Harmonic oscillations of a longitudinal shear infinite hollow cylinder arbitrary cross-section with a tunnel crack

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Abstract. An effective analytical-numerical method for determining the dynamic stresses in a hollow cylindrical body of arbitrary cross-section with a tunnel crack under antiplane strain conditions is proposed. The method allows separately solving the integral equations on the crack faces and satisfying the boundary conditions on the body boundaries. It provides a convenient numerical scheme. Approximate formulas for calculating the dynamic stress intensity factors in a neighborhood of the crack are obtained and the influence of the crack geometry and wave number on these quantities is investigated, especially from the point of view of the resonance existence.

1. Introduction
Research of the stress conditions of finite bodies with cracks under harmonic loads remains relevant nowadays. This actuality is stipulated by the need for determining the conditions of fracture through evaluation of the stress intensity factor (SIF) of dynamic stresses near the cracks and diagnosing defects by analyzing their influence on the resonance frequency [1]. Preliminary results in this area mainly appertain to infinite bodies with cracks or thin inclusions [2–10]. The situations where the body occupies a finite area is considered much less. This is primarily due to the fact that the application of the method of boundary integral equations for the source of the problems reduces the problem to solving systems of integral equations and satisfying conditions on the surface defects and on the boundary of the body [11–15], which significantly complicates the numerical implementation of the method. We propose an approach that allows solving separately integral equations on the surface cracks and satisfying the loading conditions on the boundary of the body.

2. Statements of the problem
Consider an elastic cylinder with generators parallel to the $Oz$ axis (figure 1) whose intersection by the plane $xOy$ is a doubly-connected domain bounded by arbitrary closed smooth curves. In the polar coordinates $Or\varphi$, whose center is located at the origin of coordinates, these curves are determined by the equations

$$r = r_0\psi_0(\varphi), \quad r = r_1\psi_1(\varphi), \quad 0 \leq \varphi < 2\pi.$$
Figure 1. Hollow cylinder with a tunnel crack.

The cylinder contains a through crack. In the plane \(xOy\), the defect does not fall outside the boundary of cross-section and occupies a segment of length \(2a\). The oscillations of longitudinal shear go on in the lateral surface of the cylinder due to the action of a harmonic load \(P(\varphi)e^{-i\omega t}\), which is directed along the axis \(Oz\). We omit the multiplier \(e^{-i\omega t}\), which determines the time dependence, and consider only the amplitude values below. Under such conditions, only the \(z\)-component of displacement vector, which satisfies the Helmholtz equation \([16]\), is different from zero. This equation in the polar coordinates has the form

\[
\Delta w + \kappa_2^2 w = 0, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2},
\]

where \(\kappa_2^2 = \omega^2 c_2^{-2},\) \(c_2^{-2} = G\rho^{-1}\), \(G\) and \(\rho\) are the shear module and the density of the cylinder.

Now we formulate conditions on the outer and inner surfaces of the cylinder. According to the conditions of loading of the outer surface, the stress must satisfy the equality on it

\[
\tau_{n_z}(r_0\psi_0(\varphi), \varphi) = GP(\varphi), \quad 0 \leq \varphi < 2\pi,
\]

where \(\vec{n}\) is the vector of normal on the cylinder surface. The inner surface of the cylinder is considered to be unmovable

\[
w(r_1\psi_1(\varphi), \varphi) = 0, \quad 0 \leq \varphi < 2\pi.
\]

To formulate the boundary conditions on the defect, we connect a coordinate system \(O_1x_1y_1\) with its center, as is shown in figure 1. The relation between the system \(O_1x_1y_1\) and the polar coordinates is given by

\[
x_1 = r \cos(\varphi - \alpha) - c \cos\alpha - d \sin\alpha, \quad y_1 = r \sin(\varphi - \alpha) - d \cos\alpha + c \sin\alpha,
\]

where \(\alpha\) is the angle between the axes \(O_1x_1\) and \(Ox\).

