AMBENT OBSTRUCTION FLOW

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Abstract. We establish fundamental results for a parabolic flow of Riemannian metrics introduced by Bahuaud-Helliwell in \cite{3} which is based on the Fefferman-Graham ambient obstruction tensor. First, we obtain local $L^2$ smoothing estimates for the curvature tensor and use them to prove pointwise smoothing estimates for the curvature tensor. We use the pointwise smoothing estimates to show that the curvature must blow up for a finite time singular solution. We also use the pointwise smoothing estimates to prove a compactness theorem for a sequence of solutions with bounded $C^0$ curvature norm and injectivity radius bounded from below at one point. Finally, we use the compactness theorem to obtain a singularity model from a finite time singular solution and to characterize the behavior at infinity of a nonsingular solution.

1. Introduction

1.1. Introduction. The uniformization theorem ensures that for a compact two dimensional Riemannian manifold $(M, g)$, there is a metric $\tilde{g}$ conformal to $g$ for which $(M, \tilde{g})$ has constant sectional curvature equal to $K$. Moreover, the sign of $K$ can be determined via the Gauss-Bonnet theorem. In higher dimensions, curvature functionals have been used with great success to define and locate optimal metrics in higher dimensions; see \cite{28}. One conformally invariant curvature functional for a 4-dimensional Riemannian manifold $(M, g)$ is given by

$$F^4_W(g) = \int_M |W_g|^2 dV_g,$$

where $W_{ijkl}$ is the Weyl tensor. The negative gradient of $F^4_W$ is the Bach tensor $B_{ij}$ defined as

$$B_{ij} = -\nabla^k \nabla^l W_{kijl} - \frac{1}{2} R^{kl} W_{kijl}.$$

The study of critical metrics for $F^4_W$, ie. Bach-flat metrics, has been fruitful. The class of Bach-flat metrics contains, as shown in \cite{5}, familiar metrics such as locally conformally Einstein metrics, scalar flat (anti) self-dual metrics.

Another conformally invariant functional for a 4-dimensional Riemannian manifold $(M, g)$ is given by

$$F^4_Q(g) = \int_M Q(g) dV_g,$$

where $Q(g)$ is a scalar quantity introduced by Branson in \cite{7} called the $Q$ curvature. Via the Chern-Gauss-Bonnet theorem, this functional is related to $F^4_W$ by $F^4_Q = 8\pi^2 \chi(M) - \frac{1}{4} F^4_W$. The Bach tensor is also the gradient of $F^4_Q$. Unlike the Weyl tensor, the $Q$ curvature is not pointwise conformally covariant.

One can generalize the $Q$ curvature to a scalar quantity defined on $n$ dimensional Riemannian manifolds $(M, g)$, where $n$ is even. Consider the functionals defined for $n$ even by

$$F^n_Q(g) = \int_M Q(g) dV_g.$$
These functionals are conformally invariant. The gradient of $F^n_Q$ is a symmetric 2-tensor $O$, introduced by Fefferman and Graham in [15], called the ambient obstruction tensor. This tensor arises in physics: for example, Anderson and Chruściel use $O$ in [1] to construct global solutions of the vacuum Einstein equation in even dimensions. In dimension 4, $O$ is just the Bach tensor. The ambient obstruction tensor is conformally covariant in $n$ dimensions. This is in contrast to the $n$ dimensional generalization of the Bach tensor, which is only conformally covariant in dimension 4. This fact follows from a result in Graham-Hirachi [18] stating that in even dimensions 6 and greater, the only conformally covariant tensors essentially are $W$ and $O$. Extending the 4-dimensional case, Fefferman and Graham showed in [16] that $O$ vanishes for Einstein metrics for all even dimensions. However, there also exist non conformally Einstein metrics for which $O = 0$, as shown by Gover and Leitner in [17]. The conformal covariance of $O$ and the fact that obstruction flat metrics generalize conformally Einstein metrics suggest that studying the critical points of $F^n_Q$ via its gradient flow may aid in the study of optimal metrics on $M$. Our main goal is to establish fundamental results for this gradient flow.

1.2. Main Results. We will continue the study of a variant of the gradient flow of $F^n_Q$, that was introduced by Bahuaud and Helliwell in [3], establishing fundamental results. This flow, which we will refer to as the ambient obstruction flow (AOF), is defined for a family of metrics $g(t)$ on a smooth manifold $M$ by

$$\begin{cases}
\partial_t g = (-1)^{\frac{n}{2}} O + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^\frac{n}{2} - 1)R)g \\
g(0) = h.
\end{cases}$$

The conformal term involving the scalar curvature was added in order to counteract the invariance of $O$ under the action of the conformal group on the space of metrics on $M$. In the papers [3], [4] they proved the short time existence and uniqueness, respectively, of solutions to AOF given by (1). Kotschwar recently has given in [26] an alternate uniqueness proof via a classical energy argument without using the DeTurck trick.

Gradient flows have been studied extensively since Hamilton in [20, 21, 22] and Perelman in [32, 33, 34] (expositions are given in [9, 24, 31]) used the Ricci flow to study the geometry of 3-manifolds. In the past fifteen years, these have begun to include higher order flows. Mantegazza studied a family of higher order mean curvature flows in [30], Kuwert-Schätzle studied the gradient flow of the Willmore functional in [27], Streets studied the gradient flow of $\int_M |Rm|^2$ in [37], Chen-He studied the Calabi flow in [11, 12], and Kisisel-Sarioglu studied the Cotton flow in [25]. Bour studied the gradient flows of certain quadratic curvature functionals in [6], including some variants of $\int_M |W|^2$.

Our first result gives pointwise smoothing estimates for the $C^0$ norms of the derivatives of the curvature. Since the AOF PDE (1) is of order $n$, the maximum principle cannot be used to obtain these estimates. Instead, we first use interpolation inequalities derived by Kuwert and Schätzle in [27] in order to derive local integral Bernstein-Bando-Shi-type smoothing estimates. Then, we use a blowup argument adapted from Streets [38] in order to convert the integral smoothing estimates to pointwise smoothing estimates, as stated in the following theorem. During the proof, we use the local integral smoothing estimates to take a local subsequential limit of the renormalized metrics.

**Theorem 1.1.** Let $m \geq 0$ and $n \geq 4$. There exists a constant $C = C(m, n)$ so that if $(M^n, g(t))$ is a complete solution to AOF on $[0, T]$ satisfying

$$\max \left( 1, \sup_{M \times [0,T]} |Rm| \right) \leq K,$$
then for all \( t \in (0, T] \),
\[
\sup_M |\nabla^{\alpha} Rm|_{g(t)} \leq C \left( K + \frac{1}{t^2} \right)^{1+\frac{\mu}{2}}.
\]

We obtain from the pointwise smoothing estimates two additional theorems. The first theorem gives an obstruction to the long time existence of the flow. Since the pointwise smoothing estimates do not require that the Sobolev constant be bounded on \([0, T)\), we rule out that the manifold collapses with bounded curvature.

**Theorem 1.2.** Let \( g(t) \) be a solution to the AOF on a compact manifold \( M \) that exists on a maximal time interval \([0, T)\) with \( 0 < T \leq \infty \). If \( T < \infty \), then we must have
\[
\limsup_{t \uparrow T} \|Rm\|_{C^0(g(t))} = \infty.
\]

The second theorem allows us to extract convergent subsequences from a sequence of solutions to AOF with uniform \( C^0 \) curvature bound and uniform injectivity radius lower bound. We prove this in section 7 by using the Cheeger-Gromov compactness theorem to obtain subsequential convergence at one time. Then, after extending estimates on the covariant derivatives of the metrics from one time to the entire time interval, we obtain subsequential convergence over the entire time interval.

**Theorem 1.3.** Let \( \{(M^*_k,g_k(t),O_k)\}_{k \in \mathbb{N}} \) be a sequence of complete pointed solutions to AOF for \( t \in (\alpha, \omega) \), with \( t_0 \in (\alpha, \omega) \), such that
1. \( |Rm(g_k)|_{g_k} \leq C_0 \) on \( M_k \times (\alpha, \omega) \) for some constant \( C_0 < \infty \) independent of \( k \)
2. \( \text{inj}(g_k(t_0))(O_k) \geq \iota_0 \) for some constant \( \iota_0 > 0 \).

Then there exists a subsequence \( \{j_k\}_{k \in \mathbb{N}} \) such that \( \{(M_{j_k},g_{j_k}(t),O_{j_k})\}_{k \in \mathbb{N}} \) converges in the sense of families of pointed Riemannian manifolds to a complete pointed solution to AOF \( (M^*_\infty,g_\infty(t),O_\infty) \) defined for \( t \in (\alpha, \omega) \) as \( k \to \infty \).

We use this compactness theorem to prove two corollaries. For a compact Riemannian manifold \((M,g)\), let \( C_S(M,g) \) denote the \( L^2 \) Sobolev constant of \((M,g)\), defined as the smallest constant \( C_S \) such that
\[
\|f\|_{L^2}^2 \leq C_S \left( \|\nabla f\|_{L^2}^2 + V^{-\frac{2}{n}} \|f\|_{L^2}^2 \right),
\]
where \( V = \text{vol}(M,g) \). The following result states that if the Sobolev constant and the integral of \( Q \)-curvature are bounded along the flow, there exists a sequence of renormalized solutions to AOF that converge to a singularity model.

**Theorem 1.4.** Let \((M^n,g(t))\), \( n \geq 4 \), be a compact solution to AOF on a maximal time interval \([0, T)\). Suppose that \( \sup\{C_S(M,g(t)) : t \in [0, T)\} < \infty \). Let \( \{(x_i,t_i)\}_{i \in \mathbb{N}} \subset M \times [0, T) \) be a sequence of points satisfying \( t_i \to T \), \( |Rm(x_i,t_i)| = \sup\{|Rm(x,t)| : (x,t) \in M \times [0,t_i]\} \), and \( \lambda_i \to \infty \), where \( \lambda_i = |Rm(x_i,t_i)| \). Then the sequence of pointed solutions to AOF given by \( \{(M,g(t),x_i)\}_{i \in \mathbb{N}} \), with
\[
g_i(t) = \lambda_i g(t_i + \lambda_i^{-\frac{n}{2}} t), \quad t \in [-\lambda_i^\frac{n}{2} t_i, 0]
\]
subsequentially converges in the sense of families of pointed Riemannian manifolds to a nonflat, noncompact complete pointed solution \((M_\infty,g_\infty(t),x_\infty)\) to AOF defined for \( t \in (\infty, 0] \). Moreover, if \( n = 4 \) or
\[
\sup_{t \in [0, T]} \int_M Q(g(t)) \, dV_{g(t)} < \infty,
\]
then \( \mathcal{O}(g_\infty(t)) \equiv 0 \) for all \( t \in (-\infty, 0] \).
The next result states that if a nonsingular solution to AOF does not collapse at time $\infty$ and the integral of $Q$-curvature is bounded along the flow, there exists a sequence of times $t_i \to \infty$ for which $g(t_i)$ converges to an obstruction flat metric. We note that in cases (2) and (3), the boundedness of the integral of the $Q$ curvature along the flow implies that $g_\infty(t)$ is obstruction flat. However, this does not imply that $\partial_t g_\infty = 0$. Rather, $\partial_t g_\infty = (-1)^{n/2}C(n)(\Delta^{\frac{2}{n^2}}-1)g$, i.e. the metric is still flowing by the conformal term of AOF within the conformal class of $g_\infty(0)$.

**Theorem 1.5.** Let $(M, g(t))$ be a compact solution to AOF on $[0, \infty)$ such that

$$\sup_{t \in [0, \infty)} \|\text{Rm}\|_{C^0(g(t))} < \infty.$$  

Then exactly one of the following is true:

1. $M$ collapses when $t = \infty$, i.e.

$$\lim_{t \to \infty} \inf_{x \in M} \text{inj}_g(t)(x) = 0.$$

2. There exists a sequence $\{(x_i, t_i)\}_{i \in \mathbb{N}} \subset M \times [0, \infty)$ such that the sequence of pointed solutions to AOF given by $\{(M, g_i(t), x_i)\}_{i \in \mathbb{N}}$, with

$$g_i(t) = g(t_i + t), \quad t \in [-t_i, \infty)$$

subsequentially converges in the sense of pointed Riemannian manifolds to a complete non-compact finite volume pointed solution $(M_\infty, g_\infty(t), x_\infty)$ to AOF defined for $t \in (-\infty, \infty)$. If $n = 4$ or

$$\sup_{t \in [0, \infty)} \int_M Q_g(t) dV_{g(t)} < \infty,$$

then $g_\infty(t)$ is obstruction flat for all $t \in (-\infty, \infty)$.

3. There exists a sequence $\{(x_i, t_i)\}_{i \in \mathbb{N}} \subset M \times [0, \infty)$ such that the sequence of pointed solutions to AOF given by $\{(M, g_i(t), x_i)\}_{i \in \mathbb{N}}$, with

$$g_i(t) = g(t_i + t), \quad t \in [-t_i, \infty)$$

subsequentially converges in the sense of pointed Riemannian manifolds to a compact pointed solution $(M_\infty, g_\infty(t), x_\infty)$ to AOF defined for $t \in (-\infty, \infty)$, where $M_\infty$ is diffeomorphic to $M$. If $n = 4$ or

$$\sup_{t \in [0, \infty)} \int_M Q_g(t) dV_{g(t)} < \infty,$$

then $g_\infty(t)$ is obstruction flat for all $t \in (-\infty, \infty)$ and there exists a family of metrics $\hat{g}_\infty(t)$ conformal to $g_\infty(t)$ for all $t \in (-\infty, \infty)$, with $\hat{g}_\infty(t) = \hat{g}_\infty(0)$ for all $t \in (-\infty, \infty)$, such that $\hat{g}_\infty(0)$ is obstruction flat and has constant scalar curvature.

2. **Background**

2.1. **$Q$ Curvature.** Here we recall a description of $Q$ curvature given by Chang et al. in [10]. The $Q$ curvature was introduced in 4 dimensions by Riegert in [36] and Branson-Orsted in [8] and in even dimensions by Branson in [7]. It is a scalar quantity defined on an even dimensional Riemannian manifold $(M^n, g)$. If $n = 2$, we define $Q$ to be $Q = -\frac{1}{2}R = -K$, where $K$ is the Gaussian curvature of $M$. The Gauss-Bonnet theorem gives $\int Q dV = -2\pi \chi(M)$. The $Q$ curvature of a metric $\hat{g} = e^{2f}g$ is given by $e^{2f}Q = Q + \mathcal{P}f$, where the Paneitz operator $\mathcal{P}$ introduced by Graham-Jenne-Mason-Sparling in [19] is given by $\mathcal{P}f = \Delta f$. If $n = 4$, we define $Q$ to be

$$Q = -\frac{1}{6}\Delta R - \frac{1}{2}R^{ab}R_{ab} + \frac{1}{6}R^2.$$
The Chern-Gauss-Bonnet theorem gives
\[ \int Q \, dV = 8\pi^2 \chi(M) - \frac{1}{4} \int |W|^2. \]
In particular, if \( M \) is conformally flat, then \( \int Q \, dV = 8\pi^2 \chi(M) \). The \( Q \) curvature of a metric \( \tilde{g} = e^{2f}g \) is given by \( e^{Af} \tilde{Q} = Q + Pf \), where the Paneitz operator \( P \) is given by
\[ Pf = \nabla_a [\nabla^a \nabla^b + 2R^{ab} - \frac{2}{n} Rg^{ab} \nabla_b f]. \]
In general when \( n \) is even, we are only able to write down the highest order terms of \( Q \) and \( P \):
\[ Q = -\frac{1}{n-2} \Delta^{\frac{n}{2} - 1} R + \text{lots}, \quad Pf = \Delta^\frac{n}{2} f + \text{lots}. \]
Nonetheless, \( Q \) still has nice conformal properties. Under a conformal change of metric \( \tilde{g} = e^{2f}g \), we have \( e^{nf} \tilde{Q} = Q + Pf \). The integral of \( Q \) is conformally invariant. In particular, if \( M \) is locally conformally flat, we have an analogue of the Gauss-Bonnet theorem:
\[ \int Q \, dV = (-1)^{\frac{n}{2}} \left( \frac{n}{2} - 1 \right)! 2^{n-1} \pi^{\frac{n}{2}} \chi(M). \]

2.2. Ambient Obstruction Tensor. Fefferman and Graham proposed in [15] a method to determine the conformal invariants of a manifold from the pseudo-Riemannian invariants of an ambient space it is embedded into. They introduced the ambient obstruction tensor \( O \) as an obstruction to such an embedding. They subsequently provided a detailed description of the properties of \( O \) in their monograph [16].

We define several tensors that we will use to express \( O \). The Schouten tensor \( A \), Cotton tensor \( C \), and Bach tensor \( B \) are defined as
\[ A_{ij} = \frac{1}{n-2} (R_{ij} - \frac{1}{2(n-1)} R g_{ij}), \quad C_{ijk} = \nabla_k A_{ij} - \nabla_j A_{ik}, \quad B_{ij} = \nabla^k C_{ijk} - A^{kl} W_{kij}. \]
We obtain via the identity \( \nabla^i \nabla^k W_{kij} = (3-n) \nabla^k C_{ijk} \) that
\[ B_{ij} = \frac{1}{3-n} \nabla^i \nabla^k W_{kij} + \frac{1}{2-n} R_{kij}. \]
We define the notation \( P^n_k(A) \) for a tensor \( A \) by
\[ P^n_k(A) = \sum_{i_1 + \cdots + i_k = m} \nabla^{i_1} A \ast \cdots \ast \nabla^{i_k} A. \]
The following result describes \( O \). The form of the lower order terms is implied by the proofs.

**Theorem 2.1.** (Fefferman-Graham [16], Theorem 3.8; Graham-Hirachi [15], Theorem 2.1) Let \( n \geq 4 \) be even. The obstruction tensor \( O_{ij} \) of \( g \) is independent of the choice of ambient metric \( \tilde{g} \) and has the following properties:

1. \( O \) is a natural tensor invariant of the metric \( g \); i.e., in local coordinates the components of \( O \) are given by universal polynomials in the components of \( g \), \( g^{-1} \), and the curvature tensor of \( g \) and its covariant derivatives, and can be written just in terms of the Ricci curvature and its covariant derivatives. The expression for \( O_{ij} \) takes the form
   \[ O_{ij} = \Delta^{n/2-2} (\Delta A_{ij} - \nabla_j \nabla_i A_k^k) + \sum_{j=2}^{n/2} P^{n-2j}_j(Rm) \]
   \[ = \frac{1}{3-n} \Delta^{n/2-2} \nabla^i \nabla^k W_{kij} + \sum_{j=2}^{n/2} P^{n-2j}_j(Rm), \]
   where \( \Delta = \nabla^i \nabla_i \) and lots denotes quadratic and higher terms in curvature involving fewer derivatives.
(2) One has $O^i_i = 0$ and $\nabla^j O_{ij} = 0$.
(3) $O_{ij}$ is conformally invariant of weight $2 - n$; ie. if $0 < \Omega \in C^\infty(M)$ and $\hat{g}_{ij} = \Omega^2 g_{ij}$, then $\hat{O}_{ij} = \Omega^{-n} O_{ij}$.
(4) If $g_{ij}$ is conformal to an Einstein metric then $O_{ij} = 0$.

