A Relativistic Toda Lattice Hierarchy, Discrete Generalized $(m, 2N - m)$-Fold Darboux Transformation and Diverse Exact Solutions

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Abstract: This paper investigates a relativistic Toda lattice system with an arbitrary parameter that is a very remarkable generalization of the usual Toda lattice system, which may describe the motions of particles in lattices. Firstly, we study some integrable properties for this system such as Hamiltonian structures, Liouville integrability and conservation laws. Secondly, we construct a discrete generalized $(m, 2N - m)$-fold Darboux transformation based on its known Lax pair. Thirdly, we obtain some exact solutions including soliton, rational and semi-rational solutions with arbitrary controllable parameters and hybrid solutions by using the resulting Darboux transformation. Finally, in order to understand the properties of such solutions, we investigate the limit states of the diverse exact solutions by using graphic and asymptotic analysis. In particular, we discuss the asymptotic states of rational solutions and exponential-and-rational hybrid solutions graphically for the first time, which might be useful for understanding the motions of particles in lattices. Numerical simulations are used to discuss the dynamics of some soliton solutions. The results and properties provided in this paper may enrich the understanding of nonlinear lattice dynamics.

Keywords: relativistic Toda lattice system; discrete generalized $(m, 2N - m)$-fold Darboux transformation; soliton and rational solutions; hybrid solutions; asymptotic analysis

1. Introduction

In recent years, the study of nonlinear lattice equations (NLEs), viewed as spatially discrete counterparts of nonlinear partial differential equations, has attracted increasing interest. The NLEs may well model some physical phenomena in various fields such as nonlinear optics, lattice dynamics, electric circuits, population dynamics and so on [1–9]. The Toda lattice (TL) system is the first example of Lax integrable NLEs, which can describe a lattice of particles interacting with their nearest neighbors via forces exponentially dependent on distances [1–3], as shown in Figure 1. In the picture, the wavy lines represent springs, and small spheres represent particles, which are connected graphically.

Figure 1. A one-dimensional lattice of particles with fixed ends (see also the first figure in Ref. [3]).

TL system is also used in studying nonlinear waves in nonlinear lattice dynamics [3]. In order to better investigate the physical phenomena described by the TL system [10], proposed a new Hamiltonian function $H = \sum_{n=1}^{N} \frac{e^{-\alpha \Delta x_n \Delta n} - 1 + \alpha \Delta p_n}{\alpha^2} + \sum_{n=1}^{N} e^{\Delta x_n \Delta n - 1 - \Delta p_n}$ for which
its Hamilton’s motion equation is the following relativistic Toda lattice (RTL) equation (see also Equation (8.2.28) in Ref. [10]):

\[
\begin{align*}
  x_{n,t} &= \alpha^{-1}(1 - \exp^{-ap_n}) - \alpha\exp^{x_n-x_{n-1}-ap_n}, \\
  p_{n,t} &= \exp^{x_n-x_{n-1}-ap_n} - \exp^{x_n-x_{n-1}-ap_n},
\end{align*}
\]

(1)

where \(x_n = x(n,t)\) and \(p_n = p(n,t)\) are the real functions of discrete variable \(n\) and time variable \(t\). When \(\alpha \to 0\), Equation (1) reduces to the famous TL system as a very remarkable generalization of TL system; thus, the study of Equation (1) is of great significance for understanding the physical phenomena described by the TL system. Equation (1) can also be considered as a one-parameter perturbation of TL system, and in a certain physical interpretation, the small parameter \(\alpha\) has the meaning of the inverse speed of light [10,11]. The author has used the abbreviated name RTL\(_\alpha\) for Equation (1). For convenience, hereafter we still use RTL\(_\alpha\) for Equation (1). According to Ref. [10], the Lax pair of Equation (1) admits the following:

\[
E\phi_n = U_n\phi_n = \begin{pmatrix} z\alpha\exp^{x_n} - 1 \\
-\alpha\exp^{-x_n+ap_n} \end{pmatrix} \phi_n,
\]

(2)

\[
\phi_{n,t} = V_n\phi_n = \begin{pmatrix} 0 \\
\alpha\exp^{-x_n+ap_n} \end{pmatrix} \phi_n,
\]

(3)

where \(z\) is the spectral parameter independent of \(t\), \(E\) is the shift operator defined by \(Ef(n,t) = f(n+1,t)\), \(E^{-1}f(n,t) = f(n-1,t)\) and \(\phi_n = (\phi_n, \psi_n)^T\) is an eigenfunction vector. Equation (1) can be recovered from the integrability condition \(U_{n,1} = (EV_n)U_n - U_nV_n\) between the spatial part (2) and time evolution part (3), and they have been verified to be correct. It should be noted here that we have made some small changes to the Lax pair, (2) and (3), for the convenience of later discussion. We divide every element of matrix \(U_n\) in the spatial part (2) by \(\frac{1}{z}\), which has no effect on the compatibility condition of Equation (1) because each term in the compatibility condition has \(U_n\). The Lax pairs (2) and (3) are inconvenient for constructing Darboux transformations (DTs); thus, we need to simplify them before performing this. Here, we take \(q_n = \exp^{ap_n}, r_n = \exp^{ap_n}, \lambda = z\), then the Lax pair (2) and (3) can be transformed into the following.

\[
E\phi_n = U_n\phi_n = \begin{pmatrix} \lambda^2 q_n - 1 \\
-\alpha\lambda q_n \end{pmatrix} \phi_n,
\]

(4)

\[
\phi_{n,t} = V_n\phi_n = \begin{pmatrix} 0 \\
\lambda q_n \end{pmatrix} \phi_n.
\]

(5)

At this time, Equation (1) is equivalent to the following equation.

\[
\begin{align*}
  q_{n,t} &= \frac{\alpha(qnr_{n-1}r_{n+1}-q_{n+1}r_{n}^2)}{q_{n+1}r_n-r_{n-1}}, \\
  r_{n,t} &= -\frac{r_n(\alpha^2 r_n-q_{n+1}r_{n-1}+r_{n-1})}{\alpha^2 q_{n-1}r_n}.
\end{align*}
\]

(6)

Below, we first investigate Equation (6), then we use transformations \(p_n = \frac{1}{r_n}\ln q_n\) and \(x_n = \ln r_n\) to study Equation (1) when \(q_n > 0, r_n > 0\). There are some known research results on the RTL system such as Lax pair [12–15], Hamiltonian structures [12,16,17], DT and exact solutions [17–21] and so on. In Ref. [11], Suris proposed some discrete time Toda systems including the RTL\(_\alpha\) system (see System (8.2) in Ref. [11]). Although Equation (1) has the same abbreviation as system (8.2) in the literature [11], by careful comparison, we found that they are different from each other. Here, we need to point out that the RTL\(_\alpha\) system that we study here is different from the RTL equation in
we have the following recursion relations:

\[(i) \text{ constructing the lattice hierarchy associated with Equation (6) and then studying its}\]

\[\text{provides soliton solutions, this generalized method is a useful technique that not only can}\]

\[\text{pairs, infinitely many conservation laws, Hamiltonian structure and Liouville integrability}\]

\[\text{by means of the Tu scheme [29–31]. Furthermore, explicit exact solutions to NLEs}\]

\[\text{play an essential role in understanding complex natural phenomena; thus, searching for exact}\]

\[\text{solutions, particularly soliton solutions, is an important subject in the study of NLEs.}\]

\[\text{Recently, a discrete generalized } (m, 2N - m)\text{-fold DT based on Lax pair was proposed}\]

\[\text{to solve Lax integrable NLEs [17,32]. Compared with the usual } N\text{-fold DT, which only}\]

\[\text{provides soliton solutions, this generalized method is a useful technique that not only can}\]

\[\text{present soliton solutions but also provide rational and semi-rational solutions and hybrid}\]

\[\text{solutions of exponential and rational functions [17,32].}\]

\[\text{Therefore, the aim of this paper can be summarized in the following two points:}\]

\[\text{(i) constructing the lattice hierarchy associated with Equation (6) and then studying its}\]

\[\text{relevant integrable properties including Hamiltonian structure, Liouville integrability}\]

\[\text{and infinite conservation laws; (ii) establishing the discrete generalized } (m, 2N - m)\text{-fold}\]

\[\text{DT for Equation (6) to provide various exact solutions such as discrete soliton solutions,}\]

\[\text{rational and semi-rational solutions and hybrid solutions and then carrying out the asymp-}\]

\[\text{totic analysis technique in order to analyze the limit states of such exact solutions. It}\]

\[\text{should be noted here that the properties mentioned above are studied for the first time for}\]

\[\text{Equation (6) or (1).}\]

\[\text{The structure of this paper is as follows. Section 2 constructs relativistic Toda lattice}\]

\[\text{hierarchies associated with Equation (6) and studies some integrable properties including}\]

\[\text{Hamiltonian structures and Liouville integrability via the Tu scheme [29–31]. Section 3}\]

\[\text{constructs the discrete generalized } (m, 2N - m)\text{-fold DT of Equation (6). Various exact}\]

\[\text{solutions such as multi-soliton solutions, rational and semi-rational solutions and hybrid}\]

\[\text{solutions for Equation (6) are given by using the special cases of the}\]

\[\text{resulting generalized DT in Section 4. Some conclusions are provided in Section 5.}\]

2. The Integrable Properties of Equation (6)

In this section, we will study the integrable properties of Equation (6) including its

lattice hierarchy, Hamiltonian structures, Liouville integrability and infinite conservation

laws.

2.1. A Discrete Integrable Lattice Hierarchy Associated with Equation (6)

First, we will construct the lattice hierarchy associated with Equation (6) from the

spectral problem (4) by using the Tu scheme technique [29–31]. Let the following be the case.

\[P_n = \begin{pmatrix} A_n & B_n \\ C_n & -A_n \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{\infty} A_n^{(j)} \lambda^{-2j} & \sum_{j=0}^{\infty} B_n^{(j)} \lambda^{-2j+1} \\ \sum_{j=0}^{\infty} C_n^{(j)} \lambda^{-2j+1} & -\sum_{j=0}^{\infty} A_n^{(j)} \lambda^{-2j+1} \end{pmatrix}.\]

By solving the following stationary discrete zero-curvature equation

\[P_{n+1} U_n - U_n P_n = 0,\]

we have the following recursion relations:

\[
\begin{align*}
q_n (A_{n+1}^{(j+1)} - A_n^{(j+1)}) - A_{n+1}^{(j)} + A_n^{(j)} &+ \frac{\sigma_{n+1}}{r_n} B_{n+1}^{(j)} - \alpha r_n C_n^{(j)} = 0, \\
\alpha r_n (A_{n+1}^{(j+1)} + A_n^{(j+1)}) - q_n B_{n+1}^{(j+1)} + B_n^{(j+1)} &= 0, \\
\frac{\sigma_{n+1}}{r_n} (A_{n+1}^{(j+1)} + A_n^{(j+1)}) - C_{n+1}^{(j)} + q_n C_n^{(j+1)} &= 0, \\
\alpha r_n C_{n+1}^{(j)} + \frac{\sigma_{n+1}}{r_n} B_n^{(j)} &= 0,
\end{align*}
\]

(7)
where $A_n^{(j)}, B_n^{(j)}$ and $C_n^{(j)}$ are functions of $q_n, r_n$. Let $A_n^{(0)} = -\frac{1}{\alpha}$ be the initial values, the recursion relations (7) will determine the rest $A_n^{(j)}, B_n^{(j)}, C_n^{(j)}$ uniquely and the first few coefficients are listed below.

\[
\begin{align*}
B_n^{(0)} &= \frac{r_n}{\eta_n} \phi_n^{(0)}, \\
B_n^{(1)} &= -\frac{1}{\eta_n} \left( \frac{\alpha^2 r_{n+1}}{\eta_{n+1}} + \frac{\alpha^2 r_n}{\eta_n} + \frac{r_n}{\eta_n} + 2 r_n \right), \\
C_n^{(1)} &= \frac{1}{\eta_n} \left( \frac{\alpha^2 r_{n+1}}{\eta_{n+1}} + \frac{\alpha^2 r_n}{\eta_n} + \frac{r_n}{\eta_n} + 2 r_n \right), \\
A_n^{(2)} &= \frac{\alpha}{\eta_n} \left( \frac{\alpha^2 r_{n+1}}{\eta_{n+1}} + \frac{\alpha^2 r_n}{\eta_n} + \frac{r_n}{\eta_n} + 2 r_n \right), \\
B_n^{(2)} &= \frac{1}{\eta_n} \left( \frac{\alpha^2 r_{n+1}}{\eta_{n+1}} + \frac{\alpha^2 r_n}{\eta_n} + \frac{r_n}{\eta_n} + 2 r_n \right), \\
C_n^{(2)} &= \frac{1}{\eta_n} \left( \frac{\alpha^2 r_{n+1}}{\eta_{n+1}} + \frac{\alpha^2 r_n}{\eta_n} + \frac{r_n}{\eta_n} + 2 r_n \right),
\end{align*}
\] (8)

For any integer $m \geq 0$, we define the following.

\[
\begin{align*}
r_n^{(m)} &= \lambda^{2m+2} P_n = \left( \sum_{j=0}^{m} A_n^{(j)} \lambda^{-2j+2m+2} + \sum_{j=0}^{m} B_n^{(j)} \lambda^{-2j+2m+1} - \sum_{j=0}^{m} C_n^{(j)} \lambda^{-2j+2m} \right), m \geq 0.
\end{align*}
\]

By means of the coefficients in (8), we obtain the following.

\[
\begin{align*}
EV_n^{(m)} U_n - U_n V_n^{(m)} &= \left( -q_n (A_n^{(m+1)} - A_n^{(m+1)}) \lambda B_n^{(m)} + \alpha \lambda r_n A_n^{(m+1)} \right), \\
&\quad -\lambda C_n^{(m+1)} 0.
\end{align*}
\] (9)

From (4), we know that the following is the case.

\[
U_{n,t} = \begin{pmatrix}
\lambda^2 q_{n,t} \frac{\alpha}{r_n} & \alpha \lambda r_{n,t} \\
\frac{\alpha \lambda q_{n,t} - \alpha \lambda q_{n,t}}{r_n} & 0
\end{pmatrix}.
\] (10)

In order to render (9) and (10) compatible, we need to modify (9); thus, we here take a modification matrix:

\[
\Delta_n^{(m)} = \begin{pmatrix}
0 & 0 \\
0 & -A_n^{(m+1)}
\end{pmatrix},
\] (11)

and define $v_n^{(m)} = r_n^{(m)} + \delta_n$ such that the following is obtained.

\[
\begin{align*}
EV_n^{(m)} U_n - U_n V_n^{(m)} &= \left( -q_n (A_n^{(m+1)} - A_n^{(m+1)}) \lambda B_n^{(m)} + \alpha \lambda r_n A_n^{(m+1)} \right), \\
&\quad -\lambda C_n^{(m+1)} + \alpha \lambda q_n A_n^{(m+1)},
\end{align*}
\] (12)

Assume that the time evolution of $\phi_n$ meets equations $\phi_{n,tm} = v_n^{(m)} \phi_n$ for which its compatibility condition with (4) is as follows:

\[
U_{n,tm} = (EV_n^{(m)}) U_n - U_n V_n^{(m)}, m \geq 0,
\] (13)

which yields the following discrete integrable lattice hierarchy.

\[
\begin{align*}
q_{n,tm} &= -q_n (A_n^{(m+1)} - A_n^{(m+1)}), \\
r_{n,tm} &= B_n^{(m)} + \alpha \lambda r_{n,t} A_n^{(m+1)},
\end{align*}
\] (14)
(i) When \( m = 0 \), Equation (14) reduces to (6) with \( t_0 = t \) as follows.

\[
\begin{align*}
q_{n,t_0} &= -q_n(A_{n+1}^{(1)} - A_n^{(1)}) = \frac{\alpha (q_n r_{n+1} \tau_{n+1} - q_{n+1} r_{n+1}^2)}{q_n + 1 r_{n+1}^2}, \\
r_{n,t_0} &= \frac{B_n^{(0)} + \alpha r_n A_n^{(1)}}{\alpha} = \frac{r_n (\alpha^2 r_n - q_n r_{n-1} + r_{n-1})}{\alpha q_n r_{n-1}}.
\end{align*}
\]

