Relations for certain symmetric norms and anti-norms before and after partial trace

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Changes of some unitarily invariant norms and anti-norms under the operation of partial trace are examined. The norms considered form a two-parametric family, including both the Ky Fan and Schatten norms as particular cases. The obtained results concern operators acting on the tensor product of two finite-dimensional Hilbert spaces. For any such operator, we obtain lower bounds on norms of its partial trace in terms of the corresponding dimensionality and norms of this operator. Similar inequalities, but in the opposite direction, are obtained for certain anti-norms of positive matrices. Applications of the results to generalized quantum entropies are discussed.

Keywords: Ky Fan norm, Schatten norm, symmetric anti-norms, singular values, Rényi entropy, Tsallis entropy

I. INTRODUCTION

In quantum theory, the state of a subsystem of composite quantum system is described by a reduced density operator. So, the operation of partial trace is common in studying properties of multipartite quantum systems. Hence we are interested in how used quantitative measures may be changed by this operation. Many of frequently used operator. So, the operation of partial trace is common in studying properties of multipartite quantum systems. Hence

II. PRELIMINARIES

Let \( \mathcal{L}(\mathcal{H}) \) be the space of linear operators on \( m \)-dimensional Hilbert space \( \mathcal{H} \). By \( \mathcal{L}_+(\mathcal{H}) \) and \( \mathcal{L}_{++}(\mathcal{H}) \), we respectively denote the set of positive semidefinite operators and the set of strictly positive ones. A unitarily invariant norm, in signs \( \| \cdot \| \), is a norm on square matrices that enjoys \( \| Q U V \| = \| U Q V \| \) for all \( Q \in \mathcal{L}(\mathcal{H}) \) and for unitary \( U, V \) [7]. For any \( Q \in \mathcal{L}(\mathcal{H}) \), we define \( | Q | \in \mathcal{L}_+(\mathcal{H}) \) as the positive square root of \( Q^\dagger Q \). The singular values \( \sigma_j(Q) \) are defined as the eigenvalues of \( | Q | [7] \). The Schatten and Ky Fan norms both form especially important families of unitarily invariant norms. For each real number \( p \geq 1 \), the Schatten \( p \)-norm is defined as [3, 22]

\[
\| Q \|_p := \left( \text{Tr}(| Q |^p) \right)^{1/p} = \left( \sum_{j=1}^m \sigma_j(Q)^p \right)^{1/p}.
\] (2.1)

This family includes the trace norm \( \| Q \|_1 = \text{Tr}| Q | \) for \( p = 1 \), the Frobenius norm \( \| Q \|_2 = \sqrt{\text{Tr}(Q^\dagger Q)} \) for \( p = 2 \), and the spectral norm \( \| Q \|_\infty = \max\{\sigma_j(Q) : 1 \leq j \leq m\} \) for \( p = \infty \). Note that the right-hand side of (2.1) is actually the ordinary \( p \)-norm of the vector \( \sigma(Q) \). Here the vector \( p \)-norm is defined for \( p \in [1; \infty] \) as

\[
\| x \|_p := \left( \sum_{j=1}^m | x_j |^p \right)^{1/p}.
\] (2.2)

The function (2.2) is an example of symmetric gauge function. For each integer \( k = 1, \ldots, m \), the Ky Fan \( k \)-norm is defined as

\[
\| Q \|_{(k)} := \sum_{j=1}^k \sigma_j(Q)^{1/2},
\] (2.3)
where the non-increasing order $\sigma_1(Q)^\downarrow \geq \sigma_2(Q)^\downarrow \geq \ldots \geq \sigma_m(Q)^\downarrow$ is assumed. The $k$-norm is related to the symmetric gauge function

$$G_{(k)}(x) := \sum_{j=1}^{k} |x_j|^k. \quad (2.4)$$

Note that the family (2.3) includes both the spectral norm $||Q||(1) \equiv ||Q||_\infty$ and trace one $||Q||(m) \equiv ||Q||_1$. It is known that every unitarily invariant norm can be defined via the corresponding symmetric gauge function (for details, see sect. IV.2 in [7] or 7.4 in [22]). Let $G(x)$ be a symmetric gauge function. Then the function

$$G^{(p)}(x) := \left(G(|x|^p)\right)^{1/p}, \quad (2.5)$$
defined for $p \geq 1$, is symmetric gauge as well [7]. Applying this to (2.4), we obtain the corresponding symmetric gauge function and unitarily invariant norm defined for integer $k = 1, \ldots, m$ and real $p \geq 1$ by

$$G_{(k)}^{(p)}(x) := \left(\sum_{j=1}^{k} (|x_j|^i)^p\right)^{1/p}, \quad (2.6)$$

and

$$||Q||_{(p)}^{(k)} := G_{(k)}^{(p)}(\sigma(Q)) = \left\{ \sum_{j=1}^{k} (\sigma_j(Q)^i)^p \right\}^{1/p}. \quad (2.7)$$

This two-parametric family includes both the Ky Fan and Schatten norms as particular cases. In the following, we will study changes of unitarily invariant norms of the form (2.7) under the operation of partial trace.

