ANALYSIS OF A DIFFUSIVE CHOLERA MODEL INCORPORATING LATENCY AND BACTERIAL HYPERINFECTIVITY

WEI YANG AND JINLIANG WANG*

School of Mathematical Science, Heilongjiang University
Harbin 150080, China

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ABSTRACT. In this paper, we are concerned with the threshold dynamics of a diffusive cholera model incorporating latency and bacterial hyperinfectivity. Our model takes the form of spatially nonlocal reaction-diffusion system associated with zero-flux boundary condition and time delay. By studying the associated eigenvalue problem, we establish the threshold dynamics that determines whether or not cholera will spread. We also confirm that the threshold dynamics can be determined by the basic reproduction number. By constructing Lyapunov functional, we address the global attractivity of the unique positive equilibrium whenever it exists. The theoretical results are still hold for the case when the constant parameters are replaced by strictly positive and spatial dependent functions.

1. Introduction. In recent years, the dynamics of cholera epidemics has been studied extensively by using mathematical modelling. The interactions between human populations, pathogens, and environments construct the basic and specific objects of investigation, which allow us to get more useful information on quantitative understanding of cholera epidemics and designing control measures for this disease. There are several outbreaks have been reported in Africa, America, and Asia [1].

It is well-known that cholera is caused by the bacterium *Vibrio cholerae*. Cholera gives rise to some typical symptoms such as severe copious watery diarrhea, vomiting, muscle cramps and diarrhea. Once vibrios in the contaminated environment are ingested by a person, after complexity chemical and genomic interactions within a few hours, human cholera occurs and results in above mentioned symptoms and even death. When vibrios are freshly shed from infected humans, they can survive in the aquatic environment and can be transmitted by human-to-human pathways. During the past decades, a large number of research has been devoted to studying the cholera dynamics involving two infection routes: indirect pathway
(environment-to-human) and/or direct pathway (human-to-human) transmission of cholera, see, for example, [4, 5, 7, 25, 14] and references therein.

Recent laboratory findings [16] suggest that the freshly shed vibrios can stay alive for hours, and exhibit up to 700-fold infectivity than environmental vibrios (see, for example, [1, 7]), which may largely contribute to cholera dynamics in the aspect of the transmission through human-to-human transmission of cholera. In [7, 25], the freshly shed vibrios and environmental vibrios are distinguished by the infectivity, and termed as hyperinfectious (HL vibrios) and lower-infectious (LI vibrios) state of V. cholerae, respectively to get a better understanding of cholera dynamics.

It is worth mentioning that many contributions have been devoted to investigating the cholera dynamics. Mukandavire et al. [14] estimated the reproductive numbers for the 10 provinces in Zimbabwe and revealed that spatial heterogeneity remarkably affected the underlying transmission pattern for the 2008-2009 cholera outbreaks. After that, Tuite et al. [20] estimated the reproductive numbers by using the data from 10 administrative departments, and also revealed that spatial heterogeneity brings the difficulties in guiding practical control strategies for the 2010 cholera epidemic in Haiti. For other cholera epidemic models by ordinary differential equations (ODEs) modelling, we refer the readers to [4, 7] and ODEs models with patch/network structures [5]. For the aspects of partial differential equations (PDEs) of cholera epidemic models, Shuai et al. [2] proposed the model incorporating infection age for infectious individuals and vibrios. They also established the threshold dynamics, which determines whether cholera will die out or persist. Subsequently, Wang et al. [22] and Yang et al. [27] carried out completely analysis on the model in [2] with distinct approaches and supplied necessary mathematical arguments including the existence of global compact attractor. Recently, spatial heterogeneity has been considered as one of main factors in understanding the spatial spread of infectious diseases. Reaction-diffusion models involving environmental heterogeneity, mobility of human individuals and the spatial dispersal of vibrios have been formulated to get threshold dynamics of cholera epidemics and find practical control strategies, see for example, [3, 23, 25].

This paper aims to improve the model in [25] by incorporating the mobility of the human individuals in the latent period. For simplicity, we shall ignore the convection coefficient, which is used to describe the drift of HL (resp. LI) state of vibrios. Further, we do not consider any loss of immunity of the cholera epidemic (that is, a flux from recovered individuals to susceptible individuals) since it usually lasts more than two years [9]. For $x \in \Omega$, $t > 0$, the system in [25] reduces to,

$$
\begin{align*}
\frac{\partial S}{\partial t} - D_s \Delta S &= \Lambda - dS - S \left( \alpha I + \beta_1 \frac{B_1}{B_1 + K_1} + \beta_2 \frac{B_2}{B_2 + K_2} \right), \\
\frac{\partial I}{\partial t} - D_I \Delta I &= S \left( \alpha I + \beta_1 \frac{B_1}{B_1 + K_1} + \beta_2 \frac{B_2}{B_2 + K_2} \right) - (d + \gamma + m)I, \\
\frac{\partial B_1}{\partial t} - D_1 \Delta B_1 &= \xi I - \delta_1 B_1, \\
\frac{\partial B_2}{\partial t} - D_2 \Delta B_2 &= \delta_1 B_1 - \delta_2 B_2,
\end{align*}
$$

with the boundary condition

$$
\frac{\partial U}{\partial n} = 0, \ U = S, I, B_1, B_2, \ x \in \partial \Omega, \ t > 0.
$$
The meaning of symbols and parameters used in (1.1) is explained as follows: \( S(t,x) \) (resp. \( I(t,x) \)) is the densities of susceptible (infectious) individuals at location \( x \) and time \( t \). \( B_1(t,x) \) (resp. \( B_2(t,x) \)) stands for the concentration of HL (resp. LI) state of V. cholera at location \( x \) and time \( t \) in the water environment. \( D_i(i = S,I,1,2) \) stands for the diffusion rate of \( S,I,B_1 \) and \( B_2 \), respectively. \( \Lambda \) is the recruitment rate. \( d \) is the natural mortality rate of human. \( \gamma, m \) and \( \xi \) stand for the recovery rate, the disease-induced mortality rate, and the shedding rate of vibrios of infectious individuals, respectively. \( \alpha \) (resp. \( \beta_i(i = 1,2) \)) represents the transmission rate for direct pathway (resp. indirect pathway). \( K_1 \) (resp. \( K_2 \)) represents the half saturation constant of HL (resp. LI) state of V. cholera. \( \delta_1 \) (resp. \( \delta_2 \)) represents the (natural) death rate of HL (resp. LI) state of V. cholera, respectively. The label \( \frac{\partial}{\partial n} \) is the outward normal derivative on \( \partial\Omega \).

Up to now, few works have considered simultaneously the bacterial hyperinfectivity and latency in cholera dynamics. Several important questions remains unknown. For example, How do the mobility of hosts and spatial diffusion of vibrios affect the cholera dynamics? and What’s role of latency of hosts in the spread of cholera? In reality, however, the model proposed in [25] does not consider the mobility of latency individual in the transmission of an infectious disease. On the other hand, our motivation of this paper base on recent works on nonlocal and delayed reaction-diffusion system in bounded domain [11], where the joint effects of the latency and the mobility of human hosts on the spatial spread of infectious diseases are discussed. In fact, when considering infectious disease with a latency, the mobility of infectious individuals in the latent period will leads to non-local infection [11]. To make things not too complicated, we adopt the bilinear incidence rate between susceptible and infectious hosts, and saturation incidence rate between susceptible and vibrios. Our goal of this paper is to determine the threshold dynamics of diffusive cholera model with latency and bacterial hyperinfectivity and to understand how the hyperinfectivity affect the dynamics of cholera epidemics.

The plan of this paper is proceed as follows. In next section, we describes the model used in detail. The well-possedness of the model is proved in Section 3. In Section 4, we study the eigenvalue problem by linearization of system in disease free steady state and investigate the threshold dynamics with principal eigenvalue. We confirm that the basic reproduction number can also be regarded as a sharp threshold determining whether or not the disease dies out or persist in Section 5. In section 6, by adopting the bilinear incidences and extra additional condition besides the basic reproduction number, we obtain the global attractivity of the positive equilibrium in the case where all the coefficients are constants.

2. Model formulation. Assume that \( \Omega \) is a connected, bounded domain in \( \mathbb{R}^n \), in which human hosts and vibrios are diffusing with smooth boundary \( \partial \Omega \). We suppose that the average infection period is fixed by \( \tau \). In such a setting, the full population is divided into four classes: susceptible (\( S = S(t,x) \)), latent (\( L = L(t,x) \)), infectious (\( I = I(t,x) \)) and recovered (\( R = R(t,x) \)) class.

Like the method used in [11], we use variable \( a \) to denote the infection age (the time since infection began). With infection age \( a \), we use \( i(t,x,a) \) to denote the density of infectious individuals at location \( x \) and time \( t \). By appealing to the standard system of structured population and spatial diffusion [15], we have

\[
\frac{\partial i(t,a,x)}{\partial t} + \frac{\partial i(t,a,x)}{\partial a} - D_i \Delta i(t,a,x) = -(d + \gamma + m(a))i(t,a,x).
\]  

(2.1)
Here $m(a)$ is the disease-induced mortality rate with infection age $a$ associated with following assumptions,

$$
m(a) = \begin{cases} 
m_1, & \text{for } t \geq 0 \text{ and } a \in [0, \tau], \\
m_2, & \text{for } t \geq 0 \text{ and } a \in [\tau, \infty].
\end{cases}
$$

We use $L(t, x) = \int_0^x i(t, a, x) \, da$ and $I(t, x) = \int_0^\infty i(t, a, x) \, da$ to denote the latent and infectious individuals, respectively. From biological point of view, we assume that $i(t, \infty, x) = 0$. From (2.1), we can integrate it from 0 to $\tau$, and from $\tau$ to $\infty$, respectively, obtaining that

$$
\frac{\partial L(t, x)}{\partial t} - D_1 \Delta L = -(d + \gamma + m_1)L(t, x) - i(t, \tau, x) + i(t, 0, x),
$$

and

$$
\frac{\partial I(t, x)}{\partial t} - D_1 \Delta I = -(d + \gamma + m_2)I(t, x) - i(t, \infty, x) + i(t, \tau, x),
$$

respectively. Based on the infection mechanism in model (1.1), we have the following condition,

$$
i(t, 0, x) = S \left( \alpha I + \beta_1 \frac{B_1}{B_1 + K_1} + \beta_2 \frac{B_2}{B_2 + K_2} \right).
$$

