Tropical geometric interpretation of ultradiscrete singularity confinement

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Abstract
Using the interpretation of the ultradiscretization procedure as a non-Archimedean valuation, we use results of tropical geometry to show how roots and poles manifest themselves in piece-wise linear systems as points of non-differentiability. This will allow us to demonstrate a correspondence between singularity confinement for discrete integrable systems and ultradiscrete singularity confinement for ultradiscrete integrable systems.

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(Some figures may appear in colour only in the online journal)

1. Introduction
The ultradiscretization procedure takes a subtraction-free rational function to a function defined over the max-plus semifield via a non-analytic limit [33]. This procedure famously linked integrable soliton equations to integrable cellular automata, hence, the ultradiscretization procedure was thought to preserve properties associated with integrability [32, 31, 33]. The ultradiscretization procedure has been used to create various new and novel ultradiscrete systems such as the ultradiscrete Painlevé equations [28, 24], ultradiscrete Quispel–Roberts–Thompson maps [21] and ultradiscrete lattice equations [33].

Many of the key properties of the original systems seemed to survive the ultradiscretization procedure; the Lax integrability [26, 14], symmetries [34], conserved quantities [17] and special solutions [23]. Recently, the interesting observation was made by Hone and Fordy that the ultradiscrete versions of mappings govern the growth of degrees of (subtraction-free) mappings in the initial conditions [8].

The crucial point this note makes is that one may interpret the ultradiscretization procedure as a map that sends singular points, such as poles and roots, to points of non-differentiability. This involves considering the ultradiscretization of arbitrary rational functions, not just subtraction-free rational functions. With this in mind, the ultradiscrete analogue of singularity confinement, proposed by Lafortune and Joshi [13], may be understood as the image of
singularity confinement [9, 27]. While we are unable to show a general correspondence, a correspondence holds for a large class of examples. We attempted to capture the essence of this correspondence with a few simple cases.

For this to make sense, we first interpret the ultradiscretization procedure as a non-Archimedean valuation, which allows the ultradiscretization procedure to be understood in terms of tropical geometry [29]. By realizing the ultradiscretization procedure as a non-Archimedean valuation, the image of a singularity makes sense. This will also allow us utilize the results of tropical geometry [19], or more specifically, we use the original results of Bieri and Groves [5] and Richter-Gebert et al [29], to show how roots and poles are mapped to points of non-differentiability via the ultradiscretization procedure. One of the implications of this interpretation is that we are able to shed some light on the correspondence between singularity confinement for discrete integrable systems and the proposed ultradiscrete version of singularity confinement [13] over the max-plus semifield [25].

There have been many attempts at reconciling the role of subtraction in the ultradiscretization procedure, such as the so-called inversible max-plus algebra [22], the $s$-ultradiscretization [12] and more analytic approaches [18]. The approach that is closest to our framework is that of Kasman and Lafortune [18], yet they do not make any tropical geometric connections in their work. Algebraically, this approach is actually closer to the arithmetic integrability of Kanki et al [16].

This brings us to our second underlying point; we wish to strengthen the argument that a fundamental understanding of the geometry of these piece-wise linear systems should be phrased in terms of the points of non-differentiability. There are some who believe that the concept of a singularity is lost when passing to the ultradiscrete, however, we argue through the realization of the ultradiscretization procedure as a non-Archimedean valuation that the singularities have just manifested themselves in a different way.

In section 2, we will introduce the ultradiscretization procedure and show how to realize the ultradiscretization procedure as a lift of the system to a extension of the field of algebraic functions followed by the application of a non-Archimedean valuation. In section 3 we will study some particular cases of both ultradiscrete ordinary difference and partial difference equations to see how using the valuations around various singularities of the lifted equations map to points of discontinuity of the ultradiscrete systems.

2. Ultradiscretization

The algebraic domain of ultradiscrete systems are the tropical semifields, which are the min-plus and max-plus semirings (which are isomorphic to each other) [25]. From an integrable perspective, we will use the max-plus semifield, which we set to be $S = \mathbb{R} \cup \{-\infty\}$, with operators, $\otimes$ and $\oplus$, defined as

\[
a \otimes b = a + b,
\]

\[
a \oplus b = \max(a, b),
\]

referred to as tropical multiplication and tropical addition respectively [29]. The element $-\infty$ is a tropical additive identity and is adjoined for convenience. These operators may be extended to matrices over the max-plus semiring in a natural way [26, 14] and also to the field of tropically rational functions, $\mathbb{S}(X_1, \ldots, X_n)$, in a natural manner. A tropical semifield possesses a tropical division operator, $\oslash$, but there is no tropical analogue of the subtraction operation [29].
2.1. Ultradiscretization as a non-analytic limit

Let us first recount the definition of the ultradiscretization procedure as it appears in the literature [33]. Let us start with a rational function of a number of positive variables, $f(x_1, \ldots, x_n)$, which does not require the operation of subtraction to be expressed (i.e., the function is subtraction-free). That is to say that the $x_i$ and the coefficients of the $x_i$ in the numerator and denominator are positive and real. We introduce a number of ultradiscrete variables, $X_i$, via the relation $x_i = e^{X_i/\epsilon}$. The ultradiscretization of $f$, denoted $F(X_1, \ldots, X_n)$, is defined as the non-analytic limit

$$F(X_1, \ldots, X_n) := \lim_{\epsilon \to 0} \epsilon \log f(e^{X_1/\epsilon}, \ldots, e^{X_n/\epsilon}).$$

