BPS COHOMOLOGY FOR RANK 2 DEGREE 0 HIGGS BUNDLES (AND MORE)

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Abstract. We give a formula comparing the E-series of the moduli stacks of rank 2 degree 0 semistable Higgs bundles in genus $g \geq 2$ to intersection E-polynomials of its coarse moduli space. A parallel formula holds in various 2-Calabi–Yau settings, for example for sheaves on K3 surfaces, or preprojective algebras of $g$-loop quivers. As a consequence we provide evidence for a conjecture of Davison on the BPS cohomology of Higgs bundles, which has implications for non-abelian Hodge theory for stacks. We apply the formula to cohomological $\chi$-independence tests for BPS cohomology of Higgs bundles and K3 surfaces.

1. Introduction

The Borel–Moore homology of moduli stacks $\mathcal{M}$ of objects in 2-Calabi–Yau categories as well as the compactly supported intersection cohomology of their coarse moduli spaces $\overline{\mathcal{M}}$ have been at the intersection of many recent advances in enumerative invariants in algebraic geometry and geometric representation theory [Dav21b; Dav20; Dav21a; MS21; MS20a; KV22; KK21; Kin22; KM21; SS20; SV13; Sch16].

This paper is an attempt to explicitly describe the topology of the morphism $p: \mathcal{M} \to \overline{\mathcal{M}}$ to the coarse moduli space whenever it is locally modelled on the semi-simplification morphism $\mathcal{M}_2(\Pi_{S_g}) \to \mathcal{M}_2(\Pi_{S_g})$ of two-dimensional representations of the preprojective algebra of the $g$-loop quiver $S_g$ for $g \geq 2$.

More concretely, we prove a formula relating the compactly supported cohomology (the dual of the Borel–Moore homology) of the moduli stack $\mathcal{M}$ with the compactly supported intersection cohomology of the coarse moduli space $\overline{\mathcal{M}}$ for the following examples

1. certain moduli of semistable Higgs bundles on a smooth projective complex curve $C$ of genus $g \geq 2$
2. certain moduli of $\pi_1(S_g \setminus \{p\})$-representations with prescribed monodromy around the puncture $p$ for a closed Riemann surface $S_g$ of genus $g \geq 2$
3. certain moduli of sheaves on K3 or abelian surfaces
4. moduli of two-dimensional representations of the preprojective algebra $\Pi_{S_g}$ of the $g$-loop quiver $S_g$ for $g \geq 2$.

See Theorem 1.6 and Corollary 1.11 for a precise formula.

1.1. Cohomology of moduli spaces of rank 2 Higgs bundles. Let $C$ be a smooth connected projective complex curve of genus $g \geq 2$. Let $\mathcal{M}_{r,d}^{\text{Dol}}$ be the moduli stack of rank $r$ degree $d$ semistable Higgs bundles on $C$ and let $p_{r,d}: \mathcal{M}_{r,d}^{\text{Dol}} \to \mathcal{M}_{r,d}^{\text{Dol}}$ be its coarse moduli space.

When $\gcd(r,d) = 1$, the moduli space $\mathcal{M}_{r,d}^{\text{Dol}}$ and moduli stack $\mathcal{M}_{r,d}^{\text{Dol}}$ are smooth and the morphism $p_{r,d}$ is the trivial $\mathbb{G}_m$-gerbe over $\mathcal{M}_{r,d}^{\text{Dol}}$.

When $\gcd(r,d) \neq 1$, the moduli space $\mathcal{M}_{r,d}^{\text{Dol}}$ and moduli stack $\mathcal{M}_{r,d}^{\text{Dol}}$ are singular and the morphism $p_{r,d}$ is more complicated than in the coprime case. In the literature the cohomology of the moduli stack and moduli space have been studied extensively (see for example [GHS14]).
The compactly supported intersection cohomology of the moduli spaces $IH_c(\mathcal{M}^\text{Dol}_{r,d})$ has been determined for rank 2 and degree 0 in genus 2 by Felisetti [Fel21] and for arbitrary genus $g \geq 2$ by Mauri [Mau21]. Building on work of Schiffmann [Sch16] and Mozgovoy–Schiffmann [MS20b] on point counts of the stacks $\mathcal{M}^\text{Dol}_{r,d}$ over finite fields, Fedorov–Soibelman–Soibelman in [FSS19] find a formula for the class $[\mathcal{M}^\text{Dol}_{r,d}]$ in the Grothendieck ring of stacks $K_0(\text{St})$.

Set $d = 0$, guaranteeing gcd$(r, d) \neq 1$ for $r > 1$. Consider the compactly supported cohomology (with rational coefficients) for all ranks
\[
\mathcal{A}_0 = \bigoplus_{r \geq 0} H^*_c(\mathcal{M}^\text{Dol}_{r,0}) \otimes \mathbb{L}^{(1-g)r^2}
\]
as an object in the symmetric monoidal abelian category of (graded) mixed Hodge structures. Here $\mathbb{L} = H^*_c(\mathbb{A}^1)$ is the mixed Hodge structure given by the compactly supported cohomology of the affine line.

Motivated by cohomological Donaldson–Thomas theory and non-abelian Hodge theory for stacks, the original motivation for this work was to provide evidence towards this conjecture. Conjecture 7.7 in [Dav20] is a related conjecture in the setting of preprojective algebras of quivers.

Conjecture 1.1 ([Dav21a, Conjecture 5.6]). There is an isomorphism of mixed Hodge structures
\[
\mathcal{A}_0 \cong \text{Sym} \left( \bigoplus_{r \geq 1} H^*_c(\mathcal{M}^\text{Dol}_{r,0}) \otimes \mathbb{L}^{(1-g)r^2-1} \right) \otimes H^*_c(BG_m) \otimes \mathbb{L}
\]
where Sym is taken in the graded sense and $\text{Sym}(V^*)$ denotes the free graded Lie algebra generated by the graded (super) vector space $V^*$.

Remark 1.2. Conjecture 4.6 in [Dav21a] is the corresponding conjecture on the other side of non-abelian Hodge theory. Conjecture 7.7 in [Dav20] is a related conjecture in the setting of preprojective algebras of quivers.

The original motivation for this work was to provide evidence towards this conjecture. Conjecture 1.1 predicts the following formula for the compactly supported cohomology of the moduli stack of rank 2 degree 0 Higgs bundles.

Theorem 1.3. There is an isomorphism of graded mixed Hodge structures
\[
H^*_c(\mathcal{M}^\text{Dol}_{2,0}) \otimes \mathbb{L}^{4-g} \cong H^*_c(\mathcal{M}^\text{Dol}_{1,0}) \otimes \mathbb{L}^{3-g} \otimes H^*_c(BG_m) \otimes \mathbb{L}
\]
\[
\oplus \Lambda^2(H^*_c(\mathcal{M}^\text{Dol}_{1,0}) \otimes \mathbb{L}^2) \otimes H^*_c(BG_m) \otimes \mathbb{L}
\]
\[
\oplus \text{Sym}^2(H^*_c(\mathcal{M}^\text{Dol}_{1,0}) \otimes \mathbb{L}^2 \otimes H^*_c(BG_m) \otimes \mathbb{L}).
\]

where the alternating square $\Lambda^2$ and the symmetric square $\text{Sym}^2$ are taken in the symmetric monoidal abelian category of graded mixed Hodge structures.

Let $q = E(\mathbb{L})$. We deduce Theorem 1.3 from the following equality of E-series (see Corollary 1.11).

Theorem 1.4.
\[
\frac{E(\mathcal{M}^\text{Dol}_{2,0})}{q^{4g-4}} = \frac{1}{q^{4g-3}}E(IH^*_c(\mathcal{M}^\text{Dol}_{2,0}))E(H^*_c(BG_m) \otimes \mathbb{L})
\]
\[
+ \frac{1}{q^{2g}}E(\Lambda^2(H^*_c(\mathcal{M}^\text{Dol}_{1,0})))E(H^*_c(BG_m) \otimes \mathbb{L})
\]
\[
+ \frac{1}{q^{2g}}E(\text{Sym}^2(H^*_c(\mathcal{M}^\text{Dol}_{1,0}) \otimes H^*_c(BG_m) \otimes \mathbb{L})).
\]

Remark 1.5. The RHS can be made more explicit using $E(H^*_c(BG_m) \otimes \mathbb{L}) = q/q - 1$ and Lemma 2.8.

1.2. Other settings. Although first intended for moduli of semistable rank 2 degree 0 Higgs bundles, the calculation we give in Section 3 works in many other settings as well.

To get an analogue of the formula in Theorem 1.4 for other settings we need as input an integer $g \geq 2$, a stack $\mathcal{M}^\text{ss}_2$, and spaces $\mathcal{M}^\text{ss}_2, \mathcal{M}^\text{ss}_1$ which play the role of $\mathcal{M}^\text{Dol}_{2,0}$ and $\mathcal{M}^\text{Dol}_{1,0}$, respectively.
Setting 1 (More Higgs bundles). Let \((r, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\) such that \(\gcd(r, d) = 1\). We consider rank 2\(r\) and degree 2\(d\) semistable Higgs bundles on a smooth connected projective curve \(C\) of genus \(g_0 \geq 2\). In this setting we take \(g = r^2(g_0 - 1) - 1\), \(\mathcal{M}_{2}^{ss} = \mathcal{M}_{2r, 2d}^{\text{Del}}\), \(\mathcal{M}_{1}^{ss} = \mathcal{M}_{2r, 2d}^{\text{Del}}\), \(\mathcal{M}_{1}^{s} = \mathcal{M}_{r, d}^{\text{Del}}\).

Setting 2 (Character stacks). Let \(\Sigma_{g_0}\) be a closed connected Riemann surface of genus \(g_0 \geq 2\) and fix a point \(p \in \Sigma_{g_0}\). Let \(\mathcal{M}_{g_0, r, d}^{\text{sst}}\) be the moduli stack or \(r\)-dimensional representations of \(\pi_1(\Sigma_{g_0} \setminus \{p\})\) with monodromy around the puncture given by \(e^{2\pi i r}\) and let \(\mathcal{M}_{g_0, r, d}\) denote its coarse moduli space. Let \((r, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\) such that \(\gcd(r, d) = 1\). In this setting we take \(g = r^2(g_0 - 1) - 1\), \(\mathcal{M}_{2}^{ss} = \mathcal{M}_{g_0, 2r, 2d}^{\text{Betti}}\), \(\mathcal{M}_{2}^{ss} = \mathcal{M}_{g_0, 2r, 2d}^{\text{Betti}}\), \(\mathcal{M}_{1}^{s} = \mathcal{M}_{g_0, r, d}^{\text{Betti}}\).

