Some remarks on orthogonally additive operators

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Abstract
We study orthogonally additive operators on Riesz spaces. Our first result gives necessary and sufficient conditions on a pair of Riesz spaces \((E, F)\) for which every orthogonally additive operator from \(E\) to \(F\) is laterally-to-order bounded. Second result extends an analogue of Pitt’s compactness theorem obtained by the second and third named authors for narrow linear operators to the setting of orthogonally additive operators. Third result provides sufficient conditions on a pair of orthogonally additive operators \(S\) and \(T\) to have \(S \lor T\), as well as to have \(S \land T\) without any assumption on the domain and range spaces. Finally we prove an analogue of Meyer’s theorem on the existence of modules of disjointness preserving operator for orthogonally additive operators.

Keywords Riesz space · Orthogonally additive operator · Disjointness preserving operator

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1 Introduction

Orthogonally additive operators (OAOs in short) between Riesz spaces generalize linear operators. Nonlinear OAOs naturally appeared in different fields of analysis, including Uryson and Nemytsky integral operators. Fortunately, lots of tools applied for linear operators, which use the additivity only for disjoint vectors, maintain applicability to OAOs (see [14] for a recent survey on OAOs). For notions and facts on Riesz spaces that we use in the paper we refer the reader to [2].

Let $E, F$ be Riesz spaces. A function $T : E \to F$ is called an OAO provided $T(x + y) = T(x) + T(y)$ for all disjoint vectors $x, y \in E$. An OAO $T : E \to F$ is said to be

- **Positive** (write $T \geq 0$) provided $T(x) \geq 0$ for all $x \in E$;
- **Order bounded** or an abstract Uryson operator provided $T$ sends order bounded subsets of $E$ to order bounded subsets of $F$.

The symbols $\mathcal{O}(E, F)$, $\mathcal{O}^+(E, F)$ and $\mathcal{U}(E, F)$ denote the sets of all OAOs, the set of all positive OAOs and the set of all abstract Uryson operators respectively.

Obviously, $\mathcal{O}(E, F)$ is an ordered vector space with respect to the order $S \leq T$ provided $T - S \geq 0$, that is, $S(x) \leq T(x)$ for all $x \in E$. Remark that the standard order on the vector space $\mathcal{L}(E, F)$ of all linear operators from $E$ to $F$, which is a linear subspace of $\mathcal{O}(E, F)$, is different, and the only linear operator which is positive as an OAO is zero.

Let $E$ be a Riesz space. The notation $x = y \sqcup z$ for $x, y, z \in E$ means that $x = y + z$ and $y \perp z$. An element $x \in E$ is called a fragment\(^1\) of $y \in E$ provided $x \perp y - x$. In this case we write $x \sqsubseteq y$. The binary relation $\sqsubseteq$ is a (non-strict) partial order on $E$, called the lateral order. The set of all fragments of a given element $e \in E$ will be denoted by $\mathfrak{F}_e$. The set $\mathfrak{F}_e$ is a Boolean algebra with zero 0, unit $e$ with respect to the operations $\bigvee$ and $\bigwedge$ of taking the lateral supremum and infimum respectively. Moreover, $x \bigvee y = (x^+ \lor y^+) - (x^- \lor y^-)$ and $x \bigwedge y = (x^+ \land y^+) - (x^- \land y^-)$ for all $x, y \in \mathfrak{F}_e$ (see [8] for details).

Let $E, F$ be Riesz spaces. An OAO $T : E \to F$ is said to be *laterally-to-order bounded* provided for every $e \in E$ the image $T(\mathfrak{F}_e)$ under $T$ of the set $\mathfrak{F}_e$ of all fragments of $e$ is order bounded in $F$. The set of laterally-to-order bounded OAOs from $E$ to $F$ is denoted by $\mathcal{P}(E, F)$. The notion of laterally-to-order bounded OAOs was introduced and first studied by Pliev and Ramdane in [15].

Obviously, $\mathcal{U}(E, F) \subseteq \mathcal{P}(E, F) \subseteq \mathcal{O}(E, F)$.

An element $u \neq 0$ of a Riesz space $E$ is called

- An atom provided $0 \leq x \leq |u|$, $0 \leq y \leq |u|$ and $x \land y = 0$ imply that either $x = 0$ or $y = 0$;
- A weak atom, if $\mathfrak{F}_u = \{0, u\}$.

Every atom is a weak atom and if $E$ has the principal projection property then the converse assertion holds [9, Proposition 1.3].

