Quantization of the inhomogeneous Bianchi I model: quasi-Heisenberg picture

Cherkas S.L. *
Institute for Nuclear Problems, Bobruiskaya str. 11, Minsk, 220050, Belarus

Kalashnikov V.L.†
Institut für Photonik, Technische Universität Wien, Gusshausstrasse 27/387, Vienna A-1040, Austria

Abstract

The quantization scheme is suggested for a spatially inhomogeneous 1+1 Bianchi I model. The scheme consists in quantization of the equations of motion and gives the operator (so-called quasi-Heisenberg) equations describing an explicit evolution of a system. Some particular gauge suitable for quantization is proposed. The Wheeler-DeWitt equation is considered in the vicinity of zero scale factor and it is used to construct a space, where the quasi-Heisenberg operators act. Spatial discretization as a UV regularization procedure is suggested for the equations of motion.

1 Introduction

Spatially homogeneous minisuperspace models [1–3] are often used as a testbed for the quantum gravity [4–9]. Inhomogeneous 1+1 Bianchi I model belongs to the so-called midisuperspace models [10] and has more rich properties, in particular, it admits an existence of gravitational waves. This model can be reduced to the Gowdy one [11] for which the solution of the Wheeler-DeWitt equation has been obtained in a closed form [12,13]. However, the solution of the Wheeler-DeWitt equation does not resolve the problem of the gravity quantization completely. An interpretation of the Wheeler-DeWitt equation encounters the absence of a variable, which would play the role of time, and all approaches to the quantum gravity face, as a rule, this challenge. Some approaches regarding the relation of the Wheeler-DeWitt equation to dynamics have been suggested. For instance, the approach of Ref. [14] explores the notion of “arrival time” from the non-relativistic quantum mechanics to build the incoherent histories. Also, there exists a more straightforward approach consisting

---

*E-mail:cherkas@inp.bsu.by
†E-mail:kalashnikov@tuwien.ac.at
in quantization of the equations of motion [15–17]. The result of quantization is the so-called quasi-Heisenberg operators.

Below we apply this approach to the quantization of the Bianchi I model. The reason why we investigate the Bianchi I model instead the Gowdy one is that the former has a Hamiltonian, which is diagonal on the momentums. Besides, it divides naturally a spatial geometry into the scale factor and the remaining conformal geometry [18], while the Gowdy model suggests other separation. Also, the Wheeler-DeWitt equation will be used below since the quasi-Heisenberg operators act in a space of solutions of this equation.

2 Nonuniform Bianchi model

Let us consider the metric given by the interval

\[ ds^2 = e^{2\alpha} \left( d\eta^2 - e^{-4B} dx^2 - e^{2B+2\sqrt{3}V} dy^2 - e^{2B-2\sqrt{3}V} dz^2 \right), \]

(1)

where the functions \( \alpha, B, V \) depend on the spatial coordinate \( x \) and the conformal time \( \eta \). The spatially homogeneous metric of such a type has been considered in Ref. [19].

Substitution of this metric into the Einstein equations allows obtaining a set of five independent equations. Two of them are the Hamiltonian and the momentum constraints

\[ \mathcal{H} = \frac{1}{2} e^{2\alpha} \left( -\alpha'' + B'' + V'' \right) + e^{2\alpha+4B} \left( \frac{1}{6} (\partial_x \alpha)^2 + \frac{1}{3} \partial_{xx} \alpha + \frac{7}{6} (\partial_x B)^2 + \frac{1}{3} \partial_{xx} B + \frac{4}{3} \partial_x \alpha \partial_x B + \frac{1}{2} (\partial_x V)^2 \right) = 0, \]

(2)

\[ \mathcal{P} = e^{2\alpha} \left( -\frac{1}{3} \partial_x \alpha \alpha' + \partial_x BB' + \frac{2}{3} \partial_x \alpha B' + \partial_x V V' + \frac{1}{3} \partial_x B' + \frac{1}{3} \partial_x \alpha' \right) = 0, \]

(3)

and the rest are the equations of motion

\[ \alpha'' - e^{4B} \left( \partial_{xx} \alpha + (\partial_x \alpha)^2 + (\partial_x V)^2 + \frac{7}{3} (\partial_x B)^2 + \frac{2}{3} \partial_{xx} B + 4 \partial_x B \partial_x \alpha \right) + \alpha'^2 + V'^2 + B'^2 = 0, \]

(4)

\[ B'' + \frac{1}{3} e^{4B} \left( \partial_{xx} B + 2(\partial_x B)^2 + 6(\partial_x V)^2 - 2(\partial_x \alpha)^2 + 2 \partial_x \alpha \partial_x B + 2 \partial_{xx} \alpha \right) + 2 B' \alpha' = 0, \]

(5)

\[ V'' + 2 V' \alpha' - e^{4B} \left( \partial_{xx} V + 2 \partial_x V \partial_x \alpha + 4 \partial_x B \partial_x V \right) = 0. \]

(6)

The full Hamiltonian \( H = \int \mathcal{H} dx \) has to be zero during an evolution of system.

