Application of Pareto Distribution in Wage Models*

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This paper deals with the use of Pareto distribution in models of wage distribution. Pareto distribution cannot generally be used as a model of the whole wage distribution, but only as a model for the distribution of higher or of the highest wages. It is usually about wages higher than the median. The parameter $b$ is called the Pareto coefficient and it is often used as a characteristic of differentiation of fifty percent of the highest wages. Pareto distribution is so much the more applicable model of a specific wage distribution, the more specific differentiation of fifty percent of the highest wages will resemble to differentiation that is expected by Pareto distribution. Pareto distribution assumes a differentiation of wages, in which the following ratios are the same: ratio of the upper quartile to the median; ratio of the eighth decile to the sixth decile; ratio of the ninth decile to the eighth decile. This finding may serve as one of the empirical criterions for assessing, whether Pareto distribution is a suitable or less suitable model of a particular wage distribution. If we find only small differences between the ratios of these quantiles in a specific wage distribution, Pareto distribution is a good model of a specific wage distribution. Approximation of a specific wage distribution by Pareto distribution will be less suitable or even unsuitable when more expressive differences of mentioned ratios. If we choose Pareto distribution as a model of a specific wage distribution, we must reckon with the fact that the model is always only an approximation. It will describe only approximately the actual wage distribution and the relationships in the model will only partially reflect the relationships in a specific wage distribution.

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Pareto distribution is usually used as a model of the distribution of the largest wages, not for the whole wage distribution. In this article, we will consider using the Pareto distribution to model wages higher than median.

The $100\cdot P\%$ quantile of the wage distribution will be denoted by $x_P$, $0 < P < 1$. This value represents the upper bound of $100\cdot P\%$ lowest wages and also the lower bound of $100(1 - P)\%$ highest wages. A particular quantile (denoted as $x_{P_0}$) which will be the lower bound of some small number of the highest wages is usually set to be the maximum wage. If the following formula (1) holds for any quantile $x_P$, the wage distribution is Pareto distribution.

$$\frac{x_{P_0}}{x_P} = \left(\frac{1 - P}{1 - P_0}\right)^b$$

(1)

The parameter $b$ of the Pareto distribution (1) is called the Pareto coefficient. It can be used as a characteristic of differentiation of $50\%$ highest wages.

We will now consider a pair of quantiles $x_{P_1}$ and $x_{P_2}$, $P_1 < P_2$. It follows from equation (1) that:

$$\frac{x_{P_0}}{x_{P_1}} = \left(\frac{1 - P_1}{1 - P_0}\right)^b$$

(2)

and

$$\frac{x_{P_0}}{x_{P_2}} = \left(\frac{1 - P_2}{1 - P_0}\right)^b$$

(3)

From what we can derive for the rate of $x_{P_2}$ to $x_{P_1}$ that:

$$\frac{x_{P_2}}{x_{P_1}} = \left(\frac{1 - P_1}{1 - P_2}\right)^b$$

(4)

The rate $\frac{x_{P_2}}{x_{P_1}}$ is an increasing function of the Pareto coefficient $b$. If the rate of quantiles increases, the relative differentiation of wages increases too. If only absolute differences between quantiles increase, only the absolute differentiation of wages increases.

It follows from the equation (1) that once the values $x_{P_0}$ and $b$ are chosen, we can determine the quantile $x_P$ for any chosen $P$ or the other way around for any value $x_P$ we can find the corresponding value of $P$. In the first case, it is advantageous to write the equation (1) as:

$$x_P = \frac{x_{P_0}}{\left(\frac{1 - P}{1 - P_0}\right)^b}$$

(5)

or after logarithmic transformation as:

$$\log x_P = \log x_{P_0} - b[\log(1 - P) - \log(1 - P_0)]$$

(6)

in the second case:

$$1 - P = (1 - P_0)^b \sqrt[1 - P_0]{x_{P_0}}$$

(7)
or after logarithmic transformation as:

\[ \log (1 - P) = \log (1 - P_0) + \frac{1}{b} (\log x_{p_0} - \log x_p) \]  

(8)

