Exponential-Binary State-Space Search

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Abstract

Iterative deepening search is used in applications where the best cost bound for state-space search is unknown. The iterative deepening process is used to avoid overshooting the appropriate cost bound and doing too much work as a result. However, iterative deepening search also does too much work if the cost bound grows too slowly. This paper proposes a new framework for iterative deepening search called exponential-binary state-space search. The approach interleaves exponential and binary searches to find the desired cost bound, reducing the worst-case overhead from polynomial to logarithmic. Exponential-binary search can be used with bounded depth-first search to improve the worst-case performance of IDA* and with breadth-first heuristic search to improve the worst-case performance of search with inconsistent heuristics.

1 Introduction and Motivation

Iterative deepening has been a common approach in state-space search when the correct depth of search is unknown. With alpha-beta pruning in games, for instance, iterative deepening is broadly used, among other reasons [Marsland, 1986], to ensure the best action from a completed iteration is always available when the move clock runs out. In single-agent search, iterative deepening is used in IDA* [Korf, 1985] to find an optimal solution when the running time will be dominated by the last iteration, which assumes the underlying search tree grows exponentially. If it does not grow exponentially, a tree with $N$ nodes could require as many as $\Theta(N^2)$ node expansions if each successive iteration only expands one new node. As a result, IDA* is known to have poor performance in problem domains with non-unit edge costs.

This paper introduces a new approach, exponential-binary (EB) state-space search, which can be used to reduce the worst-case number of node expansions. If the optimal solution has cost $C^*$, EB search reduces the worst-case running time from $O(N^2)$ to $O(N \log C^*)$ while still returning an optimal solution. It achieves this by no longer expanding states in a strict best-first order. Like iterative deepening, exponential-binary state-space search is a general technique that can be applied to a range of different problems and algorithms. In addition to reducing the worst-case overhead of IDA*, this paper also demonstrates that exponential-binary state-space search can be used in problems with inconsistent heuristics to reduce the overhead of node re-expansions.

EB search uses an $f$-cost bound to limit the cost of any path that is searched. Additionally, it also uses lower and upper node expansion bounds during the search to ensure that the $f$-cost bounds increase neither too slowly (potentially causing quadratic overhead as in IDA*) nor too quickly (causing large overhead by overshooting the optimal solution cost). The key idea is to use a combination of exponential and binary searches to determine an $f$-cost bound that falls into the desired expansion number window.

Next, we introduce the necessary theory on state-space search to establish our formal results, followed by a general description of the exponential-binary state-space search framework and its application to IDA*-style and A*-style search. We prove that our new algorithms achieve better guarantees on the number of node expansions than previous algorithms from the literature and experimentally validate the performance claims.

2 State-Space Search

The problem we consider in this paper is heuristic state-space search with a black-box problem representation. A state space $\Theta = (S, A, T, c, s_0, S^\star)$ has a finite set of states $S$, a finite set of actions $A$, and a set of (state) transitions $T \subseteq S \times A \times S$. Moreover, there is a cost function $c : A \to \mathbb{N}_0$ that associates non-negative integer costs with actions. Finally, $s_0 \in S$ is called the initial state and $S^\star \subseteq S$ is the set of goal states.

We are concerned with optimal state-space search, where the objective is to find a path (sequence of transitions) of minimum cost from $s_0$ to any goal state $s^\star \in S^\star$. The cost

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$^1$This paper and another independent IJCAI 2019 submission have been merged into a single paper that subsumes both of them [Helmert et al., 2019]. This paper is placed here only for historical context. Please only cite the subsuming paper.

$^2$Using integer costs simplifies presentation. Rational numbers can be represented by raising them to a common denominator and then dropping the denominator, as scaling all costs by a constant factor has no effect on semantics. Irrational costs can be approximated to any desired finite precision and then scaled to integers.
of a path \( \pi = \langle (s_0, a_1, s_1), (s_1, a_2, s_2), \ldots, (s_{n-1}, a_n, s_n) \rangle \) is defined as the sum of the action costs involved: \( c(\pi) = \sum_{i=1}^{n} c(a_i) \). We assume throughout the paper that a state space \( \Theta = (S, A, T, c, s_0, S_0) \) is given and that \( C^\ast \) is the cost of an optimal solution\(^1\).