The relevant Hamiltonian and the momentum constraints written in the terms of momentums \( \pi_V(x) \equiv \frac{\delta H}{\delta V'(x)} = e^{2\alpha} V', \pi_B(x) \equiv \frac{\delta H}{\delta B'(x)} = e^{2\alpha} B' \) and \( p_\alpha(x) \equiv -\frac{\delta H}{\delta \alpha'(x)} = e^{2\alpha} \alpha' \) take the form...
\[ \mathcal{H} = \frac{1}{2} e^{-2\alpha} \left(-p_\alpha^2 + \pi_B^2 + \pi_V^2\right) + e^{2\alpha + 4B} \left(\frac{1}{6} (\partial_x \alpha)^2 + \frac{1}{3} \partial_{xx} \alpha + \frac{7}{6} (\partial_x B)^2 + \frac{1}{3} \partial_{xx} B + \frac{4}{3} \partial_x \alpha \partial_x B + \frac{1}{2} (\partial_x V)^2\right), \]  

(7)

\[ \mathcal{P} = -p_\alpha \partial_x \alpha + \pi_B \partial_x B + \pi_V \partial_x V + \frac{1}{3} \partial_x \pi_B + \frac{1}{3} \partial_x p_\alpha. \]  

(8)

Using the Poisson brackets

\[ \{F(x), G(x')\} = \int \left( \frac{\delta F(x) \delta G(x')}{\delta p_\alpha(\xi) \delta \alpha(\xi)} + \frac{\delta F(x) \delta G(x')}{\delta \alpha(\xi) \delta p_\alpha(\xi)} + \frac{\delta F(x) \delta G(x')}{\delta \pi_V(\xi) \delta V(\xi)} + \frac{\delta F(x) \delta G(x')}{\delta V(\xi) \delta \pi_V(\xi)} + \frac{\delta F(x) \delta G(x')}{\delta \pi_B(\xi) \delta B(\xi)} + \frac{\delta F(x) \delta G(x')}{\delta B(\xi) \delta \pi_B(\xi)} \right) d\xi \]  

(9)

one can obtain the constraint algebra:

\[ \{\mathcal{H}(x), \mathcal{H}(x')\} = (\mathcal{P}(x)e^{AB(x)} + \mathcal{P}(x')e^{AB(x')})\delta'(x' - x), \]  

\[ \{\mathcal{P}(x), \mathcal{P}(x')\} = (\mathcal{P}(x) + \mathcal{P}(x'))\delta'(x' - x), \]  

\[ \{\mathcal{H}(x), \mathcal{P}(x')\} = \frac{2}{3} (\mathcal{H}(x) + \mathcal{H}(x'))\delta'(x' - x) - \frac{1}{3} \mathcal{H}'(x)\delta(x' - x). \]  

(10)

It is also possible to find the evolution of constraints by calculation of their Poisson brackets with the Hamiltonian \(H\):

\[ \partial_{\xi} \mathcal{P}(\eta, x) = \{H, \mathcal{P}(\eta, x)\} = \frac{1}{3} \partial_x \mathcal{H}(\eta, x), \]  

(11)

\[ \partial_{\eta} \mathcal{H} = \partial_x \left(e^{AB} \mathcal{P}\right). \]  

(12)

### 3 Quantization

The quantization procedure consists in the formulation of initial conditions for the quasi-Heisenberg operators. Thereafter it is permissible for operators to evolve in accordance with the equations of motion, considered as the operator equations. To determine the initial commutation relations, we will use the Dirac quantization procedure [20][22]. Besides the constraints, an additional gauge condition is needed.

Let us take the following gauge at the initial moment of time

\[ A = \alpha - \alpha_0, \]  

(13)

\[ B = \partial_x \pi_B, \]  

(14)

where \(\alpha_0\) should be tended to \(-\infty\).

For quantization by means of the Dirac brackets [20], one has to calculate the matrix \(M_{ij}(x, x') = \{\Phi_i(x), \Phi_j(x')\}\), where a set of constraints is

\[ \Phi_i = (\mathcal{H}, \mathcal{P}, A, B). \]
At the shell of constraints $\Phi_i(x) = 0$, the matrix $M_{ij}(x, x')$ in the vicinity $\alpha_0 \to -\infty$ has the form
\[
M(x, x') = \begin{pmatrix}
0 & 0 & \frac{p_0(x)\delta(x-x')}{\exp(2\alpha_0)} & 0 \\
0 & 0 & \frac{-\delta(x-x')}{\pi_B\delta''(x-x')} & 0 \\
-\frac{p_0(x)\delta(x-x')}{\exp(2\alpha_0)} & -\frac{\delta'(x-x')}{3} & 0 & 0 \\
0 & 0 & -\pi_B\delta''(x-x') & 0 \\
\end{pmatrix}. \quad (15)
\]

Due to antisymmetry of the Poison brackets, $M_{ij}(x, x')$ obeys the identity $M_{ij}(x, x') = -M_{ji}(x', x)$. The inverse matrix satisfying $\int M_{ij}(x, x'')M^{-1}_{jk}(x'', x')dx'' = \delta_{ik}\delta(x-x')$ is given by
\[
M^{-1}(x, x') = \begin{pmatrix}
0 & 0 & -e^{2\alpha_0}\frac{\delta(x-x')}{p_0(x)} & e^{2\alpha_0}\frac{\theta(x-x')}{\pi_B} \\
0 & 0 & 0 & -\frac{\Delta'(x-x')}{3p_0(x)\pi_B} \\
-e^{2\alpha_0}\frac{\delta(x-x')}{p_0(x)} & -\frac{\delta'(x-x')}{\pi_B} & 0 & 0 \\
e^{-2\alpha_0}\frac{\delta(x-x')}{p_0(x)} & \frac{\Delta(x-x')}{\pi_B} & 0 & 0 \\
\end{pmatrix},
\]
where $\theta(x)$ is an antiderivative of the Dirac delta-function: $\theta'(x) = \delta(x)$, and $\Delta(x)$ is an antiderivative of $\theta(x)$: $\Delta'(x) = \theta(x)$. It is assumed, that the antiderivatives have the following symmetry properties: $\theta(-x) = -\theta(x)$, $\Delta(-x) = \Delta(x)$.

