Vorticity cutoff in nonlinear photonic crystals

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Using group theory arguments, we demonstrate that, unlike in homogeneous media, no symmetric vortices of arbitrary order can be generated in two-dimensional (2D) nonlinear systems possessing a discrete-point symmetry. The only condition needed is that the non-linearity term exclusively depends on the modulus of the field. In the particular case of 2D periodic systems, such as nonlinear photonic crystals or Bose-Einstein condensates in periodic potentials, it is shown that the realization of discrete symmetry forbids the existence of symmetric vortex solutions with vorticity higher than two.

Vortices are particular higher-order stationary solutions present in many different nonlinear systems, ranging from fluid dynamics to photonics. A vortex is characterized by a typical phase dislocation determined by an integer number, that we refer to as vorticity (also known as winding-number, “topological charge” or even spin). An optical vortex with a rotationally invariant amplitude in a nonlinear Kerr medium, experimentally observed in homogeneous self-defocussing media [1], can be understood as an eigenmode of the equivalent rotationally invariant waveguide generated by itself [2]. Thus, a vortex appear as an object carrying well-defined angular momentum: \( \phi_l = e^{il\theta} f(r) \). In this case, angular momentum and vorticity are the same integer number; a consequence of the continuous \( O(2)-\)symmetry of the operator defining the equivalent waveguide. However, in systems such as 2D nonlinear photonic crystals or Bose-Einstein condensates in 2D periodic traps this \( O(2)\)-symmetry is replaced by a discrete point-symmetry. Angular momentum is no longer well defined and thus the angular-momentum-vorticity equivalence is lost. Nevertheless, optical vortices have been predicted to exist in 2D periodic photonic crystals [3, 4] and in photonic crystal fibers [5] and experimentally observed in optically-induced photonic lattices [6]. Although these solutions cannot longer have well-defined angular momentum, certainly all of them present neat phase dislocations that can be characterized by an integer vorticity value. In this paper, we will prove how to re-interpret vorticity in terms of the rotational properties of vortex solutions without resorting to the angular-momentum concept. As a result, severe restrictions on vorticity values will be found using group-theory arguments.

Let us consider the following general nonlinear equation for stationary states:

\[
[L_0 + L_{\text{NL}}(\phi)] \phi(x, y) = -E \phi(x, y),
\]

(1)

where \( L_0 \) is a linear field-independent self-adjoint operator (normally dependent on gradients and functions of the transverse coordinates) and \( L_{\text{NL}}(\phi) \) is the nonlinear field-dependent piece of the full operator acting on the field \( \phi \). This equation is valid for all type of 2D systems in which the nonlinearity depends on the field through its modulus. Many different systems can be modeled using an equation that can be written in the form given by Eq. (1). We are interested in systems that, besides being described by Eq. (1), are invariant under some discrete-symmetry group \( G : [L, G] = 0 \) \( (L \equiv L_0 + L_{\text{NL}}) \). This means that we assume that all linear and nonlinear coefficients appearing in the operators defining Eq. (1) are invariant under the \( G \) group. Our goal is to study the implications that the realization of discrete symmetry have on the characterization of vortex solutions of Eq. (1).

The key concept in our approach is the so-called group self-consistency condition. This condition establishes that if a system described by Eq. (1) is invariant under some discrete-symmetry group \( G \) then any of its solutions either belongs to one representation of the group \( G \) or to one of its subgroups \( G' \) \( (G' \subset G) \). Note that the identity group is always a subgroup of any group and, therefore, asymmetric solutions also satisfy the group self-consistency condition [6]. In this letter, however, we will focus on symmetric solutions exclusively.

The elements of point-symmetry groups in a plane are rotations through integral multiples of \( 2\pi/n \) about some axis (called an \( n \)-fold rotation axis), reflections on a mirror plane containing the axis of rotation and combinations of both. Groups containing a \( n \)-fold rotation axis constitute the \( C_n \) groups. When, in combination with the \( n \)-fold rotation axis, these groups have mirror planes, one generates the so-called \( C_{nv} \) groups. In Fig. 1 we give two examples of structures exhibiting \( C_{6v} \) and \( C_{8v} \) point-symmetries.

