EXPONENTIAL DECAY RATES FOR THE DAMPED KORTEWEG-DE VRIES TYPE EQUATION.

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Abstract. The exponential decay rate of $L^2$–norm related to the Korteweg-de Vries equation with localized damping posed on the whole real line is established. In addition, using classical arguments we determine that the solutions associated to the fully damped Korteweg-de Vries equation do not decay in $H^1$–level for arbitrary initial data.

1. Introduction

The present paper sheds new light on the study of exponential decay rate of the energy associated with mild solutions of the nonlinear damped Korteweg-de Vries type equation given by

\[
\begin{cases}
  u_t + uu_x + u_{xxx} + a(x)u = 0, & (x,t) \in \mathbb{R} \times [0, +\infty), \\
  u(x,0) = u_0(x), & x \in \mathbb{R},
\end{cases}
\]

with the following assumptions:

(H1) $a \in W^{2,\infty}(\mathbb{R})$ is a nonnegative function in $\mathbb{R}$, satisfying $a(x) \geq \alpha_0 > 0$ for $x > R_1$, $R_1 > 0$ (or $a(x) \geq \lambda_0 > 0$ for $x < -R_2$, $R_2 > 0$).

The function $a(x)$ presented in equation (1.1) is responsible for the localized effect of the dissipative mechanism.

Equation (1.1) is a generalization of the well-known Korteweg-de Vries equation, that is

\[
u_t + uu_x + u_{xxx} = 0,
\]

and was established by Boussinesq in 1877, and later, in 1895, Korteweg and de Vries proved that this equation is an approximate description of surface water waves propagating in a canal. An important characteristic of this equation is its travelling-wave solutions, that is, its special solutions of the form $u(x,t) = \varphi(x - ct)$, where $\varphi$ denotes the wave profile and $c > 0$ is the wave speed. The study of the nonlinear orbital stability or instability for equation (1.2) based on travelling waves has had a considerable development and refinement in recent years (e.g [2],

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In particular, when it is considered in equation (1.1), \( a(x) = \mu > 0 \), where \( \mu \) is a sufficiently small positive constant, the stability of the solitary wave can be studied using asymptotic methods which were developed for perturbed solitons (see [13]). The variation of the solitary wave amplitude (weak-amplitude) is given by the energy balance equation:

\[
\frac{dE_0(t)}{dt} = -2\mu \int u^2 dx,
\]

where \( E_0(t) = \|u(t)\|^2_{L^2(\mathbb{R})} \) is the momentum (energy), substituting the expression for a solitary wave directly into (1.3).

Equation (1.1) has attracted considerable attention, particularly when the exponential decay rates are studied for equations posed on a finite interval \((0, L)\), instead of the whole line; for instance see [6], [12], [19], [20], [21] and references therein. According to our best knowledge, to deal with dissipative effects in the whole line and assuming that the function \( a(\cdot) \) does not possess compact support, as in the present paper, has never been treated so far in the literature. This brings up new difficulties when establishing the uniform stabilization of the energy, namely, \( E_0(t) := \|u(t)\|^2_{L^2(0, L)}, t \geq 0 \).

To overcome these difficulties, we have to combine sharp estimates due to Kenig, Ponce and Vega [16] in order to establish the well-posedness of problem (1.1), with unique continuation properties proved in Zhang [25] and new tools, in order to derive exponential and uniform decay rates of the energy. It is important to observe that the majority of the papers in the literature regarding this subject (decay rates estimates for \( E_0 \)), deals with regular solutions in order to employ unique continuation properties, for instance see Kenig, Ponce and Vega [16], Zhang [25] and references therein. In the present case, using the continuous dependence and arguments of density we are going to work with smooth solutions which enable us to use arguments of unique continuation in order to obtain the desired decay rate estimates for mild solutions.

We observe that a mild solution \( u \) of (1.1) belongs to \( C_B([0, T]; H^1(\mathbb{R})) \), then it is natural to look for exponential stability in \( H^1 \)-level instead of \( L^2 \)-level; that is, it is natural to investigate if the energy of first order \( E_1(t) = \frac{1}{2}\|u(t)\|^2_{H_1} - \frac{1}{3} \int \mathbb{R} u^3 dx \), or at least, the \( H^1 \)-norm: \( \|u(t)\|^2_{H^1} \), decays exponentially. But, even if \( E_1'(t) \leq 0 \), for all \( t \geq 0 \), the exponential decay does not hold for arbitrary mild solutions, according to section 3. The last statement is a new approach in the context of dispersive equation and it is proved using the well-known potential well theory, see for instance references [8], [9], [10] or [24].

In order to show the current findings, this paper is organized as follows. Section 2 establishes the well-posedness and exponential decay of the energy related to equation (1.1). Section 3 is devoted to prove that the decay rate estimates in \( H^1 \)-level is not expected for arbitrary initial data.
2. Well-Posedness and Exponential Decay for Equation (1.1).

