How often does the Unruh-DeWitt detector click beyond four dimensions?

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Abstract

We analyse the response of an arbitrarily-accelerated Unruh-DeWitt detector coupled to a massless scalar field in Minkowski spacetimes of dimensions up to six, working within first-order perturbation theory and assuming a smooth switch-on and switch-off. We express the total transition probability as a manifestly finite and regulator-free integral formula. In the sharp switching limit, the transition probability diverges in dimensions greater than three but the transition rate remains finite up to dimension five. In dimension six, the transition rate remains finite in the sharp switching limit for trajectories of constant scalar proper acceleration, including all stationary trajectories, but it diverges for generic trajectories. The divergence of the transition rate in six dimensions suggests that global embedding spacetime (GEMS) methods for investigating detector response in curved spacetime may have limited validity for generic trajectories when the embedding spacetime has dimension higher than five.

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1 Introduction

The conventional notion of a particle in Minkowski spacetime quantum field theory is related physically to measurements made by inertial observers and characterised mathematically by positive frequencies with respect to the Minkowski time translation Killing vector. This notion generalises to a certain extent to noninertial observers whose worldline is the orbit of a more general timelike Killing vector, a celebrated application being the thermality that uniformly linearly accelerated observers see in Minkowski vacuum [1]. The notion of a particle seen by a generic noninertial observer in Minkowski space is however significantly more subtle, as a generic timelike trajectory need not be the orbit of any Killing vector. The notion of a particle seen by an observer in curved spacetime is even more subtle, as a generic curved spacetime need not admit any timelike Killing vectors [2, 3].

The notion of “a particle seen by an observer” can be given an operational meaning by coupling the quantum field to a pointlike particle detector that follows the observer’s worldline, known as an Unruh-DeWitt detector [1, 4], and by regarding transitions between the detector’s quantum states as absorptions and emissions of particles. This method is applicable whenever the quantum state is sufficiently regular to render the transition probabilities well defined, and in particular it does not assume existence of any Killing vectors. The method confirms the thermality seen by uniformly linearly accelerated observers in Minkowski vacuum [1], and its other applications include the thermality seen by static detectors in Schwarzschild spacetime [5, 6] and by inertial detectors in de Sitter space [7].

In first-order perturbation theory, the transition probability of the Unruh-DeWitt detector is proportional to the response function. This function is obtained by pulling the Wightman distribution of the quantum field back to the detector worldline, smearing with a switching function that specifies how the interaction is turned on and off, and Fourier transforming from the detector’s proper time to the detector’s energy level separation. The response function is well defined provided the quantum state of the field is regular in the sense of the Hadamard property [8, 9] and the switching function is smooth and has compact support [10, 11, 12, 13]. The physical situation here is that the initial states of the field and the detector have been prepared before the interaction begins, the interaction is switched smoothly on and off, and the final state of the detector is read after the interaction has ceased.

It may be tempting to ask what the detector’s transition probability is while the interaction is still ongoing, and to define the transition rate per unit time by differentiating the transition probability with respect to the time at which the detector is read. The pull-back of the Wightman function is however too singular to provide a mathematically well-defined expression for these quantities, and seemingly inconspicuous treatments can lead to paradoxical results: in particular, a naïve regularisation in four-dimensional Minkowski space yields a transition rate that is not Lorentz covariant [14, 15, 16, 17]. In the special cases where the trajectory is stationary, the initial state of the field is invariant under the Killing vector generating the trajectory and the
detector has been switched on in the infinite past \[1, 4, 5, 7\], regularisation can be bypassed by integrating formally over the whole trajectory and factoring out the infinite total proper time, as the transition rate should be time-independent on grounds of stationarity. This line of argument is however not available if the trajectory or the initial state of the field are not stationary, as would be the case for example for a detector that falls into a black hole, or for a detector that responds to the onset of Hawking radiation in the spacetime of a collapsing star.

A mathematically consistent way to formulate the question of detector’s response while the interaction is still ongoing is to start with a smooth switching function of compact support and then attempt to take the limit in which the switching function approaches a step-function of specified duration. In four dimensions, this limit leads to a diverging total transition probability within the first-order perturbation theory treatment \[14, 18, 19, 20, 21\], but the divergent piece depends just on the details of the switching and neither on the quantum state of the field nor on the detector’s trajectory, and the derivative of the transition probability with respect to the interaction duration remains finite \[22, 23\]. This means that the time derivative of the transition probability provides a well-defined notion of a transition rate even in the sharp switching limit. Physically, this notion is meaningful when the switching is rapid compared with the overall duration of the interaction, and it then extracts from the transition probability the piece that depends on the trajectory and on the quantum state, disregarding the numerically dominant piece that only depends on the details of the switching, and the whole procedure remains reliable provided the coupling constant is so small that first-order perturbation theory is still valid despite the large switching-dependent contribution. Observationally, the transition rate is meaningful in terms of consequent measurements in identical ensembles of detectors \[23\]. In Minkowski space and with the field prepared in Minkowski vacuum, this definition of the transition rate agrees with the transition rate obtained from the pointlike limit of a spatially smeared but sharply-switched detector \[15, 16, 17, 24, 25\].

In this paper we shall examine the response of an Unruh-DeWitt detector coupled to a massless scalar field, in first-order perturbation theory, in the limit of sharp switch-on and switch-off in Minkowski spacetimes of dimension two to six. Mathematically, departing from the macroscopically observed spacetime dimension \(d = 4\) is prompted by the observation that the divergence of the \(d = 4\) transition probability in the sharp switching limit is due to the singularity of the Wightman function \[22, 23\], and the strength of this singularity increases with spacetime dimension. In \(d = 2\) the singularity is merely logarithmic in proper time, and the transition probability remains finite; in \(d = 4\) the singularity is inverse square in proper time and the transition probability diverges but the transition rate remains finite \[22, 23\]. Does the divergence of the total transition probability start already at \(d = 3\) or only at \(d = 4\)? Does the instantaneous transition rate become divergent in sufficiently high dimension, and if yes, is the divergence sensitive to the non-stationarity of the trajectory?

Physically, one motivation to go above \(d = 4\) is that our spacetime may have di-
dimensions not yet observed, or it may admit a holographic description in terms of a higher-dimensional theory. It is hence of interest to clarify the usefulness of pointlike particle detectors in spacetimes that arise in these higher-dimensional theories, including the various black hole, black brane and black ring spacetimes that appear in string theory and in the black hole solutions that appear in brane world cosmology [26].

Another physical motivation to go above $d = 4$ is that quantum fields in curved spacetimes have been modelled by embeddings in a higher-dimensional flat spacetime, and these global embedding spacetime (GEMS) methods have yielded reasonable results for the temperature seen by observers on stationary trajectories in spacetimes of high symmetry. A selection of original papers are [27, 28, 29, 30] and a review with further references can be found in [24, 25]. Could GEMS methods be expected to provide a reasonable model for the response of an Unruh-DeWitt detector in the nonstationary setting, for example for a detector falling into a Schwarzschild black hole, allowing a comparison with predictions obtained by Bogoliubov transformation techniques [31]?

As a final motivation, recent investigations of entanglement within relativistic quantum information theory [32, 33, 34] have introduced non-inertial model detectors in a time-dependent setting on an effectively two-dimensional Minkowski spacetime, where the switch-on and switch-off can safely be taken sharp. How would the switching effects need to be handled when the effective spacetime has higher dimension?

We work in Minkowski spacetime of dimension $d$ and assume the initial state of the quantum field to be the Minkowski vacuum. Following the $d = 4$ analysis of [22], we start by taking the switching function to be smooth and compactly supported, and we express the response function (proportional to the transition probability by a factor that only depends on the internal properties of the detector) as a manifestly regular and Lorentz-covariant integral formula from which all $i \epsilon$ regulators have been removed. We then investigate the sharp switching limit, in which the switching function approaches a theta-step of prescribed duration in a controlled way. For $d = 2$ and $d = 3$ the response function remains finite in this limit; for $d = 2$ this is immediate already from the fact that the singularity in the Wightman function is merely logarithmic. For $d = 4$ it was found in [22] that the response function diverges but its derivative with respect to the duration remains finite, and we find that the same holds for $d = 5$. For $d = 6$, the response function again diverges, but a new phenomenon emerges in its derivative with respect to the duration: the derivative diverges for precisely those trajectories on which the scalar proper acceleration is not constant. This means that for $d = 6$ the notion of instantaneous transition rate is well defined for trajectories of constant scalar proper acceleration, including all trajectories that are stationary in the sense of being the orbit of a timelike Killing vector, but is ill defined for more general trajectories.

As a consistency check, we verify that for uniformly linearly accelerated motion our instantaneous transition rate results agree in each dimension with those obtained by Takagi [35] by a regularisation that relies on the stationary at the outset. For $d = 6$, we also verify that our transition rate result agrees with that obtained by Schlicht’s sharply-switched but spatially smeared detector model [15, 16, 17], in particular agreeing about
the presence of a divergent term that is proportional to the proper time derivative of the scalar proper acceleration. This gives us confidence that the obstacle to defining an instantaneous transition rate for generic trajectories in $d = 6$ is genuine.

Finally, we address the response of a detector in the Schwarzschild spacetime by the GEMS method. The Kruskal-Szekeres extension of Schwarzschild [36] has a global embedding in six-dimensional Minkowski space [37], and the GEMS method suggests that detector response in the Hartle-Hawking-Israel vacuum [6, 38] could be modelled by the response of the embedded detector in six-dimensional Minkowski space with the field in Minkowski vacuum. The instantaneous transition rate obtained from the sharp switching limit of the genuinely four-dimensional detector is finite [23]; by contrast, the GEMS method predicts a divergent instantaneous transition rate for all trajectories in which the six-dimensional scalar proper acceleration is not constant. Four-dimensional trajectories with constant six-dimensional scalar proper acceleration include all equatorial trajectories that have constant area-radius and constant angular velocity; as special cases, these include the static trajectories and the circular geodesics. However, we find that the only four-dimensional timelike geodesics with constant six-dimensional scalar proper acceleration are the circular geodesics. The GEMS method does therefore not provide a way to model the transition rate of detectors on generic geodesics in Schwarzschild, including detectors falling freely into the black hole.

