THE EQUIVALENCE THEORY FOR INFINITE TYPE
HYPERSURFACES IN $\mathbb{C}^2$

PETER EBENFELT, ILYA KOSSOVSKIY, AND BERNHARD LAMEL

ABSTRACT. We develop the classification theory for real-analytic hypersurfaces in $\mathbb{C}^2$ in the case when the hypersurface is of infinite type at the reference point. This completes the solution of the Problème local raised by H. Poincaré in 1907 in complex dimension 2. To accomplish this, we use the recent CR – DS technique in CR-geometry.

Contents

1. Introduction 2
   1.1. Overview 2
   1.2. Historical background for the Problème local 3
   1.3. The classification results 5
   1.4. Regularity results 9
   1.5. Principal method 10
   1.6. Concluding remark 11
2. Preliminaries 11
   2.1. Infinite type real hypersurfaces 11
   2.2. Real hypersurfaces and second order differential equations. 13
   2.3. Complex differential equations with an isolated singularity 14
3. Reduction to the classification problem for ODEs 16
4. The general approach to the classification of ODEs 20
   4.1. Reduction to a Cauchy problem 20
   4.2. A singular system of ODEs for the Cauchy data 23
5. Case $m = 1$ 26
6. Case $m > 1$ 30
7. Solution of the equivalence problem 33
8. Fuchsian type ODEs 39
   8.1. The normal form problem for Fuchsian type hypersurfaces 39
   8.2. Regularity of mappings between Fuchsian type hypersurfaces 44
References 47

The first author was supported in part by the NSF grant DMS-1600701. The second author was supported in part by the Czech Grant Agency (GACR) and the Austrian Science Fund (FWF). The third author was supported in part by the Austrian Science Fund (FWF).
1. Introduction

1.1. Overview. In 1907, Poincaré [41] initiated the study of the (bi)holomorphically invariant geometry of real hypersurfaces in complex spaces. He formulated the classification problem in the following form:

**Problème local (H. Poincaré, 1907).** Given two germ \((M,p)\) and \((M^*,p^*)\) of real-analytic hypersurfaces in \(\mathbb{C}^2\), find all local biholomorphic maps \(F: (\mathbb{C}^2, p) \mapsto (\mathbb{C}^2, p^*)\) mapping the hypersurfaces into each other: \(F(M) \subset M^*\).

Even though he did not succeed in solving this problem, Poincaré discovered that germs of real hypersurfaces have local invariants and possess strong rigidity properties. He showed that given any real hypersurface \(M\), most other real hypersurface \(M^*\) are not holomorphically equivalent to the given one. As a model for real hypersurfaces in \(\mathbb{C}^2\), he employed the (unit) sphere \(S^3 \subset \mathbb{C}^2\). He thought of a generic hypersurface in \(\mathbb{C}^2\) as a certain perturbation of the model, and showed that the dimension of the automorphism group of the model is an upper bound for that of a perturbed hypersurface. This work of Poincaré is often considered as a starting point of Cauchy-Riemann geometry, or CR-geometry for short, and has inspired intensive research in this subject in the intervening century. In particular, the original **Problème local** of Poincaré mentioned above has attracted a lot of attention starting with E. Cartan’s solution [9, 8] of the problem for strictly pseudoconvex hypersurfaces in \(\mathbb{C}^2\). The general problem, however, has remained open. In this paper, we complete the solution to the **Problème local** for real hypersurfaces in a complex euclidean space of dimension 2. We do so by developing a classification theory for real-analytic hypersurfaces in \(\mathbb{C}^2\) at points of infinite type.

Before giving a historical background for the **Problème local** and stating our main results, we should point out that the theory we develop shares many distinctive traits with the Poincaré-Dulac classification theory for vector fields. The latter was also initiated by H. Poincaré as a way to classify (and analyze the dynamics of) vector fields at their singular points. Poincaré suggested first bringing the germ of a vector field at a singularity to a normal form, known as the Poincaré-Dulac normal form, and then integrating the vector field in the new, normalized coordinates. For constructing the normal form, Poincaré developed the homological method: one compares the Taylor expansion of a given vector field \(X\) at a singular point \(p\) with that of its model, namely, the linearization of \(X\) at \(p\), and attempts to approximate \(X\) by the model as closely as possible. By using the homological method, Poincaré discovered resonances — obstructions for linearizing a vector field, existence of which depends on the the spectrum of the linear part of a vector field at a singularity. The Poincaré-Dulac normal form can be either convergent (in case the spectrum of the linear part lies in the Poincaré domain) or divergent (the case of the Siegel domain). In the latter case, the normal form is merely formal and gives only a proper subset of the complete set of invariants, and therefore there are analytic invariants of a singularity supplementing the formal normal form. Existence of such invariants forms the content of the Stokes phenomena for the classification problem under discussion. For more details here we refer to the excellent book [22] of Ilyashenko-Yakovenko on the subject, and to Lombardi-Stolovitch [36] for some important recent developments.
We emphasize that even though the Poincaré-Dulac theory provides a satisfactory and in a sense complete solution to the classification problem for singularities of vector fields, some aspects of the holomorphic classification (such as the situation of vanishing linear part at a singularity or various Stokes phenomena) are still not understood completely. Similar exceptions occur in our resolution of the Problème local.

1.2. Historical background for the Problème local. As discussed above, the classification problem for real-analytic hypersurfaces in complex space was first considered by Poincaré, and followed by Cartan’s solution in the case of Levi-nondegenerate real hypersurfaces in $\mathbb{C}^2$. The classification of Levi-nondegenerate hypersurfaces in $\mathbb{C}^N$, $N \geq 2$, has been completed by Tanaka [45] and Chern-Moser [11]. These works used two principal approaches to the equivalence problem. Tanaka and Chern employed a differential geometric approach, extending and developing E. Cartan’s method of equivalence. This approach relies heavily on a certain uniformity of the geometric structure under consideration. In contrast, Moser’s approach in [11] used another method to solve the classification problem (adapting and developing the homological method of Poincaré), which has been more successfully further adapted to the Levi-degenerate situation. This approach is via normal forms: instead of comparing to a geometric model situation, one tries to find a unique coordinate choice by prescribing normalization conditions of the defining equation of $M \subset \mathbb{C}^N$ in these coordinates. The key ingredient in the standard application of this approach is finding the right model, for which Moser used real hyperquadrics

$$\text{Im } w = H(z, \bar{z}),$$

where $H$ is a nondegenerate Hermitian form on $\mathbb{C}^{N-1})$. The normal form is then obtained by approximating, in a suitable sense, a general hypersurface by its model as closely as possible.

We shall remark at this point that, in what follows, we will be distinguishing between three different notions of equivalence. We say that two germs of real-analytic hypersurfaces $(M, p)$, $(M', p')$ in $\mathbb{C}^N$ are formally equivalent if there exists a germ of a formal map $H \in \mathbb{C}[Z - p]^N$ which satisfies $H(M) \subset M'$ and $H(p) = p'$. We say that $(M, p)$ and $(M', p')$ are biholomorphically equivalent if there exists a germ of a biholomorphism $H \in \mathbb{C}\{Z - p\}^N$ such that $H(M) \subset M'$ and $H(p) = p'$. And lastly, we say that $(M, p)$ and $(M', p')$ are CR equivalent if there exists a smooth CR diffeomorphism $h: U \to M'$ defined in a neighbourhood $U$ of $p$ in $M$ with $h(p) = p'$. Here we recall that $h$ is CR if its component functions are smooth CR functions on $M$, or equivalently, if the differential $dh$ restricts to a complex linear map of the complex tangent spaces $T_p^c M = T_p M \cap iT_p M$, $dh_p: T_p^c M \to T_p^c M'$. We note that a biholomorphic equivalence restricts to a CR equivalence. Moreover, a CR equivalence induces a formal equivalence. Coming back to the normal form problem, we distinguish between formal and convergent normal forms. A formal normal form is given by the construction of formal coordinates $(z^*, w^*) = H(z, w) \in \mathbb{C}\{z, w\}^N$, while a convergent normal form is given by the construction of (local) holomorphic coordinates $H(z, w) \in \mathbb{C}\{z, w\}^N$. A formal normal form solves
the formal equivalence problem, whereas a convergent normal form solves the biholomorphic equivalence problem. Notably, Moser’s normal form is convergent, as is proved in [11].

We next outline the progress on the classification problem in the *Levi-degenerate case*, when some (and sometimes all) ingredients of both the Cartan-Tanaka-Chern approach and Moser’s approach are not applicable. Let us recall first that a real-analytic hypersurface $M \subset \mathbb{C}^2$ is of finite type (at a point $p \in M$) if it does not contain a holomorphic curve through $p$. (Such a hypersurface also satisfies the Hörmander-Kohn bracket-generating condition at $p$ [3]). A construction of a formal normal form for finite type (but Levi-degenerate) hypersurfaces in $\mathbb{C}^2$ was carried out by Kolář in [25]. He employed models of the type

$$\text{Im } w = P(z, \bar{z}),$$

where $P(z, \bar{z})$ is a nonzero homogeneous polynomial without harmonic terms of degree $k \geq 3$. Notably, for the class of finite type hypersurfaces, it is known by the work of Baouendi-Ebenfelt-Rothschild [1] that every formal holomorphic map actually converges. Therefore, remarkably, Kolář’s formal normal form for finite type hypersurfaces provides a solution to the biholomorphic equivalence problem for these hypersurfaces. It also provides a solution to the CR equivalence problem, because every CR diffeomorphism of two real-analytic hypersurfaces of finite type in $\mathbb{C}^2$ extends to a biholomorphic map by a result of Baouendi, Jacobowitz, and Treves [2]. Kolář’s normal form has been shown to be convergent under certain geometric conditions, see the work [33] of Kossovskiy-Zaitsev, but is divergent in general [26]. For Levi-degenerate hypersurfaces in $\mathbb{C}^N$, $N \geq 3$ satisfying certain special conditions (in addition to the Hormander-Kohn bracket-generating condition) normal form constructions were carried out by Ebenfelt [15, 14] and by Kossovskiy-Zaitsev [33]. For results on normal forms for real submanifolds of higher codimension as well as CR-singular submanifolds we refer to Ezhov-Schmalz [18], Beloshapka [4], Lamel-Stolovitch [35], Moser-Webster [39], Huang-Yin [20, 21], Gong [19], Coffman [12], Burcea [7]. See also Zaitsev [48] for normal forms in the non-integrable setting. More references and discussion of the normal form problem can be found in the survey [24].

The situation changes dramatically when one considers *infinite type* hypersurfaces in $\mathbb{C}^2$, that is, hypersurfaces $M \subset \mathbb{C}^2$ which contain a complex curve $X \subset M$ through the reference point $p$. We first remark that the automorphism aspect of the *Problème local* (i.e., describing possible automorphism algebras of real-analytic hypersurfaces in $\mathbb{C}^2$) in the infinite type setting was addressed in the paper [31] by Kossovskiy-Shafikov. However, the equivalence problem in this setting appears to be more difficult, and classification results are available only for some particular classes of infinite type hypersurfaces. Even the existence of a formal normal form is only known in a particular setting (the 1-infinite type case) by work of Ebenfelt-Lamel-Zaitsev [17]. We shall outline some of the difficulties that arise. First of all, one of the main difficulties for providing even a formal normal form here is perhaps the absence of polynomial models for the problem. For example, in the class of infinite type hypersurfaces

$$\text{Im } w = (\text{Re } w)\psi(|z|^2), \quad \psi(0) = 0, \psi'(0) \neq 0,$$
all of which contain the complex hypersurface \( X = \{ w = 0 \} \), any polynomial model has an automorphism (isotropy) group of dimension 2, while the hypersurface \( \text{Im } w = (\text{Re } w) \tan \left( \frac{1}{2} \arcsin |z|^2 \right) \) has an automorphism group of dimension 5 (see [3], [27]). This fact completely rules out the concept of a model in the sense of Poincaré-Moser, and thus the strategy of using polynomial models entirely fails in the infinite type setting. Secondly, and probably even more importantly, the connection between different notions of equivalence in the infinite type setting is more subtle and only became clearer over the last few of years, by work of Kossovskiy, Lamel and Shafikov: There exist infinite type hypersurfaces \( M \) and \( M^* \) in \( \mathbb{C}^2 \) that are formally but not biholomorphically equivalent [32], and there also exist \( M \) and \( M^* \) that are CR equivalent but not biholomorphically equivalent [29]. On the other hand, it was shown by Kossovskiy-Lamel-Stolovitch [30] that in this setting, every formal equivalence arises as the Taylor series of a CR equivalence and, hence, \( M \) and \( M^* \) are formally equivalent if and only if they are CR equivalent.

Before describing the main results of this paper in more detail, we mention that there is a powerful analogy explaining the distinction between finite type and infinite type hypersurfaces by comparing with the situation of regular and singular ordinary differential equations (ODEs): While formal solutions of regular (analytic) ODEs converge, solutions of singular (analytic) ODEs might diverge, but often extend to actual (smooth) solutions of the singular ODEs in sectors. (Actually, this is more than just an analogy, as we shall discuss in more detail below.)

1.3. The classification results. In light of the discussion above concerning the classification at finite type points, we need to deal with the classification of germs of Levi-nonflat real-analytic hypersurfaces \( M \subset \mathbb{C}^2 \), considered near a point of infinite type \( p \in M \). If \( M \) is such a hypersurface, there is a unique germ of a complex hypersurface (complex curve) \( X \subset M \) passing through \( p \). The complex hypersurface \( X \) consists of all infinite type points in \( M \) near \( p \), it is nonsingular and we will also refer to it as the infinite type locus of \( M \). We say that \((M, p)\) is of \textit{generic infinite type} if the canonical extension of the Levi form from \( M \) to its complexification \( M^C \subset \mathbb{C}^2 \times \mathbb{C}^2 \) locally vanishes only on the complexification \( X^C \subset \mathbb{C}^2 \times \mathbb{C}^2 \) of \( X \). (We refer the reader to Section 2 for details). If \( M \) is a Levi-nonflat real-analytic hypersurface with infinite type locus \( X \), then \( M \) must be of generic infinite type at points \( p \) lying outside of a proper real-analytic subset of \( X \). In what follows, we shall consider only germs \((M, p)\) of generic infinite type.

We say that local holomorphic coordinates \((z, w)\), where \( w = u + iv \), near \( p \) are \textit{admissible} (for \( M \)) if in these coordinates, \( p \) becomes the origin and \( M \) is given by

\[
(1.1) \quad v = \frac{1}{2} u^m \left( \varepsilon |z|^2 + \sum_{k,l \geq 2} h_{kl}(u) z^k \bar{z}^l \right) =: h(z, \bar{z}, u), \quad \varepsilon = \pm 1
\]

(such admissible coordinates always exist under the generic infinite type assumption, see [31]); in particular, in these coordinates \( X = \{ w = 0 \} \). The integer \( m \geq 1 \) is an important invariant of an infinite type hypersurface; if we want to be explicit, we are going to say that a hypersurface \( M \) as defined by (1.1) is of \textit{m}-infinite type (see Section 2 for details). For an even \( m \), we can further normalize \( \varepsilon \) to be equal to 1, while for an odd \( m \), \( \varepsilon \) is a
biholomorphic invariant. Note that the form (1.1) is stable under the group of dilations
(1.2)
\[ z \mapsto \lambda z, \quad w \mapsto \mu w, \quad \mu^{1-m} = e|\lambda|^2, \quad \lambda \in \mathbb{C} \setminus \{0\}, \quad \mu \in \mathbb{R}. \]

We shall construct a (formal or holomorphic) normal form for hypersurfaces of the form (1.1). By this, we mean that we find a choice of (formal or holomorphic) admissible coordinates which become essentially unique by requiring more conditions on the defining function \( h \) in (1.1). As already noted above, we exploit a link between mappings of infinite type hypersurfaces with the theory of singular ODEs. As a result, very analogously to the situation in the Poincaré-Dulac theory, we get a distinction between nonresonant and resonant hypersurfaces, and between classes of hypersurfaces leading to either convergent or divergent normal forms. (We shall note that the existence of possible resonances for the problem became clear already from the above mentioned work [17] of Ebenfelt-Lamel-Zaitsev). Although we need to refer the reader to Sections 5-6 for a detailed discussion of the resnonancy conditions, we point out here that we will give explicit polynomials \( p \), defined on the space of 7-jets of defining functions \( h \) in (1.1) at 0, such that the vanishing of these polynomials corresponds to the resonant hypersurfaces.

It turns out that for the case \( m = 1 \), the non-resonancy condition alone guarantees that normal form we construct is convergent (which is somehow expected by the convergence result for CR-maps due to Juhlin-Lamel [23]). In contrast, in the non-resonant \( m > 1 \) case we construct a formal normal form, which is in general divergent. By the result of the Kossovskiy-Shafikov cited above, we can not expect any better here. We refer the reader to Section 6 for details of the non-resonancy condition in the case \( m > 1 \). This formal normal form still solves the smooth CR-equivalence problem (see Theorem 2 below). Finally, within the hypersurfaces with \( m > 1 \), we are able to distinguish a subclass, in some sense optimal, of regular (Fuchsian type) hypersurfaces for which a convergent normalization is possible, again under an appropriate non-resonancy condition (see Section 8 for details). We shall now describe the normalization conditions in our normal form.

We introduce, for each \( m \geq 1 \), the space \( \mathcal{N}_m \) of (formal) power series \( h(z, \bar{z}, u) \), as in (1.1), satisfying, in addition,

(1.3) \[ h'_{22}(u) = h'_{23}(u) = h'_{32}(u) \equiv 0, \quad h^{(j)}_{33}(0) = 0, \quad j \notin \{0, m-1\}. \]

In other words, we have

(1.4) \[ h_{22}(u) = h_{22}(0), \quad h_{23}(u) = h_{23}(0), \quad h_{32}(u) = h_{32}(0), \quad h_{33}(u) = r + su^{m-1}, \]

where \( r, s \in \mathbb{R} \) are constants.

**Definition 1.1.** We say that a hypersurface \( M \) of the form (1.1) is in normal form if its defining function \( v = h(z, \bar{z}, u) \) satisfies (1.3).

Obviously, the dilations (1.2) preserve the property of being in normal form. We now formulate our normalization results in the cases \( m = 1 \) and \( m > 1 \), respectively.

**Theorem 1.** Let \( M \subset \mathbb{C}^2 \) be a real-analytic Levi-nonflat hypersurface, which is of 1-infinite type at a point \( p \in M \), and assume that that \( p \) is a non-resonant point for \( M \). Then there exists a biholomorphic transformation

(1.5) \[ H : (\mathbb{C}^2, p) \mapsto (\mathbb{C}^2, 0) \]
bringing $M$ into normal form. The normalizing transformation $H$ is uniquely determined by the restriction of its differential $dH|_p$ to the complex tangent $T^\mathbb{C}_pM$.

**Remark 1.2.** We remark that the normal form in Theorem 1 solves the holomorphic equivalence problem in the following way: Two hypersurfaces $M, M^*$ as above are biholomorphically equivalent at their points $p, p^*$ respectively, if and only if in some (and hence any) normal forms at these points there exists a dilation (1.2) transforming the normal form of one into the other.

Note that the normal form in Theorem 1 coincides with the formal normal form obtained in [17]. Thus, Theorem 1 proves convergence of the normal form in [17]. (We point out, however, that the normal forms in [17] and in the present paper are obtained in completely different ways).

A typical application of any normal form is a bound on the dimension of the isotropy algebras of the hypersurfaces under consideration, which we also have here.

**Corollary 1.3.** For a hypersurface $(M, p)$ satisfying the assumptions of Theorem 1, we have $\dim_{\mathbb{R}} \text{aut}(M, p) \leq 2$.

Another application that our normal form allows for is the unique determination of a CR-map between hypersurfaces by its finite jet at a point.

**Corollary 1.4.** Let $H : (M, p) \mapsto (M^*, p^*)$ be a local biholomorphism of two hypersurfaces satisfying the assumptions of Theorem 1. Then $H$ is uniquely determined by its $1$-jet at the point $p$.

Note that Corollary 1.4 gives a specific bound for the jet order needed for determination, thereby improving the general finite jet determination result of Ebenfelt-Lamel-Zaitsev in [13] in this specific case.

For $m > 1$, we first give the general formal normal form.

**Theorem 2.** Let $M \subset \mathbb{C}^2$ be a real-analytic Levi-nonflat hypersurface, which is of $m$-infinite type at a point $p \in M$ for some $m \geq 1$, and assume that $p$ is a non-resonant point for $M$. Then there exists a formal invertible transformation

$$H : (\mathbb{C}^2, p) \mapsto (\mathbb{C}^2, 0)$$

(1.6)

bringing $M$ into normal form. The normalizing transformation $H$ is uniquely determined by the restriction of its differential $dH|_p$ to the complex tangent $T^\mathbb{C}_pM$ and the transverse $m$-th order derivative of the transverse component of $H$.

