Numerical simulation of vortex streets behind triangular objects

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Abstract

In a recent experimental study of the formation of vortex streets behind triangular objects in flowing soap films (Kim, J. Fluid Mech. 841, pp. 216-226, 2019), it was reported that an exotic structure of vortex street can emerge when $\delta/D$ is smaller than a critical value. Under the condition, the spatial arrangement of vortices is morphologically distinctive from the wake behind circular cylinders in that the two vortex rows are separated from each other by a thin layer of irrotational fluid. In this paper, we further investigate the vortex formation behind triangular objects numerically and theoretically. In our study covering a broader parameter space, we find that the experimental observation is reproduced in the simulation, indicating that the observed phenomenon is not an artifact of a specific system. We also model the flow around triangular objects as a double shear layer and solve for the linear dispersion relation using the Rayleigh’s equation. The theoretical analysis shows that the critical value of the ratio $\delta/D$ is approximately 0.3, which is in agreement with the experiment.

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I. INTRODUCTION

When a steady stream of fluid meets a solid boundary, vorticity is created and dispatched downstream. The vorticity is then organized into a spatio-temporal pattern which is named after the discoverer and pioneer of the subject von Kármán [1]. The von Kármán vortex street has been studied for many years in various context. In early years, the spatial structure of the vortex street has been investigated by considering vortices as singularities in an ideal fluid [1–3]. Those mathematical investigations assumed the vorticity a priori because the vorticity cannot be created in an ideal fluid (d’Alembert paradox), and the stability and dynamic behavior of the pre-existing vortices are analyzed.

However, from a practical point of view, the creation mechanism of the vorticity cannot be ignored. It is well known that many aspects of the vortex street, such as the shedding frequency [4–6] and drag coefficient [7, 8], depend on Reynolds number $Re$, and such dependencies are not rationalized only using the ideal fluid model. In reality, as $Re$ changes, the thickness of the boundary layer surrounding the vortex-creating object changes, and so does the intensity and the total amount of vorticity discharged into the downstream fluid. The change of the initial vorticity distribution affects the downstream vortex structure as well as various physical quantities that are associated with the flow. The importance of the role of the boundary layer in this topic can be further emphasized by pointing out that the $Re$-dependences are essentially originated from the scale separation, which is commonly observed in many singular perturbation problems [9] like the Navier-Stokes equation.

In this work, we aim to understand how the spatial structure of a vortex street is changed by the initial setting of the boundary layer. For this purpose, we present a more complex problem than the traditional cylinder wake by replacing a pole whose cross-section is triangular for the circular cylinder. Most studies of vortex street concern the wake behind a circular cylinder [10], but such setting is not necessarily attractive to study the role of the boundary layer. In two-dimension, the circular cylinder has a single length scale (diameter of the cylinder), and therefore the resultant wake structure is characterized by a single parameter, that is, $Re$. The use of the circular cylinder is favored for the simplicity, but it is, in fact, a limited setting because the thickness of the boundary layer, that scales with the square root of $Re$, cannot be controlled independently. We note that there exist investigations using non-circular objects [11–15], but the boundary layer was not controlled in these
studies.

Here, by using triangular objects, we vary the thickness of the boundary layer \( \delta \) independent of \( \text{Re} \). Under this setting, the variation of \( \delta \) is achieved by changing the aspect ratio \( r_a = H/D \), where \( H \) and \( D \) are the height and the base of the triangle. When \( H \) is longer, the flow interact with the boundary of the triangle for a longer distance, the boundary layer is developed thicker. Therefore, we have an additional control knob to vary \( \delta \) independent of \( D \), and two length scales of the problem can be varied independent of each other by controlling \( D \) and \( r_a \). In this manner, we tailor initial flow profiles, from which the development of the wake structure is calculated numerically.

