FEEDBACK STABILIZATION AND BOUNDARY CONTROLLABILITY OF THE KORTEWEG-DE VRIES EQUATION ON A STAR-SHAPED NETWORK

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Abstract. We propose a model using the Korteweg-de Vries (KdV) equation on a finite star-shaped network. We first prove the well-posedness of the system and give some regularity results. Then we prove that the energy of the solutions of the dissipative system decays exponentially to zero when the time tends to infinity. Lastly we show an exact boundary controllability result.

Résumé. On propose dans cet article un modèle de l’équation de Korteweg-de Vries (KdV) sur un réseau sous forme d’une étoile. On prouve que le problème est bien posé et on établit quelques propriétés de régularité. De plus, on montre que l’énergie du système décroît d’une manière exponentielle vers 0 quand le temps tend vers l’infini. À la fin, on déduit un résultat de contrôlabilité frontière du système associé.

Contents

1. Introduction 2
2. Well-posedness and regularity results 4
2.1. Well-posedness of the (LKdV) system and regularity results 5
2.2. Well-posedness of (KdV) and regularity results 10
3. Exponential stability 11
3.1. Exponential stability of (LKdV) 11
3.2. Stabilization of the (KdV) system on a star-shaped network in the critical or non-critical case 15
4. Controllability results 20
References 21

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1. Introduction

In the last few years various physical models of multi-link flexible structures consisting of finitely many interconnected flexible elements such as strings, beams, plates, shells have been mathematically studied. For details about some physical motivation for the models, see [11, 3, 4, 1] and the references therein.

In [10], the Korteweg-de Vries equation (KdV) is designed for modeling the pressure in an arterial compartment. Indeed, the Korteweg-de Vries equation models usually long waves in a channel of relatively shallow depth. Thus we propose a new model using this nonlinear dispersive partial differential equation on a network to be used to model the pressure on the arterial tree.

Numerous papers on the stability or the exact controllability of the KdV equation on a finite length interval have already been studied, see for example [15, 13] for the stability and [16, 8, 6, 7] for the control problem. In [6], a tutorial of both problems is presented.

To our knowledge, there is no work about the KdV equation on a star-network but we can cite the article [9] where the controllability of the KdV equation on a compartment with nodes is presented.

Now, let us first introduce some notations and definitions which will be used throughout the rest of the paper, in particular some which are linked to the notion of $C^\nu$- networks, $\nu \in \mathbb{N}$ (as introduced in [11]).

Let $\Gamma$ be a connected topological graph embedded in $\mathbb{R}$, with $N$ edges ($N \in \mathbb{N}^*$). Let $K = \{k_j : 1 \leq j \leq N\}$ be the set of the edges of $\Gamma$. Each edge $k_j$ is a Jordan curve in $\mathbb{R}$ and is assumed to be parametrized by its arc length $x_j$ such that the parametrization $\pi_j : [0, \ell_j] \to k_j : x_j \mapsto \pi_j(x_j)$ is $\nu$-times differentiable, i.e. $\pi_j \in C^\nu([0, \ell_j], \mathbb{R})$ for all $1 \leq j \leq N$. The $C^\nu$- network $\mathcal{T}$ associated with $\Gamma$ is then defined as the union

$$\mathcal{T} = \bigcup_{j=1}^{N} k_j.$$

We define by $L := \sup_{j=1,...,N} \ell_j$, the maximal length of the network.

We study here the stabilization problem and the controllability one of a KdV system on a star-shaped network as in the following figure 1 for $N = 3$. 


More precisely, we study a system which is in connection with the mathematical modeling of the human cardiovascular system. For each edge $k_j$, the scalar function $u_j(t, x)$ for $x \in (0, \ell_j)$ and $t > 0$ contains the information on the displacement of the wave at location $x$ and time $t$, $1 \leq j \leq N$.

We consider the evolution problems ($KdV$) and ($LKdV$) described by the following systems:

\[
\begin{align*}
(KdV) \quad & \left\{ \begin{array}{l}
(\partial_t u_j + \partial_x u_j + u_j \partial_x u_j + \partial_x^3 u_j)(t, x) = 0, \\
u_j(t, 0) = u_k(t, 0), \\
\sum_{j=1}^{N} \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0), \\
u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, \\
u_j(0, x) = u_0^j(x),
\end{array} \right. \\
& \quad \forall x \in (0, \ell_j), t \in (0, +\infty), j = 1, ..., N, \\
& \quad \forall j, k = 1, ..., N, t > 0, \\
& \quad \forall t > 0, j = 1, ..., N, \\
& \quad \forall x \in (0, \ell_j), j = 1, ..., N,
\end{align*}
\]

and

\[
\begin{align*}
(LKdV) \quad & \left\{ \begin{array}{l}
(\partial_t u_j + \partial_x u_j + \partial_x^3 u_j)(t, x) = 0, \\
u_j(t, 0) = u_k(t, 0), \\
\sum_{j=1}^{N} \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0), \\
u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, \\
u_j(0, x) = u_0^j(x),
\end{array} \right. \\
& \quad \forall x \in (0, \ell_j), t \in (0, \infty), j = 1, ..., N, \\
& \quad \forall j, k = 1, ..., N, t > 0, \\
& \quad \forall t > 0, j = 1, ..., N, \\
& \quad \forall x \in (0, \ell_j), j = 1, ..., N,
\end{align*}
\]

where $\alpha > \frac{N}{2}$.

We define the natural energy $E(t)$ of a solution $u = (u_1, ..., u_N)$ of ($KdV$) or ($LKdV$) system by

\[
E(t) = \frac{1}{2} \sum_{j=1}^{N} \int_0^{\ell_j} |u_j(t, x)|^2 \, dx.
\]
We can easily check that every sufficiently smooth solution of \((KdV)\) satisfies the following dissipation law

\[
E'(t) = -\left(\alpha - \frac{N}{2}\right) |u_1(t, 0)|^2 - \frac{1}{2} \sum_{j=1}^{N} |\partial_x u_j(t, 0)|^2 \leq 0,
\]

and therefore, the energy is a nonincreasing function of the time variable \(t\).

This paper is organized as follows: In Section 2, we give the proper functional setting for both systems \((LKdV)\) and \((KdV)\) and prove that those systems are well-posed. We also give some regularity results. In Section 3, we prove our main results, namely the stabilization problem of the systems given by \((LKdV)\) and \((KdV)\). For doing this, we derive first an observability inequality for the linear system and then we apply a fixed point theorem for the non-linear one. In the last Section 4, we prove that the observability inequality also gives the controllability result in the case where the network is non-critical.

2. Well-posedness and regularity results

In order to study both systems on the network, we need a proper functional setting. We define the following spaces:

\[
H^s_r(0, \ell_j) = \left\{ v \in H^s(0, \ell_j), \left( \frac{d}{dx} \right)^{i-1} v(\ell_j) = 0, 1 \leq i \leq s \right\}, \ s = 1, 2,
\]

\[
H^s_e(T) = \left\{ u = (u_1, ..., u_N) \in \prod_{j=1}^{N} H^s_r(0, \ell_j), u_j(0) = u_k(0), \forall j, k = 1, \ldots, N \right\}, \ s = 1, 2,
\]

and

\[
L^2(T) = \prod_{j=1}^{N} L^2(0, \ell_j),
\]

equipped with the inner product

\[
(u, v)_{L^2(T)} = \sum_{j=1}^{N} \int_0^{\ell_j} u_j \overline{v}_j \, dx, \forall u, v \in L^2(T).
\]

We also define the following space \(\mathcal{B} := C([0, T], L^2(T)) \cap L^2(0, T; H^1_e(T))\) endowed with the norm

\[
\|u\|_{\mathcal{B}} := \|u\|_{C([0, T], L^2(T))} + \|u\|_{L^2(0, T; H^1_e(T))} = \max_{t \in [0, T]} \|u(t, .)\|_{L^2(T)} + \left( \int_0^T \|u(t, .)\|_{H^1_e(T)}^2 \, dt \right)^{1/2}.
\]
2. Well-posedness of the (LKdV) system and regularity results. The system (LKdV) can be rewritten as the first order evolution equation

\[ \begin{cases} U' = AU, \\ U(0) = (u^0)^T = U_0, \end{cases} \]

where $U$ is the vector $u^T$ and the operator $A: D(A) \subset L^2(T) \rightarrow L^2(T)$ is defined by

\[ A u^T := - \left( D_T + D_T^3 \right) u^T, \]

\[ \forall u \in D(A) = \left\{ u = (u_1, \ldots, u_N) \in H^2_c(T) \cap \prod_{j=1}^N H^3(0, \ell_j), \sum_{j=1}^N \frac{d^2 u_j}{dx^2}(0) = -\alpha u_1(0) \right\}, \]

and

\[ D_T u^T := \left( \begin{array}{c} \partial_x u_1 \\ \vdots \\ \partial_x u_N \end{array} \right), \forall u \in \prod_{j=1}^N H^1(0, \ell_j). \]

Now we can prove, according to the linear semi-group theory (see [14]), the well-posedness of system (LKdV) and that the solution satisfies the dissipation law (1.2).

