A $p$-ADIC GENERALIZED GIBBS MEASURE FOR THE ISING MODEL ON A CAYLEY TREE

M. M. Rahmatullaev,*† O. N. Khakimov,† and A. M. Tukhtaboev‡

We consider a $p$-adic Ising model on the Cayley tree of order $k \geq 2$. We completely describe all $p$-adic translation-invariant generalized Gibbs measures for $k = 3$. Moreover, we show the existence of a phase transition for the $p$-adic Ising model for any $k \geq 3$ if $p \equiv 1 \pmod{4}$.

Keywords: $p$-adic number, Ising model, Gibbs measure, phase transition

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1. Introduction

A central problem in statistical mechanics is to study infinite-volume Gibbs measures corresponding to a given Hamiltonian. This problem includes studying phase transitions, which occur for a Hamiltonian if there exist at least two distinct Gibbs measures. But a complete analysis of the set of all Gibbs measures for a specific Hamiltonian is often a difficult problem. Therefore, most work on this topic is devoted to studying Gibbs measures on Cayley trees [1], [2].

It is known [3], [4] that $p$-adic models in physics cannot be described with the usual probability theory. In [3], an abstract $p$-adic probability theory was developed using the theory of non-Archimedean measures. Probabilistic processes in the field of $p$-adic numbers have been studied by many authors (see [5]–[7]). A non-Archimedean analogue of Kolmogorov’s theorem was proved in [8].

We note that $p$-adic Gibbs measures have been studied for several $p$-adic models of statistical mechanics [9]–[22]. It was proved in [23] that for a $\lambda$-model on the Cayley tree of order $k$, there is no phase transition. We recall [11] that the Ising model is a special case of the $\lambda$-model. A new set of real-valued Gibbs measures for the Ising model on the Cayley tree was constructed in [24], [25]. New non-translation-invariant ($k_0$)-translation-invariant) Gibbs measures were constructed in [26] for the Ising model on the Cayley tree of order $k$.

Here, we study the existence a phase transition for a $p$-adic Ising model on the Cayley tree of order three. Using the Akin–Rozikov–Temir (ART) construction, we derive new $p$-adic generalized Gibbs measures for the Ising model based on the results in [20] and show that the model has a phase transition for arbitrary $k \geq 3$.

*Namangan State University, Namangan, Uzbekistan, e-mail: mrahmatullaev@rambler.ru.
†Institute of Mathematics, Tashkent, Uzbekistan, e-mail: hakimovo@mail.ru.
‡Namangan Construction Institute, Namangan, Uzbekistan, e-mail: akbarxoja.toxtaboyev@mail.ru.

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2. Preliminaries

2.1. p-Adic numbers and measures. Let \( \mathbb{Q} \) be the field of rational numbers. For a fixed prime number \( p \), every rational number \( x \neq 0 \) can be represented in the form \( x = p^r n/m \), where \( r, n \in \mathbb{Z}, m \) is a positive integer, and \( n \) and \( m \) are relatively prime to \( p \). The \( p \)-adic norm of \( x \) is given by

\[
|x|_p = \begin{cases} 
  p^{-r}, & x \neq 0, \\
  0, & x = 0.
\end{cases}
\]

This norm is non-Archimedean, i.e., it satisfies the strong triangle inequality \( |x + y|_p \leq \max\{|x|_p, |y|_p\} \) for all \( x, y \in \mathbb{Q} \). From this property, it immediately follows that

1. if \( |x|_p \neq |y|_p \), then \( |x - y|_p = \max\{|x|_p, |y|_p\} \), and

2. if \( |x|_p = |y|_p \), then \( |x - y|_p = |x|_p \).

The completion of \( \mathbb{Q} \) with respect to the \( p \)-adic norm defines the \( p \)-adic field \( \mathbb{Q}_p \) (see [27]). The completion of the field of rational numbers \( \mathbb{Q} \) is either the real number field \( \mathbb{R} \) or one of the \( p \)-adic number fields \( \mathbb{Q}_p \) (Ostrowski’s theorem).

Any \( p \)-adic number \( x \neq 0 \) can be uniquely represented in the canonical form

\[
x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \ldots),
\]

where \( \gamma = \gamma(x) \in \mathbb{Z} \) and the integers \( x_j \) satisfy \( x_0 > 0, 0 \leq x_j \leq p - 1 \) (see [27], [28], [4]). In this case, \( |x|_p = p^{-\gamma(x)} \).

We have the following theorem and corollary [4].