**Remark 1.5.**

- For any coordinates $(z, w)$ where $p = 0$ and $T^\mathbb{C}_p = \{w = 0\}$, a normalizing transformation $H = (F, G)$ in Theorem 2 is uniquely determined by the complex parameter $\lambda := F_z(0, 0)$ and by the real parameter

$$\tau := \frac{1}{m!} \frac{\partial^m G}{\partial w^m}(0, 0).$$

(1.7)

Alternatively, a normalizing transformation is determined uniquely, up to the right action of the group of dilations (1.2) and the flow of the vector field

$$w^m \frac{\partial}{\partial w}.$$
The normal form in Theorem 2 solves the $C^\infty$ CR equivalence problem as follows: two hypersurfaces $M, M^*$ as above are $C^\infty$ CR equivalent at their points $p, p^*$ respectively, if and only if some normal forms of them at these points coincide. This fact follows from [30]. Furthermore, it is possible to verify from [30] that the normal form solves the equivalence problem even in the stronger class of multi-summable (in particular, Gevrey regular) transformations. We will not provide the details of this here and refer an interested reader to [30].

Corollary 1.6. For a hypersurface $(M, p)$ satisfying the assumptions of Theorem 2, we have $\dim_{\mathbb{R}} \text{aut}(M, p) \leq 3$.

Corollary 1.7. Let $H : (M, p) \mapsto (M^*, p^*)$ be a formal biholomorphism of two hypersurfaces satisfying the assumptions of Theorem 1. Then $H$ is uniquely determined by its $m$-jet at the point $p$, where $m$ is the nonminimality order of $M$ at $p$.

Again, Corollary 1.7 gives a specific jet bound for the general jet determination result of Ebenfelt-Lamel-Zaitsev [13] in this particular setting; in addition, it is also valid for formal biholomorphism (which is not the case in [13]).

It turns out that even for $m > 1$ it is possible to distinguish a class of real hypersurfaces (1.1) (Fuchsian type hypersurfaces) with a convergent normal form.

Definition 1.8. A hypersurface (1.1) is called a hypersurface of Fuchsian type, if its defining function $h(z, \bar{z}, u)$ satisfies

\[
\begin{align*}
\ord h_{22}(w) &\geq m - 1; \quad \ord h_{23}(w) \geq 2m - 2; \quad \ord h_{33}(w) \geq 2m - 2; \\
\ord h_{2l}(w) &\geq 2m - l + 2, \quad 4 \leq l \leq 2m + 1; \\
\ord h_{kl}(w) &\geq 2m - k - l + 5, \quad k \geq 3, \quad l \geq 3, \quad 7 \leq k + l \leq 2m + 4.
\end{align*}
\]

(here $m \geq 1$).

We point out that

- The Fuchsian condition requires vanishing of an appropriate part of the $(2m + 4)$-jet of the defining function $h$ at 0;
- It is easy to see from (1.9) that for $m = 1$ the Fuchsian type condition holds automatically, while for $m > 1$ it fails to hold in general;
- As will be shown in Section 8, the Fuchsian type property is holomorphically invariant.

Remark 1.9. The property of being Fuchsian extends earlier versions of this property given respectively in the work [31] Kossovskiy-Shafikov, and the work [28] of Kossovskiy-Lamel. In the paper [31], a Fuchsian property of generically spherical hypersurfaces (1.1) was introduced. It is possible to check that for a generically spherical hypersurface the two notions of being Fuchsian coincide. In the paper [28], general hypersurfaces (1.1) were considered, but the notion of Fuchsian type considered there is different from that given in the present paper; it serves to guarantee the regularity of infinitesimal CR-automorphisms, while the property (1.9) guarantees regularity of arbitrary CR-maps
(see next subsection). The property introduced in [28] is more appropriately addressed as weak Fuchsian type, while the property (1.9) as the (actual) Fuchsian type.

Let us introduce, for each \( m > 1 \), the space \( N^F_m \) of power series \( h(z, \bar{z}, u) \), as in (1.1), that are of Fuchsian type and satisfy, in addition,

\[
(1.10) \quad h^{(m)}_{22}(u) = h^{(2m-1)}_{23}(u) = h^{(2m-1)}_{33}(u) \equiv 0.
\]

In other words, we have

\[
(1.11) \quad h_{22}(u) = \text{const} \cdot u^{m-1}, \quad h_{23}(u) = \text{const} \cdot u^{2m-2}, \quad h_{33}(u) = \text{const} \cdot u^{2m-2}.
\]

Now, the normalization result for Fuchsian type hypersurfaces is as follows.

**Theorem 3.** Let \( M \subset \mathbb{C}^2 \) be a real-analytic Levi-nonflat hypersurface, which is of \( m \)-infinite type at \( p \in M \) for some \( m > 1 \) and of Fuchsian type at \( p \), and assume that \( p \) is a non-resonant point for \( M \). Then there exists a biholomorphic transformation

\[
(1.12) \quad H : (\mathbb{C}^2, p) \mapsto (\mathbb{C}^2, 0)
\]

bringing \( M \) into the normal form (1.10). Furthermore, a normalizing transformation \( H \) is uniquely determined by the restriction of its differential \( dH|_p \) to the complex tangent \( T_p \mathbb{C} \) \( M \).

**Corollary 1.10.** For a hypersurface \((M, p)\) satisfying the assumptions of Theorem 3, we have \( \dim_{\mathbb{R}} \text{aut}(M, p) \leq 2 \).

**Corollary 1.11.** Let \( H : (M, p) \mapsto (M^*, p^*) \) be a local biholomorphism of two hypersurfaces satisfying the assumptions of Theorem 3. Then \( H \) is uniquely determined by its 1-jet at the point \( p \).

**Remark 1.12.** As can be seen from the description of the resonances in the Fuchsian type case (either \( m = 1 \) or \( m > 1 \)), a Fuchsian type hypersurface can admit only finitely many resonances (up to 7 for \( m = 1 \) and up to 8 for \( m > 1 \)). This means that a resonant normal form (which will still provide a finite-dimensional reduction of the equivalence problem) can be produced and used to solve the equivalence problem in the Fuchsian type case without the non-resonancy assumption. The corresponding construction is, however, a bit technical, and we will not provide it in this paper.

### 1.4. Regularity results.

As discussed above, as a biproduct of the normalization procedure, we are able to prove regularity of respectively formal and \( C^\infty \) smooth CR-mappings between hypersurfaces of Fuchsian type in \( \mathbb{C}^2 \) (even without assuming non-resonancy). For some discussions of state-of-the-art in the regularity subject see e.g. Mir [38] and Kossovskiy-Lamel [28]. The results are formulated below.

**Theorem 4.** Let \( M, M^* \subset \mathbb{C}^2 \) be Fuchsian type hypersurfaces at their respective points \( p, p^* \). Then any formal invertible power series map \( H : (M, p) \to (M^*, p^*) \) is convergent.

**Theorem 5.** Let \( M, M^* \subset \mathbb{C}^2 \) be Fuchsian type hypersurfaces at their respective points \( p, p^* \). Then any \( C^\infty \) CR-map \( H : (M, p) \to (M^*, p^*) \) is holomorphic.
Theorem 4 and Theorem 5 extend earlier results in this direction obtained in [23, 16] in the case $m = 1$. They also extend, in a certain sense, the result in [28] on the regularity of infinitesimal CR-automorphisms of Fuchsian type hypersurfaces to the case of general maps (not necessarily appearing as flows of infinitesimal CR-automorphisms). However, as discussed above, the Fuchsian type condition in [28] is more mild and involves only vanishing conditions on the coefficient functions $h_{kl}$, $k + l \leq 7$ (unlike the conditions in (1.9)). As arguments in Section 8 below show, the case of a general CR-mapping requires considering all the coefficients $h_{kl}$ in (1.9), as they appear in the complete (singular) system of ODEs determining a CR-map.

1.5. Principal method. The main tool of the paper is the recent CR $\rightarrow$ DS (Cauchy-Riemann manifolds $\rightarrow$ Dynamical Systems) technique developed by Kossovskiy and Lamel in the recent work [32, 31, 29, 28, 30] (partially in their joint work with Shafikov and Stolovitch). The technique involves replacing a given degenerate CR-submanifold $M$ by an appropriate holomorphic dynamical system $E(M)$, and then study mappings of CR-submanifolds through mappings of the associated dynamical system. The approach to replace a real-analytic CR-manifold by a complex dynamical system is based on the fundamental parallel between CR-geometry and the geometry of completely integrable PDE systems. This parallel was first observed by E. Cartan and Segre [10, 42] (see also Webster [47]), and was revisited, modernized and further developed in the work of Sukhov [44, 43]. The “mediator” between a CR-manifold and the associated PDE system is the Segre family of the CR-manifold. Unlike the nondegenerate setting in the cited work [10, 42, 44, 43], the CR-DS technique deals systematically with the degenerate setting, providing a sort of a dictionary between CR-geometry and dynamical systems.

In this paper, we give a rigorous formal description of this process in terms of categories of real hypersurfaces $\mathcal{R}_m^\pm$, Segre varieties $\mathcal{S}_m^\pm$, and singular holomorphic ODEs $\mathcal{E}_m$. We shall use the language of category theory for this. We obtain faithful functors $\mathcal{R}_m^\pm \rightarrow \mathcal{S}_m^\pm \rightarrow \mathcal{E}_m$. After completing the formal classification of holomorphic singular ODEs in $\mathcal{E}_m$ satisfying a nonresonance condition, we will show that a suitable choice of normal form descends along this chain and gives rise to a normal form for hypersurfaces that way.

The paper is organized as follows. In Section 2, we provide preliminaries. In Section 3, we explain how the initial equivalence problem can be (partially) reduced to that for the associated ODEs. In Section 4, we explain our general approach to the equivalence problem, and demonstrate that bringing an ODE to a normal form amounts to solving a precise system of 4 singular second order ODEs with a meromorphic singularity. In Section 5, we provide a natural normal form for the ODEs under consideration in the case $m = 1$, and in Section 6 we provide a family of normal forms in the quite different general $m > 1$ case. In Section 7, we complete the last step in solving the equivalence problem in the latter two cases, by establishing a criterion for extracting the fact of equivalence of hypersurfaces from that for associated ODEs. This allows for the desired normal forms for infinite type hypersurfaces. Lastly, in Section 8 we treat the Fuchsian type case (both the normal form construction and the regularity results).
1.6. Concluding remark. We shall emphasize that our classification theory does not close the equivalence problem for real hypersurfaces in $\mathbb{C}^2$ entirely. Describing the Stokes phenomenon for the case $m > 1$ in detail is still open, as is the case of infinite type points of nongeneric kind. We shall, however, observe again the parallel with the Poincaré-Dulac theory, where certain Stokes phenomena are still not well understood, and the classification of degenerate singularities of vector fields (which can be seen as analogues of nongeneric infinite type points) is understood only in very particular situations (we refer again to the paper [36] of Lombardi-Stolovitch). We believe that the solution of the problem provided in the present paper forms the optimal general theory one can hope for.

Acknowledgements

Peter Ebenfelt is supported by the NSF (the National Science Foundation of the USA). Ilya Kossovskiy and Bernhard Lamel are supported by the Austrian Science Fund (FWF). Ilya Kossovskiy is as well supported by GACR (the Grant Agency of Czech Republic).

2. Preliminaries

2.1. Infinite type real hypersurfaces. We recall that if $M \subset \mathbb{C}^2$ is a real-analytic hypersurface, then for any $p \in M$ there exist so-called normal coordinates $(z, w)$ centered at $p$ for $M$. The coordinates being normal means that $(z, w)$ is a local holomorphic coordinate system near $p$ in which $p = 0$ and for which near 0, $M$ is defined by an equation of the form

$$v = F(z, \bar{z}, u)$$

for some germ $F$ of a holomorphic function on $\mathbb{C}^3$ which satisfies the normality condition

$$F(z, 0, u) = F(0, \bar{z}, u) = 0$$

and the reality condition $F(z, \bar{z}, u) \in \mathbb{R}$ for $(z, u) \in \mathbb{C} \times \mathbb{R}$ close to 0 (see e.g. [3]). Equivalently, $v = F(z, \bar{z}, u)$ defines a real hypersurface, and in the coordinates $(z, w)$, we have $Q_{0, u} = \{(0, w) \in U : \text{Re } w = u\}$.

We also recall that $M$ is of infinite type at $p$ if there exists a germ of a nontrivial complex curve $X \subset M$ through $p$. It turns out that in normal coordinates, such a curve $X$ is necessarily defined by $w = 0$ (because $X = Q_0 = \{w = 0\}$); in particular, any such $X$ is nonsingular.

It also turns out that $M$ is Levi-flat if and only if in normal coordinates, it is defined by $v = 0$. Thus a Levi-nonflat real-analytic hypersurface $M$ is of infinite type at $p$ if and only if in normal coordinates $(z, w)$ as above, the defining function $F$ satisfies $F(z, \bar{z}, 0) = 0$. In other words, $M$ is of infinite type if and only if it can be defined by an equation of the form

$$(2.1) \quad v = u^m \psi(z, \bar{z}, u), \text{ with } \psi(z, 0, u) = \psi(0, \bar{z}, u) = 0 \text{ and } \psi(z, \bar{z}, 0) \neq 0,$$

where $m \geq 1$. 
It turns out that the integer \( m \geq 1 \) is independent both of the choice of \( p \in X \) and also of the choice of normal coordinates for \( M \) at \( p \) (see [37]), and we say that \( M \) is \( m \)-infinite type along \( X \) (or at \( p \)).

We are going to utilize a number of different ways to write a defining function. Throughout this paper, we use the complex defining function \( \Theta \) in which \( M \) is defined by

\[
 w = \Theta(z, \bar{z}, \bar{w});
\]

it is obtained from \( F \) by solving the equation

\[
 \frac{w - \bar{w}}{2i} = F\left(z, \bar{z}, \frac{w + \bar{w}}{2}\right)
\]

for \( w \), and it agrees with the function \( h \) defining the Segre varieties in those coordinates, that is, \( Q_z = \{(z, \Theta(z, \bar{Z}): z \in U^z\} \). We are going to make extensive use of the Segre varieties and refer the reader to [3] for a discussion of their properties in the general case, and to [29] for specific properties in the infinite type setting.

The complex defining function (in normal coordinates) satisfies the conditions

\[
 \Theta(z, 0, \tau) = \Theta(0, \chi, \tau) = \tau, \quad \Theta(z, \chi, \bar{\Theta}(\chi, z, w)) = w.
\]

If \( M \) is of \( m \)-infinite type at \( p \), then \( \Theta(z, \chi, \tau) = \tau \theta(z, \chi, \tau) \) and thus \( M \) is defined by the equation \( w = \bar{w} \theta(z, \bar{z}, \bar{w}) = \bar{w} + \bar{w}^m \bar{\theta}(z, \bar{z}, \bar{w}) \), where \( \theta \) satisfies \( \bar{\theta}(z, 0, \tau) = \bar{\theta}(0, \chi, \tau) = 0 \) and \( \bar{\theta}(z, \chi, 0) \neq 0 \).

We also note that the external complexification \( M^C \) of \( M \), which is the hypersurface in \( \mathbb{C}^2 \times \mathbb{C}^2 \) defined by \( M^C = \{(Z, \zeta) \in U \times \bar{U}: Z \in Q_\zeta\} \), is conveniently defined as the graph of the complex defining function \( \Theta \), i.e.

\[
 w = \Theta(z, \chi, \tau).
\]

We also introduce the real line

\[
 (2.2) \quad \Gamma = \{(z, w) \in M: z = 0\} = \{(0, u) \in M: u \in \mathbb{R}\} \subset M,
\]

and recall that

\[
 Q_{(0,u)} = \{w = u\}, \quad (0, u) \in \Gamma
\]

for \( u \in \mathbb{R} \). This property, as already mentioned, is actually equivalent to the normality of the coordinates \((z, w)\). More exactly, for any real-analytic curve \( \gamma \) through \( p \) one can find normal coordinates \((z, w)\) in which a small piece of \( \gamma \) corresponds to \( \Gamma \) in (2.2) (see e.g. []).

We finally notice that a real-analytic Levi-nonflat hypersurface \( M \subset \mathbb{C}^2 \) has infinite type points of two kinds, which we will refer to as generic and exceptional infinite type points, respectively. A generic point \( p \in M \) is characterized by the condition that the complexified Levi form of \( M \) only degenerates on the complexified infinite type locus \( w = \tau = 0 \) near \( p \). (The complexified Levi form is defined similarly to the classical Levi form, but instead the \((1, 0)\) and the \((0, 1)\) vector fields are considered on the complexification \( M^C \), see e.g. [3]). We refer to a non-generic point \( p \) as exceptional. We note that the set of exceptional points is a proper real-analytic subvariety of \( X \) and that \( p \in X \) is generic if
and only if the Levi-determinant of $M$ vanishes to order $m$ along any real curve $\gamma$ passing through $p$ which is transverse to $X$ at $p$.

A generic infinite type point is characterized in normal coordinates by requiring in addition to (2.1) the condition $\psi_{z\bar{z}}(0,0,0,\bar{\zeta}) \neq 0$. If $p$ is a generic infinite type point, we can further simplify $M$ to the form (1.1) above, or alternatively to the exponential form

\begin{equation}
(2.3) \quad w = \bar{w}e^{i\bar{w}^{-1}s(z,\bar{z},\bar{w})}, \quad \text{where} \quad s(z,\bar{z},\bar{w}) = \pm z\bar{z} + \sum_{k,l \geq 2} \varphi_{kl}(\bar{w})z^k\bar{z}^l
\end{equation}

(see, e.g., \[31\]).

### 2.2. Real hypersurfaces and second order differential equations.

There is a natural way to associate to a Levi nondegenerate real hypersurface $M \subset \mathbb{C}^N$ a system of second order holomorphic PDEs with 1 dependent and $N-1$ independent variables by using the Segre family of the hypersurface $M$. This remarkable construction goes back to E. Cartan [10] and Segre [42] (see also a remark by Webster [47]), and was recently revisited in the work of Sukhov [44, 43] in the nondegenerate setting, and in the work of Kossovskiy, Lamel and Shafikov in the degenerate setting (see [32, 31, 29, 28]). For the convenience of the reader, we recall this procedure in the case $N = 2$, but refer to the above references for more details.

So assume that $M \subset \mathbb{C}^2$ is a smooth real-analytic hypersurface passing through the origin and $U = U^z \times U^w$ is chosen small enough. The second order holomorphic ODE associated to $M$ is uniquely determined by the condition that for every $\zeta \in U$, the function $h(z,\zeta) = w(z)$ defining the Segre variety $Q_\zeta$ as a graph is a solution of this ODE.

To be more precise, one can show that the Levi-nondegeneracy of $M$ (at 0) implies that near the origin, the Segre map $\zeta \mapsto Q_\zeta$ is injective and the Segre family has the so-called transversality property: if two distinct Segre varieties intersect at a point $q \in U$, then their intersection at $q$ is transverse (actually it turns out that, again due to the Levi-nondegeneracy of $M$, the Segre varieties passing through a point $p$ are uniquely determined by their tangent spaces $T_p Q_\zeta$. Thus, \( \{Q_\zeta\}_{\zeta \in U} \) is a 2-parameter family of holomorphic curves in $U$ with the transversality property, depending holomorphically on $\zeta$. It follows from the holomorphic version of the fundamental ODE theorem (see, e.g., [22]) that there exists a unique second order holomorphic ODE $w'' = \Phi(z,w,w')$ such that for each $\zeta \in U$, $w(z) = h(z,\zeta)$ is one of its solutions.

We can carry out the construction of this ODE concretely by utilizing the complex defining equation $w = \Theta(z,\chi,\tau)$ introduced above. Recall that the Segre variety $Q_\zeta$ of a point $\zeta = (a,b) \in U$ is now given as the graph

\begin{equation}
(2.4) \quad w(z) = \rho(z,a,b).
\end{equation}

Differentiating (2.4) once, we obtain

\begin{equation}
(2.5) \quad w' = \rho_z(z,a,b).
\end{equation}
The system of equations (2.4) and (2.5) can be solved, using the implicit function theorem, for \( \bar{a} \) and \( \bar{b} \). This gives us holomorphic functions \( A \) and \( B \) such that

\[
\bar{a} = A(z, w, w'), \quad \bar{b} = B(z, w, w').
\]

The application of the implicit function theorem is possible since the Jacobian of the system consisting of (2.4) and (2.5) with respect to \( \bar{a} \) and \( \bar{b} \) is just the Levi determinant of \( M \) for \( (z, w) \in M(3) \). Differentiating (2.5) once more, we can substitute \( \bar{a} = A(z, w, w') \) and \( \bar{b} = B(z, w, w') \) to obtain

\[
\tag{2.6}
w'' = \rho_{zz}(z, A(z, w, w'), B(z, w, w')) =: \Phi(z, w, w').
\]

Now (2.6) is a holomorphic second order ODE, for which all of the functions \( w(z) = h(z, \zeta) \) are solutions by construction. We will denote this associated second order ODE by \( E = E(M) \).

