

STABLE EXISTENCE OF INCOMPRESSIBLE
3-MANIFOLDS IN 4-MANIFOLDS

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Abstract. Given an injective amalgam at the level of fundamental groups and a specific 3-manifold, is there a corresponding geometric-topological decomposition of a given 4-manifold, in a stable sense? We find an algebraic-topological splitting criterion in terms of the orientation classes and universal covers. Also, we equivariantly generalize the Lickorish–Wallace theorem to regular covers.

1. Introduction

In this paper, we shall develop a cutting tool to better understand the stable geometric topology of 4-dimensional manifolds. Here, stable means diffeomorphism up to connected sums with finitely many copies of $S^2 \times S^2$, which is ‘inert’ at levels of fundamental group, signature, and spin. Just like in homotopy theory, the stable classification is more tractible and reflects what properties of a space persist in the long run. The above stabilization bypasses hangups with the Whitney trick [CS71, 2.1], which in higher dimensions allows for mirroring between topology and algebra.

1.1. Bistable results. Our results on stable embeddings vary according to the so-called $w_2$-type of the 4-manifold. So we first consider a weaker equivalence relation.

We call 4-manifolds bistably diffeomorphic if they become diffeomorphic after connecting sum each with finitely many copies of the complex-projective plane $\mathbb{C}P_2$ (nonspin) and its orientation-reversal $\mathbb{C}P_2$. For any oriented 4-manifold $X$, denote $X(r) := X \# r(S^2 \times S^2)$, $X(a,b) := X \# a(\mathbb{C}P_2) \# b(\overline{\mathbb{C}P_2})$.

Stable implies bistable, as $(S^2 \times S^2) \# (\mathbb{C}P_2) \cong 2(\mathbb{C}P_2) \# (\overline{\mathbb{C}P_2})$ [Wal64, Cor 1, Lem 1].

Given a nonempty connected CW-complex $A$, by a continuous map $u : A \to B\Gamma$ classifying the universal cover $\tilde{A}$, we mean the induced map $u_\#$ on fundamental groups is an isomorphism, for some basepoints. The map $u$ is uniquely determined up to homotopy and composition with self-homotopy equivalences $B\alpha : B\Gamma \to B\Gamma$ for all automorphisms $\alpha$ of $\Gamma$. By a connected subcomplex being incompressible, we shall mean that the inclusion induces a monomorphism on fundamental groups.

Theorem 1.1. Let $X$ be a oriented closed smooth 4-manifold. Let $c : X \to BG$ classify its universal cover. Let $X_0$ be a connected oriented closed 3-manifold with fundamental group $G_0$. Suppose $G = G_- *_{G_0} G_+$ with $G_0 \subset G_\pm$. There exists an incompressible embedding of $X_0$ in some bistabilization $X(a,b)$ inducing the given injective amalgamation of fundamental groups, if and only if there exists a map $d : X_0 \to BG_0$ classifying its universal cover and satisfying the equation

$$d_\ast [X_0] = \partial c_\ast [X] \in H_3(G_0; \mathbb{Z}),$$

with $\partial$ the boundary in a Mayer–Vietoris sequence in group homology [Bro94, III:6a].
The simplest case of $G_0 = 1$ was done transparently by J Hillman [Hi95], whose hands-on approach with direct manipulation of handles we generalize in this paper.

**Corollary 1.2** (Hillman). Let $X$ be a connected orientable closed smooth 4-manifold whose fundamental group is a free product $G_- * G_+$. Some bistabilization $X(a, b)$ is diffeomorphic to a connected sum $X_- # X_+$ with $X_+ \in$ of fundamental group $G_+$.

Similarly, when $G_0 = \mathbb{Z}$, note $X$ is bistably diffeomorphic to some $X_- \cup_{S^1 \times S^2} X_+$.

**Proof.** Here $G_0 = 1$, hence $H_3(G_0) = 0$. Take $X_0 = S^3$ and $d$ the constant map. □

The proof of Theorem 1.1 generalizes Hillman’s strategy for proving Corollary 1.2 and employs an equivariant generalization of the Lickorish–Wallace theorem ([LM89]).

Theorem 2.1 is a bordism version that slides 1-handles then does Wallace’s trick. Wallace’s proof of Corollary 2.2 relied upon the Rohlin–Thom theorem ($\Omega^{SO}_3 = 0$).

### 1.2. Stable results.

Shortly after Hillman’s result, Kreck–Lück–Teichner offered an alternative proof, using Kreck’s machinery of modified surgery theory ([Kre99]). They were able to replace bistabilization with stabilization, due to a careful analysis of $w_2$-types and triviality of 3-plane bundles over embedded 2-spheres in certain 5-dimensional cobordisms ([KLT95b]). In general, stabilization is required ([KLT95a]).

Regarding the removal of $S^2 \times S^2$ factors (destabilization), see [HK93] and [Kha17].

Recall that a (stable) spin structure $s$ on a smooth oriented manifold $M$ is a homotopy-commutative diagram (reduction of structure groups in [LM89, II:1.3]):

$$
\begin{array}{ccc}
M & \xrightarrow{r_M} & BSO \\
\downarrow{s} & & \downarrow{\text{BSpin}} \\
\end{array}
$$

**Theorem 1.3** (totally nonspin). Let $X$ be a oriented closed smooth 4-manifold whose universal cover has no spin structure. Let $c : X \to BG$ classify this cover. Let $X_0$ be a connected oriented closed 3-manifold with fundamental group $G_0$. Suppose $G = G_- * G_0 G_+ \subset G_\pm$. There exists an incompressible embedding of $X_0$ in some $X(r)$ inducing the given injective amalgam of fundamental groups, if and only if there is $d : X_0 \to BG_0$ classifying its universal cover satisfying (1.1).

Any oriented manifold has a spin structure if and only if $w_2$ of its tangent bundle vanishes [LM89]. So any oriented 3-manifold has a spin structure (as $w_2 = w_2 = 0$).

**Theorem 1.4** (spinable). Let $X$ be a oriented closed smooth 4-manifold that admits some spin structure. Let $c : X \to BG$ classify the universal cover. Let $X_0$ be a connected oriented closed 3-manifold with fundamental group $G_0$. Suppose $G = G_- * G_0 G_+ \subset G_\pm$. There exists an incompressible embedding of $X_0$ in a stabilization $X(r)$ inducing the given injective amalgam of fundamental groups, if and only if there exist a map $d : X_0 \to BG_0$ classifying its universal cover and spin structures $s$ on $X$ and $t$ on $X_0$ satisfying, with transverse $M := c^{-1}(BG_0)$:

$$
[X_0, t, d] = \partial[X, s, c] := [M, s | M, c | M] \in \Omega^3_{BG_0}(BG_0),
$$

with $\partial$ the boundary map in a Mayer–Vietoris sequence in spin bordism [CF64, 5.7].

Observe that (1.2) is a lift of (1.1), via the cobordism-Hurewicz homomorphism

$$
\Omega^3_{BG_0} \xrightarrow{\text{epi}} \Omega^3_{BG_0} \xrightarrow{\text{iso}} H_3(BG_0) : [M, \sigma, f] \mapsto [M, f] \mapsto f_*[M].
$$
Finally, we generalize Theorem 1.4 to only require that $\tilde{X}$ admits a spin structure. In order to understand the more delicate criterion, we state a lemma and definition.