We will prove next how vorticity is affected by the finite order of the \( n \)-fold rotation axis defining the \( C_n \) (or \( C_{nv} \)) group. In order to do so, we need first to properly characterize the different representations of a \( C_n \) group. Since \( C_n \) groups are abelian, its representations are one-dimensional and given by a single scalar complex number (the character of the representation) \( \epsilon \). This scalar is nothing but a root of unity of order \( n \) and thus the representations of the \( C_n \) group are given by \{1, \epsilon^{\pm 1}, \ldots, \epsilon^{\pm t}, \ldots, \epsilon^{n/2}\} for even \( n \) and \{1, \epsilon^{\pm 1}, \ldots, \epsilon^{\pm t}, \ldots, \epsilon^{(n-1)/2}\} for odd \( n \), where \( \epsilon = e^{2\pi i/n} \).
exp(2πi/n). In Fig. 2 we present, as an example, the construction of the roots of unity for the \(C_6\) (even \(n\)) and \(C_3\) (odd \(n\)) groups. Each representation can be labeled by the natural number \(l\) and, when present, by its sign. We denote it by \(D_{l,s}\) \((l \in \mathbb{N}, s = \pm)\). No sign is needed for the identity representation \(D_0\) \((l = 0)\) nor for \(D_{n/2}\) \((l = n/2, \text{even } n)\). A state belonging to representations with \(l \neq 0, n/2\) can be written as \(|l, s\rangle\) with \(0 < l < n/2\) (if \(n\) is even) or \(0 < l \leq (n - 1)/2\) (if \(n\) is odd). When we act with a group operator \(G\) (representing a discrete rotation of angle \(2\pi/n\)) on a function belonging to a representation \(D_{l,s}\), it transforms as \(G\phi_l = e^{l\phi_l}\), where \(l \equiv sl\) \((s = \pm, l \in \mathbb{N})\). Clearly too, \(G\phi_0 = \phi_0\) and \(G\phi_{n/2} = e^{n/2\phi_{n/2}}\) \((\text{even } n)\). If there is no other symmetry involved, \([L, G]\) = 0 implies that every one-dimensional representation is characterized by a different \(L\)-eigenvalue: \(L\phi_l = -\bar{\epsilon}_l\phi_l\). Representations are thus non-degenerated.

We proceed now to explicitly construct functions belonging to \(l \neq 0\) representations. Let us consider the complex coordinate vector \(u = x + iy = re^{i\theta}\). Integer powers of \(u\) have well-defined transformation properties under a \(2\pi/n\) rotation: \(u \rightarrow e^{i\theta + 2\pi/n} u \rightarrow e^{l} u'\). Therefore, we can easily construct a function in the \(D_{l,s}\) representation of \(C_n\) as

\[
\phi_l(u) = u^{\bar{I}} \phi_0^{(l)}(u),
\]

\(\phi_0^{(l)}\) being a function in the \(D_0\) representation of \(C_n\). Clearly, \(G\phi_l = e^{l\phi_l}\).

The representations of \(C_{nv}\) (discrete rotations plus reflexions) are easily obtained from those of \(C_n\) groups. The existence of the extra symmetries provided by mirror reflexions yields to degeneracies for \(l \neq 0\) representations. High-order states are now doubly degenerated; they form pairs of complex-conjugated functions \((\phi_l, \phi_0^l)\) with the same \(L\)-eigenvalue: \(L\phi_l = -\bar{\epsilon}_l\phi_l\). Remarkable exceptions are the \(D_0\) and \(D_{n/2}\) representations. Because of their different behavior under mirror reflexions there are two distinct non-degenerated one-dimensional \(l \neq 0\) representations: \(|0; +\rangle\) and \(|0; -\rangle\). They transform differently under reflexions with respect to \(x\) and \(y\) axis: \(R_{x,y} |0; +\rangle = + |0; +\rangle\) and \(R_{x,y} |0; -\rangle = - |0; -\rangle\). The \(|0; +\rangle\) state has maximal symmetry. Fundamental solitons belong to this identity representation of \(C_{nv}\). In the same way, there are also two different non-degenerated one-dimensional representations with \(l = n/2\) \((\text{even } n)\): \(|n/2; +\rangle\) and \(|n/2; -\rangle\). The distinction is made by \(R_{x,y}\)-reflexions. In Fig. 3 we show the lowest order eigenfunctions of the spectrum of a \(C_{nv}\)-invariant operator self-consistently generated by a fundamental soliton solution \((\phi_{\text{fund}} = \phi_{0; +})\):

\[
L = L_0 + L_{NL}(|\phi_{\text{fund}}|) \text{ (see the final section of the paper for details on the physical system associated to } L).
\]

We easily recognize, from lower to higher values of \(E\), the \(|0; +\rangle\) self-consistent state \((\text{i.e., the fundamental soliton})\), the doubly-degenerated \(|1; \pm\rangle\) and \(|2; \pm\rangle\) states and the non-degenerated \(|3; +\rangle\) state. The rest of the spectrum, including continuum de-localized states, systematically falls into the representations described above.

