Verification of indefinite-horizon POMDPs

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Abstract The verification problem in MDPs asks whether, for any policy resolving the nondeterminism, the probability that something bad happens is bounded by some given threshold. This verification problem is often overly pessimistic, as the policies it considers may depend on the complete system state. This paper considers the verification problem for partially observable MDPs, in which the policies make their decisions based on (the history of) the observations emitted by the system. We present an abstraction-refinement framework extending previous instantiations of the Lovejoy-approach. Our experiments show that this framework significantly improves the scalability of the approach.

1 Introduction

Markov decision processes are the model to reason about systems involving nondeterministic choice and probabilistic branching. They have widespread usage in planning and scheduling, robotics, and formal methods. In the latter, the key verification question is whether for any policy, i.e., for any resolution of the nondeterminism, the probability to reach the bad states is below a threshold [3]. The verification question may be efficiently analysed using a variety of techniques such as linear programming, value iteration, or policy iteration, readily available in mature tools such as Storm [15], Prism [22] and Modest [13].

However, those verification results are often overly pessimistic. They assume that the adversarial policy may depend on the specific state. Consider a game like mastermind, where the adversary has a trivial strategy if it knows the secret they have to guess. Intuitively, to analyse an adversary that has to find a secret, we must assume it cannot observe this secret. For a range of privacy, security, and robotic domains, we may instead assume that the adversary must decide based on system observations. Consider, e.g., surveillance problems, where the aim is to compute the probability that an intruder accesses a (physical or cyber) location with critical information or infrastructure.

Partially observable MDPs [19,29] cater to this need. They extend MDPs with observation labels, and restrict policies to be observation-based: paths with the

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same observation traces are indistinguishable and yield the same decisions. The verification problem for POMDPs with *indefinite horizon specifications* such as unbounded undiscounted reachability is whether all observation-based policies satisfy this specification, e.g., whether for each policy, a bad state is reached with a probability less than 0.1. This problem is undecidable [24]. Intuitively, undecidability follows from the fact that optimal policies require the full history.

Nevertheless, the analysis of POMDPs is a vibrant research area. Traditionally, the focus has been on finding some “good” policy, in planning, control, and robotics [31,35,20] and in software verification [9]. Many works have been devoted to finding a policy that behaves “almost optimal” for *discounted or bounded* reachability, most prominently (variants of) point-based solvers [30,28,21,4,33]. These methods can be exploited to find policies for temporal specifications [6]. *Error bounds provided by those methods do require a discounting factor (or a finite horizon).* A notable exception is the recent Goal-HSVI [16], which explores the computation tree and cuts off exploration using sound bounds. Another popular approach to overcome the hardness of the problem is to limit the policies, i.e., by putting a (small) p-a-priori bound on the memory of the policy [25,12,8,1,27,34,18]. We remark that it is often undesirable to assume small memory bounds on adversarial policies.

Orthogonally, we focus on the *undiscounted and unbounded* (aka the *indefinitive horizon*) case. Reachability in this case is the key question to soundly support temporal logic properties [3]. Discounting is optimistic about events in the future, i.e., it under-approximates the probability that a bad state is reached after many steps, and is therefore inadequate in some safety analyses. Furthermore, we do not make assumptions on the amount of memory the policies may use. This means that we give absolute guarantees about the performance of an optimal policy. *While techniques for discounting, finite horizons, or finite memory policies may yield policies that are almost optimal in the unbounded case, they are inadequate to prove the absence of better policies.*

Like [26], we use a result from Lovejoy [23]. Whereas [20] focuses on supporting a wider range of properties and partially-observable probabilistic timed automata, we focus on the performance of the basic approach. In this paper, we discuss a method constructing a finite MDP such that the optimal policy in this MDP over-approximates the optimal observation-based policy in the POMDP. Thus, model checking this MDP may be used to prove the absence of POMDP policies. We use ideas similar to Goal-HSVI [16] in providing cut-offs: instead of the computation tree, we do these cut-offs on top of the MDP.

**Contributions.** We provide a concise method for the verification problem that builds upon the Lovejoy construction [23]. Contrary to [23,20], we describe a flexible variant of the approach in terms of the underlying MDP. Among other benefits, this enables an on-the-fly construction of this MDP, enables further (tailored) abstractions on this MDP, and clarifies how to analyse this MDP using standard methods. The approach is embedded in an automated abstraction-refinement loop. Our implementation is part of the next release of the open-source model checker STORM. Experiments show superior scalability over [20].
We introduce partially observable MDPs by first considering MDPs.

**Definition 1 (MDP).** A Markov decision process (MDP) is a tuple $M = \langle S, \text{Act}, P, s_1 \rangle$ with a countable set $S$ of states, an initial state $s_1 \in S$, a finite set $\text{Act}$ of actions, and a transition function $P : S \times \text{Act} \times S \rightarrow [0,1]$ with $\sum_{s' \in S} P(s, \alpha, s') \in \{0,1\}$ for all $s \in S$ and $\alpha \in \text{Act}$.

**Definition 2 (POMDP).** A partially observable MDP (POMDP) is a tuple $M = \langle M, Z, O \rangle$ where $M = \langle S, \text{Act}, P, s_1 \rangle$ is the underlying MDP with finite $S$, $Z$ is a finite set of observations, and $O : S \rightarrow Z$ is an observation function.

We fix a POMDP $M := \langle M, Z, O \rangle$ with underlying MDP $M := \langle S, \text{Act}, P, s_1 \rangle$. For $s \in S$ and $\alpha \in \text{Act}$, let $\text{post}^M(s, \alpha) = \{s' \in S | P(s, \alpha, s') > 0\}$. The set of enabled actions for $s$ is given by $\text{Act}(s) = \{\alpha \in \text{Act} | \text{post}^M(s, \alpha) \neq \emptyset\}$. W.l.o.g., we assume that states with the same observation have the same set of enabled actions, i.e. $\forall s, s' \in S : O(s) = O(s') \implies \text{Act}(s) = \text{Act}(s')$. Therefore, we can also write $\text{Act}(z) = \text{Act}(s)$ for observation $z$ and state $s$ with $O(s) = z$.

**Policies.** We want to make a statement about each possible resolution of the nondeterminism. Nondeterminism is resolved using policies that map paths to distributions over actions. A (finite) path is a sequence of states and actions, i.e., $\hat{\pi} = s_0 \xrightarrow{\alpha_0} s_1 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_{n-1}} s_n$, such that $\alpha_i \in \text{Act}(s_i)$ and $s_{i+1} \in \text{post}^M(s_i, \alpha_i)$ for all $0 \leq i < n$. Let $\text{last}(\hat{\pi})$ denote the last state of $\hat{\pi}$, and $\text{Paths}^M_{\text{fin}}$ denote the set of all paths in an MDP. We may (by slight misuse of notation) lift the observation function to paths: $O(\hat{\pi}) = O(s_0) \xrightarrow{\alpha_0} O(s_1) \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_{n-1}} O(s_n)$. Two paths $\hat{\pi}_1, \hat{\pi}_2$ with $O(\hat{\pi}_1) = O(\hat{\pi}_2)$ are observation-equivalent.

