Graphs with many independent vertex cuts

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Abstract

The cycles are the only 2-connected graphs in which any two nonadjacent vertices form a vertex cut. We generalize this fact by proving that for every integer \( k \geq 3 \) there exists a unique graph \( G \) satisfying the following conditions: (1) \( G \) is \( k \)-connected; (2) the independence number of \( G \) is greater than \( k \); (3) any independent set of cardinality \( k \) is a vertex cut of \( G \). The edge version of this result does not hold.

We also consider the problem when replacing independent sets by the periphery.

Key words. Vertex cut; connectivity; independent set

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We consider finite simple graphs. For terminology and notations we follow the books [2, 5]. It is known [4, p.46] that the cycles are the only 2-connected graphs in which any two nonadjacent vertices form a vertex cut. We will generalize this fact and consider two related problems.

We denote by \( V(G) \) the vertex set of a graph \( G \). The order of \( G \), denoted by \(|G|\), is the number of vertices of \( G \). For \( S \subseteq V(G) \), the notation \( G[S] \) denotes the subgraph of \( G \) induced by \( S \). Let \( K_{s,t} \) denote the complete bipartite graph whose partite sets have cardinality \( s \) and \( t \), respectively.

Notation. The notation \( K_{s,s} - PM \) denotes the graph obtained from the balanced complete bipartite graph \( K_{s,s} \) by deleting all the edges in a perfect matching of \( K_{s,s} \).
Note that $K_{s,s} - PM$ is an $(s - 1)$-connected $(s - 1)$-regular graph, $K_{3,3} - PM$ is the 6-cycle $C_6$ and $K_{4,4} - PM$ is the cube $Q_3$.

**Theorem 1.** Let $k \geq 3$ be an integer. Then $K_{k+1,k+1} - PM$ is the unique graph $G$ satisfying the following three conditions: (1) $G$ is $k$-connected; (2) the independence number of $G$ is greater than $k$; (3) any independent set of cardinality $k$ is a vertex cut of $G$.

**Proof.** Let $G$ be a graph satisfying the three conditions in Theorem 1. We first assert that $G$ has order at least $2k + 2$. Let $S$ be an independent set of $G$ with cardinality $k + 1$. Since $G$ is $k$-connected, every vertex has degree at least $k$. Let $T$ be the neighborhood of one vertex in $S$. Then $|T| \geq k$. Thus $|G| \geq |S| + |T| \geq 2k + 1$. If $|G| = 2k + 1$, then $T$ would be the common neighborhood of all the vertices in $S$. But now any $k$ vertices in $S$ do not form a vertex cut, contradicting condition (3). This shows that $|G| \geq 2k + 2$.

Choose an arbitrary but fixed independent set $A = \{x_1, x_2, \ldots, x_{k+1}\}$ of cardinality $k+1$ in $G$. By condition (3), for every $i$ with $1 \leq i \leq k+1$, the graph $H_i \triangleq G - (A \setminus \{x_i\})$ is disconnected. Let $G_i$ denote the union of all the components of $H_i$ except the component containing $x_i$. Note that each $G_i$ is disjoint from the set $A$.

Let $Q$ and $W$ be subgraphs of $G$ or subsets of $V(G)$. We say that $Q$ and $W$ are **adjacent** if there exists an edge with one endpoint in $Q$ and the other endpoint in $W$; otherwise $Q$ and $W$ are **nonadjacent**. Next we prove three claims.

Claim 1. $V(G_i) \cap V(G_j) = \phi$, $G_i$ and $G_j$ are nonadjacent for $1 \leq i < j \leq k+1$.

In the sequel, for notational simplicity, a vertex $v$ may also mean the set $\{v\}$. We will use the fact that if $T$ is a minimum vertex cut of $G$, then every vertex in $T$ has a neighbor in every component of $G - T$. Clearly, $G$ has connectivity $k$. Since $A \setminus x_j$ is a minimum vertex cut of $G$, the subgraph $G[x_i \cup V(G_j)]$ is connected and it is contained in the component of $H_i$ containing $x_i$. By the definition of $G_i$, we deduce that $(x_i \cup V(G_j)) \cap V(G_i) = \phi$, implying $V(G_i) \cap V(G_j) = \phi$.