(15)

At this time, its time part of the Lax pair is as follows.

\[
\phi_{n,t_0} = V_n^{(0)} \phi_n = \left( \frac{-\lambda^2}{2\alpha} - \frac{\lambda r_n}{q_n r_{n-1}} - \frac{\lambda^2}{2\alpha} - \frac{1}{\alpha} + \frac{\alpha r_n}{r_n - 1} \right) \phi_n.
\]

(16)

(ii) When \( m = 1 \), Equation (14) reduces to a new higher-order RTL\(_m\) (\(\alpha\)) system.

\[
\begin{align*}
q_{n,t_1} &= -q_n(A_{n+1}^{(2)} - A_n^{(2)}) = q_n\left[ \frac{\alpha}{q_n} \left( \frac{2r_n}{r_n - 1} - \frac{r_n}{r_n - 9.q_n - 1} \right) - \frac{\alpha^2 r_n}{q_n r_{n-1}^2} \right], \\
r_{n,t_1} &= \frac{B_n^{(1)} + \alpha r_n A_n^{(2)}}{\alpha} = \frac{r_n}{q_n r_{n-1}} \left( \frac{\alpha^2 r_n}{q_n r_{n-1}^2} + \frac{\alpha^2 r_n}{q_n r_{n-1}^2} - \frac{2r_n}{r_n - 1} \right).
\end{align*}
\]

(17)

Moreover, its time part of the Lax pair is as follows:

\[
\phi_{n,t_1} = V_n^{(1)} \phi_n = \left( \lambda^4 A_n^{(0)} + \lambda^2 A_n^{(1)} - \lambda^3 B_n^{(0)} + \lambda B_n^{(1)} \right) \phi_n = \left( \begin{array}{c} V_{11}(n) \\ V_{21}(n) \\ V_{22}(n) \end{array} \right) \phi_n.
\]

(18)

with the following case.

\[
\begin{align*}
V_{11}(n) &= -\frac{\alpha^2 r_n}{q_n r_{n-1}^2} + \frac{\lambda^2}{\alpha}, \\
V_{12}(n) &= -\frac{\alpha^2 r_n}{q_n r_{n-1}^2} + \frac{\lambda^2}{\alpha} \left( \frac{\alpha^2 r_n}{q_n r_{n-1}^2} + \frac{\alpha^2 r_n}{q_n r_{n-1}^2} - \frac{2r_n}{r_n - 1} \right), \\
V_{21}(n) &= \frac{\lambda^3 r_n}{q_n r_{n-1}} + \frac{\lambda^2}{\alpha} \left( \frac{\alpha^2 r_n}{q_n r_{n-1}^2} + \frac{\alpha^2 r_n}{q_n r_{n-1}^2} - \frac{2r_n}{r_n - 1} \right), \\
V_{22}(n) &= \frac{\lambda^3 r_n}{q_n r_{n-1}} + \frac{\lambda^2}{\alpha} \left( \frac{\alpha^2 r_n}{q_n r_{n-1}^2} + \frac{\alpha^2 r_n}{q_n r_{n-1}^2} - \frac{2r_n}{r_n - 1} \right).
\end{align*}
\]

Remark 1. It should be pointed out that the new higher-order RTL\(_m\) (\(\alpha\)) system (17) has more nonlinear terms than Equation (6) and may have some novel properties that are worthy of further study. The main purpose of this paper is to study system (6); thus, we do not specifically study system (17) here.

2.2. Discrete Hamiltonian Structures

Next, we will construct the Hamiltonian structures of Equation (14) by using the trace identity technique [29–31]. Let \( U, V > tr(UV) \), where \( U \) and \( V \) are arbitrary square matrices, and \( tr \) means the trace of a matrix [29–31].

By defining the following:

\[
V_n = P_n U_n^{-1} = \left( \begin{array}{cc} A_n & B_n \\ C_n & -A_n \end{array} \right) \left( \begin{array}{cc} 0 & -\frac{r_n}{\alpha q_n} \\ 1 & \frac{\lambda^2 q_n}{a^2 q_n} \end{array} \right) = \left( \begin{array}{cc} B_n & \frac{A_n}{\lambda q_n} - \frac{r_n}{\alpha q_n} \frac{(\lambda^2 q_n - 1)B_n}{\alpha^2 q_n} \\ \frac{A_n}{\lambda q_n} - \frac{r_n}{\alpha q_n} \frac{(\lambda^2 q_n - 1)A_n}{\alpha^2 q_n} & -B_n \end{array} \right),
\]

(19)

we obtain the following case.

\[
\begin{align*}
< V_n, \frac{\partial H_n}{\partial q_n} > &= \frac{\partial H_n}{\partial q_n} + B_n \frac{\partial V_n}{\partial q_n}, \\
< V_n, \frac{\partial H_n}{\partial r_n} > &= \frac{2A_n}{r_n} + \frac{\lambda^2 B_n}{a^2} - \frac{B_n}{\alpha \lambda r_n}.
\end{align*}
\]

(20)
Substituting them into the trace identity [29–31]:

$$\frac{\delta}{\delta u} \sum_{n \in Z} < V_n, \frac{\partial U_n}{\partial \lambda} > = (\lambda^{-1} \frac{\partial}{\partial \lambda} \lambda^c) < V_n, \frac{\partial U_n}{\partial u} >, \quad i = 1, 2,$$

results in the following case.

$$\frac{\delta}{\delta u} \sum_{n \in Z} (\eta_n B_n + \frac{B_n}{\alpha r_n}) = (\lambda^{-1} \frac{\partial}{\partial \lambda} \lambda^c) \left( -2 \frac{\delta}{\partial u} \right) + \frac{B_n}{\alpha \lambda r_n} - \frac{B_n}{\alpha r_n^2} \right).$$

Comparing the coefficients of $\lambda^{-2m-3}$ on both sides of Equation (22), we obtain the following.

$$\left( \frac{\delta}{\delta u} \right) \sum_{n \in Z} (\frac{\eta_n B_n^{(m+1)}}{\alpha r_n} + \frac{B_n^{(m)}}{\alpha^2 r_n^2}) = (\epsilon - 2m - 2) \left( \frac{A_n^{(m+1)}}{\alpha q_n} + \frac{B_n^{(m)}}{\alpha^2 r_n^2} \right).$$

Setting $m = 0$ in the above equation, we can fix $\epsilon = 0$. Let $H_n^{(m)} = \sum_{n \in Z} \frac{q_n B_n^{(m+1)} + B_n^{(m)}}{2\alpha (m-1)r_n}$, thus the following is obtained.

$$\frac{\delta H_n^{(m)}}{\delta u} = \left( \frac{A_n^{(m+1)}}{\alpha q_n} + \frac{B_n^{(m)}}{\alpha^2 r_n^2} \right) \equiv f_n^{(m+1)} \eta_n^{(m+1)},$$

with the following case.

$$J = \begin{pmatrix} 0 & -q_n r_n \\ -q_n r_n & 0 \end{pmatrix}.$$
involuntary concerning the Poisson bracket. Here, we will not provide a detailed derivation of their involution, and the readers can refer to Refs. [29–31] for the detailed process. Therefore, we can say that the lattice hierarchy (14) or the Hamiltonian form (27) has Liouville integrability.

2.3. An Infinite Number of Conservation Laws of Equation (6)

Existing infinite conservation laws in an NLE play an important role in further verification of its integrability [33–35]. Therefore, the study of infinitely many conservation laws for NLE has important theoretical significance and practical values. In this subsection, we will derive the infinitely many conservation laws of Equation (6).

Setting \( \theta_n = \frac{\psi_n}{\psi} \), from the space part of Lax pair (4) of Equation (6), we have the following.

\[
\frac{\psi_{n+1}}{\psi_n} = \lambda^2 q_n - 1 + a\lambda r_n \theta_n,
\]

\[
\frac{\psi_{n+1}}{\psi_n} = - \frac{a\lambda q_n}{r_n \theta_n}.
\]

(28)

Let the above two expressions be divided by each other, and we obtain the following.

\[
(\lambda^2 q_n - 1 + a\lambda r_n \theta_n)\theta_{n+1} + \frac{a\lambda q_n}{r_n} = 0.
\]

(29)

Inserting \( \theta_n = \sum_{j=0}^{\infty} \theta_n^{(j)} \lambda^{-j} \) into (29) and collecting the coefficients of same powers of \( \lambda \) yield the following.

\[
\begin{align*}
\theta_n^{(0)} &= 0, & \theta_n^{(1)} &= - \frac{a}{q_n}, & \theta_n^{(2)} &= 0, & \theta_n^{(3)} &= - \frac{a}{q_n} \left( \frac{1}{r_{n-1}} + \frac{a^2}{r_n} \right), & \theta_n^{(4)} &= 0, \\
\theta_n^{(5)} &= - \frac{a^3 r_n}{q_n^2} \left( \frac{1}{r_{n-1}} + \frac{a^2}{r_n} \right), & \theta_n^{(6)} &= - \frac{a^3 r_n}{q_n^2} \left( \frac{1}{r_{n-1}} + \frac{a^2}{r_n} \right), & \theta_n^{(7)} &= - \frac{a^3 r_n}{q_n^2} \left( \frac{1}{r_{n-1}} + \frac{a^2}{r_n} \right), & \theta_n^{(8)} &= 0, \\
\theta_n^{(2j)} &= 0, & \theta_n^{(2j+1)} &= \frac{a^j r_n}{q_n^2} \left( \frac{1}{r_{n-1}} + \frac{a^2}{r_n} \right), & j & \geq 3.
\end{align*}
\]

(30)

Combining the time part (16) of the Lax pair, we can obtain the following.

\[
[\ln(\lambda^2 q_n + a\lambda r_n \theta_n - 1)]_t = (E - 1) \left( - \frac{\lambda^2}{2a^2} - \frac{\lambda r_n}{q_n} \theta_n \right).
\]

(31)

Substitution of (30) into (31) and comparing the same powers of \( \lambda \) on both sides of (31) can provide an infinite number of conservation laws for Equation (6). The first three conservation laws are listed below:

\[
(T_k)_t = (E - 1)(X_k), \quad (k = 1, 2, 3),
\]

(32)

with the following being the case:

\[
T_1 = \ln q_n, \quad X_1 = \frac{ar_n}{q_n^2}, \quad T_2 = - \frac{1}{q_n} \left( \frac{a^2 r_n}{q_n} - 1 \right), \quad X_2 = \frac{ar_n}{q_n^2} \left( \frac{1}{r_{n-1}} + \frac{a^2}{r_n} \right),
\]

\[
T_3 = - \frac{a^2 r_n}{q_n} \left( \frac{1}{r_{n-1}} + \frac{a^2}{r_n} \right) + \frac{2a^2}{q_n^2} \left( \frac{1}{r_{n-1}} + \frac{a^2}{r_n} \right), \quad X_3 = \frac{a^2 r_n}{q_n^2} \left( \frac{1}{r_{n-1}} + \frac{a^2}{r_n} \right) + \frac{a^2}{q_n} \left( \frac{1}{r_{n-1}} + \frac{a^2}{r_n} \right) + \frac{a^2}{r_{n-1}} \left( \frac{1}{q_n^2} + \frac{a^2}{r_n} \right),
\]

where \( T_k \) denotes the conserved densities, and \( X_k \) denotes associated fluxes. The above three conservation laws usually represent energy conservation, momentum conservation and Hamiltonian conservation, respectively [35]. Existing infinitely many conservation laws means that Equation (6) is a discrete integrable system.
3. Discrete Generalized \((m, 2N - m)\)-Fold DT

In the previous section, we have discussed the integrability of Equation (6). In this part, we will construct the discrete generalized \((m, 2N - m)\)-fold DT of Equation (6). For this purpose, we consider the following gauge transformation:

\[
\hat{\phi}_n = T_n \phi_n,
\]

which can transform the Lax pair (4) and (16) into the following forms.

\[
\hat{\phi}_{n+1} = \hat{U}_n \hat{\phi}_n = T_{n+1} U_n T_n^{-1} \hat{\phi}_n, \quad \hat{\phi}_{n,t} = V_n^{(0)} \hat{\phi}_n = (T_{n,t} + V_n^{(0)} T_n) T_n^{-1} \hat{\phi}_n.
\]

In order to construct DT, we require that \(\hat{U}_n, V_n^{(0)}\) have the same forms as \(U_n, V_n^{(0)}\) in addition to replacing the new potential functions \(\hat{q}_n, \hat{r}_n\) with the old potential functions \(q_n, r_n\). In order to achieve this special purpose, we need to construct a particular Darboux matrix \(T_n\) in the following form:

\[
T_n = \left( \begin{array}{cc} a_n & b_n \\ c_n & d_n \end{array} \right) = \left( \begin{array}{cc} 1 + \sum_{j=1}^{N} a_n^{(2N-2j)} \lambda^{2j} & \sum_{j=1}^{N} b_n^{(2j-1)} \lambda^{2j-1} \\ \sum_{j=1}^{N} c_n^{(2j-1)} \lambda^{2j-1} & \lambda^{2N} + \sum_{j=0}^{N-1} d_n^{(2N-2j)} \lambda^{2j} \end{array} \right),
\]

in which the number \(N\) is a positive integer representing the order of DT, and \(a_n^{(2N-2j)}, b_n^{(2j-1)}, c_n^{(2j-1)}\) and \(d_n^{(2N-2j)}\) \((j = 1, 2, \ldots, N)\) are functions of the variables \(n, t\) that are determined later. According to (34), we can obtain the following transformations between the new potentials and old ones as follows.

\[
\hat{q}_n = \frac{a_n^{(0)} + \hat{q}_n}{a_n^{(0)}}, \quad \hat{r}_n = \frac{ar_n + b_n^{(2N-1)}}{ad_n^{(2N)}}.
\]

In order to provide the new solutions of Equation (6), the unknown functions \(a_n^{(0)}, b_n^{(2N-1)}, c_n^{(2j-1)}, d_n^{(2N-2j)}\) must be determined.

Next, we will determine their concrete expressions by constructing them directly. According to Darboux matrix \(T_n\), we know that det \(T_n\) is a \(4N\)-th order polynomial of \(\lambda\), which should have \(4N\) roots. Let us assume that \(\phi_{i,n}(\lambda) = (\phi_{i,n}(\lambda_{i,n}), \phi_{i,n}(\lambda_i))^{T} \equiv (\phi_{i,n}, \phi_{i,n})^{T}\) are \(n\) solutions of Lax pair (4) and (16) for \(n\) different spectral parameters \(\lambda_i\) \((k = 1, 2, \ldots, n)\). Moreover, we also want them to be \(n\) roots of det \(T_n\). It should be noted that this can easily meet the conditions. In fact, we only need to solve Lax pairs at \(n\) known roots of det \(T_n\). Moreover, for these roots, we must guarantee that \(T(\lambda_i) \phi_{i,n}(\lambda_i) = 0\) \((i = 1, 2, \ldots, m)\), which will only produce \(2m\) algebraic equations such that we cannot determine the \(4N\) unknown functions \(a_n^{(2N-2j)}, b_n^{(2j-1)}, c_n^{(2j-1)}, d_n^{(2N-2j)}\) when \(m < N\).

