Reduced density matrix of permutational invariant many-body systems

Mario Salerno\textsuperscript{1} and Vladislav Popkov\textsuperscript{2}
\textsuperscript{1} Dipartimento di Fisica “E.R. Caianiello” and Istituto Nazionale di Fisica Nucleare (INFN), Gruppo Collegato di Salerno, Università di Salerno, via Ponte don Melillo, I-84084 Fisciano (SA), Italy
\textsuperscript{2}Dipartimento di Fisica “E.R. Caianiello” and Consorzio Nazionale per le Scienze Fisiche della Materia (CNISM), Università di Salerno, via Ponte don Melillo, I-84084 Fisciano (SA), Italy

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We consider density matrices which are sums of projectors on states spanning irreducible representations of the permutation group of \(L\) sites (eigenstates of permutational invariant quantum system with \(L\) sites) and construct reduced density matrix \(\rho_n\) for blocks of size \(n < L\) by tracing out \(L - n\) sites, viewed as environment. Explicit analytic expressions of the elements of \(\rho_n\) are given in the natural basis and the corresponding spectrum is derived. Results apply to all quantum many-body systems with permutational symmetry for which the mean field theory is exact.

\section*{I. INTRODUCTION}

Reduced density matrices contain complete information about open quantum i.e. quantum systems in contact with an environment such as a thermal bath or a larger system of which the original system constitute a part (subsystem). In many cases one is interested in the spectrum of the reduced density matrix (RDM) because it reveals intrinsic properties (sometimes universal) of the subsystem. By definition, the spectrum of the RDM is real and nonnegative with all eigenvalues summing up to 1. The relative importance of a state of the subsystem can then be measured by the weight the corresponding eigenvalue has in the RDM spectrum. Thus, for instance, the fact that the eigenvalues \(\lambda_i\) of the RDM for a one dimensional quantum interacting subsystems decay exponentially with \(i\) implies that the properties of the subsystems are determined by only a few states. This property is crucial for the success of the density-matrix renormalization group (DMRG) method \cite{white} in one dimension. In two dimensions this property is lost \cite{moessner} and the DMRG method fails.

For a subsystem consisting of \(n\) sites (or \(n\) q-bits) the RDM is of rank \(2^n\) so that for large \(n\) the calculation of the spectrum becomes a problem of exponential difficulty. While the spectrum of the full RDM for subsystems with a small number of sites (e.g. \(n \leq 6\)) has been calculated \cite{salerno}, the full RDM for arbitrary \(n\), to our knowledge, is exactly known only for the very special case of systems of free fermions (see e.g. \cite{pollet}).

The aim of the present paper is to analytically calculate the elements of the RDM of permutationally invariant quantum systems of arbitrary size \(L\) for arbitrary permutation symmetry of the state of the system (labeled by an integer number \(0 < r < L/2\)) and for arbitrary sizes \(n\) (number of q-bits) of the subsystem. We remark that the invariance under the permutational group physically implies that the interactions among sites have infinite range. From this point of view our results may apply to all quantum systems for which mean field theory becomes exact. As an example we consider a system of Heisenberg spins 1/2 on a full graph consisting of \(L\) sites, with fixed value of magnetization \(S_z = L/2 - N\). For this system we calculate the RDM for a subsystem of arbitrary \(n \geq 1\) sites for arbitrary \(L, N\).

The plan of the paper is the following. The formulation of the main problem and the basic definitions are given Section 2. In Section 3 we discuss general properties of the elements of the RDM while in Section 4 we use symmetry properties of the system to decompose the RDM into block diagonal form. The main results of the paper are presented in Sect. 5 in the form of a theorem giving the analytical expressions of the RDM elements for arbitrary \(L, n, r\). For simplicity, we provide a proof of this theorem only in the thermodynamic limit. In Sec. 6 we characterize the spectrum of the RDM and discuss some of its main properties. Finally, in the last section we briefly summarize the main results of the paper.

\section*{II. MODEL EQUATION AND MAIN DEFINITIONS}

Consider a permutationally invariant system of \(L\) spins 1/2 on a complete graph with fixed total magnetization \(S_z = L/2 - N\) and described by the Hamiltonian

\begin{equation}
H = -\frac{J}{2L} \left( S^z - \frac{L}{2} \left( \frac{L}{2} + 1 \right) \right) + hS_z \quad (1)
\end{equation}

Here \(S = (S_x, S_y, S_z)\), \(S_\alpha = \frac{1}{2}\sum_{i=1}^L \sigma^\alpha_i\), with \(\sigma^\alpha_i\) Pauli matrices acting on the factorized \(L \otimes C_2^2\) space. This Hamiltonian is invariant under the action of the symmetric group \(S_L\) and conserves the total spin polarization \(S_z\), \([H, S_z] = 0\). A complete set of eigenstates of \(H\) are states \(|\Psi_{L,N,r}\rangle\) associated to filled Young Tableau (YT) of type

\cite{E-mail: salerno@sa.infn.it}

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\[ \{L - r, r\}_{(N)} \text{ (see \cite{7}),} \]
\[
H|\Psi_{L,N,r}\rangle = E_{L,N,r}|\Psi_{L,N,r}\rangle, \tag{2}
\]
\[
E_{L,N,r} = \frac{1}{2} \left( \frac{Jr}{L} (L - r + 1) + h(L - 2N) \right), \tag{3}
\]
\[
S_z |\Psi_{L,N,r}\rangle = \left( \frac{L}{2} - N \right) |\Psi_{L,N,r}\rangle. \tag{4}
\]

