Non-locality and time-dependent boundary conditions: 
a Klein-Gordon perspective

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The dynamics of a particle in an expanding cavity is investigated in the Klein-Gordon framework in a regime in which the single particle picture remains valid. The cavity expansion represents a time-dependent boundary condition for the relativistic wavefunction. We show that this expansion induces a non-local effect on the current density throughout the cavity. Our results indicate that a relativistic treatment still contains apparently spurious effects traditionally associated with the unbounded velocities inherent to non-relativistic solutions obtained from the Schrödinger equation. Possible reasons for this behaviour are discussed.

I. INTRODUCTION

Every prediction we can make in a quantum system is encoded in the quantum state, which is generally not localized, but extended over all the available space. The quantum state evolves deterministically by way of an evolution equation: the Schrödinger equation in non-relativistic quantum mechanics, the Klein-Gordon or Dirac equations in single particle relativistic quantum mechanics. Because the quantum state is extended all over space, it has features that link together instantaneously properties in different regions of space. However this non-locality cannot be used to communicate. If signaling were possible, we could communicate with the future and modify the past. It turns out however that the Schrödinger equation displays in some cases signaling [1, 2]. This is not necessarily worrying because the Schrödinger equation being non-relativistic there is no bound on the velocity associated with its solutions. But sometimes it is not obvious to assert whether this form of signaling is an artifact (employing a non-relativistic equation), or is a genuine feature of quantum non-locality.

A concrete example was investigated in [3] where the authors considered the case of a non-relativistic particle placed in a one-dimensional cavity whose right wall can be set in uniform motion. This system – an infinite well with a moving wall – has often been suspected of displaying some form of single-particle non-locality [4–7] and has very recently been found to be relevant in atomic [8, 9] or neutron [10] spectroscopy. Assume a particle is prepared in an energy eigenstate of a static cavity of length $L_0$. Then, at $t = 0$, the right wall starts to expand. This induces a current change $\Delta j(t, x)$ inside the cavity. As shown in [3], this change in the current can in principle be measured by performing a weak measurement (as will be recalled below). The current change was found to be non-local, in the sense that $\Delta j(t, \epsilon) \neq 0$ for $t < \frac{L_0 - \epsilon}{c}$ and $x = \epsilon$ a position near the left wall. This can lead to a protocol based on monitoring the current by which it is possible to send information faster-than-light. However the results reported in [3] were obtained with solutions of the Schrödinger equation. Indeed, solutions of the Schrödinger equation with a moving wall have been known analytically for some time for specific wall motions [11, 12]. But since the Schrödinger equation is non-relativistic, any expansion over the energy eigenstates will formally include states associated with arbitrarily high energies, corresponding to supraluminal velocities. The question is therefore whether the supraluminal results constitute an artifact of the description of the system by the Schrödinger equation.

The goal of the present article is precisely to answer this question by investigating the same system in a relativistic setting based on the Klein-Gordon (KG) equation. We will rely on the solutions of the KG equation for a particle inside a uniformly expanding cavity that were recently obtained [13–15]. As is well-known, the KG equation is not free of interpretation problems, but we will consider a regime in which these problems do not appear.

The structure of the paper is as follows. We first recall the solutions for the KG equation with a boundary condition expanding linearly in time that were obtained recently [13–15]. We also derive expressions for the current density. We show that the relativistic wavefunctions and current density reduce to the non-relativistic solutions used in [3] in the limit where the wall is slowly expanding and the length of the cavity is large with respect to the Compton wavelength of the particle. We then compare the current change as a function of time when the outer wall moves or remains fixed. We will see that this current change appears instantaneously inside the cavity. We discuss these results and present our conclusions concerning single-particle non-locality in the relativistic case.
II. KG EQUATION WITH A MOVING BOUNDARY

A. Analytical solutions for a KG particle inside a linearly expanding cavity

A KG particle of mass $m$ is initially trapped in a fixed infinite well (the potential is equal to zero for $x \in [0, L_0]$ and to $+\infty$ elsewhere). The solutions of the KG equation

$$\frac{\partial^2 \Phi}{c^2} - \partial_x^2 \Phi + \frac{m^2 c^2}{h^2} \Phi = 0$$

are given by

$$\Phi_{\pm,n}(t, x) = \frac{1}{\sqrt{2E_n L_0}} e^{i \pm \frac{E_n t}{\hbar}} \sin(\frac{p_n x}{\hbar})$$