**Example 1.** We depict a POMDP in Fig.1. The following two paths are observation-equivalent:

$s_0 \xrightarrow{a} s_1 \xrightarrow{b} s_4 \xrightarrow{a} \otimes$ and $s_0 \xrightarrow{a} s_2 \xrightarrow{b} s_4 \xrightarrow{a} \otimes$

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3 More general observation functions can be efficiently encoded in this formalism [10].
For finite set $A$ let $\text{Dist}(A) = \{\mu : A \rightarrow [0, 1] \mid \sum_{a \in A} \mu(a) = 1\}$ be the set of distributions over $A$ and for $\mu \in \text{Dist}(A)$ let $\text{supp}(\mu) = \{a \in A \mid \mu(a) > 0\}$.

**Definition 3 (Policies).** A policy is a mapping $\sigma : \text{Paths}^M_{\text{fin}} \rightarrow \text{Dist}(\text{Act})$ that for path $\pi$ yields a distribution over actions with $\text{supp}(\sigma(\pi)) \subseteq \text{Act}(\text{last}(\pi))$. A policy $\sigma$ is observation-based, if for paths $\hat{\pi}'$, $\hat{\pi}'$

$$O(\hat{\pi}) = O(\hat{\pi}') \implies \sigma(\hat{\pi}) = \sigma(\hat{\pi}').$$

A policy $\sigma$ is memoryless, if for paths $\hat{\pi}$, $\hat{\pi}''$

$$\text{last}(\hat{\pi}) = \text{last}(\hat{\pi}') \implies \sigma(\hat{\pi}) = \sigma(\hat{\pi}').$$

Let $\Sigma^M_{\text{obs}}$ denote the set of observation-based policies for a POMDP $\mathcal{M}$, and $\Sigma^M$ all policies for an MDP $M$.

**Reachability probability.** The reachability probability $\Pr^\sigma_\mathcal{M}(s \models \diamond \text{Bad})$ to reach a set of states $\text{Bad}$ from $s$ using a policy $\sigma$ is defined as standard, by considering the probability in the induced Markov chain (with state space $\text{Paths}^M_{\text{fin}}$). For details, consider e.g. [3]. We write $\Pr^\sigma_\mathcal{M}(\diamond \text{Bad})$ to denote $\Pr^\sigma_\mathcal{M}(s_1 \models \diamond \text{Bad})$.

**Problem 1.** For a given POMDP $\mathcal{M}$, a set $\text{Bad} \subseteq S$ of bad states, and a rational threshold $\lambda \in (0, 1)$, decide whether $\sup_{\sigma \in \Sigma^M_{\text{obs}}} \Pr^\sigma_\mathcal{M}(\diamond \text{Bad}) \leq \lambda$.

We emphasise that the techniques in this paper are applicable to upper and lower bounds, and to expected rewards properties. LTL properties can be supported by the standard encoding of the corresponding automaton into the MDP state space. The technique also applies (but is inefficient) for $\lambda \in \{0, 1\}$.

**Example 2.** Consider the POMDP in Fig. 1. Using the (memoryless) policy $\sigma = \{s_3, s_6 \rightarrow a, s_i \rightarrow b (i \neq 3, 6)\}$, state $\otimes$ is reached with probability one, but this policy is not observation-based: e.g. $\sigma(s_5) \neq \sigma(s_6)$. Now consider the policy $\{s_i \rightarrow a\}$, which is memoryless and observation-based. Indeed, this policy is optimal among the memoryless observation policies (the probability to reach $\otimes$ is $37/64 \approx 0.57$). A policy taking $b$ in the first step and then resorting to the memoryless policy $\{s_0, s_5, s_6 \rightarrow a, s_1, s_2, s_3, s_4 \rightarrow b\}$ is better: the induced probability to reach $\otimes$ is $23/26 \approx 0.639$. The questions we aim to answer is whether there exists a strategy that achieves probability $65/100$ (yes), or even $7/10$ (no).

### 3 Belief MDPs and their Approximation

A central notion in the analysis of POMDPs is belief: A distribution over the states that describes the likelihood of being in a particular state given the observation-based history $O(\hat{\pi})$. We reformulate our problem in terms of the belief MDP, a standard way of defining operational semantics of POMDPs, discuss some essential properties, and discuss abstractions of this infinite belief MDP.

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4 The implementation discussed in Sect. 3 supports all these combinations.
We start with the belief that $POMDP$ is in state $s_0$, and with $\frac{4}{7}$ that $M$ is in state $s_0$, or $s_2$. In the first case, based on the observations, we surely are in state $s_0$. In the latter case, the belief is computed by normalising the transition probabilities on the observation: The belief $b_1$ indicates that $M$ is in $s_2$ with probability $\frac{1}{7}$, and in $s_1$ with probability $\frac{3}{7}$. Upon executing action $a$ again after observing that $M$ is in $s_1$ or $s_2$, we reach state $s_3$ with probability $b_1(s_1) \cdot P(s_1, a, s_3) + b_1(s_2) \cdot P(s_2, a, s_3) = \frac{3}{7} + \frac{1}{7} \cdot \frac{3}{7} = \frac{15}{16}$.

In the following, let $P(s, \alpha, z) := \sum_{s' \in S} [O(s') = z] \cdot P(s, \alpha, s')$ denote the probability of moving from (some state with) observation $z$ from state $s$ using action $\alpha$. Then, $P(b, \alpha, z) := \sum_{s \in S} b(s) \cdot P(s, \alpha, z)$ is the probability to observe $z$ after taking $\alpha$ in $b$. We define the belief obtained by taking $\alpha$ from $b$, conditioned on observing $z$:

$$[b][\alpha, z](s') := \frac{[O(s') = z] \cdot \sum_{s \in S} b(s) \cdot P(s, \alpha, s')}{P(b, \alpha, z)}.$$ 

Using these ingredients, the belief MDP is defined as follows.

