Abstract

A Valued Constraint Satisfaction Problem (VCSP) provides a common framework that can express a wide range of discrete optimization problems. A VCSP instance is given by a finite set of variables, a finite domain of labels, and an objective function to be minimized. This function is represented as a sum of terms where each term depends on a subset of the variables. To obtain different classes of optimization problems, one can restrict all terms to come from a fixed set $\Gamma$ of cost functions, called a language.

Recent breakthrough results have established a complete complexity classification of such classes with respect to language $\Gamma$: if all cost functions in $\Gamma$ satisfy a certain algebraic condition then all $\Gamma$-instances can be solved in polynomial time, otherwise the problem is NP-hard. Unfortunately, testing this condition for a given language $\Gamma$ is known to be NP-hard. We thus study exponential algorithms for this meta-problem. We show that the tractability condition of a finite-valued language $\Gamma$ can be tested in $O(\sqrt[3]{3}^{|D|} \cdot \text{poly}(\text{size}(\Gamma)))$ time, where $D$ is the domain of $\Gamma$ and $\text{poly}(\cdot)$ is some fixed polynomial. We also obtain a matching lower bound under the Strong Exponential Time Hypothesis (SETH). More precisely, we prove that for any constant $\delta < 1$ there is no $O(\sqrt[3]{3}^{\delta |D|})$ algorithm, assuming that SETH holds.

1 Introduction

Minimizing functions of discrete variables represented as a sum of low-order terms is a ubiquitous problem occurring in many real-world applications. Understanding complexity of different classes of such optimization problems is thus an important task. In a prominent VCSP framework (which stands for Valued Constraint Satisfaction Problem) a class is parameterized by a set $\Gamma$ of cost functions of the form $f : D^n \to \mathbb{Q} \cup \{\infty\}$ that are allowed to appear as terms in the objective. Set $\Gamma$ is usually called a language.

Different types of languages give rise to many interesting classes. A widely studied type is crisp languages $\Gamma$, in which all functions $f$ are $\{0, \infty\}$-valued. They correspond to Constraint Satisfaction Problems (CSPs), whose goal is to decide whether a given instance has a feasible solution. Feder and Vardi conjectured in [11] that there exists a dichotomy for CSPs, i.e. every crisp language $\Gamma$ is either tractable or NP-hard. This conjecture was refined by Bulatov, Krokhin and Jeavons [7], who proposed a specific algebraic condition that should separate tractable languages from NP-hard ones. The conjecture was verified for many special cases [28, 5, 31, 2, 8], and was finally proved in full generality by Bulatov [6] and Zhuk [32].

At the opposite end of the VCSP spectrum are the finite-valued CSPs, in which functions do not take infinite values. In such VCSPs, the feasibility aspect is trivial, and one has to deal only with the optimization issue. One polynomial-time algorithm that solves tractable finite-valued CSPs is based on the so-called basic linear programming (BLP) relaxation, and its applicability (also for the general-valued case) was fully characterized by Kolmogorov, Thapper and Živný [22]. The complexity of finite-valued CSPs was completely classified by Thapper and Živný [30], where it is shown that all finite-valued CSPs not solvable by BLP are NP-hard.
The dichotomy is also known to hold for general-valued CSPs, i.e. when cost functions in $\Gamma$ are allowed to take arbitrary values in $\mathbb{Q} \cup \{\infty\}$. First, Kozik and Ochremiak showed [23] that languages that do not satisfy a certain algebraic condition are NP-hard. Kolmogorov, Krokhin and Rolínek then proved [20] that all other languages are tractable, assuming the (now established) dichotomy for crisp languages conjectured in [7].

In this paper languages $\Gamma$ that satisfy the condition in [23] are called solvable. Since optimization problems encountered in practice often come without any guarantees, it is natural to ask what is the complexity of checking solvability of a given language $\Gamma$. We envisage that an efficient algorithm for this problem could help in theoretical investigations, and could also facilitate designing optimization approaches for tackling specific tasks.

Checking solvability of a given language is known as a meta-problem or a meta-question in the literature. Note it can be solved in polynomial time for languages on a fixed domain $D$ (since the solvability condition can be expressed by a linear program with $O(|D|^{|D|d})$ variables and polynomial number of constraints, where $m = 2$ if the language is finite-valued and $m = 4$ otherwise). This naive solution, however, becomes very inefficient if $D$ is a part of the input (which is what we assume in this paper).

The meta-problem above was studied by Thapper and Živný for finite-valued languages [30], and by Chen and Larose for crisp languages [10]. In both cases it was shown to be NP-complete. We therefore focus on exponential-time algorithms. We obtain the following results for the problem of checking solvability of a given finite-valued language $\Gamma$:

- An algorithm with complexity $O(\sqrt[3]{3}^{|D|} \cdot \text{poly}(\text{size}(\Gamma)))$, where $D$ is the domain of $\Gamma$ and $\text{poly}(\cdot)$ is some fixed polynomial.
- Assuming the Strong Exponential Time Hypothesis (SETH), we prove that for any constant $\delta < 1$ the problem cannot be solved in $O(\sqrt[3]{3}^{|\delta|D|} \cdot \text{poly}(\text{size}(\Gamma)))$ time.

We also present a few weaker results for general-valued languages (see Section 3).

Other related work There is a vast literature devoted to exponential-time algorithms for various problems, both on the algorithmic side and on the hardness side. Hardness results usually assume one of the following two hypotheses [15, 16, 9].

**Conjecture 1** (Exponential Time Hypothesis (ETH)). Deciding satisfiability of a 3-CNF-SAT formula on $n$ variables cannot be solved in $O(2^{o(n)})$ time.

**Conjecture 2** (Strong Exponential Time Hypothesis (SETH)). For any $\delta < 1$ there exists integer $k$ such that deciding satisfiability of a $k$-CNF-SAT formula on $n$ variables cannot be solved in $O(2^{\delta n})$ time.

Below we discuss some results specific to CSPs. Let $(k,d)$-CSP be the class of CSP problems on a $d$-element domain where each constraint involves at most $k$ variables. The number of variables in an instance will be denoted as $n$. A trivial exhaustive search for a $(k,d)$-CSP instance runs in $O^*(d^n)$ time, where notation $O^*(\cdot)$ hides factors polynomial in the size of the input. For $(2,d)$-CSP instances the complexity can be improved to $O^*((d-1)^n)$ [27]. Some important subclasses of $(2,d)$-CSP can even be solved in $O^*(2^{O(n)})$ time. For example, [25] and [4] developed respectively $O(2.45^n)$ and $O^*(2^n)$ algorithms for solving the $d$-coloring problem. On the negative side, ETH is known to have the following implications:

- The $(2,d)$-CSP problem cannot be solved in $d^{O(n)} = 2^{o(n \log d)}$ time [31].
- The **Graph Homomorphism** problem cannot be solved in $2^{O\left(\frac{\log d}{\log \log d}\right)}$ time [12]. (This problem can be viewed as a special case of $(2,d)$-CSP, in which a single binary relation is applied to different pairs of variables).
Recently, exponential-time algorithms for crisp NP-hard languages have been studied using algebraic techniques \[17,18,24\]. For example, \[18\] showed that the following conditions are equivalent, assuming the (now proved) algebraic CSP dichotomy conjecture: (a) ETH fails; (b) there exists a finite crisp NP-hard language \(\Gamma\) that can be solved in subexponential time (i.e. all \(\Gamma\)-instances on \(n\) variables can be solved in \(O(2^{o(n)})\) time); (c) all finite crisp NP-hard languages \(\Gamma\) can be solved in subexponential time.

The rest of the paper is organized as follows: Section 2 gives a background on the VCSP framework, and Section 3 presents our results. All proofs are given in Section 4 and Appendices A-C.

2 Background

We denote \(\mathbb{Q} = \mathbb{Q} \cup \{\infty\}\), where \(\infty\) is the positive infinity. A function of the form \(f: D^n \to \mathbb{Q}\) will be called a cost function over \(D\) of arity \(n\). We will always assume that the set \(D\) is finite. The effective domain of \(f\) is the set \(\text{dom}\ f = \{x \mid f(x) < \infty\}\). Note that \(\text{dom}\ f\) can be viewed both as an \(n\)-ary relation over \(D\) and as a function \(D^n \to \{0, \infty\}\). We assume that \(f\) is represented as a list of pairs \((x, f(x)) : x \in \text{dom}\ f\). Accordingly, we define \(\text{size}(f) = \sum_{x \in \text{dom}\ f} n \log |D| + \text{size}(f(x))\), where the size of a rational number \(p/q\) (for integers \(p, q\)) is \(\log(|p| + 1) + \log q\).

Definition 1. A valued constraint satisfaction language \(\Gamma\) over domain \(D\) is a set of cost functions \(f: D^n \to \mathbb{Q}\), where the arity \(n\) depends on \(f\) and may be different for different functions in \(\Gamma\). The domain of \(\Gamma\) will be denoted as \(\text{D}_{\Gamma}\). For a finite \(\Gamma\) we define \(\text{size}(\Gamma) = |D| + \sum_{f \in \Gamma} \text{size}(f)\).

Language \(\Gamma\) is called finite-valued if all functions \(f \in \Gamma\) take finite (rational) values. It is called crisp if all functions \(f \in \Gamma\) take only values in \(\{0, \infty\}\). We denote \(\text{Feas}(\Gamma) = \{\text{dom}\ f \mid f \in \Gamma\}\) to be the crisp language obtained from \(\Gamma\) in a natural way. Throughout the paper, for a subset \(A \subseteq D\) we use \(u_A\) to denote the unary function \(D \to \{0, \infty\}\) with \(\arg\min u_A = A\). (Domain \(D\) should always be clear from the context). For a label \(a \in D\) we also write \(u_a = u_{\{a\}}\) for brevity.

Definition 2. An instance \(I\) of the valued constraint satisfaction problem (VCSP) is a function \(D^V \to \mathbb{Q}\) given by

\[
\begin{align*}
  f_I(x) &= \sum_{t \in T} f_t(x_{v(t,1)}, \ldots, x_{v(t,n_t)}) \\
  &\text{It is specified by a finite set of variables } V, \text{ finite set of terms } T, \text{ cost functions } f_t: D^{n_t} \to \mathbb{Q}, \text{ of arity } n_t \text{ and indices } v(t, k) \in V \text{ for } t \in T, k = 1, \ldots, n_t. \text{ A solution to } I \text{ is a labeling } x \in D^V \text{ with the minimum total value. The size of } I \text{ is defined as } \text{size}(I) = |V| + |D| + \sum_{t \in T} \text{size}(f_t).

  \text{The instance } I \text{ is called a } \Gamma\text{-instance if all terms } f_t \text{ belong to } \Gamma.
\end{align*}
\]

The set of all \(\Gamma\)-instances will be denoted as \(\text{VCSP}(\Gamma)\). A finite language \(\Gamma\) is called tractable if all instances \(I \in \text{VCSP}(\Gamma)\) can be solved in polynomial time, and it is \(NP\)-hard if the corresponding optimization problem is \(NP\)-hard. A long sequence of works culminating with recent breakthrough papers \[6,32\] has established that every finite language \(\Gamma\) is either tractable or \(NP\)-hard.

2.1 Polymorphisms and cores

Let \(O_D^{(m)}\) denote the set of all operations \(g: D^m \to D\) and let \(O_D = \bigcup_{m \geq 1} O_D^{(m)}\). When \(D\) is clear from the context, we will sometimes write simply \(O^{(m)}\) and \(O\).

Any language \(\Gamma\) defined on \(D\) can be associated with a set of operations on \(D\), known as the polymorphisms of \(\Gamma\), which allow one to combine (often in a useful way) several feasible assignments into a new one.
Definition 3. An operation \( g \in \mathcal{O}_D^{(m)} \) is a polymorphism of a cost function \( f : D^n \to \overline{\mathbb{Q}} \) if, for any \( x^1, x^2, \ldots, x^m \in \text{dom} \ f \), we have that \( g(x^1, x^2, \ldots, x^m) \in \text{dom} \ f \) where \( g \) is applied component-wise.

For any valued constraint language \( \Gamma \) over a set \( D \), we denote by \( \text{Pol}^{(m)}(\Gamma) \) the set of all operations on \( \mathcal{O}_D^{(m)} \) which are polymorphisms of every \( f \in \Gamma \). We also let \( \text{Pol}(\Gamma) = \bigcup_{m \geq 1} \text{Pol}^{(m)}(\Gamma) \).

Clearly, if \( g \) is a polymorphism of a cost function \( f \), then \( g \) is also a polymorphism of \( \text{dom} \ f \). For \( \{0, \infty\} \)-valued functions, which naturally correspond to relations, the notion of a polymorphism defined above coincides with the standard notion of a polymorphism for relations. Note that the projections (aka dictators), i.e., operations of the form \( e_i^n(x_1, \ldots, x_n) = x_i \), are polymorphisms of all valued constraint languages. Polymorphisms play the key role in the algebraic approach to the CSP, but, for VCSPs, more general constructs are necessary, which we now define.

Definition 4. An \( m \)-ary fractional operation \( \omega \) on \( D \) is a probability distribution on \( \mathcal{O}_D^{(m)} \). The support of \( \omega \) is defined as \( \text{supp}(\omega) = \{ g \in \mathcal{O}_D^{(m)} | \omega(g) > 0 \} \).

For an operation \( g \in \mathcal{O}^{(m)} \) we will denote \( \chi_g \) to be characteristic vector of \( g \), i.e., the fractional operation with \( \chi_g(g) = 1 \) and \( \chi_g(h) = 0 \) for \( h \neq g \).

Definition 5. A \( m \)-ary fractional operation \( \omega \) on \( D \) is said to be a fractional polymorphism of a cost function \( f : D^n \to \overline{\mathbb{Q}} \) if, for any \( x^1, x^2, \ldots, x^m \in \text{dom} \ f \), we have

\[
\sum_{g \in \text{supp}(\omega)} \omega(g)f(g(x^1, \ldots, x^m)) \leq \frac{1}{m}(f(x^1) + \ldots + f(x^m)).
\] (2)

For a constraint language \( \Gamma \), \( \mathcal{FPol}^{(m)}(\Gamma) \) will denote the set of all \( m \)-ary fractional operations that are fractional polymorphisms of each function in \( \Gamma \). Also, let \( \mathcal{FPol}(\Gamma) = \bigcup_{m \geq 1} \mathcal{FPol}^{(m)}(\Gamma) \), \( \text{supp}^{(m)}(\Gamma) = \bigcup_{\omega \in \mathcal{FPol}^{(m)}(\Gamma)} \text{supp}^{(m)}(\omega) \) and \( \text{supp}(\Gamma) = \bigcup_{m \geq 1} \text{supp}^{(m)}(\Gamma) \).