In an experimental study using the same setting [16], it was suggested that the spatial structure of the vortex arrangement is determined by the dimensionless thickness of the boundary layer \( \delta' = \delta/D \). In the experiment using a flowing soap film channel, the vortex structure is found to be “conventional” when \( \delta' > 0.4 \), but an exotic structure called “separated rows” structure emerges when \( \delta' < 0.4 \). In the separated rows structure, two rows of the vortex street are physically separated by the thin layer of irrotational fluid, which prevents the mixing of vorticity and delays the breakdown of the vortex street. However, this vortex arrangement is unstable and persists only a finite downstream distance before it is transformed to the secondary wake structure, as similarly reported by a number of experimental [17–19] and computational [20–23] studies using circular cylinders.

This study is a sequel of the aforementioned experimental study [16] (denoted as K2019 hereafter) to discuss the unresolved issues using the numerical simulation. The first issue is the universality of the phenomenon. The flow on soap films are strongly influenced by the interfacial forces, e.g. Marangoni stress or frictional damping, and therefore the general applicability of experimental results on them have been occasionally doubted. In current investigation, we find that the appearance of the separated rows structure is not specific to the medium of experiment but is rather universal in two-dimensional Navier-Stokes systems. The second issue is the speciality of the criteria. In experiment, the transition between the conventional and the separated rows structures occured at \( \delta' = 0.4 \), but it had not been discussed why this particular ratio is special. We approximate the base flow profile around the triangles to a double shear layer and solve for the dispersion relation using Rayleigh’s stability equation. The solved dispersion relation is compared with that of a single shear layer profile, and we find that the fastest growing modes of two cases differs prominently.
only when $\delta' > 0.3$. We interpret the result that the boundary layers on either side of the object does not form a coupled oscillation when $\delta' < 0.3$. The result indicates that the change in the base flow profile actually affects the spatial structure of the downstream wake.

II. METHODOLOGY

For the direct numerical simulation, we solve for the free Navier-Stokes equation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{u}$$

(1)

and the continuity equation

$$\nabla \cdot \vec{u} = 0,$$

(2)

where $\vec{u}$ is the velocity field, $p$ is the pressure, $\rho$ is the density, and $\nu$ is the kinematic viscosity of the fluid. The fluid is assumed to be two-dimensional and incompressible.

The flow geometry is designed to match the experimental setting of K2019. As depicted in Fig. 1, a triangular object of base length $D$ and height $H$ is placed at the origin. The computational domain is set large enough to cover the entire flow structure, covering $x' \in [-20, 20]$ and $y' \in [-200, 200]$, where $x' = x/D$ and $y' = y/D$. Throughout this writing, the primed variables denote the dimensionless lengths that are normalized by $D$. The fluid flows in $-\hat{y}$ direction, from the top to the bottom. The flow enters the domain at $U_0 = 65 \text{ cm/s}$, which is set constant for all cases, and exits the domain through the bottom boundary. The fluid properties are taken from those of soapy water, slightly different from those of water, having the density $\rho = 1 \text{ g/cm}^3$ and the kinematic viscosity $\nu = 0.013 \text{ cm}^2/\text{s}$. We vary $D$ from 0.02 cm to 0.1 cm, and $H$ is calculated by the aspect ratio, i.e. $H = r_a D$, which varies from from 0.1 to 3. The Reynolds number is defined as $Re = UD/\nu$ and ranges from 100 to 700.

For the configured geometry, Eqs. (1) and (2) are solved for the laminar vortex street. The equations are solved by finite element method using a commercial software (COMSOL Multiphysics 5.3). We use triangular meshes that are created by using the same software. The size of a mesh element is approximately 100 $\mu$m far from the object and is refined as small as 10 $\mu$m near the object. A brief overview of the meshes are shown in Fig. 1. The schematic diagram in Fig. 1, including the meshes in the background, is not in scale.
FIG. 1. Computational domain and meshes. A triangle of base $D$ and height $H = r_a D$, where $D$ ranges from 0.02 cm to 0.1 cm and $0.1 < r_a < 3$, is placed at the origin of the coordinate system. The fluid flows in $-\hat{y}$ direction, from the top to the bottom, at 65 cm/s. The computational domain is $x \in [-20D, 20D]$ and $y \in [-200D, 20D]$. The figure is not in scale.

