Hamilton-Jacobi approach for Regge-Teitelboim cosmology

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Abstract. The Hamilton-Jacobi formalism for a geodetic brane-like universe described by the Regge-Teitelboim model is developed. We focus on the description of the complete set of Hamiltonians that ensure the integrability of the model in addition to obtaining the Hamilton principal function $S$. In order to do this, we avoid the second-order in derivative nature of the model by appropriately defining a set of auxiliary variables that yields a first-order Lagrangian. Being a linear in acceleration theory, this scheme unavoidably needs an adequate redefinition of the so-called Generalized Poisson Bracket in order to achieve the right evolution in the reduced phase space. Further this Hamilton-Jacobi framework also enables us to explore the quantum behaviour under a semi-classical approximation of the model. A comparison with the Ostrogradski-Hamilton method for constrained systems is also provided in detail.

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1. Introduction

The Regge-Teitelboim (RT) model, also named geodetic brane gravity, deals with our 4-dimensional universe as an extended object geodesically floating in a higher-dimensional flat Minkowski spacetime \cite{RT1}. The underlying motivation of this model was to develop General Relativity (GR) following the principles that ordinarily are used to determine the behaviour of either the worldline of a particle or the worldvolume of an extended
object by considering the embedding functions instead of the metric as the fundamental geometric objects. The theory exhibits a built-in Einstein limit which sets this model as an attractive theoretical tool that deserves further development. Both the RT model and GR have in common that the action that describe them depend linearly on second-order derivatives of the fields variables that, respectively, characterize them. Within the mini-superspace brane-like cosmology framework, such dependence is more evident and manageable. This last issue has been worked at greater length at both the classical and the quantum levels in many contributions [2, 3, 4, 5, 6, 7, 8].

From the viewpoint of the Dirac-Bergmann approach for constrained systems [9, 10, 11], the RT model is described by a singular Lagrangian system that has been analyzed by using different strategies, all of them interesting in their own. However, there is a variational alternative to study the aforementioned singular nature of the theory based on the equivalent Lagrangians method introduced by Carathéodory [12] for regular systems, and further developed for the treatment of singular systems by Güler [13, 14], and shortly afterwards extended to higher-order derivative theories by Pimentel and collaborators [15, 16, 17, 18]. Even though this approach has as a starting point the Lagrangian scheme, it brings into play a suitable shortcut to the Hamilton-Jacobi (HJ) framework without going through the usual approximation of the canonical transformations of the Hamiltonian formalism even in the singular case. Indeed, within this geometrical HJ framework the constraints emerge as a set of partial differential equations (PDE) that must obey certain conditions that ensure their integration. Contrary to the situation in the Dirac-Bergmann approach, in this formalism it is not mandatory to classify the constraints as first- and second-class constraints. In fact, in this HJ approach a singular physical system is viewed as a many variables system which leads to replace the usual Hamilton equations by a set of total differential equations in those variables. On mathematical grounds, for singular systems it is not possible to solve such equations unless we meet certain geometrical conditions known as integrability conditions which could lead to more constraints. Certainly, one of the central roles of this approach is to complete the analysis on the integrability of singular systems through the analysis of the conditions under which the canonical field equations are integrable or not. The canonical equations naturally appear as total differentials expanded in terms of the parameters of the theory, and whenever these form a fully integrable set, their simultaneous solutions will determine the generating function $S(\tau, z^A)$, the Hamilton principal function, uniquely by imposing some initial conditions.

The purpose of this paper is to offer a novel geometrical alternative to analyze the integrability of the geodetic brane gravity within the minisuperspace cosmological scenario. This is performed in terms of the HJ framework. As we will see, in order to construct the HJ structures for the RT model, it is convenient to introduce auxiliary variables in order to reduce the complexity of the analysis by considering a first-order in derivatives Lagrangian. As a consequence of this extension in the configuration variables, the number of integrability conditions to be studied increases. Nevertheless, the purpose behind the reduction of the order is twofold. First, to use a HJ approach
developed for first-order actions, derived from higher-order derivative theories and thus apply the standard HJ method and, second, to detect the local gauge symmetries of the theory through a purely geometrical approach in order to achieve the correct gauge transformations. Furthermore, unlike the existing distinct Hamiltonian approaches, we claim that we are able to obtain the Hamilton principal $S$-function, which in turn will allow us to explore its quantum semi-classical approximation. Indeed, the knowledge of this $S$-function has a direct implication when one tries to obtain some information on quantum singular systems by investigating the semi-classical WKB approximation. In this case, the constraints are promoted to conditions that the wave function must satisfy at the semi-classical limit, in addition to satisfy the Schrödinger equation.

The layout of the paper is as follows. In section 2 we provide an overview of the HJ formalism, adapted to first-order actions. In section 3 we implement this formalism for the RT model, adapted to a FRW geometry, by introducing auxiliary variables. We elucidate in detail the conditions under which the integrability conditions emerge by making contact with the Ostrogradski-Hamiltonian approach. We also address the local gauge symmetries for this model. In section 4, we obtain the Hamilton principal function and briefly discuss the semi-classical approximation of the quantum approach for the RT cosmological model. Conclusions are presented in section 5.

2. Hamilton-Jacobi framework for first-order actions

Consider that the dynamical behaviour of a physical system is described by the action

$$S[z^A] = \int d\tau L(z^A, \dot{z}^A, \tau), \quad A, B = 1, 2, 3, \ldots, N;$$

where $z^A$ are the coordinates of the associated $N$-dimensional configuration manifold $C_N$, and $\dot{z}^A$ denote their associated velocities. Here an overdot stands for the derivative with respect to the time parameter $\tau$. Since our interest lies in cosmological models with a linear dependence on accelerations $^{19}$, we first proceed to construct a HJ framework for first-order actions i.e., theories with a linear dependence in the velocities, and then we will implement this for the RT cosmological model. The explicit form of the Lagrangian function to be considered is

$$L(z^A, \dot{z}^A, \tau) = K_A(z^B) \dot{z}^A - V(z^B).$$

Summation over repeated indices is henceforth assumed. The variational principle selects the optimal trajectory $z^A = z^A(\tau)$ parametrized by $\tau$. Here, $K_A(z)$ and $V(z)$ are assumed to be smooth functions defined on $C_N$. The Euler-Lagrange equations of motion (eom) are

$$\left(\frac{\partial K_B}{\partial z^A} - \frac{\partial K_A}{\partial z^B}\right) \ddot{z}^B - \frac{\partial V}{\partial z^A} = 0.$$  

Following Carathéodory’s equivalent Lagrangians approach $^{12}$ $^{13}$ $^{14}$, in order to have an extreme configuration of the action $^{11}$ the necessary and sufficient conditions
are associated to the existence of a family of surfaces defined by a generating function or Hamilton’s principal function, $S(z^A, \tau)$, such that it satisfies

$$\frac{\partial S}{\partial z^A} = \frac{\partial L}{\partial \dot{z}^A} =: p_A, \quad (4)$$

$$\frac{\partial S}{\partial \tau} + \frac{\partial S}{\partial z^A} \dot{z}^A - L = 0, \quad (5)$$

where $p_A$ denotes the conjugate momenta to the coordinates $z^A$. Note that in our particular case the conjugate momenta is given by $p_A = K_A(z^B)$.

The HJ framework in which we are interested arises from (4) and (5) considered as partial differential equations (PDE) for the generating function $S$. Indeed, for non-singular systems it is relatively straightforward to obtain the function $S$ by expressing the velocities $\dot{z}^A$ in terms of the coordinates $z^A$ and partial derivatives of $S$ what is provided by appropriately inverting Eq. (4). However, for singular physical systems this may not be as direct, as we will see below. Even worse, for affine in velocity (or acceleration) theories [19] this turns out rather intricate as the Hessian matrix of the system vanishes identically,

$$H_{AB} = \frac{\partial^2 L}{\partial \dot{z}^A \partial \dot{z}^B} = 0. \quad (6)$$

Clearly, for the Lagrangian we are considering, the rank of the Hessian matrix is zero which causes the phase space to be non-isomorphic to the tangent bundle of the configuration manifold, $T^*C_N$. This means that the manifold $C_N$ is fully spanned by the $R = N - 0 = N$ variables where the $z^A$ play the role of parameters from the HJ viewpoint [13, 14]. Indeed, these parameters are related to the null space of the Hessian $H_{AB}$. Of course, we can not invert the velocities $\dot{z}^A$ in favor of the coordinates and partial derivatives of the function $S$, that is, $\dot{z}^A \neq f^A(\tau, z^B, \partial S/\partial z^B)$.

