AXIAL ANOMALY IN THE PRESENCE OF
THE AHARONOV–BOHM GAUGE FIELD

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ABSTRACT: We investigate on the plane the axial anomaly for euclidean
Dirac fermions in the presence of a background Aharonov–Bohm gauge potential.
The non perturbative analysis depends on the self–adjoint extensions of the Dirac
operator and the result is shown to be influenced by the actual way of under-
standing the local axial current. The role of the quantum mechanical parameters
involved in the expression for the axial anomaly is discussed. A derivation of the
effective action by means of the stereographic projection is also considered.
1. Introduction

The behaviour of matter fields in the presence of Aharonov-Bohm (AB), or anyon, gauge fields has recently attracted a lot of interest owing to its possible application to condensed matter physics. Clearly, the origin of the much studied bidimensional anyon model [1] is intimately connected with topological AB interactions. From the field theoretical point of view, the coupling of 2-dimensional fermions with such a kind of gauge fields exhibits some interesting features which are worthwhile to be carefully investigated.

In this note we shall deal with the problem of computing the axial anomaly [2], as well as the effective action, for an euclidean 2-dimensional Dirac field interacting with a background AB potential, whose flux intensity is given by $\alpha$, with $-1 < \alpha < 0$. Besides its own theoretical interest, in Ref.[3] the axial anomaly has been shown to be related to the second virial coefficient of an anyon gas, thereby giving a connection with in principle measurable thermodynamical properties of anyonic matter. Moreover it has also been observed that the 1+1 axial anomaly is connected with measurable effect in solid state physics [4]. We shall more precisely concentrate on the dependence of the axial anomaly on boundary conditions at the location of the flux tube, i.e. on possible self–adjoint extensions of the Dirac hamiltonian. The result of our analysis will be fully exhibited, on the plane, for the special value $\alpha = -\frac{1}{2}$, owing to merely technical limitations. As a matter of fact, it should be hopefully clear that there is no reason, in principle, to doubt that our main statements still hold true for the whole range of $\alpha$.

We first treat the model in the 2-dimensional plane and later on we shall study its compactification by means of the stereographic projection.

The starting point is the classical euclidean action,

$$S = \int d^2x \bar{\psi}(x)(i\not{D})\psi(x) \ ,$$

(1.1)

where

$$i\not{D} = i\gamma_\mu(\partial_\mu - ieA_\mu) \ ,$$

(1.2)
the AB gauge potential being

\[ A_\mu \equiv \frac{\alpha}{e} \epsilon_{\mu\nu} \partial_\nu \ln \sqrt{x_1^2 + x_2^2} = \frac{\alpha}{e} \epsilon_{\mu\nu} \frac{x_\nu}{x_1^2 + x_2^2}, \]  

(1.3)

Here, \( \gamma_1 = \sigma_1 \), \( \gamma_2 = \sigma_2 \), \( \gamma_3 = i\gamma_1\gamma_2 = \sigma_3 \), where the sigma’s are the Pauli matrices. As it is well known the field strength is

\[ F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu} F, \quad F = -\frac{4\pi\alpha}{e} \delta^{(2)}(x), \]  

(1.4)

which corresponds to an infinitely thin solenoid at the origin; \( \psi \) and \( \bar{\psi} \) are independent euclidean spinors.

The basic tool which allows for the non-perturbative definition of the axial current and anomaly is the complex power of the Dirac operator. In general, complex powers of pseudo-elliptic invertible operator on compact manifolds without boundary do indeed exist under very general hypotheses [5]. Since, however, we are on the whole 2-dimensional plane and in the presence of field strengths with \( \delta \)-like singularities, the only way available to set up the complex powers is by means of the spectral theorem. To this aim let us consider the eigenvalues and eigenfunctions of the Dirac operator. The crucial point is that, in the presence of an AB gauge potential, a symmetric Dirac Hamiltonian is selected if one considers a domain spanned by spinors which are regular at the origin, but then completeness of the eigenstate basis is lost. In order to find a complete orthonormal basis which diagonalizes the Dirac operator it is necessary to consider its self-adjoint extensions [6][7].

