Superintegrable systems in non-Euclidean plane: hidden symmetries leading to linearity

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Abstract

Nineteen classical superintegrable systems in two-dimensional non-Euclidean spaces are shown to possess hidden symmetries leading to their linearization. They are the two Perlick systems [A. Ballesteros, A. Enciso, F.J. Herranz and O. Ragnisco, Class. Quantum Grav. 25, 165005 (2008)], the Taub-NUT system [A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, and D. Riglioni, SIGMA 7, 048 (2011)], and all the seventeen superintegrable systems for the four types of Darboux spaces as determined in [E.G. Kalnins, J.M. Kress, W. Miller, P. Winternitz, J. Math. Phys. 44, 5811–5848 (2003)].

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1 Introduction

In [1] we have shown that all classical superintegrable systems (and their generalizations not necessary superintegrable) in two-dimensional real Euclidean space $E_2$\textsuperscript{2} possess hidden symmetries leading to their linearization, as well as the Tremblay-Turbiner-Winternitz system [3], and a superintegrable system that is separable in cartesian coordinates and admits a third-order integral of motion as derived by Gravel in [4]. Then, we have conjectured that also superintegrable systems in two-dimensional non-Euclidean space can be reduced to linear equations by means of their hidden symmetries. In this paper, we consider the two Perlick systems on two-dimensional non-Euclidean spaces [5], [6] [7], [8], the two-dimensional Taub-NUT system [9], [10], [11], and all the superintegrable systems for the four types of Darboux spaces as determined in [12], [13]. We show that they are all intrinsically linear by determining their hidden Lie symmetries. As in [14], [15], [16], [11], [17] we also make use of the reduction method [18]. More details on superintegrable systems and their hidden linearity have been described in [1]. In particular, it is regardless of the separability of the corresponding Hamilton-Jacobi equation as shown in [15] for Kepler problem in cartesian coordinates, and in [16] for a superintegrable system in $E_2$ that does not allow separation of variables [19].

2 Perlick Type I

We consider the so-called Hamiltonian of Perlik Type I [8], i.e.:

$$H_I = \frac{(1 + kr^2)^2}{2} \left( \frac{p_r^2 + \frac{p_0^2}{r^2}}{r^2} \right) + A \frac{1 - kr^2}{r}$$

(1)
that generates the Hamiltonian equations:

\[
\begin{align*}
\dot{r} &= p_r (1 + kr^2)^2, \\
\dot{\theta} &= \frac{p_\theta}{r^2} (1 + kr^2)^2, \\
\dot{p}_r &= \frac{(1 - kr^2) p_\theta^2 - 2kr^4 p_r^2 + Ar}{r^3} (1 + kr^2), \\
\dot{p}_\theta &= 0.
\end{align*}
\]

(2)

The last equation can be easily integrated to give \( p_\theta = w = \text{constant}. \) If we apply the reduction method developed in [18] to the three remaining equations of system (2), and choose \( \theta \) as a new independent variable \( y \), then we obtain the following two equations:

\[
\begin{align*}
\frac{dr}{dy} &= \frac{p_r r^2}{w}, \\
\frac{dp_r}{dy} &= \frac{(1 - kr^2) w^2 - 2kr^4 p_r^2 + Ar}{rw(1 + kr^2)}.
\end{align*}
\]

(3)

If we derive \( p_r \) from the first equation of system (3) and replace it into the second equation, then we obtain the following second-order equation in \( r \):

\[
\frac{d^2 r}{dy^2} = \frac{Ar^3 + (1 - kr^2) w^2 r^2 + 2w^2 \left( \frac{dr}{dy} \right)^2}{w^2 r (1 + kr^2)}.
\]

(4)

This equation admits an eight-dimensional Lie symmetry algebra isomorphic to \( \mathfrak{sl}(3, \mathbb{R}) \), and thus is linearizable. A two-dimensional abelian intransitive subalgebra is that generated by the two operators

\[
\begin{align*}
\Gamma_7 &= \cos(y) r^2 \frac{\partial}{1 + kr^2}, \\
\Gamma_8 &= \sin(y) r^2 \frac{\partial}{1 + kr^2},
\end{align*}
\]

(5)

that can be put into the canonical form [20] \( \partial \hat{r}, \hat{y} \partial \hat{r} \) by means of the transformation

\[
\hat{y} = \tan(y), \quad \hat{r} = \frac{kr - r^{-1} - A/w^2}{\cos(y)}.
\]

(6)

Then equation (4) becomes the free-particle equation

\[
\frac{d^2 \hat{r}}{d\hat{y}^2} = 0.
\]

Instead if we make the transformation of the dependent variable \( u = kr - r^{-1} - A/w^2 \) only, then equation (4) becomes the equation of the harmonic oscillator

\[
\frac{d^2 u}{d\hat{y}^2} = -u.
\]

3 Perlick Type II

We consider the so-called Hamiltonian of Perlik Type II [8], i.e.:

\[
H_{II} = \frac{(1 - \lambda^2 r^4)^2}{2(1 + \lambda^2 r^4 - 2\delta^2)} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) + \frac{Br^2}{1 + \lambda^2 r^4 - 2\delta^2}
\]

(7)
that generates the Hamiltonian equations:

\[
\begin{aligned}
\dot{r} &= \frac{p_r(1 - \lambda^2 r^4)^2}{1 + \lambda^2 r^4 - 2\delta r^2}, \\
\dot{\theta} &= \frac{p_\theta(1 - \lambda^2 r^4)^2}{r^2(1 + \lambda^2 r^4 - 2\delta r^2)}, \\
\dot{p}_r &= \frac{1 - \lambda^2 r^4}{r^3(1 + \lambda^2 r^4 - 2\delta r^2)^2} \left(2r^2 p_r^2 \left(\lambda^4 r^6 + 3\lambda^2 r^2 - \delta - 3\delta\lambda^2 r^4\right) + p_\theta^2 \left(1 + \lambda^4 r^8 + 6\lambda^2 r^4 - 4\delta\lambda^2 r^6 - 4\delta r^2\right) - 2Br^4\right), \\
\dot{p}_\theta &= 0.
\end{aligned}
\] (8)

The last equation can be easily integrated to give \(p_\theta = w = \text{constant} \). If we apply the reduction method developed in [21] to the three remaining equations of system (8), and choose \(\theta\) as a new independent variable \(y\), then we obtain the following two equations:

\[
\begin{aligned}
\frac{dr}{dy} &= \frac{p_r r^2}{w}, \\
\frac{dp_r}{dy} &= \frac{1}{w r (1 + \lambda^2 r^4 - 2\delta r^2)(1 - \lambda^2 r^4)} \left(2r^2 p_r^2 \left(\lambda^4 r^6 + 3\lambda^2 r^2 - \delta - 3\delta\lambda^2 r^4\right) + w^2 \left(1 + \lambda^4 r^8 + 6\lambda^2 r^4 - 4\delta\lambda^2 r^6 - 4\delta r^2\right) - 2Br^4\right),
\end{aligned}
\] (9)

If we derive \(p_r\) from the first equation of system (9) and replace it into the second equation, then we obtain the following second-order equation in \(r\):

\[
\frac{d^2 r}{dy^2} = \frac{w^2 \left(2 \left(\frac{dr}{dy}\right)^2 \left(\lambda^4 r^6 + 3\lambda^2 r^2 - \delta - 3\delta\lambda^2 r^4\right) + 1 + \lambda^4 r^8 + 6\lambda^2 r^4 - 4\delta\lambda^2 r^6 - 4\delta r^2\right) - 2Br^4}{w r (1 + \lambda^2 r^4 - 2\delta r^2)(1 - \lambda^2 r^4)},
\] (10)

that admits a three-dimensional symmetry algebra \(\text{sl}(2, \mathbb{R})\), unless \(B = 2w^2(\lambda^2 - \delta^2)\) in which case it admits an eight-dimensional Lie symmetry algebra \(\text{sl}(3, \mathbb{R})\) and thus it is linearizable. We now use the general method described in [21] and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra \(\text{sl}(2, \mathbb{R})\). If we solve equation (10) with respect to \(B\) and derive once with respect to \(y\), then we obtain the following third-order equation:

\[
\frac{d^3 r}{dy^3} = \frac{dr}{dy} \left(3r(3 + \lambda^2 r^4) \frac{d^2 r}{dy^2} - 12 \left(\frac{dr}{dy}\right)^2 - 4r^2(1 - \lambda^2 r^4)\right),
\] (11)

which admits a seven-dimensional Lie symmetry algebra, and therefore is linearizable. Indeed, the new dependent variable \(\tilde{r} = \frac{1 + \lambda^2 r^4}{2r^4}\) transforms equation (11) into the linear equation

\[
\frac{d^3 \tilde{r}}{dy^3} = -4 \frac{d\tilde{r}}{dy},
\]

which is once-derived linear harmonic oscillator with frequency equal to 2.

### 3.1 Taub-NUT

The following Taub-NUT Hamiltonian [10], [11]:

\[
H_{\text{TN}}(\eta) = \frac{1}{2} \frac{r}{\eta + r} \left(p_r^2 + \frac{1}{r^2} p_\theta^2\right) - \frac{\alpha}{\eta + r}
\] (12)
yields the Hamiltonian equations:

\[
\begin{align*}
\dot{r} &= \frac{rp_r}{\eta + r}, \\
\dot{\varphi} &= \frac{p_\varphi}{(\eta + r)r}, \\
\dot{p}_r &= - \frac{2(\alpha r - p_\varphi^2)}{2(\eta + r)^2} r + \eta \left( r^2 p_r^2 - p_\varphi^2 \right), \\
\dot{p}_\varphi &= 0.
\end{align*}
\] (13)

The last equation can be easily integrated to give \( p_\varphi = w_0 = \text{constant} \). If we apply the reduction method developed in [18] to the three remaining equations of system (13), and choose \( \varphi \) as a new independent variable \( y \), then we obtain two equations:

\[
\begin{align*}
\frac{dr}{dy} &= \frac{r^2}{w_0 p_r}, \\
\frac{dp_r}{dy} &= \frac{2(\alpha r - w_0^2) r + \eta (r^2 p_r^2 - w_0^2)}{2(\eta + r)rw_0}.
\end{align*}
\] (14)

Solving the first equation for \( p_r \) and substituting into the second yields:

\[
\frac{d^2u}{dy^2} = \frac{3\eta + 4u}{2u(\eta + u)} \left( \frac{du}{dy} \right)^2 - u \left( \frac{2\alpha u^2 - \eta u_0^2 - 2w_0^2 u}{2w_0^2 (\eta + u)} \right).
\] (15)

with \( u \equiv r \). This equation (15) admits a three-dimensional Lie symmetry algebra spanned by the following operators:

\[
\Theta_1 = \partial_y, \quad \Theta_2 = \cos(y)\partial_y + \frac{u(\eta + u)}{\eta} \sin(y)\partial_u, \quad \Theta_3 = \sin(y)\partial_y - \frac{u(\eta + u)}{\eta} \cos(y)\partial_u,
\] (16)

However if \( \alpha = 0 \) then the equation admits an eight-dimensional Lie symmetry algebra. Therefore if we solve equation (15) with respect to \( \alpha \) and derive once with respect to \( y \), then we get the following third-order equation:

\[
u^2 \frac{d^3u}{dy^3} + \frac{du}{dy} \left[ u^2 - 6u \frac{d^2u}{dy^2} + 6 \left( \frac{du}{dy} \right)^2 \right] = 0,
\] (17)

which is linearizable since it admits a seven-dimensional Lie symmetry algebra spanned by the following operators:

\[
\Pi_1 = \partial_y, \quad \Pi_2 = \cos(y)\partial_y + u \sin(y)\partial_u, \quad \Pi_3 = \sin(y)\partial_y - u \cos(y)\partial_u, \quad \Pi_4 = u\partial_u, \quad \Pi_5 = u^2\partial_u, \quad \Pi_6 = u^2 \cos(y)\partial_u, \quad \Pi_7 = u^2 \sin(y)\partial_u.
\] (18)

A two-dimensional Abelian intransitive subalgebra is that generated by the operators \( \Pi_6 \) and \( \Pi_7 \). If we put them into the canonical form \( \partial_U, Y \partial_U \), then the transformation

\[
Y = \frac{\sin(y)}{\cos(y)}, \quad U = -\frac{1}{u \cos(y)}
\] (19)

changes equation (15) into the following linear equation:

\[
\frac{d^3U}{dy^3} = -3 \frac{Y^2}{1 + Y^2} \frac{d^2U}{dy^2}.
\] (20)

Moreover, if we consider the transformation \( v = -\frac{1}{u} \) then equation (15) becomes the once-derived linear harmonic oscillator with frequency equal to 1, i.e.:

\[
\frac{d^3v}{dy^3} = -\frac{dv}{dy}.
\] (21)

This shows the connection between the Taub-NUT Hamiltonian (12) and the harmonic oscillator.
4 Darboux I

Three superintegrable systems were determined in [12], where the problem of superintegrability of the Hamiltonian
\[ H_{DI} = \frac{1}{4u} (p_u^2 + p_v^2) + V(u, v) \]  
was addressed, namely finding the potentials \( V(u, v) \) such that \( H_{DI} \) admits at least two extra quadratic integrals. We show that all of three systems have hidden symmetries that make them linear.

