Multi-Product Dynamic Pricing in High-Dimensions with Heterogenous Price Sensitivity

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Abstract

We consider the problem of multi-product dynamic pricing in a contextual setting for a seller of differentiated products. In this environment, the customers arrive over time and products are described by high-dimensional feature vectors. Each customer chooses a product according to the widely used Multinomial Logit (MNL) choice model and her utility depends on the product features as well as the prices offered. Our model allows for heterogenous price sensitivities for products. The seller a-priori does not know the parameters of the choice model but can learn them through interactions with the customers. The seller’s goal is to design a pricing policy that maximizes her cumulative revenue. This model is motivated by online marketplaces such as Airbnb platform and online advertising. We measure the performance of a pricing policy in terms of regret, which is the expected revenue loss with respect to a clairvoyant policy that knows the parameters of the choice model in advance and always sets the revenue-maximizing prices.

We propose a pricing policy, named M3P, that achieves a $T$-period regret of $O(\sqrt{\log(T)T})$ under heterogenous price sensitivity for products with features dimension of $d$. We also prove that no policy can achieve worst-case $T$-regret better than $\Omega(\sqrt{T})$.

1 Introduction

Online marketplaces offer very large number of products described by a large number of features. This contextual information creates differentiation among products and also affects the willingness-to-pay of buyers. To provide more context, let us consider the Airbnb platform: the products sold in this market are “stays.” In booking a stay, the user first selects the destination city, dates of visit, type of place (entire place, 1 bedroom, shared room, etc) and hence narrows down her choice to a so-called consideration set. The platform sets the prices for the products in the consideration set. Notably, the products here are highly differentiable. Each product can be described by a high-dimensional feature vector that encodes its properties, such as space, amenities, walking score, house rules, reviews of previous tenants, etc. We study a model where the platform aims to maximize its revenue.

In setting prices, there is a clear tradeoff. A high price may drive the user away (decreases the likelihood of a sale) and hence hurts the revenue. A low price, on the other hand encourages the user to purchase the product; however, it results in a smaller revenue from that sell. Therefore, in order for the seller to maximize its revenue, it must try to learn the purchase behavior of the
users. Using the users’ interactions and purchasing decisions, the seller can learn how users weighs different features in their purchasing decisions.

In this work, we study a setting where the utility from buying a product is a linear function of the product features and its price. Let \( u(\theta_0, x, p) \) be the utility obtained from buying a product with feature vector \( x \), at price \( p \) where the parameter vector \( \theta_0 \) represents the users’ purchase behavior. Namely, \( \theta_0 \) captures the contribution of each feature to the user’s valuations of the products. Similar to \([2, 24, 22]\), we focus on a linear utility model:

\[
  u(\theta_0, x, p) = \langle x, \theta_0 \rangle - \langle x, \gamma_0 \rangle p + z, 
\]

where \( \langle a, b \rangle \) indicates the inner product of two vectors \( a \) and \( b \). The term \( z \), a.k.a. market noise, captures the idiosyncratic change in the valuation of each user, and \( \beta_x = \langle x, \gamma_0 \rangle \) is the price sensitivity parameter. We encode the “no-purchase” option as a new product with zero utility. We emphasize that the parameters of the utility model, \( \theta_0 \) and \( \gamma_0 \), are a priori unknown to the seller.

In our model, given a consideration set, the customer chooses the products that result in the highest utility. We study the widely used Multinomial Logit (MNL) choice model \([27]\) which corresponds to having the noise terms, Eq (1), drawn independently from a standard Gumbel distribution.

We propose a dynamic pricing policy, called M3P, for Multi-Product Pricing Policy in high-dimensional environments. Our policy uses an \( \ell_1 \) regularized maximum likelihood method to estimate the true parameters of the utility model based on previous purchasing behavior of the users.

We measure the performance of a pricing policy in terms of the regret, which is the difference between the expected revenue obtained by the pricing policy and the revenue gained by a clairvoyant policy that has full information of the utility model and always offers the revenue-maximizing price. Our policy achieves a \( T \)-regret of \( O(\sqrt{\log(dT)}T) \), where \( d \) and \( T \) respectively denote the features dimension and the length of time horizon. Furthermore, we also prove that our policy is almost optimal in the sense that no policy can achieve worst-case \( T \)-regret better than \( \Omega(\sqrt{T}) \).

In the next section, we briefly review the related work to ours. We would like to highlight that our work is distinguished from the previous literature in two major aspects: i) Multi-product pricing that should take into account the interaction of different products as changing the price for one product may shift the demand of other products and this makes the pricing problem even more complex. ii) heterogeneity and uncertainty in price sensitivity parameters. We point out that our methods can obtain logarithmic cumulative regret in \( T \) if the price sensitivity parameters \( \gamma_0 \) in Eq (1) were a-priory known, cf., \([22]\).

**Related Work**

There is a vast literature on dynamic pricing as one of the central problems in revenue management. We refer the reader to \([12, 4]\) for extensive surveys on this area. A popular theme in this area is dynamic pricing with learning where there is uncertainty about the demand function, but information about it can be obtained via interaction with customers. A line of work \([3, 16, 20, 8, 17, 10]\) took Bayesian approach. Another related line of work assumes parametric models for the demand function with a small number of parameters, and proposes policies to learn these parameters using statistical procedures such as maximum likelihood \([6, 7, 14, 13, 9]\) or least square estimation \([6, 18, 23]\).
Recently, there has been an interest in dynamic pricing in contextual setting. The work \cite{1,11,24,22,5} consider single-product setting where the seller receives a single product at each step to sell (corresponding to $N = 1$ in our setting) and assume equal price sensitivities $\beta = 1$ for all products. In \cite{1}, the authors consider a noiseless valuation model with strategic buyer and propose a policy with $T$-period regret of order $O(T^{2/3})$. This setting has been extended to include market noise and also a market of strategic buyers who are utility maximizers \cite{19}. In \cite{11}, authors propose a pricing policy based on binary search in high-dimension with adversarial features that achieves regret $O(d^2 \log(T/d))$. The work \cite{22} studies the dynamic pricing in high-dimensional contextual setting with sparsity structure and propose a policy with regret $O(s_0 \log(d) \log(T))$ but in a single-product scenario. The problem has been also studied under time-varying coefficient valuation models \cite{21} to address the time-varying purchase behavior of customers and the perishability of sales data. Very recently, \cite{25} studied high-dimensional multi-product pricing, with a low-dimensional linear model for the aggregate demand. In this model, the demand vector for all the products at each step is observed, while in our work the seller only sees the product index that is chosen from the buyer’s consideration set at each step. Similarly, \cite{26} studies a model where the seller can observe the aggregate demand and proposes a myopic policy based on least-square estimations that obtains a logarithmic regret.