**Proposition 2.1.** For an initial datum $U_0 \in L^2(T)$, there exists a unique solution $U(t) := e^{tA}U_0 \in C([0, +\infty), L^2(T))$ to problem (2.4). Moreover, the solution $u$ satisfies (1.2). Therefore the energy is decreasing.

**Proof.** The operator $A$ is clearly closed. Let $u \in D(A)$, then by using some integration by parts, we get,

\[ (u^T, Au^T) = \sum_{j=1}^N \int_{\ell_j}^0 u_j(-\partial_x u_j - \partial_{xxx} u_j)dx \]

\[ = \left( \frac{N}{2} - \alpha \right) |u_1(0)|^2 - \frac{1}{2} \sum_{j=1}^N |\partial_x u_j(0)|^2 \leq 0. \]

Thus $A$ is dissipative.

The adjoint operator of $A$ is defined by $A^* v^T := (D_T + D_T^3) v^T$, with

\[ D(A^*) = \left\{ v = (v_1, \ldots, v_N) \in H^1_c(T) \cap \prod_{j=1}^N H^3(0, \ell_j), \right. \]

\[ \left. \frac{dv_j}{dx}(0) = 0, \sum_{j=1}^N \frac{d^2 v_j}{dx^2}(0) = (\alpha - N) v_1(0) \right\}. \]

In the same manner, we obtain,

\[ (v^T, A^* v^T) = \left( \frac{N}{2} - \alpha \right) |v_1(0)|^2 - \frac{1}{2} \sum_{j=1}^N |\partial_x v_j(\ell_j)|^2 \leq 0, \]

hence $A^*$ is also dissipative and then $A$ generates a strongly semi-group of contractions on $L^2(T)$. We denote by $S$ this semi-group.
We also need some regularity results for the solution of the linear equation with some extra boundary conditions,

\[
\begin{cases}
\partial_t u_j + \partial_x u_j + \partial^2_x u_j(t, x) = 0, & \forall x \in (0, \ell_j), t \in (0, \infty), j = 1, \ldots, N, \\
u_j(t, 0) = u_k(t, 0), & \forall j, k = 1, \ldots, N, t > 0, \\
\sum_{j=1}^N \partial^2_x u_j(t, 0) = -\alpha u_1(t, 0) + g(t), & \forall t > 0, \\
u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, & \forall t > 0, j = 1, \ldots, N, \\
u_j(0, x) = \nu^0_j(x), & \forall x \in (0, \ell_j), j = 1, \ldots, N.
\end{cases}
\]

(2.5)

**Proposition 2.2.** Let \((\nu^0, g) \in D(A) \times C^2_0([0, T]), \) where \(C^2_0([0, T]) := \{ \varphi \in C^2([0, T]), \varphi(0) = 0 \}. Then there exists a unique solution \(u \in C([0, T], D(A)) \cap C^1([0, T]), L^2(T) \)) of (2.5).

**Proof.** We first define the functions \(\phi_j(x) := \frac{(x - \ell_j)^2}{\ell_j^2 \left( 2 \sum_{i=1}^N \ell_i^{-2} + \alpha \right)} \). Thus \(\phi_j \in C^\infty([0, \ell_j])\) and satisfies,

\[
\begin{cases}
\phi_j(\ell_j) = \phi_j'(\ell_j) = 0, & \forall j = 1, \ldots, N, \\
\phi_j(0) = \frac{1}{2 \sum_{i=1}^N \ell_i^{-2} + \alpha} \phi_k(0), & \forall j, k = 1, \ldots, N, \\
\phi''_j(0) = 1 - \alpha \phi_1(0).
\end{cases}
\]

We define \(\tilde{z} := u - g\phi\), then \(\tilde{z}\) satisfies the system:

\[
\begin{cases}
(\partial_t z_j + \partial_x z_j + \partial^2_x z_j)(t, x) = -\phi_j(x)g'(t) - (\phi_j' + \phi_j''(x))g(t), & \forall x \in (0, \ell_j), t > 0, j = 1, \ldots, N, \\
z_j(t, 0) = z_k(t, 0), & \forall j, k = 1, \ldots, N, t > 0, \\
\sum_{j=1}^N \partial^2_x z_j(t, 0) = -\alpha z_1(t, 0), & \forall t > 0, \\
z_j(t, \ell_j) = \partial_x z_j(t, \ell_j) = 0, & \forall t > 0, j = 1, \ldots, N, \\
z_j(0, x) = \nu^0_j(x), & \forall x \in (0, \ell_j), j = 1, \ldots, N.
\end{cases}
\]

(2.6)

Thus, as \(-\phi'g' + (\phi' + \phi''g) \in C^1([0, T], L^2(\mathcal{T}))\), we deduce from Proposition 2.1 and classical results on semi-group theory, that system (2.6) admits a unique classical solution \(\tilde{z} \in C([0, T], D(A)) \cap C^1([0, T]), L^2(T)\). Hence we can easily prove that problem (2.5) admits a unique classical solution \(u \in C([0, T], D(A)) \cap C^1([0, T]), L^2(T)\).

□

Now, we study the same system but with less regular data.
Proposition 2.3. Let \((u^0, g) \in L^2(T) \times L^2(0, T)\), then there exists a unique mild solution of \((2.20)\), \(u \in B\). Furthermore \(u(\cdot, 0)\) and \(\partial_x u(\cdot, 0)\) belong to \(L^2(0, T)\) and we have the following estimates,

\[
\|u\|_{L^2(T)}^2 \leq C(T, N, L, \alpha)(\|u^0\|_{L^2(T)}^2 + \|g\|_{L^2(0, T)}^2),
\]

\[
\|u_1(\cdot, 0)\|_{L^2(0, T)}^2 \leq \frac{1}{\alpha - \frac{N}{2}} \|u^0\|_{L^2(T)}^2 + \frac{1}{\alpha - \frac{N}{2}}^2 \|g\|_{L^2(0, T)}^2,
\]

\[
\|\partial_x u(\cdot, 0)\|_{L^2(0, T)}^2 \leq \|u^0\|_{L^2(T)}^2 + \frac{1}{\alpha - \frac{N}{2}} \|g\|_{L^2(0, T)}^2,
\]

\[
\|u^0\|_{L^2(T)}^2 \leq \frac{1}{T} \|u\|_{L^2(0, T), L^2(T)}^2 + 3(\alpha - \frac{N}{2})\|u_1(\cdot, 0)\|_{L^2(0, T)}^2 + \|\partial_x u(\cdot, 0)\|_{L^2(0, T)}^2 + \frac{1}{\alpha - \frac{N}{2}} \|g\|_{L^2(0, T)}^2.
\]

Proof. The proof of this result is obtained by a density argument and the multiplier method.

We first suppose that \((u^0, g) \in D(A) \times C_0^\infty([0, T])\) and thus the solution of \((2.5)\) satisfies \(u \in C([0, T], D(A)) \cap C^1([0, T], L^2(T))\).

