ALTERNATIVE STRUCTURES AND BIHAMILTONIAN SYSTEMS

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Abstract

In the study of bi-Hamiltonian systems (both classical and quantum) one starts with a given dynamics and looks for all alternative Hamiltonian descriptions it admits.

In this paper we start with two compatible Hermitian structures (the quantum analog of two compatible classical Poisson brackets) and look for all the dynamical systems which turn out to be bi-Hamiltonian with respect to them.
1 Introduction.

It is by now well known that the general structures of classical and quantum systems are not essentially different. When considered as abstract dynamical systems on infinite or finite dimensional vector spaces of states, both are "Hamiltonian vector fields" when considered in the Schrödinger picture, and both are "inner derivations" on the Lie algebra of observables with respect to the Poisson brackets and the commutator brackets respectively [1]. Moreover, in some appropriate limit, quantum mechanics should reproduce classical mechanics. From this point of view, it is a natural question to ask which alternative quantum descriptions of a given quantum system would reproduce the alternative classical descriptions known as bi-Hamiltonian descriptions of integrable systems [2]. This question has been addressed recently in several collaborations involving some of us in different combinations [3].

When we consider composite systems and interactions among them it is interesting to analyze to what extent these alternative quantum descriptions survive. This paper is a preliminary step in this direction.

The specific problem we would like to address can then be stated as follows:

"How many different quantum systems may, at the same time, admit of biHamiltonian descriptions with respect to the same alternative structures?"

We will show that for generic, compatible, alternative structures the different admissible quantum systems are pairwise commuting; moreover, in finite dimensions, they close on a maximal torus.

To tackle the stated problem [4], we construct a quantum system out of a given Hermitian structure and then we consider compatible systems out of two Hermitian structures. The "quantum systems" associated with a given Hermitian structure corresponds to the infinitesimal generator of the "phase group". The compatibility condition amounts to the requirement of compatibility (commutativity) of the "phase groups" with respect to both Hermitian structures.

All of our considerations will be carried over on finite-dimensional vector spaces, and only in the final section we shall consider the extension of our results to infinite dimensional Hilbert spaces.

2 Hermitian structures on $\mathbb{R}^{2n}$.

In view of our interest in quantum systems, we will focus here our attention on linear systems. Also, the tensorial structures that will be described below will be assumed to be represented by constant matrices. We will consider, to start with, three relevant tensor structures that can be introduced in $\mathbb{R}^{2n}$, namely a metric tensor of the form:

$$g = g_{jk}dx^j \otimes dx^k$$

(1)
a symplectic structure:

\[ \omega = \omega_{jk} dx^j \wedge dx^k \]  

(2)

and a complex structure \( J \), i.e. a \((1,1)\)-type tensor satisfying:

\[ J^2 = -\mathbb{I} \]  

(3)

We will be interested in the case in which the above structures are admissible. By this we mean the following:

i) Suppose we are given \( g \) and \( J \). We will say that they are admissible, or that \( g \) is ”Hermitian” iff:

\[ g(Jx,Jy) = g(x,y) \quad \forall x,y \]  

(4)

We will always assume this to be the case.

Now, by virtue of \( J^2 = -\mathbb{I} \): \( g(x,Jy) = -g(J(Jx),Jy) = -g(Jx,y) \). Hence:

\[ g(Jx,y) + g(x,Jy) = 0 \]  

(5)

Notice that the previous two equations imply that \( J \) generates finite as well as infinitesimal rotations at the same time. Moreover, Eq.(5) implies that the \((0,2)\)-type tensor:

\[ \omega =: g(J,.) \]  

(6)

is a symplectic form on \( \mathbb{R}^{2n} \) and that:

\[ J = g^{-1} \circ \omega \]  

(7)

By proceeding as before one can prove that:

\[ \omega(Jx,Jy) = \omega(x,y) \]  

(8)

and:

\[ \omega(Jx,y) + \omega(x,Jy) = 0 \]  

(9)

\( J \) will generate then both finite and infinitesimal symplectic transformations.

ii) Alternatively, one could start from \( g \) and \( \omega \) and say that they are admissible iff: \( J = g^{-1} \circ \omega \) satisfies: \( J^2 = -\mathbb{I} \). Written explicitly in components this condition reads:

\[ g^{jk} \omega_{kl} g^{lm} \omega_{mn} = -\delta^j_i \]  

(10)

Remarks.

i) Notice that if the condition (4) does not hold, we can always build a Hermitian structure out of a given \( g \) by substituting it with the symmetrized metric tensor:

\[ g_s(.,.) =: \frac{1}{2} \{g(J,.) + g(.,.)\}. \]  

(11)
which will be positive and nondegenerate if \( g \) is.

Quite similarly\(^5\), if the condition (10) does not hold, then Riesz’s theorem tells us that there exists a nonsingular linear operator \( A \) such that:

\[
\omega(x, y) = g(Ax, y) \tag{12}
\]

and the antisymmetry of \( \omega \) implies:

\[
g(Ax, y) = -g(x, Ay) \tag{13}
\]

i.e. that \( A \) is skew-hermitian: \( A^\dagger = -A \), and: \( -A^2 > 0 \). Let then \( P \) be a (symmetric) nonnegative square root of \( A \). \( P \) will be injective, and so \( P^{-1} \) will be well defined\(^1\). We define then: \( J = : AP^{-1} \) and:

\[
g_{\omega}(., .) =: g(P ., .) \tag{14}
\]

