ON REALIZATION OF THE KREIN-LANGER CLASS $N_\kappa$ OF MATRIX-VALUED FUNCTIONS IN HILBERT SPACES WITH INDEFINITE METRIC

YURI ARLINSKII, SERGEY BELYI, VLADIMIR DERKACH, AND EDUARD TSEKANOVSKII

Dedicated to Henk de Snoo on the occasion of his 60th birthday

Abstract. In this paper the realization problems for the Krein-Langer class $N_\kappa$ of matrix-valued functions are being considered. We found the criterion when a given matrix-valued function from the class $N_\kappa$ can be realized as linear-fractional transformation of the transfer function of canonical conservative system of the M. Livsic type (Brodskii-Livsic rigged operator colligation) with the main operator acting on a rigged Pontryagin space $\Pi_\kappa$ with indefinite metric. We specify three subclasses of the class $N_\kappa(R)$ of all realizable matrix-valued functions that correspond to different properties of a realizing system, in particular, when the domains of the main operator of a system and its conjugate coincide, when the domain of the hermitian part of a main operator is dense in $\Pi_\kappa$. Alternatively we show that the class $N_\kappa(R)$ can be realized as transfer matrix-functions of some canonical impedance systems with self-adjoint main operators in rigged spaces $\Pi_\kappa$. The case of scalar functions of the class $N_\kappa(R)$ is considered in details and some examples are presented.

1. Introduction

Realizations and corresponding operator models of different classes of holomorphic matrix-valued functions in the open right half-plane, unit circle and upper half-plane play important role in spectral analysis of different classes of linear operators in Hilbert spaces, interpolation problems and system theory, and we refer in this matter to [1], [2], [5]-[8], [10], [11]-[13], [16], [21], [22], [24], [26], [33], [37]-[40]. In this paper we continue the investigation of various problems that arise in the study of linear stationary conservative dynamic systems (operator colligations). Relying on the results and technique developed in [11], [12] we keep dealing with linear stationary conservative dynamic systems (l.s.c.d.s) $\theta$ of the form

$$\begin{cases}
(\mathbb{A} - zI) = KJ \varphi_- \\
\varphi_+ = \varphi_- - 2iK^*x
\end{cases} \quad (\text{Im} \mathbb{A} = KJ^*)$$

or

$$\theta = \begin{pmatrix}
\mathbb{A} & K \\
\mathcal{H}^+ \subset \Pi_\kappa \subset \mathcal{H}^- & E
\end{pmatrix}.$$

In the system $\theta$ above $\mathbb{A}$ is a bounded linear operator acting from $\mathcal{H}^+$ into $\mathcal{H}^-$, where $\mathcal{H}^+ \subset \Pi_\kappa \subset \mathcal{H}^-$ is a rigged Pontryagin space, $K$ is a linear bounded operator from a Hilbert space $E$ into $\mathcal{H}^-$, $J = J^* = J^{-1}$ is acting in $E$, $\varphi_\pm \in E$, $\varphi_-$ is an
input vector, \( \varphi_+ \) is an output vector, and \( x \in \mathcal{H}_+ \) is a vector of the inner state of the system \( \theta \). The operator-valued function
\[
W_\theta(z) = I - 2iK^*(A - zI)^{-1}KJ
\]
is the transfer operator-valued function of the system \( \theta \). It was shown in [12] that a Herglotz-Nevanlinna matrix-valued function \( V(z) \) acting on a Hilbert space \( E \) can be represented and realized in the form
\[
V(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J = K^*(A_R - zI)^{-1}K,
\]
where \( W_\theta(z) \) is a transfer function of some canonical system \( \Theta \),
\[
\Theta = \begin{pmatrix}
A & K \\
\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & E
\end{pmatrix},
\]
if certain conditions on integral representation of \( V(z) \) are met. Alternatively, one can realize a Herglotz-Nevanlinna matrix-valued \( V(z) \) as a transfer mapping of an impedance system \( \Delta \) of the form
\[
\begin{cases}
(D - zI)x = K \varphi_-, \\
\varphi_+ = K^*x,
\end{cases}
\]
where \( D \) is a self-adjoint operators acting from \( \mathcal{H}_+ \) into \( \mathcal{H}_- \) (see [12], [13]). In this case associated transfer function is given by
\[
V_\Delta(z) = K^*(D - zI)^{-1}K.
\]
In this paper we study similar realization problems but utilize a new type of realizing systems whose main operator is acting on a rigged Pontryagin space \( \Pi_\kappa \). The set of realizable functions appears to be a subclass of the well known Krein-Langer’s class \( N_\kappa \) also known as generalized Nevanlinna functions. In Section 6 we specify three subclasses of the class \( N_\kappa(R) \) of all realizable matrix-valued functions that yield different properties of operators in the realizing systems. It is worth mentioning that in the case when \( \kappa = 0 \) all the subclasses coincide with the similar subclasses of realizable Herglotz-Nevanlinna functions described in [12], [13]. Section 7 uses a factorization formula from [25] to provide applications of \( N_\kappa(R) \) realizations to the scalar case when \( E = \mathbb{C} \) while establishing a connection with the class \( N(R) \) of realizable Herglotz-Nevanlinna functions. The paper is concluded with several examples.

2. Operators in Pontryagin spaces \( \Pi_\kappa \)

We start with the basic construction following some results from the theory of operators in \( \Pi_\kappa \) spaces [9], [29], [32], [33]. Let \( \Pi_\kappa \) be a Pontryagin space \( \mathcal{H} \), i.e., a Hilbert space \( \mathcal{H} \) where along with the usual scalar product \( (x, y) \) there is an indefinite scalar product
\[
[x, y] = (Jx, y),
\]
where \( J = P_+ - P_- \) is a bounded linear operator such that \( J = J^*, J^2 = I \), and \( P_+ \) and \( P_- \) are complementary orthoprojections, \( P_+ + P_- = I \). Putting \( \Pi_\pm = P_\pm \Pi_\kappa \) we have
\[
\Pi_\kappa = \Pi_+ \boxplus \Pi_-,
\]
\( \dim \Pi_- = \kappa \).
Here and below the direct orthogonal sum with respect to an indefinite scalar product \([\mathbf{2}]\) is denoted by \( \boxplus \) and called \( \pi \)-orthogonal sum. Similarly, the \( \pi \)-orthogonal
complement of a lineal \( L \) will be denoted by \( L^{[\perp]} \). The positive definite \((x, y)\) and indefinite \([x, y]\) scalar products are related by

\[
(x, y) = [x_+, y_+] - [x_-, y_-], \\
[x, y] = (x_+, y_+) - (x_-, y_-),
\]

where \( x = x_+ + x_- \), \( y = y_+ + y_- \), \( x_+, y_+ \in \Pi_+ \), and \( x_-, y_- \in \Pi_- \).

The set of vectors \( f \in L \) that are \( \pi \)-orthogonal to \( L \), i.e. \( f^{[\perp]}L \) is called the isotropic part of the linear manifold \( L \). If the isotropic part of \( L \) has non-zero elements we say that the scalar product \([\cdot, \cdot]\) is degenerate \( \pi \)-symmetric operator \( A \) if \( f^{[\perp]}L \) (respectively, \( L_-, L_0 \)) will denote the set of all \( x \in \Pi_\kappa \) for which \([x, x] > 0 \) (respectively, \([x, x] < 0 \), \([x, x] = 0 \)) and is called positive \( \pi \)-symmetric operator \( A \).

Every subspace \( L \subseteq \Pi_\kappa \) can be decomposed into a direct sum of \( \pi \)-orthogonal subspaces

\[
L = L_+ \oplus L_0 \oplus L_-,
\]

where \( L_+ \), \( L_0 \), and \( L_- \) are, respectively, positive, neutral, and negative subspaces, some of which may degenerate into null subspaces. For a subspace \( L \) above we write \( \text{sign} L = (l_+, l_0, l_-) \) where \( l_\pm = \dim L_\pm \) and \( l_0 = \dim L_0 \).

Recall \([1]\) that a linear relation in \( \Pi_\kappa \) is a subspace \( \mathcal{A} \) in \( \Pi_\kappa \times \Pi_\kappa \). The domain of a linear relation \( \mathcal{A} \) is

\[
\mathcal{D}(\mathcal{A}) = \{ f \in \Pi_\kappa : (f, f') \in \mathcal{A} \text{ for some } f' \in \Pi_\kappa \},
\]

and the range of \( \mathcal{A} \) is

\[
\mathcal{R}(\mathcal{A}) = \{ f' \in \Pi_\kappa : (f, f') \in \mathcal{A} \text{ for some } f \in \Pi_\kappa \}.
\]

The subspace

\[
\text{mul} \mathcal{A} = \{ g \in \Pi_\kappa : (0, g) \in \mathcal{A} \}
\]

is called the multivalued part of a linear relation \( \mathcal{A} \). A linear relation \( \mathcal{A} \) is the graph of a linear operator in \( \Pi_\kappa \) if and only if \( \text{mul} \mathcal{A} = \{0\} \).

Let us associate with a linear operator \( A \) in \( \Pi_\kappa \) the linear relation \( \mathcal{A} := \text{Gr}(A) \), the graph of the operator \( A \).

For a linear relation \( \mathcal{A} \) in \( \Pi_\kappa \), its \( \pi \)-adjoint \( \mathcal{A}^+ \) is defined by

\[
\mathcal{A}^+ = \{ (h, h') \in \Pi_\kappa \times \Pi_\kappa : [f', h] = [f, h'] \text{ for all } (f, f') \in \mathcal{A} \}.
\]

A linear relation \( \mathcal{A} \) (operator \( A \)) is called \( \pi \)-symmetric if \( \mathcal{A} \subseteq \mathcal{A}^+ \) and \( \pi \)-selfadjoint if \( \mathcal{A} = \mathcal{A}^+ \).

We recall \([20]\) that a \( \pi \)-symmetric operator \( A \) in \( \Pi_\kappa \) can not have more than \( \kappa \) eigenvalues, counting multiplicities, in the upper (lower) half-plane. If the operator \( A \) is \( \pi \)-self-adjoint, then these non-real eigenvalues are located symmetrically with respect to the real axis. For an arbitrary complex number \( z \) and a \( \pi \)-symmetric operator \( A \) in \( \Pi_\kappa \) we set \( \mathcal{M}_z = (A - z)\mathcal{D}(A), \quad \mathfrak{M}_z = \mathcal{M}_z^{[\perp]} \).