An OAO $T : E \to X$ from a vector lattice $E$ to a Banach space $X$ is called narrow if for every $e \in E$ and every $\varepsilon > 0$ there is a decomposition $e = e' \sqcup e''$ such that

\(^1\) Component in the terminology of [2].
\[ \| T(e') - T(e'') \| < \varepsilon. \] Obviously, a narrow operator sends weak atoms to zero. Narrow linear operators to some extent generalize compact operators [17], and narrow OAOs naturally generalize narrow linear operators [13].

2 The largest subclass of all OAOs having the Riesz space structure

The first study of the Riesz space structure of the ordered vector space \( \mathcal{O}(E, F) \) of all OAOs between Riesz spaces \( E \) and \( F \) was presented by Mazón and Segura de León in [6] and [7]. One of the main structural results in [6] asserts that, if the Riesz space \( E \) is Dedekind complete then \( \mathcal{U}(E, F) \) is a Dedekind complete Riesz space as well, and some natural formulas for lattice operations on \( \mathcal{U}(E, F) \) were provided. In 2018 Pliev and Ramdane [15] generalized the above mentioned result of Mazón and Segura de León to the class \( \mathcal{P}(E, F) \) of laterally-to-order bounded OAOs:

Let \( E, F \) be Riesz spaces with \( F \) Dedekind complete. Then \( \mathcal{P}(E, F) \) is a Dedekind complete Riesz space and the following assertions hold.

1. For every \( S, T \in \mathcal{P}(E, F) \) and every \( x \in E \) one has
   
   (a) \( (S \lor T)(x) = \sup\{S(u) + T(v) : x = u \uplus v\} \);
   
   (b) \( (S \land T)(x) = \inf\{S(u) + T(v) : x = u \uplus v\} \);
   
   (c) \( T^+(x) = \sup\{T(u) : u \subseteq x\} \);
   
   (d) \( T^-(x) = -\inf\{T(u) : u \subseteq x\} \);
   
   (e) \( |T(x)| \leq |T|(x) \).

2. The set \( \mathcal{U}(E, F) \) is an order ideal of \( \mathcal{P}(E, F) \) and hence itself is a Dedekind complete Riesz space with properties (a)–(e).

It is interesting to note that formulas (a) and (b) coincide with Abramovich’s formulas for join and meet of linear operators, which are true under the additional assumption on \( E \) to have the principal projection property and false without this assumption [2, Theorem 1.50].

So the Riesz space \( \mathcal{P}(E, F) \) is a more natural object for the study of OAOs than \( \mathcal{U}(E, F) \). Remark that \( \mathcal{U}(E, F) \) need not be a band of \( \mathcal{P}(E, F) \) [15].

The linear subspace \( \mathcal{P}(E, F) \) of \( \mathcal{O}(E, F) \) is so “large” that these two ordered vector spaces have equal positive cones, that is, every positive OAO is laterally-to-order bounded, which is easy to see. However, the inclusion \( \mathcal{P}(E, F) \subset \mathcal{O}(E, F) \) can be strict due to [15, Example 3.4]:

Let \( \ell_0^\infty \) be the Riesz space of all real eventually constant sequences \( x = (x_n)_{n=1}^\infty \), that is, \( (\exists k \in \mathbb{N})(\forall n \geq k) x_n = x_k \) with the coordinate-wise order. Then the map \( T : \ell_0^\infty \to \mathbb{R} \) defined by setting \( T(x) = \sum_{n=1}^\infty \frac{(-1)^n x_n}{n} \), \( x \in \ell_0^\infty \), belongs to \( \mathcal{O}(\ell_0^\infty, \mathbb{R}) \setminus \mathcal{P}(\ell_0^\infty, \mathbb{R}) \).

Observe that the Riesz space \( \ell_0^\infty \) is not Dedekind complete, and the idea used in [15, Example 3.4] cannot help to construct an example of this kind with Dedekind complete domain space. We show that the inclusion \( \mathcal{P}(E, F) \subset \mathcal{O}(E, F) \) is strict for all cases except for some trivial ones.

The following result characterizes pairs of Riesz spaces \( (E, F) \) for which the inclusion \( \mathcal{P}(E, F) \subset \mathcal{O}(E, F) \) is strict.
Theorem 2.1 Let $E$, $F$ be Riesz spaces with $F$ Archimedean. Then the following assertions are equivalent.

1. $\mathcal{P}(E, F) = \mathcal{O}(E, F)$.
2. Either $F = \{0\}$ or for every $e \in E$ the set $\mathcal{F}_e$ is finite.

Proof (1) $\Rightarrow$ (2). Assuming the contrary, let $f \in F \setminus \{0\}$ and let $e$ be an element of $E$ with infinite $\mathcal{F}_e$. Our goal is to construct an OAO $T: E \to F$ which is not laterally-to-order bounded (it is going to be even linear).