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces of dimensionality $\dim(\mathcal{H}_A) = m$ and $\dim(\mathcal{H}_B) = n$. By $\{e_i\}$ and $\{f_j\}$, where $1 \leq i \leq m$ and $1 \leq j \leq n$, we respectively denote some orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$. Consider operators of a kind $|a\rangle \langle a'| \otimes |b\rangle \langle b'|$ with any vectors $|a\rangle', |a\rangle'' \in \mathcal{H}_A$ and $|b\rangle', |b\rangle'' \in \mathcal{H}_B$. For such operators, the partial trace over $\mathcal{H}_A$ and the partial trace over $\mathcal{H}_B$ are defined as

$$\text{Tr}_A \left(|a\rangle \langle a'| \otimes |b\rangle \langle b'|\right) := \langle a''|a'\rangle \langle b'|b''\rangle, \quad \text{Tr}_B \left(|a\rangle \langle a'| \otimes |b\rangle \langle b'|\right) := \langle b''|b'\rangle \langle a'|a''\rangle. \quad (2.8)$$

The definition is completed by requiring that the partial trace be linear in its input [32]. Such a definition is rather physicist-friendly. For description of partial trace as a linear map, see, e.g., sect. 5.4 in [11]. For arbitrary operator $Q \in \mathcal{L} (\mathcal{H}_A \otimes \mathcal{H}_B)$, we can write the expression

$$\tilde{Q} = \sum_{i,j=1}^{m} |e_i\rangle \langle e_j| \otimes Q_{ij}. \quad (2.9)$$

With respect to chosen bases, this operator is represented as the $m$-by-$m$ block matrix $[Q_{ij}]$, in which the submatrices $Q_{ij}$ are of size $n \times n$. Such representation assumes the Kronecker product. Then the partial traces of any matrix $\tilde{Q}$ are expressed as

$$\text{Tr}_A (\tilde{Q}) = \sum_{j=1}^{m} Q_{jj}, \quad \text{Tr}_B (\tilde{Q}) = [[\text{Tr}(Q_{ij})]]. \quad (2.10)$$

That is, tracing-out $\mathcal{H}_A$ leads to the sum of $m$ diagonal submatrices; tracing-out $\mathcal{H}_B$ gives the $m$-by-$m$ matrix such that each submatrix $Q_{ij}$ has been replaced with its trace.

Let $X_B$ and $Z_B$ be generalized Pauli operators on $\mathcal{H}_B$. These operators act as $X_B |f_j\rangle = |f_{j+1}\rangle$ and $Z_B |f_j\rangle = \exp(i2\pi j/m) |f_j\rangle$ [24]. The following relations can immediately be checked:

$$\frac{1}{n} \sum_{j=1}^{n} (1 \otimes Z_B^j) \tilde{Q} (1 \otimes Z_B^{-j}) = [[D_{ij}]], \quad \frac{1}{n} \sum_{j=1}^{n} (1 \otimes X_B^j) [[D_{ij}]] (1 \otimes X_B^{-j}) = [[\text{Tr}(D_{ij})]] \otimes 1_B. \quad (2.11)$$

Here the diagonal matrix $D_{ij}$ is obtained from $Q_{ij}$ by replacing all its off-diagonal entries with zeros. Since $\text{Tr}(D_{ij}) = \text{Tr}(Q_{ij})$, the set of $n^2$ unitary matrices $X_B^l Z_B^l$, where $l, j = 1, \ldots, n$, enjoy the formula

$$\frac{1}{n} \sum_{l=1}^{n} \sum_{j=1}^{n} (1 \otimes X_B^l Z_B^l) \tilde{Q} (1 \otimes X_B^l Z_B^l)^\dagger = \text{Tr}_B (\tilde{Q}) \otimes 1_B. \quad (2.13)$$

