An Algorithm to Compute the Topological Euler Characteristic, the Chern-Schwartz-MacPherson Class and the Segre class of Subschemes of Some Smooth Complete Toric Varieties

Martin Helmer

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Abstract

Let $X_\Sigma$ be a complete smooth toric variety of dimension $n$ defined by a fan $\Sigma$ where all Cartier divisors in $\text{Pic}(X_\Sigma)$ are nef and let $V$ be a subscheme of $X_\Sigma$. We show a new expression for the Segre class $s(V, X_\Sigma)$ in terms of the projective degrees of a rational map associated to $V$. In the case where the number of primitive collections of rays in the fan $\Sigma$ is equal to the number of generating rays in $\Sigma(1)$ minus the dimension of $X_\Sigma$ we give an explicit expression for the projective degrees which can be easily computed using a computer algebra system. We apply this to give effective algorithms to compute the Segre class $s(V, X_\Sigma)$, the Chern-Schwartz-MacPherson class $c_{SM}(V)$ and the Euler characteristic $\chi(V)$ of $V$. These algorithms can, in particular, compute the Segre class, Chern-Schwartz-MacPherson class and Euler characteristic of arbitrary subschemes of any product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_j}$ (over an algebraically closed field of characteristic zero). Running time bounds for several of the algorithms are given and the algorithms are tested on a variety of
examples. In all cases the algorithm to compute the Segre class is found to offer significantly increased performance over other known algorithms. At present we know of no other algorithms which compute Chern-Schwartz-MacPherson classes and Euler characteristics in this setting.

1 Introduction

The Euler characteristic has a long and storied history in mathematics. Beginning with observations of Decartes (circa 1639) and first formalized in Euler’s polyhedral formula (circa 1751) this invariant has become an important tool for the consideration of a diverse selection of mathematical problems. Modern realizations of the Euler characteristic have proved especially important in algebraic geometry and algebraic topology, enabling, among other things, the classification of orientable surfaces. The Euler characteristic has also proved to be of utility in applied mathematics; recently several authors have noted applications of the Euler characteristic of projective varieties to problems in statistics and physics. For example, Huh [23] and Rodriguez and Wang [31] apply the Euler characteristic of projective varieties to study problems of maximum likelihood estimation in algebraic statistics. Applications to string theory in physics include Aluffi and Esole [6] and Collinucci, Denef, and Esole [8].

One of the first computational approaches to calculate the Euler characteristic of a subscheme of projective space was to do so by computing Hodge numbers and using the fact that the Euler characteristic is an alternating sum of Hodge numbers. This approach has been implemented by the Macaulay2 [18] where the Hodge numbers are found by computing the ranks of the appropriate cohomology rings. This approach, however, has significant drawbacks in both applicability and performance. Specifically, this method may only be used for smooth subschemes and the calculation of the required cohomology rings and their respective ranks is computationally expensive.

Let $M$ be a smooth variety and let $V$ be some subscheme of $M$. The Euler characteristic of $V$ may, in fact, be obtain directly from the Chern-Schwartz-MacPherson class of $V$, $c_{SM}(V)$. More specifically, if we consider $c_{SM}(V)$ as
an element of the Chow ring of $M$, $A^*(M)$, we have that $\chi(V)$ is equal to the degree of the zero dimensional component of $c_{SM}(V)$. It is this method that we shall use to obtain the Euler characteristic. In the case of subschemes of a projective space $\mathbb{P}^n$ this approach has been used by several authors (e.g. Aluffi [3], Jost [24], the author [21]) to construct different algorithms which are capable of calculating Euler characteristics of complex projective varieties.

Aside from containing the Euler characteristic, $c_{SM}$ classes are an important algebraic geometric invariant in their own right as they provide a generalization of the Chern class, and its functorial properties, to singular schemes. While there are other generalizations of the Chern class to singular schemes (i.e. the Chern-Fulton and Chern-Fulton-Johnson classes, see [4] for a discussion of these), the $c_{SM}$ class is unique in that it preserves the relation between Chern classes and the Euler characteristic. In addition to this the $c_{SM}$ class has been employed in the context of string theory in physics, see for example Aluffi and Esole [7].

In this note we will consider the computation of the Segre class, and the $c_{SM}$ class (and hence the Euler characteristic) of subschemes in a much more general setting which encompasses previous work to compute these invariants for subschemes of $\mathbb{P}^n$. More specifically we will significantly generalize all of the results and algorithms presented by the author in [21, 22] from the setting of subschemes of projective varieties to the setting of subschemes $V$ of certain smooth complete toric varieties without torus factors $X_\Sigma$, including arbitrary products of projective spaces. The conditions we will impose on the toric varieties we will consider here are the following: first we will require that all the divisors $D_\rho$ associated to a generating ray $\rho \in \Sigma(1)$ are nef and second we will require that the number of primitive collections in $\Sigma$ is equal to the number of generating rays in $\Sigma(1)$ minus the dimension of $X_\Sigma$.

As such, throughout this note $X_\Sigma$ will always denote a smooth complete toric variety without torus factors and with all Cartier divisors corresponding to rays in $\Sigma(1)$ being nef. In this setting we will give an algorithm to compute the Segre class $s(V, X_\Sigma)$, the Chern-Schwartz-MacPherson class $c_{SM}(V)$ and the Euler characteristic $\chi(V)$, additionally we will give a second algorithm to compute the $c_{SM}$ class in the special case where $V$ is a global complete intersection. This second algorithm offers performance improvements in some
cases. This will, in particular, allow us to compute characteristic classes of subschemes of some product of projective spaces \( \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_j} \) (over an algebraically closed field of characteristic zero).

We now give an example of the computation of the Segre class, the \( c_{SM} \) class and the Euler characteristic for a singular variety in \( \mathbb{P}^4 \times \mathbb{P}^2 \). Note that since the variety \( V \) considered in the example is singular the results could not be obtained with standard Chern class computations.

**Example 1.1.** Let \( k \) be an algebraically closed field of characteristic zero and let \( V = V(I) \) be the subvariety of \( \mathbb{P}^4 \times \mathbb{P}^2 \cong \text{Proj}(k[x_0, \ldots, x_4]) \times \text{Proj}(k[y_0, \ldots, y_2]) \) defined by the ideal

\[
I = (5x_0y_0, 9x_2y_1y_2 - 4x_1y_2^2)
\]

in \( R = k[x_0, x_1, x_2, x_4, y_0, y_1, y_2] \). Also let \( A^*(\mathbb{P}^4 \times \mathbb{P}^2) \cong \mathbb{Z}[h_1, h_2]/(h_1^5, h_2^3) \) be the Chow ring of \( \mathbb{P}^4 \times \mathbb{P}^2 \).

**Using Algorithm 1 with input** \( I \) **we obtain the Segre class**

\[
s(V, \mathbb{P}^4 \times \mathbb{P}^2) = 170h_1^4h_2^2 - 30h_1^4h_2 - 90h_1^3h_2^2 + 3h_1^4 + 18h_1^3h_2 + 40h_1^2h_2^2 - 2h_1^3 + 9h_1^2h_2 - 13h_1h_2^2 + h_1^2 + 3h_1h_2 + 2h_2^2 \in A^*(\mathbb{P}^4 \times \mathbb{P}^2).
\]

**Using Algorithm 2 with input** \( I \) **we obtain the Chern-Schwartz-MacPherson class**

\[
c_{SM}(V) = 13h_1^4h_2^2 + 11h_1^4h_2 + 23h_1^3h_2^2 + 3h_1^4 + 16h_1^3h_2 + 21h_1^2h_2^2 + 3h_1^3 + 11h_1^2h_2 + 10h_1h_2^2 + h_1^2 + 3h_1h_2 + 2h_2^2 \in A^*(\mathbb{P}^4 \times \mathbb{P}^2)
\]

**and/or the Euler characteristic** \( \chi(V) = 13 \).

This note will be organized as follows.

In §2 begin by precisely stating the problem to be considered and the setting in which we shall work. Following this we review several previous results and constructions which are important for this work. Previous work is also reviewed in §2.5.

The main results of this chapter are presented in §3. Let \( V \) be a subscheme of a smooth complete toric variety \( X_\Sigma \) without torus factors and with the
property that all divisors $D_\rho$ associated to a generating ray $\rho \in \Sigma(1)$ are nef. In Theorem 3.2 we prove an expression for the Segre class $s(V, X_\Sigma)$ in terms of classes in the Chow ring which depend solely on the so-called projective degrees (see (30)). These projective degrees generalize the notion of projective degrees of a rational map between projective spaces, see for example Harris [19, Example 19.4]. Now assume that number of primitive collections in $\Sigma$ is equal to the number of generating rays in $\Sigma(1)$ minus the dimension of $X_\Sigma$; Theorem 3.3 gives a method to compute the projective degrees in this setting and hence can be used to compute the Segre class $s(V, X_\Sigma)$. In Theorem 3.5 we give an expression for the $c_{SM}$ class of certain types of complete intersection subschemes of smooth complete toric varieties; this result generalizes the main theorem of the author in [22]. Note that these results are, in particular, applicable for subschemes of some product of projective spaces.

Let $X_\Sigma$ be a toric variety such that all divisors $D_\rho$ associated to a generating ray $\rho \in \Sigma(1)$ are nef and such that the number of primitive collections in $\Sigma$ is equal to the number of generating rays in $\Sigma(1)$ minus the dimension of $X_\Sigma$. In §4 we apply the results of §3 to construct algorithms to compute the Segre and Chern-Schwartz-MacPherson classes and the Euler characteristic of subschemes of $X_\Sigma$. Our algorithm to compute Segre classes of arbitrary subschemes of $X_\Sigma$ is given in Algorithm 1. In Algorithm 2 we present an algorithm to compute the $c_{SM}$ class and Euler characteristic in the toric setting using the inclusion/exclusion property of $c_{SM}$ classes (see Proposition 2.2). In Algorithm 3 we present an algorithm to compute the $c_{SM}$ class of certain complete intersection subschemes of $X_\Sigma$ without using inclusion/exclusion.

In §5 we discuss the performance of these algorithms. The running times of our test implementation on a variety of examples are given in §5.1 and are compared with those of other known algorithms where possible. Running time bounds for Algorithm 1 and Algorithm 2 are given in §5.2.

The Macaulay2 [18] implementation of the algorithms for computing Segre classes, $c_{SM}$ classes and Euler characteristics of subschemes of smooth complete toric varieties (without torus factors and satisfying the assumptions of Theorem 3.3) presented in this note can be found at https://github.com/Martin-Helmer/char-class-calc-toric.
2 Setting, Review and Problem

Throughout this note we only consider ambient spaces which are smooth complete toric varieties without torus factors. A toric variety is said to have no torus factors if it is equivariantly isomorphic to the product of a nontrivial torus and a toric variety of smaller dimension, see Proposition 3.3.9 of Cox, Little, and Schenck [11].

We will also (primarily) consider subschemes of toric varieties over the complex numbers and take $k = \mathbb{C}$ throughout the document. This is done because several of the results of Cox, Little, and Schenck [11] which we shall need to make use of are stated in this setting. We note, however, that if the toric variety $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_j}$ is the ambient space for our characteristic class computations we could work instead over $k$ any algebraically closed field (of sufficiently large characteristic), in the case of Segre class computations and over $k$ any algebraically closed field of characteristic zero for $c_{SM}$ class computations (the characteristic zero assumption is present and is needed in MacPherson’s [29] construction of the $c_{SM}$ class).