First we construct recursively an infinite disjoint sequence $(e_n)_{n=1}^{\infty}$ of nonzero elements of $\mathcal{F}_e$. To do it, observe that, if $z \in E$ is such that the Boolean algebra $\mathcal{F}_z$ is infinite then the following assertions hold:

(i) $z$ is not a weak atom;
(ii) If $z = x \cup y$ then either $\mathcal{F}_x$ or $\mathcal{F}_y$ is an infinite Boolean algebra.

At the first step, following (i) we decompose $e = u \cup v$ with $u, v \neq 0$. Then by (ii), at least one of $\mathcal{F}_u$, $\mathcal{F}_v$ is an infinite Boolean algebra, say $\mathcal{F}_v$. Then we set $e_1 = u$. Fix any $k \in \mathbb{N}$. Assume $e_1, \ldots, e_k \in \mathcal{F}_e \setminus \{0\}$ are chosen so that $e_i \perp e_j$ for $i \neq j$ and the Boolean algebra $\mathcal{F}_e$ is infinite, where $z = e - \sum_{j=1}^{k} e_j$. By (i) decompose $z = x \cup y$ with $x, y \neq 0$. Then by (ii), at least one of $\mathcal{F}_x$, $\mathcal{F}_y$ is an infinite Boolean algebra, say $\mathcal{F}_y$. Then set $e_{k+1} = x$. Now we have that $(e_j)_{j=1}^{k+1}$ are disjoint elements of $\mathcal{F}_e \setminus \{0\}$ and the Boolean algebra $\mathcal{F}_y$ is infinite, where $y = e - \sum_{j=1}^{k+1} e_j$. By that, the recursive construction is finished and we obtain an infinite disjoint sequence $(e_n)_{n=1}^{\infty}$ of nonzero elements of $\mathcal{F}_e$.

Consequently, $(e_n)_{n=1}^{\infty}$ is a linearly independent system of elements of $E$. Extend $(e_n)_{n=1}^{\infty}$ to a Hamel basis $(e_i)_{i \in I}$, $\mathbb{N} \subseteq I$ in the linear space $E$. Then define a linear operator $T: E \to F$ using the Hamel basis as follows. First define $Te_n = nf$ for all $n \in \mathbb{N}$ and somehow define $T$ on the rest of the basis, say, $Te_i = 0$ for $i \in I \setminus \mathbb{N}$. Finally we extend $T$ from the basis to the entire $E$ by linearity, that is, we set

$$Tx = \sum_{i \in I} a_i Te_i = \sum_{n \in \mathbb{N}} a_n nf = \left(\sum_{n \in \mathbb{N}} a_n n\right) f$$

for all $x \in E$, where $\sum_{i \in I} a_i e_i \in E$ is the expansion of $x$ to the Hamel basis $(e_i)_{i \in I}$, which has finitely many nonzero summands.

It remains to observe that $T(\mathcal{F}_e)$ is not order bounded in $F$, because $nf \in T(\mathcal{F}_e)$ for all $n \in \mathbb{N}$ and $F$ is Archimedean.

(2) $\Rightarrow$ (1) is obvious. \hfill $\square$

So the inclusion $\mathcal{P}(E, F) \subset \mathcal{O}(E, F)$ is strict, except for somewhat trivial cases, including the following one.

Remark 1 The infinite dimensional Dedekind complete Riesz space $c_{00}$ of all eventually zero sequences possesses the second part of (2) in Theorem 2.1, and so $\mathcal{P}(c_{00}, F) = \mathcal{O}(c_{00}, F)$ for every Archimedean Riesz space $F$.
3 An analogue of Pitt’s theorem for orthogonally additive operators on atomless $L_p$-spaces

The classical Pitt’s theorem claims that if $1 \leq p < r$ then every linear bounded operator $T : \ell_r \to \ell_p$ is compact. An analogue of Pitt’s compactness theorem for narrow linear operators asserts that if $1 \leq p < 2$ and $p < r < \infty$ then every linear bounded operator $T : L_p \to L_r$ is narrow [3], [17, Theorem 9.7]. Moreover, the latter result is false for any other values of $p$, $r$ [17, p. 216].

We extend this result to absolutely norm bounded OAOs. On the other hand, we show that this is no longer true for order continuous OAOs.

3.1 Extension to absolutely norm bounded OAOs

First we provide necessary information on OAOs acting between normed lattices. Assume $E$ is a normed lattice and $F$ is a Dedekind complete Banach lattice. An OAO $T \in \mathcal{U}(E, F)$ is called absolutely norm bounded provided there is $M \in [0, +\infty)$ such that $\|T(x)\| \leq M\|x\|$ for every $x \in E$.

