NODAL DOMAIN COUNT FOR THE GENERALIZED GRAPH
\(p\)-LAPLACIAN

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Abstract. Inspired by the linear Schrödinger operator, we consider a generalized \(p\)-Laplacian operator on discrete graphs and present new results that characterize several spectral properties of this operator with particular attention to the nodal domain count of its eigenfunctions. Just like the one-dimensional continuous \(p\)-Laplacian, we prove that the variational spectrum of the discrete generalized \(p\)-Laplacian on forests is the entire spectrum. Moreover, we show how to transfer Weyl’s inequalities for the Laplacian operator to the nonlinear case and prove new upper and lower bounds on the number of nodal domains of every eigenfunction of the generalized \(p\)-Laplacian on generic graphs, including variational eigenpairs. In particular, when applied to the linear case \(p = 2\), in addition to recovering well-known features, the new results provide novel properties of the linear Schrödinger operator.

1. Introduction

The study of nodal domains dates back to Sturm’s oscillation theorem that states that the zeros of the \(k\)-th mode of vibration of an oscillating string are the endpoints of a partition of the string into \(k\) subdomains where the mode has constant sign. Later Courant extended this result to higher dimensions, proving that the \(k\)-th eigenfunction of an oscillating membrane admits no more than \(k\) subdomains, called nodal domains [15]. The count of the nodal domains of the Laplacian operator, and its generalized version called the Schrödinger operator, has been shown to hold important information about the geometry of the system [8, 26, 27]. In particular, nodal domains are tightly connected to higher-order isoperimetric constants of graphs, making the study of nodal domains particularly relevant in the context of data clustering, expander graphs, and mixing time of Markov chains [1, 16, 22, 30]. Because of these reasons, the estimation of the number of nodal domains of the Schrödinger eigenfunctions both on continuous manifolds and on discrete and metric graphs has been an active field of research in recent years.

In the discrete graph setting, it was proved in [3, 6] that trees behave like strings and that the \(k\)-th eigenvector \(f_k\) of the Schrödinger operator, if everywhere non-zero, induces exactly \(k\) nodal domains. Moreover, again under the assumption that the \(k\)-th eigenvector \(f_k\) of the graph Schrödinger operator is everywhere non-zero, it was proved in [6] that for general graphs the following inequality holds for the number of nodal domains \(\nu(f_k)\) of \(f_k\), provided the corresponding eigenvalue is simple:

\[
k - \beta + l(f_k) \leq \nu(f) \leq k.
\]

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Here $\beta$ is the total number of independent loops of the graph and $l(f_k)$ is the number of independent loops where $f_k$ has constant sign. In the general case of eigenvalues with any multiplicity and eigenvectors with possibly some zero entry, it was proved in [17, 19, 41] that the following inequality holds:

$$k + r - 1 - \beta - z \leq \nu(f_k) \leq k + r - 1,$$

where $f_k$ is an eigenvector of the eigenvalue $\lambda_k$, $z$ is the number of zeros of $f_k$ and $r$ is the multiplicity of $\lambda_k$.

In recent years, there has been a surge in interest towards extensions of the above results to the nonlinear $p$-Laplacian and $p$-Schrödinger operators and their spectral properties, including the nodal domain count, have been widely investigated both in the continuous and in the discrete cases [3, 18, 31, 32]. This interest is prompted by applications to biological and physical processes often connected to complex networks and data clustering, as well as semi-supervised learning and machine learning in general [12, 13, 20, 21, 24, 35, 36], where the limiting cases $p = 1$ and $p = \infty$ are especially noteworthy. In particular, similarly to the linear case, an important relation connects the nodal domains of the $p$-Laplacian on graphs and the $k$-th order isoperimetric constant $h_k$ of the graph. Indeed, it is shown in [40] that this fundamental graph invariant can be bounded from above and from below using the variational spectrum $\lambda_k$ of the $p$-Laplacian and its nodal domain count via the Cheeger-like inequality

$$\lambda_{\nu(f_k)} \leq h_{\nu(f_k)} \leq c(p)\lambda_1^{1/p},$$

where $c(p) \to 1$ as $p \to 1$ and $f_k$ is any eigenfunction of $\lambda_k$. To our knowledge, this result is the tightest available connection between $h_k$ and the spectrum of the graph $p$-Laplacian and clearly highlights the importance of the nodal domain count in connection to, for example, the quality of $p$-Laplacian graph embeddings for data clustering, for which there is a wealth of empirical evidence [10–12, 21]. A nodal domain theorem for the graph $p$-Laplacian is provided in [40], where it is shown that, for any eigenfunction $f_k$ of the $p$-Laplacian, the number of nodal domains is bounded above as $\nu(f_k) \leq k + r - 1$, where $r$ is the multiplicity of the corresponding eigenvalue. Analogous results are proved in [14] for the case $p = 1$. However no lower bounds for $\nu(f_k)$ are known in the general case.

The main aim of this paper is to provide lower bounds on the number of nodal domains of the generic eigenfunction of the $p$-Laplacian. To this end, we will introduce a class of generalized $p$-Laplacian operators, already addressed in [34]. Such operators, inspired by the generalized linear Laplacian or Schrödinger operator [5, 41], are also largely related to $p$-Laplacian problems with zero Dirichlet boundary conditions [28]. Thus, all our results apply to both the classical $p$-Laplacian and the generalized $p$-Schrödinger operators. We prove a classical characterization of the first and the second variational eigenpairs and nonlinear Weyl’s like inequalities. These are fundamental instruments to study the nodal domains of the generic eigenfunction. Our general strategy, inspired by the work of [5], consists in defining appropriate rules to remove nodes or edges from the graph without changing an eigenpair. Repeated applications of this procedure allows us to arrive to a structured graph (e.g. a tree or the disjoint union of the nodal domains) for which nodal domain numbers or other spectral quantities can be fully characterized. This characterization can be brought back to the original graph by reversing the proposed procedure. This strategy allows us to find new lower bounds as well as retrieve known upper bounds.
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for the number of nodal domains of any eigenfunction, as a function of the position of the corresponding eigenvalue in the variational spectrum. In addition, our estimates, with $p = 2$, provide an improvement on the known results for the linear case.

An important side result of our work is that we are able to prove that, if the graph is a tree, the variational eigenvalues are all and only the eigenvalues of the $p$-Laplacian operator and that the $k$-th eigenfunction, if everywhere nonzero, admits exactly $k$ nodal domains. This result extends what is already known in the particular cases of the path graph $[40]$ and the star graph $[3]$, and is a generalization to the $p$-Laplacian of analogous findings known in the linear case $[5, 7]$. This is of independent interest for its potential applications to nonlinear spectral graph sparsification, expander graphs, and graph clustering $[38]$. In particular, note that our findings in combination with (1) show that the $k$-th order isoperimetric constant of a tree coincides with the $k$-th variational eigenvalue of the 1-Laplacian.

The paper is organized as follows. In Section 2 we introduce notations and needed definitions. Our main results are collected without proofs in Section 3. The strategy for the proofs is organized into several steps. In Section 4 we provide some preliminary results about the eigenfunctions of the generalized graph $p$-Laplacian, including the characterization of the first two eigenvalues. In Section 5 we develop the procedures that remove nodes and edges maintaining an eigenpair, and, along the way, we prove Weyl’s like inequalities. In section 6 we analyze the particular case of a tree. Finally, in Section 7 we provide the proofs of the main results, namely new inequalities on the number of the nodal domains of the eigenfunctions of the graph $p$-Laplacian operator.

2. Notation

Let $G = (V, E)$ be a connected undirected graph, where $V$ and $E$ are the sets of nodes and edges endowed with positive measures $\varrho : V \to \mathbb{R}^+$ and $\omega : E \to \mathbb{R}^+$, respectively. Given a function $f : V \to \mathbb{R}$, for any $p > 1$ consider the $p$-Laplacian operator:

$$(\Delta_p f)(u) := \sum_{v \sim u} \omega_{uv} |f(u) - f(v)|^{p-2}(f(u) - f(v)) \quad \forall u \in V,$$

where $v \sim u$ denotes the presence of an edge between $v$ and $u$. We point out that, while the $p$-Laplacian is well-defined also for $p = 1$, throughout this work we will not consider this limit case and we will always implicitly assume that $p > 1$. Throughout the paper, we will often use the function $\phi_p(x) := |x|^{p-2}x$, so that the above equation reads simply:

$$(\Delta_p f)(u) := \sum_{v \sim u} \omega_{uv} \phi_p(f(u) - f(v)) \quad \forall u \in V.$$ 

In analogy to the linear case, where the generalized Laplacian is defined as the Laplacian plus a diagonal matrix $[41]$, we define the generalized $p$-Laplacian (or $p$-Schrödinger) operator as

$$(\mathcal{H}_p f)(u) := (\Delta_p f)(u) + \kappa_u |f(u)|^{p-2}f(u) \quad \forall u \in V,$$

where $\kappa_u$ is a real coefficient. We say that $f$ is an eigenfunction of $\mathcal{H}_p$ if there exists $\lambda \in \mathbb{R}$ such that

$$(\mathcal{H}_p f)(u) = \lambda \varrho_u |f(u)|^{p-2}f(u) \quad \forall u \in V.$$
Similarly to the linear case, generalized $p$-Laplacians and their eigenfunctions are directly connected with the solutions of Dirichlet problems on graphs for the $p$-Laplacian operator. In fact, assume we have a graph $G = (V, E)$ with node and edge sets that can be partitioned as the disjoint union of internal and boundary sets $V = (V_I \sqcup V_B)$ and $E = (E_I \sqcup E_B)$, defined as $E_I = \{uv \in E : u, v \in V_I\}$ and $E_B = \{uv \in E : u \in V_I, v \in V_B\}$. This is the definition used for example in [25]. Then, if $f$ is a solution to the Dirichlet problem
\begin{align}
(\Delta_p f)(u) = \lambda \varrho_u |f(u)|^{p-2} f(u) & \quad \forall u \in V_I \\
f(u) = 0 & \quad \forall u \in V_B,
\end{align}
we deduce that $f$ is automatically also solution to the following eigenvalue equation for a generalized $p$-Laplacian where the information about the boundary nodes has been condensed in the nodal weights:
\[
\sum_{v \in V_I} \omega_{uv}|f(u) - f(v)|^{p-2}(f(u) - f(v)) + \left( \sum_{v \in V_B} \omega_{uv}\right) |f(u)|^{p-2} f(u) = \lambda \varrho_u |f(u)|^{p-2} f(u).
\]
In other words, the $p$-Laplacian Dirichlet problem with zero boundary conditions is equivalent to the eigenvalue problem [24] for the generalized $p$-Laplacian $\mathcal{H}_p$ with $\kappa_u = \sum_{v \in V_B} \omega_{uv}$.

Finally, the following definition introduces the concept of strong nodal domains of $G$, corresponding to a given function $f : V \to \mathbb{R}$.

**Definition 2.1 (Nodal domains).** Consider a graph $G = (V, E)$ and a function $f : V \to \mathbb{R}$. A set of vertices $A \subseteq V$ is a nodal domain induced by $f$ if the subgraph $G_A$ with vertices in $A$ is a maximal connected subgraph of $G$ where $f$ is nonzero and has constant sign. For convenience, in the following we will refer interchangeably to both $A$ and $G_A$ as the nodal domain induced by $f$.

Sometimes it is useful to distinguish between maximal subgraphs where the sign is strictly defined and those where zero entries are allowed. In particular, when zero entries of $f$ are allowed in the definition above, the maximal subgraphs are called weak nodal domains, whereas the maximal subgraphs with strictly positive or strictly negative sign as in Definition 2.1 are called strong nodal domains. However, as in this work we are not interested in weak nodal domains, throughout we shall simply use the term “nodal domain” to refer to the strong nodal domains, as defined above.

3. Variational spectrum and main results

In this section we state our main results and will devote the remainder of the paper to their proof. We first need to introduce the notion of variational spectrum. A set of $N$ variational eigenvalues of the generalized $p$-Laplacian on the graph $G = (V, E)$ can be defined via the Lusternik–Schnirelman theory and the min-max procedure based on the Krasnoselskii genus, which we review below [24].

**Definition 3.1 (Krasnoselskii genus).** Let $X$ be a Banach space and consider the class $A$ of closed symmetric subsets of $X$, $A = \{A \subseteq X \mid A$ closed, $A = -A\}$. For any $A \in A$ consider the space of the Krasnoselskii test maps on $A$ of dimension $k$:
\[
\Lambda_k(A) = \{\varphi : A \to \mathbb{R}^k \text{ continuous and such that } \varphi(x) = -\varphi(-x)\}.
\]
The Krasnoselskii genus of $A$ is the number $\gamma(A)$ defined as
\[
\gamma(A) = \begin{cases} 
\inf \{ k \in \mathbb{N} : \exists \varphi \in \Lambda_k(A) \text{ s.t. } 0 \notin \varphi(A) \} \\
\infty & \text{if } \not\exists k \text{ as above} \\
0 & \text{if } A = \emptyset
\end{cases}.
\]

Our reference Banach space is the space of vertex states $X = \{ f : V \to \mathbb{R} \} = \mathbb{R}^N$ and $\mathcal{A}$ denotes the family of all closed symmetric subsets of $\mathbb{R}^N$. Let $\mathcal{S}_p = \{ f \in X : \|f\|_p = 1 \}$ be the $p$-unit sphere on $X$ and for $1 \leq k \leq N$ consider the family of closed symmetric subsets of $\mathcal{S}_p$ of genus greater than $k$
\[
\mathcal{F}_k(\mathcal{S}_p) := \{ A \subseteq \mathcal{A} \cap \mathcal{S}_p | \gamma(A) \geq k \}.
\]

In order to define the variational eigenvalues of $\mathcal{H}_p$, we consider the Rayleigh quotient functional
\[
R_{\mathcal{H}_p}(f) = \frac{\sum_{uv \in E} \omega_{uv}|f(u) - f(v)|^p + \sum_{u \in V} \kappa_u |f(u)|^p}{\sum_{u \in V} \vartheta_u |f(u)|^p}.
\]
As $R_{\mathcal{H}_p}$ is positively scale invariant, i.e. $R_{\mathcal{H}_p}(\alpha f) = R_{\mathcal{H}_p}(f)$ for all $\alpha > 0$, it is not difficult to observe that the eigenvalues and eigenfunctions of the generalized $p$-Laplacian operator are the critical values and the critical points of $R_{\mathcal{H}_p}$ on $\mathcal{S}_p$.

The Lusternik-Schnirelman theory allows us to define a set of $N$ variational such critical values, via the following principle
\[
(4) \quad \lambda_k = \min_{A \in \mathcal{F}_k(\mathcal{S}_p)} \max_{f \in A} R_{\mathcal{H}_p}(f).
\]

We emphasize that the Krasnoselskii genus is a homeomorphism-invariant generalization to symmetric sets of the notion of dimension. In particular, if $A \in \mathcal{A}$ is the intersection of any subspace of dimension $k$ with $\mathcal{S}_p$, then $\gamma(A) = k$. Moreover, note that any $A$ such that $\gamma(A) \geq k$ contains at least $k$ mutually orthogonal functions (see e.g. [37]). Therefore, the definition in (4) is a generalization of the Courant-Fisher min-max characterization of the eigenvalues of a symmetric matrix, as $\mathcal{F}_k(\mathcal{S}_p)$ contains all subspaces of dimension greater than $k$. However, while Courant-Fisher applies directly to the case $p = 2$, linear subspaces alone are not sufficient to provide critical points in the general case $p \neq 2$.

3.1. Multiplicity and $\gamma$-multiplicity. Similarly to the case of symmetric matrices, we note that the variational eigenvalues $\{\lambda_k\}$ are by definition an increasing sequence. This allows us to define a notion of multiplicity for variational eigenvalues:

**Definition 3.2.** Let $\lambda_k$ be a variational eigenvalue of $\mathcal{H}_p$. If $\lambda_k$ appears $m$ times in the sequence of the variational eigenvalues
\[
(5) \quad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+m-1} < \lambda_{k+m} \leq \cdots \leq \lambda_N,
\]
we say that $\lambda_k$ has multiplicity $m$ and we write $\text{mult}_{\mathcal{H}_p}(\lambda_k) = m$ or simply $\text{mult}(\lambda_k) = m$ when no ambiguity may occur.