**Definition 4 (Belief MDP).** The belief MDP of $POMDP$ $M = (M, Z, O)$ is the MDP $\text{bel}(M) := \langle B, \text{Act}, P^B, b_I \rangle$ with $B$ as above, initial belief state $b_I := \{s_I \mapsto 1\}$, and transition function $P^B$ given by

$$P^B(b, \alpha, b') := \begin{cases} P^B(b, \alpha, O(b')) & \text{if } b' = [b][\alpha, O(b')], \\ 0 & \text{otherwise.} \end{cases}$$

In the formula, we use Iverson brackets: $[x] = 1$ if $x$ is true and 0 otherwise.
To ease further notation, we denote $\mathcal{Bad} := \{ b \mid \sum_{s \in \mathcal{Bad}} b(s) = 1 \}$, and we define the (standard notion of the) value of a belief $b$,

$$V(b) := \sup_{\sigma \in \Sigma^{bel(M)}} \Pr^\sigma_{bel(M)}(b \models \Diamond \mathcal{Bad})$$

and for action $\alpha$:

$$V_\alpha(b) := \sup_{\sigma \in \Sigma^{bel(M)}, \sigma(b) = \alpha} \Pr^\sigma_{bel(M)}(b \models \Diamond \mathcal{Bad}).$$

**Theorem 1.** For any POMDP $M$ and $b_I$, the initial state of $\text{bel}(M)$:

$$V(b_I) = \sup_{\sigma \in \Sigma^{M}_{\text{bel}}} \Pr^\sigma_{M}(\Diamond \mathcal{Bad}).$$

We can now restrict ourselves to memoryless deterministic schedulers, but face a potentially infinite MDP. Instead of solving Problem 1, we consider:

**Problem 2.** Given a belief MDP $\text{bel}(M)$, a set $\mathcal{Bad}$ of bad beliefs, and a threshold $\lambda \in (0, 1)$, decide whether $V(b_I) \leq \lambda$.

In the remainder of this section, we discuss two types of approximations, but not before reviewing an essential property of the value in belief MDPs. We discuss how we combine these abstractions in Sect. 4.

**Value function.** Assuming a fixed total order on the POMDP states $s_1 < \cdots < s_n$, we interpret belief states as vectors $b \in [0, 1]^n$ where the $i$th entry corresponds to $b(s_i)$. In particular, we can encode a belief by a tuple $(z, [0, 1]^{n_z})$, where $n_z$ denotes the number of states with observation $z$. This encoding also justifies the representation of beliefs in Figure 3 and 4.

Figure 3(a) contains a typical belief-to-value plot for $z = O(s_3) = O(s_4)$. On the x-axis, we depict the belief to be in state $s_3$ (from 1 to 0), and thus, the belief to be in state $s_4$ (from 0 to 1). On the y-axis, we denote the value of the belief. This value is constructed as follows: A policy takes action $a$ or action $b$ (or randomise, more about that later). We have plotted the corresponding $V_a$ and $V_b$.

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6 In general, the set of states of the belief MDP is uncountable. However, a given belief state $b$ only has a finite number of successors for each action $\alpha$, i.e. $\text{post}^{\text{bel}(M)}(b, \alpha)$ is finite, and thus the belief MDP is countably infinite. Acyclic POMDPs always give rise to finite belief MDPs (but may be exponentially large).
In Fig. 3(b) we depict the same functions for observing that we are in either \(s_1\) or \(s_2\). This plot can be constructed as the maximum of four policy applications. Formally, the following relations hold (from the Bellman equations):

**Lemma 1.** Let \( \text{Zero} := \{b \mid P_{\mathcal{M}}(b \models \Diamond \text{Bad}) = 0\} \). For each \(b \notin (\text{Bad} \cup \text{Zero})\):

\[
V_{\alpha}(b) = \sum_{b'} P^B(b, \alpha, b') \cdot V(b'), \quad \text{with} \quad V(b) = \max_{\alpha \in \text{Act}(O(b))} V_{\alpha}(b).
\]

Furthermore: \(V(b) = 0\) for \(b \in \text{Zero}\), and \(V(b) = 1\) for \(b \in \text{Bad}\).

**Remark 1.** As we are over-approximating \(V\), we do not need to precompute \(\text{Zero}\).

Note that the function \(V\) is convex iff for each \(b_1, b_2 \in B\) and for each \(\alpha \in [0, 1]\), it holds that \(V(\alpha \cdot b_1 + (1-\alpha) \cdot b_2) \leq \alpha \cdot V(b_1) + (1-\alpha) \cdot V(b_2)\).

For \(b \in (\text{Bad} \cup \text{Zero})\), the value function is constant and thus convex. The \(n\)-step reachability for a particular action is a linear combination over the \((n-1)\)-step reachabilities, and we take the maximum over these values to get the \(n\)-step reachability. The value \(V(b)\) is the limit for \(n\) towards infinity. As convex functions are closed under linear combinations with non-negative coefficients, under taking the maximum, and under taking the limit, we obtain:

**Theorem 2.** For any POMDP, the value-function \(V\) is convex.

### 3.2 Finite Exploration Approximation

One way to circumvent building the complete state space is to cut-off its exploration after some steps, much like we depicted part of the belief POMDP in Fig. 2. To ensure that the obtained finite MDP over-approximates the probability to reach a bad state, we simply assume that all transitions we cut go to a bad state immediately. Elaborate techniques for this approach (on general MDPs) have been discussed in the context of verification \([7]\), and have been successfully adapted to other models \([2,32,17]\). It shares many ideas with the SARSOP and GOAL-HSVI approaches for POMDPs \([21,16]\). This approach may be applied directly to belief MDPs, and we may use the POMDP \(\mathcal{M}\) to guide the cut-off process. In particular, using Theorem 2 and that the maximising policy over all policies is necessarily overapproximating the maximum over all observation-based policies, we obtain the following inequality:

\[
V(b) \leq \sum_{s\in S} b(s) \cdot V(\{s \rightarrow 1\}) \leq \sum_{s\in S} b(s) \cdot \sup_{\sigma \in \Sigma^B} P_{\mathcal{M}}(s \models \Diamond \text{Bad}) \tag{1}
\]

We may use this inequality to cut-off with a less pessimistic value than assuming that we reach the bad states with probability one.

Nevertheless, this approach has limited applicability on its own. It may well get stuck in regions of the belief space that are not near the goal. From state \(s_5, s_6\) in Fig. 1 the maximal reachability according to the underlying MDP is 1, which is too pessimistic to provide a good cut-off. Another issue is that the belief converges slowly along \(b_1, b_5, b_9\) in Fig. 2 and that cut-offs do not immediately allow to reason that the belief converged.
3.3 Discretised Belief Approximation

The idea of this approach is to select a finite set $F \subseteq B$ of beliefs, and construct an approximation of the belief MDP using only $F$ as states. We refer to $F$ as the foundation. (Reachable) beliefs $b$ not in $F$ are approximated using beliefs in $N_F(b)$, where $N_F(b) \subseteq F$ is the neighbourhood of $b$. We clarify the selection of these neighbourhoods later, and we omit the subscript $F$ whenever possible.

**Definition 5.** A neighbourhood $N(b)$ of belief $b$ is convex-containing, if there exists $\delta_b \in \text{Dist}(N(b))$ such that $b = \sum_{b' \in N(b)} \delta_b(b') \cdot b'$.

