SOLVING A PROBLEM OF ANGIOGENESIS OF DEGREE THREE

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Dedicated to Professors Dr. Alexander O. Ivanov and Dr. Alexey A. Tuzhilin for their contributions on minimal trees

Abstract. An absorbing weighted Fermat-Torricelli tree of degree four is a weighted Fermat-Torricelli tree of degree four which is derived as a limiting tree structure from a generalized Gauss tree of degree three (weighted full Steiner tree) of the same boundary convex quadrilateral in $\mathbb{R}^2$. By letting the four variable positive weights which correspond to the fixed vertices of the quadrilateral and satisfy the dynamic plasticity equations of the weighted quadrilateral, we obtain a family of limiting tree structures of generalized Gauss trees which concentrate to the same weighted Fermat-Torricelli tree of degree four (universal absorbing Fermat-Torricelli tree). The values of the residual absorbing rates for each derived weighted Fermat-Torricelli tree of degree four of the universal Fermat-Torricelli tree form a universal absorbing set. The minimum of the universal absorbing Fermat-Torricelli set is responsible for the creation of a generalized Gauss tree of degree three for a boundary convex quadrilateral derived by a weighted Fermat-Torricelli tree of a boundary triangle (Angiogenesis of degree three). Each value from the universal absorbing set contains an evolutionary process of a generalized Gauss tree of degree three.

1. Introduction

We shall describe the structure of a generalized Gauss tree with degree three and a weighted Fermat-Torricelli tree of degree four with respect to a boundary convex quadrilateral $A_1A_2A_3A_4$ in $\mathbb{R}^2$.

Definition 1. [5, Section 2, pp. 2] A tree topology is a connection matrix specifying which pairs of points from the list $A_1, A_2, A_3, A_4, A_0'$ have a connecting linear segment (edge).

Definition 2. [2, Subsection 1.2, pp. 8, 9] The degree of a vertex is the number of connections of the vertex with linear segments.

Let $A_1, A_2, A_3, A_4$ be four non-collinear points in $\mathbb{R}^2$ and $B_i$ be a positive number (weight) which corresponds to $A_i$ for $i = 1, 2, 3, 4$.

The weighted Fermat-Torricelli problem for four non-collinear points (4wFT problem) in $\mathbb{R}^2$ states that:

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Problem 1 (4wFT problem). Find a point (weighted Fermat-Torricelli point) \( A_0 \in \mathbb{R}^2 \), which minimizes

\[
f(A_0) = \sum_{i=1}^{4} B_i \| A_0 - A_i \|,
\]

where \( \| \cdot \| \) denotes the Euclidean distance.

By letting \( B_1 = B_2 = B_3 = B_4 \) in the 4wFT problem we obtain the following two cases:

(i) If \( A_1 A_2 A_3 A_4 \) is a convex quadrilateral, then \( A_0 \) is the intersection point of the two diagonals \( A_1 A_3 \) and \( A_2 A_4 \).

(ii) If \( A_i \) is an interior point of \( \triangle A_j A_k A_l \), then \( A_0 \equiv A_i \), for \( i, j, k, l = 1, 2, 3, 4 \) and \( i \neq j \neq k \neq l \).

The characterization of the (unique) solution of the 4wFT problem in \( \mathbb{R}^2 \) is given by the following result which has been proved in [1] and [10]:

**Theorem 1.** [1, Theorem 18.37, p. 250], [10]

Let \( A_0 \) be a weighted minimum point which minimizes (1.1).

(a) Then, the 4wFT point \( A_0 \) uniquely exists.

(b) If for each point \( A_i \in \{ A_1, A_2, A_3, A_4 \} \)

\[
\| \sum_{j=1, i \neq j}^{4} B_j \bar{u}_{ij} \| > B_i,
\]

for \( i, j = 1, 2, 3, 4 \) holds, then

(b1) \( A_0 \) does not belong to \( \{ A_1, A_2, A_3, A_4 \} \) and

(b2)

\[
\sum_{i=1}^{4} B_i \bar{u}_{0i} = \vec{0},
\]

where \( \bar{u}_{kl} \) is the unit vector from \( A_k \) to \( A_l \), for \( k, l \in \{ 0, 1, 2, 3, 4 \} \) (Weighted Floating Case).

(c) If there is a point \( A_i \in \{ A_1, A_2, A_3, A_4 \} \) satisfying

\[
\| \sum_{j=1, i \neq j}^{4} B_j \bar{u}_{ij} \| \leq B_i,
\]

then \( A_0 \equiv A_i \). (Weighted Absorbed Case).