2 Model

We consider a firm which sells a set of products to customers that arrive over time. The products are differentiated and each is described by a wide range of features.

At each step $t$, the customer selects a consideration set $C_t$ of size at most $N$ from the available products. This is the set the customer will actively consider in her purchase decision. The seller sets the price for each of the products in this set, after which the customer may choose (at most) one of the products in $C_t$. If he chooses a product, a sale occurs and the seller collects a revenue in the amount of the posted price; otherwise, no sale occurs and seller does not get any revenue.

Each product $i$ is represented by an observable vector of features $x_i \in \mathbb{R}^d$. Products offered at different round can be highly differentiated (their features vary) but we assume that the feature vectors are sampled independently from a fixed, but unknown, distribution $\mathcal{D}$.

We assume that $\|x_i\|_\infty \leq 1$ for all $x_i$ in the support of $\mathcal{D}$, and $\|\theta_0\|_1 + \|\gamma_0\|_1 \leq W$, for an arbitrarily large but fixed constant $W$. Throughout the paper, we use $\|\cdot\|$ to indicate the $\ell_2$ norm.

If an item $i$ (at period $t$) is priced at $p_{it}$, then the customer obtains utility $u_{it}$ from buying it, where\footnote{In general the offered price not only depends on the feature vectors $x_i$ but also the period $t$, as the estimate of the model parameters may vary across time $t$. We make this explicit in the notation $p_{it}$ by considering both $i$ and $t$ in the subscript.}

$$u_{it} = \langle x_i, \theta_0 \rangle - \langle x_i, \gamma_0 \rangle p_{it} + z_{it}.$$ 

Here, $\theta_0, \gamma_0 \in \mathbb{R}^d$ are the parameters of the demand curve and are unknown a priori to the seller. The term $\langle x_i, \theta_0 \rangle$ is the product-based utility, and the $z_{it}$ component represents market shocks and are modeled as zero mean random variables drawn independently and identically from a standard Gumbel distribution. This noise distribution gives us the well-known multinomial logit (MNL) choice model that has been widely used in academic literature and practice \cite{27,15}. Under the
MNL model, the probability of choosing an item $i$ from set $C_t$ is given by

$$q_{it} \equiv P(i_t = i|C_t) = \frac{\exp(u^0_{it})}{1 + \sum_{\ell \in C_t} \exp(u^0_{\ell t})}, \text{ for } i \in C_t,$$

(2)

where $u^0_{it} = \langle x_i, \theta_0 \rangle - \langle x_i, \gamma_0 \rangle p_{it}$, for $i \in C_t$.

We refer to the term $\beta_i = \langle x_i, \gamma_0 \rangle$ in the utility model as the price sensitivity of product $i$. Note that our model allows for heterogeneous price sensitivities. We also encode the no-purchase option by item $\emptyset$, with market utility $z_{0t}$, drawn from zero mean Gumbel distribution. The random utility $z_{0t}$ can be interpreted as the utility obtained from choosing an option outside the offered ones. This is equivalent to $u^0_{0t} = 0$. Having the utility model established as above, at all steps the user chooses the item with maximum utility from her consideration set; in case of equal utilities, we break the tie randomly.

To summarize, our setting is as follows. At each period $t$:

1. The user narrows down her options by forming a consideration set $C_t$ of size at most $N$.
2. For each product $i \in C_t$, the seller offers a price $p_{it}$.
3. The user chooses item $i_t \in C_t \cup \{\emptyset\}$ where $i_t = \arg\max_{i \in C_t \cup \{\emptyset\}} u_{it}$.
4. The seller observes what product is chosen from the consideration set and uses this information to set the future prices.

We make the following assumption that ensures positivity of the products price sensitivity parameters.

**Assumption 2.1.** We have $\min_{x \in \mathcal{D}} \langle x, \gamma_0 \rangle \geq L_0 > 0$, for some constant $L_0$.

Before proceeding with the policy description, we will discuss the benchmark policy which is used in defining the notion of regret and measuring the performance of pricing policies.

## 3 Benchmark policy

The seller’s goal is to minimize her regret, which is defined as the expected revenue loss against a clairvoyant policy that knows the utility model parameters $\theta_0, \gamma_0$ in advance and always offers the revenue-maximizing prices. We next characterize the benchmark policy. Let $p^*_t = (p^*_{it})_{i \in C_t}$, $\beta_t = (\beta_i)_{i \in C_t}$, where $\beta_i = \langle x_i, \gamma_0 \rangle$ and $X_t \in \mathbb{R}^{||C_t|| \times d}$ be the feature matrix, which is obtained by stacking $x_i$, $i \in C_t$ as its rows (Recall that $|C_t| \leq N$). The proposition below gives an implicit formula to write the vector of optimal prices $p^*_t$ as a function $p^*_t = g(\beta_t, X_t \theta_0)$. We refer to $g$ as the pricing function.

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2Equivalently, the seller can determine all the prices in advance and reveal them after the user determines the consideration set. We note that the consideration set of the user does not depend on the prices, but the choice she makes from the consideration set depends on the prices. In addition, recall that all the users share the same $\theta_0$ and $\gamma_0$ and the choice of consideration set does not reveal information about these parameters.
Proposition 3.1. The benchmark policy that knows the utility model parameters $\theta_0, \gamma_0$, sets the optimal prices as follows: For product $i \in C_t$, the optimal price is given by $p^*_it = \frac{1}{\langle x_i, \gamma_0 \rangle} + B^0_t$, where $B^0_t$ is the unique value of $B$ satisfying the following equation:

$$B = \sum_{t \in C_t} \frac{1}{\langle x_t, \gamma_0 \rangle} e^{-(1+(x_t, \gamma_0)B)} e^{(x_t, \theta_0)}.$$  

(3)

Proof of Proposition 3.1 is similar to that in [28, Theorem 3.1] and is deferred to Appendix A.1.

