Universal entanglement signatures of quantum liquids as a guide to fermionic criticality

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Abstract

An outstanding challenge involves understanding the many-particle entanglement of liquid states of quantum matter that arise in systems of interacting electrons. The Fermi liquid (FL) shows a violation of the area-law in real-space entanglement entropy of a subsystem, believed to be a signature of the ground state of a gapless quantum critical system of interacting fermions. Here, we apply a $T = 0$ renormalization group approach to the FL, unveiling the long-wavelength quantum fluctuations from which long-range entanglement arises. A similar analysis of non-FL liquids such as the 2D marginal Fermi liquid (MFL) and the 1D Tomonaga–Luttinger liquid reveals a universal logarithmic violation of the area-law in gapless electronic liquids, with a proportionality constant that depends on the nature of the underlying Fermi surface. We extend this analysis to classify the gapped quantum liquids emergent from the destabilisation of the Fermi surface by renormalisation group relevant quantum fluctuations arising from backscattering processes.

1. Introduction

There has been a surge recently in applying quantum information-theoretic tools towards understanding the nature of entanglement and quantum correlation encoded within the ground states and low-lying excitations of strongly correlated systems [1–3]. For a system with local interactions among its constituents, the entanglement of a subsystem is expected to follow an area-law (i.e. scale with the size of its boundary) [2]. There are, however, some notable departures. First, a gapped topologically ordered system possesses, in addition to the area-law term, a sub-dominant piece that arises from its topological properties and encodes true long-range entanglement [4–6]. Another exception involves the finding of signatures of volume-law entanglement in some quantum critical systems arising from the existence of gapless quantum fluctuations at all length-scales [7–11]. A final exception is found in gapless quantum liquids of interacting electrons in spatial dimensions $d > 1$, and forms the focus of our work. The best understood gapless electronic liquids are metals belonging to the Fermi liquid (FL) paradigm [12–14]. These quantum critical systems are known to possess long-range entanglement in the form a modified area-law: $S_{EE} \sim L^{d-1} \ln L$ for $d$-dimensional subsystem of spatial length $L$ [15–21]. The extra $\ln L$ is conjectured to arise from the presence of gapless long-wavelength quasiparticle excitations lying proximate to a Fermi surface [15, 16, 22–24]. Indeed, this result is believed to be the higher dimensional analogue of the entanglement scaling observed in quantum critical phases of interacting quantum spin and electron systems in 1D: $S^{1D}_{EE} \sim \ln L$ [25–28]. A similar violation of the area-law is also expected in other quantities of such gapless liquids, such as the fluctuation in the number of particles within a subsystem [29–44].

In addition to the FL, we will also focus here on a particular variant of the 2D non-FL liquid known as the marginal Fermi liquid (MFL). Proposed on phenomenological grounds as the parent metal of high-temperature superconductivity in the cuprates [45–47], a first-principles derivation of the effective theory for the MFL (e.g. low-energy Hamiltonian, nature of gapless excitations etc) was obtained only recently from a detailed renormalisation group (RG) study of the 2D Hubbard model [48–50]. Interestingly,
the modified area-law discussed above for the real-space entanglement entropy of the FL is also proposed to hold for non-Fermi liquids such as the MFL [51], hinting at a possible universality connecting various gapless electronic quantum liquids despite the obvious differences in some of their key properties (e.g. nature of their low-energy excitations etc).

1.1. Gapless electronic liquids are different
A deeper understanding of the entanglement properties of gapless quantum liquids such as the FL and MFL necessitates an approach that lies beyond the well-established Ginzburg–Landau–Wilson (GLW) paradigm. This can be argued as follows: the effective theories for phases falling within the GLW paradigm describe the quantised excitations of classical scalar (or bosonic) field degrees of freedom that lie above symmetry broken ground states that are short-range entangled at best. They are characterised by order parameters that correspond to the ground state expectation values of real-space local operators, and correlation functions that satisfy the cluster decomposition property. On the other hand, the excitations of a system of interacting fermions carry sign factors that arise from the exchange of particles and do not have a classical origin. Further, a filled Fermi volume at $T = 0$ is the ground state for gapless fermionic quantum liquids, owes its origin to the Pauli exclusion principle, and is characterised by a topological quantum number called the Luttinger volume that is robust against the inclusion certain kinds of inter-particle interactions [52–58]. Indeed, these liquids are described by effective theories that contain only the physics of the low-energy long-wavelength degrees of freedom proximate to the Fermi surface [59, 60]. The remarkable simplicity of these theories is that they are comprised purely of terms that are number-diagonal in the momentum (or, $k$)-space single-electron Fock state occupation number operators $n_k$ (see equations (7) and (8) below). This indicates that the $k$-space many-particle wavefunctions of these quantum liquids are direct product in form, i.e. they are separable in terms of the single-electron Fock states and do not encode any entanglement among them. Instead, as mentioned above, these states of electronic quantum matter display long-range entanglement in real-space. From an RG perspective, these low-energy (IR) theories correspond to universal scale-invariant quantum critical fixed points obtained from the coarse-graining of bare (UV) theories of interacting electrons possessing translation invariance.

1.2. Setting out the goals
We first seek the quantum fluctuations in the UV from which the real-space long-range entanglement of these universal IR theories is emergent. As the low-energy degrees of freedom in the IR are effectively decoupled from their UV counterparts, meeting this challenge necessitates an investigation of the forward and tangential scattering related RG relevant quantum fluctuations that are resolved under the flow from UV to IR. In keeping with the Wilsonian approach to critical phenomena, this involves understanding the signatures of universality that are likely encoded within the many-particle entanglement wrought from such quantum fluctuations. The answer to this question holds the potential to offer crucial insight on whether certain aspects of the physics of candidate non-Fermi liquids (e.g. the MFL) show commonalities with metals belonging to the FL paradigm, even if some other aspects (e.g. the nature of the low-lying excitations) are qualitatively different. To the best of our knowledge, such a study of the RG evolution of the $k$-space entanglement has been attempted only for the case of scalar (or bosonic) field theories [61].

Systems of interacting fermions, on the other hand, involve constraints on the inter-particle scattering mechanisms arising from the Pauli exclusion principle as well as phase space constraints due to the existence of a well-defined Fermi volume and a bounding Fermi surface [59, 60]. We recall that the quantum fluctuations in such systems thus contain fermion-exchange related signs, rendering their study difficult. Recent studies indicate that the sign structure of the many-particle wavefunction significantly affect the nature of the entanglement encoded in them [62, 63]. This is of particular significance to the study of non-Fermi liquid gapless states that are emergent at QCPs related to the collapse of a FL metal: a recent numerical study based on an ansatz wavefunction incorporating correlations from long-ranged backflow effects of the interacting fermions revealed a crossover from a volume-law scaling of the subsystem entanglement entropy to the modified area-law familiar for the FL effective theory [64]. The crossover takes place across a distance scale related to the physics of critical backflow interactions. In addition, it is pertinent to enquire on whether there exist similar imprints of universality hidden within the backscattering related RG relevant quantum fluctuations that destabilise the Fermi surface of these gapless metallic states and lead to the emergence of topologically ordered gapped liquid states of quantum matter (e.g. the 2D Mott liquid found in the 2D Hubbard model on the square lattice at $1/2$-filling [48] and the Cooper pair insulator for the reduced BCS model with a circular Fermi surface [65]). This amounts to an attempt at a systematic classification of the states of quantum matter based on their entanglement content, and looks beyond the short range entangled states that belong to the GLW paradigm of broken symmetries and real-space local order parameters.
1.3. The strategy
In order to meet these goals, we adopt a RG approach that we have formulated recently for the investigation of criticality observed in systems of interacting fermions [48–50, 57, 58, 65–68]. Formulated using only many-particle unitary transformations to the bare Hamiltonian, the procedure reveals IR fixed point theories and their low-energy wavefunctions. An iterative application of the unitaries to the IR wavefunctions then generates a family of states spanning towards the UV. This facilitates a quantitative study of the RG evolution of many-body correlations, as well as several many-particle entanglement features, from the ground state and lowest lying excited states [58, 65, 67–69]. Thus, we first employ this strategy to obtain the UV wavefunctions of several gapless and gapped quantum liquid states that are unitarily connected to their IR counterparts. We then compute the scaling of the entanglement entropy of a block of states in k-space (lying proximate to the IR Fermi energy) with the block size (Λ).

We note that there is a small, but growing, body of work on adopting a field theoretic approach towards understanding the dependence of many-particle entanglement on scale in systems with inter-particle interactions. While most of these works apply the Wilsonian RG towards obtaining entanglement measures from the effective action of interacting scalar theories in real-space [70–74] and in momentum-space [61, 75–85], some recent works along these lines have also been devoted to the study of interacting fermions at finite densities in momentum-space [86, 87]. Our work, on the other hand, takes the Hamiltonian renormalisation route in momentum-space towards the same goal. As discussed in the concluding section, our methods corresponds to a tensor network that readily admits an exact holographic mapping.

2. Methods
We now present the main methods employed by us. As these methods have been employed by us in several recent works [48–50, 57, 58, 65–69], we present only an overview here and refer the reader to appendix A as well as our earlier works for further details.