**BBHS Algorithms.** Rather than assuming that \( \Theta \) is represented as an explicit graph (which is often not feasible due to the large number of states), we consider black-box heuristic search (BBHS) algorithms. BBHS algorithms do not require a declarative state-space representation such as those used in classical planning \([\text{Ghallab et al.}, 2004]\), which makes them more generally applicable but also means that they can only reason about the state space in limited ways. Specifically, a BBHS algorithm may only access the components of a state and cannot reason about the state space in limited ways. Specifically, a BBHS algorithm may only access the components of a state space through the following methods: 1) \textit{init}(), returning the initial state; 2) \textit{is_goal}(s), returning a boolean result indicating if \( s \) is a goal state; 3) \textit{succ}(s), returning a sequence of pairs \((a, s')\) representing all outgoing transitions \((s, a, s')\) of state \( s \); 4) \textit{cost}(a), returning the cost of action \( a \); and 5) \textit{h}(s), computing a numerical value called the \textit{heuristic value} for state \( s \). (We use \( h \) to denote both the algorithm that computes the heuristic value for a state and the underlying mathematical mapping from states to numbers. More on heuristics below.)

BBHS algorithms are a similar class to \textit{DXBB} algorithms as defined by Eckerle et al. \([2017]\) except that Eckerle et al. consider bidirectional search and only allow unidirectional search and we make no assumption of determinism.

**Cost Measures.** The computational complexity of BBHS algorithms is often measured using one of three abstract cost measures: \textit{node expansions}, \textit{generated states}, or \textit{heuristic evaluations}. The number of node expansions is the number of calls to \textit{succ}; the number of heuristic evaluations is the number of calls to \( h \). When measuring the number of generated states, a call to \textit{init} counts as 1, and a call to \textit{succ} counts as \( N \) if the given state has \( N \) outgoing transitions.

In more fine-grained analyses, it can be useful to take the cost of data structure operations within a search algorithm into account. When we speak of the \textit{runtime} of a search algorithm, we count all above-mentioned abstract operations as 1 unit of cost except \textit{succ}, which we count as 1 unit for each returned transition, and we assume that states can be copied, compared and inserted into/removed from a hash table in 1 unit of time.

**Search Paths, Search Nodes, Graph and Tree Search.** We define a \textit{search path} \( \pi \) as a sequence of transitions that defines a path in the state space beginning at the initial state \( s_0 \). BBHS algorithms often keep track of the states they have explored so far via so-called \textit{search nodes} \( n \), and because BBHS algorithms can only generate states via sequences of node expansions starting from the initial state, each such node can be associated with a generating search path. However, the concept of paths explored by a search algorithm can also be applied to BBHS algorithms like IDA* that do not use explicit node data structures. Given a search path \( \pi \), we write \textit{state}(\( \pi \)) for the state at the end of the path, i.e., the next state to be expanded if the path is to be explored further. The cost of a search path is traditionally denoted as \( g(\pi) \), so we define \( g(\pi) \) as a synonym for \( c(\pi) \).

We say that a BBHS algorithm is a \textit{graph search} if it includes tests that two states generated by the algorithm are the same. (Such tests can eliminate paths leading to the same state as other paths from consideration, a technique called \textit{duplicate elimination}.) A BBHS algorithm that does not perform such tests is called a \textit{tree search}. The most famous BBHS graph search is A* \([\text{Hart et al.}, 1968]\), and the most famous BBHS tree search is IDA* \([\text{Korf}, 1985]\).

**f-Values and Properties of Heuristics.** The efficiency of BBHS algorithms largely hinges on the properties of the heuristic \( h \) it uses. BBHS algorithms generally require that \( h \) has certain properties in order to guarantee that only optimal solutions are returned. A heuristic \( h \) is called \textit{admissible} if \( h(s) \leq h^\ast(s) \) for all states \( s \), where \( h^\ast(s) \) denotes the cost of an optimal (minimum-cost) path from \( s \) to the nearest goal state, or \( \infty \) if no such path exists. A heuristic \( h \) is called \textit{consistent} if \( h(s) \leq c(a, s') + h(s') \) for all state transitions \((s, a, s')\). The \textit{f-value} of a search path \( \pi \) is defined as \( f(\pi) = g(\pi) + h(\text{state}(\pi)) \) and provides an estimate of the cost of a solution that extends the path \( \pi \).

**Runtime Bounds.** A search path \( \pi \) is called \textit{highly promising} if \( f(\pi) < C^\ast \) and \textit{promising} if \( f(\pi) \leq C^\ast \) and \( \pi \) does not include a goal state. A state \( s \) is called highly promising if there exists a highly promising search path leading to \( s \) and promising if there exists a promising search path leading to \( s \). We write \( P_s \) for the number of highly promising paths, \( P^+_s \) for the number of promising paths, \( S_s \) for the number of highly promising states and \( S^+_s \) for the number of promising states of the state space.

In the following, we only consider BBHS algorithms that are guaranteed to produce optimal solutions without requiring any properties of \( h \) other than (possibly) admissibility and consistency.

**Bounds for Tree Search.** A tree search \textit{must} perform a separate node expansion for each highly promising path, and consequently \( \Omega(P_s) \) is a lower bound on the number of node expansions for any tree search.

