Finite-size scaling in anisotropic systems

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Abstract

We present analytical results for the finite-size scaling in $d$-dimensional $O(N)$ systems with strong anisotropy where the critical exponents (e.g. $\nu_{||}$ and $\nu_{\perp}$) depend on the direction. Prominent examples are systems with long-range interactions, decaying with the interparticle distance $r$ as $r^{-d-\sigma}$ with different exponents $\sigma$ in corresponding spatial directions, systems with space-"time"a anisotropy near a quantum critical point and systems with Lifshitz points. The anisotropic properties involve also the geometry of the systems. We consider systems confined to a d-dimensional layer with geometry $L^m \times \infty^n$; $m + n = d$ and periodic boundary conditions across the finite $m$ dimensions. The arising difficulties are avoided using a technics of calculations based on the analytical properties of the generalized Mittag-Leffler functions.

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I. INTRODUCTION

Anisotropic systems are omnipresent in soft matter and solid state physics. Prominent examples are liquid crystals, dipolar-coupled uniaxial ferromagnets, systems with Lifshitz points, systems with space "time" anisotropy near a quantum critical point, systems with long-range interactions decaying with the interparticle distance $r$ as $r^{-d-\sigma}$ with different exponents $\sigma$ in corresponding spatial directions and some dynamical systems \cite{1,2}. Specific problems arise in the consideration of critical phenomena in such systems. In any of the cases the fundamental idea of scaling must to be modified in an appropriate way. In terms of the correlation length one can distinguish two type of anisotropy: weak and strong. In weakly anisotropic systems, the correlation length has spatially dependent amplitude. In the strongly anisotropic systems, in addition the critical exponents (e.g. $\nu_{||}$ and $\nu_{\perp}$) depend on the direction. A more general definition based on an anisotropic scale covariance of the n-point correlators and different exhaustive examples one can see in ref \cite{2}. In \cite{2} a general approach to scale invariance in infinite volume systems with strong anisotropy has been developed.

The object of the present paper is scaling in finite - size systems. In contrast to the theory of finite-size scaling in isotropic systems (see e.g. \cite{3,4}) and weakly anisotropic systems (see e.g. \cite{5} and refs. therein) the theory of finite - size scaling in strongly anisotropic systems (see \cite{6,7,8,9,10,11,12,13}) is still a field where the lack of results obtained in the framework of simplified and analytically tractable models are noticeable. There exist by now quite a few examples \cite{9,10,13} where the predictions of anisotropic finite-size scaling hypothesis have been reproduced analytically. In \cite{9,10} anisotropy appears near a quantum critical point as a result of mapping of a "time" dependent problem (in $d$ dimensions) to a "static" problem (in $d + 1$ dimensions). In \cite{13} it is due to the spatial direction dependence of the interactions .

Recently \cite{14} a recipe for studying finite-size effects based on some useful properties of the generalized Mittag-Leffler functions is suggested. It permits to consider isotropic and some strongly anisotropic systems (including long-range quantum systems) on an equal footing. The interest in Mittag-Leffler functions has grown up because of their applications in some finite-size scaling problems (see e.g. \cite{3,4,14,15,16}). The present study (see also \cite{17}) is an illustration of the rare possibility to handle the final expressions of the scaling equations for strongly anisotropic systems analytically.

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II. THE MODEL

We restrict our attention to the N-vector spin model defined at the sites of the lattice. The Hamiltonian of the model reads

\[ H = -N \sum_{x,y} J(x-y) \vec{\sigma}_x \cdot \vec{\sigma}_y, \]  

(2.1)

where \( \vec{\sigma}_x \) is a classical N-component unit vector defined at site \( x \) of the lattice and the spin-spin coupling decays with different power-laws in different lattice directions. We assume a d-dimensional system with mixed geometry \( L^m \times \infty^n \) under periodic boundary conditions in the finite dimensions. The interaction between spins enters the expressions of the theory only through its Fourier transform. We will consider the following anisotropic small \( q \) expansion of the Fourier transform of the spin-spin coupling:

\[ J(q) \simeq J(0) + a_{||} |q_{||}|^{2\sigma} + a_{\perp} |q_{\perp}|^{2\rho}, \]

