Multiple Complex Symbol Golden Code

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ABSTRACT The Golden code is a full-rate full-diversity (FRFD) space-time block code (STBC). The conventional Golden code takes four complex symbols and generates two pairs of Golden codewords. Each pair of Golden codewords contains two complex symbols. In this paper, a new type of Golden code is proposed, where we extend the two complex symbols in each pair of Golden codewords into multiple complex symbols. The scheme hereinafter referred to as the multiple complex symbol Golden code (MCS-Golden code) preserves the FRFD property but achieves a diversity order of $2^n N_r$ compared to the conventional Golden code, where $N_r$ is the number of receive antennas, $n$, $n \geq 1$, is an integer, and $2^n$ is the number of multiple complex symbols in the MCS-Golden code system. An equivalent model of the MCS-Golden code is constructed, then used to derive a closed-form bound on the average bit error probability (ABEP) for the MCS-Golden code system. We further propose a low complexity detection scheme, sphere decoding with sorted detection subset (SD-SDS) for the MCS-Golden code. Finally, we discuss the detection complexity for the proposed SD-SDS. Both simulation and theoretical results show that the MCS-Golden code significantly improves error performance compared to the conventional Golden code. For example, at an average bit error rate of $3 \times 10^{-6}$, a four complex symbol Golden code system with three receive antennas achieves approximately 2.5 dB signal-to-noise ratio (SNR) gain compared to the conventional Golden code.

INDEX TERMS Golden code, golden codeword, maximum likelihood detection, multiple complex symbol golden code, sphere decoding, sphere decoding with sorted detection subset.

I. INTRODUCTION

There is a ubiquitous demand for increased data rates and reliability in next-generation wireless communication systems. On this note, multiple-input multiple-output (MIMO) techniques can provide either multiplexing gain and/or diversity order and is one of the key techniques for 5G and 6G wireless communication systems. From theory, there exists a trade-off between transmission rate and communication reliability in terms of the diversity order-multiplexing gain in MIMO systems [1]. Either multiplexing gain and/or diversity order achieved in MIMO systems depends on the encoding scheme employed in the MIMO system.

The Alamouti space-time block code (STBC) is a MIMO system with two transmit antennas [2]. The encoding of the Alamouti STBC is to construct an orthogonal codeword matrix. The orthogonal codeword matrix in the Alamouti STBC is able to decouple the two transmitted symbols, which in-turn allows simple linear maximum-likelihood (ML) detection in quasi-static frequency-flat fading channels. Thus, the Alamouti STBC has a very low complexity detection.

However, the Alamouti STBC only achieves full-diversity and no multiplexing gain [2]. More specifically, the Alamouti scheme is only a full-diversity half-rate STBC, where the code rate is defined as the number of transmitted symbols per antenna per transmission time slot.

The conventional Golden code also represents a MIMO system with two transmit antennas [3], [4]. However, unlike the Alamouti codeword matrix, the Golden codeword matrix is not orthogonal. The encoding of the Golden code involves the construction of two pairs of Golden codewords. Each Golden codeword is a combination of two complex input symbols and the two Golden codewords in a pair are transmitted in different time slots. Compared to the Alamouti STBC scheme, the Golden code not only achieves full-multiplexing gain, but also achieves full-diversity. Thus the Golden code is a full-rate full-diversity (FRFD) STBC.

Meanwhile, by analyzing the signal constellation of Golden codewords, [5] found that all Golden codewords are actually rotated. The rotated angles represent the exact angles, which can be derived based on the product Euclidean distance (ED) criteria for signal space diversity (SSD) systems [6], [7]. Motivated by the rotated signal constellation of the Golden code, [5] further proposed a new
type of Golden code, component-interleaved Golden code (CI-Golden code), which not only achieves FRFD, but also achieves SSD. The achieved SSD in the CI-Golden code can be regarded as a complex input symbol is transmitted in four different time slots.

Motivated by the work of [5], and the encoding of the Golden code, in this paper, a new type of Golden code, multiple complex symbol Golden code (MCS-Golden code) is proposed. Note, in this paper, 2CS-Golden code denotes the conventional Golden code, while KCS-Golden code denotes a K complex symbol Golden code. The proposed MCS-Golden code not only achieves full-rate, but also achieves multiple diversity order compared to the conventional Golden code. A closed-form bound on the average bit error probability (ABEP) for the MCS-Golden code system is further derived in this paper.

The Golden code has an important application in the IEEE 802.16e WiMAX standard [4]. However, a serious disadvantage of the Golden code is the extremely high detection complexity of the maximum likelihood (ML) detection. The detection complexity of the ML detection is proportional to $O(M^4)$, where $M$ is the modulation order.