Next, we determine $i(t, \tau, x)$ by using the method of the integration along characteristics. To this end, let us define $Z(r, a, x) = i(a + r, a, x)$, with $r \geq 0$, which implies that $i(t, \tau, x) = Z(t - \tau, \tau, x)$, $\forall t \geq \tau$. Based on this setting, for $a \in [0, \tau]$, we get

$$
\frac{\partial Z(r, a, x)}{\partial a} = \left[ \frac{\partial i(t, a, x)}{\partial t} + \frac{\partial i(t, a, x)}{\partial a} \right]_{t=a+r} = D_1 \Delta i(a + r, a, x) - (d + \gamma + m_1)i(a + r, a, x)
$$

and

$$
Z(r, 0, x) = i(r, 0, x).
$$

Solving the above equation with respect to $r$ yields

$$
Z(r, a, x) = e^{-d_1 a} \int_\Omega \Gamma(D_1 \tau, x, y)S(t - \tau, y) \left( \alpha I(r, y) + \beta_1 \frac{B_1(r, y)}{B_1(r, y) + K_1} + \beta_2 \frac{B_2(r, y)}{B_2(r, y) + K_2} \right) \, dy,
$$

where $\Gamma(\cdot, \cdot, \cdot)$ is the Green function for the operator $D_1 \Delta$ with the Neumann boundary condition, and $d_1 := (d + \gamma + m_1)$. Hence, $i(t, \tau, x)$ can be described as follows:

$$
i(t, \tau, x) = e^{-d_1 \tau} \int_\Omega \Gamma(D_1 \tau, x, y)S(t - \tau, y) \times \left( \alpha I(t - \tau, y) + \beta_1 \frac{B_1(t - \tau, y)}{B_1(t - \tau, y) + K_1} + \beta_2 \frac{B_2(t - \tau, y)}{B_2(t - \tau, y) + K_2} \right) \, dy.
$$

Hence we have

$$
\frac{\partial L}{\partial t} - D_1 \Delta L = -(d + \gamma + m_1)L + S \left( \alpha I + \beta_1 \frac{B_1}{B_1 + K_1} + \beta_2 \frac{B_2}{B_2 + K_2} \right)
$$

$$
- e^{-d_1 \tau} \int_\Omega \Gamma(D_1 \tau, x, y)S(t - \tau, y) \times \left( \alpha I(t - \tau, y) + \beta_1 \frac{B_1(t - \tau, y)}{B_1(t - \tau, y) + K_1} + \beta_2 \frac{B_2(t - \tau, y)}{B_2(t - \tau, y) + K_2} \right) \, dy.
$$
and
\[ \frac{\partial I(t, x)}{\partial t} - D_1 \Delta I = - (d + \gamma + m_2) I + e^{-d_1 \tau} \int_{\Omega} \Gamma(D_1 \tau, x, y) S(t - \tau, y) \times \left( \alpha I(t - \tau, y) + \beta_1 \frac{B_1(t - \tau, y)}{B_1(t - \tau, y) + K_1} + \beta_2 \frac{B_2(t - \tau, y)}{B_2(t - \tau, y) + K_2} \right) dy, \]
respectively.

Note that \( L(t, x) \) can be determined if \( S(t, x), I(t, x), B_1(t, x) \) and \( B_2(t, x) \) are known. We consider the nonlocal reaction-diffusion system,
\[
\begin{align*}
\frac{\partial S}{\partial t} - D_S \Delta S &= \Lambda - dS - S \left( \alpha I + \beta_1 \frac{B_1}{B_1 + K_1} + \beta_2 \frac{B_2}{B_2 + K_2} \right), \\
\frac{\partial I}{\partial t} - D_I \Delta I &= -(d + \gamma + m_2) I + e^{-d_1 \tau} \int_{\Omega} \Gamma(D_1 \tau, x, y) S(t - \tau, y) \times \left( \alpha I(t - \tau, y) + \beta_1 \frac{B_1(t - \tau, y)}{B_1(t - \tau, y) + K_1} + \beta_2 \frac{B_2(t - \tau, y)}{B_2(t - \tau, y) + K_2} \right) dy, \\
\frac{\partial B_1}{\partial t} - D_1 \Delta B_1 &= \xi I - \delta_1 B_1, \\
\frac{\partial B_2}{\partial t} - D_2 \Delta B_2 &= \delta_1 B_1 - \delta_2 B_2,
\end{align*}
\] (2.3)

with boundary condition (1.2). Throughout this paper, we will use the following notations:
- \( \mathbb{X} := C(\overline{\Omega}, \mathbb{R}^4) \) and \( \mathbb{Y} := C(\overline{\Omega}, \mathbb{R}) \) represent the Banach space with the supremum norm \( \| \cdot \|_{\mathbb{X}} \) and \( \| \cdot \|_{\mathbb{Y}} \), respectively.
- \( C_\tau := C([\tau, 0], \mathbb{X}) \) represents the Banach space with the norm
\[
\| \phi \| = \max_{\theta \in [-\tau, 0]} \| \phi(\theta) \|_{\mathbb{X}}.
\]

We denote by \( \mathbb{X}^+ := C(\overline{\Omega}, \mathbb{R}^4^+) \), \( \mathbb{Y}^+ := C(\overline{\Omega}, \mathbb{R}^+) \) and \( C_\tau^+ := C([\tau, 0], \mathbb{X}^+) \) the cones of \( \mathbb{X}, \mathbb{Y} \) and \( C_\tau \), respectively. Given a function \( u : [\tau, \sigma) \rightarrow \mathbb{X} \) for \( \sigma > 0 \), define \( u_\sigma \in C_\tau \) by \( u_\sigma(\theta) = u(t + \theta), \theta \in [-\tau, 0] \).

For convenience, we denote
\[
(u_1, u_2, u_3, u_4) = (S, I, B_1, B_2).
\]

Then in the sequel, we arrive at the main model to be studied in this paper,
\[
\begin{align*}
\frac{\partial u_1}{\partial t} - D_S \Delta u_1 &= -d u_1 - u_1 \left( \alpha u_2 + \beta_1 \frac{u_3}{u_3 + K_1} + \beta_2 \frac{u_4}{u_4 + K_2} \right), \\
\frac{\partial u_2}{\partial t} - D_I \Delta u_2 &= -(d + \gamma + m_2) u_2 + e^{-d_1 \tau} \int_{\Omega} \Gamma(D_1 \tau, x, y) u_1(t - \tau, y) \times \left( \alpha u_2(t - \tau, y) + \beta_1 \frac{u_3(t - \tau, y)}{u_3(t - \tau, y) + K_1} + \beta_2 \frac{u_4(t - \tau, y)}{u_4(t - \tau, y) + K_2} \right) dy, \\
\frac{\partial u_3}{\partial t} - D_1 \Delta u_3 &= \xi u_2 - \delta_1 u_3, \\
\frac{\partial u_4}{\partial t} - D_2 \Delta u_4 &= \delta_1 u_3 - \delta_2 u_4,
\end{align*}
\] (2.4)

with boundary condition
\[
\frac{\partial U}{\partial n} = 0, \quad U = u_1, u_2, u_3, u_4, \quad x \in \partial \Omega, \quad t > 0.
\] (2.5)
3. Well-posedness of system (2.4). Suppose that $T_i(t)(i = 1, 2, 3, 4) : \mathbb{Y} \rightarrow \mathbb{Y}$, $t \geq 0$, is the strongly continuous semigroups associated with $D_S \Delta - d$, $D_I \Delta - (d + \gamma + m_2)$, $D_1 \Delta - \sigma_1$ and $D_2 \Delta - \sigma_2$ subject to the Neumann boundary condition, respectively. It then follows from [19, Section 7.1 and Corollary 7.2.3] that for each $t > 0$, $T_i(t)(i = 1, 2, 3, 4)$ is compact and strongly positive.

For any $x \in \Omega$, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in C^+_\tau$, we define the nonlinear operator $F = (F_1, F_2, F_3, F_4) : C^+_\tau \rightarrow \mathbb{K}$ as

\[
\begin{align*}
F_1(\phi)(x) &= \Lambda - \phi_1(0, x) \left( \alpha \phi_2(0, x) + \beta_1 \frac{\phi_3(0, x)}{\phi_3(0, x) + K_1} + \beta_2 \frac{\phi_4(0, x)}{\phi_4(0, x) + K_2} \right), \\
F_2(\phi)(x) &= e^{-d_I \tau} \int_{\Omega} \Gamma(D_I \tau, x, y) \phi_1(-\tau, y) \\
& \quad \times \left( \alpha \phi_2(-\tau, y) + \beta_1 \frac{\phi_3(-\tau, y)}{\phi_3(-\tau, y) + K_1} + \beta_2 \frac{\phi_4(-\tau, y)}{\phi_4(-\tau, y) + K_2} \right) dy, \\
F_3(\phi)(x) &= \xi \phi_2(0, x), \\
F_4(\phi)(x) &= \delta_1 \phi_3(0, x).
\end{align*}
\]

Hence, by using $u(t) = (u_1, u_2, u_3, u_4)^T$ and $\bar{T}(t) = (T_1(t), T_2(t), T_3(t), T_4(t))^T$, we can describe system (2.4) as the following integral equation:

\[
u = \bar{T}(t)\phi + \int_0^t \bar{T}(t-s)F(u_\tau(s))ds, \quad t > 0.
\]

Further we can check that for any $\phi \in C^+_\tau$ and $\kappa \geq 0$,

\[
\phi(0, x) + \kappa F(\phi)(x)
\]

\[
= \left( \begin{array}{c}
\phi_1(0, x) + \kappa \left( \Lambda - \phi_1(0, x) \left( \alpha \phi_2(0, x) + \beta_1 \frac{\phi_3(0, x)}{\phi_3(0, x) + K_1} + \beta_2 \frac{\phi_4(0, x)}{\phi_4(0, x) + K_2} \right) \right) \\
\phi_2(0, x) + \kappa \left( \alpha \phi_2(-\tau, y) + \beta_1 \frac{\phi_3(-\tau, y)}{\phi_3(-\tau, y) + K_1} + \beta_2 \frac{\phi_4(-\tau, y)}{\phi_4(-\tau, y) + K_2} \right) dy \\
\phi_3(0, x) + \kappa \xi \phi_2(0, x) \\
\phi_4(0, x) + \kappa \delta_1 \phi_3(0, x)
\end{array} \right), \quad \text{for } x \in \Omega.
\]

It follows that $\phi(0, x) + \kappa F(\phi)(x) \in C^+_\tau$ can be achieved by choosing sufficiently small $\kappa$. Hence, we have

\[
\lim_{\kappa \to 0^+} \text{dist}(\phi(0) + \kappa F(\phi), C^+_\tau) = 0, \forall \phi \in C^+_\tau.
\]

According to the assertions from [13, Corollary 4], we can conclude that for each $\phi \in C^+_\tau$, there exists a unique mild solution $u(t, \cdot, \phi)$ on its maximal existence interval $[0, \sigma_\phi)$ with $u_0 = \phi$, where $\sigma_\phi \leq \infty$. Moreover, $u(t, \cdot, \phi) \in C^+_\tau$, $\forall t \in [0, \sigma_\phi)$, and $u(t, \cdot, \phi)$ is a classical solution of (2.4), $\forall t > \tau$.