**Example 4.** In Fig. 4(a) we depict various neighbourhoods. In Fig. 4(a) the belief $\{s_3 \mapsto \frac{2}{3}, s_4 \mapsto \frac{1}{3}\}$ lies in the neighbourhood $\{s_3 \mapsto 1, s_3, s_4 \mapsto \frac{1}{2}\}$. All other subfigures depict belief-spaces for observations where three states have this observation (the third dimension implicitly follows). For the belief state $b_5 = \{s_0 \mapsto \frac{1}{2}, s_5 \mapsto \frac{1}{6}, s_6 \mapsto \frac{1}{6}\}$ from Fig. 2 and a neighbourhood as in Fig. 4(b) the vertex-weights $\delta_b$ follow straightforwardly from the belief. Observe that a small distance to a vertex induces a large weight. In Fig. 4(c) we adapt the neighbourhood to $x = \{s_5 \mapsto 1\}, y = \{s_0 \mapsto 1\}, z = \{s_5 \mapsto \frac{1}{4}, s_6 \mapsto \frac{3}{4}\}$. Then, the vertex weights follow from the following linear equations:

$$
\delta_b(x) = \frac{1}{2}, \quad \delta_b(y) + \frac{1}{4} \cdot \delta_b(z) = \frac{1}{6}, \quad \text{and} \quad 3/4 \cdot \delta_b(z) = \frac{1}{3}.
$$

From the convexity of the value function $V$ (Theorem 2), it follows that:

**Lemma 2.** Given $b$, $N(b)$ and $\delta_b$ as in Definition 5 it holds:

$$
V(b) \leq \sum_{b' \in N(b)} \delta_b(b') \cdot V(b').
$$

We emphasise that this inequality also holds if one over-approximates the values of the beliefs in the neighbourhood.

**Example 5.** Fig. 3(c) depicts the belief-to-value from Fig. 3(a) and (in blue) depicts the over-approximation based on Lemma 2. As neighbourhood, we use $\{s_3 \mapsto 1\}$ and $\{s_4 \mapsto 1\}$. In Fig. 3(d) we depict the over-approximation using a partitioning into three neighbourhoods, using the foundation $\{s_3 \mapsto 1\}$, $\{s_3 \mapsto \frac{1}{4}, s_4 \mapsto \frac{3}{4}\}$, $\{s_3 \mapsto \frac{3}{4}, s_4 \mapsto \frac{1}{4}\}$ and $\{s_4 \mapsto 1\}$. We see that the outer neighbourhoods now yield a tight over-approximation, and the inner neighbourhood yields a much better approximation compared to Fig. 3(c).
We select some finite foundation $F$ such that for each reachable $b$ in $\text{bel}(\mathcal{M})$, there exists a convex containing neighbourhood $\mathcal{N}(b)$. We call such a foundation adequate. One small adequate $F$ is $\{s \mapsto 1 \mid s \in S\}$. Practically, we use a tiling of the belief space into convex hyper-triangles, see below.

**Definition 6 (Discretised Belief MDP).** Let $F \subseteq B$ be an adequate foundation. Let $\mathcal{N}$ be arbitrarily fixed such that $\mathcal{N}(b) \subseteq F$ is convex-containing for any $b$. The discretised belief MDP of POMDP $\mathcal{M} = \langle M, Z, O \rangle$ is the MDP $db_F(\mathcal{M}) := (F, \text{Act}, P_F, b_1)$ with initial belief state $b_1 = \{s_1 \mapsto 1\}$, and—using the auxiliary notation from before—transition function $P_F$ given by

$$P_F(b, \alpha, b') := \begin{cases} \delta_{[b|\alpha,z]}(b') \cdot P^B(b, \alpha, z) & \text{if } b' \in \mathcal{N}([b|\alpha,z]) , \\ 0 & \text{otherwise,} \end{cases}$$

**Example 6.** Consider Fig. 5. We fixed $F = \{s \mapsto 1 \mid s \in S\} \cup \{s_3, s_4 \mapsto 1/2\} \cup \{s_5 \mapsto 3/4, s_6 \mapsto 3/4\}$. The weights for $\text{post}(b_2, b)$ and $\text{post}(b_1, b)$ follow from the computations in Example 4. Observe that $b_8$ is not reachable. The optimal policy in this MDP induces probability $3/4$, which is an upper bound on $V(b_1)$.

**Theorem 3.** For POMDP $\mathcal{M}$ with discretised belief MDP $db_F(\mathcal{M})$ and $b \in F$

$$V(b) \leq \sup_{\sigma \in \Sigma_{db_F(\mathcal{M})}} \Pr_{\sigma}^{db_F(\mathcal{M})}(b \models \Box \text{Bad}) .$$

As the MDP is finite and fully observable, the supremum is achieved by a memoryless policy, and we use MDP model checking to compute these values.

## 4 Abstraction-Refinement

In this section, we discuss a framework that combines the two types of abstraction discussed before. Roughly, the approach is a typical abstraction-refinement loop.
We start with an abstraction of the belief MDP; model checking this abstraction yields an upper bound on the values $V(b)$. In every iteration, we update the MDP and then obtain more and more accurate bounds. The abstraction applies cut-offs on a discretised belief MDP with some foundation $F$. For the refinement, we either explore beliefs that were previously cut off, we extend the foundation $F$, or we rewire the successors $b' \in \text{post}^{\text{bel}(M)}(b, \alpha)$ of some belief $b$ and action $\alpha$ to a new $N_F(b')$. Thus, rewire updates neighbourhoods, typically after refining the foundation. We give an example and then clarify the precise procedure, along with some technical details.

**Example 7.** In Fig. 6(a) we used a foundation as in Fig. 5 but with $b_{10}$ replacing $b_9$. Furthermore, we used cut-offs in $b_2$ and $b_3$ with the overapproximation from Eq. (1). In Fig. 6(b) we refined as follows: We **extended the foundation** with $b_9 = \{s_5 \mapsto 1/2, s_6 \mapsto 1/2\}$, we **explored** from $b_2, b_9$, and we rewired **only** $(b_{10}, b)$.

Alg. 4 sketches the abstraction-refinement loop. The algorithm iteratively constructs an abstraction MDP $A$ via a breath-first-search on the state space of the discretised belief MDP $db_F(M)$ (Lines 3 to 21). In Line 7 a heuristic decide decides for each visited belief to either explore or cut-off. If we explore, we may encounter a state that was previously explored. Heuristic rewire decides in Line 9 whether we rewire, i.e., whether we explore the successors again (to account for potentially updated neighbourhoods) or whether we keep the existing successor states. When cutting off, we use Eq. (1) to obtain an upper bound $U(b)$ for $V(b)$ and add a transition to some bad state with probability $U(b)$ and a transition to a sink state with probability $1 - U(b)$. The foundation is extended in Line 20. This only has an effect in the next refinement step.