In order to derive a linear algebraic system of \(4N\) equations with \(4N\) unknown functions \(a_n^{(2N-2j)}, b_n^{(2j-1)}, c_n^{(2j-1)}, d_n^{(2N-2j)}\), for every \(\lambda_{i}\), we need to expand the following:

\[
T(\lambda_i + \epsilon) \phi_{i,n}(\lambda_i + \epsilon) = \sum_{K=0}^{N-1} \sum_{j=0}^{k} T^{(j)}(\lambda_i) \phi_{i,n}^{(k-j)}(\lambda_i) \epsilon^k,
\]

where \(\epsilon\) is a small parameter, \(\phi_{i,n}^{(k)}(\lambda_i) = \frac{1}{k!} \phi_{i,n}^{(k)}(\lambda_i)\) is given by the Taylor series expansion of \(\phi_{i,n}(\lambda_i + \epsilon) = \phi_{i,n}^{(0)}(\lambda_i) + \phi_{i,n}^{(1)}(\lambda_i) \epsilon + \phi_{i,n}^{(2)}(\lambda_i) \epsilon^2 + \phi_{i,n}^{(3)}(\lambda_i) \epsilon^3 + \cdots\) around \(\epsilon = 0\), \(T_i^{(j)}\) is given by a binomial expansion of \(T(\lambda + \epsilon) = T_n^{(0)} + T_n^{(1)} \epsilon + \cdots + T_n^{(m)} \epsilon^m\). Thus, the formula \(\lim_{\epsilon \to 0} \frac{T(\lambda_i + \epsilon) \phi_{i,n}(\lambda_i + \epsilon)}{\epsilon^k} = 0\) \((i = 1, 2, \ldots, m, k_i = 0, 1, \ldots, v_i, 2N = m + \sum_{i=1}^{m} v_i)\) can pro-
duce 4N algebraic equations for 4N unknown functions $a_n^{(2N-2i)}$, $b_n^{(2j-1)}$, $c_n^{(2j-1)}$, $d_n^{(2N-2i)}$. In other words, we have the following:

$$
\begin{align*}
T^0((\lambda_i)\phi^{(0)}_{i,m}(\lambda_i) = 0, \\
T^0((\lambda_i)\phi^{(1)}_{i,m}(\lambda_i) + T^1((\lambda_i)\phi^{(0)}_{i,m}(\lambda_i) = 0, \\
T_n^0((\lambda_i)\phi^{(0)}_{i,m}(\lambda_i) + T_n^1((\lambda_i)\phi^{(1)}_{i,m}(\lambda_i) + T_n^2((\lambda_i)\phi^{(2)}_{i,m}(\lambda_i) = 0, \\
\ldots, \\
\sum_{j=0}^{N^i} T^j((\lambda_i)\phi^{(N^i-j)}_{i,m}(\lambda_i) = 0. 
\end{align*}
$$

from which the determinant of the coefficients for system (4) is nonzero when the m spectral parameters $\lambda_i$ are suitably chosen so that $a_n^{(2N-2i)}$, $b_n^{(2j-1)}$, $c_n^{(2j-1)}$, $d_n^{(2N-2i)}$ in the Darboux matrix $T_n$ are uniquely determined by (38). Thus, we can provide the undetermined functions $a_n^{(0)}$, $b_n^{(2N-1)}$, $d_n^{(2N)}$ in (36) as follows:

$$
a_n^{(0)} = \frac{\Delta a_n^{(0)}}{\Delta_1^{(1)}},
\quad b_n^{(2N-1)} = \frac{\Delta b_n^{(2N-1)}}{\Delta_1^{(1)}},
\quad d_n^{(2N)} = \frac{\Delta d_n^{(2N)}}{\Delta_2^{(1)}},
$$

with

$$
\Delta_1 = (\Lambda_1^{(1)}, \Lambda_2^{(2)}, \ldots, \Lambda_1^{(m)})^T,
\quad \Delta_2 = (\Lambda_2^{(1)}, \Lambda_2^{(2)}, \ldots, \Lambda_2^{(m)})^T,
\quad \Lambda_1^{(i)} = (\Lambda_{1,i,s}^{(i)})_{2(v+1)\times 2N},
\quad \Lambda_2^{(i)} = (\Lambda_{2,i,s}^{(i)})_{2(v+1)\times 2N},
$$

in which $\Lambda_{1,i,s}^{(i)}$, $\Lambda_{2,i,s}^{(i)}$ ($1 \leq j \leq 2(v+1)$, $1 \leq s \leq 2N$, $i = 1, 2, \ldots, m$) given as follows:

$$
\Lambda_{1,i,s}^{(i)} = \left\{ \begin{array}{ll}
\sum_{k=0}^{j-1} c_{k}^{2N-2s+2} \Lambda_{1,i,s}^{(i-j-k)} & \text{for } l + \sum_{i=1}^{j-1} v_i \leq l + \sum_{i=1}^{j} \sum_{i=1}^{j} (1 \leq l \leq m), 1 \leq s \leq N, \\
\sum_{k=0}^{j-1} c_{k}^{2N-2s+1} \Lambda_{1,i,s}^{(i-j-k)} & \text{for } l + \sum_{i=1}^{j-1} v_i \leq l + \sum_{i=1}^{j} \sum_{i=1}^{j} (1 \leq l \leq m), 1 \leq s \leq 2N,
\end{array} \right.
$$

$$
\Lambda_{2,i,s}^{(i)} = \left\{ \begin{array}{ll}
\sum_{k=0}^{j-1} c_{k}^{2N-2s+1} \Lambda_{2,i,s}^{(i-j-k)} & \text{for } l + \sum_{i=1}^{j-1} v_i \leq l + \sum_{i=1}^{j} \sum_{i=1}^{j} (1 \leq l \leq m), 1 \leq s \leq 2N,
\end{array} \right.
$$

where $\Delta a_n^{(0)}$ and $b_n^{(2N-1)}$ are given from the determinant $\Delta_1$ by replacing their first and last columns by the column vector $(f_1^{(i)}, f_2^{(i)}, \ldots, f_{(v+1)}^{(i)}, f_1^{(i)}, f_2^{(i)}, \ldots, f_{(v+1)}^{(i)}, r_1^{(i)}, r_2^{(i)}, \ldots, r_{(v+1)}^{(i)}, r_1^{(i)}, r_2^{(i)}, \ldots, r_{(v+1)}^{(i)})$ with $f_j^{(i)} = -\phi_{i,n}^{(i-j)} (1 \leq j \leq (v+1), 1 \leq i \leq m)$, respectively, while $\Delta d_n^{(2N)}$ is obtained from the determinant $\Delta_2$ by replacing the last columns by the column vector $(f_1^{(i)}, f_2^{(i)}, \ldots, f_{(v+1)}^{(i)}, r_1^{(i)}, r_2^{(i)}, \ldots, r_{(v+1)}^{(i)}, r_1^{(i)}, r_2^{(i)}, \ldots, r_{(v+1)}^{(i)}, r_1^{(i)}, r_2^{(i)}, \ldots, r_{(v+1)}^{(i)})$ with $r_j^{(i)} = -\phi_{i,n}^{(i-j)} (1 \leq j \leq (v+1), 1 \leq i \leq m)$.

Different from the previous usual N-fold DT using N spectral parameters, we call the transformations (33) and (36) using m spectral parameters as the generalized $(m, 2N - m)$-fold DT. Here, the following notation is used: m in the name of the generalized $(m, 2N - m)$-fold DT stands for the number of spectral parameters used, while $2N - m$ represents the sum of the used Taylor expansion order number of the eigenfunction $\phi_{i,n}$.

**Remark 2.** When $m = 2N$ and $m_i = 0$, the transformations (33) and (36) reduce to the discrete generalized $(2N, 0)$-fold DT which includes the discrete 2N-fold DT if we do not use the Taylor series expansion for every $\phi_{i,n}(\lambda_i)$. If $m = 1$ and $m_i = 2N - 1$, the transformations (33) and (36) reduce to the discrete generalized $(1, 2N - 1)$-fold DT, which can provide higher-order rational and semi-rational solutions of Equation (6). If $m = 2$ and $m_i = 2N - 2$, the transformations (33) and (36) reduce to the discrete generalized $(2, 2N - 2)$-fold DT, which are hybrid solutions of soliton and rational or semi-rational solutions. If $2 < m < 2N$, the transformations (33) and (36) can also provide some other new discrete mixed solutions that will not be discussed here.

**4. Explicit Exact Solutions and Their Asymptotic Analysis**

In this section, we will apply the special cases of the discrete generalized $(m, 2N - m)$-fold DT to provide some explicit exact solutions including soliton, rational and semi-
rational solutions and their hybrid solutions. Then, we will analyze the limit states of such solutions by asymptotic analysis technique.

4.1. Multi-Soliton Solutions and Dynamics

When \( m = 2N \), the discrete generalized \((m, 2N - m)\)-fold DT reduces to the discrete generalized \((2N, 0)\)-fold DT. If we do not make a Taylor series expansion for every eigenfunction \( \phi_{i,n}(\lambda_i) (i = 1, 2, ..., 2N) \), at this time the discrete generalized \((2N, 0)\)-fold DT is just the usual \(2N\)-fold DT, which can provide multi-soliton solutions. Substitution of the initial seed solutions \( \eta_n = 1 + a^2, r_n = 1 \) into (4) and (5) yields one basic solution with \( \lambda = \lambda_i (i = 1, 2, ..., 2N) \) as follows:

\[
\phi_{i,n} = \left( \frac{\phi_{i,n}}{\psi_{i,n}} \right) = \left( -\alpha \lambda_i (a^2 + 1) C_{n1} \tau_1^{-1} e^{\varphi t} - \alpha \lambda_i (a^2 + 1) C_{n2} \tau_2^{-1} e^{\varphi t} \right) \tag{39}
\]

with

\[
\begin{align*}
\tau_1 &= \frac{1}{2} (\alpha^2 \lambda_i^2 + \lambda_i^2 - 1 - \sqrt{\alpha^4 \lambda_i^4 - 4 \alpha^2 \lambda_i^2 + 2 \alpha^2 \lambda_i^2 + 6 \lambda_i^4 - 2 \lambda_i^4 + 1}), \\
\tau_2 &= \frac{1}{2} (\alpha^2 \lambda_i^2 + \lambda_i^2 - 1 - \sqrt{\alpha^4 \lambda_i^4 - 4 \alpha^2 \lambda_i^2 + 2 \alpha^2 \lambda_i^2 + 6 \lambda_i^4 - 2 \lambda_i^4 + 1}), \\
\rho_{i1} &= \frac{1}{\alpha (a^2 + 1)} (\alpha^2 \lambda_i^2 - \lambda_i^2 + 1 + \sqrt{\alpha^4 \lambda_i^4 - 4 \alpha^2 \lambda_i^2 + 2 \alpha^2 \lambda_i^2 + 6 \lambda_i^4 - 2 \lambda_i^4 + 1}), \\
\rho_{i2} &= \frac{1}{\alpha (a^2 + 1)} (\alpha^2 \lambda_i^2 - \lambda_i^2 + 1 - \sqrt{\alpha^4 \lambda_i^4 - 4 \alpha^2 \lambda_i^2 + 2 \alpha^2 \lambda_i^2 + 6 \lambda_i^4 - 2 \lambda_i^4 + 1}).
\end{align*}
\]

According to (36), we can obtain exact \(2N\)-soliton solutions of Equation (6). By direct calculation, we find that even soliton solutions of higher orders can degenerate into odd soliton solutions of lower order when we take one of \( \lambda_i \) equal to \( 1 + \frac{a}{\sqrt{n+1}} \), \( 1 - \frac{a}{\sqrt{n+1}} \), \( -1 + \frac{a}{\sqrt{n+1}} \), \( -1 - \frac{a}{\sqrt{n+1}} \). Hereafter, in the degenerate case, we uniformly choose \( \lambda = 1 + \frac{a}{\sqrt{n+1}} \). Next, as an example, we discuss the two cases: \( N = 1, 2 \).

Case (1) When \( N = 1 \), \( \lambda = \lambda_i (i = 1, 2) \) based on the \(2\)-fold DT from (36), we can provide the two-fold exact solutions as follows:

\[
\begin{align*}
\tilde{\eta}_n &= \left( 1 + \alpha^2 \right) a_{n+1}^{(0)} \eta_n^{(0)}, \\
\tilde{\rho}_n &= \alpha + b_{n+1}^{(1)} \rho_n^{(1)}.
\end{align*}
\tag{40}
\]

where \( a_n^{(0)} = \frac{\Delta \eta_n^{(0)}}{\Delta_1}, b_n^{(1)} = \frac{\Delta \rho_n^{(1)}}{\Delta_2} \) and \( a_n^{(2)} = \frac{\Delta \rho_n^{(2)}}{\Delta_2} \), in which the following is the case.

\[
\begin{align*}
\Delta_1 &= \begin{vmatrix}
\lambda_1^2 & \lambda_1 \delta_{1,n} \\
\lambda_2 & \lambda_2 \delta_{2,n}
\end{vmatrix}, \\
\Delta_2 &= \begin{vmatrix}
\lambda_1 & \lambda_2 \\
\delta_{1,n} & \delta_{2,n}
\end{vmatrix}, \\
\Delta \rho_n^{(0)} &= \begin{vmatrix}
-1 & \lambda_1 \delta_{1,n} \\
-1 & \lambda_2 \delta_{1,n}
\end{vmatrix}, \\
\Delta \rho_n^{(2)} &= \begin{vmatrix}
\lambda_1 & -\lambda_2 \delta_{1,n} \\
\lambda_2 & -\lambda_2 \delta_{2,n}
\end{vmatrix}.
\end{align*}
\]

For the sake of analysis, the analytical expressions of solution (36) can be rewritten as follows:

\[
\begin{align*}
\tilde{\eta}_n &= \left( 1 + \alpha^2 \right) \frac{[\lambda_1 \cosh(\xi_1 + X_1) \cosh(\xi_1 - \lambda_1 \cosh(\xi_1) \cosh(\xi_2 + X_2)]/[\lambda_2 \cosh(\xi_1 - X_1) \cosh(\xi_1 - \lambda_2 \cosh(\xi_1 - X_2))] - 1}{[\lambda_1 \cosh(\xi_1 + X_1) \cosh(\xi_1 - \lambda_1 \cosh(\xi_1) \cosh(\xi_2 + X_2)]/[\lambda_2 \cosh(\xi_1 - X_1) \cosh(\xi_1 - \lambda_2 \cosh(\xi_1 - X_2))]}, \\
\tilde{\rho}_n &= \frac{\lambda_1 \cosh(\xi_1) \cosh(\xi_1 - X_1) \cosh(\xi_1 - \lambda_1 \cosh(\xi_1) \cosh(\xi_2 + X_2)]}{\lambda_2 \cosh(\xi_1 - X_1) \cosh(\xi_1 - \lambda_2 \cosh(\xi_1 - X_2))}
\end{align*}
\tag{41}
\]

in which the following is the case.

\[
\begin{align*}
R_1 &= [\alpha \sqrt{1 + \alpha^2 \lambda_1^2 \lambda_2} \cosh(\xi_1 - X_1) \cosh(\xi_2 - \alpha \sqrt{1 + \alpha^2 \lambda_1^2 \lambda_2} \cosh(\xi_1) \cosh(\xi_2 - X_2)) \\
&+ (\lambda_2^2 - \lambda_1^2) \cosh(\xi_1) \cosh(\xi_2)]/[\alpha \cosh(\xi_1 - X_1) \cosh(\xi_2 - \lambda_2 \cosh(\xi_1) \cosh(\xi_2 - X_2))], \\
\tau_i &= \frac{1}{2} (\ln \tau_1 - \ln \tau_2) n + \frac{1}{2} (\rho_{i1} - \rho_{i2}) t + \frac{1}{2} (\ln C_{i1} - \ln C_{i2}), \quad X_i = \frac{1}{2} (\ln \tau_1 - \ln \tau_2), \quad i = 1, 2.
\]
The parameters are suitably chosen so that the solutions (41) may be one-soliton or two-soliton solutions, for which their evolution plots are indicated in Figures 2 and 3. When both $\lambda_1$ and $\lambda_2$ are not equal to $1 + \frac{\alpha}{\sqrt{\alpha^{2} + 1}}$, the solutions (41) are two-soliton ones. It is worth mentioning that when one of the $\lambda_1, \lambda_2$ is $1 + \frac{\alpha}{\sqrt{\alpha^{2} + 1}}$, without loss of generality, we set $\lambda_1 = 1 + \frac{\alpha}{\sqrt{\alpha^{2} + 1}} = A$, and the solutions (41) reduce to one-soliton ones of the following form:

$$\begin{align}
\bar{q}_n &= \left(1 + a^2\right)\left[\lambda_1 \cosh(\xi_2 - A \cosh(\xi_2 + X_2)) - \lambda_2 \cosh(\xi_2 - A \cosh(\xi_2 + X_2))\right] \\
\bar{p}_n &= a \sqrt{1 + a^2 \lambda_1^2} A \cosh^2(\xi_2) + \lambda_2 \cosh(\xi_2 - A \cosh(\xi_2 + X_2)) - \lambda_2 \cosh(\xi_2 - A \cosh(\xi_2 + X_2))
\end{align} \tag{42}$$

for which their corresponding evolution plots are shown in Figure 2. Figure 2a1,a2 show dark one-soliton structures of the component $\bar{q}_n$ under nonzero background (i.e., $\bar{q}_n = 1.6$). Figure 2b1,b2 show the kink-shaped one-soliton propagation structure for solution $\bar{p}_n$. We know that $\bar{q}_n = \frac{\ln |\bar{q}_n|}{\bar{q}_n}$, $\bar{p}_n = \ln |\bar{p}_n|$ are the solutions of Equation (1), and Figure 2c1,c2 show dark one-soliton propagation structures on nonzero background. Figure 2d1,d2 show the kink-shaped one-soliton structures for solution $\bar{p}_n$. From Figure 2, we can clearly observe that one soliton retains the same amplitude and shape during its propagation.

