Subdiffusive continuous time random walks with power-law resetting

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Abstract
In the present work we revisit the problem of the behavior of a subdiffusive continuous time random walk (CTRW) under resetting. The resetting process is considered as a renewal process with power-law distribution of waiting times between the resetting events. We consider both the case of complete resetting, when an ordinary CTRW starts anew after the resetting event, and the case of incomplete resetting, when the internal memory of CTRW is not erased by the resetting event, and the CTRW restarts as an aged one. Using a special representation for the waiting time distribution in resetting, we obtained closed-form expressions for the probability density of displacements and for the mean first passage time to a given point under complete resetting, and asymptotic forms for the probability density of displacements (including prefactors) in the case of incomplete resetting.

Keywords: continuous time random walk, resetting, power law

1. Introduction

Random processes under resetting adequately describe a wide range of phenomena and processes in physics, chemistry, biophysics and biochemistry, in movement ecology, and also in computer science, see [1] for a review, and the beginning of [2] for a succinct account. Additional interesting information and an almost exhaustive list of references is contained in [3].
A random process under resetting (a reset process) is a superposition of two random processes: the displacement process $x(t)$ between the resetting events, considered as a continuous-time process, and a resetting process being a point process $\xi$, with a realization $\{t_i\}$, on the non-negative real line. In the simplest variant of the reset process, the coordinate $x(t)$ is set back to $x = 0$ at $t = t_i$. For non-Markovian displacement processes, the resetting of the coordinate may or may not be accompanied by erasing the internal memory of $x(t)$ [4–6]. In the case when the internal memory is erased, one speaks about the complete resetting, while if the memory is not erased, the resetting is said to be incomplete. In what follows, we investigate both cases for the situation when the displacement process is a continuous time random walk (CTRW). This process was considered e.g. in [6, 7]. The problem of resetting in CTRW, especially in the case, when the internal memory is not erased, is a complicated one. The situation we consider here is close to the one discussed in [6]: we consider a CTRW with power-law distribution of waiting times, and a resetting process being a renewal process with power-law waiting time densities (WTDs) for resetting events.

Our approach to the problem follows the general lines of analysis in [6], but combines this approach with representations used in [7]. This approach, especially under a special choice of WTDs for resetting proposed in [7], allows for performing calculations explicitly, getting either closed expressions, or asymptotic expressions including the prefactors. In the present work we concentrate on the derivation of the asymptotic forms of the probability density functions (PDFs) of the walker’s position $p(x,t)$, and note that several other results, e.g. the ones for the mean first passage times (MFPTs) to a given point, may be obtained along the similar lines (these results are explicitly stated for the case of complete resetting). The discussion in the present work is much more elegant than the one in [6], where the results were achieved by continuously switching between different representations, and no closed expressions could be obtained. We also note that alternative approach based on the Markovian embedding, i.e. following the lines of [7], is also possible, and, as we checked, leads to the same results giving an additional confirmation. Here we, however, refrain from reproducing this alternative approach and follow a single, and relatively simple, line of thought.

We also note, that several related problems were already considered in the literature. Thus, the CTRW under Poissonian resetting was considered in [8–12] while the non-Poissonian resetting problems for normal diffusion were discussed in [13–16]. We will use these works for verification of our results in the corresponding limiting cases.

2. Main formulae

Let the displacement process be an ordinary CTRW starting at time $t = 0$ at position $x = 0$ with the waiting time (an ordinary, wait-first CTRW, following the Montroll–Weiss scheme [17]). Let our reset process be observed at time $t$. Then there was either no resetting events between $t = 0$ and the observation time $t$ (i.e. observed is an uninterrupted CTRW), or there was at least one such event. The two situations are mutually exclusive. In the last case observed is a position of a walker performing a CTRW, provided the position of the walker set to $x = 0$ at the time $t'$ of the last resetting before the observation. Depending on whether the memory is erased or not, this CTRW may be an ordinary, or an aged one [18]. The first situation corresponds to the case, when an ordinary CTRW (a Montroll–Weiss CTRW starting with the waiting time) started anew at time $t'$ at position $x = 0$, with the new waiting time counted from $t'$, while last situation corresponds to the case when the ordinary CTRW, started at $t = 0$, was not interrupted by a resetting, but what is measured is the displacement of the walker in the time interval
between \( t' \) and \( t \). Let \( p_{\text{OW}}(x,t) \) denote the PDF of displacements (propagator) of the ordinary CTRW at time \( t \), and \( p_{2}(x,t-t',t') \) the PDF of displacements in the CTRW during the time interval \( t-t' \) after the last resetting, which took place at time \( t' \). For the case of complete resetting we have

\[
p_{2}(x,t-t',t') = p_{\text{OW}}(x,t-t')
\]

(with OW denoting an ordinary walk) while for incomplete resetting

\[
p_{2}(x,t-t',t') = p_{\lambda W}(x,t-t',t'),
\]

where the subscript AW indicates the aged random walk, and the time of last resetting \( t' \) corresponds to the aging time.

According to the discussion above, the PDF of the reset process at time \( t \) is given by

\[
P(x,t) = \Psi(t)p_{\text{OW}}(x,t) + \int_{0}^{t} \kappa(t')\Psi(t-t')p_{2}(x,t-t',t')dt'.
\]

(1)

Here, \( \Psi(t) \) is the survival probability of the resetting process, i.e. the probability that no new resetting event followed during the time \( t \) after the last one, and \( \kappa(t) \) is the resetting rate, so that \( \kappa(t)dt \) is the probability that a resetting event takes place in the time interval \((t,t+dt)\). This resetting rate is connected with the WTD \( \psi_{r}(t) \) of the resetting process, \( \psi_{r}(t) = -\frac{d}{dt}\hat{\psi}_{r}(t) \) : it is namely the sum of the multiple convolutions of WTD with itself, \( \kappa(t) = \psi_{r}(t) + [\psi_{r} * \psi_{r}](t) + [\psi_{r} * \psi_{r} * \psi_{r}](t) + \ldots \) (here the asterisk denotes the convolution of the functions). In the Laplace domain one thus has

\[
\kappa(s) = \frac{\psi_{r}(s)}{1 - \psi_{r}(s)}
\]

with \( \psi_{r}(s) = 1 - s\Psi_{r}(s) \), see e.g. [6] for a more detailed discussion. Here and below we continuously use the notation in which using the variables \( s \) or \( u \) implies that the corresponding functions are the Laplace transforms: \( f(s) = \mathcal{L}[f(t)] \), where \( \mathcal{L}[f] \) denotes the operator of the Laplace transform. Similarly, using the variable \( k \) implies the Fourier-transform with respect to the spatial coordinate \( x \).

The PDF in a standard (Montroll–Weiss) CTRW in the Fourier-Laplace domain is given by [18]

\[
p_{\text{OW}}(k,s) = \frac{1 - \psi_{W}(s)}{s} \frac{1}{1 - \lambda(k)\psi_{W}(s)}
\]

(2)

where \( \psi_{W}(s) \) is the Laplace transform of the WTD for the jumps of CTRW (the subscript \( W \) stands for ‘walk’), and \( \lambda(k) \) is the characteristic function of the distribution of displacements in a single jump. The WTD will be assumed to possess a power-law asymptotics, \( \psi_{W}(t) \sim t^{-1-\alpha} \).