The inverse weighted Fermat-Torricelli problem for four non-collinear points (Inverse 4wFT problem) in \( \mathbb{R}^2 \) states that:

**Problem 2.** Inverse 4wFT problem Given a point \( A_0 \) which belongs to the convex hull of \( A_1 A_2 A_3 A_4 \) in \( \mathbb{R}^2 \), does there exist a unique set of positive weights \( B_i \), such that

\[
B_1 + B_2 + B_3 + B_4 = c = \text{const},
\]

for which \( A_0 \) minimizes

\[
\sum_{i=1}^{4} B_i \| A_0 - A_i \|.
\]
By letting $B_4 = 0$ in the inverse $4wFT$ problem we derive the inverse $3wFT$ problem which has been introduced and solved by S. Gueron and R. Tessler in [6, Section 4, p. 449].

In 2009, a negative answer with respect to the inverse $4wFT$ problem is given in [13, Proposition 4.4, p. 417] by deriving a dependence between the four variable weights in $\mathbb{R}^2$. In 2014, we obtain the same dependence of variable weights on some $C^2$ surfaces in $\mathbb{R}^3$ and we call it the "dynamic plasticity of convex quadrilaterals" ([16, Problem 2, Theorem 2, Definition 12, Theorem 1, p. 92, p. 97-98]).

An important generalization of the Fermat-Torricelli problem is the generalized Gauss problem (or full weighted Steiner tree problem) for convex quadrilaterals in $\mathbb{R}^2$ which has been studied on the K-plane (Sphere, Hyperbolic plane, Euclidean plane) in [15].

We mention the following theorem which provide a characterization for the solutions of the (unweighted) Gauss problem in $\mathbb{R}^2$.

**Theorem 2.** [1, Theorem(*), pp. 328] Any solution of the Gauss problem is a Gauss tree (equally weighted full Steiner tree) with at most two (equally weighted) Fermat-Torricelli points (or Steiner points) where each Fermat-Torricelli point has degree three, and the angle between any two edges incident with a Fermat-Torricelli point is of $120^\circ$.

We need to mention all the necessary definitions of the weighted Fermat-Torricelli tree and weighted Gauss tree topologies, in order to derive some important evolutionary structures of the Fermat-Torricelli trees (Absorbing Fermat-Torricelli trees) and Gauss trees (Absorbing Gauss trees) which have been introduced in [17, Definitions 1-7, p. 1070-1071].

**Definition 3.** A weighted Fermat-Torricelli tree topology of degree three is a tree topology with all boundary vertices of a triangle having degree one and one interior vertex (weighted Fermat-Torricelli point) having degree three.

**Definition 4.** A weighted Fermat-Torricelli tree topology of degree four is a tree topology with all boundary vertices of a convex quadrilateral having degree one and one interior vertex (4wFT point) having degree four.

**Definition 5.** [5, Subsection 3.7, pp. 6] A weighted Gauss tree topology (or full Steiner tree topology) of degree three is a tree topology with all boundary vertices of a convex quadrilateral having degree one and two interior vertices (weighted Fermat-Torricelli points) having degree three.

**Definition 6.** A weighted Fermat-Torricelli tree topology of degree four is a Fermat-Torricelli tree topology of degree four is called a weighted Fermat-Torricelli tree of degree four.

**Definition 7.** [3, Subsection 3.7, pp. 6] A weighted Gauss tree of weighted minimum length with a Gauss tree topology of degree three is called a generalized Gauss tree of degree three or a full weighted minimal Steiner tree.

In 2014, we study an important generalization of the weighted Gauss (tree) problem that we call a generalized Gauss problem for convex quadrilaterals in $\mathbb{R}^2$ by using a mechanical construction which extends the mechanical construction of Gueron-Tessler in the sense of Pólya and Varigon ([17, Problem 1, Theorem 4, pp. 1073-1075]).
We state a generalized Gauss problem for a weighted convex quadrilateral $A_1A_2A_3A_4$ in $\mathbb{R}^2$, such that the weights $B_i$ which correspond to $A_i$ and $B_{00'} = x_G$, satisfy the inequalities

$$|B_i - B_j| < B_k < B_i + B_j,$$

and

$$|B_i - B_m| < B_n < B_i + B_m$$

where $x_G$ is the variable weight which corresponds to the given distance $l \equiv \|A_0 - A_{0'}\|$, for $i, j, k \in \{1, 4, 00'\}, t, m, n \in \{2, 3, 00'\}$ and $i \neq j \neq k, t \neq m \neq n$.