2.1. Unitary renormalization group (URG) method
The URG method iteratively decouples quantum fluctuations in higher energy Fock states by applying a many-body unitary operator, generating thereby a low-energy effective Hamiltonian. For a general electronic system with Hamiltonian $H_{(N)}$ with total N Fock states, mutually non-commuting terms in the Hamiltonian are the source of quantum fluctuation in the occupation of the electronic Fock states. We first label the Fock states according to the eigenvalues of the diagonal part of the bare Hamiltonian, $\epsilon_N > \epsilon_{N-1} > \cdots$. The unitary operator $U_N$ then decoupled the Nth Fock state from all other states, i.e. it removes quantum fluctuation present in this state. The low-energy effective Hamiltonian after the first step of URG is

$$H_{(N-1)} = U_N H_{(N)} U_N^\dagger.$$  

Subsequently, the unitary operator $U_{N-1}$ is used to remove the quantum fluctuations in the Fock state $N - 1$, and so on. The general form of the unitary operator for $j$th step of URG is given as

$$U_j = (1 + \eta_j - \eta_j^\dagger)/\sqrt{2}, \quad \eta_j^\dagger = \frac{1}{\omega - \text{Tr}(H_j \eta_j)} c_j^\dagger \text{Tr}(H_j c_j),$$

where $\omega$ characterises the energy scale for quantum fluctuations, and an equivalence of $\omega$ has been established with that for thermal fluctuations at a finite-temperature [48–50, 57]. Iterative application of the unitaries generates a series of effective Hamiltonians with progressively lower RG energy scale, $H_{(N)}, H_{(N-1)}, \cdots$. The fixed point effective theory is reached at the effective Hamiltonian $H_{(\omega^*)}$, when further decoupling is not possible due to the vanishing of the denominator of $\eta_j^\dagger$.

At any step of the URG [48–50, 57, 58, 65, 66, 69], the Hamiltonian in the rotated basis is given by

$$U_{\sigma\sigma} H_{\sigma\sigma} U_{\sigma\sigma}^\dagger = \frac{1}{2} \text{Tr}_{\sigma\sigma} (\tilde{H}) + \tau_{\lambda\sigma} \text{Tr}_{\lambda\sigma} (H_{\lambda\sigma}) + \tau_{\kappa\sigma} \{ c_{\sigma} \text{Tr}_{\kappa\sigma} (H_{\kappa\sigma}), \hat{n}_{\kappa\sigma} \}. $$

It is important to note that while $\hat{n}_{\kappa\sigma} \tilde{H} (1 - \hat{n}_{\kappa\sigma}) \neq 0$ (i.e. there exist non-trivial quantum fluctuations in the occupation of single-particle Fock state given by $n_{\kappa\sigma}$) prior to the application of the unitary operator, the unitary operation removes such quantum fluctuation.

$$\hat{n}_{\kappa\sigma} U_{\sigma\sigma} \tilde{H} U_{\sigma\sigma}^\dagger (1 - \hat{n}_{\kappa\sigma}) = 0 \Rightarrow [\hat{n}_{\kappa\sigma}, U_{\sigma\sigma} \tilde{H} U_{\sigma\sigma}^\dagger] = 0,$$

upon the application of $U$. The degree of freedom $n_{\kappa\sigma}$ is thus rendered an integral of motion (IOM) of the RG flow. The RG equations can then be obtained from the condition equation (4). The diagonal part of the
Hamiltonian $H^D_{(k\sigma)}$ with respect to the Fock state $(k\sigma)$ has the property $[H^D_{(k\sigma)}, \hat{n}_{k\sigma}] = 0$. Then, the operators $\eta^\dagger_{k\sigma}$ and $\eta_{k\sigma}$ are given as
\[
\eta^\dagger_{k\sigma} = \frac{1}{\omega_{k\sigma} - \text{Tr}_\sigma[H^D_{(k\sigma)}] \hat{n}_{k\sigma}} T^\dagger_{k\sigma}, \quad \eta_{k\sigma} = \frac{1}{\omega_{k\sigma} - \text{Tr}_\sigma((1 - \hat{n}_{k\sigma}) H^D_{(k\sigma)}) (1 - \hat{n}_{k\sigma})} \hat{T}_{k\sigma},
\]
where $T^\dagger_{k\sigma} = \text{Tr}_\sigma[H_{(k\sigma)}(\sigma)] c^\dagger_{k\sigma}$, and $\hat{T}_{k\sigma} = c_{k\sigma} \text{Tr}_\sigma(\eta_{k\sigma} H_{(k\sigma)})$. Such decoupling leads to the emergence of a lower energy effective Hamiltonian $U_{k\sigma} H^D_{(k\sigma)}$. Further iterative application of unitary operators on effective Hamiltonians leads to the decoupling of further higher energy (UV) Fock states and the emergence of lower energy effective Hamiltonians. Importantly, an equivalence of the quantum fluctuation scale $\omega$ with an energy scale for finite-temperature thermal fluctuations has also been established in [48–50, 57]. RG flow equations for various couplings are derived from the series of effective Hamiltonians. The RG flow attains a stable effective Hamiltonian at the IR fixed point when further decoupling is not possible. The URG has been applied in deriving analytic expressions for the low-energy effective theories of several gapless (e.g. FL, MFL, Tomonaga–Luttinger liquid (TLL)) as well as gapped quantum liquid states of matter discussed in the introduction (e.g. Mott liquid, Cooper Pair Insulator, etc.) [48–50, 57, 58, 65, 66]. We will next see that the URG motivates a RG scheme for the study of many-particle entanglement in these quantum liquids.

2.2. Momentum-space entanglement renormalisation group (MERG)

Having set out the URG flow to the IR fixed point Hamiltonian, we can now sketch the MERG algorithm by which to study the RG flow of entanglement in momentum-space. First, we solve for the wavefunctions of the FL and MFL for the square lattice:

\[
H_{\text{FL}} = \sum_{k, \sigma} \epsilon_k \hat{n}_{k\sigma} + \sum_{kk' \in \langle i, j \rangle} V_{kk', \sigma \sigma'} \hat{n}_{k\sigma} \hat{n}_{k'\sigma'},
\]

where $\hat{s}$ and $\hat{s}'$ correspond to excitations in directions normal to the Fermi surface, and an analytic expression for the RG evolution of the interaction coupling $V_{kk' \sigma \sigma'}$ has been obtained from a URG study of the 2D Hubbard model on the square lattice at large hole doping [49, 50]. Further, the IR fixed point effective Hamiltonian for the MFL phase is obtained from the URG analysis of the 2D Hubbard model at half-filling, as well as optimal doping, as as [48–50].

3. Models of quantum matter

We introduce the various states of quantum matter whose entanglement has been studied using the methods outlined above. In order to focus the discussion, we present all technical details (e.g. the derivation of URG flow equations that lead to various IR fixed point theories) in appendix A.

3.1. Gapless quantum liquids

We investigate the momentum-space entanglement features of a variety of gapless states of fermionic quantum matter in this work, each of which is described briefly below.

1. FL and MFL for the square lattice:

The IR fixed point Hamiltonian describing the excitations of the FL phase is [12–14]

where $\hat{s}$ and $\hat{s}'$ correspond to excitations in directions normal to the Fermi surface, and an analytic expression for the RG evolution of the interaction coupling $V_{kk' \sigma \sigma'}$ has been obtained from a URG study of the 2D Hubbard model on the square lattice at large hole doping [49, 50]. Further, the IR fixed point effective Hamiltonian for the MFL phase is obtained from the URG analysis of the 2D Hubbard model at half-filling, as well as optimal doping, as as [48–50].
\[ H_{\text{MFL}} = \sum_{k,\sigma} \epsilon_k n_{k\sigma} + \sum_{k' \neq k} \mathcal{R}_{k,k'} n_{k\sigma} n_{k'\sigma} (1 - n_{k\sigma} n_{k'\sigma}) , \tag{8} \]

where the RG evolution for the interaction coupling \( \mathcal{R}_{k} \) shows the emergence of an excitation different from the FL kind. We can see from equation (8) that the low-lying excitations of the MFL are described by three-particle composite objects comprised of two electrons and a hole. Further, various features conjectured for the MFL phenomenology (e.g. the vanishing of the Landau quasiparticle residue at energies proximate to the Fermi surface \([45-47]\)) have been established from the MFL effective Hamiltonian equation \( (8) \) \([48-50]\). Importantly, all backscattering related quantum fluctuations in \( k \)-space are rendered irrelevant under the RG flows to the FL and MFL fixed point theories. On the other hand, quantum fluctuations arising from RG relevant forward and tangential scattering processes of interacting electrons are of interest in understanding the nature of the entanglement. Finally, in unveiling the entanglement features of the gapped quantum liquids (e.g. Mott liquid, Cooper pair insulator) obtained by destabilising the FL and MFL fixed points, we will consider the quantum fluctuations arising from RG relevant backscattering processes.

2. FL with circular Fermi surface in 2D:
In contrast to the electronic differentiation in the electronic dispersion that is present for the case of the FL and MFL phases on the square lattice, the FL phase for the case of a circular Fermi surface is isotropic in the angular direction. Thus, in order to understand the dependence of various entanglement and scattering signatures of the FL phase on the electronic differentiation, we have also studied the momentum space entanglement scaling of an FL phase proximate to a circular Fermi surface \([65]\). Though the nature of the forward scattering for both cases is similar, the difference in the nature of tangential scatterings makes them different. Further, the symmetry of the circular Fermi surface allows one to focus on the momentum space scaling for any one radial direction; this renders possible the simulation of a larger number of steps of the MERG, and thereby obtain a larger family of wavefunctions ranging between the IR and UV.

