Radiatively-Induced First-Order Phase Transitions:  
The Necessity of the Renormalization Group

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Abstract

We advocate a (Wilson) renormalization-group (RG) treatment of finite-temperature first-order phase transitions, in particular those driven by radiative corrections such as occur in the standard model, and other spontaneously-broken gauge theories. We introduce the scale-dependent coarse-grained free energy $S_\Lambda[\phi]$ which we explicitly calculate, using the Wilson RG and a $(4-\varepsilon)$-expansion, for a scalar toy model that shares many features of the gauged case. As argued by Langer and others, the dynamics of the phase transition are described by $S_\Lambda[\phi]$ with $\Lambda$ of order the bubble wall thickness, and not by the usual (RG-improved) finite-temperature effective action which is reproduced by $S_\Lambda[\phi]$ for $\Lambda \to 0$. We argue that for weakly first-order transitions (such as that in the standard model) the $(4-\varepsilon)$-expansion is necessary to control an inevitable growth of the effective scale-dependent coupling towards the strong-coupling regime, and that diagrammatic resummation techniques are unlikely to be appropriate.
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1 Introduction

Systems that undergo first-order phase transitions as a function of temperature have been extensively discussed in the particle physics literature during the last decade. The initial motivation was the role of GUT-scale first-order phase transitions in driving an inflationary period of expansion in the very early universe \cite{1}. Lately the primary motivation has been to show that the observed baryon asymmetry of the universe can be produced through the baryon and lepton number anomalies, in concert with the out-of-equilibrium evolution of a first-order electroweak phase transition \cite{2}.

Most of the systems studied in these two contexts have been gauge theories where the first-order transition is \textit{radiatively induced} (\textit{fluctuation-driven} in the condensed-matter terminology we will employ in this paper). This means that the tree-level scalar potential has a form that implies a second-order phase transition, but when fluctuations in the fields are taken into account (at one-loop or higher) the potential indicates a first-order phase transition. If the transition is strongly first-order then the effective potential may be calculated by the standard loop expansion. However if it is weakly first-order (generically, when \((M_h/M_v)^2 \gtrsim 1\), where \(M_h\) and \(M_v\) are the Higgs and gauge boson masses) then the naive loop expansion breaks down, we must use the renormalization group (RG) to obtain reliable results \cite{3,4}. Ginsparg \cite{5} has investigated the Gell-Mann–Low RG flows of Yang-Mills-Higgs systems in the heavy Higgs regime, and found that the phase transition remains first order (specifically, there were no stable fixed points within the \((4 - \varepsilon)\)-expansion) for reasonable scalar sectors.

In this paper we propose to study the properties of weakly first-order phase transitions by the systematic use of the Wilson RG, together with an expansion in the number of spatial dimensions around four (the \((4 - \varepsilon)\)-expansion) \cite{6}. Following earlier work in the condensed matter literature, we will take the view that the appropriate quantity for the analysis of a first-order phase transition is the \textit{scale dependent} ("coarse-grained") effective Hamiltonian defined by successively integrating out momentum modes. In the limit of infinite coarse-graining scale, this coincides with the effective action, and correctly describes the static properties of the system. However for a correct description of the dynamics we must coarse-grain only up to a physical scale, typically the correlation length. This approach has been advocated by Langer and coworkers \cite{7}, and by Kawasaki, Imaeda, and Gunton \cite{8}, and is also, of course, the strategy proposed by Wilson in his work on critical phenomena. We will explicitly calculate the coarse-grained effective potential for a toy model which possesses a fluctuation-induced first-order phase transition similar to that of the standard model.

An important motivation for our RG study is that the current experimental lower
bound on the Higgs mass is around 60 GeV, so that the naive loop expansion for
the electroweak phase transition is breaking down. Over the last few years there
have been many attempts in the particle physics literature to remedy this situation
[2]. They have mainly consisted of attempts to sum certain infinite subclasses of
diagrams (for example “daisy” or “super-daisy” graphs, and “ring-improvement”) and
thus improve perturbation theory. We believe that the RG analysis is an advance over
these diagrammatic methods for three reasons.

Firstly, the RG analysis that we will discuss below generically leads to results for the
properties of the phase transition (such as latent heats, bubble wall tensions etc.) that are
highly non-analytic as a function of initial couplings (and ε). This non-analytic behaviour
is quite easily handled by the RG, but is very difficult to reproduce by diagrammatic
methods.

Second, as we will see in detail below, the typical flows of the effective couplings are
such as to take them first towards, and then away from, an unstable fixed point. These
unstable fixed points occur at values of the couplings of order $2\pi^2\varepsilon$. As $\varepsilon \to 1$ (the
physical value) this is far into the region of strong coupling where even the (non-Borel-
resummed) renormalization group improved loop expansion is invalid. Diagrammatic
methods obscure this problem rather than solving it. In this paper we will take $\varepsilon \ll 1$ for
most of our analysis, only setting $\varepsilon = 1$ at the final stage. The motivation, and dangers
[4], of this approach we will discuss at the end of the Introduction, and in Section 5.

Thirdly, the Wilson RG flow of a theory down to a scale $\Lambda$ is by definition the
procedure which gives us the correct Hamiltonian for the dynamics of the modes of
momentum $|k| < \Lambda$, when fluctuations on scales $|k| > \Lambda$ are taken into account
(integrated out). Thus, at a sensibly chosen scale, it is the relevant object for studying
the dynamics of the phase transition. For example, the properties of the critical bubbles
that nucleate the phase transition, such as the bubble wall tension, are determined by
the effective Hamiltonian with coarse-graining length of order the correlation length [7].
We show that the effective Hamiltonian at this coarse-graining scale calculably differs
from the usual effective action (which corresponds to an infinite coarse-graining length).

Rather than confront the full complication of the phase transition of a GUT or the
electroweak model, in this paper we will study the following purely scalar system:

$$
\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi) \cdot (\partial^\mu \varphi) - \frac{\mu^2}{2} \varphi \cdot \varphi + \frac{h_1}{4!} (\varphi \cdot \varphi)^2 + \frac{h_2}{4!} \sum_{i=1}^{N} (\varphi_i \cdot \varphi_i)^4.
$$

(1.1)

Here $\varphi$ is an $N$-component real scalar field, and $\varphi \cdot \varphi = \sum_{i=1}^{N} \varphi_i \cdot \varphi_i$. Inclusion of the second
quartic coupling $h_2$ breaks the original continuous $O(N)$ symmetry of the model down to
the discrete hypercubic group in $N$-dimensions. This system exhibits a close analog to the

weakly-first-order fluctuation driven phase transition that occurs in the standard model (and most other Yang-Mills-Higgs systems) for suitable Higgs mass. This will enable us to discuss some very important qualitative features of the physics of such phase transitions that have been missed by previous authors, without the inessential complication of gauge invariance. In the final section of this paper we will explain why our toy system is indeed a good theoretical laboratory for the general, gauged, case.

There is one caveat that we should mention at this point. As one of has argued in an earlier publication [4] the results of a $4-\varepsilon$ expansion can be unreliable; in particular, new fixed points of the RG can appear at $\varepsilon = 1$ that are not accessible by such an expansion, rendering the transition second-order after all. However very useful information can still be extracted if the phase transition really is first-order, since its properties (such as the latent heat, critical bubble parameters etc.) can be calculated as an (asymptotic) expansion in $\varepsilon$. Of course, the leading order calculation that we undertake in this work is not sufficient to accurately determine the properties at $\varepsilon = 1$. However we are hopeful that higher-order calculations, in concert with asymptotic estimates of large-order $\varepsilon$-perturbation theory, can supply accurate parameters in $D = 3$, as is the case for second-order phase transitions [9]. Whether or not this succeeds, we have still extracted information about the limit $\varepsilon \rightarrow 0$ which other proposed calculations must be able to reproduce if they are correct.

Before starting, we note that the RG has been previously applied to the cubic anisotropic model (1.1) by Rudnick [10], however he specialized to $N = 2$, used classical statistical mechanics rather than finite temperature field theory, and used specific methods whose generalization to more complicated (e.g. gauge) theories is not immediate; and also by Amit [11], who used the Gell-Mann–Low RG in which there is no finite coarse-graining scale. We have derived our evolution equations from Polchinski’s general formulation [12], using the perturbative expansion of Bonini, D’Attanasio, and Marchesini [13]. This makes it relatively straightforward to push our results to higher order in the couplings, and even to include the evolution of derivative terms.

Finally, we should mention that the philosophy of coarse-graining has also been advocated recently by Tetradis and Wetterich [14], whose “average potential” is a smoothly cut-off version of the Wilson coarse-grained free energy (however, for a critique of the smooth cutoff see Ref. [21]). They obtain accurate critical exponents for the second-order phase transition in an $O(N)$ model, but do not study first-order phase transitions.
2 The coarse-grained potential

The Wilson RG [3] is based on the idea of gradually integrating out degrees of freedom in a field theory by lowering the ultraviolet cutoff $\Lambda$, and at the same time varying the action functional in such a way that all Green functions of the remaining modes are invariant. We first describe how this is accomplished for zero temperature Euclidean quantum field theories, or equivalently classical statistical mechanical systems, and then generalize to finite temperature quantum field theories.