C.R. Graham and K. Hirachi express the gradient of $Q$ in terms of $O$:

**Theorem 2.2.** ([18], Theorem 1.1) If $g(t)$ is a 1-parameter family of metrics on a compact manifold $M$ of even dimension $n \geq 4$ and $h = \partial_t|_{t=0} g(t)$, then

$$\frac{\partial}{\partial t}|_{t=0} \int_M Q(g(t)) \, dV_{g(t)} = (-1)^{\frac{n}{2}} \frac{n-2}{2} \int_M \langle O(g(0)), h \rangle \, dV_{g(0)}.$$

Define the adjusted ambient obstruction tensor $\hat{O}$ to be

$$\hat{O} = (-1)^{\frac{n}{2}} O + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)} (\Delta^\frac{n}{2} - 1 R) g.$$

We rewrite $\hat{O}$ in terms of the Ricci and scalar curvatures.

**Proposition 2.3.** If $(M, g)$ is a Riemannian manifold, then

$$O = \Delta^\frac{n}{2} - 1 A - \frac{1}{2(n-1)} \Delta^\frac{n}{2} - 2 \nabla^2 R + \sum_{j=2}^{n/2} P_{n-2j}^j (Rm)$$

$$\hat{O} = \frac{(-1)^{\frac{n}{2}}}{n-2} \Delta^\frac{n}{2} - 1 Rc + \frac{(-1)^{\frac{n}{2}}}{2(n-1)} \Delta^\frac{n}{2} - 2 \nabla^2 R + \sum_{j=2}^{n/2} P_{n-2j}^j (Rm).$$

**Proof.** First, we reexpress $O$:

$$A^k_k = \frac{1}{n-2} \left[ g^{jk} R_{kj} - \frac{1}{2(n-1)} R g^{jk} g_{kj} \right]$$

$$= \frac{1}{n-2} \left[ R - \frac{n}{2(n-1)} R \right]$$

$$= \frac{1}{2(n-1)} R$$

and

$$O_{ij} = \Delta^\frac{n}{2} - 2 (\Delta A_{ij} - \nabla_j \nabla_i A^k_k) + \sum_{j=2}^{n/2} P_{n-2j}^j (Rm)$$

$$= \Delta^\frac{n}{2} - 1 A_{ij} - \frac{1}{2(n-1)} \Delta^\frac{n}{2} - 2 \nabla_j \nabla_i R + \sum_{j=2}^{n/2} P_{n-2j}^j (Rm).$$
Next, we reexpress $\hat{\Omega}$ using (3):

\[ \hat{\Omega} = (-1)^{\frac{n}{2}} \hat{\Omega} + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}}-1)Rg \]

\[ = (-1)^{\frac{n}{2}} \Delta^{\frac{n}{2}}-1A + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}}-1) - \frac{\Delta^{\frac{n}{2}}-1}{2(n-1)}g \]

\[ = (-1)^{\frac{n}{2}} \Delta^{\frac{n}{2}}-1A + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}}-1)Rg + \frac{(-1)^{\frac{n}{2}}}{2(n-1)}(\Delta^{\frac{n}{2}}-1) - \frac{\Delta^{\frac{n}{2}}-1}{2(n-1)}g \]

\[ = \frac{(-1)^{\frac{n}{2}}}{n-2} \Delta^{\frac{n}{2}}-1A + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}}-1) - \frac{\Delta^{\frac{n}{2}}-1}{2(n-1)}g + \frac{(-1)^{\frac{n}{2}}}{2(n-1)} \sum_{j=2}^{n/2} P_j^{n-2j}(Rm) \]

\[ = \frac{(-1)^{\frac{n}{2}}}{n-2} \Delta^{\frac{n}{2}}-1A + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}}-1)Rg + \frac{(-1)^{\frac{n}{2}}}{2(n-1)} \sum_{j=2}^{n/2} P_j^{n-2j}(Rm). \]

\[ \square \]

3. Short Time Existence and Uniqueness

In this section, we derive the evolution equations for the covariant derivatives of the curvature tensor. We then give a theorem asserting the short time existence and uniqueness of solutions to AOF.

3.1. Preliminaries. We collect some facts about Riemannian manifolds that will be used to derive the evolution equations.

Lemma 3.1. (Hamilton [20], Lemma 7.2) On any Riemannian manifold, the following identity holds:

\[ \Delta R_{jklm} = \nabla_j \nabla_m R_{lk} - \nabla_j \nabla_l R_{mk} + \nabla_k \nabla_l R_{mj} - \nabla_k \nabla_m R_{lj} + R_{ml} \nabla^2 \nabla_{lm}. \]

Proposition 3.2. If $A$ is a tensor on a Riemannian manifold and $k, l \geq 1$, then

\[ \nabla^k \Delta^l A = \Delta^l \nabla^k A + \sum_{i=0}^{2l+k-2} \nabla^{2l+k-2-i}Rm \ast \nabla^i A. \]

Proof. First we claim that $\nabla \Delta^l A = \Delta^l \nabla A + \sum_{i=0}^{l} \nabla^{2l+1-i}Rm \ast \nabla^i A$. For any tensor $A$,

\[ \nabla A = \nabla_i \nabla_j A \]

\[ = \nabla_j \nabla_i A + Rm \ast \nabla A \]

\[ = \nabla_j \nabla_i A + \nabla Rm \ast A + Rm \ast \nabla A \]

\[ = \Delta A + \nabla Rm \ast A + Rm \ast \nabla A. \]
Suppose the claim is true for $l - 1$. Then
\[
\nabla^2 \sum_{i=0}^{2l-3} \nabla^{2l-3-i} \text{Rm} \ast \nabla^i A = \nabla \sum_{i=0}^{2l-3} \left( \nabla^{2l-2-i} \text{Rm} \ast \nabla^i A + \nabla^{2l-3-i} \text{Rm} \ast \nabla^{i+1} A \right)
\]
\[
= \nabla \left( \sum_{i=0}^{2l-3} \nabla^{2l-2-i} \text{Rm} \ast \nabla^i A + \text{Rm} \ast \nabla^{2l-2} A \right)
\]
\[
= \sum_{i=0}^{2l-3} \left( \nabla^{2l-1-i} \text{Rm} \ast \nabla^i A + \nabla^{2l-2-i} \text{Rm} \ast \nabla^{i+1} A \right)
\]
\[
+ \nabla \text{Rm} \ast \nabla^{2l-2} A + \text{Rm} \ast \nabla^{2l-1} A
\]
\[
= \sum_{i=0}^{2l-3} \nabla^{2l-1-i} \text{Rm} \ast \nabla^i A + \nabla \text{Rm} \ast \nabla^{2l-2} A + \nabla \text{Rm} \ast \nabla^{2l-2} A + \text{Rm} \ast \nabla^{2l-1} A
\]
\[
= \sum_{i=0}^{2l-3} \nabla^{2l-1-i} \ast \nabla^i A.
\]

Next,
\[
\nabla \Delta^l A = \nabla \Delta \Delta^{l-1} A
\]
\[
= \Delta \nabla \Delta^{l-1} A + \nabla \text{Rm} \ast \nabla^{2l-2} A + \text{Rm} \ast \nabla^{2l-1} A
\]
\[
= \Delta \left( \Delta^{l-1} \nabla A + \sum_{i=0}^{2l-1} \nabla^{2l-1-i} \text{Rm} \ast \nabla^i A \right) + \nabla \text{Rm} \ast \nabla^{2l-2} A + \text{Rm} \ast \nabla^{2l-1} A
\]
\[
= \Delta^l \nabla A + \sum_{i=0}^{2l-1} \nabla^{2l-1-i} \ast \nabla^i A + \nabla \text{Rm} \ast \nabla^{2l-2} A + \text{Rm} \ast \nabla^{2l-1} A
\]
\[
= \Delta^l \nabla A + \sum_{i=0}^{2l-1} \nabla^{2l-1-i} \ast \nabla^i A.
\]

Assume the proposition holds for $k - 1$. Then
\[
\nabla \sum_{i=0}^{2l+k-3} \nabla^{2l+k-3-i} \text{Rm} \ast \nabla^i A = \sum_{i=0}^{2l+k-3} \nabla^{2l+k-2-i} \text{Rm} \ast \nabla^i A + \text{Rm} \ast \nabla^{2l+k-2} A
\]
\[
= \sum_{i=0}^{2l+k-2} \nabla^{2l+k-2-i} \text{Rm} \ast \nabla^i A.
\]

Lastly,
\[
\nabla^k \Delta^l A = \nabla \nabla^{k-1} \Delta^l A
\]
\[
= \nabla \left( \Delta^l \nabla^{k-1} A + \sum_{i=0}^{2l+k-3} \nabla^{2l+k-3-i} \text{Rm} \ast \nabla^i A \right)
\]
\[
= \Delta^l \nabla \nabla^{k-1} A + \sum_{i=0}^{2l-1} \nabla^{2l-1-i} \text{Rm} \ast \nabla^i \nabla^{k-1} A + \nabla \sum_{i=0}^{2l+k-3} \nabla^{2l+k-3-i} \text{Rm} \ast \nabla^i A
\]
\[
= \Delta^l \nabla \nabla^{k-1} A + \sum_{i=0}^{2l-1} \nabla^{2l-1-i} \text{Rm} \ast \nabla^i \nabla^{k-1} A + \sum_{i=0}^{2l+k-3} \nabla^{2l+k-3-i} \text{Rm} \ast \nabla^i A
\]
Proposition 3.3. Let $A$ be a tensor on $M$ and $g(t)$ be a one-parameter family of metrics on $M$. If $k \geq 1$, then

$$\partial_t \nabla^k A = \nabla^k \partial_t A + \sum_{j=0}^{k-1} \nabla^j (\nabla \partial_t g \ast \nabla^{k-j} A).$$

Proof. First

$$\partial_t \nabla A = \partial_t \partial_t A_{j_1 \cdots j_r} - \sum_{m=1}^{r} \partial \Gamma^m_{ijm} A^{k_1 \cdots k_s} A^{j_1 \cdots j_m + 1} A^{j_1 \cdots j_s} + \Gamma^m_{ijm} \partial \partial_t A^{k_1 \cdots k_s} A^{j_1 \cdots j_m + 1} A^{j_1 \cdots j_s} + \partial \partial_t A^{k_1 \cdots k_s} A^{j_1 \cdots j_m + 1} A^{j_1 \cdots j_s}$$

$$+ \sum_{p=1}^{s} \partial \Gamma_{il}^{k_1 \cdots k_p} A^{j_1 \cdots j_r} + \Gamma_{il}^{k_1 \cdots k_p} \partial \partial_t A^{k_1 \cdots k_p} A^{j_1 \cdots j_r} + \partial \partial_t A^{k_1 \cdots k_p} A^{j_1 \cdots j_r}$$

$$= \nabla \partial_t A + \partial_t \Gamma \ast A$$

$$= \nabla \partial_t A + \nabla \partial_t g \ast A,$$

so the proposition is true when $k = 1$. Assume the proposition holds for $k - 1$. Then

$$\partial_t \nabla^k A = \partial_t \nabla \nabla^{k-1} A$$

$$= \nabla \partial_t \nabla^{k-1} A + \nabla \partial_t g \ast \nabla^{k-1} A$$

$$= \nabla^k \partial_t A + \nabla \partial_t g \ast \nabla^{k-1} A + \sum_{j=0}^{k-2} \nabla^j (\nabla \partial_t g \ast \nabla^{k-2-j} A)$$

$$= \nabla^k \partial_t A + \nabla \partial_t g \ast \nabla^{k-1} A + \nabla \sum_{j=0}^{k-2} \sum_{i=0}^{j} \nabla^{i+1} \partial_t g \ast \nabla^{k-2-i} A$$

$$= \nabla^k \partial_t A + \nabla \partial_t g \ast \nabla^{k-1} A + \sum_{j=0}^{k-2} \sum_{i=0}^{j} (\nabla^{i+2} \partial_t g \ast \nabla^{k-2-i} A + \nabla^{i+1} \partial_t g \ast \nabla^{k-1-i} A)$$

$$= \nabla^k \partial_t A + \sum_{j=0}^{k-2} \sum_{i=0}^{j} \nabla^{i+1} \partial_t g \ast \nabla^{k-1-i} A + \nabla \partial_t g \ast \nabla^{k-1} A + \sum_{j=0}^{k-2} \nabla^{j+2} \partial_t g \ast \nabla^{k-2-j} A$$

$$= \nabla^k \partial_t A + \sum_{j=0}^{k-2} \sum_{i=0}^{j} \nabla^{i+1} \partial_t g \ast \nabla^{k-1-i} A + \nabla \partial_t g \ast \nabla^{k-1} A + \sum_{j=0}^{k-1} \nabla^{j+1} \partial_t g \ast \nabla^{k-1-j} A$$

$$= \nabla^k \partial_t A + \sum_{j=0}^{k-2} \sum_{i=0}^{j} \nabla^{i+1} \partial_t g \ast \nabla^{k-1-i} A + \sum_{j=0}^{k-1} \nabla^{j+1} \partial_t g \ast \nabla^{k-1-j} A$$

$$= \nabla^k \partial_t A + \nabla^j (\nabla \partial_t g \ast \nabla^{k-1-j} A) + \nabla^{k-1} (\nabla \partial_t g \ast A).$$
\[= \nabla^k \partial_t A + \sum_{j=0}^{k-1} \nabla^j (\nabla \partial_t g * \nabla^{k-1-j} A).\]

3.2. Evolution Equations. We derive the equations for \( \partial_t \nabla^k \text{Rm} \) for every \( k \geq 0 \).

**Proposition 3.4.** If \((M, g(t))\) is a solution to AOF, then

\[\partial_t \text{Rm} = \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^\frac{n}{2} \text{Rm} + \sum_{j=2}^{n/2+1} P_j^{n-2j+2}(\text{Rm}).\]

**Proof.** Let \( \hat{g}(t) \) be a one-parameter family of metrics on \( M \) and \( h = \partial_t \hat{g} \). The evolution of \( \text{Rm} \) is given by \([20], \text{Theorem 7.1}\)

\[\partial_t R_{ijkl} = \frac{1}{2}\left[\nabla_i \nabla_k h_{jl} + \nabla_j \nabla_l h_{ik} - \nabla_i \nabla_l h_{jk} - \nabla_j \nabla_k h_{il}\right] + \text{Rm} * h.\]

If \( h = \Delta^\frac{n}{2} \text{Rc} \) then, using Proposition 3.2 in the second line and Lemma 3.1 in the third line,

\[\partial_t R_{ijkl} = \frac{1}{2}\left[\nabla_i \nabla_k \Delta^\frac{n}{2} \text{Rc} + \nabla_j \nabla_l \Delta^\frac{n}{2} \text{Rc} - \nabla_i \nabla_l \Delta^\frac{n}{2} \text{Rc} - \nabla_j \nabla_k \Delta^\frac{n}{2} \text{Rc} - \nabla_i \nabla_l \Delta^\frac{n}{2} \text{Rc} - \nabla_j \nabla_k \Delta^\frac{n}{2} \text{Rc} \right] + \text{Rm} * \Delta^\frac{n}{2} \text{Rc}
\]

\[= \frac{1}{2} \Delta^\frac{n}{2} \left[\nabla_i \nabla_k \text{Rc} + \nabla_j \nabla_l \text{Rc} - \nabla_i \nabla_l \text{Rc} - \nabla_j \nabla_k \text{Rc} \right] + \sum_{i=0}^{n-2} \nabla^{n-2-i} \text{Rm} * \nabla^i \text{Rc} + P_2^{n-2} (\text{Rm})
\]

\[= \frac{1}{2} \Delta^\frac{n}{2} R_{ijkl} + P_2^{n-2} (\text{Rm}).\]

If \( h = \Delta^\frac{n}{2} \nabla^2 \text{R} \) then, using Proposition 3.2 in the second and fourth lines,

\[\partial_t R_{ijkl} = \frac{1}{2}\left[\nabla_i \nabla_k \Delta^\frac{n}{2} \nabla^2 \text{R} + \nabla_j \nabla_l \Delta^\frac{n}{2} \nabla^2 \text{R} - \nabla_i \nabla_l \Delta^\frac{n}{2} \nabla^2 \text{R} - \nabla_j \nabla_k \Delta^\frac{n}{2} \nabla^2 \text{R} - \nabla_i \nabla_l \Delta^\frac{n}{2} \nabla^2 \text{R} - \nabla_j \nabla_k \Delta^\frac{n}{2} \nabla^2 \text{R} \right]
\]

\[+ \text{Rm} * \Delta^\frac{n}{2} \nabla^2 \text{R}
\]

\[= \frac{1}{2} \Delta^\frac{n}{2} \left[\nabla_i \nabla_k \nabla^2 \text{R} + \nabla_j \nabla_l \nabla^2 \text{R} - \nabla_i \nabla_l \nabla^2 \text{R} - \nabla_j \nabla_k \nabla^2 \text{R} \right] + \sum_{i=0}^{n-2} \nabla^{n-2-i} \text{Rm} * \nabla^i \nabla^2 \text{R} + P_2^{n-2} (\text{Rm})
\]

\[= \frac{1}{2} \Delta^\frac{n}{2} \left[\nabla_i \nabla_k \nabla^2 \text{R} + \nabla_j \nabla_l \nabla^2 \text{R} - \nabla_i \nabla_l \nabla^2 \text{R} - \nabla_j \nabla_k \nabla^2 \text{R} \right] + \sum_{i=0}^{n-2} \nabla^{n-2-i} \text{Rm} * \nabla^i \nabla^2 \text{R} + P_2^{n-2} (\text{Rm})
\]

\[= P_2^{n-2} (\text{Rm}).\]

If \( h = \sum_{j=2}^{n/2} P_j^{n-2j} (\text{Rm}) \), then

\[\partial_t \text{Rm} = \nabla^2 \sum_{j=2}^{n/2} P_j^{n-2j} (\text{Rm}) + \text{Rm} * \sum_{j=2}^{n/2} P_j^{n-2j} (\text{Rm})
\]

\[= \sum_{j=2}^{n/2} P_j^{n-2j} (\text{Rm}) + \sum_{j=2}^{n/2} P_j^{n-2j} (\text{Rm}).\]
Combining these results, we conclude that if \( h = \hat{O} \) then

\[
\partial_t Rm = \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^\frac{n}{2} Rm + P_2^n - 2(Rm) + \sum_{j=2}^{n/2} P_j^{n-2j+2}(Rm) + \sum_{j=2}^{n/2} P_j^{n-2j}(Rm)
\]

\[
= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^\frac{n}{2} Rm + \sum_{j=2}^{n/2+1} P_j^{n-2j+2}(Rm).
\]

\[\square\]