After building the MDP $A$, it is analysed in Line 19 using model checking. This analysis yields a new upper bound $U(b_f) \geq V(b_f)$. The loop can be stopped at any time, e.g., when threshold $\lambda$ is shown as upper bound. Next, we describe how the foundation $F$ is initialised, extended, and iteratively explored.

**Picking foundations** The **initial foundation**. We discretise the beliefs using the foundation $F$. The choice of this foundation is driven by the need to easily de-

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1. The implementation actually still connects $b$ with already explored successors and only redirects the ‘missing’ probabilities w.r.t. $U(b')$, $b' \in \text{post}^{db_F(M)}(s, \alpha) \setminus S_{expl}$. 

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Figure 6. Beliefs as in in Fig. 5 with $b_{10} = \{s_5 \mapsto 1/2, s_6 \mapsto 1/2\}$. 

(a) First abstraction (b) Adding $b_9$, rewire $b_{10}$, exploring $b_2$. 


The set \( Z_{\text{extend}} \) of observations for which the foundation will be extended is determined by assigning a score \( : Z \rightarrow [0, 1] \). Low scoring observations are refined first. Intuitively, the score is assigned such that a score close to 0 indicates that one of the approximated beliefs with observation \( z \) is far away from all points in its neighbourhood, and a high score (close to 1) then means that all approximated beliefs are close to one of their neighbours. We set \( Z_{\text{extend}} = \{ z \in Z | \text{score}(z) \leq \rho_2 \} \) for some threshold \( \rho_2 \in [0, 1] \). When the value of \( \rho_2 \) is iteratively increased towards 1, each observation is eventually considered for refinement. Details are given in Appendix A.
Iterative exploration

The iterative exploration is guided using an estimate of how coarse the approximation is for the current belief state $b$, and by an estimate of how likely we reach $b$ under the optimal policy (which is unknown). If either of these values is small, then the influence of a potential cut-off at $b$ is limited.

Bounds on reaching the bad state. We use a lower bound $L(b)$ and an upper bound $U(b)$ for the value $V(b)$. Eq. 1 yields an easy-to-compute initial over-approximation $U(b)$. Running the refinement-loop improves this bound. For the lower bound, we exploit that any policy on the POMDP under-approximates the performance of the best policy. Thus, we guess some set of observation-based policies on the POMDP and evaluate them. If these policies are memoryless, the induced Markov chain is in the size of the POMDP and is typically easy to evaluate. Using a better under-approximation (e.g., by picking better policies, possibly exploiting the related work) is a promising direction for future research.

Estimating reachability likelihoods. As a naive proxy for this likelihood, we consider almost optimal policies from the previous refinement step as well as the distance of $b$ to the initial belief $b_I$. Since the algorithm performs a breadth-first exploration, the distance from $b_I$ to $b$ is reflected by the number of beliefs explored before $b$.

State exploration. In Line 7 of Alg. 1, `explore` decides whether the successors of the current belief $b$ are explored or cut off. We only explore the successors of $b$ if: (1) the approximation is coarse, i.e., if the relative gap between $U(b)$ and $L(b)$ is above (a decreasing) $\rho_{\text{gap}}$, (2) the state is likely relevant for the optimal scheduler, i.e., if (i) at most $\rho_{\text{step}}$ beliefs were explored (or rewired) before and (ii) $b$ is reachable under a $\rho_{\Sigma}$-optimal policy from the previous refinement step.

Rewiring. We apply the same criteria for `rewire` in Line 9. In addition, we only rewire the successors for action $\alpha$ if (i) $\alpha$ is selected by some $\rho_{\Sigma}$-optimal policy and (ii) the rewiring actually has an effect, i.e., for at least one successor the foundation has been extended since the last exploration of $b$ and $\alpha$.

5 Experiments

Implementation. We integrated the abstraction-refinement framework in the model checker STORM [15]. The implementation constructs the abstraction MDP as detailed in Alg. 1 using sparse matrices. The computation in Line 19 is performed using STORM’s implementation of optimistic value iteration [14], yielding sound precision guarantees up to relative precision $\varepsilon = 10^{-6}$. Our implementation supports arbitrary combinations of minimisation and maximisation of reachability and reach-avoid specifications, and indefinite-horizon expected rewards. For minimisation, lower and upper bounds are swapped.

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8 We guess policies in $\Sigma^M_{\text{obs}}$ by distributing over actions of optimal policies for MDP $M$.
9 $\rho_{\text{gap}}$ is set to 0.1 initially and after each iteration we update it to $\rho_{\text{gap}}/4$.
10 $\rho_{\text{step}}$ is set to $\infty$ initially and after each iteration we update it to $4 \cdot |S^A|$.
11 A policy $\sigma$ is $\rho_{\Sigma}$-optimal if $\forall b: V_{\sigma(b)}(b) + \rho_{\Sigma} \geq V(b)$. We set $\rho_{\Sigma} = 0.001$. 
### Verification of indefinite-horizon POMDPs

| Model | $\phi$ | Data | $S/Z$ | MDP | $\eta=4$ | $\eta=12$ | refine |
|-------|-------|------|--------|-----|---------|---------|--------|
| Drone | 4-1   | $1226$ | 384   | $0.98 \geq 0.98 -6.67$ | TO | MO | $\leq 0.97 \leq 0.97$ | $\leq 0.97 \leq 0.97$ |
| Drone | 4-2   | $1226$ | 761   | $0.98 \geq 0.98 -1$ | TO | MO | $\leq 0.97 \leq 0.97$ | $\leq 0.97 \leq 0.97$ |
| Grid-av | 4-0.1 | 17   | 4     | $0.93 \geq 0.21 -2.05$ | TO | MO | $\leq 0.94 \leq 0.94$ | $\leq 0.94 \leq 0.94$ |
| Grid | 4-0.1 | 62   | 3     | $0.94 \leq 4.74 -2.05$ | TO | MO | $\leq 0.94 \leq 0.94$ | $\leq 0.94 \leq 0.94$ |
| Maze2 | 0.1   | 62   | 3     | $6.32 \leq 0.94 -1$ | TO | MO | $\leq 6.18 \leq 6.18$ | $\leq 6.18 \leq 6.18$ |
| Refuel | 0.1   | 54   | 8     | $6.53 \leq 6.25 -1$ | TO | MO | $\leq 6.25 \leq 6.25$ | $\leq 6.25 \leq 6.25$ |
| Refuel | 0.1   | 54   | 8     | $6.53 \leq 6.25 -1$ | TO | MO | $\leq 6.25 \leq 6.25$ | $\leq 6.25 \leq 6.25$ |
| Rocks | 12    | 1496  | 384   | $16.5 \leq 10 -1$ | TO | MO | $\leq 20 \leq 20$ | $\leq 20 \leq 20$ |
| Rocks | 16    | 1496  | 2701  | $16.5 \leq 10 -1$ | TO | MO | $\leq 20 \leq 20$ | $\leq 20 \leq 20$ |

Additionally, our implementation may compute lower bounds by iteratively exploring (a fragment of) the belief MDP, without the discretisation. The state-space exploration is cut off after exploring an increasing number of states.

