Adoption of innovations with contrarians and repentant agents

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Abstract The dynamics of adoption of innovations is an important subject in many fields and areas, like technological development, industrial processes, social behavior, fashion or marketing. The number of adopters of a new technology generally increases following a kind of logistic function. However, empirical data provide evidences that this behavior may be more complex, as many factors influence the decision to adopt an innovation. On the one hand, although some individuals are inclined to adopt an innovation if many people do the same, there are others who act in the opposite direction, trying to differentiate from the “herd”. People who prefer to behave like the others are called mimetic, whereas individuals who resist adopting new products, the stronger the greater the number of adopters, are named contrarians. On the other hand, new adopters may have second thoughts and change their decisions accordingly. Agents who regret and abandon their decision will be denominated repentant. In this paper we investigate a simple model for the adoption of an innovation for a society composed by mimetic and contrarian individuals whose decisions depend mainly on three elements: the appeal of the novelty, the inertia or resistance to adopt it, and the social interactions with other agents. In the process, agents can repent and turn back to the old technology. We present analytic calculations and numerical simulations to determine the conditions for the establishment of the new technology. The inclusion of repentant agents modify the balance between the global incentive to adopt and the number of contrarians who prevent full adoption, generating a rich landscape of temporal evolution that includes cycles of adoption.

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1 Introduction

Innovation is at the core of the changing in living conditions all along the human history. It is also one of the main driving forces of sustainable economical development in modern societies. However, even when innovations may represent a clear improvement over existing technologies, its adoption is not guaranteed because it depends on other factors that can restrain the adoption process, like the individual resistance or a high price. Besides, the adoption may be boosted by means of advertising or interpersonal influence. Rogers\cite{1} was the first one to address the problem of innovation adoption. In his qualitatively description, he claims that adoption curves are $S$-shaped (logistic) as a function of time: there are few early adopters, and only when their number becomes larger than a threshold, adoption develops up to a saturation point.

Systems of heterogeneous interacting individuals are complex systems whose properties have been studied in the contexts of economics (see \cite{2,3,4} and references therein), criminality \cite{5}, game theory \cite{6}, and in many other social and biological systems. It has been shown that when the individuals are mimetic, i.e. they choose to imitate the behavior of the others (also called herding or conglomerator behavior), the possible equilibria have well known properties \cite{7,8}. If some individuals do not exhibit a mimetic behavior, adoption dynamics is more involved, but also more interesting. Such individuals, called contrarians, have been described in different contexts in the literature. Galam\cite{9} introduced contrarian agents in voter models, in such a way that they adopt opinions that are systematically opposite to the one of the majority of their neighbors. Other possibilities have been proposed more recently by Masuda\cite{10}, who considered different models in which the decision of each contrarian depends on its neighborhood (made of contrarians and/or mimetics). It is also possible to have indifferent agents, who are not aware of the social tendencies. As we want to focus on the role of repentants in the dynamics of innovation adoption we do not consider indifferent agents in the present contribution.
With or without contrarians, the time evolution of the fraction of adopters in a Markov chain: \( n(t+1) = \mathcal{F}(n(t)) \). The fixed points attractors of the dynamics satisfy \( n = \mathcal{F}(n) \). However, as demonstrated by Golun et al. [11], systems with interacting binary agents evolve toward fixed points only when the interactions are symmetric and positive. Negative symmetric interactions may lead either to fixed points or to cycles of length 2, depending on details of the dynamics and on the initial state. These results rely on the existence of an energy function that is a decreasing (more rigorously, non-increasing) function of time under the system’s dynamics. However, a system with contrarians does not necessarily have an underlying energy function, because the interactions between mimetic agents and contrarian agents are not symmetric. Being an individual property, contrarians have negative interactions with all other agents. Thus, interactions between contrarians are symmetric — both being negative — but interactions between a contrarian and a mimetic agent are anti-symmetric. Consequently, the existence of fixed points is not guaranteed.

