Hyperstar Decomposition of r-partite complete, Knodel and Fibonacci Hypergraphs

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Abstract. A relation in a set of elements is a subset of the Cartesian product $V \times V$ called edge set. A pictorial representation of a relation between a pair of objects in the set is given by a graph. If a relation is treated as a subset of $V \times V \times \ldots \times V$, $n$ times, instead of $V \times V$ the corresponding representation is a hypergraph. The $r$-uniform hypergraph is a Steiner system if every pair of points is contained in precisely one edge. Some optimal structures in communication properties are Knodel graphs, Fibonacci graphs, and r-partite complete graphs. Knodel hypergraph and Fibonacci hypergraph are established in this article. The edge set partition is the decomposition of a given graph into sub-structures. If each of such sub-structures is isomorphic to each other the decomposition is isomorphic. The isomorphic decomposition of an r-partite hypergraph, Knodel hypergraph, and Fibonacci hypergraph into hyper stars is furnished in this article.

1. Introduction

Hypergraphs can be served as a generalization of ordinary graphs. If an edge is an unordered pair of vertices in a graph, a hyperedge is a subset of vertices with any cardinality $0 \leq |e_i| \leq n$; where $n$ is the order of the hypergraph. The studies in hypergraph theory have their origin in the late 1950s but were formally defined by Claude [1] Berge. Hypergraphs are range spaces and hyperedges as ranges in computational geometry. In social choice theory, simple co-operative games can be modeled as hypergraphs. Hyper-edges are also known as hyperlinks and connectors. The fundamentals of hypergraph theory include specific hypergraphs, duality in hypergraphs, the neighborhood of a vertex in a hypergraph, weighted hypergraph, hyperpath, etc were explained by Ouvrard [2]. He discusses Hypergraph features like distance and connectivity in [2].

The topology of several problems can be expressed with hypergraphs rather than graphs. The Indian railway system is modeled using a hypergraph. Stations are vertices and trains are hyperedges. Ravindran and Nageswar [3] detected that the diameter of 247 in a base graph model was modified to 5 in a hypergraph model. 75 latent communities had decreased to 5. In Social Networks, group memberships can be used for learning clustering. The groups show multi-way relationships.

An r-uniform hypergraph is a Steiner system; every pair of points is accommodated in a single edge. Hypergraphs have been broadly used as the data model and classifier regularization in machine learning. They have implementations in image retrieval [4], recommender systems [5],...
and bioinformatics [6]. For substantial hypergraphs, a disseminated framework [7] model using Apache Spark is convenient.

The subdivision of a hypergraph into two or more parts such that the cost of the hyperedges is minimum requires Partitioning of hypergraphs. A detailed survey on the hypergraph partitioning problem is accessible in [8]. hMetis is a hypergraph partitioning algorithm formulated by Karypis [9], depend on the multi-level nourish of the sparse matrix. Linear Algebra Calculus uses Hypergraph partitioning of rows of the sparse matrix. The convenience of graph partitioning on test data from range areas such as DNA electrophoresis, polymer self-assembly, Markov chains, electrical circuit simulation, information retrieval, and sensor placement is demonstrated by Karen in [10]. Many software packages have been accessible for the partitioning of hypergraphs such as G METIS, PaToH, MLpart, and Mondriaan.

Clustering is integrating vertices into larger groups from an input hypergraph to compute a rough hypergraph. Diverse applications of graph clustering in VLSI design, Numerical Linear Algebra, formal verification, and automated theorem proving were given by Papa and Markov [11]. Social network analysis makes use of Hypergraphs especially in collaboration networks, Recommender systems, chemical reactions, Neural networks, and machine learning.

The partitioning of the edge set is Graph decomposition. Each subset induces a subgraph in the decomposition whose union is the given graph. The decomposition can be interpreted as a collection of disjoint subgraphs; disjoint in the sense of the Edge set. The decomposition is isomorphic if each subgraph is isomorphic to one another. If at least one subgraph violates isomorphism the decomposition is non-isomorphic. Reji and Jasmine [12] evince the isomorphic path and star decomposition of Fibonacci graphs. The decomposition of Complete bipartite graphs and complete graphs are shown with the help of the decomposition of Fibonacci graphs. Reji and Jasmine succeed in path and star decomposition of Knodel and Fibonacci digraphs [13].

A review of various graph parameters that describe the communication properties of various graph structures is given in [14]. Important graph structures and models along with their communication properties are discussed. The more general digraph representation and hypergraph representation that enables a reduction of huge graph parameters in large networks are also discussed. Several graph parameters of undirected graphs that improve the competency of communication structures are explained.

