ON FACTORIALITY OF COX RINGS

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Abstract. Generalized Cox’s construction associates with an algebraic variety a remarkable invariant – its total coordinate ring, or Cox ring. In this note we give a new proof of factoriality of the Cox ring when the divisor class group of the variety is finitely generated and free. The proof is based on a notion of graded factoriality. We show that if the divisor class group has torsion, then the Cox ring is again factorially graded, but factoriality may be lost.

1. Introduction

Let $X$ be an irreducible normal algebraic variety over an algebraically closed field $K$ with a free finitely generated divisor class group $\text{Cl}(X)$. Denote by $\text{WDiv}(X)$ the group of Weil divisors on $X$ and fix a sublattice $K \subset \text{WDiv}(X)$ which maps onto $\text{Cl}(X)$ isomorphically. Following famous D. Cox’s construction [4] from toric geometry, define the Cox ring of the variety $X$ as

$$R(X) = \bigoplus_{D \in K} \mathcal{O}(X, D), \text{ where } \mathcal{O}(X, D) = \{ f \in K(X) \mid \text{div}(f) + D \geq 0 \}. $$

Multiplication on graded components of $R(X)$ coincides with multiplication in the field $K(X)$ of rational functions, and extends to other elements by distributivity. It can be easily checked that the ring $R(X)$ depends on the choice of the lattice $K$ only up to isomorphism (for a more general statement, see Proposition 3.2). An important property of $R(X)$ is that it is a factorial ring, see [2], [5]. Here we give a new proof of this result.

Theorem 1.1. The ring $R(X)$ is factorial.

Our main aim is to show that factoriality of $R(X)$ reflects the fact that any effective Weil divisor on $X$ is a unique non-negative integral combination of prime divisors. This observation immediately implies that the multigraded ring $R(X)$ is ”factorial on the set of homogeneous elements” (we call this a priori weaker property ”graded factoriality”). Further, we prove that factoriality follows from graded factoriality. This approach is realized in Section 2.

In Section 3 the Cox ring $R(X)$ is defined in the case when the divisor class group has torsion. Following [2] and [7], we check that $R(X)$ is well-defined. Here the ring $R(X)$ is also factorially graded, but factoriality may be lost. The corresponding examples are given in Section 4 where we describe the Cox ring of a homogeneous space of an affine algebraic group.

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2. Proof of Theorem 1.1

We start with some elementary properties of multigraded algebras. Let $R$ be a commutative associative algebra with unit over a field $\mathbb{K}$. Assume that $R$ is graded by the lattice $\mathbb{Z}^n$,

$$R = \bigoplus_{u \in \mathbb{Z}^n} R_u.$$  

Denote by $R^\times$ (resp. $R^+$) the multiplicative semigroup of invertible (resp. homogeneous) elements of $R$.

Lemma 2.1. 
(i) Suppose that for any $a, b \in R^+$ the condition $ab = 0$ implies $a = 0$ or $b = 0$. Then $R$ has no zero-divisors.
(ii) If $R$ has no zero-divisors and for any $a, b \in R^+$ the condition $ab = 1$ implies $a, b \in R_0$, then $R^\times = R_0^\times$.
(iii) Suppose that $R$ has no zero-divisors. If $a \in R^+$ and $a = bc$, then $b, c \in R^+$.

Proof. Let us fix the lexicographic order on the lattice $\mathbb{Z}^n$. With any element $a \in R$ one associates two homogeneous elements $L(a)$ and $l(a)$, namely its leading and lowest terms. Clearly, $L(ab) = L(a)L(b)$ and $l(ab) = l(a)l(b)$. Now the statements of the lemma follow easily. 

Corollary 2.2.
(i) The Cox ring $R(X)$ has no zero-divisors.
(ii) The semigroup $R(X)^\times$ coincides with $\mathcal{O}(X)^\times$, where $\mathcal{O}(X)$ is the algebra of regular functions on the variety $X$.