With relevant modifications, such relation can be recast for $1_A \otimes \text{Tr}_A (\tilde{Q})$. In sect. 7.2 of [11], one presents convexity properties of some functionals defined in terms of partial trace. The formula (2.13) is one of basic tools in proving
the monotonicity under partial trace for a family of quantum relative entropies [24]. Hence the monotonicity under trace-preserving completely positive maps can be derived. For a summary of this issue, see the review [21], in which many general results are posed for quantum f-divergences. These quantities are a quantum analog of Csiszar’s f-divergences [12], and are a special case of Petz’ quasi-entropies [23]. The monotonicity of the standard relative entropy is equivalent to the strong subadditivity of the von Neumann entropy. This celebrated result was first proved in [28] on the base of Lieb’s concavity theorem [27]. Some extensions of Lieb’s theorem are obtained in [3, 18]. The strong subadditivity is widely used in quantum information [32]. So, it is applied in proving that the Holevo quantity is not larger than an exchange entropy [41]. Studies of the strong subadditivity were closely related to properties of the Wigner–Yanase entropy. The subadditivity was assumed to be true for the Wigner–Yanase entropy [31], until Hansen gave a counterexample [19]. Conditions for subadditivity of the Wigner-Yanase entropy is still the subject of active research [9]. We will use [21,13] in studying changes of unitarily invariant norms (2.7) under partial trace.

III. MAIN RESULTS FOR NORMS

In this section, we prove the desired relation between corresponding norms of operator \( \tilde{Q} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \) itself and one of its partial traces. The following statement takes place.

Proposition 1 Let \( \tilde{Q} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B), \) \( \dim(\mathcal{H}_A) = m, \) \( \dim(\mathcal{H}_B) = n, \) and \( Q_A = \text{Tr}_B(\tilde{Q}). \) For all \( k = 1, \ldots, m \) and \( p \geq 1, \) there holds

\[
\|Q_A\|^{(p)}_{(k)} \leq n^{(p-1)/p} \|\tilde{Q}\|^{(p)}_{(kn)} .
\]

Proof. Applying the triangle inequality and the homogeneity of norms to (2.7), we first observe that

\[
\|Q_A \otimes 1_B\|^{(p)}_{(kn)} \leq 1/n \sum_{l=1}^{n} \sum_{j=1}^{n} \left\| (\mathbb{1}_A \otimes X_B^j Z_B^j) \tilde{Q} (\mathbb{1}_A \otimes X_B^l Z_B^l) \right\|^{(p)}_{(kn)} = n \|\tilde{Q}\|^{(p)}_{(kn)} ,
\]

since the norms (2.7) are all unitarily invariant. It is easy to check that \( |Q_A \otimes 1_B| = |Q_A| \otimes 1_B \) for any \( Q_A \in \mathcal{L}(\mathcal{H}_A). \) Repeating the spectrum of \( |Q_A| \) \( n \) times, one then obtains the spectrum of \( |Q_A \otimes 1_B| \). Hence we write

\[
\|Q_A \otimes 1_B\|^{(p)}_{(kn)} = \left\{ n \sum_{j=1}^{k} (\sigma_j(Q_A))^{1/p} \right\}^{1/p} = n^{1/p} \|Q_A\|^{(p)}_{(k)} .
\]

Combining (3.2) and (3.3) gives the inequality

\[
n^{1/p} \|Q_A\|^{(p)}_{(k)} \leq n \|\tilde{Q}\|^{(p)}_{(kn)} ,
\]

which is equivalent to the claim (3.1). ■

The inequality (3.1) provides an upper bound on the norms (2.7) of partial trace \( Q_A = \text{Tr}_B(\tilde{Q}) \) in terms of similar norms of \( \tilde{Q} \) and traced-out dimensionality. This inequality is sharp in the following sense. It is saturated for all \( k = 1, \ldots, m \) and \( p \geq 1, \) when the operator \( \tilde{Q} \) is a multiple of \( R_A \otimes 1_B \) with some \( R_A \in \mathcal{L}(\mathcal{H}_A). \) We then write

\[
\tilde{Q} = c R_A \otimes 1_B , \quad Q_A = \text{Tr}_B(\tilde{Q}) = c n R_A ,
\]

where \( c \) denotes a complex number. Substituting these expressions into (3.2), we have actually arrived at the equality. Setting \( k = m \) in (3.1), we obtain relations between the Schatten norms in the form

\[
\|Q_A\|_p \leq n^{(p-1)/p} \|\tilde{Q}\|_p ,
\]

where \( p \geq 1. \) Taking \( p = 1, p = 2, \) and \( p \to \infty, \) we further obtain the inequalities

\[
\|Q_A\|_1 \leq \|\tilde{Q}\|_1 , \quad \|Q_A\|_2 \leq \sqrt{n} \|\tilde{Q}\|_2 , \quad \|Q_A\|_{\infty} \leq n \|\tilde{Q}\|_{\infty} ,
\]

