QCD condensates and the pion wave functions in the nonlocal chiral model

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We use the simple instanton motivated Nambu Jona-Lasinio - type model to calculate a twist 3 pseudo-scalar pion light cone wave function $\psi_{PS}^{\pi}$. Using the normalization condition for $\psi_{PS}^{\pi}$ we calculate the quark condensate and also the gluon condensate, which agree with the phenomenological values for these quantities. Since we can compute also the $k_T^2$ dependence of the light cone wave functions, we calculate $k_T^2$ moments of the pseudo-scalar and axial-vector wave functions which are related to the mixed vacuum condensates. This allows us to extract the condensates and compare them with existing estimates.

I. INTRODUCTION

Soft pion theorems provide link between dynamical objects like the light cone wave functions [1–4] and static properties of the physical vacuum [5,6] (for review see [7]). Customarily in QCD pion properties are expressed in terms of the nonperturbative condensates, which are estimated by some theoretical model, like the QCD sum rules [8] (see for example [9]), or the instanton model of the QCD vacuum [10,11] (for recent review see e.g. [12]). Here we propose to invert this logic. In Ref. [13] we have discussed a simple and tractable model, which describes very well the leading twist pion wave function $\psi_{AV}^{\pi}$ not only in longitudinal variable $u$, but also in $k_T$. In this paper we calculate the twist 3 pion wave function $\psi_{PS}^{\pi}$ [9,14–16] and try to extract from both $\psi_{AV}^{\pi}$ and $\psi_{PS}^{\pi}$ the quark condensate and mixed condensates of dimension 5 and 7, and the gluon condensate as well. The model, which we use is a simple nonlocal chiral pion-quark model [17,18], which in principle can be obtained from the instanton model of the QCD vacuum [12]. However, since we want to work directly in the Minkowski space, we have to modify the form of the nonlocality [13,17,18] in order to make the calculations feasible. The price is that, of course, we loose the direct contact with the original model. Therefore our aim is twofold. Apart from giving estimates of the condensates we want to perform tests in order to gain confidence in the model as well as to find its limitations.

In most applications of the effective quark-meson theory described above, an approximation of the constant constituent quark mass was used. This approach has an impressive success in describing baryon (which emerge as solitons) properties (see [12] and references therein). Recently an attempt to include the momentum dependence in calculating solitons in the nonlocal NJL model has been reported in Refs. [19].

The instanton model of the QCD vacuum is formulated in the Euclidean space-time, whereas the light cone wave functions are naturally defined in the Minkowski space-time. One can in principle Wick rotate the integrals and perform the calculations in the Euclidean metric, as has been for example done in Refs. [20–22] (for early attempts to calculate the pion wave function in the instanton model see Ref. [23]), however in this work we shall perform all calculations directly in the Minkowski space-time.

The instantons induce both gluon condensate, which is proportional to the density of instantons, and quark condensate, which emerges due to the delocalization of the quark zero modes [12,24]. Therefore both quark and gluon condensation occurs at the same scale $Q_0$, which is associated with the average instanton size $1/Q_0 = 600$ MeV. After integrating out gluons and performing the bosonization, the model reduces to the simple Nambu-Jona Lasinio type model where the quarks interact nonlocally with an external meson field $U$ [12,24]:

$$S_I = \int \frac{d^4k d^4l}{(2\pi)^8} \bar{\psi}(k)\sqrt{M(k)}U^{\gamma_5}(k-l)\sqrt{M(l)}\psi(l)$$ (1)

and $U^{\gamma_5}$ can be expanded in terms of the pion fields:

$$U^{\gamma_5} = 1 + \frac{i}{F_\pi}\gamma_5\tau^A\pi^A - \frac{1}{2F_\pi^2}\pi^A\pi^A + \ldots$$ (2)
Here \( F_2 = 93 \text{ MeV} \) and \( M(k) \) is a momentum dependent constituent quark mass. In principle \( M(k) \) has been calculated in the instanton model in Euclidean space time. Here, following Refs. \([5,8]\), we wish to perform calculations directly in the Minkowski space. To this end we shall choose a simple pole formula \([13]\)

\[
M(k) = M \left( \frac{\Lambda^2}{k^2 - \Lambda^2 + i\epsilon} \right)^{2n}
\]

which for \( n \sim 2 - 3 \) and for \( k^2 < 0 \) reproduces the \( k \) dependence obtained from the instantons reasonably well \([3]\). In order to check model sensitivity to the specific form of the cutoff function \([3]\) we shall vary \( M \) from 325 to 400 MeV and \( n \) from 1 to 5. Notice that \( M(k) \) of Eq.\([3]\) provides an UV cutoff for the loop integrals entering the expressions for the pion wave functions. This is true both in Euclidean and in Minkowski space.

Using this simple prescription we study various condensates, which are related to the normalization and \( k_T^2 \) moments of leading twist axial-vector (AV) and nonleading twist pseudo-scalar (PS) pion light cone wave functions (for definitions see e.g. Ref. \([15,16]\)) as well as the PS wave function itself.

From the point of view of QCD the quantities we calculate depend on a nonperturbative scale \( Q_0 \) which, however, must not be confused neither with the constituent mass \( M \) nor with an auxiliary parameter \( \Lambda \), which is fixed by the normalization condition for \( \psi_{AV}^\psi \). For \( k^2 < 0 \) the Ansatz \([3]\) should imitate \( M(k) \) obtained from the instantons. And for the latter, as explained above, \( Q_0 \sim 1/\pi = 600 \text{ MeV} \). It is therefore natural to assume that \( Q_0 \) is of the order of a few hundred MeV irrespectively of \( M \) and \( \Lambda \). The precise definition of \( Q_0 \) is only possible within QCD and in all effective models one can only use only qualitative order of magnitude arguments to estimate \( Q_0 \). A more practical way to determine \( Q_0 \) will be discussed in Sect. \( \nu \) where we will associate \( Q_0 \) with transverse integration cutoff \( K_T \) which, as we shall see, is of the order of \( 700 < K_T < 1100 \text{ MeV} \).