(It is easy to check that \( \text{supp}(\Gamma) \subseteq \text{Pol}(\Gamma) \), and \( \text{supp}(\Gamma) = \text{Pol}(\Gamma) \) if \( \Gamma \) is crisp).

Next, we will need the notion of cores.

Definition 6. Language \( \Gamma \) on domain \( D \) is called a core if all operations \( g \in \text{supp}^{(1)}(\Gamma) \) are bijections. Subset \( B \subseteq D \) is called a core of \( \Gamma \) if \( B = g(D) \) for some operation \( g \in \text{supp}^{(1)}(\Gamma) \) and the language \( \Gamma[B] \) is a core, where \( \Gamma[B] \) is the language on domain \( B \) obtained by restricting each function \( f : D^n \to \overline{\mathbb{Q}} \) in \( \Gamma \) to \( B^n \).

The following facts are folklore knowledge. We do not know an explicit reference (at least in the case of general-valued languages), so we prove them in Appendix A for completeness.

Lemma 7. Let \( B \) be a subset of \( D = D_\Gamma \) such that \( B = g(D) \) for some \( g \in \text{supp}^{(1)}(\Gamma) \).

(a) Set \( B \) is a core of \( \Gamma \) if and only if and \( |B| = \text{core-size}(\Gamma) \) is defined as \( \min \{|g(D)| : g \in \text{supp}^{(1)}(\Gamma)\} \).

(b) There exists vector \( \omega \in \mathcal{FPol}^{(1)}(\Gamma) \) such that \( g(D) \subseteq B \) for all \( g \in \text{supp}(\omega) \). Furthermore, if \( B \) is a core of \( \Gamma \) then such \( \omega \) can be chosen so that \( g(a) = a \) for all \( g \in \text{supp}(\omega) \) and \( a \in B \).

(c) Let \( I \) be a \( \Gamma \)-instance on variables \( V \). Then \( \min_{x \in B^V} f_I(x) = \min_{x \in D^V} f_I(x) \).

For a language \( \Gamma \) we denote \( \mathcal{B}_\Gamma^{\text{core}} \) to be the set of subsets \( B \subseteq D \) which are cores of \( \Gamma \), and \( \mathcal{O}_\Gamma^{\text{core}} \) to be set of operations \( g \in \text{supp}^{(1)}(\Gamma) \) such that \( g(D) \in \mathcal{B}_\Gamma^{\text{core}} \) (or equivalently such that \( |g(D)| = \text{core-size}(\Gamma) \)).

2.2 Dichotomy theorem

Several types of operations play a special role in the algebraic approach to (V)CSP.

Definition 8. An operation \( g \in \mathcal{O}_D^{(m)} \) is called
• idempotent if \( g(x, \ldots, x) = x \) for all \( x \in D \);
• cyclic if \( m \geq 2 \) and \( g(x_1, x_2, \ldots, x_m) = g(x_2, \ldots, x_m, x_1) \) for all \( x_1, \ldots, x_m \in D \);
• symmetric if \( m \geq 2 \) and \( g(x_1, x_2, \ldots, x_m) = g(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(m)}) \) for all \( x_1, \ldots, x_m \in D \), and any permutation \( \pi \) on \([m]\);
• Siggers if \( m = 4 \) and \( g(r, a, r, e) = g(a, r, e, a) \) for all \( a, e, r \in D \).

A fractional operation \( \omega \) is said to be idempotent/cyclic/symmetric if all operations in \( \text{supp}(\omega) \) have the corresponding property.

Note, the Siggers operation is traditionally defined in the literature as an idempotent operation \( g \) satisfying \( g(r, a, r, e) = g(a, r, e, a) \). Here we follow the terminology in [1] that does not require idempotency. (In [10] such operation was called quasi-Siggers).

We can now formulate the dichotomy theorem.

**Theorem 9.** Let \( \Gamma \) be a language. If the core of \( \Gamma \) admits a cyclic fractional polymorphism then \( \Gamma \) is tractable [20, 16, 32]. Otherwise \( \Gamma \) is NP-hard [23].

We will call languages \( \Gamma \) satisfying the condition of Theorem 9 solvable. The following equivalent characterizations of solvability are either known or can be easily be derived from previous work [29, 19, 22, 23] (see Appendix B):

**Lemma 10.** Let \( \Gamma \) be a language and \( g \in \text{supp}(\Gamma) \). The following conditions are equivalent:

(a) \( \Gamma \) is solvable.

(b) \( \Gamma \) admits a cyclic fractional polymorphism of some arity \( m \geq 2 \).

(c) \( \text{supp}(\Gamma) \) contains a Siggers operation.

(d) \( \Gamma \cup \{u_a \mid a \in B\} \) is solvable for any core \( B \) of \( \Gamma \).

(e) \( \Gamma[g(D_\Gamma)] \) is solvable.

Furthermore, a finite-valued language \( \Gamma \) is solvable if and only if it admits a symmetric fractional polymorphism of arity 2.

Note that checking solvability of a given language \( \Gamma \) is a decidable problem. Indeed, condition (c) can be tested by solving a linear program with \( |\text{supp}(\Gamma)| = |D|^4 \) variables and \( O(\text{poly(size(\Gamma)))} \) constraints, where we maximize the total weight of Siggers operations subject to linear constraints expressing that \( \omega \in \mathbb{R}^{O(D^4)} \) is a fractional polymorphism of \( \Gamma \) of arity 4.

### 2.3 Basic LP relaxation

Symmetric operations are known to be closely related to LP-based algorithms for CSP-related problems. One algorithm in particular has been known to solve many VCSPs to optimality. This algorithm is based on the so-called basic LP relaxation, or BLP, defined as follows.

Let \( \mathbb{M}_n = \{ \mu \geq 0 \mid \sum_{x \in D^n} \mu(x) = 1 \} \) be the set of probability distributions over labelings in \( D^n \). We also denote \( \Delta = \mathbb{M}_1 \); thus, \( \Delta \) is the standard \((|D| - 1)\)-dimensional simplex. The corners of \( \Delta \) can be identified with elements in \( D \). For a distribution \( \mu \in \mathbb{M}_n \) and a variable \( v \in \{1, \ldots, n\} \), let \( \mu[v] \in \Delta \) be the marginal probability of distribution \( \mu \) for \( v \):

\[
\mu[v](a) = \sum_{x \in D^n : x_v = a} \mu(x) \quad \forall a \in D.
\]
Given a VCSP instance $\mathcal{I}$ in the form (1), we define the value $\text{BLP}(\mathcal{I})$ as follows:

$$
\text{BLP}(\mathcal{I}) = \min \sum_{t \in T} \sum_{x \in \text{dom } f_t} \mu_t(x) f_t(x) 
$$

(3)

s.t. $(\mu_t)[k] = \alpha_{v(t,k)} \quad \forall t \in T, k \in \{1, \ldots, n_t\}$

$\mu_t \in M_{n_t} \quad \forall t \in T$

$\mu_t(x) = 0 \quad \forall t \in T, x \notin \text{dom } f_t$

$\alpha_v \in \Delta \quad \forall v \in V$

If there are no feasible solutions then $\text{BLP}(\mathcal{I}) = \infty$. The objective function and all constraints in this system are linear, therefore this is a linear program. Its size is polynomial in $\text{size}(\mathcal{I})$, so $\text{BLP}(\mathcal{I})$ can be found in time polynomial in $\text{size}(\mathcal{I})$.

We say that $\text{BLP}$ solves $\mathcal{I}$ if $\text{BLP}(\mathcal{I}) = \min_{x \in D^V} f_\mathcal{I}(x)$, and $\text{BLP}$ solves VCSP($\Gamma$) if it solves all instances $\mathcal{I}$ of VCSP($\Gamma$). The following results are known.

**Theorem 11** ([22]). (a) $\text{BLP}$ solves VCSP($\Gamma$) if and only if $\Gamma$ admits a symmetric fractional polymorphism of every arity $m \geq 2$. (b) If $\Gamma$ is finite-valued then $\text{BLP}$ solves VCSP($\Gamma$) if and only if $\Gamma$ admits a symmetric fractional polymorphism of arity 2 (i.e. if it is solvable).

$\text{BLP}$ relaxation also plays a key role for general-valued languages, as the following result shows. Recall that $u_A$ for a subset $A \subseteq D$ is the unary function $D \rightarrow \{0, \infty\}$ with $\text{dom } u_A = A$.

**Definition 12.** Consider instance $\mathcal{I}$ with the set of variables $V$ and domain $D$. For node $v \in V$ denote $D_v = \{a \in D \mid \exists x \in D^V \text{ s.t. } f_\mathcal{I}(x) < \infty, x_v = a\}$. We define $\text{Feas}(\mathcal{I})$ and $\mathcal{I} + \text{Feas}(\mathcal{I})$ to be the instances with variables $V$ and the following objective functions:

$$
f_{\text{Feas}(\mathcal{I})}(x) = \sum_{v \in V} u_{D_v}(x_v) \quad f_{\mathcal{I} + \text{Feas}(\mathcal{I})}(x) = f_\mathcal{I}(x) + f_{\text{Feas}(\mathcal{I})}(x)
$$

It is easy to see that $f_\mathcal{I}(x) = f_{\mathcal{I} + \text{Feas}(\mathcal{I})}(x)$ for any $x \in D^V$. However, the $\text{BLP}$ relaxations of these two instances may differ.

**Theorem 13** ([20]). If $\Gamma$ is solvable and $\mathcal{I}$ is a $\Gamma$-instance then $\text{BLP}$ solves $\mathcal{I} + \text{Feas}(\mathcal{I})$.

If $\Gamma$ is solvable and we know a core $B$ of $\Gamma$, then an optimal solution for every instance $\Gamma$-instance can be found by using the standard self-reducibility method, in which we repeatedly add unary terms of the form $u_a(x_v)$ to the instance for different $v \in V$ and $a \in B$ and check whether this changes the optimal value of the $\text{BLP}$ relaxation. A formal description of the method is given below. (Notations $\mathcal{I}[B]$ and $\mathcal{I} + u_a(x_v)$ should be self-explanatory; in particular, the former is the instance obtained from $\mathcal{I}$ by restricting each term to domain $B$).

**Algorithm 1: LP-Probe($\mathcal{I}, B$).** Input: instance $\mathcal{I}$ with variables $V$ and domain $D$, set $B \subseteq D$

Output: either a labeling $x^* \in \arg \min_{x \in D^V} f_\mathcal{I}(x)$ or “FAIL”

1. compute value $LP^* = BLP(\mathcal{I} + \text{Feas}(\mathcal{I}))$, then update $\mathcal{I} \leftarrow \mathcal{I}[B]$
2. for each variable $v \in V$ in some order do
3.  
4.  
5.  
6.  
7. return labeling $x^* \in B^V$ where $x^*_v$ equals the label $a$ for which term $u_a(x_v)$ has been added

**Lemma 14.** (a) If LP-Probe($\mathcal{I}, B$) returns a labeling $x^*$ then $x^* \in \arg \min_{x \in D^V} f_\mathcal{I}(x)$.

(b) Suppose that $\mathcal{I}$ is a $\Gamma$-instance where $\Gamma$ is solvable and $B \in B^\text{core}_I$. Then LP-Probe($\mathcal{I}, B$) $\neq$ FAIL.
Proof. Part (a) holds by construction, and part (b) can be derived from the following two facts (which hold under the preconditions of part (b)):

- \[ \min_{x \in D} V f_I(x) = \min_{x \in D} V f_I(x) \] by Lemma 7(c).
- BLP solves all instances to which it is applied during the algorithm. Indeed, by Lemma 10 the language \[ \Gamma' = \Gamma[B] \cup \{ u_a \mid a \in B \} \] is solvable. The initial instance is a \( \Gamma \)-instance, and all instances in line 4 are \( \Gamma' \)-instances. The claim now follows from Theorem 13.

2.4 Meta-questions and uniform algorithms

In the light of the previous discussion, it is natural to ask the following questions about a given language \( \Gamma \): (i) Is \( \Gamma \) solvable? (ii) Is \( \Gamma \) a core? (iii) What is a core of \( \Gamma \)? Such questions are usually called meta-questions or meta-problems in the literature. For finite-valued languages their computational complexity has been studied in [30].

Theorem 15 ([30]). Problems (i) and (ii) for \( \{0,1\} \)-valued languages are NP-complete and co-NP-complete, respectively.

Theorem 16 ([30]). There is a polynomial-time algorithm that, given a core finite-valued language \( \Gamma \), either finds a binary idempotent symmetric fractional polymorphism \( \omega \) of \( \Gamma \) with \( |\text{supp}(\omega)| = O(\text{poly}(\text{size}(\Gamma))) \), or asserts that none exists.

For crisp languages the following hardness results are known.

Theorem 17 ([14]). Deciding whether a given crisp language \( \Gamma \) with a single binary relation is a core is a co-NP-complete problem. (Equivalently, testing whether a given directed graph has a non-bijective homomorphism onto itself is an NP-complete problem).

Theorem 18 ([10]). Deciding whether a given crisp language \( \Gamma \) with binary relations is solvable is an NP-complete problem.

It is still an open question whether an analogue of Theorem 16 holds for crisp languages, i.e. whether solvability of a given core crisp language \( \Gamma \) can be tested in polynomial time. However, it is known [10] that the answer would be positive assuming the existence of a certain uniform polynomial-time algorithm for CSPs.

Definition 19. Let \( F \) be a class of languages. A uniform polynomial-time algorithm for \( F \) is a polynomial-time algorithm that, for each input \( (\Gamma,I) \) with \( \Gamma \in F \) and \( I \in \text{VCSP}(\Gamma) \), computes \[ \min_{x} f_I(x) \].

Theorem 20 ([10]). Suppose that there exists a uniform polynomial-time algorithm for the class of core crisp languages. Then there exists a polynomial-time algorithm that decides whether a given core crisp language is solvable (or equivalently admits a Siggers polymorphism).

