IRREDUCIBLE COMPONENTS OF HILBERT SCHEME OF POINTS ON NON-REDUCED CURVES

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Abstract. We classify the irreducible components of the Hilbert scheme of \( n \) points on non-reduced algebraic plane curves. The irreducible components are indexed by partitions and all have dimension \( n \).

1. Introduction

Let \( C \) denote an algebraic curve on the plane defined by the equation \( f(x, y) = 0 \). We assume that \( f(x, y) \) can be factored into \( f(x, y) = \prod_{1 \leq j \leq r} f_j^{\beta_j}(x, y) \) where each \( f_j \) is irreducible. Let \( S_j := \{f_j^{\beta_j}(x, y)\} \) denote a non-reduced curve with multiplicity \( \beta_j \in \mathbb{Z}_{\geq 0} \). Denote the underlying reduced curve of \( S_j \) by \( C_j := \{f_j = 0\} \). We study the irreducible components of the Hilbert scheme of \( n \) points on \( C \), denoted by \( \text{Hilb}^n(C) \).

When \( C \) is irreducible and reduced (in other words, it has exactly one component \( S_1 \) of multiplicity \( \beta_1 = 1 \)), the Hilbert scheme of \( n \) points on it is irreducible. This was proven in \([1]\) and \([2]\).

When \( C \) has several components \( S_j \), and all the components are reduced (in other words, all the \( \beta_j \) are equal to 1), the irreducible components of \( \text{Hilb}^n(C) \) are classified in \([6, \text{Proposition 2.7}]\). See also \([7, \text{Fact 2.4}]\).

We generalize these results to the case of arbitrary non-reduced plane curves.

We compose the embedding \( \text{Hilb}(C) \hookrightarrow \text{Hilb}(\mathbb{C}^2) \) and the Hilbert-Chow map \( \text{Hilb}(\mathbb{C}^2) \to S^n\mathbb{C}^2 \) to obtain the Hilbert-Chow map on \( \text{Hilb}(C) \). The image of \( \text{Hilb}^n(C) \) is a collection of points on \( \bigcup_j C_j \). We use \( x_i^j \) to denote the \( i \)th point on the reduced curve \( C_j \), and use \( m_i^j \) to denote the multiplicity of the point \( x_i^j \).

Theorem 1.1. For an algebraic curve \( C \), the irreducible components of \( \text{Hilb}^n(C) \) have dimension exactly \( n \). The components are indexed by partitions \( \{m_1^1, \ldots, m_s^1, \ldots, m_1^r, \ldots, m_s^r\} \) of \( n \) where \( m_i^j \) are positive integers and \( m_i^j \leq \beta_j \) for all \( i \) and \( j \).
The irreducible components are given by the closures of the following strata: take the preimage in Hilb^n(C) under the Hilbert-Chow map of a collection of points where x_i^j is on the smooth part of C_j, and the multiplicities m_i^j satisfy 1 ≤ m_i^j ≤ β_j for all i and j.

In the proof of Theorem 1.1 we show in Lemma 8.2 that all the irreducible components of Hilb^n(C) have dimension at least n. Before that, we look locally at the curve \{y^β = 0\} and find that the irreducible components of Hilb^n(\{y^β = 0\}) all have dimension n and are the closures of the strata containing ideals whose image under the Hilbert-Chow map are points with multiplicities all less than or equal to β (Theorem 7.8). In the proof of Theorem 7.8 we first prove that all components intersect the affine chart U(n) and use U(n) to describe a coordinate system on Hilb^n(\{y^β = 0\}), so that we explicitly characterize the strata and their dimensions.

In the study of the strata of Hilb^n(C), when the image under the Hilbert-Chow map of ideals in the strata are points on the smooth parts of \bigcup_j C_j, locally they are the same as points on the curve \{y^β = 0\}, and the argument of dimension follows from the local case. When some points in the image land on the singularities or the intersection points of \bigcup_j C_j, we prove that the dimensions of the spaces of these ideals are strictly less than n, so they cannot be irreducible components.

2. Acknowledgments

I would like to thank Professor Eugene Gorsky for introducing me to the problem and the Hilbert scheme of points, and being extremely supportive throughout the process of producing this paper.

3. Background

Definition 3.1. The Hilbert scheme of n points on the plane \( \mathbb{C}^2 \), denoted as Hilb^n(\( \mathbb{C}^2 \)), is the set of ideals I \( \subset \mathbb{C}[x, y] \) such that \( \mathbb{C}[x, y]/I \) has dimension n.

Definition 3.2. Let \( \mu = (\mu_1, \ldots, \mu_k) \) denote a partition of n, where \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \). We write the Young diagram corresponding to \( \mu \) using the French notation, i.e. put \( \mu_1 \) boxes at the bottom row, \( \mu_2 \) boxes at the second to bottom row, etc. We label the boxes with integer valued Cartesian coordinates: the bottom left corner box is denoted as (0,0), and (i, j) corresponds to the box in the ith column and jth row. We fill the Young diagrams with monomials, such that \( x^{i-j}y^i \) is filled in the box (i, j). (See example below.)

The set of all such monomials in the Young diagram of \( \mu \) is denoted by \( B_\mu \).

Example 3.3. Let n = 7 and \( \mu = (4, 2, 1) \).

The Young diagram of (4, 2, 1) has the form:

\[
\begin{array}{ccc}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\end{array}
\]

Therefore \( B_{(4,2,1)} = \{1, x, x^2, x^3, y, xy, y^2\} \).

Definition 3.4. We define \( U_\mu \) as the set of ideals I in Hilb^n(\( \mathbb{C}^2 \)) such that \( B_\mu \) is a basis of \( \mathbb{C}[x, y]/I \).
Remark 3.5. \(U_\mu\) is an open subset of \(\text{Hilb}^n(\mathbb{C}^2)\) for all partitions \(\mu\) of \(n\).

Theorem 3.6. \[5\] The collection of all \(U_\mu\), where \(\mu\) is a partition of \(n\), forms an open cover of \(\text{Hilb}^n(\mathbb{C}^2)\).

Definition 3.7. We define coordinates on \(U_\mu\) as in \[5\]. For any fixed \(I \in U_\mu\), because the monomials in \(B_\mu\) form a basis of \(\mathbb{C}[x, y]/I\), we can write any monomial \(x^r y^s\) in \(\mathbb{C}[x, y]/I\) as a linear combination of monomials in \(B_\mu\) mod \(I\), and the variables \(C_{hk}^{rs}\) are defined as the coefficients in this expansion:

\[
x^r y^s = \sum_{(h,k) \in \mu} C_{hk}^{rs} x^h y^k \mod I.
\]

We require that \((h, k) \in \mu\) while \((r, s)\) can be arbitrary. Note that \(C_{hk}^{rs}\) is a complex valued function of \(I\), and the collection of all \(C_{hk}^{rs}(I)\) determines the ideal \(I\) uniquely.

Remark 3.8. For \((r, s)\) and \((h, k)\) both in \(\mu\), the coefficients \(C_{hk}^{rs}\) equal to 0 if \((r, s) \neq (h, k)\), and \(C_{hk}^{rs} = 1\) if \((r, s) = (h, k)\).