Vorticity \(v\) can be defined as the integer variation \((2\pi\text{ units})\) that the phase of a complex field experiments under a \(2\pi\) rotation around a rotation axis. Solutions with non-zero vorticity are called vortices of order \(v\). They are characterized by their rotation axis, whose intersection with the 2D plane defines the vortex center, where their amplitude vanishes. If \(\Phi(r, \theta)\) represents the phase of a complex vortex field of order \(v\) given by \(f_v = |f_v|e^{i\Phi}\), then \(\Phi(r, \theta + 2\pi) - \Phi(r, \theta) = 2\pi v\), where the polar coordinates are referred to a reference frame centered on the rotation axis. For systems enjoying a 2D point symmetry, this axis is naturally given by the \(n\)-fold rotation axis of the corresponding \(C_n\) (or \(C_{nv}\)) group.

According to the group self-consistency condition, all symmetric solutions of Eq.1 in a system with \(C_n\) symmetry have to lie on the representations of \(C_n\) or of any of its subgroups. Let us consider now a solution \(\phi_l\) in the \(D_{l,s}\) representation of \(C_n\) given by Eq.2. Its phase will be given by \(\arg\phi_l(r, \theta) = \bar{l}\theta + \arg\phi_0^l(r, \theta)\). Since \(\phi_0^l(r, \theta)\) is invariant under rotations, \(\arg\phi_0^l(r, \theta + 2\pi) = \arg\phi_0^l(r, \theta) + 2\pi \bar{l}\). Therefore we find the important relation between the index representation and vorticity:

\[v = \bar{l}.\]
Vortices are thus solutions belonging to $D_l$ generated by a soliton solution in the identity (fundamental) representation of $C_{6v}$. The symmetry of the full operator is $C_{6v}$: $[L, C_{6v}] = 0$. 

Figure 3: Lowest order eigenfunctions of a nonlinear operator $L$ generated by a soliton solution in the identity (fundamental) representation of $C_{6v}$. The symmetry of the full operator is $C_{6v}$: $[L, C_{6v}] = 0$.

Vortices are thus solutions belonging to $D_{l,s}$ representations with $l \neq 0$. There is, however, no vortex associated to $l = n/2$ (even $n$). It can be proved that $\phi_{n/2}$ is a real field, so that its argument is a function that can only take the values 0 or $\pi$. More explicitly, from Eq. (2), $\phi_{n/2} \sim \cos (n\theta/2 + \arg \phi_0^{(n/2)}(r, \theta))$, which has the phase behavior of alternating signs typical of a nodal soliton and not of a vortex. In $C_{nv}$, the behavior of the $\phi_{n/2,+}$ and $\phi_{n/2,-}$ functions is also of the nodal-soliton type, as one can check by observing the phase of the $|3; +\rangle$ state in Fig. 4.

Let us summarize now our main conclusions. Firstly, if a system is invariant under a $C_n$ or $C_{nv}$ point-symmetry group, the solutions of Eq. (1) belong to representations of these groups or of their corresponding subgroups. Secondly, symmetric solutions of Eq. (1) are characterized by the representation index $l$, which has an upper bound fixed by the order of the group: $l \leq n/2$ (even $n$) and $l \leq (n - 1)/2$ (odd $n$). Thirdly, the vorticity $v$ of the vortex solutions of such a system has a cutoff due to Eq. (3) and the upper bound for $l$:

$$|v| < n/2 \text{ (even n) and } |v| \leq (n - 1)/2 \text{ (odd n).} \tag{4}$$

Note that the group of continuous rotations on a plane can be understood as the limiting case $O(2) = \lim_{n \to \infty} C_n$ and, thus, Eq. (4) correctly establishes the absence of a cutoff for it ($|v| < \infty$).

