Another approach to volume of matroid polytopes
University of Western Ontario
Ahmed Umer Ashraf
December 27, 2018

Abstract

Ardila, Benedetti and Doker [ABD10] showed that matroid polytopes are examples of generalized permutohedra. Then employing the work of Postnikov [Pos09] they gave an expression of volume of a matroid polytope, in terms of a sum of products of Crapo’s beta invariants of certain contractions of the given matroid. Derksen and Fink [DF10] have defined a universal valuative invariant for matroids. Each valuative invariant of a matroid is a specialization of their invariant. Recently, motivated by tropical geometry, Hampe [Ham17] invented the notion of the matroid intersection ring and showed that it is generated additively by Schubert matroids. More importantly, he showed that Derksen-Fink invariant is a $\mathbb{Z}$-module homomorphism from this ring. Using this along with the Bidkhori-Sullivant formula [Bid12] for volume of lattice path matroid polytope, we come up with another approach to find volume of any matroid polytope from lattice of cyclic flats of its respective matroid. As an application, we give a closed formula for the volumes of connected sparse paving matroid polytopes.

Contents

1 Matroids
  1.1 Introduction to matroids ...................................... 2
  1.2 The Lattice of flats of a matroid .............................. 4
  1.3 The Lattice of cyclic flats of a matroid ....................... 6

2 Matroid Polytopes
  2.1 The base polytope of a matroid ............................... 7
  2.2 The matroid base polytope as a generalised permutohedron 9
  2.3 Matroid valuations .............................................. 10
  2.4 The Derksen-Fink invariant ................................. 12
1 Matroids

This section covers some basics of matroid theory, and fixes some terminology that persists throughout the paper. We keep this presentation to minimum, in the sense that only those constructs are being considered that will play some role in the sections to follow.

1.1 Introduction to matroids

Matroids are combinatorial abstractions of different notions of independence in mathematics. This includes linear independence and algebraic independence. Here we give some basic definitions. To get more details on matroid theory, we suggest the standard reference in this subject [Oxl11] to our readers. There are many equivalent definitions of a matroid and each one of them come with its own advantages and sophistications. For the purpose of this thesis, we are content with the basis definition of a matroid.

Definition 1.1. A matroid $M$ on an underlying set $E$ is a nonempty collection $\mathcal{B} \subseteq 2^E$ of subsets of $E$, called bases of $M$, which satisfies the basis exchange property: for every $A, B \in \mathcal{B}$, and $a \in A \setminus B$, there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

In reference to the matroid $M$, the set $E$ is called the ground set of $M$ and $\mathcal{B} = \mathcal{B}(M)$ is called the set of bases of $M$. We denote this by writing $M = (E, \mathcal{B})$. The operation of deleting an element of $A \setminus B$ from the basis $A$, and adjoining an element of $B \setminus A$ to $A$ is called basis exchange. Two bases that are related by a basis exchange are called adjacent bases. This relation is symmetric and therefore give rise to a graph called basis exchange graph $G(M)$ of $M$, where the vertices are bases and the edges are only between adjacent bases.

Example 1.2. Let $n$ be a positive integers and $0 \leq r \leq n$, then all the $r$ size subsets of $[n] = \{1, 2, \cdots, n\}$ are the bases of a matroid on $[n]$, called the uniform matroid $U_{r,n}$. The basis exchange graph $G(U_{r,n})$ in this case consists of complete graph $K_{\binom{n}{r}}$ on vertex set $V = \binom{[n]}{k}$. 

3 Schubert Matroids

3.1 The Schubert matroid $R_\rho$  
3.2 Lattice path matroids  
3.3 The Schubert matroid $R_\rho$ as a lattice path matroid  
3.4 Hampe’s matroid intersection ring

4 Volume Calculations

4.1 Volume of the Schubert matroid $R_\rho$  
4.2 From the lattice of cyclic flats to the volume of connected matroid polytope  
4.3 Volume of connected sparse paving matroid polytopes
Matroids occur naturally in many other areas of mathematics, for example graph theory, algebraic geometry and optimization theory. These occurrences also act as rich sources of examples of matroids. We here consider the one coming from graph theory. Given a finite graph $G = (V, E)$ on a vertex set $V$ with edge multiset $E$, it is a result of [Whi32] that the set of spanning trees of $G$ forms the collection of bases of a matroid on $E$. Matroids arising from this construction are called graphic matroids.

**Example 1.3.** Consider the graph $G_\square$ shown in figure 1 on vertex set $V = \{a, b, c, d\}$ with edge set $\{1, 2, 3, 4, 5\}$. The spanning trees of this graph form bases of the graphic matroid $M_{G_\square}$ associated to $G_\square$. The bases sets of matroid $M_{G_\square}$ are given by

$$B(M_{G_\square}) = \{\{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}\}$$

The basis exchange graph for $M$ is shown in figure 2.

We will now review some basic definitions. For a matroid $M = (E, B)$, an element $e \in E$ which is contained in each basis $B \in B(M)$ is called an isthmus, and an element $f \in E$ which is not in any of the bases is called a loop. A subset $I$ of a base $B \in B(M)$ is called an independent set. We denote the collection of independent sets of a matroid $M$ by $I(M)$. Any subset of $E$, that is not an independent is called dependent, and minimal dependent sets are called circuits. Rank of a subset $S \subseteq E$, denoted by $\text{rk}_M(S)$, is the size of an inclusion-maximal independent subset of $S$. A matroid $M$ is said to be of rank $r$ if the rank of the underlying set is $r$.

The operation of basis exchange also induces the following equivalence relation on the elements of $E$: We write $a \sim b$ if either $a = b$ or there exists bases $A, B \in B(M)$ such that $(A \setminus \{a\}) \cup \{b\} = B$. The equivalence classes with respect to this relation are called connected components of $M$. In other words, connected components of a matroid $M$ are the connected components of the basis exchange graph $G(M)$. The number of connected components of $M$ is denoted by $c(M)$, and we call $M$ connected if $c(M) = 1$.

We can also make new matroids out of old ones. Given a matroid $M = (E, B)$, we see that if the nonempty family $B$ satisfies the basis exchange property, than so does the (nonempty) family

$$B^* = \{E \setminus B : B \in B\}$$
The matroid \( M^* := (E, \mathcal{B}^*) \) is called the dual of \( M \). Given a subset \( S \subseteq E \), let \( J \) be the maximal independent set contained in \( S \), then

- the deletion \( M \setminus S \) is the matroid with the ground set \( E \setminus S \), and the independent sets \( \{I \subseteq E \setminus S : I \in \mathcal{I}(M)\} \).

- the contraction \( M / S \) is defined as the matroid with the ground set \( E \setminus S \) and the independent sets \( \{I \subseteq E \setminus S : I \cup J \in \mathcal{I}(M)\} \).

- the restriction \( M|_S \) is defined as the matroid with the ground set \( S \) and the independent sets \( \{I \subseteq S : I \in \mathcal{I}(M)\} \)

If a matroid \( N \) is constructed by a sequence of deletion and contraction on a given matroid \( M \), then \( N \) is called a minor of \( M \). Minors play a major role in the classification of matroids. If \( M \) is graphical then these operations corresponds to the usual deletion and contraction of the edges, respectively.

### 1.2 The Lattice of flats of a matroid

The inclusion-maximal subsets of \( E \) of rank \( k \) are called flats of rank \( k \). These can be thought of as subsets \( F \subseteq E \) such that \( \text{rk}(F \cup x) > \text{rk}(F) \) for any \( x \notin F \). This motivates the definition of closure operation \( \text{cl}_M \) of a matroid. For a subset \( S \subseteq E \), the closure of \( S \) is defined as

\[
\text{cl}(S) = \{x \in E : \text{rk}(S) = \text{rk}(S \cup x)\}
\]
In this terminology, flats are closed subsets of $E$. The set of flats $L(M)$ of a matroid form a ranked lattice under inclusion, where the rank function is given by the rank function $\text{rk}_M$ of the matroid $M$. For $F, G \in L(M)$, the join $F \vee G$ is given by $\text{cl}(F \cup G)$, and the meet $F \wedge G$ is given by the intersection $F \cap G$. The top element $\hat{1}_{L(M)}$ is given by the underlying set $E$, and the bottom $\hat{0}_{L(M)}$ is given by the closure of empty set $\text{cl}(\emptyset)$. The closure $\text{cl}(\emptyset)$ is precisely the set of loops of $M$. Each flat in $L(M)$ is generated as a join of atoms, hence $L(M)$ is atomic. Furthermore, the rank function $\text{rk}_M$ is semimodular, i.e. for $F, G \in L(M)$

$$\text{rk}(F \vee G) + \text{rk}(F \wedge G) \leq \text{rk}(F) + \text{rk}(G)$$

Having these two properties in the lattice $L(M)$, makes it a geometric lattice. It is a theorem due to Birkhoff [Dil90] that every geometric lattice is isomorphic to the lattice of flats of some loopless matroid.

