AN EQUATION LINKING \( \mathscr{W} \)-ENTROPY WITH REDUCED VOLUME

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Abstract. \( \mathscr{W} \)-entropy and reduced volume for Ricci flow were introduced by Perelman, which had proved their importance in the study of Ricci flow. Lei Ni studied the analogous concepts for heat equation on static manifolds, and proved \( \lim_{t \to \infty} \mathscr{W} = \ln \left( \lim_{t \to \infty} \bar{V} \right) \), which links the large time behavior of these two. Due to the surprising similarity between those concepts in the Ricci flow and the linear heat equation, a natural question whether such equation holds for the Ricci flow ancient solution was asked by Lei Ni. In this note, we gave an alternative proof to Lei Ni’s equation based on a new method. And following the same philosophy of this method, we answer Lei Ni’s question positively for type I \( \kappa \)-solutions of Ricci flow.

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1. Introduction

In the celebrated paper [19], Perelman introduced \( \mathscr{W} \)-entropy and reduced volume for the Ricci flow, which turn out to be of fundamental importance in the study of Ricci flow.

In [16], [17] and [18], Lei Ni studied the similar entropy for the linear heat equation and the reduced volume for the static metric. More concretely, let \((M^n, g)\) be a complete Riemannian manifold with \(Rc \geq 0\) and maximum volume growth, namely \(\theta_\infty = \lim_{r \to \infty} \frac{V(x)}{\omega_n r^n} > 0\), where \(\omega_n\) is the volume of the unit ball of \(\mathbb{R}^n\) and \(V(x)\) denotes the volume of the ball \(B(x, r)\). He defined Nash entropy and \( \mathscr{W} \)-entropy for linear heat equation as the following:

\[
N(H, t) = \int_{M^n} (fH) d\mu - \frac{n}{2}, \quad \mathscr{W}(f, t) = \int_{M^n} \left(t|\nabla f|^2 + f - n\right)H d\mu
\]

where \(H(x, y, t) = (4\pi t)^{-\frac{n}{2}} e^{-f(x, y, t)}\) and \(H\) is the heat kernel on \((M^n, g)\). The monotonicity of \(N(H, t)\) and \( \mathscr{W}(f, t) \) was proved, and the more interesting thing is the following equation:

\[
\lim_{t \to \infty} \mathscr{W}(f, t) = \lim_{t \to \infty} N(H, t) = \ln \theta_\infty
\]

(1.1)

It is easy to show \(\theta_\infty = \lim_{t \to \infty} \bar{V}(g, t)\), where \(\bar{V}(g, t) = \int_{M^n} (4\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(x, y)}{4t}} dx\) (see Section 8.1 of [5]). \(\bar{V}(g, t)\) is called the reduced volume for the static metric \(g\).

To prove (1.1), Lei Ni used the sharp pointwise bounds for the heat kernel proved by Li, Tam and Wang (see [14]), which is closely related the large time behavior of heat kernel studied by Li in [13].

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On the other side, as observed by T. H. Colding, \( \lim_{t \to \infty} t \frac{\partial}{\partial t} N(H, t) = 0 \) can follow from the cone structure at infinity of the manifold (see Section 5 of [8]). Also note \( \mathcal{W}(f, t) = t \frac{\partial}{\partial t} N(H, t) + N(H, t) \), this motivated our alternate proof of (1.1) in Section 2 following Colding’s observation.

The key point of our proof of (1.1) is that we only need the uniform (not sharp) Gaussian bounds of heat kernel, which were proved by Li and Yau in [15]. In our case, uniform means the coefficients in the bounds do not depend on time \( t \). Using such uniform bounds of heat kernel, we can reduce the proof of (1.1) to the tangent cone at infinity of manifold by sort of “Dominated Convergence Theorem”. From the work of Cheeger-Colding the tangent cone is a metric cone \( C(X) \) (see [4]), and the explicit formula of heat kernel on \( C(X) \) was given by Ding in [9]. (1.1) on \( C(X) \) follows from these facts explicitly.

Due to the surprising similarity between the entropy formula for the Ricci flow and the entropy formula for the linear heat equation, the following question was asked by Lei Ni in [17]:

**Question 1.1.** Let \( (M^n, g(t))_{t \in [0, \infty)} \) be a non-flat backward Ricci flow solution with bounded nonnegative curvature operator \( Rm \), and \( g(t) \) is \( \kappa \)-noncollapsed at all scales (see definition of \( \kappa \)-noncollapsed in Section 4 of [19]). One may ask if

\[
\lim_{t \to \infty} \mathcal{W}(g, f(x, y, t), t) = \lim_{t \to \infty} N(g, H(x, y, t), t) = \ln \hat{V}_\infty(y, 0)?
\]

where \( H(x, y, t) \) is the conjugate heat kernel, \( \mathcal{W}(g, f, t) \) is define in (4.1), \( N(g, H, t) \) is defined in (4.17) and \( \hat{V}_\infty(y, 0) \) is defined in (4.8).

If we try to prove (1.2) imitating Ni’s method mentioned above, we need to get a sharp bound of the conjugate heat kernel in the Ricci flow case. But it is much harder to get the sharp Gaussian bounds than the uniform Gaussian bounds for the conjugate heat kernel in the Ricci flow case, and we do not know the sharp bounds so far.

On the other hand, by modifying the argument of Cao and Zhang in [2], we succeed in getting the uniform Gaussian bounds of the conjugate heat kernel \( H(x, y, t) \) for a large class of ancient solutions, where the coefficients of the bounds do not depend on time \( t \). We believe that such uniform Gaussian bound has its own interest.

Also from Perelman’s result, the asymptotic backward limit of \( \kappa \)-noncollapsed ancient solution with bounded nonnegative curvature operator is shrinking soliton (see [19]). When \( (M^n, g(t))_{t \in (0, \infty)} \) are shrinking soliton solutions to backward Ricci flow and \( g(t) \) converges in the Gromov-Hausdorff sense as \( t \to 0 \) to a metric cone \( C \) which is smooth except at the vertex, the above equation (1.2) was proved by Cao, Hamilton and Ilmanen (see Section 3 of [11]). In the Ricci flow case, these provide us the analogue of Cheeger, Colding and Ding’s results about heat equations on static manifolds.

Following the similar strategy as the linear heat equation case, we get our main theorem as the following:

**Theorem 1.2.** Let \( (M^n, g(t))_{t \in [0, \infty)} \) be a non-flat Type I \( \kappa \)-solution to the backward Ricci flow for some \( \kappa > 0 \), and \( Rm(x, t) \geq 0 \) for all \( (x, t) \in M^n \times [0, \infty) \). Then for
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$y \in M^n$,

$$\lim_{t \to \infty} \mathcal{W}(g, f(x, y, t), t) = \lim_{t \to \infty} N(g, H(x, y, t), t) = \ln \hat{V}_\infty(y, 0)$$

The definition of Type I $\kappa$-solution is given in Section 3 (see the definition 3.1 and 3.2). Because shrinking soliton solutions are obviously type I ancient solutions, our theorem can be thought as a kind of generalization of the result of Cao, Hamilton and Ilmanen in Section 3 of [1].

