Definable versions of Menger’s conjecture

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Abstract

Menger’s conjecture that Menger spaces are $\sigma$-compact is false; it is true for analytic subspaces of Polish spaces and undecidable for more complex definable subspaces of Polish spaces. For non-metrizable spaces, analytic Menger spaces are $\sigma$-compact, but Menger continuous images of co-analytic spaces need not be. The general co-analytic case is still open, but many special cases are undecidable, in particular, Menger co-analytic topological groups. We also prove that if there is a Michael space, then productively Lindelöf Čech-complete spaces are $\sigma$-compact. We also give numerous characterizations of proper K-Lusin spaces. Our methods include the Axiom of Co-analytic Determinacy, non-metrizable descriptive set theory, and Arhangel’skii’s work on generalized metric spaces.

1 Menger co-analytic groups

We shall assume all spaces are completely regular.

Definition. A topological space is analytic if it is a continuous image of $\mathbb{P}$, the space of irrationals. A space is Lusin if it is an injective continuous image of $\mathbb{P}$. (This is the terminology of [24]. This term is currently used

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A space is **K-analytic** if it is a continuous image of a Lindelöf Čech-complete space. A space is **K-Lusin** if it is an injective continuous image of a Lindelöf Čech-complete space.

**Definition.** A space is **co-analytic** if $\beta X \setminus X$ is analytic. In general, we call $\beta X \setminus X$ the **remainder** of $X$. $bX \setminus X$, for any compactification $bX$ of $X$, is called a **remainder** of $X$.

**Definition.** A space is **Menger** if whenever $\{U_n : n < \omega\}$ is a sequence of open covers, there exist finite $V_n$, $n < \omega$, such that $V_n \subseteq U_n$ and $\bigcup\{V_n : n < \omega\}$ is a cover.

Arhangel’skii [4] proved that Menger analytic spaces are $\sigma$-compact, generalizing Hurewicz’s classic theorem that Menger completely metrizable spaces are $\sigma$-compact. Menger’s conjecture was disproved in [19], where Miller and Fremlin also showed it undecidable whether Menger co-analytic sets of reals are $\sigma$-compact. In [27] we proved that Menger Čech-complete spaces are $\sigma$-compact and obtained various sufficient conditions for Menger co-analytic topological spaces to be $\sigma$-compact. We continue that study here. In [27] we observed that $\Pi^1_1$-determinacy – which we also call CD: the Axiom of Co-analytic Determinacy – implies Menger co-analytic sets of reals are $\sigma$-compact. Indeed, PD (the Axiom of Projective Determinacy) implies Menger projective sets of reals are $\sigma$-compact [25], [27]. When one goes beyond co-analytic spaces in an attempt to generalize Arhangel’skii’s theorem, one runs into ZFC counterexamples, but it is not clear whether there is a ZFC co-analytic counterexample. Assuming $V = L$, there is a counterexample which is a subset of $\mathbb{R}$ [19], [27]. Here we prove:

**Theorem 1.1.** CD implies every Menger co-analytic topological group is $\sigma$-compact.

**Remark.** CD follows from the existence of a measurable cardinal [17].

We first slightly generalize the CD result quoted above.

**Lemma 1.2.** CD implies every separable metrizable Menger co-analytic space is $\sigma$-compact.

In order to prove this, we need some general facts about analytic spaces and perfect maps.
Lemma 1.3. Metrizable perfect pre-images of analytic spaces are analytic.

Proof. Rogers and Jayne [24, 5.8.9] prove that perfect pre-images of metrizable analytic spaces are $K$-analytic, and that $K$-analytic metrizable spaces are analytic [24, 5.5.1]. □

Lemma 1.4 [11, 3.7.6]. If $f : X \to Y$ is perfect, then for any $B \subseteq Y$, $f_B : f^{-1}(B) \to B$ is perfect.

Lemma 1.5 [24, 5.2.3]. If $f$ is a continuous map of a compact Hausdorff $X$ onto a Hausdorff space $Y$ and the restriction of $f$ to a dense subspace $E$ of $X$ is perfect, then $f^{-1} \circ f(E) = E$.

Lemma 1.6. Metrizable perfect pre-images of co-analytic spaces are co-analytic.