A useful specific choice of \( \psi_{W}(t) \) may correspond to the generalized Mittag–Leffler distribution, \( \psi_{W}(t) = \frac{\alpha-1}{\Gamma(\alpha)} t^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{t}{t_{0}}\right)^{\alpha} \), [19], with \( t_{0} \) being the the characteristic waiting time for a step. In the case when the distribution of jump lengths does possess the second moment, and when only the spatial scales \( x \) larger than the mean squared displacements (MSDs) in a single step \( \alpha \), and times longer than the typical time of a single step \( t_{0} \) are considered, one can use the small \( k \), small \( s \) asymptotics of the expression above, in which case

\[
p_{\text{OW}}(k,s) = \frac{s^{\alpha-1}}{DK^{2} + s^{2}},
\]
with subdiffusion exponent $0 < \alpha \leq 1$, governing the MSD as a function of time, $\langle x^2(t) \rangle = \frac{2D}{\alpha t(\alpha t)^\alpha}$. The prefactor $D$ depends on the MSD in a single step and on a characteristic time of the step, $D \sim a^2/t_0^\alpha$. All asymptotic regimes discussed below correspond to $|x| \gg a$ and $t \gg t_0$ or $s \ll t_0^{-1}$, respectively. Performing the back Fourier transform to the space domain, one obtains the standard representation of the propagator in the Laplace domain in time:

$$p_{\text{OW}}(x, s) = s^{\frac{\alpha}{2} - 1} \frac{2}{\sqrt{D}} \exp \left( -\frac{s^{\alpha} |x|}{D} \right).$$

(3)

A representation for the PDF in an aged CTRW will be discussed later, in section 4, where the corresponding situation with incomplete resetting will be considered. We also note that using a fluid limit for the function $p_{\text{OW}}(x, t)$ implies that the characteristic time of a step of a CTRW is much shorter than the characteristic waiting time for the resetting event. According to [6] this is a necessary condition for observing of some of the intermediate asymptotics discussed below.

The trick allowing for obtaining closed analytical expressions for many properties of interest consists in assuming an integral form for the survival probability for the resetting process

$$\Psi_r(t) = \int_0^\infty \rho(\nu) \exp(-\nu t) d\nu,$$  

(4)

which allows for a representation of a large class of functions $\Psi(t)$. Equation (4) implies the following form of the Laplace transform of $\Psi_r(t)$:

$$\Psi_r(s) = \int_0^\infty \frac{\rho(\nu)}{s + \nu} d\nu.$$  

(5)

In what follows we concentrate on the case when $\rho(\nu)$ (essentially, the inverse Laplace transform of $\Psi_r(t)$ with $t$ considered as a Laplace variable) represents the density of the Laplace frequencies, i.e. it is a nonnegative function normalized to unity, as proposed in [7]. This assumption, of course, narrows the possible class of waiting time distributions for reset events to functions which can be represented as Laplace transforms of non-negative ones. For example, distributions which decay at infinity faster than exponential are excluded (cf appendix A of [20]), so that such important cases as resetting at a constant pace, cannot be considered within this scheme even as limiting ones. The class of $\Psi_r(t)$ which can be represented by equation (4) corresponds to completely monotonic functions, see e.g. [21]. The WTD $\psi_r(t)$, being the negative derivative of a completely monotonic function $\Psi_r(t)$, also has to be completely monotonic.

The representation, equation (4), considerably simplifies the overall structure of calculations making unnecessary cumbersome switching between different representations, like in [6]. If one is only interested in the long-time asymptotic behavior of relevant quantities for subdiffusive CTRW under power-law resetting, the distribution, equation (4) covers all possible behaviors, and therefore does not pose unnecessary restrictions.

Specifically, we choose $\rho(\nu)$ in the form of a generalized gamma density

$$\rho(\nu) = \frac{\alpha \sigma^\beta}{2 \Gamma \left( \frac{\alpha}{2} \beta \right)} \nu^{\beta - 1} \exp \left[ -\frac{(\sigma \nu)^{\frac{\alpha}{2}}}{\Gamma \left( \frac{\alpha}{2} \beta \right)} \right]$$

(6)

with positive parameters $\alpha$, $\beta$ and $\sigma$. The parameter $\alpha$ coincides with the one in equation (3). The somewhat awkward representation of $\Psi_r(t)$ in equations (4) and (6) as a Laplace transform of a non-negative function $\rho(\nu)$ has an advantage that many of expressions ensuing later can be
evaluated in a closed form. For incomplete resetting we still will have to resort to asymptotic expressions.

Substituting equation (6) into equation (4) one obtains a survival function which decays at infinity as a power law:

$$\Psi(t) \simeq \frac{\alpha \Gamma(\beta)}{2 \Gamma(\frac{2}{\alpha} \beta)} \left(\frac{s}{t}\right)^{\beta}. \quad (7)$$

To see this, one changes the integration variable in equation (4) to $z = \nu t$:

$$\Psi(t) = \frac{\alpha \sigma^{\beta}}{2 \Gamma(\frac{2}{\alpha} \beta)} \int_0^{\infty} z^{\beta - 1} \exp \left\{ - \left[ z + \left( \frac{\sigma}{t} \right)^{\frac{2}{\alpha}} \right] \right\} dz. \quad (8)$$

For $t \to \infty$ the integral in this expression tends to $\Gamma(\beta)$, so that $\Psi(t) \propto t^{-\beta}$. Equation (7) clarifies the physical meaning of parameters $\beta$ and $\sigma$: The parameter $\beta$ is the exponent of the power-law asymptotic decay of the survival probability for a resetting process, and $\sigma$, having the dimension of time, is a characteristic time of resetting. Since, as it was already noticed, the fluid limit, equation (3), is attained between the resettings only if the characteristic time $t_0$ of the CTRW step, the condition $\sigma \gg t_0$ will be assumed to hold.

We note that the representation essentially implies that the resetting process does possess a Markovian embedding, which allows for another line of discussion, the one followed in [7]. As stated in the introduction, in the present work, we decided to follow only a single line of derivations, the one starting with equation (1).