2.1. Well-Posedness. The usual spaces $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ will be considered endowed with the followings norms

$$||u||_{L^2} := \left(\int_{\mathbb{R}} |u(x)|^2 \, dx \right)^{1/2},$$

and

$$||u||_{H^1} := ||u||_{H^1(\mathbb{R})} = \left(||u||_{L^2}^2 + ||u_x||_{L^2}^2\right)^{1/2}.$$

For $1 \leq p < \infty$ we define the norm in $L^p(\mathbb{R})$

$$||u||_{L^p} := ||u||_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |u(x)|^p \, dx \right)^{1/p}$$

and for $p = \infty$ we define

$$||u||_{L^\infty} := ||u||_{L^\infty(\mathbb{R})} = \text{ess sup}_{x \in \mathbb{R}} |u(x)|.$$

Next, we shall consider similar sets introduced by Kenig, Ponce and Vega in [16]. For each $T > 0$ and each measurable function $u : \mathbb{R} \times [-T, T] \rightarrow \mathbb{R}$ we define

(2.4) \hspace{1cm} \gamma_1(u, T) = \text{ess sup}_{t \in [-T, T]} ||u(t)||_{H^1(\mathbb{R})},

(2.5) \hspace{1cm} \gamma_2(u, T) = \left(\int_{-T}^{T} ||u_{xx}(t)||_{L^\infty(\mathbb{R})}^2 \, dt \right)^{1/2},

(2.6) \hspace{1cm} \gamma_3(u, T) = \left(\int_{-T}^{T} ||u_x(t)||_{L^\infty(\mathbb{R})}^6 \, dt \right)^{1/6},

(2.7) \hspace{1cm} \gamma_4(u, T) = \frac{1}{1 + T} \left(\text{ess sup}_{t \in [-T, T]} ||u(t)||_{L^2(\mathbb{R})}^2 \, dx \right)^{1/2}

and

(2.8) \hspace{1cm} \Gamma(u, T) = \max_{i=1,2,3,4} \gamma_i(u, T).

Let us define

(2.9) \hspace{1cm} X_T = \{ u \in C([-T, T], H^1(\mathbb{R})), \Gamma(u, T) < +\infty \},

and

(2.10) \hspace{1cm} X^k_T = \{ u \in C([-T, T], H^1(\mathbb{R})), \Gamma(u, T) \leq \kappa < +\infty \},

where $\kappa > 0$ will be chosen later.

It is straightforward to prove that $X_T$ defined in (2.10) is a Banach space when endowed with the norm $|| \cdot ||_{X_T} := \Gamma(\cdot, T)$. 
2.2. Linear Estimates. In this subsection the estimates related to the linear Korteweg-de Vries equation is considered, that is, estimates related to problem

\[
\begin{cases}
    v_t + v_{xxx} = 0, & (x, t) \in \mathbb{R} \times \mathbb{R} \\
    v(x, 0) = v_0(x), & x \in \mathbb{R}.
\end{cases}
\]  

(2.11)

The proof of the following result can be found in Kenig, Ponce and Vega [16].

Lemma 2.1. Let us denote by \( \{S(t)\}_{t \in \mathbb{R}} \) the group associated with the linear equation (2.11). There is a constant \( c_1 > 1 \) such that

\[
\left( \int_{-\infty}^{+\infty} \| S(t)v_0 \|_{L^\infty}^6 dt \right)^{1/6} \leq c_1 \| v_0 \|_{L^2},
\]

(2.12)

for all \( v_0 \in L^2(\mathbb{R}) \), and

\[
\left( \int_{\mathbb{R}} \sup_{t \in [-T,T]} |S(t)v_0(x)|^2 dx \right)^{1/2} \leq c_1 (1 + T) \| v_0 \|_{H^1},
\]

(2.13)

for all \( v_0 \in H^1(\mathbb{R}) \) and \( T > 0 \). Moreover, for all \( v_0 \in L^2(\mathbb{R}) \) we have,

\[
\left( \sup_{x \in \mathbb{R}} \int_{-T}^{T} |\partial_x S(t)v_0(x)|^2 dt \right)^{1/2} \leq c_1 \| v_0 \|_{L^2}.
\]

(2.14)

Since \( S(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) and \( S(t) : H^1(\mathbb{R}) \to H^1(\mathbb{R}) \) are isometries for all \( t \in \mathbb{R} \), if the initial condition \( v_0 \) belongs to \( H^1(\mathbb{R}) \), the solution associated to (2.11) belongs to \( C_B([-T,T]; H^1(\mathbb{R})) \), that is,

\[
v_0 \in H^1(\mathbb{R}) \mapsto v \in C_B([-T,T]; H^1(\mathbb{R})),
\]

(2.15)

for all \( T > 0 \), where \( v(t) = S(t)v_0 \).

From Lemma 2.1 and (2.15) we are able to present the following result,

Lemma 2.2. Let \( T > 0 \) and \( v_0 \in H^1(\mathbb{R}) \). Defining \( v(\cdot, t) = S(t)v_0 \), we obtain that \( v \in X_T \) and,

\[
\| v \|_{X_T} \leq c_1 \| v_0 \|_{H^1},
\]

(2.16)

where \( c_1 > 1 \) is the constant given in Lemma 2.1 and does not depend on \( T > 0 \) and \( v_0 \).

Proof. Indeed, since \( w \in H^1(\mathbb{R}) \mapsto S(t)w \in C_B([-T,T]; H^1(\mathbb{R})) \) is a linear isometry, we get from (2.15) that \( \gamma_1(v, T) \leq c_1 \| v_0 \|_{H^1} \).

(2.17)

From (2.14) we deduce

\[
\gamma_2(v, T) = \left( \sup_{x \in \mathbb{R}} \int_{-T}^{T} |\partial_x (S(t) \partial_x v_0(x,t))|^2 dt \right)^{1/2}
\]

\[
\leq c_1 \| \partial_x v_0 \|_{L^2} \leq c_1 \| v_0 \|_{H^1}.
\]
Using (2.12) and (2.13) we conclude
\[ \gamma_3(v, T) \leq c_1 ||v_0||_{H^1} \quad \text{and} \quad \gamma_4(v, T) \leq c_1 ||v_0||_{H^1}. \]

This arguments establish the lemma.

