CONE-VOLUME MEASURES OF POLYTOPES

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Abstract. The cone-volume measure of a polytope with centroid at the origin is proved to satisfy the subspace concentration condition. As a consequence a conjectured (a dozen years ago) fundamental sharp affine isoperimetric inequality for the U-functional is completely established — along with it’s equality conditions.

1. Introduction

Let $K^o_n$ be the set of all convex bodies in $\mathbb{R}^n$ having the origin in its interior, i.e., $K \in K^o_n$ is a convex compact subset of the $n$-dimensional Euclidean space $\mathbb{R}^n$ with $0 \in \text{int}(K)$. For $K \in K^o_n$ the cone-volume measure, $V_K$, of $K$ is a Borel measure on the unit sphere $S^{n-1}$ defined for a Borel set $\omega \subseteq S^{n-1}$ by

$$V_K(\omega) = \frac{1}{n} \int_{x \in \nu^{-1}(\omega)} \langle x, \nu_K(x) \rangle \, d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial'K \to S^{n-1}$ is the Gauss map of $K$, defined on $\partial'K$, the set of points of the boundary of $K$ having a unique outer normal, $\langle x, \nu_K(x) \rangle$ is the standard inner product on $\mathbb{R}^n$, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. In recent years, cone-volume measures have been appeared and studied in various contexts, see, e.g., [2, 4, 5, 9, 16, 17, 20, 21, 22, 26].

In particular, in the very recent and groundbreaking paper [5] on the logarithmic Minkowski problem, B"or"oczky Jr., Lutwak, Yang & Zhang characterize the cone-volume measures of origin-symmetric convex bodies as exactly those non-zero finite even Borel measures on $S^{n-1}$ which satisfy the subspace concentration condition. Here a finite Borel measure $\mu$ on $S^{n-1}$ is said to satisfy the subspace concentration condition if for every subspace $L \subseteq \mathbb{R}^n$

$$\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \mu(S^{n-1}),$$

and equality holds in (1.2) for a subspace $L$ if and only if there exists a subspace $\overline{L}$, complementary to $L$, so that also

$$\mu(\overline{L} \cap S^{n-1}) = \frac{\dim \overline{L}}{n} \mu(S^{n-1}),$$

i.e., $\mu$ is concentrated on $S^{n-1} \cap (L \cup \overline{L})$.

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This concentration condition is at the core of different problems in Convex Geometry; it provides not only the solution to the logarithmic Minkowski problem for origin-symmetric convex bodies [5], but, for instance, in [4] Theorem 1.2, it was shown that the subspace concentration condition is also equivalent to the property that a finite Borel measure has an affine isotropic image.

Now let \( P \in \mathcal{K}_n \) be a polytope with facets \( F_1, \ldots, F_m \), and let \( a_i \in S^{n-1} \) be the outer unit normal of the facet \( F_i \), \( 1 \leq i \leq m \). For each facet we consider \( C_i = \text{conv}\{0, F_i\} \), i.e., the convex hull of \( F_i \) with the origin, or in other words, \( C_i \) is the cone/pyramid with basis \( F_i \) and apex 0.

The cone-volume measure of \( P \) is given by (cf. (1.1))

\[
V_P = \sum_{i=1}^{m} V(C_i) \delta_{a_i},
\]

where \( V(C_i) \) is the volume, i.e., \( n \)-dimensional Lebesgue measure, of \( C_i \) and \( \delta_{a_i} \) denotes the delta measure concentrated on \( a_i \). Hence, \( P \) satisfies the subspace concentration condition (cf. (1.2)) if for every subspace \( L \subseteq \mathbb{R}^n \)

\[
\sum_{a_i \in L} V(C_i) \leq \frac{\dim L}{n} V(P),
\]

and equality holds in (1.3) for a subspace \( L \) if and only if there exists a subspace \( \overline{L} \), complementary to \( L \), so that \( \{a_j : a_j \notin L\} \subset \overline{L} \). In other words, \( A = (A \cap L) \cup (A \cap \overline{L}) \), where \( A = \{a_1, \ldots, a_m\} \).

In general, the cone-volume measure depends on the position of the origin and not every \( K \in \mathcal{K}_n \) meets the subspace concentration condition. In order to extend results from the origin-symmetric case, in [4, Problem 8.9] it is asked whether the cone-volume measure of convex bodies having the centroid at the origin satisfies the subspace concentration condition and our main result gives an affirmative answer in the case of polytopes.

**Theorem 1.1.** Let \( P \in \mathcal{K}_n \) be a polytope with centroid at the origin. The cone-volume measure of \( P \) satisfies the subspace concentration condition.

As mentioned before, in [5] it was even shown that the subspace concentration condition for even measures characterizes cone-volume measures of origin-symmetric convex bodies. Regarding polytopes, the inequality (1.3) was proven in the special case \( \dim L = 1 \) by Hernández Cifre (2007, private communication), in the cases \( \dim L = 1, n-1 \) by Xiong [27], and for origin-symmetric polytopes independently by Henk, Schürmann & Wills [14] and by He, Leng & Li [13].

