CONTRASTS ON THE OPACITY PROFILE OF THE SUN FROM HELIOSEISMIC OBSERVABLES AND SOLAR NEUTRINO FLUX MEASUREMENTS

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ABSTRACT

Motivated by the solar composition problem and by using the recently developed linear solar model approach, we analyze the role of opacity and metals in the Sun. After a brief discussion of the relation between the effects produced by a variation of composition and those produced by a modification of the radiative opacity, we calculate numerically the opacity kernels that, in a linear approximation, relate an arbitrary opacity variation to the corresponding modification of the solar observable properties. We use these opacity kernels to discuss the present constraints on opacity (and composition) provided by helioseismic and solar neutrino data.

Key words: Sun: fundamental parameters – Sun: general – Sun: interior

Online-only material: color figures

1. INTRODUCTION

In the last few years, a new solar problem has emerged. Recent determinations of the photospheric heavy element abundances (Asplund et al. 2005, 2009; Caffau et al. 2010) indicate that the Sun’s metallicity is lower than previously assumed (Grevesse & Sauval 1998). Solar models that incorporate these lower abundances are no more able to reproduce the helioseismic results. As an example, the sound speed predicted by standard solar models (SSMs) implementing the heavy element admixture of Asplund et al. (2005) disagrees at the bottom of the convective envelope by ~10σ with the value inferred by helioseismic data, see, e.g., Bahcall et al. (2005) and the black dashed line in Figure 1. In addition, the predicted surface helium abundance is lower by ~6σ and the inner radius of the convective envelope is larger by ~15σ with respect to the helioseismic results. Detailed studies have been done to resolve this controversy, see, e.g., Basu & Antia (2008). The latest determinations of the solar photospheric composition (Asplund et al. 2009; Caffau et al. 2010) alleviate the discrepancies but a definitive solution of the “solar composition problem” still has to be obtained.

The main effect of changing the heavy element admixture is to modify the opacity profile of the Sun. It is, thus, evident that the comprehension of the solar composition problem is intimately related to understanding the role of opacity in solar modeling. Several authors have investigated the effects of opacity changes on the solar structure by using different methods and assumptions, see, e.g., Tripathy & Christensen-Dalsgaard (1998), Christensen-Dalsgaard et al. (2009), and Bahcall et al. (2005). Here, we continue their work, completing and extending the analysis of Tripathy & Christensen-Dalsgaard (1998) by using a different and original approach. The final goal is to provide the instruments to analyze in a transparent and efficient way the role of the opacity in the Sun and to perform a critical “step-by-step” discussion of the present constraints on opacity (and composition) provided by observable properties of the Sun.

In order to calculate the effects of arbitrary opacity changes on the Sun, we use the linear solar model (LSM) approach, presented in Villante & Ricci (2010), Villante (2010), and briefly summarized in Appendices A, B, and C. In this approach, the structure equations of the present Sun are linearized and, by estimating the variation of the present solar composition from the variation of the nuclear reaction rates and elemental diffusion efficiency in the present Sun, we obtain a linear system of ordinary differential equations that can be easily solved and that completely determines the physical and chemical properties of the Sun. It was shown in Villante & Ricci (2010) that this kind of approach reproduces with good accuracy the results of nonlinear evolutionary solar models and, thus, can be used to study the role of parameters and assumptions in solar model construction.

By considering localized opacity changes in LSM approach, we determine numerically the kernels that, in a linear approximation, relate an arbitrary opacity variation to the corresponding modification of the solar observable properties. These opacity kernels are useful in several respects. First, they allow us to individuate the region of the Sun whose opacity is probed with maximal sensitivity by each observable quantity. Then, they permit us to show that effects produced by variations of opacity in different regions of the Sun can compensate among each other. Finally, they will be used to discuss how the different pieces of observational information cooperate to determine the present constraints on the opacity profile of the Sun.

The plan of the paper is the following. In Section 2, we discuss the relation between the effects produced by a variation of the heavy element admixture and those produced by a modification of the radiative opacity. We show that the relevant quantity is the variation of the opacity profile δκ(r) defined in Equation (2), which is approximately given by the superposition of the intrinsic opacity change δκI(r) and the composition opacity change δκZ(r), defined in Equations (3) and (5), respectively. In Section 3, we define the opacity kernels and describe the method adopted to calculate them. In Section 4, we calculate numerically the kernels for the squared isothermal sound speed u(r) = P(r)/ρ(r), the surface helium abundance Yb, the inner boundary of the convective envelope Rb, and the various neutrino fluxes Φν. Moreover, we discuss the constraints on δκ(r) provided by the present observational data. Finally, in Section 5 we summarize our results. A conclusive view of the constraints on δκ(r) is provided in Figure 8.

2. THE RELATION BETWEEN OPACITY AND METALS

We consider a modification of the opacity κ(ρ, T, Z) and/or of the heavy element photospheric admixture {zi}, expressed
Figure 1. Left panel: the difference between the squared isothermal sound speed inferred from helioseismic data and the predictions of solar models implementing AGS05 (black), GS98 (red), and AGSS09 (blue) heavy element admixtures. Right panel: the composition opacity changes \( \delta \kappa_Z(r) \) that corresponds to using the GS98 (red) and the AGSS09 (blue) composition in place of the AGS05 composition.

(A color version of this figure is available in the online journal.)

here in terms of the quantities \( z_i \equiv Z_{i,b}/X_b \) where \( Z_{i,b} \) is the surface abundance of the \( i \)-element and \( X_b \) is that of hydrogen. If we neglect the role of metals in the equation of state and in the energy generation coefficient, the only effect of these changes is to modify radiative energy transport in the Sun. The relevant parameter, in this respect, is the total variation of the opacity in the shell \( r \) of the present Sun, given by

\[
\delta \kappa^{\text{tot}}(r) = \frac{\kappa(\rho(r), T(r), Y(r), Z_i(r))}{\kappa(\bar{\rho}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r))} - 1, \tag{1}
\]

where \( \kappa(\bar{\rho}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r)) \) is the opacity profile of the SSM, while \( \kappa(\rho(r), T(r), Y(r), Z_i(r)) \) is the opacity profile of the solar model that implements the modified opacity and photospheric composition.\(^1\) We note that the quantity \( \delta \kappa^{\text{tot}}(r) \) is not related in a direct way to the performed variations of opacity and composition. In order to calculate it, we have to take into account that the “perturbed” Sun has different density \( \rho \), temperature \( T \), and chemical composition profiles \( Y \) and \( Z_i \) with respect to the SSM. These are not known a priori but have to be obtained as a result of numerical solar modeling.