Proof. Statement (i) follows from Lemma 2.1 (i), because $\mathbb{K}(X)$ has no zero-divisors. To prove (ii), note that if $f_1 \in \mathcal{O}(X, D_1), f_2 \in \mathcal{O}(X, D_2)$ and $f_1f_2 = 1$, then $D_1 + D_2 = 0$, $\text{div}(f_1) + \text{div}(f_2) = 0$, and the sum of the effective divisors $\text{div}(f_1) + D_1$ and $\text{div}(f_2) + D_2$ equals zero. Therefore $\text{div}(f_i) + D_i = 0$, $i = 1, 2$, and $D_i = 0$. Now one uses Lemma 2.1 (ii) and the equality $R(X)_0 = \mathcal{O}(X)$. 

Definition 2.3. Let $A$ be a finitely generated abelian group, and $R = \oplus_{u \in A} R_u$ be an $A$-graded algebra.

- A non-zero element $a \in R^+ \setminus R^\times$ is called $h$-irreducible, if the condition $a = bc$, $b, c \in R^+$ implies that either $b$ or $c$ is invertible.

- An $A$-graded algebra $R$ is said to be factorially graded, if any its non-zero non-invertible homogeneous element may be expressed as a product of $h$-irreducible elements, and such an expression is unique up to association and renumbering.

Remark 2.4. Assume that $R = \oplus_{u \in \mathbb{Z}^n} R_u$ has no zero-divisors. It follows from Lemma 2.1 (iii) that if $R$ is factorial, then it is factorially graded.

Proposition 2.5. The Cox ring $R(X) = \bigoplus_{D \in K} \mathcal{O}(X, D)$ is factorially graded.

Proof. Effective Weil divisors on $X$ are in one-to-one correspondence with classes of associated elements of $R(X)^+$, and the product of homogeneous elements corresponds to the sum of divisors. This shows that classes of $h$-irreducible elements of the ring $R(X)$ correspond to prime divisors. Since any effective Weil divisor is a unique non-negative integral combination of prime divisors, the ring $R(X)$ is factorially graded. 

Below we shall need the following well-known lemma. For convenience of the reader we give a short proof, cf. [11] Prop. 17.1.
Lemma 2.6. Let $T$ be an algebraic torus and $Z$ a normal algebraic variety with a regular $T$-action. Then any Weil divisor on $Z$ is linearly equivalent to a $T$-invariant Weil divisor.

Proof. By normality, the singular locus of $Z$ has codimension $\geq 2$, and one may assume that $Z$ is smooth. Then any Weil divisor $D$ on $Z$ is Cartier. We may assume that $D$ is effective. There exists a $T$-linearization of the corresponding line bundle, which defines a structure of a rational $T$-module on $\mathcal{O}(Z, D)$ [8 2.4]. Any $T$-eigenvector in $\mathcal{O}(Z, D)$ represents a $T$-invariant divisor equivalent to $D$. \qed

Again by normality, the passage from $X$ to its smooth locus does not change the Cox ring. Further we shall assume that $X$ is smooth. Following [2], let us introduce the ”universal torsor” $\hat{X} \to X$ over $X$. Consider the $K$-graded sheaf of $\mathcal{O}_X$-algebras

$$\mathcal{R}_X = \bigoplus_{D \in K} \mathcal{O}(D), \quad \text{where} \quad \mathcal{O}(U, D) = \{ f \in K(X) \mid |\text{div}(f) + D||_U \geq 0 \},$$

and the relative spectrum $\hat{X} = \text{Spec}_X(\mathcal{R}_X)$ of this sheaf over $X$. Clearly, $R(X) = H^0(X, \mathcal{R}_X) = \mathcal{O}(\hat{X})$. The $K$-grading on $\mathcal{R}_X$ defines a regular action of the torus $T = \text{Spec}(K[\hat{K}])$ on $\hat{X}$, and the canonical affine morphism $p : \hat{X} \to X$ given by inclusion $\mathcal{O}_X \subset \mathcal{R}_X$ is $T$-invariant. Since all divisors on $X$ are Cartier, $p$ is a locally trivial fibration with $T$ as a fiber. In particular, $\hat{X}$ is smooth. Fix an open affine covering $\{U_i\}$ of $X$. Each divisor $D_i = X \setminus U_i$ corresponds to an element $f_i \in R(X)^\times$. The section $f_i$ of the sheaf $\mathcal{R}_X$ is invertible exactly over $U_i$. Therefore $p^{-1}(U_i)$ coincides with $\hat{X}_{f_i} = \{ z \in \hat{X} : f_i(z) \neq 0 \}$. We get an open affine covering $\{\hat{X}_{f_i}\}$ of $\hat{X}$, where $f_i \in \mathcal{O}(\hat{X})$. It is easy to deduce from this that $\hat{X}$ is a quasi-affine variety, see. [6 Ch. 2, App., Lemma 8].

