The Principle-Agent Conflict Problem in a Continuous-Time Delegated Asset Management Model

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This paper considers the principle-agent conflict problem in a continuous-time delegated asset management model when the investor and the fund manager are all risk-averse with risk sensitivity coefficients $c_f$ and $c_m$, respectively. Suppose that the investor entrusts his money to the fund manager. The return of the investment is determined by the manager’s effort level and incentive strategy, but the benefit belongs to the investor. In order to encourage the manager to work hard, the investor will determine the manager’s salary according to the terminal income. This is a stochastic differential game problem, and the distribution of income between the manager and the investor is a key point to be solved in the custody model. The uncertain form of the incentive strategy implies that the problem is different from the classical stochastic optimal control problem. In this paper, we first express the investor’s incentive strategy in term of two auxiliary processes and turn this problem into a classical one. Then, we employ the dynamic programming principle to solve the problem.

1. Introduction

Since professional asset management institutions can make efficient investment decisions, save investors’ time and effort, and simplify the investment process, more and more investors now entrust their money to fund managers, securities firms, and other asset management organizations. Nowadays, scholars pay more and more attention to asset management problems. We can refer to [1–5] to name just a few.

The whole asset management process involves two parties: the investor and the manager. The return of the investment is closely related to the manager’s effort level and investment strategy, but the interests belong to the investor. So, the investor and manager’s relation poses a principal-agent conflict. An important part of discussing the asset management problem is finding the investor’s optimal incentive mode under the principle agent conflict.

There are many papers committed to solving principal-agent conflict problems. Most of the early literature studies investigate the discrete-time case (we can refer to [6–8] or a summary book [9]). The problem in continuous-time models is discussed for the first time in [10]. It points out that the investor’s optimal incentive mode is linear. See references [11–14] for further work. In recent years, the maximum principle or the martingale representation theorem is often used to solve this problem in continuous-time models. For the literature using the maximum principle, we can refer to [15, 16], and for the literature of using the martingale representation theorem, we can refer to [17, 18]. However, since this problem often needs to solve a backward stochastic differential equation (BSDE) that rarely has explicit solutions, there are few articles which give analytical solutions to this problem. In order to get explicit solutions of principal-agent conflict problems, the authors of [19] express the investor’s incentive strategy in terms of two auxiliary processes and turn the principle agent problem into a classical stochastic differential game problem.

Although there are many papers committed to solving principal-agent conflict problems in continuous-time models, the delegated asset management problems are usually investigated in discrete-time models for the sake of simplicity. Thus, there are some contributions in this paper:
(i) This paper considers the delegated asset management problem in a continuous-time model.
(ii) Learning from [19], this paper gives explicit value functions and the optimal strategies of both sides by expressing the investor’s incentive strategy in terms of two auxiliary processes and turning the problem into a classical stochastic differential game problem.
(iii) In order to make the model more realistic, this paper brings in risk sensitivity coefficients to represent the subjects’ risk aversion attitudes.

This paper is organized as follows. In Section 2, we establish a continuous-time model of the fund management problem. In Section 3, we discuss the manager’s optimization problem under fixed investor’s incentive strategy. By substituting the manager’s optimal strategy into the investor’s optimal problem, both the investor and the manager’s optimal strategies are obtained in Section 4.

2. The Principal-Agent Conflict Model

Similar to the model in [20], let us assume that the investor employs a professional fund controller (manager) to invest and the investor will get a profit and pay the manager at the terminal moment $T$. Since the manager’s effort level cannot be observed, the investor will determine the manager’s salary according to the terminal profit of the investment. The investor’s return is determined by the terminal investment profit and the manager’s salary. The terminal investment profit is related to the manager’s investment strategy and effort level, and the incentive mechanism largely determines the manager’s strategy. Therefore, the investor needs to find the optimal incentive mechanism (the manager’s salary) to maximize his terminal net income. Meanwhile, according to the investor’s incentive mechanism, the manager shall decide his investment strategy and the best effort level to maximize his net salary (terminal salary minus effort cost). This is a non-cooperative game problem. Next, let us build a mathematical model of this problem in probability space $(\Omega, \mathcal{F}, P)$.

Similar to the model in [18], we suppose that the manager’s effort will affect the fund income $R^n_t$ which satisfies

$$dR^n_t = R^n_t \left[ (r + \mu + n_t)dt + \sigma dW(t) \right], \quad (1)$$

where $\mu \geq 0$, $\sigma \geq 0$, and $r > 0$ is the risk-free interest rate, $W(t)$ is a Brownian motion on $(\Omega, \mathcal{F}, P)$, and $[n_t]_{t \geq 0}$ is the manager’s effort level. Here, for the convenience of calculation, we assume that the drift coefficient of $R^n_t$ is a linear function of the manager’s effort level. In fact, as long as the drift coefficient of $R^n_t$ has the form of $R^n_t (r + f(n_t))$ for some function $f(n)$, the same method in this paper can be used after replacing $n$ with $f(n)$. For more general forms of the drift coefficient of $R^n_t$, the existence of the time value makes it hard to obtain explicit solutions.