A relevant simplification is obtained in the LSM approach presented in Villante & Ricci (2010), where one assumes that: (1) the performed changes of opacity and heavy element admixture are small; (2) the variation of the chemical composition of the Sun can be estimated from the variation of nuclear reaction rate and diffusion efficiency of the present Sun; (3) the variation of the metal admixture has a negligible direct effect on nuclear production of helium and on diffusion efficiency. In this case, the relation between \( \delta \kappa^{\text{tot}}(r) \) and the performed variation of opacity and admixture can be worked out explicitly, as it is described in Appendix A. It can be shown, moreover, that the source term \( \delta \kappa(r) \) that drives the modification of the solar properties\(^2\) and that can be constrained by observational data can be written as the sum of two contributions:

\[
\delta \kappa(r) = \delta \kappa_I(r) + \delta \kappa_Z(r). \tag{2}
\]

The first term \( \delta \kappa_I(r) \), which we refer to as intrinsic opacity change, represents the fractional variation of the opacity along the SSM profile and it is given by

\[
\delta \kappa_I(r) = \frac{\kappa(\bar{\rho}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r))}{\kappa(\bar{\rho}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r))} - 1. \tag{3}
\]

This contribution is obtained when we revise the opacity function \( \kappa(\rho, T, Y, Z_i) \) and/or we introduce new effects, like, e.g., the accumulation of few GeVs WIMPs in the solar core that mimics a decrease of the opacity at the solar center, see, e.g., Bottino et al. (2002) and references therein.

The second term \( \delta \kappa_Z(r) \), which we refer to as composition opacity change, describes the effects of a variation of \( \{z_i\} \). It takes into account that a modification of the photospheric admixture implies a different distribution of metals inside the Sun and, thus, a different opacity profile, even if the function \( \kappa(\rho, T, Y, Z_i) \) is unchanged. The contribution \( \delta \kappa_Z(r) \) is given by (see Appendices A and B)

\[
\delta \kappa_Z(r) = \frac{\kappa(\bar{\rho}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r))}{\kappa(\bar{\rho}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r))} - 1, \tag{4}
\]

where \( \bar{Z}_i(r) = Z_i(r)/(Z_i/Z_i) \) and can be calculated as

\[
\delta \kappa_Z(r) \simeq \sum_i \frac{\partial \ln \kappa}{\partial \ln Z_i} \bigg|_{\text{SSM}} \delta z_{i,b}, \tag{5}
\]

\(^1\) The notation \( \bar{\ } \) indicates, here and in the following, the SSM’s value for the generic quantity \( Q \).

\(^2\) In mathematical terms, the quantity \( \delta \kappa(r) \) represents the inhomogeneous term in the linearized structure equation of the present Sun.
3. THE METHOD

In the following, we consider the effects produced by a generic variation of the opacity profile \( \delta \kappa (r) \), without discussing whether this is due to a change of the function \( \kappa (\rho, T, Y, Z_r) \) or to a change of the admixture \( [z_i] \). As a result of this modification, we obtain a solar model that deviates from SSM predictions. If the opacity variation is sufficiently small (i.e., \( \delta \kappa (r) \ll 1 \)), the Sun responds linearly. In this case, the fractional variation of a generic quantity \( Q \), defined as

\[
\delta Q = \frac{Q}{Q} - 1, \tag{6}
\]

can be related to \( \delta \kappa (r) \) by the linear relation:

\[
\delta Q = \int dr \ K_Q(r) \ \delta \kappa (r). \tag{7}
\]

The kernel \( K_Q(r) \) represents the functional derivative with respect to opacity and allows to quantify the sensitivity of \( Q \) to opacity variations in different zones of the Sun.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Composition} & \delta R_0 & \Delta Y_0 & \delta \Phi_{\rho \rho} & \delta \Phi_{\nu \nu} & \delta \Phi_{U} & \delta \Phi_{\delta} \\
\hline
\text{GS98} & -0.019 & 0.019 & -0.011 & 0.14 & 0.26 & 0.56 & 0.49 \\
\text{AGSS09} & -0.0056 & 0.0028 & -0.0014 & 0.018 & 0.034 & 0.13 & 0.12 \\
\hline
\end{array}
\]

Notes. The above results have been estimated within the LSM approach, by applying the opacity changes \( \delta \kappa (r) \) shown in the right panel of Figure 1. For CNO neutrinos, we also considered that the fluxes scale proportionally to the total CN-abundance. Note that the absolute variation is reported for the surface helium abundance, whereas the relative variations are shown for all the other quantities.

and of those produced by a modification of the heavy element admixture explicit. In this paper, we take as a reference the Asplund et al. (2005) composition (AGS05) and we refer with “SSM predictions” to the numerical results obtained by using this composition as input for evolutionary solar model calculations.\(^3\) Other compilations can be considered, like, e.g., the Grevesse & Sauval (1998, hereafter GS98) or the more recent Asplund et al. (2009, hereafter AGSS09). The red and blue dashed lines in the right panel of Figure 1 correspond to the opacity change \( \delta \kappa (r) \) that are obtained when we use the GS98 and the AGSS09 admixture in place of the AS05 composition, as calculated by applying relation (5) and by using the logarithmic derivatives \( \partial \ln \kappa / \partial \ln Z_r \), presented in Figure 12 of Basu & Antia (2008). We observe that the variation from AGS05 to GS98 (AGSS09) heavy element admixture corresponds to increasing the opacity by about 5% (1%) at the center of the Sun and by about 20% (5%) at the bottom of the convective region. The effects of these opacity changes on helioseismic and solar neutrino observables, calculated in the LSM approach, are described in Table 1 and in the left panel of Figure 1. They can be compared with the results of full nonlinear evolutionary codes reported, e.g., in Serenelli (2009), obtaining a satisfactory agreement.

\(\text{\(^4\) Some of the opacity kernels presented in this paper are also calculated in} \ \text{Tripathy & Christensen-Dalsgaard (1998) by using static solar models and/or} \ \text{evolutionary models with simplified equation of state and without elemental} \ \text{diffusion. Where comparison is possible, a very good agreement is achieved} \ \text{showing that the linearization procedure adopted here and the simplifying} \ \text{assumptions implied in Tripathy & Christensen-Dalsgaard (1998) do not} \ \text{introduce relevant errors. In order to make the comparison, one should note} \ \text{that the definition of the kernels given in Tripathy & Christensen-Dalsgaard} \ \text{(1998) differs from that adopted in this paper.} \)

\(\text{\(^5\) We remark that the LSMs are linear “by construction.” The validity of} \ \text{relation (7) is, thus, not limited by the condition} \ \delta \kappa (r) \ll 1. \)
where \( r_c = 0.3 R_\odot \), \( A = 0.01 \), while \( R_b = 0.730 R_\odot \) is the radius of the convective envelope predicted by SSM, see Villante & Ricci (2010). The functions \( \delta \kappa_{\text{in}}(r) \) and \( \delta \kappa_{\text{out}}(r) \) correspond to a constant increase of the opacity in the energy producing zone \( (r \leq 0.3 R_\odot) \) and in the outer radiative region \( (r \geq 0.3 R_\odot) \), see Figure 2. They have been defined in such a way that \( \delta \kappa_{\text{in}}(r) + \delta \kappa_{\text{out}}(r) \approx 1 \). The function \( \delta \kappa_0(r) \) and \( \delta \kappa_1(r) \) correspond to a global rescaling and to a linear tilt of the opacity profile.