Therefore:

\[
\omega(x, y) = g(Ax, y) = g_{\omega}(Jx, y) \tag{15}
\]

and: \( J^\dagger = -J, J^2 = -I \). The triple \( (g_{\omega}, J, \omega) \) will be then an admissible triple, Eq.(5) will hold true for \( g_{\omega} \) and, moreover:

\[
g_{\omega}(Jx, Jy) = g(Ax, Jy) = -g(AJx, y) = g_{\omega}(x, y) \tag{16}
\]

and Eq. (4) will be satisfied as well.

ii) The adjoint \( A^\dagger \) of any linear operator \( A \) w.r.t. a metric tensor \( g \) is defined by the standard relation:

\[
g(A^\dagger x, y) =: g(x, Ay) \tag{17}
\]

and we can read Eq. (5) as saying that the complex structure \( J \) is skew-adjoint w.r.t. the metric tensor \( g \). Although it may seem elementary, it is worth stressing here that, despite the fact that we are working in a real vector space, the adjoint of \( A \) does not coincide with the transpose \( A^T \) for a general \( g \). Indeed, spelling out explicitly Eq. (17) in terms of matrices leads to:

\[
A^\dagger = g^{-1} A^T g \tag{18}
\]

and therefore, even for real matrices: \( A^\dagger = A^T \) only if the metric is standard Euclidean one and, in general, symmetric matrices need not be self-adjoint.

A linear structure on \( R^{2n} \) is associated with a given dilation (or Liouville) vector field \( \Delta \). Given then a linear structure on \( R^{2n} \), we can associate with every matrix \( A \equiv||A^i_j|| \in \mathfrak{gl}(2n, \mathbb{R}) \) both a \((1, 1)\)-type tensor field:

\[
T_A = A^i_j dx^j \otimes \frac{\partial}{\partial x^i} \tag{19}
\]

\(^1\) In the infinite-dimensional case, it can be proved\(^\dagger\) that \( A \) is bounded and injective, and that \( P \) is also injective and densely defined, so that \( P^{-1} \) is well defined in the infinite-dimensional case as well.
and a linear vector field:

\[ X_A = A^j_i x^j \frac{\partial}{\partial x^i} \]  (20)

The two are connected by:

\[ T_A(\Delta) = X_A \]  (21)

and are both homogeneous of degree zero, i.e.:

\[ L_\Delta X_A = L_\Delta T_A = 0 \]  (22)

The correspondence: \( A \rightarrow T_A \) is (full) associative algebra and Lie algebra isomorphism. The correspondence: \( A \rightarrow X_A \) is instead only a Lie algebra (anti)isomorphism, i.e.:

\[ T_A \circ T_B = T_{AB} \]  (23)

while:

\[ [X_A, X_B] = -X_{[A,B]} \]  (24)

Moreover, for any \( A, B \in \mathfrak{gl}(2n, \mathbb{R}) \):

\[ L_{X_A} T_B = -T_{[A,B]} \]  (25)

This implies that all statements that can be proved at the level of vector fields and/or at that of \((1,1)\) tensors can be rephrased into equivalent statements in terms of the corresponding representative matrices, and vice versa. That is why we will work mostly directly with the representative matrices in what follows.

Out of the Liouville field and the metric tensor we can construct the quadratic function:

\[ g = \frac{1}{2} \eta(\Delta, \Delta) \]  (26)

and the associated Hamiltonian vector field \( \Gamma \) via:

\[ i_\Gamma \omega = -dg \]  (27)

In a given coordinate system \((x^1, ..., x^{2n})\): \( \Delta = x^i \partial/\partial x^i, \Gamma = \Gamma^k \partial/\partial x^k \) and, explicitly: \( \omega_{jk}\Gamma^k = -g_{jk} x^k \), implying:

\[ g^j \omega_{jk} \Gamma^k = -x^j \] i.e.: \( \Gamma^k = J^k_\Gamma x^j \), or:

\[ J(\Gamma) = -\Delta \Leftrightarrow \Gamma = J(\Delta) \]  (28)

\( \Gamma \) will preserve both \( g \) and \( \omega \), and hence \( J \) (more generally, it will preserve any third structure if it preserves the other two):

\[ L_{\Gamma} \omega = L_{\Gamma} g = L_{\Gamma} J = 0 \]  (29)
Given a metric tensor and an admissible symplectic form, an **hermitian structure** on $\mathbb{R}^{2n}$ is a map: $h: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^2$ defined as:

$$h: (x, y) \mapsto (g(x, y), \omega(x, y)).$$

Equivalently (and having in mind quantum systems) one can exploit the fact that $\mathbb{R}^{2n}$ can be given a complex vector space structure by defining, for $z = \alpha + i\beta \in \mathbb{C}$ and $x \in \mathbb{R}^{2n}$:

$$z \cdot x = (\alpha + i\beta) x =: \alpha x + J\beta x;$$

then $h$ will become an hermitian scalar product, linear in the second factor, on a complex vector space, and we can write now:

$$h(x, y) = g(x, y) + ig(Jx, y) \quad (31)$$

and of course a statement equivalent to the previous ones will be:

$$L_\Gamma h = 0 \quad (32)$$

The vector field $\Gamma$ will be therefore a generator of the unitary group on $\mathbb{C}^n$, and will be an instance of what we will call a **quantum system**. More generally, a quantum system will be any linear vector field:

$$\Gamma_A = A^j_k x^k \frac{\partial}{\partial x^j} \quad (33)$$

associated with a matrix: $A = [\Gamma^j_k]$ that preserves both $g$ and $\omega$ or, equivalently, $h$:

$$L_\Gamma h = 0 \quad (34)$$

This defining requirement on $\Gamma_A$ implies that the matrix $A$ in the description of $\Gamma_A$ be at the same time an infinitesimal generator of a realization of the symplectic group $Sp(n)$ and of a realization of the rotation group $SO(2n)$. The intersection of these two Lie algebras yields a realization of the Lie algebra of the unitary group.