When equipped with an oracle that gives the optimal solution cost \( C^\ast \), the IDA* algorithm could be implemented to perform between \( P_s \) and \( P^+_s \) node expansions. In the absence of such an oracle, there exist infinite families of state spaces for which IDA* requires \( \Omega(P^+_s) \) node expansions. The key to constructing such quadratic examples is to ensure that every round of the algorithm only leads to a small (bounded by a constant) number of newly explored paths compared to the previous round. While such bad scenarios for IDA* are rare in unit-cost state spaces \((c(a) = 1 \text{ for all actions } a)\), they can arise naturally when action costs vary widely.

This shortcoming of IDA* has been observed before, and there have been attempts to address it by modifying the way in which the next \( f \)-cost bound of IDA* is computed \([\text{Sarkar et al.}, 1991]; \text{Vempaty et al.}, 1991]; \text{Wah and Shang}, 1994]; \text{Burns and Ruml}, 2013\). However, to the best of our knowledge no
tree search in the literature is known to have a better than quadratic worst-case bound in terms of $P_n$. As one of our contributions, we present a tree search performing at most $O(P_n \log C^*)$ node expansions.

**Bounds for Graph Search.** For graph search, the situation more subtly depends on the properties of the heuristic. Any graph search that requires an admissible and consistent heuristic and makes no further assumptions on the properties of the heuristic must expand all highly promising states, so $\Omega(S_1)$ is a lower bound on the number of node expansions. With such a heuristic, the A* algorithm always performs between $S_1$ and $S_\infty$ node expansions.

If the heuristic is admissible but not consistent, there exist infinite families of state spaces, first described by Martelli [1977], for which A* requires $\Omega(2^{S_n})$ node expansions, making A* exponentially less efficient than blind search in the worst case. There exist algorithms that improve this worst-case behavior, such as the B’ algorithm by Mérő [1984], but to the best of our knowledge no BBHS algorithm described in the literature is known to have a worst-case bound better than $O(S_\infty^2).$ As one of our contributions, we present an algorithm performing at most $O(S_1 \log C^*)$ node expansions.

### 3 Exponential-Binary State-Space Search

In this section we introduce exponential-binary-state-space search (EBSSS). We think of EBSSS as a family of algorithms because it uses another state-space search algorithm as a component, resulting in different flavors of EBSSS depending on which component algorithm is chosen.

Specifically, EBSSS delegates most of its work to an algorithm that performs a bounded state-space search, denoted by $\text{BoundedSearch}$ in the following. $\text{BoundedSearch}$ must accept two parameters: an $f$-value bound ($f_{\text{max}}$) and a bound on the number of permitted node expansions ($N_{\text{max}}$). Any search algorithm can be used for $\text{BoundedSearch}$ if it satisfies the following properties:

1. It never performs more than $N_{\text{max}}$ state expansions. If after $N_{\text{max}}$ state expansions the search has not yet been completed, $\text{BoundedSearch}$ signals with its return value that the bounded search remained incomplete and returns no solution.
2. If the search was completed and $f_{\text{max}} \geq C^*$, $\text{BoundedSearch}$ returns an optimal solution.
3. If the search was completed and $f_{\text{max}} < C^*$, $\text{BoundedSearch}$ signals with its return value that no solution of cost at most $f_{\text{max}}$ exists.
4. The return value of $\text{BoundedSearch}$ includes the information how many nodes were expanded.

#### High-Level Description of the EBSSS Algorithm

We now describe EBSSS in detail. Pseudo-code is shown in Algorithm 1. The main function $\text{EBSSS}$ (lines 1–3) controls the overall search by performing a sequence of bounded state-space searches. Each iteration of the main loop (line 3–11) performs a search with a given $f$-bound and unbounded (= bound $\infty$) node expansions (line 4). This follows the basic structure of algorithms like IDA* [Korf, 1985] and BFIDA* [Zhou and Hansen, 2006], and from the above description of the properties of $\text{BoundedSearch}$ it should be clear that this results in an optimal BBHS algorithm as long as the $f$-bounds