Currently it is not known whether a uniform polynomial-time algorithm for core crisp languages exists. (Algorithms in [15, 32] assume that needed polymorphisms of the language are part of the input; furthermore, the worst-case bound on the runtime is exponential in \( |D| \)).

We remark that [10] considered a wider range of meta-questions for crisp languages. In particular, they studied the complexity of deciding whether a given \( \Gamma \) admits polymorphism \( g \in O_D^m \) satisfying a given strong linear Mal'tsev condition specified by a set of linear identities. Examples of such identities are \( g(x,\ldots,x) \approx x \) (meaning that \( g \) is idempotent), \( g(x_1,x_2,\ldots,x_m) \approx g(x_2,\ldots,x_m,x_1) \) (meaning that \( g \) is cyclic), and \( g(r,a,r,e) \approx g(a,r,e,a) \) (meaning that \( g \) is Siggers). We refer to [10] to further details.
3 Our results

Let $\mathbb{I}_{n, \Gamma}$ be the set of $\Gamma$-instances $\mathcal{I}$ on $n$ variables with $\text{size}(\mathcal{I}) = O(\text{poly}(\text{size} \Gamma))$ (for some fixed polynomial). We denote $T_{n, \Gamma}$ to be the running time of a procedure that computes $\text{Feas}(\mathcal{I})$ for $\mathcal{I} \in \mathbb{I}_{n, \Gamma}$. Also, let $T^*_{n, \Gamma}$ be the combined running times of computing $\text{Feas}(\mathcal{J})$ for instances $\mathcal{J}$ during a call to LP-Probe$(\mathcal{I}, B)$ for $\mathcal{I} \in \mathbb{I}_{n, \Gamma}$ and some subset $B$. Note, if $\Gamma$ is finite-valued then computing $\text{Feas}(\mathcal{I})$ is a trivial problem, so $T_{n, \Gamma}$ and $T^*_{n, \Gamma}$ would be polynomial in $n + \text{size}(\Gamma)$.

In the results below $D$ is always assumed to be the domain of language $\Gamma$. The size of $D$ is denoted as $d = |D|$.

Conditional cores First, we consider the problem of computing a core $B \in \mathcal{B}^\text{core}_\Gamma$ of a given language $\Gamma$. A naive solution is to solve a linear program with $|\mathcal{O}(\Gamma)| = d^d$ variables. We will present an alternative technique that runs more efficiently (in the case of finite-valued languages) but is allowed to output an incorrect result if $\Gamma$ is not solvable. It will be convenient to introduce the following terminology: language $\Gamma$ is a conditional core if either $\Gamma$ is a core or $\Gamma$ is not solvable. Similarly, set $B$ is a conditional core of $\Gamma$ if either $B \in \mathcal{B}^\text{core}_\Gamma$ or $\Gamma$ is not solvable. Note, $B = \emptyset$ is a conditional core of $\Gamma$ if and only if $\Gamma$ is not solvable.

To compute a conditional core of $\Gamma$, we will use the following approach. Consider a pair $(\Gamma, \sigma)$ where $\sigma$ is a string of size $O(\text{poly}(\Gamma))$ that specifies set $\mathcal{B}_\sigma$ of candidate cores of $\Gamma$. Formally, $\mathcal{B}_\sigma = \{B_1, \ldots, B_N\}$ where $\emptyset \neq B_i \subseteq D$ for each $i \in [N]$. We assume that elements of $\mathcal{B}_\sigma$ can be efficiently enumerated, i.e. there exists a polynomial-time procedure for computing $B_1$ from $\sigma$ and $B_{i+1}$ from $(\sigma, B_i)$. If $\mathcal{B}$ is a set of subsets $B \subseteq D$, we will denote

\[ O[\mathcal{B}] = \{g \in \mathcal{O}(\Gamma) \mid g(D) = B \text{ for some } B \in \mathcal{B}\} \]
\[ \bar{O}[\mathcal{B}] = \{g \in \mathcal{O}(\Gamma) \mid g(D) \subseteq B \text{ for some } B \in \mathcal{B}\} \]

**Theorem 21.** There exists an algorithm that for a given input $(\Gamma, \sigma)$ does one of the following:

(a) Produces a fractional polymorphism $\omega \in \text{fPol}(\Gamma)$ with $\text{supp}(\omega) \subseteq \bar{O}[\mathcal{B}_\sigma]$ and $|\text{supp}(\omega)| \leq 1 + \sum_{f \in \Gamma} |\text{dom} f|$.

(b) Asserts that there exists no vector $\omega \in \text{fPol}(\Gamma)$ with $\text{supp}(\omega) \subseteq \bar{O}[\mathcal{B}_\sigma]$.

(c) Asserts that one of the following holds: (i) $\Gamma$ is not solvable; (ii) $\mathcal{B}_\sigma \cap \mathcal{B}^\text{core}_\Gamma = \emptyset$.

It runs in $(|\mathcal{B}_\sigma| + O(\text{poly}(\text{size}(\Gamma)))) + (T^*_{d, \Gamma} + O(\text{poly}(\text{size}(\Gamma))))$ time and uses $O(\text{poly}(\text{size}(\Gamma)))$ space.

The algorithm in the theorem above is based on the ellipsoid method \[13\], which tests feasibility of a polytope using a polynomial number of calls to the separation oracle. In our case this oracle is implemented via one or more calls to LP-Probe$(\mathcal{I}, B)$ for appropriate $\mathcal{I}$ and $B$.

One possibility would be to use Theorem 21 with the set $\mathcal{B}_\sigma = \{B \subseteq D \mid B \neq \emptyset, B \neq D\}$. If the algorithm returns result (a) then we can take operation $g \in \text{supp}(\omega)$ and call the algorithm recursively for the language $\Gamma[g(D)]$ on a smaller domain. If we get result (b) or (c) then one can show that $\Gamma$ is a conditional core, so we can stop. For finite-valued languages this approach would run in $O(2^d \cdot \text{poly}(\text{size}(\Gamma)))$ time. We will pursue an alternative approach with an improved complexity $O(\sqrt{3}^d \cdot \text{poly}(\text{size}(\Gamma)))$.

This approach will use partitions $\Pi = \{D_1, \ldots, D_k\}$ of domain $D$. For such $\Pi$ we denote

\[ \mathcal{O}_\Pi = \{g \in \mathcal{O}(\Gamma) \mid g(a) = g(b) \quad \forall a, b \in A \in \Pi\} \]
\[ \Pi^\perp = \{B \subseteq D \mid |B \cap A| = 1 \quad \forall A \in \Pi\} \]

We say that $\Pi$ is a partition of $\Gamma$ if the set $\mathcal{O}_\Pi \cap \text{supp}(\Gamma)$ is non-empty. In particular, the partition $\Pi = \{\{a\} \mid a \in D\}$ of $D$ into singletons is a partition of $\Gamma$, since $\text{supp}(\Gamma)$ contains the identity
mapping $D \rightarrow D$. We say that $\Pi$ is a maximal partition of $\Gamma$ if $\Pi$ is a partition of $\Gamma$ and no coarser partition $\Pi' \supset \Pi$ (i.e., $\Pi'$ with $O_{\Pi'} \subset O_{\Pi}$) has this property. Clearly, for any $\Gamma$ there exists at least one $\Pi$ which is a maximal partition of $\Gamma$. By analogy with cores, we say that $\Pi$ is a conditional (maximal) partition of $\Gamma$ if either $\Pi$ is a (maximal) partition of $\Gamma$ or $\Gamma$ is not solvable.

In the results below $\Pi$ is always assumed to be a partition of $\Gamma$.

**Lemma 22.** (a) If $\Pi$ is a maximal partition of $\Gamma$ then $B^\text{core}_\Pi \subseteq \Pi^\perp$ and $O^\text{core}_\Pi = O[\Pi^\perp] \cap \text{supp}(\Gamma)$. (b) If $\Pi$ is a partition of $D$ then $|\Pi^\perp| \leq \sqrt[3]{3}^d$.

**Theorem 23.** There exists an algorithm with runtime $T_{d,\Gamma} + T_{|\Pi|,\Gamma} + O(\text{poly}(\text{size}(\Gamma)))$ that for a given input $(\Gamma, \Pi)$ does one of the following:

(a) Asserts that $\Pi$ is a maximal partition of $\Gamma$.

(b) Asserts that $\Pi$ is not a partition of $\Gamma$.

As before, the algorithm in Theorem 23 is based on the ellipsoid method. However, now we cannot use procedure $\text{LP-Probe}(\mathcal{I}, B)$ to implement the separation oracle, since a candidate core $B$ is not available. Instead, we solve the BLP relaxation of instance $\mathcal{I}$ and derive a separating hyperplane from a (fractional) optimal solution of the relaxation.

**Corollary 24.** (1) A conditional maximal partition $\Pi$ of $\Gamma$ can be computed in $O(d^2) \cdot T_{d,\Gamma} + O(\text{poly}(\text{size}(\Gamma)))$ time. (2) Once such $\Pi$ is found, a conditional core $B$ of $\Gamma$ can be computed using $(|\Pi^\perp| + O(\text{poly}(\text{size}(\Gamma)))) \cdot (T_{d,\Gamma} + O(\text{poly}(\text{size}(\Gamma))))$ time and $O(\text{poly}(\text{size}(\Gamma)))$ space. If $B \neq \emptyset$ then the algorithm also produces a fractional polymorphism $\omega \in \text{fPol}(\Gamma)$ such that supp($\omega$) $\subseteq O[\Pi^\perp]$, $|\text{supp}(\omega)| \leq 1 + \sum_{f \in \Gamma} |\text{dom} f|$ and supp($\omega$) contains operation $g$ with $g(D) = B$.

In part (1) we use a greedy search that starts with $\Pi = \{\{a\} \mid a \in D\}$ and then repeatedly calls the algorithm in Theorem 23 for coarser partitions $\Pi$. In part (2) we call the algorithm from Theorem 21 with $\sigma = \Pi$ and $B_\sigma = \Pi^\perp$. For further details we refer to Appendix C.

**Testing solvability of a conditional core** Once we found a conditional core $B$ of $\Gamma$, we need to test whether language $\Gamma[B]$ is solvable. This problem is known to be solvable in polynomial-time for finite-valued languages [30], see Theorem 16. Their result can be extended as follows.

**Theorem 25.** There exists an algorithm that for a given language $\Gamma$ does one of the following:

(a) Produces an idempotent fractional polymorphism $\omega \in \text{fPol}(\Gamma)$ certifying solvability of $\Gamma$:

- $\omega$ has arity $m = 2$ and is symmetric, if $\Gamma$ is finite-valued;
- $\omega$ has arity $m = 4$ and contains a Siggers operation in the support, if $\Gamma$ is not finite-valued.

Furthermore, in each case vector $\omega$ satisfies $|\text{supp}(\omega)| \leq 1 + \sum_{f \in \Gamma} \left(\frac{|\text{dom} f|}{m}\right)$.

(b) Asserts that one of the following holds: (i) $\Gamma$ is not solvable; (ii) $\Gamma$ is not a core.

Its runtime is $O(\text{poly}(\text{size}(\Gamma)))$ if $\Gamma$ is finite-valued, and $O(T_{d,\Gamma}^* \cdot \text{poly}(\text{size}(\Gamma)))$ otherwise.

Combining procedures in Corollary 21 and the algorithm in Theorem 25 yields our main algorithmic result.

**Corollary 26.** Solvability of a given finite-valued language $\Gamma$ can be tested in $O(\sqrt[3]{3}^d \cdot \text{poly}(\text{size}(\Gamma)))$ time. If the answer is positive, the algorithm also returns a fractional polymorphism $\omega_1 \in \text{fPol}^{(1)}(\Gamma)$ with supp($\omega$) $\subseteq O^\text{core}$ and a symmetric idempotent fractional polymorphism $\omega_2 \in \text{fPol}^{(2)}(\Gamma[B])$ where $B = g(D)$ for some $g \in \text{supp}(\omega_1)$; furthermore, $|\text{supp}(\omega_m)| \leq 1 + \sum_{f \in \Gamma} \left(\frac{|\text{dom} f|}{m}\right)$ for $m \in \{1, 2\}$. 
Hardness results Let us fix a constant $L \in \{1, \infty\}$. As Theorems 15, 17 and 18 state, testing whether $\Gamma$ is (i) solvable and (ii) is a core are both NP-hard problems for $\{0, L\}$-valued languages. We now present additional hardness results under the Exponential Time Hypothesis (ETH) and the Strong Exponential Time Hypothesis (SETH) (see Conjectures 1 and 2). Note that for Theorem 27 we simply reuse the reductions from [10]. We say that a family of languages $\mathcal{F}$ is $k$-bounded if each $\Gamma \in \mathcal{F}$ satisfies $\text{size}(\Gamma) = O(\text{poly}(d))$ for some fixed polynomial, and arity($f$) $\leq k$ for all $f \in \Gamma$.

Theorem 27. Suppose that ETH holds. Then there exists a 2-bounded family $\mathcal{F}$ of $\{0, L\}$-valued languages such that the following problems cannot be solved in $O(2^{d(d)})$ time:
(a) Deciding whether language $\Gamma \in \mathcal{F}$ is solvable.
(b) Deciding whether language $\Gamma \in \mathcal{F}$ is a core.

Theorem 28. Suppose that SETH holds. Then for any $\delta < 1$ there exists an $O(1)$-bounded family $\mathcal{F}$ of $\{0, L\}$-valued languages such that the following problems cannot be solved in $O(\sqrt[3]{3}^{\delta d})$ time:
(a) Deciding whether language $\Gamma \in \mathcal{F}$ is solvable (assuming the existence of a uniform polynomial-time algorithm for core crisp languages, in the case when $L = \infty$).
(b) Deciding whether language $\Gamma \in \mathcal{F}$ satisfies $\text{core-size}(\Gamma) \leq d/3$.

4 Proofs

4.1 Ellipsoid method

Using the ellipsoid method, Grötschel, Lovász and Schrijver [13] established a polynomial-time equivalence between linear optimization and separation problems in polytopes. We will need one implication of this result, namely that efficient separation implies efficient feasibility testing. A formal statement is given below.