The following result comes from [11, Lemma 1].

**Lemma 3.1.** [11, Lemma 1] Considering the following system:

\[
\begin{align*}
\frac{\partial \tilde{u}(t, x)}{\partial t} - D_S \Delta \tilde{u}(t, x) &= \Lambda - D \tilde{u}(t, x), & x \in \Omega, & t > 0, \\
\frac{\partial \tilde{u}(t, x)}{\partial n} &= 0, & x \in \partial \Omega, & t > 0.
\end{align*}
\]
System (3.2) admits a unique positive steady state \( \hat{\omega} = \frac{A}{4} \), which is globally asymptotically stable in \( \mathcal{Y} \).

The well-posedness of system (2.4) is stated in the following result.

**Theorem 3.2.** For each \( u_0 = \phi \in C^+_1 \), system (2.4) admits a unique solution \( u(t, \cdot, \phi) \) on \([0, \infty)\), and the solution semiflow \( \Phi(t) = u_t(\cdot) : C^+_1 \to C^+_1 \) generated by (2.4), that is,

\[
(\Phi(t)(\phi))(\theta, x) = u(t + \theta, x, \phi), \quad \forall x \in \Omega, \ t \geq 0, \ \theta \in [-\tau, 0],
\]

admits a global compact attractor in \( C^+_1 \).

**Proof.** It is easy to see that \( u_1 \) equation of (2.4) satisfies

\[
\frac{\partial u_1}{\partial t} - D \Delta u_1 \leq \Lambda - d u_1. \tag{3.3}
\]

From Lemma 3.1 and the comparison theorem, there exist \( \mathbb{B}_1 > 0 \) and \( t_1 = t_1(\phi) > 0 \) such that for any \( \phi \in C^+_1 \), \( u_1(t_1, \cdot, \phi) \leq \mathbb{B}_1 \) for all \( t \geq t_1 \).

Following the boundedness of \( u_1 \) and using the property of \( \Gamma(\cdot, \cdot, \cdot) \), we have

\[
\frac{\partial u_2}{\partial t} - D \Delta u_2 \leq (d + \gamma + m_2) u_2 + c \mu_2(t - \tau) + \mathbb{B}_1 (\beta_1 + \beta_2), \quad \text{for } c \geq 0, \quad (3.4)
\]

where \( \mu_2(t) = \int_\Omega u_2(t, x) dx \). We next wish to prove the boundedness of \( \mu_2(t) \). In fact, denoting \( \mu_1(t) = \int_\Omega u_1(t, x) dx \) and integrating the first equation in (2.4) over \( \Omega \) give

\[
\frac{d \mu_1(t)}{dt} = \Lambda |\Omega| - \int_\Omega u_1 \left( \alpha u_2 + \beta_1 \frac{u_3}{u_3 + K_1} + \beta_2 \frac{u_4}{u_4 + K_2} \right) dx - d \mu_1(t), \quad t \geq 0,
\]

where \( |\Omega| \) is the volume of \( \Omega \). Thus

\[
\int_\Omega u_1 \left( \alpha u_2 + \beta_1 \frac{u_3}{u_3 + K_1} + \beta_2 \frac{u_4}{u_4 + K_2} \right) dx = \Lambda |\Omega| - d \mu_1(t) - \frac{d \mu_1(t)}{dt}, \quad t > 0. \tag{3.5}
\]

Similarly, integrating the second equation of (2.4) over \( \Omega \) (with the help of (3.5)) gives

\[
\frac{d \mu_2(t)}{dt} \leq -(d + \gamma + m_2) \mu_2(t) - \Theta_1 \mu_1(t - \tau) - \Theta_2 \frac{d \mu_1(t - \tau)}{dt} + \Theta_3, \quad \forall t \geq \tau,
\]

where \( \Theta_1, \Theta_2 \) and \( \Theta_3 \) are some positive numbers. By choosing \( \Theta_1 \leq (d + \gamma + m_2) \Theta_2 \) in (3.5), we have

\[
\frac{d}{dt} [\mu_2(t) + \Theta_2 \mu_1(t - \tau)] \leq -(d + \gamma + m_2) \mu_2(t) - \Theta_1 \mu_1(t - \tau) + \Theta_3
\]

\[
\leq -\frac{\Theta_1}{\Theta_2} \mu_2(t) - \Theta_1 \mu_1(t - \tau) + \Theta_3
\]

\[
= -\frac{\Theta_1}{\Theta_2} [\mu_2(t) + \Theta_2 \mu_1(t - \tau)] + \Theta_3, \quad \forall t \geq \tau. \tag{3.6}
\]

Hence, there exists \( \Theta_4 > 0 \) depending on \( \phi \) such that

\[
\mu_2(t) \leq \mu_2(t) + \Theta_2 \mu_1(t - \tau) \leq \Theta_4(\phi) \mu_1(t - \tau) + \Theta_5, \quad \forall t \geq \tau,
\]

where \( \Theta_5 \) a positive constant independent of \( \phi \). Hence the boundedness of \( \mu_2(t) \) directly follows. By appealing the theory of delayed parabolic equation (see, for example, [19, 26]), we arrive at the conclusion that there exists \( \mathbb{B}_2 > 0 \), independent
of $\phi$, and $t_2 = t_2(\phi) > t_1(\phi) + \tau$ such that $u_2(t, \cdot, \phi) \leq B_2$, $\forall t \geq t_2$. From $u_3$ equation of (2.4), we have

$$\begin{cases}
\frac{\partial u_3}{\partial t} - D_1 \Delta u_3 \leq \xi B_2 - \delta_1 u_3, & x \in \Omega, \ t \geq t_2, \\
\frac{\partial u_3}{\partial n} = 0, & x \in \partial \Omega, \ t \geq t_2.
\end{cases}$$

Due to Lemma 3.1, we know that there is a $t_3(\phi) > t_2$ such that $\forall t \geq t_3$, $u_3(t, \phi) \leq 2 \frac{\xi B_2}{\delta_1}$. Further form the $u_4$ equation of (2.4),

$$\begin{cases}
\frac{\partial u_4}{\partial t} - D_2 \Delta u_4 \leq 2 \xi B_2 - \delta_2 u_4, & x \in \Omega, \ t > t_3, \\
\frac{\partial u_4}{\partial n} = 0, & x \in \partial \Omega, \ t > t_3,
\end{cases}$$

and Lemma 3.1, we know that there is a $t_4(\phi) > t_3$ such that $\forall t > t_4$, $u_4(t, \phi) \leq 4 \frac{\xi B_2}{\delta_2}$. As a consequence, the well-posedness of system (2.4) is confirmed. Further, according to the assertions from [26, Theorem 2.1.8], we know that for each $\tau > 0$, $\Phi(t) : C^+_\tau \rightarrow C^+_\tau$ is point dissipative and compact. Thus, we can follow the results in [6, Theorem 3.4.8] to conclude that $\Phi(t) : C^+_\tau \rightarrow C^+_\tau$, $t \geq 0$, admits a global compact attractor.

In what follows, we use $u(t, x, \phi)$ to denote the solution of system (2.4) with $u_0 = \phi \in C^+_\tau$.

**Lemma 3.3.** For each $\phi \in C^+_\tau$, $u_1(t, x, \phi)$ is strictly positive $\forall t > 0, x \in \overline{\Omega}$. Further,

$$\liminf_{t \to \infty} u_1(t, x) \geq \hat{u}^*_1 := \frac{\Lambda}{\alpha B_2 + \beta_1 + \beta_2 + d}, \textup{ uniformly for } x \in \overline{\Omega}.$$

**Proof.** From Theorem 3.2, we know that $u_1(t, x) \geq 0$, $\forall t \geq 0$ and $x \in \Omega$. Suppose that there exists $x_1 \in \overline{\Omega}$ and $t^*_1 \in (0, \infty)$ such that $u_1(t^*_1, x_1) = 0$. Choosing large enough number $T_1 > 0$ such that $t^*_1 < T_1$ and $(t^*_1, x_1) \in [0, T_1] \times \overline{\Omega}$. Then $u_1(t, x)$ achieves its local minimum at $(t^*_1, x_1)$. In view of equation (2.4), we directly have

$$\begin{cases}
\frac{\partial u_1}{\partial t} - D_S \Delta u_1 \geq -u_1 \left( \alpha u_2 + \beta_1 \frac{u_3}{u_3 + K_1} + \beta_2 \frac{u_4}{u_4 + K_2} + d \right), & x \in \Omega, \ t \in (0, t^*_1], \\
\frac{\partial u_1}{\partial n} = 0, & x \in \partial \Omega, \ t \in (0, t^*_1].
\end{cases}$$

If $x_1 \in \partial \Omega$, with the help of Hopf boundary Lemma [17, Theorem 3], we have $\frac{\partial u_1}{\partial n} < 0$, which leads to a contradiction. If $x_1 \in \Omega$, according to the strong maximum principle [17, Theorem 7], we directly have $u_1(t, x) = u_1(t^*_1, x_1) = 0$, for all $(t^*_1, x_1) \in [0, T_1] \times \overline{\Omega}$. Again from the $u_1$ equation of (2.4), it leads to $\Lambda = 0$, $\forall x \in \overline{\Omega}$, a contradiction. Thus, $u_1(t, x, \phi) > 0$, $\forall t > 0, x \in \Omega$.