#### Algorithm 1: Exponential-Binary State-Space Search

```plaintext
1 function EBSSS(c1, c2, \Delta)
2 \quad f_{\text{max}} \leftarrow h(s_0)
3 \quad \text{loop do}
4 \quad \quad \text{search_result} \leftarrow \text{BoundedSearch}(f_{\text{max}}, \infty)
5 \quad \quad \text{if search_result.solution_found then}
6 \quad \quad \quad \text{extract solution and terminate}
7 \quad \quad end
8 \quad \quad N_{\text{min}} \leftarrow \lfloor c_1 \cdot \text{search_result.expanded_nodes} \rfloor
9 \quad \quad N_{\text{max}} \leftarrow \lceil c_2 \cdot \text{search_result.expanded_nodes} \rceil
10 \quad \quad f_{\text{max}} \leftarrow \text{NextFBound}(f_{\text{max}}, \Delta, N_{\text{min}}, N_{\text{max}})
11 \quad end
12 function TestFBound(f_{\text{max}}, N_{\text{min}}, N_{\text{max}})
13 \quad \text{search_result} \leftarrow \text{BoundedSearch}(f_{\text{max}}, N_{\text{max}})
14 \quad \text{if search_result.solution_found then}
15 \quad \quad \text{extract solution and terminate}
16 \quad \quad \text{else if search_result.expanded_nodes < N_{\text{min}} then}
17 \quad \quad \quad \text{return "bound too low"}
18 \quad \quad \text{else if search_result.is_incomplete then}
19 \quad \quad \quad \text{return "bound too high"}
20 \quad \quad \text{else}
21 \quad \quad \quad \text{return "bound is good"}
22 end
23 function NextFBound(f_{\text{old}}, \Delta, N_{\text{min}}, N_{\text{max}})
24 \quad f_{\text{low}} \leftarrow f_{\text{old}} + 1
25 \quad f_{\text{high}} \leftarrow f_{\text{old}} + \Delta
26 \quad \text{loop do}
27 \quad \quad \text{status} \leftarrow \text{TestFBound}(f_{\text{high}}, N_{\text{min}}, N_{\text{max}})
28 \quad \quad \text{if status = "bound is good" then}
29 \quad \quad \quad \text{return } f_{\text{high}}
30 \quad \quad \text{else if status = "bound too high" then}
31 \quad \quad \quad \text{break}
32 \quad \quad \quad \Delta \leftarrow 2 \cdot \Delta
33 \quad \quad \quad f_{\text{low}} \leftarrow f_{\text{high}} + 1
34 \quad \quad \quad f_{\text{high}} \leftarrow f_{\text{low}} + \Delta
35 \quad \quad end
36 \quad \text{while } f_{\text{low}} \neq f_{\text{high}}
37 \quad \quad f_{\text{mid}} \leftarrow (f_{\text{low}} + f_{\text{high}}) / 2
38 \quad \quad \text{status} \leftarrow \text{TestFBound}(f_{\text{mid}}, N_{\text{min}}, N_{\text{max}})
39 \quad \quad \text{if status = "bound too low" then}
40 \quad \quad \quad f_{\text{low}} \leftarrow f_{\text{mid}} + 1
41 \quad \quad \text{else if status = "bound too high" then}
42 \quad \quad \quad f_{\text{high}} \leftarrow f_{\text{mid}}
43 \quad \quad \quad \text{else}
44 \quad \quad \quad \quad \text{return } f_{\text{mid}}
45 \quad \quad \quad end
46 \quad \quad end
47 \quad return f_{\text{low}}
48 end
```

An algorithm with an $O(S_\infty^{1.5})$ bound has been described [Stutevant et al., 2008], but the authors noted in a personal communication that the algorithm is not general – it relies on an assumption unstated in that paper.
increase from iteration to iteration.

The distinguishing characteristic of EBSSS is in how the sequence of \( f \)-bounds is determined. On the one hand, we want the number of nodes expanded in each iteration of the main loop to grow at an exponential rate, so that the effort for the unsuccessful early iterations can be amortized. On the other hand, we want to avoid growing the \( f \)-bound so fast that the last iteration performs an unreasonably large amount of wasted work considering \( f \)-values larger than \( C^* \).

After an iteration that expanded \( N \) nodes, we aim to set the next \( f \)-bound in such a way that the number of expansions of the next bounded search in the main loop falls into a "target interval" of at least \( N_{\text{min}} = \lceil c_1 \cdot N \rceil \) and at most \( N_{\text{max}} = \lceil c_2 \cdot N \rceil \) expansions (lines [28][30]). Here, \( c_1 \) and \( c_2 \) are real-valued parameters that must satisfy \( c_2 \geq c_1 > 1 \). For example, one might choose \( c_1 = 2 \) and \( c_2 = 10 \) to aim for at least \( 2N \) and at most \( 10N \) expansions in the next iteration.

Finding a new \( f \)-bound that falls into this expansion range is not always possible: it may be the case that no such \( f \)-bound exists. Therefore, we also allow choosing an \( f \)-bound that results in more than \( N_{\text{max}} \) expansions, but only if we can guarantee that the new bound is no larger than \( C^* \), so that there is no wasted work considering nodes with \( f \)-values larger than \( C^* \).

### Computing the New \( f \)-Bound

Determining a suitable \( f \)-bound is the task of function \( \text{NextFBound} \) (lines [25][51]), which receives the current \( f \)-bound in parameter \( f_{\text{old}} \), the target expansion interval in parameters \( N_{\text{min}} \) and \( N_{\text{max}} \), and an integer parameter \( \Delta \geq 1 \) that affects how aggressively we attempt to increase the bound.