As a first step we calculate the quark condensate, which enters the normalization of the PS pion light cone wave function. The numerical result depends upon \( M \) and \( n \) and varies between \(-(258 \text{ MeV})^3\) and \(-(328 \text{ MeV})^3\). The commonly used phenomenological value of \( \langle \bar{q}q \rangle \) is lower and reads approximately \(-(250 \text{ MeV})^3\). Therefore our highest value for the quark condensate overshoots the phenomenological value by a factor of 2. The reason for this rather too high values of \( \langle \bar{q}q \rangle \) is a relatively slow convergence of the \( dk_T^2 \) integral for this quantity. This is even more visible in the case of the \( k_T^2 \) moments of the PS wave function. Indeed, one can relate both \( \langle k_T^2 \rangle_{AV} \) and \( \langle k_T^2 \rangle_{PS} \) to the same mixed condensate of dimension 5, \( \langle ig\bar{q}\sigma \cdot Gq \rangle \), which yields \( \langle k_T^2 \rangle_{AV} / \langle k_T^2 \rangle_{PS} = 5/9 \) \([3]\). In our case this ratio is two times smaller. It is, however, interesting to observe, that if we cut the \( dk_T^2 \) at some \( k_T^2_{max} = K_T^2 \), which we choose in such a way that \( \langle \bar{q}q \rangle_{k_T^2_{max}} = -(250 \text{ MeV})^3 \), then the ratio of \( k_T^2 \) is stable and approximately equal to 5/9. The mixed condensate \( \langle ig\bar{q}\sigma \cdot Gq \rangle \) comes out between \(-(427 \text{ MeV})^5\) and \(-(443 \text{ MeV})^5\) in a fair agreement with the estimates from the QCD sum rules Ref. \([25]\). Unfortunately the higher moments of the PS wave function are still too large even with a \( k_T^2 \) cutoff.

One of the advantages of our method is that the analytical expression for the quark condensate is given in terms of a Minkowskian integral, which in a limit of a constant \( M(k) \) and \( k_T^2 \rightarrow k_F^2 \) reduces to the well known Euclidean form. By comparing the two expressions one can by inspection guess a continuation prescription, which allows to rewrite certain Euclidean integrals as the Minkowskian ones. We use this, in some respect \( ad \text{ hoc} \), prescription to calculate the gluon condensate with very encouraging result: \( \langle \alpha/\pi GG \rangle = (378 \text{ MeV})^4 \) to \( (383 \text{ MeV})^4 \).

Finally we plot the PS pion light cone wave function for our set of model parameters. We find that \( \phi_{PS}^\psi(u) \) vanishes at the end points \( u = 0 \) and \( u = 1 \) except for the case of \( n = 1 \). The latter behavior is in agreement with the results obtained in Ref. \([14]\) within the QCD sum rules. It is interesting to observe, that vanishing of \( \phi_{PS}^\psi(u) \) at the end points is correlated with the fact that \( \phi_{AV}^\psi(u) \) is concave at \( u \sim 1 \) and \( u \sim 0 \) for \( n > 1 \).

The paper is organized as follows: in Section \[1\] we shortly summarize the results of Refs. \([3,8]\) concerning the relation of the \( k_T^2 \) moments of the pion wave functions to the mixed quark-gluon condensates. In Sect. \[11\], following Ref. \([13]\), we explain how to evaluate the loop integrals with the momentum dependent quark mass \( M(k) \) given by Eq.\([3]\). In Sect. \[\nu\] we construct the continuation prescription discussed above and calculate the gluon condensate. In Sect. \[\nu\] we give the numerical results and Sect. \[\nu\] contains summary and conclusions. Some of the results presented here have been partially reported in Ref. \([26]\).
II. PION WAVE FUNCTIONS AND THEIR RELATION TO THE VACUUM CONDENSATES

We shall be dealing with the leading twist axial-vector (AV) and twist 3 pseudo-scalar (PS) wave functions defined as follows \cite{13,15}:

\[
\phi_{\pi}^{AV}(u) = \frac{1}{i\sqrt{2F_{\pi}}} \int_{-\infty}^{\infty} \frac{d\tau}{\pi} e^{-i(2u-1)(nP)} \langle 0 | \bar{\psi}(n\tau) \gamma_5 \psi(-n\tau) | \pi^+(P) \rangle,
\]
\[
\phi_{\pi}^{PS}(u) = -(nP) \frac{F_{\pi}}{\sqrt{2} \langle \bar{q}q \rangle} \int_{-\infty}^{\infty} \frac{d\tau}{\pi} e^{-i(2u-1)(nP)} \langle 0 | \bar{\psi}(n\tau) i\gamma_5 \psi(-n\tau) | \pi^+(P) \rangle. \tag{4}
\]

where we have chosen \( n = (1,0,0,-1) \) as a light-cone vector parallel to \( z_\mu = \tau n_\mu \) and \( \bar{n} = (1,0,0,1) \) parallel to \( P_\mu \). For any 4-vector \( v \) we have:

\[
v^+ = n \cdot v, \quad v^- = \bar{n} \cdot v, \quad v_\mu = \frac{v^+}{2} n_\mu + \frac{v^- - \bar{n} \cdot v}{2} n_\mu + \frac{v_\perp}{2}.
\]