Setting 3 (K3 and abelian surfaces). Let \(S\) be a K3 or abelian surface and \(H\) an ample class on \(S\). For every Mukai vector \(v \in \mathcal{H}_{\text{alg}}^{\bullet}(S, \mathbb{Z})\) let \(\mathcal{M}_{S, v}^{\text{H-sat}}\) and \(\mathcal{M}_{S, v}^{H, \text{sat}}\) be the moduli stack and moduli space, respectively, of \(H\)-semistable sheaves with Mukai vector \(v\) on \(S\). Consider a primitive Mukai vector \(w \in \mathcal{H}_{\text{alg}}^{\bullet}(S, \mathbb{Z})\) with \(w^2 \geq 0\) such that \(H\) is generic with respect to \(w\). In this setting we take \(g = \frac{w^2 + 2}{2}\), \(\mathcal{M}_{2}^{ss} = \mathcal{M}_{S, 2w}^{\text{H-sat}}\), \(\mathcal{M}_{2}^{ss} = \mathcal{M}_{S, 2w}^{H, \text{sat}}\), \(\mathcal{M}_{1}^{s} = \mathcal{M}_{S, w}^{H, \text{sat}}\).

Setting 4 (Preprojective algebra of the \(g\)-loop quiver). Let \(S_g\) be the \(g\)-loop quiver. Let \(\mathcal{M}_{d}(\Pi_{S_g})\) be the moduli stack and \(\mathcal{M}_{d}(\Pi_{S_g})\) the coarse moduli space of \(d\)-dimensional representations of the preprojective algebra of \(S_g\). In this setting we take \(g = g\), \(\mathcal{M}_{2}^{ss} = \mathcal{M}_{2}(\Pi_{S_g})\), \(\mathcal{M}_{2}^{ss} = \mathcal{M}_{2}(\Pi_{S_g})\), \(\mathcal{M}_{1}^{s} = \mathcal{M}_{1}(\Pi_{S_g})\).

Theorem 1.6. Let \((g, \mathcal{M}_{2}^{ss}, \mathcal{M}_{2}^{ss}, \mathcal{M}_{1}^{s})\) be as in one of Settings 1-4. Then

\[
\frac{E(\mathcal{M}_{2}^{ss})}{q^{g_0-1}} = \frac{1}{q^{g_0-3}} E(H^{\bullet}(\mathcal{M}_{2}^{ss})) E(H^{\bullet}(BG_m) \otimes \mathbb{L})
+ \frac{1}{q^{g_0}} E(H^{\bullet}(\mathcal{M}_{1}^{ss})) E(H^{\bullet}(BG_m) \otimes \mathbb{L})
+ \frac{1}{q^{g_0}} E(\text{Sym}^2(H^{\bullet}(\mathcal{M}_{1}^{ss}) \otimes H^{\bullet}(BG_m) \otimes \mathbb{L})).
\]

Remark 1.7. A key ingredient for the proof is Theorem 1.3 in [Mau21] which provides a “uniform” formula for the intersection \(E\)-polynomial of the moduli spaces \(\mathcal{M}_{2}^{ss}\) for each of the settings. See Proposition 3.16.

1.3. BPS cohomology. As mentioned, the motivation for Conjecture 1.1 comes from cohomological Donaldson–Thomas theory for 2CY-categories. In particular Conjecture 1.1 can be interpreted as a statement about the BPS-cohomology for Higgs bundles. In this section we give a short introduction to BPS-cohomology of 2CY-categories.

The foundations of cohomological Donaldson–Thomas theory for 2-Calabi–Yau categories have been laid out in [Dav21b; Dav16b; Dav20; Dav21a; Dav16a; KK21] and we refer the reader to these papers for more details.

Let \(\mathcal{C}\) be a 2-Calabi–Yau (2CY) abelian category, i.e., there are natural non-degenerate graded-symmetric pairings

\[
\text{Ext}^{\bullet}_{\mathcal{C}}(E, F) \times \text{Ext}^{2\bullet}_{\mathcal{C}}(F, E) \rightarrow \mathbb{C}
\]

for all \(E, F \in \mathcal{C}\).

By the 2CY-property, the Euler pairing \(\chi(E, F) = \chi(\text{Ext}^{\bullet}_{\mathcal{C}}(E, F))\), for \(E, F \in \mathcal{C}\), defines a symmetric pairing on the Grothendieck group \(K_0(\mathcal{C})\). Assume the Euler pairing is the pullback along a group homomorphism \(\gamma: K_0(\mathcal{C}) \rightarrow \mathbb{Z}^n\) of a symmetric bilinear form on \(\mathbb{Z}^n\), which by abuse of notation we denote by \(\chi\).

Let \(v \in \mathbb{Z}^n\) be a primitive element. For every \(r \geq 0\) let \(\mathcal{M}_{rv}\) be the moduli stack of objects in \(E \in \mathcal{C}\) of class \(rv\), that is, \(\gamma(E) = rv\).
We require a notion of semistable objects in the category $\mathcal{C}$, which is additive in short exact sequences: if $E' \hookrightarrow E \twoheadrightarrow E''$ is a short exact sequence with $E', E''$ semistable, then $E$ is also semistable. Imposing semistability must define open substacks of finite type $\mathcal{M}^\text{ss}_{r,v} \subseteq \mathcal{M}_{r,v}$ that admit a good moduli space $\mathcal{M}^\text{ss}_{r,v} \to \mathcal{M}^\text{ss}_{r,v}$, and have virtual dimension $\text{vdim}(\mathcal{M}^\text{ss}_{r,v}) = r^2 \chi(v, v)$.

The total cohomology 

$$A^\text{ss}_{r,v} = \bigoplus_{r \geq 0} H_c^\bullet(\mathcal{M}^\text{ss}_{r,v}) \otimes \mathbb{L}^{\text{vdim}(\mathcal{M}^\text{ss}_{r,v})/2}$$

is an important object of study in the subject of cohomological Donaldson–Thomas theory.

**Definition 1.8.** We say that $A^\text{ss}_{r,v}$ satisfies the cohomological integrality conjecture if there exists mixed Hodge structures $\text{BPS}_{r,v}$ of finite total dimension and an isomorphism of $\mathbb{Z}_{\geq 0}$-graded mixed Hodge structures

$$A^\text{ss}_{r,v} \cong \text{Sym} \left( \bigoplus_{r \geq 1} \text{BPS}_{r,v} \otimes H_c^\bullet(BG_m) \otimes \mathbb{L} \right).$$

The graded mixed Hodge structure $\bigoplus_{r \geq 1} \text{BPS}_{r,v}$ is the BPS cohomology.

**Remark 1.9.** The cohomological integrality conjecture is known in Setting 1 by [KK21], Setting 2 by [Dav16a], and Setting 4 by [Dav16b].

Theorem 1.6 can be rephrased as a formula for the $E$-polynomial of BPS cohomology.

**Corollary 1.10.** Suppose $A^\text{ss}_{r,v}$ satisfies the cohomological integrality conjecture. Then

$$E(\text{BPS}_{2v}) = \frac{E(IH_c^\bullet(\mathcal{M}^\text{ss}_{2v}))}{q^{4v-3}} + E(\Lambda^2(H_c^\bullet(\mathcal{M}^\text{ss}_{1}) \otimes \mathbb{L}^{-g})).$$

Moreover, assuming purity of $A^\text{ss}_{r,v}$, we can upgrade the statement for $E$-polynomials to a statement for cohomology.

**Corollary 1.11.** Suppose $A^\text{ss}_{r,v}$ satisfies the cohomological integrality conjecture. If $H_c^\bullet(\mathcal{M}^\text{ss}_{2v})$ is pure, then there is a canonical isomorphism of mixed Hodge structures

$$H_c^\bullet(\mathcal{M}^\text{ss}_{2v}) \otimes \mathbb{L}^{4-4g} \cong IH_c^\bullet(\mathcal{M}^\text{ss}_{2v}) \otimes \mathbb{L}^{3-4g} \otimes H_c^\bullet(BG_m) \otimes \mathbb{L} \otimes \Lambda^2(H_c^\bullet(\mathcal{M}^\text{ss}_{1}) \otimes \mathbb{L}^{-g}) \otimes \text{Sym}^2(H_c^\bullet(\mathcal{M}^\text{ss}_{1}) \otimes H_c^\bullet(BG_m) \otimes \mathbb{L}),$$

equivalently, there is a canonical isomorphism

$$\text{BPS}_{2v} \cong IH_c^\bullet(\mathcal{M}^\text{ss}_{2v}) \otimes \mathbb{L}^{3-4g} \otimes \Lambda^2(H_c^\bullet(\mathcal{M}^\text{ss}_{1}) \otimes \mathbb{L}^{-g}).$$

**Proof.** By the cohomological integrality theorem $H_c^\bullet(\mathcal{M}^\text{ss}_{2v}) \otimes \mathbb{L}^{4-4g}$ must be the second graded piece of $\text{Sym}(\bigoplus_{r \geq 1} \text{BPS}_{r,v} \otimes H_c^\bullet(BG_m) \otimes \mathbb{L})$. Thus by [Dav21b, Theorem 6.6] there is a canonical inclusion of the RHS into $H_c^\bullet(\mathcal{M}^\text{ss}_{2v}) \otimes \mathbb{L}^{4-4g}$. Purity and equality of $E$-polynomials (Theorem 1.6) implies that the inclusion is in fact an isomorphism of mixed Hodge structures. \hfill $\Box$

**Example 1.12.** In Settings 1, 3, and 4 the compactly supported cohomology $H_c^\bullet(\mathcal{M}^\text{ss}_{2v})$ is pure ([Dav16b; Dav21b]). However, in Setting 2 the compactly supported cohomology $H_c^\bullet(\mathcal{M}^\text{ss}_{2v})$ is not pure.

**1.4. Notation.** We work over the complex numbers. For an algebraic group $G$ acting on a scheme $X$ denote by $X/G$ the quotient stack.

