System response analysis in wavenumber domain for linear space-invariant time-varying problems

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Being a powerful tool for linear time-invariant (LTI) systems, system response analysis can also be applied to the so-called linear space-invariant (LSI) but time-varying systems, which is a dual of the conventional LTI problems. In this paper, we propose a system response analysis method for LSI problems by performing a Fourier transform of the field distribution on the space instead of time coordinate. Specifically, input and output signals can be expressed in the wavenumber (spatial frequency) domain. In this way, the system function in wavenumber domain can also be obtained for LSI systems. Given an arbitrary input and temporal profile of the medium, the output can be easily predicted using the system function. Moreover, for a complex temporal system, the proposed method allows for decomposing it into multiple simpler subsystems that appear in sequence in time. The system function of the whole system can be efficiently calculated by multiplying those of the individual subsystems.

1. INTRODUCTION

As an emerging field, time-varying metamaterials have attracted the eyes of researchers in the past several years, because they unlock a new degree of freedom to control EM waves propagation [1-6]. Specifically, there exists one class of problems where electromagnetic (EM) wave propagates in an infinite and homogeneous medium whose permittivity or permeability changes in time. This topic was first explored by Morganthaler [7], and in recent years some researchers have revisited it to come up with the concept of a ‘temporal boundary’ [8], which has been explored both theoretically and numerically. These include temporal effective medium [9], temporal coating [10, 11], temporal aiming [12], polarization conversion [13], energy pumping [14], photonic time crystals [15, 16], etc [17,18]. We refer to them as linear space-invariant (LSI) systems, which serve as a dual of the conventional linear time-invariant (LTI) systems where the material properties vary in space.

In order to analyze LSI systems, researchers have proposed several analytical formalisms, such as temporal transfer matrix methods (TTMM) [19-20] and d’Alembert solution [21]. These methods rely on solving temporal boundary conditions at the time instances when the material properties change abruptly. Some researchers have proposed solutions to waves propagating in a continuously-changing media using the well-known Wentzel–Kramers–Brillouin (WKB) method [22]. However, it is intrinsically approximate and may not be accurate in many cases. Other researchers have developed closed-form solutions to the wave equations for some special cases of materials’ temporal profile, but they could not generalize the formulations for all possible scenarios [23]. To this end, a systematic solution strategy for general LSI problems is still lacking.

In light of the shortcomings of the above-mentioned theories, in this paper, we propose a system response analysis method. By retrieving the system function of LSI problems from only one set of test input and corresponding output signals (via simulation or measurement), the response to any arbitrary input can be efficiently computed. This system function computation is just a one time exercise done using numerical tools (or measurements). Once extracted, any arbitrary input profile changes, the output response can be obtained in real-time, without any further simulations or measurements. Section 2 demonstrates the procedure of the proposed method, And then in Section 3, we apply this approach to several scenarios with different temporal profiles of materials.

2. FORMALISM

In order to demonstrate the design procedure, first, let us consider a 1D harmonic wave with electric field: \( E(x,t) = E_0 e^{i(kx-\omega t + \phi)} \), where \( k \) is the wavenumber, \( \omega \) is the angular frequency, and \( \phi \) is the phase constant. The EM wave varies in both space and time, as shown in Fig. 1. The electric field is projected as a curve varying in time at a specific location (red), or is projected as a curve varying in space at a specific time (black).
For conventional LTI problems, Fourier transform is conducted on the time coordinate and the resultant relationship is shown in Eq. (1). The input (at \( x = x_i \)) and output (at \( x = x_o \)) curves can be transformed into frequency domain as \( E_i(\omega) \) and \( E_o(\omega) \), respectively. These parameters are related to the system function \( S(\omega) \) in the frequency domain by:

\[
E_o(\omega) = S(\omega) \cdot E_i(\omega)
\]  \( \text{(1)} \)

In order to prove the effectiveness of the proposed method, we apply it to several LSI systems, and compare the results with simulated ones for validation. The whole process is illustrated in Fig. 2: First, we conduct a test process (the 1st row in blue background) to obtain the system function (the 2nd row in orange background) using Eq. (3). Next, we apply the system function to a different input, which we refer to as ‘application input’. Then the output can be computed by Eq. (4) (the 3rd row in white background).