The motivation for studying the relation (1.3) for the class of polytopes in [13, 27] stems from the U-functional of a polytope \( P \) given by

\[
U(P)^n = \sum_{a_{i_1} \wedge \ldots \wedge a_{i_n} \neq 0} V(C_{i_1}) \cdot \ldots \cdot V(C_{i_n}),
\]

where \( a_{i_1} \wedge \ldots \wedge a_{i_n} \neq 0 \) means that the vectors are linearly independent. This centro-affine functional, i.e., it is invariant with respect to volume preserving linear transformations, was introduced by Lutwak, Yang & Zhang in [19] and has been proved to be an important functional in order to get strong
inequalities on the volume of projection bodies of polytopes. For information on projection bodies we refer to the books by Gardner [7] and Schneider [25], and for more information on the importance of centro-affine functionals we refer to [12, 17] and the references within.

Here we are mainly interested in the relation of $U(P)$ to the volume of $P$. Obviously, $U(P) \leq V(P)$ and in [19] the problem was posed that for polytopes with centroid at the origin $U(P)$ is bounded from below by

$$(1.4) \quad U(P) \geq \frac{(n!)^{1/n}}{n} V(P),$$

and equality holds if and only if $P$ is a parallelotope (see also [4] Problem 8.6 for an extension to convex bodies). Observe, for $n \to \infty$ the factor in the lower bound becomes $1/e$ and so it is independent on the dimension. This is one feature of the $U$-functional making it so useful.

In [13] it was shown by He, Leng & Li that (1.4) can be deduced from (1.3) and they proved (1.4) (including the equality case) for origin-symmetric polytopes. Analogously, the results on (1.3) in [27] were used in order to establish (1.4) (including the equality case) for arbitrary two- and three-dimensional polytopes with centroid at the origin. Following the lines of these results we prove (1.4) in full generality.

**Theorem 1.2.** Let $P \in K_n^o$ be a polytope with centroid at the origin. Then

$$(1.4) \quad U(P) \geq \frac{(n!)^{1/n}}{n} V(P),$$

and equality holds if and only if $P$ is a parallelotope.

Finally, we remark that the logarithmic Minkowski problem is a particular case of the $L_p$-Minkowski problem, one of the central problems in convex geometric analysis.

**$L_p$-Minkowski problem.** Find necessary and sufficient conditions on a finite Borel measure $\mu$ on the unit sphere $S^{n-1}$ so that $\mu$ is the $L_p$-surface area measure of a convex body in $\mathbb{R}^n$.

For details we refer to [5, 18] and the references within. Here we just want to mention that for $p = 0$ the $L_0$-surface area measure is the cone-volume measure, and the subspace concentration condition seems to be the right condition in this case. Theorem 1.1 shows its necessity in the case of polytopes with centroid at the origin.

The proof of Theorem 1.1, which will be given in the last section, is based on the Gaussian divergence theorem applied to the log-concave function measuring the volume of slices of $P$ by parallel planes.

2. Preliminaries

In order to keep the paper largely self-contained, we collect here some basic facts from Convex Geometry and Polytope Theory needed in our investigations. Good general references on these topics are the books by Barvinok [3], Gardner [7], Gruber [10], Schneider [25] and Ziegler [28].

As usual, for two subsets $C, D \subseteq \mathbb{R}^n$ and reals $\nu, \mu \geq 0$ the Minkowski combination is defined by

$$\nu C + \mu D = \{\nu c + \mu d : c \in C, d \in D\}.$$
By the celebrated Brunn-Minkowski-inequality we know that the \( n \)-th root of the volume of the Minkowski combination is a concave function. More precisely, for two convex bodies \( K_0, K_1 \subset \mathbb{R}^n \) and for \( \lambda \in [0,1] \) we have
\[
V((1-\lambda)K_0 + \lambda K_1)^{1/n} \geq (1-\lambda)V(K_0)^{1/n} + \lambda V(K_1)^{1/n}
\]
with equality for some \( 0 < \lambda < 1 \) if and only if \( K_0 \) and \( K_1 \) lie in parallel hyperplanes or are homothetic, i.e., there exist a \( t \in \mathbb{R}^n \) and \( \mu \geq 0 \) such that \( K_1 = t + \mu K_0 \) (see, e.g., [6, Sect. 6.1]).