Considering the manager’s strategy $\pi = (b^n_t, n^n_t)$, where $b^n_t$ represents the wealth that the manager decides to operate at moment $t$ (The manager may not want to operate all the wealth since the cost of the effort will increase with the wealth operated increases. The money left will get a risk-free return.) and $n^n_t$ represents the manager’s effort level at $t$. By some simple calculations, we can get that the investment income under this strategy satisfies

$$dX^n_t = \left( rX^n_t + b^n_t (\mu + n^n_t) \right) dt + b^n_t \sigma dW(t). \quad (2)$$

Define the natural filtration produced by $W(t)$ as $\{\mathcal{F}_t \}_{t \geq 0}$. Now, let us give the definition of both the manager and the investor’s admissible strategies. Considering the manager’s strategy $\pi = (b^n_t, n^n_t)$, if $b^n_t$ and $n^n_t$ are bounded positive predictable stochastic processes, under the strategy $\pi$, (2) has a unique solution.

We call that strategy $\pi = (b^n_t, n^n_t)$ is admissible. Denote the set of all the manager’s admissible strategies by $\Pi$.

**Remark 1.** Here, we do not consider the case when $b = 0$ or $n = 0$ since in that case, the model is meaningless.

Suppose that the investor’s incentive strategy is a function of the investment income at $T$ and denote it by $w(\cdot)$. If $\sup_{\pi \in \Pi} E[w(X^n_T)] < \infty$, the manager’s value function under $w(\cdot)$ is a decreasing convex function with respect to the initial wealth, we say that $w(\cdot)$ is the investor’s admissible strategy. Denote the set of all the investor’s admissible strategies by $\Pi$.

Now, let us analyze the whole game process. Referring to [15], we know that investors play a leading role in the game. Managers need to decide their effort level and investment strategy according to the investors’ incentive strategy. Therefore, first, we need to fix $w(\cdot)$ and investigate the manager’s optimal problem. We can get the manager’s optimal effort and investment strategy in terms of $w(\cdot)$ as a byproduct. Then, by substituting the manager’s optimal strategy into the wealth process, we can solve the investor’s optimal problem by using the dynamic programming principle.

Therefore, firstly, we fix the investor’s incentive strategy $w(\cdot)$ and consider the manager’s optimal problem. Suppose that the manager is risk-averse and denote his risk sensitivity coefficient by $\gamma_m < 0$. Referring to [18], we suppose that the manager needs to pay $(\delta r^2 b/2)$ to manage $b$ units of capital in unit time under the effort level $n$. Here, $\delta > 0$ is a constant which represents the effort cost parameter. The objective of the manager is to find the optimal effort level and investment strategy to maximize his net income (salary minus effort cost), which is equivalent to minimize

$$f^m_m(t, x; w) = E \left\{ e^{-\theta(t-T)} \left( \gamma_m w(x^n_T) - \int_t^T e^{\theta(t-s)} \left( \theta (n^n_s)^2 / 2 \right) b^n_s dt \right) \right\} \left| X^n_T = x \right. \right. , \quad (3)$$

Denote the manager’s optimal strategy by $\pi^m$, then the value function is

$$V_m(t, x; w) = \inf_{\pi \in \Pi} f^m_m(t, x; w) = f^{\pi^m}_m(t, x; w). \quad (4)$$
3. The Manager’s Optimization Problem

Define \( D_t = e^{r(T-t)} \), \( \beta(t, \pi) = \gamma_m D_t (\theta y_n^2 / 2) b^*, \) and \( \Gamma(t, T, \pi) = e^{-\int_t^T \beta(u, \pi) du} \). Then, \( J_m^* (t, x; w) \) can be denoted by

\[
J_m^* (t, x; w) = E \left[ \Gamma(t, T, \pi) e^{y_m w(X^*_t)} | X^*_t = x \right].
\]

Using the results of Section 3.4 in [21], we know that, under the incentive strategy \( w(\cdot) \), the manager’s value function \( V_m(t, x; w) \) satisfies the HJB equation:

\[
-V_m(t, x; w) = \inf_{\pi \in \Pi} \left[ -\beta(t, \pi) V_m(t, x; w) + [rx + b^*(\mu + n_t)] \right]
\]

\[
V_{mx}(t, x; w) + \frac{b^*_t n_t^2}{2} V_{mxx}(t, x; w) \right] \}
\]

and the boundary condition

\[
V_m(T, x; w) = e^{y_m w(x)}.
\]

