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Maximal automorphisms of Calabi-Yau manifolds versus maximally unipotent monodromy

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Abstract. Let \( \alpha \) be an automorphism of the local universal deformation of a Calabi-Yau 3-manifold \( X \), which does not act by \( \pm \text{id} \) on \( H^3(X, \mathbb{C}) \). We show that the bundle \( F^2(H^3) \) in the VHS of each maximal family containing \( X \) is constant in this case. Thus \( X \) cannot be a fiber of a maximal family with maximally unipotent monodromy, if such an automorphism \( \alpha \) exists. Moreover we classify the possible actions of \( \alpha \) on \( H^3(X, \mathbb{C}) \), construct examples and show that the period domain is a complex ball containing a dense set of CM points given by a Shimura datum in this case.

1. Introduction

Due to their importance in theoretical physics, we are interested in Calabi-Yau 3-manifolds. We construct some examples of Calabi-Yau 3-manifolds \( X \) with degree 3 automorphisms, which extend to the local universal deformation. Here such automorphisms are called maximal. Our examples of maximal automorphisms do not act by \( \pm 1 \) on \( H^3(X, \mathbb{C}) \). The subbundle \( F^2(H^3) \) of the variation of Hodge structures of the local universal deformation is constant, if a maximal automorphism exists and does not act by \( \pm 1 \) on \( H^3(X, \mathbb{C}) \). Moreover we give an additional example of a Calabi-Yau 3-manifold, which does not necessarily have a maximal automorphism, but satisfies the condition that \( F^2(H^3) \) is constant in the VHS of the local universal deformation. We will see that \( F^2(H^3) \) is constant for each maximal family containing \( X \) as fiber, if this holds true with respect to the local universal deformation of \( X \). This forbids \( X \) to be a fiber of a maximal family with maximally unipotent monodromy. Thus the assumptions of the formulation of the mirror conjecture in [11] cannot be satisfied by \( X \), if \( F^2(H^3) \) is constant in the local universal deformation of \( X \).

Moreover we show that the period domain is a complex ball and the local universal deformation of \( X \) has a dense set of complex multiplication (CM) fibers, if \( X \) has a maximal automorphism, which does not act by \( \pm 1 \) on \( H^3(X, \mathbb{C}) \). Theoretical physicists are interested in Calabi-Yau 3-manifolds with CM—in particular if there exists a mirror pair of Calabi-Yau 3-manifolds with CM (see [10]).

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Moreover we will see that the quotient of the maximal automorphisms of $X$ by the automorphisms acting trivially on $H^3(X, \mathbb{Z})$ is given by

$$\{e\}, \ Z/(2), \ Z/(3), \ Z/(4) \text{ or } Z/(6).$$

2. Examples with maximal automorphisms

Here a Calabi-Yau 3-manifold $X$ is a compact Kähler manifold of dimension 3 such that

$$\omega_X \cong \mathcal{O}_X \text{ and } h^{k,0}(X) = 0 \text{ for } k = 1, 2.$$

Let $\mathcal{X} \to B$ be the local universal deformation of $X = X_0$. We say that a family $f : \mathcal{Y} \to Z$ of Calabi-Yau 3-manifolds is maximal, if for each $z \in Z$ there exists an open neighborhood $U$ of $z$ such that $\mathcal{Y}_U$ is isomorphic to the Kuranishi family of $\mathcal{Y}_z$. Recall that

$$H^3 := R^3f_* (\mathbb{Q})$$

is a local system and that

$$\mathcal{H}^3 := H^3 \otimes \mathbb{Q} \mathcal{O}_Z$$

is a holomorphic bundle. The variation of Hodge structures of weight 3 is given by the filtration

$$0 \subset F^3(\mathcal{H}^3) \subset F^2(\mathcal{H}^3) \subset F^1(\mathcal{H}^3) \subset \mathcal{H}^3$$

by holomorphic subbundles.

Recall that a marked $K3$ surface is a pair $(S, \mu)$ consisting of a $K3$ surface $S$ and a marking $\mu$, that is an isometry $\mu : L \to H^2(S, \mathbb{Z})$ of lattices, where

$$L = U \oplus U \oplus U \oplus -E_8 \oplus -E_8.$$

The marked $K3$ surfaces $(S, \mu)$ and $(S', \mu')$ are isomorphic, if there exists an isomorphism $f : S \to S'$ such that $\mu = f^* \circ \mu'$. By gluing marked local universal deformations of $K3$ surfaces, we obtain the complex analytic moduli space $M$ of marked $K3$ surfaces with universal family $f : S \to M$. Moreover let $\phi$ denote an isometry of order 3 on $L$ and $(L_\mathbb{C})_\eta$ denote the eigenspace on $L_\mathbb{C}$ with eigenvalue $\eta$ with respect to $\phi$. For this section we fix

$$\xi = \exp\left(\frac{2\pi i}{3}\right) \text{ and } r = \dim(L\mathbb{C})_\xi - 1.$$