(2.2)

where the first \( n \) directions (called “parallel” and denoted by the subscript \( || \) ) are extended to infinity and the remaining \( m \) directions (called “transverse” and denoted by \( \perp \) ) are kept finite, with \( m + n = d \) and \( a_{\perp} \) and \( a_{||} \) are metric factors and \( \rho, \sigma > 0 \). In finite directions the corresponding summations are over the vector \( q_{\perp} = \{ q_{\perp 1}, ..., q_{\perp m} \} \) that takes values in \( \Lambda^m \) defined by \( q_{\perp \nu} = 2\pi n_{\nu}/(aN_0) \) and \(- (N_0 - 1)/2 \leq n_{\nu} \leq (N_0 - 1)/2, \nu = 1, ..., m \). In infinite directions the sums are substituted with normalized integrals over the corresponding part of the first Brillouin zone \( [-\pi/a, \pi/a]^n \). For our further purposes let us recall that the finite linear dimension \( L = N_0 a \), in the continuous limit, means that the lattice spacing \( a \to 0 \) and simultaneously \( N_0 \to \infty \). In our analysis we accept \( a_{\perp} = a_{||} = -1/2 \). Finite-size scaling behavior of such type of systems are considered in reference [13] (with \( 0 < \rho, \sigma < 1 \)). In the large-N limit, the theory is solved in term of the gap equation for the parameter \( \lambda_V \) related with the finite-volume correlation length of the system. The bulk system is characterized by a vanishing \( \lambda_{\infty} \), so that the appropriately scaled inverse critical temperature

\[ \beta_c = \frac{1}{(2\pi)^d} \int_{[-\pi/a]^d} d\mathbf{q} \frac{d\mathbf{q}}{|\mathbf{q}_{\perp}|^{2\rho} + |\mathbf{q}_{||}|^{2\sigma}} \]

(2.3)

is finite whenever the effective dimensionality \( D = m/\rho + n/\sigma \) is greater than 2. The corresponding critical exponents \( \nu_{||} \) and \( \nu_{\perp} \) associated with the behavior of the infinite system are

\[ \nu_{||} = \frac{1}{\sigma(D - 2)}, \quad \nu_{\perp} = \frac{1}{\rho(D - 2)}. \]

(2.4)

For more details see ref. [13].
III. THE GAP EQUATION FOR THE REFERENCE SYSTEM

For the system with mixed geometry $\infty^n \times L^m$ the gap equation has the form:

$$\beta = \frac{1}{(2\pi)^n} \frac{1}{L^m} \int_{[-\pi/a]^n}^{[\pi/a]^n} \sum_{\mathbf{q}_\perp \in \Lambda^m} \frac{d^n\mathbf{q}_\parallel}{|\mathbf{q}_\perp|^2\rho + |\mathbf{q}_\parallel|^{2\sigma} + \lambda_V}. \quad (3.1)$$

Our analysis will be limited to system below the upper critical dimension $D_u = 4$ and lower critical one $D_l = 2$.

From the physical point of view, the infinite n-dimensional system, which has a finite size $L$ in the remaining $m$ dimensions, can be found in three qualitatively different situations depending on the value of $\frac{n}{\sigma}$: (i) If $2 < \frac{n}{\sigma}$, then the system is above its lower critical dimension $d_l = 2\sigma$ and, therefore, it exhibits a true critical behavior. A crossover from $n$-dimensional to $D$-dimensional critical behavior takes place when $L \to \infty$. (ii) In the borderline case of $n = 2\sigma$, the system is at its lower critical dimension and may have only a zero-temperature critical point. (iii) When $\frac{n}{\sigma} < 2$, the system is below its lower critical dimension and a (D-dimensional) critical behavior appears only in the thermodynamic limit $L \to \infty$.

In the present study we assume that there is no phase transition for finite $L$, hence $n < 2\sigma$. For $n < 2\sigma$ and $\lambda_V \to 0$, due to the convergence of the integral in (3.1) over $d^n\mathbf{q}_\parallel$, one can extend the integration over all $R^n$ in consistence with the underlying continuum field theory.

Further the corresponding $n$ - dimensional integral can be presented as

$$\frac{1}{(2\pi)^n} S_n \int_0^\infty \sum_{\mathbf{q}_\perp \in \Lambda^m} \frac{p^{n-1}dp}{|\mathbf{q}_\perp|^{2\rho} + p^{2\sigma} + \lambda_V}, \quad (3.2)$$

where $S_n = 2(\pi)^{n/2}/\Gamma(n/2)$ is the surface of the $n$-dimensional unit sphere. With the help of the identity:

$$\int_0^\infty \frac{p^{\alpha-1}dp}{t + p^n + |\mathbf{q}_\perp|^{\tau}} = \frac{\Gamma(1 - \frac{\alpha}{\eta})\Gamma(\frac{\alpha}{\eta})}{\eta} \frac{1}{(t + |\mathbf{q}_\perp|^{\tau})^{1-\frac{\alpha}{\eta}}}, \quad \eta > \alpha > 0, \quad (3.3)$$

if we choose $\alpha = n$, $\tau = 2\rho$ and $\eta = 2\sigma$, for Eq.(3.2) we end up with the result