Since \( V_m(t, x; w) \) is a decreasing convex function of \( x \), for \( \forall (t, x, y, z, y) \in [0, T] \times \mathbb{R} \times [0, \infty) \times (-\infty, 0) \times (0, \infty) \), we can define the Hamiltonian function:

\[
H(t, x, y, z, y, n, b) = \inf_{\pi \in \Pi, b > 0} h(t, x, y, z, y, n, b),
\]

where

\[
h(t, x, y, z, y, n, b) = -D_t \frac{\gamma_m n_t^2 b}{2} y + (rx + b(\mu + n))z + \frac{b^2 \sigma^2}{2} y.
\]

Theorem 1.

\[
n_t^* y, x, y = -\frac{z}{\theta y_m D_t}, \quad b_t^* y, x, y = \frac{(\mu + (n_t y, x, y, z, y))}{\sigma^2} z.
\]

is the minimum point of \( h \) in (10).

Proof. According to the definition, we know that \( h \) is a convex function of \( (n, b) \). So, the minimum point of \( h \) in (10) is the stable point under constraint conditions \( n > 0, b > 0 \). By some simple calculations, we have

\[
h_n(n, b; t, x, y, z, y) = -\theta D_t bny_m y + bx,
\]

\[
h_b(n, b; t, x, y, z, y) = \sigma^2 yb + (\mu + n)z - \frac{D_t bny_m y}{2}.
\]

Combining the above two equations, we can obtain the stable point of \( h \):

\[
n_t^* y, x, y = -\frac{z}{\theta y_m D_t} > 0,
\]

\[
b_t^* y, x, y = \frac{-(\mu + (n_t y, x, y, z, y)) z}{\sigma^2} > 0.
\]

The proof is done.

Remark 3. In this case, the optimal investment strategy is similar to that without principal-agent relationships. The only difference is that the numerator of the optimal investment strategy is changed from \( (\mu + n_t^* y, x, y) \) into \( (\mu + (n_t y, x, y, z, y) / 2) \). Clearly, this is due to the existence of the agency relationship.

Apparently, the investor’s incentive strategy and the manager’s value function are one-to-one. In the following, we will use auxiliary stochastic processes \( (Z, \Gamma) \) to determine the manager’s value function and transform the investor’s incentive strategy into \( (Z, \Gamma) \). Then, the problem in Section 2 can be translated into a classical stochastic optimal control problem.

First, let us give the space of auxiliary stochastic processes \((Z, \Gamma)\). Fix \( t \in [0, T] \), let \( \mathbb{Z} : [t, T] \times \Omega \rightarrow (-\infty, 0), \Gamma : [t, T] \times \Omega \rightarrow (0, \infty) \) be \( \mathcal{F}^W \)-predictable processes which satisfy

\[
E \left[ \int_t^T (Z_s^2 \sigma_s^2 + \Gamma_s \sigma_s^2) ds \right] < +\infty.
\]
Denote the set of all the processes satisfying the above conditions by \( \mathcal{Y}(t) \).

For some \( (Z, \Gamma) \in \mathcal{Y}(t) \) and \( Y_t \geq 0 \), define the \( \mathcal{F}_t \)-progressively measurable process \( Y_t \) on the filtration space \( (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{\text{t \geq 0}}) \):

\[
Y_t = Y_t - \int_t^s H(r, X_r, Z_r, \Gamma_r) \, dr + \int_t^s Z_r \, dX_r + \frac{1}{2} \int_t^s \Gamma_r \, d\langle X \rangle_r.
\]

where \( X_t \) is the investment income process. Clearly, for fixed \( Y_t, Z_t, \Gamma_t, Y_t \) is only related to the investment income process and \( \mathcal{F}_t \) measurable, suppose that it is an incentive strategy (we prove it in Corollary 1). In the following, we give the relationship between \( Y_t \) and the manager’s value function. First, we give the following lemma.

**Lemma 1.** Define

\[
\pi^{*, Y_t}(t, x; Y_t) = \left\{ \left( b^{*, Y_t}(t, x; Y_t), n^{*, Y_t}(t, x; Y_t) \right) \right\}_{t \geq 0},
\]

and then we have \( \pi^{*, Y_t} \in \Pi \).

**Proof.** On the one hand, since \( Z, \Gamma, Y_t \) are all predictable stochastic processes, referring to (12) and (13), we can get that \( b^{*, Y_t} \) and \( n^{*, Y_t} \) are bounded positive predictable stochastic processes. On the other hand, \( b^{*, Y_t} \) and \( n^{*, Y_t} \) are independent of \( x \). Taking \( b^{*, Y_t} \) and \( n^{*, Y_t} \) into (2), we can get the Lipschitz continuity and linear growth of the coefficients in (2) with respect to \( X_t \); then, (2) has a unique solution. The proof is done.