The paper is organized as follows: In Section 2, we give the alternative proof of (1.1), which is described above. We prove the uniform (but not sharp) Gaussian bounds for conjugate heat kernel in Section 3, which is crucial for the later results. In Section 4, by similar argument as in Section 3 of [1], (1.2) is firstly proved on shrinking soliton solutions. Then we use the uniform Gaussian bound got in Section 3 to reduce the proof of Theorem 1.2 about ancient solutions to shrinking soliton solutions.

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2. $\mathcal{W}$-ENTROPY AND REDUCED VOLUME FOR LINEAR HEAT EQUATION

In this section, $(M^n, g)$ is a complete Riemannian manifold with $Rc \geq 0$ and maximum volume growth, namely $\theta_\infty = \lim_{r \to \infty} \frac{\hat{V}(r)}{\omega_n r^n} > 0$.

For any increasing sequence $\{t_i\}$ with $\lim_{i \to \infty} t_i = \infty$, from Gromov’s compactness theorem (see [11]), there exists a subsequence, also denoted as $\{t_i\}$, such that

$$\lim_{i \to \infty} (M^n, g_{t_i}, y) \to (M_\infty, y)$$

where $g_i = t_i^{-1}g$, $M_\infty$ is some length space with measure $d\mu_\infty$ and the convergence is in the pointed Gromov-Hausdorff sense. From Theorem 7.6 of [11], $M_\infty$ is a metric cone, denoted as $C(X)$, where $X$ is a compact length space with measure $d\mu$.

Define $H_i = H_i(x, y, s) = \frac{1}{t_i^2} H(x, y, t_i s)$, note that $H_i$ is a positive fundamental solution of heat equation on $(M^n, g_i)$. From Theorem 5.54 and Theorem 6.20 of [9], we get that for any $t > 0$,

$$\lim_{i \to \infty} H_i(x, y, t) = H_\infty(x, y, t)$$

the convergence is uniform in $C^0$-topology and also in $L^1$. And from (6.23) of [9], we have

$$H_\infty(x, y, t) = (4\pi t)^{-\frac{n}{2}} e^{-f_\infty}, \quad f_\infty(x, y, t) = \frac{r^2}{4t} + \ln \left(\frac{V(X)}{n\omega_n}\right)$$

where $r = d(x, y)$, $V(X)$ is the volume of $X$, and it is easy to get

$$\frac{V(X)}{n\omega_n} = \frac{V_{M_\infty}(B_\infty(y, 1))}{\omega_n} = \lim_{i \to \infty} \frac{V_{M_i}(B(y, t_i))}{\omega_n t_i^n} = \theta_\infty$$
From (2.3) and the above,

\[ f_\infty = \frac{r^2}{4t} + \ln \theta_\infty \]  

(2.4)

On \( M_\infty = C(X) \), we have the following lemma.

**Lemma 2.1.**

\[ \int_{M_\infty} (f_\infty H_\infty)(x, y, 1) \, d\mu_\infty(x) = \frac{n}{2} + \ln \theta_\infty \]

**Proof:** From (2.4),

\[ \left( \int_{M_\infty} f_\infty H_\infty \right)(1) = \int_{C(X)} \left[ \frac{r^2}{4} + \ln \theta_\infty \right] H_\infty(1) = \ln \theta_\infty + \int_{C(X)} \frac{r^2}{4} \cdot H_\infty(1) \]

On the other side from (2.3),

\[ \int_{C(X)} \frac{r^2}{4} \cdot H_\infty = \int_0^\infty \int_x \left[ \frac{r^2}{4} \cdot (4\pi)^{-\frac{n}{2}} \exp \left( - \frac{r^2}{4} \right) \right] \, d\mu \, dr = (n\omega_n \pi) \cdot (4\pi)^{-\frac{n}{2}} \int_0^\infty r^{p+1} \exp \left( - \frac{r^2}{4} \right) \, dr \]

where \( \Gamma \) is the Gamma function.

By all the above, the conclusion is proved. \( \square \)

**Proposition 2.2.**

\[ \lim_{t \to \infty} \mathcal{N}(f, t) = \lim_{t \to \infty} N(H, t) = \ln \theta_\infty \]

**Proof:** We firstly show that \( \lim_{t \to \infty} N(H, t) = \ln \theta_\infty \). From [15],

\[ \frac{C^{-1}(n)}{V_y(\sqrt{t})} \exp \left( - \frac{d^2(x, y)}{3t} \right) \leq H(x, y, t) \leq \frac{C(n)}{V_y(\sqrt{t})} \exp \left( - \frac{d^2(x, y)}{5t} \right) \]

Recall \( \lim_{t \to \infty} \frac{V_y(t)}{\omega_n t^{\frac{n}{2}}} = \theta_\infty > 0 \) and the volume comparison theorem, from (2.5) we have

\[ C^{-1}(n) \exp \left( - \frac{d^2(x, y)}{3t} \right) \leq t^2 H \leq C(n) \exp \left( - \frac{d^2(x, y)}{5t} \right) \]

Hence for any \( b \geq 1 \),

\[ \left| \int_{M^n \setminus B(y, b \sqrt{t})} (fH)(x, y, t) \, d\mu_\infty(x) \right| \leq \int_{M^n \setminus B(y, b \sqrt{t})} \left| \log [(4\pi t)^{\frac{n}{2}} H] \right| \cdot H \]

(2.7)

\[ \leq \int_{M^n \setminus B(y, b \sqrt{t})} \left( C + \frac{d^2(x, y)}{3t} \right) \cdot H(x, y, t) \, d\mu_\infty(x) \]

where \( C = C(n) \), and in the last inequality above we used (2.6). In the rest of the proof \( C = C(n) \) if not specifically mentioned,
Using (2.5) and do integration by parts, we obtain that
\[
\int_{M^n \setminus B(y, b \sqrt{t})} \left( C + \frac{d^2(x,y)}{3t} \right) \cdot H(x,y,t) \, d\mu_g(x) \leq C \int_{b \sqrt{t} \setminus \{y\}} \exp \left( - \frac{r^2}{5t} \right) (C + \frac{r^2}{t}) \, dV_y(r)
\]
\[
\leq C \int_{b \sqrt{t} \setminus \{y\}} \frac{V_y(r)}{V_y(\sqrt{t})} \exp \left( - \frac{r^2}{5t} \right) \left( \frac{C}{r} + \frac{r^2}{5t} - 1 \right) d\left( \frac{r^2}{t} \right)
\]
(2.8)
in the last equality above the volume comparison theorem is used.

