\alpha n}} \mapsto \hbar \partial_{t_{\alpha n}},$$

where $\hbar$ is treated as a deformation parameter. Next, assuming the WKB form of the eigenfunctions and that dynamical fields have quasi-classical counterparts, i.e.

$$\Psi_\alpha = \exp \left( \frac{1}{\hbar} S_\alpha + O(\hbar) \right)$$

$$u_{\alpha,l} \left( \frac{1}{\hbar} t \right) = \tilde{u}_{\alpha,l}(t) + O(\hbar),$$

one takes the limit $\hbar \to 0$. Then, the fields $\tilde{u}_{\alpha,l}(t)$ can be identified with those from Lax functions (2.1-2.2). Hence, the zero-curvature equations (3.8) can be considered as dispersive analogues of the Whitham equations (2.4) without the case related to $n = 0$. However, such an approach with the use of the formalism of pseudo-differential operators has fundamental disadvantages. The first one is that the coefficients of $\Omega_{i,n}$ for $n \geqslant 2$ do not have 'limited' or 'compact' form, which means that they are differential polynomials in all, infinitely many, coefficients from Lax operators $L_i$. It makes particular problems when one tries to define counterparts of algebraic orbits for the original Whitham hierarchy.

**Example 3.1** Consider equations from the dispersive Whitham hierarchy related to the times: $t_{\infty 1} =: x$, $t_{\infty 2} =: t$, $t_{i_1} =: y_i$ for which the related operators (3.4) have the form

$$\Omega_{\infty 1} \Psi_\alpha = \partial \Psi_\alpha, \quad \Omega_{\infty 2} \Psi_\alpha = \left( \partial^2 + 2w \right) \Psi_\alpha, \quad \Omega_{i_1} \Psi_\alpha = -a_i(\partial - q_i)^{-1} \Psi_\alpha,$$

where $w := u_{\infty,1}$, $a_i := u_{i,-1}$. Equations for $\Omega_{\infty 1}$ are satisfied trivially. Then, for $t \leftrightarrow y_i$ one finds

$$(a_i)_t = (a_i)_x + 2(q_i q_i)_x \quad (q_i)_t = -(q_i)_x + 2q_i(q_i)_x + 2w_x \quad w_{y_i} = (a_i)_x. \quad (3.9)$$
i.e. the dispersive counterpart of the generalized Benney gas system (2.7). To calculate the equation related to $y_i \leftarrow y_j$ it is better to rewrite $\Omega_1$ in the form $\Omega_1 \Psi_\alpha = -\psi_i \partial^{-1} \phi_i \Psi_\alpha$, where $a_i = \psi_i \phi_i$ and $q_i = -(\ln \phi_i)_x$. Then, the dispersive versions of (2.8) are

$$(\psi_j)_{yi} = -\psi_i \partial^{-1}_x (\psi_j \phi_i), \quad (\phi_j)_{yi} = -\phi_i \partial^{-1}_x (\psi_j \phi_i),$$

where $\partial^{-1}_x$ means the formal integration operation which cannot be confused with $\partial^{-1}$. With the use of $\psi_i$ and $\phi_i$ the system (3.9) becomes

$$(\psi_i)_t = (\psi_i)_x + 2w \psi_i, \quad (\phi_i)_t = -(\phi_i)_x - 2w \phi_i, \quad w_{yi} = (\psi_i \phi_i)_x.$$  

### 3.2 Finite-field reductions

After quantization we would like to construct reductions of the above dispersive Whitham hierarchy being counterparts to algebraic orbits for the Whitham hierarchy, i.e. we are looking for a Lax operator $L$ satisfying the following linear equation

$$L \Psi = L_{n_\infty} \Psi = L^{n_i} \Psi = E \Psi \quad n_\infty, n_i \geq 1,$$  

(3.10)

where now the eigenfunctions $\Psi_\alpha$ reduce to one $\Psi$ and we have only one eigenvalue $E$. The related Lax hierarchy takes the form

$$[\partial_{\beta n} - \Omega_{\beta n}, L] \Psi = 0.$$  

(3.11)

The disadvantage mentioned above practically makes it impossible to construct $L$ related to $n_i \geq 2$ with the use of pseudo-differential operators. For $n_i = 1$ this operator can be defined as

$$L \Psi = \left( \partial^{n_\infty} + \sum_{i=0}^{n_\infty-2} u_i \partial^i + \sum_{i=1}^{N} a_i (\partial - q_i)^{-1} \right) \Psi.$$  