We say that a candidate \( f \)-bound is too low if a search with that bound would require fewer than \( N_{\text{min}} \) node expansions and too high if it would require more than \( N_{\text{max}} \). If it is neither too low nor too high, it is good. The contract of \( \text{NextFBound} \) is to return a new \( f \)-bound \( f_{\text{new}} \) that is good, or one that is too high but satisfies \( f_{\text{new}} \leq C^* \).

\( \text{NextFBound} \) internally uses a straightforward function \( \text{TestFBound} \) (lines [13][22]) to test if a given bound is too low, too high, or good by performing bounded searches with an expansion limit of \( N_{\text{max}} \). If these trial searches result in a solution, we terminate the overall algorithm immediately (line [16]), which implies that whenever \( \text{TestFBound} \) returns "bound too low" or "bound is good" for a candidate bound \( f_{\text{max}} \), we know that \( f_{\text{max}} < C^* \). (These are the cases in which \( \text{TestFBound} \) completed the search for the given \( f \)-bound.)

In order to compute \( f_{\text{new}} \), \( \text{NextFBound} \) follows a strategy from an algorithm called \textit{exponential search} that was originally designed to search for values in infinitely large sorted arrays [Bentley and Yao, 1976]. In brief, \( \text{NextFBound} \) tests the potential new \( f \)-bounds \( f_{\text{old}} + \Delta, f_{\text{old}} + 2 \Delta, f_{\text{old}} + 4 \Delta, f_{\text{old}} + 8 \Delta, f_{\text{old}} + 16 \Delta \) and so on until it encounters a bound that is not too low (lines [26][33]). If we encounter a bound that is good, we can return it directly (line [31]). Otherwise we break out of the loop the first time we encounter a bound that is too high (line [35]).

In this case we perform a binary search (lines [39][50]) between the last tested bound that was too low and the first tested bound that was too high until we either find a bound that is good (line [37]) or the binary search window collapses to a single value (line [50]).

In the latter case, we might return a bound \( f_{\text{new}} \) which is not good. If this happens, to satisfy the contract of \( \text{NextFBound} \) we must guarantee that (1) \( f_{\text{new}} \) is too high (rather than too low) and (2) \( f_{\text{new}} \leq C^* \).

To verify (1), observe that the returned value equals the value of the variable \( f_{\text{high}} \), and the binary search part of \( \text{NextFBound} \) has the invariant that \( f_{\text{high}} \) always holds an \( f \)-bound that is too high.

To verify (2), observe that the returned value also equals the value of the variable \( f_{\text{low}} \), and every time \( \text{NextFBound} \) assigns a value to \( f_{\text{low}} \), a search to the \( f \)-bound \( f_{\text{low}} - 1 \) has previously been completed without finding a solution (in line [25] because \( f_{\text{low}} - 1 = f_{\text{old}} \), and in lines [36] and [33] because \( f_{\text{low}} - 1 \) was just tested by \( \text{TestFBound} \), which returned "bound too low"). This implies \( f_{\text{low}} - 1 < C^* \) and hence \( f_{\text{new}} = f_{\text{low}} \leq C^* \) because \( f_{\text{low}} \) and \( C^* \) are integers.

### Caching Search Results

If implemented naively, EBSSS may perform searches that are clearly redundant. For example, if a search triggered by \( \text{NextFBound} \) finds a good \( f \)-bound, exactly the same \( f \)-bound will immediately be searched again by the next iteration of the main loop of EBSSS. While such redundancies do not affect the following worst-case performance results, they are clearly undesirable in an efficient implementation.

These inefficiencies can be addressed without changing the pseudo-code of the algorithm by making the component algorithm \( \text{BoundedSearch} \) remember the arguments and results of its last invocation and return the cached result if it can detect that the result for the current invocation must be the same. If \( \text{BoundedSearch} \) keeps track of the lowest pruned \( f \)-values and highest expanded \( f \)-values encountered during its search, the cached result can also be used in cases where the \( f \)-bound only changes so slightly between searches that the set of expanded nodes must necessarily be the same. Our implementation is able to avoid such redundancies.

### Computational Complexity

We now turn to the computational complexity of EBSSS. The parameters \( c_1, c_2 \) and \( \Delta \) can be chosen arbitrarily, but will be considered fixed for the purposes of the analysis. In terms of big-O complexity, it is sufficient to consider the time spent inside \( \text{BoundedSearch} \), as every loop of the algorithm calls \( \text{BoundedSearch} \) at least once in every iteration (either directly or via \( \text{TestFBound} \)) and otherwise only performs \( O(1) \) work per iteration.

