On 4-dimensional cosmological models locally embedded in a 5-dimensional Ricci-flat space

A. G. Agnese* and M. La Camera†

Dipartimento di Fisica dell’Università di Genova
Istituto Nazionale di Fisica Nucleare, Sezione di Genova
Via Dodecaneso 33, 16146 Genova, Italy

Abstract

We employ a theorem due to Campbell to build some simple 4-dimensional cosmological models which originate from solutions describing waves propagating along the extra-dimension of a 5-dimensional Ricci-flat space. The dimensional reduction is performed in the Jordan frame according to the induced-matter theory of Wesson.

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*Email: agnese@ge.infn.it
†Email: lacamera@ge.infn.it
An interesting version of 5-dimensional General Relativity has been developed in recent years by Wesson [1,2,3]. The central thesis of his induced-matter theory is that the 4D Einstein’s field equations with matter

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$  \hspace{1cm} (1)

are a subset of the 5D field equations for vacuum in terms of the Ricci tensor

$$R_{AB} = 0$$  \hspace{1cm} (2)

The theory also allows to obtain flat cosmological solutions containing the usual 4D perfect fluid energy-momentum tensor [4].

Due to its primary significance in the present context, it is worth quoting a theorem due to Campbell [5], which states as follows:

**Theorem:** Any analytic Riemannian space $V_n(s,t)$ can be locally embedded in a Ricci-flat Riemannian space $V_{n+1}(s+1,t)$ or $V_{n+1}(s,t+1)$.

This theorem has been recently brought to light by Romero et al. [6,7] and employed both in applying Wesson’s method [1] and in investigating the embedding of lower-dimensional spacetimes.

In this Brief Note we consider the cosmological solution describing waves propagating in the extra-dimension of a (4+1)-dimensional Ricci-flat space and, after selecting particular modes, we obtain the corresponding (3+1)-dimensional cosmologies. The final result will be written in the Jordan frame,
according to the dimensional reduction prescribed by the “induced-matter theory of Wesson”.

We start from the 5D line element

\[ ds_5^2 = e^\omega (dr^2 + d\Omega^2) - e^\nu dt^2 + e^\mu dl^2 \]  

where \( d\Omega^2 = d\theta^2 + \sin \theta d\phi^2 \) and \( l \) is the extra coordinate.

The metric coefficients \( \omega, \nu \) and \( \mu \) will depend in general on both \( t \) and \( l \), the dependence on \( r \) being ruled out in the absence of sources. The case when sources are present and the metric coefficients depend only on \( r \), has been treated in a more general context in ref. [8]. Moreover we can put, by simmetry considerations, \( \nu = \mu \).

The relevant Ricci equations are:

\[ 3 \nu' \omega' - 3 \omega'^2 - 2 \nu'' - 6 \omega'' + 3 \nu \dot{\omega} + 2 \ddot{\nu} = 0 \]  

(4a)

\[ 3 \nu' \omega' + 2 \nu'' + 3 \nu \ddot{\omega} - 3 \dot{\omega}^2 - 2 \ddot{\nu} - 6 \ddot{\omega} = 0 \]  

(4b)

\[ \omega' \dot{\nu} + \nu' \dot{\omega} - \omega' \ddot{\omega} - 2 \dot{\omega} = 0 \]  

(4c)

\[ 3 \omega'^2 + 2 \omega'' - 3 \dot{\omega}^2 - 2 \ddot{\omega} = 0 \]  

(4d)

Here partial derivatives with respect to \( t \) and \( l \) are denoted by an overdot and a prime respectively.

One can immediately see that equation (4d) admits 3-brane wave solutions, propagating along the fifth dimension of a 5-dimensional bulk, of the form
\[ \omega_\pm = \omega(t \pm l) \text{ and } \nu_\pm = \nu(t \pm l). \] Denoting by an asterisk derivatives with respect to \( t \pm l \), equations (4a), (4b) and (4c) all become, after substitution:

\[
2 \dot{\nu}_\pm \dot{\omega}_\pm - \ddot{\omega}_\pm^2 - 2 \dddot{\omega}_\pm = 0
\] (5)

Therefore, selecting a particular form of \( \omega_\pm \), the other metric coefficients are given by

\[
e^{\nu_\pm} = e^{\mu_\pm} = L \dot{\omega}_\pm e^{\frac{\omega_\pm}{2}}
\] (6)

where \( L \) is a suitable constant of integration.

Starting from the above solutions, which describe waves propagating along the extra dimension of a 5-dimensional Ricci-flat spacetime with line element

\[
ds_5^2 = e^{\omega_\pm} (dr^2 + r^2 d\Omega^2) + L \dot{\omega}_\pm e^{\frac{\omega_\pm}{2}} (-dt^2 + dl^2)
\] (7)

we build some simple cosmological models in a 4-dimensional spacetime with line element

\[
ds_4^2 = e^{\omega(t)} (dr^2 + r^2 d\Omega^2) - L \dot{\omega}(t) e^{\frac{\omega(t)}{2}} dt^2
\] (8)

obtained from (7) by a section with a hypersurface at constant \( l \) chosen, without loss of generality, as \( l = 0 \).