4.1 Case (1)

The Hamiltonian
\[ H_{DI1} = \frac{1}{4u} (p_u^2 + p_v^2) + b_1 \frac{4u^2 + v^2}{4u} + \frac{b_2}{u} + \frac{b_3}{uv^2} \]  
yields the Hamiltonian equations
\[
\begin{align*}
\dot{u} &= \frac{p_u}{2u}, \\
\dot{v} &= \frac{p_v}{2v}, \\
\dot{p}_u &= \frac{v^2(p_u^2 + p_v^2) + b_1v^2(u^2 - 4u^2) + 4b_2v^2 + 4b_3}{4u^2v^2}, \\
\dot{p}_v &= \frac{4b_3 - b_1v^4}{2uv^3}.
\end{align*}
\]  

If we apply the reduction method developed in [18] and choose \( v \) as a new independent variable \( y \), then we obtain the following three equations:
\[
\begin{align*}
\frac{du}{dy} &= \frac{p_u}{p_v}, \\
\frac{dp_u}{dy} &= \frac{y^2(p_u^2 + p_v^2) + b_1y^2(y^2 - 4u^2) + 4b_2y^2 + 4b_3}{2uy^2p_v}, \\
\frac{dp_v}{dy} &= \frac{4b_3 - b_1y^4}{y^3p_v}.
\end{align*}
\]  
The last equation can be easily integrated, i.e.:
\[ p_v = \pm \sqrt{\frac{2w_0b_1y^2 - b_1y^4 - 8w_0b_3y^2 - 4b_3}{y}}, \]  
with \( w_0 \) an arbitrary constant. Moreover, if we derive \( p_u \) from the first equation of system (25) and replace it into the second equation, then we obtain the following second-order equation in \( u \):
\[ \frac{d^2u}{dy^2} = \frac{1}{2u} \left( \frac{du}{dy} \right)^2 + \frac{u(b_1y^4 - b_3)}{2u(b_1y^4 - b_3)\frac{du}{dy} + y^3w_0b_1y^2 - b_1y^4 - 8w_0b_3y^2 - 4b_3}, \]  
that admits a three-dimensional symmetry algebra \( sl(2, \mathbb{R}) \), unless \( b_2 = 0 \) in which case it admits an eight-dimensional Lie symmetry algebra \( sl(3, \mathbb{R}) \) and thus it is linearizable. We now use the general method described in [21] and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra \( sl(2, \mathbb{R}) \). If we solve equation (27) with respect to \( b_2 \) and derive once with respect to \( y \), then we obtain the following linear third-order equation:
\[ \frac{d^3u}{dy^3} = \frac{(b_1y^4 - 4b_3)\left( \frac{du}{dy} - y \frac{d^2u}{dy^2} \right)}{y^2(2w_0b_1y^2 - b_1y^4 - 8w_0b_3y^2 - 4b_3)}. \]
4.2 Case (2)

The Hamiltonian

\[
\mathcal{H}_{DI2} = \frac{1}{4u} (p_u^2 + p_v^2) + \frac{a_1}{u} + \frac{a_2 v}{u} + \frac{a_3 u^2 + v^2}{u}
\]

yields the Hamiltonian equations

\[
\begin{align*}
\dot{u} &= \frac{p_u}{2u}, \\
\dot{v} &= \frac{p_v}{2u}, \\
\dot{p}_u &= \frac{p_u^2 + p_v^2 + 4a_1 + 4a_2v^2 - 4a_3(u^2 - v^2)}{4u^2}, \\
\dot{p}_v &= -\frac{a_2 + 2a_3v}{u}.
\end{align*}
\]  

If we apply the reduction method developed in [18] and choose \(v\) as a new independent variable \(y\), then we obtain the following three equations:

\[
\begin{align*}
\frac{du}{dy} &= \frac{p_u}{p_v}, \\
\frac{dp_u}{dy} &= \frac{4a_1 + 4a_2y - 4a_3u^2 + 4a_3y^2 + p_u^2 + p_v^2}{2ap_v}, \\
\frac{dp_v}{dy} &= -\frac{a_2 + 2a_3y}{p_v}.
\end{align*}
\]  

The last equation can be easily integrated, i.e.:

\[p_v = \pm 2\sqrt{a_2w_0 - a_2y - a_3y^2},\]

with \(w_0\) an arbitrary constant. Moreover, if we derive \(p_u\) from the first equation of system (31) and substitute it into the second equation, then we obtain the following second-order equation in \(u\):

\[
\frac{d^2u}{dy^2} = \frac{(a_2w_0 - a_2y - a_3y^2) \left( \frac{du}{dy} \right)^2 + (a_2 + 2a_3y)u \frac{du}{dy} - a_3u^2 + a_1 + a_2w_0}{2u(a_2w_0 - a_2y - a_3y^2)},
\]

that admits a three-dimensional symmetry algebra \(\text{sl}(2, \mathbb{R})\), unless \(a_1 + a_2w_0 = 0\) in which case it admits an eight-dimensional Lie symmetry algebra \(\text{sl}(3, \mathbb{R})\) and thus it is linearizable. We now use the general method described in [21] and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra \(\text{sl}(2, \mathbb{R})\). If we solve equation (33) with respect to \(a_1\) and derive once with respect to \(y\), then we obtain the following linear third-order equation:

\[
\frac{d^3u}{dy^3} = \frac{3(a_2 + 2a_3y)}{2(a_2w_0 - a_2y - a_3y^2)} \frac{d^2u}{dy^2}.
\]

4.3 Case (3)

The Hamiltonian

\[
\mathcal{H}_{DI3} = \frac{1}{4u} (p_u^2 + p_v^2) + \frac{a}{u}
\]

(35)
yields the Hamiltonian equations
\[
\begin{align*}
\dot{u} &= \frac{p_u}{2u}, \\
\dot{v} &= \frac{p_v}{2u}, \\
\dot{p}_u &= \frac{4a + p_u^2 + p_v^2}{4u^2}, \\
\dot{p}_v &= 0.
\end{align*}
\] (36)

The last equation can be easily integrated, i.e. \( p_v = w_0 \), with \( w \) an arbitrary constant. If we apply the reduction method developed in [18] and choose \( v \) as new independent variable, then system (36) reduces to the following two equations:
\[
\begin{align*}
\frac{du}{dy} &= \frac{p_u}{w_0}, \\
\frac{dp_u}{dy} &= \frac{4a + p_u^2 + w_0^2}{2uw_0}.
\end{align*}
\] (37)

If we derive \( p_u \) from the first equation of system (37) and replace it into the second equation, then we obtain the following second-order equation in \( u \):
\[
\frac{d^2u}{dy^2} = \frac{1}{2u} \left( \frac{du}{dv} \right)^2 + \frac{4a + w_0^2}{2w_0u},
\] (38)

that admits a three-dimensional symmetry algebra \( \text{sl}(2,\mathbb{R}) \), unless \( 4a + w_0^2 = 0 \) in which case it admits an eight-dimensional Lie symmetry algebra \( \text{sl}(3,\mathbb{R}) \) and thus it is linearizable. We now use the general method described in [21] and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra \( \text{sl}(2,\mathbb{R}) \). If we solve equation (38) with respect to \( a \) and derive once with respect to \( y \), then we obtain the following linear third-order equation:
\[
\frac{d^3u}{dy^3} = 0.
\] (39)

5 Darboux II

Four superintegrable systems were determined in [13], where the problem of superintegrability for the Hamiltonian
\[
H_{DII} = \frac{w_1^2}{w_1^2 + 1} \left( w_3^2 + w_4^2 \right) + V(w_1, w_2)
\] (40)

was addressed, namely finding the potentials \( V(w_1, w_2) \) such that \( H_{DII} \) admits at least two extra quadratic integrals.

5.1 Case (A)

The Hamiltonian
\[
H_{DIIA} = \frac{w_1^2}{w_1^2 + 1} \left( w_3^2 + w_4^2 + a_1 \left( \frac{w_1^2}{4} + w_2^2 \right) + a_2 w_2 + \frac{a_3}{w_1^2} \right)
\] (41)
yields the Hamiltonian equations

\[
\begin{align*}
\dot{w}_1 &= \frac{2w_1^2 w_3}{w_1^2 + 1}, \\
\dot{w}_2 &= \frac{2w_1^2 w_4}{w_1^2 + 1}, \\
\dot{w}_3 &= -w_1 \frac{4w_3^2 + 4w_4^2 + a_1 w_1^4 + 2a_1 w_1^2 + 4a_1 w_1^2 + 4a_2 w_2 - 4a_3}{2(w_1^2 + 1)^2}, \\
\dot{w}_4 &= -w_1^2 \frac{2a_1 w_2 + a_2}{w_1^2 + 1}.
\end{align*}
\]

(42)

If we apply the reduction method developed in [18] and choose \( w_1 \) as a new independent variable \( y \), then we obtain the following second-order equation in \( w_3 \):

\[
\begin{align*}
\frac{dw_2}{dy} &= \frac{w_4}{w_3}, \\
\frac{dw_3}{dy} &= -\frac{4w_3^2 + 4w_4^2 + a_1 y^4 + 2a_1 y^2 + 4a_1 w_1^2 + 4a_2 w_2 - 4a_3}{2yw_3(w_1^2 + 1)}, \\
\frac{dw_4}{dy} &= -\frac{2a_1 w_2 + a_2}{w_3}.
\end{align*}
\]

(43)

If we solve the second equation with respect to \( a_3 \) and then derive once with respect to \( y \), then we obtain the following second-order equation in \( w_3(y) \):

\[
w_3'' = -\frac{w_3'(y w_3' + 3w_3) + a_1 y}{y w_3},
\]

(44)

that admits an eight-dimensional Lie symmetry algebra \( \text{sl}(3, \mathbb{R}) \) and therefore it is linearizable. In this case, Lie canonical transformation is such:

\[
\dot{w}_3 = \frac{y^2 w_3}{2} + \frac{a_1 y^4}{8}, \quad \dot{y} = y^2 \implies \frac{d^2 \tilde{w}_3}{dy^2} = 0,
\]

(45)

and consequently

\[
w_3 = \pm \frac{\sqrt{8C_2 y^2 + 8C_1 - a_1 y^4}}{2y},
\]

(46)

with \( C_1, C_2 \) arbitrary integration constants, although only one is really arbitrary since there is a relationship between them and \( a_3 \). Then, if we solve the first equation in (43) with respect to \( w_4 \), and replace it into the third equation, we obtain the following linear second-order equation in \( w_2(y) \):

\[
w_2'' = -\frac{(a_1 y^4 + 8C_1)w_2' + 4a_1 y^3 w_2 + 2a_2 y^3}{a_1 y^3 - 8C_1 y - 8C_2 y^3},
\]

(47)

and its general solution is

\[
w_2 = (a_1 y^2 - 4C_2)C_3 + \sqrt{a_1 y^4 - 8C_1 - 8C_2 y^2}C_4 - \frac{a_2}{2a_1}.
\]

(48)

with \( C_3, C_4 \) arbitrary integration constants.