**Proposition 3.5.** If \((M, g(t))\) is a solution to AOF, then

\[
\partial_t \nabla^k Rm = \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^\frac{n}{2} \nabla^k Rm + \sum_{l=2}^{n/2+1} P_l^{n-2l+k+2}(Rm).
\]

**Proof.** We compute:

\[
k-1 \sum_{j=0} \nabla^j (\nabla \partial_t g \ast \nabla^{k-1-j} Rm) = \sum_{j=0} \nabla^j \left( \sum_{l=1}^{n/2} P_l^{n-2l+1}(Rm) \ast \nabla^{k-1-j} Rm \right)
\]

\[= \sum_{j=0} \nabla^j \sum_{l=1}^{n/2} P_l^{n-2l+k-j}(Rm)
\]

\[= \sum_{j=0} \sum_{l=1}^{n/2} P_l^{n-2l+k-j}(Rm)
\]

\[= \sum_{l=1}^{n/2} P_l^{n-2l+k}(Rm)
\]

\[= \sum_{l=2}^{n/2+1} P_l^{n-2l+k+2}(Rm).
\]

Then, using Proposition 3.3 in the first line, Proposition 3.4 in the second line, and Proposition 3.2 in the third line, we get

\[
\partial_t \nabla^k Rm = \nabla^k \partial_t Rm + \sum_{j=0}^{k-1} \nabla^j (\nabla \partial_t g \ast \nabla^{k-1-j} Rm)
\]

\[= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \nabla^k \Delta^\frac{n}{2} R_{ijkl} + \nabla^k \sum_{j=2}^{n/2+1} P_j^{n-2j+2}(Rm) + \sum_{l=2}^{n/2+1} P_l^{n-2l+k+2}(Rm)
\]

\[= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^\frac{n}{2} \nabla^k R_{ijkl} + \sum_{j=2}^{n/2+1} P_j^{n-2j+k+2}(Rm) + \sum_{l=2}^{n/2+1} P_l^{n-2l+k+2}(Rm)
\]

\[= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \nabla^k R_{ijkl} + \sum_{l=2}^{n/2+1} P_l^{n-2l+k+2}(Rm).
\]

\[\square\]
3.3. Short Time Existence and Uniqueness. We recall the short time existence and uniqueness theorems for the AOF. E. Bahuaud and D. Helliwell have shown in their papers [3], [4] the following result:

**Theorem 3.6.** Let $h$ be a smooth metric on a compact manifold $M$ of even dimension $n \geq 4$. Then there is a unique smooth short time solution to the following flow:

\[
\begin{align*}
\partial_t g &= \tilde{\mathcal{O}} = (-1)^{\frac{n}{4}} \mathcal{O} + \frac{(-1)^{\frac{n}{4}}}{2(n-1)(n-2)} (\Delta^{\frac{n}{4}} - 1) R g \\
g(0) &= h,
\end{align*}
\]

where $\mathcal{O}$ is the ambient obstruction tensor on $M$ and $R$ is the scalar curvature of $M$.

**Proof.** We outline a proof of the existence theorem. Due to the diffeomorphism invariance of $M$, the system (4) is not strongly parabolic. However, by choosing a vector field $W$ given by

\[
W = \frac{(-1)^{\frac{n}{4}}}{2(n-2)} \Delta^{\frac{n}{4}} - 1 X + \frac{(-1)^{\frac{n}{4}}}{4(n-1)} \left( \nabla \Delta^{\frac{n}{4}} - 1 R \right),
\]

where $X^k = g^{ij} (\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij})$ and $\tilde{\Gamma}$ is the connection of $h$, we obtain a strongly parabolic system:

\[
\begin{align*}
\partial_t g &= \tilde{\mathcal{O}} + L_W g \\
g(0) &= h.
\end{align*}
\]

We show this by computing the principal symbol $\sigma$ of the system (5). Let $A = -2Rc + \mathcal{L}_X g$. We know from Proposition 2.3 that

\[
\begin{align*}
\tilde{\mathcal{O}} &= \frac{(-1)^{\frac{n}{4}}}{n-2} \Delta^{\frac{n}{4}} - 1 Rc + \frac{(-1)^{\frac{n}{4}}}{2(n-1)} \Delta^{\frac{n}{4}} - 1 R + \sum_{j=2}^{n/2} P^{n-2j}(Rm).
\end{align*}
\]

We can linearize the first term of (6) as follows:

\[
\begin{align*}
\partial_t \left[ \frac{(-1)^{\frac{n}{4}}}{n-2} \Delta^{\frac{n}{4}} - 1 Rc + \frac{(-1)^{\frac{n}{4}}}{2(n-2)} \Delta^{\frac{n}{4}} - 1 \mathcal{L}_X g \right] &= \frac{(-1)^{\frac{n}{4}}}{2(n-2)} \partial_t \Delta^{\frac{n}{4}} - 1 A \\
&= \frac{(-1)^{\frac{n}{4}}}{2(n-2)} [g^{*(1-\frac{n}{4})} \partial_t \nabla^{\frac{n}{4}} - 1 A \text{ + lots}] \\
&= \frac{(-1)^{\frac{n}{4}}}{2(n-2)} \Delta^{\frac{n}{4}} - 1 \partial_t A \text{ + lots} \\
&= \frac{(-1)^{\frac{n}{4}}}{2(n-2)} \Delta^{\frac{n}{4}} - 1 \partial_t g \text{ + lots}.
\end{align*}
\]

We used Proposition 3.3 in the third line and the fact from Ricci flow (Chow-Knopf [13], Theorem 3.13) that $\sigma[D A](\zeta) = |\zeta|^2$ in the fourth line. Let $Y = \frac{(-1)^{\frac{n}{4}}}{4(n-1)} (\nabla \Delta^{\frac{n}{4}} - 1 R)$. The second term of (6) can be absorbed into $L_W g$:

\[
\begin{align*}
\frac{(-1)^{\frac{n}{4}}}{2(n-1)} \Delta^{\frac{n}{4}} - 2 \nabla_i \nabla_j R + (\mathcal{L}_Y g)_{ij} &= \frac{(-1)^{\frac{n}{4}}}{4(n-1)} \Delta^{\frac{n}{4}} - 2 \nabla_i \nabla_j R + \frac{(-1)^{\frac{n}{4}}}{4(n-1)} \Delta^{\frac{n}{4}} - 2 \nabla_j \nabla_i R + \frac{(-1)^{\frac{n}{4}}}{4(n-1)} \Delta^{\frac{n}{4}} - 2 \nabla_i \nabla_j R \\
&\quad + \frac{(-1)^{\frac{n}{4}}}{4(n-1)} \Delta^{\frac{n}{4}} - 2 \nabla_j \nabla_i R \text{ + lots}. \\
&= \text{lots}.
\end{align*}
\]
We commuted $\nabla_i$ and $\nabla_j$ and used Proposition 3.2 to commute $\Delta^{\frac{n}{2}-1}$ and $\nabla_i \nabla_j$. So the principal symbol of the system (5) is
\[ \sigma[D(\bar{\mathcal{O}} + \mathcal{L}g)](\zeta) = \frac{(-1)^{\frac{n}{2}} - 1}{2(n-2)}|\zeta|^\frac{n}{2}. \]
Since the PDE has order $n$ with respect to $g$, (5) is strongly parabolic.

So there exists $\epsilon > 0$ for which the solution to (5) exists for $t \in [0, \epsilon)$ via parabolic PDE theory. Next, there exists a family $\varphi_t : M \to M$ of diffeomorphisms satisfying
\[ \begin{cases} \frac{\partial \varphi_t}{\partial t} = -W(\varphi_t, t) \\ \varphi_0 = \text{id}_M. \end{cases} \]
for $t \in [0, \epsilon)$. The existence of the $\varphi_t$ follows from the existence and uniqueness theorem for nonautonomous ODE on manifolds, and the uniform $\epsilon$ follows from bounds on $W$ that result from the compactness of $M$. We now show that $\partial_t(\varphi_t^* g) = \varphi_t^* \partial_t g$. First, if $p \in M$ and $v_1, v_2 \in T_p M$,
\[ \lim_{s \to 0} \frac{\varphi_{s+t}^* g(s+t)(v_1, v_2) - \varphi_s^* g(t)(v_1, v_2)}{s} = \lim_{s \to 0} \frac{g(s+t)_{ij} - g(t)_{ij}}{s} \lim_{s \to 0} \left( \left( \varphi_{s+t}^* v_1 \right)^i \left( \varphi_{s+t}^* v_2 \right)^j \right) \]
\[ = (\partial_t g)_{ij} \left( \varphi_t^* v_1 \right)^i \left( \varphi_t^* v_2 \right)^j \]
\[ = \varphi_t^* \partial_t g(v_1, v_2). \]
So
\[ \partial_t(\varphi_t^* g) = \lim_{s \to 0} \frac{\varphi_{s+t}^* g(s+t) - \varphi_s^* g(t)}{s} \]
\[ = \lim_{s \to 0} \frac{\varphi_{s+t}^* g(s+t) - \varphi_s^* g(t)}{s} + \lim_{s \to 0} \frac{\varphi_s^* g(t) - \varphi_s^* g(t)}{s} \]
\[ = \varphi_t^* \partial_t g + \partial_s |_{s=0}(\varphi_{t+s}^* g(t)) \]
\[ = \varphi_t^* \left[ (1)^{\frac{n}{2}} \mathcal{O}(g) + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}[\Delta^{\frac{n}{2}} - 1 R(g(t))] g(t) + \mathcal{L}W(g(t)) \right] + \partial_s |_{s=0}(\varphi_t^{-1} \circ \varphi_{t+s}^* \varphi_t^* g(t)) \]
\[ = (-1)^{\frac{n}{2}} \mathcal{O}(\varphi_t^* g(t)) + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}[\Delta^{\frac{n}{2}} - 1 R(g(t))] \varphi_t^* g(t) + \varphi_t^* (\mathcal{L}W(g(t)) - \mathcal{L}[(\varphi_t^{-1} \circ \varphi_{t+s}^* g(t))] \varphi_t^* g(t) \]
\[ = \varphi_t^* \partial_t g(t) + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}[\Delta^{\frac{n}{2}} - 1 R(g(t))] \varphi_t^* g(t). \]
Since $\varphi_t^* g(0) = g(0) = h$, $\varphi_t^* g(t)$ satisfies (1). Therefore these diffeomorphisms pull back the short time solution of (5) to give a solution of (1) that exists for $t \in [0, \epsilon)$.

4. Local Integral Estimates

In this section, let $(M^n, g)$ be a Riemannian manifold that is a solution to the AOF on a time interval $[0, T)$. We give local $L^2$ estimates for $\nabla^k Rm$ for all $k \in \mathbb{N}$. We need to use local $L^2$ estimates since we can only convert $L^2$ estimates to pointwise estimates locally. These local pointwise estimates are used in the proof of the pointwise smoothing estimates given in Theorem 1.1. Specify the Laplace operator by $\Delta = -\nabla^i \nabla_i$. Let $\varphi \in C^\infty_c(M)$ be a cutoff function with constants $\Lambda, \Lambda_1 > 0$ such that
\[ \sup_{t \in [0, T)} |\nabla \varphi| \leq \Lambda_1, \max_{0 \leq i \leq \frac{n}{2}} \sup_{t \in [0, T)} |\nabla^i \varphi| \leq \Lambda. \]

**Lemma 4.1.** Suppose $M, \varphi$ satisfy the above hypotheses. Let $A$ be any tensor and $p \geq 1, q \geq 2$. Then
\[ \int_M \varphi^p (\Delta^q A, A) = (-1)^q \int_{|\varphi| > 0} \sum_{i = 0}^q P_{2p-2i}(\varphi) * \nabla^i A * \nabla^q A + \int_M \sum_{i = 0}^{2q-2} \varphi^p \nabla^{2q-2-i} \text{Rm} * \nabla^i A * A. \]
Proof. We first claim that if $q \geq 2$, then

$$\Delta^q A = (-1)^q (\nabla^*)^q \nabla^q A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} Rm \ast \nabla^i A.$$  

If $q = 2$, we get, using Proposition 3.2, that

$$\Delta^2 A = -\nabla^* \nabla \Delta A$$

$$= -\nabla^* \Delta \nabla A + \nabla^* [\nabla Rm \ast A + Rm \ast \nabla A]$$

$$= (\nabla^*)^2 \nabla^2 A + \nabla^2 Rm \ast A + \nabla Rm \ast \nabla A + Rm \ast \nabla^2 A,$$

which agrees with the claim. Suppose the claim is true for every integer less than $q$. First,

$$\Delta^q A = -\nabla^* \nabla \Delta^{q-1} A$$

$$= -\nabla^* \left[ \Delta^{q-1} \nabla A + \sum_{i=0}^{2q-3} \nabla^{2q-3-i} Rm \ast \nabla^i A \right]$$

$$= -\nabla^* \Delta^{q-1} \nabla A + \sum_{i=0}^{2q-3} \nabla^{2q-2-i} Rm \ast \nabla^i A + \sum_{i=1}^{2q-2} \nabla^{2q-2-i} Rm \ast \nabla^i A$$

$$= -\nabla^* \Delta^{q-1} \nabla A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} Rm \ast \nabla^i A.$$

Applying the last equation above and then the inductive hypothesis,

$$\Delta^q A = -\nabla^* \Delta^{q-1} \nabla A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} Rm \ast \nabla^i A$$

$$= -\nabla^* \left[ (-1)^q (\nabla^*)^q \nabla^{q-1} \nabla A + \sum_{i=0}^{2q-4} \nabla^{2q-4-i} Rm \ast \nabla^i \nabla A \right] + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} Rm \ast \nabla^i A$$

$$= (-1)^q (\nabla^*)^q \nabla^q A + \sum_{i=0}^{2q-3} \nabla^{2q-3-i} Rm \ast \nabla^{i+1} A + \sum_{i=0}^{2q-2} \nabla^{2q-4-i} Rm \ast \nabla^{i+2} A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} Rm \ast \nabla^i A$$

$$= (-1)^q (\nabla^*)^q \nabla^q A + \sum_{i=1}^{2q-3} \nabla^{2q-2-i} Rm \ast \nabla^i A + \sum_{i=2}^{2q-2} \nabla^{2q-2-i} Rm \ast \nabla^i A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} Rm \ast \nabla^i A$$

$$= (-1)^q (\nabla^*)^q \nabla^q A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} Rm \ast \nabla^i A.$$
This proves the claim. We compute

\[
(-1)^{q+1} \int_M \nabla^q A \ast \nabla^q (\varphi^p A) = (-1)^{q+1} \int_M \nabla^q A \ast \sum_{i=0}^q \nabla^{q-i} (\varphi^p) \ast \nabla^i A
\]

\[
= (-1)^{q+1} \int_{[\varphi > 0]} \sum_{i=0}^q \sum_{|\alpha| = q-i} \nabla^{\alpha_1} \varphi_1 \ast \cdots \ast \nabla^{\alpha_p} \varphi_p \ast \nabla^i A \ast \nabla^q A
\]

\[
= (-1)^{q+1} \int_{[\varphi > 0]} \sum_{i=0}^q \mathcal{P}_p^{q-i} (\varphi) \ast \nabla^i A \ast \nabla^q A.
\]

Finally, applying the claim,

\[
\int_M \varphi^p (\Delta^q A, A) = \int_M \varphi^p \left( (-1)^q (\nabla^*)^q \nabla^q A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} Rm \ast \nabla^i A, A \right)
\]

\[
= (-1)^q \int_M \nabla^q A \ast \nabla^q (\varphi^p A) + \int_M \sum_{i=0}^{2q-2} \varphi^p \nabla^{2q-2-i} Rm \ast \nabla^i A \ast A
\]

\[
= (-1)^q \int_{[\varphi > 0]} \sum_{i=0}^q \mathcal{P}_p^{q-i} (\varphi) \ast \nabla^i A \ast \nabla^q A + \int_M \sum_{i=0}^{2q-2} \varphi^p \nabla^{2q-2-i} Rm \ast \nabla^i A \ast A.
\]

**Proposition 4.2.** Suppose \( M, \varphi \) satisfy the above hypotheses. If \( p \geq 1, k \geq 0 \), then

\[
(7) \quad \frac{\partial}{\partial t} \int_M \varphi^p |\nabla^k Rm|^2 = -\frac{1}{n-2} \int_M \varphi^p |\nabla^{\frac{n}{2}+k} Rm|^2 + \int_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} \mathcal{P}_p^{2l} (\varphi) \ast \nabla^{k+l} Rm \ast \nabla^{k+l} Rm
\]

\[
+ \int_{[\varphi > 0]} \sum_{i=0}^{\frac{n}{2}-1} \mathcal{P}_p^{2-i} (\varphi) \ast \nabla^{k+i} Rm \ast \nabla^{k+i} Rm.
\]

**Proof.** First, we have

\[
\frac{\partial}{\partial t} \int_M \varphi^p |\nabla^k Rm|^2 \, dV_g = 2 \int_M \varphi^p \left( \frac{\partial}{\partial t} \nabla^k Rm, \nabla^k Rm \right) \, dV_g + \int_M \varphi^p |\nabla^k Rm|^2 \frac{\partial g}{\partial t} \, dV_g.
\]