**Theorem 2.1.** The equation \( x^2 = a, 0 \neq a, a = p^{\gamma(a)}(a_0 + a_1 p + a_2 p^2 + \ldots) \), \( a_0 > 0, 0 \leq a_j \leq p - 1 \), has a solution in \( x \in \mathbb{Q}_p \) iff

1. \( \gamma(a) \) is even and

2. \( x^2 \equiv a_0 \pmod{p} \) is solvable if \( p \neq 2 \) and the equality \( a_1 = a_2 = 0 \) is satisfied if \( p = 2 \).

**Corollary 2.1.** The equation \( x^2 = -1 \) has a solution in \( \mathbb{Q}_p \) iff \( p \equiv 1 \pmod{4} \).

For \( a \in \mathbb{Q}_p \) and \( r > 0 \), we set

\[
B(a, r) = \{ x \in \mathbb{Q}_p : |x - a|_p \leq r \}.
\]

The \( p \)-adic logarithm is defined by the series

\[
\log_p(x) = \log_p(1 + (x - 1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x - 1)^n}{n!},
\]

which converges for \( x \in B(1, 1) \). The \( p \)-adic exponential is defined by

\[
\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},
\]

which converges for \( x \in B(0, p^{-1/(p-1)}) \).

We set

\[
\mathcal{E}_p = \{ x \in \mathbb{Q}_p : |x - 1|_p < p^{-1/(p-1)} \}.
\]

This set is the range of the \( p \)-adic exponential function. The following fact is known [19].
Lemma 2.1. Let $p \geq 3$. Then the set $\mathcal{E}_p$ has the properties that

1. $\mathcal{E}_p$ is a group under multiplication,
2. $|a - b|_p < 1$ for all $a, b \in \mathcal{E}_p$,
3. $|a + b|_p = 1$ for all $a, b \in \mathcal{E}_p$, and
4. if $a \in \mathcal{E}_p$, then there is an element $h \in B_{p^{-1}(p-1)}(0)$ such that $a = \exp_p(h)$.

More details of p-adic calculus and p-adic mathematical physics can be found in [27], [28], [4].

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets $X$. A function $\mu: \mathcal{B} \to \mathbb{Q}_p$ is a p-adic measure if for any $A_1, A_2, \ldots, A_n \in \mathcal{B}$ such that $A_i \cap A_j = \emptyset$, $i \neq j$,

$$
\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).
$$

A p-adic measure $\mu$ is said to be bounded if $\sup_{A \in \mathcal{B}} |\mu(A)|_p < \infty$ (see [3]). A p-adic measure is said to be probabilistic if $\mu(X) = 1$ [8].

2.2. Cayley tree. The Cayley tree $\Gamma^k$ of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $V$ denote the set of vertices and $L$ denote the set of edges of the Cayley tree $\Gamma^k$. Two vertices $x$ and $y$ are nearest neighbors, denoted by $l = (x, y)$, if there exists an edge $l \in L$ connecting them.

We fix $x_0 \in \Gamma^k$. For a given vertex $x$, let $|x|$ denote the number of edges in the shortest path connecting $x_0$ and $x$. For $x, y \in \Gamma^k$, let $d(x, y)$ denote the number of edges in the shortest path connecting $x$ and $y$. For $x, y \in \Gamma^k$, we write $x \leq y$ if $x$ belongs to the shortest path connecting $x_0$ with $y$, and we write $x < y$ if $x \leq y$ and $x \neq y$. If $x \leq y$ and $|y| = |x| + 1$, then we write $x \to y$. We call vertex $x_0$ the root of the Cayley tree $\Gamma^k$.

We set

$$
W_n = \{x \in V: |x| = n\}, \quad V_n = \{x \in V: |x| \leq n\}, \quad L_n = \{l = (x, y) \in L: x, y \in V_n\},
$$

$$
S(x) = \{y \in V: x \to y\}, \quad S_1(x) = \{y \in V: d(x, y) = 1\}.
$$

The set $S(x)$ is called the set of direct successors of the vertex $x$.

3. Construction of p-adic Gibbs measures for the Ising model

We consider a p-adic Ising model on the Cayley tree $\Gamma^k$. Let $\mathbb{Q}_p$ be a field of $p$-adic numbers and $\Phi = \{-1, 1\}$. A configuration $\sigma$ on $V$ is defined by a function $V \ni x \to \sigma(x) \in \Phi$. Similarly, we can define the respective configurations $\sigma_n$ and $\sigma^{(n)}$ on $V_n$ and $W_n$. The set of all configurations on $V$, $V_n$, or $W_n$ is respectively denoted by $\Omega = \Phi^V$, $\Omega_{V_n} = \Phi^{V_n}$, or $\Omega_{W_n} = \Phi^{W_n}$.

