Functions generating \((m,M,\Psi)\)-Schur-convex sums

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Dedicated to Professor Karol Baron on his 70th birthday.

Abstract. The notion of \((m,M,\Psi)\)-Schur-convexity is introduced and functions generating \((m,M,\Psi)\)-Schur-convex sums are investigated. An extension of the Hardy–Littlewood–Polya majorization theorem is obtained. A counterpart of the result of Ng stating that a function generates \((m,M,\Psi)\)-Schur-convex sums if and only if it is \((m,M,\psi)\)-Wright-convex is proved and a characterization of \((m,M,\psi)\)-Wright-convex functions is given.

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1. Introduction

Let \((X,\| \cdot \|)\) be a real normed space. Assume that \(D\) is a convex subset of \(X\) and \(c\) is a positive constant. A function \(f : D \to \mathbb{R}\) is called:

- strongly convex with modulus \(c\) if
  \[
  f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2
  \]
  for all \(x, y \in D\) and \(t \in [0, 1]\);

- strongly Wright-convex with modulus \(c\) if
  \[
  f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y) - 2ct(1-t)\|x - y\|^2
  \]
  for all \(x, y \in D\) and \(t \in [0, 1]\);

- strongly Jensen-convex with modulus \(c\) if (1) is assumed only for \(t = \frac{1}{2}\), that is
  \[
  f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2} - \frac{c}{4}\|x - y\|^2, \quad x, y \in D.
  \]
The usual concepts of convexity, Wright-convexity and Jensen-convexity correspond to the case $c = 0$, respectively. The notion of strongly convex functions was introduced by Polyak [22] and they play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see, for instance, [10,15,19,22–24,27]). Let us also mention the paper [18] by the second author which is a survey article devoted to strongly convex functions and related classes of functions.

In [1] the first author introduced the following concepts of $(m, \psi)$-lower convex, $(M, \psi)$-upper convex and $(m, M, \psi)$-convex functions (see also [2–4]): Assume that $D$ is a convex subset of a real linear space $X$, $\psi : D \to \mathbb{R}$ is a convex function and $m, M \in \mathbb{R}$. A function $f : D \to \mathbb{R}$ is called $(m, \psi)$-lower convex if the function $f - m\psi$ (the function $M\psi - f$) is convex. We say that $f : D \to \mathbb{R}$ is $(m, M, \psi)$-convex if it is $(m, \psi)$-lower convex and $(M, \psi)$-upper convex. Denote the above classes of functions by:

$$
\mathcal{L}(D,m,\psi) = \{f : D \to \mathbb{R} \mid f - m\psi \text{ is convex}\},
$$

$$
\mathcal{U}(D,M,\psi) = \{f : D \to \mathbb{R} \mid M\psi - f \text{ is convex}\},
$$

$$
\mathcal{B}(D,m,M,\psi) = \mathcal{L}(D,m,\psi) \cap \mathcal{U}(D,M,\psi).
$$

Let us observe that if $f \in \mathcal{B}(D,m,M,\psi)$ then $f - m\psi$ and $M\psi - f$ are convex and then $(M - m)\psi$ is also convex, implying that $M \geq m$ whenever $\psi$ is not trivial (i.e. is not the zero function).

If $m > 0$ and $(X, \| \cdot \|)$ is an inner product space (that is, the norm $\| \cdot \|$ in $X$ is induced by an inner product: $\|x\| = \sqrt{\langle x, x \rangle}$) the notions of $(m, \| \cdot \|^2)$-lower convexity and strong convexity with modulus $m$ coincide. Namely, in this case the following characterization was proved in [19]: A function $f$ is strongly convex with modulus $c$ if and only if $f - c\| \cdot \|^2$ is convex (for $X = \mathbb{R}^n$ this result can be also found in [8, Prop. 1.1.2]). However, if $(X, \| \cdot \|)$ is not an inner product space, then the two notions are different. There are functions $f \in \mathcal{L}(D,m,\| \cdot \|^2)$ which are not strongly convex with modulus $m$, as well as there are functions strongly convex with modulus $m$ which do not belong to $\mathcal{L}(D,m,\| \cdot \|^2)$ (see the examples given in [6]).