We begin in section 2 by reviewing the Unruh-DeWitt detector model, emphasising how the regularisation of the Wightman distribution needs to be addressed prior to the sharp switching limit. In section 3 we write the $d = 6$ response function in a form in which the regulator in the Wightman distribution has been eliminated, adapting the $d = 4$ techniques of [22]. In sections 4 and 5 we perform a similar analysis for respectively $d = 3$ and $d = 5$, where new technical issues arise from the fractional powers of squared geodesic distance that occur in odd spacetime dimensions. The sharp switching limit is addressed in section 6. Section 7 presents an overview of the divergences that emerge as the spacetime dimension increases, giving consistency checks with earlier work and discussing how the structure might continue for $d > 6$. Section 8 applies the results to extended Schwarzschild as embedded in $d = 6$ Minkowski. Section 9 presents a summary and concluding remarks.

We work in Minkowski spacetime of dimension $d \geq 2$ and of metric signature $(-++\cdots+)$. The units are such that $\hbar = c = 1$. Sans-serif letters denote Minkowski $d$-vectors, the Minkowski scalar product of $d$-vectors $k$ and $x$ is denoted by $k \cdot x$, and we write $x^2 := x \cdot x$. Complex conjugation is denote by overline. $O(x)$ denotes a quantity for which $O(x)/x$ is bounded as $x \to 0$, and $O(1)$ denotes a quantity that remains bounded when the parameter under consideration approaches zero.

## 2 Particle detector model

We take as our detector model a point-like two-state ‘atom’. The detector Hilbert space is spanned by the orthonormal basis states $|0\rangle_d$ and $|E\rangle_d$ whose respective energy
eigenvalues are 0 and $E \neq 0$. For $E > 0$, $|0\rangle_d$ is the ground state and $|E\rangle_d$ is the excited state; for $E < 0$, the roles are reversed.

The detector moves in $d$-dimensional Minkowski space along the timelike world-line $x(\tau)$, where the parameter $\tau$ is the detector’s proper time. The detector is coupled to a real, massless scalar field $\phi$ by the interaction Hamiltonian

$$H_{\text{int}} = c \chi(\tau) \mu(\tau) \phi(x(\tau)),$$

where $c$ is a small coupling constant, $\mu(\tau)$ is the atom’s monopole moment operator and $\chi(\tau)$ is a switching function, positive during the interaction and vanishing elsewhere.

We assume the switching function $\chi(\tau)$ to be smooth and of compact support, and we assume the trajectory $x(\tau)$ to be smooth on the support of $\chi$. We shall return to the smoothness of the trajectory in section 9.

We take the detector to be initially in the state $|0\rangle_d$ and the field to be in the Minkowski vacuum $|0\rangle$. After the interaction has been turned on and off, we are interested in the probability for the detector to have made a transition to the state $|E\rangle_d$, regardless the final state of the field. In first order perturbation theory in $c$, this probability factorises as

$$P(E) = c^2 |d\langle 0|\mu(0)|E\rangle_d|^2 \mathcal{F}(E),$$

where the factor $|d\langle 0|\mu(0)|E\rangle_d|^2$ depends only on the internal structure of the detector but neither on the trajectory nor on the switching, while the dependence on the trajectory and on the switching is encoded in the response function $\mathcal{F}(E)$. The formula for the response function reads

$$\mathcal{F}(E) = \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau'' e^{-iE(\tau' - \tau'')} \chi(\tau') \chi(\tau'') W(\tau', \tau''),$$

where the correlation function $W(\tau', \tau'') := \langle 0|\phi(x(\tau'))\phi(x(\tau''))|0\rangle$ is the pull-back of the Wightman function to the detector world line. Using $W(\tau', \tau'') = W(\tau'', \tau')$ and changing the integration variables from $(\tau', \tau'')$ to $(u, s)$ where $u := \tau'$ and $s := \tau'' - \tau'$, a useful alternative expression for the response function is

$$\mathcal{F}(E) = 2 \Re \int_{-\infty}^{\infty} du \chi(u) \int_{0}^{\infty} ds \chi(u - s) e^{-iEs} W(u, u - s).$$

In summary, we may regard the response function as answering the question “What is the probability of the detector to be observed in the state $|E\rangle_d$ after the interaction has ceased?” With mild abuse of terminology, we shall drop the constant prefactors in (2.2) and refer to the response function simply as the transition probability.

As the detector’s worldline is by assumption timelike, it follows from the general properties of the wave front sets of Hadamard state Wightman functions that the correlation function $W$ is a well-defined distribution on $\mathbb{R} \times \mathbb{R}$. As the switching function is by assumption smooth with compact support, formulas (2.3) and
are hence mathematically well defined. They are however not as such well suited for discussing the sharp switching limit. While the distribution $W$ may be represented by a family $W_\epsilon$ of functions that converge to $W$ as $\epsilon \to 0_+$, the limit $\epsilon \to 0_+$ may not necessarily be taken in (2.3) and (2.4) pointwise under the integrals. Our first task is therefore to express the response function in a form in which the regulator $\epsilon$ does not appear.

The case $d = 2$ is exceptional. The Wightman function of a massless scalar field in two dimensions is infrared divergent and should be understood in some appropriate limiting sense, such as the $m \to 0$ limit of a scalar field of mass $m > 0$. Given this understanding, the singularity in the correlation function $W(\tau', \tau'')$ is logarithmic in $\tau' - \tau''$, that is, integrable. In this case it follows by dominated convergence that the sharp switching limit in (2.4) can be taken immediately by setting $\chi(u) = \theta(u - \tau_0)\theta(\tau - u)$, where $\theta$ is the Heaviside function, $\tau_0$ is the moment of switch-on and $\tau$ is the moment of switch-off. The result is

$$F_\tau(E) = 2 \text{Re} \int_{\tau_0}^{\tau} du \int_0^{u-\tau_0} ds \, e^{-iEs}W(u, u-s).$$

(2.5)

The instantaneous transition rate can then be defined as the derivative of $F_\tau(E)$ (2.5) with respect to $\tau$, with the result [25, 39]

$$\dot{F}_\tau(E) = 2 \text{Re} \int_0^{\Delta\tau} ds \, e^{-iEs}W(\tau, \tau-s),$$

(2.6)

where $\Delta\tau := \tau - \tau_0$.

For $d > 2$, the singularity in $W(\tau', \tau'')$ is proportional to $(\tau' - \tau'')^{2-d}$, and the regulator must be removed more carefully. The case $d = 4$ was addressed in [22]. In the following sections we shall address the cases $d = 6, d = 3$ and $d = 5$ in turn.

3 Response function for $d = 6$

In this section we remove the regulator from the response function formula (2.4) for $d = 6$. The method is an adaptation of that introduced in [22] for $d = 4$.

3.1 Regularisation

The regularised $d = 6$ Wightman function reads [8, 9, 25]

$$W_\epsilon(u, u-s) = \frac{1}{4\pi^3} \frac{1}{[(\Delta x)^2 + 2i\epsilon \Delta t + \epsilon^2]^2},$$

(3.1)
where $\epsilon > 0$ is the regulator, $\Delta x := x(u) - x(u - s)$ and $\Delta t := t(u) - t(u - s)$. From (2.4) we obtain

$$
\mathcal{F}(E) = \lim_{\epsilon \to 0} \frac{1}{2\pi^3} \int_{-\infty}^{\infty} du \chi(u) \int_0^\infty ds \frac{\chi(u - s)}{R^4} \times \left[ \cos (Es) \left( (\Delta x)^2 + \epsilon^2 \right)^2 - 4\epsilon^2 (\Delta t)^2 \right] - 4\epsilon \sin (Es) \Delta t \left( (\Delta x)^2 + \epsilon^2 \right) \right]$$

with

$$
R := \sqrt{\left( (\Delta x)^2 + \epsilon^2 \right)^2 + 4\epsilon^2 (\Delta t)^2},
$$

where in (3.3) the quantity under the square root is positive and the square root is taken positive.

We record here inequalities that will be used repeatedly below. First, as geodesics maximise the proper time on timelike curves in Minkowski space, it follows that $|(\Delta x)^2| \geq s^2$. Second, as $\chi$ has compact support, the contributing interval of $s$ in (3.2) is bounded above, uniformly under the integral over $u$. From the small $s$ expansions $(\Delta x)^2 = -s^2 + O(s^4)$ and $\Delta t = O(s)$ it hence follows that $|(\Delta x)^2| \leq Ks^2$ and $|\Delta t| \leq sM$, where $K$ and $M$ are positive constants, independent of $u$.

We need to address first the integral over $s$ in (3.2). Working under the expression $(2\pi^3)^{-1} \int_{-\infty}^{\infty} du \chi(u)$, we write this integral over $s$ as the sum $I_{\geq}^{\text{even}} + I_{\geq}^{\text{odd}} + I_{>}^{\text{even}} + I_{>}^{\text{odd}}$, where the superscript even (odd) refers to the factor $\cos(Es)$ (respectively $\sin(Es)$) and the subscript $<$ (>) indicates that the range of integration is $(0, \eta)$ (respectively $(\eta, \infty)$), where $\eta := \epsilon^{1/4}$. We remark that this choice for $\eta$ differs from the choice $\eta = \epsilon^{1/2}$ that was made for $d = 4$ in [17, 22] and will be made for $d = 3$ in section 4 below, for reasons that stem from the increasing singularity of the Wightman function with increasing $d$.

We consider the two intervals of $s$ in the next two subsections.

### 3.2 Subinterval $\eta < s < \infty$

Consider $I_{\geq}^{\text{even}}$. When $\epsilon$ is set to zero, the integrand in $I_{\geq}^{\text{even}}$ reduces to $\chi(u - s) \cos (Es)/[(\Delta x)^2]$. This replacement creates in $I_{>}^{\text{even}}$ an error that can be arranged in the form

$$
\int_\eta^\infty ds \chi(u - s) \frac{\cos (Es)}{[(\Delta x)^2]^2} \times \left[ \left( 1 + \frac{\epsilon^2}{(\Delta x)^2} \right)^2 - 4\epsilon^2 \frac{(\Delta t)^2}{[(\Delta x)^2]^2} \right] - \left( 1 + \frac{\epsilon^2}{(\Delta x)^2} \right)^2 + 4\epsilon^2 \frac{(\Delta t)^2}{[(\Delta x)^2]^2} \right] \times \left[ \left( 1 + \frac{\epsilon^2}{(\Delta x)^2} \right)^2 + 4\epsilon^2 \frac{(\Delta t)^2}{[(\Delta x)^2]^2} \right]^{-2} \right] \right].
$$

Using $|(\Delta x)^2| \geq s^2$ and $s \geq \eta = \epsilon^{1/4}$, we have $\epsilon^2/(\Delta x)^2 \leq \epsilon^2/s^2 \leq \epsilon^2/\sqrt{\epsilon} = O(\epsilon^{3/2})$. Using $|\Delta t| \leq sM$, we similarly have $\epsilon^2(\Delta t)^2/(\Delta x)^2 = O(\epsilon^{3/2})$. The integrand in
is hence bounded in absolute value by a constant times $\epsilon^{3/2} / |(\Delta x)^2| \leq \epsilon^{3/2} / s^4$. It follows that the integral is of order $O \left( \epsilon^{3/2} / \eta^3 \right) = O(\eta^3)$.