The notion of multiplicity defined above applies only to variational eigenvalues. In the case of a generic eigenvalue $\lambda$, we can use the Krasnoselskii genus to extend the notion of geometric multiplicity to the nonlinear setting:
Definition 3.3. Let $\lambda$ be an eigenvalue of $H_p$. If
\[ \gamma \left( \{ f \in S_p : H_p(f) = \lambda |f|^{p-2} f \} \right) = m \]
we say that $\lambda$ has $\gamma$-multiplicity $m$ and we write $\gamma$-mult$_{H_p}(\lambda) = m$, or simply $\gamma$-mult($\lambda$) = $m$ when no ambiguity may occur.

Finally, we define simple eigenvalues

Definition 3.4. We say that $\lambda$ is a simple eigenvalue of $H_p$ if $\lambda$ has a unique eigenfunction $f \in S_p$.

Notice that the notions of multiplicity and $\gamma$-multiplicity do not coincide and an eigenvalue with $\gamma$-multiplicity equal to one is not necessarily simple. Viceversa, if $\lambda$ is a simple eigenvalue, then necessarily $\gamma$-mult($\lambda$) = 1 and, if $\lambda$ is variational then also mult($\lambda$) = 1. This result is a direct consequence of the next lemma, whose proof follows directly from Lemma 5.6 and Proposition 5.3, Chapter II of [39].

Lemma 3.5. If $\lambda$ is a variational eigenvalue, then
\[ \gamma \text{-mult}(\lambda) \geq \text{mult}(\lambda). \]

Note that the inequality above implies, in particular, that, to any variational eigenvalue $\lambda$, there correspond at least mult($\lambda$) orthogonal eigenfunctions. Finally, we remark the following direct consequence of Lemma 3.5

Corollary 3.6. Let $H_p$ be the generalized $p$-Laplacian operator on a graph $G$ with $N$ nodes. Let $\{\lambda_i\}_{i=1}^n$ be the variational eigenvalues of $H_p$ counted without multiplicity, i.e. $\lambda_i \neq \lambda_j \forall i \neq j$. Then
\[ \sum_{i=1}^n \gamma \text{-mult}(\lambda_i) \geq \sum_{i=1}^n \text{mult}(\lambda_i) = N, \]
with the equality holding if and only if $\gamma$-mult($\lambda_i$) = mult($\lambda_i$), for all $i = 1, \ldots, n$.

3.2. Main results. We present below our main results. Recalling the idea summarized in the introduction, our strategy for counting nodal domains of generalized $p$-Laplacians is to come up with algorithmic steps to remove vertices and edges from the original graph in such a way that the original eigenpairs can be recovered from the eigenpairs of the new graph. Since the proofs of our main results require relatively long arguments, we state the results here and devote the remainder of the paper to their proofs. In particular, after discussing in Sections 4 and 5 a number of preliminary observations and results, which are of independent interest, Section 6 will provide proofs for Theorems 3.7 and 3.8 which deal with the special case of trees and forests, whereas Section 7 will present the proofs of Theorems 3.9 and 3.10 which address the case of general graphs.

Notice that, unlike linear operators, the variational spectrum does not cover the entire spectrum of the generalized $p$-Laplacian and, in general, establishing whether a certain eigenvalue is variational or not is still an open problem. For example, Amghibech shows in [2] that the $p$-Laplacian on a complete graph admits more than just the variational eigenvalues. Another simple example for the setting $\omega \equiv 1$, $q \equiv 1$ and $\kappa \equiv 0$ is provided by Figure 3.2, while a more refined analysis of non-variational eigenvalues is recently provided by Zhang in [42].
Our first main result shows that the situation is different for the special case of trees and, more in general, forests. In fact, as for the standard linear case, we prove that when $G$ is a forest, the variational spectrum covers all the eigenvalues of the generalized $p$-Laplacian. Here and in the following, we use the symbol $\sqcup$ to denote disjoint union.

**Theorem 3.7.** Let $G = \sqcup_{i=1}^{k} T_i$ be a forest, $\mathcal{H}_p$ a generalized $p$-Laplacian operator on $G$, $p > 1$, and $\mathcal{H}_p(T_i)$ the restriction of $\mathcal{H}_p$ to the $i$-th tree $T_i$. Then $\mathcal{H}_p$ admits only variational eigenvalues and for any such eigenvalue $\lambda$ it holds

$$\text{mult}_{\mathcal{H}_p}(\lambda) = \gamma \cdot \text{mult}_{\mathcal{H}_p}(\lambda) = \sum_{i=1}^{k} \text{mult}_{\mathcal{H}_p(T_i)}(\lambda)$$

where $\text{mult}_{\mathcal{H}_p(T_i)}(\lambda) = 0$ if $\lambda$ is not an eigenvalue of $\mathcal{H}_p(T_i)$.

In addition, we are able to prove the following theorem about the number of nodal domains induced on a forest, which generalizes well-known results for the case of the linear Laplacian [5, 7, 23].

**Theorem 3.8.** Let $G = \sqcup_{i=1}^{m} T_i$ be a forest and consider the generalized $p$-Laplacian operator $\mathcal{H}_p$, $p > 1$, on $G$. If $f_k$ is an everywhere nonzero eigenfunction associated to the eigenvalue $\lambda_k = \cdots = \lambda_{k+m-1}$ of $\mathcal{H}_p$, then $f_k$ changes sign on exactly $k-1$ edges. In other words, $f_k$ induces exactly $k-1 + m$ nodal domains.

Next, we address the case of general graphs. A tight upper bound for the number of nodal domains of the eigenfunctions of the $p$-Laplacian on graphs is provided in [40]. It is not difficult to observe that the same upper bound carries over unchanged to the generalized $p$-Laplacian case. This is summarized in the following result.

**Theorem 3.9.** Suppose that $G$ is connected and $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ are the variational eigenvalues of $\mathcal{H}_p$, $p > 1$. Let $\lambda$ be an eigenvalue of $\mathcal{H}_p$ such that $\lambda < \lambda_k$. Any eigenfunction associated to $\lambda$ induces at most $k-1$ nodal domains.

Finally, the following theorem provides novel lower bounds for the number of nodal domains of $\mathcal{H}_p$ in the case of general graphs. Moreover, when tailored to the case $p = 2$, it provides improved estimates of the nodal domain count that are strictly tighter than the currently available results [6, 41]. We will discuss these properties in more details below.
Theorem 3.10. Suppose that $\mathcal{G}$ is a connected graph with $\beta = |E| - |V| + 1$ independent loops, and let $\lambda_1 \leq \cdots \leq \lambda_N$ be the variational eigenvalues of $\mathcal{H}_p$, $p > 1$. For a function $f : V \to \mathbb{R}$, let $\nu(f)$ be the number of nodal domains induced by $f$, $l(f)$ the number of independent loops in $\mathcal{G}$ where $f$ has constant sign and $\{v_i\}_{i=1}^{z(f)}$ the nodes such that $f(v_i) = 0$, with $z(f)$ being the number of such nodes. Let $\mathcal{G}' = \mathcal{G} \setminus \{v_i\}_{i=1}^{z(f)}$ be the graph obtained by removing from $\mathcal{G}$ all the nodes where $f$ is zero as well as all the edges connected to those nodes. Let $c(f)$ be number of connected components of $\mathcal{G}'$ and $\beta'(f) = |E'| - |V'| + c(f)$ the number of independent loops of the graph $\mathcal{G}'$. Then:

P1. If $f$ is an eigenfunction of $\mathcal{H}_p$ with eigenvalue $\lambda$ such that $\lambda > \lambda_k$, then $f$ induces strictly more than $k - \beta + l(f) - z(f)$ nodal domains. Precisely, it holds $\nu(f) \geq k - \beta'(f) + l(f) - z(f) + c(f)$.

P2. If $f$ is an eigenfunction of $\mathcal{H}_p$ corresponding to the variational eigenvalue $\lambda_k > \lambda_{k-1}$ with $\text{mult}_{\mathcal{H}_p}(\lambda_k) = m$, then $\nu(f) \geq k + m - 1 - \beta'(f) + l(f) - z(f)$.

Before moving on, we would like to briefly comment on the above results and provide a comparison with respect to lower bounds available for the linear case $p = 2$. First, note that both P1 and P2 in Theorem 3.10 apply to variational eigenvalues of $\mathcal{H}_p$. However they are not corollaries of each other in the sense that there are settings where P1 is more informative than P2 and vice versa. Indeed, if $\lambda_k$ is a variational eigenvalue of multiplicity equal to one, then $\lambda_k > \lambda_{k-1}$ and from P1 we obtain

\[ \nu(f) \geq k - \beta'(f) + l(f) - z(f) + (c(f) - 1) \]

for any eigenfunction $f$ of $\lambda_k$, which is strictly tighter than the lower bound in P2. However, in P2, when $\lambda_k$ has multiplicity $m > 1$, we have $\lambda_k > \lambda_{k-1}$ and the two lower bounds in P1 and P2 cannot be compared a-priori. Instead, their combination leads to

\[ \nu(f) \geq \max \left\{ \left( k - \beta'(f) + l(f) - z(f) + (c(f) - 1) \right), \left( k - \beta'(f) + l(f) - z(f) + (m - 1) \right) \right\} \]

for any eigenfunction $f$ of $\lambda_k$. These observations allow us to draw new lower bounds for the eigenvalues of $\mathcal{H}_2$, which are all variational. In fact, for a simple eigenvalue $\lambda_k$ of $\mathcal{H}_2$ with an everywhere nonzero eigenfunction $f$, it was proved in [4] that $\nu(f) \geq k - \beta + l(f)$. Point P1 of Theorem 3.10 improves this result by allowing eigenfunctions with zero nodes via inequality (7). Note that this implies in particular $\nu(f) \geq k - \beta + l(f) - z(f)$, as $c > 1$ and $\beta'(f) \leq \beta$. Similarly, when $\lambda_k$ is a multiple eigenvalue of multiplicity $m$ and $f$ is any corresponding eigenfunction, it was proved in [41] for the linear case that $\nu(f) \geq k + m - 1 - \beta - z(f)$. Combining P1 and P2 allows us to improve this bound via the sharper version given in (8), which further accounts for the number of independent loops of $f$, the number of connected components of $\mathcal{G}'$ and its number of independent loops.

4. Preliminary properties of the eigenfunctions of the generalized $p$-Laplacian

In this section we present a brief review of the main results about $p$-Laplacian eigenpairs and discuss how to extend them to the generalized $p$-Laplacian case. We start with the characterization of the first and the last variational eigenvalues. Classical results available for the $p$-Laplacian equation in the continuous case [31].
have been extended to the discrete case in \[28, 40]. In the following we present analogous results for the generalized \( p \)-Laplacian operator on graphs.

4.1. The smallest variational eigenvalue. We consider in this section the first (smallest) variational eigenvalue \( \lambda_1 \) of \( \mathcal{H}_p \), defined as:

\[
\lambda_1 = \min_{f \in S_g} \mathcal{R}_{\mathcal{H}_p}(f).
\]

Since obviously \( \mathcal{R}_{\mathcal{H}_p}(f) \geq \mathcal{R}_{\mathcal{H}_p}(|f|) \) for all \( f \in \mathcal{S}_p \), we can assume that the first eigenfunction \( f_1 \) is always greater then or equal to zero. On the other hand, if \( f_1(u) = 0 \) for some \( u \in V \), then from the eigenvalue equation (2) we get

\[
\mathcal{H}_p(f_1)(u) = -\sum_{v \in V} (\omega_{uv}|f_1(v)|^{p-2}f_1(v)) = 0,
\]

which shows that \( f_1 \) assumes both positive and negative values, contradicting the previous assumption. We deduce that any eigenfunction corresponding to \( \lambda_1 \) must be everywhere strictly positive, i.e., \( f_1(u) > 0 \) \( \forall u \). This observation generalizes a well-known result for the standard \( p \)-Laplacian \( (\kappa_u = 0) \) on a graph with no boundary for which \( \lambda_1 = 0 \) and any corresponding eigenfunction is positive and has constant values \( 2 \). We formalize the characterization of the first eigenfunction of the generalized \( p \)-Laplacian in the following theorem.

**Theorem 4.1.** Let \( \lambda_1 \) be the first eigenvalue of \( \mathcal{H}_p \) on a connected graph \( G \) as in \( 9 \). Then

1. \( \lambda_1 \) is simple and the corresponding eigenfunction \( f_1 \) is strictly positive, i.e., \( f_1(u) > 0 \) \( \forall u \in V \);
2. if \( g \) is an eigenfunction associated to an eigenvalue \( \lambda \) of \( \mathcal{H}_p \) and \( g(u) > 0 \) \( \forall u \in V \), then \( \lambda = \lambda_1 \).

**Proof.** We have already observed that any eigenfunction \( f \) of \( \lambda_1 \) must be strictly positive so it remains to prove that for any strictly positive eigefunction \( g \) associated to an eigenvalue \( \lambda \), it holds \( g = f_1 \) and \( \lambda = \lambda_1 \). From the eigenvalue equation, we have

\[
\sum_{u \sim v} \omega_{uv} \phi_p(f_1(u) - f_1(v)) = (\lambda_1 g_u - \kappa_u) f_1(u)^{p-1},
\]

\[
\sum_{u \sim v} \omega_{uv} \phi_p(g(u) - g(v)) = (\lambda g_u - \kappa_u) g(u)^{p-1},
\]

where \( \phi_p \) is defined in Section 2. If we multiply both sides of (10) by the function \( f_1(u) - g(u)^p f_1(u)^{1-p} \) and both sides of (11) by \( g(u) - f_1(u)^pg(u)^{1-p} \), we obtain

\[
\sum_{u \sim v} \omega_{uv} \phi_p(f_1(u) - f_1(v))\left(f_1(u) - g(u)^p f_1(u)^{1-p}\right) = (\lambda_1 g_u - \kappa_u) \left(f_1(u)^p - g(u)^p\right),
\]

\[
\sum_{u \sim v} \omega_{uv} \phi_p(g(u) - g(v))\left(g(u) - f_1(u)^pg(u)^{1-p}\right) = (\lambda g_u - \kappa_u) \left(g(u)^p - f_1(u)^p\right).
\]

Summing the two equations first together and then over all the vertices, we obtain

\[
S(f_1, g) + S(g, f_1) = (\lambda_1 - \lambda) \sum_{u \in V} g_u \left(f_1(u)^p - g(u)^p\right)
\]
with
\[ S(f, g) = \sum_{uv \in E} \omega_{uv} \left( |g(u) - g(v)|^p - \phi_p(f(u) - f(v)) \left( \frac{g(u)^p}{f(u)^{p-1}} - \frac{g(v)^p}{f(v)^{p-1}} \right) \right). \]

If we apply Lemma A.1 to the above sums first with \( \alpha = 1 \) and, following [3], one can provide an upper bound to the magnitude of the Rayleigh quotient:

\[ 4.2. \] the largest variational eigenvalue.

to an eigenvalue different from \( \lambda \) simple. This allows us to conclude that \( f \) is the operator defined on a connected graph is simple, i.e. \( \lambda > \lambda \).

