Reflection Factors for the Principal Chiral Model

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Abstract

We consider the $SU(2)$ Principal Chiral Model (at level $k = 1$) on the half-line with scale invariant boundary conditions. By looking at the IR limiting conformal field theory and comparing with the Kondo problem, we propose the set of permissible boundary conditions and the corresponding reflection factors.
1 Introduction

The principal chiral model $PCM_k$ is defined by the action:

$$S_{PCM_k} = \frac{1}{2\lambda^2} \int_{\partial B} Tr \left\{ (g^{-1} \partial_{\mu} g)(g^{-1} \partial_{\mu} g) \right\} d^2 x + i k \Gamma(g),$$  \hspace{1cm} (1)

where $\Gamma(g)$ is the Wess-Zumino-Witten (WZW) term [1]:

$$\Gamma(g) = \frac{1}{24\pi} \int_B Tr \left\{ (g^{-1} \partial_{\mu} g)(g^{-1} \partial_{\nu} g)(g^{-1} \partial_{\lambda} g) \right\} \epsilon^{\mu\nu\lambda} d^3 x.$$  \hspace{1cm} (2)

In eq.(1) $g$ is a Lie group valued field defined on a two-dimensional compact spacetime surface $\partial B$. The region of integration $B$ in eq.(2) is a three-dimensional simply-connected manifold whose boundary is $\partial B$. Topological arguments show that the ambiguity in this definition amounts to the WZW functional (2) being determined up to a positive integer, which can be reabsorbed into the constant $k$ in eq.(1) [2]. If the Lie group is simple this ensures the positivity of the action (1). Here we shall consider the group $G = SU(2)$. The action (1) hence enjoys a $SU(2)_L \times SU(2)_R$ global symmetry.

For $k = 0$ the theory corresponds to a $O(4) \approx SU(2) \times SU(2)$ nonlinear sigma model. Its behaviour is massive. If $k \neq 0$ the renormalization group (RG) analysis reveals that it interpolates between two fixed points [3]. The ultraviolet (UV) fixed point is controlled essentially by the first term in eq.(1). The RG flow of the coupling $\lambda^2$ terminates at the infrared (IR) fixed point $\lambda^2 = 8\pi/k$ where the theory becomes massless at all distances. At this point the theory is characterized by a conformal field theory based on two $SU(2)_k$ Kac-Moody algebras (at level $k$). For a generic $k$ the RG trajectory arrives at the IR fixed point along the direction defined by the irrelevant field $Tr(g^{-1} \partial g g^{-1} \partial g)$ of dimension $1 + 2/(k+2)$ [4]. For $k = 1$, this field does not exist in the conformal theory and the incoming direction is defined by the operator $T \bar{T}$, composed from the components of the stress tensor of the IR conformal field theory. The point where the model crosses over from the region of one fixed point to the other introduces a mass scale that breaks the scale invariance.

The $PCM_k$ was argued to be integrable in refs. [5]-[9] and its thermodynamic Bethe Ansatz (TBA) equations proposed in [10]. Zamolodchikov’s $c$-function was shown to take the values $c_{UV} = 3$ and $c_{IR} = 3k/(k+2)$ at the fixed points. The latter is in agreement with the central charge of the $SU(2)_k$ conformal field theory [11], [12]. A.B.Zamolodchikov and A.B.Zamolodchikov subsequently proposed the background scattering in terms of massless particles that leads to the correct TBA equations for $k = 1$ [3]. Following a prescription developed by Smirnov and Kirillov [23] in the context of the $SU(2)$-invariant Thirring model, they also showed that the form factors associated with the chiral currents obey the correct commutation relations. However the central term remains undetermined. Notwithstanding this, it can be shown to take the correct value by TBA analysis, [3]. Mejean and Smirnov [27] derived the form factors for the trace of the stress tensor.

In this work, we study the model in the presence of reflecting boundaries. This problem may appear awkward at the first sight due to the very definition of the action (1). This is because the boundary of a boundary is evidently an empty set. However we shall circumvent this obstacle by ignoring the classical action (1) altogether and going directly to the quantum theory. The drawback of this approach is that we are not able to apply useful information from the classical theory, [30], [31], [29]. The determination of the boundary conditions compatible with integrability and the corresponding reflection amplitudes will involve some amount of guesswork. We shall use as guideline some knowledge
coming from the symmetries of the problem, the limiting IR conformal field theory and a related problem (Kondo). The difference between this and the Kondo problem lies in the fact that in the former the scale invariance is broken in the bulk by a mass scale associated with a very unstable $O(4)$-isovector resonance $[1]$, whereas in the Kondo problem the scale invariance is broken at the boundary.

