Holomorphic Engel Structures

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Abstract: Recently there has been renewed interest among differential geometers in the theory of Engel structures. We introduce holomorphic analogues of these structures, and pose the problem of classifying projective manifolds admitting them. Besides providing the basic properties of these varieties, we present two series of examples and characterize them by certain positivity conditions on the Engel structure. AMS MSC:53C15; 14E35.

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1 Introduction.

A class of distributions of a manifold is called open if it is open as a set inside the parameter space of distributions of the manifold. For instance the class of contact distributions is open, in the real and holomorphic cases. Moreover, the class is called topologically stable if it is open and it admits a unique local model in the neighborhood of each point. Again contact distributions are topologically stable since any of them admits a holomorphic chart $(z, x_1, y_1, \ldots, x_n, y_n)$ at a given point in which it becomes \( \ker(dz - \sum x_idy_i) \). In the real case, it has been proved (cf. [Mo1, Mo2]) that the list of topologically stable distributions is given by:

- line fields,
• contact structures in odd dimensional manifolds,
• even contact structures in even dimensional manifolds,
• Engel structures in 4-dimensional manifolds.

The definitions of the last two ones will be provided later. We must remark that in the holomorphic case a similar proof provides the same list. The name of "Engel structure" comes from the fact that H. Engel gave the first proof of the topological stability of this kind of distributions in the 19th century, basically by showing that they follow a canonical local model, i.e. a kind of Darboux theorem. By completeness, in Subsection 2.2 we offer the proof of the equivalent result in the holomorphic case, which guarantees the topological stability of the holomorphic Engel structures.

An Engel structure on a (real/complex) variety $X$ is a 2-dimensional distribution $D \subset T_X$ satisfying that $[D, D] = E$ is a rank 3 distribution and $[E, E] = T_X$. It is simple to check that there is a one dimensional subbundle $L \subset E$ that is invariant for the bracket operation, i.e. $[L, E] \subset E$. Moreover, we have $L \subset D$. This provides a flag

$$L \subset D \subset E \subset T_X$$

of vector bundles inside the tangent bundle. In the real orientable case, this immediately implies that the tangent bundle is parallelizable. A long standing conjecture in differential geometry has been that the converse is also true: any parallelizable 4-fold admits an Engel structure. This has been recently proved by A. Vogel [V], lighting the interest for (real) Engel structures.

We want to start in this note the study of holomorphic Engel structures on projective varieties. The main result we show is

**Theorem 1.1.** Let $X$ be a projective variety admitting an Engel structure $L \subset D \subset E$. Then either:

• $L^{-1}$ is not pseudoeffective and then $X$ is the Cartan prolongation of a contact 3-fold, or

• $(D/L)^{-1}$ is not pseudoeffective. In this case, if we assume moreover that $D \cong L \oplus D/L$, then $X$ is a Lorentzian tube.

Moreover the two classes have a unique common element which is the universal family of lines contained in a quadric hypersurface in $\mathbb{P}^4$.

The definitions of Cartan prolongation and Lorentzian tube will be given in section 3. It is enough for now to understand that the Engel structures described in the previous list are classified by 3-dimensional contact structures and 3-dimensional conformal structures. Remark that 3-dimensional contact structures are completely classified, since the abundance conjecture [KM] is true in dimension 3 and then by [KPSW] $X$ is either a projectivized tangent bundle or a Fano 3-fold. On the other hand holomorphic conformal structures also have been classified in a recent article [JR]. So, the Theorem 1.1 provides a very concrete classification result.

The structure of this note is as follows. In Section 2 we will provide the basic definitions and properties we will play with, also proving the local stability result.
mentioned as “Darboux Theorem”. In Section 3 the two classical examples of Theorem 1.1 will be explained. Finally we will provide the proof of Theorem 1.1 in Section 4.

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2 Basic properties

2.1 Global properties.

Along this paper $X$ will denote a smooth projective complex variety of dimension $2n$, and $T_X$ its tangent bundle. Given any subsheaf $E \subset T_X$, the composition of the usual Lie-bracket of $E$ with the projection onto $T_X/E$ $E \otimes E \to T_X/E$ is an $\mathcal{O}_X$-linear morphism, that we will call O’Neill tensor of $E$.

Definition 1. Let $E \subset T_X$ denote a codimension 1 subbundle, defined as the kernel of a surjective morphism:

$$T_X \xrightarrow{\theta} L' \to 0.$$  

Locally we can compute the exterior differential $d\theta$, but these data do not glue together to define a 2-form. Nevertheless a direct computation shows that $\theta \wedge (d\theta)^{n-1}$ is a well defined $(2n-1)$-form with values in $(L')^n$. We say that $E$ defines an even contact structure if $\theta \wedge (d\theta)^{n-1} \in H^0(\Omega^{2n-1}_X \otimes (L')^n)$ is everywhere non-zero.