Let \( f : C \to \mathbb{R}_{>0} \) be a positive function on an open convex subset \( C \subset \mathbb{R}^n \) with the property that there exists a \( k \in \mathbb{N} \) such that \( f^{1/k} \) is concave. Then by the (weighted) arithmetic-geometric mean inequality
\[
f((1-\lambda)x + \lambda y) = \left(f^{1/k}((1-\lambda)x + \lambda y)\right)^k \geq \left((1-\lambda)f^{1/k}(x) + \lambda f^{1/k}(y)\right)^k \geq f^{1-\lambda}(x) \cdot f^{\lambda}(y).
\]
This means that \( f \) belongs to the class of log-concave functions which by the positivity of \( f \) is equivalent to
\[
\ln f((1-\lambda)x + \lambda y) \geq (1-\lambda)\ln f(x) + \lambda \ln f(y),
\]
for \( \lambda \in [0,1] \). Assuming in addition that \( f \) is differentiable on \( C \) with gradient \( \nabla f \) the concavity condition on \( \ln f \) is equivalent to
\[
(2.2) \quad \ln f(x) - \ln f(y) \leq \langle \nabla \ln f(y), x - y \rangle = \left\langle \frac{1}{f(y)} \nabla f(y), x - y \right\rangle
\]
for all \( x, y \in C \) (cf. e.g., [24, Sect. 25]).

For a subspace \( L \subseteq \mathbb{R}^n \), let \( L^\perp \) be its orthogonal complement, and for \( X \subseteq \mathbb{R}^n \) we denote by \( X|L \) its orthogonal projection onto \( L \), i.e., the image of \( X \) under the linear map forgetting the part of \( X \) belonging to \( L^\perp \).

Here, for a convex body \( K \in \mathcal{K}_0^n \) and a \( k \)-dimensional subspace \( L, 0 < k < n \), we are interested in the function measuring the volume of \( K \) intersected with planes parallel to \( L \), i.e., in the function
\[
(2.3) \quad f_L : K|L \to \mathbb{R}_{\geq 0} \text{ with } x \mapsto V_{n-k}(K \cap (x + L^\perp)),
\]
where \( V_{n-k}(\cdot) \) denotes the \((n-k)\)-dimensional volume. By the Brunn-Minkowski inequality and the remark above, \( f_L \) is a log-concave function which is positive at least in the (relative) interior of \( K|L \) (cf. [1]). \( f_L \) is also called the \((n-k)\)-dimensional X-ray of \( K \) parallel to \( L^\perp \) (see [24 Chapter 2]).

Next we want to consider this function for a polytope \( P \in \mathcal{K}_0^n \). Such a polytope may be represented as
\[
P = \{ x \in \mathbb{R}^n : \langle a_i, x \rangle \leq b_i, 1 \leq i \leq m \},
\]
where \( a_i \in S^{n-1} \) are the outer unit normals of the facets of \( P \), i.e., all \((n-1)\)-dimensional faces of \( P \) are given by \( F_i = P \cap \{ x \in \mathbb{R}^n : \langle a_i, x \rangle = b_i \}, 1 \leq i \leq m \). Since \( 0 \in \text{int } P \) we have \( b_i > 0 \), and since \( a_i \in S^{n-1}, b_i \) is the distance of \( F_i \) from the origin. Thus
\[
V(P) = \frac{1}{n} \sum_{i=1}^{m} V_{n-1}(F_i) b_i = \sum_{i=1}^{m} V(C_i),
\]
where \( C_i = \text{conv}\{0, F_i\} \). The boundary \( \partial P \) of \( P \) is the union of the facets of \( P \). In general, for a fixed \( k \in \{0, \ldots, n-1\} \), the union of all \( k \)-faces of \( P \) is called the \( k \)-skeleton of \( P \). The orthogonal projection of the \((k-1)\)-skeleton onto a \( k \)-dimensional plane \( L \) induces a polytopal subdivision \( \mathcal{D}(P)_L \) of \( P|L \), i.e., \( \mathcal{D}(P)_L \) is a collection of \( k \)-dimensional polytopes having pairwise disjoint interiors, the intersection of any two of them is a face of both, the union covers \( P|L \) and the preimage of the boundary of a polytope in \( \mathcal{D}(P)_L \) is contained in a \((k-1)\)-face of \( P \).

It was used, observed and proved in different contexts that \( f_L : P|L \to \mathbb{R}_{\geq 0} \) is a piecewise polynomial function, actually a spline for a generic subspace \( L \). Here we will only use the following result as stated by Gardner and Gritzmann \[8, \text{Proposition 3.1} \]

**Proposition 2.1.** Let \( L \) be a \( k \)-dimensional subspace, \( 0 < k < n \). The function \( f_L : P|L \to \mathbb{R}_{\geq 0} \) is a piecewise polynomial function of degree at most \( n-k \); more precisely, on every \( k \)-dimensional polytope of the subdivision \( \mathcal{D}(P)_L \) it is a polynomial of degree at most \( n-k \).