Denote the investment income process under \( \pi^{*, Y_t} \) by \( X^{*, Y_t} \). We also have the following theorem.

**Theorem 2.** Denote the manager’s value function with a terminal return \( (\ln Y_{T}^{*, Y_t}/\gamma_m) \) by \( V_m(t, x; Y_T^{*, Y_t}) \). We can obtain that

\[
Y_t = V_m(t, x; Y_T^{*, Y_t}).
\]

Furthermore, the manager’s optimal strategy is \( \pi^{*, Y_t} \).

**Proof.** \( \forall \pi \in \Pi, s \in [t, T] \), we have

\[
Y_s^{*, Y_t} = Y_t - \int_t^s H(r, X_r^{*, Y_t}, Z_r, \Gamma_r) \, dr + \int_t^s Z_r \, dX_r^{*, Y_t} + \frac{1}{2} \int_t^s \Gamma_r \, d\langle X \rangle_r.
\]

Using Ito’s formula, we have

\[
de - \int_t^s \beta(u, \pi) \, du \quad Y_r^{*, Y_t} = e^{- \int_t^s \beta(u, \pi) \, du} \left[ -H(r, X_r^{*, Y_t}, Z_r, \Gamma_r) + \left( p Y_r^{*, Y_t} + b_r^{*, Y_t} \right) Z_r + \frac{b_r^{*, Y_t} Z_r^2}{2} - \beta(r, \pi) \right] \, dr + e^{\int_t^s \beta(u, \pi) \, du} \, \sigma Z_r \, dW_r(r).
\]

It follows from (16) that \( e^{- \int_t^s \beta(u, \pi) \, du} \sigma Z_r \, dW_r(r) \) is a martingale. Integrating and taking expectations on both sides of (21), we can get

\[
Y_t \geq E \left[ e^{- \int_t^T \beta(u, \pi) \, du} \left( Y_T^{*, Y_t} \right) \mid X_t^{*, Y_t} = x \right] = J_m^T(t, x; Y_T^{*, Y_t}).
\]

Furthermore, by simple calculations, under \( \pi^{*, Y_t} \in \Pi \), we have

\[
dY_t^{*, Y_t} = \beta(t, \pi^{*, Y_t}) Y_t^{*, Y_t} \, dt + b_t^{*, Y_t} Y_t^{*, Y_t} \, dZ_t + \sigma Y_t^{*, Y_t} \, dW_t.
\]

Using (23) and Ito’s formula, we can obtain

\[
de - \int_t^s \beta(u, \pi^{*, Y_t}) \, du \quad Y_r^{*, Y_t} = e^{- \int_t^s \beta(u, \pi^{*, Y_t}) \, du} \left( b_r^{*, Y_t} Y_r^{*, Y_t} + \sigma Y_r^{*, Y_t} \right) \, dW_r(r).
\]

With similar methods, integrating and taking expectations on both sides of (24), we have

\[
Y_t = E \left[ e^{- \int_t^T \beta(u, \pi^{*, Y_t}) \, du} \left( Y_T^{*, Y_t} \right) \mid X_t^{*, Y_t} = x \right] = J_m^{*, Y_t}(t, x; Y_T^{*, Y_t}).
\]

This implies that \( \pi^{*, Y_t} \) is the manager’s optimal strategy and

\[
Y_t = V_m(t, x; Y_T^{*, Y_t}).
\]

Up till now, fixing \( (Z, \Gamma) \in \mathcal{Y}(t) \), we can get the manager’s optimal strategy and represent the manager’s value function. In Section 4, we begin to consider the investor’s optimization problem. That is, finding the optimal \( (Z, \Gamma) \in \mathcal{Y}(t) \) to maximize the investor’s net profit.

**4. The Investor’s Optimization Problem**

Suppose that the investor’s wealth is \( x \) at \( t \). Apparently, the investor’s value function is uniquely determined by the wealth process and the manager’s value function. So, the
objective of the investor is to find the optimal \((Z, \Gamma) \in \mathcal{V}(t)\) to minimize his value function. Define

\[
v(t, x, y) = \inf_{(Z, \Gamma) \in \mathcal{V}(t)} E \left[ e^{\gamma_f (X_t^{Z, \Gamma} - (\log Y_t^{Z, \Gamma}))} \right| X_t^n = x, Y_t^n = y].
\]  

(27)

Referring to Theorem 4.1 in [19], we know that if Assumption 3.2, Assumption 3.3, and Assumption 4.4 in [19] hold, the investor’s value function satisfies

\[
V_f (t, x, y) = \inf_{y \in [0, e^{\gamma_f n}]} v(t, x, y).
\]  

(28)

Here, \(R\) is the minimum pay in order to make sure that the manager takes the job. Section 4.1 gives the verification of the three assumptions.