We always take (compactly supported) cohomology and intersection cohomology with rational coefficients. We overload the notation $\mathbb{L}$ which denotes the mixed Hodge structure $H_c^\bullet(\mathbb{A}^1)$ or the class $[\mathbb{A}^1]$ in the Grothendieck ring of varieties. The $E$-polynomial $q := E(\mathbb{L}) = uv$ is unambiguous.
2. E-series

2.1. E-series of mixed Hodge structures. A (cohomologically graded) mixed Hodge structure is a triple $H = (H^\bullet, W, F)$ consisting of

- a $\mathbb{Z}$-graded vector space $H$ over $\mathbb{Q}$
- an increasing weight filtration $W$ on the graded vector space $H$
- a decreasing Hodge filtration $F$ on the complexified graded vector space $H_{\mathbb{C}}$

such that the filtration $F$ on the complexification of each associated graded piece $W_k H/W_{k-1} H$ endows said piece with a rational weight $k$ pure Hodge structure.

A cohomologically graded mixed Hodge structure $H$ is said to be a (cohomologically graded) pure Hodge structure if $H^n = (H^n, W \cap H^n, F \cap H^n)$ is a (non-graded) pure Hodge structure of weight $n$ for all $n \in \mathbb{Z}$.

Henceforth all Hodge structures are assumed to be cohomologically graded.

For every cohomologically graded mixed Hodge structure $H = (H^\bullet, W, F)$ such that

$$\dim(\text{Gr}_p^W \text{Gr}_{p+q}^W H^n) < \infty, H^n = 0 \text{ for } n \gg 0, \text{ and } \text{Gr}_p^W H^n = 0 \text{ for } p > n$$

we define the E-series of $H$ to be

$$E(H) = \sum_{p,q \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} (-1)^n \dim(\text{Gr}_p^W \text{Gr}_{p+q}^W H^n) u^p v^q \in \mathbb{Z}[(u^{-1}, v^{-1})].$$

If $H$ is finite dimensional, then $E(H)$ is a Laurent-polynomial in $u, v$ and so we call it the E-polynomial of $H$.

Remark 2.1. The boundedness condition (2) is satisfied for the compactly supported cohomology for all finite type Artin stacks with affine stabilisers [Dav20, Lemma 4.6]. Moreover it guarantees that the E-series is well-defined.

Let $\text{MHS}^-$ be the category of mixed Hodge structures $H = (H^\bullet, W, F)$ satisfying (2). It is a symmetric monoidal abelian category. The E-series defines a ring homomorphism

$$E: K_0(\text{MHS}^-) \rightarrow \mathbb{Z}[(u^{-1}, v^{-1})].$$

2.2. E-series of varieties. For every finite type separated scheme $X$ over $\mathbb{C}$ its compactly supported cohomology $H^\bullet(X)$ is endowed with a mixed Hodge structure by [Del71; Del74]. The E-polynomial of a finite type separated scheme $X$ is the E-polynomial of its compactly supported cohomology $E(X) = E(H^\bullet(X))$. The E-polynomial is a motivic invariant: for every closed subscheme $Z \subseteq X$ we have $E(X) = E(Z) + E(X \setminus Z)$ and $E(X \times Y) = E(X)E(Y)$.

Example 2.2. $q := E(\mathbb{L}) = uv, E(G_m) = q - 1, E(\mathbb{P}^1) = q + 1, E(\text{GL}_2) = q(q + 1)(q - 1)^2$, and $E(\text{GL}_n) = \prod_{i=0}^{n-1} (q^n - q^i)$.

2.3. E-series of stacks. There are two equivalent approaches to defining the E-series of a quotient stack. The first uses the Grothendieck ring of stacks. The second uses an algebrao-geometric approximation of the Borel construction to define a mixed Hodge structure on the compactly supported cohomology of a quotient stack, see [Dav21b, Section 2.3] for details. We take the approach via the Grothendieck ring of stacks.

Let $K_0(\text{Var})$ and $K_0(\text{St})$, denote the Grothendieck ring of varieties and finite type Artin stacks with affine stabilizers, respectively. See [Eke09; FSS19; GHS14] for introductions to these rings. Let $L = [\mathbb{A}^1]$ be the class of the affine line.
Proposition 2.3 ([Eke09, Theorem 1.2]). The classes $\mathbb{L} = [\mathbb{G}_a]$, $[\mathbb{GL}_n]$ in $K_0(\text{St})$ are invertible with inverses $[B\mathbb{G}_a]$, resp., $[B\mathbb{GL}_n]$, and the natural inclusion

$$\mathcal{I} : K_0(\text{Var})[L^{-1}], [\mathbb{GL}_n]^{-1} \hookrightarrow K_0(\text{St})$$

is an isomorphism of rings.

We use this isomorphism to define the E-series of a finite type stack with affine stabilizers.

Definition 2.4. By the Künneth formula and the long exact sequence in compactly supported cohomology, the E-polynomial of a variety defines a ring homomorphism

$$E : K_0(\text{Var}) \rightarrow \mathbb{Z}[u,v],$$

which induces the E-series homomorphism

$$E = E \circ \mathcal{I}^{-1} : K_0(\text{St}) \rightarrow \mathbb{Z}[u,v][(uv)^{-1}, E(\mathbb{GL}_n)^{-1}] \subseteq \mathbb{Z}((u^{-1}, v^{-1})), $$

where we overload $E$ in our notation.

Example 2.5. In general $E(X/G) \neq E(X)/E(G)$. Let $G$ be a finite group acting on a smooth variety $X$. Then $E(X/G) = E(H^*_G(X)^G)$.

Example 2.6. An algebraic group $G$ is special if every $G$-torsor is Zariski-locally trivial. Let $G$ be a special algebraic group acting on a variety $X$. The E-series of the quotient stack $X/G$ is $E(X/G) = E(X)/E(G)$.

Example 2.7. Suppose $G$ is a special algebraic group and $X \rightarrow X$ is a $G$-gerbe. Then $E(X) = E(X)/E(G)$.

2.4. E-series of a symmetric square via $\lambda$-rings. For our computations in the sequel it is convenient to use the language of $\lambda$-rings. We refer to [Kum73] for the basics of $\lambda$-rings.

The ring $\mathcal{R} = \mathbb{Z}((u^{-1}, v^{-1}))$ carries two natural $\lambda$-ring structures coming from the isomorphism $\mathcal{R} \cong K_0(\mathbf{Vect}_{\mathbb{Z}^2})$ where $\mathbf{Vect}_{\mathbb{Z}^2}$ is the symmetric monoidal abelian category of $\mathbb{Z}^2$-graded, bounded above vector spaces with finite dimensional graded pieces. The first is the $\lambda$-ring structure $\lambda(t) = \sum_{n \geq 0} \lambda^n t^n$ induced by taking alternating powers $\Lambda^n V$ for $V \in \text{Ob}(\mathbf{Vect}_{\mathbb{Z}^2})$. The second is the symmetric power $\lambda$-ring structure $\sigma(t) = \sum_{n \geq 0} \sigma^n t^n$ induced by taking symmetric powers $\text{Sym}^n V$ for $V \in \text{Ob}(\mathbf{Vect}_{\mathbb{Z}^2})$. The $\lambda$-ring structures $\lambda$ and $\sigma$ in $\mathcal{R}$ are opposite.

In the sequel we only require $\sigma^2$ and $\lambda^2$, which are explicitly given for all $f \in \mathcal{R} = \mathbb{Z}((u^{-1}, v^{-1}))$ by

$$\sigma^2(f) = \frac{1}{2}(f(u,v)^2 + f(u^2, v^2)) \quad \text{and} \quad \lambda^2(f) = \frac{1}{2}(f(u,v)^2 - f(u^2, v^2)).$$

The ring $K_0(\text{St})$ admits a pre-$\lambda$-ring structure $\text{Sym}^i[X](t) = \sum_{n \geq 0} \text{Sym}^n[X]t^n$ given by the (stacky) symmetric powers $\text{Sym}^i[X] = \text{Sym}^i(\mathcal{X})$. The opposite pre-$\lambda$-ring structure $\Lambda^i[X](t) = \sum_{n \geq 0} \Lambda^n[X]t^n$ is given by $\Lambda[X](t) = (\text{Sym}^i[X])(-t)^{-1}$. We think of $\Lambda^i[X]$ as the class of the $n$th alternating power of $[\mathcal{X}]$. Indeed, for all varieties $X$ we have the equality

$$E(\Lambda^i[X]) = E(\Lambda^i H_i(X)).$$

The E-series ring homomorphism of Definition 2.4 is a homomorphism of pre-$\lambda$-rings, because $\mathcal{I}$ is an isomorphism of pre-$\lambda$-rings (see Example 3.5 and Proposition 3.6 in [DM11]).

With the formalism of $\lambda$-rings we easily find a concise expression for the E-series of the symmetric square of a quotient by $\mathbb{G}_m$.

Lemma 2.8. Let $\mathbb{G}_m$ act on a separated scheme of finite type $X$. Then

$$E(\text{Sym}^2(X/\mathbb{G}_m)) = \frac{qE(\text{Sym}^2(X)) + E(\Lambda^2[X])}{(q-1)^2(q+1)}.$$
Proof.

\[
E(\text{Sym}^2(X/G_m)) = \sigma^2(E(X/G_m)) = \frac{1}{2} \left( (E(X/G_m)(u, v)^2 + E(X/G_m)(u^2, v^2) \right) \\
= \frac{1}{2} \left( \frac{E(X)^2}{(q-1)^2} + \frac{E(X)(u^2, v^2)}{q^2-1} \right) = \frac{q \sigma^2(E(X)) + \lambda^2(E(X))}{(q-1)^2(q+1)}
\]

\[\Box\]

Remark 2.9. Upon reading the calculation in Section 3, the reader might notice that most of the steps are valid more generally in the Grothendieck ring of stacks. The main reason we do not work in \(K_0(\text{St})\) is to have access to the identity of Lemma 2.8.

3. The Computation

We continue to use the terminology and notation from Section 1.3. In this section we compute the E-series of certain moduli stacks \(\mathcal{M}_{2v}^{\text{ss}}\) of semistable objects in certain 2CY-categories \(\mathcal{C}\). First we spell out the assumptions we make on the category \(\mathcal{C}\), the class \(v\), semistability, and the moduli stacks and spaces, all of which are satisfied by each of our Settings 1-4.

We consider the moduli stack \(\mathcal{M}_1 = \mathcal{M}_v\) of semistable objects \(L \in \mathcal{C}\) of a primitive class \(v\). In all of our cases \(\mathcal{M}_1\) is smooth and admits a smooth good moduli space \(p_1: \mathcal{M}_1 \rightarrow \mathcal{M}_1\). Since \(v\) is primitive, an object \(L \in \mathcal{C}\) of class \(v\) is necessarily simple.