The equations (2)-(4) will after logarithmic transformation have the following forms:

\[
\begin{align*}
\log \frac{x_{P_0}}{x_{P_1}} &= \log \frac{1 - P_1}{1 - P_0} \quad \text{(9)} \\
\log \frac{x_{P_1}}{x_{P_2}} &= \log \frac{1 - P_1}{1 - P_0} \quad \text{(10)}
\end{align*}
\]

It follows from the equation (9) that instead of the Pareto coefficient \( b \) we can use any other quantile \( x_{P_1} \) of the Pareto distribution and it follows from the equation (10) that the Pareto coefficient \( b \) can be calculated using any known quantiles \( x_{P_1} \) and \( x_{P_2} \). Then we can also determine the value \( x_{P_0} \) using the formulas:

\[
\begin{align*}
x_{P_0} &= x_{P_1} \left( \frac{1 - P_1}{1 - P_0} \right)^b \quad \text{(11)} \\
x_{P_0} &= x_{P_2} \left( \frac{1 - P_2}{1 - P_0} \right)^b \quad \text{(12)}
\end{align*}
\]

The model characterized with the relationship (1) will be practically applicable if the following is known:

- The value of the quantile that characterizes the assumed wage maximum and the value of the Pareto coefficient \( b \);
- The value of the quantile that characterizes the assumed wage maximum and the value of any other quantile;
- The values of any two quantiles of the Pareto distribution.

Any two quantiles can be written as \( x_P \) and \( x_{P+k} \), where \( 0 < k < 1 - P \). Using the equation (4), we can derive for the rate of these two quantiles:

\[
\frac{x_{P+k}}{x_P} = \left( \frac{1 - P}{1 - P - k} \right)^b
\]

(13)

The rate (13) will be equal for such pairs of quantiles for which the following formula holds:

\[
\frac{1 - P}{1 - P - k} = c,
\]

(14)

where \( c \) is a constant, i.e., the rate will be the same for all pairs of quantiles for which:

\[
k = \frac{c - 1}{c} (1 - P)
\]

(15)

We will use the constant \( c = 2 \) in equation (15) and we will choose gradually \( P = 0.5; 0.6; 0.8 \). Then using the equation (13) we can show the equality of rates of some frequently used quantiles:
From the relationship (16) we can conclude that Pareto distribution assumes such a wage differentiation for which the rate of the upper quartile to median is the same as:

- The rate of the 8th to the 6th decile;
- And as the rate of the 9th to the 8th decile.

If in a particular case, the observed differences of the rates of the above mentioned quantiles are negligible, Pareto distribution will be an appropriate model of the considered wage distribution. In the case, the differences are quite material, the approximation of the considered wage distribution with Pareto distribution will be more or less inappropriate. More about the theory of Pareto distribution is described in statistical literature (Forbes, Evans, Hastings, & Peacock, 2011; Johnson, Kotz, & Balakrishnan, 1994; Kleiber & Kotz, 2003; Krishnamoorthy, 2006).

**Parameter Estimates**

If the Pareto distribution is chosen as a model for a particular distribution we have to keep in mind that this model is only an approximation. The wage distribution will be only approximated and the relations derived from the model will also hold for the “true distribution” only approximately. Which relations will hold more precisely and for which the precision will be lower will be mostly dependent on the method of parameter estimates.

There are many possibilities to choose from. In the following text the quantiles of Pareto distribution will be denoted as $x_P$ and the quantiles of the observed wage distribution will be denoted as $y_P$.

First we need to decide which quantile to choose as $x_P$. In this article we will assume that $x_P = x_{0.99}$. From the equation (1) we can see that the considered Pareto distribution will be defined by the equation:

\[
x_{0.99} = \frac{1 - P}{0.01}
\]  
(17)

Then we need to determine the value $x_{0.99}$ and the value of the Pareto coefficient $b$. Because it is necessary to estimate the values of two parameters we need to choose two equations to estimate from.