The set of all absolutely norm bounded abstract Uryson operators is denoted by $\mathcal{AB}(E, F)$ and endowed with the following nonnegative value

$$\|T\|_{ab} := \sup_{x \in E \setminus \{0\}} \frac{\|T(x)\|}{\|x\|},$$
calling the absolute norm. The set $\mathcal{AB}(E, F)$ is a normed lattice with respect to the absolute norm, which is a sublattice of $\mathcal{U}(E, F)$ [16, Theorem 3.2]. Remark that the normed lattice $\mathcal{AB}(E, F)$ need not be norm complete even for the class of AL-spaces [16, Example 3.3]. However, there is a bigger norm on a suitable sublattice $\mathcal{UB}(E, F)$ of $\mathcal{AB}(E, F)$, with respect to which $\mathcal{UB}(E, F)$ is a Dedekind complete Banach lattice (see [16] for details).

Recall that a Banach space $X$ is said to have infratype $q > 1$ if there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$,

$$\min_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k x_k \right\| \leq C \left( \sum_{k=1}^n \|x_k\|^q \right)^{1/q}.$$

The following theorem, which is the main result of the section, generalizes and modifies [17, Theorem 9.8] to the notion of a narrow operator between Riesz spaces.

**Theorem 3.1** Let $1 \leq p < 2$. Let $E$ be an atomless abstract $L_p$-space and $F$ be a Dedekind complete Banach lattice with infratype $q > p$. Then every absolutely norm bounded OAO $T : E \to F$ is narrow.

**Proof** By the Kakutani-Bohnenblust-Nakano Theorem [2, Theorem 4.27], we may and do assume that $E = L_p(\mu)$ for some (atomless) measure space $(\Omega, \Sigma, \mu)$. Fix any $T \in \mathcal{AB}(E, F)$ and $e \in F \setminus \{0\}$. Given any $n \in \mathbb{N}$, using the atomlessness of the measure $\nu : \mathcal{F}_e \to [0, \|e\|^{p} ]$ defined by setting
\[ v(x) = \|x\|^p = \int_{\Omega} |x|^p \, d\mu, \quad x \in \mathcal{F}, \]

find a decomposition \( e = e_1 \sqcup \ldots \sqcup e_n \) with \( \|e_i\| = n^{-1/p} \|e\| \). Then

\[
\min_{\theta_k = \pm 1} \left\| \sum_{k=1}^{n} \theta_k T(e_k) \right\| \leq C \left( \sum_{k=1}^{n} \|T(e_k)\|^q \right)^{\frac{1}{q}}
\]

\[
\leq C \left( \sum_{k=1}^{n} \|T(e_k)\|^q \right)^{\frac{1}{q}}
\]

\[
\leq C \|T\|_{\text{abs}} \left( \sum_{k=1}^{n} \|e_k\|^q \right)^{\frac{1}{q}}
\]

\[
= C \|T\|_{\text{abs}} n^{\frac{1}{q} - \frac{1}{r}} \|e\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Hence, for every \( \varepsilon > 0 \) one can find \( n \in \mathbb{N} \) and signs \( \theta_1, \ldots, \theta_n \) so that \( \| \sum_{k=1}^{n} \theta_k T(e_k) \| < \varepsilon \). Set \( J = \{ k \leq n : \theta_k = 1 \}, \) \( e' = \sum_{j \in J} e_j \) and \( e'' = e - e' \). Then \( e = e' \sqcup e'' \) and, by the orthogonal additivity of \( T \),

\[
\|T(e') - T(e'')\| = \left\| \sum_{k=1}^{n} \theta_k T(e_k) \right\| < \varepsilon.
\]

By [5], every \( L_q(\mu) \)-space with \( q > 1 \) has infratype \( \min\{q, 2\} \). Taking into account the Kakutani-Bohnenblust-Nakano Theorem, we obtain the following consequence.

**Corollary 3.2** Let \( 1 \leq p < 2 \) and \( p < r < \infty \). Let \( E \) be an atomless abstract \( L_p \)-space and \( F \) be an abstract \( L_r \)-space. Then every absolutely norm bounded OAO \( T : E \rightarrow F \) is narrow.

By the corresponding example of a linear operator [17, p. 216], Corollary 3.2 is false for any other values of \( p, r \).

### 3.2 A non-narrow order continuous OAO \( T : L_p \rightarrow L_r \) for any given \( p, r \in [0, +\infty) \)

The following example shows that Theorem 3.1 cannot be extended to the set of all order continuous abstract Uryson operators.

**Proposition 3.3** Let \( p, r \in [0, +\infty] \). Then there exists a non-narrow\(^2\) positive disjointness preserving order continuous OAO \( T : L_p \rightarrow L_r \).