We can now formally define the notion of regret. Let $\pi$ be a pricing policy that sets the vector of prices $p^*_t = (p^*_it)_{i \in C_t}$ at time $t$ for the products in the consideration set $C_t$. Then, the seller’s expected revenue at period $t$, under such policy will be

$$\text{rev}^\pi_t = \sum_{i \in C_t} q_{it} p^*_it,$$

(4)

with $q_{it}$ being the probability of buying product $i$ from the set $C_t$ as given by Eq (2).\footnote{More precisely, $\text{rev}^\pi_t$ is the expected revenue conditional on filtration $\mathcal{F}_{t-1}$, where $\mathcal{F}_t$ is the sigma algebra generated by feature matrices $X_1, \ldots, X_{t+1}$ and market shocks $z_1, \ldots, z_t$.} Similarly, we let $\text{rev}^\pi_t$ be the seller’s expected revenue under the benchmark policy that sets price vectors $p^*_t$, at period $t$. The worst-case cumulative regret of policy $\pi$ is defined as

$$\text{Regret}^\pi(T) \equiv \max \left\{ \sum_{t=1}^T (\text{rev}^*_t - \text{rev}^\pi_t) : \theta_0 \in \Omega, \supp(D) \subseteq [-1, 1]^d \right\}.$$  

(5)

4 Multi-Product Pricing Policy (M3P)

In this section, we provide a formal description of our multi-product dynamic pricing policy (M3P). The policy sees the time horizon in an episodic structure, where the length of episodes grow geometrically (episode $k$ is of length $\ell_k = 2^{k-1}$). Throughout, we use notation $E_k$ to refer to periods in episode $k$, i.e., $E_k = \{\ell_k, \ldots, \ell_{k+1} - 1\}$. The policy updates its estimate of the model parameters $\psi_0 = (\theta^T_0, \gamma^T_0)^T$ at the beginning of each episode and adhere to that estimate throughout the episode when setting the prices. At each period during one episode, our policy sets the price vectors as $p_t = g(X_t \hat{\gamma}_t, X_t \hat{\theta}^k)$, where $\hat{\theta}^k, \hat{\gamma}^k$ are respectively the estimates of $\theta_0$ and $\gamma_0$, which are obtained by solving a regularized maximum-likelihood minimization problem using solely the observations (the products sold) in the previous episode. Note that the seller can only observe which products were sold in the previous episode. Formally, the estimate $\hat{\psi}^k$ is obtained by minimizing the negative log-likelihood function given by

$$\mathcal{L}_k(\psi) = -\frac{1}{\ell_{k-1}} \sum_{t \in E_{k-1}} \log \exp(u^0_{it}(\psi)) \sum_{t \in C_t \cup \{\emptyset\}} \exp(u^0_{it}(\psi)),$$

where $i_t$ denotes the product purchased at time $t$, and

$$u^0_{it}(\psi) = \langle x_i, \theta \rangle - \langle x_i, \gamma \rangle p_{it},$$

(9)

with $\psi = (\theta^T, \gamma^T)^T$. We adopt the convention that $i_t = \emptyset$ for the “no-purchase” case with $u^0_{it}(\cdot) = 0$. 

Input: \( g \), regularizations \( \lambda_k \), \( W \) (bound on \( \|\psi_0\|_1 \))

Input: (arrives over time) covariate matrices \( \{ X_t \}_{t \in [T]} \)

Output: \( \{ p_t \}_{t \in [T]} \)

1: \( \tau_1 \leftarrow 1, \ p_1 \leftarrow 0, \ \theta^1 \leftarrow 0 \)

2: for each episode \( k = 2, 3, \ldots \) do

3: Set the length of \( k \)-th episode: \( \ell_k \leftarrow 2^{k-1} \)

4: Update the model parameter estimate \( \hat{\psi}^k \) using the regularized ML estimator obtained from observation during the previous exploration periods:

\[
\hat{\psi}^k = \arg \min_{\|\psi\|_1 \leq W} \left\{ L_k(\psi) + \lambda_k \|\psi\|_1 \right\}, \tag{6}
\]

with

\[
L_k(\psi) = -\frac{1}{\ell_{k-1}} \sum_{t \in E_{k-1}} \log \frac{\exp(u^0_{it}(\psi))}{\sum_{\ell \in C_t \cup \{\emptyset\}} \exp(u^0_{lt}(\psi))},
\]

where \( \emptyset \) corresponds to “no-purchase” with \( u^0_{it}(\cdot) = 0 \).

5: Offer prices based on the current estimate \( \hat{\psi}^k = \begin{bmatrix} \hat{\theta}^k \\ \hat{\gamma}^k \end{bmatrix} \) as

\[
p_{it} \leftarrow \frac{1}{\langle x_{it}, \hat{\gamma}^k \rangle} + B_t, \tag{7}
\]

where \( B_t \) is the unique value of \( B \) satisfying the following equation:

\[
B = \sum_{\ell \in C_t} \frac{1}{\langle x_{\ell}, \hat{\gamma}^k \rangle} e^{-(1 + \langle x_{\ell}, \hat{\gamma}^k \rangle) B} e^{\langle x_{\ell}, \hat{\theta}^k \rangle}. \tag{8}
\]

Algorithm 1: M3P policy for multi-product dynamic pricing
The log-likelihood loss can be written in a more compact form. We let \( y_t = (y_{it})_{i \in C_t} \) be the response vector that indicates which product is purchased at time \( t \):

\[
y_{it} = \begin{cases} 
1 & \text{product } i \text{ is chosen}, \\
0 & \text{otherwise}.
\end{cases}
\]

We also let \( u_t^0 = (u_{it}^0)_{i \in C_t \cup \{0\}} \). Then, the log-likelihood loss can be written as

\[
L_k(\mu) = -\frac{1}{\ell_k} \sum_{t \in E_k - 1} \sum_{i \in C_t} y_{it} \cdot \exp(u_{it}^0) \frac{1}{1 + \sum_{\ell \in C_t} \exp(u_{\ell t}^0)}.
\]

(10)

We also add the \( \ell_1 \) regularization \( \lambda_k \| \mu \|_1 \) in the cost function to promote sparsity structure in the estimator

\[
\hat{\psi}_k = \arg \min_{\| \mu \|_1 \leq W} \left( L(\mu) + \lambda_k \| \mu \|_1 \right),
\]

(11)

with \( \lambda_k = M \log(\ell_{k-1} d)/\ell_{k-1} \), for a constant \( M = M(W, L_0, N) \).

The policy terminates at time \( T \) but note that the policy does not need to know \( T \) in advance. Further, in our policy exploration and exploitation are mixed. In the beginning of each episode, the policy exploits the observations in the previous episode to update its estimates of the model parameters. Meanwhile, the market shocks in the utilities gives us sufficient amount of exploration and hence we do not need to actively randomize prices to learn the parameters. Also, by the design when the policy does not have much information about the model parameters it updates its estimates frequently (since the length of episodes are small) but as time proceeds the policy gathers more information about the parameters and updates its estimates less frequently, and use them over longer episodes.

5 Regret Analysis for M3P

We next state our result on the regret of M3P policy.

**Theorem 5.1. (Regret upper bound)** Consider the choice model (2). Then, the \( T \)-period regret of the M3P is of \( O(\sqrt{\log(dT)T}) \), with \( d \) and \( T \) being the feature dimension and the length of time horizon. Further, regret of any pricing policy in this case is \( \Omega(\sqrt{T}) \).