**Models.** We use all sets of POMDPs from [26]. Small versions of these benchmarks are omitted. We additionally introduced some variants, e.g., added uncertainty to the movement in the grid examples. Finally, we consider three scalable variants of typical grid-world planning domains in artificial intelligence.

**Set-up.** We evaluate our implementation with and without the refinement loop. In the former case, the refinement loop runs a given amount of time and we report the results obtained so far. In the latter case, a single iteration of Alg. 1 is performed with a fixed triangulation resolution $\eta$—a set-up as in [26]. We compare with the implementatation [26] in Prism. We used a simple SCC analysis to find POMDPs where the reachable belief MDP is finite. All POMDPs from [26] are in this category. We refer to the remaining POMDPs as infinite belief POMDPs.

All experiments were run on 4 core of an Intel® Xeon® Platinum 8160 CPU with a time limit of 1 hour (unless indicated otherwise) and 32 GB RAM.

**Results.** We consider the infinite belief POMDPs in Table 1. The first columns indicate the POMDP model instance, the type of the checked property (probabilities ($P$) or rewards ($R$), minimising or maximising policies), as well as the number of states, state-action pairs, and observations of the POMDP. The column ‘MDP’ shows the model checking result on the underlying, fully-observable MDP.
Table 2. Results for POMDPs with finite belief MDP.

| Benchmark | Model | Data | S/Act | Z | MDP | Data | CEO(M) | Storm | PRISM | Torus | MDP | Data | CEO(M) | Storm | PRISM | Torus |
|-----------|-------|------|-------|---|-----|------|--------|-------|-------|-------|-----|------|--------|-------|-------|-------|
| Crypt     | 4     | 514  | ≤3    | ≤3 | ≤2  | 10   | ≥1    | 20    | ≤1    | 1.46  | 6.12| 64   | ≤0    | ≤1    | 1.46  | 6.12  |
| Crypt     | 6     | 47   | ≥1    | 20 | ≤2  | 514  | ≤1    | 17.8  | ≤1    | 10    | 150 | 2    | ≤0    | ≤1    | 10    | 150   |
| Grid-av   | 4.0   | 34   | ≤1    | 10 | ≤1  | 46   | ≤1    | 1.51  | ≤1    | 1.43  | 4   | 8    | ≤0    | ≤1    | 1.43  | 4     |
| Maze2     | 0     | 54   | ≤1    | 4.5 | ≤0  | 502  | ≤1    | 0     | ≤1    | 4.5   | 0   | 8    | ≤0    | ≤1    | 0     | 8     |
| Maze2     | 1     | 58   | ≤1    | 8   | ≤0  | 502  | ≤1    | 0     | ≤1    | 8     | 0   | 8    | ≤0    | ≤1    | 0     | 8     |
| Netw-p    | 5.2-20| 4909 | 0     | 10 | ≤0  | 557  | ≤1    | 2.17  | ≤1    | 2.17  | 10  | 10   | ≤0    | ≤1    | 2.17  | 10    |
| Netw-p    | 5.2-20| 104  | ≤0    | 612| ≤0  | 557  | ≤1    | 2.17  | ≤1    | 2.17  | 10  | 10   | ≤0    | ≤1    | 2.17  | 10    |
| Netw-p    | 5.2-20| 2.106| ≤0    | 10 | ≤0  | 557  | ≤1    | 2.17  | ≤1    | 2.17  | 10  | 10   | ≤0    | ≤1    | 2.17  | 10    |
| Netw-p    | 5.2-20| 2.106| ≤0    | 10 | ≤0  | 557  | ≤1    | 2.17  | ≤1    | 2.17  | 10  | 10   | ≤0    | ≤1    | 2.17  | 10    |
| Netw     | 5.2-20| 2.106| ≤0    | 10 | ≤0  | 557  | ≤1    | 2.17  | ≤1    | 2.17  | 10  | 10   | ≤0    | ≤1    | 2.17  | 10    |
| Nrp       | 8     | 125  | ≤0    | 10 | ≤0  | 557  | ≤1    | 2.17  | ≤1    | 2.17  | 10  | 10   | ≤0    | ≤1    | 2.17  | 10    |
| Maze2     | 9     | 161  | ≤0    | 10 | ≤0  | 557  | ≤1    | 2.17  | ≤1    | 2.17  | 10  | 10   | ≤0    | ≤1    | 2.17  | 10    |

The column ‘bel(M)’ considers the refinement loop for the non-discretised belief MDP as discussed above and lists the best result obtained within 60 seconds, and the number of iterations. The subsequent columns show our result for a single approximation step with fixed resolution η and cut-off threshold ρgap, as well as the results of PRISM when invoked with resolution η. ‘TO’ and ‘MO’ indicate a time-out (> 1 hour) and a memory-out (> 32 GB), respectively. Each cell contains the obtained bounds on the result and the analysis time in seconds. Finally, the last two columns report on running the refinement loop for at most t (60 and 1800) seconds. The cells contain the best bound on the result and the number of loop iterations of Alg. 1. In addition, * indicates that no further refinement was possible (in this case the model-checking result corresponds to the precise value) and † indicates that an MO occurred before t seconds.

Table 2 provides the experimental results for benchmark models with finite belief MDP. The columns are similar as in Table 1 except that column ‘bel(M)’ indicates the model checking result and analysis time in seconds for the complete finite belief MDP. Appendix C contains further experiments.

Discussion. We start with some observations and focus on Table 1. First, our implementation outperforms the implementation of [20] by several orders of magnitude, most likely due to the on-the-fly state-space construction, and by an engineering effort. This difference cannot be explained by the currently implemented cut-offs; indeed, when choosing a static foundation, cut-offs do not improve performance noticeably. Second, our refinement loop avoids the need for a user-picked resolution, but a hand-picked resolution is sometimes faster (e.g., for Maze) or yields better results (e.g., for Grid). On the other hand, the refinement loop might find finite abstractions that concisely represent the belief MDP reachable under the optimal policy (e.g., for Rocks). Here, cut-offs are essential. Third, on many benchmarks, the refinement loop finds the crucial part of the abstraction within a minute, but e.g., Refuel profits from additional time.
We want to share three further observations: First, it seems interesting to investigate finite-belief POMDPs as these occur quite frequently (see Table 2) and can be analysed straightforwardly. Second, the current bottleneck is the bookkeeping of the belief states and the computation of neighbourhoods, not the model checking. Finally, even more than for MDPs, the size of the POMDP (or the number of observations) is not at all a proxy for the difficulty of verification.

Data Availability. The implementation, models, and log files are available at [5].