(3.12)

It is useful to replace $a_i (\partial - q_i)^{-1}$ by the operator $\psi_i \partial^{-1} \phi_i$, where $a_i = \psi_i \phi_i$ and $q_i = -(\ln \phi_i)_x$. Then, (3.12) takes the form

$$L \Psi = \left( \partial^{n_\infty} + \sum_{i=0}^{n_\infty-2} u_i \partial^i + \sum_{i=1}^{N} \psi_i \partial^{-1} \phi_i \right) \Psi.$$  

(3.13)

The Lax operators in the form (3.13) have been considered in [6] as reductions of the KP hierarchy, i.e. only for $\alpha = \infty$. In a more general context, i.e. for all $\alpha$, the theory of such operators together with Hamiltonian structures is presented in [5]. The fields $\psi_i, \phi_i$ are the so-called source terms as $\psi_i$ and $\phi_i$ are eigenfunctions and adjoint-eigenfunctions of the Lax hierarchy (3.11), respectively, i.e. $\psi_i$ satisfy linear equations (3.6) on eigenfunction $\Psi$.

**Example 3.2** The multi-component AKNS hierarchy.

Consider the case $n_\infty = 1$ of (3.12) or equivalently (3.13), that is

$$L \Psi = \left( \partial + \sum_i a_i (\partial - q_i)^{-1} \right) \Psi = \left( \partial + \sum_i \psi_i \partial^{-1} \phi_i \right) \Psi.$$  

(3.14)
Then for $\Omega_\infty^2 \Psi = (\partial^2 + 2 \sum_j a_j) \Psi = (\partial^2 + 2 \sum_j \psi_j \phi_j) \Psi$ one calculates the multi-component AKNS system in a form similar to its dispersionless limit (2.16)

\[
(a_i)_t = (a_i)_{2x} + 2(a_i q_i)_x \\
(q_i)_t = -(q_i)_{2x} + 2 \sum_j (a_j)_{x} + 2q_i(q_i)_x
\]

or in the representation with source fields

\[
(\psi_i)_t = (\psi_i)_{2x} + 2\psi_i \sum_j \psi_j \phi_j \\
(\phi_i)_t = -(\phi_i)_{2x} - 2\phi_i \sum_j \psi_j \phi_j.
\]

For $\Omega_k \Psi = -\psi_k \partial^{-1} \phi_k \Psi$ one finds a dispersive counterpart of (2.17)

\[
(\psi_i)_{yk} = -\psi_k \partial^{-1}_x (\psi_i \phi_k) \\
(\phi_i)_{yk} = -\phi_k \partial^{-1}_x (\psi_i \phi_k) \\
(\psi_k)_{yk} = (\psi_k)_{x} + \sum_{j \neq k} \psi_j \partial^{-1}_x (\psi_k \phi_j) \\
(\phi_k)_{yk} = (\phi_k)_{x} + \sum_{j \neq k} \phi_j \partial^{-1}_x (\psi_j \phi_k), \tag{3.15}
\]

where $i \neq k$. Obviously the above systems are compatible with the equations from Example 3.1 as $w = 2 \sum_i a_i$.

**Example 3.3** Let us take dispersionful analogue of the Lax function from Example 2.4, then we have

\[\mathcal{L} \Psi = (\partial^2 + u + a(\partial - q)^{-1}) \Psi.\]

We have that $\mathcal{L} \Psi = (\Omega_{\infty^2} - \Omega_{11}) \Psi$. So, as before $\mathcal{L}_t = \mathcal{L}_y$, i.e. $t \equiv y$. Thus,

\[
\Omega_{\infty^2} \Psi = (\partial^2 + u) \Psi \implies u_t = 2a_x \\
a_t = (a_x + 2aq)_x \\
q_t = (-q_x + q^2 + u)_x,
\]

which is the dispersionful version of (2.22). For $\Omega_{\infty^3} \Psi = (\partial^3 + \frac{3}{2} u \partial + \frac{3}{4} (u_x + 2a)) \Psi$ one obtains the dispersionful version of (2.23)

\[
u = \left(\frac{1}{4}u_{2x} + \frac{3}{2}a_x + \frac{3}{4}u^2 + 3aq\right)_x \\
a = \left(a_{2x} + 3a_x q + 3aq^2 + \frac{3}{2}ua\right)_x \\
q = \left(q_{2x} - 3qq_x - \frac{3}{4}u_x + q^3 + \frac{3}{2}uq + \frac{3}{2}a\right)_x.
\]
3.3 Multi-component Toda-like formulation