**Lemma 1.5.** Let $u : Y \to B\Gamma$ classify the universal cover of an oriented connected smooth manifold $Y$. The universal cover $\tilde{Y}$ admits a spin structure if and only if there is a unique class $w^3_2 \in H^2(\Gamma; \mathbb{Z}/2)$ satisfying the equation $w_2(\tilde{Y}) = u^*(w^3_2)$.

The secondary characteristic class $w^3_2$ will vanish if $Y$ admits a spin structure. The following definition is rather delicate due to two explicit choices of homotopies. For $H : A \times [0, 1] \to B$ and $a \in A$, the $a$-track is $H^a := (t \mapsto H(a, t)) \in B^{[0, 1]}$.

**Definition 1.6.** Let $Y$ be an oriented connected smooth manifold whose universal cover $\tilde{Y}$ admits a spin structure. Let $u : Y \to B\Gamma$ classify the universal cover. Fix homotopy representatives $\tau_Y : Y \to BSO, w_2 : BSO \to K(\mathbb{Z}/2, 2), w^3_2 : B\Gamma \to K(\mathbb{Z}/2, 2)$. By Lemma 1.5, there is a homotopy $\eta$ from $w_2 \circ \tau_Y$ to $w^3_2 \circ u$. Suppose $\Gamma = \Gamma^- * \Gamma_0 \Gamma_+$ with $\Gamma_0 \subset \Gamma_\pm$. Write $i_0 : \Gamma_0 \to \Gamma$ for the inclusion homomorphism. Assume $Bi_0 : B\Gamma_0 \to B\Gamma$ is the inclusion of a biconnected subspace, with $u$ transverse to $Bi_0$. If there exists a nullhomotopy $\theta$ of the map $w^3_2 \circ Bi_0$, then we define the induced spin structure $s^\theta_\eta$ on the submanifold $N := u^{-1}(B\Gamma_0)$ of $Y$ by

$$s^\theta_\eta : N \to BSpin : x \mapsto \left(\tau_Y(x), \eta^* \circ \sigma^\theta(x)\right),$$

where we identify $BSpin$ with the homotopy fiber of $w_2$ and $*$ denotes join of paths.

We arrive at a generalization of Theorem 1.4 which further requires $i_0^*(w^3_2) = 0$.

**Theorem 1.7** (pre-spinnable). Let $X$ be a oriented closed smooth 4-manifold whose universal cover admits a spin structure. Let $c : X \to BG$ classify this cover. Let $X_0$ be a connected oriented closed 3-manifold with fundamental group $G_0$. Suppose $G = G_- * G_0 G_+$ with $G_0 \subset G_\pm$. There exists an incompressible embedding of $X_0$ in some $X(r)$ inducing the given injective amalgam of fundamental groups, if and only if there exist a map $d : X_0 \to BG_0$ classifying its universal cover and a spin structure $t$ on $X_0$ and a nullhomotopy $\theta$ of $w^3_2 \circ Bi_0$ satisfying, with $M := c^{-1}(BG_0)$:

$$[X_0, t, d] = [M, s^\theta_\eta, c|M] \in \Omega^{Spin}_3(BG_0).$$

The special case [KLT95b] is now a mutual corollary of Theorems 1.3 and 1.7.

**Corollary 1.8** (Kreck–Lück–Teichner). Let $X$ be a nonempty connected orientable closed smooth 4-manifold whose fundamental group is a free product $G_- * G_+$. Some $X(r)$ is diffeomorphic to a sum $X_- \# X_+$ with each $X_\pm$ of fundamental group $G_\pm$.

**Proof.** Here $G_0 = 1$, so $H_3(G_0) = 0 = \Omega^{Spin}_3(BG_0)$. Take $X_0 = S^3$, $d$ constant. □

Albeit that Kreck’s modified surgery theory [Kre99] is a powerful formalism, by which we were inspired and against which we checked our progress, we sought to write this paper from first principles, to be accessible to low-dimensional topologists. In particular, we avoid ‘subtraction of solid tori’ and ‘stable s-cobordism theorem.’

2. Surgery on a link and regular covers

We generalize the notion of classifying a universal cover. For a nonempty connected CW-complex $A$, a continuous map $u : A \to B\Gamma$ classifies a regular cover means that the induced map $u_\#$ on fundamental groups is an epimorphism, for a choice of basepoints. The (connected) regular cover $\tilde{A}$ corresponds to the kernel of $u_\#$, and its covering group is identified with $\Gamma$, which acts transitively on the fibers.
2.1. Oriented version. This development is used to prove Theorems 1.1 and 1.3.

**Theorem 2.1.** Let $M$ and $M'$ be connected oriented closed 3-manifolds. Let $f : M \to B\Gamma$ and $f' : M' \to B\Gamma$ classify regular covers. Then there exists a framed oriented link $L$ in $M$ that transforms $(M, f)$ into $(M', f')$ by surgery if and only if
\[
f_\ast[M] = f'_\ast[M'] \in H_3(\Gamma; \mathbb{Z}).
\]

In other words, this is an algebraic-topological criterion for whether or not there is a link in $M$ whose preimage in $\tilde{M}$ has a $\Gamma$-equivariant surgery resulting in $\tilde{M}'$.

The original version is simply without reference maps; see [Wal60] and [Lic62].

**Corollary 2.2** (Lickorish–Wallace). Any nonempty connected oriented closed 3-manifold $N$ is the result of surgery on a framed oriented link $L$ in the 3-sphere.

Lickorish also obtained each component is unknotted with $\pm 1$ Dehn coefficients.

**Proof.** Here $\Gamma = 1, M = S^3, M' = N$. Note $B\Gamma$ is a point, hence $H_3(\Gamma) = 0$. \hfill $\square$

Here is a more general, technical version of Theorem 2.1 that we shall use later.

**Lemma 2.3.** Let $M$ and $M'$ be connected oriented closed 3-manifolds, and let $B$ be a connected CW-complex. Suppose $f : M \to B$ and $f' : M' \to B$ are continuous maps that induce epimorphisms on fundamental groups, for some basepoints. Then
\[
f_\ast[M] = f'_\ast[M'] \in H_3(B; \mathbb{Z})
\]
if and only if there is a 4-dimensional smooth connected oriented compact bordism
\[
(F; f, f') : (W; M, M') \to (B \times [0,1]; B \times \{0\}, B \times \{1\})
\]
such that $W$ has no 1-handles with respect to $M$ and no 1-handles with respect to $M'$, for a certain handle decomposition of the 4-dimensional cobordism $(W; M, M')$.

**Proof of Theorem 2.1.** By Lemma 2.3, use only 2-handles: surger along a link. \hfill $\square$

The argument below, after the preliminary three paragraphs, can be perceived in two geometric steps, even though it is combined into a single surgical move. The first step is to slide 1-handles, along with the map data, so that they become trivial. The second step is a reference-maps version of Wallace’s trick to exchange oriented 1-handles for trivial 2-handles [Wal60, 5.1]. (If $\dim M > 3$, see [RS72, 6.15] and subsequent remark to replace 1-handles for 3-handles in certain cobordisms on $M$.)