Let \(q = (q_j)_{j=1, \ldots, N} = \prod_{j=1}^N C^\infty([0, T] \times [0, \ell_j]; \mathbb{R}).\) Then by multiplying \((LKdV)\) by \(q_j \bar{u}_j\), integrating on \([0, s] \times [0, \ell_j]\) with \(s \in [0, T]\) and using some integrations by parts we get the following equation,

\[
\sum_{j=1}^N \int_0^{\ell_j} [q_j |u_j|^2]_0^s dx - \sum_{j=1}^N \int_0^s \int_0^{\ell_j} (\partial_t q_j + \partial_x q_j + \partial_x^2 q_j) |u_j|^2 dx dt + 3 \sum_{j=1}^N \int_0^s \int_0^{\ell_j} \partial_x q_j |\partial_x u_j|^2 dx dt
\]

\[
= \sum_{j=1}^N \int_0^s \left((q_j + \partial_x^2 q_j)|u_j|^2 + 2q_j u_j \partial_x^2 \bar{u}_j - 2\partial_x q_j u_j \partial_x \bar{u}_j - q_j |\partial_x u_j|^2\right)(t, 0) dt.
\]

(1) Taking first \(q = 1\), then \((2.10)\) becomes,

\[
\sum_{j=1}^N \int_0^{\ell_j} |u_j(s, x)|^2 dx + \int_0^s \sum_{j=1}^N |\partial_x u_j(t, 0)|^2 dt + (2\alpha - N) \int_0^s |u_1(t, 0)|^2 dt =
\]

\[
\sum_{j=1}^N \int_0^{\ell_j} |u_j(0, x)|^2 dx + 2 \int_0^s \bar{u}_1(t, 0) g(t) dt
\]

\[
\leq \sum_{j=1}^N \int_0^{\ell_j} |u_j(0, x)|^2 dx + \left(\alpha - \frac{N}{2}\right) \int_0^s |u_1(t, 0)|^2 dt + \frac{1}{\alpha - \frac{N}{2}} \int_0^s g^2(t) dt.
\]

So we have,

\[
\sum_{j=1}^N \int_0^{\ell_j} |u_j(s, x)|^2 dx + \int_0^s \sum_{j=1}^N |\partial_x u_j(t, 0)|^2 dt + (\alpha - \frac{N}{2}) \int_0^s |u_1(t, 0)|^2 dt \leq
\]

\[
\sum_{j=1}^N \int_0^{\ell_j} |u_j(0, x)|^2 dx + \frac{1}{\alpha - \frac{N}{2}} \int_0^T g^2(t) dt.
\]
Thus, \( u(s, \cdot) \in L^2(T) \) and we have the estimate,

\[
\max_{s \in [0,T]} \| u(s, \cdot) \|_{L^2(T)}^2 \leq \| u^0 \|_{L^2(T)}^2 + \frac{1}{\alpha - N} \| g \|_{L^2(0,T)}^2.
\]

Taking \( s = T \) in inequality (2.11) gives that \( u_1(\cdot,0) \in L^2(0,T) \) and \( \partial_x u_j(\cdot,0) \in L^2(0,T) \), for all \( j = 1, \ldots, N \) and we have the estimates,

\[
\| u_1(\cdot,0) \|^2_{L^2(0,T)} \leq \frac{1}{\alpha - N} \| u^0 \|^2_{L^2(T)} + \frac{1}{(\alpha - N)^2} \| g \|^2_{L^2(0,T)},
\]

\[
\| \partial_x u(\cdot,0) \|^2_{L^2(0,T)} \leq \| u^0 \|^2_{L^2(T)} + \frac{1}{\alpha - N} \| g \|^2_{L^2(0,T)}.
\] (2.13)

(2) Secondly, we take \( q_j(t,x) = x \) for \( j = 1, \ldots, N \) and \( s = T \) then equation (2.10) gives us,

\[
\sum_{j=1}^N \int_0^T \int_0^{T_j} x|u_j|^2 \, dx 
- \sum_{j=1}^N \int_0^T \int_0^{T_j} |u_j|^2 \, dx 
+ 3 \sum_{j=1}^N \int_0^T \int_0^{T_j} |\partial_x u_j|^2 \, dx dt
= \sum_{j=1}^N \int_0^T \left( -2u_j \partial_x u_j \right)(t,0) dt.
\]

Then we have,

\[
\sum_{j=1}^N \int_0^T \int_0^{T_j} |\partial_x u_j|^2 \, dx dt \leq \frac{L}{3} \sum_{j=1}^N \int_0^{T_j} |u_j(0,x)|^2 \, dx
+ \frac{1}{3} \sum_{j=1}^N \int_0^T |u_j|^2 \, dx dt
+ \frac{1}{3} \sum_{j=1}^N \int_0^T \left( -2u_j \partial_x u_j \right)(t,0) \, dt.
\]

Using estimates (2.12) and (2.13), we can deduce the following estimate,

\[
\| \partial_x u \|^2_{L^2([0,T],L^2(T))} \leq C(T, L, N, \alpha) \left( \| u^0 \|^2_{L^2(T)} + \| g \|^2_{L^2(0,T)} \right).
\] (2.14)

(3) Lastly, we choose \( q_j(t,x) = T - t \) for \( j = 1, \ldots, N \) and \( s = T \) then we obtain the equation,

\[
T \sum_{j=1}^N \int_0^{T_j} |u_j(0,x)|^2 \, dx
= \sum_{j=1}^N \int_0^T \int_0^{T_j} |u_j|^2 \, dx dt + (2\alpha - N) \int_0^T (T - t)|u_1(t,0)|^2 \, dt
+ \sum_{j=1}^N \int_0^T (T - t)|\partial_x u_j(t,0)|^2 \, dt - 2 \int_0^T (T - t)\bar{u}_1(t,0) g(t) \, dt,
\]

and then we easily get

\[
\| u^0 \|^2_{L^2(T)} \leq \frac{1}{T} \| g \|^2_{L^2(0,T),L^2(T)} + 3(\alpha - N) \| u_1(\cdot,0) \|^2_{L^2(0,T)} + \| \partial_x u(\cdot,0) \|^2_{L^2(0,T)} + \frac{1}{\alpha - N} \| g \|^2_{L^2(0,T)}.
\]

By the density of \( D(A) \) in \( L^2(T) \) and of \( C^2_0([0,T]) \) in \( L^2(0,T) \), and by using inequalities (2.12), (2.13) and (2.14), we get the desired result.
Before proving the well-posedness of (KdV) we need also a result of regularity for the linear system with a source term.

**Proposition 2.4.** Let \( (u_0, f, g) \in L^2(T) \times L^1(0, T, L^2(T)) \times L^2(0, T) \), then there exists a unique mild solution \( u \in \mathcal{B} \) of

\[
\begin{aligned}
& (\partial_t u_j + \partial_x u_j + \partial^3_x u_j)(t, x) = f_j, \quad \forall x \in (0, \ell_j), \; t > 0, \; j = 1, \ldots, N, \\
& u_j(t, 0) = u_k(t, 0), \quad \forall j, k = 1, \ldots, N, \; t > 0, \\
& \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0) + g(t), \quad \forall t > 0, \\
& u_j(t, \ell_j) = \partial_x u_j(t, \ell_j) = 0, \quad \forall t > 0, \; j = 1, \ldots, N, \\
& u_j(0, x) = u_j^0(x), \; \forall x \in (0, \ell_j), \; j = 1, \ldots, N,
\end{aligned}
\]

and it satisfies,

\[
\|u\|^2_{\mathcal{B}} \leq C(T, N, L, \alpha)(\|u_0\|^2_{L^2(T)} + \|f\|^2_{L^1(0, T, L^2(T))} + \|g\|^2_{L^2(0, T)}).
\]

**Proof.** Thanks to Proposition 2.3 we consider that \((u^0, g) = (0, 0)\). By using standard semigroup theory, we get that if \( f \in L^1(0, T, L^2(T)) \) then the solution of (2.15) verifies \( u \in C([0, T], L^2(T)) \) and

\[
\|u\|_{C([0, T], L^2(T))} \leq C\|f\|_{L^1(0, T, L^2(T))}.
\]

As before we multiply the PDE in (2.15) by \( \bar{u}_j \) and we integrate by parts on \([0, T] \times (0, \ell_j)\). We easily obtain that,

\[
\sum_{j=1}^N \int_0^\ell_j |u_j(T, x)|^2 dx + (2\alpha - N) \int_0^T |u_1(t, 0)|^2 dt + \sum_{j=1}^N \int_0^T |\partial_x u_j(t, 0)|^2 dt \leq
\]

\[
2 \sum_{j=1}^N \int_0^T \|f_j(t, .)||u_j(t, .)||_{L^2(0, \ell_j)} dt \leq 2\|u\|_{C([0, T], L^2(T))}\|f\|_{L^1(0, T, L^2(T))} \leq C\|f\|^2_{L^1(0, T, L^2(T))}.
\]

Thus

\[
|u_1(t, 0)|^2_{L^2(0, T)} + |\partial_x u(t, 0)|^2_{L^2(0, T)} \leq C\|f\|^2_{L^1(0, T, L^2(T))}.
\]

Next, we multiply the PDE in (2.15) by \( x\bar{u}_j \) and integrate by parts. We obtain,

\[
\sum_{j=1}^N \int_0^\ell_j x|u_j(T, x)|^2 dx - \sum_{j=1}^N \int_0^T \int_0^{\ell_j} |u_j|^2 dx dt + 3 \sum_{j=1}^N \int_0^{\ell_j} |\partial_x u_j|^2 dx dt
\]

\[
= 2 \sum_{j=1}^N \int_0^T \int_0^{\ell_j} x\bar{u}_j f_j dx dt + \sum_{j=1}^N \int_0^T \left(-2u_j \partial_x \bar{u}_j\right)(t, 0) dt.
\]
Thanks to \[(2.17),\] we obtain that,
\[
3 \sum_{j=1}^{N} \int_{0}^{T} \int_{0}^{T} |\partial_x u_j|^2 dx dt \leq T \|u\|_{C([0,T], L^2(T))}^2 + 2L \|u\|_{C([0,T], L^2(T))} \|f\|_{L^1(0,T,L^2(T))}
+ \|u_1(t,0)\|_{L^2(0,T)}^2 + \|\partial_x u_1(t,0)\|_{L^2(0,T)}^2
\leq C \|f\|_{L^1(0,T,L^2(T))}^2.
\]
Which ends the proof. \[
\square
\]

2.2. **Well-posedness of (KdV) and regularity results.** In order to prove the well-posedness of the nonlinear KdV equation, we need some regularity on the nonlinearity appearing in the equation and at the central node.