We are ready to finish with the proof of Theorem 1.1. Assume that the ring $R(X)$ is not factorial. Then $R(X)$ contains a non-principal prime ideal of height one. Since $R(X) = \mathcal{O}(\hat{X})$, it is a Krull ring, and its prime ideals of height one are in bijection with essential discrete valuations of the ring $R(X)$, see [10 I.3]. On the other hand, prime divisors on a normal quasi-affine variety are in bijection with essential discrete valuations of the ring of regular functions. Consequently, there is a non-principal divisor on $\hat{X}$. By Lemma 2.6 one may assume that there is a prime $T$-invariant non-principal divisor on $\hat{X}$. The corresponding ideal $I \subset R(X)$ contains a homogeneous, and even an $h$-irreducible element. Suppose that for some $a, b \in R(X)$ the product $ab$ is divisible by $p$, but any homogeneous component of both $a$ and $b$ is not divisible by $p$. Since $R(X)$ is factorially graded, one considers the product of leading components of $a$ and $b$ and comes to contradiction. Thus the ideal $(p) \subset R(X)$ is prime. The inclusion $(p) \subset I$ implies the equality $(p) = I$. This contradiction completes the proof of Theorem 1.1.

We finish this section with the following observation.

Proposition 2.7. Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a multigraded finitely generated $K$-algebra without zero divisors. Assume that $R$ is factorially graded. Then $R$ is factorial.

Proof. Let us show that $R$ integrally closed. The multigrading defines actions of an $n$-dimensional torus $T$ on the algebra $R$ and its quotient field $\text{Quot}(R)$ by automorphisms. It is known that the integral closure $\overline{R}$ of $R$ in $\text{Quot}(R)$ is a $T$-invariant subalgebra, and the $T$-action defines a structure of a rational $T$-module
Remark 2.8. As in the proof of Proposition 2.7 one may show that in characteristic zero a factorial graded finitely generated algebra \( R = \oplus_{a \in A} R_a \) without zero divisors is integrally closed for any finitely generated abelian group \( A \).

3. TORSION IN THE DIVISOR CLASS GROUP

In this section we define the Cox ring \( R(X) \) for a variety \( X \) with arbitrary finitely generated divisor class group and check that \( R(X) \) is well-defined, compare [2], [7].

Let \( S \subset \text{WDiv}(X) \) be a finitely generated subgroup that projects to \( \text{Cl}(X) \) surjectively. Consider a ring

\[
T_S(X) = \bigoplus_{D \in S} \mathcal{O}(X, D).
\]

Let \( S^0 \subset S \) be the kernel of the projection \( S \to \text{Cl}(X) \). Take compatible bases \( D_1, \ldots, D_s \) in \( S \) and \( D_0^i = d_i D_1, \ldots, D_0^r = d_r D_1 \) in \( S^0, r \leq s \).

We call a family of rational functions \( F = \{F_D \in \mathbb{K}(X)^\times : D \in S^0\} \) coherent, if \( \text{div}(F_D) = D \) and \( F_{D+D'} = F_D F_{D'} \) (the term "shifting family" was used in [2] for a similar notion). Obviously, the family \( F \) is defined by \( F_{D_0^i}, i = 1, \ldots, r \); if \( D = a_1 D_0^1 + \cdots + a_r D_0^r \), then \( F_D = F_{D_0^1}^{a_1} \cdots F_{D_0^r}^{a_r} \). Let us fix a coherent family \( F \).