Consider a family of instances where an instance $\Lambda$ can be described by a string of length $\text{size}(\Lambda)$ over a fixed alphabet. Suppose that for each $\Lambda$ we have an integer $n$ and a finite set $\mathcal{G}$, where each element $g \in \mathcal{G}$ corresponds to a hyperplane $c_g \in \mathbb{Q}^{n+1}$. This hyperplane encodes linear inequality $\langle c_g, [y, 1] \rangle \geq 0$ on vector $y \in \mathbb{R}^n$. Let us denote $\langle g \rangle = \{y \in \mathbb{R}^n \mid \langle c_g, [y, 1] \rangle \geq 0\}$ and $\langle \mathcal{G} \rangle = \bigcap_{g \in \mathcal{G}} \langle g \rangle$ for a subset $\mathcal{G} \subseteq \mathcal{G}$.

We make the following assumptions: (i) each $g \in \mathcal{G}$ can be described by a string of size $O(\text{poly}(\text{size}(\Lambda)))$; (ii) vector $c_g$ can be computed from $\Lambda$ and $g$ in polynomial time (implying that $\text{size}(c_g) = O(\text{poly}(\text{size}(\Lambda)))$, where the size of a vector in $\mathbb{Q}^{n+1}$ is the sum of sizes of its $n+1$ components); (iii) set $\mathcal{G}$ can be constructed algorithmically from input $\Lambda$. Note that quantities $n$, $\mathcal{G}$, $\{(c_g, \langle g \rangle) \mid g \in \mathcal{G}\}$ all depend on $\Lambda$; for brevity this dependence is not reflected in the notation.

Theorem 29 ([13], Lemma 6.5.15)). Consider the following problems:

- [Separation] Given instance $\Lambda$ and vector $y \in \mathbb{Q}^n$, either decide that $y \in \langle \mathcal{G} \rangle$, or find a separating hyperplane $c \in \mathbb{Q}^{n+1}$ with $\text{size}(c) = O(\text{poly}(\text{size}(\Lambda) + \text{size}(y)))$ satisfying $\langle c, [y, 1] \rangle < 0$ and $\langle c, [z, 1] \rangle \geq 0$ for all $z \in \langle \mathcal{G} \rangle$.
- [Feasibility] Given instance $\Lambda$, decide whether $\langle \mathcal{G} \rangle = \emptyset$.

There exists an algorithm for solving [Feasibility] that makes a polynomial number of calls to the oracle for [Separation] plus a polynomial number of other operations.

Note that a (possibly inefficient) algorithm for solving [Separation] always exists: if $y \notin \langle \mathcal{G} \rangle$ then one possibility is to find an element $g \in \mathcal{G}$ with $y \notin \langle g \rangle$ and return hyperplane $c_g$. (Its size is polynomial in $\text{size}(\Lambda)$ by assumption).

For some parts of the proof we will also need the following variation.

Theorem 30. Consider the following problems:
• **[Separation+]** Given instance $\Lambda$ and vector $y \in \mathbb{Q}^n$, either decide that $y \notin \langle \mathcal{G} \rangle$, or find an element $g \in \mathcal{G}$ with $y \notin \langle g \rangle$ (i.e. element $g$ with $\langle cg, [y \ 1] \rangle < 0$).

• **[Feasibility+]** Given instance $\Lambda$, decide whether $\langle \mathcal{G} \rangle = \emptyset$. If $\langle \mathcal{G} \rangle = \emptyset$, find subset $\mathcal{G}' \subseteq \mathcal{G}$ such that $|\mathcal{G}'| = O(poly(size(\Lambda)))$ and $\langle \mathcal{G}' \rangle = \emptyset$.

There exists an algorithm for solving **[Feasibility+]** that makes a polynomial number of calls to the oracle for **[Separation+]** plus a polynomial number of other operations.

This result can be deduced from Theorem 28: the desired subset $\mathcal{G}'$ can be taken as the set of all elements in $\mathcal{G}$ returned by the oracle during the algorithm.

### 4.2 Farkas lemma for fractional polymorphisms

Let us fix integer $m \geq 1$ and sets $\Omega^-, \Omega^+$ with $\Omega^- \subseteq \Omega^+ \subseteq \text{Pol}(\Gamma) \cap O^{(m)}$. These choices will be specified later (they will depend on the specific theorem that we will be proving). Let $\Gamma^+$ be the set of tuples $(f, x)$ such that $f : D^n \to \mathbb{Q}$ is an $n$-ary function in $\Gamma$ and $x \in [\text{dom } f]^m$. Note, $x$ can be viewed as a matrix of size $m \times n$:

$$x = \begin{pmatrix} x_1^1 & \ldots & x_n^1 \\ \vdots & \ddots & \vdots \\ x_1^n & \ldots & x_n^n \end{pmatrix}$$

For such $x$ we will write $x^i = (x_1^i, \ldots, x_n^i) \in D^n$ and $x_j = (x_1^j, \ldots, x_m^j) \in D^m$. For an operation $g \in O^{(m)}$ we denote $g(x) = (g(x_1), \ldots, g(x_n)) \in D^n$, and for a cost function $f : D^n \to \mathbb{Q}$ we denote $f^m(x) = \frac{1}{m}(f(x^1) + \ldots + f(x^m))$.

Next, we define various hyperplanes in $\mathbb{Q}^{\Gamma^+} \cup \mathbb{Q}$ as follows:

• For $g \in \Omega^+$ let $c_g$ be the hyperplane corresponding to the inequality

$$\sum_{(f, x) \in \Gamma^+} [f(g(x)) - f^m(x)] \cdot y(f, x) \geq [g \in \Omega^-]$$

where we used the Iverson bracket notation: $[\phi] = 1$ if $\phi$ is true, and $[\phi] = 0$ otherwise.

• For $(f, x) \in \Gamma^+$ let $c_{(f, x)}$ be the hyperplane corresponding to the inequality $y(f, x) \geq 0$.

• Introduce a special element $\perp$, and let $c_{\perp} = (0, \ldots, 0, 1)$ be the hyperplane corresponding to the (unsatisfiable) inequality $0 \geq 1$.

For a subset $\Omega \subseteq \Omega^+$ it will be convenient to denote $Y[\Omega] = \langle \Omega \cup \Gamma^+ \rangle$. In other words, $Y[\Omega]$ is the set of vectors $y \in \mathbb{R}^{|\Gamma^+|}$ satisfying

$$\sum_{(f, x) \in \Gamma^+} [f(g(x)) - f^m(x)] \cdot y(f, x) \geq [g \in \Omega^-] \quad \forall g \in \Omega$$

**Lemma 31.** Suppose that $\Omega \subseteq \Omega^+$. Then $Y[\Omega] = \emptyset$ if and only if $\Gamma$ admits an $m$-ary fractional polymorphism $\omega$ such that $\text{supp}(\omega) \subseteq \Omega$ and $\text{supp}(\omega) \cap \Omega^- \neq \emptyset$.

If $Y[\Omega] = \emptyset$ then it is possible to compute such $\omega$ in $O(poly(size(\Gamma) + |\Omega|))$ time (given $\Gamma$ and $\Omega$) so that it additionally satisfies $|\text{supp}(\omega)| \leq 1 + |\Gamma^+| = 1 + \sum_{f \in \Gamma} \binom{|\text{dom } f|}{m}$.

**Proof.** Introducing slack variables $\{y(g) \mid g \in \Omega\}$, we have $Y[\Omega] = \emptyset$ if and only if the following system does not have a solution $y \in \mathbb{R}^{|\Omega|} \cup \mathbb{Q}$:

$$y(g) - \sum_{(f, x) \in \Gamma^+} [f(g(x)) - f^m(x)] \cdot y(f, x) = -[g \in \Omega^-] \quad \forall g \in \Omega$$

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By Farkas Lemma this happens if and only if the following system has a solution \( \omega \in \mathbb{R}^\Omega \):

\[
\begin{align*}
- \sum_{g \in \Omega} [f(g(x)) - f^m(x)] \cdot \omega(g) & \geq 0 \quad \forall (f, x) \in \Gamma^+ \\
\omega(g) & \geq 0 \quad \forall g \in \Omega \\
- \sum_{g \in \Omega \cap \Omega^-} \omega(g) & < 0
\end{align*}
\]  

(7a) (7b) (7c)

If a solution exists, then it can be chosen to satisfy \( \sum_{g \in \Omega} \omega(g) = 1 \) (since multiplying feasible solutions to (7) by a positive constant gives feasible solutions). We obtain that \( Y[\Omega] = \emptyset \) if and only if there exists vector \( \omega \in \mathbb{R}^\Omega \) satisfying

\[
\begin{align*}
\sum_{g \in \Omega} \omega(g) f(g(x)) & \leq f^m(x) \quad \forall (f, x) \in \Gamma^+ \\
\omega(g) & \geq 0 \quad \forall g \in \Omega \\
\sum_{g \in \Omega} \omega(g) & = 1 \\
\sum_{g \in \Omega \cap \Omega^-} \omega(g) & > 0
\end{align*}
\]  

(8a) (8b) (8c) (8d)

This establishes the first claim of Lemma 31. Now suppose that \( Y[\Omega] = \emptyset \), so that system (5) has a solution. Inequalities (5a)-(5c) can be written in a matrix form as \( A \omega \leq b \) with \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \) where \( n = |\Omega| \) is the number of variables and \( m = |\Gamma^+| + |\Omega| + 2 \) is the number of constraints. The equality (5d) is represented as two inequalities) Let \( P \) be the polytope \( \{ \omega \in \mathbb{R}^n \mid A \omega \leq b \} \), and \( \omega \) be a vertex of \( P \) that maximizes \( \sum_{g \in \Omega \cap \Omega^-} \omega(g) \). (By standard results from linear programming, such \( \omega \) can be computed in polynomial time). It now suffices to show that \( \text{supp}(\omega) \leq |\Gamma^+| + 1 \).

We have \( \text{rank}(A) = n \), since \( -A \) contains the identity submatrix of size \( n \times n \). Since \( \omega \) is a vertex of \( P \), \( A \) has a non-singular submatrix \( A' \in \mathbb{R}^{n \times n} \) such that \( A' \omega = b' \), where \( b' \) is the corresponding subvector of \( b \). Matrix \( A' \) has at most \( |\Gamma^+| + 1 \) rows corresponding to constraints (5a) and (5d) (note that the two rows of \( A \) corresponding to constraint (5d) are linearly dependent and thus cannot be both present in \( A' \)). Thus, \( A' \) has at least \( n - |\Gamma^+| - 1 \) rows corresponding to constraints (5c). This constraints are tight for \( \omega \), and so at least \( n - |\Gamma^+| - 1 \) components of \( \omega \) are zeros. Thus, \( \omega \) has at most \( n - (n - |\Gamma^+| - 1) = |\Gamma^+| + 1 \) non-zero components.

4.3 Proof of Theorem 21 (enumeration of candidate cores)

The input to the desired algorithm is a pair \( \Lambda = (\Gamma, \sigma) \). Let us define \( m = 1 \) and \( \Omega^- = \Omega^+ = \partial[B_\sigma] \cap \text{Pol}(\Gamma) \). Note that condition “\( \omega \in f\text{Pol}(\Gamma) \) and \( \text{supp}(\omega) \subseteq \partial[B_\sigma] \)” used in Theorem 21(a,b) is equivalent to the condition “\( \omega \in f\text{Pol}(\Gamma) \) and \( \text{supp}(\omega) \subseteq \Omega^+ \)”, since \( \text{supp}(\Gamma) \subseteq \text{Pol}(\Gamma) \). By Lemma 31, vector \( \omega \) satisfying these conditions exists if and only if \( Y[\Omega^+] = \emptyset \).

Theorem 32. There exists an algorithm with complexity \( T_{\mathcal{G}[D]}^\ast + O(\text{poly(size}(\Gamma) \text{size}(y))) \) that given a tuple \( (\Gamma, B) \) with \( B \subseteq D \) and a vector \( y \in \mathbb{Q}^\Gamma^+ \) does one of the following:

(a) Produces element \( g \in \Omega^+ \cup \Gamma^+ \) such that \( \langle c_g, [y, 1] \rangle < 0 \).

(b) Asserts that one of the following holds: (i) \( \Gamma \) is not solvable; (ii) \( B \notin B_\Gamma^{\text{core}} \).

Before proving this lemma, let us describe how it implies Theorem 21. We use the ellipsoid method where for the input \( \Lambda = (\Gamma, \sigma) \) we define \( \mathcal{G} = \Omega^+ \cup \Gamma^+ \). Theorem 30 gives a class of algorithms for solving \([\text{Feasibility}^+]\) (that depend on the implementation of separation oracles). If \( \langle \mathcal{G} \rangle = Y[\Omega^+] = \emptyset \) then we have a subset \( \mathcal{G}' \subseteq \mathcal{G} \) of polynomial size with \( \langle \mathcal{G}' \rangle = 0 \). Let \( \Omega' = \)
We claim that this algorithm has the complexity stated in Theorem 21. Indeed, each call to the oracle for Separation+ can involve several passes of steps 1-3. Each pass has complexity $T_{|D|,\Gamma} + O(poly(size(\Gamma)))$ (note that size($y$) would be polynomial in size($\Gamma$)). Call the last pass successful, and the other passes unsuccessful. The number of successful passes is polynomial in size($\Gamma$) (since the number of oracle calls is polynomial), and the number of unsuccessful passes is at most $|B_\sigma|$. This establishes the claim about the complexity.

**Proof of Lemma 32** We can assume that the input vector $y \in \mathbb{Q}^{\Gamma^+}$ is non-negative (otherwise we can easily find element $(f, x) \in \Gamma^+$ with $\langle c, f, x \rangle, [y 1] < 0$). Let $I(y)$ be the instance with variables $D$ and the following objective function:

$$f_I(y)(g) = \sum_{(f,x) \in \Gamma^+} y(f, x)f(g_{x_1}, \ldots, g_{x_n}) \quad \forall g : D \rightarrow D$$

Note, we wrote $g_a$ instead of $g(a)$ to emphasize that $g$ is now treated as a labeling of instance $I(y)$, though mathematically it is the same object as an operation in $O^{(1)}$. (This is the convention that we use for instances, see eq. (1)).