In what follows, it is convenient to introduce the notation

$$t^A := z^A \quad \text{and} \quad H_A := -\frac{\partial L}{\partial \dot{t}^A} = -K_A(z^B), \quad (7)$$

which, by relation (4), follows

$$\frac{\partial S}{\partial t^A} + H_A \left(\tau, t^B, \frac{\partial S}{\partial H^B}\right) = 0. \quad (8)$$

Notice that $H_A$ does not depend on $\dot{t}^A = \dot{z}^A$. Similarly, taking into account (4) and by introducing the Hamilton function

$$H_0 := \frac{\partial S}{\partial t^A} \dot{t}^A - L(\tau, t^A, \dot{t}^A), \quad (9)$$

one finds that the expression (5) becomes

$$\frac{\partial S}{\partial \tau} + H_0 \left(\tau, t^B, \frac{\partial S}{\partial H^B}\right) = 0, \quad (10)$$

which is most commonly known as the Hamilton-Jacobi equation.
Now, we are able to collect (8) and (10) into a single equation expressing a unified set of PDE for the generating function $S$. To perform this, we simply introduce the notation $t^0 := \tau$, and thus we find

$$\frac{\partial S}{\partial t^I} + H_I\left(t^J, \frac{\partial S}{\partial t^A}\right) = 0, \quad I, J = 0, 1, 2, \ldots, N. \quad (11)$$

In the following, these $N + 1$ relations will be referred to as the Hamilton-Jacobi partial differential equations (HJPDE). Bearing in mind (4), it is very useful to express the HJPDE in the Hamiltonian fashion

$$H'_I(t^J, p_J) := p_I + H_I(t^J, p_J) = 0, \quad (12)$$

where we have considered $p_0 := \frac{\partial S}{\partial \tau}$. These relationships have thus acquired the well-known form of canonical Dirac constraints. In other words, to get a clearer picture, this HJ approach replaces the analysis of the $N$ canonical constraints, $H'_A = 0$, with the analysis of the ($N + 1$) HJPDE given by relations (11).

Within this framework the equations of motion are written as total differential equations. These are often known as the characteristic equations associated to the Hamiltonian set (12). In our particular case these characteristic equations are given by

$$dz^I = \frac{\partial H'_I}{\partial p_I} dt^I, \quad (13)$$

$$dp_I = -\frac{\partial H'_I}{\partial z^I} dt^I. \quad (14)$$

At this initial stage we must notice that all coordinates $z^A$ have a status of independent evolution parameters, in a similar fashion as the time parameter $\tau$. On mathematical grounds, within this HJ formalism it is said that $t^I = z^I$ are the independent variables or parameters of the theory. On physical grounds, this may be confusing but we may consider that these parameters encode the local symmetries and gauge transformations, as we will see below. We have thus enlarged the configuration space to be $\mathcal{C}_{N+1}$. Moreover, another important equation is provided by the function $S$. Indeed, we have that

$$dS = \frac{\partial S}{\partial \tau} d\tau + \frac{\partial S}{\partial z^A} dz^A = -H_I dt^I, \quad (15)$$

where (7) and (12) have been considered.

For two arbitrary functions defined on the cotangent bundle of the configuration manifold, $F, G \in \Gamma_{N+1} := T^*\mathcal{C}_{N+1}$, that is, functions in the extended phase space spanned by the variables $z^I = (t^0, t^A)$ and their conjugate momenta $p_A = (p_0, p_A)$, we introduce the extended Poisson bracket (PB)

$$\{F, G\} = \frac{\partial F}{\partial z^I} \frac{\partial G}{\partial p_I} - \frac{\partial F}{\partial p_I} \frac{\partial G}{\partial z^I}, \quad I, J = 0, 1, 2, \ldots, N. \quad (16)$$

We may therefore express evolution in $\Gamma_{N+1}$ as follows

$$dF = \{F, H'_I\} dt^I, \quad (17)$$
where the role that the $t^I$ play as parameters of the Hamiltonian flows generated by the constraints $H'_I$ is more evident. It is worth mentioning that the characteristic equations \[ \text{[13]} \] and \[ \text{[14]} \] also may be obtained from \[ \text{[17]} \]. In this HJ framework, the dynamical evolution provided by \[ \text{[17]} \] is referred to as the fundamental differential, \[ \text{[14]} \].

2.1. Integrability conditions

With the intention of integrating the HJPDE \[ \text{[12]} \], it is convenient to rely in the method of characteristics \[ \text{[12]} \]. On physical grounds, it is not clear whether or not all coordinates are relevant parameters of the theory, so it is crucial to find a subspace among the parameters where the system becomes integrable. Regarding this, the matrix occurring in \[ \text{[3]} \]

\[ M_{AB} := \{ H'_A, H'_B \} = \partial K_B / \partial z^A - \partial K_A / \partial z^B, \]  

(18)

enters the game in order to unravel under what conditions first-order actions will have integrability. Geometrically, this matrix may be interpreted as the “curl” of the vector $K_A$.

The complete solution of \[ \text{[12]} \] is given by a family of surfaces orthogonal to the characteristic curves. The fulfillment of the Frobenius integrability conditions \[ \text{[12, 20]} \],

\[ \{ H'_I, H'_J \} = C^K_I J H'_K; \]  

(19)

ensures the existence of such a family where $C^K_I J$ are the structure coefficients of the theory. This means that the Hamiltonians must close as a Lie algebra. Hence, it must be imposed that both $dH'_0$ and $dH'_A$ are vanishing identically

\[ dH'_I = 0. \]  

(20)

In view of this, we shall discuss the possible scenarios \[ \text{[17, 18]} \]

- First, if both $\{ H'_0, H'_A \} = 0$ and $M_{AB} = 0$ identically, we will have $dH'_I = 0$ so that $dt^I$ are independent. Accordingly, the equations of motion are all integrable.
- Second, $M_{AB} \neq 0$ and $\det(M_{AB}) \neq 0$, that is, the regular case. The realization of $dH'_I = 0$ leads to consider that $dt^0$ and $dt^A$ are dependent. In such case it is often enough to consider that $t^0 = \tau$ is the independent parameter of the theory and the evolution of $F \in \Gamma_{N+1}$ is provided by

\[ dF = \{ F, H'_0 \}^* \, dt^0, \]  

(21)

where

\[ \{ F, G \}^* := \{ F, G \} - \{ F, H'_A \} (M^{-1})^{AB} \{ H'_B, G \}. \]  

(22)

Here, $(M^{-1})^{AB}$ denotes the inverse matrix of $M_{AB}$ such that $M_{BC}(M^{-1})^{CA} = \delta^A_B$ or $(M^{-1})^{AC} M_{CB} = \delta^A_B$. 
systems \[9, 10, 11\]. On the other hand, it may happen that the constraints the Dirac bracket arising in the Dirac-Bergmann Hamiltonian approach for constrained Hamilton-Jacobi approach for Regge-Teitelboim cosmology do not satisfy the condition of the generalized Poisson bracket. This redefine the dynamics by eliminating some of the coordinates \(\{A_{\alpha}\}\) related to the kernel of \(M_{AB}\) and \(C_P\) spanned by \(P\) coordinates, namely \(z^{A'}\), with \(A' = A + 1, A + 2, \ldots, N\), associated to the regular part of \(M_{AB}\). Clearly, the fact that \(P \neq 0\) leads to the existence of a \(P \times P\) submatrix of \(M_{AB}\), say \(M_{A'B'}\), such that \(\det(M_{A'B'}) \neq 0\) indicates the existence of an inverse matrix \((M^{-1})^{A'B'}\) satisfying \(M_{B'C'}(M^{-1})^{C'A'} = \delta_{B'}^{A'}\) or \((M^{-1})^{A'C'}M_{C'B'} = \delta_{B'}^{A'}\). Therefore, the condition \(dH_A' = 0\) causes the presence of \(t^{\alpha'}\) independent parameters such that the evolution is given by

\[
dF = \{F, H_{A'}^0\}^* dt^{\alpha'}, \quad \alpha' = 0, 1, 2, \ldots, R;
\]

where

\[
\{F, G\}^*: = \{F, G\} - \{F, H_{A'}^0\}(M^{-1})^{A'B'}\{H_{B'}^0, G\}.
\]