Definition 3.1. For convenience we call simple and semistable objects stable. (This is why we write \(\mathcal{M}_1\) and \(\mathcal{M}_1\) instead of \(\mathcal{M}_{2v}^{\text{ss}}\) and \(\mathcal{M}_{2v}^{\text{ss}}\).) For a primitive class \(v\), direct sums of stable objects all of which are of a class which is a multiple of \(v\) are called polystable of slope \(v\).

We also consider the moduli stack \(\mathcal{M}_2 = \mathcal{M}_{2v}\) of semistable objects of class \(2v\). The moduli space \(\mathcal{M}_2\) is singular and parametrizes objects of class \(2v\) which are both semistable and polystable. Semistable objects \(E\) of class \(2v\) are either stable or there is a stable subobject \(K \subset E\) of class \(v\). In the second case, the isomorphism class of the direct sum \(K \oplus E/K\) does not depend on the choice of the subobject \(K\) of class \(v\). The morphism \(p_2: \mathcal{M}_2 \rightarrow \mathcal{M}_2\) sends an object \(E \in \mathcal{M}_2\) to the factors of the filtration by stable objects of class \(v\) or \(2v\):

\[
p_2(E) = \begin{cases} 
E & \text{if } E \text{ is stable} \\
K \oplus E/K & \text{if } K \subset E \text{ is of class } v
\end{cases}
\]

We assume that for every non-zero object \(E \in \mathcal{C}\) the first self-Ext-group does not vanish \(\text{Ext}^1_{\mathcal{C}}(E, E) \neq 0\). The 2CY-pairing (1) restricts to a non-degenerate alternating pairing on \(\text{Ext}^1_{\mathcal{C}}(E, E)\) and so \(\text{Ext}^1_{\mathcal{C}}(E, E)\) is even-dimensional. For all objects \(L \in \mathcal{C}\) of class \(v\), let \(g > 1\) be the integer such that \(\dim \text{Ext}^1_{\mathcal{C}}(L, L) = 2g\). This implies that for all non-isomorphic objects \(Q, K \in \mathcal{C}\) of class \(v\) we have \(\dim \text{Ext}^1_{\mathcal{C}}(Q, K) = 2g - 2\).

Remark 3.2 (Ext-quivers). All semistable objects of class \(2v\) have one of the following three Ext-quivers.

![Ext-quivers](image)

The numbers inside the vertices represent the corresponding dimension vector determined by the object. All of the objects of class \(v\) have the following Ext-quiver.

![Ext-quivers](image)
Remark 3.3 (Isosingularity of the moduli problems). Each of the Settings 1-4 appear as examples in [Dav21b, §7]. Thus by the étale Ext-quiver neighborhood theorem [Dav21b, Theorem 5.11] and Remark 3.2 we have that the morphisms $\mathcal{M}_2^{ss} \to \mathcal{M}_2^{ss}$ for each of the Settings 1-4 (and $\mathcal{M}_1 \to \mathcal{M}_1$) are pairwise étale locally isomorphic. Thus it suffices to check local properties for all settings by checking it for Setting 4. This implies that the moduli spaces $\mathcal{M}_2$ (and $\mathcal{M}_1$) are pairwise stably isosingular for each of the Settings 1-4. See [Mau21, §2.4] for a definition and further discussion of stable isosingularity.

3.1. The stratification. The standard strategy to compute the motivic invariant of a space, such as the E-series of a stack, is to stratify the space into locally-closed pieces for which the E-series is known or easy to determine and then add everything up by the cut-and-paste relation. The calculation below is an execution of this strategy for $\mathcal{M}_2^{ss}$.

3.1.1. The stratification of the good moduli space. The points of the good moduli space $\mathcal{M}_2^{ss}$ correspond to polystable objects of class $v$. We stratify $\mathcal{M}_2^{ss}$ by polystability type.

First we distinguish between stable and strictly polystable objects of class $2v$. Let $\mathcal{M}_2^{ss} \subseteq \mathcal{M}_2^{ss}$ be the locus of stable objects of class $2v$. Its complement $\Sigma = \mathcal{M}_2^{ss} \setminus \mathcal{M}_2^{ss}$ is the locus of strictly polystable objects

$$\Sigma = \{L_1 \oplus L_2 \mid L_1, L_2 \text{ objects of class } v\}.$$ 

More precisely $\Sigma$ is the image of the direct sum map

$$\oplus: \mathcal{M}_1^s \times \mathcal{M}_1^s \to \mathcal{M}_2^s$$

which is isomorphic to the symmetric square of $\mathcal{M}_1^s$.

Indeed, the direct sum map is a quasi-finite map onto a normal target (which follows from the normality in the case of preprojective algebras [Cra03] and Remark 3.3). Thus by Zariski’s Main Theorem the induced map $\text{Sym}^2(M_1) \to M_2^{ss}$ is an isomorphism onto its image. Similarly, by [Le 02, Theorem 3.2], Remark 3.2, and Remark 3.3, we deduce that $\Sigma$ is precisely the singular locus of $\mathcal{M}_2^{ss}$.

By the isomorphism (3) the singular locus $\Omega$ of $\Sigma$ is identified with the image of $\mathcal{M}_1^s$ by the diagonal embedding $\mathcal{M}_1^s \to \text{Sym}^2(M_1^s)$. Explicitly $\Omega$ is given by the locus of polystable objects that are direct sums of two copies of the same object of class $v$

$$\Omega = \{L^\oplus 2 \mid L \text{ object of class } v\} \subseteq \Sigma.$$

These loci yield the stratification by polystability type of $\mathcal{M}_2^{ss}$

$$\mathcal{M}_2^{ss} = \mathcal{M}_2^{ss} \cup (\Sigma \setminus \Omega) \cup \Omega,$$

where the stratum $\Sigma \setminus \Omega$ has the explicit description

$$\Sigma \setminus \Omega = \{L_1 \oplus L_2 \mid L_1, L_2 \text{ distinct objects of class } v\}.$$ 

This stratification has already appeared in the literature and is applied in the works [Fel21] [Mau21] to compute the intersection E-polynomials of $\mathcal{M}_2^{ss}$.

Remark 3.4. The three loci $\mathcal{M}_2^s$, $\Sigma \setminus \Omega$, and $\Omega$ correspond, in order, to the first three Ext-quivers in Remark 3.2. We emphasize that the deepest stratum $\Omega$ corresponds to the g-loop quiver with dimension vector 2.

3.1.2. Pulling back the stratification. A natural stratification of the stack $\mathcal{M}_2^s$ is the pull-back of the stratification (4) along the morphism $p = p_2: \mathcal{M}_2^{ss} \to \mathcal{M}_2^s$ to the good moduli space. Write

$$\mathcal{S} = p^{-1}(\Sigma), \mathcal{Z} = p^{-1}(\Omega), \text{ and } \mathcal{G} = p^{-1}(\Sigma \setminus \Omega) = \mathcal{S} \setminus \mathcal{Z}.$$ 

The stack $\mathcal{S}$ is the singular locus of $\mathcal{M}_2^{ss}$. We call $\mathcal{G}$ the off-diagonal locus and $\mathcal{Z}$ the diagonal locus. Additionally, we have the stable locus $\mathcal{M}_2^s = p^{-1}(\mathcal{M}_2^{ss})$, which is a $G_m$-gerbe over $\mathcal{M}_2^s$.

The pullback stratification is

$$\mathcal{M}_2^{ss} = \mathcal{M}_2^s \cup \mathcal{G} \cup \mathcal{Z}.$$
The E-series of the stable loci $M^s_1$ and $M^s_2$ of the stacks is calculated from the E-polynomial of the stable loci of the $M^s_1$ and $M^s_2$ of the good moduli spaces.

**Lemma 3.5.** We have

$$E(M^s) = E(M^s/G_m) = E(M^s)/(q-1)$$

**Proof.** We apply [Hei12, Lemma 3] to the $G_m$-gerbe $M^s 	o M^s$. Thus it suffices to construct a vector bundle on $M^s$ of $G_m$-weight 1.

In Settings 2 and 4, we take the tautological bundle $E$ on $M^s$ which records the underlying vector space of the representation. The $G_m$-weight of $E$ is given by the weight of the scaling action which is equal to 1.

For Settings 1 and 3 see [HL10, Proposition 4.6.2] and its proof, which applies to Setting 1 by the BNR correspondence [BNR89].

Therefore, to compute $E(M^s_2)$ it remains to compute the E-polynomials of the strata $Y$ and $Z$.

### 3.1.3. Stratification of the strictly semistable locus.

For every strictly semistable object $E$ of class $2v$ there exists stable objects $K$ and $Q$ of class $v$ and a short exact sequence

$$0 \to K \to E \to Q \to 0$$

that witnesses the strict semistability of $E$. We call $W(E)$ the **semistabilizing** short exact sequence.

There are at most two isomorphism classes of objects of class $v$ that can appear as the subobject or quotient in a semistabilizing short exact sequence.

If $W(E)$ is non-split, then the semistabilizing short exact sequence $W(E)$ is unique up to (non-unique) isomorphism. On the other hand, if $W(E)$ is split, that is, if $E$ is polystable, then all semistabilizing short exact sequences are split.

Altogether, using the short exact sequences $W(E)$ we can distinguish four types of strictly semistables

- $W(E)$ is non-split, $K \not\cong Q$
- $W(E)$ non-split, $K \cong Q$
- $W(E)$ split, $K \not\cong Q$
- $W(E)$ split, $K \cong Q$

We stratify the moduli stack $M^s_2$ according to these four cases.

Let $\tilde{S} \subseteq S$ be the image of the direct-sum morphism

$$s: M^s_1 \times M^s_1 \to S$$

$$(K, Q) \mapsto K \oplus Q.$$  

This locus parametrizes those strictly semistables admitting a split semistabilizing short exact sequence. Denote the complement of $\tilde{S}$ in $S$ by $\bar{S}$. The stack $\tilde{S}$ parameterizes those strictly semistable objects with non-split semistabilizing short exact sequence.

These are two new loci that we intersect with the strata $Y$ and $Z$ to obtain our final stratification. Set

$$\tilde{Y} = Y \cap \tilde{S}, \quad \tilde{Z} = Z \cap \tilde{S},$$

$$\bar{Y} = Y \cap \bar{S}, \quad \bar{Z} = Z \cap \bar{S}.$$  

This defines the stratification

$$M^s_2 = M^s_2 \cup \tilde{Y} \cup \tilde{Z} \cup \bar{Y} \cup \bar{Z}$$

that we ultimately use to compute the E-series of $M_2$. 

