Possible constraints on string theory in closed space with symmetries

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Abstract

It is well known that certain quadratic constraints have to be imposed on linearized gravity in closed space with symmetries. We review this phenomenon and discuss one of the constraints which arise in linearized gravity on static flat torus in detail. Then we point out that the mode with negative kinetic energy, which is necessary for satisfying this constraint, appears to be missing in the free bosonic string spectrum.
I. INTRODUCTION

(Super)string theory is the leading candidate for a unified theory including gravity. In particular, it contains and generalizes Einstein’s general relativity [1, 2, 3]. Therefore, it is natural to expect that the theory incorporates diffeomorphism invariance. However, this invariance is not manifest in the perturbative definition of string theory starting from non-interacting string. Now, it is well known that a solution of linearized Einstein equations (with or without matter fields) in compact background space with Killing symmetries cannot be extended to an exact solution unless the linearized solution satisfies certain quadratic constraints [4, 5]. This phenomenon, called linearization instability, is a consequence of diffeomorphism invariance of the full theory. (This fact can be seen most clearly in the quantum context.) Therefore, one may gain some insight into how diffeomorphism invariance is incorporated in string theory by investigating the way linearization instabilities manifest themselves.

In this article we review the phenomenon of linearization instability in general relativity with emphasis on the case with static flat torus space. In particular, we point out that in this space a mode with negative kinetic term is essential in satisfying one of the constraints and that this mode seems to be missing in the spectrum of free bosonic string theory. The rest of the article is organized as follows. In Section 2 the phenomenon of linearization instability in classical and quantum general relativity is reviewed. In Section 3 one of the constraints occurring in flat torus space is discussed in detail and the importance of a mode with negative kinetic term is emphasized. In Section 4 it is pointed out that this mode is absent in a seemingly natural treatment of the zero-momentum sector of closed bosonic string in this space. In Section 5 a summary of this article is given. The metric signature is \((-+\cdots+)\) throughout this article.

II. LINEARIZATION INSTABILITIES IN GENERAL RELATIVITY

Consider classical general relativity with any bosonic matter fields. Suppose we want to find a solution in this theory order by order in perturbation theory starting from a (globally-hyperbolic) background spacetime satisfying the vacuum Einstein equations \(R_{ab} = 0\). To do so we write the metric \(g_{ab}\) and the matter fields \(\phi_i\) as

\[
\begin{align*}
g_{ab} & = g^{(0)}_{ab} + h^{(1)}_{ab} + h^{(2)}_{ab} + \cdots, \\
\phi_i & = \phi^{(1)}_i + \phi^{(2)}_i + \cdots,
\end{align*}
\]

where \(g^{(0)}_{ab}\) is the background metric and where \(h^{(k)}_{ab}\) and \(\phi^{(k)}_i\) are the fields obtained as the \(k\)-th order approximation. (The fields \(\phi_i\) are assumed to vanish at zero-th order for simplicity.) The first-order approximation \((h^{(1)}_{ab}, \phi^{(1)}_i)\) corresponds to non-interacting waves in the background spacetime. The second-order perturbation of the metric, \(h^{(2)}_{ab}\), can be regarded as the gravitational field generated by the free fields \(h^{(1)}_{ab}\) and \(\phi^{(1)}_i\).

Let the stress-energy tensor of the fields \(h^{(1)}_{ab}\) and \(\phi^{(1)}_i\) in the background spacetime with metric \(g^{(0)}_{ab}\) be \(T^{(1)}_{ab}\). We note first that the linear contribution to the Einstein tensor

\[
E_{ab} = R_{ab} - \frac{1}{2} g_{ab} R
\]
with \( g_{ab} = g_{ab}^{(0)} + h_{ab} \) is

\[
E_{ab}^{(L)}(h) = \frac{1}{2} (\nabla_c \nabla_b h^c_a + \nabla_c \nabla_a h^c_b - \nabla_c \nabla^c h_{ab} - \nabla_a \nabla_b h^c_c)
\]
\[
- \frac{1}{2} g_{ab}^{(0)} (\nabla_c \nabla_d h^{cd} - \nabla_c \nabla^c h^d_d).
\]