3. Complete resetting

Let us return to our equation (1), and take the Laplace of its both sides:

$$P(x, s) = \hat{L}[\Psi(t)p_{OW}(x, t)] + \kappa(s)\hat{L}[\Psi(t)p_{OW}(x, t)].$$

Noting that $1 + \kappa(s) = \frac{1}{1 - \psi(s)} = \frac{1}{\psi(s)}$, we obtain the expression for $p(x, s)$:

$$P(x, s) = \frac{\hat{L}[\Psi(t)p_{OW}(x, t)]}{\psi(s)}. \quad (9)$$

Here our representation, equation (6), is of an advantage. The Laplace transform of $\Psi(t)$ is in the denominator is given by equation (5). Substituting equation (4) into the expression in the numerator we get

$$\hat{L}[\Psi(t)p_{OW}(x, t)] = \int_0^{\infty} \left[ \int_0^{\infty} \rho(\nu)e^{-\nu t}d\nu \right] p_{OW}(x, t)e^{-\nu t}dt = \int_0^{\infty} \int_0^{\infty} \rho(\nu)p_{OW}(x, t)e^{-(\nu + \nu)t}d\nu dt = \int_0^{\infty} \rho(\nu)p_{OW}(x, s + \nu)d\nu. \quad (10)$$

Substituting the expressions for $\rho(\nu)$, equation (6), and for the propagator of the free motion, equation (2), and performing evident manipulations, we get for $P(x, s)$ the expression of the form

$$P(x, s) = \frac{A(x, s)}{B(s)} \quad (11)$$
with
\[ A(x, s) = \int_0^\infty \frac{\nu^{\beta-1} e^{-(\sigma \nu)\frac{x}{D}}}{(s + \nu)^{1 - \frac{\nu}{\alpha}}} \exp \left[ -\frac{(s + \nu)\frac{x}{D}}{\sqrt{D}} \right] d\nu, \]  
(12)
and
\[ B(s) = 2s\sqrt{D} \int_0^\infty \frac{\nu^{\beta-1} e^{-(\sigma \nu)\frac{x}{D}}}{s + \nu} d\nu. \]  
(13)

We note that the structure of expressions equations (11)–(13) guarantees the normalization of \( p(x, t) \) at any time. To see this we note that the \( x \)-dependence of \( p(x, s) \) stems from the one in \( A(x, s) \), and performing integration in \( x \) we get
\[ \int_{-\infty}^{\infty} A(x, s) dx = 2 \sqrt{D} \int_0^\infty \frac{\nu^{\beta-1} e^{-(\sigma \nu)\frac{x}{D}}}{s + \nu} \frac{\nu}{\alpha} d\nu = s^{-1} B(s). \]  
(14)

This note will be of importance when discussing some limiting transitions below.

For \( \beta > 1 \) the propagator equation (11) at long times tends to a stationary state. Indeed, the integrals in both expressions, equations (12) and (13), possess finite limits for \( s \to 0 \), so that
\[ A(x, 0) = \frac{2}{\alpha} \left( \sigma + \frac{\nu}{\sqrt{D}} \right)^{-\frac{1}{2}(\beta - 1)} \Gamma \left[ 1 - \frac{1}{\alpha}(\beta - 1) \right], \]  
(15)
\[ B(0) = s \sqrt{D} \frac{4}{\alpha} e^{\frac{1}{\alpha}(\beta - 1)} \Gamma \left[ 2 - \frac{1}{\alpha}(\beta - 1) \right], \]  
(16)
from which the stationary probability profile follows:
\[ P_{ct}(x) = \frac{\beta - 1}{\alpha} \frac{1}{\sqrt{\sigma D}} \frac{1}{\left( 1 + \frac{\nu}{\sqrt{\sigma D}} \right)^{1 + \frac{1}{\alpha}(\beta - 1)}}. \]  
(17)

a kind of a two-sided Lomax distribution. The normalization of the expression can be proved by its direct integration. The exponent of the power-law decay of this PDF at it wings coincides with the one obtained in [6].

If the parameter \( \beta \) is smaller than unity, \( \beta < 1 \), but the condition \( \beta + \frac{2}{\alpha} > 1 \) is fulfilled, then the integral in equation (12) is still converging for \( s \to 0 \) for any given \( x \), and is still given by equation (15), but the integral in equation (13) diverges (and therefore shows an explicit \( s \)-dependence for small \( s \)). Changing the variable of integration to \( z = \frac{x}{D} \) we get
\[ B(s) = 2s^\beta \sqrt{D} \int_0^\infty \frac{z^{\beta-1} \exp \left[ -(\sigma z)\frac{x}{D} \right]}{1 + z} dz. \]  
(18)

The integral in this expression now converges for \( s \to 0 \), and is equal to \( \pi / \sin(\pi \beta) \). Substituting the corresponding result and the expression for \( A(x, s) \), equation (12), into equation (11) we get for \( s \) small
\[ P(x, s) \simeq \frac{C}{s^\beta \left( 1 + \frac{\nu}{\sqrt{\sigma D}} \right)^{1 + \frac{1}{\alpha}(\beta - 1)}}. \]  
(19)
with
\[ C = \frac{\sin(\pi \beta) \Gamma \left[ 1 + \frac{1}{\alpha}(\beta - 1) \right]}{\alpha \pi \sqrt{\sigma D}} \sigma^{1-\beta}. \]  
(20)
We note that here and in what follows we will denote by $C$ the ‘local’ multiplicative constants in different expressions. These constants are different, and may possess different dimension. The constants in the final expressions will be numbered ($C_1, C_2,$ etc) to simplify the comparisons. All these numbered constants are dimensionless.

Passing to the time domain we obtain for long times

$$P(x,t) \simeq C_1 \frac{1}{\sqrt{\sigma^\alpha D}} \left( \frac{\sigma}{t} \right)^{1-\beta} \frac{1}{\left( 1 + \frac{|x|}{\sqrt{\sigma^\alpha D}} \right)^{1 + \frac{1}{2} (\beta - 1)}} \tag{21}$$

with

$$C_1 = \frac{\sin(\pi \beta) \Gamma \left[ 1 + \frac{2}{\beta} (\beta - 1) \right]}{\alpha \pi \Gamma(\beta)}, \tag{22}$$

where the power-laws in $x$- and $t$-dependence again coincide with the expressions obtained in [6]. The terms in equation (21) are grouped in the way which shows the correct dimension of the result.

We note that at difference with equation (17) the expression, equation (21), is not normalized per se, since it gives only the intermediate asymptotic behavior of the corresponding PDF, see [6]. This is clear from the fact that while for any fixed $x$ the expression for $A(x,s)$ does converge to a finite limit for $s \to 0$, its $x$-integral, equation (14) does not converge (its divergence is compensated by the one of $B(s)$). This means that for any given time equation (21) is not able to describe far tails of the distribution, which follow a different law.

We also note that the representation used allows for obtaining a closed expression for the first passage times to a given point $x_0$. To evaluate the MFPT, we use the general expression given in section IV of [7], which in our present notation reads

$$T = \frac{\int_0^\infty \rho(\nu) \left[ 1 - \exp \left( -\sqrt{\frac{\nu}{D}} |x| \right) \right] d\nu}{\int_0^\infty \rho(\nu) \exp \left( -\sqrt{\frac{\nu}{D}} |x| \right) d\nu}, \tag{23}$$

where $x$ is the distance between the starting point after resetting (which in our case corresponds to the origin of the coordinates) and the sink at $x_0$. Substituting the expression for $\rho(\nu)$ one can evaluate the integrals explicitly:

$$T = \sigma \frac{\Gamma \left[ \frac{2}{\beta} (\beta - 1) \right]}{\Gamma \left( \frac{2}{\beta} \right)} \left[ \left( 1 + \frac{|x|}{\sqrt{\sigma^\alpha D}} \right)^{\frac{2}{\beta}} - \left( 1 + \frac{|x|}{\sqrt{\sigma^\alpha D}} \right)^2 \right]. \tag{24}$$

This expression holds for $\beta > 1$. As a function of $\beta$, the MFPT $T$ monotonically decays, and when $\beta$ approaches unity, it diverges. For $\alpha$ and $\beta$ fixed, the MFPT $T(\sigma)$ as a function of parameter $\sigma$ is non-monotonic, and possesses a minimum. For $\alpha = 1$ and $\beta = 1.5$ the minimal value of $T$ is $T_{\text{min}} = 2 \frac{\sigma^2}{D}$, for $\alpha = 1$ and $\beta = 2$ it is $T_{\text{min}} = 1.846 \frac{\sigma^2}{D}$. For $\alpha = 1$ and $\beta \to \infty$, the minimum tends to a value $T_{\text{min}} = 1.544 \frac{\sigma^2}{D}$ (the same as for a simple random walk under Poissonian resetting).