We know that \( \text{BoundedSearch}(f_{\text{max}}, N_{\text{max}}) \) performs at most \( N_{\text{max}} \) node expansions, so we study the complexity of EBSSS in terms of the overall number of node expansions. For a given implementation of \( \text{BoundedSearch} \) on a given state space, let \( E_{\text{max}} \) be the maximum number of node expansions that \( \text{BoundedSearch}(f_{\text{max}}, \infty) \) performs for any bound \( f_{\text{max}} \leq C^* \), i.e., the worst-case cost of one call to \( \text{BoundedSearch} \) for \( f \)-bounds that do not overshoot \( C^* \).

### Expansions within the Main Loop

We first consider the node expansions in the calls to \( \text{BoundedSearch} \) in the main loop of \( \text{NextFBound} \). For each call to \( \text{BoundedSearch} \), a search is done using the current bound. If the returned value is too low, the search is repeated with the new bound. This process continues until a bound is found that is too high. The total number of node expansions performed in these searches is \( E_{\text{max}} \).
loop (line 4), ignoring the costs incurred by NextFBound. We prove by induction that each iteration performs at most $E_{\text{max}}$ expansions, except possibly the last one, which may perform up to $\lceil c_2 E_{\text{max}} \rceil$ expansions. For the induction basis, the first iteration uses a bound of $f_{\text{max}} = h(s_0) \leq C^*$ (we assume an admissible heuristic) and therefore performs at most $E_{\text{max}}$ expansions. For the induction step, consider an iteration other than the first. If $f_{\text{max}} \leq C^*$ in this iteration, it performs at most $E_{\text{max}}$ expansions. If $f_{\text{max}} > C^*$, we must be in the final iteration. From the contract of NextFBound, the previous iteration must have found a good $f$-bound (otherwise it is not allowed to return $f_{\text{max}} > C^*$), which means that the number of expansions is at most $\lceil c_2 N_{\text{prev}} \rceil$, where $N_{\text{prev}}$ is the number of expansions in the previous iteration. From $N_{\text{prev}} \leq E_{\text{max}}$ (induction hypothesis), we get a bound of $\lceil c_2 E_{\text{max}} \rceil$ expansions for the last iteration, completing the induction proof.

For the total number of expansions of the main loop, the contract of NextFBound guarantees that the number of expansions grows exponentially (by a factor of at least $c_1$) between iterations, and therefore all expansions in iters other than the last one disappear in big-O notation, giving us an overall bound of $O(\lceil c_2 E_{\text{max}} \rceil) = O(E_{\text{max}})$ expansions in the main loop. (Note that $c_2$ is a constant.)

**Expansions within NextFBound.** Consider an invocation of NextFBound in the main loop after $\text{BoundedSearch}(f_{\text{max}}, \infty)$ expanded $N$ nodes. All calls to $\text{BoundedSearch}$ in this invocation use an expansion limit of $N_{\text{max}} = \lceil c_2 N \rceil$, so the total number of expansions is bounded by $N_{\text{max}} \cdot K$, where $K$ is the number of calls to TestFBound made by this invocation of NextFBound.

The first $f$-bound tested by TestFBound is $f_{\text{old}} + \Delta < C^* + \Delta$ (if we had $f_{\text{old}} \geq C^*$, the search would already have terminated). All further $f$-bounds tested by TestFBound in the exponential search phase are bounded from above by $2C^*$ because the $f$-bound at most doubles from one test to the next, and if it ever reaches $C^*$, we must break out of the loop. (If the bounded search for such a bound completes, we have found the solution and terminate. If it does not complete, we break out of the loop because we have found a bound that is too high.) Therefore, the highest value reached by $f_{\text{high}}$ is bounded from above by $M = \max(C^* + \Delta, 2C^*)$.

From the analysis of the original algorithm for exponential search in sorted arrays [Bentley and Yao, 1976], it follows that $K = O(\log M) = O(\log(\max(C^* + \Delta, 2C^*) \cdot \log C^*)) = O(\log C^*)$, where we use that $\Delta$ is a constant. Therefore, the total number of expansions of this invocation of NextFBound is $O(K \cdot N_{\text{max}}) = O(\lfloor c_2 N \rfloor \cdot \log C^*) = O(N \log C^*)$.

In summary, whenever the main loop performs $N$ expansions, NextFBound performs $O(N \log C^*)$ expansions, so the total number of expansions of the complete algorithm can be bounded by the total number of expansions in the main loop multiplied with $O(\log C^*)$, for a total bound of $O(E_{\text{max}} \log C^*)$.

**EBTS and EBGS**

We conclude this section by considering two concrete instances of EBSSS: a tree search algorithm called exponential-binary tree search (EBTS) and a graph search algorithm called exponential-binary graph search (EBGS).

EBTS is EBSSSS where $\text{BoundedSearch}$ is implemented like the $f$-bounded search within IDA*, except that the $f$-bounded search does not terminate immediately after finding solution but continues searching for cheaper solutions, using solutions found so far for pruning. (This modification is necessary because, unlike IDA*, EBSSS does not increase the $f$-bound in minimal steps.) In other words, $\text{BoundedSearch}$ is a depth-first branch-and-bound search using $f$ for pruning.