3. TLL in 1D:
In order to understand the dependence of the nature of the many-particle entanglement on the dimensionality of the underlying electronic system, we have studied the gapless quantum liquid phase in one spatial dimension known as the TLL \([58]\). While this phase corresponds to a non-Fermi liquid, the logarithmic entanglement scaling of this phase in real-space \([28, 43]\) hints at an underlying universality across dimensions. Due to the simplicity of the physics being confined to one spatial dimension, we can once again simulate a larger family of wavefunctions from the MERG flow.

4. Quantum fluctuations proximate to a quantum critical point:
An important class of gapless quantum matter are those found at quantum critical points (QCPs). Here, we study the entanglement features of the gapless quantum liquid obtained at the QCP obtained recently by some of us at optimal hole-doping in the 2D Hubbard model \([50, 57]\). The four nodal directions of the Fermi surface were observed to be gapless at this critical point, whereas other directions perpendicular to the Fermi surface were found to be gapped (and comprised of the 2D Mott liquid). We expect a volume law behaviour for the entanglement entropy arising from the strong quantum fluctuations present at such a QCP.

3.2. Gapped quantum liquids
In order to contrast the entanglement features of various gapless quantum liquids with their gapped counterparts, we have performed MERG calculations of two examples of the latter as well.

1. 2D Cooper Pair Insulator:
The Cooper pair insulator is an insulating phase emergent from a RG relevant pairing instability of the FL phase. The momentum space symmetry of the emergent CPI phase (\( C_4 \) or \( U(1) \)) \([49, 50, 65]\) corresponds to the symmetry of the Fermi surface of the parent metallic phase (optimally hole-doped Fermi surface of the square lattice and circular Fermi surface respectively).

2. 2D Mott liquid:
The 2D Mott liquid phase is an insulating phase emergent from an instability of the MFL phase of the 2D Hubbard model on the square lattice at 1/2-filling. It arises from RG relevant backscattering of charge pseudospin degrees of freedom \([48, 50, 69]\), and possesses the \( C_4 \) symmetry of the square Fermi surface of a half-filled tight-binding problem on the square lattice.
Figure 1. Momentum-space construction of reduced system studied under MERG. Four directions ($\hat{s}_0$, $\hat{s}_1$, $\hat{T}\hat{s}_0$, $\hat{T}\hat{s}_1$) are chosen normal to the Fermi surface in the 2D Brillouin zone; two ($\hat{s}_0$, $\hat{T}\hat{s}_0$, lines of red circles) are labelled as the nodal directions, and two ($\hat{s}_1$, $\hat{T}\hat{s}_1$, lines of blue circles) as the near-anti-nodal. Each $\hat{s}$-direction is comprised of 28 electronic Fock states (red and blue circles). Pairs of electronic Fock states with opposite momentum and spin ($\vec{k}^\uparrow$, $-\vec{k}^\downarrow$) form Anderson pseudospins. The bold blue and red lines show the two opposite momentum present at two ends, participating in the pseudospin formation.

4. Results

4.1. Forming a reduced system

In order to study the $T=0$ FL and MFL phases obtained recently for the Hubbard model \cite{50,57}, we consider the square Fermi volume of the $1/2$-filled tight-binding problem on the square lattice and divide the momentum-space window of low-momenta around the square Fermi surface into several directions normal to it (see figure 1). We note that electrons among all the directions perpendicular to the Fermi surface interact with one other, and will therefore contribute to the renormalisation group computation. We consider scattering processes among all such electronic states in implementing the URG and MERG formalisms.

Our investigations of entanglement involve simulating the RG evolution of many-body wavefunctions. For this, we apply iteratively the inverse many-particle unitary transformations of the URG method to the IR ground state wavefunction of, say, the FL (see Method). This generates quantum fluctuations by coupling the electronic Fock states near the Fermi surface to those farther away via forward and tangential scattering processes. Note that any two-particle scattering process can always be decomposed in forward, backward, and tangential scattering. In this way, we obtain a family of wavefunctions ranging from IR to UV, and involving a systematic growth of the size of the effective Hilbert space of interacting electrons. This $k$-space entanglement renormalisation group (MERG) process is, however, rendered very challenging due to the fact that the many-particle Hilbert space scales exponentially in the system size. In order to make the MERG analysis tractable, we construct a reduced subspace of the FL and MFL systems in $k$-space for an underlying 2D square lattice (figure 1). We consider only the quantum fluctuations generated by scattering processes along only four such directions ($\hat{s}_0$, $\hat{s}_1$, $\hat{T}\hat{s}_0$, $\hat{T}\hat{s}_1$). The direction $\hat{s}_0$ and its inverse ($\hat{T}\hat{s}_0$) are along the nodal direction of the square Brillouin zone, while $\hat{s}_1$ and its inversion ($\hat{T}\hat{s}_1$) are near the anti-nodal direction. The Fock states residing on the four $\hat{s}$-directions considered form a minimal set that captures all the possible (forward, backward, and tangential) scattering processes. We have used this construction to capture difference and universality among various gapless and gapped quantum liquids.

As a study of the 112 fermionic states (i.e. a Hilbert space dimension of $\sim 10^{33}$) is intractable, we focus the study on the quantum fluctuations of the holon (pairs of unoccupied electronic Fock states) and doublon (pairs of occupied electronic Fock states) degrees of freedom in the hole-doped 2D Hubbard model. As shown in figure 1, this corresponds to a smaller sub-space defined in terms 56 pseudospin degrees of freedom (defined in terms of electronic Fock states with opposite momenta) and a reduced Hilbert space size $\sim 10^{16}$ \cite{48,49,69,88}. A further drastic simplification is made by ignoring the RG irrelevant backscattering related quantum fluctuations: this allows us to study a reduced system of 28 pseudospins residing on only
one side of the Brillouin zone, say, the $\hat{s}_0$ and $\hat{s}_1$ directions. In each of these directions, we have chosen 14 pseudospin states, seven outside and seven inside the Fermi surface. We have labelled all 28 pseudo-spins using integers in figure 2. Specifically, Fock states residing outside or inside the Fermi surface along a particular $\hat{s}$—direction are labelled such that states nearest/farthest to the Fermi surface are associated with highest/lowest integer respectively. Further, along a particular $\hat{s}$—direction, states inside the Fermi surface are labelled by a higher integer value than those outside the Fermi surface.

The MERG proceeds via the iterative application of a set of unitaries $(U^j_{(j)}, j \in \{1, 2, \ldots, 6\})$ to the IR fixed point ground state and lowest-lying excited state wavefunctions of the gapless FL and MFL liquids that systematically couple the 4 electronic Fock states $(6, 13, 20, 27)$ lying within the window proximate to the Fermi surface to the 24 Fock states lying outside.

The IR wavefunctions for the ground state and lowest-lying excited states of the reduced system taken for the FL and MFL phases are shown in figure 3. The IR fixed-point ground state involving the four Fock states $(6, 13, 20, 27)$ are identical for both the FL and MFL (see figures 3(a) and (d)): all state below the Fermi surface are occupied, and those above it are unoccupied. This is a consequence of the fact that both these gapless liquids have been established as satisfying Luttinger’s theorem \[52–58\]. The lowest lying excited states of the FL and MFL involve configurations with holes below the Fermi surface and occupied states above it. We consider two such possible excited states in the reduced model: the near-anti-node (figure 3(b) for the FL and (e) for the MFL) and nodal (figure 3(c) for the FL and (f) for the MFL) excited states. The IR ground state of the FL is written in terms of the pseudospin degrees of freedom as $|\Psi_{FL,0}\rangle = |\phi_{FL}^{E}\rangle \otimes |\Psi_{IOM}\rangle_{(0)}$, where $|\phi_{FL}^{E}\rangle = |\uparrow_{20}\uparrow_{27}\rangle \otimes |\downarrow_{6}\downarrow_{13}\rangle$ represents the window of Fock states $(6, 13, 20, 27)$ proximate to the Fermi surface and the configuration of the Fock states lying outside these windows (whose occupation numbers correspond to IOMs of the URG method) is given by
Figure 4. Scaling of entanglement entropy ($S_{EE}$) and number fluctuations ($\Delta N$) within a window in $k$-space with window size ($\Lambda$) for gapless liquids. (a) $S_{EE}$ and (b) $\Delta N$ for FL ground state along nodal direction $\delta_0$, (c) $S_{EE}$ and (d) $\Delta N$ for MFL ground state along nodal direction $\delta_0$, (e) and (f) show scaling of $S_{EE}$ for excited states of FL along nodal ($\delta_0$) and antinodal ($\delta_1$) directions respectively. Similarly, (g) and (h) show scaling of $S_{EE}$ for excited states of MFL along nodal ($\delta_0$) and antinodal ($\delta_1$) directions respectively. While four MERG steps where performed for case (a), five were performed in all other cases (b), (e), (f), (c), (d), (g), (h).

![Image of Figure 4](image_url)

The first step of the MERG then couples the four Fock states ($5, 12, 19, 26$) with ($6, 13, 20, 27$) via the application of the unitary operator $U_i^\dag$, $\langle \Psi_4^\dag \rangle = U_i^\dag \langle \Psi_0^\dag \rangle$, and so onwards till we obtain a UV wavefunction that is unitarily connected to the IR state we started from.