A “microscopic” model of adoption dynamics has been proposed recently [12]. This model considers heterogeneous individuals in the presence of advertising. Mimetic individuals have a positive interaction with the adopters, increasing their pay-off function with the fraction of them. Contrarians, instead, have the opposite behavior, with a preference to adopt that decreases when the fraction of adopters increases. In that model, adopters, both mimetics and contrarians, cannot change their minds, so that the fraction of adopters is a non-decreasing function of time. The dynamics of adoption in models with only mimetic individuals has been studied by Bass [4] and also by Phan et al. [13] for different types of networks. In Ref. [13] authors have shown that the fraction of adopters increases with time through avalanches that depend on the underlying network structure. Moreover, the fraction of adopters at equilibrium in the absence of contrarians has been obtained analytically in [12] for a uniform distribution of the resistance to adopt and small values of the interaction weights. Numerical results have been obtained when contrarians are included: in the context of innovation the most important consequence of the inclusion of contrarians is the non-trivial restraining effect on the adoption curves, i.e., a small fraction of contrarians produce a large reduction on the final fraction of adopters. Gonçalves et al. model [12] is suitable for situations where users can not change their decisions, such as the case of expensive technologies. But there are other situations, as for example the choice of an operating system, a software, or an internet supplier, where the decision can be revised periodically. In such cases, adopters may change their minds and abandon the innovation.

In this article we will focus on a society where individuals exhibit a mimetic or contrarian behavior (kept fixed during the whole adoption process), but they can repent for their decisions, going back to a non-adopter state. We will study this model using analytical and numerical approaches, and considering different distributions of the idiosyncratic resistance to adopt. We explore the parameters space by comparing the results of simulations with a mean field analytic approach, analyzing the phase diagram of the system for different proportions of mimetics and contrarians. The paper is organized as follows: In section I we present the model, in section II we consider a uniform distribution of the resistance to adopt (analytically and numerically), and in section III we analyze the case of a logistic distribution. Conclusions are presented in section IV.

2 The model of adoption with social interactions

We consider a system of \( N \) individuals that must decide whether to adopt or not a novelty. We define a parameter \( A \geq 0 \) as a global incentive to adopt, the same for all agents. This incentive is proportional to the advantages introduced by the new technology, to advertising, and to eventual social values associated with the possession of the new product. On the other hand, each individual has a resistance to adopt the innovation given by a value \( R + r_i \) \((1 \leq i \leq N)\), where \( R \) is the population’s average and the \( r_i \) are (quenched) idiosyncratic deviations distributed among the population according to a probability density function \( P(r) \) of zero mean and variance \( s \). In [12] \( P(r) \) is uniform in \([-r_0, r_0]\) with \( R = 0.5, r_0 = 0.5 \) and \( s = r_0^2 / 3 \). This resistance can be associated to suspicion against the novelty, to a certain laziness that induces to remain with the old technology or to limited resources for the acquisition of the new technology. When confronted with the decision to adopt or not the new technology, we assume that there are two kinds of individuals: a fraction \( f \) of the population is composed by contrarian agents, i.e., they resist to imitate what others do, while a fraction \( 1 - f \) of the population is mimetic, so they tend to follow the herd. We assume these attitudes also remain fixed during the adoption process. At each time step, each agent weights the expected decisions of the others with a strength \( J_i \), which represents the social influence on his own decision. In [12], mimetic individuals increase, while contrarians decrease, their willingness to adopt the innovation proportionally to the fraction of adopters \( n \), with weight \( J_i = J = 1 \) for all \( i \). In other models [9,10], individuals are susceptible to the majority, meaning that the willingness to adopt are proportional to \( n - 1/2 \) instead of \( n \).

As it stands, the model has four parameters: \( A, R, J, \) and \( s \). We can get rid of \( J \) by using it as a normalization factor for the others, as in [28]. In the following we measure all the values in terms of the strength of the social interactions, \( J \), and define

\[
d \equiv \frac{A - R}{J}, \quad u_i \equiv \frac{r_i}{J}, \quad u_0 \equiv \frac{r_0}{J}, \quad \sigma \equiv \frac{s}{J} \quad (1)
\]

Notice that this normalization was implicit in [12], where most of the time it was assumed that \( J=1 \).
The pay-offs of the model \[12\] are:
\[
\begin{align*}
\pi_i^M &= d - u_i + n_{-i} \quad \text{if } i \text{ is mimetic,} \\
\pi_i^C &= d - u_i - n_{-i} \quad \text{if } i \text{ is contrarian,}
\end{align*}
\]
where \(n_{-i}\) is the fraction of adopters without counting individual \(i\):
\[
n_{-i} = \frac{1}{N-1} \sum_{k \neq i} \omega_k,
\]
with \(\omega_k = 1\) if \(k\) is an adopter and zero otherwise.

When calculating the expected pay-offs we can check each individual at random and immediately update his decision according to the sign of his pay-off. In this case the expected pay-off is equal to the actual one. On the other hand we can perform a synchronous updating of all agents at once. In the later case agents determine their pay-offs taking into account the last value of the number of adopters, \(n\), but this value is refreshed after all agents have been checked. So, the expected and the actual pay-offs may be different after the updating. We will discuss this point in further detail in the next section.