Proof. Let \( \lambda \) be an eigenfunction associated with \( \lambda \) of \( \mathcal{H}_p \). Thus, if \( \lambda = \lambda_1 \), in which case \( S(f_1, g) = S(g, f_1) = 0 \), again using Lemma A.1, we obtain

\[ \frac{g(u)}{g(v)} = \frac{f_1(u)}{f_1(v)^{1/p}}, \]

which shows that, since the graph is connected, \( g \) is proportional to \( f_1 \), implying \( \lambda_1 \) simple. This allows us to conclude that \( f_1 \) and \( g \) are the same eigenfunction. Assume now that there exists an eigenvalue \( \lambda > \lambda_1 \) with the associated eigenfunction \( g \) being strictly positive. For any \( \varepsilon > 0 \), the function \( \varepsilon g \) is also a strictly positive eigenfunction associated with \( \lambda \). Thus we can find a \( \varepsilon > 0 \) such that \( f_1(u) > \varepsilon g(u) \) for all \( u \in V \). This yields an absurd in (12) as the left hand side term is strictly positive and the right hand side is strictly negative. Thus, every eigenfunction that does not change sign has to be necessarily associated to the first eigenvalue and this concludes the proof.

The following corollary is a direct consequence of Theorem 4.1 and generalizes to \( \mathcal{H}_p \) a well-known property of the eigenfunctions of the standard \( p \)-Laplacian (see e.g. [40, Cor. 3.6])

**Corollary 4.2.** The first eigenvalue \( \lambda_1 \) of the generalized \( p \)-Laplacian operator defined on a connected graph is simple, i.e. \( \lambda_1 < \lambda_2 \), and any eigenvector associated to an eigenvalue different from \( \lambda_1 \) has at least two nodal domains.

### 4.2. The largest variational eigenvalue.

Opposite to the case of the first variational eigenvalue, the last variational eigenvalue realizes the maximum of the Rayleigh quotient:

\[ \lambda_N = \max_{f \in S_p} \mathcal{R}_{\mathcal{H}_p}(f) \]

and, following [3], one can provide an upper bound to the magnitude of \( \lambda_N \) in terms of \( \omega, g \) and the potential \( \kappa \).

**Proposition 4.3.** The largest variational eigenvalue \( \lambda_N \) of the generalized \( p \)-Laplacian operator \( \mathcal{H}_p \) defined on a connected graph satisfies:

\[ |\lambda_N| \leq \max_{u \in V} \left( 2^{p-1} \sum_{u \sim u} \frac{\omega_{uu}}{\mathcal{Q}_u} + \frac{|\kappa_u|}{\mathcal{Q}_u} \right). \]

Proof. Let \( f_N \) be an eigenfunction associated to \( \lambda_N \) and let \( u_0 \) be a node where \( g f_N \) assumes the maximal absolute value \( |\mathcal{Q}_{u_0} f_N(u_0)| = \max_{v \in V} |\mathcal{Q}_v f_N(v)| \). Then, from the eigenvalue equation, we have

\[ \mathcal{Q}_{u_0} |f_N(u_0)|^{p-1} = \sum_{u \sim u_0} \omega_{uu} \phi_p \left( f_N(u_0) - f_N(v) \right) + \kappa_{u_0} \phi_p(f_N(u_0)) \]

from which we obtain

\[ |\lambda_N| \leq \sum_{u \sim u_0} \frac{\omega_{uu}}{\mathcal{Q}_{u_0}} |f_N(u_0)|^{p-1} + \frac{|\kappa_{u_0}|}{\mathcal{Q}_{u_0}} \leq \max_{v \in V} \left( 2^{p-1} \sum_{u \sim u} \frac{\omega_{uu}}{\mathcal{Q}_u} + \frac{|\kappa_u|}{\mathcal{Q}_u} \right). \quad \square \]
As done for the first eigenfunction, we provide here a characterization of the sign pattern of the last (maximal) eigenfunction in the particular case of bipartite graphs. Our result extends to the generalized \( p\)-Laplacian the analogous results obtained in the linear case in \([8, 33]\) and in the case of the \( p\)-Laplacian with Dirichlet boundary conditions in \([25]\).

**Theorem 4.4.** If \( \mathcal{G} \) is a bipartite connected graph, then the largest eigenvalue \( \lambda_N \) of \( \mathcal{H}_p \) is simple and the corresponding unique eigenfunction \( f_N \) is such that \( f_N(u)f_N(v) < 0 \), for any \( u \sim v \).

*Proof.* We start by proving that if \( f \in \mathcal{S}_p \) is a maximizer of the Rayleigh quotient, necessarily \( f(u)f(v) < 0 \), \( \forall u \sim v \). Indeed, since \( \mathcal{G} \) is a bipartite graph we can decompose \( V \) into two subsets \( V = V_1 \sqcup V_2 \), such that if \( u, v \in V_i \), \( i = 1, 2 \), then \( u \not\sim v \). Thus, starting from \( f \), we define \( f' \) such that \( f'(u) = |f(u)|, \forall u \in V_1 \) and \( f'(u) = -|f(u)|, \forall u \in V_2 \). Now observe that

\[
\mathcal{R}_{\mathcal{H}_p}(f) = \sum_{uv \in E} \omega_{uv} |f(u) - f(v)|^p + \sum_{u \in V} \kappa_u |f(u)|^p
\]

\[
\leq \sum_{uv \in E} \omega_{uv} |f(u)| + |f(v)|^p + \sum_{u \in V} \kappa_u |f(u)|^p = \mathcal{R}_{\mathcal{H}_p}(f')
\]

where the equality holds if and only if \( f = \pm f' \). Since \( f \) is a maximal eigenfunction, then \( f = f' \) up to a sign and thus \( f(u)f(v) \leq 0 \), \( \forall u \sim v \). To conclude, if \( f'(u) = 0 \) then, for \( u \in V_1 \) we have \( \lambda_n f'(u) = \mathcal{H}_p(f')(u) \leq 0 \) and the equality holds only if \( f'(v) = 0 \) for every \( v \sim u \). Since the graph is connected this would lead to the absurd \( f' \equiv 0 \). Thus, we have that \( f'(u) \neq 0 \), \( \forall u \), implying \( f(u)f(v) < 0 \), \( \forall u \sim v \).

We now prove uniqueness of the minimizer. Given two maximizers \( f, g \in \mathcal{S}_p \) such that

\[
\mathcal{R}_{\mathcal{H}_p}(f) = \lambda_n = \mathcal{R}_{\mathcal{H}_p}(g),
\]

up to a sign as above, \( f \) and \( g \) must be strictly greater than zero on \( V_1 \) and strictly smaller than zero on \( V_2 \). Then, similarly to the proof of Theorem 4.1 we first multiply the eigenvalue equations for \( f \) and \( g \) by \( f(u) - |g(u)|^p/\phi_p(f(u)) \) and \( g(u) - |f(u)|^p/\phi_p(g(u)) \), respectively. Then, we sum the two equations together and over all the nodes to obtain:

\[
\sum_{uv \in E} \omega_{uv} \left( |g(u) - g(v)|^p - \phi_p((f(u) - f(v)) \left( |g(u)|^p/\phi_p(f(u)) - |g(v)|^p/\phi_p(f(v)) \right) + \right.
\]

\[
\sum_{uv \in E} \omega_{uv} \left( |f(u) - f(v)|^p - \phi_p(g(u) - g(v)) \left( |f(u)|^p/\phi_p(g(u)) - |f(v)|^p/\phi_p(g(v)) \right) \right) = 0
\]

From Lemma A.1 both the sums above are smaller than zero unless \( f = g \), thus showing uniqueness of the maximizer and hence of the maximal eigenfunction \( f_N \).

\( \square \)

**Corollary 4.5.** Consider a graph \( \mathcal{G} \) and the generalized \( p\)-Laplacian operator \( \mathcal{H}_p \). Then, the graph \( \mathcal{G} \) is bipartite and connected if and only if the maximal eigenfunction \( f_N \) of \( \mathcal{H}_p \) induces exactly \( N \) nodal domains.

*Proof.* If the graph is bipartite, by Theorem 4.4 the \( N \)-th variational eigenfunction is unique and induces \( N \) nodal domains. Vice-versa, let \( f_N \) be an eigenfunction such that \( f_N \) induces exactly \( N \) nodal domains. Then, considering \( V_1 = \{ v | f_N(v) > 0 \} \)
and \( V_2 = \{ v | f_N(v) < 0 \} \), we have \( V = V_1 \sqcup V_2 \) and each node in \( V_1 \) is connected only to nodes in \( V_2 \), showing that the graph is bipartite.

4.3. Further properties of \( \mathcal{H}_p \) and its eigenfunctions. Observe that, similarly to the linear Schrödinger operator and unlike the \( p \)-Laplacian case, the eigenvalues of the generalized \( p \)-Laplacian depend on the potential \( \kappa_u \) and may attain both positive and negative values. This follows directly from the eigenvalue equation (2) for \((\lambda_1, f_1)\):

\[
\sum_{u \sim v} \left( \omega_{uv} |f_1(u) - f_1(v)|^{p-2} (f_1(u) - f_1(v)) \right) + \kappa_u f_1(u)^{p-1} = \lambda_1 \varrho_u f_1(u)^{p-1}.
\]

In fact, summing over all the vertices \( u \in V \) yields

\[
\lambda_1 = \frac{\sum_{u \in V} \kappa_u f_1(u)^{p-1}}{\sum_{u \in V} \varrho_u f_1(u)^{p-1}} = \frac{\sum_{u \in V} \kappa_u \varrho_u f_1(u)^{p-1}}{\sum_{u \in V} \varrho_u f_1(u)^{p-1}}
\]

which shows that \( \lambda_1 \) is in the convex hull of the coefficients \( \{ \frac{\kappa_u}{\varrho_u} \} \) and, since \( \kappa_u \) may be negative, \( \mathcal{H}_p \) may not be positive definite.

The next lemmas extend to the generalized \( p \)-Laplacian the results proved in \([40]\) for the standard \( p \)-Laplacian, and provide partial orderings for the given eigenpairs. In particular, Lemma 4.6 below follows directly by replacing the standard \( p \)-Laplacian with the generalized operator \( \mathcal{H}_p \) in the proof of \([40, \text{Lemma 3.8}] \) and, for this reason, its proof is omitted.

**Lemma 4.6.** If \( f \) is an eigenfunction relative to an eigenvalue \( \lambda \) and \( A_1, \ldots, A_m \) are the nodal domains of \( f \), consider \( f|_{A_i} \) the function that is equal to \( f \) on \( A_i \) and zero on \( V \setminus A_i \). Then

\[
\max \left\{ \mathcal{R}_{\mathcal{H}_p}(f) : f \in \text{span}\{f|_{A_1}, \ldots, f|_{A_m} \} \right\} \leq \lambda.
\]

**Corollary 4.7.** If \( f \) is an eigenfunction relative to an eigenvalue \( \lambda \) and \( f \) induces \( k \) nodal domains, then \( \lambda \geq \lambda_k \).

**Proof.** If \( A_1, \ldots, A_k \) are the nodal domains of \( f \), consider \( f|_{A_i} \) the function that is equal to \( f \) on \( A_i \) and zero on \( V \setminus A_i \). If \( A = \text{span}\{f|_{A_1}, \ldots, f|_{A_k} \} \), then notice that the Krasnoselskii genus of \( A \) is \( k \), i.e., \( \gamma(A) = k \). Thus, from Lemma 4.6 we have that \( \lambda_k = \min_{A \in \mathcal{F}_k} \max_{f \in A} \mathcal{R}_{\mathcal{H}_p} \leq \max_{f \in A} \mathcal{R}_{\mathcal{H}_p}(f) \leq \lambda \).

We conclude by noticing that, combining Corollaries 4.2 and 4.7, one immediately obtains that, as for the standard \( p \)-Laplacian, the second variational eigenvalue \( \lambda_2 \) of the generalized \( p \)-Laplacian is the smallest eigenvalue larger than \( \lambda_1 \). Precisely, it holds:

\[
\lambda_2 = \min \{ \lambda : \lambda > \lambda_1 \text{ is an eigenvalue of } \mathcal{H}_p \}
\]

5. Graph perturbations and Weyl’s-like inequalities

In this section we show how to modify the graph and, consequently, the associated generalized \( p \)-Laplacian operator, maintaining eigenpairs. In particular, we will show how to remove edges and nodes obtaining a new generalized \( p \)-Laplacian operator on a simpler graph written as a “small” perturbation of the initial operator \( \mathcal{H}_p \). For this perturbed operator, we will prove Weyl’s like inequalities relating its variational eigenvalues to those of the starting operator.
5.1. Removing an edge. Consider a graph $\mathcal{G}$ and the generalized $p$-Laplacian operator $\mathcal{H}_p$ on $\mathcal{G}$. Let $\lambda$ and $f$ be an eigenvalue and a corresponding eigenfunction of $\mathcal{H}_p$ and let $e_0 = (u_0, v_0)$ be an edge of the graph such that $f(u_0) f(v_0) \neq 0$. We want to define a new generalized $p$-Laplacian operator $\mathcal{H}_p'$ on the graph $\mathcal{G}' := \mathcal{G} \setminus e_0$, such that $(f, \lambda)$ is also an eigenpair of $\mathcal{H}_p'$.

Our strategy extends to the nonlinear case the work of [5], where the new operator $\mathcal{H}_p'$ is written as a rank-one variation of the starting Laplacian. To this end, we write $\mathcal{H}_p' = \mathcal{H}_p + \Xi_p$ where

$$\Xi_p(g)(u) = \begin{cases} 0 & \text{if } u \neq u_0, v_0 \\ \omega_{u_0v_0} \phi_p(1 - \alpha) \phi_p(g(u_0)) - \phi_p(g(u_0) - g(v_0)) & \text{if } u = u_0 \\ \phi_p(1 - \frac{1}{\alpha}) \phi_p(g(v_0)) - \phi_p(g(v_0) - g(u_0)) & \text{if } u = v_0 \end{cases}$$

where, for $g \in S_p$,

$$\mathcal{R}_{\mathcal{H}_p'}(g) = \mathcal{R}_{\mathcal{H}_p}(g) + \mathcal{R}_{\Xi_p}(g),$$

where, for $g \in S_p$:

$$\mathcal{R}_{\Xi_p}(g) = \frac{1}{\omega_{u_0v_0}} \left( \frac{|g(u_0)|^p}{\phi_p(f(u_0))} - \frac{|g(v_0)|^p}{\phi_p(f(v_0))} \right) \phi_p(f(u_0) - f(v_0))$$

$$- \left( g(u_0) - g(v_0) \right) \phi_p \left( g(u_0) - g(v_0) \right).$$

A direct application of Lemma A.1 shows that $R_{\Xi_p}$ is positive if $\frac{f(u_0)}{f(v_0)}$ is negative and negative if $\frac{f(v_0)}{f(u_0)}$ is positive. Moreover, if we assume that $g(v_0)$ and $g(u_0)$ are non-zero, we can write $\Xi_p g$ in the following equivalent way

$$\Xi_p(g)(u) = \begin{cases} 0 & \text{if } u \neq u_0, v_0 \\ \omega_{u_0v_0} \phi_p(g(u_0)) \left( \phi_p(1 - \alpha) - \phi_p(1 - \frac{g(u_0)}{g(v_0)}) \right) & \text{if } u = u_0 \\ \phi_p(1 - \frac{1}{\alpha}) \phi_p(g(v_0)) \left( \phi_p(1 - \frac{1}{\alpha}) - \phi_p(1 - \frac{g(u_0)}{g(v_0)}) \right) & \text{if } u = v_0 \end{cases}.$$

From this last equation we can easily see that, if $g(v_0) = \alpha g(u_0)$, then $\Xi_p g = \mathcal{R}_{\Xi_p}(g) = 0$.

To continue, we need the following lemma from [37], reported here without proof, which provides a bound on the Krasnoselskii genus of the intersection of different subsets.

**Lemma 5.1.** [37, Prop. 4.4] Let $X$ be a Banach space and $A$ the class of the closed symmetric subsets of $X$. Given $A \in A$, consider a Krasnoselskii test map $\varphi \in \Lambda_k(A)$ with $k < \gamma(A)$. Then, $\gamma(\varphi^{-1}(0)) \geq \gamma(A) - k$. 
Using the fact that \( R_{\Xi} \) is zero on the hyperplane \( \pi = \{ g : g(u_0)f(v_0) - g(v_0)f(u_0) = 0 \} \), we obtain the following ordering of the \( k \)-th eigenvalue of \( H_p \) within the spectrum of \( H'_p \).