The next step is to analyze the non-homogenous equation associated with (2.11), given by
\[
\begin{aligned}
\varphi_t + \varphi_{xxx} &= g, \quad (x, t) \in \mathbb{R} \times \mathbb{R} \\
\varphi(x, 0) &= \varphi_0(x), \quad x \in \mathbb{R}.
\end{aligned}
\]

In Kenig, Ponce and Vega [16] they proved the following result:

**Lemma 2.3.** Let \( T > 0 \) and \( g \in L^1([−T, T]; H^1(\mathbb{R})) \). If we define
\[
\varphi(\cdot, t) = \int_0^t S(t − \tau)g(\cdot, \tau)d\tau,
\]
for \( t \in [0, T] \), then \( \varphi \) belongs to \( X_T \) and
\[
||\varphi||_{X_T} \leq c_1 ||g||_{L^1([−T, T]; H^1(\mathbb{R}))},
\]
where \( c_1 > 1 \) is the constant given in Lemma [2.1].

**Proof.** We denote by \( \psi(t) \) the characteristic function over the interval \([0, t]\) for \( 0 \leq t < T \). Therefore, from (2.19) we can write
\[
\varphi(\cdot, t) = \int_{−T}^T \psi(t)S(t − \tau)g(\cdot, \tau)d\tau.
\]

Then, from (2.14) we conclude that
\[
\begin{aligned}
\gamma_2(\varphi, T) &\leq \int_{−T}^T \left( \text{ess sup}_{x \in \mathbb{R}} \int_{−T}^T \left( \partial_x^2 \left( \psi(t)S(t − \tau)g(\cdot, \tau) \right) \right)^2 dt \right)^{\frac{1}{2}} d\tau \\
&\leq \int_{−T}^T \left( \text{ess sup}_{x \in \mathbb{R}} \int_{−T}^T \left( \partial_x ((t)S(t')\partial_x g(\cdot, \tau)) \right)^2 dt' \right)^{\frac{1}{2}} d\tau \\
&\leq \int_{−T}^T ||g(\cdot, \tau)||_{H^1} d\tau = \frac{\sqrt{3}}{3} ||g||_{L^1([−T, T]; H^1(\mathbb{R}))}
\end{aligned}
\]

On the other hand, since \( S(t) : H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R}) \) is an isometry, we deduce from Bochner Theorem that
\[ \| \varphi(t) \|_{H^1(\mathbb{R})} = \left\| \int_{-T}^{T} \psi(\tau) S(t - \tau) g(\cdot, \tau) d\tau \right\|_{H^1(\mathbb{R})} \]
\[ \leq \int_{-T}^{T} \| S(t - \tau) g(\cdot, \tau) \|_{H^1(\mathbb{R})} d\tau \leq c_1 \| g \|_{L^1([-T, T]; H^1(\mathbb{R}))}. \]

Thus\( \gamma_1(\varphi, T) \leq c_1 \| g \|_{L^1([-T, T]; H^1(\mathbb{R}))}. \)

Considering similar ideas used to prove (2.22), we deduce from (2.12) and (2.14) that
\[ \gamma_i(\varphi, T) \leq c_1 \| g \|_{L^1([-T, T]; H^1(\mathbb{R}))}, \]
where \( i = 3, 4. \)

We turn our attention to the non-linear term \( uu_x \) present in equation (1.1). In fact, from [16] we have the following statement:

\[ \text{Lemma 2.4. Let } T > 0 \text{ and let } u, w \in X_T. \text{ Then,} \]
\[ \| (uw)_x \|_{L^1([-T, T]; H^1(\mathbb{R}))} \leq 4(1 + \sqrt{2})T^{1/2}(1 + T) \| u \|_{X_T} \| w \|_{X_T}. \]

As a consequence of Lemmas 2.3 and 2.4 if we define
\[ \varphi(\cdot, t) = \int_0^t S(t - \tau)(uw)_x d\tau, \]
for \( t \in [-T, T], T > 0 \) and \( u, v \in X_T, \) we obtain that
\[ \| \varphi \|_{X_T} \leq 4(1 + \sqrt{2})c_1 T^{1/2}(1 + T) \| u \|_{X_T} \| w \|_{X_T}. \]

In order to obtain an analogous estimate as in (2.20), it is necessary to analyze the term \( a(\cdot)u. \) In fact,
\[ \int_{-T}^{T} \| a(\cdot)u(t) \|_{L^2} dt \leq \| a \|_{W^{2,\infty}} \int_{-T}^{T} \| u(t) \|_{L^2} dt \leq \sqrt{2} \| a \|_{W^{2,\infty}} T \| u \|_{X_T}, \]
and
\[ \int_{-T}^{T} \| (a(\cdot)u(t))_x \|_{L^2} dt \leq \sqrt{2} \int_{-T}^{T} \| a_x u(t) \|_{L^2} dt + \sqrt{2} \int_{-T}^{T} \| au_x(t) \|_{L^2} dt \]
\[ \leq \sqrt{2} \| a \|_{W^{2,\infty}} T \| u \|_{X_T}. \]

Therefore,
\[ \| a(\cdot)u \|_{L^1([-T, T]; H^1(\mathbb{R}))} \leq 2\sqrt{2} \| a \|_{W^{2,\infty}} T \| u \|_{X_T}. \]

From (2.29) we obtain the following result:
Lemma 2.5. Let $T > 0$ and $u \in X_T$. Consider the integral equation given by
\begin{equation}
\varphi(\cdot,t) = \int_0^t S(t-\tau)(a(\cdot)u(\tau))d\tau, \quad t \in [0,T].
\end{equation}
Then
\begin{equation}
||\varphi||_{X_T} \leq 2\sqrt{2}c_1||a||_{W^{2,\infty}}T||u||_{X_T},
\end{equation}
where $c_1 > 1$ is the constant given in Lemma 2.1.