To this concern let us choose polar coordinates \((r, \phi)\) on the plane; the eigenvalue problem becomes

\[
(\gamma_1[\cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \phi(\frac{\partial}{\partial \phi} + i\alpha)] + \gamma_2[\sin \phi \frac{\partial}{\partial r} + \frac{1}{r} \cos \phi(\frac{\partial}{\partial \phi} + i\alpha)])\psi_\lambda(r, \phi)
= -i\mu \lambda \psi_\lambda(r, \phi),
\]

(1.5)

where \( \mu \) is a suitable mass parameter to fix the scale of the eigenvalues. If we
rescale the spinor wave functions as

\[ \frac{1}{\sqrt{\mu}} \psi_{\lambda}(r, \phi) \mapsto \psi_{\lambda,n}(\xi, \phi) \equiv \begin{vmatrix} \psi_{\lambda}^{(L)}(\xi) e^{in\phi} \\ \psi_{\lambda}^{(R)}(\xi) e^{i(n+1)\phi} \end{vmatrix}, \tag{1.6} \]

with \( n \in \mathbb{Z} \), \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( \xi = \mu r \), we get the eigenspinors regular at the origin when \( n \neq 0 \), namely

\[ \psi_{\lambda, \pm n}(\xi, \phi) = \sqrt{\frac{|\lambda|}{4\pi}} \begin{vmatrix} (\pm i) J_{\pm \nu}(|\lambda|\xi) e^{\pm in\phi} \\ J_{\pm (\nu+1)}(|\lambda|\xi) e^{i(1 \pm n)\phi} \end{vmatrix}; \tag{1.7} \]

here \( n \in \mathbb{N} \), \( J_\nu \) being the Bessel function of order \( \nu(\pm n) \equiv \pm n + \alpha \).

On the other hand, the partial waves corresponding to \( \nu(0) \equiv \alpha \) can not be both regular at the origin unless completeness of the eigenfunctions is lost [6]. Then one has to consider the self-adjoint extensions of the Dirac operator by means of the standard Von Neumann method of the deficiency indices [8]. This leads to a one-parameter family \( \mathcal{D}_\omega \), \( \omega \in \mathbb{R} \), whose domain is given by \( \mathcal{D}(\mathcal{D}_\omega) = \{ \psi + \beta(\psi^{(+)} + e^{i\omega} \psi^{(-)}) | \psi \in C^{(0)}[0, +\infty] \cap H^1_2([0, \infty]), \beta \in \mathbb{C}, \omega \in \mathbb{R} \} \), where

\[ \psi^{(\pm)} \equiv \sqrt{\frac{\mu}{\mathcal{N}}} \begin{vmatrix} K_{\alpha}(\mu \xi) \\ \pm K_{(1+\alpha)}(\mu \xi) e^{i\phi} \end{vmatrix}; \tag{1.8} \]

\( K_\nu \) being the Basset–McDonald function, \( \mathcal{N} \) a normalization constant and \( H^1_2 \) the Sobolev space

\[ H^1_2([0, \infty]) = \{ \psi | \int_0^{2\pi} d\phi \int_0^{\infty} \xi d\xi \psi^\dagger(\xi, \phi)\psi(\xi, \phi) < \infty \}. \tag{1.9} \]

The corresponding eigenfunctions for \( \nu = \alpha \) can be written in the form

\[ \psi_{\lambda,0}^{(\omega)}(\xi, \phi) = \sqrt{\frac{|\lambda|}{4\pi(1 + \sin \theta(|\lambda|) \cos \nu \pi)}} \times 
\begin{vmatrix} i \cos \frac{\theta(|\lambda|)}{2} J_\alpha(|\lambda|\xi) - i \sin \frac{\theta(|\lambda|)}{2} J_{-\alpha}(|\lambda|\xi) \\ \text{sgn}(\lambda) \left[ \cos \frac{\theta(|\lambda|)}{2} J_{(1+\alpha)}(|\lambda|\xi) + \sin \frac{\theta(|\lambda|)}{2} J_{-(1+\alpha)}(|\lambda|\xi) \right] e^{i\phi} \end{vmatrix}; \tag{1.10} \]