If $M > 0$ and $f \in \mathcal{U}(D,M,\psi)$, then $f$ is a difference of two convex functions. Such functions are called d.c. convex or $\delta$-convex and play an important role in convex analysis (cf. e.g. [26] and the reference therein). Functions from the class $\mathcal{U}(D,M,\| \cdot \|^2)$ with $M > 0$ were also investigated in [13] under the name approximately concave functions.

In [5] Dragomir and Ionescu introduced the concept of $g$-convex dominated functions, where $g$ is a given convex function. Namely, a function $f$ is called $g$-convex dominated, if the functions $g + f$ and $g - f$ are convex. Note that this concept can be obtained as a particular case of $(m, M, \psi)$-convexity by choosing $m = -1$, $M = 1$ and $\psi = g$. Observe also (cf. [1]), that in the case
where \( I \subset \mathbb{R} \) is an open interval and \( f, \psi : I \to \mathbb{R} \) are twice differentiable, \( f \in \mathcal{B}(I, m, M, \psi) \) if and only if
\[
  m \psi''(t) \leq f''(t) \leq M \psi''(t), \quad \text{for all} \ t \in I.
\]
In particular, if \( I \subset (0, \infty) \), \( f : I \to \mathbb{R} \) is twice differentiable and \( \psi(t) = -\ln t \), then \( f \in \mathcal{B}(I, m, M, -\ln) \) if and only if
\[
  m \leq t^2 f''(t) \leq M, \quad \text{for all} \ t \in I, \tag{4}
\]
which is a convenient condition to verify in applications.

Let \( I \subset \mathbb{R} \) be an interval and \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in I^n \), where \( n \geq 2 \). Following I. Schur (cf. e.g. \cite{12,25}) we say that \( x \) is majorized by \( y \), and write \( x \preceq y \), if there exists a doubly stochastic \( n \times n \) matrix \( P \) (i.e. a matrix containing nonnegative elements with all rows and columns summing up to 1) such that \( x = y \cdot P \). A function \( F : I^n \to \mathbb{R} \) is said to be Schur-convex if \( F(x) \leq F(y) \) whenever \( x \preceq y, \ x, y \in I^n \).

It is known, by the classical works of Schur \cite{25}, Hardy et al. \cite{7} and Karanmata \cite{9} that if a function \( f : I \to \mathbb{R} \) is convex then it generates Schur-convex sums, that is the function \( F : I^n \to \mathbb{R} \) defined by
\[
  F(x) = F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n)
\]
is Schur-convex. It is also known that the convexity of \( f \) is a sufficient but not necessary condition under which \( F \) is Schur-convex. A full characterization of functions generating Schur-convex sums was given by Ng \cite{16}. Namely, he proved that a function \( f : I \to \mathbb{R} \) generates Schur-convex sums if and only if it is Wright-convex (cf. also \cite{17}). Recently Nikodem et al. \cite{20} obtained similar results in connection with strong convexity in inner product spaces. Let us also mention the paper by Olbryš \cite{21} in which delta Schur-convex mappings are investigated.

The aim of this paper is to present some generalizations and counterparts of the above mentioned results related to \((m, \psi)\)-lower convexity, \((M, \psi)\)-upper convexity and \((m, M, \psi)\)-convexity. We introduce the notion of \((m, M, \Psi)\)-Schur-convex functions and give a sufficient and necessary condition for a function \( f \) to generate \((m, M, \Psi)\)-Schur-convex sums. As a corollary we obtain a counterpart of the classical Hardy–Littlewood–Pólya majorization theorem. Finally we introduce the concept of \((m, M, \psi)\)-Wright-convex functions, prove a representation theorem for them and present an Ng-type characterization of functions generating \((m, M, \Psi)\)-Schur-convex sums. It is worth underlining, that our results concern a few different classes of functions related to convexity and are formulated in vector spaces, that is in a much more general setting than the original ones.
2. Main results

Let $X$ be a real vector space. Similarly as in the classical case we define majorization in the product space $X^n$. Namely, given two $n$-tuples $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X^n$ we say that $x$ is majorized by $y$ written $x \preceq y$, if

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \cdot P$$

for some doubly stochastic $n \times n$ matrix $P$.