We can expand the first integral by substituting Proposition 3.5 which states that for our flow,

\[
\frac{\partial}{\partial t} \nabla^k Rm = \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^k Rm + \sum_{i=2}^{\frac{n}{2}+1} \mathcal{P}_i^{n-2i+k+2} (Rm).
\]

\[
+ \sum_{i=0}^{\frac{n}{2}-1} \mathcal{P}_i^{n-i} (\varphi) \ast \nabla^{k+i} Rm \ast \nabla^{k+i} Rm.
\]

\[
\int \nabla^2 \nabla^k Rm \, dV_g = \int \nabla^2 \left( \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^k Rm \right) \, dV_g + \int \sum_{i=2}^{\frac{n}{2}+1} \mathcal{P}_i^{n-2i+k+2} (Rm) \, dV_g + \sum_{i=0}^{\frac{n}{2}-1} \mathcal{P}_i^{n-i} (\varphi) \ast \nabla^{k+i} Rm \ast \nabla^{k+i} Rm.
\]

\[
\int \nabla^2 \nabla^k Rm \, dV_g = \int \left( \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^k Rm \right) \, dV_g + \int \sum_{i=2}^{\frac{n}{2}+1} \mathcal{P}_i^{n-2i+k+2} (Rm) \, dV_g + \sum_{i=0}^{\frac{n}{2}-1} \mathcal{P}_i^{n-i} (\varphi) \ast \nabla^{k+i} Rm \ast \nabla^{k+i} Rm.
\]

\[
\int \nabla^2 \nabla^k Rm \, dV_g = \int \left( \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^k Rm \right) \, dV_g + \int \sum_{i=2}^{\frac{n}{2}+1} \mathcal{P}_i^{n-2i+k+2} (Rm) \, dV_g + \sum_{i=0}^{\frac{n}{2}-1} \mathcal{P}_i^{n-i} (\varphi) \ast \nabla^{k+i} Rm \ast \nabla^{k+i} Rm.
\]

\[
\int \nabla^2 \nabla^k Rm \, dV_g = \int \left( \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^k Rm \right) \, dV_g + \int \sum_{i=2}^{\frac{n}{2}+1} \mathcal{P}_i^{n-2i+k+2} (Rm) \, dV_g + \sum_{i=0}^{\frac{n}{2}-1} \mathcal{P}_i^{n-i} (\varphi) \ast \nabla^{k+i} Rm \ast \nabla^{k+i} Rm.
\]
Applying Lemma 4.1 to the first term of \( \frac{\partial}{\partial t} \nabla^k R \) gives that

\[
\frac{(-1)^{\frac{n}{2}+1}}{n-2} \int_M \varphi^p \langle \Delta_{\frac{n}{2}} \nabla^k R, \nabla^k R \rangle = \frac{(-1)^{n+1}}{n-2} \int_M \sum_{i=0}^{\frac{n}{2}} P_{\frac{n}{2}-i}^n (\varphi) \ast \nabla^{k+i} R \ast \nabla^{k+\frac{n}{2}} R
\]

\[
+ \int_M \sum_{i=0}^{n-2} \varphi^p \nabla^{n-2-i} R \ast \nabla^{k+i} R \ast \nabla^k R
\]

\[
= - \frac{1}{n-2} \int_M \varphi^p |\nabla^{\frac{n}{2}+k} R|^2
\]

\[
+ \int_M \sum_{i=0}^{\frac{n}{2}-1} P_{\frac{n}{2}-i}^n (\varphi) \ast \nabla^{k+i} R \ast \nabla^{k+\frac{n}{2}} R
\]

\[
+ \int_M \varphi^p P_{\frac{n}{4}+2k-2}^n (R).
\]

Substituting the second term of \( \frac{\partial}{\partial t} \nabla^k R \) into the inner product gives that

\[
\int_M \varphi^p \left( \nabla^k R, \sum_{i=2}^{\frac{n}{2}+1} P_i^{n-2i+k+2} (Rm) \right) = \int_M \varphi^p \sum_{i=3}^{\frac{n}{2}+2} P_i^{n-2i+k+2} (Rm)
\]

\[
= \int_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P_{\frac{n}{2}+k-l+2}^l (Rm).
\]

Since

\[
\frac{\partial g}{\partial t} = \Delta_{\frac{n}{2}-1} R + \Delta_{\frac{n}{2}-2} \nabla^2 R + \sum_{i=2}^{\frac{n}{2}} P_i^{n-2i} (Rm)
\]

\[
= \nabla^{n-2} R + \nabla^{n-4+2} R + \sum_{i=2}^{\frac{n}{2}} P_i^{n-2i} (Rm)
\]

\[
= \sum_{i=1}^{\frac{n}{2}} P_i^{n-2i} (Rm),
\]

we have

\[
\int_M \varphi^p |\nabla^k R|^2 \frac{\partial g}{\partial t} = \int_M \varphi^p (\nabla^k R)^2 \sum_{i=1}^{\frac{n}{2}} P_i^{n-2i} (Rm)
\]

\[
= \int_M \varphi^p \sum_{i=1}^{\frac{n}{2}} P_{i+2}^{n-2i+2k} (Rm)
\]

\[
= \int_M \varphi^p \sum_{i=3}^{\frac{n}{2}+2} P_i^{n-2i+2k} (Rm)
\]

\[
= \int_M \varphi^p \sum_{i=3}^{\frac{n}{4}+2} P_i^{n-2i+2k+4} (Rm)
\]
Lemma 4.5. Suppose Proposition 4.4. tensor. Let inequality is true for all integers at most \( l \) where \( C \). Then\

\[
\frac{\partial}{\partial t} \int_M \varphi^p |\nabla^k Rm|^2 = - \frac{1}{n-2} \int_M \varphi^p |\nabla^{k+2} Rm|^2 + \int_{|\varphi| > 0} \sum_{l=0}^{\frac{n}{2}+k-1} p^{\frac{n}{2}+\frac{k}{2}} \varphi^p (\varphi) * \nabla^{k+1} Rm * \nabla^{k+\frac{n}{2}} Rm
\]

\[
+ \int_M \varphi^p P_{\frac{n}{2}+2k-2}^{|2k-2}(Rm) + \int M \varphi^p \sum_{l=0}^{\frac{n}{2}+k-1} P_{\frac{n}{2}+k-2}^{|2k-2} \varphi^p (\varphi) * \nabla^{k+1} Rm * \nabla^{k+\frac{n}{2}} Rm
\]

\[
= - \frac{1}{n-2} \int_M \varphi^p |\nabla^{k+2} Rm|^2 + \int_{|\varphi| > 0} \sum_{l=0}^{\frac{n}{2}+k-1} p^{\frac{n}{2}+\frac{k}{2}} \varphi^p (\varphi) * \nabla^{k+1} Rm * \nabla^{k+\frac{n}{2}} Rm
\]

\[
+ \int_M \varphi^p \sum_{l=0}^{\frac{n}{2}+k-1} P_{\frac{n}{2}+k-2}^{|2k-2}(Rm).
\]

We estimate the last two terms of (7). First, we recall two corollaries from the paper [27] of E. Kuwert and R. Schätzle.

Proposition 4.3. [27], Corollary 5.2 Suppose \( M, \varphi \) satisfy the above hypotheses. Let \( A \) be a tensor. For \( 2 \leq p < \infty, s \geq p, \) and \( c = c(n, p, s, \Lambda_1), \)

\[
\left( \int_M |\nabla A|^p \varphi^s \right)^{\frac{1}{p}} \leq \epsilon \left( \int_M |\nabla^2 A|^p \varphi^{s+p} \right)^{\frac{1}{p}} + \frac{c}{\epsilon} \left( \int_{|\varphi| > 0} |A|^p \varphi^{s-p} \right)^{\frac{1}{p}

Proposition 4.4. [27], Corollary 5.5 Suppose \( M, \varphi \) satisfy the above hypotheses. Let \( A \) be a tensor. Let \( 0 \leq i_1, \ldots, i_r \leq k, i_1 + \cdots + i_r = 2k, \) and \( s \geq 2k. \) Then we have

\[
\left| \int_M \varphi^s \nabla^{i_1} A * \cdots * \nabla^{i_r} A \right| \leq c \|A\|_{r-2}^2 \left( \int_M \varphi^s |\nabla^k A|^2 dV + \|A\|_{2,|\varphi| > 0} \right),
\]

where \( c = c(k, n, r, s, \Lambda_1). \)

We estimate the last term of (7).

Lemma 4.5. Suppose \( M, \varphi \) satisfy the above hypotheses. If \( l \geq 1, q \geq 0, \) then for every \( \epsilon > 0, \)

\[
\int_M \varphi^{2l+q} |\nabla^l Rm|^2 \leq \epsilon \int_M \varphi^{2l+q+2} |\nabla^{l+1} Rm|^2 + \frac{C}{\epsilon} \int_{|\varphi| > 0} \varphi^q |Rm|^2.
\]

where \( C = C(n, l, \Lambda_1, q). \)

Proof. If \( l = 1, \) the inequality follows immediately from Proposition 4.3 Assume that the the inequality is true for all integers at most \( l. \) Then

\[
\int \varphi^{2l+2+q} |\nabla^{l+1} Rm|^2 \leq \frac{\epsilon}{2} \int \varphi^{2l+4+q} |\nabla^{l+2} Rm|^2 + \frac{C}{\epsilon} \int \varphi^{2l+q} |\nabla^l Rm|^2
\]

\[
\leq \frac{\epsilon}{2} \int \varphi^{2l+4+q} |\nabla^{l+2} Rm|^2 + \frac{C}{\epsilon} \frac{\epsilon}{2C} \int \varphi^{2l+q+2} |\nabla^{l+1} Rm|^2 + \frac{C}{\epsilon} \frac{C}{\epsilon} \int \varphi^q |Rm|^2
\]

\[
= \frac{\epsilon}{2} \int \varphi^{2l+4+q} |\nabla^{l+2} Rm|^2 + \frac{1}{2} \int \varphi^{2l+q+2} |\nabla^{l+1} Rm|^2 + \frac{C}{\epsilon+1} \int \varphi^q |Rm|^2.
\]
Collecting terms, we see that the statement is also true for \( l + 1 \).

**Lemma 4.6.** Suppose \( M, \varphi \) satisfy the above hypotheses. If \( q \geq 0 \) and \( 0 \leq l \leq q \),
\[
\int_M \varphi^{2l+r} |\nabla^l R_m|^2 \leq \epsilon^{q-l} \int_M \varphi^{2q+r} |\nabla^q R_m|^2 + C \epsilon^{m-q} \int_{[\varphi > 0]} \varphi^r |R_m|^2.
\]
where \( C = C(n, l, \Lambda_1, r) \).

**Proof.** Let \( m = q - l \). The desired inequality is equivalent to
\[
\int_M \varphi^{2q-2m+r} |\nabla^{q-m} R_m|^2 \leq \epsilon^m \int_M \varphi^{2q+r} |\nabla^q R_m|^2 + C \epsilon^{m-q} \int_{[\varphi > 0]} \varphi^r |R_m|^2.
\]
We prove this inequality by induction on \( m \). If \( m = 0 \) the inequality is true:
\[
\int_M \varphi^{2q+r} |\nabla^q R_m|^2 \leq \int_M \varphi^{2q+r} |\nabla^q R_m|^2 + C \epsilon^{m-q} \int_{[\varphi > 0]} \varphi^r |R_m|^2.
\]
Assume the inequality (8) is true for every integer less than \( m \). Then
\[
\int_M \varphi^{2q-2m+r} |\nabla^{q-m} R_m|^2 \leq \epsilon^{m-1} \int_M \varphi^{2q-2m+2} |\nabla^{q-m+2} R_m|^2 + C \epsilon^{m-q} \int_{[\varphi > 0]} \varphi^r |R_m|^2
\]
\[
\leq \epsilon^m \int_M \varphi^{2q+r} |\nabla^q R_m|^2 + \epsilon C \epsilon^{m-q} \int_{[\varphi > 0]} \varphi^r |R_m|^2
\]
\[
= \epsilon^m \int_M \varphi^{2q+r} |\nabla^q R_m|^2 + C \epsilon^{m-q} \int_{[\varphi > 0]} \varphi^r |R_m|^2.
\]
We applied Lemma 4.5 in the first line and the inductive hypothesis in the second line.

**Lemma 4.7.** Suppose \( M, \varphi \) satisfy the above hypotheses. Let \( 0 \leq i \leq \frac{n}{2} - 1 \) and \( p \geq n + 2k \). Then for every \( \delta > 0 \),
\[
\int_M P_{p-1}^n (\varphi) \ast \nabla^{i+k} R_m \ast \nabla^{\frac{n}{2}+k} R_m \leq C \delta \int_M \varphi^p |\nabla^{\frac{n}{2}+k} R_m|^2 + C \delta^{\frac{n-2i-4k}{n-2i}} \int_{[\varphi > 0]} \varphi^{p-n-2k} |R_m|^2,
\]
where \( C = C(n, k, p, \Lambda, i) \).

**Proof.** We apply the Cauchy-Schwarz inequality:
\[
\int_M P_{p-1}^n (\varphi) \ast \nabla^{i+k} R_m \ast \nabla^{\frac{n}{2}+k} R_m \leq C(\Lambda) \int_M |\varphi^{p-i-(\frac{n}{2}-i)} \ast \nabla^{i+k} R_m \ast \nabla^{\frac{n}{2}+k} R_m|
\]
\[
\leq C \epsilon^\beta \int_M \varphi^p |\nabla^{\frac{n}{2}+k} R_m|^2 + C \epsilon^{-\beta} \int_{[\varphi > 0]} \varphi^{p-n+2i} |\nabla^{i+k} R_m|^2.
\]
The second term can be estimated using Lemma 4.6
\[
\int_{[\varphi > 0]} \varphi^{p-n+2i} |\nabla^{i+k} R_m|^2 = \int_{[\varphi > 0]} \varphi^{2(i+k)+(p-n-2k)} |\nabla^{i+k} R_m|^2
\]
\[
\leq \epsilon^{\frac{n}{2}-i} \int_M |\nabla^{\frac{n}{2}+k} R_m|^2 + C \epsilon^{-i-k} \int_{[\varphi > 0]} \varphi^{p-n-2k} |R_m|^2.
\]
If \( \beta = \frac{n}{2} - i - \beta \), then \( \beta = \frac{n-2i}{4} \). If we set \( \delta = \epsilon^{\frac{n-2i}{2}} \), then \( \epsilon = \delta^{\frac{4}{n-2i}} \), and
\[
\epsilon^{-i-k} = \delta^{\frac{4}{n-2i}} (\frac{2i+n-i-k}{n-2i}) = \delta^{\frac{n-2i-4k}{n-2i}}.
\]
Therefore
\[
\int_M P_{\frac{\alpha}{2} - i}^\star (\varphi) \ast \nabla^{i + k} R_m \ast \nabla^{\frac{n}{2} + k} R_m \leq C\epsilon^\beta \int_M \varphi^p |\nabla^{\frac{n}{2} + k} R_m|^2 + C\epsilon^{-\beta - \frac{\alpha}{2} - i} \int_M |\nabla^{\frac{n}{2} + k} R_m|^2
+ C\epsilon^{-\beta - i - k} \int_{[\varphi > 0]} \varphi^{p - n - 2k} |R_m|^2
\leq C\delta \int_M \varphi^p |\nabla^{\frac{n}{2} + k} R_m|^2 + C\delta^{\frac{n - 2k}{n - 2k}} \int_{[\varphi > 0]} \varphi^{p - n - 2k} |R_m|^2.
\]

We estimate the penultimate term of \( (7) \).

**Lemma 4.8.** Suppose \( M, \varphi \) satisfy the above hypotheses. Let \( K = \max\{1, \|R_m\|_\infty\} \). If \( p \geq n + 2k \) and \( k \leq l \leq \frac{n}{2} + k - l \), then for every \( \delta \) satisfying \( 0 < \delta \leq 1 \),
\[
\int_M \varphi^p P_{\frac{\alpha}{2} + k - l + 2} (R_m) \leq C\delta \int_M \varphi^{p + n + 2k - 2l} |\nabla^{\frac{n}{2} + k} R_m|^2 + CK^{\frac{n}{2} + k} \delta^{\frac{2l}{21 - n - 2k}} \|R_m\|^2_{2,[\varphi > 0]},
\]
where \( C = C(n, k, p, \Lambda, l) \).

**Proof.** Since \( p \geq n + 2k \geq n + 2k - 2 = 2(\frac{n}{2} + k - 1) \), Proposition 4.4 implies
\[
\int_M \varphi^p P_{\frac{\alpha}{2} + k - l + 2} (R_m) \leq C\|R_m\|_\infty^{\frac{n}{2} + k - l} \left( \int_M \varphi^p |\varphi'| R_m|^2 + \|R_m\|^2_{2,[\varphi > 0]} \right).
\]
Let \( \epsilon = K^{-1} \delta^{\frac{2l}{21 - n - 2k}} \). We have \( p - 2l \geq n + 2k - (n + 2k - 1) = 1 \). Via Lemma 4.6
\[
C\|R_m\|_\infty^{\frac{n}{2} + k - l} \int_M \varphi^p |\nabla\varphi| R_m|^2 \leq CK^{\frac{n}{2} + k - l} \epsilon^{\frac{n}{2} + k - l} \int_M \varphi^{n + 2k + p - 2l} |\nabla^{\frac{n}{2} + k} R_m|^2
+ CK^{\frac{n}{2} + k - l} \epsilon^{-l} \int_{[\varphi > 0]} \varphi^{p - 2l} |R_m|^2
= C\delta \int_M \varphi^{n + 2k + p - 2l} |\nabla^{\frac{n}{2} + k} R_m|^2 + CK^{\frac{n}{2} + k} \epsilon^{\frac{2l}{21 - n - 2k}} \int_{[\varphi > 0]} \varphi^{p - 2l} |R_m|^2.
\]
Since \( k \leq l \leq \frac{n}{2} + k - l \) and \( 0 < \delta \leq 1 \), we get \( \delta^{\frac{2l}{21 - n - 2k}} \geq \delta^{-\frac{2k}{n}} \geq 1 \) and \( K^{\frac{n}{2} + k - l} \leq K^{\frac{n}{2}} \). Therefore
\[
\int_M \varphi^p P_{\frac{\alpha}{2} + k - l + 2} (R_m) \leq C\delta \int_M \varphi^{n + 2k + p - 2l} |\nabla^{\frac{n}{2} + k} R_m|^2 + CK^{\frac{n}{2} + k} \delta^{\frac{2l}{21 - n - 2k}} \int_{[\varphi > 0]} \varphi^{p - 2l} |R_m|^2
+ K^{\frac{n}{2} + k - l} \|R_m\|^2_{2,[\varphi > 0]}
\leq C\delta \int_M \varphi^{p + n + 2k - 2l} |\nabla^{\frac{n}{2} + k} R_m|^2 + CK^{\frac{n}{2} + k} \delta^{\frac{2l}{21 - n - 2k}} \|R_m\|^2_{2,[\varphi > 0]}.
\]

**Proposition 4.9.** Suppose \( M, \varphi \) satisfy the above hypotheses. Let \( K = \max\{1, \|R_m\|_\infty\} \). If \( p \geq n + 2k \), then for every \( \delta \) satisfying \( 0 < \delta \leq 1 \),
\[
\partial_t \|\varphi^\frac{p}{2} \nabla^k R_m\|^2_2 \leq -\frac{1}{2(n - 2)} \|\varphi^\frac{p}{2} \nabla^{\frac{n}{2} + k} R_m\|^2_2 + CK^{\frac{n}{2} + k} \|R_m\|^2_{2,[\varphi > 0]},
\]
where \( C = C(n, k, p, \Lambda) \).
Proof. Applying the estimates from Lemmas 4.8 and 4.7 to the equation (7) in Proposition 4.2 we obtain

\[
\partial_t \| \nabla^k Rm \|_2^2 \leq - \frac{1}{n-2} \| \nabla^\frac{n}{2+k} Rm \|_2^2 \\
+ \sum_{l=k}^{\frac{n}{2}+k-1} C_1 \delta \| \nabla^\frac{n}{2+k-l} \nabla^\frac{n}{2+k} Rm \|_2^2 + C_1 K \nabla^\frac{n}{2+k} \delta \| Rm \|_2^2,
\]

where \( C_1 = C_1(n, k, p, \Lambda, l) \) and \( C_2 = C_2(n, k, p, \Lambda_1, i) \). From the inequalities

\[
1 - n - 2k \leq 1 - \frac{2n + 4k}{n - 2i} \leq - \frac{n + 4k}{n}, \quad \frac{2 - n - 2k}{2} \leq 1 + \frac{n + 2k}{2l - n - 2k} \leq - \frac{2k}{n}
\]

we conclude

\[
\max \left\{ \{ \delta \frac{n}{2+k-l} : 0 \leq l \leq \frac{n}{2} + k - 1 \} \cup \{ \delta \frac{n-2i-4k}{n-2} : 0 \leq i \leq \frac{n}{2} - 1 \} \right\} = \delta^{1-n-2k}.
\]

Therefore

\[
\partial_t \| \nabla^k Rm \|_2^2 \leq - \frac{1}{n-2} \| \nabla^\frac{n}{2+k} Rm \|_2^2 + C \delta \| \nabla^\frac{n}{2+k} Rm \|_2^2 + \frac{C K n^{\frac{n}{2+k}} \delta^{1-n-2k} \| Rm \|_2^2,}
\]

where

\[
C \equiv \sum_{l=k}^{\frac{n}{2}+k-1} C_1 + \sum_{i=0}^{\frac{n}{2}-1} C_2, \quad \delta \equiv \min \left\{ \frac{n}{2(n-2)}, C \right\}.
\]

\[
\Box
\]

Proposition 4.10. Suppose \( M, \varphi \) satisfy the above hypotheses. Suppose \( \max \{ \| Rm \|_\infty, 1 \} \leq K \) for all \( t \in [0, \alpha K^{-\frac{n}{2}}] \). Then

\[
\| \nabla^{(m+1)} Rm \|_2 \leq C t^{-m} \sup \| Rm \|_{L^2(t, [\varphi > 0])},
\]

where \( C = C(m, n, \alpha, \Lambda) \), for all \( t \in (0, \alpha K^{-\frac{n}{2}}] \).