In [18] a unified approach to systematic construction of field and lattice soliton systems was presented. It was shown that Lie algebras of pseudo-differential operators can be considered as Weyl-Moyal-like deformations of algebras with Poisson structure given by canonical Poisson bracket (2.6). The Weyl-Moyal-like deformation is a special case of the deformation quantization. On the other hand, Lie algebras of shift operators in the same procedure can be considered as deformations of algebras with Poisson bracket in the form (2.25), which suggests that shift operators and similar relation to (1.2) can be helpful in solving problems appearing in the pseudo-differential representation of dispersionful analogue of the Whitham hierarchy. Hence, instead of constructing the operator counterpart of the logarithm functions \( \Omega_{0} \) we will use, following [13], the ‘quantum’ analogue of the relation (2.24), given at this moment ad-hoc as

\[
\frac{\partial \Psi_{\alpha}}{\partial x} = (e^{\partial s_{k}} + q_{k}(s_{k} + 1)) \Psi_{\alpha}, \tag{3.16}
\]

where \( e^{\partial s_{k}} = \sum_{n=0}^{\infty} \frac{1}{n!} \partial^{n} s_{k} \) is the contracted formula for Taylor expansion. Then, the quasi-classical limit of (3.16) is exactly the relation (2.24). Let us denote by \( \Lambda_{k} := e^{\partial s_{k}} \) the shift operators that when acting on dynamical fields satisfy

\[
\Lambda_{k}^{m} u(s_{k}) = u(s_{k} + m) \Lambda_{k}^{m} \quad m \in \mathbb{Z}.
\]

The field \( q_{k} \) has shifted the argument in (3.16), so then it follows that

\[
\Lambda_{k} \Psi_{\alpha} = (\partial - q_{k}(s_{k} + 1)) \Psi_{\alpha} \implies \Lambda_{k}^{-1} \Psi_{\alpha} = (\partial - q_{k})^{-1} \Psi_{\alpha}. \tag{3.17}
\]

From (3.17) we have also the relation between different shift operators

\[
\Lambda_{i} \Psi_{\alpha} = (\Lambda_{k} + q_{k}(s_{k} + 1) - q_{i}(s_{i} + 1)) \Psi_{\alpha}. \tag{3.18}
\]

It is important to remember that the above rules are valid only when acting on eigenfunctions \( \Psi_{\alpha} \). Other relations which may be useful are

\[
\Lambda_{k}(\Lambda_{k} - v)^{-1} = v(\Lambda_{k} - v)^{-1} + 1
\]

\[
(\Lambda_{k} - v)^{-1} \Lambda_{k} = (\Lambda_{k} - v)^{-1} v + 1
\]

\[
(\Lambda_{k} - v)^{-1} u = u(s_{k} - 1)(\Lambda_{k} - v)^{-1} - (\Lambda_{k} - v)^{-1} v [u - u(s_{k} - 1)] (\Lambda_{k} - v)^{-1}
\]

\[
(\Lambda_{k} - v)^{-1} u = u(s_{k} - 1) \left( \Lambda_{k} - v \frac{u(s_{k} - 1)}{u} \right)^{-1}
\]

\[
[ (\Lambda_{k} - v)^{-1} ]_{v} = (\Lambda_{k} - v)^{-1} v [ (\Lambda_{k} - v)^{-1} ]
\]

We can add the linear relations (3.16) to the auxiliary linear equations (3.6) and complete the set of zero-curvature equations (3.8) by

\[
s_{i} \mapsto t_{am} : \quad [\partial - \Lambda_{i} - q_{i}(s_{i} + 1), \partial_{t_{am}} - \Omega_{am}] \Psi_{\gamma} = 0 \tag{3.19}
\]

\[
s_{i} \mapsto s_{j} : \quad [\partial - \Lambda_{i} - q_{i}(s_{i} + 1), \partial - \Lambda_{j} - q_{j}(s_{j} + 1)] \Psi_{\gamma} = 0. \tag{3.20}
\]