6 Conclusion and Future Work

We presented an abstraction-refinement for solving the verification problem for indefinite-horizon properties in POMDPs, e.g., for proving that all policies reach a bad state with at most probability $\lambda$. As the original problem is undecidable, we compute a sequence of over-approximations by iteratively refining an abstraction of the belief MDP. Our prototype shows superior performance over [26] in PRISM. The next step is to integrate better under-approximations.

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A Details for Selecting and Extending the Foundation

Initializing and refining the foundation. As mentioned in Sect. 4, we initialize the foundation $\mathcal{F}$ in Line 1 of Alg. 1 by applying Freudenthal Triangulation [11], with a fixed resolution $\eta_{\text{init}} > 0$ i.e.

$$\mathcal{F} = \set{b \in B | \forall s \in S: b(s) \in \{i/\eta_z | z = O(b), with \ i \in \mathbb{N}, 0 \leq i \leq \eta_z\}}.$$

Where initially $\eta_z = \eta_{\text{init}}$ for all $z \in Z$. To extend $\mathcal{F}$ (Line 20 of Alg. 1), we heuristically pick a set of observations $Z_{\text{extend}}$ (details below) and increase the resolutions $\eta_z$ for $z \in Z_{\text{extend}}$ by a factor $f_\eta > 1$. By default, our implementation assumes $\eta_{\text{init}} = 3$ and $f_\eta = 2$.

We also implemented a more dynamic triangulation scheme that attempts to minimize the cardinality of the neighbourhoods (and thus the branching of the approximation MDP). For belief state $b$ let $\mathcal{N}_{\eta}(b)$ be the neighbourhood obtained with Freudenthal triangulation when using resolution $\eta$. For our dynamic triangulation approach we triangulate belief $b$ with observation $z = O(b)$ using the neighbourhood $\mathcal{N}_{\eta}(b)$, where $\eta$ is the largest resolution satisfying

$$\eta \leq \eta_z \quad \text{and} \quad |\mathcal{N}_{\eta}(b)| = \min_{\eta' \leq \eta_z} |\mathcal{N}_{\eta'}(b)|.$$

An experimental evaluation of this dynamic approach is given in Appendix C.

Selecting observations to refine. To determine the set $Z_{\text{extend}}$ of observations that will be refined, we assign the following score to each observation $z.

For belief state $b$ let $\mathcal{N}_{\mathcal{F}}(b)$ be the triangulation neighbourhood with respect to the current foundation $\mathcal{F}$. Further, let $\delta_b \in \text{Dist}(\mathcal{N}_{\mathcal{F}}(b))$ be the vertex distribution as in Definition 4 and let $n = |\text{supp}(b)|$.

We use the following score to evaluate how good $b$ is approximated by its neighbourhood. If $n = 1$, $b$ is a Dirac belief and gets a score of 1 (the best possible score). Otherwise,

$$\text{score}(b) = \max \set{n \cdot \delta_b(b') - 1 | b' \in \mathcal{N}_{\mathcal{F}}(b)}.$$

Intuitively, if the score of $b$ is close to 1, $b$ is close to one of the beliefs $b'$ in its neighbourhood ($\delta_b(b') \approx 1$). If the score is close to 0, it has a large distance to all $b'$ in its neighbourhood ($\delta_b(b') \approx 1/n$). The score of an observation is obtained by taking the minimum score of any triangulated belief with that observation times the current (relative) resolution for $z$, more precisely

$$\text{score}(z) = \min_{b, \alpha} \left(\text{score}(b|\alpha, z)\right) \cdot \frac{\eta_z}{\max_{z' \in Z} \eta_{z'}},$$

where $[b|\alpha, z]$ is as in Definition 3. To make sure that irrelevant parts of the abstraction MDP do not affect the score, we only consider belief states $b$ and actions $\alpha$ that are reachable under some $\rho\Sigma$-optimal policy $\sigma$. 
We set \( Z_{\text{extend}} = \{ z \in \mathbb{Z} \mid \text{score}(z) \leq \rho_Z \} \) for some threshold \( \rho_Z \in [0,1] \). In our implementation, we start with \( \rho_Z = 0.1 \) and add \( 0.1 \cdot (1 - \rho_Z) \) for each refinement step. This way, \( \rho_Z \) approaches 1 and thus every observation is eventually refined (unless it already has score 1, i.e. does not need refinement).

## B Benchmarks

**Input for STORM.** Our implementation constructs POMDPs either from an explicit description or from a POMDP-extension of the PRISM language\(^{15} \)\(^{26} \). We have further extended the language, such that besides observing variable values, one can observe the values of arbitrary predicates.

**Differences in models.** The model for crypt and maze are slightly different from the original due to a modelling error in the original formulation.

**New models.** Our newly introduced models are grid-world based planning tasks.
- In drone we search for a drone-plan to arrive at a target location, while avoiding a randomly moving obstacle. The obstacle is only visible within a limited radius.
- In refuel we also search for a plan to arrive at a target. Movement is uncertain, and the own position is not observable. Obstacles are static. Additionally, any movement requires some energy. Energy can be refilled at recharging stations.
- rocks describes a resource collection task. Some rocks need to be collected, and it is a-priori unknown which rocks to collect. Sensing is noisy and both sensing and collection is costly, so this yields an intricate trade-off.

Details can be found

https://github.com/moves-rwth/indefinite-horizon-pomdps

## C Additional Experiments

We have done some further experiments which we omitted in the tight page limit. We used more models, used an alternative method to determine the resolution, and used an alternative set of ‘magic’ constants in our implementation. We report on the results below.

### C.1 Additional Benchmark instances and Approximation Sizes

Tables 3 and 4 report on our experiments on some additional model instances. The experimental set-up is as in Sect. 5. The displayed data is similar to Tables 1 and 2 except that we now also report on the size of the approximation MDP. More precisely, the number of states \( |S^A| \) of the approximation MDP is denoted after the \(|\) at the bottom line of each table cell. In case of PRISM, this is the number of unknown grid points as reported by the tool.

\(^{15}\) STORM rejects some POMDPs where action identifiers are missing: Whereas model checking MDPs does not require action names, these are essential in POMDPs.
We observe that several millions of belief states can be explored within the
\(T\) and memory limit. We also note that the implementation in PRISM often
considers far more grid points, which is a possible explanation for the superior
performance of STORM in many cases.

C.2 Evaluation of the Dynamic Triangulation Approach

Tables 5 and 6 show the experimental results for the dynamic approach for
triangulating beliefs as discussed in Appendix A. Again, the set-up is as in
Sect. 5 except for the different triangulation scheme.

Comparing with the results for the standard triangulation approach with
static resolutions (Tables 3 and 4), we often observe that the dynamic approach
yields smaller approximations for the finite belief MDPs (Table 6) but larger
approximations for the infinite ones (Table 5).

