Effective Scenario of Loop Quantum Cosmology

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Semiclassical states in isotropic loop quantum cosmology are employed to show that the improved dynamics has the correct classical limit. The effective Hamiltonian for the quantum cosmological model with a massless scalar field is thus obtained, which incorporates also the next to leading order quantum corrections. The possibility that the higher order correction terms may lead to significant departure from the leading order effective scenario is revealed. If the semiclassicality of the model is maintained in the large scale limit, there are great possibilities for $k = 0$ Friedmann expanding universe to undergo a collapse in the future due to the quantum gravity effect. Thus the quantum bounce and collapse may contribute a cyclic universe in the new scenario.

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Theoretical search for a quantum theory of gravity has been rather active. The expectation that the singularities predicted by classical general relativity would be resolved by some quantum gravity theory has been confirmed by the recent study of certain isotropic models in loop quantum cosmology (LQC) [1, 2, 3], which is a simplified symmetry-reduced model of a full background independent quantum theory of gravity [4], known as loop quantum gravity (LQG) [5, 6, 7, 8]. The basic purpose of LQG is to merge the conceptual insight of general relativity (GR) into quantum mechanics [9]. To achieve this purpose, one only makes use of the general tools of a quantum theory. The Hilbert space and and operators are obtained from classical GR following certain quantization strategy. In contrast to the initial Wheeler-DeWitt canonical quantization of GR [10], the classical algebra that one wants to represent on the Hilbert space of LQG is based on the holonomies of the gravitational connection. Physically, holonomies are natural variables representing Faraday’s “lines of force”, that do not refer to what happens at a point, but rather refer to the relation between different points connected by a line. Mathematically, the quantum configuration space of LQG can be constructed by the concept of holonomy, since its definition does not depend on an extra background. Then the kinematical framework of LQG can be established with mathematical rigour [11].

The idea that one should view holonomies rather than connections as basic variables for the quantization of gravity is successfully carried on in the models of LQC [1, 12, 13]. In a LQC scenario for a universe filled with a massless scalar field, the classical singularity gets replaced by a quantum bounce [3, 12]. Various features of the bounce have been revealed through different considerations [12, 13, 14]. While the model shows that the quantum effect played the key role in Planck scale to cure the big bang singularity as one expected, the question whether quantum gravity effect can also be manifested in large scale cosmology remains open. This question is crucial since, besides overcoming the difficulties of a classical theory, to predict phenomena which are dramatically different from those of the classical theory is also a hallmark to identify a quantum theory.

The framework that we are considering is the so-called improved dynamics of LQC [3]. In the kinematical setting, one has to introduce an elementary cell $\mathcal{V}$ and restricts all integrations to this cell. Fix a fiducial flat metric $\gamma_{ab}$ and denote by $V_0$ the volume of $\mathcal{V}$ in this geometry. The gravitational phase space variables — the connections and the density weighted triads — can be expressed as $A_i^a = c V_0^{(1/3)} \omega_i^a$ and $E_i^a = p V_0^{-(2/3)} \sqrt{V} \epsilon_i^a$, where $\{\omega_i^a, \epsilon_i^a\}$ are a set of orthonormal co-triads and triads compatible with $\gamma_{ab}$ and adapted to $\mathcal{V}$. $p$ is related to the scale factor $a$ via $|p| = V_0^{2/3} a^2$. The fundamental Poisson bracket is given by: $\{c, p\} = 8\pi G \gamma/3$, where $G$ is the Newton’s constant and $\gamma$ the Barbero-Immirzi parameter. The gravitational part of the Hamiltonian constraint reads $\hat{C}_{grav} = -6 c^2 \sqrt{|p|}/\gamma^2$. It is convenient to introduce new conjugate variables by a canonical transformation:

$$b := \frac{\sqrt{\Delta}}{2} \frac{c}{\sqrt{|p|}}, \quad \nu := \frac{4}{3\sqrt{\Delta}} \text{sgn}(p)|p|^{1/2},$$

where $\Delta \equiv (2\sqrt{3\pi} \gamma) \ell_P^2$ is the smallest non-zero eigenvalue of area operator in full LQG and $\ell_P^2 = G\hbar$.