Adequately the Lax hierarchies (3.7) are completed by

\[
[\partial - \Lambda_{i} - q_{i}(s_{i} + 1), L_{a}] \Psi_{\alpha} = 0.
\]
They differ from the rest of equations belonging to the dispersionful Whitham hierarchy as they are partially lattice systems in variables $s_i$. With supplementary equations \((3.19, 3.20)\) the dispersionful Whitham hierarchy is in one to one correspondence to the original one. These equations are well defined as using the relation \((3.16)\) we can pass from pseudo-differential operators to shift ones and formulate the dispersive analogue of the Whitham hierarchy entirely by means of shift operators. This is a counterpart of the alternative formulation performed in the Section 2.3.

We define the following Lie algebras of shift operators

$$a_i = a_{(i,+)} \oplus a_{(i,-)} = \left\{ \sum_{l \geq 0} a_l \Lambda^l_i \right\} \oplus \left\{ \sum_{l < 0} a_l \Lambda^l_i \right\} \quad i = 1, 2, ..., N,$$

that are decomposed into Lie subalgebras with respect to the commutator. The different Lie algebras $a_i$ are mutually connected through the relations \((3.17, 3.18)\) due to which also the rules of expansion can be formulated. Thus, the algebras of pseudo-differential and shift operators act compatibly on the eigenfunctions $\Psi_\alpha$, i.e. $g_{(i, \pm)} \Psi_\alpha = a_{(i, \pm)} \Psi_\alpha$ and $g_{(\infty, \pm)} \Psi_\alpha = a_{(i, \pm)} \Psi_\alpha$, for arbitrary $i$. Hence, the rules of expansion and $R$-matrices related to decomposition of Lie algebras are invariant under the passage between both alternative formulations. Moreover, the form of Lax hierarchies \((3.7)\) and zero-curvature equations \((3.8)\) is preserved. Then, the set of linear equations \((3.5)\) following from \((3.2-3.3)\) is given by

$$L_\infty \Psi_\alpha = \left( \Lambda_k + q_k (s_k + 1) + \sum_{l=1}^{\infty} a_{\infty, l} \Lambda^{-l}_k \right) \Psi_\alpha = z_\infty \Psi_\alpha$$

and the definition of generating operators \((3.4)\) remain unchanged. Thus, equation \((3.16)\) follows from \((3.6)\) for $\beta = \infty, n = 1$. The zero-curvature equations \((3.8)\) together with Lax hierarchies \((3.7)\) defined for Lax operators \((3.21)\) can be extracted from the 'charged' multi-component KP hierarchy \cite{13}. As we are using simultaneously shift operators $\Lambda_i$ with respect to different $s_i$ we call such a formulation of the dispersionful Whitham hierarchy as the multi-component Toda-like.

In the above formulation there is still missing the 'quantum' counterpart of $\Omega_{k0} = \ln \lambda_k$ from Section 2.3. The natural logarithm of the shift operator can be defined in the way $\ln \Lambda_k := \frac{d}{d\alpha} \Lambda^\alpha_k \bigg|_{\alpha=0}$, where $\alpha$ is arbitrary. Over the logarithm $\ln \Lambda_k$, for fixed $k$, the algebra $a_k$ can be extended in such a way that the Lie algebra splitting is be preserved. Thus, we have

$$\frac{\partial \Psi_\alpha}{\partial t_{k0}} = \Omega_{k0} \Psi_\alpha = \ln \Lambda_k \Psi_\alpha = \frac{d}{d\alpha} e^{\alpha \partial_{s_k}} \bigg|_{\alpha=0} \Psi_\alpha = \partial_{s_k} e^{\alpha \partial_{s_k}} \bigg|_{\alpha=0} \Psi_\alpha = \partial_{s_k} \Psi_\alpha.$$