It is easy to get

\[
\int_{b^2} s^2 \left( \frac{S}{S} + C \right) e^{-\frac{s^2}{10}} \, ds \leq C \int_{b^2} e^{-\frac{s^2}{10}} \, ds \leq C \cdot e^{-\frac{b^2}{10}}
\]
(2.9)

From (2.7), (2.8) and (2.9), it follows that

\[
\left| \int_{M^n \setminus B(y, b \sqrt{t})} (fH)(x,y,t) \, d\mu_g(x) \right| \leq C \cdot \exp \left( - \frac{1}{10} b^2 \right)
\]
(2.10)

Then

\[
\lim_{i \to \infty} \left( \int_{M^n} fH \right)(t_i) = \lim_{i \to \infty} \left( \int_{M^n \setminus B(y, b \sqrt{t})} fH \right)(t_i) + \lim_{i \to \infty} \left( \int_{B_{t_i}(y,b)} f_i \mu_{i} \right)
\]
\[
\geq -C \exp \left( - \frac{1}{10} b^2 \right) + \left( \int_{B_{t_i}(y,b)} f_\infty \mu_{\infty} \right)
\]
(1)

where \( f_i \) is defined by \( H_i(s) = (4\pi s)^{-\frac{n}{2}} \exp \left( - f_i \right) \), and in the last inequality we used (2.10) and (2.2). Let \( b \to \infty \) in the above, we deduce

\[
\lim_{i \to \infty} \left( \int_{M^n} fH \right)(t_i) \geq \left( \int_{M^n} f_\infty \mu_{\infty} \right)
\]
(1)

On the other hand, similarly using (2.10) and (2.2), we arrive at

\[
\lim_{i \to \infty} \left( \int_{M^n} fH \right)(t_i) \leq \left( \int_{M^n} f_\infty \mu_{\infty} \right)
\]
(1)

By all the above and Lemma 2.1

\[
\lim_{i \to \infty} \left( \int_{M^n} fH \right)(t_i) = \left( \int_{M^n} f_\infty \mu_{\infty} \right) = \frac{n}{2} + \ln \theta_\infty
\]
(1)

We know that \( \theta_\infty \) is independent of the choice of the sequence \( \{t_i\} \), hence

\[
\lim_{i \to \infty} \left( \int_{M^n} fH \right)(t) = \frac{n}{2} + \ln \theta_\infty
\]
and it is equivalent to

\[
\text{lim}_{t \to \infty} N(H,t) = \ln \theta_\infty
\]
(2.11)
From (2.11), we can get that $|N(H, 2t) - N(H, t)| \leq \epsilon$ for $t \gg 1$. This implies that there exists $\{t_i\}$ such that $t_i \frac{\partial}{\partial t} N(H, t_i) \to 0$ as $t_i \to \infty$. From the monotonicity of $\mathcal{W}(f, t)$ shown in [16] and (2.11), we finally get

$$\lim_{t \to \infty} \mathcal{W}(f, t) = \lim_{i \to \infty} \mathcal{W}(f, t_i) = \lim_{i \to \infty} \left[ t_i \frac{\partial}{\partial t} N(H, t_i) + N(H, t_i) \right] = \ln \theta_{\infty}$$

The conclusion is proved. \qed

3. Gaussian bounds of the conjugate heat kernel in the Ricci flow

Before we state our main result, we first recall the definition of $\kappa$-solution and type I solution.

**Definition 3.1.** A complete, non-flat backward Ricci flow solution $(M^n, g(t))_{t \in [0, \infty)}$ is a $\kappa$-solution if it is $\kappa$-noncollapsed at all scales for some positive constant $\kappa$.

Note our definition of $\kappa$-solution is different from the definition in [19], although we use the same definition of $\kappa$-noncollapsed as in Section 4 of [19].

**Definition 3.2.** A $\kappa$-solution on $[0, \infty)$ is called Type I if there exists a positive constant $C_1$ such that $|Rm(x, t)| \leq \frac{C_1}{1+t}$ for any $t \in [0, \infty)$.

In the rest of this section, we assume that $(M^n, g(t))_{t \in [0, \infty)}$ is a non-flat Type I $\kappa$-solution to the backward Ricci flow for some $\kappa > 0$ and $|Rm(x, t)| \leq \frac{C_1}{1+t}$. Let $H(x, y, t)$ be the fundamental solution to

$$\frac{\partial}{\partial t} H(x, y, t) = \Delta_x H(x, y, t) - R(x, t)H(x, y, t), \quad (x, t) \in M^n \times (0, \infty)$$

where $y \in M^n$ is fixed, $\Delta_x$ is the Laplacian operator with respect to $x$ and $g(t)$.

We will use the following result due to Cao and Zhang repeatedly.

**Lemma 3.3** (Lemma 4.1 of [2]).

$$C^{-1}t^{-\frac{n}{2}} \leq H(x, y, t) \leq Ct^{-\frac{n}{2}}$$

where $C = C(C_1, n, \kappa)$ and $(x, y, t) \in M^n \times M^n \times (0, \infty)$.

**Lemma 3.4.**

$$|\nabla Rm(x, t)| \leq Ct^{-\frac{n}{2}}, \quad \text{for } (x, t) \in M^n \times (0, \infty)$$

where $C = C(C_1, n)$.

**Proof:** Fix $T > 0$, define $\tilde{g}(t) = g(2T - t)$, then

$$|Rm|_{\tilde{g}(t)} \leq \frac{C_1}{1 + (2T - t)} \leq \frac{C_1}{T}, \quad \text{for } t \in [0, T]$$

We use Shi’s global derivative estimates Theorem 14.5 in [6], choose $\alpha = C_1$, $K = \frac{C_1}{T}$ there, we have

$$|Rm|_{\tilde{g}(t)} \leq K, \quad \text{for } t \in [0, \frac{\alpha}{K}]$$
hence there exists $C = C(n, \max\{\alpha, 1\}) = C(C_1, n)$, such that

$$|\nabla Rm|_{\bar{g}(t)} \leq \frac{C}{\sqrt{t}}, \quad \text{for} \quad t \in (0, \frac{\alpha}{K}] = (0, T]$$

Let $t = T$, we get

$$|\nabla Rm|_{\bar{g}(T)} = |\nabla Rm|_{\bar{g}(T)} \leq \frac{C}{T^\frac{3}{2}}$$

Because $T > 0$ is chosen freely, the lemma is proved. 

**Proposition 3.5.**

(3.3) \[ \left| \frac{\nabla}{H} \right|_{(x, y, t)} \leq \frac{C}{\sqrt{t}} \]

where $\nabla$ is with respect to $x$, and $C = C(C_1, n, \kappa)$.

**Proof:** Fix $T > 0$, define $\bar{g}(t) = g(\frac{T}{2} + t)$, use Lemma 3.4 we get

$$|\nabla Rm|_{\bar{g}(t)} \leq \frac{C}{(t + \frac{T}{2})^\frac{3}{2}} \leq \left( \frac{C}{T} \right)^\frac{3}{2}, \quad t \in [0, \frac{T}{2}]$$

Define $\bar{H}(x, y, t) = H(x, y, t + \frac{T}{2})$, then $\bar{H}(t)$ is the solution to (3.1) with respect to metric $\bar{g}(t)$. From Lemma 3.3

$$C^{-t^{-\frac{3}{2}}} \leq \bar{H}(x, y, t) \leq CT^{-\frac{3}{2}}, \quad t \in [0, \frac{T}{2}]$$

where $C = C(n, C_1, \kappa)$.