**Proof of Lemma 2.3.** $\iff$ is due to $\Omega_3^{SO}(B) \cong H_3(B)$. Consider the $\implies$ direction.

Clearly $\Omega_0^{SO} = \mathbb{Z}$ and $\Omega_1^{SO} = \Omega_2^{SO} = 0$; recall that $\Omega_3^{SO} = 0$ by Rohlin–Thom [Tho54, IV.13]. Then note, for the CW-complex $B$, by the Atiyah–Hirzebruch spectral sequence, that the cobordism-Hurewicz map is an isomorphism:
\[
\Omega_3^{SO}(B) \to H_3(B) ; \quad [M, f : M \to B] \mapsto f_\ast[M].
\]

Thus the criterion (2.2) transforms into the equation: $[M, f] = [M', f'] \in \Omega_3^{SO}(B)$.

In other words, there exists a 4-dimensional smooth oriented compact bordism
\[
(F_0; f, f') : (W_0; M, M') \to (B \times [0,1]; B \times \{0\}, B \times \{1\}).
\]

Since $M$ and $M'$ are connected, by joining their two possibly different components in $W_0$ via connected sum and ignoring the rest, we may assume that $W_0$ is connected. Hence, in the handle decomposition of a Morse function $(W_0; M, M') \to ([0,1]; \{0\}, \{1\})$, $W_0$ has no 0-handles with respect to $M$ and no 0-handles with
respect to $M'$. Therefore, it remains to eliminate the 1-handles of $W_0$ with respect to $M$ and $M'$. For simplicity of notation, we assume that $W_0$ has a single 1-handle.

Let $h : (D^1 \times D^3, S^0 \times D^3) \rightarrow (W_0, M)$ be the 1-handle, preserving orientation. Since $M$ is connected, there is a path $\alpha_0 : [-1, 1] \rightarrow M$ with $\alpha_0(\pm 1) = h(\mp 1, 0)$. Concatenation yields a loop $\beta_0 = h(-, 0) * \alpha_0 : S^1 \rightarrow W_0$. Since $f_\#$ is an epimorphism, there exists a loop $\beta : (S^1, 1) \rightarrow (M, \alpha_0(1))$ such that $f_\#[\beta] = F_0[\beta_0]$. By general position, there is a normally framed embedded arc $\alpha_1 : [-1, 1] \times D^2 \rightarrow M$ such that $\alpha_1(\pm 1, 0) = h(\mp 1, 0)$ and $\alpha_1(-, 0)$ is homotopic rel boundary to $\alpha_0 * \beta^{-1}$.

Push off the 1-handle core $h(-, 0)$ to obtain a normally framed embedded arc $h_0 : [-1, 1] \times D^2 \rightarrow M'$. A matching isotopy takes $\alpha_1$ to $\alpha_1' : [-1, 1] \times D^2 \rightarrow M'$ with $\alpha_1'(\pm 1) = h_1(\mp 1, 0)$. Concatenation yields a normally framed embedded loop
\[
\lambda := \{h_0(-, r) * \alpha_1'(-, r)\}_{r \in D^2} : S^1 \times D^2 \rightarrow M'.
\]

Write $W_1 := M' \times [0, 1] \cup_{\lambda} D^2 \times D^2$ for the trace of the surgery along $\lambda$ in $M'$. Since $F_0 \circ \lambda(-, 0)$ is nullhomotopic, choose a nullhomotopy to yield a bordism
\[
(F_1 : f', f'') : (W_1, M', M'') \rightarrow (B \times [0, 1]; B \times \{0\}, B \times \{1\}).
\]

Observe that $W_1$ is the trace of surgery along the framed belt sphere $S^1 \times D^2 \hookrightarrow M''$. By the cancellation lemma [RST72, 6.4], $W_0 \cup M' W_1$ is diffeomorphic to $M \times [0, 2]$ relative to $M \times \{0\}$. In particular, there is $\delta : M \approx M''$ with $f'' \circ \delta \approx f$. Write $W'_0 := M \times [0, 2] \cup_{f''} W_1$. So we have a new bordism with $h$ replaced by a 2-handle:
\[
(\delta \cup f'' : F_1 : f, f') : (W'_0, M, M') \rightarrow (B \times [0, 1]; B \times \{0\}, B \times \{1\}).
\]

By iteration, we kill all 1-handles of $W'_0$ relative to $M$. Similarly, repeat relative to $M'$. Thus, we obtain the desired bordism $(W, F)$ with only 2-handles rel $M$. \hfill \square

2.2. Spin version. We shall need this development to prove Theorem 1.4.

Theorem 2.4. Let $(M, s)$ and $(M', s')$ be spin closed 3-manifolds. Let $f : M \rightarrow B\Sigma$ and $f' : M' \rightarrow B\Sigma$ classify regular covers. There exists a framed oriented link $L$ in $M$ that transforms $(M, s, f)$ into $(M', s', f')$ by a spin bordism if and only if
\[
[M, s, f] = [M', s', f'] \in \Omega_3^{\text{Spin}}(B\Sigma).
\]

Lemma 2.5. Let $(M, s)$ and $(M', s')$ be connected spin closed 3-manifolds, and let $B$ be a connected CW-complex. Suppose $f : M \rightarrow B$ and $f' : M' \rightarrow B$ are continuous maps that induce epimorphisms on fundamental groups. Then
\[
[M, s, f] = [M', s', f'] \in \Omega_3^{\text{Spin}}(B)
\]
if and only if there is a 4-dimensional smooth connected spin compact bordism
\[
(F : f, f') : (W, t ; M, s, M', s') \rightarrow (B \times [0, 1]; B \times \{0\}, B \times \{1\})
\]
such that $W$ has no 1-handles with respect to $M$ and no 1-handles with respect to $M'$, for a certain handle decomposition of the 4-dimensional cobordism $(W; M, M')$.

Proof of Theorem 2.4. By Lemma 2.5, use only 2-handles: surger along a link. \hfill \square

Proof of Lemma 2.5. The $\Leftarrow$ implication is obvious. Consider the $\Rightarrow$ implication.

Recall the proof of Lemma 2.3. We reconstruct $W_1$ to admit a spin structure extending the spin structure $\overline{s}$ on $\partial_- W_1 = \overline{M}$, since by gluing along $M''$ this will induce a spin structure on $W'_0$ extending the spin structure $\overline{s} \sqcup s'$ on $\partial W'_0 = \overline{M} \sqcup M'$. 

\[
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\]
Since $H^{i+1}(M'; \pi_i(\text{Spin}_4)) = 0$ for all $i \geq 0$, by obstruction theory, the spin structure $s'$ lifts to a framing $\phi$ of the tangent bundle $TM'$. The sole obstruction to extending the stable framing $\phi \oplus \text{id}$ of $TM' \oplus \mathbb{R}$ to the tangent bundle $\tau$ of $W_1$ is $\phi(\tau) \in H^2(W_1, M'; \pi_1(\text{SO}_4)) = H^2(D^2, S^1; \pi_1(\text{SO}_4)) = \pi_1(\text{SO}_4) \cong \mathbb{Z}/2$.

Let $\eta \in \pi_1(\text{SO}_2) \cong \mathbb{Z}$. Reframe the normal bundle of the surgery circle $\lambda_1(-, 0)$ as $\lambda^n : S^1 \times D^2 \to M' \setminus \{(z, r) \mapsto \lambda(z, \eta_1(r))\}$.