Let $X_\Sigma$ be a smooth complete toric variety (without torus factors) of dimension $n$ with homogeneous coordinate ring $R$. Let $V = V(I)$ be any subscheme of $X_\Sigma$ defined by an ideal $I$ in $R$ which is homogeneous with respect to the grading on $R$. Additionally we assume that all divisors in Pic($X_\Sigma$) are nef and that there are $m - n$ primitive collections in the fan $\Sigma$ where $\Sigma(1)$ contains $m$ generating rays (these assumptions are satisfied by the toric varieties $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_j}$, for example). The problem we consider in this note is the following: determine the Segre class of $V$ in $X_\Sigma$, $s(V, X_\Sigma)$, the Chern-Schwartz-MacPherson class of $V$, $c_{SM}(V)$, and the Euler characteristic of $V$, $\chi(V)$, in a time efficient manner on a computer algebra system.

We will represent all characteristic classes as elements of the Chow ring of $X_\Sigma$, $A^*(X_\Sigma)$. Proposition 2.4 gives the concrete realization of this Chow ring which will be used for all algorithms described in this note. Note that we will abuse notation and write $s(V, X_\Sigma)$, $c_{SM}(V)$ and $c(V)$ for the pushforwards to $X_\Sigma$ of the Segre class of $V$, the $c_{SM}$ class of $V$ and the total Chern class of $V$ (that is the total Chern class of the tangent bundle of $V$), respectively.

To establish the setting for this work we review the definitions of the Segre
class of a subscheme in §2.1 and of the Chern-Schwartz-MacPherson class in §2.2.

2.1 Segre Class

The Segre class is an important invariant in intersection theory, both because it contains important intersection theoretic information and because it can be used to construct other commonly studied structures and invariants. In particular the Chern-Fulton class (see (3)) and the Chern-Schwartz-MacPherson class (see Proposition 2.7) may be defined in terms of Segre classes.

For $V$ a proper closed subscheme of a variety $W$, we may define the Segre class of $V$ in $W$ as

$$s(V, W) = \sum_{j \geq 1} (-1)^{j-1} \eta_*(\tilde{V}^j) = \eta_* \left( \frac{[\tilde{V}]}{1 + [V]} \right) \in A^*(V)$$

(1)

where $\tilde{V}$ is the exceptional divisor of the blow-up of $W$ along $V$, $\text{Bl}_V W$, $\eta : \tilde{V} \to V$ is the projection (and $\eta_*$ is its pushforward), the class $\tilde{V}^k$ is the $k$-th self intersection of $\tilde{V}$ and $[\tilde{V}]$ is the class of $\tilde{V}$ in the Chow ring of the blow-up, $A^*(\text{Bl}_V W)$. See Fulton [16, §4.2.2] for further details.

The total Chern class of a $j$-dimensional nonsingular variety $V$ is defined as the Chern class of the tangent bundle $T_V$; we express this as $c(V) = c(T_V) \cdot [V]$ in the Chow ring of $V$, $A^*(V)$. A definition of the Chern class of a vector bundle can be found in Fulton [16, §3.2]. Following from the Hirzebruch-Riemann-Roch theorem, we have that the degree of the zero dimensional component of the total Chern class of a smooth variety is equal to the Euler characteristic, that is

$$\int c(T_V) \cdot [V] = \chi(V).$$

(2)

Here we let $\int \alpha$ denote the degree of the zero dimensional component of the class $\alpha \in A_*(V)$, that is the degree of the part of $\alpha$ in $A^0(V)$.

We note that any algorithm to compute the Segre class will immediately give us an algorithm to compute the Chern-Fulton class $c_F$ (refered to as the Canonical class by Fulton [16]) of a subscheme $V$ of a smooth variety $M$ over
an algebraicly closed feild. Specifically we have that
\[ c_F(V) = c(T_M) \cdot s(V, M) \in A^*(M). \] (3)

The Chern-Fulton class \( c_F \) is a generalization of the Chern class to singular schemes, see, for example, Fulton [16, Examples 4.2.6, 19.1.7]. In particular then, any method to compute the Segre class will also give the Chern class
\[ c(V) = c(T_V) \cdot [V] = c(T_M) \cdot s(V, M) \]
in the case where \( V \) is a smooth subscheme of \( M \), also see Eklund, Jost and Peterson [13, Remark 4.2].

2.2 The Chern-Schwartz-MacPherson Class, Inclusion/Exclusion and the Euler Characteristic

While there are several generalizations of the total Chern class to singular varieties, all of which agree with \( c(T_V) \cdot [V] \) for nonsingular \( V \), the Chern-Schwartz-Macpherson class is the only generalization which satisfies a property analogous to (2) for any \( V \), i.e.
\[ \int c_{SM}(V) = \chi(V). \] (4)

Here we review the construction of the \( c_{SM} \) classes, giving the definition in the manner considered by MacPherson [29]. For a scheme \( V \), we take \( \mathcal{C}(V) \) to be the abelian group of finite linear combinations \( \sum_W m_W 1_W \), where \( W \) are (closed) subvarieties of \( V \), \( m_W \in \mathbb{Z} \), and \( 1_W \) denotes the function that is 1 in \( W \), and 0 outside of \( W \). Elements \( f \in \mathcal{C}(V) \) are referred to as constructible functions and the group \( \mathcal{C}(V) \) is termed the group of constructible functions on \( V \). We may make \( \mathcal{C} \) into a functor by letting \( \mathcal{C} \) map a scheme \( V \) to the group of constructible functions on \( V \) and a proper morphism \( f : V_1 \to V_2 \) is mapped by \( \mathcal{C} \) to
\[ \mathcal{C}(f)(1_W)(p) = \chi(f^{-1}(p) \cap W), \quad W \subset V_1, \quad p \in V_2 \text{ a closed point.} \]

The Chow group functor \( A_* \) is also a functor from algebraic varieties to albeian groups. The \( c_{SM} \) class may be realized as a natural transformation between these two functors.
Definition 2.1. The Chern-Schwartz-MacPherson class is the unique natural transformation between the constructible function functor and the Chow group functor, that is $c_{SM} : \mathcal{C} \rightarrow \mathbb{A}_*$ is the unique natural transformation satisfying:

- (Normalization) $c_{SM}(1_V) = c(T_V) \cdot [V]$ for $V$ non-singular and complete.
- (Naturality) $f_*(c_{SM}(\phi)) = c_{SM}(\mathcal{C}(f)(\phi))$, for $f : X \rightarrow Y$ a proper transformation of projective varieties, $\phi$ a constructible function on $X$.

Let $V_{\text{red}}$ denote the support of a scheme $V$, the notation $c_{SM}(V)$ is taken to mean $c_{SM}(1_V)$ and hence, since $1_V = 1_{V_{\text{red}}}$, we denote $c_{SM}(V) = c_{SM}(V_{\text{red}})$.

The $c_{SM}$ class satisfies the same inclusion/exclusion relation as the Euler characteristic. That is for $V_1, V_2$ subschemes of a smooth variety $M$ we have that

$$c_{SM}(V_1 \cap V_2) = c_{SM}(V_1) + c_{SM}(V_2) - c_{SM}(V_1 \cup V_2). \quad (5)$$

Note that this relation for $c_{SM}$ classes will allow us to reduce all computation of $c_{SM}$ classes to the case of hypersurfaces. From this property we obtain the following proposition, discussed informally by Aluffi [3]; Proposition 2.2 follows directly from (5).

**Proposition 2.2.** Let $V = X_0 \cap \cdots \cap X_r = V(f_0) \cap \cdots \cap V(f_r)$ be a subscheme of a smooth variety $M$ with coordinate ring $R$. Write the polynomials defining $V$ as $F = (f_0, \ldots, f_r) \in R$ and let $F(S) = \prod_{i \in S} f_i$ for $S \subset \{1, \ldots, r\}$. Then,

$$c_{SM}(V) = \sum_{S \subset \{1, \ldots, r\}} (-1)^{|S|+1} c_{SM}(V(F(S)))$$

where $|S|$ denotes the cardinality of the integer set $S$. 


2.3 The Homogeneous Coordinate Ring of a Smooth Complete Toric Variety

In this subsection we briefly review some notation and results of Cox [9, 10] which will be used throughout this note.

In this subsection $X_\Sigma$ will denote a smooth complete toric variety (without torus factors) defined by a fan $\Sigma \subset N \cong \mathbb{Z}^l$. Let $\Sigma(1)$ denote the set of one dimensional cones, also referred to as rays, in the fan $\Sigma$. The homogeneous coordinate ring of $X_\Sigma$ is given by

$$R = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)],$$

(6)

$R$ is also referred to as the Cox ring or total coordinate ring. We will grade the ring $R$ by defining the degree of a monomial

$$x = \prod_{\rho \in \Sigma(1)} x_\rho^{a_\rho}$$

(7)

to be

$$\deg(x) = \left[ V \left( \prod_{\rho \in \Sigma(1)} x_\rho^{a_\rho} \right) \right] \in A^1(X_\Sigma).$$

(8)

With this grading, and considering $x$ to be monomials of (7), if we set

$$R_\alpha = \bigoplus_{\deg(x) = \alpha} \mathbb{C} \cdot x$$

(9)

then we have that

$$R = \bigoplus_{\alpha \in A^1(X_\Sigma)} R_\alpha.$$

(10)

Additionally $R_\alpha \cdot R_\beta = R_{\alpha + \beta}$.

For a cone $\sigma \in \Sigma$ take $\sigma(1) = \{ \rho \in \Sigma(1) \mid \rho \text{ is a face of } \sigma \}$ to be the set of one-dimensional faces of $\sigma$. We may define the irrelevant ideal of the coordinate ring $R$ as the ideal

$$B = \left( \prod_{\rho \not\in \sigma(1)} x_\rho \mid \sigma \in \Sigma \right) \subset R.$$

(11)
The ideal $B$ describes the combinatorial structure of the fan $\Sigma$.

Take $\mathbb{C}^{\Sigma(1)} = \text{Spec}(\mathbb{C}[x_\rho \mid \rho \in \Sigma(1)])$ to be an affine space. Define the group $G = \text{Hom}_\mathbb{Z}(A^1(X_\Sigma), \mathbb{C}^*)$.

Cox [10, Theorem 2.1] (see also Cox [9, Theorem 3.3]) shows the following theorem, which we specialize to the case of a smooth complete toric variety.

**Theorem 2.3** (Cox [10]). Let $X_\Sigma$ be a smooth complete toric variety. Then $X_\Sigma$ is the geometric quotient of $\mathbb{C}^{\Sigma(1)} - V(B)$ by $G$.

In light of this result we may regard elements of $\mathbb{C}^{\Sigma(1)} - V(B)$ as “homogeneous coordinates” for $X_\Sigma$.

We now consider the structure of $V(B)$ and $G$ in greater detail. We say the collection $v_{\rho_1}, \ldots, v_{\rho_s}$ of ray generators (i.e. $\rho_i = \langle v_{\rho_i} \rangle$) is a primitive collection (see Cox [9]) if the collection $v_{\rho_1}, \ldots, v_{\rho_s}$ does not lie in any cone $\sigma \in \Sigma$ but every proper subset does. We may write an irreducible decomposition of $V(B)$ as

$$V(B) = \bigcup_{v_{\rho_1}, \ldots, v_{\rho_s} \text{ primitive}} V(x_{\rho_1}, \ldots, x_{\rho_s}).$$

(12)

### 2.4 The Chow Ring of a Smooth Complete Toric Variety

Let $k = \mathbb{C}$. The following proposition gives us a simple method to compute the Chow ring of a smooth, complete toric variety $X_\Sigma$. We will use this result to compute the Chow ring $A^*(X_\Sigma)$ in the algorithms of §4.