Note that as stated by the theorem, the regret of M3P scales logarithmically in \( d \), making the algorithm applicable for high dimensional setting. Below, we state the key lemmas in the proof of Theorem 5.1 and refer to Appendices for the proof of technical steps.

Let \( p_t = (p_{it})_{i \in C_t} \) be the vector of prices posted at time \( t \) for products in the consideration set \( C_t \). Recall that M3P sets the prices as \( p_t = g(X_t \hat{\gamma}_k, X_t \hat{\theta}_k) \), where \( g(\cdot, \cdot) \) is the pricing function whose implicit characterization is given by Proposition 3.1.

Our next lemma shows that the pricing function \( g(\cdot, \cdot) \) is Lipschitz.

**Lemma 5.2.** Suppose that \( p_1 = g(X_t \gamma_1, X_t \theta_1) \) and \( p_2 = g(X_t \gamma_2, X_t \theta_2) \). Then, there exists a constant \( C = C(W, L_0) > 0 \) such that the following holds

\[
\| p_1 - p_2 \| \leq CN^{3/2} \left( \| X_t(\gamma_1 - \gamma_2) \|^2 + \| X_t(\theta_1 - \theta_2) \|^2 \right)^{1/2}.
\]

(12)
We next upper bound the right-hand side of Eq (12) by bounding the estimation error of the proposed regularized estimator. Denote by $X^{(k)}$ the matrix obtained by putting all the feature matrices $X_t$ corresponding to $t$ belonging to episode $k$.

**Proposition 5.3.** Let $\hat{\psi}^k$ be the solution of optimization problem (6), with $\lambda_k \geq M \sqrt{\log(d\ell_{k-1})/\ell_{k-1}}$ for a constant $M = M(W, L_0, N) > 0$. Then, with probability at least $1 - 1/(d\ell_{k-1})$, we have

$$ \frac{1}{N\ell_{k-1}} \|X^{(k-1)}(\hat{\theta}^k - \theta_0)\|^2 \leq \frac{4W}{c_0} \lambda_k, \quad \frac{1}{N\ell_{k-1}} \|X^{(k-1)}(\hat{\gamma}^k - \gamma_0)\|^2 \leq \frac{4W}{c_0} \lambda_k, $$

where $c_0 = c_0(W, L_0, N)$ is a constant depending on $W$ and $L_0$.

The last part of the proof is to relate the regret of the policy at each period $t$ to the distance between the posted price vector $p_t$ and the price vector $p_t^*$ posted by the benchmark. Recall the definition of revenue $\text{rev}^*_t$ from (4) and define the regret as $\text{reg}_t \equiv \text{rev}^*_t - \text{rev}_t$.

**Lemma 5.4.** Let $p_t^* = g(X_t \gamma_0, X_t \theta_0)$ be the optimal price vector posted by the benchmark policy that knows the model parameters $\theta_0$ and $\gamma_0$ in advance. There exists a constant $C > 0$ (depending on $W$) such that the following holds,

$$ \text{reg}_t \leq c_1 N \|p_t^* - p_t\|^2, $$

for some constant $c_1 = c_1(W, L_0)$.

The reason that in Lemma 5.4, the revenue gap depends on the squared of the difference of the price vectors is that $p_t^* = \arg \min \text{rev}_t(p)$ is the optimal price and hence $\nabla \text{rev}_t(p) = 0$. Therefore, by Taylor expansion of function $\text{rev}_t(p)$ around $p_t^*$, we see that the first order term vanishes and the second order term $O(\|p_t - p_t^*\|^2)$ matters.

The proof of Theorem 5.1 follows by combining Lemma 5.2, Proposition 5.3 and Lemma 5.4. We refer to Appendix B.1 for its proof.

Our next theorem provides a lower bound on the $T$-regret of any pricing policy. The proof of Theorem 5.5 is given in Appendix B.2 and employs the notion of ‘uninformative prices’, introduced by [7].

**Theorem 5.5.** (Regret lower bound) Consider the choice model (2). Then, the $T$-period regret of any pricing policy in this case is $\Omega(\sqrt{T})$.

Theorem 5.5 implies that M3P has optimal cumulative regret up to logarithmic factor.

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A Proof of Technical Lemmas

A.1 Proof of Proposition 3.1

In the benchmark policy, the seller knows the model parameters $\theta_0, \gamma_0$. For simplicity, we use the shorthands $\beta_i = \langle x_i, \gamma_0 \rangle$, $e_{it} = \exp(\langle x_i, \theta_0 \rangle - \beta_i p_{it})$, and the sum as $G(e_t) = \sum_{\ell \in C_t} e_{\ell t}$. The revenue function can be written in terms of $e_{it}$ as

$$\text{rev}_t(p_t) = \sum_{i \in C_t} p_{it} \mathbb{P}(i_t = i | C_t) = \sum_{i \in C_t} p_{it} \frac{e_{it}}{1 + G(e_t)},$$

where we used (2). Writing the stationarity condition for the optimal price vector $p^*_t$, we get that for each $i \in C_t$:

$$\frac{\partial \text{rev}_t(p^*_t)}{\partial p_{it}} = \frac{e_{it} - e_{it} \beta_i p^*_t}{1 + G(e_t)} + \frac{\sum_{\ell \in C_t} p^*_\ell e_{\ell t} e_{it} \beta_i}{(1 + G(e_t))^2} = 0,$$

which is equivalent to

$$\beta_i \frac{e_{it}}{1 + G(e_t)} \left\{ \frac{1}{\beta_i} - p^*_t + \sum_{\ell \in C_t} p^*_\ell e_{\ell t} \frac{1}{1 + G(e_t)} \right\} = 0.$$

Since $e_{it} > 0$, the above equation implies that

$$p^*_t = \frac{1}{\beta_i} + \text{rev}_t(p^*_t). \quad (14)$$

Define $B_0^t \equiv \text{rev}_t(p^*_t)$. We next show that $B_0^t$ is the solution to Equation (3). By multiplying both sides of (14) by $e_{it}$ and summing over $\ell \in C_t$, we have

$$\sum_{\ell \in C_t} e_{\ell t} p^*_\ell = \sum_{\ell \in C_t} \frac{e_{\ell t}}{\beta_\ell} + B_0^t \left( \sum_{\ell \in C_t} e_{\ell t} \right) = \sum_{\ell \in C_t} \frac{e_{\ell t}}{\beta_\ell} + B_0^t G(e_t).$$

By definition of $B_0^t$, the left-hand side of the above equation is equal to $B_0^t(1 + G(e_t))$. By rearranging the terms we obtain

$$B_0^t = \sum_{\ell \in C_t} \frac{e_{\ell t}}{\beta_\ell} = \sum_{\ell \in C_t} \frac{1}{\beta_\ell} e^{\langle x_\ell, \theta_0 \rangle - \beta_\ell p_{\ell t}} = \sum_{\ell \in C_t} \frac{1}{\beta_\ell} \exp \left\{ \langle x_\ell, \theta_0 \rangle - \beta_\ell \left( \frac{1}{\beta_\ell} + B_0^t \right) \right\} = \sum_{\ell \in C_t} \frac{1}{\beta_\ell} e^{\langle x_\ell, \theta_0 \rangle - (1 + \beta_\ell B_0^t)} ,$$

where the second line follows from Equation (14).

