Renormalized $g$-$\log(g)$ double expansion for the invariant $\phi^4$-trajectory in three dimensions

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Abstract
We study the invariant unstable manifold of the trivial renormalization group fixed point tangent to the $\phi^4$-vertex in three dimensions. We parametrize it by a running $\phi^4$-coupling with linear step $\beta$-function. It is shown to have a renormalized double expansion in the running coupling and its logarithm.
1 Introduction

We study massless $\phi^4$-theory on three dimensional Euclidean space-time by means of Wilson’s renormalization group, see [W70, W71, WF72, WK74]. Its renormalization is analyzed as a dynamical system on a space of effective actions. The renormalization group acts on this space as a semi-group of scale transformations. The renormalized theory comes as a special one-dimensional curve of effective actions. It is one-dimensional as a renormalization group orbit. We parametrize it by a running $\phi^4$-coupling.

We consider a discrete renormalization group of transformations $R_L$, which scale by $L > 1$, built from a momentum space decomposition of a massless Gaussian random field by means of an exponential regulator. See Gallavotti’s review [G85] for its application to renormalized perturbation theory; and Brydges’ review [B92] for its use in constructive field theory. The renormalization group acts here on a space of potentials $V(\Phi)$ with suitable properties.

The free massless field corresponds to the trivial fixed point $V_*(\Phi) = 0$. The linearized renormalization group at this trivial fixed point is diagonalized by normal ordering. The normal ordered local $\phi^4$-vertex is a particular eigenvector. Its scaling dimension is one in three dimensions. It defines an unstable perturbation of the trivial fixed point, which extends tangentially to an invariant curve in the unstable manifold of the trivial fixed point. This curve is the renormalized $\phi^4_3$-trajectory. We parametrize it by $g$ such that

$$V(\Phi|g) = \frac{g}{4!} \int d^3x : \Phi(x)^4 : + O(g^2).$$

(1)

log($g$) corrections are neglected for the moment. The program of this paper is a study of this curve as a function of $g$ for small couplings, $g << 1$, beyond the linear approximation.

Our main dynamical principle is invariance under the renormalization group. As in [W97], we look for a pair, given by the potential $V(\Phi|g)$ and a step $\beta$-function $\Delta \beta_L(g)$, such that

$$R_L(V(\Phi|g) = V(\Phi| \Delta \beta_L(g)).$$

(2)

To first order in $g$, $R_L$ is to be linearized. Consequently, $\Delta \beta_L(g) = Lg + O(g^2)$ on the renormalized $\phi^4_3$-trajectory. The meaning of $g$ is twofold. It is both a coordinate of the renormalized $\phi^4_3$-trajectory and a running coupling. We select it by the condition that the step $\beta$-function be exactly linear,

$$\Delta \beta_L(g) = Lg.$$  

(3)

$^1$ $V(\Phi)$ corresponds to a perturbation $d\mu_{C_{\infty}}(\Phi) \exp(-V(\Phi))$ of a Gaussian measure $d\mu_{C_{\infty}}(\Phi)$, with mean zero and covariance $C_L = \int_1^{L^2} d\alpha \exp(\alpha \triangle)$, in the limit $L \rightarrow \infty$; where $\triangle$ denotes the Laplace operator.

$^2$ The renormalization flow is therefore asymptotically free in the backward direction, yielding ultraviolet asymptotic freedom.

$^3$ Infrared properties are then encoded in the strong coupling limit $g \rightarrow \infty$. 

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The linear step $\beta$-function defines a normal form of the renormalization group at the trivial fixed point; in the sense of Ecalle’s theory of resurgent functions. Its use in renormalization theory was proposed by Eckmann and Wittwer, [EW84]. In this representation, we look for $V(\Phi|g)$ as a fixed point of the composed transformation $R_L \times \Delta \beta_L^{*}$. We will then show that the local $\phi^2$-vertex has a resonance in the order $g^2$, analogous to the hierarchical approximation [RW96]; there exists no solution $V(\Phi|g)$ of (3) as a formal power series in $g$. As in the hierarchical approximation, we will then resolve the resonance by a local $\phi^2$-vertex of the type $g^2 \log(g)$. This leads us to investigate a double expansion in $g$ and $(g^2 \times \log(g))$. We will show that there exists a one parameter set of fixed points $V(\Phi|g)$ in formal double expansion. The present paper extends the previous study [W97] of the renormalized $\phi^4$-trajectory. The closest relative to our method in the literature is the $\beta$-functional method in the tree expansion of Gallavotti and Nicolò. See [BG95] for an introduction to this technology. However, we do not begin from a regularized theory defined by a bare action. This scheme is here replaced by a construction of an effective theory on all length scales in units of the coupling.