### 3 Bihamiltonian descriptions.

Consider now two different Hermitian structures on $\mathbb{R}^{2n}$: $h_1 = g_1 + i\omega_1$ and: $h_2 = g_2 + i\omega_2$, with the associated quadratic functions: $g_1 = g_1(\Delta, \Delta)$, $g_2 = g_2(\Delta, \Delta)$ and Hamiltonian vector fields $\Gamma_1$ and $\Gamma_2$. Then:

**Definition:** $h_1$ and $h_2$ will be said to be **compatible** iff:

$$L_{\Gamma_1} h_2 = L_{\Gamma_2} h_1 = 0 \quad (35)$$
This will imply of course:

\[ L_{\Gamma_1}\omega_2 = L_{\Gamma_1}g_2 = 0 \]  \hspace{1cm} (36)

separately, and similar equations with the indices interchanged.

**Remark.** Notice that, if: \( \omega = \frac{1}{2} \omega_i dx^i \wedge dx^j \) is a constant symplectic structure and: \( X = A^i_j x^j \partial/\partial x^i \) is a linear vector field, then the condition: \( L_X \omega = 0 \) can be written in terms of the representative matrices as the requirement that the matrix \( \omega A \) be symmetric, i.e.:

\[ \omega A = (\omega A)^T \iff \omega A + A^T \omega = 0 \]  \hspace{1cm} (37)

while the condition: \( L_X g = 0 \) implies that the matrix \( g A \) be skew-symmetric, i.e.:

\[ g A + (g A)^T = g A + A^T g = 0 \]  \hspace{1cm} (38)

Notice now that, from: \( i_{\Gamma_1} \omega_2 = -d g_2 \) and: \( L_{\Gamma_1}g_2 = 0 \) we obtain:

\[ 0 = L_{\Gamma_1}(i_{\Gamma_2} \omega_2) =: L_{\Gamma_1} \omega_2(\Gamma_2,. \Gamma_2) = (L_{\Gamma_1}\omega_2)(\Gamma_2,. \Gamma_2) + \omega_2([\Gamma_1, \Gamma_2],.) \]

and as \( L_{\Gamma_1}\omega_2 = 0 \):

\[ i_{[\Gamma_1, \Gamma_2]}\omega_2 = 0 \]  \hspace{1cm} (39)

and similarly for \( \omega_1 \). As neither \( \omega_1 \) nor \( \omega_2 \) is degenerate, this implies that \( \Gamma_1 \) and \( \Gamma_2 \) commute, i.e. that:

\[ [\Gamma_1, \Gamma_2] = 0 \]  \hspace{1cm} (40)

Moreover, remembering that, given a symplectic structure \( \omega \), the Poisson bracket of any two functions \( f \) and \( g \), \( \{ f, g \} \) is defined as: \( \{ f, g \} := \omega(X_g, X_f) \), where \( X_f \) and \( X_g \) are the Hamiltonian vector fields associated with \( f \) and \( g \) respectively, we find: \( 0 = L_{\Gamma_1}g_2 = dg_2(\Gamma_1) = -\omega_2(\Gamma_2, \Gamma_1) \). Hence we find:

\[ \{ g_1, g_2 \}_2 = 0 \]  \hspace{1cm} (41)

where \( \{ .,. \}_2 \) is the Poisson bracket associated with \( \omega_2 \) and similarly for \( \omega_1 \).

**Remark**. The four real conditions: \( \{ g_1, g_2 \}_1,2 = 0 \) and: \( L_{\Gamma_1}\omega_2 = L_{\Gamma_2}\omega_1 = 0 \) are actually equivalent to those stated in complex form in Eq.(35).

Remembering what has already been said about the fact that statements concerning linear vector fields translate into equivalent statements for the (1,1)-type tensor fields having the same representative matrices, and recalling that
the defining matrices of $\Gamma_1$ and $\Gamma_2$ are precisely those of the corresponding complex structures, we see at once that:

$$[\Gamma_1, \Gamma_2] = 0 \iff [J_1, J_2] = 0$$

(42)
i.e. that the two complex structures will commute as well.

In general, given any two $(0, 2)$ (or $(2, 0)$) tensor fields $h$ and $g$ one (at least) of which, say $h$, is invertible, the composite tensor $h^{-1} \circ g$ will be a $(1, 1)$-type tensor. Then, out of the two compatible structures we can build up the two $(1, 1)$-type tensor fields:

$$G = g_1^{-1} \circ g_2$$

and:

$$T = \omega_1^{-1} \circ \omega_2$$

(44)

Actually one can prove at once that the two are related, and indeed direct calculation proves that:

$$G = J_1 \circ T \circ J_2^{-1} \equiv -J_1 \circ T \circ J_2 \iff T = -J_1 \circ G \circ J_2$$

(45)

It turns out that $T$ (and hence $G$) commutes with both complex structures i.e.:

$$[G, J_a] = [T, J_a] = 0, \ a = 1, 2$$

(46)

This follows from the fact that both $G$ and $T$ are $\Gamma$-invariant, i.e.:

$$L_{\Gamma_{1,2}} G = L_{\Gamma_{1,2}} T = 0$$

(47)

and from Eq. (25).