On the other hand, Theorem 3.2 ensures that $u_2(t, x) \leq B_2$ for $t > t_2 = t_2(\phi)$. Hence $u_1$ equation of (2.4) satisfies

$$\begin{cases}
\frac{\partial u_1}{\partial t} - D_S \Delta u_1 \geq \Lambda - (\alpha B_2 + \beta_1 + \beta_2 + d) u_1, & x \in \Omega, \ t \geq t_2, \\
\frac{\partial u_1}{\partial n} = 0, & x \in \partial \Omega, \ t \geq t_2.
\end{cases}$$

This combines with Lemma 3.1 and comparison principle yield that

$$\liminf_{t \to \infty} u_1(t, x) \geq \hat{u}^*_1,$$
where \( \hat{u}_i^* := \frac{\Lambda_i}{\alpha \beta_2 + \beta_1 + \gamma + d} \) is the positive equilibrium of

\[
\begin{cases}
\frac{\partial \hat{u}_1}{\partial t} - D_S \Delta \hat{u}_1 = \Lambda - (\alpha \beta_2 + \beta_1 + \gamma + d) \hat{u}_1, & x \in \Omega, \ t \geq t_2, \\
\frac{\partial \hat{u}_1}{\partial n} = 0, & x \in \partial \Omega, \ t \geq t_2,
\end{cases}
\tag{3.12}
\]

and globally asymptotically stable in \( \mathcal{Y} \) due to Lemma 3.1. \( \square \)

**Lemma 3.4.** If there exists some \( t_0 \geq 0 \) such that \( u_2(t_0, \cdot, \phi) \neq 0 \) or \( u_3(t_0, \cdot, \phi) \neq 0 \) or \( u_4(t_0, \cdot, \phi) \neq 0 \), then

\[
u_i(t, \cdot, \phi) > 0 \ (i = 2, 3, 4), \ \forall t > t_0, x \in \Omega.
\]

**Proof.** We first prove the case of \( u_2(t_0, \cdot, \phi) \neq 0 \). From Theorem 3.2 and the \( u_2 \) equation of (2.4), we can see that

\[
\begin{cases}
\frac{\partial u_2}{\partial t} - D_1 \Delta u_2 \geq -(d + \gamma + m_2) u_2, & x \in \Omega, \ t > 0, \\
\frac{\partial u_2}{\partial n} = 0, & x \in \partial \Omega, \ t > 0.
\end{cases}
\tag{3.13}
\]

If \( u_2(t_0, \cdot, \phi) \neq 0 \) for some \( t_0 \geq 0 \), with the help of the strong maximum principle [17] and the Hopf boundary lemma [17], and using the initial time at \( t = t_0 \) instead of \( t = 0 \), we have \( u_2(t, \cdot, \phi) > 0 \), \( \forall t > 0, x \in \Omega \). Further, \( u_3 \) equation of (2.4) satisfies

\[
\begin{cases}
\frac{\partial u_3}{\partial t} - D_1 \Delta u_3 \geq -\delta_1 u_3, & x \in \Omega, \ t > 0, \\
\frac{\partial u_3}{\partial n} = 0, & x \in \partial \Omega, \ t > 0.
\end{cases}
\tag{3.14}
\]

Similarly to above arguments, we have \( u_3(t, \cdot, \phi) > 0 \), \( \forall t > 0, x \in \Omega \). Consequently, \( u_4 \) equation of (2.4) satisfies

\[
\begin{cases}
\frac{\partial u_4}{\partial t} - D_2 \Delta u_4 \geq -\delta_2 u_4, & x \in \Omega, \ t > 0, \\
\frac{\partial u_4}{\partial n} = 0, & x \in \partial \Omega, \ t > 0.
\end{cases}
\tag{3.15}
\]

Similarly to above arguments, we have \( u_4(t, \cdot, \phi) > 0 \), \( \forall t > 0, x \in \Omega \).

We next wish to prove the case of \( u_3(t_0, \cdot, \phi) \neq 0 \). From (3.14) and (3.15), \( u_3(t, \cdot, \phi) > 0 \) and \( u_4(t, \cdot, \phi) > 0 \) directly follow for \( \forall t > 0, x \in \Omega \), which in turn implies that \( u_2(t_0, \cdot, \phi) > 0 \). We finally prove the case of \( u_4(t_0, \cdot, \phi) \neq 0 \). From (3.15), we know that \( u_4(t, \cdot, \phi) > 0 \). Consequently, the strong maximum principle and the Hopf boundary lemma ensures that \( u_2(t_0, \cdot, \phi) > 0 \) and \( u_3(t_0, \cdot, \phi) > 0 \). \( \square \)

4. **Eigenvalue problem and threshold dynamics.** Let \( u_2 = u_3 = u_4 = 0 \) in system (2.4). Then the densities of the susceptible individuals \( u_1 \) satisfies (3.2). It follows that (2.4) admits a disease free equilibrium \( E_0 = (u_1^*, 0, 0, 0)^T \), where \( u_1^* = \frac{\Lambda}{d} \). Linearizing system (2.4) at \( E_0 \) gives the linear system for the infectious
components $u_2$, $u_3$ and $u_4$:

$$\begin{align*}
\frac{\partial u_2}{\partial t} - D_1 \Delta u_2 &= -(d+\gamma+m_2)u_2 + u_1^* e^{-d_1 \tau} \int_{\Omega} \Gamma(D_1 \tau, x, y) \\
&\quad \times \left( \alpha u_2(t, y) + \frac{\beta_1}{K_1} u_3(t, y) + \frac{\beta_2}{K_2} u_4(t, y) \right) dy,
\end{align*}$$

$$\begin{align*}
\frac{\partial u_3}{\partial t} - D_1 \Delta u_3 &= \xi u_2 - \delta_1 u_3,
\end{align*}$$

$$\begin{align*}
\frac{\partial u_4}{\partial t} - D_2 \Delta u_4 &= \delta_1 u_3 - \delta_2 u_4,
\end{align*}$$

with (2.5). Let us first pay attention to the following linear nonlocal system,

$$\begin{align*}
\frac{\partial u_2}{\partial t} - D_1 \Delta u_2 &= -(d+\gamma+m_2)u_2 + u_1^* e^{-d_1 \tau} \int_{\Omega} \Gamma(D_1 \tau, x, y) \\
&\quad \times \left( \alpha u_2(t, y) + \frac{\beta_1}{K_1} u_3(t, y) + \frac{\beta_2}{K_2} u_4(t, y) \right) dy,
\end{align*}$$

$$\begin{align*}
\frac{\partial u_3}{\partial t} - D_1 \Delta u_3 &= \xi u_2 - \delta_1 u_3,
\end{align*}$$

$$\begin{align*}
\frac{\partial u_4}{\partial t} - D_2 \Delta u_4 &= \delta_1 u_3 - \delta_2 u_4,
\end{align*}$$

with (2.5). Plugging $(u_2, u_3, u_4) = e^{\lambda t}(\psi_2(x), \psi_3(x), \psi_4(x))$ into (4.2) yields

$$\begin{align*}
\lambda \psi_2(x) - D_1 \Delta \psi_2(x) &= -(d+\gamma+m_2)\psi_2(x) + u_1^* e^{-d_1 \tau} \int_{\Omega} \Gamma(D_1 \tau, x, y) \\
&\quad \times \left( \alpha \psi_2(y) + \frac{\beta_1}{K_1} \psi_3(y) + \frac{\beta_2}{K_2} \psi_4(y) \right) dy,
\end{align*}$$

$$\begin{align*}
\lambda \psi_3(x) - D_1 \Delta \psi_3(x) &= \xi \psi_2(x) - \delta_1 \psi_3(x),
\end{align*}$$

$$\begin{align*}
\lambda \psi_4(x) - D_2 \Delta \psi_4(x) &= \delta_1 \psi_3(x) - \delta_2 \psi_4(x),
\end{align*}$$

with

$$\frac{\partial \psi_2(x)}{\partial n} = \frac{\partial \psi_3(x)}{\partial n} = \frac{\partial \psi_4(x)}{\partial n} = 0, \quad x \in \partial \Omega, \quad t > 0.$$  

Note that system (4.2) is linear in $C(\Omega, \mathbb{R}_+^3)$. Additionally, from Theorem 3.2, the semigroup generated by system (4.2) is strongly positive and compact. The following result on the nonlocal eigenvalue problems (4.3) is based on [19, Theorem 7.6.1].

**Lemma 4.1.** The eigenvalue problem (4.3) has a principal eigenvalue $\lambda(u_1^*)$ with a unique strongly positive eigenfunction.

We next consider the linear system (4.1). Substituting

$$(u_2, u_3, u_4) = e^{\lambda t}(\psi_2(x), \psi_3(x), \psi_4(x))$$

into (4.1) yields

$$\begin{align*}
\lambda \psi_2(x) - D_1 \Delta \psi_2(x) &= -(d+\gamma+m_2)\psi_2(x) \\
&\quad + e^{-\lambda \tau} u_1^* e^{-d_1 \tau} \int_{\Omega} \Gamma(D_1 \tau, x, y) \left( \alpha \psi_2(y) + \frac{\beta_1}{K_1} \psi_3(y) + \frac{\beta_2}{K_2} \psi_4(y) \right) dy,
\end{align*}$$

$$\begin{align*}
\lambda \psi_3(x) - D_1 \Delta \psi_3(x) &= \xi \psi_2(x) - \delta_1 \psi_3(x),
\end{align*}$$

$$\begin{align*}
\lambda \psi_4(x) - D_2 \Delta \psi_4(x) &= \delta_1 \psi_3(x) - \delta_2 \psi_4(x),
\end{align*}$$

$$\frac{\partial \psi_2(x)}{\partial n} = \frac{\partial \psi_3(x)}{\partial n} = \frac{\partial \psi_4(x)}{\partial n} = 0.$$  

(4.4)
Theorem 4.3. \( \lambda \) has a unique strongly positive eigenfunction. Furthermore, for any \( \tau \geq 0 \), \( \tilde{\lambda}(u^*_1, \tau) \) and \( \lambda(u^*_1) \) have the same sign.

The following results indicate that \( \tilde{\lambda}(u^*_1, \tau) \) (or \( \lambda(u^*_1) \)) plays the role in determining the disease extinction.