2.3. Local Well-Posedness. Let us consider an initial data $u_0 \in H^1(\mathbb{R})$. The aim in this subsection is to find a local solution of \((1.1)\) in the mild sense (see \([22]\)), that is, to find the unique fixed point of the map $\Psi : X_T \to X_T$ defined by
\begin{equation}
\Psi(u)(t) = S(t)u_0 - \int_0^t S(t-\tau)\left( a(\cdot)u(\tau) + \frac{1}{2}(u(\tau)^2)x \right) d\tau,
\end{equation}
for $t \in [-T,T]$. Indeed, first of all it is necessary to show that $\Psi$ is well-defined. Let $u \in X_T$, then from Lemmas 2.4 and 2.5 we get
\begin{equation}
||\Psi(u)(t)||_{X_T} = \left\| S(t)u_0 + \int_0^t S(t-\tau)\left( -a(\cdot)u(\tau) - \frac{1}{2}(u(\tau)^2)x d\tau \right) \right\|_{X_T}
\end{equation}
\begin{equation}
\leq c_1||u_0||_{X_T} + 2\sqrt{2}c_1||a||_{W^{2,\infty}}T||u||_{X_T} + \frac{c_2}{2}T^{1/2}(1+T)||u||_{X_T}^2,
\end{equation}
where $c_2 = 4(1 + \sqrt{2})c_1$. Since $u \in X_T$ we have for $\kappa = 2c_1||u_0||_{H^1}$ that
\begin{equation}
||\Psi(u)(t)||_{X_T} \leq c_1||u_0||_{H^1} \left( 1 + 4\sqrt{2}c_1||a||_{W^{2,\infty}}T + 2c_1c_2T^{1/2}(1+T)||u_0||_{H^1} \right).
\end{equation}
Taking $0 < T_\kappa < 1$ verifying
\begin{equation}
4\sqrt{2}c_1||a||_{W^{2,\infty}}T\kappa + 2c_1c_2T_\kappa^{1/2}(1+T_\kappa)||u_0||_{H^1} < 1,
\end{equation}
it is possible to conclude $||\Psi(u)(t)||_{X_T} < 2c_1||u_0||_{H^1} = \kappa$.

Next, we prove that $\Psi$ in \((2.32)\) is a strict contraction. In fact, let $u, w \in X_T$; then
\begin{equation}
||\Psi(u) - \Psi(w)||_{X_T} \leq \left\| \int_0^t S(t-\tau)(a(\cdot)(u-w)(\tau))d\tau \right\|_{X_T} + 
\end{equation}
\begin{equation}
+ \frac{1}{2} \left\| \int_0^t S(t-\tau)(u(\tau)^2 - w(\tau)^2)x d\tau \right\|_{X_T}.
\end{equation}

From the equality $(u-w)^2 = (u-w)(u+w)$, Lemmas 2.4 and 2.5 and since $\kappa = 2c_1||u_0||_{H^1}$, we obtain,
\begin{equation}
||\Psi(u) - \Psi(w)||_{X_T} \leq 2\sqrt{2}c_1||a||_{W^{2,\infty}}T||u-w||_{X_T} + \frac{c_2}{2}T^{1/2}(1+T)||u-w||_{X_T}||u+w||_{X_T}
\end{equation}
\begin{equation}
\leq \left( 2\sqrt{2}c_1||a||_{W^{2,\infty}}T + 2c_1c_2T^{1/2}(1+T)||u_0||_{H^1} \right)||u-w||_{X_T}.
\end{equation}
Since $0 < T\kappa < 1$ was chosen verifying

$$4\sqrt{2}c_1||a||_{W^{2,\infty}}T\kappa + 2c_1c_2T^{1/2}\kappa(1 + T\kappa)||u_0||_{H^1} < 1,$$

we guarantee the existence of $0 < \alpha < 1$ such that

$$||\Psi(u) - \Psi(w)||_{X_T} < \alpha||u - w||_{X_T}.$$

Then, $\Psi : X_{T\kappa} \to X_{T\kappa}$ is a strict contraction and therefore, from Banach fixed point Theorem, it is possible to conclude that problem (1.1) possesses a unique (mild) solution.

This argument proves the following Theorem.

**Theorem 2.1.** Suppose that $a \in W^{2,\infty}(\mathbb{R})$ satisfies the assumption (H1) and consider that $u_0 \in H^1(\mathbb{R})$. Given $\kappa > 0$ and $0 < T\kappa < 1$ defined in (2.35), then, there is a unique mild solution $u \in X_{T\kappa}$ of equation (1.1). Moreover, the map

$$u_0 \in H^1(\mathbb{R}) \mapsto u \in C_B([-T', T']; H^1(\mathbb{R})),$$

is continuous for an appropriate $0 < T' < T\kappa < 1$.

**Remark 2.1.** It is possible to deduce, taking the duality product in equation (1.1) that if $u$ is a solution in the mild sense, then $u$ is also a solution in the weak sense, and vice-versa, the weak solution which belongs to $X_T$ and is the fixed point of $\Psi$, is the unique mild solution.

### 2.4. Global Well-Posedness.

A result of global well-posedness associated to equation (1.1) for given initial data $u_0 \in H^1(\mathbb{R})$, will be established in this subsection. Indeed, from local well-posedness result (see Theorem 2.1), it is possible to obtain some estimates in $H^1(\mathbb{R})$ in order to get the required result. Before going on, we restrict ourselves on mild solutions, obtained above, such that $u \in C_B([0, T']; H^1(\mathbb{R}))$ instead of $u \in C_B([-T', T']; H^1(\mathbb{R}))$, for an appropriate $0 < T' < T\kappa < 1$ (see Theorem 2.1) to obtain the exponential decay in $L^2$-norm (which we shall present in the next section).