Dominant graph models that are used in communication systems, dominant graph topologies, and their communication parameters in terms of graphs are discussed in the second section. The third section deals with more general representation including the direction of communication called digraphs. The fourth section is devoted to another generalization known as hypergraphs. We have discussed this in the second section. The literature of parametric data of digraphs and hypergraphs is in its infancy. Researchers can find a wide variety of problems such as the diameter of Knodel digraphs are open.

An induction of Tutte’s disjoint tree theorem for hypergraphs accomplished by Andras Frank, Tamas Kiraly, and Matthias Kriesell [15]. For positive integers $k$ and $q$ each $kq$-edge connected hypergraph of rank $q$ is decomposable into $k$ connected sub-hypergraphs.

Using hypergraph decomposition G.D.Crescenzo and C.Galdi [16] explored the construction of systematic secret sharing schemes. Using information rate and average information rate the effectiveness of secret sharing schemes can be measured. The storage complexity is directly attached to these parameters, the quantity of communication complexity, and secret information. The secret sharing schemes for various classes of access structures have acquired the implementation of hypergraph decomposition like hyper stars, hypercycles, hyperpaths, and acyclic hypergraphs with greater efficiency. Some fundamental characterization of the perfect access structures is appended between the hyper stars. This paper look into the decomposition of Knodel and Fibonacci hypergraphs.
Y.M Chee et al. [17] explored the minimum number of hypergraphs without cycles that decompose a given hypergraph. The count is termed as the arboricity of hypergraphs. The arboricity of complete \( k \)-uniform hypergraph when \( k = n - 3 \), is available. The arboricity of complete \( k \)-uniform hypergraph is determined for order \( n \) when \( k = n = O(\log^{1-\delta}n) \), \( \delta \) positive.

Peter Jeavons, Marc Gyssens, and David Cohen [18] came out by a systematic Decomposition procedure for hypergraphs concerning \( k \)-hinges and \( k \)-hinge trees. Jeong-Ok Choi and Douglas B. West [19] considered the decomposition of regular hypergraphs. Heather Jordon and Genevieve Newkirk [20] gave inevitable conditions for the decomposition into 4-cycles of a complete 3-uniform hypergraph of order \( n \). Madeline V. Brandt [21] explored decompositions of complete uniform hypergraphs and the intersecting hypergraphs in her thesis. Hypertree decompositions were surveyed by Gottlob, Leone, and Scarcello [22].

H. Verrall [25] outlines the Hamilton decompositions of complete 3-uniform hypergraphs. Hamilton decomposition of a uniform hypergraph with the help of generalized Hamilton cycles of Katona and Kierstead [24] was interpreted by Robert and Brett [23]. They constructed Hamilton decomposition of complete uniform hypergraphs and narrated its relationship with design theory. J.C Bermond [26] has done a correspondence between Hamilton decompositions of graphs, directed graphs, and hypergraphs in Advances in graph theory. The decomposition of complete 3-uniform hypergraphs into small 3-uniform hypergraphs specified by D.Bryant et al. [27]. The complete uniform hypergraphs were decomposed by Kuhn and Osthus [28] into Hamilton Berge cycles.

In this article, the authors try to convey the notion of Knodel and Fibonacci hypergraphs. The isomorphic hyper star decomposition of r-partite hypergraphs, Knodel hypergraphs, and Fibonacci hypergraphs are also presented.

Berge’s definitions of [1] hypergraphs were followed throughout the article.

2. Hypergraphs

Consider the set \( V = \{v_1, v_2, \cdots, v_n\} \), and \( H = \{e_1, e_2, \cdots, e_m\} \) of subsets of \( V \) in such a way that

(i) \( e_j \neq \phi \), for \( j = 1, 2, \cdots, m \)

(ii) \( \bigcup_{j=1}^{m} e_j = V \) The pair \((V, H)\) is termed as a hypergraph. The elements of \( H \) and \( V \) are hyperedges and hyper nodes respectively. If

(iii) \( e_i \subseteq e_j \Rightarrow i = j \)

then a Hypergraph is a simple hypergraph. If \( |e_j| \leq 2 \), for every \( j = 1, 2, \cdots, m \), then the simple hypergraph becomes a simple graph.

The empty sets are also included as hyperedges by Voloshin [29] and broke the second condition that the union covers the vertex set. A transitional definition was used by Brettos 30, by relaxing the context of the covering of vertex set by edges.

The hypergraphs were presented by Euler diagram. A set of points representing the nodes may be drawn as dots in a hypergraph \( H \). The hyperedge \( e_j \) is depicted by a continuous curve joining the two hypernodes if \( |e_j| = 2 \), by a loop if \( |e_j| = 1 \) and by a simple closed curve surrounding the hypernodes if \( |e_j| \geq 3 \).