We then employ the UV wavefunction to study the scaling of the von-Neumann entanglement entropy of a subsystem in $k$-space with its size. This is done by first selecting a block in $k$-space of size ($2\Lambda$), chosen symmetrically about the Fermi surface and along the $\delta_0$ and $\delta_1$ directions. Small $k$-space blocks contain degrees of freedom corresponding to the largest real-space lengthscales and vice versa. By tracing over all degrees of freedom except those within the chosen block, a reduced density matrix is then obtained for the block ($\rho_{\Lambda}$). The von-Neumann entanglement entropy ($S_{EE} = -\text{Tr}[\rho_{\Lambda} \ln \rho_{\Lambda}]$) is computed for the $k$-space block. Understanding the entanglement signatures in the UV wavefunctions of FL and MFL ($\langle |\Psi_j\rangle, j \geq 0 \rangle$) is the primary goal of this work.

### 4.2. Scaling of $S_{EE}$ for gapless quantum liquids

Figures 4(a) and (c) show the scaling of $S_{EE}$ $k$-space block size ($\Lambda$) along the nodal direction of the UV ground states of the FL and MFL obtained from the optimally doped and overdoped 2D Hubbard model respectively. Both gapless quantum liquids show logarithmic scaling of block entanglement in $k$-space with decrease in block size $\Lambda$, i.e. upon focusing on quantum fluctuations lying proximate to the Fermi surface: $S_{EE} \sim \ln \Lambda$. The logarithmic scaling is clear for a smaller $\Lambda$ window near the Fermi surface, and deviations away from the logarithmic scaling are observed as $\Lambda$ is increased to large values. The deviations are, however, different for various types of phases. The momentum space entanglement signature suggests a corresponding logarithmic scaling in real-space as well (in large system size limit), with deviations for small and intermediate real-space scales [89, 90].

Further, the number fluctuations ($\Delta \Lambda = \sqrt{<N^2> - <N>^2}$) within the block $\Lambda$ also show similar logarithmic scaling for both the FL and MFL: $\Delta N \sim \ln \Lambda$, as shown in figures 4(b) and (d) respectively. These results correspond to a modified area-law for these gapless liquids (upon accounting for all directions normal to the Fermi surface): 

$$S_{EE}^\text{gapless} \sim \Lambda \ln \Lambda.$$  

Further, this confirms the existence of long-range entanglement in $k$-space encoded within the UV wavefunctions of both these gapless quantum liquids, arising from long-wavelength quantum fluctuations at low energies.

The results for the scaling of $S_{EE}$ with $\Lambda$ for the two excited state configurations (i.e. along the nodal ($\delta_0$) and near-anti-nodal ($\delta_1$) directions normal to the Fermi surface) of the FL and MFL phases are shown in figures 4(e), (f) and (g), (h) respectively. While all results show a logarithmic scaling for $S_{EE}$ with $\Lambda$, $S_{EE} \sim (1/\alpha) \ln \Lambda$, the values of the coefficient $\alpha_{\text{FL}}$ are similar for the two excited states of the FL ($\alpha_{\text{FL}} \sim 2$). This likely arises from the presence of strong forward scattering normal to the Fermi surface, as well as tangential scattering between the two $\delta$ directions. On the other hand, the values of $\alpha_{\text{MFL}}$ are quite different for the two excited states of the MFL, and $\alpha_{\text{MFL}} \gg \alpha_{\text{FL}}$. This displays the strong electronic differentiation among the $\delta$ directions in the MFL phase, and the fact that only strong forward scattering processes
determine the entanglement of this gapless liquid. Further, we have also found that both the 2D FL with circular Fermi surface and the 1D TLL show logarithmic scaling behaviour of $S_{\text{EE}}$ with $\Lambda$. In figures 5(a) and (b), we show that the coefficient $\alpha \rightarrow 2$ for both these gapless quantum liquids upon computing the $S_{\text{EE}}$ from wavefunctions after a sufficiently large number of MERG steps. These results suggest that while the FL and several non-Fermi liquids show the same modified area-law for the scaling of $S_{\text{EE}}$ with $\Lambda$, the value of the proportionality constant $1/\alpha$ is sensitive to the nature of the underlying Fermi surface.

4.3. Scaling of $S_{\text{EE}}$ for gapped quantum liquids

In order to understand whether $k$-space entanglement is a good diagnostic of fermionic criticality, we need to study RG relevant quantum fluctuations of various kinds. Having accounted above for the forward and tangential scattering related quantum fluctuations that lead to gapless quantum liquids, we now turn to the gapped quantum liquids that arise from backscattering. The ground state wavefunctions for the states studied below have been obtained from [48, 49, 58, 65], and adopted to the reduced system shown in figures 1 and 2 similarly to that discussed earlier for gapless liquids. Figure 6(a) shows the linear scaling of $S_{\text{EE}}$ with $\Lambda$ for the gapped 2D Mott liquid along the nodal direction $\hat{s}_0$. As we now argue, this can be understood as an area-law scaling in momentum space. The nature of the effective Hamiltonian for the 2D Mott liquid [48] and 2D Cooper pair insulator [49] obtained from instabilities of a strongly nested Fermi surface involves a strong coupling between all the electronic degrees of freedom within the gapped low-energy window lying proximate to the Fermi surface. As a result, the ground state wavefunction of these gapped quantum liquid phases cannot be decomposed into a direct product of states along each $\hat{s}$-hat direction normal to the Fermi surface [48, 91], leading to the area-law momentum space scaling of $S_{\text{EE}} \sim \Lambda$ for the 2D Mott liquid observed in figure 6(a).

On the other hand, the momentum space scaling of $S_{\text{EE}} \sim \ln \Lambda$ (with a proportionality coefficient $\alpha \sim 2$) for a gapped 2D Cooper pair insulator phase along a given direction normal to a circular Fermi surface observed in figure 6(b) likely arises from it is non-nested nature [65], while the area law scaling (similar to figure 6(a)) is expected for the Cooper pair insulator arising from the strongly nested 2D Fermi surface [49]. Further, the nature of the parent metal (from which these gapped phases are emergent) also appears to be an important factor: the MFL is the parent metal for the 2D Mott liquid [48, 50], while the FL is the parent meal for the CPI studied in figure 6(b) [65]. The Mott liquid and Cooper pair insulating phases are special types of gapped quantum liquids comprised of pseudo-spin degrees of freedom.

In order to understand the dependence of these scaling laws of $S_{\text{EE}}$ on the nature of the underlying Fermi surface, we studied the case of the QCP obtained recently in the optimally-doped 2D Hubbard model [49]: here, the four nodal directions contain gapless MFL liquids while the rest of the Fermi surface is composed of a gapped 2D Cooper pair insulator (CPI) liquid. Thus, in figures 6(c) and (d), we present the scaling of $S_{\text{EE}}$ with $\Lambda$ for the nodal MFL (along $\hat{s}_0$) and near-anti-nodal CPI liquid (along $\hat{s}_1$) respectively. While the CPI again shows the logarithmic scaling behaviour (albeit with a coefficient $\alpha < 1$), the MFL shows linear scaling of $S_{\text{EE}}$ with $\Lambda$ and reflects on an enhanced long-range entanglement along the nodal directions due to an underlying quantum critical Fermi surface.

![Figure 5](image-url)
5. Discussions

5.1. A modified area-law in $k$-space

By studying a reduced subspace containing the relevant $k$-space scattering processes (i.e. along and between certain directions normal to the Fermi surface), our results reveal a modified area-law scaling in both the block entanglement ($S_{EE}$) and number fluctuations ($\Delta N$) for the UV ground states and lowest lying excited states unitarily connected to the FL, MFL and TLL gapless quantum liquids (figures 4 and 5): $S_{EE} \propto \log(\Lambda/\Lambda_F)$ and $\Delta N \propto \log(\Lambda/\Lambda_F)$ (where $\Lambda_F$ is the Fermi momentum, and which we set to unity hereafter). This is in agreement with findings for the real-space block entanglement entropy and number fluctuations computed from the IR state of the FL, and reveals that the forward and tangential scattering related RG relevant quantum fluctuations near the Fermi surface encode the long-range $k$-space entanglement of the modified area-law even at UV scales. This is consistent with the conjectured duality mapping between the real-space and $k$-space subsystem entanglement entropies of a system of non-interacting fermions in $d$-spatial dimensions [92]. Further, it shows that the $k$-space quantum fluctuations of the UV theory are systemically converted into their real-space counterparts along the URG flow.

5.2. Universality of gapless liquids

Importantly, the similarity of the results obtained for both FL, MFL and TLL offer striking evidence on the universality of the entanglement properties of various gapless quantum liquids that arise from systems of interacting electrons. This suggests a unification of the FL paradigm with non-Fermi liquid metals that are qualitatively different in terms of their low-lying excitations. Interestingly, we find that the proportionality constant $\alpha$ in the relation $S_{EE} \propto (1/\alpha)\ln(\Lambda)$ is very different for the different metallic systems: $\alpha \sim 2$ for the FL (whether obtained from a 2D Fermi volume with a $U(1)$ symmetric circular Fermi surface (figure 5(a)), or from the strongly rounded $C(4)$ symmetric Fermi surface of the FL obtained in the 2D Hubbard at large hole-doping [49] (figures 4(a), (b), (e) and (f))) and the TLL (with a two point Fermi surface, figure 5(b)). On the other hand, we find $\alpha \gg 1$ for the MFL obtained in the 2D Hubbard at optimal hole-doping [49] (figures 4(c), (d), (g) and (h)). We believe that this difference arises from the fact that the MFL is governed by strongly RG relevant forward scattering in directions normal to the Fermi surface, and that MFL states on different normalizes show variations arising from the differentiation in the electronic dispersion everywhere along the $C(4)$ symmetric Fermi surface of the 2D square lattice [48, 49]. By contrast, the presence of subdominant RG relevant tangential scattering processes in the FL between different directions normal to the Fermi surface remove the electronic differentiation inherent in the MFL [49].

