Real scalar field stars of the EKG equations including matter

A. Cabo Montes de Oca and D. Suarez Fontanella

Theoretical Physics Department, Instituto de Cibernética,
Matemática y Física, Calle E, No. 309, Vedado, La Habana, Cuba.

Static (not stationary) solutions of the Einstein-Klein-Gordon (EKG) equations including matter are obtained for real scalar fields. The scalar field interaction with matter is considered. The introduced coupling allows the existence of static solutions in contraposition with the case of the simpler EKG equations for real scalar fields and gravity. Surprisingly, when the considered matter is a photon-like gas, it turns out that the gravitational field intensity at large radial distances becomes nearly a constant, exerting an approximately fixed force to small bodies at any distance. The effect is clearly related with the massless character of the photon-like field. It is also argued that the gravitational field can generate a bounding attraction, that could avoid the unlimited increase in mass with the radius of the obtained here solution. This phenomenon, if verified, may furnish a possible mechanism for explaining how the increasing gravitational potential associated to dark matter, finally decays at large distances from the galaxies. A method for evaluating these photon bounding effects is just formulated in order to be further investigated.

I. INTRODUCTION

The search for boson and fermion-boson stars is a subject of current interest in modern Astrophysics. The interest in the theme is intensified the circumstances related with the search for purely boson stars. As it is known, the Einstein-Klein-Gordon (EKG) equations have no static solutions for real scalar fields. Also, although the complex EKG equations show centrally symmetric solutions, they need to be time dependent in a stationary form (harmonic). In the classic references on fermion-boson stars, it was investigated the existence and stability of such systems. The discussion in these works defined conditions for the existence of such objects for a general case, but in which still the interactions between the scalar fields and matter where not considered.

In the present work, we relax the assumption of the lack of interaction between the scalar field and matter for the considered systems. This is done in order to inspect the possibility for the appearance of solutions in which the scalar field becomes static, that is, time independent when real scalar field are considered. In order to simplify the discussion, we write the EKG equations by considering matter described by simple constituent relations including the photon gas one. The interaction between the scalar field and the gas is introduced by assuming that the scalar field source is proportional to the matter energy density.

Firstly, it is considered a matter being close to the limit of pressureless gas. The initial conditions for the energy density at the coordinate origin, for a centrally symmetric solution was then fixed to a given value. Further, the equations were solved by specifying a tentative scalar field value at the origin. This first step solutions was then inspected for the behavior of the scalar field at large radial distance. This behavior emerged in only two types: a) One in which the scalar field tends to be singular an positive at some radial distance, and b) Another, in which the field tended to be also singular but negative at some radius value. Then, we noted that, if the field tends to be positive (negative) valued, the reduction (increasing) of its initial value at origin, reduced the tendency to positive (negative) values, by at the same time always augmenting the radius value at which the field become singular. Therefore, after properly selecting the initial values by iterating the above described process, a solution in which the scalar field attains a Yukawa like behavior at infinity can be approached. The resulting energy and pressure of the matter became centered in a bounded spherical vicinity of the origin. As for the temporal $v$ and the radial $u$ diagonal components of the metric and its inverse, respectively, they both approach constants at large distances, thus reproducing the Minkowski space-time. At short distances the temporal metric defines a gravitational potential which bounds the matter to the origin, at which its minimum sits. These results indicate the existence of static solutions of the EKG equations in presence of matter, when it interacts with the scalar field. The stability of the solutions is preliminarily investigated. For this purpose, a usual stability criterion for simpler stars constituted by fluids is verified to be valid.

In second place, we also examined the solution associated to a matter related to a photon-like gas. The procedure for obtaining the solution was identical. As before, after fixing the matter density and the scalar field at the origin, the before described process to define a decreasing scalar field at large distances became closely similar. However, a different outcome arose in connection with the temporal component of the metric at large distances. It, surprisingly got a nearly linear radial dependence at large distance, in place of the expected constant behavior associated to the Minkowski space-time. This potential behavior corresponds to a approximately constant gravitational force over any small body at large distances. Such a behavior should be associated to the special massless character of the photon-like...
field, which might not allow the gravitational forces to fully confine the massless particles as they bound massive ones.

The above results strongly motivates to consider a photon-like field to describe the velocity vs radius curves of the galaxies, being associated to dark-matter. In addition, although it not corresponds to the main current point of view that real photons can act as dark-matter ([7],[8]), we also examine this question, seeking for unnoticed possibilities.

In a preliminary study of these issues, we considered the determination of the parameters of the considered star, in order to reproduce the experimentally observed velocity vs radius curves. For this purpose the mass of the scalar field was fixed to a value defining a galaxy of a typical size. The rotation curve following was of the type C of the three A,B and C kinds in which the galaxies rotation curves are classified [4]. That is, the velocity curve after rapidly growing for small radial distances, continue to grow, but with a smaller slope. Then, the value of the photon like particles energy density at the radial distances near outside of the galaxy was evaluated and compared with the CMB energy density. The obtained result for the energy density at such points resulted enormously higher than the CMB energy density. Therefore, this first checking of the possibility that dark-matter were constituted by real photons and not dark-matter analogous massless particles gave a negative result.

However, some causes for the discrepancy of the result for the energy density can be thought. One them, is that EKG equations solved here consider the photons as obeying a free dispersion curve, which leads to the employed equation of state: $\epsilon = 3 p$. In this sense, it can be noted that the photons under the gravitational attraction, can be imagined to acquire some of the properties of confined photons in a resonant cavity. By example a discrete spectrum, that could have similar effects as a mass for photons. In order to discuss this possibility a simple model is presented in which a Newtonian gravitational potential is able to create mass like terms in the Maxwell equations. They, explicitly makes the photon wave to decrease in the direction in which the potential increases. In a coming work, we expect to consider a classicalstatistical model for photons in each space-time point, but which statistics will be controlled by the local values of the metric. This consideration, we expect that can introduce an attracting action of the gravitational field upon the photon density. This effect could bound them, by consequently stopping the growing of the potential and leading to an asymptotic Minkowski space. Upon this, it could be the case that the energy density of the modified galaxy solution can perhaps tend to the CMB density far form the galaxy [10].