with $n = 1, 2, 3, \ldots$, $p_n = \frac{2n\pi}{L_0}$ and $E_n = \sqrt{m^2 c^4 + p_n^2 c^2}$. The +/- index refers to the signs in the exponent, opposite to the sign of the energies (+/- refers to anti-particles/particles respectively). These solutions are orthonormal with respect to the KG scalar product

$$(\Phi, \Xi)_{KG} = \int dx (\Phi^* \hbar \partial_t \Xi - i \hbar (\partial_x \Phi^*) \Xi),$$

the KG probability density for a given state $\Phi$ being defined as

$$\rho_{KG}(t, x) = (\Phi^* i \hbar \partial_t \Phi - i \hbar (\partial_x \Phi^*) \Phi).$$

At $t = 0$, the right wall starts to move with constant velocity $v = \beta c$ and its position at time $t$ is therefore given by $L = L_0 + vt$ (the left wall remains fixed). The analytical solutions of this problem (a KG particle in an infinite square-well potential with a linearly expanding wall) were obtained by Koehn (see eq. 11 in [13]); other authors proposed a generalization shortly after [14], and gave an alternative method in [15]. These solutions can be written as linear superpositions of

$$\Psi_{J,n} = N_{J,n} J_{ik_n} \left( \frac{\sqrt{L^2 - \beta^2 x^2}}{\lambda_C \beta} \right) \sin \left( \frac{k_n}{2} \ln \left( \frac{L + \beta x}{L - \beta x} \right) \right)$$

and

$$\Psi_{Y,n} = N_{Y,n} Y_{ik_n} \left( \frac{\sqrt{L^2 - \beta^2 x^2}}{\lambda_C \beta} \right) \sin \left( \frac{k_n}{2} \ln \left( \frac{L + \beta x}{L - \beta x} \right) \right)$$

where $\lambda_C = \frac{A}{mc}$ (the Compton wavelength of the KG particle), $k_n = \frac{2n\pi}{L_0 + vt}$ with $n = 1, 2, 3, \ldots$, $N_{J,n}$ and $N_{Y,n}$ being normalization constants. $J$ and $Y$ are Bessel functions respectively regular and irregular at the origin [18]. Alternatively, one can use the basis of solutions

$$\Psi_{+,n} = N_{+,n} J_{ik_n} \left( \frac{\sqrt{L^2 - \beta^2 x^2}}{\lambda_C \beta} \right) \sin \left( \frac{k_n}{2} \ln \left( \frac{L + \beta x}{L - \beta x} \right) \right)$$

with $\Psi_{+,n} = \Psi_{J,n}$.

B. The non-relativistic limit

We will now assume $\beta \ll 1$ and $\beta x \ll L$. Indeed typically (in particular if we have experiments in mind) the wall motion will be non-relativistic, and recall, as mentioned in the Introduction, that we will be interested in values of the current density near the fixed wall. Thus the orders of the Bessel functions of interest will be very large in magnitude, with $k_n \approx \frac{\pi n}{L}$, and the sine part of the above solutions can be approximated as

$$\sin \left( \frac{k_n}{2} \ln \left( \frac{L + \beta x}{L - \beta x} \right) \right) \approx \sin \left( \frac{k_n \beta x}{L} \right) \approx \sin(n \pi x \beta) .$$

To keep the expressions short in the following, we will use

$$\phi_n = \frac{k_n}{2} \ln \left( \frac{L + \beta x}{L - \beta x} \right).$$

The argument of the Bessel functions, denoted $z$ for short,

$$z = \frac{\sqrt{L^2 - \beta^2 x^2}}{\lambda_C \beta},$$

can also be very large (firstly $\beta$ is small, secondly $L_0 \gg \lambda_C$). Therefore we will consider the approximation for the Bessel functions of imaginary order in the limit where the argument is large and positive.