5.3. Classification of gapped quantum liquids

We also analyse the $k$-space block entanglement entropy ($S_{EE}$) of the UV states that are unitarily connected to the gapped topologically ordered quantum liquids emergent from the destabilisation of the Fermi surface of the FL and MFL gapless metals [48–50, 65]. As shown in figures 6(a) and (b), we find that while the CPI state for the 2D FL with circular Fermi surface [65] shows logarithmic scaling of $S_{EE}$ with $\Lambda$ (with the inverse coefficient $\alpha \sim 2$), the Mott liquid state in the 2D Hubbard model at 1/2-filling [48, 50] shows an area-law scaling with $\Lambda$. We believe that this striking difference arises from the fact that the 2D Mott liquid is emergent from the backscattering related instability of a strongly nested singular Fermi surface of the underlying 2D tight-binding model (i.e., containing van Hove singularities), while the 2D CPI state emergent from the minimal nesting of a circular Fermi surface (i.e. that of diametrically opposite Fermi points).
Further, the parent metal of the former is the 2D MFL [48], while that of the latter is the 2D FL [65]. Thus, we conclude that the entanglement features of a gapped quantum liquid appear to be determined by the nature of the Fermi surface and parent metal from which it emerges. Further evidence of this dependence is seen by considering the scaling behaviour of the $k$-space block entanglement entropy ($S_{EE}$) of the quantum critical state of matter found recently in the 2D Hubbard model on the square lattice at optimal doping [49]. At this novel QCP, only the four nodal directions of the 2D Brillouin zone for the square lattice are found to be comprised of gapless metallic MFLs, while there exists a gapped 2D CPI quantum liquid in the antinodal regions of the rounded C(4) symmetric hole-doped Fermi surface of the underlying 2D tight-binding model. We find that $S_{EE}$ scales logarithmically with $A$ for this 2D CPI state (figure 6(d)), while the nodal MFLs (about four point-like singular quantum critical Fermi surfaces) show a volume-law scaling (figure 6(c)). Indeed, this appears to be consistent with the findings of [64] for $S_{EE}$ of a gapless metal at a quantum critical Fermi surface.

5.4. Holographic evolution of entanglement

Finally, the URG method has also been shown to provide an explicit demonstration of the holographic principle [93, 94]. By this, we mean that the evolution of the Hamiltonian from UV to IR via the unitaries corresponds to a tensor network that admits an exact holographic mapping, i.e. it provides a precise relationship between the coupled degrees of freedom within one layer of the network with those that are emergent in the next [95, 96]. Further, this generates a eigenstate coefficient tensor network possessing an entanglement metric [57]. The renormalisation of the entanglement corresponds to the evolution of the many-particle Hilbert space geometry. By employing this method, we unveil several distinct holographic signatures of Fermi and non-Fermi liquids in measures of entanglement (see appendix B) and many-body correlations (see appendix C). An important challenge for the future will be to characterise the entanglement properties of gapless liquids using multipartite quantum information measures, as has been achieved recently for topologically ordered gapped liquids [97]. Other future studies include analysing the deviations from logarithmic scaling for any subtle signatures of universality, and investigating the microscopic origin of long-ranged entanglement in quantum critical liquid studied above as well as elsewhere [98, 99].

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Details of the URG flow equations

We obtain the effective Hamiltonian for the FL and MFL IR fixed point theories for the case of the half-filled square lattice by performing a URG analysis of the 2D Hubbard model [48, 49]. The Hamiltonian RG flow equation is

$$\Delta H_{(j)} = \sum_l \text{Tr}_{(j)}(\epsilon_{l,j}^0)\tilde{G}_{(j),l} \epsilon_{l,j}^0 \text{Tr}_{(j)}(H_{(j)}\tilde{G}_{(j),l}) ,$$  

(A.1)

where $\tilde{G}_{(j),l} = [\omega - \tilde{n}_{l,j}\text{Tr}_{(j)}(H_{(j)}\tilde{n}_{l,j})]^{-1}$ and $H_{(j)}^{\text{D}}$ is the diagonal part of the Hamiltonian at the RG step $j$. This change in the Hamiltonian is then decomposed in different scattering channels: namely, forward scattering $\Delta H_{(j)}^{F}$, backscattering $\Delta H_{(j)}^{B}$, tangential scattering $\Delta H_{(j)}^{T}$ and the three-particle interaction term $\Delta H_{(j)}^{I}$. This leads to the following RG equations

$$\Delta H_{(j)}^{F} = \sum_{k,k',l'} c_{k',l'}^{\dagger}c_{k,l} \frac{4V(\delta_{l,j}^{(j)})^2\tau_{l,j}^T\tau_{l,j}^T}{G_{l,j}^{-1} - V(\delta_{l,j}^{(j)})^2\tau_{l,j}^T\tau_{l,j}^T} , \quad \text{where } \tau_{l,j} = \left( \tilde{n}_{l,j} - \frac{1}{2} \right) ,$$

(A.2)

$$\Delta H_{(j)}^{B} = \sum_{k,k',l'} c_{k',l'}^{\dagger}c_{k,l} \frac{4V(\delta_{l,j}^{(j)})^2K(\delta_{l,j}^{(j)})\tau_{l,j}^T\tau_{l,j}^T}{G_{l,j}^{-1} - V(\delta_{l,j}^{(j)})^2\tau_{l,j}^T\tau_{l,j}^T} ,$$

(A.3)
\[ \Delta H^{\gamma}_{(j)} = \sum_{k,k',m} \epsilon_{k,m} \epsilon_{k',m'} \frac{(\Gamma^{(j)}_2 L_{ij}^2 - L_{ij}^2 - L_{ij}^2)}{\omega - \epsilon_{ij}} \]  

\[ \Delta H^{\beta}_{(j)} = \sum_{k',k''} \epsilon_{k',k''} \epsilon_{k',k''} \frac{V^{(j)}_1 (\delta) V^{(j)}_1 (\delta') \tau_{k,k''}}{\omega - \epsilon_{ij}} \times c_{k',k''} c_{k',k''} \]  

Further, the components of the pseudospin \( \vec{L} \) and \( L_+ = \sum_{m} \epsilon_{j,m} \epsilon_{j,m'} \)  

\[ L_{ij}^2 = \frac{1}{2} \sum_{m} (\tilde{\xi}_{j,m} + \tilde{\xi}_{j,m'}) - 1 \) etc follow the standard spin-algebra. Further, the electronic Fock states are labelled as \( i_1 : (\tilde{K}_{\Lambda_i,\sigma}), i_2 : (\tilde{K}_{\Lambda_i+\delta,,\tilde{\gamma}_{i},-\sigma}), i_3 : (\tilde{K}_{\Lambda_i,+\sigma}). \)  

\[ G^{(j)}_{i,j} = \left( \epsilon_{ij} + \epsilon_{ij} / 2 - \Delta \mu_{\text{eff}} - \omega \right) \] where \( \omega < 0 < \tilde{\omega} \) and \( \mu_{\text{eff}} = - \Delta U / 2 \).
Figure B1. Three choices of momentum-space blocks to compute block entanglement entropy from each member of the family of wavefunctions obtained from the MERG. **Block-1** includes all the nodes on the direction $\hat{s}_1$. **Block-2** includes the nodes outside the Fermi surface on $\hat{s}_1$ and inside the Fermi surface on $\hat{s}_0$. **Block-3** includes all the nodes residing outside the Fermi surface.

As shown in the figure B1, we have chosen three blocks of states in $k$-space blocks comprising a total of 14 states. Subsystem choice 1 consists of states along the near-anti-nodal direction $\hat{s}_1$ normal to the Fermi surface. Thus, the entanglement entropy measured here involves the entanglement between the states in $\hat{s}_1$ and those in $\hat{s}_0$ (i.e. the nodal direction normal to the Fermi surface), thus offering a measure of the entanglement driven by quantum fluctuations generated via tangential scatterings between the two $\hat{s}$ directions. Subsystem choice 2, on the other hand, consists of states outside the Fermi surface in $\hat{s}_1$ but in $\hat{s}_0$; the entanglement entropy measures here the entanglement arising from quantum fluctuations involving both tangential and forward scatterings. Another subsystem choice is that shown in block 3, involving states outside the Fermi surface in both the $\hat{s}$ directions. The entanglement entropy here measures the $k$-space entanglement driven due to excitations that lead from within states the Fermi surface to those outside it, arising from quantum fluctuations involving both forward and tangential scattering events.

The results presented in figures B2–B4 of the variation of the MBE with the number of MERG steps show distinctions between the FL and MFL states. As block 1 is very sensitive to entanglement arising from quantum fluctuations due to tangential scattering events, the stark difference between the numbers achieved by the MBE in the UV for the case of block 1 in the FL and MFL reveals the dominance of tangential scattering events in the former over the latter. On the other hand, the similar values of the MBE achieved in the UV for the blocks 2 and 3 shows the importance of forward scattering events in both the FL and MFL phases. Taken together, these results show that while the IR ground state of the FL is obtained from the decoupling of quantum fluctuations that arise from both forward and tangential scattering, the MFL is obtained from the decoupling of quantum fluctuations that arise almost completely from forward scattering.