### 4. Incomplete resetting

Let us now turn to the case of incomplete resetting. We return to equation (1) and note that in this case the PDF $p_2(x,t,t') = p_{\text{AW}}(x,t,t')$ has a Fourier-Laplace representation [18] (in the
Laplace domain with respect to the temporal variable \( t \) corresponding to the time elapsed from the resetting event, and the aging time \( t' \) fixed)

\[
p_{\text{AW}}(k, s, t') = \frac{1 - \psi_1(s, t')}{s} + \psi_1(s, t') \frac{1 - \psi_W(s)}{s} \lambda(k) \frac{1}{1 - \lambda(k) \psi_W(s)},
\]

where \( \psi_1(s, t') \) is the Laplace transform of the forward waiting time for the first step in the aged process. The expression multiplying \( \psi_1(s, t') \) in the second term is a Fourier-Laplace representation of the PDF of a jump first CTRW (not of a Montroll–Weiss wait-first one),

\[
p_{\text{JW}}(k, s) = \lambda(k) \frac{1 - \psi_W(s)}{s} \frac{1}{1 - \lambda(k) \psi_W(s)},
\]

where JW stands for jump-first walk. The PDF of the jump-first CTRW in Fourier-Laplace domain can be expressed via the one of the ordinary walk, equation (2):

\[
p_{\text{JW}}(k, s) = \frac{1}{\psi_W(s)} p_{\text{OW}}(k, s) - \frac{1 - \psi_W(s)}{\psi_W(s)}.
\]

Substituting equation (25) we get

\[
p_{\text{AW}}(k, s, t') = \frac{\psi_W(s) - \psi_1(s, t')}{s \psi_W(s)} + \psi_1(s, t') \psi_W(s) p_{\text{OW}}(k, s).
\]

Therefore, in real space (and in the Laplace domain with respect to its temporal variable \( t \)) \( p_{\text{AW}}(x, s, t') \) reads

\[
p_{\text{AW}}(x, s, t') = \frac{\psi_W(s) - \psi_1(x, s, t')}{s \psi_W(s)} \delta(x) + \psi_1(x, s, t') \psi_W(s) p_{\text{OW}}(x, s).
\]

The first term gives the singular part of \( p_{\text{AW}}(x, s, t') \), and the second one its regular part \( p_{\text{AW, reg}}(x, s, t') \), essentially, the only one of interest.

Returning to equation (1) we note that the singular contribution does not affect the PDF of positions at any finite \( x \) (and can be restored by requesting the normalization of the overall PDF). Therefore, knowing the regular part is absolutely sufficient, since the singular part of the PDF, with a \( \delta \)-function peak at zero can be easily restored from normalization of the overall expression

\[
P_{\text{sing}}(x, s) = \delta(x) \left[ \frac{1}{s} - \int_{-\infty}^{\infty} P_{\text{reg}}(x, s) dx \right].
\]

The regular part is

\[
P_{\text{reg}}(x, t) = \Psi_1(t) p_{\text{OW}}(x, t) + \int_0^t \kappa(t') \Psi_1(t - t') p_{\text{AW, reg}}(x, t - t', t') dt'
\]

where the first term corresponds to no resettings and was essentially already discussed in the previous section, while the second term, which we will denote by \( J(x, t) \), contains \( p_{\text{AW, reg}}(x, t_1, t_2) \) being the inverse Laplace transform \( (s \to t_1) \) of

\[
p_{\text{AW, reg}}(x, s, t_2) = \frac{\psi_1(s, t_2)}{\psi_W(s)} p_{\text{OW}}(x, s).
\]

Let us now discuss the overall structure of the Laplace transform of \( P_{\text{reg}}(x, t) \) as given by equation (27), \( P_{\text{reg}}(x, s) \), where the first term is given by equation (10), and concentrate on the second term
Let us use our representation, equation (5), for the Laplace transform of $\Psi_1(t)$, and assume a representation for the of $\kappa(t)$ similar to equation (4):

$$\kappa(t) = \int_{0}^{\infty} \gamma(\lambda) e^{-\lambda t} d\lambda,$$

so that $\gamma(\lambda)$ is the inverse Laplace transform of $\kappa(t)$. This can be formally obtained by using Bromwich integral for Laplace inversion and the Fourier representation for $\kappa(t)$ which follows by a Fourier inversion. Alternatively, one can pass to the Laplace representation,

$$\kappa(s) = \frac{\psi_1(s)}{1 - \psi_1(s)} = \int_{0}^{\infty} e^{-\mu t} \int_{0}^{\infty} \gamma(\lambda) e^{-\lambda t} d\lambda dt = \int_{0}^{\infty} \frac{\gamma(\lambda)}{s + \lambda} d\lambda,$$

and express $\psi_1(t)$ via $\Psi_1(s)$, which is known. The function $\gamma(\lambda)$ then follows by the solution of the Fredholm integral equation of the first kind.

Even without a formal solution at hands, one can show that the solution $\gamma(\lambda)$ exists and is non-negative (which is important for our approach to getting asymptotic expressions for $\gamma(\lambda)$), as we proceed to show. Assuming the non-negativity of $\gamma(\lambda)$ we see its two important properties which follow from the properties of $\kappa(s)$. Since $\kappa(s)$ is finite for any finite $s$ and diverges for $s \to 0$ as $\kappa(s) \sim s^{-\beta}$, the function $\gamma(\lambda)/\lambda$ must decay for $\gamma \to \infty$ and be integrable on the upper limit, and behave as a power law (and diverge) for $\lambda \to 0$. The positive parameter $s$ then regularizes the corresponding divergence.