Using $Rc \geq -\frac{c}{1 + t}$ and Theorem 16.52 in [6] for $(M^n, \bar{g}(t))$, $t \in [0, \frac{T}{2}]$, we get

$$\left| \frac{\nabla}{H} \right|_{\bar{g}(t)}^2 (x, y, t) \leq \left[ 1 + \ln \left( \frac{C^{-t^{-\frac{3}{2}}}}{H} \right) \right] \left( \frac{C}{t} + 2C \frac{T}{T} \right)$$

where $\nabla$ is with respect to $x$. Let $t = \frac{T}{2}$ in the above inequality, we get

$$\left| \frac{\nabla}{H} \right|_{\bar{g}(t)}^2 (x, y, T) = \left| \frac{\nabla}{H} \right|_{\bar{g}(t)}^2 (x, y, \frac{T}{2}) \leq \frac{C}{T}$$

where $C = C(C_1, n, \kappa)$. Because $T > 0$ is chosen freely, the conclusion is proved. 

**Corollary 3.6.**

(3.4) \[ H(x, y, t) \leq H(z, y, t) \exp \left( \frac{Cd(x, z, t)}{\sqrt{t}} \right) \]

where $C = C(C_1, n, \kappa)$.

**Proof:** Fix $x$ and $z$, then integrate (3.3) along a minimal geodesic connecting $x$ and $z$. 

**Theorem 3.7.** There exist positive constants $\Lambda_i = \Lambda_i(C_1, n, \kappa), i = 1, 2, 3, 4$, such that

(3.5) \[ \frac{\Lambda_1}{V_{\bar{g}(t)}(B(x, \sqrt{t}, t))} \exp \left( -\frac{\Lambda_2 d^2}{t} \right) \leq H \leq \frac{\Lambda_3}{V_{\bar{g}(t)}(B(x, \sqrt{t}, t))} \exp \left( -\frac{\Lambda_4 d^2}{t} \right) \]
where $B(x, r, t) = \{ z | d_{g(t)}(z, x) \leq r \}$ for any $r > 0$, $H = H(x, y, t)$, $d = d(x, y, t)$ and $(x, y, t) \in M^n \times M^n \times (0, \infty)$. 

**Remark 3.8.** We use Grigor’yan’s method (see [10]) as the proof of Theorem 3.1 in [2], but we estimate the conjugate heat kernel directly, which is different from Cao and Zhang’s strategy in [2].

**Proof.** **Step 1.** Pick a weight function $\epsilon^{(x,t)}$ which will be specified later. Using integration by parts, we can get

$$
\frac{\partial}{\partial t} \int_{M^n} H^2(x, y, t)e^{\epsilon^{(x,t)}}d\mu_{g(t)}(x) = \int_{M^n} H^2e^{\epsilon^{(x,t)}}\xi_t + (2H\Delta H - RH^2)e^{\epsilon^{(x,t)}}
$$

(3.6)

Because $M^n$ can be non-compact, one needs to justify integration by parts near infinity in (3.6). For fixed $t$, $H(x, y, t)$ has a generic Gaussian upper bound with coefficients depending on $t$, curvature tensor and their derivatives, as shown in [3]. Since the curvatures are all bounded, from Proposition 3.5 and volume comparison theorem, the term $\int_{\partial B(r)} \nabla H e^{\epsilon^{(x,t)}} \to 0$ as $r \to \infty$, this justifies the integration by parts in (3.6).

We choose

$$
\xi_t = \left\{ \begin{array}{ll}
\frac{(t-d(x,y,t))^2}{4(s_0-t)} , & d(x, y, t) \leq t
\\
0 , & d(x, y, t) > t
\end{array} \right.
$$

where $t > 0$ and $s_0 > t > 0$.

Then for $x \in B(y, t, t)$, we have

$$
\xi_t + \frac{1}{2} |\nabla \xi|^2 = \frac{1}{8} \left( \frac{t - d}{s_0 - t} \right)^2 + \frac{1}{2} \frac{t - d}{s_0 - t} \frac{\partial d(x, y, t)}{\partial t}
$$

By Lemma 8.3 (b) of [19] and $|\text{Rc}(z, t)| \leq \frac{C}{t^{1+\gamma}}$ for any $z \in M^n$, where $C = C(C_1, n)$, we get

$$
\frac{\partial}{\partial t}d(x, y, t) \leq 2(n-1)\left( \frac{C}{2} \frac{1}{1+t} r_0 + r_0^{-1} \right), \text{ for any } r_0 > 0
$$

we can choose $r_0 = \sqrt{1 + t}$, then $\frac{\partial d}{\partial t} \leq \frac{C}{\sqrt{1+t}}$, where $C = C(C_1, n)$. It follows that

$$
\xi_t + \frac{1}{2} |\nabla \xi|^2 \leq -\frac{1}{8} \left( \frac{t - d}{s_0 - t} \right)^2 + \frac{1}{2} \frac{t - d}{s_0 - t} \frac{C}{\sqrt{1 + t}}
$$

$$
= \frac{1}{8} \left( \frac{t - d}{s_0 - t} \right)^2 + 1 + \frac{C^2}{2} \frac{1}{1 + t} \leq \frac{C}{1 + t}
$$

Combining with $\text{Rc} \geq -\frac{C}{1+t}$, from (3.6) we get

$$
\frac{\partial}{\partial t} \left( \int_{M^n} H^2 e^{\epsilon^{(x,t)}} \right) \leq \frac{C_2}{1+t} \left( \int_{M^n} H^2 e^{\epsilon^{(x,t)}} \right)
$$

where $C_2 = C_2(C_1, n)$. Hence

$$
\int_{M^n} H^2 e^{\epsilon^{(x,t)}}(s_1) \leq \left( \int_{M^n} H^2 e^{\epsilon^{(x,t)}} \right)(s_2) \left( \frac{1 + s_1}{1 + s_2} \right)^{C_2}
$$

(3.7)
where \( s_0, s_1, s_2 \) are any positive constants satisfying \( s_0 > s_1 > s_2 > 0 \).

**Step 2.** Define

\[
I_i(t) = \int_{M^n \setminus B(y,t)} H^2(x, y, t) d\mu_{g(t)}(x)
\]

Choose any \( 0 < \rho < \iota \), using (3.7), we have

\[
I_i(s_1) = \int_{M^n \setminus B(y,t)} H^2(x, y, s_1) \leq \int_{M^n} H^2(x, y, s_1) e^{\xi(x,s_1)}
\]

\[
\leq \left[ \int_{M^n} H^2 e^{\xi(s_2)} \right] \left( \frac{1 + s_1}{1 + s_2} \right)^C_2
\]

\[
\leq I_p(s_2) + \exp\left( -\frac{(\iota - \rho)^2}{4(s_0 - s_2)} \right) \int_{B(y,s_2)} H^2(s_2) \left( \frac{1 + s_1}{1 + s_2} \right)^C_2
\]

\[
\leq I_p(s_2) + C_3 s_2^{-\frac{\xi}{2}} \exp\left( -\frac{(\iota - \rho)^2}{4(s_0 - s_2)} \right) \left( \frac{1 + s_1}{1 + s_2} \right)^C_2
\]

where \( C_3 = C(C_1, n, \kappa) \), and in the last inequality we used Lemma 3.3 and \( \int_{M^n} H \equiv 1 \).

Let \( s_0 \to s_1 \) in the above inequality,

\[
I_i(s_1) \leq \left[ I_p(s_2) + C_3 s_2^{-\frac{\xi}{2}} \exp\left( -\frac{(\iota - \rho)^2}{4(s_0 - s_2)} \right) \left( \frac{1 + s_1}{1 + s_2} \right)^C_2
\]

Note (3.8) holds for any \( s_1 > s_2 > 0 \), \( \iota > \rho > 0 \).