Let us first assemble some known results about this model, $[3]$. The spectrum of the theory consists of stable massless particles: left-movers and right-movers. It is convenient to parametrise the on-mass-shell momenta of the particles in terms of the rapidity variables $-\infty < \beta, \beta' < \infty$:

$$e = p = \frac{M}{2} e^\beta, \text{ for right-movers,}$$
$$e = -p = \frac{M}{2} e^{-\beta'}, \text{ for left-movers.} \quad (3)$$

With this parametrisation opposite momenta still correspond to opposite rapidities, $[2]$. For left-left and right-right scattering all Mandelstam variables vanish and since the scattering depends only on the dimensionless ratios of the momenta, the mass scale $M$ is arbitrary. The right-left scattering on the other hand distinguishes some preferable scale. The forward and backward amplitudes are:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (5)$$

We shall denote the antiparticles of $R_a(\beta)$ and $L_a(\beta')$ by $\bar{R}_{\bar{a}}(\beta)$ and $\bar{L}_{\bar{a}}(\beta')$, respectively. Let us now consider the general $2 \to 2$ scattering of a particle $R_a(\beta_1)$ with its antiparticle $\bar{R}_{\bar{a}}(\beta_2)$. The S-matrix element is given by $[21]$

$$\begin{align*}
R_c(\beta_1') \bar{R}_{\bar{a}}(\beta_2') | R_a(\beta_1) \bar{R}_{\bar{a}}(\beta_2) > & = \langle \delta(\beta_1' - \beta_1) \delta(\beta_2' - \beta_2) F_{ab}^{cd}(\beta) \\
- \delta(\beta_1' - \beta_2) \delta(\beta_2' - \beta_1) B_{ab}^{dc}(\beta),
\end{align*} \quad (6)$$

where $\beta \equiv \beta_1 - \beta_2$. The forward and backward amplitudes are:

$$\begin{align*}
F_{ab}^{cd}(\beta) &= \delta_a^c \delta_b^d u_1(\beta) + \delta_{ab} \delta^{cd} v_1(\beta), \\
B_{ab}^{dc}(\beta) &= \delta_a^d \delta_b^c u_2(\beta) + \delta_{ab} \delta^{cd} v_2(\beta).
\end{align*} \quad (7)$$

$\text{We assume that the boundary introduces no additional mass scale.}$
However, for massless particles backward scattering is unacceptable and we therefore set $u_2(\beta) = v_2(\beta) = 0$.

The particle-particle S-matrix element is given by:

$$R_a(\beta_1)R_b(\beta_2) = S_{ab}^{cd}(\beta_1 - \beta_2)R_d(\beta_2)R_c(\beta_1),$$

with

$$S_{ab}^{cd}(\beta) = \sigma_T(\beta)\delta_a^c\delta_b^d + \sigma_R(\beta)\delta_a^d\delta_b^c.$$  \hspace{1cm} (9)

$\sigma_T(\beta)$ and $\sigma_R(\beta)$ are the transition and reflection amplitudes, respectively. It is also convenient to introduce the 2-particle amplitude in the isovector and isoscalar channels:

$$\begin{align*}
S_V(\beta) &= \sigma_T(\beta) + \sigma_R(\beta), \\
S_0(\beta) &= \sigma_T(\beta) - \sigma_R(\beta).
\end{align*}$$  \hspace{1cm} (10)

Using the requirements of factorizability, unitarity and crossing symmetry, the following minimal solution was suggested in ref.\[3\]:

$$\begin{align*}
u_1(\beta) &= -\sigma_T(\beta) - \sigma_R(\beta), \\
v_1(\beta) &= \sigma_R(\beta), \\
\sigma_T(\beta) &= \frac{1}{\pi} \beta \sigma_R(\beta), \\
\sigma_R(\beta) &= -\frac{i\pi}{\beta - i\pi} S_V(\beta),
\end{align*}$$  \hspace{1cm} (11)

where

$$S_V(\beta) = \frac{\Gamma \left( \frac{1}{2} + \frac{\beta}{2\pi} \right) \Gamma \left( -\frac{\beta}{2\pi} \right)}{\Gamma \left( \frac{1}{2} - \frac{\beta}{2\pi} \right) \Gamma \left( \frac{\beta}{2\pi} \right)}.$$  \hspace{1cm} (12)