If \( \lambda = (\text{Im} \lambda \neq 0) \) is not an eigenvalue of \( A \), then \( \mathcal{M}_\lambda \) is a subspace of \( \Pi_\kappa \) and \( \mathfrak{M}_\lambda \) is called \( \lambda \)-deficiency subspace corresponding to \( \lambda \) with \( \dim \mathfrak{M}_\lambda \) maintaining a constant value as a deficiency index of \( A \) in \( \Pi_\kappa \). Let \( \Delta_A \) be the set of all non-real \( \lambda \) for which the scalar product \([\cdot, \cdot]\) is degenerate on \( \mathfrak{M}_\lambda \). According to \([32]\) the set \( \Delta_A \) of a \( \pi \)-symmetric operator \( A \) contains no interior points, its complement \((\mathbb{C}_+ \cup \mathbb{C}_-) \setminus \Delta_A \) is an open set, and on every component of this open set \( \text{sign} \mathfrak{M}_\lambda \) is constant.
It was shown in [15] that every \( \pi \)-symmetric operator \( A \) in the space \( \Pi_\kappa \) admits \( \pi \)-self-adjoint extensions in \( \Pi_\kappa \) if and only if its deficiency indices coincide. An operator \( A \) is called \textit{prime} if it has no non-real eigenvalues and

\[
\text{c.l.s.} \{ \mathfrak{R}_z, \ z \neq \bar{z} \} = \Pi_\kappa. \tag{3}
\]

In what follows we denote \( \text{Re}(T) = (T + T^+)/2, \text{Im}(T) = (T - T^+)/2i \) for linear operators \( T \) in \( \Pi_\kappa \) with \( \mathcal{D}(T) = \mathcal{D}(T^+) \). Similarly, for a linear operator \( Q \) with \( \mathcal{D}(Q) = \mathcal{D}(Q^*) \) in a Hilbert space we use the same notation to denote \( \text{Re}(Q) = (Q + Q^*)/2 \) and \( \text{Im}(Q) = (Q - Q^*)/2i \).

3. Bi-extensions in Rigged Pontryagin Space

Let consider \( A \) as an operator from the Hilbert space \( \mathcal{D}(A) \) into the Hilbert space \( \mathcal{H} \). Then its adjoint \( A^* \) is defined on a set \( \mathcal{D}(A^*) \) that is dense in \( \mathcal{H} \) and has the range in \( \overline{\mathcal{D}(A)} \). This allows us to introduce the Hilbert spaces \( \mathcal{H}_+ = \mathcal{D}(A^*) \) and \( \mathcal{H}_+ = J\mathcal{D}(A^*) \) with corresponding inner products

\[
(f, g)_+ = (f, g) + (A^*f, A^*g), \quad f, g \in \mathcal{H}_+,
\]

\[
(f, g)_+ = (f, g) + (A^*f, A^*g), \quad f, g \in \mathcal{H}_+.
\]

Next we construct two rigged Hilbert spaces [15, 18, 12]

\( \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \) and \( \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \).

Let \( \mathcal{R}_1 \in [\mathcal{H}_+, \mathcal{H}_-] \) and \( \mathcal{R}_2 \in [\mathcal{H}_+, \mathcal{H}_-] \) be the isometric Riesz-Berezanskii operators [15] corresponding to the above triplets. We introduce

\[
J_+ = J \big|_{\mathcal{H}_+},
\]

the linear operator mapping \( \mathcal{H}_+ \) isometrically onto \( \mathcal{H}_+ \) and let \( J_+^\times \in [\mathcal{H}_-, \mathcal{H}_+] \) be its dual, which isometrically maps \( \mathcal{H}_- \) onto \( \mathcal{H}_- \). It is easy to see that

\[
(J_+^\times)^{-1} = \mathcal{R}_2 J_+ \mathcal{R}_1^{-1}.
\]

Since \( \mathcal{H}_- \) is isomorphic to \( \mathcal{H}_- \) it can be considered as the space of anti-linear functionals on \( \mathcal{H}_+ \) defined by

\[
\alpha(f) = (\alpha, Jf) = [\alpha, f], \quad \alpha \in \mathcal{H}_-, f \in \mathcal{H}_+.
\]

Thus, we can form a rigged \( \Pi_\kappa \) space

\( \mathcal{H}_+ \subset \Pi_\kappa \subset \mathcal{H}_- \).

Consequently, if \( \mathcal{A} \in [\mathcal{H}_+, \mathcal{H}_-] \) then \( \mathcal{A}^\times \in [\mathcal{H}_+, \mathcal{H}_-] \) and \( (\mathcal{A}f, g) = [f, \mathcal{A}^\times g] \) for all \( f, g \in \mathcal{H}_+ \).

**Definition 1.** An operator \( \mathcal{A} \in [\mathcal{H}_+, \mathcal{H}_-] \) is called \textit{bi-extension} of a closed \( \pi \)-symmetric operator \( A \) if \( \mathcal{A} \supset A \) and \( \mathcal{A}^\times \supset A \). A bi-extension \( \mathcal{A} \) is called \textit{\( \pi \)-self-adjoint} if \( \mathcal{A} = \mathcal{A}^\times \).

Let \( \mathcal{A} \) be a bi-extension of a \( \pi \)-symmetric operator \( A \). The operator \( \hat{A} = \mathcal{A} \) with \( \mathcal{D}(\hat{A}) = \{ f \in \mathcal{H}_+ \mid \mathcal{A} f \in \Pi_\kappa \} \) is called \textit{quasi-kernel} [14] of the operator \( \mathcal{A} \). If \( \mathcal{A} = \mathcal{A}^\times \) and \( \hat{A} \) is a quasi-kernel of \( \mathcal{A} \) such that \( A \neq \hat{A} \), \( \hat{A}^+ = \hat{A} \) then \( \mathcal{A} \) is said to be a \textit{strong} \( \pi \)-self-adjoint bi-extension of \( A \).
In what follows we will assume that the closed and non-densely defined \( \pi \)-symmetric operator \( A \) is \( J \)-regular \([5]\), i.e., the operator \( PA \) is closed, where \( P \) is the orthogonal projection onto \( JD(A) \) in \( H \). The analog of von Neumann’s formula for the operator \( JA \) (see \([10]\))

\[
\mathfrak{H}_+ = D(A) \oplus \mathfrak{M}_+ \oplus \mathfrak{M}_- \oplus \mathfrak{R}
\]

holds in the space \( \mathfrak{H}_+ \), where \( \mathfrak{M}_{\pm} \) are the semi-deficiency subspaces of the operator \( JA \) \([31]\), i.e.

\[
\mathfrak{M}'_{\pm} = JD(A) \oplus (PA \mp iI)D(A),
\]

\( \mathfrak{R} = J^+ \mathfrak{R}, \mathfrak{M} = \mathfrak{R}^{-1} \mathfrak{S}, \) and \( \mathfrak{S} = I_A \mathfrak{D}(A) \). The condition of \( A \) being \( J \)-regular is equivalent to the subspace \( \mathfrak{S} \) being closed in \( \mathfrak{H}_- \) and a sufficient condition of \( J \)-regularity is \( \dim \mathfrak{S} < \infty \) (see \([20], [3]\)). Let \( P_1^+, P_{\mathfrak{M}_+}^+, P_{\mathfrak{M}_-}^+, P_{\mathfrak{R}^+}^+, \) and \( P_{\mathfrak{R}^-}^+ \) be the orthogonal projections in \( \mathfrak{H}_+ \) onto \( D(A), \mathfrak{M}_+, \mathfrak{M}_-, \mathfrak{R}, \) and \( \mathfrak{M} = \mathfrak{H}_+ \oplus D(A) \), respectively. Then the set of all bi-extensions of a \( J \)-regular operator \( A \) is described by the formula \([40]\)

\[
\mathfrak{K} = AP_1^+ + (A^+ + \mathfrak{R} \mathfrak{S}_+^+) P_{\mathfrak{R}^+}^+,
\]

where \( \mathfrak{S}_+^+ = [\mathfrak{M}, \mathfrak{M}] \). Moreover, \( \mathfrak{K} \) is \( \pi \)-self-adjoint if and only if

\[
\mathfrak{S}_+^+ = -iP_{\mathfrak{M}^-}^+ + iP_{\mathfrak{M}_-}^+.
\]

The Hilbert space version of the class \( \Omega_A \) in the definition below is found in \([3], [4], [10], [11]\).

**Definition 2.** We say that a closed densely defined linear operator \( T \) acting in a Pontryagin space \( \Pi_k \) belongs to the class \( \Omega_A \) if:

1. \( T \supset A, T^+ \supset A \) where \( A \) is a closed Hermitian operator;
2. \( T \) has a regular point in the lower half-plane;
3. \( PT \) and \( PT^+ \) are closed operators.

Note that a closed and non-densely defined \( J \)-regular \( \pi \)-symmetric operator \( A \) admits \( \pi \)-selfadjoint extensions of the class \( \Omega_A \) if and only if its semi-deficiency indices coincide \([7], [10]\).

An operator \( \mathfrak{K} \) in \([\mathfrak{H}_+, \mathfrak{H}_-] \) is called a \((\Gamma)\)-extension \([11]\) of an operator \( T \) of the class \( \Omega_A \) if both \( \mathfrak{K} \supset T \) and \( \mathfrak{K}^\times \supset T^+ \). This \((\Gamma)\)-extension is called correct \([11]\) (regular \([1]\)), if an operator \( \text{Re} \mathfrak{K} = \frac{1}{2}(\mathfrak{K} + \mathfrak{K}^\times) \) is a strong \( \pi \)-self-adjoint bi-extension of an operator \( A \). It is easy to show that if \( \mathfrak{K} \) is a \((\Gamma)\)-extension of \( T \), the \( T \) and \( T^+ \) are quasi-kernels of \( \mathfrak{K} \) and \( \mathfrak{K}^\times \), respectively.

**Definition 3.** We say the operator \( T \) of the class \( \Omega_A \) belongs to the class \( \Lambda_A \) if

1. \( T \) admits a correct \((\Gamma)\)-extension;
2. \( \text{Gr}(A) = \text{Gr}(T) \cap \text{Gr}(T^+) \).

It can be shown (see \([5]\)) that if \( T \in \Lambda_A \) then the equation

\[
(A - \lambda I)x = g,
\]

is solvable for all \( \lambda \in \rho(T) \) and all \( g \in \text{Im} \mathfrak{K}^\times = \frac{i}{2}(\mathfrak{K} - \mathfrak{K}^\times) \).