Regarding the uniqueness of the solution of (3), note that the left-hand side of (3) is strictly increasing in $B$ and is zero at $B = 0$, while the right hand side is strictly decreasing in $B$ and is positive at $B = 0$. Therefore, Equation (3) has a unique solution.
A.2 Proof of Lemma 5.2

Define function \( f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R} \) as

\[
f(B, \delta, \beta) \equiv B - \sum_{\ell \in \mathcal{C}_t} \frac{1}{\beta^\ell} e^{\delta^\ell} e^{-(1+\beta^\ell)B}.
\]

(15)

By characterization of the pricing function \( g \), given in Proposition 3.1, we have \( p^{(1)}_{it} = \frac{1}{\beta^i} + B(t)^{(1)} \) and \( p^{(2)}_{it} = \frac{1}{\beta^i} + B(t)^{(2)} \), where \( B(t)^{(1)} \) and \( B(t)^{(2)} \) are the solution of \( f(B, X_t\theta_1, X_t\gamma_1) = 0 \) and \( f(B, X_t\theta_2, X_t\gamma_2) = 0 \).

By implicit function theorem for a point \((B, \delta, \beta)\) that satisfies \( f(B, \delta, \beta) = 0 \), there exists an open set around \((\delta, \beta)\), and a unique differentiable function \( h : U \mapsto \mathbb{R} \) such that \( g(\delta, \beta) = B \) and \( f(g(z_1, z_2), z_1, z_2) = 0 \) for all \((z_1, z_2) \in U \). Furthermore, the partial derivative of \( g \) can be computed as

\[
\frac{\partial g}{\partial \delta}(\delta, \beta) = -\left[\frac{\partial f}{\partial B}(\delta, \beta)\right]^{-1} \frac{\partial f}{\partial \delta}(g(\delta, \beta), \delta, \beta)
\]

\[
= -\left(1 + \sum_{\ell \in \mathcal{C}_t} e^{\delta^\ell} e^{-(1+\beta^\ell)B}\right)^{-1} \left( -\frac{1}{\beta^i} e^{-(1+\beta^i)B} e^{\delta^i} \right)
\]

\[
< \frac{e^{\delta^i}}{\beta^i} < \frac{e^W}{L_0},
\]

where in the last step we use the normalization \(|\delta^i| = |\langle x_i, \theta_1 \rangle| \leq W \), and \(0 < L_0 < \min \beta^i\) is the lower bound on the price sensitivities. Likewise, we have

\[
\frac{\partial g}{\partial \beta}(\delta, \beta) = -\left[\frac{\partial f}{\partial B}(\delta, \beta)\right]^{-1} \frac{\partial f}{\partial \beta}(g(\delta, \beta), \delta, \beta)
\]

\[
= -\left(1 + \sum_{\ell \in \mathcal{C}_t} e^{\delta^\ell} e^{-(1+\beta^\ell)B}\right)^{-1} \left( \left(\frac{B}{\beta^i} + \frac{1}{\beta^i} \right) e^{-(1+\beta^i)B} e^{\delta^i} \right)
\]

\[
< \left(\frac{B}{L_0} + \frac{1}{L_0^2}\right) e^{\delta^i} < \left( N e^{W-1} + 1 \right) \frac{e^W}{L_0^2},
\]

where we used the fact that the solution \( B \) of \( f(B, \delta, \beta) \) satisfies \( B \leq N e^{W-1}/L_0 \). (This follows readily by noting that the right-hand side of (15) is non-increasing in \( B \).) This shows that \( g(\delta, \beta) \) is a Lipschitz function of \( \delta \), with Lipschitz constant \( C N \), where \( C \equiv N e^{2W} / \min(L_0^2, 1) \). Therefore,

\[
|p_{it}^{(1)} - p_{it}^{(2)}| = |B(t)^{(1)} - B(t)^{(2)}| + \left| \frac{1}{\beta^i(t)^{(1)}} - \frac{1}{\beta^i(t)^{(2)}} \right|
\]

\[
= |g(X_t\theta_1, X_t\gamma_1) - g(X_t\theta_2, X_t\gamma_2)| + \frac{1}{\beta^i(t)^{(1)}} \beta^i(t)^{(2)} |\langle x_i, \gamma_1 - \gamma_2 \rangle|
\]

\[
\leq C N (\|X_t(\theta_1 - \theta_2)\|_1 + \|X_t(\gamma_1 - \gamma_2)\|_1) + \frac{1}{L_0} |\langle x_i, \gamma_1 - \gamma_2 \rangle|. \quad (16)
\]

Hence, \( \|p_{it}^{(1)} - p_{it}^{(2)}\| \leq \tilde{C} \sqrt{N^3} (\|X_t(\theta_1 - \theta_2)\|_1 + \|X_t(\gamma_1 - \gamma_2)\|_1) \), for some constant \( \tilde{C} \). This completes the proof.
A.3 Proof of Proposition 5.3

We start by recalling the notation \( \psi_0 = (\theta_0^T, \gamma_0^T)^T \) and define \( \tilde{X}_t = [X_t, -\text{diag}(p_t)X_t] \). To prove Proposition 5.3, we first rewrite the loss function in terms of the augmented parameter vector \( \psi \). (Recall our convention that \( \emptyset \) corresponds to “no-purchase” with \( u^{0}_0(\cdot) = 0 \).