for the trace, Frobenius, and spectral norms, respectively. The formulas (3.7) were obtained in [29] on base of a certain integral representation for partial trace. Following from Shur’s lemma [14], this representation uses a group integration with respect to the normalized Haar measure. So we have arrived at (3.7) with use of the elementary representation (2.13). In addition, the results of [29] do not hold for the Ky Fan norms, except for the trace and
spectral ones. Recent advances in quantum information have lead to renewed interest in matrix inequalities and simple proofs for them. For instance, the author of [6] presented new tracial inequality for matrix absolute values and, further, a short proof for the tracial analog of Hölder’s inequality. So, the above reasons can be regarded as a short and rather simple proof for (3.7). Choosing \( p = 1 \) in (3.1), we have the inequality in terms of the Ky Fan norms, namely

\[
\|Q_A\|_{(k)} \leq \|\tilde{Q}\|_{(mn)}. \tag{3.8}
\]

For \( k = m \), the relation (3.8) leads to the first inequality of (3.7) with the trace norm. For \( k = 1 \), we obtain

\[
\|Q_A\|_{\infty} \leq \|\tilde{Q}\|_{(n)}. \tag{3.9}
\]

This is stronger than the third inequality of (3.7) (except when multiplicity of the largest singular value of \( \tilde{Q} \) is not less than \( \dim(\mathcal{H}) = n \)). Note that the relation (3.8) was previously obtained in [36] with use of the Ky Fan maximum principle [13]. We shall now prove another statement, which can be applied in combination with (3.1).

**Proposition 2** Let \( R \in \mathcal{L}(\mathcal{H}) \) and \( \dim(\mathcal{H}) = m \). For all \( k = 1, \ldots, m \) and \( p, q \geq 1 \), there holds

\[
\|R\|_{(k)}^{(p)} \leq k^{(q - 1)/(pq)} \|R\|_{(k)}^{(pq)} \tag{3.10}
\]

with equality if and only if multiplicity of the largest singular value of \( R \) is not less than \( k \).

**Proof.** In line with the Hölder inequality for vector norms of \( k \)-tuples \( x \) and \( y \), we have [20]

\[
|\langle x, y \rangle| \leq \|x\|_q \|y\|_r. \tag{3.11}
\]

Here the conjugate indices \( q \) and \( r \) obey \( 1/q + 1/r = 1 \). Let us put \( x_j = (\sigma_j(R)^i)^p \) and \( y_j = 1 \) for all \( 1 \leq j \leq k \). It then follows from (3.11) that

\[
\sum_{j=1}^k (\sigma_j(R)^i)^p \leq \left\{ \sum_{j=1}^k (\sigma_j(R)^i)^{pq} \right\}^{1/q}k^{1 - 1/q}. \tag{3.12}
\]

Raising both the sides to the power \( 1/p \), we finally obtain (3.10). The equality in (3.11) takes place if and only if the \( k \)-tuples \( x \) and \( y \) are linearly related (see, e.g., theorem 14 in [20]). Hence the equality in (3.12) is equivalent to that \( \sigma_1(R)^i = \sigma_2(R)^i = \ldots = \sigma_k(R)^i \), i.e. multiplicity of the largest singular value of \( R \) is not less than \( k \). \( \blacksquare \)

Proposition 2 is an extension of the statement proved for Schatten norms in Appendix of the paper [39]. Combining (3.10) and (3.12) leads to the inequality

\[
\|Q_A\|_{(k)}^{(p)} \leq [k^{q - 1}n^{pq - 1}]^{1/(pq)} \|\tilde{Q}\|_{(mn)}^{(pq)}. \tag{3.13}
\]

in which \( k = 1, \ldots, m \) and \( p, q \geq 1 \). In some respects, these results are complementary to the inequalities given in section IV of [4]. Upper bounds on unitarily invariant norms of a traceless Hermitian operator in terms of the trace norm are provided therein. The relation (3.10) holds for all operators and gives an upper bound on the norm (2.7) in terms of similar norms with the same \( k \) and larger real parameter \( pq \). Relations with traceless Hermitian operators are especially useful in deriving continuity estimates of Fannes type. Recall that Fannes’ inequality gives an upper bound on a potential change of the von Neumann entropy for an alteration of its argument [16]. Continuity bounds of such a kind have been obtained for the quantum conditional entropy [2] as well as for the standard relative entropy [4, 5] and its \( q \)-extension [38]. Although inequalities of Fannes type are usually posed in terms of the trace norm distance, their reformulations with other norms may be of interest [39]. The above results characterize relations between some unitarily invariant norms of an operator and one of its partial traces. In the next section, we obtain relations of such a kind for symmetric anti-norms of positive operators.