Both wave functions are normalized to 1. The normalization condition for \( \phi_{\pi}^{PS} \) yields therefore the expression for \( \langle \bar{q}q \rangle \). Evaluation of the norm of \( \phi_{\pi}^{PS} \) in the present model gives:

\[
\langle \bar{q}q \rangle = -4iN_c \int \frac{d^4k}{(2\pi)^4} \frac{M(k)M(k-P)k(k-P)-M(k)M(k-P)}{(k^2-M^2)(k^2-(k-P)^2-M^2(k-P))}. \tag{6}
\]

For \( P \to 0 \) in the local case \( M(k) = M \) and in Euclidean metric Eq.(5) reduces to

\[
\langle \bar{q}q \rangle = -4iN_c \int \frac{d^4k}{(2\pi)^4} \frac{M}{k^2-M^2} = -4N_c \int \frac{d^4k_F}{(2\pi)^4} \frac{M}{k_F^2 + M^2}. \tag{7}
\]

The last expression in (6) is well defined only in Euclidean metric and Eq.(6) may serve as a prescription how it should be continued to the Minkowski space. In fact, as we shall explicitly see, \( \langle \bar{q}q \rangle \) of Eq.(6) does not depend on \( P \).

Our approach enables to calculate not only the distributions in \( u \) but in fact full wave functions \( \psi_{\pi} \) both in \( u \) and \( k_T^2 \):

\[
\phi_{\pi}(u) = \int_{0}^{\infty} dk_T^2 \psi_{\pi}(u,k_T^2) \tag{8}
\]

both for AV and PS channels. Integrating first over \( u \) gives the \( k_T^2 \) distribution

\[
\bar{\phi}_{\pi}(k_T^2) = \int_{0}^{1} du \psi_{\pi}(u,k_T^2). \tag{9}
\]

It has been shown in Refs. \cite{13,15} that moments of \( \bar{\phi}_{\pi}(k_T^2) \) are given in terms of the mixed quark-gluon condensates

\[
\langle k_T^2 \rangle_{AV} = \frac{5}{36} \frac{\langle ig \bar{q} \gamma \cdot G q \rangle}{\langle \bar{q}q \rangle}, \quad \langle k_T^2 \rangle_{PS} = \frac{1}{4} \frac{\langle ig \bar{q} \gamma \cdot G q \rangle}{\langle \bar{q}q \rangle}. \tag{10}
\]

Here \( G_{\mu\nu}^a \) is a gluon field strength and

\[
\sigma \cdot G = \sigma_{\mu\nu} G^{\mu\nu}, \quad G_{\mu\nu} = \frac{\lambda_a}{2} G_{\mu\nu}^a.
\]

Note that formula for \( \langle k_T^2 \rangle_{AV} \) is an approximate one since the soft pion theorems used to derive \cite{10} apply strictly only in the pseudo scalar channel. Equations (11) predict

\[
\langle k_T^2 \rangle_{AV} = \frac{5}{9} \langle k_T^2 \rangle_{PS}. \tag{12}
\]
For \( \langle k_T^4 \rangle \) we have:

\[
\langle k_T^4 \rangle_{AV} = -\frac{3}{32} \frac{\langle g^2 \bar{\psi} \sigma \cdot G \sigma \cdot G \psi \rangle}{\langle \psi \bar{\psi} \rangle} + \frac{13}{72} \frac{\langle g^2 \bar{\psi} G^2 \psi \rangle}{\langle \psi \bar{\psi} \rangle},
\]

\[
\langle k_T^4 \rangle_{PS} = -\frac{1}{12} \frac{\langle g^2 \bar{\psi} \sigma \cdot G \sigma \cdot G \psi \rangle}{\langle \psi \bar{\psi} \rangle} + \frac{1}{6} \frac{\langle g^2 \bar{\psi} G^2 \psi \rangle}{\langle \psi \bar{\psi} \rangle},
\]

(13)

where for \( \langle k_T^4 \rangle_{AV} \), the same reservations as for \( \langle k_T^4 \rangle_{AV} \) hold. Unfortunately equations (13) are almost linearly dependent. Indeed, coefficients in front of the condensates are of the order of 1/10, whereas the determinant of (13) is of the order \( 6 \times 10^{-4} \). Therefore it is difficult to disentangle the condensates from the knowledge of \( \langle k_T^4 \rangle \), unless the latter are known with very high accuracy.

III. PION WAVE FUNCTION WITH MOMENTUM DEPENDENT CONSTITUENT QUARK MASS

In our previous paper [13] we have shown how to handle the loop integral, like the one in Eq. (6), with momentum dependent mass \( M(k) \) given by Eq. (3). To this end one introduces the light cone integration variables

\[ d^4k = \frac{1}{2} dk^+ dk^- d^2k_T, \quad k^+ = uP^+. \]

(14)

The key point of this analysis consists in finding the integration contour in the complex \( k^- \) plane. The poles of the fermion propagator are given by the zeros of the following function:

\[ k^2 - M^2 \left[ \frac{\Lambda^2}{k^2 - \Lambda^2 + i\epsilon} \right]^{4n} + i\epsilon = 0 \]

which after pulling out the factor \( \Lambda^2 [k^2/\Lambda^2 - 1 + i\epsilon]^{-4n} \) reduces to

\[
\left( \frac{k^2}{\Lambda^2} - 1 + i\epsilon \right)^{4n+1} + \left( \frac{\mu^2}{\Lambda^2} - 1 + i\epsilon \right)^{4n} - \frac{M^2}{\Lambda^2} = 0
\]

(15)

and similarly for \( k \to k - P \). Here

\[ k^2 = uP^+ k^- - \vec{k}_T^2 \quad \text{and} \quad (k - P)^2 = -(1 - u)P^+ k^- - \vec{k}_T^2 \]

(16)

Eq. (15) should be understood as an equation for \( k^- \).