The high detection complexity may limit the applications of the Golden code. Meanwhile, several researchers have worked on low complexity detection for the Golden code. Sinnokrot and Barry [4] firstly proved that the Golden code is fast-decodable, and consequently proposed a fast ML detection scheme for the code. Furthermore, [4] presented an efficient implementation of the ML detector with a worst-case complexity proportional to $O(M^{2.5})$. Based on [4], an efficient decoding technique based on the dimensionality reduction of the search tree in sphere decoding was also proposed in [8]. The worst-case complexity of the proposed scheme in [8], is $O(M^{1.5})$. However, the error performance suffers a 1 dB signal-to-noise ratio (SNR) loss compared to the optimal decoding. Based on the structure of the Golden codeword matrix, a fast essentially ML detection scheme was proposed in [9]. The fast essentially ML detection scheme partitions four complex-valued symbols into two pairs of symbols. Given one pair of symbols, the likelihood maximization function can be easily solved. The detection complexity of the fast essentially ML algorithm is $O(M^2)$. However, the fast essentially ML detection algorithm [9] is only applicable for low-order modulation. For high-order modulation, $M \geq 16$, the detection complexity remains impractically high. Recently, two low-complexity detection schemes, fast essentially ML with detection subset (FEML-DS) and sphere decoding with detection subset (SD-DS), were proposed for the conventional Golden code [10]. The complexity of the proposed fast essentially ML with detection subset is only $O(2 \times 4^2)$ and $O(2 \times 4.5^2)$ for $M$-ary quadrature amplitude modulation (MQAM) with $M = 16$ and $M = 64$, respectively. The average cardinality of the signal set used in SD-DS is reduced from $16^2$ and $64^2$ to $4^2$ and $4.5^2$ for 16QAM and 64QAM, respectively. Very recently, the concept of the FEML-DS and SD-DS have been applied in the CI-Golden code [5].

Based on the above, low complexity detection is also important for the proposed MCS-Golden code. Suppose there are $2^n$ multiple complex symbols in a vector of MCS-Golden codewords (MCSGCs), where $n, n \geq 1$, is an integer. If we apply the above FEML-DS into the MCS-Golden code, the detection complexity is proportional to $O(2 \times 4^{2n} - 1)$ and $O(2 \times 4.5^{2n} - 1)$ for 16QAM and 64QAM, respectively. Again, if we apply the above SD-DS into the MCS-Golden code, the average cardinality of the signal set used in SD-DS is $4^{2n-1}$ and $4.5^{2n-1}$ for 16QAM and 64QAM, respectively. Obviously, either the detection complexity of the FEML-DS or the average cardinality of the signal set used in SD-DS exponentially increases as the number of multiple complex symbols in each vector of MCSGCs increases. We also know that the SD-DS has low detection complexity at high SNR for a given average cardinality of the signal set [5]. Based on these observations, we propose a modified SD-DS scheme, sphere decoding with sorted detection subset (SD-SDS) to further reduce detection complexity for the proposed MCS-Golden code. The detection complexity of the SD-SDS is also discussed.

The remainder of the paper is organized as follows: In Section II, the system model, which includes the Golden code and the MCS-Golden code, is presented. In Section III, we present the derivation of the ABEP for the MCS-Golden code system. The sphere decoding with sorted detection subset is described in Section IV followed by the analysis of computational complexity for the proposed detection scheme in Section V. In Section VI, the simulation results are demonstrated. Finally, the paper is concluded in Section VII.

Notation: Bold letters are used to denote vectors and matrices. $[\cdot]^T$, $(\cdot)^H$, $\cdot$, $\|\cdot\|$ and $\|\cdot\|_F$ represent the transpose, Hermitian, Euclidean and Frobenius norm operations, respectively. $D(\cdot)$ is the constellation demodulator function. $(\cdot)^{-1}$ is the inverse. $E[\cdot]$ is the expectation operation. $j$ is a complex number.

II. SYSTEM MODEL

The key component of the proposed MCS-Golden code is the conventional Golden code. In this section, we briefly present the concepts of the Golden code, and then describe the proposed MCS-Golden code in detail.

A. THE GOLDEN CODE

Consider an $N_t \times N_r$ Golden code system, where $N_r$ and $N_t$ are the number of receive and transmit antennas, $N_r = 2$ and $N_r \geq N_t$ [3], [4]. Let $u_{0,i}^j$ be the input of the Golden code encoder, $i \in [1 : 2], j \in [1 : N_t]$, $E[|u_{0,i}^j|^2] = 1$ and $u_{0,i}^j \in \Omega_M$, where $\Omega_M$ is the signal set of $M$-ary quadrature amplitude modulation (MQAM) or $M$-ary phase shift keyed (MPSK).

In this paper, we only consider MQAM. The Golden encoder takes the four complex-valued symbols $u_{0,i}^j$ and generates
four super-symbols, \( u_{1,1}^1 = \frac{1}{\sqrt{3}} \alpha (u_{0,1}^1 + u_{0,2}^1 \theta) \), \( u_{1,2}^1 = \frac{1}{\sqrt{3}} \tilde{\alpha} (u_{0,1}^1 + u_{0,2}^1 \tilde{\theta}) \), \( u_{1,1}^2 = \frac{1}{\sqrt{3}} \alpha (u_{0,1}^2 + u_{0,2}^2 \theta) \) and \( u_{1,2}^2 = \frac{1}{\sqrt{3}} \tilde{\alpha} (u_{0,1}^2 + u_{0,2}^2 \tilde{\theta}) \), where \( \theta = \frac{1+i}{\sqrt{2}} \), \( \tilde{\theta} = 1 - \theta \), \( \alpha = 1 + j \tilde{\theta} \), and \( \tilde{\alpha} = 1 + j (1 - \tilde{\theta}) \). Let \( u_{1,j}^i \in \Omega_G \), where \( \Omega_G \) is the signal set of \( u_{1,j}^i \). Then Appendix A in [10], shows that \( u_{1,i}^1 \in \Omega_G \), \( u_{1,1}^1 \in \Omega_G \) and \( u_{1,2}^2 \in \Omega_G \). In this paper, we refer to these four super-symbols as the Golden codewords. The Golden codewords form two pairs \( \{ \frac{1}{\sqrt{3}} \alpha (u_{0,1}^1 + u_{0,2}^1 \theta), \frac{1}{\sqrt{3}} \tilde{\alpha} (u_{0,1}^1 + u_{0,2}^1 \tilde{\theta}) \} \) and \( \{ \frac{1}{\sqrt{3}} \alpha (u_{0,1}^2 + u_{0,2}^2 \theta), \frac{1}{\sqrt{3}} \tilde{\alpha} (u_{0,1}^2 + u_{0,2}^2 \tilde{\theta}) \} \). Each pair of Golden codewords is a combination of two input symbols and conveys the same information. In this paper, we further refer to the conventional Golden code as two complex symbol Golden code (2CS-Golden code).