In linear approximation, the fractional variation \( \delta Q \) produced by the opacity profiles (11) and (12) can be expressed as

\[
\delta Q = A_{\text{in}} \delta Q_{\text{in}} + A_{\text{out}} \delta Q_{\text{out}},
\]

\[
\delta Q = A_0 \delta Q_0 + A_1 \delta Q_1,
\]

where the coefficients \( \delta Q_j \) are given by

\[
\delta Q_j = \int dr K_\nu(r) \delta \kappa_j(r)
\]

with \( j = \text{in, out, 0, 1} \). We report these coefficients in Table 2 for helioseismic observables and solar neutrino fluxes.

### 4. THE OPACITY KERNELS

We calculate the opacity kernels for following observable quantities: the squared isothermal sound speed \( u(r) \equiv P(r)/\rho(r) \), the surface helium abundance \( Y_b \), the depth \( R_b \) of the convective envelope, and the solar neutrino fluxes \( \Phi_\nu \), where \( \nu = pp, \text{Be, B, N, O} \) refers to the neutrino producing reactions.\(^6\) Our results are presented in the following subsections, together with a discussion of the present observational constraints on \( \delta \kappa(r) \).

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\(^6\) For completeness, we recall that: (1) \( pp, \text{Be and B} \) neutrinos are produced in the \( pp \)-chain by reactions \( p + p \rightarrow d + e^+ + \nu_e \) and \( \text{Be} \rightarrow ^7 \text{Li} + e^+ + \nu_e \); and

\(^8\) \( \text{Be} \rightarrow ^7 \text{Be} + e^+ + \nu_e \), respectively; (2) \( \text{N and O} \) neutrinos are produced in the CN-cycle by the decays \( ^{13} \text{N} \rightarrow ^{13} \text{C} + e^+ + \nu_e \) and \( ^{15} \text{O} \rightarrow ^{15} \text{N} + e^+ + \nu_e \), respectively.
almost perfect compensation between the positive contribution from the region \( r' \simeq r \) and the negative contribution from the other regions of the Sun.

A qualitative argument to explain the stability of the sound speed is the following. The virial theorem connects the gravitational energy, \( E_g = -\int dm \, Gm/r \), and the thermal energy content of a given star, \( E_t = 3/2 \int dm \, (P/\rho) = 3/2 \int dm \, u \), being \( E_g = -2E_t \). For a generic star, a global rescaling of the opacity reflects into a global rescaling of the radial profile and, thus, taking into account the virial theorem, also into a global rescaling of the sound speed. This is not the case for the Sun because the solar radius is observationally determined. We are forced to re-adjust the free parameters in the model in order to keep the solar radius fixed, with the effect of stabilizing the radial profile and, thus, the sound speed profile \( u(r) \).

The above result has relevant implications. In particular, the statement that the “sound speed problem” requires an increase of the opacity at the bottom of the convective region is, strictly speaking, not correct because other solutions are possible. We can consider, e.g., the sound speed profiles produced by the opacity variations parameterized by Equations (11) and (12). In linear approximation, we have that

\[
\delta u(r) = A_{in} \delta u_{in}(r) + A_{out} \delta u_{out}(r) \\
\delta u(r) = A_0 \delta u_0(r) + A_1 \delta u_1(r). 
\]

The functions \( \delta u_{in}(r) \), \( \delta u_{out}(r) \), \( \delta u_0(r) \), and \( \delta u_1(r) \) are shown in the left panel of Figure 4 and are reported in Table 2 for the selected values \( r = 0.1, 0.2, 0.4, 0.6, 0.65 R_{\odot} \). We see that \( \delta u_0(r) \ll \delta u_1(r) \), indicating that the “tilt” and not the scale of the opacity is fixed by the sound speed. Moreover, we have \( \delta u_{in}(r) \simeq -\delta u_{out}(r) \) that shows that the effect of an enhancement of the opacity in the external radiative region can be equally produced by a decrease of the opacity at the solar center.

This is confirmed in the right panel of Figure 4 where we show the sound speed profiles obtained by implementing four different opacity modifications, described by:

- **Model A**: 15% decrease of opacity in the energy producing region \( A_{in} = -0.15 \) and \( A_{out} = 0 \);
- **Model B**: 15% increase of the opacity in the outer radiative region \( A_{in} = 0 \) and \( A_{out} = 0.15 \);
- **Model C**: a linear tilt of opacity corresponding to a 20% increase at the bottom of the convective envelope \( (A_0 = 0 \) and \( A_1 = 0.2) \); and
- **Model D**: a linear tilt plus a global rescaling of opacity corresponding to a 20% decrease at the solar center \( (A_0 = -0.2 \) and \( A_1 = 0.2) \).

The sound speed profiles obtained in all the considered cases reproduce equally well the result inferred by helioseismic data, showing that the helioseismic determination of \( u(r) \) translates into a bound on the differential increase \( A_{out} - A_{in} \simeq 0.15 \) and on the tilt \( A_1 \simeq 0.2 \), with no relevant constraint on \( A_0 \) (or, equivalently, \( A_{in} + A_{out} \)) that fix the global scale of opacity.

In light of this observation, it is intriguing the possibility that non-standard effects that mimic a decrease of the opacity at the solar center, like, e.g., the accumulation of few GeVs WIMPs in the solar core (see, e.g., Bottino et al. 2002), could have a role in the solution of the solar composition puzzle. We will see, in the following, that this possibility is disfavored by the determination of the surface helium abundance and by the measurement of the \(^7\)Be and \(^8\)B neutrino fluxes.

### 4.2. The Surface Helium Abundance

In the left panel of Figure 5, we show with solid line the functional derivative \( K_Y(r) \) of the surface helium abundance \( Y_b \), defined according to equation

\[
\Delta Y_b = \int dr \, K_Y(r) \, \delta \kappa(r),
\]

where we considered the absolute variation \( \Delta Y_b \) to conform with the notations adopted in Villante & Ricci (2010).

The surface helium abundance depends on the initial chemical composition and on the effects of elemental diffusion. In Villante

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7. The idea that the accumulation of few GeV WIMPs in the solar core could alleviate the “solar composition problem” was originally proposed by Villante (2009). The recent paper (Frandsen & Sarkar 2010) presented a qualitative implementation of this idea. The effect on boron neutrinos is discussed in Taoso et al. (2010).

8. Here and in the following, we use the notation \( Q_b \) to indicate that a given quantity \( Q(r) \) is evaluated at the bottom of the convective region, i.e., \( Q_b \equiv Q(\bar{R}_b) \), where \( \bar{R}_b = 0.730 \) R⊙.
& Ricci (2010), by assuming that the variation of the time-integrated effect of diffusion can be estimated from the variation of diffusion efficiency in the present Sun, we have obtained the following relation:

$$\Delta Y_b = A_Y \Delta Y_{ini} + A_C \delta C$$

(20)

with $A_Y = 0.838$ and $A_C = 0.033$, which gives $\Delta Y_b$ as a function of the absolute variation of the initial helium abundance, $\Delta Y_{ini}$, and of the fractional variation of pressure at the bottom of the convective region, $\delta C = \delta P_b$. The above equation allows us to obtain the functional derivative $K_Y(r)$ as the sum of two contributions, one related to the term $A_Y \Delta Y_{ini}$ and the other to $A_C \delta C$. These are shown in Figure 5 with red and blue dashed lines, respectively.