In this HJ spirit, the remaining variables \(z^{A'}\) are referred to as dependent variables. In passing, in this particular case, the condition \(dH_A' = 0\) yields

\[
C_{(\alpha)} := \frac{\partial H_0}{\partial z^A} \lambda_{(\alpha)}^A = 0,
\]

where the explicit value of \(\{H_A', H_0^0\}\) has been used. This orthogonality condition among the zero-modes and \(H_A := \partial H_0/\partial z^A\), induces a very convenient strategy to identify a totally equivalent set of constraints to those emerging in the integrability procedure behind this HJ formalism adapted for first-order actions. Likewise, these Lagrangian constraints can be obtained straightforwardly from the eom \([3]\).

The new bracket structure, introduced either in \([22]\) or in \([24]\), is referred to as the generalized Poisson bracket (GPB) which has all the known properties of the standard Poisson bracket. This redefine the dynamics by eliminating some of the coordinates with exception of the \(t^{\alpha'}\). As a matter of fact, this structure is closely related to the Dirac bracket arising in the Dirac-Bergmann Hamiltonian approach for constrained systems \([3, 10, 11]\). On the other hand, it may happen that the constraints \(H_I'^0 = 0\) do not satisfy the condition \(dH_I' = 0\) identically when \([21]\) or \([23]\) are considered as fundamental differentials. In such case, this condition leads us to obtain equations of the form \(f(z^A, p_A) = 0\) which should also be considered as constraints of the system. These in turn, must obey the established integrability condition, \(df = 0\), which can lead to more constraints. Once all the constraints have been found, it is mandatory to incorporate them within the HJ framework where some of them must be considered as generators of the dynamics. It should be remarked that this incorporation must be accompanied by the introduction of more parameters to the theory, these related to the new constraints that generate dynamics, derived from the imposition of integrability. Therefore, the space of parameters has been increased where, every arbitrary parameter is in relation to the generators of the dynamics \([18, 20]\). Accordingly, in this new scenario,
the fundamental differential becomes modified once the complete set of parameters has been identified and incorporated to the theory

\[ dF = \{ F, H'_\alpha \}^* \, d\bar{t}. \]  

(26)

Here, \( t_\alpha \) denotes the complete set of independent parameters of the theory, where the index \( \alpha \) covers the entire set of these parameters. As a result, the fundamental differential (26) must be used to obtain the right evolution in the reduced phase space through the GPB. In a like manner, consistency of conditions (25) requires to analyse the relation

\[ dC(\alpha) = \{ C(\alpha), H'_\alpha \}^* \, dt' =: C(\alpha, \alpha') \, dt'. \]

This could bring further conditions of the type

\[ C(\alpha, \alpha')(z, p) = 0 \]

that should be also treated as constraints of the theory on an equal footing to the former set of constraints. One must continue with this iterative algorithm until no further independent constraints emerge.

Hereinafter, unless otherwise stated, we confine ourselves to the third case above. On the technical side, the matrix \((M^{-1})^{A'B'}\) defines a reduced symplectic structure on the phase space \( T^*C_{P+1} \), where \( M_{AB} \) is singular. This structure provides the appropriate dynamics of the system as we can observe from (23). Regarding this, when considering \( F = z^A \) in (23), it is straightforward to show that

\[ dz^A = (M^{-1})^{A'B'} \left( \frac{\partial K_{B'}'}{\partial z'^\alpha} - \frac{\partial K_{B'}'}{\partial z'^{B'}} \right) \, dt', \quad \alpha' = 0, 1, 2, \ldots, R; \]  

(27)

where one must consider the definition \( K_0 := -H_0 \) with \( H_0 \) given by (7). In arriving to these characteristic equations we have used the explicit value of \( \{ H'_A, H'_\alpha' \} \). We would like to stress some points. First, from the latter expression we readily infer that the solutions to (27) will be of the form \( z^A = z^A(t'^\alpha) \), which represent congruences of \((R + 1)\)-parameter curves where the \( t'^\alpha \) play the role of parameters. Second, by inserting (27) into (15) one finds

\[ dS = - \left[ H'_\alpha + H'_A(M^{-1})^{A'B'} \left( \frac{\partial K_{B'}'}{\partial z'^\alpha} - \frac{\partial K_{B'}'}{\partial z'^{B'}} \right) \right] \, dt', \]  

(28)

where \( H'_A \) is introduced by (7). Third, the characteristic equations (27) will be considerable simplified for the case of the RT gravity as a consequence of the affine in acceleration property of the model, as we will see shortly.

2.2. Generator of gauge symmetries

In reference [20] it has been argued that once the complete set of involutive Hamiltonians \( H'_\alpha = 0 \) of the theory has been found, i.e., \( \{ H'_\alpha, H'_\beta \}^* = C_{\alpha,\beta} H'_\gamma \), these Hamiltonians must be considered as generators of infinitesimal canonical transformations in \( T^*C_{P+1} \) as follows

\[ \delta z^A = \{ z^A, H'_\alpha \}^* \, \delta t' \].

(29)

These are referred to as the characteristic flows of the system. Here, \( \delta t' := t' - \bar{t} = \delta t'(z^A) \). In particular, when one set \( \delta t^0 = 0 \), expression (29) defines a special class of transformations

\[ \delta z^A = \{ z^A, H'_\alpha \}^* \, \delta t'^\alpha, \]  

(30)
which, by imposing that they remain in the reduced phase space, \( T^*C_P \), form the so-called infinitesimal contact transformations in the spirit of the constrained Hamiltonian framework by Dirac, [9]. In this sense, they do not alter the physical states of the system. In (30), \( t^\alpha \) denotes the set of all independent parameters where \( t^0 \) is excluded. Clearly, the transformations (30) are generated by

\[
G := H'_\alpha \delta t^\alpha, \tag{31}
\]

so that (30) is equivalent to

\[
\delta G_z^A = \{z^A, G\}^*. \tag{32}
\]

Thus, \( \delta G_z^A \) is the specialization of (29) to \( T^*C_P \) where \( G \) is the generating function of the infinitesimal canonical transformation. In the spirit of the theory of gauge fields, transformations (32) set the gauge transformations of the theory.

3. RT cosmological brane theory

The original RT gravity, including a cosmological constant term \( \Lambda \), is described by the action [1]

\[
S[X^\mu] = \frac{\alpha}{2} \int_m d^{3+1}x \sqrt{-g} \mathcal{R} - \int_m d^{3+1}x \sqrt{-g} \Lambda, \tag{33}
\]

where \( X^\mu \), the embedding functions, are the field variables describing the 4-dim trajectory, \( m \), spanned by a 3-dim extended object in its evolution in a flat Minkowski spacetime with metric \( \eta_{\mu\nu}, \mu, \nu = 0, 1, 2, \ldots, 4 \); \( \mathcal{R} \) stands for the Ricci scalar defined on \( m \), \( g = \det(g_{ab}) \) with \( g_{ab} = \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \) being the induced metric with \( a, b = 0, 1, 2, 3 \), and \( \alpha \) is a constant with appropriate dimensions.