We first recall the following Proposition whose proof can be found in [10, Proposition 4.1] or [6, Proposition 4].

**Proposition 2.5.** Let \( T, L > 0 \), let \( y \in L^2(0,T; H^1(0,L)) \). Then \( yy_x \in L^1(0,T; L^2(0,L)) \) and the map \( y \in L^2(0,T; H^1(0,L)) \to yy_x \in L^1(0,T; L^2(0,L)) \) is continuous. Moreover, we have
\[(2.18) \quad \|yy_x\|_{L^1(0,T; L^2(0,L))} \leq C \|y\|_{L^2(0,T; H^1(0,L))}^2.
\]

We also need the following proposition,

**Proposition 2.6.** Let \( u \in \mathfrak{B} \), then \( |u_1(\cdot,0)|^2 \in L^2(0,T) \) and the map \( u \in \mathfrak{B} \to |u_1(\cdot,0)|^2 \in L^2(0,T) \) is continuous. Moreover, we have the estimate,
\[(2.19) \quad \|u_1^2(\cdot,0)\|_{L^2(0,T)} \leq \frac{1}{\sqrt{2}} \|u\|_{\mathfrak{B}}^2.
\]

**Proof.** Let \( u, v \in \mathfrak{B} \). As \( u_1(t,\ell_1) = v_1(t,\ell_1) = 0 \) we have
\[
|u_1^2(t,0) - v_1^2(t,0)| = \left| \frac{1}{2} \int_{\ell_1}^{T} u_1(t,x)\partial_x u_1(t,x) - v_1(t,x)\partial_x v_1(t,x) dx \right|
\leq \frac{1}{2} \int_{\ell_1}^{T} |(u_1 - v_1)\partial_x u_1 + v_1(\partial_x u_1 - \partial_x v_1)| dx
\leq \frac{1}{2} \left( \|u_1(t,\cdot) - v_1(t,\cdot)\|_{L^2(\ell_1,t)} \|u_1(t,\cdot)\|_{H^1(0,\ell_1)} + \|v_1(t,\cdot)\|_{L^2(\ell_1,t)} \|u_1(t,\cdot) - v_1(t,\cdot)\|_{H^1(0,\ell_1)} \right).
\]

Thus,
\[
\int_{0}^{T} |u_1^2(t,0) - v_1^2(t,0)|^2 dt
\leq \int_{0}^{T} \frac{1}{2} \left( \|u_1(t,\cdot) - v_1(t,\cdot)\|_{L^2(0,\ell_1)} \|u_1(t,\cdot)\|_{H^1(0,\ell_1)} + \|v_1(t,\cdot)\|_{L^2(0,\ell_1)} \|u_1(t,\cdot) - v_1(t,\cdot)\|_{H^1(0,\ell_1)} \right) dt
\leq \frac{1}{2} \|u_1 - v_1\|_{C([0,T]; L^2(0,\ell_1))}^2 \|u_1\|_{L^2(0,T; H^1(0,\ell_1))}^2 + \frac{1}{2} \|v_1\|_{C([0,T]; L^2(0,\ell_1))}^2 \|u_1 - v_1\|_{L^2(0,T; H^1(0,\ell_1))}^2
\leq \frac{1}{2} \|u\|_{\mathfrak{B}}^2 + \|v\|_{\mathfrak{B}}^2 ||u - v||_{\mathfrak{B}}^2.
\]

We get the desired result and estimate \[(2.19).\] \[
\square
\]
With those both previous Propositions, we can prove the well-posedness of the non-linear KdV system.

**Theorem 2.7.** Let \((\ell_i)_{i=1..N} \in (0, +\infty)^N\) and \(T > 0\). Then there exist \(\epsilon > 0\) and \(C > 0\) such that for all \(u^0 \in L^2(T)\) with \(\|u^0\|_{L^2(T)} < \epsilon\), then there exists a unique solution of (KdV) that satisfies
\[
\|u\|_B \leq C\|u^0\|_{L^2(T)}.
\]

**Proof.** Let us fix \(u^0 \in L^2(T)\) such that \(\|u^0\|_{L^2(T)} < \epsilon\) where \(\epsilon > 0\) will be chosen later. We prove this theorem by using the Banach fixed point Theorem on the following map, \(F : u \in B \mapsto v \in B\), where \(v\) is the solution of,
\[
\begin{align*}
(\partial_t v_j + \partial_x v_j + \partial_x^2 v_j)(t, x) &= -u_j \partial_x u_j, & \forall x \in (0, \ell_j), t > 0, j = 1, \ldots, N, \\
v_j(t, 0) &= v_k(t, 0), & \forall j, k = 1, \ldots, N, t > 0, \\
\sum_{j=1}^N \partial_x^2 v_j(t, 0) &= -\alpha v_1(t, 0) - \frac{N}{3} (u_1(t, 0))^2, & \forall t > 0, \\
v_j(t, \ell_j) &= \partial_x v_j(t, \ell_j) = 0, & \forall t > 0, j = 1, \ldots, N, \\
v_j(0, x) &= u^0_j(x), & \forall x \in (0, \ell_j), j = 1, \ldots, N.
\end{align*}
\]

Clearly, \(u \in B\) is a solution of (KdV) is equivalent to \(u\) is a fixed point of \(F\). By using the previous regularity results, namely Propositions 2.4, 2.5 and 2.6, we get that for all \(u \in B\),
\[
\|Fu\|_B \leq C\left(\|u^0\|_{L^2(T)} + \|u\|_B^2\right),
\]
and for all \(u^1, u^2 \in B\),
\[
\|Fu^1 - Fu^2\|_B \leq C\left(\|u^1\|_B + \|u^2\|_B\right)\|u^1 - u^2\|_B.
\]

Let us choose \(R > 0\) to be defined later and \(u, u^1\) and \(u^2 \in B_B(0, R)\), then we have
\[
\|Fu\|_B \leq C(\epsilon + R^2)
\]
\[
\|Fu^1 - Fu^2\|_B \leq C(2R)\|u^1 - u^2\|_B.
\]

Thus by taking \(R > 0\) such that \(R < \frac{1}{2\epsilon}\) and \(\epsilon > 0\) such that \(C(\epsilon + R^2) < R\) we get the well-posedness result with the Banach fixed point Theorem. 

3. Exponential stability

3.1. **Exponential stability of \((L\text{KdV})\).** In this section we will study two cases. First when the number of lengths which are in the space of critical lengths, namely \(N := \{2\pi \sqrt{k^2 + l^2 + kl}, k, l \in \mathbb{N}^*\}\), is strictly less than two. And in the second case when this number is larger than two.
3.1.1. Observability inequality and stability in the non critical case.

**Theorem 3.1.** Let \((\ell_i)_{i=1..N} \in (0, +\infty)^N\) such that \(#\{\ell_i \in \mathcal{N}\} \leq 1\). Then for all \(T > 0\), there exists \(C > 0\) such that for all \(u_0^0 \in L^2(\mathcal{T})\) we have,

\[
\|u_0\|_{L^2(\mathcal{T})}^2 \leq C \left( \|\partial_x u_0(., 0)\|_{L^2(0,T)}^2 + \left( \alpha - \frac{N}{2} \right) \|u(., 0)\|_{L^2(0,T)}^2 \right),
\]

where \(u \in \mathbb{B}\) is the solution of \((LKdV)\).