The function $K_Y(r)$ is positive everywhere, showing that an increase of opacity in an arbitrary shell of the Sun translates into an increase of the helium abundance. Moreover, the kernel $K_Y(r)$ has a rather broad profile. This implies that the determination of $Y_b$ effectively constrains the opacity scale and breaks the degeneracy between the possible solutions of the sound speed problem presented in the previous section. The SSM that implements the AGS05 heavy element admixture predicts the value

$\mu$ is the mean molecular weight. In order to reproduce the observed solar luminosity $L_\odot$, we are forced to readjust the chemical composition of the Sun by increasing the helium abundance. This has the simultaneous effects of increasing $\mu$ and decreasing $\kappa$.

Figure 4. Left panel: the functions $\delta u_{ini}(r)$, $\delta u_{tel}(r)$, $\delta u_{0}(r)$, and $\delta u_{1}(r)$ defined in Equation (18), which allow to calculate the sound speed response to opacity changes parameterized by Equations (11) and (12). Right panel: the difference between the sound speed inferred from helioseismic data and that obtained by solar models implementing AGS05 heavy element admixture (dotted line) with suitably modified opacity profile (the solid lines correspond to models A, B, C, and D described in the text).

(A color version of this figure is available in the online journal.)

Figure 5. Left panel: the solid line corresponds to the kernel $K_Y(r)$ defined in Equation (19). The red and blue dashed lines describe the contributions to surface helium variations provided by the terms $A_Y \Delta Y_{ini}$ and $A_C \delta C$, respectively. See the text for details. Right panel: the solid line corresponds to the kernel $K_R(r)$ defined in Equation (23). The red, blue, and green dashed lines describe the contributions to convective radius variations provided by the terms $\Gamma_Y \Delta Y_{ini}$, $\Gamma_C \delta C$, and $\Gamma_\kappa \delta \kappa_b$, respectively. See the text for details.

(A color version of this figure is available in the online journal.)
The discrepancy in the above result, we have considered that \( \delta C = \delta P_b \simeq \delta \rho_b \), since the fractional variation of the sound speed \( \delta u(r) \) is expected to vanish at the bottom of the convective region, i.e., \( \delta u_b = \delta P_b \simeq 0 \). As it is discussed in Villante & Ricci (2010), the discrepancy between the helioseismic determinations of \( R_b \) and \( Y_b \) and the predictions of SSMs implementing AS05 admixture is quantified as \( \delta R_b = -0.0053 \pm 0.0015 \) and \( \delta Y_b = 0.0195 \pm 0.0034 \). The density at the bottom of the convective region deviates from value inferred by helioseismology by \( \delta \rho_b = 0.08 \), as it is discussed, e.g., in Serenelli (2009). We obtain \( \delta \rho_b \simeq 0.24 \pm 0.03 \), where errors have been combined in quadrature and we have neglected the (sub-dominant) contribution to the total error budget due to the uncertainties in the density determination. We remark that the obtained result is model-independent, since it does not rely on any assumption or parameterization for the function \( \delta(k) \).

As a final application, we consider the response of \( R_b \) to the opacity changes parameterized by Equations (11) and (12). We obtain the relations

\[ \delta R_b = -0.01 A_0 - 0.10 A_1 \]

which show that the inner radius of the convective envelope, just like the sound speed, provides bounds on the differential increase \( A_{\text{out}} - A_{\text{in}} \) and on tilt \( A_1 \), with no relevant constraints on \( A_0 \) (or equivalently \( A_{\text{in}} + A_{\text{out}} \)) that fix the global scale of opacity. By considering \( \delta R_b = -0.0053 \pm 0.0015 \), we obtain \( A_{\text{out}} - A_{\text{in}} \simeq 0.15 \) and \( A_1 \simeq 0.2 \) in substantial agreement (and complete degeneracy) with the information provided by the sound speed measurement. By combining this information with the constraints provided by the surface helium abundance and performing a simple \( \chi^2 \) analysis, we obtain \( A_{\text{in}} = 0.07 \pm 0.04 \) and \( A_{\text{out}} = 0.21 \pm 0.04 \) for the parameterization given in Equation (11) and \( A_0 = 0.056 \pm 0.040 \) and \( A_1 = 0.187 \pm 0.023 \) for that given in Equation (12). The corresponding bounds on the opacity change \( \delta \kappa \) are shown in Figure 8 and commented upon in the conclusive section.

4.4. Neutrino Fluxes

In the left panel of Figure 6, we show the functional derivatives \( K_\nu(r) \) of the neutrino fluxes \( \Phi_\nu \) defined according to relation:

\[ \delta \Phi_\nu = \int dr \ K_\nu(r) \delta \kappa(r) \]

where the index \( \nu = pp, \ Be, B, N, O \) refers to the neutrino producing reactions. The kernel \( K_\nu(r) \) have been calculated by using the LSM approach and by taking into account that the fractional variations of the fluxes \( \delta \Phi_\nu \) are related to physical and chemical properties of the Sun by

\[ \delta \Phi_\nu = \int dr[\phi_{\nu,p}(r) \delta \rho(r) + \phi_{\nu,T}(r) \delta T(r) + \phi_{\nu,Z}(r) \delta Z(r)] \]
The functions $\phi_{\nu,i}(r)$ have been defined and calculated in Figure 10 of Villante & Ricci (2010).

Our results show that neutrino fluxes probe the opacity of the Sun in the region $r \leq 0.45 R_\odot$ which is larger than the neutrino producing zone.\footnote{The $pp$ neutrinos are produced in the region $r \leq 0.25 R_\odot$. The other components of the solar neutrino flux are produced in the more internal region $r \leq 0.15 R_\odot$. See, e.g., Figure 5 of Castellani et al. (1997).} This is not surprising. A modification of the opacity translates, indeed, into a different temperature profile (and into a readjustment of the parameters of the model in such a way that the observed solar luminosity is recovered). Thus, even an opacity change localized outside the energy producing region can lead to a different central temperature and to modification of the solar neutrino fluxes.