Example 3.9. Let \(n = 2\), \(I = (x^2 - 3x + 1, y)\), and \(\mathbb{C}[x, y]/I\) has the monomial basis \(B_\mu = \{1, x\}\). We can write any monomial \(x^r y^s\) as a linear combination of \(\{1, x\}\) mod \(I\). For example, \(x^2 = 3x - 1\) mod \(I\), so \(C_{00}^{20} = -1\) and \(C_{10}^{00} = 3\). Here’s another example: \(x^3 = x \cdot x^2 = x \cdot (3x - 1) = 3x^2 - x = 3(3x - 1) - x = 8 \cdot x + (-3) \cdot 1\) mod \(I\). So \(C_{00}^{20} = -3\) and \(C_{10}^{00} = 8\).

Remark 3.10. For each ideal \(I \in U_\mu\), there are infinitely many monomials \(x^r y^s\) producing infinitely many coefficients \(C_{hk}^{rs}(I)\). But it turns out that the coefficients \(C_{hk}^{rs}(I)\) have dependence relationship among themselves, and \(2n\) of them determine the others completely.

The reason is that when \(r \leq r'\) and \(s \leq s'\), a monomial \(x^{r'} y^{s'}\) of higher power is divisible by the monomial \(x^r y^s\) of lower power. This means that the expansion of the monomial \(x^{r'} y^{s'}\) is dependent on the expansion of the monomial \(x^r y^s\). When we correspond the monomials to boxes, the coefficients associated to the box \((r', s')\) depend on the boxes to the left and bottom of \((r', s')\).

Therefore, the expansion of monomials at the corners of the complement of \(\mu\), i.e. \((\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}) \setminus \mu\), are enough to determine all the other expansion of monomials. Therefore all \(U_\mu\) are finite dimensional. In fact, the following stronger theorem is true.

Theorem 3.11. \[5\] Each \(U_\mu\) is affine, irreducible, smooth and has dimension \(2n\).

Remark 3.12. When the Young diagram of a partition \(\mu\) has the shape of a size \(M \times N\) rectangle, we get exactly \(2n\) variables \(C_{hk}^{rs}\) by expanding the 2 monomials \(x^M\) and \(y^N\) at the corners. There are no relations among these \(C_{hk}^{rs}\) and they give all the degrees of freedom of \(U_\mu\). See Examples 3.1 and 3.2.

When the shape of the Young diagram of a partition is not rectangular, the complement of the Young diagram has more than 2 corners. So we get more than \(2n\) variables \(C_{hk}^{rs}\) in the expansion of monomials at those corners, and there will be relations among those variables \(C_{hk}^{rs}\) that reduces the number of variables to \(2n\). See Example 3.3.

Corollary 3.13. \[4\] \(\text{Hilb}^n(\mathbb{C}^2)\) is a smooth algebraic variety of dimension \(2n\).
This corollary was first proved in [4], but also directly follows from Haiman’s Theorem 3.11.

**Definition 3.14.** The Hilbert-Chow map is the map \( \pi : \text{Hilb}^n(\mathbb{C}^2) \to S^n \mathbb{C}^n \) that sends an ideal \( I \) to its zero locus \( V(I) \) with multiplicities. In particular, if \( I \) vanishes at points \( x_1, \ldots, x_k \in \mathbb{C}^2 \) with multiplicities \( d_1, \ldots, d_k \), then we denote the image under the Hilbert-Chow map by \( \pi(I) = d_1 \cdot x_1 + \cdots + d_k \cdot x_k \).

**Definition 3.15.** The punctual Hilbert scheme is \( \text{Hilb}^n(\mathbb{C}^2, 0) := \pi^{-1}(n \cdot (0,0)) \), the preimage under \( \pi \) of the point \((0,0)\) with multiplicity \( n \).

**Theorem 3.16.** \( \text{Hilb}^n(\mathbb{C}^2, 0) \) is irreducible and has dimension \( n - 1 \).

4. Local case: Hilbert scheme of \( n \) points on \( C_\beta := \{ y^\beta = 0 \} \)

**Definition 4.1.** Fix a positive integer \( \beta \), we consider the non-reduced curve \( C_\beta := \{ y^\beta = 0 \} \). The ring of functions on \( C_\beta \) is \( \mathbb{C}[C_\beta] := \mathbb{C}[x, y]/(y^\beta) \). Define the Hilbert scheme of \( n \) points on \( C_\beta \), denoted by \( \text{Hilb}^n(\{ y^\beta = 0 \}) \), as set of ideals \( I \) in \( \mathbb{C}[C_\beta] \) such that \( \mathbb{C}[C_\beta]/I \) has dimension \( n \).

**Remark 4.2.** \( \text{Hilb}^n(\{ y^\beta = 0 \}) \) is isomorphic to the set of ideals \( I \) in \( \mathbb{C}[x, y] \) containing \( y^\beta \) such that \( \mathbb{C}[x, y]/I \) has dimension \( n \). This is a subvariety of \( \text{Hilb}^n(\mathbb{C}^2) \) whose points are ideals \( I \) that contain \( y^\beta \). We define

\[
Z_\beta := \{ I \in \text{Hilb}^n(\mathbb{C}^2) \mid I \text{ contains } y^\beta \}
\]

as an alternative definition of \( \text{Hilb}^n(\{ y^\beta = 0 \}) \).

**Remark 4.3.** The collection of subsets \( \{ Z_\beta \cap U_\mu \mid \mu \text{ is a partition of } n \} \) forms an open cover of \( Z_\beta \). This follows immediately from the Theorem 3.10.

The question studied in the next few sections is understanding the irreducible components of \( Z_\beta \) and their dimensions.

We first make the following observations:

**Lemma 4.4.** The dimension of any irreducible component of \( Z_\beta \) is greater than or equal to \( n \).

**Proof.** Each irreducible component \( E \) of \( Z_\beta \) must intersect some \( U_\mu \), because the collection of \( U_\mu \) forms an open cover of \( \text{Hilb}^n(\mathbb{C}) \) (Theorem 3.10). And the non-empty intersection of \( E \) with some \( U_\mu \) must be an irreducible component of the intersection \( Z_\beta \cap U_\mu \) and have the same dimension as \( E \) itself, because \( U_\mu \) is open.

Therefore, to study the dimensions of the irreducible components of \( Z_\beta \), we can look at the irreducible components of \( Z_\beta \cap U_\mu \).

Take a chart \( U_\mu \) whose intersection with \( Z_\beta \) is nonempty. This imposes a new condition \( y^\beta \in I \) for ideals \( I \in U_\mu \), which is equivalent to \( y^\beta = 0 \) mod \( I \). So when writing \( y^\beta \) as a linear combination of the basis \( B_\mu \) mod \( I \), the coefficients should be 0. This gives us \( n \) equations \( C_{h,i}^\beta = 0 \) mod \( I \) for each \((h, k)\) in \( \mu \). We know from Theorem 3.11 that \( U_\mu \) has dimension \( 2n \), so the irreducible components of \( Z_\beta \cap U_\mu \) have dimension \( \geq 2n - n = n \).