For the Dirac quantization, one needs to calculate the Dirac brackets which give the commutation relations for the corresponding operators at an initial moment of time after multiplication by $-i$.

Calculation of the Dirac brackets
\[
\{G(x), F(x')\}_D = \{G(x), F(x')\} - \sum_{i,j} \int \{G(x), \Phi_i(x'')\}M^{-1}_{ij}(x'', x')\{\Phi_j(x''), F(x')\}dx''dx''
\]
leads to
\[
\{\pi_V(x), V(x')\}_D = \delta(x-x'), \\
\{\pi_B(x), B(x')\}_D = 0, \\
\{\pi_\alpha(x), \alpha(x')\}_D = 0, \\
\{\pi_B(x), \alpha(x')\}_D = 0, \\
\{\pi_B(x), \pi_\alpha(x')\}_D = 0, \\
\{B(x), V(x')\}_D = -\frac{\pi_V(x)}{3p_0(x)\pi_B}\delta(x-x'), \\
\{B(x), \pi_\alpha(x')\}_D = -\frac{1}{\pi_B}(\partial_{x'}\pi_\alpha(x')\theta(x'-x) + p_\alpha(x')\delta(x'-x)), \\
\{\pi_B(x), \pi_\alpha(x')\}_D = 0, \\
\{\pi_\alpha(x), V(x')\}_D = \frac{\pi_V(x)}{p_\alpha(x)}\delta(x'-x). \quad (16)
\]

From the foregoing it follows that $\alpha$ and $\pi_B$ are initially $c$-numbers in the gauge considered.
In fact we use some time-dependent gauge, which is known only at an initial moment of time, and it is permissible for the commutation relations in the model under consideration to evolve.

Operator realization of the commutation relations at the initial moment of time (corresponding to $\alpha_0 \to -\infty$) may be written as

$$\hat{\pi}_V(x) = -i \frac{\delta}{\delta V(x)},$$

$$\hat{p}_\alpha(x) = \sqrt{\hat{\pi}_V^2(x) + \pi_B^2},$$

$$\hat{B}(x) = -\frac{1}{\pi_B} \left( \int_0^x S(\hat{\pi}_V(x') \partial_{x'} V(x')) dx' + \frac{1}{3} \hat{p}_\alpha(x) \right),$$

where the symbol $S$ denotes symmetrization of the noncommutative operators, i.e.

$$S(\hat{A}\hat{B}) = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}) \text{ or } S(\hat{A}\hat{B}\hat{C}) = \frac{1}{6}(\hat{A}\hat{B}\hat{C} + \hat{B}\hat{A}\hat{C} + \hat{A}\hat{C}\hat{B} + \ldots).$$

Thus the equation of motion (4),(5),(6) should be considered as the operator equations of motion with the initial conditions

$$\hat{V}(0, x) = V(x), \quad \hat{V}'(0, x) = -ie^{2\alpha_0} \frac{\delta}{\delta V(x)},$$

$$\hat{B}(0, x) = -\frac{1}{\pi_B} \left( \int_0^x S(\hat{\pi}_V(x') \partial_{x'} V(x')) dx' \right) + \frac{1}{3} \hat{p}_\alpha(x),$$

$$\hat{B}'(0, x) = e^{2\alpha_0} \pi_B, \quad \hat{\alpha}(0, x) = \alpha_0, \quad \hat{\alpha}'(0, x) = -e^{2\alpha_0} \sqrt{-\frac{\delta^2}{\delta V^2(x)} + \pi_B^2},$$

where $\pi_B$ is some constant and $\alpha_0$ should be tended to $-\infty$.

Let us suggest a space, where these operators act. Let’s consider the Wheeler-DeWitt equation in the vicinity of $\alpha \to -\infty$

$$\left( \frac{\delta^2}{\delta \alpha(x)} - \frac{\delta^2}{\delta V^2(x)} + \pi_B^2 \right) \Psi[\alpha, V] = 0,$$  \hspace{1cm} (18)

where we take into account that $\pi_B$ is some constant.

Solution of Eq. (18) is of the form of the wave packet

$$\Psi[\alpha, V] = \int C[\pi_V] e^{\int \left( -i \alpha(x) \sqrt{\pi_B^2 + \pi_V^2(x) + i \pi_V(x)V(x)} \right) dx} \mathcal{D}_{\pi_V},$$

$\mathcal{D}_{\pi_V}$ denotes functional integration over $\pi_V(x)$.

A mean value of an arbitrary operator can be evaluated as

$$<\Psi|\hat{A}[\alpha, -i \frac{\delta}{\delta V(x)}]|\Psi> = i \int \left( \Psi^*[\alpha, V] \hat{A} \hat{D}^{-1/4} \hat{A} \hat{D}^{-1/4} \frac{\delta}{\delta \alpha(x)} \Psi[\alpha, V] \right. \left. - \left( \frac{\delta}{\delta \alpha(x)} \Psi^*[\alpha, V] \right) \hat{D}^{-1/4} \hat{A} \hat{D}^{1/4} \Psi[\alpha, V] \right) \mathcal{D}V \bigg|_{\alpha(x)=\alpha_0 \to -\infty},$$

where
where $\hat{D}(x) = -\frac{\delta^2}{\delta V^2(x)} + \pi_B$ and $\mathcal{D}V$ denotes functional integration over $V(x)$.