More generally it is possible to associate a completely integrable PDE to any of a wide range of CR-submanifolds (see [44, 43]) such that the correspondence \( M \to E(M) \) has the following fundamental properties:

1. Every local holomorphic equivalence \( F: (M, 0) \to (M', 0) \) between CR-submanifolds is an equivalence between the corresponding PDE systems \( E(M), E(M') \);
2. The complexification of the infinitesimal automorphism algebra \( \mathfrak{ho}^{\omega}(M, 0) \) of \( M \) at the origin coincides with the Lie symmetry algebra of the associated PDE system \( E(M) \) (see, e.g., [40] for the details of the concept).

In contrast to the case of a finite type real hypersurface described above, if \( M \subset \mathbb{C}^2 \) is of infinite type at the origin one cannot associate to \( M \) a second order ODE or even a more general PDE system near the origin such that the Segre varieties are graphs of solutions. However, in [31] and [28], Kossovskiy, Lamel and Shafikov found an injective correspondence associating to a hypersurface \( M \subset \mathbb{C}^2 \) at a generic infinite type point a certain singular complex ODEs \( E(M) \) with an isolated singularity at the origin. We are going to base our normal form construction on this construction, which is therefore extensively used in the paper (more details are given in Section 3).

We finally point out that at exceptional infinite type points, one can still associate a system of singular complex ODEs to a real-analytic hypersurface \( M \subset \mathbb{C}^2 \) (although possibly of higher order \( k \geq 2 \)) as in the paper [30] Kossovskiy-Lamel-Stolovitch.

### 2.3. Complex differential equations with an isolated singularity.

We will again just gather the facts from the classical theory of singular (complex) differential equations, and refer the reader to e.g. [22], [46], [31] for any details.

A linear system \( \mathcal{L} \) of (holomorphic) first order ODEs on a domain \( G \subset \mathbb{C} \) (or simply a linear system in a domain \( G \)) is an equation of the form \( y'(x) = A(x)y(x) \), where \( A: G \to \mathbb{C}^{n \times n} \) is a matrix-valued holomorphic map on \( G \) and \( y(x) = (y_1(x), ..., y_n(x)) \) is an \( n \)-tuple of (unknown) functions. The set of solutions of \( \mathcal{L} \) near a point \( p \in G \) is isomorphic to \( \mathbb{C}^n \) by \( y \mapsto y(p) \). Because every germ \( y \) of a solution of \( \mathcal{L} \) at \( p \in G \) extends analytically along any path \( \gamma \subset G \) starting at \( p \), any solution \( y(x) \) of \( \mathcal{L} \) is defined in all of \( G \) as a (possibly multi-valued) analytic function. If \( G \) is a punctured disc, centered at 0, we say that \( \mathcal{L} \) has an isolated singularity (at \( x = 0 \)). If \( A(x) \) has a pole at the
isolated singularity $x = 0$, we say that the system has a meromorphic singularity. As the solutions of $\mathcal{L}$ are holomorphic in any proper sector $S \subset G$ of a sufficiently small radius with vertex at $x = 0$, it is important to study the behaviour of the solutions as $x \to 0$. If for every sector $S = \{x \in G : |x| < \delta, \alpha < \arg x < \beta\}$ there exist constants $C > 0$ and $a \in \mathbb{R}$ such that for every solution $y$ of $\mathcal{L}$ defined in $S$ we have that $||y(x)|| \leq C|x|^a$ holds for $x \in S$, then we say that $x = 0$ is a regular singularity, otherwise we say it is an irregular singularity.

An important condition ensuring regularity of a singularity is due to L. Fuchs: We say that the singular point $x = 0$ is Fuchsian if $A(x)$ has a pole of order at most 1 at $x = 0$. If 0 is a Fuchsian singularity, then $x = 0$ is a regular singular point. Another important property of Fuchsian singularities is that every formal power series solution (at $x = 0$) of the equation is actually convergent. The dynamical system associated to a Fuchsian singularity corresponds to the dynamical system of the vector field

$$x \frac{\partial}{\partial x} + A(x)y \frac{\partial}{\partial y},$$

which is “almost” non-resonant in the sense of Poincaré-Dulac.

However, in the non-Fuchsian case we encounter very different behaviours, both of solutions and of mappings between linear systems with such a singularity. A generic solution of a non-Fuchsian system

$$y' = \frac{1}{x^m} B(x)y, \quad m \geq 2$$

does not have polynomial growth in sectors, and generic formal power series solutions of such a system (as well as formal equivalences between generic non-Fuchsian systems) are divergent. The dynamics associated to a non-Fuchsian singularity correspond to the dynamics of the vector field

$$x^m \frac{\partial}{\partial x} + A(x)y \frac{\partial}{\partial y},$$

which is always resonant, in the sense of Poincaré-Dulac.

Further information on the classification of isolated singularities can be found in e.g. [22] or [16].

Fuchsianity admits a certain extention to the non-linear case as well, giving rise to the notion of Briot-Bouquet type ODEs, that is, ODEs of the form

$$(2.7) \quad xy' = F(x, y),$$

where $x$ lies in a neighborhood of 0 in $\mathbb{C}$, $y$ is $n$-dimensional and $F$ is holomorphic in a neighborhood of 0 in $\mathbb{C}^{n+1}$. Briot-Bouquet ODEs are similar to linear systems of ODEs with a Fuchsian singularity in many respects; for example, their formal power series solutions are necessarily convergent (see, e.g., [34]). Dynamics associated to a Briot-Bouquet type ODE corresponds to the dynamics of the vector field

$$x \frac{\partial}{\partial x} + F(x, y) \frac{\partial}{\partial y}. $$
We also note that a Briot-Bouquet type ODE whose principal matrix $F_y(0,0)$ has no positive integer eigenvalues has at least one holomorphic solution (see [34]).

3. Reduction to the classification problem for ODEs

We consider a real-analytic hypersurface with defining equation as in (1.1). The complex defining function of such a hypersurface is given by

$$(3.1) \quad w = \bar{w} + i\bar{w}^m \left( \epsilon |z|^2 + \sum_{k,\ell \geq 2} \Theta_{k\ell}(\bar{w}) z^k \bar{z}^\ell \right).$$

We recall from subsection 2.1 that this means that the Segre family $S = \{ \Theta_{\xi,\eta} \}$ of $M$ is given by:

$$(3.2) \quad w = \bar{\eta} e^{i\eta m - 1} \varphi(z, \bar{\xi}, \bar{\eta}), \quad \text{where} \quad \varphi(z, \bar{\xi}, \bar{\eta}) = \epsilon z \bar{\xi} + \sum_{k,\ell \geq 2} \varphi_{k\ell}(\bar{\eta}) z^k \bar{\xi}^\ell$$

We still think about (3.2) as a parameterized family of planar complex curves, depending on the parameters $\xi, \eta$ anti-holomorphically. Differently to the case of the Segre family of a Levi-nondegenerate hypersurface, this parameterized family does not satisfy the transversality condition. We can therefore not expect it to satisfy a regular ODE, and we will recall later that we can find a singular ODE which it satisfies.

Our first step however is going to be an observation which will allow us to restrict the class of maps we need to consider in our equivalence problem by a fair amount, since formal transformations between hypersurfaces of the form we are interested satisfy a class of maps we need to consider in our equivalence problem by a fair amount, since formal transformations between hypersurfaces of the form we are interested satisfy a number of restrictions on the low order terms in their Taylor expansion.

**Lemma 3.1.** Let $H(z, w) = (F(z, w), G(z, w))$ be a formal transformation vanishing at the origin, with invertible Jacobian $H'(0)$, which maps a hypersurface defined by (1.1) or equivalently (3.2) into another such hypersurface. Then $H$ satisfies

$$(3.3) \quad F_z(0, 0) = \lambda, \quad G_w(0, 0) = \mu, \quad G = O(w),$$

$$G_z = O(w^{m+1}), \quad \mu^{1-m} = |\lambda|^2, \quad \lambda \in \mathbb{C} \setminus \{0\}, \mu \in \mathbb{R}.$$  

In addition, we have

$$(3.4) \quad G_w^\ell(0, 0) \in \mathbb{R}, \quad \text{for} \ \ell \leq m.$$  

**Proof.** The content of the first part of the Lemma, (3.3) is a well-known consequence of being in normal coordinates. Indeed, since in normal coordinates, we have $S_0 = \{w = 0\}$, and $H(S_0) \subset S_0'$, necessarily $G(z, 0) = 0$, and so $G(z, w) = O(w)$. Since $H'(0)$ is invertible, we therefore necessarily have $F_z(0, 0) G_w(0, 0) \neq 0$, and we can write $F(z, w) = \lambda z + \ldots$ and $G(z, w) = \mu w + \ldots$ with $\lambda \mu \neq 0$.

Since $H$ maps a hypersurface of the form (3.1) into another such, we have

$$(3.5) \quad G(z, \bar{w} + \bar{w}^m \cdot \bar{z} \cdot O(1)) = \bar{G}(\bar{z}, \bar{w}) + \bar{G}(\bar{z}, \bar{w})^m \bar{F}(\bar{z}, \bar{w}) F(z, w) \cdot O(1)$$

for all $z, \bar{z}$, and $\bar{w}$. Evaluating (3.5) for $\bar{z} = 0$ gives $G(z, \bar{w}) = G(0, \bar{w}) + O(\bar{w}^{m+1})$ and therefore $G_z(0, \bar{w}) = O(\bar{w}^{m+1})$ as claimed in (3.3), and we also obtain the claim (3.4).
If we compare the coefficient of $z \bar{z} \bar{w}^m$ on both sides of (3.5), we obtain that $\mu = \mu_m |\lambda|^2$ and therefore the missing claim in (3.3).

Lemma 3.1 implies in particular that any transformation $H$ between hypersurfaces defined by equations of the form (1.1) can be factored as

$$H = H_0 \circ \psi,$$

for some dilation $\psi$ of the form (1.2) and where $H_0$ is a transformation of the form:

$$z \mapsto z + f(z, w), \quad w \mapsto w + wg_0(w) + w^m g(z, w)$$

with

$$f_z(0, 0) = 0, \quad g_0(0) = 0, \quad g(z, w) = O(zw), \quad g^{(\ell)}_0(0) \in \mathbb{R}, \quad \ell \leq m - 1.$$  

(for $m = 1$ the last condition is void). In fact, one can also represent $H$ as

$$H = \psi \circ H_0$$

(with a different $H_0$). We therefore consider the classification problem only under transformations (3.6).

We now recall that [31, 28] showed that we can associate to a hypersurface in the form (1.1) a second order singular holomorphic ODE $E(M)$ given by

$$w'' = w^m \Phi \left( z, w, \frac{w'}{w^m} \right),$$

where $\Phi(z, w, \zeta)$ is holomorphic near the origin in $\mathbb{C}^3$, and satisfies $\Phi = O(\zeta^2)$. This ODE is characterized by the condition that any of the functions $w(z) = \Theta(z, \xi, \eta)$, for $(\xi, \eta) \in \bar{U}$, is a solution of the ODE (3.7). We will decompose $\Phi$ as

$$\Phi(z, w, \zeta) = \sum_{j,k\geq 0, \ell \geq 2} \Phi_{j\ell} z^j w^k \zeta^\ell.$$  

The key step in our normalization procedure for hypersurfaces (1.1) is the reduction of the classification problem of hypersurfaces to the classification of ODEs of the form (3.7) which have some additional properties (derived from being associated to a real hypersurface), which we will carry out in Section 5. However, we start by considering general parameterized families of planar complex curves which are given by equations of the kind (3.2), instead of Segre families of hypersurfaces (1.1). If no risk of confusion arises, we will also call such a parametrized family of planar curves a Segre family.

**Definition 3.2.** A parameterized family of planar (formal or holomorphic) complex curves (a Segre family), given by $S_\varphi = \{ w = \bar{\eta} e^{i\varphi(z, \xi, \eta)} \}$ is said to be $m$-admissible if $\varphi$ satisfies $\varphi(z, \xi, \eta) = \epsilon z \xi + \sum_{k,\ell \geq 2} \varphi_{k\ell}(\eta) z^k \xi^\ell$ with $\epsilon = \pm 1$. Depending on the sign of $\epsilon$, we say that $S_\varphi$ is positive ($\epsilon = +1$) or negative ($\epsilon = -1$). We call the equation $w = \bar{\eta} e^{i\varphi(z, \xi, \eta)}$ the defining equation of the Segre family.

We will be dealing with a number of ways of associating objects and maps with one another. Even though there is some additional structure in our setting, we will not use it, and it will be sufficient to use (elementary) language of category theory. For a given $m \in \mathbb{N}$, we introduce the category $\mathcal{E}_m$ whose objects are ODEs of the form (3.7); that is,
as a set, the set of objects of $\mathcal{E}$ can be identified with power series (formal or convergent) $\Phi \in C[z, w, \zeta]$ of the form (3.3). If $m$ is fixed, we will usually drop it from the notation; we fix an arbitrary $m \geq 1$ for now. If $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{E}$, then $H \in \text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ is given by an invertible formal map $H(z, w) = (F(z, w), G(z, w))$, where $F, G \in C[z, w]$ transforming graphs of solutions of $\mathcal{E}_1$ into graphs of solutions of $\mathcal{E}_2$ and satisfying (3.6).

For any given $m \in \mathbb{N}$, we also introduce the categories $\mathcal{S}^+_m$ and $\mathcal{S}^-_m$ of positive and negative Segre families, respectively, whose objects are positive (resp. negative) Segre families as defined in [Definition 3.2]. We will again drop the $m$ from the notation if it is fixed. The objects of $\mathcal{S}^+_\varepsilon$ and $\mathcal{S}^-_{\varepsilon}$ can therefore be identified with power series $\varphi$ as in (3.2), with $\varepsilon = 1$ or $\varepsilon = -1$, respectively. A transformation $H \in \text{Hom}(\mathcal{S}_1, \mathcal{S}_2)$ is an invertible formal map $H(z, w, \xi, \eta) = (F(z, w), G(z, w), \Lambda(\xi, \eta), \Omega(\xi, \eta))$ where $F, G \in C[z, w]$ and $\Lambda, \Omega \in C[\xi, \eta]$ which satisfies that it maps the family $\mathcal{S}_1$ into $\mathcal{S}_2$, and which satisfies the normalization conditions (3.6) for $(F, G)$ and the analogous conditions for $(\Lambda, \Omega)$.

**Proposition 3.3.** The categories $\mathcal{E}$ and $\mathcal{S}^\pm$ are equivalent, that is, there exists a functor $\mathcal{E}: \mathcal{S}^\pm \to \mathcal{E}$ and a functor $\mathcal{S}^\pm: \mathcal{E} \to \mathcal{S}^\pm$ such that $\mathcal{E} \circ \mathcal{S}^\pm = \text{id}_\mathcal{E}$ and $\mathcal{S}^\pm \circ \mathcal{E} = \text{id}_\mathcal{S}^\pm$. The functors satisfy the following conditions:

i) $\mathcal{E}(\mathcal{S}_0)$ is determined by the condition that the defining function of $\mathcal{S}_0$ satisfies $\mathcal{E}(\mathcal{S}_0)$, and $\mathcal{S}^\pm(\mathcal{E}_0)$ is a positive Segre family all of whose elements satisfy $\mathcal{E}_0$.

ii) Convergent elements and morphisms are transformed into convergent elements and morphisms via $\mathcal{E}$, $\mathcal{S}^\pm$, and $\mathcal{S}^\mp$.

The functors constructed in Proposition 3.3 allow us in particular to translate the equivalence problem for complexifications of hypersurfaces (1.1) under the transformation group (3.6) into the classification of the class of ODEs (3.7) under transformations (3.6).

**Proof of Proposition 3.3.** We shall carry out the proof for $\mathcal{S}^+$, the case of negative Segre families being completely analogous.

We first show that, if the equation

$$\mathcal{E}_0: w'' = w^m \Phi \left( z, w, \frac{w'}{w^m} \right)$$

is given, then condition i) uniquely determines a positive segre family $\mathcal{S}^+(\mathcal{E}_0)$. Indeed, substituting the defining equation of a positive Segre family $\mathcal{S}^+ = \{ w = \bar{\eta}e^{i\eta m-1} \varphi(z, \xi, \bar{\eta}) \}$ into the equation $\mathcal{E}_0$ yields the following (formal or holomorphic) ODE for $\varphi$:

$$\varphi_{zz} = -ie^{i(m-1)\eta m-1} \varphi \cdot \Phi \left( z, \bar{\eta}e^{i\eta m-1} \varphi, ie^{i(1-m)\eta m-1} \varphi_{z} \right) - \bar{\eta}^{m-1}(\varphi_z)^2.$$  

We let $\varphi$ denote the unique solution of the (resp. formal or holomorphic) ODE (3.9) with Cauchy data

$$\varphi(0, \xi, \bar{\eta}) = 0, \quad \varphi_z(0, \xi, \bar{\eta}) = \bar{\xi}. $$

We claim that $\varphi$ satisfies furthermore

$$\varphi(z, 0, \bar{\eta}) = 0, \quad \varphi_{\xi}(z, 0, \bar{\eta}) = z,$$

so that $\mathcal{S}^+(\mathcal{E}_0) \in \mathcal{S}^+$ determines an $m$-admissible family as required.
To prove the claim, let us recall that $\Phi(z, w, \zeta) = O(\zeta^2)$. Hence the substitution

$$\varphi(z, \bar{\xi}, \bar{\eta}) = \bar{\xi}\psi(z, \bar{\xi}, \bar{\eta})$$

turns (3.9) into a (resp. formal or holomorphic) ODE for $\psi$. If $\psi$ is its solution with the initial data

$$\psi(0, \bar{\xi}, \bar{\eta}) = 0, \quad \psi_0(0, \bar{\xi}, \bar{\eta}) = 1,$$

then by uniqueness we have $\varphi = \bar{\xi}\psi$, which proves the first identity in (3.11). Substituting again $\varphi = \bar{\xi}\psi$ into (3.9) we see that $\varphi_{zz}$ is divisible by $\bar{\xi}$, which implies the second condition in (3.11).

The proof that one can find an ODE for a positive $m$-admissible family is very analogous to the proof of the same fact for Segre families of real hypersurfaces, which is in detail carried out in [28], and the reader can easily make the changes needed to accomodate the slightly more general situation considered here. This proves (i).

We now prove (ii). We fix two $m$-admissible families $S_1, S_2 \in \mathcal{S}^+$ and their associated ODEs $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{E}$. If there is a (resp. formal or holomorphic) map $(z, w, \xi, \eta) \mapsto (F(z, w), G(z, w), \Lambda(\xi, \eta), \Omega(\xi, \eta)) \in \text{Hom}(S_1, S_2)$, then the map $(z, w) \mapsto (F(z, w), G(z, w)) \in \text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ by definition.

Conversely, if a map $(z, w) \mapsto H(z, w) = (F(z, w), G(z, w)) \in \text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$, let us consider (yet undefined) maps $(\Lambda(\xi, \eta), \Omega(\xi, \eta))$ and the the preimage $\tilde{\mathcal{S}}$ of $S_2$ under the map $H(z, w, \xi, \eta) = (F(z, w), G(z, w), \Lambda(\xi, \eta), \Omega(\xi, \eta))$.

If we denote the defining function of $S_2$ by $\rho_2$, then the equation $w = \rho(z, \xi, \eta)$ defining $\tilde{\mathcal{S}}$ is given by solving the equation

$$G(z, \rho(z, \bar{\xi}, \bar{\eta})) = \rho_2 \left( F(z, \rho(z, \bar{\xi}, \bar{\eta})), \bar{\Lambda}(\bar{\xi}, \bar{\eta}), \bar{\Omega}(\bar{\xi}, \bar{\eta}) \right)$$

for $\rho$. The conditions (3.6) ensure that this is possible using the implicit function theorem.