If adopters are not allowed to change their minds, as in the model considered in \[12\], only the actual pay-offs of non-adopters are important for the dynamics; however, the equilibrium properties of the present model depend both on adopters and non-adopters pay-offs. In the limit of large populations with large numbers of adopters we may replace \(N - 1 \approx N\), drop down the constraint \(k \neq i\) in the equation above and approximate \(n_{-i}\) by the bare fraction of adopters \(n\):
\[
n_{-i} \approx n \equiv \frac{1}{N} \sum_k \omega_k.
\]

Individuals adopt the new technology whenever their expected pay-offs are positive. The adoption dynamics may become quite complex upon introduction of contrarians that decide according to the majority rule \[12\]. Here we include the possibility of coming back from previous decisions, thus, individuals will abandon innovation if the pay-off is negative. No doubts or delays are allowed in this version of the model. The results of the present and the previous \[12\] models are compared and discussed in the following, that is, when adoption can be reverted or not.

When considering a large number of agents we can take the limit \(N \to \infty\). Introducing the fraction of adopters \(f\) in equations \[2\], and assuming that the idiosyncratic normalized resistance to adopt \(u_i\), are quenched random variables of probability density \(P(u)\), the adoption probability (the probability of positive pay-off) in the limit \(N \to \infty\) is
\[
P(\omega = 1|M) = \int_{-\infty}^{d+n} P(u)du \\
P(\omega = 1|C) = \int_{-\infty}^{d+n} P(u)du
\]
where \(M\) stands for mimetic and \(C\) for contrarian agents.

If the fraction of adopters at time \(t\) is \(n(t)\), the adopters’ dynamics is given by the following equation:
\[
n(t+1) = (1 - f) \int_{-\infty}^{d+n(t)} P(u)du + f \int_{-\infty}^{d-n(t)} P(u)du
\]
\[
= \int_{-\infty}^{d+n(t)} P(u)du - f \int_{d-n(t)}^{d+n(t)} P(u)du
\]
In the absence of contrarians, \(f = 0\), the phase diagram of the model is well known \[2,3\]. The stationary states satisfy
\[
n = \int_{-\infty}^{d+n} P(u)du
\]
which is easily solved (see \[3\]).

In the following sections we include contrarians and repentants, we assume pay-offs given by equations \[2\] and we include the possibility of changing decisions for both mimetic and contrarian agents. Agents adopt if the expected pay-off is positive and do not adopt otherwise. Moreover, those that have adopted previously may go back to no-adopter (repentants) if the actual pay-off turns out to be negative. We analyze two particular distributions \(P(u)\), the uniform distribution and the logistic one.

### 3 Uniform distribution

In this section, we present first results of simulations and then we discuss analytic results for a uniform distribution, i.e. \(P(u) = (2u_0)^{-1}\) in \([-u_0,u_0]\) and \(P(u) = 0\) elsewhere. We will compare the results of this model with the ones presented in \[12\], thus we adopt the same value of the parameters of that paper: i.e. we consider \(u_0 = 0.5\) so that \(P(u) = 1\). We also restrict the comparison to the results in ref. \[12\] where \(J = 1\), so we can consider the variables already normalized.

#### 3.1 Numerical results

When performing the simulations we consider two different dynamics, corresponding to synchronous and non-synchronous updates. In the case of synchronous Parallel Dynamics (PD), the pay-offs are evaluated for all agents at the same time, then the status of each agent is changed or not accordingly to their pay-off and thereafter the new fraction of adopters, \(n\), is updated. In the Monte Carlo sequential dynamics (MC), one agent is selected at random and its status is updated depending on its pay-off, then the number of adopters is immediately adjusted. This process is repeated \(N\) times, which corresponds to one MC step. The reason for considering these two dynamics is that the first one (PD) is better adapted to be compared with analytical results, while MC simulations probably provides a better description of the changes in real societies. The difference between the two dynamics is that in the MC method a sequential update is performed, which means that the number of adopters changes in a continuous way.
during each MC step, while in the PD case the number of adopters is updated at the end of each step, after evaluating the pay-offs of all agents. A second difference between the two types of dynamics is that with the PD procedure all agents are visited, while in MC dynamics they may not.

