On Some Properties of Functions on Convex Galaxies

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ABSTRACT

In this paper, we define and study extensively a new type of external sets in $\mathbb{R}$, we call it "convex galaxies". We show that these convex external sets may be classified in some definite types. More precisely, we obtain the following:

(1) Let $G \subseteq \mathbb{R}$ be a convex galaxy which is symmetric with respect to zero, then

(i) $G$ is an $\alpha$-galaxy (0) if and only if there exists an internal strictly increasing sequence of strictly positive real numbers $\{a_n\}_{n \in \mathbb{N}}$ with $G = \bigcup_{n \in \mathbb{N}} [-a_n, a_n]$ such that

$$a_0 = \alpha \quad \text{and} \quad \frac{a_{n+1}}{a_n} = c, \quad \text{for all} \quad n \in \mathbb{N},$$

where $c$ is some limited real number such that $c > 1$.

(ii) $G$ is a non-linear galaxy if and only if there exists an internal strictly increasing sequence of strictly positive real numbers $\{a_n\}_{n \in \mathbb{N}}$ with $G = \bigcup_{n \in \mathbb{N}} [-a_n, a_n]$ such that $\frac{a_{n+1}}{a_n}$ is unlimited for all $n \in \mathbb{N}$.

(2) Let $G \subseteq \mathbb{R}$ be a convex galaxy which is symmetric with respect to zero, then

(i) $G$ is an $\alpha$-galaxy (0) iff there exists a real internal strictly increasing $C^\infty$-function $f$, such that $f(G) = G$, and $\frac{f'(t)}{f(t)} = c$ for all limited $t \geq 1$, where $c$ is a positive real number.

(ii) $G$ is a non-linear galaxy if and only if there exists a real internal strictly increasing $C^\infty$-function $f$, such that $f(G) = G$ and $\frac{f'(t)}{f(t)}$ is positive unlimited, for all appreciable $t \geq 1$.

Keywords: Convex, Galaxy, External Sets.

حول بعض خواص الدوال في الكالكسيات المحدبة

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الملخص

في هذا البحث، تم تعريف نوع جديد من المجموعات الخارجية في $\mathbb{R}$ سميت بـ "الكالكسيات المحدبة" كما تم دراستها بشكل مستغفيض. يمكن أن تصفف هذه المجموعات المحدبة الخارجية إلى بعض الأنواع المحددة. وعلى نحو أدق حصلنا على ما يلي:

(1) لتكن $G \subseteq \mathbb{R}$ كل كالكسي محدبة متواضعة بالنسبة للصغر، عندها...
1. Introduction

An application of this classification may be found in non-standard analysis approach. The study of slow-fast vector fields as shown by Diener F. [2]. For example, the notion width of jump may be defined in terms of a convex galaxy. Further, this classification can be used in approximations. Thus, the set of points on the real line, where two real functions $f$ and $g$ are infinitely close on $R$, that is the set \{ $x \in R : f(x) \cong g(x)$ \} will often be a convex monad. [1]

For practice reasons we start with the study of convex galaxies which are symmetric with respect to zero.

The following definitions and notations are needed throughout this paper. See [4], [5], [6], [7], [8], [9] and [10].

Every concept concerning sets or elements defined in the classical mathematics is called \textit{standard}.

Any set or formula which does not involve new predicates “standard, infinitesimals, limited, unlimited … etc” is called \textit{internal}, otherwise it is called \textit{external}.

A real number $x$ is called \textit{unlimited} if and only if $|x| > r$ for all positive standard real numbers $r$ ; otherwise it is called \textit{limited}.

The set of all unlimited real numbers is denoted by $\overline{R}$, and the set of all limited real numbers is denoted by $R$.

A real number $x$ is called \textit{infinitesimal} if $|x| < r$ for all positive standard real numbers $r$.

A real number $x$ is called \textit{appreciable}, if $x$ is limited but not infinitesimal.

Two real numbers $x$ and $y$ are said to be \textit{infinitely close} if and only if $x - y$ is infinitesimal and denoted by $x \equiv y$.

The set of all limited real numbers is called \textit{principal galaxy}, (denoted by $G$).
For any real number \( a \), the set of all real numbers \( x \) such that \( x - a \) limited is called the \textit{galaxy of} \( a \) (denoted by \( \text{gal}(a) \)).

Let \((\alpha \neq 0)\) \( \alpha \) and \( x \in \mathbb{R} \), we define the \( \alpha \)-\textit{galaxy} \( (x) \) as follows:

\( \alpha \)-\textit{galaxy}(\( x \))\( = \{ y \in \mathbb{R} : \frac{y - x}{\alpha} \text{ is limited} \} \) and denoted by \( \alpha \)-\textit{G}(\( x \)).

A subset \( G \) of \( \mathbb{R} \) is a \textit{convex galaxy} which is asymmetric with respect to zero iff there exists an internal strictly increasing sequence of strictly positive real numbers \( \{t_n\}_{n \in \mathbb{N}} \) such that \( G = \bigcup_{n \in X} [-t_n, t_n] \).