We also examined the bounds posed on the real photons acting as dark-matter, by the fact that the photon matter should not be observable. In this sense, the Tolman theorem expressing the temperature a black-body radiation subject to gravitational field becomes helpful. Let us assume as above, that photons are able to be trapped in galaxies by the mentioned self-consistent effects. Then, the natural value of the radiation temperature in external zones to the galaxies is the Cosmic-Microwave-Background one $T_{ cmb}$. Therefore, from Tolman theorem it follows that no matter the high temperatures the photon radiation can attain at some interior point of the structure, the radiation coming to the Earth from that point, will always have nearly 3 Kelvin degrees of temperature. Thus, it could not be easily observed. This is a conclusion not ruling out the possibility for the real photons playing the role of dark-matter. A deeper investigation should however be pursued in order to check, if there is still some way for the CMB could be in thermal equilibrium with an internal to the galaxy photon gas defining the halo. It should be recognized that the enormous difference between the energy densities allowing the photons to reproduce the rotation curves, and the CMB energy density suggests that the answer should be negative. However, the mechanism discussed here has not clear restrictions to work if a photon like but really dark matter is constituting the dark matter.

In section 2, the Lagrangian of the system is presented and the equations for the time independent spherically symmetric solutions written. Section 3, exposes the derivation of the static solution for nearly pressureless matter. Next, section 4, is devoted to describe the second solution associated to the photon-like matter. Section 5, considers the preliminary inspection of the possibility that the second solution might describe the velocities vs radius curves in galaxies. The conclusions resume the content of the work and describe possible extensions.

II. THE FIELD EQUATIONS

Let us consider the metric defined by the following squared interval and coordinates

$$ds^2 = v(\rho)dx^2 - u(\rho)^{-1}d\rho^2 - \rho^2(\sin^2\theta \, d\varphi^2 + d\theta^2), \quad (1)$$
$$x^0 = c \, t, \quad x^1 = \rho, \quad (2)$$
$$x^2 = \varphi, \quad x^3 = \theta. \quad (3)$$
where the CGS unit system is employed. Therefore, the Einstein tensor $G_{\mu \nu}$ components in terms of functions $u, v$ and the radial variable $\rho$ are evaluated in the form

$$G^0_0 = \frac{u'}{\rho} - \frac{1 - u}{\rho^2}, \quad (4)$$

$$G^1_1 = \frac{u v'}{v \rho} - \frac{1 - u}{\rho^2}, \quad (5)$$

$$G^2_2 = G^3_3 = \frac{u}{2v} v'' + \frac{u u'}{2v} - \frac{u'}{v}, \quad (6)$$

The physical system interacting with gravity that will be considered is composed of a scalar field and gas of matter. The scalar field will be assumed to also linearly interact with an external source associated to it. The action of the field will have the form

$$S_{\text{mat-\phi}} = \int L \sqrt{-g} \, d^4x, \quad (7)$$

with a Lagrangian density given by

$$L = \frac{1}{2} (g^{\alpha \beta} \Phi_{,\alpha} \Phi_{,\beta} + m^2 \Phi^2 + 2 J(\rho) \Phi), \quad (8)$$

in which the $A_{\alpha}$ mean the derivative of $A$ over the variable $\alpha$.

This Lagrangian determines an energy momentum of the form

$$(T_{\text{mat-\phi}})^{\nu}_{\mu} = \frac{\delta^{\nu}_{\mu}}{2} (g^{\alpha \beta} \Phi_{,\alpha} \Phi_{,\beta} + m^2 \Phi^2 + 2 J(\rho) \Phi), \quad (9)$$

which afterwards can be added to the energy momentum tensor of the matter ([12]):

$$(T_{\text{e,p}})^{\nu}_{\mu} = P \delta^{\nu}_{\mu} + u^{\nu} u_{\mu}(P + e), \quad (10)$$

to write the total energy momentum tensor as

$$T^{\nu}_{\mu} = \frac{\delta^{\nu}_{\mu}}{2} (g^{\alpha \beta} \Phi_{,\alpha} \Phi_{,\beta} + m^2 \Phi^2 + 2 J(\rho) \Phi) + g^{\alpha \nu} \Phi_{,\alpha} \Phi_{,\mu} + P \delta^{\nu}_{\mu} + u^{\nu} u_{\mu}(P + e). \quad (11)$$

Since static field configurations are being searched, the four-velocity reduces to the form in the local rest system

$$u^{\mu} = (1, 0, 0, 0). \quad (12)$$

After these definitions, the Einstein equations can be written as follows

$$G^{\nu}_{\mu} = \kappa T^{\nu}_{\mu}, \quad (13)$$

in which both of the tensors appearing are diagonal and the gravitational constant has the value

$$\kappa = 8\pi l_P^2, \quad (14)$$

in terms of the Planck length $l_P = 1.61 \times 10^{-33}$ cm. Their explicit forms are

$$\frac{u'}{\rho} - \frac{1 - u}{\rho^2} = -\kappa \left[ \frac{1}{2} (u \Phi_{,\rho} + m^2 \Phi^2 + 2 j \Phi) + e \right], \quad (15)$$

$$\frac{u v'}{v \rho} - \frac{1 - u}{\rho^2} = \kappa \left[ \frac{1}{2} (u \Phi_{,\rho} - m^2 \Phi^2 - 2 j \Phi) + P \right], \quad (16)$$

$$\frac{\rho^2 u''}{2} + \frac{\rho^2}{4} u' \left( \frac{u'}{u} - \frac{v'}{v} \right) + \frac{\rho}{2} u v' = \kappa \left[ \frac{1}{2} (u \Phi'' + m^2 \Phi^2 + 2 j \Phi) + P \right], \quad (17)$$

$$\frac{\rho^2 v''}{2} + \frac{\rho^2}{4} v' \left( \frac{u'}{u} - \frac{v'}{v} \right) + \frac{\rho}{2} u v' = \kappa \left[ \frac{1}{2} (u \Phi'' + m^2 \Phi^2 + 2 j \Phi) + P \right]. \quad (18)$$
These are four equations, the last two of which are identical. Thus, there are three independent Einstein equations in the problem. But, the third and the equivalent fourth ones can be substituted by a simpler relation. It comes from the Bianchi identities (12):

\[ G^\nu_{\mu \nu} = 0, \] (19)

where the semicolon indicates the covariant derivative of the tensor \( G^\nu_{\mu \nu} \). After assuming the satisfaction of the Einstein equations (13) the \( G^\nu_{\mu \nu} \) tensor in (19) can be substituted by the energy momentum tensor \( T^\nu_{\mu} \) leading to the relation

\[- \Phi^J' + P' + \frac{v'}{2c}(P + e) = 0. \] (20)