where

\[ \tan \theta(|\lambda|) = |\lambda|^{2\alpha+1} \tan \omega. \tag{1.11} \]
We would like to notice that the eigenfunctions in eq.s (1.7), (1.10) are improper eigenfunctions, since they belong to eigenvalues of the continuous spectrum. They are suitably normalized according to theory of the distributions, \( \text{viz.} \)

\[
\lim_{R \to \infty} \int_0^{\mu R} \xi d\xi \int_0^{2\pi} d\phi \psi^\dagger_{n_1}(|\lambda_1|, \phi) \psi_{n_2}(|\lambda_2|, \phi) = \delta_{n_1 n_2} \delta(\lambda_1 - \lambda_2) . \quad (1.12)
\]

Moreover, in order to obtain the correct normalization as in eq. (1.12), one has to put the contribution at the origin equal to zero, thereby finding the relationship of eq. (1.11). It should be stressed that, rather than the requirement on the domain \( D(\Omega) \), which actually involves normalizable states, it is just the condition (1.12) of being a complete orthonormal family of improper eigenfunctions, leading eventually to eq.s (1.10), (1.11) (for the angular momentum component \( n = 0 \)). As a matter of fact, the property of having a complete orthonormal basis turns out to be necessary and sufficient for an operator to be self-adjoint. Furthermore, we see that, for any value of \( \omega \), the spectrum is purely continuous and is provided by the whole real line, but the point \( \lambda = 0 \), the zero modes being absent since they are not orthonormalizable.

2. The axial anomaly on the plane

Once the eigenvalue problem has been solved, we are able to set up the complex power by means of the spectral theorem. The complex power of the dimensionless operator \( I_{\omega}^{-s} \equiv \left( \frac{\Omega}{\mu} \right)^{-s} \) is defined by the kernel

\[
< \xi_1, \phi_1 | I_{\omega}^{-s} | \xi_2, \phi_2 > \equiv K_{-s}(I_{\omega}; \xi_1, \phi_1, \xi_2, \phi_2)
\]

\[
= 2 \sum_{n=1}^{\infty} \left[ \int_0^{\infty} d\lambda \, \lambda^{-s} \psi_{\lambda,n}(\xi_1, \phi_1) \psi^\dagger_{\lambda,n}(\xi_2, \phi_2) + (n \to -n) \right] + 2 \int_0^{\infty} d\lambda \, \lambda^{-s} \psi_{\lambda,0}^{(\omega)}(\xi_1, \phi_1) \psi_{\lambda,0}^{(\omega)}(\xi_2, \phi_2) ,
\]

which can be analytically extended to a meromorphic function of the complex variable \( s \); the key property is that the kernel of the complex power is regular at \( s = 0 \). In particular, the value of its trace over spinor indices, on the diagonal
\((\xi_1, \phi_1) = (\xi_2, \phi_2)\), can be explicitly evaluated either in the case \(\omega = 0, \pi, \ -1 < \alpha < 0\) or in the case \(\alpha = -\frac{1}{2}, \ \omega \in \mathbb{R}\). Actually, to the aim of computing the anomaly, it is more useful to write down the explicit form of the traces of the axial kernels, namely

\[
tr[\gamma_3 K_s(I_{\omega=0,\pi}; \xi, \xi)] = \Sigma_{-s}(\alpha, \xi) + \Sigma_{-s}(-\alpha, \xi) + \frac{\mu}{2\pi} \xi^{s-2} \frac{\Gamma\left(s + \frac{1}{2}\right) \Gamma(\alpha + 1 - \frac{s}{2})}{\sqrt{\pi} (s - 1) \Gamma(\frac{5}{2}) \Gamma(\alpha + \frac{s}{2})} + (\alpha \mapsto \alpha + 1),
\]

where we have set

\[
\Sigma_{-s}(\alpha, \xi) = \frac{\mu}{2\pi} \xi^{s-2} \frac{\Gamma\left(s + \frac{1}{2}\right) \Gamma(-\alpha + 2 - \frac{s}{2})}{\sqrt{\pi} (s - 1) (s - 2) \Gamma(\frac{5}{2}) \Gamma(-\alpha + \frac{s}{2})},
\]

whereas

\[
tr[\gamma_3 K_s(I_{\omega}; \xi, \xi)]|_{\alpha = -\frac{1}{2}} = \frac{\mu}{\pi \sqrt{\pi}} \xi^{s-2} \frac{\Gamma\left(\frac{1-s}{2}\right) \Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \left(1 + \sin \omega \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\right).
\]