**Lemma 5.2.** Assume that there exist an eigenfunction \( f \) of \( H_p \) and an edge \( e_0 = (u_0, v_0) \) such that \( f(u_0), f(v_0) \neq 0 \) and consider the operator \( H'_p = H_p + \Xi_p \), where \( \Xi_p \) is defined as in \( \Xi \). Let \( \eta_k \) be the variational eigenvalues of \( H'_p \) and \( \lambda_k \) those of \( H_p \). The following inequalities hold:

- If \( \frac{f(v_0)}{f(u_0)} < 0 \), then \( \eta_{k-1} \leq \lambda_k \leq \eta_k \);
- If \( \frac{f(v_0)}{f(u_0)} > 0 \), then \( \lambda_k \leq \lambda_{k+1} \).

**Proof.** Let \( F_\pi \) be the Krasnoselskii family \( F_\pi = \{ A \subseteq A \cap S_p | \gamma(A) \geq k \} \) as defined in Section 5. Let \( A_k \in F_\pi \) be such that \( \lambda_k = \max_{f \in A_k} R_{H_p}(f) \), and let

\[
\pi = \{ g : g(u_0)f(v_0) - g(v_0)f(u_0) = 0 \}.
\]

Then \( A_k \cap \pi = \phi|_{A_k}(0) \), and from Lemma 5.1, since \( \phi|_{A_k} \in \Lambda_1(A_k) \), we have

\[
\gamma(A_k \cap \pi) \geq \gamma(A_k) - 1 \geq k - 1.
\]

Thus, \( A_k \cap \pi \in F_{k-1} \) and

\[
\eta_{k-1} = \min_{A_k \in F_{k-1}} \max_{f \in A_k} R_{H'_p}(f) = \max_{f \in A_k \cap \pi} R_{H_p}(f) \leq \max_{f \in A_k \cap \pi} R_{H_p} + R_{\Xi_p} \leq \lambda_k.
\]

This implies that \( \eta_{k-1} \leq \lambda_k \). Moreover, since \( R_{\Xi_p} \geq 0 \) we have that \( R_{H'_p}(f) \geq R_{H_p} \), which implies

\[
\lambda_k = \min_{A_k \in F_{k-1}} \max_{f \in A_k} R_{H'_p}(f) = \min_{A_k \in F_{k-1}} \max_{f \in A_k} R_{H_p}(f) = \eta_k,
\]

and this concludes the proof of the first inequality. The second inequality can be proved analogously, by exchanging the roles of \( H'_p \) and \( H_p \).

\( \square \)

5.2. **Removing a node.** Consider a generalized \( p \)-Laplacian operator \( H_p \) defined on a graph \( G \) and let \( u_0 \) be a node of \( G \). We want to define a new operator \( H'_p \) on the graph \( G' := G \setminus \{u_0\} \) that behaves on \( G' \) like \( H_p \) behaves on the hyperplane \( \{ f : f(u_0) = 0 \} \). Note that, in the linear case, this operation is equivalent to considering the principal submatrix of the generalized Laplacian matrix obtained by removing the row and the column relative to \( u_0 \). If we remove a node \( u_0 \), we have to remove also all its incident edges from the graph \( G \). Thus, on the graph \( G' \) we can define the generalized \( p \)-Laplacian:

\[
H'_p(f)(u) := \sum_{v \in V'} \omega_{uv} \phi_p(f(u) - f(v)) + \kappa'_u \phi_p(f(u)),
\]

where \( V' = V \setminus \{u_0\} \), \( \omega_{uv} = \omega_{vu} \) and \( \kappa'_u = \kappa_u + \omega_{ uu_0 } \).

**Remark 5.3.** If \( f \) is an eigenfunction of the generalized \( p \)-Laplacian \( H_p \) on \( G \), with eigenvalue \( \lambda \) and such that \( f(u_0) = 0 \), then the restriction \( f' \) of \( f \) on the graph \( G' = G \setminus \{u_0\} \) is automatically an eigenfunction of \( H'_p \) with eigenvalue \( \lambda \). Indeed, for each \( u \neq u_0 \) we have

\[
\lambda \varphi_u \phi_p(f(u)) = \sum_{v \neq u_0} \omega_{uv} \phi_p(f(u) - f(v)) + \omega_{ uu_0 } \phi_p(f(u)) + \kappa_u \phi_p(f(u)) = H'_p(f)(u).
\]

Next, we provide an ordering for the variational eigenvalues of \( H'_p \), in comparison with those of \( H_p \), as stated in the following lemma.
Proof. Let $S'_p = \{ f : V' \to \mathbb{R} : \|f\|_p = 1 \}$ and consider $A'_k \in F_k(S'_p)$ such that

$$\eta_k = \max_{f \in A'_k} \max_{v \in V'} w'(uv)|f(u) - f(v)|^p + \sum_{u \in V'} \kappa'_u |f(u)|^p,$$

where $E'$ is the set of edges of $G'$. Consider now $A_k$, the immersion of $A'_k$ in the $N - 1$ dimensional hyperplane $\pi = \{ f : V \to \mathbb{R} : f(u_0) = 0 \}$, i.e. the set of functions $f$ that, when restricted to the nodes different from $u_0$, belong to $A'_k$ and are such that $f(u_0) = 0$. Thus, $A_k$ belongs to $F_k(S_p)$ since $A_k$ and $A'_k$ are homeomorphic, and we obtain:

$$\lambda_k = \min_{A_k \in F_k} \max_{f \in A_k} \mathcal{R}_{H_p}(f) \leq \max_{f \in A_k} \mathcal{R}_{H_p}(f) = \max_{f \in A_k} \mathcal{R}_{H'_p}(f) = \eta_k.$$

To prove the other inequality, consider $A_{k+1} \in F_{k+1}(S_p)$ such that

$$\lambda_{k+1} = \max_{f \in A_{k+1}} \mathcal{R}_{H_p}(f).$$

Because of Lemma 5.1, we have that $\gamma(A_{k+1} \cap \{ f : f(u_0) = 0 \}) \geq k$, which implies that $A_{k+1} \cap \{ f : f(u_0) = 0 \} \in F_k(S_p)$. Thus

$$\eta_k \leq \max_{f \in A'_k} \mathcal{R}_{H'_p}(f) = \max_{f \in A_{k+1} \cap \pi} \mathcal{R}_{H_p}(f) \leq \max_{f \in A_{k+1}} \mathcal{R}_{H_p}(f) = \lambda_{k+1},$$

where $A'_k$ is the set of functions $f' : V' \to \mathbb{R}$ obtained as the restriction of functions from $A_{k+1} \cap \pi$ to $G'$, i.e., $f' \in A'_k$ if the lifting $f : V \to \mathbb{R}$ defined as $f(u_0) = 0$ and $f(u) = f'(u)$, $\forall u \neq u_0$, belongs to $A_{k+1} \cap \pi$. \qed

This result can be generalized by induction to the case of $n$ removed nodes, obtaining the main theorem of this section.

Theorem 5.5. Let $H_p$ be the generalized $p$-Laplacian operator defined on the graph $G$ and let $G'$ be the graph obtained from $G$ by deleting the $n$ nodes $u_1, u_2, \ldots, u_n$. Consider the generalized $p$-Laplacian operator on $G'$ defined as

$$H'_p(u) := \sum_{v \in V'} \omega_{uv} \phi_p(f(u) - f(v)) + \kappa'_u \phi_p(f(u)),$$

where $V' = V \setminus \{ u_1, \ldots, u_n \}$, $\omega_{uv} = \omega_{uv}$ and $\kappa'_u = \kappa_u + \sum_{i=1}^n \omega_{uu_i}$. Let $\{ \lambda_k \}$ denote the variational eigenvalues of $H_p$ and $\{ \eta_k \}$ those of $H'_p$. Then

$$\lambda_k \leq \eta_k \leq \lambda_{k+n},$$

for any $k \in \{1, \ldots, |V| - n\}$.

Proof. The proof follows directly from Lemma 5.4, removing recursively the nodes $u_1, \ldots, u_n$. \qed
6. Nodal domain count on trees (proofs of Theorems 3.7 and 3.8)

In this section we deal with the case in which $T := G = (V,E)$ is a tree and we provide proofs of the two Theorems 3.7 and 3.8. In particular, we will prove that the eigenvalues of the generalized $p$-Laplacian on a tree are all and only the variational ones. Moreover, again restricting ourselves to trees, we will show that, if an eigenfunction of the $k$-th variational eigenvalue is everywhere non zero, then it induces exactly $k$ nodal domains. This generalizes to the nonlinear case a well-known result for the linear Schrödinger operator.

In the following, given a tree $T = (V,E)$, we assume a root $r \in V$ is chosen arbitrarily. This provides a partial ordering of the nodes so that a precise root is automatically assigned to any subtree of $T$. In particular, we write $v < u$ if $v$ is a descendant of $u$ and $v \prec u$ if $v$ is a direct child of $u$. Moreover, for each node $u \in V$, we let $T_u$ denote the subtree of $T$ having $u$ as root and formed by all the descendants of $u$. On this subtree we can define a new operator $H_u$ obtained as follows: starting from $T$, we remove all the nodes that do not belong to $T_u$ and, for each deleted node, we modify the original operator $H_p$ on $T$ as in Section 5.2.

We also consider the operator $H^u_p$, obtained by removing from $T_u$ also the root node $u$ and by modifying $H_p$ accordingly. This latter operator is defined on a subforest, $T_u = \sqcup_i T_i$, that has as many connected components as the number of children of $u$. From the generalized Weyl’s inequalities of Section 5 we have that

\[ \cdots \leq \lambda_i(H^u_p) \leq \lambda_i(H^\bar{u}_p) \leq \lambda_{i+1}(H^u_p) \leq \cdots \]

where $\lambda_i(H^u_p)$ and $\lambda_i(H^\bar{u}_p)$ denote the $i$-th variational eigenvalue of $H^u_p$ and $H^\bar{u}_p$, respectively. Observe also that $H_p = \bigoplus_{v \prec u} H_p(T_i)$, where $H_p(T_i)$ is the generalized $p$-Laplacian of $T_i$ and $v \prec u$ indicates that $v$ is a direct child of $u$.

6.1. Generating functions. Consider now an eigenfunction $f$ of $H_p$ with eigenvalue $\lambda$ and assume that $f \neq 0$ everywhere. For each $u$ different from the root $r$, we denote by $u_F$ the parent of $u$ in $T$. Then, the following quantity

\[ g(u) := \frac{f(u_F)}{f(u)} \]

is well defined for all $u \neq r$ and we can rewrite the eigenvalue equation $H_p(f)(u) = \lambda g(u) \phi_p(f(u))$ as

\[ \omega_{uu_F} \phi_p(1 - g(u)) = \lambda \phi_u - \kappa_u - \sum_{v \prec u} \omega_{uv} \phi_p \left(1 - \frac{1}{g(v)}\right), \]

for each $u \neq r$.

Now, if $u$ is a leaf, Equation (18) allows us to write $g(u)$ explicitly as a function of $\lambda$:

\[ g_u(\lambda) = 1 + \phi^{-1}_p \left( \frac{\kappa_u - \phi_u \lambda}{\omega_{uu_F}} \right). \]

Similarly, for a generic node $u$ different from the root, we can use (18) to characterize $g(u)$ implicitly as a function of the variable $\lambda$:

\[ g_u(\lambda) = 1 + \phi^{-1}_p \left( \frac{\kappa_u - \phi_u \lambda + \sum_{v \prec u} \omega_{uv} \phi_p \left(1 - \frac{1}{g(v)}\right)}{\omega_{uu_F}} \right). \]
Finally, for the root $u = r$, we define

\begin{equation}
 g_r(\lambda) := 1 + \phi_p^{-1}(\kappa_r - 1 - \lambda \phi_r + \sum_{v < r} \omega_{rv} \phi_p(1 - \frac{1}{g_v(\lambda)})).
\end{equation}

We call the functions $g_u$ defined in (19), (20), (21) the generating functions of the eigenfunction $f$. In fact, we will show in Section 6.2 that $g_u(\lambda)$ characterizes the ratio $f(u_F)/f(u)$ for any eigenfunction of $\lambda$ such that $f(u) \neq 0$. To this end, we need a number of preliminary results to unveil several properties of the generating functions $g_u$.

First, observe that when $f \neq 0$ everywhere, the claimed characterizing property follows directly from the definition of $g_u$. We highlight this statement in the following remark.

**Remark 6.1.** If $\lambda$ is an eigenvalue of $\mathcal{H}_p^{u_0}$ for some $u_0 \in V$, and $f$ is an associated eigenfunction such that $f(u) \neq 0$, $\forall u \in \mathcal{T}_{u_0}$, then by the definition of the functions $g_u(\lambda)$ one directly obtains that

\[ \frac{f(u_F)}{f(u)} = g_u(\lambda) \neq 0, \quad \forall u \in \mathcal{T} \setminus \{u_0\} \quad \text{and} \quad g_{u_0}(\lambda) = 0. \]

On the other hand, it is not difficult to observe that also the opposite property holds, namely

**Remark 6.2.** Assume that $\lambda$ is a zero of $g_{u_0}(\lambda)$ and $g_u(\lambda) \neq 0$, for all $u < u_0$, i.e., for all the descendents of $u_0$ and not only the direct children. Then $\lambda$ is an eigenvalue of $\mathcal{H}_p^{u_0}$ and a corresponding eigenfunction $f$ can be defined on the subtree $\mathcal{T}_{u_0}$ by setting $f(u_0) = 1$ and $f(u) = \frac{f(u_F)}{g_u(\lambda)}$, for all $u < u_0$. Indeed, with these definitions, (19) and (20) imply that $\lambda$ and $f$ are solutions of the system of equations

\[
\begin{align*}
\sum_{v < u_0} \omega_{uv} \phi_p(f(u_0) - f(v)) + \omega_{u_0 u} \phi_p(f(u_0)) + \kappa_u \phi_p(f(u_0)) &= \lambda \phi_u \phi_p(f(u_0)), \\
\sum_{v \in \mathcal{V}} \omega_{uv} \phi_p(f(u) - f(v)) + \kappa_u \phi_p(f(u)) &= \lambda \phi_u \phi_p(f(u)) \quad \forall u < u_0,
\end{align*}
\]

which shows that $\lambda$ and $f$ are an eigenvalue and an eigenfunction of $\mathcal{H}_p^{u_0}$.

We have observed already that it is possible to relate the eigenpairs of the subtrees of $\mathcal{T}$ with the values of the functions $g_u(\lambda)$. Then, we show that it is always possible to immerse the tree $\mathcal{T}$ in a larger tree for which the values of the functions $g_u(\lambda)$ do not change.

**Remark 6.3.** Let $\mathcal{H}_p$ be the generalized $p$-Laplacian operator defined on a tree $\mathcal{T} = (V, E)$. We can always immerse $\mathcal{T}$ in a tree $\mathcal{T}$ obtained adding a parent $r_F$ to the root $r$. Next, we define the generalized $p$-Laplacian operator $\mathcal{H}_p$ on $\mathcal{T}$ by setting $\widetilde{\omega}_{uv} = \omega_{uv}$, $\forall (u, v) \in E$, $\widetilde{\omega}_{rr_F} = 1$, $\widetilde{\kappa}_u = \kappa_u$, $\forall u \in V \setminus \{r\}$, and $\widetilde{\kappa}_r = \kappa_r - 1$. Considering $\mathcal{T}_{r_F}$ and $\mathcal{H}_p^{r_F}$, the subtree and the operator obtained removing the root $r_F$ from $\mathcal{T}$, it is straightforward to observe that $\mathcal{T} = \mathcal{T}_{r_F}$ and $\mathcal{H}_p = \mathcal{H}_p^{r_F}$. Moreover, working on $\mathcal{T}$ and the associated operator $\mathcal{H}_p$, it is possible to introduce the functions $\widetilde{g}_u(\lambda)$ as in (19), (20), (21).

\[ g_u(\lambda) = \widetilde{g}_u(\lambda), \quad \forall u \in \mathcal{T}. \]
Thus, the generalized \( p \)-Laplacian eigenvalue problem on a tree can always be studied as the generalized \( p \)-Laplacian eigenvalue problem on a subtree of a suitable larger tree.