For $f$-bounds of at most $C^*$, this implementation of $\text{BoundedSearch}$ never expands more than $P_+^*$ nodes. Therefore, we have $E_{\text{max}} \leq P_+^*$ for EBTS, and hence our general bound for EBSSS shows that this algorithm never performs more that $O(P_+^* \log C^*)$ node expansions. As discussed in Section 2, earlier tree search algorithms have worst-case expansion bounds of $\Omega(P_+^*)$.

EBGS is EBSSSS where $\text{BoundedSearch}$ is implemented like the $f$-bounded search in the BFIDA* algorithm [Zhou and Hansen, 2006], except that we replace the breadth-first search of BFIDA* with Dijkstra search. (The original BFIDA* algorithm was only described for the unit-cost setting, but replacing breadth-first search with Dijkstra search is the natural generalization to general costs.)

For $f$-bounds of at most $C^*$, this implementation of $\text{BoundedSearch}$ never expands more than $S_+^*$ nodes. Therefore, we have $E_{\text{max}} \leq S_+^*$ for EBGS, and hence our general bound for EBSSS shows that this algorithm never performs more that $O(S_+^* \log C^*)$ node expansions. As discussed in Section 2, A* requires $\Omega(2^{S_+^*})$ expansions in the worst case, and earlier graph search algorithms designed to eliminate this worst case require $\Omega(S_+^*)$ expansions.

## 4 Experimental Results

This section evaluates the EBSS approach experimentally in three domains, the sliding tile puzzle, the pancake puzzle, and in a class of graphs due to [Merid, 1984] for which A* and variants such as $B^*$ require $\Omega(S_+^*)$. The code for EBSS and each of the experimental results that appear in this paper is publicly available (url redacted); each experiment can be run through the use of command-line parameters.

Experiments are run in a distributed manner on a research cluster which contains 2.1GHz Intel Skylake and Broadwell processors. Each job is run on a single processor. EBIDA* and IDA* were limited to 32 MB of memory per process and 24 hours of computation time. A* was limited to 64 GB of memory. Whenever A* could not solve a problem it was due to running out of memory.

To mitigate floating point rounding errors, IDA* and A* use a tolerance of $10^{-5}$ for all comparison and equality checks. The EBSS implementation uses 64-bit integers to represent heuristics and edge costs and is parameterized by the constant used to discretize costs. Using a multiple of $10^6$ gives approximately the same precision as IDA* and A*, while a multiple of $10^8$ uses higher precision.

EBSS has four parameters: the node upper bound and node lower bound, the resolution used when converting to an internal integer representation, and the $\Delta$ used for the exponential portion of the search. Each experimental result with EBSS indicates which parameters were used. In addition, we compare
to an oracle that is provided with the solution cost and only needs to prove that there are no solutions with lesser cost.

### 4.1 Sliding-Tile Puzzle

The first experiment studies EBTS in the sliding-tile puzzle where the cost of moving tile \( t \) is \( 1 + \frac{1}{t + t^T} \). This cost function provides a variety of unique edge costs within a small range (1 to 1.5), and thus does not introduce a large number of cycles not present in the unit-cost problem. The heuristic is based on Manhattan distance, where the heuristic incorporates the cost of moving the different tiles. The problem instances are the standard 100 instances \([Korf, 1985]\).

Experimental results are found in Table 4. EBTS is the only algorithm that could solve all 100 problems within the resource limits. EBTS significantly outperforms IDA*, and with the best settings EBTS is within a factor of three of the oracle. A* expands fewer states than the oracle because the oracle must expand all promising paths, not just all promising states. Using \( \Delta = 1 \) introduces extra overhead because the initial steps of the exponential portion of the search are too small. Using a \( \Delta \) which is close to the average action cost mitigates this.

### 4.2 Pancake Puzzle

The second experiment studies EBTS in the pancake puzzle where the cost of flipping a stack of \( f \) pancakes is \( 1.0 + \frac{f}{2f + 2} \). As with the sliding-tile puzzle this cost function falls into a small range, but the costs grow by increments of 0.005 as more pancakes are flipped. The heuristic used is the GAP heuristic \([Helmert, 2010]\) without accounting for the new cost function. Experiments are run on 100 hard instances of the 20-pancake puzzle \([Valenzano and Yang, 2017]\).

Experimental results are found in Table 2. A* is able to solve 99 of the problem instances without running out of memory, EBTS* and IDA* both solve all problems. The same trends for EBTS hold in the pancake puzzle as in the sliding-tile puzzle. Using a \( \Delta \) that is too small hurts performance, but EBTS is robust to different resolutions of the cost function.