**Proposition 2.4** (Theorem 12.5.3 of Cox, Little, and Schenck [11]). Let $N$ be an integer lattice with dual lattice $M$ and let $X_\Sigma$ be a complete and smooth toric variety with generating rays $\Sigma(1) = \rho_1, \ldots, \rho_m$ where $\rho_j = \langle v_j \rangle$ for $v_j \in N$. Then the Chow ring of $X_\Sigma$ has the following presentation

$$A^*(X_\Sigma) \cong \mathbb{Z}[x_1, \ldots, x_r]/(\mathcal{I} + \mathcal{J}),$$

(13)

with the isomorphism map specified by $[V(\rho_i)] \mapsto [x_i]$. Here $\mathcal{I}$ denotes the Stanley-Reisner ideal of the fan $\Sigma$, that is the ideal in $\mathbb{Z}[x_1, \ldots, x_r]$ specified
by
\[ \mathcal{I} = (x_i \cdots x_s \mid i, j \text{ distinct and } \rho_{i_1} + \cdots + \rho_{i_s} \text{ is not a cone of } \Sigma) \] (14)
and \( \mathcal{J} \) denotes the ideal of \( \mathbb{Z}[x_1, \ldots, x_r] \) generated by linear relations of the rays, that is \( \mathcal{J} \) is generated by linear forms
\[ \sum_{j=1}^{r} \langle m, v_j \rangle x_j \] (15)
for \( m \) ranging over some basis of \( \mathcal{M} \).

The following lemma gives us a generating set for the Chow ring.

**Lemma 2.5** (Lemmas 12.5.1 and 12.5.2 of Cox, Little and Schenck [11]). Let \( X_\Sigma \) be a smooth complete toric variety of dimension \( n \). The \( [V(\sigma)] \) for \( \sigma \in \Sigma \) generate \( A^*(X_\Sigma) \) as an abelian group. Specifically \( [V(\sigma)] \) for \( \sigma \) of dimension \( n-j \) generate the dimension \( j \) Chow group \( A_j(X_\Sigma) \).

Additionally if \( \rho_1, \ldots, \rho_d \in \Sigma(1) \) are distinct then we have the following equality in \( A^*(X_\Sigma) \),
\[ [V(\rho_1)] \cdots [V(\rho_d)] = \begin{cases} [V(\sigma)] & \text{if } \sigma = \rho_1 + \cdots + \rho_d \in \Sigma \\ 0 & \text{otherwise}. \end{cases} \] (16)

In particular each generating ray \( \rho_1, \ldots, \rho_m \) in \( \Sigma(1) \) can be associated to a divisor \( D_\rho = [V(\rho)] \) in \( A^1(X_\Sigma) \cong \text{Pic}(X_\Sigma) \) where Pic denotes the Picard group. These divisors can in turn be associated with an \( x_i \) via Proposition 2.4 above.

In §3 we will need an additional property for the elements of \( A^1(X_\Sigma) \cong \text{Pic}(X_\Sigma) \). Let \( X \) be a normal toric variety; a Cartier divisor \( D \) on \( X \) is termed *numerically effective* or *nef* if \( D \cdot C \geq 0 \) for every irreducible complete curve \( C \subset X \).

**Theorem 2.6** (Theorem 6.3.12 of Cox, Little and Schenck [11]). Let \( D \) be a Cartier divisor on a complete toric variety \( X_\Sigma \). The following are equivalent:

- \( D \) is basepoint free, i.e., \( \mathcal{O}_{X_\Sigma}(D) \) is generated by global sections.
- \( D \) is nef.
- \( D \cdot C \geq 0 \) for all torus-invariant irreducible complete curves \( C \subset X_\Sigma \).
2.5 Previous Algorithm to Compute the Segre Class of a Subscheme of a Smooth Projective Toric Variety

In [30] Moe and Qviller give an algorithm to compute the Segre class of a subscheme of certain smooth projective toric varieties. The algorithm of Moe and Qviller [30] is based on a result which gives an expression for the Segre class of a subscheme of certain smooth projective toric variety in terms of the Chow ring classes of certain residual sets which are computed via saturation. This result of Moe and Qviller [30] generalizes a previous result of Eklund, Jost and Peterson [13] which gave an expression for the Segre class of a subscheme of \( \mathbb{P}^n \) in terms of residual sets having a similar structure. For both these theorems the residual sets are in the sense of Fulton’s residual intersection theorem/formula (Theorem 9.2 and Corollary 9.2.3 of Fulton [16]).

We note that the result Moe and Qviller [30] also uses the results of Section 4.4 of Fulton [16] in the proof of their main theorem (Theorem 2 of [30]) to obtain an explicit expression for the class of the exceptional divisor of a blow-up. This explicit expression is needed to construct their subsequent algorithm, as such their result may only be applied in cases where the subscheme being considered is the zero scheme of some set of global sections. Thus to be sure of being able to use this result for any subscheme of a given toric variety \( X_\Sigma \) we must have that all divisors are nef (as this is equivalent to being able to define all subschemes as the zero scheme of global sections for smooth complete toric varieties). In particular their algorithm for computing Segre classes has the same restrictions as the algorithm presented in this note.

Moe and Qviller [30] describe their algorithm which uses this result to obtain the Segre classes by computing saturations to find the Chow ring class of residual sets in Section 5 of [30].
2.6 \( c_{SM} \) Class of a Hypersurface For a Subscheme of any Smooth Variety

We now give Theorem I.4 of Aluffi [2] which will allow us to compute \( c_{SM} \) classes by computing Segre classes.

**Proposition 2.7** (Theorem I.4 of Aluffi [2]). Let \( V \) be a hypersurface in a nonsingular variety \( M \) and let \( Y \) be the singularity subscheme of \( V \). Then we have

\[
c_{SM}(V) = c(T_M) \cdot \left( s(V, M) + \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n-i}{j} [V]^j \cdot (-1)^{n-i} s_{i+j}(Y, M) \right)
\]

where \([V]\) is the class of \( V \) in \( A^*(M) \). Here \( s_{i+j}(Y, M) \) denotes the dimension \( i+j \) component of \( s(Y, M) \) and \( T_M \) denotes the tangent bundle to \( M \).

2.7 The Milnor Class of Certain Complete Intersection Subschemes of a Smooth Variety

Let \( X \) be a closed locally complete intersection subscheme of a smooth ambient variety \( M \) and let \( T_M \) denote the tangent bundle of \( M \). Recall from (3) that the Chern-Fulton class is given by:

\[
c_F(X) = c(T_M) \cdot s(X, M).
\]  

(18)

Also note that since we assume that \( X \) is a locally complete intersection (meaning there exists a regular embedding \( i : X \to M \)) then by Proposition 4.1 of Fulton [16] we have

\[
c_F(X) = c(T_M) \cdot s(X, M) = c(T_M) \cdot (c(N_X M)^{-1} \cdot [X]) .
\]

Here \( N_X M \) is the normal bundle to \( X \) in \( M \) (that is the vector bundle with sheaf of sections \((I/I^2)\) where \( I \) is the ideal sheaf of \( X \)). Finally, let \( V \) be a locally complete intersection subscheme of \( M \); we define the Milnor class of \( V \) as

\[
\mathcal{M}(V) = (-1)^{\text{codim}(V)} (c_F(V) - c_{SM}(V)).
\]  

(19)
Note that if $V$ is not a locally complete intersection one must use the Chern-Fulton-Johnson class in place of the Chern-Fulton class in (19), also note that other sign conventions may be used in definition of the Milnor class, we use the sign convention used by [15]. For a more in depth discussion see Fullwood [15] or Aluffi [4].

Let $M$ be a smooth algebraic variety and let $V$ be a locally complete intersection subscheme of $M$. From the definition of the Milnor class in (19) we have the following formula for the class $c_{SM}(V)$ in $A^*(M)$:

$$c_{SM}(V) = c_F(V) - (-1)^{\text{codim}(V)} \mathcal{M}(V).$$  

(20)

We now state several notations of Aluffi [1, §1.4] for operations in the Chow ring. Let $\alpha = \sum_{i \geq 0} \alpha^{(i)}$ be a cycle class in $A^*(M)$ with $\alpha^{(i)}$ denoting the piece of $\alpha$ of codimension $i$ in $A^*(M)$, that is $\alpha^{(i)} \in A^i(M)$. Also take $\mathcal{L}$ to be some line bundle on $M$. Throughout this note we will occasionally use a notation of Aluffi [1, §1.4], which we give below

$$\alpha^\vee = \sum_{i \geq 0} (-1)^i \alpha^{(i)}, \quad \text{and} \quad \alpha \otimes_M \mathcal{L} = \sum_{i \geq 0} \frac{\alpha^{(i)}}{c(\mathcal{L})^i}.$$  

(21)

Fullwood [15, §1.1] gives a formula for the Milnor class of a subscheme $V \subset M$ which is a global complete intersection of any codimension with an additional assumption on the structure of $V$; we state this result below.

**Theorem 2.8** (Theorem 1.1 of Fullwood [15]). Let $M$ be a smooth algebraic variety over an algebraically closed field of characteristic zero. Let $V$ be a possibly singular global complete intersection corresponding to the zero scheme of a vector bundle $\mathcal{E} \to M$. Let $j = \text{rk}(\mathcal{E})$. Additionally assume that $V = M_1 \cap \cdots \cap M_j$ for some hypersurfaces $M_1, \ldots, M_j$ and assume that, for some ordering of the hypersurfaces, $M_1 \cap \cdots \cap M_{j-1}$ is smooth. Let $\mathcal{L} \to M$ denote the line bundle associated to the divisor $M_j$ and let $Y$ denote the singularity subscheme of $V$. Then we have

$$\mathcal{M}(V) = \frac{c(T_M)}{c(\mathcal{E})} \cdot (c(\mathcal{E}^\vee \otimes \mathcal{L}) \cdot (s(Y, M)^\vee \otimes_M \mathcal{L})).$$  

(22)

Note that if $V$ is non-singular we will have that $\mathcal{M}(V) = 0$. 

15
3 Main Results

In this section we present the main results of this note. Throughout this section and in the following sections we will take $X_\Sigma$ to be a smooth complete toric variety without torus factors and with the additional property that all the divisors $D_\rho$ associated to a generating ray $\rho \in \Sigma(1)$ are nef. As noted above, a product of projective spaces is one example of such a toric variety.

In §3.2 we prove Theorem 3.2 which extends the result of Proposition 3.1 of Aluffi [3] to subschemes of toric varieties of the form $X_\Sigma$. For a subscheme $V$ of $X_\Sigma$ this result gives us an expression for the Segre class $s(V, X_\Sigma)$ in terms of the projective degrees of a rational map $\phi : X_\Sigma \dashrightarrow \mathbb{P}^r$. We then prove Theorem 3.3 which gives us a method to compute the projective degrees using a computer algebra system, with the additional assumption that the number of primitive collections of the fan $\Sigma$ is equal to the number of generating rays minus the dimension of $X_\Sigma$. This result generalizes Theorem 4.1 of the author in [21].

In Theorem 3.5 we give an expression for the $c_{SM}$ class of certain types of complete intersection subschemes of toric varieties of the form $X_\Sigma$; this result generalizes Theorem 3.3 of the author [22] and is proved using the expression of Fullwood [15] for the Milnor class given in Theorem 2.8.

Before giving the main theorems of this work, however, we make explicit a construction which gives us a concrete procedure to compute the number of points in a dimension zero subscheme of the types of smooth complete toric varieties for which the main results of this note are applicable. This is necessary for the proof of Theorem 3.3.