Let \(w^{(1)}(x_1, y_1)\) be the \(z\) component of the displacement vector \(w(r, \varphi)\) in passing from the polar coordinates to local coordinates, according to (4). Considering a crack, we assume that its surface is free from stresses:

\[
\tau_{xy_1}(x_1, 0) = 0, \quad |x_1| < a.
\]

In addition, the displacement \(w^{(1)}_1(x_1, y_1)\) on the crack surface has a discontinuity, whose jump is denoted by \(\chi(x_1)\):

\[
w^{(1)}_1(x_1, +0) - w^{(1)}_1(x_1, -0) = \chi(x_1), \quad \chi(\pm a) = 0.
\]

Thus, determining of the stress state of the considered body with crack under conditions of oscillations is reduced to solving equation (1) with conditions (2), (3), (5), and (6).
3. Satisfaction of the conditions on the crack surface

To solve our problem, we first, in the system $O_1x_1y_1$, construct a discontinuous solution of the Helmholtz equation [2, 17] with a jump $\chi(x_1)$ determined by (6):

\[
w_1^{(1)}(x_1, y_1) = \frac{\partial}{\partial y_1} \int_{-a}^{a} \chi(\eta)r(\eta - x_1, y_1) \, d\eta, \quad r(\eta - x_1, y_1) = -i \frac{H_0^{(1)}(\kappa_2 \sqrt{(\eta - x_1)^2 + y_1^2})}{4},
\]

where $H_0^{(1)}(z)$ is the Hankel function. The stresses corresponding to displacement (7) are determined as

\[
\tau_{y_1z}^{(1)} = G \frac{\partial w_1^{(1)}(x_1, y_1)}{\partial y_1} = -G \left[ \int_{-a}^{a} \chi(\eta) \frac{\partial^2}{\partial \eta^2} r(\eta - x_1, y_1) \, d\eta + \kappa_2^2 \int_{-a}^{a} \chi(\eta)r(\eta - x_1, y_1) \, d\eta \right].
\]

Then, in the system $Or\varphi$, we represent the displacement as

\[
w(r, \varphi) = w_0(r, \varphi) + w_1(r, \varphi),
\]

where $w_1(r, \varphi)$ is the discontinuous solution (7) after passage to the polar coordinates according to (4) and $w_0(r, \varphi)$ is a unknown solution of the Helmholtz equation (1) such that the boundary conditions (2), (3) are satisfied for (9). We represent this unknown function as a linear combination of particular solutions of equation (1):

\[
w_0(r, \varphi) = r_0 \sum_{k=1}^{N} [A_k g_k(r, \varphi) + B_k h_k(r, \varphi)],
\]

where $J_m(z)$, $H_0^{(1)}(z)$ are cylindrical functions, $A_k$ and $B_k$ are unknown coefficients.

The functions $g_k(r, \varphi)$, $h_k(r, \varphi)$ are linearly independent and form a complete closed system of functions in the domain of cross-section [18].

After substitution of expressions (9) and (10) for displacements, the boundary condition (5) on the crack becomes

\[
\tau_{y_1z}^{(1)}(x_1, 0) + \tau_{y_1z}^{(0)}(x_1, 0) = 0,
\]

where the first term $\tau_{y_1z}^{(1)}(x_1, 0)$ is given by (8), and

\[
\tau_{y_1z}^{(0)}(x_1, 0) = G r_0 \sum_{k=1}^{N} \left( A_k \frac{\partial g_k^{(1)}(x_1, y_1)}{\partial y_1} \bigg|_{y_1=0} + B_k \frac{\partial h_k^{(1)}(x_1, y_1)}{\partial y_1} \bigg|_{y_1=0} \right),
\]

where $g_k^{(1)}(x_1, y_1)$, $h_k^{(1)}(x_1, y_1)$ are the function $g_k(r, \varphi)$ and $h_k(r, \varphi)$ in the coordinate system $O_1x_1y_1$. We use relation (11) to obtain an integral equation, which, after integrating by parts and isolating the singular part in its kernel, can be written as

\[
(2\pi)^{-1} \int_{-1}^{1} \chi(a\tau)[(\tau - \zeta)^{-1} + Q(\tau - \zeta)] \, d\tau = f(\zeta), \quad -1 < \zeta < 1.
\]