Write $W^n_1 := M' \times [0, 1] \cup_{\lambda_1} D^2 \times D^2$ with tangent bundle $\tau^n$. By [KM63, Lemma 6.1],

$$\phi(\tau^n) = \phi(\tau) + \sigma_\#(\eta) \in \pi_1(\text{SO}_4),$$

where $\sigma : \text{SO}_2 \to \text{SO}_4$ denotes the inclusion. Since the induced map $\sigma_\#$ on fundamental groups is surjective, find $\eta$ so that $\tau^n$ has a framing extending $\phi \oplus \text{id}$. Hence $W^n_1$ has a spin structure extending $s'$ on its lower oriented boundary $\overline{M'}$.

By gluing, we obtain an induced spin structure on $W_0 \cup_{M'} W^n_1 \approx M \times [0, 2]$ relative to $M \times \{0\}$. Modifying Proof 2.3, redefine $W^n_0 := M \times [0, 2] \cup_{\eta} W^n_1$ with spin structure the union of this one and the orientation-reversal of the one on $W^n_1$. Therefore, the spin structure on $W_0$ restricts to $\overline{s'} \cup s'$ on $\partial W^n_0 = \overline{M} \cup M'$. $\square$

3. AMBIENT SURGERY ON PAIRS OF POINTS

Let $G = G_- *_{G_0} G_+$ be an injective amalgam of groups. The corresponding **double mapping cylinder model** of its classifying space is the homotopy colimit

$$BG := BG_- \cup_{BG_0 \times \{-1\}} BG_0 \times [-1, +1] \cup_{BG_0 \times \{+1\}} BG_+$$

with respect to the maps $BG_0 \to BG_\pm$ induced from the inclusions $G_0 \to G_\pm$.

Akin to Stalling’s thesis, here is a folklore fact proven in [Bow99, 1.1] (cf. [BS90]).

**Theorem 3.1** (Bowditch). If $G$ and $G_0$ are finitely presented, so are $G_-$ and $G_+$.

Instead of $G_0$ being finitely presented, the proof of the next statement can work assuming $G_-, G_0, G_+$ are finitely generated, but we prefer the former hypothesis.

**Proposition 3.2.** Let $X$ be a connected oriented closed smooth 4-manifold. Suppose $f : X \to BG$ classifies a regular cover. Assume $G_0$ is finitely presented. Then $f$ can be re-chosen up to homotopy so that $f$ is transverse to the bicollared subspace $BG_0 \times \{0\}$ in the model (3.1), the 3-submanifold preimage $M$ in $X$ is connected, and the restriction $f : M \to BG_0$ also classifies a regular cover.

This is proven after three lemmas. The first is an apparatus to recalibrate paths.

**Lemma 3.3.** Let $X$ be a connected oriented smooth $n$-manifold with $n > 2$. Consider a space $B = B_- \cup_{B_0} B_+$ with $B, B_\pm - B_0$ path-connected and $B_0 = B_- \cap B_+$.

Suppose $f : X \to B$ is $\pi_1$-surjective. Assume $\pi_1(B_\pm - B_0)$ are finitely generated, by $r_\pm$ elements. There are disjoint 1-handlebodies $\Lambda_\pm \approx \# r_\pm (S^1 \times D^{n-1}) \subset X$ and $f' : X \to B$ homotopic to $f$ having $\pi_1$-surjective restrictions $f' : \Lambda_\pm \to B_\pm - B_0$.

**Proof.** We may homotope $f$ so that its image contains some points $b_\pm$ in $B_\pm - B_0$. Then there are $x_\pm \in X$ such that $f(x_\pm) = b_\pm$. There are based loops $\mu_\pm^1, \ldots, \mu_\pm^n : (S^1, 1) \to (B_\pm - B_0, b_\pm)$ whose based homotopy classes generate $\pi_1(B_\pm - B_0, b_\pm)$. Since $f_\# : \pi_1(X, x_\pm) \to \pi_1(B, b_\pm)$ is surjective and $n > 2$, there exist disjoint smoothly embedded based loops $\lambda_\pm^1, \ldots, \lambda_\pm^n : (S^1, 1) \to (X, x_\pm)$ and a based homotopy $H^1_\pm : S^1 \times [0, 1] \to B$ from $f \circ \lambda_\pm^1$ to $\mu_\pm^1$. Since $X$ is oriented, for each
i, there is a tubular neighborhood $\Lambda^\pm_\ast$ of $\lambda^\pm_\ast(S^1)$ and a diffeomorphism $\Lambda^\pm_\ast \approx S^1 \times D^{n-1}$. Taking the radii of the tubes sufficiently small, we find that the $\Lambda^\pm_\ast$ pairwise intersect in a fixed $D^n$-neighborhood of $x_\pm$. Thus we obtain disjoint embeddings of $\#_\pm^{-}(S^1 \times D^{n-1})$ and $\#_\mp^{-}(S^1 \times D^{n-1})$ in $X$, say with images called $\Lambda_-$ and $\Lambda_+$.

Finally, using these NDR neighborhoods $\Lambda_\pm$ of $\nu_\ast \lambda_\pm$ and specific homotopies $\nu_\ast H_\pm$ as the data for the homotopy extension property [Bre97, Theorem VII:1.5], we obtain a homotopy $H : X \times [0, 1] \to B$ from $f$ to a map $f'$ such that $f' \circ \lambda_\pm = \mu^\pm_\ast$. Hence $f'(\Lambda_\pm) \subset B_\pm - B_0$ and $(f'|\Lambda_\pm)_\ast \pi_1(\Lambda_\pm, x_\pm) = \pi_1(B_\pm - B_0, b_\pm)$. □

Given a continuous map $f : X \to B$ from a smooth manifold $X$ to a topological space $B$, and given a subspace $B_0$ that admits a tubular neighborhood $E(\xi) \subset B$, W Browder defines $f$ to be transverse to $B_0$ to mean that the conclusion of the implicit-function theorem holds: the preimage $X_0 = f^{-1}(B_0)$ is a smooth submanifold of $X$ with normal bundle $\nu(X_0 \to X) = (f|X_0)_\ast(\xi)$ [Bro72, II:2.1]. Since the proof is omitted for Browder’s generalization [Bro72, II:2.1] of Thom’s transversality theorem [Tho54, I:5], we give details for the trivial line bundle $\xi = \mathbb{R}$.

**Lemma 3.4.** Let $f : X \to B$ be a continuous map from a smooth manifold to a space $B$. For any bicollared subspace $B_0$ of $B$ (i.e., $B_0$ has a neighborhood in $B$ homeomorphic to $B_0 \times \mathbb{R}$), there exists a map $f' : X \to B$ transverse to $B_0$ and homotopic to $f$, relative to the complement of an open neighborhood of $f^{-1}(B_0)$.

**Proof.** We have an open embedding $\beta : B_0 \times \mathbb{R} \to B$ with $\beta(B_0 \times \{0\}) = B_0 \subset B$. Write $N \subset B$ for the image of $\beta$, and write $\pi_2 : B_0 \times \mathbb{R} \to \mathbb{R}$ for the projection. Note $f^{-1}(N) \subset X$ is a smooth manifold, since it is an open set in a smooth manifold. By Whitney’s approximation theorem [Bre97, II:11.7], the $C^0$ function $\pi_2 \circ \beta^{-1} \circ f : f^{-1}(N) \to \mathbb{R}$ is 0.5-close to a $C^\infty$ function $f : f^{-1}(N) \to \mathbb{R}$. Define $H : f^{-1}(N) \times [0, 1] \to B$ by $(x, t) \mapsto \beta((\pi_1 \beta^{-1} f)(x), (1 - t)(\pi_2 \beta^{-1} f)(x) + t g(x))$.