A natural choice is the equation $x_{P0} = y_{P0}$; that is in our case $x_{0.99} = y_{0.99}$. As the other equation we set a quantile $x_{P1}$ equal to the corresponding observed quantile, i.e., $x_{P1} = y_{P1}$. In this case, the parameters of the model will be:

\[
x_{P0} = y_{P0}
\]  
(18)

and using equation (9):

\[
b = \frac{\log y_{P0}}{\log y_{P1}}
\]  
(19)

We can get different modifications using different choice of the maximum wage and the second quantile. If we use equation $x_{0.99} = y_{0.99}$ and we use the median in the second equation, i.e., $x_{0.5} = y_{0.5}$ we get a model with
parameters:

\[ x_{0.99} = y_{0.99} \]  
\[ \log \frac{y_{0.99}}{y_{0.5}} = b \]

Another possibility is setting any two quantiles of the model equal to the quantiles of the observed distribution:

\[ x_{p_1} = y_{p_1} \]
\[ x_{p_2} = y_{p_2} \]

Using the formula (10), we get the following parameter estimates:

\[ \log \frac{y_{p_2}}{y_{p_1}} = b \]
\[ = b \log \frac{1 - P_1}{1 - P_2} \]

and from equations (11) and (12) we get:

\[ x_{p_0} = y_{p_1} \left( \frac{1 - P_1}{1 - P_0} \right)^b = y_{p_2} \left( \frac{1 - P_2}{1 - P_0} \right)^b \]

With this alternative we can also get numerous modifications depending on the choice of quantiles \( y_{p_1} \) and \( y_{p_2} \) that are used.

The third possibility is based on the request that \( x_{p_0} = y_{p_0} \) and that the rate of some other two quantiles of the Pareto distribution \( x_{p_2} / x_{p_1} \) is equal to the rate \( y_{p_2} / y_{p_1} \) of corresponding quantiles of the wage distribution observed. In this case we will estimate the parameters using equation (10):

\[ x_{p_0} = y_{p_0} \]
\[ \log \frac{y_{p_2}}{y_{p_1}} = b \log \frac{1 - P_1}{1 - P_2} \]

In this case, notwithstanding that \( x_{p_2} / x_{p_1} = y_{p_2} / y_{p_1} \) holds, the equality of quantiles itself, \( x_{p_1} \neq y_{p_1} \) and \( x_{p_2} \neq y_{p_2} \), does not hold. In this case, we can also arrive to numerous modifications depending on what maximum wage is chosen and what quantiles \( y_{p_1} \) and \( y_{p_2} \) are chosen.

For all of the above methods the equality of two characteristics of the model and the observed distribution was required. There are also different approaches to the parameter estimates.

The least squares method is frequently used for the Pareto distribution parameter estimates as well. We will consider the following quantiles of the observed wage distribution \( y_{p_1}, y_{p_2}, \ldots, y_{p_k} \) and corresponding quantiles of the
Pareto distribution \( x_{p1}, x_{p2}, \ldots, x_{pk} \). The model distribution will be most precise when the sum of squared differences:

\[
\sum_{i=1}^{k} (y_{pi} - x_{pi})^2
\]

is minimized. In this case closed formula solution does not exist. Therefore sum of squared differences of logarithms of quantiles is often considered:

\[
\sum_{i=1}^{k} (\log y_{pi} - \log x_{pi})^2
\]

Minimizing the objective function (29), it is possible to derive the following estimates:

\[
b = \frac{\sum_{i=1}^{k} \log y_{pi} \log \frac{1 - P_0}{1 - P_i} - \sum_{i=1}^{k} \log y_{pi} \sum_{i=1}^{k} \log \frac{1 - P_0}{1 - P_i}}{\sum_{i=1}^{k} \log^2 \left( \frac{1 - P_0}{1 - P_i} \right)}
\]

\[
\log x_{pi} = \frac{\sum_{i=1}^{k} \log y_{pi} \log \frac{1 - P_0}{1 - P_i}}{k} - b \frac{\sum_{i=1}^{k} \log \frac{1 - P_0}{1 - P_i}}{k}
\]

In the case we use this estimating method, it is needed to keep in mind that the equality of model quantiles and observed quantiles is not guaranteed for any \( P \). Again we can arrive to different results depending on what quantiles \( y_{p1}, y_{p2}, \ldots, y_{pk} \) are used for the calculations. Furthermore the parameter estimates also depend on the choice of the maximum wage.