Now we are ready to obtain the local forms of the axial anomaly, as it arises from the non perturbative definition of the fermionic axial current associated to the invertible operators \(I_{\omega}\). As a matter of fact we notice that, since the kernel \(K_{-s}(I_{\omega}; \xi, \xi)\) of the complex power is a well defined tempered distribution, depending meromorphically upon the complex variable \(s\), one can properly define the euclidean averages of the vector and axial currents, respectively, by means of point-splitting as well as analytic continuation [9], namely

\[
< j^{(\omega)}_{\mu}(x) > = e < tr[\gamma_{\mu} \psi(x) \psi^{\dagger}(x)] > \equiv \lim_{s \to 1, \epsilon \to 0} tr[\gamma_{\mu} K_{-s}(I_{\omega}; x, x + \epsilon)] = \lim_{s \to 1, \epsilon \to 0} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\lambda \lambda^{-s} e tr[\gamma_{\mu} \psi^{(\omega)}_{n,\lambda}(x) \psi^{(\omega)}_{n,\lambda}(x + \epsilon)],
\]

(2.5a)

where \(< \cdot >\) means euclidean average and

\[
< j^{(\omega)}_{\mu 3}(x) > = e < tr[\gamma_{\mu 3} \psi(x) \psi^{\dagger}(x)] > \equiv \lim_{s \to 1, \epsilon \to 0} tr[\gamma_{\mu 3} K_{-s}(I_{\omega}; x, x + \epsilon)] = \lim_{s \to 1, \epsilon \to 0} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\lambda \lambda^{-s} e tr[\gamma_{\mu 3} \psi^{(\omega)}_{n,\lambda}(x) \psi^{(\omega)}_{n,\lambda}(x + \epsilon)],
\]

(2.5b)
From the above definitions of the averaged local currents it is straightforward to show, taking eq. (1.5) into account, the quantum balance equations, namely

\[ < \partial_\mu j_\mu^\omega (x) > = 0 , \]

(2.6a)

testing the gauge invariance of the definition in eq. (2.5a), whereas

\[ < \partial_\mu j_\mu^3 (x) > = 2ie \lim_{s \to 0} tr[\gamma_3 K_s(I_{\omega};x,x)] \equiv A^{(\omega)}(x) \]

(2.6b)

leads to the definition of the local axial anomaly, once the topology has been chosen in taking the limit \( s \to 0 \); we shall discuss below this delicate matter.

First we consider the \( S' \)-topology, namely we study

\[ \lim_{s \to 0} \int d^2x \ tr[\gamma_3 K_s(I_{\omega};x,x)] f(x) , \]

for any rapidly decreasing function \( f \in S(\mathbb{R}^2) \).

If we take the limit \( s \to 0 \) in the sense of the distributions, from eq. (2.2) we get

\[ \int d^2x \ A^{(0,\pi)}(x)f(x) = -\alpha , \]

(2.7)

where \( f \) is a suitable test function belonging to \( S(\mathbb{R}^2) \) normalized to \( f(0) = \mu \); we notice that the above result, in full agreement with the one of Ref.[3], actually corresponds to the usual result, \( \text{viz.} A^{(0,\pi)}(x) = -\frac{ie^2}{2\pi} \epsilon_{\mu\nu} F_{\mu\nu}(x) \) as a distribution.