Also note that we are now using rational weights in the definition of an instance, where we adopt the following convention: if the weight $y(f, x)$ is zero then expression $y(f, x)f(\ldots)$ means $\text{dom } f(\ldots)$. To be consistent with the original definition, one could multiply everything by a constant to make all weights integers, and then treat these integers as the number of occurrences of the corresponding terms. However, this would make the notation cumbersome, so we avoid this.

Let us run procedure LP-Probe($I(y), B$). If it returns “FAIL” then by Lemma 14 either $\Gamma$ is not solvable or $B \notin B_{\Gamma}^{\text{core}}$. Thus, we can output result (b) in Lemma 32. Now suppose that LP-Probe($I(y), B$) returns labeling $g : D \rightarrow B$. We will show next that $g \in \Omega^+$ and $\langle c, g, [y 1] \rangle < 0$, and thus element $g$ can be returned as the output of the algorithm in Lemma 32.

The following equation can be easily verified:

$$f_I(y)(g) = \begin{cases} \sum_{(f,x) \in \Gamma^+} y(f, x)f(g(x)) < \infty & \text{if } g \in O^{(1)} \cap \text{Pol}(\Gamma) \\ \infty & \text{if } g \in O^{(1)} - \text{Pol}(\Gamma) \end{cases} \quad \text{(9)}$$
Let $\textbf{1}$ be the identity mapping $D \to D$. We have $f_{\{y\}}(g) \leq f_{\{y\}}(\textbf{1}) < \infty$, since $g$ is a minimizer of $f_{\{y\}}$ and $\textbf{1} \in \text{Pol}(\Gamma)$. Thus, $g \in \text{Pol}(\Gamma)$. By construction, $g(D) \subseteq B \in \text{B}_\sigma$, and so $g \in \hat{O}[B_\sigma] \cap \text{Pol}(\Gamma) = \Omega^+$. We can now prove the claim:

$$\langle cg, [y \ 1] \rangle = f_{\{y\}}(g) - f_{\{y\}}(\textbf{1}) - 1 \leq -1 < 0$$

4.4 Proof of Theorem 23 (testing partition $\Pi$ of $D$)

The general structure of the proof will be same as in the previous section, but some details will differ. In particular, we will not be able to apply procedure $\text{LP-Probe}(\cal{I}, B)$ since we do not know a core $B$, and will not be able to obtain an integer solution of the BLP relaxation. Instead, we will derive a separating hyperplane from an optimal (fractional) solution of the BLP relaxation.

The input to the desired algorithm is a pair $(\Gamma, \Pi)$ where $\Pi$ is a partition of $D = D_\Gamma$. We set $m = 1$, $\Omega^- = \cal{O}_\Pi \cap \text{Pol}(\Gamma)$ and $\Omega^+ = \cal{O}(1) \cap \text{Pol}(\Gamma)$. Using Lemma 31 and the fact that $\text{supp}(\Gamma) \subseteq \text{Pol}(\Gamma)$, we conclude that $\Pi$ is a partition of $\Gamma$ if and only if $\text{supp}(\Gamma) = \emptyset$.

For an element $a \in D$ let $[a]$ be the unique set $A \in \Pi$ containing $a$. Let $\cal{J}_\sigma$ and $\cal{J}'_\sigma$ be the instances with variables $D$ and $\Pi$, respectively, that have the following objective functions:

$$f_{\cal{J}_\sigma}(g) = \sum_{(f, x) \in \Gamma^+} \text{dom} f(g_{x_1}, \ldots, g_{x_n}) \quad \forall g : D \to D$$

$$f_{\cal{J}'_\sigma}(g) = \sum_{(f, x) \in \Gamma^+} \text{dom} f(g_{[x_1]}, \ldots, g_{[x_n]}) \quad \forall g : \Pi \to D$$

We denote $\cal{J} = \text{Feas}(\cal{J}_\sigma)$ and $\cal{J}' = \text{Feas}(\cal{J}'_\sigma)$.

**Lemma 33.** There exists a polynomial-time algorithm that given a tuple $(\Gamma, \Pi, \cal{J}, \cal{J}')$ and a vector $y \in \mathbb{Q}^{\Gamma^+}$ does one of the following:

(a) Asserts that $y \in \text{supp}(\Gamma)$.

(b) Produces hyperplane $c \in \mathbb{Q}^{\Gamma^+} \times \mathbb{Q}$ such that $\langle c, [y \ 1] \rangle < 0$ and one of the following holds:

(i) $\langle c, [z \ 1] \rangle \geq 0$ for all $z \in \text{supp}(\Pi)$; (ii) $\Gamma$ is not solvable.

Before proving this lemma, let us describe how it implies Theorem 23. We use the ellipsoid method with the input $\Lambda = (\Gamma, \Pi)$ where the set $\cal{G}$ is defined as follows: if $\Gamma$ is solvable then $\cal{G} = \Omega^+ \cup \Gamma^+$ (in which case $\langle \cal{G} \rangle = Y[\Omega^+]$), otherwise $\cal{G} = \{\perp\}$ (in which case $\langle \cal{G} \rangle = \emptyset$). Theorem 20 gives a class of algorithms for solving [Feasibility]. If $\langle \cal{G} \rangle \neq \emptyset$ then $\text{supp}(\Gamma)$ and thus $\Pi$ is a partition of $\Gamma$, so we can output result (b) in Theorem 23. If $\langle \cal{G} \rangle = \emptyset$ then either $\text{supp}(\Gamma) = \emptyset$ or $\Pi$ is not a partition of $\Pi$ or $\Gamma$ is not solvable. Thus, we can output result (a). We obtained a correct but possibly inefficient algorithm for solving the problem in Theorem 23. Next, we will modify this algorithm so that it remains correct and has the desired complexity.

As the first step, we compute instances $\cal{J} = \text{Feas}(\cal{J}_\sigma)$ and $\cal{J}' = \text{Feas}(\cal{J}'_\sigma)$, and store the result. Now suppose that the ellipsoid method calls the oracle for [Separation] with vector $y \in \mathbb{Q}^{\Gamma^+}$. We implement this call as follows. First, call the algorithm from Lemma 33. If it asserts that $y \in \text{supp}(\Pi)$ then $\text{supp}(\Gamma)$ and so $\Pi$ is not a partition of $\Gamma$. Thus, we can immediately terminate the ellipsoid method and output result (b) in Theorem 23. Otherwise we have a hyperplane $c \in \mathbb{Q}^{\Gamma^+} \times \mathbb{Q}$ that satisfies either condition (i) or (ii) in Lemma 33(b). We return $c$ as the output of the oracle; clearly, in both cases it satisfies the required condition. It can be seen that the modified algorithm is still correct and performs a polynomial number of operations (excluding the time for computing $\cal{J}$ and $\cal{J}'$).
Proof of Lemma 33: We can assume w.l.o.g. that the input vector \( y \in \mathbb{Q}^{\Gamma^+} \) is non-negative (otherwise it is easy to find a separating hyperplane \( c \)). For a vector \( z \in \mathbb{Q}^{\Gamma^+}_\geq \) let \( I(z) \) and \( I'(z) \) be the instances with variables \( D \) and \( \Pi \), respectively, that have the following objective functions (we use the same convention for weights as in Section 4.3):

\[
f_{I(z)}(g) = \sum_{(f,x) \in \Gamma^+} z(f,x) f(g(x_1, \ldots, g_{x_n})) + f_J(g) \quad \forall g : D \rightarrow D
\]

\[
f_{I'(z)}(g) = \sum_{(f,x) \in \Gamma^+} z(f,x) f(g_{[x_1]}, \ldots, g_{[x_n]}) + f_J'(g) \quad \forall g : \Pi \rightarrow D
\]

Let \( \mu \) be an optimal solution of the BLP relaxation of instance \( I(y) \), assuming that a feasible solution exists. (The BLP relaxation in eq. 3 also uses vector \( \alpha \); however, it can be easily computed from \( \mu \), so we omit it). Note, \( \mu \) can be computed in polynomial time given inputs \((\Gamma, J)\) and \( y \in \mathbb{Q}^{\Gamma^+}_\geq \). Vector \( \mu \) has components \( \mu_{f,x}(x') \in \mathbb{Q}^{\Gamma^+} \) for \((f,x) \in \Gamma^+ \) and \( x' \in \text{dom} f \), with \( \sum_{x' \in \text{dom} f} \mu_{f,x}(x') = 1 \). Define vector \( c \in \mathbb{Q}^{\Gamma^+} \) as follows:

\[
c(f,x) = \left[ \sum_{x' \in \text{dom} f} \mu_{f,x}(x') f(x') \right] - f(x) \quad \forall (f,x) \in \Gamma^+
\]

Similarly, let \( \mu' \) be an optimal solution of the BLP relaxation of \( I'(y) \) (if exists), and define vector \( c' \in \mathbb{Q}^{\Gamma^+} \) via

\[
c'(f,x) = \left[ \sum_{x' \in \text{dom} f} \mu'_{f,x}(x') f(x') \right] - f(x) \quad \forall (f,x) \in \Gamma^+
\]

Now do the following:

1. if \( \mu \) exists and \( \langle c, y \rangle < 0 \) then return \([c, 0] \in \mathbb{Q}^{\Gamma^+} \times \mathbb{Q} \) as a separating hyperplane;
2. if \( \mu' \) exists and \( \langle c', y \rangle < 0 \) then return \([c', -1] \in \mathbb{Q}^{\Gamma^+} \times \mathbb{Q} \) as a separating hyperplane;
3. if (1) and (2) do not apply then report that \( y \in \mathcal{Y}[\Omega^+] \).

If (1) and (2) are both applicable then we pick one of them arbitrarily. This concludes the description of the algorithm in Lemma 33; it remains to show that it is correct.

Observing that \( f_{I + \text{Feas}(I)}(x) = f_I(x) \) for any instance \( I \) and labeling \( x \), we obtain

\[
f_{I(z)}(g) = \begin{cases} \sum_{(f,x) \in \Gamma^+} z(f,x) f(g(x)) < \infty & \text{if } g \in \mathcal{O}^{(1)} \cap \text{Pol}(\Gamma) = \Omega^+ \\ \infty & \text{if } g \in \mathcal{O}^{(1)} - \text{Pol}(\Gamma) \end{cases} \quad (10)
\]

Clearly, there is a natural isomorphism between sets \( \mathcal{O}_\Pi \) and \( D^{\Pi} \). Thus, operations \( g \in \mathcal{O}_\Pi \) can be equivalently viewed as labelings \( g : \Pi \rightarrow D \). With this convention, we can evaluate expression \( f_{I'(z)}(g) \) for an operation \( g \in \mathcal{O}_\Pi \). It can be seen that \( f_{I'(z)}(g) = f_{I(z)}(g) \) for any \( g \in \mathcal{O}_\Pi \), and so

\[
f_{I'(z)}(g) = \begin{cases} \sum_{(f,x) \in \Gamma^+} z(f,x) f(g(x)) < \infty & \text{if } g \in \mathcal{O}_\Pi \cap \text{Pol}(\Gamma) = \Omega^- \\ \infty & \text{if } g \in \mathcal{O}_\Pi - \text{Pol}(\Gamma) \end{cases} \quad (11)
\]

Let \( \text{BLP}(I(z), \mu) \) be the cost of solution \( \mu \) in the BLP relaxation of \( I(z) \), and \( \text{BLP}(I'(z), \mu') \) be the cost of solution \( \mu' \) in the BLP relaxation of \( I'(z) \). By checking [3] we obtain

\[
\text{BLP}(I(z), \mu) = \sum_{(f,x) \in \Gamma^+} z(f,x) \sum_{x' \in \text{dom} f} \mu_{f,x}(x') f(x')
\]

\[
\text{BLP}(I'(z), \mu') = \sum_{(f,x) \in \Gamma^+} z(f,x) \sum_{x' \in \text{dom} f} \mu'_{f,x}(x') f(x')
\]
Denoting $h(z) = \sum_{(f,x)\in \Gamma^+} z(f,x)f(x)$, we have

$$\langle c, z \rangle = BLP(I(z), \mu) - h(z) \quad \quad \langle c', z \rangle = BLP(I'(z), \mu') - h(z)$$

We are now ready to prove algorithm’s correctness. We need to consider three cases:

1. **\( \mu \) exists and \( \langle c, y \rangle < 0 \).** Consider vector \( z \in \mathbb{Q}^\Gamma^+ \) with \( \langle c, z \rangle < 0 \). We will show that either \( z \notin Y[\Omega^+] \) or \( \Gamma \) is not solvable. Clearly, this will imply a similar claim for vectors \( z \in \mathbb{R}^\Gamma^+ \).

   We can assume that \( z \) is non-negative and \( \Gamma \) is solvable, otherwise the claim holds trivially. By Theorem 13 BLP solves instance \( I(z) \), therefore there exists mapping \( g \in O^{(1)} \) such that \( f_{I(z)}(g) \leq BLP(I(z), \mu) \). From (10) we conclude that \( g \in \Omega^+ \). We can write

   $$\sum_{(f,x)\in \Gamma^+} z(f,x) \cdot [f(g(x)) - f(x)] = f_{I(z)}(g) - h(z) \leq BLP(I(z), \mu) - h(z) = \langle c, z \rangle < 0$$

   Equivalently, \( \langle c_y, [z 1] \rangle < 0 \). Vector \( z \) violates inequality (4) for \( g \), and therefore \( z \notin Y[\Omega^+] \).

2. **\( \mu' \) exists and \( \langle c', y \rangle < 1 \).** Consider vector \( z \in \mathbb{Q}^\Gamma^+ \) with \( \langle c', z \rangle < 0 \). Similar to the previous case, we will show that either \( z \notin Y[\Omega^+] \) or \( \Gamma \) is not solvable.

   We can assume that \( z \) is non-negative and \( \Gamma \) is solvable, otherwise the claim holds trivially. By Theorem 13 BLP solves instance \( I'(z) \), therefore there exists mapping \( g \in O^{\Omega^+} \) such that \( f_{I'(z)}(g) \leq BLP(I'(z), \mu') \). From (11) we conclude that \( g \in \Omega^- \). We can write

   $$\sum_{(f,x)\in \Gamma^+} z(f,x) \cdot [f(g(x)) - f(x)] = f_{I'(z)}(g) - h(z) \leq BLP(I'(z), \mu') - h(z) = \langle c', z \rangle < 1$$

   We showed that \( \langle c_y, [z 1] \rangle < 0 \). This means that \( z \notin Y[\Omega^+] \).