III. RESULTS AND DISCUSSION

A. Formation of the boundary layer

Using the vorticity maps acquired from the numerical simulation, we measure the thickness of the boundary layer $\delta$ with respect to $r_a$ and Re. The thickness of the boundary layer is the key control parameter of the hypothesis of the current study, and therefore its quantification is of great importance for the ensuing discussion. In K2019, the thickness of the boundary layer $\delta$ is assumed to be proportional to the square root of the time that takes for a flow of speed $U$ to pass a length of the hypotenuse $L$ of the triangle. This assumption renders the relation for the non-dimensional thickness of the boundary layer $\delta' \equiv \delta/D$, 

$$
\delta' = \alpha \sqrt{\nu L'/U'/D} = \alpha \sqrt{L'} \text{Re}^{-1/2},
$$

where $\alpha$ is a proportionality coefficient and $L' = L/D = \sqrt{r_a^2 + 0.25}$. In soap film flows, the relation $\delta \sim \sqrt{\nu t}$ was observed in experiments with $\alpha \simeq 6$ using a thin plate, but the full verification of Eq. (3) for the hypotenuse of triangles was not present in K2019. Since
the derivation of Eq. (3) requires an unrealistic assumption that the flow speed does not change over the course along the hypotenuse, it is desirable to check the validity under more realistic setting.

Our investigation shows that Eq. (3) is valid for \( r_a > 0.5 \). In Fig. 2a, the measurement of \( \delta' \) is plotted with respect to \( r_a \) at \( \text{Re} = 300 \). The presented measurement shows that it monotonically increases for \( r_a \gtrsim 0.5 \) and follows Eq. (3) well, which is shown by the solid curve in the figure, but it deviates from Eq. (3) for \( r_a < 0.5 \). It is inferred that when \( r_a \) is small, the obstruction effect is substantial and decreases the flow speed significantly, and therefore the interaction time between the flow and the boundary is increased to form the thicker boundary layer.

For \( r_a \gtrsim 0.5 \), we determine the proportionality constant \( \alpha = 3.3 \). This value of \( \alpha \) is smaller than 6 for the soap film flows, and the primary source of the discrepancy is the surface viscosity of the soap films. In soap films, the interfacial action adds the surface viscosity to the bulk viscosity, and the total viscosity of the fluid is expressed as the Trapeznikov approximation \( \nu_{\text{tot}} = \nu_b + 2\nu_s/h \), where \( \nu_b \) is the bulk viscosity, \( \nu_s \) is the surface viscosity and \( h \) is the thickness of the film [24]. Using the difference in the value of \( \alpha \), we estimate the value of the kinematic viscosity of the soap film system used in K2019 is approximately 3 times greater than the bulk viscosity. Considering that \( h \approx 3 \mu \text{m} \) of the considered system, it is implied that the surface dynamic viscosity is approximately \( \nu_s \approx 4.5 \times 10^{-6} \text{ g/s} \). This estimated value of \( \nu_s \) is in accordance with other measurement, lying between \( 1.2 \times 10^{-6} \text{ g/s} \) by Vivek and Weeks [25] and \( 10 \times 10^{-6} \text{ g/s} \) by Couder and Basdevant [26].

We also find that Eq. (3) is valid for extensive range of \( \text{Re} > 250 \). In Fig. 2b, \( \delta'/\sqrt{L} \) is plotted with respect to \( \text{Re} \) in the log-log coordinates. The plotted data are from \( 0.9 < r_a < 1.5 \), and those data all collapse into Eq. (3) for \( \text{Re} > 250 \). The power law scaling with exponent -0.5 further confirms that Eq. (3) is a fair approximation that is applicable for most cases of our study.