Here \( N = 0, 1, ..., L \) determines possible values of the spin polarization and \( r \) takes values \( r = 0, 1, ..., \max(N, L - N) \) (the explicit form of the state \( |\Psi_{L,N,r}\rangle \) is given below in Eq. \( 23 \)). The degeneracies of the eigenvalues \( E_{L,N,r} \) are given by the dimension of the corresponding \( \Gamma \)\( T \)s,
\[
\deg_{L,r} = \binom{L}{r} - \binom{L}{r-1}. \tag{5}
\]

**Definition 1** Consider a set of vectors \( |\Psi_u\rangle, \ u = 1, ..., \deg_{L,r} \), forming an orthonormal basis in the eigenspace of \( H \) with eigenvalue \( E_{L,N,r} \). We define the density matrix of the whole system as
\[
\sigma_{L,N,r} = \frac{1}{\deg_{L,r}} \sum_{u=1}^{\deg_{L,r}} |\Psi_u\rangle\langle\Psi_u|. \tag{6}
\]

\( \sigma_{L,N,r} \) possesses the following properties:

i) The matrix \( \sigma_{L,N,r} \) has eigenvalues \( \lambda_1 = \lambda_2 = ... = \lambda_{\deg_{L,r}} = \frac{1}{\deg_{L,r}} \left( \frac{1}{\deg_{L,r}} \right)^{-1} \), with remaining \( 2^L - \deg_{L,r} \) eigenvalues all equal to zero. This follows from the fact that each vector \( |\Psi_u\rangle \) is an eigenvector of \( \sigma_{L,N,r} \) with eigenvalue \( \frac{1}{\deg_{L,r}} \). Since the spectrum of \( \sigma_{L,N,r} \) is real and nonnegative with all eigenvalues summing up to 1, the remaining \( 2^L - \deg_{L,r} \) eigenvalues must vanish.

ii) Matrix \( \sigma_{L,N,r} \) satisfies: \( (\sigma_{L,N,r})^2 = \frac{1}{\deg_{L,r}} \sigma_{L,N,r} \). This follows from the definition \( 19 \) and the orthonormality condition \( \langle \Psi_u | \Psi_u \rangle = \delta_{uu} \).

iii) Introduce the operator \( P_{ij} \), permuting subspaces \( i \) and \( j \) of the Hilbert space \( \prod_{L} C_2 \) on which the matrix \( \sigma_{L,N,r} \) acts. Then \( \sigma, P_{ij} = 0 \) for any \( i, j \).

Proof.

Let us consider
\[
P_{ij} \sigma_{L,N,r} P_{ij} = \frac{1}{\deg_{L,r}} \sum_{u=1}^{\deg_{L,r}} P_{ij} |\Psi_u\rangle\langle\Psi_u| P_{ij} = \frac{1}{\deg_{L,r}} \sum_{u=1}^{\deg_{L,r}} |\Psi'_u\rangle\langle\Psi'_u|. \tag{7}
\]

The vectors \( |\Psi'_u\rangle = P_{ij} |\Psi_u\rangle \) form an orthonormal basis. Indeed, \( \langle \Psi'_u | \Psi'_v \rangle = \langle \Psi_u | P_{ij}^\dagger P_{ij} | \Psi_v \rangle = \langle \Psi_u | \Psi_v \rangle = \delta_{uv} \), because \( P_{ij}^\dagger = P_{ij} \), and \( (P_{ij})^2 = I \). Now, the sum \( \sum_{u=1}^{\deg_{L,r}} |\Psi_u\rangle\langle\Psi_u| = I_{\deg_{L,r}} \) is a unity operator in a subspace of dimension \( \deg_{L,r} \), and therefore it does not depend on the choice of the basis. Note that vector \( |\Psi'_u\rangle \) belongs to the same subspace as \( |\Psi_u\rangle \), because permutation \( P_{ij} \) only results in different enumeration. Consequently, \( P_{ij} \rho P_{ij} = \rho \), or \( [\rho, P_{ij}] = 0 \). \tag{8}

The latter property implies that in Eq. \( 6 \) the sum over the orthogonalized set of basis vector in \( 6 \) can be replaced by the symmetrization of the density matrix directly, namely \( \sigma_{L,N,r} = \frac{1}{\deg_{L,r}} \sum_{\rho} |\Psi_{12...L}\rangle\langle\Psi_{12...L}| \), where the sum is over all \( L! \) permutations of indexes 1, 2, ..., \( L \), and \( |\Psi_{12...L}\rangle \) is some unit eigenvector of \( H \) with eigenvalue \( E_{L,N,r} \). In particular, it is convenient to choose \( |\Psi_{12...L}\rangle \equiv |\Psi_{L,N,r}\rangle \),
\[
\sigma_{L,N,r} = \frac{1}{\deg_{L,r}} \sum_{\rho} |\Psi_{L,N,r}\rangle\langle\Psi_{L,N,r}|. \tag{9}
\]

It is evident that such a sum is invariant with respect to permutations and that \( \sigma_{L,N,r} \) is properly normalized: \( Tr \sigma_{L,N,r} = 1 \).