Here the covariant derivatives are compatible with the metric \( g_{ab}^{(0)} \) and indices are raised and lowered by this metric. The field \( h_{ab}^{(2)} \) must satisfy

\[
E_{ab}^{(L)}(h) = \kappa T_{ab}^{(1)},
\]

where \( \kappa \) is a constant. The stress-energy tensor \( T_{ab}^{(1)} \) is divergence-free, i.e., \( \nabla^a T_{ab}^{(1)} = 0 \), if the linear equations of motion are satisfied. On the other hand the equation

\[
\nabla^a E_{ab}^{(L)}(h) = 0
\]

holds for any \( h_{ab} \). This is a consequence of the Bianchi identity \( \tilde{\nabla}^a E_{ab} = 0 \), where \( \tilde{\nabla}_a \) is the covariant derivative compatible with the full metric \( g_{ab} \). For this reason Eq. (2) is called the background Bianchi identity.

Now, suppose that there is a Killing vector field \( X^a \) satisfying

\[
\nabla_a X_b + \nabla_b X_a = 0.
\]

Then, it is easy to verify that the current \( j^a_X \equiv T^{(1)ab} X_b \) is conserved. The corresponding conserved Noether charge is given by

\[
Q_X \equiv \int d\Sigma n^a_X j^a_X,
\]

where the integration is over any Cauchy surface \( \Sigma \) and \( n_a \) is the unit normal to the Cauchy surface. (Since \( Q_X \) comes from a stress-energy tensor of the free fields \( h_{ab}^{(1)} \) and \( \phi_{1}^{(1)} \), it is quadratic in these fields.) If the vector \( X^a \) is a time-translation Killing vector, then the charge \( Q_X \) is nothing but the energy. If it is a space-translation Killing vector, then \( Q_X \) is a component of the momentum. We note that

\[
E^{(L)ab}(h) X_b = \frac{1}{2} \nabla_b K^{ab}(h),
\]

where \( K_{ab}(h) \) is an anti-symmetric tensor given by

\[
K_{ab}(h) = X_a \nabla_b h^c_c - X_b \nabla_a h^c_c + X^c \nabla_a h_{bc} - X^c \nabla_b h_{ac}
\]
\[
+ X^c \nabla_a h_{bc} - X^c \nabla_b h_{ac} + h_{ca} \nabla_b X^c - h_{cb} \nabla_a X^c.
\]

Hence, the integral of \( E^{(L)ab}(h) X_b \) over the Cauchy surface can be expressed as a surface integral as

\[
\int d\Sigma n_a E^{(L)ab}(h) X_b = \frac{1}{2} \int_{\partial\Sigma} dS n_a r_b K^{ab}(h),
\]
where $\partial \Sigma$ is the “boundary” of the Cauchy surface at infinity and $r_a$ is the unit vector normal to the boundary along the Cauchy surface. By using this expression and Eq. (1), one can write the Noether charge $Q_X$ as a surface integral:

$$Q_X = \frac{1}{2\kappa} \int_{\partial \Sigma} dS n_a r_b K^{ab}(h^{(2)}) .$$

(3)

In asymptotically-flat spacetime this equation allows us to express energy and momentum of an isolated system as surface integrals at spacelike infinity.

Now, suppose that the Cauchy surface is compact, i.e., that the space is “closed”. Then, the right-hand side of Eq. (3) must vanish for any $h_{ab}$ because there is no surface term. Hence,

$$Q_X = 0 .$$

(4)

Thus, the conserved charge $Q_X$ is constrained to vanish. Note that this constraint cannot be derived from the linearized theory alone. It arises in the full theory when we try to find the correction to the linear theory. Solutions of the linearized field equations are not extendible to exact solutions unless they satisfy this constraint. (The background spacetime here is said to be linearization unstable because of the existence of spurious solutions to the linearized equations. The constraint (4) is sometimes called a linearization stability condition.)