In order to illustrate the dynamics let us first consider a very simple case with just two agents, $N = 2$, and the three possible combinations: two mimetics, one mimetic and one contrarian, and two contrarians. We choose parameters such that for both agents $d$ is slightly higher than $u_1$, for example $d = 0.01$ and $u_1 = u_2 = 0$. The results are exhibited on Fig. 1 (black filled squares correspond to PD and red open circles to MC dynamics). In the case of two mimetic agents, both of them will adopt immediately and no further changes are observed, the system arrives at a fixed point. With one mimetic and one contrarian, both agents adopt in the first time step but then only the mimetic remains as adopter. The system gets to a fixed point with $n = 0.5$. In the cases above the description corresponds strictly to PD while some random variations are possible with MC dynamics. An interesting time evolution arises when there are two contrarians because both of them adopt when $n = 0$, but as soon as they adopt, $n_{i-1} = 1/2$ (notice that for small values of $N$, the approximation given by Eq. (4) is not valid), so the contrarian’s pay-off becomes negative and at the next evaluation both become non-adopters. Therefore, the system exhibits a strictly periodic behavior in parallel simulations. In MC simulations, however, there are no oscillations: after the first agent adopts, when the second is selected, it will not adopt because its pay-off will be negative, so the evolution stops at 50% of adopters.

For an even larger number of agents, $N = 10^7$, oscillations always disappear in the long term. Even for a high proportion of contrarians, $f = 0.9$, and with parallel dynamics, oscillations decay after a short transient, as can be seen in Fig. 3. For a lower fraction of contrarians, oscillations are very short lived; for $f = 0.5$, for instance, no more than three oscillations are seen in Fig. 3b. The previous results, all put together, suggest that for uniform distribution of resistance to adoption—which is a raw simplification of the society representation—the system exhibits oscillations. Black squares correspond to PD and open red circles to MC dynamics. In all cases, $d = 0.4$ and $u_0 = 0.5$.

Figure 2. Fraction of adopters as a function of time for $N = 100$ agents and different values of the fraction of contrarians $f$: (a) $f = 0.2$, no oscillations; (b) $f = 0.5$, small amplitude oscillations; and (c) $f = 0.9$, large amplitude sustained oscillations. Black squares correspond to PD and open red circles to MC dynamics. In all cases, $d = 0.4$ and $u_0 = 0.5$.

Let us now consider the case with an intermediate number of agents, $N = 100$; henceforth we use the approximation given by Eq. (4). In Fig. 2 it can be verified that when the fraction of contrarians is large, $f = 0.9$, the system exhibits oscillations in the parallel dynamics case, while there are no oscillations with Monte Carlo dynamics. For a lower fraction of contrarians, $f = 0.5$, oscillations are of smaller amplitude and disappear for $f < 0.2$.

Figure 1. Case of two agents: (a) Two mimetics, the final value of $n$ is $n = 1$; (b) one mimetic and one contrarian, $n$ quickly converges to $n = 0.5$; (c) two contrarians, the system exhibits oscillations. Black squares correspond to PD and open red circles to MC dynamics. In all cases, $d = 0.01$ and $u_1 = u_2 = 0$. 

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It is also interesting to investigate the effect of the advertising. The values considered, $d = 0.4$ may be too high. It can be argued that a too high value of the incentive to adopt could have a role in the appearance, or not, of oscillations. And it has. In order to check this we try a lower value of the incentive, $d = -0.2$. As $u$ goes from $-0.5$ to $+0.5$ that value of $d$ implies that 30% of the agents have an idiosyncrasy below $d$, i.e. 30% are potential early adopters. As we are interested in possible oscillations we focus on a high concentration of contrarians, $f = 0.9$, and three system sizes, $N = 2$, $N = 100$, and $N = 10^7$. The results are shown in Fig. 4, where a clear feature can be seen: the asymptotic values for the average fraction of adopters are much lower ($n \approx 0.15$) than in the case of $d = 0.4$. This is expected because the advertising is what promotes the
Figure 3. Fraction of adopters as a function of time for $N = 10^7$ agents and different values of the fraction of contrarians $f$: (a) $f = 0.2$, no oscillations; (b) $f = 0.5$, very short lived oscillations; and (c) $f = 0.9$, transient oscillations. Black squares correspond to PD and open red circles to MC dynamics. The other parameters in all cases are $d = 0.4$ and $u_0 = 0.5$.

Figure 4. Fraction of adopters as a function of time for a large fraction of contrarians ($f = 0.9$), but a low value of advertising ($d = -0.2$). Results for different system sizes: (a) $N = 2$, (b) $N = 100$, and (c) $N = 10^7$. Black squares correspond to PD and open red circles to MC dynamics. The qualitative behavior is the same as in Fig. 3 with the same value of $f$, but a bigger value of $d$.

adoption in the first place. But on the other hand the oscillatory behavior is very similar to previous results with a bigger value of $d$, so our conclusions regarding the oscillatory behavior (and the existence or not of oscillations) are robust against a change of the advertising. However, if the width of the distribution is narrower, no oscillations appear for low values of $d$.