Now we define

\[
t_k = t a^{-k}, \quad r_k = \left( \frac{1}{2} + \frac{1}{k + 2} \right) r, \quad k = 0, 1, 2, \ldots
\]

where \( a > 1 \) is a constant to be chosen later. Let \( s_1 = t_k, s_2 = t_{k+1}, \iota = r_k \) and \( \rho = r_{k+1} \), applying (3.8), we can get

\[
I_r(t_k) \leq \left[ I_{r_k+1}(t_{k+1}) + C_3 t_{k+1}^{-\frac{\xi}{2}} \exp\left( -\frac{(r_{k+1} - r_k)^2}{4(t_{k} - t_{k+1})} \right) \right] \cdot \left( \frac{1 + t_k}{1 + t_{k+1}} \right)^C_2
\]

After applying iteration to (3.9), we obtain

\[
I_r(t) = I_r(t_0) \leq \left( \frac{1 + t_0}{1 + t_{k+1}} \right)^C_2 \cdot I_{r_k+1}(t_{k+1}) + C_3 t_{k+1}^{-\frac{\xi}{2}} \sum_{j=1}^{k+1} a^{\left( \frac{j}{2} + C_2 \right)n} \exp\left( -\frac{r^2 a^j}{4(j + 3)^4(a - 1)t} \right)
\]

When \( k \to \infty, t_k \to 0 \) and \( H(x, y, t_k) \to \delta_y(x) \) which is concentrated at the point \( y \). Hence \( \lim_{k \to \infty} I_r(t_k) = 0 \). Let \( k \to \infty \) in (3.10), we get

\[
I_r(t) \leq C_3 t_{k+1}^{-\frac{\xi}{2}} \sum_{j=1}^{\infty} a^{\left( \frac{j}{2} + C_2 \right)n} \exp\left( -\frac{r^2 a^j}{4(j + 3)^4(a - 1)t} \right)
\]

By taking \( r^2 \geq \frac{4}{\xi} t \) and making the constant \( a \) sufficiently large, it leads to

\[
\int_{M^n \setminus B(y,t)} H^2(t) = I_r(t) \leq C_4 t^{-\frac{\xi}{2}} \exp\left( -\frac{C_5 r^2}{t} \right)
\]

where \( C_4 = C_4(C_1, n, \kappa) \) and \( C_5 = C_5(C_1, n) \).
Using Lemma 3.3 we can get

\[ \int_{M \setminus B(y, r, t)} H(x, y, t) \leq C_6 \exp \left( -\frac{C_5 r^2}{t} \right) \]  

(3.11)

where \( C_6 = C_6(C_1, n, \kappa) \) and \( r \geq \frac{1}{2} \sqrt{t} \).

**Step 3.** Let \( x_0 \in M^n \), there are two cases.

If \( d(x_0, y, t) \geq \sqrt{t} \), then \( B(x_0, \frac{1}{2} \sqrt{t}, t) \subset M^n \setminus B(y, r, t) \), where \( r = \frac{1}{2}d(x_0, y, t) \).

Then from (3.11), there exists \( z_0 \in B(x_0, \frac{1}{2} \sqrt{t}, t) \) such that

\[ H(z_0, y, t)V_{g(t)}(B(x_0, \frac{1}{2} \sqrt{t}, t)) \leq C_6 \exp \left( -\frac{C_5 r^2}{t} \right) \]

(3.12)

Note the following fact

\[ \frac{V_K(r_1)}{V_K(r_2)} \leq \left( \frac{r_1}{r_2} \right)^n \exp \left( \sqrt{K} r_1 \right) \]

where \( V_K(r_i) \) denotes the volume of a ball of radius \( r_i \) in the constant curvature \(-K\) \( n \)-dimensional space form, and \( K > 0 \).

From the above fact, by the classical volume comparison theorem and \( Rc \geq -\frac{1}{2\sqrt{t}} \), these imply that

\[ H(z_0, y, t) \leq C_7 \left[ V_{g(t)}(B(x_0, \sqrt{t}, t)) \right]^{-1} \exp \left( -\frac{C_8 d^2(x_0, y, t)}{t} \right) \]

(3.13)

where \( C_7 = C_7(C_1, n, \kappa) \) and \( C_8 = C_8(C_1, n) \).

By Corollary 3.6 using \( d(x_0, z_0, t) \leq \frac{1}{2} \sqrt{t} \) we have

\[ H(x_0, y, t) \leq H(z_0, y, t) \cdot \exp \left( \frac{C_9 d(x_0, z_0, t)}{\sqrt{t}} \right) \]

(3.14)

\[ \leq C_{10} \left[ V_{g(t)}(B(x_0, \sqrt{t}, t)) \right]^{-1} \exp \left( -\frac{C_{10} d^2(x_0, y, t)}{t} \right) \]

where \( C_{10} = C_{10}(C_1, n, \kappa) \).

If \( d(x_0, y, t) \leq \sqrt{t} \), by Lemma 3.3 using the volume comparison theorem and \( Rc \geq -\frac{1}{2\sqrt{t}} \) again, there exists \( C_{11} = C_{11}(C_{10}, C_8, n) = C_{11}(C_1, n, \kappa) \) such that

\[ H(x_0, y, t) \leq C_{11} \left[ V_{g(t)}(B(x_0, \sqrt{t}, t)) \right]^{-1} \exp \left( -\frac{C_{11} d^2(x_0, y, t)}{t} \right) \]

(3.15)

Define \( C_{12} = \max \{ C_{10}, C_{11} \} \), then from (3.14) and (3.15) we get

\[ H(x_0, y, t) \leq C_{12} \left[ V_{g(t)}(B(x_0, \sqrt{t}, t)) \right]^{-1} \exp \left( -\frac{C_{12} d^2(x_0, y, t)}{t} \right) \]

Since \( x_0 \) is arbitrary, this proves the desired upper bound.

**Step 4.** Next we show that a lower bound follows from the upper bound. From (3.11), we get

\[ \int_{B(y, \sqrt{t}, t)} H \geq 1 - C_6 \exp \left( -\frac{C_5 b^2}{t} \right) \]