B. Structure of vortex streets

From the numerical simulation, we find that the emergence of the exotic structure of the vortex streets behind triangular objects is a general phenomenon in two-dimensional Navier-Stokes systems. In our numerical investigation, the key features of the real flow are
FIG. 2. The thickness of the boundary layer. (a) The measurement of $\delta'$ is plotted with respect to $r_a$ for $Re = 300$. (b) The $Re^{-1/2}$ dependence is valid for $Re > 250$. The solid curves in (a) and (b) represent Eq. (3) with $\alpha = 3.3$. The measurement error mainly comes from the frame-by-frame variation of the unsteady flow.

reproduced at a considerable degree of the similarity. These features are: (i) when $Re$ is sufficiently large, the vortex rows are separated by a thin layer of irrotational fluid, and the resultant flow pattern becomes the exotic structure which is referred as the separated rows structure, and (ii) when $r_a$ is higher, or the triangle is more acute, higher $Re$ is required for having the separated rows structure. These similarities are the clear indication of that the emergence of the separated rows structure is a generic phenomenon in two-dimensional Navier-Stokes system and is not specific to the medium of experiment.

In Fig. 3, we present the vorticity maps acquired from the numerical simulation both at $Re = 300$ but using different $r_a$. In the first image in Fig. 3a, the presented vortex street is produced using $r_a = 0.9$, which is nearly equilateral triangle. Here, the two rows of the vortex street are separated by a thin layer of the irrotational fluid soon after the vorticity is shed from the object ($\sim 5D$). The separation layer spans over a short downstream distance, approximately $10D$, and then the vortex street is reorganized into the secondary wake. In comparison, we present the vortex street produced using $r_a = 1.5$ in the second image 3b. While the overall topology of the vortex structure is very similar to the one in Fig. 3a, the thin separation layer of irrotational fluid appears around $10D$ from the object, and it persists a lengthy distance of $\sim 60D$. Because these two vorticity maps are calculated using the
same flow speed and the base length of the triangle, the morphological difference is solely attributed to the difference in the height of the triangle.

For quantitative measurement of the changes in the vortex structures, we adopt the measurements of two length scales $L_{CM}$ and $L_{SR}$ from K2019. The length of the conventional structure in the first wake zone, $L_{CM}$, is defined as the distance between two points, one being where the initial expansion of the vortex street is complete (point P in Fig. 3b) and the other being where two vortex rows are separated by a thin layer of irrotational fluid (point Q in Fig. 3b). And the length of the thin layer of irrotational fluid, $L_{SR}$, is defined as the distance between Q and R (also marked in the Fig. 3b), where the secondary wake region begins. These points may look ambiguously defined when they are displayed in a snapshot of the simulated flow, but the ambiguity can be removed by taking a time average of the unsteady flow.

In Fig. 4, we present the measurement of $L'_{CM}$ and $L'_{SR}$, which are $L_{CM}/D$ and $L_{SR}/D$ respectively, with respect to Re. Our measurement is consistent to K2019 in that both $L'_{CM}$ and $L'_{SR}$ decrease and approach each respective asymptotic value as Re increases. We find that for the simulated flows $L'_{CM} \to 0$ and $L'_{SR} \to 10$. We note that for the real flow $L'_{CM} \to 0$ and $L'_{SR} \to 20$ was observed.

Our result is in accordance with the proposition of K2019 that the broadening mechanism of the vortex street depends on the structure of the vortex street and vice versa. Based on
FIG. 4. Spatial extent of the vortex structures. For \( r_a=0.5, 0.9, \) and 1.2, (a) \( L'_{CM} \) and (b) \( L'_{SR} \) are plotted with respect to \( Re \). The curves in (a) are the calculations of Eq. (4) using \( f = 12, 10 \) and 9 and \( g = 35, 650 \) and 2000 for \( r_a = 0.5, 0.9 \) and 1.2, respectively.

The proposition, it was suggested that \( L'_{CM} \) should follow a scaling relation

\[
L'_{CM} = f Re^{-1/2} (1 + g Re^{-1}) ,
\]

where \( f \) and \( g \) are parameters which depend on \( r_a \). In Fig. 4a, the calculation of Eq. (4) are shown using \( f = 12, 10 \) and 9 and \( g = 35, 650 \) and 2000, for \( r_a=0.5, 0.9 \) and 1.2, respectively. These parameters are determined by using the least square method while the ratios between the values of \( f \) are fixed because \( f \propto (r_a^2 + 0.25)^{-1/4} \). While the noisy nature of the measured quantity prevents us to make a concrete and full conclusion, the result in Fig. 4 can be considered as another case that supports the scaling relation in Eq. (4). As a corollary, the data support that the broadening mechanism is coupled with the spatial arrangement.