Of course we get exactly the same expression for the L-L scattering. The non-trivial right-left scattering is defined by the commutation relations:

$$R_a(\beta)L_b(\beta') = U_{ab}^{\beta\beta'}(\beta - \beta')L_b(\beta')R_a(\beta).$$  \hspace{1cm} (13)

As we discussed before, this scattering breaks the scale invariance thus spoiling the $SU(2) \times SU(2)$ current algebra symmetry. However action (1) is invariant under the global $SU(2)_L \times SU(2)_R$ isotopic symmetry at all distances. The only form of $U_{ab}^{\beta\beta'}$ preserving this symmetry is:

$$U_{ab}^{\beta\beta'}(\beta) = U_{RL}(\beta)\delta_a^b\delta_a^b.$$  \hspace{1cm} (14)

The factorization constraint is trivially met for this choice. For massless particles there is a combined unitarity-crossing restriction, \[24\]:

$$U_{RL}(\beta + i\pi)U_{RL}(\beta) = 1.$$  \hspace{1cm} (15)

The simplest non-trivial solution proposed in ref.\[3\] is\[2\]:

$$U_{RL}(\beta) = \frac{1}{U_{LR}(-\beta)} = \tanh \left( \frac{\beta}{2} - \frac{i\pi}{4} \right).$$  \hspace{1cm} (16)

\[2\]This solution has also the virtue of yielding a kernel of the form $-i\partial/\partial\beta\log U_{RL}(\beta) = 1/cosh\beta$, which coincides with that of the TBA equations for the magnons, \[3\], \[22\], \[10\].
It is worth noting that both amplitudes (12) and (16) have no poles on the physical sheet. Also, we see from the soft scattering ($\beta \to -\infty$) in eq.(16) that the fields behave as fermions. And since $S_{aa}^a(0) = 1$, we conclude that we have a selection rule preventing any two particles of the same type to be in exactly the same quantum state.

2 Boundary interactions and the Kondo problem

Let us assume that our system is now restricted to the truncated line $x < 0$ with a boundary located at $x = 0$. We can interpret the boundary as an impenetrable particle sitting at the origin. Furthermore we consider that the boundary conditions satisfy the following prerequisites:

1. the integrability is preserved,
2. the boundary does not introduce any additional mass scale,
3. at the IR fixed point conformal invariance is conserved.

The last statement can be made more precise. If $|B>$ is the boundary state in the Hilbert space of the conformal field theory, then [11], [12]:

$$(J^n_a + \bar{J}^a_n)|B> = 0,$$

where $J^n_a$ are the generators of the current algebra. If the Virasoro modes, $L_n$, are constructed according to Sugawara’s prescription, then:

$$(L_n - \bar{L}_n)|B> = 0.$$

Cardy [13] [14] argued that these boundary states are in one-to-one correspondence with the set of conformal blocks. This has the immediate consequence that, since the conformal towers are labeled by the isospin $l = 0, 1/2, \cdots, k/2$ [15] of the corresponding primary operator, so must the boundary states. The modular properties of the Kac-Moody characters $\chi^{(k)}_l$ are encoded in the matrix $S^{(k)}$,

$$\chi^{(k)}_l(\tau) = \sum_{l' = 0}^{k/2} S^{(k)}_{ll'} \chi^{(k)}_{l'}(-1/\tau), \quad (l = 0, 1/2, \cdots, k/2),$$

where:

$$S^{(k)}_{ll'} = \sqrt{\frac{2}{k + 2}} \sin \left\{ \frac{\pi(2l + 1)(2l' + 1)}{k + 2} \right\}.$$

The partition function on an annulus of width $R$ with boundary conditions $(l, p)$ on each side and the points along the periodic direction identified modulo $L$ is given by:

$$Z_{lp}(\tau) = \sum_{l' = 0}^{k/2} N^{kl}_{lp} \chi^{(k)}_l(\tau),$$

where $\tau = L/R$. $N^{kl}_{lp}$ are the structure constants of the fusion rules, which can be obtained from the Verlinde formula [16]:

$$\sum_{i=0}^{k/2} S^{(k)}_{lij} N_{pl} = \frac{S^{(k)}_{lp} S^{(k)}_{lj}}{S^{(k)}_{0lj}},$$
with $j, p, l = 0, 1/2, \ldots, k/2$. All these results hold because the conformal field theory $\hat{SU}(2) \times \hat{SU}(2)$ is diagonal in the sense that the partition function of the theory on the torus takes the diagonal form

$$Z_{\text{torus}}(\tau) = \sum_{l=0}^{k/2} \chi_l^{(k)}(\tau) \chi_l^{(k)}(\tau).$$