Similar estimates show that $I_{\geq}^{\text{odd}} = O(\eta)$.

Collecting, we have

$$I_{\leq}^{\text{even}} + I_{\leq}^{\text{odd}} = \int_{\eta}^{\infty} ds \frac{\chi(u - s) \cos (Es)}{|(\Delta x)^2|} + O(\eta) \quad (3.5)$$

### 3.3 Subinterval $0 < s < \eta$

Consider $I_{<}^{\text{odd}}$, given by

$$I_{<}^{\text{odd}} = -4\epsilon \int_{0}^{\eta} ds \chi(u - s) \frac{\sin (Es) \Delta t (\Delta x)^2 + \epsilon^2}{R^4} \quad (3.6)$$

The delicate task is to estimate the denominator in (3.6).

By Taylor’s theorem, $(\Delta t)^2$, $(\Delta x)^2$ and $[(\Delta x)^2]^2$ have the asymptotic small $s$ expansions

$$(\Delta t)^2 = \sum_{n=0}^{n_1-1} T_n s^n + O(s^{n_1}) \quad (3.7a)$$

$$(\Delta x)^2 = \sum_{n=0}^{n_2-1} X_n s^n + O(s^{n_2}) \quad (3.7b)$$

$$[(\Delta x)^2]^2 = \sum_{n=0}^{n_3-1} F_n s^n + O(s^{n_3}) \quad (3.7c)$$

where the expansion coefficients $T_n$, $X_n$ and $F_n$ are functions of $u$, satisfying $T_0 = T_1 = X_0 = X_1 = X_3 = 0$, $X_2 = -1$ and the consequences for $F_n$, and the positive integers $n_1$, $n_2$ and $n_3$ may be chosen arbitrarily. (Note that as the trajectory is assumed smooth but not necessarily analytic, the error terms in (3.7) are not guaranteed to vanish for fixed $s$ as $n_i \to \infty$.) With this notation, the key insight is the rearrangement

$$R^2 = \epsilon^4 P \left[ 1 - \frac{4\epsilon r^3}{P} + \sum_{n=0}^{n_4-1} (2X_{(n+4)} + 4T_{(n+4)}) \frac{\epsilon^{n+2} r^n + 4}{P} + \sum_{n=0}^{n_5-1} F_{(n+6)} \frac{\epsilon^{n+2} r^n + 6}{P} \right] + O(\epsilon^{n_4+2} r^{n_4+4}) + O(\epsilon^{n_5+2} r^{n_5+6}) P \quad (3.8)$$

where we have written $s = \epsilon r$,

$$P := 1 + 2(2\tilde{t}^2 - 1)r^2 + r^4 \quad (3.9)$$
the dots denote derivatives with respect to \( u \), and the positive integers \( n_4 \) and \( n_5 \) may be chosen arbitrarily.

Since \( l \geq 1 \), \( P \) is positive for \( r \geq 0 \). We wish to regard the external factor \( \epsilon^4 P \) in (3.8) as the dominant part and the terms in the square brackets as a leading 1 plus sub-leading corrections. To this end, we rewrite (3.8) as

\[
R^2 = \epsilon^4 P \left[ 1 - \frac{4^l \tilde{r} r^3}{P} z + \sum_{n=0}^{n_4-1} \left( 2X_{(n+4)} + 4T_{(n+4)} \right) \frac{\epsilon^{n+2} r^{n+4}}{P} z^{2+(n/4)} \right.
\]

\[
+ \sum_{n=0}^{n_5-1} F_{(n+6)} \frac{\epsilon^{n+2} r^{n+6}}{P} z^{(n+2)/4} + \frac{O(\epsilon^{n_4+2} r^{n_4+4})}{P} z^{2+(n_4/4)} + \frac{O(\epsilon^{n_5+2} r^{n_5+6})}{P} z^{(n_5+2)/4} \right] ,
\]

(3.10)

where the book-keeping parameter \( z \), with numerical value 1, indicates what order in \( \epsilon \) the term in question is uniformly over the full range of \( r \), \( 0 \leq r \leq \epsilon^{-3/4} \). The term \(-4^l \tilde{r} r^3/P\) is assigned the factor \( z \) because \( r^3/P \) is bounded by a constant. The \( z \)-factors in the other terms follow because \( r^{n+4}/P \) and \( r^{n+6}/P \) are respectively bounded by a constant times \( \epsilon^{-3n_4} \) and a constant times \( \epsilon^{-3(n_5+2)/4} \).

Inserting (3.10) in the denominator of (3.6), performing a similar expansion in the numerator and changing the integration variable to \( r \) we obtain

\[
I_\text{odd} = -\frac{1}{\epsilon^2} \int_0^{\eta^{-3}} dr \frac{4E r^2}{P} \left[ \chi(u) - \epsilon r \chi(u) z^{1/4} + \cdots + \frac{1}{8!} \chi^{(8)}(u) \epsilon^8 r^8 z^2 + O(z^{9/4}) \right] \times
\]

\[
\times \left[ 1 - \frac{1}{3!} \epsilon^2 E^2 r^2 \sqrt{z} + \cdots + \frac{1}{9!} \epsilon^8 E^8 r^8 z^2 + O(z^{5/2}) \right] \times
\]

\[
\times \left[ i - \frac{1}{2} \epsilon r z^{1/4} + \cdots + \frac{1}{9!} \epsilon^{(9)} r z^2 + O(z^{9/4}) \right] \left[ \frac{1 - r^2}{P} + X_4 \epsilon^2 r^4 \frac{P}{z^2} + O(z^{9/4}) \right] \times
\]

\[
\times \left[ 1 - \frac{4^l \tilde{r} r^3}{P} z + (2X_4 + 4T_4) \frac{\epsilon^2 r^4}{P} z^2 + \sum_{n=0}^{n_5} F_{(n+6)} \frac{\epsilon^{n+2} r^{n+6}}{P} z^{(n+2)/4} + O(z^{9/4}) \right]^{-2},
\]

(3.11)

where in each bracket factor in the integrand we have kept terms to order \( z^2 \) because of the factor \( \epsilon^{-2} \) outside the integral and because \( \int_0^{\eta^{-3}} (r^2/P) \, dr \) remains bounded as \( \eta \to 0 \). We may now Taylor expand the integrand in (3.11) in \( z^{1/4} \), keeping terms to order \( z^2 \); the dropped terms are of order \( z^{9/4} \) and their contribution to \( I_\text{odd} \) is \( O(\eta) \). After this expansion \( z \) can be replaced by its numerical value 1, and we obtain for \( I_\text{odd} \) an expression that consists of elementary integrals of rational functions plus the error term \( O(\eta) \).
Consider then \( I_{<}^{\text{even}} \), rearranged as

\[
I_{<}^{\text{even}} = \int_0^\eta ds \, \chi(u - s) \frac{\cos(Es)}{R^2} - 8\epsilon^2 \int_0^\eta ds \, \chi(u - s) \frac{\cos(Es)(\Delta t)^2}{R^4}. \tag{3.12}
\]

Proceeding as above, we find

\[
I_{<}^{\text{even}} = \frac{1}{\epsilon^3} \int_0^{\eta - 3} dr \, \frac{\chi(u) - \epsilon r \chi(u) z^{1/4} + \cdots + \frac{1}{12!} \chi^{(12)}(u) \epsilon^{12} r^{12} z^3 + O(z^{13/4})}{P} \times
\]

\[
\left[ 1 - \frac{1}{2!} \epsilon^2 E^2 r^2 \sqrt{z} + \cdots + \frac{1}{12!} \epsilon^{12} E^{12} r^{12} z^3 + O(z^{7/2}) \right] \times
\]

\[
\left[ 1 - \frac{4 t \epsilon r^3}{P} z + \sum_{n=0}^4 \left( 2 X_{(n+4)} + 4 T_{(n+4)} \right) \frac{\epsilon^{(n+2)} r^{(n+4)}}{P} z^{2+(n/4)} + \frac{10}{1} \sum_{n=0}^{10} F_{(n+6)} \frac{\epsilon^{n+2} r^{n+6}}{P} z^{(n+2)/4} + O(z^{13/4}) \right]^{-1}
\]

\[
- \frac{8}{\epsilon^3} \int_0^{\eta - 3} dr \, \frac{\chi(u) - \epsilon r \chi(u) z^{1/4} + \cdots - \frac{1}{15!} \chi^{(15)}(u) \epsilon^{15} r^{15} z^{15/4} + O(z^{4})}{P} \times
\]

\[
\left[ 1 - \frac{1}{2!} \epsilon^2 E^2 r^2 \sqrt{z} + \cdots - \frac{1}{14!} \epsilon^{14} E^{14} r^{14} z^{7/2} + O(z^{4}) \right] \times
\]

\[
\left[ \frac{T_2 r^2}{P^2} + \frac{T_3 \epsilon r^3}{P^2} z + \frac{T_4 \epsilon^2 r^4}{P^2} z^2 + \frac{T_5 \epsilon^3 r^5}{P^2} z^3 + O(z^{4}) \right] \times
\]

\[
\left[ 1 - \frac{4 t \epsilon r^3}{P} z + \sum_{n=0}^7 \left( 2 X_{(n+4)} + 4 T_{(n+4)} \right) \frac{\epsilon^{(n+2)} r^{(n+4)}}{P} z^{2+(n/4)} + \frac{13}{1} \sum_{n=0}^{13} F_{(n+6)} \frac{\epsilon^{n+2} r^{n+6}}{P} z^{(n+2)/4} + O(z^{4}) \right]^{-2}. \tag{3.13}
\]

We Taylor expand the integrands in (3.13) in \( z^{1/4} \), keeping in the first (respectively second) integrand terms to order \( z^3 (z^{15/4}) \), at the expense of an error of order \( O(\eta) \) in \( I_{<}^{\text{even}} \). Replacing \( z \) by its numerical value 1, we then obtain for \( I_{<}^{\text{even}} \) an expression that consists of elementary integrals of rational functions plus the error term \( O(\eta) \).