The centroid \( c(S) \) of a set \( S \subset \mathbb{R}^n \) with \( V(S) > 0 \) is defined as

\[
c(S) = \frac{1}{V(S)} \int_S x \, dx,
\]

where \( dx \) is the abbreviation for \( d\mathcal{H}^n(x) \). Thus, for \( K \in \mathcal{K}_P^n \) with \( c(K) = 0 \) and a subspace \( L \) with \( 0 < \dim L < n \) we have by Fubini’s theorem with respect to the decomposition \( L \oplus L^\perp \)

\[
0 = \int_K x \, dx = \int_{K|L} \left( \int_{(\hat{x} + L^\perp) \cap K} \hat{x} \, d\hat{x} \right) d\hat{x}
= \int_{K|L} f_L(\hat{x}) c((\hat{x} + L^\perp) \cap K) \, d\hat{x}.
\]

Writing \( c((\hat{x} + L^\perp) \cap K) = \hat{x} + \tilde{y} \) with \( \tilde{y} \in L^\perp \) gives

\[
\int_{K|L} f_L(\hat{x}) \, \hat{x} \, d\hat{x} = 0,
\]

i.e., the first moment of \( f_L \) vanishes. This will be the main property of the centroid used later on. Indeed, we will need it in order to apply the following lemma on log-concave functions.

**Lemma 2.2.** Let \( C \in \mathcal{K}_P^n \), and let \( f : \text{int} \, C \to \mathbb{R}_{\geq 0} \) be a log-concave function with \( \int_C f(x) \, x \, dx = 0 \). Furthermore, assume that \( \nabla f(x) \) exists almost everywhere on \( \text{int} \, C \), and that also \( \int_C \langle x, \nabla f(x) \rangle \, dx \) exists. Then

\[
\int_C \langle x, \nabla f(x) \rangle \, dx \leq 0,
\]

with equality if and only if there exist \( c \in \mathbb{R}^n \), \( \gamma \in \mathbb{R}_{\geq 0} \) with \( f(x) = \gamma e^{c \langle x, \cdot \rangle} \).

**Proof.** By the concavity of \( \ln f(x) \) we have for all \( x, y \in \text{int} \, C \) (cf. \[2.2\])

\[
\ln f(x) - \ln f(y) \leq \left\langle \frac{1}{f(y)} \nabla f(y), x - y \right\rangle.
\]
Interchanging the role of $x$ and $y$ and adding up leads to
\[ 0 \leq \left\langle \frac{1}{f(y)} \nabla f(y) - \frac{1}{f(x)} \nabla f(x), x - y \right\rangle. \]
Setting $y = 0$ gives for $x \in \text{int } C$
\[ \left\langle \nabla f(x), x \right\rangle \leq \left\langle \frac{1}{f(0)} \nabla f(0), f(x) \right\rangle. \]
Hence in view of our assumption on the first moment of $f$ on $C$ we get
\[ \int_C \left\langle x, \nabla f(x) \right\rangle \, dx \leq \int_C \left\langle \frac{1}{f(0)} \nabla f(0), f(x) \right\rangle \, dx = \left\langle \frac{1}{f(0)} \nabla f(0), \int_C f(x) \, dx \right\rangle = 0. \]
If the inequality holds with equality, we must have almost everywhere equality in (2.10) for $y = 0$. Hence, $\ln f(x)$ is an affine function. Together with the positivity of $f$ on $\text{int } C$ there exist $c \in \mathbb{R}^n$, $\gamma \in \mathbb{R}_{>0}$ with $f(x) = \gamma e^{\langle c, x \rangle}$. On the other hand, if $f$ is of this form then $\nabla f(x) = f(x) \, c$ and so
\[ \int_C \left\langle x, \nabla f(x) \right\rangle \, dx = \left\langle c, \int_C f(x) \, dx \right\rangle = 0. \]
\[ \square \]

3. Constant volume Sections

In order to treat the equality case in the subspace concentration condition of Theorem 1.1 we need the following characterization.

**Lemma 3.1.** Let $P \in \mathcal{K}_0^n$ be a polytope, let $A$ be the set of its outer unit normals, and let $L \subset \mathbb{R}^n$ be a $k$-dimensional subspace, $0 < k < n$.

Then $f_L : P|L \to \mathbb{R}_{\geq 0}$ is a constant function if and only if there exists a subspace $\overline{L}$, complementary to $L$, such that
\[ A = (A \cap L) \cup (A \cap \overline{L}). \]