To show the non-negativity of $\gamma(\lambda)$ we note that, for a continuous non-negative function $\rho(\nu)$, the integral in equation (4) may, with any desirable accuracy, be represented as a sum of $N$ exponentials

$$\Psi_1(t) = \sum_{i=1}^{N} \rho_i \exp(-\nu_i t)$$

with positive coefficients $\rho_i$ and different $\nu_i$, which can be taken equidistant on a finite interval. We note that since the function $\Psi_1(t)$ is a decaying one, all $\nu_i$ are strictly positive. The Laplace transform of $\Psi_1(t)$ reads

$$\Psi_1(s) = \sum_{i=1}^{N} \frac{\rho_i}{s + \nu_i}.$$

This function has $N$ poles on the negative real axis (and no other singularities), and changes its sign (on the real axis) once between each two poles, since its derivative on the real axis is negative wherever it is defined. Therefore the function has $N-1$ distinct simple roots $-\lambda_j$, all on negative real axis, which can be considered as roots of the polynomial of degree $N-1$ appearing in the numerator when bringing the expression for $\Psi_1(s)$ to a common denominator. Therefore, $\Psi_1(s)$ does not possess any other zeroes. The zeroes, all laying between the equidistant poles, cannot accumulate or come together to form a root of a higher order. These roots of $\Psi_1(s)$ correspond to simple poles of $1/\Psi_1(s)$. In this case the function

$$\kappa(s) = \frac{\psi_1(s)}{1 - \psi_1(s)} = \frac{1}{s\Psi_1(s)} - 1$$
will be a rational function allowing for a partial fraction decomposition
\[
\kappa(s) = \sum_{i=1}^{N} \frac{\gamma_i}{s + \lambda_i},
\]
with poles at \(-\lambda_i\) being the roots of \(\Psi(t)_i\), and with an additional pole at \(s = 0\). The derivative
\[
\kappa'(s) = \frac{\psi'_r(s)}{[1 - \psi_r(s)]^2} = -\sum_{i=1}^{N} \frac{\gamma_i}{(s + \lambda_i)^2},
\]
wherever it is defined, has to have the same sign as the derivative
\[
\psi'_r(s) = -\sum_{i=1}^{N} \rho_i \mu_i \frac{1}{(s + \nu_i)^2},
\]
i.e. has to be negative, also on the negative real axis, which implies that all \(\gamma_i\) are positive. Otherwise, the function \(\kappa'(s)\) will be positive for \(s\) in vicinity of \(\lambda_n\), with number \(n\) corresponding to the one of the negative coefficient \(\gamma_n\). Therefore, considering approximations for \(\Psi(t)_i\) with growing numbers \(N\) of the terms, we will obtain better and better approximations for \(\kappa(s)\) in form of equation (31) with positive coefficients. These expressions tend to an integral given by equation (30). We note that this demonstration holds both for continuous \(\rho(\nu)\) and for discrete distributions containing only a finite number of Laplace frequencies \(\nu_i\).

Provided the solution \(\gamma(\lambda)\) exists, we can give a more formal way to see that it is non-negative, and even provide an explicit (although formal) expression for it in terms of \(\rho(\nu)\). One notes that equation (30) states that \(\kappa(s)\) is a Stieltjes transform of the function \(\gamma(\lambda)\). Assuming \(\gamma(\lambda)\) being continuous, one then may write
\[
\gamma(\lambda) = \lim_{\epsilon \to 0} \frac{\kappa(-\lambda - i\epsilon) - \kappa(-\lambda + i\epsilon)}{2i\pi},
\]
where \(\epsilon\) is a small positive number, see e.g. Chap. VIII of [22] (if \(\gamma(\lambda)\) is discontinuous at \(\lambda\) the corresponding limit gives the mean value of its left and right limits at \(\lambda\)). Substituting \(\kappa(s) = \frac{\psi_r(s)}{1 - \psi_r(s)}\) and \(\psi_r(s) = \int_{0}^{\infty} \frac{\nu \rho(\nu)}{(\nu - \lambda)^2} d\nu\), we get that
\[
\gamma(\lambda) = \frac{1}{2i\pi} \lim_{\epsilon \to 0} \left( \frac{\int_{0}^{\infty} \frac{\nu \rho(\nu)}{(\nu - \lambda - i\epsilon)^2} d\nu}{\int_{0}^{\infty} \frac{\nu \rho(\nu)}{(\nu - \lambda)^2} d\nu} - \frac{\int_{0}^{\infty} \frac{\nu \rho(\nu)}{(\nu - \lambda + i\epsilon)^2} d\nu}{\int_{0}^{\infty} \frac{\nu \rho(\nu)}{(\nu - \lambda)^2} d\nu} \right)
\]
\[
= \frac{1}{2i\pi} \lim_{\epsilon \to 0} \left( \frac{I_- - I_+}{1 - I_- (1 - I_+)} \right),
\]
with \(I_{\pm} = \int_{0}^{\infty} \frac{\nu \rho(\nu)}{(\nu - \lambda)^2} e^{\pm i\epsilon} d\nu\). Since the integrals \(I_+\) and \(I_-\) are complex conjugate, we may write
\[
\gamma(\lambda) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\text{Im } I_-}{|1 - I_-|^2} = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\text{Im } I_-}{(1 - \text{Re } I_-)^2 + (\text{Im } I_-)^2},
\]
from which we get
\[
\gamma(\lambda) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\int_{0}^{\infty} \frac{\exp(\nu \epsilon) \rho(\nu)}{(\nu - \lambda)^2 + \epsilon^2} d\nu}{1 - \int_{0}^{\infty} \frac{(\nu - \lambda) \exp(\nu \epsilon) \rho(\nu)}{(\nu - \lambda)^2 + \epsilon^2} d\nu}^2 + \left( \int_{0}^{\infty} \frac{\exp(\nu \epsilon) \rho(\nu)}{(\nu - \lambda)^2 + \epsilon^2} d\nu \right)^2.
\]
The non-negativity of this expression is evident. A more convenient expression which holds for continuous \(\rho(\nu)\) and gives a formal expression for \(\gamma(\lambda)\) can be obtained by using in
equation (33) the representation \( \lim_{x \to 0} \frac{1}{x^{2 \epsilon}} = \text{p.v.} \frac{1}{x^2} + i \pi \delta(x) \) with \( \text{p.v.} \) denoting the principal value in the corresponding integral. One gets

\[
\gamma(\lambda) = \frac{\lambda \rho(\lambda)}{(1 - \text{p.v.} \int_0^\infty \frac{\nu \rho(\nu) d\nu}{\nu - \lambda})^2 + \pi^2 \lambda^2 \rho^2(\lambda)},
\]

which is nonnegative for all non-negative continuous \( \rho(\lambda) \) and represents a solution of equation (30).

Using equations (4) and (29) we write for the Laplace transform \( J(x,s) \)

\[
J(x,s) = \int_0^\infty e^{-st} \int_0^t \left\{ \int_0^\infty \gamma(\lambda)e^{-\lambda't} d\lambda \right\} \int_0^\infty \rho(\nu)e^{-\nu(t-t')} d\nu \]

\[
\times p_{AW,\text{reg}}(x,t-t',t') dt \]

\[
= \int_0^\infty \int_0^\infty d\lambda d\nu \gamma(\lambda)\rho(\nu) \]

\[
\times \left[ \int_0^\infty e^{-\pi} \int_0^t e^{-\lambda't'} e^{-\nu(t-t')} p_{AW,\text{reg}}(x,t-t',t') dt' d\nu \right].
\]

The expression in the square brackets in the last line has a form

\[
I(x,s) = \int_0^\infty e^{-st} \int_0^t F(x,t-t',t') dt' dt.
\]

To proceed further, we first interchange the sequence of integrations,

\[
I(x,s) = \int_0^\infty dt' \int_0^\infty \int_0^t e^{-st} F(x,t-t',t') d\nu dt,
\]

and introduce the new integration variable \( t'' = t - t' \), so that

\[
I(x,s) = \int_0^\infty \int_0^\infty e^{-i(t''+t')} F(x,t'',t') dt' dt''.
\]