Finally, the following lemma summarizes several relevant structural properties of the functions \( g_u(\lambda) \).

**Lemma 6.4.** For each \( u \in V \), consider the function \( g_u(\lambda) \) defined as in \((19)-(21)\). Then:

1. the poles of \( g_u(\lambda) \) are the zeros of the functions \( \{g_v(\lambda)\}_{v \prec u} \);
2. \( g_u \) is strictly decreasing between each two consecutive poles;
3. \( \lim_{\lambda \to -\infty} g_u = +\infty, \lim_{\lambda \to +\infty} g_u = -\infty, \lim_{\lambda \to p^-} g_u = -\infty, \lim_{\lambda \to p^+} g_u = +\infty \) where \( p \) is any of the poles.

**Proof.** Let \( u \in V \), if \( u \) is a leaf then the three properties follow immediately from \((19)\). Otherwise, assume by induction the thesis holds for each \( v \prec u \). From \((20)\), it immediately follows that the poles of \( g_u \) are the zeros of \( \{g_v(\lambda)\}_{v \prec u} \). To show that the function \( \sum_{v \prec u} \omega_{uv} \phi_p \left( 1 - \frac{1}{g_v(\lambda)} \right) \) is strictly decreasing between any couple of neighboring poles, observe that \( x \mapsto \phi_p(x) \) is strictly increasing and, by induction, \( \forall v \prec u, \lambda \mapsto g_v(\lambda) \) is strictly decreasing between any two of its zeros (i.e. the poles of \( g_u \)). Moreover, since \( \lambda \mapsto -g_u(\lambda) \) is decreasing and \( \phi_p^{-1} \) increasing, we can conclude that the function \( \lambda \mapsto g_u(\lambda) \) is strictly decreasing between any two of its poles. Finally, the limits of \( g_u(\lambda) \) for \( \lambda \to \pm \infty \) in the third statement follow as a consequence of the previous observations, while the limits for \( \lambda \to \pm \infty \) can be proved directly by the induction assumption. \( \square \)

### 6.2. Eigenfunction characterization via generating functions.

The following result shows that the generating functions \( g_u(\lambda) \) always characterize the eigenfunctions of \( \lambda \), generalizing what observed earlier in Remark \( 6.1 \).

**Theorem 6.5.** Let \((f, \lambda)\) be an eigenpair of a generalized \( p \)-Laplacian operator, \( \mathcal{H}_p \), defined on a tree \( T = (V, E) \) with root \( r \). For any node \( u \in V \setminus \{r\} \) such that \( f(u) \neq 0 \), it holds

\[
\frac{f(u_F)}{f(u)} = g_u(\lambda),
\]

where \( u_F \) is the parent of \( u \) in \( T \).

**Proof.** If \( f(v) \neq 0 \) \( \forall v \in T \) we have already observed in Remark \( 6.4 \) that the thesis holds. Assume thus that there exist \( v_1, \ldots, v_k \in V \) such that

\[
f(v_i) = 0, \quad i = 1, \ldots, k \quad \text{and} \quad f(u) \neq 0, \quad \forall u \not\in \{v_i\}_{i=1}^k.
\]

and let \( T' = \bigsqcup_{i=1}^h T_i \) and \( \mathcal{H}'_p = \bigoplus_{i=1}^h \mathcal{H}_p(T_i) \) be the forest and the corresponding operator obtained from \( G \) removing the nodes \( v_1, \ldots, v_k \) as in Section \( 5.2 \). From Remark \( 5.3 \) \( \forall i = 1, \ldots, h \), the pair \((f|_{T_i}, \lambda)\) is an eigenpair of \( \mathcal{H}_p(T_i) \) such that \( f|_{T_i}(u) \neq 0, \forall u \in T_i \). Denoting with \( r_i \) the root of \( T_i \) and using \((19)-(21)\) and Remark \( 6.1 \) \( \forall T_i \), starting from the leaves, we can define functions \( g^T_i(u) \) such that

\[
\begin{cases}
g^T_u(\lambda) = \frac{f|_{T_i}(u_F)}{f|_{T_i}(u)} \neq 0 & \forall u \in T_i \setminus \{r_i\} \\
g^T_{r_i}(\lambda) = 0
\end{cases}
\]

\((22)\)
We claim that $\forall i = 1, \ldots, h$ and $\forall u \in \mathcal{T}_i$, then $g_u(\lambda) = g_u^{\mathcal{T}_i}(\lambda)$. The thesis follows directly from this claim since

$$g_u(\lambda) = g_u^{\mathcal{T}_i}(\lambda) = \frac{f(u_F)}{f(u)} \quad \forall u \in \mathcal{T}_i.$$ 

To prove the claim, first we introduce a partial ordering on $\{\mathcal{T}_i\}_{i=1}^{h}$ and $\{v_j\}_{j=1}^{k}$ so that $\mathcal{T}_i \prec v_j$ if $v_j$ is the parent of the root of $\mathcal{T}_i$, while $v_j \prec \mathcal{T}_i$ if $v_j$ is a child of some node of $\mathcal{T}_i$. Then, if $v_j \prec \mathcal{T}_i$ there exists a subtree $\mathcal{T}_i \prec v_j$. In fact, considering the generalized $p$-Laplacian eigenvalue equation in $v_j$ with $u_i = v_j F \in \mathcal{T}_i$, we can write

$$\omega_{v_j u_i} \phi_p(f(u_i)) + \sum_{u \prec v_j} \omega_{v_j u} \phi_p(f(u)) = 0.$$ 

Since $f(u_i) \neq 0$, there exists a node $u_i \prec v_j$ such that $f(u_i) \neq 0$ i.e. $u_i \in \mathcal{T}_i \prec v_j$. Similarly, one observes that if $f(v_j) = 0$, and $v_j$ is a leaf, then also $f(v_j F) = 0$. Because of these two facts, there exists some $\mathcal{T}_{i_0}$ in the set $\{\mathcal{T}_i\}_{i=1}^{h}$ such that a node $v_j$ with $v_j \prec \mathcal{T}_{i_0}$ cannot exists. In addition, the leaves of $\mathcal{T}_{i_0}$ are all and only the leaves of $\mathcal{T}$ that are connected to $\mathcal{T}_{i_0}$. It is then easy to observe that, for any such $\mathcal{T}_{i_0}$, by definition, $g_u^{\mathcal{T}_{i_0}}(\lambda) = g_u(\lambda) \forall u \in \mathcal{T}_{i_0}$, $u \neq r_{i_0}$. Moreover, when $u = r_{i_0} \prec v_j$ we have

$$g_{r_{i_0}}(\lambda) : = 1 + \phi_p^{-1}\left(\kappa_{r_{i_0}} - \lambda \theta_{r_{i_0}} + \sum_{v \prec r_{i_0}} \omega_{r_{i_0} v} \phi_p\left(1 - \frac{1}{g_v(\lambda)}\right)\right) = 0,$$

which implies

$$\kappa_{r_{i_0}} + \omega_{r_{i_0} v_j} - \lambda \theta_{r_{i_0}} + \sum_{v \prec r_{i_0}} \omega_{r_{i_0} v} \phi_p\left(1 - \frac{1}{g_v(\lambda)}\right) = 0,$$

where, since $v_j$ is one of the removed nodes, we have used the expression $\kappa_{r_{i_0}} = \kappa_{r_{i_0}} + \omega_{r_{i_0} v_j}$ that we obtain when moving from $\mathcal{H}_p$ to $\mathcal{H}_p$ as in Section 5.2.

Thus, (24) implies

$$g_{r_{i_0}}(\lambda) = 1 + \phi_p^{-1}\left(\kappa_{r_{i_0}} - \theta_{r_{i_0}} \lambda + \sum_{v \prec r_{i_0}} \omega_{r_{i_0} v} \phi_p\left(1 - \frac{1}{g_v(\lambda)}\right)\right)$$

$$= 1 + \phi_p^{-1}\left(\omega_{r_{i_0} v_j} - \omega_{r_{i_0} v_j} \phi_p\left(1 - \frac{1}{g_v(\lambda)}\right)\right) = 0,$$

that is $g_{r_{i_0}}(\lambda) = g_{r_{i_0}}(\lambda) = 0$ and $\lambda$ is a pole of $g_{v_j}$, due to Lemma 6.4.

Now, given a general subtree $\mathcal{T}_{i_0}$, w.l.o.g. we can assume that the claim is true for any $\mathcal{T}_i \prec v_j \prec \mathcal{T}_{i_0}$. Then if $u$ is a leaf of $\mathcal{T}_{i_0}$ that is also a leaf of $\mathcal{T}$, clearly

$$g_u^{\mathcal{T}_{i_0}}(\lambda) = g_u(\lambda).$$

Consider now the case of a leaf, $u$, of $\mathcal{T}_{i_0}$ that is not a leaf of $\mathcal{T}$. Since $u$ is not a leaf of $\mathcal{T}$, by construction, there exist some node $v_j \prec u$ and some subtree $\mathcal{T}_i \prec v \prec \mathcal{T}_{i_0}$. For any such $v_j$, by the inductive assumption, $\lambda$ has to be a pole of
the corresponding \( g_{\nu_j} \), leading to the following equation:

\[
g_u(\lambda) = 1 + \phi_p^{-1} \left( \frac{\kappa_u - \varrho_u \lambda + \sum_{\nu_j \prec u} \omega_{uv} \varphi_p \left( 1 - \frac{1}{g_{\nu_j}(\lambda)} \right)}{\omega_{uu}} \right) \tag{26}
\]

\[
= 1 + \phi_p^{-1} \left( \frac{\kappa_u - \varrho_u \lambda + \sum_{\nu_j \prec u} \omega_{uv} \varphi_p(1)}{\omega_{uu}} \right)
\]

\[
= 1 + \phi_p^{-1} \left( \frac{\kappa_u' - \varrho_u \lambda}{\omega_{uu}} \right) = g^T_u(\lambda).
\]

Here we have used as before the fact \( \kappa'_u = \kappa_u + \sum_{\nu_j \prec u} \omega_{uv} \), see Section 5.2.

The case of \( u \) a generic node of \( T_u \) can be proved analogously assuming, w.l.o.g., the claim true for any \( w \prec u, w \in T_u \). Indeed, recalling \( \kappa'_u = \kappa_u + \sum_{\nu_j \prec u} \omega_{uv} \) and that, by the inductive assumption, \( \lambda \) is a pole of \( g_{\nu_j} \) for any \( \nu_j \prec u \), we get

\[
g_u(\lambda) = 1 + \phi_p^{-1} \left( \omega_{uu}^{-1} \left( \kappa_u - \varrho_u \lambda + \sum_{\nu_j \prec u} \omega_{uv} + \sum_{w \prec u} \omega_{uw} \varphi_p \left( 1 - \frac{1}{g_{\nu_j}(\lambda)} \right) \right) \right)
\]

\[
= 1 + \phi_p^{-1} \left( \omega_{uu}^{-1} \left( \kappa_u' - \varrho_u \lambda + \sum_{w \prec u} \omega_{uw} \varphi_p \left( 1 - \frac{1}{g_{\nu_j}(\lambda)} \right) \right) \right) = g^T_u(\lambda)
\]

The case of \( u = r_{\nu_0} \) can be finally dealt with as done in (28), concluding the proof.

\[\square\]

Corollary 6.6. Let \( (f, \lambda) \) be an eigenpair of \( \mathcal{H}_p \), then, if \( g_u(\lambda) = 0 \), necessarily \( f(u_F) = 0 \).

\textbf{Proof.} First, notice that Remark 6.3 allows us to assume that, given any \( u \in T \setminus r \), also the node \( u_F \) has a parent, since we can always think of \( T \) as immersed in a larger tree with a suitably defined generalized \( p \)-Laplacian. Assume by contradiction that \( f(u_F) \neq 0 \), then by Theorem 6.5 we would have that

\[
f(u_F) = f(u_F) = g_u(\lambda).
\]

At the same time, Lemma 6.4 implies that \( \lambda \) is a pole of the function \( g_{u_F} \), leading to a contradiction.

\[\square\]

6.3. Multiplicity via generating functions. Theorem 6.5 shows that given any eigenpair \( (f, \lambda) \), the generating functions \( \{g_u(\lambda)\}_u \) characterize the value of \( f \) up to a scaling factor. In this section we observe that counting the number of generating functions that vanishes on the eigenvalue \( \lambda \) provides several insights about its multiplicity.

First, we obtain the following sufficient result for simple eigenvalues, which directly follows from Theorem 6.5.

\textbf{Proposition 6.7.} Let \( \mathcal{H}_p \) be the generalized \( p \)-Laplacian operator defined on a tree \( T = (V,E) \), and let \( u_0 \in V \). If \( g_u(\lambda) \neq 0, \forall u < u_0 \) and \( g_{u_0}(\lambda) = 0 \), then \( \lambda \) is a simple eigenvalue of \( \mathcal{H}_p^{u_0} \) associated to an everywhere nonzero eigenfunction.
Proof. We have already observed in Remark 6.2 that such a non-zero eigenfunction $f$ exists. Assume by absurd that there exist also an eigenfunction $f^*$ of $H^w_p$ associated to $\lambda$ with $f^* \neq cf$, $\forall c \in \mathbb{R}$. Then, due to Theorem 6.5 there has to exist a node $v$ such that $f^*(v) = 0$. Since $f^*(v) = 0$ and for any node $u$ such that $f^*(u) \neq 0$ it holds that $f^*(u) = f^*(u)g_\nu(\lambda) \neq 0$, then necessarily we get that $f^*(u) = 0$, $\forall u < v$. On the other hand, by the generalized $p$-Laplacian eigenvalue equation, if $f^*(v) = 0$ and $f^*(u) = 0$, then $\forall u < v$ we have in addition that $f^*(u_F) = 0$. Thus, if $f^*$ is zero in some node then necessarily $f^* = 0$ everywhere, yielding a contradiction. \hfill $\Box$

Next, in the following lemma, we establish a more general condition for $\lambda$ to be an eigenvalue, counted with its $\gamma$-multiplicity, in terms of zeros of the generating functions $g_\nu(\lambda)$.

Lemma 6.8. Let $\mathcal{H}_p$ be a generalized $p$-Laplacian on a forest $\mathcal{G}$ and, for any $u \in V$ and any tree of the forest, let $g_\nu(\lambda)$ be the function defined in (19), (20), (21). Given $\lambda$, assume there exist $v_1, \ldots, v_k \in V$ such that

$$g_\nu(\lambda) = 0 \quad \forall i = 1, \ldots, k.$$ 

Let $\{v_i\}^h_{j=1}$ be the set of the parents of the nodes $\{v_i\}^k_{j=1}$, where roots do not have parents. Then, $\lambda$ is an eigenvalue of $\mathcal{H}_p$ if and only if $k - h > 0$, and

$$\gamma\text{-mult}(\lambda) = k - h.$$ 

Proof. From Theorem 6.5 and Corollary 6.6 we know that if $\lambda$ is an eigenvalue, any corresponding eigenfunction $f$ is such that

$$f(u) = 0 \quad \text{and} \quad \frac{f(w_F)}{f(w)} = g_\nu(\lambda) \quad \text{if} \quad f(w) \neq 0.$$ 

Following the strategy of Section 5.2 remove the nodes $\{u_j\}^h_{j=1}$ from the forest $\mathcal{G}$ ending with a forest $\mathcal{G}'$ and an associated operator $\mathcal{H}'_p$ of the form

$$(29) \quad \mathcal{G}' = \bigcup_{l=1}^{n} \mathcal{T}_l \quad \mathcal{H}'_p = \bigoplus_{l=1}^{n} \mathcal{H}_p(\mathcal{T}_l).$$ 

for some $n \geq 1$. Then, from Remark 6.3 any eigenfunction of $\mathcal{H}_p$ with eigenvalue $\lambda$ corresponds to an eigenfunction of $\mathcal{H}'_p$. In particular, given any subtree $\mathcal{T}_l$ and corresponding operator $\mathcal{H}_p(\mathcal{T}_l)$ it is easy to observe that

$$g'^{\mu}_\nu(\lambda) = g_\nu(\lambda) \quad u \in \mathcal{T}_l,$$

where $g'^{\mu}_\nu$ are the generating functions defined starting from $\mathcal{H}_p(\mathcal{T}_l)$ via equations (19), (20), (21) (see the proof of Theorem 6.5 for a similar construction). Among the $\{\mathcal{T}_l\}^n_{l=1}$, let $\{\mathcal{T}'_l\}^k_{l=1}$ be the subtrees with root $r_i = v_i$. Due to Proposition 6.7 for any such $\mathcal{T}'_l$ and corresponding $\mathcal{H}_p(\mathcal{T}'_l)$ there exists a unique everywhere nonzero eigenfunction $f'_l$ of $\mathcal{H}_p(\mathcal{T}'_l)$ with eigenvalue $\lambda$ whose ratios $f'_l(w_F)/f'_l(w)$ are induced by the functions $g_\nu(\lambda)$, $\forall w \in \mathcal{T}'_l$. Moreover, notice that for any $f$ eigenfunction of $\mathcal{H}_p$ with eigenvalue $\lambda$, since $f$ is also an eigenfunction of $\mathcal{H}'_p$, we have $f|_{\mathcal{T}'_l} = \alpha_l f'_l$, for some $\alpha_l \in \mathbb{R}$.