**Proof.** Suppose $f_L(x) = f_L(0)$ for all $x \in P|L$. Then, in particular,
\[ f_L((1-\lambda)x + \lambda 0)^{1/(n-k)} = (1-\lambda)f_L(x)^{1/(n-k)} + \lambda f_L(0)^{1/(n-k)} \]
for all $\lambda \in [0,1]$ and $x \in P|L$. Hence, by the equality case of the Brunn-Minkowski inequality (cf. (2.1)), for every $x \in P|L$ there exists a $t(x) \in \mathbb{R}^n$ such that
\[ (x + L^\perp) \cap P = t(x) + (L^\perp \cap P). \]
This translation vector $t(x)$ is uniquely determined and moreover, the function $T : P|L \to \mathbb{R}^n$ mapping $x \mapsto t(x)$ is injective and convex linear, i.e., $T((1-\lambda)x + \lambda y) = (1-\lambda)T(x) + \lambda T(y)$ for $x, y \in P|L$ and $\lambda \in [0,1]$. Thus it is an affine function, and since $T(0) = 0$ we conclude that $T$ is linear. Hence, $\widetilde{L} = \text{lin } T(P|L)$, i.e., the linear hull of $T(P|L)$, is a $k$-dimensional linear subspace and we have
\[ P = (P \cap \widetilde{L}) + (P \cap L^\perp). \]
Since $P \cap \tilde{L}$ is a $k$-dimensional polytope and $(P \cap L^\perp)$ an $(n-k)$-dimensional polytope, the facets of $P$ are given by
\[ \tilde{F} + (P \cap L^\perp) \text{ or } F + (P \cap \tilde{L}), \]
where $\tilde{F}$ is a facet, i.e., a $(k-1)$-face of $P \cap \tilde{L}$ and $F$ is a facet, i.e., a $(n-k-1)$-face of $P \cap L^\perp$. In the first case the outer unit normal of such a facet is contained in $(L^\perp)^\perp = L$, and in the latter case in $\tilde{L}^\perp$. Hence $A = (A \cap L) \cup (A \cap \tilde{L}^\perp)$, and since $P$ is bounded we also know that $\tilde{L}^\perp$ is complementary to $L$.

On the other hand, if we have $A = (A \cap L) \cup (A \cap \tilde{L})$ for complementary subspaces $L, \tilde{L}$, then it is easy to see that
\[ P = (P \cap L^\perp) + (P \cap \tilde{L}^\perp). \]

In particular, by the complementarity of the subspaces we know that for every $x \in P|L$ there exists an unique $\tilde{t}(x) \in P \cap \tilde{L}^\perp$ with $\tilde{t}(x)|L = x$. Hence, $P \cap (x + L^\perp) = \tilde{t}(x) + (P \cap L^\perp)$ for every $x \in P|L$, which shows $f_L(x) = f_{\tilde{L}}(0)$. \hfill $\Box$

Lemma 3.1 also allows us to give a weak generalization of a characterization of parallelotopes due to Guggenheimer & Lutwak [11]. It will be used for the discussion of the equality case in Theorem 1.2.

Here we need the following notation: for $n$ linearly independent unit vectors $V = \{v_1, \ldots, v_n\}$, $v_i \in S^{n-1}$, and a $k$-subset $I \subset \{1, \ldots, n\}$, $0 < k < n$, we denote by $L_I(V) = \text{lin} \{v_j : j \in I\}$ the $k$-dimensional subspace generated by this selection of vectors.

**Lemma 3.2.** Let $P \in K^n_0$ be a polytope and let $0 < k < n$. There exist $n$-linearly independent unit vectors $V = \{v_1, \ldots, v_n\}$ such that the function $f_{L_I(V)} : P|L_I(V) \to \mathbb{R}_{\geq 0}$ is constant for every $k$-subset $I \subset \{1, \ldots, m\}$ if and only if $P$ is a parallelotope.

Before giving the proof we want to remark that for arbitrary convex bodies and $k = 1$ the result was shown by Guggenheimer & Lutwak [11]. In fact, they only assumed that the function is constant for $(n-1)$-many 1-subsets.

**Proof.** The outer unit normals $\pm v_i$, $1 \leq i \leq n$, of a parallelotope are always contained in complementary subspaces. Hence the sufficiency follows from Lemma 3.1.

Now let vectors $V = \{v_1, \ldots, v_n\}$ be given such that $f_{L_I(V)} : P|L_I(V) \to \mathbb{R}_{\geq 0}$ is a constant function for any $k$-subset $I \subset \{1, \ldots, n\}$. For short we will write $L_I$ instead of $L_I(V)$. According to Lemma 3.1 there exists for any $k$-subset $I$ a complementary subspace $\tilde{L}_I$ such that
\[ A = (A \cap L_I) \cup (A \cap \tilde{L}_I), \]
where $A$ is the set of outer unit normals of the polytope $P$. Since $\dim A = n$ we have
\[ \dim(A \cap L_I) = k \text{ and } \dim(A \cap \tilde{L}_I) = n - k, \]
where, in general, $\dim X$ is the dimension of the affine hull of $X \subset \mathbb{R}^n$. 

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For a subset $I$ let $I_c$ be its complement with respect to \{1, \ldots, n\}, i.e.,
$I_c = \{1, \ldots, n\} \setminus I$. We claim
\begin{equation}
L_I = L_{I_c}.
\end{equation}
Assume first $k \leq n/2$, and let $j \in I_c$. Then we may complement \{j\} to a
$k$-subset $J$ with $I \cap J = \emptyset$ and thus $L_I \cap L_J = \{0\}$. By (3.2), applied to $J$,
there exists $a_j, \ldots, a_{jk} \in A \cap L_J$ with $v_j \in \text{lin} \{a_j, \ldots, a_{jk}\}$. On the other
hand, since $L_I \cap L_J = \{0\}$ we know by (3.1) that $a_j, \ldots, a_{jk} \in A \cap L_I$. Thus
$v_j \in \overline{L_I}$ and so $L_{I_c} \subset \overline{L_I}$. Since both subspaces are of dimension $n - k$ we
are done.