Remark 2.1. Furthermore $d\theta$ defines a global section of $\bigwedge^2 \mathcal{E}^\vee \otimes L'$, that we denote by the same symbol. In fact $d\theta$ coincides with the O’Neill tensor via the Cartan formula

$$d\theta(V_1, V_2) = \theta(V_1)V_2 - \theta(V_2)V_1 - \theta[V_1, V_2].$$

Remark 2.2. By the non-vanishing condition, the kernel of the skew-symmetric morphism $d\theta : \mathcal{E} \to \mathcal{E}^\vee \otimes L'$ is a line subbundle of $\mathcal{E}$, that we denote by $L$. We call it the kernel of the even contact structure. With this notation $d\theta$ provides an everywhere nondegenerate skew-symmetric morphism $\bigwedge^2(\mathcal{E}/L) \to L'$. In particular, if dim $X = 4$, det($\mathcal{E}$) = $L' \otimes L^{-1}$ and $\omega_X^{-1} = (L')^2 \otimes L^{-1}$.

Definition 2. An Engel structure is a rank 2 vector subbundle $D \subset T_X$ on a 4-dimensional complex manifold $X$ satisfying that $\mathcal{E} = [D, D]$ is an even contact structure. Here $[\cdot, \cdot]$ denotes the usual Lie bracket.

Lemma 2.3. An Engel structure provides a filtration of $T_X$:

$$L \subset D \subset \mathcal{E} \subset T_X.$$ 

Moreover the O’Neill tensor of $D$ induces an isomorphism $L \otimes D/L \cong \mathcal{E}/D$.  

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Proof. The statement is local, hence we assume that $D$ is generated by two vector fields $V_1$ and $V_2$, where $\{V_1, V_2, V_3 := [V_1, V_2]\}$ generates $\mathcal{E}$. We may assume that $[V_2, V_3] \notin \mathcal{E}$. A vector field $V = aV_1 + bV_2 + cV_3$ belongs to $L$ iff $[V_i, V] \in \mathcal{E}$, $i = 1, 2, 3$. Taking $i = 2$ we obtain $c = 0$, therefore $L \subset D$.

We may now assume that $L$ is generated by $V_1$, and the class of $V_2$ generates $D/L$. Then the class of $V_3 = [V_1, V_2]$ generates $\mathcal{E}/D$.

We will usually identify an Engel structure with its associated filtration. The elements of this filtration fit into the following commutative diagram with exact rows and columns

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & & & & & & \\
0 & \longrightarrow & L & \longrightarrow & L & \longrightarrow & 0 \\
\downarrow & & & & & & \\
0 & \longrightarrow & D & \longrightarrow & \mathcal{E} & \longrightarrow & L \otimes D/L \longrightarrow 0 \\
\downarrow & & & & & & \\
0 & \longrightarrow & D/L & \longrightarrow & \mathcal{E}/L & \longrightarrow & L \otimes D/L \longrightarrow 0 \\
\downarrow & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{array}
\]

(1)

Remark 2.4. It does not seem true in general that an even contact structure determines an Engel structure. Moreover, there are examples of even contact structures not supporting an Engel structure at all (see Corollary 3.3).

Lemma 2.5. With the above notation, the anticanonical line bundle $\omega^{-1}_X$ is isomorphic to $L^3 \otimes (D/L)^4$.

Proof. It follows directly from Remark 2.2 and the commutative diagram above:

\[
\omega^{-1}_X = L \otimes \det(\mathcal{E}/L) \otimes L^1 = L \otimes \det(\mathcal{E}/L)^2 = L \otimes (L \otimes (D/L)^2)^2 = L^3 \otimes (D/L)^4
\]

\]

(2) \]

\[
(3)
\]

2.2 Local properties.

Now, we will prove the existence of a local canonical form for any Engel structure.

Theorem 2.6 (Darboux Theorem). Let $D$ be an Engel structure on $X$ and let $x$ be a point on $X$. Then, there exists a neighborhood $U$ of $x$ in $X$ and a chart $\phi : V \subset \mathbb{C}^4 \to U$ such that $\phi^*D = D_0$, where $D_0$ is the Engel structure in $\mathbb{C}^4$ given by the vanishing conditions

\[
dx - ydz = 0 \tag{2}
\]

\[
dy - wdx = 0 \tag{3}
\]
We need a couple of holomorphic contact geometry results before addressing the proof of this Theorem.