(3.16)
where \( b \geq 1 \) is a constant to be chosen later. Hence there exists \( x_1 \in B(y, b \sqrt{t}, t) \) such that
\[
H(x_1, y, t) \geq \left[ V_{g(t)}(B(y, b \sqrt{t}, t)) \right]^{-1} \left[ 1 - C_6 \exp \left( -\frac{C_5 b^2}{\sqrt{t}} \right) \right]
\]
For any \( x_2 \in M^n \), from Corollary 3.6
\[
H(x_2, y, t) \geq H(x_1, y, t) \exp \left( -\frac{C_9 d(x_1, x_2, t)}{\sqrt{t}} \right)
\]
(3.17) \[
\geq \left[ V_{g(t)}(B(y, b \sqrt{t}, t)) \right]^{-1} \left[ 1 - C_6 \exp \left( -\frac{C_5 b^2}{\sqrt{t}} \right) \right] \exp \left( -\frac{C_9 d(x_1, x_2, t)}{\sqrt{t}} \right)
\]
Note
\[
d^2(x_1, x_2, t) \leq \left[ d(x_1, y, t) + d(x_2, y, t) \right]^2 \leq 2b^2 t + 2d^2(x_2, y, t)
\]
then
(3.18) \[
\frac{d(x_1, x_2, t)}{\sqrt{t}} \leq \frac{d^2(x_1, x_2, t)}{t} + 1 \leq \frac{2d^2(x_2, y, t)}{t} + (2b^2 + 1)
\]
From (3.12), combining with the volume comparison theorem and \( Rc \geq -\frac{C}{1+t} \), we can get
\[
V_{g(t)}(B(y, b \sqrt{t}, t)) \leq V_{g(t)}(B(x_2, d(x_2, y, t) + b \sqrt{t}, t))
\]
(3.19) \[
\leq C_{13} \exp \left( \frac{C_{14} d(x_2, y, t)}{\sqrt{t}} \right) \cdot V_{g(t)}(B(x_2, \sqrt{t}, t)) \cdot \left( \frac{d(x_2, y, t) + b \sqrt{t}}{\sqrt{t}} \right)^n
\]
where \( C_{13} = C_{13}(C_1, n, b) \) and \( C_{14} = C_{14}(C_1, n) \).

Choose the constant \( b \) large enough such that \( 1 - C_6 \exp \left( -\frac{C_5 b^2}{\sqrt{t}} \right) \geq \frac{1}{2} \), then from (3.17), (3.18) and (3.19),
\[
H(x_2, y, t) \geq C_{15} \left[ V_{g(t)}(B(x_2, \sqrt{t}, t)) \right]^{-1} \cdot \left( \frac{d(x_2, y, t) + b \sqrt{t}}{\sqrt{t}} \right)^n \exp \left( -\frac{C_{16} d^2(x_2, y, t)}{t} \right)
\]
\[
\geq C_{17} \left[ V_{g(t)}(B(x_2, \sqrt{t}, t)) \right]^{-1} \cdot \exp \left( -\frac{C_{18} d^2(x_2, y, t)}{t} \right)
\]
Since \( x_2 \) is arbitrary, this is a lower bound which matches the upper bound except for constant coefficients \( \Lambda_i \).

\( \square \)

4. THE LIMIT OF REDUCED VOLUME AND \( \mathcal{W} \)-ENTROPY FOR THE RICCI FLOW

In this section, \((M^n, g(t))_{t \in [0, \infty)}\) is a non-flat Type I \( \kappa \)-solution to the backward Ricci flow for some \( \kappa > 0 \), and the curvature operator \( Rm \) is nonnegative. \( H(x, y, t) \) is the fundamental solution to (3.1), \( y \in M^n \) is a fixed point.

By the \( \kappa \)-noncollapsed assumption and curvature bound \( |Rm(\cdot, t)| \leq \frac{C_1}{1+t} \), it follows from Hamilton’s compactness theorem (see [12]): for any increasing sequence \( \{t_i\} \) with \( \lim_{i \to \infty} t_i = \infty \), there exist a subsequence, also denoted as \( \{t_i\} \), such that the following statement holds:

The pointed manifolds \((M^n, g_i(s), y)\) with metrics \( g_i(s) \equiv t_i^{-1} g(t_i, s) \) converges to a pointed manifold \((M_\infty, g_\infty(s), y)\) in \( C^\infty_{\text{loc}} \)-topology, where \( s \in (0, \infty) \).
It was shown in [2] that \((M^n, g_\infty(s))\) is a gradient shrinking soliton. For completeness and later use, we give the details here following the argument in [2].

Define \(H_i \equiv H_i(x, y, s) \equiv i^2 H(x, y, tf_i)\). By Lemma 3.3 for fixed \(s > 0\), there exists a uniform positive constant \(U_0\), such that \(H_i(x, y, s) \leq U_0\) for all \(i = 1, 2, \ldots\) and \(x \in M^n\). Note that \(H_i\) is a positive fundamental solution of the conjugate heat equation on \((M^n, g_i(s))\), i.e.,

\[
\frac{\partial}{\partial t} H_i = \Delta_{g_i} H_i - R_{g_i} H_i
\]

For any compact time interval in \((0, \infty)\), \(H_i\) are uniformly bounded, moreover, \(R_{g_i}\) and \(Rm_{g_i}\) are uniformly bounded. It follows from the standard parabolic theory that \(H_i\) is Hölder continuous uniformly with respect to \(g_i\). Hence there exist a subsequence, still denoted as \([H_i(x, y, s)]\), which converges to a \(C^\alpha_{loc}\)-topology sense.

It is easy to see that \(H_\infty\) is a weak solution of the conjugate heat equation on \((M^n, g_\infty(s))\). By standard parabolic theory and the boundedness of \(H_\infty\) on compact time interval, \(H_\infty\) is a smooth solution of the conjugate heat equation on \((M^n, g_\infty(s))_{s \in (0, \infty)}\).

By Lemma 3.3 \(H(x, y, tf_i) \geq \frac{C}{(t_i s)^{\frac{\alpha}{2}}}\), which is equivalent to \(H_i(x, y, s) \geq Cs^{-\frac{\alpha}{2}}\).

Hence \(H_\infty(s) \geq Cs^{-\frac{\alpha}{2}} > 0\), it yields that \(H_\infty\) is positive everywhere when \(s > 0\).

Now for each \(H\), Perelman’s \(\mathcal{W}\)-entropy is defined as

\[
\mathcal{W}(g, f, t) = \int_{M^n} \left( |\nabla f|^2 + R - n \right) H d\mu_{g(t)}(x)
\]

where \(H = (4\pi t)^{-\frac{\alpha}{2}} e^{-f}\).

We define

\[
\mathcal{W}(f, s) = \mathcal{W}(g_i, f_i, s) = \int_{M^n} \left[ s |\nabla f_i|^2 + R_i - f_i - n \right] H_i d\mu_{g_i(s)}(x)
\]

where \(H_i(x, y, s) = (4\pi s)^{-\frac{\alpha}{2}} e^{-f_i(x, y, s)}\), and \(R_i\) is the scalar curvature with respect to \(g_i\).

Because \(M^n\) may be noncompact, one needs to justify that the integral \(\mathcal{W}(f, s)\) is finite. This can be deduced from Lemma 3.3 Proposition 3.3 and \(|Rm(\cdot, t)| \leq \frac{C}{t^n}\) easily.

By \(Rm \geq 0\), for fixed \(s > 0\),

\[
\mathcal{W}(f, s) \geq \left( \int_{M^n} f_i H_i \right) - n = -n - \int_{M^n} \ln \left[ (4\pi)^{\frac{\alpha}{2}} s^\frac{\alpha}{2} H_i(s) \right] H_i
\]

\[
\geq \int_{M^n} C \cdot H_i - n = C
\]

(4.2)

where \(C\) is independent of \(i\), in the last inequality we used Lemma 3.3.