![Figure 2](image_url)

Figure 2. (Color online) One-soliton structures via solutions (42) with parameters $\lambda_1 = \frac{8}{3}, \lambda_2 = 2, \alpha = \frac{3}{4}$ and $C_{11} = C_{12} = C_{21} = -C_{22} = 1$. (a1–d1) The profiles of one-soliton solutions are $\bar{q}_n, \bar{p}_n, \bar{x}_n$ and $\bar{z}_n$. (a2–d2) The propagation processes for $\bar{q}_n, \bar{p}_n$ and $\bar{x}_n$ at $t = 15$ (dashed line) and $t = 15$ (solid line).

When neither of the two parameters $\lambda_1$ and $\lambda_2$ are $1 + \frac{\alpha}{\sqrt{\alpha^{2} + 1}}$, the solutions (41) are two-soliton ones. In order to further study the elastic interaction properties of two-soliton solutions $\bar{q}_n, \bar{p}_n$ in (41), we perform an asymptotic analysis to investigate their limit states, which are listed as follows.

Before collision ($t \to -\infty$), the following cases are observed:

(i) If $\xi_1$ is fixed, $\xi_2 \to +\infty$, we can calculate the initial limit states of the first soliton as follows:

$$\begin{align}
\bar{q}_n &\to \bar{q}_{n1} = \frac{(1 + a^2)\left[\lambda_1 e^{-\xi_2} \cosh(\xi_1 - X_1) - \lambda_2 e^{-\xi_2} \cosh(\xi_1 + X_1)\right]}{\lambda_1 e^{-\xi_2} \cosh(\xi_1 - X_1) - \lambda_2 e^{-\xi_2} \cosh(\xi_1 + X_1)} \\
\bar{p}_n &\to \bar{p}_{n1} = \frac{a \sqrt{1 + a^2 \lambda_1^2} \lambda_2 \cosh(\xi_1 - X_1) + \lambda_2 \cosh(\xi_1 + X_1) - \lambda_2 \cosh(\xi_1 - X_1)}{a \sqrt{1 + a^2 \lambda_1^2} \lambda_2 \cosh(\xi_1 - X_1) - \lambda_1 \cosh(\xi_1 + X_1)}
\end{align}$$

(ii) If $\xi_2$ is fixed, $\xi_1 \to +\infty$, we can calculate the initial limit states of the second soliton as follows:

$$\begin{align}
\bar{q}_n &\to \bar{q}_{n2} = \frac{(1 + a^2)\left[\lambda_1 e^{-\xi_2} \cosh(\xi_1 - X_1) - \lambda_2 e^{-\xi_2} \cosh(\xi_1 + X_1)\right]}{\lambda_1 e^{-\xi_2} \cosh(\xi_1 - X_1) - \lambda_2 e^{-\xi_2} \cosh(\xi_1 + X_1)} \\
\bar{p}_n &\to \bar{p}_{n2} = \frac{a \sqrt{1 + a^2 \lambda_2^2} \lambda_1 \cosh(\xi_1 - X_1) + \lambda_1 \cosh(\xi_1 + X_1) - \lambda_1 \cosh(\xi_1 - X_1)}{a \sqrt{1 + a^2 \lambda_2^2} \lambda_1 \cosh(\xi_1 - X_1) - \lambda_2 \cosh(\xi_1 + X_1)}
\end{align}$$
(ii) If $\zeta_2$ is fixed, $\zeta_1 \to +\infty$, we can calculate the initial limit states of the second soliton as follows.

$$
\tilde{q}_n \to q_{n2}^- = \frac{(1-a^2)[\lambda_1e^{X_1}\cosh(\zeta_2-\lambda_1\cosh(\zeta_2+X_2))]\left[\lambda_2e^{X_1}\cosh(\zeta_2-\lambda_1\cosh(\zeta_2-X_2))\right]}{\left[\lambda_2\cosh(\zeta_2+X_2)\right]-\lambda_1e^{-X_1}\cosh(\zeta_2+X_2)}.
$$

$$
\tilde{r}_n \to r_{n2}^- = \frac{[\sqrt{1+a^2}\lambda_2^2e^{X_1}\cosh(\zeta_2-\lambda_1\cosh(\zeta_2+X_2)) + (\lambda_2^2-\lambda_1^2)cosh(\zeta_2)]\left[\lambda_2e^{X_1}\cosh(\zeta_2-\lambda_1\cosh(\zeta_2-X_2))\right]}{\sqrt{1+a^2}\lambda_2^2e^{X_1}\cosh(\zeta_2-\lambda_1\cosh(\zeta_2+X_2)) - \lambda_1e^{-X_1}\cosh(\zeta_2+X_2)}.
$$

After collision ($t \to +\infty$), we observe the following:

(iii) If $\zeta_1$ is fixed, $\zeta_2 \to -\infty$, we can calculate the end limit states of the first soliton as follows;

$$
\tilde{q}_n \to q_{n1}^- = \frac{(1+a^2)[\lambda_2\cosh(\zeta_1+X_1) - \lambda_1e^{X_2}\cosh(\zeta_1-X_1)]\left[\lambda_2\cosh(\zeta_1+X_1) - \lambda_1e^{-X_2}\cosh(\zeta_1-X_1)\right]}{\left[\lambda_2\cosh(\zeta_1+X_1)\right]-\lambda_1e^{-X_1}\cosh(\zeta_1-X_1)}.
$$

$$
\tilde{r}_n \to r_{n1}^- = \frac{[\sqrt{1+a^2}\lambda_1^2\cosh(\zeta_1+X_1) - \alpha\sqrt{1+a^2}\lambda_1^2e^{-X_2}\cosh(\zeta_1) + (\lambda_1^2-\lambda_2^2)cosh(\zeta_1)]\left[\lambda_1\cosh(\zeta_1-X_1) - \lambda_2e^{-X_2}\cosh(\zeta_1-X_2)\right]}{\sqrt{1+a^2}\lambda_1^2\cosh(\zeta_1+X_1) - \alpha\sqrt{1+a^2}\lambda_1^2e^{-X_2}\cosh(\zeta_1) - \lambda_2e^{-X_2}\cosh(\zeta_1-X_2)}.
$$

(iv) If $\zeta_2$ is fixed, $\zeta_1 \to -\infty$, we can calculate the end limit states of the second soliton as follows:

$$
\tilde{q}_n \to q_{n2}^- = \frac{(1+a^2)[\lambda_1e^{X_1}\cosh(\zeta_2-\lambda_1\cosh(\zeta_2+X_2))]\left[\lambda_2e^{-X_1}\cosh(\zeta_2-\lambda_1\cosh(\zeta_2-X_2))\right]}{\left[\lambda_2\cosh(\zeta_2+X_2)\right]-\lambda_1e^{X_1}\cosh(\zeta_2+X_2)}.
$$

$$
\tilde{r}_n \to r_{n2}^- = \frac{[\sqrt{1+a^2}\lambda_2^2\cosh(\zeta_2+X_2) - \sqrt{1+a^2}\lambda_2^2e^{-X_2}\cosh(\zeta_2) + (\lambda_2^2-\lambda_1^2)cosh(\zeta_2)]\left[\lambda_2\cosh(\zeta_2-\lambda_1\cosh(\zeta_2-X_2))\right]}{\sqrt{1+a^2}\lambda_2^2\cosh(\zeta_2+X_2) - \sqrt{1+a^2}\lambda_2^2e^{-X_2}\cosh(\zeta_2) - \lambda_1e^{-X_1}\cosh(\zeta_2-X_2)}.
$$

where $q_{n1}^, q_{n2}^, r_{n1}^, r_{n2}^,$ stand for the asymptotic state expressions of $\tilde{q}_n$ and $\tilde{r}_n$, the ‘−’ sign indicates the limit states before collision and the ‘+’ sign denotes the limit states after collision.

According to the above analysis, we know that the interactions between two solitons for $\tilde{q}_n$ and $\tilde{r}_n$ are elastic. When the parameters are given by $\lambda_1 = \frac{1}{3}, \lambda_2 = 3, \alpha = 1, C_{11} = C_{12} = C_{21} = C_{22} = \frac{2}{3}$, the corresponding evolution structure plots are shown in Figure 3. Figure 3a1,a2 demonstrate the head-on elastic interactions between one dark soliton and one anti-bell-shaped soliton of the component $\tilde{q}_n$ on nonzero constant background for $\tilde{q}_n$. Figure 3b1,b2 display the head-on inelastic interactions between two kink solitons for solution $\tilde{r}_n$. Figure 3c1,c2 demonstrate the head-on elastic interactions between one dark soliton and one anti-bell-shaped soliton of the component $\tilde{p}_n$. Figure 3d1,d2 exhibit the head-on elastic interactions between two kink solitons for solution $\tilde{r}_n$. From Figure 3, we can clearly observe that the amplitudes and shapes of two solitons do not change before and after the interactions, which is consistent with the results of our asymptotic analysis.

Next, we implement numerical simulations in order to display the dynamical behaviors of the previous one-soliton and two-soliton solutions of Equation (6) by using the finite difference method [36]. Figures 4 and 5 exhibit exact one-soliton and two-soliton solutions (42) and (41) of Equation (6), with numerical solutions without any noises and the perturbed numerical solutions with small noises 2% and 10%. Figures 4a1,a2,b1,b2 and 5a1,a2,b1,b2 show that time evolutions of one-soliton solutions (42) and two-soliton solutions (41) without any noise are almost consistent with one of the corresponding exact one-soliton and two-soliton solutions. We can clearly observe that the numerical solutions almost recover the analytical exact solutions. In other words, these solutions have stable evolutions without any noise, which also indicates the accuracy of our numerical scheme. The numerical results in Figures 4c1,c2,d1,d2 and 5c1,c2,d1,d2 are the numerical solutions obtained by adding 2% and 10% noises to both the initial exact solutions $\tilde{q}_n$ and $\tilde{r}_n$, respectively. For 2% noise, they almost evolve as before without any noise, while for 10% noise, the evolutions show obvious fluctuation. That is to say that the exact solutions (42) and (41) have almost steady evolutions against a small noise in a relatively short time.
Figure 3. (Color online) Two-soliton interaction structures via solutions (41) with parameters $\lambda_1 = \frac{1}{4}, \lambda_2 = 3, \alpha = 1$, $C_{11} = C_{12} = C_{21} = 1$ and $C_{22} = \frac{1}{2}$. (a1–d1) The profiles of two-soliton solutions $\tilde{q}_n, \tilde{r}_n, \tilde{p}_n$ and $\tilde{x}_n$. (a2–d2) The propagation processes for $\tilde{q}_n, \tilde{r}_n, \tilde{p}_n$ and $\tilde{x}_n$ at $t = -10$ (dashdot line), $t = 0$ (longdash line) and $t = 10$ (solid line).

Figure 4. (Color online) One-soliton solutions (42) with the same parameters as Figure 2. (a1,a2) Exact solutions. (b1,b2) Numerical solutions without any noise. (c1,c2) Numerical solutions with a 2% noise. (d1,d2) Numerical solutions with a 10% noise.

Figure 5. (Color online) Two-soliton solutions (42) with the same parameters as Figure 3. (a1,a2) Exact solutions. (b1,b2) Numerical solutions without any noise. (c1,c2) Numerical solutions with a 2% noise. (d1,d2) Numerical solutions with a 10% noise.
Case (2) When \( N = 2, \lambda = \lambda_i \) \((i = 1, 2, 3, 4)\), from (36), the 4-fold DT will provide the exact solutions as follows:
\[
\tilde{q}_n = \frac{(1 + a^2) a_n^{(0)}}{a_n^{(0)}}, \quad \tilde{r}_n = \frac{\alpha + b_n^{(3)}}{\alpha d_n^{(4)}} \tag{43}
\]
where \( a_n^{(0)} = \frac{\Delta a_n^{(0)}}{\Delta_1}, b_n^{(3)} = \frac{\Delta b_n^{(3)}}{\Delta_2} \) and \( d_n^{(4)} = \frac{\Delta d_n^{(4)}}{\Delta_2} \), in which the following is the case.

\[
\begin{align*}
\Delta_1 &= \left| \begin{array}{cccc}
\lambda_4 & \lambda_2 & \lambda_2 & \lambda_1 \\
\lambda_4 & \lambda_2 & \lambda_2 & \lambda_1 \\
\lambda_4 & \lambda_2 & \lambda_2 & \lambda_1 \\
\lambda_4 & \lambda_2 & \lambda_2 & \lambda_1 \\
\end{array} \right|, \\
\Delta_2 &= \left| \begin{array}{cccc}
\lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 \\
\lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 \\
\lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 \\
\lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 \\
\end{array} \right|, \\
\Delta a_n^{(0)} &= \left| \begin{array}{cccc}
-1 & \lambda_2 & \lambda_2 & \lambda_1 \\
-1 & \lambda_2 & \lambda_2 & \lambda_1 \\
-1 & \lambda_2 & \lambda_2 & \lambda_1 \\
-1 & \lambda_2 & \lambda_2 & \lambda_1 \\
\end{array} \right|, \\
\Delta d_n^{(4)} &= \left| \begin{array}{cccc}
-1 & \lambda_2 & \lambda_2 & \lambda_1 \\
-1 & \lambda_2 & \lambda_2 & \lambda_1 \\
-1 & \lambda_2 & \lambda_2 & \lambda_1 \\
-1 & \lambda_2 & \lambda_2 & \lambda_1 \\
\end{array} \right|
\end{align*}
\]

when the appropriate parameters are selected, the solutions (43) may be the three-soliton or four-soliton solutions for which its corresponding evolution plots are shown in Figure 6. If one of \( \lambda_i \) \((i = 1, 2, 3, 4)\) is equal to \( 1 + \frac{a}{\sqrt{\pi + 1}} \), the solutions reduce to three-soliton ones. We here choose \( a = \frac{3}{4} \), that is to say that one component of \( \lambda \) is \( \frac{\pi}{4} \). Without loss of generality, we permit \( \lambda_1 = \frac{\pi}{4}, \lambda_2 = \frac{\pi}{4}, \lambda_3 = 3, \lambda_4 = \frac{\pi}{4} \), \( C_{11} = C_{12} = C_{21} = -C_{22} = C_{31} = C_{32} = C_{41} = 1 \) and \( C_{42} = \frac{1}{2} \), the corresponding evolution plots are displayed in Figure 6a1,b1,a2,b2. Figure 6a1,a2 demonstrate the elastic interactions between one bell-shaped soliton and two dark solitons of the component \( \tilde{q}_n \) on nonzero constant background. Figure 6b1,b2 display the collisions among three anti-kink solitons of the component \( \tilde{r}_n \). When none of the parameters \( \lambda_i \) \((i = 1, 2, 3, 4)\) are equal to \( 1 + \frac{a}{\sqrt{\pi + 1}} \), the solutions (43) are four-soliton solutions. By choosing the parameters \( \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = \frac{\pi}{4}, \lambda_4 = \frac{\pi}{4}, C_{11} = C_{12} = C_{21} = C_{22} = C_{31} = C_{32} = C_{41} = 1 \) and \( C_{42} = \frac{1}{2} \), we plot the corresponding evolution in Figure 6c1,d1,c2,d2. Figure 6c1,c2 demonstrate the elastic interactions between one bell-shaped soliton and three dark solitons of the component \( \tilde{q}_n \) on nonzero constant background. Figure 6d1,d2 display the collision interactions among four anti-kink solitons of the component \( \tilde{r}_n \). It should be noted here that for the component \( \tilde{r}_n \), no matter how we choose the parameters, there will always be one anti-kink soliton with high amplitude, which makes the other three solitons look shorter. In fact, they represent four anti-kink solitons if we look through local magnification. Similarly, similarly to one-soliton or two-soliton solutions, we can draw the components \( \tilde{p}_n \) and \( \tilde{x}_n \) of Equation (1), and here we omit their figures.
4.2. Rational and Semi-Rational Solutions and Their Mathematical Characteristics