In the kinematic Hilbert space $\mathcal{H}_{kin}$ of the quantum theory, eigenstates of $\hat{\nu}$, which are labelled by real numbers $v$, constitute an orthonormal basis as: $|v_1 v_2\rangle = \delta_{v_1, v_2}$. The fundamental operators act on $|v\rangle$ as: $\hat{\nu} |v\rangle = (8\pi \gamma \ell_P^2/3) |v\rangle$ and $e^{ib} |v\rangle = |v+1\rangle$. In the improved LQC treatments [8], the gravitational part of the Hamiltonian operator is given in the $v$ representation by:

$$\hat{C}_{grav} |v\rangle = f_+(v) |v+4\rangle + f_o(v) |v\rangle + f_-(v) |v-4\rangle \quad (1)$$

where

$$f_+(v) = \frac{27}{16} \sqrt{\frac{8\pi}{6}} K \ell_P \gamma^{3/2} |v+2\rangle \left| \langle v+1 \rangle - |v+3\rangle \right|,$$

$$f_o(v) = \frac{27}{16} \sqrt{\frac{8\pi}{6}} K \ell_P \gamma^{3/2} \left| \langle v+1 \rangle - |v+3\rangle \right|,$$

$$f_-(v) = \frac{27}{16} \sqrt{\frac{8\pi}{6}} K \ell_P \gamma^{3/2} \left| \langle v+1 \rangle - |v+3\rangle \right|.$$

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\[ f_-(v) = f_+(v - 4), \quad f_o(v) = -f_+(v) - f_-(v), \]

here \( K \equiv \frac{2\sqrt{7}}{3\sqrt{3} \sqrt{3}} \). As in \( 2, 3 \), to identify a dynamical matter field as an internal clock, we take a massless scalar field \( \phi \) with Hamiltonian \( C_{\phi} = |p|^{-2} \rho_{\phi}^2/2 \), where \( \rho_{\phi} \) denotes the momentum of \( \phi \). While we choose the standard Schrödinger representation for \( \phi \), the operator \( 1/|p|^{3/2} \) is diagonal in the \( v \) representation with action:

\[
|p|^{-\frac{3}{2}}|v| = \left( \frac{27}{16\pi \gamma \ell_P^2} \right)^{3/2} K |v| |v + 1|^{1/3} - |v - 1|^{1/3} |v|.
\]

Collecting these results we can express the matter part of the quantum Hamiltonian constraint as \( \hat{C}_{\phi} = \frac{1}{2} |p|^{-2} \rho_{\phi}^2 - \hat{p}_\phi^2 \) and the total constraint as \( \hat{C} = \frac{1}{16\pi G} \hat{C}_{\text{grav}} + \hat{C}_{\phi} \).

To ensure that the Hamiltonian constraint operator is a viable quantization, one needs to show that its expectation value with respect to suitable semiclassical states does reduce to the classical constraint. Let us first consider the gravitational part. Since there are uncountable basis vectors, the natural Gaussian semiclassical states live in the algebraic dual space of some dense set in \( \mathcal{H}_{\text{kin}} \). A semiclassical state \( |\Psi_{(b_o, v_o)}\rangle \) peaked at a point \( (b_o, v_o) \) of the gravitational classical phase space reads:

\[
|\Psi_{(b_o, v_o)}\rangle = \sum_{v \in \mathbb{R}} e^{-\frac{\|v - v_o\|^2}{4\epsilon}} e^{ib_o(v - v_o)} |v\rangle,
\]

where \( \epsilon = 1/d \) and we choose \( v_o = N \in \mathbb{Z} \). Since we consider large volumes and late times, the relative quantum fluctuations in the volume of the universe must be very small. Therefore we have the restrictions: \( 1/N < \epsilon < 1 \) and \( b_o \ll 1 \). To check that the state \( |\Psi_{(b_o, v_o)}\rangle \) is indeed semiclassical, we have to calculate the expectation values and the fluctuations of the fundamental variables. Although there is no operator corresponding to \( b \) in loop quantization, one may define an approximating operator \( \hat{b} := (e^{ib} - e^{-ib})/2i \) for \( b \ll 1 \). Using the shadow state scheme \( 13 \), the expectation values and the fluctuations in the state \( |\Psi_{(b_o, v_o)}\rangle \) are calculated as:

\[
\langle \hat{b} \rangle = e^{-\frac{\epsilon^2}{2}} \sin b_o (1 + O(e^{-\frac{\epsilon^2}{2^2}})), \quad \langle \hat{v} \rangle = v_o,
\]

\[
(\Delta b)^2 = \frac{\epsilon^2}{2} \cos^2 b_o + O(\epsilon^4) + O(e^{-\frac{\epsilon^2}{2^2}}),
\]

\[
(\Delta v)^2 = \frac{1}{2\epsilon^2} (1 + O(e^{-\frac{\epsilon^2}{2^3}})).
\]