Therefore, all the irreducible components of \( Z_\beta \) have dimension greater than or equal to \( n \). \( \Box \)

**Lemma 4.5.** \( Z_\beta \) only intersects charts \( U_\mu \) where \( \mu \) has at most \( \beta \) rows.
Proof. Suppose for contradiction that there exists a partition \( \mu \) with more than \( \beta \) rows and the intersection \( Z_{\beta} \cap U_{\mu} \) is non-empty. Then the box \((0, \beta)\) in the \( \beta \)th row has to be in \( \mu \) which implies \( \gamma^\beta \in B_{\mu} \). Also, \( Z_{\beta} \cap U_{\mu} \neq \emptyset \) implies that there has to be an ideal \( I \) in both \( Z_{\beta} \) and \( U_{\mu} \). So this ideal \( I \) must contain \( \gamma^\beta \) to be in \( Z_{\beta} \) and also satisfy that \( B_{\mu} \) is the set of basis of \( \mathbb{C}[x, y]/I \) for this ideal \( I \) to be in \( U_{\mu} \). But \( I \) contains \( \gamma^\beta \) means \( \gamma^\beta \equiv 0 \mod I \), contradiction. \( \square \)

Now, we begin the stratification of \( Z_{\beta} \) by introducing the stratification of a bigger space, namely \( \text{Hilb}^n(\mathbb{C}^2, \mathbb{C}) \).

Definition 4.6. The Hilbert scheme of \( n \) points on the line \{ \( y = 0 \) \}, denoted by \( \text{Hilb}^n(\mathbb{C}^2, \mathbb{C}) \), is the preimage under the Hilbert-Chow map \( \pi \) of any arbitrary collection of points on the line \{ \( y = 0 \) \} with the total multiplicity \( n \), i.e.

\[
\text{Hilb}^n(\mathbb{C}^2, \mathbb{C}) := \{ I | \pi(I) = m_1(x_1, 0) + \cdots + m_s(x_s, 0) \text{ where } \sum_i m_i = n \}
\]

Lemma 4.7. The variety \( Z_{\beta} \) is a subset of \( \text{Hilb}^n(\mathbb{C}^2, \mathbb{C}) \).

Proof. Because \( Z_{\beta} \) is the set of ideals \( I \) in \( \text{Hilb}^n(\mathbb{C}^2) \) containing \( \gamma^\beta \), such ideals \( I \) must vanish at points where \( \gamma^\beta = 0 \). And \( \gamma^\beta = 0 \) only when \( y = 0 \). So the ideals \( I \) in \( Z_{\beta} \) maps to the points on the line \{ \( y = 0 \) \} under the Hilbert-Chow map, which implies that \( Z_{\beta} \subset \text{Hilb}^n(\mathbb{C}^2, \mathbb{C}) \). \( \square \)

Remark 4.8. Not every ideal in \( \text{Hilb}^n(\mathbb{C}^2, \mathbb{C}) \) is an ideal in \( Z_{\beta} \). For example, when \( \beta = 3, n = 5 \), \((x, \gamma^3) \in \text{Hilb}^5(\mathbb{C}^2, \mathbb{C}) \), but \((x, y^3) \notin Z_{\beta} \).

Definition 4.9. We stratify \( \text{Hilb}^n(\mathbb{C}^2, \mathbb{C}) \) by the multiplicities of points in the image of the Hilbert-Chow map \( \pi \). Let \( m_1, \ldots, m_s \) be a partition of \( n \), and each partition indexes a stratum. Define the stratum \( \Sigma_{m_1, \ldots, m_s} \) as the preimage under the Hilbert-Chow map of all \( s \) points on the line \{ \( y = 0 \) \} with fixed multiplicities \( m_1, \ldots, m_s \).

Definition 4.10. Fix \( s \) distinct points \((x_1, 0), \ldots, (x_s, 0)\) on the line \{ \( y = 0 \) \} with fixed multiplicities \( m_1, \ldots, m_s \), namely \( m_1(x_1, 0) + \cdots + m_s(x_s, 0) \). We denote their preimage of under the Hilbert-Chow map to be \( \Sigma_{m_1, \ldots, m_s}(x_1, \ldots, x_s) \).

Lemma 4.11. The set \( \Sigma_{m_1, \ldots, m_s}(x_1, \ldots, x_s) \) is an irreducible variety, isomorphic to \( \text{Hilb}^{m_1}(\mathbb{C}^2, 0) \times \cdots \times \text{Hilb}^{m_s}(\mathbb{C}^2, 0) \), and has dimension \( n - s \).

Proof. The preimage of each point \( m_i(x_i, 0) \) under the Hilbert-Chow map is isomorphic to \( \text{Hilb}^{m_i}(\mathbb{C}^2, 0) \). The ideals that vanish at all of the points \( m_1(x_1, 0) + \cdots + m_s(x_s, 0) \) must be a point in the product \( \text{Hilb}^{m_1}(\mathbb{C}^2, 0) \times \cdots \times \text{Hilb}^{m_s}(\mathbb{C}^2, 0) \). And each punctual Hilbert scheme \( \text{Hilb}^{m_i}(\mathbb{C}^2, 0) \) is irreducible and of dimension \( m_i - 1 \) (Theorem 3.16), so their product is also irreducible, and has dimension \( (m_1 - 1) + (m_2 - 1) + \cdots + (m_s - 1) = m_1 + m_2 + \cdots + m_s - s = n - s \). \( \square \)

We explore some examples of \( Z_{\beta} \) before we study its general properties.
5. Examples

In this section, we study the Hilbert schemes of 2 and 3 points on the line \( \{y^2 = 0\} \) where \( \beta = 2 \). We intersect \( Z_2 \) with different charts \( U_\mu \) and compute the dimension of the irreducible components of the intersection.

We first set \( n = 2 \) and consider the Hilbert scheme of 2 points. Both charts \( U_{(1,1)} \) and \( U_{(2)} \) have nonempty intersection with \( Z_2 \).

**Example 5.1.** \( \mu = (1,1) \), \( B_\mu = \begin{bmatrix} y \\ 1 \end{bmatrix} \)

We write down the expansion of \( x \) and \( y^2 \) in the basis \( \{1, y\} \) with coefficients \( C_{hk}^{\tau_k} : x = C_{00}^{10} \cdot 1 + C_{01}^{10} \cdot y \mod I \), and \( y^2 = C_{00}^{02} \cdot 1 + C_{01}^{02} \cdot y \mod I \).

There are no relations among those coefficients \( C_{hk}^{\tau_k} \). So \( y^2 = 0 \mod I \) is equivalent to the conditions \( C_{00}^{02} = C_{01}^{02} = 0 \mod I \), and \( C_{00}^{02} \) and \( C_{01}^{02} \) are free variables. \( Z \cap U_{(1,1)} \) is therefore 2 dimensional and isomorphic to \( \mathbb{C}^2 \).

**Example 5.2.** \( \mu = (2) \), \( B_\mu = \begin{bmatrix} 1 \\ x \end{bmatrix} \)

We omit the notation \( C_{hk}^{\tau_k} \) for simplicity and denote the expansion of \( x^2 \) and \( y \) in the basis \( B_\mu = \{1, x\} \) by \( y = a + bx \) and \( x^2 = c + dx \). Again there are no relations among the coefficients \( a, b, c \) and \( d \).

Now \( y^2 = (a + bx)^2 = a^2 + 2abx + b^2x^2 = a^2 + 2abx + b^2(c + dx) = (a^2 + b^2c) + (2ab + b^2d)x \).