The ultraviolet limit of $\phi^4$-theory has been rigorously studied in constructive field theory. See Glimm and Jaffe [GJ73, GJ87], Feldman and Osterwalder [FO76], Magnen and Seneor [MS77], Balaban [B83], Benfatto et al. [BCGNOPS80], Brydges et al. [BDH93], and references therein. In view of this knowledge, our contribution is modest, clarifying $\log(g)$ corrections from the dynamical systems point of view. Double expansions in $g$ and $\log(g)$ appeared also in Symanzik’s work on non-renormalizable $\phi^4$-theory in $4 + \epsilon$ dimensions [S73].

## 2 Renormalization group

We study perturbations of a scalar Gaussian massless random field by means of a renormalization group with Gaussian regulator. See [G85, FFS92, BG95, K91, GK84] for background material. We will use the setup as in [W97] with the difference that the dimension of Euclidean space-time will be three rather than four.

### 2.1 Renormalization group transformation

We consider the renormalization group of transformations $R_L$, given by

$$ R_L(V)(\Phi) = -\log \left( \int d\mu_L(\phi) \exp (-V(S_L(\Phi) + \phi)) \right) - \text{const.}, $$

where $\phi$ is a Gaussian random field with mean zero and covariance

$$ C_L(x, y) = \int \frac{d^3p}{(2\pi)^3} e^{ip(x-y)} \frac{\hat{\chi}(p) - \hat{\chi}(Lp)}{p^2}, \quad \hat{\chi}(p) = e^{-p^2}, $$

and where $S_L(\Phi)(x) = L^{-1/2} \Phi(L^{-1}x)$. A few remarks on the covariance (5) are collected in the appendix 6.1. The scale $L$ will be kept fixed at some value $L > 1$, say $L = 2$. We thus consider a discrete version of the renormalization group.
The covariance $C_L$ and the dilatation $S_L$ are related to an $L$-independent covariance $C_\infty$, given by

$$C_\infty(x, y) = \int \frac{d^3p}{(2\pi)^3} e^{ip(x-y)} \frac{\hat{\chi}(p)}{p^2},$$

through $C_L = C_\infty - S_L C_\infty S_L^T$. It will serve as normal ordering covariance. The covariance (6) has a unit ultraviolet cutoff but no infrared cutoff.

### 2.2 Space of potentials

We consider potentials $V(\phi)$, which are given by power series

$$V(\Phi) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int d^3x_1 \cdots \int d^3x_{2n} : \Phi(x_1) \cdots \Phi(x_{2n}) : V_{2n}(x_1, \ldots, x_{2n})$$

in $\Phi$. The real space kernels $V_{2n}(x_1, \ldots, x_{2n})$ will be required to be symmetric and Euclidean invariant distributions, which are given by Fourier integrals

$$V_{2n}(x_1, \ldots, x_{2n}) = \int \frac{d^3p_1}{(2\pi)^3} \cdots \int \frac{d^3p_{2n}}{(2\pi)^3} (2\pi)^3 \delta(p_1 + \cdots + p_{2n}) e^{ip_1 x_1 + \cdots + p_{2n} x_{2n}} \hat{V}_{2n}(p_1, \ldots, p_{2n})$$

of smooth momentum space kernels $\hat{V}_{2n}(p_1, \ldots, p_{2n})$. Smoothness will be required in particular at the point $p_1 = \cdots = p_{2n} = 0$.

The matter of the convergence of (7) as a power series in $\Phi$ will be left aside. We will restrict our attention to polynomial approximations of finite orders, leaving aside the large field problem. Potentials will always be normalized such that $V(0) = 0$. The transformation (9) is understood to be supplemented by a subtraction of a normalization constant, proportional to the volume.

### 2.3 Localization operators

We define linear localization operators $\mathcal{L}_{2n}$ by

$$\mathcal{L}_{2n}(V)(\Phi) = \lambda_{2n}(V) \mathcal{O}_{2n}(\Phi), \quad \lambda_{2n}(V) = \hat{V}_{2n}(0, \ldots, 0),$$

where $\mathcal{O}_{2n}(\Phi)$ is the local $\Phi^{2n}$-vertex

$$\mathcal{O}_{2n}(\Phi) = \frac{1}{(2n)!} \int d^3x : \Phi(x)^{2n} : .$$

This completes our setup.
3 \( \phi^4 \)-trajectory

The \( \phi^4 \)-trajectory is a renormalization invariant curve in the space of potentials, which emerges from the trivial fixed point tangent to the \( \phi^4 \)-vertex. The \( \phi^4 \)-vertex is an unstable perturbation in three dimensions, whereas it is a marginally stable perturbation in four dimensions. This difference is encoded in the respective step \( \beta \)-functions. In three dimensions, the power counting of vertices becomes order dependent, since the coupling has a scaling dimension different from zero. Peculiar to three dimensions is a second order mass resonance, whose resolution requires logarithmic couplings. See [RW96] for a treatment of its hierarchical approximation.