When we deal with 2D periodic systems, the realization of discrete symmetry has particular features. This is a well-known problem in crystallography [2]. The crystal structure is constructed according to a pattern that repeats itself to “tessellate” the 2D plane. Patterns, unlike objects, are invariant under translations (defined by the periodicity of the crystal). This translation property, inherent to periodicity, determines that only certain collections of symmetry elements are possible for patterns. In other words, only patterns that exhibit a selected set of symmetries can “tessellate” the 2D plane. The important result for us here is that pattern periodicity establishes a restriction on the order of discrete rotations allowed in plane groups. Only $n$-fold rotations of order 2, 3, 4 and 6 are permitted in a 2D periodic crystal [2].

The previous group analysis has important implications for 2D nonlinear periodic systems. According to the group self-consistent condition, if a periodic system described by Eq. (1) is invariant under a discrete group $G$, its stationary solutions cannot belong to representations of groups with higher symmetry than $G$. If $G'$ is the symmetry group of the solution, then $G' \subseteq G$. On the other hand, as seen above, the maximum $n$-fold rotation symmetry compatible with periodicity is a sixth-fold rotation, which means that the maximum value for the order $n$ of $C_n$ and $C_{nv}$ point-symmetry groups in 2D periodic systems is $n = 6$. Consequently, the point-symmetry group of a solution cannot exceed this order: $n \leq 6$. Since vorticity is restricted by the order of the point-symmetry group according to Eq. (4), we come up to the conclusion that in 2D nonlinear periodic systems of the type described by Eq. (1) vorticity has a strict bound: $|v| \leq 2$. Putting this into words, there are no vortices of order higher than two in 2D nonlinear periodic systems described by Eq. (1).

In order to illustrate our previous theoretical results, we have numerically studied a realistic system, namely, a photonic crystal fiber (PCF). A PCF is a type of 2D photonic crystal consisting on a regular lattice of holes in silica (characterized by the hole radius $a$ and the lattice period $-\pi$) extending along the entire fiber length. When one considers that the silica response is nonlinear (nonlinearity represented by the nonlinear coefficient $\gamma$, defined in Ref. [4]), a PCF becomes a 2D nonlinear photonic crystal. The nonlinear propagation modes of a PCF for monochromatic illumination in the scalar approximation verify Eq. (1) with $\varepsilon = -\beta^2$, $\beta$ being the mode propagation constant (see [4]). Among possible hole-distribution geometries we choose that based on a triangular lattice with $C_{6e}$ symmetry (see Fig. 1(a)). The reason of our symmetry choice is simple. As proved before, the $C_{6e}$ group provides the highest vorticity solu-
Figure 4: Higher-order solitons for a periodic $C_6$ PCF: (a)-(b) First- and second-order vortex pairs, $|1; \pm\rangle$ and $|2; \pm\rangle$; (c) nodal soliton of order three, $|3; +\rangle$.

Figure 5: Same as in Fig. (4) but in a PCF with defect.

solutions since it corresponds to the maximal point-symmetry achievable in a 2D nonlinear photonic crystal. In Fig. 4 we find the first three (from lowest to highest value of $\beta^2$) higher-order solitons of a perfectly periodic PCF (without defect) calculated for the values $a = 5 \mu m$, $\Lambda = 26 \mu m$, and $\lambda = 1064 \text{ nm}$ at $\gamma = 0.01$. In Fig. 5 we present the same first three higher-order solitons but for a PCF with periodicity broken by the presence of a defect (absence of a hole). Note that in both cases the symmetry group is $C_6$ and that, in agreement with our previous result, the maximum vorticity allowed is two. The soliton solution with $l = 3$ is not a vortex. As predicted by group theory, it presents a binary phase structure (corresponding to a $|3; +\rangle$ state) of the nodal-soliton type [7]. It is interesting to check the generality and accuracy of the group-theory approach using these numerical examples. The spectrum of higher-order soliton solutions is perfectly explained by our previous group-theory arguments nevertheless the periodic (Fig. 4) and non-periodic (Fig. 5) photonic crystal structures present notable differences. Despite they share the same $C_6$ symmetry, a description in terms of weakly interacting localized fundamental solitons on lattice sites (the equivalent of the tight-binding approximation in solid state physics) [4] can only be valid in the perfectly periodic case. As it is apparent in Fig. 4 this localization feature is clear in the amplitude and phase of vortex and nodal soliton solutions in the periodic PCF. However, single fundamental solitons are no longer recognizable in the vortex and nodal solitons of Fig. 5 due to the presence of the periodicity-breaking defect. One can think of a situation of strongly interacting solitons causing the “tight-binding approximation” to stop being valid. Despite this fact, our main results concerning the nature of solutions and, more specifically, the restrictions on vorticity remain valid with complete generality.

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