2.1 The coarse-grained potential at zero temperature

An elegant account of Wilson coarse-graining was given by Polchinski [12] (see also Morris [21]). Following his analysis we consider a simple scalar field theory defined by the functional integral

$$Z[J] = \int \left[ \mathcal{D}\phi \right] \exp \left\{ \int_k \left[ -\frac{1}{2} \phi(k)(k^2 + m^2)K^{-1}(k^2/\Lambda^2)\phi(-k) + J(k)\phi(-k) \right] + L_\Lambda[\phi] \right\}$$

$$\equiv \int \left[ \mathcal{D}\phi \right] \exp S_\Lambda[\phi],$$

(2.1)

where $K(k^2/\Lambda^2)$ is a smooth UV cutoff function that falls quickly to zero for $k^2 > \Lambda^2$. The generating functional, $Z$, is independent of $\Lambda$ (up to physically irrelevant field-independent changes of overall normalization) if the interaction Lagrangian $L$ satisfies

$$\Lambda \frac{dL[\phi]}{d\Lambda} = -\frac{1}{2} \int_k \frac{(2\pi)^8}{(k^2 + m^2)^2} \Lambda \frac{dK(k^2/\Lambda^2)}{d\Lambda} \left\{ \frac{\delta L}{\delta\phi(k)} \frac{\delta L}{\delta\phi(-k)} + \frac{\delta^2 L}{\delta\phi(k)\delta\phi(-k)} \right\}$$

(2.2)

together with $J(k) = 0$ for $k^2 > \Lambda^2$. The Green’s functions of all the modes $\phi(k)$ with $k^2 \leq \Lambda^2$ are then independent of $\Lambda$. No subtleties arise in the verification of these formulae since, with a physical cutoff, no divergences are present. The resultant coarse-grained effective action $S_\Lambda[\phi]$ is the action that describes the dynamics of the degrees of freedom with momentum $k \leq \Lambda$. In the infinitely coarse-grained limit $\Lambda \to 0$ it becomes the standard effective action.

Note that the evolution equation (2.2) is an exact functional equation for the coarse-grained effective action. We wish to use this equation (actually a generalization to finite temperature) to study the RG flow of our model system Eq. (1.1). It is, however, in an inconvenient form for our purposes – in particular the derivation of the “beta-functions” for the flow of the couplings is rather involved in this formulation.

We will therefore employ the Wilson RG in a formulation originally constructed by Wegner and Houghton [13] in a statistical mechanical context, and further elaborated by Nicoll, Chang, and Stanley [16]. This formulation was then used to give the first
logically complete derivation of a coarse-grained effective action by Kawasaki, Imaeda, and Gunton (KIG) \cite{8}. (There are other discussions of this topic in the condensed-matter literature, see \cite{17} and \cite{18}, but overall there has been very little work on the derivation of the coarse-grained free-energy from a microscopic Hamiltonian.) Our choice of toy system Eq. (1.1) allows one major technical simplification in the passage from the Wilson-Polchinski equation (2.2) to the KIG form. Since our model involves only scalar fields with quartic interaction terms in the initial action, wavefunction renormalization only arises at two loops and above in the perturbative (coupling constant) expansion. This in turn implies that wavefunction renormalization is an $O(\varepsilon^2)$ effect. However we will see that a calculation to $O(\varepsilon)$ is sufficient to uncover the properties of the phase transition, and therefore, we can neglect the renormalization of the kinetic term to the order in which we work. Furthermore, all higher derivative terms generated as a result of the RG procedure are similarly higher order in the $\varepsilon$-expansion. (Of course at very large momentum – in other words for field configurations with large spatial derivatives – the higher derivative terms can dominate the kinetic term. However for the configurations of interest, such as true vacuum bubbles, the gradients are such that this estimate is correct.) We can therefore ignore the renormalization of all momentum dependent terms – only the potential terms are renormalized to $O(\varepsilon)$. This is a significant technical simplification relative to, say, scalar QED, or any other gauged system, where we would have to include wavefunction renormalization. We note in passing that it is possible to generalize our analysis to this case – this work will be reported in a future publication.

In the case where we can ignore the flow of the derivative terms, the Wilson-Polchinski equation (2.2) can be recast in the form:

$$\frac{\partial V_\Lambda(\varphi)}{\partial \Lambda} = \frac{K_{(D+1)}}{2} \Lambda^D \text{tr}_{ij} \ln \left[ \Lambda^2 \delta_{ij} + \frac{\partial^2 V_\Lambda}{\partial \varphi_i \partial \varphi_j} \right].$$

(2.3)

This is our master equation for the flow of the potential of an $N$-component scalar field theory in $(D + 1)$ Euclidean dimensions. Here $K_{(D+1)} = S_D/(2\pi)^{(D+1)}$, and $S_D$ is the surface area of a unit $D$-sphere. This equation was first explicitly written down in Ref. \cite{8}, so we will refer to it as the KIG-equation. In Appendix A we show that it can be derived from Ref.’s. \cite{13} perturbative analysis of Polchinski’s exact RG flow equation. Intuitively, however, Eq. (2.3) seems very reasonable: it states that the contribution to the effective potential from integrating out the degrees of freedom in an infinitesimal shell $\Lambda - \delta \Lambda < |k| < \Lambda$ in momentum space is just the standard loop result \cite{19} for the effective potential, with the momentum integral restricted to that shell.

The KIG equation as formulated above describes the flow of statistical mechanical systems, which correspond to zero temperature Euclidean quantum field theories. Given
the potential at some high cutoff $\Lambda_0$, it shows us how to flow down to some scale of interest $\Lambda_1$ and obtain $V_{\Lambda_1}(\phi)$. Following KIG, we proceed in two stages:

1. Expand $V_\Lambda(\phi)$ as a truncated power series in $\phi$, including all the relevant operators (here defined to include the marginal operators as well). Recall that relevant operators are those whose couplings do not have large negative mass dimension for small $\varepsilon$. We obtain coupled differential equations for the relevant couplings (only), which in the case of $\phi^4$ theory near four space-time dimensions are the mass-squared $m^2$ and the $\phi^4$ coupling(s) $h_i$, which we solve to obtain $m^2(\Lambda)$ and $h_i(\Lambda)$.

2. Substitute the solutions of these equations into the RHS of the KIG equation, and integrate from $\Lambda_0$ to $\Lambda_1$ to obtain the higher order corrections to $V_{\Lambda_1}$, i.e. $\phi^6$ and higher. In general one must check for self-consistency—the “back-reaction” effect of these truncated couplings on the differential equations for the relevant couplings must be higher order in some small parameter.

Note the standard 1-loop effective potential (with ultraviolet cutoff $\Lambda_0$ and infra-red cutoff $\Lambda_1$) may be obtained by the crude approximation of skipping the first step, and substituting $m(\Lambda_0)$ and $g(\Lambda_0)$ directly into the RHS in the second step.

In practical applications the theory would be specified by renormalization conditions on $m^2$ and the $h_i$ at a low momentum scale, so that at stage 1 we would choose values of $m^2(\Lambda_0)$ and $h_i(\Lambda_0)$ that give the right values when flowed down using Eq. (2.3), then at stage 2 we would calculate the remaining terms in the low-momentum potential. This is done because experimental measurements are typically made at energy scales low relative to the UV cutoff. In this paper, however, we are dealing with a toy model, and hence will express our results in terms of couplings defined at a high energy scale $\Lambda_t$, defined in the next subsection.

### 2.2 Coarse-grained potential at finite temperature

The generalization of the KIG flow equation to finite-temperature becomes obvious if we recall the rules for constructing the finite-temperature (equilibrium) generating functional for a field theory: We compactify the Euclidean time direction on a circle of circumference $\beta = 1/T$. This results in a discrete set of fourier modes $\varphi_{i,n}(k)$, with $n \in \mathbb{Z}$ — the Matsubara modes. The $(D+1)$-momentum becomes $k = (2\pi n T, k)$, and integrals over $k_0$ are replaced by a sum over $n$. For example, the finite-temperature generating functional
for a simple $\lambda\phi^4$ theory, is, in momentum space:

$$Z[J,T] = \int [D\phi] \exp \left\{ -T \sum_n \int_{k} \left( \frac{1}{2} (k^2 + (2\pi nT)^2 - m^2) \phi_n(k) \phi_{-n}(-k) + J \cdot \phi \right) - \frac{\lambda}{4!} T^3 \sum_{n,n',n''} \int_{k,k',k''} \phi_n(k) \phi_{n'}(k') \phi_{n''}(k'') \phi_{-n-n'-n''}(-k-k'-k'') \right\}.$$  

(2.4)

It will be convenient to simplify this expression so that we have a conventionally normalized kinetic term – this is achieved by rescaling the field $\phi_n(k)$ by a factor of $\sqrt{T}$. With this choice, the quartic term is proportional to $\lambda T$. This is convenient since a field theory in three dimensions has a quartic coupling of mass dimension one, which in this case is soaked up by $T$, so $\lambda$ stays dimensionless.