Let \( D_1, D_2 \in S \) and \( D_1 - D_2 \in S^0 \). A map \( f \to F_{D_1-D_2} f \) is an isomorphism between vector spaces \( \mathcal{O}(X, D_1) \) and \( \mathcal{O}(X, D_2) \). One easily checks, that the linear span of elements \( f - F_{D_1-D_2} f \) over all \( D_1, D_2 \) with \( D_1 - D_2 \in S^0 \) and all \( f \in \mathcal{O}(X, D_1) \) is an ideal \( I(S, F) \) of \( T_S(X) \). Define the Cox ring of the variety \( X \) as

\[
R_{S,F}(X) = T_S(X)/I(S, F).
\]

Since \( D/D^0 \cong \text{Cl}(X) \), the ring \( R_{S,F}(X) \) carries a natural \( \text{Cl}(X) \)-grading.

Lemma 3.1. Assume that \( \mathcal{O}(X)^\times = \mathbb{K}^\times \). Then the ring \( R_{S,F}(X) \) does not depend on a choice of \( F \) up to isomorphism.

Proof. The set of functions \( F_{D_0^1}, \ldots, F_{D_0^r} \) is defined up to transformations \( F_{D_0^i} \to \gamma_i F_{D_0^i}, \gamma_i \in \mathbb{K}^\times \). Fix elements \( \alpha_i \in \mathbb{K}^\times, i = 1, \ldots, r \), such that \( \alpha_i a_i = 1 \), and set \( \alpha_{r+1} = \cdots = \alpha_s = 1 \). Then the desired isomorphism of the quotient-rings is induced by an automorphism \( T_S(X) \to T_S(X) \), which acts on the component \( \mathcal{O}(X, D) = a_1 D_1 + \cdots + a_s D_s \), via multiplication by \( \alpha_{a_1}^{\alpha_{a_1}} \cdots \alpha_{a_s}^{\alpha_{a_s}} \).

Proposition 3.2. Assume that \( \mathcal{O}(X)^\times = \mathbb{K}^\times \). Then the ring \( R_S(X) \) does not depend on a choice of \( S \) up to isomorphism.

Proof. Let \( M \subset \text{WDiv}(X) \) be another finitely generated subgroup that projects to \( \text{Cl}(X) \) surjectively. One may assume that \( \text{rk}(S) \geq \text{rk}(M) \).
Lemma 3.3. There is a surjective homomorphism $\phi : S \to M$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
S & \xrightarrow{\phi} & M \\
\downarrow & & \downarrow \\
\text{Cl}(X) & & \\
\end{array}
\]

Proof. Take compatible bases $D_1, \ldots, D_s$ in $S$ and $d_1 D_1, \ldots, d_r D_r$ in $S^0$, and also $M_1, \ldots, M_k$ in $M$ and $m_1 M_1, \ldots, m_p M_p$ in $M^0$, such that $d_i$ (resp. $m_i$) is divisible by $d_{i+1}$ (resp. $m_{i+1}$). The condition $S/S^0 \cong \text{Cl}(X) \cong M/M^0$ implies that $s - r = k - p$, and the sets $(d_1, \ldots, d_r)$ and $(m_1, \ldots, m_p)$ may differ only in the final group of units. It remains to put $\phi(D_i) = M_i$ for $i = 1, \ldots, p$, $\phi(D_i) = M_{i-r+p}$ for $i = r + 1, \ldots, s$, and $\phi(D_i) = 0$ for other $i$. \hfill \Box

We return to the proof of Proposition 3.2. Fix a coherent family $\mathcal{F}$ for the subgroup $(S+M)^0 \subset S + M$, and define a surjective homomorphism $\Phi : T_S(X) \to T_M(X)$ that sends a homogeneous component $\mathcal{O}(X, D)$ to $\mathcal{O}(X, \phi(D))$ as $\Phi(f) = F_{D-\phi(D)} f$. The kernel of $\Phi$ is contained in the ideal $I(S, \mathcal{F}|_{S^0})$, and the ideal $I(S, \mathcal{F}|_{S^0})$ itself maps surjectively to $I(M, \mathcal{F}|_{M^0})$. This shows that $\Phi$ defines a homomorphism $R_S(X) \to R_M(X)$ that, in fact, is an isomorphism. \hfill \Box

Note that homogeneous elements of the ring $T_S(X)$ from different components corresponding to $D_1$ and $D_2$ with $D_1 - D_2 \in S^0$ may define the same effective divisor on $X$. However, after factorization by the ideal $I(S, \mathcal{F})$ effective Weil divisors on $X$ are again in bijection with association classes in $R(X)^+$.  