The four super-symbols form the Golden codeword matrix which is given by [3]:

\[
U = \begin{bmatrix}
  u_{1,1}^1 & u_{1,2}^1 \\
  u_{1,1}^2 & u_{1,2}^2
\end{bmatrix}.
\]  

The 2CS-Golden code uses two time slots to transmit these four Golden codewords. The received signal may be written as:

\[
y_1 = h_{1,1} u_{1,1}^1 + h_{1,2} u_{1,2}^1 + w_1, \quad (2.1)
\]

\[
y_2 = h_{2,1} u_{1,1}^2 + h_{2,2} u_{1,2}^2 + w_2, \quad (2.2)
\]

where \( y_i \in \mathbb{C}^{N_x \times 1} \) is the \( i \)-th received signal vector. \( h_{l,i} \in \mathbb{C}^{N_x \times 1}, i \in [1 : 2], l \in [1 \quad 2] \), is the channel gain vector. The channels \( h_{l,i} \) between the transmitter and receiver are assumed as Rayleigh frequency-flat fast fading, i.e. the channels remain constant during one time slot and take on independent values in another time slot. \( w_i \in \mathbb{C}^{N_x \times 1} \) is the additive white Gaussian noise (AWGN) vector. The entries of \( h_{l,i} \) and \( w_i \) are independent and identically distributed (i.i.d.) complex Gaussian random variables (RVs) with distribution \( CN(0, 1) \) and \( CN(0, \frac{1}{\rho}) \), respectively. \( \rho \) is the average SNR at each receive antenna.

From (2.1) and (2.2), it is easily seen that each transmit antenna transmits two input symbols in two time slots. Hence, 2CS-Golden code is a full-rate code. Further, it is easily seen that each transmitted input symbol experiences two different fadings in two time slots. Therefore, 2CS-Golden code also achieves full-diversity order \( 2N_f \).

In the next subsection, we extend the two complex symbols in each pair of Golden codewords into multiple complex symbols in each vector of Golden codewords, which is named as multiple complex symbol Golden code (MCS-Golden code). The proposed MCS-Golden code is also a full-rate code. However, the proposed MCS-Golden code achieves diversity order \( 2^n N_f \) compared to the 2CS-Golden code, where \( n \) is a positive integer, \( n \geq 1 \) and \( 2^n \) is the number of complex symbols in each MCSGC. Note, for convenience, in this paper, we only use the pair of Golden codewords, \( \{ \frac{1}{\sqrt{3}} \alpha (u_{0,1}^1 + u_{0,2}^1 \theta), \frac{1}{\sqrt{3}} \tilde{\alpha} (u_{0,1}^1 + u_{0,2}^1 \tilde{\theta}) \} \) to construct the MCS-Golden code.

### B. Multiple Complex Symbol Golden Code (MCS-Golden Code)

Consider an \( N_f \times N_f \) MCS-Golden code system, where \( N_f \) and \( N_t \) are the numbers of receive and transmit antennas, with \( N_f = 2 \) and \( N_t \geq N_f \). Let the input information bit stream be \( \mathbf{b}_i^t = [b_{i,1}^t b_{i,2}^t \cdots b_{i,n}^t] \), \( t = \log_2 M, l \in [1 \quad N_t], i \in [1 \quad 2^l] \), where \( M \) is the modulation order and \( n \) is a positive integer, \( n \geq 1 \). Bit stream \( \mathbf{b}_i^t \) is fed into a mapper which maps \( n \) input bits onto constellation points from the signal set \( \Omega_M \) in the Argand plane, and yields symbol \( u_{i,1}^t, u_{i,0}^t \in \Omega_M \). It is also assumed that \( E\{|u_{i,1}^t|^2\} = 1 \). Based on the encoding of the conventional Golden code, the encoder of the MCS-Golden code iteratively encodes \( n \) times. For the \( k \)-th, \( k \in [1 \quad n] \) encoding, the outputs are given by:

\[
u_{k,2m-1} = \frac{1}{\sqrt{5}} \alpha (u_{k-1,m}^t + u_{k-1,m+2^n-1}^t), \quad (3.1)
\]

\[
u_{k,2m} = \frac{1}{\sqrt{5}} \tilde{\alpha} (u_{k-1,m}^t + u_{k-1,m+2^n-1}^t), \quad (3.2)
\]

where \( m \in [1 \quad 2^{n-1}] \).

Since \( E\{|u_{i,1}^t|^2\} = 1 \), we have \( E\{|u_{i,1}^t|^2\} = 1, k \in [1 \quad n] \) and \( i \in [1 \quad 2^n] \).

As an example of (3.1) and (3.2), the encoding for 8 complex symbol Golden code (8CS-Golden code) is shown in Table 1. From Table 1 it is easily found that the output codeword \( u_{i,1}^t, i \in [1 \quad 8] \) is a combination of all input symbols \( u_{i,1}^t, i \in [1 \quad 8] \). In this paper, we refer to \( u_{i,1}^t, i \in [1 \quad 2^n] \) as MCSGCs. Since each transmit antenna in the MCS-Golden code system takes \( 2^n \) time slots to transmit \( 2^n \) MCSGCs, the MCS-Golden code is also a full-rate STBC. Through the MCS-Golden encoding every input symbol experiences \( 2^n \) different fadings in \( 2^n \) time slots. Hence, the MCS-Golden code achieves diversity order \( 2^n N_f \), and further improves error performance compared to 2CS-Golden code.