**Remark 4.** A survey of theory of bi-extensions of symmetric operator in a Hilbert space and its application to characteristic functions of operators of the class \( \lambda_A \) is presented in \([10]\). Bi-extensions of \( \pi \)-symmetric operators in Pontryagin spaces were studied in \([18]\).
4. Operator colligations in $\Pi_\kappa$

In this section we consider linear stationary conservative dynamic systems (l. s. c. d. s.) $\theta$ of the form

$$\begin{cases}
(\mathbb{A} - zI) = K\mathcal{J}\varphi_-
\varphi_+ = \varphi_- - 2iK^+x
\end{cases}$$

(Im $\mathbb{A} = K\mathcal{J}K^*$).

In the system $\theta$ above $\mathbb{A}$ is a bounded linear operator acting from $H^+\subset \Pi_\kappa \subset H^-$, $H^\pm$ is a rigged Pontryagin space, $K$ is a linear bounded operator from a Hilbert space $E$ into $H^-$, $J = J^* = J^{-1}$ is acting in $E$, $\varphi_\pm \in E$, $\varphi_-$ is an input vector, $\varphi_+$ is an output vector, and $x \in \mathcal{F}^+$ is a vector of the inner state of the system $\theta$.

For our purposes we need the following more precise definition:

**Definition 5.** The array

$$\theta = \begin{pmatrix}
\mathbb{A} & K \\
\mathcal{F}^+ \subset \Pi_\kappa \subset \mathcal{F}^- & \mathcal{J}
\end{pmatrix}$$

is called a **linear stationary conservative dynamic system** (l.s.c.d.s.) or Brodskiǐ-Livšic rigged operator colligation if

1. $\mathbb{A}$ is a correct ($\ast$)-extension of an operator $T$ of the class $\Lambda_A$ for some $J$-regular operator $A$ with finite and equal deficiency indices;
2. $\mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1} \in [E, E]$, $\dim E < \infty$;
3. $\mathbb{A} - \mathbb{A}^\ast = 2iK\mathcal{J}K^+$, where $K \in [E, \mathcal{F}^-]$ $\ker K = \{0\}$ ($K^+ \in [\mathcal{F}^+, E]$).

In this case, the operator $K$ is called a **channel operator** and $\mathcal{J}$ is called a **direction operator** [11]. We associate with the system $\theta$ an operator-valued function

$$W_\theta(z) = I - 2iK^+(\mathbb{A} - zI)^{-1}K\mathcal{J}$$

which is called a **transfer operator-valued function** of the system $\theta$ or a **characteristic operator-valued function** of Brodskiǐ-Livšic rigged operator colligations [11].

Following [10], [12] we call an l.s.c.d. system $\theta$ **minimal** if the $\pi$-symmetric operator $A$ is such that there are no nontrivial invariant subspaces on which $A$ induces $\pi$-self-adjoint operators. Clearly, the l.s.c.d. system $\theta$ is minimal if the operator $A$ satisfies the condition [13].

Let $\theta$ be a l.s.c.d.s. of the form (4). We consider an operator-valued function

$$V_\theta(z) = K^+(\mathbb{A}_R - zI)^{-1}K$$

The transfer operator-function $W_\theta(z)$ of the system $\theta$ and an operator-function $V_\theta(z)$ of the form (5) are connected by the relation

$$V_\theta(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]\mathcal{J}$$

5. Class $\mathcal{N}_\kappa$. Realization Theorems.

Let $E$ be a Hilbert space with an inner product $(\cdot, \cdot)$ and an operator-valued function $Q(z)$ belong to $[E, E]$. 

Definition 6. An operator-valued function $V(z) \in [E, E]$ belongs to the class $N_\kappa$ if it is meromorphic in $\mathbb{C} \setminus \mathbb{R}$ and such that $V(\bar{z}) = V(z)^*$, $z \in Z_V$, and the kernel

$$N_V(z, \zeta) = \frac{V(\zeta) - V(z)^*}{\zeta - \bar{z}}, \quad z, \zeta \in Z_V, \quad \zeta \neq \bar{z},$$

has $\kappa$ negative squares, i.e. for all $z_j$ in the domain of holomorphy $Z_V$ of the meromorphic (in $\mathbb{C} \setminus \mathbb{R}$) function $V(z)$ and $h_j \in E$ ($j = 0, 1, ..., n$) the form

$$\sum_{j, k=0}^{n} \left(N_V(z_j, z_k)h_j, h_k \right) \xi_j \bar{\xi}_k$$

contains at most $\kappa$ negative squares and for one such a set exactly $\kappa$ negative squares.

Mention, that the kernel $N_V(z, \zeta)$ for a function $V \in N_\kappa$ restricted to the upper half-plane has the same number $\kappa$ of negative squares, see \cite{32}. Class $N_\kappa$ was introduced in \cite{33} and studied further in \cite{34}, \cite{17}. Different operator models corresponding to $N_\kappa$-functions are constructed in \cite{33}, \cite{21}, \cite{22}, \cite{23}, \cite{28}.

Definition 7. An operator-valued function $V(z)$ in a finite-dimensional Hilbert space $E$ is called realizable if, in some domain $D \subset \mathbb{C} - \mathbb{R}$, $V(z)$ can be represented in the form

$$V_\theta(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I],$$

where $W_\theta(z)$ is a transfer operator-function of some l.s.c.d.s. $\theta$ with the direction operator $J = I$.

Definition 8. An operator-function $V(z) \in [E, E]$ ($\dim E < \infty$) belongs to the class $N_\kappa(R)$ if the following conditions are met:

1. $V \in N_\kappa$;
2. for all $f \in E$

$$\lim_{y \to \infty} \frac{(V(iy)f, f)}{y} = 0;$$

3. for all $z \in Z_V$

$$\bigcap_{\zeta \in Z_V} \ker N_V(\zeta, z) = \{0\};$$

4. for all $f \in \mathcal{B} = \{f \in E \mid \lim_{y \to \infty} y(\text{Im} V(iy)f, f) < \infty\}$

$$\lim_{y \to \infty} V(iy)f = 0.$$

Theorem 9. Let $\theta$ be a minimal l.s.c.d.s. of the form \cite{41} with $J = I$ in $E$, $\dim E < \infty$. Then the operator-function $V_\theta(z)$ of the form \cite{41} admits a holomorphic continuation to a function $V(z)$ which belongs to the class $N_\kappa(R)$.

Proof. For a l.s.c.d.s $\theta$ of the form \cite{41} consider

$$V(z) = K^+(\hat{A}_R - zI)^{-1}K, \quad z \in \rho(\hat{A}_R),$$

where $\hat{A}_R$ is the quasi-kernel of the operator $A_R$. As follows from \cite{45} $V(z)$ is a holomorphic continuation of the function $V_\theta(z)$. We set

$$\Gamma_z = (\hat{A}_R - zI)^{-1}K, \quad z \in \rho(\hat{A}_R).$$
It can be seen that the operator $\Gamma_z$ is invertible and the following relation is valid

$$\Gamma_z = (\hat{A}_R - \zeta I)(\hat{A}_R - zI)^{-1}\Gamma_z, \quad z, \quad z \in \rho(\hat{A}_R).$$

Let $z_0 \in \rho(\hat{A}_R)$ and let $\hat{A}$ be a $\pi$-symmetric operator defined as follows

$$D(\hat{A}) = \left\{ f \in D(\hat{A}_R) \mid \left[ (\hat{A}_R - z_0I)f, g \right] = 0, \forall g \in E \right\}, \quad \hat{A}f = \hat{A}_Rf, \quad f \in D(\hat{A}_R),$$

Since

$$\left( A_R - \zeta I \right)^{-1}K - K^+\left( A_R - \zeta I \right)^{-1}K$$

$$= K^+(A_R - zI)^{-1}(A_R - \zeta I)^{-1}K$$

$$= \Gamma_z^+\Gamma_z, \quad z, \quad z \in \rho(\hat{A}_R),$$

the operator-function $V(z)$ is a Krein-Langer $Q$-function for the $\pi$-symmetric operator $\hat{A}$ and its $\pi$-self-adjoint extension $\hat{A}_R$. Let $f \in D(\hat{A})$. Then for any $g \in \mathcal{H}^+$

$$[\text{Im} \hat{A}f, g] = [f, \text{Im} \hat{A}g] = [f, (A_R - z_0I)\Gamma_{\xi_0}K^+g] = [(A_R - z_0I)f, \Gamma_{\xi_0}K^+g] = 0.$$ 

Thus, $\text{Im} \hat{A}f = 0$ for $f \in D(\hat{A})$ and $\hat{A} \supset T \supset \hat{A}, \hat{A}^\times \supset T^+ \supset \hat{A}$. But this is possible only when $\hat{A} = \hat{A}$ since $\hat{A}$ is the maximal $\pi$-symmetric part of the operator $T$. Consequently, $\Gamma_zE = \mathfrak{N}_z$ and $\mathcal{V}(z)$ is a Krein-Langer $Q$-function for a prime $\pi$-symmetric operator $A$. It was shown in [33] that a $Q$-function of a prime $\pi$-symmetric operator (in the upper half-plane) belongs to the class $N_{\kappa}$ and condition [10] holds.

In order to prove [11] we note that since the operator $A$ is prime then for all $f \in \bigcap_{\zeta \in \mathcal{Z}_V} \ker N_V(\zeta, z)$ and all $g \in E$ we have

$$[\Gamma_z f, \Gamma_\zeta g] = \frac{V_\theta(z) - V_\theta^*(\zeta)}{z - \zeta} f, g = 0,$$

and hence $f = 0$.

It was shown in [33] that

$$\mathfrak{N}_z \cap D(\hat{A}_R) = \Gamma_z \mathcal{B}, \quad z \in \rho(\hat{A}_R).$$

(14)

It is easy to see that

$$(\hat{A}_R - zI)(\mathfrak{N}_z \cap D(\hat{A}_R)) = \mathcal{L},$$

(15)

where $\mathcal{L} = D(A)^{[1]}$. Clearly, $\dim \mathcal{B} < \infty$, since $\mathcal{B} \subset E$.