As $L$ tends to infinity the dominant contribution to the partition function (18) is that of the state with lowest energy given by:

$$E_{lp}^0 = \frac{\pi}{R} \left( L_0^{(0)} - \frac{c}{24} \right),$$

where $L_0^{(0)}$ is the lowest eigenvalue of $L_0$ in the representations $l'$ for which $N_{lp}' \neq 0$. In the case $k = 1$, we have two integrable representations, corresponding to isospins $l = 0$ and $l = 1/2$. They have been identified with the identity operator and the fundamental field $g$ in the WZW action, respectively [23]. The modular matrix $S^{(1)}$ is given by, [33]:

$$S^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$  

From (18), (19) we get:

$$\begin{cases} Z_{00}(\tau) = Z_{\frac{1}{2}+\frac{1}{2}}(\tau) = \chi_0^{(1)}(\tau), \\ Z_{0\frac{1}{2}}(\tau) = Z_{\frac{1}{2}0}(\tau) = \chi^{(1)}_{\frac{1}{2}}(\tau). \end{cases}$$

So we look for two distinct boundary conditions that lead to the two above in the IR limit. The $\hat{SU}(2) \times \hat{SU}(2)$ conformal field theory with a boundary can also be seen as the IR limit of another integrable model where this time the RG flow is controlled by the boundary interaction. This situation is connected to the Kondo problem.

The Hamiltonian of the multi-channel Kondo problem is that of a $k$-tuple of free massless fermions $\psi_i (i = 1, \ldots, k)$ antiferromagnetically coupled to a fixed impurity of spin $S$ in 3 dimensional Euclidean space. The interaction term is of the form $\lambda \delta(x) \sum_{i=1}^{k} S_j \psi_i^\dagger \sigma^j \psi_i$ where $\sigma^j$ are the Pauli matrices. A variety of arguments show that this problem can be described by a (1+1)-dimensional theory on the half line with the impurity sitting at the boundary. Moreover the RG flow interpolates between an unstable UV fixed point where the impurity is decoupled ($\lambda = 0$) and a strongly coupled IR one where the spin of the impurity is ‘screened’ ($\lambda = 2/(k+2)$). As before the crossover introduces a scale $T_K$ called the Kondo temperature. In the bulk the spectrum of particles is the same as for the $PCM_k$. However due to the bulk scale invariance, the R-L scattering must be trivial. For $k = 1$, the L-L and R-R scattering are given by (9), (11) and (12). Actually in this description we are only considering the $SU(2)_k$ spin symmetry in $SU(2)_k \otimes SU(2)_k \otimes U(1)$ and discarding the additional charge ($U(1)$) and “flavour” ($SU(k)_2$) symmetries. This is because the impurity only couples to the spin degrees of freedom. The particles are again $SU(2)$ doublets with a kink structure.

Several cases have to be distinguished, [18], [7]. In the underscreened case ($k < 2S$) one electron from each species binds to the impurity effectively reducing its spin to $q \equiv 3$This is a consequence of $SU(2)$ being a simply-connected group (cf.[15])

4For a review see ref.[17] and references therein.
$S-k/2$. The impurity particle is thus a member of a $(2q+1)$-dimensional $SU(2)$ multiplet. In the exactly screened case ($k = 2S$) the electrons completely screen the impurity and it behaves like a $SU(2)$ singlet. Finally, in the overscreened case ($k > 2S$) the boundary particle develops a kink structure.

For $k = 1$, the overscreened case is not allowed, as this would imply $S = 0$, which means that there is no coupling to the impurity. The exactly screened case corresponds to $q = 0$ and hence the impurity behaves effectively as a $SU(2)$ singlet. In this case the reflection amplitude was found to be, \[^{18}\] :

$$S_{RB} = \frac{\tanh \left( \frac{\beta}{2} - \frac{i\pi}{4} \right)}{\beta}.$$  \(^{(24)}\)

The underscreened case arises when $S > 1/2$. If $S = 1$, then $q = 1/2$ and the impurity is a $SU(2)$ doublet. There is also the possibility $S > 1$, which we shall not consider here. In the underscreened case $S = 1$ the boundary has an isotopic spin index associated with it. The reflection matrix is given by eq.(9)\[^{5}\].