**Definition 2** The Reduced Density Matrix (RDM) of a subsystem of \( n \) sites is defined by tracing out \( L - n \) degrees of freedom from the density matrix of the whole system:
\[
\rho(n) = Tr_{L-n} \sigma_{L,N,r}. \tag{10}
\]

Due to the properties \( 3 \) and \( 10 \), \( \rho(n) \) does not depend on the particular choice of the \( n \) sites, and satisfies the property \( 1 \) in its subspace (we omit the explicit dependence of \( \rho(n) \) on \( L, N, r \) for brevity of notations).

### III. PROPERTIES OF THE RDM

The RDM can be calculated in the natural basis by using its definition in terms of observables: \( \langle \hat{f} \rangle = Tr(\rho(n) \hat{f}) \) where \( \hat{f} \) is a physical operator acting on the Hilbert space of the \( 2^n \times 2^n \) subsystem. The knowledge of the full set of observables determines the RDM uniquely. Indeed, if we introduce the natural basis in the Hilbert space of the subsystem, \( \prod_{k=1}^{n} C_2 \), the elements of the RDM in this basis are
\[
\rho_{ij12...jn}^{i_1j_12...i_n} = \langle \hat{e}_{i_1j_12...i_n} | \hat{e}_{ij12...jn} \rangle = Tr \left( \rho(n) \hat{e}_{i_1j_12...i_n} \right), \tag{11}
\]
with \( \hat{e}_{i_1j_12...i_n} = \prod_{k=1}^{n} \hat{e}_{i_kj_k} \) and \( \hat{e}_{i_kj_k} \) a \( 2 \times 2 \) matrix with elements \( \hat{e}_{i_kj_k} = \delta_{i_kj_k} \). The matrix \( \hat{e}_{i_1j_12...i_n} \) has only one nonzero element, equal to 1, at the crossing of the row \( 2^{n-1} + 2^{n-2} + ... + i_n + 1 \) and the column \( 2^{n-1} + 2^{n-2} + ... + j_n + 1 \) (all indices \( i, j \) take binary values \( i_k = 0, 1 \) and \( j_k = 0, 1 \)). To determine all the RDM elements one must find a complete set of observables and compute the averages \( \langle \hat{e}_{i_1j_12...i_n} \rangle \). Note that a generic property of the RDM elements, which follows directly from \( 3 \), is that any permutation between pairs of indices \( (i_m, j_m) \) and \( (i_k, j_k) \) does not change its value, e.g.
\[
\rho_{ij12...jn}^{i_1j_12...i_n} = \rho_{ij12...jn}^{i_1j_12...i_n} = \rho_{ij12...jn}^{i_1j_12...i_n} = ... = \rho_{ij12...jn}^{i_1j_12...i_n} \tag{12}
\]
Another property of the RDM follows from the $S_z$ invariance

$$\rho_{j_1,i_2,...,i_n}^{i_1,j_2,...,j_n} = 0, \quad \text{if } i_1 + i_2 + ... + i_n \neq j_1 + j_2 + ... + j_n.$$  

Thus, for instance, the RDM for $n = 2$ has only three different nonzero elements, $\rho_{00}^{00}, \rho_{00}^{01}, \rho_{00}^{11}$. It is convenient to introduce the operators

$$e_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \equiv \sigma^-, \quad e_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \equiv \sigma^+,$$

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \equiv \hat{h}, \quad e_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \equiv \hat{h}.$$

If we represent a site spin up with the vector $(0,1)$ and a site spin down with the vector $(1,0)$ then $\hat{p}_k$ and $\hat{h}_k$ are spin up and spin down number operators on site $k$, while $\sigma^-, \sigma^+$ represent spin lowering and rising operators, respectively. Thus, for instance, the observable $\langle \hat{p}_1 \hat{p}_2 \hat{h}_3 \hat{h}_4...\hat{h}_n \rangle = \rho_{0011}^{0011}$ gives the probability to find spins down at sites $3, 4, ... n$, and spins up at sites $1, 2$, while the observable $\langle \sigma^+ \sigma^+ \sigma^- \hat{h}_5...\hat{h}_n \rangle = \rho_{10011}^{0011}$ gives the probability to find spins down at sites $5, 6, ... n$, spin lowering at sites $3, 4$ and spin rising at sites $1, 2$. Note that the latter operator conserves the total spin polarization since the number of lowering and rising operators is the same. Also note that the correlation functions with a non conserved polarization vanish, e.g.

$$\langle \sigma^+ \sigma^+ \sigma^- \hat{h}_5...\hat{h}_n \rangle = \rho_{10011}^{0011} = 0,$$

in accordance with (13).

IV. BLOCK DIAGONAL FORM OF THE RDM

One can take advantage of the $S_z$ invariance (e.g. Eq. (13)) to block diagonalize the RDM into independent blocks $B_k$ of fixed polarization $k = i_1 + i_2 + ... + i_n = j_1 + j_2 + ... + j_n$, (here $k$ gives the number of spin up present in the subsystem). Note (see Fig. 1) that the block diagonal form in the natural basis becomes evident after a number of permutations of rows and columns of the RDM have been performed. Also notice that the $n + 1$ blocks correspond to the values $s_z = (n - 2k)/2, k = 0, 1, ..., n$ the subsystem polarization can assume. The dimension of a block $B_k$ coincides with the number of possible configurations that $k$ spin up can assume on $n$ sites, e.g. $\dim B_k = \binom{n}{k}$. One can easily check that the sum of the dimensions of all blocks gives the full RDM dimension, i.e. $\sum_k \dim B_k = 2^n.$

Blocks $B_{i_1} \otimes B_{j_2} \otimes B_{j_3} \otimes ... \otimes B_{j_n}$ consist of elements $e_{j_1} \otimes e_{j_2} \otimes e_{j_3} \otimes ... \otimes e_{j_n}$ of the original matrix, with $\sum_{k=1}^{n} i_p = \sum_{k=1}^{n} j_p = k$ and $i_p = 0, 1, j_p = 0, 1$. In its turn, all elements $e_{j_1} \otimes e_{j_2} \otimes e_{j_3} \otimes ... \otimes e_{j_n}$ of the block $B_k$ can be further block diagonalized according to the irreducible representations of the permutation group $S_n$ of the subsystem (see also section 6 below).