In any case, the important point about Eq. (2.4) is that it is of the form of (infinitely) many coupled “species” of scalar field, indexed by $n$, living in one lower dimension. Therefore we can directly apply the KIG equation (2.3) to this system – actually the $N$-component version – to find,

$$\frac{\partial V_T^\Lambda(\varphi)}{\partial \Lambda} = \frac{K_D}{2} \Lambda^{(D-1)} \text{tr}_{i,j}(n,m) \ln \left[ \Lambda^2 \delta_{i,j} \delta_{n-m} + \frac{\partial^2 V_T^\Lambda}{\partial \phi_i \partial \phi_j} \right],$$  

(2.5)

as the appropriate generalization of the KIG equation to finite-temperature. Remember that in this equation the potential now includes a temperature-dependent effective (mass)$^2$ term, $(2\pi nT)^2$, for the $n \neq 0$ modes. (We employ the tr$_{(n,m)}$ notation as a shorthand for a sum over Matsubara modes where the discretized energy is conserved at each vertex.)

There are two interesting limits of this formula. For $\Lambda \gg 2\pi T$, the Matsubara frequencies are so close that the “trace” over $n$ approximates the original $(D + 1)$-dimensional integral, so Eq. (2.3) is valid. For $\Lambda \ll 2\pi T$, the sum is dominated by the $n = 0$ mode – the very massive $n \neq 0$ modes decouple leaving behind temperature-dependent corrections to the potential, and the system looks $D$-dimensional. In this limit Eq. (2.3) thus simplifies to

$$\frac{\partial V_T^\Lambda(\varphi_0)}{\partial \Lambda} = \frac{K_D}{2} \Lambda^{(D-1)} \text{tr}_{i,j} \ln \left[ \Lambda^2 \delta_{i,j} + \frac{\partial^2 V_T^\Lambda}{\partial \phi_i \partial \phi_j} \right],$$  

(2.6)

with the addition of a particular, temperature dependent, $V_T^\Lambda(\varphi_0)$ as a boundary condition at the starting scale $\Lambda_t < T$.

We must remember, however, that for a finite temperature field theory the renormalization conditions are still imposed on the zero temperature theory. The correct procedure is therefore to select some high scale $\Lambda_0 \gg T$, and choose $m^2(\Lambda_0)$ and $g(\Lambda_0)$
such that the renormalization conditions are obeyed when we flow down to low momentum in the zero temperature theory, i.e. using (2.3). To calculate the finite temperature coarse-grained effective potential we should in principle run down from $\Lambda_0$ using Eq. (2.5), however the sum over Matsubara frequencies is difficult to perform analytically.

Therefore in this paper we will instead arbitrarily choose some scale $\Lambda_t$ somewhat below $T$, so that we can make the reasonable approximation of completely integrating out the massive $n \neq 0$ modes. The running is then $D$-dimensional (in the physical case three-dimensional) below $\Lambda_t$ and described by Eq. (2.6). We will use the results of Ginsparg’s perturbative calculation [5] of the effect of eliminating the non-zero modes to approximately calculate the resulting “boundary condition” potential at $\Lambda_t$. Note that no infra-red difficulties arise in integrating out the $n \neq 0$ modes as their propagators are cut off by large effective masses $m_n^2 = (2\pi n T)^2$. Also note that we should not take $\Lambda_t$ too much smaller than $T$ or else the couplings of the effective Lagrangian will flow significantly due to fluctuations in $\phi_i(0)$, for $k > \Lambda_t$. Note that if greater accuracy is required, it is (numerically) straightforward to perform the correct procedure outlined above.

3 Application to the cubic anisotropy model

After this long preamble we are finally ready to calculate the appropriate initial three-dimensional action at $\Lambda_t$ for our toy system, and then give a general account of its fluctuation-driven first-order phase transition. Detailed analysis will be given in the next section.

We are working at finite temperature $T$, which enters the problem in two ways. Firstly, as we mentioned above, we rescale the fields by a factor of $\sqrt{T}$ so that all couplings in the action have their usual engineering dimension for a three-dimensional field theory, the four-dimensional couplings being modified by factors of $T$. The mass term is unaffected, but the $\phi^4$ couplings are redefined

$$g_i(\Lambda) \equiv h_i(\Lambda)T,$$

where $h_i$ are the dimensionless (4D) couplings.

Second, the elimination of the $n \neq 0$ modes shifts the values of the couplings $\mu^2$ and $h_i$ in Eq. (1.1), and also introduces higher dimension terms. However, only the correction to the value of $\mu^2$ is not suppressed by powers of the small couplings $h_i$ [3]. For instance the shift in the $\phi^4$ terms goes like $h_i^2$ — suppressed by one power of $h_i$.

For our model the
correction to the mass term is approximately
\[ m^2(\Lambda_t) = -\mu^2 + \frac{(N + 2)h_1 + 3h_2 T^2}{72} \]  
(3.2)
where as we will see the first-order transition temperature \( T_c \) is very close to the temperature where \( m^2(T) = 0 \).

Our project is therefore to analyse the phase transitions of the \( N \) component scalar field theory defined by its truncated potential \( V^{\text{tr}} \) at \( \Lambda_t \) (Eqs. (3.3), (3.2)) whose flow into the IR is given by Eq. (2.6). We will now follow stages 1 and 2 of the procedure suggested at the end of Sec. 2.1, first examining the flow of \( V^{\text{tr}} \), then evaluating the full scale-dependent potential.

The “truncated potential”, whose flows we study, is
\[ V^{\text{tr}}(\varphi) = \frac{1}{2} m^2(l) \varphi \cdot \varphi + \frac{g_1(l)}{4!} (\varphi \cdot \varphi)^2 + \frac{g_2(l)}{4!} \sum_{i=1}^{N} (\varphi_i)^4, \]
(3.3)
where we have replaced the coarse-graining scale \( \Lambda \) with a dimensionless variable \( l, \)
\[ \Lambda = \exp(-l) \Lambda(0), \]
(3.4)
so \( l \) goes to infinity in the IR, and we choose our units of energy such that \( \Lambda(0) = \Lambda_t = 1 \).

From the high temperature KIG equation (2.6) we obtain the flow equations
\[
\frac{d m^2}{d l} = \frac{K_D e^{-Dl}}{e^{-2l} + m^2} \frac{1}{6} \left( (N + 2)g_1 + 3g_2 \right), \\
\frac{d g_1}{d l} = -\frac{K_D e^{-Dl}}{(e^{-2l} + m^2)^2} \left( g_1 g_2 + (N + 8)g_1^2 / 6 \right), \\
\frac{d g_2}{d l} = -\frac{K_D e^{-Dl}}{(e^{-2l} + m^2)^2} \left( 2g_1 g_2 + (3/2)g_2^2 \right).
\]
(3.5)
A more useful set of variables is obtained by rescaling by the floating cutoff,
\[ r(l) \equiv m^2(l)/\Lambda(l)^2, \]
\[ \lambda_i(l) \equiv g_i(l)/\Lambda(l)^{(4-D)}. \]
(3.6)
These are the variables in which a second order phase transition is described by a fixed point. Writing \( (4 - D) \equiv \varepsilon \), the RG equations for \( \{ r, \lambda_1, \lambda_2 \} \) are:
\[
\frac{d r}{d l} = 2r(l) + \frac{K_D}{1 + r} \frac{1}{6} \left( (N + 2)\lambda_1 + 3\lambda_2 \right), \\
\frac{d \lambda_1}{d l} = \varepsilon \lambda_1(l) - \frac{K_D}{(1 + r)^2} \left( \lambda_1 \lambda_2 + (N + 8)\lambda_1^2 / 6 \right), \]
(3.7)
\[ \frac{d \lambda_2}{d l} = \varepsilon \lambda_2(l) - \frac{K_D}{(1 + r)^2} \left( 2\lambda_1 \lambda_2 + (3/2)\lambda_2^2 \right). \]
Note that this is where we introduce the \((4 - \varepsilon)\)-expansion. The physical value is \(\varepsilon = 1\), but we will soon find it necessary to take \(\varepsilon\) small in order to obtain any sensible results. For \(r \ll 1\) we find four fixed points of (3.7) (taking \(N > 4\) for the sake of simplicity – the \(N < 4\) case is entirely similar); they are (see Fig. 1),

1. The doubly unstable Gaussian fixed point at \(\lambda_1 = 0, \lambda_2 = 0\).
2. The doubly stable Wilson-Fisher fixed point at \(\lambda_1 = 2\varepsilon/(NK_D), \lambda_2 = 2(N - 4)\varepsilon/(3NK_D)\).
3. The Ising fixed point at \(\lambda_1 = 0, \lambda_2 = 2\varepsilon/(3K_D)\).
4. The Heisenberg fixed point at \(\lambda_1 = 6\varepsilon/((N + 8)K_D), \lambda_2 = 0\).

The last two are both unstable in one direction in the \(\{\lambda_1, \lambda_2\}\) subspace.