**Example 1.4.** For the uniform matroid $U_{r,n}$ of rank $r$ over a set $E$ of size $n$, each set of size $k$ where $k < n$ is a flat of rank $k$, and there is a unique flat of rank $r$, namely $E$. Therefore, the lattice of flats of $U_{r,n}$ is a truncated boolean lattice at level $r - 1$, adjoined by $E$ at the top.

**Example 1.5.** For the matroid $M_{G_{\emptyset}}$, the lattice of flats is given in figure 3.

There are some natural constructions on matroids. We mention one here. Given two matroids $M_1 = (E_1, B_1), M_2 = (E_2, B_2)$, the direct sum $M_1 \oplus M_2$ of $M_1$ and $M_2$ is defined as a matroid on disjoint union $E_1 \sqcup E_2$ with bases $B(M_1 \oplus M_2) = B_1 \times B_2$. The lattice of flats $L(M_1 \oplus M_2)$ is isomorphic to $L(M_1) \times L(M_2)$. In general, a matroid $M$ is a direct sum of its connected components. This highlights the importance of determining whether a given matroid is connected or not. Crapo in [Cra67] came up with a criterion to do exactly that. He defined the beta invariant $\beta(M)$ of a matroid $M$ as

$$\beta(M) = (-1)^{\text{rk}_M(E)} \sum_{X \subseteq E} (-1)^{|X|} \text{rk}_M(X)$$
Crapo showed in [Cra67] that $\beta(M)$ is always a non-negative integer and it is positive if and only if $M$ is connected (Crapo’s connectivity criterion). Note that $\beta(M_{G_2}) = 1$, and hence it is connected. On the other other hand, for the uniform matroid $U_{n,n}$, the beta invariant $\beta(U_{n,n}) = 0$.

1.3 The Lattice of cyclic flats of a matroid

Given a flat $F$ of a matroid $M$, the cyclic part $\text{cyc}(F)$ of $F$ is defined to be the union of circuits of $M$ contained in $F$ i.e

$$\text{cyc}(F) := \bigcup_{\gamma \text{ circuit of } M \subseteq F} \gamma$$

We call a flat cyclic if it is equal to its cyclic part. The empty set is regarded as a cyclic flat. Equivalently, a flat $F$ is cyclic if the restriction $M|_F$ has no isthmuses. The set of cyclic flats, denoted as $Z(M)$, forms a lattice under set inclusion. Given $X, Y \in Z(M)$, the join $X \vee Y$ is given by $\text{cl}(X \cup Y)$, and the meet $X \wedge Y$ is given by $\text{cyc}(X \cap Y)$. Brylawski [Bry75] showed that the lattice $Z(M)$ together with its rank function, determines $M$. It is shown by Bonin and De Mier [BdM08] that every finite lattice is a lattice of cyclic flats of some bitransversal matroid. This means that the lattice of cyclic flats of a matroid can be as wild as possible. Given a collection $Z$ of subsets of a ground set $E$, and a rank function $\text{rk}$ on $Z$, Bonin and De Mier also gave an axiomatic scheme to determine when is $Z$ the lattice of cyclic flats of a matroid, along with the given rank function $\text{rk}$. As an example, we show the lattice of flats of $M_{K_4}$ in figure 4 and lattice of cyclic flats of $M_{K_4}$ in figure 5.
2 Matroid Polytopes

After going through some basics of matroid theory, we can define the fundamental object of this paper, the matroid (base) polytope. The vertex picture of a matroid polytope is given by bases, and the hyperplane picture (with some redundancies) is given by flats of the underlying matroid. Feichtner and Sturmfels [FS05] gave a description of matroid polytopes as irredundant intersection of half spaces. The facet-defining hyperplanes, in this case, correspond to a special class of flats. Matroid polytopes were also realised by Ardila, Benedetti and Doker [ABD10] as generalised permutohedra. This realisation is used to derive their volume from Postnikov’s theory of generalised permutohedron [Pos09]. Derksen defined a valuative invariant on matroids [Der09] and conjectured it to be universal. This was proved subsequently by Derksen and Fink [DF10].

2.1 The base polytope of a matroid

Let \( M = (E, B) \) be a matroid of rank \( r \), and let \( e_i \) denote the standard basis vector in \( \mathbb{R}^E \). For every base \( B \in B \), the indicator vector \( e_B \in \mathbb{R}^E \) is defined to be

\[
e_B = \sum_{i \in B} e_i
\]

**Definition 2.1.** The matroid base polytope of \( M \) is defined as the convex hull of incidence vectors of bases of \( M \), that is,

\[
P_M = \text{conv}\{ e_B : B \in B \} \subseteq \mathbb{R}^E
\]

Before going further, let us look at a classical example.

**Example 2.2.** Consider the uniform matroid \( U_{2,4} \) on \( [4] \), then

\[
P(U_{2,4}) = \text{conv}\{ (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1) \}
\]

This gives us a regular octahedron embedded in \( \mathbb{R}^4 \) as shown in figure 6.
Feichtner and Sturmfels have studied matroid polytopes in \[\text{[FS05]}\]. They have determined the dimension and a combinatorial description of these polytopes as polyhedras.

**Theorem 2.3.** \[\text{[FS05]}\] Let M be a rank r matroid on ground set E, then

\[
P_M = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^E : \sum_{i \in F} x_i \leq \text{rk}(F) \text{ for all flats } F \subseteq E \right\}
\]

and \(\dim(P(M)) = |E| - c(M)\), where \(c(M)\) denote the number of connected components of \(M\).

In the above description, each flat of our matroid gives rise to a linear inequality. Some of these linear inequalities might be redundant. One may ask for an irredundant representation of \(P_M\), i.e. which flats give rise to facet-determining hyperplanes. Feichtner and Sturmfels have determined them as flats \(F\) such that \(M|_F\) and \(M/F\) are both connected.

**Example 2.4.** Given positive integers \(0 < r < n\), the hypersimplex \(\triangle_{r,n}\) is defined as the matroid polytope of the uniform matroid \(U_{r,n}\). It is classically known (see for example Exercise 4.59(b) in \[\text{[Sta12]}\]) and can be found in the work of Laplace that

\[
\text{Vol}(\triangle_{r,n}) = \frac{A_{n-1,r-1}}{(n-1)!}
\]

where \(A_{n-1,r-1}\) are Eulerian numbers, counting the number of permutations \(w \in \mathfrak{S}_{n-1}\) with \(r - 1\) descents.

Gelfand, Goersky, MacPherson and Serganova studied matroid polytopes in their extremely impactful paper \[\text{[GiGMS87]}\]. They proved the following fundamental result:

**Theorem 2.5.** A convex polytope \(P \subseteq \triangle_{r,n}\) with vertices in \(\{0, 1\}^n\) is the matroid polytope of a matroid of rank \(r\) on the ground set \(E\) if and only if every edge of \(P\) is parallel to \(e_i - e_j\) for some \(i, j \in E, i \neq j\).
Furthermore, they showed that two vertices $e_{B_1}$ and $e_{B_2}$ for $B_1, B_2 \in \mathcal{B}(M)$ are adjacent if and only if $e_{B_1} - e_{B_2} = e_i - e_j$ for some $i, j \in E$. This implies that, the edges of $P_M$ corresponds to basis exchange in $M$. Therefore, the vertex-edge graph of $P_M$ is precisely the basis exchange graph $G(M)$ of the matroid $M$.

Another implication of the above is that the faces of matroid polytopes are again matroid polytopes. On these lines, the facial structure of matroid polytopes has been described combinatorially by Ardila and Klivans in [AK06].

This also implies that the class of matroid polytopes is a subclass of generalised permutohedra. We will see the definition of a generalised permutohedron in the following subsection.

### 2.2 The matroid base polytope as a generalised permutohedron

Recall that given two convex polytopes $P$ and $Q$ in $\mathbb{R}^n$, the Minkowski sum $P + Q$ of $P$ and $Q$ is defined as the set

$$P + Q = \{p + q : p \in P, q \in Q\}$$

We say that $P$ is a Minkowski summand of $R$ if there is a polytope $Q$ such that $P + Q = R$. In that case, we call $Q$ the Minkowski difference of $R$ and $P$, and we write $Q = R - P$. A convex polytope $P$ in $\mathbb{R}^n$ is said to be a generalised permutohedron if it satisfies one of the following equivalent conditions [Zie95]

- Every edge of $P$ is parallel to $e_i - e_j$ for some $1 \leq i < j \leq n$.
- The normal fan of $P$ in $(\mathbb{R}^n)^*$ is refined by the braid arrangement.
- The polytope $P$ is a Minkowski summand of a permutohedron.