In the natural basis, this block diagonalization is done according to the number of pairs of type $(e_{i_1} \otimes e_{i_2})$ present in the elements. In the following we denote by $G_Z$ the part of the block associated to elements with $Z$ pairs $(e_{i_1} \otimes e_{i_2})$ in it. The sub-block $G_0$ of the block $B_k$ is formed by the elements containing $k$ terms $e_1^2$ and $(n - k)$ terms $e_0^2$ in the product, i.e. $e_1^2 \otimes ... e_1^2 \otimes e_0^2 \otimes ... \otimes e_0^2$ and all permutations. All such elements lie on the diagonal, and vice versa, each diagonal element of $B_k$ belongs to $G_0$. Consequently, the sub-block $G_0$ consists of $(\binom{n}{k})$ elements.

The number of elements, $\deg G_1(k)$, in the sub-block $G_1$ is equal to the number of elements of the type $e_1^0 \otimes e_1^1 \otimes e_1^2 \otimes ... \otimes e_{n-2}^2$, such that $1 + 0 + i_1 + i_2 + ... + i_{n-2} = k$. Using elementary combinatorics we obtain:

$$\deg G_1(k) = \binom{n}{1} \binom{n}{2} \frac{(n - 2)}{k - 1}.$$  

Analogous calculations for arbitrary sub-block $G_Z$ yields

$$\deg G_Z(k) = \binom{n}{Z} \binom{n - 2Z}{n - 2Z} \binom{n - 2Z}{k - Z}.$$  

From the restriction $\sum_{p=1}^{n} i_p = \sum_{p=1}^{n} j_p = k$ we deduce that the block $B_k$ contains non-empty parts $G_0, G_1, ..., G_{\min(k, n - k)}$, leading to the following decomposition:

$$B_k = \bigcup_{Z=0}^{\min(k,n-k)} G_Z.$$  

Indeed, the normalization condition following from (16),
We shall prove Eq. (19) first for the case \( r = 0 \) corresponding to the symmetric ground states in Eq. (20) and then for the case \( r > 0 \) corresponding to excited states.

A. Case \( r = 0 \).

If \( r = 0 \), the Young Tableau \( Y_{L,N,r} \) is nondegenerate and the state of the initial system is pure: \( \rho = |\Psi_{L,N}\rangle\langle \Psi_{L,N}| \) with \( |\Psi_{L,N}\rangle \) the symmetric state

\[
|\Psi_{L,N}\rangle = \left( \begin{array}{c} L \end{array} \right)^{-1/2} \sum_{P} \left( \begin{array}{c} N \end{array} \right)^{-1/2} \sum_{Q} \left( \begin{array}{c} N \end{array} \right)^{-1/2} \sum_{L} \left( \begin{array}{c} N \end{array} \right)^{-1/2} \sum_{N} \left( \begin{array}{c} 1 \end{array} \right)
\]

where the sum is over all possible permutations, and the prefactor takes care of normalization. Since all \( L \) sites are equivalent due to permutational invariance, any choice of \( n \) sites \( i_1, i_2, \ldots, i_n \) within \( L \) sites gives the same RDM, which we denote by \( \rho_{(n)}^{L,N,0} = T_{r=L-n} \rho \). It has been shown in Eq. (10) that \( \rho_{(n)}^{L,N,0} \) takes form

\[
\rho_{(n)}^{L,N,0} = \sum_{k=0}^{n} \left( \begin{array}{c} L-n \end{array} \right)_{k} \left| \Psi_{n,k} \right\rangle \left\langle \Psi_{n,k} \right|.
\]

In the natural basis the matrix elements of RDM are given by the above discussed values of observables. Using Eq. (21), one explicitly computes all RDM elements as

\[
\left( \rho_{(n)}^{L,N,0} \right)_{P,Q} = \left( \rho_{(n)}^{L,N,0} \right)_{ij\ldots m}^{i'j'\ldots m'} = \delta_{i+j+\ldots+m} \left( \begin{array}{c} L-n \end{array} \right)_{w} \left( \begin{array}{c} L \end{array} \right)
\]

with \( w = i + j + \ldots + m \) (the sets \( ij\ldots m \) and \( i'j'\ldots m' \) are binary representation of numbers \( P-1, Q-1 \)). So, all the elements of a given block \( B_k \) are equal. This property does not hold in the general case \( r \geq 0 \). Note that in the thermodynamic limit \( \eta = 1 \), and \( \left( \begin{array}{c} L-n \end{array} \right)_{w} \left( \begin{array}{c} L \end{array} \right) \rightarrow p^{n-k}(1-p)^k \).