When \( m = 1 \), the discrete generalized \((m, 2N - m)\)-fold DT reduces to the discrete generalized \((1, 2N - 1)\)-fold DT, which can investigate some rational or semi-rational solutions of Equation (6). In order to obtain a more general solution, we rewrite the solution (39) as follows:

\[
\phi_{1,n} = \left( \phi_{1,n}^r, \phi_{1,n}^s \right) = \left( \frac{C_1 \tau_1^{n+1} e^{i \lambda_1 t + i \zeta(e)} + C_2 \tau_2^{n+1} e^{i \lambda_2 t - i \zeta(e)}}{-a \lambda_1 (a^2 + 1) C_1 \tau_1^{n+1} e^{i \lambda_1 t + i \zeta(e)} - a \lambda_2 (a^2 + 1) C_2 \tau_2^{n+1} e^{i \lambda_2 t - i \zeta(e)}} \right), \tag{44}
\]

where \( \zeta(e) = \sqrt{\alpha^4 \lambda_1^4 - 4a^4 \lambda_1^2 + 2a^2 \lambda_1^4 - 6a^2 \lambda_1^2 + \lambda_1^4} \) are real parameters. \( e \) is an artificially introduced small parameter, while \( \tau_1, \tau_2, \rho_1, \rho_2 \) are the same as ones defined in the previous subsection. We expand the vector function \( \phi_{1,n} \) with \( \lambda = \lambda_1 + \epsilon \) in (44) as two Taylor series around \( \epsilon = 0 \) given by the following:

\[
\phi_{1,n}(\epsilon) = \phi_{1,n}^{(0)} + \phi_{1,n}^{(1)} \epsilon + \phi_{1,n}^{(2)} \epsilon^2 + \phi_{1,n}^{(3)} \epsilon^3 + \phi_{1,n}^{(4)} \epsilon^4 + \phi_{1,n}^{(5)} \epsilon^5 + \cdots. \tag{45}
\]

In fact, by choosing different parameters \( C_{11} \) and \( C_{12} \), we can obtain different rational or semi-rational solutions, and we here enumerate two types of Taylor series expansions of the vector function \( \phi_{1,n} \) by choosing two sets of different parameters \( C_{11} \) and \( C_{12} \):

- **The first kind of expansion.** By setting \( \alpha = \frac{1}{4} \), \( C_{11} = C_{12} = 1 \), one can obtain \( \phi_{1,n}^{(i)} \) as follows.

\[
\phi_{1,n}^{(0)} = \left( \begin{array}{c} \phi_{1,n}^{(0)} \\ \phi_{1,n}^{(0)} \end{array} \right) = \left( \begin{array}{c} 2(4)^n e^{\frac{27}{4} t} \\ -10 \frac{1}{3} (4)^n e^{\frac{27}{4} t} \end{array} \right), \quad \phi_{1,n}^{(1)} = \left( \begin{array}{c} \phi_{1,n}^{(1)} \\ \phi_{1,n}^{(1)} \end{array} \right), \quad \phi_{1,n}^{(2)} = \left( \begin{array}{c} \phi_{1,n}^{(2)} \\ \phi_{1,n}^{(2)} \end{array} \right). \tag{46}
\]

Since the expressions of \( \phi_{1,n}^{(1)} \) and \( \phi_{1,n}^{(2)} \) are relatively long, we list them in the Appendix A. The rest \( \left( \phi_{1,n}^{(j)} \phi_{1,n}^{(j)} \right)^T \) are omitted here.

- **The second kind of expansion.** By setting \( \alpha = \frac{1}{4} \), \( C_{11} = -C_{12} = \frac{1}{2} \), one can obtain the new expansions, and we here only list the following first two expansions:
\[
\phi_1^{(0)} = \left( \begin{array}{c} \phi_{1,n}^{(0)} \\ \psi_{1,n}^{(0)} \end{array} \right) = \left( \begin{array}{c} \frac{\sqrt{2} 4^n e^{\frac{2\pi}{3}} t (25n - 27t + 200e_0)}{\sqrt[3]{2} 4^n e^{\frac{2\pi}{3}} t (25n - 27t + 200e_0 - 25)} \\ \frac{\sqrt{2} 4^n e^{\frac{2\pi}{3}} t (25n - 27t + 200e_0)}{\sqrt[3]{2} 4^n e^{\frac{2\pi}{3}} t (25n - 27t + 200e_0 - 25)} \end{array} \right), \quad \phi_1^{(1)} = \left( \begin{array}{c} \phi_{1,n}^{(1)} \\ \psi_{1,n}^{(1)} \end{array} \right),
\]

where the following is the case.

\[
\phi_1^{(1)} = \frac{\sqrt{2}}{10\sqrt{3} \pi} 4^n e^{\frac{2\pi}{3} t} (62,500n^3 - 202,500n^2t + 1,500,000n^2e_0 + 218,700nt^2 - 3,240,000nte_0 + 12,000,000n^2e_0^2 - 78,732t^3 + 1,749,600t^2e_0 - 12,960,000t^2e_0^2 + 32,000,000n^2e_0^3 + 150,000n^2 + 202,500nt + 1,200,000nte_0 - 393,660t^2 + 2,916,000te_0 - 23,125n - 204,525t + 1,515,000e_0 + 2,160,000e_0^2),
\]

\[
\psi_1^{(1)} = \frac{\sqrt{2}}{9\sqrt{3} \pi} 4^n e^{\frac{2\pi}{3} t} (62,500n^3 - 202,500n^2t + 1,500,000n^2e_0 + 218,700nt^2 - 3,240,000nte_0 + 12,000,000n^2e_0^2 - 78,732t^3 + 1,749,600t^2e_0 - 12,960,000t^2e_0^2 + 32,000,000n^2e_0^3 - 37,500n^2 + 607,500nt - 1,800,000nte_0 - 612,360t^2 + 6,156,000nte_0 - 12,000,000n^2e_0^3 + 14,375n - 771,525t + 3,015,000e_0 + 2,160,000e_0^2 - 39,375).
\]

In what follows, for rational or semi-rational solutions, we also discuss two cases: \(N = 1, 2\).

**Case (1)** When \(N = 1\), using the first kind expansion, the discrete generalized (1, 1)-fold DT yields the first-order rational solutions of Equation (6) as the follows:

\[
\bar{q}_n = \frac{(1 + a^2) \alpha_{n+1}}{\beta_{n+1}}, \quad \bar{r}_n = \frac{\alpha + b_n^{(1)}}{\alpha \beta_{n+1}},
\]

where \(a_n^{(0)} = \frac{\Delta_1}{\lambda_{1,n}}, b_n^{(1)} = \frac{\Delta_2}{\lambda_{1,n}}\) and \(d_n^{(2)} = \frac{\Delta_1}{\lambda_{1,n}}\) in which the following is the case.

\[
\Delta_1 = \left| \begin{array}{cc} \lambda_1^{(0)} & \lambda_1^{(0)} \\ \lambda_1^{(1)} + 2\lambda_1^{(0)} & \lambda_1^{(1)} + \lambda_1^{(0)} \end{array} \right|, \quad \Delta_2 = \left| \begin{array}{cc} \lambda_1^{(0)} & \lambda_1^{(0)} \\ \lambda_1^{(1)} + \lambda_1^{(0)} & \lambda_1^{(1)} \end{array} \right|, \quad \Delta_{1n}^{(0)} = \left| \begin{array}{cc} \lambda_1^{(0)} & \lambda_1^{(0)} \\ \lambda_1^{(1)} & \lambda_1^{(1)} \end{array} \right|, \quad \Delta_{1n}^{(1)} = \left| \begin{array}{cc} \lambda_1^{(0)} & \lambda_1^{(0)} \\ \lambda_1^{(1)} & \lambda_1^{(1)} \end{array} \right|.
\]

Direct calculation results in the simplification expressions of solutions (48) below.

\[
\bar{q}_n = \frac{25}{3} + \frac{50,000}{3^n(50n-54t+400e_0-45)(50n-54t+400e_0-45)},
\]

\[
\bar{r}_n = 1 + \frac{5300n-5724t+42400e_0-3520}{4050n-4374t+32400e_0-3645}.
\]

We can see that \(\bar{q}_n\) possesses singularities at two parallel straight lines \(50n - 54t + 400e_0 + 45 = 0\) and \(50n - 54t - 400e_0 - 45 = 0\), while \(\bar{r}_n\) has singularity at one straight line \(4050n - 4374t + 32400e_0 - 3645 = 0\) from which we can move these lines with singularities parallel to any position by changing the parameter \(e_0\). Moreover, we can conclude that \(\bar{q}_n \to \frac{25}{3}, \bar{r}_n \to 1\) as \(n \to \pm \infty, t \to \pm \infty\). What we need to explain here is that, in the first-order rational solutions, we have an arbitrary parameter \(e_0\) that can control the position of the rational solutions; that is to say that we can move the first-order rational solutions to any position we need. Through the transformations \(p_n = \frac{\ln |x_n|}{\pi}, \chi_n = \ln |r_n|\), we can provide the solutions of Equation (1) as follows.

\[
\bar{p}_n = \frac{1}{3} \ln \left| \frac{25}{3} + \frac{50,000}{3^n(50n-54t+400e_0-45)(50n-54t+400e_0-45)} \right|,
\]

\[
\bar{x}_n = \ln \left| 1 + \frac{5300n-5724t+42400e_0-3520}{4050n-4374t+32400e_0-3645} \right|.
\]

It should be noted that if we permit \(\lambda_1 \neq \frac{9}{5}\) (e.g., \(\lambda_1 = 3\)) and expand \(\phi_{1,n}\) in (39) around \(\lambda_1 = 3\), then by using the above similar process we can obtain the hybrid solutions of polynomial functions and exponential functions. Compared with the above rational
solutions, we here call this kind of hybrid solutions the semi-rational solutions. From (48), the generalized \((1,1)\)-fold DT yields the first-order semi-rational solutions as follows:

\[
\tilde{q}_n = \frac{Q_1}{Q_2}, \quad \tilde{r}_n = \frac{R_1}{R_2},
\]

where the following is the case:

\[
Q_1 = 350|140,625\epsilon^{5\frac{1}{2}} + 48\sqrt{14}(584,200e_0 + 682,752c_1 + 14,859\eta - 78,867t + 27\sqrt{14}e^{\frac{1}{2}} - 48\sqrt{14}(-584,200c_0 - 268,752c_1 - 14,859\eta + 78,867t + 27\sqrt{14}e^{\frac{1}{2}} + 2(6,132,672n^2 - 65,100,672nt + 482,227,200n_0 + 563,576,832n_1 + 172,767,168t^2 - 2,5,59,513,600t_0 - 2,991,292,416t_1 + 9,479,680,000c_0^2 + 22,157,721,600c_0c_1 + 12,947,816,448e^{\frac{1}{2}} - 158,769n_1 + 9e^{-\frac{1}{2}}|^{\frac{1}{2}} - 75(-84 + 23\sqrt{14})e^{\frac{1}{2}} + 75(-84 + 23\sqrt{14})e^{\frac{1}{2}} + 56(234n - 1242t - 9200c_0 + 10,752c_1 - 225)| - 75(-84 + 23\sqrt{14})e^{\frac{1}{2}} + 75(-84 + 23\sqrt{14})e^{\frac{1}{2}} + 56(234n - 1242t - 9200c_0 + 10,752c_1 - 225)),
\]

\[
R_1 = -56(9200e_0 + 10,752c_2 + 234n - 1242t + 9) + 3(127\sqrt{14} - 84)e^{\frac{1}{2}} - 3(127\sqrt{14} + 84)e^{-\frac{1}{2}},
\]

\[
R_2 = 504(234n - 1242t - 9200c_0 + 10,752c_1 - 225) - 675(84 + 23\sqrt{14})e^{\frac{1}{2}} + 675(-84 + 23\sqrt{14})e^{-\frac{1}{2}}.
\]

With the aid of symbolic computation Maple, one can verify the solutions (48) and (49) by substituting them into Equation (6).

**Case (2)** When \(N = 2\), using the first kind expansion, the generalized \((1,3)\)-fold DT yields the second-order rational solutions of Equation (6) as follows:

\[
\tilde{q}_n = \frac{(1 + a^2)a_1^{(0)}}{a_1^{(0)}}, \quad \tilde{r}_n = \frac{a + b^{(3)}}{a_1^{(0)}}, \quad \frac{R_1}{R_2} = \frac{a_1^{(0)}}{a_1^{(0)}},
\]

where \(a_1^{(0)} = \frac{\Delta_1^{(0)}}{\Delta_2^{(0)}}\), \(b^{(3)} = \frac{\Delta_1^{(3)}}{\Delta_2^{(0)}}\), and \(d_1^{(4)} = \frac{\Delta_1^{(4)}}{\Delta_2^{(0)}}\), in which the following is the case.

\[
\Delta_1^{(0)} = \begin{pmatrix}
\lambda_1^{(0)} & \lambda_2^{(0)} & \lambda_3^{(0)} & \lambda_4^{(0)} & \lambda_5^{(0)} \\
\lambda_1^{(1)} & \lambda_2^{(1)} & \lambda_3^{(1)} & \lambda_4^{(1)} & \lambda_5^{(1)} \\
\lambda_1^{(2)} & \lambda_2^{(2)} & \lambda_3^{(2)} & \lambda_4^{(2)} & \lambda_5^{(2)} \\
\lambda_1^{(3)} & \lambda_2^{(3)} & \lambda_3^{(3)} & \lambda_4^{(3)} & \lambda_5^{(3)} \\
\lambda_1^{(4)} & \lambda_2^{(4)} & \lambda_3^{(4)} & \lambda_4^{(4)} & \lambda_5^{(4)}
\end{pmatrix}
\]

\[
\Delta_2^{(0)} = \begin{pmatrix}
\lambda_1^{(0)} & \lambda_2^{(0)} & \lambda_3^{(0)} & \lambda_4^{(0)} & \lambda_5^{(0)} \\
\lambda_1^{(1)} & \lambda_2^{(1)} & \lambda_3^{(1)} & \lambda_4^{(1)} & \lambda_5^{(1)} \\
\lambda_1^{(2)} & \lambda_2^{(2)} & \lambda_3^{(2)} & \lambda_4^{(2)} & \lambda_5^{(2)} \\
\lambda_1^{(3)} & \lambda_2^{(3)} & \lambda_3^{(3)} & \lambda_4^{(3)} & \lambda_5^{(3)} \\
\lambda_1^{(4)} & \lambda_2^{(4)} & \lambda_3^{(4)} & \lambda_4^{(4)} & \lambda_5^{(4)}
\end{pmatrix}
\]

\[
\Delta_1^{(3)} = \begin{pmatrix}
\lambda_1^{(0)} & \lambda_2^{(0)} & \lambda_3^{(0)} & \lambda_4^{(0)} & \lambda_5^{(0)} \\
\lambda_1^{(1)} & \lambda_2^{(1)} & \lambda_3^{(1)} & \lambda_4^{(1)} & \lambda_5^{(1)} \\
\lambda_1^{(2)} & \lambda_2^{(2)} & \lambda_3^{(2)} & \lambda_4^{(2)} & \lambda_5^{(2)} \\
\lambda_1^{(3)} & \lambda_2^{(3)} & \lambda_3^{(3)} & \lambda_4^{(3)} & \lambda_5^{(3)} \\
\lambda_1^{(4)} & \lambda_2^{(4)} & \lambda_3^{(4)} & \lambda_4^{(4)} & \lambda_5^{(4)}
\end{pmatrix}
\]