The kernels $K_B(x), K_{Be}(x), K_{N}(x)$, and $K_O(x)$ are positive-valued almost everywhere, while the kernel $K_{pp}(r)$ is negative. This indicates that an increase of opacity generally translates into an enhancement of $^8$B, $^7$Be, and CNO neutrino fluxes and into a (slight) decrease of the $pp$-neutrino component. In Table 2, we show the coefficients $\delta \Phi_{v,in}, \delta \Phi_{v,out}, \delta \Phi_{v,0}$, and $\delta \Phi_{v,1}$ that allow to describe the effects of opacity changes parameterized by Equations (11) and (12), through the simple relations:

$$\delta \Phi_v = A_{in} \delta \Phi_{v,in} + A_{out} \delta \Phi_{v,out}$$

(33)

$$\delta \Phi_v = A_0 \delta \Phi_{v,0} + A_1 \delta \Phi_{v,1}.$$  

(34)

We see that $^8$B and CNO neutrinos are extremely sensitive to opacity changes, as it is expected since they strongly depend on the temperature of the central regions of the Sun. We also see that $\delta \Phi_{v,in} \geq \delta \Phi_{v,out}$ as a consequence of the fact that the kernels are peaked in the most internal region of the Sun. We finally note, in the right panel of Figure 6, that the normalized neutrino kernels $K_v(x)/\Phi_{v,0}$ have a common behavior, with two maxima at $r \sim 0.1 R_\odot$ and $r \sim 0.3 R_\odot$, and one minimum at $r \sim 0.2 R_\odot$. This shows that different fluxes basically probe the same quantity, constraining the opacity profile with the maximal sensitivity in the two regions of the Sun $r \sim 0.05 \div 0.15$ and $r \sim 0.2 \div 0.45$.

Some insights on the above results can be obtained by recalling that the total neutrino flux is essentially fixed by the solar luminosity constraint. We, thus, expect that $\Delta \Phi_{tot} = \sum_v \Phi_v \cdot \delta \Phi_v \approx 0$, where $\Phi_v$ are the SSM predictions for the various neutrino components. Considering that about 99% of the total flux is provided by $pp$ and Be neutrinos, the luminosity constraint implies $\delta \Phi_{pp} \approx -(\Phi_{Be}/\Phi_{pp}) \delta \Phi_{Be} = -0.075 \delta \Phi_{Be}$ which explains the smallness of the $pp$-neutrino kernel, the ratio between the $pp$ and Be-neutrino coefficients in Table 2 and the equality between the normalized $pp$ and Be-neutrino kernels observed in the right panel of Figure 6. The wavy shape of the kernels $K_v(r)$ reflects, instead, the response of the central temperature to localized opacity modifications. This was first noted and discussed by Tripathy & Christensen-Dalsgaard (1998) and it is seen in the right panel of Figure 6, where we show with a dashed line the functional derivative of the central temperature $T_c$, defined by

$$\delta T_c = \int dr \ K_T(r) \ \delta k(r).$$  

(35)

For convenience, we plot the normalized kernel $K_T(x)/\delta T_{c,0}$, where the normalization factor is $\delta T_{c,0} = 0.138$.

The peculiar behavior with $r$ of the kernel $K_T(r)$ cannot be explained in simple terms. A qualitative comprehension can be obtained from Figure 7, where we show the temperature profiles $\delta T(r)$ produced by the opacity changes $\delta k(r) = G(r - r_0)$ with $r_0 = 0.1, 0.2, 0.3 R_\odot$. We see that the performed opacity modifications translate into a large increase of temperature close to $r_0$, that necessarily alters nuclear burning rates since $r_0$ is inside or close to the energy producing region. The free parameters of the solar model are readjusted in such a way that the same luminosity is obtained with a different temperature profile.

In the two cases $r_0 = 0.1 R_\odot$ (i.e., opacity increase well inside the energy producing region) and $r_0 = 0.3 R_\odot$ (i.e., opacity increase just outside the energy producing region), the entire energy producing zone is affected. A new “equilibrium”
situation is achieved in which nuclear burnings occur at higher temperatures with a larger helium abundance, favoring $^8$B, CNO, and $^7$Be neutrinos at the expense of $pp$ neutrinos. For $r_0 = 0.2 R_\odot$, we have a peculiar situation since the maximal effect is produced in a region of the Sun where only $pp$ neutrinos are produced, as it is seen from Figure 10 of Villante & Ricci (2010). This necessarily shifts energy production outward with respect to the SSM case. In order to avoid overproduction of energy and to recover the observed luminosity, the central temperature $T_c$ has to be (slightly) decreased so that the burning rates at the center of the Sun are suppressed.

At present, the best studied components of the solar neutrino flux are the $^8$B neutrino flux which is determined by the SNO neutral current measurement with about 6% accuracy, $\Phi_{8B} = (5.18 \pm 0.29) \times 10^6$ cm$^{-2}$ s$^{-1}$ (Ahmad et al. 2002; Aharmim et al. 2005, 2008), and the $^7$Be neutrino flux which is measured by Borexino, $\Phi_{7Be} = (5.18 \pm 0.51) \times 10^6$ cm$^{-2}$ s$^{-1}$ (Arpesella et al. 2008). These fluxes have to be compared with the results of theoretical calculations. SSMs implementing AGS05 heavy elements admixture predict values for $\Phi_{8B}$ and $\Phi_{7Be}$ which are about 10% lower than the experimental results. We take, for definiteness, the values $\Phi_{8B} = 4.66 \times 10^6$ cm$^{-2}$ s$^{-1}$ and $\Phi_{7Be} = 4.54 \times 10^6$ cm$^{-2}$ s$^{-1}$ obtained in Serenelli (2009) which are affected by −9% and −5% theoretical uncertainties, respectively. The difference between the theoretical predictions and the experimental data points toward a moderate increase of the central opacity of the Sun. As an example, a 5% increase of the opacity in the region $r \leq 0.3 R_\odot$ would produce a 9.7% (4.3%) increase of the $^8$B($^7$Be) neutrino flux. Models that account for a reduction of the "effective" central opacity, due, e.g., to WIMP accumulation, increase the disagreement and are, thus, disfavored by solar neutrino data.

We, finally, discuss the constraints provided by solar neutrinos on opacity changes parameterized by Equations (11) and (12). We note that the coefficients $A_{in}$ and $A_{out}$ and/or $A_0$ and $A_1$ required to fit helioseismic results (see the previous section) produce enhancements of the $^8$B and $^7$Be neutrino fluxes which are equal to 28% and 14%, respectively. While the $^7$Be component would be consistent with the observational data, the $^8$B neutrino flux is too large with respect to the SNO measurement. When we fit simultaneously helioseismic and solar neutrino data, we obtain a slight reduction of the required opacity change. A simple $\chi^2$ analysis gives $A_{in} = 0.05 \pm 0.03$, $A_{out} = 0.19 \pm 0.03$ for the parameterization (11) and $A_0 = 0.038 \pm 0.034$, $A_1 = 0.192 \pm 0.023$ for the parameterization (12) with the best-fit values corresponding to $\chi^2_{min}/d.o.f. = 2.1/2$ and $\chi^2_{min}/d.o.f. = 1.7/2$, respectively. The corresponding bounds on $\delta\kappa(r)$ are displayed in Figure 8.

5. SUMMARY

Motivated by the solar composition problem and by using the recently developed LSM approach (Villante & Ricci 2010), in this paper we have discussed the effects of arbitrary opacity changes on the Sun. Our main results can be summarized as follows.