For given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\varphi^{(n)} \in \Omega_{W_n}$, we define a configuration in $\Omega_{V_n}$ as

$$
(\sigma_{n-1} \lor \varphi^{(n)})(x) = \begin{cases} 
\sigma_{n-1}(x), & x \in V_{n-1}, \\
\varphi^{(n)}(x), & x \in W_n.
\end{cases}
$$

A formal p-adic Hamiltonian $H: \Omega \to \mathbb{Q}_p$ of the Ising model is defined as

$$
H(\sigma) = J \sum_{(x, y) \in L} \sigma(x)\sigma(y), \quad (3.1)
$$
where $|J|_p < p^{-1/(p-1)}$ for any $(x, y) \in L$.

Let $h : x \to h_x \in \mathbb{Q}_p \setminus \{0\}$ be a $p$-adic function on $V$. We consider a $p$-adic probability distribution $\mu_h^{(n)}$ on $\Omega_{V_n}$ defined as

$$\mu_h^{(n)}(\sigma_n) = Z_{n,h}^{-1} \exp_p \{H_n(\sigma_n)\} \prod_{x \in W_n} h_x^{(\sigma_n)}, \quad n = 1, 2, \ldots,$$

(3.2)

where $Z_{n,h}$ is the normalizing constant

$$Z_{n,h} = \sum_{\varphi \in \Omega_{W_n}} \exp_p \{H_n(\varphi)\} \prod_{x \in W_n} h_x^{\varphi(x)}, \quad H_n(\sigma_n) = J \sum_{(x,y) \in L_n} \sigma_n(x)\sigma_n(y).$$

(3.3)

A $p$-adic probability distribution $\mu_h^{(n)}$ is said to be consistent if for all $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$, we have

$$\sum_{\varphi \in \Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \lor \varphi) = \mu_h^{(n-1)}(\sigma_{n-1}).$$

(3.4)

In this case, by the $p$-adic analogue of Kolmogorov’s theorem [8], there exists a unique measure $\mu_h$ on the set $\Omega$ such that $\mu_h(\{\sigma | V_n \equiv \sigma_n\}) = \mu_h^{(n-1)}(\sigma_{n-1})$ for all $n$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$.

The measure $\mu_h$ is called a $p$-adic generalized Gibbs measure corresponding to the function $h : x \to h_x \in \mathbb{Q}_p$ if the restriction of $\mu_h$ to $V_n$ is measure (3.2). We note that if $h_x \in \mathcal{P}$ for all $x \in V$, then the corresponding measure is called a $p$-adic Gibbs measure. We say that a phase transition occurs for a given Hamiltonian if there exist at least two measures such that one of them is unbounded and the other is bounded. If there are two different measures that are both bounded, then we say that there is a quasi-phase transition [15].

We have the following proposition [20].

**Proposition 3.1.** A sequence of $p$-adic probability distributions $\mu_h^{(n)}$, $n = 1, 2, \ldots$, determined by formula (3.2) is consistent iff for any $x \in V \setminus \{x_0\}$, we have the equality

$$h_x^2 = \prod_{y \in S(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta}, \quad \theta = \exp_p \{2J\}.$$  

(3.5)

**Remark 3.1.** It is easy to see that if a function $h_x$ is a solution of Eq. (3.5), then the function $-h_x$ is also a solution. If we consider the Ising model on the Cayley tree of order $k$, then these solutions define the same measure $\mu_h$.

### 4. $p$-Adic translation-invariant generalized Gibbs measure

We consider $p$-adic translation-invariant generalized Gibbs measures for the Ising model on the Cayley tree $\Gamma^3$. Because of Proposition 3.1, to find all translation-invariant measures, it suffices to consider the equation

$$h_x^2 = \left(\frac{\theta h_y^2 + 1}{h_y^2 + \theta}\right)^3.$$  

(4.1)

Setting $z = h^2$, from this equation, we obtain

$$z = \left(\frac{\theta z + 1}{z + \theta}\right)^3,$$  

(4.2)
which is equivalent to
\[(z^2 - 1)(z^2 + (3\theta - \theta^3)z + 1) = 0.\]  \hspace{1cm} \text{(4.3)}

This equation has at least two solutions \( z_1 = 1 \) and \( z_2 = -1 \). If there exists \( \sqrt{\theta^2 - 4} \in \mathbb{Q}_p \), then Eq. (4.3) has exactly four solutions, which are \( z_1, z_2, \) and
\[
z_3 = \frac{\theta^3 - 3\theta + (\theta^2 - 1)\sqrt{\theta^2 - 4}}{2}, \quad z_4 = \frac{\theta^3 - 3\theta - (\theta^2 - 1)\sqrt{\theta^2 - 4}}{2}.
\]

We define \( \Delta(\theta) = \theta^2 - 4 \).