Note that the effect of the “repentants” with the associated cycle dynamics is only possible if $f > 0$. If there are no contrarians, even if it is possible to repent, no agent will do it because for a mimetic the pay-off can not decrease. In other words, cycles are only possible if both, contrarians and repentants, are present in the system. Moreover, the effect of contrarians modifies the number of adopters, and this change may induce mimetics to abandon the innovation. We finally remark that in this section we have restrained our simulations to the parameters utilized in ref. 12. In particular, we have chosen $u_0 = 0.5$. If a narrower distribution of the idiosyncratic resistance to adopt is considered, stable oscillations may appear for relatively high values of the advertising. In the next section we show, as an example, that such is the case for $u_0 = 0.25$ and $\mathcal{P}(u) = 2$ (see Fig. 5). Moreover, the effect of the width of the distribution on the existence of oscillations will be discussed in detail in section 3.3 as the logistic distribution is simpler to be treated.

### 3.2 Analytic Results

In this section we present analytic mean field results and compare them with numerical simulations of the preceding section. As the analytic calculations implicitly assume $N \to \infty$ we will compare them with the numerical results for the big size system, i.e., $N = 10^7$.

Due to the compact support of $\mathcal{P}$, the possible (normalized) pay-offs as a function of $n$ (see Eq. 2 and Fig. 5) are bounded by

$$
\pi^M_{\text{max}}(n) = d + u_0 + n \\
\pi^M_{\text{min}}(n) = d - u_0 + n \\
\pi^C_{\text{max}}(n) = d + u_0 - n \\
\pi^C_{\text{min}}(n) = d - u_0 - n
$$

and the fixed point equations of the dynamics, $n(t + 1) = n(t) + n$ are:

$$
n = n^M + n^C; \tag{9a}$$

$$
n^M = (1 - f) \int_{\pi^{M}_{\text{min}}(n)}^{\pi^{M}_{\text{max}}(n)} \mathcal{P}(u) du; \tag{9b}$$

$$
n^C = f \int_{\pi^{C}_{\text{min}}(n)}^{\pi^{C}_{\text{max}}(n)} \mathcal{P}(u) du \tag{9c}
$$

that must be solved for $n$. We call $n^M$ the number of adopters who are mimetic and $n^C$ those who are contrarian.

There are different regimes that have to be analyzed separately, depending on the signs of the extreme pay-offs at $n = 0$ and at $n = 1$. To illustrate this point we show on Fig. 4 the extreme pay-offs as a function of the number of adopters, $n$, for $d = 0.4$ and $u_0 = 0.5$. Blue lines are extreme pay-offs for mimetics and red lines for contrarians. The area with positive pay-off correspond to the number of mimetics (blue ones) and contrarians (red ones) but one should make attention to the fact that these areas are weighted with the factors $(1 - f)$ for mimetics, and $f$ for contrarians. When $f = 0.5$ both areas have the same weight. In this case, starting with $n = 0$ the number of adopters after the first step is of the order of $n = 0.9$. But with such a high number of adopters most of
the contrarians will defect the innovation and the number of adopters will fall to \( n = 0.5 \) in the second step. After that, the number of adopters increases again, and then decreases to finally converge to an intermediary value of \( n = 0.6 \). These decaying oscillations converging to \( n = 0.6 \) are also observed in the numerical results, on Fig. 3(b).

We consider now the points where the extreme payoffs change sign by solving Eqs. 9 for the pay-off equal to zero:

\[
\begin{align*}
  n_1 &= -d - u_0 \\
  n_2 &= -d + u_0 \\
  n_3 &= d + u_0 = -n_1 \\
  n_4 &= d - u_0 = -n_2
\end{align*}
\]  

(10)

Two of these points, \( n_2 \) and \( n_3 \), are also indicated on Fig. 5. \( \pi_{max}(n) \) is always positive and \( \pi_{min}(n) \) is always negative, so \( n_1 \) and \( n_4 \) are both negative and are not solutions. In the general case we can state that \( u_0 > 0, n_2 > n_1 \) and \( n_3 > n_4 \), but depending on the relative values of \( d \) and \( u_0 \), \( n_3 \) may be larger or smaller than \( n_4 \). Here we would like just to analyze the two cases that we have simulated numerically: \( d = 0.4 \) and \( d = -0.2 \), both with \( u_0 = 0.5 \). In the first case (\( d = 0.4 \)) one has the following boundaries:

\[
\begin{align*}
  \pi_{max}(n) &= 0.9 + n \\
  \pi_{min}(n) &= -0.1 + n \\
  \pi_{max}^C(n) &= 0.9 - n \\
  \pi_{min}^C(n) &= -0.1 - n
\end{align*}
\]  