This is a dynamic equations for the energy, the pressure and the scalar field, substituting the two equivalent Einstein equations being associated to the both angular directions.

The last of the equations of movement for the system is the Klein-Gordon one for the scalar field. It can be obtained by imposing the vanishing of the functional derivative of the action \( S_{\text{mat-}\phi} \) with respect to the scalar field

\[
\frac{\delta S_{\text{mat-}\phi}}{\delta \Phi(x)} = \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial \Phi_{,\mu}} - \frac{\partial L}{\partial \Phi} \right) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} g^\mu\nu \Phi_{,\nu}) - m^2 \Phi - J = 0, \] (21)

a relation that after employing the temporal and radial Einstein equations in (13) can be rewritten in the form

\[
J(\rho) + m^2 \Phi(\rho) - u(\rho) \Phi''(\rho) = \Phi'(\rho)\left(\frac{u(\rho)}{\rho} + 1 - \rho \kappa \left(\frac{m^2 \Phi(\rho)^2}{2} + J(\rho) \Phi(\rho) + \frac{e(\rho)}{2}\right)\right) \] (22)

Therefore, the three relevant for the problem EKG equations are resumed as

\[
\frac{u'(\rho)}{\rho} - \frac{1 - u(\rho)}{\rho^2} = - \kappa \left[ \frac{1}{2} (u(\rho) \Phi'^2(\rho) + m^2 \Phi(\rho)^2 + 2 J(\rho) \Phi(\rho) + e(\rho)) \right], \] (23)

\[
\frac{u(\rho)}{v(\rho)} \frac{v'(\rho)}{\rho} - \frac{1 - u(\rho)}{\rho^2} = \kappa \left[ \frac{1}{2} (u(\rho) \Phi'^2(\rho) - m^2 \Phi(\rho)^2 - 2 J(\rho) \Phi(\rho) + p(\rho)) \right], \] (24)

\[
J(\rho) + m^2 \Phi(\rho) - u(\rho) \Phi''(\rho) = \Phi'(\rho)\left(\frac{u(\rho)}{\rho} + 1 - \rho \kappa \left(\frac{m^2 \Phi(\rho)^2}{2} + J(\rho) \Phi(\rho) + \frac{e(\rho) - p(\rho)}{2}\right)\right). \] (25)

In order to work with dimensionless forms of the equations, let us define the new radial variable, scalar field and parameters as follows

\[
r = m\rho, \] (26)

\[
\phi(r) = \sqrt{8\pi l_\rho r} \Phi(\rho), \] (27)

\[
j(r) = \sqrt{8\pi l_\rho m^2} J(\rho), \] (28)

\[
\epsilon(r) = \frac{8\pi l_\rho^2}{m^2} e(\rho), \] (29)

\[
p(r) = \frac{8\pi l_\rho^2}{m^2} P(\rho). \] (30)

It should noted that the new variable \( r \) has dimension of \( gr \times cm \).
Therefore, the to be worked EKG equations in the new coordinates will be

\[
\begin{align*}
\frac{u'(r)}{r} - \frac{1 - u(r)}{r^2} &= -\frac{1}{2}(u(r)\phi'(r)^2 + \phi(r)^2 + 2j(r)\phi(r)) - \epsilon(r), \\
\frac{u(r)}{v(r)} \frac{v'(r)}{r} - \frac{1 - u(r)}{r^2} &= -\frac{1}{2}(-u(r)\phi'(r)^2 + \phi(r)^2 + 2j(r)\phi(r)) + p(r), \\
(\epsilon(r) + p(r)) \frac{v'(r)}{2v(r)} - \phi(r) j'(r) &= 0, \\
j(r) + \phi(r) - u(r) \phi''(r) &= \phi'(r)(\frac{u(r) + 1}{r} - r \frac{\phi(r)^2}{2} + j(r)\phi(r) + \frac{\epsilon(r) - p(r)}{2}).
\end{align*}
\]

Note that in order to simplify the notation, the same letter \( u \) and \( v \) had been used to indicates the metric components in the new variables. That is, we will write \( u(r) = u(\rho) \) and \( v(r) = v(\rho) \) in spite of the fact that functional forms of the two quantities can not be equal. This should not create confusion.

A. The constitutive relations and the matter-field interaction

Let us now define the functional form of the constitutive relation for matter as expressed in the new coordinate system

\[\epsilon(r) = n \ p(r),\]

where the parameter \( n \) defines the ratio between the energy density and the pressure. This simple form allows to consider the cases of nearly pressureless dust for \( n \) large and the case of a photon gas for \( n = 3 \).

As mentioned before, the discussion will consider the interaction between matter and the scalar field. It will be taken into account by assuming the source of the scalar field \( j(r) \) becomes proportional to the matter energy \( \epsilon(r) \)

\[j(r) = g \ \epsilon(r).\]

It should be stressed that we have considered the matter in a simpler approach with respect to the more detailed one employed in references [2, 3]. This was done in order to concentrate the discussion in the relevant issue addressed in this paper: the role of the inclusion of matter scalar field interaction.