1. We have discussed the relation between the effects produced by a variation of the heavy element admixture and those produced by a modification of the radiative opacity. We have shown that the relevant quantity is the variation of the opacity profile $\delta\kappa(r)$ defined in Equation (2), that is approximately given by the superposition of the intrinsic opacity change $\delta\kappa_I(r)$ and the composition opacity change $\delta\kappa_Z(r)$, defined in Equations (3) and (5), respectively.

2. We have studied the response of the Sun to an arbitrary modification of the opacity $\delta\kappa(r)$. Namely, we have calculated numerically the kernels that, in a linear approximation, relate the opacity change $\delta\kappa(r)$ to the corresponding modifications of the solar observable properties. We have considered the following observable quantities: the squared isothermal sound speed $u(r)$, the surface helium abundance $Y_s$, the inner boundary of the convective envelope $R_b$, and the solar neutrino fluxes $\Phi_{\nu}$. We have shown that different observable quantities probe different regions of the Sun. Moreover, effects produced by variations of opacity in distinct zones of the Sun may compensate among each other. In this respect, we noted that the sound speed $u(r)$ and the depth of the convective envelope $R_b$ are practically insensitive to a global rescaling of the opacity.

3. As a consequence of the above result, we have seen that the discrepancy between the SSM predictions for $u(r)$ and $R_b$ and the helioseismic inferred values can be equally solved by a ∼15% decrease of the opacity at the center of the Sun or by a ∼15% increase of the opacity in the external radiative region. The degeneracy between these two possible solutions is broken by the “orthogonal” information provided by the measurements of the surface helium abundance and of the boron and beryllium neutrino fluxes, that fix the scale of opacity and indicate that only the second possibility can effectively solve the solar composition problem.

4. We have derived a model-independent relation, Equation (25), that allows us to obtain a direct determination of the opacity change required at the bottom of the convective envelope $\delta\kappa_b$ from quantities that are determined by helioseismic observations, i.e., the surface helium abundance, the depth of the convective region, and

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11 The quoted value is the weighed average of the three phases, used in the recent analysis (Serenelli 2009).

12 The quoted theoretical uncertainties are obtained from Pena-Garay & Serenelli (2008). In our estimate, we have not included the contribution due to opacity, since this is considered as a free parameter in our analysis.
been discussed in Figure 4. The right panel also includes the helioseismic determination of the sound speed, as it is has recently proposed by Frandsen & Sarkar (2010). We predict a decrease of the opacity at the solar center, like, e.g., results disfavor solutions of the solar composition problem that correspond well with the composition opacity change that is different calculations (see Figure 7 of Badnell et al. 2005), and at the few percent level, as can be estimated by comparing with respect uncertainties in opacity calculations which are increased by about 25% while a more moderate increase is obtained at the solar center. The required variation is large with respect to uncertainties in opacity calculations which are at the few percent level, as can be estimated by comparing different calculations (see Figure 7 of Badnell et al. 2005), and corresponds well with the composition opacity change that is obtained by using the GS98 heavy element admixture in place of the AGS05 composition. Finally, we note that the obtained results disfavor solutions of the solar composition problem that predict a decrease of the opacity at the solar center, like, e.g., that recently proposed by Frandsen & Sarkar (2010).

A conclusive view of the present constraints on $\delta \kappa (r)$ is contained in Figure 8. The red (blue) dashed lines correspond to the composition opacity changes obtained when we replace the AGS05 composition with the GS98 (AGSS09) admixture. The blue tick at $r = R_b$ is the value of the opacity at the bottom of the convective region that is obtained by applying the model-independent relation (25) to helioseismic data. The dark and light areas individuate the opacity changes $\delta \kappa(r)$ that, for each value of $r$, are obtained at 68.33% (1σ) and 95.4% (2σ) C.L., by applying a simple $\chi^2$ analysis to the opacity modification parameterized by Equation (12). The left panel takes into account only helioseismic observables. Namely, it is obtained by using the present observational determinations of the surface helium abundance and of the inner radius of the convective zone. The selected opacity profile $\delta \kappa(r)$ is also consistent with the helioseismic determination of the sound speed, as it is has been discussed in Figure 4. The right panel also includes the observational information on boron and beryllium neutrinos. These data move the required opacity change $\delta \kappa(r)$ toward slightly smaller values.

In summary, we see that helioseismic and solar neutrino data require a tilt of the opacity profile of the Sun, in such a way that the opacity at the bottom of the convective region is increased by about 25% while a more moderate increase is obtained at the solar center. The required variation is large with respect to uncertainties in opacity calculations which are at the few percent level, as can be estimated by comparing different calculations (see Figure 7 of Badnell et al. 2005), and corresponds well with the composition opacity change that is obtained by using the GS98 heavy element admixture in place of the AGS05 composition. Finally, we note that the obtained results disfavor solutions of the solar composition problem that predict a decrease of the opacity at the solar center, like, e.g., that recently proposed by Frandsen & Sarkar (2010).

It is a pleasure for me to thank B. Ricci for collaboration on the subject presented in this paper, for precious suggestions, and very useful comments.

APPENDIX A

THE TOTAL VARIATION OF OPACITY $\delta \kappa_{\text{tot}}(r)$

In order to calculate the total variation of opacity $\delta \kappa_{\text{tot}}(r)$ at a given radius $r$, we have to take into account that the perturbed Sun has different temperature, density, and chemical composition profiles with respect to SSM. We define

$$\delta \kappa_{\text{tot}}(r) = \frac{\kappa(\rho(r), T(r), Y(r), Z_i(r))}{\kappa(\bar{\rho}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r))} - 1 \quad (A1)$$

and, by expanding the above equation to linear order, we obtain

$$\delta \kappa_{\text{tot}}(r) = \kappa_T \delta T(r) + \kappa_\rho \delta \rho(r) + \kappa_Y \delta Y(r) + \sum_i \kappa_i \delta Z_i(r) + \delta \kappa_I(r), \quad (A2)$$

where

$$\kappa_T(r) = \left. \frac{\partial \ln \kappa}{\partial \ln T} \right|_{\text{SSM}} \quad \kappa_\rho(r) = \left. \frac{\partial \ln \kappa}{\partial \ln \rho} \right|_{\text{SSM}}$$

$$\kappa_Y(r) = \left. \frac{\partial \ln \kappa}{\partial Y} \right|_{\text{SSM}} \quad \kappa_i(r) = \left. \frac{\partial \ln \kappa}{\partial \ln Z_i} \right|_{\text{SSM}}$$

and the symbol $|_{\text{SSM}}$ indicates that we calculate the derivatives $\kappa_i(r)$ along the density, temperature, and chemical composition profiles predicted by the SSM. The quantity $\delta \kappa_I(r)$ represents the intrinsic opacity change and it is given by

$$\delta \kappa_I(r) = \frac{\kappa(\bar{\rho}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r))}{\kappa(\bar{\rho}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r))} - 1. \quad (A3)$$