For that purpose, the functions $J_\nu(z)$ and $Y_\nu(z)$ (with real $\nu$ and positive $z$) are introduced (see [16] section 3 for a starting point, then [17] and [18]) as

$$\tilde{J}_\nu = \text{sech}(\frac{\pi \nu}{2}) \Re(J_\nu), \quad \tilde{Y}_\nu = \text{sech}(\frac{\pi \nu}{2}) \Im(Y_\nu).$$

Thanks to the relations

$$\Re(Y_\nu) = \coth(\frac{\pi \nu}{2}) \Re(J_\nu), \quad \Im(Y_\nu) = -\tanh(\frac{\pi \nu}{2}) \Re(J_\nu),$$

the functions $J$ and $Y$ can be expressed as

$$J_\nu(z) = \cosh(\frac{\pi \nu}{2}) \tilde{J}_\nu(z) + i \sinh(\frac{\pi \nu}{2}) \tilde{Y}_\nu(z)$$

and

$$Y_\nu(z) = \cosh(\frac{\pi \nu}{2}) \tilde{Y}_\nu(z) - i \sinh(\frac{\pi \nu}{2}) \tilde{J}_\nu(z).$$

Employing the asymptotic expansions given in [17] (see eqs. (3.17) and (3.18) of [17]; details are given in the
\[ J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \left( \cos \left( z - \frac{\pi}{4} \right) + \frac{4 \nu^2 + 1}{8z} \sin \left( z - \frac{\pi}{4} \right) \right), \]  

\[ \tilde{J}_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \left( \sin \left( z - \frac{\pi}{4} \right) - \frac{4 \nu^2 + 1}{8z} \cos \left( z - \frac{\pi}{4} \right) \right). \]  

Using these forms in Eq. (14), we obtain the following expressions

\[ J_{i\nu}(z) \approx \cosh \left( \frac{\pi}{2} \nu \right) \sqrt{\frac{2}{\pi z}} \left( e^{i(z - \frac{\pi}{4})} \left[ 1 - \frac{4 \nu^2 + 1}{8z} \right] \right), \]  

\[ \tilde{J}_{i\nu}(z) \approx \cosh \left( \frac{\pi}{2} \nu \right) \sqrt{\frac{2}{\pi z}} \left( e^{i(z - \frac{4\nu^2 + 1}{8z} - \frac{\pi}{4})} \right), \]  

and the analytical solutions (Eqs. (8) and (7)) become

\[ \Psi_{+,n} \approx N_{+,n} \cosh \left( \frac{\pi}{2} k_n \right) \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{4k_n^2 + 1}{8z} - \frac{\pi}{4})} \sin(\phi_n) \]  

and

\[ \Psi_{-,n} \approx N_{-,n} \cosh \left( \frac{\pi}{2} k_n \right) \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{4k_n^2 + 1}{8z} - \frac{\pi}{4})} \sin(\phi_n). \]  

Now let us show that the last expression (20), corresponding to particles (positive energies) indeed reduces to the solution of the Schrödinger equation given by

\[ \psi_n(t, x) = \sqrt{\frac{2}{L}} \exp \left( -i \frac{\pi}{2} \frac{h^2 n^2 t}{2mL_0 \varepsilon} \right) \exp \left( \frac{i m v x^2}{2 \hbar \varepsilon} \right). \]  

First we note that the factor \( \sqrt{\frac{2}{\pi z}} \approx \sqrt{\frac{2}{\pi L}} \) when \( \beta \ll 1 \) (justifying the factor \( \sqrt{\frac{2}{L}} \) in (21)). Secondly we have already pointed out that \( \sin(\phi_n) \approx \sin(\frac{n\pi x}{L}) \) (hence the presence of \( \sin(\frac{n\pi x}{L}) \) in (21)). Finally the imaginary exponential term \( z - \frac{4k_n^2 + 1}{8z} \) becomes in that limit \( \frac{mc^2}{\hbar^2} - \frac{m^2 v^2 + h^2 n^2 L_0^2}{2mL_0 L} \) (the proof is given in the Appendix).

**C. Normalization**

We want to compute the KG density (3) for the state

\[ \Psi_{-,n}(t, x) = N_{-,n} J_{-ik_n}(z) \sin(\phi_n). \]  

We first note that the following relation

\[ \frac{dJ_{-i\nu}(z)}{dz} = \frac{1}{2} (J_{-i\nu-1}(z) - J_{-i\nu+1}(z)) \]  

holds although the orders of the Bessel functions are imaginary. Then, from the definition of the density (1), the previous relation and the following ones

\[ \partial_t z = \frac{c}{\lambda C \sqrt{L^2 - \beta^2 x^2}} = \frac{cL}{\beta \lambda C^2 z}, \]  

\[ \partial_t \phi_n = -k_n \beta v x \frac{L^2 - \beta^2 x^2}{\lambda C^2 z^2}, \]  

obtained from (11) and (10), we find the density

\[ \rho_{KG}(t, x) = N_{n, n, h} \sin^2(\phi_n) \frac{\partial z}{\partial t} \left[ 2 \Re (J_{-ik_n}(z)) \frac{dJ_{+ik_n}(z)}{dz} \right]. \]  

