THE OPTIMAL STABILIZATION OF ORBITAL MOTION IN A NEIGHBORHOOD OF COLLINEAR LIBRATION POINT

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Abstract. In this paper we consider the special problem of stabilization of controllable orbital motion in a neighborhood of collinear libration point $L_2$ of Sun-Earth system. The modification of circular three-body problem – nonlinear Hill’s equations, which describe orbital motion in a neighborhood of libration point is used as a mathematical model. Also, we used the linearized equations of motion. We investigate the problem of spacecraft arrival on the unstable invariant manifold. When a spacecraft reaches this manifold, it does not leave the neighborhood of $L_2$ by long time. The distance to the unstable invariant manifold is described by a special function of phase variables, so-called “hazard function”. The control action directed along Sun-Earth line.

1. The problem formulation and equations of motion. We control the orbital motion in a neighborhood of collinear libration point for the purpose of stabilization. In this case, different versions of the problem formulation arise. Usually [2], control keeps a spacecraft or some celestial body close to the prescribed halo-orbit. Perhaps, however, the expansion of such a formulation with the aim of achieving an invariant variety containing halo-orbits. This formulation gives some advantage in respect to energy costs. In practice, this gain may not be large, however, when controllable body has a large mass, the proposed weakening of the boundary conditions can be significant.

In the study of orbital motion in a neighborhood of collinear libration points of the Earth-Sun $L_1$ and $L_2$ is comfortably to use Hill’s equations. This mathematical model is non-linear approximation of equations of circular three-body problem and it adequately describes the motion of a spacecraft in a neighborhood of collinear libration points. The equations in the rotating coordinate system with control, that acting on the Sun-Earth line, is of the form [5, 6, 8]

\[
\begin{align*}
\dot{x}_1 &= y_1 + x_2, \quad \dot{y}_1 = -\frac{3x_1}{(x_1^4 + x_2^4 + x_3^4)^{3/2}} + 2x_1 + y_2 + u; \\
\dot{x}_2 &= y_2 - x_1, \quad \dot{y}_2 = -\frac{3x_2}{(x_1^4 + x_2^4 + x_3^4)^{3/2}} - x_2 - y_1; \\
\dot{x}_3 &= y_3, \quad \dot{y}_3 = -\frac{3x_3}{(x_1^4 + x_2^4 + x_3^4)^{3/2}} - x_3,
\end{align*}
\]

where $x = (x_1, x_2, x_3)$ – spacecraft positions, $y = (y_1, y_2, y_3)$ – impulses, $u$ – control.
The unit of distance is nearly 0.01 AU (the distance from center of Earth to \(L_2\)), 2\(\pi\) time units is one year. Another variant of Hills equations is published in [4]. Libration point \(L_2\) has coordinates
\[
(2)
\]
\[
x^* = (-1, 0, 0), \quad y^* = (0, -1, 0).
\]

The linearized system of equations (1) (at \(u = 0\)) is of the form
\[
(3)
\]
\[
\dot{x}_1 = x_2 + y_1, \quad \dot{y}_1 = 8(x_1 + 1) + (y_2 + 1);
\]
\[
\dot{x}_2 = -x_1 + y_2, \quad \dot{y}_2 = -4x_2 - y_1;
\]
\[
\dot{x}_3 = y_3, \quad \dot{y}_3 = -4x_3.
\]

The matrix of system (3) has the following set of eigenvalues
\[
(5)
\]
\[
\lambda_1 = \sqrt{1 + 2\sqrt{7}}, \quad \lambda_2 = -\sqrt{1 + 2\sqrt{7}}, \quad \lambda_3 = i\sqrt{2\sqrt{7} - 1}
\]
\[
\lambda_4 = -i\sqrt{2\sqrt{7} - 1}, \quad \lambda_5 = 2i, \quad \lambda_6 = -2i.
\]
The eigenvalue \(\lambda_1\) is a real positive number, which leads to instability of the uncontrolled motion in a neighborhood of collinear libration point. The real parts of all other eigenvalues are either negative or equal to zero. That is why the goal of control is to achieve the invariant manifold corresponding to the eigenvalue \(\lambda_1\), which ensures a long stay of a spacecraft in a neighborhood of collinear libration point.

In linear case the space variables \((x_3, y_3)\) separated from the variables \((x_1, x_2, y_1, y_2)\), which describe the motion in ecliptic plane. These variables describe the vibrations of spacecraft along the normal of ecliptic plane with the period of about six months, and do not participate in solving of problem of stabilization in linear case.

The linearized system for the plane variables is of the form
\[
(4)
\]
\[
\dot{z} = Az + bu, \quad A = \begin{pmatrix}
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
8 & 0 & 0 & 1 \\
0 & -4 & -1 & 0
\end{pmatrix}, \quad b = \begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix},
\]
where \((z_1 = x_1, z_2 = x_2, z_3 = y_1, z_4 = y_2)\). The system (4) is the system in deviations [9] for (3) in ecliptic plane. In [7] we studied the problem of stability of linear system (4) with help of so called "hazard function" \(l_1(z)\), which is defined as
\[
(5)
\]
\[
l_1(z) \overset{\Delta}{=} d_1 z, \quad d_1 A = \lambda_1 d_1, \quad (d_1, b) = 1,
\]
therefore
\[
(6)
\]
\[
\dot{l}_1 = \lambda_1 l_1 \Rightarrow l_1(t) = l_1(t_0)e^{\lambda_1(t-t_0)}.
\]
The last expression shows that spacecraft leaves a neighborhood of libration points, when hazard function increases. Hazard function \(l_1\) shows how quickly a spacecraft can leave a neighborhood of libration point, when control action is not operate. On the other hand, it is desirable to make the control costs low enough. This leads to the idea to consider the optimal control problem with functional
\[
(6)
\]
\[
J_1(u(\cdot)) = \int_{t_0}^{\infty} [s_1^2 l_1^2(x, y) + u^2] dt \rightarrow \min.
\]
where \(s_1\) – weight coefficient.
2. **The construction of optimal control.** We solve the problem of getting into a sufficiently small neighborhood of unstable invariant manifolds \[ l_1(z) = 0, \]
when functional \[ \text{(6)} \] is minimized. As well as in \[ \text{(7)} \] we apply sufficient conditions of optimality for problems \[ \text{(4)-(6)}. \]