**Lemma 2.7 (Holomorphic Gray stability lemma).** Let \( X \) be a smooth variety, let \( U \subset \mathbb{C} \) be a neighborhood of \( 0 \), let \( x \) be a point in \( X \) and assume that there is a family of contact forms

\[
\mathcal{F}_t \to T_X \theta_t \to \mathcal{O},
\]

with \( t \in U \subset \mathbb{C} \). Then there is a neighborhood \( V \) of \( x \in X \) and family of holomorphic flows \( \phi_t : V \to X \) such that \( \phi_t^* \mathcal{F}_t = \mathcal{F}_0 \), for \( |t| \) small enough.

The flow is not well-defined for all \( t \in U \), but the holomorphic vector fields generating it are. The point is that since we do not assume compactness the fields do not always integrate. However, this will not be a problem in the applications. We are assuming that \( L = \mathcal{O} \) to make the proof simpler; this case is enough for our purposes, though the proof can be probably extended to other situations.

**Proof.** The condition is equivalent to the following one

\[
\phi_t^* \theta_t = f_t \theta_0, \quad (4)
\]

for some \( f_t \in H^0(\mathcal{O}^t) \). The flow \( \phi_t \) is uniquely represented by a family of holomorphic vector fields \( V_t \). Differentiating in the previous equation we obtain

\[
\phi_t^* \left( \frac{d}{dt} \theta_t + \mathcal{L}_{V_t} \theta_t \right) = \frac{df}{dt} \theta_0,
\]

which can be rewritten as

\[
\frac{d}{dt} \theta_t + \mathcal{L}_{V_t} \theta_t = h_t \theta_t, \quad (5)
\]

with \( h_t = \left( \frac{1}{f_t} \frac{df}{dt} f_t \right) \circ \phi_t^{-1} \). There is a well-defined non-zero holomorphic vector field associated to any contact form, whenever \( L = \mathcal{O} \), that is called the Reeb field \( R \) and it is defined by the following two equations

\[
i_R d\theta = 0
\]
\[
i_R \theta = 1.
\]

Recall that \( \ker d\theta \) is a 1-dimensional space, so the first condition determines a 1-dimensional foliation and the second one is just a normalization. The Reeb fields associated to our family of forms \( \theta_t \) will be denoted by \( R_t \). We impose \( h_t = i_{R_t} \frac{d}{dt} \theta_t \).

Note that \( \mathcal{F} \bigoplus R_t = T_X \). It is simple to check that the previous assignation makes the equation \( (5) \) true for any vector in the direction \( R_t \). So, we have to check it just for vectors \( Y \in \mathcal{F}_t \). Use Cartan’s formula

\[
\mathcal{L}_Y = di_X + i_X d,
\]

and assume that we check it against \( Y \in \mathcal{F}_t \), then we have

\[
\frac{d}{dt} \theta_t + d_{V_t} \theta_t + i_{V_t} d \theta_t = 0.
\]
We just look for a vector $V_t \in F_t$ and this further reduces the expression to

$$\frac{d}{dt} \theta_t + i_{V_t} d\theta_t = 0.$$ 

Now, recall that $d\theta_t$ is non degenerate on $F_t$, so this equation has a unique solution $V_t$. This ends the proof.

**Corollary 2.8.** Let $(X, \mathcal{F})$ be a contact manifold. For any point $x \in X$, there exists a neighborhood $U$ and a chart $\phi : V \subset \mathbb{C}^{2n+1} \to U$ such that $\phi^* \mathcal{F} = \mathcal{F}_0$, where $\mathcal{F}_0$ is the standard contact form in $\mathbb{C}^{2n+1}$ defined by the vanishing condition

$$dx - \sum_{j=1}^n y_j dz_j = 0.$$ 

**Proof.** Take any chart $\phi : V \subset \mathbb{C}^{2n+1} \to U$ centered at $x$. Choose an element of $A \in GL(2n+1, \mathbb{C})$ and construct $\Phi = A \circ \phi$ in such a way that $\Phi^* \theta(x) = \theta_0(0)$ and $\Phi^* d\theta(x) = d\theta_0(0)$. This can be done just by choosing symplectic basis at $\mathcal{F}(x)$ and $\mathcal{F}_0(0)$ and choosing $A$ in order to map the first one into the second one. Now, to be contact is an open condition so $\theta_t = (1-t)\theta_0 + t\Phi^* \theta$ is a family of contact structures for some small neighborhood $W$ of $0 \in \mathbb{C}^{2n+1}$. We are in the hypothesis of Lemma 2.7 and therefore there exists a flow, for $|t|$ small enough, $\Psi_t : W \to W$ such that $\Psi_t \ker(\theta_t) = \mathcal{F}_0$. In fact, probably shrinking $W$ again, the flow is well defined for $t = 1$. We conclude that $\Psi_1 \circ \Phi$ is the needed map.