In order to prove [12] we need to use the spectral decomposition of the $\pi$-self-adjoint operator $\hat{A}_R$. It was shown in [15] that for all $g \in D(\hat{A}_R)$

$$\hat{A}_R g = \lim_{z \rightarrow z_0 \in \mathcal{C}} (-z^2) \left( (\hat{A}_R - zI)^{-1} + \frac{1}{z} \right) g.$$ 

(16)

It follows from [13] that

$$(-z^2)\Gamma_{z_0}^+ \left( (\hat{A}_R - zI)^{-1} + \frac{1}{z} \right) \Gamma_{z_0} = -\frac{z^2}{z_0} \left[ V_\theta(z) - V_\theta^*(z_0) \right] + \frac{zz_0}{z - z_0} \Gamma_{z_0}^+ \Gamma_{z_0}.$$

(17)

Taking the limit in (17) and using (16), we obtain for all $f \in \mathcal{B}$

$$\Gamma_{z_0}^+\hat{A}_R\Gamma_{z_0}f = -\lim_{z \rightarrow z_0 \in \mathcal{C}} V_\theta(z)f + V_\theta^*(z_0)f + z_0 \Gamma_{z_0}^+ \Gamma_{z_0}f$$

$$= -\lim_{z \rightarrow z_0 \in \mathcal{C}} V_\theta(z)f + \text{Re} V_\theta(z_0)f + \text{Re} z_0 \Gamma_{z_0}^+ \Gamma_{z_0}f.$$
Subtracting (24) from (23) one obtains due to (20) and the linear relation
\[ \text{with the domain } \text{dom} A \text{ follows:} \]

\[ \text{continuous in } \Pi \]

In particular, it follows from (b) that the evaluation operator
\[ A \]

is a self-adjoint extension of
\[ h \]

hence, the operator
\[ V \]

is a prime symmetric operator in \( \Pi \). Therefore, it is possible to construct a Pontryagin space \( \Pi(V) \) of symmetric operators and a symmetric operator \( A \) with a \( \pi \)-self-adjoint extension \( A_V \) in \( \Pi(V) \). The strictness condition 3) guarantees that the function \( V(z) \) is a Krein-Langer Q-function of a pair \( A \) and \( A_V \). We will use a reproducing kernel space model for the operators \( A, A_V \) elaborated in \[2, 20\] and \[22\].

**Proof.** It was shown in \[33\] that if an operator-function \( V(z) \) in \([E, E]\) satisfies the conditions 1) and 2) of Definition \[8\] then it is possible to construct a Pontryagin space \( \Pi(V) \) and a symmetric operator \( A \) with a \( \pi \)-self-adjoint extension \( A_V \) in \( \Pi(V) \). Let us consider the reproducing kernel Pontryagin space \( \Pi(V) \) corresponding to the kernel \( N(z, \zeta) := N_V(z, \zeta) \). The latter means see \[11\] that \( \Pi(V) \) consists of functions holomorphic on \( Z_V \) and for each \( z \in Z_V \) and \( h \in E \) the followings hold:

(a) \( N(z, \zeta)h \) belongs to \( \Pi(V) \) as a function on \( \zeta \);
(b) \( [f(\cdot), N(z, \zeta)h] = (f(z), h)_E \) for every \( f(\cdot) \) in \( \Pi(V) \).

In particular, it follows from (b) that the evaluation operator \( f \rightarrow f(z)h_E \) is continuous in \( \Pi(V) \). The multiplicity operator

\[ A : f(\zeta) \rightarrow \zeta f(\zeta), \quad (19) \]

with the domain \( \text{dom} A = \{ f \in \Pi(V) : \zeta f(\zeta) \in \Pi(V) \} \) is a prime closed symmetric operator in \( \Pi(V) \). Let \( A := \text{Gr}(A) \) as was shown in \[22\] the adjoint linear relation \( A^+ \) takes the form

\[ A^+ = \{ \{ f, \tilde{f} \} \in \Pi(V)^2 : \tilde{f}(\zeta) - \zeta f(\zeta) = h_1 - V(\zeta)h_0 ; \ h_0, h_1 \in E \} \]

and the linear relation

\[ A_V = \{ \{ f, \tilde{f} \} \in \Pi(V)^2 : \tilde{f}(\zeta) - \zeta f(\zeta) = h_1 \in E \} \quad (20) \]

is a self-adjoint extension of \( A \) with \( \rho(A_V) = Z_V \).

The deficiency subspace \( \mathfrak{N}_2 \) of \( A \) \((z \in Z_V)\) consists of vector-functions \( \gamma(z)h := N(z, \cdot)h, h \in E \). The mapping \( \gamma(z) : E \rightarrow \mathfrak{N}_2 \) is injective since the assumption \( \gamma(z)h = 0 \) and the equality

\[ [N(z, w)h, N(w, \cdot)g] = \left( N(z, w)h, g \right)_E = \left( \frac{V(z) - V(w)^*}{z - w}, h \right) \quad (21) \]

imply that \( h \in \bigcap_{w \in Z_V} \ker(V(z) - V(w)^*) \). It follows from the hypothesis 3) that \( h = 0 \). The linear span of deficiency subspaces \( \mathfrak{N}_2 \) \((z \in Z_V)\) is dense in \( \Pi(V) \) and, hence, the operator \( V \) is a prime symmetric operator in \( \Pi(V) \).

Let us show that \( \gamma(z) \) satisfies the identity

\[ \gamma(z) = (A_V - z_0)(A_V - z)^{-1}\gamma(z_0), \quad z, z_0 \in Z_V. \quad (22) \]

This is straightforward from the identities

\[ zN(\bar{z}, \zeta)h - \zeta N(\bar{z}, \zeta)h = V(z)h - V(\zeta)h, \quad (23) \]

\[ z_0N(\bar{z}_0, \zeta)h - \zeta N(\bar{z}_0, \zeta)h = V(z_0)h - V(\zeta)h. \quad (24) \]

Subtracting (24) from (23) one obtains due to (20)

\[ \{ N(\bar{z}, \zeta)h - N(\bar{z}_0, \zeta), zN(\bar{z}, \zeta)h - z_0N(\bar{z}_0, \zeta)h \} \in A_V. \quad (25) \]
The latter equality is equivalent to (22).

The identities (22) and (21) show that \( V(z) \) is the Krein-Langer \( Q \)-function of the pair \( \mathcal{A}, \mathcal{A}_V \). The assumption (10) implies that \( \mathcal{A}_V = \{0\} \), see [32], and, therefore, \( \mathcal{A}_V \) is the graph of an operator \( \mathcal{A}_V \).

**Step 2.** The operator \( \mathcal{A}_V \) need not be densely defined. Let us calculate vector-functions from the subspace \( \mathcal{L} = D(\mathcal{A})^{[1]} \) explicitly. As follows from (14) and (15),

\[
\gamma(z) B = 9l_z \cap D(A_V) = (A_V - z)^{-1} L.
\]

This implies that for every \( h \in B \) there exists a strong limit of the vector-function

\[
h_\infty(\cdot) := \lim_{y \to \infty} (iy)\gamma(-iy)h = \lim_{y \to \infty} (iy)N(iy, \cdot)h
\]

as \( y \to \infty \). Since the evaluation operator is continuous in \( \Pi_k(V) \) one obtains from the hypothesis 4) for every \( z \in Z_V \)

\[
h_\infty(z) = \lim_{y \to \infty} (iy)N(iy, z)h
\]

\[
= \lim_{y \to \infty} (iy) \frac{V(z) - V(-iy)}{z + iy} h = V(z)h.
\]

Therefore, the subspace \( \mathcal{L} \) takes the form

\[
\mathcal{L} = \{ V(\cdot)h : h \in \mathcal{B} \}.
\]

It follows from (20) and (26) that for every \( h \in \mathcal{B} \)

\[
(A_V - z)^{-1} V(\cdot)h = N(z, \cdot)h \quad (z \in Z_V).
\]

One can derive the same equality from (20), (26), (28) and (22).

**Step 3.** Let us show that for every \( g \in E \) the function \( V(\cdot)g \) generates a functional on \( D(A^+) = D(A_V) + \mathcal{N} \) by the formulas

\[
[(A_V - z_0)^{-1} f(\cdot), V(\cdot)g] = (f(z_0), g)_E, \quad f \in \Pi_k(V),
\]

\[
[N(z_0, \cdot)h, V(\cdot)g] = (V(z_0)h, g)_E, \quad h \in E,
\]

which is continuous in the norm of \( \mathcal{H}^+ = D(\mathcal{A}^+) \).

Mention first that the formulas (30) and (31) are consistent since for \( f(\cdot) = V(\cdot)h \) (\( h \in \mathcal{B} \)) one obtains from (20) and (26) the formula

\[
[(A_V - z_0)^{-1} V(\cdot)h, V(\cdot)g] = (V(z_0)h, g)_E,
\]

which agrees with (31).

Next, it follows from (30), (31), (22) and the identity

\[
(A_V - z)^{-1} = (A_V - z_0)^{-1}(A_V - z_0)(A_V - z)^{-1}
\]

that

\[
[N(z_0, \cdot)h, V(\cdot)g] = \left( V(z_0)v + (z - z_0)\frac{V(z) - V(z_0)}{z - z_0} h \right)_E = (V(z)h, g)_E.
\]

Let now a sequence \( \varphi_n \in D(A_V) \) converges to \( \varphi \in D(A_V) \) in \( \mathcal{H}^+ \)-norm. Then

\[
f_n(\cdot) = (A_V - z_0)\varphi_n(\cdot) \to f(\cdot) = (A_V - z_0)\varphi(\cdot) \quad \text{strongly in } \Pi_k(V)
\]

and by continuity of the evaluation operator one has

\[
f_n(z) \to f(z) \quad \forall z \in Z_V.
\]
Then it follows from (30) that
\[
[\varphi_n(\cdot), V(\cdot)g] \to [\varphi(\cdot), V(\cdot)g] \quad (n \to \infty)
\]
and, therefore, the functional generated by \( V(\cdot)g \) via (30) and (31) is continuous, since \( \dim \mathcal{F}^+(\text{mod } D(A_V)) < \infty \).