Proof. Let $M$ be a metrizable perfect pre-image of a co-analytic $X$. Let $p$ be the perfect map. Extend $p$ to $P$ mapping $\beta M$ onto $\beta X$. Then by Lemma 1.5 $P^{-1} \circ P(M) = M$, i.e. $P^{-1}(X) = M$. Then $P(\beta M \setminus M) = \beta X \setminus X$, since $P$ is onto and points in $M$ map into $X$. By Lemma 1.4 $P|P^{-1}(\beta X \setminus X)$ is perfect. But then $\beta M \setminus M$ is analytic by Lemma 1.3 so $M$ is co-analytic. □

Proof of Lemma 1.2. Let $X$ be separable metrizable Menger co-analytic. It is folklore (see e.g. [13]) that every separable metrizable space $X$ is a perfect image of a 0-dimensional one, and hence of a subspace $M$ of the Cantor space $\mathbb{K} \subseteq \mathbb{R}$. Then $M$ is Menger co-analytic, so by CD is $\sigma$-compact. But then so is $X$. □

Lemma 1.7 [5]. A topological group with Lindelöf remainder is a perfect pre-image of a metrizable space.

Since analytic spaces are Lindelöf, a co-analytic group is a perfect pre-image of a metrizable space. Since Menger spaces are Lindelöf, a Menger co-analytic topological group $G$ is a perfect pre-image of a separable metrizable space $M$. In [27], we proved perfect images of co-analytic spaces are co-analytic, so $M$ is co-analytic and Menger and therefore $\sigma$-compact by CD and Lemma 1.2. Then $G$ is $\sigma$-compact as well. □

After hearing about Theorem 1.1 S. Tokgöz [29] proved:

Proposition 1.8. $V = L$ implies there is a Menger co-analytic group which is not $\sigma$-compact.
2 Productively Lindelöf co-analytic spaces

Definition. A space $X$ is **productively Lindelöf** if for every Lindelöf space $Y$, $X \times Y$ is Lindelöf.

We have extensively studied productively Lindelöf spaces \([1, 8, 9, 10, 25, 26, 28]\), as have other authors. Since productively Lindelöf spaces consistently are Menger \([26, 1, 25, 23]\) it is natural to ask:

**Problem 1.** Are productively Lindelöf co-analytic spaces $\sigma$-compact?

Definition. A **Michael space** is a Lindelöf space whose product with the space $\mathbb{P}$ of irrationals is not Lindelöf.

It is consistent that there is a Michael space, but it is not known whether there is one from ZFC. If there is no Michael space, then the space $\mathbb{P}$ of irrationals is productively Lindelöf, co-analytic, nowhere locally compact, but not $\sigma$-compact. We shall prove:

**Theorem 2.1.** $\text{CH}$ implies productively Lindelöf co-analytic spaces which are nowhere locally compact are $\sigma$-compact.

I do not know whether the unwanted “nowhere locally compact” clause can be removed. It assures us that $\beta X \setminus X$ is dense in $\beta X$. Laying the groundwork for proving Theorem 2.1 we need some definitions and previous results.

**Definition** \([3]\). A space is of **countable type** if each compact set is included in a compact set of countable character.

**Lemma 2.2** \([14]\). A completely regular space is of countable type if and only if some (all) remainder(s) are Lindelöf.

**Definition** \([2]\). A space is **Alster** if each cover by $G_\delta$’s such that each compact set is included in the union of finitely many members of the cover has a countable subcover.

**Lemma 2.3** \([1, 25]\). Alster spaces of countable type are $\sigma$-compact.

**Lemma 2.4** \([2]\). $\text{CH}$ implies productively Lindelöf spaces of weight $\leq \aleph_1$ are Alster.
We can now prove Theorem 2.1. Let $X$ be productively Lindelöf, co-analytic, and nowhere locally compact. $\beta X \setminus X$ is analytic and hence Lindelöf and separable. It is dense in $\beta X$, so $w(\beta X)$ and hence $w(X) \leq 2^{\aleph_0} = \aleph_1$. Then $X$ is Alster. Since $\beta X \setminus X$ is Lindelöf, $X$ has countable type, so it is $\sigma$-compact.

For metrizable spaces, Repovš and Zdomskyy [23] proved:

**Proposition 2.5.** If there is a Michael space and $\text{CD}$ holds, then every co-analytic productively Lindelöf metrizable space is $\sigma$-compact.