\[
\mathcal{L}_k(\psi) = -\frac{1}{\ell_{k-1}} \sum_{t \in E_{k-1}} \log \frac{\exp(u^{0}_0)}{\sum_{\ell \in \mathcal{C}_t \cup \{\emptyset\}} \exp(u^{0}_\ell)}
\]

\[
= \frac{1}{\ell_{k-1}} \sum_{t \in E_{k-1}} \left( \log(1 + \sum_{\ell \in \mathcal{C}_t} e^{\langle x_t, \theta \rangle - p_{lt}(x_t, \gamma)} ) - \begin{cases} 0 & i_t = \emptyset \\ \langle x_{it}, \theta \rangle - p_{it}(x_{it}, \gamma) & \text{otherwise} \end{cases} \right) 
\]

\[
= \frac{1}{\ell_{k-1}} \sum_{t=1}^{\ell_{k-1}} \left( \log(1 + \sum_{\ell \in \mathcal{C}_t} e^{\langle \tilde{x}_t, \psi \rangle}) - \begin{cases} 0 & i_t = \emptyset \\ \langle \tilde{x}_{it}, \psi \rangle & \text{otherwise} \end{cases} \right),
\]

where \( \tilde{x}_t = [x_t^T, -p_{lt}x_t^T]^T \). The gradient and the hessian of \( \mathcal{L}_k \) are given by

\[
\nabla \mathcal{L}_k(\psi) = \frac{1}{\ell_{k-1}} \sum_{t \in E_{k-1}} \left( \frac{\sum_{\ell \in \mathcal{C}_t} \exp(u^{0}_\ell)u^{0}_\ell \tilde{x}_t}{1 + \sum_{\ell \in \mathcal{C}_t} \exp(u^{0}_\ell)} - \tilde{x}_{it} \right),
\]

\[
\nabla^2 \mathcal{L}_k(\psi) = \frac{1}{\ell_{k-1}} \sum_{t \in E_{k-1}} \frac{(1 + \sum_{\ell \in \mathcal{C}_t} \exp(u^{0}_\ell))((\sum_{\ell \in \mathcal{C}_t} \exp(u^{0}_\ell))(u^{0}_\ell)^2 + 1)\tilde{x}^{\otimes 2}_t - (\sum_{\ell \in \mathcal{C}_t} u^{0}_\ell \tilde{x}_t)\otimes 2}{(1 + \sum_{\ell \in \mathcal{C}_t} \exp(u^{0}_\ell))^2}.
\]

We proceed by bounding the gradient and the hessian of the loss function. Before that, we establish an upper bound on the prices that are set by the pricing function \( g \).

**Lemma A.1.** Suppose that \( \|x_t\|_\infty \leq 1 \) and \( \|\psi_0\|_1 \leq W \). Let \( B^u = B^u(W, L_0, N) \) be the solution to the following equation:

\[
B = N \frac{1}{L_0} e^{-(1+L_0 B)} e^W.
\]

Then, the prices set by the pricing function \( p_t = g(X_t, X\theta) \), where \( \psi_0 = (\theta_0^T, \gamma_0^T)^T \), are bounded by \( P = 1/L_0 + B^u \).

The proof of above Lemma follows readily by noting that the right-hand side of (20) is an upper bound for the right hand side of (3) and therefore \( B^u \leq B^u \). The results then follows by recalling that the pricing function sets prices as \( p_{lt} = 1/\beta_i + B^0_l \).

To bound the gradient of the loss function at the true model parameters, note that

\[
\|\nabla \mathcal{L}_k(\psi_0)\|_\infty = \left\| \frac{1}{\ell_{k-1}} \sum_{t \in E_{k-1}} \left( \frac{\sum_{\ell \in \mathcal{C}_t} \exp(u^{0}_\ell)u^{0}_\ell \tilde{x}_t}{1 + \sum_{\ell \in \mathcal{C}_t} \exp(u^{0}_\ell)} - \tilde{x}_{it} \right) \right\|_\infty \leq \left\| \frac{1}{\ell_{k-1}} \sum_{t \in E_{k-1}} \sum_{\ell \in \mathcal{C}_t} u^{0}_\ell \tilde{x}_t \right\|_\infty.
\]

(21)
We also have
\[ |u_{t,t}^0| \leq |\langle x_t, \theta_0 \rangle| + |\langle x_t, \gamma_0 \rangle| p_{t,t} \leq W(1 + P) \equiv M, \] (22)
for a constant \( M = M(W, L_0, N) > 0 \). We also note that by (18), \( \nabla \mathcal{L}_k(\psi_0) \) is written as some of \( \ell_{k-1} \) terms. In each term, the index \( i_t \) has randomness coming from the market noise distribution. By a straightforward calculation, one can verify that each of these terms has zero expectation. Using (22) and by applying Azuma-Hoeffding inequality to the right-hand side of (A.3), followed by union bounding over \( d \) coordinates of feature vectors, we obtain
\[ \| \nabla \mathcal{L}_k(\psi_0) \|_{\infty} \leq 2MN \sqrt{\frac{\log(d\ell_{k-1})}{\ell_{k-1}}} \leq \frac{\lambda_k}{2}, \] (23)
with probability at least \( 1 - 1/(d\ell_{k-1}) \). (Note that we can absorb \( N \) in constant \( M \) since it already depends on \( N \)).

We next pass to lower bounding the hessian of the loss. For \( \| \psi_0 \|_1 \), we write
\[
\langle \psi_0 - \hat{\psi}_k, \nabla^2 \mathcal{L}_k(\hat{\psi})(\psi_0 - \hat{\psi}_k) \rangle
= \frac{1}{\ell_{k-1}} \sum_{t \in E_{k-1}} \left( 1 + \sum_{e \in \mathcal{E}_t} \exp(\tilde{u}_{e,t}^0) \right) \frac{(\sum_{e \in \mathcal{E}_t} \exp(\tilde{u}_{e,t}^0)((\tilde{u}_{e,t}^0)^2 + 1)(\tilde{x}_t(\psi_0 - \hat{\psi}_k))^2}{(1 + \sum_{e \in \mathcal{E}_t} \exp(\tilde{u}_{e,t}^0))^2}
\]
\[
\geq \frac{1}{\ell_{k-1}} \sum_{t \in E_{k-1}} \frac{(\sum_{e \in \mathcal{E}_t} \exp(\tilde{u}_{e,t}^0)(\tilde{x}_t(\psi_0 - \hat{\psi}_k))^2)}{1 + \sum_{e \in \mathcal{E}_t} \exp(\tilde{u}_{e,t}^0)}
\]
\[
\geq \frac{e^{-M}}{\ell_{k-1}} \left( \frac{1}{1 + N e^M} \right) \left( \left\| X^{(k-1)}(\gamma_0 - \hat{\gamma}_k) \right\|_2^2 + \left\| \text{diag}(p^{(k-1)}) X^{(k-1)}(\theta_0 - \hat{\theta}_k) \right\|_2^2 \right)
\]
\[
= c_0(W, L_0, N) \left( \left\| X^{(k-1)}(\gamma_0 - \hat{\gamma}_k) \right\|_2^2 + \left\| X^{(k-1)}(\theta_0 - \hat{\theta}_k) \right\|_2^2 \right),
\] (24)
where (a) and (b) follow from Jensen’s Inequality. (c) is because \( |u_{e,t}^0| \leq M \) as per (22); in (d), we used the notation \( X^{(k-1)} = (X_{1,t}^T, X_{2,t}^T, \ldots, X_{\ell_{k-1},t}^T)^T \) with \( X_t \in \mathbb{R}^{N \times d} \) being the covariate matrix at period \( t \). Likewise, \( p^{(k-1)} \) is defined by stacking all the prices posted in episode \( k \).