**Problem 3.** [17, Problem 1, p. 1073] Given $l, B_1, B_2, B_3, B_4$, find a generalized Gauss tree of degree three with respect to $A_1A_2A_3A_4$ which minimizes

$$B_1\|A_1 - A_0\| + B_2\|A_2 - A_0\| + B_3\|A_3 - A_{0'}\| + B_4\|A_4 - A_{0'}\| + x_G l.$$ (1.5)

For $l = 0$, we obtain a weighted Fermat-Torricelli tree of degree four.

**Definition 8.** [17, Definition 8, p.1076] We call the variable $x_G$ which depend on $l$, a generalized Gauss variable.

**Definition 9.** [17, Definition 9, p.1080] The residual absorbing rate of a generalized Gauss tree of degree at most four with respect to a boundary convex quadrilateral is

$$\sum_{i=1}^{4} B_i - x_G.$$

**Definition 10.** [17, Definition 10, p.1080] An absorbing generalized Gauss tree of degree three is a generalized Gauss tree of degree three with residual absorbing rate

$$\sum_{i=1}^{4} B_i - x_G.$$
equations, we obtain a family of limiting tree structures of generalized Gauss trees of degree three which concentrate to the same weighted Fermat-Torricelli tree of degree four in a geometric sense (Universal absorbing Fermat-Torricelli tree).

Furthermore, we calculate the values of the universal rates of a Universal absorbing tree regarding a fixed boundary quadrilaterals (Section 5, Examples 2,3).

In section 6, we introduce a class of Euclidean minimal tree structures that we call steady trees and evolutionary trees (Section 6, Definitions 15,16). Thus, the minimum of the universal absorbing Fermat-Torricelli set (Universal Fermat-Torricelli minimum value) leads to the creation of a generalized Gauss tree of degree three for the same boundary convex quadrilateral which is derived by a weighted Fermat-Torricelli tree of degree four. A universal absorbing Fermat-Torricelli minimum value corresponds to the intersection point (4wFT point). This quantity is of fundamental importance, because by attaining this value the absorbing Fermat-Torricelli tree start to grow and will be able to produce a generalized Gauss tree of degree three (Evolutionary tree). Each specific value from the universal absorbing set gives an evolutionary process of a generalized Gauss tree of degree three regarding a fixed boundary quadrilateral by spending a positive quantity from the storage of the universal Fermat-Torricelli quantity which stimulates the evolution at the 4wFT point (Section 6, Example 4, Angiogenesis of degree three).

2. Extending Torricelli-Engelbrecht’s solution for convex quadrilaterals

The weighted Torricelli-Engelbrecht solution for a triangle $\triangle A_1A_2A_3$ in the weighted floating case is given by the following proposition:

**Lemma 1.** [6,14] If $A_0$ is an interior weighted Fermat-Torricelli point of $\triangle A_1A_2A_3$, then

$$\angle A_iA_0A_j \equiv \alpha_{ij} = \arccos \left( \frac{B_i^2 - B_j^2 - B_k^2}{2B_iB_j} \right).$$

Let $A_1A_2A_3A_4$ be a convex quadrilateral in $\mathbb{R}^2$, $O$ be the intersection point of the two diagonals and $B_i$ be a given weight which corresponds to the vertex $A_i$, $A_0$ be the weighted Fermat-Torricelli point in the weighted floating case (Theorem ) and $\vec{U}_{ij}$ be the unit vector from $A_i$ to $A_j$, for $i,j = 1,2,3,4$.

We mention the geometric plasticity principle of quadrilaterals in $\mathbb{R}^2$,

**Lemma 2.** [13,16] Definition 13,Theorem 3, Proposition 8, Corollary 4 p. 103-108] Suppose that the weighted floating case of the weighted Fermat-Torricelli point $A_0$ point with respect to $A_1A_2A_3A_4$ is satisfied:

$$\left\| B_i\vec{U}_{ki} + B_j\vec{U}_{kj} + B_m\vec{U}_{km} \right\| > B_k,$$

for each $i,j,k,m = 1,2,3,4$ and $i \neq j \neq k \neq m$. If $A_0$ is connected with every vertex $A_k$ for $k = 1,2,3,4$ and we select a point $A_k'$ with non-negative weight $B_k$ which lies on the ray $A_kA_0$ and the quadrilateral $A_i'A_i'A_j'A_j'$ is constructed such that:

$$\left\| B_i\vec{U}_{k'i'} + B_j\vec{U}_{kj'} + B_m\vec{U}_{km'} \right\| > B_k,$$

for each $i',j',k',m' = 1,2,3,4$ and $i' \neq j' \neq k' \neq m'$. Then the weighted Fermat-Torricelli point $A_0'$ is identical with $A_0$. 