The positivity of the variables and subtraction-free nature of the function plays a crucial role in making the ultradiscretization procedure a homomorphism of the semifield of subtraction-free rational functions to the tropical semifield, $\mathbb{S}(X_1, \ldots, X_n)$. The way in which this homomorphism may be computed is simply via the following replacement of binary operations:

$$\begin{align*}
    x_1 + x_2 &\to \max(x_1, x_2) = X_1 \oplus X_2, \\
    x_1 x_2 &\to x_1 + x_2 = X_1 \odot X_2, \\
    x_1 / x_2 &\to x_1 - x_2 = X_1 \ominus X_2.
\end{align*}$$

**Lemma 2.1.** The ultradiscretization procedure, defined by (2.1), of any subtraction-free rational function is equivalent to the replacement of the $x_i$ with $X_i$ and the replacement of binary operations defined by (2.2).

To obtain an ultradiscrete integrable system, we take a known equation that defines a discrete integrable system and perform the ultradiscretization procedure to both sides. For example, let us take the QRT mapping defined by the recurrence relation

$$x_{n-1}x_{n+1} = a_3a_4(x_n + a_1)(x_n + a_2),$$

which is related to the map

$$\phi \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} y \\ x(a_3 + y)(y + a_1) \end{array} \right),$$

which preserves the biquadratic

$$I = \frac{(a_1 + y)(a_2 + y)}{xy} + \frac{x(a_3 + y)(a_4 + y)}{a_3a_4y} + \frac{a_1 + a_2 + \left(\frac{1}{a_1} + \frac{1}{a_2}\right)y^2}{y}.$$

By following (2.2), we obtain the ultradiscrete analogue, defined by

$$X_{n-1}X_{n+1} = A_3 + A_4 + \max(X_n, A_1) + \max(X_n, A_2) - \max(X_n, A_3) - \max(X_n, A_4),$$

which is related to the map

$$\Phi \left( \begin{array}{c} X \\ Y \end{array} \right) = \left( \begin{array}{c} A_3 + A_4 + \max(Y, A_1) + \max(Y, A_2) \\ - \max(Y, A_3) - \max(Y, A_4) \end{array} \right)$$

and preserves the tropical biquadratic

$$I = \max(\max(Y, A_1) + \max(Y, A_2) - Y - Y, \max(A_1, A_2)) - Y,$$

$$Y - \min(A_3, A_4), X - Y + \max(Y - A_3, 0) + \max(Y - A_4, 0)).$$

If the initial conditions, $X_n$ and $X_{n-1}$, and the parameters, $A_i$, are integers, the resulting system may be defined over the integers, hence, systems such as this are often referred to as cellular automata [14].
2.2. Ultradiscretization as a non-Archimedean valuation

Tropical geometry is a skeletonization of algebraic geometry where the main objects of interest are convex polyhedra [29]. A tool that is ubiquitous in this theory is the realization of the image of a variety under a non-Archimedean valuation as a polyhedral complex, which was initially established by Bieri and Groves [5] and Richter-Gebert et al [29]. To elucidate the link between tropical geometry and the ultradiscretization process, we must first realize the ultradiscretization procedure as a non-Archimedean valuation.

The base field that is often used in tropical geometry is the field of the Puiseux series [29] in an indeterminant, \( t \), given by

\[
\mathbb{K} = \bigcup_{n=1}^{\infty} \mathbb{C}\left(\left(t^{\frac{1}{n}}\right)\right).
\]

Every element, by definition, admits a representation of the form

\[
f(t) = a_0 t^{q_0} + a_1 t^{q_1} + \cdots
\]

where \( a_0 \neq 0 \) and the \( q_i \in \mathbb{Q} \) are ordered such that \( q_i < q_{i+1} \) for all \( i \), and the set of \( q_i \) possess a common denominator [29]. The Puiseux series over \( \mathbb{C} \) is an algebraically closed field of characteristic 0. It may also be thought of as the algebraic closure of the field of Laurent series.

There is a natural valuation, \( \nu \), on this field, defined by

\[
\nu(a_0 t^{q_0} + \cdots) = -q_0.
\] (2.5)

In general, a valuation possesses the following properties:

1. \( \nu(f) = -\infty \) if and only if \( f = 0 \).
2. \( \nu(fg) = \nu(f) + \nu(g) \).
3. \( \nu(f + g) \leq \nu(f) + \nu(g) \).

The valuation, \( \nu \), also satisfies the non-Archimedean triangle inequality:

3a. \( \nu(f + g) \leq \max(\nu(f), \nu(g)) \).