On the other hand, on the subtrees $\{\mathcal{T}''_j\}^{n-k}_{j=1}$ whose root $r_j$ is such that $r_j \neq v_i \forall i = 1, \ldots, k$, since $g_\nu(\lambda) \neq 0$, $\forall w \in \mathcal{T}''_j$, any eigenfunction associated to $\lambda$ of $\mathcal{H}_p$ has to be such that $f|_{\mathcal{T}''_j}(w) = 0$ because of Theorem 6.5. Indeed, suppose by contradiction that $f$ is an eigenfunction associated to $\lambda$ such that $f|_{\mathcal{T}''_j} \neq 0$, then $f$
should be an eigenfunction of $H_p(\mathcal{T}_j')$ with same eigenvalue $\lambda$. However, $g_w(\lambda) \neq 0$, $\forall w \in \mathcal{T}_j'$ implies that $f_i|_{\mathcal{T}_j'}(w) \neq 0$, $\forall w \in \mathcal{T}_j'$ and thus, by Remark 6.1 we would have that $g_{r_j}(\lambda) = 0$, which is absurd.

Now, let $f_i$ be the immersion of $f_i'$ into $\mathbb{R}^N$ such that $f_i|_{\mathcal{T}_j'} = f_i'$ and $f_i(w) = 0$ for all $w \notin \mathcal{T}_j'$. Define $\Omega := \text{span}\{f_i\}_{i=1}^k$ the $k$-dimensional linear space spanned by the $f_i$.

The observations above together with Corollary 6.1 imply that if $f$ is an eigenfunction of $H_p$ with eigenvalue $\lambda$, then $f \in \Omega$. Starting from $\Omega$, we want to recover all the possible eigenfunctions of $H_p$ relative to $\lambda$. To this end, we select among the functions $f \in \Omega$ all those functions that satisfy the eigenvalue equation for $H_p$ also in the removed points $\{u_j\}_{j=1}^b$. For any node $u_j$, let $w_{i,j}$ be the node in the neighborhood of $u_j$ such that $w_{i,j} \in \mathcal{T}_j'$. Then, the $H_p$ eigenvalue equation on a node $u_j$ reads

$$\Theta_j(f) := \sum_i \omega_{w_{i,j},u_j} \phi_p(\beta_{w_{i,j}}) \phi_p(f_i(r_i))$$

$$= \sum_i \omega_{w_{i,j},u_j} \phi_p(f_i(w_{i,j})) = \left(\lambda \theta_{u_j} - \kappa_{u_j}\right) \phi_p(f(u_j)) = 0 \quad \forall j = 1, \ldots, h$$

where we have used the fact that on any $\mathcal{T}_j'$ the ratios between the components of $f_i$ are fixed by the functions $g_w(\lambda)$, $w \in \mathcal{T}_j'$ and thus, for every $w \in \mathcal{T}_j'$ there exists $\beta_w \neq 0$ such that $f_i(w) = \beta_w f_i(r_i)$.

We continue by defining the set $A = \{f \mid \Theta_j(f) = 0, \forall j = 1, \ldots, h\}$. It is clear that $f$ is an eigenfunction of $H_p$ relative to $\lambda$ if and only if $f \in A \cap \Omega$. Thus, let us now study the genus of such a set. Observe that $\gamma(A \setminus \{0\}) = N - h$, since $A$ is diffeomorphic to a linear subspace of dimension $N - h$ through the homeomorphism of $\mathbb{R}^N$ given by $x_i \mapsto \phi_p(x_i)$, $i = 1, \ldots, N$ (the set of equations $\{\Theta_j(f) = 0\}$ is transformed into a set of $h$ linearly independent equation by the change of variable $y_i := \phi_p(f_i(r_i))$). Thus, if $k > h$ then the intersection is always nonempty because of Lemma 5.1 and in particular

$$\gamma(A \cap \Omega \setminus \{0\}) \geq \gamma(A \setminus \{0\}) - (N - k) = N - h - N + k = k - h.$$ 

Now we claim that it is possible to define a function $\tilde{\psi}$ in the set of Krasnoselskii test maps $\Lambda_{k-h}(\Omega \cap A \setminus \{0\})$ such that $0 \notin \tilde{\psi}(\Omega \cap A \setminus \{0\})$. This implies $\gamma(\Omega \cap A \setminus \{0\}) \leq k - h$, from which the statement follows. To construct such $\tilde{\psi}$, consider the function $\psi \in \Lambda_k(\Omega)$ given by:

$$f = \sum_{i=1}^k \alpha_i f_i \rightarrow \psi(f) := \left(f(r_1), \ldots, f(r_k)\right) = \left(\alpha_1 f_1(r_1), \ldots, \alpha_k f_k(r_k)\right).$$

It is easy to verify that $0 \notin \psi(\Omega \setminus \{0\})$, as $f_i(r_i) \neq 0$, $\forall i$. Since we want to define the function $\tilde{\psi}$ on $A \cap \Omega$, we define $\tilde{\psi}$ as the restriction to $\mathbb{R}^{k-h}$ of $\psi$. To define such a restriction, note that among the $\{\mathcal{T}_j'\}$ it is possible to select $h$ distinct subtrees $\{\mathcal{T}_{j_l}'\}_{l=1}^h$ such that any node $u_j$ is incident to some $\mathcal{T}_{j_l}'$. As before, let $w_{i,j}$ be the neighbor of $u_j$ in $\mathcal{T}_{j_l}'$. Then consider the function $\tilde{\psi} : \Omega \cap A \rightarrow \mathbb{R}^{k-h}$, entrywise defined as

$$\left(\tilde{\psi}(f)\right)_i = \left(\psi(f)\right)_i \quad i \neq i_l, \ l = 1, \ldots, h.$$ 

It is easily proved that $\tilde{\psi} \in \Lambda_{k-h}(\Omega \cap A)$. Finally, we show that if $\tilde{\psi}(f) = 0$, for some $f \in \Omega \cap A$, then necessarily $f = 0$. To this end, write $f = \sum_{i=1}^k \alpha_i f_i$. If
\[ \psi(f) = 0, \text{ then (up to a reordering of the indices of the chosen subtrees)} \]
\[ \alpha_i f_i(r_i) = 0 \quad \forall i \neq i_l, l = 1, \ldots, h. \]

Thus, \( f = \sum_{i=1}^{h} \alpha_i f_i. \) Then, observe that, since \( f \in A, \) we have
\[ (32) \quad \Theta_j(f) = \sum_l \omega_{w_{i,j}, j} \phi_p(\alpha_i) \phi_p(\beta_{w_{i,j}}) \phi_p(f_i(r_i)) = 0 \quad \forall j = 1, \ldots, h. \]

Consider a node \( u_{j_0} \) that is incident only to one of the subtrees \( \{ T_{i, j} \}_{i=1}^{h} \), say \( T'_{i_0} \), observe that such a node necessarily exists because there are no loops in a forest. Then \( \Theta_{j_0}(f) \) for \( j = j_0 \) reads (up to a reordering of the indices)
\[ (33) \quad \Theta_{j_0}(f) = \omega_{u_{j_0}w_{i_0}, j_0} \phi_p(\alpha_{i_0}) \phi_p(\beta_{w_{i_0}}) \phi_p(f_{i_0}(r_{i_0})) = 0. \]

This means that \( \alpha_{i_0} = 0 \), i.e. \( f = \sum_{i=1}^{h-1} \alpha_i f_i. \) Repeating this procedure for all the \( h \) nodes \( u_j \), we obtain that all the \( \alpha_i \) have to be zero. In particular, this implies that, if \( k = h \), then all the \( \alpha_i \) are zero and thus \( A \cap \Omega = \{0\} \) i.e. \( \lambda \) is not an eigenvalue, thus concluding the proof.

To conclude this preparatory section needed to tackle the proofs of Theorems 3.7 and 3.8, we show in the next result how the eigenvalues and the corresponding \( \gamma \)-multiplicities change when moving from \( \mathcal{H}_p^u \) to \( \mathcal{H}_p^w \) (recall that \( \mathcal{H}_p^u \) is the operator obtained by removing all the nodes different from \( u \) and its descendants while \( \mathcal{H}_p^w \) is the one obtained by removing also the node \( u \)). We state this result as a corollary of the previous lemma, recalling that \( \lambda \) is not an eigenvalue if and only if \( \gamma \)-mult(\( \lambda \)) = 0.

**Corollary 6.9.**

1. Let \( \lambda \) be such that \( g_u(\lambda) = 0 \), then \( \lambda \) is an eigenvalue of \( \mathcal{H}_p^u \) and \( \gamma \)-mult\( \mathcal{H}_p^u(\lambda) = \gamma \)-mult\( \mathcal{H}_p^w(\lambda) + 1. \)

2. Let \( \lambda \) be an eigenvalue of \( \mathcal{H}_p^w \) such that \( g_u(\lambda) \neq 0 \) for all \( u \prec u \) and \( g_u(\lambda) \neq 0 \), then \( \lambda \) is an eigenvalue of \( \mathcal{H}_p^u \) such that \( \gamma \)-mult\( \mathcal{H}_p^u(\lambda) = \gamma \)-mult\( \mathcal{H}_p^w(\lambda) \).

3. Let \( \lambda \) be an eigenvalue of \( \mathcal{H}_p^w \) and assume there exist \( w_1, \ldots, w_h \prec u \) with \( g_{w_i}(\lambda) = 0 \), then \( \gamma \)-mult\( \mathcal{H}_p^u(\lambda) = \gamma \)-mult\( \mathcal{H}_p^w(\lambda) - 1. \)

**Proof.** From Lemma 6.8 we know that \( \lambda \) is an eigenvalue if and only if \( k - h > 0 \), where \( k \) is the number of nodes \( v \) such that \( g_v(\lambda) = 0 \) and \( h \) is the number of their parents. In particular, we have that \( \gamma \)-mult\( \mathcal{H}_p^u(\lambda) = k - h \). To prove point 1, we observe that, since \( u \) is the root of the subtree \( T_u \), \( u \) has no parents and thus necessarily \( h < k \). Moreover, by Lemma 6.4 \( g_u(\lambda) \neq 0 \), for all \( v \prec u \) implying that \( h \) does not change when moving from \( T_u \) to \( T_u \), while \( k \) increases by one. This implies the statement. To prove point 2, it is enough to observe that the number \( k - h \) does not change going from \( T_u \) to \( T_u \). Finally, in order to prove point 3 observe that in this case, when moving from \( T_u \) to \( T_u \), \( k \) does not change while \( h \) increases by one.

**6.4. Proofs of Theorems 3.7 and 3.8.** We are finally ready to prove the main Theorems 3.7 and 3.8.

**Proof of Theorem 3.7.** We first observe that if the thesis holds for trees, then it holds as well for forests. To prove this fact, we assume that all the eigenvalues on trees are variational and the multiplicity matches the \( \gamma \)-multiplicity. Then,
we note that if \( G = \bigcup_i T_i \), with \( T_i = (V_i, E_i) \) trees, then \( \mathcal{H}_p = \oplus_i \mathcal{H}_p(T_i) \), where \( \mathcal{H}_p(T_i) \) is a suitable generalized \( p \)-Laplacian operator defined on \( T_i \), and hence \( \sigma(\mathcal{H}_p) = \bigcup_i \sigma(\mathcal{H}_p(T_i)) \). In other words, the spectrum of \( \mathcal{H}_p \) is the union of the spectra of the operators defined on the trees forming \( G \). Next, we observe that, by the same assumption on trees, \( \sigma(\mathcal{H}_p(T_i)) \) is formed only by variational eigenvalues and thus it contains at most \( |V_i| \) distinct elements, implying that \( \sigma(\mathcal{H}_p) \) is formed by at most \( N \) different eigenvalues. Now, let \( \lambda \in \sigma(\mathcal{H}_p) \). By the previous assumption, \( \lambda \) is a variational eigenvalue of \( \mathcal{H}_p(T_i) \), for some \( i \in \{1, \ldots, k\} \) and \( \gamma\text{-mult}_{\mathcal{H}_p(T_i)}(\lambda) = \gamma\text{-mult}_{\mathcal{H}_p(T_i)}(\lambda) = m_i(\lambda) \). Then, for any \( i \in \{1, \ldots, k\} \) there exists \( \varphi_i \in \Lambda_{m_i(\lambda)}(A^i_\lambda) \) s.t. \( 0 \notin \varphi_i(A^i_\lambda \cap S_p) \), where

\[
A^i_\lambda = \{ f : V_i \to \mathbb{R} | \mathcal{H}_p(T_i)(f) = \lambda|f|^{p-2}f \}.
\]

Let

\[
A_\lambda = \{ f : V \to \mathbb{R} | \mathcal{H}_p(f) = \lambda|f|^{p-2}f \}.
\]

Then, we can consider the extensions of the functions \( \varphi_i \) to \( A_\lambda \) and, given \( m(\lambda) = \sum_i m_i(\lambda) \), define the function \( \varphi_\lambda \in \Lambda_{m(\lambda)}(A_\lambda) \) as a linear combination of \( \varphi_i \) such that \( 0 \notin \varphi_\lambda(A_\lambda \cap S_p) \). This implies that \( \gamma\text{-mult}_{\mathcal{H}_p}(\lambda) \leq m(\lambda) \). Noting that \( N = \sum_\lambda \sum_i m_i(\lambda) \), we have \( \sum_\lambda \gamma\text{-mult}_{\mathcal{H}_p}(\lambda) \leq N \). Thus, by Corollary 3.6 we conclude that all the eigenvalues of \( \mathcal{H}_p \) are variational and \( \gamma\text{-mult}_{\mathcal{H}_p}(\lambda) = \sum_{i=1}^k \gamma\text{-mult}_{\mathcal{H}_p(T_i)}(\lambda) \).