**Theorem 1.1:** (Extension Principle) [3]

Let \( X \) and \( Y \) be two standard sets, \( \{X\} \) and \( \{Y\} \) be the subsets constitute of the standard elements \( X \) and \( Y \), respectively. If we can associate with every \( S \in X \) a unique \( \{f_y \in Y : f_x = f_y \} \), then there exists a unique standard \( \star \)\( y \in Y \) such that \( \forall x \in X \), \( \star f_x = \star f_y \).

**Theorem 1.2:** (Principal of External Induction) [3]

If \( E \) is an internal or external property such that \( E(0) \) is true and \( E(n) \rightarrow E(n + 1) \) true for all \( n \in \mathbb{N} \). Then, \( E(n) \) is true for all \( n \in \mathbb{N} \).

Consider the following characterization of the galaxies:

2. Examples: Let \( G \) and \( \{a_n\}_{n \in \mathbb{N}} \) be as mentioned previously, we may always assume that \( a_0 = 1 \)

(1) Suppose that \( \frac{a_{n+1}}{a_n} = 2 \) for all \( n \in \mathbb{N} \), then \( G \) is an additive group. Because \( x, y \in G \)
and \( |x|, |y| \) being less than \( a_n \) then
\[
|x + y| \leq |x| + |y| \leq 2a_n \leq a_{n+1}.
\]
So \( x + y \in G \).
Since, \( a_0 = 1 \), it follows that \( G = \mathbb{G} \). If, on the contrary, we had \( a_0 = \alpha \), then \( G \) will be \( \alpha \)-\textit{galaxy}(0).

(2) Suppose that \( \frac{a_{n+1}}{a_n} = \omega \), for all \( n \in \mathbb{N} \), where \( \omega \) is a positive unlimited real number,
then the galaxy \( G \) is again additive group. However, \( G \) is not an \( \alpha \)-\textit{galaxy}(0) because if \( \alpha > 0 \) be such that \( \alpha \)-\textit{galaxy} \((0) \subseteq G \), then there is \( n \in \mathbb{N} \) such that \( \alpha < a_n \) so \( \alpha < a_{n+1} \) which implies that \( a_{n+1} \in \alpha \)-\textit{galaxy}(0). Hence, \( \alpha \)-\textit{galaxy}(0) \( \subseteq \mathbb{G} \).

(3) If \( \frac{a_{n+1}}{a_n} = 1 + \varepsilon \) for all \( n \in \mathbb{N} \), where \( \varepsilon \) is a positive infinitesimal, then \( G \) is not additive group because \( 2 \notin G \).
A convex galaxy which is a group, but not an \( \alpha \)-\textit{galaxy} will be called non-linear. Informally, the \( \alpha \)-\textit{galaxy} is the set of all real numbers of order \( \alpha \), while a non-linear galaxy cannot be the set of real numbers of the order of one of its element.
We generalize the connection between the convex galaxy $G$ and the ratio of consecutive terms $\frac{a_{n+1}}{a_n}$ of the sequence $\{a_n\}_{n \in \mathbb{N}}$ such that $G = \bigcup_{n \in \mathbb{N}} [-a_n, a_n]$ of above examples. Consider the following theorem

**Theorem 2.1:**

Let $G \subset R$ be a convex galaxy which is symmetric with respect to zero, then

(i) $G$ is an $\alpha$-galaxy(0) if and only if there exists an internal strictly increasing sequence of strictly positive real numbers $\{a_n\}_{n \in \mathbb{N}}$ with $G = \bigcup_{n \in \mathbb{N}} [-a_n, a_n]$ such that $a_0 = \alpha$ and $\frac{a_{n+1}}{a_n} = c$, for all $n \in \mathbb{N}$, where $c$ is some limited real number such that $c \geq 1$.

(ii) $G$ is a non-linear galaxy if and only if there exists an internal strictly increasing sequence of strictly positive real numbers $\{a_n\}_{n \in \mathbb{N}}$ with $G = \bigcup_{n \in \mathbb{N}} [-a_n, a_n]$ such that $\frac{a_{n+1}}{a_n}$ is unlimited for all $n \in \mathbb{N}$.

**Proof:**

(i) Let $G$ be $\alpha$-galaxy(0), then every sequence $\{\alpha^k\}_{k \in \mathbb{N}}, k > 1$, clearly satisfies the relation. Conversely let $\{a_n\}_{n \in \mathbb{N}}$ be an internal strictly in an increasing sequence of strictly positive real numbers such that $\frac{a_{n+1}}{a_n} = c$, for all $n \in \mathbb{N}$, where $c$ some limited real number such that $c \geq 1$, then it is clear that $\bigcup_{n \in \mathbb{N}} [-a_n, a_n]$ is the $a_0$-galaxy(0).