Any valuation field is metrizable, where the metric is

\[
d(f, g) = e^{\nu(f-g)}.
\] (2.6)

The Puiseux series have two minor deficiencies; firstly that the valuation group (i.e., the image of \( \nu \)) is \( \mathbb{Q} \), hence, tropical geometers require an additional closure to obtain polyhedral complexes in \( \mathbb{R}^n \). Secondly, the field is not complete with respect to the metric. One way to overcome this is to deal with a more general field, which comprises of the formal series expansions in an indeterminant \( t \) whose powers are elements of \( \mathbb{R} \). One may find a suitable field extension, from which one may lift the valuation, to a valuation field that is algebraically closed and complete;

\[
\mathbb{R} = \left\{ f = \sum_i a_i t^{q_i} \mid \text{the } q_i \text{ are well ordered} \right\}.
\]

This was introduced in [19], however, an isomorphic algebraic construction appeared in [15]. The valuation is still defined by (2.5), and the metric is still given by (2.6). This field also has the advantage that its valuation group is \( \mathbb{R} \), making it an excellent candidate as a base field for the use in tropical geometry [19]. To express the ultradiscretization as a non-Archimedean valuation, for every positive variable, \( x_j \), we introduce ultradiscrete variables, \( X_j \), via the relation \( x_j = t^{-X_j} \). The ultradiscretization of a subtraction-free function \( f(x_1, \ldots, x_n) \), denoted \( F(X_1, \ldots, X_n) \), is redefined as

\[
F(X_1, \ldots, X_n) := \nu(f(t^{-X_1}, \ldots, t^{-X_n})).
\] (2.7)
Lemma 2.2. The ultradiscretization procedures defined by (2.1) and (2.7) coincide.

One of the advantages of viewing the ultradiscretization in this way is that the valuation is defined for all rational functions. However, the ultradiscretization procedure as a map defined on all rational functions to tropical rational functions, \( \nu : \mathbb{C}(x_i) \to \mathbb{S}(X_i) \), is not a homomorphism of semifields. In defining the ultradiscretization of a rational function, we replace the equality we had for the image of a subtraction-free rational function, with an inequality for the images of the numerator and denominator of a rational function. We are not at a total loss, as we have at our disposal a well known simple lemma.

Lemma 2.3. For \( f, g \in K \), if \( \nu(f) \neq \nu(g) \), then \( \nu(f + g) = \max(\nu(f), \nu(g)) \).

We used this lemma to show that a hypergeometric solution over \( K \) faithfully mapped to a hypergeometric solution of u-P\(_{III}\) over \( S \) [23]. We now state the main theorem that we will use.

Theorem 2.4. Given a principle ideal, \( I = (f) \), of the polynomial ring, \( I \subset K[x_1, \ldots, x_n] \), the tropical variety, \( \text{Trop}(I) := \nu(V(I)) \), coincides with the points where the ultradiscretization of \( f \) is not differentiable.

Despite the integrable wording, this is a result that should be rightly attributed to those in tropical geometry [5, 29, 19]. We have assumed a bit of notation here, \( V(I) \subset K^n \) is the variety associated with an ideal:

\[
V(I) = \bigcap_{g \in I} \{ g(x) = 0, x \in K \},
\]

and \( S \), for any set \( S \), is the topological closure with respect to the Euclidean metric on \( \mathbb{R}^n \). Naturally, if we deal with a field whose valuation group is \( \mathbb{R} \) itself (rather than \( \mathbb{Q} \)), then this topological closure is not necessary [19], however, we include it in keeping with standard work on tropical geometry [29]. One final result required for this correspondence to be complete is that the tropical variety is a function only of the valuations of the various coefficients [7]. This correspondence provides tropical geometers with a straightforward way of computing tropical varieties [29].

Let us demonstrate the power of this theorem by considering a line defined by \( f(x, y) = 0 \), where

\[
f(x, y) = ax + by + 1.
\]

Let us lift this function to the field, \( K \), in which \( \nu(a) = A \) and \( \nu(b) = B \). Solving \( f(x, y) = 0 \) in \( y \) gives

\[
y = -\frac{1 + ax}{b}.
\]

By applying the valuation and using lemma 2.3, if \( \nu(x) + A \neq 0 \), then \( \nu(y) = \max(0, A + \nu(x)) - B \). Conversely, expressing \( x \) as a function of \( y \), we obtain \( \nu(x) = \max(0, B + \nu(y)) - A \). This allows us to determine \( \text{Trop}(f) \), which we characterize as a node at \((-A, -B)\) with three semi-infinite rays, as depicted in figure 1.

As for what the correspondence is with the ultradiscretization procedure, since \( f \) is subtraction free, we apply our valuation, where \( \nu(x) = X \) and \( \nu(y) = Y \), which are assumed to be subtraction-free, to give an ultradiscretized function

\[
F(X, Y) := \max(A + X, B + Y, 0).
\]

This defines three regions of \( \mathbb{R}^2 \) defined by which of the three arguments of the max expression is dominant, and the points in which there is a switch between one of the arguments being
dominant in the max expression to another argument being dominant is clearly a point in which $F$ is not differentiable. The remainder is easy to see.

In addition to the correspondence, one may also compute the stable intersection\(^1\) of two lines fairly easily from this framework. Let us consider another line over $\mathbb{K}$, given by $g(x, y) = cx + dy + 1 = 0$. It may be associated with a tropical line defined as the points of non-differentiability of

$$G(X, Y) := \max(C + X, D + Y, 0),$$

where $\nu(c) = C$ and $\nu(d) = D$. If we wish to find the stable intersection of these two tropical lines, we find that the point of intersection of the two lines over $\mathbb{K}$ is given by

$$p = \left( \frac{b - d}{ad - bc}, \frac{a - c}{bc - ad} \right),$$

whose valuation, under the assumption that

$$(\nu(b) - \nu(d)) (\nu(a) - \nu(c)) (\nu(ad) - \nu(bc)) \neq 0,$$

gives us the (stable) intersection point,

$$P = (\max(B, D) - \max(A + D, B + C), \max(A, C) - \max(A + D, B + C)).$$

One can verify this is indeed the intersection point using elementary arguments.