Generally an equation of the form

\[ z^{4n+1} + z^{4n} - \mu^2 = 0 \]

(17)

with \( \mu^2 = M^2/\Lambda^2 \) has \( 4n + 1 \) complex solutions, which in the following will be denoted as \( z_i \) where \( i = 1, \ldots, 4n + 1 \).

These solutions depend on the specific value of \( \mu^2 \) and have to be calculated numerically. Then the poles of the quark propagators in \( k^- \) can be found from the following relations:

\[
\frac{k^2}{\Lambda^2} - 1 + i\epsilon = z_i,
\]

(18)

\[
\frac{(P - k)^2}{\Lambda^2} - 1 + i\epsilon = z_i.
\]

(19)

We have shown in Ref. [13] that all the poles corresponding to Eq. (18) should lie below the integration contour whereas the poles corresponding to Eq. (19) above. As a result the \( dk^- \) integrals yield real wave functions, which vanish for \( u \) outside the region \( 0 < u < 1 \). Moreover for \( \Lambda \to \infty \), i.e. for a constant \( M(k) \), this prescription reduces in a continuous way to the standard one of Feynman.

With this prescription the calculations are rather straightforward and we obtain:

\[
\psi_{\pi}^{AV}(u, k_T^2) = \frac{1}{\Lambda^2} \frac{N_c M^2}{(2\pi)^2 F_\pi^2} \sum_{i,k=1}^{4n+1} f_i f_k \frac{z_i^{n+3n} u + z_k^{n+3n} (1 - u)}{t + 1 + z_i u + z_k (1 - u)},
\]

(20)
\[ \psi_{\pi}^{PS}(u, k_T^2) = \frac{N_c M}{(2\pi)^2 \langle \bar{q}q \rangle} \sum_{i,k=1}^{4n+1} f_i f_k \frac{z_i^3 n + z_k^3 n (1 + z_i z_k) - \mu^2 z_i^2 z_k^2}{t + 1 + u z_i + (1 - u) z_k} \]  

(21)

(where \( t = k_T^2 / \Lambda^2 \)). Factors \( f_i \)

\[ f_i = \prod_{k=1 \atop k \neq i}^{4n+1} \frac{1}{z_i - z_k} \]  

(22)

obey the following properties

\[ \sum_{i=1}^{4n+1} z_i^m f_i = \begin{cases} 0 & \text{for } m < 4n \\ 1 & \text{for } m = 4n \end{cases} \]  

(23)

which are crucial for the convergence of the \( dt \) integrals. The \( dt \) integration gives:

\[ \phi_{\pi}^{AV}(u) = \frac{N_c M^2}{(2\pi)^2 F_\pi^2} \sum_{i,k} f_i f_k \left[ z_i^3 n + z_k^3 n (1 - u) \right] \ln \left( 1 + z_i u + z_k (1 - u) \right) \]  

(24)

and

\[ \phi_{\pi}^{PS}(u) = -\frac{N_c \Lambda^2}{(2\pi)^2 \langle \bar{q}q \rangle} \sum_{i,k} f_i f_k \left[ z_i^3 n + z_k^3 n (1 + \frac{z_i + z_k}{2}) - \mu^2 z_i^2 z_k^2 \right] \ln \left( 1 + u z_i + (1 - u) z_k \right). \]  

(25)

The normalization condition for \( \phi_{\pi}^{AV} \) is used to fix the value of the cutoff parameter \( \Lambda \) for given \( n \) with \( F_\pi \) fixed at the physical value of 93 MeV. Then the normalization condition for \( \phi_{\pi}^{PS} \) can be used to calculate the quark condensate.

**IV. GLUON CONDENSATE**

The Euclidean formula for the gluon condensate in the instanton model of the QCD vacuum reads \[12,24\]:

\[ \langle \frac{\alpha}{\pi} GG \rangle = 32 N_c \int \frac{d^4 k_E}{(2\pi)^4} \frac{M^2(k_E)}{k_E^2 + M^2(k_E)}. \]  

(26)

In the instanton model, assuming that the gluon condensate is known, Eq. (26) should be understood as a gap equation for the value \( M = M(0) \), since the shape of the \( k \) dependence is uniquely determined as a Fourier transform of a fermion zero mode in the presence of the instanton configuration. Alternatively, for given \( M \), one can use Eq. (26) to calculate the gluon condensate.

A comparison of equations (7) and (26) for a constant constituent mass \( M \) (assuming some Euclidean cutoff) suggests the following relation between the quark and gluon condensates:

\[ \mathcal{M} \langle \bar{q}q \rangle = -c \langle \frac{\alpha}{\pi} GG \rangle. \]  

(27)