### 3.4 Combining the subintervals

Evaluating the numerous elementary integrals obtained from (3.11) and (3.13) and combining the results with (3.5), we find from (3.2) that the response function takes the
form

\[
\mathcal{F}(E) = \lim_{\eta \to 0} \frac{1}{2\pi^3} \int_{-\infty}^{\infty} du \chi(u) \left[ -\frac{\chi(u)}{3\eta^3} - \frac{E\pi}{12} \left[ \chi(u)(E^2 + \ddot{x}^2) - 3\ddot{\chi}(u) \right] \right. \\
+ \frac{1}{6\eta} \left[ \chi(u)(3E^2 + \ddot{x}^2) - 3\ddot{\chi}(u) \right] + \int_{\eta}^{\infty} ds \frac{\chi(u-s)\cos(Es)}{[(\Delta x)^2]^2} \\
\left. \right] \left. \right] \right] \\
\tag{3.14}
\]

where \( \dddot{x}^2 \) is evaluated at \( u \). The uniformity of the \( O(\eta) \) error terms in \( u \) has been used to control the errors, and all terms involving the Lorentz-noncovariant quantities \( T_n \) have cancelled on integration over \( u \) (cf. section 3 of \[22\] for a similar cancellation in four dimensions). Taking the inverse powers of \( \eta \) under the \( s \)-integral, we have

\[
\mathcal{F}(E) = -\frac{E}{24\pi^2} \int_{-\infty}^{\infty} du \left[ \chi^2(u)(E^2 + \ddot{x}^2) + 3\dot{x}^2(u) \right] \\
+ \lim_{\eta \to 0} \frac{1}{2\pi^3} \int_{-\infty}^{\infty} du \chi(u) \int_{\eta}^{\infty} ds \left( \frac{\chi(u-s)\cos(Es)}{[(\Delta x)^2]^2} - \frac{\chi(u)}{s^4} + \frac{\chi(u)(3E^2 + \ddot{x}^2)}{6s^2} - \ddot{\chi}(u) \right). \\
\tag{3.15}
\]

To take the limit \( \eta \to 0 \) in \[3.15\], we add and subtract under the \( s \)-integral that disentangle the small \( s \) divergences of \( \cos(Es)/[(\Delta x)^2]^2 \) from the small \( s \) behaviour of \( \chi(u-s) \). Following \[22\], we obtain

\[
\mathcal{F}(E) = -\frac{E}{24\pi^2} \int_{-\infty}^{\infty} du \left[ \chi^2(u)(E^2 + \ddot{x}^2) + 3\dot{x}^2(u) \right] \\
- \frac{E^2}{4\pi^3} \int_{0}^{\infty} ds \int_{-\infty}^{\infty} du \chi(u) \left[ \chi(u-s) - \chi(u) \right] \\
+ \frac{1}{2\pi^3} \int_{0}^{\infty} ds \int_{-\infty}^{\infty} du \chi(u) \left[ \chi(u-s) - \chi(u) - \frac{1}{2}s^2\ddot{\chi}(u) \right] \\
- \frac{1}{12\pi^3} \int_{0}^{\infty} ds \int_{-\infty}^{\infty} du \chi(u) \left\{ [\chi(u-s) - \chi(u)]\dddot{x}^2 - s\chi(u-s)\dddot{x}^2 \right\} \\
+ \frac{1}{2\pi^3} \int_{-\infty}^{\infty} du \chi(u) \int_{0}^{\infty} ds \chi(u-s) \left( \frac{\cos(Es)}{[(\Delta x)^2]^2} - \frac{1}{s^4} + \frac{3E^2 + \ddot{x}^2}{6s^2} - \frac{\dddot{x}^3}{6s} \right), \\
\tag{3.16}
\]

where \( \dddot{x}^2 \) and \( \dddot{x} \cdot \dddot{x} \) are evaluated at \( u \). The interchanges of the integrals before taking the limit \( \eta \to 0 \) are justified by absolute convergence of the double integrals, and taking the limit \( \eta \to 0 \) under the outer integral is justified by dominated convergence: in each integral over \( s \) in \[3.16\], the integrand is regular as \( s \to 0 \).

Equation \[3.16\] is our final, regulator-free expression for the response function. In section \[6\] we shall consider its behaviour when the switching approaches the step-function.
4 Response function for \( d = 3 \)

In this section we remove the regulator from the response function formula (2.4) for \( d = 3 \). The qualitatively new feature is that the techniques of section 3 need to be adapted to the fractional power in the Wightman function.

4.1 Regularisation

The regularised \( d = 3 \) Wightman function reads \[^{8, 9, 25}\]

\[
W_\epsilon(u, u-s) = \frac{1}{4\pi} \frac{1}{\sqrt{(\Delta x)^2 + 2i\Delta t + \epsilon^2}},
\]

where the branch of the square root is such that the \( \epsilon \to 0 \) limit of the square root is positive when \( (\Delta x)^2 > 0 \). Separating the real and imaginary parts gives

\[
W_\epsilon(u, u-s) = \sqrt{R + (\Delta x)^2 + \epsilon^2} - i\sqrt{R - (\Delta x)^2 - \epsilon^2}
\]

\[
\frac{4\sqrt{2}}{\pi R}
\]

(4.2)

where \( R \) is given by (3.3), the quantities under the square roots are positive and the square roots are taken positive. From (2.4) we now obtain

\[
F(E) = \lim_{\epsilon \to 0} \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} du \chi(u) \int_{0}^{\infty} ds \frac{\chi(u-s)}{R} \times
\]

\[
\left[ \sqrt{R + (\Delta x)^2 + \epsilon^2 \cos (Es) - \sqrt{R - (\Delta x)^2 - \epsilon^2 \sin (Es)}} \right].
\]

(4.3)

We proceed as in section 3. Working under the expression \((2\sqrt{2}\pi)^{-1} \int_{-\infty}^{\infty} du \chi(u)\), we write the integral over \( s \) as the sum \( I_{>}^{\text{even}} + I_{>}^{\text{odd}} + I_{<}^{\text{even}} + I_{<}^{\text{odd}} \), where the notation follows section 3 with the exception that we now choose \( \eta := \epsilon^{1/2} \). We consider the two intervals of \( s \) in the next two subsections.

4.2 Subinterval \( \eta < s < \infty \)

Consider \( I_{>}^{\text{odd}} \). When \( \epsilon \) is set to zero, the integrand in \( I_{>}^{\text{odd}} \) reduces to \(-\chi(u-s)\sqrt{-2/(\Delta x)^2}\), where the quantity under the square root is positive and the square root is taken positive. This replacement creates in \( I_{>}^{\text{odd}} \) an error that can be arranged in the form

\[
\int_{\eta}^{\infty} ds \frac{\chi(u-s)\epsilon^2}{(\Delta x)^2 \sqrt{-2/(\Delta x)^2}} \times
\]

\[
\left[ \frac{2 + \left( \frac{\epsilon^2}{(\Delta x)^2} + 4(\Delta t)^2 - 2 \frac{S}{(\Delta x)^2} - \epsilon^2 \frac{S^2}{(\Delta x)^4} \right)}{\frac{R}{(\Delta x)^2} \left( 1 + \frac{\epsilon^2}{(\Delta x)^2} - \frac{R}{(\Delta x)^2} - \sqrt{2R} \frac{\sqrt{S}}{(\Delta x)^2} \left( \frac{R}{(\Delta x)^2} - 1 - \frac{\epsilon^2 S}{(\Delta x)^2} \right) \right)} \right],
\]

(4.4)
where \( S := 3(\Delta x)^2 + 2\epsilon^2 + 8(\Delta t)^2 \).

Using bounding arguments similar to those in section 3, we find that \( R/(\Delta x)^2 = -1 + O(\eta^2) \), \( \epsilon^2/(\Delta x)^2 = O(\eta^2) \) and \( (\Delta t)^2/(\Delta x)^2 = O(1) \), and as a consequence \( S/(\Delta x)^2 = O(1) \). The integrand in (4.4) is hence bounded in absolute value by a constant times \( \epsilon^2/[-(\Delta x)^2]^{3/2} \leq \epsilon^2/s^3 \), from which it follows that the integral is of order \( O(\epsilon^2/\eta^2) = O(\eta^2) \).

Similar estimates show that

\[
I_{\text{even}} = O(\eta) \quad \text{and} \quad I_{\text{odd}} = O(\eta).
\]

Collecting, we have

\[
I_{\text{even}} + I_{\text{odd}} = -\int_\eta^\infty ds \chi(u-s) \sin(\sqrt{R} s) \left[ -2/(\Delta x)^2 \right] + O(\eta). \tag{4.5}
\]

### 4.3 Subinterval \( 0 < s < \eta \)

Consider \( I_{\text{odd}} \), given by

\[
I_{\text{odd}} = -\int_0^\eta ds \chi(u-s) \frac{\sin(\sqrt{R} s)}{R} \sqrt{R - (\Delta x)^2 - \epsilon^2}. \tag{4.6}
\]

Writing \( s = \epsilon r \) and introducing the book-keeping parameter \( z \) as in section 3, the counterpart of (3.10) reads

\[
R^2 = \epsilon^4 P \left[ 1 - \frac{4i\epsilon r^3}{P} z + \sum_{n=0}^{n_4-1} (2X_{(n+4)} + 4T_{(n+4)}) \frac{\epsilon^{n+2} r^{n+4}}{P} z^{2+(n/2)} \right.
\]

\[
+ \sum_{n=0}^{n_5-1} F_{(n+6)} \frac{\epsilon^{n+2} r^{n+6}}{P} z^{1+(n/2)} + \frac{O(\epsilon^{n_4+2} r^{n_4+4})}{P} z^{2+(n_4/2)} + \frac{O(\epsilon^{n_5+2} r^{n_5+6})}{P} z^{1+(n_5/2)} \left. \right], \tag{4.7}
\]

where the powers of \( z \) differ from those in (3.10) because the range of \( r \) is now \( 0 \leq r \leq \epsilon^{-1/2} \). It follows that in the denominator in (4.6) we have \( R = \epsilon^2 \sqrt{P} [1 + O(z)] \), and in the numerator we have \( \chi(u-s) \sin(\sqrt{R} s) = \epsilon Er [\chi(u) + O(\sqrt{z})] \).