Under the parametrization \( x^\mu = X^\mu(\tau, \chi, \theta, \phi) = (t(\tau), a(\tau), \chi, \theta, \phi) \) and by assuming that the background spacetime is expressed by \( ds_5 = -dt^2 + da^2 + a^2 d\Omega_3^2 \) with \( d\Omega_3^2 \) being the unit three-sphere metric, the induced metric gets the FRW geometry \( ds_4 = -N^2d\tau^2 + a^2d\Omega_3^2 \). Accordingly, within the mini-superspace framework the action (33) reduces to \( S = 6\pi^2 \int d\tau L(t, a, ȧ, ̇a, ȧ, ̇t, \tau) \) where the Lagrangian is explicitly given by [4]

\[
L(t, a, ȧ, ̇a, ȧ, ̇t, \tau) = \frac{ȧt}{N^3} \left( a ȧ ȧ - a ȧ̇ + N^2 ȧ \right) - Nα^3 ȧ^2. \tag{34}
\]

Here \( ȧ^2 := \Lambda/3\alpha \) is a constant, and \( N := \sqrt{\dot{t}^2 - ȧ^2} \) represents the lapse function that commonly appears when we perform an ADM decomposition of the RT action (33), [4].

Further, the dot stands for derivative with respect to the parameter \( \tau \). Notice that this is a second-order derivative theory where now \( t \) and \( a \) span the configuration space. The Hessian matrix with respect to the second-order derivatives associated to the Lagrangian (34) vanishes identically

\[
H_{bc} := \frac{\partial^2 L}{\partial \dot{x}^b \partial \dot{x}^c} = 0, \quad b, c = 1, 2. \tag{35}
\]
Thus, the extended Lagrangian $L$ for thebrane-like model (34) is given by

$$L = L_1(t, a, \dot{t}, \dot{a}, \tau) + L_2(t, a, \dot{t}, \dot{a}, \dot{\tau}, \ddot{\tau}),$$

where

$$L_1 := -\frac{aa^2}{N} + aN \left(1 - a^2H^2\right),$$

$$L_2 := \frac{d}{d\tau} \left(\frac{a^2\dot{a}}{N}\right).$$

It is quite evident that accelerations $\ddot{t}$ and $\ddot{a}$ enter in the Lagrangian only through the total derivative term (38), and thus the equations of motion describing the system will be of second-order.

3.1. Auxiliary variables

When treating the velocities and accelerations as independent variables the introduction of Lagrange multipliers is required to get a Lagrangian with only first-order derivatives. This allows us to implement the HJ framework developed above. The obvious choice is

$$x^b = \{x^b, \dot{x}^b\} = \{t, a; \dot{t}, \dot{a}\}, \quad b = 1, 2 \quad s = 0, 1;$$

$$v^b = \{\dot{x}^b\} = \{\ddot{t}, \ddot{a}\},$$

where the index $b$ ranges over the number of field variables while the index $s$ keeps track of the order of the derivatives of the field variables. Explicitly,

$$x^1 = t \quad \quad X^1 = \dot{x}^1 =: T \quad \implies \quad \dot{x}^1 - X^1 = 0,$$

$$x^2 = a \quad \quad X^2 = \dot{x}^2 =: A \quad \implies \quad \dot{x}^2 - X^2 = 0,$$

$$v^1 = \dot{x}^1 =: \dot{T} \quad \implies \quad \dot{X}^1 - v^1 = 0,$$

$$v^2 = \dot{x}^2 =: \dot{A} \quad \implies \quad \dot{X}^2 - v^2 = 0.$$

The introduction of (41) into (34), $L \rightarrow L^{(x,v)} = L^{(x,v)}(x^1, x^2, X^1, X^2, v^1, v^2)$, yields

$$L^{(x,v)} = -\frac{a^2T}{N^3}(Av^1 - Tv^2) + \frac{aT^2}{N} - Na^3\ddot{\Lambda}. \quad (42)$$

Henceforth, for the sake of simplicity we shall use the short notation $N := \sqrt{T^2 - A^2}$ for the lapse function. In order to have a well defined Lagrangian, it is mandatory to introduce a set of Lagrange multipliers enforcing the constraints (41), say

$$\pi^s_b := \{\pi^s_b; \Pi^s_b\} = \{\pi_1, \pi_2; \Pi_1, \Pi_2\}.$$

Thus, the extended Lagrangian $L_E = L_E(x^a_s, v^a, \pi^s_a)$ replacing the RT cosmological brane-like model (34) is given by

$$L_E = L^{(x,v)}(x^a_s, v^a) + \pi_1(\dot{x}^1 - X^1) + \pi_2(\dot{x}^2 - X^2) + \Pi_1(\dot{X}^1 - v^1) + \Pi_2(\dot{X}^2 - v^2). \quad (44)$$

From this new viewpoint, we have an enlarged 10-dimensional configuration space $C_{10}$, spanned by the variables $\{x^a_s, v^a, \pi^s_a\}$. Specifically, the extended Lagrangian (44) reads

$$L_E = \pi_1\dot{x}^1 + \pi_2\dot{x}^2 + \Pi_1\dot{X}^1 + \Pi_2\dot{X}^2 - H^{(x,v, \pi)}(x^a_s, v^a, \pi^s_a), \quad (45)$$
where
\[
H^{(x_a, v^a, \pi^a)}(x_a^s, v^a, \pi^a_a) = \pi_1 X^1 + \pi_2 X^2 + \Pi_1 v^1 + \Pi_2 v^2 - L^{(x_a, v)},
\]
and the Lagrangian \( L^{(x_a, v)} \) is defined by (42). We infer that this function is the canonical Hamiltonian associated to this cosmological brane-like model, now described by the extended Lagrangian \( L_E \) but, with the peculiarity that at this level it solely depends on the coordinates \( z^A \) and not on their conjugate momenta.

As we already mentioned, the purpose of extending the configuration space is to bring the Lagrangian into the form provided by (2) in order to implement a HJ analysis of it as a first-order model. To do this, we suitably choose the following ordered coordinates with the aim to make transparent the relation to the first-order Lagrangian in \( C_{10} \),
\[
z^A = \{ v^a, x^a_s, \pi^a \} = \{ T, A, t, a, T, A, \pi_t, \pi_a, \Pi_T, \Pi_A \}, \quad A = 1, 2, \ldots, 10. \quad (47)
\]
We are able to readily identify the values for \( K_A \) from the Lagrangian (45)
\[
K_A = (K_\alpha, K_{A'}) = \{ 0, 0, \pi_1, \pi_2, \Pi_1, \Pi_2, 0, 0, 0, 0 \}, \quad (48)
\]
which allows us to explicitly split the values for the \( A \) index: \( A = (\alpha, A') \), where \( \alpha = 1, 2 \) and \( A' = 3, 4, \ldots, 10 \). In the same spirit, \( V \) turns to \( H^{(x_a, v^a, \pi^a)} \) as given by (46). In passing, from (3), (45), (47) and (48), the solely eqm of this theory reads
\[
\mathcal{E} := \frac{d}{d\tau} \left( \frac{A}{T} \right) + \frac{N^2}{aT} \Theta \Phi = 0, \quad (49)
\]
where
\[
\Theta := T^2 - 3N^2a^2\Lambda^2, \quad (50)
\]
\[
\Phi := 3T^2 - N^2a^2\bar{\Lambda}^2. \quad (51)
\]
As dictated by (12), the HJPDE of our theory are explicitly given by
\[
H'_{\alpha'} = \begin{cases} 
H_0' = p_T + H^{(x_a, v^a, \pi^a)} = 0, \\
H_v^1 = p_T = 0, \\
H_v^2 = p_A = 0, \\
H_t^1 = p_t - \pi_1 = 0, \\
H_a^\alpha = p_a - \pi_2 = 0, \\
H_T^t = p_T - \Pi_1 = 0, \\
H_A^t = p_A - \Pi_2 = 0, \\
H_{\pi_t}^\alpha = p_{\pi_t} = 0, \\
H_{\pi_a}^\alpha = p_{\pi_a} = 0, \\
H_{\Pi_T}^t = p_{\Pi_T} = 0, \\
H_{\Pi_A}^t = p_{\Pi_A} = 0.
\end{cases} \quad (52)
\]
\[
H'_{A'} = \begin{cases} 
H_{10} = 0, \\
H_{40} = 0, \\
H_{20} = 0, \\
H_{30} = 0, \\
H_{40} = 0, \\
H_{20} = 0.
\end{cases} \quad (52)
\]
Note that \( H_0' \) is the only constraint depending on the \( v^a \) coordinates, that is, on the coordinates associated to the second-order derivatives. Further, one may
straightforwardly check that the only non-vanishing PB among the $H'_A$ are
\[
\{H'_3, H'_7\} = \{H'_t, H'_{\pi_1}\} = -1 \quad \{H'_5, H'_6\} = \{H'_T, H'_{\Pi_1}\} = -1, \\
\{H'_4, H'_8\} = \{H'_a, H'_{\pi_2}\} = -1 \quad \{H'_6, H'_{10}\} = \{H'_A, H'_{\Pi_2}\} = -1,
\]
and thus we can construct the matrix (18)
\[
(M_{AB}) = \begin{pmatrix}
0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\
0_{2\times2} & 0_{4\times4} & -I_{4\times4} \\
0_{2\times2} & I_{4\times4} & 0_{4\times4}
\end{pmatrix}
A, B = 1, 2, \ldots, 10. \tag{53}
\]
The rank of $M_{AB}$ being $P = 8$ implies the existence of 2 null-eigenvectors, $\lambda^A_{(\alpha)}$, $\alpha = 1, 2, \ldots$ as well as an invertible submatrix and its associated inverse given by
\[
(M_{A'B'}) = \begin{pmatrix}
0_{4\times4} & -I_{4\times4} \\
I_{4\times4} & 0_{4\times4}
\end{pmatrix}
\quad \text{and} \quad
(M^{-1})^{A'B'} = \begin{pmatrix}
0_{4\times4} & I_{4\times4} \\
-I_{4\times4} & 0_{4\times4}
\end{pmatrix}, \tag{54}
\]
respectively. Clearly, $\det (M_{A'B'}) \neq 0$. Notice that this is a symplectic matrix belonging to the symplectic group $Sp(8)$.