3. Singularity confinement

The singularity confinement property, for both maps and lattice equations, is often used as evidence for the integrability of a system [9, 27]. It is not a definition of integrability, in fact, there is no one definition [35]. There are some systems that possess the singularity confinement property that are even chaotic, hence, would not be considered integrable [4]. Nonetheless, in the integrable systems community, possessing the singularity confinement property is still considered evidence for integrability.

The singularity confinement property for maps has been used in many contexts; to de-autonomize known integrable autonomous maps [27], to define new integrable interesting mappings and to distinguish integrable maps from non-integrable maps [9]. The singularity

\(^1\) For a definition of stable intersection, we refer to [29].
confined property is very useful in defining connections between the theory of integrable mappings and rational surfaces [30, 6].

For lattice equations, the singularity confinement property is not as widely used, but has been used to provide evidence for integrability [9] and to de-autonomize lattice equations [10].

3.1. Tropical singularity confinement for OΔE’s

In terms of the definition of singularity confinement, the original presentation was via a series of examples. This definition can be made more concrete if we are able to phrase it in terms of injectivity. Let us consider a map, \( \phi : \mathbb{C}^n \to \mathbb{C}^n \), which defines a discrete dynamical system by letting \( x_{k+1} = \phi(x_k) \). A point is called a singularity if \( \phi \) is not injective at it. These typically occur at roots and poles of the components of the map.

**Definition 3.1.** We say that a map has the singularity confinement property if for all \( x \) such that \( \phi \) is singular at \( x \), there exists an \( n \) such that \( \phi^n \) is non-singular at \( x \).

Let us consider a class of maps that are specified by the recurrence relation

\[
\begin{align*}
x_{n+1} = & a + bx_n, \\
& \text{where } \sigma = 0, 1, 2 \text{ and } 3.
\end{align*}
\]

These maps are associated with the class of mappings

\[
\phi(x, y) = \left( \frac{y}{a + by}, \frac{y}{x} \right).
\]

This is a class of systems that has been considered numerous times to test whether a integrability criteria distinguishes integrable equations from non-integrable ones. When \( \sigma = 0, 1 \text{ or } 2 \), this is a QRT map, hence, it is an integrable difference equation, in fact, they preserve the integrals

\[
\begin{align*}
I_0 &= \frac{y(a + b^2 + x^2) + (b + x)(a + bx) + y^2(b + x)}{bxy}, \\
I_1 &= \frac{a + bx + by + x^2y + xy^2}{xy}, \\
I_2 &= \frac{a + bx + by + x^2y^2}{xy},
\end{align*}
\]

respectively, whereas the case in which \( \sigma = 3 \) is not integrable. Each map sends the line, \((x_0, -a/b)\), parameterized by \( x_0 \), to the point \((-a/b, 0)\), hence, in each case, \( \phi \) is not injective.

If we let \( x_1 = -a/b + \epsilon \), then the iterates of the map in the limit as \( \epsilon \to 0 \) are

| \( \sigma \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | \( x_6 \) | \( x_7 \) | \( x_8 \) | \( x_9 \) |
|---|---|---|---|---|---|---|---|---|
| 0 | -\frac{a}{b} | 0 | -b | \infty | \infty | \infty | -b | 0 | -\frac{a}{b} |
| 1 | -\frac{a}{b} | 0 | \infty | \infty | 0 | -\frac{a}{b} | x_0 | * | * |
| 2 | -\frac{a}{b} | 0 | \infty | 0 | -\frac{a}{b} | x_0 | * | * | * |
| 3 | -\frac{a}{b} | 0 | \infty | 0 | \infty | 0 | \infty | 0 | \infty |

We readily find that one of the iterates for the integrable cases contains the data from the initial condition, \( x_0 \) (we have omitted iterates after that point). The non-integrable case, when \( \sigma = 3 \), alternates between \((x_{k-1}, x_k) = (0, \infty)\) and \((\infty, 0)\) ad infinitum. Another way to calculate this singularity pattern, which is relevant to the correspondence between singularity confinement and tropical singularity confinement, is to start with \( x_1 = -b/ae^\delta \) where one is concerned with the limit as \( \delta \to 0 \). The calculations follow analogously. The above constitutes the way in which this procedure is usually viewed [9]. Alternatively, one may calculate the determinants of the Jacobians of the powers of \( \phi \).

We may now turn to the tropical analogue of singularity confinement, as discussed by Joshi and Lafortune [13].
**Definition 3.2.** We say that a piece-wise linear map has the tropical singularity confinement property if for all $x$ such that $\Phi$ is not differentiable at $x$, there exists an $n$ such that $\Phi^n$ that is differentiable at $x$.

Making a correspondence between singularity confinement and tropical singularity confinement is still not so straightforward. We cannot use theorem 2.4 directly as the theorem concerns ideals generated by a polynomial, not a rational function, in which an equivalent statement is not true. Problems arise when two singularities have the same valuation, for example, the function $f(x) = (1 + 2x)/(1 + x)$ has a root and pole whose valuations are both 0. This means, one could devise a rational subtraction-free map that passes the singularity confinement test that does not pass the ultradiscrete singularity confinement test. Furthermore, because there are systems that possess the same ultradiscretization, if a map’s ultradiscrete analogue possesses tropical singularity confinement, this does not mean the map itself possesses singularity confinement.