For better comparison, both the test input \( (E_i(x)) \) and application input \( (E_i'(x)) \) are set to be of the same form for all examples. It is defined in space domain at \( t_i = 0 \) using a modulated Gaussian pulse:

\[
E(x) = -\exp\left[-\frac{4\pi(x-x_0)^2}{x_n^2}\right] \cos[k_0 x], \quad \text{(5)}
\]

where \( x_0 \) denotes the reference point, \( x_n \) is a parameter deciding the bandwidth, and \( k_0 \) is the center wavenumber of the modulated signal. In this paper, \( k_0 \) is set to \( 1.05 \times 10^7 / \text{m} \) (wavelength \( \lambda_0 = 600 \text{ nm} \)), which corresponds to a period \( T_o = 4 \text{ fs} \) in free space. To obtain a system response that is valid over a broadband regime, we choose a relatively small value of \( x_n = 3.33A_0 \). The test input and its Fourier transform is shown in Fig. 2 (a) and (b), where the green block indicates its effective bandwidth.

As for application input, it can be arbitrarily chosen in the ideal case. However, to ensure that the input lies inside the effective bandwidth of the system function, we still used a modulated Gaussian pulse in Eq. (5), but with narrower bandwidth \( (x_n = 16.67\lambda_0) \). The application input and its Fourier transform is shown in Fig 2(h) and (i).

A. A single temporal boundary

First, we investigate a simple system where there is only a single temporal boundary, denoted as 3T1. The temporal boundary is set at \( t_1 = 60T_o \) when the relative permittivity \( \varepsilon_r \) of the medium changes abruptly from 1 to 4, as shown in Fig. 2(e). The output field is captured at a time \( t_o = 90T_o \), which can be computed using FDTD method as depicted in Fig. 2(c). As can be seen, the output curve is composed of a forward (FW) and a backward (BW) term, which propagate in opposite directions. We studied two different sub-cases depending on whether the FW and BW are considered separately or together.

1. FW and BW terms decoupled

After decoupling \( E_{t_0} \) into FW \( (E_{t_0-fw}) \) and BW \( (E_{t_0-bw}) \) terms, we get their Fourier transform in wavenumber domain, which are shown in Fig 2 (d). Applying Eq. (3), the system function can be calculated as \( S_{fw}(k) = E_{t_0-fw}(k)/E_{t_0}(k) \) and \( S_{bw}(k) = E_{t_0-bw}(k)/E_{t_0}(k) \), for the FW and BW terms, respectively. The magnitudes and phases of \( S_{fw}(k) \) and \( S_{bw}(k) \) are plotted in Fig 2 (f) and (g). It can be observed that their magnitudes are constant over the whole bandwidth. This is as expected since the dispersion of the medium is not considered. Moreover, their magnitudes are...
numerically equal to the reflection and transmission coefficients, which match well with the theoretical results [7, 8].

As for the phase plot in Fig. 2(g), we know from Eq (3) that the phase of the system function is the phase difference between the output and input. For FW term, the phase is identical to that of the input, therefore, \( \angle S_{fw}(k) \) is almost constantly zero. On the other hand, as shown in Fig. 2(c), at time \( t_o \), the BW term has a constant spatial distance \( \Delta x \) to the FW term, which leads to a phase difference between FW and BW:

\[
\varphi(k) = k \cdot \Delta x. \quad (5)
\]

Therefore, the phase difference between BW and input is also \( k \)-dependent (Eq. (5)), which leads to the oscillations in the \( |S_{bw}(k)| \) (the black curve in Fig 2(g)).

Once the system function is extracted, we now excite this LSI system with an arbitrary application input and compute its output using the retrieved system function. The corresponding Fourier transform \( E_i(k) \) is shown in Fig. 2 (f). By multiplying \( E_i(k) \) with the obtained system functions \( S_{fw}(k) \) and \( S_{bw}(k) \), the output in wavenumber domain \( E_o(k) \) can be computed. Their phases and magnitudes are plotted in Fig 2(j) and (k). The results from the theory agree well with the simulated ones within the effective bandwidth of the system function.