$$\frac{A_{n,\sigma}}{L^m} \sum_{\mathbf{q}_\perp \in \Lambda^m} \frac{1}{(\lambda_V + |\mathbf{q}_\perp|^{2\rho})^{1-\frac{n}{2\sigma}}}, \quad 2\sigma > n, \quad (3.4)$$

where

$$A_{n,\sigma} = \frac{S_n}{(2\pi)^n} \frac{\Gamma(1 - \frac{n}{2\sigma})\Gamma(\frac{n}{2\sigma})}{2\sigma}. \quad (3.5)$$
Now the gap equation \( (3.1) \) may be presented in the equivalent form

\[
\beta = A_{n,\sigma} \frac{1}{L^m} \sum_{\mathbf{q}_\perp \in \Lambda^m} \frac{1}{(\lambda_V + |\mathbf{q}_\perp|^{2\rho})^{1-\frac{n}{2\sigma}}}, \quad 2\sigma > n.
\] (3.6)

Let us emphasize that one can relate Eq. \((3.6)\) with a fictitious fully finite isotropic \(m\) dimensional reference system in which the memory of the extended to infinity dimensions and of the anisotropy of the system is retained only in the parameter

\[
\gamma := 1 - \frac{n}{2\sigma}, \quad 0 < \gamma < 1,
\] (3.7)

and in the multiplier \(A_{n,\sigma}\) in front of the sum.

The normalized \(m\) dimensional sum in Eq. \((3.6)\)

\[
W^{1-\frac{n}{2\sigma}}_{m,2\rho}(\lambda_V, L) := \frac{1}{L^m} \sum_{\mathbf{q}_\perp \in \Lambda^m} \frac{1}{(\lambda_V + |\mathbf{q}_\perp|^{2\rho})^{1-\frac{n}{2\sigma}}}, \quad 2\sigma > n.
\] (3.8)

can be evaluated with the help of the identity \([14]\)

\[
\frac{1}{(\lambda_V + y^\alpha)^\gamma} = \int_0^\infty dt e^{-yt^\alpha\gamma^{-1}} E_{\alpha,\alpha\gamma}^\gamma(-\lambda_V t^\alpha),
\] (3.9)

in terms of the generalized Mittag-Leffler function \(E_{\alpha,\alpha\gamma}^\gamma(z)\) (see appendix \(A\)). If one chooses \(\alpha = \rho, \gamma = 1 - \frac{n}{2\sigma}\) and \(y = |\mathbf{q}_\perp|^2\) the needed result is

\[
W^{\gamma}_{m,2\rho}(\lambda_V, L) = \int_0^\infty dx x^{\gamma\rho - 1} E_{\rho,\gamma\rho}^\gamma(-\lambda_V x^\rho) \left[ \frac{1}{L} \sum_{\mathbf{q} \in \Lambda^1} \exp(-q^2 x) \right]^m, \quad \gamma > 0.
\] (3.10)

Now let us define

\[
Q_{N_0}(x) := \frac{1}{L} \sum_{\mathbf{q} \in \Lambda^1} \exp(-q^2 x) = \frac{1}{aN_0} \sum_{l=-N_0/2}^{N_0/2-1} \exp \left( -\frac{4\pi^2 l^2 x}{a^2 N_0^2} \right)
\] (3.11)

and using the approximating formula \((5.5)\) of ref.\([18]\), we have the expression

\[
Q_{N_0}(x) \simeq \frac{1}{\sqrt{4\pi x}} \left[ \text{erf} \left( \frac{\pi x^{1/2}}{a} \right) \right] - \frac{2\pi^2 x}{3a} \exp \left[ -\left( \frac{\pi^2}{a^2} \right) x \right] + \frac{1}{\sqrt{\pi x}} \left\{ \sum_{l=1}^\infty \exp[-(laN_0)^2/4x] \right\}
\] (3.12)

valid in the large \(N_0\) asymptotic regime. The first and the second terms in the above equation are size independent and are precisely the infinite volume limit of \(Q_{N_0}(x)\). The remainder of the calculation involves the insertion of \((3.12)\) into \((3.10)\).
IV. FINITE-SIZE SCALING FORM OF THE GAP EQUATION

In order to illustrate the further calculations we will consider in more details the case \( m = 1 \).

We can represent the right-hand side of Eq. (3.10) as a sum of three terms.

The first one is given by

\[
\frac{1}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \int_{0}^{\infty} dx x^{\gamma-1} E_{\rho,\gamma \rho}(-\lambda V x^\rho) \exp(-x k^2) = \frac{1}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \frac{1}{(\lambda V + k^2)^{1-\frac{2\rho}{\gamma}}}. \tag{4.1}
\]

where the definition of the \( \text{erf} \)-function

\[
\text{erf}(\Lambda \sqrt{x}) = \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} \exp(-x k^2) \, dk \tag{4.2}
\]

and the identity (3.9) have been used.