\(^2\) Actually, every non-zero disjointness preserving OAO is non-narrow.
For the construction, we need ε-shading operator, which was introduced and studied in [10]. Let $E$ be a Riesz space, $0 < e \in E$ and $0 < \varepsilon < 1$. For every $x \in E$ set

$$Q^e_\varepsilon(x) = e - e \wedge \frac{1}{\varepsilon}|x - e|. \quad (3.1)$$

**Lemma 3.4** ([10, Lemma 3.1 and part of Theorem 3.2]) Let $E$ be a Riesz space, $0 < e \in E$ and $0 < \varepsilon < 1$. Then the map $Q^e_\varepsilon$ defined by (3.1) is a positive disjointness preserving order continuous OAO possessing the following properties.

(i) $(\forall x \in E) \ 0 \leq Q^e_\varepsilon(x) = Q^e_\varepsilon(x^+) \leq x^+ \wedge e.$

(ii) $(\forall \tilde{t} \in \mathfrak{F}_e) \ Q^e_\varepsilon(t) = \tilde{t};$

**Proof of Proposition 3.3** Let $1$ denote the function constantly equal to 1, which we consider as an element of both $L_p$ and $L_r$. Then define an operator $T : L_p \to L_r$ by setting for all $x \in L_p$

$$T(x) = Q^1_1(x) = 1 - 1 \wedge 2|x - 1|. \quad (3.2)$$

Since by (i) of Lemma 3.4 $0 \leq T(x) \leq 1$, we can consider $T(x)$ as an element of $L_r$. By Lemma 3.4, $T$ is a positive disjointness preserving OAO.

Show that $T$ is order continuous. Since $L_p$ has the countable sup property, the order continuity of $T$ can be verified on sequences only [2, p. 56]. Let $x_n \in L_p$ satisfy $x_n \xrightarrow{o} 0$. By Lemma 3.4, $T(x_n) \xrightarrow{o} 0$ in $L_p$, that is, $|T(x_n)| \leq u_n$, $n \in \mathbb{N}$ for some sequence $(u_n)_{n=1}^\infty$ in $L_p$ with $u_n \downarrow 0$. By (i) of Lemma 3.4, $|T(x_n)| \leq v_n$, $n \in \mathbb{N}$ and $v_n \downarrow 0$, where $v_n := 1 \wedge u_n$. Since $v_n \in L_r$, one has $T(x_n) \xrightarrow{o} 0$ in $L_r$ and thus $T$ is order continuous.

One can easily show that a disjointness preserving operator cannot be narrow. But we prove the non-narrowness of $T$ directly. Indeed, for any decomposition $1 = x \sqcup y$ one has by (ii) of Lemma 3.4, $\|T(x) - T(y)\| = \|x - y\| = 1$ and so $T$ is not narrow. \hfill \Box

### 4 Existence of a supremum $S \vee T$ for OAOs $S$ and $T$

In this section, we generalize Pliev and Ramdane’s Theorem [15] mentioned at the beginning of Sect. 2 and find sufficient condition on OAOs $S$ and $T$ to have $S \vee T$ as well as to have $S \wedge T$, and consequently, for an OAO $T$ to have $T^+$, $T^-$ or $|T|$ without any assumption on the domain and range spaces.

**Theorem 4.1** Let $E$, $F$ be Riesz spaces and $S, T \in \mathcal{O}(E, F)$. If for every $x \in E$ the supremum $R(x) := \sup\{S(u) + T(v) : x = u \sqcup v\}$ exists in $F$ then $S \vee T$ exists in $\mathcal{O}(E, F)$ and $S \vee T = R$. Moreover, in case of existence the following hold.

1. If $S, T \in \mathcal{P}(E, F)$ then $S \vee T \in \mathcal{P}(E, F)$.
2. If $S, T \in \mathcal{U}(E, F)$ then $S \vee T \in \mathcal{U}(E, F)$.
Proof Show that $R$ is an OAO. Fix any $x, y \in E$ with $x \perp y$. Let $x + y = u \cup v$ be any decomposition into disjoint fragments. According to Pliev’s Lemma [11, Proposition 3.11], there are disjoint vectors $w_i, i = 1, \ldots, 4$ so that $x = w_1 \cup w_3$, $y = w_2 \cup w_4$, $u = w_1 \cup w_2$ and $v = w_3 \cup w_4$ (in notations of the Boolean algebra $\mathcal{F}_e$ the vectors $w_1 = x \cap u$, $w_2 = y \cap u$, $w_3 = x \cap v$, and $w_4 = y \cap v$ work). Then

$$S(u) + T(v) = S(w_1) + S(w_2) + T(w_3) + T(w_4) \leq R(x) + R(y).$$

By the arbitrariness of a decomposition of $x + y$, we obtain $R(x + y) \leq R(x) + R(y)$. On the other hand,

$$R(x) + R(y) = \sup\{S(u) + T(v) : x = u \cup v\} + \sup\{S(w) + T(z) : y = w \cup z\}
= \sup\{S(u + w) + T(v + z) : x = u \cup v, y = w \cup z\}
\leq R(x + y).$$