**Proof.** We follow the proof of Lemma 3.5 in [16] or Proposition 8 in [6]. Let us suppose that the result is false. Then we could find a sequence \((u_0^{n,n})_{n \in \mathbb{N}} \in L^2(\mathcal{T})\) such that \(\|u_0^{n,n}\|_{L^2(\mathcal{T})} = 1\) and such that

\[
\|\partial_x u^n(., 0)\|_{L^2(0,T)}^2 + \|u^n(., 0)\|_{L^2(0,T)}^2 \to 0
\]

where \(u^n := S(.)u_0^{0,n}\).

By using estimates (2.7) we have

\[
\|u^n\|_{L^2(0,T; \mathbb{H}_e^2(\mathcal{T}))} \leq \|u^n\|_{\mathbb{B}} \leq C(T, L, N, \alpha).
\]

Thus \((u^n)\) is bounded in \(L^2(0,T; \mathbb{H}_e^1(\mathcal{T}))\) and then \((u^n)\) is bounded in \(L^2(0,T; \mathbb{H}_e^{-2}(\mathcal{T}))\). Thanks to the Aubin-Lions Lemma, we can deduce that \(u^n\) is relatively compact in \(L^2(0,T; L^2(\mathcal{T}))\) and we can assume that \(u^n\) converges in \(L^2(0,T; L^2(\mathcal{T}))\).

With inequality (2.9), we have

\[
\|u_0^{n,n}\|_{L^2(\mathcal{T})}^2 \leq \frac{1}{T} \|u^n\|_{L^2(0,T; L^2(\mathcal{T}))}^2 + 3 \left( \alpha - \frac{N}{2} \right) \|u^n(., 0)\|_{L^2(0,T)}^2 + \|\partial_x u^n(., 0)\|_{L^2(0,T)}^2.
\]

As the two last terms tends to 0 as \(n\) tends to infinity, \((u_0^{0,n})\) is a Cauchy sequence in \(L^2(\mathcal{T})\) and then converges to a function \(u^0\) satisfying \(\|u^0\|_{L^2(\mathcal{T})} = 1\). Then, we have \(u = S(.)u^0\), \(u_1(t, 0) = 0\) and \(\partial_x u(t, 0) = 0\).

With the same type of proof as in [16], we have to prove the following Lemma:

**Lemma 3.2.** Let \((\ell_i)_{i=1..N} \in (0, +\infty)^N\). Let us consider the following assertion:

\[
\exists (\lambda_i)_{i=1..N} \in \mathbb{C}^N, \exists y \in \prod_{i=1}^N H^3(\mathcal{O}, \ell_i) \setminus \{0\} \text{ s.t.} \left\{
\begin{array}{l}
\lambda_i y_i + y_i' + y_i''' = 0, \forall i = 1 \ldots N, \\
y_i(\ell_i) = 0, y_i'(\ell_i) = 0, \forall i = 1 \ldots N, \\
y_i(0) = 0, y_i'(0) = 0, \forall i = 1 \ldots N,
\end{array}
\right.
\]

\[
\sum_{i=1}^N y_i''(0) = 0.
\]

Then (3.22) \(\iff \#\{\ell_i \in \mathcal{N}\} \geq 2\).

**Proof.** Let us first recall Lemma 3.5 in [16]:

\[
(3.22)
\]
Lemma 3.3. [16] Lemma 3.5] Let $L \in (0, +\infty)$. Consider the following assertion,

\begin{equation}
\exists \lambda \in \mathbb{C}, \exists y \in H^3(0, L) \setminus \{0\} \text{ s.t.} \left\{ \begin{array}{l}
\lambda y + y' + y''' = 0, \\
y(0) = 0, y'(0) = 0, y(L) = 0, y'(L) = 0.
\end{array} \right.
\end{equation}

Then \(3.23\) $\Rightarrow L \in \mathcal{N}$.

(1) If $\forall i, \ell_i \notin \mathcal{N}$ then Lemma 3.3 gives us $y = 0$.

(2) If $\#\{\ell_i \in \mathcal{N}\} = 1$, then we can suppose that $\ell_1 \in \mathcal{N}$ and for all $i = 2 \ldots N$, $\ell_i \notin \mathcal{N}$.

Then Lemma 3.3 gives us that for all $i = 2 \ldots N$, $\ell_i \notin \mathcal{N}$, $y_i = 0$ and then $y_1$ has to satisfy,

\begin{equation}
\left\{ \begin{array}{l}
\lambda_1 y_1 + y'_1 + y'''_1 = 0, \\
y_1(\ell_1) = 0, y'_1(\ell_1) = 0, \\
y_1(0) = 0, y'_1(0) = 0, \\
y''_1(0) = 0.
\end{array} \right.
\end{equation}

Due to the three null conditions at the spatial origin 0, the unique solution of this system is $y_1 = 0$.

Thus $y = 0$.

(3) If $\#\{\ell_i \in \mathcal{N}\} \geq 2$, then we can suppose that $\ell_1, \ell_2 \in \mathcal{N}$. Then we can take $y_i = 0$ for $i = 3 \ldots N$. Lemma 3.3 gives us two non null functions $z_1$ and $z_2$ that satisfy

\begin{equation}
\left\{ \begin{array}{l}
\lambda_1 z_1 + z'_1 + z'''_1 = 0, \\
z_1(\ell_1) = 0, z'_1(\ell_1) = 0, \\
z_1(0) = 0, z'_1(0) = 0, \\
z''_1(0) = 0.
\end{array} \right.
\end{equation}

\begin{equation}
\left\{ \begin{array}{l}
\lambda_2 z_2 + z'_2 + z'''_2 = 0, \\
z_2(\ell_2) = 0, z'_2(\ell_2) = 0, \\
z_2(0) = 0, z'_2(0) = 0.
\end{array} \right.
\end{equation}

We then define

$$y = (z''(0)z_1, -z''(0)z_2, 0, \ldots, 0).$$

As $z_1$ and $z_2$ are non null satisfy an ODE of order 3 and $z_1(0) = z'_1(0) = 0$ and $z_2(0) = z'_2(0) = 0$ then $z''_1(0) \neq 0$ and $z''_2(0) \neq 0$. Then $y$ is non null and satisfies the system given in \(3.22\).

From this Lemma, we easily deduce the observability inequality \(3.21\) and this ends the proof of Theorem 3.1.

We can now prove the result of stability.

**Theorem 3.4.** Let $(\ell_i)_{i=1..N} \in (0, +\infty)^N$ such that $\#\{\ell_i \in \mathcal{N}\} \leq 1$, then there exists $C > 0$ and $\mu > 0$ such that for all $u^0 \in L^2(T)$ the solution of $(LKdV)$ satisfies,

$$\|u(t, \cdot)\|_{L^2(T)} \leq C \|u^0\|_{L^2(T)} e^{-\mu t}.$$
Proof. We follow the proof given in [15]. With (1.2) we have by integration and using the previous observability inequality (3.21),
\[
\|u(T,.)\|_{L^2(T)}^2 = \|u^0\|_{L^2(T)}^2 - \left(\alpha - \frac{N}{2}\right) \|u_1(.,0)\|^2_{L^2(0,T)} + \|\partial_x u(.,0)\|^2_{L^2(0,T)}
\]
\[
\leq \frac{C-1}{C} \|u^0\|_{L^2(T)}^2
\]
Thus we get easily the stability result.

\[\square\]

3.1.2. Stability in the critical case. We suppose in this section that \(\#\{\ell_i \in \mathcal{N}\} \geq 2\) then adding a damping mechanism on the critical branches except at most one gives the stability of the system.

Let us define \(I_c = \{i \in \{1, \ldots, N\}, \ell_i \in \mathcal{N}\}\), the set of critical indexes, and \(I_c^*\) equals to \(I_c\) minus one index. We study the following problem,

\[(L K d V_{damped}) \begin{cases}
(\partial_t u_j + \partial_x u_j + \partial_x^3 u_j + a_j(x) u_j)(t,x) = 0, & \forall x \in (0, \ell_j), t > 0, j = 1, \ldots, N, \\
u_j(t,0) = u_k(t,0), & \forall j, k = 1, \ldots, N, t > 0, \\
\sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0), & \forall t > 0, \\
u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & \forall t > 0, j = 1, \ldots, N, \\
u_j(0,x) = u^0_j(x), & \forall x \in (0, \ell_j), j = 1, \ldots, N,
\end{cases}\]

where \(\alpha > \frac{N}{2}\) and the damping \((a_j)_{j=1,N} \in \prod_{j=1}^{N} L^\infty(0, \ell_j)\) is defined by

\[
(3.24) \begin{cases}
a_j = 0 \text{ for } j \in \{1, \ldots, N\} \setminus I_c^*, \\
a_j \geq c_j \text{ in an open nonempty set } \omega_j \text{ of } (0, \ell_j), \text{ for all } j \in I_c^*, \\
\text{and } c_j > 0 \text{ is a constant.}
\end{cases}
\]

We can prove the well-posedness of this system as in [15], by considering it as a perturbation of (LKdV). With same types of arguments we get the stability result.