From Theorem 2.5, it is known that a matroid polytope $P_M$ is a generalised permutohedron. Postnikov in [Pos09] showed that every generalised permutohedron can be written uniquely as a signed Minkowski sum of simplices. Recall that the signed beta invariant of $M$ is defined as

$$\tilde{\beta}(M) = (-1)^{rk_M(E)} \beta(M)$$

Ardila, Benedetti and Doker in [ABD10] gave the following signed Minkowski sum decomposition of $P_M$

$$P_M = \sum_{A \subseteq E} \tilde{\beta}(M/A) \Delta_{E \setminus A}$$

Using the theory developed by Postnikov [Pos09], they have given an elegant formula for volume of matroid polytope $P_M$. 

Theorem 2.6. Let \( M \) be a connected matroid on a ground set \( E \), then the volume of the matroid polytope \( P_M \) is given by

\[
\text{Vol}(P_M) = \frac{1}{(n-1)!} \sum_{(J_1, \cdots, J_{n-1})} \hat{\beta}(M/J_1) \cdots \hat{\beta}(M/J_{n-1})
\]

summing over the ordered collections of sets \( J_1, \cdots, J_{n-1} \subseteq E \) such that if \( i_1, \cdots, i_k \) are pairwise distinct, then \( |J_{i_1} \cap \cdots \cap J_{i_k}| < |E| - k \).

The condition on the ordered collection of sets indexing the terms in the sum above is known as dragon marriage condition [Pos09]. The enumeration of ordered sequences of sets satisfying dragon marriage condition is computationally very expensive. So we here only present a small scale example.

Example 2.7. Consider the matroid \( U_{2,3} \) on edge set \( E = \{1, 2, 3\} \). We have grouped all subsets satisfying the dragon marriage condition according to their cardinality type in the following table:

| \((|J_1|, |J_2|)\) | count | \(\hat{\beta}(M/J_1)\) | \(\hat{\beta}(M/J_2)\) | contribution to \((n-1)!\text{Vol}(P_M)\) |
|----------------|-------|-----------------|-----------------|---------------------|
| (0, 0)         | 1     | -1              | -1              | 1                   |
| (1, 0)         | 3     | 1               | -1              | -3                  |
| (0, 1)         | 3     | -1              | 1               | -3                  |
| (2, 0)         | 3     | -1              | -1              | 3                   |
| (0, 2)         | 3     | -1              | -1              | 3                   |
| (1, 1)         | 9     | 1               | 1               | 9                   |
| (2, 1)         | 9     | -1              | 1               | -9                  |
| (1, 2)         | 9     | 1               | -1              | -9                  |
| (2, 2)         | 9     | -1              | -1              | 9                   |

Table 1: Volume computation for \( P_{U_{2,3}} \)

As we can see in the table that the \( \sum_{(J_1, J_2)} \hat{\beta}(M/J_1)\hat{\beta}(M/J_2) \) can be simplified to

\[1 - 3 - 3 + 3 + 3 + 9 - 9 - 9 + 9 = 1\]

equals the Eulerian number \( A_{2,1} \). Therefore the volume of \( P_{U_{2,3}} \) is \( \frac{1}{2!} \). The simplification of the sum into Eulerian number is not evident from the formula itself.

2.3 Matroid valuations

Billera, Jia and Reiner [BJR09] introduced the notion of matroid base polytope decomposition. As the name suggests, it is a decomposition of a matroid base polytope into polytopes that in turn are also matroid base polytopes.

Definition 2.8. A matroid polytope decomposition of a matroid polytope \( P = P_M \) is a set of matroid polytopes \( \{P_{M_1}, \cdots, P_{M_m}\} \) such that
• $P_{M_1} \cup \cdots \cup P_{M_m} = P$.

• for all $1 \leq i < j \leq m$, the intersection $P_{M_i} \cap P_{M_j}$ is a face of both $P_{M_i}$ and $P_{M_j}$.

For the special case when $m = 2$ i.e. when $P_M$ has a matroid polytope decomposition into $P_{M_1}$ and $P_{M_2}$, we call such a decomposition hyperplane split. Since the intersections $P_{M_i} \cap P_{M_j}$ are faces of matroid polytopes, these are themselves matroid polytopes. Given a matroid polytope $P \subseteq \mathbb{R}^n$, we denote by $M(P)$ the matroid whose bases $B$ correspond to vertices $e_B$ of $P$. Matroid polytope decomposition is reminiscent of the idea of polytopal subdivision, but now we are only using a specific class of polytopes for the subdivision. One can then define the notion of matroid valuation. We denote the class of all matroids by Mat, and denote the class of all matroids on the ground set $[n]$ by Mat$_n$.

**Definition 2.9.** Let $G$ be an abelian group. A function $f : \text{Mat} \rightarrow G$ is called a matroid polytope valuation, or valuation for short, if for any matroid polytope decomposition $\{P_{M_1}, P_{M_2}, \cdots, P_{M_m}\}$ of a matroid polytope $P_M$ of $M \in \text{Mat}$, we have

$$\sum_{A \subseteq [m]} (-1)^{\#A} f \left( M \left( \bigcap_{i \in A} P_{M_i} \right) \right) = 0$$

Many examples of matroid valuations were studied and discussed in detail by Ardila, Fink and Rincon in [AFR10]. We mention here some valuations that are relevant in our context.

**Example 2.10.** For a positive integer $n$, the volume function $\text{Vol}$, that gives the $(n-1)$-dimensional volume of the matroid polytope $P_M$ of $M$ for each matroid $M \in \text{Mat}_n$ is a matroid polytope valuation.

**Example 2.11.** For a polytope $P$ in $\mathbb{R}^d$, the Ehrhart function $\text{Ehr}_P(x)$ is defined as the number of lattice points contained in the $n$-th dilate $nP$ of $P$, that is,

$$\text{Ehr}_P(n) = \left| \left( nP \cap \mathbb{Z}^d \right) \right|$$

This is known to be a polynomial function with rational coefficients [Ehr62] for lattice polytopes. Matroid polytopes are lattice polytopes, hence we have a well-defined function

$$\text{Ehr} : \text{Mat} \rightarrow \mathbb{Q}[t]$$

$$M \mapsto \text{Ehr}_{P_M}(t)$$

The Ehrhart polynomial satisfies the inclusion-exclusion property for lattice polytopes, which implies that the function $\text{Ehr}$ above is a matroid polytope valuation.
2.4 The Derksen-Fink invariant

Here we study a valuative invariant of matroids that was first introduced by Derksen in [Der09]. In the literature, it is usually referred as the $G$-invariant. We are opting for the name Derksen-Fink invariant to credit Derksen who defined it in [Der09] and Fink who proved a universality result for it in [DF10] along with Derksen. This also avoids confusion with Speyer’s $g$-invariant [Spe08] in verbal exchanges. Let $M$ be a matroid of rank $r$ on the set $E = \{e_1, \cdots, e_n\}$. Each permutation $w$ of the set $E$ gives rise to a flag of sets:

$$S(w) : \emptyset = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n = E$$

where $S_i = \{w(e_1), \cdots, w(e_i)\}$ for $i = 1, \cdots, n$. The rank sequence $\rho(w) = (\rho_1, \rho_2, \cdots, \rho_n)$ of $w$ is defined as

$$\rho_i = \text{rk}(S_i) - \text{rk}(S_{i-1})$$

Notice that for a matroid, $\rho \in \{0, 1\}^n$. Let $[\rho]$ be a formal symbol, one for each possible rank sequence $\rho$. The Derksen-Fink invariant $G(M)$ of a matroid $M$ is defined as

$$G(M) = \sum_{w \in \Theta_n} [\rho(w)] = \sum_{\rho \in \{0, 1\}^n} g(M, \rho)[\rho]$$

where $g_M(\rho)$ is the number of permutations $w$ whose rank sequence is $\rho$. We skip the subscript $M$, whenever the underlying matroid is understood. Note that originally the invariant $G(M)$ is defined as a quasisymmetric function by Derksen [Der09], this definition is equivalent once we identify the symbol $\rho$ with a chosen bases of space of quasisymmetric functions.

Example 2.12. For the uniform matroid $U_{r,n}$, all $r$-sized subsets are bases, and therefore the rank sequence for any permutation is

$$\underbrace{1 \cdots 1}_{r} \underbrace{0 \cdots 0}_{n-r}$$

This implies

$$G(U_{r,n}) = n! [\underbrace{1 \cdots 1}_{r} \underbrace{0 \cdots 0}_{n-r}]$$

Derksen in [Der09] showed that the invariant $G(M)$ is a valuative invariant, and asked whether it is universal. This was answered affirmatively later by Derksen and Fink [DF10]. A valuation $v$ is called universal if every valuation $f$ can be written as $f = f' \circ v$ where $f'$ is a group homomorphism.