5.2 Case (B)

The Hamiltonian

\[
H_{DIB} = \frac{w_2^2}{w_1^2 + 1} \left( w_3^2 + w_4^2 + b_1(w_1^2 + w_2^2) + \frac{b_2}{w_1} + \frac{b_3}{w_2} \right)
\]

(49)

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yields the Hamiltonian equations

\[
\begin{align*}
\dot{w}_1 &= 2\frac{w_1^2 w_3}{w_1^2 + 1}, \\
\dot{w}_2 &= 2\frac{w_1^2 w_4}{w_1^2 + 1}, \\
\dot{w}_3 &= -2w_1\frac{w_2^2 w_4^2 + w_2^2 w_1^2 + b_1 w_1^4 w_2^2 + 2b_1 y_2 w_2^2 + b_1 w_4^2 - b_2 w_2^2 + b_3}{w_2^2(w_1^2 + 1)^2}, \\
\dot{w}_4 &= -2w_1^2\frac{b_1 w_4^2 - b_3}{w_2^2(w_1^2 + 1)}.
\end{align*}
\] (50)

If we apply the reduction method developed in [18] and choose \( w_1 \) as a new independent variable \( y \), then we obtain the following three equations:

\[
\begin{align*}
\frac{dw_2}{dy} &= \frac{w_4}{w_3}, \\
\frac{dw_3}{dy} &= -\frac{w_2^2 w_3^2 + w_2^2 w_1^2 + b_1 y_1^2 w_2^2 + 2b_1 y_2 w_2^2 + b_1 w_4^2 - b_2 w_2^2 + b_3}{w_2^2 w_3 y(y^2 + 1)}, \\
\frac{dw_4}{dy} &= -\frac{b_1 y_4^2 - b_3}{w_2^2 w_3}.
\end{align*}
\] (51)

If we solve the second equation with respect to \( b_2 \) and then derive once with respect to \( y \), then we obtain the following second-order equation in \( w_3(y) \):

\[
\dot{w}_3 = -\frac{w_3'(y w_3' + 3w_3) + 4b_1 y}{yw_3},
\] (52)

which is exactly the linearizable equation [14] if \( a_1 \) is replaced with \( 4b_1 \), and consequently

\[
w_3 = \pm \sqrt{8C_2 y^2 + 8C_1 - 4b_1 y^4},
\] (53)

with \( C_1, C_2 \) arbitrary integration constants.

Another reduction would also lead to linearity. If we choose \( w_2 \) as a new independent variable \( y \), then we obtain the following three equations:

\[
\begin{align*}
\frac{dw_1}{dy} &= \frac{w_4}{w_3}, \\
\frac{dw_3}{dy} &= -\frac{y^2 w_3^2 + y^2 w_1^2 + b_1 y_1^4 w_2^2 + 2b_2 y_2^2 w_2^2 + b_1 y_4^2 - b_2 y_2^2 + b_3}{w_1 y_4 w_2^2(w_1^2 + 1)}, \\
\frac{dw_4}{dy} &= -\frac{b_1 y_4^2 - b_3}{y^2 w_4}.
\end{align*}
\] (54)

and we can easily integrate the third equation, i.e.

\[
w_4 = \pm \frac{1}{y} \sqrt{2b_1 w_0 y^2 - b_1 y^4 - 2b_3 w_0 y^2 - b_3},
\] (55)

with \( w_0 \) an arbitrary integration constant. Then, if we solve the first equation in [14] with respect to \( w_3 \), and replace it into the second equation, we obtain the following second-order equation in \( w_1(y) \):

\[
w_1'' = \frac{-w_1^2}{w_1(w_1^2 + 1)} + \frac{(b_1 y_1^4 - b_3) w_1'}{y(2(b_1 - b_3) w_0 y^2 - b_1 y^4 - b_3)} - y^2 \frac{b_1(w_1^4 + 2w_1^2 + 2w_0) - b_2 - 2b_3 w_0}{w_1(w_1^2 + 1)(2(b_1 - b_3) w_0 y^2 - b_1 y^4 - b_3)},
\] (56)
that admits a three-dimensional symmetry algebra \( \text{sl}(2, \mathbb{R}) \), unless \( b_2 = 2(b_1 - b_3)w_0 - b_1 \), in which case it admits an eight-dimensional Lie symmetry algebra \( \text{sl}(3, \mathbb{R}) \) and thus it is linearizable. We now use the general method described in [21] and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra \( \text{sl}(2, \mathbb{R}) \). If we solve equation (56) with respect to \( b_2 \) and derive once with respect to \( y \), then we obtain the following third-order equation:

\[
 w_1''' = -3\frac{w_1w_2'}{w_1} + 3\frac{b_1y^4 - b_3}{2(b_1 - b_3)w_0y^2 - b_1y^4 - b_3} \left( \frac{w_1''}{y} + \frac{w_1'^2}{y^2} - \frac{w_1'}{y^2} \right),
\]

which admits a seven-dimensional Lie symmetry algebra, and therefore is linearizable. Indeed, the new dependent variable \( U = \frac{1}{2w_1} \) and independent variable \( Y = b_1y^2 + w_0(b_3 - b_1) \) transform equation (57) into the linear equation

\[
 \frac{d^3U}{dY^3} = -3Y\frac{d^2U}{dY^2}\frac{1}{Y^2 - b_5w_0^2 + b_1b_3 + 2b_1b_3w_0^2 - b_1'w_0^2}.
\]

### 5.3 Case (C)

The Hamiltonian

\[
 \mathcal{H}_{DIIIC} = \frac{w_3^2 + w_4^3 + \frac{a_1}{2w_1} + \frac{a_2}{w_2}}{w_1^2 + w_2^2 + \frac{1}{w_1} + \frac{1}{w_2}}
\]

yields the Hamiltonian equations

\[
 \begin{align*}
 \dot{w}_1 &= 2\frac{w_1^2w_2^3w_3}{(w_1^2w_2^2 + 1)(w_1^2 + w_2^2)}, \\
 \dot{w}_2 &= 2\frac{w_1^2w_2^3w_4}{(w_1^2w_2^2 + 1)(w_1^2 + w_2^2)}, \\
 \dot{w}_3 &= 2w_1w_2^2 \left( a_1w_1^2 + a_3 + (w_3^2 + w_2^3)w_2^2 \right)(w_1^4 - 1) + (w_2^4 + 1 + 2w_1^2w_3^3)a_2, \\
 \dot{w}_4 &= 2w_1w_2^2 \left( a_1w_1^2 + a_2 + (w_3^2 + w_2^3)w_2^2 \right)(w_1^4 - 1) + (w_2^4 + 2w_1^2w_3^2 + 1)a_3.
\end{align*}
\]

Before applying the reduction method [18], we introduce the following transformations of the dependent variables in order to avoid the mishandling of formulas with square roots by either REDUCE or MAPLE, i.e.

\[
 w_1 = \sqrt{r_1}, \quad w_2 = \sqrt{r_2}, \quad w_3 = \sqrt{r_3}, \quad w_4 = \sqrt{r_4},
\]

and then choose \( r_2 \) as a new independent variable \( y \) which gives rise to the following three equations:

\[
 \begin{align*}
 \frac{dr_1}{dy} &= \sqrt{\frac{r_1r_3}{y^4r_4}}, \\
 \frac{dr_3}{dy} &= \sqrt{\frac{r_3}{y^4r_4}} \left( a_1y + a_3 + (r_3 + r_4)y(y^2 - 1) + (y^2 + 1 + 2r_1y)a_2 \right) \left( r_4 + 1 \right) \left( r_1 + y \right), \\
 \frac{dr_4}{dy} &= \left( a_1r_1 + a_2 + (r_3 + r_4)r_1 \right)(y^2 - 1) + \left( r_2^2 + 1 + 2r_1y \right)a_3 \left( r_4 + 1 \right) \left( r_1 + y \right).
\end{align*}
\]

From the Hamiltonian \( \mathcal{H}_{DIIIC} \), i.e.

\[
 \mathcal{H}_{DIIIC} = \frac{r_3 + r_4 + a_1 + \frac{a_2}{r_1} + \frac{a_3}{y}}{r_1 + y + \frac{1}{r_1} + \frac{1}{y}} = h_0,
\]

\[
10
\]
we can derive:

\[ r_3 = \frac{(r_1 y + 1)(r_1 + y)h_0 - r_1 r_4 y - a_3 r_1 - a_2 y - a_1 yr_1}{yr_1}, \] (63)

with \( h_0 \) an arbitrary constant. Consequently, the third equation in (61) becomes:

\[ \frac{dr_4}{dy} = \frac{a_3 + (y^2 - 1)h_0}{y^2}, \] (64)

that can be easily integrated, i.e.:

\[ r_4 = \frac{w_0 y - a_3 + (y^2 + 1)h_0}{y^2}, \] (65)

with \( w_0 \) an arbitrary constant. Finally, we are left with the first equation in (61), i.e.

\[ 2 \left( a_3 - w_0 y - (y^2 + 1)h_0 \right) \frac{dr_1}{dy^2} - \left( w_0 + 2h_0 y \right) \frac{dr_1}{dy} + 2h_0 r_1 - w_0 - a_1 = 0. \] (67)

5.4 Case (D)

The Hamiltonian

\[ H_{DIID} = \frac{w_1^2}{w_1^2 + 1} \left( w_3^2 + w_4^2 + d \right) \] (68)

yields the Hamiltonian equations

\[
\begin{align*}
\dot{w}_1 &= 2 \frac{w_1^2 w_3}{w_1^2 + 1}, \\
\dot{w}_2 &= 2 \frac{w_1^2 w_4}{w_1^2 + 1}, \\
\dot{w}_3 &= -2w_1 \frac{w_3^2 + w_4^2 + d}{(w_1^2 + 1)^2}, \\
\dot{w}_4 &= 0.
\end{align*}
\] (69)

The last equation yields \( w_4 = w_0 \). If we apply the reduction method developed in [18] and choose \( w_2 \) as a new independent variable \( y \), then we obtain the following two equations:

\[
\begin{align*}
\frac{dw_1}{dy} &= \frac{w_3}{w_0}, \\
\frac{dw_4}{dy} &= -\frac{w_3^2 + w_0^2 + d}{w_0 w_1 (w_1^2 + 1)}.
\end{align*}
\] (70)

Then, if we solve the first equation in (70) with respect to \( w_3 \), and replace it into the second equation, we obtain the following second-order equation in \( w_1(y) \):

\[ w_1'' = \frac{w_0^2 w_1'^2 + w_0^2 + d}{w_0^2 w_1 (w_1^2 + 1)}. \] (71)

that admits a three-dimensional symmetry algebra \( \text{sl}(2, \mathbb{R}) \), unless \( d = -w_0^2 \), in which case it admits an eight-dimensional Lie symmetry algebra \( \text{sl}(3, \mathbb{R}) \) and thus it is linearizable. We now use the general method described
yields the Hamiltonian equations

\[ \text{From the Hamiltonian} \ H, \ \text{the following three equations:} \]

\[ \begin{align*}
    w''_1 &= -\frac{3w'_1 w''_1}{w'_1}, \\
    r'' &= 0.
\end{align*} \tag{72} \]

which admits a seven-dimensional Lie symmetry algebra, and therefore is linearizable. Indeed, the new dependent variable \( r_1 = w'_1 \) transforms equation \( \tag{72} \) into the linear equation

\[ r'' = 0. \tag{73} \]

6 Darboux III

Five superintegrable cases were determined in \[13\], where the problem of superintegrability for the Hamiltonian

\[ H_{DIII} = \frac{e^{2a}(p^2 + p^2_e)}{4e^{u+1}} \tag{74} \]

was addressed.