We conclude that the state \( |\Psi_{(b_o, v_o)}\rangle \) is sharply peaked at \( (b_o, v_o) \) and the fluctuations are within specified tolerances. The semiclassical state of matter part is given by the standard coherent state

\[
|\Psi_{(\phi_{\sigma}, p_{\sigma})}\rangle = \int d\phi e^{-\frac{(\phi - \phi_{\sigma})^2}{2\sigma^2}} e^{i\phi} \rho_{\phi} \phi (\phi_{\sigma}) (\phi),
\]

where \( \sigma \) is the width of the Gaussian. Thus the whole semiclassical state reads \(|\Psi_{(b_o, v_o)}\rangle \otimes |\Psi_{(\phi_{\sigma}, p_{\sigma})}\rangle\).

The task is to use this semiclassical state to calculate the expectation value of the Hamiltonian operator to a certain accuracy. In the calculation of \( \langle C_{\phi} \rangle \), one gets the expression with the absolute values, which is not analytical. To overcome the difficulty we separate the expression into a sum of two terms: one is analytical and hence can be calculated straightforwardly, while the other becomes exponentially decayed out. In the calculation of \( \langle C_{\phi} \rangle \), one has to calculate the expectation value of the operator \(|p|^{-\frac{3}{2}}\). Using the Poisson resummation formula and the steepest descent approximation, we obtain

\[
\langle |p|^{-\frac{3}{2}} \rangle = \left( \frac{6}{8\pi \gamma \ell_P^2} \right)^{3/2} \frac{K}{N} \left( 1 + \frac{1}{2N^2\epsilon^2} + \frac{5}{9N^2} + O\left( \frac{1}{N^4 \epsilon^4} \right) \right) + O(e^{-N^2\epsilon^2}) + O(e^{-\frac{\epsilon^2}{N^4}}).
\]

Collecting these results we can express the expectation value of the total Hamiltonian constraint, up to corrections of order \( 1/(N^4 \epsilon^4) \) and \( e^{-\pi^2/\epsilon^2} \), as follows:

\[
\langle \hat{C} \rangle = -\frac{27}{32\pi} \sqrt{\frac{8\pi K\ell_P}{6G}} |v_o| (e^{-4\epsilon^2} \sin^2(2b_o) - \frac{1}{2} (e^{-4\epsilon^2} - 1))
\]

\[
+ \frac{1}{2} \left( \frac{6}{8\pi \gamma \ell_P^2} \right)^{3/2} \frac{K}{|v_o|} \left( \rho_{\phi}^2 + \frac{\hbar}{2\sigma^2} \right) \left( 1 + \frac{1}{2|v_o|^2 \epsilon^2} + \frac{5}{9|v_o|^2} \right).
\]

It is easy to write Eq. (9) in terms of \( (c, p) \) and see that the classical constraint is reproduced up to small corrections of order \( b_o^2, \epsilon^2, 1/|v|^2 \epsilon^2 \) and \( \hbar/\sigma^2 \). Hence, the improved Hamiltonian operator is a viable quantization of the classical expression. Using the expectation value \( (9) \) of the Hamiltonian operator \( 10 \), we can further obtain an effective Hamiltonian with the relevant quantum geometry corrections of order \( c^2, 1/|v|^2 c^2 \) as

\[
\mathcal{H}_{\text{eff}} = -\frac{27}{32\pi} \sqrt{\frac{8\pi K\ell_P}{6G\gamma^3/2}} v \left( \sin^2(2b) + 2\epsilon^2 \right)
\]

\[
+ \left( \frac{6}{8\pi \gamma \ell_P^2} \right)^{3/2} v^{3/2} \frac{K}{K \rho} \left( 1 + \frac{1}{2v^2 \epsilon^2} \right),
\]