We would like to drop the metrizability assumption, using:

**Lemma 2.6 [23].** If there is a Michael space, then productively Lindelöf spaces are Menger.

As in [27], we run up against the unsolved problem:

**Problem 2.** Is it consistent that co-analytic Menger spaces are $\sigma$-compact?

However, we can apply the various partial results in the previous section and [27] to obtain:

**Theorem 2.7.** Suppose there is a Michael space and $\text{CD}$ holds. Then if $X$ is co-analytic and productively Lindelöf, then $X$ is $\sigma$-compact if either:

1. closed subspaces of $X$ are $G_\delta$’s,
   
   or

2. $X$ is a $\Sigma$-space,
   
   or

3. $X$ is a $p$-space,
   
   or

4. $X$ is a topological group.

**Proof.** These conditions all imply under $\text{CD}$ that Menger co-analytic spaces are $\sigma$-compact. $\Sigma$-spaces and $p$-spaces are discussed in Section 3. \qed
The two hypotheses of Theorem 2.7 are compatible, since it is well-known that CH is compatible with the existence of a measurable cardinal, and that CH implies the existence of a Michael space [18]. Various other hypotheses about cardinal invariants of the continuum also imply the existence of a Michael space – see e.g. [20]. These are all compatible with CD.

We also have:

**Theorem 2.8.** There is a Michael space if and only if productively Lindelöf Čech-complete spaces are \( \sigma \)-compact.

**Proof.** If there is no Michael space, the space of irrationals is productively Lindelöf, and of course it is Čech-complete but not \( \sigma \)-compact. If there is a Michael space, productively Lindelöf spaces are Menger, but we showed in [27] that Menger Čech-complete spaces are \( \sigma \)-compact.

Repovš and Zdomskyy [23] prove:

**Proposition 2.9.** Suppose \( \text{cov}(\mathcal{M}) > \omega_1 \), and there is a Michael space. Then every productively Lindelöf \( \Sigma^1_2 \) subset of the Cantor space is \( \sigma \)-compact.

**Example.** The metrizability condition cannot be removed; Okunev’s space is a productively Lindelöf continuous image of a co-analytic space, but is not \( \sigma \)-compact (see [9]). In more detail, consider the Alexandrov duplicate \( A \) of the space \( \mathbb{P} \) of irrationals. \( A \) is co-analytic, since it has a countable remainder with a countable base. A countable metrizable space is homeomorphic to an \( F_{\sigma} \) in the Cantor space, and so is analytic. Okunev’s space is obtained by collapsing the non-discrete copy of \( \mathbb{P} \) in \( A \) to a point. Note that Okunev’s space is not co-analytic. To see this, if it were, it would be of countable type by Lemma 2.2. In [9] we showed that this space is Alster but not \( \sigma \)-compact, which would contradict Lemma 2.3.

### 3 K-analytic and K-Lusin spaces

We take the opportunity to make some observations about K-analytic, K-Lusin, absolute Borel, Frolik, and what Arhangel’skii [6] calls Borelian of the first type spaces. These are all attempts to generalize concepts of Descriptive Set Theory beyond separable metrizable spaces.

**Definition** [9]. A space is Frolik if it is homeomorphic to a closed subspace of a countable product of \( \sigma \)-compact spaces.
Definition. A space $X$ is absolute Borel if it is in the $\sigma$-algebra generated by the closed sets of $\beta X$. A space $X$ is Borelian of the first type if it is in the $\sigma$-algebra generated by the open sets of $\beta X$.

Definition. A space is projectively $\sigma$-compact (projectively countable) if its continuous images in separable metrizable spaces are all $\sigma$-compact (countable).

Frolík [12] showed that each Frolík space is absolute $K_{\sigma\delta}$ (and therefore Lindelöf), i.e. the intersection of countably many $\sigma$-compact subspaces of its Čech-Stone compactification (and conversely), and also is the continuous image of a Čech-complete Frolík space, so that Frolík spaces are absolute Borel and $K$-analytic. $K$-Lusin spaces are clearly $K$-analytic; $K$-Lusin spaces are also Frolík [24, 5.8.6]. Since $K$-analytic metrizable spaces are analytic and analytic Menger spaces are $\sigma$-compact [1], we see that Menger $K$-analytic spaces are projectively $\sigma$-compact [9]. In [26] we proved that projectively $\sigma$-compact Lindelöf spaces are Hurewicz, so we conclude:

Theorem 3.1. Menger $K$-analytic spaces are Hurewicz.