**Lemma 4.1.** A number \( \sqrt{\Delta(\theta)} \) exists in \( \mathbb{Q}_p \) if and only if \( p \equiv 1 \) (mod 6).

**Proof.** Because \( \theta \in \mathbb{F}_p \), by virtue of Lemma 2.1, we have \( |\theta^2 - 1|_p < 1 \), which yields \( |\Delta(\theta) + 3|_p < 1 \).

Then according to Theorem 2.1, a number \( \sqrt{\Delta(\theta)} \) exists in \( \mathbb{Q}_p \) iff \( \sqrt{\theta^2 - 3} \in \mathbb{Q}_p \). Again because of Theorem 2.1, the existence of \( \sqrt{\theta^2 - 3} \) is equivalent to the solvability of \( x^2 + 3 \equiv 0 \) (mod \( p \)). It is known (see [29]) that the congruence \( x^2 + 3 \equiv 0 \) (mod \( p \)) is solvable iff \( p \equiv 1 \) (mod 6).

Because \( z_3z_4 = 1 \), we can conclude that the existence of \( \sqrt{z_3} \) implies the existence of \( \sqrt{z_4} \in \mathbb{Q}_p \). Therefore, to describe the set of all \( p \)-adic translation-invariant generalized Gibbs measures for the homogeneous Ising model, it suffices to check the existence of \( \sqrt{z_3} \).

**Lemma 4.2.** The number \( \sqrt{z_3} \) exists in \( \mathbb{Q}_p \) iff \( p \equiv 1 \) (mod 12).

**Proof.** We assume that \( z_3 \) is a solution of (4.3). Using Lemma 4.1, we then obtain \( p \equiv 1 \) (mod 6) and
\[
z_3 + 1 = \frac{\theta^3 - 3\theta + 2 + (\theta^2 - 1)\sqrt{\Delta(\theta)}}{2} = \frac{(\theta - 1)(\theta^2 + \theta - 2 + (\theta + 1)\sqrt{\Delta(\theta)})}{2}.
\]

Because \( p \neq 2 \) and \( |\theta - 1|_p < 1 \), we have \( |z_3 + 1|_p < 1 \). Again keeping in mind that \( p \neq 2 \) and using Theorem 2.1, we find that the existence of \( \sqrt{z_3} \) is equivalent to the existence of the number \( \sqrt{\theta^2 - 4} \). Using Corollary 2.1, we then conclude that \( \sqrt{z_3} \in \mathbb{Q}_p \) iff \( p \equiv 1 \) (mod 4). After combining \( p \equiv 1 \) (mod 6) and \( p \equiv 1 \) (mod 4), we obtain \( p \equiv 1 \) (mod 12), which is the necessary and sufficiently condition for the existence of \( \sqrt{z_3} \in \mathbb{Q}_p \).

**Proposition 4.1.** Let \( \mathcal{N}_p \) be the number of solutions of (4.1). Then
\[
\mathcal{N}_p = \begin{cases} 
2, & p \not\equiv 1 \pmod{4}, \\
4, & p \equiv 1 \pmod{4}, p \not\equiv 1 \pmod{3}, \\
8, & p \equiv 1 \pmod{12}.
\end{cases}
\]

**Proof.** We suppose that \( p \not\equiv 1 \pmod{4} \). By Corollary 2.1, we then have \( \sqrt{z_2} \not\in \mathbb{Q}_p \). Moreover, according to Lemma 4.2, we have \( \sqrt{z_3}, \sqrt{z_4} \not\in \mathbb{Q}_p \). Hence, in this case, Eq. (4.1) has exactly two solutions: \( h_1 = \sqrt{z_2} \) and \( -h_1 \).

We now suppose that \( p \equiv 1 \pmod{4} \) and \( p \not\equiv 1 \pmod{3} \). Again by Corollary 2.1, there then exists \( \sqrt{z_2} \in \mathbb{Q}_p \), and by Lemma 4.2, we obtain \( z_3, z_4 \not\in \mathbb{Q}_p \). Consequently, (4.1) has exactly four solutions: \( \pm h_1, h_2 = \sqrt{z_2}, \) and \( -h_2 \).