By taking advantage of the equation of state of the stellar plasma, we can eliminate the density from Equation (A2). We
use the relation
\[ \delta \rho(r) = \delta P(r) - \delta T(r) - P_Y(r) \Delta Y(r), \tag{A4} \]
where \( P_Y(r) \approx -\partial \ln \mu / \partial Y \approx -5/[8.5Y(r)] \) and \( \mu \) represents the mean molecular weight, and we obtain
\[
\delta \kappa^{\text{tot}}(r) = (\kappa_T - \kappa_p) \delta T(r) + \kappa_p \delta P(r) + (\kappa_Y - P_Y \kappa_p) \Delta Y(r) \\
+ \sum_i \kappa_i \delta z_i(r) + \delta \kappa_i(r). \tag{A5}
\]

In order to evaluate Equation (A5), we need to estimate the chemical composition of the perturbed Sun, i.e., the quantities \( \delta Z_i(r) \) and \( \Delta Y(r) \). By using the approximate method introduced in Villante & Ricci (2010) and reviewed in the next section, we can relate the chemical composition profiles to the modification of the photospheric heavy element admixture, to the present values of \( \delta T(r) \) and \( \delta P(r) \), and to the parameters \( \Delta Y_{\text{ini}} \) and \( \delta C \) which represent the absolute variation of the initial helium abundance and the fractional variation of the pressure at the bottom of the convective region, respectively. We obtain
\[
\delta \kappa_Z(r) = \frac{\kappa_T - \kappa_p}{\kappa_T} \delta T(r) + \frac{\kappa_p}{\kappa_T} \delta P(r) + \frac{\kappa_Y - P_Y \kappa_p}{\kappa_T} \Delta Y_{\text{ini}} \\
+ \frac{\kappa_C}{\kappa_T} \delta C + [\delta \kappa_i(r) + \delta \kappa_Z(r)], \tag{A6}
\]
where \( \delta \kappa_Z(r) \) is the composition opacity change given by
\[ \delta \kappa_Z(r) = \sum_i \kappa_i \delta z_i, \tag{A7} \]
where \( \delta z_i \) is the fractional variation of \( z_i = Z_{i,b}/X_b \), while
\begin{align*}
\kappa_T & = \kappa_T - \kappa_p (1 + P_Y \xi_T) - K_Y \xi_T \\
\kappa_p & = \kappa_p (1 - P_Y \xi_p) - K_Y \xi_p \\
\kappa_Y & = (K_Y - P_Y) \xi_Y + Q_Y \xi_Z \\
\kappa_C & = Q \xi_Z
\end{align*}
with the coefficients \( Q_h \) and \( \xi_h \) defined in the next section. In the derivation of the above equation, we took into account that \( \sum_i \kappa_i = \kappa_Z \) where \( k_z = \partial \ln \kappa / \partial \ln Z \) is the partial derivative of opacity with respect to the total metal abundance (i.e., calculated by rescaling all the heavy element abundances by a constant factor, so that the metal admixture remains fixed).

It is useful to define the opacity change \( \delta \kappa(r) \) given by
\[ \delta \kappa(r) = \delta \kappa_Z(r) + \delta \kappa_C(r) \tag{A8} \]
which groups together the contributions to \( \delta \kappa^{\text{tot}}(r) \) which are directly related to the variation of the input parameters. The other terms in Equation (A6) represent derived quantities that have to be determined by solving the structure equations and/or by fitting the observed properties of the Sun.

APPENDIX B
THE CHEMICAL COMPOSITION OF THE SUN

The chemical composition of the perturbed Sun should be calculated by integrating the perturbed structure and chemical-evolution equations starting from an ad hoc chemical homogeneous ZAMS model. In Villante & Ricci (2010), we proposed a simplified approximate procedure that allows to estimate with sufficient accuracy the helium and metal abundances of the modified Sun, without requiring to follow explicitly its time-evolution. We review this procedure and we extend it to take into account the effect of a variation of the photospheric composition.

In order to quantify the relevance of the different mechanisms determining the present composition of the Sun, we express the helium and metal abundance according to
\[
Y(r) = Y_{\text{ini}} [1 + D_Y(r)] + \Delta Y_{\text{ini}}(r) \\
Z_i(r) = Z_{i,\text{ini}} [1 + D_Z(r)]. \tag{B1}
\]
Here, \( Y_{\text{ini}} \) and \( Z_{i,\text{ini}} \) are the initial values for the abundances, the terms \( D_Y(r) \) and \( D_Z(r) \) describe the effects of elemental diffusion, and \( \Delta Y_{\text{ini}}(r) \) represents the total amount of helium produced in the shell \( r \) by nuclear reactions. Note we, implicitly, assumed that heavy elements have all the same diffusion velocity by introducing a common diffusion term \( D_Z(r) \) for all metals.

We are interested in describing how the chemical composition is modified when we perturb the SSM. In the radiative core \( r < R_b \), we neglect the effects produced by variations of the diffusion terms\(^{13}\) and we write
\[
\Delta Y(r) = \Delta Y_{\text{ini}} [1 + \bar{D}_Y(r)] + \Delta Y_{\text{nucl}}(r) \\
\Delta Z_i(r) = \Delta Z_{i,\text{ini}}. \tag{B2}
\]
where \( \Delta Y_{\text{nucl}}(r) \) is the absolute variation of the amount of helium produced by nuclear reactions. A better accuracy is required in the convective region, because the surface helium abundance \( Y_b \) is an observable quantity. We, thus, discuss explicitly the role of diffusion and we write
\[
\Delta Y_b = (1 + \bar{D}_{Y,b}) \Delta Y_{\text{ini}} + \bar{Y}_{\text{ini}} \bar{D}_{Y,b} \delta D_{Y,b} \\
\Delta Z_{i,b} = \Delta Z_{i,\text{ini}} + \frac{D_{Z,b}}{1 + \bar{D}_{Z,b}} \delta D_{Z,b}, \tag{B3}
\]
where \( \delta D_{Y,b} \) and \( \delta D_{Z,b} \) are the fractional variations of the diffusion terms \( D_{Y,b} \) and \( D_{Z,b} \). The quantities \( \Delta Y_b \) and \( \delta Z_{i,b} \) are related among each other, since the metals-to-hydrogen ratios at the surface of the Sun are observationally fixed. If we indicate with \( z_i = Z_{i,b}/X_b \) the surface abundance of the \( i \)-element (rescaled to that of hydrogen), we obtain
\[
\delta Z_{i,b} = -\frac{1}{1 - Y_b} \Delta Y_b + \delta z_i, \tag{B4}
\]
where we considered that \( X_b \approx 1 - Y_b \), while \( \delta z_i \) is defined by
\[ \delta z_i = \frac{(Z_{i,b}/X_b) - (Z_{i,\text{ini}}/X_{\text{ini}})}{(Z_{i,b}/X_b)}. \tag{B5} \]

The above relation can be rewritten in terms of the initial helium and metal abundances, obtaining:
\[
\delta Z_{i,\text{ini}} = Q_0 \Delta Y_{\text{ini}} + Q_1 \delta D_{Y,b} + Q_2 \delta D_{Z,b} + \delta z_i. \tag{B6}
\]

The coefficients \( Q \) have been calculated explicitly in Villante & Ricci (2010) and are given by \( Q_0 = -1.141 \), \( Q_1 = 0.041 \), and \( Q_2 = +0.118 \), respectively.