As we will see below, first-order transitions occur when the renormalization group flows take the couplings out of the region where the quartic part of the potential is positive definite. There are two types of instability:

1. Hypercubic diagonal instability. If \(\lambda_1 < 0\) then the potential is minimized for \(\varphi\) in a diagonal direction, namely \(\varphi = (\pm 1, \pm 1, ..., \pm 1)\phi/\sqrt{N}\) (of magnitude \(\phi\)). The quartic part of the potential is then negative for \(\lambda_1 + \lambda_2/N < 0\).

2. Hypercubic axis instability. If \(\lambda_2 < 0\) then the potential is minimized for \(\varphi\) along an axis, namely \(\varphi = (0, ..., 1, ..., 0)\phi\). The quartic part of the potential is now negative for \(\lambda_1 + \lambda_2 < 0\).

The unstable regions are indicated by shading in Fig. 1.

Solving the flow equations (3.7) with \(r \ll 1\) yields the flow pattern shown in Fig. 1. Note that there are regions of the coupling constant space that define sensible models (i.e. that are bounded from below), that have flows that \textit{inevitably take the couplings across the stability lines}, and into regions without fixed points. For \(N > 4\) these regions are the two wedges inside the stability region, but outside the quadrant \(\lambda_1 \geq 0, \lambda_2 \geq 0\) which is the domain of attraction of the Wilson-Fisher fixed point. Systems with \(\{\lambda_1(0), \lambda_2(0)\}\) in these wedges will undergo fluctuation-induced first-order phase transitions. Since the analysis of the phase transition for the two wedges is nearly identical, we will choose to concentrate on the lower wedge where \(\lambda_2 < 0\), for which the RG flows take \(\lambda_i(l)\) across the stability line \(\lambda_2(l) = -\lambda_1(l)\). In this case the phase transition is from a disordered phase to one with ordering along a hypercubic axis in \(\varphi\)-space.

Also note that there exist regions with arbitrarily small initial couplings \((\lambda_i \ll \varepsilon\) at \(\Lambda_r)\), that are first drawn towards the unstable fixed points, where the couplings are
of order $\varepsilon$, before veering away and then crossing one of the stability lines. *Weakly* first-order transitions correspond precisely to such values of the initial couplings where the amount of flow is large. Of course, there are also initial couplings that flow very little before crossing these lines, never becoming very large. These values correspond to strongly first-order transitions – for which it is not necessary to employ the RG to get a reasonable description of the transition, at least on infinite length scales. Ideally we would like to investigate the solution to the flow equations in the case $\varepsilon = 1$. However the closed set of flow equations (3.7) was derived by truncation of an otherwise infinite set of coupled equations for the couplings in an arbitrary potential. Such a truncation is under control if the theory is weakly coupled for all scales $\Lambda < \Lambda_t$. For the weakly first-order case at $D = 3$ the couplings would quickly flow into a region of strong coupling where the loop-expansion and the closely associated truncation fails. This is the reason why a $4 - \varepsilon$ expansion is necessary if we are to describe in a controlled way the weakly first-order case.

In any event, for any given $\{r(0), \lambda_1(0), \lambda_2(0)\}$, all three of order $\varepsilon$ or less, we can follow the RG flows using Eq. (3.7). Fixing $\lambda_1(0)$ and $\lambda_2(0)$, we can then fine-tune $r(0)$ so that the completely coarse-grained potential $V_{\infty}(\phi)$ has two degenerate minima. This means we have chosen the parameters such that $T$ is the temperature of a first-order phase transition. It turns out that we must tune $r(0)$ to be very close to a critical surface $r_c(\lambda_1, \lambda_2)$, where $r_c \sim O(\varepsilon)$ (see Eq. (4.1)). As we evolve up in $l$, $r(l)$ stays close to the critical surface, and hence the evolution follows the $r \ll 1$ trajectories of Fig. 1. At a value of $l$ that we call $l_*$, we cross the stability line:

$$\lambda_1(l_*) + \lambda_2(l_*) = 0$$

(3.8)

We will see in Section 4 that in order to get the degenerate minima in $V_{\infty}$, we must have chosen $r(0)$ such that $r$ begins to deviate from $r_c$ at $l_*$. $(r(l_*) - r_c(\lambda_1(l_*), \lambda_2(l_*)) \sim O(\varepsilon))$. This deviation grows very fast, and at $l \equiv l_f$ (see Eq. (4.27)) $r$ becomes of order 1:

$$r(l_f) = 1 + O(\varepsilon), \quad l_f = l_* + \frac{1}{2} \ln(1/\varepsilon) + O(1)$$

(3.9)

At this point the anomalous scaling of $r$ (the second term in the first equation of (3.7)) is suppressed relative to the canonical scaling (the first term) by a factor of $\varepsilon$. Since the anomalous term in the $r$ flow equation has an factor of $1 + r(l)$ in the denominator, it is now doomed: $r$ grows like $\exp(2l)$ for all $l \gtrsim l_f$. The $\lambda$’s suffer the same fate – their anomalous scaling is suppressed by powers of $1 + r(l)$ too, so for $l > l_f$, $\{r, \lambda_1, \lambda_2\}$ flow canonically (or equivalently the physical parameters $\{m^2, g_1, g_2\}$ are independent of $l$). This is illustrated in Fig. 1 by the straight line flow of the couplings $\{\lambda_1, \lambda_2\}$ in the
“unstable” regions. It does not matter that the $\lambda$’s will ultimately become large: the much faster growth of $r$ suppresses one and higher-loop effects completely, and we can calculate reliably. These flows can then be substituted into the RHS of (2.6) to determine $V_l$ for any $l$. This is the topic to which we now turn.

If we take $\varphi$ to have magnitude $\phi$ along a hypercubic axis, the quadratic and quartic parts of the renormalized potential are

$$V^r_l(\phi) = \frac{1}{2}m^2(l)\phi^2 + \frac{1}{24}(g_1(l) + g_2(l))\phi^4,$$

(3.10)

expressed in dimensionful variables. When we substitute this into the RHS of (2.6) we get an integral expression for the higher-dimension terms that are generated as a result of the coarse-graining process:

$$V_l(\phi) = V^r_l(\phi) + \Delta V_l\left[(g_1(s) + g_2(s))\phi^2\right] + (N - 1)\Delta V_l\left[\frac{1}{3}g_1(s)\phi^2\right],$$

(3.11)

where, again we have specialized to an axis direction. Here $\Delta V$ is a functional, whose argument is a function of $s$ over the range $0$ to $l$:

$$\Delta V_l[w(s)] \equiv \frac{1}{2}K_D\int_0^l e^{-(D-\epsilon)s} ds \left\{ \ln \left[ 1 + \frac{w(s)}{P(s)} \right] - \frac{w(s)}{P(s)} + \frac{1}{2} \left( \frac{w(s)}{P(s)} \right)^2 \right\}$$

$$P(s) \equiv e^{-2s} + m^2(s).$$

(3.12)

Note that we have subtracted off the quadratic and quartic parts of $\Delta V_l$, since these constitute the truncated potential, whose running we have already calculated.

Iterating the KIG equation once again (before taking $\varphi$ to lie along the axis) we would find that the couplings of the induced higher-dimension terms change the RG equations for two and four-point terms. This change is perturbatively small when the four-point coupling is itself small. As we noted above, this is in general only the case when we work in a $(4 - \epsilon)$-expansion with $\epsilon$ small.

4 The phase transition

We now give the detailed proof of the qualitative results sketched above, and calculate the infinitely coarse-grained ($l = \infty$) effective potential near the phase transition, finding what value of $m^2(0)$ yields degenerate minima for given $g_1(0), g_2(0)$.

4.1 The solution of the RG equations

In view of the discussion at the end of Section 3.1, we only need solve the flow equations for $l < l_f$, for which the $\lambda$’s are always of order $\epsilon$. Ultimately we will show that the phase transition occurs at values of $m^2(0)$ etc. such that (3.9) is satisfied.
We start with the flow of $r$. Defining

$$r_c(l) \equiv -\frac{K_D}{12} \left( (N + 2)\lambda_1(l) + 3\lambda_2(l) \right),$$

(note that $r_c \sim \mathcal{O}(\varepsilon)$) we can rewrite the flow of $r$ as

$$\frac{dr}{dl} = 2r - \frac{2r_c}{1 + r} + \mathcal{O}(\varepsilon^2).$$

As long as $r \sim \mathcal{O}(\varepsilon)$ we can ignore the $(1 + r)$ denominator; When $r \sim \mathcal{O}(1)$ the whole second term is subleading. Thus

$$\frac{dr}{dl} = 2r - 2r_c + \mathcal{O}(\varepsilon r) \quad \forall l \gtrsim l_f$$

Now $dr_c/dl \sim \mathcal{O}(\varepsilon^2)$, so to leading order in $\varepsilon$,

$$r(l) = r_c(l) + \bar{t}e^{2l}$$

i.e.

$$m^2(l) = e^{-2l}r_c(l) + \bar{t}$$

$$\bar{t} = e^{-2l_f}(1 + \mathcal{O}(\varepsilon))$$

Putting this into words, we have found that there is an unstable “critical surface” $r_c(l)$. If $r$ starts off on the critical surface ($\bar{t} = 0$) then it never leaves, its flow within the surface being determined by the flow of the $\lambda$’s. In order to see a subsidiary minimum in the potential we will have to choose $\bar{t}$ exceedingly small, so that $l_f$ is large. Specifically, we must ensure that $r$ does not grow to $\mathcal{O}(1)$ until (just) after the couplings cross the stability line, i.e. at $l_* + \mathcal{O}(\ln(1/\varepsilon))$.