The second term is

\[
-\frac{2\pi^2}{3a} \int_{0}^{\infty} dx x^{\gamma-1} E_{\rho,\gamma \rho}(-\lambda V x^\rho) \exp \left[ -\left( \frac{\pi}{a} \right)^2 x \right] = -2\gamma \rho \frac{(\frac{\pi}{a})^{2\rho-1}}{3} \left[ \lambda V + (\frac{\pi}{a})^{2\rho} \right]^{\gamma+1}. \tag{4.3}
\]

The third one equals

\[
\int_{0}^{\infty} dx x^{\gamma-1} E_{\rho,\gamma \rho}(-\lambda V x^\rho) \frac{1}{\sqrt{\pi x}} \left\{ \sum_{j=1}^{\infty} \exp\left[ -(jN_0 a)^2 / 4x \right] \right\}. \tag{4.4}
\]

The first term is exactly the bulk limit \( W_{\gamma,1,2\rho}(\lambda V, \infty) \). The second one in the considered regime \( \lambda V \to 0 \) and \( a \to 0 \) is of order \( O \left( \frac{1}{\pi/a} \right) \) and can be omitted.

It is convenient to write the third term, Eq. (4.4), in terms of the function (particular case of the Jacobi \( \Theta_3 \) function)

\[
A(x) \equiv \sum_{n=-\infty}^{+\infty} e^{-x n^2}. \tag{4.5}
\]

and the universal finite-size scaling function \[19\]

\[
F^{\gamma,2\rho}_{m,2\rho}(y) = \frac{1}{(2\pi)^{2\rho}} \int_{0}^{\infty} dx x^{\gamma-1} E_{\rho,\gamma \rho} \left( -\frac{x^\rho}{(2\pi)^{2\rho}} y \right) \left[ A^m(x) - 1 - \left( \frac{\pi}{x} \right)^m \right]. \tag{4.6}
\]

This can be done with the help of the Poisson transformation formula

\[
A(x) = \sqrt{\pi x} A \left( \frac{\pi^2}{x} \right) \tag{4.7}
\]

and the identity \[A4\].

After some algebra the result for the third term is

\[
L^{2\gamma-1} \left[ F^{\gamma}_{1,2\rho}(\lambda V L^{2\rho}) + \frac{1}{(\lambda V L^{2\rho})^{\gamma}} \right]. \tag{4.8}
\]
Collecting the above results for Eq. (3.10), if \( m = 1 \), we obtain

\[
W_{\gamma, 2}^{\gamma}(\lambda V, L) = \frac{1}{2\pi} \int_{-\frac{\pi}{\sigma}}^{\frac{\pi}{\sigma}} \frac{dk}{(\lambda V + k^2)^{1-\frac{n}{2\sigma}}} - \frac{2\gamma \rho}{3} \frac{(\frac{x}{a})^{2\rho - 1}}{[\lambda V + (\frac{x}{a})^{2\rho}]^{\gamma + 1}} + L^{2\gamma - 1} \left[ F_{\gamma, 2}^{\gamma}(\lambda V L^2) + \frac{1}{(\lambda V L^2)^\gamma} \right], \quad \gamma \equiv 1 - \frac{n}{2\sigma} > 0. \tag{4.9}
\]

In the last term of Eq. (4.12) apart from the factor \( L^{2\gamma - 1} \) the intrinsic scaling combination

\[
y = \lambda V L^2 = (L/\xi_{L})^{2\rho} \tag{4.10}
\]

emerges, where \( \xi_{L} \) is the finite-size transverse correlation length (see [13]). The limitation \( m = 1 \) is not principal. If \( m > 1 \), in view of Eq. (3.12), the product \([Q N_0(x)]^m\) in Eq. (3.10) contains sums of terms of the form

\[
\left\{ \frac{1}{\sqrt{4\pi x}} \right\}^m \left\{ \text{erf} \left( \frac{\pi x^{1/2}}{a} \right) \right\}^{m'} \left\{ \exp \left[ - \sum_{i=1}^{m-m'} (l_i a N_0)^2 / 4x \right] \right\} \tag{4.11}
\]

with \( 1 \leq m' \leq m - 1 \) and \( l_i \neq 0, i = 1, ..., m - m' \). In such terms the error function \( \text{erf} \left( \frac{\pi x^{1/2}}{a} \right) \) can be replaced by unity, since the exponential function in the right-side of Eq. (4.11) cuts off the contribution from values of \( x^{1/2} \ll a N_0 \). Note that all the other terms that contain as a multiplier \( \frac{2\pi^2 x}{3 a} \exp \left[ - \left( \frac{x}{a} \right)^2 \right] \) can be estimated. They are of order \( O \left( \frac{1}{\pi a} \right) \) and must be omitted in the considered continuum limit. As a result instead of Eq. (4.9) we get

\[
W_{m, 2}^{\gamma}(\lambda V, L) \simeq \frac{1}{(2\pi)^m} \int_{[-\pi]^{m}} \frac{d^m k}{(\lambda V + |k|^{2\rho})^{1-\frac{n}{2\sigma}}} + L^{2\gamma - 1} \left[ F_{m, 2}^{\gamma}(\lambda V L^2) + \frac{1}{(\lambda V L^2)^\gamma} \right], \quad \gamma \equiv 1 - \frac{n}{2\sigma} > 0. \tag{4.12}
\]