III. THE EKG INCLUDING MATTER STATIC SCALAR FIELD SOLUTION

Once the physical system had been defined, let us present in this section a particular kind of solutions of the EKG equations for a real scalar field interacting with matter. The objective of the presentation is to show the existence of static configurations of a centrally symmetric star formed by scalar field and matter contents. The found solution tends to reproduce the Schwarzschild space-time with a Yukawa like scalar field decaying at large distances [1].

We start to search for it by fixing initial conditions for the radial evolution defined at very small radial distance \( \delta = 10^{-6} \)

\[
\begin{align*}
u(\delta) &= 1, \\
v(\delta) &= 1, \\
\phi(\delta) &= \phi_0 = 0.65, \\
p(\delta) &= p_0 = 0.0595725.
\end{align*}
\]

These initial conditions were fixed not exactly at the coordinate axis, but at a very small non vanishing radial distance \( \delta \), because the equations for \( u(r) \) and \( v(r) \) are singular at \( r = 0 \). This singularity also enforces the value of \( u \) to tend to one in the limit \( r \rightarrow 0 \) assumed that the solution for this quantity is regular. To solve the equations we programmed the differential equations by using the software Mathematica. The proportionally constant between the scalar field source and the energy density of the matter was chosen as

\[g = 0.9.\]
Figure 1: The figure shows the evolution with the radial coordinate \( r \) for the two fields \( u(r) \) and \( v(r) \). Note that at large radial distances the metric components coincide, which indicates that the metric tensor tends to be the Minkowski one, faraway from the symmetry center. At small distances at which the matter and field energy densities start to grow, \( u \) and \( v \) deviates one from another: \( u \) tends to the unit at the origin and \( v \) reduces its value to a minimum at this point. Note that the trapping of the matter near the origin is compatible with the interpretation of \( v \) as a gravitational potential, which attracts matter to the region in which its minimum appears.

The procedure for obtaining the solutions was as follows. Firstly, it was fixed the specified value for the pressure at the point \( r = \delta \). The required value of \( u(\delta) = 1 \) was imposed. Arbitrarily we fixed \( v(\delta) = 1 \). This arbitrary character of the initial condition for \( v(\delta) \) is associated to the fact that for any solution of the general equations, the multiplication of the function \( v(r) \) by an arbitrary constant is also a solution. Thus \( v(\delta) = 1 \) can be always chosen. The employed property directly follows from the fact that the EKG equations only depend on \( v(r) \) through the ratio \( \frac{v'(r)}{v(r)} \).

Figure 2: The plot show the radial behavior of the scalar field. In the large radial distance region, in which the equations become linear ones, the scalar field gets a decaying value corresponding with the Yukawa solution.

Afterwards, in a first step, the equations were solved by fixing an arbitrary value of the scalar field near the origin \( \phi(\delta) = \phi_0 \). The possible results in this first stage were two fold: a first one in which the scalar field grew to positive singular value at a given radial distance; a second one in which the singular values resulted to be negative also at a specific radial distance. Then, it was possible to note that decreasing (increasing) the value of \( \phi_0 \), reduced the positive (negative) singular values, while at the same time, in both cases the radial distance at which the singularity appears increased. Repeating iteratively this process of adjustment of the values of \( \phi_0 \), the singularity position was fixed each time at larger distances. After this, it became clear that the scalar field solution tended to reproduce the small field Yukawa like solution of the Klein-Gordon equation. Upon this, the matter and scalar field energy densities and pressures became concentrated in vicinity of the origin.

The figure\(^1\) shows the values of the metric components. As it can be noticed, at large distances from the symmetry center, the metric tends to be the flat space Minkowski one. It should be here remarked that the solution for \( v(r) \) was multiplied by a constant in order to enforce the equality between \( u \) and \( v \) at large distance, which reproduces the Minkowski space in the faraway region. At small distances, the value of \( u \) starts deviating from the value of \( v \). This occurs in the region in which the matter and scalar field densities are mainly concentrated. While \( u \) tends to the unit in approaching the symmetry point, the temporal component of the metric tends to a minimum value. This should be the case, if this metric component takes the role of gravitational potential attracting the matter to the zone in which its minimum value appear.

The resulting solution for the scalar field is shown in figure\(^2\). Far form the center, the behavior is exponential as it should be, because in Minkowski space the only real decaying radially symmetric solution of the Klein-Gordon (KG) equation is the Yukawa potential one.
Figure 3: The two plots show the radial behavior of the total energy density and the total pressure of the solution. The central role of the scalar field-matter interaction in allowing the solution to exist, is compatible with the fact that the matter energy density closely overlaps with the scalar field one.

As for the total energy and pressure, defined as

\[ \epsilon_t(r) = \frac{1}{2} (u(r) \phi'(r)^2 + \phi(r)^2 + 2j(r)\phi(r)) + \epsilon(r), \]  
\[ p_t(r) = \frac{1}{2} (u(r) \phi'(r)^2 - \phi(r)^2 - 2j(r)\phi(r)) + p(r), \]

they show to be concentrated in spatial regions being similar in size to the ones in which the scalar field energy is localized. Their radial dependence are shown in figure 3. This property is compatible with the determining role of the matter-field interaction in allowing the existence of the solution.

A. Stability analysis

The full stability analysis of the considered physical system including a scalar field, should follow from the general linearized equations of the systems. However, this complete discussion requires an involved mathematical discussion which is out of the context of this work. However, we will consider a simpler preliminary analysis. It will be assumed that the inclusion of the scalar field in addition with matter, allows to justify that stability implies the total mass of the solution should grow when the initial condition for the density of matter at the origin is also increased [6].