Proof. Define

\[
G(t) \equiv t^m \| \nabla^{(m+1)} Rm \|_2^2 + \sum_{k=0}^{m-1} \beta_k t^k \| \nabla^{(k+1)} Rm \|_2^2,
\]

where

\[
\beta_k = \frac{\gamma_k}{\alpha K^{\frac{n}{2}}}.
\]

\[
\Box
\]
Using Proposition 4.9
\[
\frac{dG}{dt} \leq mt^{m-1}\|\varphi^{(m+1)}\|_{2}^{2} \left(-\frac{1}{2(n-2)}\|\varphi^{(m+1)}\|_{2}^{2} + C_{\frac{n}{2}}mK^{\frac{n}{2}(m+1)}\|Rm\|_{2,[\varphi>0]}^{2}\right)
\]
\[
+ t^{m}\left(-\sum_{k=1}^{m-1} \beta_2k\|\varphi^{(k+1)}\|_{2}^{2} + C_{\frac{n}{2}}mK^{\frac{n}{2}(k+1)}\|Rm\|_{2,[\varphi>0]}^{2}\right)
\]
\[
+ \sum_{k=0}^{m-1} \beta_2k^{k-1}\|\varphi^{(k)}\|_{2}^{2} + C_{\frac{n}{2}}mK^{\frac{n}{2}(k+1)}\|Rm\|_{2,[\varphi>0]}^{2}\right)
\]
\[
\leq mt^{m-1}\|\varphi^{m}\|_{2}^{2} + t^{m}\left(C_{\frac{n}{2}}mK^{\frac{n}{2}(m+1)}\|Rm\|_{2,[\varphi>0]}^{2}\right)
\]
\[
+ \sum_{k=0}^{m-2} \beta_{k+1}(k+1)t^{k}\|\varphi^{(k+1)}\|_{2}^{2} + C_{\frac{n}{2}}mK^{\frac{n}{2}(k+1)}\|Rm\|_{2,[\varphi>0]}^{2}\right)
\]
\[
+ \sum_{k=0}^{m-1} \beta_2k^{k-1}\|\varphi^{(k)}\|_{2}^{2} + C_{\frac{n}{2}}mK^{\frac{n}{2}(k+1)}\|Rm\|_{2,[\varphi>0]}^{2}\right).
\]
If
\[
mt^{m-2}\|\varphi^{m}\|_{2}^{2} - \beta_{m-1}t^{m-1}\|\varphi^{m}\|_{2}^{2} = 0
\]
then \(\beta_{m-1} = 2(n-2)m\). If for \(k\) satisfying \(0 \leq k \leq m - 2\),
\[
\beta_{k+1}(k+1)t^{k}\|\varphi^{(k+1)}\|_{2}^{2} - \beta_2k^{k-1}\|\varphi^{(k)}\|_{2}^{2} = 0,
\]
then
\[
\beta_k = 2(n-2)(k+1)\beta_{k+1}
\]
\[
= (2n-4)^{m-k-1}(m-1)\cdots(k+1)\beta_{m-1}
\]
\[
= (2n-4)^{m-k}m!/k!.
\]
Also define \(\beta_m = 1\). Using these choices for \(\beta_k, 0 \leq k \leq m\) and choosing \(t_0 \in [0, \alpha K^{-\frac{1}{2}}]\) such that
\[
\|Rm\|_{L^2(t_0),[\varphi>0]} = \sup_{t \in [0, \alpha K^{-\frac{1}{2}}]}\|Rm\|_{L^2(t),[\varphi>0]},
\]
we have
\[
\frac{dG}{dt} \leq \alpha^m K^{-\frac{n}{2}}mC_{\frac{n}{2}}mK^{\frac{n}{2}(m+1)}\|Rm\|_{2,[\varphi>0]}^{2} + \sum_{k=0}^{m-1} \beta_k\alpha^k K^{-\frac{n}{2}}mC_{\frac{n}{2}}mK^{\frac{n}{2}(k+1)}\|Rm\|_{2,[\varphi>0]}^{2}
\]
\[
= \sum_{k=0}^{m} \beta_kC_{\frac{n}{2}}m\alpha^k K^{\frac{n}{2}}\|Rm\|_{2,[\varphi>0]}^{2}
\]
\[
= C\|Rm\|_{L^2(t_0),[\varphi>0]}^{2},
\]
Therefore
\[
\|\varphi^{(m+1)}\|_{2}^{2} \leq G \leq \beta_0\|Rm\|_{L^2(t_0),[\varphi>0]}^{2} + C\|Rm\|_{L^2(t_0),[\varphi>0]}^{2}t^t
\]
\[
\leq (\beta_0 + \alpha C)\|Rm\|_{L^2(t_0),[\varphi>0]}^{2},
\]
proving the proposition. \(\square\)
Proposition 4.11. Let \((M^n, g(t))\) be a solution to the AOF for \(t \in [0, T]\). Let \(\varphi \in C^\infty_c(M)\) be a cutoff function such that

\[
\max_{0 \leq i \leq \frac{n}{2}} \sup_{t \in [0, T]} \|\nabla^i \varphi\|_{C^0(M, g(t))} \leq \Lambda.
\]

Suppose \(\max\{\|R_m\|_{C^0(M, g(t))}, 1\} \leq K\) for all \(t \in [0, \alpha K^{-\frac{n}{2}}]\). Then, for every \(l \geq 0\) and all \(t \in (0, \alpha K^{-\frac{n}{2}}]\),

\[
\|\varphi^{l+\frac{n}{2}} \nabla^l R_m\|_{L^2(M, g(t))} \leq C(1 + t^{-[2l/n]/2}) \sup_{t \in [0, \alpha K^{-\frac{n}{2}}]} \|R_m\|_{L^2(\text{supp}(\varphi), g(t))},
\]

where \(C = C(l, n, \alpha, \Lambda)\).

Proof. Let \(l = \frac{n}{2} m + r, 1 \leq r \leq \frac{n}{2}\). Then, applying Lemma 4.6 and Proposition 4.10, we get

\[
\int_M \varphi^{n(m+1)+2r} |\nabla^r m + r R_m|^2 \leq \int_M \varphi^{n(m+2)} |\nabla^r m + 1 R_m|^2 + C' \int_{[\varphi > 0]} \varphi^n |R_m|^2
\]

\[
\leq t^{-(m+1)} C \Theta^2 + C'' \Theta^2
\]

\[
\|\varphi^{l+\frac{n}{2}} \nabla^l R_m\|_{L^2(t)} \leq \Theta(C t^{-\frac{m+1}{2}} + C'),
\]

where

\[
\Theta = \sup_{t \in [0, \alpha K^{-\frac{n}{2}}]} \|R_m\|_{L^2(t), [\varphi > 0]}.
\]

\[\square\]

5. Pointwise Smoothing Estimates

Let \((M, g(t))\) be a solution to AOF and let \(\varphi\) be a cutoff function on \(M\). We give estimates of \(|\nabla^i \varphi|_{g(t)}\) for \(1 \leq i \leq \frac{n}{2}\) that depend on spacetime derivatives of the metric and \(|\nabla^i \varphi|_{g(0)}\) for \(0 \leq i \leq \frac{n}{2}\). We then give a proof of the pointwise smoothing estimates given in Theorem 1.11.

Lemma 5.1. Let \(M\) be a manifold and \(g(t)\) be a one-parameter family of metrics on \(M\). For a function \(\varphi \in C^i(M)\) and \(i \geq 2\),

\[
\partial_t \nabla^i \varphi = \sum_{j=1}^{i-1} \nabla^{i-j} \partial_t g * \nabla^j \varphi.
\]

Proof. If \(i = 2\), the statement is true since via Proposition 3.3, we have

\[
\partial_t \nabla^2 \varphi = \nabla \partial_t \nabla \varphi + \nabla \partial_t g * \nabla \varphi = \nabla \partial_t g * \nabla \varphi.
\]
Suppose \( i > 2 \) and the statement is true for every \( j \) such that \( 2 \leq j \leq i - 1 \). Then, using Proposition 5.3 in the second line,

\[
\partial_t \nabla^i \varphi = \partial_t \nabla^{i-1} \varphi
= \nabla \partial_t \nabla^{i-1} \varphi + \nabla \partial_t g * \nabla^{i-1} \varphi
= \nabla \sum_{j=1}^{i-2} \nabla^{i-1-j} \partial_t g * \nabla^j \varphi + \nabla \partial_t g * \nabla^{i-1} \varphi
= \sum_{j=1}^{i-2} \nabla^{i-j} \partial_t g * \nabla^j \varphi + \nabla \sum_{j=1}^{i-2} \nabla^{i-1-j} \partial_t g * \nabla^j \varphi + \nabla \sum_{j=1}^{i-2} \nabla^{i-1-j} \partial_t g * \nabla^j \varphi
= \sum_{j=1}^{i-1} \nabla^{i-j} \partial_t g * \nabla^j \varphi.
\]

\( \square \)

**Proposition 5.2.** Let \( M \) be a manifold and \( g(t) \) be a one-parameter family of metrics on \( M \). For a function \( \varphi \in C^i(M) \) and \( i \geq 1 \),

\[
\partial_t |\nabla^i \varphi|_{g(t)}^2 = \sum_{j=1}^{i} \nabla^{i-j} \partial_t g * \nabla^j \varphi * \nabla^i \varphi.
\]

**Proof.** We compute, using the preceding Lemma 5.1 in the second line:

\[
\partial_t |\nabla^i \varphi|_{g(t)}^2 = \partial_t g * \nabla^i \varphi * 2 + \partial_t \nabla^i \varphi * \nabla^i \varphi
= \partial_t g * \nabla^i \varphi * 2 + \nabla^{i-j} \partial_t g * \nabla^j \varphi * \nabla^i \varphi
= \sum_{j=1}^{i} \nabla^{i-j} \partial_t g * \nabla^j \varphi * \nabla^i \varphi.
\]

\( \square \)

**Proposition 5.3.** Let \( M \) be a Riemannian manifold with a one-parameter family of metrics \( \{g(t)\}_{t \in [0,T]} \) and \( \varphi \in C^\infty_c(M) \). Fix \( i \geq 1 \). Suppose that, for each \( j \) satisfying \( 0 \leq j \leq i - 1 \), there exists \( K_j > 0 \) such that \( |\nabla^j \partial_t g(x,t)|_{g(t)} \leq K_j \) on \( \text{supp} \varphi \times [0,T] \) and, for each \( j \) satisfying \( 1 \leq j \leq i \), there exists \( C_j' > 0 \) such that \( |\nabla^j \varphi|_{g(0)} \leq C_j' \) on \( \text{supp} \varphi \). Then there exists a constant \( C_i \) such that

\[
|\nabla^i \varphi|_{g(t)}^2 \leq C_i = C_i(K_0, \ldots, K_{i-1}, C_1', \ldots, C_i', T).
\]

**Proof.** Let \( i = 1 \). Then Proposition 5.2 gives

\[
\partial_t |\nabla \varphi|_{g(t)}^2 = \partial_t g * \nabla \varphi * 2 \leq C K_0 |\nabla \varphi|_{g(t)}^2.
\]

Solving the differential inequality, we get

\[
|\nabla \varphi|_{g(t)}^2 \leq |\nabla \varphi|_{g(0)}^2 e^{CT} \equiv C_1^2
\]

which proves the proposition for \( i = 1 \).
Fix \( i \geq 2 \) and suppose that the proposition is true for every \( j \) satisfying \( 1 \leq j \leq i - 1 \). Let
\[
 f(t) = |\nabla^i \varphi|_{g(t)}^2.
\]
Then, via Proposition 5.2,
\[
 \frac{df}{dt} \leq \sum_{j=1}^{i} |\nabla^{i-j} \partial_1 g \ast \nabla^j \varphi \ast \nabla^i \varphi|
\]
\[
 \leq \sum_{j=1}^{i-1} |\nabla^{i-j} \partial_1 g||\nabla^j \varphi||\nabla^i \varphi| + |\partial_1 g||\nabla^i \varphi|^2
\]
\[
 \leq \sum_{j=1}^{i-1} C K_{i-j} C_j f^j + C K_0 f
\]
\[
 \leq \tilde{C}(K_0, \ldots, K_{i-1}, C_1, \ldots, C_{i-1})(1 + f)
\]
\[
 = \tilde{C}(K_0, \ldots, K_{i-1}, C'_1, \ldots, C'_{i-1}, T)(1 + f).
\]
Solving the differential inequality, we get
\[
 1 + f(t) \leq (1 + f(0)) \tilde{C} t
\]
\[
 |\nabla^i \varphi|_{g(t)}^2 \leq (1 + |\nabla^i \varphi|_{g(0)}^2) e^{\tilde{C} T}
\]
\[
 \leq (1 + (C'_i)^2) e^{\tilde{C} T} \equiv C''_i.
\]

**Proposition 5.4.** Let \((M^n, g(t))\) solve AOF on \([0, T]\), where \( n \geq 4 \). Fix \( r > 0 \). Suppose there exist \( x \in M, r > 0, \) and \( K > 0 \) such that
\[
 \max_{0 \leq j \leq \frac{n}{2}} \|\nabla^j \varphi\|_{C^0(M, g(T))} \leq C'(n, K, \frac{1}{T}).
\]

Then for all \( l \geq 0 \) and \( t \in (0, T] \),
\[
 \|\nabla^l Rm\|_{L^2(B_g(T)(x, 2r), g(t))} \leq C(1 + t^{-\lceil 2l/n \rceil/2}) \sup_{t \in [0, T]} \|Rm\|_{L^2(B_g(T)(x, 2r), g(t))},
\]
where \( C = C(n, l, K, T, \frac{1}{T}) \).

**Proof.** Let \( \varphi \) be a cutoff function that is equal to 1 on \( B_g(T)(x, r) \) and supported on \( B_g(T)(x, 2r) \). The inequality \((9)\) provides \( C^0 \) bounds for the first \( \frac{n}{2} - 2 \) covariant derivatives of \( Rm \), so that
\[
 \max_{0 \leq j \leq \frac{n}{2}} \|\nabla^j \varphi\|_{C^0(M, g(T))} \leq C'(n, K, \frac{1}{T}).
\]

The inequality \((11)\) provides bounds for the first \( \frac{n}{2} \) covariant derivatives of \( \varphi \) at time \( T \), and the inequality \( (9) \) includes bounds on the first \( \frac{n}{2} - 1 \) covariant derivatives of \( \partial_1 \). We therefore are able to, for each \( t \in [0, T] \) and \( j \) satisfying \( 0 \leq j \leq \frac{n}{2} \), to obtain via Proposition 5.3 bounds given by
\[
 \|\nabla^j \varphi\|_{C^0(M, g(t))} \leq \tilde{C}_j(n, K, \frac{1}{T}, T).
\]

Therefore, via Proposition 5.3
\[
 \|\nabla^l Rm\|_{L^2(B_g(T)(x, r), g(t))} \leq \|\varphi^{1 + \frac{l}{2}} \|\nabla^l Rm\|_{L^2(M, g(t))}
\]
\[
 \leq C(1 + t^{-\lceil 2l/n \rceil/2}) \sup_{t \in [0, T]} \|Rm\|_{L^2(B_g(T)(x, 2r), g(t))}
\]
\[
 = C(1 + t^{-\lceil 2l/n \rceil/2}) \sup_{t \in [0, T]} \|Rm\|_{L^2(B_g(T)(x, 2r), g(t))}.
\]
where $C = C(n, l, K, T, \frac{1}{r})$. □

We are now able to prove the pointwise smoothing estimates given in Theorem 1.1.