3.2. The stack of strictly semistables and the stack of short exact sequences. Let $X$ be the stack of short exact sequences $$0 \rightarrow K \rightarrow E \rightarrow Q \rightarrow 0$$ where $K, Q$ are stable objects of class $v$. The following convolution diagram relates $X$ to the moduli stacks $\mathcal{M}_1$ and $\mathcal{M}_2$.

$\xymatrix{ \mathcal{X} \ar[dr]_{\pi} \ar[rr]^\epsilon \ar[rrd] & & \mathcal{M}_1 \times \mathcal{M}_1 \\ \mathcal{S} \ar[rr]_\pi & & \mathcal{M}_2}$

In the diagram the morphism $\epsilon: \mathcal{X} \rightarrow \mathcal{M}_1$ maps a short exact sequence to the pair consisting of the quotient object and the subobject $\epsilon(K \rightarrow E \rightarrow Q) = (Q, K)$.

The morphism $\pi: \mathcal{X} \rightarrow \mathcal{M}_2$ maps a short exact sequence to the middle term $\pi(K \rightarrow E \rightarrow Q) = E$.

An extension of two objects of class $v$ is necessarily strictly semistable, thus the morphism $\pi$ indeed factors through $\mathcal{S}$.

Let $\tilde{X} \subseteq X$ be the closed substack of split short exact sequences and let $\tilde{X} \subseteq X$ be its complement, which is the open substack of non-split short exact sequences. To compute the E-series of $\tilde{X}$ and $\tilde{X}$ we identify $\tilde{X}$ as a Picard stack over $(\mathcal{M}_1) \times (\mathcal{M}_2)$ as its zero-section.

Aside on Picard stacks. For the convenience of the reader we recall the definition and computation of some E-series of associated to a Picard stack. For more details see [SGA4, Exposé XVIII, Section 1.4] and [LM00, (14.4), (14.5)].

Let $\mathcal{B}$ be a finite type Artin stack with affine stabilizers and let $\mathcal{F}$ be a coherent sheaf on $\mathcal{B}$. Recall that the total space of $\mathcal{F}$ is the relative spectrum of the symmetric algebra $\text{Sym}_{\mathcal{B}}(\mathcal{F}^\vee)$, $$\text{Tot}_{\mathcal{B}}(\mathcal{F}) = \text{Spec}_{\mathcal{B}}(\text{Sym}_{\mathcal{B}}(\mathcal{F}^\vee)) \rightarrow \mathcal{B}.$$ Let $\mathcal{F}^* = \mathcal{F}^{-1} \rightarrow \mathcal{F}^0$ be a two-term complex of coherent sheaves on $\mathcal{B}$. Define the Picard stack $\text{Tot}_{\mathcal{B}}(\mathcal{F}^*)$ associated to the two-term complex $\mathcal{F}^*$ explicitly as follows. For every morphism $u: U \rightarrow \mathcal{B}$ from an affine scheme $U$ we define $$\text{Tot}_{\mathcal{B}}(\mathcal{F}^*)_U = \left\{ \begin{array}{c} \text{objects} = H^0(U, u^* \mathcal{F}^*)(U) \\ \text{morphisms} = H^{-1}(U, u^* \mathcal{F}^*)(U) \end{array} \right\}.$$ By interpreting $d: \mathcal{F}^{-1} \rightarrow \mathcal{F}^0$ as an action of the group stack $\text{Tot}_{\mathcal{B}}(\mathcal{F}^{-1})$ on the stack $\text{Tot}_{\mathcal{B}}(\mathcal{F}^0)$ we have the description $$\text{Tot}_{\mathcal{B}}(\mathcal{F}^*) = \text{Tot}_{\mathcal{B}}(\mathcal{F}^0)/\text{Tot}_{\mathcal{B}}(\mathcal{F}^{-1}).$$ A chain map $\mathcal{F}^* \rightarrow \mathcal{G}^*$ of two-term complexes of coherent sheaves which is a quasi-isomorphism induces an isomorphism of Picard stacks $\text{Tot}_{\mathcal{B}}(\mathcal{F}^*) \rightarrow \text{Tot}_{\mathcal{B}}(\mathcal{G}^*)$.

In general the Picard stack $\text{Tot}_{\mathcal{B}}(\mathcal{F}^*)$ need not be an Artin stack. However, if $\mathcal{F}^{-1}$ is locally free, then $\text{Tot}_{\mathcal{B}}(\mathcal{F}^*)$ is an Artin stack with affine stabilizers ([LM00, page 143]).

The zero-section of the Picard stack $\text{Tot}_{\mathcal{B}}(\mathcal{F}^*)$ is the closed immersion of Picard stacks $$\text{Tot}_{\mathcal{B}}(\ker(d)[1]) = \text{Tot}_{\mathcal{B}}(\tau_{\leq -1}^* \mathcal{F}^*) \hookrightarrow \text{Tot}_{\mathcal{B}}(\mathcal{F}^*),$$ where $\tau_{\leq i}$ denotes the standard truncation.

**Lemma 3.6.** Suppose $\mathcal{F}^* = \mathcal{F}^{-1} \rightarrow \mathcal{F}^0$ is a two-term complex of coherent sheaves which is quasi-isomorphic to a complex of locally free sheaves with amplitude non-positive degrees. Then $$[\text{Tot}_{\mathcal{B}}(\mathcal{F}^*)] = [\mathcal{F}]_{q^0(\mathcal{F})}$$

If $\mathcal{B}$ has the resolution property, then every two-term complex of coherent sheaves satisfies the assumption. See [Tho87; Tot04; Gro17] for general criteria for stacks to have the resolution property.
Proof. 1 Up to stratifying with respect to an open cover, we can assume without loss of generality that there is a resolution $E^\bullet \to F^\bullet$ such that $E^1 \cong \mathcal{O}_{\mathfrak{S}_1}^n$ are trivial vector bundles and $E^i = 0$ for $i > 0$. Taking inspiration from the proof of [GHS14, Lemma 3.3] we stratify further along loci $Z_{r_1,r_2}$ for which the differential $d_{E}^{r_2}$ of $E^\bullet$ has constant rank $r_i$ for $i = 1,2$. Along the strata $Z_{r_1,r_2}$, the truncation $\tau E_{r_1}^\bullet$, which is quasi-isomorphic to $E^\bullet$, is a two-term complex of vector bundles. After applying [GHS14] to this complex, we deduce the result by consolidating the stratifications. □

We compute the E-polynomial of the zero-section via the cut-and-paste relation.

Let $(Q, K)$ be the tautological pair of objects over $(\mathcal{W}_1)^{\times 2}$. We have the tautological Hom-sheaf $\text{Hom}(Q, K)$ on $(\mathcal{W}_1)^{\times 2}$, defined as follows. For every morphism $t : U \to (\mathcal{W}_1)^{\times 2}$ out of an affine scheme $U$ the coherent sheaf $t^* \mathcal{H}om(Q, K)$ is defined to be the sheaf associated to the coherent sheaf $\mathcal{H}om_{\mathcal{W}_1}(t^*Q, t^*K)$ on $U$. We define the complex of coherent sheaves $R\mathcal{H}om(Q, K)$ similarly.

We give a few more details for how to construct these complexes in each of the Settings 1-4. In Settings 2 and 4 the moduli stacks $M_{\mathcal{W}_1}$ parametrize pairs of isomorphic stable objects of class $s$. For every morphism $p : v \to (\mathcal{W}_1)^{\times 2}$ out of a point in $\mathcal{W}_1$ there is a morphism of algebras $M_1 \to \mathcal{O}_{\mathcal{W}_1}^n$ which endows $V$ with the structure of a $A \otimes \mathcal{O}_{\mathcal{W}_1}$-module. Pulling back to $M_1 \times M_2$ we obtain the tautological objects $Q := pr_1^*V$ and $K := pr_2^*V$ which are $A \otimes \mathcal{O}_{(M_1)^{\times 2}}$-modules. The complex $R\mathcal{H}om(Q, K)$ is the complex $R\mathcal{H}om_{A \otimes \mathcal{O}_{(M_1)^{\times 2}}}(Q, K)$.

Lemma 3.7. The edge term morphism $\epsilon : x \to (\mathcal{W}_1)^{\times 2}$ is isomorphic to the Picard stack over $(\mathcal{W}_1)^{\times 2}$ associated to the two-term complex

\[ \tau \mathcal{E}xt = \tau \mathcal{S}et \mathcal{H}om(Q, K)[1]. \]

Under this isomorphism $\tilde{x} \hookrightarrow x$ is the zero-section and $\tilde{x}$ is its complement.

Proof. See (the proof of) [KV22, Proposition 2.3.4]. □

3.3. The locus of non-split semistabilizing short exact sequences. Over the non-split locus $\mathfrak{S}$ the middle-term morphism $\pi$ is an isomorphism.

Lemma 3.8. The morphism of stacks mapping a strictly semistable non-polystable object to its semistabilizing short exact sequence

\[ W : \mathfrak{S} \to \tilde{x}, \]
\[ E \mapsto W(E) \]

is an isomorphism with inverse given by the projection to the middle term $\pi : x \to \mathfrak{S}$.

Consider the four Cartesian squares

\[ \begin{array}{ccc} \mathfrak{S} & \to & x \\ \downarrow^u & & \downarrow^\epsilon \\ \mathfrak{U} & \to & (\mathcal{W}_1)^{\times 2} & \to & \mathcal{D} \\ & & \downarrow & & \downarrow \Delta \\ (M_1)^{\times 2} \setminus \Delta(M_1) & \to & (M_1)^{\times 2} & \to & M_1, \end{array} \]

where $\Delta : M_1 \to (M_1)^{\times 2}$ is the diagonal. The stack $\mathfrak{U}$ parametrizes pairs of non-isomorphic stable objects of class $v$. The stack $\mathcal{D}$ parametrizes pairs of isomorphic stable objects of class $v$.