In this HJ point of view the coordinates $z^A$ are separated in two sets
\[
t^a = \{v^a\} = \{T, A\} \quad \text{and} \quad t^A = \{x_s^a, \pi_a^s\} = \{t, a, T, A, \pi_t, \pi_a, \Pi_T, \Pi_A\}, \tag{55}
\]
where $t^A = z^A$ are the true dynamical variables while $t^a$ are the parameters of the theory. Therefore, by expanding (27), and observing from (18) that none of the $K_{A'}$ depend on the $t^{a'}$, the characteristic equations transform into
\[
dz^A = (M^{-1})^{A'B'} \partial H^{(x_s, v, \pi^s)} \partial z^{B'} dt^0, \tag{56}
\]
while the remaining characteristic equation (28) reads
\[
dS = - \left[ H^{(x_s, v, \pi^s)} + H_A (M^{-1})^{A'B'} \partial H^{(x_s, v, \pi^s)} \partial z^{B'} \right] dt^0. \tag{57}
\]
The characteristic equations (56) look like the standard Hamilton equations of motion, as expected, where $(M^{-1})^{A'B'}$ plays the role of a symplectic structure. Even though the characteristic equations (27) should depend, in principle, on the three parameters $t^{a'}$, we realize that for our model the characteristic equation (56) dictates that, at this stage, $t^0$ is indeed the only relevant parameter. Hence, the characteristic equations (27) have been replaced by the simplified set of equations (56). On physical terms, the only real parameter results to be $\tau$ while $T$ and $A$ are arbitrary, that is, they are pure gauge. This completely results as a consequence of the fact that these parameters belong to the second-order nature of our model which, however, may be effectively avoided by considering decomposition (56).

3.2. Integrability analysis

We proceed to analyse the integrability conditions on the Hamiltonians $H'_t$, (52). The evolution of $H'_t$ dictated by (23) with the GPB (24), conveniently denoted at this stage
by \{F,G\} for reasons that will be clarified later, when using \(M_{AB'}\) and its inverse given by (54), leads to the only non-vanishing terms

\[
\begin{align*}
\frac{dH_0'}{dt} &= \left( \Pi_1 - \frac{\partial L(x_s,v)}{\partial v^1} \right) dt^1 + \left( \Pi_2 - \frac{\partial L(x_s,v)}{\partial v^2} \right) dt^2, \\
\frac{dH_{v1}'}{dt} &= - \left( \Pi_1 - \frac{\partial L(x_s,v)}{\partial v^1} \right) dt^0, \\
\frac{dH_{v2}'}{dt} &= - \left( \Pi_2 - \frac{\partial L(x_s,v)}{\partial v^2} \right) dt^0,
\end{align*}
\]

By imposing the conditions \(dH_1' = \{H_1', H_{a'}\} \cdot dt^a = 0\) we readily identify two new Hamiltonian constraints

\[
\begin{align*}
h_1' &= \Pi_1 - \frac{\partial L(x_s,v)}{\partial v^1} = \Pi_1 + a^2 T A N^3 = 0, \\
h_2' &= \Pi_2 - \frac{\partial L(x_s,v)}{\partial v^2} = \Pi_2 - a^2 T^2 N^3 = 0,
\end{align*}
\]

where \(L(x_s,v)\), given by (12), has been used. On the other hand, as discussed in section 2, the orthogonality condition (25) is totally equivalent to the integrability conditions (58) and (59). In this sense, we strategically opt to consider the orthogonality condition in order to continue with the integrability analysis. To achieve this, we must compute the basis of the kernel of the matrix (53) and then, construct suitable zero-modes. Firstly, we find that

\[
\begin{align*}
u_{(1)}^A &= (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0), \\
u_{(2)}^A &= (0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0),
\end{align*}
\]

span a basis for the null space of the matrix \(M_{AB}\). Accordingly, by considering

\[
\lambda_{(1)}^A := X^1 u_{(1)}^A + X^2 u_{(2)}^A,
\]

the internal product between \(\lambda_{(1)}^A\) and \(H_{A} := \partial H_{0}/\partial x^{A}\), guided by (25), yields

\[
\frac{\partial H(x_s,v,\pi^s)}{\partial v^1} X^1 + \frac{\partial H(x_s,v,\pi^s)}{\partial v^2} X^2 = \Pi_1 X^1 + \Pi_2 X^2 - \frac{\partial L(x_s,v)}{\partial v^1} X^1 - \frac{\partial L(x_s,v)}{\partial v^2} X^2 = 0.
\]

Now, from (12) we readily obtain the identity \((\partial L(x_s,v)/\partial v^1) X^1 + (\partial L(x_s,v)/\partial v^2) X^2 = 0\). Thus, when substituting this into (63), one deduces directly

\[
C_1 = \Pi_1 X^1 + \Pi_2 X^2 = 0.
\]

In a like manner, by constructing the zero-mode

\[
\lambda_{(2)}^A := X^2 u_{(1)}^A + X^1 u_{(2)}^A,
\]

the internal product between \(\lambda_{(2)}^A\) and \(H_{A}\), following (25), reads

\[
\frac{\partial H(x_s,v,\pi^s)}{\partial v^1} X^2 + \frac{\partial H(x_s,v,\pi^s)}{\partial v^2} X^1 = \Pi_1 X^2 + \Pi_2 X^1 - \frac{\partial L(x_s,v)}{\partial v^1} X^2 - \frac{\partial L(x_s,v)}{\partial v^2} X^1 = 0.
\]
As before, from (42) it is fairly easy to verify the identity $(\partial L(x,s,v)/\partial v^1)X^2 + (\partial L(x,s,v))/\partial v^2)X^1 = a^2T/N$. Hence, the insertion of this relationship into (66) allows us to deduce

$$C_2 = \Pi_1 X^2 + \Pi_2 X^1 - \frac{a^2T}{N} = 0. \quad (67)$$

The constraints (64) and (67) are nothing but the primary constraints arising from an Ostrogradski-Hamilton treatment of the original second-order Lagrangian function (34). In fact, on physical grounds, the zero-modes $\lambda_A^{(1)}$ and $\lambda_A^{(2)}$ correspond to the velocity and the normal vectors, respectively, associated to the brane-like universe under study [4].