PATATE, clique expansion, Venn diagrams, pie-chart mode approach, radial approach, using extra-node, and Steiner tree representation of hyperedges are other means of representation. The hypergraphs’ planarity is explained by Zykov’s representation. The planarity of the incidence graph of the hypergraph is termed Zykko-planarity. The visualization of the Incidence representation of broad hypergraphs is used for decreasing the cognitive load. The cognitive load is reduced in the count of hyperedges that is to be composed in real networks [2].
Consider the matrix $A = (a_{ij})$; where $a_{ij} = 1$ if $v_i \in e_j$, otherwise zero. $v_1, v_2, \ldots, v_n$ represent vertices and $e_1, e_2, \ldots, e_m$ are the edges.

The order of $H$ is denoted by $n = n(H)$, which is the number of vertices. The size of $H$ is the number of edges denoted by $m = m(H)$. $r(H) = \max_j |e_j|$ is the rank and anti-rank is $s(H) = \min_j |e_j|$. If $r(H) = s(H)$, then $H$ is a uniform hypergraph. For a uniform hypergraph, $|e_j| = r, \forall j = 1, 2, \ldots, m$.

**Definition 2.1.** The family $H' = \{e_i/i \in I\}$; where $I \subseteq \{1, 2, \ldots, m\}$ is a partial hypergraph initiated by $I$. The family $H_A = \{e_i/e_i \cap A \neq \emptyset, 1 \leq i \leq m\}$ for a set $A \subset V$, is the subgraph induced by the set $A$.

Jenson defined a star in a hypergraph as follows.

**Definition 2.2.** Let $v \in V$, be a vertex of a hypergraph. The partial hypergraph obtained by the vertices containing $v$ is the star $H(v)$ with center $v$. The degree $d_H(v)$ of $v$ is the number of edges of $H(v)$, so $d_H(v) = m(H(v))$. The maximum degree of a hypergraph $H$ is denoted by $\Delta(H) = \max_{v \in H} d_H(v)$. A regular hypergraph is the one with every node has equal degree.

Let $1 \leq r \leq n$ where $r, n$ are integers. If the edge set contains all $r$-subsets of a set $V$ of cardinality $n$ the corresponding hypergraph is called $r$-complete hypergraph of order $n$ or $r$-uniform complete hypergraph. If the intersection of the edges containing $v$ is the singleton $\{v\}$ for every vertex $v$, then it is separable. i.e, if $\bigcap_{E \in H(v)} E = \{v\}$.

**Remark 1.** If every pair of nodes contained in exactly one edge the corresponding $r$-uniform hypergraph on $V$ with $|V| = n$, is a Steiner system $S(2, r, n)$.

A set of edges of a hypergraph with non-empty pairwise intersection is an intersecting family. For every node $v$ of $H$, the star $H(v) = \{e/e \in H, v \in e\}$ is an intersecting family.

$r$-partite complete hypergraph denoted by $K_r^n$ is given by $H = (V_h, E_h)$. Let $V^h = \{V^i\}$ be the vertex set, with $|V| = n_i$, for $i = 1, 2, \ldots, r$ and the collection of edges $E^h = \{v^1, v^2, \ldots, v^r\}$ with each $v^i \in V^i$, for all $i = 1, 2, \ldots, r$.

**Traversals Hypergraphs** Let $H = \{e_1, e_2, \ldots, e_r\}$ be a hypergraph with vertex set $V$. A transversal of $H$ is a collection $T \subset V$ if it contains all edges. i.e, $T \cap e_i \neq \emptyset, i = 1, 2, \ldots, m$. If a simple hypergraph is constituted by the family of minimal traversals of $H$ on $V$, it is known as the traversal hypergraph of $H$. It is represented by $Tr(H)$.

$Tr(K_r)^H = K_r^{n-1}$ and $Tr(K_r^n) = \{X^1, X^2, \ldots, X^r\}$.

A matching is a family of pairwise disjoint edges in a hypergraph $H$, and largest size of a matching is denoted by $\nu(H)$. A partial hypergraph with $\Delta(H_0) = 1$ is a matching. A fan of rank $r$ is a hypergraph $F_r$ contains $r$ edges of size $2$ and one edge of size $r$.

$r$-partite complete hypergraph is decomposed into stars in the upcoming theorem.