where \( \rho = \frac{1}{2} \left( \frac{6}{8\pi \gamma \ell_P^2} \right)^{3} \left( \frac{K}{v} \right)^{2} \rho_{\phi}^2 \) is the density of the matter field. Note that the neglected quantum fluctuations of matter field would not qualitatively change the following discussions. Then we obtain the Hamiltonian evolution...
equation for \( v \) by taking its Poisson bracket with \( \mathcal{H}_{\text{eff}} \) as:

\[
\dot{v} = 3v \left( \sqrt{\frac{8\pi G}{3}} \rho_c \right) \sin(2b) \cos(2b),
\]

where \( \rho_c \equiv \sqrt{3/\left(16\pi^2 G^2 \hbar^3\right)} \). Note that a direct calculation shows that Eq. (11) coincides with the expectation value of \( \langle \dot{v}, \hat{C} \rangle/\hbar \) in the shadow state \( |3\rangle \). The vanishing of the effective Hamiltonian constraint (10) gives rise to

\[
\sin^2(2b) = \frac{\rho}{\rho_c} \left( 1 + \frac{1}{2v^2 \epsilon^2} \right) - 2\epsilon^2.
\]

The modified Friedmann equation can then be derived from Eqs. (11) and (12) as:

\[
H^2 = \frac{8\pi G}{3} \left[ 1 - \frac{\rho}{\rho_c} \left( 1 + \frac{1}{2v^2 \epsilon^2} \right) - \frac{1}{2v^2 \epsilon^2} - 2\epsilon^2 \rho_c \rho \right].
\]

It is obvious that Eq. (13) reduces to the leading order effective Friedmann equation

\[
H^2 = \frac{8\pi G}{3} \rho(1 - \rho/\rho_c),
\]

if the terms of order \( 1/(v^2 \epsilon^2) \) and \( \epsilon^2 \) are neglected. However, as we will see, the minus sign in front of the \( \epsilon^2 \) term in Eqs. (12) and (13) may lead to a qualitatively different scenario from the leading order effective theory. Thus these subleading terms cannot be neglected at will. Since all the quantum geometry corrections come from \( \langle \hat{C}_{\text{grav}} \rangle \) and Eq. (8), the higher order corrections are only higher order terms at the same place of \( \epsilon^2 \) or \( 1/(v^2 \epsilon^2) \). Hence, those neglected higher order corrections cannot lead to qualitatively different effect.

It is not difficult to see that the modified Friedmann equation (13) implies significant departure from classical general relativity, which is manifested in the bounce or collapse point determined by \( \dot{v} = 0 \). For a contracting universe, Eq. (11) ensures that the so-called quantum bounce will occur when \( \cos(2b) = 0 \). Around this point the departure of our effective scenario from the leading order effective one is slight. On the other hand, for an expanding universe, while the collapse point given by \( \sin(2b) = 0 \) will never occur in the leading order effective scenario, it may come on stage in our scenario. The quantum fluctuations or the Gaussian spread \( \epsilon \) plays a key role here. Thus its concrete form becomes rather relevant. One usually sets the innocent condition that the relative spreads of the basic conjugate variables are small for semiclassical states, i.e., \( \frac{\Delta v}{v} \sim \frac{\epsilon}{\sqrt{2v}} \ll 1 \) [12, 14]. A simple setting could be

\[
\epsilon = \lambda(r) v^{-r(\phi)}, \quad \text{where} \ 0 \leq r(\phi) \leq 1 \quad \text{and the parameter} \ \lambda(r) \ \text{has to be suitably chosen for different value of} \ r.
\]