Theorem 4.3. If \( \tilde{\lambda}(u^*_1, \tau) < 0 \), then \( E_0 \) is globally attractive in \( C^+ \) in the sense that
\[
\lim_{t \to \infty} ||u(t, x, 0) - E_0||_x = 0, \text{ uniformly for all } x \in \bar{\Omega}.
\]

Proof. Assume that \( \tilde{\lambda}(u^*_1, \tau) < 0 \). By the continuity of \( \tilde{\lambda}(u^*_1, \tau) \) in \( u^*_1 \), there is a \( \epsilon_0 > 0 \) such that \( \tilde{\lambda}(u^*_1 + \epsilon_0, \tau) < 0 \). Due to the existence of \( \tilde{\lambda}(u^*_1, \tau) \), we know that \( \tilde{\lambda}(u^*_1 + \epsilon_0, \tau) < 0 \) is the principal eigenvalue of
\[
\begin{align*}
\lambda \psi_2(x) - D_1 \Delta \psi_2(x) &= -(d + \gamma + m_2) \psi_2(x) + e^{-\lambda \tau} (u^*_1 + \epsilon_0) e^{-d \tau} \int_{\Omega} \Gamma(D_1 \tau, x, y) \\
\lambda \psi_3(x) - D_1 \Delta \psi_3(x) &= \xi \psi_2(x) - \delta_1 \psi_3(x), \\
\lambda \psi_4(x) - D_2 \Delta \psi_4(x) &= \delta_1 \psi_3(x) - \delta_2 \psi_4(x), \\
\frac{\partial \psi_2(x)}{\partial n} &= \frac{\partial \psi_3(x)}{\partial n} = \frac{\partial \psi_4(x)}{\partial n} = 0.
\end{align*}
\]
\( (4.5) \)

From \( (3.3) \), we can fix this \( \epsilon_0 \) such that
\[
u_1(t, \cdot, \phi) < u^*_1 + \epsilon_0, \text{ } \forall t \geq t_0, \text{ } x \in \bar{\Omega}, \text{ for some } t_0 > 0.
\]
\( (4.6) \)

Using \( (4.6) \), we know that \( u_2 \), \( u_3 \) and \( u_4 \) equation in system \( (2.4) \) satisfies
\[
\begin{align*}
\frac{\partial u_2}{\partial t} - D_1 \Delta u_2 &\leq -(d + \gamma + m_2) u_2 + (u_1^* + \epsilon_0) e^{-d \tau} \int_{\Omega} \Gamma(D_1 \tau, x, y) \\
\frac{\partial u_3}{\partial t} - D_1 \Delta u_3 &\leq \xi u_2 - \delta_1 u_3, \\
\frac{\partial u_4}{\partial t} - D_2 \Delta u_4 &\leq \delta_1 u_3 - \delta_2 u_4, \\
\frac{\partial u_2(t, x)}{\partial n} = \frac{\partial u_3(t, x)}{\partial n} = \frac{\partial u_4(t, x)}{\partial n} = 0.
\end{align*}
\]
\( (4.7) \)

In order to use comparison principle, we fix a \( \zeta > 0 \) such that for any \( \phi \in C^+_r \), we have
\[
(u_2(t, \cdot, \phi), u_3(t, \cdot, \phi), u_4(t, \cdot, \phi)) \leq \zeta(\tilde{\psi}_2(\cdot), \tilde{\psi}_3(\cdot), \tilde{\psi}_4(\cdot)), \text{ } \text{for } t \in [t_0 - \tau, t_0],
\]
where \( (\tilde{\psi}_2(\cdot), \tilde{\psi}_3(\cdot), \tilde{\psi}_4(\cdot)) \) is the positive eigenfunction with respect to \( \tilde{\lambda}(u^*_1 + \epsilon_0, \tau) \). Consequently, the comparison principle ensures that
\[
(u_2(t, \cdot, \phi), u_3(t, \cdot, \phi), u_4(t, \cdot, \phi)) \leq \zeta e^{\lambda(u^*_1 + \epsilon_0, \tau)(t - t_0)}(\tilde{\psi}_2(\cdot), \tilde{\psi}_3(\cdot), \tilde{\psi}_4(\cdot)), \text{ } t \geq t_0.
\]
From \( \lambda(u^*_1 + \epsilon_0, \tau) < 0 \), we know that
\[
\lim_{t \to \infty} (u_2(t, \cdot, \phi), u_3(t, \cdot, \phi), u_4(t, \cdot, \phi)) = (0, 0, 0).
\]
Hence, $u_1$ equation is asymptotic to system (3.2), which allow us to use the standard theory for asymptotical autonomous semiflows obtaining that $\lim_{t \to \infty} u_1(t, \cdot) = u_1^*$ uniformly for $x \in \Omega$. This proves Theorem 4.3.

Before going into proving the persistence results of system (2.4), we first introduce a useful assertion.

**Theorem 4.4.** Let $\bar{u} = u_2$ or $u_3$ or $u_4$ be given. If there is a $\varsigma > 0$ such that
\[
\liminf_{t \to \infty} \bar{u}(t, x, \phi) \geq \varsigma, \quad \text{uniformly for } x \in \overline{\Omega},
\]
then there is a $\varsigma > 0$ such that
\[
\liminf_{t \to \infty} \bar{U}(t, x, \phi) \geq \varsigma, \quad \text{for } \bar{U} = u_1, u_2, u_3, u_4,
\]
uniformly for $x \in \overline{\Omega}$.

**Proof.** We prove the assertion by three cases. For the case where $\bar{u} = u_2$, it is obvious that there is a time $\tilde{t}_1 > 0$ such that $u_2(t, x) \geq \frac{1}{2} \varsigma$, $\forall t \geq \tilde{t}_1$, $x \in \overline{\Omega}$. Hence, $u_3$ equation of (2.4) satisfies
\[
\begin{aligned}
\frac{\partial u_3}{\partial t} - D_1 \Delta u_3 &\geq \frac{1}{2} \varsigma \xi - \delta_1 u_3, \quad x \in \Omega, \ t \geq \tilde{t}_1, \\
\frac{\partial u_3(t, x)}{\partial n} &= 0, \quad x \in \partial \Omega, \ t \geq \tilde{t}_1.
\end{aligned}
\]
This together with Lemma 3.1 and comparison principle indicate that
\[
\liminf_{t \to \infty} u_3(t, x, \phi) \geq u_3^*,
\]
where $u_3^*$ is a positive number. Thus, there is a time $\tilde{t}_2 > 0$ such that $u_3(t, x, \phi) \geq \frac{1}{2} u_3^*$, $\forall t \geq \tilde{t}_2$, $x \in \overline{\Omega}$. Hence, $u_4$ equation of (2.4) satisfies
\[
\begin{aligned}
\frac{\partial u_4}{\partial t} - D_2 \Delta u_4 &\geq \frac{1}{2} u_3^* \delta_1 - \delta_2 u_4, \quad x \in \Omega, \ t \geq \tilde{t}_2, \\
\frac{\partial u_4(t, x)}{\partial n} &= 0, \quad x \in \partial \Omega, \ t \geq \tilde{t}_2.
\end{aligned}
\]
This together with Lemma 3.1 and comparison principle indicate that
\[
\liminf_{t \to \infty} u_4(t, x, \phi) \geq u_4^*,
\]
where $u_4^*$ is a positive number. This proves (4.9) with $\bar{u} = u_2$.

Next, we prove the case where $\bar{u} = u_3$. By Lemma 3.3, and (4.8) with $\bar{u} = u_3$, we have that there is a time $\tilde{t}_3 > 0$ and $\forall t \geq \tilde{t}_3$ and $x \in \overline{\Omega},$
\[
u_1(t, x, \phi) \geq \frac{1}{2} \bar{u}_3^* \quad \text{and} \quad u_3(t, x, \phi) \geq \frac{1}{2} \varsigma.
\]
Hence, $u_2$ equation of (2.4) satisfies
\[
\begin{aligned}
\frac{\partial u_2}{\partial t} - D_2 \Delta u_2 &\geq \frac{1}{2} \bar{u}_3^* e^{-d_1 \tau} \int_{\Omega} \Gamma(D(x, y)) \beta_1 \frac{1}{2} \varsigma \kappa + K_1 \ dy \\
&\quad - (d + \gamma + m_2) u_2, \quad x \in \Omega, \ t \geq \tilde{t}_3, \\
\frac{\partial u_2(t, x)}{\partial n} &= 0, \quad x \in \partial \Omega, \ t \geq \tilde{t}_3.
\end{aligned}
\]
This together with Lemma 3.1 and comparison principle indicate that
\[
\liminf_{t \to \infty} u_2(t, x, u_0) \geq u_2^*.
\]
where $u_2^*$ is a positive number.

In view of (4.13) and the case of $\bar{u} = u_2$, we can deduce that (4.9) with $\bar{u} = u_3$ eventually holds. Similarly to the arguments above, the case of (4.9) with $\bar{u} = u_3$ can be verified. This proves Theorem 4.4.

We next wish to establish the persistence result for (2.4).

Let

$$W_0 = \{u(\cdot) = (u_1, u_2, u_3, u_4)(\cdot) \in C^+_{\tau} : u_2(\cdot) \neq 0\},$$

and

$$\partial W_0 = X^+ \setminus W_0 = \{u(\cdot) = (u_1, u_2, u_3, u_4)(\cdot) \in C^+_{\tau} : u_2(\cdot) = 0\}.$$

Further let us define

$$M_{\theta} := \{\phi \in \partial W_0 : \Phi(t)\phi \in \partial W_0, \forall t \geq 0\},$$

and denote by $\omega(\phi)$ the omega limit set of the orbit $O^+(\phi) := \{\Phi(t)\phi : t \geq 0\}$.

**Theorem 4.5.** If $\tilde{\lambda}(u_1^*, \tau) > 0$, then there is an $\eta > 0$ such that for any $\phi \in C^+_{\tau}$ with $\phi_2 \neq 0$ or $\phi_3 \neq 0$ or $\phi_4 \neq 0$, we have

$$\liminf_{t \to \infty} U(t, x, \phi) \geq \eta, \text{ for } U = u_1, u_2, u_3, u_4,$$

uniformly for all $x \in \Omega$. Further, (2.4) admits at least one (componentwise) positive equilibrium $\bar{u}$.

**Proof.** We only verify the case where $\phi_2 \neq 0$. From the definition of $\Phi(t)$ (see Theorem 3.2) and Lemma 3.4, we know that for any $\phi_2 \in W_0$, we have $u_i(t, \cdot, \phi) > 0(i = 2, 3, 4), \forall t > 0, x \in \Omega$. Further, with the help of Lemma 3.3, we directly have $\Phi(t) W_0 \subseteq W_0, \forall t \geq 0$.

We prove the assertion by the following claims.