Note $H$ is a homotopy from $H(x, 0) = f(x)$ to a map $f' := H(-, 1) : f^{-1}(N) \to B$. Then $f'$ is transverse to $B_0$ with $(f')^{-1}(B_0) = g^{-1}(0)$ a smooth submanifold of $X$; where by Sard’s theorem and a tiny homotopy, we assume 0 is a regular value of $g$.

It remains to extend $H$ to $X \times [0, 1]$ so that $H(x, t) = f(x)$ for all $x \in X - N$. Using explicit formulas derived from the tubular neighborhood structure $\beta$, this is achieved by the homotopy extension property for the neighborhood deformation retract $f^{-1}(\beta(B_0 \times [-1, 1]) \cup (X - N))$ closed in the $T_3$ space $X$; see [Bre97, Theorem VII:1.5]. The desired map $f' : X \to B$ is again $H(-, 1)$ of this extension. □

We perform 1-handle exchanges in dimension 4 by an obstruction-theoretic argument. This is not in the literature, but see [Hem76, p67] and [Cap76, Lemma I:3]. Recall the frontier $\text{Fr}_X(A) := C_\ast X(A) \cap C_\ast X(X - A)$ for $A \subset X$, a topological space.

**Lemma 3.5.** Let $f : X \to B$ be a continuous map from a smooth 4-manifold to a path-connected space $B$, transverse to a path-connected separating subspace $B_0$ of $B = B_+ \cup B_0 \cup B_-$. Decompose $X = X_- \cup X_0 \cup X_+$ by the $f$-preimages. Let $\alpha : (D^1, \partial D^1) \to (X_\pm, X_0)$ be a smoothly embedded arc with $[f \circ \alpha] = 0 \in \pi_1(B_\pm, B_0)$. Suppose $\pi_3(B_\pm) = 0 = \pi_4(B)$. Then $f$ is homotopic to a $B_0$-transverse map $g : X \to B$ whose preimage of $B_0$ is the result of adding a 1-handle with core $\alpha$. Namely, for some open-tubular neighborhood $U \approx D^1 \times D^3$, the arc $\alpha$ in $X_\pm$:

$$g^{-1}(B_0) = (X_0 \cup \text{Fr}_X(U)) - (X_0 \cap U).$$
Proof. Let $T$ be a closed-tubular neighborhood of $\alpha(D^1)$ in $X_\pm$. There is a framing diffeomorphism $\phi : (D^1 \times D^3, \partial D^1 \times D^3) \rightarrow (T, T \cap X_0)$ with $\phi(s, 0) = \alpha(s)$. Define

\[
O := \phi \{ (s, x) \in D^1 \times D^3 \mid \frac{2}{3} \leq \| x \| \leq 1 \}
\]

\[
M := \phi \{ (s, x) \in D^1 \times D^3 \mid \frac{1}{3} \leq \| x \| \leq \frac{2}{3} \}
\]

\[
I := \phi \{ (s, x) \in D^1 \times D^3 \mid 0 \leq \| x \| \leq \frac{1}{3} \},
\]

which is a decomposition of $T = O \cup M \cup I$ into three closed subsets. Define a map

\[
g : O \rightarrow B_\pm ; \phi(s, x) \mapsto (f \circ \phi)(s, (3\|x\|-2)x).
\]

Since $[f \circ \alpha] = 0 \in \pi_1(B_\pm, B_0)$, there exists a map $H : D^1 \times [0, 1] \rightarrow B_\pm$ such that

\[
H(s, 1) = (f \circ \alpha)(s) \quad \forall s \in D^1
\]

\[
H(\pm 1, t) = \alpha(\pm 1) \quad \forall t \in [0, 1]
\]

\[
H^{-1}(B_0) = \partial D^1 \times [0, 1] \cup D^1 \times \{0\}.
\]

By the pasting lemma, we can extend $g$ from $O$ to $O \cup M$ by

\[
g : M \rightarrow B_\pm ; \phi(s, x) \mapsto H(s, 3\|x\|-1).
\]

Next, there exist both a neighborhood $C$ of the attaching 0-sphere $\alpha(\partial D^1)$ in $X_\mp$ and a diffeomorphism $\psi : \partial D^1 \times D^4 \rightarrow X_\mp$ such that $\psi|\partial D^1 \times D^3 = \phi|\partial D^1 \times D^3$. Extend $g$ from Fr$_X T = \phi(\partial D^1 \times \partial D^3)$ to Fr$_X C = \psi(\partial D^1 \times \partial D^4)$ by $g = f$. Then $g$ is defined on the ‘riveted’ 3-sphere $S := \text{Fr}_X (C \cup I)$. Since $g(S) \subset B_\pm$ and $\pi_3(B_0) = 0$, we may extend $g$ to the ‘riveted’ 4-disc $C \cup I$; using the collar of $B_0$ in $B_\mp$, we can guarantee that $g(C \cup I - S) \subset B_\mp - B_0$. Lastly, extend $g$ to the complement $X - (C \cup T)$ by $g = f$. Therefore $g : X \rightarrow B$ is transverse to $B_0$ with

\[
g^{-1}(B_0) = (X_0 - I) \cup (M \cap I).
\]

Finally, note Fr$_X (C \cup T)$ is a 3-sphere in $X$, so we obtain a 4-sphere $X \times [0, 1]$:

\[
\Sigma := (C \cup T) \times \{0\} \cup \text{Fr}_X (C \cup T) \times [0, 1] \cup (C \cup T) \times \{1\}.
\]

Since $\pi_4(B) = 0$, we may fill in $(f \circ \text{proj}_X)|\Sigma$ to obtain a homotopy from $f$ to $g$. □

We adapt to dimension 4, and simplify, ‘arc-chasing’ arguments of [Hem76, p67] and [Cap76, p88]. Further, we generalize the sliding of 1-handles trick of Proof 2.3.

Proof of Proposition 3.2. By Theorem 3.1 and by Lemma 3.3 with respect to the model (3.1), we homotope $f$ so that there are disjointly embedded 1-handlebodies $\Lambda_\pm \approx \#r_{\pm}(S^1 \times D^3) \subset X$ satisfying $f(\Lambda_\pm) \subset BG_\pm - BG_0$ and $f_\# \pi_1(\Lambda_\pm, x_\pm) = G_\pm$. Hence $f(\Lambda_- \cup \Lambda_\pm)$ is disjoint from the bicollar neighborhood $BG_0 \times [-1, 1]$ in $BG$.

Next, by Lemma 3.4, we further re-choose $f$ up to homotopy relative to $\Lambda_- \cup \Lambda_\pm$, so that $f$ is also transverse to $BG_0 \times \{0\}$, say with $f$-preimage $K$. Write $V_\pm$ for the $X$-closure of the path-component neighborhood of $\Lambda_\pm$ in the open subset $X - K$.