**TABLE 1. Encoding for the 8CS-Golden code.**

| \( u_{i,1}^t \) | 1st Encoding | 2nd Encoding | 3rd Encoding |
|---------------|-------------|-------------|-------------|
| \( u_{1,1}^t \) | \( u_{1,2}^t \) | \( u_{1,3}^t \) | \( u_{1,4}^t \) |
| \( u_{1,2}^t \) | \( u_{1,3}^t \) | \( u_{1,4}^t \) | \( u_{1,5}^t \) |
| \( u_{1,3}^t \) | \( u_{1,4}^t \) | \( u_{1,5}^t \) | \( u_{1,6}^t \) |
| \( u_{1,4}^t \) | \( u_{1,5}^t \) | \( u_{1,6}^t \) | \( u_{1,7}^t \) |
| \( u_{1,5}^t \) | \( u_{1,6}^t \) | \( u_{1,7}^t \) | \( u_{1,8}^t \) |
| \( u_{1,6}^t \) | \( u_{1,7}^t \) | \( u_{1,8}^t \) | \( u_{1,9}^t \) |
| \( u_{1,7}^t \) | \( u_{1,8}^t \) | \( u_{1,9}^t \) | \( u_{1,10}^t \) |
| \( u_{1,8}^t \) | \( u_{1,9}^t \) | \( u_{1,10}^t \) | \( u_{1,11}^t \) |

The MCSGC matrix is given by:

\[
[U_1 \cdots U_{2^n}] = \begin{bmatrix}
  u_{1,1}^t & \cdots & u_{1,2^n}^t \\
  u_{1,1}^t & \cdots & u_{1,2^n}^t
\end{bmatrix},
\]

where \( U_i = [u_{i,1}^t \ u_{i,2}^t]^T, i \in [1 \quad 2^n]. \)
The received signal in (2.1) and (2.2) for 2CS-Golden code may be rewritten as:

\[ y_i = H_i U_i + w_i, \]

where \( H_i = [h_{i,1} \ h_{i,2}] \). The received signal in (5) is with respect to the transmitted MQAM symbols, \( u_{0,i,l} \), \( i \in [1 : 2^n] \) and \( l \in [1 : 2] \) and \( w_i \) are the same as the discussion in (2.1) and (2.2).

Let \( U_0 = \begin{bmatrix} U_{0,1} \ U_{0,2} \end{bmatrix} \), \( \tilde{H}_i = \begin{bmatrix} \tilde{h}_{i,1} \ \tilde{h}_{i,2} \end{bmatrix} \), where

\[ U_{0,i} = \begin{bmatrix} u_{0,1,i} \ u_{0,2,i} \end{bmatrix}, \]

then (5) may be rewritten as:

\[ y_i = \tilde{H}_i U_0 + w_i, \quad i \in [1 : 2^n]. \]  

(6)

The equivalent received signals in (6) will be used to perform error performance analysis of the MCS-Golden code, while the equivalent received signals in (7) will be used to perform signal detection of the MCS-Golden code.

**III. ERROR PERFORMANCE ANALYSIS OF THE MCS-GOLDEN CODE**

The error performance of the conventional Golden code (2CS-Golden code) has been analyzed in [10]. Appendix B in [10] proves that the bounded conditional pairwise error probability (PEP) for 2CS-Golden code at high SNR is equivalent to assume that only one input MQAM symbol is detected with errors, while other input MQAM symbols are detected correctly. The simulated bit error rate (BER) and the theoretical average bit error probability (ABEP) derived in [10] validated that the assumptions work very well at high SNR for the 2CS-Golden code.

In this paper, we extend the above assumption for 2CS-Golden code into MCS-Golden code. In the following discussion, we firstly present the equivalent analysis model for the error performance analysis of 2CS-Golden code [10], then apply the approach of [10] to derive the equivalent analysis model for the error performance of the 4CS-Golden code and the MCS-Golden code. In the following discussion, we also use the following relationships: \( \theta \theta = -1, \phi \theta = -j \alpha \) and \( \theta \theta = -j \alpha \).

**A. EQUIVALENT MODEL FOR ERROR PERFORMANCE ANALYSIS OF 2CS-GOLDEN CODE [10]**

The received signal in (2.1) and (2.2) for 2CS-Golden code may be rewritten as:

\[ y_1 = \frac{1}{\sqrt{5}} \tilde{a} (h_{1,1} u_{0,1}^1 + u_{0,2}^1 \theta + h_{1,2} u_{0,1}^2 + u_{0,2}^2 \theta) + w_1, \]

(8.1)

\[ y_2 = \frac{1}{\sqrt{5}} \tilde{a} (h_{2,1} (u_{0,1}^1 + u_{0,2}^1 \theta) + j h_{2,2} (u_{0,1}^2 + u_{0,2}^2 \theta)) + w_2. \]

(8.2)

Suppose \( u_{0,1}^1 \) is detected with error, while other input symbols \( u_{0,1}^i \) are detected correctly. Then (8.1) and (8.2) can be simplified as:

\[ y_1 = \beta_1 h_{1,1} u_{0,1}^1 + w_1, \]

(9.1)

\[ y_2 = \beta_2 h_{2,1} u_{0,1}^1 + w_2. \]

(9.2)

The equivalent analysis model for error performance of 2CS-Golden code in (9.1) and (9.2) can be regarded as the transmission of \( u_{0,1}^1 \) over two identical fading channels with different transmit power \( |\beta_1|^2 \) and \( |\beta_1|^2 + |\beta_2|^2 = 1 \).