Our next task is to solve the flow of $g_1$ and $g_2$, or equivalently $\lambda_1$ and $\lambda_2$. For $l < l_*$, (3.5) becomes

$$\varepsilon \frac{dg_1}{dx} = -\frac{K_D}{6} \left( 6g_1 g_2 + (N + 8)g_1^2 \right),$$

$$\varepsilon \frac{dg_2}{dx} = -K_D \left( 2g_1 g_2 + (3/2)g_2^2 \right),$$

where $x = \exp(\varepsilon l)$

It is convenient, following Rudnick [10], to define

$$F \equiv -\frac{g_1}{g_2},$$

($F$ is positive for the case we consider) so that

$$\frac{dg_1}{dg_2} = \frac{(N + 8)/3F^2 - 2F}{3 - 4F},$$
\[ \frac{dF}{dg_2} = \frac{dg_1}{dg_2} + F = \frac{(N - 4)F^2/3 + F}{3 - 4F} \] (4.8)

which is readily integrated to yield

\[ g_2(l) = g_2(0) \left( \frac{F(0)}{F(l)} \right)^3 \left( \frac{(N - 4)F(l) + 3}{(N - 4)F(0) + 3} \right)^{3N/(N - 4)} \] (4.9)

Now all we have to do is find \( F \). By definition (4.6) and the RG equations (3.5),

\[ \frac{dF}{dx} = \frac{K_Dg_2}{6\varepsilon} (3F + (N - 4)F^2) \] (4.10)

so by (4.9),

\[ \frac{dF}{dx} = A F^2 (3 + (N - 4)F)^{\alpha + 1} \]

\[ A \equiv \frac{K_Dg_2(0)}{6\varepsilon} \left( \frac{F(0)^3}{3 + (N - 4)F(0)^\alpha} \right) \]

\[ \alpha \equiv \frac{3N}{N - 4} \]

Note that \( A \) is negative. Solving, we find the dependence of \( F \) on \( l \):

\[ A(N - 4)^3(e^{\varepsilon l} - 1) = \left\{ Z(3 + (N - 4)F(l)) - Z(3 + (N - 4)F(0)) \right\} \]

\[ Z(y) \equiv \frac{y^{2-\alpha}}{2 - \alpha} - \frac{6y^{1-\alpha}}{1 - \alpha} - \frac{9y^{-\alpha}}{\alpha} \] (4.12)

Armed with Eqs. (4.9) and (4.12) we can now obtain \( g_1(l) \) and \( g_2(l) \) for any \( l < l_* \) given \( g_1(0) \) and \( g_2(0) \).

### 4.2 The effective potential near the phase transition

We have managed to obtain the flows in closed form. Stage 2 of our procedure now calls for us to substitute them in to Eq. (3.11) and send \( l \to \infty \) to get the effective potential, the form of which will tell us whether there is a first-order phase transition. In order to obtain understandable (and self-consistent) results we must again make the approximation of keeping only the leading order non-trivial terms in the \( \varepsilon \)-expansion.

We have previously argued (below Eq. (3.9)) that \( r, \lambda_1, \lambda_2 \) scale canonically for \( l > l_f \), which means that \( m^2, g_1, \) and \( g_2 \) can be taken to be constant. Thus the old non-RG-improved effective potential formula is valid in this region. It is therefore convenient to define the following variants of \( \Delta V \) (see Eq. (3.12)) where the couplings are evaluated at
some particular $l$ and do not flow:
\[
\Delta U_{IR}^I(w) \equiv \frac{1}{2}K_D \int_{l_f}^{\infty} e^{-(D-\varepsilon)s} ds \ln \left[ 1 + \frac{w}{P(s)} \right] \\
\Delta U_{IV}^I(w) \equiv \frac{1}{2}K_D \int_{0}^{l_f} e^{-(D-\varepsilon)s} ds \left\{ \ln \left[ 1 + \frac{w}{P(s)} \right] - \frac{w}{P(s)} + \frac{1}{2} \left( \frac{w}{P(s)} \right)^2 \right\} \\
P(s) \equiv e^{-2s} + m^2(l)
\]
so that the potential becomes
\[
V_\infty = V_{IR}^I(\phi) \quad = \quad \Delta V_{l_f} \left[ \frac{1}{2}(g_1(l) + g_2(l))\phi^2 \right] + (N - 1) \Delta V_{l_f} \left[ \frac{1}{2}g_1(l)\phi^2 \right] \\
+ \Delta U_{IR}^I \left( \frac{1}{2}(g_{1f} + g_{2f})\phi^2 \right) + (N - 1) \Delta U_{IR}^I \left( \frac{1}{2}g_{1f}\phi^2 \right) \\
g_{1f} \equiv g_1(l_f), \quad \text{etc.}
\]

The first three terms represent the “ultraviolet” contribution, which requires integrating over the flowing couplings from $l = 0$ to $l_f$. The last two terms are the “infrared” contribution, which are easy to evaluate, since nothing flows in the integrand.

Actually, to the leading non-trivial order in $\varepsilon$ to which we work, one can also avoid the complications of integrating over flowing mass and couplings in the expression for $\Delta V$ (Eq. (3.12)), which is integrated from $l = 0$ to $l_f$. To see this, first note that the flow of the mass in $\Delta V$ can be ignored, since it only enters through the propagator $P(l) = \exp(-2l) + m^2(l) = \exp(-2l)(1 + O(\varepsilon)) + \bar{t}$, so we can set $m^2(l) = m^2(l_f) = \bar{t}$ to leading order in $\varepsilon$, with $\bar{t}$ given in Eq. (4.4). Note that $\Delta V$ only supplies the $\phi^6$ and higher terms – the flow of $m^2$, $g_1$, $g_2$ is explicitly taken into account in the lower order terms in $V_{\text{tr}}$. To see that the flows of $g_1$ and $g_2$ can be ignored, look a typical term in $\Delta V_{l_f}$:
\[
\frac{1}{2}K_D \int_{0}^{l_f} e^{-(D-\varepsilon)s} w(s)^n ds \quad (n \geq 3)
\]
where $w(s) \sim g(s)\phi^2$. Because $\bar{t} = \exp(-2l_f)$ this integral is dominated by the region close to $l_f$: the domain $l_f - \frac{1}{2} \ln(1/\varepsilon) < l < l_f$ gives the dominant contribution to order $\varepsilon$. In this relatively small range, the $g$’s hardly flow at all, since $(1/g)dg/dl \sim \varepsilon$, so we can take them to have their values at $l_f$ throughout the whole range. Thus $\Delta V_{l_f}[w(s)] = \Delta U_{IV}^I(w(l_f))$ to lowest order in $\varepsilon$, and
\[
V_\infty(\phi) = V_{IR}^I(\phi) \quad + \quad \Delta U_{IV}^I \left( \frac{1}{2}(g_{1f} + g_{2f})\phi^2 \right) + (N - 1) \Delta U_{IV}^I \left( \frac{1}{2}g_{1f}\phi^2 \right) \\
+ \Delta U_{IR}^I \left( \frac{1}{2}(g_{1f} + g_{2f})\phi^2 \right) + (N - 1) \Delta U_{IR}^I \left( \frac{1}{2}g_{1f}\phi^2 \right)
\]
We have now expressed the effective potential entirely in terms of simple integrals that depend on the relevant couplings at $l_f$. 

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Performing the integrals, we find that
\[
\Delta U_l(\bar{w}) \equiv \Delta U^R_l(w) + \Delta U^U_l(w) \\
= \frac{K_D}{8} e^{(\varepsilon-4)l} \left\{ 2\bar{w} - \frac{r_l}{1 + r_l} \bar{w}^2 + f(r_l + \bar{w}) - f(r_l) - (\bar{w}^2 + 2r_l\bar{w}) \ln(1 + r_l) \right\}
\]
\[
f(x) \equiv x^2 \left( \ln(x) - \frac{1}{2} \right), \quad r_l \equiv r(l), \quad \bar{w} \equiv e^{2l}w,
\]
and defining a rescaled field
\[
\bar{\phi}_f^2 \equiv \phi^2 / \Lambda(l_f)^{(2-\varepsilon)} = e^{(2-\varepsilon)l_f \phi^2},
\]
we find that the potential is
\[
V_\infty(\bar{\phi}) = \exp[(\varepsilon - 4)l_f] \left\{ \frac{1}{2} r_f \bar{\phi}_f^2 + \frac{1}{24}(\lambda_{1f} + \lambda_{2f})\bar{\phi}_f^4 + 2 \Delta U_l \left( \frac{1}{2}(\lambda_{1f} + \lambda_{2f})\bar{\phi}_f^2 \right) \right\} \\
+ (N - 1) \Delta U_l \left( \frac{1}{6}\lambda_{1f}\bar{\phi}_f^2 \right)
\]
(4.19)
\[
\lambda_{1f} \equiv \lambda_1(l_f), \quad \text{etc.,} \quad r_{lf} = 1.
\]