We suppose that \( p \equiv 1 \pmod{12} \). In this case, by Corollary 2.1 and Lemma 4.2, there exist \( \sqrt{z_2}, \sqrt{z_3}, \) and \( \sqrt{z_4} \) in \( \mathbb{Q}_p \), which imply that Eq. (4.1) has exactly eight solutions: \( \pm h_1, \pm h_2, h_3 = \sqrt{z_3}, \) and \( -h_3 \).
We let $p$-TIGGM$(H)$ denote the set of all $p$-adic translation-invariant generalized Gibbs measures for a Hamiltonian $H$. The cardinality of a set $A$ is denoted by $|A|$.

**Theorem 4.1.** Let $H$ be the Ising model on the Cayley tree of order three. Then

$$|p\text{-TIGGM}(H)| = \begin{cases} 
1, & p \notin 1 \pmod{4}, \\
2, & p \equiv 1 \pmod{4}, \ p \equiv 2 \pmod{3}, \\
4, & p \equiv 1 \pmod{12},
\end{cases}$$

**Proof.** The theorem follows from Proposition 3.1, Remark 3.1, and Proposition 4.1.

We now study the boundedness of $p$-adic translation-invariant generalized Gibbs measures $\mu_{h_i}$, $i = 1, 2, 3, 4$. We need some auxiliary lemmas. The first of them was proved in [20].

**Lemma 4.3.** Let $h$ be a translation-invariant solution of Eq. (4.1) and $\mu_h$ be the corresponding $p$-adic translation-invariant generalized Gibbs measure. Then for the normalizing constant $Z_{n,h}$, we have

$$Z_{n+1,h} = A_{n,h} Z_{n,h}, \quad (4.4)$$

where

$$A_{n,h} = \left( \frac{\left( \theta h^2 + 1 \right) \left( h^2 + \theta \right)}{\theta h^2} \right)^{2^{n-1}}.$$

**Lemma 4.4.** The norms of the solutions $h_i$, $i = 1, 2, 3, 4$, are equal to one.

**Proof.** Because $|z_i|_p = 1$, we can immediately calculate $|\sqrt{z_i}|_p = 1$ for each $i = 1, 2, 3, 4$.

**Lemma 4.5.** For the normalizing constants $Z_{n+1,h_i}$, $i = 1, 2, 3, 4$, we have

$$|Z_{n+1,h_i}|_p = \begin{cases} 
1, & p \neq 2, \\
2^{-2^n+2}, & p = 2,
\end{cases} \quad |Z_{n+1,h_i}|_p \leq |\theta - 1|_p^{2^n-2} \quad \text{for } i = 2, 3, 4.$$

**Proof.** For the normalizing constant $Z_{n,h_1}$ in (4.4), we obtain

$$Z_{n,h_1} = A_{n-1,h_1} A_{n-2,h_1} \cdots A_{1,h_1} Z_{1,h_1} = \left( \frac{\left( \theta + 1 \right)^2}{\theta} \right)^{2^{n-1}-1}.$$

Because $\theta \in \mathcal{E}_p$, we have $|Z_{n+1,h_i}|_p = 1$ if $p \neq 2$ and $|Z_{n+1,h_i}|_p = 2^{-2^n+2}$ if $p = 2$.

Let $h_2$ be a solution of (4.1). Then by virtue of the non-Archimedean norm property, we have

$$Z_{n,h_2} = \left( \frac{\left( \theta h_2^2 + 1 \right) \left( h_2^2 + \theta \right)}{\theta h_2^2} \right)^{2^{n-1}-1} = \left( \frac{\left( \theta - 1 \right)^2}{\theta} \right)^{2^{n-1}-1}.$$}

We hence have $|Z_{n+1,h_1}|_p = |\theta - 1|_p^{2^n-2}$.

Let $h_3$ be a solution of (4.1). Then we have

$$|h_3^2 + 1|_p = |z_3 + 1|_p = \left| \frac{\theta^3 - 3\theta + (\theta^2 - 1) \sqrt{\Delta(\theta)}}{2} + 1 \right|_p = \left| \frac{(\theta - 1)(\theta - 1)(\theta + 2) + (\theta + 1) \sqrt{\Delta(\theta)}}{2} \right|_p = |\theta - 1|_p.$$
Using the strong triangle inequality, we then obtain
\[ Z_{n,h_3} = \left( \frac{\theta h_3^2 + 1}{\theta h_3^2 + \theta} \right)^{2n-1-1} = \left( \frac{\theta(z_3 + 1) + 1 - \theta}{\theta z_3} \right)^{2n-1-1}. \]

Again noting that \( \theta \in \mathcal{E}_p \) and \( |h_3|_p = 1 \) (see Lemma 4.5), we can obtain \( |Z_{n+1,h_3}|_p \leq |\theta - 1|_p^{2n-2} \). Using \( z_3 z_4 = 1 \), we obtain
\[
Z_{n,h_4} = \left( \frac{\theta h_4^2 + 1}{\theta h_4^2 + \theta} \right)^{2n-1-1} = \left( \frac{\theta z_4 + 1}{\theta z_4} \right)^{2n-1-1} = \left( \frac{z_4^2 (\theta + z_4)(1 + \theta z_4)}{\theta z_4} \right)^{2n-1-1} = \left( \frac{\theta + h_4^2}{\theta h_4^2} \right)^{2n-1-1} = Z_{n,h_3}.
\]

Consequently, \( |Z_{n+1,h_4}|_p \leq |\theta - 1|_p^{2n-2} \).