6.1 Case (A)

The Hamiltonian

\[ H_{DIIIA} = \frac{w^2 + w^2_2 + a_1 w_1 + a_2 w_2 + a_3}{4 + w^2_1 + w^2_2} \tag{75} \]

yields the Hamiltonian equations

\[ \begin{align*}
    \dot{w}_1 &= \frac{2w_3}{w^2_1 + w^2_2 + 4}, \\
    \dot{w}_2 &= \frac{2w_4}{w^2_1 + w^2_2 + 4}, \\
    \dot{w}_3 &= \frac{2a_2 w_1 w_2 + a_1 (w^2_1 - w^2_2 - 4) + 2a_3 w_1 + 2w_1 (w^2_3 + w^2_4)}{(w^2_1 + w^2_2 + 4)^2}, \\
    \dot{w}_4 &= \frac{2a_1 w_1 w_2 - a_2 (w^2_1 - w^2_2 - 4) + 2a_3 w_2 + 2w_2 (w^2_3 + w^2_4)}{(w^2_1 + w^2_2 + 4)^2}.
\end{align*} \tag{76} \]

We apply the reduction method \[18\] by choosing \( w_2 \) as a new independent variable \( y \) which gives rise to the following three equations:

\[ \begin{align*}
    \frac{dw_1}{dy} &= \frac{w_3}{w_4}, \\
    \frac{dw_3}{dy} &= \frac{2a_2 w_1 y + a_1 (w^2_1 - y^2 - 4) + 2a_3 w_1 + 2w_1 (w^2_3 + w^2_4)}{2w_4(w^2_1 + y^2 + 4)}, \\
    \frac{dw_4}{dy} &= \frac{2a_1 w_1 y - a_2 (w^2_1 - y^2 - 4) + 2a_3 y + 2w_2 (w^2_3 + w^2_4)}{2w_4(w^2_1 + y^2 + 4)}.
\end{align*} \tag{77} \]

From the Hamiltonian \( H_{DIIIA} \), i.e.

\[ H_{DIIIA} = \frac{w^2 + w^2_2 + a_1 w_1 + a_2 y + a_3}{4 + w^2_1 + y^2} = h_0, \tag{78} \]

we can derive:

\[ w_3 = \pm \sqrt{h_0(w^2_1 + y^2) + 4h_0 - a_1 w_1 - a_2 y - a_3 - w^2_4}. \tag{79} \]
with \( h_0 \) an arbitrary constant. Consequently, the third equation in (77) becomes:

\[
\frac{dw_4}{dy} = \frac{2h_0y - a_2}{2w_4},
\]

that can be easily integrated, i.e.:

\[
w_4 = \pm \sqrt{a_2(w_0 - y) + h_0y^2},
\]

with \( w_0 \) an arbitrary constant. Finally, we are left with the first equation in (77), i.e.

\[
\frac{dw_1}{dy} = \frac{\sqrt{h_0(w_1^2 + 4)} - a_1w_1 - a_2w_0 - a_3}{\sqrt{a_2(w_0 - y) + h_0y^2}}
\]

which can be solved by quadratures. However, if we solve it with respect to \( a_3 \), and derive once by \( y \), then the following linear second-order equation is obtained:

\[
2 \left( (w_0 - y)a_2 + h_0y^2 \right) \frac{d^2w_1}{dy^2} + (a_2 - 2h_0y) \frac{dw_1}{dy} + 2h_0w_1 - w_0 - a_1 = 0.
\]

### 6.2 Case (B)

The Hamiltonian

\[
\mathcal{H}_{DIII} = \frac{w_3^2 + w_4^2 + \frac{b_1}{w_1} + \frac{b_2}{w_2} + b_3}{4 + w_1^2 + w_2^2}
\]

yields the Hamiltonian equations

\[
\begin{aligned}
 w_1 &= \frac{2w_3}{w_1^2 + w_2^2 + 4}, \\
 w_2 &= \frac{2w_4}{w_1^2 + w_2^2 + 4}, \\
 w_3 &= 2 \left( b_2 + b_3w_2^2 + (w_3^2 + w_4^2)w_3^4 \right) w_3^2 + \frac{w_3^2 + w_4^2 + 2(2w_3^2 + 2) + w_3^2}{w_1^2 + w_2^2 + 4} b_2 w_3^2,
\end{aligned}
\]

Before applying the reduction method [13], we introduce the following transformations of dependent variables, in order to render the next calculations more amenable to a computer algebraic software such REDUCE, i.e.:

\[
w_1 = \sqrt{r_1}, \quad w_2 = \pm \sqrt{r_2},
\]

and then choose \( r_2 \) as a new independent variable \( y \) which gives rise to the following three equations:

\[
\begin{aligned}
 \frac{dr_1}{dy} &= \frac{\sqrt{r_1}w_3}{\sqrt{2w_4}}, \\
 \frac{dw_3}{dy} &= \frac{(b_2 + b_3y + (w_3^2 + w_4^2)y)r_1^2 + (y + 4 + 2r_1)b_1y}{2yr_1\sqrt{r_1}r_1 + y + 4)w_4}, \\
 \frac{dw_4}{dy} &= \frac{(b_1 + b_3r_1 + (w_3^2 + w_4^2)r_1)y^2 + (2(y + 2) + r_1)b_2r_1}{2y^2r_1r_1 + y + 4)w_4}.
\end{aligned}
\]

From the Hamiltonian \( \mathcal{H}_{DIII} \), i.e.

\[
\mathcal{H}_{DIII} = \frac{w_3^2 + w_4^2 + \frac{b_1}{w_1} + \frac{b_2}{w_2} + b_3}{4 + r_1 + y} = h_0,
\]
we can derive:

\[ w_3 = \pm \frac{\sqrt{(h_0(y + 4r_1) - b_3 - w_1^2)yr_1 - b_2r_1 - b_1y}}{yr_1}, \] (89)

with \( h_0 \) an arbitrary constant. Consequently, the third equation in (87) becomes:

\[ \frac{dw_4}{dy} = \frac{b_2 + h_0y^2}{2w_4y^2}, \] (90)

that can be easily integrated, i.e.:

\[ w_4 = \pm \sqrt{h_0y(y - w_0) - b_2(1 + w_0y)}, \] (91)

with \( w_0 \) an arbitrary constant. Finally, we are left with the first equation in (87), i.e.

\[ \frac{dr_1}{dy} = \sqrt{\frac{b_1 + (b_3 - b_2w_0)r_1 - h_0r_1(r_1 + w_0 + 4)}{b_2(1 + w_0y) + h_0y(w_0 - y)}}, \] (92)

which can be solved by quadratures. However, if we solve it with respect to \( b_1 \), and derive once by \( y \), then the following linear second-order equation is obtained:

\[ \frac{d^2r_1}{dy^2} = -\frac{(b_2w_0 + h_0w_0 - 2h_0y)\frac{dr_1}{dy} + 2h_0r_1 + b_2w_0 - b_3 + h_0w_0 + 4h_0}{2(b_2(1 + w_0y) + h_0y(w_0 - y))}. \] (93)

6.3 Case (C)

The Hamiltonian

\[ H_{DHHC} = \frac{w_1^2}{w_3} - \frac{w_2^2}{w_4} + c_1(w_1 + w_2) + c_2\frac{w_1 + w_2}{w_1w_2} + c_3\frac{w_1^2 - w_2^2}{w_1^2w_2^2} \] (94)

yields the Hamiltonian equations

\[
\begin{align*}
\dot{w}_1 &= \frac{2w_1^2w_3}{(w_1 + w_2)(2 + w_1 - w_2)}, \\
\dot{w}_2 &= -\frac{2w_2^2w_4}{(w_1 + w_2)(2 + w_1 - w_2)}, \\
\dot{w}_3 &= 2\left(\frac{w_3^2 - w_2^2}{w_1^2w_2^2}w_1w_2 - \frac{w_2^2}{w_1^2}w_1w_2 - \frac{w_3^2}{w_2^2}w_1w_2\right) + \frac{c_1}{(w_1 + w_2)^2(2 + w_1 - w_2)^2} \\
&\quad + c_2\frac{2w_1 - w_2 + 2}{(2 + w_1 - w_2)^2w_1^2w_2^2} + 2c_3\frac{w_1^2 - 2w_1w_2 + w_1 + w_2^2 - 2w_2}{(2 + w_1 - w_2)^2w_1^2w_2^2}, \\
\dot{w}_4 &= -2\left(\frac{w_3^2 - w_2^2}{w_1^2w_2^2}w_1w_2 - \frac{w_2^2}{w_1^2}w_1w_2 - (2w_1 + w_2)w_2w_1^2\right) - \frac{c_1}{(w_1 + w_2)^2(2 + w_1 - w_2)^2} \\
&\quad + c_2\frac{w_1 - 2w_2 + 2}{(2 + w_1 - w_2)^2w_1^2w_2^2} + 2c_3\frac{w_1^2 - 2w_1w_2 + 2w_1 + w_2^2 - 2w_2}{(2 + w_1 - w_2)^2w_1^2w_2^2}.
\end{align*}
\] (95)
We apply the reduction method [18] by choosing \( w_2 \) as a new independent variable \( y \) which gives rise to the following three equations:

\[
\begin{align*}
\frac{dw_1}{dy} &= -\frac{w_1^3 w_3}{y^2 w_4}, \\
\frac{dw_3}{dy} &= -\left(\frac{w_1^2 - w_3^2}{2(2 + w_1 - y)} y^2 w_4 - 2w_1^2 w_1 y - w_1^2 w_3^2 - y^2 w_4^2\right) - c_1 \frac{w_1 + y}{2(2 + w_1 - y)y^2 w_4} \\
&\quad - c_2 \frac{(w_1 + y - 2)(w_1 + y)}{2(2 + w_1 - y)w_1 y^2 w_4} - c_3 \frac{w_1^2 - 2w_1 y + w_1^2 y - 2y w_1 y}{(2 + w_1 - y)w_1 y^2 w_4}, \\
\frac{dw_4}{dy} &= \frac{w_1^2 y (w_1^2 - w_3^2) - w_1^2 w_3^2 - 2w_1 w_1 y - w_1^2 w_1^2 y^2 - w_1^2 y^2}{(w_1 + y)(2 + w_1 - y)w_4 y^2} + c_1 \frac{w_1 + y}{2(2 + w_1 - y)w_4 y^2} \\
&\quad - c_2 \frac{(w_1 + y)(w_1 - 2y + 2)}{2(2 + w_1 - y)w_1 y w_4 y^2} - c_3 \frac{(w_1^2 - 2w_1 y + w_1^2 + 2y^2 - y)(w_1 + y)}{(2 + w_1 - y)w_1 y w_4 y^2}. 
\end{align*}
\]

From the Hamiltonian \( \mathcal{H}_{DIIC} \), i.e.

\[
\mathcal{H}_{DIIC} = \frac{w_1^2 w_3^2 - y^2 w_1^2 + c_1(w_1 + y) + c_2 w_1 + w_1 y + c_3 \frac{w_1^2 - y^2}{w_1 y}}{(w_1 + y)(2 + w_1 - y)} = h_0, 
\]

we can derive:

\[
w_3 = \pm \sqrt{w_1^2 w_4^2 y^4 - w_1 y (w_1 + y) (c_1 w_1 y + c_2) + c_3 (y^2 - w_1^2)} + h_0 (w_1 + y)(2 + w_1 - y)w_1 y^2, 
\]

with \( h_0 \) an arbitrary constant. Consequently, the third equation in (96) becomes:

\[
\frac{dw_4}{dy} = \frac{2 w_1^2 y^4 - c_1 y^2 + c_2 y + 2 c_3 + 2 y^3 h_0 (1 - y)}{2 w_4 y^5}, 
\]

that can be easily integrated, i.e.:

\[
w_4 = \pm \sqrt{w_0 y^2 + c_1 y^3 + c_2 y + c_3 + h_0 y^3 (y - 2)} y^2, 
\]

with \( w_0 \) an arbitrary constant. Finally, we are left with the first equation in (96), i.e.