In the non-relativistic limit, the above expression can be further approximated to

\[ \rho_{KG}(t, x) = N_{n, n, h} \sin^2(\phi_n) \left( \frac{cL}{\beta \lambda C^2} \right) \left[ \frac{4}{\pi z} \left( \frac{1}{\cosh \frac{\pi k_n}{2}} \right)^2 \right]. \]  

Using \( z \approx \beta^{-1} \lambda C^{-1} L \), we find that

\[ \int dx \rho_{KG}(t, x) = N_{n, n, h} \left( \frac{2\hbar c \beta}{\pi} \right) \left( \frac{1}{\cosh \frac{\pi k_n}{2}} \right)^2. \]  

Therefore, if we introduce the \( C_{\pm,n} \) thanks to \( N_{\pm,n} = C_{\pm,n} \int \frac{\pi}{2\beta \hbar c} \cosh(\frac{\pi k_n}{2}) \), the normalized states will read

\[ \Psi_{\pm,n}(t, x) = C_{\pm,n} \sqrt{\frac{\pi}{2\beta \hbar c} \cosh(\frac{\pi k_n}{2})} J_{\pm ik_n}(z) \sin(\phi_n), \]  

and we will have that \( C_{\pm,n} \approx 1 \) in the non-relativistic limit.

**D. Current density**

The KG current density for a given state \( \Phi \) is defined as

\[ j_{KG}(t, x) = -\hbar c^2 (\Phi^* \partial_x \Phi - i(\partial_t \Phi^* \Phi)). \]  

Its computation for the state \( \Psi_{-,n}(t, x) \) is not much different from that of the KG probability density. First we have that

\[ \partial_x z = -\frac{1}{\lambda C \sqrt{L^2 - \beta^2 x^2}} = -\frac{x}{\lambda C^2 z}, \]  

\[ \partial_x \phi_n = \frac{k_n \beta L}{L^2 - \beta^2 x^2} = \frac{k_n}{\beta \lambda C^2 z^2}. \]  

Next, thanks to (23), (28) and (30), (29) becomes

\[ -C_{n, n} \frac{\pi}{2\beta \hbar c} \cosh(\frac{\pi k_n}{2}) \times \frac{1}{\sqrt{\frac{\pi}{2}}} \]  

\[ e^{2\hbar \sin^2(\phi_n) \frac{\partial z}{\partial x}} \left[ 2 \Re (J_{-ik_n}(z)) \frac{dJ_{+ik_n}(z)}{dz} \right]. \]  

In the non-relativistic limit, the last expression becomes

\[ j_{KG}(t, x) \approx 2C_{n, n} x \beta c \frac{L^2 \sin^2(\frac{n\pi x}{L})}{L}, \]  

which is the expression found in [3], provided \( C_{n, n} \to 1. \)
III. NON-LOCALITY IN THE KG EXPANDING CAVITY

A. Current change

We examine here the effect of the expanding wall on the current density at the opposite side of the cavity. The most straightforward manner to proceed would be to prepare a KG particle in a positive energy eigenstate of the fixed walls cavity given by Eq. (2), say $\Phi_{-1}(t,x)$ for $t \leq 0$. The corresponding current is equal to zero everywhere for negative times. That state, at $t = 0$, can be decomposed along the basis given by the $\Psi_{-n}$ (see Eq. (7)): 

$$\Phi_{-1}(t = 0, x) = \sum_n a_n \Psi_{-n}(t = 0, x).$$  \hspace{1cm} (33)

If the wall is indeed set in motion for $t > 0$, the solution of the KG equation would be 

$$\Psi(t, x) = \sum_n a_n \Psi_{-n}(t, x)$$  \hspace{1cm} (34)

and the current would change inside the cavity. This method however is computationally demanding.

