MATRIX-VALUED CORONA THEOREM FOR
MULTIPLY CONNECTED DOMAINS

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January 17, 1999

Abstract

Let $D \subset \mathbb{C}$ be a bounded domain, whose boundary $B$ consists of $k$ simple closed continuous curves and $H^\infty(D)$ be the algebra of bounded analytic functions on $D$. We prove the matrix-valued corona theorem for matrices with entries in $H^\infty(D)$.

1. Introduction.

Let $D \subset \mathbb{C}$ be a bounded domain, whose boundary $B$ consists of $k$ simple closed continuous curves. Let $H^\infty(D)$ be the uniform algebra of bounded analytic functions on $D \subset \mathbb{C}$ with the pointwise multiplication and with the norm

$$||f|| = \sup_{z \in D} |f(z)|.$$ 

The maximal ideal space of $H^\infty(D)$ is defined by

$$M(H^\infty(D)) = \{ \phi : \phi \in \text{Hom}(H^\infty, \mathbb{C}), \phi \neq 0 \}$$

equipped with the weak * topology induced by the dual space of $H^\infty$. It is a compact Hausdorff space. A function $f \in H^\infty(D)$ can be thought of as a continuous function on $M(H^\infty(D))$ via the Gelfand transform $\hat{f}(\phi) = \phi(f)$ ($\phi \in M(H^\infty(D))$). It is well known (see, e.g. [St]) that if $f_1, f_2, ..., f_n$ are functions in $H^\infty(D)$ such that

$$|f_1(z)| + |f_2(z)| + ... + |f_n(z)| \geq \delta > 0 \quad \text{for all } z \in D$$

1991 Mathematics Subject Classification. Primary 46J15 Secondary 30H05.

Key words and phrases. Bounded holomorphic function, dimension, complement of matrices.
then there are $H^\infty(D)$ functions $g_1, g_2, ..., g_n$ so that

$$f_1(z)g_1(z) + f_2(z)g_2(z) + ... + f_n(z)g_n(z) = 1.$$ 

This result is a generalization of the famous Carleson’s corona theorem ([C]) and is equivalent to the statement that $D$ is dense in $M(H^\infty(D))$. In this paper we consider the following generalization of Carleson’s theorem.

Let $f = (f_{ij})$ be a $(k \times n)$-matrix, $k \leq n$, whose entries belong to $H^\infty(D)$ and \{\(F_s\)\}_{s \in S} be the family of minors of $f$ of order $k$. Assume that

$$\sum_{s \in S} |F_s(z)| \geq \delta > 0 \quad \text{for all } z \in D. \quad (1.1)$$

**Theorem 1.1** There exists a unimodular $(n \times n)$-matrix $\tilde{f} = (\tilde{f}_{ij})$, $\tilde{f}_{ij} \in H^\infty$ for all $i, j$, so that $\tilde{f}_{ij} = f_{ij}$ for $1 \leq i \leq k$.

Our proof heavily relies upon the fact that the topological dimension of $M(H^\infty(D))$ equals 2, that is a consequence of the similar result of Su’arez ([S]) for $M(H^\infty(D))$, where $D \subset \mathbb{C}$ is the unit disc.

**2. Proof of Theorem 1.1.**

We begin with

**Lemma 2.1** Let $B_1, ..., B_k$ be the components of the boundary of $D$, with $B_1$ forming the outer boundary. Let $D_1$ be the interior of $B_1$, and $D_2, ..., D_k$ the exteriors of $B_2, ..., B_k$, including the point at infinity. For every $f \in H^\infty(D)$ there exists a decomposition

$$f(z) = f_1(z) + ... + f_k(z) \quad (z \in D), \quad (2.1)$$

such that $f_n \in H^\infty(D_n)$ for $n = 1, ..., k$.

For similar but more general statement see [R, Th. 3.3]. For the sake of completeness we present here a simple proof of the lemma.