1Thank you to the referee for suggesting this argument.
There are isomorphisms of stacks
\[ \mathcal{U} \cong ((\mathcal{M}_1^g \times \mathcal{M}_1^g) \smallsetminus \Delta(\mathcal{M}_1^g))/\mathbb{G}_m^2, \]
\[ \mathcal{D} \cong \mathcal{M}_1^g/\mathbb{G}_m^2. \]
Thus their E-series are
\[ E(\mathcal{U}) = \frac{E(\mathcal{M}_1^g)^2 - E(\mathcal{M}_1^g)}{(q-1)^2}, \]
\[ E(\mathcal{D}) = \frac{E(\mathcal{M}_1^g)}{(q-1)^2}. \]

By Lemmas 3.8 and 3.7 we have for the non-split loci of the stack of short exact sequences
\[ \mathfrak{Y} \cong \tilde{\mathcal{X}}_{\mathcal{U}} \cong \mathcal{X}_{\mathcal{U}} \setminus (\mathcal{X}_{\mathcal{U}} \cap \mathcal{X}_0), \]
\[ \mathfrak{Z} \cong \tilde{\mathcal{X}}_{\mathcal{D}} \cong \mathcal{X}_{\mathcal{D}} \setminus (\mathcal{X}_{\mathcal{D}} \cap \mathcal{X}_0). \]
Thus to compute the E-series of \( \mathfrak{Y} \) and \( \mathfrak{Z} \), it remains to compute the E-series of \( \tilde{\mathcal{X}}_{\mathcal{U}} \) and \( \tilde{\mathcal{X}}_{\mathcal{D}} \).

**Lemma 3.9.** The restriction of the complex \( \tau \mathcal{E} \mathcal{X} \mathcal{E} \mathcal{T} \) to \( \mathcal{U} \) is quasi-isomorphic to a rank \( 2g - 2 \) vector bundle supported in degree 0. Hence
\[ E(\mathfrak{Y}) = \frac{(E(\mathcal{M}_1^g)^2 - E(\mathcal{M}_1^g))(q^{2g-2} - 1)}{(q-1)^2}. \]

**Proof.** For all non-isomorphic stable objects of class \( v \) \( L_1, L_2 \) we have
\[ \text{Hom}(L_1, L_2) = 0 \quad \text{and} \quad \text{dim Ext}^1(L_1, L_2) = 2g - 2. \]
Thus over \( \mathcal{U} \), the complex \( \tau \mathcal{E} \mathcal{X} \mathcal{E} \mathcal{T} \) is concentrated in degree 0. The rank of the degree 0 term is constant and equal to \( 2g - 2 \). Since \( \mathfrak{M}_1^g \) is smooth, we deduce the lemma. \( \square \)

**Lemma 3.10.** The degree \( -1 \) cohomology of the restriction of the complex \( \tau \mathcal{E} \mathcal{X} \mathcal{E} \mathcal{T} \) to \( \mathcal{D} \) is a rank \( 1 \) vector bundle and the degree 0 cohomology is a rank \( 2g \) vector bundle. Hence
\[ E(\mathfrak{Z}) = \frac{E(\mathcal{M}_1^g)(q^{2g} - 1)}{q(q-1)^2}. \]

**Proof.** For all stable objects \( L \) of class \( v \) we have
\[ \text{Hom}(L, L) \cong \mathbb{C} \quad \text{and} \quad \text{dim Ext}^1(L, L) = 2g. \]
Thus over \( \mathcal{D} \), the cohomology sheaf of the complex \( \tau \mathcal{E} \mathcal{X} \mathcal{E} \mathcal{T} \) in degree \( -1 \) has constant rank equal to 1 and in degree 0 has constant rank equal to \( 2g \). For every Setting 1-4, the stacks \( \mathfrak{M}_1^g \) have the resolution property. Thus we can apply Lemma 3.6 to deduce the identity for \( E(\mathfrak{Z}) \). \( \square \)

**Corollary 3.11.** By the cut-and-paste relation for \( \tilde{\mathfrak{S}} = \mathfrak{Y} \sqcup \mathfrak{Z} \) we have
\[ E(\tilde{\mathfrak{S}}) = E(\mathfrak{Y}) + E(\mathfrak{Z}) = \frac{E(\mathcal{M}_1^g)^2}{(q-1)^2} (q^{2g-2} - 1) + \frac{E(\mathcal{M}_1^g)}{q(q-1)} (q^{2g-1} + 1). \]

### 3.4. The locus of polystables.
Over the split off-diagonal locus \( \mathfrak{Y} \) the morphism \( \pi \) is a double cover: a direct sum \( K \oplus Q \) with non-isomorphic summands, arises as the middle term of precisely two different (isomorphism classes of) short exact sequences with edge terms stable objects of class \( v \).

**Lemma 3.12.** There is a commutative diagram with horizontal maps isomorphisms
\[
\begin{array}{ccc}
((\mathcal{M}_1^g) \times 2 \smallsetminus \Delta(\mathcal{M}_1^g))/\mathbb{G}_m^2 & \xrightarrow{h} & \pi^{-1}(\mathfrak{Y}) \\
\downarrow & & \downarrow \pi \\
((((\mathcal{M}_1^g) \times 2 \smallsetminus \Delta(\mathcal{M}_1^g))/\mathbb{G}_m^2)/(\mathbb{Z}/2) & \xrightarrow{h} & \mathfrak{Y}
\end{array}
\]
Theorem 3.15. We have
\[
E(\mathfrak{g}) = \frac{qE(\text{Sym}^2(M'_1)) + E(\Lambda^2[M'_1])}{(q-1)^2(q+1)} - \frac{qE(M'_1)}{(q-1)^2(q+1)}.
\]

Proof. The stack \(\pi^{-1}(\mathfrak{g})\) is the stack of split short exact sequences
\[
0 \to K \to E \to Q \to 0
\]
such that \(K\) and \(Q\) are non-isomorphic stable objects of class \(v\). The morphism \(h\) maps a pair \((Q, K)\) to the short exact sequence
\[
0 \to K \to K \oplus Q \to Q \to 0.
\]

\(\pi\) (quasi-)inverse is induced from the projection \(\pi^{-1}(\mathfrak{g}) \to \mathfrak{g} \to (M'_1) \setminus \Delta(M'_1)\). There is a \(\mathbb{Z}/2\)-action on the stack \(\pi^{-1}(\mathfrak{g})\) which swaps the roles of subobject and quotient in the split short exact sequence and \(\mathfrak{g} \cong \pi^{-1}(\mathfrak{g})/(\mathbb{Z}/2)\). Under the isomorphism \(h\) this agrees with the \(\mathbb{Z}/2\)-action on \((M'_1)^{\times 2} \setminus \Delta(M'_1)/G_m^2\). Thus we have the desired diagram.

By the cut and paste relation we have
\[
E((((M'_1)^{\times 2} \setminus \Delta(M'_1))/G_m^2)/(\mathbb{Z}/2)) = E(\text{Sym}^2(M'_1/G_m)) - E((\Delta(M'_1)/G_m^2)/\mathbb{Z}/2)).
\]

Note that both the \(G_m^2\)-action and \(\mathbb{Z}/2\)-action on \(\Delta(M'_1)\) are trivial, hence
\[
(\Delta(M'_1)/G_m^2)/(\mathbb{Z}/2) \cong M'_1 \times (BG_m)^2/\mathbb{Z}/2 \cong M'_1 \times \text{Sym}^2(BG_m).
\]

The required expression for \(E(\mathfrak{g})\) follows from Lemma 2.8. \(\square\)

Over \(\tilde{\mathfrak{g}}\) the morphism \(\pi\) is the Zariski-locally trivial \(\mathbb{P}^1\)-fibration given by the quotient morphism \(M'_1/B \to M'_1/GL_2\), where \(B \subseteq GL_2\) is the subgroup of upper-triangular matrices.

Lemma 3.13. The split diagonal locus \(\tilde{\mathfrak{g}}\) is isomorphic to the quotient stack \(M'_1/GL_2\). Thus
\[
E(\tilde{\mathfrak{g}}) = \frac{E(M'_1)}{q(q-1)^2(q+1)}.
\]

Corollary 3.14. By the cut-and-paste relation for \(\tilde{\mathfrak{g}} = \mathfrak{g} \cup \tilde{\mathfrak{g}}\) we have
\[
E(\tilde{\mathfrak{g}}) = \frac{qE(\text{Sym}^2(M'_1)) + E(\Lambda^2[M'_1])}{(q-1)^2(q+1)} - \frac{E(M'_1)}{q(q-1)}.
\]

3.5. Adding it all up.

Theorem 3.15. We have
\[
E(\mathfrak{g})_2 = E(\mathfrak{g})_2 + \frac{q^{2g-1} + q^{2g-2} - 1}{(q-1)^2(q+1)}E(\text{Sym}^2(M'_1)) + \frac{q^{2g-1} + q^{2g-2} - q}{(q-1)^2(q+1)}E(\Lambda^2[M'_1]) + \frac{q^{2g-2}E(M'_1)}{(q-1)^2(q+1)}.
\]

Thus if the cohomological integrality conjecture (Definition 1.8) holds
\[
E(\text{BPS}_2) = \frac{E(M'_2)}{q^{4g-3}} + \frac{E(\mathfrak{g}_2^{2g-3})}{q^{4g-3}(q+1)}E(\text{Sym}^2(M'_1))
\]
\[
+ \frac{E(\mathfrak{g}_2^{2g-3})}{q^{4g-3}(q+1)}E(\Lambda^2[M'_1]) + \frac{E(M'_1)}{q^{2g-1}}.
\]
Proof. First using Corollaries 3.11 and 3.14 we gather the terms which are linear in $E(M_1^s)$.

$$E(\mathcal{S}) = E(\tilde{\mathcal{S}}) + E(\tilde{\mathcal{S}})$$
$$= \frac{E(M_1^s)^2}{(q-1)} - E([P^2g-3]) + \frac{qE(Sym^2(M_1^s)) + E(\Lambda^2[M_1^s])}{(q-1)^2(q+1)} + \frac{q^{2g-2}E(M_1^s)(q-1)}{(q-1)}$$

Using $E(M_1^s)^2 = E(Sym^2(M_1^s)) + E(\Lambda^2[M_1^s])$ we have

$$E(\mathcal{S}) = \frac{E(M_1^s)^2}{(q-1)} - E([P^2g-3]) + \frac{qE(Sym^2(M_1^s)) + E(\Lambda^2[M_1^s])}{(q-1)^2(q+1)} + \frac{q^{2g-2}E(M_1^s)(q-1)}{(q-1)}$$

By the cut-and-paste relation $E(M_2^s) = E(M_1^s) + E(\tilde{\mathcal{S}})$ we have

$$E(M_2^s) = E(M_1^s) + \frac{q^{2g-1} + q^{2g-2} - 1}{(q-1)^2(q+1)} E(Sym^2(M_1^s))$$
$$+ \frac{q^{2g-1} + q^{2g-2} - q}{(q-1)^2(q+1)} E(\Lambda^2[M_1^s]) + \frac{q^{2g-2}E(M_1^s)}{(q-1)}$$

Multiply $E(M_2^s)$ by $\frac{q^{4(1-g)}}{q}$ and subtract

$$E(Sym^2(H_c(M_1^s) \otimes L^{-g} \otimes H^*_c(BG_m) \otimes \mathbb{L})) = \frac{E(Sym^2(M_1^s))}{q^{2g-3}(q-1)^2(q+1)} + \frac{E(\Lambda^2[M_1^s])}{q^{2g-2}(q-1)^2(q+1)}$$

to deduce the required expression for $\frac{E}{q}E(BPS_2)$. □

3.6. Comparison to intersection cohomology. We recall Mauri’s computation of the intersection cohomology of the coarse moduli spaces $M_2^s$.