If we require that $\rho(0, \bar{\xi}, \bar{\eta}) = \bar{\eta}$ and $\rho_2(0, \bar{\xi}, \bar{\eta}) = i\bar{\xi}\bar{\eta}^m$, then we obtain the following system of equations:

$$G(z, 0, \bar{\eta}) = \rho_2 \left( F(0, \eta), \bar{\Lambda}(\bar{\xi}, \bar{\eta}), \bar{\Omega}(\bar{\xi}, \bar{\eta}) \right)$$

$$G_x(0, \bar{\eta}) + G_w(0, \bar{\eta})i\bar{\xi}\bar{\eta}^m = \frac{\partial \rho_2}{\partial z} \left( F(0, \eta), \bar{\Lambda}(\bar{\xi}, \bar{\eta}), \bar{\Omega}(\bar{\xi}, \bar{\eta}) \right) \left( F_z(0, \bar{\eta}) + F_w(0, \bar{\eta})i\bar{\xi}\bar{\eta}^m \right).$$

Using (3.6) we write $G(0, w) = w g_0(w)$, $G_x(0, w) = w^m g(w)$ and set $\Omega(\xi, \eta) = \eta\bar{\Omega}(\bar{\xi}, \bar{\eta})$. By Definition 3.2 we can write $\rho_2(z, \xi, \eta) = \eta\tilde{\rho}_2(z, \bar{\xi}, \bar{\eta})$ and $\rho_2(0, \bar{\xi}, \bar{\eta}) = i\bar{\xi}\bar{\eta}^m \sigma(z, \bar{\xi}, \bar{\eta})$. Note that $g_0(0) = 1$, $\tilde{\rho}_2(0, 0, 0) = 1$, and $\sigma(0, 0, 0) = 1$ by (3.6) and Definition 3.2. The system can therefore be rewritten as

$$g_0(\eta) = \tilde{\rho}_2 \left( F(0, \eta), \bar{\Lambda}(\bar{\xi}, \bar{\eta}), \eta\bar{\Omega}(\bar{\xi}, \bar{\eta}) \right) \bar{\Omega}(\bar{\xi}, \bar{\eta})$$

$$\frac{g(\eta) + (g_0(\eta) + \tilde{\rho}_2(\eta) \xi)\bar{\xi}}{F_z(0, \bar{\eta}) + F_w(0, \bar{\eta})i\bar{\xi}\bar{\eta}^m} = \sigma \left( F(0, \eta), \bar{\Lambda}(\bar{\xi}, \bar{\eta}), \eta\bar{\Omega}(\bar{\xi}, \bar{\eta}) \right) i\bar{\Lambda}(\bar{\xi}, \bar{\eta})\bar{\Omega}(\bar{\xi}, \bar{\eta})^m.$$
The implicit function theorem now ensures that (3.14) has a unique solution \((\bar{\Lambda}(\bar{\xi}, \bar{\eta}), \bar{\Omega}(\bar{\xi}, \bar{\eta}))\) satisfying \(\bar{\Lambda}(0, 0) = 0\) and \(\bar{\Omega}(0, 0) = 1\). One checks also that (3.14) implies that \((\Lambda, \Omega)\) satisfies the normalization conditions (3.6).

By the first part of the proof, the defining equation \(w = \rho_1(z, \bar{\xi}, \bar{\eta})\) for the Segre family \(\mathcal{S}_1\) is uniquely determined by the Cauchy data \(\rho_1(0, \bar{\xi}, \bar{\eta})\) and \(\rho_{1,z}(0, \bar{\xi}, \bar{\eta})\) from which we conclude that actually \(\rho = \rho_1\), and so \(\mathcal{H} \in \text{Hom}(\mathcal{S}_1, \mathcal{S}_2)\).

Lastly, if we define the category \(\mathfrak{R}_m^\pm\) of \(m\)-infinite type hypersurfaces for which the fixed coordinate system \((z, w)\) is admissible, and where for \(M, M' \in \mathfrak{R}_m^\pm\) we define \(\text{Hom}(M, M')\) to consist of all germs of holomorphic (or all formal) maps \(h: M \to M'\), then the associated ODE and the associated Segre family give rise to faithful functors \(\mathfrak{R}_m^\pm \hookrightarrow \mathfrak{C}_m\) and \(\mathfrak{R}_m^\pm \hookrightarrow \mathcal{S}_m^\pm\).

4. The general approach to the classification of ODEs

4.1. Reduction to a Cauchy problem. Based on Proposition 3.3 we now proceed with the holomorphic classification of ODEs (3.7). According to the classical formulas for prolonging maps \(\mathbb{C} \mapsto \mathbb{C}\) to the space of 2-jets [9], for two given ODEs \(\mathcal{E} = \{w'' = \Psi(z, w, w')\}\) and \(\mathcal{E}^* = \{w'' = \Psi^*(z, w, w')\}\) and a diffeomorphism \((z, w) \mapsto (\tilde{f}, \tilde{g})\) transforming \(\mathcal{E}\) to \(\mathcal{E}^*\) we have the following basic identity:

\[
(4.1) \quad \Psi(z, w, w') = \frac{1}{J} \left( (\tilde{f}_z + w'\tilde{f}_w)^3 \Psi^* \left( \tilde{f}(z, w), \tilde{g}(z, w), \frac{\tilde{g}_z + w'\tilde{g}_w}{\tilde{f}_z + w'\tilde{f}_w} \right) + \right. \\
I_0(z, w) + I_1(z, w)w' + I_2(z, w)(w')^2 + I_3(z, w)(w')^3,
\]

where \(J := \tilde{f}_z\tilde{g}_w - \tilde{f}_w\tilde{g}_z\) is the Jacobian determinant of the transformation and

\[
(4.2) \quad I_0 = \tilde{g}_z\tilde{f}_{zz} - \tilde{f}_z\tilde{g}_{zz} \\
I_1 = \tilde{g}_w\tilde{f}_{zz} - \tilde{f}_w\tilde{g}_{zz} - 2\tilde{f}_z\tilde{g}_{zw} + 2\tilde{g}_z\tilde{f}_{zw} \\
I_2 = \tilde{g}_z\tilde{f}_{ww} - \tilde{f}_z\tilde{g}_{ww} - 2\tilde{f}_w\tilde{g}_{zw} + 2\tilde{g}_w\tilde{f}_{zw} \\
I_3 = \tilde{g}_w\tilde{f}_{ww} - \tilde{f}_w\tilde{g}_{ww}.
\]

Similarly, for a pair of singular ODEs of the kind (3.7) we get:

\[
(4.3) \quad \Phi \left( z, w, \frac{w'}{w_m} \right) = \frac{1}{J} \left( (\tilde{f}_z + w'\tilde{f}_w)^3 \tilde{g}^m \Phi^* \left( \tilde{f}(z, w), \tilde{g}(z, w), \frac{\tilde{g}_z + w'\tilde{g}_w}{\tilde{g}^m(\tilde{f}_z + w'\tilde{f}_w)} \right) + \right. \\
I_0(z, w) + I_1(z, w)w' + I_2(z, w)(w')^2 + I_3(z, w)(w')^3,
\]

where the expressions \(J, I_0, \ldots, I_3\) are identical to the ones in (4.1). In what follows we treat the ODE \(\mathcal{E}\) as the target and \(\mathcal{E}^*\) as the initial ODE, respectively.
Recall that we are looking for transformations of the kind (3.6). Rewriting \( \tilde{f}(z, w) = z + f(z, w) \), and \( \tilde{g}(z, w) = w + wg_0(w) + wg_1(z, w) \), (4.3) (after dividing by \( w^m \)) becomes:

\[
(4.4) \quad \Phi(z, w, \zeta) = \frac{1}{J} \left[ (1 + f_z + w^m f_w \cdot \zeta)^3 (1 + g_0(w) + w^{m-1}g)^m \right] \cdot \Phi^\ast(z + f, w + wg_0(w) + w^m g, g_z + \zeta (1 + w g'_0 + g_0 + mw^{m-1}g + w^m g_w)) + I_0(z, w) + I_1(z, w) + I_2(z, w) + I_3(z, w) w^2m \zeta^3 \right],
\]

where \( \zeta := \frac{w^m}{w^m} \) and

\[
(5.5) \quad J = (1 + f_z)(1 + wg'_0 + g_0 + mw^{m-1}g + w^m g_w) - w^m f_w g_z,
\]

\[
I_0 = g_z f_{zz} - (1 + f_z) g_{zz},
\]

\[
I_1 = (1 + wg'_0 + g_0 + mw^{m-1}g + w^m g_w) f_{zz} - w^m f_w g_{zz} - 2(1 + f_z)(mw^{m-1}g_z + w^m g_{zw}) + 2w^m g_z f_{zw},
\]

\[
I_2 = w^m g_z f_{ww} - (1 + f_z)(wg'_0 + 2g'_0 + m(m - 1)w^{m-2}g + 2mw^{m-1}g_w + w^m g_{ww}) - 2f_w (mw^{m-1}g_z + w^m g_{zw}) + 2(1 + wg'_0 + g_0 + mw^{m-1}g + w^m g_w) f_{zw},
\]

\[
I_3 = (1 + wg'_0 + g_0 + mw^{m-1}g + w^m g_w) f_{ww} - f_w (wg''_0 + 2g'_0 + m(m - 1)w^{m-2}g + 2mw^{m-1}g_w + w^m g_{ww}).
\]

Importantly, (4.4) is an identity in the free variables \( z, w, \zeta \), where the latter triple runs a suitable open neighborhood of the origin in \( \mathbb{C}^3 \).

Though the expressions (4.4), (4.5) look cumbersome, we will shortly be able to work out (4.4) elegantly.

After that, let us equalize in (4.4) terms with a fixed degree \( k, 0 \leq k \leq 3 \) in \( \zeta \). We expand

\[
(4.6) \quad \Phi(z, w, \zeta) = \sum_{k \geq 0} \Phi_k(z, w) \zeta^k,
\]

and similarly \( \Phi^\ast \), where we assume that \( \Phi^\ast \) satisfies (3.8), and obtain:

\[
I_0 = \Phi_0 + \ldots
\]

\[
I_1 = \Phi_1 + \ldots
\]

\[
(4.7) \quad w^m I_2 = \Phi_2 - \Phi^\ast_2 + \ldots
\]

\[
\quad w^{2m} I_3 = \Phi_3 - \Phi^\ast_3 + \ldots,
\]

where \( \ldots \) signify convergent power series, without constant terms, in the 1-jet of \( (f, g_0, g) \), whose coefficients only depend on the source defining function \( \Phi^\ast \) (and are independent of the target \( \Phi \)). We have

**Proposition 4.1.** Let \( E^\ast \in \mathcal{E} \), and \( f_0(w), f_1(w), g_0(w), g_1(w) \) be power series satisfying \( f_0(0) = g_0(0) = g_1(0) = f_1(0) = 0 \). Then there exist unique \( E \in \mathcal{E} \) and \( H = (\tilde{f}, \tilde{g}) \in \mathcal{H} \).
Hom(\mathcal{E}, \mathcal{E}^*) such that

\begin{align}
  f_0(w) &= \tilde{f}(0, w), & f_1(w) &= \tilde{f}_z(0, w) - 1, \\
  wg_0(w) &= \tilde{g}(0, w) - w, & w^m g_1(w) &= \tilde{g}_z(0, w).
\end{align}

**Proof.** Because we are looking for a singular ODE with right hand side \( \Phi \) for which \( \Phi_0 = \Phi_1 = 0 \), we need to prove that the first two equations in \((4.7)\), with \( \Phi_0 = \Phi_1 = 0 \), have a solution \( f, g \). We are going to show that this is uniquely possible with the given initial data \((4.8)\).

In order to study the system of PDEs under discussion, let us use the first two formulas in \((4.7)\) as a system of equations determining \( \Phi_0 \) and \( \Phi_1 \), respectively, which we require to be zero. Using the first two equations in \((4.5)\), this yields system of PDEs, linear in the second derivatives \( f_{zz}, g_{zz} \), which can be written as

\[
\begin{pmatrix}
g_z \\
(1 + wg_0 + g_0 + mw^m g + w^m g_w)
\end{pmatrix}
\begin{pmatrix}
(1 + f_z) \\
-w^m f_w
\end{pmatrix}
\begin{pmatrix}
f_z \\
g_z
\end{pmatrix} = \ldots.
\]

In view of \((4.8)\), the determinant of the matrix is nonzero at the point \((z, w) = (0, 0)\). Hence, applying Cramer’s rule, we obtain an analytic system of the kind

\begin{align}
f_{zz} &= U(z, w, g_0, g_0', f, g, f_z, g_z, f_w, g_w, f_{zw}, g_{zw}), \\
g_{zz} &= V(z, w, g_0, g_0', f, g, f_z, g_z, f_w, g_w, f_{zw}, g_{zw})
\end{align}

for some germs of holomorphic functions \( U, V \) at the origin, which depend only on \( \Phi^* \). The Cauchy-Kowalevskaya theorem guarantees the existence of a (unique) solution to \((4.9)\) with initial data

\[
f(0, w) = f_0(w), \quad f_z(0, w) = f_1(w), \quad g(0, w) = 0, \quad g_z(0, w) = g_1(w),
\]

we determine a unique (resp. formal or holomorphic near the origin) solution \( f, g \) for \((4.9)\).

The associated functions \( \tilde{f}(z, w) = z + f(z, w), \tilde{g}(z, w) = w + wg_0(w) + g(z, w) \) transform \( \mathcal{E}^* \) to the (up to the initial data unique) \( \mathcal{E} \). The initial conditions also imply that \( (\tilde{f}, \tilde{g}) \) is of the form required in \((3.6)\). \( \square \)

In view of Proposition 4.1, the unique determination of a map \((\tilde{f}, \tilde{g})\) is equivalent to determining the Cauchy data for \( H \), i.e. the vector

\begin{align}
Y(w) := (f_0(w), f_1(w), g_0(w), g_1(w)).
\end{align}

In order to uniquely determine the Cauchy data \( Y(w) \), we have to put further normalization conditions on the function \( \Phi \). Any such collection of normalization condition will be referred to as a normal form of an ODE \((3.7)\). We describe them below in three different cases, based on the decomposition

\begin{align}
\Phi(z, w, \zeta) = \sum_{a, b, c} \Phi_{a, b, c} z^a w^b \zeta^c.
\end{align}
For $m = 1$, we define the space $D_1$ to consist of all formal power series $\Phi(z, w, \zeta) \in \zeta^2 \mathbb{C}[z, w, \zeta]$ which in addition to satisfying $\Phi = O(\zeta^2)$ also satisfy
\begin{equation}
\Phi_{0,j,2} = \Phi_{1,j,2} = \Phi_{0,j,3} = \Phi_{1,j,3} = 0, \quad j > 0,
\end{equation}
or equivalently,
\begin{equation}
\frac{\partial^{j+k}\Phi}{\partial z^j \zeta^k}(0, w, 0) = \frac{\partial^{j+k}\Phi}{\partial z^j \zeta^k}(0, w, 0), \quad j = 0, 1, \quad k = 2, 3.
\end{equation}

For $m > 1$, we need, for technical reasons that will become clear later, to first fix a real vector
\[ \sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2. \]
For each such fixed $\sigma = (\sigma_1, \sigma_2)$, we define the space $D_m^\sigma$ to consist of all formal power series $\Phi(z, w, \zeta) \in \zeta^2 \mathbb{C}[z, w, \zeta]$ which in addition to satisfying $\Phi = O(\zeta^2)$ also satisfy
\begin{equation}
\begin{aligned}
\Phi_{0,j,2} &= 0 \quad j > 0, j \neq m - 1 \\
\Phi_{1,j,2} &= \Phi_{0,j,3} = 0, \quad j > 0; \\
\Phi_{1,j,3} &= \begin{cases} 
0 & j > 0, j \not\in \{m - 1, 2m - 2, 3m - 3\} \\
\sigma_1 & j = 2m - 2 \\
\sigma_2 & j = 3m - 3.
\end{cases}
\end{aligned}
\end{equation}

We will denote the corresponding subcategories of equations whose right hand side is in $D_m^\sigma$ or $D_1$, respectively, by $D_m^\sigma \subset \mathcal{E}_m$ and $D_1 \subset \mathcal{E}_1$.

### 4.2. A singular system of ODEs for the Cauchy data.

We will show that under a generic condition on the defining function $\Phi^*$ of an ode $\mathcal{E}^*$, which we will call the non-resonancy of the ODE $\mathcal{E}^*$, the condition
\[ \text{Hom}(\mathcal{E}, \mathcal{E}^*) \neq \emptyset, \quad \mathcal{E} \in D_1 \text{ (or } D_m^\sigma \text{, respectively)} \]
determines the Cauchy data $f_0(w), f_1(w), g_0(w), h_1(w)$ (almost) uniquely. However, the analytic regularity of the resulting functions for a system arising from an $m$-infinite type hypersurface as well as the non-resonancy condition itself depend significantly on $m$ and subsequently also (for $m > 1$) on whether the ODE $\mathcal{E}$ is of Fuchsian type or it is not (this condition is introduced and discussed in Section 8 below). Accordingly, we consider the normalization procedure in the latter three distinct cases, two of which are justified below and the last one considered in Section 8. The goal of this section is to collect the results which apply to all the systems, regardless of the specific $m$.

In the sequel, we consider the expansions
\begin{equation}
\Phi_2(z, w) = \sum_{j \geq 0} A_j(w) z^j, \quad \Phi_3(z, w) = \sum_{j \geq 0} B_j(w) z^j, \quad \Phi_4(z, w) = \sum_{j \geq 0} C_j(w) z^j
\end{equation}
and
\begin{equation}
f = f_0 + f_1 z + f_2 z^2 + \ldots, \quad g = g_1 z + g_2 z^2 + \ldots
\end{equation}
In order to obtain uniqueness conditions for the collection (4.8), we will make use of the last two equations in (4.7). Some of the terms can actually not be changed by transformations of the form we consider:

**Lemma 4.2.** Let \( \mathcal{E}, \mathcal{E}^* \in \mathcal{E} \) and assume that \( \text{Hom}(\mathcal{E}, \mathcal{E}^*) \neq \emptyset \). Then

\[
\Phi_{002} = \Phi_{002}^*, \quad \Phi_{102} = \Phi_{102}^*, \quad \Phi_{003} = \Phi_{003}^*, \quad \Phi_{103} = \Phi_{103}^*.
\]

The proof of this Lemma is a straightforward computation of the corresponding terms in the transformation rule (4.4) by using the vanishing of the Cauchy data at \( w = 0 \) and (4.7).

**Observation 4.3.** For \( m = 1 \), the normal form condition \( \Phi \in \mathcal{D}_1 \) determines the four coefficient functions \( A_0(w), A_1(w), B_0(w), B_1(w) \) completely, modulo their constant terms (4.17). Unlike that, if \( m > 1 \) the normal form condition \( \Phi \in \mathcal{D}_m^* \) determines (for each fixed \( \sigma \)) the four coefficient functions \( A_0(w), A_1(w), B_0(w), B_1(w) \) completely, modulo their constant terms and the coefficient

\[
\Phi_{1,m-1,3} = \frac{1}{(m-1)!} \frac{\partial^{m-1} B_1}{\partial w^{m-1}}(0).
\]

We will discuss the actual freedom in a possible determination of this parameter later in Section 6.

**Convention 4.4.** We denote the undetermined parameter \( \Phi_{1,m-1,3} \) as above in what follows by \( * \).

The preliminary result for preparing the ODEs for the Cauchy data can be summarized in the following proposition. After this preliminary result, the classification procedure becomes different depending on \( m \) and the defining function \( \Phi^* \).

**Proposition 4.5.** Let \( \mathcal{E}^* \in \mathcal{E}_m \). There exists a germ of an analytic map \( T = (T_1, T_2, T_3, T_4) \) defined near the origin in \( \mathbb{C}^0 \), vanishing at the origin, and depending on \( \mathcal{E}^* \) (and in case \( m > 1 \) also the undetermined coefficient \( * \)) such that if \( (f, g) \in \text{Hom}(\mathcal{E}, \mathcal{E}^*) \), where \( \mathcal{E} \in \mathcal{D}_1 \) (or \( \mathcal{D}_m^* \), respectively), then

\[
\begin{align*}
    w^{m+1} g'' &= T_1(w, g_0, g_1, f_0, f_1, w g_0^0, w^m g_1^0, w^m f_0, w^m f_1), \\
    w^{2m} g_1'' &= T_2(w, g_0, g_1, f_0, f_1, w g_0^0, w^m g_1^0, w^m f_0, w^m f_1), \\
    w^{2m} f_0'' &= T_3(w, g_0, g_1, f_0, f_1, w g_0^0, w^m g_1^0, w^m f_0, w^m f_1), \\
    w^{2m} f_1'' &= T_4(w, g_0, g_1, f_0, f_1, w g_0^0, w^m g_1^0, w^m f_0, w^m f_1).
\end{align*}
\]

On the other hand, if \( (f_0, f_1, g_0, g_1) \) satisfy (4.19), then there exist a map \( (f, g) \) and \( \mathcal{E} \in \mathcal{D}_1 \) (or \( \mathcal{D}_m^* \), respectively) such that \( (f, g) \in \text{Hom}(\mathcal{E}, \mathcal{E}^*) \).