B. General case \( r > 0 \).

To calculate the RDM, we shall use the representation \( |\Psi_{L,N,r}\rangle \) for the density matrix of the whole system \( \sigma \), re-writing it in the following form

\[
\rho_{(n)} = T_{r=L-n} \left\{ \frac{1}{L!} \sum_{P} \left| \Psi_{L,N,r}\right\rangle \left\langle \Psi_{L,N,r}\right| \right\}
\]

\[
= T_{r=L-n} \left\{ \frac{1}{n! (L-n)!} \sum_{P_{(n)}} \sum_{r_{(n)}} \sum_{i_1 \neq i_2 \ldots \neq i_n} \left| \Psi_{L,N,r}\right\rangle \left\langle \Psi_{L,N,r}\right| \right\}.
\]

V. ANALYTICAL EXPRESSION FOR RDM ELEMENTS

The main result of the paper is provided by the following theorem.

**Theorem** Elements of RDM \( g_z \), belonging to a subblock \( G_z \) of a block \( B_k \) of the reduced density matrix \( \rho_{(n)} \) are given, for arbitrary \( L, N, r, n, \) by

\[
g_z = \left( \begin{array}{c} L \end{array} \right)^{n} \sum_{m=0}^{Z} (-1)^{m} \left( \begin{array}{c} N-r \end{array} \right)_{Z-m} \left( \begin{array}{c} Z-m \end{array} \right)_{m} \left( \begin{array}{c} N \end{array} \right)^{Z}.
\]

This expression defines completely all elements of the RDM in the natural basis. In practice, to find the element \( P, Q \) of the RDM, \( \left( \rho_{(n)} \right)_{P,Q} \), we have to take the binary representation of numbers \( P-1, Q-1 \) (which gives us the sets \( \{i_p\} \) and \( \{j_q\} \) respectively), find the corresponding number \( Z \), and use Eq. (18).

The proof of this theorem will be given in the thermodynamic limit, \( L \rightarrow \infty, \frac{N}{L} = p, \frac{r}{L} = \mu \), in which the expression simplifies to

\[
g_z = p^{n-k}(1-p)^k \eta^z,
\]

with

\[
\eta = \frac{(p-\mu)(1-p-\mu)}{p(1-p)}.
\]
Note that the $L!$ permutations can be done in three steps: first, choose at random $n$ sites $i_1 \neq i_2 \neq \ldots \neq i_n$ among the $L$ sites. There are $\binom{L}{n}$ such choices. Then, permute the chosen $n$ sites, the total number of such permutations being $n!$. Finally, permute the remaining $L - n$ sites, the total number of such permutations being $(L - n)!$. The latter step (c) under the trace operation is irrelevant because these degrees of freedom will be traced out. The operation permuting $n$ sites commutes with the trace operation since $Tr_{L-n}$ does not touch the respective subset of $n$ sites. Consequently, (24) can be rewritten as

$$\rho(n) = \frac{1}{n!} \sum_{P(n)} Tr_{L-n} \frac{1}{(L/n)!} \sum_{i_1 \neq i_2 \neq \ldots \neq i_n} |\Psi_{L,N,r}\rangle \langle \Psi_{L,N,r}|. \tag{24}$$

The filled Young tableau $\{L - r, r\}_N$ consists of the antisymmetric part containing $2r$ sites and $r$ spin up, and the symmetric part containing the remaining $L - 2r$ sites and $N - r$ spin up, see Fig. 3 (in the following we adopt an equivalent terminology appropriate for non spin systems by identifying spins up with particles and spins down with holes). The wave function $|\Psi_{L,N,r}\rangle$, corresponding to this Young tableau factorizes into symmetric and antisymmetric parts

$$\Psi = |\phi_{12}\rangle \otimes |\phi_{34}\rangle \otimes \ldots \otimes |\phi_{2r-1,2r}\rangle \otimes |\Psi_{L-2r,N-r}\rangle_{2r+1,2r+2\ldots L} \tag{25}$$

with the antisymmetric part consisting of the first $r$ factors of the type

$$|\phi_{12}\rangle = \frac{1}{\sqrt{2}}(|10\rangle_{12} - |01\rangle_{12}) \tag{26}$$

and with the symmetric part, $|\Psi_{L-2r,N-r}\rangle$, given by (21). A general property of factorized states implies that if the global wave function is factorized, $|\Phi\rangle = |\psi\rangle_I |\phi\rangle_{II}$ and out of $n$ sites of the subsystem, $n_1$ sites belong to subset $I$, and the remaining $n_2 = n - n_1$ sites belong to the subset $II$, then the reduced density matrix factorizes as well:

$$\rho(n) = \rho(n_1) \otimes \rho(n_2). \tag{27}$$

To do the averaging, we note that among total number of choices $\binom{L}{n}$ there are (a) $\binom{L-2r}{n_2}$ possibilities to choose $n$ sites inside the symmetric part of the tableau, containing $N - r$ particles, (b) $2r \binom{L-2r}{n - r_2}$ possibilities to choose $n - 1$ sites inside the symmetric part of the tableau and one site in the antisymmetric part (c) $\binom{2r}{r} \binom{L-2r}{n-2r}$ possibilities to choose $n - 2$ sites inside the symmetric part of the tableau and two sites in the antisymmetric part and so on. The contributions given by (a) and (b) to the right hand side of (24) for $\rho_{L,N,r}^{L,n}$ are given, according to (27), by

$$\left( \langle F \rangle_n \rho_{\psi,M,0}^{F,M,0} + 2r \langle F \rangle_{n-1} \rho_{\psi,M,0}^{F,M,0} \otimes \rho_{\psi,M,0}^{F,M,0} \right), \tag{28}$$