Proposition 3.4. The ring $R(X)$ is factorially graded.

Proof. The statement follows from the proof of Proposition 2.5. \hfill \Box

Remark 3.5. By [5, Cor. 1.2], the ring $T_S(X)$ is factorial. As we shall see in the next section, this property may be lost for the ring $R(X)$ if the group $\text{Cl}(X)$ has torsion.

4. Homogeneous spaces of an algebraic group

In this section we assume that the ground field $\mathbb{K}$ has characteristic zero. Let $G$ be a connected affine algebraic group with $\text{Cl}(G) = 0$ and without nontrivial characters. Note that the first condition may be achieved by passing to a finite covering of a given group $G$ [5, Prop. 4.6]. Denote by $\mathbb{X}(F)$ the group of characters of an algebraic group $F$. By Rosenlicht’s Theorem, the condition $\mathbb{X}(G) = 0$ is equivalent to $\mathcal{O}(G)^\times = \mathbb{K}^\times$.

Let $H$ be a closed subgroup of $G$. The homogeneous space $G/H$ admits a canonical structure of a smooth quasi-projective algebraic $G$-variety. In [4], it was proved that $\text{Cl}(G/H) \cong \text{Pic}(G/H) \cong \mathbb{X}(H)$. Let us recall how to establish the last isomorphism.

Any character $\chi \in \mathbb{X}(H)$ defines a one-dimensional $H$-module $\mathbb{K}_\chi$. Consider a homogeneous fiber bundle

$$L_\chi = G \times_H \mathbb{K}_\chi := (G \times \mathbb{K}_\chi)/H, \quad h \cdot (g, a) := (gh^{-1}, \chi(h)a).$$
It is shown in [3] that the projection \( L_\chi \to G/H \) is a \( G \)-linearized line bundle over \( G/H \), and \( L_\chi \otimes L_{\chi^2} \cong L_{\chi^3} \). Moreover, the map \( \chi \to L_\chi \) defines an isomorphism between \( \mathcal{X}(H) \) and \( \text{Pic}(G/H) \).

Since \( \text{Cl}(G) = 0 \), the pull-back of the line bundle \( L_\chi \) with respect to the projection \( G \to G/H \) is a trivial \( G \)-linearized line bundle on \( G \). This allows to identify the space of sections \( H^0(X, L_\chi) \) with the following subspace of \( \mathcal{O}(G) \):

\[
\mathcal{O}(G)^{(H)}_\chi := \{ f \in \mathcal{O}(G) : f(gh^{-1}) = \chi(h)f(g) \text{ for all } h \in H, g \in G \}.
\]

The tensor product of sections corresponds to the product in \( \mathcal{O}(G) \).

Consider a subgroup \( H_1 = \cap_{\chi \in \mathcal{X}(H)} \text{Ker}(\chi) \). The next theorem delivers an effective description of the Cox ring associated with the homogeneous space \( G/H \), see also [1, Lemma 3.14].

**Theorem 4.1.** Let \( G \) be a connected affine algebraic group with \( \mathcal{X}(G) = 0 \) and \( \text{Cl}(X) = 0 \), and \( H \) be a closed subgroup of \( G \). Then

\[
R(G/H) \cong \mathcal{O}(G/H_1).
\]

**Proof.** The diagonalizable group \( Q = H/H_1 \) acts on \( G/H_1 \) by right multiplication. Since \( \mathcal{X}(H) = \mathcal{X}(Q) \) and any rational \( Q \)-module is a direct sum of one-dimensional submodules, we get \( \mathcal{O}(G/H_1) = \oplus_{\chi \in \mathcal{X}(H)} \mathcal{O}(G)^{(H)}_\chi \).