To estimate the square root in the numerator in (4.6), we note first that all the terms with a positive power of \( z \) in (4.7) are at small \( r \) asymptotic to a power of \( r \) that is greater than 2. It follows that the same powers of \( z \) are retained if these terms are multiplied by any positive function of \( r \) that is bounded at small \( r \) by a constant times \( r^{-3} \) and at large \( r \) by a constant.

Now, we rearrange the quantity under the square root in (4.6) as

\[
R - (\Delta x)^2 - \epsilon^2 = \epsilon^2 Q \left[ 1 - \sum_{n=0}^{n_6-1} X_{(n+4)} \frac{\epsilon^{n+2} r^{n+4}}{Q} \right. + \frac{\sqrt{P}}{Q} \left( \frac{R}{\epsilon^2 \sqrt{P}} - 1 \right)
\]

\[
+ \frac{O(\epsilon^{n_5+2} r^{n_5+4})}{Q}. \tag{4.8}
\]
where
\[ Q := \sqrt{P} + r^2 - 1 \] (4.9)
and the positive integer \( n_6 \) may be chosen arbitrarily. Note that \( Q \) is positive for \( r > 0 \), its small \( r \) behaviour is \( Q = 2t^2r^2 + O(r^4) \) where the coefficient of \( r^2 \) is positive, and its behaviour at large \( r \) is \( Q/r^2 = 2 + O(r^{-2}) \). We wish to regard the external factor \( \epsilon^2 Q \) in (4.8) as the dominant part and the terms in the square brackets as a leading 1 plus sub-leading corrections. In the terms proportional to \( X_{(n+4)} \), this is accomplished by inserting the book-keeping factors \( z^{1+(n/2)} \). From the asymptotic behaviour of \( \sqrt{P}/Q \) at small and large \( r \) we see that in the term involving \( \sqrt{P}/Q \) this is accomplished by taking \( R^2 \) to be given by (4.7), with the \( z \)-factors therein. A Taylor expansion in \( z^{1/2} \) then shows that \( R - (\Delta x)^2 - \epsilon^2 = \epsilon^2 Q \left[ 1 + O(z) \right] \).

Collecting, we find
\[ I_{<}^{\text{odd}} = -\epsilon E \int_0^{1/\eta} \, dr \frac{r\sqrt{Q}}{\sqrt{P}} \left[ \chi(u) + O\left(\sqrt{z}\right) \right] = O(\eta) \] (4.10)
where the final form follows because the integrand asymptotes to a constant at large \( r \).

Consider then \( I_{<}^{\text{even}} \), given by
\[ I_{<}^{\text{even}} = \int_0^{\eta} \, ds \chi(u-s) \frac{\cos(Es)}{R} \sqrt{R + (\Delta x)^2 + \epsilon^2} \] (4.11)

We now rearrange the quantity under the square root in (4.11) as
\[ R + (\Delta x)^2 + \epsilon^2 = \epsilon^2 N \left[ 1 + \sum_{n=0}^{n_7-1} X_{(n+4)} \frac{\epsilon^{n_7+2}r^{n_7+4}}{N} + \frac{\sqrt{P}}{N} \left( \frac{R}{\epsilon^2\sqrt{P}} - 1 \right) \right. \]
\[ \left. + O\left(\epsilon^{n_7+2}r^{n_7+4}\right) \frac{N}{N} \right] \] (4.12)
where
\[ N := \sqrt{P} + 1 - r^2 \] (4.13)
and the positive integer \( n_7 \) may be chosen arbitrarily. Note that \( N \) is positive, its small \( r \) behaviour is \( N = 2 + O(r^2) \) and its large \( r \) behaviour is \( N = 2t^2 + O(r^{-2}) \). We wish to regard the external factor \( \epsilon^2 N \) in (4.12) as the dominant part. In the square brackets, the terms proportional to \( X_{(n+4)} \) can be given the book-keeping factors \( z^{n/2} \), while in the term involving \( \sqrt{P}/N \), the large \( r \) behaviour of \( \sqrt{P}/N \) implies that the powers of \( z \) inherited from (4.7) must be appropriately decreased. Using \( F_6 = -2X_4 \), we find
\[ R + (\Delta x)^2 + \epsilon^2 = \epsilon^2 N \left[ 1 + X_4 \frac{\epsilon^{n_7+4}}{N} \left( 1 - \frac{r^2}{\sqrt{P}} \right) + O\left(\sqrt{z}\right) \right] \] (4.14)
Although the term proportional to $X_4$ in the square brackets in (4.14) has arisen as a combination of two individual terms that came with $z$-factors $z^0$, a cancellation between these individual terms at large $r$ implies that the term as a whole can now be reassigned the factor $z$. We hence have $\sqrt{R + (\Delta x)^2 + \epsilon^2} = \epsilon \sqrt{N} [1 + O(\sqrt{z})]$. Using this and (4.7) in (4.11), we obtain

$$
I_{\text{even}}^\eta = \int_0^{1/\eta} dr \frac{\sqrt{N}}{\sqrt{P}} [\chi(u) + O(\sqrt{z})]
$$

$$
= \frac{\pi \chi(u)}{\sqrt{2}} + O(\eta), \quad (4.15)
$$

where the final form comes by evaluating the elementary integral.

### 4.4 Combining the subintervals

Combining (4.5), (4.10) and (4.15), we obtain from (4.3) for the response function the final form

$$
F(E) = \frac{1}{4} \int_{-\infty}^{\infty} du \chi^2(u) - \frac{1}{2\pi} \int_{-\infty}^{\infty} du \chi(u) \int_0^\infty ds \frac{\chi(u-s)\sin(Es)}{\sqrt{-(\Delta x)^2}}. \quad (4.16)
$$

The limit $\eta \to 0$ has been taken by just setting the lower limit of the $s$-integral to zero, as the small $s$ behaviour of the numerator cancels the singularity in the denominator.

### 5 Response function for $d = 5$

In this section we remove the regulator from the response function formula (2.4) for $d = 5$, extending the technique of section 4.

#### 5.1 Regularisation

The regularised $d = 5$ Wightman function reads [8, 9, 25]

$$
W_{\epsilon}(u, u-s) = \frac{1}{8\pi^2} \frac{1}{[(\Delta x)^2 + 2i\epsilon \Delta t + \epsilon^2]^{3/2}}. \quad (5.1)
$$

Separating the real and imaginary parts and substituting in (2.4) yields

$$
F(E) = \lim_{\epsilon \to 0} \frac{1}{4\sqrt{2} \pi^2} \int_{-\infty}^{\infty} du \chi(u) \int_0^\infty ds \frac{\chi(u-s)}{R^3} \times
$$

$$
\times \left[ \cos(Es) \left( ((\Delta x)^2 + \epsilon^2) \sqrt{R + (\Delta x)^2 + \epsilon^2} - 2\epsilon(\Delta t) \sqrt{R - (\Delta x)^2 - \epsilon^2} \right) \right.
$$

$$
- \sin(Es) \left( ((\Delta x)^2 + \epsilon^2) \sqrt{R - (\Delta x)^2 - \epsilon^2} + 2\epsilon(\Delta t) \sqrt{R + (\Delta x)^2 + \epsilon^2} \right) \right]. \quad (5.2)
$$
Working under the expression \((4\sqrt{2} \pi^2)^{-1} \int_{-\infty}^{\infty} du \chi(u)\), we again write the integral over \(s\) as the sum \(I_{<}^{\text{even}} + I_{<}^{\text{odd}} + I_{>}^{\text{even}} + I_{>}^{\text{odd}}\), choosing \(\eta := \epsilon^{1/4}\) as in section 3.

### 5.2 Subinterval \(\eta < s < \infty\)

\(I_{>}^{\text{odd}}\) and \(I_{>}^{\text{even}}\) can be estimated as in section 4. We merely record here the outcome,

\[
I_{>}^{\text{odd}} + I_{>}^{\text{even}} = \int_{\eta}^{\infty} ds \chi(u - s) \sin(ES) \sqrt{\frac{-2}{[(\Delta x)^2]^3}} + O(\eta),
\]

where the integral term has arisen from replacing the integrand in \(I_{>}^{\text{odd}}\) by its pointwise \(\epsilon \to 0\) limit.

### 5.3 Subinterval \(0 < s < \eta\)

Consider \(I_{<}^{\text{even}}\) and \(I_{<}^{\text{odd}}\), given by

\[
I_{<}^{\text{even}} = \int_{0}^{\eta} ds \chi(u - s) \frac{\cos(ES)}{R^3} \times \left[ ((\Delta x)^2 + \epsilon^2) \sqrt{R + (\Delta x)^2 + \epsilon^2 - 2\epsilon(\Delta t)\sqrt{R - (\Delta x)^2 - \epsilon^2}} \right], \tag{5.4a}
\]

\[
I_{<}^{\text{odd}} = -\int_{0}^{\eta} ds \chi(u - s) \frac{\sin(ES)}{R^3} \times \left[ ((\Delta x)^2 + \epsilon^2) \sqrt{R - (\Delta x)^2 - \epsilon^2 + 2\epsilon(\Delta t)\sqrt{R + (\Delta x)^2 + \epsilon^2}} \right]. \tag{5.4b}
\]

In the \(R^3\) in the denominator, we use (3.10). In the square root \(\sqrt{R - (\Delta x)^2 - \epsilon^2}\) in the numerator, we use (4.8), inserting the factors \(z^{(n+2)/4}\) in the terms proportional to \(X_{(n+4)}\) and using in the last term (3.10) for \(R\); by the asymptotic behaviour of \(\sqrt{P}/Q\) and the observations made in section 4, this makes \(z\) into an appropriate parameter for organising the square brackets in (4.8) into a Taylor expansion in \(z^{1/4}\) with the leading term 1.