**Theorem 3.5.** Assume that the damping \(a\) is defined as in (3.24), then there exist \(C > 0\) and \(\mu > 0\) such that for all \(u^0 \in L^2(T)\), the solution of \((L K d V_{damped})\) satisfies,

\[
\|u(t,\cdot)\|_{L^2(T)} \leq C\|u^0\|_{L^2(T)} e^{-\mu t}.
\]

**Proof.** We first multiply the PDE of \((L K d V_{damped})\) by \(x u_j\) and we easily get the following estimate,

\[
\|\partial_x u\|_{L^2([0,T], L^2(T))}^2 \leq \frac{1}{3} \left( L + T + N + \frac{1}{2\alpha - N} \right) \|u^0\|_{L^2(T)}^2.
\]
Then we multiply the PDE of \((LKdV_{\text{damped}})\) by \((T-t)\varpi_j^2\) to get,
\[
\|u^0\|_{L^2(T)}^2 \leq \frac{1}{T} \|u\|_{L^2(0,T;L^2(T))}^2 + (2\alpha - N)\|u(.,0)\|_{L^2(0,T)}^2 + \|\partial_x u(.,0)\|_{L^2(0,T)}^2 + 2\sum_{j \in I_c} \int_0^T \int_0^{l_j} a_j(x) |u_j|^2 \, dx \, dt.
\]

We argue by contradiction to prove the following inequality,
\[
\|u^0\|_{L^2(T)}^2 \leq C \left( (2\alpha - N)\|u(.,0)\|_{L^2(0,T)}^2 + \|\partial_x u(.,0)\|_{L^2(0,T)}^2 + 2\sum_{j \in I_c} \int_0^T \int_0^{l_j} a_j(x) |u_j|^2 \, dx \, dt \right).
\]

By following the same arguments as for the proof of Theorem \ref{t1} we can construct a sequence \((u^n,0) \in L^2(T)\) such that the corresponding solution of \((LKdV_{\text{damped}})\) satisfies
\[
\begin{align*}
\|u^n(.,0)\|_{L^2(0,T)} &\to 0, \\
\|\partial_x u^n(.,0)\|_{L^2(0,T)} &\to 0, \\
\sum_{j \in I_c} \int_0^T \int_0^{l_j} a_j(x) |u_j^n|^2 \, dx \, dt &\to 0.
\end{align*}
\]

By passing to the limit we obtain a non trivial solution \(u \in \mathbb{B}\) of \((LKdV_{\text{damped}})\) such that
\[
\begin{align*}
u(.,0) &= 0, \\
\partial_x u(.,0) &= 0, \\
\int_0^T \int_0^{l_j} a_j(x) |u_j|^2 \, dx \, dt &= 0, \forall j \in I_c^*.
\end{align*}
\]

(1) For all \(j \in \{1, \ldots, N\} \setminus I_c\), \(u_j\) is solution of \((LKdV)\) and such that \(u_j(.,0) = \partial_x u_j(.,0) = 0\). Then thanks to Lemma \ref{L2}, \(u_j = 0\).

(2) For all \(j \in I_c^*\), \(\int_0^T \int_0^{l_j} a_j(x)|u_j|^2 \, dx \, dt = 0\), thus \(a_j u_j = 0\) and \(u_j = 0\) in \((0,T) \times \omega_j\). Then \(\partial_t u_j + \partial_x u_j + \partial^2_x u_j = 0\) and thanks to Holmgren’s Theorem, \(u_j = 0\).

(3) For \(j \in I_c \setminus I_c^*\), \(u_j\) satisfies,
\[
\begin{align*}
\partial_t u_j + \partial_x u_j + \partial^2_x u_j &= 0, \\
\partial_x u_j(t,0) = 0, \partial_t u_j(t,0) = 0, \partial^2_x u_j(t,0) = 0, \\
u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0.
\end{align*}
\]

Due to the three null conditions at the central node, we obtain that \(u_j = 0\).

Thus \(u = 0\) and we get a contradiction which ends the proof of Theorem \ref{t5}.

\boxed{}

3.2. Stabilization of the \((KdV)\) system on a star-shaped network in the critical or non critical case.
3.2.1. Stability for small amplitude solutions. In this section we study the stabilization of the non linear (KdV) system for the critical and the non critical case.

We define as before $I_c = \{ i \in \{ 1, \ldots, N \}, \ell_i \in \mathcal{N} \}$, the set of critical indexes, and $I_c^* \equal{} \{ 1, \ldots, N \} \setminus I_c$ equals to $I_c$ minus one index. Eventually, $I_c^* = \emptyset$. We study the following problem,

\[
\begin{align*}
\left( \text{KdV}_{\text{damped}} \right) & \quad \left\{ \begin{array}{ll}
(\partial_t u_j + \partial_x u_j + \partial_x^2 u_j + a_j u_j + u_j \partial_x u_j)(t, x) = 0, & \forall x \in (0, \ell_j), t > 0, j = 1, \ldots, N, \\
u_j(t, 0) = u_k(t, 0), & \forall j, k = 1, \ldots, N, t > 0,
\end{array} \right. \\
\sum_{j=1}^{N} \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} (u_1(t, 0))^2, & \forall t > 0,
\end{align*}
\]

where $\alpha > \frac{N}{2}$ and the damping $(a_j)_{j=1}^{N} \in \prod_{j=1}^{N} L^\infty(0, \ell_j)$ is defined by

\[
\begin{align*}
\begin{cases}
\quad a_j = 0 \text{ for } j \in \{ 1, \ldots, N \} \setminus I_c^*, \\
\quad a_j \geq c_j \text{ in an open nonempty set } \omega_j \text{ of } (0, \ell_j), \text{ for } j \in I_c^*,
\end{cases}
\end{align*}
\]

and $c_i > 0$ is a constant.

Let $u^0 \in L^2(\mathcal{T})$ such that $\|u^0\|_{L^2(\mathcal{T})}$ is sufficiently small in order to have with Theorem 2.7 the existence and unicity of a solution $u \in \mathcal{B}$ of the perturbation $(KdV_{\text{damped}})$ which is a perturbation of $(KdV)$. Then we can decompose $u$ into $u^1 + u^2$ respective solutions of

\[
\begin{align*}
\left( \text{KdV} \right) & \quad \left\{ \begin{array}{ll}
(\partial_t u_j^1 + \partial_x u_j^1 + \partial_x^2 u_j^1 + a_j u_j^1)(t, x) = 0, & \forall x \in (0, \ell_j), t > 0, j = 1, \ldots, N, \\
u_j^1(t, 0) = u_k^1(t, 0), & \forall j, k = 1, \ldots, N, t > 0,
\end{array} \right. \\
\sum_{j=1}^{N} \partial_x^2 u_j^1(t, 0) = -\alpha u_1^1(t, 0), & \forall t > 0,
\end{align*}
\]

\[
\begin{align*}
\left( \text{KdV} \right) & \quad \left\{ \begin{array}{ll}
(\partial_t u_j^2 + \partial_x u_j^2 + \partial_x^2 u_j^2 + a_j u_j^2)(t, x) = -u_j \partial_x u_j, & \forall x \in (0, \ell_j), t > 0, j = 1, \ldots, N, \\
u_j^2(t, 0) = u_k^2(t, 0), & \forall j, k = 1, \ldots, N, t > 0,
\end{array} \right. \\
\sum_{j=1}^{N} \partial_x^2 u_j^2(t, 0) = -\alpha u_1^2(t, 0) - \frac{N}{3} (u_1(t, 0))^2, & \forall t > 0,
\end{align*}
\]

Then thanks to Theorems 3.4 and 3.5 we have the existence of $\gamma < 1$ such that for all $t \in [0, T]$,

\[
\|u^1(t, \cdot)\|_{L^2(\mathcal{T})} \leq \gamma \|u^0\|_{L^2(\mathcal{T})}.
\]
Thanks to Propositions 2.4, 2.5 and 2.6 we can deduce that
\[
\|u^2(t,\cdot)\|_{L^2(T)} \leq C(\|u\|_{L^4(0,T,L^2(T))} + \|u_1(t,0)\|_{L^2(0,T)}) \\
\leq C\|u\|_2^3.
\]
We need some estimates on this last right term.