**Claim 1.** $\omega(\phi) = \{E_0\}$, for $\forall \phi \in M_{\theta}$. From $\phi \in M_{\theta}$ and the definition of $M_{\theta}$, we know that $u_2(t, \cdot, \phi) \equiv 0, t \geq 0$. Hence, $u_2$ equation of (2.4) satisfies

$$e^{-d_1 \tau} \int_\Omega \Gamma(D_1\tau, x, y) u_1(t - \tau, y) \left(\beta_1 \frac{u_3(t - \tau, \cdot, \phi)}{u_3(t - \tau, \cdot, \phi) + K_1} + \beta_2 \frac{U_4(t - \tau, \cdot, \phi)}{u_4(t - \tau, \cdot, \phi) + K_2}\right) dy \equiv 0,$$

which in turn implies that $u_3(t - \tau, \cdot, \phi) \equiv u_4(t - \tau, \cdot, \phi) \equiv 0, \forall t \geq 0$. Then $u_1$ equation of (2.4) is asymptotic to system (3.2), which allow us to obtain that $\lim_{t \to \infty} u_1(t, \cdot) = u_1^* \text{ uniformly for } x \in \Omega$. This proves Claim 1.

Since $\tilde{\lambda}(u_1^*, \tau) > 0$, by the continuity of $\tilde{\lambda}(u_1^*, \tau)$ in $u_1^*$, there is an $\epsilon_1 > 0$ such that $\tilde{\lambda}(u_1^* - \epsilon_1, \tau) > 0$. Here $\tilde{\lambda}(u_1^* - \epsilon_1, \tau)$ is the principal eigenvalue of

$$\begin{cases}
\lambda \psi_1(x) - D_1 \Delta \psi_1(x) = -(d + \gamma + m_2) \psi_1(x) + (u_1^* - \epsilon_1) e^{-d_1 \tau} \int_\Omega \Gamma(D_1\tau, x, y) \left(\alpha \psi_1(y) + \beta_1 \frac{1}{K_1} - \epsilon_1 \psi_2(y) + \beta_2 \frac{1}{K_2} - \epsilon_1 \psi_3(y)\right) dy, \\
\lambda \psi_2(x) - D_1 \Delta \psi_2(x) = \xi \psi_1(x) - \delta_1 \psi_2(x), \\
\lambda \psi_3(x) - D_1 \Delta \psi_3(x) = \delta_1 \psi_2(x) - \delta_2 \psi_3(x), \\
\frac{\partial \psi_1(x)}{\partial n} = \frac{\partial \psi_2(x)}{\partial n} = \frac{\partial \psi_3(x)}{\partial n} = 0.
\end{cases}$$

Using this $\epsilon_1$, we can see that there exists $\hat{\rho} > 0$ such that for $i = 3, 4$,

$$\frac{1}{u_i(t, x) + K_{i-2}} > \frac{1}{K_{i-2}} - \epsilon_1, \text{ for } |u_i| < \hat{\rho}.$$
Claim 2. \((u_1^*, 0, 0, 0)\) forms a uniform weak repeller for \(W_0\), that is, for \(\forall \phi \in W_0\),
\[
\limsup_{t \to \infty} \| \Phi(t)\phi - (u_1^*, 0, 0, 0) \| \geq \varrho, \quad \text{where} \quad \varrho = \min\{\hat{\rho}, \varepsilon_1\} > 0.
\]
If it is not true, we assume that there is an initial value \(\phi_0 \in W_0\) satisfies
\[
\limsup_{t \to \infty} \| \Phi(t)\phi_0 - (u_1^*, 0, 0, 0) \| < \varrho.
\]
With the definition of \(\Phi(t)\), we know that there is a time \(\hat{t} > 0\) such that for \(t > \hat{t}\),
\[
u_1(t, \cdot, \phi_0) > u_1^* - \varrho \quad \text{and} \quad |u_i(t, \cdot, \phi_0)| < \varrho, \quad i = 2, 3, 4. \tag{4.17}
\]
This together with (4.16) imply that for \(t \geq \hat{t}\),
\[
\begin{align*}
\frac{\partial \hat{u}_2}{\partial t} - D_1 \Delta \hat{u}_2 &= -((d+\gamma+m_2)u_2 + (u_1^* - \varepsilon_1)e^{-d_1 t} \int_\Omega \Gamma(D_1 \tau, x, y) \\
&\quad \times \left( \alpha u_2(t, \tau, y) + \beta_1 \left( \frac{1}{K_1} - \varepsilon_1 \right) u_3(t, \tau, y) + \beta_2 \left( \frac{1}{K_2} - \varepsilon_1 \right) u_4(t, \tau, y) \right) dy, \\
\frac{\partial \hat{u}_3}{\partial t} - D_1 \Delta \hat{u}_3 &= \xi u_2 - \delta_1 \hat{u}_3, \\
\frac{\partial \hat{u}_4}{\partial t} - D_2 \Delta \hat{u}_4 &= \xi u_3 - \delta_2 \hat{u}_4, \\
\frac{\partial \hat{u}_2}{\partial n} &= \frac{\partial \hat{u}_3}{\partial n} = \frac{\partial \hat{u}_4}{\partial n} = 0.
\end{align*}
\]
Denote by \(\hat{\psi} = (\hat{\psi}_2(\cdot), \hat{\psi}_3(\cdot), \hat{\psi}_4(\cdot))\) the positive eigenfunction with respect to \(\hat{\lambda}(u_1^* - \varepsilon_1, \tau)\). Then, for \(t \geq \hat{t}\), system
\[
\begin{align*}
\frac{\partial \hat{u}_2}{\partial t} - D_1 \Delta \hat{u}_2 &= -((d+\gamma+m_2)u_2 + (u_1^* - \varepsilon_1)e^{-d_1 t} \int_\Omega \Gamma(D_1 \tau, x, y) \\
&\quad \times \left( \alpha u_2(t, \tau, y) + \beta_1 \left( \frac{1}{K_1} - \varepsilon_1 \right) u_3(t, \tau, y) + \beta_2 \left( \frac{1}{K_2} - \varepsilon_1 \right) u_4(t, \tau, y) \right) dy, \\
\frac{\partial \hat{u}_3}{\partial t} - D_1 \Delta \hat{u}_3 &= \xi \hat{u}_2 - \delta_1 \hat{u}_3, \\
\frac{\partial \hat{u}_4}{\partial t} - D_2 \Delta \hat{u}_4 &= \xi \hat{u}_3 - \delta_2 \hat{u}_4, \\
\frac{\partial \hat{u}_2}{\partial n} &= \frac{\partial \hat{u}_3}{\partial n} = \frac{\partial \hat{u}_4}{\partial n} = 0,
\end{align*}
\]
admits a solution \((\hat{u}_2(t, x), \hat{u}_3(t, x), \hat{u}_4(t, x)) = e^{\hat{\lambda}(u_1^* - \varepsilon_1, \tau)(t-\hat{t})} \hat{\psi}\). Hence, from Lemma 3.4, we can find a \(\zeta_1 > 0\) such that
\[
(u_2(t, \cdot, \phi_0), u_3(t, \cdot, \phi_0), u_4(t, \cdot, \phi_0)) \geq \zeta_1 (\hat{\psi}_2(\cdot), \hat{\psi}_3(\cdot), \hat{\psi}_4(\cdot)), \quad \text{for} \quad t \in [\hat{t} - \tau, \hat{t}].
\]
Consequently,
\[
(u_2(t, \cdot, \phi), u_3(t, \cdot, \phi), u_4(t, \cdot, \phi)) \geq \zeta_1 e^{\hat{\lambda}(u_1^* - \varepsilon_1, \tau)(t-\hat{t})} \hat{\psi}, \quad t \geq \hat{t},
\]
directly follows from the comparison principle. Due to the fact that \(\hat{\lambda}(u_1^* - \varepsilon_1, \tau) > 0\), we arrive at the conclusion that \((u_2(t, \cdot, \phi), u_3(t, \cdot, \phi), u_4(t, \cdot, \phi))\) is unbounded, which results in a contradiction. This completes the proof of the claim 2.

We define the function \(D(\cdot) : C^+ \to [0, \infty)\) by
\[
D(\phi) := \min_{x \in \Pi} u_2(0, \cdot, \phi), \forall \quad \phi \in C^+.
\]
Due to Lemma 3.4, we have that \(D^{-1}(\mathbb{R}_+) \subseteq W_0\). In addition, \(D(\cdot)\) satisfies that if \(D(\varphi) > 0\) or \(\varphi \in W_0\) with \(D(\varphi) = 0\), then \(D(\Phi(t)\varphi) > 0, \forall \quad t > 0\). This combined with arguments as those in [18] imply that \(D(\cdot)\) is indeed a distance function for \(\Phi(t) : C^+ \to C^+\).
Consequently, $\Phi(t)$ is a cycle in $\mathcal{F}$, and its stable set $E_0$ satisfies $W^s(E_0) \cap W^s(E_0) = \emptyset$, where $W^s(E_0)$ is the stable manifold of $E_0$. Moreover, no subset of $\{(u_1^0, 0, 0, 0)\}$ forms a cycle in $\partial S_0$. Consequently, $\Phi(t) : C^+ \to C^+$ indeed has a global compact attractor in $C^+$, $\forall t \geq 0$.

According to [18, Theorem 3]), we know that there exists an $\eta > 0$ such that

$$\min_{\phi \in \omega(\phi)} D(\phi) > \eta, \quad \forall \phi \in S_0.$$ 

Hence, $\liminf_{t \to \infty} u_2(t, \cdot, \phi) \geq \eta, \quad \forall \phi \in S_0$. This together with Lemma 3.3 and Theorem 4.4 imply that (4.14) holds by choosing a sufficiently small $\eta$. The existence of an endemic equilibrium of (2.4) directly follows from related theory in [12, Theorem 4.7]). The positivity of a steady state is provided by Lemma 3.4. □

5. Basic reproduction number and threshold dynamics. In this section, we mainly follow notions and procedures as those in [24] to define the basic reproduction number $R_0$ for (2.4). Further, we shall confirm that $R_0$ acts as the role in determining whether or not disease will spread as stated in Sect. 4 with $\lambda(u_1^0, \tau)$. We define the $R_0$ obeying the following procedures:

(i) For $t \in [-\tau, 0)$, we assume that $u_2(\theta, \cdot) = u_3(\theta, \cdot) = u_4(\theta, \cdot) = 0$, whose are near $E_0$.

(ii) At time $t = 0$, we denote by $(\psi_2(x), \psi_3(x), \psi_4(x))^T$ the spatial initial distribution for infectious components, that is, $u_2(0, x) = \psi_2(x)$, $u_3(0, x) = \psi_3(x)$ and $u_3(0, 0) = \psi_4(x)$.