In the square root \(\sqrt{R + (\Delta x)^2 + \epsilon^2}\) in the numerator, we wish to use (4.12). Attempting to regard the 1 in the square brackets as the dominant term can at first sight seem problematic because the terms proportional to \(X_{(n+4)}\) acquire the \(z\)-factors \(z^{-1+(n/4)}\), where the exponent is nonpositive for \(n \leq 4\), and when the last term is Taylor expanded in \(z^{1/4}\) using (3.10), the asymptotic behaviour of the factor \(\sqrt{P}/N\) implies that the exponents of \(z\) must be appropriately decreased and some of these decreased exponents are nonpositive. However, the nonpositive powers of \(z\) coming from the last term and from the terms proportional to \(X_{(n+4)}\) can be grouped into combinations that can be reassigned positive powers of \(z\), similarly to what happened for \(d = 3\) in (4.14).
After these reassignments we therefore obtain for \( \sqrt{R + (\Delta x)^2 + \epsilon^2} \) a Taylor expansion in \( z^{1/4} \) that starts as \( \epsilon \sqrt{N [1 + O (z^{1/4})]} \).

The remaining factors in (5.4) are expanded in \( z \) as in section 3.

We can now perform the Taylor expansion in \( z^{1/4} \) under the integrals. Dropping powers of \( z \) that are too high to contribute in the \( \epsilon \to 0 \) limit, and setting then \( z = 1 \), we obtain for \( I_{\text{even}} \) and \( I_{\text{odd}} \) formulas that consist of sums of finitely many elementary integrals plus an error term that vanishes in the \( \epsilon \to 0 \) limit. The elementary integrals are of the form

\[
\int_{r^n}^{r^n} dr^n N^{\pm 1/2} P_{-m}, \int_{r^n}^{r^n} dr^n N^{-3/2} P_{-m}, \int_{r^n}^{r^n} dr^n Q^{1/2} P_{-m} \text{ and } \int_{r^n}^{r^n} dr^n Q^{-3/2} P_{-m}, \text{ where } n \text{ is a positive integer and } m \text{ is a positive integer or half-integer.}
\]

5.4 Combining the subintervals

Evaluating the numerous elementary integrals that came from (5.4), combining the results with (5.3) and proceeding as in section 3, we find from (5.2) that the response function is given by

\[
\mathcal{F}(E) = \frac{1}{64\pi} \int_{-\infty}^{\infty} du \left[ \chi^2 \left( 4E^2 + \bar{x}^2 \right) + 4\bar{\chi}^2 \right] + \lim_{\eta \to 0} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} du \chi(u) \int_{\eta}^{\infty} ds \left( \frac{\chi(u - s) \sin(Es)}{\sqrt{[-(\Delta x)^2]^{3}}} - \frac{E\chi(u)}{s^2} \right), \tag{5.5}
\]

To take the limit \( \eta \to 0 \), we add and subtract under the \( s \)-integral terms that disentangle the small \( s \) divergence of \( \sin(Es)[- (\Delta x)^2]^{-3/2} \) from the small \( s \) behaviour of \( \chi(u - s) \).

Proceeding as in section 3 we find

\[
\mathcal{F}(E) = \frac{1}{64\pi} \int_{-\infty}^{\infty} du \left[ \chi^2(u) \left( 4E^2 + \bar{x}^2 \right) + 4\bar{\chi}^2(u) \right] + \frac{E}{4\pi^2} \int_{0}^{\infty} \frac{ds}{s^2} \int_{-\infty}^{\infty} du \chi(u) \left[ \chi(u - s) - \chi(u) \right] \tag{5.6}
\]

\[
+ \frac{1}{4\pi^2} \int_{-\infty}^{\infty} du \chi(u) \int_{0}^{\infty} ds \chi(u - s) \left( \frac{\sin(Es)}{\sqrt{[-(\Delta x)^2]^{3}}} - \frac{E}{s^2} \right).
\]

6 Sharp switching limit

In this section we consider the limit in which the switching function approaches a step-function of unit height and fixed duration. Concretely, we take \( \chi(u) = h_1 \left( \frac{u - \tau_0 + \delta}{\delta} \right) \times h_2 \left( \frac{-u + \tau + \delta}{\delta} \right), \tag{6.1} \)
where the parameters $\tau$, $\tau_0$ and $\delta$ satisfy $\tau > \tau_0$ and $\delta > 0$, and $h_1$ and $h_2$ are smooth non-negative functions satisfying $h_1(x) = 0$ for $x \leq 0$ and $h_i(x) = 1$ for $x \geq 1$. In words, the detector is switched on over an interval of duration $\delta$ just before proper time $\tau_0$, it stays on until proper time $\tau$, and it is switched off over an interval of duration $\delta$ just after proper time $\tau$. The manner of the switch-on and switch-off is specified respectively by the functions $h_1$ and $h_2$. The limit of sharp switching is then $\delta \to 0$, with $\tau_0$ and $\tau$ fixed.

We denote the response function by $\mathcal{F}_{\tau}$, where the subscript serves as an explicit reminder of the dependence on the switch-off moment $\tau$. We are interested both in $\mathcal{F}_{\tau}$ and in its derivative with respect to $\tau$, which we denote by $\dot{\mathcal{F}}_{\tau}$. As mentioned in section 1, $\mathcal{F}_{\tau}$ can be regarded as the detector’s instantaneous transition rate per unit proper time, observationally meaningful in terms of consequent measurements in identical ensembles of detectors [23].

The case of two-dimensional Minkowski spacetime, $d = 2$, was discussed in section 2. We shall address the cases from $d = 3$ to $d = 6$ in the following subsections.

6.1 $d = 3$

For $d = 3$, $\mathcal{F}_{\tau}$ is given by (4.16). The limit $\delta \to 0$ is well defined and can be taken directly in (4.16), with the result

$$\mathcal{F}_{\tau}(E) = \frac{\Delta \tau}{4} - \frac{1}{2\pi} \int_{\tau_0}^{\tau} du \int_0^{u-\tau_0} ds \frac{\sin(ES)}{\sqrt{-(\Delta x)^2}}, \quad (6.2)$$

and differentiation with respect to $\tau$ gives

$$\dot{\mathcal{F}}_{\tau}(E) = \frac{1}{4} - \frac{1}{2\pi} \int_0^{\Delta \tau} ds \frac{\sin(ES)}{\sqrt{-(\Delta x)^2}}. \quad (6.3)$$

6.2 $d = 4$

The case $d = 4$ was addressed in [22]. The expression for the response function with a general switching function reads

$$\mathcal{F}(E) = -\frac{E}{4\pi} \int_{-\infty}^{\infty} du \chi^2(u) + \frac{1}{2\pi^2} \int_0^{\infty} \frac{ds}{s^2} \int_{-\infty}^{\infty} du \chi(u) [\chi(u) - \chi(u-s)]$$

$$+ \frac{1}{2\pi^2} \int_{-\infty}^{\infty} du \chi(u) \int_0^{\infty} ds \chi(u-s) \left( \frac{\cos(ES)}{(\Delta x)^2} + \frac{1}{s^2} \right). \quad (6.4)$$

The first and third terms in (6.4) have well-defined limits as $\delta \to 0$. The second term in (6.4) takes at small $\delta$ the form $(2\pi^2)^{-1} \ln(\Delta \tau / \delta) + C + O(\delta / \Delta \tau)$, where $C$ is a constant determined by the functions $h_1$ and $h_2$, and this term hence diverges logarithmically as
\( \delta \to 0 \). However, the \( \tau \)-derivative of this term remains finite as \( \delta \to 0 \), and the transition rate has the well-defined limit

\[
\dot{\mathcal{F}}_\tau (E) = -\frac{E}{4\pi} + \frac{1}{2\pi^2} \int_0^{\Delta \tau} ds \left( \frac{\cos(Es)}{(\Delta x)^2} + \frac{1}{s^2} \right) + \frac{1}{2\pi^2 \Delta \tau} .
\] (6.5)

6.3 \( d = 5 \)

For \( d = 5 \), \( \mathcal{F}_\tau \) is given by (5.6). The last term in (5.6) has a well-defined limit as \( \delta \to 0 \), and so has the part of the first term containing \( \chi^2 \). The part of the first term containing \( \dot{\chi}^2 \) equals \( \frac{C'}{\delta} \), where \( C' \) is a positive constant determined by the functions \( h_1 \) and \( h_2 \). This part diverges as \( \delta \to 0 \) but is independent of \( \tau \) and does therefore not contribute to \( \dot{\mathcal{F}}_\tau \). Finally, the second term in (5.6) is similar to the second term in the \( d = 4 \) formula (6.4), being logarithmically divergent as \( \delta \to 0 \) but having a \( \tau \)-derivative that has a well-defined limit as \( \delta \to 0 \).

Collecting, we find that the transition rate has a well-defined \( \delta \to 0 \) limit, given by

\[
\dot{\mathcal{F}}_\tau (E) = \frac{4E^2 + \ddot{x}^2(\tau)}{64\pi} + \frac{1}{4\pi^2} \int_0^{\Delta \tau} ds \left( \frac{\sin(Es)}{\sqrt{[-(\Delta x)^2]^3}} - \frac{E}{s^2} \right) - \frac{E}{4\pi^2 \Delta \tau} .
\] (6.6)

6.4 \( d = 6 \)

For \( d = 6 \), \( \mathcal{F}_\tau \) is given by (3.16). The last term in (3.16) remains finite as \( \delta \to 0 \). The first and second terms are similar to those encountered in \( d = 5 \), with contributions that diverge in the \( \delta \to 0 \) limit proportionally to \( 1/\delta \) and \( \ln \delta \), but with \( \tau \)-derivatives that remain finite in this limit.