In many cases, it is more convenient to use the momentum representation $\hat{\pi}_V(x) = \pi_V(x)$, $\hat{V}(x) = i\frac{\delta}{\delta \pi_V(x)}$, where the wave function $\psi$ is

$$\psi[\alpha, \pi_V] = C[\pi_V] \exp \left( -i \int \alpha(x) \sqrt{\pi_V^2(x) + \pi_B^2} \, dx \right). \quad (21)$$

Then, a mean value of an operator becomes

$$<\psi | \hat{A}[\alpha, \pi_V(x), i \frac{\delta}{\delta \pi_V(x)}] | \psi > =$$

$$\int \mathcal{D}^*[\pi_V] e^{-i \int \alpha(x) \sqrt{\pi_V^2(x) + \pi_B^2} \, dx} \hat{A} e^{i \alpha(x) \sqrt{\pi_V^2(x) + \pi_B^2} \, dx} C[\pi_V] \mathcal{D} \pi_V \bigg|_{\alpha(x)=\alpha_0 \to -\infty}. \quad (22)$$

Thus, one has an exact quantization scheme consisting of the Wheeler-DeWitt equation in the vicinity of small scale factor (15), the operator initial conditions (17) for the equations of motion and the expressions (20), (22) for calculation of the mean values of operators.

4 Discretization of the operator equations

At least two generations of physicists can not overcome divergencies arising in rigorous operator formulation of the ordinary QFT in the 4-dimensional Minkowsky space, although some success for the 1+1 and 1+3 dimensional models has been reached [23]. Here we consider a discretization as a method of regularization of the functional operator equations, which eliminates the infinite quantities. An alternative is to use the discretized action initially, however, we do not consider such a possibility here. The discretization consists in choosing of some spatial box of length $L$ and granulation of it by points $x_i$ separated by distance $\Delta x$. Periodicity condition is implied at $x_1 = x_N$.

Continuous oscillators of $V$-field should be replaced by the discrete momentums $\pi_V(x) \to \pi_{Vj}/\sqrt{\Delta x}$, $V(x) \to V_j/\sqrt{\Delta x}$ [24], where $\pi_{Vj}$ and $V_j$ posses an ordinary commutation relation $[\pi_{Vn}, V_m] = -i\delta_{nm}$ including the Kronecker symbol $\delta_{mn}$. Really in this case for $\pi_V(x_n)$ and $V(x_m)$, we have $[\pi_V(x_n), V(x_m)] = -i\delta_{nm}/\Delta x$, that turns into the Dirac delta-function $\delta_{nm}/\Delta x \to \delta(x_n - x_m)$ in the limit $\Delta x \to 0$.

However, it is more convenient to use straightforwardly the quantities $\pi_{Vj} = \pi_V(x_j)$, $V_j = V(x_j)$ for which the commutation relations $[\pi_{Vn}, V_m] = -i\delta_{nm}/\Delta x$ are satisfied. These commutation relations can be realized by the operators $\hat{V}_m = V_m$, $\hat{\pi}_{Vn} = -\frac{\Delta x}{\delta V_n}$ or $\hat{\pi}_{Vn} = \pi_{Vn}$, $\hat{V}_n = -\frac{\Delta x}{\delta \pi_{Vn}}$.

The discrete Wheeler-DeWitt equation in the vicinity of $\alpha_j = \alpha_0 \to -\infty$ has the following form in the momentum representation:

$$\left( -\frac{1}{(\Delta x)^2} \frac{\partial^2}{\partial^2 \alpha_i} + \pi_{V1}^2 + \pi_B^2 \right) \psi(\alpha_i, \ldots, \alpha_N, \pi_{V1}, \ldots, \pi_{VN}) = 0, \quad (23)$$
with a solution
\[
\psi(\alpha_1 \ldots \alpha_N, \pi_{V1}, \ldots, \pi_{VN}) = C(\pi_{V1}, \ldots, \pi_{VN}) \exp \left( -i\Delta x \sum_{j=1}^{N} \alpha_j \sqrt{\pi_{Vj}^2 + \pi_{Bj}^2} \right),
\]
where \( \Delta x \) is the discretization length.

A mean value of an arbitrary operator can be evaluated as
\[
\langle \psi | \hat{A}(\alpha_1 \ldots \alpha_N, \pi_{V1}, \ldots, \pi_{VN}) \frac{i}{\Delta x} \frac{\partial}{\partial \pi_{V1}}, \ldots, \frac{i}{\Delta x} \frac{\partial}{\partial \pi_{VN}} | \psi \rangle = \int C^*(\pi_{V1} \ldots \pi_{VN}) e^{-i\Delta x \sum_{j=1}^{N} \alpha_j \sqrt{\pi_{Vj}^2 + \pi_{Bj}^2}} \hat{A} e^{i\Delta x \sum_{j=1}^{N} \alpha_j \sqrt{\pi_{Vj}^2 + \pi_{Bj}^2}} C(\pi_{V1} \ldots \pi_{VN}) d\pi_{V1} \ldots d\pi_{VN} \bigg|_{\alpha_1 \ldots \alpha_N = 0 \to -\infty}.
\]

For the equations of motion, one has to take some operator ordering and then to come to its discrete version. A question arises about the conservation of constraints during evolution. Constraints can be violated by both noncommutativity of operators and discretization of the equations of motion. The last question is analogous to that regarding the energy conservation in a system of the discretized magnetic hydrodynamic equations [25]. Special discretization schemes were suggested for the magnetic hydrodynamic equations, which conserve the energy [25].