To avoid this issue, we prepare instead the KG particle in the state $\Psi_{-1}(t = 0, x)$, for which we know the later current if the well is expanding. We then compare this current ($j_\varepsilon$) to the one ($j_s$) corresponding to the situation in which the well is static. $j_s$ is obtained by expanding $\Psi_{-1}(0, x)$ on the energy eigenstates of the fixed-walls cavity

$$\Psi_{-1}(t = 0, x) = \sum_n c_n \Phi_{-n}(t = 0, x)$$  \hspace{1cm} (35)

where

$$c_n = \langle \Phi_{-n}, \Psi_{-1} \rangle_{KG}$$

$$= \int dx (\Phi_{-n}^* \partial_t \Psi_{-1} + E_n \Phi_{-n}^* \Psi_{-1}).$$  \hspace{1cm} (36)

At later times, the wavefunction is given by $\Psi_{-1}(t, x)$ when the wall is expanding and by $\sum_n c_n \Phi_{-n}(t, x)$ for a stationary cavity. The resulting density currents $j_\varepsilon$ and $j_s$ will therefore be different. Note that since we are in a non-relativistic regime, the anti-particles contribution are not expected to be significant. The next aspect to discuss is the choice of parameters and some related numerical aspects.

1 In order to obtain the $a_n$, we would need to compute the KG scalar products between $\Phi_{-1}$ and $\Psi_{-n}$ for every $n < n_{max}$, where $n_{max}$ is the dimension of the reduced basis used for the numerical simulation, and the computation of the Bessel functions to high precision is time consuming.

B. Numerical aspects

To avoid any problem of interpretation of the Klein-Gordon equation, we work in the non-relativistic regime characterized by

$$\beta \ll 1 \text{ and } z \approx \frac{L_0}{\lambda_C} \gg 1.$$  \hspace{1cm} (37)

As explained in the Appendix, we cannot numerically use the approximations for the Bessel functions but need instead to rely on the exact forms. Since their values grow like $\cosh(\nu z) \propto e^{\nu z}$ with $\nu \approx n\pi/\beta$, $\beta$ cannot be too small in order to keep the computations tractable. For this reason, we will set $\beta = 0.01$, which might be somewhat higher than a typical non-relativistic case but still abides by $\beta \ll 1$. We also need $z$ to be large, implying $L_0/(\beta \lambda_C)$ must be large.

C. Example

We consider the case

$$m = 10^{-30} \text{ kg}, L_0 = 10^{-6} \text{ m}, \beta = 0.01, \text{ hence } z \approx 10^8.$$  \hspace{1cm} (38)

We assume that the initial state is $\Psi_{-1}(t = 0, x)$. It can be expanded over the static cavity eigenstates as per Eq. (30). We can check that, as mentioned above, the anti-particles do not contribute: the scalar products $b_n = \langle \Phi_{+n}, \Psi_{-1} \rangle_{KG}$ are negligible (we have found that $|b_n| < 10^{-14}$ for all $n < 15000$). In Fig. 1 we plot the norm of the expansion coefficients $c_n$.

The presence probability density for $\Psi_{-1}(0, x)$ is almost identical to the one for $\Phi_{-1}(0, x)$ but contrary to $\Phi_{-1}(0, x)$, $\Psi_{-1}(0, x)$ has internal oscillations depending on the system mass and cavity properties (see Fig. 2).

Then we compare both currents in a space-time region which is spacelike separated from the event $(0, L_0)$. This illustrated in Fig. 3 at $t = 10^{-15}$ where the difference $j_s(t = 10^{-15}, x) - j_\varepsilon(t = 10^{-15}, x)$ is plotted in a small

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The norm of the expansion coefficients $c_n$ (Eq. (36)) as a function of $n$. The maximum value is reached at $n = 4551$. We have that $|c_{9000}/c_{4551}| = 0.0312$, $|c_{10000}/c_{4551}| = 2.6148 \times 10^{-5}$ and $|c_{15000}/c_{4551}| = 5.7616 \times 10^{-7}$.}
\end{figure}
FIG. 3. Difference between the currents $j_s$ and $j_e$ at $t = 10^{-15}$ s in a region $[0, 10^{-8}$ m] causally disconnected from the event $(0, L_0)$.

region representing the leftmost $10^{-8}/10^{-6}$ fraction of the cavity near the fixed wall. A light signal emitted from the right wall would take at least $(10^{-8} - 10^{-8})/c = 3.3 \times 10^{-15}$ seconds to reach this region.

For the figure, we have varied the number of steps for the numerical integration needed for the computation of the $c_n$ (the largest number of steps being 200000) and $n_{max}$ (we have used 10000 and 15000); all runs gave almost identical curves. The relative difference between the currents $j_e$ and $j_s$ at $x = 10^{-8}$ m is of the order of 2% while the relative difference between the box lengths at $t = 0$ and $t = 10^{-15}$ s is of the order of 0.3%.