We require an extra condition for a version of theorem 2.4 for rational functions; that the collection of roots and poles have distinct valuations, which is generically true, but not generally true. Under this condition, we simply apply theorem 2.4 to the numerator and denominator to obtain the point that the function is discontinuous. Applying this to mappings with singularities that have distinct valuations, then a singularity will map to a tropical singularity.

Given a subtraction-free rational map, $\phi$, whose singularities all have distinct valuations, and point, $p$, which is a confined singularity of $\phi$ (i.e., is not a singularity of some $\phi^n$), what we can say that the ultradiscretization, $\Phi$, has a singularity at the ultradiscretization of $p$, which we denote $P$, and that $p$ does not contribute to $\Phi^n$ being singular at $P$. What this cannot guarantee is that $\phi^n$ possesses a singularity, $p'$, such that $v(p) = v(p') = P$. This could make $\Phi^n$ tropically singular at $P$ despite $\phi^n$ not being singular at $p$.

This situation does not occur generically, and certainly does not occur for many interesting integrable systems in the literature, such as ultradiscrete QRT mappings and ultradiscrete Painlevé equations (in cases where the singularities have distinct valuations). These two classes encompasses the examples of maps studied by Joshi and Lafoutre [13]. This means that tropical singularity confinement, for many examples (such as those shown below), is indeed a direct image of singularity confinement for maps. That is to say, we can see that a direct correspondence generically holds, but may not generally hold. Unfortunately, we can only conjecture that singularity confinement for an arbitrary subtraction-free mapping, whose singularities have distinct valuations, implies the tropical singularity confinement of its ultradiscretization.

Let us illustrate how the image of singularity confinement can map to tropical singularity confinement for the class of mappings above. We first consider the tropical singularity confinement test for the ultradiscretization of (3.1), which is

$$X_{n-1} + x X_n + X_{n-1} = \max(A, B + X_n).$$

(3.2)

For $\sigma = 0, 1$ and 2, this system is in the class of ultradiscrete QRT equations studied by Nobe [21]. The invariants are

$$I_0 = \max(Y + \max(A, B, 2X), \max(B, X) + \max(A, B + X),$$
$$2Y + \max(B, X)) - B - X - Y,$$

$$I_1 = \max(A, B + X, B + Y, 2X + Y, X + 2Y) - X - Y,$$

$$I_2 = \max(A, B + X, B + Y, 2X + 2Y) - X - Y.$$

At this point, we simplify the calculations by setting $A = B = 0$, in which the level sets are depicted in figure 2.
Figure 2. The level sets for the invariants, $I_0$, $I_1$ and $I_2$ (left to right), of (3.1) where $A = B = 0$.

We now calculate the difference between the left and right derivatives (left minus the right) where a non-zero discrepancy indicates non-differentiability. These differences, for each of the iterates, are given by

$$
\begin{array}{c|c|c|c|c|c}
\sigma & X_3 & X_4 & X_5 & X_6 & X_7 \\
0 & \{0,1\}x_0 & 0 & 0 & 0 & 1 \\
1 & \{-1, 0\}x_0 & \{-1, 0\}x_0 & \{1, 0\}x_0 & 0 & 0 \\
2 & \{-2, -1\}x_0 & 1 & 0 & 0 & 0 \\
3 & \{-3, -2\}x_0 & 5 & \{-12, -8\}x_0 & \{19, -45, 30\}x_0 & \\
\end{array}
$$

where we have used the notation $\{d_1, d_2\}x_0$ to mean there is a discrepancy (left derivative minus the right derivative) of $d_1$ if $X_0 < 0$ and $d_2$ if $X_0 > 0$ (we omit the case $X_0 = 0$ for concision).

There are more complicated formulas for more general $A$ and $B$ that we omit. The point of this exercise is that at some point, we have that the left derivative minus the right derivative is 0 (underlined above), indicating that the functions are continuous in each of the integrable cases, whereas it is possible to show for $\sigma = 3$ that $X_k$ is discontinuous for all $k > 1$.

To understand this pattern, we first consider the system (3.1) as being defined over $\mathbb{K}$, with the specialization $a = b = 1$. To consider the tropical singularity confinement of the tropical map, we calculated the derivative from first principles, which involved an additive factor of $\delta$. As addition in the max-plus semifield is analogous to multiplication over $\mathbb{K}$, we take a perturbed initial condition of $x_1 = -t^{-\delta}$. If we take the map in the case that $\sigma = 2$, the few iterates are

$$
\begin{align*}
x_2 &= -\frac{\Delta t^\delta}{x_0}, \\
x_3 &= \frac{x_0(\Delta - x_0t^{-\delta})}{\Delta^2}, \\
x_4 &= \frac{\Delta(x_0^2 - \Delta t^\delta(\Delta + x_0))}{x_0(x_0 - \Delta t^\delta)^2}, \\
x_5 &= \frac{t^\delta(\Delta^3 - x_0)(\Delta^3 t^\delta + x_0(\Delta^2 + x_0)(\Delta t^\delta - x_0))}{(x_0^2 - \Delta t^\delta(\Delta + x_0))^2}, \\
x_6 &= \frac{x_0(-\Delta^3 x_0 t^{3\delta} + \Delta^3 t^{2\delta}(\Delta^2 + 3x_0(x_0 + 1)) - \Delta x_0^2(3x_0 + 2)t^\delta + x_0^4)}{t^{2\delta}(\Delta^3 t^\delta + x_0(\Delta^2 + x_0)(\Delta t^\delta - x_0))^2} \\
&\quad\times (x_0^2 - \Delta t^\delta(\Delta + x_0))
\end{align*}
$$

where we have introduced notation for the factor $
\Delta = 1 - t^\delta$.
whose valuation is the discontinuous function
\[ v(\Delta) = \begin{cases} 
\max(\delta, 0) & \text{for } \delta \neq 0, \\
-\infty & \text{for } \delta = 0.
\end{cases} \]