In what follows we will assume that $D$ is a convex subset of a real vector space $X$, $\psi : D \to \mathbb{R}$ is a convex function and $m, M \in \mathbb{R}$. For any $n \geq 2$ define $\Psi_n : D^n \to \mathbb{R}$ by

$$\Psi_n(x_1, \ldots, x_n) = \psi(x_1) + \cdots + \psi(x_n), \quad x_1, \ldots, x_n \in D. \quad (5)$$

We say that a function $F : D^n \to \mathbb{R}$ is $(m, \Psi)$-Schur-convex if for all $x, y \in D^n$

$$x \preceq y \implies F(x) \leq F(y) - m(\Psi(y) - \Psi(x)) \quad (6)$$

and

$$x \preceq y \implies F(x) \geq F(y) - M(\Psi(y) - \Psi(x)) \quad (7)$$

If only condition (6) [condition (7)] is satisfied, we say that $F$ is $(m, \Psi_n)$-lower (($M, \Psi_n$)-upper) Schur-convex.

Note that if $x \preceq y$ then $\Psi_n(x) \leq \Psi_n(y)$. It follows from the fact that the function $\psi$ is convex and so it generates Schur-convex sums $\Psi_n$.

Given a function $f : D \to \mathbb{R}$ and an integer $n \geq 2$ we define the function $F_n : D^n \to \mathbb{R}$ by

$$F_n(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n), \quad x_1, \ldots, x_n \in D. \quad (8)$$

Now, let $D$ be a convex subset of a real vector space $X$, and let $m, M \in \mathbb{R}$. Assume that $\psi : D \to \mathbb{R}$ is a convex function and $\Psi_n : D^n \to \mathbb{R}$ is defined by (5). We will prove now that $(m, \Psi_n)$-convex functions generate $(m, \Psi_n)$-Schur-convex sums.

Theorem 1. (i) If $f \in L(D, m, \psi)$, then the function $F_n$ defined by (8) is $(m, \Psi_n)$-lower Schur-convex;

(ii) If $f \in U(D, M, \psi)$, then the function $F_n$ defined by (8) is $(M, \Psi_n)$-upper Schur-convex;

(iii) If $f \in B(D, m, M, \psi)$, then the function $F_n$ defined by (8) is $(m, M, \Psi_n)$-Schur-convex.

Proof. To prove (i) fix $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in D^n$ with $x \preceq y$. There exists a doubly stochastic $n \times n$ matrix $P = [t_{ij}]$ such that $x = y \cdot P$. Then

$$x_j = \sum_{i=1}^{n} t_{ij} y_i, \quad j = 1, \ldots, n.$$
Since \( f \in \mathcal{L}(D, m, \psi) \), the function \( g = f - m\psi \) is convex and hence
\[
g(x_1) + \cdots + g(x_n) = \sum_{j=1}^{n} g \left( \sum_{i=1}^{n} t_{ij} y_i \right) \leq \sum_{j=1}^{n} \sum_{i=1}^{n} t_{ij} g(y_i)
= \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} g(y_i) = \sum_{i=1}^{n} g(y_i) \sum_{j=1}^{n} t_{ij} = g(y_1) + \cdots + g(y_n).
\]
Consequently,
\[
F_n(x) = f(x_1) + \cdots + f(x_n)
= g(x_1) + \cdots + g(x_n) + m(\psi(x_1) + \cdots + \psi(x_n))
\leq g(y_1) + \cdots + g(y_n) + m(\psi(x_1) + \cdots + \psi(x_n))
= f(y_1) + \cdots + f(y_n) - m(\psi(y_1) + \cdots + \psi(y_n))
+ m(\psi(x_1) + \cdots + \psi(x_n))
= F_n(y) - m(\Psi_n(y) - \Psi_n(x)).
\]
This shows that \( F_n \) satisfies (6), i.e. it is \((m, \Psi_n)\)-lower Schur-convex.