Up to this point, the derived relations have a general validity, since the only assumption implied by our analysis is that the heavy elements have all the same diffusion velocity (we take iron

\(^{13}\) The diffusion terms \( D_Y(r) \) and \( D_Z(r) \) are at the few per cent level in the radiative region. Their variations are, thus, expected to produce very small effects on the solar composition.
as representative for all metals). To complete our calculation, we have to estimate the term \( \Delta Y_{\text{ini}}(r) \) in Equation (B2) and the quantities \( \delta D_{Y,b} \) and \( \delta D_{Z,b} \) in Equations (B3) and (B6). We use the procedure adopted in the LSM approach, where we assumed that the helium produced by nuclear reactions scales proportionally to the energy generation coefficient (and, thus, the helium production rate) in the present Sun, i.e.,

\[
\Delta Y_{\text{ini}} = \nabla_{\text{ini}}(r) \delta \epsilon^{\text{tot}}(r),
\]

where \( \delta \epsilon^{\text{tot}}(r) \) is the fractional variation of the energy generation rate. The effect of elemental diffusion is modeled by assuming that the terms \( D_{\text{b},Y} \) vary proportionally to the efficiency of diffusion in the present Sun, obtaining (see Section 6.3 and Appendix C of Villante & Ricci (2010))

\[
\delta D_{Y,b} = \Pi_Y \delta T_b + \Pi_P \delta P_b \quad \delta D_{Z,b} = \Pi_Z \delta T_b + \Pi_P \delta P_b,
\]

where \( \Pi_Y = 2.05, \Pi_Z = 2.73, \) and \( \Pi_P = -1.10. \)

By following the calculations described in Section 6.2 of Villante & Ricci (2010), we obtain the following expression for the variation of the helium abundance in the radiative region:

\[
\Delta Y(r) = \xi_Y(r) \Delta Y_{\text{ini}} + \xi_T(r) \delta T(r) + \xi_P(r) \delta P(r).
\]

The coefficients \( \xi_Y(R) \) are defined in Equation (31) of Villante & Ricci (2010) and are shown in their Figure 4.

By taking into account relations (B8) and by considering the conditions that hold at the bottom of the convective region (expressed in Equation (21) of Villante & Ricci 2010), we can estimate the variation of metal abundances in the radiative region, obtaining:

\[
\delta Z_i(r) = \delta Z_i,\text{ini} = Q_Y \Delta Y_{\text{ini}} + Q_C \delta C + \delta \zeta_i,
\]

where \( Q_Y = -0.887, Q_C = -0.164, \) and \( \delta C = \delta P_b \) represent the variation of pressure at the bottom of the convective envelope.

Finally, we can calculate the abundances in the convective region obtaining

\[
\begin{align*}
\delta Y_b &= A_Y \Delta Y_{\text{ini}} + A_C \delta C \\
\delta Z_{i,b} &= B_Y \Delta Y_{\text{ini}} + B_C \delta C + \delta \zeta_i,
\end{align*}
\]

where \( A_Y = 0.838, A_C = 0.033, B_Y = -1.088, \) and \( B_C = -0.043. \)

We remark that, while the quantities \( \Delta Y_{\text{ini}} \) and \( \delta C \) are parameters which are univocally determined by imposing the appropriate integration conditions (see the next section), the quantities \( \delta \zeta_i \) represent input parameters for solar model calculations.

**APPENDIX C**

**LINEAR SOLAR MODELS**

By expanding to linear order the structure equations of the present Sun close to the SSM solution and by assuming that the variation of the chemical abundances of the Sun can be estimated by the procedure outlined in the previous section, we obtain a linear system of ordinary differential equations that completely determine the physical and chemical properties of the “perturbed” Sun, see Villante & Ricci (2010) for details. Namely, we obtain

\[
\begin{align*}
\frac{d \delta m}{dr} &= \frac{1}{l_m} \left[ \gamma_P \delta P + \gamma_T \delta T - \delta m + \gamma_Y \Delta Y_{\text{ini}} \right] \\
\frac{d \delta P}{dr} &= \frac{1}{l_P} \left[ (\gamma_P - 1) \delta P + \gamma_T \delta T + \delta m + \gamma_Y \Delta Y_{\text{ini}} \right] \\
\frac{d \delta l}{dr} &= \frac{1}{l_l} \left[ \beta_P \delta P + \beta_T \delta T - \delta l + \beta_Y \Delta Y_{\text{ini}} + \beta_C \delta C \right] \\
\frac{d \delta T}{dr} &= \frac{1}{l_T} \left[ \alpha_P \delta P + \alpha_T \delta T + \delta l + \alpha_Y \Delta Y_{\text{ini}} + \alpha_C \delta C + \delta \kappa \right].
\end{align*}
\]

The coefficients \( \gamma_Y, \beta_Y, \) and \( \alpha_Y \) and the scale heights \( l_b \equiv \left[ d \ln(n) / dr \right]^{-1} \) have been calculated in Villante & Ricci (2010) and are shown in their Figures 1, 5, and 6. The parameters \( \Delta Y_{\text{ini}} \) and \( \delta C \) represent the absolute variation of the initial helium abundance and the relative variation of pressure at the bottom of the convective envelope and can be univocally determined by imposing the appropriate integration conditions. At the center of the Sun \( (r = 0) \), we have

\[
\begin{align*}
\delta m &= \gamma P_0 \delta P_0 + \gamma T_0 \delta T_0 + \gamma Y_0 \Delta Y_{\text{ini}} \\
\delta P &= \delta P_0 \\
\delta l &= \beta P_0 \delta P_0 + \beta T_0 \delta T_0 + \beta Y_0 \Delta Y_{\text{ini}} + \beta C_0 \delta C \\
\delta T &= \delta T_0,
\end{align*}
\]

where the subscript “0” indicates that a given quantity is evaluated at \( r = 0 \). At the bottom of the convective envelope \( (r = R_b) \), we have instead

\[
\begin{align*}
\delta m &= - \overline{m}_{\text{conv}} \delta C \\
\delta P &= \delta P_b \\
\delta l &= 0 \\
\delta T &= A_Y \Delta Y_{\text{ini}} + A_C \delta C,
\end{align*}
\]

where \( A_Y = 0.626 \) and \( A_C = 0.025 \) and \( \overline{m}_{\text{conv}} = \overline{M}_{\text{conv}} / M_\odot = 0.0192 \) is the fraction of solar mass contained in the convective region.

The term \( \delta \kappa(r) \) contains the contributions to the modification of the opacity profile of the Sun that are directly related to the variation of the input parameters. It is given by

\[
\delta \kappa(r) = \delta \kappa_Y(r) + \delta \kappa_C(r),
\]

where \( \delta \kappa_Y(r) \) and \( \delta \kappa_C(r) \) are the intrinsic and composition opacity changes, defined in Equations (3) and (5), respectively (see Appendix A). It represents the source term that drives the modification of the solar properties and that can be bounded by observational data.

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