If we introduce the notations

\[
K := K(\sigma, n, m) \equiv A_{n, \sigma}^{-1} \beta \tag{4.13}
\]

Eq. (3.6) can be rewritten as

\[
K - K_{\infty}^c = W_{m, 2}^{\gamma}(\lambda V, L) - W_{m, 2}^{\gamma}(0, \infty), \tag{4.14}
\]

where

\[
K_{\infty}^c := K_{\infty}(\sigma, \rho, n, m) \equiv W_{1, 2}^{\gamma}(0, \infty) = \frac{1}{(2\pi)^m} \int_{[-\pi]^{m}} d^m k \frac{1}{(|k|^{2\rho})^\gamma} \tag{4.15}
\]
is the inverse critical temperature (normalized with $A_{n,\sigma}$) of the "isotropic" bulk system.

The first term in the right hand side of Eq.(4.12) can be presented in the form

$$W_{m,2\rho}(\lambda_V, \infty) = \frac{1}{(2\pi)^m} \int_{[-\pi]}^\pi \frac{d^m k}{(\lambda_V + |k|^{2\rho})^\gamma} \approx K^c_\infty + \frac{S_m}{2(2\pi)^m} \lambda^{\frac{m}{2\rho} - \gamma} \int_0^\infty dx \frac{x^{\gamma \rho} - (1 + x^\rho)^\gamma}{x^{\gamma \rho + 1 - m/2} (1 + x^\rho)^\gamma}$$

(4.16)

valid for $\xi_{\perp, L} \gg a$. The integral over $x$ converges, provided $m > 2\gamma \rho > m - 2\rho$, and

$$\int_0^\infty dx \frac{x^{\gamma \rho} - (1 + x^\rho)^\gamma}{x^{\gamma \rho + 1 - m/2} (1 + x^\rho)^\gamma} = \frac{1}{\rho} \frac{\Gamma\left(\frac{m}{2\rho}\right)}{\Gamma\left(1 - \frac{m}{2\rho}\right)} \frac{\Gamma\left(1 - \frac{n}{2\sigma} - \frac{m}{2\rho}\right)}{\Gamma\left(\frac{m}{2\rho} - \frac{n}{2\sigma}\right)}$$

(4.17)

By substitution Eq.(4.12) into Eq.(4.14), taking into account the small-argument expansion Eq.(4.16), for the gap equation (3.6) we obtain the scaling form:

$$x \approx -a(n, m; \rho, \sigma) y^{D/2 - 1} + F_{m,2\rho}(y) + \frac{1}{y^\gamma}, \quad 4 > D > 2,$$

(4.18)

where

$$x = L^{2\rho(D/2 - 1)} (K - K_c), \quad y = (L/\xi_{\perp, L})^{2\rho}$$

(4.19)

and

$$a(n, m; \rho, \sigma) = -\frac{1}{2\rho} \frac{S_m}{(2\pi)^m} \frac{\Gamma\left(\frac{m}{2\rho}\right)}{\Gamma\left(1 - \frac{m}{2\rho}\right)} \frac{\Gamma\left(1 - \frac{n}{2\sigma} - \frac{m}{2\rho}\right)}{\Gamma\left(\frac{m}{2\rho} - \frac{n}{2\sigma}\right)}.$$

(4.20)

Our model study confirm the phenomenological assumption [11] that the finite-size scaling behavior in systems with mixed geometry $L^m \times \infty^n$ is governed by the "perpendicular" correlation length only.

V. FINITE-SIZE CORRECTIONS

Given the gap equation in scaling form, we are now in a position to explore the various finite-size corrections. Here, we look at the different regimes: the finite-size scaling regime defined by the condition $y \sim 1$, crossover to the thermodynamic critical behavior $y \gg 1$, and the regime $y \ll 1$. In this section for the sake of simplicity we will consider the important particular case of slab geometry, $m = 1$. 