The proof of part (ii) is similar. Since \( f \in \mathcal{U}(D, M, \psi) \), the function \( h = M\psi - f \) is convex. Hence, for \( x \) and \( y \) as previously, we have
\[
F_n(x) = f(x_1) + \cdots + f(x_n)
= +M(\psi(x_1) + \cdots + \psi(x_n)) - h(x_1) - \cdots - h(x_n)
\geq M(\psi(x_1) + \cdots + \psi(x_n)) - h(y_1) - \cdots - h(y_n)
= M(\psi(x_1) + \cdots + \psi(x_n)) - M(\psi(y_1) + \cdots + \psi(y_n))
+ f(y_1) + \cdots + f(y_n)
= F_n(y) - M(\Psi_n(y) - \Psi_n(x)).
\]
Part (iii) follows from (i) and (ii). 

As an immediate consequence of the above theorem, we obtain the following counterpart of the classical Hardy–Littlewood–Pólya majorization theorem \([7]\).

**Corollary 2.** Let \( I \subset \mathbb{R} \) be an interval and \( n \geq 2 \). Assume that \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \) \( \in I^n \) satisfy:
(a) \( x_1 \leq \cdots \leq x_n, \ y_1 \leq \cdots \leq y_n; \)
(b) \( y_1 + \cdots + y_k \leq x_1 + \cdots + x_k, \ k = 1, \ldots, n-1; \)
(c) \( y_1 + \cdots + y_n = x_1 + \cdots + x_n. \)
Assume also that \( f, \psi : I \to \mathbb{R} \) and \( \psi \) is convex.

(i) If \( f \in \mathcal{L}(D, m, \psi) \), then
\[
f(x_1) + \cdots + f(x_n) \leq f(y_1) + \cdots + f(y_n) - m(\Psi_n(y) - \Psi_n(x));
\]
(ii) If $f \in \mathcal{U}(D, M, \psi)$, then
\[ f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n) - M(\Psi_n(y) - \Psi_n(x)) ; \]

(iii) If $f \in \mathcal{B}(D, m, M, \psi)$, then
\[ f(y_1) + \cdots + f(y_n) - M(\Psi_n(y) - \Psi_n(x)) \leq f(x_1) + \cdots + f(x_n) \]
\[ \leq f(y_1) + \cdots + f(y_n) - m(\Psi_n(y) - \Psi_n(x)) . \]

**Proof.** Note that assumptions (a)–(c) imply $x \leq y$ (see e.g. [12]) and apply Theorem 1. \qed

**Remark 3.** Specifying the functions $\psi$ and $f$ in Corollary 2 above, one can get various analytic inequalities. For example, if $I \subset (0, \infty)$ and $f \in \mathcal{B}(I, m, M, -\ln)$, then for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in I^n$ satisfying conditions (a)–(c), we get

\[ m \ln \prod_{i=1}^n \left( \frac{x_i}{y_i} \right) \leq \sum_{i=1}^n f(y_i) - \sum_{i=1}^n f(x_i) \leq M \ln \prod_{i=1}^n \left( \frac{x_i}{y_i} \right) , \]

or, equivalently,

\[ \prod_{i=1}^n \left( \frac{x_i}{y_i} \right)^m \leq \frac{\exp \left( \sum_{i=1}^n f(y_i) \right)}{\exp \left( \sum_{i=1}^n f(x_i) \right)} \leq \prod_{i=1}^n \left( \frac{x_i}{y_i} \right)^M . \quad (9) \]

If we take, for instance, $I = [k, K] \subset (0, \infty)$ and $f(t) = \frac{1}{p(p-1)} t^p$, with $p > 0$, $p \neq 1$, then $t^2 f''(t) = t^p \in [k^p, K^p]$, which means [cf. (4)] that $f \in \mathcal{B}(I, k^p, K^p, -\ln)$. Therefore, by (9), we then have

\[ \prod_{i=1}^n \left( \frac{x_i}{y_i} \right)^{p(p-1)k^p} \leq \frac{\exp \left( \sum_{i=1}^n y_i^p \right)}{\exp \left( \sum_{i=1}^n x_i^p \right)} \leq \prod_{i=1}^n \left( \frac{x_i}{y_i} \right)^{p(p-1)K^p} . \]

One can give other examples by choosing $f(t) = t^q$ with $q < 0$, $f(t) = t \ln t$, etc.