Theorem 2.13. (Derksen-Fink theorem) The Derksen-Fink invariant $G(M)$ is a universal valuative invariant on matroids.
It is easy to see that the function $b = b(M)$ that assigns the number of bases to each matroid $M$ is a matroid valuation. Universality of $G(M)$ implies $b(M)$ should be an evaluation, which it is as following equality suggests:

$$g(\underbrace{1\cdots 1}_{k}\cdots \underbrace{0\cdots 0}_{n-k}) = k!(n-k)!b(M)$$

Speyer [Spe08] proved that Tutte polynomial is also a matroid valuation. Another polynomial associated to matroid is the corank-nullity polynomial $S(M; x, y)$, which is defined as

$$S(M; x, y) = \sum_{A \subseteq E} x^{\text{rk}(E) - \text{rk}(A)} y^{\text{rk}(A)}$$

where $s_{i,j}$ is the number of subsets $A \subseteq E$ of size $i$ and rank $j$. One can show using induction that

$$T(M; x, y) = S(M; x-1, y-1)$$

This shows that corank-nullity polynomial of a matroid $M$, determines its Tutte polynomial. The significance of the universality can now be depicted in the following theorem of Derksen that determines the coefficients of corank-nullity polynomial using the coefficients of the Derksen-Fink invariant.

**Theorem 2.14.** (Derksen) Given a matroid $M = (E, \mathcal{B})$ with corank nullity polynomial $S(M; x, y) = \sum_{i,j} s_{i,j} x^{r-1}y^{i-j}$, then the coefficients $s_{i,j}$ equal the following evaluation of the Derksen-Fink invariant $G(M)$

$$s_{i,j} = \frac{1}{i!(n-i)!} \sum_{\rho} g(\rho)$$

where the sum on the right is over all $\rho$ such that $\sum \rho = \rho_1 + \cdots + \rho_i = j$.

**Example 2.15.** The Derksen-Fink invariant for $M_{K_4}$ is given by

$$G(M_{K_4}) = 144[110100] + 576[111000]$$

and the corank-nullity polynomial is given by

$$x^3 + y^3 + 6x^2 + 4xy + 6y^2 + 15x + 15y + 16$$

We illustrate the above theorem by computing these coefficients from Derksen-Fink invariant in table 2.
### Table 2: Computing coefficients $s_{ij}$ of corank-nullity polynomial from the Derksen-Fink invariant

Table 2: Computing coefficients $s_{ij}$ of corank-nullity polynomial from the Derksen-Fink invariant

| $(i,j)$ | $s_{ij}$ from equation[1] |
|--------|--------------------------|
| (0,0)  | $144 + 5/6 = 1$          |
| (1,1)  | $144 + 5/6 = 6$          |
| (2,2)  | $144 + 3/6 = 15$         |
| (3,2)  | $144 + 3/6 = 4$          |
| (3,3)  | $144 + 3/6 = 16$         |
| (4,3)  | $144 + 3/6 = 15$         |
| (5,3)  | $144 + 3/6 = 6$          |
| (6,3)  | $144 + 3/6 = 1$          |

### 3 Schubert Matroids

We study an important class of matroids called Schubert matroids. They have been discovered and rediscovered many times. This also explains why we have several alternative names for them in the literature, such as freedom matroids, nested matroids, PI-matroids, shifted matroids, counting matroids and generalized Catalan matroids. Interestingly, these can all be realized as lattice path matroids. They played a role in Derksen and Fink’s work [DF10]. Schubert matroids also arise as additive bases of Hampe’s matroid intersection ring in [Ham17].

#### 3.1 The Schubert matroid $R_\rho$

Let $\rho \in \{0,1\}^n$ be a binary sequence with 1’s in positions $b_1 < b_2 < \cdots < b_r$. The Schubert matroid $R_\rho$ defined by $\rho$ is the matroid of rank $r$ on the set $[n] = \{1, 2, \cdots, n\}$ such that

- $1, 2, \cdots, b_1-1$ are loops.
- for $1 \leq j \leq r-1$, $b_j$ is added as a coloop and $b_j, b_j+1, \cdots, b_{j+1}-1$ are freely positioned in $\text{cl}(\{b_1, b_2, \cdots, b_j\})$, and
- $b_r, b_{r+1}, \cdots, n$ are freely positioned in the entire matroid.

In other words, the Schubert matroid $R_\rho$ is obtained from a loop or a coloop by a sequence of free extensions and coloop additions. The Schubert matroid $R_\rho$ has a distinguished flag of flats

$$F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r = [n]$$

where $F_i = \{1, 2, \cdots, b_{i+1} - 1\}$ is a flat of rank $i$ for $i = 1, \cdots, r$. As pointed out in [CS05], the independent sets of $R_\rho$ can be constructed from such a flag as

$$I(R_\rho) = \{I \subseteq [n] : \#(I \cap F_i) \leq i \text{ for all } i\}$$
Since collection of independent sets uniquely define a matroid, we have an alternative construction of the Schubert matroid $R_\rho$. Sometimes, this construction will be more convenient, therefore we would like to fix a notation for it. The Schubert matroid coming from applying this construction to a flag of sets

$$F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r = [n]$$

will be denoted as $R[F_0, \cdots, F_r]$. Note that this gives rise to a matroid on $[n]$ which is isomorphic to the Schubert matroid $R_0 \cdots 0 \# \cdots R_1 \cdots 10 \cdots 0$

We have constructed $R_\rho$ as a matroid on $[n]$, but such a matroid can be constructed over any finite set $E$ of size $n$. In what follows, $R_\rho$ is considered as a matroid up to isomorphism.

**Example 3.1.** The uniform matroid $U_{r,n}$ can be constructed by adding first $r$ elements in $[n]$ as coloops and then putting $n - r$ remaining elements freely in the span of the first $r$ elements. This means every $r$-sized subset is independent. The distinguished flag of flats in this case is

$$\emptyset \subsetneq \{1\} \subsetneq \cdots \subsetneq \{1, \cdots, r - 1\} \subsetneq [n]$$

This implies

$$U_{r,n} \simeq R_1 \cdots 10 \cdots 0$$

Let $V(n,r)$ be the vector space of formal $\mathbb{Q}$-linear combinations of all symbols $[\rho]$ such that $\rho \in \{0,1\}^n$ have exactly $r$ 1’s. The vector space $V(n,r)$ has dimension $\binom{n}{r}$. One natural choice of basis for $V(n,r)$ is $\{[\rho] : \rho \in \{0,1\}^n \text{ such that } \sum \rho_i = r\}$. Kung recently has determined another choice of basis for $V(n,r)$ in [Kun17], that makes Schubert matroid $R_\rho$ all more relevant.

**Proposition 3.2.** The set

$$\{R_\rho : \rho \in \{0,1\}^n \text{ such that } \sum \rho_i = r\}$$

forms a basis for the vector space $V(n,r)$.

In the table below we show the Derksen-Fink invariant $G(M)$ for all connected loopless Schubert matroids on $[6] = \{1,2,3,4,5,6\}$ of rank 3.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$R_\rho$ & $G(R_\rho)$ \\
\hline
$R_{111000}$ & 720$[111000]$ \\
$R_{110100}$ & 36$[110100]$ + 684$[111000]$ \\
$R_{110010}$ & 48$[101100]$ + 96$[110100]$ + 576$[111000]$ \\
$R_{101100}$ & 48$[110010]$ + 96$[110100]$ + 576$[111000]$ \\
$R_{101010}$ & 8$[101010]$ + 40$[101100]$ + 40$[110010]$ + 128$[110100]$ + 504$[111000]$ \\
$R_{100110}$ & 36$[100110]$ + 36$[101010]$ + 72$[101100]$ + 72$[110010]$ + 144$[110100]$ + 360$[111000]$ \\
\hline
\end{tabular}
\caption{The Derksen-Fink invariants $G(M)$ of all connected loopless Schubert matroids on $[6]$ of rank 3}
\end{table}

### 3.2 Lattice path matroids

A set system $A := A_j$ over a set $E$ is by definition a multiset of subsets of $E$, indexed by a set $J$: that is,

$$A_j := \{ A_j \subseteq E : j \in J \}$$

A set system $B := \{ A_k \subseteq E : k \in K \}$ where $K \subseteq J$ is called a subsystem of $A$. A transversal of the set system $A_j$ is a set $T = \{ x_j \in E : j \in J \}$ of $|J|$ distinct elements such that $x_j \in A_j$ for all $j \in J$. A partial transversal of $A$ is a transversal of a subsystem of $A$. We recall the following result due to Edmonds and Fulkerson [EF65].