Let us now return to our equation (35), with \( F(x,t-t',t') = e^{-\nu(t-t')} e^{-\lambda't'} p_{AW,\text{reg}}(x,t-t',t') \), and perform the final integration:

\[
J(x,s) = \int_0^\infty e^{-st} \int_0^t e^{-\lambda't'} e^{-\nu(t-t')} p_{AW,\text{reg}}(x,t-t',t') dt' d\nu
\]

\[
= \int_0^\infty \int_0^\infty e^{-i(t''+t')} e^{-i(\lambda - \nu)t'} p_{AW,\text{reg}}(x,t'',t') dt' d\nu,
\]

i.e. \( J(x,s) \) is a double Laplace transform of \( p_{AW,\text{reg}}(x,t'',t') \) in its both temporal variables, as if these were independent:

\[
J(x,s) = p_{AW,\text{reg}}(x,s+i\nu,s+i\lambda).
\]

The final result reads:

\[
J(x,s) = \int_0^\infty \int_0^\infty \gamma(\lambda)\rho(\nu)p_{AW,\text{reg}}(x,s+i\nu,s+i\lambda) d\lambda d\nu.
\]

Now, we know that

\[
p_{AW,\text{reg}}(x,s,t_2) = \frac{\psi_1(x,t_2)}{\psi_W(x)} \tilde{p}_{OW}(x,s),
\]
and that the double Laplace transform of $\psi_1(t_1, t_2)$ in its both time variables (forward waiting time $t_1$ and aging time $t_2$) reads [18]

$$
\psi_1(s, u) = \frac{1}{1 - \psi_W(u)} \frac{\psi_W(u) - \psi_W(s)}{s - u},
$$

we get for $J(x, s)$

$$
J(x, s) = \int_0^\infty d\nu \rho(\nu) \int_0^{\infty} d\lambda \gamma(\lambda) \frac{1}{\psi_W(s + \nu)(1 - \psi_W(s + \lambda))} \frac{\psi_W(s + \nu) - \psi_W(s + \lambda)}{(s + \nu) - (s + \lambda)} p_{OW}(x, s +\nu).
$$

Rewriting the integrand of the second integral in the form

$$
\frac{\gamma(\lambda)}{s + \lambda} \psi_W(s + \nu) - \psi_W(s + \lambda)\frac{\psi_W(s + \nu) - \psi_W(s + \lambda)}{(s + \nu) - (s + \lambda)} p_{OW}(x, s +\nu)
$$

one can recognize its relation to equation (26) in [23], where

$$
P_{\text{reg}}(x, s +\nu) = \int_0^\infty d\nu \rho(\nu) p_{OW}(x, s +\nu)\frac{\psi_W(s + \nu) - \psi_W(s + \lambda)}{(s + \nu) - (s + \lambda)} p_{OW}(x, s +\nu)
$$

corresponds to the regular component of the distribution function $P_{AW}$ in equation (26) of [23], and the factor $\frac{\gamma(\lambda)}{s + \lambda}$ replaces to the factor $P_{AW}$ in the same equation. The Laplace variable $s + \nu$ corresponds to a current time, and the variable $s + \lambda$ to the lag time. The final result reads:

$$
P_{\text{reg}}(x, s +\nu) = \int_0^\infty d\nu \rho(\nu) p_{OW}(x, s +\nu)\frac{\psi_W(s + \nu) - \psi_W(s + \lambda)}{(s + \nu) - (s + \lambda)} p_{OW}(x, s +\nu)\frac{\gamma(\lambda)}{\lambda - \nu} d\lambda d\nu.
$$

The first term on the r.h.s. corresponds to the situation when no resetting events took place up to the current time. Keeping in mind the existence of this term, we note that it can be neglected when discussing the long-time asymptotic behavior. Omitting of this term can be justified \textit{a posteriori}, after we get asymptotic results for the second term on the r.h.s. According to equation (1), the no-resetting term in the time domain is given by $\Psi_1(t)p_{OW}(x, t)$, with $\Psi_1(t) \sim t^{-\beta}$ at long times, and $p_{OW}(x, t) \sim t^{-\frac{x^2}{2}}$ for $t >> \left(\frac{\nu}{\beta}\right)$. Therefore, for $t >> \left(\frac{\nu}{\beta}\right)$

the first term behaves as a function of time as $t^{-(\beta + \frac{x^2}{2})}$. As we proceed to show, for $\beta > 1$, the asymptotics of $P_{\text{reg}}$ at long times is $P_{\text{reg}}(x, t) \sim f(x) t^{-\left(\frac{x^2}{\beta} - \alpha\right)}$, equation (44), corresponding to a slower decay, since for $\beta > 1$ one has $\beta + \frac{x^2}{2} > 1 - \alpha$. For $\beta < 1$ the case in which the universal intermediate asymptotics does exist corresponds to the restriction $\beta + \frac{x^2}{2} > 2$, in which case $P_{\text{reg}}(x, t) \sim f(x) t^{\left(\frac{2 - \alpha - \beta}{\beta}\right)}$, equation (48), which again corresponds to a slower decaying, and therefore dominant, contribution.

Let us now return to the second term, and assume that the values of $\nu$ and $\lambda$ dominating the integrals in the second term are small, so that the approximation $\psi_W(s) \simeq 1 - (t_0 s)^\alpha$ (with $t_0$ being the characteristic time of a step of CTRW, assumed small compared to the typical
resetting time, as discussed above) is sufficient, and use the expression for \( \frac{\psi_{W}(x+\nu)-\psi_{W}(x+\lambda)}{\psi_{W}(x+\nu)-\psi_{W}(x+\lambda)} \) in the lowest non-vanishing order. Since in the asymptotics of long times only the second term in equation (27) plays the role, the asymptotic expression for \( P_{\text{reg}}(x,s) \) takes the universal form

\[
P_{\text{reg}}(x,s) \simeq \int_{0}^{\infty} \rho(\nu) \int_{0}^{\infty} \frac{\gamma(\lambda)}{\lambda-\nu} \left[ 1 - \frac{(s+\nu)^{\alpha}}{(s+\lambda)^{\alpha}} \right] p_{\text{OW}}(x,s+\nu)d\nu d\lambda.
\] (40)

We note that the assumption of smallness of \( \nu \) and \( \lambda \) is not necessary if the waiting times in the CTRW follow a generalized Mittag–Leffler distribution [19], for which \( \psi_{W}(s) = [1 + (t_{0} s)^{\alpha}]^{-1} \). In this case equation (40) is exact.

If the waiting times in CTRW do not follow a generalized Mittag–Leffler distribution our lowest-order approximation is not an independent one, but is consistent with the ones done previously. In general, if we assume that the fluid-limit form for the PDF in CTRW, equation (3), is applicable, the values of \( \nu > (D/x^{2})^{\alpha} \) are strongly suppressed by the exponential decay of \( p_{\text{OW}}(x,s) \). Noting that \( D \sim a^{2}/t_{0}^{2} \) we see, that the relevant values of \( \nu \) are small (compared to \( t_{0}^{-1} \)) for all \( x^{2} \gg a^{2} \), i.e. when the fluid form of the PDF in the ordinary walk holds. The additional suppression of large values of \( \nu \) comes from the form of \( \rho(\nu) \), equation (6). The dominance of the small values of \( \lambda \sim \nu \) is then guaranteed by the divergence of \( \gamma(\lambda) \) at zero. An additional hint onto the validity of this equation is given by the fact that in the limiting case \( \beta \rightarrow 1 \), i.e. for Poissonian resetting, it coincides, after restoring the contribution of the no-resetting realizations, with the one obtained in [7, 11]. We also note that equation (40) can be derived by an alternative method based on the Markovian embedding, in which case the assumptions of smallness of \( \nu \) and \( \lambda \) do not have to be made in advance.