**Proof of Theorem 1.1.** We adapt the proof of Theorem 1.3 in Streets [38]. We will show that if this inequality fails, we can construct a blowup limit that is flat and has nonzero curvature. Consider the function given by

$$f_m(x, t, g) = \sum_{j=1}^{m} |\nabla^j Rm(g(x, t))|_{g(t)}^{\frac{2}{j+2}} + 2g(t).$$

It suffices to show that

$$f_m(x, t, g) \leq C \left( K + \frac{1}{t^{\frac{2}{n}}} \right)$$

since for every $l$ satisfying $1 \leq l \leq m$,

$$|\nabla^l Rm(g(x, t))|_{g(t)}^{\frac{2}{l+2}} \leq \sum_{j=1}^{m} |\nabla^j Rm(g(x, t))|_{g(t)}^{\frac{2}{j+2}} = f_m(x, t, g) \leq C \left( K + \frac{1}{t^{\frac{2}{n}}} \right)$$

and

$$|\nabla^l Rm(g(x, t))|_{g(t)} \leq C \left( K + \frac{1}{t^{\frac{2}{n}}} \right)^{\frac{l+2}{2}} \leq C \left( K + \frac{1}{t^{\frac{2}{n}}} \right)^{\frac{m+2}{2}}.$$

Suppose that the inequality (12) fails. It suffices to take

$$m \geq \frac{3n}{2} - 3.$$  

Without loss of generality, for each $i \in \mathbb{N}$ there exists a solution to AOF $(M^n_i, g_i(t))$ and $(x_i, t_i) \in M_i \times (0, T]$ such that

$$i < f_m(x_i, t_i, g_i) = \frac{\sum_{j=1}^{m} |\nabla^j Rm(g(x_i, t_i))|_{g_i(t_i)}^{\frac{2}{j+2}} + 2g_i(t_i)}{K + t_i^{-\frac{2}{n}}} = \sup_{M_i \times (0, T]} f_m(x, t, g) < \infty.$$  

and define a new sequence of blown up metrics by

$$\tilde{g}_i(t) = \lambda_i g_i(t_i + \lambda_i^{-\frac{2}{n}} t_i),$$

where $\lambda_i = f_m(x_i, t_i, g_i)$. We will show in the next section that these metrics also solve AOF. These metrics, which are defined for $t \in [-\lambda_i^{-\frac{2}{n}} t_i, 0]$, are eventually defined on $[-1, 0]$ since as $i \to \infty$,

$$t_i^2 \lambda_i = \frac{f_m(x_i, t_i, g_i)}{t_i^{-\frac{2}{n}}} \geq \frac{f_m(x_i, t_i, g_i)}{K + t_i^{-\frac{2}{n}}} \to \infty.$$  

Replace the sequence of AOF solutions $\{(M_i, \tilde{g}_i(t))\}_{i \in \mathbb{N}}$ with the tail subsequence for which $\lambda_i^2 t_i > 1$. The curvatures of these manifolds converge to 0 since as $i \to \infty$,

$$|\text{Rm}(\tilde{g}_i)|_{\tilde{g}_i} \leq \frac{K}{\lambda_i} = \frac{K}{f_m(x_i, t_i, g_i)} \leq \frac{K + t_i^{-\frac{2}{n}}}{f_m(x_i, t_i, g_i)} \to 0.$$  

\[25]
Furthermore, there is a uniform $C^m$ estimate on the curvature given by

$$f_m(x, t, \tilde{g}_i) = \frac{f_m(x_i, t_i + t\lambda_i^{-\frac{n}{2}}, g_i)}{\lambda_i}$$

$$= \frac{f_m(x_i, t_i + t\lambda_i^{-\frac{n}{2}}, g_i)}{f_m(x_i, t_i, g_i)}$$

$$\leq \frac{K + (t_i + t\lambda_i^{-\frac{n}{2}})^{-\frac{2}{n}}}{K + t_i^{-\frac{n}{2}}}$$

$$\leq \frac{K + t_i^{-\frac{n}{2}}(1 + \frac{1}{2})^{-\frac{2}{n}}}{K + t_i^{-\frac{n}{2}}}$$

$$\leq 2^2$$

for all $i \in \mathbb{N}$ and $(x, t) \in M_i \times [-1, 0]$.

Let $\varphi_i : B(0, 1) \to M_i$ be given by $\exp_{x_i}$ with respect to $g_i(0)$ for each $i \in \mathbb{N}$ and $h_i(t) \equiv \varphi_i^* g_i(t)$. The uniform $C^0$ bound on $\text{Rm}(\tilde{g}_i(t))$ given by (14) induces a uniform bound on $(\varphi_i)_*$ (see Petersen [35]) which permits the uniform $C^m$ estimate on $\text{Rm}(\tilde{g}_i(t))$ to lift to a uniform $C^m$ estimate on $\text{Rm}(h_i(t))$. Furthermore, $h_i(t)$ solves AOF for all $i$ since $\varphi_i$ does not depend on $t$.

Since $m \geq \frac{3n}{2} - 3$, we have uniform $C^0$ bounds on $\nabla^j \hat{O}(g(t))$ for $0 \leq j \leq \frac{n}{2} - 1$. Via Proposition 6.4, we obtain uniform bounds on the $L^2(B_{h_i}(0)(0, \frac{1}{2}))$-norms of all covariant derivatives of $\text{Rm}(h_i(t))$. Since the metrics $h_i(0)$ are uniformly equivalent to the Euclidean metric, the Sobolev constant of $B_{h_i(0)}(0, \frac{1}{2})$ is uniformly bounded for all $i$. Via the Kondrakov compactness theorem, we thus obtain uniform bounds on the $C^0(B_{h_i(0)}(0, \frac{1}{2}))$-norms of all covariant derivatives of $\text{Rm}(h_i(t))$. The Taylor expansion for $\tilde{h}_i$ in terms of geodesic coordinates about 0 with curvature coefficients can then be used to obtain uniform bounds on the $C^0(B_{h_i(0)}(0, \frac{1}{2}))$-norms of all covariant derivatives of $h_i(0)$. Finally, by the Arzelà-Ascoli - type Proposition 7.10 after taking a subsequence, still named $\{h_i(0)\}_{i \in \mathbb{N}}$, we get $h_i(0) \to h_\infty$ in $C^\infty(B(0, \frac{1}{2}))$ for some Riemannian metric $h_\infty$. We have already shown with inequality (13) that $(B(0, \frac{1}{2}), h_\infty)$ is flat. However, for all $i \in \mathbb{N}$,

$$f_m(x_i, 0, g_i) = \sum_{j=1}^{m} \left| \nabla^j \text{Rm}(\tilde{g}_i)(x_i, 0) \right|^{\frac{2}{2j}}$$

$$= \sum_{j=1}^{m} \left( \lambda_i^{-\frac{2}{j}} \left| \nabla^j \text{Rm}(x_i, t_i) \right|_{g(t_i)} \right)^{\frac{2}{2j}}$$

$$= \sum_{j=1}^{m} \lambda_i^{-1} \left| \nabla^j \text{Rm}(x_i, t_i) \right|_{g(t_i)}^{\frac{2}{2j}}$$

$$= \lambda_i^{-1} \lambda_i = 1.$$ 

Also, $f_m(0, 0, h_i) = 1$ for all $i$ since $(\varphi_i)_*$ is the identity map at 0. Therefore $f_m(0, 0, h_\infty) = 1$. This is a contradiction, thereby proving the the inequality (12).

6. LONG TIME EXISTENCE

In this section, we prove that if a solution $(M, g(t))$ to the AOF only exists for a finite time $T$, then $\|\text{Rm}\|_\infty$ becomes unbounded along a sequence $\{(x_n, t_n)\}_{n=1}^{\infty} \subset M \times [0, T)$ with $t_n \uparrow T$. We
Lemma 6.2. \( \|g\|_{C^0(T)} = K < \infty \),
then the solution \( g(t) \) exists past the time \( T \). In order to show this, we show that \( \|\nabla^k Rm\|_{g(t)} \) and the pointwise smoothing estimates on \( |\nabla^k g(t)|_g \) induce bounds on \( |\nabla^k g(t)|_g \) with respect to some fixed background metric \( g \) and connection \( \nabla \). We also show that \( \|\nabla^k g(t)\|_g \) imply that \( g(T) \) is smooth, so that we can extend the solution \( g(t) \) past the time \( T \) via the short time existence theorem 3.6.

We first show that if \( \|\nabla^k g(t)\|_g \) holds, the metrics \( g(t) \) converge uniformly as \( t \to T \) to a continuous metric \( g(T) \) equivalent to each \( g(t) \). The following lemma is from Chow-Knopf [13];

**Lemma 6.1.** Let \( M \) be a closed manifold. For \( 0 \leq t < T \leq \infty \), let \( g(t) \) be a one-parameter family of metrics on \( M \) depending smoothly on both space and time. If there exists a constant \( C < \infty \) such that

\[
\int_0^T \left| \frac{\partial}{\partial t} g(x,t) \right|_{g(t)} dt \leq C
\]

for all \( x \in M \), then

\[
e^{-C} g(x,0) \leq g(x,t) \leq e^C g(x,0)
\]

for all \( x \in M \) and \( t \in [0,T) \). Furthermore, as \( t \to T \), the metrics \( g(t) \) converge uniformly to a continuous metric \( g(T) \) such that for all \( x \in M \),

\[
e^{-C} g(x,0) \leq g(x,T) \leq e^C g(x,0)
\]

**Lemma 6.2.** Let \( M \) be a compact manifold and let \( (M,g(t)) \) be a solution to AOF on \( [0,T) \) such that

\[
\sup_{t \in [0,T)} \|Rm\|_{C^0(g(t))} = K < \infty.
\]

Then \( g(t) \) converges uniformly as \( t \to T \) to a continuous metric \( g(T) \) that is uniformly equivalent to \( g(t) \) for every \( t \in [0,T] \).

**Proof.** Since Proposition 2.3 states that

\[
\frac{\partial g}{\partial t} = \frac{(-1)^{n/2}}{n-2} \Delta^{n/2-1} R + \frac{(-1)^{n-1}}{2(n-1)} \Delta^{n/2-2} R + \sum_{j=2}^{n/2} P_{j}^{n-2j}(Rm),
\]

in order to apply the preceding Lemma 6.1 it suffices to show that \( |\nabla^k Rm|_{g(t)} \) is bounded on \( M \times [0,T) \) for all \( k \) satisfying \( 0 \leq k \leq n-2 \). Using the smoothing estimate provided in Theorem 1.1, we get

\[
\max_{0 \leq k \leq n-2} \sup_{M \times [0,T)} |\nabla^k Rm|_{g(t)} \leq \max_{0 \leq k \leq n-2} \sup_{M \times [0,T]} |\nabla^k Rm|_{g(t)} + C(K + \left(\frac{\bar{K}}{\bar{T}}\right)^2)^{n/2},
\]

where \( C = C(n) \) and \( \bar{K} = \max\{K, 1\} \).

So \( \frac{\partial g}{\partial t} \) is bounded on \( M \times [0,T) \) and the metrics \( g(t) \) converge uniformly as \( t \to T \) to a continuous metric \( g(T) \) uniformly equivalent to each \( g(t) \). \( \square \)

Since \( M \) is a compact manifold, we can obtain bounds on \( |\nabla^k g(t)|_g \) by taking the maximum of bounds taken on finitely many coordinate patches. On such a coordinate patch, we can assume that the fixed metric is just the Euclidean one. Thus we will only need to bound the partial derivatives of \( g \) and \( \bar{O} \).
Lemma 6.3. Let $M$ be a compact manifold and let $(M,g(t))$ be a solution to AOF on $[0,T)$. Fix $m \geq 0$. Suppose that for $0 \leq i \leq m+n-1$, there exist constants $C_i$ such that $|\nabla^i_{g(t)}\text{Rm}(g(t))|_{g(t)} \leq C_i$ on $M \times [0,T)$. Then for all $t \in [0,T)$,

$$
|\partial^m g(t)|_{g(t)} < C_1(g(0),C_0,\ldots,C_{m+n-1})
$$

$$
|\partial^m \tilde{O}(t)|_{g(t)} < C_2(g(0),C_0,\ldots,C_{m+n-1}).
$$

Proof. We prove this by induction. First we bound $\partial t \partial\tilde{g}$. We have

$$
\partial_t \partial\tilde{g} = \partial\partial g = (\nabla + \Gamma) * \partial\tilde{g} = \nabla \tilde{O} + \Gamma * \tilde{O}.
$$

From the definition of $\tilde{O}$, we obtain the bound $\nabla \tilde{O} < C(C_0,\ldots,C_{n-1})$. Then, since $\partial_t \Gamma = \nabla \partial_t g = \nabla \tilde{O}$, $\Gamma$ can be bounded in terms of the initial metric and $\nabla \tilde{O}$ after integrating. So $\partial_t \tilde{O} = \partial_t \partial\tilde{g}$ is uniformly bounded by $C(g(0),C_0,\ldots,C_{n-1})$, and so is $\partial\tilde{g}$ after integrating.

Assume that

$$
|\partial^i \tilde{g}| < C(g(0),C_0,\ldots,C_{i+n-1}) \text{ for } 0 \leq i \leq m-1,
$$

$$
|\partial^i \tilde{O}| < C(g(0),C_0,\ldots,C_{i+n-1}) \text{ for } 0 \leq i \leq m-1,
$$

$$
|\partial^i \Gamma| < C(g(0),C_0,\ldots,C_{i+n-1}) \text{ for } 0 \leq i \leq m-2.
$$

We wish to bound $\partial^m \tilde{g}$. It suffices to bound $\partial^m \tilde{O}$ since $\partial_t \partial^m g = \partial^m \partial_t g = \partial^m \tilde{O}$. We can express $\partial^m \tilde{O}$ as

$$
(16) \quad \partial^m \tilde{O} = \nabla^m \tilde{O} + \sum_{i=0}^{m-1} \partial^i \tilde{O} * \mathbb{P}^{m-i}(\Gamma),
$$

where $\mathbb{P}^k(A)$ is defined to be some polynomial in $A$ such that for each term the sum of the number of partial derivatives of $g$ in each factor is at most $k$. The following is a proof by induction. First, the equation holds when $m = 1$: $\partial \tilde{O} = (\nabla + \Gamma) * \tilde{O} = \nabla \tilde{O} + \Gamma * \tilde{O}$. Assume the equation (16) holds for $0 \leq i \leq m$. Then

$$
\nabla^{m+1} \tilde{O} = (\partial + \Gamma) \nabla^m \tilde{O}
$$

$$
= \partial^{m+1} \tilde{O} + \partial^m \tilde{O} * \mathbb{P}^1(\Gamma) + \sum_{i=0}^{m-1} \left[ \partial^{i+1} \tilde{O} * \mathbb{P}^{m-i}(\Gamma) + \partial^i \tilde{O} * \mathbb{P}^{m+1-i}(\Gamma) + \Gamma * \partial^i \tilde{O} * \mathbb{P}^{m-i}(\Gamma) \right]
$$

$$
= \partial^{m+1} \tilde{O} + \partial^m \tilde{O} * \mathbb{P}^1(\Gamma) + \sum_{i=0}^{m-1} \left[ \partial^{i+1} \tilde{O} * \mathbb{P}^{m-i}(\Gamma) + \partial^i \tilde{O} * \mathbb{P}^{m+1-i}(\Gamma) \right]
$$

$$
= \partial^{m+1} \tilde{O} + \partial^m \tilde{O} * \mathbb{P}^1(\Gamma) + \sum_{i=1}^{m} \partial^i \tilde{O} * \mathbb{P}^{m+1-i}(\Gamma) + \sum_{i=0}^{m-1} \partial^i \tilde{O} * \mathbb{P}^{m+1-i}(\Gamma)
$$

$$
= \partial^{m+1} \tilde{O} + \sum_{i=0}^{m} \partial^i \tilde{O} * \mathbb{P}^{m+1-i}(\Gamma).
$$

From the equation (16), we see that in order to bound $\partial^m \tilde{O}$, we only need to bound $\partial^{m-1} \Gamma$. We have

$$
(17) \quad \partial_t \partial^{m-1} \Gamma = \partial^{m-1} \partial_t \Gamma = \sum_{i=0}^{m-1} \partial^i \nabla \tilde{O}.
$$
We bound $\partial^i \nabla \hat{O}$ via the equation

$$\partial^i \nabla \hat{O} = \nabla^{i+1} \hat{O} + \sum_{j=1}^{i} \nabla^j \hat{O} \ast P^{i-j+1}(\Gamma).$$

(18)

In order to verify this via induction, we have that for $i = 1$, $\partial \nabla \hat{O} = \nabla^2 \hat{O} + \Gamma \ast \nabla \hat{O}$. If the equation holds for the $i$th partial derivatives of $\Gamma$, then the highest partial derivative of $\Gamma$ that appears in equation (18) is of order $-2$, so $\nabla^2 \hat{O} + \Gamma \ast \nabla \hat{O}$.

If $0 \leq i \leq m - 1$, then the highest partial derivative of $\Gamma$ that appears in equation (18) is of order at most $m - 2$, so $\partial^i \nabla \hat{O}$ is bounded in terms of covariant derivatives of $\nabla \hat{O}$ and previously bounded partial derivatives of $\Gamma$. Therefore, via equation (14), $\partial^{m-1} \Gamma$ and $\partial^m \hat{O}$ are bounded.

Proof of Theorem 1.2: Suppose that equation (15) holds. By Lemma 6.2, the metrics $g(t)$ converge uniformly to a continuous metric $g(T)$ as $t \uparrow T$. We show that $g(T)$ is $C^\infty$ on $M$. It suffices to show for each $k \in \mathbb{N}$ that $g(T)$ is $C^k$ on any coordinate patch since we can take a maximum over finitely many of them to show that $g(T)$ is $C^k$ on $M$. We have

$$g(t) = g(0) + \int_0^t \nabla \hat{O}(\tau) \, d\tau.$$

Taking limits as $t \uparrow T$, we get

$$g(T) = g(0) + \int_0^T \nabla \hat{O}(\tau) \, d\tau.$$

This permits us to take the $k$th partial derivative:

$$\partial^k g(T) = \partial^k g(0) + \int_0^T \partial^k \nabla \hat{O}(\tau) \, d\tau.$$

The bounds on $\partial^k g$ and $\partial^k \nabla \hat{O}$ from Lemma 6.3 therefore imply a bound on $\partial^k g(T)$. So $g(T)$ is $C^\infty$ on $M$. Furthermore, since

$$|\partial^k g(T) - \partial^k g(t)| \leq \int_t^T |\partial^k \nabla \hat{O}(\tau)| \, d\tau \leq C_k(T - t),$$

the metrics $g(t)$ converge in $C^\infty$ to $g(T)$. So $g(t)$ is a $C^\infty$ solution to AOF on $[0, T]$. Then the short time existence Theorem 3.6 applied to $g(t)$ with initial metric $g(T)$ allows us to extend $g(t)$ past $T$. This contradicts the assumption that $T$ was the maximal time for the solution $(M, g(t))$. \qed
7. Compactness of Solutions

In this section, we give compactness results for AOF similar to Hamilton’s compactness theorem for solutions of the Ricci flow. We first prove a proposition that states that for a sequence of metrics, uniform bounds on the spacetime derivatives of curvature and the derivatives of the metric at one time extend to uniform bounds on the spacetime derivatives of the metric. This is used to prove the compactness Theorem \[ \text{[1.3]} \] for a sequence of complete pointed solutions of AOF. We then give the proofs of Theorem 1.4, which allows us to obtain a singularity model from a singular solution, and Theorem \[ \text{[1.5]} \] which describes the behavior at time \( \infty \) of a nonsingular solution.

We quote some definitions and results from Chow et. al.’s text \[ \text{[14]} \] on Ricci flow.

**Definition 7.1.** (\[14\] Definition 3.1) Let \( K \subset M \) be a compact set and let \( \{ g_k \}_{k \in \mathbb{N}}, g_\infty, \) and \( g \) be Riemannian metrics on \( M \). For \( p \in \{0\} \cup \mathbb{N} \) we say that \( g_k \) converges in \( C^p \) to \( g_\infty \) uniformly on \( K \) if for every \( \epsilon > 0 \) there exists \( k_0 = k_0(\epsilon) \) such that for \( k \geq k_0 \),

\[
\sup_{0 \leq |x| \leq p \in K} |\nabla^p g_k - g_\infty|_g < \epsilon.
\]

**Definition 7.2.** (\[14\] Definition 3.5) \((C^\infty\text{-Cheeger-Gromov convergence})\) A sequence \( \{(M^n_k, g_k, O_k)\}_{k \in \mathbb{N}} \) of complete pointed Riemannian manifolds converges (in the Cheeger-Gromov topology) to a complete pointed Riemannian manifold \((M^n_\infty, g_\infty, O_\infty)\) if there exist

1. an exhaustion \( \{ U_k \}_{k \in \mathbb{N}} \) of \( M_\infty \) by open sets with \( O_\infty \subset U_k \),
2. a sequence of diffeomorphisms \( \Phi_k : U_k \rightarrow V_k := \Phi_k(U_k) \subset M_k \) with \( \Phi_k(O_\infty) = O_k \), such that \( (U_k, \Phi_k^* [g_k|_{V_k}]) \) converges in \( C^\infty \) to \( (M_\infty, g_\infty) \) uniformly on compact sets in \( M_\infty \).