Here \( \mathcal{M} = M \) (constituent quark mass) and \( c = 1/8 \). Let us remind that relation (27) follows from the instanton model of the QCD vacuum in the zero mode approximation and in the chiral limit \( (m = 0) \) [24]. It would have to be substantially modified for heavy current quark mass \( m \) [27,28]. In Ref. [29] a similar relation has been obtained for an arbitrary self-dual gluonic field (in a dilute gas approximation) with, however, \( \mathcal{M} = m \) (current quark mass of a light quark). In contrast to the previous case such a relation is not satisfied phenomenologically and moreover it cannot be true in the chiral limit. On the other hand an expansion for large current quark mass \( m_q \) [30] yields in the leading order relation (27) with \( \mathcal{M} = m_q \) and \( c = 1/12 \) [31]. It has been argued in Ref. [31] that this relation holds also for \( \mathcal{M} = m \) (constituent quark mass). Whether this identification can be theoretically justified is not obvious, since \( M \) being of the order of 300 MeV is probably not large enough to be considered as heavy. It is, however, beyond the scope of this paper to discuss here the differences between the approaches of Refs. [29,31] and [24] (see, however [24]). We simply use Eq. (26) as derived in Ref. [24] and calculate the gluon condensate.
The first estimate of the gluon condensate given already in Ref. [8], (330 MeV)$^4$, is nowadays believed to be too small. In fact different authors advocated different values for this quantity (see e.g. Fig. 4 of Ref. [33] and references therein). For the purpose of this work we adopt more recent estimate from Refs. [34] and [35]:

$$\langle \frac{\alpha}{\pi} GG \rangle = (393^{+29}_{-38} \text{ MeV})^4. \quad (28)$$

In our case we shall simply use Eq. (28) to calculate $\langle \frac{\alpha}{\pi} GG \rangle$ and compare with Eq. (29).

It is tempting to use our experience from the wave function calculation to continue Eq. (26) to the Minkowski space. Indeed equations (26) and (7) differ only by one power of $\sqrt{M(k)M(k-P)}$ (apart from trivial numerical factors). This suggests the following generalization of (26):

$$\langle \frac{\alpha}{\pi} GG \rangle = i32N_c \int \frac{d^4k}{(2\pi)^4} M(k)M(k-P) \frac{k(k-P) - M(k)M(k-P)}{(k^2 - M^2)((k-P)^2 - M^2(k-P))}. \quad (29)$$

For a constant $M$ Eq. (29) can be easily obtained from (26) by first continuing to the Minkowski metric and second by shifting the integration variable by a constant fourvector $P$ (with $P^2 = 0$):

$$\langle \frac{\alpha}{\pi} GG \rangle = i32N_c \int \frac{d^4k}{(2\pi)^4} M^2 = i16N_c \int \frac{d^4k}{(2\pi)^4} \left( \frac{M^2}{k^2 - M^2} + \frac{M^2}{(k-P)^2 - M^2} \right)$$

Accepting Eq. (29) as a continuation from the Euclidean to the Minkowski metric we can calculate the gluon condensate for the cutoff function of Eq. (3):

$$\langle \frac{\alpha}{\pi} GG \rangle = \frac{8N_cM^2\Lambda^2}{(2\pi)^2} \int du \int d\tau \sum_{i,k} f_i f_k \frac{z_i^{2n}z_k^{2n}(1 + \frac{z_i + z_k}{2}) - \mu^2}{t + 1 + u z_i + (1 - u)z_k} \ln(1 + z_i)$$

The $dt$ integration is trivial (note that the large $t$ part vanishes due to the properties of the sums of the $f_i$ factors). Also the $du$ integral is straightforward. Finally we obtain

$$\langle \frac{\alpha}{\pi} GG \rangle = \frac{8N_cM^2\Lambda^2}{(2\pi)^2} \left( \sum_i f_i^2 \left[ z_i^4(1 + z_i) - \mu^2 \right] \ln(1 + z_i) + \sum_{i \neq k} f_i f_k \left[ z_i^{2n}z_k^{2n}(1 + \frac{z_i + z_k}{2}) - \mu^2 \right] \left[ \frac{(1+z_i)\ln(1+z_i) - (1+z_k)\ln(1+z_k)}{z_i - z_k} - 1 \right] \right) \quad (31)$$

![FIG. 1. Pion self-energy $-i\Sigma_\pi(P)$](image)

It is interesting to observe that the Minkowskian integral (29) is in fact proportional to the pion self-energy due to the quark loop calculated in the present effective model. Indeed

$$\Sigma_\pi(P) = -\frac{2N_c}{F_\pi} \int d^4k M(k)M(k-P) \text{Tr} \left[ \gamma_\pi \frac{1}{k - M(k)} \gamma_\pi \frac{1}{(k-P) - M(k-P)} \right] \quad (32)$$

which yields an interesting relation

$$\langle \frac{\alpha}{\pi} GG \rangle = 4F_\pi^2 \Sigma_\pi(P). \quad (33)$$

Let us note that in the nonlinear model which we are dealing with, there exists another loop diagram with two pions coupled to a fermion line at the same point, which cancels out (33) so that the total pion self-energy is zero as it should be.
V. NUMERICAL RESULTS

A. Quark and gluon condensates

Our numerical results are presented in Table I. For 4 different values of the constituent mass $M = M(0)$ and different $n$ (see Eq.(3)) we have adjusted the cutoff parameter $\Lambda$ by imposing the normalization condition on $\phi_{AV}^\pi$. In fact, as discussed at length in Ref. [13] the leading twist pion wave function $\phi_{AV}^\pi(u)$ does not change any more if we increase $n$ above 5. On the other hand for $n \geq 5$ the cutoff function (3), if continued to the Euclidean metric, starts to deviate significantly from the one obtained in the instanton model. Therefore we have chosen to work with $n_{\text{max}} = 5$. From our previous study it seems that the best agreement with the recent analysis of the CLEO data is obtained for $M = 325$ MeV and $n = 2 - 5$ or $M = 350$ MeV and $n = 2$.

In the last two columns of Table I we have displayed the values of the quark and gluon condensates. The gluon condensate is quite insensitive to the parameters of the cutoff function and also to the value of the constituent quark mass $M$. The numerical value being of the order $(380)^4 - (400)^4$ MeV$^4$ agrees perfectly with the phenomenological value. In contrast, the quark condensate is sensitive to the value of $n$ and the numerical value lies between $-(258)^3$ and $-(338)^3$ MeV$^3$. The reason for this sensitivity is relatively broad $k_T^2$ distribution of $\phi_{PS}^\pi(k_T^2)$. We shall come back to this point in Sect. V C.

