Robust Performance Analysis of Source-Seeking Dynamics with Integral Quadratic Constraints

Adwait Datar\textsuperscript{1} and Herbert Werner\textsuperscript{1}

Abstract—We analyze the performance of source-seeking dynamics involving vehicles embedded in an underlying scalar field with gradient based forcing terms. We leverage the recently developed framework of \(\alpha\)-integral quadratic constraints (IQCs) to obtain convergence rate estimates. We first present the hard Zames-Falb (ZF) \(\alpha\)-IQCs involving general non-causal multipliers and show that a parameterization of the ZF multiplier, suggested in the literature for the standard version of the ZF IQCs, can be adapted to the \(\alpha\)-IQCs setting. Owing to the time-domain arguments, we can seamlessly extend these results to linear parameter varying (LPV) vehicles possibly opening the doors to non-linear vehicle models with quasi-LPV representations. We illustrate the theoretical results on a linear time invariant (LTI) model of a quadrotor, a non-minimum phase LTI example and an LPV example of a quadrotor with two modes which show a clear benefit of using general non-causal dynamic multipliers to drastically reduce conservatism.

I. INTRODUCTION

Source-seeking problems are motivated by practical problems such as finding the source of an oil spill \cite{1}. The abstract problem involves one or more vehicles located at arbitrary locations in an underlying scalar field with the goal of moving towards the minimum (or maximum based on convention) of the field which is called the source. These vehicles are typically assumed to be able to estimate the gradient of the field at their respective locations (see \cite{2}, \cite{3}, \cite{4} for details). Integral quadratic constraints (IQCs) \cite{5} is a framework for analyzing the stability of an interconnection of two systems; a linear and stable "nominal" system and a possibly non-linear "uncertain" system. See \cite{6},\cite{7} for a tutorial. An alternative to the theory of IQCs is the framework of dissipativity which allows one to analyze the stability of the interconnection by searching for non-increasing energy-like functions called storage functions. The results obtained from hard-IQCs \cite{8} can be interpreted with dissipativity based arguments and vice-versa\textsuperscript{1}. This equivalence connects our earlier work \cite{4}, \cite{9} on analyzing stability of source-seeking dynamics with dissipativity based arguments to this paper where we use IQCs.

Related work and contributions

Stability analysis for source-seeking dynamics within the flocking framework is studied in \cite{4}, \cite{9} albeit without any performance guarantees. Our main motivation in this paper is to extend these results to performance analysis. Moreover, the stability analysis in \cite{4}, \cite{9} involves dissipativity based arguments and construction of storage functions from physical energy-like functions. These physically motivated storage functions have a diagonal structure (such as the one used in \cite{10}). The idea is to automate the search for storage functions (not necessarily diagonal) and include a less conservative stability analysis certificate along with performance guarantees. The starting points for the theoretical work in this paper are \cite{11} and \cite{12} where the exponential versions of the IQCs are introduced for systems in discrete-time (\(\rho\)-IQCs) and in continuous-time (\(\alpha\)-IQCs), respectively. \cite{12} introduces the soft Zames-Falb (ZF) \(\alpha\)-IQCs corresponding to causal multipliers. While \cite{13} extends \cite{11} to less conservative non-causal ZF multipliers in the discrete-time setting, \cite{14} presents the extension of \cite{12} to non-causal multipliers in the continuous-time setting. The theory developed in \cite{14} is in a very general setting of Bochner spaces and covers Lemma 1 in this paper. \cite{14} focuses on the derivation of the IQCs and does not consider parameterizations of the multiplier to arrive at a quasi-convex optimization problem for performance analysis. The present paper extends these results by considering a parameterization proposed in \cite{6} adapted to the \(\alpha\)-IQCs setting. \cite{13} compares different multiplier factorizations which include the discrete-time analogue of the parameterization in \cite{6}. While \cite{13} and \cite{14} present examples showing the benefit of non-causal multipliers in the discrete-time case, we present an example of a continuous-time system with an integrator demonstrating the benefit of non-causal multipliers over causal ones. A closely related work is \cite{15} where causal and static multipliers are used to obtain non-exponential convergence rates when the dynamics are not exponentially stable. Finally, previous works (such as \cite{11}, \cite{12} and \cite{14}) on the exponential version of IQCs present results for linear time invariant (LTI) systems. Since we consider the hard-IQCs with purely time-domain arguments, we show that it is rather straightforward to extend these results to linear parameter varying (LPV) \cite{16} systems as is done in \cite{17} for the standard IQCs. This opens the doors for considering non-linear vehicle models with quasi-LPV representations. The framework developed in this paper can be applied to extremum-seeking control problems \cite{18} or problems involving realtime-optimization \cite{19}. Although the setup considered in \cite{19} is similar to the one considered in this paper (except for the LPV extension), the focus there is on designing an optimizer and obtaining conditions for optimality and stability with static \(\alpha\)-IQCs.