**Definition 7.3.** (\[14\] Definition 3.6) A sequence \( \{(M^n_k, g_k(t), O_k)\}_{k \in \mathbb{N}} \) of one-parameter families of complete pointed Riemannian manifolds converges to a one-parameter family of complete pointed Riemannian manifolds \((M^n_\infty, g_\infty(t), O_\infty), t \in (\alpha, \omega)\), if there exist

1. an exhaustion \( \{ U_k \}_{k \in \mathbb{N}} \) by open sets with \( O_\infty \subset U_k \)
2. a sequence of diffeomorphisms \( \Phi_k : U_k \rightarrow V_k := \Phi_k(U_k) \subset M_k \) with \( \Phi_k(O_\infty) = O_k \), such that \( (U_k \times (\alpha, \omega), \Phi_k^* [g_k(t)|_{V_k}]) \subset (M_\infty \times (\alpha, \omega), g_\infty(t) + dt^2) \) converges in \( C^\infty \) on compact subsets in \( M_\infty \times (\alpha, \omega) \).

**Theorem 7.4.** \((\text{Cheeger-Gromov compactness theorem}) \) \((\text{Hamilton, [21]} \text{ Theorem 2.3})\) Let \( \{(M^n_k, g_k, O_k)\}_{k \in \mathbb{N}} \) be a sequence of complete pointed Riemannian manifolds that satisfy

\[
|\nabla^p_k Rm_k|_{k} \leq C_p \text{ on } M_k
\]

for all \( p \geq 0 \) and \( k \) where \( C_p < \infty \) is a sequence of constants independent of \( k \) and \( \text{inj}_{g_k}(O_k) \geq \iota_0 \)

for some constant \( \iota_0 > 0 \). Then there exists a subsequence \( \{j_k\}_{k \in \mathbb{N}} \) such that \( \{M_{j_k}, g_{j_k}, O_{j_k}\}_{k \in \mathbb{N}} \) converges to a complete pointed Riemannian manifold \((M^n_\infty, g_\infty, O_\infty) \) as \( k \rightarrow \infty \).

The following proposition allows us to extend bounds on the derivatives of a sequence of metrics at one time to bounds that are uniform over an interval.

**Proposition 7.5.** Let \((M, g)\) be a Riemannian manifold and \( L \) be a compact subset of \( M \). Let \( \{g_i\}_{i \in \mathbb{N}} \) be a collection of Riemannian metrics that are solutions of AOF on neighborhoods containing \( L \times [\beta, \psi] \). Let \( t_0 \in [\beta, \psi] \) and fix \( k \geq n - 2 \). Let unmarked objects such as \( \nabla \) and \( | \cdot | \) be taken with respect to \( g \), and let objects such as \( \nabla_k \) and \( | \cdot |_k \) be taken with respect to \( g_k \). Suppose that:

1. The metrics \( g_i(t_0) \) are uniformly equivalent to \( g \) for every \( i \in \mathbb{N} \): for some \( B_0 > 0 \), \( B_0^{-1} g \leq g_i(t_0) \leq B_0 g \).
2. For each \( 1 \leq p \leq k \), there exists a uniform bound \( C_p \) on \( L \) independent of \( i \) such that \( |\nabla^p g_i(t_0)| \leq C_p \).
(3) For each $0 \leq p + q \leq k + n - 2$, there exists a uniform bound $C'_{p,q}$ on $L \times [\beta, \psi]$ independent of $i$ such that $|\partial_i^p \nabla_i^q \text{Rm}(g_i)|_{g_i} \leq C'_{p,q}$.

Then:

1. The metrics $g_k(t)$ are uniformly equivalent to $g$ for every $i \in \mathbb{N}$ and $t \in [\beta, \varphi]$; for some $B = B(t, t_0) > 0$, $B^{-1} g \leq g_k(t) \leq Bg$.
2. For every $p, q$ satisfying $0 \leq p + q \leq k$, there is a uniform bound $\tilde{C}_{p,q}$ on $L \times [\beta, \psi]$ independent of $i$ such that $|\partial_i^p \nabla_i^q g_i| \leq \tilde{C}_{p,q}$.

**Lemma 7.6.** The metrics $g_k(t)$ in the above proposition are uniformly equivalent to $g$ on $L \times [\beta, \psi]$:

$$B(t, t_0)^{-1} g(V, V) \leq g_k(t)(V, V) \leq B(t, t_0) g(V, V).$$

**Proof.** We show that $|\partial_t \log g_k(t)(V, V)|$ is bounded uniformly in $k$. Fix $k \in \mathbb{N}$. First,

$$\left| \frac{\partial}{\partial t} \log g_k(t)(V, V) \right| \leq \left| \frac{\partial}{\partial t} g_k(t)(V, V) \right|.$$

Since the numerator and denominator are bilinear, it suffices to show the above is bounded when $g_k(V, V) = 1$, in which case the right hand side reduces to $|\partial_t g_k(t)(V, V)|$. In order to show this is bounded, we use the flow equation (4) and the expression for the gradient given by (2).

$$\left| \frac{\partial}{\partial t} g_k(t)(V, V) \right| \leq \left| \frac{\partial}{\partial t} g_k(t)(V, V) \right| = \left| \frac{\partial}{\partial t} g_k(t)(V, V) \right|.$$

Then,

$$\tilde{C}_0 |t_1 - t_0| \geq \int_{t_0}^{t_1} \left| \frac{\partial}{\partial t} \log g_k(t)(V, V) \right| dt,$$

which yields

$$e^{-\tilde{C}_0 |t_1 - t_0|} g_k(t_0)(V, V) \leq g_k(t_1)(V, V) \leq e^{\tilde{C}_0 |t_1 - t_0|} g_k(t_0)(V, V),$$

$$B_0^{-1} e^{-\tilde{C}_0 |t_1 - t_0|} g(V, V) \leq g_k(t_1)(V, V) \leq B_0 e^{\tilde{C}_0 |t_1 - t_0|} g(V, V).$$

\[\square\]

**Lemma 7.7.** (Chow et al. [14] Lemma 3.13). Suppose that the metrics $g$ and $h$ are equivalent: $C^{-1} g \leq h \leq Cg$. Then for any $(p, q)$-tensor $T$, we have $|T|_h \leq C' (p+q)/2 |T|_g$. 

We will need the following two lemmas in the next proof.
Lemma 7.8. (Chow et al. [14] Lemma 3.11) Let \((M, g)\) be a Riemannian manifold, and let \(\{g_k(t)\}_{k \in \mathbb{N}}\) be a collection of metrics on \(M\). Then for each \(k\), \(\nabla g_k(t)\) and \(\Gamma_k(t) - \Gamma\) are equivalent:
\[
\frac{1}{2} |\nabla g_k(t)|_k \leq |\Gamma_k(t) - \Gamma|_k \leq \frac{3}{2} |\nabla g_k(t)|_k.
\]

Lemma 7.9. For every \(p, q \geq 0\), there is a constant \(\tilde{C}_{p,q}\) independent of \(k\) such that \(|\partial_t \nabla^p g_k(t)| \leq \tilde{C}_{p,q}\) on \(L \times [\beta, \psi]\).

Proof. Define the bounds \(\overline{C}_j\) for \(j\) satisfying \(0 \leq j \leq j - n + 2\) by
\[
|\nabla^j \partial_k| \leq \sum_{p=j}^{n-2+j} \alpha_p \overline{C}_p,0 \equiv \overline{C}_j.
\]

We first prove the lemma for \((p, q) = (1, 0)\). Hamilton showed in Theorem 7.1 of [20] that \(\partial_t \Gamma = g^{-1} \nabla \partial_t g\). Then
\[
|\partial_t (\Gamma_k - \Gamma)|_k \leq C|\nabla_k \partial_k|_k \leq C \overline{C}_1.
\]

So
\[
C \overline{C}_1 |t_1 - t_0| \geq \int_{t_0}^{t_1} |\partial_t (\Gamma_k(t) - \Gamma)|_k dt
\leq \int_{t_0}^{t_1} \partial_t (\Gamma_k(t) - \Gamma) dt \bigg|_k
\geq |\Gamma_k(t_1) - \Gamma|_k - |\Gamma_k(t_0) - \Gamma|_k.
\]

This gives
\[
|\Gamma_k(t) - \Gamma|_k \leq C \overline{C}_1 |t_1 - t_0| + |\Gamma_k(t_0) - \Gamma|_k
\leq C \overline{C}_1 |t_1 - t_0| + \frac{3}{2} |\nabla g_k(t_0)|_k
\leq C \overline{C}_1 |t_1 - t_0| + \frac{3}{2} B_0^{3/2} C_1
\leq C \overline{C}_1 |\psi - \beta| + \frac{3}{2} B_0^{3/2} C_1.
\]

We used Lemma 7.8 in the second line and Lemma 7.4 in the third line. Then
\[
|\nabla g_k(t)| \leq B(t, t_0)^{3/2} |\nabla g_k(t)|_k
\leq B(\psi, \beta)^{3/2} |\Gamma_k(t) - \Gamma|_k
\leq B(\psi, \beta)^{3/2} (C \overline{C}_1 |\psi - \beta| + 3 B_0^{3/2} C_1) \equiv \tilde{C}_{1,0},
\]
where we used Lemma 7.8 in the second line.

Next, we prove the lemma for \(p\) satisfying \(p \leq k\) when \(q = 0\). We will show that for \(p \geq 1\), \((19)\)
\[
|\nabla^p \partial_t g_k| \leq C''_p |\nabla^p g_k| + C'''_p, \quad |\nabla^p g_k| \leq \tilde{C}_{p,0}.
\]

If \(p = 1\), then
\[
|\nabla^p \partial_t g_k(t)| \leq B(t, t_0)^{3/2} |(\nabla - \nabla_k) \partial_t g_k + \nabla_k \partial_t g_k|_k
\leq B(t, t_0)^{3/2} C \Gamma_k G_k |\partial_t g_k|_k + |\nabla_k \partial_t g_k|_k
\leq B(t, t_0)^{3/2} C |\nabla g_k| \overline{C}_0 + \overline{C}_1
\]
and we have already shown that \(|\nabla g_k| \leq \tilde{C}_{1,0}\).

Let \(N \geq 2\) and assume that \((19)\) is true for \(0 \leq p \leq N - 1\). The telescoping identity
\[
\nabla^N A - \nabla^N_k A = \sum_{i=1}^{N} \nabla^{N-i}(\nabla - \nabla_k) \nabla^{i-1} k A
\]
We estimate the second term of (20):

\[
|\nabla^N \partial_t g_k| = \left| \sum_{i=1}^N \nabla^{N-i} (\nabla - \nabla_k) \nabla_k^{i-1} \partial_t g_k + \nabla_k^N \partial_t g_k \right| \\
\leq \sum_{i=1}^N |\nabla^{N-i} (\nabla - \nabla_k) \nabla_k^{i-1} \partial_t g_k| + |\nabla_k^N \partial_t g_k| \\
= |\nabla^{N-1} (\nabla - \nabla_k) \partial_t g_k| + \sum_{i=2}^N |\nabla^{N-i} (\nabla - \nabla_k) \nabla_k^{i-1} \partial_t g_k| + |\nabla_k^N \partial_t g_k|.
\]

We estimate the first term of (20):

\[
|\nabla^{N-1} (\nabla - \nabla_k) \partial_t g_k| = |\nabla^{N-1} (\nabla g_k \ast \partial_t g_k)| \\
\leq \sum_{j=0}^{N-1} b_j |\nabla^{N-j} g_k| |\nabla^j \partial_t g_k| \\
\leq b_0 C_0 |\nabla^N g_k| + \sum_{j=1}^{N-1} b_j \tilde{C}_{N-j,0} (C_j'' \tilde{C}_{j,0} + C_j''').
\]

The first equality is due to the identity

\[
(g_k)_{ec} (\nabla_a (g_k)_{bc} + \nabla_b (g_k)_{ac} - \nabla_c (g_k)_{ab}) = 2 (\Gamma_k)_{ab} - 2 \Gamma_{ab}.
\]

We estimate the second term of (20):

\[
\sum_{i=2}^N |\nabla^{N-i} (\nabla - \nabla_k) \nabla_k^{i-1} \partial_t g_k| = \sum_{i=2}^N |\nabla^{N-i} (\nabla g_k \ast \nabla_k^{i-1} \partial_t g_k)| \\
\leq \sum_{i=2}^N \sum_{j=0}^{N-i} b'_j |\nabla^{N-i-j+1} g_k| |\nabla^j \nabla_k^{i-1} \partial_t g_k| \\
\leq \sum_{i=2}^N \sum_{j=0}^{N-i} b'_j |\nabla^{N-i-j+1} g_k| \sum_{l=0}^{j} |(\nabla - \nabla_k)^l \nabla_k^{j-l} \partial_t g_k| \\
= \sum_{i=2}^N \sum_{j=0}^{N-i} b'_j |\nabla^{N-i-j+1} g_k| \sum_{l=0}^{j} |(\nabla g_k)^l \ast \nabla_k^{j-l+i-1} \partial_t g_k| \\
\leq \sum_{i=2}^N \sum_{j=0}^{N-i} b'_j |\nabla^{N-i-j+1} g_k| \sum_{l=0}^{j} b''_l |\nabla g_k| |\nabla_k^{j-l+i-1} \partial_t g_k| \\
\leq \sum_{i=2}^N \sum_{j=0}^{N-i} \sum_{l=0}^{j} b'_j \tilde{C}_{N-i+j-1,0} b''_l \tilde{C}_{l,0} \tilde{C}_{j-l+i-1}.
\]

We applied (21) in the first and fourth lines. The last term of (20) is also bounded: \(|\nabla_k^N \partial_t g_k| \leq C_N|

Collecting the three previous estimates, we obtain

\[
|\nabla^N \partial_t g_k| \leq C''_N |\nabla^N g_k| + C'''.
\]
Applying the preceding inequality, we get
\[
\partial_t |\nabla^N g_k|^2 = 2 \langle \partial_t \nabla^N g_k, \nabla^N g_k \rangle \\
\leq |\partial_t \nabla^N g_k|^2 + |\nabla^N g_k|^2 \\
\leq (1 + 2(C''_N)^2) |\nabla^N g_k|^2 + 2(C''_N)^2.
\]
(22)

The solution of the ODE \(dA/d\sigma = c_1 A + c_2\) is given by
\[
A(\sigma) = e^{c_1(\sigma-\sigma_1)} \left[ A(\sigma_1) + \frac{c_2}{c_1} (1 - e^{-c_1(\sigma-\sigma_1)}) \right].
\]
(23)

Applying (23) to (22), we get
\[
|\nabla^N g_k|^2(t) \leq e^{(1+2(C''_N)^2)(t-t_0)} \left[ |\nabla^N g_k|^2(t_0) + \frac{2(C''_N)^2}{1 + 2(C''_N)^2}(1 - e^{(1+2(C''_N)^2)(t_0-t)}) \right] \\
\leq e^{(1+2(C''_N)^2)(\psi-t_0)} \left[ C_N + \frac{2(C''_N)^2}{1 + 2(C''_N)^2}(1 - e^{(1+2(C''_N)^2)(t_0-\beta)}) \right] \equiv \tilde{C}^2_{N,0}.
\]

This completes the inductive proof of (19) and the proof of the lemma for any \(p\) when \(q = 0\). Since \(\partial_t^q \nabla^p g_k = \nabla^p \partial_t^q g_k\), a similar procedure may be used to prove the lemma when \(q > 0\).

\[\square\]

**Proof of Proposition 7.10.** The two preceding Lemmas 7.3 and 7.4 prove the proposition.

We are now able to prove the compactness Theorem 1.3 for solutions of the AOF. We need the following lemma.

**Proposition 7.10.** (Chow et al. [14] Corollary 3.15) Let \((M^n, g)\) be a Riemannian manifold and let \(L \subset M^n\) be compact. Furthermore, let \(p\) be a nonnegative integer. If \(\{g_k\}_{k \in \mathbb{N}}\) is a sequence of Riemannian metrics on \(L\) such that
\[
\sup_{0 \leq |\alpha| \leq p+1} \sup_{x \in L} |\nabla^\alpha g_k| \leq C < \infty
\]
and if there exists \(\delta > 0\) such that \(g_k(V, V) \geq \delta g(V, V)\) for all \(V \in T_M\), then there exists a subsequence \(g_k\) and a Riemannian metric \(g_\infty\) on \(L\) such that \(g_k\) converges in \(C^p\) to \(g_\infty\) as \(k \to \infty\).

**Proof of Theorem 7.3.** Since we are given a uniform bound on \(|\text{Rm}(g_k)|_{g_k}\), the pointwise smoothing estimates given by Theorem 1.1 furnish uniform bounds on \(\|\nabla^m_{g_k(t_0)} \text{Rm}(g_k(t_0))\|_{C^0(g_k(t_0))}\) for all \(m \in \mathbb{N}\). Therefore, since the \((M_k, g_k)\) are complete, the Cheeger-Gromov compactness Theorem 1.3 yields a subsequence of \(\{(M_k, g_k(t), O_k)\}_{k \in \mathbb{N}}\), also called \(\{(M_k, g_k(t), O_k)\}_{k \in \mathbb{N}}\), for which \(\{(M_k, g_k(t_0), O_k)\}_{k \in \mathbb{N}}\) converges to a complete pointed Riemannian manifold \((M'_\infty, h, O_\infty)\).

Fix a compact subset \(L \subset M_\infty\) and a closed interval \([\beta, \psi]\), with \(t_0 \in (\beta, \psi)\) of \((\alpha, \omega)\). Since \(\{(M_k, g_k(t_0), O_k)\}_{k \in \mathbb{N}}\) converges to \((M_\infty, h, O_\infty)\), by definition there exists an exhaustion \(\{U_k\}_{k \in \mathbb{N}}\) of \(M_\infty\) by open sets with \(O_\infty \in U_k\) and a sequence of diffeomorphisms \(\Phi_k : U_k \to V_k \equiv \Phi_k(U_k) \subset M_k\) with \(\Phi_k(O_\infty) = O_k\), such that \(h_k \equiv \Phi_k^* [g_k(t_0)]|_{V_k}\), then \((U_k, h_k)\) converges in \(C^\infty\) to \((M_\infty, h)\) on compact sets in \(M_\infty\). Since the \(U_k\) exhaust \(M_\infty\), \(L \subset U_k\) for some \(k\). So the metrics \(h_k\) are uniformly equivalent to \(h\) on \(L\). We also obtain from the \(C^\infty\) convergence that for each \(p \geq 1\), there exists a \(C_p\) independent of \(x \in L\) and \(k\) such that \(|\nabla^p h_k|_h \leq C_p\).