3.1 Counting Points in Zero Dimensional Subschemes of Certain Smooth Complete Toric Varieties

In this subsection we will assume that $X_\Sigma$ is an $n$ dimensional smooth complete toric variety without torus factors. The result is this subsection is essentially a consequence of Cox’s construction, [11], of the homogeneous coordinate ring of a toric variety. We shall make use of the result of Theorem...
3.1 in the proof of Theorem 3.3 below.

Let $\Sigma(1) = \{\rho_1, \ldots, \rho_m\}$. We will additionally assume that all the divisors $D_{\rho_1}, \ldots, D_{\rho_m}$ associated to the rays in $\Sigma(1)$ are nef and we will assume that the number of primitive collections of the fan $\Sigma$ is equal to $m - n$ (or equivalently that there are $m - n$ primary components in a primary decomposition of the irrelevant ideal $B$ of $R$). We note that these assumptions are, for example, satisfied by products of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_j}$.

**Theorem 3.1.** Let $V$ be a (reduced) subscheme of $X_\Sigma$ having dimension zero. Suppose that $R$ is the homogeneous coordinate ring of $X_\Sigma$ and suppose that the scheme $V$ is defined by the homogeneous ideal (with respect to the grading on $R$) $I = (f_0, \ldots, f_r)$.Also let $B$ be the irrelevant ideal of $R$ and let $p_1, \ldots, p_u$ be the primary ideals in primary the decomposition of $B$, further assume that $p_l = (x_{\rho(l)}^{\lambda_{j}^{(l)}} - 1, \ldots, x_{\rho(l)}^{(c)} - 1)$, The number of points in $V \subset X_\Sigma$ is equal to the number of points in the affine set $V(f_0, \ldots, f_r) \cap V(L_A) \subset \mathbb{C}^m$ where

$$L_A = \left(\sum_{j=1}^{\nu_1} \lambda_j^{(1)} x_{\rho_j}^{(1)} - 1, \ldots, \sum_{j=1}^{\nu_w} \lambda_j^{(c)} x_{\rho_j}^{(c)} - 1\right),$$

for general $\lambda_j^{(l)}$.

**Proof.** Because the $\lambda_j^{(l)}$ are general and by our assumption on the number of primitive collections, we have that $V(L_A)$ will have codimension $m - n$ in $\mathbb{C}^m$.

From Theorem 2.3 we have a geometric quotient $\pi : \mathbb{C}^m - Z(B) \rightarrow X_\Sigma$. Following the terminology of Cox, Little and Schenck [11], given a point $p \in X_\Sigma$ we say a point $x \in \pi^{-1}(p)$ gives homogeneous coordinates for $p$. Since $\pi$ is a geometric quotient we have $\pi^{-1}(p) = G \cdot x$.

Given a homogeneous polynomial $f \in R$ we have that if $f(x) = 0$ for one choice of homogeneous coordinates of $p \in X_\Sigma$ then $f(x) = 0$ for all choices of homogeneous coordinates. Hence to count points in $V$ we may fix a choice of homogeneous coordinates for our points $p_1, \ldots, p_l \in X_\Sigma$. We may do this as follows.

Since $X_\Sigma$ is smooth and complete and since all line bundles are generated by global sections (from the nef assumption), the affine open sets $U_\sigma$ for $\sigma$
a maximal cone are a torus invariant affine covering of \( X_\Sigma \). By Proposition 5.2.10 of Cox, Little and Schenck [11] (also see the remark following Proposition 5.2.10 of [11]) \( V \cap U_\sigma \), the affine piece of \( V \) for each maximal cone \( \sigma \), may be obtained by setting \( x_\rho = 1 \) for some \( \rho \notin \sigma(1) \) in each of the polynomials defining \( I \); this gives local coordinates on \( X_\Sigma \). In our case we may patch this together to give global coordinates by choosing a unique ray \( \rho \) from each primitive collection and setting \( x_\rho = 1 \) for each of these \( \rho \).

More specifically consider a primitive collection \( C \), we know that \( C \) is not contained in \( \sigma(1) \) for all \( \sigma \in \Sigma \) and specifically if we are considering some maximal cone \( \sigma \) then \( C \) is not in this \( \sigma(1) \). Hence there is some ray in \( C \) which is not in \( \sigma(1) \). Now suppose that we have maximal cones \( \sigma_1, \ldots, \sigma_j \) and primitive collections \( C_1, \ldots, C_{m-n} \) then we may choose one ray \( \rho_i \) from each primitive collection \( C_i \) such that for at least one of the maximal cones does not this contain \( \rho_i \); further we know from the structure of the irrelevant ideal that all rays not in a maximal cone will appear in some primitive collection meaning that we may choose appropriate rays from each primitive collection so that we can give compatible local coordinates on each maximal cone. Further, again from the structure of the irrelevant ideal, we see that for each maximal cone \( \sigma \) there exists exactly one \( \rho \) in each primitive collection which is not in \( \sigma \).

Hence by setting \( x_\rho = 1 \) for some such \( \rho \) in each primitive collection we obtain affine sets \( V \cap U_\sigma \) for each maximal cone \( \sigma \) which cover \( V \). Taking the intersection of these sets we must obtain all points in \( V \) and we may not obtain points which are not in \( V \), hence the intersection of these affine sets must have the same number of points as \( V \).

If we instead take a general linear combination of the \( x_\rho \) for \( \rho \in C_i \) and set this linear combination equal to 1 for each \( C_i \) and work in the larger ambient space \( \mathbb{C}^m \), since the linear combination is general, then the vanishing set of this equation will not contain points which lay in \( Z(B) \) (since by the construction of \( B \), given a point in our homogeneous coordinates for \( X_\Sigma \), if there is at least one coordinate \( x_\rho \neq 0 \) for some \( \rho \) in each primitive collection then this point is not in \( Z(B) \)). Hence by taking such linear combinations we would expect to obtain a new set of affine spaces covering \( V \) as a subset of \( \mathbb{C}^m \), provided the linear combination is sufficiently general. Taking the intersection of the new affine covering spaces gives the set \( V(f_0, \ldots, f_r) \cap V(L_A) \subset \mathbb{C}^m \), and by
above this space will have dimension zero and hence will consist of points in $\mathbb{C}^m$. Further since it is obtained from affine pieces which cover $V$ the number of points in $V(f_0, \ldots, f_r) \cap V(L_A) \subset \mathbb{C}^m$ will be the same as the number of points in $V$.

\[ \square \]

3.2 The Segre Class of Subschemes of a Smooth Complete Toric Variety with only Nef Divisors

Let $R$ be the graded homogeneous coordinate ring of $X_\Sigma$. Let $I$ be an ideal in $R$ which is homogeneous with respect to the grading. Then, since $I$ is homogeneous with respect to the grading (by Cox [10, §3]) we may choose generators $I = (f_0, \ldots, f_r)$ so that $[V(f_i)] = \alpha \in A^1(X_\Sigma)$ for all $i$. Also let $V = V(I)$ be the closed subscheme of $X_\Sigma$ defined by $I$.

Define a rational map $\phi : X_\Sigma \dashrightarrow \mathbb{P}^r$ given by

\[ \phi : p \mapsto (f_0(p) : \cdots : f_r(p)). \]  

(24)

Let

\[ \Gamma_I \subset X_\Sigma \times \mathbb{P}^r \]  

(25)

denote the closure of the graph of $\phi$. Let $h$ denote the pullback to $X_\Sigma$ of the hyperplane class in $\mathbb{P}^r$ and let $\pi : \Gamma_I \to X_\Sigma$ be the projection. The shadow of the graph $\Gamma_I$ is the class

\[ G = \sum_{i=0}^{n} [Y_i] \in A^*(X_\Sigma), \]  

(26)

where $[Y_i] = \pi_*(h^i \cdot [\Gamma_I])$. Note that by definition $[Y_i] = \left[ \phi^{-1}(\mathbb{P}^{r-i}) \right]$ where $\mathbb{P}^{r-i}$ denotes a general hyperplane of dimension $r - i$ in $\mathbb{P}^r$. Put another way $[Y_i]$ is the class of the closure of the inverse image under $\phi$ of a general codimension $i$ hyperplane in $\mathbb{P}^r$. Hence we may also write

\[ [Y_i] = \left[ V(P_1 + \cdots + P_i) - V(I) \right] \]  

(27)
with the $P_i$ being general linear combinations of $(f_0, \ldots, f_r)$. Also note that $[Y_i] = \alpha^i$ for $i < \text{codim}(V)$ since $V$ has no components of codimension less than $\text{codim}(V)$, i.e. for $i < \text{codim}(V)$

$$[Y_i] = [V(P_1 + \cdots + P_i)].$$

(28)

Further note

$$[Y_i] = 0 \text{ for } i > r.$$  

(29)

Observe that $[Y_i]$ has pure codimension $i$. Take $\omega_1^{(i)}, \ldots, \omega_m^{(i)}$ to be a basis for $A^i(X_\Sigma)$, then the class $[Y_i] \in A^*(X_\Sigma)$ will have the form

$$[Y_i] = \sum_{i=1}^{m} \gamma_i^{(i)} \omega_i^{(i)},$$

(30)

we will refer to the $\gamma_i^{(i)}$ as the projective degrees of the rational map $\phi$. Note that these projective degrees reduce to the usual projective degrees when $X_\Sigma = \mathbb{P}^n$ is a single projective space. We will, however, often find it notationally simpler to work with the classes $[Y_i]$ and the class $G$ of (26).

The tensor notation in Theorem 3.2 below is that defined in (21) above.

**Theorem 3.2.** Let $R$ be the graded coordinate ring of a smooth complete toric variety $X_\Sigma$ and let $I = (f_0, \ldots, f_r)$ be an ideal which is homogeneous with respect to the grading. Assume that all of the divisors $D_\rho$ associated to the rays $\rho$ in $\Sigma(1)$ are nef. Consider the $g$-dimensional scheme $V = V(I)$, and assume, without loss of generality, that all the polynomials $f_i$ generating $I$ are such that $[V(f_i)] = \alpha \in A^1(X_\Sigma)$. With $G$ as in (26) we have

$$s(V, X_\Sigma) = 1 - \frac{G \otimes O_{X_\Sigma}(\alpha)}{c(O_{X_\Sigma}(\alpha))}.$$ 

Proof. By construction the graph $\Gamma_I$ is isomorphic to the blow-up of $X_\Sigma$ along $V$, $\text{Bl}_V(X_\Sigma)$. Note that $V$ is the zero scheme of a section of $O(\alpha)^{r+1}$. Let $E = \pi^{-1}(V)$ be the exceptional divisor of the blow-up $\text{Bl}_V(X_\Sigma)$. From Fulton [16, §4.4] (which we may apply since $\alpha$ is nef and hence generated by global sections) we have that $\sigma^*(O_{\mathbb{P}^r}(1)) = \pi^*(O_{X_\Sigma}(\alpha)) \otimes O(-E)$ where
\[ \sigma : Bl_V(X_\Sigma) \to \mathbb{P}^r \] is the projection; let \([E]\) be the class of the exceptional divisor in the Chow ring of \(X_\Sigma \times \mathbb{P}^r\). From this we have \(h = \alpha - [E]\) and hence \([E] = \alpha - h\). Applying Fulton [16, Corollary 4.2.2] (given in (1) above) we have

\[ s(V, X_\Sigma) = \pi_* \left( \frac{[E]}{1 + [E]} \right) = \pi_* \left( \frac{\alpha - h}{1 + \alpha - h} \right). \]

We may simplify this expression as follows:

\[
\pi_* \left( \frac{\alpha - h}{1 + \alpha - h} \right) = \pi_* \left( \frac{[\Gamma]}{1 + \alpha - h} \cdot [\Gamma] \right) \\
= 1 - \pi_* \left( \frac{1}{1 + \alpha - h} \cdot [\Gamma] \right) \\
= 1 - \frac{1}{c(O(\alpha))} \cdot \pi_* \left( \left( \frac{1}{1 - \frac{h}{1 + \alpha}} \cdot [\Gamma] \right) \otimes O(\alpha) \right) \\
= 1 - \frac{G \otimes O(\alpha)}{c(O(\alpha))}.
\]

This concludes the proof. \(\square\)

We remark that Theorem 3.2 generalizes the result of Aluffi [3, Proposition 3.1].