### IV. RELATIONS FOR SYMMETRIC ANTI-NORMS

In the recent article [8], Bourin and Hiai examined symmetric anti-norms of positive operators. They form a class of functionals containing the right-hand side of (2.1) for \( p \in (0, 1) \) and, with strictly positive matrices, for \( p < 0 \). For arbitrary \( Q \in \mathcal{L}_+(\mathcal{H}) \), we consider a functional

\[
Q \mapsto \|Q\|_r, \tag{4.1}
\]
taking values on \([0; \infty)\). If this functional enjoys the homogeneity, the symmetry \[\|Q\| = \|UQU^\dagger\|\] for all unitary \(U\), and the superadditivity

\[\|Q + R\| \geq \|Q\| + \|R\|,\]  

we call it a symmetric anti-norm \([8]\). In general, anti-norms may vanish for non-zero operators. An important class of symmetric anti-norms is formed by the Ky Fan ones. For integer \(k = 1, \ldots, m\), where \(m = \dim(\mathcal{H})\), the Ky Fan \(k\)-anti-norm is defined as the sum of the \(k\) smallest eigenvalues of \(Q \in \mathcal{L}_+(\mathcal{H})\), namely

\[\|Q\|_{\{k\}} := \sum_{j=1}^k \lambda_j(Q)^\dagger = \text{Tr}(Q) - \|Q\|_{\{m-k\}}.\]  

Here the non-decreasing order of positive eigenvalues, \(\lambda_1(Q)^\dagger \leq \lambda_2(Q)^\dagger \leq \ldots \leq \lambda_m(Q)^\dagger\), is assumed. Combining (4.3) with \(\text{Tr}(Q + R) = \text{Tr}(Q) + \text{Tr}(R)\) and the triangle inequality for the norms

\[\|Q + R\|_{\{m-k\}} \leq \|Q\|_{\{m-k\}} + \|R\|_{\{m-k\}},\]  

we obtain the superadditive property, that is

\[\|Q + R\|_{\{k\}} \geq \|Q\|_{\{k\}} + \|R\|_{\{k\}}.\]  

Note that the formula (4.4) directly follows from the Ky Fan maximum principle \([13]\). In paper \([14]\), Uhlmann examined so-called partial fidelities. So, the \(k\)-th partial fidelity of density operators \(\rho\) and \(\omega\) is actually the Ky Fan \((m - k)\)-anti-norm of \([\sqrt{p\rho}, \sqrt{\omega}]\). For \(p \in (0; 1)\), the Schatten \(p\)-anti-norm is defined as \([8]\)

\[\|Q\|_p := \left(\text{Tr}(Q^p)\right)^{1/p} = \left(\sum_{j=1}^m \lambda_j(Q)^p\right)^{1/p}.\]

When \(Q \in \mathcal{L}_+(\mathcal{H})\), i.e. the \(Q\) is also invertible, the right-hand side of (4.6) determines a symmetric anti-norm with negative exponent \(p < 0\) \([8]\). Using given anti-norm (4.1), for any \(p \in (0; 1)\) we can construct another anti-norm

\[Q \mapsto \|Q\|_p^{1/p}.\]  

This result, posed in proposition 3.7 of \([8]\), is based on the fact that the function \(t \mapsto t^p\) with \(p \in (0; 1)\) is matrix concave on positive matrices (see, e.g., theorem V.1.9 in \([13]\)). Applying (4.7) to (4.3) leads to anti-norms

\[\|Q\|_p^{(p)} := \left(\|Q^p\|_{\{p\}}\right)^{1/p} = \left\{\sum_{j=1}^k \left(\lambda_j(Q)^\dagger\right)^p\right\}^{1/p}.\]  

In a certain sense, the two-parametric family (4.8) is an anti-norm counterpart of the norm family (2.7). Other connections between symmetric norms and anti-norms are discussed in \([8]\). We shall now characterize changes of the anti-norms (4.8) under the operation of partial trace.