We first multiply the equation of \(KdV_{damped}\) by \(\bar{u}_j\) and integrate in space and time over \((0,s)\) to obtain
\[
\|u(s,\cdot)\|_{L^2(T)}^2 + \int_0^s \sum_{j=1}^N |\partial_x u_j(t,0)|^2 dt + (2\alpha-N) \int_0^s |u(t,0)|^2 dt + 2 \sum_{j=1}^N \int_0^T \int_0^{l_j} a_j(x)|u_j|^2 dx dt = \|u_0\|_{L^2(T)}^2.
\]
Secondly, we multiply \(KdV_{damped}\) by \(x\bar{u}\) and integrate in space and time and obtain with the previous result,
\[
\|\partial_x u\|_{L^2(0,T,L^2(T))} \leq C(T,L,N,\alpha)\|u_0\|_{L^2(T)}^2 + \frac{2}{9} \int_0^T \int_T (u)^3 dx dt.
\]
As for all \(i = 1,\ldots,N\), \(u_i \in L^2(0,T,H^1(0,\ell_i))\) and \(H^1(0,\ell_i)\) embeds into \(C([0,\ell_i])\), we have as in [6] or [15],
\[
\sum_{i=1}^N \int_0^T \int_0^{\ell_i} |u_i|^3 dx dt \leq CT^{1/2}\|u_0\|_{L^2(T)}^2 \|u\|_{L^2(0,T;H^1(T))}.
\]
We obtain with (3.27),
\[
\|u\|_{L^2(0,T;H^1(T))} \leq C(T,L,N,\alpha) \left(\|u_0\|_{L^2(T)}^2 + \|u_0\|_{L^2(0,T)}^4\right).
\]
This gives with the previous inequalities, the estimate,
\[
\|u(s,\cdot)\|_{L^2(T)} \leq \|u_0\|_{L^2(0,T)}(\gamma + C\|u_0\|_{L^2(T)} + C\|u_0\|_{L^2(T)}^2).
\]
Thus by taking \(\epsilon > 0\) small enough such that \(\gamma + C\epsilon + C^3\epsilon < 1\) if \(u_0\) satisfies \(\|u_0\|_{L^2(T)} < \epsilon\) we have
\[
\|u(s,\cdot)\|_{L^2(T)} \leq (\gamma + C\epsilon + C^3\epsilon)\|u_0\|_{L^2(T)},
\]
and we get the stability result.

3.2.2. Semi-global stability result. In this section we prove a semi-global result, provided that the damping is applied on all branches.

Let \(a \in L^\infty(T)\) with,
\[
\begin{cases}
a_i(x) \geq a_0 > 0, \forall x \in \omega_i, \forall i = 1,\ldots,N, \\
\text{with } \omega_i \text{ a nonempty open subset of } (0,\ell_i).
\end{cases}
\]
Then our main result of this section is:
Theorem 3.6. Let $\ell_i = 1, \ldots, N \in (0, +\infty)^N$, let $a \in L^\infty(T)$ satisfying (3.29), and let $R > 0$. Then for all $u^0 \in L^2(T)$ with $\|u^0\|_{L^2(T)} \leq R$ there exist $C = C(R) > 0$ and $\mu = \mu(R) > 0$ such that the solution $u$ of (KdV$_{damped}$) satisfies,

$$\|u(t, \cdot)\|_{L^2(T)} \leq C e^{-\mu t} \|u^0\|_{L^2(T)}, \forall t \geq 0.$$  

Proof. To prove this result we follow the article of Pazoto [13]. Our result is based on this Unique Continuation Property of Saut and Sheurer [17].

Theorem 3.7. ([17] Theorem 4.2) Let $L > 0$ and let $y \in L^2(0, T, H^3(0, L))$ be a solution of

$$y_t + y_x + y_{xxx} + yy_x = 0,$$

such that $y(t, x) = 0$, $\forall t \in (t_1, t_2)$ and $x \in \omega$ where $\omega$ is a nonempty open subset of $(0, L)$. Then $y(t, x) = 0$, $\forall t \in (t_1, t_2)$ and $x \in (0, L)$.

By multiplying (KdV$_{damped}$) by $\bar{u}_j$ and integrating on time and space, we have,

$$\|u(s, \cdot)\|_{L^2(T)}^2 + \int_0^s \sum_{j=1}^N |\partial_x u_j(t, 0)|^2 dt + (2\alpha - N) \int_0^s |u_1(t, 0)|^2 dt + 2 \sum_{j=1}^N \int_0^T \int_0^\ell_j a_j(x) |u_j|^2 dx dt = \|u^0\|_{L^2(T)}^2.$$  

By integrating (3.30) over $(0, T)$ we have,

$$T \|u^0\|_{L^2(T)}^2 \leq \int_0^T \|u(s, \cdot)\|_{L^2(T)}^2 ds + T \sum_{j=1}^N |\partial_x u_j(t, 0)|^2 dt + (2\alpha - N) T \int_0^T |u_1(t, 0)|^2 dt + 2T \sum_{j=1}^N \int_0^T \int_0^\ell_j a_j(x) |u_j|^2 dx dt.$$  

Thus we just have to prove that there exists $C = C(T, R)$ such that

$$\int_0^T \|u(t, \cdot)\|_{L^2(T)}^2 dt \leq C \left( \int_0^T \sum_{j=1}^N |\partial_x u_j(t, 0)|^2 dt + (2\alpha - N) \int_0^T |u_1(t, 0)|^2 dt + 2 \sum_{j=1}^N \int_0^T \int_0^\ell_j a_j(x) |u_j|^2 dx dt \right).$$  

We assume that this inequality is false. Then we can find a sequence $(u^n) \in B$ solution of (KdV$_{damped}$) with $\|u^{0,n}\|_{L^2(T)} \leq R$ and such that

$$\lim_{n \to \infty} \frac{\|u^n\|_{L^2(0, T, L^2(T))}^2}{\|\partial_x u^n(\cdot, 0)\|_{L^2(0, T)}^2 + (2\alpha - N) \|u^n_1(\cdot, 0)\|_{L^2(0, T)}^2 + 2 \sum_{j=1}^N \int_0^T \int_0^\ell_j a_j(x) |u^n_j|^2 dx dt} = \infty.$$
Let us define $\lambda^n := \|u^n\|_{L^2(0,T;L^2(\Omega))}$ and $v^n := \frac{\lambda^n}{\sqrt{T}}$. Then $v^n$ satisfies the following problem,

$$
\begin{align*}
\partial_t v^n_i + \partial_x v^n_i + \partial_{xxx} v^n_i + a_i v^n_i + \lambda^n v^n_i \partial_x v^n_i &= 0, & i = 1, \ldots, N, \\
v^n_i(t, \xi_i) &= 0, & i = 1, \ldots, N, \\
\sum_{i=1}^{N} \partial_{xx} v^n_i(t,0) &= -\alpha v^n_i(t,0) - \frac{N}{2} (v^n_i(t,0))^2, & i = 1, \ldots, N, \\
\|v^n\|_{L^2(0,T;L^2(\Omega))} &= 1.
\end{align*}
$$

(3.32)

By multiplying the PDE in (3.32) by $\overline{v}^n_i$ and integrating on $(0, T) \times (0, \xi_i)$ we get

$$
T \|v^n(0, .)\|_{L^2(\Omega)}^2 \leq \int_0^T \|v^n(s, .)\|_{L^2(\Omega)}^2 ds + T \int_0^T \sum_{j=1}^{N} |\partial_x v^n_j(t,0)|^2 dt \\
+ (2\alpha - N)T \int_0^T |v^n(t,0)|^2 + 2T \sum_{j=1}^{N} \int_0^{\xi_j} a_j(x)|v^n_j|^2 dx dt.
$$

Thus $(v^n(0, .))$ is bounded in $L^2(\Omega)$.

By using (3.30), we see that $\lambda^n := \|u^n\|_{L^2(0,T;L^2(\Omega))} \leq \sqrt{T} \|u^{0,n}\|_{L^2(\Omega)} \leq \sqrt{T} R$. Thus $(\lambda^n)$ is bounded in $\mathbb{R}$.