Step 4. Using the operator \( A \) defined in (32) we construct a rigged Pontryagin space \( \mathcal{F}^+ \subset \Pi_\kappa(V) \subset \mathcal{F}^- \) the way it was described in the section 3. The functional \( V(\cdot)h, \ h \in E \) considered above can be viewed as an element from \( \mathcal{F}^- \). Let us define a linear operator \( K : E \to \mathcal{F}^- \) by the equality
\[
Kh := V(\cdot)h, \quad h \in E.
\]  
(33)
Clearly the operator \( K \) is invertible, otherwise \( V(z) \equiv 0 \) and definition 3 is violated. Let us extend the operator \( A_V \) to the linear operator \( A_R : \mathcal{F}^+ = D(A_V) + \mathcal{F}_z \to \mathcal{F}^- \) by the equality
\[
A_R N(\tilde{z}_0, \cdot)g = z_0 N(\tilde{z}_0, \cdot)g + V(\cdot)g, \quad g \in E.
\]  
(34)
This definition agrees with (32) for \( g \in \mathcal{B} \subset E \). The operator \( A_R \) is a \( \pi \)-self-adjoint bi-extension of \( A \) with the quasi-kernel \( A_V \). It follows from (29) and (34) that
\[
(A_R - z)N(\tilde{z}, \cdot)g = V(\cdot)g, \quad g \in E.
\]  
(35)
Making use of (33) and (35) one obtains
\[
K^+(A_R - z)^{-1}Kg = K^+ N(\tilde{z}, \cdot)g, \quad z \in Z_V.
\]
An application of (32) and (35) implies
\[
(K^+ N(\tilde{z}, \cdot)g, h) = [N(\tilde{z}, \cdot)g, V(\cdot)h] = (V(z)g, h)_E,
\]
and, therefore,
\[
K^+(A_R - z)^{-1}K = V(z), \quad z \in Z_V.
\]
Step 5. We set \( \bar{A}_I = K + \bar{K} \) and define
\[
\bar{A} = A_R + iA_I.
\]
It is obvious that the i.s.c.d.s.
\[
\theta = \begin{pmatrix}
\bar{A} & K \\
\mathcal{F}^+ \subset \Pi_\kappa(V) & \mathcal{F}^- \\
K & I
\end{pmatrix}
\]  
(36)
satisfies conditions (2) and (3) of Definition 5. What remains to show then is that the quasi-kernel \( T \) of the operator \( \bar{A} \) belongs to the class \( \Lambda_A \). To do this it suffices to show that \( \rho(T) \cap \mathbb{C}_- \) is nonempty and check that \( A \) is the maximal symmetric part of \( T \) and \( T^+ \). Since \( V \in \mathcal{N}_\kappa \) it follows from (31) Theorem 2.2] that \( V(z) - iI \) has at most \( \kappa \) zeros in \( \mathbb{C}_- \). Let us assume without loss of generality that the operator \( (V(z_0) - iI) \) is invertible. Consequently, the operator
\[
H = I + iK^+(A_R - z_0I)^{-1}K = I + iV(z_0) \in [E, E],
\]
is invertible as well. It follows from (34), (36), (35) that
\[
\bar{A}_N(\tilde{z}_0, \cdot)g = z_0 N(\tilde{z}_0, \cdot)g + V(\cdot)Hg, \quad g \in E.
\]
Since \( H \) is invertible this implies \( \mathcal{R}(\bar{A} - z_0I) \supset \mathcal{R}(K) \).
Let \( f \in \mathcal{N}_- \). Then \( (\bar{A} - z_0I)f = g + (z - z_0)f, \) where \( g = (A_R - zI)f + iA_I f \). Since \( \mathcal{R}(A_R) \subset \mathcal{R}(K) \) and \( (A_R - zI)\mathcal{N}_z = \mathcal{R}(K) \), we have \( g \in \mathcal{R}(K) \) and therefore there is an \( x \) such that \( (\bar{A} - z_0I)x = g \). Thus
\[
(A - z_0I)(f - x) = (z - z_0)f,
\]
i.e., $\Re(\mathcal{A} - z_0 I) \supset \mathcal{M}_2$. Since the operator $A$ is prime, c. 1. s. $\{\mathcal{M}_2, \ z \neq \bar{z}\} = \Pi_\kappa(V)$, we have $\Re(T - z_0 I) = \Pi_\kappa(V)$. On the other hand, the relations

$$\Re(T - z_0 I) \supset \mathcal{M}_{z_0}, \quad \dim \Re(T - z_0 I) \equiv \dim (\mathcal{M}_{z_0}) < \infty,$$

imply that $\Re(T - z_0 I)$ is closed and hence coincides with $\Pi_\kappa(V)$. Similarly, one proves that $\Re(T^* - z_0 I) = \Pi_\kappa(V)$, $T^*$ is a quasi-kernel of $A^\times$, and concludes that $A$ is a correct ($\ast$)-extension of $T$.

To prove that $A$ is the maximal symmetric part of $T$ and $T^+$, we assume the contrary. Then there exists such a $\pi$-symmetric operator $A_0$ that

$$T \supset A_0 \supset A, \quad T^+ \supset A_0 \supset A.$$

Consequently, for all $f \in D(A_0)$, $\mathcal{A} f = \mathcal{A}^\times f = A_0 f$,

$$\mathcal{A}_T f = \frac{(\mathcal{A} - \mathcal{A}^\times)}{2i} f = 0, \quad \mathcal{A}_R f = (\mathcal{A} - \mathcal{A}_I) f = A_0 f \in \Pi_\kappa(V).$$

Thus, $f \in D(A_V)$ and $A_V f = A_0 f$. For any $g \in \mathfrak{H}^+$ we have

$$[(A_V - z_0 I)f, \mathcal{A}_V g] = [f, \mathcal{A}_V g] = [\mathcal{A}_I f, g] = 0.$$

But since $\Re(\mathcal{A}(z_0)K^+) = \mathcal{M}_{z_0}$, then $f \in D(A)$ and $D(A_0) = D(A)$. Thus $A$ is the maximal symmetric part of $T$ and $T^+$.

\[\square\]

**Remark 11.** We should mention that the realization results obtained in Theorem \[\text{10}\] can be interpreted as realization with impedance systems $\Delta$ of the form \[\text{11}\] with $D = \mathcal{A}_R$ and

$$V_\Delta(z) = K^+(\mathcal{A}_R - zI)^{-1}K.$$

**Remark 12.** In the recent paper \[\text{35}\] the authors derive an alternative integral representation for matrix-functions of the class $N_\kappa$. It can be easily shown that all functions of the class $N_\kappa(R)$ fall into the special class described in Theorem 4.1 of \[\text{35}\] and permit a reduced Krein-Langer integral representation developed in that theorem.

6. Subclasses of the class $N_\kappa(R)$

In this section we follow \[\text{11}\] and introduce three distinct subclasses of the class of realizable operator-valued functions $N_\kappa(R)$.

**Definition 13.** An operator-function $V(z)$ of the class $N_\kappa(R)$ belongs to the subclass $N^0_\kappa(R)$ if in the definition \[\text{5}\] the subspace $\mathcal{B}$ is trivial, i.e.,

$$\lim_{y \to \infty} y(\Im V(iy)f, f) = \infty, \quad \forall f \in E, \ f \neq 0. \quad (37)$$

**Definition 14.** An operator-function $V(z)$ of the class $N_\kappa(R)$ belongs to the subclass $N^1_\kappa(R)$ if in the definition \[\text{5}\] the subspace $\mathcal{B} = E$, i.e.,

$$\lim_{y \to \infty} y(\Im V(iy)f, f) < \infty, \quad \forall f \in E, \ f \neq 0. \quad (38)$$

**Definition 15.** An operator-function $V(z)$ of the class $N_\kappa(R)$ belongs to the subclass $N^2_\kappa(R)$ if in the definition \[\text{5}\] the subspace $\mathcal{B}$ is neither trivial nor equals $E$, i.e.

$$\{0\} \subsetneq \mathcal{B} \subsetneq E.$$
One may notice that \( N(R) \) is a union of three distinct subclasses \( N_0(R), N_1(R) \) and \( N_{01}(R) \). Now we prove the direct and inverse realization theorems in each of the subclasses.

**Theorem 16.** Let \( \theta \) be a l.s.c.d.s. of the form \( 4 \) such that \( \mathcal{J} = I \) and \( A \) is an operator with dense domain. Then operator-function \( V_0(z) \) of the form \( 5, 6 \) has a holomorphic continuation \( V(z) \) which belongs to the class \( N^0(R) \).

Conversely, let an operator-valued function \( V(z) \) belong to the class \( N^0(R) \). Then \( V(z) \) admits a minimal realization by a system \( \theta \) of the form \( 4 \) with a densely defined operator \( A \).

**Proof.** Since the operator \( A \) is densely defined \( \mathcal{D}(\tilde{A}) \cap \mathcal{R}_z = \{0\} \) for every self-adjoint extension \( \tilde{A} \) of \( A \). In particular, one obtains \( \mathcal{D}(\tilde{A}_R) \cap \mathcal{R}_z = \{0\}. \) Due to \( 13 \) this implies \( \mathcal{B} = \{0\} \).

Conversely, if \( \mathcal{B} = \{0\} \) then it follows from \( 14 \) and \( 15 \) that \( \mathcal{L} = \{0\} \), and hence, the operator \( A \) is densely defined. \( \square \)

In order to proceed with the similar results in the class \( N^1(R) \) we need to recall the definition of \( O \)-operator [12] and give its analogue for the spaces with indefinite metric. A \( J \)-regular \( \pi \)-symmetric operator \( A \) is called an \( O \)-operator if its semi-deficiency indices are equal to zero. As it was shown in [5] for an operator \( T \in \Lambda_A \) there exist linear operators

\[
M_T : \mathcal{M}_T \oplus \mathcal{M} \to \mathcal{M}_{T-}\oplus \mathcal{M},
\]

\[
M_{T+} : \mathcal{M}_T \oplus \mathcal{M} \to \mathcal{M}_{T+}\oplus \mathcal{M},
\]

such that \( M_T = Gr(M_T) \) and \( M_{T+} = Gr(M_{T+}) \) have trivial intersections with the manifold \( \{x, x, x, \in \mathcal{M}\} \) and

\[
\mathcal{D}(T) = \mathcal{D}(A) \oplus \{x - y \mid \langle x, y \rangle \in \mathcal{M}_T\}, \quad (39)
\]

\[
\mathcal{D}(T+) = \mathcal{D}(A) \oplus \{x - y \mid \langle x, y \rangle \in \mathcal{M}_{T+}\}. \quad (40)
\]

If \( A \) is an \( O \)-operator then \( M_T \) and \( M_{T+} \) are operators in \( \mathcal{M} \) (\( \dim \mathcal{M} < \infty \)) with \( 1 \in \rho(M_T) \cap \rho(M_{T+}) \) and the relations \( 39, 40 \) imply that \( \mathcal{D}(T) = \mathcal{D}(T+) = \mathcal{M}, \) \( \text{Im}(T) \) is a bounded self-adjoint operator in \( \Pi_\kappa, \) and \( \text{Re}(T) \) is a \( \pi \)-self-adjoint extension of \( A \) in \( \Pi_\kappa. \)

**Theorem 17.** Let \( \theta \) be a l.s.c.d.s. of the form \( 4 \) such that \( \mathcal{J} = I, A \) is an \( O \)-operator, and \( \mathcal{D}(T) = \mathcal{D}(T^+) \). Then operator-function \( V_0(z) \) of the form \( 5, 6 \) has a holomorphic continuation \( V(z) \) which belongs to the class \( N^1(R) \).