On the other hand, we can start from eq. (2.4), with \( \alpha = -\frac{1}{2} \), and try to take the limit \( s \to 0 \), for any \( \omega \), in the \( S' \)-topology. Let us consider indeed in the RHS of eq. (2.4) the term proportional to \( \sin \omega \). Then from the identity

\[ S' = \lim_{s \to 0} \left( \xi^{s-2} - \frac{2\pi}{s} \delta^{(2)}(\xi) \right) = \]

\[ \frac{1}{2\xi} \frac{d}{d\xi} \left( \xi \frac{d(\ln \xi)^2}{d\xi} \right) , \]

(2.8)

it immediately follows that the \( S' \)-limit does not exist unless \( \omega = 0, \pi \) and, in this case, eq. (2.7) still holds with \( \alpha = -\frac{1}{2} \). The above analysis strongly suggests that the definition of the anomaly, as the limit in the sense of distributions of the axial kernel, leads to select \( \omega = 0, \pi \) as the correct physical choices for the self-adjoint
extensions of the Dirac operator. The same conclusion has been claimed in the literature [7] from a quite different point of view, namely by demanding that one does not have any additional contact interaction at the origin beyond the point-like magnetic field. Nevertheless it is worthwhile to mention that some dual point of view leads to a different outcome, when we consider the limit $s \to 0$ in the natural topology of $\mathbb{R} - \{0\}$ and only afterwards continue the result to $\mathcal{S}'(\mathbb{R}^2)$. As a matter of fact, a non vanishing result is obtained in this case for $\omega \neq 0, \pi$ and, when $\alpha = -\frac{1}{2}$, we can compute explicitly

$$
\lim_{s \to 0} \lim_{\epsilon \to 0} e \, \text{tr} [\gamma_3 K_{-s}(I; x, x + \epsilon)]|_{\alpha = -\frac{1}{2}} = \frac{i e \sin \omega}{2\pi^2 r^2}, \quad r \neq 0 .
$$

As a consequence, there is a unique continuation in $\mathcal{S}'(\mathbb{R}^2)$ which reads ($\alpha = -\frac{1}{2}$)

$$
\int_0^\infty \xi d\xi \int_0^{2\pi} d\phi \, f(\xi, \phi) A^{(\omega)}(\xi) = \int_0^\infty \xi d\xi \int_0^{2\pi} d\phi \, \frac{i e \sin \omega}{2\pi^2 [\xi^2]} ,
$$

where $f$ is a test function belonging to $\mathcal{S}(\mathbb{R}^2)$. We recall the definition of the tempered distribution

$$
\frac{1}{[\xi^2]} \equiv \frac{1}{2\mu^2} \left( \partial_1^2 + \partial_2^2 \right) \left( \ln r \right)^2 + C(\mu) \delta^{(2)}(x) ,
$$

$C(\mu)$ being an arbitrary function of the regularization mass parameter. It should be stressed that the freedom (actually up to $\ln \mu$) within $C(\mu)$ amounts to the natural requirement

$$
\frac{1}{[\xi^2]} = \frac{1}{\mu^2} \cdot \frac{1}{[r^2]} ;
$$

consequently we can think about $C(\mu)$ in terms of a scaling function, in the sense that, when $\mu \to p\mu$, we have $C(\mu) \to C(p\mu) = C(\mu) + 2\pi \ln p$.

Now some remarks are in place to comment this result. On the one hand, we observe that eq. (2.10) involves two further quantum mechanical arbitrary parameters, $\omega$ and the scaling factor $C(\mu)$, beyond the flux intensity $\alpha = -\frac{1}{2}$ of the classical background infinitely thin solenoid. In particular, if the test function vanishes at the origin, where the field strength is concentrated, still a nonvanishing contribution survives, of a purely quantum mechanical nature, which depends upon the parameter of the self-adjoint extensions. On the other hand, we recall
that in the case of smooth gauge fields the local axial anomaly turns out to be proportional to the field strength itself, up to the presence of zero modes (see Refs. [3],[9],[10] and the last part of the present paper). In the presence of the singular AB gauge potential, the local axial anomaly still exhibits the above feature if we define the axial current as a limit in the \( S' \)-topology, and, moreover, the self–adjoint extension is fixed. Alternatively, the local axial anomaly is different from zero even outside the infinitely thin solenoid, and the parameter of the self–adjoint extensions is free, if we define the axial current and anomaly as in eq.s (2.9),(2.10).

We could say that this latter quite interesting feature is closely reminiscent of the AB effect.