In equation (12), we used the notation

\[
\eta = a\tau, \quad x_1 = a\varsigma, \quad \kappa_0 = \kappa_2 r_0, \quad ar_0^{-1} = \gamma, \quad \kappa_2 a = \gamma\kappa_0, \quad r_1 r_0^{-1} = \lambda.
\]
Then, from (16) and (17), we obtain systems of linear algebraic equations for the values of the unknown functions $\psi_k^{(1)}(\tau)$ and $\psi_k^{(2)}(\tau)$:

$$f(\varsigma) = \sum_{k=1}^{N} [A_k \psi_k^{(1)}(\varsigma) + B_k \psi_k^{(2)}(\varsigma)], \quad f_k^{(1)}(\varsigma) = -\frac{\partial q_k^{(1)}(a\varsigma,0)}{\partial y_1}, \quad f_k^{(2)}(\varsigma) = -\frac{\partial h_k^{(1)}(a\varsigma,0)}{\partial y_1}$$  \hspace{1cm} (13)

and its kernel has the asymptotics

$$Q(Z) = O(|Z| \ln |Z|), \quad Z \to 0.$$

We should supplement equation (12) with the equality

$$\int_{-1}^{1} \chi'(a\tau) \, d\tau = 0$$  \hspace{1cm} (14)

which follows from condition (6) of the crack closure.

Since equations (12) and (14) are linear, we can write the solution of equation (12) with right-hand side (13) in the form

$$\chi'(a\tau) = \sum_{k=1}^{N} [A_k S_k^{(1)}(\tau) + B_k S_k^{(2)}(\tau)].$$  \hspace{1cm} (15)

This representation allows us to transform equation (12) into a sequence of equations for the functions $S_k^{(1)}(\tau)$ and $S_k^{(2)}(\tau)$:

$$(2\pi)^{-1} \int_{-1}^{1} S_k^{(i)}(\tau)(\tau - \varsigma)^{-1} + Q(\tau - \varsigma) \, d\tau = f_k^{(i)}(\varsigma), \quad i = 1, 2, \quad k = 1, N.$$  \hspace{1cm} (16)

As can be seen, these equations differ only in the right-hand sides and do not depend on the boundaries of the cross-section domain. We should supplement equations (16) with the equality arising from (14):

$$\int_{-1}^{1} S_k^{(i)}(\tau) \, d\tau = 0, \quad i = 1, 2, \quad k = 1, N.$$  \hspace{1cm} (17)

To construct an approximate solutions of equations (16), we represent the unknown functions $S_k^{(i)}(\tau)$ as

$$S_k^{(i)}(\tau) = \psi_k^{(i)}(\tau)(1 - \tau^2)^{-1/2}, \quad i = 1, 2, \quad k = 1, N.$$  \hspace{1cm} (18)

where each function $\psi_k^{(i)}(\tau)$ is assumed to satisfy the H"older condition on $[-1, 1]$. To determine them, we apply the method of mechanical quadratures [19], choosing the roots of the second-kind Chebyshev polynomial $U_{n-1}(\varsigma)$ as the collocation points $\varsigma_j = \cos(\pi j/n), \quad j = 1, 2, \ldots, n - 1$. Then, from (16) and (17), we obtain systems of linear algebraic equations for the values of the function $\psi_k^{(i)}(\tau)$ at the interpolation nodes:

$$(2\pi)^{-1} \sum_{m=1}^{n} a_m \psi_{km}^{i}[(\tau_m - \varsigma_j)^{-1} + Q(\tau_m - \varsigma_j)] = f_k^{(i)}(\varsigma_j), \quad i = 1, 2, \quad k = 1, n - 1;$$  \hspace{1cm} (19)