Hurewicz is a property strictly between $\sigma$-compact and Menger. A space is Hurewicz if every Čech-complete space including it includes a $\sigma$-compact subspace including it (This is equivalent to the usual definition – see [25]). This theorem may give some inkling as to why it seems to be hard to find topological properties that imply Hurewicz spaces are $\sigma$-compact which don’t in fact imply Menger spaces are $\sigma$-compact. There are, however, Hurewicz subsets of $\mathbb{R}$ which are not $\sigma$-compact — see e.g. [30].

There is a projectively $\sigma$-compact Frolík space which is not $\sigma$-compact (Okunev’s space – see [9]). Okunev’s space is also not Čech-complete, since Menger Čech-complete spaces are $\sigma$-compact [24]. There is a Frolík subspace of $\mathbb{R}$ which is not Čech-complete, since “Čech-complete” translates into being a $G_\delta$, and we know the Borel hierarchy is non-trivial. There are of course analytic subsets of $\mathbb{R}$ which are not absolute Borel and hence not Frolík. Moore’s L-space [21] is projectively countable but not K-analytic. The reason is that all its points are $G_\delta$’s, which contradicts projectively countable for K-analytic spaces [24, 5.4.3].

Since K-Lusin spaces are Frolík, it is worth mentioning that:

Proposition 3.2 [9]. There are no Michael spaces if and only if every Frolík space is productively Lindelöf.
We could add to this “if and only if every K-Lusin space is productively Lindelöf”.

Proof. $\mathbb{P}$ is K-Lusin. 

Also of interest is:

**Proposition 3.3** [24, 2.5.5]. $K$-analytic spaces are powerfully Lindelöf, i.e. their countable powers are Lindelöf – in fact they are $K$-analytic.

**Theorem 3.4.** Co-analytic Menger $K$-analytic spaces are $\sigma$-compact.

**Corollary 3.5.** Suppose there is a Michael space. Then co-analytic productively Lindelöf $K$-analytic spaces are $\sigma$-compact.

Compare with 2.5.

The Corollary follows from 2.6. In order to prove 3.4 we need to know:

**Definition** [6]. A completely regular space is called an $s$-space if there exists a countable open source for $X$ in some compactification $bX$ of $X$, i.e. a countable collection $\mathcal{S}$ of open subsets of $bX$ such that $X$ is a union of some family of intersections of non-empty subfamilies of $\mathcal{S}$.

We also need to know about $p$-spaces and $\sum$-spaces, but do not need their internal characterizations. What we need are:

**Lemma 3.6** [3]. A completely regular space is Lindelöf $p$ if it is the perfect pre-image of a separable metrizable space.

**Lemma 3.7** [22]. A completely regular space is Lindelöf $\sum$ if and only if it is the continuous image of a Lindelöf $p$-space.

**Lemma 3.8.** An analytic space has a countable network and hence (see e.g [13]) is Lindelöf and a $\sum$-space.

**Lemma 3.9** [6]. $X$ is a Lindelöf $p$-space if and only if it is a Lindelöf $\sum$-space and an $s$-space.

**Lemma 3.10** [6]. $X$ is Lindelöf $\sum$ if and only if its remainder is an $s$-space.

**Lemma 3.11** [5]. $X$ is a Lindelöf $p$-space if and only if its remainder is.
Proof of Theorem 3.4. Such a space $X$ is a Lindelöf $p$-space, since both it and its remainder are Lindelöf $Σ$. Let $X$ map perfectly onto a metrizable $M$. Then $M$ is analytic and Menger, so is $σ$-compact, so $X$ is also.

Theorem 3.12. Co-analytic Menger absolute Borel spaces are $σ$-compact.

To see this, we introduce:

Definition. Given a family of sets $S$, Rogers and Jayne \[24\] say that a set is a Souslin $S$-set if it has a representation in the form

$$\bigcup S(\sigma|n)$$

with $S(\sigma|n) \in S$ for all finite sequences of positive integers.