Assume that $V_\pm \cap K$ has at least two components, say $K_0 \cup K_1$. Since $V_\pm$ is connected, there is a properly and smoothly embedded arc $\alpha : [0, 1] \rightarrow V_\pm$, satisfying: $\alpha(i) \in K_i$ if $0 \leq i \leq 1$, $\alpha(\frac{1}{2})$ is near-but-not $x_\pm$, and $\alpha^{-1}(V_\pm) = (0, 1)$.

Since the composite map $\pi_1(\Lambda_\pm) \xrightarrow{f_{\#}} \pi_1(BG_\pm) \rightarrow \pi_1(BG_\pm, BG_0)$ is surjective, upon midpoint-concatenation of some based loop $(S^1, 1) \to (\partial \Lambda_\pm, \alpha(\frac{1}{2}))$, we may assume that $[f \circ \alpha] = 0 \in \pi_1(BG_\pm, BG_0)$. Then, by Lemma 3.5, we re-choose $f$ up to homotopy relative $\Lambda_- \cup \Lambda_\pm$ so that the new component neighborhood $V_\pm$ of $\Lambda_\pm$ in
$X - f^{-1}(BG_0)$ contains $K_0\#K_1$. Since $X$ is compact, so is $K$, so we repeat finitely many steps until $V_+ \cap K$ becomes connected. Similarly, make $V_- \cap K$ connected.

Write $L := V_- \cap V_+$, a connected 3-submanifold of $X$. Let $x_0 \in L$. Assume there exists $x_1 \in K - L$. Since $X$ is connected, there exists a path $\gamma : [0, 1] \to X$ from $x_0$ to $x_1$. Define $s_0 := \sup \gamma^{-1}(V_- \cup V_+)$. Since $0 < s_0 < 1$, we must have $\gamma(s_0) \in Fr_X(V_-) \cup Fr_X(V_+) = L$. Then $\gamma(s_0)$ is in the interior of $V_- \cup V_+$. So there exists $s_1 > s_0$ with $\gamma(s_1)$ also in the interior of $V_- \cup V_+$. This contradicts the maximality of $s_0$. Therefore $K - L$ is empty. Hence $K = L$ and so it is connected.

Finally, since $G_0 \subset G_+$ is finitely generated and since $(f|\Lambda_+)\# : \pi_1(\Lambda_+) \to G_+$ is surjective, there exist based loops $x_0 \in G_+$, generated and $c$ may re-choose $V$ since each $j$ is also a monomorphism. So both $\pi_1(\Lambda_+)\# : \pi_1(M, c_0) \to \pi_1(\Lambda_+)\#$ is a necessary condition. So now, assume $(f|\Lambda_+)\#$ is a regular cover, by Proposition 3.2, we may re-choose $c$ up to homotopy so that: $c$ is transverse $BG_0$, the 3-submanifold preimage $M$ is connected, and the restriction $c_0 : M \to BG_0$ classifies a regular cover. Write $X = X_- \cup M X_+$ and $c = c_- \cup c_+ with restrictions c_\pm : X_\pm \to BG_\pm$.

Next, consider the commutative square, with horizontal maps being connecting homomorphisms induced from (3.1) and with vertical maps being of Hurewicz type:

\[
\begin{array}{ccc}
\Omega^3 SO(BG) & \to & \Omega^3 SO(BG_0) \\
\downarrow & & \downarrow \\
H_4(BG) & \to & H_3(BG_0)
\end{array}
\]

\[
\begin{array}{ccc}
[M, c] & \downarrow & [M, c_0] \\
\downarrow & & \downarrow \\
c_0[X] & \to & c_0[M].
\end{array}
\]

Hence the criterion (1.1) implies: there is a classifying map $d : X_0 \to BG_0$ with $d_*[X_0] = c_0_*[M] \in H_3(G_0)$.

Since $d_\#$ and $c_0_\#$ are surjective, by Lemma 2.3, there is a 4-dimensional oriented smooth bordism $e : V \to BG_0$ from $(M, c_0)$ to $(X_0, d)$ made with only 2-handles. Since $V$ is obtained from $X_0$ using only 2-handles, the inclusion $j : X_0 \to V$ induces an epimorphism on fundamental groups. Since $d_\# = e_\# \circ j_\#$ is a monomorphism, note that $j_\#$ is also a monomorphism. So both $j_\#$ and $e_\#$ are isomorphisms.

Now, we obtain a connected 5-dimensional oriented compact smooth cobordism

\[T := X \times [0, 1] \cup M \times [-1, 1] \cup V \times [-1, 1],\]

where we regard $M \times [-1, 1]$ in $X \times \{1\}$ and we smooth the corners at $M \times \{-1, 1\}$.

The resultant 4-manifold and map are $X' := \partial T - X \times \{0\}$ and $c' := D|_{X'}$, where

\[D : T \to BG ;\]

\[
\begin{cases}
(x, t) \in X \times [0, 1] & \mapsto c(x) \\
(v, s) \in V \times [-1, 1] & \mapsto (e(v), s).
\end{cases}
\]

Decompose the space $X' = X'_0 \cup \cup X'_1$ with $X'_\pm = X_\pm \cup M V \times \{\pm 1\} \cup X_0 \times \{\pm 1, 0\}$, as well as the map $c' = c'_- \cup c'_+ : X' \to BG$ with $c'_\pm = c_\pm \cup c_0 : e : X'_\pm \to BG_\pm$.

Write $i : M \to X$ for the inclusion. Since $V$ is the trace of a surgery on a framed oriented link $L$ in $M$, correspondingly note $T$ is the trace of a surgery
on \( i \circ L \) in \( X \). By a similar argument as earlier, we find that the kernel of \( c_{0\#} \)
equals the kernel of the map induced by the inclusion \( M \to V \), which is generated by the (unbased) components \( L_k \) of \( L \), upon anchoring them to the basepoint with choices of connecting paths. In addition, since \( c_{0\#} = c_{\#} \circ i_{\#} \) and \( c_{\#} \) is an isomorphism, the kernel of \( c_{0\#} \) equals the kernel of \( i_{\#} \). In particular, each embedded circle \( L_k \) is nullhomotopic in \( X \), bounding an immersed disc with transverse double points, which can be isotoped away using finger-moves [FQ90, 1.5]; thus each \( L_k \) bounds an embedded disc in \( X \). Another consequence is that \( D \cong c \cup_{c_0} e : X \cup_M V \to BG \) induces an isomorphism on fundamental groups. Then, by an argument with alternating words, each \( c^'_{\#} : X'_{\#} \to BG_{\#} \) also does so. So, since \( d_{\#} \) is an isomorphism, \( c' \) induces an isomorphism on fundamental groups.