C. Stability criterion

Many features of the experimental observation are reproduced in current numerical investigation. Based on the observational parallel in previous sections, we seek to verify the stability criterion. In K2019, the suggested criterion for the separated rows structure was \( \delta' < 0.4 \), i.e. the boundary layers on both side of the triangle are thinner than 40% of their separation distance.
FIG. 5. The map of the vortex structures behind triangular objects. When \( \text{Re} < \text{Re}_{c2} \), the vortex street is conventional. When \( \text{Re} > \text{Re}_{c2} \), the exotic “separated rows” structure appears. We find that \( \text{Re}_{c2} \) is an increasing function with respect to \( r_a \). The solid curve shows the calculation of Eq. (5).

In current work, a similar but slightly different criterion \( \delta' < 0.25 \) is discovered. In Fig. 5, the structures of the vortex streets are presented on \( \text{Re} \) and \( r_a \) plane using the simulated flows. We first assume that \( \delta' = \delta'_c \) is a criterion for the emergence of the exotic vortex arrangement, and solved for \( r_a \) in terms of \( \text{Re} \) using Eq. (3). Then the critical ratio \( \delta'_c = 0.25 \) is estimated to best fit the measurements. Simply, the following equation sets the boundary between two regimes,

\[
\text{Re}_{c2} = \frac{\alpha L'}{\delta'_c} \approx \left( \frac{3.3}{0.25} \right)^2 L'.
\]  

In Fig. 5, Eq. (5) is shown with a solid curve.

D. Linear stability analysis

In this section, we attempt to investigate the physical meaning of \( c \approx 0.3 \) by performing a linear stability analysis. We employ a simplified model in which the flow around the triangular objects is approximated to a double shear layer, as depicted in Fig. 6. In this model, the flow profile is expressed by \( D \) and \( \delta \), and we investigate how the dispersion relation changes with respect to their ratio. Physically, we expect that when \( D \sim \delta \), the two
FIG. 6. A double shear layer model. We model the base flow profile behind a triangular object as a double shear layer. When $H$ is longer for the same $D$, the boundary layer is thicker and forms a gradual shear profile.

Shear layers are coupled and oscillates together. However, when $D \gg \delta$, they are too apart to produce a coupled oscillation, but each shear layer individually produces a Kelvin-Stuart cat’s eye pattern. We speculate that a separated rows structure may appear when $D$ is large enough than $\delta$ to produce each individual instability on each side but those oscillations are still bound together as a vortex street.

The assumptions for this modeling are the followings. First, the vorticity is evenly distributed inside the boundary layer. Under this assumption, the shear rate in the boundary layer is approximated $U_0/\delta$, where $U_0$ is the mean speed of the flow outside the boundary layer. Second, the vortex shedding occurs at the trailing edge of the triangle. Using this assumption, we focus on $u(x)$ at $y = 0$, where the base of the triangle is located. Then, our base flow profile is formally

$$u(x) = \begin{cases} 
-U_0 & \text{for } -\infty < x < -\frac{D}{2} - \delta \\
\frac{U_0}{\delta} (x + \frac{D}{2}) & \text{for } -\frac{D}{2} - \delta < x < -\frac{D}{2} \\
0 & \text{for } -\frac{D}{2} < x < \frac{D}{2} \\
-\frac{U_0}{\delta} (x - \frac{D}{2}) & \text{for } \frac{D}{2} < x < \frac{D}{2} + \delta \\
-U_0 & \text{for } \frac{D}{2} + \delta < x < \infty.
\end{cases}$$

(6)