Theorem 3. The weighted Torricelli-Engelbrecht solution for \( A_1A_2A_3A_4 \) is given by the following system of four equations w.r. to the variables \( \alpha_{102}, \alpha_{203}, \alpha_{304} \) and \( \alpha_{401} \):

\[
\csc^2 \alpha_{102} \csc^2 \alpha_{304} \csc^2 \alpha_{401} \left( \cos \alpha_{102} - \sin \alpha_{102} \right) \left( \cos \alpha_{304} - \sin \alpha_{304} \right) \\
\left( \cos \left( \alpha_{102} - \alpha_{304} \right) - \cos \left( \alpha_{102} + \alpha_{304} + 2 \alpha_{401} \right) - 2 \sin \left( \alpha_{102} + \alpha_{304} \right) \right) = 0, \tag{2.2}
\]

\[-B_1^2 - 2B_1B_2 \cos \alpha_{102} - B_2^2 + B_3^2 - 2B_1B_4 \cos \alpha_{401} - 2B_2B_4 \cos \left( \alpha_{102} + \alpha_{401} \right) - B_4^2 = 0, \tag{2.3}\]

\[
\alpha_{304} = \arccos \left( \frac{B_1^2 + 2B_1B_2 \cos \alpha_{102} + B_2^2 - B_3^2 - B_4^2}{2B_3B_4} \right) \tag{2.4}
\]

and

\[
\alpha_{203} = 2\pi - \alpha_{102} - \alpha_{304} - \alpha_{401}. \tag{2.5}
\]

Proof. Assume that we select \( B_1, B_2, B_3, B_4 \), such that \( A_0 \) is an interior point of \( \triangle A_1O A_2 \). By applying the geometric plasticity principle of Lemma 2 we could choose a transformation of \( A_1A_2A_3A_4 \) to the square \( A'_1A'_2A'_3A'_4 \), where \( A'_0 = A_0 \) and \( A_0 \) is an interior point of \( A'_1O A'_2 \) where \( O' \) is the intersection of \( A'_1A'_3 \) and \( A'_2A'_4 \).

We consider the equations of the three circles which pass through \( A_1, A_2, A_0, A_1, A_4, A_0 \) and \( A_3, A_4, A_0 \), respectively, which meet at \( A_0 \):

\[
\left( x - \frac{a}{2} \right)^2 + \left( y - \frac{1}{2}a \cot \alpha_{102} \right)^2 = \frac{1}{4} a^2 \csc^2 \alpha_{102} \tag{2.6}
\]

\[
\left( y - \frac{a}{2} \right)^2 + \left( x - \frac{1}{2}a \cot \alpha_{401} \right)^2 = \frac{1}{4} a^2 \csc^2 \alpha_{401} \tag{2.7}
\]

\[
\left( x - \frac{a}{2} \right)^2 + \left( y - \left( a - \frac{1}{2}a \cot \alpha_{304} \right) \right)^2 = \frac{1}{4} a^2 \csc^2 \alpha_{304} \tag{2.8}
\]

By subtracting (2.7) from (2.6), (2.8) from (2.6) and solving w.r. to \( x, y \) we get:

\[
x = \frac{a(-1 + \cot \alpha_{102})(-1 + \cot \alpha_{304})}{(-2 + \cot \alpha_{102} + \cot \alpha_{304})(-1 + \cot \alpha_{401})} \tag{2.9}
\]

and

\[
y = -\frac{-a \cot \alpha_{304} + a}{\cot \alpha_{102} + \cot \alpha_{304} - 2} \tag{2.10}
\]

By substituting (2.9) and (2.10) in (2.6), we obtain (2.2).

Taking into account the weighted floating equilibrium condition, we get:

\[
-B_3\bar{u}_{30} = B_1\bar{u}_{10} + B_2\bar{u}_{20} + B_4\bar{u}_{40} \tag{2.11}
\]

or

\[
B_1\bar{u}_{10} + B_2\bar{u}_{20} = -B_3\bar{u}_{30} - B_4\bar{u}_{40}. \tag{2.12}
\]

By squaring both parts of (2.11) we derive (2.8) and by squaring both parts of (2.12) we derive (2.4).
Figure 1. A generalized Gauss Menger tree of degree three regarding a boundary convex quadrilateral

Remark 1. A different approach was used in [13, Solution 2.2, Example 2.4, p. 413-414], in order to derive a similar system of equations w.r. to $\alpha_{102}$ and $\alpha_{401}$.