Multiplying (1.1) by $u$ and integrating over $\mathbb{R} \times (0, T')$, where $0 < t < T'$ and $T' > 0$ is given in Theorem 2.1 we get

\begin{equation}
\int_{\mathbb{R}} |u|^2 dx = -2 \int_0^t \int_{\mathbb{R}} a(x)|u|^2 dx ds + ||u_0||_{L^2}^2 \leq ||u_0||_{L^2}^2.
\end{equation}

Therefore we can conclude that

\begin{equation}
u is bounded in $L^\infty([0, +\infty); L^2(\mathbb{R})).$
\end{equation}

Next, multiplying the first equation in (1.1) by $u^2 + 2u_{xx}$ and integrating in $x \in \mathbb{R}$ we have the following equality:

\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}} \left[ \frac{u^3}{3} - u_x^2 \right] dx + \int_{\mathbb{R}} a(x)u^3 dx + 2 \int_{\mathbb{R}} a(x)uu_{xx} dx = 0.
\end{equation}
But
\[ \int_{\mathbb{R}} a(x) uu_{xx} \, dx = \frac{1}{2} \int_{\mathbb{R}} a_{xx}(x) u^2 \, dx - \int_{\mathbb{R}} a(x) u_x^2 \, dx, \]
and therefore, from (2.40) we obtain after integrating once in \( t \in [0, T') \) that
\[ \int_{\mathbb{R}} \left[ u_x^2 - \frac{u^3}{3} \right] \, dx - \int_{0}^{t} \int_{\mathbb{R}} a(x) u^3 \, dx \, ds - \int_{0}^{t} \int_{\mathbb{R}} a_{xx}(x) u^2 \, dx \, ds \]
\[ + 2 \int_{0}^{t} \int_{\mathbb{R}} a(x) u_x^2 \, dx \, ds = \int_{\mathbb{R}} \left[ u_0^2 - \frac{u_0^3}{3} \right] \, dx. \]

Combining (2.38) and (2.41) and using Gagliardo-Nirenberg and Young inequalities we deduce that
\[ \|u(t)\|^2_{H^1} \leq (3\|a\|_{W^{2,\infty}} + \frac{1}{2} + C_2 \|u_0\|^4_{H^1} + \|u_0\|_{H^1}) \int_{0}^{t} \|u(s)\|^2_{H^1} \, ds + C_5 (\|u_0\|^2_{H^1} + \|u_0\|^\frac{10}{3}\|u_0\|^\frac{10}{3}_{L^2} + \|u_0\|^2_{L^2}). \]

From Gronwall inequality we have,
\[ \|u(t)\|^2_{H^1} \leq C_5 (\|u_0\|^2_{H^1} + \|u_0\|^\frac{10}{3}\|u_0\|^\frac{10}{3}_{L^2} + \|u_0\|^2_{L^2}) e^{(3\|a\|_{W^{2,\infty}} + \frac{1}{2} + C_2 \|u_0\|^4_{H^1})T}. \]

Inequality (2.43) able us to enunciate the next result,

**Theorem 2.2.** Suppose that \( a \in W^{2,\infty}(\mathbb{R}) \) satisfies assumption (H1) and consider \( u_0 \in H^1(\mathbb{R}) \).
Then, there exists a unique mild solution \( u \) which belongs to
\[ L^\infty_{loc}([0, +\infty); H^1(\mathbb{R})) \cap X_T, \]
for all \( T > 0 \), and it satisfies the inequality (2.42). Moreover, the map
\[ u_0 \in H^1(\mathbb{R}) \mapsto u \in C_B([0, T]; H^1(\mathbb{R})), \]
is continuous for all \( T > 0 \).

\[ \square \]

**Remark 2.2.** The equalities obtained in (2.38), (2.40), and the estimate (2.43) were deduced considering regular solutions for equation (1.1), for instance, if we take an initial data \( u_0 \in C_0^\infty(\mathbb{R}) \) and \( a \in W^{s,\infty}(\mathbb{R}) \) for \( s > 0 \) large enough, the local solution \( \tilde{u} \in X_T \) given by Theorem 2.1 for some \( T > 0 \), coincides with the classical solution, which exists globally and belongs to \( C^\infty(\mathbb{R} \times \mathbb{R}) \). By density arguments we get identities (2.38) and (2.40) for a mild solution \( u \in X_T \) belonging to \( C_B(\mathbb{R}; H^1(\mathbb{R})) \) and satisfying the inequality (2.43).
2.5. Exponential Decay for equation \((1.1)\). In this subsection we are interested in obtaining an exponential decay rate for the energy norm in \(L^2(\mathbb{R})\) related with the Korteweg-de Vries equation \((1.1)\) with localized damping.

In fact, multiplying the first equation in \((1.1)\) by \(u\) and integrating over \(\mathbb{R} \times (0,t)\), \(t \in (0, +\infty)\) (see remark 2.2) we get

\[
E_0(t) := \frac{1}{2} \int_{\mathbb{R}} |u(x,t)|^2 \, dx = - \int_0^t \int_{\mathbb{R}} a(x)|u(x,t)|^2 \, dx \, ds + \frac{1}{2} \|u_0\|^2_{L^2} \leq \frac{1}{2} \|u_0\|^2_{L^2}.
\]

Since \(\frac{d}{dt}E_0(t) \leq 0\), it makes sense to look for exponential decay rates. From equation \((1.1)\) we have the following energy estimate

\[
\int_0^T E_0(t) \, dt \leq \frac{c_0^{-1}}{2} \left[ \int_0^T u^2 \, dx \right]_0^T + \int_0^T \int_{x \leq R_1} u^2 \, dx \, dt.
\]

The following result establishes the exponential decay rate for the solution \(u\) obtained in Theorem 2.2.