**Proposition 3** Let \(\tilde{Q} \in \mathcal{L}_+(\mathcal{H}_A \otimes \mathcal{H}_B)\), \(\dim(\mathcal{H}_A) = m\), \(\dim(\mathcal{H}_B) = n\), and \(Q_A = \text{Tr}_B(\tilde{Q})\). For all \(k = 1, \ldots, m\) and \(p \in (0; 1)\), there holds

\[\|Q_A\|_{\{k\}}^{(p)} \geq n^{(p-1)/p} \|\tilde{Q}\|_{\{kn\}}^{(p)} .\]

Assuming \(\tilde{Q} \in \mathcal{L}_+(\mathcal{H}_A \otimes \mathcal{H}_B)\) and \(p < 0\), the Schatten \(p\)-anti-norm enjoys

\[\|Q_A\|_p \geq n^{(p-1)/p} \|\tilde{Q}\|_p .\]

**Proof.** For positive \(\tilde{Q}\), each of \(n^2\) summands in the left-hand side of (2.13) is also positive. Using the subadditivity inequality (4.2), the homogeneity, and the symmetry of anti-norms, we then obtain

\[\|Q_A \otimes I_B\|_{\{kn\}}^{(p)} \geq \frac{1}{n} \sum_{l=1}^n \sum_{j=1}^n \|\mathbb{I}_A \otimes X_B^l Z_B^j\| \tilde{Q} (\mathbb{I}_A \otimes X_B^l Z_B^j)^\dagger\|_{\{kn\}}^{(p)} = n \|\tilde{Q}\|_{\{kn\}}^{(p)}.\]  

Repeating the spectrum of \(Q_A\) by \(n\) times, one then obtains the spectrum of \(Q_A \otimes I_B\). So we can write

\[\|Q_A \otimes I_B\|_{\{kn\}}^{(p)} = \left\{n \sum_{j=1}^k \left(\lambda_j(Q_A)^\dagger\right)^p\right\}^{1/p} = n^{1/p} \|Q_A\|_{\{k\}}^{(p)}.\]
Combining (4.11) and (4.12) finally leads to (4.9). For strictly positive \( \bar{Q} \), each summand in the left-hand side of (4.8) is strictly positive as well. By a parallel argument, for \( p < 0 \) we have the relations

\[
\|A^{1/p} \|_p = \| A \otimes 1_B \|_p \geq n \| B \|_p ,
\]

whence the claim (4.10) is provided. \( \square \)

The inequality (4.9) provides a lower bound on the anti-norms (4.8) of partial trace \( A = \text{Tr} B(\bar{Q}) \) in terms of similar anti-norms of \( \bar{Q} \) and traced-out dimensionality. This relation is sharp in the sense that it is always saturated in the case (3.5) with any \( R_A \in L_+(\mathcal{H}_A) \). Setting \( k = m \) in (4.9), we obtain the inequality (4.10) for \( p \in (0; 1] \) and positive \( \bar{Q} \). The further choice \( p = 1 \) actually gives the trace norm. It must be stressed that the trace norm is justly an anti-norm on positive matrices. Indeed, the subadditivity inequality (4.2) is fulfilled here with equality. For the trace norm, the relations (3.1) and (4.9) give \( \text{Tr}(A) \leq \text{Tr}(\bar{Q}) \) and \( \text{Tr}(A) \geq \text{Tr}(\bar{Q}) \), respectively. In fact, the matrices \( A = \text{Tr} B(\bar{Q}) \) and \( \bar{Q} \) have the same trace. Choosing \( p = 1 \) in (4.9), we have the relation in terms of the Ky Fan anti-norms, namely

\[
\|A\|_{(k)} \geq \| \bar{Q} \|_{(kn)} .
\]

In view of \( \text{Tr}(A) = \text{Tr}(\bar{Q}) \) and (4.3), the inequality (4.14) is actually equivalent to the inequality (3.8), in which the \( k \) is replaced with \( (m-k) \). We will use the above results for norms and anti-norms in studying relations between the unified entropies of a composite quantum system and one of its subsystems.

V. INEQUALITIES FOR UNIFIED ENTROPIES

In this section, we derive some inequalities between the \((\alpha, s)\)-entropies of density operator \( \rho \in L_+(\mathcal{H}_A \otimes \mathcal{H}_B) \) and one of its partial traces \( \rho_A = \text{Tr}_B(\rho) \) and \( \rho_B = \text{Tr}_A(\rho) \). We first recall the definitions of used entropic measures. Let \( \mathcal{H} \) be \( m \)-dimensional Hilbert space, i.e. \( \dim(\mathcal{H}) = m \). For \( \alpha > 0 \neq 1 \), the Rényi entropy of density operator \( \rho \in L_+(\mathcal{H}) \) is defined as

\[
R_\alpha(\rho) := \frac{1}{1-\alpha} \ln[\text{Tr}(\rho^\alpha)] = \frac{\alpha}{1-\alpha} \ln \|\rho\|_\alpha .
\]