Recall that \(\mathcal{W}\) is invariant under proper scaling,

\[
\mathcal{W}(f, s) = \mathcal{W}(g_i, f_i, s) = \mathcal{W}(g, f, t_i s)
\]
According [19], for fixed $s > 0$, $\mathcal{W}_i(s) = \mathcal{W}(g, f, t_i s)$ is a non-increasing sequence of $i$. By (4.2), there exists a function $\mathcal{W}_\infty(s)$ such that
\[
\mathcal{W}_\infty(s) \doteq \lim_{i \to \infty} \mathcal{W}_i(s) = \lim_{i \to \infty} \mathcal{W}(g, f, t_i s)
\]
Note that $\mathcal{W}_\infty(s)$ is independent of the choice of $\{t_i\}$ by the monotonicity of $\mathcal{W}(g, f, s)$.

For any fixed $s_0 \in (0, \infty)$, we can find a subsequence $\{t_{m_i}\}_{i=1}^{\infty}$ tending to infinity such that
\[
\mathcal{W}(g, f, t_{m_i} s_0) \geq \mathcal{W}(g, f, t_{m_i} (s_0 + 1)) \geq \mathcal{W}(g, f, t_{m_i+1} s_0)
\]
Since $\lim_{i \to \infty} \mathcal{W}(g, f, t_i s_0) = \lim_{i \to \infty} \mathcal{W}(g, f, t_{m_i+1} s_0) = \mathcal{W}_\infty(s_0)$, we get
\[
\lim_{i \to \infty} \left[ \mathcal{W}(g, f, t_i s_0) - \mathcal{W}(g, f, t_i (s_0 + 1)) \right] = 0
\]
That is
\[
\text{(4.3)} \quad \lim_{i \to \infty} \left[ \mathcal{W}_{m_i}(s_0) - \mathcal{W}_{m_i}(s_0 + 1) \right] = 0
\]
According [19],
\[
\frac{d}{ds} \mathcal{W}_k(s) = -2 \int_{M^n} \left| R_{g_k} + Hess_{g_k} f_k - \frac{1}{2s} g_k \right|^2 H_k d\mu_{g_k}(s)
\]
Integrate it from $s_0$ to $s_0 + 1$, we use (4.3) to conclude that
\[
\lim_{k \to \infty} \int_{s_0}^{s_0 + 1} \int_{M^n} \left| R_{g_{m_k}} + Hess_{g_{m_k}} f_{m_k} - \frac{1}{2s} g_{m_k} \right|^2 H_{m_k} d\mu_{g_{m_k}}(s) = 0
\]
Therefore we have
\[
\text{(4.4)} \quad R_{g_{\infty}(s)} + Hess_{g_{\infty}(s)} f_{\infty} = \frac{1}{2s} g_{\infty}
\]
for $s \in (s_0, s_0 + 1)$, where $f_{\infty}$ is defined by $H_{\infty} = (4\pi s)^{-\frac{1}{2}} e^{-f_{\infty}}$. Because $s_0$ is arbitrary positive number, (4.4) holds for any $s \in (0, \infty)$. So $(M_\infty, g_\infty(s))_{s \in (0, \infty)}$ is a gradient shrinking soliton solution to backward Ricci flow.

Define $\ell_i(q, \theta) = \ell^\infty(q, \theta)$, where $\ell^\infty(q, \theta)$ is the reduced distance with base point $(y, 0)$ with respect to the backward Ricci flow solution $(M^n, g_{i}(t))_{t \in [0, \infty)}$.

**Lemma 4.1.**
\[
\text{(4.5)} \quad \lim_{i \to \infty} \ell_i(q, \theta) \doteq \ell_\infty(q, \theta)
\]
exists in the Cheeger-Gromov sense on $M_\infty \times (0, \infty)$. And $\ell_\infty(q, \theta)$ is the reduced distance of backward Ricci flow $(M_\infty, g_\infty(s))_{s \in (0, \infty)}$ with the base point $(y, 0)$.

**Remark 4.2.** Note $(M_\infty, g_\infty(0))$ is the unique tangent cone at infinity of $(M^n, g(0))$, and it is a Euclidean metric cone in fact. Although $(M_\infty, g_\infty(0))$ is not a smooth manifold, from the definition of reduced distance and $(M_\infty, g_\infty(s))_{s \in (0, \infty)}$ are shrinking soliton solutions, the reduced distance of $(M_\infty, g_\infty(s))_{s \in (0, \infty)}$ with base point $(y, 0)$ is still well defined.
Proof: Because \((M_\infty, g_\infty(s))_{s \in (0, \infty)}\) is a shrinking soliton solution, \(R(y, s) \leq \frac{C}{s}\). Then it is easy to get \(\ell(y, s) \leq C\) from the definition of reduced distance. The similar argument as in the proof of Lemma 8.35 and 8.36 in [5] leads to (4.5). Combining the definition of reduced distance and (4.5), we get that \(\ell_\infty(q, \theta)\) is the reduced distance of \((M_\infty, g_\infty(s))_{s \in (0, \infty)}\) with the base point \((y, 0)\). \(\Box\)

Lemma 4.3.

\[
(\nabla \ell_\infty(q, 1))^2 + R_{g_\infty(1)}(q) = \ell_\infty(q, 1)
\]

and

\[
\ell_\infty(q, 1) = f_\infty(q, 1) + \beta
\]

where \(\beta\) is a constant.

Proof: By (4.4) and Lemma 4.1, from the argument in section 7.3 of [5], the conclusion follows. \(\Box\)

The limit of the reduced volume is defined as

\[
\hat{V}_\infty(y, 0) = \lim_{t \to \infty} \bar{V}_{(y, 0)}(t), \quad \bar{V}_{(y, 0)}(t) = \int_{M_\infty} (4\pi t)^{-\frac{n}{2}} e^{-\ell(q, t)} d\mu_{g(t)}(q)
\]

where \(\ell(q, t)\) is the reduced distance with base-point \((y, 0)\).

Proposition 4.4.

\[
\hat{V}_\infty(y, 0) = e^{-\beta}
\]

Remark 4.5. From (4.9), we know that \(\beta\) is independent of the choice of sequence \(\{t_i\}_{i=1}^\infty\).

Proof: From Lemma 4.3, we can use Lemma 8.38 of [5] and (4.7) to conclude

\[
\hat{V}_\infty(y, 0) = \int_{M_\infty} (4\pi)^{-\frac{n}{2}} e^{-\ell_\infty(q, 1)} d\mu_{g_\infty(1)}(q)
\]

\[
= e^{-\beta} \int_{M_\infty} H_\infty(q, y, 1) d\mu_{g_\infty(1)}(q)
\]

From (3.16) and the definitions of \(H_i, g_i\), we obtain

\[
\int_{B_{g_i(y, b, 1)}} H_i(x, y, 1) d\mu_{g_i(1)}(x) \geq 1 - C \exp(-Cb^2)
\]

where \(B_{g_i}(y, b, 1) = \{x | d_{g_i(1)}(x, y) < b, \ x \in M^n\}, b \geq 1\) and \(C = C(C_1, n, \kappa)\).