**Theorem 2.3.** Consider the potential \(a(\cdot)\) satisfying hypothesis \((H1)\). For any \(L > 0\), there are \(c = c(L) > 0\) and \(\omega = \omega(L)\) such that

\[
E_0(t) \leq ce^{-\omega t},
\]

for all \(t \geq 0\) and for any solution \(u\) of \((1.1)\) determined in Theorem 2.2 provided that \(\|u_0\|_{H^1(\mathbb{R})} \leq L\).

**Proof.** Before establishing the exponential decay related to equation \((1.1)\), we need to obtain a preliminary estimate for the integral \(A\) (a similar bound can be deduced considering the integral over \(\{x \in \mathbb{R}; x \geq -R_2\}\)). In fact, we have the following lemma, where \(T_0\) is a positive constant and where we consider that the initial data lies in a bounded set of \(H^1(\mathbb{R})\).

**Lemma 2.6.** Let \(u \in L^\infty_{\text{loc}}(0, +\infty; H^1(\mathbb{R})) \cap X_T\) be the mild solution associated to the Korteweg-de Vries equation \((1.1)\), obtained from Theorem 2.2. Then, we have that for all \(T > T_0\) there exists a positive constant \(c_5 = c_5(T_0, E_0(0))\) such that if \(u\) is the mild solution of \((1.1)\) with initial data \(u_0 \in H^1(\mathbb{R})\), the following inequality holds

\[
\int_0^T \int_{x \leq R_1} u^2 \, dx \, dt \leq c_5 \int_0^T \int_{\mathbb{R}} a(x)u^2 \, dx \, dt,
\]

provided that \(u_0\) belongs to a bounded set of \(H^1(\mathbb{R})\).

**Proof.** Consider \(P_{R_1} := \{x \in \mathbb{R}; x \leq R_1, R_1 > 0\}\). First of all we note that according to the continuous dependence of the initial data and arguments of density it is sufficient to establish the Lemma for smooth solutions (for instance, those ones which belong to \(C([0,T], H^s(\mathbb{R}))\)), \(s > 0\) large enough, related to problem \((1.1)\) (see Remark 2.2).

We argue by contradiction. Let us suppose that \((2.46)\) is not verified and let \(\{u_k(0)\}_{k \in \mathbb{N}} =
\( \{u_k^0\}_{k \in \mathbb{N}} \) be a bounded sequence of initial data in \( H^1 \)-norm, where the corresponding solutions \( \{u_k\}_{k \in \mathbb{N}} \) of (1.1) verifies

\[
\lim_{k \to +\infty} \frac{\int_0^T ||u_k(t)||^2_{L^2(P_{R_1})} \, dt}{\int_0^T \int_\mathbb{R} (a(x) u_k^2) \, dx \, dt} = +\infty,
\]

that is,

\[
\lim_{k \to +\infty} \frac{\int_0^T \int_\mathbb{R} (a(x) u_k^2) \, dx \, dt}{\int_0^T ||u_k(t)||^2_{L^2(P_{R_1})} \, dt} = 0.
\]

Since,

\[
E_k^0(t) \leq E_k^0(0) \leq L,
\]

we obtain a subsequence of \( \{u_k\}_{k \in \mathbb{N}} \), still denoted by \( \{u_k\}_{k \in \mathbb{N}} \) from now on, which verifies the convergence:

\[
u_k \rightharpoonup u \quad \text{weakly in} \quad L^\infty([0,T];L^2(\mathbb{R})).
\]

From the boundedness of \( \{u_k\}_{k \in \mathbb{N}} \) in \( L^\infty(0,T;L^2(\mathbb{R})) \) and (2.47) we deduce

\[
\lim_{k \to +\infty} \int_0^T \int_\mathbb{R} a(x)u_k^2 \, dx \, dt = 0,
\]

consequently, from the hypothesis made on \( a(x) \) we have

\[
\lim_{k \to +\infty} \int_0^T \int_{\mathbb{R}\setminus P_{R_1}} u_k^2 \, dx \, dt = 0.
\]

Let us consider \( R \in \mathbb{R} \) such that \( R < R_1 \). Employing (2.43) and compactness results we obtain

\[
u_k \to u_R \quad \text{strongly in} \quad L^2([0,T];L^2([R,R_1])); \quad \forall \ R < R_1.
\]

Therefore, from (2.49) and (2.52) we obtain, from the uniqueness of weak limit, that \( u_R = u \) in \( L^2([0,T];L^2([R,R_1])); \) for all \( R < R_1 \) and then, we infer,

\[
u_k \to u \quad \text{strongly in} \quad L^2([0,T];L^2([R,R_1])); \quad \forall \ R < R_1,
\]

Having in mind (2.51) and (2.53), we conclude that

\[
u_k \to \tilde{u} \quad \text{strongly in} \quad L^2([0,T];L^2(R, +\infty)); \quad \forall \ R < R_1,
\]

where

\[
\tilde{u} = \begin{cases} 
    u, & \text{a.e. in } [R,R_1], \forall R < R_1 \\
    0, & \text{a.e. in } \mathbb{R}\setminus P_{R_1}.
\end{cases}
\]
In addition, from (2.12) it is possible to conclude that,

\[(2.56)\]
\[u_{k,x} \rightharpoonup u_x \text{ weakly in } L^2([0,T]; L^2(\mathbb{R}))\]

At this point we will divide our proof into two cases, namely: when \(u \neq 0\) and \(u = 0\).

Case (I): \(u \neq 0\).