\textsuperscript{1} Institute of Control systems, Hamburg University of Technology, Eißendorfer Str. 40, 21073 Hamburg, Germany. \{adwait.datar, h.werner\}@tuhh.de

\textsuperscript{1} The link between the soft-IQCs and dissipativity has only recently been fully understood \cite{8}
Notation

The condition number of a matrix $X$ is denoted as $\text{cond}(X)$. For any $x \in \mathbb{R}^n$, let $\text{diag}(x)$ denote the diagonal matrix formed by placing the entries of $x$ along the diagonal. For block matrices, we use $*$ to denote required entries to make the matrix symmetric. Let $0$ and $1$ denote the vectors or matrices of all zeroes and ones of appropriate sizes, respectively. Let $I_d$ be the identity matrix of dimension $d$ and we remove the subscript $d$ when the dimension is clear from context. Let $\otimes$ represent the Kronecker product. Let $S(m, L)$ denote the set of continuously differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ which are strongly convex with parameter $m$, and have Lipschitz gradients with parameter $L$ for some given $0 < m \leq L$, i.e.,

$$m||y_1 - y_2||^2 \leq (\nabla f(y_1) - \nabla f(y_2))^T(y_1 - y_2) \leq L||y_1 - y_2||^2$$

holds for all $y_1, y_2 \in \mathbb{R}^d$. The set of vector valued functions which are square-integrable over $[0, T]$ for any finite $T$ is denoted by $L_2([0, T])$. For any signal $q \in L_2([0, \infty))$, and a fixed $T \geq 0$ let us define the extension

$$q_T(t) = \begin{cases} q(t), & \text{if } t \in [0, T], \\ 0, & \text{if } t \in \mathbb{R} \setminus [0, T]. \end{cases}$$

(1)

Let $L_1(-\infty, \infty)$ denote the set of functions $h : \mathbb{R} \to \mathbb{R}$, such that $\int_{-\infty}^{\infty} |h(t)|dt < \infty$. We use $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ to represent an LTI system with state-space realization given by matrices $A, B, C$ and $D$.

Outline:

After discussing the problem setup in section II, the main theoretical results are presented in section III followed by the extension to LPV systems in section IV. Numerical results are presented in section V leading to conclusions and open directions in section VI.

II. PROBLEM SETUP

We consider source-seeking scenarios where one or more vehicles moving in $\mathbb{R}^d$ space (typically $d \in \{1, 2, 3\}$) are embedded in an underlying differentiable scalar field $f : \mathbb{R}^d \to \mathbb{R}$ with some convexity and smoothness properties (made precise later). We assume that each vehicle has access to the gradient $\nabla f$ evaluated at the vehicle’s position. Since this paper deals primarily with performance analysis, we assume that a local tracking controller has been designed and the closed-loop dynamics of the vehicle with reference input denoted by $r(t) \in \mathbb{R}^d$ with $r(t) = [r_{\text{pos}}(t) \ r_{\text{vel}}(t)]^T$, state $x(t) \in \mathbb{R}^n$ and position output $y(t) \in \mathbb{R}^d$ can be represented by

$$\dot{x}(t) = Ax(t) + Br(t), \quad x(0) = x_0,$n

$$y(t) = Cx(t).$$

(2)

We propose to augment these closed-loop dynamics by the second order dynamics,

$$\dot{r}_{\text{pos}}(t) = r_{\text{vel}}(t),$$

$$\dot{r}_{\text{vel}}(t) = -k_d \cdot r_{\text{vel}}(t) - k_p \cdot u(t),$$

(3)

where $u(t) \in \mathbb{R}^d$ denotes external input. We consider a control law with $u(t) = \nabla f(y(t))$. The intuition behind this control law is to see that if the tracking controller is doing a good job ($y(t) \approx r_{\text{pos}}(t)$), we end up with a second order dissipative Hamiltonian system, the stability of which can be analyzed as in [4] and [9]. The overall vehicle dynamics from $u$ to $y$ denoted by $G$, along with the feedback law $u(t) = \nabla f(y(t))$, can be represented by

$$\dot{\eta}(t) = A_G\eta(t) + B_Gu(t), \quad \eta(0) = \eta_0,$n

$$y(t) = C_G\eta(t),$$

(4)

$$u(t) = \nabla f(y(t)),$$

where $\eta \in \mathbb{R}^{\eta_0}$ denotes the concatenated state vector $[x^T \ t^T]^T$.