To show the second conclusion, just note that any vertex in $G_i$ and any vertex in $G_j$ lie in different components of the graph $G - (A \setminus x_i)$.

Claim 2. $A \cup (\bigcup_{i=1}^{k+1} V(G_i)) = V(G)$.

To the contrary, suppose that $F \triangleq V(G) \setminus \{A \cup (\bigcup_{i=1}^{k+1} V(G_i))\}$ is not empty. Let $F_1, F_2, \ldots, F_s$ be the components of $G[F]$.
Recall that by definition, for $1 \leq i \leq k+1$, $G_i$ denotes the union of all the components of $G - (A \setminus x_i)$ except the component $R_i$ that contains $x_i$. Hence, for every $p$ with $1 \leq p \leq s$, $F_p$ is a subgraph of $R_i$, implying that $G_i$ is nonadjacent to $F_p$. Note that

$$R_i = G[x_i \cup F \cup (\cup_{j \neq i} V(G_j))].$$

Since $R_i$ is connected, $x_i$ is adjacent to every component of $G_j$ with $j \neq i$ and $x_i$ is adjacent to each $F_p$ for $1 \leq p \leq s$. Thus, every $F_p$ is adjacent to every vertex in $A$.

We choose one vertex $y_i$ from $G_i$ for each $1 \leq i \leq k$. Then $B \triangleq \{y_1, y_2, \ldots, y_k\}$ is an independent set of $G$. We assert that every component of $(\bigcup_{i=1}^{k+1} G_i) - B$ is adjacent to $A$, since otherwise $G$ would have a cut-vertex. It follows that $G - B$ is connected, contradicting condition (3). This shows that $F$ is empty and claim 2 is proved.

Claim 3. $|G_i| = 1$ for every $1 \leq i \leq k+1$.

To the contrary, suppose some $G_i$ has order at least 2. Without loss of generality, suppose $|G_k| \geq 2$. Let $z_j$ be a neighbor of $x_{k+1}$ in $G_j$ for $j = 1, \ldots, k-1$. Since $x_{k+1}$ is adjacent to $G_k$, $x_{k+1}$ has a neighbor $w \in G_k$. The condition $|G_k| \geq 2$ ensures that $G_k$ has a vertex $z_k$ distinct from $w$. Denote $C = \{z_1, z_2, \ldots, z_k\}$. Then $C$ is an independent set.

We assert that every component of $(G_1 \cup G_2 \cup \cdots \cup G_k) - C$ is adjacent to $A \setminus x_{k+1}$, since otherwise some $z_j$ and $x_{k+1}$ would form a vertex cut of $G$, contradicting the condition that $G$ is $k$-connected and $k \geq 3$. Also, every component of $G_{k+1}$ is adjacent to every vertex in $A \setminus x_{k+1}$. It follows that the graph $G - (C \cup x_{k+1})$ is connected. But $x_{k+1}$ is adjacent to $w$, a vertex in $G_k - z_k$. Hence $G - C$ is connected, contradicting condition (3). This shows that each $G_i$ consists of one vertex.

Combining the information in the above three claims, we deduce that $|G| = 2k + 2$ and the neighborhood of $x_i$ is $\{G_1, G_2, \ldots, G_{k+1}\} \setminus \{G_i\}$ for $1 \leq i \leq k+1$. It follows that $G = K_{k+1,k+1} - PM$.

Conversely, it is easy to verify that the graph $K_{k+1,k+1} - PM$ indeed satisfies the three conditions in Theorem 1. This completes the proof.

Mr. Feng Liu [3] asked whether the edge version of Theorem 1 holds. The following result shows that the answer is negative.