In conclusion some physical, \emph{a priori} measurable quantity \([3],[4]\), such as the local axial anomaly, in the present singular case, appears to be reproduced by the standard Schwinger’s \([11]\) form \( A(x) = -\frac{i e^2}{2\pi} \epsilon_{\mu\nu} F_{\mu\nu}(x) \), the parameter of the self–adjoint extensions being fixed to \( \omega = 0,\pi \), if the regularized axial current is defined as a limit in the topology of tempered distributions. Nonetheless, from the mathematical point of view, some inequivalent construction of the singular axial current might be considered, which eventually leads to a non–standard form of the axial anomaly and to the freedom in the choice of the self–adjoint extensions. Strictly speaking, the above statements holds true in the special cases we have explicitely worked out. Nevertheless it is highly presumable that the same features still appear in general, although the explicit proof does not seem to be presently available. The physical content of this last approach will be discussed elsewhere.

3. The effective action

Another important issue is the calculation of the effective action. To this purpose it should be noticed that the standard gauge invariant Schwinger formula on the plane \([11]\) does not make sense, owing to the singular nature of the AB gauge potential. Nevertheless, a meaningful definition of the effective action may be obtained \([9],[12]\), through the transition to the compact case by means of the stereographic map from the plane to the open 2–sphere (without the north–pole), the origin of the plane coinciding with the south–pole. The open 2–sphere is
embedded in $\mathbb{R}^3$ in such a way that a convenient choice of the coordinates reads

$$
X_1 = 2a \frac{y \cos \phi}{1 + y^2}, \\
X_2 = 2a \frac{y \sin \phi}{1 + y^2}, \\
X_3 = a \frac{1 - y^2}{1 + y^2}, \\
X_\mu X_\mu + X_3^2 = a^2, \quad y = \frac{r}{a}.
$$

(3.1)

It is immediate to obtain the form of the zwei–beine, namely

$$
e_{a\mu}(y) = \frac{2}{1 + y^2}\delta_{a\mu}, \\
E^\mu_a(y) = \frac{1 + y^2}{2}\delta_{a\mu}.
$$

(3.2)

The invariant measure becomes

$$
\int d\mu \equiv \int_0^{2\pi} d\phi \int_0^\infty y dy \frac{4a^2}{(1 + y^2)^2}.
$$

(3.3)

Within the above conformal coordinate system $x_\mu$, the eigenvalue problem for the covariant Dirac operator takes the form

$$
\frac{1 + y^2}{2} \gamma_\mu \left( \nabla_\mu - \frac{1}{2} \partial_\mu \ln \frac{1 + y^2}{2} \right) \psi_\lambda(x) + \frac{\lambda}{a} \psi_\lambda(x) = 0,
$$

(3.4)

where

$$
\nabla_\mu \equiv \partial_\mu - ieA_\mu(x).
$$

(3.5)

If we pass to polar coordinates and perform a conformal mapping on the spinors, namely if we set

$$
\sqrt{a} \psi_{\lambda,n}(x) \mapsto \sqrt{\frac{2}{1 + y^2}} \begin{vmatrix}
\psi^{(L)}_\lambda(y)e^{in\phi} \\
\psi^{(R)}_\lambda(y)e^{i(n+1)\phi}
\end{vmatrix},
$$

(3.6)

the eigenvalues problem becomes equivalent to the system of coupled differential equations

$$
y \frac{d\psi^{(L)}_\lambda}{dy} - \nu \psi^{(L)}_\lambda = \frac{2\lambda y}{i(1 + y^2)} \psi^{(R)}_\lambda,
$$

(3.7a)

$$
y \frac{d\psi^{(R)}_\lambda}{dy} + (\nu + 1) \psi^{(R)}_\lambda = \frac{2\lambda y}{i(1 + y^2)} \psi^{(L)}_\lambda,
$$

(3.7b)
where $\nu(\pm n) = \pm n + \alpha$, $\nu(0) \equiv \alpha$. In order to solve eqs. (3.7) it turns out to be useful to perform the change of variables

$$\rho \equiv \frac{y^2 - 1}{y^2 + 1}, \quad \int d\hat{\mu}(\rho, \phi) = \int_0^{2\pi} d\phi \int_{-1}^1 \frac{2d\rho}{1 - \rho};$$