Example 1. By substituting $B_1 = 3.5$, $B_2 = 2.5$, $B_3 = 2$, $B_4 = 1$, $a = 10$ in (2.2) and (2.3) and solving this system of equations numerically by using for instance Newton method and choosing as initial values $\alpha_{102}^0 = 2.7$ rad, $\alpha_{401}^0 = 1.2$ rad, we obtain $\alpha_{102} = 2.30886$ and $\alpha_{401} = 1.57801$ rad. By substituting $\alpha_{102} = 2.30886$ and $\alpha_{401} = 1.57801$ rad in (2.4) we get $\alpha_{304} = 1.12492$ rad. From (2.5), we get $\alpha_{203} = 1.2714$ rad. By substituting the angles $\alpha_{i0j}$ in (2.9) and (2.10), we derive $x = 4.0700893$ and $y = 2.146831$.

Theorem 4. There does not exist an analytical solution for the 4wFT problem in $\mathbb{R}^2$.

Proof. The system of the two equations (2.2) and (2.3) taking into account (2.4) cannot be solved explicitly w.r to $\alpha_{102}$ and $\alpha_{401}$. Therefore, by considering (2.4) and (2.10) we deduce that the location of the 4wFT point $A_0$ cannot also be expressed explicitly via the angles $\alpha_{i0j}$, for $i, j = 1, 2, 3, 4$, for $i \neq j$.

Thus, from Theorem 4 the position of a weighted Fermat-Torricelli tree of degree four cannot be expressed analytically and may be found by using numerical methods (see also in [13]).

3. AN ABSORBING GENERALIZED GAUSS-MENGER TREE IN $\mathbb{R}^2$

Let $A_1A_2A_3A_4$ be a boundary weighted convex quadrilateral of a generalized Gauss tree of degree three in $\mathbb{R}^2$ and $A_0$, $A_0'$ are the two weighted Fermat-Torricelli (3wFT) points of degree three which are located at the convex hull of the boundary quadrilateral.

We denote by $l \equiv \|A_0 - A_0'\|$, $a_{ij} \equiv \|A_i - A_j\|$, $\alpha_{ijk} \equiv \angle A_iA_jA_k$, $a_{10} \equiv a_1$, $a_{40} \equiv a_4$, $a_{20'} \equiv a_2$, $a_{30'} \equiv a_3$ (Fig. 1) and by $B_i \equiv \frac{B_i'}{\sum_{i=1}^{n} B_i'}$, for $i, j, k \in \{0, 0', 1, 2, 3, 4\}$ and $i \neq j \neq k$. 
We proceed by giving the following two lemmas which have been proved recently in [15, Theorem 1] and [14].

**Lemma 3.** [15, Theorem 1], [14, Theorem 2.1, p. 485]
A generalized Gauss tree of degree three (full weighted Steiner minimal tree) of \( A_1A_2A_3A_4 \) consists of two weighted Fermat-Torricelli points \( A_0, A'_0 \) which are located at the interior convex domain with corresponding given weights \( B_0, B_0' \) and minimizes the objective function:

\[
B_1a_1 + B_2a_2 + B_3a_3 + B_4a_4 + \frac{B_0 + B_0'}{2}l \to \min,
\]

such that:

\[
|B_i - B_j| < B_k < B_i + B_j,
\]

\[
|B_t - B_m| < B_n < B_t + B_m
\]

where

\[
B_{00'} = \frac{B_0 + B_0'}{2},
\]

for \( i, j, k \in \{1, 4, 00'\}, t, m, n \in \{2, 3, 00'\} \) and \( i \neq j \neq k, t \neq m \neq n \).

Suppose that \( B_1, B_2, B_3, B_4, B_{00'} \) satisfy the inequalities (3.2) and (3.3).