$$\sum_{m=1}^{n} a_m \psi_{km}^{i} = 0,$$

where $\psi_{km}^{i} = \psi_k^{(i)}(\tau_m)$ and $\tau_m = \cos(\pi (2m - 1)/2n), \quad m = 1, \ldots, n$, are the roots of the first-kind Chebyshev polynomial $T_n(\tau)$. If system (19) is solved, then we obtain an approximation of each function $\psi_k^{(i)}(\tau)$ by the interpolation polynomial

$$\psi_k^{(i)}(\tau) \approx \sum_{m=1}^{n} \psi_{km}^{i} \frac{T_n(\tau)}{(\tau - \tau_m)T_n'(\tau_m)}.$$  \hspace{1cm} (20)
4. Satisfaction of the conditions on the cylinder surfaces

The unknown coefficients $A_k$ and $B_k$ in equality (10) are determined from condition (2), (3) on the cylinder surfaces. For this, we write the stresses and the displacement as

$$
\begin{align*}
\tau_{nz}(r_0 \psi_0(\varphi), \varphi) &= \tau_{nx}(r_0 \psi_0(\varphi), \varphi)c_x + \tau_{ny}(r_0 \psi_0(\varphi), \varphi)c_y, \\
w(r_1 \psi_1(\varphi), \varphi) &= w_0(r_1 \psi_1(\varphi), \varphi) + w_1(r_1 \psi_1(\varphi), \varphi),
\end{align*}
$$

(21)

where $c_x$, $c_y$ are the direction cosines of the normal vector $\vec{n} = (c_x, c_y)$ of the outer surfaces of the cylinder. Substituting these expressions (21) for the stresses and the displacement in (2), (3) and using formulas (7)–(10), we obtain

$$
\begin{align*}
\sum_{k=1}^{N} A_k \left[ \int_{-1}^{1} \Phi_k^1(\tau) G(\tau, \varphi) \, d\tau + F_k^1(\varphi) \right] \\
+ \sum_{k=1}^{N} B_k \left[ \int_{-1}^{1} \Phi_k^2(\tau) G(\tau, \varphi) \, d\tau + F_k^2(\varphi) \right] = P(\varphi), \quad 0 \leq \varphi < 2\pi,
\end{align*}
$$

(22)

$$
\begin{align*}
\sum_{k=1}^{N} A_k \int_{-1}^{1} \Phi_k^1(\tau) E(\tau, \varphi) \, d\tau + \sum_{k=1}^{N} B_k \int_{-1}^{1} \Phi_k^2(\tau) E(\tau, \varphi) \, d\tau \\
+ \sum_{k=1}^{N} A_k g_k(r_1(\varphi), \varphi) + \sum_{k=1}^{N} B_k h_k(r_1(\varphi), \varphi) = 0.
\end{align*}
$$

The functions $G(\tau, \varphi)$, $E(\tau, \varphi)$ are here expressed through the values of the function $r(n - x_1, y_1)$ from (7) and its derivatives on the cylinder surfaces, and the functions $F_k^1(\varphi)$, $F_k^2(\varphi)$ are expressed in terms of the values of function $g_k(r, \varphi)$, $h_k(r, \varphi)$ on these surfaces given by formulas (10). In derivation of equations (22), the relation

$$
\chi(a\tau) = a \sum_{k=1}^{N} [A_k \Phi_k^1(\tau) + B_k \Phi_k^2(\tau)], \quad [\Phi_k^i(\tau)]' = S_k^i(\tau), \quad i = 1, 2.
$$

follows from formula (15). For calculating the integrals in (22), we use the following approximation [10] for the functions $\Phi_k^i(\tau)$, which is obtained from (18) and (20):

$$
\Phi_k^i(\tau) \approx (1 - \tau^2)^{1/2} S_{kn}^i(\tau), \quad S_{kn}^i(\tau) = -\frac{2}{\pi} \sum_{m=1}^{n} \psi_{km}^i \sum_{p=1}^{n} \tau^{p-1} T_p(\tau) U_p-1(\tau).
$$