4.1. The Verification of Assumptions

Assumption 1 (Assumption 3.2 in [19]). \(H\) has at least one extreme point \((b_t^{\ast, z, \gamma, y, z, b}, n_t^{\ast, Z, \Gamma})\). For any \(t \in [0, T]\), \((Z, \Gamma) \in \mathcal{V}(t)\), we have \(n_t^{\ast, Z, \Gamma} \in \Pi\).

Proof. This is the result of Theorem 1 and Lemma 1. The Hamiltonian function can be expressed as

\[
H(t, x, y, z, b) = \inf_{b > 0} \left\{ F(t, x, y, z, b) + \frac{b_t \sigma^2}{2} y^2 \right\}. \tag{29}
\]

Here,

\[
F(t, x, y, z, b) = \inf_{n > 0} \left\{ -D_t \frac{\gamma_n \theta b^2}{2} y + (rx + b(\mu + n))z \right\}. \tag{30}
\]

Define

\[
Y_s^Z = Y_t - \int_t^s F(r, X_r, Y_r^Z, Z_r)dr + \int_t^s Z_r dX_r, \quad s \in [t, T],
\]  

(31)

and we have the following assumption. \(\Box\)

Assumption 2 (Assumption 4.3 in [19]). \(F\) has at least one extreme point \(n_t^{\ast, y, z, b} \); furthermore, \((b, n_t^{\ast, y, z, b}) \in \Pi\).

Proof. On the one hand, the right hand of \(F\) is a parabola with an opening up with respect to \(n\); so, the minimum point is attained at the axis of the parabola \((z/D_t Y_{tm} \theta y)\), that is, \(n_t^{\ast, y, z, b} = (z/D_t Y_{tm} \theta y)\). On the other hand, since \(Z < 0\) is predictable, we can get that \(n_t^{\ast, y, z, b} = (bZ_t/D_t Y_{tm} \theta y_\gamma Z_t)\) is a positive predictable process. Furthermore, \(b\) and \(n_t^{\ast, y, z, b}\) are independent of \(x\). This implies the Lipschitz continuity and linear growth of the coefficients in (2) with respect to the investment income process; then, (2) has a unique solution. \(\Box\)

Assumption 3 (Assumption 4.4 in [19]). \(\forall b > 0\), \((1/b^2 \sigma^2)\) is bounded.

4.2. The Investor’s Value Function. Clearly, as soon as we get \(v(t, x, y)\), we can obtain \(V_f (t, x, y)\). The following theorem gives the partial differential equation satisfied by \(v(t, x, y)\).

Theorem 3. \(v(t, x, y)\) is the viscosity solution of

\[
-v_t (t, x, y) = \inf_{(Z, \Gamma) \in \mathcal{V}(t)} G(t, x, y, Z, \Gamma), \tag{32}
\]

\[
v(T, x, y) = e^{\gamma_f x}(-y/\gamma_n), \tag{33}
\]

where

\[
G(t, x, y, Z, \Gamma) = \left[ rx + b_t^* Z_t \Gamma (\mu + n_t^* Z_t) \right] v_x
\]

\[
+ \frac{\sigma^2(b_t^* Z_t)^2}{2} v_{xx} + D_t Y_{tm} \theta (n_t^* Z_t)^2 b_t^* Z_t v_{yy} + \frac{\sigma^2(b_t^* Z_t)^2}{2} Z^2 v_{yy} + \frac{\sigma^2(b_t^* Z_t)^2}{2} Z v_{xy} - \frac{\sigma^2(b_t^* Z_t)^2}{2} Z^2 v_{xy}.
\]

(34)

Proof. By the definition of \(v(t, x, y)\), we can obtain that it satisfies (33). Furthermore, according to the dynamic programming principle, we have

\[
v(t, x, y) = \inf_{(Z, \Gamma) \in \mathcal{V}(t)} v(t + h, X_{t+h}^*, Y_{t+h}^*). \tag{35}
\]

By using Ito’s formula with respect to \(v(s, X_s^*, Y_s^*)\) from \(t\) to \(t + h\), we have

\[
v(t + h, X_{t+h}^*, Y_{t+h}^*) = v(t, x, y) + \int_t^{t+h} v_t(s, X_s^*, Y_s^*) ds + G(s, X_s^*, Y_s^*, Z_s, \Gamma_s) ds.
\]

(36)

Combining with the above two equations, we can get

\[
v_t (t, x, y) + \inf_{(Z, \Gamma) \in \mathcal{V}(t)} G(t, x, y, Z, \Gamma) = 0. \tag{37}
\]

That is, \(v(t, x, y)\) satisfies (32). The proof is done. Next, we are going to solve (32) and (33). Considering the boundary condition, we guess

\[
v(t, x, y) = v_f (X_t^*, Y_t^*), \tag{38}
\]

where \(E(t)\) is a function of \(t\) which satisfies \(E(T) = 1\).