Without loss of generality, we take $U_0 = 1$ and non-dimensionalize $x$ with respect to $\delta$. We
note that the current non-dimensionalization scheme uses $\delta$ because we later compare the result with the dispersion relation of the single shear layer case, in which $D$ is not defined. In dimensionless form, Eq. (6) becomes

$$u(z) = \begin{cases} 
-1 & \text{for } -\infty < z < -\frac{1}{2\delta'} - 1 \\
 z + \frac{1}{2\delta'} & \text{for } -\frac{1}{2\delta'} - 1 < z < -\frac{1}{2\delta'} \\
 0 & \text{for } -\frac{1}{2\delta'} < z < \frac{1}{2\delta'} \\
 -z + \frac{1}{2\delta'} & \text{for } \frac{1}{2\delta'} < z < \frac{1}{2\delta'} + 1 \\
 -1 & \text{for } \frac{1}{2\delta'} + 1 < z < \infty, 
\end{cases}$$

(7)

where $z = x/\delta$.

To solve for the dispersion relation, we use Eq. (7) and the Rayleigh’s stability equation

$$0 = (U - c) \left( \frac{\partial^2}{\partial x^2} - \alpha^2 \right) \phi - U''\phi,$$  

(8)

where the non-dimensional wavenumber $\alpha = k\delta$ and $k$ is the dimensional wave number. Here, $\phi$ is the stream function of the perturbation, i.e. $\vec{u}' = \nabla \times (\psi \hat{k})$ and $\psi = \phi(x) \exp[i\alpha(y-ct)]$. Therefore $u = i\alpha\phi$ and $v = -\partial\phi/\partial x$.

Using the general solution of Eq. (8), $\phi = \{e^{i\alpha z'}, e^{-i\alpha z'}\}$, we propose the trial solution

$$\phi(z) = \begin{cases} 
 A_3 e^{i\alpha z} + B_3 e^{-i\alpha z} & \text{for } z < \frac{1}{2\delta'} \\
 A_2 e^{i\alpha z} + B_2 e^{-i\alpha z} & \text{for } \frac{1}{2\delta'} < z < \frac{1}{2\delta'} + 1 \\
 B_1 e^{-i\alpha z} & \text{for } z > \frac{1}{2\delta'} + 1. 
\end{cases}$$

(9)

The five unknown coefficients, $A_1$, $A_2$, $B_1$, $B_2$, and $B_3$ are determined by applying boundary conditions. The first boundary condition is that the $\phi$ is a symmetric function of $z$ so that the resultant oscillation of the flow is sinuous, which implies $A_3 = B_3$. The second boundary condition is that the pressure is continuous at the interfaces, i.e. $\Delta[(u - c)\phi' - u'\phi] = 0$ across $z = 1/2\delta'$ and $z = 1/2\delta' + 1$. The third condition is that the normal velocity is continuous, i.e. $\Delta[\phi/(u - c)] = 0$ across $z = 1/2\delta'$ and $z = 1/2\delta' + 1$.

By using the boundary conditions and following the arithmetics, a secular equation $Ac^2 + Bc + C = 0$ is derived, and the dispersion relation of the double shear profile is derived in the following form

$$c = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \quad \text{(10)}$$
where
\[ A = \alpha^2 e^\alpha (1 + \tanh \alpha/2\delta'), \]
\[ B = \tanh \alpha/2\delta'(\alpha^2 e^\alpha - \alpha \sinh \alpha) + \alpha^2 e^\alpha + \alpha \sinh \alpha, \]
\[ C = \alpha e^\alpha - \sinh \alpha. \]

We derive the dispersion relation for double and single shear layers using that \( \psi \sim \exp[-i\alpha ct] \) and Eq. (10). For the double layer, it is required that \( B^2 - 4AC < 0 \) for an unstable mode exists, because \( \psi \sim \exp[-i\alpha(c_r + ic_i)t] = \exp[-i(\alpha c_r)t] \exp[\alpha c_i t], \) where \( c_r = -B/2A \) and \( c_i = \sqrt{4AC - B^2/2A}. \) For the double layer, the growth rate \( \omega_d \) of the perturbation is
\[ \omega_d = \alpha \frac{\sqrt{4AC - B^2}}{2A}. \]