**Theorem 3.3.** The partial transversals of a set system over $E$ are the independent sets of a matroid over $E$.

We refer to such a matroid as transversal matroid $M = (E, A)$. $A$ is called a presentation of $M$. We would now define a set system from a pair of lattice paths on $\mathbb{N} \times \mathbb{N}$. We only consider lattice paths that start from $(0, 0)$ and that only use two kind of steps: East step $E = (1, 0)$ and North step $N = (0, 1)$. For short, we call them $E$-step and $N$-step, respectively. For us, paths are just words in alphabet $\{E, N\}$. Let $P = p_1 \cdots p_n$ and $Q = q_1 \cdots q_n$ be two paths from $(0, 0)$ to $(n-r, r)$, with $P$ never going above $Q$. Let $p_i, \ldots , p_r$ be the north steps of $P$ and let $q_{i_1}, \ldots , q_{i_r}$ be the north steps of $Q$. Define the intervals $N_k = [p_{i_k}, q_{i_k}]$. This gives us a set system $A = \{ N_k : k \in [r] \}$ over the set $\{1, \cdots , n\}$. The corresponding transversal matroid is denoted as $M[P, Q]$. A lattice path matroid is a matroid isomorphic to a transversal matroid $M[P, Q]$. We introduce the following shorthand for the extreme paths: $\perp_{n,r} = E^{n-r}N^r$ and $\cap_{n,r} = N^r E^{n-r}$.

**Example 3.4.** Consider the paths $P = EEENN$ and $Q = NENENE$, then $M[P, Q]$ is a matroid on $[6]$ of rank 3. The bases of $M[P, Q]$ are given by sets $\{a_1, a_2, a_3\}$ such that $1 \leq a_1 \leq 4$, $3 \leq a_2 \leq 5$ and $5 \leq a_3 \leq 6$. The bounds on $a_i$’s are determined by the $N$-steps of extremal paths $P$ and $Q$, as shown in the diagram:
Figure 7: Lattice-path diagram for \( P = E^3N^3 \) and \( Q = (NE)^3 \)

Bonin and De Mier studied the structural properties of lattice path matroid polytopes in [BdM06]. Beside other results, they showed the following connectivity criterion for lattice-path matroids:

**Theorem 3.5.** A lattice path matroid \( M[P, Q] \) of rank \( r \) on \([n]\) is connected if and only if \( P \) and \( Q \) intersect only at \((0, 0)\) and \((n - r, r)\).

This implies that any lattice path matroid of the form \( M[E^{n-r}N^r, P] \) is a connected matroid as long as \( P \) is a North-East path whose first step is an \( N \)-step and whose last step is an \( E \)-step. We do not need to worry about what happens in between the first and the last step for such path \( P \).

### 3.3 The Schubert matroid \( R_\rho \) as a lattice path matroid

We would like to realize Schubert matroid \( R_\rho \) as a lattice path matroid. Given a \( \rho \in \{0, 1\}^n \), we define path \( P_\rho \in \{E, N\}^n \) by replacing 0 with \( E \) and 1 with \( N \) in \( \rho \). The following proposition seems to be known, but we are unable to find a reference.

**Proposition 3.6.** Let \( \rho \in \{0, 1\}^n \), then

\[
R_\rho \cong M[E^{n-r}N^r, P_\rho]
\]

**Proof.** Consider the flag of sets

\[
F_0 \subsetneq \cdots \subsetneq F_r
\]

where \( F_0 = \{1, \ldots, a_1 - 1\} \) where \( a_1 \) is the position of first north step. \( F_{i-1} = \{a_i + 1, \ldots, a_i - 1\} \) where \( a_i \) is the \( i \)th north step. This gives a distinguish flag of sets, such that the independent sets of \( M = M[E^{n-r}N^r, P_\rho] \) satisfy

\[
\#(I \cap F_i) \leq i
\]

for all \( i \).

The above proposition gives a description of \( R_\rho \) as a transversal matroid, with the following presentation

\[
\{N_k = [p_{ik}, q_{jk}] : k \in [r]\}
\]

We have realised \( R_\rho \) as a lattice path matroid. This determines the lattice of cyclic flats of \( R_\rho \), as per the following proposition:
Proposition 3.7. [Bon10] The lattice $\mathcal{Z}(M)$ of cyclic flats of the lattice-path matroid $M = M[E^n - r N^r, P_p]$ is a chain. Furthermore, the proper non-trivial cyclic flats of the matroid $M$ are given by $F_i = \{1, 2, \ldots, c_i\} \subseteq [n]$, where $c_i$ is the $i$th incident of an $E$ step that is followed by an $N$ step.

Example 3.8. As an example, we show all the connected loopless Schubert matroids on $[6] = \{1, 2, 3, 4, 5, 6\}$ of rank 3, and their lattice path diagram along with the chain of cyclic flat that characterize them.

| Matroid $M$ | $[\lambda]$ | $C$ |
|-------------|--------------|-----|
| $R_{111000}$ | $\emptyset \subsetneq [6]$ | $\emptyset \subsetneq [3]^2 \subsetneq [6]$ |
| $R_{110100}$ | $\emptyset \subsetneq [4]^2 \subsetneq [6]$ | $\emptyset \subsetneq [2]^1 \subsetneq [6]$ |
| $R_{110010}$ | $\emptyset \subsetneq [3]^1 \subsetneq [6]$ | $\emptyset \subsetneq [2]^1 \subsetneq [4]^2 \subsetneq [6]$ |
| $R_{101100}$ | $\emptyset \subsetneq [2]^1 \subsetneq [4]^2 \subsetneq [6]$ | $\emptyset \subsetneq [3]^1 \subsetneq [6]$ |

Table 4: connected loopless Schubert matroids on $[6]$ of rank 3 along with their respective lattice path diagrams.

Brylawski [Bry75] showed that the cyclic flats together with their ranks uniquely determined the matroid. Therefore, we can reconstruct $R_p$ from its respective chain of cyclic flats. If $R$ is a Schubert matroid coming from such a chain $F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = [n]$ of cyclic flats, where $r_i = \text{rk}(F_i)$, then we denote it by $R(F_0, F_1, \cdots, F_{k-1}, F_k)$.

Example 3.9. The uniform matroid $U_{r,n}$ on $[n]$ of rank $r$ can be realised as a lattice path matroid $M = M[E^n - r N^r, N^r E^n - r] = M[\underbrace{n, n, \ldots, n}_{r}]$. The bases in this case corresponds to all $N - E$ paths from $(0, 0)$ to $(n, r)$. We show the diagram $[\lambda]$ in Figure for the case $U_{3,5}$. The chain of cyclic flats is $\emptyset \subsetneq [n]^r$, that is, $U_{r,n} = R(\emptyset \subsetneq [n]^r)$.

Figure 8: Lattice-path diagram for $U_{3,5}$.

3.4 Hampe’s matroid intersection ring

Let us fix our ground set $E = [n] := \{1, 2, \ldots, n\}$ for this section. For $r : 1 \leq r \leq n$, let $C_{r,n}$ be the set of all chains of subsets of $[n]$ of length $r$:

$$\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_r = [n]$$
We denote by $V_{r,n} = \mathbb{Z}^{C_{r,n}}$ the free $\mathbb{Z}$-module whose coordinates are indexed by elements of $C_{r,n}$.

Let $M_{r,n}^\text{free}$ be the free $\mathbb{Z}$-module with generators the set of all loopless matroids of rank $r$ on the ground set $[n] = \{1, 2, \ldots , n\}$. Define a homomorphism

$$\Phi_{r,n} : M_{r,n}^\text{free} \rightarrow V_{r,n} \quad M \mapsto v_M$$

where for each chain $C$,

$$(v_M)_C := \begin{cases} 1 & \text{if } C \text{ is a chain of flats in } M \\ 0 & \text{otherwise} \end{cases}$$

The $\mathbb{Z}$-module $M_n$ is defined as

$$M_n = \bigoplus_{r=1}^{n} M_{r,n}$$

with $M_{r,n} = M_{r,n}^\text{free} / \ker \Phi_{r,n}$. One way to think of it as a way to identify matroids with the set of saturated chains of their flats.