Define parameter vectors \( \psi_{0,\land} = (\theta_0; \hat{\gamma}_k), \psi_{\land,0} = (\hat{\theta}_k; \gamma_0) \), and \( \hat{\psi}_k = (\hat{\theta}_k; \hat{\gamma}_k) \) all in \( \mathbb{R}^{d+1} \). By optimality of \( \hat{\psi}_k \), we have
\[ \mathcal{L}(\hat{\psi}_k) + \lambda_k \| \hat{\psi}_k \|_1 \leq \mathcal{L}(\psi_{0,\land}) + \lambda_k \| \psi_{0,\land} \|_1. \] (25)
By the second-order Taylor’s Theorem, expanding around \( \psi_{0,\land} \) we have
\[ \mathcal{L}_k(\psi_{0,\land}) - \mathcal{L}_k(\hat{\psi}_k) = -\langle \nabla \mathcal{L}_k(\gamma_0), (\gamma_0 - \hat{\gamma}_k) \rangle - \frac{1}{2} (\gamma_0 - \hat{\gamma}_k, \nabla^2 \mathcal{L}_k(\hat{\gamma}_0)(\gamma_0 - \hat{\gamma}_k) \rangle \] (26)
for some \( \hat{\gamma} \) on the line segment between \( \gamma_0 \) and \( \hat{\gamma}^k \).

Combining (25) with (26), and applying the bounds on the gradient and the hessian of the loss function given by (23) and (24), we arrive at

\[
\frac{2c_0(W, L_0, N)}{N^{t_{k-1}}} \|X^{(k-1)}(\gamma_0 - \hat{\gamma}^k)\|^2 + 2\lambda_k \|\hat{\psi}^k\|_1 \leq 2\lambda_k \|\gamma_0 - \hat{\gamma}^k\|_1 + 2\lambda_k \|\psi_0\|_1.
\]

Applying triangle inequality and rearranging the terms, this yields to the following inequality:

\[
\frac{2c_0(W, L_0, N)}{N^{t_{k-1}}} \|X^{(k-1)}(\gamma_0 - \hat{\gamma}^k)\|^2 \leq 4\lambda_k \|\gamma_0 - \hat{\gamma}^k\|_1 \leq 8W\lambda_k.
\]  

(27)

This completes the proof of the second part in Claim (13). The first part of the claim also follows by a similar argument.

**A.4 Proof of Lemma 5.4**

In this lemma, we aim at bounding the revenue loss (against the clairvoyant policy) in terms of the distance between the posted price vector and the optimal one posted by the clairvoyant policy. By Taylor expansion,

\[
\text{rev}_t(p_t) = \text{rev}_t(p^*_t) + \nabla \text{rev}_t(p^*_t)(p_t - p^*_t) + \frac{1}{2}(p_t - p^*_t)^T \nabla^2 \text{rev}_t(\tilde{p})(p_t - p^*_t),
\]  

(28)

for some \( \tilde{p} \) between \( p_t \) and \( p^*_t \). Note that \( p_t^* = \arg \max \text{rev}_t(p) \), thus \( \nabla \text{rev}_t(p_t^*) = 0 \) and the first term in the Taylor expansion vanishes.

In order to prove the result, it suffices to show that the operator norm \( \|\nabla^2 \text{rev}_t(\tilde{p})\|_2 \) is bounded.

Fix \( i, j \in \mathcal{C}_t \). We have

\[
\frac{\partial \text{rev}_t(p)}{\partial p_i} = \frac{e^{u^0_{it}}(1 - \beta_ip_i)}{1 + \sum_{\ell \in \mathcal{C}_t} e^{u^0_{i\ell}}} + \sum_{k \in \mathcal{C}_t} \frac{p_k e^{u^0_{kt}}}{(1 + \sum_{\ell \in \mathcal{C}_t} e^{u^0_{i\ell}})^2},
\]  

(29)

with \( \beta_i = \langle x_i, \gamma_0 \rangle \). Taking derivative with respect to \( p_j \), we get

\[
\frac{\partial^2 \text{rev}_t(p)}{\partial p_i \partial p_j} = \frac{e^{u^0_{it}}}{(1 + \sum_{\ell \in \mathcal{C}_t} e^{u^0_{i\ell}})^2} \left[ \beta_j(1 - \beta_ip_i)e^{u^0_{jt}} + 1 - p_j\beta_j \right] + 2\beta_j e^{u^0_{jt}} \sum_{k \in \mathcal{C}_t} \frac{p_k e^{u^0_{kt}}}{(1 + \sum_{\ell \in \mathcal{C}_t} e^{u^0_{i\ell}})^3}.
\]  

(30)

By Lemma A.1, we have \( p_{it} \leq P \). Also, by (22), we have \( |u^0_{it}| \leq M \). In addition, \( 0 \leq \beta_i \leq \|x_i\|\|\gamma_0\| \leq W \). Since \( P \) and \( M \) are constants depending only on \( W \) and \( L_0 \), there exists a constant \( c_1(W, L_0) > 0 \), such that

\[
\left| \frac{\partial^2 \text{rev}_t}{\partial p_i \partial p_j} \right| \leq c_1(W, L_0),
\]  

(31)

uniformly over \( i, j \in \mathcal{C}_t \). We next bound the operator norm of \( \nabla^2 \text{rev}_t(p) \). Note that for a matrix \( A \in \mathbb{R}^{N \times N} \), we have

\[
\|A\|_2 = \sup_{\|u\| \leq 1} \|u^TAu\| \leq \sup_{\|u\| \leq 1} \left\{ \sum_{i,j=1}^{N} |A_{i,j}| |u_i||u_j| \right\} \leq |A|_\infty \sup_{\|u\| \leq 1} \|u\|_1 \leq N\|A\|_\infty,
\]  

(32)

where \( |A|_\infty = \max_{1 \leq i,j \leq N} |A_{i,j}| \). Therefore, the result follows by using (31).
B \textbf{Proof of main theorems}

B.1 Proof of Theorem 5.1

By Lemma 5.4, we have

\[ \text{reg}_t \leq c_1 N \| p_t - p^*_t \|^2 \leq c_1 L^2 N^4 \left( \| X_t (\hat{\gamma}_k - \gamma_0) \|^2 + \| X_t (\hat{\theta}_k - \theta_0) \|^2 \right), \tag{33} \]

where we used Lemma 5.2. Note that at this point, we cannot directly use Proposition 5.3 because here \( t \) belongs to episode \( k \), while in the statement of Proposition 5.3, we have \( X^{(k-1)} \) (product features arrived in episode \( k - 1 \)). Instead, we use the assumption that the feature vectors are independent and identically distributed.