Let \( \tilde{h}_{1,1} = \beta_1 h_{1,1} \) and \( \tilde{h}_{2,1} = \beta_2 h_{2,1} \). Then (9.1) and (9.2) may also be rewritten as:

\[ y_1 = \tilde{h}_{1,1} u_{0,1}^1 + w_1, \]

(10.1)

\[ y_2 = \tilde{h}_{2,1} u_{0,1}^1 + w_2. \]

(10.2)

In (10.1) and (10.2), the entries of \( \tilde{h}_{1,1} \) and \( \tilde{h}_{2,1} \) are i.i.d. complex Gaussian RVs with distribution \( CN(0, |\beta_1|^2) \) and \( CN(0, |\beta_2|^2) \), respectively. Alternatively, the equivalent analysis model for error performance of 2CS-Golden code in (10.1) and (10.2), can be regarded as the transmission of \( u_{0,1}^1 \) over two non-identical fading channels with variance \( |\beta_1|^2 \) and \( |\beta_2|^2 \), respectively.

**B. EQUIVALENT MODEL FOR ERROR PERFORMANCE ANALYSIS OF 4CS-GOLDEN CODE**

The received signal in (6) for 4CS-Golden code may be rewritten as:

\[ y_i = \tilde{H}_{i,1} U_{0,1} + \tilde{H}_{i,2} U_{0,2} + w_i, \quad i \in [1 : 4]. \]

(11)

where

\[ U_{0,1} = \begin{bmatrix} u_{0,1}^1 \ u_{0,1}^2 \ u_{0,1}^3 \ u_{0,1}^4 \end{bmatrix}^T, \]

\[ U_{0,2} = \begin{bmatrix} u_{0,2}^1 \ u_{0,2}^2 \ u_{0,2}^3 \ u_{0,2}^4 \end{bmatrix}^T, \]

\[ \tilde{H}_{1,1} = \begin{bmatrix} [\alpha^2 h_{1,1}^1 \ j \alpha \tilde{h}_{1,1}^1 \ j \alpha \tilde{h}_{1,1}^1 \ - \alpha^2 h_{1,1}^1] \end{bmatrix}, \]

\[ \tilde{H}_{1,2} = \begin{bmatrix} [\alpha \tilde{h}_{2,1}^1 \ - \alpha^2 h_{2,1}^1 \ j \alpha \tilde{h}_{2,1}^1 \ - \alpha^2 h_{2,1}^1] \end{bmatrix}, \]

\[ \tilde{H}_{2,1} = \begin{bmatrix} [\alpha \tilde{h}_{3,1}^1 \ - \alpha^2 h_{3,1}^1 \ j \alpha \tilde{h}_{3,1}^1 \ - \alpha^2 h_{3,1}^1] \end{bmatrix}, \]

\[ \tilde{H}_{3,1} = \begin{bmatrix} [\alpha^2 h_{4,1}^1 \ j \alpha \tilde{h}_{4,1}^1 \ - \alpha^2 h_{4,1}^1 \ - \alpha^2 h_{4,1}^1] \end{bmatrix}. \]

Again, assume that only \( u_{0,1}^1 \) is detected with error, while other input symbols \( u_{0,1}^i, i \in [2 : 4] \) and \( u_{0,2}^i, i \in [1 : 4] \) are detected correctly. Then (11) is simplified as:

\[ y_i = \beta_1 h_{1,1} u_{0,1}^1 + \tilde{w}_i, \quad i \in [1 : 4]. \]

(12)

where \( \beta_1 = \frac{1}{2} \alpha^2, \beta_2 = \beta_3 = \frac{1}{2} \tilde{\alpha} \alpha \) and \( \beta_2 = \frac{1}{2} \tilde{\alpha}^2 \).

The equivalent analysis model for error performance in (12) can be regarded as the transmission of \( u_{0,1}^1 \) over four
identical fading channels with different transmit power $|\beta_i|^2$, and $\sum_{i=1}^n |\beta_i|^2 = 1$.

Similar to the discussion in 2CS-Golden code, the equivalent analysis model for error performance of 4CS-Golden code can alternatively be regarded as the transmission of $u_{0,1}$ over four non-identical fading channels with variance $|\beta_i|^2$, $i \in [1 : 4]$.

### C. EQUIVALENT MODEL FOR ERROR PERFORMANCE ANALYSIS OF THE MCS-GOLDEN CODE

In the MCS-Golden code system, we again assume that only $u_{0,1}$ is detected with error, while other input symbols $u_{0,i}$, $i \in [2 : 2^n]$ and $u_{1,j}$, $i \in [1 : 2^n]$, are detected correctly. The equivalent received signal is still (12), but $i \in [1 : 2^n]$. Based on the encoding of the MCS-Golden code in (3.1) and (3.2), it is easily found that $\beta_i$ are the terms of $\frac{1}{(\sqrt{5}n)}(\alpha + \bar{\alpha})^n$.

So we regard the transmission of $u_{0,1}$ over $2^n$ identical fading channels with different transmit power $|\beta_i|^2$, and $\sum_{i=1}^{2^n} |\beta_i|^2 = 1$.