### 3.3 Computing the Projective Degrees

We now prove a result which will allow us to compute the classes \([Y_i]\) of (26), and hence to compute the class \(G\) appearing in Theorem 3.2, using a computer algebra system by calculating the projective degrees \(\gamma_i\) as in (30).

**Theorem 3.3.** Let \(R\) be the homogeneous graded coordinate ring of a smooth complete toric variety \(X_\Sigma\) of dimension \(n\) with generating rays \(\rho_1, \ldots, \rho_m \in \Sigma(1)\). Assume that \(X_\Sigma\) is without torus factors and has the property that all the divisors \(D_\rho\) defined by generating rays \(\rho \in \Sigma(1)\) are nef. Additionally assume that the number of primitive collections of the fan \(\Sigma\) is equal to \(m - n\).
Take \( \iota = 0, \ldots, n \). Suppose that \( I = (f_0, \ldots, f_r) \) is an ideal in \( R \) which is homogeneous with respect to the grading on \( R \), without loss of generality we may assume that all generators \( f_i \) have the same degree, i.e. that \( [V(f_i)] = \alpha \in A^*(X_\Sigma) \) for some fixed \( \alpha \in A^1(X_\Sigma) \) and for all \( i \). Also let \( V = V(I) \) be the subscheme of \( X_\Sigma \) defined by \( I \).

Further suppose that \( Y_\iota = V(P_1 + \cdots + P_\iota) - V(I) \) with the \( P_i \) being general linear combinations of \( (f_0, \ldots, f_r) \), i.e. \( Y_\iota \) is as in (26). Let \( \{\omega_1^{(\iota)}, \ldots, \omega_m^{(\iota)}\} \) be a monomial basis for the Chow group \( A^*(X_\Sigma) \), we have that \( Y_\iota \) is a subscheme of \( X_\Sigma \) having pure codimension \( \iota \) so that

\[
[Y_\iota] = \sum_i^m \gamma_i^{(\iota)} \omega_i^{(\iota)}
\]

in the Chow ring \( A^*(X_\Sigma) \). Take \( [V(\sigma)] \in A_0(X_\Sigma) \) to be the basis of the dimension zero Chow group which is generated by the class of any point \( V(\sigma) \) where \( \sigma \in \Sigma \) is a cone of dimension \( n \). Also let

\[
a_i^{(\iota)} = \frac{[V(\sigma)]}{\omega_i^{(\iota)}} \in A^*(X_\Sigma),
\]

we have that the projective degrees are given by

\[
\gamma_i^{(\iota)} = \dim_k \left( R[T]/(P_1 + \cdots + P_\iota + L_{a_i^{(\iota)}} + L_A + S) \right),
\]

where:

- \( P_1, \ldots, P_\iota \) are ideals defined by general linear combinations of the generators of \( I \), i.e. for general \( \lambda_{j,l} \)

\[
P_j = \left( \sum_{l=0}^r \lambda_{j,l} f_l \right),
\]

- \( S \) is an ideal given by

\[
S = \left( 1 - T \sum_{l=0}^r \vartheta_l f_l \right)
\]

for general \( \vartheta_l \),
• let $B$ be the irrelevant ideal of $R$ and let $p_1, \ldots, p_\nu$ be the primary ideals in primary the decomposition of $B$, further assume that $p_l = (x_{\rho_1^{(l)}}, \ldots, x_{\rho_\nu^{(l)}})$. Then $L_A$ is the ideal

$$L_A = \left(\sum_{j=1}^{\nu_1} \lambda_j^{(1)} x_{\rho_j^{(1)}} - 1, \ldots, \sum_{j=1}^{\nu_\nu} \lambda_j^{(\nu)} x_{\rho_j^{(\nu)}} - 1\right),$$

(31)

for general $\lambda_j^{(l)}$.

• Factor $a_i^{(c)}$ so that $a_i^{(c)} = b_1^{j_1} \cdots b_q^{j_q}$ for some $(j_1, \ldots, j_q)$, and for some $b_1, \ldots, b_q \in A^1(X_\Sigma)$. Let $\ell(b)$ be a general linear form in $R$ such that $[\ell(b)] = b$ in $A^*(X_\Sigma)$ for $b \in A^1(X_\Sigma)$. Then $L_{a_i^{(c)}}$ is the ideal generated by $j_1$ linear forms $\ell(b_1)$, $j_2$ linear forms $\ell(b_2)$, \ldots, and $j_q$ linear forms $\ell(b_q)$.

Further

$$[Y_\iota] = \alpha_\iota \in A^*(X_\Sigma) \text{ for } \iota = 0, \ldots, \text{codim}(V) - 1, \text{ and }$$

$$[Y_\iota] = 0 \in A^*(X_\Sigma) \text{ for } \iota > \min(n, r).$$

Proof. The statement for $\iota < \text{codim}(V)$ is given in (28) and the statement for $\iota > \min(n, r)$ is given in (29).

Now take $\iota$ such that $\text{codim}(V) \leq \iota \leq \min(n, r)$. We wish to compute the class $[Y_\iota]$ in the Chow ring $A^*(X_\Sigma)$ where $Y_\iota$ is the closure of the open set

$$\tilde{Y}_\iota = V(P_1 + \cdots + P_\iota) - V(I).$$

Fixing a monomial basis $[V(\sigma)]$ for $A_0(X_\Sigma)$ (for $\sigma \in \Sigma$ a cone of dimension $n$) we see that

$$[Y_\iota] \cdot a_i^{(c)} = \gamma_i^{(c)}[V(\sigma)]$$

since we will have $a_i^{(c)} \omega_j^{(c)} = 0$ for $i \neq j$. Now if we choose sufficiently general linear forms (so that all intersections are transverse, which is possible since all line bundles are generated by global sections due to our nef assumption) then the zero dimensional set associated to $\gamma_i^{(c)}[V(\sigma)]$ is given by

$$\tilde{Y}_\iota \cap V(L_{a_i^{(c)}}) = (V(P_1 + \cdots + P_\iota) - V(I)) \cap V(L_{a_i^{(c)}})$$

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and hence to find $\gamma_i^t$ we must find the degree of $\tilde{Y}_i \cap V(L_{\omega_i^t})$, i.e. the number of points in $\tilde{Y}_i \cap V(L_{\omega_i^t})$ since $\tilde{Y}_i \cap V(L_{\omega_i^t})$ has dimension zero. Hence we wish to compute

$$\gamma_i^t = \text{card} \left( \bigcap_{l=1}^{t} V \left( \sum_{j=0}^{r} \lambda_{t,j} f_j \right) \cap V(L_{\omega_i^t}) - V(f_0, \ldots, f_r) \right),$$

where card denotes the number of points in a zero dimensional set. Let

$$W = \bigcap_{l=1}^{t} V \left( \sum_{j=0}^{r} \lambda_{t,j} f_j \right) \cap V(L_{\omega_i^t}).$$

By the Kleiman-Bertini Theorem (see Kleiman [25, Theorem 2] or Vakil [33, §25.3.8]) we have that there exists Zariski open dense sets $U_1, U_2$ so that for constants $\lambda_{t,j}$ and linear forms $\ell(b_j)$ chosen in $U_1$ and $U_2$ respectively we have that

$$\tilde{W} = \bigcap_{l=1}^{t} V \left( \sum_{j=0}^{r} \lambda_{t,j} f_j \right) \cap V(L_{\omega_i^t}) - V(f_0, \ldots, f_r)$$

is smooth (scheme-theoretically) and has dimension 0. In what follows we assume that $\lambda_{t,j}$ and $\ell(b_j)$ lay in the desired sets $U_1$ and $U_2$. Hence we may write the set $\tilde{W}$ as a finite collection of points, that is we may write $W - V(f_0, \ldots, f_r) = \{p_0, \ldots, p_s\}$. Then

$$U_3 = \mathbb{P}^r - \bigcup_{i=0}^{s} V \left( f_0(p_i)x_0 + \cdots + f_r(p_i)x_r \right)$$

is open and dense in $\mathbb{P}^r$, because $(f_0(p_i), \ldots, f_r(p_i)) \neq (0, \ldots, 0)$ for all $i$. Take $\vartheta = (\vartheta_0, \ldots, \vartheta_r) \in U_3$; then

$$W \cap V \left( \sum_{j=0}^{r} \vartheta_j f_j \right) - V(f_0, \ldots, f_r)$$

is empty. Now consider the ideals $L_{\omega_i^t}$ and $\left( \sum_{j=0}^{r} \lambda_{t,j} f_j \right)$ as ideals in the ring $R[T]$, and define $V_S = V(S)$ where

$$S = \left( 1 - T \sum_{j=0}^{r} \vartheta_j f_j \right)$$

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is an ideal in \( R[T] \).

For a point \( p \in V(f_0, \ldots, f_r) \) we have that
\[
f_j(p) = 0, \quad j = 0, 1, \ldots, r
\]
which implies that \( p \) is not in \( V_S \) since \( p \) cannot be a solution to the equation
\[
1 - T \cdot \sum_{j=0}^r \vartheta_j f_j = 0.
\]
Now take \( p \in W - V(f_0, \ldots, f_r) \) then
\[
\overline{T}_p = \frac{1}{\sum_{j=0}^r \vartheta_j f_j(p)}
\]
is well defined since for \( \vartheta \in U_3 \) we have that \( W \cap V\left(\sum_{j=0}^r \vartheta_j f_j\right) - V(f_0, \ldots, f_r) \) is empty, so \((p, \overline{T}_p) \in V_S\). Now let \( \tilde{W} \subset X_\Sigma \times \mathbb{A}^1 \) be the variety given by a linear embedding of \( W \) in \( X_\Sigma \times \mathbb{A}^1 \), where \( \mathbb{A}^1 = \text{Spec}(k[T]) \). We have
\[
\pi(\tilde{W} \cap V_S) = W - V(f_0, \ldots, f_r), \quad (32)
\]
where \( \pi \) is the projection \( \pi : X_\Sigma \times \mathbb{A}^1 \to X_\Sigma \), and in particular
\[
\text{card}(\tilde{W} \cap V_S) = \text{card}(W - V(f_0, \ldots, f_r)).
\]

Rather than considering the intersection \( \tilde{W} \cap V_S \) in \( X_\Sigma \times \mathbb{A}^1 \) we take \( W \subset \mathbb{A}^n \) i.e. we dehomogenize using the method of Theorem 3.1 in §3.1 by taking
\[
W = \bigcap_{\ell=0}^t V\left(\sum_{j=0}^r \lambda_{\ell,j} f_j\right) \cap V(L_{q_0}) \cap V(L_A) \subset \mathbb{A}^n
\]
and we then consider the intersection \( \tilde{W} \cap V_S \) in \( \mathbb{A}^{n+1} \). Here \( L_A \) is the collection of affine linear forms given above in (31). Adding the affine hyperplanes \( V(L_A) \) will ensure that we do not count any points which lay in \( V(B) \), the vanishing set of the irrelevant ideal. As the points in \( \tilde{W} \) have multiplicity one by the a Klienman-Bertini Theorem (see Kleiman [25, Theorem 2] or Vakil [33, §25.3.8]) the cardinality of the zero dimensional set
\[
\bigcap_{\ell=0}^t V\left(\sum_{j=0}^r \lambda_{\ell,j} f_j\right) \cap V(L_{q_0}) \cap V(L_A) \cap V_S \subset \mathbb{A}^{n+1}
\]
is given by the vector space dimension of
\[ R[T]/(P_1 + \cdots + P_t + L_{a_i} + L_A + S). \]

The following is a conjecture which seems to be true, and for which a proof may exist in the literature, however we were not able to find such a proof.