Proposition 3.16 ([Mau21, Theorem 1.3]).

$$IE(M_2^s) = E(M_2^s) + \frac{q^{2g-4} - 1}{q^2 - 1} (q^2E(Sym^2(M_1^s)) + qE(\Lambda^2[M_1^s])) + q^{2g-2}E(M_1^s)$$

Proof. [Mau21, Theorem 1.3] is applicable by the stable isosingularity of the moduli spaces $M_2^s$ (Remark 3.3).

The form we give here follows from the identities

$$E(\Sigma) = E(Sym^2(M_1^s))$$
$$E(\Sigma) = E(\Lambda^2[M_1^s])$$

which themselves are deduced by considering the ramified double cover

$$\Sigma = M_1^s \times M_1^s \to Sym^2(M_1^s) \cong \Sigma.$$ □

Proof of Theorem 1.6. By Proposition 3.16 we have

$$IE(M_2^s) = E(M_2^s) + \frac{q^{2g-4} - 1}{q^2 - 1} (q^2E(Sym^2(M_1^s)) + qE(\Lambda^2[M_1^s])) + q^{2g-2}E(M_1^s)$$

$$= E(M_2^s) + \frac{q^{2g-2} - 1}{q^2 - 1} E(Sym^2(M_1^s)) + \frac{q(q^{2g-4} - 1)}{q^2 - 1} E(\Lambda^2[M_1^s]) + q^{2g-2}E(M_1^s)$$

where in the second line we use $E(M_2^s) = E(M_1^s) + E(Sym^2(M_1^s))$. 

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We shift by $\mathbb{L}^{4(g-1)+1}$ (i.e. divide by $q^{4g-3}$) and add $E(\Lambda^2([\mathcal{M}^1_r] \otimes \mathbb{L}^{-g})) = q^{-2g}E(\Lambda^2[\mathcal{M}^1_r])$ to obtain

$$\frac{IE(\mathcal{M}^m_r)}{q^{4g-3}} + \frac{E(\Lambda^2[\mathcal{M}^1_r])}{q^{2g}} = \frac{E(\mathcal{M}^m_2)}{q^{4g-3}} + \frac{q^{2g-2} - 1}{q^{4g-3}(q^2 - 1)}E(Sym^2(\mathcal{M}^1_r))$$

$$+ \frac{q^{2g-2} - 1}{q^{4g-4}(q^2 - 1)}E(\Lambda^2[\mathcal{M}^1_r]) + \frac{E(\mathcal{M}^m_1)}{q^{2g-1}}.$$  

The RHS of (7) is equal to the RHS of (6) in Theorem 3.15. Thus the contribution of $2\mathcal{E}$-polynomial from the $\text{SL}_2$ bundles moduli space. We apply the method explained in [Mau21, Section 4.2] to convert the $\mathcal{E}$-polynomial of the stable locus

4.1. Rank 2 degree 0 BPS cohomology

Gopakumar–Vafa invariants $n_{g,\beta}$ as defined by Maulik–Toda [MT18] are enumerative invariants of one-dimensional sheaves $F$ on a Calabi–Yau 3-fold $X$, which a priori depend on the full Chern character $\text{ch}(F) = (0, 0, \beta, \chi)$ of $F$. The Gopakumar–Vafa invariants $n_{g,\beta}$ are expected to only depend on the curve class $\beta$, i.e., we expect independence of the Euler-characteristic $\chi$, see [MT18, Section 3.3] for more details.

For a curve $C$, respectively a K3 surface $S$, by considering the local curve $T^*C \times \mathbb{A}^1$, respectively, the local surface $S \times \mathbb{A}^1$ one defines Gopakumar–Vafa invariants for Higgs bundles on $C$, respectively for the K3 surface $S$, for which $\chi$-independence should hold.

Similarly, BPS-cohomology (when defined) for one-dimensional sheaves on $X$ is expected to be independent of the Euler-characteristic. In fact, $\chi$-independence for BPS-cohomology conjecturally implies $\chi$-independence for Gopakumar–Vafa invariants. In this section we show how the formula of Theorem 1.6 can be applied to check $\chi$-independence for $E$-polynomials of BPS-cohomology.

For work on $\chi$-independence phenomena for BPS-invariants see [COW21; MS20a; Mel20; KK21].
\[ \frac{E(M_{2,0}^{\text{st}})}{q^{4g-3}} = \left( \frac{(1-u^2v)^g(1-v)^2 - q^{g+1}E(\text{Jac}(C))}{(q-1)^2(q+1)} \right) E(\text{Jac}(C)) \]

\[ + \frac{1}{(q-1)(q+1)}(qE(\Lambda^2[\text{Jac}(C)]) + E(\text{Sym}^2(\text{Jac}(C)))) \]

\[ + \frac{(g^9 - 1)(g^{9-1} - 1)}{q^2g-3(q-1)(q+1)}E(\text{Sym}^2(\text{Jac}(C))) + \frac{(g^{g-1} - 1)(g^{g-2} - 1)}{q^{2g-2}(q-1)(q+1)}E(\Lambda^2[\text{Jac}(C)]) \]

\[ + \frac{(g^{g-1} - 1)(g^{g-2} - 1)}{q^{2g-2}(q-1)}E(\text{Jac}(C))^2 - E(\text{Jac}(C)) + \frac{(g^g - 1)(g^{g-1} - 1)}{q^{2g-1}(q-1)}E(\text{Jac}(C)) \]

\[ + \frac{1}{2}E(\text{Jac}(C))((1-u)^{g-1}v^{g-1} + (1+u)^{g-1}v^{g-1} - 2q^{g-1}) \]

\[ + E(\text{Jac}(C)) \left( \frac{q^{g-1}E(\text{Jac}(C))}{(q-1)^2(q+1)} \right) - \frac{(1+u)^{g-1}(1+v)^{g-1}(1-u)(1-v)}{4(q+1)} - \frac{4g-7}{q-1}E(\text{Jac}(C)) \]

\[ - \frac{qE(\text{Jac}(C))}{2(q-1)^2} \]

Now separately gathering terms with factors \(E(\text{Jac}(C))^2\), and a single factor of \(E(\text{Jac}(C)) = (1-u)^g(1-v)^g\) we have

\[ \frac{E(M_{2,0}^{\text{st}})}{q^{4g-3}} = \left( \frac{1 - q^{g-1}}{q^{2g-2}(q-1)} - \frac{4g - 3}{4(q-1)} - \frac{q}{2(q-1)^2} \right) E(\text{Jac}(C))^2 \]

\[ + \frac{1}{(q-1)(q+1)}(qE(\Lambda^2[\text{Jac}(C)]) + E(\text{Sym}^2(\text{Jac}(C)))) \]

\[ + \frac{(g^9 - 1)(g^{9-1} - 1)}{q^2g-3(q-1)(q+1)}E(\text{Sym}^2(\text{Jac}(C))) + \frac{(g^{g-1} - 1)(g^{g-2} - 1)}{q^{2g-2}(q-1)(q+1)}E(\Lambda^2[\text{Jac}(C)]) \]

\[ + \frac{(1-u^2v)^g(1-v)^2 - q^{g+1}E(\text{Jac}(C))}{(q-1)^2(q+1)} \]

\[ - \frac{(1+u)^{g-1}(1+v)^{g-1}(1-u)(1-v)}{4(q+1)} \]

\[ - \frac{4g-7}{q-1}E(\text{Jac}(C)) \]

\[ - \frac{qE(\text{Jac}(C))}{2(q-1)^2} \]
Substituting into (6) and gathering the $E(\text{Sym}^2(\text{Jac}(C)))$ and $E(A^2[\text{Jac}(C)])$ terms we have

$$E(\text{BPS}_{2,0}^{\text{Dol}}) = \left( \frac{-q^{4g-1} - q^{4g-2} + q^{4g} + q^{3g-1}}{q^{4g-3}(q-1)(q+1)} - \frac{4g-3}{4(q-1)} - \frac{q}{2(q-1)^2} \right) E(\text{Jac}(C))^2$$

$$+ \frac{q^{4g-1} + q^{4g-2} + q^{4g-3} - q^{3g} - q^{3g-1}}{q^{4g-3}(q-1)(q+1)} E(\text{Sym}^2(\text{Jac}(C)))$$

$$+ \frac{q^{4g-1} + 2q^{4g-2} - q^{3g} - q^{3g-1}}{q^{4g-3}(q+1)(q-1)} E(A^2[\text{Jac}(C)])$$

$$+ \left( \frac{(1 - u^2)q(1 - u^2)^g}{(q-1)^2(q+1)} \right)$$

$$+ \frac{1}{2} \left( \frac{1}{(1 - u)^{g-1}(1 - v)^{g-1} + (1 + u)^{g-1}(1 + v)^{g-1}} \right)$$

$$- \frac{(1 + u)^{g-1}(1 + v)^{g-1}(1 - u)(1 - v)}{4(q+1)}$$

$$- \frac{g - 1}{q - 1} \frac{(1 + u + v - 2uv)(1 - u)^{g-1}(1 - v)^{g-1}}{q - 1} \right) E(\text{Jac}(C))$$

where two $E(\text{Jac})$ terms cancelled out with the contribution $E(M_{1,0})/q^{2g-1} = E(\text{Jac}(C))/q^{g-1}$.