**Characteristics of the Appropriateness of the Pareto Distribution**

For the application of Pareto distribution as a model of the wage distribution, it is crucial that the model fits the observed distribution as close as possible. It is important that the observed relative frequencies in particular wage intervals are as close to the theoretical probabilities assigned to these intervals by the model as possible.

It is needed to note that the same parameter estimation method does not always lead to the best results. It is of particular importance in “what direction” is the observed wage distribution different from Pareto distribution. Pareto distribution assumes such wage differentiation that the relations (16) hold. With real data we can encounter many different situations:

\[
\begin{align*}
y_{0.75} &< y_{0.8} < y_{0.9} \\
y_{0.5} &< y_{0.6} < y_{0.8}
\end{align*}
\]

\[
\begin{align*}
y_{0.75} &> y_{0.8} > y_{0.9} \\
y_{0.5} &> y_{0.6} > y_{0.8}
\end{align*}
\]

\[
\begin{align*}
y_{0.75} &< y_{0.9} < y_{0.8} \\
y_{0.5} &< y_{0.8} < y_{0.6}
\end{align*}
\]

\[
\begin{align*}
y_{0.75} &> y_{0.9} > y_{0.8} \\
y_{0.5} &> y_{0.8} > y_{0.6}
\end{align*}
\]

\[
\begin{align*}
y_{0.8} &< y_{0.75} < y_{0.9} \\
y_{0.5} &< y_{0.75} < y_{0.9}
\end{align*}
\]
It follows from equations (32)-(37) that the observed distributions will more or less systematically differ from the Pareto distribution. In the case of equation (32) the differentiation of the observed wage distribution is higher; in the case of equation (33) the differentiation will be lower than in the case of Pareto distribution. Some bias occurs in cases equations (34)-(37) as well (but cannot be so specified). Systematical bias should be a signal for potential adjustment of the model which could be based for example on adding one or more parameters into the model. These adjustments usually lead to more complicated models. Therefore, the above mentioned bias is often neglected and simple models are preferred even though they lead to some bias.

**Wage Distribution of Males and Females in the Czech Republic in 2001-2008**

The data used in this article is the gross monthly wage of male and female in CZK in the Czech Republic in the years 2001-2008. Data were sorted in the table of interval distribution with opened lower and upper bound in the lowest and highest interval respectively. The source is the web page of the Czech statistical office. The following quantiles were calculated (see Table 1).