with $F = L - 2r$, $M = N - r$ and with $\rho_{\psi,M,0}^{F,M,0} \otimes \rho_{\psi,M,0}^{F,M,0}$ the density matrix corresponding to a single site in the antisymmetric part of the tableau. Brackets $\langle \rangle$ denote the average with respect to permutations of $n$ elements. The contribution due to (c) to (24) splits into two parts since the $\binom{2r}{r}$ possibilities to choose two sites in the antisymmetric part of the tableau consists of $4 \binom{r}{2}$ choices with two sites into different columns and the remaining $r$ choices with both sites belonging to a same column. For the former choice, the corresponding density matrix is $\rho_{\psi,M,0}^{F,M,0} \otimes \rho_{\psi,M,0}^{F,M,0} \otimes \rho_{\psi,M,0}^{F,M,0}$, while for the latter case is given by $\rho_{\psi,M,0}^{F,M,0} \otimes \rho_{\psi,M,0}^{F,M,0}$, with

$$\rho_{\psi,M,0} = \frac{1}{\sqrt{2}}(10-01) \langle 10-01| = \frac{1}{2} \left[ \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{array} \right]. \tag{29}$$

Proceeding in the same manner for arbitrary partitions of $Z$ sites in the antisymmetric part of the tableau and $n - Z$ sites in the symmetric part, we get

$$\rho_{F,M}^{L,N,r}(n) = \left\langle \begin{array}{l}
\sum_{Z=0}^{\min(2r,n)} \langle F \rangle_{n-Z} \rho_{\psi,M,0}^{F,M,0} \sum_{i=0}^{Z/2} \binom{Z}{i} \\
\prod_{i=1}^{Z/2} \rho_{\psi,M,0} \end{array} \right\rangle \otimes \otimes_{Z/2} \left( \begin{array}{c}
Z - 2i \\
Z - 2i
\end{array} \right) \langle Z - 2i | \prod_{i=1}^{Z/2} \otimes \rho_{\psi,M,0} \rangle \tag{30}\right\\
$$

From this the general scheme for the decomposition of the general RDM becomes evident. In the above formula, the products $\prod_i$ with $Q < i$ are discarded. The matrix elements $\rho_{F,M,0}^{F,M,0}$ are given by (22).

For simplicity of presentation, we prove Eq. (19) for the case $Z = k$ and then outline the proof for arbitrary $Z$.

In the thermodynamic limit one can neglect the difference between factors like $4 \binom{r}{2}$ and $\binom{r}{2}$ in Eq. (30). The latter then can be then rewritten in a simpler form as

$$\rho_{F,M,0}^{L,N,r}(n) = \left( \begin{array}{c}
\langle F \rangle_n \\
\binom{r}{2}
\end{array} \right) \rho_{\psi,M,0}^{F,M,0} \otimes \rho_{\psi,M,0}^{F,M,0} \otimes \rho_{\psi,M,0}^{F,M,0} + \ldots \right). \tag{31}$$

Note that one can omit all terms in (30) containing $\rho_{\psi,M,0}^{F,M,0}$ since the respective coefficients correspond to