**Proof.** Our normalization conditions \( \Phi \in \mathcal{D}_1 \), and \( \Phi \in \mathcal{D}_m^* \), respectively, translate the equations coming from the corresponding terms in (4.7) into four ODEs in \( H = (f_0, f_1, g_0, g_1) \) by the following process.

(i) We set in the last two equations in (4.7) either

\[
\begin{align*}
    \Phi_{002} &= \Phi_{002}^*, \quad \Phi_{102} = \Phi_{102}^*, \quad \Phi_{003} = \Phi_{003}^*, \quad \Phi_{103} = \Phi_{103}^*, \\
    \Phi_{0j2} &= 0 = \Phi_{1j2} = \Phi_{0j3} = \Phi_{1j3} = 0, \quad j \geq 1.
\end{align*}
\]

if \( m = 1 \)}
or

\[ \Phi_{002} = \Phi_{102}^*, \Phi_{102} = \Phi_{202}^*, \Phi_{003} = \Phi_{103}^*, \Phi_{103} = \Phi_{203}^*; \]

\[
\Phi_{0j2} = \begin{cases} 
0 & j \in \mathbb{N} \setminus \{m - 1\} \\
m & j = m
\end{cases}
\]

(4.21)  
\[
\Phi_{1j2} = \Phi_{0j3} = 0, \ j \geq 1,
\]

\[
\Phi_{1j3} = \begin{cases} 
0 & j \in \mathbb{N} \setminus \{m - 1, 2m - 2, 3m - 3\} \\
\sigma_1 & j = 2m - 2 \\
\sigma_2 & j = 3m - 3.
\end{cases}
\]

if \( m > 1. \)

(ii) After performing (i), we collect the terms of degrees 0 and 1 in \( z \) in the last two equations in (4.7). Note that, for \( m > 1 \), the coefficient (4.18) remains undetermined and thus becomes an unknown parameter for the system of four ODEs under discussion.

By performing (i),(ii), we obtain four second order ODEs, depending on \( f_2, g_2 \), as well as \( f_0(w), f_1(w), g_0(w), g_1(w) \) and their derivatives of order \( \leq 2 \) (because \( f, g \) appear in the last two equations of (4.7) differentiated in \( z \) at most once), and the ODEs depend only on \( \Phi^* \) and the parameter * (because the right hand side of (4.7) only depended on \( \Phi^* \)).

In order to eliminate \( f_2, g_2 \) from the four ODEs under discussion, we use the equations (4.9), evaluated at \( z = 0 \), which allow us to express \( f_2, g_2 \) as analytic functions of \( f_0(w), f_1(w), g_0(w), g_1(w) \) and their derivatives of order \( \leq 1 \). After substituting these two expressions into the four second order ODEs under discussion, we obtain a collection of four second order ODEs in \( f_0(w), f_1(w), g_0(w), g_1(w) \) and their derivatives of order \( \leq 2 \) only. The coefficient (4.21) (for \( m > 1 \)) remains a parameter in this system. We denote the system of ODEs by \( \mathcal{F} \). Before we discuss the form of the system of ODEs obtained by the procedure outlined above, let us note that we have proved the statement about any \( (f, g) \in \text{Hom}(\mathcal{E}, \mathcal{E}^*) \) having the property that \((f_0, f_1, g_0, g_1)\) is a solution of \( \mathcal{F} \). On the other hand, if we have a formal or analytic solution \((f_0, f_1, g_0, g_1)\) of \( \mathcal{F} \), then we can apply Proposition 4.1 to see that it uniquely defines a (formal or analytic) equation \( \mathcal{E} \in \mathcal{D}_1 \) or \( \mathcal{E} \in \mathcal{D}_m^0 \text{ and a (resp. formal or analytic) transformation } (f, g) \in \text{Hom}(\mathcal{E}, \mathcal{E}^*). \)

Now, let us discuss how one obtains the exact form of the ODEs in (4.19). First we note that when solving the first two equations in (4.7) for \( f_{zz}, g_{zz} \) in order to obtain (4.9), all the derivatives \( f_w, g_w, f_wz, g_wz \) appear with the factor \( w^m \), and the derivative \( g_0 \) appear with the factor \( w \). In accordance with that, when we express \( f_2, g_2 \) as analytic functions of \( f_0(w), f_1(w), g_0(w), g_1(w) \) and their derivatives of order \( \leq 1 \) as discussed above, the derivatives \( f_0'', g_0'' \) appear with the factor \( w^m \), and the derivative \( g_0'' \) appears with the factor \( w \).

Further, (4.5) implies that the second order derivatives \( g_0'', g_1', f_0'', f_1'' \) appear in \( \mathcal{F} \) linearly. Now, using (4.3), (4.7), as well as our observation above for \( f_2 \) and \( g_2 \) we can see that \( \mathcal{S} \) is a linear system in

\[ w^{m+1}g_0'', w^mg_1'', w^{2m}f_0'', w^{2m}f_1''. \]
in which the derivatives $f'_0, f'_1, g'_1$ appear with the factor $w^m$, and the derivative $g'_0$ appears with the factor $w$.

The matrix of this linear system evaluated at the origin in the space of 1-jets of maps $w \mapsto (g_0(w), g_1(w), f_0, f_1)$ equals

$$\text{diag}\{-1, -1, 1, 1\}.$$  

Hence we can solve the linear system under discussion in $w^{m+1}g''_0, w^{2m}g''_1, w^{2m}f''_0, w^{2m}f''_1$ and obtain the system of singular holomorphic ODEs claimed in (4.19). Note that if the system has a solution, then the $T$'s have no constant term, and so we can just assume that the right hand side $T$ vanishes at $0$. □

5. Case $m = 1$

In order to define our non-resonancy condition, we start by constructing a first-order system from (4.19). Writing

$$H := (g_0, g_1, f_0, f_1),$$

the system (4.19), in case $m = 1$, takes the following form:

$$w^2 H'' = T(w, H, wH')$$

for a holomorphic map $T$ defined near and vanishing at the origin. We can rewrite (5.1) as a first order ODE using the substitution

$$G := wH', \quad H := (H, G).$$

We write the resulting first order ODE as

$$wH' = T(w, H)$$

where $T$ is a holomorphic map defined near the origin which satisfies $T(0, 0) = 0$. Note that $H(0) = 0$. The singular holomorphic ODE (5.3) is within the class of Briot-Bouquet ODEs (see Section 2), hence every formal solution of (5.3) which satisfies the condition $H(0) = 0$ is convergent. For the existence of formal (and hence holomorphic) solutions, it is known that such a solution with $H(0) = 0$ exists and is unique if the linearization matrix

$$M := \frac{\partial T}{\partial H} \bigg|_{w=H=G=0}$$

has no positive integer eigenvalues, or equivalently, if the intimately related Euler system

$$wH' = MH$$

has only the trivial solution.

Note that the matrix $M$ defined by (5.4), is actually given by

$$M = \left( \begin{array}{cc} 0 & I \\ T_H & I + TG \end{array} \right) \bigg|_{w=H=G=0},$$

with $T$ defined in Proposition 4.5 $T$, and therefore $M$ only depend on $\mathcal{E}^*$. We can therefore make the following definition.
**Definition 5.1.** An ODE $\mathcal{E} \in \mathcal{E}_1$ is called **non-resonant** if the matrix given in (5.4) does not have positive integer eigenvalues. A real hypersurface (1.1) is called **non-resonant**, if its associated ODE $\mathcal{E}(M)$ (see (3.7)) is non-resonant.

Consider now our system (5.1). As discussed above, under the non-resonancy condition, it has a unique holomorphic solution with $H(0) = 0$. Therefore, Proposition 4.5 implies the following.

**Theorem 6.** If $\mathcal{E} \in \mathcal{E}_1$ is nonresonant, then there exists a unique normal form $\mathcal{N} \in \mathcal{D}_1$ such that $\text{Hom}(\mathcal{N}, \mathcal{E}) \neq \emptyset$. Indeed, in that case, there is a unique normalizing transformation: $|\text{Hom}(\mathcal{N}, \mathcal{E})| = 1$. If $\mathcal{E}$ is convergent, so is $\mathcal{N}$ and the normalizing transformation.

For the rest of this section, we will discuss how to write down the non-resonancy condition explicitly, using the linearization of the system (5.3) in $H$, i.e. the linear system

$$wH' = L(w)H,$$

where $L(w) := \frac{\partial T}{\partial H}_{|H=0}$.

This linearized system can be obtained by entirely repeating the above computational scheme of identifying a map (3.6) under consideration, but at the same time ignoring all the non-linear terms in the variables $f, g, g_0$ within the basic identity (4.4). This latter procedure precisely corresponds to searching for **infinitesimal automorphisms** (Lie symmetries)

$$L = P(z, w) \frac{\partial}{\partial z} + Q(z, w) \frac{\partial}{\partial w}$$

of the initial ODE $\mathcal{E}^\ast$ with

$$P(z, w) = f_0(w) + f_1(w)z + f_2(w)z^2 + \cdots,$$

$$Q(z, w) = wg_0(w) + wg_1(w)z + wg_2(w)z^2 + \cdots$$

(Note that, according to Lemma 3.1, any infinitesimal automorphism $L$ of a hypersurface (1.1) has the form (5.8)).

Second, we observe that the linearization matrix $M$ in (5.4) is given by

$$M = L(0).$$

In view of that, we can explicitly determine the matrix, and therefore the non-resonancy condition, in the following way.

(i) we write down the condition that a vector field $L$ satisfying (5.8) is an infinitesimal automorphism (Lie symmetry) of the ODE $\mathcal{E}^\ast$.

(ii) we extract from the respective basic identity terms with

$$z^k w^l(w')^0, z^k w^l(w')^1, z^0 w^l(w')^2, z^1 w^l(w')^2, z^0 w^l(w')^3, z^1 w^l(w')^3, \quad k, l \geq 0$$

and obtain a system of linear PDEs for $P, Q$, which gives us a system of four second order linear ODEs for the initial components $g_0, g_1, f_0, f_1$ of $L$. A substitution identical to (5.2) into the latter system gives us the first order system of eight ODEs (5.3).

(iii) finally, we investigate formal solutions of the intimately related Euler system (5.5) (obtained by putting $w = \text{into the right hand side}$), and write down the non-resonancy
condition. In fact, the latter can be done either for the first order order system of 8 ODEs or for the second order system of four ODEs.

To perform step (i), we proceed similarly to [28] and search for a Lie symmetry $L$ of an ODE $\mathcal{E}$ (satisfying (5.8)) via the jet prolongation method (see Section 2). That is, we consider the ODE $\mathcal{E}^*$ as a submanifold in the space $J^2(\mathbb{C}, \mathbb{C})$ of 2-jets of functions $\mathbb{C} \mapsto \mathbb{C}$ with the coordinates $z, w, w_1, w_2$. After that, we write down the fact that the second jet prolongation

$$L^{(2)} = P(z, w) \frac{\partial}{\partial z} + Q(z, w) \frac{\partial}{\partial w} + Q^{(1)}(z, w, w_1) \frac{\partial}{\partial w_1} + Q^{(2)}(z, w, w_1, w_2) \frac{\partial}{\partial w_2},$$

of a Lie symmetry $L$ is tangent to this submanifold. We have

$$w_1 := w', \quad w_2 := w'',$$

$$Q^{(1)} = Q_z + (Q_w - P_z)w_1 - P_w(w_1)^2,$$

$$Q^{(2)} = Q_{zz} + (2Q_{zw} - P_{zz})w_1 + (Q_{ww} - 2P_{zw})(w_1)^2 - P_{ww}(w_1)^3 + (Q_w - 2P_z)w_2 - 3P_w w_1 w_2,$$

and the tangency condition means

$$Q^{(2)} \bigg|_{w_2 = \Phi(z, w, \frac{w_1}{w_m})} = \Phi_z \left(z, w, \frac{w_1}{w_m}\right) \cdot P + \Phi_w \left(z, w, \frac{w_1}{w_m}\right) \cdot Q + \frac{1}{w_m} \Phi_\xi \left(z, w, \frac{w_1}{w_m}\right) \cdot Q^{(1)}$$

for all $(z, w, w_1)$ lying in an appropriate open set $V \subset J^1(\mathbb{C}, \mathbb{C})$.

To perform step (ii), we first gather in (5.10) terms with $(w_1)^0, (w_1)^1, (w_1)^2, (w_1)^3$. Using the notations

$$a := \frac{1}{w_m} \Phi_2, \quad b := \frac{1}{w_{2m}} \Phi_3, \quad c := \frac{1}{w_{3m}} \Phi_4,$$

we get, respectively:

$$Q_{zz} = 0,$$

$$2Q_{zw} - P_{zz} = 2aQ_z,$$

$$Q_{ww} - 2P_{zw} = a(-Q_w + 2P_z) + a_z P + a_w Q + 3bQ_z + 2a(Q_w - P_z),$$

$$P_{ww} = b(Q_w - 2P_z) - aP_w - b_z P - b_w Q - 4cQ_z + 3b(P_z - Q_w).$$

By employing the expansion (5.8), from the first equation in (5.11) we get

$$Q(z, w) = w g_0(w) + w^m g_1(w) z,$$

and from the second

$$P(z, w) = f_0(w) + f_1(w) z + (w^m g'_1(w) + mw^{m-1} g_1) z^2 - 2w^m \tilde{a}(z, w) g_1(w),$$

where $\tilde{a}_{zz} = a, \tilde{a} = O(z^2)$. For $m = 1$ this gives, in particular,

$$g_2 = 0, \quad f_2 = -A_0 g_1 + w g'_1 + g_1$$
The non-resonancy condition is satisfied if and only if each of the equations (5.17) for (5.18)

\[ z^0 w'(w')^2, z^1 w'(w')^2, z^0 w'(w')^3, z^1 w'(w')^3, \]

respectively, and substitute (5.14) into the result. This gives us a system of four second order Fuchsian linear ODEs. It is not difficult to compute that this system looks as follows (we use the notations in (4.15)):

\[
\begin{align*}
(5.15) \\
& w^2 g''_0 + w(2 - A_0)g'_0 - 2wf'_1 - A'_0 wg_0 - 3B_0g_1 - A_1f_0 = 0 \\
& 3w^2 g''_1 + A_1wg'_0 + 3wA_0g'_1 + wA'_1g_0 + (3B_1 - 3wA'_0)g_1 + 2A_2f_0 + A_1f_1 = 0 \\
& w^2 f''_0 + 2B_0wg'_0 + A_0wf'_0 + wB'_0g_0 + 4C_0g_1 + B_1f_0 - B_0f_1 = 0 \\
& w^2 f''_1 + 2B_1wg'_0 + A_1wf'_0 + A_0wf'_1 + wB'_1g_0 + (2B_0(A_0 - 1) + wB'_0 + 4C_1)g_1 + 2B_2f_0 = 0.
\end{align*}
\]

For step (iii), we have to evaluate the coefficients of the system (5.15) at \( w = 0 \), and search for a formal solution \( H(w) \) of the arising system with constant coefficients. If we expand a formal solution \( H(w) \) of it as

\[ H(w) = \sum_{k \geq 1} h_kw^k \]

and the coefficients of (5.15) as

\[ (5.16) \quad \alpha_j := A_j(0), \quad \beta_j := B_j(0), \quad \gamma_j := C_j(0), \quad j = 0, 1, 2. \]

Then we obtain for the Taylor coefficients \( h_k, k \geq 1 \) the following set of equations:

\[ (5.17) \quad \begin{pmatrix}
(k + 1 - \alpha_0) & -3\beta_0 & -\alpha_1 & -2k \\
2k\beta_0 & 3k(1 + \alpha_0) - 3\beta_1 & 2\alpha_2 & \alpha_1 \\
2k\beta_1 & 2\beta_0(\alpha_0 - 1) + 4\gamma_1 & k\alpha_1 + 2\beta_2 & k(1 + \alpha_0)
\end{pmatrix} h_k = 0. \]

The non-resonancy condition is satisfied if and only if each of the equations (5.17) for \( k \geq 1 \) has only the trivial solution, that is, if the equation

\[ (5.18) \quad \det \begin{pmatrix}
(k + 1 - \alpha_0) & -3\beta_0 & -\alpha_1 & -2k \\
2\beta_0 & 3k(1 + \alpha_0) - 3\beta_1 & 2\alpha_2 & \alpha_1 \\
2\beta_1 & 2\beta_0(\alpha_0 - 1) + 4\gamma_1 & k\alpha_1 + 2\beta_2 & k(1 + \alpha_0)
\end{pmatrix} = 0 \]

has no solutions \( k \in \mathbb{N} \).

We summarize this result as following.

**Proposition 5.2.** An ODE (3.7) is non-resonant if and only if the algebraic equation (5.18) of degree 7 (where \( \alpha_j, \beta_j, \gamma_j \) are taken from (5.16)) has no positive integer solutions.

In particular, a hypersurface will have at most finitely many resonances (i.e. integer solutions to the equation (5.18)).
Corollary 5.3. There can exist only finitely many resonances for a hypersurface \((1.1)\) when \(m = 1\) (in fact, at most 7 of them).

Using the explicit characterization of ODE (3.7) associated with generically spherical hypersurfaces \((1.1)\), it is not difficult to prove also

Proposition 5.4. There exist real hypersurfaces \((1.1)\) which are non-resonant at \(0\). Accordingly, a generic hypersurface \((1.1)\) is non-resonant at \(0\) (in the sense of the jet topology in the space of defining functions \(h(z, \bar{z}, u)\)).

Proof. Fix \(k \geq 1\) and consider the determinant in (5.18) as a linear expression in \(\alpha_2, \gamma_0,\) and the product \(\alpha_2 \gamma_0\). Then the coefficient of \(\alpha_2 \gamma_0\) equals, up to a nonzero factor, to

\[
(k - 1 + \alpha_0)(k + 1 - \alpha_0) + 4\beta_1.
\]

Fix a generic pair \(\alpha_0, \beta_1\) such that the expression (5.19) does not vanish for all natural \(k\) (by using formulas (7.1), (7.11) below, this is accomplished by an appropriate choice of the Taylor coefficients \(h_{22}(0), h_{33}(0)\) of the real defining function). On the other hand, one can see again from (7.1), (7.11) that, by varying \(h_{24}(0)\), the coefficients \(\alpha_2, \gamma_0\) are (up to the constant factor 6) arbitrary complex conjugated complex numbers. Hence, if we fix \(h_{23}(0), h_{34}(0)\) in an arbitrary way, then for each fixed natural \(k\) one can avoid vanishing of the determinant in (5.18) by erasing (at most) a real curve from the complex plane parameterized by \(h_{24}(0)\). This proves that there is a choice of \(\alpha_2, \gamma_0\) making the determinant in (5.18) nonvanishing for all natural \(k\) at once, as required. □

6. Case \(m > 1\)

In this section, we investigate formal power series solutions of the system (4.19) in the case \(m > 1\). In order to do so, we make a power series ansatz, and expand a candidate for a formal vector solution \(H := (g_0, g_1, f_0, f_1)\) as

\[
H = \sum_{k \geq 1} h_k w^k
\]

After we plug this ansatz into (4.19), the coefficients of \(w^k, k \geq 1\) in the resulting equation give rise to equations \(E_k\), and we will study try to solve \(E_k\) for \(h_k\). That is, we aim to determine uniquely \(h_1\) from \(E_1\), \(h_2\) from \(E_2\), etc.

It is not difficult to see that

(i) each \(E_k\) only involves \(h_1, h_2, ..., h_k\);

(ii) because \(m > 1\), the left hand side of the equation \(E_k\) does not involve \(h_k\), while on the right hand side only linear terms in \(g_0, g_1, f_0, f_1, wg_0^k\) can give rise to \(h_k\). Moreover, in the respective linear expressions

\[
\lambda_1(w)g_0 + \lambda_2(w)g_1 + \lambda_3(w)f_0 + \lambda_4(w)f_1 + \lambda_5(w)g_0^k w
\]

one has to evaluate the coefficients \(\lambda_1(w), ..., \lambda_5(w)\) at \(w = 0\).

In view of the above, each of the equations \(E_k\) has the form:

\[
M_k h_k = R_k(h_1, ..., k_{k-1}),
\]
where $M_k$ is a $4 \times 4$ matrix, and $R_k$ is a polynomial in its variables (its concrete form depends on $\Phi^*$ and is of no interest to us here).