(11)

Those boundaries are the ones plotted on Fig. 5. The boundary \( \pi_{max}^C(n) \) is always positive, \( \pi_{min}^C(n) \) is positive for \( n > 0.1 \), \( \pi_{max}^C(n) \) is positive if \( n < 0.9 \) and \( \pi_{min}^C(n) \) is always negative, so in the corresponding contrarian integral (eq. 12) the lower bound is always 0. The fixed point can be evaluated using Eqs. 9. Considering \( f = 0.5 \) in those equations, it is easy to verify that the equilibrium is in the region \( 0.1 \leq n \leq 0.9 \) and the result is \( n \approx 0.63 \), that coincides very well with the asymptotic limit showed in Fig. 3(b). For \( f = 0.9 \) the asymptotic value is \( n \approx 0.48 \) that also coincides with the numerical result (Fig. 3). The figure also explains the oscillations before attaining the fixed point, as described above.

In the case with \( d = -0.2 \), the boundaries are:

\[
\begin{align*}
  \pi_{max}^M(n) &= 0.3 + n \\
  \pi_{min}^M(n) &= -0.7 + n \\
  \pi_{max}^C(n) &= 0.3 - n \\
  \pi_{min}^C(n) &= -0.7 - n
\end{align*}
\]  

(12)

Now, the boundary \( \pi_{max}^M(n) \) is always positive, \( \pi_{min}^M(n) \) is positive for \( n > 0.7 \), \( \pi_{max}^C(n) \) is positive if \( n < 0.3 \) and \( \pi_{min}^C(n) \) is always negative. Let’s examine the case \( f = 0.9 \). If one assumes a trial value, \( n_t \), restricted to \( n_t > 0.3 \) there are no contrarians adopting and the number of adopters should be \( n = (1 - f)n_t \) that is always lower than \( n_t \), in contradiction with the hypothesis. So, \( n \) must be lower than 0.3 and the solution is \( n \approx 0.16 \) again in agreement with the simulations (See Fig. 4).

Finally, and as we said, sustained oscillations can be obtained for a set of parameters such that \( 0 < n_4 \equiv d - u_0 < n_3 \equiv d + u_0 < 1 \), provided that both, \( u_0 \) and \( d \) be small enough. We show in Fig. 6 the extreme pay-offs obtained analytically for \( d = 0.6 \) and \( u_0 = 0.25 \). The interpretation of this figure is the same as the one done for Fig. 5. Moreover, the inset of Fig. 6 shows the numerical result for the same values of parameters and \( f = 0.7 \). It can be observed that the fraction of adopters as a function of time for parallel update dynamics present oscillations, in accordance with the analytical result.

## 4 Logistic distribution

While the uniform distribution is simpler than other distributions, the discontinuity at the borders generates some complications particularly for the analytic calculations. Also, one may imagine that the distribution of idiosyncrasies in a real society exhibit a concentration of values around the mean value and a relatively low concentration in the extremes. Taking these points into consideration one could envisage the use of a Gaussian distribution, as most of the values of the resistance to adopt will be distributed within a bell-shape of a few standard deviations width. However, the integral of the Gaussian is not analytic. So, to avoid the cumbersome complications raised by the uniform and Gaussian distributions, we consider hereafter a bell-shaped logistic distribution of the resistances to adopt, \( P(u_i) \), that is continuous and has infinite support. The probability density of the logistic distribution is:
with its variance given by $\sigma$.

with $\sigma$.

Following the practice of the previous section, we present first the numerical results.

### 4.1 Numerical Results

We have performed different simulations with the logistic distribution given by Eq. (13). The results are presented in Figs. 6 and 7. When the simulation is performed in parallel (PD), permanent oscillations may appear. This is the case for intermediate values of the advertising, $d$, and relatively high values of the fraction of contrarians, $f$. This can be verified in Fig. 6 where we have represented a threshold value of the fraction of contrarians $f_c$, above which oscillations appear, as a function of the advertising $d$. Notice that there are no oscillations for $d < 0$ or for high values of $d$. Oscillations are present in a region of values of $d$ around $d = 1$, i.e. when the advertising is as strong as the social interaction; oscillations are cycles of period two and arise because contrarians adopt when the number of adopters is low, but abandon the innovation when the number of adopters is high. However some mimics may follow the contrarian’s behavior. The amplitude of the oscillations decreases when decreasing $f$ or when $d$ is smaller or bigger than 1 (See Fig. 6). The region where stable oscillations occur is larger the narrower the distribution of idiosyncrasies, $\sigma$.