**Example 3.10.** For $r = 2$ and $n = 4$, the following identity holds in $M_{2,4}$, where matroids are identified with their respective lattice of flats.

$$\begin{align*}
\{1, 2, 3, 4\} & + \{1, 2, 3, 4\} = \{1, 2, 3, 4\} \\
\{3\} & + \{1\} = \{2\} \\
\{4\} & + \{2\} = \{1\}
\end{align*}$$

Table 5: An identity in $M_{2,4}$

Recall that, given matroids $M$ and $N$ on the same ground set $E$, the union $M \vee N$ is a matroid on $E$ which is defined in terms of its independent sets

$$\mathcal{I}(M \vee N) = \{I \cup J : I \in \mathcal{I}(M), J \in \mathcal{I}(N)\}$$

The intersection $M \wedge N$ is defined as

$$M \wedge N = (M^* \vee N^*)^*$$

It is shown by Hampe [Ham17] that $M_n$ forms a ring under the product defined as

$$M \cdot N := \begin{cases} M \wedge N & \text{if } M \wedge N \text{ is loopfree} \\ 0 & \text{otherwise} \end{cases}$$
extended to linear combinations of matroids via distributivity. $\mathcal{M}_n$ is called the intersection ring of matroids on $[n]$. In our discussion, the multiplicative structure of $\mathcal{M}_n$ is not needed, though it is of independent importance.

For a loopless matroid $M$ on $[n]$ of rank $r$, let $Z = Z(M)$ be its lattice of cyclic flats. We denote by $\Delta = \Delta(Z)$ the (reduced) order complex of $Z$. The complex $\Delta$ consists of chains of cyclic flats of $M$, without top $[n]$ or bottom $\emptyset$. Let $F = F(\Delta)$ be the face lattice of $\Delta$. Let $\Gamma = \Gamma(M)$ be the lattice we get by adjoining a formal bottom to the dual of $F(\Delta)$, that is, $\Gamma = F^* \cup \{\emptyset\}$. The (nonbottom) elements $C$ of $\Gamma$ are chains of cyclic flats of $M$ without top $E$ or bottom $\emptyset$. Let $F = F(\Delta)$ be the face lattice of $\Delta$. With each such chain $C = (F_1 \subseteq \cdots \subseteq F_{k-1})$, we can associated a Schubert matroid $R(C) = R(\emptyset \subseteq F_1 \subseteq \cdots \subseteq F_{k-1} \subseteq E)$, where $r_i = \text{rk}_M(F_i)$ for $i = 1, \cdots, r - 1$. Again keeping in mind our shorthand, we write $R(F_1, \cdots, F_{k-1})$ for such a matroid. Note that the top element in $\Gamma(M)$ is the empty face, and the corresponding Schubert matroid for it is the uniform matroid $R(\emptyset \subseteq [n])$. An important result in \cite{Ham17} is the following:

**Theorem 3.11.** \cite{Ham17} Let $M$ be a loopfree matroid of rank $r$ on $[n]$. Then the following identity holds in $\mathcal{M}_{r,n}$:

$$M = - \sum_{C \in \Gamma(M)} \mu(\emptyset, C) R(C)$$

**Example 3.12.** The example 3.10 is a special case of the above identity. Let $M$ be a matroid whose lattice of flats is given by

\[
\begin{align*}
\{1,2,3,4\} & \quad \{2,3\} \quad \{1,4\} \\
& \quad \{2,3\} \quad \emptyset \\
& \quad \emptyset
\end{align*}
\]

Notice that in this case, the lattice of cyclic flats of $M$ is the same as lattice of flats, since each flat is cyclic. The $\Gamma(M)$ is given by

\[
\begin{align*}
\emptyset & \quad \{2,3\} \quad \{1,4\} \\
& \quad \emptyset
\end{align*}
\]

This implies that

$$M = R(\emptyset \subseteq [4] \subseteq \emptyset) - R(\emptyset \subseteq \{2,3\} \subseteq [4] \subseteq \emptyset) - R(\emptyset \subseteq \{1,4\} \subseteq \{2\} \subseteq [4] \subseteq \emptyset)$$

$$= R(\emptyset) - R(\{2,3\}) - R(\{1,4\})$$

which is precisely the equality in example 3.10.
4 Volume Calculations

Recall that the hypersimplex $\triangle_{r,n}$ is the matroid polytope of the uniform matroid $U_{r,n}$, and its volume is given by $\frac{1}{(n-1)!} A_{n-1,r-1}$ where $A_{n-1,r-1}$ is the Eulerian number, that counts the number of permutations of $[n-1]$ with $r-1$ descents. We can ask whether there is a combinatorial formula for the volumes of other matroid polytopes. This is already answered by Ardila, Benedetti and Doker in [ABD10], as we saw in section 1. However, their formula has its pros and cons. For example, it is not obvious how their formula implies that volume of a uniform matroid polytope (hypersimplex $\triangle_{r,n}$) is given by $\frac{1}{(n-1)!} A_{n-1,r-1}$. Another class of matroid polytopes for which the answer is known is lattice-path matroid polytopes. Bidkhor and Sullivant [Bid12] gave an expression for the volume of a lattice-path matroid polytope in terms of number of standard skew Young tableaux corresponding to certain lattice paths. We have seen in section 2 that the Schubert matroid $R_\rho$ is isomorphic to a lattice-path matroid. This enables us to compute the volume of its base polytope. From section 3, we know that volume of a matroid polytope can be written as a linear combination of volume of Schubert matroid polytopes, given by the identity 3.11. Combining these two facts, we get a combinatorial algorithm to compute volume of any matroid polytope. We use this algorithm to give a formula for volumes of connected sparse paving matroid polytopes.

4.1 Volume of the Schubert matroid $R_\rho$

Bidkhor [Bid12] generalized Stanley’s approach for computing volumes of hypersimplices to all lattice path matroid polytopes. She gave an expression in terms of sum of number of standard skew Young tableaux of certain shapes. These can be computed combinatorially by famous hook formula. Bidkhor decomposed a connected lattice path matroid polytope into a certain type of lattice-path matroid polytopes, called border strip matroid polytopes.

**Definition 4.1.** Let $M[P,Q]$ be a connected lattice path matroid polytope such that the boxes between $P$ and $Q$ form a border strip $\lambda$: that is, if $P = p_1 \cdots p_n$ and $Q = q_1 \cdots q_n$, then $p_1 = q_n = E$, $p_n = q_1 = N$ and $p_i = q_i$ for $1 < i < n$. Let $p$ be a path whose vertices are boxes of a border strip $\lambda$ and whose edges are connected boxes. We call such a matroid $M[P,Q]$ a border strip matroid polytope and denote it by $M[\lambda(p)]$.

The border strip $\lambda$ can be seen as a skew Young diagram coming from $p$. To highlight this perspective, we denote such a shape as $\lambda(p)$. The following result of Bidkhor and Sullivant uses Stanley’s triangulation of hypersimplex.

**Proposition 4.2.** [Bid12] The volume of the border strip matroid polytope $P_{M[\lambda(p)]}$ is given by the number $f^{\lambda(p)}$ of standard skew Young tableaux of shape $\lambda(p)$.

For a path $p$, the number $f^{\lambda(p)}$ can also be thought of as number of permutations of the set $\{1, 2, \cdots, n-1\}$ which have descents exactly where we have a horizontal step in our path. The number of standard Young tableaux of shape $\lambda$ can be computed combinatorially by Frame-Robinson-Thrall hook-length formula [FRT54]. For standard skew Young tableaux of (skew) shape $\lambda$, we can compute $f^{\lambda}$ using the Naruse formula (that generalizes hook-length formula [Nar14]).
Given a lattice path matroid polytope \( M[P, Q] \), Bidkhori [Bid12] used Alfonsin and Chanclain’s work [CRA11] on hyperplane splits to come up with the following matroid polytope decomposition of \( M[P, Q] \).

**Proposition 4.3.** If \( M[P, Q] \) be a connected lattice path matroid polytope, then

\[
\{ M[\lambda(p)] : p \text{ path of boxes between } P \text{ and } Q \}
\]

is a matroid polytope decomposition of \( M[P, Q] \).

This matroid polytope decomposition arises via iterative hyperplane splits. If we only consider the volume of lattice path matroid polytope, the above decomposition implies the following corollary.

**Corollary 4.4.** [Bid12] The volume of a lattice path matroid polytope corresponding to connected lattice path matroid \( M[Q] := M[-_n r, Q] \) on \([n]\) of rank \( r \) is given by

\[
\text{Vol}(P_{M[Q]}) = \frac{1}{(n-1)!} \sum_p f^{\lambda(p)}
\]

where the sum is over all paths \( p \) under \( Q \), and \( f^{\lambda(p)} \) denote the set of standard skew Young tableaux of shape \( \lambda(p) \).