Here, $U_p-1(\tau)$, $T_p(\tau)$ are the first- and second-kind Chebyshev polynomials, respectively. Now we can calculate the integral in (22) by the quadrature formula [20]

$$
\int_{-1}^{1} \Phi_k(\tau) K(\tau, \varphi) \, d\tau = \sum_{m=1}^{n} a_m \psi_{km} \sum_{l=1}^{n} D_{lm} K(Z_l, \varphi),
$$

$$
D_{lm} = -\frac{2}{n + 1} \sin \frac{l\pi}{n + 1} \sum_{p=1}^{n-1} \tau^{p-1} \cos \frac{p(2m - 1)\pi}{2n} \sin \frac{\pi lp}{n + 1}, \quad Z_l = \cos \frac{\pi l}{n + 1}.
$$
Finally, the boundary condition (22) becomes
\begin{align}
\sum_{k=1}^{N} A_k & \left[ \sum_{m=1}^{n} a_m \psi_{km}^1 \sum_{l=1}^{n} D_{lm} G(Z_l, \varphi) + F_k^1(\varphi) \right] \\
+ & \sum_{k=1}^{N} B_k \left[ \sum_{m=1}^{n} a_m \psi_{km}^2 \sum_{l=1}^{n} D_{lm} G(Z_l, \varphi) + F_k^2(\varphi) \right] = P(\varphi), \\
\sum_{k=1}^{N} A_k & \left[ \sum_{m=1}^{n} a_m \psi_{km}^1 \sum_{l=1}^{n} D_{lm} E(Z_l, \varphi) + g_k(\varphi) \right] \\
+ & \sum_{k=1}^{N} B_k \left[ \sum_{m=1}^{n} a_m \psi_{km}^2 \sum_{l=1}^{n} D_{lm} E(Z_l, \varphi) + h_k(\varphi) \right] = 0.
\end{align}
(23)

Now we obtain a system of linear equations for determining the coefficients $A_k$ and $B_k$ by applying the collocation method to (23) and taking the quantities $\sigma_l = 2\pi l/N$, $l = 1, \ldots, N$, as the nodes:
\begin{align}
\sum_{k=1}^{N} A_k & \left[ \sum_{m=1}^{n} a_m \psi_{km}^1 \sum_{l=1}^{n} D_{lm} G(Z_l, \sigma_l) + F_k^1(\sigma_l) \right] \\
+ & \sum_{k=1}^{N} B_k \left[ \sum_{m=1}^{n} a_m \psi_{km}^2 \sum_{l=1}^{n} D_{lm} G(Z_l, \sigma_l) + F_k^2(\sigma_l) \right] = P(\sigma_l), \\
\sum_{k=1}^{N} A_k & \left[ \sum_{m=1}^{n} a_m \psi_{km}^1 \sum_{l=1}^{n} D_{lm} E(Z_l, \sigma_l) + g_k(\sigma_l) \right] \\
+ & \sum_{k=1}^{N} B_k \left[ \sum_{m=1}^{n} a_m \psi_{km}^2 \sum_{l=1}^{n} D_{lm} E(Z_l, \sigma_l) + h_k(\sigma_l) \right] = 0, \quad l = 1, \ldots, N.
\end{align}
(24)

As is well known, the possibility of crack propagation is determined by the stress intensity factors (SIF), which in our case are given by
$$K^\pm = \sqrt{a} \lim_{\epsilon \to \pm 1 \pm 0} (\epsilon^2 - 1)^{1/2} r_{yy}(a\tau, 0).$$