Now let $k > n/2$ and assume that there exists a $a^* \in A \cap \overline{L_I}$ such that
$a^* \notin L_{I_c}$. Without loss of generality let $I = \{1, \ldots, k\}$, and let
\begin{equation}
a^* = \sum_{i=1}^{k} \alpha_i v_i + \sum_{j=k+1}^{n} \beta_j v_j
\end{equation}
with some $\alpha_i, \beta_j \in \mathbb{R}$. Since $a^* \notin L_{I_c} \cup L_I$ we may assume $\alpha_1 \cdot \beta_n \neq 0$. Then,
for $J = \{n - k + 1, \ldots, n\}$ we get by (3.1) that $a^* \in A \cap \overline{L_J}$ and thus
\begin{equation}
a^* \in (A \cap \overline{L_I}) \cap (A \cap \overline{L_J}).
\end{equation}
On the other hand, (3.1) also implies that each $a \in A \setminus ((A \cap \overline{L_I}) \cup (A \cap \overline{L_J}))$
belongs to \{\(v_{n-k+1}, \ldots, v_k\)\} which finally gives
\[A = (A \cap \text{lin} \{v_{n-k+1}, \ldots, v_k\}) \cup (A \cap \overline{L_I}) \cup (A \cap \overline{L_J}).\]
By (3.4) and (3.2) the union of the last two sets is contained in a linear
subspace of dimension $2(n - k) - 1$ which yields the contradiction that
dim $A \leq n - 1$. Hence, $(A \cap \overline{L_I}) \subset L_{I_c}$ and on account of (3.3) we get (3.3).

Now let $a \in A$ and let $a = \sum_{i=1}^{n} \alpha_i v_i$ for some scalars $\alpha_i \in \mathbb{R}$. Suppose
two of them are non zero, and let $j_1, j_2$ be the corresponding indices. Let
$J$ be a $k$-subset containing $j_1$ but not $j_2$. Then $a \notin (A \cap L_J) \cup (A \cap L_{I_c})$
contradicting (3.3) and (3.1). Hence, $A \subset \{\pm v_i : 1 \leq i \leq n\}$, and since none
strict subset of the latter set can be the outer unit normals of a bounded set
we conclude
\[A = \{\pm v_i : 1 \leq i \leq n\},\]
and $P$ is a paralleloptope. \hfill \Box

4. Proof of theorems

The proof of Theorem 1.1 relies on the Gaussian divergence theorem,
which is usually stated in the form (cf. e.g. [15])
\[\int_V \text{div} F(x) \, dx = \int_{bd V} \langle F(x), \nu(x) \rangle \, d\mathcal{H}^{n-1}(x),\]
where $V \subset \mathbb{R}^n$ is a compact subset with a piecewise smooth boundary,
$F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable vector field in an open
neighborhood of $V$, $bd V$ is that part of the boundary of $V$ admitting a unique
outer normal $\nu(x)$ in $x \in bd V$, and $\text{div} F$ is the divergence of the vector field
$F$, i.e., $\text{div} F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$, where $F(x) = (F_1(x), \ldots, F_n(x))^T$.

Here we want to apply this theorem to the vector field
\begin{equation}
F_L : P|L \to \mathbb{R}^k \text{ with } x \mapsto f_L(x) x,
\end{equation}
where $P \in \mathcal{K}_0^n$ is a polytope, $L$ is a $k$-dimensional linear subspace, $0 < k < n$, and $f_L: P|L \to \mathbb{R}_{\geq 0}$ is the volume intersection function $f_L(x) = V_{n-k}(x + L^+ \cap P)$. This vector field is, in general, not continuously differentiable. There are, however, numerous extensions of the divergence theorem to much more general sets than compact sets and to functions with certain singularities which also cover the case we need (see, e.g., [23]). On the other hand, our vector field \textbf{is} “just” a piecewise polynomial vector field, and so we briefly state how the Gaussian divergence theorem can be applied in our setting.

**Lemma 4.1.** Let $P$ be a polytope, and let $L \subset \mathbb{R}^n$ be a $k$-dimensional linear subspace, $0 < k < n$. Let $\mathcal{D}(P)_L = \{P_1, \ldots, P_r\}$ be the polytopal subdivision induced by the orthogonal projection of the $(k-1)$-skeleton $S$ of $P$ onto $L$. Let $F: P|L \to \mathbb{R}^k$ be a vector field which is a polynomial in each component and on each polytop $P_i \in \mathcal{D}(P)_L$. Then

$$\int_{(P|L)\setminus(S|L)} \text{div} F(X) \, dx = \int_{\text{bd}'(P|L)} \langle F(x), a(x) \rangle \, d\mathcal{H}^{k-1}(x).$$

Here $a(x)$ is the unique outer normal in the boundary point $x \in \text{bd}'(P|L)$.