This criterion can be easily checked for the here obtained solution, by considering the total mass formula

\[ M(\epsilon_0) = \int dr \ 4\pi r^2 \epsilon_t(r), \]  
\[ \epsilon_t(r) = \frac{1}{2} (u(r) \phi'(r)^2 + \phi(r)^2 + 2j(r)\phi(r)) + \epsilon(r), \]

as a function of the initial condition for the matter energy density \( \epsilon(0) = \epsilon_0 = 40 \ p_0 \). Note that the initial value of the scalar field at the origin \( \phi_0 \) is in fact a function of \( \epsilon_0 \), which value was found by imposing the Yukawa behavior of the scalar field at large radius. The new value of the total mass after increasing the matter energy density at origin \( \epsilon_0 \), was then derived by solving the equations for two nearly values of \( \epsilon_0 \), the original one \( \epsilon_0 =2.3829 \) and the closer one \( \epsilon_1 =2.3929 \). After modifying \( \epsilon_0 \) the new value of \( \phi_0 \) was determined by correspondingly modifying the old value in order to assure the Yukawa dependence faraway from the origin. Therefore, the approximate evaluation of the derivative of the mass respect to the central density of matter resulted in

\[ \frac{dM(\epsilon_0)}{d\epsilon_0} \simeq \frac{M(\epsilon_0 + 0.01) - M(\epsilon_0)}{0.01} = 0.03504 > 0. \]

Thus, the obtained solution satisfies this assumed stability criterion, which is valid for standard stars being constituted only by fluids of matter. As mentioned before, the inclusion of the scalar field should imply more general linear equations for the oscillation modes of the system. However, it can be expected that the considered here criterion remains being valid.
It should be noted that the solution derived is valid for any value of the scalar field mass. Thus, its stability criterion works for the whole family of solutions for arbitrary values of this mass. It should be however, still investigated the stability for the star configurations by varying the initial condition for the energy density at the origin and the intensity of the coupling of the matter with the scalar field. This will allow to determine whether there are bounds for the total mass of the obtained solutions. We expect to consider this question in coming works.

In ending this section, it should be remarked that the found static solution had been allowed to exist thanks to the assumed interaction between the scalar field and matter, as defined by the source of the field being proportional to the matter density. It might be also helpful to note, that in reference [11] the same type of interaction was employed to identify a static Universe in which dark-energy interacts with matter. As well as it happened here, the matter-scalar field interaction became central in the existence of the static solution.

**IV. THE SOLUTION FOR PHOTON-LIKE MATTER.**

In this section we will consider the important special case in which the assumed matter constitutive relation is associated to the photon-like gas. That is, massless vector particles are assumed. They could be the real photons or other dark-matter analogous massless vector particles. For these kind of particles the traceless of the energy momentum tensor implies

$$
\epsilon(r) = 3 p(r).
$$

(45)

Figure 4: The plots of the corresponding metric functions $u$ and $v$ with the radial coordinates. They show how the gravitational potential, reflected by $v$ grows nearly linearly with the coordinates at large radii.

Since the considered particles are massless, it is possible that the solution could show special characteristics, as it will be effectively the case.

Figure 5: The radial dependence of the derivative of the function $v(r)$. This quantity defines the gravitational force in the Newtonian limit. The behavior indicates that the force over small massive bodies tends to be nearly constant at large distances.
The method of solving the EKG equations became identical to the one employed for the case of nearly pressureless matter in the past section. The initial conditions for the fields again were defined at the small radial distance $\delta = 10^{-6}$ in the form

\begin{align*}
u(\delta) &= 1, \quad (46) \\
\phi(\delta) &= \phi_0 = 0.623048, \quad (47) \\
p(\delta) &= p_0 = 0.7943. \quad (48)
\end{align*}

The interaction between the scalar field and the matter was fixed at the same value employed in the past section $g = 0.9$.

The EKG equations were then also solved using the operation NDSolve of the programme Mathematica. The general procedure for obtaining the solutions was exactly the same that was followed in the past section. After that, the iterative process led to a definite solution in which the scalar field also decreased at large radial values. However, the net results became radically different from the physical point of view, as it will be described in what follows.

The figure depicts the two metric components. As it can be observed, at large distances from the symmetry center the field $v(r)$ in place of becoming the constant Minkowski value, tends to get linear radial dependence. This linear dependence implies that the gravitational field intensity of the system is not decreasing with the distance. That is, the system is able to produce an attractive nearly constant force on bodies situated faraway from the symmetry center. The plot of this force vs radius is shown in figure 5. The appearance of this force means that the gravitation is unable to trap the photon-like particles close to the centre. This effect is not out of place to occur because the considered particles are massless, which move at the velocity of light. This interpretation is clearly supported by the graph in figure 6 which plots the total mass density up to a radial distance $r$

\begin{align*}
\rho_t(r) &= 4\pi r^2 \epsilon_t(r), \quad (50) \\
\epsilon_t(r) &= \frac{1}{2}(u(r)\phi'(r)^2 + \phi(r)^2 + 2j(r)\phi(r)) + \epsilon(r). \quad (51)
\end{align*}

Figure 6: The figure show the mass density by unit radial distance, as a function of the radius, for the photon-like matter interacting with the scalar field. It shows that the photons are able to escape to relatively faraway regions from the symmetry center.

The picture indicates that the photon-like energy at large distances is not decaying, but slowly increases with the radius. It can remarked that this photon-like energy is what effectively defines this effect, because the radial decaying of the scalar field energy density makes the radial density of scalar field energy ($4\pi r^2$ times this energy density) to rapidly vanish with the radius.

The radial behavior of the scalar field is shown in figure. Far from the center, the field again tends to rapidly decrease. As for the matter density, for this alternative solution there is a region close to the symmetry center showing the highest values of the density. This vicinity is close to the one in which the scalar gets its highest values. The radial dependence of the energy density is shown in figure 8. The vanishing of the energy density at large radii can
give the impression that the total mass of the structure is concentrated near the origin. However, this shown to be not true, by the plot in figure 6.

In the following section, it will be argued that the found solution suggests the possibility that photon-like matter can determine the velocity vs radius curves indicating the presence of dark-matter in the Universe.