We say that a function $f : D \to \mathbb{R}$ is $(m, \psi)$-lower Jensen-convex ($(M, \psi)$-upper Jensen-convex) if the function $f - m \psi$ (the function $M \psi - f$) is Jensen-convex, i.e. satisfies (3) with $c = 0$. We say that $f : D \to \mathbb{R}$ is $(m, M, \psi)$-Jensen-convex if it is $(m, \psi)$-lower Jensen-convex and $(M, \psi)$-upper Jensen-convex.

In the next theorem we show that functions generating $(m, M, \Psi_n)$-Schur-convex sums must be $(m, M, \psi)$-Jensen–convex.

**Theorem 4.** Let $f : D \to \mathbb{R}$.

(i) If for some $n \geq 2$ the function $F_n$ defined by (8) is $(m, \Psi_n)$-lower Schur-convex, then $f$ is $(m, \psi)$-lower Jensen-convex;

(ii) If for some $n \geq 2$ the function $F_n$ defined by (8) is $(M, \Psi_n)$-upper Schur-convex, then $f$ is $(M, \psi)$-upper Jensen-convex;

(iii) If for some $n \geq 2$ the function $F_n$ defined by (8) is $(m, M, \Psi_n)$-Schur-convex, then $f$ is $(m, M, \psi)$-Jensen–convex.
Proof. To prove (i) take $y_1, y_2 \in D$ and put $x_1 = x_2 = \frac{1}{2}(y_1 + y_2)$. Consider the points

$$y = (y_1, y_2, y_2, \ldots, y_2), \quad x = (x_1, x_2, y_2, \ldots, y_2)$$

(if $n = 2$, then we take $y = (y_1, y_2), \ x = (x_1, x_2)$). One can check easily that $x \preceq y$. Therefore, by (6),

$$F_n(x) \leq F_n(y) - m(\Psi_n(y) - \Psi_n(x)),$$

that is

$$2f\left(\frac{y_1 + y_2}{2}\right) \leq f(y_1) + f(y_2) - m(\psi(y_1) + \psi(y_2) - 2\psi\left(\frac{y_1 + y_2}{2}\right)).$$

Hence, for $g = f - m\psi$ we have

$$2g\left(\frac{y_1 + y_2}{2}\right) = 2f\left(\frac{y_1 + y_2}{2}\right) - 2m\psi\left(\frac{y_1 + y_2}{2}\right)$$

$$\leq f(y_1) + f(y_2) - m\left((\psi(y_1) + \psi(y_2)) = g(y_1) + g(y_2),$$

which means that $f$ is $(m, \psi)$-lower Jensen-convex.

The proof of part (ii) is similar. Part (iii) follows from (i) and (ii). \qed

We say that a function $f : D \rightarrow \mathbb{R}$ is $(m, \psi)$-lower Wright-convex ($(M, \psi)$-upper Wright-convex) if the function $f - m\psi$ (the function $M\psi - f$) is Wright-convex, i.e. satisfies (2) with $c = 0$. We say that $f : D \rightarrow \mathbb{R}$ is $(m, M, \psi)$-Wright-convex if it is $(m, \psi)$-lower Wright-convex and $(M, \psi)$-upper Wright-convex.

As was shown above in Theorems 1 and 2, if a function $f : D \rightarrow \mathbb{R}$ is $(m, M, \psi)$-convex, then for every $n \geq 2$ the corresponding function $F_n$ defined by (8) is $(m, M, \Psi_n)$-Schur-convex and if for some $n \geq 2$ the function $F_n$ is $(m, M, \Psi_n)$-Schur-convex, then $f$ is $(m, M, \psi)$-Jensen-convex. The next theorem characterizes all the functions $f$ for which $F_n$ are $(m, M, \Psi_n)$-Schur-convex. It is a counterpart of the result of Ng [16] on functions generating Schur-convex sums.

Recall also that a subset $D$ of a vector space $X$ is said to be algebraically open if for every $x \in D$ and for every $y \in X$ there exists $\varepsilon > 0$ such that

$$\{ty + (1 - t)x \mid t \in (-\varepsilon, \varepsilon)\} \subset D.$$
(iii) If $f$ is $(m,M,\psi)$-Wright-convex, then for every $n \geq 2$ the function $F_n$ defined by (8) is $(m,M,\Psi_n)$-Schur-convex. Conversely, if for some $n \geq 2$ the function $F_n$ is $(m,M,\Psi_n)$-Schur-convex, then $f$ is $(m,M,\psi)$-Wright-convex.