**Proof.** By definition each $B_i$ can be approximated by a sequence $\{C_{ip}\}_{p \geq 1}$ of simple smooth closed curves containing in $D$. Then $C_{1p}, ..., C_{kp}$ for sufficiently big $p$ are nonintersecting smooth closed curves in $D$ which bound a domain $D_p \subset D$. Put

$$f_{np}(z) := \frac{1}{2\pi i} \int_{C_{np}} \frac{f(w)}{w - z} dw \quad (n = 1, ..., k; \ z \in D_p).$$

Then $(2.1)$ holds in $D_p$. Moreover, we can choose $D_p$ so closed to $D$ that the following inequalities

$$||f_s|_{C_{np}}||_{L^\infty(C_{np})} \leq A ||f|| \quad (1 \leq s, n \leq k, \ s \neq n)$$

hold with $A$ depending only on the distances between curves $B_1, ..., B_k$. Here $|| \cdot ||$ stands for the $H^\infty$ norm of $f$. This follows directly from estimates of Cauchy’s
kernels restricted to \( C_{np} \). The above inequality and decomposition \( 2.1 \) combined imply

\[
||f_{np}||_{L^\infty(C_{np})} \leq A'||f||
\]

for any \( n = 1, \ldots, k \) with an absolute constant \( A' \). Letting \( p \to \infty \), one easily sees that there is \( f_n = \lim_{p \to \infty} f_{np} \) such that \( f_n \in H^\infty(D_n), 1 \leq n \leq k \), and

\[
f(z) = f_1(z) + \ldots + f_k(z) \quad (z \in D)
\]

which completes the proof of the lemma. \( \square \)

By Osgood-Caratheodory theorem for each \( i = 1, \ldots, k \) there is a biholomorphic map \( T_i : D_i \to \mathbb{D} \) that can be extended to continuous map \( \overline{T_i} \to \mathbb{D} \). Thus we obtain isomorphism \( T_i^* : H^\infty(\mathbb{D}) \to H^\infty(D_i) \) which maps \( H^\infty(\mathbb{D}) \cap C(\mathbb{D}) \) into \( H^\infty(D_i) \cap C(\overline{D_i}) \). Let \( M(H^\infty) \) be the maximal ideal space of \( H^\infty(\mathbb{D}) \) and \( M(H^\infty(D_i)) \) be the maximal ideal space of \( H^\infty(D_i) \), \( i = 1, \ldots, k \). Then \( T_i \) can be extended to homeomorphism \( T_i : M(H^\infty(D_i)) \to H^\infty(\mathbb{D}) \).

**Lemma 2.2** Let \( f \in H^\infty(D) \). Then \( f \) admits a continuous extension to \( (M(H^\infty(D_i)) \setminus D_i) \cup D \) for \( i = 1, \ldots, k \).

**Proof.** First, notice that for any \( i = 1, \ldots, k \) there is a natural continuous surjective mapping \( P_i : M(H^\infty(D_i)) \setminus D_i \to B_i \) defined by the embedding homomorphism \( H^\infty(D_i) \cap C(\overline{D_i}) \to H^\infty(D_i) \). Further, according to Lemma \( 2.1 \), \( f = f_1 + \ldots + f_k \) with \( f_n \in H^\infty(D_n) \) for all \( n \). Moreover, \( f_s \mid D_i \) is continuous for \( s \neq i \). Let \( \{z_\alpha\} \) be a net in \( D \) converging to a point \( \xi \in M(H^\infty(D_i)) \setminus D_i \). In particular, \( \{z_\alpha\} \) converges to a point \( P_i(\xi) \in B_i \) in the topology defined on \( \mathbb{C} \). Since \( f_s, s \neq i \), is continuous on \( B_i \) we obtain (for such \( s \))

\[
\lim_{\alpha} f_s(z_\alpha) = f_s(P_i(\xi))
\]

We stress that this limit does not depend on the choice of a net converging to \( \xi \).

Let us define

\[
f(\xi) = f_i(\xi) + \sum_{s \neq i} f_s(P_i(\xi))
\]

where by definition

\[
f_i(\xi) = \lim_{\alpha} f_i(z_\alpha).
\]

It is easy to see that such extension of \( f \) determines a continuous function on \( (M(H^\infty(D_i)) \setminus D_i) \cup D \). Note also that the algebra of extended functions separates points of \( (M(H^\infty(D_i)) \setminus D_i) \cup D \).