**Assumption 1:** For some $0 < m \leq L$, $f \in S(m, L)$ and let $u_*$ minimize $f$, i.e., $f(y) \geq f(y_*) \forall y \in \mathbb{R}^d$ and $\nabla f(y_*) = 0$.

**Remark 1:** The assumption on continuous differentiability can be relaxed by using sub-gradient maps, but the assumption on strong convexity is essential. Since we consider vehicle models, we assume that $A_G$ has an eigenvalue at 0 (integral action) which is enforced here by the control architecture of $G$ with the first block having an integrator in series with a stable block. Since this implies that $\exists \eta_* \neq 0, A_G\eta_* = 0$, we get a feasible equilibrium $(\eta_*, u_*, y_*)$ for (4) satisfying

$$0 = A_G\eta_* + B_Gu_* = A_G\eta_*,$$

$$y_* = C_G\eta_*,$n

$$u_* = \nabla f(y_*) = 0.$$n

The dynamics in the deviation variables $\tilde{\eta}(t) = \eta(t) - \eta_*$, $\tilde{u}(t) = u(t) - u_*$, and $\tilde{y}(t) = y(t) - y_*$ can be represented by

$$\dot{\tilde{\eta}}(t) = A_G\tilde{\eta}(t) + B_G\tilde{u}(t), \quad \tilde{\eta}(0) = \eta_0 - \eta_*,$$

$$\dot{\tilde{y}}(t) = \nabla f(\tilde{y}(t)),$$

(6)

and

$$\dot{\tilde{u}}(t) = \nabla f(\tilde{y}(t) + y_*).$$

(7)

A. Flocking protocols under quadratic fields

We now consider a group of $N$ identical LTI agents located in the scalar field with interactions among agents captured by flocking dynamics and along with a gradient-based forcing term driving them towards the source. We use the notation and terminology from [10], [4] and [9] and refer the reader to these sources for details. Let $\eta_i(t), u_i(t), y_i(t)$ denote the state, input and output of the $i^{th}$ agent respectively. We stack all the states of the agents to form a long state vector $\eta(t) = [\eta_1(t)^T \cdot \eta_N(t)^T]^T$ and similarly define $u(t)$ and $y(t)$. Let $f : \mathbb{R}^{Nd} \to \mathbb{R}$ be defined as $\sum_{i=1}^{N} f(y_i)$ which leads to $\nabla f(y) = [\nabla f(y_1)^T \cdot \nabla f(y_N)^T]^T$. As introduced in [10], let $V : \mathbb{R}^{Nd} \to \mathbb{R}$ denote the interaction potential among agents and let $L$ denote the graph-Laplacian matrix representing the communication topology between agents. The interaction potential is defined such that there is force balance in every interacting pair of agents and so, the
net force acting on the center of mass due to pair-wise interactions is zero. Similarly, the force arising from velocity alignment also satisfies this pair-wise symmetry and averages out to zero at the center of mass. This is summarized in
\[
(1^T_N \otimes I_d) \cdot \nabla \mathbf{V}(\mathbf{y}) = 0 \quad \forall \mathbf{y} \in \mathbb{R}^{Nd},
\]
\[
(1^T_N \otimes I_d)(\mathbf{L} \otimes I_d) = (1^T_N \mathbf{L}) \otimes I_d = 0.
\]
(8)
The overall flocking dynamics with gradient based forcing terms could be written as
\[
\dot{\mathbf{y}}(t) = (I_N \otimes A_G)\mathbf{y}(t) + (I_N \otimes B_G)\mathbf{u}(t),
\]
\[
\dot{\mathbf{u}}(t) = \nabla \mathbf{V}(\mathbf{y}(t)) - (\mathbf{L} \otimes I_d)\dot{\mathbf{y}}(t).
\]
(9)
Let \( \eta_c = \frac{1}{N}(1^T_N \otimes I_{n_d})\mathbf{y}, y_c = \frac{1}{N}(1^T_N \otimes I_d)\mathbf{y} \) and \( u_c = \frac{1}{N}(1^T_N \otimes I_d)\mathbf{u} \) be the center of mass (COM) state, input and output, respectively.