We have represented in Figs. 8 and 9 the time evolution of the number of adopters exhibiting the oscillations, when they happen, or the convergence to a fixed point when there are no oscillations. When performing Monte Carlo simulations there are no oscillations in none of the cases. Figure 10 summarizes the numerical results for the logistic distribution. Red curves (dashed) correspond to the final number of adopters (fixed points) when performing MC simulations, while black curves correspond to Parallel Dynamics (PD). In the later case, oscillations may be observed above a critical value of the fraction of contrarians. When there are no oscillations, the results for PD and MC simulations coincide. When oscillations are present, PD results are different from MC results, and the plot shows both the extreme amplitude of the oscillations and, in between, the average value of the number of adopters.

**Figure 6.** Extreme pay-offs as a function of the number of adopters, $n$, for the uniform distribution with $d = 0.6$, $u_0 = 0.25$. $\pi$ lines are extreme pay-offs, blue lines for mimetics and red lines for contrarians. In general they are given by Eq. (8). Inset: Numerical results of the fraction of adopters as a function of time in the parallel update dynamics for $N = 10^7$ ($d = 0.6$, $u_0 = 0.25$, and $f = 0.7$).

**Figure 7.** Thresholds value of the fraction of contrarians above which oscillations appear. The curves correspond to different values of the width of the logistic distribution $\sigma$, as indicated. We have represented just positive values of $d$ as there are no oscillations for negative values. The curves go through a minimum that is lower the narrower the distribution. Notice that the oscillations disappear when the advertising is slightly higher than $d = 1$.
Figure 8. Temporal behavior of the fraction of adopters for the logistic distribution with $\sigma = 0.25$, $d = 0.4$, and different values of the fraction of contrarians $f$: (a) $f = 0.2$, (b) $f = 0.5$, and (c) $f = 0.9$. Results are for a large number of agents, $N = 10^5$. Open red circles correspond to the MC simulations and black squares to PD simulations. In the PD case it is possible to see the oscillations in the number of adopters for a high concentration of contrarians. We have considered much longer times than those represented in the figure and the oscillations are stable.

Figure 9. Temporal behavior for the logistic distribution with $\sigma = 0.25$, $f = 0.9$ and two different values of the parameter $d$ (the normalized effective marketing): (a) $d = -0.2$ and (b) $d = 0.1$. Oscillations in the number of adopters are obtained if $d > 0$.

Figure 10. This figure summarizes the numerical results for the logistic distribution of idiosyncrasies with $\sigma = 0.25$. All four panels exhibit the fraction of adopters as a function of the fraction of contrarians for four different values of $d$: (a) $d = 0.1$, (b) $d = 0.4$, (c) $d = 0.7$, and (d) $d = 1.0$. As expected, the number of adopters decreases when the number of contrarians increases. The red curves (dashed) correspond to Monte Carlo simulations and the black ones to a parallel dynamics. An oscillatory behavior is obtained only for parallel dynamics and the black lines correspond to the average value of the oscillations, while the shadowed areas indicates the amplitude of the oscillations. Both dynamics exhibit identical results for low and intermediate values of $f$, but there exists a critical value of $f$ when the parallel dynamics exhibits period two oscillations. When increasing $d$ the region of oscillations increases up to $d = 0.7$ and then decreases for $d = 1.0$. When $d < 0$ there are no oscillations and both dynamics produce the same results.

through the intersections of the function $y_1(n)$ with the line $y_2(n) = n$. Fig. 11 presents some examples for different parameter values.

Fig. 11 represents a plot of $y_1(n)$ and the line $y_2(n) = n$. The intersections correspond to the fixed points and are stable solutions provided that $y'_1 \equiv \frac{dy_1}{dn} < 1$. However, solutions with $|y'_1| \equiv |\frac{dy_1}{dn}| > 1$ are unstable, and we are then obliged to consider a second iteration, i.e., $y_1(y_1(n))$.

The solutions for this second iteration are represented on Fig. 12 if more than one intersection is present, the upper and lower intersections correspond to the extreme value of the oscillations.

The comparison between numerical and analytical solutions is discussed in detail in the caption of Figs. 11 and 12. We find a very good agreement of both solutions, then, there is no need of further discussion of this point. We will concentrate in the next section in the discussion of the results and comparison with a previous model 12.