The proof is complete. \( \square \)

Let us consider now compact topological space

\[
K := D \cup \bigcup_{i=1}^{k} (M(H^\infty(D_i)) \setminus D_i)
\]

with the topology induced from the corresponding maximal ideal spaces. Then according to Lemma \( 2.2 \), \( H^\infty(D) \) admits a continuous extension to \( K \) and the extended algebra separates points of \( K \). Since \( D \) is dense in \( K \) the corona theorem for \( H^\infty(D) \) obviously implies that \( K \) is homeomorphic to \( M(H^\infty(D)) \).
Definition 2.3 For a normal space $X$ we say that $\dim X \leq n$ if every finite open covering of $X$ can be refined by an open covering whose order $\leq n + 1$. If $\dim X \leq n$ and the statement $\dim X \leq n - 1$ is false, we say that $\dim X = n$. For the empty set we put $\dim \emptyset = -1$.

Lemma 2.4 $\dim M(H^\infty(D)) = 2$.

Proof. We recall first

Definition 2.5 Let $X$ be a normal space and $F \subset X$ be a closed subset. We say that $\dim (X \mod F) \leq n$ if $\dim M(H^\infty(D_i)) \mod F = 2$ and since $\dim M(H^\infty(D_i)) \mod F \leq 2$ for $i = 1, \ldots, k$ due to Su'arez' theorem [S], application of Proposition 2.6 yields equality $\dim M(H^\infty(D)) = 2$. 

Proposition 2.6 Let $X$ be a normal space and $F$ be a closed subset of $X$. If $\dim F \leq n$ and $\dim (X \mod F) \leq n$, then $\dim X \leq n$.

Set $F := \bigcup_{i=1}^{k}(M(H^\infty(D_i))) \mod F_i$. Since by Definition 2.3 $\dim M(H^\infty(D)) \mod F = 2$ and since $\dim M(H^\infty(D_i)) \mod F \leq 2$ for $i = 1, \ldots, k$ due to Su'arez' theorem [S], application of Proposition 2.6 yields equality $\dim M(H^\infty(D)) = 2$. 

We are now in position to prove Theorem 1.1.

Let $f = (f_{ij})$ be a $(k \times n)$-matrix, $k \leq n$, with entries in $H^\infty(D)$ and let the family of minors of $f$ of order $k$ satisfy inequality (1.1). According to the corona theorem we can extend $f$ to $M(H^\infty(D))$ in such a way that the extended matrix satisfies (1.1) for all $\phi \in M(H^\infty(D))$. Now we will prove a general statement which implies immediately our result.

Let $b$ be a $(k \times n)$-matrix whose entries belong to a commutative Banach algebra $A$ of complex-valued functions defined on maximal ideal space $M(A)$. Assume that $\dim M(A) = 2$. Assume also that condition (1.1) holds at each point of $M(A)$ for the family of minors of $b$ of order $k$ (this means that these minors do not belong together to a maximal ideal).

Proposition 2.7 There is an invertible $(n \times n)$-matrix with entries in $A$ which completes $b$.

According to [L, Th.3], it suffices to complete $b$ in the class of continuous matrix-functions on $M(A)$. Thus we have to find an $(n \times n)$-matrix $\tilde{b} = (\tilde{b}_{ij})$ with entries $\tilde{b}_{ij}$ in $C(M(A))$ with $\det \tilde{b} = 1$ and $\tilde{b}_{ij} = b_{ij}$ for $1 \leq i \leq k$. Matrix $b$ determines trivial subbundle $\xi$ of complex rank $k$ in trivial vector bundle $\theta^n = M(A) \times \mathbb{C}^n$. Let $\eta$ be an additional to $\xi$ subbundle of $\theta^n$. Then clearly $\xi \oplus \eta$ is topologically trivial. We will prove that $\eta$ is also topologically trivial. Then a trivialization $s_1, s_2, \ldots, s_{n-k}$ of $\eta$ will determine the required complement of $b$.

Lemma 2.8 Let $X$ be a 2-dimensional Hausdorff compact and let $p : E \to X$ be a locally trivial continuous vector bundle over $X$ of complex rank $m$. If $m \geq 2$ then there is a continuous nowhere vanishing section of $E$. 