If \( f \) is a quadratic field of the form \( f(y) = \frac{1}{2} y^T Q y + c^T y + d \), we get, \( \nabla f(y) = Q y + c \), which is linear. Therefore, \( \nabla f(y) = (I_N \otimes Q)\mathbf{y} + 1_N \otimes c \) and \( (1^T_N \otimes I_d)\nabla f(y(t)) = \nabla f(y_c(t)). \)

Using this fact as well as (8), we can derive the reduced COM dynamics as
\[
\dot{\eta}_c(t) = A_G \eta_c + B_G u_c,
\]
\[
\dot{y}_c(t) = C_G \eta_c,
\]
\[
\dot{u}_c(t) = \nabla f(y_c).
\]
(10)
This has the same form as (4) and hence, the analysis of a single agent can be used to analyze the performance of center of mass of multiple agents flocking under quadratic scalar fields.

**Remark 2:** It is important to note that the analysis of the complete system involves a further analysis of the structural dynamics (disagreement dynamics) such as in [10]. The availability of gradient information at all agent locations may be too restrictive and is relaxed in future work.

### III. Theory

The fundamental idea in the framework of IQCs is to find properties (integral inequalities) satisfied by the input and output signals \( \mathbf{u} \) and \( \mathbf{y} \) of the gradient map \( \dot{\mathbf{y}}(t) = \nabla \mathbf{V}(\mathbf{y}(t)) + \mathbf{u}(t) \). It is convenient to derive properties of some new signals (defined below) related to the input and output signals \( \mathbf{u} \) and \( \mathbf{y} \). For the constants \( m, L \) from Assumption 1, for any \( \mathbf{u}, \mathbf{y} \in \mathcal{L}_{2e}(0, \infty) \), define for \( t \in [0, \infty) \):
\[
p(t) = \mathbf{u}(t) - m\mathbf{y}(t),
\]
\[
q(t) = L\mathbf{y}(t) - \mathbf{u}(t).
\]
(11)
With any \( h \in \mathcal{L}_1(-\infty, \infty) \) satisfying
\[
h(s) \geq 0 \quad \forall s \in \mathbb{R},
\]
\[
\int_{-\infty}^{\infty} h(s)ds \leq H,
\]
(12)
let
\[
w_1(t) = \int_0^t e^{-2\alpha(t-\tau)} h(t-\tau)q(\tau)d\tau,
\]
\[
w_2(t) = \int_0^t e^{-2\alpha(t-\tau)} h(-(t-\tau))p(\tau)d\tau.
\]
(13)
We first present a key technical result adapted to our setting which is covered by Lemma 3 from [14] where the result is presented in a very general setting of Bochner spaces.

**Lemma 1:** Let \( \alpha \geq 0 \) be fixed and let \( \beta(\tau) = \min\{1, e^{-2\alpha \tau}\} \) for \( \tau \in \mathbb{R} \). Then for all \( \tilde{u}, \tilde{y} \in \mathcal{L}_{2e}(0, \infty) \) that satisfy (7) with \( f \) satisfying Assumption 1, the signals \( p \) and \( q \) as defined in (11) satisfy
\[
\int_0^T e^{2\alpha t} p(t)^T(q(t) - \beta(\tau)q_T(t-\tau))dt \geq 0,
\]
(14)
where, \( q_T \) denotes the extension defined in (1).

**Proof:** See [20] for the proof.

Since (14) holds for all \( \tau \in \mathbb{R} \), we can conically combine the inequality (14) by multiplying it by a positive function \( h(\tau) \) satisfying (12) and integrating out \( \tau \). This leads to the next result.

**Theorem 2:** Let \( h \) satisfying (12) be fixed and let \( \alpha \geq 0 \).

For any \( \tilde{u}, \tilde{y} \in \mathcal{L}_{2e}(0, \infty) \) that satisfy (7) with \( f \) satisfying Assumption 1, the signals defined in (11), (13) satisfy
\[
\int_0^T e^{2\alpha t} (H p(t)^T q(t) - p(t)^T w_1(t) - q(t)^T w_2(t))dt \geq 0,
\]
(15)
\( \forall T \geq 0 \).