If systems (19) and (24) are solved, then we obtain the following relations for the dimensionless SIF value:
\begin{align*}
k^+ &= \frac{K^+}{G\sqrt{a}}, \\
k^+ &= \frac{1}{2n} \left[ \sum_{k=1}^{N} A_k \sum_{m=1}^{n} (-1)^{m} \psi_{km}^1 \cot \frac{\gamma m}{2} + \sum_{k=1}^{N} B_k \sum_{m=1}^{n} (-1)^{m} \psi_{km}^2 \cot \frac{\gamma m}{2} \right], \\
k^- &= \frac{(-1)^{n+1}}{2n} \left[ \sum_{k=1}^{N} A_k \sum_{m=1}^{n} (-1)^{m} \psi_{km}^1 \tan \frac{\gamma m}{2} + \sum_{k=1}^{N} B_k \sum_{m=1}^{n} (-1)^{m} \psi_{km}^2 \tan \frac{\gamma m}{2} \right].
\end{align*}

5. Numerical analysis of the intensity of dynamic stresses and resonance phenomena

As an example, we consider a cylinder with the cross-section bounded by two ellipses with a common center, as is shown in figure 2.
Figure 2. Cross-section of the cylinder with a crack.

Figure 3. Dependence of absolute values on the wave number for different angles of crack inclination.

Figure 4. Dependence of absolute values of the dimensionless SIF on the wave number for increasing crack length.

The eccentricities of the both ellipses are the same $\varepsilon = 0.5$ and the main semiaxes are related as $r_1/r_0 = 0.5$. The load acting on the outer surface is determined by the formula. The center of the crack is located on the main axis of the outer ellipse. The results of numerical investigation of the SIF behavior in the frequency range and the transition to the resonance mode are shown in figures 3 and 4. The graphs in figure 3 are constructed in the case of a crack of fixed length inclined at an angle $\alpha$.

The length of crack is equal to one third of the distance $AB$ between the vertices of the ellipse and its center is located on its axis. The graphs correspond to the angles of the crack inclination: $1 - \alpha = 0^\circ$, $2 - \alpha = 30^\circ$, $3 - \alpha = 45^\circ$, $4 - \alpha = 60^\circ$, $5 - \alpha = 90^\circ$. An analysis of these graphs shows that, at frequencies less than the frequency of the first resonance when the angle of the crack inclination increases, the value of SIF decreases. The SIF attains maximum values in this frequency range in the case where the crack is located on the axis of the ellipse ($\alpha = 0^\circ$). At all values of the crack inclination angle, the resonance occurs at the frequency with $\kappa_0 \approx 3.8$.

The graphs in figure 4 are constructed for the angle $\alpha = 0^\circ$ and a crack of variable length. The length varied so that the left end $C$ of the crack was fixed, and the right end $D$ approached to the outer boundary of the cross-section. This approximation is attained when the parameter $\gamma = a/r_0$ changes from 0.0945 to 0.189. At the last value of this parameter, the crack touches the outer boundaries of the cross-section. Graphs 1–3 in figure 4 correspond to the next parameter values: $1 - \gamma = 0.0946$, $2 - \gamma = 0.1412$, $3 - \gamma = 0.188$. The absence of influence of the boundary on the SIF value $k^-$ at the internal end of the crack is established. The SIF value $k^+$ at the opposite cracks end increases when it approaches the boundary as the crack length increases. The resonance phenomena are observed in the frequency range $3 < \kappa_0 < 4$. 
Conclusions
The presence of cracks in an elastic hollow cylinder under harmonic load is accompanied both by the intensity of the dynamic stresses in a neighborhood of the defect and the resonant nature of their changes as a result of generation of a wave process in a bounded area. The most convenient for such an analysis is to consider the longitudinal shear strain describing the contribution of the crack SIFs. The SIF determination in the frequency range is based on the SIF relation to the function of the dynamic displacement jump at the crack, while this function is determined by separately solving the integral equations for the crack and satisfying the boundary conditions of dynamic load of the cylindrical body on the lateral surfaces. In this frequency range, one or two resonances can occur depending on the crack angle made with the body boundaries. The changes in the angle, as well as the crack approach to the outer surface, significantly influence the SIF values and the speed of their exit to the resonant mode in the low-frequency region.