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A. \( y \sim 1 \)

In order to consider the case of \( y \sim 1 \) we will use a new representation for \( F_{1,2}^\gamma(y) \) (see appendix [B])

\[
F_{1,2}^\gamma(y) = F_{1,2}^\gamma(0) + a(n,1;\rho,\sigma)y^{1-2\gamma \rho} + 2 \sum_{l=1}^{\infty} \frac{\left(4\pi^2 l^2\right)^{\rho \gamma} - [y + (4\pi^2 l^2)^{\rho}]^{\gamma \rho}}{(4\pi^2 l^2)^{\rho \gamma}[y + (4\pi^2 l^2)^{\rho}]^{\gamma}}, \quad 1 > 2\gamma \rho.
\]

(5.1)

and rewrite Eq.(4.18) in a form suitable for obtaining the shift of the bulk critical temperature.

Substituting Eq.(5.1) in Eq.(4.18) \((m = 1)\) we obtain the gap equation in the form

\[
x \simeq F_{1,2}^\gamma(0) + 2 \sum_{l=1}^{\infty} \frac{\left(4\pi^2 l^2\right)^{\rho \gamma} - [y + (4\pi^2 l^2)^{\rho}]^{\gamma \rho}}{(4\pi^2 l^2)^{\rho \gamma}[y + (4\pi^2 l^2)^{\rho}]^{\gamma}} + \frac{1}{y^\gamma}.
\]

(5.2)

Therefore, when \( K \to K_\infty^c \), simultaneously with \( L \to \infty \), in the way prescribed by the equation

\[
K = K_\infty^c + \frac{x}{L^{2\rho(D/2-1)}}
\]

(5.3)

with \( x = O(1) \), the leading-order asymptotic form of \( \lambda_V \) is given by

\[
\lambda_V \simeq \frac{y(x)}{L^{2\rho}},
\]

(5.4)

where \( y(x) \) is the positive solution of Eq.(5.2). Hence, at the critical point \( x = 0 \), we obtain

\[
\xi_{\perp,L} = A(n,\rho,\sigma)L
\]

(5.5)

where \( A(n,\rho,\sigma) = 1/[y(0)]^{1/2\rho} \) is an universal amplitude. In systems with mixed geometry the existence of the universal amplitude of the (finite-size) correlation length \( \xi_{\perp,L} \) on the level of the phenomenological scaling has been suggested in [11]. Here this qualitative statement is made quantitative being a model confirmation of the generalized in [11] Privman-Fisher hypothesis (c.f. with Eq.(23) in [11]).

When \( y \to \infty \), for \( 2 < D < 4 \), we may approximate the sum in Eq.(5.2) by an integral:

\[
\sum_{l=1}^{\infty} \frac{\left(4\pi^2 l^2\right)^{\rho \gamma} - [y + (4\pi^2 l^2)^{\rho}]^{\gamma \rho}}{(4\pi^2 l^2)^{\rho \gamma}[y + (4\pi^2 l^2)^{\rho}]^{\gamma}} \simeq \frac{1}{(2\pi z)^{2\rho}} \int_0^\infty dx \frac{\left(\frac{x}{z}\right)^{2\rho} - \left(1 + (\frac{x}{z})^{2\rho}\right)^2}{\left(\frac{x}{z}\right)^{2\rho+1/2} \left(1 + (\frac{x}{z})^{2\rho}\right)^{\gamma}}, \quad z := y^{1/2\rho}/(2\pi).
\]

(5.6)

With the use of Eqs.(4.17) and (5.6) one finds the asymptotic solution of Eq.(5.2), which to the leading order in \( x \gg 1 \) is

\[
x \approx -a(n,1;\rho,\sigma)y^{D/2-1}
\]

(5.7)

i.e. it recovers exactly the familiar bulk high-temperature result.
B. $y \gg 1$

The finite-size correction to the bulk critical behavior can be extracted from the asymptotic form of the functions $F_{d,\sigma}^\gamma(y)$ at large argument $y \gg 1$ (see [14, 17]). The result is

$$F_{1,2\rho}^\gamma(y) \simeq -y^{-\gamma} + \left[2\gamma(2\pi)^{2\rho}\zeta(-2\rho)\right]y^{-(1+\gamma)},$$

(5.8)

where $\zeta(\rho)$ is Riemann’s Zeta function.

Using Eq.(5.8) for the gap equation (3.1) we obtain:

$$x \approx -a(n, 1; \rho, \sigma)y^{D/2-1} + \left[2\gamma(2\pi)^{2\rho}\zeta(-2\rho)\right]y^{-(1+\gamma)}, \quad y \gg 1.$$  

(5.9)

As one can see the finite-size effects governed by the second term in the right hand side of Eq.(5.8) vary as an algebraic power of the variable $y$. Since $\zeta(-2\rho) = 0$ for $\rho = k, k$ - a natural number, there are not power low dependent finite size corrections if $\rho = k$. The case $0 < \rho < 1$ corresponds to the long-range interaction. For $\rho = 1$ corresponding to the short-range interaction the result for the universal finite-size scaling function is

$$F_{1,2}^\gamma(y) \simeq -y^{-\gamma} + \left[\frac{1}{2\gamma\Gamma(\gamma)}\right]y^{-\frac{\gamma}{2}}e^{-\sqrt{y}}$$

(5.10)

that reflects with exponential fall of finite-size corrections in Eq.(5.9). And as long as $\rho > 1$ and if it is not an integer, the power low corrections take place in the case of so-called subleading LR interaction [18] but with strong anisotropy.