Finally, we show that the embedding solution \( X' \) is bistably diffeomorphic to \( X \). For each \( L_k \), consider embedded in \( \tilde{T} \) the 2-sphere \( S_k \) with equator \( L_k \), with northern hemisphere the core of the bounding 2-handle in \( V \times \{0\} \), and with southern hemisphere the bounding 2-disc in \( X \times \{1\} \). Write \( N_k \) for the 5-dimensional closed-tubular neighborhood of \( S_k \) in \( \tilde{T} \). Observe that \( T \) is diffeomorphic to the boundary-connected sum \((X \times [0,1]) \# (\bigcup_k N_k) \). Each \( N_k \) is diffeomorphic to either \( D^3 \times S^2 \) or \( D^3 \times S^2 \), where the latter is the nontrivial (nonspin) disc bundle. Thus, we obtain \( X' \cong X \# p(S^2 \times S^2) \# q(S^2 \times S^2) \) for some \( p \geq r \) and \( q \geq 0 \). Since \((S^2 \times S^2) \# (CP_2) \cong 2(CP_2) \# (\overline{CP_2}) \) and \( S^2 \times S^2 \cong (CP_2) \# (\overline{CP_2}) \) [Wal64, C1, L1],

\[
X'(1,0) \cong X(1 + p + q, p + q).
\]

**Proof of Theorem 1.3.** Since \( w_2(\tilde{X}) \neq 0 \), by the Hurewicz theorem, there exists a spherical class \( \bar{\alpha} : S^2 \to \tilde{X} \) such that \( \langle w_2(\tilde{X}), \bar{\alpha}_s[S^2] \rangle \neq 0 \). Write \( p : \tilde{X} \to X \) for the covering map, and write \( \alpha := p \circ \bar{\alpha} : S^2 \to X \). Since on tangent bundles \( T\tilde{X} = p^*(TX) \), as one obtains the smooth structure on \( \tilde{X} \) by even-covering, note

\[
\langle w_2(X), \alpha_s[S^2] \rangle = \langle w_2(X), p_\# \bar{\alpha}_s[S^2] \rangle = \langle p^* w_2(X), \bar{\alpha}_s[S^2] \rangle = \langle w_2(\tilde{X}), \bar{\alpha}_s[S^2] \rangle = 1.
\]

Do the same as in the proof of Theorem 1.1, until the construction of the 2-sphere \( S_k \). In the case that the normal bundle of \( S_k \) is nontrivial, replace the southern hemisphere with its one-point union with \( \alpha \), smooth rel \( L_k \) into immersion then an embedding by finger-moves, to obtain \( S'_k \). Since \( [S'_k] = [S_k] + [\alpha] \in \pi_2(X) \), note

\[
\langle w_2(X), S'_k[S^2] \rangle = \langle w_2(X), S_k[S^2] + \alpha_s[S^2] \rangle = 1 + 1 = 0 \in \mathbb{Z}/2.
\]

Hence the normal 3-plane bundle of the new embedded 2-sphere \( S'_k \) in \( \tilde{T} \) is trivial. So \( T \) is diffeomorphic to the boundary-connected sum \((X \times [0,1]) \# (\bigcup_{k=1}^r D^3 \times S^2) \). Therefore, we obtain \( X' \) is diffeomorphic to \( X(r) = X \# r(S^2 \times S^2) \).

**Proof of Theorem 1.4.** Do the same as in the proof of Theorem 1.1, except using (1.2) and Lemma 2.5 instead of (1.1) and Lemma 2.3, until the construction of the 2-sphere \( S_k \). Here, the spin structure \( s_M \) on \( M \) is the restriction of the spin structure \( s \) on \( X \times \{1\} \), where the spin structure on the normal line bundle is induced from its pullback orientation [LM89, II:2.15]. Since \( s_M \) is the restriction of the spin structure on \( V \times \{0\} \), we obtain that \( T \) has an induced spin structure.

Then, since \( w_2(T) = 0 \), the normal 3-plane bundle of each \( S_k \) in \( T \) is trivial. So \( T \) is diffeomorphic to the boundary-connected sum \((X \times [0,1]) \# (\bigcup_{k=1}^r D^3 \times S^2) \). Therefore, we obtain \( X' \) is diffeomorphic to \( X(r) = X \# r(S^2 \times S^2) \).
For clarity, we repeat the following proof from [KLT95b, p258] and [Kre99, p713]. The statement shall be applied in Proof 1.7 for manifolds Y of dimensions 3, 4, 5.

**Proof of Theorem 1.5.** Since Y is 1-connected, by the Leray–Serre spectral sequence for the homotopy fibration sequence \(Y \xrightarrow{\pi} BG\), we obtain an exact sequence

\[
0 \longrightarrow H^2(BG; \mathbb{Z}/2) \xrightarrow{\pi^\ast} H^2(Y; \mathbb{Z}/2) \xrightarrow{\pi^\ast} H^2(\tilde{Y}; \mathbb{Z}/2)^G.
\]

Then, since \(w_2(TY) = w_2(p^\ast(TY)) = p^\ast(w_2(TY))\), the oriented smooth manifold \(Y\) admits a spin structure if and only if there exists \(w^g_2 \in H^2(BG; \mathbb{Z}/2)\) such that \(u^\ast(w^g_2) = w_2(TY)\). Further by exactness, this class \(w^g_2\) is unique if it exists. \(\square\)

For \(r \geq 0\), the pinch map \(p : X(r) \to X \vee \#_r(S^2 \times S^2)\) gives a degree-one map

\[k := (\text{id} \vee \text{const}) \circ p : X(r) \to X.\]

The \(\pi_1\)-isomorphism \(c : X \to BG\) induces the \(\pi_1\)-isomorphism \(c \circ k : X(r) \to BG\).

**Proof of Theorem 1.7: necessity of (1.3).** Assume for some \(r \geq 0\) that there exists an incompressible embedding \(j_0 : X_0 \to X(r)\) such that \((c \circ k \circ j_0) \neq \pi_1 X_0 = G_0\). Then \(X(r) = X_0 \vee X_0 \times \mathbb{R} \times \mathbb{R}\), with inclusions \(j_\pm : X_0 \times \mathbb{R} \to X(r)\). Since \(X(r)\) and \(X_0\) are connected, so are \(X_\pm\). Furthermore, since \((c \circ k)\#\) and \((c \circ k \circ j_\pm)\#\) are isomorphisms, by a basic observation on normal form [Sm17], so are \((c \circ k \circ j_\pm)\# : \pi_1 (X_\pm) \to G_\pm\).

Consider the double mapping cylinder model (3.1) of \(BG\), where \(B_0 : BG_0 \to BG\) is the inclusion of a bicollared subspace. Since \(X_0\) is a CW-complex, there is a homotopically unique map \(d : X_0 \to BG_0\) such that \(B_0 \circ d \simeq c \circ k \circ j_0\). Furthermore, since \(X_\pm\) are CW-complexes, \(d\) extends to maps \(c_\pm : X_\pm \to BG_0 \simeq X_0 \times [0, 1]\) with \(B_0 \circ c_\pm \simeq c \circ k \circ j_\pm\). Therefore, \((c \circ k)\#\) is homotopic to a \(BG_0\)-transverse map \(c' := c_\pm \cup d : X(r) \to BG\) satisfying \((c')^{-1}(BG_0 \times \{0\}) = X_0\).

Next, since \(X\) admits a spin structure, by Lemma 1.5, there is a unique class \(w^g_2 \in H^2(BG; \mathbb{Z}/2)\) such that \(w_2(TX) = c^\ast(w^g_2)\). Since \(S^2\) is stably parallelizable, so is \(S^2 \times S^2\). Then the tangent bundle \(TX(r)\) is stably isomorphic to the pullback \(k^\ast TX\). The corresponding statement is false for a bistabilization \(X(a, b)\) unless \(a = 0 = b\). Hence \(w_2(TX(r)) = k^\ast w_2(TX)\).