Let us emphasize that our problem is not the Kondo problem. Nevertheless, when we derive the boundary consistency equations, in the limit when the bulk theory becomes scale invariant, the scattering amplitudes of the Kondo problem should solve these equations. This will be an important consistency check. This program is depicted in fig.1.

With these considerations in mind let us first admit that the boundary impurity has effectively no spin. This would correspond to the exactly screened case in the Kondo problem. We shall call this “fixed” boundary condition.

The reflection matrix $R^b_a$ is defined by \[^{24}\] :

$$R^b_a(\beta) = \sum_{b=\pm} R^b_a(\beta) L^b(-\beta) B.$$  \(^{(25)}\)

$B$ is formally an operator which, when acting on the vacuum, creates a boundary state $|B\rangle$, i.e.

$$|B\rangle = B|0\rangle.$$  

The fact that the boundary impurity is an $SU(2)$ singlet implies the following diagonal form:  

$$R^b_a(\beta) = \delta^b_a R_{RL}(\beta).$$  \(^{(26)}\)
The diagonal matrix (26) automatically satisfies the boundary Yang-Baxter equation irrespective of $U_{RL}$. Let us now consider the boundary crossing unitarity condition:

$$K^{ab}(\beta) = \sum_{\tilde{c},d=\pm} K^{\tilde{c}d}(\beta) U_{\tilde{c}d}^{ab}(2\beta), \quad (27)$$

where $K^{ab}(\beta) = R^b_\beta(i\pi/2 - \beta)$. From eqs.(14) and (26) we get:

$$R_{LR}(-\beta) = -\frac{R_{RL}(i\pi + \beta)}{U_{RL}(2\beta)}. \quad (28)$$

Next we consider the boundary unitarity condition:

$$\sum_{\tilde{c}=\pm} R^\tilde{c}_a(\beta) R^\tilde{c}_b(-\beta) = \delta^b_a. \quad (29)$$

Using (26), we get:

$$R_{RL}(\beta) R_{LR}(-\beta) = 1. \quad (30)$$

Note that we have been using the quantity $R_{LR}$ which would correspond to a left-moving particle being reflected into a right-moving one. This does not seem to make much sense given that our system is defined on the half line $(-\infty, 0]$. However, as we will see, it will prove useful to ignore this and see it only as a formal tool to derive consistency equations for the boundary reflection factors.

Substituting eq. (28) into (30), we get:

$$R_{RL}(\beta) R_{RL}(i\pi + \beta) = -U_{RL}(2\beta). \quad (31)$$

Notice that we cannot take $R_{RL} = R_{LR}$ as can readily be verified if we substitute $\beta = -i\pi/2$ in eqs.(28) and (30). As we discussed before if we take the bulk R-L scattering to be trivial ($U_{RL} \rightarrow -\infty$), then the exactly screened amplitude (24) of the Kondo problem is a solution of eq.(31). Let us now consider (31) with nontrivial R-L scattering. This has the minimal solution:

$$R_{RL}(\beta) = \exp\left(-\frac{i\pi}{4}\right) \left\{ \frac{\sinh\left(\frac{\beta}{2} - \frac{i\pi}{4}\right)}{\sinh\left(\frac{\beta}{2} + \frac{i\pi}{4}\right)} \right\}, \quad (32)$$

with $R_{LR}(-\beta) = [R_{RL}(\beta)]^*$. This amplitude has no poles on the physical sheet. The only pole lies in the second sheet $-\pi < \text{Im}\beta < 0$ at $\beta = -i\pi/4$ and is associated with the mass scale of the bulk theory.