Thus, $R$ is an OAO. Show that $R = S \vee T$ in $\mathcal{O}(E, F)$. Indeed, obviously $S \leq R$ and $T \leq R$. Let $P \in \mathcal{O}(E, F)$ satisfy $S \leq P$ and $T \leq P$. Fix any $x \in E$ and consider any decomposition $x = u \cup v$. Then $S(u) + T(v) \leq P(u) + P(v) = P(x)$. By the arbitrariness of the decomposition of $x$, $R(x) \leq P(x)$ and so $R \leq P$. Thus, $R = S \vee T$.

Items (1) and (2) easily follow from the obtained formula for $S \vee T$. □

Theorem 4.1 gives the following consequences for the existence of meet of two OAOs, positive and negative parts and modulus of an OAO.

**Corollary 4.2** Let $E, F$ be Riesz spaces and $S, T \in \mathcal{O}(E, F)$. If for every $x \in E$ the infimum $R(x) := \inf\{S(u) + T(v) : x = u \cup v\}$ exists in $F$ then $S \wedge T$ exists in $\mathcal{O}(E, F)$ and $S \wedge T = R$. Moreover, in case of existence the following hold.

(1) If $S, T \in \mathcal{P}(E, F)$ then $S \wedge T \in \mathcal{P}(E, F)$.
(2) If $S, T \in \mathcal{U}(E, F)$ then $S \wedge T \in \mathcal{U}(E, F)$.

Proof For every $x \in E$ one has

$$(S \wedge T)(x) = -(\neg S) \vee (-T))(x) = -\sup\{-S(u) - T(v) : x = u \cup v\}
= \inf\{S(u) + T(v) : x = u \cup v\}.$$ 

Items (1) and (2) follow from the obtained formula for $S \wedge T$. □

Next three corollaries directly follow from Theorem 4.1.

**Corollary 4.3** Let $E, F$ be Riesz spaces and $T \in \mathcal{O}(E, F)$. If for every $x \in E$ the supremum $R(x) := \sup\{T(u) : u \subseteq x\}$ exists in $F$ then $T^+$ exists in $\mathcal{O}(E, F)$ and $T^+ = R$. Moreover, in case of existence the following hold.

(1) If $T \in \mathcal{P}(E, F)$ then $T^+ \in \mathcal{P}(E, F)$.
(2) If $T \in \mathcal{U}(E, F)$ then $T^+ \in \mathcal{U}(E, F)$.
Corollary 4.4  Let $E, F$ be Riesz spaces and $T \in \mathcal{O}(E, F)$. If for every $x \in E$ the infimum $-R(x) := \inf \{T(u): u \subseteq x\}$ exists in $F$ then $T^- \in \mathcal{O}(E, F)$ and $T^- \equiv R$. Moreover, in case of existence the following hold.

(1) If $T \in \mathcal{P}(E, F)$ then $T^- \in \mathcal{P}(E, F)$.

(2) If $T \in \mathcal{U}(E, F)$ then $T^- \in \mathcal{U}(E, F)$.

Corollary 4.5  Let $E, F$ be Riesz spaces and $T \in \mathcal{O}(E, F)$. If for every $x \in E$ the supremum $R(x) := \sup \{T(u) - T(v): x = u \cup v\}$ exists in $F$ then $|T|$ exists in $\mathcal{O}(E, F)$ and $|T| = R$. Moreover, in case of existence the following hold.

(1) If $T \in \mathcal{P}(E, F)$ then $|T| \in \mathcal{P}(E, F)$.

(2) If $T \in \mathcal{U}(E, F)$ then $|T| \in \mathcal{U}(E, F)$.

Remark that, as we promised above, Theorem 4.1 together with all its corollaries entirely yield Pliev-Ramdane Theorem [15].

5 The existence of modules of a disjointness preserving OAO

As usual, $\mathcal{L}_b(E, F)$ denotes the ordered vector space of all order bounded linear operators between Riesz spaces $E$ and $F$, which is a Riesz space once $F$ is Dedekind complete [2, Theorem 1.18]. Recall that an OAO $T: E \to F$ is said to preserve disjointness provided $T(x) \perp T(y)$ for all $x, y \in E$ with $x \perp y$.