3.1 Renormalization invariance

We investigate the following renormalization problem. We look for a pair, consisting of a potential \( V(\Phi|g) \) and a differential \( \beta \)-function \( \beta(g) \), such that

\[
R_L(V)(\Phi|g) = V(\Phi| \Delta \beta_L(g)),
\]

where \( \Delta \beta_L(g) \) is the step \( \beta \)-function, defined as the solution \( \Delta \beta_L(g) = g(L) \) of the ordinary differential equation

\[
L g'(L) = \beta(g(L)), \quad g(1) = g.
\]

A few remarks on the \( \beta \)-function are appended in 3.2. The \( \phi^4 \)-trajectory is a particular pair, which is distinguished by the first order condition

\[
V(\Phi|g) = V^{(1)}(\Phi) \ g + O(g^2), \quad V^{(1)}(\Phi) = O_4(\Phi).
\]

We postpone the matter of log\( (g) \) corrections in \( V(\Phi|g) \) and \( \beta(g) \) for the moment.

3.2 Order \( g \)

To first order in \( g \), (11) becomes an eigenvector problem

\[
\langle V^{(1)} \rangle_{C_L S_L \Phi} - L^{\beta^{(1)}} V^{(1)}(\Phi) = 0
\]

for the linearized renormalization group

\[
\langle O \rangle_{C_L S_L \Phi} = \int d\mu_{C_L}(\phi) \ O(S_L \Phi + \phi)
\]

with the eigenvalue related to

\[
\beta(g) = \beta^{(1)} g + O(g^2), \quad \Delta \beta_L(g) = L^{\beta^{(1)}} g + O(g^2).
\]

The normal ordered \( \phi^4 \)-vertex \( O_4(\Phi) \) is an eigenvector of the linearized renormalization group (13). Its eigenvalue is \( \beta^{(1)} = 1 \) in three dimensions.
3.3 \( \beta \)-function

To higher orders in \( g \), our renormalization problem has to be supplemented either by a condition on \( V(\Phi|g) \), by a condition on \( \beta(g) \), or by a condition on both to define \( g \). A canonical condition on \( V(\Phi|g) \) is

\[
\mathcal{L}_4(V)(\Phi|g) = \mathcal{O}_4(\Phi) \ g, \quad \mathcal{V}_4(0,0,0,0|g) = g.
\]  

(17)

It defines \( g \) to be the local \( \phi^4 \)-coupling. With this choice, the \( \beta \)-function has to be computed from (11). A canonical condition on \( \beta(g) \) is

\[
\beta(g) = \beta^{(1)} g, \quad \triangle \beta_L(g) = L \beta^{(1)} g,
\]  

(18)

provided that \( \beta^{(1)} \) is not zero. With this choice, the local \( \phi^4 \)-coupling, we name it \( \lambda_4(g) \), has to be computed from (11). Mixed conditions will not be considered here. The linear \( \beta \)-function has technical advantages in the below double perturbation theory. It can be viewed as a normal coordinate in the vicinity of the trivial fixed point. Therefore, we choose to work with the second condition (18). Some remarks on this issue have been added in the appendix 6.2. Our renormalization problem has now turned into a fixed point problem. We seek to compute \( V(\Phi|g) \) as a fixed point of \( R_L \times \triangle \beta_{L-1}^* \), for all \( L > 1 \).

3.4 Order \( g^2 \) mass resonance

There exists no second order potential \( V^{(2)}(\Phi) \) such that the fixed point \( V(\Phi|g) \) is of the form

\[
V(\Phi|g) = \mathcal{O}_4(\Phi) \ g + V^{(2)}(\Phi) \frac{g^2}{2} + O(g^3).
\]  

(19)

The reason is the following. Suppose that there existed a fixed point (19). To second order in \( g \), (11) would require that

\[
L^{-2} \left\langle V^{(2)} \right\rangle_{C_L S_L \Phi} - V^{(2)}(\Phi) = L^{-2} \left\langle V^{(1)}; V^{(1)} \right\rangle^T_{C_L S_L \Phi},
\]  

(20)

where the right hand side is a truncated expectation value. Eq. (20) implies that

\[
L^{-2} \left\langle V^{(1)}; V^{(1)} \right\rangle^T_{C_L S_L \Phi} = 0,
\]  

(21)

because the local \( \phi^2 \)-couplings on the left hand side of (20) cancel. This proves to be false. See 5.3. We have a contradiction.