3. **Cases (1) and (2) do not hold.** We need to show that \( y \in Y[\Omega^+] \). Suppose not, then there exists an element \( g \in \Omega^+ \) such that inequality (11) is violated for \( g \). If \( g \in \Omega^+ - \Omega^- \) then the BLP relaxation of \( I(y) \) has feasible solutions (since \( f_{I(y)}(g) < \infty \)) and we can write

   $$\langle c, y \rangle = BLP(I(y), \mu) - h(y) \leq f_{I(y)}(g) - h(y) = \sum_{(f,x)\in \Gamma^+} y(f,x) \cdot [f(g(x)) - f(x)] < 0$$

   where the first inequality holds since \( \mu \) is an optimal solution of the BLP relaxation of \( I(y) \). Thus, case (1) holds - a contradiction. If \( g \in \Omega^- \) then we obtain in a similar way that case (2) holds:

   $$\langle c', y \rangle = BLP(I'(y), \mu') - h(y) \leq f_{I'(y)}(g) - h(y) = \sum_{(f,x)\in \Gamma^+} y(f,x) \cdot [f(g(x)) - f(x)] < 1$$

### 4.5 Proof of Theorem 25 (testing solvability of core languages)

Let \( O_{id}^{(m)} \subseteq O^{(m)} \) be the set of idempotent operations of arity \( m \). Given an input language \( \Gamma \), we make the following definitions:

- If \( \Gamma \) is finite-valued then \( m = 2 \) and \( \Omega^- = \Omega^+ = \{ g \in O_{id}^{(2)} \cap \text{Pol}(\Gamma) \mid g \text{ is symmetric} \} \).
- Otherwise \( m = 4 \), \( \Omega^+ = O_{id}^{(4)} \cap \text{Pol}(\Gamma) \) and \( \Omega^- = \{ g \in \Omega^+ \mid g \text{ is Siggers} \} \).
Using Lemmas 10 and 31, we conclude that \( \Gamma \) is solvable if and only if \( Y[\Omega^+] = \emptyset \).

Define undirected graph \((D^m, E)\) via \( E = \{(x, y), (y, x)\} \mid x, y \in D\) in the case of \( m = 2 \) or \( E = \{(r, a, r, e), (a, r, e, a)\} \mid a, e, r \in D\) in the case of \( m = 4 \). Let \( \Pi \) be the set of connected components of \((D^m, E)\) (it is a partition of \( D^m \)). It can be seen that operation \( g \in O^{(m)} \) is symmetric (if \( m = 2 \)) / Siggers (if \( m = 4 \)) if and only if \( g(x) = g(y) \) for all \( x, y \in A \in \Pi \). In this section the set of such operations will be denoted as \( O^+ \subseteq O^{(m)} \).

**Lemma 34.** There exists an algorithm with complexity \( 2 \cdot T_{[D^m, \Gamma]}^* + O(\text{poly}(\text{size}(\Gamma) + \text{size}(y))) \) that given a language \( \Gamma \) and a vector \( y \in \mathbb{Q}^{\Gamma^+} \) does one of the following:

(a) Produces element \( g \in \Omega^+ \cup \Gamma^+ \) such that \( \langle c_y, [y \setminus 1] \rangle < 0 \).

(b) Asserts that one of the following holds: (i) \( \Gamma \) is not solvable; (ii) \( \Gamma \) is not a core.

Before proving this lemma, let us describe how it implies Theorem 21. We use the ellipsoid method where for the input \( \Lambda = \Gamma \) we define \( \mathcal{G} = \Omega^+ \cup \Gamma^+ \). Theorem 30 gives a class of algorithms for solving \([\text{Feasibility}^+]\). If \( \langle \mathcal{G} \rangle = Y[\Omega^+] = \emptyset \) then we have a subset \( \mathcal{G}' \subseteq \mathcal{G} \) of polynomial size with \( \langle \mathcal{G}' \rangle = 0 \). Let \( \Omega' = \mathcal{G}' \cap \Omega^+ \subseteq \Omega^+ \), then \( Y[\Omega'] = \emptyset \). Lemma 31 gives a desired \( m \)-ary fractional polymorphism \( \omega \) of \( \Gamma \), which we return as the result. If \( \langle \mathcal{G} \rangle = Y[\Omega^+] \neq \emptyset \) then \( \Gamma \) is not solvable (by the observation in the beginning of this section) and so we can output result (b) in Theorem 25. Next, we will modify this algorithm so that it remains correct and has the desired complexity.

Suppose that the ellipsoid method calls the oracle for \([\text{Separation}^+]\) with vector \( y \in \mathbb{Q}^{\Gamma^+} \). We implement this call as follows. First, call the algorithm from Lemma 31. If it returns \( g \in \Omega^+ \cup \Gamma^+ \) with \( \langle c_y, [y \setminus 1] \rangle < 0 \) then we return \( g \) as the output of the oracle. Otherwise it asserts that \( \Gamma \) is not solvable or not a core; we can then terminate the ellipsoid method and output result (b) in Theorem 25.

**Proof of Lemma 34.** We can assume that the input vector \( y \in \mathbb{Q}^{\Gamma^+} \) is non-negative (otherwise we can easily find element \( (f, x) \in \Gamma^+ \) with \( \langle c_{(f, x)}, [y \setminus 1] \rangle < 0 \)). Let \( I(y) \) and \( I'(y) \) be the instances with variables \( D^m \) and \( \Pi \), respectively, that have the following objective functions (we use the same convention for weights as in Section 4.3.3):

\[
\begin{align*}
    f_{I(y)}(g) &= \sum_{(f, x) \in \Gamma^+} y(f, x) f(g(x_1, \ldots, x_n)) + \sum_{a \in D} u_a(g(a, \ldots, a)) \quad \forall g : D^m \to D \\
    f_{I'(y)}(g) &= \sum_{(f, x) \in \Gamma^+} y(f, x) f(g(x_1, \ldots, x_n)) + \sum_{a \in D} u_a(g(a, \ldots, a)) \quad \forall g : \Pi \to D
\end{align*}
\]

Note that \( I(y), I'(y) \in \text{VCSP}(\Gamma \cup \{u_a \mid a \in D\}) \). Similar to previous sections, we have

\[
\begin{align*}
    f_{I(y)}(g) &= \begin{cases} 
    \sum_{(f, x) \in \Gamma^+} y(f, x) f(g(x)) < \infty & \text{if } g \in \mathcal{O}_{1d}^{(m)} \cap \text{Pol}(\Gamma) \\
    \infty & \text{if } g \in \mathcal{O}^{(m)} \cap (\mathcal{O}_{1d}^{(m)} \cap \text{Pol}(\Gamma))
    \end{cases} \\
    f_{I'(y)}(g) &= \begin{cases} 
    \sum_{(f, x) \in \Gamma^+} z(f, x) f(g(x)) < \infty & \text{if } g \in \mathcal{O}_{\Pi} \cap \mathcal{O}_{1d}^{(m)} \cap \text{Pol}(\Gamma) = \Omega^- \\
    \infty & \text{if } g \in \mathcal{O}_{\Pi} \cap (\mathcal{O}_{1d}^{(m)} \cap \text{Pol}(\Gamma))
    \end{cases}
\end{align*}
\]

where we used the natural isomorphism between sets \( \mathcal{O}_{\Pi} \) and \( D^\Pi \) to evaluate expression \( f_{I'(y)}(g) \) for \( g \in \mathcal{O}_{\Pi} \). Let us denote \( h(y) = \sum_{(f, x) \in \Gamma^+} y(f, x) f^{\Pi}(x) \). We now run the following steps.
1. Compute \( g' = \text{LP-Probe}(\mathcal{I}'(y), D) \). If \( g' = \text{FAIL} \) then \( \Gamma \) is either not solvable or not a core (by Lemmas 10 and 14), so we output result (b) in Lemma 34 and terminate. If \( f_{\mathcal{I}'(y)}(g') < h(y) + 1 \) then \( g' \in \Omega^- \) and \( \langle c_{g'}, [y \, 1] \rangle = f_{\mathcal{I}'(y)}(g') - h(y) - 1 < 0 \), so we can output element \( g' \) in Lemma 34 and terminate. From now on we assume that \( g' \neq \text{FAIL} \) and \( f_{\mathcal{I}'(y)}(g') \geq h(y) + 1 \). This means that for any \( \hat{g} \in \Omega^- \) we have
\[
\langle c_{\hat{g}}, [y \, 1] \rangle = f_{\mathcal{I}'(y)}(\hat{g}) - h(y) - 1 \geq f_{\mathcal{I}'(y)}(g') - h(y) - 1 \geq 0
\]

2. (This step is run only if \( m = 4 \)) Compute \( g = \text{LP-Probe}(\mathcal{I}(y), D) \). If \( g = \text{FAIL} \) then we output result (b) in Lemma 34 and terminate (by the same argument as above). If \( f_{\mathcal{I}(y)}(g) < h(y) \) then \( g \in \mathcal{O}_{\text{id}}^\wedge \cap \text{Pol}(\Gamma) = \Omega^+ \) and \( \langle c_g, [y \, 1] \rangle = f_{\mathcal{I}(y)}(g) - h(y) - [g \in \Omega^-] < 0 \), so we can output element \( g \) in Lemma 34 and terminate. From now on we assume that \( g \neq \text{FAIL} \) and \( f_{\mathcal{I}(y)}(g) \geq h(y) \). This means that for any \( \hat{g} \in \Omega^+ - \Omega^- \) we have
\[
\langle c_{\hat{g}}, [y \, 1] \rangle = f_{\mathcal{I}(y)}(\hat{g}) - h(y) \geq f_{\mathcal{I}(y)}(g) - h(y) \geq 0
\]

3. If we reached this point, we know that \( \langle c_{\hat{g}}, [y \, 1] \rangle \geq 0 \) for all \( \hat{g} \in \Omega^+ \), and therefore \( y \in Y[\Omega^+] \). Thus, \( Y[\Omega^+] \neq \emptyset \) and so \( \Gamma \) is not solvable (by the observation in the beginning of this section), therefore we can output result (b) in Lemma 34.

### 4.6 Proof of Theorems 27 and 28 (hardness under ETH/SETH)

Let us fix constant \( L \in \{1, \infty\} \), and denote \( \mathcal{L} = \{0, L\} \). For a function \( f : D^n \rightarrow \{0, \infty\} \) over some domain \( D \) let \( f^\mathcal{L} : D^n \rightarrow \mathcal{L} \) be the function with \( f^\mathcal{L}(x) = \min \{f(x), L\} \). In particular, \( u^\mathcal{L}_A \) for a subset \( A \subseteq D \) is the function \( u^\mathcal{L}_A : D \rightarrow \mathcal{L} \) with \( \arg \min u^\mathcal{L}_A = A \).

The proof will be based on the following construction. Consider a CSP instance \( \mathcal{I} \) with variables \( V \), domain \([c]\) and the following objective function:
\[
f_{\mathcal{I}}(x) = \sum_{t \in T} f_t(x_{u(t,1)}, \ldots, x_{v(c,nt)}) \quad \forall x : V \rightarrow [c]
\]  

where all terms \( f_t \) are \( \{0, \infty\} \)-valued functions satisfying \( \min_{x \in [c]^n} f_t(x) = 0 \). To such \( \mathcal{I} \) we associate the following language \( \Gamma = \Gamma^\mathcal{L}(\mathcal{I}) \) on domain \( D = V \times [c] \):
\[
\Gamma = \{ f^\mathcal{L}(t_{v(1),\ldots,v(n,t)}) \mid t \in T \} \cup \{ u^\mathcal{L}_{\mathcal{V}_v} \mid v \in V \}
\]

where for an \( n \)-ary function \( f \) over \([c]\) and variables \( v_1, \ldots, v_n \in V \), function \( f^{(v_1,\ldots,v_n)} : D^n \rightarrow \mathcal{L} \) is defined via
\[
f^{(v_1,\ldots,v_n)}(y) = \begin{cases} 
L & \text{if } y = ((v_1, x_1), \ldots, (v_n, x_n)) \\
0 & \text{otherwise}
\end{cases} \quad \forall y \in D^n
\]

Note, this construction is similar to the technique of “lifting a language” introduced in [21]; it was also used in [20] in a different context.

**Theorem 35** ([11]). Suppose that instance \( \mathcal{I} \) expresses a 3-coloring problem in an undirected graph \((V, E)\), i.e. \( c = 3 \) and \( \mathcal{I} \) has the following objective function for labeling \( x : V \rightarrow \{1, 2, 3\} \):
\[
f_{\mathcal{I}}(x) = \sum_{(u,v) \in E} F(x_u, x_v) \quad \text{where} \quad F(a, b) = \begin{cases} 
0 & \text{if } a \neq b \\
\infty & \text{if } a = b
\end{cases}
\]

Then language \( \Gamma = \Gamma^\mathcal{L}(\mathcal{I}) \) has the following properties:
(a) If \( \min_x f_{\mathcal{I}}(x) = 0 \) then \( \text{core-size}(\Gamma) = |V| \) and \( \Gamma \) is solvable.
(b) If \( \min_x f_{\mathcal{I}}(x) \neq 0 \) then \( \text{core-size}(\Gamma) = 3 \cdot |V| \) (i.e. \( \Gamma \) is a core) and \( \Gamma \) is not solvable.
Remark 1. Chen and Larose [11] showed this statement for the case \( L = \{0, \infty\} \) and for the language \( \Gamma = \Gamma - \{uvF_v | v \in V\} \) without unary terms. However, the proof can easily be extended to our setting. Indeed, part (a) is an “easy” direction (see e.g. Lemma 37 below). Part (b) holds for language \( \Gamma^{(0,\infty)}(I) \) and thus for language \( \Gamma^{(0,\infty)}(I) \). Observing that \( \text{supp}(\Gamma^{(0,1)}(I)) \subseteq \text{supp}(\Gamma^{(0,\infty)}(I)) = \text{Pol}(\Gamma^{(0,\infty)}(I)) \) yields the claim for language \( \Gamma^{(0,1)}(I) \).

Theorem 36 (Theorem 3.2]). Suppose that ETH holds. Then the 3-coloring problem for a graph on \( n \) nodes cannot be solved in \( 2^{o(n)} \) time.