To use the expression, equation (40), for evaluating the asymptotic forms of the PDFs we need an estimate for the function \( \gamma(\lambda) \). To circumvent using the formal solution of equations (30) and (34), we can proceed less formally, and note that the representation, equation (29) implies that \( \gamma(\lambda) \) is the inverse Laplace transform of \( \kappa(t) \). Thus we can proceed in two steps: obtain the long-time asymptotics of \( \kappa(t) \) from the small-\( s \) asymptotics of

\[
\kappa(s) = \frac{1}{s \Psi_{r}(s)} - 1,
\]

which is dominated by the first term, and then obtain the small-\( \lambda \) asymptotics of \( \gamma(\lambda) \).

For \( \beta > 1 \) the rate \( \kappa(t) \) tends for long times to a constant \( \kappa_{0} = \frac{1}{\sigma} \) with \( \tau \) being the mean waiting time for a resetting event. This is given by \( \tau = \Psi_{r}(s)_{s=0} = \int_{0}^{\infty} \frac{\rho(\nu)}{\nu} d\nu \). Using our expression for \( \rho(\nu) \), equation (6), we obtain

\[
\tau = \sigma \Gamma[2(\beta-1)/\alpha]/\Gamma(2/\beta) \alpha.
\]

The inverse Laplace transform of this constant \( \kappa_{0} \) is a delta-function: \( \gamma(\lambda) = \frac{1}{\sigma} \delta(\lambda) \).

Substituting the expressions for \( \rho(\nu) \) and \( p_{\text{OW}}(x,s) \) in equation (40) we get

\[
P_{\text{reg}}(x,s) = \frac{1}{\pi x^{2\alpha}} \frac{\alpha \sigma^{\beta}}{2 \Gamma \left( \frac{\beta}{\alpha} \right)} \frac{1}{2 \sqrt{D}} \int_{0}^{\infty} \nu^{\beta-2} [(s+\nu)\alpha - s^{\alpha}](s+\nu)^{\beta-2} - 1 \times \exp \left\{ - \left( (\sigma \nu) \frac{\alpha}{\sqrt{D}} + (s+\nu) \frac{\beta}{\sqrt{D}} \right) \right\} d\nu.
\] (41)

If the condition \( \beta + \frac{\beta}{\alpha} - 2 > 0 \) is fulfilled, the integral stays finite for \( s \rightarrow 0 \). In this case we can take \( s = 0 \) in the integrand, evaluate
For $0 < \beta < 1$ the long-time asymptotics of $\kappa(t)$ is a power law. To find its parameters it is enough to note that for $s \to 0$ the Laplace-transformed rate is $\kappa(s) \simeq |s\Psi_t(s)|^{-1}$. The expression for $s\Psi_t(s)$ can be obtained in analogy with equation (18), so that

$$\kappa(s) \simeq C s^{-\beta}$$

with $C = 2 \sin(\pi\beta)\Gamma\left(\frac{2\beta}{\alpha}\right)/(\pi\alpha\beta)$. The asymptotic expression for $\gamma(\lambda)$ follows by applying twice the Tauberian theorem, and gives

$$\gamma(\lambda) \simeq \frac{C}{\Gamma(\beta)\Gamma(1-\beta)} \lambda^{-\beta} = 2 \Gamma\left(\frac{2\beta}{\alpha}\right) \frac{1}{\alpha} \left(\frac{\sin(\pi\beta)}{\pi}\right)^2 \sigma^{-\beta} \lambda^{-\beta}.$$ 

For $0 < \beta < 1$ the propagator $P_{\text{reg}}$ assumes the form

$$P_{\text{reg}}(x,x) \simeq \frac{\alpha K}{2 \sqrt{D}} \int_0^\infty \int_0^\infty \frac{\nu^{\beta-1}D^{1-\beta}}{\lambda - \nu} \left[ 1 - \left(\frac{s + \nu}{s + \lambda}\right)^\alpha \right] \times (s + \nu)^{\frac{\alpha}{2} - 1} \exp \left\{ - \left[ \left(\frac{s + \nu}{\sqrt{D}}\right)^\alpha + (s + \nu)^\frac{\alpha}{2} \left(\frac{|\lambda|}{\sqrt{D}}\right) \right] \right\} d\lambda d\nu$$

with $K$ being an additional constant comprising the prefactors of $\lambda^{-\beta}$ in the expression for $\gamma(\lambda)$. Changing to the integration variable $z = \lambda/s$ we get for the r.h.s.
For normal diffusion, the power-law agrees with equations (10). We note that the power-law expression tends to $\frac{\alpha K}{2\sqrt{D}} \int_0^\infty \int_0^\infty \frac{\nu^{\beta-1}}{z^{\frac{1}{2}}} \left[ 1 - \frac{(1 + \frac{z}{\nu})^\alpha}{(1 + z)^\alpha} \right] \times (s + \nu)^{\frac{\nu}{2} - 1} \exp \left\{ - \left[ (\sigma \nu)^\frac{\nu}{2} + (s + \nu)^\frac{\nu}{2} \left| \frac{x}{\sqrt{D}} \right| \right] \right\} \, dz \, d\nu.$

For $s \to 0$ the expression tends to

$$P_{\text{reg}}(x,s) \simeq \frac{\alpha K}{2\sqrt{D}} x^{1-\beta-\alpha} \int_0^\infty \int_0^\infty \frac{\nu^{\beta+\alpha-2}}{(1 + z)^\alpha} \times \nu^{\frac{\nu}{2} - 1} \exp \left\{ - \left[ (\sigma \nu)^\frac{\nu}{2} + \nu^{\frac{\nu}{2}} \left| \frac{x}{\sqrt{D}} \right| \right] \right\} \, dz \, d\nu \simeq \frac{\alpha K}{2\sqrt{D}} x^{1-\beta-\alpha} \left\{ \int_0^\infty \frac{z^{-\beta}}{(1 + z)^\alpha} \, dz \right\} \times \left\{ \int_0^\infty \nu^{\beta+\alpha-3} \exp \left[ -\nu^{\frac{\nu}{2}} \left( \sigma^{\frac{\nu}{2}} + \left| \frac{x}{\sqrt{D}} \right| \right) \right] \, d\nu \right\}. \quad (47)$$