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{image.png}
\caption{Graphical representation of a filled Young Tableau \{L - r, r\}_N with L = 11, r = 4, N = 6. The 1 inside a box denote a spin up (particle) while the 0 denote a spin down (hole).} \label{fig:young_tableau}
\end{figure}
probabilities of finding two adjacent sites in the asymmetric part of the YT (proportional to $r$), which vanish in the thermodynamic limit, respect to the total number of choices which is of order of $r^2$. A sub-block $G_Z$ of a block $k$ consists of all elements of the matrix $p_{(n)}$ having $Z$ pairs of $e_1^g, e_0^g$ in its tensor representation, like e.g. \((e_1^g \otimes e_0^g) \otimes Z \otimes e_1^g \otimes e_0^g \otimes \ldots \otimes e_1^g \otimes e_0^g\), such that $Z + i_1 + i_2 + \ldots + i_{n-2Z} = k$. The total number of elements $g_Z \subset G_Z$ in $\rho_{(n)}^{L,N,r}$ is equal to the number of distributions of $Z$ objects $e_1^g$, $Z$ objects $e_0^g$, and $(k-Z)$ objects $e_1^g$ on $n$ places, given by
\[
\deg G_Z = \frac{n!}{Z!Z!(n-k-Z)!} (32)
\]
(this is another way of writing (15)). Each term $W$ in the sum (31) after averaging will acquire the factor
\[
\Gamma(W) = \frac{\deg G_Z(W)}{\deg G_Z} (33)
\]
where $\deg G_Z(W)$ is a total number of $g_Z$ elements in the term $W$, provided all of them are equal. For instance, $\deg G_Z(\rho_{(n)}^{F,M,0}) = \deg G_Z$, $\deg G_Z(\rho_{(n-m)}^{F,M,0} \otimes (\rho_{2Z}^Z)^{\otimes m}) = (2Z)^{(n-m)} \sum_{m_1=0}^{m} (m-nZ-m_1)\) (the last formula is only true for $k = Z$, otherwise elements constituting $G_Z(W)$ are not all restricted). Restricting to the case $k = Z$ and denoting $W_m = \rho_{(n-m)}^{F,M,0} \otimes (\rho_{2Z}^Z)^{\otimes m}$, we have
\[
\Gamma(W_m) = \Gamma_m = \frac{(n-m)!}{(2Z)!}. (34)
\]
It is worth to note that the element $g_Z \subset G_Z$ is simply given by
\[
\binom{L}{n} g_Z = \Gamma_0 \binom{F}{n} g_0^{(n,k)} + \Gamma_1 \binom{2r}{n} \binom{F}{n-1} g_0^{(n-1,k)} + \Gamma_2 \binom{2r}{n} \binom{F}{n-2} g_0^{(n-2,k)} + \ldots (35)
\]
with $q = 1-p$ and $g_0^{(n,k)} = \binom{F-n}{k} q^k$, it is the element of a $\rho_{(n)}^{F,M,0}$ corresponding to a block with $k$ particles (the factors $\Gamma_n$ are due to the averaging while the factors $\frac{1}{\Gamma_n}$ come from $\binom{\rho_{2Z}^Z}{\otimes m}$). Restricting to the case $k = Z$, and taking into account
\[
\binom{L}{n} \approx \frac{n!}{(n-m)!} \frac{1-2\mu}{F_m}, \quad \frac{2r}{m} \approx \frac{(2\mu)^m}{m!} L_m, (36)
\]
so that
\[
\binom{F}{n} \frac{2r}{m} \frac{1}{2m} \approx \frac{n}{m} \frac{(2\mu)^m}{m!} \frac{1-2\mu}{F_m}, (36)
\]
we finally obtain, using (35), that
\[
g_Z = \sum_{m=0}^{n-2Z} \mu^m \binom{n-xZ}{m} (p-p)^{n-m-Z} (q-\mu)^Z = \binom{n-xZ}{m} (p-p)^{n-m} + \binom{n-xZ}{m} \sum_{m=0}^{n-2Z} \frac{\mu^m}{m!} (n-2Z) (37)
\]
\[
= (p-p)^{n-Z} (q-\mu)^Z \left( \frac{p}{p-\mu} \right)^{n-2Z}
\]
\[
= p^{n-Z} \eta Z \left( \frac{(p-p)(q-\mu)}{pq} \right)^Z
\]
with $\eta = \frac{g_0}{g}$ the diagonal element in the same block $k = Z$. In the last calculation we used the relation $\binom{\eta}{\frac{m-\eta}{m}} = \binom{\eta}{\frac{m-\eta}{m}}$. This proves formula (15) for the particular case $k = Z$ and arbitrary $n$.

For arbitrary $k, Z$, one proceeds in similar manner as for the case $k = Z$ case. Since the respective calculations are tedious and not particularly illuminating, we omit them and give only the final result:
\[
g_Z = \sum_{m=0}^{n-2Z} \mu^m \min(m,k-Z) \sum_{i=\max(0, Z+k-n+m)}^{\min(m,k-Z)} (p-p)^{n-m-k+i} (q-\mu)^k-i \binom{k-Z}{i} \binom{n-k-Z}{m-i}, (38)
\]
which, after some algebraic manipulation, can be rewritten in the form
\[
g_Z = (p-p)^{n-k} (q-\mu)^k \sum_{j=0}^{n-k-Z} \left( \frac{\mu}{p-p} \right)^j \binom{n-k-Z}{j} \sum_{i=0}^{k-Z} \left( \frac{\mu}{q-\mu} \right)^i \binom{k-Z}{i} = \eta Z p^{n-k} q^k. (39)
\]
This concludes the proof of the theorem in the thermodynamic limit $L \to \infty$. For finite $L$ the expression of (15) can be obtained by solving a recurrence relation obtained from the analytical expressions obtained from (31) for the cases $n = 2, 3, 4$. The correctness of this expression can be checked indirectly through the expression for the eigenvalues (see (15)).

VI. SPECTRAL PROPERTIES OF RDM

Since RDM is block diagonalized with respect to quantum number $k$ into blocks $B_k$, of size $\binom{n}{k} \times \binom{n}{k}$, to diagonalize the complete RDM we can diagonalize each block $B_k$, $k = 0, 1, ..., n$ separately. As mentioned before, the permutational invariance of the subsystem of size $n$ permits to further block diagonalize each block $B_k$ with respect to a quantum number $s$ which labels the
irreducible representations of \( S_n \) (e.g. the YT of type \( (n-s,s) \)) which are compatible with the block polar-
ization. This implies that for a block \( k \) the number \( s \) can take only the values \( s = 0, \ldots \min(k, n-k) \). We refer
to the respective eigenvalues as \( \lambda_{(s)}(n, k) \), omitting
for brevity the explicit dependence on \( L, N, r \). Note that
the eigenvalues \( \lambda_{(s)}(n, k) \) have degeneracies \( \binom{n}{s} - \binom{n}{s-1} \) which are equal to the dimension of the corresponding
\( S_n \) representation (e.g. the dimension of the YT of type
\( (n-s,s) \)) \(^{[11]} \). These properties can be easily checked on
the particular example given in Fig. 2). By diagonaliz-
ing matrices \( B_k \) for small \( n \) one finds that all eigenvalues
\( \lambda_{(s)}(n, k) \) are linear combinations of the matrix elements
\( g_z \) of the form:

\[
\lambda_{(s)}(n, k) = \sum_{Z=0}^{|\min(k,n-k)|} \alpha_Z^{(s)}(n,k) g_z, \tag{40}
\]