\[
\Delta_2^{(3)} = \begin{pmatrix}
\lambda_1^{(0)} & \lambda_2^{(0)} & \lambda_3^{(0)} & \lambda_4^{(0)} & \lambda_5^{(0)} \\
\lambda_1^{(1)} & \lambda_2^{(1)} & \lambda_3^{(1)} & \lambda_4^{(1)} & \lambda_5^{(1)} \\
\lambda_1^{(2)} & \lambda_2^{(2)} & \lambda_3^{(2)} & \lambda_4^{(2)} & \lambda_5^{(2)} \\
\lambda_1^{(3)} & \lambda_2^{(3)} & \lambda_3^{(3)} & \lambda_4^{(3)} & \lambda_5^{(3)} \\
\lambda_1^{(4)} & \lambda_2^{(4)} & \lambda_3^{(4)} & \lambda_4^{(4)} & \lambda_5^{(4)}
\end{pmatrix}
\]

\[
\Delta_1^{(4)} = \begin{pmatrix}
\lambda_1^{(0)} & \lambda_2^{(0)} & \lambda_3^{(0)} & \lambda_4^{(0)} & \lambda_5^{(0)} \\
\lambda_1^{(1)} & \lambda_2^{(1)} & \lambda_3^{(1)} & \lambda_4^{(1)} & \lambda_5^{(1)} \\
\lambda_1^{(2)} & \lambda_2^{(2)} & \lambda_3^{(2)} & \lambda_4^{(2)} & \lambda_5^{(2)} \\
\lambda_1^{(3)} & \lambda_2^{(3)} & \lambda_3^{(3)} & \lambda_4^{(3)} & \lambda_5^{(3)} \\
\lambda_1^{(4)} & \lambda_2^{(4)} & \lambda_3^{(4)} & \lambda_4^{(4)} & \lambda_5^{(4)}
\end{pmatrix}
\]

\[
\Delta_2^{(4)} = \begin{pmatrix}
\lambda_1^{(0)} & \lambda_2^{(0)} & \lambda_3^{(0)} & \lambda_4^{(0)} & \lambda_5^{(0)} \\
\lambda_1^{(1)} & \lambda_2^{(1)} & \lambda_3^{(1)} & \lambda_4^{(1)} & \lambda_5^{(1)} \\
\lambda_1^{(2)} & \lambda_2^{(2)} & \lambda_3^{(2)} & \lambda_4^{(2)} & \lambda_5^{(2)} \\
\lambda_1^{(3)} & \lambda_2^{(3)} & \lambda_3^{(3)} & \lambda_4^{(3)} & \lambda_5^{(3)} \\
\lambda_1^{(4)} & \lambda_2^{(4)} & \lambda_3^{(4)} & \lambda_4^{(4)} & \lambda_5^{(4)}
\end{pmatrix}
\]
Since the simplified expressions are very complex, we list only their analytical expressions with \( c_0 = c_1 = c_2 = 0 \) of solutions \( \hat{q}_n = \frac{Q_1}{Q_2} \), \( \hat{r}_n = \frac{R_1}{R_2} \) with the following:

\[
Q_1 = \frac{25}{9} (-16\xi^6 + 240\xi^5 - 2,125,000n^2\xi + 120,000\xi^3 + 15,937,500n^2\xi^2 - 907,500\xi^3 + 56,445,312,500n^2\xi^2 + 907,500\xi^3 + 56,445,312,500n^2\xi^2 + 2,711,718,750n^2\xi^2 + 2,728,125\xi^2 - 9,357,421,875n + 455,625,000\xi) (-16\xi^6 - 240\xi^5 - 2,125,000n^2\xi + 120,000\xi^3 - 15,937,500n^2\xi^2 + 907,500\xi^3 + 56,445,312,500n^2\xi^2 - 2,711,718,750n^2\xi^2 + 2,728,125\xi^2 + 9,357,421,875n + 455,625,000\xi),
\]

\[
Q_2 = (-16\xi^6 - 2160\xi^5 - 2,125,000n^2\xi^3 + 5,467,500\xi^2 + 56,445,312,500n^2\xi^2 - 5,899,218,750n^2\xi + 184,528,125\xi^2 + 12,498,046,875n) (-16\xi^6 + 2160\xi^5 - 2,125,000n^2\xi^3 + 5,467,500\xi^2 + 56,445,312,500n^2\xi^2 - 5,899,218,750n^2\xi + 184,528,125\xi^2 - 12,498,046,875n),
\]

\[
R_1 = -10,000\xi^6 - 150,000\xi^5 + 75,000,000n^2\xi + (-1,328,125,000n + 567,187,500)\xi^3 + (-9,960,937,500n + 1,423,828,125)\xi^2 + (-1,694,824,218,750n - 284,765,625,000)\xi + 35,278,320,312,500n^2 + 5,848,386,671,875n,
\]

\[
R_2 = -104,976\xi^6 + 14,171,760\xi^5 + (-13,942,125,000n - 35,872,267,500)\xi^3 + (941,093,437,500n + 1,210,689,028,125)\xi^2 - 38,704,774,218,750n^2 + 370,337,695,312,500n - 81,999,685,546,875n,
\]

in which \( \xi = 25n - 271 \). Next, we implement asymptotic analysis to study the second-order rational solutions \( \hat{q}_n \) and \( \hat{r}_n \). Using \( \xi_1 = \xi + (\frac{459}{100} + \frac{1377}{300} \sqrt{5}) \frac{1}{2} t^\frac{1}{2} \), \( \xi_2 = \xi + (\frac{459}{100} - \frac{1377}{300} \sqrt{5}) \frac{1}{2} t^\frac{1}{2} \) and \( c = (\frac{459}{100} + \frac{1377}{300} \sqrt{5}) \frac{1}{3} - (\frac{459}{100} - \frac{1377}{300} \sqrt{5}) \frac{1}{3} > 0 \), we obtain the solutions \( \hat{q}_n \) and \( \hat{r}_n \) with two different asymptotic states when \( |t| \to \infty \), which are listed as follows:

(i) If \( \xi_1 = \xi + (\frac{459}{100} + \frac{1377}{300} \sqrt{5}) \frac{1}{2} t^\frac{1}{2} \) is fixed, from \( \xi_2 = \xi_1 - ct^\frac{1}{2} \) we have \( \hat{q}_1 \to \pm \infty \) when \( t \to \pm \infty \). Calculating the limits of solutions \( \hat{q}_n \) and \( \hat{r}_n \) in (50) then yields the following:

\[
\hat{q}_n \to q_1^\pm = \frac{25}{9} + \frac{50,000}{9(442^2 - 2025)}, \quad \hat{r}_n \to r_1^\pm = 1 - \frac{11,872\xi_1^2 + 298,370}{13,122\xi_1^2 - 295,245}. \quad (51)
\]

(ii) If \( \xi_2 = \xi + (\frac{459}{100} - \frac{1377}{300} \sqrt{5}) \frac{1}{2} t^\frac{1}{2} \) is fixed, from \( \xi_1 = \xi_2 + ct^\frac{1}{2} \) we have \( \hat{q}_1 \to \pm \infty \) when \( t \to \pm \infty \). Then calculating the limits of solutions \( \hat{q}_n \) and \( \hat{r}_n \) in (50) has the following asymptotic expressions in the following form:

\[
\hat{q}_n \to q_2^\pm = \frac{25}{9} + \frac{50,000}{9(442^2 - 2025)}, \quad \hat{r}_n \to r_2^\pm = 1 - \frac{11,872\xi_2^2 + 298,370}{13,122\xi_2^2 - 295,245}. \quad (52)
\]

It can be observed that \( q_1^\pm \) and \( q_2^\pm \) possess singularities at four curves \( 2\xi_1 \pm 45 = 0 \), \( 2\xi_2 \pm 45 = 0 \), which also are the four center trajectories of solution \( \hat{q}_n \), while \( r_1^\pm \) and \( r_2^\pm \) possess singularities at two curves \( 13,122\xi_1 - 295,245 = 0, 13,122\xi_2 - 295,245 = 0 \), which are also the two center trajectories of solution \( \hat{r}_n \). In order to show the correctness of our analysis results, we draw the density plots of the rational solutions and the trajectory plots obtained after asymptotic analysis, respectively, which are shown in Figure 7. Through comparison, we find that the singularity of rational solutions is completely consistent with these trajectories, which also shows the correctness of our asymptotic analysis results of second-order rational solutions. In addition, from asymptotic expressions (51) and (52), we can clearly observe that the asymptotic expressions of second-order rational solutions are consistent with the first-order rational solutions. The main difference is that first-order rational solutions’ trajectories are straight lines, while higher-order rational solutions’ trajectories are curves.
Next, we will no longer calculate higher-order rational solutions, but we will summarize some mathematical characteristics of the higher-order rational solutions for Equation (6) listed in Tables 1 and 2. In the two tables, the first column represents the order number of the rational solutions, the second and fourth columns represent the highest power of numerator (HPN) polynomials involved in solutions \( q_n, r_n \), the third and fifth columns represent the highest power of denominator (HPD) polynomials involved in solutions \( q_n, r_n \) and the sixth and seventh columns mean the background levels of solutions \( q_n, r_n \).

In Table 1, these rational solutions are given by using the first kind of expansion that was represented in the previous calculation of rational solutions, while these rational solutions in Table 2 are obtained by using the second kind of expansion, which is omitted from the calculation. From Table 1, we can easily observe that for the rational solution \( q_n \) of order \( j \), HPN and HPD are both \( 2j(2j - 1) \), whereas for the rational solution \( r_n \) of order \( j \), HPN and HPD are \( j(2j - 1) \). From Table 2, we know that HPN and HPD of the rational solutions \( q_n, r_n \) of order \( j \) are \( 2j(2j + 1) \) and \( j(2j + 1) \), respectively.

### Table 1. Main mathematical characteristics of the \( j \)-th rational order solutions \( q_n, r_n \).

| \( j \) | HPN of \( q_n \) | HPD of \( q_n \) | HPN of \( r_n \) | HPD of \( r_n \) | Background of \( q_n \) | Background of \( r_n \) |
|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 1 | 1 | \( 1 + a^2 \) | 1 |
| 2 | 12 | 12 | 6 | 6 | \( 1 + a^2 \) | 1 |
| 3 | 30 | 30 | 15 | 15 | \( 1 + a^2 \) | 1 |
| ... | ... | ... | ... | ... | ... | ... |
| \( j \) | \( 2j(2j - 1) \) | \( 2j(2j - 1) \) | \( j(2j - 1) \) | \( j(2j - 1) \) | \( 1 + a^2 \) | 1 |

### Table 2. Main mathematical characteristics of the \( j \)-th rational order solutions \( q_n, r_n \).

| \( j \) | HPN of \( q_n \) | HPD of \( q_n \) | HPN of \( r_n \) | HPD of \( r_n \) | Background of \( q_n \) | Background of \( r_n \) |
|---|---|---|---|---|---|---|
| 1 | 6 | 6 | 3 | 3 | \( 1 + a^2 \) | 1 |
| 2 | 20 | 20 | 10 | 10 | \( 1 + a^2 \) | 1 |
| 3 | 42 | 42 | 21 | 21 | \( 1 + a^2 \) | 1 |
| ... | ... | ... | ... | ... | ... | ... |
| \( j \) | \( 2j(2j + 1) \) | \( 2j(2j + 1) \) | \( j(2j + 1) \) | \( j(2j + 1) \) | \( 1 + a^2 \) | 1 |

**Remark 3.** It should be noted here that, unlike our previous research in [17,32], we have obtained rational and semi-rational solutions with arbitrary controllable parameters. By using these parameters, we can control the positions where the rational and semi-rational solutions appear, which is not available in our previous researches in Refs. [17,32]. The introduction of these controllable parameters is an important innovation of this paper.
4.3. Hybrid Solutions of Exponential Function and Rational Solutions

When \( m = 2 \), the discrete generalized \((m,2N-m)\)-fold DT reduces to the discrete generalized \((2,2N-2)\)-fold DT, which can provide hybrid solutions of exponential function and rational solutions of Equation (6). We will also discuss two cases: \( N = 1, 2 \).

**Case (1)** When \( N = 1 \), for \( \lambda_1 = \frac{q}{2} \) (i.e., \( \alpha = \frac{q}{2} \)), by using the second kind expansion for \( \lambda_2 \neq \frac{q}{2} \) (e.g., \( \lambda_2 = 3 \)) without using Taylor expansion, the discrete generalized \((2,0)\)-fold DT will yield hybrid collision solutions of exponential function and first-order rational solutions of Equation (6) as follows:

\[
\tilde{q}_n = \frac{(1 + a^2)q_n^{(0)}}{Q_2}, \quad \tilde{r}_n = \frac{a + b_n^{(1)}}{\Delta d_n^{(2)}} = \frac{R_1}{R_2},
\]

where \( a_n^{(0)} = \frac{\Delta a_n^{(0)}}{\Delta^1}, b_n^{(1)} = \frac{\Delta b_n^{(1)}}{\Delta^1} \) and \( d_n^{(2)} = \frac{\Delta d_n^{(2)}}{\Delta^2} \), in which the following is the case.

\[
\begin{align*}
\Delta_1 &= \left| \begin{array}{cc} \lambda_1 \psi_1^{(0)} & \lambda_1 \psi_1^{(0)} \\
\lambda_2 \psi_2^{(0)} & \lambda_2 \psi_2^{(0)} \end{array} \right|, \\
\Delta_2 &= \left| \begin{array}{cc} \lambda_1 \psi_1^{(0)} & \psi_1^{(0)} \\
\lambda_2 \psi_2^{(0)} & \psi_2^{(0)} \end{array} \right|, \\
\Delta d_n^{(2)} &= \left| \begin{array}{cc} \lambda_1 \psi_1^{(0)} & -\lambda_2 \psi_1^{(0)} \\
\lambda_2 \psi_2^{(0)} & -\lambda_2 \psi_2^{(0)} \end{array} \right|.
\end{align*}
\]

Direct calculation provides the simplified analytic expressions of the solutions (53) as follows.

\[
Q_1 = \left[ \left( \frac{400}{9} \right)^n e^{\frac{36\sqrt{14}}{25}t} \right] \times \left( 1 - 2,700,000n^2 + 2,743,200nt - 20,320,000n^2e_0 - 1,481,328t^2 + 21,945,600t^2e_0 - 81,280,000e_0^2 
+ 1,562,500 \right) - 125(\frac{2102}{9} - 64\sqrt{14})n^2 e^{\frac{36\sqrt{14}}{25}t} \left( 400e_0 + 5 + 15\sqrt{14} + 50t - 54t^2 \right) \left( 400e_0 + 5 + 15\sqrt{14} + 50t - 54t^2 \right) 
- 500(\frac{2102}{9} + 64\sqrt{14})n \left( -50n + 5 - 15\sqrt{14} + 54t - 400e_0 \right) \left( -50n + 5 + 15\sqrt{14} + 54t - 400e_0 \right),
\]

\[
Q_2 = \left[ \left( \frac{400}{9} \right)^n e^{\frac{36\sqrt{14}}{25}t} \right] \times \left( 9(\sqrt{14} + 3)(50n - 54t + 400e_0 + 45 + 15\sqrt{14})(12 - \frac{8}{3}\sqrt{14})n^2 e^{\frac{36\sqrt{14}}{25}t} + 2(3 + \sqrt{14})(-50n + 54t - 400e_0 + 15\sqrt{14} + 45)(12 - \frac{8}{3}\sqrt{14})n^2 e^{\frac{36\sqrt{14}}{25}t} + 2(\sqrt{14} + 3)(-50n + 54t - 400e_0 - 45 + 15\sqrt{14})(12 - \frac{8}{3}\sqrt{14})n, \right.
\]

\[
R_1 = \left[ \left( \frac{400}{9} \right)^n e^{\frac{36\sqrt{14}}{25}t} \right] \times \left( (1 - 3\sqrt{14})(15\sqrt{14} + 400e_0 + 50n - 54t + 5)(12 - \frac{8}{3}\sqrt{14})n^2 e^{\frac{36\sqrt{14}}{25}t} - 2(1 + 3\sqrt{14})(-400e_0 - 50n + 54t + 15\sqrt{14} - 5)(12 - \frac{8}{3}\sqrt{14})n, \right.
\]

\[
R_2 = \left[ \left( \frac{400}{9} \right)^n e^{\frac{36\sqrt{14}}{25}t} \right] \times \left( 27(\sqrt{14} + 3)(50n - 54t + 400e_0 + 15\sqrt{14} - 45)(12 - \frac{8}{3}\sqrt{14})n^2 e^{\frac{36\sqrt{14}}{25}t} - 54(\sqrt{14} - 3)(-400e_0 - 50n + 54t + 15\sqrt{14} + 45)(12 - \frac{8}{3}\sqrt{14})n. \right.
\]