Now, we address the proof of the assumption and consider the case in which \( G = T \) is a tree. The proof proceeds by induction on the number of nodes \( N \). If \( N = 1 \), from [13] and Proposition 6.7 we can conclude that there exists only one eigenvalue, \( \lambda_1 \), with \( \gamma\text{-mult}_{\mathcal{H}_p}(\lambda_1) = \gamma\text{-mult}_{\mathcal{H}_p}(\lambda_1) = 1 \). Assume now that \( N > 1 \) and that the theorem holds up to \( N - 1 \). First note that the inductive assumption and the result derived in the previous paragraph imply that the thesis holds for any forest composed by trees, each one, with less then \( N \) nodes. Then, fix a root \( r \) for \( T \) and consider \( T = \bigcup_{\ell=1}^m T_\ell \) and \( \mathcal{H}_p = \bigoplus \mathcal{H}_p(T_\ell) \). Proceed by dividing the eigenvalues of \( \mathcal{H}_p \) into two sets \( \{\nu_j\}_{j=1}^k \) and \( \{\xi_i\}_{i=1}^h \), where each \( \nu_j \) is a zero of some \( g_v \) for some \( v < r \), whereas \( \xi_i \) is not. By the inductive assumption, we have that

\[
\sum_{j=1}^k \gamma\text{-mult}_{\mathcal{H}_p}(\nu_j) + \sum_{i=1}^h \gamma\text{-mult}_{\mathcal{H}_p}(\xi_i) = N - 1.
\]

Now, let us divide the eigenvalues of \( \mathcal{H}_p \) in a similar way. Let \( \{\mu_i\}_{i=1}^{k+1} \) the eigenvalues that are zeros of \( g_r \). By Lemma 6.4 we know that they are exactly \( k+1 \). Lemma 6.8 ensures that all the other eigenvalues of \( \mathcal{H}_p \) are also eigenvalues of \( \mathcal{H}_p^* \) and, in particular, they must be either in the set \( \{\nu_j\}_{j=1}^k \) or in the set \( \{\xi_i\}_{i=1}^h \). Moreover, from Lemma 6.4 we deduce that \( \{\nu_j\}_{j=1}^k \cap \{\mu_i\}_{i=1}^{k+1} = \emptyset \) while \( \{\xi_i\}_{i=1}^h \cap \{\mu_i\}_{i=1}^{k+1} \) could be non empty. In particular, let us set

\[
\{\xi_i\}_{i=1}^h = \{\xi_i\}_{i=1}^h \setminus \{\xi_i\}_{i=1}^h \cap \{\mu_i\}_{i=1}^{k+1}.
\]
Then,

\[
\sum_{i=1}^{k+1} \gamma_{\text{mult}}_{\mathcal{H}_p}(\mu_i) + \sum_{j=1}^{k} \gamma_{\text{mult}}_{\mathcal{H}_p}(\nu_j) + \sum_{l=1}^{h_1} \gamma_{\text{mult}}_{\mathcal{H}_p}(\xi_l)
\]

\[
= \sum_{i=1}^{k+1} \left( \gamma_{\text{mult}}_{\mathcal{H}_p}(\mu_i) + 1 \right) + \sum_{j=1}^{k} \left( \gamma_{\text{mult}}_{\mathcal{H}_p}(\nu_j) - 1 \right) + \sum_{l=1}^{h_1} \gamma_{\text{mult}}_{\mathcal{H}_p}(\xi_l)
\]

\[
= k + 1 - k + \sum_{j=1}^{h} \gamma_{\text{mult}}_{\mathcal{H}_p}(\nu_j) + \sum_{l=1}^{h} \gamma_{\text{mult}}_{\mathcal{H}_p}(\xi_l) = N - 1 + 1 = N
\]

where we have used Corollary 5.9 and the fact that \(\{\xi_l\}_{l=1}^{h} \subseteq \{(\mu_i)_{i=1}^{h_1} \cup \nu_{i=1}^{k+1}\}\), with \(\gamma_{\text{mult}}_{\mathcal{H}_p}(\mu_i) = 0\) if \(\mu_i \notin \{\xi_l\}_{l=1}^{h}\). Together with Corollary 5.10, the latter equality concludes the proof.

Before moving on to the proof of Theorem 3.8, several observations are in order.

**Remark 6.10.** Suppose \(G = \bigcup_{i=1}^{m} T_i\) is a forest and let \(\mathcal{H}_p = \bigoplus_{i=1}^{m} \mathcal{H}_p(T_i)\) as before. If we consider an eigenfunction \(f_k\) of \(\mathcal{H}_p\) that is everywhere nonzero Theorem 3.7 ensures that the corresponding eigenvalue \(\lambda_k\) has multiplicity exactly equal to \(m\). Indeed, necessarily \(f_k|_{T_i}\) is an eigenfunction of \(\mathcal{H}_p(T_i)\) and, since it is everywhere non-zero, its corresponding eigenvalue is simple because of Proposition 6.7.

In addition, observe that \(e_0 = (u_0, v_0)\) is an edge of some \(T_i\) such that \(f_k(u_0)f_k(v_0) < 0\) if and only if \(e_0\) separates two distinct nodal domains. This means that the number of nodal domains induced by \(f_k\) on \(T_i\) is equal to the number of edges where \(f_k|_{T_i}\) changes sign, plus one. Thus the total number of nodal domains induced by \(f_k\) on \(G\) is equal to \(m\) plus the total number of edges where \(f_k\) changes sign. Combining all these observations, we eventually obtain the following proof.

**Proof of Theorem 3.8.** We prove by induction on \(k\) that, if \(f_k\) is an eigenfunction everywhere nonzero associated to the multiple eigenvalue \(\lambda_k = \cdots = \lambda_{k+m-1}\), then \(f_k\) changes sign on exactly \(k - 1\) edges, implying that \(f_k\) induces exactly \(k - 1 + m\) nodal domains. If \(k = 1\), \(f_1|_{T_i}\) is an eigenfunction related to the first eigenvalue of each operator \(\mathcal{H}_p(T_i)\), \(i = 1, \ldots, m\). Thus, as a consequence of Theorem 4.4, \(f_1\) is strictly positive or strictly negative on every tree \(T_i\) and overall it induces \(m\) nodal domains. Moreover, it does not change sign on any edge. Now we assume the statement to be true for every \(h < k\) and prove it for \(h = k\). If \(k > 1\), then \(f_k\) cannot be a first eigenfunction on every tree \(T_i\). Then, by Theorem 4.4 there exists at least one edge \(e_0 = (u_0, v_0)\) in some \(T_{i_0}\) such that \(f_k(u_0)f_k(v_0) < 0\). Thus, we operate as in Section 5.1 and remove edge \(e_0\) to disconnect \(T_{i_0}\) into the two subtrees \(T'_{i_0}\) and \(T''_{i_0}\), so that the reduced graph \(G'\) is the union of the \(m + 1\) subtrees:

\[
G' = \left( \bigcup_{i=1}^{m} T_i \right) \sqcup T'_{i_0} \sqcup T''_{i_0}.
\]

Similarly, the new operator \(\mathcal{H}'_p\), obtained after removing \(e_0\), can be decomposed as:

\[
\mathcal{H}'_p = \left( \bigoplus_{i=1}^{m} \mathcal{H}_p(T_i) \right) \oplus \mathcal{H}'_p(T'_{i_0}) \oplus \mathcal{H}'_p(T''_{i_0}).
\]
Now we can compare the eigenvalues $\{\eta_k\}$ of $H'_p$ with the ones $\{\lambda_k\}$ of $H_p$. From Lemma 5.2 we have:

$$\eta_{k-1} \leq \lambda_k \leq \cdots \leq \lambda_{k+m-1} \leq \eta_{k+m-1} \leq \lambda_{k+m}.$$ 

Due to Remark 6.10 and Theorem 3.7 the multiplicity of $\eta_k$ has to be exactly $m + 1$ and, by assumption, $\lambda_{k-1} < \cdots < \lambda_{k+m-1} < \lambda_{k+m}$, i.e. $\eta_{k-1} = \cdots = \eta_{k+m-1}$. Moreover, by the inductive assumption, $f_k$ changes sign on $k - 1$ edges of the graph $G'$. Thus, on the original graph $G$, $f_k$ changes sign $k - 1 + 1 = k$ times, concluding the proof.

\section{Nodal domain count on generic graphs (proofs of Theorems 3.9 and 3.10)}

In this final section we prove Theorems 3.9 and 3.10, providing upper and lower bounds for the number of nodal domains of $f$ and for the number of edges where $f$ changes sign. To this end, we need first a few preliminary results.

Consider an eigenpair $(f, \lambda)$ of the generalized $p$-Laplacian operator $H_p$ on a generic graph $G$. Suppose we remove from $G$ an edge $e_0 = (u_0, v_0)$, obtaining the graph $G' = G \setminus \{e_0\}$. Modifying accordingly the generalized $p$-Laplacian $H_p$, as in Section 6.1 the new operator $H'_p$ on $G'$ is such that the pair $(f, \lambda)$, restricted to $G'$, remains an eigenpair of $H'_p$.

Let us denote by $\Delta l(e_0, f)$ the variation of the number of independent loops of constant sign, namely the difference between the number of loops of constant sign of $f$ in $G'$ minus the number of those in $G$. Similarly, let $\Delta \nu(e_0, f)$ be the variation between the number of nodal domains induced by $f$ on $G'$ and on $G$. We can characterize the difference $\Delta \nu(e_0, f) - \Delta l(e_0, f)$ in terms of $\text{sign}_{e_0}(f) = f(u_0)f(v_0)$, i.e. whether or not $f$ changes sign over $e_0$. In fact, note that, by definition, if $\text{sign}_{e_0}(f) < 0$, then neither the number of loops of constant sign nor the number of nodal domains changes. If, instead, $\text{sign}_{e_0}(f) > 0$, then either the number of independent loops decreases by one ($\Delta l(e_0, f) = -1$) or the number of nodal domains increases by one ($\Delta \nu(e_0, f) = +1$). Overall, we have

$$\Delta \nu(e_0, f) - \Delta l(e_0, f) = \begin{cases} 0 & \text{sign}_{e_0}(f) < 0 \\ 1 & \text{sign}_{e_0}(f) > 0 \end{cases}.$$

Based on the above formula, the following lemma provides a relation between the number of nodal domains induced by an eigenfunction and the number of edges where the sign changes. It is a generalization of a result from [4], which was proved for linear Laplacians and for the case of everywhere nonzero functions.

\textbf{Lemma 7.1.} Consider $f : V \to \mathbb{R}$, a function on the graph $G$. Denote by $\zeta(f)$ the number of edges where $f$ changes sign, by $z(f)$ the number of nodes where $f$ is zero, by $l(f)$ the number of independent loops in $G$ where $f$ has constant sign, and by $|E_z|$ the number of edges incident to the zero nodes. Then

$$\zeta(f) = |E| - |E_z| + z(f) - |V| + \nu(f) - l(f) \leq |E| - |V| + \nu(f) - l(f).$$

\textbf{Proof.} Operating as in Section 5.2 we start by removing from $G$ all the $z(f)$ nodes where $f$ is zero, thus obtaining a new graph $G'$ with the corresponding new generalized $p$-Laplacian $H'_p$. Since $|E_z|$ is the number of edges incident to the zero nodes that have been removed, the number of edges in $G'$ can be estimated as $|E'| = |E| - |E_z| \leq |E| - z(f)$. Moreover, the edges incident to the zero nodes
neither connect different nodal domains nor belong to constant sign loops. Hence, the restriction of $f$ to $G'$ remains an eigenfunction of $\mathcal{H}'_p$, having the same number of nodal domains and the same number of constant sign loops as $f$.

Next, we proceed by removing from $G'$ all the edges $e_1, \ldots, e_{\tau(f)}$ where $f$ does not change sign (i.e., such that $\text{sign}_{e_i}(f) > 0$) and modify consequently the operator $\mathcal{H}'_p$ as in Section 5.1. We obtain a new graph $G''$ and the corresponding new operator $\mathcal{H}''_p$ in such a way that $(\lambda, f)$ remains an eigenpair. Since the number of edges where $f$ does not change sign is $\tau(f) = |E'| - \zeta(f)$, thanks to (35), we have that

$$ \sum_{i=1}^{h} \left( \Delta \nu(e_i, f) - \Delta l(e_i, f) \right) = |E'| - \zeta(f) \leq |E| - \zeta(f) - \zeta(f). \tag{36} $$

In the final graph $G''$, only edges connecting nodes of different sign are present, so that each node is a nodal domain of $f$. As a consequence, there are a total $|V| - z$ nodal domains of $f$ and no loops with constant sign. Then:

$$ \sum_{i=1}^{h} \Delta \nu(e_i, f) = |V| - z(f) - \nu(f) \quad \text{and} \quad \sum_{i=1}^{h} \Delta l(e_i, f) = -l(f), $$

and, by (34), $\tau(f) = \sum_{i=1}^{h} \Delta \nu(e_i, f) - \Delta l(e_i, f) = |V| - z(f) - \nu(f) + l(f)$. Hence, using (36) and the fact that $\tau(f) = |E| - |E_z| - \zeta(f)$, we obtain

$$ \zeta(f) = |E| - |E_z| + z(f) - |V| + \nu(f) - l(f) \leq |E| - |V| + \nu(f) - l(f) $$

thus concluding the proof.

The above lemma allows us to prove our third and fourth main results given in Theorems 3.9 and 3.10 which provide new upper and lower bounds for the number of nodal domains of the eigenfunctions of the generalized $p$-Laplacian, extending and generalizing previous results for the standard $p$-Laplacian and the linear Schrödinger operators [5, 40, 41]. As the proof of the two claims in Theorem 3.10 requires different arguments, we subdivide it into two parts, each addressing one of the two points P1 and P2 in the statement.

**Proof of Theorem 3.9.** For a connected graph $G$, let $f$ be an eigenfunction of $\mathcal{H}_p$ relative to $\lambda$. Let $\nu(f)$ denote the number of nodal domains of $f$ and let $G_1, \ldots, G_{\nu(f)}$ be such domains. Furthermore, let $e_1, \ldots, e_\zeta$ be the edges where $f$ changes sign and $v_1, \ldots, v_\zeta$ the nodes where $f$ is zero, with $z = z(f)$ the number of such nodes. The proof proceeds as follows.

According to Section 5.2 we start by removing the nodes $v_1, \ldots, v_\zeta$ from $G$ obtaining a new graph $G'$. Operator $\mathcal{H}_p$ is then modified to form the operator $\mathcal{H}'_p$ in such a way that the restriction of $f$ to $G'$ is an eigenfunction of $\mathcal{H}'_p$ with the same eigenvalue $\lambda$. Moreover, as all the zero nodes that are not part of any nodal domain are now removed, we observe that $f$ restricted to $G'$ has no zeros and induces the same nodal domains that $f$ induces on $G$. From Lemma 5.2 we conclude that $\lambda < \lambda_k \leq \lambda'_k$, where $\lambda'_k$ denotes the $k$-th variational eigenvalue of $\mathcal{H}'_p$.