If the variables in the solution can be separated from each other, (32) can be easily solved. However, (32) contains \(e^{\gamma_f Z_t}\), which is a cross term of \(t\) and \(x\). To cancel the cross term, we introduce \(z_t = D_t X_t^* Z_t\). Using Ito’s formula, we can get

\[
dz_t = -Dr_t X_t^* Z_t dt + dX_t^* Z_t
\]

\[
= D_t b_t^* Z_t \left[ (\mu + n_t^* Z_t) dt + \sigma dW(t) \right]. \tag{39}
\]
We can also obtain \( z_T = X_T^* z^T \). Define 
\[
V(t, z, y) = \inf_{(Z^T)\in\mathbb{F}(t)} \left\{ \left[ e^{\gamma_t z_t - (\ln\gamma)^2 y_m} \right] | z_t = z \right\}
\]
\[
= \left( \frac{\gamma t}{D_t} \right)^{1/2} y V_y (t, z, y)
\]

(40)

Obviously, solving \( v(t, x, y) \) is equivalent to solving \( V(t, z, y) \). Using a similar method as the one in Theorem 3, we can get that 
\[
-V_t = \left\{ \left( \frac{\mu + n^*_1/2}{\sigma^2} \right)^2 \right\} \frac{D_t Z^2_t}{\Gamma^2 V_z}
\]
\[
- \frac{y^2_m \theta^2 (n^*_1)^2 (\mu + (n^*_1/2)^2)}{2 \sigma^2} y D_t V_y
\]
\[
+ \left( \frac{\mu + (n^*_1/2)^2}{\sigma^2} \right)^2 y^2 M^2 V_y y^2
\]
\[
- \left( \frac{\mu + (n^*_1/2)^2}{\sigma^2} \right)^2 y_m^2 \theta d y D_t^2 V_y y^2
\]
\[
+ \left( \frac{\mu + (n^*_1/2)^2}{\sigma^2} \right)^2 y^2 M^2 V_y y^2
\]
\[
- \frac{n^*_1 (\mu + (n^*_1/2)^2)}{\sigma^2} y^2 M^2 V_y y^2
\]
\[
(41)
\]

\[
V(T, z, y) = e^{\gamma_T z} y^{-\gamma y_m}
\]
\[
(42)
\]

The first step in solving (41) is to find its minimum point. Define \( M^2 z^T = (Z^T) \), it is shown in Section 3 that \( (Z^T, \Gamma^2) \) and \( (M^2 z^T, n^*_1) \) are one-to-one. Then, (41) is transformed into
\[
-V_t = \inf_{(n, M)\in\mathbb{R}^* \times \mathbb{R}^*} \left\{ \left( \frac{\mu + (n/2)}{\sigma^2} \right)^2 D_t M^2 V_z
\]
\[
+ \left( \frac{\mu + (n/2)^2}{\sigma^2} \right)^2 D_t^2 M^2 V_z
\]
\[
- \left( \frac{\mu + (n/2)^2}{\sigma^2} \right)^2 y D_t V_y y^2
\]
\[
+ \left( \frac{\mu + (n/2)^2}{\sigma^2} \right)^2 y_m^2 \theta d y D_t^2 M^2 V_y y^2
\]
\[
+ \left( \frac{\mu + (n/2)^2}{\sigma^2} \right)^2 y^2 M^2 V_y y^2
\]
\[
- \frac{n^*_1 (\mu + (n/2)^2)}{\sigma^2} y^2 M^2 V_y y^2
\]
\[
(43)
\]

Now, the problem of finding the minimum point in (41) is changed into a problem of finding the minimum point in (43). According to (38), we suppose that \( V(t, z, y) = E(t)e^{\gamma_T z} y^{-\gamma y_m} \). By some simple calculations, we can get that 
\[
V_z (t, z, y) = \gamma_T V_z (t, z, y)
\]
\[
V_{zz} (t, z, y) = \gamma_T^2 V_z (t, z, y)
\]
\[
y V_y (t, z, y) = -\gamma_T V_y (t, z, y)
\]
\[
y^2 V_{yy} (t, z, y) = \gamma_T^2 V_{yy} (t, z, y)
\]
\[
y^2 V_{zy} (t, z, y) = \gamma_T V_{zy} (t, z, y)
\]
\[
(44)
\]

Taking them into (43), we have 
\[
-E'(t)V(t, z, y) = \inf_{(n, M)\in\mathbb{R}^* \times \mathbb{R}^*} \left\{ \left( \frac{\mu + (n/2)}{\sigma^2} \right)^2 D_t M^2 y^2
\]
\[
+ \left( \frac{\mu + (n/2)^2}{\sigma^2} \right)^2 D_t^2 M^2 y^2
\]
\[
- \left( \frac{\mu + (n/2)^2}{\sigma^2} \right)^2 y D_t V_y y^2
\]
\[
+ \left( \frac{\mu + (n/2)^2}{\sigma^2} \right)^2 y_m^2 \theta d y D_t^2 M^2 V_y y^2
\]
\[
- \frac{n^*_1 (\mu + (n/2)^2)}{\sigma^2} y^2 M^2 V_y y^2
\]
\[
(45)
\]