Using the identity $E(\text{Jac}(C))^2 = E(A^2[\text{Jac}(C)]) + E(\text{Sym}^2(\text{Jac}(C)))$ cancels out the first $E(\text{Jac}(C))^2$ term.

$$E(\text{BPS}_{2,0}^{\text{Dol}}) = \left( \frac{4g-3}{4(q-1)} - \frac{q}{2(q-1)^2} \right) E(\text{Jac}(C))^2$$

$$+ \frac{1}{(q-1)(q+1)} E(\text{Sym}^2(\text{Jac}(C)))$$

$$+ \frac{q}{(q+1)(q-1)} E(A^2[\text{Jac}(C)])$$

$$+ \left( \frac{(1 - u^2)q(1 - u^2)^g}{(q-1)^2(q+1)} \right)$$

$$+ \frac{1}{2} \left( \frac{1}{(1 - u)^{g-1}(1 - v)^{g-1} + (1 + u)^{g-1}(1 + v)^{g-1}} \right)$$

$$- \frac{(1 + u)^{g-1}(1 + v)^{g-1}(1 - u)(1 - v)}{4(q+1)}$$

$$- \frac{g - 1}{q - 1} \frac{(1 + u + v - 2uv)(1 - u)^{g-1}(1 - v)^{g-1}}{q - 1} \right) E(\text{Jac}(C))$$

Expanding

$$E(A^2[\text{Jac}(C)]) = \frac{1}{2} (1 - u)^g (1 - v)^g ((1 - u)(1 - v)(1 - u)^{g-1}(1 - v)^{g-1})$$

$$- (1 + u)(1 + v)(1 + u)^{g-1}(1 + v)^{g-1}$$

$$E(\text{Sym}^2(\text{Jac}(C))) = \frac{1}{2} (1 - u)^g (1 - v)^g ((1 - u)(1 - v)(1 - u)^{g-1}(1 - v)^{g-1})$$

$$+ (1 + u)(1 + v)(1 + u)^{g-1}(1 + v)^{g-1}$$

and combining with the second, third and fourth $E(\text{Jac}(C)) = (1 - u)^g (1 - v)^g$ terms yields

$$E(\text{BPS}_{2,0}^{\text{Dol}}) = \left( \frac{(1 - u)^g (1 - v)^g (1 - u^2)v(1 - u^2)^g}{(q-1)^2(q-1)} \right) + (1 - u)^g (1 - v)^g \left( \frac{- (1 + u)^g (1 + v)^g}{4(q+1)} \right)$$

$$- \frac{g}{2} \frac{(u + v - 2uv)(1 - u)^{g-1}(1 - v)^{g-1}}{q - 1} - \frac{4g - 3}{4} \frac{(1 - u)^g (1 - v)^g}{q - 1}$$

$$- \frac{1}{2} \frac{q(1 - u)^g (1 - v)^g}{(q-1)^2(q-1)}.$$
4.1.2. Rank 2 degree 1 BPS cohomology. In degree 1 the BPS cohomology is the shifted cohomology of the rank 2 degree 1 coarse moduli space. A formula for its E-polynomial can be extracted from [GHS14, Appendix] and is given by

\[ E(\text{BPS}_{2,1}^{\text{Dol}}) = E(M_{2,1}) \]

\[ = E(\text{Jac}(C)) \left( \frac{(1-u^2v^2)(1-uv^2)^g - q^gE(\text{Jac})}{(q^2-1)(q-1)} + \sum_{d=1}^{g-1} \frac{E(\text{Sym}^{2g-2d-1}(C))}{(q^2-1)(q-1)} \right) \]

\[ = \frac{(1-u)^g(1-v)^g(1-u^2v^2)^g(1-uv^2)^g}{(q^2-1)(q-1)} - \frac{q^g(1-u)^{2g}(1-v)^{2g}}{(q^2-1)(q-1)} \]

\[ + (1-u)^g(1-v)^g \frac{\sum_{d=1}^{g-1} \text{Coeff}_{2g-2d-1} \left( \frac{(1-uv)(1-v)^g(1-v)^g}{(1-t)(1-qt)} \right)}{\left( \frac{(1-uv)(1-v)^g(1-v)^g}{(1-t)(1-qt)} \right)} . \]

The third equality follows from Macdonald’s computation of the cohomology of symmetric powers of curves [Mac62]. We evaluate

\[ \sum_{d=1}^{g-1} \text{Coeff}_{2g-2d-1} \left( \frac{(1-ut)(1-v^g)}{(1-t)(1-qt)} \right) = \frac{q^g(1-u)^g(1-v)^g}{(q-1)^2(q+1)} - \frac{(1+u)^g(1+v)^g}{4(q+1)} \]

\[ - g \left( \frac{u+v-2uv}{2} \right) (1-u)^{g-1}(1-v)^{g-1} \frac{4g-3(1-u)^g(1-v)^g}{4(q-1)} \]

following Hitchin [Hit87, Proof of Theorem 7.6] (see also [KY08, Section 3.3]). Altogether we have

\[ E(\text{BPS}_{2,1}^{\text{Dol}}) = \frac{(1-u)^g(1-v)^g(1-u^2v^2)^g(1-uv^2)^g}{(q^2-1)(q-1)} - \frac{q^g(1-u)^{2g}(1-v)^{2g}}{(q^2-1)(q-1)} \]

\[ + (1-u)^g(1-v)^g \left( \frac{q^g(1-u)^g(1-v)^g}{(q-1)^2(q+1)} - \frac{(1+u)^g(1+v)^g}{4(q+1)} \right) \]

\[ - \frac{g}{2} \frac{(u+v-2uv)(1-u)^{g-1}(1-v)^{g-1}}{q-1} - \frac{4g-3(1-u)^g(1-v)^g}{4} \]

\[ - \frac{1}{2} \frac{q(1-u)^g(1-v)^g}{(q-1)^2} \left( \frac{(1-uv)(1-v)^g(1-v)^g}{(1-t)(1-qt)} \right) . \]

This simplifies slightly to

\[ E(\text{BPS}_{2,1}^{\text{Dol}}) = \frac{(1-u)^g(1-v)^g(1-u^2v^2)^g(1-uv^2)^g}{(q^2-1)(q-1)} + (1-u)^g(1-v)^g \left( \frac{(1+u)^g(1+v)^g}{4(q+1)} \right) \]

\[ - \frac{g}{2} \frac{(u+v-2uv)(1-u)^{g-1}(1-v)^{g-1}}{q-1} - \frac{4g-3(1-u)^g(1-v)^g}{4} \]

\[ - \frac{1}{2} \frac{q(1-u)^g(1-v)^g}{(q-1)^2} \left( \frac{(1-uv)(1-v)^g(1-v)^g}{(1-t)(1-qt)} \right) , \]

which agrees with the expression (8) for \( E(\text{BPS}_{2,0}^{\text{Dol}}) \).

4.1.3. Betti side. Via non-abelian Hodge theory for stacks, as developed in [Dav21a], one similarly expects a \( \chi \)-independence phenomenon on the Betti side.

Since \( \gcd(2,1) = 1 \), we have \( E(\text{BPS}_{2,1}^{\text{Betti}}) = q^{g-1} E(M_{2,1}^{\text{Betti}}) \). The E-polynomial \( E(M_{2,1}^{\text{Betti}}) \) was determined in [HR08, Corollary 3.6.1] and the intersection E-polynomial \( I E(M_{2,0}^{\text{Betti}}) \) was determined in [Mau21, Theorem 1.4]. Using Theorem 1.4 one can directly check \( E(\text{BPS}_{2,0}^{\text{Betti}}) = E(\text{BPS}_{2,1}^{\text{Betti}}) \).

4.2. Sheaves on K3 surfaces. We can reinterpret some results in [dCRS21] as a cohomological \( \chi \)-independence check for sheaves on K3 surfaces.

Let \( S \) be a K3 surface. Let \( H \) be a sufficiently general polarization of \( S \). Suppose \( (S, H) \) is of genus 2, i.e., the curves in the linear system \( |H| \) are of genus 2. Let \( v \in H^0_{\text{alg}}(S, \mathbb{Z}) \) be a primitive
Mukai vector with $v^2 = 2$. Consider the moduli stack and moduli spaces of $H$-Gieseker-semistable sheaves $\mathcal{M}^{H\text{-sst}}_{S,2v}$, $\mathcal{M}^{H\text{-sst}}_{S,2v}$, and $\mathcal{M}^{H\text{-sst}}_{S,v}$.

Let $O'G_{10}$ be O’Grady’s ten-dimensional sporadic example of a hyper-Kähler manifold. By [dCRS21, Lemma 4.1.3] we have

$$IE(\mathcal{M}^{H\text{-sst}}_{S,2v}) = E(O'G_{10}) - qE(\text{Sym}^2(\mathcal{M}^{H\text{-sst}}_{S,v})) - q^3H(\mathcal{M}^{H\text{-sst}}_{S,v}).$$

and by (6) we have

$$E(\text{BPS}_{S,2v}) = \frac{E(O'G_{10})}{q^5} - \frac{E_{w^2,v^2}(\mathcal{M}^{H\text{-sst}}_{S,v})}{q^4} - \frac{E(\mathcal{M}^{H\text{-sst}}_{S,v})}{q^2}.$$ (9)

Pick a primitive Mukai vector $w \in H_{\text{alg}}^\bullet(S,\mathbb{Z})$ satisfying $w^2 = 4 = (2v)^2$. The moduli space $\mathcal{M}^{H\text{-sst}}_{S,w}$ is smooth and deformation equivalent to the Hilbert scheme of 5 points on a K3 surface.

**Corollary 4.1.** Suppose the cohomological integrality conjecture is true for $A_{\text{Coh}(S),v}$, then

$$E(\text{BPS}_{S,2v}) = E(\text{BPS}_{S,w}).$$

**Proof.** By definition we have $E(\text{BPS}_{S,w}) = q^{-5}E(\mathcal{M}^{H\text{-sst}}_{S,w})$. By [dCRS21, Proposition 6.1.2] and (9) we have $E(\text{BPS}_{S,2v}) = q^{-5}E(\mathcal{M}^{H\text{-sst}}_{S,w})$. □

Therefore, the polynomial $E(\text{BPS}_{S,2v})$ is determined by the Hodge numbers of $\mathcal{M}^{H\text{-sst}}_{S,w}$, which are recorded in [dCRS21, (103)].

**Remark 4.2.** If we assume the $\chi$-independence conjecture for BPS cohomology, then equation (9) (which follows from a simple application of the decomposition theorem [dCRS21, Lemma 4.1.3] and Theorem 3.15) together with $\chi$-independence, yields a conjectural computation of the Hodge numbers of $O'G_{10}$.

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