Conversely, let an operator-valued function \( V(z) \) belongs to the class \( N^1(R) \). Then \( V(z) \) admits a minimal realization by a system \( \theta \) of the form \( 4 \) with a non-densely defined \( O \)-operator \( A \).

**Proof.** Once again using Theorem 9 we have that \( V(z) \in N_k(R) \) and need to show \( 38 \). Since \( A \) is an \( O \)-operator, \( \mathcal{M}_z = 0 \), and hence \( \dim \mathcal{M}_z = \dim \mathcal{M} \). Using \( 14 \) we have

\[
\dim (\Gamma_x B) = \dim (\mathcal{D}(\tilde{A}_R) \cap \mathcal{M}_x) = \dim \mathcal{M}.
\]

On the other hand, \( \mathcal{M}_x = \Gamma_x E \) and thus \( \dim \Gamma_x E = \dim \mathcal{M} \). Consequently, since \( \mathcal{B} \) is a subspace of \( E \) and \( \dim E = \dim \mathcal{B} \) we have \( \mathcal{B} = E \).

Conversely, if \( \mathcal{B} = E \), then using \( 14 \) we get \( \Gamma_x B = \Gamma_x E = \mathcal{D}(\tilde{A}_R) \cap \mathcal{M}_x = \mathcal{M}_x \).

Applying \( 13 \) yields

\[
\dim ((\tilde{A}_R - zI) \mathcal{M}_z) = \dim \mathcal{L} = \dim \mathcal{M}, \quad z \in \rho(\tilde{A}_R).
\]
Since \((\mathcal{A} - zI)\) is an invertible operator for \(z \in \rho(\mathcal{A})\) we conclude that \(\dim \mathfrak{N}_z = \dim \mathfrak{N}\). It can be shown (see [5], [40]) that the operator \(P^2\mathfrak{N}\) described in the section 3 is a bijective mapping from \(\mathfrak{N}_{\pm i}\) onto \(\mathfrak{N}_{\pm i} \oplus \mathfrak{N}\). Considering the above we have then \(\dim \mathfrak{N}_{\pm i} \oplus \mathfrak{N} = \dim \mathfrak{N}\). This proves that \(\mathfrak{N}_{\pm i} = 0\) and thus \(A\) is an \(O\)-operator. □

Remark 18. If \(V \in \mathfrak{N}^0_\kappa(R)\) then the operator \(A\) in the realization (4) is densely defined and this implies that \(D(T) \neq D(T^+\) for the operator \(T\) mutually disjoint with \(T^+\). When the operator \(A\) is nondensely defined even mutually disjoint operators \(T\) and \(T^+\) may have the same domain. In fact the equality \(D(T) = D(T^+\) holds if \(V \in \mathfrak{N}^1_\kappa(R)\). In this case we may not consider the bi-extensions of \(T\) in the rigged Pontryagin space and the corresponding l.s.c.d.s. can be written as follows

\[
\theta = \begin{pmatrix} T & K & J \\ \Pi_\kappa & E \end{pmatrix}.
\]

Theorem 19. Let \(\theta\) be a l.s.c.d.s. of the form (41) such that \(J = I, \overline{D(A)} \neq \Pi_\kappa,\) and \(D(T) \neq D(T^+\). Then operator-function \(V_\theta(z)\) of the form (5), (6) has a holomorphic continuation \(V(z)\) which belongs to the class \(N^0_\kappa(R)\).

Conversely, let an operator-valued function \(V(z)\) belongs to the class \(N^0_\kappa(R)\). Then \(V(z)\) admits a minimal realization by a system \(\theta\) of the form (41) with a non-densely defined operator \(A\) and \(D(T) \neq D(T^+\).

The proof is immediate from Theorem 16 and Theorem 17.

7. Applications to the scalar case

In this section we consider scalar functions of the class \(N_\kappa\). We will establish the link between scalar \((E = \mathbb{C})\) realizable functions of the class \(N_\kappa(R)\) and a class of realizable Nevanlinna functions [12].

The realization problems of the present type for Nevanlinna operator-valued functions were studied in details in [12] and [13] where similar subclass structure was developed. In particular, it was shown that any realizable Nevanlinna operator-function \(V(z)\) admits an integral representation

\[
V(z) = Q + F \cdot z + \int_{-\infty}^{+\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2}\right) dG(t),
\]

(41)

in the Hilbert space \(E\). In this representation \(Q = Q^*, F = 0, G(t)\) is non-decreasing operator-function on \((-\infty, +\infty)\) for which

\[
\int_{-\infty}^{+\infty} \frac{dG(t)}{1 + t^2} \in [E, E],
\]

and

\[
Qe = \int_{-\infty}^{+\infty} \frac{t}{1 + t^2} dG(t)e
\]

for all \(e \in E_\infty\) where

\[
E_\infty = \left\{ e \in E : \int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty \right\}.
\]

(42)
The class of all realizable Nevanlinna operator-functions is called $N(R)$ (see [12]). The three subclasses of the class $N(R)$ were introduced in [13] and are called $N^0(R)$, $N^1(R)$, and $N^{01}(R)$. Each subclass is described in terms of the subspace $E_\infty$ in [12] determined by the variation of measure $G(t)$ in the representation [11]. In particular, $E_\infty = \{0\}$ for the class $N^0(R)$, $E_\infty = E$ for $N^1(R)$, and $\{0\} \subsetneq E_\infty \subsetneq E$ for $N^{01}(R)$.

Now let us recall the following factorization result from [25] (see also [22]). Every scalar function $V(z) \in N_\kappa$ admits a unique factorization

$$V(z) = \frac{p(z)p^*(z)}{q(z)q^*(z)} V_0(z),$$

(43)

where $V_0$ belongs to the class $N_0$, $p(z)$ and $q(z)$ are relatively prime monic polynomials such that $\max(\deg p, \deg q) = \kappa$, $p^*(z) = \overline{p(z)}$. Recall also that the point $\infty$ is called a generalized pole of nonpositive type of $Q$ if

$$-\infty \leq \lim_{z \to \infty} \frac{Q(z)}{z} < 0,$$

(44)

and the point $\infty$ is called a generalized zero of nonpositive type of $Q$ if

$$0 \leq \lim_{z \to \infty} zQ(z) < \infty.$$

(45)

In terms of the factorization (43) one can consider the following three possibilities:

1. $\infty$ is a generalized pole of non-positive type of the function $V(z)$ if and only if $\deg p > \deg q$;
2. $\infty$ is a generalized zero of non-positive type of the function $V(z)$ if and only if $\deg p < \deg q$;
3. $\infty$ is neither the generalized pole of non-positive type nor generalized zero of non-positive type of the function $V(z)$ if and only if $\deg p = \deg q$.

In the first case $V$ does not belong to the class $N_\kappa(R)$, in the second case $V$ is definitely in the class $N_\kappa(R)$, and in the third case the inclusion $V \in N_\kappa(R)$ can be characterized in terms of the function $V_0$.

**Theorem 20.** Let $V(z) \in N_\kappa$ with $E = \mathbb{C}$ and let $\infty$ be neither the generalized pole nor generalized zero of non-positive type of the function $V(z)$. Then $V(z)$ belongs to the class $N_\kappa(R)$ if and only if the function $V_0$ in the factorization (43) belongs to the class $N(R)$.

**Proof.** Suppose $V_0(z) \in N(R)$ and let $V$ admits the factorization (43) with $p$ and $q$ such that $\deg p = \deg q = \kappa$. Then $V$ belongs to the class $N_\kappa$ (see [25]). Also, since $V_0(z) \in N(R)$ then $F = 0$ in (41). It follows from the representation (41) (see [20], [27]) that in this case

$$\lim_{y \to \infty} \frac{V_0(iy)}{y} = F = 0.$$

(46)

Combining (43) and (46) we get (10). In order to prove the third item in the definition of the class $N_\kappa(R)$ we first notice that it is equivalent to the function $V(z)$ not being an identical constant. Let us assume the contrary, i.e.,

$$V(z) = \frac{p(z)p^*(z)}{q(z)q^*(z)} V_0(z) \equiv k, \quad k \in \mathbb{C}.$$
This immediately contradicts that $V_0(z)$ is holomorphic in the upper half-plane. Therefore, the condition (3) of the definition of the class $N_\kappa(R)$ is satisfied.

It was shown in [30] that if $V_0(z) \in N(R)$ then

$$\lim_{y \to \infty} yV_0(iy) = \int_{-\infty}^{\infty} dG(t), \quad (47)$$

where $G(t)$ is the function from the representation (11) of $V_0(z)$. Then we see that the subspace $B$ defined by (12) for the class $N_\kappa(R)$ coincides with the definition of the subspace $E_\infty$ in (13) written for the function $V_0(z)$. We notice that since $E = \mathbb{C}$ then either $E_\infty = \{0\}$ or $E_\infty = E = \mathbb{C}$. Finally, if $E_\infty = \mathbb{C}$

$$\lim_{y \to \infty} V(iy) = \lim_{y \to \infty} V_0(iy) = 0,$$

(see [30]) and hence $V(z) \in N_\kappa(R)$.

Conversely, let $V(z) \in N_\kappa(R)$. Then (11) will provide us with $F = 0$ in the integral representation (11) of $V_0(z)$. Furthermore, since $V(z) \in N_\kappa(R)$ then

$$\lim_{y \to \infty} yV(iy) = \lim_{y \to \infty} yV_0(iy), \quad (48)$$

and is either finite or infinite. If the limit is infinite then $E_\infty = \{0\}$ for $V_0(z)$ and $V_0(z) \in N(R)$. If the limit is finite then $E_\infty = E = \mathbb{C}$, (17) holds for $V_0(z)$, and (see [30])

$$Q = \int_{-\infty}^{+\infty} \frac{t}{1 + t^2} dG(t).$$

Therefore, $V_0(z) \in N(R)$. \qed

**Corollary 21.** A function $V(z)$ belongs to the class $N^0_\kappa(R)$ with $E = \mathbb{C}$ if and only if the function $V_0$ in the equation (13) belongs to the class $N^0(R)$.

**Corollary 22.** A function $V(z)$ belongs to the class $N^1_\kappa(R)$ with $E = \mathbb{C}$ if and only if the function $V_0$ in the equation (13) belongs to the class $N^1(R)$.

**Proof.** The proofs of both corollaries immediately follow from (17) and (18). \qed

### 8. Examples

We conclude the paper with simple illustrations.