Clearly, Theorem 27 follows immediately from Theorems 25 and 26 (Note that in the construction above we have \( \text{size}(\Gamma^L(I)) = O(\text{poly}(\text{size}(I))) \), \( \text{size}(I) = O(\text{poly}(|V|)) \) and \( d = 3 \cdot |V| \)). We now turn to the proof of Theorem 28. First, we establish some connections between instance \( I \) and language \( \Gamma^L(I) \).

Lemma 37. Language \( \Gamma = \Gamma^L(I) \) has the following properties:
(a) \( \text{core-size}(\Gamma) \geq |V| \). Furthermore, \( \text{core-size}(\Gamma) = |V| \) if and only if \( \min_x f_I(x) = 0 \).
(b) If \( \min_x f_I(x) = 0 \) then \( \Gamma \) is solvable.
(c) There exists an algorithm for deciding whether \( \min_x f_I(x) = 0 \) that works as follows:

- If \( L = 1 \) then it makes one oracle call to test solvability of \( \Gamma \) and performs a polynomial number of other operations.
- If \( L = \infty \) then it makes \( |D| + 1 \) oracle calls to test solvability of languages of the form \( \Gamma \oplus B = \Gamma \cup \{u_a | a \in B\} \) for subsets \( B \subseteq D \) and performs a polynomial number of other operations, assuming the existence of a uniform polynomial-time algorithm for core crisp languages.

Before giving a proof of this lemma, let us show how it implies Theorem 28. Let \( \delta \) be the constant from Theorem 28. Fix constant \( \delta', \delta'' \) and positive integers \( p, q \) such that \( \delta < \delta' < \delta'' < 1 \) and \( \frac{1}{\delta} \log_2 3 \leq \frac{p}{q} \leq \log_2 3 \). Assume that SETH holds, then there exists \( k \) such that satisfiability of a \( k \)-CNF-SAT formula \( \varphi = C_1 \land \ldots \land C_m \) with \( n \) Boolean variables and \( m \) clauses cannot be decided in \( O(2^{|\varphi|} n) \) time. Note that \( m = O(\text{poly}(n)) \), since \( k \) is a constant. Next, we construct CSP instance \( I = I(\varphi) \) such that \( \varphi \) is satisfiable if and only if \( I \) has a feasible solution. The idea is to encode labelings in \( \{0,1\}^p \) via labelings in \( \{1,2,3\}^q \). Specifically, let us fix an injective mapping \( \sigma : \{0,1\}^p \to \{1,2,3\}^q \) (it exists since \( 2^p \leq 3^q \)). Denote \( r = \lceil \frac{q}{p} \rceil \). We can treat \( \varphi \) as a formula over \( rp \) Boolean variables (by adding dummy variables, if necessary). Thus, we can name variables of \( \varphi \) as follows: \( x = (x^1, \ldots, x^r) \) where \( x^i = (x^i_1, \ldots, x^i_p) \in \{0,1\}^p \) for each \( i \in [r] \). Instance \( I \) will have \( rq \) variables \( y \in \{1,2,3\}^r \). It will be convenient to write \( y = (y^1, \ldots, y^r) \) where \( y^i = (y^i_1, \ldots, y^i_p) \in \{1,2,3\}^q \) for each \( i \in [r] \). To construct the objective function of \( I \), we do the following for each clause \( C_i \) of \( \varphi \):

- Clause \( C_i \) can be viewed as a function \( f_i(x^{i_1}, \ldots, x^{i_k'}) \in \{0,\infty\} \) of \( kp \) variables (some of which may be “dummy”), where \( k' \leq k \). Add the following term to \( I \):
  \[
  f_i'(y^{i_1}, \ldots, y^{i_k}) = \begin{cases} 
 f_i(\sigma^{-1}(y^{i_1}), \ldots, \sigma^{-1}(y^{i_k})) & \text{if } y^{i_1}, \ldots, y^{i_k'} \in \sigma(\{0,1\}^p) \subseteq \{1,2,3\}^q \\
 \infty & \text{otherwise}
  \end{cases}
  \]

By construction, formula \( \varphi \) is satisfiable if and only if instance \( I \) has a feasible solution.

Let \( \mathcal{F} \) be the family of languages of the form \( \Gamma = \Gamma^L(I(\varphi)) \) for \( k \)-CNF-SAT formulas \( \varphi \). Clearly, family \( \mathcal{F} \) is \( O(1) \)-bounded. (Note that we have \( d = 3rq = 3q \lceil \frac{q}{p} \rceil = \Theta(n) \) and \( \text{size}(I) = O(\text{poly}(m + n)) = O(\text{poly}(n)) \)). We claim that Theorem 28 holds for family \( \mathcal{F} \). Indeed, if there exists an algorithm with complexity \( O(3^{3d}) \) for either problem (a) or problem (b) then we can use the algorithm in Lemma 37(c) to test feasibility of instance \( I(\varphi) \) (and thus satisfiability of \( \varphi \)) in time \( O(3^{3d} \cdot \text{poly}(d)) \). We have \( 3^{3d} = 3^{3q \lceil \frac{q}{p} \rceil} = 3^{3q \lceil \frac{q}{p} \rceil + o(1)} \leq 2^{\delta' n + o(1)} \), so we obtain an algorithm for \( k \)-CNF-SAT with complexity \( O(2^{\delta' n}) \) - a contradiction.
Proof of Lemma 37. For any node \( v \in V \) and core \( B \in B^\text{core}_\Gamma \) we have \(|B \cap D_v| \geq 1\) (since the presence of function \( u^{D_v}_I \) in \( \Gamma \) implies that \( g(D_v) \subseteq D \) for any \( g \in \text{supp}(\Gamma) \)). Therefore, we have \( \text{core-size}(\Gamma) = |B| \geq |V| \). We make the following claims:

- If \( \min_x f_I(x) = 0 \) then \( \text{core-size}(\Gamma) = |V| \) and \( \Gamma \) is solvable. Indeed, pick \( x^* \in [c]^V \) with \( f_I(x^*) = 0 \), and let \( g : D \rightarrow D \) be the operation with \( g((v, a)) = x^* \) for all \( a \in [c] \). It can be checked that vector \( \chi_g \) is a fractional polymorphism of \( \Gamma \). (Note, if \( f_I(v_1, \ldots, v_n)(y) = 0 \) then \( y = ((v_1, x^*_1), \ldots, (v_n, x^*_n)) \) and so \( g(y) = ((v_1, x^*_1), \ldots, (v_n, x^*_n)) \) and \( f_I(v_1, \ldots, v_n)(g(y)) = 0 \). This implies that set \( B = \{(v, x^*_v) \mid v \in V\} \) is a core of \( \Gamma \) and \( \text{core-size}(\Gamma) = V \).

To show that \( \Gamma \) solvable, it suffices to show that \( \Gamma' = \Gamma[B] \) is solvable (by Lemma 10). Language \( \Gamma' \) contains unary functions \( \{u^L_\ell \mid a \in B\} \) and functions of the form

\[
\begin{align*}
f_{(v_1, \ldots, v_n)}(y) &= \begin{cases} 0 & \text{if } y = ((v_1, x^*_1), \ldots, (v_n, x^*_n)) \\ L & \text{otherwise} \end{cases} \quad \forall x \in B^n
\end{align*}
\]

It can be checked that vector \( \frac{1}{3}(\chi_{g_1} + \chi_{g_2} + \chi_{g_3}) \) is a fractional polymorphism of \( \Gamma' \), where \((g_1, g_2, g_3) : B^3 \rightarrow B^3\) are ternary operations defined via

\[
(g_1, g_2, g_3)(a, b, c) = \begin{cases} (\text{mjrt}(a, b, c), \text{mjrt}(a, b, c), \text{mnrt}(a, b, c)) & \text{if } \{|a, b, c| \leq 2 \\ (a, b, c) \text{ otherwise} \end{cases}
\]

and functions \( \text{mjrt}, \text{mnrt} \) return respectively the majority and the minority element among its arguments. We obtain that \( \text{mjrt} \in \text{supp}(\Gamma') \). It is well-known that such \( \Gamma' \) is solvable.\(^1\)

- If \( \text{core-size}(\Gamma) = |V| \) then \( \min_x f_I(x) = 0 \). Indeed, pick a core \( B \in B^\text{core}_\Gamma \). Since \(|B \cap D_v| \geq 1\) for each \( v \in V \), we have \(|B \cap D_v| = 1\) for each \( v \in V \). Define labeling \( x^* \in [c]^V \) so that \( x^*_v \) is the unique label in \( B \cap D_v \). Pick vector \( \omega \in \text{fPol}^{(1)}(\Gamma) \) such that \( g(D) = B \) for all \( g \in \text{supp}(\omega) \) (it exists by Lemma 7(b)). Consider \( g \in \text{supp}(\omega) \). We must have \( g(D_v) \subseteq D_v \) due to the presence of function \( u^{D_v}_I \) in \( \Gamma \). Thus, \( g(D_v) = \{x^*_v\} \) and so \( \text{supp}(\omega) = \{g\} \). We showed that \( \chi_g \in \text{fPol}^{(1)}(\Gamma) \). Now consider function \( f_I(v_1, \ldots, v_n) \in \Gamma \). By assumption, there exists tuple \( y \) with \( f_I(v_1, \ldots, v_n)(y) = 0 \). We can write

\[
f_I(v_1, \ldots, v_n)((v_1, x^*_1), \ldots, (v_n, x^*_n)) = f_I(v_1, \ldots, v_n)(g(y)) \leq f_I(v_1, \ldots, v_n)(y) = 0
\]

Thus, \( f_I(x^*_1, \ldots, x^*_n) = 0 \). This means that \( f_I(x^*) = 0 \), as claimed.

We now describe the algorithm for part (c). First, we test whether language \( \Gamma \) is solvable. Assume (otherwise we can report that \( \min_x f_I(x) \neq 0 \) and terminate). Let \( J \) be the instance with variables \( V \) and domain \( D \) obtained from \( I \) by replacing each term \( f_t \) with \( f_t^{(v(t,1), \ldots, v(t,n_t))} \).

\[
f_J(y) = \sum_{t \in T} f_t^{(v(t,1), \ldots, v(t,n_t))}(y_{v(t,1), \ldots, v(t,n_t)}) \quad \forall y : V \rightarrow D
\]

It can be verified that \( \min_x f_I(x) = 0 \) if and only if \( \min_y f_J(y) = 0 \). Also, \( J \) is a \( \Gamma \)-instance with \( \text{size}(J) = O(\text{poly}(\text{size}(I))) \). If \( L = 1 \) then we have \( \min_y f_J(y) = BLP(J) \) by Theorem 11, so this minimum can be computed in polynomial time via linear programming. Now suppose that \( L = \infty \). We initialize \( B = \emptyset \) and then go through labels \( b \in D \) in some order and do the following:

- Test whether language \( \Gamma \oplus B \oplus \{b\} \) is solvable. If it is then add \( b \) to \( B \).

\(^1\)One can use, for example, Lemma 39 and the fact that a clone of operations containing a near-unanimity operation (such as \text{mjrt}) also contains a Siggers operation, see 11.
Let $B \subseteq D$ be the set upon termination. By construction, $\Gamma \oplus B$ is solvable. We claim that $B$ is a core of $\Gamma \oplus B$. Indeed, consider core $B' \in \mathcal{B}_{\Gamma \oplus B}^{\text{core}}$. Clearly, we have $B \subseteq B'$. Suppose that there exists $b \in B' - B$. By Lemma 10(d), language $\Gamma \oplus B \oplus \{b\}$ is solvable. By construction, there exists subset $\tilde{B} \subseteq B$ such that language $\Gamma \oplus \tilde{B} \oplus \{b\}$ is not solvable. We obtained a contradiction.

Since $B$ is a core of $\Gamma \oplus B$, we have $\min_{y \in B'} f_{\tilde{T}}(y) = \min_{y \in B'} f_{\tilde{T}}(y)$. The latter minimum can be obtained by calling the assumed uniform polynomial-time algorithm for a core crisp language $(\Gamma \oplus B)[B]$.

### A Proof of Lemmas 7 and 22 (properties of cores)

In this section we fix language $\Gamma$ on a domain $D$, and denote $G = \text{supp}(1)(\Gamma)$. It is well-known that set $G$ is closed under compositions. Indeed, for vectors $\omega, \omega' \in \text{fPol}(1)(\Gamma)$ we can write

$$f(x) \geq \sum_{g \in \text{supp}(\omega)} \omega(g)f(g(x)) \geq \sum_{g \in \text{supp}(\omega)} \omega(g) \sum_{h \in \text{supp}(\omega')} \omega'(h)f(h(g(x))) \quad \forall f \in \Gamma, x \in \text{dom } f \quad (17)$$

This means that vector $\nu = \sum_{g \in \text{supp}(\omega), h \in \text{supp}(\omega')} \omega(g)\omega'(h)\chi_{\text{hog}}$ is a unary fractional polymorphism of $\Gamma$ with $\text{supp}(\nu) = \{h \circ g \mid g \in \text{supp}(\omega), h \in \text{supp}(\omega')\}$. The claim follows.

We will use a tool from [22] that allows to construct new fractional polymorphisms from existing ones using superpositions. The following claim is a special case of the “Expansion Lemma” in [22] for unary fractional polymorphisms.

**Lemma 38 ([22]).** Let $G^* \subseteq G = \text{supp}(1)(\Gamma)$ such that for any $g \in G$ there exists a sequence $h_1, \ldots, h_r, r \geq 0$ with $h_r \circ \ldots \circ h_1 \circ g \in G^*$. Then there exists a unary fractional polymorphism $\omega$ of $\Gamma$ with $\text{supp}(\omega) \subseteq G^*$.

We are now ready to prove Lemmas 7 and 22.

**Proof of Lemma 7(a)** Consider operation $g \in G$, set $B = g(D)$ and language $\Gamma' = \Gamma[B]$. Using the same argument as in [17], we conclude the following:

(i) If $h' \in \text{supp}(\Gamma')$ then $h' \circ g \in \text{supp}(\Gamma)$.

(ii) If $h \in \text{supp}(\Gamma)$ then $g \circ h \in \text{supp}(\Gamma')$.