Here the quantity \( \|\rho\|_\alpha = [\text{Tr}(\rho^\alpha)]^{1/\alpha} \) is an anti-norm for \( \alpha \in (0; 1) \) and a norm for \( \alpha \in (1; \infty) \). The quantity (5.1) is a quantum counterpart of the classical entropy introduced by Rényi. The maximal value \( \ln m \) of (5.1) is reached with the completely mixed state \( \rho_0 = 1/m \) on \( \mathcal{H} \). Another extension of the standard entropy is the nonextensive entropy, also called the Tsallis entropy. The concept of nonextensive entropy is widely used in many many topics of physics and other sciences. In quantum regime, this entropy is defined as

\[
H_\alpha(\rho) := \frac{1}{1-\alpha} \text{Tr}(\rho^\alpha - \rho) = -\text{Tr}(\rho^\alpha \ln \rho) ,
\]

where \( \alpha > 0 \neq 1 \) and the \( \alpha \)-logarithmic function is \( \ln_\alpha x = (x^{1-\alpha} - 1)/(1-\alpha) \). The maximal value \( \ln_\alpha m \) of (5.2) is reached with \( \rho_0 = 1/m \) as well. Estimates of Fannes type were derived for the Tsallis entropy itself \( 17, 47 \) and its partial sums \( 32 \). Such estimates are required in studying stability properties of various entropies. The stability issue was inspired by Lesche, who showed that the Rényi entropy is not stable in the thermodynamic limit for all entropy \( \alpha > 0 \neq 1 \). In the limit \( \alpha \to 1 \), the definitions (5.1) and (5.2) both lead to the von Neumann entropy \( S(\rho) = -\text{Tr}(\rho \ln \rho) \). General references on the von Neumann entropy are the review \( 10 \) and the comprehensive book \( 33 \). As it is shown in \( 23 \), the Rényi and Tsallis entropies can both be treated as particular cases of the following entropic functional. For \( \alpha > 0 \neq 1 \) and \( s \neq 0 \), the unified \((\alpha, s)\)-entropy is defined by

\[
E_\alpha^{(s)}(\rho) := \frac{1}{(1-\alpha)s} \left\{ [\text{Tr}(\rho^\alpha)]^s - 1 \right\} = \frac{\|\rho\|_\alpha^s - 1}{(1-\alpha)s} ,
\]

The formula (5.3) gives the Tsallis entropy (5.2) for \( s = 1 \) and the Rényi entropy (5.1) in the limit \( s \to 0 \). It is considered in \( 31, 32 \) that the quantum \((\alpha, s)\)-entropy (5.3) enjoy many properties similarly to the von Neumann entropy. In particular, uniform estimates obtained in a wide parametric range \( 37 \). Entropies of the form (5.3) have been used for a unification of monogamy inequalities in multi-qubit systems \( 22 \). In the finite-dimensional case, the \((\alpha, s)\)-entropy is bounded from above for \( \alpha > 0 \) and all real \( s \), namely

\[
E_\alpha^{(s)}(\rho_\alpha) = \frac{n^{(1-\alpha)s} - 1}{(1-\alpha)s} .
\]
Classical entropies are obtained by replacing the traces with the proper sums over a probability distribution. In quantum theory, various entropies of classical distributions are used for measuring uncertainties in quantum measurements [12]. Entropic uncertainty relations have found applications in quantum cryptography [13]. In this section, we deal only with the quantum entropies. Our first relation with entropies is posed as follows.

**Proposition 4** Let \( \tilde{\rho} \in L_+(\mathcal{H}_A \otimes \mathcal{H}_B) \) be a density matrix, \( \dim(\mathcal{H}_B) = n \), and \( \rho_A = \text{Tr}_B(\tilde{\rho}) \). For all \( \alpha > 0 \neq 1 \) and \( s \neq 0 \), there holds

\[
E_\alpha^{(s)}(\tilde{\rho}) \leq n^{(1-\alpha)s} E_\alpha^{(s)}(\rho_A) + \frac{1}{s} \ln_\alpha(n^s) .
\]  

**Proof.** Raising (4.10) and (3.6), with \( \alpha \) instead of \( p \), to the power \( \alpha \), we have arrived at the relations

\[
\|\tilde{\rho}\|_\alpha \begin{cases} \leq, \quad \alpha \in (0;1) \\ \geq, \quad \alpha \in (1;\infty) \end{cases} n^{1-\alpha} \|\rho_A\|_\alpha^{\alpha s} .
\]