Note \(d^\ast (i_0^\ast w^g_2) = (i_0 \circ d)^\ast (w^g_2) = (c \circ k \circ j_0)^\ast (w^g_2) = j_0^\ast k^\ast(c^\ast w^g_2)\)

\[= j_0^\ast k^\ast w_2(TX(r)) = j_0^\ast w_2(TX(r)) = w_2(TX_0) = v_2(X_0) = 0,
\]

with \(v_2 = w_2 + w^g_2\) the second Wu class [MS74, 11.14] and \(\text{Sq}^2 = 0\) on \(H^2(X_0; \mathbb{Z}/2)\).

The exact sequence (4.1) holds analogously for \(X_0\), so \(\text{Ker}(d^\ast) = 0\) hence \(i_0^\ast(w^g_2) = 0\). Thus, there is a nullhomotopy \(\theta\) of \(w^g_2 \circ B_0\). By Lemma 3.4, we may assume \(c\) is transverse to \(BG_0\) in model (3.1), with 3-submanifold \(M := c^{-1}(BG_0 \times \{0\})\) of \(X\).

Now, \(k : X(r) \to X\) extends to a retraction \(K : X[r] \simeq X \vee rS^2 \to X\), where \(X[r] := (X \times [0, 1]) \vee rD^3 \times S^2\) is the canonical cobordism from \(X\) to \(X(r)\). Since \(c\) is transverse to \(BG_0\), so is \(c \circ K\). Recall there is homotopy \(H : X(r) \times [1, 2] \to BG\) such that \(H(-, 1) = c \circ k\) and \(H(-, 2) = c'\). These unite to a \(BG_0\)-transverse map

\[C := (c \circ K) \cup_{c \circ k} H : W := X[r] \cup (X(r) \times [1, 2]) \to BG.
\]

The preimage 4-manifold \(V := C^{-1}(BG_0 \times \{0\})\) fits into an oriented bordism \((V, C[V])\) from \((M, c[M])\) to \((X_0, d)\). Furthermore, this enhances to a spin bordism, as Definition 1.6 produces a spin structure \(s^g_\mu\) on \(V\) defined by the formula

\[s^g_\mu : V \to BSpin = \text{hofib}(w_2) ; x \mapsto \left(\pi_V(x), \mu^\ast \theta^C(x)\right),\]
with \( \mu : W \times [0, 1] \to K(\mathbb{Z}/2, 2) \) a homotopy from \( w_2 \circ \tau_W \) to \( w_2^C \circ C \). Indeed, \( w_2^C \) exists by Lemma 1.5, since \( \tilde{W} \) has a spin structure as \( T \tilde{W} \cong \tilde{K}^*TX \oplus K \). Define \( \eta : X \times [0, 1] \to K(\mathbb{Z}/2, 2) \) as a restriction of \( \mu \). So \( s^0_\mu \) on \( V \) restricts to spin structures \( s^\mu_\eta \) on \( M \) and \( t := s^\mu_\eta|X_0 \) on \( X_0 \). Therefore, Equation (1.3) holds. \( \square \)

Recall the homotopy fiber of a map \( f : A \to B \) with respect to \( b_0 \in B \) is
\[
\text{hofib}(f) := \{(a, p) \in A \times B^{[0,1]} \mid p(0) = f(a) \text{ and } p(1) = b_0 \}.
\]

**Proof of Theorem 1.7: sufficiency of (1.3).** Assume Equation (1.3) holds, where the transverse 3-submanifold \( M := c^{-1}(BG_0) \) of \( X \) exists by Lemma 3.4, upon altering \( c \) by a homotopy. Furthermore, by Proposition 3.2, we can further homotope \( c \) so that \( M \) is connected and its restriction \( c_0 : M \to BG_0 \) is a \( \pi_1 \)-epimorphism. Then the spin bordism \((V, \Sigma, e)\) from \((M, s^\mu_\eta, c|M)\) to \((X_0, t, d)\), by Lemma 2.5, can be assumed to only have 2-handles relative to \( X_0 \). From the proof of Theorem 1.1, \( e : V \to BG_0 \) is a \( \pi_1 \)-isomorphism, and the map \( D \simeq c \cup \omega e : T \to BG \) is also.

Observe that the spin structure \( \Sigma : V \to B\text{Spin} = \text{hofib}(w_2) \) is of the form
\[
\Sigma = \left( \tau_V : V \to BSO, \sigma : V \to K(\mathbb{Z}/2, 2)^{[0,1]} \right),
\]
with \( \sigma(x) \in K(\mathbb{Z}/2, 2)^{[0,1]} \) a path from \( w_2(\tau_V(x)) \) to the basepoint \( \omega \) of \( K(\mathbb{Z}/2, 2) \).

Recall that \( \theta : BG_0 \times [0, 1] \to K(\mathbb{Z}/2, 2) \) is a homotopy from \( w_2^C \circ B \) to \( const_\omega \). Then define a homotopy \( \xi : V \times [0, 1] \to K(\mathbb{Z}/2, 2) \) from \( w_2 \circ \tau_V \) to \( w_2 \circ B \circ e \) by
\[
\xi^\tau := \sigma(x) \ast \theta^\tau(x).
\]

Recall that \( \eta : X \times [0, 1] \to K(\mathbb{Z}/2, 2) \) is a homotopy from \( w_2 \circ \tau_X \) to \( w_2^C \circ c \). This restricts to a homotopy \( \eta_0 : M \times [0, 1] \to K(\mathbb{Z}/2, 2) \) from \( w_2 \circ \tau_M \) to \( w_2^C \circ B_0 \) and \( c_0 \).

Note \( \xi \) extends \( \eta_0 \), since \( \tau_V \) extends \( \tau_M \) and \( e \) extends \( c_0 \). Thus, since \( T \simeq X \cup_M V \),
we obtain a homotopy \( \eta \cup_M \xi \) from \( w_2 \circ \tau_T \) to \( w_2^C \circ D \). Since \( D \) classifies the universal cover of the 5-manifold \( T \), by Lemma 1.5, the universal cover \( \tilde{T} \) has a spin structure.

Consider the embedded 2-spheres \( S_k : S^2 \to T \), in the proof of Theorem 1.1. Write \( P : \tilde{T} \to T \) for the universal covering map. As \( S^2 \) is simply connected, by the lifting theorem, there is an embedding \( \tilde{S}_k : S^2 \to \tilde{T} \) with \( S_k = P \circ \tilde{S}_k \). Note \( \langle w_2T, S_k[S^2] \rangle = \langle w_2T, P_k \tilde{S}_k[S^2] \rangle = \langle P^\ast(w_2T), \tilde{S}_k[S^2] \rangle = \langle w_2\tilde{T}, \tilde{S}_k[S^2] \rangle = 0 \).

Then, although \( T \) need not be spin, nonetheless the normal 3-plane bundle of each \( S_k \) in \( \tilde{T} \) is trivial. So \( T \) is diffeomorphic to the boundary-connected sum \((X \times [0, 1]) \# (\bigcup_{k=1}^c D^3 \times S^2) \). Therefore \( X' \) is diffeomorphic to \( X(r) = X \# r(S^2 \times S^2) \). \( \square \)

A final remark on (1.1) is that \( H_3(G_0) = \mathbb{Z} \) if \( X_0 \) is irreducible with infinite fundamental group, as \( X_0 \) models \( BG_0 \), a consequence of the sphere theorem [Hem76].

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