The third and fourth terms can be handled by breaking the integrations into subintervals as in [22]. The third term diverges proportionally to \( \delta^{-2} \) as \( \delta \to 0 \), but its \( \tau \)-derivative has a well-defined limit as \( \delta \to 0 \). The fourth term resembles the second term in that the divergence at \( \delta \to 0 \) is logarithmic in \( \delta \), but the presence of \( \ddot{x}^2 \) and \( \dot{x} \cdot x^{(3)} \) in the integrand has the consequence that the coefficient of the divergent logarithm depends on the trajectory and does not vanish on differentiation with respect to \( \tau \). We find that the transition rate is given by

\[
\dot{\mathcal{F}}_\tau (E) = \frac{\dot{x}(\tau) \cdot x^{(3)}(\tau)}{12\pi^3} \left( \ln \left( \frac{\Delta \tau}{\delta} \right) + C'_+ \right) - \frac{E(E^2 + \ddot{x}^2(\tau))}{24\pi^2} + \frac{1}{2\pi^3} \int_0^{\Delta \tau} ds \left( \frac{\cos(Es)}{[(\Delta x)^2]^2} - \frac{1}{s^4} + \frac{3E^2 + \ddot{x}^2(\tau)}{6s^2} - \frac{\ddot{x}(\tau) \cdot x^{(3)}(\tau)}{6s} \right) + \frac{3E^2 + \ddot{x}^2(\tau)}{12\pi^3 \Delta \tau} - \frac{1}{6\pi^3 \Delta \tau^3} + O \left( \delta \ln \left( \frac{\Delta \tau}{\delta} \right) \right) ,
\] (6.7)
where the constant $C'_+$ is determined by the switch-off function $h_2$ by

$$C'_+ = -2 \int_0^1 dr \frac{1}{r^2} \left( \int_0^1 dv \, h_2(1-v) \left[ h_2(1-v+r) - h_2(1-v) \right] - \frac{1}{2}r \right)$$

$$-2 \int_0^1 dv \, h_2(v) \left[ 1 - h_2(v) \right].$$

(6.8)

The qualitatively new feature is that the transition rate (6.7) does not have a well-defined limit for generic trajectories as $\delta \to 0$, because the coefficient of $\dddot{x}(\tau) \cdot x^{(3)}(\tau)$ diverges in this limit; further, even if $\delta$ is kept finite, the coefficient of this term depends on the details of the switch-off profile through the constant $C'_+$ (6.8). The limit exists only for trajectories whose scalar proper acceleration $\sqrt{\dddot{x}^2}$ is constant over the trajectory, in which case the coefficient of the divergent term in (6.7) vanishes. Note that this special class includes all trajectories that are uniformly accelerated, in the sense of following an orbit of a timelike Killing vector.

### 7 Spacetime dimension versus sharp switching

We have found that the sharp switching limit of the detector response function becomes increasingly singular as the spacetime dimension $d$ increases from 2 to 6. In this section we discuss further aspects of this singularity.

First, we have seen that the sharp switching limit of the response function diverges for $d \geq 4$. For $d = 4$ and $d = 5$ the divergent term is independent of the total detection time, and the limit of the instantaneous transition rate is still finite. For $d = 6$, however, the instantaneous transition rate diverges for generic trajectories. We summarise this behaviour in Table 1.

Second, we re-emphasise that when the Wightman distribution $W$ in (2.3) or (2.4) is represented as the $\epsilon \to 0$ limit of the regularised Wightman function $W_\epsilon$, the $\epsilon \to 0$ limit needs to be taken before considering the sharp switching limit: this is the only way one is guaranteed to be implementing the technical definition of the Wightman function correctly. With the regulator that we have used in this paper [equations (3.1),

| $d$ | $\mathcal{F}_\tau$ | $\dot{\mathcal{F}}_\tau$ |
|-----|-----------------|-----------------|
| 2   | finite          | finite          |
| 3   | finite          | finite          |
| 4   | $\ln \delta$   | finite          |
| 5   | $1/\delta$     | finite          |
| 6   | $1/\delta^2 \cdot x(3) \ln \delta$ | finite |

Table 1: The divergent pieces of the total transition probability $\mathcal{F}_\tau$ and the instantaneous transition rate $\dot{\mathcal{F}}_\tau$ for spacetime dimensions $d = 2, \ldots, 6$ in the sharp switching limit.
it is known that attempting to reverse the limits naïvely for \( d = 4 \) would yield an incorrect, and even Lorentz-noncovariant, result for the transition rate for all noninertial trajectories \[15, 16, 17\]. We have verified that attempting to reverse the limits naïvely would be incorrect also for \( d = 3 \), \( d = 5 \) and \( d = 6 \). For \( d = 3 \), substituting the regularised Wightman function \(4.1\) in \(2.6\) and evaluating the limit by the method of section 4 does give the correct result \(6.3\), but attempting to take the limit \( \epsilon \rightarrow 0 \) in \(2.6\) naïvely under the integral would miss the first of the two terms in \(6.3\). For \( d = 5 \), substituting the regularised Wightman function \(5.1\) in the naïve transition rate formula \(2.6\) and evaluating the limit \( \epsilon \rightarrow 0 \) by the methods of section 5 yields for the transition rate an expression that consists of \(6.6\) plus the Lorentz-noncovariant terms

\[
\frac{i(2 + \dot{t})E}{8\pi^2(1 + \dot{t})^2} - \frac{i}{8\pi^2(1 + \dot{t})^2} \epsilon,
\]

of which the second diverges as \( \epsilon \rightarrow 0 \). For \( d = 6 \), starting with the regularised Wightman function \(3.1\) yields for the transition rate a formula that is similar to \(6.6\), with the logarithmically divergent term replaced by a term that is logarithmically divergent in \( \epsilon \), plus a number of Lorentz-noncovariant terms.

Third, since the sharp switching divergence of \( \dot{\mathcal{F}}_\tau \) for \( d = 6 \) is perhaps surprising, we have verified that a similar divergence occurs also in the pointlike detector model where the switching is sharp at the outset but the detector is initially spatially smeared, having the Lorentz-function spatial profile with an overall size parameter \( \epsilon \), and the pointlike detector is recovered in the limit \( \epsilon \rightarrow 0 \) \[15, 24\]. (The model can be alternatively regarded as that of a sharply-switched pointlike detector whose Wightman function is regularised in terms of the frequency measured in the detector’s instantaneous rest frame, rather than in terms of the frequency measured in an externally-specified Lorentz frame \[24\].) Adapting the methods of section 3 and proceeding as in \[17\], we find that the expression for \( \dot{\mathcal{F}}_\tau \) is obtained from \(6.7\) by the replacement \( \ln(\tau/\delta) + C'_+ \rightarrow \ln(\tau/\epsilon) - \frac{4}{3} - \ln 2 \), so that the pointlike detector limit \( \epsilon \rightarrow 0 \) is again divergent unless the trajectory has constant scalar acceleration.

Fourth, for a trajectory of uniform linear acceleration \( a \), switched on in the infinite past, the transition rate formulas \(6.3\), \(6.5\), \(6.6\) and \(6.7\) yield

\[
\dot{\mathcal{F}}_{d=3}(E) = \frac{1}{2} \frac{1}{e^{2\pi E/a} + 1}, \quad \dot{\mathcal{F}}_{d=5}(E) = \frac{1}{32\pi} \frac{(4E^2 + a^2)}{e^{2\pi E/a} + 1},
\]

\[
\dot{\mathcal{F}}_{d=4}(E) = \frac{1}{2\pi} \frac{E}{e^{2\pi E/a} - 1}, \quad \dot{\mathcal{F}}_{d=6}(E) = \frac{1}{12\pi^2} \frac{E(E^2 + a^2)}{e^{2\pi E/a} - 1}.
\]

This was verified for \( d = 4 \) in \[17\], and we have used the same contour deformation method for the other values of \( d \). The results \(7.2\) agree with those found in \[35\], equation (4.1.27), where they were obtained from a definition of transition rate that relies at the outset on the stationarity of the trajectory.
Finally, we would like to speculate on how the response function and transition rate patterns that we have found for \( d \leq 6 \) might continue to \( d > 6 \), and specifically to \( d = 7 \).

Recall that the formula (2.4) gives the response function in terms of the distributional Wightman function \( W \). If \( W \) is to be replaced by the unregularised Wightman function \( W \) under the integrals, then the negative powers of \( s \) in \( \text{Re} \left[ e^{-iEs} W(u, u-s) \right] \) must be subtracted. The last term in our formulas (3.16), (4.16), (5.6), and (6.4) is precisely of this form. The corresponding term can be constructed for any \( d \), and for \( d = 7 \) it reads

\[
\frac{3}{8\pi^3} \int_{-\infty}^{\infty} du \, \chi(u) \int_0^\infty ds \, \chi(u-s) \left( \frac{\sin(Es)}{\sqrt{-[(\Delta x)^2]^5}} - \frac{E}{s^4} + \frac{E (4E^2 + 5\ddot{x}^2)}{24s^2} - \frac{5E \ddot{x} \cdot x^{(3)}}{24s} \right).
\]

(7.3)

Next, observe that our formulas (3.16), (5.6), and (6.4) contain terms in which the subtracted negative powers of \( s \) are combined with similar powers of \( s \) multiplied by quadratic combinations of \( \chi \) and its derivatives evaluated at \( u \) rather than at \( u-s \). All the negative powers of \( s \) that appear in (7.3) have already appeared in this fashion in (3.16), and comparison of the coefficients shows that the corresponding terms for \( d = 7 \) read

\[
-\frac{E^3}{16\pi^3} \int_0^{\infty} \frac{ds}{s^2} \int_{-\infty}^{\infty} du \, \chi(u) \left[ \chi(u-s) - \chi(u) \right]
+ \frac{3E}{8\pi^3} \int_0^{\infty} \frac{ds}{s^4} \int_{-\infty}^{\infty} du \, \chi(u) \left[ \chi(u-s) - \chi(u) - \frac{1}{2}s^2\ddot{\chi}(u) \right]
- \frac{5}{64\pi^3} \int_0^{\infty} \frac{ds}{s^2} \int_{-\infty}^{\infty} du \, \chi(u) \left\{ \left[ \chi(u-s) - \chi(u) \right] \dddot{x}^2 - s \chi(u-s) \dddot{x} \cdot x^{(3)} \right\}.
\]

(7.4)

The remaining term in (3.16), (4.16), (5.6), and (6.4) is a single integral involving derivatives of \( x \). We are not aware of pattern arguments that might fix this term fully for general \( d \), but we note that if this term for \( d = 7 \) contains the piece

\[
\frac{1}{2048\pi^2} \int_{-\infty}^{\infty} du \, \chi^2(u) \left( 4E^2 + \dddot{x}^2 \right) \left( 4E^2 + 9\dddot{x}^2 \right),
\]

(7.5)

then the transition rate computed from (7.3), (7.4), and (7.5) for a uniformly linearly accelerated trajectory agrees with that found in \[35\]. We further note that the power of \( E \) in the single integral term in (3.16), (4.16), (5.6), and (6.4) fits the empirical formula

\[
\frac{\Gamma(d/2 - 1)}{(d-3)!} \left( \frac{-E}{4\pi^{(d/2-1)}} \right)^{(d-3)},
\]

(7.6)

and so does the highest power of \( E \) in (7.5).