Components of the Einstein tensor in mixed form, derived from the above metric (8), are

\[
G^r_t = G^\phi_\phi = C^\rho_\rho = - \frac{e^{-\frac{3}{2} \frac{\omega}{2}} (\dot{\omega}^2 + \dddot{\omega})}{2 L \dot{\omega}}
\]

\[
G^r_\tau = \frac{3 e^{-\frac{3}{2} \frac{\omega}{2}} \dddot{\omega}}{4 L}
\] (9)
We wish to match the terms in (9) with the components of the usual 4D perfect fluid energy-momentum tensor $T_{\alpha\beta} = (p + \rho) u_\alpha u_\beta + pg_{\alpha\beta}$. In our case the pressure and density are given by $T^r_r = p$ and $T^t_t = -\rho$, and therefore we can simply identify $G^r_r$ with $8\pi p$ and $G^t_t$ with $-8\pi \rho$.

Of course the choice of the function $\omega(t)$ is to a large extent arbitrary so we suggest, to make some physically meaningful examples, the following one:

$$\omega(t) = \alpha \ln \left( 1 + \frac{t}{\alpha L} \right)$$

(10)

where $\alpha$ is an assignable constant.

As a consequence, the line element (8) becomes

$$ds_4^2 = \left( 1 + \frac{t}{\alpha L} \right)^\alpha (dr^2 + r^2 d\Omega^2) - \left( 1 + \frac{t}{\alpha L} \right)^\frac{\alpha + 2}{2} dt^2$$

(11)

and clearly characterizes a conformally flat spacetime when $\alpha + 2 = 0$.

To go further, it is useful to make in (11) the change of variable

$$\tau = \begin{cases} 
\frac{4 \alpha L}{\alpha + 2} \left[ \left( 1 + \frac{t}{\alpha L} \right)^{\frac{\alpha + 2}{\alpha + 2}} - 1 \right] & \text{if } \alpha + 2 \neq 0 \\
-2L \ln \left( 1 - \frac{t}{2L} \right) & \text{if } \alpha + 2 = 0 
\end{cases}$$

(12)

thus obtaining

$$ds_4^2 = \left( 1 + \frac{(\alpha + 2) \tau}{4 \alpha L} \right)^{\frac{\alpha + 2}{\alpha + 2}} (dr^2 + r^2 d\Omega^2) - d\tau^2$$

if $\alpha + 2 \neq 0$ (13)

and

$$ds_4^2 = e^{\tau} (dr^2 + r^2 d\Omega^2) - d\tau^2$$

if $\alpha + 2 = 0$ (14)
where in both cases $1/(2L)$ represents the Hubble constant $H_0$. Accordingly, pressure and density of the perfect fluid can be rewritten respectively as

$$8\pi p = \frac{2(1 - \alpha) H_0^2}{\left(1 + \frac{(\alpha + 2) H_0\tau}{2\alpha}\right)^2}$$

and

$$8\pi \rho = \frac{3H_0^2}{\left(1 + \frac{(\alpha + 2) H_0\tau}{2\alpha}\right)^2}$$

(15)

and

$$8\pi p = -3H_0^2$$

$$8\pi \rho = 3H_0^2$$

(16)

It is apparent that the case $\alpha + 2 = 0$ describes a de-Sitter Universe with cosmological constant $\Lambda = 3H_0^2$. On the other hand, the case $\alpha + 2 \neq 0$, provides the equation of state of radiation $3p = \rho$ when $\alpha = 2/3$, and the equation of state of matter $p = 0$ when $\alpha = 1$. 
References

[1] Wesson P. S., *Phys. Lett. B*, 276 (1992) 299.

[2] Overduin J. M. and Wesson P. S., *Phys. Reports*, 283 (1997) 304.

[3] Wesson P. S., *Space - Time - Matter. Modern Kaluza-Klein theory*, (World Scientific, Singapore) 1999.

[4] Billyard A. and Wesson P. S., *Gen. Rel. Grav.*, 28 (1996) 129.

[5] Campbell J. E., *A Course of differential Geometry*, (Clarendon Press, Oxford) 1926.

[6] Romero C., Tavakol R. and Zalaletdinov R., *Gen. Rel. Grav.*, 28 (1996) 365.

[7] Lidsey J. E., Romero C., Tavakol R. and Rippl S., gr-qc 9907040. Preprint 1999.

[8] Agnese A. G. and La Camera M., *Phys. Rev. D*, 58 (1998) 087504.