V. DARK-MATTER AND PHOTON-LIKE-PARTICLES

In this Section, it will be preliminarily examined the possibility of applying the photon-like solutions to explain the dark-matter properties. For this purpose, lets us assume that the considered solution is furnishing the attraction required to define a rotational galaxy motion. In this situation the circular equation of motion (in the original CGS units) of a given star of mass $\delta m$, after assuming that its motion is defined by the non-relativistic Newton equation, will be

$$-\delta m \frac{d}{d\rho} \Phi_G(\rho) = -\delta m \frac{V(\rho)^2}{\rho}. \quad (52)$$

That is, the gravitational force generated by the potential $\Phi_G$ produces the centripetal acceleration of the circular motion. But, assuming the Newtonian approximation for the gravitation potential it follows

$$\Phi_G(\rho) = \frac{c^2}{2}(v(\rho) - 1), \quad (53)$$

which after substituted in (52) leads for the radial dependence of the velocity the expression

$$V(\rho) = \sqrt{\frac{m c^2}{2} \rho \frac{dv(r)}{dr}} = \sqrt{\frac{c^2}{2} \rho v'(r)}. \quad (54)$$

$$r = m \rho, \quad (55)$$
Figure 9: The expression of the velocity $V(\rho)$ of the rotating star being at radial distance $\rho$ as a function of the variable $r$.

where the derivative of $v$ over the radius $\rho$ measured in cm has been expressed in terms of derivative relative to the radial variables $r = m \rho$. Thus, the velocity at some radial distance $\rho$ is only a function of the $r$ variable. This dependence is illustrated for the solution being considered in the figure 9.

It should be now recalled that we have found the solution for a particular values of the energy density and scalar field at the origin of coordinates. Therefore, the energy of the obtained field configuration is fixed, once its parameters $m$ and $\kappa$ had been specified. Therefore the rotation curve for the particular solution being investigated is only a function of $m$ (assumed the gravitational constant $\kappa$ is given). For illustrative purposes, let us select a value of $m$ (the mass of the scalar field) such that the radial position corresponding the knee in the curve shown in the figure 9 at $r = r_c = 2.9469$ corresponds to a radial distance of

$$\rho_c = 7.517 \times 10^3 \text{ ly.}$$

This is a distance of the order of the sizes of the galaxies. Then, the value of the scalar field mass turns to be

$$m = \frac{r_c}{\rho_c} = 4.1484 \times 10^{-22} \text{ gr.}$$

It should be noted that this is a mass for the boson of nearly one hundred time larger than the proton mass. However, we have only chosen this value for exemplifying the theoretical possibility that solutions of the considered form could be associated to dark-matter. It is clear that many other possibilities for the matter being coupled with the massless field can be considered in substitution of the assumed scalar field. This type of field was relevant here in connection with the first part of the work illustrating the existence of static boson stars under coupling of this field with matter.

Further, the velocity vs radius curve for the considered solution, assumed that the scalar field mass is given by $m = 4.1484 \times 10^{-22}$ gr, is shown in figure 10. Note that the star velocities in km/s are smaller than the light velocity, but not very much: only nearly an order of magnitude. The considered solution defines a mass for the system which is larger than the total mass of galaxies of the same size (by example Messier 33).

Therefore, in conclusion, it can be argued that particular forms of the considered solution turn to be able in furnishing the total mass required for galaxies of the observed types in Nature. It may be also expected that similar effects can be obtained by coupling photon-like matter with the more usual fermion matter.

A. Photon-like dark-matter

It should be remarked that the currently most accepted view about the nature of dark matter is that it is constituted by massive cold particles moving non relativistically. However, the dark matter problem is so relevant that we estimate that it allows to explore all sort of ideas for clarifying it. Since the here discussed solutions indicates the possibility of generating the complementary attractive forces acting in galaxies, below we will consider some attention to examine the possibility that photon-like or real photons could describe the dark matter properties.
Figure 10: The figure shows the rotation curve for an hypothetical galaxy which is generated by the photon-like matter in interaction with a scalar field for the parameters chosen and the initial conditions for the matter at the origin specified. The curve shows a type C of the three sorts (A, B and C) in the classification of the galaxies according to the form of such curves [9]. The type B corresponds almost constant velocities and the A ones are associated to curves in that the velocities firstly rapidly increases and then start diminishing almost linearly. The galaxy was defined by assuming that the value $r_c$ of the variable $r$ at the ”knee” of the curve in figure 9 is defining a radial distance through $\rho_c = \frac{r_c}{m}$.

B. A restriction for real photon matter

Let us firstly consider a required condition for the solution examined here (after assuming that it corresponds to real photons) can describe the galaxy observations associated to dark-matter This requirement for photons to generate dark-matter is that energy distribution of the photons should approach the value associated to the CMB, at large distances from the centre of the galaxy. This is a concrete requirement that can be easily checked for the here obtained solution. By example, the CMB energy density in CGS units has the formula and value

$$\epsilon_{CMB}^{\text{cgs}} = \frac{4}{c} \sigma T^4$$

$$= 6.1236 \times 10^{-13} \text{ erg cm}^{-3},$$

$$T = 3.0 K,$$ (58)

where $\sigma$ is the Stefan-Boltzman constant. But, expressed in the special units defined in this work, it gets the value

$$\epsilon_{CMB} = \frac{8\pi l_p^2 \epsilon_{CMB}^{\text{cgs}}}{m^2}$$

$$= 2.9417 \times 10^{-34}.$$ (59)

This result should be compared with the energy density of the considered solution determining the galaxy size [56]. This size after expressed in the $r$ spatial units used here gets the value

$$r = m \rho_c$$

$$= 19.6016,$$

where $m$, is the mass of the scalar field determining a solution with the required total mass for generating a galaxy of similar size as the Messier 33 one. Therefore the energy density at distances from the symmetry center of the order of the radius of the galaxy, takes the value

$$\epsilon(19.6016) = 1.071 \times 10^{-3} \gg \epsilon_{CMB} = 2.9417 \times 10^{-34}.$$ (60)

This result indicates that the photon like particles energy density of the here investigated solution is enormously much larger than the CMB energy density. Therefore, it allows to conclude that real photons are not able to describe dark-matter in the galaxy, if the solution considered is taken as it is.