(iii) As time evolves, let $S(t) = (T_2(t) \psi_2, T_3(t) \psi_3, T_4(t) \psi_4)^T$ be the distribution for infectious components.

(iv) Define a positive linear operator

$$\forall \psi(x) = (\mathcal{V}_1(\psi)(x), \mathcal{V}_2(\psi)(x), \mathcal{V}_3(\psi)(x)), \quad \forall \psi \in \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y}, \quad x \in \mathcal{D},$$

by

$$\begin{cases}
\mathcal{V}_1(\psi)(x) = u_1^* e^{-d_1 \tau} \int_{\Omega} \Gamma(D_1 \tau, x, y) \left( \alpha \psi_2(y) + \frac{\beta_1}{K_1} \psi_3(y) + \frac{\beta_2}{K_2} \psi_4(y) \right) dy, \\
\mathcal{V}_2(\psi)(x) = \xi \psi_2(x), \\
\mathcal{V}_3(\psi)(x) = \delta \psi_3(x),
\end{cases}$$ (5.2)

which account for the sums of new infection caused by $(\psi_2(x), \psi_3(x), \psi_4(x))$ over domain $\Omega$ that survive the latent period.

(v) For $t \in [0, \tau)$, there will be no new actively infected cells available. For $t \geq \tau$, the new infectious individuals can be calculated as

$$\mathcal{V}_1(S(t- \tau) \psi)(x) = u_1^* e^{-d_1 \tau} \int_{\Omega} \Gamma(D_1 \tau, x, y) \left( \alpha (T_2(t- \tau) \psi_2)(y) + \frac{\beta_1}{K_1} (T_3(t- \tau) \psi_3)(y) + \frac{\beta_2}{K_2} (T_4(t- \tau) \psi_4)(y) \right) dy.$$

Thus, the total distribution of new infectious individuals is

$$\int_{\tau}^{\infty} \mathcal{V}_1(S(t- \tau) \psi)(x) dt = \int_{0}^{\infty} e^{-d_1 \tau} \int_{\Omega} \Gamma(D_1 \tau, x, y) \left( \alpha (T_2(t) \psi_2)(y) + \frac{\beta_1}{K_1} (T_3(t) \psi_3)(y) + \frac{\beta_2}{K_2} (T_4(t) \psi_4)(y) \right) dy dt.$$
Similarly, for \( t > 0 \), the total distribution of new vibrios are
\[
\int_{-\infty}^{\infty} \mathcal{V}_2(S(t - \tau)\psi)(x)\,dt = \xi \int_{-\infty}^{\infty} (T_2(t)\psi_2)(x)\,dt,
\]
and
\[
\int_{-\infty}^{\infty} \mathcal{V}_3(S(t - \tau)\psi)(x)\,dt = \delta_1 \int_{-\infty}^{\infty} (T_3(t)\psi_3)(x)\,dt,
\]
respectively.

(vi) Consequently, we can define the next infection operator (the distribution of the total infection caused by the initial distribution \( \psi \)) as follows:
\[
\mathcal{L}(\psi) := \int_{-\infty}^{\infty} V(S(t)\psi)(x)\,dt.
\]

By [24], we can define \( R_0 \) for model (2.4) as
\[
R_0 := r(\mathcal{L}),
\]
where \( r(\mathcal{L}) \) is the spectral radius of \( \mathcal{L} \).

The following result concerns the relationship between \( R_0 \) and \( \lambda(u^*_1) \).

**Lemma 5.1.** \( R_0 - 1 \) has the same sign as \( \lambda(u^*_1) \).

This together with Theorem 4.3 and Theorem 4.5 allow us to have the natural results in terms of \( R_0 \).

**Theorem 5.2.**

(i) If \( R_0 < 1 \), then \( E_0 \) is globally attractive in \( C^+_\tau \);

(ii) If \( R_0 > 1 \), then there is an \( \eta > 0 \) such that for any \( \phi \in C^+_\tau \) with \( \phi_2 \neq 0 \) or \( \phi_3 \neq 0 \) or \( \phi_4 \neq 0 \), we have
\[
\liminf_{t \to \infty} u_i(t, x, \phi) \geq \eta, \quad \text{for } i = 1, 2, 3, 4,
\]
uniformly for all \( x \in \Omega \).

Further system (2.4) admits at least one (component-wise) positive equilibrium \( \hat{u} \).

**Remark 1.** Note that Wang et al. [25] considered model (1.1) in an bounded domain of one dimension, where the model allowed the parameters to be space dependent functions. As we can see, after slight modifications, our main results on model (2.4) still hold if the constant parameters are replaced by strictly positive and spatial dependent functions in a bounded domain.

### 6. Global attractivity for positive equilibrium

In this section, we shall show the global attractivity for the positive equilibrium of system (2.4). Let
\[
[R_0] = \frac{e^{-d_1 \tau} A}{d(d + \gamma + m_2)} \left( \alpha + \frac{\xi \beta_1}{\delta_1 K_1} + \frac{\xi \beta_2}{\delta_2 K_2} \right).
\]

It then follows that system (2.4) has a unique constant positive equilibrium \( \hat{u} = (u^*_1, u^*_2, u^*_3, u^*_4) \) if \( [R_0] > 1 \), which satisfies
\[
\begin{align*}
\Lambda - du^*_1 - u^*_1 \left( \alpha u^*_2 + \beta_1 \frac{u^*_3}{u^*_3 + K_1} + \beta_2 \frac{u^*_4}{u^*_4 + K_2} \right) &= 0, \\
-(d + \gamma + m_2)u^*_2 + e^{-d_1 \tau} u^*_1 \left( \alpha u^*_2 + \beta_1 \frac{u^*_3}{u^*_3 + K_1} + \beta_2 \frac{u^*_4}{u^*_4 + K_2} \right) &= 0, \\
\xi u^*_2 - \delta_1 u^*_3 &= 0, \\
\delta_1 u^*_3 - \delta_2 u^*_4 &= 0.
\end{align*}
\]
(6.1)
In fact, the last two equations of (6.1) give us
\[ u_3^* = \frac{\xi}{\delta_1} u_2^* \quad \text{and} \quad u_4^* = \frac{\xi}{\delta_2} u_2^*, \]
together with the first two equations of (6.1), we have the following formula on \( u_2^* \):
\[ \Lambda \left( \alpha + \frac{a\beta_1}{au + K_1} + \frac{b\beta_2}{bu + K_2} \right) = e^{\tau \xi_2} d_I \left( d + \alpha u_2^* + \frac{a\beta_1 u_2^*}{au + K_1} + \frac{b\beta_2 u_2^*}{bu + K_2} \right), \]
where \( d_I = d + \gamma + m_2 \), \( a = \frac{\xi}{\delta_1} \) and \( b = \frac{\xi}{\delta_2} \). Denote
\[ \varpi(u) = \Lambda \left( \alpha + \frac{a\beta_1}{au + K_1} + \frac{b\beta_2}{bu + K_2} \right) - e^{\tau \xi_2} d_I \left( d + \alpha u + \frac{a\beta_1 u}{au + K_1} + \frac{b\beta_2 u}{bu + K_2} \right). \]
It is easy to check that \( \lim_{u \to -\infty} \varpi(u) = -\infty \) and
\[ \varpi(0) = \Lambda \left( \alpha + \frac{a\beta_1}{K_1} + \frac{b\beta_2}{K_2} \right) - e^{\tau \xi_2} d_I d > 0 \quad \text{if} \quad |R_0| > 1. \]
Moreover,
\[ \varpi'(u) = -\frac{\Lambda \beta_1 a^2}{(au + K_1)^2} - \frac{\Lambda \beta_2 b^2}{(bu + K_2)^2} - e^{\tau \xi_2} d_I \left( \alpha + \frac{a\beta_1 K_1}{(au + K_1)^2} + \frac{b\beta_2 K_2}{(bu + K_2)^2} \right) < 0. \]

Thus, there exists a unique positive constant \( u_2^* \) such that \( \varpi(u) = 0 \). Hence, we obtain the existence of unique constant positive equilibrium \( \hat{u} \) for system (2.4) if \( |R_0| > 1 \).

Next, we shall show that \( \hat{u} \) is globally attractive by appealing to establish a suitable Lyapunov functional. With the help of Theorem 3.2, we know that there exist two positive constants \( B_1, B_2 > 0 \) such that the set
\[ A = \{ u(t, x, \phi) \in C_+^r : u_1(\theta, x) \leq B_1, u_2(\theta, x) \leq B_2, \]
\[ u_3(\theta, x) \leq \frac{2\xi B_2}{\delta_1}, u_4(\theta, x) \leq \frac{4\xi B_2}{\delta_2}, \forall \theta \in [-\tau, 0], x \in \Omega \} \]
is positively invariant. The main results of this section is the following theorem.

**Theorem 6.1.** If \( |R_0| > 1 \), then \( \hat{u} \) with initial value \( \phi_2 \neq 0 \) or \( \phi_3 \neq 0 \) or \( \phi_4 \neq 0 \), is globally attractive in \( C_+^r \), that is,
\[ \lim_{t \to \infty} u(t, x, \phi) = \hat{u}, \quad \text{uniformly for} \quad x \in \Omega. \]

**Proof.** For simplicity, we use \( u_i \) short for \( u_i(t, x) \) \( (i = 1, 2, 3, 4) \), and denote
\[ f_1(u_3) = \frac{u_3}{u_3 + K_1} \quad \text{and} \quad f_2(u_4) = \frac{u_4}{u_4 + K_1}. \]
Let \( g(\varsigma) = \varsigma - 1 - \ln \varsigma \) for \( \varsigma > 0 \). Clearly, \( g(\varsigma) \geq 0 \) for \( \varsigma > 0 \). Define a Lyapunov functional as follows,
\[ V(t) = \int_{\Omega} \left( \sum_{i=1}^{4} L_i(t, x) + \sum_{i=1}^{3} W_i(t, x) \right) dx \]
with
\[ L_1 = u_3^* g \left( \frac{u_1}{u_4^*} \right), \quad L_2 = e^{\tau \xi_2} u_3^* g \left( \frac{u_2}{u_4^*} \right), \quad L_3 = A u_3^* g \left( \frac{u_3}{u_4^*} \right), \quad L_4 = B u_4^* g \left( \frac{u_4}{u_4^*} \right) \]
and
\[ W_1 = \alpha u^*_1 u_2^* \int_{-\tau}^{0} \int_{\Omega} \Gamma(DI(-\theta), x, y) g \left( \frac{u_1(t + \theta, y) u_2(t + \theta, y)}{u_1^* u_2^*} \right) dy d\theta, \]
We then calculate the derivatives of $W$:

$$W_2 = \beta_1 u_1^* f_1(u_1^*) \int_{-\tau}^{0} \int_{\Omega} \Gamma(D_1(-\theta), x, y) g \left( \frac{u_1(t + \theta, y) f_1(u_2(t + \theta, y))}{u_1^* u_2^*} \right) \, dy \, d\theta,$$

$$W_3 = \beta_2 u_1^* f_2(u_1^*) \int_{-\tau}^{0} \int_{\Omega} \Gamma(D_1(-\theta), x, y) g \left( \frac{u_1(t + \theta, y) f_2(u_4(t + \theta, y))}{u_1^* u_4^*} \right) \, dy \, d\theta,$$