We will now determine the form of the matrices $M_k$, and formulate the arising nonresonancy condition. For doing so, in view of the observations (i),(ii) above, we may use a procedure similar to that in the case $m = 1$. We, however, should ignore in the arising system of second order linear differential equations for $g_0, g_1, f_0, f_1$ analogous to (5.15) all the second order derivatives, as well as the derivatives $g'_1, f'_0, f'_1$. Now the system of equations analogous to (5.15) becomes:

\[
(6.2)
(2w^m - wA_0)g'_0 + ((m - 1)A_0 - A'_0w)g_0 - 3B_0g_1 - A_1f_0 = mw^{m-1}
\]

\[
A_1wg' + [A_1(1 - m) + wA'_1]g_0 + [(3B_1 - 3w^mA'_1 + 3(m - 1)w^{m-1}g_1 + 2A_2f_0 + A_1f_1 = 0
\]

\[
2B_0wg' + [(2 - 2m)B_0 + wB'_0]g_0 + 4C_0g_1 + B_1f_0 - B_0f_1 = 0
\]

\[
2B_1wg' + [(2 - 2m)B_1 + wB'_1]g_0 + (2B_0A_0 + w^mB'_0 - 2mw^{m-1}B_0 + 4C_1)g_1 + 2B_2f_0 =
\]

\[
= *w^{m-1} + \sigma_1 w^{m-2} + \sigma_2 w^{3m-3}
\]

(we recall that we are using Convention 4.4). To investigate the formal solvability of (6.2), we (i) evaluate the coefficients of the system (6.2) at $w = 0$; (ii) substitute a formal power series $H(w)$ into the resulting system of ODEs with constant coefficients; (iii) put zero for the right hand side (that is, switching to the respective homogeneous system).

In the notation of (5.16) this gives

\[
(6.3)
\begin{pmatrix}
\alpha_0(k + 1 - m) & 3\beta_0 & \alpha_1 & 0 \\
\alpha_1(k + 1 - m) & 3\beta_1 & 2\alpha_2 & \alpha_1 \\
2\beta_0(k + 1 - m) & 4\gamma_0 & \beta_1 & -\beta_0 \\
2\beta_1(k + 1 - m) & 2\beta_0\alpha_0 + 4\gamma_1 & 2\beta_2 & 0
\end{pmatrix}

\cdot h_k = 0.
\]

for $k \neq m - 1$, and

\[
(6.4)
\begin{pmatrix}
0 & 3\beta_0 & \alpha_1 & 0 \\
0 & 3\beta_1 & 2\alpha_2 & \alpha_1 \\
0 & 4\gamma_0 & \beta_1 & -\beta_0 \\
0 & 2\beta_0\alpha_0 + 4\gamma_1 & 2\beta_2 & 0
\end{pmatrix}

\cdot h_k = 0
\]

for $k = m - 1$. From (6.3), we can see that, unlike the case $m = 1$, for the special value $k = m - 1$ the respective matrix $M_{m-1}$ is always degenerate. That is why, for any given source ODE $\mathcal{E}$, there is no hope to guarantee the uniqueness of a formal normalizing transformation (3.6). We then proceed as follows.

The special form of the matrices $M_k$ arising from (6.3),(6.4) leads to

**Definition 6.1.** An ODE (3.7) with $m > 1$ is called *nonresonant*, if both the matrix

\[
(6.5)
M := \begin{pmatrix}
\alpha_0 & 3\beta_0 & \alpha_1 & 0 \\
\alpha_1 & 3\beta_1 & 2\alpha_2 & \alpha_1 \\
2\beta_0 & 4\gamma_0 & \beta_1 & -\beta_0 \\
2\beta_1 & 2\beta_0\alpha_0 + 4\gamma_1 & 2\beta_2 & 0
\end{pmatrix}
\]
and its upper right $3 \times 3$ minor

\[
\tilde{M} := \begin{pmatrix}
3\beta_0 & \alpha_1 & 0 \\
3\beta_1 & 2\alpha_2 & \alpha_1 \\
4\gamma_0 & \beta_1 & -\beta_0
\end{pmatrix}
\]

are invertible. A real hypersurface $M$ is called \textit{non-resonant}, if its associated ODE is non-resonant.

Let us again check that non-resonant hypersurfaces with $m > 1$ actually exist.

**Proposition 6.2.** Let $m > 1$. There exists a real hypersurface $M \in \mathcal{R}_m^\pm$ which is non-resonant at $0$. Accordingly, a generic hypersurface in $\mathcal{R}_m^\pm$ (in the sense of the jet topology in the space of defining functions $h(z, \bar{z}, u)$) is non-resonant at $0$.

**Proof.** Let us first choose hypersurfaces $M$ for which the coefficient $\alpha_1$ (in the notations (5.16)) is non-zero. To achieve this condition, we apply the formulas (7.1) below and see that the condition under discussion is achieved by varying $\varphi_{32}(0)$, which can be chosen arbitrary complex thanks to the formulas (7.12) below. Then, for a generic $\gamma_0$ (and fixed $\beta_0, \beta_1$) the determinant of (6.6) is nonzero. However, the coefficient $\gamma_0$ can be as well chosen arbitrary complex, since it is proportional to $\varphi_{24}(0)$ (see (7.1)), and $\varphi_{24}(0)$ is proportional to $h_{24}(0)$ (the proof of the latter fact is identical to the proof of (7.12), and we do not provide details). This finally proves that (6.6) can be achieved by choosing a generic collection $h_{23}(0), h_{33}, h_{24}(0)$.

We further consider the determinant of (6.5) as a linear expression in $\gamma_1$. The coefficient of $\gamma_1$ then equals up to a constant nonzero factor:

\[
(6.7) \quad 3\alpha_1^2\beta_0 - \alpha_0(\alpha_1\beta_1 + 2\alpha_2\beta_0).
\]

We can still vary the above triple $h_{23}(0), h_{33}, h_{24}(0)$ in order to keep the coefficient of $\alpha_0$ nonzero in (6.7). As can be seen from (7.1), possible values of $\alpha_0$ run a (real) line in complex plane, hence we can finally choose $\alpha_0$ in such a way that (6.7) is nonzero, and this shows that an appropriate choice of $\gamma_1$ (which is proportional to $h_{25}$) allows to finally make the determinant of (6.5) nonzero simultaneously with that of (6.6), as required. \(\Box\)

In the non-resonant case, we fix the parameter

\[
(6.8) \quad \tau := \frac{1}{(m-1)!}g_0^{(m-1)}(0) = \frac{1}{m!} \frac{\partial^m G}{\partial w^m}(0, 0)
\]

(it corresponds to the first component of the vector $h_{m-1}$, and as well corresponds to the parameter (1.7)). This parameter is the only one we can not determine uniquely, and its choice is with one-to-one correspondence with the choice of the coefficient (4.21). After that, in view of the non-resonancy condition, for each fixed $\tau$ we can uniquely solve the sequence of linear systems (6.3) with $k \neq m - 1$, while for $k = m - 1$ we can solve it, modulo the last component of the right hand side (this means that this first component of the right hand side of the respective linear system becomes a precise value). This gives us a unique formal solution for the set of equations (6.1). We furthermore observe that
Remark 6.3. The coefficient (4.18) is in fact independent of \( \tau \), and thus becomes an invariant of an ODE (3.7). This follows from the fact that the coefficient \( \tau \) for the first time appears in the \( m \)-th equation in the above formal procedure, as earliest.

By the above argument, we have proved

**Theorem 7.** If \( \mathcal{E} \in \mathfrak{e}_m \), where \( m > 1 \), is non-resonant, then for every \( \tau \in \mathbb{C} \) there exists a normal form \( \mathcal{N}_\tau \in \mathcal{D}_m^\sigma \) and a formal transformation \((f_\tau, g_\tau) \in \text{Hom}(\mathcal{N}_\tau, \mathcal{E})\). The normalizing transformation and the normal form are unique (up to the complex parameter \( \tau \) in (6.8)).

### 7. Solution of the equivalence problem

In this section, we prove **Theorem 1** and **Theorem 2**. In what follows, if not otherwise stated, we treat the case of a positive real hypersurface \( M \in \mathfrak{N}_m^+ \) (i.e. \( \epsilon = 1 \) in (1.1)), as in the negative case the considerations are very analogous.

We start with transferring the normal form condition \( \mathcal{E} \in \mathfrak{D}_m^\sigma \) for ODE (3.7) to the associated \( m \)-admissible Segre families.

**Definition 7.1.** We say that an \( m \)-admissible Segre family \( \mathcal{S} \in \mathfrak{S}_m^+ \) is in normal form, if its associated ODE is in normal form, i.e. if \( \mathcal{E}(\mathcal{S}) \in \mathfrak{D}_m^+ \). We write \( \mathfrak{N}_m^\sigma = \mathfrak{E}^{-1}(\mathfrak{D}_m^\sigma) \) for the space of normal forms.

We next formulate

**Proposition 7.2.** The following relations hold between the coefficient functions (4.15) of an ODE (3.7) and its associated Segre family (3.2):

\[
\begin{align*}
A_0(w) &= w^{m-1} - 2i\varphi_{22}(w), \quad A_1(w) = \mp 6i\varphi_{32}(w), \quad B_0(w) = -2\varphi_{23}(w), \\
C_0(w) &= \pm 2i\varphi_{24}(w), \quad A_2(w) = \mp 12i\varphi_{42}(w), \\
B_1(w) &= -6\varphi_{33}(w) \mp 2i(m-1)\varphi_{22}(w)w^{m-1} + 8(\varphi_{22}(w))^2 \pm 2i\varphi'_{22}(w)w^m \\
B_2(w) &= -12\varphi_{23}(w) + 36\varphi_{22}(w)\varphi_{32}(w) \mp 6i(1-m)w^{m-1}\varphi_{32}(w) \pm 6iw^m\varphi'_{32}(w) \\
C_1(w) &= \pm 6i\varphi_{34}(w) \mp 20i\varphi_{22}(w)\varphi_{23}(w) + (4 - 4m)w^{m-1}\varphi_{23}(w) + 2w^m\varphi'_{23}(w).
\end{align*}
\]

Since the proof of Proposition 7.2 is obtained by a direct substitution of the defining equation (3.2) into an ODE (3.7), and is as well very analogous to the proof of Proposition 4.2 in [29], we leave the details to the reader.

Now Proposition 7.2 implies

**Corollary 7.3.** A Segre family \( \mathcal{S} \in \mathcal{S}_m^+ \), defined by (3.2), is in normal form (\( \mathcal{S} \in \mathfrak{N}_m^+ \)) if and only if it satisfies:

\[
\begin{align*}
\varphi_{22}(w) &= \varphi_{22}(0), \quad \varphi_{23}(w) = \varphi_{23}(0), \quad \varphi_{32}(w) = \varphi_{32}(0), \quad \varphi_{33}(w) = \varphi_{33}(0),
\end{align*}
\]
A Segre family $\mathcal{S} \in \mathfrak{S}^*_m$, where $m > 1$, is in normal form ($\mathcal{S} \in \mathfrak{R}^{m,\pm}$) if and only if it satisfies

$$
\begin{align*}
\varphi_{23}(w) &= \varphi_{23}(0), \quad \varphi_{32}(w) = \varphi_{32}(0), \quad \varphi_{22}(w) = \lambda \pm \frac{i}{2}(m-1)w^{m-1}, \\
\varphi_{33}(w) &= \mu + \nu w^{m-1} + \sigma_1 w^{m-1} + \sigma_2 w^{3m-3} + \\
&\quad + \frac{1}{3} \left( \mp i(m-1)\varphi_{22}(w)w^{m-1} + 4(\varphi_{22}(w))^2 \pm i\varphi'_{22}(w)w^m \right),
\end{align*}
$$

where $\lambda, \mu, \nu \in \mathbb{C}$, $\sigma_1, \sigma_2 \in \mathbb{R}$.

We now need a development of the technique for investigating the reality condition for parameterized families of curves introduced in [32].

**Definition 7.4.** We say that an $m$-admissible Segre family $\mathcal{S} \in \mathfrak{S}^*_m$ has a real structure if it is the Segre family of an $m$-admissible real hypersurface $M \subset \mathbb{C}^2$, i.e. if $\mathcal{S} = \mathcal{S}(M)$ for some $M \subset \mathfrak{R}^+_m$. We also say that an ODE $\mathcal{E} \in \mathfrak{E}_m^\pm$ has $m$-positive (respectively, $m$-negative) real structure, if the associated positive (respectively, negative) $m$-admissible Segre family $\mathcal{S}^+_m(\mathcal{E})$ has a real structure. We say that the corresponding real hypersurface $M$ is associated with $\mathcal{E}$.

Let $\rho(z, y, u)$ be a holomorphic function near the origin in $\mathbb{C}^3$ with $\rho(0, 0, 0) = 0$, and $d\rho(0, 0, 0) = du$. For $z, \xi \in \Delta_\delta, w, \eta \in \Delta_\epsilon$, let

$$
\mathcal{S} = \{ w = \rho(z, \bar{\xi}, \bar{\eta}) \}
$$

be a 2-parameter antiholomorphic family of holomorphic curves near the origin, parametrized by $(\xi, \eta)$. An admissible (re)-parametrization of $\mathcal{S}$ is given by a function $\bar{\rho}(z, \bar{\xi}', \bar{\eta}')$ such that

$$
\mathcal{S} = \{ w = \bar{\rho}(z, \bar{\xi}', \bar{\eta}') \}
$$

and there exists a germ of a biholomorphism $(\xi, \eta) \mapsto (\xi', \eta')$ such that $\rho(z, \bar{\xi}, \bar{\eta}) = \bar{\rho}\left(z, \xi'(\xi, \eta), \eta'(\xi, \eta)\right)$. Fixing a parametrization and considering all admissible (re)-parametrizations gives rise to the notion of a general Segre family.

For each point $p = (\xi, \eta) \in \Delta_\delta \times \Delta_\epsilon$ we call the corresponding holomorphic curve $Q_p^\rho = \{ w = \rho(z, \bar{\xi}, \bar{\eta}) \} \in \mathcal{S}$ its Segre variety. Clearly, an $m$-admissible Segre family is a particular example of a general Segre family. Note that the Segre varieties of a general Segre family do depend on the parametrization, but admissible parametrizations give rise to a (holomorphic) relabeling of the Segre varieties.

We say that two general Segre families $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are equivalent if there exists a germ of a biholomorphism $H = (f, g)$ of $(\mathbb{C}^2, 0)$ such that $\tilde{\mathcal{S}} = H^{-1}(\mathcal{S})$, and such that the solution of the implicit function problem $g(z, w) = \bar{\rho}(f(z, w)), \xi, \eta)$ for $w$ is an admissible parametrization of $\mathcal{S}$.

Further, given a (general) Segre family $\mathcal{S}$, from the implicit function theorem one concludes that the antiholomorphic family of planar holomorphic curves

$$
\mathcal{S}^{*, \rho} = \{ \bar{\eta} = \rho(\bar{\xi}, z, w) \}.
$$
is also a general Segre family for some, possibly, smaller polydisc $\Delta_\xi \times \Delta_\zeta$, which depends on the chosen parametrization $\rho$. We note that for every admissible parametrization of $S$, we obtain an equivalent Segre family.

**Definition 7.5.** The Segre family $S^{*,\rho}$ is called the **dual Segre family** for $S$ with the parametrization $\rho$.

The dual Segre family has a simple interpretation: in the defining equation of the family $S$ one should consider the parameters $\xi, \eta$ as new coordinates, and the variables $z, w$ as new parameters. If we denote the Segre variety of a point $p$ with respect to the family $S^{*,\rho}$ by $Q_p^{*,\rho}$, this just means that $Q_p^{*,\rho} = \{(z, w): p \in Q_p^\rho(z, \bar{w})\}$. In the following, we will suppress the dependence on $\rho$ from the notation whenever we make claims which hold for all admissible parametrizations of a given Segre family.

It is not difficult to see that if $S \in \mathcal{S}_m^\pm$ is a positive (respectively, negative) $m$-admissible Segre family, then $S^* \in \mathcal{S}_m^\pm$ is a negative (respectively, positive) $m$-admissible Segre family, if we are using its naturally associated defining function: Indeed, to obtain the defining function $\rho^*(z, \xi, \eta)$ of the general Segre family $S^*$ we need to solve for $w$ in the equation

$$\eta = w e^{\pm i w^{m-1}(z \xi + \sum_{k,l \geq 2} \varphi_{kl}(w) z^k \xi^l)}.$$  

(7.4)

Note that (7.4) implies

$$w = \bar{\eta} e^{\pm i w^{m-1}(\bar{z} \xi + O(z^2 \xi^2))}.$$  

(7.5)

We then obtain from (7.5) $w = \rho^*(z, \xi, \eta) = \bar{\eta}(1 + O(z \xi))$. Substituting the latter representation into (7.5) gives $w = \rho^*(z, \xi, \eta) = \bar{\eta} e^{\pm i \eta^{m-1}(z\xi + O(z^2 \xi^2))}$, as required.

We also need the following Segre family, connected with $S$:

$$\tilde{S} = \{w = \bar{\rho}(z, \xi, \eta)\},$$

where for a power series of the form

$$f(x) = \sum_{\alpha \in \mathbb{Z}^d} c_\alpha x^\alpha$$

we denote by $\tilde{f}(x)$ the series $\sum_{\alpha \in \mathbb{Z}^d} \bar{c}_\alpha x^\alpha$. Note that $\tilde{S}$ does not depend on the particular admissible parametrization, in contrast to the dual family.

**Definition 7.6.** The Segre family $\tilde{S}$ is called the **conjugated family** of $S$.

If $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$ is the antiholomorphic involution $(z, w) \to (\bar{z}, \bar{w})$, then one simply has $\sigma(Q_p) = \overline{Q_{\sigma(p)}}$. We will denote the Segre variety of a point $p$ with respect to the family $\tilde{S}$ by $\overline{Q_p^\rho}$. It follows from the definition that if $S$ is a positive (respectively, negative) $m$-admissible Segre family, then $\tilde{S}$ is a negative (respectively, positive) $m$-admissible Segre family.

In the same manner as for the case of an $m$-admissible Segre family, we say that a (general) Segre family $S = \{w = \rho(z, \xi, \eta)\}$ has a real structure, if there exists a smooth real-analytic hypersurface $M \subset \mathbb{C}^2$, passing through the origin, such that $S$ is the Segre family of $M$.

The use of the dual and the conjugated Segre families is illuminated by the fact that
A (general) Segre family $S$ has a real structure if and only if the conjugated Segre family $\bar{S}$ is also a dual family, i.e. if there exists an admissible parametrization $\rho$ such that $S^{*}\rho = \bar{S}$.

This fact proved, for example, in [32] is a corollary of the reality condition for a real-analytic hypersurface.

**Definition 7.7.** Let $E_0 \in E_m$. The $m$-dual $E_0^* \in E_m$ to $E_0$ is defined by

$$E_0^* = E((S_m^+(E_0))^*) .$$

In other words, $E_0^*$ is the $m$-dual of $E_0$ if the negative $m$-admissible Segre family which is dual to the family $S_0^+$ is associated with $E_0^*$.

The $m$-conjugated ODE $\overline{E_0}$ to $E_0$, is given by

$$\overline{E_0} = E((\overline{S_m^+(E_0)})^*).$$

In other words, $\overline{E_0}$ is $m$-conjugated to $E_0$ if the negative $m$-admissible Segre family which is conjugated to the family $S_m^+(E)$ is in fact associated with $\overline{E_0}$.

The characterization of the existence of a real structure for Segre families now implies that

**Proposition 7.8.** An equation $E \in E_m$ has a real structure if and only if

$$(7.6) \quad E^* = \overline{E}$$

holds.

**Proposition 7.9.** Let $S \in \mathfrak{M}_m$ be in normal form. Then both $S^*$ and $\overline{S}$ are in normal form.

**Proof.** Let

$$S = \left\{ w = \bar{\eta} e^{i\eta m^{-1} \varphi} \right\}, \quad \bar{S} = \left\{ w = \bar{\eta} e^{-i\eta m^{-1} \varphi} \right\}, \quad S^* = \left\{ w = \bar{\eta} e^{-i\eta m^{-1} \varphi^*} \right\} .$$

Then, according to Lemma 20 in [29],

$$(7.7) \quad \bar{\varphi}_{kl}(w) = \varphi_{kl}(w), \quad k, l \geq 2$$

$$(7.8) \quad \varphi_{22}^*(w) = \varphi_{22}(w) - i(m-1)w^{m-1}, \quad \varphi_{32}^*(w) = \varphi_{32}(w), \quad \varphi_{23}^*(w) = \varphi_{23}(w),$$

$$(7.9) \quad \varphi_{33}^* = \varphi_{33}(w) - \frac{3}{2}(m-1)^2w^{2m-2} - 2i(m-1)w^{m-1}\varphi_{22}(w) - iw^m\varphi_{22}'(w).$$

**Important:** we have hereby corrected a typo in [29], which is the sign of $\frac{3}{2}$ in the formula (7.9). Now the formula (7.7) combined with Corollary 7.3 immediately implies that the conjugated family $\overline{S}$ is in normal form.

