Collective modes and the broken symmetry of a rotating attractive Bose gas in an anharmonic trap

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We study the rotational properties of an attractively interacting Bose gas in a quadratic + quartic potential. The low-lying modes of both rotational ground state configurations, namely the vortex and the center of mass rotating states, are solved. The vortex excitation spectrum is positive for weak interactions but the lowest modes decrease rapidly to negative values when the interactions become stronger. The broken rotational symmetry involved in the center of mass rotating state induces the appearance of an extra zero-energy mode in the Bogoliubov spectrum. The excitations of the center of mass rotational state also demonstrate the coupling between the center of mass and relative motions.

I. INTRODUCTION

The appearance of quantized vortices offers a strong evidence of superfluidity in a rotating Bose-Einstein condensed system. A classical example is the quantized circulation in superfluid $^4$He. More recently, the existence of these novel quantum states has also been demonstrated in dilute atomic gases $^{11,12}$. It is found that above a certain critical rotation frequency, atomic vapor with repulsive interactions responds to rotation by forming a singly quantized vortex $^{3,4}$. As the rotation rate increases, more vortices appear and vortex lattices consisting hundreds of vortices may be detected $^{5,6}$. In addition to atomic boson gases, the experimental detection of vortex lattices in both sides of Feshbach resonance in quantum degenerate Fermi gas has just been reported $^{7}$.

The response of a Bose condensed gas to an external rotation is expected to be different in the case of attractive interactions. In a harmonic-oscillator potential, it is not the vortex state which is energetically favorable. Instead, the least-energy configuration for an attractive condensate with non-zero angular momentum is a state with a rotating center of mass (c.m.) $^{8,9,10}$. However, this state is thermodynamically unstable, for the critical rotation frequency to excite the c.m. mode is equal to the trap frequency. The condensate set on rotation thus experiences the centrifugal force being larger than the confining force of the trap and eventually escapes the trap.

Although a rotating attractive condensate is not stable in a harmonic potential, it can be stabilized by using a steeper confinement. A possible scheme is an anharmonic trap configuration in which a small quartic potential is superposed upon the harmonic trap. This potential is experimentally realistic: it has already been implemented to study the fast rotation of repulsive $^{87}$Rb condensate $^{11}$. On the attractive side it is expected that the phase-space consists of states of broken cylindrical symmetry and, for very weak interactions, multiply quantized vortices $^{12,13,14,15}$. The states of broken cylindrical symmetry involve a rotating center of mass and are closely connected with the c.m. rotating state in a harmonic trap. Even though the correspondence is never exact, in this paper we use the term c.m. rotating state for all the states with a broken cylindrical symmetry.

The rich collection of rotational states of an attractive condensate in an anharmonic trap is worth further studies. An especially interesting effect is the spontaneously broken cylindrical symmetry in the case of c.m. rotating state. In addition, the investigation of dynamical and thermodynamical processes is a necessary extension to the work performed so far. This includes the study of the lack of superfluidity in attractive condensates $^{16}$, indicating the fact that, if the external rotation is stopped, the condensate relaxes to the non-rotating ground state, independently of the initial rotational state. Recently, these topics have been studied in one dimension where even exact results have been obtained $^{17,18,19,20}$.

In the present work, we study the elementary excitations of rotating attractive Bose condensed gas confined in an anharmonic quartic + quadratic trap potential. We present our system and the relevant formalism in Sec. II. The nature of the ground state excitation spectrum is discussed in Sec. III and the results are summarized in Sec. IV.

II. THE MODEL

We assume a zero-temperature Bose gas with negative s-wave interaction. The anharmonic confining potential has the following form

$$ V(r, \theta, z) = \frac{1}{2} m \left[ \omega^2 \left( r^2 + \lambda \frac{r^4}{a_{osc}} \right) + \omega_z^2 z^2 \right], $$

which consists of the harmonic oscillator potential plus the radial quartic term described by the dimensionless parameter $\lambda$. The oscillator length is defined as $a_{osc} = (\hbar/m\omega)^{1/2}$.

Following the argumentation of Ref. $^{11}$, by ignoring the quantum fluctuations of the possible c.m. motion, the ground state behavior of a rotated attractive Bose condensate can be approximated by the ordinary single-component Gross-Pitaevskii (GP) equation in the frame
of reference rotating with frequency $\Omega$. The GP equation for a stationary state is

$$
\left[ H_0 - \Omega \cdot \hat{L} + U_0 |\Psi(r)|^2 \right] \Psi(r) = \mu \Psi(r), \tag{2}
$$

where $H_0 = \frac{-\hbar^2}{2m} \nabla^2 + V(r)$ is the Hamiltonian for a single atom in the nonrotating trap. The interaction constant $U_0$ is proportional to the scattering length by the relation $U_0 = 4\pi\hbar^2a/m$. $\hat{L} = -i\hbar(\mathbf{r} \times \nabla)$ is the angular momentum operator and $\mu$ is the chemical potential. The condensate wave function $\Psi$ is normalized to the number of atoms $N$ which is fixed here to $N = 1000$. We also fix the mass $m$ to the mass of atomic $^7\text{Li}$. For the angular frequencies of the harmonic oscilator we set the values $\omega = 2\pi \times 30 \text{ Hz}$ and $\omega_z = 2\pi \times 180 \text{ Hz}$. Here, $\Omega$ is always pointing to the $z$-direction, which, together with the strong confinement in this direction, makes the problem effectively two-dimensional.