Let us propose a solution of both problems. Let’s consider the discretized operator Hamiltonian, the momentum constraints and, besides, choose the symmetrical ordering of operators:
\[
\hat{P}_j = S \left( e^{2\hat{\alpha}_j} \left( -\frac{\hat{\alpha}_{j+1} - \hat{\alpha}_j}{3\Delta x} \hat{\alpha}_j + \frac{\hat{B}_{j+1} - \hat{B}_j}{\Delta x} \hat{B}_j + \frac{2}{3} \frac{\hat{\alpha}_{j+1} - \hat{\alpha}_j}{\Delta x} \hat{B}_j' + \frac{\hat{V}_{j+1} - \hat{V}_j}{\Delta x} \hat{V}_j' + \frac{\hat{B}_{j+1}' - \hat{B}_j'}{3\Delta x} + \frac{\hat{\alpha}_{j+1}' - \hat{\alpha}_j'}{3\Delta x} \right) \right),
\]
\[
\hat{H}_j = S \left( \frac{1}{2} e^{2\hat{\alpha}_j} \left( -\hat{\alpha}_j^2 + 3 \hat{B}_j^2 + \frac{3}{2} \hat{V}_j^2 \right) + e^{2\hat{\alpha}_j + 4\hat{B}_j} \left( \frac{1}{6} \left( \frac{\hat{\alpha}_{j+1} - \hat{\alpha}_j}{\Delta x} \right)^2 \right) + \frac{7}{6} \left( \frac{\hat{B}_{j+1} - \hat{B}_j}{\Delta x} \right)^2 + \frac{\hat{B}_{j+1} - \hat{B}_j}{3(\Delta x)^2} + \frac{4}{3} \left( \frac{\hat{\alpha}_{j+1} - \hat{\alpha}_j}{(\Delta x)^2} \right)^2 \right) \right). \]

Here, we need more careful definition of the symmetrization [26][27]. Let’s there is an arbitrary function \( f(x_1, x_2, \ldots, x_n) \) of \( n \)-variables. Then, one can define a formal Fourier transform
\[
\tilde{f}(\zeta_1, \zeta_2 \ldots \zeta_n) = \frac{1}{(2\pi)^n} \int f(x_1, x_2, \ldots, x_n) e^{-i(x_1\zeta_1 + x_2\zeta_2 + \ldots + x_n\zeta_n)} dx_1 \ldots dx_n.
\]
A symmetrized function of noncommutative operators \( \hat{A}_1, \ldots \hat{A}_n \) is defined as

\[
S(f(\hat{A}_1, \hat{A}_2 \ldots \hat{A}_n)) = \int \tilde{f}(\zeta_1, \zeta_2, \ldots \zeta_n) e^{i(\hat{A}_1 \zeta_1 + \hat{A}_2 \zeta_2 + \ldots + \hat{A}_n \zeta_n)} d\zeta_1 \ldots d\zeta_n. \tag{29}
\]

Our idea is to use the discretized version of Eqs. (11, 12) as the equations of motion:

\[
\partial_t \hat{H}_j = \frac{e^{4\hat{B}_j+1} \hat{P}_{j+1} + \hat{P}_{j+1} e^{4\hat{B}_j+1} - e^{4\hat{B}_j} \hat{P}_j - \hat{P}_j e^{4\hat{B}_j}}{2\Delta x},
\]

\[
\partial_t \hat{P}_j = \frac{\hat{H}_{j+1} - \hat{H}_j}{3\Delta x}. \tag{31}
\]

Using the formula for differentiation of a symmetrized function [27]:

\[
\frac{d}{dt} S(f(\hat{A}_1(t), \hat{A}_2(t) \ldots \hat{A}_n(t))) = S \left( \sum_{j=1}^{n} \frac{d\hat{A}_j}{dt} \partial_j f(\hat{A}_1(t), \hat{A}_2(t) \ldots \hat{A}_n(t)) \right), \tag{32}
\]

(here \( \partial_j f(x_1, x_2 \ldots x_n) \) denotes a partial derivative of a function \( f \) over the \( j \)-argument) allows calculating the time derivatives in the left hand side of Eqs. (30, 31) and rewriting Eqs. (30, 31) in the form of

\[
S \left( e^{2\hat{a}_j} \left( \hat{B}'_j \hat{B}''_j + \hat{V}'_j \hat{V}''_j - \hat{a}'_j \hat{a}''_j + \frac{1}{3(\Delta x)^2} e^{4\hat{B}_j} \left( 7\hat{B}'_j + 7\hat{B}''_j + 2\hat{B}_{j-1} + 3\hat{V}''_j \right) \right) \right)
\]

\[
-2\hat{B}_j (2 + 7\hat{B}_{j+1} - 4\hat{a}_j + 4\hat{a}_{j+1}) + 2\hat{B}_{j+1} (1 - 4\hat{a}_j + 4\hat{a}_{j+1}) - 6\hat{V}_j \hat{V}'_j + 3\hat{V}''_j + 14\hat{V}'_j \hat{V}''_j - 2\hat{a}_j - 2\hat{a}_{j+1} - 2\hat{a}_{j+1} \hat{a}_{j+1} - 2\hat{a}_j \hat{a}_{j+1}
\]

\[
+3\hat{V}'_{j+1} - 4\hat{a}_j + 2\hat{a}_{j+1} - 2\hat{a}_j \hat{a}_{j+1}
\]

\[
\hat{a}'_j (\hat{B}'_j + \hat{V}'_j - \hat{a}'_j)
\]