Weil divisors on \( G/H \) are in bijection with lines generated by \( Q \)-semiinvariants \( f \in \text{Quot}(\mathcal{O}(G/H_1)) \subseteq \mathbb{K}(G) \). Further, effective divisors correspond to semiinvariants \( f \in \mathcal{O}(G/H_1) \). Let us choose a multiplicative finitely generated group of semiinvariants in \( \text{Quot}(\mathcal{O}(G/H_1)) \) whose weights run through the whole group \( \mathcal{X}(Q) \). One may identify this subgroup with \( S \subset \text{WDiv}(G/H) \) and check easily that the Cox ring associated with \( S \) is isomorphic to \( \oplus_{\chi \in \mathcal{X}(H)} \mathcal{O}(G)^{(H)}_\chi \subset \mathcal{O}(G) \). \( \square \)

For a connected \( H \), the group \( Q \) is a torus and \( \mathcal{X}(H) \) is free. Moreover, in this case \( \mathcal{X}(H_1) = 0 \), thus \( \text{Cl}(G/H_1) = 0 \) and the ring \( R(G/H) \) is factorial (it follows also from Theorem 4.1).

For a disconnected \( H \), the character group \( \mathcal{X}(H_1) \) may be non-trivial.

**Example 4.2.** Let \( G = \text{SL}(2) \) and \( H \) be the normalizer \( N \) of a maximal torus \( T \subset \text{SL}(2) \). Here \( \mathcal{X}(H) \) is isomorphic to the cyclic group \( \mathbb{Z}_2 \) of order 2, \( H_1 = T \), and \( \mathcal{X}(H_1) \cong \mathbb{Z} \). This shows that the Cox ring \( R(\text{SL}(2)/N) \cong \mathcal{O}(\text{SL}(2)/T) \) is not factorial. The space \( \text{SL}(2)/N \) is a smooth affine surface \( X \) with \( \text{Cl}(X) \cong \mathbb{Z}_2 \), the ring \( R(X) \) is isomorphic to

\[
\mathbb{K}[x_1, x_2, x_3]/(x_2^2 - x_1 x_3 - 1),
\]

and the \( \mathbb{Z}_2 \)-grading on \( R(X) \) is given by \( \deg(x_1) = \deg(x_2) = \deg(x_3) = 1 \).

One may propose several ways to associate a factorial ring with the surface \( X \). Firstly, consider a subgroup \( S \subset \text{WDiv}(X) \) generated by a non-principal divisor. The ring \( T_S(X) \) is isomorphic to

\[
(\mathbb{K}[x_1, x_2, x_3, t, t^{-1}]/(x_2^2 - x_1 x_3 - 1))^\mathbb{Z}_2,
\]

where \( \mathbb{Z}_2 \) acts on the variables \( x_1, x_2, x_3, t, t^{-1} \) via multiplication by \(-1\).

Secondly, the space \( \text{SL}(2)/N \) admits a wonderful (the term is due to D. Luna) \( \text{SL}(2) \)-equivariant embedding in \( \mathbb{P}^2 \), and \( R(\mathbb{P}^2) \cong \mathbb{K}[x_1, x_2, x_3] \). The Cox ring of a wonderful embedding of any spherical homogeneous space is described in [3].
Example 4.3. Using the construction of Example 4.2, one may find a smooth affine variety $X$ with $\text{Cl}(X) \cong A$ and a non-factorial Cox ring for any non-free finitely generated abelian group $A$. Indeed, let $A \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_s} \oplus \mathbb{Z}^n$. Put $X = G/H$, where $G$ is a direct product of $s + n$ simple groups,

$$G = \text{SL}(d_1) \times \cdots \times \text{SL}(d_s) \times \text{SL}(2) \times \cdots \times \text{SL}(2),$$

and $H = H(1) \times \cdots \times H(s) \times T \times \cdots \times T$, where $H(i)$ is an extension of a maximal torus of the group $\text{SL}(d_i)$ by elements of its normalizer that act as degrees of one $d_i$-cycle. In this case, $X(H) \cong A$ and $X(H_1) \cong \mathbb{Z}^{d_1+\cdots+d_s-n}$.

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