**Proof.** Since $F$ is a polynomial vector field on each $P_i$, $F$ can canonically be extended to an open neighborhood of $P_i$, and hence we can use the divergence theorem of Gauss and get

$$\int_{P_i} \text{div} F(x) \, dx = \int_{\text{bd}'(P_i)} \langle F(x), a_i(x) \rangle \, d\mathcal{H}^{k-1}(x).$$

Thus, in particular, $\int_{P_i \setminus (S|L)} \text{div} F(x) \, dx$ is well defined and since $S|L$ is a set of measure 0, we have

$$\int_{(P|L)\setminus(S|L)} \text{div} F(X) \, dx = \sum_{i=1}^r \int_{\text{bd}'(P_i)} \langle F(x), a_i(x) \rangle \, d\mathcal{H}^{k-1}(x),$$

with $a_i(x)$ being the unique outer normal in $x \in \text{bd}'(P_i)$, $1 \leq i \leq r$. Every $x \in \text{bd}'(P_i) \setminus \text{bd}(P)$ is contained in exactly one more $P_j$, $j \neq i$, and it is $a_j(x) = -a_i(x)$. Hence, with $a_i(x) = a(x)$ for $x \in \text{bd}'(P_i) \cap \text{bd}'(P)$ we get

$$\sum_{i=1}^r \int_{\text{bd}'(P_i)} \langle F(x), a_i(x) \rangle \, d\mathcal{H}^{k-1}(x) = \int_{\text{bd}'(P|L)} \langle F(x), \nu(x) \rangle \, d\mathcal{H}^{k-1}(x),$$

which finishes the proof. \hfill $\Box$

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $P \in \mathcal{K}_0^n$ be a polytope with centroid at the origin. Let $F_1, \ldots, F_m$ be the facets of $P$, and let $a_i \in S^{n-1}$ be the outer unit normal of the facet $F_i$, $1 \leq i \leq m$. Let $A = \{a_1, \ldots, a_m\}$, and let $b_i > 0$ be the distance of the facet $F_i$ from the origin. Let $L$ be a $k$-dimensional subspace with $0 < k < n$. We have to show (cf. (1.3) and (2.4))

$$\sum_{a_i \in L} V_{n-1}(F_i) b_i \leq k V(P).$$

(4.2)

with equality if and only if there exists a subspace $\mathcal{L}$, complementary to $L$, so that $A = (A \cap L) \cup (A \cap \mathcal{L})$. 

According to Proposition 2.1, the vector field \( F_L(x) = f_L(x) x \) (cf. (4.1)) satisfies the assumptions of Lemma 4.1 and on account of

\[
\text{div} F_L(x) = k f_L(x) + \langle \nabla f_L(x), x \rangle
\]

we get

\[
\int_{bd'(P|L)} f_L(x) \langle x, a(x) \rangle \, d\mathcal{H}^{k-1}(x)
\]

(4.3)

\[
= k \int_{(P|L)\setminus(S|L)} f_L(x) \, dx + \int_{(P|L)\setminus(S|L)} \langle \nabla f_L(x), x \rangle \, dx
\]

\[
= k V(P) + \int_{(P|L)\setminus(S|L)} \langle \nabla f_L(x), x \rangle \, dx,
\]

where in the last step we used again that \( S|L \) is a set of measure 0.

Now let \( \tilde{a}_1, \ldots, \tilde{a}_l \) be the outer unit normals of the facets of \( P|L \), i.e., the \((k-1)\)-faces of \( P|L \), having distance \( \tilde{b}_i \) to the origin. Let \( \tilde{F}_i = P \cap \{ x \in \mathbb{R}^n : \langle \tilde{a}_i, x \rangle = \tilde{b}_i \} \), \( 1 \leq i \leq l \), be the faces of \( P \) projected onto the facets of \( P|L \). Taking into account that \( f_L \) measures the \((n-k)\)-dimensional volume we have

\[
\int_{bd'(P|L)} f_L(x) \langle x, a(x) \rangle \, d\mathcal{H}^{k-1}(x) = \sum_{i=1}^l V_{n-1}(\tilde{F}_i) \tilde{b}_i.
\]

Hence \( \tilde{F}_i \) contributes to the above sum only when it is a facet of \( P \), i.e., \( F_j = \tilde{F}_i \), \( a_i j = \tilde{a}_i \in L \) and \( b_j = \tilde{b}_i \) for a certain \( j \in \{1,\ldots,m\} \). Thus we may write (cf. (4.3))

(4.4)

\[
\sum_{a_i \in L} V_{n-1}(F_i) b_i = k V(P) + \int_{(P|L)\setminus(S|L)} \langle \nabla f_L(x), x \rangle \, dx.
\]

Since 0 is the centroid of \( P \) we have (cf. (4.5))

\[
\int_{P|L} f_L(x) \, dx = 0,
\]

and since \( S|L \) is a set of measure 0 we may apply Lemma 2.2 to \( f_L \). Thus

(4.5)

\[
\int_{(P|L)\setminus(S|L)} \langle \nabla f_L(x), x \rangle \, dx \leq 0,
\]

which yields (4.2) by (4.4).