We are now almost finished. We can now see for what values of \(\lambda_{1f}, \lambda_{2f}\) there is a phase transition at this temperature. We look for a subsidiary minimum at non-zero \(\bar{\phi}_f\), degenerate with the one at the origin. Anticipating the final result, we guess that it will occur for
\[
\lambda_{1f}, \lambda_{2f} \sim O(\varepsilon), \\
(\lambda_{1f} + \lambda_{2f}) \sim O(\varepsilon^2 \ln \varepsilon)
\]
(4.20)
\[
\bar{\phi}_f \sim O(1/\varepsilon)
\]
Using this in (4.19) and keeping only terms of order \((1/\varepsilon^2) \ln \varepsilon\) or \(1/\varepsilon^2\), we find
\[
V_\infty(\bar{\phi}_f) = \frac{1}{2}\bar{\phi}_f^2 + \frac{1}{24}(\lambda_{1f} + \lambda_{2f})\bar{\phi}_f^4 + \frac{K_D(N - 1)}{8} \left( \frac{1}{6}\lambda_{1f}\bar{\phi}_f^2 \right)^2 \left( \ln(\frac{1}{12}\lambda_{1f}\bar{\phi}_f^2) - 1 \right)
\]
(4.21)
which has the required degenerate subsidiary minimum for
\[
\lambda_{1f}^{PT} + \lambda_{2f}^{PT} = -\frac{K_D(N - 1)}{12}(\lambda_{1f}^{PT})^2 \ln \left[ \frac{12}{\lambda_{1f}^{PT}K_D(N - 1)} \right],
\]
(4.22)
(which is of order \(\varepsilon^2 \ln \varepsilon\), as expected), at
\[
(\bar{\phi}_f^{PT})^2 = \frac{144}{\lambda_{1f}^2K_D(N - 1)},
\]
(4.23)
(which is of order \(1/\varepsilon^2\) as expected). This demonstrates that indeed our model possesses a first-order phase transition of the type claimed.
We could at this stage perform the final task of re-expressing these values, and the mass $r(l_f) = 1$, in terms of the initial renormalized parameters of the theory at $\Lambda_t$ (i.e. $l = 0$) using Eqs. (4.9), (4.12), and (4.4). Let us, however, delay this until our comments and conclusions in Section 5.

It is also possible to express the infinitely coarse-grained free-energy in terms of the couplings evaluated at $l_\ast$, rather than $l_f$. The independence of the physical results (to leading order in $\varepsilon$) enables us to check that we have not made an error in our calculation. To do this we need to flow $r_{t_f}$, $\lambda_{1f}$, $\lambda_{2f}$ up to $l_\ast$. First we must find $l_\ast$, which is defined by $\lambda_{1}(l_\ast) + \lambda_{2}(l_\ast) = 0$. We therefore define

$$u(l) \equiv e^{-\varepsilon(l-l_\ast)}(\lambda_{1} + \lambda_{2})$$

and we find, by Eqs. (3.7), (4.4), to leading order in $\varepsilon$,

$$\frac{\partial u}{\partial l} = \frac{1}{6} K_D(N - 1) \lambda_{1}^{2}(l) \frac{e^{-\varepsilon(l-l_\ast)}}{(1 + e^{2l_{f}})^{2}}$$

We can integrate this equation from $l_f$ to $l_\ast$, since $\lambda_{1}$ varies by an amount subleading in $\varepsilon$ in this range. Under the correct assumption that $\exp(l_f - l_\ast) \sim O(1/\varepsilon)$, we obtain

$$u(l_f) = \frac{1}{12} K_D(N - 1) \lambda_{1}^{2}(l_f) \left(\frac{1}{2} + \ln[2e^{-2(l_f-l_\ast)}] + \cdots\right).$$

Hence

$$\exp(2(l_f - l_\ast)) = \frac{24e^{1/2}}{\lambda_{1*} K_D(N - 1)}$$

which is much greater than 1 as anticipated.

We can now go back to Eqs. (4.16) and (4.17) and re-express the potential as a function of quantities defined at $l_\ast$, obtaining an expression like Eq. (4.19) with $f \to \ast$. Let us see for what mass and couplings we expect to see the degenerate subsidiary minimum. We flow the mass up to $l_\ast$ by using (4.4), (4.1) and (4.27)

$$r_{PT}(l_\ast) = \frac{1}{12} K_D(N - 1) \lambda_{1*} \left(\frac{1}{2}e^{-1/2} - 1\right).$$

The minimum will be at

$$\left(\phi_{\ast}\right)^{2} = e^{2(l_f-l_\ast)} \left(\phi_{f}\right)^{2} = \frac{6e^{-1/2}}{\lambda_{1*}}.$$  

We conclude, analogously to (4.20), that the interesting parameter range for the potential is

$$r(l_\ast) \sim O(\varepsilon),$$

$$\lambda_{1*} = \lambda_{2*} \sim O(\varepsilon),$$

$$\overline{\phi}_{\ast} \sim O(\varepsilon^{-1/2}).$$
Keeping leading order in $\varepsilon$ we find the potential

$$V_\infty(\bar{\phi}_*) = \frac{1}{2} r(l_*)\bar{\phi}_*^2 + \frac{K_D(N-1)}{8} \left\{ \frac{1}{3} \lambda_1 \bar{\phi}_*^2 + \left( \frac{1}{6} \lambda_1 \bar{\phi}_*^2 \right)^2 \left[ \ln \left( \frac{1}{6} \lambda_1 \bar{\phi}_*^2 \right) - \frac{1}{2} \right] \right\}$$  \hspace{1cm} (4.31)

This should be exactly the same as [1.24], and in fact it is. It has a degenerate subsidiary minimum at the values of $r$ and $\bar{\phi}$ predicted by (4.28) and (4.29). Finally, note that our result is identical (to leading order in $\varepsilon$ and after suitable redefinitions) with that of Rudnick [10], who previously analyzed a Landau-Ginzburg free energy of the form Eq. (1.1) within a condensed-matter context.

5 Comments and conclusions

We now wish to make some comments concerning the formalism and calculation presented above.

1) We have shown that it is possible to systematically calculate a scale-dependent coarse-grained free energy that describes a (fluctuation-induced) first-order phase transition. As we argued in the introduction, if one wishes to explicitly follow the dynamics of the phase transition (such as critical bubble nucleation and expansion) then one must coarse-grain not out to infinity ($\Lambda \rightarrow 0$), but to some appropriate length scale. We want to re-emphasize that this procedure should be familiar from the study of effective gauge theories – there one integrates out the heavy degrees of freedom to get the effective Lagrangian describing the dynamics of the light modes. In our case we might want, for instance, to accurately calculate bubble wall properties – we should therefore stop coarse-graining when we reach the bubble-wall thickness, and use the coarse-grained free energy at that length-scale.

As an illustration of this procedure we can look at the (typical) case where this thickness is the same order as the correlation length. In terms of the variable $r_l$ used in Section 4 this is precisely when $r_l = 1$, i.e. $l = l_f$. We therefore need $V_{l_f}$. However this is nothing more than the expression for the free-energy in Eq. (4.14) without the extra infrared contributions

$$V_\infty(\phi) - V_{l_f}(\phi) = \Delta U_{l_f}^{IR} \left( \frac{1}{2} (g_{1f} + g_{2f}) \phi^2 \right) + (N-1) \Delta U_{l_f}^{IR} \left( \frac{1}{6} g_{1f} \phi^2 \right),$$  \hspace{1cm} (5.1)

where we recall from (4.13) that

$$\Delta U_{l_f}^{IR}(w) \equiv \frac{1}{2} K_D \int_l^\infty e^{-(D-\varepsilon)s} ds \ln \left[ 1 + \frac{w}{P(s)} \right].$$  \hspace{1cm} (5.2)

In general, the difference between $V_\infty(\phi)$ and $V_{l_f}(\phi)$ (given for our toy model by Eq. (5.1)) can be quite significant. In our model it turns out that within the $(4 - \varepsilon)$-expansion, and
for the orders of magnitude of $\phi$, and the couplings, of interest Eq. (4.20), this difference is actually subleading by one power of $\varepsilon$. Since we have not calculated our RG flows to higher order in $\varepsilon$ we therefore cannot give a reliable explicit expression for the values of the integrals in (5.1). Of course, despite the fact that they are subleading in $\varepsilon$, at $\varepsilon = 1$ their contribution can be important. We expect that for the gauged case we describe below the difference between $V_\infty(\phi)$ and $V_{lf}(\phi)$ will not be subleading in $\varepsilon$.