It follows also from Eqns. (25) and (26) that $G$ and $T$ commute, i.e.:

$$[G, T] = 0$$

(48)

Moreover, $G$ enjoys the property that:

$$g_a(Gx, y) = g_a(x, Gy), \ a = 1, 2$$

(49)

Indeed one can prove by direct calculation that:

$$g_1(Gx, y) = g_1(x, Gy) = g_2(x, y)$$

(50)

while:

$$g_2(Gx, y) = g_2(x, Gy) = g_1^{-1}(g_2(x, .), g_2(y, .))$$

(51)

and this completes the proof. Eq.(49) can be seen as saying that $G$ is "self-adjoint" w.r.t. both metrics.
Notice that the derivation of this result does not require the compatibility condition to hold. If the latter is assumed, however, one can prove also that $T$ is self-adjoint w.r.t. both metrics, and that both $J_1$ and $J_2$ are instead skew-adjoint w.r.t. both structures, i.e. that:

$$g_a(Tx, y) = g_a(x, Ty), \quad a = 1, 2$$

(52)

and that:

$$g_1(x, J_2y) + g_1(J_2x, y) = 0 \quad \forall x, y$$

(53)

with a similar equation with the indices interchanged.

Indeed, from, e.g.: $L_{\Gamma_1} \omega_2 = 0$ we obtain, in terms of the representative matrices and using Eq.(37) and: $J_1 = g^{-1}_1 \omega_1$:

$$\omega_2 g^{-1}_1 \omega_1 = \omega_1 g^{-1}_2 \omega_2 \Leftrightarrow \omega_2 g^{-1}_1 \omega_1 = g_1 \omega^{-1}_1 \omega_2$$

(54)

Remembering the definition of $T$, this is equivalent to: $g_1 T = (g_1 T)^T$, and this leads to:

$$T = g^{-1}_1 T^T g_1 = (T^\dagger)_1$$

(55)

where $(T^\dagger)_1$ is the adjoint of $T$ w.r.t. $g_1$. Interchanging indices, one can prove that: $(T^\dagger)_2 = T$ as well.

Concerning the $J$’s (that have already been proved to be skew-adjoint w.r.t. the respective metric tensors), consider, e.g.

$$(J^\dagger)_2 =: g_2^{-1} J^T_1 g_2 = -g_2^{-1} g_1 J^T_1 g_1^{-1} g_2 = -G^{-1} J_1 G = -J_1$$

as $G$ and the $J$’s commute. A similar result holds of course for $J_2$.

Summarizing what has been proved up to now, we have found that $G, T, J_1$ and $J_2$ are a set of mutually commuting linear operators. $G$ and $T$ are self-adjoint, while $J_1$ and $J_2$ are skew-adjoint, w.r.t. both metric tensors.

If we now diagonalize $G$, the $2n$-dimensional vector space $V = \mathbb{R}^{2n}$ will split into a direct sum of eigenspaces: $V = \bigoplus_k V_{\lambda_k}$, where the $\lambda_k$’s ($k = 1, \ldots, r \leq 2n$) are the distinct eigenvalues of $G$. According to what has just been proved, the sum will be an orthogonal sum w.r.t. both metrics, and, in $V_{\lambda_k}$, $G = \lambda_k I_k$, with $I_k$ the identity matrix in $V_{\lambda_k}$. Assuming compatibility, $T$ will commute with $G$ and will be self-adjoint. Therefore we will get a further orthogonal decomposition of each $V_{\lambda_k}$ of the form:

$$V_{\lambda_k} = \bigoplus_r W_{\lambda_k, \mu_k, r}$$

(56)

where the $\mu_k, r$’s are the (distinct) eigenvalues of $T$ in $V_{\lambda_k}$. The complex structures commute in turn with both $G$ and $T$. Therefore they will leave each one of the $W_{\lambda_k, \mu_k, r}$’s invariant.
Now we can reconstruct, using the $g$’s and the $J$’s, the two symplectic structures. They will be block-diagonal in the decomposition (47) of $V$, and on each one of the $W_{\lambda_k,\mu_k}$’s they will be of the form:

$$g_1 = \lambda_k g_2$$
$$\omega_1 = \mu_{k,\lambda_k} \omega_2$$

Therefore, in the same subspaces: $J_1 = g_1^{-1} \omega_1 = \frac{\mu_{k,\mu_k}}{\lambda_k} J_2$. It follows from: $J_1^2 = J_2^2 = -1$ that: $(\frac{\mu_{k,\mu_k}}{\lambda_k})^2 = 1$, whence: $\mu_{k,\mu_k} = \pm \lambda_k$ (and $\lambda_k > 0$).

The index $r$ can then assume only two values, corresponding to $\pm \lambda_k$ and at most $V_{\lambda_k}$ will have the decomposition of $V_{\lambda_k}$ into the orthogonal sum: $V_{\lambda_k} = W_{\lambda_k,\lambda_k} \oplus W_{\lambda_k, -\lambda_k}$. All in all, what we have proved is the following:

**Lemma.** If the two hermitian structures $h_1 = (g_1, \omega_1)$ and $h_2 = (g_2, \omega_2)$ are compatible, then the vector space $V \approx \mathbb{R}^{2n}$ will decompose into the (double) orthogonal sum:

$$\bigoplus_{k=1,\ldots,r; \alpha = \pm} W_{\lambda_k, \alpha \lambda_k}$$

where the index $k = 1, \ldots, r \leq 2n$ labels the eigenspaces of the $(1,1)$-type tensor: $G = g_1^{-1} \circ g_2$ corresponding to its distinct eigenvalues $\lambda_k > 0$, while: $T = \omega_1^{-1} \circ \omega_2$ will be diagonal (with eigenvalues $\pm \lambda_k$) on the $W_{\lambda_k, \pm \lambda_k}$’s, on each one of which:

$$g_1 = \lambda_k g_2$$
$$\omega_1 = \pm \lambda_k \omega_2$$
$$J_1 = \pm J_2$$

As neither symplectic form is degenerate, the dimension of each one of the $W_{\lambda_k, \pm \lambda_k}$’s will be necessarily even.