Let us now consider the second situation when the boundary has an effective spin $q = 1/2$. This will be denoted as “free” boundary condition. We then have:

$$R_a(\beta) B_b = \sum_{\tilde{c},d=\pm} R^\tilde{c}_a(\beta) L_{\tilde{c}}(-\beta) B_d, \quad (33)$$

where $B_d$ creates a boundary state with isotopic spin $d = \pm$:

$$|B \rangle_d = B_d |0 \rangle .$$

The boundary Yang-Baxter equation has to be slightly modified to incorporate this additional structure, [24]:

$$R_{bc'}^{\tilde{b}'}(\beta_2) U_{ab'}^{\tilde{b}'}(\beta_1 + \beta_2) R_{a'c'}^{\tilde{d}}(\beta_1) S_{\tilde{b}'a'}^{\tilde{b}}(\beta_1 - \beta_2) =$$

$$= S_{ab}^{a'b'}(\beta_1 - \beta_2) R_{a'c'}^{a'b'}(\beta_1 + \beta_2) U_{b'd'}^{a'b'}(\beta_1 + \beta_2) R_{b'd'}^{\tilde{b}'}(\beta_2). \quad (34)$$
Substituting (14), we get:

\[
R^{\beta'_c}(\beta_2)R^{\delta'd}_{ac}(\beta_1)S^{\hat{\beta}_a}_{\delta'b}(\beta_1 - \beta_2) = S^{\beta'\beta}_{ab}(\beta_1 - \beta_2)R^{\delta'c}_{\alpha_c}(\beta_1)R^{\delta d}_{\gamma c}(\beta_2).
\] (35)

We see that the bulk R-L scattering decouples as before. We will see that it only plays a role in the boundary crossing-unitarity condition. This, of course, is a consequence of the R-L scattering being diagonal. Substituting (9), this yields:

\[
\left[ R^{\delta d}_{ac}(\beta_1)R^{\delta c}_{bc}(\beta_2) - R^{\delta'c}_{ac}(\beta_1)R^{\delta d}_{bc}(\beta_2) \right] \sigma_T(\beta_1 - \beta_2) =
\]

\[
= \left[ R^{\delta'c}_{bc}(\beta_1)R^{\delta d}_{ac}(\beta_2) - R^{\delta d}_{ac}(\beta_1)R^{\delta'c}_{bc}(\beta_2) \right] \sigma_R(\beta_1 - \beta_2).
\] (36)

Since the total isospin has to be conserved, we assume the following \( SU(2) \) symmetric combination:

\[
R^{\delta d}_{ab}(\beta) = \delta_a^c \delta_b^d f_{RL}(\beta) + \delta_a^d \delta_b^c g_{RL}(\beta),
\] (37)

We then get, using (11):

\[
\frac{i}{\pi}(\beta_1 - \beta_2)g_{RL}(\beta_1)g_{RL}(\beta_2) = f_{RL}(\beta_1)g_{RL}(\beta_2) - g_{RL}(\beta_1)f_{RL}(\beta_2),
\] (38)

which implies:

\[
f_{RL}(\beta) = \frac{i}{\pi}g_{RL}(\beta).
\] (39)

The boundary unitarity condition,

\[
\sum_{\alpha', \beta'} R^{\alpha' \beta'}_{ab}(\beta)R^{\delta d}_{ab'}(\beta) = \delta_a^c \delta_b^d,
\] (40)

reads:

\[
f_{RL}(\beta)f_{LR}(-\beta) + g_{RL}(\beta)g_{LR}(-\beta) = 1,
\] (41)

\[
f_{RL}(\beta)g_{LR}(-\beta) + g_{RL}(\beta)f_{LR}(-\beta) = 0.
\]

Finally, we consider the boundary crossing-unitarity condition. We assume the following generalization:

\[
R^{\delta d}_{ac}(\beta) = R^{\delta d}_{bc}(i\pi - \beta)U_{RL}(i\pi - 2\beta),
\] (42)

where we used the fact that the R-L scattering is diagonal. Following Berg et al. [21] (cf. section 1), we define the following crossing symmetric matrix:

\[
G^{\alpha' \beta'}_{ac}(\beta) \equiv R^{\alpha' \beta'}_{bc}(\beta) = \delta_a^b \delta_c^d u_{RL}(\beta) + \delta_{bc} \delta_a^d v_{RL}(\beta).
\] (43)

As before the matrix \( H^{\alpha' \beta'}_{ac}(\beta) \equiv R^{\alpha' \beta'}_{bc}(\beta) \) vanishes, because it is associated with the exchange of momenta, which is not possible since the boundary particle has to stay at rest after the interaction. In terms of \( u \) and \( v \), (42) reads:

\[
f_{LR}(\beta) = -U_{RL}(2\beta)u_{RL}(i\pi - \beta), \quad g_{LR}(\beta) = -U_{RL}(2\beta)v_{RL}(i\pi - \beta).
\] (44)