2) We now want to return to the one loose-end of Section 4 – namely the re-expression of the properties of the potential in terms of the initial variables at $l = 0$. Using the solutions, Eqs. (4.9), (4.12), and (4.4), we derived in Section 4.1 to the RG equations this is a straightforward task. However the expressions one gets in doing this are quite lengthy. A simplification occurs if we take the initial ratio of the couplings $F_0 = -g_1(0)/g_2(0) \gg 1$. For example, in this limit the value of the field at the new (true vacuum) minimum is given by

$$
\phi^2 = e^{-(2-\varepsilon)l_\ast} \frac{N(N+2)(N+8) K_D}{N(N+5)+3} \frac{K_D}{\varepsilon},
$$

(5.3)

where

$$
e^{-(2-\varepsilon)l_\ast} = \left\{ \frac{N(N+2)(N+8)}{N(N+5)+3} \frac{g_1(0) K_D}{6\varepsilon} \left( \frac{N-1}{N-4} \right)^{3N/(N-4)} \right\}^{(2-\varepsilon)/\varepsilon} \frac{(2-\varepsilon)(N+8)}{F_0 \varepsilon(N-4)}. \quad (5.4)
$$

($l_\ast$ is the value of the flow parameter at which the system crosses the stability line $g_1(l_\ast) + g_2(l_\ast) = 0$). As advertised in the Introduction these expressions are quite non-analytic as a function of the initial couplings and $\varepsilon$. The expression for $\phi^2$ Eq. (5.3) also exemplifies the connection between weakly first-order transitions and significant flow in $l$ that we mentioned in Section 3. This is because when $l_\ast$ is large the value of $\phi^2$ in the new minimum (at the transition temperature) given by (5.3) is small compared to its zero-temperature value.

3) In the introduction we motivated the study of our toy scalar model by stating that it shared many features in common with the phase transition occurring in gauged systems. One way of understanding why this is true is to look at the RG flow equations (once again in $(4 - \varepsilon)$-spatial dimensions) for the couplings of the prototypical gauged system – the $N$-component Abelian Higgs model. Again we have two couplings that are marginal near four-dimensions, the quartic scalar self-coupling $h(l)$, and the gauge coupling $e^2(l)$. Their RG equations are

$$
\frac{de^2}{dl} = \varepsilon e^2 - \frac{NK_D}{3} e^4
$$

$$
\frac{dh}{dl} = (\varepsilon - 2\eta)h - \frac{(N+4)K_D}{2(1+r)^2} h^2 - 12K_D e^4,
$$

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\end{center}
where $\eta = -3K_D e^2/(1 + r)$ is the anomalous dimension of the scalar field induced by wavefunction renormalization, and $r(l)$ is the (dimensionless) scalar mass parameter with a RG equation we shall not display. (As in Section 3 we have implicitly included the appropriate factor of temperature $T$ induced by the rescaling of the fields, as well as then dividing out by $\Lambda(l)^\varepsilon$ so as to get dimensionless couplings.) For any reasonable number of complex scalar fields $N < 366$ these equations have only two fixed points – the doubly unstable Gaussian fixed point at the origin, and a fixed point at

$$h = \frac{2\varepsilon}{(N + 4)K_D}, \quad e^2 = 0,$$

which is unstable in the direction of the gauge coupling. The qualitative pattern of the flows in the $\{h, e^2\}$ plane is illustrated in Fig. 2.

The important point is that there are well-defined models (indeed the whole of the $e^2 > 0, h > 0$ quadrant!) whose flows inevitably take the system outside the naive stability region (the same $e^2 > 0, h > 0$ quadrant). Just as in the scalar case we have considered above, this implies that the system undergoes a fluctuation induced (finite-temperature) phase transition. Furthermore, the work of Ginsparg \cite{5} shows that the standard model and almost all other gauge systems of interest fall into this class. In a future publication we will show how the properties of these first-order phase transitions can be analyzed by the application of the Wilson RG and the $(4 - \varepsilon)$-expansion.

4) In a recent very interesting work, Shaposhnikov \cite{20} argued that to fully understand the finite-temperature phase transition that occurs in gauge theories, it is necessary to include non-perturbative contributions to the free-energy. This, he argued, is because the effective three-dimensional theory of the zero Matsubara modes is a confining theory in the far infrared – at least in the non-Abelian case. It was also argued that the non-perturbative effects can make the transition in, say, the minimal standard model considerably stronger for a given Higgs mass than a perturbative analysis would indicate. It is interesting to see what our approach says about this general question.

We have already noted that the effective $(4 - \varepsilon)$-dimensional theories that we must consider if we want to study the phase transition have the property that as we flow into the infra-red the effective couplings first flow towards the unstable fixed point, which for $\varepsilon = 1$ is well into the strong-coupling regime. The same phenomenon occurs in gauge theories, as exemplified by the RG flows of the Abelian Higgs model illustrated in Fig. 2. One is therefore tempted to say that this is almost a proof of the correctness of Shaposhnikov’s suggestion at $\varepsilon = 1$. All the theories are in their strong-coupling regime and non-perturbative effects are important. However, this does not necessarily mean that we cannot calculate.
To see this, consider the flows in Fig. 1 that take us towards the stable Wilson-Fisher fixed point at \(\lambda_1 = 2\varepsilon/(NK_D)\), \(\lambda_2 = 2(N-4)\varepsilon/(3NK_D)\), corresponding to a second-order transition. For \(\varepsilon = 1\) this is also far into the strong-coupling region, so surely we must consider non-perturbative effects if we are to get accurate predictions for the properties of this transition (such as the critical exponents). In fact, the experimentally observed critical exponents are amazingly well described by an expansion up to only \(O(\varepsilon^2)\). Even more accurate results can be obtained by Lipatov techniques for estimating large order behavior of the \(\varepsilon\)-expansion together with Borel resummation. These techniques are closely related to the usual semi-classical methods employed in dealing with weakly non-perturbative effects. Thus we might say that for this second-order case it seems to be that the non-perturbative effects that are present are fairly benign – and can be handled within more sophisticated versions of the \(\varepsilon\)-expansion.

What about the first-order case – especially for gauge theories? Here the situation is less clear, mainly as a consequence of the relatively limited amount of work that been performed so far. There has been for example, to our knowledge, no extension of the Lipatov techniques to the first-order case. One obvious difference between the first and second-order cases is that in terms of the dimensionless variables, \(\lambda_i(l)\), the flows take one out to “infinity” in coupling constant space, rather than to a fixed point. This is actually benign in the scalar case since soon after we cross the stability line we reach a point, \(l_f\), where the increase in the scalar mass cuts off any further non-canonical scaling of the couplings, so the \(\lambda_i(l)\) move along a straight line trajectory projecting out from the origin (see Fig. 1) – in other words the dimensionful couplings are to a good approximation constant after \(l_f\). However, in the gauge case the flow of the gauge coupling doesn’t seem to be cut off in quite such a simple way. This is consequence of the lack of any factors of \(1/(1+r(l))^n\) appearing in the RG equation for \(e^2\) in (5.3). Non-Abelian theories also present new difficulties. We are at present considering these issues. In any case it is certainly true that the RG techniques we have utilized in this paper lead to considerably greater insight into this question than more standard diagrammatic resummation techniques.

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Appendix A  Derivation of the evolution equation

In this appendix we will derive the KIG evolution equation (2.3) from Polchinski’s evolution equation (2.2), for the case of a single scalar field $\phi$ in 4 dimensions. The generalization to multiple fields and arbitrary dimension is straightforward. We first show that, for momenta below the cutoff $\Lambda$, Polchinski’s interaction Lagrangian $L_\Lambda[\phi]$ is the same as the 1PI generating functional $\Gamma_\Lambda[\phi]$ defined by Bonini, D’Attanasio, and Marchesini (BDM) [13] in their perturbative analysis of Polchinski’s equation. We then use their evolution equation for $\Gamma_\Lambda[\phi]$ to derive the KIG evolution equation for the effective potential.

To clarify matters we will take the momentum cutoff at $\Lambda$ to be sharp, and use superscripts $\rangle$ and $\langle$ to distinguish degrees of freedom above and below the cutoff respectively. (For a rigorous analysis see Morris’s very clear exposition [21], in which the sharp cutoff limit is taken at the end.) The propagator has two domains,

\[
\text{Infrared (IR): } D_\langle(k) = \frac{K(k^2/\Lambda^2)}{(k^2 + m^2)}, \]

\[
\text{Ultraviolet (UV): } D_\rangle(k) = \frac{1 - K(k^2/\Lambda^2)}{(k^2 + m^2)}.
\]

These add up to the full propagator $D(k)$. For most purposes it is natural to take $K(x)$ to be a step function that is 1 for $x < 1$ and zero otherwise. To make sense of BDM’s definitions, however, we will have to allow the UV propagator to have some infinitesimal value in the IR, and take the limit as it goes to zero.