Since the right hand of (45) is continuous, the minimum point can only be attained at the stable points or the boundary points, which depends on the parameter values. Denote the minimum point of (45) by \( (n_f^1, M_f^1) \), and denote the corresponding minimum point of (41) by \( (Z_f^1)^* \). It is shown from the Appendix that \( n_f^1 \) and \( D_t M_f^1 \) are constants concerning \( \mu, \theta, \gamma_f, \) and \( y_m \). Let \( n_f^1 = n^* \). \( \square \)

Remark 4. On the one hand, the exponential form of the objective function implies that \( b_f^1 \) is independent of \( X^*_f \). On the other hand, the benefit and the cost brought by the manager’s effort are only related to \( b_f^1 \), so \( n_f^1 \) is independent of \( X^*_f \). Furthermore, in this paper, we consider the discounted benefit and cost brought by the manager’s effort; so, \( n_f^1 \) is independent of \( t \).

Remark 5. It is shown from figures in the Appendix that \( n^* \) decreases with an increase in \( \mu \) (the drift coefficient of the fund wealth process), \( \theta \) (the effort cost coefficient), and \( |y_m| \) (the manager’s risk aversion level). It increases with an increase in \( |\gamma_f| \) (the investor’s risk aversion level).
Remark 6. Define \( Y^*_t = V_m(t, x; Y^*_T, y^*_T) \), considering (12) and (13), we can get that 
\[
\left( \frac{Z^*_t}{Y^*_t} \right) = \theta y_m D_t n^* \quad \text{and} \quad D_t b^*_t = \left( -\frac{(\mu + (n^*/2))/\sigma^2)}{\theta^2 D_t M_t^*} \right) \] are constants.

Taking the minimum point into (45) and solving it, we can get

\[
V(t, z, y) = e^{B(t)} e^{\gamma^*_f y^*_f (\gamma^*_f / y_m)}.
\] (46)

Here,

\[
B = \frac{\left( \mu + (n^*/2) \right) (\mu + n^*)}{\sigma^2} D_t M_t^* \gamma_f
+ \frac{(\mu + (n^*/2))^2}{2\sigma^2} D_t^2 M_t^* \gamma_f^2 + \frac{\gamma f \theta n^* (\mu + (n^*/2))}{2\sigma^2} D_t M_t^*
+ \frac{n^* (\mu + (n^*/2))^2}{2\sigma^2} \theta^2 D_t^2 M_t^* \gamma_f (\gamma_f + \gamma_m)
- \frac{n^* (\mu + (n^*/2))^2}{\sigma^2} \gamma_f \theta D_t^2 M_t^* \gamma_f^2
\] (47)
is a constant. As a consequence, we can also get the following results.

\[ v(t, x, y) = e^{B(T-t)} e^{y_j R,} (y_j y / y_m), \]

\[ V_f (t, x, y) = v(t, x, e^{y_j R}) = e^{B(T-t)} e^{y_j (D(x, x - R)).} \]

4.3. The Investor’s Excitation Mechanism. In this section, let us analyze the investor’s excitation mechanism. Denote \( Y_t^* = Y_t^{Z_t.} \). From the above analysis, we know that

\[ dY_t^* = D_t y_m \theta n^2 b_t^* b_t^* Y_t^* \frac{dt}{2} + b_t^* Z_t^* \sigma dW_t, \]

\[ Y_t^* = e^{y_m R}. \]

Using Ito’s formula, we have

\[ d \ln Y_t^* = \frac{D_t y_m \theta n^2 b_t^* b_t^*}{2} \frac{dt}{2} + \frac{b_t^* Z_t^* \sigma}{Y_t^*} \frac{dt}{2} dW_t. \]

Furthermore, we can get that the investment income under \( n^* \) and \( b_t^* \) satisfies

\[ dX_t^* = (r X_t^* + b_t^* (n^* + \mu)) dt + b_t^* \sigma dW_t, \]

which implies

\[ \text{Corollary 1} \]

\[ Y_t^* = V_m(s, X_t^*, Y_t^*) = e^{y_m [R + A (T-t) + n^* \theta (X_t^* - D_t, x)]}. \]

This implies that \( V_m(s, X_t^*, Y_t^*) \) is a decreasing convex function of \( X_t^* \). Thus, the assumption in Section 2 that \( Y_t^* \) is an incentive strategy is proved.