The equation \( y^2 = 0 \mod I \) gives that

\[
\begin{align*}
(1) & \quad a^2 + b^2c = 0 \\
(2) & \quad 2ab + b^2d = 0
\end{align*}
\]

There are 2 following cases.

Case 1: \( b = 0 \), then equation (1) gives \( a = 0 \). When \( a = b = 0 \), equation (2) automatically holds. \( c \) and \( d \) are free variables. So this component has dimension 2.

Case 2: \( b \neq 0 \), then by equation (2), \( a = -\frac{bd}{2} \).

Then from equation (1), we get that \( \frac{b^2d^2}{4} + b^2c = 0 \). Because \( b \neq 0 \), we can divide by \( b^2 \) and get \( \frac{d^2}{4} + c = 0 \), so \( c = -\frac{d^2}{4} \).

In this case, \( b \) and \( d \) are free variables, and \( a, c \) are determined by \( b \) and \( d \). So this component also has dimension 2.

In conclusion, \( Z_2 \cap U_{(2)} \) has 2 irreducible components, and each component has dimension 2 and is isomorphic to \( \mathbb{C}^2 \).

Note that \( a = b = 0 \) means that \( a + bx = 0 \), and \( c = -\frac{d^2}{4} \) means that the quadratic equation \( x^2 - c - dx = 0 \) has a root of multiplicity 2. This is a special case of Theorem \([7.8]\) in a later section.

Now set \( \beta = 2 \) and \( n = 3 \). All the partitions of 3 are \((1,1,1),(2,1),(3)\). By Lemma \([4.5]\) \( Z_2 \cap U_{(1,1,1)} = \emptyset \) because \((1,1,1)\) has 3 rows. We consider the partition \((2,1)\) and the non-empty intersection \( U_{(2,1)} \cap Z_2 \).
Example 5.3. $\mu = (2,1)$.

First, we define some explicit coordinates and equations for the chart $U_{(2,1)}$.

We start by expanding the monomials $x^2, xy$ and $y^2$ at the corners outside of partition $(2,1)$: $x^2 = a_1 + a_2x + a_3y \mod I$, $y^2 = b_1 + b_2x + b_3y \mod I$, and $xy = c_1 + c_2x + c_3y \mod I$, for some coefficients $a_i, b_i$ and $c_i$ that are secretly $C_{bh}$.

Note that now we have 9 variables for $U_{(2,1)}$.

There are 2 ways of expanding $xy^2$ in terms of these coefficients $a_i, b_i$ and $c_i$, namely either $xy^2 = x \cdot y^2$ or $xy^2 = xy \cdot y$. As the following computation shows, this gives us 3 relations among $a_i, b_i$ and $c_i$, reducing the dimension from 9 to 6.

\[ xy^2 = x \cdot y^2 = x(b_1 + b_2x + b_3y) = b_1x + b_2x^2 + b_3xy = b_1x + b_2(a_1 + a_2x + a_3y) + b_3(c_1 + c_2x + c_3y) = (a_1b_2 + c_1b_3) + (b_1 + b_2a_2 + b_3c_2)x + (b_2a_3 + b_3c_3)y. \]

\[ xy^2 = xy \cdot y = (c_1 + c_2x + c_3y)y = c_1y + c_2xy + c_3y^2 = c_1y + c_2(c_1 + c_2x + c_3y) + c_3(b_1 + b_2x + b_3y) = (c_1c_2 + c_3b_1) + (c_2^2 + c_3b_2)x + (c_1 + c_2c_3 + c_3b_3)y. \]

Comparing, equating and simplifying the coefficients give us the following relations.

\[ a_1b_2 + c_1b_3 = c_1c_2 + c_3b_1 \quad (3) \]

\[ b_1 + b_2a_2 + b_3c_2 = c_2^2 + c_3b_2 \quad (4) \]

\[ b_2a_3 = c_1 + c_2c_3 \quad (5) \]

Similarly, there are 2 ways of expanding $x^2y$, namely either $x^2y = x^2 \cdot y$ or $x^2y = xy \cdot x$.

\[ x^2y = x \cdot xy = x(c_1 + c_2x + c_3y) = c_1x + c_2x^2 + c_3xy = c_1x + c_2(a_1 + a_2x + a_3y) + c_3(c_1 + c_2x + c_3y) = (a_1c_2 + c_1c_3) + (c_1 + a_2c_2 + c_2c_3)x + (a_3c_2 + c_3^2)y. \]

\[ x^2y = x^2 \cdot y = (a_1 + a_2x + a_3y)y = a_1y + a_2xy + a_3y^2 = a_1y + a_2(c_1 + c_2x + c_3y) + a_3(b_1 + b_2x + b_3y) = (a_2c_1 + a_3b_1) + (a_2c_2 + a_3b_2)x + (a_1 + a_2c_3 + a_3b_3)y. \]

This gives us relations as follows.

\[ a_1c_2 + c_1c_3 = a_2c_1 + a_3b_1 \quad (6) \]

\[ c_1 + c_2c_3 = a_3b_2 \quad (7) \]

\[ a_3c_2 + c_3^2 = a_1 + a_2c_3 + a_3b_3 \quad (8) \]

Solving equation (3) gives us

\[ b_1 = c_2^2 + b_2c_3 - b_3c_2 - a_2b_2 \quad (9) \]

Solving equation (7) gives

\[ c_1 = a_3b_2 - c_2c_3 \quad (10) \]

Solving equation (8) gives

\[ a_1 = a_3c_2 + c_3^2 - a_2c_3 - a_3b_3 \quad (11) \]
As we will show, the chart $U_{(2,1)}$ is isomorphic to $\mathbb{C}^3$.

Next, we describe the intersection with $Z_2$. To compute $Z_2 \cap U_{(2,1)}$, we set $y^2 = 0$ which means $b_1, b_2, b_3 = 0$. Due to equation (9) above, we also have that $c_2 = 0$. Along with equation (10), we have that $c_1 = 0$. Equation (11) gives that $a_1 = c_2^3 - a_2 c_3$.

In conclusion, $a_2, a_3, c_3$ are free variables, $b_1 = b_2 = b_3 = c_1 = c_2 = 0$, and $a_1 = c_2^3 - a_2 c_3$. So $Z_2 \cap U_{(2,1)}$ is isomorphic to $\mathbb{C}^3$.

6. The special chart $U_{(n)}$

The stratification of $\text{Hilb}^n(\mathbb{C}^2, \mathbb{C})$ induces a stratification on $\text{Hilb}^n(\mathbb{C}^2, \mathbb{C}) \cap U_\beta$. As we will show, the chart $U_{(n)}$, where $(n)$ corresponds to the partition with $n$ boxes in one row, intersects all the irreducible components of $Z_\beta$.

**Lemma 6.1.** (a) The space $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s) - U_{(n)}$ has dimension strictly less than $n - s$.

(b) Each stratum’s complement $\Sigma_{m_1,\ldots,m_s} - U_{(n)}$ has dimension strictly less than $n$.