In this case, passing to the limit in the equation

\[(2.57)\]
\[u_{k,t} + u_{k,u_{k,x}} + u_{k,xxx} + a(x)u_k = 0 \quad \text{a.e. in } (R, +\infty) \times (0, T) \quad (\forall R < R_1),\]

when \(k \to +\infty\), we deduce that

\[(2.58)\]
\[
\begin{cases}
  u_t + uu_x + u_{xxx} = 0, \\
  u = 0,
\end{cases}
\quad \text{in } C([0,T]; L^2(R, +\infty)), \quad \text{a.e. in } \mathbb{R}\setminus P_{2R_1},
\]

where \(u \in C([0,T]; H^s(\mathbb{R}))\), \(s > 0\) large enough. From Unique Continuation due to Zhang [25], we conclude that \(u \equiv 0\) a.e. in \(P_{2R_1}\). Being \(u \equiv 0\) a.e. in \(\mathbb{R}\setminus P_{R_1}\) we get \(u \equiv 0\) a.e. in \(\mathbb{R}\), which is a contradiction.

Case (II): \(u = 0\).

Define,

\[(2.59)\]
\[\nu_k = ||u_k||_{L^2([0,T]; H^1(\mathbb{R}))}.\]

Then, for \(v_k = \frac{u_k}{\nu_k}\) we have

\[(2.60)\]
\[||v_k||_{L^2([0,T]; H^1(\mathbb{R}))} = 1, \quad \forall k \in \mathbb{N},\]

which implies that there exists \(v \in L^2([0,T]; H^1(\mathbb{R}))\)

\[(2.61)\]
\[v_k \rightharpoonup v \text{ weakly in } L^2([0,T]; L^2(\mathbb{R}))\] and
\[v_{k,x} \rightharpoonup v_x \text{ weakly in } L^2([0,T]; L^2(\mathbb{R})).\]

Moreover, \(v_k\) satisfies the equation

\[(2.62)\]
\[v_{k,t} + u_{k}v_{k,x} + v_{k,xxx} + a(x)v_k = 0, \quad \text{in } \mathcal{D}'((R, +\infty) \times (0, T)) \quad (\forall R < R_1).\]

From (2.47),

\[(2.63)\]
\[
\lim_{k \to +\infty} \frac{\int_0^T ||v_k(t)||_L^2(P_{R_1}) dt}{\int_0^T \int_{\mathbb{R}} (a(x)v_k^2) dx dt} = +\infty.
\]

On the other hand, being \(||v_k||_{L^2(0,T; L^2(P_{R_1}))} \leq ||v_k||_{L^2(0,T; L^2(\mathbb{R}))} \leq 1\) for all \(k \in \mathbb{R}\) we obtain from (2.63) that

\[(2.64)\]
\[
\lim_{k \to +\infty} \int_0^T \int_{\mathbb{R}} (a(x)v_k^2) dx dt = 0.
\]

Since \(a(x) \geq a_0 > 0\) for \(x \in \{x \in \mathbb{R}; x \geq R_1, R_1 > 0\}\) we obtain from (2.64),

\[(2.65)\]
\[
\lim_{k \to +\infty} \int_0^T \int_{\mathbb{R}\setminus P_{R_1}} v_k^2 dx dt = 0.
\]
Thus,
\begin{equation}
(2.66)
v_k \to 0 \text{ in } L^2([0,T];L^2(\mathbb{R}\setminus P_{R_1})).
\end{equation}

Taking (2.54), (2.55) (observe that \(u = 0\)), (2.62), (2.61), (2.64) and (2.66) into account, we obtain,
\begin{equation}
(2.67)
\begin{cases}
  v_t + v_{xxx} = 0, & \text{in } \mathcal{D}'((R, +\infty) \times (0,T)), \\
  v = 0, & \text{a.e. in } \mathbb{R}\setminus P_{2R_1}.
\end{cases}
\end{equation}

Therefore, from Holmgreen’s Theorem we conclude that \(v \equiv 0\) in \((R, +\infty) \times [0,T]\), for all \(R < R_1\), that is, \(v \equiv 0\) in \(\mathbb{R}\) which contradicts (2.60).

\[\square\]

Using the result established in Lemma 2.6, we are able to prove the result in Theorem 2.3. Indeed, taking (2.45) and (2.46) into account and making use of the identity of the energy
\[E_0(T) - E_0(0) = -\int_0^T \int_\mathbb{R} a(x)u^2 \, dx \, dt,\]
we deduce that
\begin{equation}
(2.68)
\int_0^T E_0(t) \, dt \leq C E_0(0), \quad \text{for all } T > T_0,
\end{equation}
for some positive constant \(C > 0\), which implies the exponential stability.

\[\square\]

3. NON-DECIAY IN \(H^1\)-NORM.

In this section let us consider a particular case, when equation (1.1) is fully damped, that is,
\begin{equation}
(3.69)
\begin{cases}
  u_t + uu_x + u_{xxx} + \mu u = 0, & (x,t) \in \mathbb{R} \times [0, +\infty), \\
  u(x,0) = u_0(x), & x \in \mathbb{R},
\end{cases}
\end{equation}
where \(\mu > 0\). It is straightforward to conclude that for \(u_0 \in H^1(\mathbb{R})\) equation (3.69) possesses a unique solution \(u \in C([0,T];H^1(\mathbb{R}))\), for all \(T > 0\). Moreover, the solution \(u\) satisfies a similar identity (see (2.40)), for all \(t \geq 0\), in this case given by,
\begin{equation}
(3.70)
\frac{d}{dt}\left\{\frac{1}{2}||u||^2_{H^1} - \frac{1}{3} \int_\mathbb{R} u^3 \right\} = -\mu \left\{2||u||^2_{H^1} - \int_\mathbb{R} u^3 \right\}
\end{equation}

Assuming that \(K(t) \geq 0\) occurs for all \(t \geq 0\) and considering a \(C^1\)-functional \(J : H^1(\mathbb{R}) \to \mathbb{R}\) defined by
\begin{equation}
(3.71)
J(w) = \frac{1}{2}||w||^2_{H^1} - \frac{1}{3}||w||^3_{L^3}; \quad w \in H^1(\mathbb{R});
\end{equation}
we note that
\[
E(t) \geq J(u).
\]