**Lemma 4.** [14, Theorem 2.2, p. 486] The location of \( A_0 \) and \( A'_0 \) is given by the relations:

\[
cot \varphi = \frac{B_0a_{12} + B_4a_{14}\cos(\alpha_{214} - \alpha_{400'}) + B_3a_{23}\cos(\alpha_{123} - \alpha_{300'})}{B_4a_{14}\sin(\alpha_{214} - \alpha_{400'}) - B_3a_{23}\sin(\alpha_{123} - \alpha_{300'})}, \quad (3.4)
\]

\[
a_1 = \frac{a_{14}\sin(\alpha_{214} - \varphi - \alpha_{400'})}{\sin(\alpha_{100'} + \alpha_{400'})},
\]

and

\[
a_2 = \frac{a_{23}\sin(\alpha_{123} + \varphi - \alpha_{300'})}{\sin(\alpha_{200'} + \alpha_{300'})}, \quad (3.6)
\]

where \( \varphi \) is the angle which is formed between the line defined by \( A_1 \) and \( A_2 \) and the line which passes from \( A_1 \) and it is parallel to the line defined by \( A_0 \) and \( A'_0 \).

**Definition 11.** A generalized Gauss-Menger tree is a solution of a generalized Gauss problem in \( \mathbb{R}^2 \) for a boundary quadrilateral \( A_1A_2A_3A_4 \) which depend on the Euclidean distances \( a_{ij} \) and the five given weights \( B_1, B_2, B_3, B_4, B_{00'} \).

By letting \( B_{00'} \equiv x_G \), we obtain an absorbing generalized Gauss-Menger tree for a boundary quadrilateral.

**Theorem 5.** An absorbing generalized Gauss-Menger tree w.r. to a fixed convex quadrilateral \( A_1A_2A_3A_4 \) depends only on the five given weights \( B_1, B_2, B_3, B_4, B_{00'} \equiv x_G \) and the five given lengths \( a_{12}, a_{23}, a_{34}, a_{41} \) and \( a_{13} \).

**Proof.** Consider the Caley-Menger determinant which gives the volume of a tetrahedron \( A_1A_2A_3A_4 \) in \( \mathbb{R}^3 \).
\[288V^2 = \det \begin{pmatrix} 0 & a_{12}^2 & a_{13}^2 & a_{14}^2 & 0 & 1 \\ a_{12}^2 & 0 & a_{23}^2 & a_{24}^2 & 1 & 0 \\ a_{13}^2 & a_{23}^2 & 0 & a_{34}^2 & 1 & 0 \\ a_{14}^2 & a_{24}^2 & a_{34}^2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}. \quad (3.7)\]

By letting \( V = 0 \) in (3.7), we obtain a dependence of the six distances \( a_{12}, a_{13}, a_{14}, a_{23}, a_{34} \) and \( a_{24} \). For instance, by solving a fourth order degree equation w.r. to \( a_{13} \) we derive that \( a_{13} = (a_{12}, a_{14}, a_{23}, a_{34}, a_{24}) \).

By applying the cosine law in \( \triangle A_1A_2A_4 \) and \( \triangle A_1A_2A_3 \) we get:

\[\alpha_{214} = \arccos \left( \frac{a_{12}^2 + a_{14}^2 - a_{24}^2}{2a_{12}a_{14}} \right), \quad (3.8)\]

and

\[\alpha_{123} = \arccos \left( \frac{a_{12}^2 + a_{23}^2 - a_{13}^2}{2a_{12}a_{23}} \right). \quad (3.9)\]

By Lemma \( \text{H} \) and Lemma \( \text{B} \) and considering that \( A_0 \) is the 3wFT point of \( \triangle A_1A_4A_0 \) and \( A_{0'} \) is the 3wFT point of \( \triangle A_2A_3A_0 \) we get:

\[\alpha_{100'} = \arccos \left( \frac{B_1^2 - B_2^2 - x_G^2}{2B_1x_G} \right), \quad (3.10)\]

\[\alpha_{0'04} = \arccos \left( \frac{B_2^2 - B_4^2 - x_G^2}{2B_3x_G} \right), \quad (3.11)\]

\[\alpha_{104} = \arccos \left( \frac{x_G^2 - B_1^2 - B_4^2}{2B_1B_4} \right), \quad (3.12)\]

\[\alpha_{00'3} = \arccos \left( \frac{B_2^2 - B_3^2 - x_G^2}{2B_3x_G} \right), \quad (3.13)\]

\[\alpha_{00'2} = \arccos \left( \frac{B_3^2 - x_G^2 - B_2^2}{2x_GB_2} \right), \quad (3.14)\]

and

\[\alpha_{20'3} = \arccos \left( \frac{x_G^2 - B_2^2 - B_3^2}{2B_2B_3} \right). \quad (3.15)\]