Now we can further qualify and strengthen the compatibility condition by stating the following:

**Definition:** Two (compatible) Hermitian structures will be said to be in a **generic position** iff the eigenvalues of $G$ and $T$ have minimum (i.e. double) degeneracy.

In general, two appropriate geometrical objects like two $(0,2)$ or $(2,0)$-type tensor fields are said to be in a generic position if they can be “composed” to yield a 1-1 tensor whose eigenvalues have minimum degeneracy. For instance $g_1$ and $g_2$ are in a generic position if the eigenvalues of $G = g_1^{-1} \circ g_2$ have minimum degeneracy, which possibly depends on further conditions: when the compatibility is required, this degeneracy is double. The results that we have just proved

\[\text{[2] Coming, of course, from admissible triples } (g_1, \omega_1, J_1) \text{ and } (g_2, \omega_2, J_2).\]
will imply that each one of the $W_{\lambda_k, \lambda_k}, W_{\lambda_k, -\lambda_k}$ will have the minimum possible dimension, that is two.

Denoting then by $E_k (k = 1, ..., n, \text{now})$ these two-dimensional subspaces, all that has been said up to now can be summarized in the following:

**Proposition:** If $h_1$ and $h_2$ are compatible and in a generic position, then $\mathbb{R}^{2n}$ splits into a sum of $n$ mutually "bi-orthogonal" (i.e. orthogonal with respect to both metrics $g_1$ and $g_2$) two-dimensional vector subspaces: $\mathbb{R}^{2n} = E_1 \oplus E_2 \oplus \ldots \oplus E_n$. All the structures $g_a, J_a, \omega_a$ decompose accordingly into a direct sum of structures on these two-dimensional subspaces, and on each one of the $E_k$’s they can be written as:

$$
\begin{align*}
  g_1|_{E_k} &= \lambda_k (e_1^* \otimes e_1^* + e_2^* \otimes e_2^*); \quad \lambda_k > 0 \\
  J_1|_{E_k} &= (e_2 e_1^* - e_1 e_2^*) \\
  \omega_1|_{E_k} &= \lambda_k (e_1^* \wedge e_2^*) \\
  g_2|_{E_k} &= \varrho_k g_1|_{E_k}; \quad \varrho_k > 0 \\
  J_2|_{E_k} &= \pm J_1|_{E_k} \\
  \omega_2|_{E_k} &= \pm \varrho_k \omega_1|_{E_k}
\end{align*}
$$

(60)

where $e_2 = J_1 e_1$, $e_1$ is any given vector in $E_k$ and the $e^*$’s are the dual basis of the $e$’s. 

Every linear vector field preserving both $h_1 = (g_1, \omega_1)$ and $h_2 = (g_2, \omega_2)$ will have a representative matrix commuting with those of $T$ and $G$, and it will be block-diagonal in the same eigenspaces $E_k$. Therefore, in the generic case, the analysis can be restricted to each 2-dimensional subspace $E_k$ in which the vector field will preserve both a symplectic structure and a positive-definite metric. Therefore it will be in $\text{sp}(2) \cap SO(2) = U(1)$ and, on each $E_k$, it will represent a harmonic oscillator with frequencies depending in general on the $V_k$’s.

Having discussed the general case, and to gather more insight into the problem we are discussing here, we will describe now in full details the two-dimensional case.

**A two-dimensional example.**

Starting from the observation that two quadratic forms\footnote{In other words on each subspace $g_1$ and $g_2$ are proportional, while $J_1 = \pm J_2$ and accordingly $\omega_2 = \pm \varrho \omega_1$.} can always be diagonalized simultaneously (at the price of using a non-orthogonal transformation, if necessary) we can assume from start $g_1$ and $g_2$ to be of the form:

$$
\begin{pmatrix}
  g_1 & 0 \\
  0 & g_2
\end{pmatrix}
$$

(61)

and:

$$
\begin{pmatrix}
  \sigma_1 & 0 \\
  0 & \sigma_2
\end{pmatrix}
$$

(62)

\footnote{One of which is assumed to be positive.}
The more general $J$ such that $J^2 = -1$ will be of the form:

$$J = \begin{vmatrix} a + a^2 & b \\ \frac{b}{a} & -a \end{vmatrix}$$

(63)

Compatibility with $g_1$ requires that $J$ be anti-hermitian (w.r.t. $g_1$), and this leads to:

$$J = J_{1\pm} = \begin{vmatrix} 0 & \pm \sqrt{q_2 q_1} \\ \pm \sqrt{q_2} & 0 \end{vmatrix}$$

(64)

and similarly:

$$J = J_{2\pm} = \begin{vmatrix} 0 & \pm \sqrt{\sigma_2 \sigma_1} \\ \pm \sqrt{\sigma_2} & 0 \end{vmatrix}$$

(65)

from the requirement of admissibility with $g_2$.