**Conjecture 3.4.** Let \( X_\Sigma \) be a dimension \( n \) smooth complete toric variety without torus factors with generating rays \( \rho_1, \ldots, \rho_m \). We conjecture that if all divisors \( D_{\rho_1}, \ldots, D_{\rho_m} \) are nef then the fan \( \Sigma \) will have exactly \( m - n \) primitive collections.

### 3.4 The \( c_{SM} \) Class of Complete Intersections

In this subsection we prove Theorem 3.5 which extends the result of the author [22, Theorem 3.3] to the case where \( V \) is a subscheme of a smooth complete toric variety \( X_\Sigma \) without torus factors and with only nef divisors in \( \text{Pic}(X_\Sigma) \). We will use this result to construct Algorithm 3. This result is based on Theorem 1.1 of Fullwood [15] given in (22).

**Theorem 3.5.** Let \( X_\Sigma \) be a dimension \( n \) smooth complete toric variety without torus factors over \( \mathbb{C} \) and let \( V = V(f_0, \ldots, f_r) \) be a possibly singular global complete intersection subscheme of \( X_\Sigma \). Let \( \{\rho_1, \ldots, \rho_m\} = \Sigma(1) \) be the generating rays of the cone \( \Sigma \) and let \( x_i = [V(\rho_i)] \in A^*(X_\Sigma) \) be the associated Chow ring elements.

Additionally assume that for some ordering of the hypersurfaces
\[ V_0 = V(f_0), \ldots, V_r = V(f_r) \]

we have that \( V_0 \cap \cdots \cap V_{r-1} \) is smooth. Let \( A^*(X_\Sigma) \) be the Chow ring of \( X_\Sigma \) and let \([V_i]\) denote the class of \( V_i \) in \( A^*(X_\Sigma) \), note that in particular \([V_i] \in A^1(X_\Sigma)\). Then we have

\[
c_{SM}(V) = \frac{(1 + x_1) \cdots (1 + x_m)}{(1 + [V_0]) \cdots (1 + [V_r])} \left([V_0] \cdots [V_r] + \left((-1)^r \sum_{j=0}^{r} \sum_{i=0}^{j} (-1)^j [V_i]^{j-1} c_i \right) \cdot \left(\sum_{i=0}^{n} \frac{(-1)^i c_i^{(i)}(Y, X_\Sigma)}{(1 + [V_i])^i}\right)\right),
\]

(33)
where \( c_i \) is the dimension \( i \) component of \((1 + [V_0]) \cdots (1 + [V_r])\) and \( s^{(i)}(Y, X_\Sigma) \) is the codimension \( i \) component of the Segre class of \( Y \) in \( X_\Sigma \) where \( Y \) denotes the singularity subscheme of \( V \).

**Proof.** Recall from (20) that \( c_{SM}(V) = c_F(V) + (-1)^r \mathcal{M}(V) \). Note that we have \( c(T_{X_\Sigma}) = (1 + x_1) \cdots (1 + x_m) \), see Cox [11, Proposition 13.1.2]. Since \( V \) is a complete intersection we have that
\[
s(Y, X_\Sigma) = \left[ V_0 \right] \cdots \left[ V_r \right] (1 + \left[ V_0 \right]) \cdots (1 + \left[ V_r \right]),
\]
hence the Chern-Fulton-Johnson class is
\[
c_F(V) = c(T_{X_\Sigma}) \cdot s(Y, X_\Sigma) = \frac{(1 + x_1) \cdots (1 + x_m) \left[ V_0 \right] \cdots \left[ V_r \right]}{(1 + \left[ V_0 \right]) \cdots (1 + \left[ V_r \right])}.
\]
In the notation of Theorem 1.1 of Fullwood (see (22)) we have
\[
\mathcal{M}(V) = \frac{c(T_{X_\Sigma})}{c(\mathcal{E})} \cdot (c(\mathcal{E}^\vee \otimes \mathcal{L}) \cdot (s(Y, X_\Sigma)^\vee \otimes \mathcal{L})),
\]
where \( \mathcal{E} \) is the line bundle associated to \( V_0 \cap \cdots \cap V_r \) and \( \mathcal{L} \) is the line bundle associated to \( V_r \). Hence \( c(\mathcal{E}) = (1 + [V_0]) \cdots (1 + [V_r]) \) and \( c(\mathcal{L}) = 1 + [V_r] \), using Remark 3.2.3. of Fulton [16] and (21) to expand \( c(\mathcal{E}^\vee \otimes \mathcal{L}) \) and \( s(Y, X_\Sigma)^\vee \otimes \mathcal{L} \) respectively we have
\[
\mathcal{M}(V) = \frac{(1 + x_1) \cdots (1 + x_m)}{(1 + \left[ V_0 \right]) \cdots (1 + \left[ V_r \right])} \left( \sum_{j=0}^{r} \sum_{i=0}^{r-j} \left( \begin{array}{c} r-j \cr j \end{array} \right) (-1)^j \left[ V_r \right]^{j-i} c_i \right) \cdot \left( \sum_{i=0}^{n} \frac{(-1)^i s^{(i)}(Y, X_\Sigma)}{(1 + \left[ V_r \right])^{i}} \right).
\]
Putting this together gives the expression in (33). \( \square \)

We note that the result of Theorem 3.5 can be applied to any complete intersection subscheme of \( X_\Sigma \) by using a specialized form of the inclusion/exclusion property of the \( c_{SM} \) class which considers only the defining equations which correspond to singular subschemes of \( X_\Sigma \). We give this result in Proposition 3.6 below.

**Proposition 3.6.** Let \( X_\Sigma \) be a smooth complete toric variety (without torus factors). Let \( Z \subset X_\Sigma \) be smooth (scheme-theoretically) and let \( V_1 = V(f_1) \), \( V_2 = V(f_2) \) be singular hypersurfaces in \( X_\Sigma \). If \( V = Z \cap V_1 \cap V_2 \), then we have
\[
c_{SM}(V) = c_{SM}(Z \cap V_1) + c_{SM}(Z \cap V_2) - c_{SM}(Z \cap (V_1 \cup V_2)),
\]
where \( c_i \) is the dimension \( i \) component of \((1 + [V_0]) \cdots (1 + [V_r])\) and \( s^{(i)}(Y, X_\Sigma) \) is the codimension \( i \) component of the Segre class of \( Y \) in \( X_\Sigma \) where \( Y \) denotes the singularity subscheme of \( V \).
here $V_1 \cup V_2$ is the scheme generated by $f_1 \cdot f_2$. Additionally, when $V$ is a complete intersection each of the terms in (35) can be computed using Theorem 3.5.

Proposition 3.6 follows directly from the inclusion/exclusion property of the $c_{SM}$ class.

4 Algorithms for Subschemes of Smooth Complete Toric Varieties

In this section we extend all algorithms of the author [21, 22] to the setting where $V$ is a subscheme of some smooth complete toric variety satisfying the assumptions of Theorem 3.3.

We note that given an input the assumptions of Theorem 3.3 can be easily and quickly verified by a computer algebra system, in particular methods of determining if a divisor is nef are discussed by Cox, Little and Schenck in [11, §6.3] and the number of primitive collections in a fan $\Sigma$ can be easily determined from the structure of $\Sigma$ using combinational methods (or by doing a primary decomposition of the associated irrelevant ideal). Specifically these assumptions may be checked using available methods in the “NormalToricVarieties” Macaulay2 [18] package, for example.

We first use the results of Theorem 3.2 and of Theorem 3.3 to construct Algorithm 1 which computes the Segre class of a subscheme of a smooth complete toric variety satisfying the assumptions of Theorem 3.3.

In Algorithm 2 we give an algorithm which uses Proposition 2.7 combined with the inclusion/exclusion property of $c_{SM}$ classes and Algorithm 1 to construct an algorithm to compute $c_{SM}(V)$ for $V$ a subscheme of a smooth complete toric variety satisfying the assumptions of Theorem 3.3.

We also give Algorithm 3 which computes the $c_{SM}$ class of a complete intersection subscheme of a smooth complete toric variety with a certain structure.
4.1 Segre Class

As above we consider a subscheme $V$ of a product of smooth complete toric variety $X_\Sigma$ satisfying the assumptions of Theorem 3.3 and we let $R$ denote the homogeneous graded coordinate ring of $X_\Sigma$. In Algorithm 1 we give an algorithm to compute the Segre class $s(V, X_\Sigma)$. This algorithm is based on the results of Theorem 3.2 and Theorem 3.3. More specifically we use Theorem 3.2 to give an expression for the Segre class $s(V, X_\Sigma)$ in terms of classes $[Y_i] \in A^*(X_\Sigma)$ which we compute by calculating the projective degrees using Theorem 3.2.

For $\beta \in A^1(X_\Sigma)$ let $R.random(\beta)$ be a function which creates a general polynomial $f$ in $R$ such that $[V(f)] = \beta \in A^*(X_\Sigma)$.

We will also use the tensor notation of (21) in the description of our algorithm; this notation is easily implemented in a computer algebra system.

**Algorithm 1.**

**Input:** A smooth complete toric variety $X_\Sigma$ (satisfying the assumptions of Theorem 3.3) and an ideal $I = (f_0, \ldots, f_r)$ homogeneous with respect to the grading on the coordinate ring $R$ of $X_\Sigma$ defining a subscheme $V = V(I)$ of $X_\Sigma$. Further assume, without loss of generality, that $[V(f_j)] = \alpha$ for all $j$.

**Output:** $s(V, X_\Sigma)$ in $A^*(X_\Sigma)$.

- Let $\{\rho_1, \ldots, \rho_m\} = \Sigma(1)$ be the generating rays of $\Sigma$.
- Let $x_1 = x_{\rho_1}, \ldots, x_m = x_{\rho_m}$ be the intermediates of $R$.
- Compute the Chow ring $A = A^*(X_\Sigma) \cong \mathbb{Z}[x_1, \ldots, x_m]/(I + J)$.
- Let $[V(\sigma)] = [V(\rho_1 + \cdots + \rho_m)]$ be the monomial basis for $A_0(X_\Sigma)$.
- Let $P_j = \sum_{l=0}^r \lambda_{j,l} f_l$ for $j = 1, \ldots, n$ and for general $\lambda_{j,l}$.
- Let $c = \text{codim}(V)$.
- Let $B$ be the irrelevant ideal of $R$.
- Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_v$ be the primary ideals in primary decomposition of $B$, further assume that $\mathfrak{p}_1 = (x_{\rho_{i_1}}, \ldots, x_{\rho_{i_v}})$. 