This factor of $\Delta$ encapsulates the correspondence between discontinuities and singularities. If we replace $t$ with a number, the limit as $\delta \to 0$ gives us the usual singularity pattern, and if we leave $t$ as an indeterminate and apply the non-Archimedean valuation, we obtain the pattern of non-differentiability at $\delta = 0$. One of the reasons this works is because $t^\delta$ is algebraically a generic arbitrary initial condition, hence, we may assume that lemma 2.3 holds. If one expands out the iterates, the factors of $\Delta$ explain precisely the degrees of the non-differentiability, showing how the image of the usual singularity confinement, under the ultradiscretization procedure, is tropical singularity confinement.

If one performs the same test on the case in which $\sigma = 3$, we readily find the iterates over $\mathbb{K}$ are
\[
x_2 = \frac{\Delta t^{23}}{x_0},
\]
\[
x_3 = -\frac{\Delta^3}{x_0^5 (\Delta t^{23} + x_0)}
\]
\[
x_4 = \frac{\Delta^5 t^{58}}{x_0^3 (\Delta t^{23} + x_0)}
\]
\[
x_5 = \frac{x_0^5 (\Delta t^{23} + x_0)^5}{\Delta^3 (\Delta t^{23} + x_0) \delta}
\]
\[
= \Delta^{12} (\Delta t^{23} + x_0)^3
\]
which explains precisely the degree of the non-differentiability observed above. If we allow $t$ to be some constant, the limit as $\delta \to 0$ gives us the usual singularity pattern, whereas, the non-Archimedean valuation gives us the pattern of non-differentiability. Since the singularity is not confined over the field, we will always find a prefactor of $\Delta$ to some power in the iterates, which will make the iterate not differentiable at $\delta = 0$. We have simplified the calculations by presenting the case in which there is just one point-of-non-differentiability, however, the arguments have worked equally well for many mappings with more points of non-differentiability, such as (2.4).

### 3.2. Tropical singularity confinement for PDEs

We first consider singularity confinement, as it is described in [9]. We specialize this to quad-graphs, specified by a multilinear function
\[
q(w_{l,m}, w_{l+1,m}, w_{l,m+1}, w_{l+1,m+1}; \alpha, \beta) = 0.
\] (3.3)
The multilinearity ensures that we may solve this equation for any of the desired variables. For example, in solving for $w_{l+1,m+1}$, we use the expression
\[
q = \frac{\partial}{\partial w_{l+1,m+1}} q(w_{l,m}, w_{l+1,m}, w_{l,m+1}, w_{l+1,m+1}; \alpha, \beta)
\]
\[+ q(w_{l,m}, w_{l+1,m}, w_{l,m+1}; \alpha, \beta),
\]
\[= 0,
\]
\[q(w_{l,m}, w_{l+1,m}, w_{l,m+1}; \alpha, \beta) = 0.
\]
The key property is having the first equation hold, which admits two interpretations in the literature:
The second equation does not necessarily hold and \( w_{l+1,m+1} \) involves a denominator that becomes 0, hence, the value of \( w_{l+1,m+1} \) tends to \( \infty \), hence, is singular.

The second equation necessarily holds, hence, \( w_{l+1,m+1} \) may be chosen arbitrarily.

In both cases, note that there is some loss of information, as \( w_{l+1,m+1} \) does not contain information regarding \( w_{l,m} \). We acknowledge the second interpretation, as considered by Atkinson [3], however, for this work we adopt the first interpretation, which is consistent with the original implementation of singularity confinement [9].

**Definition 3.3.** A singularity is confined if one may iterate beyond singular boundaries to reclaim initial conditions in some limit.

Let us consider the example of the modified Korteweg–de Vries equation [11, 20], or \( H3(\delta = 0) \) as it appears in [1, 2] (with the parameter \( \beta \to -\beta \)),

\[
\alpha (w_{l,m}w_{l+1,m} - w_{l,m+1}w_{l+1,m+1}) + \beta (w_{l,m}w_{l+1,m+1} - w_{l+1,m}w_{l+1,m+1}) = 0, \tag{3.4}
\]

which we solve in \( w_{l+1,m+1} \) to obtain

\[
w_{l+1,m+1} = w_{l,m} \frac{\alpha w_{l+1,m} + \beta w_{l+1,m+1}}{\beta w_{l+1,m} + \alpha w_{l+1,m+1}}. \tag{3.5}
\]

Differentiating (3.4) with respect to \( w_{l+1,m+1} \) shows us that the equation is clearly singular when \( w_{l+1,m} = -\alpha w_{l+1,m+1}/\beta \), hence, let us define initial conditions

\[
w_{0,0} = 1 + \epsilon, \quad w_{-1,1} = -\frac{\beta}{\alpha},
\]

\[
w_{-1,0} = x_1, \quad w_{-1,-1} = -\frac{\alpha}{\beta},
\]

\[
w_{0,-1} = x_2.
\]

From these initial conditions, it is clear that \( w_{1,0} \) and \( w_{0,1} \) tend to \( \infty \) as \( \epsilon \to 0 \). This configuration is depicted in figure 3.