**Theorem 2.1.** The $r$-partite complete hypergraph is star-decomposable.

**Proof.** Let $K_r^{n_1, n_2, \ldots, n_r}$ be a given $r$-partite complete hypergraph. Each part of $K_r^{n_1, n_2, \ldots, n_r}$ has $n_1$, $n_2$, $\ldots$, $n_r$ vertices. Let the $n_1$ vertices in the first part be $1^1, 2^1, \ldots, 1^{n_1}$. Let $S_i = \{e_j \in E(H)/1^i \in e_j\}$. Each $S_i$, for $i = 1, 2, \ldots, r$ is a hyperstar. Also $S_i \cap S_k = \emptyset, \forall i \neq k$ and $\bigcup S_i = E(H)$. Therefore $\{S_i\}$ is a star-decomposition of $K_r^{n_1, n_2, \ldots, n_r}$.

Select $n_k$ nodes $\{k^1, k^2, \ldots, k^{n_k}\}$ from $k^{th}$ part if we require $n_k$ disjoint stars. Let $S^k = \{e_j \in E(H)/k^i \in e_j\}$. Then $\{S^k\}$ is a decomposition of $K_r^{n_1, n_2, \ldots, n_k}$ into $n_k$ stars.

An $r$-equipartite hypergraph is a $r$-partite hypergraph, if each part has the same size, i.e, $H^{r}_{n_1, n_2, \ldots, n_r}$ if $n_1 = n_2 = \cdots = n_r = n$. 

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The vertices of the first part is labeled by \(1^1, 1^2, \ldots, 1^n\), the vertices of the second part by \(2^1, 2^2, \ldots, 2^n\), and so on. In general the nodes of \(r\)-th part are represented by \(r^1, r^2, \ldots, r^n\).

**Definition 2.3.** Knodel Hypergraph

A Knodel hypergraph is an \(r\)-equi-partite hypergraph, if each edge \(E = (1^{n_1}, 2^{n_2}, \ldots, r^{n_r})\), with \(n_m \in \{1, 2, \ldots, n\}\) is a set of \(r\) vertices fulfilling the conditions; \(\exists\) a pair \((n_i, n_j)\) in such a way that \(n_i \equiv n_j + 2^k \pmod{n}\), \(1 \leq k \leq \lceil \log_2(n) \rceil\), \(\forall\) other \(n_i\), either \(n_i = n_i\) or \(n_i = n_j\).

**Theorem 2.2.** \(W_{d,r,n}^h\) is Star decomposition of Knodel hypergraph is possible.

**Proof.** Consider the Knodel hypergraph \(W_{d,r,n}^h\) with each part has \(n\) nodes. The \(n\) vertices in the first part be \(1^1, 1^2, \ldots, 1^n\). Let \(S_i = \{E_j \in E(W_{d,r,n}^h)/1^j \in E_j\}\). Then each hyperstar is given by \(S_i\), for \(i = 1, 2, \ldots, r\). Also \(S_i \cap S_k \neq \phi\), \(\forall i \neq j\) and \(\bigcup_i S_i = E(W_{d,r,n}^h)\). Therefore the decomposition of \(W_{d,r,n}^h\) into \(n\) stars \(\{S_i\}\) is possible.

**Definition 2.4.** Fibonacci hypergraphs

A Fibonacci hypergraph is an \(r\)-equi-partite hypergraph if each edge \(E = (1^{n_1}, 2^{n_2}, \ldots, r^{n_r})\), each \(n_m \in \{1, 2, \ldots, n\}\) is a set of \(r\) vertices satisfying the upcoming conditions. \(\exists\) a pair \((n_i, n_j)\) in such a way that \(n_i \equiv n_j + F(k) \pmod{n}\), \(1 \leq k \leq F^{-1}(n)\), \(\forall\) other \(n_i\), either \(n_i = n_i\) or \(n_i = n_j\).

The star-decomposition of Fibonacci hypergraphs is provided in the next theorem.

**Theorem 2.3.** The Fibonacci hypergraph \(F_{d,r,n}^h\) is star-decomposable.

**Proof.** Consider the Fibonacci hypergraph \(F_{d,r,n}^h\) with each part has \(n\) vertices. Let the first part contains the nodes \(1^1, 1^2, \ldots, 1^n\) be the \(n\). Let \(S_i = \{E_j \in E(F_{d,r,n}^h)/1^j \in E_j\}\). Then each hyperstar is given by \(S_i\), for \(i = 1, 2, \ldots, r\). Also \(S_i \cap S_k \neq \phi\), \(\forall i \neq k\) and \(\bigcup_i S_i = E(F_{d,r,n}^h)\). The collection \(\{S_i\}\) gives a star decomposition of \(F_{d,r,n}^h\) into \(n\) stars.

3. Conclusion

The article outlines Knodel and Fibonacci hypergraphs. The star decomposition of Knodel hypergraphs, \(r\)-partite complete hypergraphs, and Fibonacci hypergraphs are also given.

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