Now we complete the proof of Theorem 2.6.

**Proof.** We will construct a special chart $(\phi, U)$ at $p \in X$ with local coordinates $(x, y, z, w)$. The first condition imposed to the chart will be to send the line bundle $L$ to the derivative $\frac{\partial}{\partial w}$ of a function $w : U \to \mathbb{C}$. The second and final one will be to choose local coordinates $(x, y, z)$ in a hypersurface $w = \epsilon$ in such a way that the coordinates $(x, y, z, w)$ are provided by the flow along the field $\frac{\partial}{\partial w}$.

Now, in these special coordinates the bundle $E$ is $w$-invariant so projects to the quotient $U/L$, which has coordinates $(x, y, z)$, as a bundle $\tilde{E}$. This bundle defines a contact structure on the quotient. Denote the projection by $\pi : U \to U/L$. We apply Corollary 2.8 to $U/L$ to obtain a map $\tilde{\phi} : V/L \to V/L$, where $V \subset U$, satisfying that $\tilde{\phi}(\tilde{E}) = \tilde{E}_0$. The last one is the standard contact structure in $\mathbb{C}^3$ and is defined by the condition (2). Denote by $\phi$ the fiberwise-constant lift of $\tilde{\phi}$ to $V$. Let $q$ be a point on $V$. Recall that $D_q \supset L_q$ can be projected to $V/L$ defining an element in $\mathbb{P}(E_{\pi(q)})$. This provides a map $\Psi : V \to W \subset \mathbb{P}(E)$, where $W$ is an open subset. The condition of being Engel is completely equivalent to the biholomorphicity of this map and this implies that the Engel structure is determined by an equation of the form

$$dy - gdx = 0,$$

with $g$ a certain holomorphic function satisfying that $\frac{\partial g}{\partial w} \neq 0$. By another change of coordinates, provided by the Inverse Function Theorem, the Engel structure gets defined by condition (3).
3 Examples

We have seen in the previous section, thanks to the Darboux Theorem, that every local Engel structures is defined, up to holomorphic change of coordinates, by the vanishing of the equations

\[
\begin{align*}
    dx - ydz &= 0 \\
    dy - wdx &= 0.
\end{align*}
\]

In this section we will present two compact examples of Engel structures. They are holomorphic analogues of two real constructions already known to Cartan. The other two known examples of real Engel structures, a construction over mapping tori [Ge] and a general construction on parallelizable 4-folds [Y], cannot be adapted to the holomorphic case.

Example 3.1 (Cartan prolongation of a contact structure). Let \( Z \) be a projective smooth threefold admitting a contact structure \( F \subset T_Z \) appearing as the kernel of a morphism \( \varphi : T_Z \to L'_Z \). Let \( X \) be the projectivization of \( F^\vee \), and denote by \( \pi \) the natural projection onto \( Z \). Composing the pull-back of \( \varphi \) with the differential of \( \pi \) we obtain a twisted 1-form

\[
\theta := \pi^* \varphi \circ d\pi : T_X \to L' := \pi^* L'_Z.
\]

Being \( F \) a contact distribution, it follows that \( \theta \) defines an even contact structure \( d\pi^{-1}(\pi^* F) \) on \( X \), that we denote by \( \mathcal{E} \), and in fact the converse is also true, as one can easily check locally: if the contact structure on \( Z \) is given locally by the 1-form \( \theta_Z \), then \( \theta \wedge d\theta = \pi^*(\theta_Z \wedge d\theta_Z) \). Note also that, by construction, \( \ker(\mathcal{E}) = T_X|_Z \).

Furthermore, we claim that it admits an Engel structure. In order to see that, consider the relative Euler sequence associated with the tautological line bundle \( \mathcal{O}_X(1) \), twisted with \( \mathcal{O}_X(-1) \):

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_X(-1) & \to & \pi^* F & \to & T_X|_Z(-1) & \to & 0.
\end{array}
\]

Denote by \( D \) the kernel of the composition \( \mathcal{E} \to \pi^* F \to T_X|_Z(-1) \). We get the following commutative diagram with exact rows and columns:
Note that, in a more general setting, a regular fibration by curves over a contact manifold supports an even contact structures whose kernel is the relative tangent bundle of the fibration. Hence it makes sense to ask whether these even contact structures are associated to Engel structures. The next proposition and its corollary show that this is only possible if the fibers have genus 0.