**Proof:** See [20] for the proof.

We now parameterize \( h \) by proceeding along the lines of [6]. Let \( A_v, B_v, Q_v, \psi_v \) be defined as in [6] with \( \lambda = -1 \). Let \( A_v^0 = A_v - 2\alpha I \), \( Q_v^0(t) = e^{-2\alpha t}Q_v(t) \) and let
\[
\begin{bmatrix}
A_v^0 & B_v^0 \\
C_v^0 & D_v^0
\end{bmatrix}
\]
be a state-space realization of \( \tilde{\psi}_v \). Let the state space realization of \( \Psi \) be as described in the appendix. Before we present the next result, let us define for any \( \tilde{u}, \tilde{y} \in \mathcal{L}_{2e}(0, \infty) \),
\[
\tilde{z}(t) = \int_0^t C_{\Psi} e^{A_{\Psi}(t-\tau)} B_{\Psi} \begin{bmatrix}
\tilde{y}(\tau) \\
\tilde{u}(\tau)
\end{bmatrix} d\tau + D_{\Psi} \begin{bmatrix}
\tilde{y}(t) \\
\tilde{u}(t)
\end{bmatrix}.
\]
(16)
Consider the constraint in variables \( H \in \mathbb{R}, P_1 \in \mathbb{R}^{1 \times \nu}, P_3 \in \mathbb{R}^{1 \times \nu} \),
\[
H + (P_1 + P_3)A_v^{-1}B_v \geq 0,
\]
(17)
and consider the condition on \( P_1, P_3 \)
\[
\exists \chi_1, \chi_3 \in \mathbb{S}^{\nu-1} \text{ such that for } i \in \{1, 3\},
\]
\[
(*) \begin{bmatrix}
\chi_i & 0 \\
0 & \chi_i
\end{bmatrix} \begin{bmatrix}
I & 0 \\
0 & A_v
\end{bmatrix} \begin{bmatrix}
B_v & 0 \\
R_v \tilde{C}_v & R_v \tilde{D}_v
\end{bmatrix} > 0.
\]
(18)
Let
\[
\mathbb{P} = \begin{bmatrix}
0 & \begin{bmatrix}
H & -P_3 \\
-P_1^T & 0
\end{bmatrix} \\
* & 0
\end{bmatrix} : H, P_1, P_3 \text{ satisfy (17),(18)}.
\]
(19)
**Theorem 3:** For any \( \tilde{u}, \tilde{y} \in \mathcal{L}_{2e}(0, \infty) \) that satisfy (7) with \( f \) satisfying Assumption 1, the signal \( \tilde{z} \) as defined in (16) satisfies the hard \( \alpha \)-IQC$
\[
\int_0^T e^{2\alpha t} \tilde{z}(t) \mathbb{P} \tilde{z}(t)dt \geq 0 \quad \forall \mathbb{P} \in \mathbb{P}, \forall T \geq 0.
\]
(20)
**Proof:** See [20] for the proof.
Remark 3: The function \( h \) is usually referred to as the multiplier and by enforcing \( P_1 = 0 \) (\( P_3 = 0 \)), we restrict the search space to causal (anti-causal) ZF multipliers. By enforcing \( P_1 = 0 \) and \( P_3 = 0 \), we specialize to the case of static multipliers which corresponds to the well-known circle-criterion (CC) [8]. The conservatism between these specializations is investigated in section V.

Remark 4: Extension of these results to cases when the map from \( \tilde{y} \) to \( \tilde{u} \) is additionally known to be odd is possible along the same lines but is not pursued in this paper.

We now present the final analysis result which leads to a computational procedure for obtaining convergence rate estimates. Let \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \Psi \begin{bmatrix} G \\ I \end{bmatrix} \). The following theorem from [12] gives the final performance analysis condition.

**Theorem 4 ([12]):** If \( \exists \lambda > 0, P \in \mathbb{P} \) such that

\[
\begin{bmatrix} A^T \lambda + \lambda A + 2\alpha \lambda^2 \lambda^2 & \lambda^2 B^T \lambda \\ B^T \lambda & 0 \end{bmatrix} + \begin{bmatrix} C^T & D^T \end{bmatrix} (P \otimes I_d) \begin{bmatrix} C & D \end{bmatrix} \preceq 0,
\]

(21)

then, under dynamics (4) with \( f \) satisfying Assumption 1, \( y \) converges exponentially to \( y_* \) with rate \( \alpha \), i.e., \( \exists \kappa \geq 0 \) such that \( \| \hat{y}(t) \| \leq \kappa e^{-\alpha t} \) holds for all \( t \geq 0 \).

Remark 5: We point out that (21) is not linear in \( \alpha \) and \( P \) due to the product \( \alpha \lambda \). It falls into the class of quasi-convex optimization problems which can be solved efficiently. We perform a bisection over \( \alpha \) as suggested in [11].