\[
\frac{1}{3(\Delta x)^2} e^{4\hat{B}_j} \left( -2 + 7\hat{B}_j + 7\hat{B}_{j+1} + 4\hat{a}_j + 4\hat{a}_{j+1} \right)
\]

\[
\hat{B}'_j (2 + 7\hat{B}_{j+1} - 4\hat{a}_j + 4\hat{a}_{j+1}) + 2\hat{B}_{j+1} (1 - 4\hat{a}_j + 4\hat{a}_{j+1}) - 6\hat{V}_j \hat{V}'_j + 3\hat{V}''_j + 14\hat{V}'_j \hat{V}''_j - 2\hat{a}_j - 2\hat{a}_{j+1} - 2\hat{a}_{j+1} \hat{a}_{j+1} - 2\hat{a}_j \hat{a}_{j+1}
\]

\[
-4\hat{B}_j \hat{a}_{j+1} + 4\hat{B}_{j+1} \hat{a}'_{j+1} - \hat{a}_j \hat{a}'_{j+1} + \hat{a}_{j+1} \hat{a}'_j + \hat{a}'_{j+1}
\]

\[
\frac{e^{4\hat{B}_j+1} \hat{P}_{j+1} + \hat{P}_{j+1} e^{4\hat{B}_j+1} - e^{4\hat{B}_j} \hat{P}_j - \hat{P}_j e^{4\hat{B}_j}}{2\Delta x}, \tag{33}
\]

\[
S \left( e^{2\hat{a}_j} \left( -3\hat{B}'_j - 3\hat{V}'_j + 2\hat{B}_{j+1} \hat{a}'_j - \hat{a}'_j + 2\hat{a}_j \hat{a}'_j - 2\hat{a}_j \hat{a}'_j \right) \right)
\]

\[
+3\hat{V}'_j (-\hat{V}'_{j+1} + 2(-\hat{V}_j + \hat{V}_{j+1}) \hat{a}'_j + \hat{a}'_j \hat{a}_{j+1} + \hat{B}'_j (3\hat{B}'_{j+1} + (-4 - 6\hat{B}_j + 6\hat{B}_{j+1} - 4\hat{a}_j + 4\hat{a}_{j+1}) \hat{a}'_j + 2\hat{a}'_{j+1}) + (-1 - 3\hat{B}_j + 3\hat{B}_{j+1}) \hat{B}''_j - 2\hat{a}_j \hat{B}''_j + 2\hat{a}_j \hat{B}''_j + \hat{B}''_{j+1}
\]

\[
+3(-\hat{V}_j + \hat{V}_{j+1}) \hat{V}'_j - \hat{a}'_j + (\hat{a}_j - \hat{a}_{j+1}) \hat{a}'_j + \hat{a}'_{j+1} \right) = \hat{H}_{j+1} - \hat{H}_j. \tag{34}
\]
where $\hat{H}_j$ and $\hat{P}_j$ are given by (26), (27). Moreover, one can consider Eqs. (33), (34) without the right hand side as the equations of motion, since $\hat{H}_j$, $\hat{P}_j$ equal to zero initially at $\alpha_0 \approx -\infty$ in accordance with the initial conditions. One must be sure only that the time derivatives $\hat{H}_j'$, $\hat{P}_j'$ equal to zero, as well.

As the third equation, the discretized version of Eq. (6) can be taken

$$
\hat{V}_j'' + S\left(2\hat{V}_j' \hat{\alpha}_j' - e^{4\hat{B}} \left(\frac{\hat{V}_{j+1} - 2\hat{V}_j + \hat{V}_{j-1}}{(\Delta x)^2}\right) + 2(\hat{V}_{j+1} - \hat{V}_j)(\hat{\alpha}_{j+1} - \hat{\alpha}_j) + 4(\hat{V}_{j+1} - \hat{V}_j)(\hat{B}_{j+1} - \hat{B}_j)\right) = 0. \tag{35}
$$

Thus we have three operator equations (33), (34), (35) (or equivalently (34), (35) without the right-hand side) which for commuting quantities and in the continuous limit $\Delta x \to 0$ are fully equivalent to the classical equations (4), (5), (6).

The equations should be solved with the following initial condition

$$
\hat{V}_j(0) = \frac{i}{\Delta x} \frac{\partial}{\partial \pi V_j}, \quad \hat{V}_j'(0) = e^{-2\alpha_0} \pi V_j, \\
\hat{B}_j(0) = -\frac{1}{\pi B} \left(\frac{1}{3} \sqrt{\frac{\pi^2 V_j + \pi^2 B}{\pi^2 V_j}} + \sum_{k=1}^j S\left(\pi V_k \frac{i}{\Delta x} \left(\frac{\partial}{\partial \pi V_{k+1}} - \frac{\partial}{\partial \pi V_k}\right)\right)\right), \\
\hat{B}_j'(0) = e^{-2\alpha_0} \pi B, \quad \hat{\alpha}_j(0) = \alpha_0, \quad \hat{\alpha}_j' = e^{-2\alpha_0} \sqrt{\frac{\pi^2 V_j + \pi^2 B}{\pi^2 V_j}}. \tag{36}
$$

In the vicinity of small scale factors, the time derivatives are much larger than the spatial ones so that the operator equations take a simple form

$$
\hat{\alpha}_j'' + \hat{\alpha}_j'^2 + \hat{V}_j'^2 + \hat{B}_j'^2 = 0, \tag{37}
\hat{B}_j'' + 2\hat{B}_j' \hat{\alpha}_j' = 0, \tag{38}
\hat{V}_j'' + 2\hat{V}_j' \hat{\alpha}_j' = 0, \tag{39}
$$

with the constraint $-\hat{\alpha}_j'^2 + \hat{B}_j'^2 + \hat{V}_j'^2 = 0$. All quantities, i.e. $\hat{\alpha}_j$, $\hat{B}_j$, $\hat{V}_j$ in these equations commute with each other.