Although we will concentrate on classical theory, it is interesting to note what the constraint (4) implies in quantum theory. In the Dirac quantization, constraints are imposed on the physical states. Thus, the quantum version of (4) reads

$$Q_X |\text{phys}\rangle = 0 ,$$

(5)

where $|\text{phys}\rangle$ is any physical state and $Q_X$ is the quantum operator corresponding to the conserved Noether charge $Q_X$. Since the operator $Q_X$ generates the spacetime symmetry associated with the Killing vector field $X^a$, the constraint (5) implies that all physical states must be invariant under this spacetime symmetry. This requirement might seem absurdly strong at first sight. For example, in linearized gravity in de Sitter spacetime all physical states are required to be de Sitter invariant. However, in the (formal) Dirac quantization of full general relativity, the states are (roughly speaking) required to be diffeomorphism invariant. The constraint (5) can be interpreted to be enforcing the part of the diffeomorphism invariance of the physical states that has not been broken by the background metric.

III. THE HAMILTONIAN CONSTRAINT OF LINEARIZED GRAVITY ON FLAT TORUS

In this section we discuss linearized gravity in static flat $(D-1)$-dimensional torus space with all directions compactified. This spacetime has space- and time-translation invariance. Therefore, the energy and momentum of linearized gravity are conserved and are both constrained to vanish. Below we concentrate on the linearization stability condition which requires that energy be zero since it will be important later in the discussion of string theory. We find that there is a mode with negative kinetic term and that there would be no excitation as a result of the linearization stability condition if it were not for this mode. We consider only pure gravity for simplicity.
Let us impose the standard (“Lorenz” or Hilbert) gauge condition

\[ \partial_a h^{ab} = \frac{1}{2} \partial^b h, \quad (6) \]

where \( h = h^c_c \). Then the Hamiltonian density reads

\[ H = \frac{1}{4} \left[ \partial_t \tilde{h}_{ab} \partial_t \tilde{h}^{ab} + \partial_i \tilde{h}_{ab} \partial^i \tilde{h}^{ab} \right] - \frac{D - 2}{4D} \left[ (\partial_t h)^2 + \partial_i h \partial^i h \right], \]

where \( \tilde{h}_{ab} = h_{ab} - \frac{1}{D} g_{ab} h \) is the traceless part of \( h_{ab} \). The index \( i \) runs from 1 to \( D - 1 \), i.e., it is a spacelike index. The field equations are simply

\[ \Box \tilde{h}_{ab} = 0, \quad \Box h = 0. \]

The modes with nonzero momentum \( k \) are proportional to \( e^{-ik^0 t + i k \cdot x} \), where \((k^0)^2 - k^2 = 0\). On the other hand, the modes with \( k = 0 \) take the form

\[ \tilde{h}_{ab}, \ h \propto At + B, \]

where \( A \) and \( B \) are constants.\[11\]

The Hamiltonian can be written as

\[ H = \int d^{D-1}x \mathcal{H} = H_0 + H', \]

where \( H_0 \) is the energy in the modes with \( k = 0 \) and where \( H' \) is the energy in the modes with \( k \neq 0 \). For the modes with \( k \neq 0 \) the trace \( h \) can be gauged away and the physical modes have the form

\[ \tilde{h}_{ab} \propto H_{ab} e^{-ik^0 t + i k \cdot x}, \]

where \( H_{ab} \) is a constant symmetric tensor satisfying \( H_{tb} = 0, H^i_i = 0 \) and \( k^i H_{ij} = 0 \). Then we can easily see that \( H' \geq 0 \). The situation is rather different for the modes with \( k = 0 \). Since these modes are constant in space, they satisfy \( \partial_i \tilde{h}_{ab} = \partial_i h = 0 \). Hence, the conditions coming from (6) are \( \partial_t \tilde{h}_{tt} = 0 \) and

\[ \partial_t \tilde{h}_{tt} = -\frac{D - 2}{2D} \partial_t h. \]

Let us write

\[ \tilde{h}_{ab} = \tilde{h}_{ab}^{(0)} + \tilde{h}_{ab}', \quad \tilde{h} = \tilde{h}^{(0)} + \tilde{h}', \]

where \( \tilde{h}_{ab}^{(0)} \) and \( \tilde{h}^{(0)} \) are the zero-momentum parts of \( \tilde{h}_{ab} \) and \( h \). Then the zero-momentum Hamiltonian \( H_0 \) is given by

\[ H_0 = \int d^{D-1}x \left[ \frac{1}{4} \partial_i \tilde{h}_{ij}^{(0)} \partial_t \tilde{h}^{(0)ij} - \frac{D^2 - 4}{8D} (\partial_t \tilde{h}^{(0)})^2 \right]. \]