Again the equivalent analysis model of error performance of MCS-Golden code can be regarded as the transmission of $u_{0,1}$ over $2^n$ non-identical fading channels with fading variances $|\beta_i|^2$, $i \in [1 : 2^n]$. Hence, the maximal ratio combining technique with non-identical fading channels [11] can be applied to derive the error performance of the above equivalent model. Based on the exact symbol error probability of MQAM in Eqn. (8.10) in [11], and the approximated expression of the Gaussian Q-function using the trapezoidal rule, the ABEP of MQAM-MCS-Golden code systems may be derived as:

$$p_e \approx \frac{a}{c \log_2 M} \left[ \frac{1}{2^n} \sum_{k=1}^{2^n} \left( \frac{2}{1 + |\beta_k|^2 b^2} \right)^{N_r} - \left( \frac{a}{2^n} \right)^{N_r} \right] + (1 - a) \frac{c - 1}{c} \left( \sum_{k=1}^{2^n} \frac{s_k}{1 + |\beta_k|^2 b^2} \right)^{1/2} + \sum_{i=1}^{2^n} \left( \frac{2^n}{s_i + |\beta_i|^2 b^2} \right)^{N_r}.$$  \hspace{1cm} (13)

where $c \geq 10$ is the number of partitioning intervals in this algorithm of numerical integration.

From the ABEP in (13), it is easy to find that the MCS-Golden code achieves diversity order $2^nN_r$. For $n = 1$ then the MCS-Golden code actually is 2CS-Golden system. 2CS-Golden code system achieves diversity order $2N_r$. From the ABEP of the CI-Golden code in (12) in [5] the CI-Golden code achieves diversity order $4N_r$. So both 4CS-Golden code and the CI-Golden code achieves the same diversity order. For $n > 2$ the MCS-Golden code will achieve more diversity order compared to the CI-Golden code.

### IV. DETECTION SCHEMES FOR MCS-GOLDEN CODE SYSTEMS

Sphere decoding is a near-ML detection scheme. In this section, we firstly present the sphere decoding of the MCS-Golden code, and then describe the proposed SD-DS detector of the MCS-Golden code.

#### A. SPHERE DECODING OF THE MCS-GOLDEN CODE

Sphere decoding for general MIMO systems has been documented in detail in [12]. Recently sphere decoding has also been applied to detect the 2CS-Golden code [10] and CI-Golden code [5]. In this paper, we apply the sphere decoding to decode the MCS-Golden code.

In the detection of the MCS-Golden code, we assume that the channel state information (CSI) is fully known at the receiver. Based on the complex QR decomposition of $H$ in (7), we have:

$$H = QR.$$  \hspace{1cm} (14)

where $Q \in \mathbb{C}^{2^nN_r \times 2^nN_r}$ is a unitary matrix and $R \in \mathbb{C}^{2^nN_r \times N}$, $N = 2 \times 2^n$. $R = [R_1 \ R_2]^T$, where $R_2$ is a zero matrix with $(2^nN_r - N) \times N$ dimension and $R_1$ is an upper-triangular matrix with $N \times N$ nonnegative real diagonal elements which is given by:

$$R_1 = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,2^n} & \cdots & r_{1,N} \\ 0 & r_{2,2} & \cdots & r_{2,2^n} & \cdots & r_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{N,2} & \cdots & r_{N,N} \end{bmatrix}.$$  \hspace{1cm} (15)

Substituting (15) into (7), and multiplying both sides by $Q^H$, we have:

$$Z = RU_0 + \hat{W},$$  \hspace{1cm} (16)

where $\hat{W} = Q^HW$ and $Z = [Z_1 \ Z_2]^T = Q^HY$. $Z_1$ is a vector with $N \times 1$ dimension denoted as $Z_1 = [Z_{1}(1) \ Z_{1}(2) \ \cdots \ Z_{1}(N-1) \ Z_{1}(N)]^T$ and $Z_2$ is a vector with $(2^nN_r - N) \times 1$ dimension.

Based on (7), the sphere decoding of the MCS-Golden code may be formulated as:

$$\|Z_1 - R_1 U_0 \|^2 \leq r^2,$$  \hspace{1cm} (17)

where $r$ is the radius of sphere decoding, which is based on Eq. (28) in [13].

For convenience, in the following discussion, let $U_0 = [U_{0,1} \ U_{0,2}]^T = [u_1 \ \cdots \ u_N]^T$ and $u_k = [u_{k1} \ \cdots \ u_{kN}]$.

Now we detect $u_N$. Let $p_N(u_N) = Z_1(N) - r_{N,N}u_N$, then we have:

$$|p_N(u_N)|^2 \leq r^2.$$  \hspace{1cm} (18)

Sphere decoding searches $u_{N}, u_N \in \Omega_M$, to meet the constraint in (18).

After $u_N$ is found to meet the constraint in (18), we detect $u_{N-1}$. Let $p_{N-1}(u_{N-1}^N) = Z_1(N - 1) - r_{N-1,N-1}u_{N-1} - r_{N-1,N}u_N$. Then we have:

$$|p_{N-1}(u_{N-1}^N)|^2 \leq r^2 - |p_N(u_N)|^2.$$  \hspace{1cm} (19)
Sphere decoding searches \( u_{N-1}, u_{N-1} \in \Omega_M \), to meet the constraint in (19).

In general, if we detect \( u_k \) then we have:

\[
|p_k(u_k^N)|^2 \leq r^2 - \sum_{l=k+1}^{N} |p_l(u_l^N)|^2.
\]  
(20)

**B. SPHERE DECODING WITH SORTED DETECTION SUBSET (SD-SDS)**

The detection complexity of the conventional sphere decoding is very high as the number of multiple symbols in MCS-Golden code increases. SD-DS has been proposed for the conventional Golden code system in [10] and the CI-Golden code in [5]. However, if the estimated input symbol or symbols are at the end of the signal subset, then the above sphere decoding will take more time to search the estimated symbols. In this paper, we propose a modified SD-DS, referred to as SD-SDS to detect the MCS-Golden code. In the proposed SD-SDS, we firstly estimate the transmitted MQAM symbols, \( u_i \), \( i \in [1:N] \), and then sort all symbols in the signal set from the most probable transmitted to the least probable transmitted.