Suppose that $\Gamma'$ is not a core. Then $\text{supp}(\Gamma')$ contains operation $h'$ with $|h'(B)| = |B|$. Operation $h = h' \circ g \in \text{supp}(\Gamma)$ satisfies $|h(D)| = |h'(B)| = |B|$, and so $|B| > \text{core-size}(\Gamma)$.

Conversely, suppose that $|B| > \text{core-size}(\Gamma)$. Then there exists operation $h \in \text{supp}(1)(\Gamma)$ with $|h(D)| = |B|$. Operation $h' = g \circ h \in \text{supp}(\Gamma')$ satisfies $|h'(B)| = |h(D)| < |B|$, and so $\Gamma'$ is not a core.

**Proof of Lemma 7(b)** Suppose that $\nu \in \text{fPol}(1)(\Gamma)$, $h \in \text{supp}(\nu)$ and $B = h(D)$. Let us apply Lemma 28 with $G^* = \{g \in G \mid g(D) \subseteq B\}$. The lemma’s precondition clearly holds (note, $h \circ g \in G^*$ for any $g \in G$). This shows the existence of $\omega \in \text{fPol}(1)(\Gamma)$ with $\text{supp}(\omega) \subseteq G^*$.

Now suppose in addition that $B$ is a core. We again use Lemma 28 but now with the set $G^* = \{h \in G \mid g(D) = B, g(a) = a \forall a \in B\}$. Let us show that the lemma’s precondition holds for $\omega = \sum_{h \in \text{supp}(\nu)} \nu(h)\chi_{\text{hog}}$. Here $\nu$ is a fractional polymorphism of $\Gamma$ with $\text{supp}(\nu) = G$. (Such $\nu$ exists due to the following observation: if $\nu', \nu'' \in \text{fPol}(1)(\Gamma)$ then $\frac{\nu'}{\nu''} \in \text{fPol}(1)(\Gamma)$ and $\text{supp}(\frac{\nu'}{\nu''}) = \text{supp}(\nu') \cup \text{supp}(\nu'')$. It can be seen that this operator $\text{Exp}$ is “valid” in the terminology of [22], i.e. every $f \in \Gamma$ and $x \in \text{dom } f$ satisfies $\sum_{h \in \text{supp}(\omega)} \omega(h)f(h(x)) \leq f(g(x))$.)

---

2 The Expansion Lemma uses the notion of an “expansion operator” $\text{Exp}$ that assigns to each $g \in G$ a probability distribution $\omega = \text{Exp}(g)$ over $G$. In the case of Lemma 28 this distribution would be defined as $\omega = \sum_{h \in \text{supp}(\nu)} \nu(h)\chi_{\text{hog}}$, where $\nu$ is a fractional polymorphism of $\Gamma$ with $\text{supp}(\nu) = G$. (Such $\nu$ exists due to the following observation: if $\nu', \nu'' \in \text{fPol}(1)(\Gamma)$ then $\frac{\nu'}{\nu''} \in \text{fPol}(1)(\Gamma)$ and $\text{supp}(\frac{\nu'}{\nu''}) = \text{supp}(\nu') \cup \text{supp}(\nu'')$. It can be seen that this operator $\text{Exp}$ is “valid” in the terminology of [22], i.e. every $f \in \Gamma$ and $x \in \text{dom } f$ satisfies $\sum_{h \in \text{supp}(\omega)} \omega(h)f(h(x)) \leq f(g(x))$.)

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Proof of Lemma 22(c) Consider \( x^* \in \arg \min_{x \in D^v} f_\Gamma(x) \), and suppose that \( f_\Gamma(x^*) < \infty \). Clearly, any vector \( \omega \in \mathcal{FPol}^1(\Gamma) \) is a fractional polymorphism of \( f_\Gamma \), so
\[
\sum_{g \in \text{supp}(\omega)} \omega(g) f_\Gamma(g(x^*)) \leq f_\Gamma(x^*)
\]
Thus, \( g(x^*) \in \arg \min_{x \in D^v} f_\Gamma(x) \) for any \( g \in \text{supp}(\omega) \) (and in fact for any \( g \in \mathbb{G} \)). The claim follows.

Proof of Lemma 22(a) For an operation \( h \in \mathbb{G} \) let \( \Pi_h \) be the partition of \( D \) induced by the following equivalence relation \( \sim_h \) on \( D \): \( a \sim_h b \) if \( h(a) = h(b) \). We have \( h \in \mathcal{O}_{\Pi_h} \) and thus \( \Pi_h \) is a partition of \( \Gamma \). Now let \( \Pi \) be a maximal partition of \( \Gamma \), and fix operation \( h \in \mathcal{O}_{\Pi \cap \mathbb{G}} \). We must have \( h(a) \neq h(b) \) if \( a, b \) belong to different components of \( \Pi \) (otherwise \( \Pi_h \) would be a coarser partition of \( \Gamma \) than \( \Pi \), a contradiction). Thus, \( |h(D)| = |\Pi| \). Also, the following holds for any \( g \in \mathbb{G} \): \( g(a) \neq g(b) \) for any distinct \( a, b \in h(D) \) (otherwise \( \Pi_{gh} \) would be a coarser partition of \( \Gamma \) than \( \Pi \)). Consequently, \( |g(D)| \geq |h(D)| \). We proved that \( |\Pi| = |h(D)| = \text{core-size}(\Gamma) \).

Now fix operation \( g \in \mathbb{G} \), and denote \( B = g(D) \). We will show that \( B \) is a core of \( \Gamma \) if and only if \( B \in \Pi^1 \). Clearly, this will imply the lemma’s claims (\( B_{\text{core}}^1 \subseteq \Pi^1 \) and \( \mathcal{O}_{\text{core}} = \mathcal{O}(\Pi^1) \cap \mathbb{G} \)).

If \( B \in \Pi^1 \) then \( |g(D)| = |B| = |\Pi| = \text{core-size}(\Gamma) \). Thus, \( B \in B_{\text{core}}^1 \) by Lemma 7(a).

Conversely, suppose that \( B \in B_{\text{core}}^1 \). By Lemma 7(a) \( |B| = \text{core-size}(\Gamma) \), and so \( |B| = |\Pi| \). If there exists component \( A \in \Pi \) with \( |B \cap A| \geq 2 \) then \( |(h \circ g)(D)| = |h(B)| < |B| = \text{core-size}(\Gamma) \), a contradiction. We showed that \( |B \cap A| \leq 1 \) for all \( A \in \Pi \). Thus, we must have \( |B \cap A| = 1 \) for all \( A \in \Pi \), i.e. \( B \in \Pi^1 \).

Proof of Lemma 22(b) It can be seen that \( |\Pi^1| = \prod_{A \in \Pi} |A| \). The lemma’s claim is thus equivalent to the following statement: \( c(d_1, \ldots, d_k) \leq \alpha^{d_1+\ldots+d_k} \) for positive integers \( d_1, \ldots, d_k \) with \( k \geq 1 \), where we defined \( c(d_1, \ldots, d_k) = d_1 \cdot \ldots \cdot d_k \) and denoted \( \alpha = \sqrt[k]{3} \).

To prove this, we use induction on \( k \). It can be checked that \( d \leq \alpha^d \) for any integer \( d \geq 1 \), so the base case \( k = 1 \) holds. For the induction step (with \( k \geq 2 \)) we can write \( c(d_1, d_2, \ldots, d_k) = c(d_1) \cdot c(d_2, \ldots, d_k) \leq \alpha^{d_1} \cdot \alpha^{d_2+\ldots+d_k} = \alpha^{d_1+\ldots+d_k} \), where the inequality holds by the induction hypothesis.
Proof. Let $\nu$ be a unary fractional polymorphism of $\Gamma$ such that $g(D) \subseteq B$ for all $g \in \text{supp}(\omega)$ (it exists by Lemma 7). We now consider the two directions; below $f$ is a function in $\Gamma$ of arity $n$.

1. Let $\omega$ be an $m$-ary fractional polymorphism of $\Gamma$. For any $x^1, \ldots, x^m \in (\text{dom } f) \cap B^n$ we have

$$
\frac{1}{m} \sum_{i=1}^{m} f(x^i) \geq \sum_{g \in \text{supp}(\omega)} \omega(g) f(g(x^1, \ldots, x^m)) \\
\geq \sum_{g \in \text{supp}(\omega)} \omega(g) \sum_{h \in \text{supp}(\nu)} \nu(h) f(h(g(x^1, \ldots, x^m)))
$$

Thus, vector $\omega' = \sum_{g \in \text{supp}(\omega), h \in \text{supp}(\nu)} \omega(g) \nu(h) \chi_{g \circ h \circ g}$ is an $m$-ary fractional polymorphism of $\Gamma'$ (where we have $g : B^n \rightarrow D$, $h : D \rightarrow B$ and $h \circ g : B^m \rightarrow B$). It can be checked that if $g$ satisfies some identity such as $g(x_1, x_2, \ldots, x_m) = g(x_2, \ldots, x_m, x_1)$ then $h \circ g$ also satisfies the same identity. The claim follows.

2. Let $\omega'$ be an $m$-ary fractional polymorphism of $\Gamma'$. For any $x^1, \ldots, x^m \in \text{dom } f$ we have

$$
\frac{1}{m} \sum_{i=1}^{m} f(x^i) \geq \frac{1}{m} \sum_{i=1}^{m} \sum_{h \in \text{supp}(\nu)} \nu(h) f(h(x^i)) \\
\geq \sum_{h \in \text{supp}(\nu)} \nu(h) \sum_{g \in \text{supp}(\omega')} \omega'(g) f(g(h(x^1), \ldots, h(x^m)))
$$

Thus, vector $\omega = \sum_{g \in \text{supp}(\omega'), h \in \text{supp}(\nu)} \omega'(g) \nu(h) \chi_{g \circ h \circ g}$ is an $m$-ary fractional polymorphism of $\Gamma$ (where we have $[h, \ldots, h] : D^m \rightarrow B^m$, $g : B^m \rightarrow B$ and $g \circ [h, \ldots, h] : D^m \rightarrow B$). It can be checked that if $g$ satisfies some identity such as $g(x_1, x_2, \ldots, x_m) = g(x_2, \ldots, x_m, x_1)$ then $g \circ [h, \ldots, h]$ also satisfies the same identity. The claim follows.

Thus, we can return a non-empty set $B$ that $\Gamma$ is solvable (otherwise any $B$ is treated as “1”, and output $(b)$ as “0”).

C Proof of Corollary 24

**Part (1)** Let $\text{TEST}(\Gamma, \Pi) \in \{0, 1\}$ be the predicate that equals 1 if $\Pi$ is a partition of $\Gamma$, and 0 otherwise. Note that $\text{TEST}(\Gamma, \Pi) \geq \text{TEST}(\Gamma, \Pi')$ if $\Pi \preceq \Pi'$. Clearly, a maximal partition $\Pi$ of $\Gamma$ can be found by a greedy search that starts with $\Pi = \{\{a\} \mid a \in D\}$ and then makes at most $|D|(|D| - 1)/2$ evaluations of $\text{TEST}(\Gamma, \Pi)$ for partitions $\Pi \neq \{\{a\} \mid a \in D\}$. To get the desired algorithm, we simply replace each evaluation of $\text{TEST}(\Gamma, \Pi)$ in this greedy search with a call to the algorithm in Theorem 23 (where output $(a)$ is treated as “1”, and output $(b)$ as “0”).

**Part (2)** Assume that we have a conditional maximal partition $\Pi$ of $\Gamma$. Let us run the algorithm from Theorem 21 with $\sigma = \Pi$ and $B_0 = \Pi^\perp$. Suppose that it returns a fractional polymorphism $\omega \in \mathcal{fPol}(\Gamma)$ with $\text{supp}(\omega) \subseteq \hat{O}[\Pi^\perp]$. If it satisfies $\text{supp}(\omega) \subseteq \hat{O}[\Pi^\perp]$ then we pick some $g \in \text{supp}(\omega)$ and return set $B = g(D)$ together with vector $\omega$. In any other case we return set $B = \varnothing$, meaning that $\Gamma$ is not solvable. Next, we show the correctness of this procedure. From now on we assume that $\Gamma$ is solvable (otherwise any $B$ is a conditional core of $\Gamma$ and so the claim is trivial). The assumption means $\Pi$ that maximal partition of $\Gamma$ and $\hat{O}[\Pi^\perp] \cap \text{supp}(\Gamma)$ (by Lemma 22).

Calling the algorithm from Theorem 21 may have four possible outcomes listed below. Note, we return a non-empty set $B$ only in the first case.

(i) *The algorithm gives vector $\omega \in \mathcal{fPol}(\Gamma)$ with $\text{supp}(\omega) \subseteq \hat{O}[\Pi^\perp]*.

Recall that in this case we pick $g \in \text{supp}(\omega)$ and return set $B = g(D)$ together with vector $\omega$. We then have $g \in \hat{O}[\Pi^\perp] \cap \text{supp}(\Gamma) = \hat{O}[\Pi^\perp]$ and thus $B$ is a core of $\Gamma$. 

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In the last three cases we will arrive at a contradiction, meaning that $\Gamma$ is not solvable.

(ii) *The algorithm gives vector $\omega \in \text{fPol}(\Gamma)$ s.t. supp$(\omega)$ contains operation $g \in \widehat{O}[\Pi^\perp] - O[\Pi^\perp]$. From definitions, $|g(D)| < |\Pi|$. However, we have $O^{core}_\Gamma \subseteq O[\Pi^\perp]$ and thus core-size$(\Gamma) = |\Pi|$ - a contradiction.*

(iii) *The algorithm asserts that there is no $\omega \in \text{fPol}(\Gamma)$ with supp$(\omega) \subseteq O^{\perp}[\Pi^\perp]$. By Lemma 7(a,b), there exists $\omega \in \text{fPol}(\Gamma)$ with supp$(\omega) \subseteq O^{core}_\Gamma \subseteq O[\Pi^\perp]$ - a contradiction.*

(iv) *The algorithm asserts that either $\Gamma$ is not solvable or $\Pi^\perp \cap B^{core}_\Gamma = \emptyset$. Since $O^{core}_\Gamma \subseteq O[\Pi^\perp]$, any $B \in B^{core}_\Gamma$ satisfies $B \in \Pi^\perp$. Thus, $\Pi^\perp \cap B^{core}_\Gamma \neq \emptyset$ - a contradiction.*

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