**Example 4.5.** As a special case to the above result, we see that for

\[
\text{Vol}(P_{R_{1, \cdots, 10, \cdots, 0}}) = \frac{1}{(n-1)!} \sum_p f^{\lambda(p)}
\]

which translates into

\[
A_{n-1,r} = \sum_p f^{\lambda(p)}
\]

The left-hand side counts the number of permutations \( w \in \mathfrak{S}_{n-1} \) with \( r \) descents and the right-hand side counts the same by indexing over all possible \( r \) descent positions. For example,

\[
26 = f^{\begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \end{array}} + f^{\begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \end{array}} + f^{\begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \end{array}} + f^{\begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \end{array}}
\]

\[
= 4 + 9 + 9 + 4
\]

**Example 4.6.** The volume of all connected loopless Schubert matroids on \([6]\) of rank 3 can then be computed by the above as follows:
| Matroid $M$       | $\sum f^A$ | $\text{Vol}(P_M)$ |
|-------------------|------------|------------------|
| $R_{111000}$      | $f^{[1]} + f^{[2]} + f^{[3]} + f^{[4]} + f^{[5]} + f^{[6]}$ | $66$          |
| $R_{110100}$      | $f^{[1]} + f^{[2]} + f^{[3]} + f^{[4]} + f^{[5]}$ | $60$          |
| $R_{110010}$      | $f^{[1]} + f^{[2]} + f^{[3]}$ | $33$          |
| $R_{101100}$      | $f^{[1]} + f^{[2]} + f^{[3]}$ | $33$          |
| $R_{101010}$      | $f^{[1]} + f^{[2]}$ | $22$          |
| $R_{100110}$      | $f^{[1]}$ | $6$            |
| $R_{100010}$      | $f^{[1]}$ | $6$            |

Table 6: Volume of connected loopless Schubert matroid polytopes on $[6]$ of rank 3

4.2 From the lattice of cyclic flats to the volume of connected matroid polytope

In this subsection, we explain how the lattice of cyclic flats of a connected matroid can be used to compute the volume of its matroid polytope. Let $M$ be a matroid of rank $r$ on $[n]$, and let $\mathcal{Z} = \mathcal{Z}(M)$ be its lattice of cyclic flats. We construct the lattice $\Gamma(M)$ as explained in section 3.4, and consider the following relation in $\mathcal{M}_{r,n}$ from Theorem 3.11

$$M = -\sum_{C \in \Gamma(M), C \neq \emptyset} \mu(\emptyset, C) R(C)$$

Since the Derksen-Fink invariant $G : \mathcal{M}_{r,n} \rightarrow V(n, r)$ is a $\mathbb{Z}$-module homomorphism, the above can be translated to a relation between Derksen-Fink invariants of matroids. Let $M$ be a connected matroid, which means that its polytope is $n - 1$ dimensional. Since the $(n - 1)$-dimensional normalised volume $\text{Vol}_{n-1}$ is a matroid invariant and a matroid valuation, and the Derksen-Fink invariant is universal valuative invariant. This gives that the volume $\text{Vol}_{n-1}(P_M)$ of the matroid base polytope $P_M$ is then an evaluation of Derksen-Fink invariant of the matroid $M$. Therefore we have an equation for $\text{Vol}_{n-1}(P_M)$ as

$$\text{Vol}_{n-1}(P_M) = -\sum_{C \in \Gamma(M), C \neq \emptyset} \mu(\emptyset, C) \text{Vol}_{n-1}(R(C))$$

Now the volumes $\text{Vol}_{n-1}(R(C))$ of Schubert matroids can now be computed by Bidkhoi-Sullivant formula 4.4. We suppress the subscript $n - 1$, and represent $(n - 1)$-dimensional normalised volume by just Vol. Let us illustrate this for the graphic matroid $M_{G_{[2]}}$.

Example 4.7. The lattice of cyclic flats of $M_{G_{[2]}}$ is given in figure 9. By Theorem 3.11, we get

$$M_{G_{[2]}} = -R(\emptyset) + R(\{1, 2, 3\}^2) + R(\{3, 4, 5\}^2)$$
Figure 9: Lattice of cyclic flats of $M_{\Box}$

From which it follows that

$$G(M_{\Box}) = -G(R_{11100}) + 2G(R_{11010})$$
$$\text{Vol}(P_{M_{\Box}}) = -\frac{1}{4!}(A_{4,2} - 2\sum_{p \subseteq \Box} f^{\lambda(p)})$$
$$= -\frac{1}{4!}(1 \cdot 11 - 2 \cdot 8)$$
$$= \frac{1}{4!}(5)$$
$$= \frac{5}{4!}$$

which we already know is the volume of $P_{M_{\Box}}$

4.3 Volume of connected sparse paving matroid polytopes

In this subsection, we study connected sparse paving matroids and the volume of their base polytopes. A matroid $M$ of rank $r$ on the ground set $E$ is called paving if all circuits have size $r$ or $r + 1$. A hyperplane of size $r - 1$ of $M$ is called a trivial hyperplane, and otherwise it is called nontrivial. The calculation of the Derksen-Fink invariant of a paving matroid is given by the following result due to Tugger [FK17].

**Proposition 4.8.** Let $M$ be a rank $r$ paving matroid on $[n]$. The Derksen-Fink invariant $G(M)$ of a paving matroid $M$ is given by

$$G(M) = \sum_{H \text{ trivial}} (r - 1)!(n - r + 1)![1'0^{n-r}]$$
$$+ \sum_{H \text{ nontrivial}} \left(\frac{|H|!}{\left(\frac{|H|}{r} - i + 1\right)!} \cdot (n - |H|)(n - i)!1^{r-1-i}0^{n-i}\right)$$

**Proof.** Every subset of size $r$ of $[n]$ is independent, and therefore the rank sequence of any permutation $w$ starts with an $(r - 1)$-string of 1’s. The last remaining 1 can occur at any position $i : i \geq r$. For a given permutation $w$, denote this position by $a_w$. Consider the set

$$S_{r-1}(w) = \{w(1), w(2), \ldots, w(r - 1)\}$$
This is a subset of \([n]\) of size \(r - 1\) and rank \(r - 1\). If \(S_{r-1}(w)\) is a flat, which makes it a trivial hyperplane, then \(a_w = r\). The number of permutations for which this happens is 

\[(r - 1)!(n - r + 1)!
\]

This is because any permutation of the set \(S_{r-1}(w)\) and its complement gives such a rank sequence. On the other hand, if \(S_{r-1}(w)\) is not a flat, then it spans a nontrivial hyperplane, say \(H\). In this case, \(a_w\) can be any integer from \(r\) to \(|H| + 1\). For \(i : r \leq i \leq |H|\), \(a_w = i\) if and only if

\[S_i(w) = \{w(1), w(2), \ldots, w(i)\} \subseteq H
\]

The number of permutations for which \(a_w = i\) for a given nontrivial hyperplane is

\[\left(\frac{|H|}{i - 1}\right)(n - |H|)(n - i)!
\]

This is because any of the \(n - |H|\) elements outside \(H\) increase the rank by 1.

A paving matroid is **sparse** if all nontrivial hyperplanes have size \(r\). All uniform matroids are sparse paving, since the hyperplanes are exactly all \(r - 1\) subsets of underlying set \(E\), and all of them are trivial. For the Derksen-Fink invariant of sparse paving matroids, the formula in Proposition 4.8 can further be simplified.

**Corollary 4.9.** If \(M\) is a rank-\(r\) sparse paving matroid on \(n\) elements with \(\alpha\) nontrivial hyperplanes, then \(M\) has \(\binom{n}{r} - \alpha\) bases. Hence

\[G(M) = \left(\binom{n}{r} - \alpha\right) r!(n-r)! [1^\alpha 0^{n-r}] + \alpha r!(n-r)! [1^{\alpha-1} 0^{n-r-1}]
\]

The above corollary suggests us to find a simple formula for the volume of matroid polytope of a connected sparse paving matroid with given number of nontrivial hyperplanes. This brings us to our main result.