**Theorem 4.2.** Let \( H \) be the Ising model on the Cayley tree of order three. Then the following statements hold:

1. If \( p = 2 \), then the unique \( p \)-adic translation-invariant generalized Gibbs measure \( \mu_{h_1} \) is unbounded.
2. If \( p \neq 2 \), then among the \( p \)-adic translation-invariant generalized Gibbs measures only \( \mu_{h_1} \) is bounded.

**Proof.** Let \( p = 2 \). In this case, by virtue of Theorem 4.1, there exists a unique \( p \)-adic translation-invariant generalized Gibbs measure \( \mu_{h_1} \). By Lemmas 4.4 and 4.5, we then have \( |\mu_{h_1}^{(n)}(\sigma)|_2 = 2^{2n-2} \) for all \( n \in \mathbb{N} \) and \( \sigma \in \Omega_{V_n} \). Hence, \( |\mu_{h_1}^{(n)}(\sigma)|_2 \to \infty \) as \( n \to \infty \). This means that the measure \( \mu_{h_1} \) is unbounded.

Let \( p \neq 2 \). According to Lemmas 4.4 and 4.5, we have \( |\mu_{h_1}^{(n)}(\sigma)|_2 = 1 \) for all \( n \in \mathbb{N} \) and \( \sigma \in \Omega_{V_n} \), which implies that the limit measure \( \mu_{h_1} \) is bounded. Moreover, if at least one of the measures \( \mu_{h_2}, \mu_{h_3}, \) and \( \mu_{h_4} \) exists, then again using Lemmas 4.4 and 4.5, we can verify that it is unbounded.

From Theorems 4.1 and 4.2, we obtain the following theorem.

**Theorem 4.3.** If \( p \equiv 1 \pmod{12} \), then a phase transition occurs for the Ising model on the Cayley tree of order three.

5. Existence of a phase transition for a \( p \)-adic Ising model on \( \Gamma^k \) for \( k \geq 3 \)

5.1. \( p \)-Adic ART generalized Gibbs measures. In [20], \( p \)-adic translation-invariant and periodic generalized Gibbs measures for the Ising model on the Cayley tree of order two were studied. In the preceding section, we showed that a phase transition occurs on the Cayley tree of order three. In this section, we describe new \( p \)-adic generalized Gibbs measures for the Ising model on the Cayley tree of order \( k \geq 3 \) using the ART method (see [24]).

We recall that each solution of (3.5) defines a \( p \)-adic generalized Gibbs measures for the Ising model on the Cayley tree of order \( k \geq 1 \). We can see that \( h_x = 1 \) for all \( x \in V \) is a solution of (3.5) for any \( k \geq 1 \). We now construct new solutions of (3.5) for \( k \geq 3 \). If \( k = 2 \), then all translation-invariant solutions of (3.5) can be found from
\[
h^2 = \left( \frac{\theta h^2 + 1}{h^2 + \theta} \right)^2. \tag{5.1}
\]
It was proved in [20] that the Eq. (5.1) has a unique solution if \( p \not\equiv 1 \pmod{4} \) and has exactly three solutions if \( p \equiv 1 \pmod{4} \). In what follows, we assume that \( p \equiv 1 \pmod{4} \). In this case, by virtue of the results in [20],

\[
\begin{align*}
\hat{h}_0^{(2)} &= 1, \\
\hat{h}_1^{(2)} &= \frac{\theta - 1 + \sqrt{(\theta - 3)(\theta + 1)}}{2}, \\
\hat{h}_2^{(2)} &= \frac{\theta - 1 - \sqrt{(\theta - 3)(\theta + 1)}}{2}
\end{align*}
\]  

(5.2)

are solutions of (5.1). For \( k \geq 3 \), we construct some solutions of (3.5) using these equations.