The key observations is that while \( w_{1,0} \to \infty \) and \( w_{0,1} \to \infty \) as \( \epsilon \to 0 \), the limit of \( w_{1,1} \) as \( \epsilon \to 0 \) is non-singular and involves both \( x_1 \) and \( x_2 \). Similarly to the situation for maps, this singularity pattern may also be observed by considering a multiplicative deviation from the singular boundary. That is, if we replace \( w_{0,0} = e^{\delta} \), the limit as \( \delta \to 0 \) gives us the same pattern.
Remark 3.4. Another way in which we may approach singular boundary conditions is if we were to let \( w_{-1,1} = -\frac{a}{\beta} + \epsilon_1 \) and \( w_{1,-1} = -\frac{\beta}{\alpha} + \epsilon_2 \). If \( \epsilon_1 = \alpha \epsilon \) and \( \epsilon_2 = \beta \epsilon \), then we may evaluate \( w_{1,1} \) in the limit as \( \epsilon \to 0 \), which is given by
\[
\lim_{\epsilon \to 0} w_{1,1} = \frac{b\beta^3 x_1 - a\alpha^3 x_2}{ab\beta^2 x_1 - a\alpha^2 \beta x_2},
\]
which not only depends on \( x_1 \) and \( x_2 \) but also depends on the way in which we approach the singular boundary. Put another way, the limiting value of \( w_{1,1} \) depends explicitly on the projective coordinate
\[
\left( w_{-1,-1} + \frac{\alpha}{\beta} : w_{-1,1} + \frac{\beta}{\alpha} \right),
\]
which is the blow-up coordinate of \((w_{-1,1}, w_{1,-1})\)-space at \((-\frac{a}{\beta}, -\frac{\beta}{\alpha})\).

If we now consider ultradiscrete lattice equations, there are many that have appeared in the literature, such as the ultradiscrete modified Korteweg–de Vries equation \([31, 26]\), ultradiscrete Lotka–Volterra equation \([33]\) and the ultradiscrete Kadomtsev–Petviashvili equation \([32]\). Our example will be from a particular class of equations, which we would consider the ultradiscrete analogue of those in the ABS-list \([1]\). We may now interpret \((3.3)\) as a function over \(\mathbb{K}\), then the image, under the valuation, is some tropically multi-affine linear function
\[
Q(W_{l,m}, W_{l+1,m}, W_{l,m+1}, W_{l+1,m+1}; A, B).
\]
(3.6)

Given the correspondence between \(0s\) and points of non-differentiability, we may interpret the evolution in terms of values taken by one of the lattice variables so that this function is not differentiable. We claim that this is a valid tropical analogue of the evolution defining lattice equations.

If we attempt to find where this function is not differentiable for one of its arguments, say \(W_{l+1,m+1}\), it is abundantly clear that \(Q\) admits a representation of the form
\[
Q = \max(W_{l+1,m+1} + F(W_{l,m}, W_{l+1,m}, W_{l,m+1}), G(W_{l,m}, W_{l+1,m}, W_{l,m+1})).
\]
It is clear that \(Q\) is not differentiable in \(W_{l+1,m+1}\) when
\[
W_{l+1,m+1} = G(W_{l,m}, W_{l+1,m}, W_{l,m+1}) - F(W_{l,m}, W_{l+1,m}, W_{l,m+1}).
\]
If \(W_{l,m}, W_{l+1,m}\) and \(W_{l,m+1}\) are chosen generically, this evolution makes sense.

Analogously to the discrete case, we may consider what it means for \(W_{l,m}, W_{l+1,m}\) and \(W_{l,m+1}\) to form a tropically singular boundary. If we just require \(Q\) to be non-differentiable in any variable, we may find \(W_{l,m}, W_{l+1,m}\) and \(W_{l,m+1}\) such that \(F\) (or \(G\)) are not differentiable, in which case \(W_{l+1,m+1}\) may take any value that makes \(F + W_{l+1,m+1}\) (or \(G\)) dominate the max expression. This multivaluedness mimics the second interpretation of a singularity.

This definition of the evolution also admits a version of the second interpretation of a singularity, for if \(F\) or \(G\) are not-differentiable, we may still assign \(W_{l+1,m+1}\) to be \(G - F\), and then \(W_{l+1,m+1}\) is just singular in the sense that it is not differentiable in the variables \(W_{l,m}, W_{l+1,m}\) and \(W_{l,m+1}\).

So defining the evolution in terms of where \((3.6)\) is not differentiable can be interpreted to mimic both notions of a singularity. In order for us to have a unique evolution equation, we choose the second interpretation to give us a direct tropical analogue of the discrete singularity confinement.

Definition 3.5. A tropical singularity is confined if, by iterating past non-differentiable boundaries, one reclaims differentiability.
Unfortunately, the same difficulties in forming a direct correspondence between singularity confinement and tropical singularity confinement for maps arise for lattice equations (in fact, these notions are related via periodic reductions). This means that we are unsure whether singularity confinement for arbitrary lattice equations implies tropical singularity confinement for their ultradiscretizations. We can say, once again, that a correspondence between the two notions of singularity confinement generically holds, but we are unsure as to whether the correspondence generally holds. The correspondence once again holds for many examples, but we can only conjecture that the correspondence holds in general.