**Example 23.** Let us define $\Pi_1$ as a set of all $L^2([0, 2\pi], dx)$ functions with the scalar product

$$[f, g] = \int_0^{2\pi} f(x)g(x) dx - \frac{1}{\pi} \int_0^{2\pi} f(x) dx \int_0^{2\pi} g(x) dx.$$

Let also $A$ be a $\pi$-symmetric operator defined by

$$Af = \frac{1}{i} \frac{df}{dx},$$

with

$$\mathcal{D}(A) = \{f \in \Pi_1 \mid f, f' \in AC_{\text{loc}}([0, 2\pi]), f(0) = f(2\pi) = 0\}.$$

Let $T$ be an operator in $\Pi_1$ defined by

$$Tf = \frac{1}{i} \frac{df}{dx} - \frac{1}{\pi i} f(2\pi),$$
where
\[ D(T) = \{ f \in \Pi_1 \mid f, f' \in AC_{loc}([0, 2\pi]), f(0) = 0 \}. \]

One can check that the operator \( T \supset A, T^+ \supset A \), and \( A \) is a maximal \( \pi \)-symmetric part of \( T \) and \( T^+ \), i.e., \( T \in \Omega_A \). The following two formulas define a (\(*)\)-extension of \( T \).
\[
\begin{align*}
A f &= \frac{1}{i} \frac{df}{dx} - \frac{1}{\pi i} (f(2\pi) - f(0)) - if(0) \left[ \delta(x - 2\pi) + \delta(x) + \frac{2}{\pi} \right], \\
A^x f &= \frac{1}{i} \frac{df}{dx} - \frac{1}{\pi i} (f(2\pi) - f(0)) + if(2\pi) \left[ \delta(x - 2\pi) + \delta(x) - \frac{2}{\pi} \right].
\end{align*}
\]

By straightforward calculations we get
\[
\frac{A - A^x}{i} f = (f(0) + f(2\pi)) \left[ \frac{2}{\pi} - \delta(x - 2\pi) - \delta(x) \right].
\]

Now we can include \( A \) into a l.s.c.d.s.
\[ \theta = \left( \begin{array}{c} A \\ \delta^+ \subset \Pi_1 \subset \delta^- \\ K \\ \mathbb{C} \end{array} \right), \]
where
\[ Ke = \frac{1}{\sqrt{2}} \left[ \frac{2}{\pi} - \delta(x - 2\pi) - \delta(x) \right] e, \quad e \in \mathbb{C}. \]

Then we can derive
\[ W_\theta(z) = 1 + 2i[(A - zI)^{-1}Ke, Ke] = \frac{(z\pi i - 1)e^{2\pi zi} + 1}{e^{2\pi zi} - z\pi i - 1}. \]

Consequently, the function
\[ V_\theta(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J = \frac{2 + \pi iz - (2 - \pi iz)e^{2\pi iz}}{\pi z(e^{2\pi zi} - 1)}, \]
belongs to the class \( N^0_1 \).

**Example 24.** Now we consider a construction which leads to examples of functions of the class \( N^0_1(\mathcal{R}) \). Let \( \mathcal{H} \) and \( \mathcal{R} \) be two Hilbert spaces. Suppose that \( A_0 \) is a possibly unbounded operator in \( \mathcal{H} \) with nonempty resolvent set \( \rho(A_0) \). Let the operator \( T \) in the Hilbert space \( \mathcal{H} = \mathcal{H} \oplus \mathcal{R} \) is given by the block-operator matrix
\[ T = \begin{pmatrix} A_0 & C \\ B & D \end{pmatrix} \]
with bounded entries \( B, C \) and \( D \). Recall that the operator valued function
\[ S(z) = D - B(A_0 - zI)^{-1}C, \quad z \in \rho(A_0) \]
is called the transfer function of the system determined by the matrix \( T \). It is well known that the number \( z \in \rho(A_0) \) belongs to \( \rho(T) \) if and only if
\[ X(z) := S(z) - zI = D - B(A_0 - zI)^{-1}C - zI \]
has bounded inverse defined on \( \mathcal{R} \). If this is a case then the resolvent \( (T - zI)^{-1} \) is given by the Schur-Frobenius formula
\[
\begin{pmatrix}
(A_0 - zI)^{-1} \left( I + CX^{-1}(z)B(A_0 - zI)^{-1} \right) & -(A_0 - zI)^{-1}CX^{-1}(z) \\
-XX^{-1}(z)B(A_0 - zI)^{-1} & XX^{-1}(z)
\end{pmatrix}
\]
(\ref{eq:49})

Note that \( X(z) \) is called the Schur complement of the block-matrix \( T - zI \). Let \( \dim \mathcal{R} = \kappa < \infty \). Suppose that \( A_0 \) is a selfadjoint operator in \( \mathcal{H} \). In this case if
[Imz] is sufficiently large then the norm of the operator $z^{-1} (D - B(A_0 - zI)^{-1}C)$ is less then one, therefore $X^{-1}(z)$ exists and $X^{-1}(z)$ is a meromorphic function in $\mathbb{C}_+ \cup \mathbb{C}_-$. Equip the Hilbert space $\mathcal{H}$ by the indefinite inner product

$$[h, g] := (P_{\mathcal{H}_0}h, P_{\mathcal{H}_0}g) - (P_{\mathcal{R}}hP_{\mathcal{R}}g), \ h, g \in \mathcal{H},$$

where $P_{\mathcal{H}_0}$ and $P_{\mathcal{R}}$ are the orthogonal projections in $\mathcal{H}$ onto $\mathcal{H}_0$ and $\mathcal{R}$, respectively. Then $\mathcal{H}$ becomes a Pontryagin space $\Pi_\kappa$. Let $A$ be a linear operator in the Hilbert space $\mathcal{H}$ defined as follows

$$\mathcal{D}(A) = \mathcal{D}(A_0), \ Ah := A_0h + Bh, \ h \in \mathcal{D}(A).$$

The operator $A$ is a non-densely defined and closed $\pi$-Hermitian operator in $\Pi_\kappa$. Since the operator $P_{\mathcal{H}_0}A = A_0$ is selfadjoint, the operator $A$ is a regular $O$-operator, where $J = P_{\mathcal{H}_0} - P_{\mathcal{R}}$. Evidently, the operator

$$T = \begin{pmatrix} A_0 & -B^* \\ B & D \end{pmatrix}$$

meets the conditions

$$\mathcal{D}(T) = \mathcal{D}(T^+) = \mathcal{D}(A_0) \oplus \mathcal{R}, \ T \supset A, \ T^+ \supset A$$

In particular, $T$ is $\pi$-selfadjoint if and only if $D$ is a selfadjoint in the Hilbert space $\mathcal{R}$.

Let $D = \text{Re}(D) + i\text{Im}(D)$. Then

$$\text{Re}(T) = \frac{1}{2}(T + T^*) = \begin{pmatrix} A_0 & -B^* \\ B & \text{Re}(D) \end{pmatrix}, \ \text{Im}(T) = \frac{1}{2i}(T - T^*) = \begin{pmatrix} 0 & 0 \\ 0 & \text{Im}(D) \end{pmatrix}$$

Assume also that $D_1 = KJK^+$, where $K$ acts from the Hilbert space $\mathcal{R}$ into the negative subspace $\mathcal{R}$ of the Pontryagin space $\Pi_\kappa$ and $K$ is invertible. Note that $K^+ = -K^*$ where $K^*$ is the Hilbert space adjoint to $K : \mathcal{R} \rightarrow \mathcal{R}$. Then

$$\theta = \begin{pmatrix} T & K \\ \Pi_\kappa & \mathcal{J} \end{pmatrix}$$

is the l.s.c.d.s. The transfer function $W(z)$ of $\theta$ is given by

$$W(z) = I - 2iK^+(T - zI)^{-1}K\mathcal{J} = I - 2iK^+P_{\mathcal{R}}(T - zI)^{-1}K\mathcal{J}, \ z \in \rho(T)$$

and its fractional-linear transformation is

$$V(z) = i[W(z) + I]^{-1}[W(z) - I]\mathcal{J} = K^+P_{\mathcal{R}}(\text{Re}(T) - zI)^{-1}K, \ z \in \rho(T) \cap \rho(\text{Re}(T)).$$

Let

$$X_T(z) = D + B(A_0 - zI)^{-1}B^* - zI, \ X_T(z) = \text{Re}(D) + B(A_0 - zI)^{-1}B^* - zI.$$ 

Then from (51) it follows that

$$W(z) = I - 2iK^+X_T^{-1}(z)K\mathcal{J}, \ V(z) = K^+X_T^{-1}(z)K.$$

Let us take $\text{Im}(D) = -I$, $K = I$ then $K^+ = -I$ and $-X_T^{-1}(z)$ belongs to the class $N_\kappa^1(R)$ in the Hilbert space $\mathcal{R}$. Thus we obtain the following theorem.

**Theorem 25.** Let $\mathcal{H}_0$ and $\mathcal{R}$ be Hilbert spaces, $\dim \mathcal{R} = \kappa < \infty$. Let $A_0$ be a selfadjoint operator in $\mathcal{H}_0$, $B$ is a bounded operator from $\mathcal{H}_0$ into $\mathcal{R}$, and let $D$ be a selfadjoint operator in $\mathcal{R}$. Then the operator valued function

$$V(z) = -(D + B(A_0 - zI)^{-1}B^* - zI)^{-1}, \ z \in \mathbb{C}_+ \cup \mathbb{C}_-$$

belongs to the class $N_\kappa^1(R)$ in the Hilbert space $\mathcal{R}$. 


Using Theorem 25 let us give some concrete example of scalar function from the class $N_1^1(R)$.