**Proposition 3.2.** Let $X$ be projective smooth 4-fold admitting an Engel structure $L \subset D \subset \mathcal{E}$. Assume that all the leaves of $L$ are compact. Then the genus of the leaves is zero.

**Proof.** Let $C$ be a general compact leaf of the foliation $L$. Since $X$ is a fibration locally around $C$, then the normal bundle of $C$ in $X$ is trivial. This implies that $\omega^{-1}\mathcal{X}|_C = T_C = L|_C$, hence by Lemma 2.5 the divisor $L'|_C = \det(\mathcal{E}/L)|_C = (L \otimes (D/L^2)|_C$ has degree zero. On the other side the exact sequence

$$0 \to (\mathcal{E}/L)|_C \to (T_X/L)|_C \cong \mathcal{O}_C^{\oplus 3} \to L'|_C = (T_X/\mathcal{E})|_C \to 0$$

tells us that $L'|_C$ is globally generated, therefore trivial. It follows that $(\mathcal{E}/L)|_C$ is trivial, too.

Next we consider the exact sequence appearing in the last row of diagram 1, restricted to $C$:

$$0 \to (D/L)|_C \to (\mathcal{E}/L)|_C \cong \mathcal{O}_C^{\oplus 2} \to (L \otimes D/L)|_C \to 0.$$

Note that the triviality of $(L \otimes (D/L^2))|_C$ implies that $\deg(L \otimes D/L)|_C = 1 - g$. But a quotient of a trivial bundle cannot have negative degree, hence $g \leq 1$.

Finally if $g = 1$ the divisors $L$ and $D/L$ are trivial along the leaves of $L$. Being $\pi$ the subscheme of Chow($X$) parameterizing these leaves and $\pi : X \to Z$ the projection, $D/L$ is the pull-back of a line subbundle $L_Z \subset T_Z$. Hence $D$ is the inverse image of $\pi^*L_Z$ by $d\pi$, and so $D$ is trivial along the fibers of $\pi$, contradicting its non integrability.

**Corollary 3.3.** Let $Z$ be a 3-dimensional contact variety. Let $X \to Z$ be a regular fibration by curves. If the genus of the curves is not zero, then the even contact structure defined on $X$ by pull-back does not support an Engel structure.

**Remark 3.4.** In the setting of Proposition 3.2, it follows that $L$ has positive degree on its leaves. In particular Proposition 4.4 below will imply that $X$ is in fact the Cartan prolongation of a contact manifold.

The following example is based on an analogous construction in (real) differential geometry due to Cartan, where an Engel structure is built upon a Lorentzian structure on a threefold. Cartan did not show that the structure was Engel but studied its properties.

We recall some definitions from the literature (see for instance [JR]).

**Definition 3.** A holomorphic conformal structure $\lambda$ on a holomorphic bundle $E$ over a complex manifold $Y$ is a non-degenerate section $\lambda \in H^0(Y, S^2E^* \otimes M)$, where $M$ is an arbitrary line bundle. A holomorphic conformal structure on a complex manifold is a holomorphic conformal structure on the tangent bundle of the manifold.
**Definition 4** (Tubes [GS]). A tube on a complex manifold $X$ is the zero set of the defining section of a holomorphic conformal structure over a rank $r$ vector bundle $E$. It has a natural structure of $Q_{r-2}$-fibration over $X$. A Lorentzian tube is a tube on the tangent bundle of the manifold.

**Example 3.5** (Lorentzian tubes). Let $Y$ be a smooth threefold equipped with a Lorentzian tube provided by a section $λ ∈ H^0(Y, S^2Ω_Y ⊗ M)$. Equivalently, we have a smooth conic fibration $ρ : X := zeroes(λ) ⊂ P(Ω_Y) → Y$. Consider the rank $2$ subbundle $D ⊂ T_X$ defined in the following way: being $i : X → P(Ω_Y)$ the natural inclusion and denoting $L := (i^*O_{P(Ω_Y)}(-1))$ consider the diagram

$$
\begin{array}{c}
T_X \\
dρ \downarrow \\
0 \longrightarrow L \longrightarrow ρ^*T_Y \longrightarrow i^*T_{P(Ω_Y),Y}(-1) \longrightarrow 0,
\end{array}
$$

where the last row is the Euler sequence of $P(Ω_Y)$, and take $D := dp^{-1}(L)$. Equivalently, we may define $D ⊂ T_X$ by the rule $D_x = dp^{-1}(⟨x⟩)$, where $⟨x⟩ ⊂ T_{X,ρ(x)}$ denotes the vector subspace determined by $x$.