Let \(G_k(t) \equiv \Phi_k^* [g_k(t)]|_{V_k}\); then \(h_k = G_k(t_0)\). From the pointwise smoothing estimates given by Theorem 1.1, for each \(p\) we obtain a bound \(C_{p,0}\) uniform on \(L \times [\beta, \psi]\) independent of \(k\) such that \(|\nabla^p_{G_k} \text{Rm}(G_k)|_{G_k} \leq C_{p,0}\) on \(L \times [\beta, \psi]\). Using the expression of \(\partial_t \nabla^p_{G_k} \text{Rm}(G_k)\) in terms of covariant derivatives of \(\text{Rm}(G_k)\) given by Proposition 7.5, for each \((p, q)\) we obtain a bound \(C'_{p,q}\) uniform on \(L \times [\beta, \psi]\) independent of \(k\) such that \(|\partial^q_t \nabla^p_{G_k} \text{Rm}(G_k)|_{G_k} \leq C'_{p,q}\) on \(L \times [\beta, \psi]\). We then conclude via Proposition 7.5 that the metrics \(G_k\) are uniformly equivalent to \(h\) on \(L \times [\beta, \psi]\) and that for every \(p, q \geq 0\), there is a constant \(C'_{p,q}\) independent of \(k\) such that \(|\partial^q_t \nabla^p_{G_k} h|_h \leq C_{p,q}\) on \(L \times [\beta, \psi]\).
The uniform equivalence of the $G_k$ to $h$ and the uniform bounds $|\partial_t^p \nabla^p G_k|h \leq \tilde{C}_{p,q}$ allow us to apply an Arzelà-Ascoli type Proposition 7.10 to the metrics $G_k(t) + dt^2$ on $L \times [\beta, \psi]$ and obtain a subsequence that converges in $C^\infty(L \times [\beta, \psi], h + dt^2)$ to a metric $g_\infty(t) + dt^2$ such that $g_\infty(0) = h$; we relabel the convergent subsequence as $\{G_k(t) + dt^2\}_{k \in \mathbb{N}}$. It follows that $g_\infty(t) + dt^2$ is uniformly equivalent to $G_1(t) + dt^2$ on $L \times [\beta, \psi]$. Then $g_\infty(t) + dt^2$ is uniformly equivalent to $h + dt^2$ on $L \times [\beta, \psi]$ since $G_1(t) + dt^2$ is uniformly equivalent to $h + dt^2$ on $L \times [\beta, \psi]$. Since $(M, h)$ is complete, $(M \times (\alpha, \omega), h + dt^2)$ is also complete. The uniform equivalence of $g_\infty(t) + dt^2$ to $h + dt^2$ on compact subsets of $M \times (\alpha, \omega)$ and the Hopf-Rinow theorem imply that $(M_\infty \times (\alpha, \omega), g_\infty(t) + dt^2)$ is complete.

Since $(M_\infty \times (\alpha, \omega), g_\infty(t) + dt^2)$ is complete, compact sets are equivalent to closed, bounded ones. A compact set in $M_\infty \times (\alpha, \omega)$ is contained in the compact set that is the product of a closed geodesic ball in $M_\infty$ and a closed interval in $(\alpha, \omega)$. So the metrics $G_k(t) + dt^2$ subsequentially converge in $C^\infty(M_\infty \times (\alpha, \omega), h + dt^2)$. Let $\{G_k(t) + dt^2\}_{k \in \mathbb{N}}$ be the convergent subsequence. Then $\{(M_k, g_k(t), O_k)\}_{k \in \mathbb{N}}$ converges to $(M_\infty, g_\infty(t), O_\infty)$. It follows that for each $p, q$, $\partial_t^p \nabla^p G_k \to \partial_t^p \nabla^p g_\infty$ and $\mathcal{O}(G_k) \to \mathcal{O}(g_\infty)$ in $C(M_\infty \times (\alpha, \omega), g_\infty(t) + dt^2)$. Therefore $(M_\infty, g_\infty, O_\infty)$ is a complete pointed solution to AOF for $t \in (\alpha, \omega)$. \hfill \Box

As our first corollary of the compactness theorem 1.3, we show that under suitable conditions, we can obtain a singularity model for the ambient obstruction flow.

**Proof of Theorem 7.4.** We first show that the $g_t$ are also solutions to AOF by showing that if $\tilde{g} = \lambda g$ and $g$ satisfies AOF, given up to constants by

$$\partial_t g = \Delta^{\frac{n}{2} - 1}Rc + \Delta^{\frac{n}{2} - 2}\nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(Rm),$$

then $\tilde{g}$ satisfies

$$\partial_t \tilde{g} = \Delta^{\frac{n}{2} - 1}\tilde{Rc} + \Delta^{\frac{n}{2} - 2}\tilde{\nabla}^2 \tilde{R} + \sum_{j=2}^{n/2} P_j^{n-2j}(\tilde{Rm}).$$

We evaluate the first term of the right side of (24):

$$\Delta^{\frac{n}{2} - 1}\tilde{Rc} = (\lambda^{-1} g^{-1} \nabla^2)^{\frac{n}{2} - 1} Rc = \lambda^{1 - \frac{n}{2}} \Delta^{\frac{n}{2} - 1} Rc.$$

Similarly, the second term is equal to $\lambda^{1 - \frac{n}{2}} \Delta^{\frac{n}{2} - 1} Rc$. The remaining terms are contractions of terms of the form

$$\tilde{\nabla}^{i_1}\tilde{Rm} \otimes \cdots \otimes \tilde{\nabla}^{i_k}\tilde{Rm}$$

with $2 \leq j \leq \frac{n}{2}$ and $i_1 + \cdots + i_j = n - 2j$. In order to contract on all but two indices of the above term, we need to contract $\frac{1}{2}(i_1 + \cdots + i_j + 3j - j - 2) = \frac{n}{2} - 1$ pairs of indices. This implies that $P_j^{n-2j}(\tilde{Rm}) = \lambda^{1 - \frac{n}{2}} P_j^{n-2j}(Rm)$. The left side of (24) is equal to $\lambda^{1 - \frac{n}{2}} \partial_t g$. So $\tilde{g}$ satisfies (24).

We have $|Rm(g_t)|_{g_t} \leq 1$ on $M \times [-\lambda^{n/2} t_i, 0]$ for each $i$ since the definition of the $\lambda_i$ implies

$$|Rm(g_t)|_{g_t}^2 = \lambda_i^{-2}|Rm|^2 \leq \lambda_i^{-2}\lambda_i^2 \leq 1.$$

Let $k \in \mathbb{N}$. There exists $i_k$ such that if $i \geq i_k$, then $\lambda_i^{n/2} t_i > k$. Then $\{g_t\}_{i \geq i_k}$ is a sequence of complete pointed solutions to AOF on $(-k, 0]$. Since the Sobolev constant is scaling invariant, the uniform bound of $C_S(M, g)$ on $[0, T)$ implies a uniform bound independent of $i$ of $C_S(M, g_i)$ on $[0, T)$. We conclude from Lemma 3.2 of Hebe 23 that there exists a uniform lower bound independent of $i$ for $\inf_{x \in M} \text{vol}(B_{g_i}(x, 1))$. This and the bound $|Rm(g_t)|_{g_t} \leq 1$ on $M \times [-\lambda_i^{n/2} t_i, 0]$. 

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for all \( i \) give a uniform lower bound independent of \( i \) for \( \text{inj}_{g_i(0)}(x_i) \) via the Cheeger-Gromov-Taylor theorem.

The proof of the compactness theorem \[1.3\] is unchanged if we replace \((\alpha, \omega)\) with \((-k, 0)\). Thus, by theorem \[1.3\] we obtain subsequential convergence of \(\{(M, g_i(t), x_i)\}_{i \geq k}\) to a complete pointed solution \((M_\infty, g_\infty(t), x_\infty)\) to AOF for \( t \in (-k, 0) \). By taking a further diagonal subsequence over the \( k \), we get that \(\{(M, g_i(t), x_i)\}_{i \geq 1}\) subsequentially converges to a complete pointed solution \((M_\infty, g_\infty(t), x_\infty)\) to AOF for \( t \in (-\infty, 0) \). The limit \((M_\infty, g_\infty(t))\) is not flat since

\[
|\text{Rm}(g_\infty(0))(x_\infty)|_{g_\infty(0)} = 1
\]

by the definition of the \( g_i(t) \).

We show that \( M_\infty \) is not compact. Lemma 3.9 of Chow-Knopf \[13\] states that for a one parameter family of Riemannian manifolds \((M, g(t))\), the volume element evolves by \( \partial_t dV_g = \frac{1}{2} g^{ij} \partial_i g_{ij} \).

Applying the fact that \( \mathcal{O} \) is traceless and the divergence theorem,

\[
\frac{\partial}{\partial t} \text{vol}(M, g(t)) = \frac{1}{2} \int_M g^{ij} \frac{\partial g_{ij}}{\partial t} dV(g(t))
\]

\[
= \frac{1}{2} \int_M [(-1)^{\frac{n}{2}} g^{ij} \mathcal{O}_{ij} + C(n)(\Delta^{\frac{n}{2}} - 1) R_g g_{ij}] dV(g(t))
\]

\[
= C(n) \int_M \Delta^{\frac{n}{2}} - 1 R dV(g(t))
\]

\[
= 0.
\]

Therefore the volume of \((M, g(t))\) is preserved along the flow. Since \( \lambda_i \to \infty \),

\[
\text{vol}(M_\infty, g_\infty(t)) = \lim_{i \to \infty} \text{vol}(M, g_i(t)) = \lim_{i \to \infty} \lambda_i^{n/2} \text{vol}(M, g(t_i + \lambda_i^{\frac{2}{n}} t)) = \infty
\]

for all \( t \in (-\infty, 0] \). So the volume of \((M, g_\infty(t))\) is infinite for all \( t \in (-\infty, 0] \). The uniform volume lower bound for the \((M, g_i)\) passes in the limit to a uniform volume lower bound for \((M, g_\infty)\). Therefore \( M_\infty \) is noncompact by Lemma 8.1 of Bour \[6\].

Next, we show that the integral of the \( Q \)-curvature is nondecreasing along the flow on \( M \). Along the flow, the derivative of \( \int_M Q \) is given by

\[
\frac{\partial}{\partial t} \int_M Q = (-1)^{\frac{n}{2}} \frac{n-2}{2} \int_M \langle \mathcal{O}, \partial_t g \rangle
\]

\[
= (-1)^{\frac{n}{2}} \frac{n-2}{2} \int_M (-1)^{\frac{n}{2}} |\mathcal{O}|^2 + C(n) \int_M \langle \mathcal{O}, (\Delta^{\frac{n}{2}} - 1) R_g \rangle
\]

\[
= \frac{n-2}{2} \int_M |\mathcal{O}|^2,
\]

where the third line holds since \( \mathcal{O} \) is traceless. So the integral of the \( Q \)-curvature does not decrease along the flow.

Suppose that

\[
\sup_{t \in [0, T]} \int_M Q(g(t)) dV(g(t)) < \infty.
\]

This is always true when \( n = 4 \) since the Chern-Gauss-Bonnet theorem gives that for all \( t \in [0, T] \),

\[
\int_M Q = 8\pi^2 \chi(M) - \frac{1}{36} \int_M |W|^2 \leq 8\pi^2 \chi(M).
\]
So if the integral of the $Q$ curvature is bounded along the flow, 
\[ \int_0^T \int_M |Q|^2 = \int_0^T \frac{\partial}{\partial t} \int_M Q = \lim_{t \to T} \int_M Q(g(t)) - \int_M Q(g(0)) < \infty. \]

Let \{\{(M, g_i(t), x_i)\}_{i \geq 1}\} be the convergent subsequence previously found in the proof. Fix $k \in \mathbb{N}$. Since $t_i \to T$ and $\lambda_i \to \infty$, we can choose a subsequence of times \{\{t_{i_j}\}_{j \in \mathbb{N}}\} as follows:

\[ i_1 = \inf \{i : t_i \geq \frac{T}{2}, \lambda_i \geq \left(\frac{2k}{T}\right)^{\frac{2}{n}}\}, \quad i_j = \inf \{i : t_i \geq \frac{1}{2}(T + t_{i_{j-1}}), \lambda_i \geq \left(\frac{2k}{T-t_{i_{j-1}}^{\frac{1}{n}}}\right)^{\frac{2}{n}}\} \]

for $j \geq 2$. We relabel \{\{t_{i_j}\}_{j \in \mathbb{N}}\} as \{\{t_i\}_{i \in \mathbb{N}}\}. Then

\[ \sum_{i=1}^\infty \int_{t_{i-1} - k\lambda_i}^{t_i} \int_M |Q|^2 < \int_0^T \int_M |Q|^2 < \infty, \]

implying that, using the scaling law $O(\lambda g) = \lambda^{\frac{2-n}{n}} O(g)$,

\[ 0 = \lim_{i \to \infty} \int_{t_{i-1} - k\lambda_i}^{t_i} \int_M |O(g_i)|^2 dV_g dt \]
\[ = \lim_{i \to \infty} \int_0^\infty \int_M \lambda_i^2 |O(g_i)|^2_g \lambda_i^{-\frac{2-n}{n}} dV_g dt \]
\[ = \lim_{i \to \infty} \int_0^\infty \int_M |O(g_i)|^2_g dV_g dt. \]

Since $O(g_i) \to O(g_\infty)$ in $C^\infty$ on compact subsets, this implies that $O(g_\infty) \equiv 0$ on $[-k, 0]$. So for each $k \in \mathbb{N}$, there exists a sequence of pointed solutions to AOF that converge to an obstruction flat pointed solution to AOF on $[-k, 0]$. By taking a further diagonal subsequence over the $k$, we obtain a sequence of pointed solutions to AOF that converge to an obstruction flat complete pointed solution to AOF on $(-\infty, 0]$.\[\square\]

Finally, we provide a corollary of the compactness theorem 1.3 characterizing limits of nonsingular solutions to AOF.

**Proof of Theorem 1.3** Suppose $M$ does not collapse at $\infty$. Then there exists a sequence \{\{(x_i, t_i)\}_{i \in \mathbb{N}} \subset M \times [0, \infty)\} such that $\inf_i \inf_{t \in (0, \infty)} \|Rm\|_{t_i} > 0$. Let $g_i(t) = g(t + t_i)$ for $t \in [-t_i, \infty)$. Let $k \in \mathbb{N}$. Then there exists $i_k \in \mathbb{N}$ such that $t_i > k$ for all $i \geq i_k$. Since $\sup_{t \in [0, \infty)} \|Rm\|_t < \infty$ and $\inf_i \inf_{t \in (0, \infty)} \|Rm\|_{t_i} > 0$, we apply Theorem 1.3 to obtain subsequential convergence in the sense of families of pointed Riemannian manifolds of \{\{(M, g_i(t), x_i)\}_{i \geq i_k}\} to a complete pointed solution \((M_\infty, g_\infty(t), x_\infty)\) to AOF on $(-k, \infty)$. By taking a further diagonal subsequence over the $k$, we get that \{\{(M, g_i(t), x_i)\}_{i \geq 1}\} subsequentially converges to a complete pointed solution \((M_\infty, g_\infty(t), x_\infty)\) to AOF on $(-\infty, \infty)$.

If $M_\infty$ is compact, then by the definition of convergence of complete pointed Riemannian manifolds, $M_\infty$ is diffeomorphic to $M$. Just as in the proof of Theorem 1.3 the volume of $(M, g(t))$ is preserved along the flow. So for all $t \in (-\infty, \infty)$,

\[ \text{vol}(M_\infty, g_\infty(t)) = \lim_{i \to \infty} \text{vol}(M, g_i(t)) = \lim_{i \to \infty} \text{vol}(M, g(t_i + t)) < \infty. \]

Suppose that

\[ \sup_{t \in [0, \infty)} \int_M Q(g(t)) dV_{g(t)} < \infty. \]
This is always true when \( n = 4 \) by the Chern-Gauss-Bonnet theorem. Using the same argument as in the proof of Theorem 1.4, we obtain

\[
\int_0^\infty \int_M |O|^2 < \infty.
\]

Let \( \{(M, g_i(t), x_i)\}_{i \geq 1} \) be the convergent subsequence previously found in the proof. Since \( t_i \to \infty \), we can choose a subsequence of times \( \{t_{i_j}\}_{j \in \mathbb{N}} \) as follows:

\[
i_1 = \inf\{i : t_i \geq k\}, \quad i_j = \inf\{i : t_i \geq t_{i_{j-1}} + 2k\}
\]

for \( j \geq 2 \). We relabel \( \{t_{i_j}\}_{j \in \mathbb{N}} \) as \( \{t_i\}_{i \in \mathbb{N}} \). Then

\[
\sum_{i=1}^\infty \int_{t_i-k}^{t_i+k} \int_M |O|^2 < \int_0^\infty \int_M |O|^2 < \infty
\]

implies that

\[
0 = \lim_{t \to \infty} \int_{t_i-k}^{t_i+k} \int_M |O(g)|^2 \, dV_g \, dt = \lim_{t \to \infty} \int_{-k}^{k} \int_M |O(g_i)|^2 \, dV_{g_i} \, dt.
\]

Since \( O(g_i) \to O(g_\infty) \) in \( C^\infty \) on compact subsets, this implies that \( O(g_\infty) \equiv 0 \) on \([-k,k]\). So for each \( k \in \mathbb{N} \), there exists a sequence of pointed solutions to AOF that converge to an obstruction flat pointed solution to AOF on \([-k,k]\). By taking a further diagonal subsequence over the \( k \), we obtain a sequence of pointed solutions to AOF that converge to an obstruction flat complete pointed solution to AOF on \((-\infty, \infty)\). Since \( g_\infty \) solves the conformal flow \( \partial_t g_\infty = (-1)^{n/2} C(n)(\Delta + 1) R \), we see that \( g_\infty(t) \) is in the conformal class of \( g_\infty(0) \) for all \( t \in (-\infty, \infty) \). If \( M_\infty \) is compact, we can solve the Yamabe problem for \( (M_\infty, [g_\infty(0)]) \); the Yamabe problem was solved by Aubin, Trudinger, and Schoen (see [2][20]). Due to the conformal covariance of \( O \), we obtain an obstruction flat, constant scalar curvature complete pointed solution \( (M_\infty, \hat{g}_\infty(t)) \) to AOF with \( \hat{g}_\infty(t) = \hat{g}_\infty(0) \) for all \( t \in (-\infty, \infty) \).

\[\square\]

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