The problem is that a local \( \phi^2 \)-vertex is generated to second order, while the eigenvalue \( L^2 \) of \( \mathcal{O}_2(\Phi) \) is the square of the eigenvalue \( L \) of \( \mathcal{O}_4(\Phi) \). This situation was called a mass resonance. See [RW96] for a list of various resonances in the hierarchical approximation, and their resolutions. Resonances have to be resolved by logarithmic couplings.
3.5 Order $g^2$

Concerning all other vertices than the local $\phi^2$-vertex, eq. (21) has a unique regular solution. It poses a linear problem for $L^2(V^{(2)}(\Phi))$, given $V^{(1)}(\Phi)$, where $L^2 = 1 - L_2$. Restricted to $L_2^\perp$, eq. (21) is equivalent to a set of difference equations for the momentum space kernels (8). We introduce the abbreviation

$$K_L(V)^{(2)}(\Phi) = L^{-2} \langle V^{(1)}; V^{(1)} \rangle^T_{C_L,S_L}\Phi.$$  (22)

to obtain a system of equations

$$L^{1-n} \hat{V}^{(2)}_{2n}(L^{-1} p_1, \ldots, L^{-1} p_{2n}) - \hat{V}^{(2)}_{2n}(p_1, \ldots, p_{2n}) = \tilde{K}_L(V)^{(2)}_{2n}(p_1, \ldots, p_{2n}).$$  (23)

The connected normal ordered contraction of two normal ordered local $\phi^4$-vertices is a polynomial of degree six in $\Phi$. The right hand side of (23) is therefore zero for $n > 3$. For $n \geq 2$, eq. (23) is solved by

$$\hat{V}^{(2)}_{2n}(p_1, \ldots, p_{2n}) = -\sum_{\alpha=0}^{\infty} L^{(1-n)}(\alpha) \tilde{K}_L(V)^{(2)}_{2n}(L^{-\alpha} p_1, \ldots, L^{-\alpha} p_{2n}).$$  (24)

This solution is regular at $p_1 = \cdots = p_{2n} = 0$. It is the only solution with this property. More details on this can be found in [W97]. The non-local $L^\perp_2$-part of the $\phi^2$-vertex is given by

$$\tilde{L}^\perp_2(V)^{(2)}_{2}(p_1, p_2) = -\sum_{\alpha=0}^{\infty} L^\perp_2(\alpha) \tilde{K}_L(V)^{(2)}_{2}(L^{-\alpha} p_1, L^{-\alpha} p_2).$$  (25)

This sum converges, despite the absence of a power counting factor, due to the subtraction of the local $\phi^2$-coupling; there exists a radius $r > 0$ and a constant $C > 0$ such that

$$\left| L^\perp_2(\tilde{K}_L(V)^{(2)}_{2}(p, -p)) \right| \leq C p^2$$  (26)

for all $p^2 \leq r^2$. Except for the local $\phi^2$-vertex $L_2(V)^{(2)}(\Phi)$, we have a unique regular solution of (21). We rename it $L^\perp_2(V)^{(2,0)}(\Phi)$.

There remains the issue of large momentum bounds. In three dimensions we have better large momentum bounds than in four dimensions, see [W97] concerning the latter. Indeed, since there are no subtractions in $L^\perp_2$, we find that the right hand side of (23), given by convergent convolution integrals, is a bounded function of the momenta. Consequently, the solutions (24) obey

$$\|\hat{V}^{(2)}_{2n}\|_\infty < \infty, \quad n \geq 2.$$  (27)

The non-local $L^\infty_2$-part (24) is the only vertex in this theory which is not a bounded function of momentum. Its accurate large momentum behavior is logarithmic. This bound is explained in the appendix 6.4. At higher orders, all vertices will be bounded. The reason is that the logarithmic growth of the order two $\phi^2$-vertex is more than compensated by the exponential decay of the propagators at large momentum.
3.6 $g^2 \log(g)$ mass term

Armed with this insight, we then consider the following modified second order,

$$
V(\Phi|g) = \mathcal{O}_4(\Phi) g + \mathcal{L}^1_2(V) (2,0)(\Phi) \frac{g^2}{2} + \mathcal{O}_2(\Phi) \left\{ \lambda_2^{(2,0)} + \lambda^{(2,1)} \log(g) \right\} \frac{g^2}{2} + O(g^3; g^3 \log(g)),
$$

where we have added a local $\phi^2$-coupling

$$
\lambda_2(g) = \lambda^{(2,0)} \frac{g^2}{2} + \lambda^{(2,1)} \frac{g^2}{2} \log(g) + O(g^3; g^3 \log(g)).
$$

The point with it is that

$$
L^{-2} \lambda_2(Lg) = \left\{ \lambda_2^{(2,0)} + \log(L) \lambda_2^{(2,1)} \right\} \frac{g^2}{2} + \lambda_2^{(2,1)} \frac{g^2}{2} \log(L) + O(g^3; g^3 \log(g)).
$$