Therefore, by replacing (3.10), (3.11), (3.14), (3.13), (3.8), (3.9) in (3.4), (3.5), (3.6) and taking into account the dependence of the six distances \( a_{ij} \), for \( i, j = 1, 2, 3, 4 \), we derive that \( \varphi, a_1 \) and \( a_2 \) depend only on \( B_1, B_2, B_3, B_4, x_G \) and \( a_{12}, a_{13}, a_{23}, a_{34}, a_{24} \). \( \square \)

4. The dynamic plasticity of convex quadrilaterals

In this section, we deal with the solution of the inverse 4wFT problem in \( \mathbb{R}^2 \) which has been introduced in [13] and developed in [16], in order to obtain a new system of equations of the dynamic plasticity of weighted quadrilaterals w.r. to the four variable weights \( (B_i)_{1234} \) for \( i = 1, 2, 3, 4 \), which cover also the case \( (B_i)_{1234} = (B_1)_{1234} \) and \( (B_2)_{1234} = (B_3)_{1234} = (B_4)_{1234} \).

First, we start by mentioning the solution of S. Gueron and R. Tessler ([6, Section 4, p. 449]) of the inverse 3wFT problem for three non-collinear points in \( \mathbb{R}^2 \).
letting \( B_i = 0 \) in the inverse 4wFT problem for convex quadrilaterals (Problem 2) we derive the inverse 3wFT problem for a triangle.

Consider the inverse 3wFT problem for \( \triangle A_1A_2A_3 \) in \( \mathbb{R}^2 \).

**Lemma 5.** [6] Section 4, p. 449] The unique solution of the inverse 3wFT problem for \( \triangle A_iA_jA_k \) is given by

\[
\left( \frac{B_i}{B_j} \right)_{ijk} = \frac{\sin \alpha_{ijk}}{\sin \alpha_{ijk}}.
\]  

**Definition 12.** [16] We call dynamic plasticity of a weighted Fermat-Torricelli tree of degree four the set of solutions of the four variable weights with respect to the inverse 4wFT problem in \( \mathbb{R}^2 \) for a given constant value \( c \) which correspond to a family of weighted Fermat-Torricelli tree of degree four that preserve the same Euclidean tree structure (the corresponding 4wFT point remains the same for a fixed boundary convex quadrilateral), such that the three variable weights depend on a fourth variable weight and the value of \( c \).

By taking into account Lemma 5 for the triangles \( \triangle A_1A_2A_3, \triangle A_1A_3A_4, \triangle A_1A_2A_4 \) and the weighted floating equilibrium condition (1.3) taken from Theorem 1 ([13], [16])

**Proposition 1.** [13] Proposition 4.4, p. 417, [16] Problem 2, Definition 12, Theorem 1, p. 92, p. 97-98] Suppose that \( A_0 \) does not belong to the intersection of the linear segments \( A_1A_3 \) and \( A_2A_4 \). The dynamic plasticity of the variable weighted Fermat-Torricelli tree of degree four in \( \mathbb{R}^2 \) is given by the following three equations:

\[
\begin{align*}
\left( \frac{B_2}{B_1} \right)_{1234} &= \left( \frac{B_2}{B_1} \right)_{123} \left[ 1 - \left( \frac{B_4}{B_1} \right)_{1234} \left( \frac{B_1}{B_4} \right)_{134} \right], \\
\left( \frac{B_3}{B_1} \right)_{1234} &= \left( \frac{B_3}{B_1} \right)_{123} \left[ 1 - \left( \frac{B_4}{B_1} \right)_{1234} \left( \frac{B_1}{B_4} \right)_{124} \right],
\end{align*}
\]  

and

\[
(B_1)_{1234} + (B_2)_{1234} + (B_3)_{1234} + (B_4)_{1234} = c = \text{const.}
\]  

It is worth mentioning that each quad of values \( \{ (B_1)_{1234}, (B_2)_{1234}, (B_3)_{1234}, (B_4)_{1234} \} \), which satisfy simultaneously (4.2), (4.3), (4.4) create a unique concentration of different families of tetrafocal ellipses (Polyellipse or Eggellipse) to the same 4wFT point \( A_0 \) of \( A_1A_2A_3A_4 \). A family of tetrafocal ellipses may be constructed by selecting a decreasing sequence of real numbers \( c_n((B_1)_{1234}, (B_2)_{1234}, (B_3)_{1234}, (B_4)_{1234}) \)

\[
c_n((B_1)_{1234}, (B_2)_{1234}, (B_3)_{1234}, (B_4)_{1234}; X) = \sum_{i=1}^{4} (B_i)_{1234} \| A_i - X \|
\]