C. $y \ll 1$

Whenever Eq.(5.2) has a solution $y \ll 1$, use can be made of the asymptotic expansion

$$\sum_{l=1}^{\infty} \frac{(4\pi^2l^2)^{\rho\gamma}}{(4\pi^2l^2)^{\rho\gamma}[y+(4\pi^2l^2)^{\rho\gamma}]^{\gamma}} = -\frac{\gamma}{(2\pi)^{2\gamma+2\rho}}\zeta(2\gamma\rho + 2\rho)y + O(y^2).$$

(5.11)

In obtaining the first term in the right-hand side of Eq.(5.11) the fulfilling of the condition $2\gamma\rho + 2\rho > 1$ is used. Taking into account Eq.(5.11), in the limit $|x - F_{1,2\rho}^\gamma(0)| \ll 1$ one finds

$$y \simeq \frac{1}{|x - F_{1,2\rho}^\gamma(0)|^{1/\gamma}}.$$  

(5.12)

Let us remind, that when the number of infinite dimensions is less than the lower critical dimension, the singularities of the bulk thermodynamic functions are rounded and no phase transition
occurs in the finite-size system. Nevertheless, one can define a pseudocritical temperature, corresponding to the position of the smeared singularities of the finite-size thermodynamic functions, and study its shift with respect to the bulk value of the critical temperature. In the case under consideration the first term in the right hand side of Eq. (5.2) is identified with the shift of the finite-size pseudocritical temperature. Actually, for the sake of convenience here we study the quantity $K$. The corresponding result for the pseudocritical $K^c_L$ is

$$K^c_L - K^c_\infty = L^{-\lambda} F_{1,2\rho}^\gamma (0),$$

(5.13)
i.e. the shift critical exponent is $\lambda = 1/\nu_\perp$ in accordance with standard finite-size scaling conjecture, see [3]. The coefficient $F_{d-n,\sigma}^\gamma (0)$ can be evaluated analytically as well as numerically for different values of the free parameters $d$, $\rho$ and $\gamma$ using the method developed in reference [20].

VI. CONCLUSIONS

The statement, that finite-size scaling in our system with mixed geometry $L^m \times \infty^n$ is the naturally expected one, is contained in [13] where different shape dependent scaling limits have been studied. In the present study we are much more interested in the explicit form of the scaling equation in different regimes. For this aim an appropriate technics of calculation is developed. We show how the mathematical difficulties that arise in the considered anisotropic model with mixed geometry can be avoided. First, using the identity the problem is effectively reduced to the corresponding isotropic one related to a fully finite reference system, Eq. (3.6). A further step is the recognition that with the help of the identity the appearance of $\gamma \neq 1$ in the summand of the gap equation (3.6) is not an obstacle for an analytical treatment. Knowledge of the properties of the generalized Mittag-Leffler function allows to carry out all calculations analytically. We show that though the system is strongly anisotropic, the corresponding gap equation Eq. (4.18) for the intrinsic scaling variable $y = \lambda V L^{2\rho}$ has a form similar to the isotropic case with geometry $L^D$ (c.f. with Eq. (4.115) in [3]). We stress that the finite-size $L$ is scaled by the perpendicular correlation $\xi_\perp$ only. This verified the Privman- Fisher hypothesis for strongly anisotropic systems formulated in [11].

Further we conclude that, the finite-size contributions to the thermodynamic behavior decay algebraically as a function of $L$ only if $0 < \rho \neq k$, where $k$ is a natural number. In the case $\rho = 1$, the finite-size contributions decay exponentially as a function of $L$. The phenomenon
that the so-called subleading terms (in our terminology the term with ρ > 1) lead to dominant finite-size contributions, being unimportant in the bulk limit, was first discussed in ref. [18]. This characteristic feature of the long-range interactions reveals also in our consideration.

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APPENDIX A: GENERALIZED MITTAG-LEFFLER FUNCTIONS

The generalized Mittag-Leffler functions are defined by the power series [21] (see also [14, 22, 23])

\[ E_{\alpha,\beta}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta) k!} z^k, \quad \alpha, \beta, \gamma \in \mathbb{C}, \quad \text{Re}(\alpha) > 0, \]  

where

\[ (\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma + 1)(\gamma + 2) \cdots (\gamma + k - 1) = \frac{\Gamma(k + \gamma)}{\Gamma(\gamma)}, \quad k = 1, 2, \cdots \]  

Let us formulate some needed properties of the generalized Mittag-Leffler functions.