Hence $t_{k0}$ can be identified with $s_k$ and the related equations from the dispersionful Whitham hierarchy, in the multi-component Toda-like formulation, are trivially satisfied as it should be. In the case of Lax hierarchies one obtains the translational symmetries $(L_\alpha)_{t_{k0}} = (L_\alpha)_{s_k}$. As before, in the case of the original Whitham hierarchy, considering the bi-Hamiltonian formulation one would see that these translational symmetries generates higher order symmetries that are not included in our construction. The way of construction of the whole hierarchy generated by logarithmic operators, defined by means of dressing operators, for the two-field lattice Toda system is presented in \cite{12}.
Notice that \( \ln \Lambda_k \Psi_\alpha \neq \ln (\partial - q_k(s_k + 1)) \Psi_\alpha \) as \( \Lambda_k^a \Psi_\alpha \neq (\partial - q_k(s_k + 1))^a \Psi_\alpha \). Besides, the action of \( \ln (\partial - q(s_k + 1)) \) on eigenfunctions \( \Psi_\alpha \) would not be well defined. Nevertheless, one would like to have the formulation of the above logarithms by means of pseudo-differential operators. To calculate it one should use a formula similar to (3.25), but the calculations are not straightforward anymore.

**Example 3.4** To have a full dispersive counterpart of Example (2.1) we have to complete the set of equations from Example (3.1) and calculate systems with the evolution parameters \( s_k \).

We will use the related operators \( (3.4) \) from auxiliary linear equations \( (3.6) \) in the form

\[
(\Psi_\alpha)_x = (\Lambda_k + q_k(s_k + 1)) \Psi_\alpha \quad (\Psi_\alpha)_yi = (-a_i \Lambda_i^{-1}) \Psi_\alpha \\
(\Psi_\alpha)_t = (\Lambda_k^2 + (q_k(s_k + 2) + q_k(s_k + 1)) \Lambda_k + (q_k(s_k + 1))^2 + a(s_k + 1) + a) \Psi_\alpha.
\]

Then, for \( t \leftrightarrow y_i \) and \( t \leftrightarrow s_i \) one finds equations \( (3.9) \) and

\[
(q_i)_x = w - w(s_i - 1).
\]

For \( s_i \leftrightarrow s_j \) and \( y_i \leftrightarrow y_j \) one finds systems

\[
q_j - q_j(s_i - 1) = q_i - q_i(s_j - 1) = (\ln |q_i - q_j|)_x \tag{3.22}
\]

\[
(a_j)_yi = (a_i)_yi = \frac{a_i a_j(s_i - 1) - a_i(s_j - 1)a_j}{q_i(s_j - 1) - q_j(s_i - 1)},
\]

respectively. Finally, for \( s_i \leftrightarrow y_j \) we have equations

\[
(q_i)_yj = a_j - a_j(s_i - 1) \quad (a_i)_x = a_i(q_i(s_i + 1) - q_i)
\]

from which eliminating \( q_i \) one finds the multi-dimensional Toda equation.

### 3.4 Finite-field reductions

Now, in the representation of shift operators we are able to construct finite field reductions \( (3.10) \) for arbitrary \( n_\alpha > 0 \) given by a Lax operator \( \mathcal{L} \) in the form

\[
\mathcal{L} \Psi = \left( \Lambda_k^n + u \Lambda_k^{n-1} + \sum_{l=0}^{n-2} u_l \Lambda_k^l + \sum_{i=1}^{N} \sum_{l=1}^{n_i} a_{i,l} \Lambda_i^{-l} \right) \Psi = E \Psi \tag{3.23}
\]

being a dispersive counterpart of \( (2.22) \), where now \( u = \sum_{j=1}^{n} q_k(s_k + j) \). Notice that some terms of \( (3.4) \) can have coefficients that are nonlocal on lattice, see Example 3.6. However, as we are interested in the construction of (1+1)-dimensional equations we have to show how to represent \( (3.23) \) only with \( \Lambda_k \) for fixed \( k \). The equation \( (3.22) \) belongs to the zero-cutvature equations that must be always satisfied. Thus, from (3.22) we have the relation

\[
q_k(s_k + 1, s_i - 1) - q_i = q_k(s_k + 1) - q_i(s_k + 1).
\]