The proposed SD-SDS consists of four steps.

**Step 1:** Perform QR decomposition and calculate \( Z_1 \) based on (16).

**Step 2:** Estimate the transmitted MQAM symbols \( u_i \), \( i \in [1:N] \).

Based on (16), we have:

\[
v_N = \frac{Z_1(N)}{r_{N,N}}.
\]  
(21)

The estimations of \( u_N \) is given by:

\[
\tilde{u}_N = D(v_N).
\]  
(22)

In general, we have:

\[
v_k = \frac{Z_1(k) - \sum_{l=k+1}^{N} r_{k,l} \tilde{u}_l}{r_{k,k}}.
\]  
(23)

The estimations of \( u_k \), \( k \in [1:N-1] \), is given by:

\[
\tilde{u}_k = D(v_k).
\]  
(24)

**Step 3:** Sort all symbols in the whole signal set and arrange these symbols from the most probable transmitted to the least probable transmitted.

The metric to estimate the possibilities of the transmitted symbols is given by:

\[
m_i(k) = |v_i - u_k|^2,
\]  
(25)

where \( k \in [1:M] \), and \( u_k \in \Omega_M \).

Let \( m_i = \{m_i(k), k \in [1:M]\} \), \( i \in [1:N] \). The most probable index estimation of \( u_i \) is obtained by evaluating:

\[
\tilde{i}_k = \text{argsort} (m_i),
\]  
(26)

where the \text{argsort} (·) operator arranges the \( M \) elements from most probable to least probable. \( \tilde{i} = [\tilde{i}_1, \cdots, \tilde{i}_M] \).

**Step 4:** Perform sphere decoding.

Given an SNR, we select \( L \) most probable transmitted symbols for each \( \tilde{u}_i \). These \( L \) most probable transmitted symbols for each \( \tilde{u}_i \) form a set \( \Omega(\tilde{u}_i, L) \), named as sorted detection subset. Now we use \( \Omega(\tilde{u}_i, L) \) to perform sphere decoding based on (18) to (20).

In order to speed up detection and guarantee that the simulation error performance approaches the theoretical ABEP, in simulations, we select small values of \( L \) for low SNRs and large values of \( L \) for large SNRs.

**V. DETECTION COMPLEXITY ANALYSIS**

In the conventional sphere decoding [12], the detection complexity of QR decomposition is ignored compared to the detection complexity to calculate the Euclidean distance (ED) for searching the transmitted symbols. The detection complexity of calculating ED depends on the constellation size of the transmitted signal set. The size to be searched in the conventional sphere decoding is \( M^N \) for MIMO systems. In order to reduce the detection complexity of the sphere decoding, the SD-DS has been proposed in [5] and [10]. The detection complexity of the SD-DS is reduced compared to the conventional sphere decoding. The proposed SD-SDS in this paper can further reduce detection complexity by arranging these symbols from the most probable transmitted to the least probable transmitted. Similar to the complexity analysis in [5], we also ignore the detection complexity of QR decomposition. We only focus on the detection complexity to calculate the ED in (18) to (20) through simulation.

**TABLE 2. Parameters setting for simulations.**

| \( M \) | \( N \) | \( K-\text{CS} \) | \( \text{SNR and } L \) |
|---|---|---|---|
| 16 | 28 | \{0\} \times 24 | \{0\} \times 24 |
| 16 | 4 | \{0\} \times 12 | \{0\} \times 12 |
| 16 | 4 | \{0\} \times 24 | \{0\} \times 24 |
| 16 | 8 | \{0\} \times 24 | \{0\} \times 24 |
| 16 | 32 | \{0\} \times 24 | \{0\} \times 24 |
| 64 | 3 | \{0\} \times 24 | \{0\} \times 24 |
| 64 | 4 | \{0\} \times 24 | \{0\} \times 24 |
| 64 | 4 | \{0\} \times 24 | \{0\} \times 24 |
| 64 | 4 | \{0\} \times 24 | \{0\} \times 24 |

In the simulation of the MCS-Golden code, the parameter \( L \) for given SNRs is tabulated in Table 2. \( K-\text{CS} \) in Table 2 denotes the number of multiple complex symbols in the MCS-Golden code. For comparison, we also simulate the 2CS-Golden code with SD-DS and the CI-Golden code with SD-DS. In the simulation of the 2CS-Golden code with SD-DS, the \( \delta_i \) in Definition 2 of [10] is set as follows: \( \delta_i = 28.8 \) for both 16QAM and 64QAM with \( N_r = 4 \), and \( \delta_i = 976 \) for both 16QAM and 64QAM with \( N_r = 3 \). In the simulation of the CI-Golden code, the \( \delta \) in Definition 2 of [5] is set as follows: \( \delta = 51.2 \), \( \delta = 92.8 \) for 16QAM and 64QAM with \( N_r = 4 \), and \( \delta = 326 \) and \( \delta = 320 \) for 16QAM and 64QAM with \( N_r = 3 \), respectively.
FIGURE 1. Detection complexity of SD-SDS for 16QAM MCS-Golden code with \( N_r = 3 \) and \( N_r = 4 \).