In this section, we show that a kind of Meyer’s theorem [2, Theorem 2.40] on the existence of a modulus $|T|$ of a disjointness preserving linear operator $T$ from a Riesz space to an Archimedean Riesz space holds true for OAOs.

We also prove an analogue of Meyer’s lemma on disjointness preserving operators [2, Lemma 2.39] for the setting of OAOs.

The proof of the next theorem, which is an analogue of Meyer’s theorem for OAOs, is based on the results of the previous section and is much simpler than the proof of the original Meyer’s theorem for linear operators. Although OAOs generalize linear operators, we cannot consider our theorem as a generalization of Meyer’s theorem for linear operators, because spaces of linear operators and OAOs have different orders.

Theorem 5.1  Let $E, F$ be Riesz spaces and let $T \in \mathcal{O}(E, F)$ preserve disjointness. Then the following assertions hold.

(1) $T \in \mathcal{P}(E, F)$.

(2) $|T|$ exists in $\mathcal{O}(E, F)$ and $|T| \in \mathcal{P}(E, F)$. Moreover, for every $x \in E$ one has $|T|(x) = |T(x)|$.

(3) $T$ is regular; $T^+$ and $T^-$ exist and both preserve disjointness. Moreover, for every $x \in E$ one has $T^+(x) = (T(x))^+$ and $T^-(x) = (T(x))^-$.

(4) Let $\{x, y\} \subset E$ be a laterally bounded set. Then $(T(x))^+ \land (T(y))^- = 0$.

Remark that Theorem 5.1 was partially proved in [12, Lemma 2.2] under the additional assumptions of Dedekind completeness of $F$, and formulas in (2) and (3) for the modulus, positive and negative parts were discovered earlier for an order bounded $T$ in [1, Lemma 3.1]. Item (4) is an analogue of Meyer’s lemma (see Lemma 5.4 below).

For the proof of Theorem 5.1, we need the following two known simple facts.
Lemma 5.2 ([8]) Let $E$, $F$ be Riesz spaces and $T : E \to F$ a disjointness preserving OAO. Then for every $e \in E$ and every $x \sqsubseteq e$ one has $T(x) \sqsubseteq T(e)$.

The proof is straightforward and easy. Note that the converse implication also holds: every lateral order preserving OAO preserves disjointness [8, Theorem 4.9].

Lemma 5.3 ([8]) Let $E$ be a Riesz spaces and $x$, $y \in E$.

1) If $x \sqsubseteq y$ then $x^+ \sqsubseteq y^+$, $x^- \sqsubseteq y^-$, $|x| \sqsubseteq |y|$.
2) If $|x| \sqsubseteq |y|$ then $|x| \leq |y|$.

See [8, Proposition 3.1] for details about Lemma 5.3.

Proof of Theorem 5.1 (1) follows from Lemma 5.2.

2) By Corollary 4.5, for the existence of $|T|$ it is enough to prove that, for every $x \in E$ the supremum $\sup D_x$ exists in $F$, where $D_x := \{T(u) - T(v) : x = u \sqcup v\}$. Fix any $x \in E$. Then for every $t \in D_x$, say, $t = T(u) - T(v)$, where $x = u \sqcup v$, since $T$ preserves disjointness, one has by Lemma 5.2

$$|t| = |T(u)| \cup |T(v)| \overset{\text{Lemma 5.3}}{\subseteq} |T(x)|.$$

By Lemma 5.3 (2), $|t| \leq |T(x)|$. On the other hand, $T(x) \in D_x$ and $-T(x) \in D_x$. By the above, $\sup D_x = |T(x)|$. By Corollary 4.5, $|T|$ exists and $|T|(x) = |T(x)|$ for all $x \in E$. The fact that $|T| \in \mathcal{P}(E, F)$ easily follows from the formula for $|T|$, see Corollary 4.5.

3) Since $T(x) = (T(x))^+ - (T(x))^-$, to prove the regularity of $T$, it is enough to show that the functions $\phi, \psi : E \to F$ defined by $\phi(x) = (T(x))^+$ and $\psi(x) = (T(x))^-$ for all $x \in E$ are OAOs. Indeed, fix any $x$, $y \in E$ with $x \perp y$. Then

$$\phi(x + y) = (T(x \sqcup y))^+ = (T(x) \sqcup T(y))^+ = (T(x))^+ \cup (T(y))^+ = \phi(x) \sqcup \phi(y).$$

So, $\phi$ is a disjointness preserving OAO. Similarly, $\psi$ is. It is left to show that $\phi = T \lor 0$ and $\psi = (-T) \lor 0$. Obviously, $T \leq \phi$ and $0 \leq \phi$. Let $S \in \mathcal{O}(E, F)$ satisfy $T \leq S$ and $0 \leq S$. Then for every $x \in E$ one has $\phi(x) = (T(x))^+ \lor 0 \leq S(x)$, that is, $\phi \leq S$ and so $T^+ = \phi$. Analogously, $T^- = \psi$.