As a consequence:

$$\omega = \omega_{1\pm} = \begin{vmatrix} 0 & \pm \sqrt{q_2 q_1} \\ \pm \sqrt{q_2} & 0 \end{vmatrix}$$

(66)

and:

$$\omega = \omega_{2\pm} = \begin{vmatrix} 0 & \pm \sqrt{\sigma_2 \sigma_1} \\ \pm \sqrt{\sigma_2} & 0 \end{vmatrix}$$

(67)

Now we have all the admissible structures, i.e. $(g_1, \omega_{1\pm}, J_{1\pm})$ and $(g_2, \omega_{2\pm}, J_{2\pm})$.

Let’s compute the invariance group for the first triple having made a definite choice for the possible signs (say: $J = J_{1+}$). The group is easily seen to be:

$$O_1(t) = \cos(t)I + \sin(t)J_1 = \begin{vmatrix} \cos(t) & \sqrt{q_2 q_1} \sin(t) \\ -\sqrt{q_2 q_1} \sin(t) & \cos(t) \end{vmatrix}$$

(68)

while for the second triple we obtain:

$$O_2(t) = \cos(t)I + \sin(t)J_2 = \begin{vmatrix} \cos(t) & \sqrt{\sigma_2 \sigma_1} \sin(t) \\ -\sqrt{\sigma_2 \sigma_1} \sin(t) & \cos(t) \end{vmatrix}$$

(69)

and in general we obtain two different realizations of $SO(2)$.

The two realizations have only a trivial intersection (coinciding with the identity) if $\rho_2/\rho_1 \neq \sigma_2/\sigma_1$, and coincide when $\rho_2/\rho_1 = \sigma_2/\sigma_1$. The latter condition is easily seen (by imposing e.g.: $[J_{1,2},T] = 0$) to be precisely the condition of compatibility of the two triples.
To conclude the discussion of the example, let’s see what happens in the complexified version of the previous discussion.

To begin with we have to define multiplication by complex numbers on \( \mathbb{R}^2 \), thus making it a complex vector space, and this can be done in two ways, namely as:

\[
(x + iy) \begin{pmatrix} a \\ b \end{pmatrix} = (xI + Jy) \begin{pmatrix} a \\ b \end{pmatrix}
\]

(70)

or as:

\[
(x + iy) \begin{pmatrix} a \\ b \end{pmatrix} = (xI + J_2y) \begin{pmatrix} a \\ b \end{pmatrix}
\]

(71)

Correspondingly, we can introduce two different hermitian structures on \( \mathbb{R}^2 \) as:

\[
(.,.)_1 = g_1 + i\omega_1
\]

(72)

or as:

\[
(.,.)_2 = g_2 + i\omega_2
\]

(73)

They are antilinear in the first factor and in each case the corresponding multiplication by complex numbers must be used. The \( O_1(t) \) and \( O_2(t) \) actions both coincide with the multiplication of points of \( \mathbb{R}^2 \) by the complex numbers \( e^{it} \) (i.e. with different realizations of \( U(1) \)), but the definition of multiplication by complex numbers is different in the two cases.

Going back to the general case, we can make contact with the theory of complete integrability of bi-hamiltonian system by observing that \( T \) plays here the role of a recursion operator. Indeed, we show now that it generates a basis of vector fields preserving both the hermitian structures \( h_a \) given by:

\[
\Gamma_1, T\Gamma_1, ..., T^{n-1}\Gamma_1
\]

(74)

To begin with, these fields preserve all the geometrical structures, commute pairwise and are linearly independent. In fact these properties follow from the observation that \( T \), being a constant 1-1 tensor, satisfies the Nijenhuis condition. Therefore, for any vector field \( X \):

\[
L_{TX}T = TLX T
\]

(75)

that, \( T \) being invertible, amounts to:

\[
L_{TX} = TLX
\]

(76)

So, \( \forall k \in \mathbb{N} \):

\[
L_{TX}^k = TL_{X^{k-1}} = ... = T^kL_{X}
\]

(77)
and
\[ T^k L_{\Gamma_1} \omega_a = 0 = T^k L_{\Gamma_1} g_a \quad ; \quad (78) \]

Moreover, \( \forall s \in \mathbb{N} : \)
\[ [T^{k+s}\Gamma_1, T^k\Gamma_1] = L_{T^{k+s}\Gamma_1} T^k\Gamma_1 = T^s L_{T^k\Gamma_1} T^k\Gamma_1 = T^s [T^k\Gamma_1, T^k\Gamma_1] = 0. \quad (79) \]

Besides, the assumption of minimal degeneracy of \( T \) implies that the minimal polynomial of \( T \) be of degree \( n \). Indeed, we have shown that the diagonal form of \( T \) is
\[ T = \bigoplus_{k=1, \ldots, n} \{ \pm \rho_k I_k \} \quad (80) \]
where \( I_k \) is the identity on \( V_k \). Any linear combination
\[ \sum_{r=0}^{m} \alpha_r T^r = 0 \quad , \quad m \leq n - 1 \quad , \quad (81) \]
yields a linear system for the \( \alpha_r \)'s of \( n \) equations in \( m + 1 \) unknowns whose matrix of coefficients is of maximal rank and that, for \( m = n - 1 \), coincides with the full Vandermonde matrix of the \( \rho_k \)'s.

Then, we can conclude that the \( n \) vector fields \( T^r\Gamma_1, r = 0, 1, \ldots, n - 1 \) form a basis.

### 4 The infinite-dimensional case

We now analyze the same kind of problems in the framework of Quantum Mechanics, taking advantage from the experience and results we have obtained in the previous Sections where we dealt with a real \( 2n \)-dimensional vector space.