Rogers and Jayne prove:

Lemma 3.13 \[24, 2.5.4\]. The family $A$ of $K$-analytic subsets of a completely regular space is closed under the Souslin operation i.e. every Souslin $A$-set is in $A$; if a family is closed under the Souslin operation, it is closed under countable intersections and countable unions.

Corollary 3.14. Absolute Borel spaces are $K$-analytic.

Proof. This is well-known. In $βX$, closed subsets are compact; compact spaces are $K$-analytic.

Theorem 3.12 now follows from 3.4.

Theorem 3.15. Every Lindelöf Borelian space of the first type is $K$-analytic.

Proof. We proceed by induction on subspaces of a fixed compact space. For the basis step, note that open subspaces of a compact space are locally compact, while Lindelöf locally compact spaces are $σ$-compact. For the successor stage, assume a Lindelöf Borelian set of the first type is the union (intersection) of countably many $K$-analytic subspaces. By Lemma 3.13, the union (intersection) is $K$-analytic and hence Lindelöf. The limit stage is trivial.

Arhangel’skii \[6\] proved that Borelian sets of the first type are $s$-spaces. This is interesting because:

Theorem 3.16. Every absolute Borel $s$-space is a Lindelöf $p$-space.
Proof. We induct on Borel order. The basis step is trivial. We need to show s-spaces which are the countable union (intersection) of Lindelöf p-spaces are Lindelöf \( \sum \). Let \( \{X_n\}_{n<\omega} \) be Lindelöf p. Let \( \sum_{n<\omega} X_n \) be the disjoint sum of the \( X_n \)'s. Then \( \sum_{n<\omega} X_n \) is clearly Lindelöf p. Consider the natural map \( \sigma \) from \( \sum_{n<\omega} X_n \) to \( \bigcup_{n<\omega} X_n \) obtained by identifying all copies of a point \( x \in \bigcup_{n<\omega} X_n \) which are in \( \sum_{n<\omega} X_n \). \( \sigma \) is continuous, so \( \bigcup_{n<\omega} X_n \) is Lindelöf \( \sum \).

Now consider \( \prod_{n<\omega} X_n \). This is also Lindelöf p \([3]\) and so then is the diagonal \( \Delta \). Define \( \pi (\langle x, x, \ldots \rangle) = x \). Then \( \pi \) is continuous and maps \( \Delta \) onto \( \bigcap_{n<\omega} X_n \), which is therefore Lindelöf \( \sum \). \( \square \)

Note Okunev’s space is Lindelöf absolute \( F_{\sigma \delta} \) but is not s, since it is not of countable type \([9]\), while s-spaces are \([6]\). By Theorem \([3.4]\) Okunev’s space is not co-analytic.

Borel sets of reals are of course analytic; Okunev’s space shows that Lindelöf absolute Borel spaces need not be analytic, since it is Menger but not \( \sigma \)-compact. Compact spaces are Borelian of the first type, so the latter spaces need not be analytic.

A somewhat smaller class of spaces than the \( K \)-analytic (\( K \)-Lusin) ones is comprised of what Rogers and Jayne call the proper \( K \)-analytic (proper \( K \)-Lusin) spaces.

Definition. A space is proper \( K \)-analytic if it is the perfect pre-image of an analytic subspace of \( \mathbb{R}^\omega \). A space is proper \( K \)-Lusin if it is the perfect pre-image of a Lusin subspace of \( \mathbb{R}^\omega \).

Rogers and Jayne \([24]\) prove that a space is proper \( K \)-Lusin if and only if both it and its remainder are \( K \)-analytic. It follows that a space is proper \( K \)-Lusin if and only if it and its remainder are \( K \)-Lusin. They also prove that \( K \)-Lusin spaces are absolute \( K_{\sigma \delta} \), i.e. what we have called Frélix. It follows that proper \( K \)-Lusin spaces are both \( K_{\sigma \delta} \) and \( G_{\delta \sigma} \), i.e. countable unions of Čech-complete spaces. We shall provide a large number of equivalences for “proper \( K \)-Lusin” below.

Proper \( K \)-analytic spaces are p-spaces, and their continuous real-valued images are analytic, so:

Theorem 3.17. Menger proper \( K \)-analytic spaces are \( \sigma \)-compact.