References
[1] Nazarchuk Z T and Skalskij V R 2009 Acoustic Emission Diagnostics of Structural Elements vol 3 (Kyev: Naukova dumka) [in Ukrainian]
[2] Popov V G 1993 Comparison of the fields of displacements and stresses at the diffraction of shear elastic waves on different defects: a crack and a thin rigid inclusion Dinam. Sist. 12 35–41
[3] Ang D D and Knopoff L 1964 Diffraction of scalar elastic waves by a finite strip Proc. Natl. Sci. USA 51 (4) 593–8
[4] Loeber J F and Sih G C 1968 Diffraction of anisotropic waves by a finite crack J. Acoust. Soc. Amer. 44 (1) 90–8
[5] Zi G, Chen H, Xu J, and Belytschko T 2005 The extended finite element method for dynamic fractures Shock Vibrat. 12 (1) 9–23
[6] Mal A K 1970 Interaction of elastic waves with a Griffith crack Int. J. Engng Sci. 8 (9) 763–6
[7] Mykhas‘kiv V V and Zhbadynskyi I Ja 2007 Solving nonstationary for composite body with crack method integral equations Fiz.-Khim. Mekh. Mater. 43 (1) 33–42
[8] Mykhas‘kiv V, Zhbadynskyi I, and Zhang Ch 2010 Elastodynamic analysis of multiple crack problem in 3-D bi-materials by a BEM Int. J. Num. Meth. Biomed. Engng 26 (12) 1934–46
[9] Popov V G 2013 Harmonic vibrations of a half-space with a surface-breaking crack under conditions of out-of-plane deformation Mech. Solids 48 (2) 194–202
[10] Popov V G 1998 Interaction of elastic waves of longitudinal shear with radially situated cracks Prisl. Mekh. 34 (2) 60–66
[11] Nistor I, Pantale O, and Caperea S 2008 Numerical implementation of the extended finite element method for dynamic crack analysis Adv. Engng Software 39 (7) 573–87
[12] Hosseini-Tehrani P, Hosseini-Godarzi A, and Tavangar M 2005 Boundary element analysis of stress intensity factor $K_I$ in some two-dimensional dynamic thermoelastic problems Engng Anal. Bound. Elem. 29 (3) 232–40
[13] Chirino F and Domingues J 1989 Dynamic analysis of cracks using boundary element method Engng. Fract.Mech. 34 (5-6) 1051–61
[14] Bobylev L L and Dobrova Ju L 2003 Application of the boundary element method to the numerical analysis of forced vibrations of the elastic bodies of finite sizes with crack Visn. Kharkiv Nats. Univ. Ser. Mat. Model. Inform. Tehnol. Avtom. Sist. Upravl. 590 (1) 49–54
[15] Zhang Ch 2002 A 2D hypersingular time-domain traction BEM for transient elastodynamic crack analysis Wave Motion 35 (1) 17–40
[16] Poruchikov V E 1986 Methods of the Dynamic Theory of Elasticity (Moscow: Nauka) p 328
[17] Popov V G 1992 A study of the fields of displacements and stresses at the diffraction of shear elastic waves on a thin rigid exfoliated inclusion Izv. Ross. Akad. Nauk. Mekh. Tverdogo Tela No. 3 139–46
[18] Vekua N P 1970 Systems of singular integral equations and some boundary-value problems (Moscow: Nauka) p 252
[19] Belotserkovskii S M and Lifanov I K 1985 Numerical Methods in Singular Integral Equations and Their Application in Aerodynamics, Theory of Elasticity, and Electrodynamics (Moscow: Nauka) p 256
[20] Krylov V I 1967 Approximate Calculation of Integrals (Moscow: Nauka) p 500