For convenience of analysis, we chose \( e_0 = 0 \) and let \( \zeta_1 = 25n - 27t, \quad \zeta_2 = n \ln \frac{\frac{q}{2} - 2\sqrt{14}}{\frac{q}{2} + 2\sqrt{14}} + \frac{36\sqrt{14}}{25}t; \) the asymptotic states for solutions (53) when \( t \to \pm \infty \) are given as follows:

**Before collision \( t \to -\infty \), we have the following:**

(i) If \( \zeta_1 \) is fixed, \( \zeta_2 \to -\infty \), we have the following.

\[
\tilde{q}_n \to q_n^{(0)} = \frac{25}{9} + \frac{12(452 - 60(\sqrt{14} + 2\sqrt{2})\zeta_1 + 25(617 + 72\sqrt{7}))}{27(452 - 60(\sqrt{14} + 2\sqrt{2})\zeta_1 + 225(13 + 8\sqrt{7}))},
\]

\[
\tilde{r}_n \to r_n^{(-)} = 1 - \frac{10(4(4\sqrt{2} - 3\sqrt{7})(12\zeta_1 - 5(159(\sqrt{14} + 372\sqrt{2} + 424))}{837(3\sqrt{2} - 2\sqrt{7})(2\zeta_1 - 15(\sqrt{14} + 2\sqrt{2} + 3))}.
\]

(ii) If \( \zeta_2 \) is fixed, \( \zeta_1 \to +\infty \), we have the following.

\[
\tilde{q}_n \to q_n^{(0)} = \frac{25}{9} - \frac{32(89 + 5\cosh(\zeta_2 - 2\ln 2))}{875(6 - 2\cosh(\zeta_2 - 2\ln 2))},
\]

\[
\tilde{r}_n \to r_n^{(-)} = 1 - \frac{20(\sqrt{14})}{27\sqrt{10}} \cosh(\frac{1}{2} \ln \frac{10^{0.95 + 24\sqrt{14}}}{31(46 + 12\sqrt{14})}) - \frac{20(\sqrt{14})}{27\sqrt{10}} \sinh(\frac{1}{2} \ln \frac{10^{0.95 + 24\sqrt{14}}}{31(46 + 12\sqrt{14})}) \coth(\frac{1}{2} \zeta_2 + \frac{1}{2} \ln \frac{23 + 6\sqrt{14}}{10}.
\]

**After collision \( t \to +\infty \), we observe the following cases:**
(iii) If $\xi_1$ is fixed, $\xi_2 \to +\infty$, we have the following.

$$
\begin{align*}
\tilde{q}_n \to q^+_{n1} &= \frac{25}{\sqrt{3}} + \frac{4(4\sqrt{3}^2 + 60(\sqrt{14} - 2\sqrt{2})\xi_1 + 25(237 - 72\sqrt{3}))}{9(4\sqrt{3}^2 + 60(\sqrt{14} - 2\sqrt{2})\xi_1 + 225(13 - 8\sqrt{3}))}, \\
\tilde{p}_n \to r^+_{n1} &= 1 - \frac{10(4\sqrt{3}^2 + 3\sqrt{7})}{837(3\sqrt{2} + 3\sqrt{7})^2} + 5(159\sqrt{14} - 372\sqrt{2} - 2424).
\end{align*}
$$

(iv) If $\xi_2$ is fixed, $\xi_1 \to -\infty$, we have the following.

$$
\begin{align*}
\tilde{q}_n \to q^+_{n2} &= \frac{25}{\sqrt{3}} - \frac{32(89 + 5 \cosh(\xi_2 - \ln 2))}{327(16 - 2 \cosh(\xi_2 - \ln 2))}, \\
\tilde{p}_n \to r^+_{n2} &= 1 - \frac{20\sqrt{31}}{27\sqrt{10}} \cosh[\frac{1}{2} \ln \frac{10(95 + 24\sqrt{14})}{31(46 + 12\sqrt{14})}] - \frac{20\sqrt{31}}{27\sqrt{10}} \sinh[\frac{1}{2} \ln \frac{10(95 + 24\sqrt{14})}{31(46 + 12\sqrt{14})}] \coth[\frac{1}{2} \xi_2 + \frac{1}{2} \ln \frac{23 + 6\sqrt{14}}{10}].
\end{align*}
$$

Due to the relationship between exponential function and hyperbolic function, in the results of asymptotic analysis, we turn the exponential function into a hyperbolic function. From the above analysis, we can observe that the hyperbolic form solutions and rational solutions possess singularities before and after collisions, as shown in Figure 8. Next, let us analyze $\tilde{q}_n$ and $\tilde{p}_n$ in (53), respectively:

- For solution $\tilde{q}_n$, before collision $\tilde{q}_n$ has four singular lines $L_{-1}^-, L_{-2}^-, L_{-3}^-, L_{-4}^-$, in which the two singular lines $L_{-1}^-, L_{-2}^-$ are obtained by solving $4\xi_1^2 - 60(\sqrt{14} + 2\sqrt{2})\xi_1 + 225(13 + 8\sqrt{7}) = 0$, while the other two singular lines $L_{-3}^-, L_{-4}^-$ can be obtained by solving cosh($\xi_2 - \ln 2$) - 8 = 0. As $t \to -\infty$, the solution $\tilde{q}_n$ possesses singularities in four lines $L_{-1}^-, L_{-2}^-, L_{-3}^-$ and $L_{-4}^-$, which also are its four trajectories. After collision, $\tilde{q}_n$ has four singular lines $L_1^+, L_2^+, L_3^+$ and $L_4^+$, in which the two singular lines $L_1^+$ and $L_2^+$ are obtained by solving $4\xi_1^2 + 60(\sqrt{14} + 2\sqrt{2})\xi_1 + 225(13 + 8\sqrt{7}) = 0$, while the other two singular lines $L_3^+$ and $L_4^+$ are still given by solving cosh($\xi_2 - \ln 2$) - 8 = 0. As $t \to +\infty$, solution $\tilde{q}_n$ possesses singularities in four lines $L_1^+, L_2^+, L_3^+, L_4^+$ which also are its four trajectories.

- For the solution $\tilde{p}_n$, before collision $\tilde{p}_n$ has two singular lines $L_1^-$ (i.e., $2\xi_1 - 15(\sqrt{14} + 2\sqrt{2} + 3) = 0$) and $L_2^-$ (i.e., $\xi_2 + \ln \frac{23 + 6\sqrt{14}}{10} = 0$); that is to say that as $t \to -\infty$, the solution $\tilde{p}_n$ possesses singularities in two lines $L_1^-, L_2^-$, which are also its two trajectories. After collision, $\tilde{p}_n$ has two singular lines $L_1^+$ (i.e., $2\xi_1 - 15(\sqrt{14} - 2\sqrt{2} - 3) = 0$) and $L_2^+$ (i.e., $\xi_2 + \ln \frac{23 + 6\sqrt{14}}{10} = 0$); in other words, as $t \to +\infty$, the solution $\tilde{p}_n$ possesses singularities in two lines $L_1^+, L_2^+$, which also are its two trajectories.

- Through the above discussion, we can observe that the hyperbolic solutions and rational solutions do not change their directions before and after collisions, and the positions of singular lines of hyperbolic solutions do not change, while the singular lines of rational solutions have changed their positions. In order to show the correctness of our asymptotic analysis results, we draw the three-dimensional plots of hybrid solutions and the trajectory plots after asymptotic analysis, respectively, as shown in Figure 8. By comparing Figure 8a1,a2 with Figure 8b1,b2, we find that the singularities in the three-dimensional plots are completely consistent with the trajectories in the two-dimensional picture, which also verifies our asymptotic analysis’ correctness of hybrid solutions.

- Here, the authors would like to say the following: In Section 4.1, when the soliton solutions are discussed separately, they are nonsingular; however, in the hybrid solutions, although taking the same parameters, these hyperbolic soliton solutions do become singular, and the possible reason is that the rational solutions in the hybrid solutions result in their singularity. This new property is worthy of further discussion.
Figure 8. (Color online) Hybrid solutions of hyperbolic and rational solutions: (a1) the three-dimensional structure of solution $\tilde{q}_n$ in (53); (b1) the trajectory plot of solution $\tilde{q}_n$; (a2) the three-dimensional structure of solution $\tilde{r}_n$ in (53); (b2) the trajectory plot of solution $\tilde{r}_n$.

Case (2) When $N = 2$, for $\lambda_1 = \frac{\alpha}{2}$ (i.e., $\alpha = \frac{\lambda_1}{2}$), by using the first kind expansion, for $\lambda_2 \neq \frac{\alpha}{2}$ (e.g., $\lambda_2 = 3$) without using Taylor expansion, the discrete generalized (2, 1)-fold DT will provide hybrid collision solutions of standard one-soliton and second-order rational solutions of Equation (6) as follows:

$$
\tilde{q}_n = \frac{(1 + \alpha^2)a^{(0)}_{n+1}}{a^{(0)}_n}, \quad \tilde{r}_n = \frac{\alpha + b^{(3)}_n}{\alpha d^{(4)}_n},
$$

where $a^{(0)}_n = \frac{\Delta a^{(0)}_n}{\Delta t}$, $b^{(3)}_n = \frac{\Delta b^{(3)}_n}{\Delta t}$, and $d^{(4)}_n = \frac{\Delta d^{(4)}_n}{\Delta t}$, in which the following is the case.

$$
\Delta_{1,n} = \begin{vmatrix}
\lambda_1^3 \psi_{1,n}(0) & \lambda_1^3 \psi_{1,n}(0) & \lambda_1^3 \psi_{1,n}(0) & \lambda_1^3 \psi_{1,n}(0) \\
\lambda_1^4 \psi_{1,n}(0) & \lambda_1^4 \psi_{1,n}(0) & \lambda_1^4 \psi_{1,n}(0) & \lambda_1^4 \psi_{1,n}(0) \\
\lambda_1^3 \psi_{1,n}(0) & \lambda_1^3 \psi_{1,n}(0) & \lambda_1^3 \psi_{1,n}(0) & \lambda_1^3 \psi_{1,n}(0) \\
\lambda_1^2 \psi_{1,n}(0) & \lambda_1^2 \psi_{1,n}(0) & \lambda_1^2 \psi_{1,n}(0) & \lambda_1^2 \psi_{1,n}(0) \\
\lambda_1 \psi_{1,n}(0) & \lambda_1 \psi_{1,n}(0) & \lambda_1 \psi_{1,n}(0) & \lambda_1 \psi_{1,n}(0) \\
\psi_{1,n}(0) & \psi_{1,n}(0) & \psi_{1,n}(0) & \psi_{1,n}(0) \\
\psi_{2,n}(0) & \psi_{2,n}(0) & \psi_{2,n}(0) & \psi_{2,n}(0) \\
\psi_{3,n}(0) & \psi_{3,n}(0) & \psi_{3,n}(0) & \psi_{3,n}(0) \\
\psi_{4,n}(0) & \psi_{4,n}(0) & \psi_{4,n}(0) & \psi_{4,n}(0)
\end{vmatrix}
$$

$$
\Delta_{2,n} = \begin{vmatrix}
\lambda_1^3 \psi_{1,n}(0) & \lambda_1^3 \psi_{1,n}(0) & \lambda_1^3 \psi_{1,n}(0) & \lambda_1^3 \psi_{1,n}(0) \\
\lambda_1^2 \psi_{1,n}(0) & \lambda_1^2 \psi_{1,n}(0) & \lambda_1^2 \psi_{1,n}(0) & \lambda_1^2 \psi_{1,n}(0) \\
\lambda_1 \psi_{1,n}(0) & \lambda_1 \psi_{1,n}(0) & \lambda_1 \psi_{1,n}(0) & \lambda_1 \psi_{1,n}(0) \\
\psi_{1,n}(0) & \psi_{1,n}(0) & \psi_{1,n}(0) & \psi_{1,n}(0) \\
\psi_{2,n}(0) & \psi_{2,n}(0) & \psi_{2,n}(0) & \psi_{2,n}(0) \\
\psi_{3,n}(0) & \psi_{3,n}(0) & \psi_{3,n}(0) & \psi_{3,n}(0) \\
\psi_{4,n}(0) & \psi_{4,n}(0) & \psi_{4,n}(0) & \psi_{4,n}(0)
\end{vmatrix}
$$

$$
\Delta a^{(0)}_n = \begin{vmatrix}
-\phi_{1,n} & -\phi_{1,n} & -\phi_{1,n} & -\phi_{1,n} \\
\phi_{2,n} & \phi_{2,n} & \phi_{2,n} & \phi_{2,n} \\
\phi_{3,n} & \phi_{3,n} & \phi_{3,n} & \phi_{3,n} \\
\phi_{4,n} & \phi_{4,n} & \phi_{4,n} & \phi_{4,n}
\end{vmatrix}
$$
\[
\begin{align*}
\Delta b_n^{(3)} &= \begin{vmatrix}
\lambda_1^4 \psi_{1,n}^{(0)} & \lambda_1^4 \psi_{1,n}^{(0)} & \lambda_1^3 \psi_{1,n}^{(0)} & -\psi_{1,n}^{(0)} \\
\lambda_1^4 \psi_{1,n}^{(0)} + 4\lambda_1^3 \phi_{1,n}^{(0)} & \lambda_1^4 \psi_{1,n}^{(0)} + 2\lambda_1^3 \phi_{1,n}^{(0)} & \lambda_1^3 \psi_{1,n}^{(0)} + 3\lambda_1^2 \phi_{1,n}^{(0)} & -\psi_{1,n}^{(0)} \\
\lambda_1^4 \phi_{1,n}^{(2)} + 4\lambda_1^3 \phi_{1,n}^{(1)} + 6\lambda_1^2 \phi_{1,n}^{(0)} & \lambda_1^4 \phi_{1,n}^{(2)} + 2\lambda_1^3 \phi_{1,n}^{(1)} + \phi_{1,n}^{(0)} & \lambda_1^3 \phi_{1,n}^{(2)} + 3\lambda_1^2 \phi_{1,n}^{(1)} + 3\lambda_1 \psi_{1,n}^{(1)} & -\psi_{1,n}^{(2)} \\
\lambda_1^4 \phi_{2,n}^{(0)} & \lambda_1^4 \phi_{2,n}^{(0)} & \lambda_1^3 \phi_{2,n}^{(0)} & -\psi_{2,n}^{(0)} \\
\lambda_1^4 \phi_{2,n}^{(0)} & \lambda_1^4 \phi_{2,n}^{(0)} & \lambda_1^3 \phi_{2,n}^{(0)} & -\psi_{2,n}^{(0)} \\
\end{vmatrix}
\end{align*}
\]

\[
\Delta d_n^{(4)} = \begin{vmatrix}
\lambda_1^3 \psi_{1,n}^{(1)} + 3\lambda_1^2 \phi_{1,n}^{(0)} & \lambda_1^3 \phi_{1,n}^{(1)} + \phi_{1,n}^{(0)} & \lambda_1^2 \phi_{1,n}^{(2)} + 2\lambda_1 \psi_{1,n}^{(1)} + \psi_{1,n}^{(0)} & -\lambda_1^3 \phi_{1,n}^{(1)} - 4\lambda_1^2 \phi_{1,n}^{(1)} - 6\lambda_1^2 \phi_{1,n}^{(1)} \\
\lambda_1^4 \phi_{1,n}^{(2)} + 3\lambda_1^3 \phi_{1,n}^{(1)} + 3\lambda_1 \phi_{1,n}^{(1)} & \lambda_1^4 \phi_{1,n}^{(2)} + \lambda_1^3 \phi_{1,n}^{(1)} + \lambda_1^2 \phi_{1,n}^{(1)} + \phi_{1,n}^{(0)} & \lambda_1^3 \phi_{1,n}^{(2)} + 2\lambda_1 \psi_{1,n}^{(1)} + \psi_{1,n}^{(0)} & -\lambda_1^4 \phi_{1,n}^{(2)} - 4\lambda_1^3 \phi_{1,n}^{(1)} - 6\lambda_1^2 \phi_{1,n}^{(1)} \\
\lambda_1^4 \phi_{2,n}^{(0)} & \lambda_1^4 \phi_{2,n}^{(0)} & \lambda_1^3 \phi_{2,n}^{(0)} & -\lambda_1^2 \phi_{32,n}^{(0)} \\
\lambda_1^4 \phi_{2,n}^{(0)} & \lambda_1^4 \phi_{2,n}^{(0)} & \lambda_1^3 \phi_{2,n}^{(0)} & -\lambda_1^2 \phi_{32,n}^{(0)} \\
\end{vmatrix}
\]

Direct calculation provides the simplified analytic expressions with \(c_0 = c_1 = 0\) as

\[
\bar{q}_n = \frac{n}{\sqrt{2}}, \quad \bar{r}_n = \frac{n}{\sqrt{2}}.
\]

The expressions of \(F, F_2, G_1\) and \(G_2\) are very complex; thus, we list them in the Appendix B.