Note we have

\[
\int_{B_{g_\infty}(y, 2b, 1)} H_\infty(x, y, 1) d\mu_{g_\infty(1)}(x) \geq \lim_{i \to \infty} \int_{B_{g_i}(y, b, 1)} H_i \]

From (4.11) and (4.12),

\[
\int_{B_{g_\infty}(y, 2b, 1)} H_\infty \geq 1 - C \exp(-Cb^2)
\]
Let $b \to \infty$ in the above inequality, we get
\[
\int_{M_\infty} H_\infty(x, y, 1) \, d\mu_{g_\infty(1)}(x) \geq 1
\]

On the other side, we get \( \int_{M_\infty} H_\infty(x, y, 1) \, d\mu_{g_\infty(1)}(x) \leq 1 \) from Fatou’s lemma. Hence
\[
(4.13) \quad \int_{M_\infty} H_\infty(x, y, 1) \, d\mu_{g_\infty(1)}(x) = 1
\]

From (4.10) and (4.13), we get our conclusion. \( \square \)

Define \( \nu(t) = \left[ t(2\Delta f - |\nabla f|^2 + R) + f - n \right] H(t) \) and
\[
\nu_\infty(x, y, 1) = \lim_{i \to \infty} \nu(t_i)_{t^2} = \left[ 2\Delta_{g_\infty(1)} f_\infty - |\nabla f_\infty|^2_{g_\infty(1)} + R_{g_\infty(1)} + f_\infty - n \right] H_\infty(x, y, 1)
\]

We do not know whether \( \nu_\infty(x, y, 1) \) is independent on the choice of sequence \( \{t_i\} \), but we have the following Lemma.

**Lemma 4.6.**
\[
(4.14) \quad \int_{M_\infty} \nu_\infty(1) \, d\mu_{g_\infty(1)} = -\beta, \quad \int_{M_\infty} f_\infty H_\infty(1) \, d\mu_{g_\infty(1)} = \frac{n}{2} - \beta
\]

**Proof:** Do integration by parts (it is easy to justify integration by parts near infinity using the results in Section 3), then use \( \Delta_{g_\infty(1)} f_\infty + R_{\infty} = \frac{n}{2} \), we can get
\[
(4.15) \quad \int_{M_\infty} \nu_\infty = \int_{M_\infty} \left[ \Delta_{\infty} f_\infty + R_{\infty} + f_\infty - n \right] H_\infty = \int_{M_\infty} \left[ f_\infty - \frac{n}{2} \right] H_\infty = \left( \int_{M_\infty} f_\infty H_\infty \right) - \frac{n}{2}
\]

On the other hand, do integration by parts, from (4.6) and (4.7), we get
\[
(4.16) \quad \int_{M_\infty} \nu_\infty = \int_{M_\infty} \left[ |\nabla f_\infty|^2 + R_{\infty} + f_\infty - n \right] H_\infty = \int_{M_\infty} \left[ |\nabla f_\infty|^2 + R_{\infty} + f_\infty - n \right] H_\infty
\]
\[
= \int_{M_\infty} \left[ 2f_\infty + \beta - n \right] H_\infty = 2\left( \int_{M_\infty} f_\infty H_\infty \right) + (\beta - n)
\]

From (4.15) and (4.16), we get our conclusion. \( \square \)

We define Nash entropy and Fisher information for Ricci flow imitating the linear heat equation case in [17].

\[
(4.17) \quad N(g, H, t) = \left( \int_{M^n} f H \right)(t) - \frac{n}{2}, \quad F(g, H, t) = \frac{\partial}{\partial t} N(g, H, t)
\]

From (4.11), it is easy to check that
\[
(4.18) \quad \mathcal{W}(g, f, t) = \left( \int_{M^n} \nu \right)(t) = t F(g, H, t) + N(g, H, t)
\]
Proposition 4.7.

\[ \lim_{t \to \infty} \mathcal{W}(g, f, t) = \lim_{t \to \infty} N(g, H, t) = -\beta \]

Proof: We firstly show \( \lim_{t \to \infty} N(g, H, t) = -\beta \).

\[ \left| \int_{M^n \setminus (y, b \sqrt{t}, t)} (fH)(x, y, t)d\mu_{g(t)}(x) \right| \leq \int_{M^n \setminus (y, b \sqrt{t}, t)} \left| \ln \left( (4\pi t)^{\frac{3}{2}} H \right) \right| H \]

(4.19)

where we used the bound of \( t^2 H \) in Lemma 3.3 and (4.11), \( b \geq \frac{1}{2} \) is a constant and \( C = C(C_1, n, \kappa) \) is independent of \( b \) and \( t \).

For any sequence \( \{t_i\} \) as in the beginning of this section, we can find a subsequence also denoted as \( \{t_i\} \), such that \( g_i \) converges to \( g_\infty \) in the Cheeger-Gromov sense, and \( H_{t_i} \) converges to \( H_\infty \) on \( M_\infty \) in \( C^{\infty}_{\text{loc}} \) topology.

Then using (4.19) and the related convergence, we can get

\[ \lim_{i \to \infty} \left( \int_{M^n} fH(t_i) \right) = \lim_{i \to \infty} \left( \int_{M^n \setminus (y, b \sqrt{t_i}, t_i)} fH(t_i) + \lim_{i \to \infty} \left( \int_{B(y, b \sqrt{t_i}, t_i)} fH(t_i) \right) \right) \]

\[ \geq -C \exp\left(\beta - CB^2\right) + \lim_{i \to \infty} \left( \int_{B(y, b \sqrt{t_i}, t_i)} (fH)_{g_i(1)} \right) \]

\[ = -C \exp\left(\beta - CB^2\right) + \int_{B(y, b \sqrt{t_i}, t_i)} (f_{g_\infty}H_{g_\infty})_{g_\infty(1)} \]

Let \( b \to \infty \) in the above, we get

\[ \lim_{i \to \infty} \int_{M^n} fH(t_i) \geq \int_{M_\infty} f_{g_\infty}H_{g_\infty}(x, y, 1)d\mu_{g_\infty(1)}(x) \]

On the other hand, similarly using (4.19) and the convergence, we have

\[ \lim_{i \to \infty} \left( \int_{M^n} fH(t_i) \right) \leq \int_{M_\infty} f_{g_\infty}H_{g_\infty}(x, y, 1)d\mu_{g_\infty(1)}(x) \]

By all the above and Lemma 4.6 we get

\[ \lim_{i \to \infty} \left( \int_{M^n} fH(t_i) \right) = \int_{M_\infty} f_{g_\infty}H_{g_\infty}(x, y, 1)d\mu_{g_\infty(1)}(x) = \frac{n}{2} - \beta \]

From Proposition 4.3 we know that \( \beta \) is independent of the choice of \( \{t_i\} \), hence

\[ \lim_{i \to \infty} \left( \int_{M^n} fH(t_i) \right) = \frac{n}{2} - \beta \]

it is equivalent to

(4.20) \[ \lim_{t \to \infty} N(g, H, t) = -\beta \]

From (4.20), we can get that \( |N(g, H, 2t) - N(g, H, t)| \leq \varepsilon \) for \( t \gg 1 \). This implies that there exists the sequence \( \{t_i\} \) such that \( t_iF(H, t_i) \to 0 \) as \( t_i \to \infty \). Hence from (4.18),

\[ \lim_{t \to \infty} \mathcal{W}(g, f, t) = \lim_{t \to \infty} \mathcal{W}(g, f, t_i) = \lim_{t \to \infty} \left[ t_iF(g, H, t_i) + N(g, H, t_i) \right] = -\beta \]
Combining Proposition 4.4 with Proposition 4.7, Theorem 1.2 is proved.

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