**Proof.** (a) The chart $U_{(n)}$ is open in each $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s)$. The intersection $U_{(n)} \cap \Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s)$ is nonempty because the ideal $I = ((x-x_1)^{m_1}, \ldots, (x-x_s)^{m_s}, y)$ is in the intersection. Therefore, $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s) - U_{(n)}$ is a closed and proper subset of $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s)$, and $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s)$ is irreducible by Lemma 4.11. So $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s) - U_{(n)}$ has dimension strictly less than $n - s$.

(b) Varying each $x_i$ adds 1 degree of freedom, and varying all the $x_1, \ldots, x_s$ adds $s$ degrees of freedom in total. By part (a), $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s) - U_{(n)}$ has dimension strictly less than $n - s$, which means that $\Sigma_{m_1,\ldots,m_s} - U_{(n)}$ has dimension strictly less than $n - s + s = n$.

**Corollary 6.2.** All irreducible components of $Z_\beta$ intersect the chart $U_{(n)}$.

**Proof.** We want to show that $Z_\beta - U_{(n)}$ does not completely contain any irreducible components of $Z_\beta$.

Suppose for contradiction that $Z_\beta - U_{(n)}$ contains some irreducible component $A$ of $Z_\beta$. We showed in Lemma 4.3 that the dimensions of all the irreducible components are greater than or equal to $n$. So $Z_\beta - U_{(n)}$ must have dimension greater than or equal to $n$. Now we consider the union $\cup_{m_1,\ldots,m_s}(\Sigma_{m_1,\ldots,m_s} - U_{(n)})$. Because the dimension of each $\Sigma_{m_1,\ldots,m_s} - U_{(n)}$ is strictly less than $n$, the union $\cup_{m_1,\ldots,m_s}(\Sigma_{m_1,\ldots,m_s} - U_{(n)})$ also has dimension strictly less than $n$. But $Z_\beta - U_{(n)}$ is contained in the union, so $\cup_{m_1,\ldots,m_s}(\Sigma_{m_1,\ldots,m_s} - U_{(n)})$ must have dimension greater than or equal to $n$. So we have a contradiction.

Now we begin the study of the coordinate system on $U_{(n)}$. The partition $(n)$ that has $n$ boxes on one row has the following Young diagram:

$$B_{(n)} = \begin{array}{cccc}
\text{1} & x & x^2 & \cdots & x^{n-1}
\end{array}$$
We start describing the coordinates on $U_{(n)}$. Similar to the examples in the previous sections, we expand $x^n$ and $y$ in the basis $1, x, x^2, \ldots, x^{n-1}$ and denote the coefficients as follows:

\[
x^n = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \mod I
\]

\[
y = b_0 + b_1 x + \ldots + b_{n-1} x^{n-1} \mod I.
\]

There are no relations among the coefficients $a_i$ and $b_j$, as noted in Remark 6.12.

Define polynomials $a(x) = x^n - a_{n-1} x^{n-1} - \ldots - a_1 x - a_0$ and $b(x) = b_0 + b_1 x + \ldots + b_{n-1} x^{n-1}$, then any ideal $I$ in $U_{(n)}$ is generated as $I = (a(x), y - b(x))$. Throughout this paper, we will use $a(x)$ and $b(x)$ to denote the polynomials defined above. As a side remark, the degree of $a(x)$ is $n$ because the highest order term $x^n$ of $a(x)$ has degree $n$. But $b(x)$ can have degree less than or equal to $n - 1$, because the coefficient $b_{n-1}$ can sometimes be 0. For example, in the case $n = 3$, we can have $I = (x^3, y)$ as a point in $U_{(3)} \in \text{Hilb}^3(\mathbb{C}^2)$, where $b(x) = 0$.

Now we stratify Hilb$^n(\mathbb{C}^2, \mathbb{C}) \cap U_{(n)}$. Because $U_{(n)}$ has a nice coordinate system, we are able to write out specifically the ideals in each stratum of $\Sigma_{m_1,\ldots,m_s} \cap U_{(n)}$.

**Lemma 6.3.** An ideal $I$ in each stratum $\Sigma_{m_1,\ldots,m_s} \cap U_{(n)}$ has the form $((x - x_1)^{m_1} \ldots (x - x_s)^{m_s}, y - (x - x_1) \ldots (x - x_s) \theta(x))$ for some polynomial $\theta(x)$.

**Proof.** An ideal $I$ in $U_{(n)}$ has the form $(a(x), y - b(x))$ where $a(x)$ can be factored into $(x - c_1)^{l_1} \ldots (x - c_q)^{l_q}$ where $c_i$ are distinct roots with multiplicities $l_i$. We know from this factorization of $a(x)$ that $I = (a(x), y - b(x))$ vanishes at each $(c_i, b(c_i))$ with multiplicity $l_i$. And if $I$ is also in $\pi^{-1}(m_1(x_1,0) + \cdots + m_s(x_s,0))$, then $I$ must vanish at precisely $(x_1,0), \ldots, (x_s,0)$ with multiplicity $m_1,\ldots,m_s$. Therefore, $c_i = x_i, l_i = m_i$ and $b(x_i) = 0$. So $a(x) = (x - x_1)^{m_1} \ldots (x - x_s)^{m_s}$ and $b(x) = (x - x_1) \ldots (x - x_s) \theta(x)$ for some $\theta(x)$ that could be 0.

\[
\square
\]

7. $Z_\beta$ in the chart $U_{(n)}$.

In the previous section, we characterized the ideals in each stratum $\Sigma_{m_1,\ldots,m_s} \cap U_{(n)}$. Now when we look at the ideals $I$ in the stratum $\Sigma_{m_1,\ldots,m_s} \cap U_{(n)}$ that are also in $Z_\beta$, (or equivalently, $I$ containing $y^\beta$), this imposes new conditions on the coordinates of $\Sigma_{m_1,\ldots,m_s} \cap U_{(n)}$ as studied in the last section.

In fact, in this section we explicitly characterize ideals $I$ in $U_{(n)} \cap Z_\beta$, and combining with the information of ideals $I$ in each stratum $\Sigma_{m_1,\ldots,m_s} \cap U_{(n)}$ that we characterized in the last section, we characterize each ideal $I$ in $\Sigma_{m_1,\ldots,m_s} \cap U_{(n)} \cap Z_\beta$.

The collection of all $\Sigma_{m_1,\ldots,m_s} \cap U_{(n)} \cap Z_\beta$ stratifies $U_{(n)} \cap Z_\beta$. And we know from Corollary 6.12 that all irreducible components of $Z_\beta$ intersect $U_{(n)}$. So studying the strata $\Sigma_{m_1,\ldots,m_s} \cap U_{(n)} \cap Z_\beta$ gives us the irreducible components of $Z_\beta$.

**Proposition 7.1.** The condition that $y^\beta$ in $I = (a(x), y - b(x)) \in U_{(n)}$ is equivalent to the condition that the polynomial $a(x)$ divides $b^\beta(x)$.

To prove one direction of the proposition that $y^\beta \in I$ implies $a(x)|b^\beta(x)$, we first prove the following lemma:

**Lemma 7.2.** Let $f(x)$ be a polynomial in $I = (a(x), y - b(x))$ which does not depend on the variable $y$. Then $f(x)$ is divisible by $a(x)$.
Lemma 7.3. \( (y \implies \text{explore the possible multiplicities of roots that} \) must also be in \( I \), so \( r(x) = 0 \mod I \), and \( r(x) \) is a nonzero linear combination of \( 1, x, \ldots, x^r \). But we also know that \( B_n = \{1, x, x^2, \ldots, x^{n-1}\} \) is a basis of \( \mathbb{C}[x,y]/I \), contradiction.

