On a sum involving
the number of distinct prime factors function
related to the integer part function

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Abstract: In this paper, we obtain asymptotic formula on the sum $\sum_{n \leq x} \omega \left( \left\lfloor \frac{x}{n} \right\rfloor \right)$, where $\omega (n)$ denote the number of distinct prime divisors of $n$ and $\lfloor t \rfloor$ denotes the integer part of $t$.

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1 Introduction

Let, as usual, for an integer $n \geq 1$, $\omega (n) : = \sum_{\substack{p|n \leq x}} 1$ denote the the number of distinct prime divisors of $n$. Many authors investigated the properties of this function. In 1917, G. H. Hardy and S. Ramanujan [4] proved the classical result,

$$\sum_{n \leq x} \omega (n) = x \log \log x + Bx + O \left( \frac{x}{\log x} \right),$$

such that $B = \gamma + \sum_{p} (\log (1 - 1/p) + 1/p)$ and $\gamma$ is Euler’s constant. The result (1) was generalized in 1970 [6] and in 1976 [3] by the following formula

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\[ \sum_{k \leq n} \omega(k) = n \log \log n + Bn + \sum_{j=1}^{m} \frac{n a_j}{(\log n)^j} + O\left( \frac{n}{(\log n)^{m+1}} \right), \]  
for all integer \( m \geq 1 \), with \( a_j = -\int_{1}^{\infty} \frac{\{t\}}{t^2} (\log t)^{j-1} \, dt \).

In [5], we find another interesting result
\[ \sum_{n \leq x} \omega(d(n)) = cx + O\left( x^{1/2} \log^5 x \right), \]  
such that \( d(n) \) is the number of divisors of \( n \) and \( c > 0 \) it’s a constant. It is easy to show that the following relationship is correct for all real \( x \geq 1 \)
\[ \sum_{n \leq x} d(n) = \sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor, \]
where \( \lfloor t \rfloor \) denotes the integer part of any \( t \in \mathbb{R} \) (see [2, example 4.18]).

The possible question is what are the similarities between the mean values of the functions \( \omega(d(n)) \) and \( \omega\left( \left\lfloor \frac{x}{n} \right\rfloor \right) \)? Since, the sum is on a less dense set than the first, it is obvious that the result will be at least with an error term lower, than what is given in the formula (3).

2 Main result

In this section, we establish a result concerning the mean value of the function \( \omega\left( \left\lfloor \frac{x}{n} \right\rfloor \right) \). More precisely, we prove the following theorem:

**Theorem 1.** For all \( x \geq 1 \) large enough, we have
\[ \sum_{n \leq x} \omega\left( \left\lfloor \frac{x}{n} \right\rfloor \right) = Cx + O\left( x^{1/2} \log x \right). \]
Such that \( C \approx 0.5918 \cdots \).

The proof of this result is based on the following lemmas:

**Lemma 1.** Let \( x \geq 1 \) be real number. For any arithmetic function \( f \) we have
\[ \sum_{n \leq x} f\left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n \leq x} f(n) \left( \left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right). \]

**Proof.** If we pose \( \left\lfloor \frac{x}{n} \right\rfloor = k \), then we have the following equivalents:
\[ \left\lfloor \frac{x}{n} \right\rfloor = k \iff \frac{x}{n} - 1 < k \leq \frac{x}{n} \iff \frac{x}{k+1} < n \leq x/k. \]
Using that, we get
\[
\sum_{n \leq x} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n \leq x} f(k) \\
= \sum_{k \leq x} f(k) \sum_{n \leq x/k} 1 - \sum_{k \leq x} f(k) \sum_{n \leq x/(k+1)} 1 \\
= \sum_{k \leq x} f(k) \left( \left\lfloor \frac{x}{k} \right\rfloor - \left\lfloor \frac{x}{k+1} \right\rfloor \right). \tag{5}
\]

**Lemma 2.** Let \( n \in \mathbb{Z}_{\geq 0} \) and \( \delta > 0 \) real. For all real \( x \geq 1 \), we have
\[
\int_{x}^{+\infty} e^{-\delta t} (\log t)^n dt \leq \frac{n!}{\delta} e^{-\delta x} \left( \log x + \frac{1}{\delta x} \right)^n.
\]

**Proof.** We put \( I_n = \int_{x}^{+\infty} e^{-\delta t} (\log t)^n dt \) and we use integration by parts, so
\[
I_n = \frac{e^{-\delta x} (\log x)^n}{\delta} + \frac{n}{\delta} \int_{x}^{+\infty} (\log t)^{n-1} \frac{e^{-\delta x}}{te^{\delta t}} dt \leq \frac{e^{-\delta x}}{\delta} (\log x)^n + \frac{n}{\delta x} I_{n-1}.
\]
And by recurrence, we get
\[
I_n \leq \frac{e^{-\delta x}}{\delta} \sum_{k=0}^{n} k! \binom{n}{k} \left( \frac{\log x}{\delta x} \right)^{n-k} \leq \frac{n!}{\delta} e^{-\delta x} \left( \log x + \frac{1}{\delta x} \right)^n. \tag{6}
\]

**Lemma 3.** Let \( x \) be sufficiently large, there is a constant \( C > 0 \) such that
\[
\sum_{n \leq x} \frac{\omega(n)}{n(n+1)} = C + O \left( \frac{\log \log x}{x} \right), \tag{4}
\]
such that \( C \approx 0.5918 \ldots \).