We anticipate that the \( d = 7 \) response function contains terms in addition to (7.3), (7.4) and (7.5); in particular, the pattern from \( d \leq 6 \) suggests that there should be a
term proportional to $\delta^{-3}$ as $\delta \to 0$, perhaps involving $\int_{-\infty}^{\infty} du \bar{\chi}^2(u)$. However, if the only terms contributing to the transition rate are (7.3), (7.4) and (7.5), then a comparison with the $d = 6$ case shows that the transition rate takes the form

$$\dot{F}_{\tau}(E) = \frac{5E \ddot{\bar{x}}(\tau) \cdot \dot{x}^{(3)}(\tau)}{64\pi^3} \left( \ln \left( \frac{\Delta \tau}{\delta} \right) + C'_+ \right) + \frac{(4E^2 + 9\ddot{x}(\tau))(4E^2 + \dot{x}^2(\tau))}{2048\pi^2}$$

$$+ \frac{3}{8\pi^3} \int_{0}^{\Delta \tau} ds \left( \frac{-\sin(Es)}{\sqrt{-[(\Delta x)^2(s)]^5}} \right) - \frac{E(4E^2 + 5\ddot{x}^2(\tau))}{24s^2} \frac{5E \ddot{\bar{x}}(\tau) \cdot \dot{x}^{(3)}(\tau)}{24s}$$

$$+ \frac{E(4E^2 + 5\ddot{x}^2(\tau))}{64\pi^3 \Delta \tau} - \frac{E}{8\pi^3 \Delta \tau^3} + O \left( \frac{\delta}{\Delta \tau} \right), \quad (7.7)$$

where $C'_+$ is again given by (6.8). While we must leave (7.7) to the status of a conjecture, we note that it shares the logarithmic divergence of the $d = 6$ transition rate (6.7) and the divergent term is again proportional to $\ddot{x} \cdot \dot{x}^{(3)}$.

8 Application: Schwarzschild embedded in $d = 6$ Minkowski

The GEMS method [27, 28, 29, 30] aims to model detector response in four-dimensional spacetime by an embedding into a higher-dimensional flat spacetime with an appropriately-chosen quantum state, typically the Minkowski vacuum. The method has yielded reasonable results for stationary trajectories in spacetimes of high symmetry. A review with references is given in [24, 25].

We wish to discuss the prospects of GEMS modelling in nonstationary situations in view of our results.

Recall that the $d = 4$ Minkowski vacuum response function formula (6.4) and instantaneous transition rate formula (6.5) generalise to an arbitrary Hadamard state on an arbitrary four-dimensional spacetime as [23]

$$\mathcal{F}(E) = -\frac{E}{4\pi} \left( \int_{-\infty}^{\infty} du \left[ \chi(u) \right]^2 + \frac{1}{2\pi^2} \int_{0}^{\infty} ds \int_{-\infty}^{\infty} du \chi(u) \left[ \chi(u) - \chi(u-s) \right] \right)$$

$$+ 2 \int_{-\infty}^{\infty} du \chi(u) \int_{0}^{\infty} ds \chi(u-s) \text{Re} \left( e^{-iEs} W_0(u, u-s) + \frac{1}{4\pi^2 s^2} \right), \quad (8.1)$$

$$\dot{\mathcal{F}}_{\tau}(E) = -\frac{E}{4\pi} + 2 \int_{0}^{\Delta \tau} ds \text{Re} \left( e^{-iEs} W_0(\tau, \tau-s) + \frac{1}{4\pi^2 s^2} \right) + \frac{1}{2\pi^2 \Delta \tau}, \quad (8.2)$$

where $W_0$ is the pointwise $i\epsilon \to 0$ limit of the Wightman function. The divergence structure at $\delta \to 0$ is exactly as in Minkowski vacuum: the response function (8.1) diverges logarithmically but the transition rate has the finite limit given by (8.2).
As a concrete example, consider a detector in the extended Schwarzschild spacetime, globally embedded in $d = 6$ Minkowski space as in $[37]$ (for further discussion see $[40]$). For static trajectories in exterior Schwarzschild, GEMS modelling with $d = 6$ Minkowski vacuum predicts a thermal response in the local Hawking temperature $[29]$. One might hence anticipate this modelling to extend to more general detector trajectories in the Hartle-Hawking-Israel vacuum $[6, 38]$.

Now, while the genuine $d = 4$ sharp switching transition rate (8.2) is finite for arbitrary trajectories in the Hartle-Hawking-Israel vacuum, the $d = 6$ Minkowski vacuum transition rate (6.7) diverges in the sharp switching limit unless the $d = 6$ scalar proper acceleration is constant. There are trajectories of constant $d = 6$ scalar proper acceleration through every point in the extended Schwarzschild spacetime, and these trajectories include all the stationary trajectories, that is, the exterior region circular trajectories that have constant (in general non-inertial) angular velocity. However, we have verified by a direct calculation that the only timelike Schwarzschild geodesics of constant $d = 6$ scalar acceleration are the exterior circular geodesics. This suggests that the GEMS method may not provide a viable model for detectors on generic geodesics in Schwarzschild.

9 Summary and discussion

We have investigated the response of an arbitrarily-accelerated Unruh-DeWitt detector coupled to a massless scalar field in Minkowski spacetimes of dimension up to six, assuming the field to be initially in the Minkowski vacuum and the detector to be switched on and off smoothly. Following the four-dimensional analysis of $[22]$, we first expressed the response function as a manifestly regular and Lorentz-covariant integral formula from which the $i\epsilon$ regulator of the Wightman function has been eliminated. We then examined the sharp switching limit of the response function and of the transition rate, defined as the derivative of the response function with respect to the total detection time.

In four dimensions it was shown in $[22]$ that the response function diverges in the sharp switching limit as $\ln\delta$, where $\delta$ is the duration of the switch-on and switch-off periods, while the transition rate remains finite in this limit. In two dimensions it is immediate from the merely logarithmic singularity of the Wightman function that both the response function and transition rate remain finite in the sharp switching limit. In three dimensions we found that both the response function and the transition rate remain finite in the sharp switching limit. In five dimensions the response function diverges in the sharp switching limit as $1/\delta^2$, but now also the transition rate contains a divergent term, proportional to $\ddot{x} \cdot x^{(3)} \ln\delta$; further, even if $\delta$ is kept nonvanishing, the coefficient of $\ddot{x} \cdot x^{(3)}$ depends on the details of the switch-off profile. The transition rate hence diverges for generic trajectories, although it remains finite for trajectories on
which the scalar proper acceleration is constant, including all stationary trajectories.

To summarise, the results show that the instantaneous transition rate of the Unruh-DeWitt detector is well defined in dimensions up to five. In dimension six the class of trajectories for which the transition rate is well defined includes all stationary trajectories, and among them in particular the Rindler trajectories of uniform linear acceleration for which the Unruh effect arises. For generic trajectories in six dimensions the transition rate is however divergent.

We have not pushed the computations beyond six dimensions, but we note that the divergences in the sharp switching limit arise from the singularity of the Wightman function, and this singularity grows in strength as the spacetime dimension increases. We therefore anticipate that the instantaneous transition rate will continue to be divergent for generic trajectories in all dimensions above five. We also anticipate that this property holds not just for Minkowski vacuum in Minkowski spacetime but for an arbitrary Hadamard state in an arbitrary spacetime. Finally, our explicit formulas for the response function and the transition rate in three dimensions are generalised to an arbitrary Hadamard state in an arbitrary spacetime in [41], using the techniques that allowed such a generalisation in four dimensions [23], and we anticipate a similar generalisation to be possible also in five and six dimensions.

While the divergence of the transition rate beyond five dimensions may be mathematically surprising, we emphasise that this does not seem to indicate any physical pathology in the system. It just tells that a pointlike detector is too singular to have a well-defined transition rate. There appears to be no difficulty with defining the transition rate of a spatially smeared detector [24, 25]: the difficulty surfaces only in the zero spatial size limit, as we discussed in section 7.

As an application of our results in six dimensions, we investigated modelling a particle detector in four-dimensional Schwarzschild by the GEMS technique, embedding Schwarzschild in six-dimensional Minkowski and setting the six-dimensional quantum field initially in Minkowski vacuum. This modelling yields a well-defined transition rate for all stationary trajectories in Schwarzschild. However, we found that the only Schwarzschild geodesics that are mapped to trajectories of constant scalar proper acceleration in six-dimensional Minkowski are the circular geodesics. The modelling does hence not yield well-defined transition rates for any noncircular geodesics in Schwarzschild. This, together with the anticipated extensions of our results beyond six dimensions, suggests that the GEMS technique may have limited validity for nonstationary trajectories whenever the embedding spacetime has dimension higher than five.

We have assumed the detector’s trajectory to be smooth on the support of the switching function. The first role of this assumption is that it ensures the pull-back of the Wightman function to the detector’s worldline to be a well-defined distribution that can be represented by a family of functions with a standard $i\epsilon$ regulator. The second role is that it allows Taylor expansions in the proper time to be used to arbitrarily high order. If the $i\epsilon$ representation of the pull-back of the Wightman function can be justified for a trajectory of lower differentiability class, then the practical constraint
on the differentiability class for our methods to remain applicable is that the Taylor expansions in proper time, such as those in (3.11) and (3.13), still hold to the required order.

On a practical note, we remark that we have made essential use of algebraic computing with the Taylor series expansions, and to a lesser extent also with the numerous elementary integrals. As a check against errors, we have performed a number of these computations independently on both Maple and Mathematica.

We have throughout the paper worked in linear perturbation theory in the coupling between the field and the detector. Going beyond perturbation theory would raise a number of new questions. Would a smooth switch-on and switch-off regularise the response of a pointlike detector also in a fully nonperturbative treatment? Could a transition rate for the detector be recovered in a sharp switching limit, at least in sufficiently low spacetime dimensions? How would the results compare with those obtained in a nonperturbative treatment with sharp switching but with a time sampling regulator [42, 43]? 

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