B. Pseudo-scalar pion wave function

The function $\phi_{PS}^\pi(u)$ has been calculated within the QCD sum rules in Refs. [15,16]. It had a $u-$ shape and did not vanish at the end points. In Fig. 2 we plot our results for $\phi_{PS}^\pi(1)$. It can be seen, that for $n = 1$ $\phi_{PS}^\pi(1) = \phi_{PS}^\pi(0) \neq 0$, whereas for $n \geq 2$ $\phi_{PS}^\pi(1) = \phi_{PS}^\pi(0) = 0$. The end point behavior is governed by the sum

$$
\phi_{PS}^\pi(1) \sim \sum_i f_i \ln(1 + uz_i) \sum_k f_k \left[ \frac{3n+1}{2}z_i^3 - \frac{3n+1}{2}z_k^3 + \frac{1}{2}z_i^3 - \frac{1}{2}z_k^3 - \mu^2 z_i z_k \right]
$$

(34)

which vanishes due to the property (23) except for $n = 1$ where $3n+1 = 4n$.

| $M(0)$ MeV | $n$ | $\Lambda$ MeV | $\langle \bar{q}q \rangle$ MeV$^3$ | $\langle \alpha_s \pi \pi \rangle$ MeV$^4$ |
|-----------|-----|--------------|-----------------|-----------------|
| 325       | 1   | 1249         | $(328)^3$       | $(403)^4$       |
|           | 2   | 1862         | $(294)^3$       | $(396)^4$       |
|           | 5   | 3033         | $(280)^3$       | $(393)^4$       |
| 350       | 1   | 1156         | $(318)^3$       | $(399)^4$       |
|           | 2   | 1727         | $(284)^3$       | $(392)^4$       |
|           | 5   | 2819         | $(271)^3$       | $(389)^4$       |
| 375       | 1   | 1081         | $(309)^3$       | $(395)^4$       |
|           | 2   | 1621         | $(277)^3$       | $(388)^4$       |
|           | 5   | 2649         | $(264)^3$       | $(386)^4$       |
| 400       | 1   | 1020         | $(303)^3$       | $(392)^4$       |
|           | 2   | 1543         | $(271)^3$       | $(386)^4$       |
|           | 5   | 2512         | $(258)^3$       | $(383)^4$       |
FIG. 2. Pseudoscalar pion wave function $\phi_{PS}^P(u)$ for $M = 325$, 350, 375 and 400 MeV, for $n = 1$ (solid), 2 (long dashed) and 5 (dashed).

C. $k_T^2$ dependence and mixed vacuum condensates

The large $k_T^2$ asymptotics of the wave functions and of the appropriately defined $k_T^2$ distribution of the gluon condensate

$$\left\langle \frac{\alpha}{\pi} GG \right\rangle = \int_0^\infty dk_T^2 \tilde{G}(k_T^2)$$

is as follows

$$\tilde{\phi}_\pi^{AV}(k_T^2) \sim \left( \frac{1}{k_T^2} \right)^{4n+1}, \quad \tilde{\phi}_\pi^{PS}(k_T^2) \sim \left( \frac{1}{k_T^2} \right)^{2n}, \quad \tilde{G}(k_T^2) \sim \left( \frac{1}{k_T^2} \right)^4$$

Therefore for example for $n = 1$ already the first moment of $\phi_{PS}^{P}(k_T^2)$ is divergent.

As a kind of remedy for this divergence let us try to introduce a cutoff for $dk_T^2$ integration $T = K_T^2/\Lambda^2$. In Table II we list $k_T^2$ moments, $F_\pi$ and $\left\langle \frac{\alpha}{\pi} GG \right\rangle$ for $T$ chosen in such a way that the quark condensate equals $\left\langle \bar{q}q \right\rangle = -250$ MeV$^3$. Interestingly, by fixing $\left\langle \bar{q}q \right\rangle$ with one parameter $T$ we are able to reproduce ratio (12) with high accuracy for almost all sets of parameters. Unfortunately the ratio $\left\langle k_T^4 \right\rangle_{AV} / \left\langle k_T^4 \right\rangle_{PS}$ is less stable: 0.27 $- 0.43$. Moreover it is more than two times smaller than the one obtained by means of Eqs.(13) in Ref. II.

One should, however, keep in mind that the absolute values of $k_T^2$ moments, especially in the PS channel, depend strongly on $T$. This is illustrated in Table III for $M = 350$ MeV and $n = 2$.

One more remark is here in order. We have not readjusted $\Lambda$ since $F_\pi(T = 0.25) \sim 92$ MeV instead of 93, a negligible change. In Fig.4 we plot $T$ dependence of the ratios $F_\pi(T)/F_\pi$, $\sqrt{\left\langle \bar{q}q \right\rangle (T)} / \sqrt{\left\langle \bar{q}q \right\rangle}$ and $\sqrt{\left\langle \frac{\alpha}{\pi} GG \right\rangle (T)} / \sqrt{\left\langle \frac{\alpha}{\pi} GG \right\rangle}$ for $n = 2$. We see that while $F_\pi$ and gluon condensate saturate relatively fast, the quark condensate (that is $\int dk_T^2 \tilde{\phi}_\pi^{PS}(k_T^2)$) saturates very slowly.
FIG. 3. Dependence on $k_T^2$ of $\phi^{PS}(u)$ for $M = 350$ MeV, for $n = 1$ (solid), 2 (long dashed) and 5 (dashed) and of $\phi^{AV}(u)$ for all $n = 1 \ldots 5$ (dot-dashed)