\[
\frac{dw_1}{dy} = \frac{-\sqrt{w_0 w_1^2 - c_1 w_1^3 - c_2 w_1 y + c_3 + h_0 w_1^3 (w_1 + 2)}}{w_0 y^2 + c_1 y^3 + c_2 y + c_3 + h_0 y^3 (y - 2)}, 
\]

which could be solved by quadratures. If we introduce new parameters in order to simplify this equation, i.e.:

\[
c_1 = C_1 + 2 h_0, \quad c_2 = C_2 C_1, \quad c_3 = C_3 C_1, \quad h_0 = H_0 C_1, \quad w_0 = W_0 C_1, 
\]

and the new dependent variable \( u = -w_1 \), then equation (101) becomes:

\[
u'(y) = \frac{du}{dy} = \frac{\sqrt{C_2 u + C_3 + H_0 u^2 + W_0 u^2 + u^4}}{\sqrt{C_2 y + C_3 + H_0 y^2 + W_0 y^2 + y^2}}, 
\]

If we solve this first-order equation with respect to \( C_3 \), and derive once by \( y \), then a second-order equation is obtained. If we solve this second-order equation with respect to \( W_0 \), and derive once by \( y \), then a third-order equation is obtained. Finally, if we solve this third-order equation with respect to \( H_0 \), and derive once by \( y \), then the following fourth-order equation is obtained:

\[
u''(u) = \frac{-\alpha_1 u'^{3} + \alpha_2 u'^{2} - \alpha_3 u' u'' - \alpha_4 u'' + \alpha_5 u''' - 6 \alpha_7 u'' + \alpha_8}{3(C_2 - u y) (u u'' - 2 u'^2 - 2 u' - u'' y)(u^2 - y^2)}, 
\]
\[ \alpha_1 = 9(u - y)[C_2(3u + 5y) - 2u^2y - 5uy^2 - y^3], \]
\[ \alpha_2 = C_2(36u + 54y + 36uu' - 54u^2y) - 30u^2u'y + 72uu'^2y^2 + 18u'^3y^3 \]
\[ + 18u^3u' - 72u^2u'y - 18uu'y^2 + 36u'y^3 - 18u^2y - 72uy^2, \]
\[ \alpha_3 = 3(u - y) \left[ C_2(13uu' + 15u'y + 5u + 7y) - 12u^2u'y - 15uu'y^2 - u'y^3 + u^3 - 5u^2y - 8uy^2 \right], \]
\[ \alpha_4 = 18u'(u' + 1) \left[ C_2(u'^2 - 1) - 4uu'^2y + u'^2y^2 + 3u^2u' - 3u'y^2 - u^2 - 4uy \right], \]
\[ \alpha_5 = 5(u + y)(u - y)^2(C_2 - uy), \]
\[ \alpha_7 = u'(u' + 1) \left[ C_2(3uu' + 5u'y - 5u - 3y) - 2u^2u'y - 5uu'y^2 - u'y^3 + u^3 + 5u^2y + 2uy^2 \right], \]
\[ \alpha_8 = 36u'^2(u' - 1)(u' + 1)^2(u - u'y). \]

It admits a fourth-dimensional Lie symmetry algebra \(2A_2\) generated by the following operators:

\[
\begin{align*}
\Gamma_1 &= \frac{3}{u - y} \left( -(C_2^2 + uy^2)\partial_y + (C_2^2 + u^2y)\partial_u \right), \\
\Gamma_2 &= \frac{u + y}{u - y} \left( -(C_2 + y^2)\partial_y + (u^2 + C_2)\partial_u \right), \\
\Gamma_3 &= \frac{1}{3(u - y)} \left( (uy - 2C_2 - y^2)\partial_y + (2C_2 + u^2 - uy)\partial_u \right), \\
\Gamma_4 &= -\frac{1}{u - y} \left( (C_2 - 2uy - y^2)\partial_y + (u^2 - C_2 + 2uy)\partial_u \right).
\end{align*}
\] (105) (106) (107) (108)

In order to follow the classification of the fourth-dimensional Lie symmetry algebra in [22], and the fourth-order equations admitting them as derived in [23], we choose another representation of the operators that generate \(2A_2\), i.e.:

\[ X_1 = \Gamma_1 - 3C_2\Gamma_3, \quad X_2 = \Gamma_2, \quad X_3 = \frac{1}{3}\Gamma_3 + \Gamma_4, \quad X_4 = \frac{1}{3}\Gamma_3 - 2\Gamma_4. \] (109)

We thank Nicola Ciccoli for his invaluable help on this issue. A two-dimensional Abelian intransitive subalgebra of the Lie symmetry algebra \(2A_2\) is that generated by \(X_1\) and \(X_2\), and the corresponding canonical transformations are:

\[ \tilde{y} = \frac{u + y}{3(uy - C_2)}, \quad \tilde{u} = -\frac{1}{uy - C_2}. \]

Then equation (104) turns into the following fourth-order equation:

\[ \frac{d^4\tilde{u}}{d\tilde{y}^4} = \left( \frac{3}{3(uy - C_2)} \right)^2 \frac{d^3\tilde{u}}{dy^3}/3\tilde{y}^2 \] (110)

If we make the substitution

\[ \frac{d^2\tilde{u}}{dy^2} = R(\tilde{y}), \]

then equation (110) becomes the following second-order equation:

\[ \frac{d^2R}{dy^2} = \left( 3R + 5\tilde{y} \right) \frac{dR}{dy}/3\tilde{y} \] (111)

which admits an eight-dimensional Lie symmetry algebra \(sl(3, \mathbb{R})\), and therefore it is linearizable [20]. Indeed, the transformation \(Y = R^{2/3}, \quad U = \frac{\tilde{y}^2}{2}R^{2/3}\) yields

\[ \frac{d^2U}{dY^2} = 0 \Rightarrow U = A_1Y + A_2, \]
with $A_1, A_2$ arbitrary constants. Consequently, the general solution of equation (111) is

$$R = \frac{2A_2\sqrt{2A_2}}{(\hat{y}^2 - 2A_1)\sqrt{\hat{y}^2 - 2A_1}},$$

that integrated twice yields the general solution of equation (110), i.e.:

$$\hat{u} = \frac{A_2}{A_1}\sqrt{2A_2(\hat{y}^2 - 2A_1)} + A_3\hat{y} + A_4,$$

with $A_3, A_4$ arbitrary constants. Finally, the general solution of equation (104) is:

$$u = \beta_2 y^2 + \beta_1 y^2 + \beta_0 + \sqrt{\gamma_3 y^4 + \gamma_2 y^2 + \gamma_1 y^2 + \gamma_0}$$

with

$$\begin{align*}
\beta_2 &= -3A_2^2 A_3 A_4, \\
\beta_1 &= 9A_1^2 A_2^2 C_2 - 3A_2^2 A_4 + 2A_3^2 - A_1^2 A_2^2 + 36A_2^2 A_1 C_2, \\
\beta_0 &= A_2^2 A_3 (3C_2 A_4 - 1), \\
\gamma_4 &= -18A_1^2 A_2^4 + 9A_1^2 A_4 + 36A_2^4, \\
\gamma_3 &= -36A_3^2 A_2^3, \\
\gamma_2 &= 18A_2^4 A_1 C_2 - 6A_3 A_1 + 72A_2^3 C_2 - 36A_1^2 A_2^3 C_2 - 18A_1^3, \\
\gamma_1 &= -36A_2^2 C_2 A_3, \\
\gamma_0 &= -18A_1^2 A_2^2 C_2 + 9A_1 A_2^2 C_2^2 - 6A_2 A_1 C_2 A_4 + A_2 + 36A_2^3 C_2^2.
\end{align*}$$

We would like to remark that if we solve the fourth-order equation (103) with respect to $C_2$, and derive once by $y$, then a fifth-order equation is obtained, which admits an eight-dimensional Lie symmetry algebra, and can be transformed into a third-order linearizable equation since it admits a seven-dimensional Lie symmetry algebra, quite similarly to the fourth-order equation that we discuss in details here.

### 6.4 Case (D)

The Hamiltonian

$$H_{DIIIID} = \frac{w_1^2 w_3 - w_2^2 w_4 + d_1 w_1 + d_2 w_2 + d_3 (w_1^2 + w_2^2)}{(w_1 + w_2)(2 + w_1 - w_2)}$$

yields the Hamiltonian equations

$$\begin{align*}
\dot{w}_1 &= \frac{2w_1^2 w_3}{(w_1 + w_2)(2 + w_1 - w_2)}, \\
\dot{w}_2 &= -\frac{2w_2^2 w_4}{(w_1 + w_2)(2 + w_1 - w_2)}, \\
\dot{w}_3 &= \frac{1}{(w_1 + w_2)^2(2 + w_1 - w_2)^2}\left(2\left(w_1 w_2^2 (w_3^2 - w_4^2) - 2w_1 w_2 w_3^2 - w_2^2 w_4^2 - w_1^2 w_3^2\right)
+ d_1 (w_1^2 + w_2^2 - 2w_2) + 2d_2 w_2 (w_1 + 1) + 2d_3 (w_3^2 - w_1^2 + 2w_1 w_2 - 2w_2)\right), \\
\dot{w}_4 &= \frac{1}{(w_1 + w_2)^2(2 + w_1 - w_2)^2}\left(2\left(w_1^2 w_2 (w_4^2 - w_3^2) + 2w_1 w_2 w_4^2 + w_2^2 w_3^2 + w_1^2 w_4^2\right)
+ 2d_1 w_1 (1 - w_2) - d_2 (w_1^2 + w_2^2 + 2w_1) - 2d_3 (w_2^2 - w_1^2 + 2w_1 w_2 + 2w_1 w_2)\right).
\end{align*}$$
We apply the reduction method \[18\] by choosing \(w_2\) as a new independent variable \(y\) which gives rise to the following three equations:

\[
\begin{align*}
\frac{dw_1}{dy} &= \frac{w_1^2 w_3}{y^2 w_4}, \\
\frac{dw_3}{dy} &= -\frac{1}{2w_4 y^2(2 + w_1 - y)} \left(2 \left(w_1 y^2(w_3^2 - w_4^2) + 2w_1yw_3^2 - y^2w_3^2 - w_1^2 w_3^2\right)
+ d_1(w_1^2 + y^2 - 2y) + 2d_2 y(w_1 + 1) + 2d_3(y^2 - w_1^2 + 2w_1 y - 2w_1 y)\right), \\
\frac{dw_4}{dy} &= -\frac{1}{2w_4 y^2(2 + w_1 - y)} \left(2 \left(w_1^2 y(w_3^2 - w_4^2) + 2w_1yw_3^2 + y^2w_4^2 + w_1^2 w_3^2\right)
+ 2d_1w_1(1 - y) - d_2(w_1^2 + y^2 + 2w_1) - 2d_3(y^2 - w_1^2 + 2w_1 y + 2w_1 y)\right).
\end{align*}
\] (117)

From the Hamiltonian \(\mathcal{H}_{\text{DDII}}\), i.e.