(3.8)

Hence, the eigenvalues problem takes the form

$$(1 - \rho^2) \frac{d\chi_l^{(L)}}{d\rho} - \nu \chi_l^{(L)} + i\lambda_l \sqrt{1 - \rho^2} \chi_l^{(R)} = 0 \quad ,$$

(3.9a)

$$(1 - \rho^2) \frac{d\chi_l^{(R)}}{d\rho} + (\nu + 1) \chi_l^{(R)} + i\lambda_l \sqrt{1 - \rho^2} \chi_l^{(L)} = 0 \quad ,$$

(3.9b)

with $\lambda_l \neq 0$ and

$$(1 - \rho^2) \frac{d\chi_0^{(L)}}{d\rho} - \nu \chi_0^{(L)} = 0 \quad ,$$

(3.10a)

$$(1 - \rho^2) \frac{d\chi_0^{(R)}}{d\rho} + (\nu + 1) \chi_0^{(R)} = 0 \quad ,$$

(3.10b)

for the zero modes.

We would like to notice that the covariant Dirac operator on the open 2–sphere, up to the conformal mapping (3.6), turns out to be essentially self–adjoint with domain given by square integrable functions on the interval $\rho \in [-1, 1]$, the derivatives at $\rho = \pm 1$ being understood in the sense of the distributions. In order to fulfil the above requirement, only the value $n = 0$ is allowed, a feature which leads to the removal of the degeneracy with respect to the orbital angular momentum. The complete set of orthonormal eigenfunctions reads

$$\chi_l^{(L)}(\rho, \phi) = \frac{1}{2\sqrt{\pi}} \left(\frac{1 + \rho}{1 - \rho}\right)^{\frac{\lambda_l}{2}} \hat{P}_l^{(-\alpha - 1, \alpha)}(\rho) \quad ,$$

(3.11a)

$$\chi_l^{(R)}(\rho, \phi) = \frac{\pm i e^{i\phi}}{2\sqrt{\pi}} (1 - \rho) \left(\frac{1 + \rho}{1 - \rho}\right)^{\frac{\lambda_l + 1}{2}} \hat{P}_l^{(-\alpha, 1 + \alpha)}(\rho) \quad ,$$

(3.11b)

with eigenvalues $\lambda_l = l^2$, $l = 1, 2, \ldots$. The sign indetermination is related to the global euclidean axial symmetry $\psi \mapsto e^{\gamma_3 \eta} \psi$. Here $\hat{P}_l^{(\alpha, \beta)}(\rho)$ stands for the normalized Jacobi polynomials [13], namely

$$\hat{P}_l^{(\alpha, \beta)}(\rho) \equiv \sqrt{\frac{l!(2l + \alpha + \beta + 1)!}{2^\alpha + \beta + 1 \Gamma(l + \alpha + \beta + 1)}} P_l^{(\alpha, \beta)}(\rho) \quad ,$$

(3.12)

$$P_l^{(\alpha, \beta)}(\rho) \equiv \frac{(-1)^l}{2^l l!} (1 - \rho)^{-\alpha} (1 + \rho)^{-\beta} \frac{d^l}{d\rho^l}[(1 - \rho)^{\alpha + l}(1 + \rho)^{\beta + l}].$$
The noteworthy feature is the appearence of zero modes, namely
\[\chi_0^{(L)}(\rho) = \frac{1}{2\pi} \sqrt{\sin(-\pi \alpha)} \left(\frac{1 + \rho}{1 - \rho}\right)^{\frac{1}{2}}, \quad (3.13a)\]
\[\chi_0^{(R)}(\rho) = 0, \quad (3.13b)\]
which indicates the non trivial topological behaviour of the AB potential. By the way, the reason for the vanishing of the right–zero–mode comes from the requirement of orthonormality for the complete set of the eigenfunctions of the self–adjoint Dirac operator on the open 2–sphere.