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For $i = c$ to $\min(n, r)$:
\begin{itemize}
  \item $J_i = R[T].\text{ideal}(P_1, \ldots, P_i)$.
  \item $K_i = J_i + R[T].\text{ideal}\left(1 - T \cdot \sum_{j=0}^{r} \vartheta_j f_j\right)$; $\vartheta_j$ a general scalar in $k$.
  \item Let $\Omega(i) = \{\omega_1^{(i)}, \ldots, \omega_{\nu}^{(i)}\}$ denote the monomial basis of $A^i(X_\Sigma)$.
  \item For $\omega$ in $\Omega(i)$:
    \begin{itemize}
      \item Let $a^{(i)} = [V(\sigma)]_\omega$.
      \item Factor $a^{(i)} = b_1^{j_1} \cdots b_q^{j_q}$ for $b_j \in A^1(X_\Sigma)$.
      \item $L = \sum_{w=0}^{j_1} R[T].\text{ideal}(R.\text{random}(b_1)) + \cdots + \sum_{w=0}^{j_q} R[T].\text{ideal}(R.\text{random}(b_q))$
      \item $L_A = R[T].\text{ideal}\left(\sum_{j=1}^{\nu_1} \lambda_j^{(1)} x_{\rho_j^{(1)}} - 1, \ldots, \sum_{j=1}^{\nu_v} \lambda_j^{(v)} x_{\rho_j^{(v)}} - 1\right)$, for general $\lambda_j^{(\ell)}$.
      \item Set $\gamma_\omega = \dim_k (R[T]/(K_i + L + L_A))$.
    \end{itemize}
  \item Set $[Y_i] = \sum_{\omega \in \Omega(i)} \gamma_\omega \cdot \omega \in A$.
\end{itemize}

Set $G = \left(1 + \sum_{i=1}^{n} \alpha_{i-1} + \sum_{i=c}^{n} [Y_i]\right) \in A$.

$s(V, X_\Sigma) = 1 - \frac{G \otimes \mathcal{O}_{X_\Sigma}(\alpha)}{1 + \alpha} \in A$.

Return $s(V, X_\Sigma)$.

### 4.2 The $c_{SM}$ Class Via Inclusion/Exclusion

In Algorithm 2 we give an algorithm to compute the $c_{SM}$ class of an arbitrary subscheme of a a smooth complete toric variety (where the toric variety satisfies the assumptions of Theorem 3.3). This algorithm will make use of
Algorithm 1 to compute Segre classes. It will also employ Proposition 2.7 (which is Theorem I.4 of Aluffi [2]) giving a relation between the $c_{SM}$ class of a hypersurface and the Segre class. To work in higher codimension we will employ the inclusion/exclusion property of the $c_{SM}$ class.

**Algorithm 2.**

**Input:** A smooth complete toric variety $X_{\Sigma}$ (satisfying the assumptions of Theorem 3.3) and an ideal $I = (f_0, \ldots, f_r)$ homogeneous with respect to the grading on the coordinate ring $R$ of $X_{\Sigma}$ defining a subscheme $V = V(I)$ of $X_{\Sigma}$.

**Output:** $c_{SM}(V)$ in $A^*(X_{\Sigma})$ and/or $\chi(V)$.

- Let $\{\rho_1, \ldots, \rho_m\} = \Sigma(1)$ be the generating rays of $\Sigma$.
- Let $x_1, \ldots, x_m$ be the intermediates of $R$ ($x_i = x_{\rho_i}$).
- Compute the Chow ring $A = A^*(X_{\Sigma}) \cong \mathbb{Z}[x_1, \ldots, x_m]/(I + J)$.
- Let $[V(\sigma)] = [V(\rho_1 + \cdots + \rho_m)]$ be the monomial basis for $A_0(X_{\Sigma})$.
- Let $c_{SM} = 0 \in A$.
- Let $S$ be the set of all distinct non-empty subsets of $\{f_0, \ldots, f_r\}$.
- For $\{f_{i_1}, \ldots, f_{i_s}\} \in S$
  - Let $g = f_{i_1} \cdots f_{i_s}$ in $R$.
  - Let $J$ be the Jacobian ideal of $g$, that is the ideal defining the singularity subscheme $Y = V(J)$ of $W = V(g)$. $J$ is generated by $g$ and by the partial derivatives of $g$ with respect to the generators of $R$.
  - Let $[W] = [V(g)]$.
  - Calculate $s(W, X_{\Sigma}) = s(V(g), X_{\Sigma}) = [W]/[W] \in A$.
  - Compute $s(Y, X_{\Sigma}) = s(V(J), X_{\Sigma})$ using Algorithm 1.
  - $c(T_{X_{\Sigma}}) = (1 + x_1) \cdots (1 + x_m)$.
  - $c_{SM} = c_{SM} + (-1)^{s+1}c(T_{X_{\Sigma}}) \cdot \left( s(W, X_{\Sigma}) + \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n-j}{l} [W]^l \cdot (-1)^{n-j}s_{j+l}(Y, X_{\Sigma}) \right)$.
- $c_{SM}(V) = c_{SM}$. 

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Set $\chi(V)$ equal to the coefficient of $[V(\sigma)]$ in $c_{SM}(V)$.

Return $c_{SM}(V)$ and/or $\chi(V)$

4.3 $c_{SM}$: The Complete Intersection Case

As above let $X_\Sigma$ be a smooth complete toric variety which satisfies the assumptions of Theorem 3.3. We now give an algorithm to compute $c_{SM}(V)$ for a complete intersection subscheme $V$ of $X_\Sigma$ which satisfies the assumptions of Theorem 3.5.

Algorithm 3. **Input:** A smooth complete toric variety $X_\Sigma$ (satisfying the assumptions of Theorem 3.3) and an ideal $I = (f_0, \ldots, f_r)$ homogeneous with respect to the grading on the coordinate ring $R$ of $X_\Sigma$ defining a subscheme $V = V(I)$ of $X_\Sigma$ such that $V$ is a complete intersection subscheme of $X_\Sigma$ with $V(f_0) \cap \cdots \cap V(f_{r-1})$ smooth.

**Output:** $c_{SM}(V)$ in $A^*(X_\Sigma)$ and/or $\chi(V)$.

- Let $\{\rho_1, \ldots, \rho_m\} = \Sigma(1)$ be the generating rays of $\Sigma$.
- Let $x_1, \ldots, x_m$ be the intermediates of $R$.
- Compute the Chow ring $A = A^*(X_\Sigma) \cong \mathbb{Z}[x_1, \ldots, x_m]/(I + J)$.
- Let $[V(\sigma)] = [V(\rho_1 + \cdots + \rho_m)]$ be the monomial basis for $A_0(X_\Sigma)$.
- Let $B$ be the irrelevant ideal of $R$.
- Let $K$ be the ideal defined by the $(r+1) \times (r+1)$ minors of the Jacobian matrix of $I$.
- Let $J = (K + I) : B^\infty$ so that $Y = V(J)$ is the singularity subscheme of $V$.
- Compute $s(Y, X_\Sigma)$ using Algorithm 1.
- For $i = 0$ to $n$: 
Set $c_i$ equal to the dimension $i$ component of $(1 + [V(f_0)]) \cdots (1 + [V(f_r)])$.

- $c_{SM}(V) = \frac{(1+x_1) \cdots (1+x_m)}{(1+[V(f_0)]) \cdots (1+[V(f_r)])} \left( [V(f_0)] \cdots [V(f_r)] + \left( -1 \right)^r \sum_{j=0}^{r} \sum_{i=0}^{r} \left( -1 \right)^j (j-i)(-1)^i [V(f_r)]^i - i \right) \left( \sum_{i=0}^{m} \left( -1 \right)^i s_i(Y, X) \right) \left( 1 + [V(f_r)]^r \right)$

- Set $\chi(V)$ equal to the coefficient of $[V(\sigma)]$ in $c_{SM}(V)$.

- Return $c_{SM}(V)$ and/or $\chi(V)$

We note that Algorithm 3 could be extended to work for any complete intersection by performing inclusion/exclusion only on the singular generators in the manner of Proposition 3.6.

## 5 Performance

In this section we discuss the real life performance of our algorithms to compute Segre classes, $c_{SM}$ classes and the Euler characteristic of subschemes of a smooth complete toric varieties. We also give running time bounds for Algorithm 1 and Algorithm 2 in §5.2.

### 5.1 Run Time Tests

In Table 5.1 we compare the run times of our algorithm to compute the Segre class of a subscheme of a smooth complete toric variety (Algorithm 1) to the run times of the algorithm of Moe and Qviller [30]. For this comparison we use the Macaulay2 implementation of Moe and Qviller linked to in [30], this implementation was obtained from http://sourceforge.net/projects/toricsegreclass/. Also note that the run times we give for the algorithm of Moe and Qviller [30] in Table 5.1 are likely less than the actual total run time of their algorithm since their implementation is broken into two parts; one part runs in Macaulay2 [18] and the second part runs in Sage [32]. We only count the running time for the Macaulay2 [18] component of their algorithm and do not add in the extra time to run the second part in Sage which would be needed.
to actually obtain the desired result using Moe and Qviller’s implementation; this is described in [30].

We note that, technically, the \( c_{SM} \) class is only defined when working over fields of characteristic zero (see, for example, Aluffi [5] for further discussion). However since the result of the computation is the same when working over \( \mathbb{Q} \) and over a finite field of large prime characteristic on all examples considered we give the run times over the finite field with 32749 elements for symbolic computations. This approach is also used for example computations of characteristic classes by Aluffi [3], Jost [24], Eklund, Jost and Peterson [13] and the author [21, 22].

As can be seen in Table 5.1 Algorithm 1 is consistently and often quite considerably faster than the algorithm of Moe and Qviller [30].

We also note that, in light of (3), when we are considering a smooth subscheme \( V \) of a smooth complete toric variety \( X_\Sigma \) (which satisfies the assumptions of Theorem 3.3) the running time to compute the Chern class \( c_{SM}(V) = c(V) \) (and hence the Euler characteristic \( \chi(V) \)) will be the same as the time required to compute the Segre class \( s(V, X_\Sigma) \).

| Input                                      | toricSegreClass ([30]) | SegreProjectiveDegree (Algorithm 1) |
|--------------------------------------------|------------------------|-------------------------------------|
| Codimension 3 in \( \mathbb{P}^2 \times \mathbb{P}^3 \) | -                      | 33.6s                               |
| Codimension 2 in \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) | 32.0s                  | 0.1s                                |
| Codimension 2 in \( \mathbb{P}^3 \times \mathbb{P}^2 \) | 2.9s                   | 0.2s                                |
| Hypersurface in \( \mathbb{P}^2 \times \mathbb{P}^3 \) | 147.4s                 | 0.5s                                |
| Codimension 2 in \( \mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^1 \) | 66.8s                  | 0.5s                                |
| Codimension 2 in \( \mathbb{P}^3 \times \mathbb{P}^2 \times \mathbb{P}^2 \) | 15.7s                  | 0.5s                                |
| Codimension 2 in \( \mathbb{P}^4 \times \mathbb{P}^3 \times \mathbb{P}^1 \) | -                      | 7.4s                                |
| Codimension 2 in \( \mathbb{P}^4 \times \mathbb{P}^3 \times \mathbb{P}^3 \) | -                      | 22.4s                               |
| Codimension 4 in \( \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1 \) | -                      | 2.7s                                |
| Codim. 1 with 2 gens. in Dim. 3 \( X_{X_1} \) | 7.6s                   | 0.1s                                |
| Codim. 1 with 3 gens. in Dim. 3 \( X_{X_1} \) | -                      | 1.0s                                |

Table 5.1: Run time comparison of different algorithms for computing the Segre class of a subscheme of a smooth complete toric variety satisfying the assumptions of Theorem 3.3. We use - to denote computations that were stopped after ten minutes (600 s). Computations were performed over \( GF(32749) \) on a computer with a 2.90GHz Intel Core i7-3520M CPU and 8 GB of RAM.
In Table 5.2 we give the running times to compute the $c_{SM}$ class and/or Euler characteristic using Algorithm 2, our algorithm to compute the $c_{SM}$ class and/or Euler characteristic of a subscheme of a smooth complete toric variety satisfying the assumptions of Theorem 3.3. Because there are no other known algorithms to compute the $c_{SM}$ class and Euler characteristic in this setting there are no other methods to compare to.