IV. DISCUSSION

The present results confirm the non-local character of the quantum state in a relativistic setting: a local change in the potential (here in the region near the right wall) instantaneously affects the current density in a space-like separated region (near the left wall). As argued in Ref. [3], this effect – if it is physical, as we discuss below – could in principle be used to communicate supraluminally. Indeed, the protocol employed in [3], based on weak measurements, remains essentially the same in the present case: a weak unitary interaction coupling the momentum of the particle in the cavity to a pointer takes place near $x \approx 0$. This unitary interaction is immediately followed by a measurement of the position at the same point. If the position measurement succeeds, the pointer has shifted by a quantity proportional to the real part of the weak value of the momentum $P^w$, defined by

$$P^w = \frac{m j_{\psi}(t, x)}{|\psi(t, x)|^2} - i\hbar \frac{\partial_x |\psi(t, x)|^2}{2|\psi(t, x)|^2}. \quad (39)$$

This definition of the weak value is characteristic of the non-relativistic formalism and cannot be extended straightforwardly to relativistic wavefunctions. However we are here in the non-relativistic regime, and we have shown above that $j_{\psi}(t, x) \approx j_{KG}(t, x)$ and $|\psi(t, x)|^2 \approx \rho_{KG}(t, x)$. So from an operational point of view, the weak measurement made on a relativistic particle in the non-relativistic regime will result in a shift proportional to $P^w$ and directly depending on the current density.

While there is no doubt that the present relativistic model gives rise to supraluminal communication and signaling, whether the model is physical can be questioned. The most obvious culprit in the non-relativistic framework was that the basis expansion akin to Eq. (35) in principle includes states with arbitrarily high energies (leading to arbitrarily high velocities). This issue does not appear in our relativistic framework: by definition the velocity associated with the states in the expansion (35) is bounded by $c$. Actually in the numerical illustration we have given, we see from Fig. 1 that the expansion should include states up to $n \approx 10000$. Since by Eq. (2)

$$p_n \approx n \pi \hbar / L_0 = m u / \sqrt{1 - (u/c)^2}$$

where $u$ is the velocity associated to the plane wave of momentum $p_n$, we have here $u/c \approx 1.1 \times 10^{-2}$ for $n = 10000$.

Another possible artifact could come from the breakdown of the single particle picture that occurs when energies are sufficiently large so that particle creation becomes possible. We do not see any obvious reason to question the single particle picture here: first, the Klein paradox does not play any role in an infinite well [19]; second, as we have just seen, even the highest contributing energy eigenstate of the fixed wall cavity ($n = 10000$) yields a value for $p/mc \approx 10^{-2}$ reasonably below the particle creation threshold.

Since there is no obvious artifact that could account for this relativistic non-local effect, it seems the model itself must be questioned. Indeed, the wavefunction, be it a relativistic wavefunction, is globally defined throughout configuration space – here throughout the entire cavity. If the potential changes at one end of the cavity, the wavefunction readjusts throughout the cavity instantaneously. This might not be the case if instead of being modeled as a potential, the moving wall is modeled by a field interacting with the system through exchange particles. A quantum field treatment of the present problem would therefore be instructive.
V. CONCLUSION

To sum up, we have investigated a Klein-Gordon particle in an expanding cavity. We have shown that a curious form of single particle non-locality previously studied with the Schrödinger equation subsists in a relativistic setting. Contrary to the case computed with the non-relativistic formalism, we have not identified any obvious artifacts that could account for the results. If we discard the possibility of this effect being physically real (given that this form of non-locality gives rise to signaling), our results lead to the conclusion that a relativistic model based on a local potential affecting a wavefunction defined over an extended region fails to capture correctly the dynamics. Investigating additional examples as well as a quantum field based treatment would be helpful in understanding the implications of the present results.

APPENDIX

1. KG Solutions : Asymptotic Expansion

The functions $\tilde{J}_\nu(z)$ and $\tilde{Y}_\nu(z)$ given by Eq. (12) admit the following expansion (see Eqs. (3.17) and (3.18) of [17])

$$
\tilde{J}_\nu(z) = \sqrt{\frac{2}{\pi z}} \left( \cos(z - \frac{\pi}{4}) \sum_{s=0}^{s=\infty} (-1)^s \frac{A_{2s} (\nu)}{z^{2s+1}} - \sin(z - \frac{\pi}{4}) \sum_{s=0}^{s=\infty} (-1)^s \frac{A_{2s+1} (\nu)}{z^{2s+2}} \right) \quad (A-1)
$$