4) Let $\{x, y\}$ be laterally bounded by $e \in E$, that is, $x \sqsubseteq e$ and $y \sqsubseteq e$. By Lemma 5.2, $T(x) \sqsubseteq T(e)$ and $T(y) \sqsubseteq T(e)$. By Lemma 5.3, $(T(x))^+ \sqsubseteq (T(e))^+$ and $(T(y))^\leq (T(e))^\leq$ and therefore $(T(x))^+ \leq (T(e))^+$ and $(T(y))^\leq (T(e))^\leq$. Taking into account that $(T(e))^+ \wedge (T(e))^\leq = 0$, we obtain that $(T(x))^+ \wedge (T(y))^\leq = 0$.

Now we come back to the following Meyer lemma, which precedes Meyer’s theorem for linear operators.
Lemma 5.4 ([2, Lemma 2.39]) Let $E, F$ be Riesz spaces with $F$ Archimedean. Then for every disjointness preserving linear operator $T \in \mathcal{L}_b(E, F)$ and every $x, y \in E^+$ one has $(Tx)^+ \wedge (Ty)^- = 0$.

The above Meyer’s lemma is not true for OAOs due to the following simple example.

Example 5.5 Define a function $T : C[0, 1] \to C[0, 1]$ by setting $1 = 1_{[0, 1]}$ and

$$T(x) = \begin{cases} 1, & \text{if } x = 1, \\ -1, & \text{if } x = 2 \cdot 1, \\ 0, & \text{if } x \in C[0, 1] \setminus \{1, 2 \cdot 1\}. \end{cases}$$

Then $T$ is a disjointness preserving OAO and $(T(1))^+ \wedge (T(2 \cdot 1))^- = 1 \neq 0$.

There are also examples of the kind of OAOs acting between Dedekind complete Riesz spaces, however somewhat involved. For instance, define a map $S : L_p \to L_p$, $0 \leq p \leq \infty$ by setting $S(x) = x \ominus 1 - x \cap (2 \cdot 1)$. By [4, Theorem 4], $S$ is an OAO. One can easily show that $|Sx| \leq |x|$ for all $x \in L_p$, and hence $S$ preserves disjointness. Moreover, $(S(1))^+ \wedge (S(2 \cdot 1))^- = 1 \wedge (2 \cdot 1) = 1 \neq 0$.

Such examples exist, because an OAO may have independent behavior on collinear vectors (see Proposition 4.14 and Theorem 6.1 of [16] for examples of OAOs with independent behavior on fragments of different elements of a Riesz space). So Meyer’s lemma for OAOs is true for laterally bounded elements only, see Theorem 5.1(4).

6 Remarks and open problems

We do not know whether the sufficient condition in Theorem 4.1 for the existence of $S \vee T$ is necessary for the case where the range space is not Dedekind complete.

Problem 6.1 Let $E, F$ be Riesz spaces and $S, T \in \mathcal{O}(E, F)$. Assume that $S \wedge T$ exists in $\mathcal{O}(E, F)$. Does for every $x \in E$ there exists $\sup\{S(u) + T(v) : x = u \uplus v\}$ in $F$?

Let $S \wedge T$ exist in $\mathcal{O}(E, F)$, and let $\hat{F}$ be a Dedekind completion of $F$ so that $F$ is an order dense ideal of $\hat{F}$. Let $J : F \to \hat{F}$ be the inclusion embedding. For every $P \in \mathcal{O}(E, F)$ we define an operator $\hat{P} \in \mathcal{O}(E, \hat{F})$ by setting $\hat{P} = J \circ P$. Then the set $\hat{S}(u) + \hat{T}(v) : x = u \uplus v$ is order bounded in $\hat{F}$ by $S \wedge T(x)$ and by the Dedekind completeness of $\hat{F}$. $R_1(x) := \sup\{\hat{S}(u) + \hat{T}(v) : x = u \uplus v\}$ exists in $\hat{F}$. Then by Theorem 4.1, $\hat{S} \wedge \hat{T}$ exists in $\mathcal{O}(E, \hat{F})$ and equals $R_1$. Moreover, it is easily seen that $R_1 \leq S \wedge T$. One can show that, if $R_1 = S \wedge T$ then $\sup\{S(u) + T(v) : x = u \uplus v\}$ exists in $F$ for all $x \in E$, and the answer to Problem 6.1 is affirmative.

Problem 6.2 Is $R_1 = S \wedge T$ true in all cases?

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