The unitarity conditions,

\[
\sum_{\alpha', \beta'} G^{\alpha' \beta'}_{ac}(\beta)G^{\alpha' \beta'}_{d'c'}(-\beta) = \delta_a^d \delta_c^{d'},
\] (45)
yield the following equations for $u$ and $v$:

$$u_{RL}(-\beta) v_{LR}(-\beta) + v_{RL}(\beta) u_{LR}(-\beta) + 2v_{RL}(\beta) v_{LR}(-\beta) = 0,$$

$$u_{RL}(\beta) u_{LR}(-\beta) = 1.$$  \hspace{1cm} (46)

Notice that if we choose,

$$f_{RL}(\beta) = -u_{RL}(\beta) - v_{RL}(\beta), \quad g_{RL}(\beta) = v_{RL}(\beta),$$  \hspace{1cm} (47)

then eq.(38) is automatically satisfied. The boundary Yang-Baxter equation for antiparticles yields:

$$u_{RL}(\beta) = \frac{i}{\pi} \beta v_{RL}(\beta),$$  \hspace{1cm} (48)

which is perfectly compatible with eq.(39) for the choice (47). Solving this whole system is tantamount to finding $g_{RL}, g_{LR}$ such that:

$$\begin{aligned}
g_{RL}(\beta) g_{LR}(-\beta) &= \frac{\pi^2}{\pi^2 + \beta^2}, \\
g_{LR}(\beta) &= -U_{RL}(2\beta) g_{RL}(i\pi - \beta).
\end{aligned}$$  \hspace{1cm} (49)

Suppose that the R-L scattering becomes trivial $U_{RL} \to -1$. In that case it is perfectly acceptable to take $g_{RL} = g_{LR} = g$:

$$g(\beta) g(-\beta) = \frac{\pi^2}{\pi^2 + \beta^2}; \quad g(i\pi - \beta) = g(\beta).$$

This system is solved by $g(\beta) = \sigma_R(\beta)$, in agreement with Fendley.

The system (49) with $U_{RL}$ given by eq.(16) is not consistent for $g_{RL} = g_{LR} = g$. Again this is checked immediately for $\beta = i\pi/2$. It has the minimal solution,

$$g_{RL}(\beta) = iR_{RL}(\beta) \sigma_R(\beta),$$  \hspace{1cm} (50)

where $R_{LR}$ is given by eq.(32) and $g_{RL}(-\beta) = [g_{RL}(\beta)]^*$.  

3 Conclusions

Let us restate our results. The $\widehat{SU}(2)_1$ WZW theory was defined in an axiomatic fashion by introducing the current algebra and constructing the conformal symmetry according to Sugawara’s procedure. Cardy’s approach shows that if we impose the conservation of these two symmetries in the presence of a boundary there will be two permissible boundary conditions which we denoted as fixed and free. There are two RG trajectories that terminate at this theory in the IR limit. One of them - the Kondo theory - suggests the interpretation of the two boundary conditions as the exactly screened and underscreened situations, respectively. The factorized scattering is well known for this theory. The second trajectory represents the principal chiral model with scale invariant boundary conditions. The symmetries of the model, the IR limiting WZW theory and comparison with the Kondo model allowed us to construct the corresponding reflection amplitudes, (32), (50). The set of equations thus obtained are also valid if we introduce an additional boundary perturbation. However it is not clear whether the integrability is still preserved.
We will present some checks for these results elsewhere. So far, we have computed the ground state energy ($E$-function) for the system defined on an annulus with fixed boundary conditions on both ends, by the technique of boundary TBA. We had to use the fact that the particles obey an exclusion principle and that the backward scattering $B_{ab}^{dc}(\beta)$ in eq.(7) is not allowed. In the IR limit the $E$-function selects the scaling dimension $\Delta = 0$ in agreement with (21) and (23). The $g$-function, on the other hand, yields different boundary entropies at the two extremes of the RG trajectory.

The free boundary condition entails a non-diagonal reflection matrix, thus rendering the computations too complicated. The best we can hope for is to conjecture the set of TBA equations.

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$$\text{IR} = \text{WZW}$$

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$$\text{M}$$

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$$\text{U}_{ RL \rightarrow -1}$$

$$\text{PCM}$$