For the single layer, the solution is readily available in literature [27]. The growth rate \( \omega_s \) is
\[ \omega_s = \frac{1}{2} \sqrt{e^{-2\alpha} - (1 - \alpha)^2}. \]

The maximum growth rate of Eq. (15) is found to be \( \omega_s^{(\text{max})} \approx 0.2 \) at the fastest growing mode \( \alpha_{\text{max}} \approx 0.79. \) We note that Eq. (15) can be acquired also by taking a \( \delta' \to 0 \) limit of Eq. (14).

We numerically calculate Eq. (14) and find the fastest growing wavelength \( \alpha_{\text{max}} \) for each \( \delta' \) between 0 and 1. As shown in Fig. 7, the double shear layer has its own characteristic wavelength when \( \delta' > 0.3. \) However, as \( \delta' \) decreases, it converges to \( \alpha_{\text{max}} \approx 0.79 \) and indifferent from the fastest growing wavelength of the single shear layer profile. The analysis indicates that the criterion in Eq. (5) is originated from the hydrodynamic coupling of the two boundary layers on each side of the object.

This analysis also explains why the separated rows structure is unstable or meta-stable at the best. In Fig. 7, a few lines that representing \( \alpha/\delta' = kD = n \) are shown. As \( \delta' \) decreases, the curve of the fastest growing mode passes the line of higher \( n, \) indicating that the wavelength \( \ell = \frac{2\pi D}{n} \) of the vortex street decreases. Because the width \( w \) of the vortex street is approximately proportional to \( D, \) i.e. \( w \approx D, \) the width-to-wavelength ratio is derived as \( w/\ell \approx n/(2\pi). \) We find that \( w/\ell \) increases as \( \delta' \) decreases, and when \( \delta' \approx 0.2, \) a vortex street with an extremely high \( w/\ell \approx 0.8 \) can be formed. However, such an exotic vortex arrangement, which we call the separated rows structure, is not stable because
FIG. 7. The fastest growing mode of the double shear layer profile in Eq. (6). When $\delta' \lesssim 0.3$, the fastest growing wavelength is approximately 0.8 and is similar to the single shear layer case, but it deviates from 0.8 as $\delta'$ increases above 0.3. As $\delta'$ decreases, $kD$ increases, indicating that the wavelength of the vortex street decreases. This makes the vortex arrangement hydrodynamically unstable, and therefore the vortex street breaks down downstream.

$w/\ell = 0.28$ is desired for the stability [1]. Therefore, the vortex arrangement breaks down by itself after a finite lifetime.

IV. CONCLUSION

We conducted a numerical and theoretical analysis of the vortex arrangement behind triangular objects. By using triangular objects, the thickness of the boundary layer and their separation distance are independently controlled.

The first half of this paper was dedicated to reproduce the experimental result of K2019 [16] in computational domain. The numerical simulation reproduced the experimental observation up to a considerable degree of the similarity. This reproductivity indicates that the emergence of exotic vortex arrangement that is reported in K2019 is not limited to a specific medium of experiment but rather a general phenomenon in two-dimensional fluid. While qualitative features are well reproduced, some of quantitative measurements do not fully agree. We speculate the difference is due to the inaccurate estimation of the film viscosity of K2019.
In the second half of this paper, we investigated why the ratio between the thickness of the boundary layers and their separation distance impacts the vortex arrangement. We simplified the flow around triangular object to a double shear layer profile. Using the Rayleigh’s equation, the dispersion relation and the fastest growing modes of the model profile were derived. The result indicates that the two shear layers oscillate together as they are coupled when $\delta' > 0.3$. However, as $\delta' < 0.3$, two shear layers oscillate individually, producing the wavelength that is same with the shear layer. We also show that when $\delta' < 0.3$, the vortex arrangement is contracted in the longitudinal direction, as consistent to the experimental observation in K2019. The contracted arrangement is not perpetually stable but decays due to its inherent instability.

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