### IV. Extension To LPV Systems

Extensions of the results obtained in the previous section to LPV/uncertain systems is straightforward and we demonstrate one such extension next. Instead of the LTI system \( G \), let \( G(\rho) \) denote an LPV system with \( \eta_0 \) scheduling parameters [16], where, for a compact set \( \mathcal{P} \subset \mathbb{R}^{n_\rho} \), the function \( \rho : [0, \infty) \to \mathcal{P} \) captures the time-dependence of the model parameters. The dynamics of \( G(\rho) \) with initial condition \( \eta(0) = \eta_0 \) can be represented by

\[
\begin{aligned}
\dot{\eta}(t) &= AC(\rho(t))\eta(t) + B_C(\rho(t))u(t), \\
y(t) &= C_G(\rho(t))\eta(t), \\
u(t) &= \nabla f(y(t)),
\end{aligned}
\]

where \( \eta \in \mathbb{R}^{n_\eta} \) is the state vector and \( \rho : [0, \infty) \to \mathcal{P} \) is an arbitrary scheduling trajectory.

Remark 6: If the rate of parameter variation \( \dot{\rho} \) is bounded and this bound is known, we could include this information and consider parameter dependent Lyapunov functions ([21]), but we will not treat this case here.

As for the linear case, we assume integral action in \( G(\rho) \), that is, \( \exists \eta_0 \neq 0 \), \( A_G(\bar{\rho})\eta_0 = 0 \) \( \forall \bar{\rho} \in \mathcal{P} \). Note again that this is implied by the control architecture. Following the same steps as in the LTI case, we can write the dynamics in the deviation variables analogous to (5) and (7). Let

\[
\begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} = \Psi \begin{bmatrix} G(\rho) \\ I \end{bmatrix}.
\]

The following theorem gives a sufficient condition for performance analysis.

**Theorem 5:** If \( \exists \lambda > 0, P \in \mathbb{P} \) such that, for any \( \bar{\rho} \in \mathcal{P} \),

\[
\begin{bmatrix} A(\bar{\rho})^T \lambda + \lambda A(\bar{\rho}) + 2\alpha \lambda^2 \lambda^2 \lambda^2 & \lambda^2 B(\bar{\rho})^T \lambda \\ B(\bar{\rho})^T \lambda & 0 \end{bmatrix} + \begin{bmatrix} C(\bar{\rho})^T & D(\bar{\rho})^T \end{bmatrix} (P \otimes I_d) \begin{bmatrix} C(\bar{\rho}) & D(\bar{\rho}) \end{bmatrix} \preceq 0,
\]

(23)

then, under dynamics (22) with \( f \) satisfying Assumption 1, \( y \) converges exponentially to \( y_* \) with rate \( \alpha \), i.e., \( \exists \kappa \geq 0 \) such that \( \| \hat{y}(t) \| \leq \kappa e^{-\alpha t} \) holds for all \( t \geq 0 \).

**Proof:** See [20] for the proof.

**Remark 7:** We note that the equivalence between the center of mass dynamics and the dynamics of a single agent illustrated for homogeneous LTI agent flocking dynamics under quadratic fields in section II extends to LPV systems only if all agents are homogeneously scheduled, i.e., \( \rho_i(t) = \rho_j(t) \quad \forall t \in [0, \infty), \quad \forall i, j \in \{1, 2, \cdots, N\} \).

### V. Numerical Results

The code used for generating results in this section is available at [22].

**A. Quadrotor**

We consider a linearized quadrotor model of order 12 and use an Linear-Quadratic-Regulator (LQR) based state-feedback controller tuned for zero steady-state error for step references. We address the following questions next to demonstrate the applicability of the theoretical results.

1. How robust is the given controller with respect to different fields \( f \in S(m, L) \)?
2. How do we design the gains \( k_p \) and \( k_d \) for the given closed-loop quadrotor system?
3. How conservative are the estimates of the convergence rates given by our analysis for different multipliers?