We then consider separately the order $g^2$-term and the order $g^2 \log(g)$-term in (28). They are given by

$$
L^{-2} \left\langle V^{(2,0)}(\Phi) \right\rangle_{C_{L,S_{L}\Phi}} - V^{(2,0)}(\Phi) - \lambda_2^{(2,1)} \log(L) \mathcal{O}_2(\Phi) = L^{-2} \left\langle V^{(1,0)}; V^{(1,0)} \right\rangle^T_{C_{L,S_{L}\Phi}},
$$

together with

$$
L^{-2} \left\langle V^{(2,1)} \right\rangle_{C_{L,S_{L}\Phi}} - V^{(2,1)}(\Phi) = 0.
$$

The flow of the logarithmic local $\phi^2$-vertex cures the resonance. Apply $\mathcal{L}_2$ to (31) to obtain

$$
\lambda_2^{(2,1)} \log(L) \mathcal{O}_2(\Phi) = L^{-2} \left\langle V^{(1,0)}; V^{(1,0)} \right\rangle^T_{C_{L,S_{L}\Phi}}.
$$

This equation determines the invariant coupling parameter $\lambda_2^{(2,1)}$. The non-logarithmic second order parameter $\lambda_2^{(2,0)}$ is not determined by (31) and (32). It is a free parameter of this theory. All higher order vertices, both the logarithmic and the non-logarithmic ones, have negative power counting. They are thus irrelevant, and therefore uniquely determined. Thus $\lambda_2^{(2,0)}$ is the only free parameter of this renormalization problem. The curve with a minimal set of vertices is $\lambda_2^{(2,0)} = 0$.

4 Perturbation theory

We added a $g^2 \log(g)$ correction to the local $\phi^2$-vertex. It suggests a double expansion in $g$ and $(g^2 \times) \log(g)$ for the $\phi^4$-trajectory. The program of this section is this double expansion. It will be formulated in terms of a recursion relation. When the recursion relation is iterated, $\log(g)$-corrections spread to all higher vertices.
4.1 $g$-log$(g)$ double expansion

We consider potentials of the general form

$$V(\Phi|g) = \sum_{r=1}^{\infty} \sum_{a=1}^{\left\lfloor \frac{r}{2} \right\rfloor} V^{(r,a)}(\Phi) \, \frac{g^r \log(g)^a}{r! \, a!}, \quad (34)$$

where $\left\lfloor \frac{r}{2} \right\rfloor$ denotes the integer part of $\frac{r}{2}$, and where $V^{(r,a)}(\Phi)$ is a polynomial

$$V^{(r,a)}(\Phi) = \sum_{n=1}^{r-2a+1} \frac{1}{(2n)!} \int d^3x_1 \cdots \int d^3x_{2n} : \Phi(x_1) \cdots \Phi(x_{2n}) : V^{(r,a)}_{2n}(x_1, \ldots, x_{2n}) \quad (35)$$

in $\Phi$. The order $r-2a+1$ is particular to $\phi^4$-theory. It equals half the maximal number of external legs of all connected Feynman graphs, which can be built from $r-2a$ local $\phi^4$-vertices and $a$ local $\phi^2$-vertices. We anticipate that this form of potentials iterates through the renormalization group. The real space vertices $V^{(r,a)}_{2n}(x_1, \ldots, x_{2n})$ are required to be given by Fourier integrals of the form (8).

4.2 Flow of $g$

The step $\beta$-function is $\Delta \beta_L(g) = L g$ in the present representation. The right hand side of (11) then becomes

$$V(\Phi|Lg) = \sum_{r=1}^{\infty} \sum_{a=1}^{\left\lfloor \frac{r}{2} \right\rfloor} \left\{ \left( L^r \sum_{b=a}^{r} \frac{\log(L)^{b-a}}{(b-a)!} \right) \right\} \frac{g^r \log(g)^a}{r! \, a!}. \quad (36)$$

Potentials, which are given by (34) and (35), thus remain of this form under our flow of $g$.

4.3 Renormalization transformation

The image of a renormalization transformation $I$, applied to the potential (34), is

$$R_L(V)(\Phi) = \sum_{r=1}^{\infty} \sum_{a=1}^{\left\lfloor \frac{r}{2} \right\rfloor} R_L(V)^{(r,a)}(\Phi), \quad (37)$$

where $R_L(V)^{(r,a)}(\Phi)$ is again a polynomial of the form (35). The maximal order $r-2a+1$ thus iterates through the renormalization transformation. The coefficients in (37) are given by sums of truncated expectation values

$$R_L(V)^{(r,a)}(\Phi) = \sum_{i=1}^{r} \frac{(-1)^{i+1}}{i!} \sum_{(s_1, \ldots, s_i) \in \mathbb{N}^i} \delta_{\sum_{j=1}^{i} s_j - r} \left( \begin{array}{c} r \\ s_1 \cdots s_{i-1} \end{array} \right) \cdot \sum_{(b_1, \ldots, b_i) \in \mathbb{N}^i} \delta_{\sum_{j=1}^{i} b_j - a} \left( \begin{array}{c} a \\ b_1 \cdots b_{i-1} \end{array} \right) \left\langle V^{(s_1,b_1)} \cdots V^{(s_i,b_i)} \right\rangle_{C_L,S_L(\Phi)}^T. \quad (38)$$

with the agreement that $V^{(s,b)}(\Phi) = 0$ for $b > \left\lfloor \frac{s}{2} \right\rfloor$ or $s = 0$. 