Appendix

Define

\[ I(n, M; t) = \frac{1}{2} (\mu + (n/2)) \frac{1}{\sigma^2} D_t M y_j f \]

\[ + (\mu + (n/2))^2 \frac{1}{2 \sigma^2} D_t M y_j \theta (y_j + y_m) \]

\[ + \frac{n^2 (\mu + (n/2))^2}{2 \sigma^2} \theta D_t^2 M y_j (y_j + y_m) \]

\[ - \frac{n (\mu + (n/2))^2}{\sigma^2} y_j^2 \theta D_t^2 M^2. \]

We know that there are three kinds of points which may be the minimum point of (45):
(i) The points which satisfy $I_n(n, M; t) = 0, I_M(n, M; t) = 0$
(ii) The points which satisfy $n = 0, I_M(0, M, t) = 0$
(iii) The points which satisfy $M = 0, I_n(n, 0, t) = 0$

With parameters fixed, we can easily decide which is the minimum point of (45). In the following, we will investigate the form of those points.

The first kind of points $(n_{1t}, M_{1t})$ is the solution of the following equations:

\[
I_n(n, M; t) = \frac{D_t \gamma_f}{\sigma^2} \left( \frac{3}{2} \mu + n \right) M
\]
\[
+ \frac{D_t \gamma_f \theta}{2\sigma^2} \left( \frac{3}{2} n^2 + 2\mu n \right) M + \frac{D_t^2 \gamma_f^2}{2\sigma^2} \left( \mu + \frac{n}{2} \right) M^2
\]
\[
+ \theta^2 D_t^2 \gamma_f (y_f + y_m) \left( n^3 + 3\mu n^2 + 2\mu^2 n \right) M^2
\]
\[
- \frac{\gamma_f^2 \theta D_t^2}{\sigma^2} \left( \frac{3n^2}{4} + 2\mu n + \mu^2 \right) M^2 = 0,
\]

\[
D_t M_{1t} = \frac{n_{1t} + \mu \left( n_{1t} \theta / 2 \right)}{(n_{1t} / 2 + \mu)} \left[ 2n_{1t} \theta y_f + (n_{1t})^2 \theta^2 (y_m + y_f) + y_f - (y_f / 2) + y_f \theta \mu - \theta y_{1t1} \right] \tag{A.4}
\]

It also follows from (A.3) that

\[
D_t M_{1t} = \frac{n_{1t} + \mu \left( n_{1t} \theta / 2 \right)}{(n_{1t} / 2 + \mu)} \left[ 2n_{1t} \theta y_f + (n_{1t})^2 \theta^2 (y_m + y_f) + y_f \right]
\]

Combining the above two equations, we can get that

\[
\left( n_{1t} + \mu \left( n_{1t} \theta / 2 \right) \left[ \frac{y_f}{2} + y_f \theta \mu - \frac{y_f \theta n_{1t}}{2} + \theta^2 y_f (y_f + y_m) n_{1t} \right] \right.
\]
\[
+ \left( \frac{\theta n_{1t}^2}{4} + \theta \mu n_{1t} - \frac{\mu}{2} \right) \left[ 2n_{1t} \theta y_f + (n_{1t})^2 \theta^2 (y_m + y_f) + y_f \right] = 0.
\]

(A.5)

(A.6)

Clearly, by solving (A.5) and (A.6), we can get that $n_{1t}$ and $D_t M_{1t}$ are constants.

Denote the second kind of point by $(0, M_{2t})$. Thus, $M_{2t}$ satisfies (A.5) with $n$ replaced with 0 and we can get that $D_t M_{2t}$ is a constant.

Denote the third kind of points by $(n_{3t}, 0)$. They satisfy (A.4). By solving it, we can get that $n_{3t}$ is a constant.

Denote the minimum point of (45) by $(n_{*t}, M_{*t})$. It follows from the above analysis that $n_{*t}$ and $D_t M_{*t}$ are all constants. For different $\mu, \theta, y_f$, and $y_m$, by calculating (A.6), (A.2), or (A.3), we can get different $n^*$. We can deduce from (A.2) that

\[
I_M(n, M; t) = \frac{D_t \gamma_f}{\sigma^2} (\mu + n) \left( \mu + \frac{n}{2} \right)
\]
\[
+ \frac{D_t \gamma_f \theta}{2\sigma^2} \left( \mu + \frac{n}{2} \right) + \frac{D_t^2 \gamma_f^2}{\sigma^2} \left( \mu + \frac{n}{2} \right) M
\]
\[
+ \theta^2 D_t^2 \gamma_f (y_f + y_m) \frac{n^2}{M} \left( \mu + \frac{n}{2} \right) M = 0.
\]

(A.3)

By using R, we plot the following figures which indicate the effect of $\mu, \theta, y_f$, and $y_m$ on $n^*$ (Figure 1).

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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