2. FW and BW terms coupled

In some scenarios, if \( \Delta x \) is comparable to or smaller than the pulse length, then the FW and the BW terms of the output may overlap in space and cannot be decoupled easily. Therefore, it is instructive to investigate the 'total' test output by adding the FW and BW terms in Fig. 2(c). The Fourier transform of the total output and test input is shown in Fig. 3 (a). The plot shows significant ripples because the
phase difference between the FW and BW term is $k$-dependent, as explained above. Therefore, the phase difference would lead to these oscillatory features in the curves of both the test output (Fig. 3(a)) and system function (Fig. 3(b)).

Similarly, the system function can be obtained by Eq. (3), whose magnitude is plotted in Fig. 3(b). Then we apply this system function to the application input. Fig. 3(c) and (d) compares the magnitude and phase of the output obtained using proposed method as compared against FDTD simulation. The results demonstrate a good agreement within the effective bandwidth of the system function.

**B. Material properties change gradually**

It is worthy to mention that this method is not limited to temporal boundary problems, but also works for systems where the material properties (such as permittivity) change gradually. The temporal system under investigation, denoted as $S_2$, has a temporal profile of the permittivity shown in Fig. 4 (a). The output field is captured at $t_0 = 50T_0$. Overall, a gradually changing permittivity tends to produce very small BW terms when compared to the temporal boundary cases. Therefore, the curve of the system function is much smoother within the effective bandwidth, as shown in Fig. 4 (b). $|S(k)|$ is nearly constant around $|S(k_0)| = 0.84$ within the effective bandwidth. This makes sense physically because the medium’s permittivity is higher at $t_0$ than $t_1$. To make $D$ field constant in time, the magnitude of $E$ field would decrease (Fig. 3(c)). One interesting observation is that, although there is nearly no reflection, $S(k_0)$ is smaller than 1. This does not violate conservation of energy because time-varying media require energy exchange with external sources [11, 14].

Once the system function is extracted, we again apply the application input to the obtained system function. The calculated output correlates well with that obtained from the FDTD simulation, as compared in Fig. 4 (c) and (d).

**C. Cascading systems**

One of the biggest advantages of the proposed approach is that it can be easily generalized to cascading systems. If $N$ subsystems are ‘cascaded’ (i.e., they appear in sequence in time), the total system response can be expressed as

$$S_{\text{total}}(k) = \prod_{i=1}^{N} S_i(k)$$

where $S_i(k)$ is the system function of the $i$th sub-system. As an example, we consider a system composing of two subsystems cascaded: $S_1$ and $S_2$, which are the same as the case A and B, respectively. The permittivity profile of the whole system is depicted in Fig. 5 (a). Its system function can be calculated as $S_{\text{total}}(k) = S_1(k) \times S_2(k)$, which is plotted in Fig. 5 (b). In Fig. 5 (c) and (d), we compare the magnitudes and phases of the output from the proposed method and the FDTD simulation, respectively. The results agree well over the effective bandwidth, where the small discrepancies are due to numerical error of FDTD method. This example demonstrates that this method would allow us to investigate a complex temporal system by decomposing it into multiple relatively simpler subsystems. Comparing case A2, B, and C (Fig. 3(b), 4(b), 5(b)), we can observe that it is the abrupt change of permittivity that leads to the fluctuations in the curve of $|S(k)|$, while the gradual change of permittivity only modulates the ‘overall’ magnitude of it.
4. CONCLUSION

In this paper, we propose a system response analysis method to address LSI problems in wavenumber domain. The system function of an LSI system can be retrieved from one set of test process, either numerically or experimentally. Once the system function is computed, the output due to any arbitrary input can be easily computed. The validity of the formalism is guaranteed by the fact that wavenumber, instead of frequency, is constant in LSI systems. Therefore, this method not only works well, but also sheds some light into the fundamental symmetry between space and time. More importantly, the test process can be conducted either numerically or experimentally, which brings numerical or measured system functions of the temporal system. This significantly broadens the capabilities of current theoretical approaches existing in the literature. Furthermore, the system function of a cascaded structure can be derived simply by multiplying those of each subsystem. This aids in the analysis of complicated temporal systems by decomposing them into simpler ones. To this end, our systematic methodology would greatly facilitate the research on LSI systems in the future. It can serve as a powerful tool for all LSI problems, analogous to frequency domain analysis for LTI counterpart.

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