The full generating functional of the theory can be split into IR and UV portions:

\[
Z[J] = Z[\phi^<, J^>] = \int[D\phi] \exp \left\{ \int_k \left[ -\frac{1}{2} \phi(k) D^{-1}(k) \phi(-k) + J(k) \phi(-k) \right] + L_{\text{bare}}[\phi] \right\} = \int[D\phi^<] \exp \left\{ \int_k \left[ -\frac{1}{2} \phi^<(k) D^<_{\langle}(k) \phi^<(-k) + J^<(k) \phi^<(-k) \right] + L_{\Lambda}[\phi^<, J^>] \right\},
\]

(A.2)

Comparing with Eq. (2.1) we see that Polchinski’s interaction Lagrangian $L_\Lambda$ is the functional integral over UV fields, with a given IR background field $\phi^<$ and in the presence of a UV source $J^>$,

\[
\exp L_\Lambda[\phi^<, J^>] = \int[D\phi^>] \exp \left\{ \int_k \left[ -\frac{1}{2} \phi^>(k) D^>_{\langle}(k) \phi^>(-k) + J^>(k) \phi^>(-k) \right] + L_{\text{bare}}[\phi^<, \phi^>] \right\}.
\]

(A.3)
Eq. (A.2) tells us that we should think of $L_\Lambda[\phi^<,0]$ as the effective action “felt” by the IR fields. We now want to show that it is the same as BDM’s effective action (for momenta below $\Lambda$)

$$\Gamma_\Lambda[\psi^<,J^\geq=0] = \int_{k<\Lambda} j^<(k)\psi^<(k) - W[j^<,0], \quad \text{with} \quad \psi^< = \frac{\delta W}{\delta j^<}, \quad (A.4)$$

which is the Legendre transform of their connected generating functional with UV propagator,

$$\exp W_\Lambda[j^<,J^\geq] = \int [D\phi] \exp\left\{ \int k^< \left[ -\frac{1}{2}\phi(k)D^{-1}_\Lambda(k)\phi(-k) + j^<(k)\phi^<(-k) + J^>(k)\phi^>(-k) \right] + L_{\text{bare}}[\phi^<,\phi^>] \right\}. \quad (A.5)$$

A peculiar (but essential) aspect of this definition is that the functional integral is over IR as well as UV fields, even though the propagator is cut off in the IR. Moreover, there is an IR source coupled to the IR field. This only makes sense if we give $D^>$ some infinitesimal value $\epsilon/(k^2 + m^2)$ in the IR, and perform the IR part of the functional integral before sending $\epsilon \to 0$. $D^{-1}_\Lambda(k < \Lambda)$ is then very large, and in combination with the $j^<\phi^<$ term gives a $\delta$-function, requiring $\phi^<(k) = \psi^<(k) = D^>(k)$.

The result is

$$\exp\left\{ -\Gamma_\Lambda[\psi^<,0] = \exp\left( -\frac{1}{2} \int k^< \psi^<(k)D^{-1}_\Lambda(k)\psi^<(-k) \right) \right. \times \left\{ \int [D\phi^>] \exp\left\{ \int k^< \left[ -\frac{1}{2}\phi^>(k)D^{-1}_\Lambda(k)\phi^>(-k) \right] + L_{\text{bare}}[\psi^<,\phi^>] \right\} \right\}. \quad (A.6)$$

We see that $\Gamma_\Lambda[\psi^<]$ is indeed the same as Polchinski’s effective action $L_\Lambda[\phi^<]$ Eq. (A.3), except that it has an additional inverse propagator term in front. As we send $\epsilon \to 0$, eliminating BDM’s infinitesimal IR component of $D^>$, this term goes to infinity ($D^>(k < \Lambda) \sim \epsilon$). This is not a disaster—it just reflects the fact that we in effect gave the IR modes infinite mass in order to treat them as a background. It only affects the 2-point function, which BDM write as

$$\Gamma_\Lambda^{(2)}(k) = D^{-1}_\Lambda(k) + \Sigma_\Lambda(k), \quad (A.7)$$

and they obtain an evolution equation for the $\epsilon$-independent part $\Sigma_\Lambda$. We therefore write

$$V_\Lambda(\phi) = (m^2 + \Sigma_\Lambda)\phi^2 + Y_\Lambda(\phi),$$

$$Y_\Lambda(\phi) = \sum_{n=2}^{\infty} \frac{1}{(2n)!} \Gamma_\Lambda^{2n} \phi^{2n}, \quad (A.8)$$

where the $\Gamma_\Lambda^{2n}$ are zero-momentum $n$-point 1PI functions.
We now begin the second stage of our calculation, showing that BDM's evolution equation for $V_\Lambda$ reproduces the KIG equation Eq. (2.3), in the approximation where momentum-dependence in the action is ignored. Expanding $V_\Lambda$ in the same way as $\Gamma_\Lambda$, so that $V_\Lambda^{(2)} = m^2 + \Sigma_\Lambda$ but for $n > 1$, $V_\Lambda^{(2n)} = \Gamma_\Lambda^{(2n)}$, BDM find

$$\Lambda \frac{\partial V_\Lambda^{(2n)}}{\partial \Lambda} = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Lambda \frac{\partial D_\Lambda}{\partial \Lambda}(q) \frac{\Gamma_\Lambda^{(2n+2)}(q)}{(1 + D_\Lambda(q)\Sigma_\Lambda(q))^2},$$  \hspace{1cm} (A.9)

where we have written $D_\Lambda(q) \equiv D_\Lambda(q)$, and

$$\Gamma_\Lambda^{(2n+2)}(q) = \Gamma_\Lambda^{(2n+2)} - \sum_{l=1}^{n-1} \frac{(2n)}{2l} \frac{\Gamma_\Lambda^{(2l+2)}(q) \Gamma_\Lambda^{(2n-2l+2)}(q)}{D_\Lambda^{-1}(q) + \Sigma_\Lambda(q)},$$ \hspace{1cm} (A.10)

so that

$$\Lambda \frac{\partial V_\Lambda^{(2n)}}{\partial \Lambda} = \frac{1}{2} K_4 \sum_{N=1}^{n} \left\{ I_N(\Lambda) \sum_{l_1=1}^{N-1} \sum_{l_2=1}^{l_1-1} \cdots \sum_{l_{N-1}=1}^{l_{N-2}-1} \sum_{l_N=1}^{l_{N-1} - \cdots - l_1 - 1} \left[ -\frac{(1)^N(2n)!}{(2l_1)! \cdots (2l_N)!} \times \Gamma_\Lambda^{2l_1+2} \cdots \Gamma_\Lambda^{2N+2} \right] \right\}.$$ \hspace{1cm} (A.11)

The momentum integral is contained in $I_N$,

$$I_N(\Lambda) = \int q^3 dq \Lambda \frac{\partial D_\Lambda}{\partial \Lambda}(q) \frac{D_\Lambda^{-2}(q)}{(D_\Lambda^{-1}(q) + \Sigma_\Lambda(q))^{N+1}}.$$ \hspace{1cm} (A.12)

Assuming that the step function $K(q/\Lambda)$ (see (A.11)) varies very quickly near $q = \Lambda$, we can evaluate it,

$$I_N(\Lambda) = \frac{\Lambda^4}{N} (\Lambda^2 + m^2 + \Sigma_\Lambda)^{-N}.$$ \hspace{1cm} (A.13)

On the RHS of Eq. (A.11), the quantity in braces contains $N - 1$ summations. If $N = 1$ then it evaluates to the quantity in square brackets, with $l_N = n$. Otherwise the summations are evaluated in the usual way, with $l_N = n - l_1 - \cdots - l_{N-1}$. Summing over $n$,

$$\Lambda \frac{\partial V_\Lambda}{\partial \Lambda} = \frac{1}{2} K_4 \sum_{N=1}^{\infty} \frac{(-1)^N}{N} (\Lambda^2 + m^2 + \Sigma_\Lambda)^{-N} \left\{ \sum_{l_1=1}^{\infty} \frac{\Gamma_\Lambda^{2l_1+2}}{(2l_1)!} \cdots \sum_{l_N=1}^{\infty} \frac{\Gamma_\Lambda^{2N+2}}{(2l_N)!} \phi^{2(l_1+\cdots+l_N)} \right\},$$ \hspace{1cm} (A.14)

so by (A.8),

$$\Lambda \frac{\partial V_\Lambda(\phi)}{\partial \Lambda} = \frac{1}{2} K_4 \Lambda^4 \ln \left( 1 + \frac{Y_\Lambda''(\phi)}{\Lambda^2 + m^2 + \Sigma_\Lambda} \right).$$ \hspace{1cm} (A.15)

Ignoring terms that are independent of $\phi$,

$$\Lambda \frac{\partial V_\Lambda(\phi)}{\partial \Lambda} = \frac{1}{2} K_4 \Lambda^4 \ln \left( \Lambda^2 + V_\Lambda''(\phi) \right),$$ \hspace{1cm} (A.16)

which was to be proved.
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Figure Captions

Figure 1. The general form of the RG flows in the \{\lambda_1, \lambda_2\} subspace are displayed, along with the four fixed points, for the case \( N > 4 \). Shading indicates the “unstable” regions where the quartic part of the potential is no longer positive definite. The values \( l = l_* \) and \( l = l_f \) of the flow parameter are where, respectively, a stability line is crossed, and \( r = \mathcal{O}(1) \) so that the subsequent flow is canonical (\{\lambda_1(l), \lambda_2(l)\} flow along a straight line projecting through the origin).

Figure 2. The general form of the RG flows in the \{h, e^2\} subspace for the \( N \)-component Abelian Higgs model for \( N < 366 \), and in \( (4 - \varepsilon) \)-dimensions. Both fixed points are unstable. The physical “unstable” region is \( h < 0 \).