In QM the Hilbert space \( \mathbb{H} \) is given as a vector space over the field of complex numbers. Now we assume that two Hermitian structures are given on it, that we will denote as \( (\cdot, \cdot)_1 \) and \( (\cdot, \cdot)_2 \) (both linear, for instance, in the second factor). As in the real case, we look for the group that leaves invariant both structure, that is the group of unitary transformations \( \text{w.r.t. both Hermitian structures} \). We call them “bi-unitary”.

In order to assure that \( (\cdot, \cdot)_1 \) and \( (\cdot, \cdot)_2 \) do not define different topologies on \( \mathbb{H} \) it is necessary that there exists \( A, B \in \mathbb{R}, \, 0 < A, B \) such that:
\[ A \| x \|_2 \leq \| x \|_1 \leq B \| x \|_2 \quad , \quad \forall x \in \mathbb{H} \quad (82) \]
The use of Riesz theorem on bounded linear functionals immediately implies that there exists an operator \( F \) defined implicitly by the equation:

\[
(x, y)_2 = (Fx, y)_1 , \quad \forall x, y \in \mathbb{H} \tag{83}
\]

\( F \) replaces the previous \( G \) and \( T \) tensors of the real vector space situation, i.e. now it contains both the real and imaginary part of the Hermitian structure and, in fact:

\[
F = (g_1 + i\omega_1)^{-1} \circ (g_2 + i\omega_2) \tag{84}
\]

It is trivial to show that \( F \) is bounded, positive, and self-adjoint with respect to both hermitian structures and that:

\[
\frac{1}{B^2} \leq \| F \|_1 \leq \frac{1}{A^2} ; \quad \frac{1}{B^2} \leq \| F \|_2 \leq \frac{1}{A^2} . \tag{85}
\]

If \( \mathbb{H} \) is finite-dimensional, \( F \) can be diagonalized, the two hermitian structures decompose in each eigenspace of \( F \), where they are proportional and we get immediately that the group of bi-unitary transformations is indeed:

\[
U(n_1) \times U(n_2) \times \ldots \times U(n_k) , \quad n_1 + n_2 + \ldots n_k = n = \dim \mathbb{H} \tag{86}
\]

where \( n_i \) denotes the degeneracy of the \( i \) -th eigenvalue of \( F \).

In the infinite dimensional case \( F \) may have a point part of the spectrum and a continuum part. From the point part of the spectrum one gets \( U(n_1) \times U(n_2) \times \ldots \) where now \( n_i \) can be also \( \infty \). The continuum part is more delicate to discuss. It will contain for sure the commutative group \( U_F \) of bi-unitary operators of the form \( \{ e^{if(F)} \} \) where \( f \) is any real valued function (with very mild properties[9]).

The concept of genericity in the infinite dimensional case can not be given as easily as in the finite dimensional case. One can say that the eigenvalues should be non degenerate but what for the continuous spectrum? We give here an alternative definition that works for the finite and infinite case as well.

Note first that any bi-unitary operator must commute with \( F \). Indeed:

\[
(x, U^1FUy)_2 = (Ux, FUy)_2 = (FUx, Uy)_2 = (Ux, Uy)_1 = (x, y)_1 = (Fx, y)_2 = (x, Fy)_2 , \quad \text{from this:} \quad U^1FU = F' , \quad [F, U] = 0 .
\]

The group of bi-unitary operators therefore belongs to the commutant \( F' \) of the operator \( F \). The genericity condition can be restated in a purely algebraic form as follows:

**Definition:** Two hermitian forms are in a generic position iff \( F'' = F' \), i.e. the bicommutant of \( F \) coincides with the commutant of \( F \).

In other words this means that \( F \) generates a complete set of observables.

This definition reduces, for the case of a pure point spectrum, to the condition of nondegeneracy of the eigenvalues of \( F \) and, in the real case, to the minimum possible degeneracy of the eigenvalues of \( T \) and \( G \), that is two.
To grasp how the definition works, we will give some simple examples. Consider: \((F\psi)(x) = x^2 \psi(x)\) on the space \(L^2([-b, -a] \cup [a, b])\) with \(0 < a < b\); then the operator \(x\), its powers \(x^n\) and the parity operator \(P\) belong to \(F'\) while \(F''\) does not contain \(x\) (and any odd power of \(x\)) because they do not commute with \(P\). So if \(F = x^2\) the two hermitian structure are not in a generic position because \(F'' \subset F'\). On the contrary, on the space \(L^2([a, b])\), \(F'' = F'\) because a parity operator \(P\) does not exist in this case, so the two hermitian structure are now in a generic position. In this case the group of bi-unitary operators is \(\{e^{if(x^2)t}\}\) for the appropriate class of functions \(f\). In some sense, when a continuous part of the spectrum is considered, there appears a continous family of \(U(1)'s\) as a counterpart of the discrete family of \(U(1)'s\) corresponding to the discrete part of the spectrum.