In general, for the type of systems

\[
\dot{x} = f(z, u), \quad t \in [t_0, T], \quad z(t_0) = z_0, \quad u(t) \in U,
\]

with quality criterion

\[
J(u(\cdot)) = \int_{t_0}^{T} f_0(z, u) dt \rightarrow \min_{u(\cdot) \in \tilde{U}(t_0, T)}
\]

the optimal control may be sought on the basis of sufficient conditions of optimality by the construction of Bellman function as the solution of Bellman equation

\[
\min_{u \in \tilde{U}} \left( \frac{\partial W}{\partial z} f(z, u) + f_0(z, u) \right) = 0.
\]

For the systems of type \[ \text{(4)} \] with the functional \[ \text{(6)}, \] Bellman function is a quadratic form with constant matrix, that must be determined \[ \text{(1)}. \] Bellman function \( W \) can be sought in the form

\[
W = W(l_1(z)) = s_0 l_1^2(z).
\]

Really, in this case

\[
\frac{\partial W}{\partial z} f(z, u) = 2s_0 l_1(z) \frac{\partial l_1}{\partial z} (Az + bu) = 2s_0 \lambda_1 l_1^2(z) + 2s_0 l_1(z) u.
\]

Finally, Bellman equation takes the form

\[
\min_{u \in \tilde{U}} \left( 2s_0 \lambda_1 l_1^2(z) + 2s_0 l_1(z) u + s_1^2 l_1^2(z) + u^2 \right) = 0, \quad (7)
\]

where the optimal control has the form

\[
u^* = -s_0 l_1(z). \quad (8)
\]

Now we define coefficient \( s_0 \) and interpret the result. Such interpretation is necessary, because formulation of control problem in this research is differed from standard one, when the Bellman function is sought with a positive definite matrix \[ \text{(1)}. \]

The control law \[ \text{(8)} \] is defined by the existence of a smooth extremum on control \[ \text{(7)}. \] Using \[ \text{(7)} \] and \[ \text{(8)}, \] we find a relation between the coefficients \( s_0 \) and \( s_1 \)

\[
s_0 = \lambda_1 \pm \sqrt{\lambda_1^2 + s_1^2}.
\]

Further, we need to determine the range of values of coefficient \( s_0 \), in which the problem of optimization \[ \text{(1)-(6)} \] is solved. This formulation differs from the standard one for Bellman function \( W(z) = s_0 l_1^2(z) \), that may appeal to zero on unstable invariant manifold and the problem of stability is solved. The function \( W \) has some properties of Lyapunov function. Generally Lyapunov function must be not negative one and its total derivative must be not positive. The function \( W(z) \) takes the zero value on an unstable invariant manifold \( l_1(z) = 0 \), but achievement of this manifold is our purpose. Here we consider its total derivative on time
\[ \dot{W}(z) = 2s_0l_1(z)\dot{l}_1(z) = 2s_0l_1(z)d_1(Az + bu) = 2s_0l_2^2(\lambda_1 - s_0). \]

It is necessary to satisfy the condition \( \dot{W}(x) < 0 \), if \( l_1 \neq 0 \). Also \( s_0 \) can not be negative, otherwise the function \( W(z) = s_0l_2^2(z) \) is less than zero. Accordingly, area value for \( s_0 \) is of the form

\[ s_0 > \lambda_1. \] (9)

Therefore, everywhere except the manifold \( l_1(z) = 0 \) and taking into account the inequality (9) we can assume that the function \( W(z) \) has the properties of Lyapunov function. The meaning of the last inequality can be seen from the behavior of hazard function on a control trajectory. Really, by the expression (5)

\[ \dot{l}_1 = (\lambda_1 - s_0)l_1, \]

and the inequality (9) means that hazard function decreases exponentially on a control trajectory. The eigenvalues of controllable system does not have positive real parts, which implies Lyapunov stability of controllable motion with control (8).

3. The numerical simulation of motion. Figure 1 shows the results of numerical simulation of controllable motion in a neighborhood of libration point \( L_2 \). The motion is described by system (1) with control (8) at \( s_0 = 5 \). The initial data for the controllable motion is of the form

\[ x_1(0) = -1.05, \ x_2(0) = 0, \ x_3(0) = 0.05, \ y_1(0) = 0, \ y_2(0) = -0.95, \ y_3(0) = 0. \]

The time interval of simulation is about two years.

Figure 2 shows the results of numerical simulation of controllable motion in a neighborhood of libration point \( L_1 \). The control law and interval of time are the same as for the simulation of motion in a neighborhood of \( L_2 \). The initial data for motion on Figure 2 is of the form

\[ x_1(0) = 1.03, \ x_2(0) = 0, \ x_3(0) = 0.04, \ y_1(0) = 0, \ y_2(0) = 0.94, \ y_3(0) = 0. \]

Figures 1 and 2 show how motion under the action of control becomes near periodic. The distance to collinear libration points remains limited to the order of several tens of thousands of kilometers. Libration points \( L_1 \) and \( L_2 \) are at the different distance from the center of Earth in circular three-body problem. In Hill's
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model these distances are same. Therefore, the properties of orbital controllable motion in neighborhoods of $L_1$ and $L_2$ are identical.

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