| Year | $y_{0.50}$ | $y_{0.60}$ | $y_{0.75}$ | $y_{0.80}$ | $y_{0.90}$ | $y_{0.99}$ |
|------|------------|------------|------------|------------|------------|------------|
| Total 2001 | 12,502 | 14,042 | 16,987 | 18,254 | 23,319 | 44,921 |
| | 2002 | 15,545 | 17,125 | 20,215 | 22,193 | 27,754 | 47,172 |
| | 2003 | 16,735 | 18,458 | 22,224 | 23,797 | 29,590 | 47,719 |
| | 2004 | 17,709 | 19,557 | 23,077 | 24,849 | 31,082 | 56,369 |
| | 2005 | 18,597 | 20,566 | 24,470 | 26,328 | 33,292 | 56,852 |
| | 2006 | 19,514 | 21,564 | 25,675 | 27,693 | 35,230 | 57,326 |
| | 2007 | 20,987 | 23,227 | 27,590 | 29,900 | 37,892 | 66,395 |
| | 2008 | 22,310 | 24,696 | 29,553 | 31,769 | 40,541 | 68,828 |
| Males 2001 | 14,152 | 15,781 | 19,037 | 20,697 | 26,264 | 46,781 |
| | 2002 | 16,985 | 18,667 | 22,604 | 24,199 | 31,101 | 48,047 |
| | 2003 | 18,240 | 20,116 | 24,145 | 26,041 | 34,564 | 48,417 |
| | 2004 | 19,344 | 21,321 | 25,306 | 27,286 | 34,819 | 57,514 |
| | 2005 | 20,281 | 22,446 | 26,822 | 28,989 | 37,211 | 57,808 |
| | 2006 | 21,199 | 23,460 | 28,090 | 30,525 | 39,381 | 58,104 |
| | 2007 | 22,933 | 25,366 | 30,284 | 32,663 | 42,815 | 70,522 |
| | 2008 | 24,498 | 27,115 | 32,343 | 35,105 | 46,375 | 72,338 |
| Females 2001 | 10,770 | 12,187 | 14,655 | 15,700 | 18,904 | 37,526 |
| | 2002 | 13,746 | 15,181 | 17,727 | 18,903 | 23,291 | 43,339 |
| | 2003 | 14,831 | 16,453 | 19,281 | 20,628 | 24,637 | 44,883 |
| | 2004 | 15,642 | 17,303 | 20,293 | 21,560 | 25,776 | 50,776 |
| | 2005 | 16,454 | 18,211 | 21,426 | 22,804 | 27,503 | 52,508 |
| | 2006 | 17,311 | 19,202 | 22,530 | 23,966 | 29,082 | 54,054 |
| | 2007 | 18,390 | 20,392 | 24,024 | 25,924 | 31,338 | 58,649 |
| | 2008 | 19,399 | 21,600 | 25,558 | 27,215 | 33,405 | 63,628 |
From Table 2, we can see that, with the exception of male in the year 2003, 2007 and 2008, all other wage distributions have lower differentiation than Pareto distribution. The systematical error occurred also in the case of male in the year 2003, 2007 and 2008. It follows from the empirical criterion (16) and from Table 2 that in all cases the differences between the rates of the considered quantiles are negligible and therefore Pareto distribution can be used as the model of the distribution.

The 99th percentile will be considered as a characteristic of the maximum wage. The parameters of the Pareto distribution are estimated using the above described methods.

First we consider the conditions $x_{P0} = y_{P0}$ a $x_{P1} = y_{P1}$ and we chose median as the second quantile, i.e., $x_{0.99} = y_{0.99}$ and $x_{0.5} = y_{0.5}$. We estimate the parameter $b$ using the formula (21). The summary of the parameter estimates is in Table 3.

Table 2
The Rates of Quantiles $y_{75}/y_{50}$, $y_{80}/y_{60}$ and $y_{90}/y_{80}$ of the Wage Distributions in the Years 2001-2008 and Its Relations

| Year | $y_{0.75}/y_{0.50}$ | $y_{0.80}/y_{0.60}$ | $y_{0.90}/y_{0.80}$ | Relations between quantile rates |
|------|---------------------|---------------------|---------------------|---------------------------------|
| Total | 2001 | 1.358815 | 1.299910 | 1.277456 | (21.2) |
|      | 2002 | 1.300422 | 1.295897 | 1.250612 | (21.2) |
|      | 2003 | 1.327994 | 1.289216 | 1.243457 | (21.2) |
|      | 2004 | 1.303112 | 1.276060 | 1.250812 | (21.2) |
|      | 2005 | 1.315815 | 1.280162 | 1.264514 | (21.2) |
|      | 2006 | 1.315734 | 1.284213 | 1.272161 | (21.2) |
|      | 2007 | 1.314623 | 1.287295 | 1.267291 | (21.2) |
|      | 2008 | 1.324653 | 1.286403 | 1.276118 | (21.2) |
| Males | 2001 | 1.345148 | 1.311556 | 1.268936 | (21.2) |
|      | 2002 | 1.330847 | 1.296386 | 1.285203 | (21.2) |
|      | 2003 | 1.323680 | 1.294561 | 1.327273 | (21.5) |
|      | 2004 | 1.308222 | 1.279734 | 1.276084 | (21.2) |
|      | 2005 | 1.322543 | 1.291532 | 1.283632 | (21.2) |
|      | 2006 | 1.325086 | 1.301135 | 1.290146 | (21.2) |
|      | 2007 | 1.320542 | 1.287669 | 1.310810 | (21.4) |
|      | 2008 | 1.320230 | 1.294671 | 1.321037 | (21.5) |
| Females | 2001 | 1.360723 | 1.288227 | 1.204113 | (21.2) |
|       | 2002 | 1.289624 | 1.245137 | 1.232163 | (21.2) |
|       | 2003 | 1.300019 | 1.253747 | 1.194319 | (21.2) |
|       | 2004 | 1.297375 | 1.246052 | 1.195526 | (21.2) |
|       | 2005 | 1.302189 | 1.252237 | 1.206076 | (21.2) |
|       | 2006 | 1.301488 | 1.248134 | 1.213470 | (21.2) |
|       | 2007 | 1.306362 | 1.271283 | 1.208841 | (21.2) |
|       | 2008 | 1.317491 | 1.259954 | 1.227448 | (21.2) |