Remarks.

i) Suppose that complex Hilbert spaces with two Hermitian structures have been constructed from a given real vector space \(V\) using two compatible and admissible triples \((g_1, \omega_1, J_1)\) and \((g_2, \omega_2, J_2)\). Then, by complexification, we get two different Hilbert space, each one with its proper multiplication by complex numbers and with its proper Hermitian structure. The previous case we have just studied is obtained if we assume \(J_1 = J_2\). it is easy to show that this is a sufficient condition for compatibility. This is the reason why in the quantum-mechanical case the group of bi-unitary transformations is never empty, and the compatibility condition is encoded already in the assumptions.

ii) If \(J_1 \neq J_2\) but the compatibility condition still holds, we know that \(V\) splits into \(V_+ \oplus V_-\), where \(J_1 = \pm J_2\) on \(V_\pm\) respectively. On \(V_+\) we have the previous case, while on \(V_-\) we get two Hermitian structures, one \(\mathbb{C}\)-linear and one anti-\(\mathbb{C}\)-linear in the second factor (which one is which depending on the complexification we have decided to use). From the point of view of the group of unitary transformations, this circumstance is irrelevant, because the set of unitary transformations does not change from being defined w.r.t. an hermitian structure or to w.r.t. its complex conjugate. We conclude from this that our analysis goes through in general, provided the compatibility condition holds.

5 Conclusions.

We will try now to summarize our main result, by restating it at the same time in a more concise group-theoretical language. What we have shown is, to begin with, that once two admissible triples: \((g_1, \omega_1, J_1)\) and \((g_2, \omega_2, J_2)\) are given on a real, even-dimensional vector space \(V \approx \mathbb{R}^{2n}\), they define two \(2n\)-dimensional real representations, \(U_r(2n; g_1, \omega_1)\) and \(U_r(2n; g_2, \omega_2)\) of the unitary group \(U(n)\), \(U_r(2n; g_a, \omega_a)\) \((a = 1, 2)\) being the group of transformations that
leave simultaneously \( g_a \) and \( \omega_a \) (and hence \( J_a \)) invariant. Their intersection:

\[
W_r =: \{ U_r(2n; g_1, \omega_1) \cap U_r(2n; g_2, \omega_2) \} \tag{87}
\]

will be their common subgroup that is an invariance group for both triples. The assumption of compatibility implies that \( W_r \) should not reduce to the identity alone.

If the two triples are in a generic position, then:

\[
W_r = SO(2) \times ... \times SO(2) \tag{88}
\]

where, here: \( SO(2) \approx U(1) \) or, more generally if the genericity assumption is dropped:

\[
W_r = U_r(2r_1; g, \omega) \times U_r(2r_2; g, \omega) \times ... \times U_r(2r_k; g, \omega) \tag{89}
\]

where: \( r_1 + r_2 + ... + r_k = n \) and \( (g, \omega) \) is any one of the two pairs \( (g_1, \omega_1) \) or \( (g_2, \omega_2) \).

The real vector space \( V \approx \mathbb{R}^{2n} \) will decompose then into a direct sum of even-dimensional subspaces that are mutually orthogonal w.r.t. both metrics, and on each subspace the corresponding (realization of the) special orthogonal group will act irreducibly.

Alternatively, we can complexify \( V \approx \mathbb{R}^{2n} \), and that in two different ways, using the two complex structures that are at our disposal. The equivalent statement in the complex framework will be then:

Given two hermitian structures \( h_a, a = 1, 2 \) on a complex \( n \)-dimensional vector space \( \mathbb{C}^n \), they define two representations \( U(n; h_a), a = 1, 2 \) of the group \( U(n) \) on the same \( \mathbb{C}^n \). \( U(h_1, n) \) (resp. \( U(h_2, n) \)) will be the group of transformations that are unitary with respect to \( h_1 \) (resp. \( h_2 \)). The group \( W \) of simultaneous invariance for both hermitian structures:

\[
W \equiv \{ U(h_1, n) \cap U(h_2, n) \} \tag{90}
\]

will be a subgroup of both \( U(h_1, n) \) and \( U(h_2, n) \), and our assumption of compatibility of the \( h_a \)’s implies that the component of \( W \) connected to the identity should not reduce to the identity alone.

The assumption of genericity implies that:

\[
W = U(1) \times ... \times U(1) \tag{91}
\]

If the assumption of genericity is dropped, one can easily show, along the same lines as in the generic case, that \( W \) will be of the form:

\[
W = U(r_1) \times U(r_2)... \times U(r_k) \tag{92}
\]

\(^5\) As the previous two-dimensional example shows explicitly, but it should be clear by now also in general.
with \( r_1 + r_2 + \ldots + r_k = n \). \( \mathbb{C}^n \) will decompose accordingly into a direct sum of subspaces that will be mutually orthogonal with respect to both \( h_a \)'s, and on each subspace the appropriate \( U(r) \) will act irreducibly.

We have also shown that these results generalize to the infinite-dimensional case as well. Some extra assumptions must be added on the Hermitian structures in order that they define the same topology in \( \mathbb{H} \) and an appropriate definition of genericity must also be given. Then, a decomposition like in Eqns. (91) and (92) is obtained, possibly with denumerable discrete terms and a continuum part as well. We note that, in the spirit of this work where two Hermitian structures are given from the very beginning, it is natural to supplement the compatibility condition, in the infinite-dimensional case, with a topological equivalence condition. However from the point of view of the study of bi-hamiltonian systems, where a fixed dynamics is given, it would be more natural to assume some weaker regularity condition, for instance that the given dynamics should be continuous with respect to both structures.

Recently, bi-Hamiltonian systems "generated" out of a pencil of compatible Poisson structures have been considered \([10]\), also in connection with the separability problem \([11]\). It should be noticed that our compatible structures would give rise to a pencil of compatible triples defined by:

\[
g_\gamma = g_1 + \gamma g_2 \quad \omega_\gamma = \omega_1 + \gamma \omega_2 \quad J_\gamma
\]

A systematic comparison with this approach is presently under consideration.
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