Let $\mathcal{H}$ be a weighted Hilbert space $L_2([-1, 1], \rho(t))$ with the weight
\[ \rho(t) = \frac{2}{\pi} \sqrt{1 - t^2}. \]
Let the operator $A_0$ in $L_2([-1, 1], \rho(t))$ be defined as follows:
\[ (A_0 f)(t) = tf(t), \quad f(t) \in L_2([-1, 1], \rho(t)). \]
Then $A_0$ is a selfadjoint contraction. Let $e_0(t) = 1, \ t \in [-1, 1]$ The function $e_0(t)$ belongs to $L_2([-1, 1], \rho(t))$ and $\|e_0\| = 1$. Let $\mathfrak{F} = \mathbb{C}$. Define the operator $B : L_2([-1, 1], \rho(t)) \to \mathbb{C}$ as follows
\[ Bf(t) = \frac{2\gamma}{\pi} \int_{-1}^{1} f(t) \sqrt{1 - t^2} \, dt, \quad f(t) \in L_2([-1, 1], \rho(t)), \]
where $\gamma \neq 0$. Then
\[ B^* c = \overline{\gamma} c e_0(t), \quad c \in \mathbb{C}. \]
Let $D$ be the operator of multiplication on a real number $d$ in the space $\mathbb{C}$. It is known [15] that
\[ \frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1 - t^2}}{t - z} \, dt = 2\sqrt{z^2 - 1}, \quad z \notin [-1, 1], \]
where the branch of the function $\sqrt{z^2 - 1}$ is taken such that $\text{Im} \sqrt{z^2 - 1} > 0$ for $\text{Im} \ z > 0$. It follows that the function $V(z) = -(D + B(A_0 - zI)^{-1}B^* - zI)^{-1}$ takes the form
\[ V(z) = \frac{1}{z - 2|\gamma|^2(\sqrt{z^2 - 1} - z) - d} = \frac{1}{(1 + 2|\gamma|^2)z - 2|\gamma|^2\sqrt{z^2 - 1} - d} \]
According to Theorem 25 the function $V(z), \ z \in \mathbb{C}_+$ belongs to the class $N_1^1(R)$. If $|\gamma|^2 > \max\{0, (d^2 - 1)/4\}$ the function $V(z)$ has a simple pole
\[ z_0 = \frac{d(1 + 2|\gamma|^2) + 2|\gamma|^2\sqrt{4|\gamma|^2 + 1} - d^2}{1 + 2|\gamma|^2} \]
in $\mathbb{C}_+$. 

REFERENCES

[1] D. Alpay, A. Dijksma, J. Rosnyak, and H.S.V. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, Oper. Theory: Adv. Appl., 96, Birkhäuser Verlag, Basel, 1997.
[2] Alpay D., Bruinisma P., Dijksma A., and de Snoo H.S.V., A Hilbert space associated with a Nevanlinna function, in: *Proc. MTNS Meeting, Amsterdam* (1989), pp. 115-122.
[3] Yu. M. Arlinski˘ı. The inverse problem of the theory of characteristic functions of unbounded operator colligations. *Dokl. Akad. Nauk Ukrain. SSR, Ser. A* (1976), no. 2, 105–109.
[4] Yu. M. Arlinski˘ı. Regular (∗)-extensions of quasi-Hermitian operators in rigged Hilbert spaces. *Izv. Akad. Nauk Arm. SSR, Ser. Mat.*, 14 (1979), no. 4, 297–312.
[5] Yu. M. Arlinskii and V. A. Derkach. Inverse problem in the theory of characteristic operator functions of unbounded operator bundles in a space $\Pi_\alpha$, *Ukrain. Math. J.*, 31, no. 2, (1979), 89–94.
[6] Yu. M. Arlinskii and V. A. Derkach. Reconstruction of an unbounded operator colligation from characteristic function in the space $\Pi_\alpha$, *Preprint VINITI, 555-79*, (1978), 2–72.
Yu.M. Arlinskiĭ and E.R. Tsekanovskiĭ. The method of rigged spaces in the theory of extensions of Hermitian operators with a nondense domain of definition. *Sib. Math. J.* 15, (1974), 169-182.

D. Arov, “Passive linear systems and scattering theory”, in Dynamical Systems, Control Coding, Computer Vision, vol.25 of Progress in Systems and Control Theory, (1999), Birkhäuser Verlag, 27–44.

T. Ya. Azizov and I.S. Iokhvidov, “Linear operators in spaces with an indefinite metric”, *Wiley*, New York, 1989.

J.A. Ball and O.J. Staffans, “Conservative state-space realizations of dissipative system behaviors”, Report No. 37, Institute Mittag-Leffler, (2002/2003), 55 pp.; EOT (to appear)

S.V. Belyi and E.R. Tsekanovskiĭ. Realization and factorization problems for J-contractive operator-valued functions in half-plane and systems with unbounded operators, *Systems and Networks: Mathematical Theory and Applications*, Akademie Verlag, 2, (1994), 621–624.

S. V. Belyi and E. Tsekanovskiĭ. Realization theorems for operator-valued R-functions, *Operator theory: Advances and Applications*, 98, Birkhäuser Verlag Basel, (1997), 55–91.

S.V. Belyi and E.R. Tsekanovskiĭ. On classes of realizable operator-valued R-functions, *Operator theory: Advances and Applications*, 115, Birkhäuser Verlag Basel, (2000), 85–112.

S.V. Belyi and E.R. Tsekanovskiĭ. On Krein’s formula in indefinite metric spaces, *Linear Algebra and Applications*, 389 C, (2004), 305-322.

Yu.M. Berezanskiĭ, Expansions in Eigenfunctions of Self-Adjoint Operators, Transl. Math. Mono. 17, Amer. Math. Soc., Providence, RI, 1968.

M.S. Brodskiĭ, *Triangular and Jordan representations of linear operators*, Moscow, Nauka, 1969 (Russian).

K. Daho and H. Langer. Matrix functions of the class $N_{\kappa}$. *Math. Nachr.*, 120 (1985), 275–294

V.A. Derkach. $\pi$-selfadjoint biextensions of $\pi$-Hermitian operators, *Dopovidi Akad. Nauk Ukraine. SSR, Ser. A.*, (1975), no. 4, 304–306.

V.A. Derkach. On signature of defect subspaces of $\pi$-Hermitian operators with nondense domains in spaces $H_{\kappa}$, *Dopovidi Akad. Nauk Ukraine. SSR, Ser. A.*, (1977), no. 7, 579–581.

V. Derkach and M. Malamud, ”The extension theory of hermitian operators and the moment problem”, *J. Math. Sci.*, 73 (1995), 141–242.

V.A. Derkach and S. Hassi, A reproducing kernel space model for $N_{\kappa}$-functions, *Proc. Amer. Math. Soc.*, 131 (2003), no. 12, 3795–3806.

V. Derkach, S. Hassi, and H.S.V. de Snoo, ”Operator models associated with Kac subclasses of generalized Nevanlinna functions”, Methods of Functional Analysis and Topology, 5 (1999), 65–87.

V.A. Derkach, S. Hassi, and H. de Snoo, Operator models associated with singular perturbations, *Methods Funct. Anal. Topology*, 7, no. 3, (2001), 1–21.

V.A. Derkach and E.R. Tsekanovskiĭ. A class of operator-functions that are realizable by accretive operator colligations as transfer mappings of linear systems. (Russian) *Theory of mappings, its generalizations and applications*, (1982) 74–83, “Naukova Dumka”, Kiev.

A. Dijksma, H. Langer, A. Luger, and Yu. Shondin. A factorization result for generalized Nevanlinna functions of the class $N_{\kappa}$, *Integr. Eq. Oper. Th.*, 36 (2000), 121–125.

F. Gesztesy, N. J. Kalton, K. A. Makarov, E. Tsekanovskiĭ, Some Applications of Operator-Valued Herglotz Functions, *Operator Theory: Advances and Applications*, 123, Birkhäuser, Basel, (2001), 271–321.

F. Gesztesy and E. Tsekanovskiĭ, On matrix-valued Herglotz functions, *Math. Nach.* 218, (2000), 61–138.

F. Gesztesy and M. Zinchenko, On Spectral Theory for Schrödinger Operators with Strongly Singular Potentials, [arXiv:math.SP/0505120](http://arxiv.org/abs/math.SP/0505120) vol 1, (2005).

I. S. Iokhvidov and M. G. Krein, Spectral theory of operators in spaces with an indefinite metric, *Amer. Math. Soc. Translations*, 2, 13, (1960), 105–175.

I. S. Kac and M. G. Krein, $R$-functions–analytic functions mapping the upper halfplane into itself, *Amer. Math. Soc. Transl.* (2) 103, 1-18 (1974).

M.A. Krasnosel’skiĭ, On selfadjoint extensions of Hermitian operators, *Ukrain. Math. J.*, 1 (1949), 21-38 (Russian).

M. G. Krein and H. K. Langer, Defect subspaces and generalized resolvents of a Hermitian operator in the space $H_{\kappa}$, *Func. Anal. Appl.*, 5, 2, (1971), 136–146.
ON REALIZATION OF THE KREIN-LANGER CLASS $N_\kappa$.

[33] M. G. Krein and H. K. Langer, Über die $Q$-Funktion eines $\pi$-hermiteschen Operators im Raume $II_\kappa$, Acta Sci. Math., 34, (1973), 191–230.

[34] M. G. Krein and H. K. Langer, Some propositions on analytic matrix functions related to the theory of operators in the space $II_\kappa$, Acta Sci. Math., 43, (1981), 181–205.

[35] J. Rovnyak and L.A. Sakhnovich, On the Krein-Langer integral representation of generalized Nevanlinna functions, Electronic Journal of Linear Algebra, 11, (2004), pp. 1–15.

[36] Ju. L. Šmul’jan, Extension theory for operators and spaces with indefinite metric, Math. USSR Izvestija, 8, no. 4, (1974), 895–907.

[37] O.J. Staffans, Well-Posed Linear Systems: Part I Book manuscript, available at http://www.abo.fi/~staffans/ 2002

[38] O.J. Staffans, “Passive and conservative continuous time impedance and scattering systems, Part IWell posed systems”, Math. Control Signals Systems, 15, (2002), 291–315.

[39] O.J. Staffans and G. Weiss, “Transfer functions of regular linear systems, Part II: the system operator and the Lax-Phillips semigroup”, Trans. Amer. Math. Soc., 354, (2002), 3229–3262.

[40] E. R. Tsekanovskii and Yu. L. Shmul’jan, The theory of bi-extensions of operators on rigged Hilbert spaces. Unbounded operator colligations and characteristic functions, Russ. Math. Surv., 32, no. 5, (1977), 73–131.

DEPARTMENT OF MATHEMATICS, EAST UKRAINIAN NATIONAL UNIVERSITY, KVARTAL MOLOYOZHY, 20-A, 91034 LUGANSK, UKRAINE
E-mail address: yma@snu.edu.ua

DEPARTMENT OF MATHEMATICS, TROY STATE UNIVERSITY, TROY, AL 36082, USA
E-mail address: sbelyi@troy.stojus.edu, URL http://spectrum.troyst.edu/~bely1

DEPARTMENT OF MATHEMATICS, DONETSK NATIONAL UNIVERSITY, UNIVERSITETSKAYA STR, 24, 83055 DONETSK, UKRAINE
E-mail address: derkach@univ.donetsk.ua

DEPARTMENT OF MATHEMATICS, NIAGARA UNIVERSITY, NY 14109, USA
E-mail address: tsekanov@niagara.edu, URL http://faculty.niagara.edu/tsekanov/