Next we apply the conditions $x_{P1} = y_{P1}$ and $x_{P2} = y_{P2}$ and we choose 6th and 9th decile for $y_{P1}$ and $y_{P2}$. We use the formulas (24) and (25) to estimate the parameters. The summary of the parameter estimates is in Table 3.

Parameters of the Pareto distribution can also be estimated using the equations $x_{P0} = y_{P0}$ and $x_{P2}/x_{P1} = y_{P2}/y_{P1}$. We choose the 9th and 6th decile in the rate $y_{P2}/y_{P1}$. In this case we use the relations (26) and (27) to estimate the parameters. The summary of the parameter estimates is also in Table 3.
### Table 3

**Estimated Parameters of Pareto Distribution for Different Choices of the Estimation Equations**

| Equations used | Parameter estimates | Parameter estimates | Parameter estimates |
|----------------|---------------------|---------------------|---------------------|
| \( x_{0.99} = y_{0.99} \), \( x_{0.5} = y_{0.5} \) |          |          |                      |
| \( x_{0.6} = y_{0.6} \), \( x_{0.9} = y_{0.9} \) |          |          |                      |
| \( x_{0.99} = y_{0.99} \), \( x_{0.6} = y_{0.6} \) |          |          |                      |

| Year | \( x_0 \) | \( b \) | \( x_0 \) | \( b \) | \( x_0 \) | \( b \) |
|------|-----------|--------|-----------|--------|-----------|--------|
| Total | 2001 | 44,921 | 0.326952 | 54,143 | 0.365843 | 44,921 | 0.365843 |
|      | 2002 | 47,172 | 0.283758 | 61,890 | 0.348293 | 47,172 | 0.348293 |
|      | 2003 | 47,719 | 0.267846 | 64,800 | 0.340425 | 47,719 | 0.340425 |
|      | 2004 | 56,369 | 0.295969 | 67,096 | 0.334192 | 56,369 | 0.334192 |
|      | 2005 | 56,852 | 0.299456 | 74,095 | 0.347455 | 56,852 | 0.347455 |
|      | 2006 | 57,326 | 0.275468 | 79,614 | 0.354083 | 57,326 | 0.354083 |
|      | 2007 | 66,395 | 0.294405 | 85,426 | 0.353045 | 66,395 | 0.353045 |
|      | 2008 | 68,828 | 0.287978 | 92,352 | 0.357552 | 68,828 | 0.357552 |

| Males | 2001 | 46,781 | 0.305624 | 54,143 | 0.365843 | 46,781 | 0.365843 |
|       | 2002 | 48,047 | 0.265814 | 61,890 | 0.348293 | 48,047 | 0.348293 |
|       | 2003 | 48,417 | 0.249540 | 64,800 | 0.340425 | 48,417 | 0.340425 |
|       | 2004 | 57,514 | 0.278536 | 74,095 | 0.347455 | 57,514 | 0.347455 |
|       | 2005 | 57,808 | 0.267749 | 79,614 | 0.354083 | 57,808 | 0.354083 |
|       | 2006 | 58,104 | 0.257739 | 85,426 | 0.353045 | 58,104 | 0.353045 |
|       | 2007 | 70,522 | 0.287153 | 92,352 | 0.357552 | 70,522 | 0.357552 |
|       | 2008 | 72,338 | 0.276777 | 113,087 | 0.387128 | 72,338 | 0.387128 |