Moreover, since \(K(t) \geq 0\) for all \(t \geq 0\) we conclude from (3.70)
\[
E'(t) \leq 0, \quad \forall \ t \in [0, +\infty),
\]
and then
\[
E(t) \leq E(0), \quad \forall \ t \in [0, +\infty).
\]

Let \(B_1 > 0\) be the sharp Sobolev constant verifying,
\[
||u||_{L^3} \leq B_1||u||_{H^1}, \quad \forall \ u \in H^1(\mathbb{R});
\]
then
\[
\frac{1}{3}||u||_{L^3}^3 \leq \frac{B_1^3}{3}, \quad \forall \ u \in H^1(\mathbb{R}).
\]

The above inequality implies that
\[
0 < k_0 = \sup_{u \in H^1(\mathbb{R}) \setminus \{0\}} \left( \frac{\frac{1}{2}||u||_{L^3}^3}{||u||_{H^1}^3} \right) \leq \frac{B_1^3}{3}.
\]

The functional \(J\) defined in (3.71) enable us to define a function \(f : [0, +\infty) \to \mathbb{R}\) given by
\[
f(\xi) = \frac{1}{2}\xi^2 - k_0\xi^3\] (see figure below), where \(k_0\) was defined in (3.74). We note that \(f(||u||_{H^1}) \leq J(u)\).

![Figure 1. Graphic of function \(f(\xi) = \frac{1}{2}\xi^2 - \frac{1}{k_0}\xi^3\).](image-url)
Since $f$ is a differentiable function in $\mathbb{R}_+$ their critical points occur in $\xi_0 = 0$ and $\xi_1 = \frac{1}{3k_0}$. The last point is, in fact, the absolute maximum of the function $f$. Moreover, we define $d := f(\xi_1) = \frac{1}{6} \xi_1^2$.

Next, from (3.72), (3.73) and (3.74) we obtain the estimate,

$$E(t) \geq J(u) \geq f(||u||_{H^1}).$$

We have the following result due to Vitillaro (see [24]),

**Lemma 3.1.** Let $u$ be a solution that exists on the interval $[0, T]$, for all $T > 0$, according to Theorem 2.2 for $a(x) \equiv \mu > 0$. Then, if $||u_0||_{H^1} > \xi_1$ and $E(0) < d$, then

$$||u(t)||_{H^1} \geq \xi_2,$$

for some $\xi_2 > \xi_1$ and all $t \geq 0$. Moreover,

$$||u||_{L^3} \geq \xi_0^{1/3} \xi_2.$$

**Proof.** Indeed, since $f$ is strictly increasing for $0 < \xi < \xi_1$ and strictly decreasing for $\xi > \xi_1$, $f(\xi_1) = d$, $f(\xi) \to -\infty$ when $\xi \to +\infty$ and $d > E(0) > f(||u_0||_{H^1}) \geq f(0) = 0$, there are $\xi'_2 < \xi_1 < \xi_2$ such that,

$$f(\xi_2) = f(\xi'_2) = E(0).$$

On the other hand, being $E(t)$ non-increasing, we have

$$E(t) \leq E(0), \quad \forall t \geq 0.$$

From (3.75) and (3.76) we deduce

$$f(||u_0||_{H^1}) \leq E(0) = f(\xi_2).$$

Since $||u_0||_{H^1}, \xi_2 \in (\xi_1, +\infty)$ we conclude from (3.78) that

$$||u_0||_{H^1} \geq \xi_2.$$

The next step is to prove that

$$||u(t)||_{H^1} \geq \xi_2, \quad \forall t \geq 0.$$

In fact, we argue by contradiction. If (3.80) does not occur, then there is $t^* \in (0, +\infty)$ such that,

$$||u(t^*)||_{H^1} \leq \xi_2.$$

It is necessary to consider two cases:

(i) If $||u(t^*)||_{H^1} > \xi_1$, then from (3.75), (3.78) and (3.81) we obtain,

$$E(t^*) \geq f(||u(t^*)||_{H^1}) > f(\xi_2) = E(0),$$

which contradicts (3.73).
(ii) If we consider $||u(t^*)||_{H^1} \leq \xi_1$, observing (3.79), we guarantee the existence of $\bar{\xi}$ which satisfies

$$
||u(t^*)||_{H^1} \leq \xi_1 < \bar{\xi} < \xi_2 \leq ||u_0||_{H^1}.
$$

Consequently, from the continuity of $||u(\cdot)||_{H^1}$ as a function of $t \geq 0$, there is a $\bar{t} \in (0, t^*)$ verifying

$$
||u(\bar{t})||_{H^1} = \bar{\xi}.
$$

From the last equality, and (3.75), (3.78) and (3.82),

$$
E(\bar{t}) \geq f(||u(\bar{t})||_{H^1}) = f(\bar{\xi}) > f(\xi_1) = E(0),
$$

which contradicts (3.73). Since $E(t) \leq E(0)$ we get

$$
||u||_{L^3} \geq k_0^{1/3} \xi_2.
$$

This proves the Lemma.

$\square$

**Remark 3.1.** From Lemma 3.1 and Sobolev’s embedding $H^1(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$ we guarantee that

$$
||u||_{H^1} \geq C ||u||_{L^3} \geq k_0^{1/3} \xi_2,
$$

for some positive constant $C > 0$ and initial data $u_0$ satisfying $||u_0||_{H^1} > \xi_1 = \frac{1}{3k_0}$. The inequality (3.83) suggests that the $H^1$–norm of the solution $u$ related to the fully damped Korteweg-de Vries equation (3.69), can not have a decay property even being the energy $E(t)$ non-increasing.

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