Let us come to the explicit evaluation of the quantum effective action and of the corresponding axial anomaly. As it is known, the 2–dimensional classical action for the Dirac spinor field on the open 2–sphere is conformally equivalent to the corresponding quantity on the complex plane, namely [12]

\[W_{\text{plane}} \equiv W_{\text{sphere}} + \frac{1}{24\pi} \int d^2x \omega(x) \Delta \omega(x), \quad (3.14)\]
\[\omega(x) = \ln \frac{1 + (\frac{x}{a})^2}{2}, \quad (3.15)\]
where \(W\) stands for the logarithm of the regularized determinant of the Dirac operator. As a consequence, taking eq. (3.6) into account, we can define the regularized determinant of the flat Dirac operator by means of the zeta–regularization of the associated self–adjoint operator of eqs. (3.4), which has a discrete spectrum. Actually, as the spectrum includes the null eigenvalue, we have to set [9]

\[\zeta(ia \hat{\nabla}; s) \equiv \sum_{l=1}^{\infty} (l^2)^{-s} = \zeta(2s), \quad (3.16)\]
where
\[\hat{\nabla} \equiv \frac{1 + (\frac{x}{a})^2}{2} \nabla_x, \quad (3.17)\]
and consequently
\[W_{\text{plane}} = -\ln \det'(ia \hat{\nabla}) \equiv \frac{d}{ds} \zeta(ia \hat{\nabla}; s)|_{s=0} = -\frac{1}{2} \ln 2\pi. \quad (3.18)\]

whereas
\[W_{\text{sphere}} = \frac{1}{6} (1 - \ln 2) - \frac{1}{2} \ln 2\pi. \quad (3.19)\]
The local axial anomaly is now arising from the variation (see Refs. [8][14])

\[
\int d^2x \ A(x) f(x) = 2ie \int d^2x \ \frac{\delta W_{\text{plane}}}{\delta A_\mu(x)} \frac{i}{\epsilon} \epsilon_{\lambda\mu} \partial_\lambda f(x),
\]

and leads, nota bene, to a one–parameter family of local expressions, the parameter being given by the compactification radius \( a \), namely

\[
A(x) = 2ie \lim_{s \to 0} \sum_{l=1}^{\infty} (l^2)^{-s} \ tr[\gamma_3 \psi_l(x/a) \psi_l^\dagger(x/a)].
\]

Unfortunately, it does not seem that the RHS of the last equation could be set into a close analytical form, just as in the general case of the continuous spectrum on the plane. It should be stressed that the standard Seeley–DeWitt asymptotic expansion can not be safely used to handle the RHS of eq. (3.17), since the original theory is strictly defined on the open 2–sphere without poles; in particular the fermion eigenfunctions are singular (not meromorphically) at the poles. Nevertheless, one can try to approximate the present singular case by means of a smooth ”vortex–like” gauge potential, namely one can consider

\[
A_\mu(x) = S' - \lim_{\sigma \to 0} \frac{\alpha}{\epsilon} e^{\epsilon_{\mu\nu}} \frac{x_\nu}{x_1^2 + x_2^2 + \sigma^2},
\]

the limit being taken at the end of the anomaly calculation (actually this corresponds to a regularization of the singular statistical AB interaction). Within this framework, the anomaly can be computed from the standard methods [9],[10] and, in the decompactification limit \( a \to \infty \), it reproduces the usual Schwinger’s result, the topological zero modes being disappeared. It should be noticed that this result is in agreement with what we have previously found on the plane, if the anomaly is coherently understood as a limit in the topology of tempered distributions (see eq. (2.7)). In this sense we argue that the results on the plane, we have previously discussed, should be of general character.

In conclusion we find that the effective action can be computed exactly (see eq. (3.17)) starting from the stereographic map, whereas the local axial anomaly can be explicitly evaluated on the plane, where the effective action does not exist in the present singular case. Here we have presented the detailed calculation of
the local axial anomaly on the plane only for particular values of the parameters. Obviously, a technical effort should be attempted in order to work out exactly the quite general cases for the effective action as well as for the anomaly. It would also be very interesting to investigate other kinds of physical situations such as, in particular, the occurrence of confining potentials and/or smooth boundaries, to further understand their influence on the axial anomaly and the effective action. Moreover we think to analyze the eventual relevance of eq. (2.10) in the evaluation of many body hamiltonian, virial coefficients and other physical effects.

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