8
4.4 Scaling equations

Equate (36) and (37) to obtain a linear equation

\[ L^{-r} \left\langle V^{(r,a)} \right\rangle_{C_L,S_L(\Phi)} - V^{(r,a)}(\Phi) = K_L(V)^{(r,a)}(\Phi) \]  

(39)

for \( V^{(r,a)}(\Phi) \), whose right hand side is the non-linear transformation

\[
K_L(V)^{(r,a)}(\Phi) = \sum_{b=a+1}^{[\frac{r}{2}]} \frac{\log(L)^{b-a}}{(b-a)!} V^{(r,b)}(\Phi) - L^{-r} \sum_{i=2}^{r} \frac{(-1)^{i+1}}{i!} \sum_{(s_1,\ldots,s_i) \in \mathbb{N}^i} \delta_{\sum_{j=1}^{i} s_j - r} \left( s_1 \cdots s_{i-1} \right) \sum_{(b_1,\ldots,b_i) \in \mathbb{N}^i} \delta_{\sum_{j=1}^{i} b_j - a} \left( b_1 \cdots b_{i-1} \right) \left\langle V^{(s_1,b_1)} ; \ldots ; V^{(s_i,b_i)} \right\rangle_{C_L,S_L(\Phi)}.
\]

(40)

The computation of the order \((r,a)\), with \( r \geq 3 \) and \( 0 \leq a \leq [\frac{r}{2}] \), requires knowledge of the lower \( g \) orders \((s,b)\), with \( 1 \leq s \leq r - 1 \) and \( 0 \leq b \leq [\frac{s}{2}] \), and of the higher \( \log(g) \) orders \((r,b)\), with \( a + 1 \leq b \leq [\frac{r}{2}] \). This is achieved by double recursion, which proceeds forward in the \( g \) orders and backwards in the \( \log(g) \) orders.

4.5 Scaling vertices

If all previous orders \((s,b)\), which needed to compute (40) in the double recursion, are of the polynomial form (33), then also (33) is of this polynomial form

\[
K_L(V)^{(r,a)}(\Phi) = \sum_{n=1}^{r-2a+1} \frac{1}{(2n)!} \int d^3x_1 \cdots \int d^3x_{2n} : \Phi(x_1) \cdots \Phi(x_{2n}) : K_L(V)^{(r,a)}_{2n}(x_1,\ldots,x_{2n}).
\]

(41)

The maximal order \( r - 2a + 1 \) thus iterates through the renormalization group. If the real space kernels of all previous vertices are given by Fourier integrals (8) of smooth momentum space kernels, then this is also true for

\[
K_L(V)^{(r,a)}_{2n}(x_1,\ldots,x_{2n}) = \int \frac{d^3p_1}{(2\pi)^3} \cdots \int \frac{d^3p_{2n}}{(2\pi)^3} (2\pi)^3 \delta(p_1 + \cdots + p_{2n}) e^{i(p_1 \cdot x_1 + \cdots + p_{2n} \cdot x_{2n})} \overrightarrow{K_L(V)^{(r,a)}_{2n}}(p_1,\ldots,p_{2n}).
\]

(42)

There is a technical point here. One has to show a large momentum bound on the momentum space kernels to ensure the convergence of the convolutions involved in (42). An exponential bound suffices, since the propagators (3) and (8) both have exponential ultraviolet cutoffs. The estimates of (W97) directly apply and yield such an exponential bound. But we have a better bound in this three dimensional case. The scaling equation (39) becomes a system of difference equations

\[
L^{3-r-n} \hat{V}^{(r,a)}_{2n}(L^{-1}p_1,\ldots,L^{-1}p_{2n}) - \hat{V}^{(r,a)}_{2n}(p_1,\ldots,p_{2n}) = \overrightarrow{K_L(V)^{(r,a)}_{2n}}(p_1,\ldots,p_{2n}).
\]

(43)
We solve it inductively. The right hand side can be assumed to consist of symmetric, Euclidean invariant, and smooth momentum space kernels. The unique solution of (43), which has again these properties, is

\[ \hat{V}_{2n}^{(r,a)}(p_1, \ldots, p_{2n}) = - \sum_{\alpha=0}^{\infty} L^\alpha (1-n) K_L(V)_{2n}^{(r,s)} (L^{-\alpha} p_1, \ldots, L^{-\alpha} p_{2n}). \]  

(44)

In other words, we have an explicit recursion relation, from which we obtain information about all orders of formal double perturbation expansion. This theory has one free parameter, the local \( \phi^2 \)-coupling at order \((2,0)\).