For all \( s \neq 0 \), the function \( y \mapsto (1-\alpha)^{-1}s^{-1}y^s \) is increasing for \( \alpha \in (0;1) \) and decreasing for \( \alpha \in (1;\infty) \) (its derivative is positive for the former and negative for the latter). Applying this with (5.6), we obtain

\[
\frac{1}{(1-\alpha)s} \|\tilde{\rho}\|_\alpha^{\alpha s} \leq \frac{n^{(1-\alpha)s}}{(1-\alpha)s} \|\rho_A\|_\alpha^{\alpha s} .
\]

Substituting this into the formula for \( E_\alpha^{(s)}(\tilde{\rho}) \) gives

\[
E_\alpha^{(s)}(\tilde{\rho}) \leq \frac{1}{(1-\alpha)s} \left\{ n^{(1-\alpha)s}\|\rho_A\|_\alpha^{\alpha s} - 1 \right\} + n^{(1-\alpha)s} - 1 .
\]

The last expression is actually the right-hand side of (5.5). ■

An important particular case of (5.5) takes place for \( s = 1 \). For all \( \alpha > 0 \), the Tsallis \( \alpha \)-entropies of joint density operator and its partial trace satisfy

\[
H_\alpha(\tilde{\rho}) \leq n^{1-\alpha} H_\alpha(\rho_A) + \ln_\alpha n .
\]

Here the case of von Neumann entropy holds in the limit \( \alpha \to 1 \). The obtained inequalities (5.5), (5.12), and (5.9), provide an upper bound on the entropies of joint density matrix in terms of the reduced density and traced-out dimensionality. If \( \dim(\mathcal{H}_A) = m \), then \( \dim(\mathcal{H}_A \otimes \mathcal{H}_B) = mn \) and for all \( \alpha > 0 \) we have

\[
E_\alpha^{(s)}(\tilde{\rho}) \leq \frac{(mn)^{(1-\alpha)s} - 1}{(1-\alpha)s} ,
\]

\[
H_\alpha(\tilde{\rho}) \leq \ln_\alpha(mn) = n^{1-\alpha} \ln_\alpha m + \ln_\alpha n .
\]

When \( \rho_A = \mathbb{1}_A/m \), i.e. the state of subsystem \( A \) is completely mixed, the right-hand sides of (5.5) and (5.9) concur with the right-hand sides of (5.10) and (5.11), respectively. So the latter inequalities are covered by the former as a very particular case. For the Rényi entropies, the following statement takes place.

**Proposition 5** Let \( \tilde{\rho} \in L_+(\mathcal{H}_A \otimes \mathcal{H}_B) \) be a density matrix, \( \dim(\mathcal{H}_B) = n \), and \( \rho_A = \text{Tr}_B(\tilde{\rho}) \). For all \( \alpha > 0 \), there holds

\[
R_\alpha(\tilde{\rho}) \leq R_\alpha(\rho_A) + \ln n .
\]

**Proof.** Taking the logarithm of (5.10), one obtains

\[
\alpha \ln \|\tilde{\rho}\|_\alpha \begin{cases} \leq, \quad \alpha \in (0;1) \\ \geq, \quad \alpha \in (1;\infty) \end{cases} \alpha \ln \|\rho_A\|_\alpha + (1-\alpha) \ln n .
\]

We have \( 1 - \alpha > 0 \) for \( \alpha \in (0;1) \) and \( 1 - \alpha < 0 \) for \( \alpha \in (1;\infty) \). Dividing (5.13) by \( (1-\alpha) \) then provides (5.12) in view of (5.11). The case of von Neumann entropy is resolved in the limit \( \alpha \to 1 \). ■

The relation (5.12) is consistent with the relation (5.5), in which the limit \( s \to 0 \) is taken. We have preferred a direct proof, since the case of Rényi entropies is especially important. For all \( \alpha > 0 \), we can write

\[
R_\alpha(\tilde{\rho}) \leq \ln(mn) = \ln m + \ln n .
\]
The right-hand sides of (5.12) concur with the right-hand sides of (5.14), when \( \rho_A = \mathbb{1}_A/m \). So, the inequality (5.14) is a particular case of (5.12). The relations (5.5), (5.9), and (5.12) are of general form without any specifications. In special cases, more detailed bounds could be done. Due to one of the Lindblad inequalities [30], the entropy exchange is bounded from above by the sum of the input and output von Neumann entropies of the principal quantum system. The entropy exchange can be posed as the output entropy of the bipartite system composed of the principal and reference ones [32]. Here the input bipartite state is pure and the reference system itself is not altered. Applications of the Lindblad inequalities in studying additivity properties of quantum channels are discussed in [42]. Extensions of Lindblad’s inequalities with some of the unified entropies have been obtained [39]. In general situation, the bounds (5.5), (5.9), and (5.12) could be used for estimating entropies of a composite quantum system.
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