\[
\mathcal{H}_{\text{DDII}} = \frac{w_1^2 w_3^2 - y^2 w_4^2 + d_1 w_1 + d_2 y + d_3(w_1^2 + y^2)}{(w_1 + y)(2 + w_1 - y)} = h_0,
\] (118)

we can derive:

\[
w_3 = \pm \sqrt{(h_0 - d_3)w_1^2 + (2h_0 - d_1)w_1 + (2h_0 - d_2)y - (d_3 + h_0)y^2 + w_0^2 y^2},
\] (119)

with \(h_0\) an arbitrary constant. Consequently, the third equation in (117) becomes:

\[
\frac{dw_4}{dy} = \frac{d_2 - 2h_0 + 2(d_3 + h_0)y - 2w_1^2 y}{2w_4 y^2},
\] (120)

that can be easily integrated, i.e.:

\[
w_4 = \pm \sqrt{(d_2 - 2h_0)y + (d_3 + h_0)y^2 + w_0},
\] (121)

with \(w_0\) an arbitrary constant. Let us introduce new parameters that simplify the formula for \(w_3\) and \(w_4\), i.e.:

\[
D_1 = 2h_0 - d_1, \quad D_2 = d_2 - 2h_0, \quad D_3 = d_3 + h_0,
\] (122)

and consequently:

\[
w_3 = \frac{\pm \sqrt{(2h_0 - D_3)w_1^2 + D_1 w_1 + w_0}}{w_1}, \quad w_4 = \pm \sqrt{D_2 y + D_3 y^2 + w_0}.
\] (123)

Finally, we are left with the first equation in (117), i.e.

\[
\frac{dw_1}{dy} = -\frac{w_1 \sqrt{(2h_0 - D_3)w_1^2 + D_1 w_1 + w_0}}{y \sqrt{D_3 y^2 + D_2 y + w_0}},
\] (124)

which can be solved by quadratures. However, if we solve it with respect to \(D_1\), and derive once by \(y\), then the following second-order equation is obtained:

\[
2y^2 w_1(D_3 y^2 + D_2 y + w_0) \frac{d^2 w_1}{dy^2} - 3y^2(D_3 y^2 + D_2 y + w_0) \left(\frac{dw_1}{dy}\right)^2 \\
+ (4D_3 y^2 + 3D_2 y + 2w_0) y w_1 \frac{dw_1}{dy} + (D_3 - 2h_0) w_1^4 + w_0 w_1^2 = 0,
\] (125)
that admits a three-dimensional symmetry algebra \(sl(2, \mathbb{R})\), unless \(D_3 = 2h_0\) in which case it admits an eight-dimensional Lie symmetry algebra \(sl(3, \mathbb{R})\) and thus it is linearizable. We now use the general method described in [21] and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra \(sl(3, \mathbb{R})\). If we solve equation \([125]\) with respect to \(h_0\) and derive once with respect to \(y\), then we obtain the following third-order equation:

\[
\frac{2yw_1^2}{3}(D_3y^2 + D_2y + w_0)\frac{d^3w_1}{dy^3} = -4y(D_3y^2 + D_2y + w_0)\left(\frac{dw_1}{dy}\right)^3 \\
+2w_1(4D_3y^2 + 3D_2y + 2w_0)\left(\frac{dw_1}{dy}\right)^2 - 2(2D_3y + D_2)w_1\frac{dw_1}{dy} \\
+\left(4yw_1(D_3y^2 + D_2y + w_0)\frac{dw_1}{dy} - (4D_3y^2 + 3D_2y + 2w_0)w_1^2\right)\frac{d^2w_1}{dy^2},
\]

which admits a seven-dimensional Lie symmetry algebra, and therefore is linearizable. Indeed, the new dependent and independent variables, i.e.

\[
\bar{w}_1 = -\frac{1}{w_1}, \quad \bar{y} = \frac{2D_3y + D_2}{y},
\]

transform equation \([126]\) into the linear equation

\[
\frac{d^3\bar{w}_1}{d\bar{y}^3} = \frac{3(4D_3w_0 - D_2^2 - 2w_0\bar{y})}{2(2w_0\bar{y}^2 + (4D_3w_0 - D_2^2)(D_3 - \bar{y}))}\frac{d^2\bar{w}_1}{d\bar{y}^2}.
\]

### 6.5 Case (E)

The Hamiltonian

\[
\mathcal{H}_{DIIIIE} = \frac{w_3^3 + w_4^3 + c}{4 + w_1^3 + w_2^3}
\]

is a subcase of Hamiltonian \(\mathcal{H}_{DIIIA}\), with \(a_1 = a_2 = 0\) and \(a_3 = c\). Consequently, its corresponding Hamiltonian equations, i.e.,

\[
\begin{align*}
\bar{w}_1 &= \frac{2w_3}{w_1^2 + w_2^2 + 4}, \\
\bar{w}_2 &= \frac{2w_3}{w_1 + w_2^2 + 4}, \\
\bar{w}_3 &= \frac{2w_1(w_1^2 + w_2^2 + w_4^2)}{(w_1^2 + w_2^2 + 4)^2}, \\
\bar{w}_4 &= \frac{2w_2(c + w_3^2 + w_4^2)}{(w_1^2 + w_2^2 + 4)^2},
\end{align*}
\]

can be reduced to the following linear equation in \(w_1 = w_1(w_2)\):

\[
(2w_0 - w_2^2)\frac{d^2w_1}{dw_2^2} - w_2\frac{dw_1}{dw_2} + w_1 = 0,
\]

with

\[
w_3 = \pm \sqrt{h_0(w_1^2 + w_2^3)} + 4h_0 - c - w_4^2,
\]

and

\[
w_4 = \pm \sqrt{h_0(w_2^2 - 2w_0)}.
\]
7  Darboux IV

Four superintegrable systems were determined in [13], where the problem of superintegrability for the Hamiltonian

\[ H_{DIV} = -\sin^2(2u) \frac{p_u^2 + p_v^2}{2\cos(2u) + a} \]  

was addressed.

7.1  Case (A)

The Hamiltonian

\[ H_{DIV,A} = -4w_1^2w_2^2 \frac{w_3^2 + w_4^2 + a_1 + a_2 \left( \frac{1}{w_1} + \frac{1}{w_2} \right) + a_3(w_1^2 + w_2^2)}{(a + 2)w_1^2 + (a - 2)w_2^2} \]

yields the Hamiltonian equations

\[
\begin{align*}
\dot{w}_1 &= -\frac{8w_1^2w_2^2w_3}{a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2)}, \\
\dot{w}_2 &= -\frac{8w_1^2w_2^2w_4}{a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2)}, \\
\dot{w}_3 &= \frac{8w_1^2w_2^2}{(a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2))^2} \left( a_3(w_1^2 + w_2^2)^2 + aw_2^2(w_3^2 + w_4^2) + a_1w_2^2(a - 2) - 4a_2 \\
&\quad + 2a_3(w_1^2 - w_2^2 - 2w_1^2w_2^2) - 2w_3^2(w_2^2 + w_4^2) \right), \\
\dot{w}_4 &= \frac{8w_1^2w_2^2}{(a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2))^2} \left( a_3(w_1^2 + w_2^2)^2 + aw_1^2(w_3^2 + w_4^2) + a_1w_1^2(a + 2) + 4a_2 \\
&\quad + 2a_3(w_1^2 - w_2^2 + 2w_1^2w_2^2) + 2w_1^2(w_2^2 + w_4^2) \right).
\end{align*}
\]

In order to simplify the calculations, we make the following substitutions of the four dependent variables:

\[ w_1 = \sqrt{r_1}, \quad w_2 = \sqrt{r_2}, \quad w_3 = \sqrt{r_3}, \quad w_4 = \sqrt{r_4}, \]

and consequently system \((136)\) is transformed into the following system:

\[
\begin{align*}
\dot{r}_1 &= -\frac{16r_1\sqrt{r_1r_3r_2}}{a(r_1 + r_2) + 2(r_1 - r_2)}, \\
\dot{r}_2 &= -\frac{16r_2\sqrt{r_2r_4r_1}}{a(r_1 + r_2) + 2(r_1 - r_2)}, \\
\dot{r}_3 &= \frac{16\sqrt{r_3r_5r_2}}{(a(r_1 + r_2) + 2(r_1 - r_2))^2} \left( a_3(r_1 + r_2)^2 + ar_2(r_3 + r_4) + a_1r_2(a - 2) - 4a_2 \\
&\quad + 2a_3(r_1^2 - r_2^2 - 2r_1r_2) - 2r_2^2(r_3 + r_4) \right), \\
\dot{r}_4 &= \frac{16\sqrt{r_2r_4r_1}}{(a(r_1 + r_2) + 2(r_1 - r_2))^2} \left( a_3(r_1 + r_2)^2 + ar_1(r_3 + r_4) + a_1r_1(a + 2) + 4a_2 \\
&\quad + 2a_3(r_1^2 - r_2^2 + 2r_1r_2) + 2r_1(r_3 + r_4) \right).
\end{align*}
\]
We apply the reduction method \cite{18} by choosing $r_2$ as a new independent variable $y$ which gives rise to the following three equations:

\[
\begin{align*}
\frac{dr_1}{dy} &= \sqrt{\frac{r_1 r_3}{y r_4}}, \\
\frac{dr_3}{dy} &= -\frac{\sqrt{r_1 r_3}}{r_1 \sqrt{y r_4} (a(r_1 + y) + 2(r_1 - y))} \left( a a_3 (r_1 + y)^2 + a y (r_3 + r_4) + a_1 y (a - 2) - 4a_2 \\
&+ 2a_3 (r_1^2 - y^2 - 2r_1 y) - 2y (r_3 + r_4) \right), \\
\frac{dr_4}{dy} &= \frac{1}{y (a(r_1 + y) + 2(r_1 - y))} \left( a a_3 (r_1 + y)^2 + a r_1 (r_3 + r_4) + a_1 r_1 (a + 2) + 4a_2 \\
&+ 2a_3 (r_1^2 - y^2 + 2r_1 y) + 2r_1 (r_3 + r_4) \right).
\end{align*}
\]

(139)

From the Hamiltonian $H_{DIVA}$, i.e.:

\[
H_{DIVA} = -4r_1 y \frac{r_3 + r_4 + a_1 + a_2 \left( \frac{1}{r_1} + \frac{1}{y} \right) + a_3 (r_1 + y)}{(a + 2)y + (a - 2)y} = h_0,
\]

(140)

we can derive:

\[
r_3 = -4r_1 r_4 y + ((a + 2)r_1 + (a - 2)y) h_0 + 4a_1 r_1 y + 4a_2 (r_1 + y) + 4a_3 r_1 (r_1 + y),
\]

(141)

with $h_0$ an arbitrary constant. Consequently, the third equation in (139) becomes:

\[
\frac{dr_4}{dy} = \frac{(a + 2) h_0 + 4a_2 - 4a_3 y^2}{4y^2},
\]

(142)

that can be easily integrated, i.e.:

\[
r_4 = \frac{4r_0 y - (a + 2) h_0 - 4a_2 - 4a_3 y^2}{4y},
\]

(143)

with $r_0$ an arbitrary constant. Finally, we are left with the first equation in (139), i.e.:

\[
\frac{dr_1}{dy} = \sqrt{\frac{(2 - a) h_0 - 4a_2 - 4r_1 (a_1 + r_0) - 4a_3 r_1^2}{-(a + 2) h_0 - 4a_2 - 4a_3 y^2 + 4r_0 y}},
\]

(144)

which could be easily solved by quadratures. If we introduce a new parameter $b_2 = -(a + 2) h_0 - 4a_2$, such that $a_2 = -(a + 2) h_0 + b_2)/4$, and then solve the first-order equation (144) with respect to $a_1$, and derive once by $y$, then a second-order equation is obtained. If we solve this second-order equation with respect to $h_0$, and derive once by $y$, then the following linear third-order equation is obtained:

\[
(b_2 + 4r_0 y - 4a_3 y^2) \frac{d^3 r_1}{dy^3} - 6(2a_3 y - r_0) \frac{d^2 r_1}{dy^2} = 0.
\]

(145)

\[7.2 \text{ Case (B)}\]

The Hamiltonian

\[
H_{DIVB} = -\frac{\sin^2(2 w_1) \left( \frac{a_2}{\sin^2(w_2)} + \frac{b_3}{\cosh^2(w_2)} \right) + b_1}{2 \cos(2 w_1) + a}
\]