**Theorem 4.10.** Let \(M_\alpha\) be a connected sparse paving matroid of rank \(r\) with \(\alpha\) nontrivial hyperplanes. Let \(P_{M_\alpha}\) denote the corresponding matroid base polytope. Then

\[\text{Vol}_{n-1}(P_{M_\alpha}) = \frac{1}{(n-1)!} \left( A_{n-1,r-1} - \alpha \binom{n-2}{r-1} \right)
\]

**Proof.** Notice first that the lattice of cyclic flats \(Z = Z(M_\alpha)\) of \(M_\alpha\) is a rank 2 lattice with \(\alpha\) atoms which are precisely the hyperplanes of \(M_\alpha\). Now using the identity from Theorem 4.11, we have

\[M_\alpha = -\left( - \sum_{H \text{ hyperplane}} R(\emptyset \subset H^{-1} \subset E') + (\alpha - 1) R(\emptyset \subset E') \right)
\]

\[= -\left( - \sum_{H \text{ hyperplane}} R_{1 \cdots 1 \ o i \cdots 0} + (\alpha - 1) U_{r,n} \right)
\]
This implies identity of the Derksen-Fink invariant of respective matroids.

\[
G(M_\alpha) = - \left( - \sum_{H \text{ hyperplane}} G(R_{-1, \ldots, -1} 0 \cdot \cdot \cdot 1) + (\alpha - 1) G(U_{r,n}) \right)
\]

By the universality of Derksen-Fink invariant the above can be translated to an equality of volumes. Appealing to Bidkhori-Sullivant formula 4.4, we can simplify the right-hand side

\[
\text{Vol}(P_{M_\alpha}) = - \frac{1}{(n-1)!} \left( (\alpha - 1) \sum_{p \subseteq [m]/[1]} f_\lambda(p) - \alpha \sum_{p \subseteq [m]/[1]} f_\lambda(p) \right)
\]

Let \( \square \) denote the path \( N^r E^{n-r} \), then we can rewrite

\[
\text{Vol}(P_{M_\alpha}) = - \frac{1}{(n-1)!} \left( (\alpha - 1) \left( \sum_{p \subseteq [m]/[1]} f_\lambda(p) + f_\lambda(\square) \right) - \alpha \sum_{p \subseteq [m]/[1]} f_\lambda(p) \right)
\]

his can be simplified to

\[
\text{Vol}(P_{M_\alpha}) = - \frac{1}{(n-1)!} \left( (\alpha - 1) f_\lambda(\square) - \sum_{p \subseteq [m]/[1]} f_\lambda(p) \right)
\]

\[
= - \frac{1}{(n-1)!} \left( \alpha f_\lambda(\square) - \sum_{p \subseteq [m]/[1]} f_\lambda(p) \right)
\]

We know \( f_\lambda(\square) = \binom{n-2}{r-1} \) by hook formula, and from example 4.5, we know the second sum equals the Eulerian number \( A_{n-1,r-1} \). Plugging this in, give us the required formula.

\[
\text{Vol}(P_{M_\alpha}) = - \frac{1}{(n-1)!} \left( \alpha \binom{n-2}{r-1} - A_{n-1,r-1} \right)
\]

We illustrate the above theorem using the graphic matroid \( M_{K_4} \) of complete graph \( K_4 \), which is a connected sparse paving matroid.

**Example 4.11.** Consider \( M_{K_4} \), the complete graphic matroid on \( \{1, 2, 3, 4, 5, 6\} \). The lattice of cyclic flats of \( M(K_4) \) is given by

\[
\begin{align*}
\{1, 2, 3, 4, 5, 6\} & \quad \{1, 2, 3\} & \quad \{1, 2, 5\} & \quad \{2, 4, 6\} \\
\{3, 4, 5\} & \quad \{1, 3, 6\} \\
\emptyset
\end{align*}
\]
By Theorem 3.11 this implies that

\[ M_{K_4} = -R(\emptyset) + R(\{3, 4, 5\}^2) + R(\{0, 1, 3\}^2) + R(\{1, 2, 5\}^2) + R(\{0, 2, 4\}^2) \]

From which it follows that,

\[
G(M_{K_4}) = -3G(R_{11000}) + 4G(R_{11010})
\]

\[
\text{Vol}(P_{M_{K_4}}) = -\frac{1}{5!} \left( 3A_{2, 5} - 4 \sum_{p} f^\lambda(p) \right)
\]

\[ = -\frac{1}{5!} (3 \cdot 66 - 4 \cdot 60) \]

\[ = \frac{42}{5!} \]

which we know is the volume of \( M_{K_4} \) by independent means.

References

[ABD10] Federico Ardila, Carolina Benedetti, and Jeffrey Doker, Matroid polytopes and their volumes, Discrete Comput. Geom. 43 (2010), no. 4, 841–854. MR 2610473

[AFR10] Federico Ardila, Alex Fink, and Felipe Rincón, Valuations for matroid polytope subdivisions, Canad. J. Math. 62 (2010), no. 6, 1228–1245. MR 2760656

[AK06] Federico Ardila and Caroline J. Klivans, The Bergman complex of a matroid and phylogenetic trees, J. Combin. Theory Ser. B 96 (2006), no. 1, 38–49. MR 2185977

[BdM06] Joseph E. Bonin and Anna de Mier, Lattice path matroids: structural properties, European J. Combin. 27 (2006), no. 5, 701–738. MR 2215428

[BdM08] ______, The lattice of cyclic flats of a matroid, Ann. Comb. 12 (2008), no. 2, 155–170. MR 2428902

[Bid12] H. Bidkhoiri, Lattice Path Matroid Polytopes, arXiv 1212.5705, December 2012.

[BJR09] Louis J. Billera, Ning Jia, and Victor Reiner, A quasisymmetric function for matroids, European J. Combin. 30 (2009), no. 8, 1727–1757. MR 2552658

[Bon10] Joseph E. Bonin, Lattice path matroids: the excluded minors, J. Combin. Theory Ser. B 100 (2010), no. 6, 585–599. MR 2718679

[Bry75] Thomas H. Brylawski, An affine representation for transversal geometries, Studies in Appl. Math. 54 (1975), no. 2, 143–160. MR 0462992

[Cra67] Henry H. Crapo, A higher invariant for matroids, J. Combinatorial Theory 2 (1967), 406–417. MR 0215744

[CRA11] Vanessa Chatelain and Jorge Luis Ramírez Alfonsín, Matroid base polytope decomposi-tion, Adv. in Appl. Math. 47 (2011), no. 1, 158–172. MR 2799617
[CS05] Henry Crapo and William Schmitt, *A free subalgebra of the algebra of matroids*, European J. Combin. 26 (2005), no. 7, 1066–1085. MR 2155892

[Der09] Harm Derksen, *Symmetric and quasi-symmetric functions associated to polymatroids*, J. Algebraic Combin. 30 (2009), no. 1, 43–86. MR 2519849

[DF10] Harm Derksen and Alex Fink, *Valuative invariants for polymatroids*, Adv. Math. 225 (2010), no. 4, 1840–1892. MR 2680193

[Dil90] R. P. Dilworth, *The arithmetical theory of Birkhoff lattices*, pp. 101–114, Birkhäuser Boston, Boston, MA, 1990.

[EF65] Jack Edmonds and D. R. Fulkerson, *Transversals and matroid partition*, J. Res. Nat. Bur. Standards Sect. B 69B (1965), 147–153. MR 0188090

[Ehr62] Eugène Ehrhart, *Sur les polyèdres rationnels homothétiques à n dimensions*, C. R. Acad. Sci. Paris 254 (1962), 616–618. MR 0130860

[FK17] M. J. Falk and J. P. S. Kung, *Algebra and valuations related to the Tutte polynomial*, arXiv 1711.08816, November 2017.

[FRT54] J. S. Frame, G. de B. Robinson, and R. M. Thrall, *The hook graphs of the symmetric groups*, Canadian J. Math. 6 (1954), 316–324. MR 0062127

[FS05] Eva Maria Feichtner and Bernd Sturmfels, *Matroid polytopes, nested sets and Bergman fans*, Port. Math. (N.S.) 62 (2005), no. 4, 437–468. MR 2191630

[GfGMS87] I. M. Gel’fand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova, *Combinatorial geometries, convex polyhedra, and Schubert cells*, Adv. in Math. 63 (1987), no. 3, 301–316. MR 877789

[Ham17] Simon Hampe, *The intersection ring of matroids*, J. Combin. Theory Ser. B 122 (2017), 578–614. MR 3575220

[Kun17] Joseph P. S. Kung, *Syzygies on Tutte polynomials of freedom matroids*, Ann. Comb. 21 (2017), no. 4, 605–628. MR 3721644

[Nar14] Hiroshi Naruse, *Schubert calculus and hook formula*, slides available at https://www.mat.univie.ac.at/~slc/wpapers/s73vortrag/naruse.pdf, p.4, 2014.

[Oxl11] James Oxley, *Matroid theory*, second ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011. MR 2849819

[Pos09] Alexander Postnikov, *Permutohedra, associahedra, and beyond*, Int. Math. Res. Not. IMRN (2009), no. 6, 1026–1106. MR 2487491

[Spe08] David E. Speyer, *Tropical linear spaces*, SIAM J. Discrete Math. 22 (2008), no. 4, 1527–1558. MR 2448909

[Sta12] Richard P. Stanley, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR 2868112
[Whi32] Hassler Whitney, *A logical expansion in mathematics*, Bull. Amer. Math. Soc. 38 (1932), no. 8, 572–579. MR 1562461

[Zie95] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR 1311028