TABLE II. $F_x$, gluon condensate and $k_T^2$ moments of axial-vector and pseudo-scalar pion wave functions with transverse integration cutoff fixed by requirement that $\langle \bar{q}q \rangle = -(250 \text{ MeV})^3$.

| $M$ | $n$ | $T$ | $K_T$ | $F_x$ | $\sqrt{\langle \frac{1}{3}GG \rangle}$ | $\sqrt{\langle k_T^2 \rangle}$ | $(\langle k_T^2 \rangle)_{AV}^{1/2}$ | $(\langle k_T^2 \rangle)_{PS}^{1/2}$ |
|-----|-----|-----|-------|-------|---------------------------------|----------------|----------------|----------------|
| MeV |     | MeV | MeV   | MeV   | MeV                             | MeV           | MeV           | MeV           |
|-----|-----|-----|-------|-------|---------------------------------|----------------|----------------|----------------|
| 325 | 1   | 0.374 | 764  | 89.5  | 378                             | 356           | 476           | 0.56           | 428           | 529           | 0.43           |
|     | 2   | 0.191 | 813  | 90.7  | 381                             | 370           | 495           | 0.56           | 445           | 554           | 0.42           |
|     | 5   | 0.079 | 852  | 91.4  | 382                             | 378           | 508           | 0.56           | 456           | 570           | 0.41           |
| 350 | 1   | 0.47 | 792  | 90    | 380                             | 363           | 487           | 0.55           | 436           | 544           | 0.41           |
|     | 2   | 0.25 | 864  | 91.6  | 382                             | 378           | 513           | 0.54           | 455           | 577           | 0.39           |
|     | 5   | 0.105 | 913  | 92    | 383                             | 385           | 527           | 0.53           | 465           | 597           | 0.37           |
| 375 | 1   | 0.58 | 823  | 91    | 380                             | 368           | 499           | 0.54           | 442           | 558           | 0.39           |
|     | 2   | 0.31 | 902  | 92    | 382                             | 381           | 524           | 0.53           | 459           | 592           | 0.36           |
|     | 5   | 0.14 | 991  | 92.6  | 383                             | 388           | 548           | 0.50           | 471           | 626           | 0.32           |
| 400 | 1   | 0.69 | 847  | 91.5  | 380                             | 371           | 507           | 0.53           | 445           | 569           | 0.37           |
|     | 2   | 0.39 | 964  | 92.5  | 382                             | 383           | 540           | 0.50           | 464           | 615           | 0.32           |
|     | 5   | 0.189 | 1092 | 92.9  | 382                             | 389           | 569           | 0.47           | 474           | 659           | 0.27           |

TABLE III. Same as in Table II for $M = 350$ MeV and $n = 2$ compared with the results with no $k_T^2$ cutoff.

| $T$ | $\sqrt{\langle \bar{q}q \rangle}$ | $F_x$ | $\sqrt{\langle \frac{1}{3}GG \rangle}$ | $\sqrt{\langle k_T^2 \rangle}$ | $(\langle k_T^2 \rangle)_{AV}^{1/2}$ | $(\langle k_T^2 \rangle)_{PS}^{1/2}$ |
|-----|-----------------|-------|---------------------------------|----------------|----------------|----------------|
| MeV | MeV             | MeV   | MeV                             | MeV           | MeV           | MeV           |
|-----|-----------------|-------|---------------------------------|----------------|----------------|----------------|
| 0.25 | 250             | 91.6  | 382                             | 378           | 513           | 0.54           | 455           | 577           |
| $\infty$ | 284             | 93    | 392                             | 420           | 921           | 0.21           | 540           | 1360          |
FIG. 4. Dependence on the cutoff $T$ of the following ratios: $F_n(T)/F_n$ (solid), $\sqrt{T}/\sqrt{\langle \bar{q}q \rangle}$ (dashed) and $\sqrt{\langle G^2 \rangle}(T)/\sqrt{\langle G^2 \rangle}$ (dash-dotted) for $n = 2$.

In order to estimate the value of the mixed condensate of dimension 5, $\langle ig\bar{q}\sigma \cdot Gq \rangle$, we choose to use the first equation of Eqs. (10). Although this equation is only an approximate one, in the case of no transverse momentum cutoff it gives relatively stable results due to the fast convergence of the $k_T^2$ integrals. In the case of the finite transverse momentum cutoff any of the two equations (10) can be used, since the relation (12) is fulfilled with good accuracy.

In the case of no cutoff (i.e. $T = \infty$) we get that (for $\langle \bar{q}q \rangle$ from Table I):

$$\langle ig\bar{q}\sigma \cdot Gq \rangle = -450^5 \text{ to } -550^5 \text{ MeV}^5.$$  \hspace{1cm} (37)

The value of (37) decreases with increasing $M$ and with increasing $n$. Interestingly, for the parameters which are closest to the original instanton model, $M = 350$ MeV and $n = 2$, we get $-(493 \text{ MeV})^5$ in perfect agreement with the direct calculation of $\langle ig\bar{q}\sigma \cdot Gq \rangle$ in the instanton model [10], which gives $-(490 \text{ MeV})^5$. In the case of the finite cutoff the value of $\langle ig\bar{q}\sigma \cdot Gq \rangle$ is smaller and almost independent of the specific set of parameters (3):

$$\langle ig\bar{q}\sigma \cdot Gq \rangle = -427^5 \text{ to } -443^5 \text{ MeV}^5.$$  \hspace{1cm} (38)

This range of values is compatible with the result obtained within the QCD sum rules [25] at low normalization point: $-(416 \text{ MeV})^5$.