### 4.3 Inconsistent Heuristics

Finally, we evaluate EBGS in problems with inconsistent heuristics. This experiment uses a graph, shown in Figure 1 adapted from \(\text{Mérö} [1984]\). This graph is parameterized by \( k \) and has \( 2k + 2 \) states. All states have a heuristic of 0 except each state \( t_i \) which has heuristic \( k + i \). Experiments are run with \( k = 100, 1000, 10000 \). Because all edge costs have integer values, we only specify the node expansion window and the initial \( \Delta \) used in the exponential search. A*, B [Martelli, 1977], and B’ [Mérö, 1984] all are expected to perform \( O(S^{\Delta/2}_0) \) expansions on this graph. The results of the experiments are found in Table 3. While A* shows polynomial growth, EBGS has much better performance.

### 5 Conclusions

We have shown that EBSS is able to address the worst-case running time of IDA* and of search inconsistent heuristics. Further research is needed to consider (1) re-using the data

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Table 1: IDA*, A*, and EBS on the weighted 4x4 Sliding Tile Puzzle. Expansions and generations are \( \times 10^6 \).

| Alg. | Solved | Exp. | Gen. | Time (s) |
|------|--------|------|------|----------|
| A*   | 97     | 17.5 | 51.9 | 72.9     |
| IDA* | 57     | 62,044.3 | 185,800.5 | 14,186.9 |
| EBTS(2.5, 1e6, 1) | 100 | 5,401.4 | 16,066.5 | 125.2 |
| EBTS(2.5, 1e6, 1e6) | 100 | 858.6 | 2,555.2 | 202.2 |
| EBTS(10, 20, 1e6) | 100 | 2,062.5 | 6,144.9 | 452.4 |
| EBTS(10, 20, 1e6, 1e6) | 100 | 704.6 | 2,097.6 | 164.7 |
| EBTS(2.5, 1e9, 1) | 100 | 4,876.8 | 14,230.0 | 1,098.9 |
| EBTS(2.5, 1e9, 1e9) | 100 | 855.6 | 2,545.9 | 200.9 |
| EBTS(10, 20, 1e9) | 100 | 2,566.6 | 6,727.0 | 495.5 |
| EBTS(10, 20, 1e9, 1e9) | 100 | 704.8 | 2,098.4 | 165.5 |
| Oracle | 100 | 258.1 | 768.9 | 68.3 |

Table 2: IDA*, A*, and EBS on the weighted Pancake Puzzle. Expansions and generations are \( \times 10^6 \).

| Alg. | Solved | Exp. | Gen. | Time |
|------|--------|------|------|------|
| A*   | 99     | 1.61 | 30.52 | 50.4 |
| IDA* | 100    | 590.73 | 11,223.95 | 1,346.8 |
| EBTS(2.5, 1e6, 1) | 100 | 110.3 | 2,096.3 | 253.4 |
| EBTS(2.5, 1e6, 1e6) | 100 | 20.7 | 393.3 | 48.3 |
| EBTS(10, 20, 1e6) | 100 | 52.0 | 987.1 | 117.8 |
| EBTS(10, 20, 1e6, 1e6) | 100 | 10.3 | 196.2 | 24.1 |
| EBTS(2.5, 1e9, 1) | 100 | 102.8 | 1,952.3 | 240.7 |
| EBTS(2.5, 1e9, 1e9) | 100 | 21.0 | 398.6 | 48.7 |
| EBTS(10, 20, 1e9, 1) | 100 | 72.6 | 1379.2 | 162.4 |
| EBTS(10, 20, 1e9, 1e9) | 100 | 10.3 | 196.2 | 24.1 |
| Oracle | 100 | 6.2 | 118.4 | 14.3 |

Figure 1: A family of graphs from Mérö for which existing algorithms perform \( \Omega(S^4_0) \) expansions.

Table 3: Results with inconsistent heuristics.

| Prob. Size | Alg. | Exp. | Time (s) |
|------------|------|------|----------|
| 100        | A*   | 7,652 | 0.0   |
| 100        | EBGS(2, 5, 1) | 2,230 | 0.0 |
| 100        | EBGS(10, 20, 3) | 1,402 | 0.0 |
| 100        | Oracle | 202 | 0.0 |
| 1000       | A*   | 175,302 | 1.3 |
| 1000       | EBGS(2, 5, 1) | 32,911 | 0.1 |
| 1000       | EBGS(10, 20, 3) | 20,920 | 0.1 |
| 1000       | Oracle | 2,002 | 0.0 |
| 10000      | A*   | 750,015,002 | 1.646.9 |
| 10000      | EBGS(2, 5, 1) | 377,112 | 33.848 |
| 10000      | EBGS(10, 20, 3) | 91,625 | 11.74 |
| 10000      | Oracle | 20,002 | 0.36 |
structures with EBGS and (2) apply the EB approach to other problems, such as suboptimal search.

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