(146)
can be written in the following equivalent form with sinh, and cosh replaced by exp, i.e.:

\[
\mathcal{H}_{\text{DIV}} = - \frac{\sin^2(2w_1)}{2 \cos(2w_1) + a} \left( w_3^2 + w_4^2 + \frac{\cos(2w_1)}{e^{2w_1}} + \frac{\cos(2w_1)}{e^{-2w_1}} \right) + b_1.
\] (147)

We apply the reduction method [18] by choosing \( r_2 = e^{w_2} \) as a new independent variable \( y \) which gives rise to the following three equations:

\[
\begin{align*}
\frac{dw_1}{dy} &= \frac{w_3}{yw_4}, \\
\frac{dw_3}{dy} &= \frac{N}{\sin(2w_1)w_4(y^4 - 1)[a + 2 \cos(2w_1)]}, \\
\frac{dw_4}{dy} &= \frac{4y^2 \left((b_2 + b_3)y^8 + 4(b_2 - b_3)y^6 + 6(b_2 + b_3)y^4 + 4(b_2 - b_3)y^2 + b_2 + b_3\right)}{w_4(y^4 - 1)^3},
\end{align*}
\] (148)

where

\[
N = - \left[4(w_3^3 + w_4^3)y^8 + 16(b_2 + b_3)y^6 - 8(w_3^3 + w_4^3 - 4b_2 + 4b_3)y^4 + 16(b_2 + b_3)y^2 + 4(w_3^2 + w_4^2)\right] \cos(2w_1)^2
\]

\[
- 2a \left[(w_3^3 + w_4^3)y^8 + 4(b_2 + b_3)y^6 - 2(w_3^3 + w_4^3 - 4b_2 - 4b_3)y^4 + 4(b_2 + b_3)y^2 + w_3^3 + w_4^3\right] \cos(2w_1)
\]

\[
+ \left[-2(w_3^3 + w_4^3)y^8 - 8(b_2 + b_3)y^6 + 4(w_3^3 + w_4^3 - 4b_2 - 4b_3)y^4 - 8(b_2 + b_3)y^2 - 2(w_3^2 + w_4^2)\right] \sin(2w_1)^2
\]

\[- 2b_1y^8 + 4b_1y^4 - 2b_1.
\]

We can solve the third equation with respect to \( w_4 \), i.e.:

\[
w_4 = \pm \frac{2}{15\sqrt{15(y^2 - 1)}} \sqrt{3375(b_3 - b_2)(y^8 + 1) - 3375(b_3 + b_2)(y^4 + 1)g^2 - 4(625b_2 + 81b_3)(y^4 - 1)^2w_0}.
\] (150)

Then the first equation in (148) yields:

\[
w_3 = yw_4 \frac{dw_1}{dy},
\] (151)

which replaced into the second equation in (148) gives rise to a second-order equation in \( w_1 \) that we solve with respect to \( b_1 \). The we derive once with respect to \( y \) and the following third-order equation is obtained:

\[
\frac{d^3w_1}{dy^3} = -6\cot(2u)\frac{dw_1}{dy} \frac{d^2w_1}{dy^2} + \frac{3}{y(y^4 - 1)^2} P_1(y, u) \frac{d^2w_1}{dy^2} + 4 \left( \frac{dw_1}{dy} \right)^3
\]

\[- 6\cot(2u) P_1(y, u) \frac{d^2w_1}{dy^2} \left( \frac{dw_1}{dy} \right)^2 + \frac{3}{y(y^4 - 1)^2} P_2(y) \frac{dw_1}{dy},
\] (152)

where:

\[
P_1 = [(2500b_2 + 324b_3)w_0 + 3375(b_2 - b_3)]y^{12} - [(7500b_2 + 972b_3)w_0 + 16875(b_2 - b_3)]y^8 - 20250(b_2 + b_3)y^6
\]

\[(7500b_2 + 972b_3)w_0 - 10125(b_2 - b_3)y^4 - 6750(b_2 + b_3)y^2 - (2500b_2 + 324b_3)w_0 - 3375(b_2 - b_3),
\] (153)

\[
P_2 = [(2500b_2 + 324b_3)w_0 + 3375(b_2 - b_3)]y^{16} - [(10000b_2 + 1296b_3)w_0 - 20250(b_2 - b_3)]y^{12}
\]

\[- 33750(b_2 + b_3)y^{10} + [(15000b_2 + 1944b_3)w_0 - 47250(b_2 - b_3)]y^8 - 67500(b_2 + b_3)y^6
\]

\[- [(10000b_2 + 1296b_3)w_0 - 47250(b_2 - b_3)]y^4 - 6750(b_2 + b_3)y^2 + (2500b_2 + 324b_3)w_0 + 3375(b_2 - b_3),
\]

\[Q = [(2500b_2 + 324b_3)w_0 + 3375(b_2 - b_3)]y^8 + 3375(b_2 + b_3)y^6 - (5000b_2 + 648b_3)w_0 y^4
\]

\[+ 3375(b_2 + b_3)y^2 + (2500b_2 + 324b_3)w_0 + 3375(b_2 - b_3).
\] (155)
Equation [152] is linearizable since it admits a seven-dimensional Lie symmetry algebra. In fact, the two-dimensional abelian intransitive subalgebra generated by the two operators

\[-\frac{\cos(2u)}{2\sin(2u)} \partial_u, \quad \frac{1}{\sin(2u)} \partial_u \tag{156}\]

when put into the canonical form \(\partial_{\hat{u}}\), \(\hat{y}\partial_{\hat{u}}\) yield the new dependent and independent variables, i.e.

\[\hat{u} = -\frac{1}{2} \cos(2u), \quad \hat{y} = \frac{-B_2y^4 - B_2 + 4B_3y^2 + 96W_0y^2}{6y^2}, \tag{157}\]

where we have introduced new constants \(B_2, B_3, W_0\) such that:

\[b_2 = \frac{B_2 - B_3 - 12W_0}{40500}, \quad b_3 = \frac{B_2 + B_3 + 12W_0}{40500}, \quad w_0 = \frac{3375(B_3 + 18W_0)}{4(353B_2 - 272B_3 - 3264W_0)}. \tag{158}\]

Then equation [174] transforms into the linear equation

\[\frac{d^3\hat{u}}{d\hat{y}^3} = \frac{9}{2} \frac{d^2\hat{u}}{d\hat{y}^2} B_2^2 + 48B_3W_0 + 1152W_0^2 - 72W_0 \hat{y} \tag{159}\]

7.3 Case (C)

The original Hamiltonian

\[\mathcal{H}_{DIVC} = \frac{w_3^2 + w_4^2 + \cos^2(w_1) + \frac{c_2}{\cosh^2(w_2)} + c_3 \left(\frac{1}{\sin^2(w_1)} - \frac{1}{\sin^2(w_2)}\right)}{\sinh^2(2w_2) + \frac{a^2}{\sin^2(2w_1)}}, \tag{160}\]

can be written in the following equivalent form with sinh, and cosh replaced by exp, i.e.:

\[\mathcal{H}_{DIVC} = -\frac{w_3^2 + w_4^2 + \cos^2(w_1) + \frac{c_2}{\sin^2(w_2)} + c_3 \left(\frac{1}{\sin^2(w_1)} - \frac{1}{\sin^2(w_2)}\right)}{\frac{\sin^2(2w_1)}{\sinh^2(2w_2) + \frac{a^2}{\sin^2(2w_1)}}}. \tag{161}\]

Before applying the reduction method [18], we introduce the following transformations of dependent variables, in order to render the next calculations more amenable to a computer algebraic softwares such REDUCE and MAPLE, i.e.:

\[w_1 = \arccos r_1, \quad w_2 = \log r_2, \quad w_3 = \sqrt{r_3}, \quad w_4 = \sqrt{r_4}. \tag{162}\]

and then choose \(r_2\) as a new independent variable \(y\) which gives rise to the following three equations:

\[
\begin{align*}
\frac{dr_1}{dy} &= -\frac{1}{y} \sqrt{\frac{r_3(1 - r_4^2)}{r_4}}, \\
\frac{dr_3}{dy} &= -\frac{2\sqrt{r_3N_3}}{y\sqrt{1 - r_4^2\sqrt{r_3}r_1D}}, \\
\frac{dr_4}{dy} &= \frac{8N_4}{(y^4 - 1)D},
\end{align*}
\tag{163}\]
where:

\[ N_3 = 16y^4 (c_1 - c_3) (a + 2) r_1^4 + [(14c_1 + 8c_2 + 10c_3) a + 36c_1 - 16c_2 - 20c_3] y^4 \]

\[ + [2 (y - 1)^2 (y + 1)^2 (y^2 + 1)^2 (a - 2) r_3 + 2 (y - 1)^2 (y + 1)^2 (y^2 + 1)^2 (a - 2) r_4 \]

\[ + 8y^2 (c_2 - c_3) (a - 2) y^4 + ((-4c_1 - 2c_2 - 2c_3) a - 8c_1 + 4c_2 + 4c_3) y^2 + (c_2 - c_3) (a - 2)) r_1^2 \]

\[ - (y - 1)^2 (y + 1)^2 (y^2 + 1)^2 (a - 2) r_3 - (y - 1)^2 (y + 1)^2 (y^2 + 1)^2 (a - 2) r_4 \]

\[ + (c_1 - c_3) (a - 2) y^8 - 4 (c_2 - c_3) (a - 2) y^6 - 4 (c_2 - c_3) (a - 2) y^2 + (c_1 - c_3) (a - 2), \]

\[ (164) \]

\[ N_4 = 8y^2 \left[ \left( y^4 + 1 \right) r_3 + \left( y^4 + 1 \right) r_4 + 2y^2 (c_2 - c_3) \right] (a + 2) r_1^4 \]

\[ + 8y^2 \left[ - \left( y^4 + 1 \right) (r_3 + r_4) + (c_1 - c_3) y^4 + (-2c_2 + 2c_3) y^2 + c_1 - c_3 \right] (a + 2) r_1^2 \]

\[ + (c_2 - c_3) (a - 2) y^8 + \left[ (-8c_1 - 4c_2 - 4c_3) a - 16c_1 + 8c_2 + 8c_3 \right] y^2 + 6 (c_2 - c_3) (a - 2) y^4 \]

\[ + \left[ (-8c_1 - 4c_2 - 4c_3) a - 16c_1 + 8c_2 + 8c_3 \right] y^2 + (c_2 - c_3) (a - 2), \]

\[ D = -16y^4 (a + 2) r_1^4 + 16y^4 (a + 2) r_1^2 + \left( y^4 - 1 \right)^2 (a - 2). \]

\[ (165) \]