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Proof. Let $\gamma_m$ be canonical vector bundle over the Stiefel manifold $V(m,k)$ of $m$-dimensional subspaces of $\mathbb{C}^k$. (Recall that $\gamma_m$ is defined as follows: the fibre over an $x \in V(m,k)$ is the $m$-dimensional complex space representing $x$.) According to the general theory of vector bundles [Hu, Ch.3, Th.5.5] there is a continuous mapping $f$ of $X$ into $V(m,k)$ such that $f^*\gamma_m = E$. Considering $V(m,k)$ as a compact submanifold of some $\mathbb{R}^s$, $f$ can be determined by a finite family of continuous on $X$ functions, $f = (f_1, \ldots, f_s)$. For $X' = f(X)$ let $U'$ be a compact polyhedron containing $X'$ and such that $\gamma_m$ has a continuous extension to a vector bundle over $U'$. Let us consider the inverse limiting system determined by all possible finite collections of continuous on $X$ functions such that the last $s$ functions of any such collection are coordinates of the mapping $f$ (in the same order). For any such family $f_\alpha := (f_1, \ldots, f_{\alpha-s}, f)$ denote $X_\alpha = f_\alpha(X) \subset \mathbb{R}^\alpha$. Further, let $p_\alpha^\beta : X_\beta \to X_\alpha$, $\beta \geq \alpha$, be the mapping induced by the natural projection $\mathbb{R}^\beta \to \mathbb{R}^\alpha$ onto the first $\alpha$ coordinates. Let $U_\alpha$ be a compact polyhedron containing $X_\alpha$ defined in a small open neighbourhood of $X_\alpha$ so that the inverse limit of the system $(U_\alpha, p_\alpha^\beta)$ coincides with $X$ and $p_\alpha^\beta : \mathbb{R}^\alpha \to \mathbb{R}^s$ maps $U_\alpha$ to $U'$. Consider bundle $E_\alpha := (p_\alpha^\beta)^*(\gamma_m)$ over $U_\alpha$. Clearly, $(p_\alpha^\beta)^*(E_\alpha) = E_\beta$, $\beta \geq \alpha$, and the pullback of each $E_\alpha$ to $X$ coincides with $E$. Now the Euler class $e_\alpha$ of each $E_\alpha$ is an element of the Čech cohomology group $H^{2m}(U_\alpha, \mathbb{Z})$. This class equals 0 if and only if $E_\alpha$ has a nowhere vanishing section. By the fundamental property of characteristic classes [Hi], $(p_\alpha^\beta)^*(e_\alpha) = e_\beta$, $\beta \geq \alpha$. Further, it is well known [Br, Ch.II, Corol.14.6] that

$$\lim_{\alpha \to \infty}(p_\alpha^\beta)^*H^k(U_\alpha, \mathbb{Z}) = H^k(X, \mathbb{Z}), \quad k \geq 0.$$  

Since $\dim X = 2$, Theorem 37-7 and Corollary 36-15 of [N] imply $H^k(X, \mathbb{Z}) = 0$ for $k > 2$. But real rank of $E$ is $2m \geq 4$ so that the Euler class $e = \lim_{\alpha \to \infty}(p_\alpha^\beta)^*(e_\alpha)$ of $E$ equals 0. From here and the above formula it follows that there is some $\beta$ such that $e_\beta = 0$. In particular, $E_\beta$ has a continuous nowhere vanishing section $s$. Then its pullback to $X$ determines the required section of $E$. □

Lemma 2.8 implies that $E$ is isomorphic to direct sum $E_{m-1} \oplus E'$, where $E_{m-1} = X \times \mathbb{C}^{m-1}$ and $E'$ is a vector bundle over $X$ of complex rank 1. In particular, if the first Chern class $c_1(E) \in H^2(X, \mathbb{Z})$ of $E$ is 0, then $E' = X \times \mathbb{C}$ and $E = X \times \mathbb{C}^m$. In our case $\dim M(A) = 2$, $\theta^n = \xi \oplus \eta$ and $\xi$ is topologically trivial. Then for the first Chern class of $\theta^n$ we have

$$0 = c_1(\theta^n) = c_1(\xi) + c_1(\eta) = c_1(\eta).$$

Therefore from Lemma 2.8 we obtain that $\eta$ is a topologically trivial bundle.

The proposition is proved. □

To complete the proof of the theorem it suffices to apply Lemma 2.4. □

Remark 2.9 It is worth noting that similarly to [B] one can characterize topology of analytical part of $M(H^\infty(D))$.

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