The phase-diagram of the ground state solutions of Eq. (2) is studied in [15]. Here, we extend our earlier work and study the spectrum of normal modes of the solutions of Eq. (2). These are obtained by solving the coupled Bogoliubov-de Gennes equations for the quasiparticles $u_j$ and $v_j$,

$$
Ku_j(r) + U_0\Psi^2(r)v_j(r) = E_ju_j(r),
$$

$$
K^*v_j(r) + U_0\Psi^2(r)u_j(r) = -E_jv_j(r), \tag{3}
$$

where $K = H_0 - \Omega \cdot \hat{L} + 2U_0 |\Psi(r)|^2 - \mu$ and $E_j$ is the energy of the $j\text{th}$ eigenmode.

The calculation of the spectrum of the eigenmodes in our numerical procedure consists of first solving the lowest energy configuration of Eq. (2) in a two-dimensional Cartesian grid by propagating the wave function in imaginary time. The second step is to transfer the obtained wave function and chemical potential to Eqs. (3) and numerically diagonalize these in order to solve $u_j$, $v_j$ and $E_j$.

### III. GROUND STATE EXCITATIONS

We analyze the numerical solutions of Eq. (2) by fixing the pair of parameters $(\lambda, \Omega/\omega)$ and varying the interaction strength $U_0$. We first choose $(\lambda = 0.15, \Omega/\omega = 1.35)$. For these parameters the phase-space was analyzed in Ref. [17]. In addition, we choose another pair $(\lambda = 0.05, \Omega/\omega = 1.15)$ to be studied, for these values are closer to the pure harmonic oscillator. The low-energy spectra are shown in Fig. 1 where we have varied the scattering length $a$ to cover the phase-space portions of vortex and c.m. states. In both cases, the ground state on the vortex side is doubly quantized. A vortex is only stable for weak interactions, and the increase in the magnitude of interaction strength leads to a phase transition at certain critical $a = a_c$. For these relatively strong interactions, the ground state is the one with broken rotational symmetry. For very strong interactions (or high $N$) the system becomes unstable.

#### A. Vortex ground state

On the vortex side, the ground state as well as the excitations are cylindrically symmetric. Consequently, the Bogoliubov amplitudes can be written as $u_q \propto e^{i(q+m)\theta}$ with $v_q \propto e^{i(q-m)\theta}$. Here $m$ (not to be confused with the mass) is the angular momentum quantum number of the condensate, which, in the case of a doubly quantized vortex is $m = 2$. The excitation winding number $q$ is the angular momentum relative to the condensate. Energy spectra in Fig. 1 show that the lowest excitations are the modes with $q = 1$ and $q = -1$. Especially, the energy eigenvalue of $q = 1$ mode is very low. Increasing $|a|$ decreases the energy of the mode quickly to negative values. The crossover determines the critical value $a_c$ below which the vortex state is not stable. At $a \approx a_c$ the mode energy is real within the numerical precision.
Goldstone mode is $\partial \phi$ function $U(1)$ symmetry of the system [23]. However, in the case of symmetry implies the existence of a zero energy mode. According to Goldstone’s theorem [21], the spontaneously broken symmetry is then

$$\Phi_{\theta} (\mathbf{r}) = \frac{\partial \Psi (\mathbf{r})}{\partial \theta}.$$  

(4)

In Fig. 2 we show an example of a c.m. rotational wave function and the corresponding Goldstone mode $\Phi_{\theta} (\mathbf{r})$. The density profile of $\Phi_{\theta} (\mathbf{r})$ is slightly more elongated, but otherwise it is rather similar to $|\Psi (\mathbf{r})|^2$. The structure of $\Phi_{\theta} (\mathbf{r})$ may be confusing at first sight, but there is an extra twist in the problem. Namely, if more than one symmetry is spontaneously broken, the corresponding Goldstone modes are not necessarily orthogonal. This means that the mode in Eq. (4) is generally a superposition of the c.m. rotational wave function $\Psi$ and a wave function $\chi$ orthogonal to $\Psi$. In this context, we write

$$\Phi_{\theta} (\mathbf{r}) = \beta \Psi (\mathbf{r}) + \chi (\mathbf{r}),$$  

(5)

where $\beta$ is a complex constant. Because of orthogonality, both $\chi$ and $\beta$ can be easily solved. In the example of Fig. 2, we present the function $\chi$ in the left lower graph. The extra zero-energy Bogoliubov solution for the same wave function can also be written as a superposition of the two orthogonal modes. Also, the mode amplitude $u_G$ has a component orthogonal to the c.m. rotational wave function. This is plotted in the right lower graph in Fig. 2. The identical structure (within numerical precision) with the function $\chi$ is clearly visible.