Then, operating as in Section 5.1 we remove from $G'$ all the edges $e_1, \ldots, e_\zeta$ obtaining a new graph $G''$ and the new operator $\mathcal{H}''_p$ such that $f$ restricted to $G''$ is an eigenfunction of $\mathcal{H}''_p$ with the same eigenvalue $\lambda$. Notice that, since we removed only nodes where $f$ is zero and edges where $f$ changes sign, then $G''$ can
be written as the disjoint union of the nodal domains, namely \( \mathcal{G}'' = \bigsqcup_{i=1}^{\nu(f)} \mathcal{G}_i \) and, as a consequence, we have

\[
\mathcal{H}_p'' = \bigoplus_{i=1}^{\nu(f)} \mathcal{H}_p''(\mathcal{G}_i)
\]

where \( \mathcal{H}_p''(\mathcal{G}_i) \) is the restriction of the generalized \( p \)-Laplacian operator onto \( \mathcal{G}_i \). Hence, from Lemma 5.2 we have

\[
\lambda < \lambda_k' \leq \lambda_k''.
\]

where \( \lambda_k'' \) denotes the \( k \)-th variational eigenvalue of \( \mathcal{H}_p'' \). Note that the restriction \( f|_{\mathcal{G}_i} \) to each of the nodal domains \( \mathcal{G}_i \) of \( f \) has constant sign and it is then necessarily the first eigenpair of \( \mathcal{H}_p''(\mathcal{G}_i) \) corresponding to \( \lambda \) (see Theorem 4.1 and Corollary 4.2). Hence, \( \lambda \) is also the first eigenvalue of \( \mathcal{H}_p'' \) and, as an eigenvalue of \( \mathcal{H}_p'' \), has multiplicity exactly equal to \( \nu(f) \). We deduce that \( \lambda = \lambda_1'' = \cdots = \lambda_{\nu(f)}'' \) which, combined with (37), implies \( k > \nu(f) \), thus concluding the proof. □

**Proof of P2 in Theorem 3.10** Using the same notation of the proof of Theorem 3.9 above, suppose that \( \lambda > \lambda_k \). Then Lemmas 5.2 and 5.4 imply that

\[
\lambda > \lambda_k \geq \lambda_k'-z \geq \lambda_k''-z-\zeta
\]

where we define \( \lambda_k' = \lambda_k'' = -\infty \) for \( h \leq 0 \) and \( \lambda_k' = \lambda_k'' = +\infty \) for \( h \geq N-\nu(f) + 1 \). As observed above, \( \lambda \) is also the first eigenvalue of \( \mathcal{H}_p'' \). Thus, the above inequality can hold only if \( k - z(f) - \zeta \leq 0 \). Using Lemma 7.1 we obtain \( k - z(f) - |E| + |E_z| - z(f) + |V| - \nu(f) + l(f) \leq 0 \), with \( l(f) \) being the number of independent loops in \( \mathcal{G} \) where \( f \) has constant sign. This implies

\[
\nu(f) \geq k - z(f) - (|E| - |E_z|) + (|V| - z(f)) + l(f) = k - l(f) - \beta'(f) - z(f) + c(f)
\]

where \( c(f) \) is the number of connected components of \( \mathcal{G}' \) and \( \beta'(f) := (|E| - |E_z|) - (|V| - z(f)) + c(f) \) the number of independent loops in \( \mathcal{G}' \).

Next, we provide a proof for P2. Theorem 3.10. The idea is similar to the one used in the proof of P1. In the latter, we reduced the starting graph to the disjoint union of the nodal domains of an eigenfunction and doing so we knew that the corresponding eigenvalue would become the first variational one on the reduced graph. Now, instead, we reduce the graph to a forest where we know from Theorem 5.7 that our eigenvalue becomes a variational one and we know by Theorem 5.8 how to relate the nodal domains induced by the eigenfunction to the index of the eigenvalue.

**Proof of P2 in Theorem 3.10** Let \( \mathcal{H}_p \) be a generalized \( p \)-Laplacian operator defined on a connected graph, \( \mathcal{G} \) and assume

\[
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+m-1} < \lambda_{k+m} \leq \cdots \leq \lambda_N,
\]

to be the variational spectrum of \( \mathcal{H}_p \) and \( f \) to be an eigenfunction relative to \( \lambda = \lambda_k = \cdots = \lambda_{k+m-1} \). Additionally, denote by \( \nu(f) \) the number of nodal domains of \( f \), by \( l(f) \) the number of independent loops where \( f \) has constant sign, and by \( v_1, \ldots, v_z \) the nodes where \( f \) is zero, with \( z = z(f) \) the number of such nodes.

Using the results of Section 5.2 we start by removing the nodes \( v_1, \ldots, v_z \) from \( \mathcal{G} \) and accordingly modifying the operator \( \mathcal{H}_p \), obtaining a graph \( \mathcal{G}' \) and an operator \( \mathcal{H}_p' \), such that the restriction of \( f \) to \( \mathcal{G}' \) is an eigenfunction of \( \mathcal{H}_p' \) with the same eigenvalue \( \lambda \). Observe that, since we have removed all and only the nodes of \( \mathcal{G} \)
where \( f \) is zero, \( f \) restricted to \( \mathcal{G}' \) has no zeros and induces the same nodal domains and constant sign loops induced on \( \mathcal{G} \). From Lemma [5.4] we have that
\[
\lambda'_{k+m-z} \leq \lambda \leq \lambda'_{k+m-1},
\]
where \( \{\lambda'_k\} \) denote the variational eigenvalues of \( \mathcal{H}'_p \). In particular \( \lambda \) is an eigenvalue of \( \mathcal{H}'_p \) i.e. \( \lambda \in [\lambda'_{1}, \lambda'_{N-\#(f)}] \) \((N - \#(f)\) the number of nodes of \( \mathcal{G}' \)). Thus, since the variational eigenvalues of \( \mathcal{H}'_p \) split its spectrum in intervals, there has to exist and index, \( h \) with \( \lambda'_h < \lambda'_{h+1} \) such that \( \lambda \in [\lambda'_h, \lambda'_{h+1}] \) where \( \lambda'_h = \infty \) if \( h > N - \#(f) \). Moreover from [5.9] we can state
\[
h \geq k + m - \#(f) - 1.
\]
Now observe that if \( c(f) \) is the number of connected components of \( \mathcal{G}' \) and \( \beta'(f) = |E'| - |V'| + c(f) \) the number of independent loops of \( \mathcal{G}' \), we can remove \( \beta'(f) \) edges from \( \mathcal{G}' \) to obtain a forest \( \mathcal{T} \) with the same number of connected components of \( \mathcal{G}' \). Every time we remove an edge, we modify the operator \( \mathcal{H}'_p \) as in Section [5.1] so that the pair \((f, \lambda)\) remains an eigenpair of the resulting operator. At each step, denote by \( e_0 \) the edge we are removing, and by \( \mathcal{G}'', \mathcal{H}'_p, \mathcal{H}'_{p}'' \) the graph and the corresponding generalized \( p \)-Laplace operator before and after cutting \( e_0 \). Denote by \( \{\lambda''_h\} \) and \( \{\lambda''_l\} \) the variational spectra of \( \mathcal{H}'_p '' \) and \( \mathcal{H}'_p ' \). Letting \( \lambda \in [\lambda''_l, \lambda''_{l+1}] \) (always with the assumption \( \lambda''_l = \infty \) if \( l > |V(\mathcal{G}'')| \)), due to Lemma [5.2] we can bound \( \lambda \) in terms of the spectrum of the new operator as:
\[
\lambda''_{l-1} \leq \lambda < \lambda''_{l+2}.
\]
Now, define the two counting functions \( \Delta n(e_0, f) \) and \( M(e_0, f) \). The first one counts how the variational interval in which \( \lambda \) is contained changes when moving from \( \mathcal{G}' \) to \( \mathcal{G}'' \), namely:
\[
\Delta n(e_0, f) = \begin{cases} -1 & \lambda < \lambda''_{l} \\ +1 & \lambda \geq \lambda''_{l+1} \\ 0 & \text{otherwise}. \end{cases}
\]
Observe that \( \Delta n(e_0, f) = -1 \) implies that \( \lambda \in (\lambda''_{l-1}, \lambda''_{l}) \), \( \Delta n(e_0, f) = 1 \) implies that \( \lambda \in (\lambda''_{l+1}, \lambda''_{l+2}) \), and \( \Delta n(e_0, f) = 0 \) implies that \( \lambda \in [\lambda''_l, \lambda''_{l+1}) \).

The second counting function, \( M(e_0, f) \), takes into account the sign of \( f \) on the removed edge \( e_0 \). Recall that, if \( \text{sign}_{e_0}(f) < 0 \) then from Section [5.1] it follows that \( \lambda \in (\lambda''_{l-1}, \lambda''_{l+1}) \), otherwise we have \( \lambda \in [\lambda''_l, \lambda''_{l+2}) \). Thus, we define
\[
M(e_0, f) := \begin{cases} -1 & \lambda < \lambda''_{l} \quad \text{if} \quad \text{sign}_{e_0}(f) < 0, \\ 0 & \text{otherwise}, \\ 0 & \lambda \geq \lambda''_{l+1} \quad \text{if} \quad \text{sign}_{e_0}(f) > 0, \\ -1 & \text{otherwise}. \end{cases}
\]
It follows by their definition and from Lemma [5.2] that
\[
\Delta n(e_0, f) - M(e_0, f) = \begin{cases} 0 & \text{sign}_{e_0}(f) < 0, \\ 1 & \text{sign}_{e_0}(f) > 0. \end{cases}
\]
Thus, thanks to (35), every time we cut an edge $e_0$ and modify consequently the operator $\mathcal{H}_p'$, we have the following identity
\begin{equation}
\Delta n(e_0, f) - M(e_0, f) = \Delta \nu(e_0, f) - \Delta l(e_0, f),
\end{equation}
where $\Delta \nu(e_0, f)$ and $\Delta l(e_0, f)$ are the difference between the number of nodal domains and the number of constant sign loops induced by $f$ in $\tilde{G}'$ and in $\tilde{G}''$, respectively.

After $\beta'$ steps, $(f, \lambda)$ will be an eigenpair of a generalized $p$-Laplacian operator $\mathcal{H}_p''$ defined on the forest $\mathcal{T}$, such that $f(u) \neq 0 \forall u \in \mathcal{G}''$. We have proved in Theorem 3.7 that $\mathcal{H}_p''$ has only variational eigenvalues, so, w.l.o.g., we can assume that $\lambda$ has become the $s$-th variational eigenvalue of $\mathcal{H}_p''$. Note that, thanks to Theorem 3.7 and Remark 6.10 we have that $\text{mult}_{\mathcal{H}_p''}(\lambda) = c(f)$, that is
\[\lambda_{s-c(f)+1}'' = \cdots = \lambda_{s+1}'' < \lambda_{s+1}''.
\]
Moreover, because of Theorem 3.8 we know that $f$ induces $s$ nodal domains on the forest $\mathcal{T}$. Thus, using (11) and the equality $\sum_{i=1}^{\beta'} \Delta \nu(e_i, f) = s - \nu(f)$, the number of nodal domains, $\nu(f)$, induced on the original graph $\mathcal{G}$ by $f$ (which is the same as the one induced on $\mathcal{G}'$), can be written as
\[\nu(f) = s - \sum_{i=1}^{\beta'} \Delta \nu(e_i, f) = s - \sum_{i=1}^{\beta'} \Delta n(e_i, f) - \sum_{i=1}^{\beta'} \Delta l(e_i, f) + \sum_{i=1}^{\beta'} M(e_i, f).
\]
Finally, observe that, by definition of $\Delta n$, it holds
\[\sum_{i=1}^{\beta'} \Delta n(e_i, f) = s - h, \quad \text{and} \quad \sum_{i=1}^{\beta'} \Delta l(e_i, f) = -l(f)
\]
because we have removed all the loops, while $\sum_{i=1}^{\beta'} M(e_i, f) \geq -\beta'(f)$ (note that the equality holds if and only if $M(e_i, f) = -1, \forall i$). Hence, using inequality (40), we obtain
\[\nu(f) \geq s - s + h + l(f) - \beta'(f) = h + l(f) - \beta'(f) \geq k + m - 1 - z(f) + l(f) - \beta'(f),
\]
which concludes the proof.

\section*{Appendix A. A Technical Lemma}

We devote this appendix to the following technical lemma, which is helpful to prove several of our main results.

\textbf{Lemma A.1.} Consider the function
\begin{equation}
R(\beta_1, \beta_2) = \left( \frac{|\beta_1|^p}{\phi_p(\alpha_1)} - \frac{|\beta_2|^p}{\phi_p(\alpha_2)} \right) \phi_p(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2) \phi_p(\beta_1 - \beta_2),
\end{equation}
where $\phi_p(x) = |x|^{p-2}x$, $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real numbers, and $\alpha = \alpha_2/\alpha_1$. Then $R(\beta_1, \beta_2)$ is positive if $\alpha$ is negative and negative if $\alpha$ is positive. Moreover $R(\beta_1, \beta_2) = 0$ if and only if $\beta = \beta_1/\beta_2 = \alpha_1/\alpha_2$.

\textbf{Proof.} We first consider the special cases where either $\beta_1$ or $\beta_2$ are zero or $\alpha = 1$. When $\beta_2 = 0$ (42) becomes
\[R(\beta_1, 0) = |\beta_1|^p \left( \phi_p(1 - \alpha) - 1 \right),
\]
and a simple computation shows that \((\phi_p(1 - \alpha) - 1) \geq 0\) if and only if \(\alpha < 0\). The case with \(\beta_1 = 0\) is similar, since \(R(0, \beta_2) = |\beta_2|^p \left(\phi_p \left(1 - \frac{1}{\alpha} \right) - 1\right)\). Next, consider the case \(\alpha = 1\). In this case (42) simplifies to \(R(\beta_1, \beta_2) = -|\beta_1 - \beta_2|^p \leq 0\) and one easily sees that the equality holds if and only if \(\beta_1 = \beta_2\).

Consider now the case where both \(\beta_1\) and \(\beta_2\) are different from zero and \(\alpha \neq 1\). Equation (42) can be written as

\[
R(\beta_1, \beta_2) = |\beta_1|^p \left(\phi_p(1 - \alpha) - \phi_p \left(1 - \frac{\beta_2}{\beta_1}\right)\right) + |\beta_2|^p \left(\phi_p \left(1 - \frac{1}{\alpha}\right) - \phi_p \left(1 - \frac{\beta_1}{\beta_2}\right)\right).
\]

Dividing (43) by \(|\beta_2|^p\) and letting \(\beta = \beta_1/\beta_2\) we get the chain of inequalities

\[
|\beta|^p \phi_p(1 - \alpha) + \phi_p \left(1 - \frac{1}{\alpha}\right) \geq |\beta| \phi_p \left(1 - \frac{1}{\beta}\right) + \phi_p \left(1 - \beta\right)
\]

\[
\iff |\beta(1 - \alpha))|^p \left(1 - \frac{1}{\alpha}\right) + \phi_p \left(1 - \frac{1}{\beta}\right) \geq |\beta| \phi_p \left(1 - \frac{1}{\alpha}\right) + \phi_p \left(1 - \beta\right)
\]

\[
(44) \iff |\beta(1 - \alpha))|^p \left(1 - \frac{1}{\alpha}\right) + \phi_p \left(1 - \frac{1}{\beta}\right) \geq |\beta - 1|^p.
\]

Now, if \(1 < \alpha < 0\), then \(0 < \frac{1}{\alpha(1 - \alpha)} < 1\) and \(\frac{1}{(1 - \alpha)} + \frac{1}{\alpha} = 1\), so we can use the convexity of \(x \mapsto |x|^p\) to obtain

\[
\frac{|\beta(1 - \alpha))|^p \left(1 - \frac{1}{\alpha}\right) + \phi_p \left(1 - \frac{1}{\beta}\right)}{1 - \frac{1}{\alpha}} \geq |\beta - 1|^p.
\]

Since \(x \mapsto |x|^p\) is strictly convex for \(p > 1\), the equality in the expression above holds if and only if \(\beta(1 - \alpha) = \frac{1}{\alpha} - 1\) showing that \(R(\beta_1, \beta_2)\) is positive if \(\alpha\) is negative.

To face the case \(\alpha > 0\), consider again equation (43). We can assume without loss of generality that \(0 < \alpha < 1\). Indeed, if \(\alpha > 1\), we can divide (43) by \(|\beta_1|^p\) to obtain an equation like (44) where \(1/\alpha\) is used in place of \(\alpha\) and the proof would follow from the argument above. Returning to the case \(0 < \alpha < 1\), from (43) and the following sequence of inequalities can be obtained following the same steps as above:

\[
|\beta|^p \phi_p(1 - \alpha) + \phi_p \left(1 - \frac{1}{\alpha}\right) \leq |\beta - 1|^p
\]

\[
\iff |\beta|^p \leq \frac{|\beta - 1|^p}{\phi_p(1 - \alpha) + \phi_p \left(1 - \frac{1}{\alpha}\right)}
\]

\[
\iff |\beta|^p \leq \frac{|\beta - 1|^p}{\phi_p(1 - \alpha)} \left(1 - \frac{1}{\alpha}\right) + \frac{1}{\alpha} |\alpha|
\]

Note that, as before, the last inequality holds due to the convexity of \(x \mapsto |x|^p\) and thus equality holds if and only if \(\frac{\beta - 1}{(1 - \alpha)} = \frac{1}{\alpha}\) which implies \(\beta = \frac{1}{\alpha}\), concluding the proof.

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