For fixed gains \( k_p \) and \( k_d \) and given closed-loop quadrotor dynamics, Fig. 1 shows the convergence rate estimates provided by different multipliers for fields \( f \in S(1, L) \) with
increasing $L$. Since increasing $L$ enlarges the set of allowable fields, i.e., $S(1, L_1) \subset S(1, L_2) \forall L_1 \leq L_2$, the estimates are non-increasing with increasing $L$. It can be seen that while we can certify stability with the circle criterion for fields $f \in S(1, 5)$, the general non-causal ZF multipliers along with the ZF multipliers restricted to the causal case ($P_3 = 0$) can certify stability for all fields $f \in \mathcal{L}(1, 8)$. Furthermore, for each $L$, we can find a field (quadratic) that achieves the convergence rate guaranteed by the analysis showing that, in this example, the estimates are tight. The conservatism incurred by restricting the search to causal multipliers is minor in this example. The stability analysis in [9] uses manually constructed diagonal storage functions together with a small gain argument and for this example, gives the sufficient condition for stability to be $L < 5$. This interestingly coincides with the stability boundary given by the circle-criterion (static multipliers). Since performance analysis was not included in [9], this example illustrates the extension of [9] to a non-conservative performance analysis.

The effect of varying the ratio of gains $\frac{k_d}{k_p}$ on the performance estimates for fixed allowable field set is shown in Fig. 2. It shows that the highest convergence rate of 0.14 can be achieved for $\frac{k_d}{k_p} = 9$ and demonstrates a method for tuning the gains for optimal convergence rates. We also observe that tuning the gains by using static multipliers (circle criterion) would lead to a rather poor performance. Sample trajectories of a quadrotor locating the source at $(-50, -50)$ is shown in Fig. 3, where, $\frac{k_d}{k_p}$ are chosen optimal with respect to the circle criterion (dashed lines) and ZF (solid lines). Although leading to a higher overshoot, the gains tuned with respect to ZF lead to faster convergence.

**B. Example showing the benefit of non-causal multipliers**

We now present an academic example that brings out the benefit of using general non-causal multipliers over causal multipliers. Let $G(s) = \frac{(s-1)}{s(s^2 + s + 25)}$ and consider fields $f \in S(1, L)$. The convergence rate estimates provided by different multipliers for increasing $L$ is shown in Fig. 4. It can be seen that while the circle criterion and causal ZF multipliers certify stability for fields $f \in S(1, 1.9)$, the anti-causal ZF multipliers can certify stability for fields $f \in S(1, 2.4)$. Furthermore, we can find example fields (quadratic) which coincide with the convergence rate estimates showing that these estimates are tight. Since $f \in S(1, L_1) \subset S(1, L_2) \forall L_1 \leq L_2$, the estimates are non-increasing.
C. Quadrotor with two modes

We now consider a scenario with a quadrotor, as in section V-A, but with two operating modes. One operating mode corresponds to the quadrotor carrying some load and the other mode corresponds to no-load. We model this by considering two masses \( m \in \{0.2, 2\} \) with LQR controllers designed as in section V-A for each mode separately. We consider an arbitrary switching between the two modes and can be modeled as an LPV (or switching) system with \( \mathcal{P} = \{1, 2\} \) and \( \rho(t) \in \mathcal{P} \ \forall t\). Fig. 5 shows the convergence rate estimates provided by different multipliers for fields \( f \in S(1, L) \). We observe that in comparison to the LTI case (Fig. 1 from section V-A), the performance is slightly reduced due to the possibility of arbitrary switching between modes. Furthermore, the estimates with first order ZF multipliers are not tight anymore and we obtain better results with second order ZF multipliers. No improvement in the estimates was observed uptil 5th order ZF multipliers.

VI. CONCLUSIONS

This paper presents an IQC based performance analysis of source-seeking dynamics using general non-causal Zames-Falb multipliers for LTI and LPV systems with tight estimates for the presented examples. Future work considers formation control and flocking dynamics.

APPENDIX

State-space realization of \( \Psi \)

\[
\Psi = \begin{bmatrix}
A_0 \otimes I_d & 0 & -B_0 \otimes mI_d & B_0 \otimes I_d \\
0 & A_0 \otimes I_d & B_0 \otimes Ld & -B_0 \otimes I_d \\
0 & 0 & -mI_d & I_d \\
I_0 \otimes I_d & 0 & 0 & 0 \\
0 & 0 & LI_d & -I_d \\
0 & I_0 \otimes I_d & 0 & 0 
\end{bmatrix}
\]

Fig. 5. Robustness against different fields \( f \in S(1, L) \) for a quadrotor with uncertain or time-varying mass \( m \in \{0.2, 2\} \)