Let us illustrate the mechanics of this correspondence with a simple example. Let us take the image of the (3.4), which, if the parameters are taken to be generic, results in the equation

\[
Q = \max(A + \max(W_{i,m} + W_{i+1,m}, W_{i,m+1} + W_{i+1,m+1}), B + \max(W_{i,m} + W_{i,m+1}, W_{i+1,m} + W_{i+1,m+1})).
\]

If we are to now write this in a manner in which the point of non-differentiability in the coordinate \(W_{i+1,m+1}\) is made clear, we write \(Q\) as

\[
Q = \max(W_{i+1,m+1} + \max(A + W_{i,m+1}, B + W_{i+1,m}), W_{i,m} + \max(A + W_{i+1,m}, B + W_{i,m+1})),
\]

hence, the ultradiscrete evolution is given by

\[
W_{i+1,m+1} = W_{i,m} + \max(A + W_{i+1,m}, B + W_{i,m+1}) - \max(A + W_{i,m+1}, B + W_{i+1,m}),
\]

which has appeared in the literature [31, 26, 24]. It is easy to show, and interesting to note, that this equation is consistent around a cube in the usual sense. Let us now consider the analogous initial conditions,

\[
W_{0,0} = \epsilon, \quad w_{-1,1} = B - A,
\]

\[
W_{-1,0} = X_1, \quad w_{-1,1} = A - B,
\]

\[
W_{0,-1} = X_2,
\]

which are designed to be singular in the sense that the functions are non-differentiable in the coordinates comprised of the staircase of initial conditions. We readily find, using (3.7), the values

\[
W_{0,1} = X_1 + B + A + \max(0, \epsilon) - \max(2B, 2A + \epsilon),
\]

\[
W_{1,0} = X_2 + B + A + \max(0, \epsilon) - \max(2A, 2B + \epsilon),
\]

\[
W_{1,1} = -\epsilon + \max(\epsilon + \max(3A + X_2, 3B + X_1), A + B + \max(A + X_1 + B + X_2))
\]

\[
- \max(\max(3A + X_1, 3B + X_2), A + B + \epsilon + \max(A + X_2, B + X_1)).
\]

The first thing to notice is that \(W_{0,1}\) and \(W_{1,0}\) clearly have discontinuities in the derivative with respect to \(\epsilon\) at \(\epsilon = 0\). While the last expression is somewhat complicated, so long as \(A + X_1 \neq B + X_2\) or \(A + X_2 \neq B + X_1\), this expression does not depend on the direction of the limit as \(\epsilon \to 0\), hence, the derivative, under these conditions, is continuous.

Now let us return to (3.4), but define the variables to be elements of \(\mathbb{K}\), such that \(v(\alpha) = A\) and \(v(\beta) = B\). We now let \(x_{0,0}\) be \(t^{-\delta}\), where the iterates are

\[
u_{0,1} = \frac{x_1(\beta^2t^\delta - \alpha^2)}{\alpha\beta\Delta},
\]

\[
u_{1,0} = \frac{x_2(\alpha^2t^\delta - \beta^2)}{\alpha\beta\Delta},
\]

\[
u_{1,1} = \frac{t^{-\delta}(t^\delta(\alpha^3x_2 + \beta^3x_1) - \alpha\beta(\alpha x_1 + \beta x_2))}{\alpha\beta t^\delta(\alpha x_2 + \beta x_1) - \alpha^3x_1 - \beta^3x_2}.
\]
Once again the factor, $\Delta = 1 - t^i$, is the term that encapsulates both the forms of singularity confinement. If we let $\delta \to 0$, where $t$ is chosen to be some number, then we reclaim singularity confinement as it appeared originally [9]. On the other hand, if we apply our valuation, $\nu(\Delta)$ is the factor that appears and disappears in accordance with the observed ultradiscrete singularity confinement. This shows how singularity confinement for this lattice equation maps to tropical singularity confinement via the ultradiscretization procedure.

4. Conclusion

There have seen a number of recent results that have reiterated the importance of the integrability of ultradiscrete equations. The Lax integrability of an ultradiscrete analogue of the sixth Painlevé equation [24], the existence of ultradiscrete hypergeometric solutions [23], the interpretation of the ultradiscrete Quispel–Roberts–Thompson system [21] and importantly, the connection between ultradiscrete equations and entropy, as observed by Fordy and Hone [8].

Another very interesting link, although not immediately obvious, is the recent arithmetic version of singularity confinement over the rational numbers, using $p$-adic valuations [16]. The $p$-adics are also non-Archimedean valuation fields, and much of the dynamics of these integrable systems under the image of the valuation of the $p$-adics are similar.

While we are still left with a conjecture regarding a direct correspondence between singularity confinement and tropical singularity confinement, what the above shows is that a correspondence between these two forms of singularity confinement generically holds, but perhaps not generally. We believe that this is still a step towards building a geometric understanding of tropical maps. In short, through a proper understanding of what the ultradiscrete analogue of singularities are, through the interpretation of an ultradiscretization of arbitrary rational functions, we have at our disposal the powerful tool of tropical algebraic geometry. With the ripening of tropical geometry, I believe there will be a renewed interest in geometry of ultradiscrete equations on the horizon, and this result presents a way of understanding ultradiscrete equations in this context.

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