Hence, by (3.18) one finds that

\[
\Lambda_i^{-1} \Psi = (\Lambda_k + q_k(s_k + 1, s_i - 1) - q_i)^{-1} \Psi = (\Lambda_k + q_k(s_k + 1) - q_i(s_k + 1))^{-1} \Psi
\]

which when use recursively leads to

\[
\Lambda_i^{-m} \Psi = (\Lambda_k + q_k(s_k + 1 - m, s_k + 1) - q_i(s_i + 1 - m, s_k + 1))^{-1} \Psi
\]

\[
\cdots \cdot (\Lambda_k + q_k(s_k + 1) - q_i(s_k + 1))^{-1} \Psi
\]

for \( m > 0 \).
Example 3.5 The multi-component Toda hierarchy.
Let us use the notation \( u(s_k + m) := u^{(m)} \). The linear equation (3.14) represented by \( \Lambda_k \) transforms to

\[
\mathcal{L} \Psi = \left( \Lambda_k + q^{(1)}_k + a_k \Lambda_k^{-1} + \sum_{i \neq k} a_i \left( \Lambda_k + q^{(1)}_k - q^{(1)}_i \right)^{-1} \right) \Psi.
\]

Then for \( \Psi_{\infty_1} = (\Lambda_k + q^{(1)}_k) \Psi \) one calculates the multi-component Toda system

\[
(a_i)_x = a_i \left( q^{(1)}_i - q^{(1)}_k \right) + a_i \left( q^{(1)}_k - q_i \right)
\]

\[
(q_i)_x = \left( q_i - q^{(-1)}_i \right) \left( q_i - q_k \right) + \sum_j \left( a_j - a^{(-1)}_j \right),
\]

i.e. the lattice analogue of (2.27). For \( \Omega_{k1} \Psi = -a_k \Lambda_k^{-1} \Psi \) we have the counterpart of (3.15) in the form

\[
(a_k)_{y_k} = a_k \left( q^{(1)}_k - q_k \right) - a_k \sum_{j \neq k} \left( \frac{a_j}{q^{(1)}_j - q^{(1)}_k} - \frac{a^{(-1)}_j}{q_j - q_k} \right)
\]

\[
(q_k)_{y_k} = a_k - a^{(-1)}_k
\]

\[
(a_i)_{y_k} = a_k \left( \frac{a_i}{q^{(1)}_i - q^{(1)}_k} - \frac{a^{(-1)}_i}{q_i - q_k} \right)
\]

\[
(q_i)_{y_k} = a_k \left( 1 - \frac{q_i - q_k}{q^{(1)}_i - q^{(1)}_k} \right),
\]

where \( i \neq k \).

Example 3.6 We will use the notation \( q := q_1, \Lambda := \Lambda_1, y := y_1 \) and \( u^{(m)} := u(s_1 + m) \). The linear equation being a dispersive analogue of (2.28) is

\[
\mathcal{L} \Psi = \left( \Lambda + q^{(1)} + u \Lambda^{-1} + v \Lambda^{-2} \right) \Psi.
\]

Then, for \( \Omega_{\infty_1} \Psi = (\Lambda + q^{(1)}) \Psi \) one finds

\[
u_x = u \left( q^{(1)} - q \right) + v^{(1)} - v
\]

\[
v_x = v \left( q^{(1)} - q^{(-1)} \right)
\]

\[
q_x = u - u^{(-1)}
\]

analogue of (2.29). Next system is

\[
\Omega_{11} \Psi = -a \Lambda^{-1} \Psi \implies u_y = a \left( q^{(1)} - q \right)
u_y = ua^{(-1)} - u^{(-1)}a
\]

\[
q_y = a - a^{(-1)}
\]

where \( a \) satisfies \( aa^{(-1)} = v \). There are two simplest possible solutions on the field \( a \):

\[
a = \frac{v^{(1)} v^{(3)} v^{(5)} \cdots}{v^{(2)} v^{(4)} \cdots} \quad \text{or} \quad a = \frac{v^{(-2)} v^{(-4)} \cdots}{v^{(-1)} v^{(-3)} \cdots}.
\]
Example 3.7 In order to complete the dispersionful version of Examples 2.4 or 2.8 we need to write the Lax operator $\mathcal{L}$ from Example 3.3 in terms of the shift operator $\Lambda := \Lambda_1$ as

$$\mathcal{L} \Psi = (\Lambda^2 + (q^{(1)} + q^{(2)})\Lambda + (q^{(1)})^2 + u + a\Lambda^{-1}) \psi.$$  

Now, from $\Omega_{\infty 1} \Psi = (\Lambda + q^{(1)}) \Psi$ we find the lattice analogue of (2.30)

$$\begin{align*}
(q^{(1)} + q^{(2)})_x &= u^{(1)} - u \\
((q^{(1)})^2 + u)_x &= a^{(1)} - a \\
ax &= a (q^{(1)} - q).
\end{align*}$$