Notice that the trace mode \( h^{(0)} \) has a negative kinetic term.
Since the Hamiltonian is the Noether charge corresponding to the time-translation symmetry of the background spacetime, the discussion in the previous section shows that

$$H = H_0 + H' = 0.$$  

The solutions of the linearized equations which do not satisfy this condition cannot be extended to exact solutions. This equation can be re-expressed as

$$-\frac{D^2 - 4}{8D} \int d^{D-1}x (\partial_t h^{(0)})^2 + H'' = 0,$$

where

$$H'' = H' + \frac{1}{4} \int d^{D-1}x \partial_t \tilde{h}^{(0)}_{ij} \partial_t \tilde{h}^{(0)}_{ij} \geq 0.$$  

Now, the quantity $\frac{1}{2} h^{(0)} V$, where $V$ is the volume of the background space, is the change in the volume of the space. Hence, Eq. (7) relates the expansion/contraction rate of space to the energy due to the excitation of the system. In fact this equation is the linearized version of a familiar equation in cosmology. Notice that the trace mode $h^{(0)}$ plays a vital role in satisfying Eq. (7). If this mode were absent, Eq. (7) would imply that there were no excitations on flat torus compactified in all directions.

IV. MASSLESS SECTOR OF BOSONIC STRING IN THE POSITION REPRESENTATION

Massless excitations of closed string include gravitons, i.e., linearized gravity is present among the modes of free closed bosonic string in Minkowski spacetime.\[12\] This fact is one of the most important features of string theory as a unified theory. It is natural to expect that this feature persists in string theory in static flat torus compactified in all directions. Therefore, the total energy and momentum in string (field) theory are expected to vanish in this spacetime. We also expect that there is a mode with negative kinetic term among the closed-string modes so that the linearization stability condition (7) can be satisfied by non-vacuum states (in string field theory). However, we will find in the “old covariant approach” that there is no massless string excitation which corresponds to the zero-momentum mode $h^{(0)}$ with negative kinetic energy if we treat the zero-momentum modes in a way which seems most natural.

Let us start with a discussion of open string in flat $(D - 1)$-dimensional torus. The massless states in the old covariant approach are denoted by

$$\alpha^a_{-1}|0;p\rangle,$$

where the state $|0;p\rangle$ with momentum $p^a$ has no string excitation (see, e.g., Ref. [9]). The creation operator $\alpha^a_{-1}$ creates the lowest harmonic-oscillator mode on the string in the $a$-direction and the annihilation operator $\alpha^a_1$ annihilates it. As is well known, the physical state conditions lead to $p^2 = 0$ and $p \cdot \alpha_1|\text{phys}\rangle = 0$, where $[\alpha^a_1, \alpha^b_{-1}] = g^{ab}$ and $p \cdot \alpha_1 \equiv p_a \alpha^a_1$. [Here, $g_{ab} = \text{diag}(-1, 1, 1, \ldots, 1).$] Let us consider a wave-packet state

$$|\psi\rangle = \int \frac{d^Dp}{(2\pi)^D} \hat{A}_a(p) \alpha^a_{-1}|0;p\rangle,$$
where \( \hat{A}_a(p) \) is a function of \( p^a \). The physical state conditions then read \( p^2 \hat{A}_a(p) = 0 \) and \( p^a \hat{A}_a(p) = 0 \). Now, define the (spacetime) position representation of this wave packet as

\[
A_a(x) = \int \frac{d^D p}{(2\pi)^D} \hat{A}_a(p)e^{-ip \cdot x}.
\]

Then the physical state conditions become \( \Box A_a = 0 \) and \( \partial^a A_a = 0 \). Thus, we recover the equations satisfied by a non-interacting \( U(1) \) gauge field in the Lorenz gauge. The zero-momentum modes in flat \((D-1)\)-dimensional torus satisfy

\[
\partial_t A_t = 0, \quad \partial_i A_i = 0.
\]

These imply that \( A_t = \text{const} \) and \( A_i = E_i(t + A_i^{(0)}) \). The constant \( A_t \) can be gauged away, but the constants \( E_i \) (the electric field) and \( A_i^{(0)} \) represent physical degrees of freedom.