Since the statement is local, we will assume that $X = Y × C$ and that the zeroes of $λ$ in $P(Ω_Y)$ can be written as the classes of elements $V = V_1 + tv_2 + t^2V_3$ in $T_Y$, $t ∈ C$, where $V_1, V_2$ and $V_3$ are independent vector fields on $Y$. We will denote by $V_i, V_2$ and $V_3$ the corresponding vector fields in $X$ obtained by the identification $T_X ≅ p_1^*T_Y ⊕ p_2^*T_C$, where $p_1(= ρ)$ and $p_2$ are the two natural projections. Let us fix the following notation:

$$[V_i, V_j] = \sum_{i=1}^{3} f_{ij}^k V_k.$$

With the above notation, trivial computations show that $D$ is generated $\partial/\partial t$ and $V_i$, $E = [D, D]$ is a rank $3$ vector bundle generated by $D$ and $V_2 + 2tV_3$, defined by the 1-form

$$θ = t^2ω_1 - 2tω_2 + ω_3,$$

where the $ω_i$’s denote $1$-forms dual to the $V_i$’s at each point.

A straightforward computations shows that $θ ∧ dθ$ takes the form:

$$θ ∧ dθ = aω_1 ∧ ω_2 ∧ ω_3 + bdt ∧ ω_1 ∧ ω_2 + cdt ∧ ω_1 ∧ ω_3 - 2dt ∧ ω_2 ∧ ω_3,$$

and in particular it is everywhere non-zero.

Easy examples of varieties $Y$ verifying the above property are the $3$-dimensional quadric $Q ∈ P^4$ and abelian varieties. In the first case, the variety $X$ coincides with the flag manifold of pairs point-line contained in the quadric $Q$, and this is nothing but the projectivization of a contact structure on $P^3$ constructed as in Example 3.1. However, the second case is of completely different nature, since $X$ does not admit a second $P^1$ bundle structure (see also Proposition 4.6).

Note that Jahnke and Radloff have recently obtained the complete list of holomorphic conformal structures on complex $3$-folds (cf. [JR]), so providing the complete list of Engel structures of this type.
Theorem 3.6. The list of projective threefolds with a holomorphic conformal structure is as follows:

1. \( Q_3 \);
2. étale quotients of abelian threefolds;
3. threefolds with universal covering space the 3-dimensional Lie ball \( D_3^{IV} \).

We denote here by \( D_3^{IV} \) the bounded symmetric domain dual to the 3-dimensional hyperquadric \( Q_3 \).

4 Uniruledness of Engel manifolds

It is well known that the existence of rational curves on a variety \( X \) depends on the positivity of its anticanonical divisor. By analogy with the contact case (see [De]) we begin by inferring positivity properties of the anticanonical divisor of an Engel manifold from the non-integrability of the structure. From that we show that under certain positivity conditions on \( L \) or \( D/L \), the examples described in the previous section are the only possible Engel structures.

4.1 The canonical class of an Engel manifold

We begin by recalling the following theorem by J.P. Demailly (cf. [De]):

Theorem 4.1. Let \( X \) be a Kähler manifold. Assume that there is a pseudo-effective line bundle \( N \) on \( X \) and a nonzero holomorphic section \( \theta \in H^0(X, \Omega_X \otimes N^{-1}) \). Then the subsheaf defined by the kernel of \( \theta \) defines a holomorphic foliation of codimension 1 in \( X \), that is, \( \theta \wedge d\theta = 0 \).

We will also make use of the characterization of pseudo-effective divisors in terms of movable curves. A curve \( C \) in \( X \) is called movable if there exists an irreducible algebraic family of curves containing \( C \) and dominating \( X \). Boucksom, Demailly, Peternell and Paun have recently shown (cf. [BDPP]) that a divisor is pseudo-effective if and only if it has non negative degree on every movable curve.

Applying this result to the section \( \theta \in H^0(X, \Omega_X \otimes L') \) defining \( \mathcal{E} \) for an Engel structure we obtain the following straightforward result:

Lemma 4.2. Let \( X \) be a complex manifold admitting a holomorphic Engel structure \( L \subset D \subset \mathcal{E} \). Then either \( L^{-1} \) or \( (D/L)^{-1} \) are not pseudo-effective. Moreover, assume that any of the following properties is fulfilled:

- \( L \) is pseudo-effective,
- \( \det(D) \) is pseudo-effective.

Then the canonical divisor of \( X \) is not pseudo-effective and, equivalently, \( X \) is uniruled.
Proof. The first assertion follows directly from Theorem 4.1 which in our case tells us that \((L')^{-1} = (L \otimes (D/L)^2)^{-1}\) is not pseudo-effective.