Therefore, the assumption must be wrong, and \( r(x) \) must be 0. So \( a(x) \) divides \( f(x) \).

Now, we are ready to prove Proposition 7.1.

Proof of Proposition 7.1. Assume \( y^\beta \in I \). Because \( y - b(x) \) is a generator of \( I \), \( y = b(x) \mod I \), which implies that \( y^\beta = b^\beta(x) \mod I \). Because \( y^\beta \in I \), \( y^\beta = 0 \mod I \). So \( b^\beta(x) = 0 \mod I \) and by Lemma 7.2 \( b^\beta(x) \) is divisible by \( a(x) \).

Suppose \( a(x)|b^\beta(x) \), then because \( a(x) \in I \), we have \( b^\beta(x) \in I \). Again, because \( (y - b(x)) \in I \), \( y = b(x) \mod I \), and \( b^\beta(x) \in I \) implies \( b^\beta(x) = y^\beta = 0 \mod I \), which implies \( y^\beta \in I \).

Because \( \mathbb{C} \) is algebraically closed, we factor \( a(x) \) and \( b(x) \) into linear terms. We explore the possible multiplicities of roots that \( a(x) \) and \( b(x) \) can have.

Lemma 7.3. Let \( x_1 \ldots x_s \) denote the distinct roots of \( a(x) \), and \( m_i \) the multiplicity of each root \( x_i \), then we can explicitly make the factorization:

\[
a(x) = (x - x_1)^{m_1} \cdot (x - x_2)^{m_2} \cdot \ldots \cdot (x - x_s)^{m_s}
\]

where \( \sum_i m_i = n \).

Then condition of when \( a(x)|b^\beta(x) \) splits into 2 cases depending on the multiplicities \( m_i \).

1. General case: If \( \left\lceil \frac{m_1}{\beta} \right\rceil + \ldots + \left\lceil \frac{m_s}{\beta} \right\rceil \leq n - 1 \),

then \( a(x)|b^\beta(x) \) if and only if

\[
b(x) = (x - x_1)^{\left\lceil \frac{m_1}{\beta} \right\rceil} \cdot (x - x_2)^{\left\lceil \frac{m_2}{\beta} \right\rceil} \cdot \ldots \cdot (x - x_s)^{\left\lceil \frac{m_s}{\beta} \right\rceil} \cdot \alpha(x)
\]

for some polynomial \( \alpha(x) \).

2. Special case: If \( \left\lceil \frac{m_1}{\beta} \right\rceil + \ldots + \left\lceil \frac{m_s}{\beta} \right\rceil > n - 1 \),

then \( a(x)|b^\beta(x) \) if and only if \( b(x) = 0 \).

Proof. Denote the multiplicity of \( (x - x_i) \) in \( b(x) \) by \( q_i \).

Because \( a(x) \) divides \( b^\beta(x) \), each factor \( (x - x_i) \) in \( b^\beta(x) \) must have a power higher than \( m_i \), or \( b(x) \) has to be 0. That is to say, considering the multiplicity \( q_i \) of each factor \( (x - x_i) \) in \( b(x) \), we must have \( \beta \cdot q_i \geq m_i \), which gives \( q_i \geq \frac{m_i}{\beta} \).

Because \( q_i \) are integers, the smallest possible value of \( q_i \) is \( \left\lceil \frac{m_i}{\beta} \right\rceil \). So \( b(x) \) must have the factor \( (x - x_1)^{\left\lceil \frac{m_1}{\beta} \right\rceil} \cdot (x - x_2)^{\left\lceil \frac{m_2}{\beta} \right\rceil} \cdot \ldots \cdot (x - x_s)^{\left\lceil \frac{m_s}{\beta} \right\rceil} \), or is equal to 0.

Recall that \( b(x) \) is constructed to have degree at most \( n - 1 \), so if \( \left\lceil \frac{m_1}{\beta} \right\rceil + \ldots + \left\lceil \frac{m_s}{\beta} \right\rceil \leq n - 1 \), then \( \alpha(x) \) is some polynomial of degree at most \( n - 1 - \left( \left\lceil \frac{m_1}{\beta} \right\rceil + \ldots + \left\lceil \frac{m_s}{\beta} \right\rceil \right) \).
Our goal now is to find the strata with dimension $n$. This implies that the dimension of the irreducible components has to be exactly $n$. The strata (our candidates for the irreducible components) have dimension at most $\sum s_i$. Now, we want to stratify the space $Z_{\beta} \cap U(n)$ and show that all the strata have dimension $\leq n$, which implies that the closures of strata (our candidates for the irreducible components) have dimension at most $n$. This implies that the dimension of the irreducible components has to be exactly $n$. Our goal now is to find the strata with dimension $n$, and the closures of them will be the irreducible components.

We now stratify $Z_{\beta} \cap U(n)$ by the strata $C_{m_1, \ldots, m_s} := \sum_{m_1, \ldots, m_s} Z_{\beta} \cap U(n)$.

Collecting the results we have got so far, we can characterize all ideals in the strata $C_{m_1, \ldots, m_s}$.

**Proposition 7.5.** Each stratum $C_{m_1, \ldots, m_s}$ contains exactly the ideals generated as $I = (a(x), y - b(x))$, where $a(x) = (x - x_1)^{m_1} \cdot (x - x_2)^{m_2} \cdot \ldots \cdot (x - x_s)^{m_s}$ and $b(x) = (x - x_1)^{\frac{m_1}{\beta}} \cdot (x - x_2)^{\frac{m_2}{\beta}} \cdot \ldots \cdot (x - x_s)^{\frac{m_s}{\beta}} \cdot \alpha(x)$ when $\sum_{i=1}^s \frac{m_i}{\beta} \leq n - 1$, and $b(x) = 0$ when $\sum_{i=1}^s \frac{m_i}{\beta} > n - 1$.

**Proof.** By Lemma 6.3, the ideals $I$ in each stratum $\Sigma_{m_1, \ldots, m_s} \cap U(n)$ have the form $(a(x), y - b(x))$, where $a(x) = (x - x_1)^{m_1} \cdot (x - x_2)^{m_2} \cdot \ldots (x - x_s)^{m_s}$ and $b(x) = (x - x_1)^{\frac{m_1}{\beta}} \cdot (x - x_2)^{\frac{m_2}{\beta}} \cdot \ldots \cdot (x - x_s)^{\frac{m_s}{\beta}} \cdot \alpha(x)$ or $b(x) = 0$, depending on the multiplicities $m_i$. The special case is when $\sum_{i=1}^s \frac{m_i}{\beta} > n - 1$, then $b(x)$ has to be 0, in which case $\alpha(x) = 0$. □
For each stratum $C_{m_1, \ldots, m_s}$, we compute it’s dimension by counting the degree of freedom given by polynomials $a(x)$, $b(x)$, and $\alpha(x)$.