The differential operation performed on the condensate wave function in Eq. 4 is just $i \hat{L}$. Although the function $\chi$ is generally small compared to $\beta \Psi$ it is clear that $\Psi$ is not an angular momentum eigenstate. On the other hand, the true many-body c.m. rotational state in a harmonic trap is an eigenstate of $\hat{L}$. The contradiction is a side effect of the broken symmetry, which, we believe, in turn is a consequence of the used GP approximation. This is supported by the quasi-one-dimensional analog [17, 19, 20], where the symmetry is broken independent of the details of the trap potential which forms the torus.

For actual, positive energy excitations the general behavior is that the energy eigenvalues increase with the interaction strength. However, interpretations of individual modes should be done with care. In one-dimensional torus, the lowest mode is a breathing mode [20]. Generally, this is not the case in 2D because the anharmonic term in the trap potential couples the c.m. and relative motions. A change in the mean size of the cloud affects the effective potential seen by the c.m. Consequently, a time dependent wave function corresponding to an excited mode is expected to show some c.m. motion also. For the lowest excitation (denoted by A in Fig. 1), the coupling can be seen in Fig. 3 where we have assumed an excited wave function of the form $\Psi (\mathbf{r}, t) = \Psi (\mathbf{r}) + u_A e^{-i E A t / \hbar} - v_A e^{i E A t / \hbar}$. As a rough description of the oscillation, at $t / T = 0$ the elongated wave function is on azimuthal motion. The c.m. motion then begins to decelerate and excitational breathing begins to compress the cloud. The compression rate is

However, in the unstable region the imaginary part of the negative mode rapidly increases as a function of the interaction strength. This indicates the dynamical instability of the vortex state. We have checked this by performing time evolution GP simulations. Indeed, we have seen the breakup of the vortex wave function when $|a| > a_c$ the vortex state is stable in time.

### B. C.m. ground state

On the left hand side of the Fig. 1 the excitations are calculated for the ground state solution which is the c.m. rotational state. The ground state stability is ensured by the fact that the excitation energies are real and non-negative. We have also propagated the GP ground state solutions in real-time and we have not seen any sign of a dynamical instability.

As seen in Fig. 1 there is an extra zero-energy mode on the c.m. side of the phase-space (denoted by G). Due to Goldstone’s theorem [21], the spontaneously broken symmetry implies the existence of a zero energy mode. Generally, the mean-field approximation breaks the global U(1) symmetry of the system [22]. However, in the case of c.m. rotational state, the cylindrical O(2) symmetry is also spontaneously broken. For a mean-field wave function $\phi_c$ with a particular broken symmetry $\varphi_0$, the Goldstone mode is $\frac{\partial \phi_c (\mathbf{r}, \varphi_0)}{\partial \varphi_0}$ [22]. In this case, the Goldstone mode corresponding to the rotational asymmetry

is then

$\Phi_{\varphi} (\mathbf{r}) = \frac{\partial \Psi (\mathbf{r})}{\partial \varphi}$.  

(4)

FIG. 2: Density plots of different wave functions related to the c.m. state (upper left graph). The relevant parameters are $\lambda = 0.15$, $\Omega / \omega = 8$ and $a = 10.0$ a.u. The upper right graph is the Goldstone mode $\Phi_{\theta}$ as defined in Eq. (4). The part of $\Phi_{\theta}$ orthogonal to the c.m. rotational wave function is denoted as $\chi$ and plotted in the lower left graph. In the lower right graph we plot the zero-energy Bogoliubov solution $u_G$. The insets show the phase profiles of the wave-functions.
highest at the classical turning point of the c.m. On the backward c.m. motion the cloud retains the squeezed shape (see $t/T = 0.35$ plot in Fig. 3). When approaching the other turning point the c.m. motion slows down and the cloud begins to enlarge again. The coupling between c.m. and shape oscillations is not characteristic only to this mode but also to all other modes we have investigated.

The investigation of the mode profiles for $u_j$ and $v_j$ also reveals that the broken rotational symmetry is not as visible for the high energy modes as it is for the low-lying ones. In the computations we have performed, the mode B corresponding to the $q = 4$ mode on the vortex side carries only a slight asymmetry on the c.m. side whereas the lowest excitation is highly localized.

IV. CONCLUSIONS

In summary, we have studied the Bogoliubov-de Gennes excitation spectrum of a rotating attractively interacting Bose-Einstein condensate trapped in an anharmonic potential. The stability of the (multiply) quantized vortex state is ensured by the positive energy of the lowest excitations. However, the eigenmodes $q = 1$ and $q = -1$ become unstable at relatively weak values of the scattering length. These modes being unstable the true ground state becomes the state with the rotating center of mass. For the c.m. state the rotational symmetry is broken and the Bogoliubov spectrum reveals an extra zero-energy eigenmode. This Goldstone mode is the rotation mode of the c.m. state. Finally, by exciting the c.m. state with lowest positive energy mode, we have illustrated the coupling between the c.m. and relative motions in anharmonic potential.

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