| Females | 2001 | 37,526 | 0.319087 | 39,196 | 0.316679 | 37,526 | 0.316679 |
|         | 2002 | 43,339 | 0.293539 | 47,418 | 0.308749 | 43,339 | 0.308749 |
|         | 2003 | 44,883 | 0.283055 | 48,172 | 0.291217 | 44,883 | 0.291217 |
|         | 2004 | 50,776 | 0.300989 | 49,971 | 0.287505 | 50,776 | 0.287505 |
|         | 2005 | 52,508 | 0.296265 | 54,551 | 0.297414 | 52,508 | 0.297414 |
|         | 2006 | 54,054 | 0.291062 | 57,954 | 0.299456 | 54,054 | 0.299456 |
|         | 2007 | 58,649 | 0.296461 | 63,977 | 0.309955 | 58,649 | 0.309955 |
|         | 2008 | 63,628 | 0.303636 | 68,917 | 0.314516 | 63,628 | 0.314516 |

In the end we also estimate the parameters of the Pareto distribution using the least squares method. We use the relations (30) and (31). In this method, we choose 5th, 6th, 7th, 8th and 9th deciles of the observed wage distribution, i.e., \( k = 5 \). Parameters estimated using the least squares method are summarized in Table 4.

### Table 4

**Parameters Estimated Using the Least Squares Method**

| Year | Total | Males | Females |
|------|-------|-------|---------|
| \( x_0 \) | \( b \) | \( x_0 \) | \( b \) | \( x_0 \) | \( b \) |
| 2001 | 56,562 | 0.379911 | 63,774 | 0.379912 | 42,520 | 0.341047 |
| 2002 | 64,026 | 0.358469 | 73,770 | 0.372825 | 49,188 | 0.320682 |
| 2003 | 67,219 | 0.351034 | 85,080 | 0.391617 | 51,125 | 0.309187 |
| 2004 | 69,311 | 0.344615 | 80,310 | 0.369086 | 52,763 | 0.303849 |
| 2005 | 76,310 | 0.359635 | 88,251 | 0.372535 | 57,413 | 0.312826 |
| 2006 | 81,721 | 0.362626 | 95,225 | 0.381012 | 60,917 | 0.315022 |
| 2007 | 88,022 | 0.362359 | 103,405 | 0.383183 | 67,572 | 0.325878 |
| 2008 | 94,849 | 0.366387 | 114,131 | 0.391293 | 72,463 | 0.330659 |

The values of the sum of absolute differences of observed and theoretical absolute frequencies of all
intervals calculated for all cases considered wage distributions are in Table 5. In the case of the theoretical
frequencies at first we determined theoretical probabilities using the formula (8). From these, we determined
theoretical absolute frequencies.

Table 5

| Year | Equations used | Least squares method |
|------|----------------|---------------------|
|      |                |                     |
|      | $x_{0.99} = y_{0.99}$ | $y_{0.9}/y_{0.6}$ |
|      | $x_{0.5} = y_{0.5}$   | $x_{0.6}$           |
|      | $x_{0.9} = y_{0.9}$   | $y_{0.9}/y_{0.6}$   |
|      | $x_{0.9} = y_{0.9}$   |                     |

| Total | 2001  | 37,459 | 23,255 | 85,795 | 23,859 |
|       | 2002  | 51,358 | 27,327 | 171,404| 31,658 |
|       | 2003  | 73,388 | 36,520 | 204,535| 39,722 |
|       | 2004  | 103,625| 64,422 | 249,348| 66,249 |
|       | 2005  | 167,946| 69,930 | 353,661| 68,679 |
|       | 2006  | 157,094| 68,849 | 426,442| 69,104 |
|       | 2007  | 268,740| 260,786| 322,437| 262,224|
|       | 2008  | 282,396| 253,373| 372,117| 257,050|