The solution of Eqs. (37), (37), (38) with the initial conditions (36) is given as

$$
\hat{V}_j(\eta) = \hat{V}_j(0) + \frac{\pi V_j}{2\sqrt{\pi^2 B + \pi^2 V_j}} \ln \left(1 + 2e^{-2\alpha_0} \sqrt{\frac{\pi^2 B + \pi^2 V_j}{\pi V_j}} \eta\right), \\
\hat{B}_j(\eta) = \hat{B}_j(0) + \frac{\pi B}{2\sqrt{\pi^2 B + \pi^2 V_j}} \ln \left(1 + 2e^{-2\alpha_0} \sqrt{\frac{\pi^2 B + \pi^2 V_j}{\pi V_j}} \eta\right), \\
\hat{\alpha}_j(\eta) = \alpha_0 + \frac{1}{2} \ln \left(1 + 2e^{-2\alpha_0} \sqrt{\frac{\pi^2 B + \pi^2 V_j}{\pi V_j}} \eta\right).
$$

Although the evolution in the vicinity of $\alpha \approx \alpha_0 \approx -\infty$ is relatively simple, the evolution governed by the general equations (33), (34), (35), when the fields begin to oscillate is very complicated and needs the numerical investigation that is beyond the aims of the present article.
5 Discussion and Conclusion

It should be said about a choice of a state of system considered. We do not identify the state as an “initial” because there is only the state $C(\pi_{V1}, \ldots, \pi_{VN})$, which describes all evolution of system and allows calculating the mean values of the quasi-Heisenberg operators. This state is not related to the notion of “vacuum” since there is no field oscillators in the limit $\alpha \to -\infty$. In Ref. [19] solution of the Wheeler-DeWitt equation for the Gowdy model is investigated. Although the another quantities were introduced in [19] instead of logarithm of scale factor $\alpha$ and $B$-field, namely

$$T = B + \alpha,$$
$$\lambda = 6(B - \alpha)$$

the asymptotic of the Wheeler-DeWitt equation in the vicinity $T \to -\infty$ has the form containing only the momentums by analogy with Eq. (18) and admits solutions of the plane wave type. Asymptotic of the Wheeler-DeWitt equation in the vicinity $T \to \infty$ is of the oscillator type [19]. Quasiclassical treatment of the Wheeler-DeWitt equation with regard to an evolution of universe allows an interpretation as a scattering problem [19] i.e. transition of the packet of plane waves at $T \to -\infty$ to the number of gravitons (i.e exited oscillators) at $T \to \infty$. It should be noted that there were no the state at $T \to -\infty$ which give no gravitons at $T \to \infty$. The later work [13] about a quantization of the Gowdy model was in some sense a step back because it considered graviton creation from vacuum in a style of Refs. [28–31]. It was suggested [13] that at some time $T_0$ where the field oscillators already have formed there is no gravitons and later the gravitons appear from vacuum during an evolution. However, our opinion is that there is no physical background for the existence of such a time $T_0$ where the universe was empty.

In our evolutionary picture, we have also the state describing evolution of universe as a packet of the “plane waves”. In a future this will give some gravitons under the vacuum. More exactly if one calculated the correlators $<\hat{V}(\eta, x)\hat{V}(\eta, x')>$ he will find that they are analogous to the correlators of the QFT corresponding to some gravitons under vacuum. There is no wave packet which gives a pure vacuum of universe in the future, i.e. appearance of matter is inevitable in this model but the matter is not created from vacuum, because at any time the universe is not empty.

As an example of the initial state one may take

$$C(\pi_{V1}, \ldots, \pi_{VN}) = C_N \exp \left( -b \sum_{j=1}^{N} \pi_{Vj}^2 \right),$$

where $C_N$ is the normalization constant. This state implies that the momentums are random and independent in spatial points. This state does not look similar to a vacuum state of the QFT, as the fields (and their momentums) in a QFT vacuum state are highly correlated in the nearest spatial points.

As an example of the mean value calculation, one can take the evolution of a
“Hubble constant” of system at small $\eta$:

$$< \frac{1}{a_j} \frac{da_j}{dt} > = < \frac{1}{a_j^2} \frac{da_j}{d\eta} > = \exp(-\dot{\alpha}_j) \frac{d\dot{\alpha}_j}{d\eta} = \frac{b^{1/4}}{2\eta^{3/2}} U(1/4, 3/4, b\pi^2),$$

(42)

where $U(a, b, z)$ is the confluent hypergeometric function.

For this state, the universe is expanded uniformly in a mean. It would be interesting to calculate the evolution of correlators $< \hat{V}_j(\eta) \hat{V}_n(\eta) >$. Initially the field $\hat{V}_j$ is uncorrelated for different $j$: $< \hat{V}_j(\eta) \hat{V}_n(\eta) > \sim \delta_{jn}$ but then some correlation should arise and some analogy with the vacuum of QFT in this sense should appear so that the fields become highly correlated in the nearest spatial points.

The question arises: why do the commutation relations related to the field $V$ have the usual form initially but do become to be broken after an evolution and, probably, look differently (the difference have to be related to the Hubble constant) at the present time? The answer is that it is a consequence of the chosen initial gauge. The gauge can be different, but the gauge considered is simplest and allows obtaining the commutation relations explicitly at the initial moment of time. As a matter of fact, the gauge choice remains to be complex issue.