In the case $m = 1$ the fact that $S^*$ is proved similarly. However, for $m > 1$ the assertion of the proposition for the dual family requires additional arguments. If we use the normal form conditions and denote

$$A_0(w) = \lambda + mw^{m-1}, \quad B_1(w) = \mu + \nu w^{m-1} + \sigma_1 w^{2m-2} + \sigma_2 w^{3m-3}$$
(in the notations of Proposition 7.2), then by using (7.1), (7.8) and (7.9) we can obtain (after a straightforward calculation) \( (7.10) \)
\[
\varphi^*_2 = -\frac{i}{2}(m-1)w^{m-1} + \frac{i}{2}\lambda,
\]
\[
\varphi^*_3 = -\frac{1}{3}(\lambda - (m-1)w^{m-1})^2 - \frac{\lambda}{6}(m-1)w^{m-1} - \frac{1}{6}(\mu + \nu w^{m-1} + \sigma_1 w^{2m-2} + \sigma_2 w^{3m-3}).
\]
Working out now the conditions (7.3) for the dual family, one can see that they are precisely equivalent to (7.10), that is why \( S^* \) is in normal form, as required. \( \square \)

Further, we have the following

**Convention 7.10.** We call two ODEs (3.7) \( \tau \)-equivalent, if there is a (formal or holomorphic) map (3.6) with the complex parameter (6.8) being equal to \( \tau \) transforming the two ODEs into each other. The same convention holds for Segre families and real hypersurfaces.

The next important step is

**Proposition 7.11.** If two ODEs \( E_1, E_2 \), as in (3.7), are \( \tau \)-equivalent, then the ODEs \( E_1^*, E_2^* \) are \( \overline{\tau} \)-equivalent and the ODEs \( E_1^*, E_2^* \) are \( \overline{\tau} \)-equivalent.

**Proof.** For the conjugated ODEs the assertion of the proposition is obvious (one has to simply bar the map between \( E_1 \) and \( E_2 \)). For the dual ODEs, we denote the given map between \( E_1 \) and \( E_2 \) by \( (F(z, w), G(z, w)) \). By Proposition 7.8, there exists a map \( (\Lambda(\xi, \eta), \Omega(\xi, \eta)) \), as in (3.6), such that the product map \( (F(z, w), G(z, w), \overline{\Lambda}(\xi, \eta), \overline{\Omega}(\xi, \eta)) \) maps the associated Segre families \( S_1, S_2 \) of respectively \( E_1 \) and \( E_2 \) into each other. Hence, by definition of the dual family, the product map \( (\Lambda(\xi, \eta), \Omega(\xi, \eta), \overline{F}(z, w), \overline{G}(z, w)) \) maps the associated Segre families into each other. Recall that both Segre families have the form
\[
w = \eta + O(z\xi\bar{\eta}^m).
\]
Also recall that \( G = O(w), \Omega = O(\eta) \) due to (3.9). In view of that, writing the transformation rule between the Segre families \( S_1, S_2 \) and collecting terms with \( z^j\xi^\beta\eta^\gamma \), \( 1 \leq j \leq m \) gives:
\[
G^{(j)}(0, 0) = \overline{\Omega^{(j)}(0, 0)},
\]
which implies the assertion of the proposition. \( \square \)

We are now in the position to prove the first key proposition leading to Theorem 1.

**Proposition 7.12.** (i) Let \( E \in \mathfrak{E}_1 \) have a real structure, that is, assume that \( E \) is associated with a real hypersurface \( M \in \mathcal{R}^+ \). Let \( \mathcal{N} \) be its normal form. Then \( \mathcal{N} \) has a real structure, i.e. it is associated with a real hypersurface \( N = N(M) \in \mathcal{R}^+ \). Furthermore, \( N(M) \) is biholomorphically equivalent to \( M \) at the origin.
(ii) Let \( m > 1 \) and \( \mathcal{E} \in \mathfrak{E}_m \) have a real structure, that is, assume that \( \mathcal{E} \) is associated with a real hypersurface \( M \in \mathbb{R}^n_+ \) and \( N_\tau \) be its normal form corresponding to some value \( \tau \in \mathbb{R} \) of the parameter \( (6.8) \). Then \( N_\tau \) has a real structure, i.e. it is associated with a real hypersurface \( N_\tau(M) \in \mathbb{R}^n_+ \). Furthermore, \( N_\tau(M) \) is formally \( \tau \)-equivalent to \( M \) at the origin.

Proof. Assume that \( m > 1 \). Let \( H(z,w) \) be the transformation mapping \( \mathcal{E} \) into \( N_\tau \). Then obviously \( H \) is a map between \( \mathcal{E} = \mathcal{E}^* \) and \( N_\tau \). That is, \( H \) brings \( \mathcal{E} = \mathcal{E}^* \) into normal form (which is equal to \( N_\tau \)), and by Proposition 7.11 its parameter \( (6.8) \) is \( \tau = \tau \) as well. Again, by Proposition 7.11 we can claim that \( \mathcal{E}^* \) is \( \tau \)-equivalent to \( N_\tau^* \). Since both ODEs \( N_\tau \) and \( N_\tau^* \) are in normal form, we conclude from the uniqueness of a normalizing map that

\[
N_\tau = N_\tau^* ,
\]

so that \( N_\tau \) has a real structure. Hence the Segre families \( S \) and \( S_\tau \) of the two ODEs \( \mathcal{E} \) and \( N_\tau \) respectively are the Segre families of real hypersurfaces \( M \) and \( N_\tau(M) \) respectively.

By Proposition 7.8 there exists a product map between them. As follows from the above argument, this product map coincides with \( (H(z,w), H(\xi,\eta)) \), and this implies that \( H \) maps \( M \) into \( N_\tau(M) \), as required for (ii).

In the case \( m = 1 \) the proof is analogous (actually, it is even simpler because the parameter \( \tau \) from \( (6.8) \) does not occur at all). \( \square \)

Remark 7.13. As follows from (3.4), when considering normal forms with a real structure for ODEs (3.7) with a real structure, one needs to restrict to \( \tau \) being real in (6.8), which explains the assumption \( \tau \in \mathbb{R} \) in Proposition 7.12.

Our remaining task is to relate the ODE defining function \( \Phi \) and the real defining function \( h \) in the case when an ODE \( \mathcal{E} \) has a real structure. For that, we relate the exponential defining function \( \varphi \) and the real defining function \( h \) by means of the identity

\[
w = \bar{w} e^{i \bar{w} m - 1 \varphi(z,\bar{z},\bar{w})} , \quad \text{when} \quad w = u + \frac{1}{2} u^m h(z, \bar{z}, u)
\]

For the terms relevant to the normal form this gives (after successively comparing terms with \( z^2 \bar{z}^2, z^3 \bar{z}^2, z^2 \bar{z}^3, z^3 \bar{z}^3 \)):

\[
\begin{align*}
\varphi_{22} &= i \frac{m - 1}{2} w^{m-1} + h_{22}, \\
\varphi_{32} &= h_{32}, \quad \varphi_{23} = h_{23}, \quad \varphi_{33} = h_{33} + i (m - 1) h_{22} w^{m-1} - \frac{(m - 1)(3m - 2)}{8} w^{2m-2} + \frac{i}{2} h_{22} w^m - \frac{1}{12} w^{3m-3}.
\end{align*}
\]

Applying now (7.11), we obtain by a straightforward calculation:

\[
\begin{align*}
A_0 &= mw^{m-1} - 2ih_{22}, \quad B_0 = -2h_{23}, \quad A_1 = -6ih_{32}, \quad (7.13) \\
B_1 &= -6h_{33} + 8h_{22}^2 - iw^m h_{22}^2 + \frac{1}{4} w^{2m-2}(m-1)(m+2) + \frac{1}{2} w^{3m-3}.
\end{align*}
\]

We are now in the position to prove Theorem 1.
Proof of Theorem 1 and Theorem 2. In the case $m = 1$, we apply Proposition 7.12 and conclude that $M$ can be mapped biholomorphically onto a real hypersurface (1.1) with the desired uniqueness property. The latter hypersurface satisfies (7.2). Now the fact that $M$ is in normal form (i.e. its defining function $h(z, \bar{z}, u) \in N_1$) follows from (7.13).

For $m > 1$, inspired by (7.13), we make the following particular choice of a normal form space $D_m^\sigma$:

\begin{equation}
\sigma_1 := \frac{1}{4}(m - 1)(m + 2), \quad \sigma_2 := \frac{1}{2}.
\end{equation}

We then use Proposition 7.12 and Remark 7.13 to obtain a formal normal form for a hypersurface (1.1) (with the above choice of $\sigma$), which is uniquely determined by the real parameter $\tau$, as in (6.8). Now the fact that $M$ is in normal form (i.e. its defining function $h(z, \bar{z}, u) \in N_m$) again follows from (7.13), and the fact that this normal form solves the smooth CR equivalence problem follows from [30].

Remark 7.14. The choice of the real parameters $\sigma$ in (7.14) and the respective normal form space for the real defining function $h$ is of course not the only one possible. However, only the above choice makes the space of normalized power series $h(z, \bar{z}, u)$ a linear space, that is why we stay with it.

8. Fuchsian type ODEs

8.1. The normal form problem for Fuchsian type hypersurfaces. In this section we consider the normal form problem for Fuchsian type ODEs. As some of the considerations are analogous to that in the case $m = 1$, we provide only a brief outline for the respective steps.

First, we translate the Fuchsian type condition for hypersurfaces (1.1) described in the introduction onto the language of associated ODEs. For the functions $\Phi, \Phi^*$, we make use of the expansion

$$
\Phi(z, w, \zeta) = \sum_{k \geq 0, l \geq 2} \Phi_{kl}(w) z^k \zeta^l,
$$

and similarly for $\Phi^*$. We now introduce

Definition 8.1. An ODE $E \in \mathcal{E}_m$, defined by (3.7), is called Fuchsian (or a Fuchsian type ODE), if $\Phi$ satisfies the conditions:

\begin{equation}
\text{ord } \Phi_{02}(w) \geq m - 1; \text{ord } \Phi_{03}(w) \geq 2m - 2; \text{ord } \Phi_{12}(w) \geq m - 1; \text{ord } \Phi_{13}(w) \geq 2m - 2;
\end{equation}

\begin{equation}
\text{ord } \Phi_{kl}(w) \geq 2m - k - l + 3, \quad k \geq 1, \quad l \geq 3, \quad 5 \leq k + l \leq 2m + 2.
\end{equation}

We shall then prove

Proposition 8.2. For a Fuchsian type hypersurface $M \subset \mathbb{C}^2$, its associated ODE $E(M)$ is of Fuchsian type as well.

Proof. For the terms $\Phi_{02}, \Phi_{03}, \Phi_{12}, \Phi_{13}$ the desired inequalities follow (in both directions) from (7.13). For the other terms $\Phi_{kl}$ under consideration, the proof of the desired fact is
obtained by a calculation very similar to the one leading to formulas (7.13), and we leave the details to the reader.

We will use Definition 8.1 to show the invariancy of the Fuchsian type condition in the end of this section.

We now proceed with the normalization procedure. We in general follow the scheme in Section 3, and have now to deduce a system of singular ODEs for the Cauchy data $Y(w)$, as in (4.10), relevant to the Fuchsian type situation. For doing so, we have to enforce normalization conditions for the target ODE. Somewhat similarly to the general $m > 1$ case, we have to fix a number $\sigma \in \mathbb{R}$ and put

$$
\begin{align*}
\Phi_{0j2} &= 0, \quad j \geq m; \quad \Phi_{1j2} = \Phi_{0j3} = 0, \quad j \geq 2m - 1; \\
\Phi_{1j3} &= 0, \quad j \geq 2m - 1, \quad j \neq 3m - 3; \quad \Phi_{1,3m-3,3} = \sigma.
\end{align*}
$$

(The other coefficients $\Phi_{0j2}, \Phi_{0j3}, \Phi_{1j2}, \Phi_{1j3}$ we so far leave as free parameters and show later that their values are in fact predetermined.) We then collect in (4.4) terms with

$$
z^k w^j \xi^l, \quad k = 0, 1, \quad l = 2, 3, \quad j \geq 0.
$$

This gives us a system of four ODEs of the kind (4.19) (with the above discussed parameters involved). For the purposes of this section, we prefer to write down the obtained system in the form

$$
\begin{align*}
w^{m+1} g'' &= S(w, Y(w), wY'(w)), \quad w^{2m} X'' = T(w, Y(w), wY'(w)),
\end{align*}
$$

where

$$
X(w) := (g_1(w), f_0(w), f_1(w)), \quad Y(w) := (g_0(w), X(w)),
$$

and $S, T$ are holomorphic near the origin.

For the functions $T, S$ we will use the expansion

$$
T(w, Y, \tilde{Y}) = \sum_{\alpha, \beta \geq 0} T_{\alpha, \beta}(w) Y^\alpha \tilde{Y}^\beta,
$$

where $\alpha, \beta$ are multiindices, and similarly for $S$. We now shall prove the following key

**Proposition 8.3.** Under the Fuchsian type condition, the coefficient functions $T_{\alpha, \beta}(w), S_{\alpha, \beta}(w)$ satisfy

$$
\begin{align*}
\text{ord } T_{\alpha, \beta} &\geq 2m - 1 - |\alpha| - |\beta|, \quad \text{ord } S_{\alpha, \beta} \geq m - |\alpha| - |\beta|, \quad |\alpha| + |\beta| > 0.
\end{align*}
$$

**Proof.** For the proof, we make use of (8.1) (applied for the source defining function $\Phi^*$), and then study carefully the contribution of terms $\Phi^*_k \xi^l$ into the basic identity (4.4). Let us fix for the moment some positive value of $|\alpha| + |\beta|$. Then it is straightforward to check, by considering (4.4) and collecting terms (8.3), that $T_{\alpha, \beta}$ as above can arise only from $\Phi^*_k \xi^l$ with $k + l \leq |\alpha| + |\beta| + 4$, while $S_{\alpha, \beta}$ as above can arise only from $\Phi^*_k \xi^l$ with $k + l \leq |\alpha| + |\beta| + 2$. (And in the latter cases a respective $\Phi^*_k \xi^l$ is a factor for $Y^\alpha(wY')^\beta$).

Now it is not difficult to verify that (8.1) implies (8.3). \hfill $\Box$

**Corollary 8.4.** For the $(0, 0)$ coefficient functions in (8.4) we have

$$
\begin{align*}
\text{ord } S_{0,0} &\geq m; \quad \text{ord } T_{0,0} \geq 2m - 1.
\end{align*}
$$
As a consequence, for the target ODE defining function $\Phi$ we have:

(8.8) \[ \Phi_{0j2} = 0, \; 0 \leq j \leq m - 2; \]
\[ \Phi_{1j2} = \Phi_{0j3} = \Phi_{1j3} = 0, \; 0 \leq j \leq 2m - 3; \]
\[ \Phi_{0m-1,2} = \Phi_{0,m-1,2}; \; \Phi_{0,2m-2,3} = \Phi_{0,2m-2,3}; \; \Phi_{1,2m-2,2} = \Phi_{1,2m-2,2}; \; \Phi_{1,2m-2,3} = \Phi_{1,2m-2,3}. \]

Proof. As follows from the definition of $S_{\alpha,\beta}$, $T_{\alpha,\beta}$ and the conditions (8.3), all terms in the first equation in (8.4) have order at least $m$ in $w$ with possibly the exception of terms arising from $S_{0,0}$, while all terms in the second equation in (8.4) have order at least $2m - 1$ in $w$ with possibly the exception of terms arising from $T_{0,0}$. This proves (8.7). To prove (8.8), we note that the $(m - 1)$-jet of $S_{0,0}$ and the $(2m - 2)$-jet of $T_{0,0}$ respectively are formed from differences between coefficients $\Phi_{kjl}$ and $\Phi_{kjl}$ apparent in (8.8), and this proves (8.8).

\[ \square \]

Corollary 8.5. The coefficients

(8.9) \[ \Phi_{0,m-1,2}, \Phi_{0,2m-2,3}, \Phi_{1,2m-2,2}, \Phi_{1,2m-2,3}. \]

are invariants of a Fuchsian type hypersurface under transformations (3.1).

Proof. In the case when the target satisfies (8.2), the assertion follows from (4.20). However, even without requiring (8.2) we similarly have the relations (8.7) and hence obtain the desired identities in the last line of (4.20).

\[ \square \]

Based on Corollary 8.4, we can finally introduce the appropriate normal spaces $F_\sigma^m$ for Fuchsian type ODEs: for each $\sigma \in \mathbb{R}$, the space $F_\sigma^m$ is the space of all power series $\Phi(z, w, \zeta)$, as in (3.7), satisfying (8.2) and the first two lines in (4.20). Our immediate goal now is to prove that under an appropriate generic condition (the non-resonancy), each Fuchsian type ODE (3.7) can be uniquely mapped, by means of a transformation (3.6), onto another ODE (3.7) satisfying, in addition,

(8.10) \[ \Phi \in F_\sigma^m \]

(while the coefficients (8.9), again, remain unchanged). We will in general follow the scheme in the case $m = 1$ above; however, the proof of convergence will require some extra arguments.

As discussed above, the normalization condition $\Phi \in F_\sigma^m$ supplemented by the condition in the last line of (4.20) amount to a system of four ODEs (8.4) satisfying (8.3) with $\alpha, \beta$ being both possibly zero (as follows from Corollary 8.4). Let us consider the latter system (8.4) in more detail.

**Step I.** We investigate the existence of formal solutions for (8.4) vanishing at the origin. Let

\[ H(w) = \sum_{k \geq 1} H_k w^k \]

be such a formal solution. We substitute such a formal solution into (8.4) and, for each fixed $k \geq 1$, collect terms of degree $m + k - 1$ in $w$ in the first equation in (8.4), and degree $2m + k - 2$ in $w$ in the second equation in (8.4), respectively. Now it follows from
that (i) only the coefficients \( h_1, \ldots, h_k \) are present in the resulting identity; (ii) the coefficient \( h_k \) comes into the latter identity linearly. This means that the \( k \)-th identity can be considered as a linear system in \( h_k \), if \( h_1, \ldots, h_{k-1} \) are considered as known. If we are able now to prove the nondegeneracy of the latter linear system for each \( k \geq 1 \) (we address the latter property as the non-resonancy of a Fuchsian type ODE), then we conclude from the above that a formal solution exists and is unique.

We investigate the nonresonancy condition here very similarly to the case \( m = 1 \) (see section 4). Arguments identical to the ones in Section 4 show that the nondegeneracy of an above \( k \)-th linear system amounts to the nondegeneracy of the matrix
\[
\begin{pmatrix}
\alpha_0(k + 1 - m) - \alpha_0^* & -3\beta_0 & -\alpha_1 & -2k \\
\alpha_1(k + 1 - m) + \alpha_1^* & 3k(k - 1 + 2m - \alpha_0) + 3(\beta_1 - \alpha_0' + m(m - 1)) & 2\alpha_2 & \alpha_1 \\
2\beta_0(k + 1 - m) + \beta_0^* & 4\alpha_0 & k(k - 1 + \alpha_0) + \beta_1 & -\beta_0 \\
2\beta_1(k + 1 - m) + \beta_1^* & 4\gamma_0 & k\alpha_1 + 2\beta_2 & k(k - 1 + \alpha_0)
\end{pmatrix}
\]
Here, in the notations (4.15) and in contrast with (5.16), the constants \( \alpha_j, \alpha_j^*, \beta_j, \beta_j^*, \gamma_j \) are
\[
\alpha_0 := \frac{1}{(m - 1)!} A_0^{(m-1)}, \quad \alpha_0^* := \frac{1}{(m - 1)!} A_0^{(m)}, \\
\alpha_j := \frac{1}{(2m - 2)!} A_j^{(2m-2)}, \quad \alpha_j^* := \frac{1}{(2m - 2)!} A_j^{(2m-1)}, \quad j = 1, 2 \\
\beta_j := \frac{1}{(2m - 2)!} B_j^{(2m-2)}, \quad \beta_j^* := \frac{1}{(2m - 2)!} B_j^{(2m-1)}, \quad \gamma_j := \frac{1}{(2m - 2)!} C_j^{(2m-2)}, \quad j = 0, 1, 2.
\]
Accordingly we give the following

**Definition 8.6.** A Fuchsian type ODE is called nonresonant, if the associated matrices (8.11) are nondegenerate for each integer \( k \geq 1 \). A Fuchsian type hypersurface (1.1) is called nonresonant at the origin, if its associated ODE (3.7) is nonresonant.