Next, we will apply the above procedure to a closed string on static flat torus and examine whether or not there is a mode with negative kinetic term. The massless excitations of a closed bosonic string are \( \alpha^{a-1}_a \alpha^{b-1}_b |0; p\rangle \). The operator \( \alpha^{a-1}_a \) (\( \tilde{\alpha}^{a-1}_a \)) creates the lowest left-moving (right-moving) mode on the string in the \( a \)-direction, and the operator \( \alpha^1_a \) and \( \tilde{\alpha}^1_a \) annihilate them. The physical state conditions lead to \( p^2 = 0 \) and \( p \cdot \alpha^1 |\text{phys}\rangle = p \cdot \tilde{\alpha}^1 |\text{phys}\rangle = 0 \), where \([\alpha^1_a, \alpha^{b-1}_b] = [\tilde{\alpha}^1_a, \tilde{\alpha}^{b-1}_b] = g^{ab} \). We again consider a wave-packet state

\[
|\Psi\rangle = \int \frac{d^D p}{(2\pi)^D} \hat{H}_{ab}(p) \alpha^{a-1}_a \alpha^{b-1}_b |0; p\rangle.
\]

(Note here that the tensor \( \hat{H}_{ab}(p) \) is not necessarily symmetric.) The physical state conditions read \( p^2 \hat{H}_{ab}(p) = 0 \) and \( p^a \hat{H}_{ab} = p^b \hat{H}_{ab} = 0 \). In the spacetime position representation,

\[
H_{ab}(x) = \int \frac{d^D p}{(2\pi)^D} \hat{H}_{ab}(p)e^{-ip \cdot x},
\]

the physical state conditions are \( \Box H_{ab} = 0 \) and

\[
\partial^a H_{ab} = \partial^b H_{ab} = 0.
\]

(8)

The equation \( \Box H_{ab} = 0 \) naturally comes from the following Lagrangian density:

\[
\mathcal{L} = -\frac{1}{4} \partial_a H_{bc} \partial^a H^{bc}.
\]

(9)

The constraints (8) can be imposed by hand. One finds the modes corresponding to gravitons, anti-symmetric tensor particles and dilatons in the nonzero momentum sector of this theory as in Minkowski spacetime. The constraints (8) for the zero-momentum sector read

\[
\partial_t H_{ta} = \partial_t H_{at} = 0
\]

for all \( a \). The energy in the zero-momentum sector is

\[
E_0 = \frac{1}{4} \int d^{D-1} x \partial_i H_{ij} \partial_i H^{ij},
\]

where \( i, j = 1, 2, \cdots D-1 \). There is no mode with negative kinetic term in this expression, and \( E_0 \) is positive definite. Thus, the negative-energy mode, which is necessary for non-vacuum states to satisfy the constraint (14), does not appear in a seemingly natural position representation of the massless sector of closed bosonic string.
V. SUMMARY

In this article, we reviewed the fact that quadratic constraints arise in linearized gravity if the background spacetime allows Killing symmetries and has compact Cauchy surfaces. This implies that the total energy and momentum in free string (field) theory should be constrained to vanish in flat torus space with all directions compactified. We examined one of these constraints in linearized gravity in this space, emphasizing that a mode with negative kinetic energy is essential in satisfying this constraint. Then we analyzed free closed bosonic string theory in this space and found that this mode does not appear in a seemingly natural treatment of the massless sector.

It is possible that the Lagrangian density (9) is wrong, and a more careful analysis may lead to a Lagrangian density describing the usual linearized gravity, anti-symmetric tensor gauge field and dilaton scalar field after all. It will be interesting to see how this can be achieved. The situation is rather puzzling, however, because string theory is formulated in terms of a physical object, i.e. a string, and does not seem to allow any mode with negative kinetic energy.

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[9] M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory: vol. 1. Introduction*, Cambridge University Press, Cambridge, 1987, pp. 113–116.
[10] The vacuum state is the only de Sitter invariant state if one insists on using the original Fock space of linearized gravity, but one can construct infinitely many invariant states by using a different Hilbert space [8].
[11] Note that the energy corresponding to these modes would be infinite for $A \neq 0$ if the space were not compactified. This is why these modes would not be present in uncompactified space.
[12] This fact goes beyond the linearized level as well known [1, 2, 3].