From the Hamiltonian \[ \mathcal{H}_{D^{IVC}}, i.e. \]

\[ H_{DIVC} = \left\{ 4 \left( y^4 - 1 \right)^2 (r_3 + r_4) + 16y^2 \left( (c_2 - c_3) \left( y^4 + 1 \right) - 2 (c_2 + c_3) y^2 \right) \right\} \]

\[ - \left\{ 4 \left( y^4 - 1 \right)^2 (r_3 + r_4) - 4 (c_1 - c_3) \left( y^4 + 1 \right) + 16 (c_2 - c_3) y^2 \left( y^4 + 1 \right) + 8 (c_1 - 4c_2 - 5c_3) y^4 \right\} \]

\[ (y^4 - 1)^2 (a - 2) - 16y^4 (a + 2) r_1^2 (r_1^2 - 1) \]

\[ = h_0, \]

\[ (166) \]

we can derive:

\[ r_3 = -r_4 - \frac{c_1}{r_1^2} - \frac{4y^2c_2}{(y^2 + 1)^2} + \frac{(4r_1^2y^2 + y^4 - 6y^2 + 1)c_3}{(y^2 - 1)^2(r_1^2 - 1)} + \frac{(a - 2)h_0}{4r_1^2(r_1^2 - 1)} - \frac{4(a + 2)h_0y^4}{(y^4 - 1)^2}, \]

\[ (167) \]

with \( h_0 \) an arbitrary constant. Consequently, the third equation in \[ (163) \] becomes:

\[ \frac{dr_4}{dy} = 8y \frac{(y - 1)^4(y + 1)^4c_2 - (y^2 + 1)^4c_3 + 2h_0y^2(y^4 + 1)(a + 2)}{(y^4 - 1)^3}, \]

\[ (168) \]

that can be easily integrated, i.e.:

\[ r_4 = 4c_2 \frac{(y^2 + 1)^2 - y^2}{(y^2 + 1)^2} + 4c_3 \frac{y^4 - y^2 + 1}{(y^2 - 1)^2} - 2(a + 2)h_0 \frac{y^8 + 1}{(y^4 - 1)^2} + w_0 \]

\[ (169) \]

with \( w_0 \) an arbitrary constant. If we introduce new constants \( C_2, C_3, C_1, A \) as follows:

\[ c_2 = C_2 + c_3, \]

\[ c_3 = \frac{4h_0 - w_0 - C_3 - 4C_2 + 2ah_0}{8}, \]

\[ c_1 = \frac{-2ah_0 - C_1 + 4h_0}{8}, a = \frac{4C_2 + 9C_3 + w_0 - A - C_1}{4h_0}, \]

\[ (170) \]

then we are left with the following simplified expression of the first equation in \[ (163): \]

\[ \frac{dr_1}{dy} = \frac{y^4 - 1}{2yr_1} \sqrt{\frac{8C_3r_1^4 - Ar_1^2 - C_1}{2C_3y^8 + 8C_2y^6 + 4w_0y^4 + 8C_2y^2 + 2C_3}}, \]

\[ (171) \]

which could be solved by quadratures. However, if we solve it with respect to \( A \), and derive once by \( y \), then a second-order equation is obtained, that admits a three-dimensional Lie symmetry algebra sl(2,\mathbb{R}), and as a particular case if \( C_1 \) is equal to zero then it is linearizable since it admits an eight-dimensional Lie symmetry
algebra $\mathfrak{sl}(3, \mathbb{R})$. If we solve this second-order equation with respect to $C_1$, and derive once by $y$, then the following third-order equation is obtained ($r_1 \equiv u$):

$$\frac{d^3 u}{dy^3} = \frac{3}{u} \frac{du}{dy} \frac{d^2 u}{dy^2} - \frac{3}{y^2} \left[ C_3 y^{12} - (5C_3 + 2w_0) y^8 - 24C_2 y^6 - (3C_3 + 6w_0) y^4 - 8C_2 y^2 - C_3 \right] \frac{d^2 u}{dy^2} + \frac{1}{u} \left( \frac{d^2 u}{dy^2} \right)^2 + \frac{3}{y^4 (y^4 - 1)(C_3 y^8 + 4C_2 y^6 + 2w_0 y^4 + 4C_2 y^2 + C_3)} \frac{du}{dy},$$

which is linearizable since it admits a seven-dimensional Lie symmetry algebra. In fact, the two-dimensional abelian intransitive subalgebra generated by the two operators

$$u^{-1} \partial_u, \quad \frac{C_3 + 2C_2 y^2 + C_3 y^4}{uy^2} \partial_u$$

when put into the canonical form $\partial_u, \tilde{y} \partial_u$ yield the new dependent and independent variables, i.e.

$$\tilde{u} = \frac{u^2}{2}, \quad \tilde{y} = \frac{C_3 + 2C_2 y^2 + C_3 y^4}{y^2},$$

that transform equation (174) into the linear equation

$$\frac{d^3 \tilde{u}}{d\tilde{y}^3} = \frac{3 \tilde{y}}{\omega^2 - \tilde{y}^2} \frac{d^2 \tilde{u}}{d\tilde{y}^2},$$

with $\omega^2 = 2(2C_2^2 + C_3^2 - C_3 w_0)$.

### 7.4 Case (D)

The Hamiltonian

$$H_{DIVD} = -4w_1^2 w_2^2 \frac{w_1^2 + w_2^2 + d \left( \frac{1}{w_1} + \frac{1}{w_2} \right)}{(a + 2) w_1^2 + (a - 2) w_2^2}$$

is a subcase of Hamiltonian $H_{DIVA}$, with $a_1 = a_3 = 0$ and $a_2 = d$. Consequently, its corresponding Hamiltonian equations, i.e.:

$$\begin{align*}
    w_1 &= -\frac{8w_1^2 w_2^2 w_3}{a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2)}, \\
    w_2 &= -\frac{8w_1^2 w_2^2 w_4}{a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2)}, \\
    w_3 &= \frac{8w_1^2 w_2^2}{(a + 2(w_1^2 + w_2^2))} \left( aw_2^2 (w_3^2 + w_4^2) - 4d - 2w_3^2 (w_3^2 + w_4^2) \right), \\
    w_4 &= \frac{8w_1^2 w_2}{(a + 2(w_1^2 + w_2^2))} \left( aw_1^2 (w_3^2 + w_4^2) + 4d + 2w_1^2 (w_3^2 + w_4^2) \right),
\end{align*}$$

(179)

can be reduced to the following system of three equations, after making the substitutions (137) and choosing $r_2$ as a new independent variable $y$:

$$\begin{align*}
    \frac{dr_1}{dy} &= \sqrt{r_1 r_3} \frac{r_1}{y r_4}, \\
    \frac{dr_3}{dy} &= -\frac{\sqrt{r_1 r_3}}{r_1 \sqrt{y r_4}} (a(r_1 + y) + 2(r_1 - y)) \left( ay(r_3 + r_4) - 4d - 2y(r_3 + r_4) \right), \\
    \frac{dr_4}{dy} &= \frac{1}{y (a(r_1 + y) + 2(r_1 - y))} \left( ar_1(r_3 + r_4) + 4d + 2r_1(r_3 + r_4) \right).
\end{align*}$$
Then,

\[ r_3 = -\frac{4r_1r_4y + ((a + 2)r_1 + (a - 2)y)h_0 + 4d(r_1 + y)}{4r_1y}, \]  

(181)

and

\[ r_4 = \frac{4r_0y - (a + 2)h_0 - 4d}{4y}, \]  

(182)

and the first equation in (180) becomes:

\[ \frac{dr_1}{dy} = \sqrt{\frac{(2 - a)h_0 - 4d - 4r_1r_0}{-(a + 2)h_0 - 4d + 4r_0y}}, \]  

(183)

which could be easily solved by quadratures. However, if we make the simplifying substitution \( d = D - (a+2)h_0/4 \), solve the first-order equation (183) with respect to \( h_0 \), and derive once with respect to \( y \), then the following linear second-order equation is obtained:

\[ 2(r_0y - D)\frac{d^2r_1}{dy^2} + r_0 \frac{dr_1}{dy} + r_0 = 0. \]  

(184)

## 8 Conclusions

In this paper, nineteen classical superintegrable system in two-dimensional non-Euclidean spaces are shown to possess hidden symmetries leading to linearity. This fulfills the conjecture that we made in [1], namely that all classical superintegrable system in two-dimensional space hide linearity regardless of the separation of variables of the corresponding Hamilton-Jacobi equation, and of the order of the first integrals.

In some cases, we have used the Hamiltonian in order to derive one of the two momenta as function of the other momentum and coordinates. None of the other two known first integrals have been used. In other cases, one of the equation of the Hamiltonian system could be integrated by quadrature, and that was all we needed in order to then find the hidden symmetries leading to linear equation of either second or third order.

As we stated in [1], it remains an open-problem to see if linear equations are hidden in (maximally?) superintegrable systems in \( N > 2 \) dimensions, regardless of the separability of the corresponding Hamilton-Jacobi equation, and the degree of the known first integrals.

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## References

[1] G. Gubbiotti and M.C. Nucci, Are all classical superintegrable systems in two-dimensional space linearizable?, *J. Math. Phys.* 58, 012902 (2017).

[2] J. Friš, V. Mandrosov, Ya.A. Smorodinski, M. Uhlíř, and P. Winternitz, On higher symmetries in Quantum Mechanics, *Phys. Lett.* 13, 354–356 (1965).

[3] F. Tremblay, A.V. Turbiner, and P. Winternitz, An infinite family of solvable and integrable quantum systems on a plane, *J. Phys. A: Math. Theor.* 42, 242001 (2009).

[4] S. Gravel, Hamiltonians separable in Cartesian coordinates and third-order integrals of motion, *J. Math. Phys.* 45, 1002–1019 (2004).
[5] V. Perlick, Bertrand Spacetimes, *Class. Quant. Grav.* 9, 1009–1021 (1992).

[6] A. Ballesteros, A. Enciso, F.J. Herranz and O. Ragnisco, Bertrand spacetimes as Kepler/oscillator potentials, *Class. Quantum Grav.* 25, 165005 (2008).

[7] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, and D. Riglioni, Superintegrable quantum oscillator and Kepler-Coulomb systems on curved spaces, in *Symmetries and Groups in Contemporary Physics*, (Eds: Chengming Bai, Jean-Pierre Gazeau, and Mo-Lin Ge), World Scientific, Singapore, 211–216 (2013).

[8] D. Riglioni, Classical and quantum higher order superintegrable systems from coalgebra symmetry, *J. Phys. A: Math. Theor.* 46, 265207 (2013).

[9] N.S. Manton, Monopole interactions at long range, *Phys. Lett. 154 B*, 397–400 (1985).

[10] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, and D. Riglioni, Superintegrable Oscillator and Kepler Systems on Spaces of Nonconstant Curvature via the Stäckel Transform, *SIGMA 7*, 048 (2011).

[11] D. Latini and O. Ragnisco, The classical Taub-Nut system: factorization, spectrum generating algebra and solution to the equations of motion, *J. Phys. A: Math. Theor.* 48, 175201 (2015).

[12] E.G. Kalnins, J.M. Kress, P. Winternitz, Superintegrability in a two dimensional space of nonconstant curvature, *J. Math. Phys.* 43, 970–983 (2002).

[13] E.G. Kalnins, J.M. Kress, W. Miller, P. Winternitz, Superintegrable systems in Darboux spaces, *J. Math. Phys.* 44, 5811–5848 (2003).

[14] M.C. Nucci and P.G.L. Leach, The harmony in the Kepler and related problems, *J. Math. Phys.* 42, 746–764 (2001).

[15] M. Marcelli and M.C. Nucci, Lie point symmetries and first integrals: the Kowalevsky top, *J. Math. Phys.* 44, 2111–2132 (2003).

[16] M.C. Nucci and S. Post, Lie symmetries and superintegrability, *J. Phys. A: Math. Theor.* 45, 482001 (2012).

[17] M.C. Nucci, Ubiquitous symmetries, *Theor. Math. Phys.* 188, 1361–1370 (2016).

[18] M.C. Nucci, The complete Kepler group can be derived by Lie group analysis, *J. Math. Phys.* 37, 1772–1775 (1996).

[19] S. Post and P. Winternitz, A nonseparable quantum superintegrable system in 2D real Euclidean space, *J. Phys. A: Math. Theor.* 44, 162001 (2011).

[20] S. Lie, *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*, Teubner, Leipzig (1912).

[21] P.G.L. Leach, Equivalence classes of second-order ordinary differential equations with only a three-dimensional Lie algebra of point symmetries and linearisation, *J. Math. Anal. Appl.* 284, 31–48 (2003).

[22] J. Patera and P. Winternitz, Subalgebras of Real Three- and Four-Dimensional Lie Algebras, *J. Math. Phys.* 18, 1449–1455 (1977).

[23] T. Cerquetelli, N. Ciccoli, M.C. Nucci, Four dimensional Lie symmetry algebras and fourth order ordinary differential equations, *J. Nonlinear Math. Phys.* 9-s2, 24–35 (2002).