**Corollary 2.** Let $k \geq 3$ be an integer. If a graph $G$ is $k$-edge-connected with matching number greater than $k$, then $G$ contains a matching $M$ of cardinality $k$ such that $G - M$ is connected.

**Proof.** To the contrary, suppose that for any matching $M$ of cardinality $k$, $G - M$
is disconnected. Consider the line graph of $G$, denoted by $H \triangleq L(G)$. Since $G$ is $k$-edge-connected, we deduce that [5, p.283] $H$ is $k$-connected. An independent set of vertices in $H$ corresponds to a matching in $G$. Applying Theorem 1 to $H$ we have $H = K_{k+1,k+1} - PM$, where we use the equality sign for graphs to mean isomorphism. It is known ([1] or [5, p.282]) that any line graph of a simple graph cannot have the claw as an induced subgraph. But for $k \geq 3$, $K_{k+1,k+1} - PM$ contains an induced claw (many in fact). This contradiction shows that $G$ contains a matching $M$ of cardinality $k$ such that $G - M$ is connected.

**Remark.** As for the case $k = 2$ of Corollary 2, using the ideas in the above proof and using the fact mentioned at the beginning of this paper, we see that cycles are the only 2-edge-connected graphs in which any two nonadjacent edges form a separating set.

Finally we consider replacing independent vertices in Theorem 1 by peripheral vertices. The **eccentricity** of a vertex $v$ in a graph $G$, denoted by $e(v)$, is the distance to a vertex farthest from $v$. A vertex $v$ is a **peripheral vertex** of $G$ if $e(v)$ is equal to the diameter of $G$. The **periphery** of $G$ is the set of all peripheral vertices. We pose the following

**Conjecture 3.** Let $k \geq 2$ be an integer. If $G$ is a $k$-connected graph whose periphery has cardinality at least $k$, then $G$ contains a set $S$ of $k$ peripheral vertices such that $G - S$ is connected.

**Observation 4.** The case $k = 2$ of Conjecture 3 is true.

**Proof.** To the contrary, suppose that any two peripheral vertices form a vertex cut of $G$. Denote by $d(u,v)$ the distance between two vertices $u, v$ and let the diameter of $G$ be $d$. We have $d \geq 2$. Choose vertices $x, y$ such that $d(x,y) = d$. Let $P$ be a shortest $(x,y)$-path, and let $y'$ be the neighbor of $y$ on $P$. Let $H$ be a component of $G - \{x,y\}$ that does not contain the path $P - \{x,y\}$.

Since $G$ is 2-connected, both $x$ and $y$ have a neighbor in $H$. Let $x'$ be a neighbor of $x$ in $H$. Then $d(x',y) \geq d - 1$. Since every $(x',y')$-path contains either $x$ or $y$, we deduce that $d(x',y') = d$. Thus $x'$ is also a peripheral vertex. By our assumption, $G - \{x,x'\}$ is disconnected. Let $R$ be the component of $G - \{x,x'\}$ containing $y$. Clearly every component of $G - \{x,x'\}$ other than $R$ is contained in $H$. Let $Q$ be an arbitrary such component. We assert that every vertex in $Q$ is adjacent to $x'$. Let $z \in V(Q)$. Any $(z,y)$-path must contain either $x$ or $x'$. Since $d(x,y) = d$, a shortest $(z,y)$-path must contain $x'$, which implies that $z$ and $x'$ are adjacent and $z$ is a peripheral vertex, since $d(x',y) \geq d - 1$. Choose a vertex $z_0$ from any component of $G - \{x,x'\}$ other than $R$.  

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Note that $x'$ is adjacent to $R$, since $\{x, x'\}$ is a minimum vertex cut of $G$. Then the graph $G - \{x, z_0\}$ is connected, contradicting our assumption.

The graph $F$ in Figure 1 shows that the connectivity condition in Conjecture 3 cannot be dropped. $F$ has diameter 4 and periphery $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. With $k = 5$, any 5 peripheral vertices of $F$ form a vertex cut.

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