C.3 Comparison of different heuristic parameters

Finally, we evaluated our refinement heuristic under different parameters in
Tables 7 and 8. We report on the best results that the refinement loop produces
within 1800 seconds (as in the last column of the previous tables). We compare
the static and the dynamic approach for triangulation as well as 6 heuristics \(h_1, \ldots, h_5\)
\(h_0\) refers to the heuristic parameters as described in Sect. 4 and Appendix A
i.e.:

- The triangulation resolutions are initialised with \(\eta_{\text{init}} = 3\) and iteratively
  increased by factor \(f_{\eta} = 2\).
- The threshold for the score of refined observations is initially set to \(\rho_Z = 0.1\)
  and \(f_Z \cdot (1 - \rho_Z)\) is added for each refinement step with \(f_Z = 0.1\).
- The number of allowed exploration steps is initially unlimited and then set to \(\rho_{\text{step}} = f_{\text{step}} \cdot |S^A|\) with \(f_{\text{step}} = 4\).
- The maximal gap for cut-offs is initialised with \(\rho_{\text{gap}} = 0.1\) and iteratively
decreased by factor \(f_{\text{gap}} = 0.25\).
- For exploration, only the reachable fragment of the approximation under a
  \(\rho_{\Sigma} = 0.001\)-optimal policy is considered.

We obtained the other heuristic parameters \(h_1, \ldots, h_5\) from \(h_0\) as follows\(^{16}\):

- For \(h_1\) we set \(f_{\eta} = 1.4142135624 \approx \sqrt{2}\).
- For \(h_2\) we set \(f_Z = 0.05\).
- For \(h_3\) we set \(f_{\text{step}} = 2.\)
- For \(h_4\) we set \(f_{\text{gap}} = 0.5\).
- For \(h_5\) we set \(\rho_{\Sigma} = 0.5\).

We observe that the different refinement heuristics often yield similar results,
suggesting that the influence of the refinement parameters is limited. A more
extensive analysis of different strategies for refinement is left for future work.

\(^{16}\) All unmentioned parameters are as \(h_0\).
**Table 3.** Results for additional POMDP instances with infinite belief MDP.

| Benchmark | Data | S/A | Act | S/A [Act] | 0-4 | 0-12 | refine |
|-----------|------|-----|-----|----------|-----|------|--------|
| Drone     | 4.1  | 1226| 384 | ≥ 0.84  | 6.18 | 6.67 | 0.2   |
| Drone     | 4.2  | 1226| 764 | ≥ 0.84  | 7.11 | 6.97 | 0.2   |
| Drone     | 5.1  | 2555| 580 | ≥ 0.84  | 5.11 | 56.61 | 2.10 |
| Drone     | 5.3  | 2557| 1848| ≥ 0.84  | 5.11 | 56.61 | 2.10 |
| Grid-art  | 4.0  | 17  | 4   | 0.25    | 5.41 | 56.61 | 2.10 |
| Grid-art  | 4.0  | 17  | 4   | 0.25    | 5.41 | 56.61 | 2.10 |
| Grid      | 4.0  | 32  | 3   | 0.25    | 5.41 | 56.61 | 2.10 |
| Maze      | 0.1  | 54  | 1   | 0.25    | 5.41 | 56.61 | 2.10 |
| Maze      | 0.3  | 54  | 1   | 0.25    | 5.41 | 56.61 | 2.10 |
| Refold    | 0.0  | 574 | 50  | 0.25    | 5.41 | 56.61 | 2.10 |
| Refold    | 0.0  | 574 | 50  | 0.25    | 5.41 | 56.61 | 2.10 |
| Rocks     | 0.0  | 30  | 11  | 0.25    | 5.41 | 56.61 | 2.10 |
| Rocks     | 0.0  | 30  | 11  | 0.25    | 5.41 | 56.61 | 2.10 |
| Rocks     | 0.0  | 30  | 11  | 0.25    | 5.41 | 56.61 | 2.10 |
| Maze      | 1.0  | 16  | 22  | 0.25    | 5.41 | 56.61 | 2.10 |

**Table 4.** Results for additional POMDP instances with finite belief MDP.

| Benchmark | Data | S/A | Act | S/A [Act] | 0-4 | 0-12 | refine |
|-----------|------|-----|-----|----------|-----|------|--------|
| Crypt     | 4    | 492 | 50  | 0.25    | 5.41 | 56.61 | 2.10 |
| Crypt     | 4    | 492 | 50  | 0.25    | 5.41 | 56.61 | 2.10 |
| Crypt     | 6    | 30  | 11  | 0.25    | 5.41 | 56.61 | 2.10 |
| Crypt     | 6    | 30  | 11  | 0.25    | 5.41 | 56.61 | 2.10 |
| Grid-art  | 4    | 17  | 4   | 0.25    | 5.41 | 56.61 | 2.10 |
| Grid-art  | 4    | 17  | 4   | 0.25    | 5.41 | 56.61 | 2.10 |
| Nettoy-P  | 0-2  | 54  | 8.08| 0.25    | 5.41 | 56.61 | 2.10 |
| Nettoy-P  | 0-2  | 54  | 8.08| 0.25    | 5.41 | 56.61 | 2.10 |
| Nettoy-P  | 0-2  | 54  | 8.08| 0.25    | 5.41 | 56.61 | 2.10 |
| Nettoy-P  | 0-2  | 54  | 8.08| 0.25    | 5.41 | 56.61 | 2.10 |
| Nep       | 4    | 125 | 41  | 0.25    | 5.41 | 56.61 | 2.10 |
| Nep       | 4    | 125 | 41  | 0.25    | 5.41 | 56.61 | 2.10 |
Table 5. Results for POMDPs with infinite belief MDP using the dynamic triangulation approach.

Table 6. Results for POMDPs with finite belief MDP using the dynamic triangulation approach.
Table 7. Comparison of different heuristic parameters for POMDPs with infinite belief MDP.

| Benchmark Model | Data Act | A | h0 | h1 | h2 | h3 | h4 | h5 | h6 | h7 | h8 | h9 |
|-----------------|----------|---|----|----|----|----|----|----|----|----|----|----|
| Cross | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| Grid | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| Maze2 | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| Refact | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| Grid | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| Maze | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| Refact | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| Maze | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| Refact | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |

Table 8. Comparison of different heuristic parameters for POMDPs with finite belief MDP.

| Benchmark Model | Data Act | A | h0 | h1 | h2 | h3 | h4 | h5 | h6 | h7 | h8 | h9 |
|-----------------|----------|---|----|----|----|----|----|----|----|----|----|----|
| Cross | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| Maze | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| Refact | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| Maze | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| Refact | Pmax | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |
| | Pmin | 2026 | 2024 | 604 | 0.54 | 0.54 | 0.53 | 0.52 | 0.51 | 0.50 | 0.49 | 0.48 |

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