**Proof.** Let \( x \geq 2 \), we have
\[
\sum_{n \leq x} \frac{\omega(n)}{n(n+1)} = \sum_{n \geq 1} \frac{\omega(n)}{n(n+1)} - \sum_{n > x} \frac{\omega(n)}{n(n+1)}. \tag{5}
\]
Now the well-known trivial bound of \( \omega(n) \), applied to the first sum on the right-hand side of (5), implies that
\[
\sum_{n \geq 1} \frac{\omega(n)}{n(n+1)} \leq \frac{1}{\log 2} \sum_{n \geq 1} \frac{\log n}{n(n+1)}.
\]

We deduce that the series \( \sum_{n \geq 1} \frac{\omega(n)}{n(n+1)} \) is convergent, and with a numerical calculation, we find
\[
C = \sum_{n \geq 1} \frac{\omega(n)}{n(n+1)} \approx 0.5918 \ldots. 
\tag{6}
\]

In the last sum in (5), we put \( g(t) = \frac{1}{t(t+1)} \), and by partial summation, we have
\[
\sum_{n > x} \frac{\omega(n)}{n(n+1)} = -g(x) \sum_{n \leq x} \omega(n) - \int_x^{+\infty} g'(t) \left( \sum_{x < n \leq t} \omega(n) \right) dt
\]
\[
= -\frac{1}{x(x+1)} \sum_{n \leq x} \omega(n) + \int_x^{+\infty} \frac{2t - 1}{t^2(t+1)^2} \left( \sum_{x < n \leq t} \omega(n) \right) dt.
\]

And from (1) we obtain,
\[
\left| \sum_{n > x} \frac{\omega(n)}{n(n+1)} \right| \leq \frac{\log \log x}{x} + B x + O \left( \frac{1}{x \log x} \right) + O \left( \int_x^{+\infty} \frac{\log \log t}{t^2} dt \right). \tag{7}
\]

So, by Lemma 2 \((n = 1, \, \delta = 1)\), and using a variable change, we find
\[
\int_x^{+\infty} \frac{\log \log t}{t^2} dt \leq \frac{\log \log x}{x} + \frac{1}{x \log x}
\]
\[
= O \left( \frac{\log \log x}{x} \right).
\]

Finally, using the last estimate in (7), we get
\[
\sum_{n > x} \frac{\omega(n)}{n(n+1)} = O \left( \frac{\log \log x}{x} \right), \tag{8}
\]
and collecting (8), (6) and (5), we get the following desired result.

**Lemma 4.** For all \( x \geq 1 \), we have
\[
\sum_{n \geq 0} \left\{ \frac{x}{n+1} \right\} - \left\{ \frac{x}{n+2} \right\} = \frac{2}{\pi} \zeta(3/2) x^{1/2} + O \left( x^{2/5} \right),
\]
where \{t\} denotes the fractional part of any \( t \in \mathbb{R} \).

**Proof.** The proof of this result is found in the paper [1]. \( \square \)

**Proof of the theorem.** For all \( x \geq 1 \), by Lemma 1, we have
\[
\sum_{n \leq x} \omega \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n \leq x} \omega(n) \left( \left\lfloor \frac{x}{n} \right\rfloor - \frac{x}{n+1} \right)
\]
\[
= x \sum_{n \leq x} \frac{\omega(n)}{n(n+1)} + \sum_{n \leq x} \omega(n) \left( \left\{ \frac{x}{n+1} \right\} - \left\{ \frac{x}{n} \right\} \right),
\]

on the other hand, by trivial bound of $\omega(n)$ and Lemma 4, we have
\[
\left| \sum_{n \leq x} \omega(n) \left( \left\{ \frac{x}{n} \right\} - \left\{ \frac{x}{n+1} \right\} \right) \right| \leq \sum_{n \leq x} \omega(n) \left| \left\{ \frac{x}{n} \right\} - \left\{ \frac{x}{n+1} \right\} \right|
\leq \frac{\log x}{\log 2} \sum_{n \geq 0} \left| \left\{ \frac{x}{n+1} \right\} - \left\{ \frac{x}{n+2} \right\} \right|
\leq \frac{\log x}{\log 2} \left( \frac{2}{\pi} \zeta(3/2) x^{1/2} + O(x^{2/5}) \right)
\leq \frac{2 \zeta(3/2)}{\pi} \log x \log 2 + O(x^{2/5} \log x).
\]
Therefore,
\[
\sum_{n \leq x} \omega(n) \left( \left\{ \frac{x}{n} \right\} - \left\{ \frac{x}{n+1} \right\} \right) = O \left( x^{1/2} \log x \right).
\]
Finally, by Lemma 3, we obtain
\[
\sum_{n \leq x} \omega \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = Cx + O \left( \log \log x \right) + O \left( x^{1/2} \log x \right)
\leq Cx + O \left( x^{1/2} \log x \right).
\]

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