VI. SUMMARY AND CONCLUSIONS

In this paper, following our previous work [13] and Refs. [17,18], we have studied a simple, nonperturbative model in which quarks acquire a momentum dependent dynamical mass (3). In Ref. [13] we have employed this model to calculate the axial-vector leading twist pion light cone wave function, whereas here we have extended our calculations to the case of the pseudo-scalar, twist 3 wave function $\phi^{PS}_\pi$, which is normalized to the quark condensate. Since the model parameters are fixed by the normalization of the axial-vector wave function we could use the normalization condition for $\phi^{PS}_\pi$ to calculate the quark condensate. Since the model parameters are fixed by the normalization of the axial-vector wave function we could use the normalization condition for $\phi^{PS}_\pi$ to calculate the quark condensate. One of the advantages of the present model is that one can calculate not only the longitudinal quark distribution in the pion, but also $k_T^2$ distributions both in the AV and PS channels. Therefore we could use the relations between $k_T^2$ moments and mixed quark gluon condensates [5,6] to estimate the latter in the present model.

One remark is here in order. The advantage of our approach is its simplicity, both conceptual and technical, which makes it possible to compute many quantities analytically. One might therefore suspect that the model is oversimplified and that our assumptions are too crude to be in general true. For example the model has no trace of confinement. It is therefore of importance to perform various tests in order to gain confidence in the model as well as to find its limitations. This study shows that in fact the model works much better than one might have initially expected. As shown in Ref. [13] $\phi^{AV}_\pi$ is compatible with recent CLEO data [24]; quark condensate calculated here from the normalization condition of $\phi^{PS}_\pi$ comes out reasonable, although bigger than the phenomenological value. The gluon condensate, which we have computed from the gap equation (26) guessing the continuation to the Minkowski space, comes out very well. The mixed quark-gluon condensate of dimension 5 calculated from the value of $\langle k_T^2 \rangle^{AV}$ (see Eq. (10)) is in a surprising agreement with the direct evaluation of $\langle ig\bar{q}\sigma \cdot Gq \rangle$ in the instanton model [10].
An obvious limitation of the model is due to the $k_T^2$ asymptotics of $\phi_{PS}^\pi$ and also $\phi_{AV}^\pi$ \((36)\). This powerlike behavior makes higher $k_T^2$ moments divergent, and therefore higher mixed condensates uncalculable. For $\phi_{AV}^\pi$ the power is still relatively high, whereas for $\phi_{PS}^\pi$ it is rather low. This is the reason why the quark condensate comes out higher that the phenomenological value and depends (within a factor of 2) on model parameters. Since there exists another wave function normalized to $\langle \bar{q}q \rangle$, namely $\phi_{PS}^\pi$ \((14)\), it would be interesting to calculate $\langle \bar{q}q \rangle$ from $\phi_{PS}^\pi$ and compare with the present results \((3)\). As a remedy for the broadness of $\phi_{PS}^\pi(k_T^2)$ we have employed the $k_T^2$ cutoff, which does not affect much the gluon condensate and $F_{\pi^-}$ (i.e. $\phi_{AV}^\pi$) but allows to reproduce the phenomenological value of $\langle \bar{q}q \rangle = -(250 \text{ MeV})^3$. Then the ratio $\langle k_T^2 \rangle_{AV} / \langle k_T^2 \rangle_{PS} \sim 5/9$ is in agreement with the analysis of Ref. \((3)\). Unfortunately one single cutoff is probably not enough to make all higher moments reliable.

Both quark and gluon condensate depend on the scale $Q_0^2$. In model calculations it is, however, hard to define precisely what value should be taken for $Q_0$. In the instanton model of the QCD vacuum the scale is determined by the average inverse size of the instanton $1/\overline{\rho} = 600 \text{ MeV}$. It is therefore natural to expect that also in our case the normalization scale is of the same order. Its connection to the scale $\Lambda$ entering Eq.\((3)\) is, however, by no means straightforward since in fact we use various values of $\Lambda$ and $n$ to approximate the same function $M(k)$ evaluated in the instanton model for $1/\overline{\rho} = 600 \text{ MeV}$. One way to determine $Q_0^2$ in the present approach is to identify $Q_0 = K_T$, where $K_T$ is the transverse cutoff displayed in Table \(\text{I}\). Indeed, this is precisely the way how the normalization scale is defined for the light cone wave functions (see e.g. Ref. \((2)\)). Numerically it comes out that $760 < Q_0 < 1100 \text{ MeV}$.

The twist 3 pion wave function $\phi_{PS}^\pi$, which we have plotted in Fig.2 is theoretically an interesting object, since it tends asymptotically to 1, and therefore might be enhanced in certain kinematical regions. The calculations of $\phi_{PS}^\pi$ from the QCD sum rules suggest that it does not vanish at the end points \((13-16)\). We find this kind of behavior only for $n = 1$ (see Eq.\((3)\)), whereas for $n > 1$ $\phi_{PS}^\pi$ vanishes at the end points. Interestingly, the vanishing of $\phi_{PS}^\pi$ for $u = 0, 1$ is correlated with the nonconvexity of the axial-vector wave function at the end points, which for $u \to 0$ (or 1) behaves like $u^N$ (or $(1 - u)^N$). It would be worthwhile to check if this correlation holds also in other models. The fact, that the axial-vector pion distribution amplitude may be concave at the end points has been already pointed out in Ref. \((36)\) and confirmed by model calculations within the framework of the nonlocal sum rules \((37)\).

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