However, there are still possibilities for overcoming the above obstacle. It is clear that the growing potential continuously rising up to infinity, which is generated in the considered solution is not observed in Nature. Therefore, some effect should restrict this unlimited increasing. One possibility can be that the photon dynamics could show
a subtlety making the photon density to decrease very much rapidly at the outside of the galaxy. This effect might
be related with the fact that the photon dynamics at any internal point of the galaxy is being assumed as the one
valid for free photons: \( \epsilon = 3 \). But, the photons in the galaxy are subject to the "confining" gravitational potential
growing with the radial coordinates. Thus, the free photons can be expected to be perturbed by the gravitational
attraction. Let us in the following subsection consider a simplified model to argue that such a confining effect can in
fact tend to trap the photons in some circumstances.

C. Mechanisms for gravitationally bounding photons

Let us consider the Maxwell equations in the presence of a special static metric only depending on one Cartesian
coordinates \([10]\)

\[
\left( \nabla^2 - \frac{\partial^2}{\partial x^0 \partial x^0} \right) \vec{E}(x) = \nabla \left( \nabla h(x), \vec{E}(x) \right) - \frac{\partial}{\partial x^0} \left( \frac{\partial}{\partial x^0} h(x) \right) \vec{E}(x) -

\nabla \times \vec{E}(x) = -\frac{\partial}{\partial x^0} \vec{B}(x),
\]

(62)

where \( \vec{h}(x) \) is a function of the determinant of the metric \( g_{\mu\nu} \), which in this subsection will be assumed to be

\[
g_{\mu\nu}(x) = \begin{pmatrix}
g_{00}(x^3) & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

(64)
in the space of the coordinates \( x = (x^0, x^1, x^2, x^3) \). That is, the metric component \( g_{00}(x^3) \), which plays the role of
the gravitational potential in the Newtonian approximation, will be assumed to vary with the coordinate \( x^3 \). The
function \( h(x) \) is given by given as

\[
h(x) = \log(\sqrt{g(x^3)}) = \log(\sqrt{-g_{00}(x^3)}),
\]

(65)

\[
\frac{\partial}{\partial x^3} h(x) = \frac{1}{2} \frac{d}{dx^3} g_{00}(x^3).
\]

(66)

Let us assume now that the metric has the form

\[
g_{00}(x^3) = -(1 + \frac{2\Phi(x^3)}{c^2}) = -(1 - 2h x^3)
\]

(67)

and the gravitational potential \( \Phi(x^3) \) is assumed to satisfies the Newtonian approximation \( \frac{\Phi(x^3)}{c^2} \ll 1 \) in a large
region of the coordinate \( x^3 \). It corresponds with a Newtonian potential attracting the masses to the negative \( x^3 \) axis.
It is possible now to write the simplifying equation for the considered constant potential

\[
\nabla (h(x)) = (0, 0, -h).
\]

(68)

After the above definitions the Maxwell equations reduces to

\[
\left( \nabla^2 - \frac{\partial^2}{\partial x^0 \partial x^0} \right) \vec{E}(x) = -\nabla (h(x)) \times (\nabla \times \vec{E}(x)),
\]

(69)

where we have also used that we will search for waves defined by fields \( \vec{E} \) and \( \vec{B} \) being orthogonal to the \( x^3 \) axis, and
among themselves, which imply

\[
\nabla h(x). \vec{E}(x) = 0,
\]

(70)
This constant form of the gradient of \( h(x) \) allows the equation for the electric field be simplified as follows

\[
(\nabla^2 - \frac{\partial^2}{\partial x^2}) \vec{E}(x) = -h \frac{\partial}{\partial x^3} \vec{E}(x). \tag{71}
\]

This systems of equations can be solved by Fourier transforming in the variables \( x^0 \) and \( x^3 \). Then, expressing for the fields

\[
\vec{E}(x) = (E, 0, 0) \exp(-\epsilon x^0 i + k^3 x^3 i), \tag{72}
\]

reduces the Maxwell equations to

\[
-(k^3)^2 + \epsilon^2 + i h k^3 \vec{E}(x) = 0. \tag{73}
\]

Then, the waves should satisfy the dispersion relation

\[
k^3 = \frac{h i}{2} + \sqrt{\epsilon^2 - \frac{h^2}{4}}. \tag{74}
\]

The explicit form of the solution for the electric field takes the form

\[
\vec{E}(x) = (E, 0, 0) \exp(-\epsilon x^0 i + \sqrt{\epsilon^2 - \frac{h^2}{4}} x^3 i) \times \exp(-\frac{h}{2} x^3). \tag{75}
\]

This expression shows that the presence of the gravitational field introduces a damping of the wave in the direction of its propagation, defined by the positive \( x^3 \) axis. The damping is proportional to the intensity of the gravitational potential \( h \). That is, the gravitational field affects the dynamics of the photons tending to trap them in the region of lower potential values.

This effect, strongly suggests that the gravitational field can play a role determining that found solutions for photon-like matter, after corrected for the photon dynamics, can show metrics tending to the Minkowski one for large values of the radius. It this happens, it becomes an open question again whether such corrected solutions could or not show photon energy densities being able to reduce at large radii to the value associated to the CMB. This was the main obstacle posed before for the photons behave as dark-matter.

In this sense, we would like to formulate a task that could help in discussing the above issue. It can be defined as consisting in formulating a classical statistical descriptions for massless particles, but subject to the local metric in each interior point of a gravitational field region. Such an analysis can correct for the neglected influence of the gravitational field over the employed free photon dispersion relation leading to the \( \epsilon = 3 p \).

In the next subsection we will also remark about an effect that can support that photons can produce the effects of the dark-matter.

\section{D. The Tolman theorem}

If the obstacles noted above can be surmounted, one important concept can play a role in the discussion of the currently mostly rejected possibility that photon can constitute the dark-matter. It is the so called Tolman theorem \cite{13}, that states that the temperature of radiation filling in thermal equilibrium a spatial region in which a static gravitational potential is acting, is scaled with the value of the temporal component of the metric \( g_{00} \) as

\[
T(x) = \frac{T_{\infty}}{g_{00}(x)}, \tag{76}
\]

where \( x = (c t, \vec{x}) \) are the space-time coordinates of a point and \( T_{\infty} \) is the temperature in the assumed faraway spatial Minkowski like regions. Note that in this subsection we have returned to the positive signature metric for \( g_{00} \).