In order to get the second, note that by Lemma 2.5 we get \( \omega_X^{-1} \sim L \otimes (L')^{-1} \sim L \otimes \det(D) \). Thus if \( L \) or \( \det(D) \) are pseudo-effective and \( L' \) has positive degree on a movable curve \( C \), then \( \omega_X \cdot C < 0 \), allowing us to conclude using [BDPP]. But the condition on \( L' \) is equivalent (again by [BDPP]) to the non pseudo-effectivity of \((L')^{-1}\), that we obtain from 4.1.

4.2 Case I: \( L^{-1} \) is not pseudoeffective

In this section we will show that the Engel structures described in Example 3.1 may be characterized by the non pseudo-effectivity of \( L^{-1} \). Note that whenever \( L \) is not numerically trivial, this property is weaker than the pseudo-effectivity of \( L \) needed in Lemma 4.2.

Proposition 4.3. Let \( X \) be a smooth complex projective variety and \( L \subset T_X \) a line subbundle of the tangent bundle of \( X \). Assume that \( L^{-1} \) is not nef. Then \( X \) is isomorphic to a \( \mathbb{P}^1 \)-bundle over a nonsingular variety \( Z \).

Proof. The line subbundle \( L \subset T_X \) defines a 1-dimensional regular foliation on \( X \). By assumption there exists a curve \( C \subset X \) such that \( L|_C \) is ample. It follows from [KST, Thm. 1] that the leaves of \( L \) passing by points of \( C \) are algebraic and the general one is rationally connected, hence a \( \mathbb{P}^1 \).

Now we claim that the general leaf of \( L \) is isomorphic to \( \mathbb{P}^1 \). Fix a leaf \( R \) passing by a general point of \( C \). Since \( R \cong \mathbb{P}^1 \) the holonomy of the foliation \( L \) along \( R \) is trivial and Reeb’s Stability Theorem [CLN, Thm. IV.3] provides a fundamental system of analytic neighborhoods of \( R \) in \( X \) that are saturated with respect to \( L \). It follows that the neighboring leaves of \( L \) are \( \mathbb{P}^1 \)’s. This defines an analytic family of \( \mathbb{P}^1 \)’s parametrized by a submanifold \( S_0 \subset \text{RatCurves}^n(X) \). Now, every element in the Zariski closure inside \( S \subset \text{RatCurves}^n(X) \) parametrizes a rational curve tangent to \( L \). Since the family parametrized by \( S \) dominates \( X \), we conclude the claim.

Now we prove that every leaf of \( L \) is a \( \mathbb{P}^1 \). Let \( Z \) be the normalization of the closure of the family \( S \) in \( \text{Chow}(X) \). Every element \( z \in Z \) determines a rational cycle \( C_z \) in \( X \). Since \( C_z \) is a limit of elements of \( S \), it must be tangent to \( L \) at a smooth point. But \( L \) is regular, hence every cycle \( C_z \) must be smooth, reduced and irreducible, and \( Z \) is smooth at \( z \).

Finally, since \( L \cdot \pi^{-1}(z) = 2 \) for every \( z \in Z \), Lemma 2.5 tells us that \((D/L)^{-1}\) has degree 1 on every fiber. This concludes the proof.

Proposition 4.4. Let \( X \) be a smooth projective \( 2n \)-fold admitting an even contact structure \( E \subset T_X \) with kernel \( L \). If \( L^{-1} \) is not nef (in particular, if \( L^{-1} \) is not pseudoeffective), then \( X \) is a \( \mathbb{P}^1 \)-bundle over a contact \( (2n-1) \)-fold \( Z \). Furthermore, if \( n = 4 \) and \( E \) is an Engel structure, then \( X \) is the projectivization of a contact distribution on \( Z \).
Proof. Consider the variety $Z$ and the morphism $\pi : X \to Z$ constructed in Proposition 4.3 by integration of the foliation $L \subset T_X$. Consider the exact sequence

$$0 \to \mathcal{E}/L \to T_X/L \cong \pi^*T_Z \to L' \to 0.$$ 

Arguing as in Prop. 3.2, $L'$ and $\mathcal{E}/L$ are trivial on every fiber of $\pi$. By [Har77, III, 12.9] it follows that $\mathcal{E}/L = \pi^*\mathcal{E}$, $L' = \pi^*L_Z'$, where $\mathcal{E}$ and $L_Z'$ are locally free sheaves of rank $2n - 2$ and 1, respectively. Applying $\pi_*$ to the exact sequence above and considering that $R^1\pi_*\mathcal{E}/L = 0$, we obtain an exact sequence

$$0 \to \mathcal{E} \to T_Z \phi \to L'_Z \to 0.$$ 

But $\phi$ defines a contact structure, because pulling it back and composing it with $d\pi$ we obtain the original even contact structure on $X$ (see Example 3.1).