Proof. To prove (i) assume that $f$ is $(m,\psi)$-lower Wright-convex and fix an $n \geq 2$. Since the function $g = f - m\psi$ is Wright-convex, it is of the form $g = g_1 + a$, where $g_1$ is convex and $a$ is additive (cf. [11]; here the assumption that $D$ is algebraically open is needed). Therefore it generates Schur-convex sums. Thus, for $x = (x_1,\ldots,x_n) \preceq y = (y_1,\ldots,y_n)$, we have

$$g(x_1) + \cdots + g(x_n) \leq g(y_1) + \cdots + g(y_n).$$

Hence

$$f(x_1) + \cdots + f(x_n) - m(\psi(x_1) + \cdots + \psi(x_n)) \leq g(y_1) + \cdots + g(y_n) - m(\psi(y_1) + \cdots + \psi(y_n)),$$

which means that

$$F_n(x) \leq F_n(y) - m(\Psi_n(y) - \Psi_n(x)),$$

that is $F_n$ is $(m,\Psi_n)$-lower Schur-convex. Now, assume that for some $n \geq 2$ the function $F_n$ is $(m,\Psi_n)$-lower Schur-convex. Take $y_1, y_2 \in D$ and $t \in (0,1)$. Put

$$x_1 = ty_1 + (1-t)y_2, \quad x_2 = (1-t)y_1 + ty_2$$

and, if $n > 2$, take additionally $x_i = y_i = z \in D$ for $i = 3,\ldots,n$. Then $x = (x_1,\ldots,x_n) \preceq y = (y_1,\ldots,y_n)$. Therefore, by (6),

$$F_n(x) \leq F_n(y) - m(\Psi_n(y) - \Psi_n(x)),$$

that is

$$f(ty_1 + (1-t)y_2) + f((1-t)y_1 + ty_2) \leq f(y_1) + f(y_2) - m(\psi(y_1) + \psi(y_2)).$$

Hence, for $g = f - m\psi$ we get

$$g(ty_1 + (1-t)y_2) + g((1-t)y_1 + ty_2) = f(ty_1 + (1-t)y_2) + f((1-t)y_1 + ty_2) - m\psi(ty_1 + (1-t)y_2) - m\psi((1-t)y_1 + ty_2) \leq f(y_1) + f(y_2) - m\psi(y_1) - m\psi(y_2) = g(y_1) + g(y_2).$$

Thus $g$ is Wright-convex, which means that $f$ is $(m,\psi)$-lower Wright-convex. The proof of part (ii) is similar. Part (iii) follows from (i) and (ii). \qed

Remark 6. In the special case where $(X,\| \cdot \|)$ is an inner product space, $\psi = \| \cdot \|^2$ and $m = c > 0$, parts (i) of the above Theorems 1, 4, 5 reduce to the results obtained in [20] for strong Schur-convexity. For $m = 0$ and $X = \mathbb{R}^n$ they coincide with the Ng theorem [16].
Finally, we give a representation theorem for \((m, M, \psi)\)-Wright-convex functions. It is known (and easy to check) that every convex function is Wright-convex, and every Wright-convex function is Jensen-convex, but not the converse (some examples can be found in [18]). In [16] Ng proved that a function \(f\) defined on a convex subset of \(\mathbb{R}^n\) is Wright-convex if and only if it can be represented in the form \(f = f_1 + a\), where \(f_1\) is a convex function and \(a\) is an additive function (see also [18]). Kominek [11] extended that result to functions defined on algebraically open subset of a vector space. An analogous result for strongly Wright-convex functions was obtained in [14]. In the next theorem we give a similar representation for \((m, M, \psi)\)-Wright-convex functions. In the proof we will use the following fact:

**Lemma 7.** Assume that \(f, g : D \to \mathbb{R}\) are convex functions, \(a : X \to \mathbb{R}\) is additive and \(a(x) = f(x) - g(x)\) for all \(x \in D\). Then \(a\) is an affine function on \(D\).