with \( \alpha_Z^{(s)} \), integer coefficients, satisfying the following
properties:

\[
\alpha^{(0)}_Z(n,k) = \binom{n}{k}, \tag{41}
\]

\[
\sum_{Z=0}^{\min(k,n-k)} \alpha_Z^{(s)}(n,k) = 0 \quad \text{for } s > 0, \tag{42}
\]

\[
\sum_{Z=0}^{\min(k,n-k)} \alpha_Z^{(s)}(n,k) \left( \frac{Z}{p} \right) = 0, \quad p = 0, 1, \ldots, s - 1, \tag{43}
\]

\[
\sum_{Z=0}^{\min(k,n-k)} \alpha_Z^{(s)}(n,k) \left( \frac{Z}{k} \right) = \alpha_k^{(s)}(n,k) = (-1)^s \frac{n - k - s}{k - s}. \tag{44}
\]

Note that the coefficients \( \alpha \) do not depend on the char-
acteristics of the original state \( L, N \) and \( r \). The dependence
of the RDM eigenvalues on these parameters en-
ters through the elements \( g_Z \) \(^{[18]} \). This implies that the
above properties of \( \alpha \) can be proved using special cases.
E.g. in case when all \( g_Z = g_0 \), we are back to ground
state problem solved in \(^{[10]} \). Each block \( B_k \) has single
nonzero nondegenerate eigenvalue \( \lambda_{(0)}(n, k) = \binom{n}{k} g_0 \),
etailing \(^{[13], [12]} \).

The property \(^{[11]} \) is easily proved in the thermody-
namic limit. In this limit, we have \( \eta = 0 \). Then, all
\( g_Z \equiv g_0 \), for \( Z > 0 \) and all \( B_k \) are diagonal implying
\( \lambda_{(s)}(n, k) = g_0 \) for any \( s \). This gives \(^{[11]} \).

We have no proof for the properties \(^{[11], [11]} \), \(^{[11]} \). Note that
the \(^{[11]} \) is a particular case of \(^{[44]} \) for \( p = 0 \).

Given the exact form of the coefficients \( \alpha_Z^{(s)}(n,k) \) for
\( Z = 0, k, \) \(^{[11]} \) and \(^{[15]} \), the remaining coefficients
are obtained recursively from \(^{[44]} \). E.g., the last eigenvalue
of block \( B_k \) with the degeneracy \( \binom{n}{k} - \binom{n}{k-1} \) is obtained
using recursively \(^{[44]} \) for \( s = k \), and \( p = s - 1, s - 2, \ldots 0 \)
which gives

\[
\lambda_{(k)}(n, k) = \sum_{Z=0}^{k} (-1)^Z \binom{k}{Z} g_z, \quad \alpha_{Z}^{(k)}(n, k) = (-1)^Z \binom{k}{Z}. \tag{47}
\]

Note that the last eigenvalue does not depend on \( n \), but
only on \( k \). Other eigenvalues do depend on \( n \) as well. For
the last but one eigenvalue we have

\[
\alpha_{Z}^{(k-1)}(n, k) = (-1)^Z \left( \binom{k-1}{Z} - \binom{n-2k+1}{1} \frac{k-1}{Z-1} \right) \tag{48}
\]

for \( Z = 0, 1, \ldots k \). Analogously one can check that

\[
\alpha_{Z}^{(k-2)}(n, k) = (-1)^Z \left[ \binom{k-2}{Z} - \binom{n-2k+2}{2} \frac{k-2}{Z-2} \right], \tag{49}
\]

\[
\alpha_{Z}^{(k-3)}(n, k) = (-1)^Z \left[ \binom{k-3}{Z} - 3 \binom{n-2k+3}{1} \frac{k-3}{Z-1} + 3 \binom{n-2k+3}{3} \frac{k-3}{Z-3} \right]. \tag{50}
\]

From these expressions the existence of a recursive rela-
tion is evident, this leading to the following general result

\[
\alpha_{Z}^{(k-m)}(n, k) = (-1)^Z \sum_{i=0}^{m} (-1)^i \binom{m}{i} \left( \binom{n-2k+m}{i} \right) \binom{k-m}{Z-i}. \tag{51}
\]

with \( m = 0, 1, \ldots, k \). Using \(^{[49]} \) together with \(^{[40], [19]} \), \(^{[18]} \),
\(^{[18]} \), \(^{[11]} \), one obtains the complete spectrum of the RDM.

VII. CONCLUSION

In summary, we have provided explicit analytical ex-
pression of the reduced density matrix of a subsystem
of arbitrary size \( n \) of a permutational invariant quantum
many body system of arbitrary size \( L \) and characterized
by a state of arbitrary permutational symmetry. We have
shown, on the specific example of the spin 1/2 Heisen-
berg model, that the RDM acquires a block diagonal form
with respect to the quantum number \( k \) fixing the polar-
ization in the subsystem (conservation of \( S_z \) or conser-
vation of the number of particles for non spin models )
and with respect to the irreducible representations of
\( S_n \) group. The main results of the paper are represented
by Eqs. \(^{[18], [19]} \) and \(^{[49]} \) given above. These results,
being based only on the permutational invariance and on
the conservation of \( S_z \) (number of particles for non spin
systems), should apply to all quantum many-body sys-
tems which possess these symmetries and for which the
mean field theory becomes exact. A more detailed analy-
sis of the properties of RDM spectrum and of its physical
implications will be given elsewhere \(^{[12]} \).
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