Corollary 3.18. Menger proper \( K \)-Lusin spaces are \( \sigma \)-compact.
Lemma 3.19. Let $Z(Y)$ be the collection of zero-sets of $Y$. Then $X$ is proper $K$-analytic if and only if $X \in S(Z(\beta X))$.

Theorem 3.20. A space is proper $K$-analytic if and only if it is a $K$-analytic $p$-space.

Proof. By definition, a proper $K$-analytic space is a $p$-space. By 3.19 and 3.18 it is $K$-analytic. Conversely, if $X$ is a $K$-analytic $p$-space, it maps perfectly onto a separable metrizable analytic space, which embeds into $\mathbb{R}^\omega$. □

Note that zero-sets are closed $G_\delta$’s, so that the absolute Baire sets, i.e. the elements of the $\sigma$-algebra generated by the zero-sets, are both Lindelöf Borelian of the first type and absolute Borel.

Corollary 3.21. Menger absolute Baire spaces are $\sigma$-compact.

Mixing Rogers and Jayne with Arhangel’skii, we have:

Theorem 3.22. The following are equivalent:

(a) $X$ is proper $K$-Lusin,

(b) $X$ and its remainder are $K$-Lusin,

(c) $X$ and its remainder are both Frolík,

(d) $X$ is Lindelöf Borelian of the first type,

(e) $X$ is absolute Borel and Lindelöf $p$,

(f) $X$ is absolute Borel and of countable type.

Proof. We have already proved that (a), (b) and (c) are equivalent. (c) implies (d), since $X$ is Lindelöf absolute $G_{\delta\sigma}$. If $X$ is Lindelöf Borelian of the first type, it is $K$-analytic, but so is its remainder, so (d) implies (b). If $X$ is absolute Borel, it is $K$-analytic and its remainder is Borelian of the first type. If $X$ is Lindelöf $p$, so is its remainder, so (e) implies (b). (b) implies a proper $K$-Lusin space and its remainder are both $K$-analytic spaces, hence Lindelöf $\sum_{\sigma}$ spaces, so they are $p$-spaces. Thus (b) implies (e). (e) implies (f) since $p$-spaces are of countable type [3]. (f) implies the remainder of $X$ is Lindelöf Borelian of the first type, and so is $K$-analytic. Thus (a) implies (f) implies (b). □
We know that Menger proper K-analytic (a fortiori, proper K-Lusin) spaces are $\sigma$-compact, but Menger K-analytic spaces may not be.

**Problem 3.** Are Menger K-Lusin spaces $\sigma$-compact?

An interesting fact about K-Lusin spaces is that:

**Lemma 3.23** [24, 5.4.3]. The following are equivalent for a K-Lusin $X$:

(a) $X$ includes a compact perfect set;

(b) $X$ admits a continuous real-valued function with uncountable range;

(c) $X$ is not the countable union of compact subspaces which include no perfect subsets. In particular, if $X$ is not $\sigma$-compact, it includes a compact perfect set.

From this, we can conclude that Okunev’s space is not K-Lusin, since it is not $\sigma$-compact but doesn’t include a compact perfect set.

Indeed we have:

**Definition.** A space is **Rothberger** if whenever $\{U_n\}_{n<\omega}$ are open covers, there exists a cover $\{U_n\}_{n<\omega}, U_n \in U_n$.

Thus Rothberger is a strengthening of Menger.

**Lemma 3.24** [7]. Rothberger spaces do not include a compact perfect set.

**Theorem 3.25.** K-analytic Rothberger spaces are projectively countable.

**Proof.** They are projectively $\sigma$-compact. □

**Corollary 3.26.** K-Lusin Rothberger spaces are $\sigma$-compact.

**Proof.** This follows from 3.23. □

**Remark.** Projectively countable Lindelöf spaces are always Rothberger [26]; thus Okunev’s space is Rothberger [9]. The assertion that Rothberger spaces are projectively countable is equivalent to Borel’s Conjecture [26].

Here are some more problems we have not been able to solve:

**Problem 4.** Does CD imply co-analytic Hurewicz spaces are $\sigma$-compact?

**Problem 5.** Are Lindelöf co-analytic projectively $\sigma$-compact spaces $\sigma$-compact?

Note $V = L$ implies there is a co-analytic Hurewicz group of reals that is not $\sigma$-compact [29].
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