Taking expectation with respect to \( X_t \), we obtain

\[ \mathbb{E}(\text{reg}_t) = c_1 L^2 N^5 \left( \| \Sigma^{1/2} (\hat{\gamma}_k - \gamma_0) \|^2 + \| \Sigma^{1/2} (\hat{\theta}_k - \theta_0) \|^2 \right), \tag{34} \]

where \( \Sigma = \mathbb{E}(x_i x_i^T) \in \mathbb{R}^{d \times d} \) is the population covariance of the features distribution. In order to bound the right-hand side, we denote the empirical covariance by \( \hat{\Sigma}^{(k-1)} = 1/(N \ell_{k-1}) [X^{(k-1)}]^T X^{(k-1)} \in \mathbb{R}^{d \times d} \). We then have

\[ \langle \hat{\gamma}_k - \gamma_0, \Sigma (\hat{\gamma}_k - \gamma_0) \rangle = \langle \hat{\gamma}_k - \gamma_0, \hat{\Sigma}^{(k-1)} (\hat{\gamma}_k - \gamma_0) \rangle + \langle \hat{\gamma}_k - \gamma_0, \Delta^{(k-1)} (\hat{\gamma}_k - \gamma_0) \rangle, \tag{35} \]

where \( \Delta^{(k-1)} \equiv \Sigma - \hat{\Sigma}^{(k-1)} \). A simple concentration bound shows that

\[ \langle v, \Delta^{(k-1)} v \rangle \leq 3 \sqrt{\log(d \ell_{k-1})/(N \ell_{k-1})} \| v \|^2, \]

for any vector \( v \in \mathbb{R}^d \), with probability at least \( 1 - 8/(\ell_{k-1} d^2) \). Therefore, by employing Proposition 5.3, we get

\[ \langle \hat{\gamma}_k - \gamma_0, \Sigma (\hat{\gamma}_k - \gamma_0) \rangle \leq \frac{8 W}{c_0} \lambda_k + 12 \sqrt{\frac{\log(d \ell_{k-1})}{N \ell_{k-1}}} W^2 = c_2 \sqrt{\frac{\log(d \ell_{k-1})}{\ell_{k-1}}}, \tag{36} \]

with probability at least \( 1 - 1/(d \ell_{k-1}) - 8/(d^2 \ell_{k-1}) \) for a constant \( c_2 = c_2(W, L_0, N) > 0 \).

Likewise, we have

\[ \langle \hat{\theta}_k - \theta_0, \Sigma (\hat{\theta}_k - \theta_0) \rangle \leq \frac{8 W}{c_0} \lambda_k + 12 \sqrt{\frac{\log(d \ell_{k-1})}{N \ell_{k-1}}} W^2 = c_2 \sqrt{\frac{\log(d \ell_{k-1})}{\ell_{k-1}}}. \tag{37} \]

Combining Equations (36) and (37), we obtain

\[ \mathbb{E}(\text{reg}_t) \leq 2c_1 c_2 L^2 N^5 \sqrt{\frac{\log(d \ell_{k-1})}{\ell_{k-1}}} + \left( \frac{1}{d \ell_{k-1}} + \frac{8}{d^2 \ell_{k-1}} \right) P, \tag{38} \]

with \( P = P(W, L_0, N) \) the upper bound on the prices given in Lemma A.1.
Since the length of episode $k$ is $\ell_k = 2^{k-1}$, letting $K = \log T$ be the number of episodes by time $T$, we have

$$\text{Regret}(T) = \sum_{t=1}^{T} \mathbb{E}(\text{reg}_t) \leq C K \sum_{k=1}^{K} \sqrt{\frac{\log(d\ell_{k-1})}{\ell_{k-1}}} P\ell_k$$

$$\leq \sqrt{2C} \sum_{k=1}^{K} \sqrt{\log(d\ell_{k-1})\ell_{k-1}} + \sum_{k=1}^{K} 2 \left( \frac{1}{d} + \frac{8}{d^2} \right) P = O(\sqrt{\log(dT)T}), \quad (39)$$

where $C = C(W, L_0, N) > 0$. This completes the proof of the upper bound.

### B.2 Proof of Theorem 5.5

The $\Omega(\sqrt{T})$ follows by existence of the so-called ‘uninformative prices’ [7]. Fix a time $t$ and denote by $d(\theta_0, \gamma_0, p)$ the demand function (with range to be $\mathbb{R}^N$), where $d_i$ represents the likelihood of choosing product $i$ when the posted price vector is $p \in \mathbb{R}_+^N$ and the model parameters are $\theta_0$ and $\gamma_0$. (Since $t$ is fixed, we drop the index $t$ to simplify notation.) An uninformative price vector $p$, is any such vector such that all the demand curves (across model parameters) intersect at that price. The name comes from the fact that such price vector does not reveal any information about the underlying model parameters and hence does not help with the learning part (exploration of the space of model parameters). Now, if an uninformative price is also the optimal price vectors for a specific choice of parameters, then we would get a clear tension between the exploitation and exploration objectives. Indeed, for a policy to learn the underlying model parameters fast enough, it must necessarily choose prices that are away from the uninformative prices and this in turns leads to accruing regret when an uninformative price is in fact the optimal price vectors for the true model parameters.

To construct uninformative prices, we let $\theta_{0,j} = 0$ and $\gamma_{0,j} = 0$ for $j > 2$ and let $x_{i,1} = 1$ for all products $i$. We then have

$$u^0_i = \langle x_i, \theta_0 \rangle - \langle x_i, \gamma_0 \rangle p_i = \theta_{0,1} - \gamma_{0,1} p_i.$$ 

We fix arbitrary $b > 0$ and set $\gamma_{0,1} = \theta_{0,1} + b$. We then have $u^0_i = \theta_{0,1}(1 - p_i) - bp_i$. Therefore, $p_i = 1$, for all $i \in C_t$, is an uninformative price, since the demand curves only depend on $u_i^0$ (see Eq. (2)) and hence all the demand curves (across $\theta_{0,1}$) intersect at the common price $p_i = 1$. In addition, for $\theta_{0,1} = Ne^{-b} + 1 - b$, it is easy to verify that $p_i = 1$, for all $i$, is the optimal price vector, by using Proposition 3.1.

Now that we have established the existence of uninformative prices, it can be shown that the worst-case $T$-regret of any policy is lower bounded by $\Omega(\sqrt{T})$, by following the same lines in the proof of [7, Theorem 3.1] and is omitted.