**B.2. Ryu–Takayanagi upper bound for entanglement**

Given that the URG method (from which the unitaries are obtained for the MERG) is known to correspond to an exact holographic mapping [57, 95], we will now check whether the entanglement entropies of various
sizes of subsystems taken from the family of many-body wavefunctions generated by the MERG satisfy the Ryu–Takayanagi upper bound \[ S_{n} \leq n S_{1}^{\text{max}} \] (B.1), where \( S_{n}^{\text{max}} \) corresponds to the largest entanglement entropy of a subsystem containing \( n \) constituents (and \( n \geq 1, n \in \mathbb{Z} \)). For this, we first compute the entanglement entropy of various block sizes across the MERG energy scales ranging from the UV to the IR fixed point, as shown in the figure B5. The blocks have been chosen to lie along the near-anti-nodal direction \( \hat{s}_1 \) and outside the Fermi surface. The smallest block size contains one pseudospin state (6), while the largest contains all seven states residing on the \( \hat{s}_1 \) direction outside the Fermi surface (see legends in inset). Figure B5 shows that the block entanglement entropies increase from the IR and increase linearly in the UV for the FL (i.e. for the largest block size (6, 5, 4, 3, 2, 1, 0)), while they saturate for the MFL. This increase in entanglement in the FL phase is likely due to the strong growth of tangential scattering in addition to forward scattering. Figure B6 shows a comparison of \( S(6)^{\text{max}} \) (associated with the largest block size (6, 5, 4, 3, 2, 1)) for both the FL and MFL with \( 6 \times S(1)^{\text{max}} \), and reveals that the Ryu–Takayanagi upper bound is indeed obeyed for both these gapless quantum liquids. The growth of the upper bound in the UV energy scale for the FL phase (as compared to the saturation witnessed for the MFL phase) is again likely due to the presence of strong tangential scattering processes.
**Figure B5.** Top: variation of entanglement entropy of blocks of momentum-space states of varying sizes with MERG step number for the FL ground state. Different coloured curves correspond to different block sizes (shown in inset). Bottom: scaling of entanglement entropy of blocks of momentum-space states of varying sizes with respect to MERG step number with MERG step number for the MFL ground state.

**Figure B6.** MERG evolution of the Ryu–Takayanagi upper bound. The blue plus symbols represent the entanglement entropy of the largest block of block size six, and the red pentagons represent the upper bound by \( nS(1)_{\max} \), where \( n = 6 \). The top and bottom figures show that the scaling of the largest block entanglement entropy is satisfied for the FL and MFL phase respectively.

### B.3. Mutual information

Mutual information between two subsystems measures the total classical as well as quantum correlations between them. Definition of mutual information between two subsystems \( A \) and \( B \) is given as

\[
I(A : B) = S(A) + S(B) - S(AB) \geq 0 ,
\]  

(B.2)
where $S(A_i)$ is the von-Neumann entanglement entropy of the subset $A_i$ with the rest etc. Here, we compute the mutual information between two momentum-space blocks. For this, we divide the entire 28 pseudospin system into four blocks ($A, B, C, D$), as shown in figure B7, and compute the mutual information between some of them for each member of the family of wavefunctions generated by the MERG. The scaling of some of these mutual informations is presented in figure B8. We are mainly interested in the mutual informations $I_2(A, B)$ and $I_2(A, D)$: which capture the correlation arising from tangential and forward scattering. Note that $I_2(A, B)$ captures the correlations between the blocks $A$ and $B$ connected via tangential and forward scattering, whereas $I_2(A, B)$ captures that between $A$ and $D$ connected only through forward scattering. Figure B8(left) shows that $I_2(A, B)$ for the ground and excited states of the FL increases with the number of MERG steps, in keeping with the steady growth in tangential scattering events as the MERG proceeds towards the UV. This is in contrast with our finding for ground and excited states of the MFL case, which remain zero throughout the MERG; this is consistent with RG irrelevant tangential scattering processes for the MFL. Further, the $I_2(A, D)$ among the blocks $A, D$ shown in figure B8(right) shows the importance of forward scattering is far greater in the UV for the MFL than it is for the FL.

**B.4. Tripartite information**

Tripartite information measures the joint entropy among three subsystems $(A, B, C)$ [5, 97, 101], and is defined as

$$I_3(A, B, C) = S(A) + S(B) + S(C) - S(A \cup B) - S(B \cup C) - S(C \cup A) + S(A \cup B \cup C) ,$$

$$= I_2(A, B) + S(C) - S(B \cup C) - S(C \cup A) + S(A \cup B \cup C) . \quad (B.3)$$

We are mainly interested in the tripartite information among the three pseudospin states $(20, 27, 6)$ proximate to the Fermi surface: the states 6, 27 and 20, 27 are connected via forward and tangential scattering respectively. The MERG scaling of $I_3(6, 20, 27)$ for the FL and MFL ground states is shown in figure B9; while $I_3(6, 20, 27)$ is found to grow monotonically for the FL as the MERG proceeds to the UV, it is found to remain strictly zero throughout for the MFL. As seen above for the mutual informations, the growth of $I_3(6, 20, 27)$ for the FL is clearly driven by the growth in tangential scattering processes in the UV. Interestingly, the strict vanishing of $I_3(6, 20, 27)$ for the MFL ground state appears to show a constraint among various entanglement measures
Appendix C. RG evolution of various many-body correlations

C.1. Correlations generated due to pseudospin-flip scattering

The low-energy window is formed by the states \((6, 13, 2, 27)\) proximate to the Fermi surface at the IR fixed point, and increases as the MERG progresses towards the UV. We define a correlation measure \(C_{sf}^{(j), \hat{h}}\) in order to capture the pseudospin-flip correlations among the pseudospin states present in the low-energy window near the near-anti-nodal direction \(\hat{s}_1\)

\[
C_{sf}^{(j), \hat{h}} = \left( \langle \Psi^{(j)} | \sum_{i<j} U_{em}^{(j)} (S_i^+ S_j^- + S_j^+ S_i^-) | \Psi^{(j)} \rangle \right) / \left( \sum_{i<j} 1 \right),
\]

where \(| \Psi^{(j)} \rangle\) is the wave function at the \(j\)th MERG step (i.e. \(j = 0\) represents the IR fixed point and increases towards the UV), \(U_{em}^{(j)}\) is the set of pseudospin states lying in the direction \(\hat{s}_1\) within the low-energy window at the \(j\)th MERG step. (Note that \(U_{em}^{(j)}\) does not include the decoupled electronic Fock states present outside the emergent window.) As shown in figure C1(left), the MERG scaling of the correlation \(C_{sf}^{(j), \hat{h}}\) for the FL ground and excited states increases and finally saturates in the UV scale. This is due to the fact that the phase space for tangential scattering grows monotonically with the MERG. On the other hand, \(C_{sf}^{(j), \hat{h}}\) correlation shows a non-monotonic behaviour for the MFL case (figure C1(right)), likely arising due to an initial increase in the phase space for forward scattering at the beginning of the MERG flow and its eventual decrease as the flow reaches the UV (and highlights the absence of tangential scattering in the MFL).

C.2. MERG scaling of the occupation of pseudospin states

Figure C2 shows the MERG flow of the average number occupation of the pseudospin states 20 and 27 (just below the Fermi surface) for both the FL and MFL ground states shown. We see that both \(\langle \hat{n}_{20} \rangle\) (blue triangle) and \(\langle \hat{n}_{27} \rangle\) decay much faster for the FL (curves with blue triangles and red circles respectively) than for the MFL (curves with blue diamonds and red squares respectively). The similar nature of decays for the FL shows the weak electronic differentiation between the directions \(\hat{s}_0\) and \(\hat{s}_1\) due to the strong tangential scattering. On the other hand, \(\langle \hat{n}_{20} \rangle\) does not decay at all under the MERG flow for the MFL ground state while \(\langle \hat{n}_{20} \rangle\) decays slowly, displaying the electronic differentiation inherent in the MFL.

C.3. Non-local strings

Another way by which to probe how tangential and forward scattering shape the FL and MFL states can be learnt by computing the MERG evolution of the expectation value of the \(k\)-space non-local operator \(Z = \prod_{i} \sigma_i^z (Z)\) for appropriately chosen strings of pseudospin states in \(k\)-space. As shown in figure C3, we have chosen two strings of length 14 nodes each for our analysis. The string \(AD\) passes across the Fermi
Figure C1. Variation of the correlations generated by pseudospin-flip scattering processes ($\langle S^+ S^- + \text{h.c.} \rangle$) within the low-energy window with MERG step number. Left: red and blue curves show variation of the correlation for the FL ground and excited state respectively. Right: red and blue curves show variation of the correlation scaling for the MFL ground and excited states respectively.

Figure C2. Variation of the expectation value of the occupation of the states 20 and 27 with MERG step number for the FL and MFL ground states.