Let \( V^k \) be the set of all vertices of the Cayley tree \( \Gamma^k \). Because \( k > 2 \), we can regard \( V^2 \) as a subset of \( V^k \). We define the function

\[
\hat{\tilde{h}}_{x}^{(i)} = \begin{cases} 
\hat{h}_i^{(2)}, & x \in V^2, \\
1, & x \in V^k \setminus V^2,
\end{cases} 
\]

(5.3)

This function on the Cayley tree of order \( k = 3 \) is shown in Fig. 1.

We now verify that (5.3) satisfies (3.5) on \( \Gamma^k \). Let \( x \in V^2 \subset V^k \). For \( i = 1, 2 \), we have

\[
\begin{align*}
(\hat{\tilde{h}}_{x}^{(i)})^2 &= \prod_{y \in S(x) \cap V^2} \frac{\theta(\hat{h}_y^{(i)})^2 + 1}{(\hat{h}_y^{(i)})^2 + \theta} = \prod_{y \in S(x) \cap V^2} \frac{\theta(\hat{h}_y^{(i)})^2 + 1}{(\hat{h}_y^{(i)})^2 + \theta} \prod_{y \in S(x) \cap (V^k \setminus V^2)} \frac{\theta(\hat{h}_y^{(i)})^2 + 1}{(\hat{h}_y^{(i)})^2 + \theta} \\
&= \prod_{y \in S(x) \setminus V^2} \frac{\theta(\hat{h}_y^{(i)})^2 + 1}{(\hat{h}_y^{(i)})^2 + \theta} = \left( \frac{(\hat{h}_1^{(2)})^2 + 1}{(\hat{h}_1^{(2)})^2 + \theta} \right)^2 = (\hat{h}_1^{(2)})^2,
\end{align*}
\]

where we use

\[
\prod_{y \in S(x) \setminus (V^k \setminus V^2)} \frac{\theta(\hat{h}_y^{(i)})^2 + 1}{(\hat{h}_y^{(i)})^2 + \theta} = 1.
\]

If \( x \in V^k \setminus V^2 \), then it is easy to see that \( S_k(x) \subset V^k \setminus V^2 \). Therefore, we have

\[
(\hat{\tilde{h}}_{x}^{(i)})^2 = \prod_{y \in S(x)} \frac{\theta(\hat{h}_y^{(i)})^2 + 1}{(\hat{h}_y^{(i)})^2 + \theta} = 1.
\]

Therefore, \( \hat{\tilde{h}}_{x}^{(i)}, i = 1, 2 \), satisfies the functional equation (3.5). We let \( \mu_{\hat{\tilde{h}}_{x}^{(i)}} \) denote the corresponding Gibbs measures, which we call \( p \)-adic ART-generalized Gibbs measures. We thus obtain the result.
Theorem 5.1. Let \( p \equiv 1 \pmod{4} \). Then there exist at least three \( p \)-adic generalized Gibbs measures for the Ising model on a Cayley tree of order \( k \geq 3 \).

Remark 5.1. By Theorem 5.1, there are three \( p \)-adic generalized Gibbs measures \( \mu_{h_1}, \mu_{\tilde{h}(1)}, \) and \( \mu_{\tilde{h}(2)} \) on the Cayley tree of order \( k \geq 3 \). The proof that \( \mu_{h_1} \) is bounded and \( \mu_{\tilde{h}(1)} \) and \( \mu_{\tilde{h}(2)} \) are unbounded is similar to the proof of Theorem 5 in [20]. This means that there exists a phase transition for the Ising model on the Cayley tree of order \( k \geq 3 \) if \( p \equiv 1 \pmod{4} \).

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Conflicts of interest. The authors declare no conflicts of interest.