**Lemma 7.6.** If $\left[\frac{m_1}{\beta}\right] + \cdots + \left[\frac{m_s}{\beta}\right] \leq n - 1$, then $\dim(C_{m_1, \ldots, m_s}) = t + s + 1$, where $t$ is the maximum degree that $\alpha(x)$ can have.

If $\left[\frac{m_1}{\beta}\right] + \cdots + \left[\frac{m_s}{\beta}\right] > n - 1$, then the only stratum satisfying this condition on the multiplicities $m_i$ is $C_{1, \ldots, 1}$, and it has dimension $n$.

**Proof.** Let’s first look at the general case: $\left[\frac{m_1}{\beta}\right] + \cdots + \left[\frac{m_s}{\beta}\right] \leq n - 1$.

Each of the distinct roots $x_i$ of $a(x)$ gives a degree of freedom, so $a(x)$ has $s$ degrees of freedom. Denote the maximum degree of $\alpha(x)$ by $t$, then we can explicitly write out $\alpha(x)$ as $\alpha(x) = \alpha_0 + \cdots + \alpha_s x^t$ for coefficients $\alpha_i \in \mathbb{C}$, and each $\alpha_i$ gives a degree of freedom. So $\alpha(x)$ gives $t + 1$ degrees of freedom. Note that $b(x)$ is completely determined by $a(x)$ and $\alpha(x)$ so it does not contribute to any degree of freedom. The dimension of stratum $C_{m_1, \ldots, m_s}$ therefore is $t + 1 + s$.

**Special case:** $\left[\frac{m_1}{\beta}\right] + \cdots + \left[\frac{m_s}{\beta}\right] > n - 1$. As discussed in Lemma 7.3, $s = n$ and all the $m_i$ has to be 1, so there is only one stratum in this special case, which is $C_{m_1=1, \ldots, m_s=1}$.

Recall from the proof of Lemma 7.3 that in the special case, $b(x) = 0$ and $\alpha(x) = 0$, so only $a(x)$ contributes to degrees of freedom. All the degrees of freedom are given by the variables $x_1, \ldots, x_n$ in $a(x)$. So the total degrees of freedom are $n$, and the dimension of $C_{1, \ldots, 1}$ is $n$. □

**Lemma 7.7.** All the strata $C_{m_1, \ldots, m_s}$ are irreducible.

**Proof.** By Proposition 7.6 and Lemma 7.3 a stratum $C_{m_1, \ldots, m_s}$ is isomorphic to $(\mathbb{C}^{t+s+1} - \{(x_1, \ldots, x_n, \alpha_0, \ldots, \alpha_s)|x_i = x_j \text{ for some } 1 \leq i, j \leq s\})/\text{Stab}\{m_1, \ldots, m_s\})$. An affine space removing a closed subvariety is irreducible, and quotient by action of a finite group again keeps the space irreducible. □

**Theorem 7.8.** All the irreducible components of $Z_\beta$ have dimension $n$ and are parametrized by partitions $\{m_1, \ldots, m_s\}$ of $n$ with at most $\beta$ parts, i.e. $\sum_{i=1}^s m_i = n$ and $1 \leq m_i \leq \beta$.

**Proof.** In the special case, the stratum $C_{1, \ldots, 1}$ has dimension $n$, which means that its closure is an irreducible component of $Z_\beta$.

Now we look at the general case and want to find the conditions on $t$ and $s$ such that $t + s = n$, and the closures of strata satisfying $t + s = n$ will be irreducible. Recall that $t$ is the maximum degree of $\alpha(x)$ can have, and $s$ is the number of distinct roots of $a(x)$.

Here are all the equations relating the dimension of a stratum, $t$, $s$ and $n$:

$$\text{dimension of a stratum} = t + s + 1.$$ (Lemma 7.3)

Because $b(x)$ has degree at most $n - 1$, we have

$$t + \left[\frac{m_1}{\beta}\right] + \cdots + \left[\frac{m_s}{\beta}\right] = n - 1.$$
Because $m_i \geq 1$, we must have $\left\lfloor \frac{m_i}{\beta} \right\rfloor \geq 1$. So
\[
\left\lfloor \frac{m_1}{\beta} \right\rfloor + \left\lfloor \frac{m_2}{\beta} \right\rfloor + \cdots + \left\lfloor \frac{m_s}{\beta} \right\rfloor \geq s.
\]

First canceling $t$ from the first 2 equations above, and then canceling $\left\lfloor \frac{m_i}{\beta} \right\rfloor + \cdots + \left\lfloor \frac{m_s}{\beta} \right\rfloor$ from the last equation gives that dimension of a stratum $= 1 + s + (n - 1 - (\left\lfloor \frac{m_1}{\beta} \right\rfloor + \cdots + \left\lfloor \frac{m_s}{\beta} \right\rfloor)) = n + s - (\left\lfloor \frac{m_1}{\beta} \right\rfloor + \cdots + \left\lfloor \frac{m_s}{\beta} \right\rfloor)\leq n$.

The equality holds when $\left\lfloor \frac{m_1}{\beta} \right\rfloor + \cdots + \left\lfloor \frac{m_s}{\beta} \right\rfloor = s$, which means $\left\lfloor \frac{m_i}{\beta} \right\rfloor = 1$, and this happens precisely when $1 \leq m_i \leq \beta$ for all $i$.

In conclusion, all the strata have dimension $\leq n$, and the closure of a stratum $C_{\beta_1,\ldots,\beta_s}$ is an irreducible component if and only if $1 \leq m_i \leq \beta$.

Note that each tuple $(m_1, \ldots, m_s)$ indexes an irreducible component. Also recall that $\sum_i m_i = n$, so we can rearrange the tuple $(m_1, \ldots, m_s)$ into a partition $(m_{i_1}, \ldots, m_{i_s})$ of $n$, where $m_{i_1} \in \{m_i | i = 1 \ldots s\}$ and $m_{i_1} \geq \cdots \geq m_{i_s}$. So each irreducible component of $Z_\beta$ is indexed by a partition of $n$.

8. Hilbert scheme of points on general non-reduced curves

Definition 8.1. Denote a non-reduced algebraic curve by $C := \{f = 0\}$ where $f = \prod f_j^{\beta_j}(x, y)$, and each $f_j$ is irreducible. The Hilbert scheme of $n$ points on $C$, denoted by $\text{Hilb}^n(C)$, is the set of all ideals $I \in \mathbb{C}(x, y)$ such that $f \in I$ and $\mathbb{C}(x, y)/I$ has dimension $n$.

The question studied in this section is understanding the irreducible components of $\text{Hilb}^n(C)$ and their dimensions.

We denote the reduced curves $\{f_j = 0\}$ by $C_j$. After removing the singularities and intersection points of $C_j$, we denote the smooth part of the curve by $C_j^{sm}$.

Lemma 8.2. The irreducible components of $\text{Hilb}^n(C)$ have dimension at least $n$.

Proof. Fix a chart $U_\mu$ which has non-empty intersection with $\text{Hilb}^n(C)$. When $f \in I$, we can write $f$ as a linear combination of the monomial basis $B_\mu \mod I$, and the coefficients in this linear combination should all be 0. There are $n$ basis elements in $B_\mu$, so there are $n$ conditions imposed on the $2n$ coordinates of $U_\mu$, making the dimension of the irreducible components of $\text{Hilb}^n(C) \cap U_\mu$ at least $n$.