Figure C3. Two choices of the non-local string parity operators. Left: string AD containing states in only one direction perpendicular to the Fermi surface ($\hat{s}_1$), and stretching across the Fermi surface. Right: string AB lying outside the Fermi surface but spread over both $\hat{s}_0$ and $\hat{s}_1$.

surface but along only the near-anti-nodal direction $\hat{s}_1$, whereas the string AB lies completely outside the Fermi surface but across the two $\hat{s}$ directions. The operator $Z$ corresponds to a parity operator defined for the pseudospin states in $k$-space, i.e. sign of $\langle Z \rangle$ depends on the number of down pseudospins: negative for an odd number of down pseudospins (odd parity) and positive for an even number of down pseudospins (even parity). First, we find from figure C4(left) that $\langle Z_{AD} \rangle$ undergoes a crossover for the FL ground state under the MERG flow from a state with a well-defined parity ($\langle Z_{AD} \rangle = -1$) to one that does not have a well-defined parity ($\langle Z_{AD} \rangle = 0$) in the UV likely due to the increased tangential scattering. On the other hand, $\langle Z_{AD} \rangle = -1$ for the MFL throughout the MERG flow, i.e. the MFL ground state maintains its parity under forward scattering. Secondly, $\langle Z_{AB} \rangle$ undergoes a crossover under the MERG flow for both the FL and the MFL from states with a well-defined parity ($\langle Z_{AB} \rangle = +1$) to one that does not have a well-defined parity ($\langle Z_{AB} \rangle = 0$). In the MFL, this arises from the absence of tangential scattering between subsystems A and B.
Figure C4. Variation of the expectation value of the non-local string operator $\langle \hat{Z} \rangle$ for two different string choices AD (left plot) and AB (right plot) (shown in figure C3) with MERG step number. The blue curve shows variation for the MFL ground state, and the red curve for the FL ground state.

Figure C5. Left: variation of four fidelities $F(\alpha\beta,i) = \langle \psi_{\alpha,0}^{FL} | \psi_{\beta,i}^{MFL} \rangle$, constructed using the overlaps of the FL and MFL ground and excited states (i.e. the indices $\alpha$ and $\beta$ can either be $G$ (ground state) or $E$ (excited state)), with MERG step number. Right: this shows the square of the absolute value of the four fidelities $F(\alpha\beta,i) = \langle \psi_{\alpha,0}^{FL} | \psi_{\beta,i}^{MFL} \rangle$.

On the other hand, the vanishing of $\langle Z_{AB} \rangle$ for the FL likely arise from the tangential scattering events that connect states inside the Fermi surface to those outside (D with B, and C with A).

C.4. Fidelity in RG scale

Fidelity is a measure of overlap between two wavefunctions defined as $F(\alpha,\beta) = |\langle \psi_\alpha | \psi_\beta \rangle|^2$. We now study the fidelity between the wavefunctions of the FL and MFL ground and excited states as the MERG proceeds from IR to UV. We define $|\psi_{G,0}^{FL} \rangle$ and $|\psi_{G,0}^{MFL} \rangle$ as the IR fixed-point ground (G) and excited (E) state wavefunctions for the FL and MFL phases respectively. Similarly, $|\psi_{G,i}^{FL} \rangle$, $|\psi_{G,i}^{MFL} \rangle$ are the ground (G) and excited (E) state wavefunctions for the FL and MFL at the $i$th MERG step. Then, we define the four possible fidelities between the ground and excited state wavefunctions of the FL and MFL phases at the 0th and $i$th MERG steps as given by $F(\alpha\beta,i) = \langle \psi_{\alpha,0}^{FL} | \psi_{\beta,i}^{MFL} \rangle$, where the indices $\alpha$ and $\beta$ can either be $G$ (ground state) or $E$ (excited state). The MERG flow of the four different fidelities $F(\alpha\beta,i) = \langle \psi_{\alpha,0}^{FL} | \psi_{\beta,i}^{MFL} \rangle$ is shown in figure C5.

We can see from figure C5(left) that the fidelity between the FL and MFL ground states is perfect at the IR fixed point, $F_{GG,i=0} = 1$. This is simply because both these gapless quantum liquids satisfy Luttinger’s theorem, and have the same ground state. However, $F_{GG}$ falls monotonically to zero. On the other hand, the IR excited states of the FL and MFL are orthogonal to the common IR ground state and to one another, as indicated by the near vanishing of the other three fidelities at the zeroth MERG step. Further, this orthogonality is maintained under the MERG flow to the UV, as the unitary operations of the MERG maintain the orthogonality structure of these states. Figure C5(right) shows that the MERG evolution of the square of the real part of the fidelities $F(\alpha\beta,i)$ follows the same trend as $F(\alpha\beta,i)$ itself, reinforcing the fact that the phases of the four fidelities do not play an important role.
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References

[1] Zeng B et al 2019 Quantum Information Meets Quantum Matter (Berlin: Springer)
[2] Eisert J, Cramer M and Plenio M B 2010 Colloquium: Area laws for the entanglement entropy Rev. Mod. Phys. 82 277
[3] Falci G and Fazio R 2002 Scaling of entanglement close to a quantum phase transition Phys. Rev. B 71 045110
[4] Gogolin A and Preskill J 2006 Topological entanglement entropy Phys. Rev. Lett. 96 110404
[5] Klich I and Levitov L 2009 Quantum noise as an entanglement meter Phys. Rev. A 79 062104
[6] Levin M A and Wen X-G 2005 String-net condensation: a physical mechanism for topological phases Phys. Rev. B 71 045110
[7] Abele S and Zimborás Z 2007 The von Neumann entropy asymptotics in multidimensional fermionic systems Phys. Rev. B 85 140301
[8] Landau L D 1959 On the theory of the fermi liquid Sov. Phys.-JETP 3 920 (available at: http://www.jetp.ras.ru/cgi-bin/e/index/e/3/6/p920a=list)
[9] Landau L D 1957 Oscillations in a fermi liquid Sov. Phys.-JETP 5 101 (available at: http://jetp.ras.ru/cgi-bin/e/index/e/5/1/p101a=list)
[10] Landau L D 1959 On the theory of the fermi liquid Sov. Phys.-JETP B 70 (available at: http://jetp.ras.ru/cgi-bin/e/index/e/8/1/p702a=list)
[11] Swingle B 2010 Entanglement entropy and the fermi surface Phys. Rev. Lett. 105 050502
[12] Swingle B 2012 Conformal field theory approach to fermi liquids and other highly entangled states Phys. Rev. B 86 035116
[13] Wolf M M 2006 Violation of the entropic area law for fermions Phys. Rev. Lett. 96 010404
[14] Goev D and Klich I 2006 Entanglement entropy of fermions in any dimension and the Widom conjecture Phys. Rev. Lett. 96 100503
[15] Li W, Ding L, Yu R, Roscilde T and Haas S 2006 Scaling behavior of entanglement in two- and three-dimensional free-fermion systems Phys. Rev. B 74 075103
[16] Barthel T, Chung M-C and Schollwöck U 2006 Entanglement scaling in critical two-dimensional fermionic and bosonic systems Phys. Rev. A 74 023239
[17] Farkas S and Zimborás Z 2007 The von Neumann entropy asymptotics in multidimensional fermionic systems J. Math. Phys. 48 102110
[18] Helling R, Leschke H and Spitzer W 2010 A special case of a conjecture by Widom with implications to fermionic entanglement entropy Int. Math. Res. Not. 2011 1451–82
[19] Leschke H, Sobolev A V and Spitzer W 2014 Scaling of Rényi entanglement entropies of the free fermi-gas ground state: a rigorous proof Phys. Rev. Lett. 112 160403
[20] Ding W, Seidel A and Yang K 2012 Entanglement entropy of fermi liquids via multidimensional bosonization Phys. Rev. X 2 011012
[21] Osterloh A, Amico L, Falci G and Fazio R 2002 Scaling of entanglement close to a quantum phase transition Nature 416 608–10
[22] Osborne T J and Nielsen M A 2002 Entanglement in a simple quantum phase transition Phys. Rev. A 66 032110
[23] Vidal G, Latorre J I, Rico E and Kitaev A 2003 Entanglement in quantum critical phenomena Phys. Rev. Lett. 90 227902
[24] Calabrese P and Cardy J 2004 Entanglement entropy and quantum field theory J. Stat. Mech. P06002
[25] Klich I, Refael G and Silva A 2006 Measuring entanglement entropies in many-body systems Phys. Rev. A 74 032306
[26] Klich I and Levitov L 2009 Quantum noise as an entanglement meter Phys. Rev. Lett. 102 100502
[27] Hsu B, Grosfeld E and Fradkin E 2009 Quantum noise and entanglement generated by a local quantum quench Phys. Rev. B 80 235124
[28] Francis Song H, Rachel S and Le Hur K 2010 General relation between entanglement and fluctuations in one dimension Phys. Rev. B 82 012405
[29] Francis Song H, Laflorencie N, Rachel S and Le Hur K 2011 Entanglement entropy of the two-dimensional Heisenberg antiferromagnet Phys. Rev. B 83 224410
[30] Francis Song H, Rachel S, Flindt C, Klich I, Laflorencie N and Le Hur K 2012 Bipartite fluctuations as a probe of many-body entanglement Phys. Rev. B 85 035409
[31] Calabrese P and Mintchev M and Vicari E 2012 Exact relations between particle fluctuations and entanglement in fermi gases Europhys. Lett. 98 20003
[32] Rachel S, Laflorencie N, Francis Song H and Le Hur K 2012 Detecting quantum critical points using bipartite fluctuations Phys. Rev. Lett. 108 110401
[33] Sütőstrunk R and Ivanov D A 2012 Free fermions on a line: asymptotics of the entanglement entropy and entanglement spectrum from full counting statistics Europhys. Lett. 100 60009
[34] Swingle B 2012 Rényi entropy, mutual information and fluctuation properties of fermi liquids Phys. Rev. B 86 045109
[35] Vicari E 2012 Entanglement and particle correlations of fermi gases in harmonic traps Phys. Rev. A 85 062104
[36] Eisler V 2013 Universality in the full counting statistics of trapped fermions Phys. Rev. Lett. 110 080402
[37] Klich I 2014 A note on the full counting statistics of paired fermions J. Stat. Mech. P11006
[38] Petrescu A, Francis Song H, Rachel S, Ristivojevic Z, Flindt C, Laflorencie N, Klich I, Regnault N and Le Hur K 2014 Fluctuations and entanglement spectrum in quantum Hall states J. Stat. Mech. P10005
[39] Calabrese P, Mintchev M and Vicari E 2011 Entanglement entropy of one-dimensional gases Phys. Rev. Lett. 107 020601
Calabrese P, Mintchev M and Vicari E 2012 Entanglement entropies in free-fermion gases for arbitrary dimension Europhys. Lett. 97 20009