Now suppose we have equality in (4.4). Then we also have equality in (4.3) and by Lemma 2.2 there exist \( \gamma > 0, c \in \mathbb{R}^n \) such that \( f_L(x) = \gamma e^{\langle c, x \rangle} \).

Since the \((n-k)\)-the root of \( f_L(x) \) is concave we must have \( c = 0 \), i.e., \( f_L(x) \) is a constant function. Thus, by Lemma 3.1, there exists a complementary subspace \( \overline{L} \) with \( A = (A \cap L) \cup (A \cap \overline{L}) \).

On the other hand, if we have such a partition of \( A \) into complementary subspaces \( L \) and \( \overline{L} \), \( \dim L = k \) and \( \dim \overline{L} = n-k \), then we may either apply Lemma 3.1 and then (4.4), or we just observe that in this case we may write
\[ k \nu(P) + (n - k) \nu(P) = n \nu(P) \]
\[ = \sum_{a_i \in A \cap L} \nu_{n-1}(F_i) b_i + \sum_{a_i \in A \cap L} \nu_{n-1}(F_i) b_i. \]

Hence, in view of the validity of the inequality (4.2) for \( L \) and \( \mathcal{T} \), we have actually equality in (4.2) for \( L \) and \( \mathcal{T} \).

Next we come to the proof of Theorem 1.2, and here we follow the approach of He, Ling & Li [13].

**Proof of Theorem 1.2.** Let \( P \in K^n \) be a polytope with centroid at the origin. Let \( F_1, \ldots, F_m \) be the facets of \( P \) with associated outer unit normals \( a_1, \ldots, a_m \) and let \( C_i = \text{conv} \{0, F_i\}, 1 \leq i \leq m \). For \( 1 \leq k \leq n \) we set

\[ \sigma_k(P)^k = \sum_{a_{i_1} \wedge \cdots \wedge a_{i_k} \neq 0} \nu(C_{i_1}) \cdots \nu(C_{i_k}). \]

We have to show \( U(P)^n = \sigma_n(P)^n \geq n! n^n \nu(P) \) with equality if and only if \( P \) is a parallelotope. On account of Theorem 1.1 (cf. (1.3)) we may write for \( 0 < k < n \)

\[ \sigma_k(P)^{k+1} = \sum_{a_{i_1} \wedge \cdots \wedge a_{i_k} \neq 0} \nu(C_{i_1}) \cdots \nu(C_{i_k}) \times \]
\[ \left( \nu(P) - \sum_{a_i \in \text{lin} \{a_{i_1}, \ldots, a_{i_k}\}} \nu(C_i) \right) \]
\[ \geq \sum_{a_{i_1} \wedge \cdots \wedge a_{i_k} \neq 0} \nu(C_{i_1}) \cdots \nu(C_{i_k}) \left( \nu(P) \left( 1 - \frac{k}{n} \right) \right) \]
\[ = \frac{n - k}{n} \nu(P) \sigma_k(P)^k. \]

Since \( \sigma_1(P) = \nu(P) \) this recursion gives

\[ U(P)^n = \sigma_n(P)^n \geq \frac{1}{n} \nu(P) \sigma_{n-1}(P)^{n-1} \geq \cdots \geq \frac{(n-1)!}{n^{n-1}} \nu(P)^n, \]

which is the desired inequality.

Having equality we must have equality in each step of the recursion (4.7). Hence for any \( k \)-subset \( I \subset \{1, \ldots, m\} \) such that \( L_I = \text{lin} \{a_j : j \in I\} \) is of dimension \( k \), we have

\[ \sum_{a_i \in L_I} \nu(C_i) = \frac{k}{n} \nu(P). \]

Thus by the equality case of Theorem 1.1 (cf. (1.3)) we can find a complementary subspace \( \mathcal{L}_I \) with \( A = (A \cap L_I) \cup (A \cap \mathcal{L}_I) \), where \( A \) is the set of all outer unit normals of \( P \). By Lemma 3.1 this shows that \( f_{L_I} \) is a constant function for any \( k \)-subset \( I \subset \{1, \ldots, m\} \) such the vectors \( a_j, j \in I \), are linearly independent. Since there are \( n \) linearly independent vectors in \( A \) we are in the position to use Lemma 3.2 which gives that \( P \) is a parallelotope.

On the other hand if \( P \) is a parallelotope then \( \nu(C_i) = \frac{1}{2n} \nu(P) \) for all cones and thus we have equality.
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