Then, rather than using (58) and (59), we will instead use $C_1$ and $C_2$ as the new Hamiltonian constraints which contain the same information of these constraints but, as we will see below, it results adequate to reproduce the correct analysis for the gauge symmetries of the theory. Also, these constraints are fundamental within the naive quantization program in order to recover the correct Wheeler-De Witt equations in our cosmological setup, as described in detail in [4].

Continuing with the iterative procedure for generating further constraints, we turn to impose $dC_{1,2} = \{C_{1,2}, H'_{\alpha'}\} \cdot dt^{\alpha'} = 0$. According to the functional dependence of (64) and (67) we have that

$$\{C_{1,2}, H'_{\alpha'}\} = \left(M^{-1}\right)^{A'B'} \frac{\partial C_{1,2}}{\partial z^A} \frac{\partial H'_{\alpha'}}{\partial z^{B'}}. \quad (68)$$

This helps to find

$$dC_1 = -\left(\pi_1 X^1 + \pi_2 X^2 + N a^3\bar{\Lambda}^2 - \frac{aT^2}{N}\right) dt^0,$n$$
$$dC_2 = -\left(\pi_1 X^2 + \pi_2 X^1\right) dt^0.$n

In arriving to these expressions we have used that $\{C_{1,2}, H'_{\alpha'}\} = 0$. As a result, we have two new Hamiltonian constraint functions

$$C_3 = \pi_1 X^1 + \pi_2 X^2 + N a^3\bar{\Lambda}^2 - \frac{aT^2}{N} = 0, \quad (69)$$
$$C_4 = \pi_1 X^2 + \pi_2 X^1 = 0. \quad (70)$$

These correspond to the secondary constraints in the Ostrogradski-Hamilton framework. Likewise by imposing $dC_{3,4} = 0$ and guided by (68), when considering the functional dependence of (69) and (70), we deduce that

$$dC_3 = 0, \quad (71)$$
$$dC_4 = \left(\frac{aT^2\Phi}{N^3}\right) E dt^0 = 0, \quad (72)$$

where $E$ is nothing but the equation of motion (49) which should not be considered as a new constraint.
The Hamiltonians $C_i$, with $i = 1, \ldots, 4$, determine a non-involutive set of constraints. Indeed, the GPB among the $C_i$ and $H'_\alpha$ reads

\begin{align}
\{C_1, C_2\}^* &= 0, \\
\{C_1, C_3\}^* &= -C_3, \\
\{C_1, C_4\}^* &= -C_4, \\
\{C_1, H'_\alpha\}^* &= 0, \\
\{C_2, C_3\}^* &= -C_4, \\
\{C_2, C_4\}^* &= -C_3 - \Phi, \\
\{C_3, C_4\}^* &= -\Theta,
\end{align}

where $\Theta := (T/N) \Theta$ and $\Phi := (a/N) \Phi$ with $\Theta$ and $\Phi$ given by (50) and (51). Therefore, we note that under the GPB we do not recover a closed Lie algebra. To ensure integrability of the system it is mandatory to redefine these constraints by constructing suitable combinations. The Hamiltonians defined by

\begin{align}
H'_3 &:= \frac{1}{2} C_1, \\
H'_4 &:= \Theta C_2 - \Phi C_3,
\end{align}

reconstruct the GPB’s as follows

\begin{align}
\{H'_3, H'_4\}^* &= -H'_4, \\
\{C_2, C_4\}^* &= -C_3 - \Phi = -\Phi,
\end{align}

where the last identity only holds in the constraint surface. Accordingly, $H'_3$ and $H'_4$, form an involutive set of Hamiltonians so that they should be considered as generators of the dynamics of the system in addition to $H'_\alpha$. This entails their incorporation, with some related parameters $t^\alpha$, to the fundamental differential to be constructed. On the other hand, $C_2$ and $C_4$ form a non-involutive set of constraints and should be treated on an equal footing to $H'_\alpha$. This fact leads to introduce the matrix $m_{AB} := \{C_A, C_B\}^*$ with $A, B = 2, 4$, necessary to redefine the GPB. Explicitly, this matrix and its inverse are

\begin{align}
(m_{AB}) &= \Phi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
(m^{-1})^{AB} &= \frac{1}{\Phi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\end{align}

respectively. Therefore, it turns out that the right evolution in the reduced phase space is provided by the fundamental differential

\begin{equation}
dF = \{F, H'_\alpha\}^* dt^\alpha + \{F, H'_\underline{\alpha}\}^* dt^\underline{\alpha}, \quad \alpha' = 0, 1, 2; \quad \underline{\alpha} = 3, 4;
\end{equation}

where the final GPB is given by

\begin{equation}
\{F, G\}^* = \{F, G\}^* - \{F, C_A\}^* (m^{-1})^{AB} \{C_B, G\}^*.
\end{equation}

This GPB possesses the usual properties similar to those of the PB. It is crucial to notice that for the HJ approach developed for this geodetic cosmological brane model, the former $\{F, G\}^*$, provided by (24), gets modified so GPB should be used instead. Note that this final GPB was introduced only to remark the role played by the appropriate involutive Hamiltonians of the theory, namely, $H'_3$ and $H'_4$ which, as mentioned before, are the correct generators of the dynamics of the system. By a mere relabeling of the constraints $H'_3 \rightarrow L_0$ and $H'_4 \rightarrow L_1$, it becomes clear from (76) that

\begin{equation}
\{L_n, L_m\}^* = (n - m)L_{n+m}, \quad n, m = 0, 1.
\end{equation}
This represents a Virasoro truncated algebra \[7, 21, 22\]. It should be remarked that this algebra is valid just on the constraint surface. We thus conclude that reparameterization invariance symmetry of the RT cosmological brane-like model is equivalent to a 2-dimensional conformal gravity described by this peculiar type of algebra.

Once we have at our disposal (80), the non-vanishing fundamental generalized brackets of the theory are

\[
\begin{align*}
\{t, T\}_b &= -\frac{N_A^2}{\phi^2}, \\
\{t, A\}_b &= -\frac{N_A T}{\phi^2}, \\
\{t, \pi_t\}_b &= 1, \\
\{t, \pi_a\}_b &= \frac{2T A}{\phi} + \frac{a A^2}{N^2 \phi}, \\
\{t, \Pi_T\}_b &= \frac{N_A A^2}{\phi^2} = \pi_T, \\
\{t, \Pi_A\}_b &= \frac{2N_A T \Pi_T - N_A T a \Pi_A}{\phi}, \\
\{T, \Pi_A\}_b &= -\frac{N_A A}{\phi}, \\
\{A, \Pi_T\}_b &= -\frac{N_A T}{\phi}, \\
\end{align*}
\]

(82)

Therefore, we find that \((t, \pi_t)\) is the unique canonical pair of the theory under the final GPB structure. This fact indicates the presence of only one physical degree of freedom.