Finally if $\mathcal{E}$ is associated with an Engel structure $D$, then $D/L$ and $L \otimes D/L$ have degree $-1$ and 1, respectively, on the fibers of $\pi$. It follows that $\pi_*D/L = R^1\pi_*D/L = 0$ and pushing down the exact sequence:

$$0 \to D/L \to \mathcal{E}/L = \pi^*E \to L \otimes D/L \to 0$$

we get $E \cong \pi_*(L \otimes D/L)$, therefore $X \cong \mathbb{P}(E)$. \qed

4.3 Case II: $(D/L)^{-1}$ is not pseudoeffective

We now study the case of $(D/L)^{-1}$ being not pseudoeffective. In this case we need to make a second assumption in order to characterize Lorentzian tubes.

**Proposition 4.5.** Let $X$ be a smooth projective 4-fold admitting an Engel structure $L \subset D \subset \mathcal{E}$ such that $(D/L)^{-1}$ is not nef. Assume that $L$ is a direct summand of $D$. Then there exist a smooth 3-fold $Y$ and a holomorphic conformal structure $\lambda \in H^0(Y, S^2\Omega_Y \otimes M)$, whose set of zeroes is isomorphic to $X$.

**Proof.** The assumption on $L$ allows us to consider $D/L$ as a line subbundle of $T_X$. Applying Prop. 4.3 we deduce that $X$ is a $\mathbb{P}^1$-bundle over some smooth variety $Y$. Denote by $\rho$ the natural projection from $X$ to $Y$. Since the relative tangent bundle to $\rho$ is $D/L$, there is an inclusion $L \to \pi^*T_Y$, which defines an inclusion $X \rightarrow \mathbb{P}(\Omega_Y)$. But then $D/L$ has degree 2 on every fiber $C$ of $\rho$, hence Lemma 2.5 tells us that $L \cdot C = -2$. This concludes the proof. \qed

Finally we show that the two classes of varieties admitting an Engel structure previously described have a unique common element, namely the flag manifold $F_Q(0, 1)$, where $Q$ is a quadric threefold. That is the universal family of lines contained in $Q$. Note that this variety admits two $\mathbb{P}^1$-bundle structures

$$F_Q(0, 1) \overset{\rho}{\longrightarrow} Q \overset{\pi}{\longrightarrow} \mathbb{P}^3$$
Proposition 4.6. Let $X$ be a smooth projective manifold admitting and Engel structure $L \subset D \subset E$. Assume that $L^{-1}$ and $(D/L)^{-1}$ are not nef, and that $D = L \oplus D/L$. Then $X$ is isomorphic to the flag manifold $F_Q(0, 1)$.

Proof. Consider the two $\mathbb{P}^1$-bundle structures $\pi : X \to Z$ and $\rho : X \to Y$ constructed in the previous propositions. We will make use of the notation of the proofs appearing there. If we prove that $Z = \mathbb{P}^3$, then $X$ is the projectivization of a contact structure on $Z$. But it is classically known that then $X \cong F_Q(0, 1)$.

In order to prove that $Z = \mathbb{P}^3$ it is enough to check that $X$ contains a family of rational curves $M = \{C_t\}$, with $M$ proper and splitting type

$$T_{Z|_{C_t}} = O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1)^{\oplus 2}$$

for all $C_t$.

If that property is fulfilled, then every two points of $Z$ might be joined by a curve of the family $M$ (cf. [De, Prop. 4.8]), and in particular [O, Lemma 1] tells us that $Z$ is a Fano threefold of Picard number 1. Moreover, $-K_Z \cdot C_t = 4$ implies that the index of $Z$ is 4, hence $Z \cong \mathbb{P}^3$.

Consider a fiber $C := \rho^{-1}(z)$ for any $z \in Z$. By the computations we have done in Examples 3.1 and 3.5, it suffices to show that $(T_{X/D}|_{C} \cong O(1)^{\oplus 2}$. But $T_{X/D} \cong (T_{\mathbb{P}^1(\mathcal{O}_Y)}|_{C}(-1)|_X$ and this is isomorphic to $O(1)^{\oplus 2}$ because $T_{\mathbb{P}^1(\mathcal{O}_Y)}|_{C} \cong T_{\mathbb{P}^2,C}$ and $C \subset \mathbb{P}^2$ is a conic.

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