$A$ and $B$ are two constants will be determined later. Using the fact that $\Lambda = du_1^* + u_1^* (\alpha u_2 + \beta_1 f_1(u_3) + \beta_2 f_2(u_4))$, the differential of $L_1(t, x)$ is calculated as follows,

$$\frac{\partial L_1}{\partial t} = \left( 1 - \frac{u_1^*}{u_1} \right) (D_S \Delta u_1 + \Lambda - du_1 - u_1(\alpha u_2 + \beta_1 f_1(u_3) + \beta_2 f_2(u_4)))$$

$$= \left( 1 - \frac{u_1^*}{u_1} \right) D_S \Delta u_1 - \frac{d}{u_1}(u_1^* - u_1)^2 + \left( 1 - \frac{u_1^*}{u_1} \right) \left[ u_1^* (\alpha u_2^* + \beta_1 f_1(u_3^*) + \beta_2 f_2(u_4^*)) - u_1^* (\alpha u_2 + \beta_1 f_1(u_3) + \beta_2 f_2(u_4)) \right].$$

We then calculate the derivatives of $L_2$, $L_3$ and $L_4$ as follows,

$$\frac{\partial L_2}{\partial t} = e^{d_1 \tau} \left( 1 - \frac{u_2^*}{u_2} \right) \left( D_1 \Delta u_2 - (d + \gamma + m_2) u_2 + e^{-d_1 \tau} \int \Gamma(D_1 \tau, x, y) u_1(t - \tau, y) \right)$$

$$\times \left( \alpha u_1^* (t - \tau, y) + \beta \frac{u_3^* (t - \tau, y) + K_1}{u_3^* (t - \tau, y) + K_2} + \beta_2 \frac{u_4^* (t - \tau, y) + K_2}{u_4^* (t - \tau, y) + K_2} \right) \, dy,$$

$$\frac{\partial L_3}{\partial t} = A \left( 1 - \frac{u_3^*}{u_3} \right) (D_1 \Delta u_3 + \xi u_2 - \delta_1 u_3)$$

and

$$\frac{\partial L_4}{\partial t} = B \left( 1 - \frac{u_4^*}{u_4} \right) (D_2 \Delta u_4 + \delta_1 u_3 - \delta_2 u_4).$$

We are now in a position to deal with $W_i$, $i = 1, 2, 3$. Firstly,

$$\frac{d}{d\theta} \left( \Gamma(D_1(-\theta), x, y) g \left( \frac{u_1(t + \theta, x) u_2(t + \theta, x)}{u_1^* u_2^*} \right) \right)$$

$$= g \left( \frac{u_1(t + \theta, x) u_2(t + \theta, x)}{u_1^* u_2^*} \right) \frac{d}{d\theta} \Gamma(D_1(-\theta), x, y)$$

$$+ \Gamma(D_1(-\theta), x, y) \frac{d}{d\theta} g \left( \frac{u_1(t + \theta, x) u_2(t + \theta, x)}{u_1^* u_2^*} \right).$$

Then

$$\frac{\partial W_1}{\partial t} = \alpha u_1^* u_2^* \int_{-\tau}^{0} \int_{\Omega} \Gamma(D_1(-\theta), x, y) \frac{\partial}{\partial \theta} g \left( \frac{u_1(t + \theta, y) u_2(t + \theta, y)}{u_1^* u_2^*} \right) \, dy \, d\theta$$

$$= \alpha u_1^* u_2^* \int_{-\tau}^{0} \int_{\Omega} \Gamma(D_1(-\theta), x, y) \frac{\partial}{\partial \theta} g \left( \frac{u_1(t + \theta, y) u_2(t + \theta, y)}{u_1^* u_2^*} \right) \, dy \, d\theta$$

$$= \alpha u_1^* u_2^* \int_{-\tau}^{0} \int_{\Omega} \frac{\partial}{\partial \theta} \left[ \Gamma(D_1(-\theta), x, y) g \left( \frac{u_1(t + \theta, y) u_2(t + \theta, y)}{u_1^* u_2^*} \right) \right] \, dy \, d\theta$$

$$- \alpha u_1^* u_2^* \int_{-\tau}^{0} \int_{\Omega} g \left( \frac{u_1(t + \theta, y) u_2(t + \theta, y)}{u_1^* u_2^*} \right) \frac{\partial}{\partial \theta} \Gamma(D_1(-\theta), x, y) \, dy \, d\theta.$$
Similarly, we have
\[
\frac{\partial W_1}{\partial t} = \alpha u_1^* u_2^* g \left( \frac{u_1 u_2}{u_1^* u_2^*} \right) - \alpha u_1^* u_2^* \int_\Omega \Gamma(D_I, x, y) g \left( \frac{u_1(t - \tau, y) u_2(t - \tau, y)}{u_1^* u_2^*} \right) dy.
\]

Due to the fact that system (2.4) subjects to Neumann boundary condition, we directly have
\[
\int_\Omega \Delta u_i \, dx = \int_\Omega \frac{\|\nabla u_i\|^2}{u_i^2} \, dx \quad \text{and} \quad \int_\Omega \Delta u_i \, dx = 0, \quad i = 1, 2, 3, 4.
\]

Now, we define two constants \( A \) and \( B \) as follows,
\[
A = \frac{(\beta_1 u_1^* f_1(u_3^*) + \beta_2 u_1^* f_2(u_4^*))}{\xi u_2^*}, \quad B = \frac{(\beta_2 u_1^* f_2(u_4^*))}{\delta_1 u_3^*}.
\]

This together with the last three equations of (6.1) and the trick that \( \ln \frac{u_i}{u_i^*} + \ln \frac{\dot{u_i}}{u_i} = 0 \) (\( i = 3, 4 \)) implies that
\[
\frac{dV}{dt} = -D_S u_1^* \int_\Omega \|\nabla u_1\|^2 \, dx - D_I u_2^* e^{d_I \tau} \int_\Omega \|\nabla u_2\|^2 \, dx - AD_1 u_3^* \int_\Omega \|\nabla u_3\|^2 \, dx
\]
\[
- BD_2 u_4^* \int_\Omega \frac{\|\nabla u_4\|^2}{u_4^2} \, dx - \int_\Omega \frac{d}{u_1^*} (u_1^* - u_1) \, dx - \int_\Omega \alpha u_1^* u_2^* g \left( \frac{u_1}{u_1^*} \right) \, dx
\]
\[
- \beta_1 u_1^* f_1(u_3^*) \int_\Omega \Gamma(D_I, x, y) g \left( \frac{u_1(t - \tau, y) f_1(u_3(t - \tau, y)) u_2}{u_1^* f_1(u_3) u_2} \right) \, dy \, dx
\]
\[
- \beta_1 u_1^* f_1(u_3^*) \int_\Omega \Gamma(D_I, x, y) g \left( f_1(u_3) u_2^* + g \left( \frac{u_2 u_2^*}{u_2^* u_3} \right) \right) \, dx
\]
\[
- \beta_2 u_1^* f_2(u_4^*) \int_\Omega \Gamma(D_I, x, y) g \left( \frac{u_1(t - \tau, y) f_2(u_4(t - \tau, y)) u_2}{u_1^* f_2(u_4) u_2} \right) \, dy \, dx
\]
\[
+ \beta_1 u_1^* f_1(u_3^*) \int_\Omega \left( \frac{f_1(u_3) u_2}{f_1(u_3)} + 1 - \frac{u_3}{u_3^*} \right) \, dx
\]
\[
+ \beta_2 u_1^* f_2(u_4^*) \int_\Omega \left( \frac{f_2(u_4) u_2}{f_2(u_4)} + 1 - \frac{u_4}{u_4^*} \right) \, dx.
\]
Since \( f_1(u_3) = \frac{u_3}{u_3 + K_1} \) and \( f_2(u_4) = \frac{u_4}{u_4 + K_2} \), we have
\[
\frac{f_1(u_3)}{f_1(u_3)} + \frac{f_1(u_3^*) u_3}{f_1(u_3) u_3^*} - 1 = -\frac{K_1(u_3 - u_3^*)^2}{K_1 u_3^*(u_3 + K_1)(u_3 + K_1)}
\]
and
\[
\frac{f_2(u_4)}{f_2(u_4^*)} + \frac{f_2(u_4^*) u_4}{f_2(u_4) u_4^*} - 1 = -\frac{K_2(u_4 - u_4^*)^2}{K_2 u_4^*(u_4^* + K_2)(u_4 + K_2)},
\]
which lead to \( \frac{dV}{dt} \leq 0 \). Recall that the solution semi-flow is defined by \( \Phi(t) \phi = u_t(\phi) \), for any \( \phi \in C^r_\tau \), we know that the map \( t \to V(u_t(\phi)) \) is non-increasing. Moreover, \( V(u_t(\phi)) \) is bounded from below and there is some constant \( V_\infty > 0 \) such that \( \lim_{t \to \infty} V(u_t(\phi)) = V_\infty \). With a similar arguments in [10], denote by \( \gamma^+(t) := \{ \Phi(t) \phi, \forall t \geq 0 \} \) the orbit of solution semi-flow \( \Phi(t) \). Let \( \omega(\phi) \) be the omega limit set of \( \gamma^+(t) \). For any \( \chi \in \omega(\phi) \), we can choose a sequence \( t_n \) with \( t_n \to \infty \) as \( n \to \infty \) such that \( \lim_{t \to \infty} u_{t_n}(\phi) = \chi \), which means that \( V(\chi) = V_\infty \).

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E-mail address: jinliangwang@hlju.edu.cn