To summarize, failure of the QFT in flat spacetime to deal with such an inherently non-linear theory as gravity and existence of “problem of time” insist on an invention of some new quantization procedures. The considered quasi-Heisenberg quantization scheme may provide a calculational framework for the investigation of the quantum evolution of a system. The goal of further investigation may be the vacuum energy problem, more exactly, its possible zero value in the quantization scheme considered, that may result from compensation of the zero point fluctuations of gravitational waves by the quantum fluctuations of the scale factor. Thereby, the fluctuations do not contribute to the mean evolution of a system. It should be noted that this will be purely quantum effect, since it is absent in classics [32]. An additional issue is calculation of the field correlators in order to determine their correspondence to the correlators of ordinary QFT in late times.

References

[1] J. B. Hartle, S.W. Hawking, Phys. Rev. D, 28 (1983), 2960.

[2] C.Kiefer, B.Sandhoefer, Quantum Cosmology, gr-qc/0804.0672.

[3] C. Rovelli, Phys. Rev. D 42 (1990), 26382646.

[4] J.A. Wheeler, Superspace and Nature of Quantum Geometrodynamics. In: De-Witt, C., Wheeler, J.A. (eds.) Battelle Rencontres, Benjamin, New York, 1968.

[5] B.S. DeWitt, Phys. Rev. 160 (1967), 1113.

[6] A. Ashtekar, J. Stachel (eds), Conceptual problems of quantum gravity. Birkhäuser, Boston, 1991.

[7] D.L. Wiltshire, An introduction to quantum cosmology, arXiv:gr-qc/0101003.
[8] T.P. Shestakova, C. Simeone, *Grav. Cosmol.* **10** (2004), 161.

[9] J.J. Halliwell, *Introductory Lectures on Quantum Cosmology*, arXiv:0909.2566.

[10] F. Barbero, E.J.S. Villasenor, *Living Rev.*, **6** (2010).

[11] R.H. Gowdy, *Ann. Phys. (N.Y.)*, **83** (1974), 203.

[12] C.W. Mizner, *Phys. Rev.*, **8** (1973), 3271.

[13] B.K. Berger, *Ann. Phys.*, **83** (1974), 458.

[14] J.J. Halliwell, *J. Phys., Conf. Ser.*, **306** (2011), 012023.

[15] S.L. Cherkas, V.L. Kalashnikov, *Quantum evolution of the Universe from τ = 0 in the constrained quasi-Heisenberg picture*, Proc. VIIIth International School-seminar ”The Actual Problems of Microworld Physics”, (Gomel, July 25-August 5) Vol. 1. pp. 208. JINR, Dubna (2006), arXiv:gr-qc/0502044.

[16] S.L. Cherkas, V.L. Kalashnikov, *Grav. Cosmol.* **12** (2006), 126, arXiv:gr-qc/0512107.

[17] S.L. Cherkas, V.L. Kalashnikov, *Gen. Rel. Grav.*, **44** (2012), 3081-3102.

[18] J. W. Jr. York, *Phys. Rev. Lett.* **26** (1971), 1656.

[19] C.W. Mizner, K.S. Torn and J.A. Wheeler. *Gravitation*. (W. H. Freeman & Company, New-York) Vol. 2., 1973.

[20] P.A.M. Dirac, *Lectures on Quantum Mechanics*, Yeshiva University, N.Y., 1964.

[21] A. Hanson, T. Regge, C. Teitelboim, *Constraint Hamiltonian Systems*, Contributi del Centro Linceo Interdisc. di Scienze Matem. e loro Applic **22** (1976).

[22] D.M. Gitman, I.V. Tyutin, *Quantization of Fields with Constraints*. Springer, Berlin, 1990.

[23] A. Jaffe, *Constructive quantum field theory*, in Mathematical Physics 2000, A. S. Fokas, A. Grigorian, T. Kibble, B. Zegarlinsky, Eds., Proceedings of the XIII International Congress on Mathematical Physics, Imperial College, London 17-22 July 2000, Imp. Coll. Press, London, 2000,111-127, posted at http://www.arthurjaffe.com/Assets/pdf/CQFT.pdf.

[24] E.M. Henley, W. Thirring, *Elementary quantum field theory*, McGraw-Hill Book Company, Inc., New York, 1962.

[25] A.A. Samarskii, Yu. P. Popov, *Raznostnye metody reshenia zadach gazovoi dinamiki*, Nauka, Moskow 1992, [in Russian].

[26] V.P. Maslov, *Operator Methods*, Nauka, Moscow (1973) [in Russian].

[27] M. V. Karasev, V. P. Maslov, *Nonlinear Poisson Brackets: Geometry and Quantization*, American Mathematical Society, Providence, 1993.
[28] L. Parker, *Phys. Rev.*, **183** (1969), 1057.

[29] R. U. Sexl, H. K. Urbantke, *Phys. Rev.*, **179** (1969), 1247.

[30] Ya. Zel’dovich, A. Starobinsky, *Zh. Eksp. Teor. Fiz.*, **61** (1971), 2161-2175, [Sov. Phys.- JETP, **34** (1972), 1159.]

[31] A.A. Grib, S.G. Mamaev, *Yad.Fiz.* **10** (1969), 1276.

[32] S. L. Cherkas, V. L. Kalashnikov, *Nonlinear Phenomena in Complex Systems* **15** (2012), 253. [arXiv:1206.5976]