### 3.3. Characteristic equations

In order to extract physical information we focus first on the characteristic equations arising from (56). The first set of equations, taking into account (16) and (54), is

\[
\begin{align*}
\frac{dz}{dt} &= X^1 d\tau, \\
\frac{dz^2}{d\tau} &= X^2 d\tau, \\
\frac{dz^3}{d\tau} &= dT = v^1 d\tau, \\
\frac{dz^4}{d\tau} &= dA = v^2 d\tau.
\end{align*}
\]

(83)

These reproduce the definitions given in (11). Similarly, the second set of characteristic equations turns out to be

\[
\begin{align*}
\frac{dz^5}{dt} &= d\pi_1 = 0, \\
\frac{dz^6}{dt} &= d\pi_2 = \frac{\partial L^{(x,v)}}{\partial x^2} dt^0, \\
\frac{dz^7}{dt} &= d\Pi_1 = \left(\frac{\partial L^{(x,v)}}{\partial X^1} - \pi_1\right) dt^0, \\
\frac{dz^8}{dt} &= d\Pi_2 = \left(\frac{\partial L^{(x,v)}}{\partial X^2} - \pi_2\right) dt^0.
\end{align*}
\]

(84)

In order to discuss these results we first note that, from the HJ viewpoint, expressions (84a) and (84d) fix the value of the Lagrange multipliers \(\pi_1\) and \(\pi_2\) through the equations of motion. In passing, the remaining Lagrange multipliers \(\Pi_1\) and \(\Pi_2\) have been fixed by (58) and (59). Explicitly, they are given by

\[
\begin{align*}
\Pi_1 &= \frac{-2a T A}{N^3}, \\
\Pi_2 &= \frac{a A^2}{N^3}.
\end{align*}
\]

(85)

In second place, from (84d), we note that \(\pi_1\) becomes a constant as a consequence of the invariance under reparametrizations of the RT cosmological model. Finally, (84a) is
nothing but the equation of motion governing the evolution of this brane-like universe as it may be easily transformed into the Euler-Lagrange form

\[
\frac{d^2}{dt^2} \frac{\partial L(x_s,v)}{\partial v^2} - \frac{d}{dt} \frac{\partial L(x_s,v)}{\partial X^2} + \frac{\partial L(x_s,v)}{\partial x^2} = 0, \tag{87}
\]

by direct substitution of relations (59) and (84). It should be remarked that this expression is totally equivalent to the eom provided by (49). Incidentally, we also note from (85) and (86) that

\[
\pi_2 = -\left(\frac{A}{T}\right) \pi_1.
\]

On the other hand, when considering evolution along the complete set of parameters, by using (79), we are able to find that the characteristic equations become

\[
\begin{align*}
\frac{dt}{T} & = T \frac{d\tau}{dt^4} - \bar{\Phi} T \frac{dT}{dt^4}, \\
\frac{da}{A} & = A \frac{d\tau}{dt^4} - \bar{\Phi} A \frac{dA}{dt^4},
\end{align*}
\tag{88}
\]

and

\[
\begin{align*}
\frac{d\pi_t}{T} & = \left(\frac{\partial L(x_s,v)}{\partial x^2} - \pi_t\right) d\tau - \frac{1}{2} \Pi_T \frac{d\Pi_T}{dt^4}, \\
\frac{d\pi_a}{A} & = \left(\frac{\partial L(x_s,v)}{\partial x^2} - \pi_a\right) d\tau - \frac{1}{2} \Pi_A \frac{d\Pi_A}{dt^4}. \tag{89}
\end{align*}
\]

From these, we readily observe that the time evolution of the coordinates is in agreement with that expressed by (83-84). The additional contributions come from the evolution along the parameters \( t^3 \) and \( t^4 \), which are related to the transformations of the system at a fixed time and that remain in the reduced phase space; that is, they are associated to the gauge transformations of the theory that will be described in short.

### 3.4. The gauge transformations

Having at our disposal the Hamiltonians, \( H'_\alpha \) and \( H'_\beta \) generating the dynamics in the RT cosmology along the directions of the parameters \( (t^\alpha, t^\beta) \), we are able to construct the gauge generator function as dictated by (31)

\[
G := H'_\alpha \delta t^\alpha + H'_\beta \delta t^\beta \quad \alpha = 1, 2, \quad \beta = 3, 4. \tag{90}
\]

In this sense, when considering (80), the infinitesimal gauge transformations are

\[
\delta_G z^A = \{z^A, G\}^* = \{z^A, H'_\alpha\}^* \delta t^\alpha + \{z^A, H'_\beta\}^* \delta t^\beta. \tag{91}
\]

These transformations leave invariant the action functional (33) with the Lagrangian given by (42). Taking into account the functional dependence of \( H'_\alpha \) we have that \( \{z^A, H'_\alpha\}^* = \delta A^\alpha \), thus for the model under study the gauge transformations (91) become

\[
\begin{align*}
\delta_G t & = \frac{\partial H'_\alpha}{\partial \pi_t} \delta t^\alpha = -\bar{\Phi} T \delta t^4, \\
\delta_G a & = \frac{\partial H'_\alpha}{\partial \pi_a} \delta t^\alpha = -\bar{\Phi} A \delta t^4, \\
\delta_G T & = \frac{\partial H'_\alpha}{\partial \Pi_T} \delta t^\alpha = \frac{1}{2} T \delta t^3 + \bar{\Theta} A \delta t^4, \\
\delta_G A & = \frac{\partial H'_\alpha}{\partial \Pi_A} \delta t^\alpha = \frac{1}{2} A \delta t^3 + \bar{\Theta} T \delta t^4. \tag{92-95}
\end{align*}
\]
It follows that the variation of the Lagrangian (2), taking into account both (46), (48) and (54), can be written in the form

$$\delta L = -(\dot{\pi}_t + \partial H(x_s,v,\pi^x)/\partial t)\delta t - (\dot{\pi}_a + \partial H(x_s,v,\pi^x)/\partial a)\delta a - (\dot{\Pi}_T + \partial H(x_s,v,\pi^x)/\partial T)\delta T - (\dot{\Pi}_A + \partial H(x_s,v,\pi^x)/\partial A)\delta A + (d/d\tau)(\pi_t\delta t + \pi_a\delta a + \Pi_T\delta T + \Pi_A\delta A).$$

(96)

Bearing in mind the independence on the parameter $t$ of the rest of the terms occurring in the model, as well as the fact that $\pi_t$ is a constant, the variation of the Lagrangian becomes

$$\delta L = \left(\frac{\partial L(x_s,v)}{\partial a} - \dot{\pi}_a\right)\delta a + \left(\frac{\partial L(x_s,v)}{\partial T} - \dot{\Pi}_T - \pi_t\right)\delta T + \left(\frac{\partial L(x_s,v)}{\partial A} - \dot{\Pi}_A - \pi_a\right)\delta A + \frac{d}{d\tau}(\pi_t\delta t + \pi_a\delta a + \Pi_T\delta T + \Pi_A\delta A),$$

(96)

where (46) has been considered. On the other hand, by considering the definitions $\epsilon_2^2(\tau) := \Phi \delta t^4$ and $2\epsilon_1 := \delta t^3$, the transformations (92-95) reduces to

$$\delta G_t = -T \epsilon_2,$$

$$\delta G_a = -A \epsilon_2,$$

$$\delta G_T = \epsilon_1 T + \left(\frac{T \Theta}{a\Phi}\right) \epsilon_2 A,$$

$$\delta G_A = \epsilon_1 A + \left(\frac{T \Theta}{a\Phi}\right) \epsilon_2 T.$$

(97-100)

By plugging these transformations in the variation (96), after a lengthy but straightforward computation, one finds that

$$\delta L = -C_3 \epsilon_1 - \left(\frac{T \Theta}{a\Phi}\right) C_4 \epsilon_2 - \left(\frac{aAT^3\Phi}{N^5}\right) \mathcal{E} \epsilon_2 + \frac{d}{d\tau} \left\{2H^3 \epsilon_1 + \frac{N}{a\Phi}H^4 \epsilon_2 + \frac{a}{N\Phi} \left[T^2\Theta - (T^2 - N^2a^2\lambda^2)\Phi\right] \epsilon_2 \right\},$$

(101)

where we recognise the eom $\mathcal{E}$ provided by (49). Therefore, under the variations (97-100) the change induced in the Lagrangian (42) left this invariant whenever $\epsilon_2(\tau)$ vanishing at the extrema located at $\tau = \tau_1$ and $\tau = \tau_2$. To be more specific, the action (33) with the Lagrangian (34) is left invariant under the gauge transformations (97-100) where one must have in mind the following parameter relationship

$$\epsilon_1 = -\lambda(\tau) \epsilon_2 - \epsilon_2,$$

(102)

where $\epsilon_2$ is subject to the conditions $\epsilon_2(\tau_1) = \epsilon_2(\tau_2) = 0$. It follows then that $\epsilon_2$ may be chosen as the independent gauge parameter confirming the existence of only one degree of freedom where $\lambda(\tau)$ is an arbitrary function. Certainly, this expression can be obtained by imposing the standard commutativity requirements provided by the usual variational principles, namely,

$$\frac{d}{d\tau}\delta t = \delta T \quad \text{and} \quad \frac{d}{d\tau}\delta a = \delta A.$$