FIGURE 2. Detection complexity of SD-SDS for 64QAM MCS-Golden code with \( N_r = 3 \) and \( N_r = 4 \).

The simulated detection complexities of the 2CS-Golden code with SD-DS [10], the CI-Golden code with SD-DS [5] and the MCS-Golden code with SD-SDS are shown in Figures 1 and 2. Note, all detection complexities in Figures 1 and 2 are in terms of calculating ED for detection of four MQAM symbols.

From Figures 1 and 2, it is observed that:

1) The detection complexity of the SD-SDS decreases as the SNR increases and the number of receive antennas increases.

2) Compared to the Golden code with SD-DS [10], the detection complexity of the SD-SDS for 2CS-Golden code is at least one order lower.

3) Compared to the CI-Golden code with SD-DS [5], the detection complexity of the SD-SDS for 4CS-Golden code is at least two orders lower.

VI. SIMULATION RESULTS

In this section, we will present simulation results for the MCS-Golden code systems in frequency-flat Rayleigh fading with AWGN as described in Section II. It is assumed that the CSI is fully known at the receiver. As the discussion in [5], the MCS-Golden code system is valid for \( N_r \geq 2 \). However, in this section, we only consider \( N_r > 2 \) to enable comparison with the theoretical ABEP derived in Section III.

In the simulation of the MCS-Golden code, the value of the \( L \) most probable transmitted symbols for the given SNRs and the number of receive antennas \( N_r \) is as presented in Table 2. For comparison, we also simulate the SD-DS for the conventional Golden code and the CI-Golden code. The \( \delta \) for the simulation of the Golden code with SD-DS and the \( \delta \) for the simulation of the CI-Golden code with SD-DS are the same as the setting in Section V. All simulation results of 16QAM and 64QAM MCS-Golden code with \( N_r = 3 \) and \( N_r = 4 \) are shown in Figures 3, 4, 5 and 6. All theoretical results based on (13) are also shown in Figures 3, 4, 5 and 6.

FIGURE 3. BER versus normalized SNR for 16QAM MCS-Golden code with \( N_r = 3 \).

From the results in Figures 3, 4, 5 and 6 it is observed that:

1) All simulation and theoretical results validate that the proposed MCS-Golden code not only achieves full rate, but also achieves diversity order \( 2^N_r \).

2) The larger the value of the multiple complex symbols, the steeper the curve of the BER, which is consistent with the theoretical ABEP derived in Section III.
with the discussion of the error performance analysis in Section III. This is because the proposed MCS-Golden code achieves diversity order $2^{2N_r}$.

3) Both the Golden code with SD-DS and the 2CS-Golden code with SD-SDS achieve the same error performance. However, as the discussion of detection complexity in Section V, the detection complexity of the 2CS-Golden code with SD-SDS is lower than the Golden code with SD-DS. The detection complexity of the proposed SD-SDS decreases as SNR increases. There is an SNR threshold at which the detection complexity of the proposed MCS-Golden code is lower than the conventional Golden code with SD-DS. But the error performance of the MCS-Golden code is much better than the conventional Golden code. For example, with $N_r = 3$ and 16QAM the threshold is around 16.5 dB. The detection complexity of the proposed 16QAM 8CS-Golden code with SD-SDS is lower than 16QAM Golden code with SD-DS when SNR is greater than 16.5 dB.

4) The CI-Golden code with SD-DS and 4CS-Golden code with SD-SDS achieve the same error performance. Again as the discussion of detection complexity in Section V, the detection complexity of the CI-Golden code with SD-DS is much higher than the 4CS-Golden code with SD-SDS.

5) As the number of receive antennas increases, the simulated BER draws closer to the theoretical bound.

6) For $N_r = 3$, the 4CS-Golden code achieves around 2.5 dB SNR gain compared to 2CS-Golden code at a BER of $3 \times 10^{-6}$, while the 8CS-Golden code achieves around 1.5 dB SNR gain compared to the 4CS-Golden code.

7) For $N_r = 4$, the 4CS-Golden code achieves around 1.5 dB SNR gain compared to the 2CS-Golden code at a BER of $3 \times 10^{-6}$, while the 8CS-Golden code achieves around 1 dB SNR gain compared to the 4CS-Golden code.

Finally, we present the comparison of the ABEP and the asymptotic bound for MCS-Golden code, where the asymptotic bound is derived from (13) for setting large SNR, such as to make $|\beta_k|^2b\gamma > 2$. The comparisons of ABEP and asymptotic bound for 16QAM 2CS- and 4CS-Golden code are shown in Figures 7 and 8, respectively. From the results in Figures 7 and 8, the following is observed:
1) Both the ABEP and asymptotic bound are merged together, which validates that the MCS-Golden code achieves the diversity order $2^n N_r$.

2) The larger the value of the multiple complex symbols, the larger the value of the SNR, where both ABEP and the asymptotic bound are merged together.

VII. CONCLUSION
In order to further improve the error performance of the conventional Golden code, we proposed a new type of Golden code, MCS-Golden code. The ABEP for the proposed MCS-Golden code system was formulated. Furthermore, to reduce detection complexity for the MCS-Golden code, an SD-SDS scheme was proposed and supported by a complexity analysis. Simulation and theoretical results validated that the proposed MCS-Golden code also achieves full-rate, but achieves multiple diversity order compared to the conventional Golden code. Both simulation and theoretical results demonstrated that the MCS-Golden code significantly improves on error performance compared to the conventional Golden code. For example, at an average BER of $3 \times 10^{-6}$, the four complex symbol Golden code system with three receive antennas, achieves a 2.5 dB SNR gain compared to the conventional Golden code.

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