REFERENCES

[1] H. Senga, N. Kato, A. Ito, H. Niou, M. Yoshie, I. Fujita, K. Igarashi, and E. Okuyama, “Development of spilled oil tracking autonomous buoy system,” in OCEANS 2007. IEEE, 2007, pp. 1–10.
[2] S. Z. Khong, Y. Tan, C. Manzie, and D. Nesić, “Multi-agent source seeking via discrete-time extremum seeking control,” Automatica, vol. 50, no. 9, pp. 2312–2320, 2014.
[3] P. Ogren, E. Fiorelli, and N. E. Leonard, “Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment,” IEEE Transactions on Automatic control, vol. 49, no. 8, pp. 1292–1302, 2004.
[4] A. Datar, P. Paulsen, and H. Werner, “Flocking towards the source: Indoor experiments with quadrotors,” in 2020 European Control Conference (ECC). IEEE, 5/12/2020 - 5/15/2020, pp. 1638–1643.
[5] A. Megretski and A. Rantzer, “System analysis via integral quadratic constraints,” IEEE Transactions on Automatic Control, vol. 42, no. 6, pp. 819–830, 1997.
[6] J. Veennan and C. W. Scherer, “Stability analysis with integral quadratic constraints: A dissipativity based proof,” in 52nd IEEE Conference on Decision and Control, 2013, pp. 3770–3775.
[7] U. Jonsson, “Lecture notes on integral quadratic constraints,” 2001.
[8] C. Scherer, “Dissipativity and integral quadratic constraints, tailored computational robustness tests for complex interconnections.” [Online]. Available: https://arxiv.org/pdf/2105.07401
[9] A. Attallah, A. Datar, and H. Werner, “Flocking of linear parameter varying agents: Source seeking application with underwater vehicles,” IFAC-PapersOnLine, vol. 53, no. 2, pp. 7305–7311, 2020.
[10] R. Olfati-Saber, “Flocking for multi-agent dynamic systems: Algorithms and theory,” IEEE Transactions on Automatic Control, vol. 51, no. 3, pp. 401–420, 2006.
[11] L. Lessard, B. Recht, and A. Packard, “Analysis and design of optimization algorithms via integral quadratic constraints,” SIAM Journal on Optimization, vol. 26, no. 1, pp. 57–95, 2016.
[12] B. Hu and P. Seiler, “Exponential decay rate conditions for uncertain linear systems using integral quadratic constraints,” IEEE Transactions on Automatic Control, vol. 61, no. 11, pp. 3631–3637, 2016.
[13] J. Zhang, P. Seiler, and J. Carrasco, “Noncausal zames-falb multiplier search for exponential convergence rate.” [Online]. Available: https://arxiv.org/pdf/1902.09473
[14] R. A. Freeman, “Noncausal zames-falb multipliers for tighter estimates of exponential convergence rates,” in 2018 Annual American Control Conference (ACC). IEEE, 6/27/2018 - 6/29/2018, pp. 2984–2989.
[15] M. Fazlyab, A. Ribeiro, M. Morari, and V. M. Preciado, “Analysis of optimization algorithms via integral quadratic constraints: Nonstrongly convex problems,” SIAM Journal on Optimization, vol. 28, no. 3, pp. 2654–2689, 2018.
[16] J. S. Shamha and J. R. Cloutier, “A linear parameter varying approach to gain scheduled missile autopilot design,” in 1992 American Control Conference. IEEE, 1992, pp. 1317–1321.
[17] H. Pifer and P. Seiler, “Robustness analysis of linear parameter varying systems using integral quadratic constraints,” International Journal of Robust and Nonlinear Control, vol. 25, no. 15, pp. 2843–2864, 2015.
[18] S. Michalowski and C. Ebenbauer, “Extremum control of linear systems based on output feedback,” in 2016 IEEE 55th Conference on Decision and Control (CDC). IEEE, 2016, pp. 2963–2968.
[19] Z. E. Nelson and E. Mallada, “An integral quadratic constraint framework for real-time, steady-state optimization of linear time-invariant systems,” in 2018 Annual American Control Conference (ACC). IEEE, 2018, pp. 597–603.
[20] A. Datar and H. Werner, “Robust performance analysis of source-seeking dynamics with integral quadratic constraints,” arXiv preprint arXiv:2110.06369, 2021. [Online]. Available: https://doi.org/10.48550/arXiv.2110.06369
[21] C. Scherer and S. Weiland. “Linear matrix inequalities in control,” Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands, vol. 3, no. 2, 2000.
[22] A. Datar and H. Werner, Robust Performance Analysis of Source-Seeking Dynamics with Integral Quadratic Constraints. Zenodo, 2021. [Online]. Available: https://doi.org/10.5281/zenodo.5564776