(103)

We therefore have systematically obtained the gauge symmetries from a purely geometrical viewpoint.
A couple of comments are in order. The gauge variations $\delta_G t$ and $\delta_G a$ reflect the presence of the invariance under reparametrizations of the model as one can notice from (97) and (98),

$$\delta_G t = -\epsilon_2 T \quad \text{and} \quad \delta_G a = -\epsilon_2 A.$$  \hfill (104)

These preserve the action (33) with $L$ provided by (34) under the transformation $\tau \to \tau + \epsilon_2 (\tau)$. Moreover, the gauge variations $\delta_G T$ and $\delta_G A$ reflect the presence of an inverse Lorentz-like transformation in the velocity sector of the coordinates $z^A$, provided by $T$ and $A$, as we can observe from (99) and (100)

$$\delta_G T = \epsilon_1 (T + \tilde{\epsilon}_2 A) \quad \text{and} \quad \delta_G A = \epsilon_1 (A + \tilde{\epsilon}_2 T),$$  \hfill (105)

with $\tilde{\epsilon}_2 := (T \Theta/a \Phi) (\epsilon_2/\epsilon_1)$.

4. On the Hamilton principal function

By virtue of the integrability analysis developed for the RT cosmological model, the conditions to obtain the Hamilton principal function, $S = S[t^\alpha', z^{A'}(t')] = S[\tau, z^{A'}(\tau)]$, are fulfilled. First of all, we focus on the second term on the RHS occurring in (57)

$$H_A' (M^{-1})^{A'B'} \frac{\partial H(x^s, v, \pi^s)}{\partial z^{B'}} = - \left( \pi_1 X^1 + \pi_2 X^2 + \Pi_1 v^1 + \Pi_2 v^2 \right),$$  \hfill (106)

where (46), (48) and (54) have been considered. When inserting this into (57) we obtain

$$dS = \left[ \pi_1 X^1 + \pi_2 X^2 + \Pi_1 v^1 + \Pi_2 v^2 - H(x^s, v, \pi^s) \right] dt^0.$$  \hfill (107)

By integrating and using once again the relationship (46), we get

$$S = \int L(x^s, v) \, dt,$$  \hfill (108)

which is nothing but the action (33) with $L(x^s, v)$ provided by (32). We therefore realize that the action $S$ is a solution of the HJPDE, as expected. On the other hand, from (15) we have

$$dS = K_A' \, dt'^A - H(x^s, v, \pi^s) \, dt^0,$$  \hfill (109)

where (7) has been considered. Now, within the minisuperspace geodetic brane scenario, by inserting the $K_A$ terms provided by (48), and integrating we obtain

$$S = \int \left[ \pi_1 \, dx^1 + \pi_2 \, dx^2 + \Pi_1 \, dX^1 + \Pi_2 \, dX^2 - H(x^s, v, \pi^s) \right] dt^0.$$  \hfill (110)

Bearing in mind that the variables $z^{A'}$ only depend on $t^0 = \tau$, we find that

$$S = \pi_1 \, x^1 + \pi_2 \, x^2 + \Pi_1 \, X^1 + \Pi_2 \, X^2 - \int H(x^s, v, \pi^s) \, dt^0,$$

$$= \pi_1 \, t + \pi_0 \, a,$$  \hfill (111)

where the constraints (64) and the fact that $H(x^s, v, \pi^s) = 0$ have been used. Indeed, the previous identity coincides exactly with the constraint $C_3 = 0$, when one rewrite the Lagrangian function $L(x^s, v)$, (12), in terms of the quantities $\Pi_T$ and $\Pi_A$, provided by (85).
and (84). One must bear in mind that this fact implies that the evolution is carried out in the reduced phase space. In order to extract physical information regarding the semi-classical quantization of the model, we prefer to write the function $S$ in terms of the constant $\Omega$, (84a). As noticed above, $\pi_t = -\Omega$ and $\pi_a = \frac{A}{T}\Omega$, so that

$$S(z^A', \tau) = -\Omega t + \frac{A}{T}\Omega a.$$  

(112)

As it is well known, the HJ equation (10) may be viewed as the classical limit of quantum field equations [23]. In particular, the HJ equation turns out to be the classical limit of the Schrödinger equation of non-relativistic quantum mechanics, and hence the interest in computing the Hamilton principal function. In such approach, the complete wave function is computed by considering the ansatz $\Psi(z^A', \tau) = e^{\frac{i}{\hbar}S(z^A', \tau)}\psi(z^A', \tau)$. Within the minisuperspace cosmological brane scenario, the important quantities are the external time $t$ and the scale factor $a$, as one can observe from (34) whereas the velocities $T$ and $A$ are considered as functions of $\tau$. Additionally, in the same context, the fact that $\pi_t = -\Omega$ arises from the feature that the theory is independent of $t$ at the classical level. In that sense, for the case under study, from (112) we note that the wave function has the ordinary time dependence $e^{-i\frac{\Omega}{\hbar}t}$, as expected. Hence, in a semi-classical approximation, we have that the brane-like wave functions acquire the form

$$\Psi(t, a; \tau) = \psi(a; \tau)e^{i\left[\frac{(A/T)\Omega}{\hbar}a - \frac{A}{T}\Omega t\right]} =: \psi(a; \tau)e^{i(k_a a - \omega t)}$$  

(113)

where $k_a := (A\Omega/T)/\hbar$ and $\omega := \Omega/\hbar$. This represents an outgoing wave whenever we assume that $\Omega > 0$. This result has been obtained from different perspectives mainly based in the naive quantization supported by the Dirac-Bergmann theory for classical constrained systems [4, 24]. However, we must emphasize here that within the HJ formalism this result naturally emerged as a consequence of the integrability conditions that allowed us to obtain, in a straightforward manner, the Hamilton principal function for geodetic brane cosmology.

5. Concluding remarks

In this paper we have derived a Hamilton-Jacobi framework for geodetic brane cosmology. Being a singular theory with a linear dependence in the accelerations of the brane, by judiciously enlarging the configuration space we were able to properly treat this type of cosmology within the HJ framework. In consequence, we obtained a set of HJ-PDE instead of a single HJ equation, as it occurs for the case of regular theories. When identifying the complete set of involutive Hamiltonians that play the role of the distinct generators of the evolution of the system, and following the integrability conditions, we realize that it was compulsory to properly modify the original GPB, (24) by a slightly modified one, (80), in order to obtain the right dynamical evolution of the system in the reduced phase space. As discussed above, this modified GPB was consequently introduced in order to consider the appropriate involutive Hamiltonians of the theory that were related to the correct generators of the dynamics. As expected, the
non-involutive Hamilton constraint functions were eliminated as dynamical generators by this modified GPB. The adequate set of involutive constraints, \((H'_\alpha, H''_\alpha)\), allowed us to become aware of the integrability of the HJ equations. We also note that there is a subset of the Hamiltonian generators that under the modified GPB structure closes as a truncated Virasoro algebra which, as discussed in the literature \([7, 21, 22]\), may codify the information of a 2-dimensional conformal symmetry that results inherent to the model of our interest here. This last issue is still under active study. Further, the HJ scheme provided a purely geometrical approach that perfectly adjusts to construct the gauge symmetries of the theory. Indeed, we were able to straightforwardly built the generator of gauge transformations in terms of the generators of the dynamical evolution along each of the directions of the independent parameters of the theory. Finally, by obtaining the generic form of the Hamiltonian principal function for the RT cosmology, we have outlined in a simpler manner the generalities of the wave function within the context of the semi-classical approximation. This last result may be compared with the equivalent results commonly found in the literature where the wave function is encountered by invoking barely geometrical principles, as those involved within our present prescription. We believe that the extensions of the HJ scheme developed here are quite natural to be implemented within the generic context of the so-called affine in acceleration theories \([19]\). Work in this direction will be reported elsewhere.

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