(ii) Let $G$ be a non-linear galaxy, let $\{k_n\}_{n \in \mathbb{N}}$ be strictly increasing sequence of strictly positive real numbers such that $G = \bigcup_{n \in \mathbb{N}} [-k_n, k_n]$. We define a subsequence $\{a_n\}_{n \in \mathbb{N}}$ of $\{k_n\}_{n \in \mathbb{N}}$ such that $\frac{a_{n+1}}{a_n}$ is unlimited for all $n \in \mathbb{N}$, by the external induction. Put $a_0 = k_0$, suppose that $a_n$ is defined to some $k_n$. Since $G$ is a convex group, which contains $a_n$-galaxy(0). Because, $G$ is non-linear $a_n$-galaxy(0) is strictly contained in $G$. So there is $k$ in $G-a_n$-galaxy(0), by putting $a_{n+1}$ the last $k_n$ such that $k_n \in G-a_n$-galaxy(0). Then $\frac{a_{n+1}}{a_n}$ is unlimited, by the principle extension there exists an internal extension $\{a_n\}_{n \in \mathbb{N}}$ of $\{a_n\}_{n \in \mathbb{N}}$ which may be assume strictly increasing. This sequence has all the required properties.

Conversely let $\{a_n\}_{n \in \mathbb{N}}$ be an internal increasing sequence of strictly positive real numbers such that $\frac{a_{n+1}}{a_n}$ is unlimited for all $n \in \mathbb{N}$.

By putting $\bigcup_{n \in \mathbb{N}} [-a_n, a_n]$, we may prove in the same way as in example (2) that $G$ is not $\alpha$-galaxy(0), if $\alpha \leq k_n$, for some $n \in \mathbb{N}$, then $\frac{k_{n+1}}{\alpha}$ is unlimited.
Hence, $\alpha$-galaxy$(x) \subseteq G$ implies $\alpha$-galaxy$(0) \subseteq G$. 

3. Convex Galaxies and Functions:

We now establish a relation between convex galaxies $G$ and internal strictly functions $C^\infty$ and internal strictly increasing $C^\infty$ functions $f$, such that $f \left( G \right) = G$.

We use the following identity of these functions.

$$f \left( n+1 \right) = \exp \int_n^{n+1} \frac{f'(t)}{f(t)} dt, \quad n \in \mathbb{N} - \{0\}$$

**Theorem 3.1:**

Let $G \subseteq R$ be a convex galaxy which is symmetric with respect to zero. Then,

(i) $G$ is an $\alpha$-galaxy$(0)$ if and only if there exists a real internal strictly $C^\infty$-function $f$, such that $f \left( G \right) = G$, and $f'(t) = c$ for all limited $t \geq 1$, where $c$ is a positive real number.

(ii) $G$ is a non-linear galaxy if and only if there exists a real internal strictly increasing $C^\infty$-function $f$, such that $f \left( G \right) = G$ and $f'(t)$ is positive unlimited, for all appreciable $t \geq 1$.

**Proof:**

(i) The part (i) follows from theorem (2.1) part (i).

(ii) Let $G$ be non-linear and $\{a_n\}_{n \in \mathbb{N}}$ be an internal increasing sequence of strictly positive real numbers such that $G = \bigcup_{n \in \mathbb{N}} [-a_n, a_n]$ and $\frac{a_{n+1}}{a_n}$ is unlimited for all $n \in \mathbb{N}$.

Now, we define the functions $f_n$ on the $[n, n+1]$ as follows:

$$f_n = \begin{cases} a_n \left( \frac{a_{n+1}}{a_n} \right)^{n-1} & \text{if } n \geq 1 \\ c \alpha t & \text{if } n = 0 \end{cases}$$

Then, $f_n$ is internal strictly increasing and $C^\infty$ on $[n, n+1]$, for all $n \in \mathbb{N}$.

Furthermore, $f_n'(t) = \log \left( \frac{a_{n+1}}{a_n} \right)$, so $\frac{f_n'(t)}{f_n(t)}$ is unlimited for all limited $t \geq 1$, while $\bigcup_{n \in \mathbb{N}} f_n$ is continuous, we can obtain a function $f$ which conserves all these properties and is $C^\infty$ on the $[0, \infty)$ we may also expect that the odd function $f \cup g$ where $g : R^- \to R^-$ is defined by $g(t) = -f(-t)$ is $C^\infty$. Then, $f \cup g$ is the required the function.

Conversely let $f$ be a real internal strictly increasing odd $C^\infty$-function such that $f \left( G \right) = G$, and $f'(t)$ is positive unlimited for every limited $t \geq 1$, then we have for every $n \in \mathbb{N}$, $(n \geq 1)$, $f \left( n+1 \right) = \exp \int_n^{n+1} \frac{f'(t)}{f(t)} dt$, So that, $\frac{f \left( n+1 \right)}{f \left( n \right)}$ is unlimited. Because $G = \bigcup_{n \in \mathbb{N}} [-f \left( n+1 \right), f \left( n+1 \right)]$, we deduce that $G$ is not linear galaxy by theorem (2.1) part (ii).
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