**Remark 8.7.** There can exist only finitely many resonances for a hypersurface (1.1) in the Fuchsian type case with \( m > 1 \) (in fact, at most 8 of them).

**Proposition 8.8.** There exist Fuchsian type hypersurfaces (1.1) which are non-resonant at 0. Accordingly, a generic Fuchsian type hypersurface (1.1) (in the sense of the jet topology in the space of defining functions \( h(z, \bar{z}, u) \)) is non-resonant at 0.

**Proof.** The proof is very analogous to that of Proposition 5.4 and Proposition 6.2, that is why we leave its details to the reader. □

The above arguments prove that, for any Fuchsian type ODE (3.7) there exists a unique formal transformation (3.1) bringing it to a normal form.
Step II. It remains to deal with the convergence of the normalizing transformation. Since
the formal transformation under discussion arises as a solution of the regular Cauchy
problem (4.9), it is sufficient to prove the convergence of a formal solution of (8.4).

Let \( H(w) \) be such a formal solution. We decompose it as

\[
H(w) = P(w) + Z(w),
\]

where \( P(w) \) is a polynomial without constant term of degree \( \leq 2m - 1 \), while where \( Z(w) \)
is a formal series of the kind \( O(w^{2m}) \). The substitution (8.13) (for a fixed \( P(w) \)) turns
(8.4) into a similar system of ODEs for the unknown function \( Z(w) \). We shall now prove

Lemma 8.9. The transformed system (in the same way as the initial system) satisfies

\[
\text{ord } \tilde{S}_{01} \geq m - 1, \quad \text{ord } \tilde{S}_{10} \geq m - 1, \quad \text{ord } \tilde{T}_{01} \geq 2m - 2, \quad \text{ord } \tilde{T}_{10} \geq 2m - 2
\]

(the tilde here stands for coefficients of the transformed system).

Proof. the proof of the lemma is obtained by putting together the expansion (8.5), the
conditions (8.3), and the fact that \( P(w) \) is vanishing at the origin. \( \square \)

Now, based on Lemma 8.9 we perform the substitution

\[
Z := w^{2m}U,
\]

which turns the ”tilde” system into a new system of four meromorphic ODEs for the
unknown function \( U \), which, according to (8.13), has a formal solution \( U(w) \) vanishing
at the origin. It is straightforward to check then, by combining (8.15) and (8.14), that
the new system system can be written in the form

\[
w^{2}U' = R(w, U, wU'),
\]

where \( R \) is a holomorphic function defined near the origin. Performing finally in the
standard fashion the substitution

\[
V := wU'
\]

and introducing the extended vector function \( U := (U, V) \), we obtain a first order ODE

(8.17)

\[
wU' = Q(w, U'),
\]

where \( Q \) is a holomorphic near the origin function. The ODE (8.17) is a Briot-Bouquet
type ODE (see Section 2), hence its formal solutions are convergent, as required.

By competing Steps I and II, we have proved

Theorem 8. For any \( \sigma \in \mathbb{R} \), a nonresonant Fuchsian type ODE (3.7) can be brought
to a normal form \( \Phi \in \mathcal{F}_{m}^{\sigma} \) by a transformation (3.6). A normalizing transformation is
defined uniquely.

Proof of Theorem 3. The way how Theorem 8 implies Theorem 3 is very analogous to
the argument in Section 6, and we leave the details to the reader. We just clarify that
the particular choice of the parameter \( \sigma \) which is suitable for achieving the normalization
conditions (1.10) is

\[
\sigma = \frac{1}{2}
\]
In the end of the section we would like to prove the invariance of the Fuchsian type condition.

**Theorem 9.** The property of being Fuchsian for a hypersurface (1.1) does not depend on the choice of (formal or holomorphic) coordinates of the kind (1.1).

**Proof.** In view of Definition 8.1 we can switch to associated ODEs and it is enough to prove the invariance of the Fuchsianity for them. As discussed above, we can restrict to transformations (3.6). Let us consider then the transformation rule (4.4) (with a fixed transformation within it), when the source ODE (with the defining function $\Phi^*$) is of Fuchsian type. We then claim the following: for all the coefficient functions $\Phi_{k\ell}$, $k \geq 0$, $\ell \geq 2$ involved in the Fuchsianity conditions (8.1), with the exception of the coefficients functions $\Phi_{k2}, \Phi_{k2}^*$, $k \geq 2$, the Fuchsian conditions (8.1) are satisfied. Indeed, we fix any $(k, \ell)$ relevant to (8.1), and from the transformation rule (4.4) we can see that the target coefficient function $\Phi_{k\ell}$ is a sum of three groups of terms: (i) terms $\Phi^*_{\alpha\beta}$ with $\alpha + \beta \geq k + \ell$ which are multiplied by a power series in $w$ with order at 0 at least $k + \ell - \alpha - \beta$; (ii) terms $\Phi^*_{\alpha\beta}$ with $\alpha + \beta < k + \ell$; (iii) terms arising from the expressions $I_{j}$, $0 \leq j \leq 3$ (relevant for $\ell = 2, 3$ only). In view of the linearity of the Fuchsianity conditions in $k, \ell$, it is not difficult to see that terms of the first kind all have order at 0 at least as the one required for the Fuchsianity. Terms of the second kind already all have order bigger than the one required for Fuchsianity. Finally, terms of the third kind automatically provide order at least $2m$ sufficient for the Fuchsianity, except for the case $\ell = 2$. For $k = 0, 1$ and $\ell = 2$ though even the automatically provided order $m$ suffices, and this proves the claim.

It remains to deal with terms $\Phi_{k2}$ with $k \geq 2$, $2 \leq k \leq 2m + 1$. We note, however, that the ODEs under consideration have a real structure, that is why (in view of the reality condition) we have

\begin{equation}
\text{ord } h_{k\ell}(w) = \text{ord } h_{\ell k}(w)
\end{equation}

for all $k, \ell$. This, in view of the transfer relations between $\Phi$ and $h$ (similar to (7.11)) gives, in particular:

\[
\text{ord } \Phi_{k2}(w) = \text{ord } h_{k+2,2}(w) = \text{ord } h_{2,k+2}(w) = \text{ord } \Phi_{0,k+2}(w) \geq 2m - k
\]

(the last inequality follows from the Fuchsianity condition for $\Phi_{0,k+2}$ being already proved). This finally proves the theorem.

\[\square\]

### 8.2. Regularity of mappings between Fuchsian type hypersurfaces

In this section we prove [Theorem 4](#) and [Theorem 5](#).

**Proof of [Theorem 4](#) and [Theorem 5](#).** We start with proving the simpler [Theorem 4](#). As was discussed above, one can restrict to hypersurfaces (1.1) and transformations (3.6) between them, and then subsequently to ODEs (3.7) and transformations (3.6) between them. We then apply the transformation rule (4.4) and collect terms in it identically.
to the normal form procedure for Fuchsian ODEs described in the previous section. The only difference in this case is that the target coefficients $\Phi_{j,k,l}$, $k = 0, 1$, $l = 2, 3$ are prescribed not necessarily zero but arbitrary given complex values. We then in an absolutely identical way obtain first a reduction to a Cauchy problem of the kind (4.19), and second a system of ODEs of the kind (8.4), which satisfies condition of the kind (8.3) in view of the Fuchsianity of ODEs. Then the convergence of the Cauchy data (and hence of the transformation) is obtained in the same way as in the proof of Theorem 3.

The proof of Theorem 5 requires an additional argument, namely, the following

**Proposition 8.10.** Consider a first order real ODE

\[(8.19) \quad xy' = F(x, y), \quad x \in [0, a],\]

with $y$ being $n$-dimensional, $n \geq 1$, and $F$ analytic. Assume it has a solution $y(x)$ which is $C^\infty$ on $[0, a]$. Then $y(x)$ is analytic everywhere on $[0, a]$.

**Remark 8.11.** A singular ODE (8.19) belongs to the classical class of Briot-Bouquet type ODEs discussed in Section 2. Their formal solutions at the singular point $x = 0$ are convergent, which, however does not say anything about the regularity of smooth solutions, that is why Proposition 8.10 requires a separate proof.

**Proof of Proposition 8.10.** The analyticity of $y(x)$ everywhere outside $x = 0$ follows from the analyticity of the given ODE, that is why we consider only the analyticity at the singularity $x = 0$. First, consider the Taylor series $\hat{y}(x)$ of $y(x)$. Since, again, (8.19) is a Briot-Bouquet ODE, $\hat{y}(x)$ is convergent. Hence, taking $y - \hat{y}(x)$ as a new unknown function, we get an ODE again of the kind (8.19) which has now a flat at $x = 0$ solution on $[0, a]$. We assume, by contradiction, that this solution is not identical zero near $x = 0$. Substituting the flat solution into the new ODE (8.19) and equalizing the Taylor series in both sides, we conclude that $F(x, 0) = 0$. Hence we conclude that the (again analytic) right hand side expands as

\[F(x, y) = A(x)y + \cdots,\]

where $A(x)$ is an analytic at the origin matrix, and dots stand for terms of degree at least 2 in $y$.

Second, let us use the notation $|y(t)|$ for the Euclidean norm, and $\|y\|$ for the sup norm of $y$ on $[0, a]$. Since $y$ is flat at 0, we may shrink the interval to make $\|y\|$ small. Using the analyticity of $F$, we then have the bound

\[(8.20) \quad |F(x, y(x))| \leq C|y(x)|,\]

where $C$ is a constant depending on $\|y\|$.

Third, we make a simple observation that $|y|$ can’t vanish for $x > 0$. Indeed, any solution with $y(x_0) = 0$, $x_0 \neq 0$ would need to be identical zero by uniqueness near $x_0$, and hence identical zero by the analyticity of the ODE.

Fourth, we do the following: we ”resolve the singularity” of (8.19) by making the substitution

\[x := e^t, \quad t \in (-\infty, \ln a].\]
Now the ODE (8.19) reads as
\begin{equation}
\frac{dy}{dt} = F(e^t, y) =: \tilde{F}(t, y).
\end{equation}
We denote the new solution by \( y(t) \) and still have
\begin{equation}
|\tilde{F}(t, y(t))| \leq C|y(t)|.
\end{equation}

Now we need to obtain certain bounds. Taking the limit in the triangle inequality, we have
\begin{equation}
\frac{d}{dt}|y(t)| \leq \left| \frac{dy}{dt} \right|.
\end{equation}
In view of this and the inequality (8.22),
\begin{equation}
\frac{d}{dt} \ln |y(t)| = \frac{1}{|y(t)|} \frac{d}{dt}|y(t)| \leq \frac{1}{|y(t)|} \left| \frac{dy}{dt} \right| \leq C,
\end{equation}
and by integrating over \([t, \ln a]\) we obtain:
\begin{equation}
\ln |y(\ln a)| - \ln |y(t)| \leq C(\ln a - t).
\end{equation}
Simplifying (8.23) and applying exp, we finally get for the initial function \( y(x) \):
\begin{equation}
|y(x)| \geq \tilde{C} \cdot x^C
\end{equation}
(\( \tilde{C} \) is some other constant, which is nonzero since \( |y(a)| \) is nonzero!). But (8.24) is a contradiction with the fact that \( y(x) \) is flat near 0, and this proves the desired analyticity statement.

**Remark 8.12.** The assertion of Proposition 8.10 holds also for a complex Briot-Bouquet ODE, i.e. when \( y(x) \) is complex-valued and \( F \) is complex analytic (one just has to split the real and imaginary parts, and this immediately gives an already real ODE (8.19) for the vector function formed from the real and imaginary parts of \( y \)).

We come back to the proof of Theorem 5. Note that a hypersurface (1.1) necessarily contains (the germ at the origin of) the real line \( L = \{ z = 0, \ \text{Im} \ w = 0 \} \). This means, in particular, that for the given map \( H(z, w) \), the vector functions \( H(0, w), H_z(0, w) \) are well defined on \( L \) and are holomorphic in its open neighborhood. Arguing now identically to the above proof of Theorem 4, we reduce the analyticity problem for the given CR-map to the analyticity of \( C^\infty \) smooth solutions of an ODE identical to (8.17). The only difference is that, instead of substituting a formal power series map into the basic identity (4.4), we substitute into (4.4) a holomorphic map in a domain \( \Omega \), containing 0 in its closure and coming from the analyticity of the map in a neighborhood of the Levi-nondegenerate part of \( M \). In view of the above, the Cauchy data (4.10) of the map \( H \) is \( C^\infty \) on the real line, and so is a solution of (8.17) under discussion. We then apply Proposition 8.10 (together with Remark 8.12) and conclude that the desired solution of (8.17) is analytic, and so is the Cauchy data (4.10) and hence the map \( H \). This completely proves the theorem.
References

[1] M. S. Baouendi, P. Ebenfelt, and L.P. Rothschild. Convergence and finite determination of formal CR mappings. *J. Amer. Math. Soc.*, 13(4):697–723 (electronic), 2000.

[2] M. S. Baouendi, H. Jacobowitz, and F. Trèves. On the analyticity of CR mappings. *Ann. of Math. (2)*, 122(2):365–400, 1985.

[3] M.S. Baouendi, P. Ebenfelt, and L.P. Rothschild. *Real submanifolds in complex space and their mappings*, volume 47 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1999.

[4] V. K. Beloshapka. Construction of the normal form of an equation of a surface of high codimension. *Mat. Zametki*, 48(2):3–9, 1990.

[5] V. K. Beloshapka. Can a stabilizer be eight-dimensional? *Russ. J. Math. Phys.*, 19(2):135–145, 2012.

[6] George W. Bluman and Sukeyuki Kumei. *Symmetries and differential equations*, volume 81 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989.

[7] Valentin Burcea. Normal forms and degenerate CR singularities. *Complex Var. Elliptic Equ.*, 61(9):1314–1333, 2016.

[8] E. Cartan. Sur la géométrie pseudo-conforme des hypersurfaces de l’espace de deux variables complexes II. *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie II*, 1(4):333–354, 1932.

[9] E. Cartan. Sur la géométrie pseudo-conforme des hypersurfaces de l’espace de deux variables complexes. *Annali di Matematica Pura ed Applicata. Series 4*, 11(1):17–90, December 1933.

[10] Élie Cartan. Sur la géométrie pseudo-conforme des hypersurfaces de l’espace de deux variables complexes II. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2)*, 1(4):333–354, 1932.

[11] S.S. Chern and J. K. Moser. Real hypersurfaces in complex manifolds. *Acta Math.*, 133:219–271, 1974.

[12] Adam Coffman. CR singularities of real fourfolds in $\mathbb{C}^3$. *Illinois J. Math.*, 53(3):939–981 (2010), 2009.

[13] P. Ebenfelt, B. Lamel, and D. Zaitsev. Finite jet determination of local analytic CR automorphisms and their parametrization by 2-jets in the finite type case. *Geom. Funct. Anal.*, 13(3):546–573, 2003.

[14] Peter Ebenfelt. New invariant tensors in CR structures and a normal form for real hypersurfaces at a generic Levi degeneracy. *J. Differential Geom.*, 50(2):207–247, 1998.

[15] Peter Ebenfelt. Normal forms and biholomorphic equivalence of real hypersurfaces in $\mathbb{C}^3$. *Indiana Univ. Math. J.*, 47(2):311–366, 1998.

[16] Peter Ebenfelt. On the analyticity of CR mappings between nonminimal hypersurfaces. *Math. Ann.*, 322(3):583–602, 2002.

[17] Peter Ebenfelt, Bernhard Lamel, and Dmitri Zaitsev. Normal form for infinite type hypersurfaces in $\mathbb{C}^2$ with nonvanishing Levi form derivative. *Doc. Math.*, 22:165–190, 2017.

[18] Vladimir V. Ežov and Gerd Schmalz. Normal form and two-dimensional chains of an elliptic CR manifold in $\mathbb{C}^4$. *J. Geom. Anal.*, 6(4):495–529 (1997), 1996.

[19] X. Gong. Existence of real analytic surfaces with hyperbolic complex tangent that are formally but not holomorphically equivalent to quadrics. *Indiana Univ. Math. J.*, 53(1):83–95, 2004.

[20] X. Huang and W. Yin. Equivalence problem for Bishop surfaces. *Sci. China Math.*, 53(3):687–700, 2010.

[21] X. Huang and W. Yin. Flattening of CR singular points and analyticity of the local hull of holomorphy II. *Adv. Math.*, 308:1009–1073, 2017.

[22] Y. Ilyashenko and S. Yakovenko. *Lectures on analytic differential equations*, volume 86 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[23] R. Juhlin and B. Lamel. On maps between non-minimal hypersurfaces. *Math. Z.*, 273(1-2):515–537, 2013.
[24] Martin Kolar, Ilya Kossovskiy, and Dmitri Zaitsev. Normal forms in Cauchy-Riemann geometry. In Analysis and geometry in several complex variables, volume 681 of Contemp. Math., pages 153–177. Amer. Math. Soc., Providence, RI, 2017.

[25] Martin Kolář. Normal forms for hypersurfaces of finite type in \(C^2\). Math. Res. Lett., 12(5-6):897–910, 2005.

[26] Martin Kolář. Finite type hypersurfaces with divergent normal form. Math. Ann., 354(3):813–825, 2012.

[27] Martin Kolář and Bernhard Lamel. Holomorphic equivalence and nonlinear symmetries of ruled hypersurfaces in \(C^2\). J. Geom. Anal., 25(2):1240–1281, 2015.

[28] I. Kossovskiy and B. Lamel. New extension phenomena for solutions of tangential Cauchy-Riemann equations. Comm. Partial Differential Equations, 41(6):925–951, 2016.

[29] I. Kossovskiy and B. Lamel. On the analyticity of CR-diffeomorphisms. Amer. J. Math., 140(1):139–188, 2018.

[30] I. Kossovskiy, B. Lamel, and L. Stolovitch. Equivalence of Cauchy-Riemann manifolds and multi-summability theory. Preprint, 31 pages, 2017.

[31] I. Kossovskiy and R. Shafikov. Analytic differential equations and spherical real hypersurfaces. J. Differential Geom., 102(1):67–126, 2016.

[32] I. Kossovskiy and R. Shafikov. Divergent CR-equivalences and meromorphic differential equations. J. Eur. Math. Soc. (JEMS), 18(12):2785–2819, 2016.

[33] I. Kossovskiy and D. Zaitsev. Convergent normal form and canonical connection for hypersurfaces of finite type in \(C^2\). Adv. Math., 281:670–705, 2015.

[34] Ilpo Laine. Complex differential equations. In Handbook of differential equations: ordinary differential equations. Vol. IV, Handb. Differ. Equ., pages 269–363. Elsevier/North-Holland, Amsterdam, 2008.

[35] Bernhard Lamel and Laurent Stolovitch. Convergence of the chern-moser-beloshapka normal forms. E. Lombardi and L. Stolovitch. Normal forms of analytic perturbations of quasihomogeneous vector fields: rigidity, invariant analytic sets and exponentially small approximation. Ann. Sci. Éc. Norm. Supér. (4), 43(4):659–718, 2010.

[36] Francine Meylan. A reflection principle in complex space for a class of hypersurfaces and mappings. Pacific J. Math., 169(1):135–160, 1995.

[37] J.K. Moser and S.M. Webster. Normal forms for real surfaces in \(C^2\) near complex tangents and hyperbolic surface transformations. Acta Mathematica, 150(3-4):255–296, 1983.

[38] Peter J. Olver. Applications of Lie groups to differential equations, volume 107 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1993.

[39] N. Tanaka. On the pseudo-conformal geometry of hypersurfaces of the space of \(n\) complex variables. Palermo Rend., 23:185–220, 1907.

[40] S. M. Webster. On the mapping problem for algebraic real hypersurfaces. Invent. Math., 43(1):53–68, 1977.
[48] Dmitri Zaitsev. Normal forms for nonintegrable almost CR structures. Amer. J. Math., 134(4):915–947, 2012.

Department of Mathematics, University of California at San Diego, La Jolla, CA 92093-0112
E-mail address: pebenfelt@ucsd.edu

Department of Mathematics, Masaryk University, Brno, Czechia// Department of Mathematics, University of Vienna, Vienna, Austria
E-mail address: kossovskiyi@math.muni.cz

Department of Mathematics, University of Vienna, Vienna, Austria
E-mail address: bernhard.lamel@univie.ac.at