This effect has to do with the possibility that real photon matter can be observed from the Earth, if playing the examined role of dark-matter. It is true, that the considered solution retains its interest if the massless particles generating it are sorts of really dark-matter particle having a photon-like massless vector nature. However, such solutions could be even more motivating if the usual photon can be responsible for creating the dark-matter effects after surpassing the already posed limitations.
For the situation under consideration, the natural setting for the temperature in the regions far from the galaxy is the CMB radiation temperature of nearly 3 K. Therefore, the radiation temperature at the interior point of the galaxy can take very much increased temperatures than 3 K as the gravitational potential reduces near the center. However, the arriving to the Earth radiation coming from an arbitrary interior point of the galaxy, will always have a frequency spectrum in the microwave region. This is because the light should “climb” the gravitational potential barrier, which reduces its frequency spectrum down again to the CMB one. Thus, since the CMB associated frequencies are very far form the observable light spectrum, in this first instance, the real photons might play the role of dark matter. However, it should be recalled that in order to perform this function, it is required to explain how the photon gas at the interior of the structure can become in equilibrium with the CMB at the galaxies exterior regions. At first sight, this possibility seems to be difficult, due to the drastic lack of balance with the CMB radiation which was evaluated before for the solution presented here.

Summary

The role of the interaction between a real scalar field and matter in the solutions of the EKG equations is investigated [1–3, 5]. Stars showing a static scalar field are argued to exist thanks to the interaction between the scalar field and matter. The field, matter and density distributions are evaluated as smooth functions of the radial distance to the symmetry center. The solutions satisfy a standard stability criterion which is obeyed by simpler stars constituted only by fluids [6]. However, to conclude their stable character, a closer investigation should be done of the more complex linearized equations for the oscillation modes, which include a scalar field in addition to a fluid.

The case of photon like matter is also examined. Surprisingly, in this case there exist star like solutions for which the radial gravitational potential at large distances grows linearly. This effect is related with the massless character of the particles being considered. The emerging behavior leads to suspect that it could be helpful for the understanding the origin of the dark-matter effects. In a preliminary examination of this question, we determine the predictions of the solutions for the rotation curves of a galaxy. The parameters of the solution are chosen to define a galaxy of size being similar to the ones observed. The form of the velocity curve obtained corresponds to a C type of galaxies according to a standard classification [9]. It is started to be explored the possibility that correcting the free photon constitutive relation $\epsilon = 3p$ for the effects of the gravitational field may stop the linear growing of the gravitational potential obtained here. A simple model is solved to inspect the action of the Newtonian gravitational potential over a plane wave propagating against it. The solutions supports that the inclusion of the metric in the classical photon statistics can produce a bounding effect over the photons. A model is then just formulated for to consider the corrected statistics. The idea is to derive the constitutive relations for photons moving at the light velocity, but in the space-time dependent metric. This problem is expected to be considered elsewhere.

It is also discussed whether or not a gas of real photons can play the role of dark-matter, or it should be described by photon like real dark-matter particles. For this purpose the evaluation of the energy density of the photon like constitutive matter at the regions outside of the modeled galaxy was evaluated. The result was very much larger than the CMB corresponding value. This evaluation does not support the identification with photons of the photon like matter determining the rotation curves, if the solution is taken as it is. However, the determination of a corrected solution after improving the photon like dynamics, as proposed before, can still allow to satisfy this criterion.

It was also presented an observation based in the Tolman theorem. This theorem defines the temperature of the photon radiation assumed to determine the solution considered here, as the CMB temperature after divided the temporal component of the metric. One important property of the theorem, gives a partial support to photons as creating the dark matter potential. It is the fact that, no matter the high the temperature of the photons will be at an interior point of the galaxy, the light coming to the Earth from this point will always arrive with a $3^\circ K$ frequency spectrum, coinciding with the CMB radiation one, coming from any other direction.

There remains an important issue to be further addressed in connection with the found solutions. It is the question about their stability. For this purpose in a coming study, the spectrum of the linearized equations of motion for small radial perturbations will be investigated.
Acknowledgments

The authors very much thank the support received from the Office of External Activities of ICTP (OEA), through the Network on Quantum Mechanics, Particles and Fields (Net-09).

[1] P. Jetzer, Phys. Rep. 220, 161-227 (1992).
[2] A. B. Henriques, A. R. Liddle and R.G. Moorhouse. Lett. B 233, 99 (1989).
[3] A. B. Henriques, A. R. Liddle and R.G. Moorhouse. Nucl. Phys. B 337, 737 (1990).
[4] A. B. Henriques, A. R. Liddle and R.G. Moorhouse. Lett. B 251, 511 (1990).
[5] S. Valdez-Alvarado, R. Becerri and L. A. Ureña-López. http://arxiv.org/abs/2001.11009v1 (2020).
[6] S. L. Shapiro and S. A. Teukolsky, Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects, Wiley-VCH Verlag GmbH and Co. KGaA (1990).
[7] P. J. E. Peebles, Astrophys. J. 263, L1 (1983).
[8] G. R. Blumenthal, S. M. Faber, J. R. Primack and M. J. Rees, Nature 311, 317 (1984).
[9] R. L. M. Corradi and M. Capaccioli, Astronomy and Astrophysics 237, 36 (1990).
[10] S. A. Diniz and C. Pinheiro, http://arxiv.org/abs/hep-th/0006133v1 (2000).
[11] A. Cabo Bizet and A. Cabo, Astron. Nachr. AN. 340, 841-846 (2019).
[12] S. Weinberg, Gravitation and Cosmology, MIT Press, Boston (1971).
[13] R. C. Tolman, Relativity, Thermodynamics and Cosmology, Clarendon Press, Oxford (1934).