And the irreducible components of $\text{Hilb}^n(C)$ should have the same dimension as the irreducible components of $\text{Hilb}^n(C) \cap U_\mu$, because intersecting an irreducible component with an open set $U_\mu$ does not change its dimension. So each irreducible component of $\text{Hilb}^n(C)$ has dimension at least $n$.

Lemma 8.3. The points in the image under the Hilbert-Chow map of an ideal $I \in \text{Hilb}^n(C)$ are contained in $\cup C_j$, i.e. $\pi(I) \subset \cup C_j$.

Proof. Pick one ideal $I \in \text{Hilb}^n(C)$. Its zero locus is a subset of the zero locus of $f = \prod f_j^{\beta_j}(x, y)$ because $f \in I$. And $f$ vanishes at $\cup C_j$, so $\pi(I) \subset (\cup C_j)$.

Definition 8.4. We stratify the space $\text{Hilb}^n(\cup C_j^{sm})$, which is the set of all ideals $I$ such that $\pi(I)$ is a collection of points $\cup C_j^{sm}$. Let $s_j$ denote the total number of points on each curve $C_j^{sm}$, and $r$ the total number of reduced curves $C_j$, define the
stratum \( \Sigma_{m_1^j,\ldots,m_{s_j}^j} \) to be the set of all ideals \( \mathcal{I} \) such that the multiplicities are exactly \( m_i^j \) for a point \( x_i^j \) in the image \( \pi(\mathcal{I}) \) on \( C_j^{sm} \).

**Definition 8.5.** We define \( \text{Hilb}^n(C) \) (smooth part of \( C \)) as \( \text{Hilb}^n(\cup C_j^{sm}) \cap \text{Hilb}^n(C) \).

The stratification of the bigger space \( \text{Hilb}^n(\cup C_j^{sm}) \) induces a stratification of the smaller space \( \text{Hilb}^n(\text{smooth part of } C) \).

**Lemma 8.6.** Each stratum \( \Sigma_{m_1^j,\ldots,m_{s_j}^j} \) is irreducible.

**Proof.** Each stratum \( \Sigma_{m_1^j,\ldots,m_{s_j}^j} \) is isomorphic to

\[
\left( \prod_{1 \leq i \leq s_j^j} \text{Hilb}^{m_i}(\mathbb{C},0) \times \left( \prod_{j=1}^{r_j} (C_j^{sm})^\beta \right) \right) / \prod_{j=1}^{r_j} \text{Stab}(x_1^j,\ldots,x_r^j).
\]

The curve \( C_j \) is irreducible, and its points of singularities is a closed set, which gives that the smooth part of the curve \( C_j^{sm} \) is irreducible. Removing a closed set from the product of all \( C_j^{sm} \) leaves the product irreducible. By Theorem 3.10 the punctual Hilbert scheme is irreducible, so the product of all these factors is also irreducible. This irreducible product taking the quotient by a finite group is again irreducible.

**Lemma 8.7.** When each \( 1 \leq m_i^j \leq \beta_j \), the closure of each stratum \( \Sigma_{m_1^j,\ldots,m_{s_j}^j} \cap \text{Hilb}^n(C) \) of \( \text{Hilb}^n(\text{smooth part of } C) \) is equal to the closure of each stratum \( \Sigma_{m_1^j,\ldots,m_{s_j}^j} \cap \text{Hilb}^n(C_j^{sm}) \) of \( \text{Hilb}^n(\cup C_j^{sm}) \). The closure of each stratum \( \Sigma_{m_1^j,\ldots,m_{s_j}^j} \cap \text{Hilb}^n(C_j^{sm}) \) is irreducible and has dimension \( n \).

**Proof.** For the proof we use the following fact: If \( Y \) is a closed subset of an irreducible finite-dimensional topological space \( X \), and if \( \dim Y = \dim X \), then \( Y = X \).

Assume \( m_i^j \leq \beta \). We know that the closures of the strata \( \Sigma_{m_1^j,\ldots,m_{s_j}^j} \) are closed, irreducible, and have dimension \( n \), which is given by Lemma 8.6.

A collection of points moving along the smooth part of \( C \) are locally the same as the points moving along \( y_j^\beta \) by an argument of change of coordinates, and therefore the dimension of the stratum \( \Sigma_{m_1^j,\ldots,m_{s_j}^j} \cap \text{Hilb}^n(C) \) is the sum of the dimension of each stratum \( \Sigma_{m_1^j,\ldots,m_{s_j}^j} \) of \( Z_{\beta_j} \). When \( 1 \leq m_i^j \leq \beta_j \), the dimension of each stratum \( \Sigma_{m_1^j,\ldots,m_{s_j}^j} \) is \( m_i^j + \cdots + m_{s_j}^j \) by Theorem 7.8. So the dimension of the stratum \( \Sigma_{m_1^j,\ldots,m_{s_j}^j} \cap \text{Hilb}^n(C) \) is \( m_1^j + \cdots + m_{s_j}^j = n \).
So the closures of strata \( \overline{\Sigma_{m_1} \cap \text{Hilb}^n(C)} \) also have dimension \( n \) (they can’t be bigger because they’re contained in the \( n \)-dimensional \( \Sigma_{m_1} \)).

So the closures of the two types of strata \( \overline{\Sigma_{m_1}} \cap \text{Hilb}^n(C) \) and \( \overline{\Sigma_{m_2}} \cap \text{Hilb}^n(C) \) satisfy the conditions of being \( X \) and \( Y \) in the fact we’re using, and therefore should be the same space. So \( \overline{\Sigma_{m_1}} \cap \text{Hilb}^n(C) \) is irreducible.

\[ \square \]

**Theorem 8.8.** The irreducible components of \( \text{Hilb}^n(C) \) are the closures of the strata \( \Sigma_{m_1} \) where \( 1 \leq m_i \leq \beta_j \).

**Proof.** There are 2 cases when we look at the image \( \pi(I) \) of each ideal \( I \) of \( \text{Hilb}^n(C) \) under the Hilbert-Chow map.

Case 1: The points in the image are all contained in \( C_{j,m_i} \).

When \( 1 \leq m_i \leq \beta_j \), we have argued in Lemma 8.7 that such strata are irreducible and have dimension \( n \). So their closures must be the irreducible components. When \( m_i > \beta_j \), such strata have dimension strictly less than \( n \), and their closures are not the irreducible components.

Case 2: Some of the points in the image \( \pi(I) \) are singularities or the intersection points of the curves \( C \).

The preimage of \( n \) points with multiplicities \( m_i \) on \( C \) is a subset of the product of the punctual Hilbert scheme \( \text{Hilb}^{m_1}(\mathbb{C}^2, 0) \times \cdots \times \text{Hilb}^{m_r}(\mathbb{C}^2, 0) \). By Theorem 3.16, the product of the punctual Hilbert schemes has dimension \( m_1 + \cdots + m_r - r = m_1 + \cdots + m_r - r \). When we allow the points on the smooth part to move, the points at the singularities or the intersections do not move. So the degree of freedom added by moving the points will be strictly less than \( r \). When this happens, the preimage has dimension strictly less than \( n \), and cannot be irreducible components by Lemma 8.2.

\[ \square \]

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