The scaling dimension \(3 - r - n\) in (43) is negative in all orders \(r \geq 3\), which we are now concerned with. Differing from the renormalizable case [W97], no subtractions are necessary in higher orders. The vertices are all irrelevant. The right hand side of (42) is a sum of convergent convolution integrals, where all vertices except for non-local second order \( \phi^2 \)-vertex are bounded. Consequently, the right hand side of (42) is bounded for all triples \((r, a, n)\) with \(r \geq 3\). Therefore,

\[ \| \hat{V}_{2n}^{(r,a)} \|_\infty < \infty \]  

(45)

iterates through the recursion (43).

5 Summary

We have computed the renormalized \( \phi_3^4 \)-trajectory in a double a double expansion in \( g \) and \( \log(g) \) with linear step \( \beta \)-function for a discrete renormalization group as a fixed point of a composed transformation, consisting of a renormalization group transformation and a backward flow of the coupling. Such a double expansion in \( g \) and \( \log(g) \) can be carried out also for the continuous renormalization group of Wilson [WK74] and Polchinski [P84]. We have chosen a discrete version as a first step towards a constructive analysis. The double expansion is expected to give accurate results for a small coupling region \( g << 1 \). It cannot be expected to tell much about the strong coupling region \( g >> 1 \). If there is no dynamical mass generation, the limit \( g \rightarrow \infty \) should approach the non-trivial Wilson fixed point. A convincing analysis of this limit would be highly desirable.

6 Appendix

6.1 Covariance \( C_L \)

The covariance \( C_L \) has both a Gaussian ultraviolet cutoff and a Gaussian infrared cutoff. It has the integral representation

\[ C_L(x, y) = \int_{-1}^{L^2} d\alpha \left( 4\pi \alpha \right)^{-\frac{3}{2}} e^{-\frac{(x-y)^2}{4\alpha}}. \]  

(46)

Consequently, \( C_L(x, y) \) is real, positive, and satisfies the bound

\[ C_L(x, y) \leq \frac{1}{4\pi^{3/2}} e^{-\frac{(x-y)^2}{4L^2}}. \]  

(47)
The Fourier transformed covariances $\hat{C}_L$ and $\hat{C}_\infty$ satisfy the bounds
\[
\|\hat{C}_L\|_\infty < \infty, \quad \|\hat{C}_L\|_1 < \infty, \quad \|\hat{C}_\infty\|_1 < \infty.
\] (48)
See [R91] for further information on (46).

6.2 $\beta$-function

Step $\beta$-functions $\triangle \beta_L$ are required to satisfy the composition law
\[
\triangle \beta_L \circ \triangle \beta_{L'} = \triangle \beta_{LL'}.
\] (49)
It follows that the associated running coupling, defined by $g(L) = \triangle \beta_L(g)$, is the solution of the ordinary differential equation
\[
L g'(L) = \beta(g(L)), \quad g(1) = g,
\] (50)
where $\beta$ is the differential $\beta$-function
\[
\beta(g) = \partial_L \triangle \beta_L(g) \bigg|_{L=1}.
\] (51)
Step $\beta$-functions and differential $\beta$-functions are in one-to-one correspondence.

Differential $\beta$-functions transform under coordinate transformations $\varphi(g)$ as vector fields,
\[
\overline{\beta}(\varphi) = \beta(g) \partial_g \varphi(g).
\] (52)
Suppose that our differential $\beta$-function behaves for small couplings as
\[
\beta(g) = \beta^{(1)} g + O(g^2)
\] (53)
with $\beta^{(1)} > 0$. This first coefficient then turns out to be universal. The normal form of this differential $\beta$-function turns out to be the linear function
\[
\overline{\beta}(\varphi) = \beta^{(1)} \varphi.
\] (54)
To see this, it suffices to consider coordinate transformations, which behave like
\[
\varphi(g) = \varphi^{(1)} g + O(g^2)
\] (55)
for small couplings. Eqs. (52) and (54) yield a differential equation for $\varphi(g)$. It is integrated to
\[
\varphi(g) = \varphi^{(1)} \exp \left( \beta^{(1)} \int_0^g \frac{dg'}{\beta(g')} \right).
\] (56)
The coefficient $\varphi^{(1)}$ is undetermined. We can choose $\varphi^{(1)} = 1$, for instance. For small couplings, the coordinate transformation then reads
\[
\varphi(g) = g + \frac{\beta^{(2)}}{\beta^{(1)}} \frac{g^2}{2} + O(g^3).
\] (57)
As a formal power series in $g$, it is computed by differentiating (56). It follows that there always exists such a coordinate transformation in the sense of a formal power series. Beyond perturbation theory there remains the question of the convergence of (54). With an additional bound on the $O(g^2)$ corrections in (52), its existence can be proved for small couplings.
6.3 Order $g^2$