| Males | 2001  | 20,603 | 10,089 | 56,291 | 9,959  |
|       | 2002  | 33,576 | 19,711 | 111,796| 20,298 |
|       | 2003  | 47,909 | 23,576 | 96,863 | 23,747 |
|       | 2004  | 60,241 | 32,457 | 178,858| 33,076 |
|       | 2005  | 81,505 | 35,349 | 220,276| 36,321 |
|       | 2006  | 96,789 | 37,737 | 250,764| 37,653 |
|       | 2007  | 140,965| 138,678| 202,143| 139,428|
|       | 2008  | 135,960| 133,953| 173,262| 135,089|

| Females | 2001  | 24,256 | 23,926 | 23,687 | 21,270 |
|         | 2002  | 23,697 | 16,716 | 42,148 | 18,595 |
|         | 2003  | 37,215 | 30,902 | 40,237 | 30,011 |
|         | 2004  | 45,429 | 41,416 | 45,460 | 40,957 |
|         | 2005  | 51,793 | 41,615 | 52,493 | 41,449 |
|         | 2006  | 58,014 | 41,137 | 74,302 | 41,812 |
|         | 2007  | 138,241| 128,854| 150,258| 127,313|
|         | 2008  | 140,955| 132,125| 155,071| 131,224|

Conclusions

The appropriateness of particular modifications of the Pareto distribution can be evaluated comparing the
theoretic and empirical frequencies. It is possible to compare both the absolute and relative differences between
the theoretic and observed empirical distributions. In this article we used the absolute differences. The values
sums these differences are in Table 5. The values seem to be relatively high. The question of appropriateness of a
given theoretic wage distribution in the case of large samples was described in statistical literature (Bílková,
2007). Some more general conclusions can be made from the values of the absolute differences of observed and
theoretic distributions.

With the exception of the wage distribution of women in 2001, the worst results are achieved using the
equation $x_{0.99} = y_{0.99}$ and setting the ratio of other two quantiles of the Pareto distribution $x_{0.9}/x_{0.6}$ equal to the ratio
$y_{0.9}/y_{0.6}$ of the corresponding empirical quantiles. This fact is less obvious for female distribution and most
obvious for total distribution. This is also due to the larger sample size of the total sample (in comparison with the
sample size of the sub-groups of male and female). Again with the exception of the wage distribution of women in 2001 the second worst model is the estimate based on the equations $x_{0.99} = y_{0.99}$ and $x_{0.5} = y_{0.5}$. This fact is again less obvious for female distribution and most obvious for total distribution. In the case of the wage distribution of women in 2001, the worst estimate is based on the equations $x_{0.99} = y_{0.99}$ and $x_{0.5} = y_{0.5}$. In the case of the total group is the third worst (second best) method the least squares method (with the exception of 2005). The best results are achieved with the method based on the equations $x_{0.6} = y_{0.6}$ and $x_{0.9} = y_{0.9}$. In the case of the total wage distribution in 2005 is the third worst method based on the equations $x_{0.6} = y_{0.6}$ and $x_{0.9} = y_{0.9}$ and the best method is the least squares method. In the case of the wage distribution of male (with the exception of the years 2001 and 2006), the third worst (second best) results are again achieved using the least squares method. The best results are achieved with the method based on the equations $x_{0.6} = y_{0.6}$ and $x_{0.9} = y_{0.9}$. In the years 2001 and 2006 (set of men) is the third worst method the method based on the equations $x_{0.6} = y_{0.6}$ and $x_{0.9} = y_{0.9}$ and the best is the least squares method. In the case of the female group (with the exception of the years 2001, 2002 and 2006) is the third worst (second best) method based on the equations $x_{0.6} = y_{0.6}$ and $x_{0.9} = y_{0.9}$ and the most precise results are achieved with the least squares method. In the years 2001, 2003, 2004 and 2005 was for the group of women the most precise the least squares method. The very best method for the group of male in 2001 was the least squares method. In this case other methods had much higher values of the above mentioned sum of absolute differences.

From the above described comparison, it is obvious that the simplest parameter estimating methods can be in the case of the Pareto distribution competing with more advanced methods.

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