**Proof.** Fix \(x, y \in D\) and consider the function \(\varphi : [0, 1] \to \mathbb{R}\) defined by

\[
\varphi(s) = a(sx + (1 - s)y) = f(sx + (1 - s)y) - g(sx + (1 - s)y), \quad s \in [0, 1].
\]

As a difference of convex functions on \([0, 1]\), \(\varphi\) is continuous on \((0, 1)\). Fix any \(t \in (0, 1)\) and take a sequence \((q_n)\) of rational numbers in \((0, 1)\) tending to \(t\). By the additivity of \(a\) we have

\[
a(q_n x + (1 - q_n)y) = q_n a(x) + (1 - q_n)a(y),
\]

whence

\[
\varphi(q_n) = q_n a(x) + (1 - q_n)a(y).
\]

Going to the limit we get

\[
\varphi(t) = ta(x) + (1 - t)a(y).
\]

Hence

\[
a(tx + (1 - t)y) = ta(x) + (1 - t)a(y),
\]

which proves that \(a\) is affine on \(D\). \(\square\)

**Theorem 8.** Let \(f : D \to \mathbb{R}\), where \(D\) is an algebraically open convex subset of a vector space \(X\). Then:

(i) \(f\) is \((m, \psi)\)-lower Wright-convex if and only if \(f = g_1 + a_1\), where \(g_1 \in L(D, m, \psi)\) and \(a_1 : X \to \mathbb{R}\) is additive;

(ii) \(f\) is \((M, \psi)\)-upper Wright-convex if and only if \(f = g_2 + a_2\), where \(g_2 \in U(D, M, \psi)\) and \(a_2 : X \to \mathbb{R}\) is additive;

(iii) \(f\) is \((m, M, \psi)\)-Wright-convex if and only if \(f = g + a\), where \(g \in B(D, m, M, \psi)\) and \(a : X \to \mathbb{R}\) is additive.
Proof. To prove (i) assume first that $f$ is $(m, \psi)$-lower Wright-convex, that is $h = f - m\psi$ is Wright-convex. By the Ng representation theorem [16] (extended by Kominek [11] to functions defined on algebraically open domains), there exist a convex function $h_1 : D \to \mathbb{R}$ and an additive function $a_1 : X \to \mathbb{R}$ such that $h = h_1 + a_1$ on $D$. Then $g_1 = h_1 + m\psi$ belongs to $\mathcal{L}(D, m, \psi)$ and $f = h + m\psi = h_1 + a_1 + m\psi = g_1 + a_1,$ which was to be proved. Conversely, if $f = g_1 + a_1$ with some $g_1 \in \mathcal{L}(D, m, \psi)$ and $a_1$ additive, then $f - m\psi = g_1 - m\psi + a_1$ is Wright-convex as a sum of a convex function and an additive function. This shows that $f$ is $(m, \psi)$-lower Wright-convex.

The proof of part (ii) is analogous.

Part (iii). If $f = g + a$, where $g \in \mathcal{B}(D, m, M, \psi)$ and $a : X \to \mathbb{R}$ is additive, then, by (i) and (ii) $f$ is $(m, \psi)$-lower Wright-convex and $(M, \psi)$-upper Wright-convex. Consequently, it is $(m, M, \psi)$-Wright-convex.

The proof in the opposite direction is more delicate. If $f$ is $(m, M, \psi)$-Wright-convex, then $f - m\psi$ and $M\psi - f$ are Wright-convex. Then

$$f - m\psi = h_1 + a_1 \quad \text{and} \quad M\psi - f = h_2 + a_2$$

with some convex functions $h_1, h_2$ and additive functions $a_1, a_2$. Hence

$$a_1 + a_2 = (M - m)\psi - (h_1 + h_2)$$

which, by Lemma 5, implies that $A = a_1 + a_2$ is affine. Denote $a = a_1$ and $g = f - a$. Then

$$g - m\psi = f - a - m\psi = h_1,$$

which implies that $g \in \mathcal{L}(D, m, \psi)$ because $h_1$ is convex. Also

$$M\psi - g = M\psi - f + a = h_2 + a_2 + a = h_2 + A,$$

which implies that $g \in \mathcal{U}(D, m, \psi)$ because $h_2 + A$ is convex. Thus $g \in \mathcal{B}(D, m, \psi)$ and $f = g + a$, which finishes the proof. \qed}

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