It might be useful to note the relationship

\[ \frac{d}{dz} E_{\alpha,1}^\gamma(-z^\alpha) = \gamma z^{\alpha - 1} E_{\alpha,\beta}^\gamma(-z^\alpha), \]  

which follows from the power-series representation. In obtaining Eq.(4.8) we have taken into account the identity

\[ \int_0^{\infty} dx x^{\gamma - 1} E_{\rho,\gamma\rho}^\gamma(-x^\rho) = 1, \quad \rho > 0 \]  

that follows by integration of Eq.(A3) over z from zero to infinity. Next, by subtracting and adding 1/Γ(αγ) to the function \( E_{\rho,\gamma\rho}^\gamma \) over z from zero to infinity. Next, by subtracting and adding 1/Γ(αγ) to the function \( E_{\rho,\gamma\rho}^\gamma \) we obtain

\[ \int_0^{\infty} dt e^{-zt^{\alpha\gamma - 1}} \left[ E_{\alpha,\alpha\gamma}^\gamma(-t^\alpha) - \frac{1}{\Gamma(\alpha\gamma)} \right] = \frac{z^{\alpha\gamma} - (1 + z^\alpha)^\gamma}{(1 + z^\alpha)^\gamma z^{\alpha\gamma}}. \]  

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APPENDIX B: DERIVATION OF EQUATION (5.1)

First, we represent the integral in Eq.(4.6) as a sum of three terms \((m = 1)\). The first term is given by

\[
(y^{-\frac{1}{2}})^{\gamma \rho}\int_{0}^{\infty} dt t^{\rho - 1 - \frac{1}{2}}\left[ E_{\rho,\rho\gamma}^{\gamma}(\gamma t) - \frac{1}{\Gamma(\rho)} \right] A\left(\frac{4\pi^2 t}{y^{\frac{1}{2}}\gamma} \right) - 1 \equiv S_{\rho,\gamma}(y^{\frac{1}{2}}), \tag{B1}
\]

the second term is

\[-\frac{1}{2}\sqrt{\pi y^{\gamma-1/2}} \int_{0}^{\infty} dt t^{\rho - 3/2 - \frac{1}{2}}\left[ E_{\rho,\rho\gamma}^{\gamma}(\gamma t) - \frac{1}{\Gamma(\rho)} \right] \equiv -\frac{1}{y^{\gamma-1/2}} C_{\gamma,\rho} \tag{B2}
\]

and the third one equals the constant (provided \(1 > 2\gamma\rho\))

\[
\frac{1}{\Gamma(\rho)} \frac{1}{(2\gamma^2)^{\rho} \pi^{1/2}} \int_{0}^{\infty} dx x^{\gamma - 1/2} \left[ A(x) - 1 - \frac{(\pi x^{\frac{1}{2}})^{\frac{1}{2}}}{\pi x} \right] \equiv F_{1,2\rho}(0). \tag{B3}
\]

Let us now calculate the function \(S_{\rho,\gamma}(y^{\frac{1}{2}})\) and the constant \(C_{\gamma,\rho}\). Making use of the identity (A5), we represent Eq.\((B1)\) as

\[
S_{\rho,\gamma}(y^{\frac{1}{2}}) = 2 \sum_{l=1}^{\infty} \frac{(4\pi^2 l^2)^{\rho \gamma} - [y + (4\pi^2 l^2)^{\rho \gamma}]^{\gamma}}{(4\pi^2 l^2)^{\rho \gamma} [y + (4\pi^2 l^2)^{\rho}]^{\gamma}}. \tag{B4}
\]

To calculate \(C_{\gamma,\rho}\), in Eq.\((B2)\) we first write

\[
t^{-1/2} = \frac{1}{\pi^{1/2}} \int_{0}^{\infty} dx x^{-1/2} e^{-tx} \tag{B5}
\]

then, by using the identity (A5) we take the integral over \(t\), and then

\[
C_{\gamma,\rho} = \frac{1}{2\pi} \int_{0}^{\infty} dx \frac{x^{\rho \gamma} - (1 + x^{\rho})^{\gamma}}{(1 + x^{\rho})^{\gamma} x^{\rho} + 1/2}, \tag{B6}
\]

i.e. \(C_{\gamma,\rho} = -a(n, 1; \rho, \sigma)\). Collecting the results for Eqs.\((B1), (B2)\) and \((B3)\) for \((4.6)\) we get Eq.\((5.1)\).

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