REFERENCES

1. P. M. Bleher, J. Ruiz, and V. A. Zagrebnov, “On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice,” J. Stat. Phys., 79, 473–482 (1995).
2. H.-O. Georgii, Gibbs Measures and Phase Transitions (De Gruyter Stud. Math., Vol. 9), Walter de Gruyter, Berlin (1988).
3. A. Yu. Khrennikov, Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems, and Biological Models, Kluwer, Dordrecht (1997).
4. V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, p-Adic Analysis and Mathematical Physics [in Russian], Nauka, Moscow (1994); English transl. (Series Sov. East Eur. Math., Vol. 10), World Scientific, Singapore (1994).
5. S. Albeverio and W. Karwowski, “A random walk on \( p \)-adics—the generators and its spectrum,” Stoch. Proc. Appl., 53, 1–22 (1994).
6. F. Mukhamedov, “On dynamical systems and phase transitions for \( q+1 \)-state \( p \)-adic Potts model on the Cayley tree,” Math. Phys. Anal. Geom., 16, 49–87 (2013).
7. U. A. Rozikov, Gibbs Measures on Cayley Trees, World Scientific, Singapore (2013).
8. N. N. Ganikhodjaev, F. M. Mukhamedov, and U. A. Rozikov, “Phase transitions in the Ising model on \( Z \) over the \( p \)-adic number field [in Russian],” Uzb. Mat. Zh., 4, 23–29 (1998).
9. G. Gandolfo, U. A. Rozikov, and J. Ruiz, “On \( p \)-adic Gibbs measures for hard core model on a Cayley tree,” Markov Process Related Fields, 18, 701–720 (2012).
10. O. N. Khakimov, “On \( p \)-adic Gibbs measures for Ising model with four competing interactions,” p-Adic Numbers Ultrametric Anal. Appl., 5, 194–203 (2013).
11. M. Khamsraev and F. M. Mukhamedov, “On \( p \)-adic \( \lambda \)-model on the Cayley tree,” J. Math. Phys., 45, 4025–4034 (2004).
12. A. Khrennikov, F. Mukhamedov, and J. F. F. Mendes, “On \( p \)-adic Gibbs measures of countable state Potts model on the Cayley tree,” Nonlinearity, 20, 2923–2937 (2007); arXiv:0710.0244v2 [math-ph] (2007).
13. A. Khrennikov and F. Mukhamedov, “On uniqueness of Gibbs measure for \( p \)-adic countable state Potts model on the Cayley tree,” Nonlinear Anal., 71, 5327–5331 (2009).
14. F. M. Mukhamedov and U. A. Rozikov, “On Gibbs measures of \( p \)-adic Potts model on the Cayley tree,” Indag. Math., 15, 85–100 (2004).
15. F. M. Mukhamedov, “On \( p \)-adic quasi Gibbs measures for \( q+1 \)-state Potts model on the Cayley tree,” p-Adic Numbers Ultrametric Anal. Appl., 2, 241–251 (2010).
16. F. Mukhamedov, “Existence of \( p \)-adic quasi Gibbs measure for countable state Potts model on the Cayley tree,” J. Inequal. Appl. Geom., 2012, 104 (2012).
17. F. Mukhamedov and H. Akin, “Phase transitions for \( p \)-adic Potts model on the Cayley tree of order three,” J. Stat. Mech., 2013, P07014 (2013).
18. F. Mukhamedov, “On strong phase transition for one dimensional countable state \( p \)-adic Potts model,” J. Stat. Mech., 2014, P01007 (2014).
19. O. N. Khakimov, “$p$-Adic Gibbs measures for the hard core model with three states on the Cayley tree,” Theor. Math. Phys., 177, 1339–1351 (2013).
20. O. N. Khakimov, “On a generalized $p$-adic Gibbs measure for Ising model on trees,” $p$-Adic Numbers Ultrametric Anal. Appl., 6, 207–217 (2014).
21. O. N. Khakimov, “$p$-Adic Gibbs quasimeasures for the Vannimenus model on a Cayley tree,” Theor. Math. Phys., 179, 395–404 (2014).
22. F. Mukhamedov and O. Khakimov, “On periodic Gibbs measures of $p$-adic Potts model on a Cayley tree,” $p$-Adic Numbers Ultrametric Anal. Appl., 8, 225–235 (2016).
23. M. Khamraev, F. Mukhamedov, and U. Rozikov, “On the uniqueness of Gibbs measures for $p$-adic nonhomogeneous $\lambda$-model on the Cayley tree,” Lett. Math. Phys., 70, 17–28 (2004).
24. H. Akin, U. A. Rozikov, and S. Temir, “A new set of limiting Gibbs measures for the Ising model on a Cayley tree,” J. Stat. Phys., 142, 314–321 (2011).
25. M. M. Rahmatullaev and U. A. Rozikov, “Ising model on Cayley trees: A new class of Gibbs measures and their comparison with known ones,” J. Stat. Mech., 2017, P093205 (2017).
26. M. Rahmatullaev, “Ising model on trees: ($k_0$)-Non translation-invariant Gibbs measures,” J. Phys.: Conf. Ser., 819, 012019 (2017).
27. N. Koblitz, $p$-Adic Numbers, $p$-Adic Analysis, and Zeta-Functions (Grad. Texts Math., Vol. 58), Springer, New York (1977).
28. W. H. Schikhof, Ultrametric Calculus: An Introduction to $p$-Adic Analysis (Cambridge Stud. Adv. Math., Vol. 4), Cambridge Univ. Press, Cambridge (1984).
29. K. H. Rosen, Elementary Number Theory and Its Applications, Addison-Westley, Reading, Mass. (1986).