and

$$
\tilde{Y}_\nu(z) = \sqrt{\frac{2}{\pi z}} \left( \sin(z - \frac{\pi}{4}) \sum_{s=0}^{s=\infty} (-1)^s \frac{A_{2s} (\nu)}{z^{2s+1}} + \cos(z - \frac{\pi}{4}) \sum_{s=0}^{s=\infty} (-1)^s \frac{A_{2s+1} (\nu)}{z^{2s+2}} \right) \quad (A-2)
$$

where $A_s$ is defined as (see eq (4.02) of Ch. 7 in [18]):

$$
A_s (\nu) = \frac{(4\nu^2 - 1)(4\nu^2 - 3^2) \cdots (4\nu^2 - (2s-1)^2)}{s! 8^s} \quad (A-3)
$$

These relations are useful as $z \to \infty$; then to lowest order, we obtain

$$
\tilde{J}_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right) \quad \text{and} \quad \tilde{Y}_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4}\right) \quad (A-4)
$$

Note that these expressions do not depend on $\nu$. Now we start from [5] we express the $J$ function in terms of $J$ and $Y$ [14] and we use the approximation (A-4). After doing that, since $\cos\left(\frac{\pi}{2} \nu\right) \approx \sinh\left(\frac{\pi}{2} \nu\right)$ for large $\nu$, we find that

$$
\Psi_{J,n} \approx N_{J,n} \cosh(\frac{\pi}{2} k_n) \sqrt{\frac{2\beta \lambda C}{\pi \sqrt{L^2 - \beta^2 x^2}}} \times e^{i(\sqrt{\frac{\lambda^2 + \pi^2}{\beta \lambda C}} - \frac{n \pi x}{L})} \sin(\frac{n \pi x}{L}) \quad (A-5)
$$

(and similarly for $\Psi_{Y,n}$). Therefore this amounts to what is called the ultra non-relativistic limit and the above solution represent a negative energy solutions for which $E \approx -mc^2$. Note that it is best to use the functions $J_{\nu}(z) \sin(\phi_n)$ and $J_{-\nu}(z) \sin(\phi_n)$, also the basis used in [13], as they form the right basis for the emergence of non-relativistic solutions.

If we take the next terms in the series (A-1) and (A-2) we will obtain the standard non-relativistic limit. The approximations for (A-1) and (A-2) become

$$
\tilde{J}_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \left( \cos(z - \frac{\pi}{4}) + \frac{4\nu^2 + 1}{8z} \sin(z - \frac{\pi}{4}) \right), \quad (A-6)
$$

$$
\tilde{Y}_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \left( \sin(z - \frac{\pi}{4}) - \frac{4\nu^2 + 1}{8z} \cos(z - \frac{\pi}{4}) \right). \quad (A-7)
$$

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Using (A-6) and (A-7) in (14), we obtain the following expression

\[ J_\nu(z) \approx \cosh(\frac{\pi}{2} \nu) \sqrt{\frac{2}{\pi z}} \left( e^{i(z^{1/2} - \frac{\nu^2}{8z})} \left[ 1 - i \frac{4\nu^2 + 1}{8z} \right] \right) \]

\[ \approx \cosh(\frac{\pi}{2} \nu) \sqrt{\frac{2}{\pi z}} \left( e^{i(z - \frac{4\nu^2 + 1}{8z} - \frac{\nu}{2z})} \right), \quad (A-8) \]

and the analytical solutions become Eqs. (19) and (20).

2. Non-relativistic limit of the KG solutions in an expanding well

In order to show that Eq. (20) reduces to the solution of the Schrödinger equation (21) in the non-relativistic limit, we need to examine the imaginary exponential of Eq. (20). What does the argument \( z - \frac{4\nu^2 + 1}{8z} \) become in the non-relativistic limit? The terms coming from \( z \) are equal to

\[ \frac{L_0}{\lambda C \beta} + \frac{mc^2 t}{\hbar} - \frac{mvz^2}{2\hbar L} + \ldots \right. \quad (A-9) \]

from which we recover the term proportional to \( z^2 \) that we have in (21). The terms coming from the other part are

\[-\left( \frac{n^2 \pi^2}{2\beta z^2} + \frac{1}{8} - 3n^2 \pi^2 \right) + \ldots \frac{1}{z} \right. \quad (A-10) \]