The transition from equation (46) to the first expression in equation (47) corresponds to substitutions $z - \frac{\nu}{\sqrt{D}} \to -\frac{\nu}{\sqrt{D}}, 1 - \frac{(1 + \frac{z}{\nu})^\alpha}{(1 + z)^\alpha} \to \frac{d}{\nu}, s + \nu \to \nu$ which are valid in the limit $s \to 0$. The integral over $\nu$ coincides with the one discussed for the previous case, equation (42), it converges if the condition $\beta + \frac{h}{2} - 2 > 0$ is fulfilled. The integral in $z$ represents a B-function

$$\int_0^\infty \frac{z^{-\beta}}{(1 + z)^\alpha} \, dz = \frac{\Gamma(1 - \beta)\Gamma(\alpha + \beta - 1)}{\Gamma(\alpha)},$$

and converges if $\alpha + \beta > 1$, which is implied by the previous inequality since $\alpha < 1$, so that

$$P_{\text{reg}}(x,s) = C \frac{x^{1-\beta-\alpha}}{\left(1 + \frac{|x|}{\sqrt{\sigma D}}\right)^{3 + \frac{\nu}{2}(\beta-2)}}$$

with

$$C = \frac{\sigma^{2-\alpha-\beta}}{\sqrt{\sigma D}} \frac{\Gamma(1 - \beta)\Gamma(\alpha + \beta - 1)}{\Gamma(\alpha)} \Gamma \left[ 3 + \frac{2}{\alpha}(\beta - 2) \right] \frac{1}{\alpha} \left( \frac{\sin(\pi \beta)}{\pi} \right)^2.$$

Therefore, the result for this case differs from the previous one only by the form of the time dependence:

$$P_{\text{reg}}(x,t) \simeq \frac{C_3}{\sqrt{\sigma D}} (\sigma^\gamma t)^{2-\alpha-\beta} \frac{1}{\left(1 + \frac{|x|}{\sqrt{\sigma D}}\right)^{3 + \frac{\nu}{2}(\beta-2)}} \quad (48)$$

with

$$C_3 = \frac{\Gamma(1 - \beta)\Gamma \left[ 3 + \frac{2}{\alpha}(\beta - 2) \right]}{\Gamma(\alpha)} \frac{1}{\alpha} \left( \frac{\sin(\pi \beta)}{\pi} \right)^2.$$

The power-law $x$-dependence in equation (48) tends to those in equation (44), and differs from the one result presented in [6], equation (107), see also table II of the work, due to a mistake (missing of a multiplier) which occurred when passing from equation (100) to table II.

The plausibility check of these expressions is given by the limiting transition to the case of normal diffusion, $\alpha = 1$, which is a non-aging, Markovian process for which there is no difference between the complete and incomplete resetting schemes. In this case equation (44) agrees with equations (17) and (48) agrees with equation (21). We note that the power-law
asymptotics in these expressions for different values of $\alpha$ and $\beta$ were checked in direct simulations in [6]. Additional checks are given by comparison of our results with known results for the case of normal diffusion [14]. Thus, the exponents characterizing the $x$ and $t$ behavior in [14] are the same as given by equations (17) and (21). The results for small $x$ in [14] differ from ours in the sense that the corresponding PDF shows a divergence for $x \to 0$. This divergence is however an artifact of approximation $\Psi_r$ by a power-law $\Psi_r(t) \sim t^{-\beta}$ in the whole time domain, while a ‘reasonable’ $\Psi_r(t)$ must tend to unity for $t \to 0$. The results of [16] differ from ours in the statement that the stationary PDF exists only for $\beta > 2$. This statement is wrong because the divergence of the second moment does not preclude the existence of the stationary distribution, see [14]. The exponential profile of the stationary PDF (when it exists) obtained in [16] is a result of uncontrolled approximations, and other works, [6, 7, 13–15] report on non-exponential stationary PDFs.

5. Conclusions

In the present work we revisited the problem of CTRW under resetting with a power-law distribution of waiting times between the resetting events. Combining the general approach of [6] with a special representation for the survival probability in resetting process used in [7] allowed us for obtaining closed-form results for the PDF of displacements in the case of complete resetting, and asymptotic forms for these PDFs (including prefactors) in the case of incomplete resetting. In addition, for complete resetting, we provide an explicit expression for the MFPT to a given point. The approach discussed is much more straightforward and transparent than ones used in previous works.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] Evans M R, Majumdar S N and Schehr G 2020 Stochastic resetting and applications J. Phys. A: Math. Theor. 53 193001
[2] Pal A and Reuveni S 2017 First passage under restart Phys. Rev. Lett. 118 030603
[3] Pal A, Kostinski S and Reuveni S 2022 The inspection paradox in stochastic resetting J. Phys. A: Math. Theor. 55 021001
[4] Bodrova A S, Chechkin A V and Sokolov I M 2019 Scaled Brownian motion with renewal resetting Phys. Rev. E 100 012120
[5] Bodrova A S, Chechkin A V and Sokolov I M 2019 Nonrenewal resetting of scaled Brownian motion Phys. Rev. E 100 012119
[6] Bodrova A S and Sokolov I M 2020 Continuous-time random walks under power-law resetting Phys. Rev. E 101 062117
[7] Shkilev V P 2017 Continuous-time random walk under time-dependent resetting Phys. Rev. E 96 012126
[8] Montero M, Masó-Puigdellosas A and Villarroel J 2017 Continuous-time random walks with reset events: historical background and new perspectives Eur. Phys. J. B 90 176
[9] Masó-Puigdellosas A, Campos D and Méndez V 2019 Stochastic movement subject to a reset-and-residence mechanism: transport properties and first arrival statistics J. Stat. Mech. 033101
[10] Masoliver J and Montero M 2019 Anomalous diffusion under stochastic resetting: a general approach Phys. Rev. E 100 042103
[11] Kuśmierz L and Gudowska-Nowak E 2019 Subdiffusive continuous-time random walks with stochastic resetting Phys. Rev. E 99 052116
[12] Méndez V, Masó-Puigdellosas A, Sandev T and Campos D 2021 Continuous time random walks under Markovian resetting Phys. Rev. E 100 042103
[13] Eule S and Metzger J J 2016 Non-equilibrium steady states of stochastic processes with intermittent resetting New J. Phys. 18 033006
[14] Nagar A and Gupta S 2016 Diffusion with stochastic resetting at power-law times Phys. Rev. E 93 060102(R)
[15] Pal A, Kundu A and Evans M R 2016 Diffusion under time-dependent resetting J. Phys. A: Math. Theor. 49 225001
[16] Singh R K, Górskia K and Sandev T 2022 General approach to stochastic resetting Phys. Rev. E 105 064133
[17] Montroll E and Weiss G H 1965 Random walks on lattices II J. Math. Phys. 6 167–81
[18] Klafter J and Sokolov I M 2011 First Steps in Random Walks: From Tools to Applications (Oxford: Oxford University Press)
[19] Hilfer R and Anton L 1995 Fractional master equations and fractal time random walks Phys. Rev. E 51 R848–51
[20] Chechkin A and Sokolov I M 2021 Relation between generalized diffusion equations and subordination schemes Phys. Rev. E 103 032133
[21] Miller K S and Samko S G 2001 Completely monotonic functions Integral Transform Spec. Funct. 12 389–402
[22] Widder D V 1946 The Laplace Transform (Princeton, NJ: Princeton University Press)
[23] Barkai E and Sokolov I M 2007 Multi-point distribution function for the continuous time random walk J. Stat. Mech. 8 P08001