On the other hand, by introducing $b := q^{(1)} + q^{(2)}$ and $c := (q^{(1)})^2 + u$, so that $\mathcal{L} \Psi = (\Lambda^2 + b\Lambda + c + a\Lambda^{-1}) \Psi$, we have also the lattice analogue of (2.31)

$$\Omega_{\infty 2} \Psi = (\Lambda^2 + b\Lambda + c) \Psi \quad \Rightarrow \quad b_t = a^{(2)} - a \quad \Rightarrow \quad c_t = a^{(1)}b - ab^{-1} \quad \Rightarrow \quad a_t = a(c - c^{-1}).$$

As we have been able with the use of expression (3.17) to construct Lax operators $\mathcal{L}$, being dispersive counterparts of the meromorphic functions with 'finite' poles of higher order, we would like to use (3.17) in the opposite way and represent $\mathcal{L}$ by means of pseudo-differential operators. Thus, one finds the useful formula

$$\Lambda_k^{-m} \Psi = (\partial - q_k(s_k - m + 1))^{-1} \cdot \ldots \cdot (\partial - q_k(s_k - 1))^{-1}(\partial - q_k)^{-1} \Psi, \quad (3.25)$$

where $m > 0$. However, shifted fields will appear, which for $\alpha = \infty$ can be understood as new independent dynamical fields. So, one is able to construct closed equations being finite-field reductions (3.10) of KP hierarchy, also for $n_i$ higher then 1, that are analogues of algebraic orbit reductions of Whitham hierarchy. The above dispersionful analogues in general will have more dynamical fields then its dispersionless limits. The reductions of this kind of KP hierarchy have been considered earlier in [8, 9] in the context of multi-matrix models.

Example 3.8 The linear equation (3.24) written by means of pseudo-differential operators takes the form

$$\mathcal{L} \Psi = (\partial + u(\partial - q)^{-1} + v(\partial - q')^{-1}(\partial - q)^{-1}) \Psi,$$

where $q' := q^{(-1)}$. Then, for $\Omega_{\infty 2} \Psi = (\partial^2 + 2u) \Psi$ one finds the following system

$$\begin{align*}
u_t &= v_{2x} + 2v_x + 2(uq)_x \\
v_t &= v_{2x} + 2vq_x + 2(vq')_x \\
q_t &= -q_{2x} + 2u_x + 2qq_x \\
q'_t &= -q'_{2x} - 2q_{2x} + 2u_x + 2qq'_x \quad . \quad (3.26)
\end{align*}$$

The evolutions of fields $q$ and $q' = q(s_k - 1)$ in the quasi-classical limit will be identical, i.e. four-field equation (3.26) is well defined dispersive counterpart of three-field (2.19). This system was constructed in [8 9].

In the forthcoming article [26] the theory presented in this section will be treated more broadly and with all necessary proofs. The aim of [26] will be a construction of (1+1)-dimensional field and lattice soliton systems being dispersive counterparts for a wider class of dispersionless equations considered in [22].
4 Comments

In [13] Takasaki provided a scheme for deriving the Whitham hierarchy as a dispersionless limit of the 'charged' multi-component KP hierarchy. The charges are introduced through an extra set of discrete variables $s_i$. The scheme is based on the $\tau$-function bilinear formalism [27, 28] where the total charge $s_\infty + \sum_i s_i$ equal to zero. In this sense the 'charged' multi-component KP hierarchy can be understood as a multi-component generalization of Toda hierarchy. As a result, the scalar auxiliary linear equations (3.5-3.6), defined by means of shift operators, leading to the Lax formalism (3.7-3.8) can be extracted. In literature there are several different formulations of the multi-component KP hierarchy, for example the Grassmannian approach [29] or the $\bar{\Omega}$-method [30]. So, it seems that a further discussion of the corresponding connections between them and the dispersionful Whitham hierarchy from the present work and [13, 14] is needed.

Besides, due to the previous comments on the bi-Hamiltonian structures of the original and dispersionful Whitham hierarchy we see that the existence of the symmetries leads to the conclusion that these hierarchies are incomplete. So, we can extend this incompleteness onto multi-component KP and Toda hierarchies. This problem seems to be worth of further investigation.

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