where we have only approximated the first part, \( (4\nu^2 + 1)/8 \approx \frac{\nu^2}{2\beta z^2} + \frac{1}{8} - \frac{1}{3} n^2 \pi^2 \). When \( \beta \) is small, the dominant term is therefore \( -\frac{\nu^2}{2\beta z^2} \). If we do a Taylor expansion of \( \frac{1}{z} \) in \( \beta \), we find that \( -\frac{1}{\beta z^2} =

\[ \frac{\lambda C}{L_0 \beta} \]

\[ + \frac{c \lambda C t}{L_0^2} \]

\[ + \frac{\lambda C}{2L_0^2} (-2c^2 t^2 - x^2) \beta \]

\[ + \frac{c \lambda C t}{2L_0^2} (2c^2 t^2 + 3x^2) \beta^2 \]

\[ + \frac{\lambda C}{8L_0^3} (-8c^4 t^4 - 24c^2 t^2 x^2 - 3x^4) \beta^3 + (\ldots) \quad (A-11) \]

The first term can be absorbed in the normalization factor, we factorize \( \frac{c \lambda C t}{L_0} \) in the resulting expression \((A-11) + \frac{\lambda C}{L_0 \beta} \) and we keep only the terms with leading powers of \( c \). Doing that we get

\[ \frac{c \lambda C t}{L_0} \left( \frac{1}{L_0} - \frac{ct}{L_0^2} \beta + \frac{c^2 t^2}{L_0^3} \beta^2 - \frac{c^3 t^3}{L_0^4} \beta^3 + (\ldots) \right), \quad (A-12) \]

which is the Taylor expansion of

\[ \frac{c \lambda C t}{L_0} = \frac{\hbar t}{mL_0 L}, \quad (A-13) \]

hence the term \( e^{-\frac{i(4\nu^2 + 1)}{8z}} \) that we have in (21).

3. Validity of the approximations and numerical simulations

If the parameter \( \frac{\nu^2}{z^2} \) is very small, then the approximation \((A-8) \) should be very good. In order to evaluate the accuracy of this approximation, we have plotted the real and imaginary parts of the quantity

\[ \frac{J_\nu(z)}{\cosh(\frac{\pi}{2} \nu) \sqrt{\frac{2}{\pi z}}} - e^{i(z - \frac{4\nu^2 + 1}{8z} - \frac{\nu}{2z})}, \quad (A-14) \]

respectively denoted by \( d_1 \) and \( d_2 \), for a given \( \nu \) (100\( \pi \)) and for various \( z \). Contrary to what we would expect intuitively, the approximation becomes worse for some critical value of \( X = \log(z) \); at about \( X = 12.3 \), there is a sudden increase in \( d_1 \) and \( d_2 \), from about \( 10^{-7} \) to \( 10^{-4} \). This is presumably due to round off errors affecting the validity of numerical routines. Therefore, if we plan on using the approximations, we must choose our parameters (mass, initial box length and so on) in such a way as to have the lowest absolute error for the Bessel functions (say something of the order of \( 10^{-7} \)).

To avoid any problem of interpretation of the Klein-Gordon equation, we work in the non-relativistic regime characterized by

\[ \beta \ll 1 \quad \text{and} \quad z \approx \frac{L_0}{\lambda C \beta} \gg 1. \quad (A-15) \]

Furthermore the approximation for the Bessel function \( J_{-ik_1}(z) \) (used in \( \Psi_{-1}(t, x) \)) will be valid provided that

\[ \frac{k_1^2}{z} \approx \frac{\pi^2}{\beta^2 z} \approx \frac{\pi^2 \lambda C}{\beta L_0} \ll 1. \quad (A-16) \]

Finally we don’t want \( n_{\text{max}} \) (the maximum index for the reduced basis, see the footnote after Eq. (31) to be too large, otherwise the numerical integration needed to obtain the \( c_n \) (see Eq. (34)) would not be feasible. To estimate \( n_{\text{max}} \), we need to estimate the number of oscillations in \( \Phi_{-1}(t = 0, x) \) between \( x = 0 \) and \( x = L_0 \). From the expression (20), we find that it is given by

\[ \frac{z(0, 0) - z(0, L_0)}{2\pi} \approx \frac{\beta L_0}{4\pi \lambda C}. \quad (A-17) \]

We see that the last two conditions are antagonistic: we can’t have a very good approximation of \( \Psi_{-1}(t = 0, x) \) by (20) unless \( n_{\text{max}} \) is very large. This explains why a numerical simulation using the approximations for the Bessel functions would be very precise only if \( n_{\text{max}} \) is very large.