Varma C M, Be Littlewood P, Schmitt-Rink S, Abrahams E and Ruckenstein A E 1989 Phenomenology of the normal state of Cu-O high-temperature superconductors Phys. Rev. Lett. 63 1996

Littlewood P B and Varma C M 1992 Phenomenology of the superconductive state of a marginal Fermi liquid Phys. Rev. B 46 405–20

Ruckenstein A E and Varma C M 1991 A theory of marginal Fermi-liquids Physica C 185–189 134–40

Mukherjee A and Lal S 2020 Scaling theory for Mott–Hubbard transitions: I. T = 0 phase diagram of the 1/2-filled Hubbard model New J. Phys. 22 063007

Mukherjee A and Lal S 2020 Scaling theory for Mott–Hubbard transitions-II: quantum criticality of the doped Mott insulator New J. Phys. 22 063008

Mukherjee A and Lal S 2020 Holographic unitary renormalization group for correlated electrons—II: insights on fermionic criticality Nucl. Phys. B 960 115163

Swingle B and Senthil T 2013 Universal crossovers between entanglement entropy and thermal entropy Phys. Rev. B 87 045123

Luttinger J M and Ward J C 1960 Ground-state energy of a many-fermion system. II Phys. Rev. 118 1417

Oshikawa M 2000 Topological approach to Luttinger’s theorem and the fermi surface of a Kondo lattice Phys. Rev. Lett. 84 3370

Dzyaloshinskii I 2003 Some consequences of the Luttinger theorem: the Luttinger surfaces in non-fermi liquids and Mott insulators Phys. Rev. B 68 085113

Seki K and Yunoki S 2017 Topological interpretation of the Luttinger theorem Phys. Rev. B 96 085124

Heath J T and Bedell K S 2020 Necessary and sufficient conditions for the validity of Luttinger’s theorem New J. Phys. 22 063011

Mukherjee A and Lal S 2020 Holographic unitary renormalization group for correlated electrons—I: a tensor network approach Nucl. Phys. B 960 115170

Mukherjee A, Patra S and Lal S 2021 Fermionic criticality is shaped by fermi surface topology: a case study of the Tomonaga-Luttinger liquid J. High Energy Phys. [JHEP04(2021)114]

Shankar R 1994 Renormalization-group approach to interacting fermions Rev. Mod. Phys. 66 129

Polchinski J 1992 Effective field theory and the fermi surface (arXiv:gr-qc/9204046)

Balasubramanian V, McDermott M B and Van Raamsdonk M 2012 Momentum-space entanglement and renormalization in quantum field theory Phys. Rev. D 86 045014

Grover T and Fisher M P A 2014 Quantum disentangled liquids J. Stat. Mech. P10010

Grover T and Fisher M P A 2015 Entanglement and the sign structure of quantum states Phys. Rev. A 92 042308

Kaplis N, Krueger F and Zaanen J 2017 Entanglement entropies and fermion signs of critical metals Phys. Rev. B 95 155102

Patra S and Lal S 2021 Origin of topological order in a cooperator-insulator Phys. Rev. B 104 144514

Pal S, Mukherjee A and Lal S 2019 Correlated spin liquids in the quantum Kagome antiferromagnet at finite field: a renormalization group analysis New J. Phys. 21 023019

Mukherjee A, Mukherjee A, Vidhyadhiraja N S, Taraphder A and Lal S 2022 Unveiling the Kondo cloud: unitary renormalization-group study of the Kondo model Phys. Rev. B 105 085119

Patra S, Mukherjee A, Mukherjee A, Vidhyadhiraja N S, Taraphder A and Lal S 2022 Frustration shapes multi-channel Kondo physics: a star graph perspective (arXiv:2205.00790)

Mukherjee A and Lal S 2022 Superconductivity from repulsion in the doped 2D electronic Hubbard model: an entanglement perspective J. Phys.: Condens. Matter 34 275601

Miqueleto J L and Landulfo A G S 2021 Exact renormalization group, entanglement entropy and black hole entropy Phys. Rev. D 103 045012

Klco N and Savage M J 2021 Entanglement spheres and a UV-IR connection in effective field theories Phys. Rev. Lett. 127 211602

Iso S, Mori T and Sakai K 2021 Non-Gaussianity of entanglement entropy and correlations of composite operators Phys. Rev. D 103 125019

Iso S, Mori T and Sakai K 2021 Wilsonian effective action and entanglement entropy Symmetry 13 1221

Nishioka T 2018 Entanglement entropy: holography and renormalization group Rev. Mod. Phys. 90 035007

Agon C, Balasubramanian V, Kasco S and Lawrence A 2018 Coarse grained quantum dynamics Phys. Rev. D 98 025019

Agón C and Lawrence A 2018 Divergences in open quantum systems J. High Energy Phys. [JHEP04(2018)008]

Kawamoto S and Kuroki T 2021 Momentum-space entanglement in scalar field theory on fuzzy spheres J. High Energy Phys. [JHEP12(2021)101]

Grignani G and Semenoff G W 2017 Scattering and momentum space entanglement Phys. Lett. B 772 699

Peschanski R and Seki S 2016 Entanglement entropy of scattering particles Phys. Lett. B 758 89

Landgren R, Blair J, Greiter M, Läuchli A, Fiebe G A and Thomale R 2014 Momentum-space entanglement spectrum of bosons and fermions with interactions Phys. Rev. Lett. 113 256404

Landgren R, Liu F, Laurell P and Fiebe G A 2019 Momentum-space entanglement after a quench in one-dimensional disordered fermionic systems Phys. Rev. B 100 241108

Balasubramanian V, Guica M and Lawrence A 2013 Holographic interpretations of the renormalization group J. High Energy Phys. [JHEP01(2013)115]

Guijosa A, Ollivier Y D and Pedraza J F J 2022 Holographic coarse-graining: correlators from the entanglement wedge and other reduced geometries J. High Energy Phys. [JHEP08(2022)118]

Martins Costa M H, van den Brink J, Nogueira F S and Krein G I 2022 Momentum space entanglement from the Wilsonian effective action Phys. Rev. D 106 066024

Martins Costa M H, van den Brink J, Nogueira F S and Krein G I 2022 Wilsonian renormalization as a quantum channel and the separability of fixed points (arXiv:2211.10238)

Hsu T-C L, McDermott M B and Van Raamsdonk M 2013 Momentum-space entanglement for interacting fermions at finite density J. High Energy Phys. [JHEP11(2013)121]

Flynn M O, Tang L-H, Chandran A and Laumann C R 2022 Momentum space entanglement of interacting fermions (arXiv:2203.08154)

Anderson P W 1958 Random-phase approximation in the theory of superconductivity Phys. Rev. 112 1900

Ju H, Kallin A B, Fendley P, Hastings M B and Melko R G 2012 Entanglement scaling in two-dimensional gapless systems Phys. Rev. B 85 165121
[90] Rogerson D, Pollmann F and Roy A 2022 Entanglement entropy and negativity in the Ising model with defects J. High Energy Phys. JHEP06(2022)165
[91] Ehlers G, Sólyom J, Legeza O and Noack R M 2015 Entanglement structure of the Hubbard model in momentum space Phys. Rev. B 92 235116
[92] Lee C H, Ye P and Qi X-L 2014 Position-momentum duality in the entanglement spectrum of free fermions J. Stat. Mech. P10023
[93] Maldacena J 1999 The large-N limit of superconformal field theories and supergravity Int. J. Theor. Phys. 38 1113–33
[94] Witten E 1998 Anti de Sitter space and holography (arXiv:hep-th/9802150)
[95] Qi X-L 2013 Exact holographic mapping and emergent space-time geometry (arXiv:1309.6282)
[96] Lee C H and Qi X-L 2016 Exact holographic mapping in free fermion systems Phys. Rev. B 93 035112
[97] Patra S, Basu S and Lal S 2022 Unveiling topological order through multipartite entanglement Phys. Rev. A 105 052428
[98] Gori G, Paganelli S, Sharma A, Sodano P and Trombettoni A 2015 Explicit Hamiltonians inducing volume law for entanglement entropy in fermionic lattices Phys. Rev. B 91 245138
[99] Ramirez G, Rodriguez-Laguna J and Sierra G 2014 From conformal to volume law for the entanglement entropy in exponentially deformed critical spin 1/2 chains J. Stat. Mech. P10004
[100] Ryu S and Takayanagi T 2006 Aspects of holographic entanglement entropy J. High Energy Phys. JHEP08(2006)045
[101] Levin M and Wen X-G 2006 Detecting topological order in a ground state wave function Phys. Rev. Lett. 96 110405