To order $g$, we selected the potential

$$V(\Phi|g) = V^{(1,0)}(\Phi) \ g + O(g^2, g^2 \log(g)), \quad V^{(1,0)}(\Phi) = \mathcal{O}_4(\Phi). \quad (58)$$

To order $g^2$, we are then led to compute the truncated expectation value

$$\langle \mathcal{O}_4; \mathcal{O}_4 \rangle_{C_L S_L} = \int d^3x \int d^3y \left\{ \frac{1}{2} : \Phi(x)\Phi(y) : L^5 \mathcal{O}_2(L(x-y)) + \frac{1}{4!} : \Phi(x)^2\Phi(y)^2 : L^4 \mathcal{O}_4(L(x-y)) + \frac{1}{6!} : \Phi(x)^3\Phi(y)^3 : L^3 \mathcal{O}_6(L(x-y)) \right\}, \quad (59)$$

with real space kernels

$$\mathcal{O}_2(x-y) = \frac{1}{3} C_L(x-y)^3 + C_L(x-y)^2 L^{-1} C_{\infty}(L^{-1}(x-y)) + C_L(x-y) L^{-2} C_{\infty}(L^{-1}(x-y))^2, \quad (60)$$

$$\mathcal{O}_4(x-y) = 3 C_L(x-y)^2 + 6 C_L(x-y) L^{-1} C_{\infty}(L^{-1}(x-y)), \quad (61)$$

$$\mathcal{O}_6(x-y) = 20 C_L(x-y). \quad (62)$$

Define $C_{\infty}^L(x-y) = L^{-1} C_{\infty}(L^{-1}(x-y))$. The momentum space kernels are then given by convolutions

$$\hat{\mathcal{O}}_2(p) = \frac{1}{3} \hat{C}_L \star \hat{C}_L \star \hat{C}_L(p) + \hat{C}_L \star \hat{C}_L \star \hat{C}_{\infty}^L(p) + \hat{C}_L \star \hat{C}_{\infty}^L \star \hat{C}_{\infty}^L(p), \quad (63)$$

$$\hat{\mathcal{O}}_4(p) = 3 \hat{C}_L \star \hat{C}_L(p) + 6 \hat{C}_L \star \hat{C}_{\infty}^L(p), \quad (64)$$

$$\hat{\mathcal{O}}_6(p) = 20 \hat{C}_L(p). \quad (65)$$

It follows in particular that $\hat{\mathcal{O}}_2(0)$ is different from zero. The $g^2 \log(g)$ local $\phi^2$-vertex is thus indeed generated.

6.4 Logarithmic bound

Let $G(p), p \in \mathbb{R}$ for simplicity, be a given function such that

$$G(0) = 0, \quad G \in C^\infty(\mathbb{R}), \quad \|G\|_\infty < \infty. \quad (66)$$

Let $F(p)$ be the solution

$$F(p) = - \sum_{\alpha=0}^\infty G(L^{-\alpha} p) \quad (67)$$

of the difference equation

$$F(L^{-1} p) - F(p) = G(p), \quad (68)$$

for a given value of $L$ such that $L > 1$. 

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The series (67) converges uniformly on compact subsets of \( \mathbb{R} \). It defines a function \( F \) with the properties

\[
F(0) = 0, \quad F \in C^\infty(\mathbb{R}),
\]

and

\[
|F(p)| \leq \begin{cases} 
A |p|, & |p| \leq 1, \\
B + C \log |p|, & |p| > 1,
\end{cases}
\]

with

\[
A = \frac{1}{1 - L^{-1}} \sup_{|q| \leq 1} |G'(q)|, \quad B = A + \|G\|_{\infty}, \quad C = \frac{\|G\|_{\infty}}{\log(L)}. \tag{71}
\]

The bound (70) follows inductively from (68). For \( |p| \leq 1 \), we have that

\[
|G(p)| \leq \sup_{|q| \leq 1} |G'(q)| |p|. \tag{72}
\]

The first case in (70) follows. For \( 1 < |p| \leq L \), then (68) implies that

\[
|F(p)| \leq A L^{-1} |p| + \|G\|_{\infty} \leq B + C \log |p|. \tag{73}
\]

We then proceed from \( L^{n-1} < |p| \leq L^{n} \) to \( L^{n} < |p| \leq L^{n+1} \) using (68) to obtain

\[
|F(p)| \leq B + C \log |p| - C \log(L) + \|G\|_{\infty} = B + C \log |p|, \tag{74}
\]

proving the second case in (70).

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