Let $V = V(f_0, f_1, f_2) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$ be the example from Table 5.2 which has codimension 3 in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$ with degree $(2, 1, 0), (0, 1, 2), (1, 2, 0)$ equations. For this example (and for the other examples as well) the majority of the running time is spent computing the $c_{SM}$ class of the hypersurface of largest degree appearing in the inclusion/exclusion procedure. In this case that is the class $c_{SM}(V(f_0 \cdot f_1 \cdot f_2))$ and around 85% of the total computation time is spent computing this class. To compute this class in practice using Algorithm 2 we must find the projective degrees associated to the ideal defining the singularity subscheme of $V(f_0 \cdot f_1 \cdot f_2)$. To find all these projective degrees we must, essentially, solve 35 different zero dimensional polynomial systems in 11 dimensional affine space each containing equations which have degrees of up to 10. The 35 zero dimensional systems we consider in this example have 2, 3, 3, 6, 6, 6, 9, 6, 4, 9, 9, 12, 18, 12, 18, 18, 12, 12, 18, 27, 18, 36, 36, 36, 24, 36, 24, 54, 54, 72, 72, 72, 108, and 144 solutions, respectively.

In Table 5.2 we give the running times to compute the $c_{SM}$ class and/or Euler characteristic using Algorithm 2, our algorithm to compute the $c_{SM}$ class and/or Euler characteristic of a subscheme of a smooth complete toric variety satisfying the assumptions of Theorem 3.3. Computations were performed over $\mathbb{GF}(32749)$ on a computer with a 2.9GHz Intel Core i7-3520M CPU and 8 GB of RAM.

| Input | Algorithm 2 |
|-------|-------------|
| Codimension 2 in $\mathbb{P}^2 \times \mathbb{P}^2$ | 0.3s |
| Codimension 2 in $\mathbb{P}^6 \times \mathbb{P}^2$ with degree $(3, 0), (0, 2)$ eqs. | 3.9s |
| Codimension 2 in $\mathbb{P}^5 \times \mathbb{P}^3$ with degree $(2, 1)$ and $(1, 1)$ eqs. | 12.4s |
| Codimension 2 in $\mathbb{P}^5 \times \mathbb{P}^2 \times \mathbb{P}^3$ with degree $(2, 1, 0)$ and $(0, 1, 2)$ eqs. | 4.8s |
| Codimension 3 in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$ with degree $(2, 1, 0), (0, 1, 2), (1, 2, 0)$ eqs. | 52.4s |
| Codimension 1 in with 2 gens. Dim. 3 $X_{\Sigma}$ | 0.3s |
| Codimension 1 in with 3 gens. Dim. 3 $X_{\Sigma}$ | 2.0s |

Table 5.2: Running times for Algorithm 2 to compute the $c_{SM}$ class and Euler characteristic of a subscheme of a smooth complete toric variety satisfying the assumptions of Theorem 3.3. Computations were performed over $\mathbb{GF}(32749)$ on a computer with a 2.9GHz Intel Core i7-3520M CPU and 8 GB of RAM.

In Table 5.3 we compare the running times of Algorithm 3, our direct algorithm to compute the $c_{SM}$ class and Euler characteristic using Theorem 3.5, to the running time of Algorithm 2, our algorithm using inclusion/exclusion in $X_{\Sigma}$. Note that Algorithm 3 is only valid when the input ideal $f_0, \ldots, f_r$
contains some $f_0, \ldots, f_{r-1}$ such that $V(f_0, \ldots, f_{r-1})$ is smooth. For the examples in the table Algorithm 3 does indeed provide a performance improvement. Note that the running time of Algorithm 3 includes the time required to compute the singularity subscheme, which is often a considerable percentage of the overall run time of the algorithm particularity in larger dimension. For the singularity subscheme computation we saturate by the irrelevant ideal, which can be a difficult computation for many toric varieties. As such a more efficient way to compute the singularity subscheme than that presented in Algorithm 3 could result in a more marked performance gain versus the inclusion/exclusion method.

| Input                                      | Algorithm 2 | Algorithm 3 |
|--------------------------------------------|-------------|-------------|
| Codimension 3 in $\mathbb{P}^2 \times \mathbb{P}^2$ | 1.6s        | 0.3s        |
| Codimension 2 in $\mathbb{P}^2 \times \mathbb{P}^3$ | 1.9s        | 1.0s        |
| Codimension 3 in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ | 5.7s        | 0.2s        |
| Codimension 2 in $\mathbb{P}^3 \times \mathbb{P}^2 \times \mathbb{P}^2$ | 3.1s        | 0.9s        |

Table 5.3: Running times for Algorithm 3, our direct algorithm to compute the $c_{SM}$ class and Euler characteristic of a subscheme of a smooth complete toric variety which satisfies the assumptions of Theorem 3.5 (and of Theorem 3.3). These running times are compared to the running times of Algorithm 2. Computations performed over $\mathbb{GF}(32749)$ on a computer with a 2.9GHz Intel Core i7-3520M CPU and 8 GB of RAM.

5.2 Running Time Bounds

We now consider running time bounds for Algorithms 1 and 2. In these running time bounds we will consider only the time to compute the Segre or $c_{SM}$ classes and not count the time to compute the Chow ring or the irrelevant ideal of the toric variety (and its primary decomposition). We note that these objects could be computed separately from the characteristics class computations and could be reused to compute the characteristic classes of different subschemes of the same toric variety.

Throughout this subsection let $\delta(D, N)$ be the number of arithmetic opera-
lations required to find the number of points in a zero dimensional affine variety $W$ defined by a polynomial system containing $N$ degree $D$ polynomials in $N$ variables.

Using the algorithm of Lecerf [28] or the algorithm of Giusti, Lecerf and Salvy [17] we have that the number of arithmetic operations to solve such a system is polynomial in $O(N^5D^{3N})$. There also exist known bounds on some Gröbner basis algorithms for zero dimensional systems. For example, in [20] Hashemi and Lazard show that several known Gröbner basis algorithms for zero dimensional systems (such as Lakshman [26], Lakschman and Lazard [27], and others) have running time complexities which are polynomial in an expression of order approximately $O(c \cdot N \cdot (3\bar{D})^{3N})$. Here $c$ is the maximum size of the coefficients of input polynomials, $N$ is the number of variables and $\bar{D}$ is the arithmetic mean value of the degrees of input polynomials defining the zero dimensional system. Run time bounds of similar order for other Gröbner basis algorithms applied to zero dimensional systems are also given by several authors see, for example, Faugere, Gianni, Lazard, and Mora [14].

Further we note that while all of the running time bounds for solving zero dimensional systems discussed above are essentially polynomial in the Bézout bound $D^N$ (for $N$ equations of degree $D$ in $N$ variables with $S$ solutions), which is the upper bound on our actual number of solutions $S$, the complexity is still exponential relative to the number of digits, $\log(S)$, in a computer representation of the number $S$. That is, for such an algorithm to be polynomial with respect to the number of solutions of our system, which is the number we wish to compute, we would need a bound polynomial in $\log(S)$ rather than polynomial in $S$ or $D^N$ as we have here.

Hence, because $S$ is exponential in $\log(S)$, these algorithms have complexity which is exponential relative to the number of digits in the value we wish to obtain from them (which is the number of solutions to our given zero dimensional polynomial system). In the context of the calculation of projective degrees this means we might expect that the time to compute a given projective degree $\gamma^{(i)}_i$ (which requires we find the number of solutions to one zero dimensional system) would be roughly exponential in the number of digits in the integer $\gamma^{(i)}_i$.

In practice, the current implementations of Algorithms 1 and 2 use the
Gröbner basis algorithms built into Macaulay2 [18] which may have different running time bounds. For this reason we present the complexity results in terms of the complexity of solving zero dimensional polynomial systems.

**Proposition 5.1.** Let \( X_\Sigma \) be a smooth complete toric variety of dimension \( n \) satisfying the assumptions of Theorem 3.3 and let \( R \) be the graded homogeneous coordinate ring of \( X_\Sigma \), also let the \( N \) be the number of generating rays in \( \Sigma(1) \). Take \( I = (f_0, \ldots, f_r) \) to be a homogeneous ideal, with respect to the grading, in \( R \) and suppose \( I \) defines a \( \rho \)-dimensional scheme \( V = V(I) \). Further assume, without loss of generality, that \( \deg(f_i) = \alpha \in A^1(X_\Sigma) \) for all \( i = 0, \ldots, r \) and let \( D \) be the sum of the exponents of the monomial in \( \alpha \) having the largest total degree. We have that the number of arithmetic operations required to compute the Segre class \( s(V, X_\Sigma) \) using Algorithm 1 is of order

\[
O \left( \delta(D + 1, N + 1) \cdot \sum_{\ell = \text{codim}(V)} \min(n, r) \left( \sum_{i=\ell}^{n} (-1)^{i-\ell} \binom{i}{\ell} |\Sigma(\ell)| \right) \right),
\]

where \( |\Sigma(\ell)| \) denotes the number of cones in \( \Sigma \) of dimension \( \ell \).

**Proof.** By Danilov [12, Theorem 10.8] a basis of the Chow group \( A^\ell(X_\Sigma) \) will contain

\[
\text{rank} \left( A^\ell(X_\Sigma) \right) = \sum_{i=\ell}^{n} (-1)^{i-\ell} \binom{i}{\ell} |\Sigma(\ell)|,
\]

elements. For each element of the basis we must solve one linear system in an affine space of dimension \( N + 1 \). The largest total degree of a polynomial appearing in the systems we consider (with respect to the usual grading of the coordinate ring of affine space, i.e. with \( \deg(x_i) = 1 \)) will be one plus the sum of the exponents of the monomial in \( \alpha \) having the largest total degree. \( \square \)

Examining Algorithm 2 we note that one Segre class, namely that of the appropriate singularity subscheme, must be calculated for each subset of the generators of \( I \) when finding \( c_{SM}(V(I)) \). In light of this we have the following corollary.

**Corollary 5.2.** Let \( X_\Sigma \) be a smooth complete toric variety of dimension \( n \) satisfying the assumptions of Theorem 3.3 and let \( R \) be the graded homogeneous coordinate ring of \( X_\Sigma \), also let the \( N \) be the number of generating rays
in $\Sigma(1)$. Take $I = (f_0, \ldots, f_r)$ to be a homogeneous ideal, with respect to the grading, in $R$ and suppose $I$ defines a $g$-dimensional scheme $V = V(I)$. Further assume, without loss of generality, that $\deg(f_i) = \alpha \in A^1(X_\Sigma)$ for all $i = 0, \ldots, r$ and let $D$ be the sum of the exponents of the monomial in $\alpha$ having the largest total degree. Let $\kappa$ be the minimum codimension of the singularity subscheme of all hypersurfaces of all products of the generators of $I$. The number of arithmetic operations required to compute $c_{SM}(V)$ using Algorithm 2 has order

$$
\mathcal{O} \left( 2^{r+1} \cdot \delta((r + 1) \cdot D + 1, N + 1) \cdot \sum_{i=\kappa}^{n} \left( \sum_{i=\kappa}^{n} (-1)^{i-t} \binom{i}{t} |\Sigma(t)| \right) \right),
$$

where $\delta$ is as in Proposition 5.1.

**Proof.** There are $2^{r+1}$ subsets of $\{f_0, \ldots, f_r\}$. The maximum total degree of elements in the Jacobian ideal of $f_0 \cdots f_r$ (considered as polynomials in the coordinate ring of the appropriate affine space) will be $(r + 1) \cdot D$. \qed

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