5 Discussion and Conclusions

One of the interesting points of the present contribution is that the temporal behavior of the adoption of innovations is very sensitive to the width of the distribution of the resistance to adopt $u_i$. In the case of a wide uniform distribution, as the one utilized in ref 12, oscillations appear as a transient state but in the end the system converges to a fixed point. On the other side, with narrower uniform distributions of $u_i$, sustained oscillations appear, which are produced by the contrarians, whereas mimetic agents hardly change their decision. It could be argued that this
obtained with MC simulations (see Fig. 10), while PD simulations indicated the existence of oscillations.

Figure 11. Fixed points of \( y_1(n) \). The fixed points correspond to the intersections of \( y_1(n) \) and \( y_2(n) \) (indicated by the dot-dashed gray line in the figure). When the absolute value of the derivative is lower than one, the solutions are stable and correspond to a fixed point of the dynamics. The three panels show different cases for different values of the parameter \( d \) of the logistic distribution of idiosyncrasies: (a) For \( d = 0.2 \) the derivatives at the intersections are always \(|y'_1| < 1\), thus no oscillations are expected. (b) When \( d = 0.1 \) three possible stable intersections appear in each case, and the values roughly correspond to the numerical results plotted on Fig. 10(a). (c) When \( d = 0.4 \) and \( f = 0.2 \) the stable solution correspond to \( n \approx 0.8 \) that coincides with the numerical solution (see Fig. 10(b)). For \( f = 0.5 \), \( n \approx 0.5 \) that also coincides with both PD and MC simulations. Finally, for \( f = 0.9 \) the solution is unstable \((|y'_1| > 1)\). However the fixed point corresponds to the value obtained with MC simulations (see Fig. 10), while PD simulations indicated the existence of oscillations.

Figure 12. Fixed points of \( y_1(n) \) with \( y_2(n) \) (gray dot-dashed line). We have plot just the case with \( d = 0.4 \). It is possible to observe that for \( f = 0.2 \) and \( f = 0.5 \) there is just one intersection, that corresponds to the stable solutions previously obtained. For \( f = 0.9 \) there are three intersections. The middle one corresponds to the fixed point of \( y_1(n) \) while the other two represent the extremes of the oscillations. These extreme values are approximately 0.12 and 0.9 and correspond to the extreme value of the oscillations in the PD simulations, see Fig. 11(c).

kind of distribution is not representative of social systems. However, when the distribution is bell shaped, as it is the case of the logistic distribution presented in section 3 we obtain similar results. That is, stable long term oscillations may appear for intermediate values of the advertising, \( d \), or a large fraction of contrarians, \( f \), as it is evident in Fig. 11. While a high number of contrarians may be unreal considering a novel technology, that could be the case regarding brand choices, for example (iPhone vs. Samsung). In any case, Fig. 12 shows that oscillations may appear with a relatively low fraction of contrarians, provided the advertising is strong; see for example that for a narrow distribution of idiosyncrasies \((\sigma = 0.05)\) and for \( d \approx 0.95 \) the threshold is of the order of \( f \approx 0.15 \). Also, the coexistence of contrarians with the possibility of changing decisions makes the final total number of adopters lower than in the case with no regrets [12]. To check this point we have represented in Fig. 13 the present results for the uniform distribution of \( u_i \) together with those of Ref. [12]. It is possible to see that the shape is similar in both cases, but when the decision is “reversible” the final adoption is lower than when not. To produce this comparison we considered the uniform distribution of idiosyncrasies because it was the one used in Ref. [12].

The presence of repentants and contrarians have the effect of reducing the final number of adopters. Also, oscillations may appear when the distribution of resistances to adopt is smaller than 1, i.e. smaller than the social interaction, \( J \). Also, such cycles are only possible if both, contrarians and repentants, are present. Concerning the oscillations, they are interesting but somehow artificial, as they mainly arise from contrarians that regret their decision. In
any case one expect that the first oscillations are the important ones, as it is not plausible that agents continue to change their minds all the time. Regarding the fact that the inclusion of contrarians and repentants reduces significantly the final fraction of adopters, a direct consequence is that a stronger advertising campaign will be needed if it seeks to impose innovation. Work in progress include the presence of impulsive agents, i.e., people that can change their technology without assessing whether it is convenient. Preliminary results show that the introduction of such agents in the model accelerates the adoption process and increases the total number of adopters. In other words, a small number of impulsive agents in a society could be as efficient as a strong advertising.

We are also considering a dynamic distribution of idiosyncrasies, the effect of distributing the agents on a network, and a non-linear term of social interaction that may describe the effects of fashion: people adopt a new fashion when there are a few followers but abandon when the number of adopters increases.

Concluding, we would like to point out that, despite its simplicity, this model reproduce some common features of the innovation adoption process while allowing the study of more realistic cases.

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