For the sake of analysis, let \(\bar{\xi}_1 = 25n - 27t, \quad \bar{\xi}_2 = n \ln \frac{9-2\sqrt{14}}{9+2\sqrt{14}} + \frac{36\sqrt{14}}{25} t\) be the asymptotic analysis for solutions (54) when \(|t| \to \pm \infty\) is given as follows:

Before collision \(t \to -\infty:

(i) If \(\bar{\xi}_2\) is fixed, \(\bar{\xi}_1 \to +\infty\), we have the following:

\[
\bar{q}_n \to \bar{q}_{n1} = \frac{25}{9} + \frac{n}{\sqrt{2}}, \quad \bar{r}_n \to \bar{r}_{n1} = 1 + \frac{n}{\sqrt{2}},
\]

where the following is the case.

\[
Q_1 = \frac{4}{45,875}[4c_1^2 - 120c_1^2 + 24,900c_1^2 - 3125(45\sqrt{14} + 68n)c_1 + 1875(1386 + 850\sqrt{14}n)],
\]

\[
Q_2 = \frac{1}{9,988,250,000}[8(\sqrt{14} - 3)c_1^3 - 900c_1^2 + 100(567 - 54\sqrt{14})c_1 + 12,500(17\sqrt{14} + 51)n + 151,875\sqrt{14}],
\]

\[
R_1 = 32(1056\sqrt{14} - 3293)c_1 - 3,561,600c_1^2 + 1200(207,959 - 20,508\sqrt{14})c_1 + 50,000(17,952\sqrt{14} - 55,981)n
\]

\[
-671,332,500\sqrt{14},
\]

\[
R_2 = -34,992(\sqrt{14} - 3)c_1^3 + 3,936,600c_1^2 + 11,809,800(21 - 2\sqrt{14})c_1 - 10,935,000(598 - 23\sqrt{14})n
\]

\[
+664,301,250\sqrt{14}.
\]

(ii) If \(\bar{\xi}_2\) is fixed, \(\bar{\xi}_1 \to +\infty\), we have the following.

\[
\bar{q}_n \to q_{n2}^- = \frac{25}{9} + \frac{448}{45\cosh(\bar{\xi}_2 - \ln 2) - 207},
\]

\[
\bar{r}_n \to r_{n2}^- = 1 - \frac{2\sqrt{953,611}}{2107} \cos h\left[\frac{1}{2} \ln \frac{971,111 \pm 49,050\sqrt{14}}{933,611}\right] - \frac{2\sqrt{953,611}}{2107} \sinh\left[\frac{1}{2} \ln \frac{971,111 \pm 49,050\sqrt{14}}{933,611}\right] \coth\left[\frac{1}{2} c_1^2 + \frac{1}{2} \ln \frac{23 + 6\sqrt{14}}{10}\right].
\]

After collision \(t \to +\infty:

(iii) If \(\bar{\xi}_1\) is fixed, \(\bar{\xi}_2 \to +\infty\), we have the following:

\[
\bar{q}_n \to q_{n2}^+ = \frac{16}{25} + \frac{n}{\sqrt{2}}, \quad \bar{r}_n \to r_{n1}^+ = 1 + \frac{n}{\sqrt{2}},
\]

with the following case.

\[
Q_1 = \frac{1}{45,875}[4c_1^2 - 120c_1^2 + 24,900c_1^2 + 3125(45\sqrt{14} + 68n)c_1 + 1875(1386 + 850\sqrt{14}n)],
\]

\[
Q_2 = \frac{1}{9,988,250,000}[8(\sqrt{14} + 3)c_1^3 + 900c_1^2 + 100(567 - 54\sqrt{14})c_1 + 12,500(17\sqrt{14} + 51)n + 151,875\sqrt{14}],
\]

\[
R_1 = -16(1056\sqrt{14} + 3293)c_1 - 1,780,800c_1^2 - 600(20,508\sqrt{14} + 207,959)c_1 - 25,000(17,592\sqrt{14} + 55,981)n
\]

\[
+335,666,250\sqrt{14},
\]

\[
R_2 = 17,460(\sqrt{14} + 3)c_1^3 + 1,968,300c_1^2 - 900(118,089\sqrt{14} + 1,240,029)c_1 + 2500(185,886\sqrt{14} + 557,685)n
\]

\[
-332,150,625\sqrt{14}.
\]

(iv) If \(\bar{\xi}_2\) is fixed, \(\bar{\xi}_1 \to -\infty\), we have the following.
\[ \tilde{q}_n \rightarrow q_{n+1}^+ = \frac{25}{9} + \frac{448}{45 \cosh \left( \xi_2 - \ln 2 \right) - 207} \]

\[ \tilde{r}_n \rightarrow r_{n+1}^+ = 1 - \frac{2\sqrt{953,611}}{2187} \cosh \left[ \frac{1}{2} \ln \left( \frac{971,111 + 49,050\sqrt{14}}{953,611} \right) \right] - \frac{2\sqrt{953,611}}{2187} \sinh \left[ \frac{1}{2} \ln \left( \frac{971,111 + 49,050\sqrt{14}}{953,611} \right) \right] \]

\[ \coth \left[ \frac{1}{2} \xi_2 + \frac{1}{2} \ln \left( \frac{23 + 6\sqrt{14}}{10} \right) \right]. \]

From the above analysis, we can observe that the standard solitons and rational solutions due to the asymptotic results of hybrid solutions are formally consistent with those of standard solitons and rational solutions alone in the previous two subsections, which also shows the correctness of our analysis from another aspect. We can also draw their structures by imitating the case of \( N = 1 \), but the figures in this case are more complex and they do not look good, and we omit the discussion here.

**Remark 4.** It should be emphasized here that in our previous research [17,32], we did not carry out the asymptotic analysis of hybrid soliton-and rational solutions. However, in this paper, we have overcome this problem and realized the asymptotic analysis of these hybrid solutions, which were not available in our previous research in [17,32]. Therefore, the asymptotic analysis of hybrid solutions also is another important innovation of this paper.

5. Conclusions

In this paper, we have investigated Equation (6), which may describe particle vibrations in lattices with an exponential interaction force. The main contributions of this paper are as follows: (i) Some integrable properties of Equation (6) have been investigated, such as lattice hierarchy, Hamiltonian structure and infinite conservation laws. (ii) The discrete \((m, 2N - m)\)-fold DT for Equation (6) has been constructed in detail for the first time. (iii) By applying the resulting discrete DT, we have obtained the soliton solutions, rational and semi-rational solutions and hybrid solutions of Equation (6), and their limit states were discussed by using the asymptotic analysis technique. Multi-soliton solutions and relevant numerical simulations are shown in Figures 2–6. (iv) Asymptotic analysis and mathematical characteristics of rational solutions are shown in Figure 7 and listed in Tables 1 and 2, from which we find that the asymptotic analysis results of rational solutions are consistent with their exact solutions. (v) We have obtained the rational solutions (48) and semi-rational solutions (49) with arbitrary controllable parameters, which can control their positions. (vi) Asymptotic states of the exponential-and-rational hybrid solutions are shown in Figure 8, from which we can observe that the trajectories of our asymptotic analysis agree with hybrid solutions.

The properties of Equation (1) or (6) discussed above are reported for the first time. We hope that these results in this paper might provide new insights in understanding lattice dynamics.

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Appendix A

\[ \phi_{1,n}^{(1)} = \frac{4}{15n} e^{\frac{2\pi}{3\sqrt{3}}} \left( (625n^2 - 1350nt + 10,000n_t + 729n^2 - 10,800nt + 40,000n_t + 500n + 1215t) \right), \]

\[ \phi_{1,n}^{(2)} = -\frac{4}{1500000} e^{\frac{2\pi}{3\sqrt{3}}} \left( 390,625n^4 - 1687,500n^2t + 12,500,000n^3 + 2,733,750n^4t^2 - 40,500,000n^2t + 150,000,000n^2t^2 \right) \]

\[ = -1968,300n^3 + 43,740,000n^2t^2 - 324,000,000nt^3 + 800,000,000n^2t^3 + 531,441t^4 - 15,746,400nt^5 \]

\[ + 174,960,000n^2t^5 + 864,000,000n^3t^4 - 1,600,000,000n^4t^5 + 1,875,000n^5 + 566,250nt^2 + 30,000,000nt^3 \]

\[ + 7,654,500n^2t^4 + 40,500,000nt^3 + 120,000,000n^2t^4 + 5,314,410n^3t^5 - 78,732,000n^2t^6 + 291,600,000n^3t^7 + 171,875n^4t^8 - 843,750nt^9 \]

\[ + 33,250,000nt^{10} + 54,000,000n_t + 9,950,850n^2 + 81,810,000nt + 58,320,000nt^2 + 303,000,000n^2t + 432,000,000n^3t^3 + 750,000n^4 + 1,822,500n_t^2 \right), \]

\[ \phi_{2,n}^{(2)} = -\frac{4}{1500000} e^{\frac{2\pi}{3\sqrt{3}}} \left( 390,625n^4 - 1687,500n^2t + 12,500,000n^3 + 2,733,750n^4t^2 - 40,500,000n^2t + 150,000,000n^2t^2 \right) \]

\[ = -1968,300n^3 + 43,740,000n^2t^2 - 324,000,000nt^3 + 800,000,000n^2t^3 + 531,441t^4 - 15,746,400nt^5 \]

\[ + 174,960,000n^2t^5 + 864,000,000n^3t^4 - 1,600,000,000n^4t^5 + 1,875,000n^5 + 566,250nt^2 + 30,000,000nt^3 \]

\[ + 7,654,500n^2t^4 + 40,500,000nt^3 + 120,000,000n^2t^4 + 5,314,410n^3t^5 - 78,732,000n^2t^6 + 291,600,000n^3t^7 + 171,875n^4t^8 - 843,750nt^9 \]

\[ + 33,250,000nt^{10} + 54,000,000n_t + 9,950,850n^2 + 81,810,000nt + 58,320,000nt^2 + 303,000,000n^2t + 432,000,000n^3t^3 + 750,000n^4 + 1,822,500n_t^2 \right), \]

Appendix B

\[ F_1 = \frac{1}{1,125,000,000} e^{\frac{2\pi}{3\sqrt{3}}} \left( \frac{299}{9} - 64 - 64\sqrt{14} \right) \left( 112,500 \sqrt{14n^2 - 243,000 \sqrt{14nt} + 131,220 \sqrt{14t^2} + 125,000n^3 - 405,000n^2t \right) \]

\[ + 437,400nt^2 - 157,464t^3 - 22,500 \sqrt{14n} + 24,300 \sqrt{14nt} - 37,500nt^2 + 81,000nt^3 - 43,740t^2 + 1125 \sqrt{14} + 385,000n^2t \]

\[ - 186,300t - 47,250 \right) \left( 112,500 \sqrt{14t^2} - 243,000 \sqrt{14nt} + 131,220 \sqrt{14t^2} + 125,000n^3 - 405,000n^2t + 437,400nt^2 \right) \]

\[ - 157,464nt + 22,500 \sqrt{14nt} - 24,300 \sqrt{14t^2} + 37,500nt^2 - 81,000nt + 43,740t^2 + 1125 \sqrt{14} + 385,000n^2t \]

\[ - 186,300t - 47,250 \right), \]

\[ F_2 = \left( \frac{125,000}{\sqrt{14} + 3} \right) e^{\frac{2\pi}{3\sqrt{3}}} \left( 12 - \frac{8}{3} \sqrt{14n^2 - 243,000 \sqrt{14nt} + 131,220 \sqrt{14t^2} + 125,000n^3 - 405,000n^2t \right) \]

\[ + 437,400nt^2 - 157,464t^3 - 22,500 \sqrt{14n} + 24,300 \sqrt{14nt} - 37,500nt^2 + 81,000nt^3 - 43,740t^2 + 1125 \sqrt{14} + 385,000n^2t \]

\[ - 186,300t - 47,250 \right) \left( 125,000 \sqrt{14n^2 - 243,000 \sqrt{14nt} + 131,220 \sqrt{14t^2} + 125,000n^3 - 405,000n^2t + 437,400nt^2 \right) \]

\[ - 157,464nt + 22,500 \sqrt{14nt} - 24,300 \sqrt{14t^2} + 37,500nt^2 - 81,000nt + 43,740t^2 + 1125 \sqrt{14} - 385,000n^2t \]

\[ - 186,300t - 47,250 \right), \]

\[ G_1 = 25 \left( \sqrt{14} - 1 \right) e^{\frac{2\pi}{3\sqrt{3}}} \left( 12 - \frac{8}{3} \sqrt{14n^2 - 243,000 \sqrt{14tn} + 131,220 \sqrt{14t^2} + 125,000n^3 - 405,000n^2t \right) \]

\[ + 437,400nt^2 - 157,464t^3 + 22,500 \sqrt{14n} - 24,300 \sqrt{14nt} + 37,500nt^2 - 81,000nt + 43,740t^2 + 1125 \sqrt{14} + 385,000n^2t \]

\[ - 186,300t + 47,250 \right) \left( 125,000 \sqrt{14n^2 - 243,000 \sqrt{14nt} + 131,220 \sqrt{14t^2} + 125,000n^3 - 405,000n^2t + 437,400nt^2 \right) \]

\[ - 157,464nt + 22,500 \sqrt{14nt} - 24,300 \sqrt{14t^2} + 37,500nt^2 - 81,000nt + 43,740t^2 + 1125 \sqrt{14} - 385,000n^2t \]

\[ - 186,300t - 47,250 \right) \right). \]
\[ G_2 = 2187(\sqrt{14} + 3)e^{\frac{2\pi i}{14}}((12 - \frac{8}{3}\sqrt{14})^n(112,500\sqrt{14}n^2 - 243,000\sqrt{14}nt + 131,220\sqrt{14}t^2 + 125,000n^3 - 405,000n^2t + 437,400n^2 - 157,464t^3 - 202,500\sqrt{14}n + 218,700t\sqrt{14} - 337,500n^2 + 729,000nt - 393,660t^2 + 91,125\sqrt{14}t^4 + 685,000n - 510,300t - 425,250) + 4374(-3 + \sqrt{14})(12 + \frac{8}{3}\sqrt{14})^n(112,500\sqrt{14}n^2 - 243,000\sqrt{14}nt + 131,220\sqrt{14}t^2 - 125,000n^3 + 405,000n^2t - 437,400n^2 + 157,464t^3 - 202,500\sqrt{14}n + 218,700t\sqrt{14} + 337,500n^2 - 729,000nt + 393,660t^2 + 91,125\sqrt{14}t^4 + 685,000n + 510,300t + 425,250). \]

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