Comparison of the hybrid and multipoint-flux method

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We compare two approaches for the efficient implementation of the mixed finite element method with applications for elliptic problems, like single phase flow, and hyperbolic problems, like the wave equations. We briefly explain the hybrid and the multipoint flux mixed finite element method (MFME) and conduct a comparison of the two methods in terms of efficiency.

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1 Porous media flow

Let us start by considering the numerical approximation of the following elliptic problem in mixed form

\begin{equation}
K^{-1} u + \nabla p = 0 \quad \text{in } \Omega,
\end{equation}
\begin{equation}
\text{div } u = f \quad \text{in } \Omega,
\end{equation}

where \( u \) and \( p \) denote the velocity and pressure. As boundary condition, we assume \( p = 0 \) on \( \partial \Omega \). Mixed finite element approximations are very attractive because of the many physical properties they fulfill, like local mass conservation and the accurate implementation of discontinuous coefficients. The discrete system reads: find \( u_h \in V_h \) and \( p_h \in Q_h \) such that

\begin{equation}
(K^{-1} u_h, v_h) - (p_h, \text{div } v_h) = 0 \quad \forall v_h \in V_h \subseteq H(\text{div}, \Omega),
\end{equation}
\begin{equation}
(\text{div } u_h, q_h) = (f, q_h) \quad \forall q_h \in Q_h \subseteq L^2(\Omega).
\end{equation}

for an appropriate inf-sup stable pair \( V_h, Q_h \). The downside of this approach is its natural saddle-point structure, which can be circumvented in different ways. In the following, we give a brief summary of two methods and present a short comparison.

The multipoint flux mixed finite element (MFME), introduced by Wheeler and Yotov in [3], uses numerical integration to reduce the mass matrix to a block-diagonal structure, which then enables the efficient elimination of the velocity component \( u_h \). The resulting system can be interpreted as a finite volume or finite difference method for the pressure. In [3], the authors used the BDM\textsubscript{1–P}0 pair together with the vertex rule to obtain first order elements. In [2], we managed to extend the theory and introduce corresponding second order elements. On simplices, the second order space coincides with the well-known pair RT\textsubscript{1–P}1. The corresponding quadrature points are comprised of the vertices and the midpoint, see Figure 1.

Another approach which seeks to alleviate the saddle-point structure is called hybridization [4], where the continuity is transferred to Lagrange multipliers. The hybridized system reads: find \( u_h \in V_h, p_h \in Q_h \) and \( \lambda_h \in M_h \) such that

\begin{equation}
(K^{-1} u_h, v_h)_{T_h} - (p_h, \text{div } v_h)_{T_h} + \sum_{T \in \mathcal{T}_h} (\lambda_h, n \cdot v_h)_{\partial T} = 0 \quad \forall v_h \in V_h,
\end{equation}
\begin{equation}
(\text{div } u_h, q_h)_{T_h} = (f, q_h)_{T_h} \quad \forall q_h \in Q_h.
\end{equation}

In order to ensure continuity across elements, we have to additionally impose that \( \sum_{T \in \mathcal{T}_h} (n \cdot u_h, \mu_h)_{\partial T} = 0 \) for all \( \mu_h \in M_h \).

The variable \( \lambda_h \) can be interpreted as the pressure values at the interfaces. The reduced system solves for an appropriate inf-sup stable pair \( V_h, Q_h \) and \( \lambda_h \) only, where the pressure and velocity variables can be recovered via local computations. Note that hybridization is an equivalent reformulation of the original discrete mixed system (1)-(2), and is therefore equivalent to it.

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2 Numerical comparison

We now investigate the computational efficiency of the two approaches. We compare both first and second order elements on triangular meshes. In order to conduct a fair comparison, we choose methods that yield the same convergence results. We compare the lumped BDM$_1$–P$_0$ method from [3] with the hybridized triple RT$^d_0$–P$_0$–P$_0$. The two methods both yield first order convergence for the velocity and second order convergence for the projected pressure. For second order elements, we compare the lumped RT$^d_1$–P$_1$ method from [2] with the hybridized triple RT$^d_1$–P$_1$–P$_1$. For these methods, we obtain second order convergence for the velocity and third order convergence for the projected pressure. We investigate the size of the reduced systems, their stencil (Figure 3 and Figure 4) and the performance in terms of computational time.

| Method      | Order | #Variables | Stencil size |
|-------------|-------|------------|--------------|
| RT$^d_0$–P$_0$–P$_0$ | 1     | $n_e$      | 5            |
| BDM$_1$–P$_0$ | 1     | $n_t \approx \frac{2}{3} n_e$ | 13 – 14      |
| RT$^d_1$–P$_1$–P$_1$ | 2     | $2 n_e$    | 10           |
| RT$^d_1$–P$_1$ | 2     | $3 n_t \approx 2 n_e$ | 39 – 42      |

Table 1: Comparison of methods in terms of number of variables and their stencil. $n_e$ and $n_t$ denote the number of edges and triangles, respectively.

Table 2: Average CPU time in seconds for solving the reduced systems, depending on the mesh size. Tests conducted on a Ryzen 2700x.

| h   | RT$^d_0$–P$_0$–P$_0$ | BDM$_1$–P$_0$ | RT$^d_1$–P$_1$–P$_1$ | RT$^d_1$–P$_1$ |
|-----|----------------------|--------------|----------------------|----------------|
| $2^{-4}$ | 0.002803s            | 0.004879s    | 0.010094s            | 0.030939s      |
| $2^{-5}$ | 0.015824s            | 0.025316s    | 0.048216s            | 0.172085s      |
| $2^{-6}$ | 0.072978s            | 0.117407s    | 0.231498s            | 0.977868s      |

We note that for the first order methods, the lumped system has about $1/3$ less variables than the hybridized version, but a bigger stencil, about 2.8 times the size. In terms of computational time, we see that the lumped method is about 60% slower than its hybrid counterpart. For second order methods, the methods are almost equivalent in terms of number of variables, but the stencil for the lumped method is about 4 times larger and the computation time is slower by a factor of about 4, see Table 1 and Table 2. For 3-dimensional discretizations, the lumped method loses by an even larger factor, since the stencil covering all neighboring elements is very large. In conclusion, the hybridized method is faster, but if we are interested in a volumetric representation for the pressure, additional local post-processing has to be conducted to obtain $p_h$ from $\lambda_h$.

3 Mass lumping for wave propagation

Another interesting field of application for mass lumping is the modeling of acoustic wave equations, e.g.,

\begin{align}
\partial_t u + \nabla p &= 0 \quad \text{in } \Omega \times (0, T), \quad (3) \\
\partial_t p + \text{div } u &= 0 \quad \text{in } \Omega \times (0, T). \quad (4)
\end{align}

We consider the mixed finite element approximation in $H(\text{div})$–$L^2$. For this problem, the hybrid method is not advantageous, since it requires the solution of linear systems at each time step. The mass lumped variant does not require these additional computations. For the first order method, we conducted an analysis for the acoustic wave propagation in [1]. In future work, we will discuss second order elements, as well as further applications to Maxwell’s equations.

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