Then we can get as for the previous inequality (3.28),

$$
(3.33) \quad \|u^n\|_{L^2(0,T;L^2(\Omega))} \leq C(T, L, N, \alpha, R) \left( \|u^{0,n}\|_{L^2(\Omega)} + \|v^{0,n}\|_{L^2(\Omega)} \right).
$$

Thus $(u^n)$ is bounded in $L^2(0, T : H^1(\Omega))$, and we can prove that for all $i = 1, \ldots, N$, $(v^n_i \partial_x v^n_i)$ is a sequence of $L^2(0, T, L^2(0, \xi_i))$ as

$$
\|v^n_i \partial_x v^n_i\|_{L^2(0,T,L^2(0,\xi_i))} \leq \|v^n\|_{C([0,T],L^2(\Omega))} \|u^n\|_{L^2(0,T,H^1(\Omega))}.
$$

Thus we can deduce that $(\partial_x u^n)$ is bounded in $L^2(0, T, H^1(\Omega))$ and then we can extract from $(u^n)$ a subsequence that converges strongly in $L^2(0, T, L^2(\Omega))$ to a limit $\varphi$ with $\|\varphi\|_{L^2(0,T;L^2(\Omega))} = 1$ and we have $v_i(t, x) = 0$, $\forall x \in \omega_i$, $v_i(t, 0) = 0$ and $\partial_x v_i(t, 0) = 0$, $\forall t \in (0, T)$, $\forall i = 1, \ldots, N$.

As $(\lambda^n)$ is bounded in $\mathbb{R}$ we can extract a sequence that converges in $\mathbb{R}$ to a limit $\lambda \geq 0$. Thus $\varphi$ satisfies the following system,

$$
\begin{align*}
\partial_t v_i + \partial_x v_i + \partial_{xxx} v_i + \lambda v_i \partial_x v_i &= 0, & \forall i = 1, \ldots, N, \\
v_i(t, \xi_i) &= 0, & \partial_x v_i(t, \xi_i) = 0, \\
v_i(t, 0) &= 0, & \partial_x v_i(t, 0) = 0, \\
\|\varphi(0, .)\|_{L^2(\Omega)} &\leq R, \\
\|\varphi\|_{L^2(0,T;L^2(\Omega))} &= 1.
\end{align*}
$$

(1) If $\lambda = 0$ then thanks to Holmgren’s Theorem, we deduce that $\varphi = 0$ which is absurd.
(2) If \( \lambda > 0 \) then we will apply the results of Saut and Sheurer [17] to get a contradiction. As \( v_i \) satisfies the same equation as in [13] we can deduce that \( v_i \in L^2(0, T, H^3(0, \ell_i)) \) for all \( i = 1, \ldots, N \). Thus by applying Theorem 3.7 we get the contradiction and then the stability result.

4. Controllability results.

We first consider the following exact boundary controllability problem for the linearized KdV equation:

For any \( T > 0, \alpha > \frac{N}{2} \) and \((\ell_i)_{i=1,\ldots,N} \in (0, +\infty)^N\), for every \( u^0, u^T \in L^2(T) \), does there exist \((N+1)\) controls \( g \in L^2(0, T) \) and \( g \in L^2(0, T) \) such that the solution \( u \in \mathbb{B} \) of the following system, \((LKdV_{control})\), satisfies \( u(0,.) = u^0 \) and \( u(T,.) = u^T \)?

\[
(LKdV_{control}) \begin{align*}
(\partial_t u_j + \partial_x u_j + \partial_x^3 u_j)(t, x) &= 0, \quad \forall x \in (0, \ell_j), t > 0, j = 1, \ldots, N, \\
 u_j(t, 0) &= u_k(t, 0), \quad \forall j, k = 1, \ldots, N, t > 0, \\
 N \sum_{j=1}^N \partial_x^2 u_j(t, 0) &= -\alpha u_1(t, 0) + g(t), \quad \forall t > 0, \\
 u_j(t, \ell_j) &= 0, \quad \forall t > 0, j = 1, \ldots, N, \\
 \partial_x u_j(t, \ell_j) &= g_j(t), \quad \forall t > 0, j = 1, \ldots, N, \\
 u_j(0, x) &= u_j^0(x), \quad \forall x \in (0, \ell_j), j = 1, \ldots, N.
\end{align*}
\]

By applying the Hilbert Uniqueness Method, [12], it is well known that the exact boundary controllability is equivalent to the inequality of observability for the following backward adjoint problem.

\[
\begin{align*}
(\partial_t \varphi_j + \partial_x \varphi_j + \partial_x^3 \varphi_j)(t, x) &= 0, \quad \forall x \in (0, \ell_j), t > 0, j = 1, \ldots, N, \\
 \varphi_j(t, 0) &= \varphi_k(t, 0), \quad \forall j, k = 1, \ldots, N, t > 0, \\
 \partial_x \varphi_j(t, 0) &= 0 \quad \forall t > 0, j = 1, \ldots, N, \\
 N \sum_{j=1}^N \partial_x^2 \varphi_j(t, 0) &= (\alpha - N)\varphi_1(t, 0), \quad \forall t > 0, \\
 \varphi_j(t, \ell_j) &= 0, \quad \forall t > 0, j = 1, \ldots, N, \\
 \varphi_j(T, x) &= \varphi_j^T(x), \quad \forall x \in (0, \ell_j), j = 1, \ldots, N.
\end{align*}
\]

By following the same steps as done for Theorem 3.1 we can prove this observability inequality,

**Theorem 4.1.** Let \((\ell_i)_{i=1,\ldots,N} \in (0, +\infty)^N\) such that \(#\{\ell_i \in \mathbb{N}\} \leq 1\). Then for all \( T > 0 \), there exists \( C > 0 \) such that for all \( \varphi^T \in \mathbb{L}^2(T) \) we have,

\[
\|\varphi^T\|_{\mathbb{L}^2(T)}^2 \leq C \left( \sum_{j=1}^N \|\partial_x \varphi_j(., \ell_j)\|_{L^2(0,T)}^2 + \int_0^T \varphi_j^2(t, 0) dt \right),
\]

where \( \varphi \in \mathbb{B} \) is the solution of the backward adjoint problem.
Thus we get the following exact boundary controllability result, provided that the network is non critical.

**Theorem 4.2.** Let $T > 0$ and $(\ell_i)_{i=1,\ldots,N} \in (0, +\infty)^N$ such that $\#\{\ell_i \in \mathcal{N}\} \leq 1$. Then for all $u^0, u^T \in L^2(T)$, there exists $g \in L^2(0,T)$ and $g \in L^2(0,T)$ such that the solution $u \in \mathcal{B}$ of $(L\text{KdV}_{\text{control}})$ satisfies $u(0,.) = u^0$ and $u(T,.) = u^T$.

By using a standard fixed point result we then prove the local exact controllability result for the non linear problem,

$$
\begin{align*}
(K\text{dV}_{\text{control}}) \quad \begin{cases}
(\partial_t u_j + \partial_x u_j + u_j \partial_x u_j + \partial_x^3 u_j)(t, x) = 0, & \forall x \in (0, \ell_j), t > 0, j = 1, \ldots, N, \\
u_j(t, 0) = u_k(t, 0), & \forall j, k = 1, \ldots, N, t > 0, \\
\sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0) + g(t), & \forall t > 0, \\
u_j(t, \ell_j) = 0, & \forall t > 0, j = 1, \ldots, N, \\
\partial_x u_j(t, \ell_j) = g_j(t), & \forall t > 0, j = 1, \ldots, N, \\
u_j(0, x) = u_j^0(x), & \forall x \in (0, \ell_j), j = 1, \ldots, N,
\end{cases}
\end{align*}
$$

**Theorem 4.3.** Let $T > 0$ and $(\ell_i)_{i=1,\ldots,N} \in (0, +\infty)^N$ such that $\#\{\ell_i \in \mathcal{N}\} \leq 1$. Then there exists $r > 0$ such that for all $u^0, u^T \in L^2(T)$ with $\|u^0\|_{L^2(T)} < r$ and $\|u^T\|_{L^2(T)} < r$, there exists $g \in L^2(0,T) := \prod_{j=1}^N L^2(0,T)$ and $g \in L^2(0,T)$ such that the solution $u \in \mathcal{B}$ of $(K\text{dV}_{\text{control}})$ satisfies $u(0,.) = u^0$ and $u(T,.) = u^T$.

**Remark 4.4.** If $\#\{\ell_i \in \mathcal{N}\} \leq 1$, there exists a finite dimensional space of $L^2(T)$ which is unreachable for the linearized system $(L\text{KdV}_{\text{control}})$. We could certainly prove the controllability of the non linear problem by using some power series expansion for the critical branches, following the same type of proof as [8], [3] or [7].

**Remark 4.5.** In this last section, we prove the controllability by using $(N+1)$ controls, acting at the external nodes and at the central node. It could be interesting to reduce the number of controls.

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