We now illustrate two extreme cases. For \( r = 1 \), it is easy to see that the subleading order corrections \( 1/v^2 \epsilon^2 \) and \( \epsilon^2 \rho_c/\rho \) in Eq. (13) are both small constant. Hence they cannot lead to significant departure from the leading order effective scenario. While for \( r = 0 \), Eq. (13) implies that the quantum fluctuation correction \( \epsilon^2 \rho_c \) acts as a negative cosmological constant. Thus, besides the quantum bounce when the matter density \( \rho \) increases to the Planck scale, the universe would also undergo a collapse when \( \rho \) decreases to \( \rho_{\text{coll}} \approx 2\epsilon^2 \rho_c \). Therefore the quantum fluctuations lead to a cyclic universe in this case. The quantum bounces in different scenarios are compared in Fig. 1. Due to the quantum corrections of order \( 1/v^2 \epsilon^2 \), the quantum bounces in \( r = 0 \) scenario would happen at a smaller critical density of matter than that of the \( r = 1 \) scenario. The trajectory of the latter almost coincides with that of the leading order effective scenario. The cyclic universe in \( r = 0 \) scenario is illustrated in Fig. 2. Since the significant departure occurs only at the large scale limit, the asymptotic behavior of \( r(\phi) \) is crucial. It
is easy to see from Eq. (12) that an expanding universe would undergo the collapse and become cyclic provided \(0 < r < 1\) asymptotically. Suppose that the semiclassicality of our coherent state is maintained in the large scale limit. This means that the quantum fluctuation \(1/\epsilon\) of \(v\) cannot increase as \(v\) unboundedly as \(v\) approaches infinity. This is another way of saying that the quantum fluctuation \(\epsilon\) of \(b\) cannot approach zero as \(b\), since otherwise the coherent state would approach an eigenstate of \(b\) and thus lose its coherence. In fact, the innocent condition \(\Delta b / b \ll 1\) is not valid when \(b\) approaches zero. This fact is obvious if one recalls that the standard coherent states of a harmonic oscillator, where the fluctuation \(\Delta x\) is a constant and hence \(\Delta x / x \ll 1\) is not valid when \(x\) approaches zero. Therefore, the assumption that the semiclassicality of the model is maintained in the large scale limit indicates a cyclic universe driven by the quantum fluctuations. This inference is in all probability as viewed from the parameter space of \(r(\phi)\). This is an amazing possibility that quantum gravity manifests itself in the large scale cosmology, which has never been realized before.

We summarize with a few remarks: (i) The main calculational result is the effective Hamiltonian \(H_{\text{eff}}\) incorporating the next to leading order quantum corrections, which are derived by strict approximation schemes rather than certain simplified treatments such as in [12, 17, 18]. The approximation schemes can also be straightforwardly applied to the models with a cosmological constant. (ii) For the scenarios of the cyclic universe, the expectation value, infinitesimal Ehrenfest and small fluctuation properties of the shadow coherent state \(|\phi\rangle\) with respect to \(\hat{b}\) and \(\hat{v}\) are all maintained at both the quantum bounce and collapse points. Thus the universe could be semiclassical all the way and present its consistency. (iii) People used to think that quantum gravity could only take effect at small (Planck) scale. While the quantum bounce looks quite natural, one may suspect how quantum effect can change the large scale behavior of the universe. The intuitive picture that we gained from this model is the following. As the universe expands unboundedly, the matter density would become so tiny that its effect could be comparable to that of quantum fluctuations of the spacetime geometry. Then the Hamiltonian constraint may force the universe to contract back. (iv) We calculate perturbatively the effect stemming from a nonperturbative theory of quantum cosmology. The result indicates that the subleading terms in the effective Hamiltonian \(H_{\text{eff}}\) cannot be arbitrarily neglected in the approximation procedure. However, our qualitative result would not change even if corrections to higher orders are incorporated. Our analysis does not include spatially anisotropic or inhomogeneous quantum gravity fluctuations. Whether such fluctuations could have a role in the inferred effect would be an interesting question. (v) Caveats may arise from our effective approach. Our confidence arise from the fact that the Planck scale quantum bounce predicted by the effective Friedmann equation \(\ddot{r} = -4\pi G \rho_m(\dot{r}^2 + \ddot{r}^2)\) has been confirmed by the numerical simulation in the full quantum difference-differential system of this model \(\mathcal{L}\) and the fact that the effective Hamiltonian is more accurate for large volumes and late times. Nevertheless, the condition that the semiclassicality is maintained in the large scale limit has not been confirmed. Hence further numerical and analytic investigations to the properties of dynamical semiclassical states in the model are desirable. It should be noted that in a different treatment of LQC (see [12]), the dynamical coherent states and complete effective equations are obtained, where \(r(\phi)\) approaches \(1\) in the large scale limit. While that treatment leads to a quantum dynamics different from ours, it raises a caveat to the inferred re-collapse.

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