which converge to the constant number

\[
c((B_1)_{1234}, (B_2)_{1234}, (B_3)_{1234}, (B_4)_{1234}) \equiv f(A_0, (B_1)_{1234}, (B_2)_{1234}, (B_3)_{1234}, (B_4)_{1234}).
\]  

or

\[
c((B_1)_{1234}, (B_2)_{1234}, (B_3)_{1234}, (B_4)_{1234}) = \sum_{i=1}^{4} (B_i)_{1234} \| A_i - A_0 \|.
\]
or
\[ c((B_1)_{1234}, (B_2)_{1234}, (B_3)_{1234}, (B_4)_{1234}) = \sum_{i=1}^{4} (B_i)_{1234}a_i. \]

The concentration of different families of tetrafocal ellipses to the same point provide a surprising connection with a problem posed by R. Descartes in 1638. According to [1, Chapter II, p. 235], in a letter from August 23, 1638, R. Descartes invited P. de Fermat to investigate the properties of tetrafocal ellipses in \( \mathbb{R}^2 \). The dynamic plasticity of weighted quadrilaterals solves the problem of concentration of tetrafocal ellipse and offers a new property to R. Descartes’ problem.

We proceed by deriving a new system of dynamic plasticity equations for a weighted Fermat-Torricelli tree of degree four which also includes a class of weighted Fermat-Torricelli trees of degree four which coincides with the two diagonals of the boundary convex quadrilateral for \((B_1)_{1234} = (B_3)_{1234}\) and \((B_2)_{1234} = (B_4)_{1234}\).

**Proposition 2.** The dynamic plasticity of the variable weighted Fermat-Torricelli tree of degree four in \( \mathbb{R}^2 \) is given by the following three equations:

\[ (B_1)_{1234}^2 + (B_2)_{1234}^2 + 2(B_1)_{1234}(B_2)_{1234}\cos\alpha_{102} = (B_3)_{1234}^2 + (B_4)_{1234}^2 + 2(B_3)_{1234}(B_4)_{1234}\cos\alpha_{304}, \quad (4.5) \]

and

\[ (B_1)_{1234}^2 + (B_4)_{1234}^2 + 2(B_1)_{1234}(B_4)_{1234}\cos\alpha_{104} = (B_2)_{1234}^2 + (B_3)_{1234}^2 + 2(B_2)_{1234}(B_3)_{1234}\cos\alpha_{203}, \quad (4.6) \]

Proof. Suppose that we select four positive weights \((B_i)_{1234}(0)\) which correspond to the vertex \(A_i\) of the boundary convex quadrilateral \(A_1A_2A_3A_4\), such that the weighted floating inequalities (1.2) of Theorem 1 hold, in order to locate the 4wFT point \(A_0\) at the interior of \(A_1A_2A_3A_4\).

From the weighted floating (variable weighted) equilibrium condition of the 4wFT point (1.3) we get

\[ (B_1)_{1234}\vec{u}_{10} + (B_2)_{1234}\vec{u}_{20} = -(B_3)_{1234}\vec{u}_{30} - (B_4)_{1234}\vec{u}_{40} \quad (4.8) \]

and

\[ (B_1)_{1234}\vec{u}_{10} + (B_3)_{1234}\vec{u}_{30} = -(B_2)_{1234}\vec{u}_{20} - (B_4)_{1234}\vec{u}_{40}, \quad (4.9) \]

which yield (4.5) and (4.6), respectively. □

**Corollary 1.** If the variable weighted 4wFT point is the intersection point of \(A_1A_3\) and \(A_2A_4\), then

\[ (B_1)_{1234} = (B_3)_{1234} \quad (4.10) \]

and

\[ (B_2)_{1234} = (B_4)_{1234}. \quad (4.11) \]

\((B_1)_{1234} = (B_3)_{1234}\) and \((B_2)_{1234} = (B_4)_{1234}\).
Proof. By letting \( \alpha_{102} = \alpha_{304} \) and \( \alpha_{104} = \alpha_{203} \), in (4.3) and (4.6) we obtain (4.10) and (4.11), respectively.

Remark 2. The dynamic plasticity equations of Theorem 1 depend on the solutions of the inverse 3FT problem for \( \triangle A_1A_2A_3 \), \( \triangle A_1A_3A_4 \) and \( \triangle A_1A_2A_4 \). Thus, the corresponding 4wFT point \( A_0 \) remains at the interior of \( \triangle A_1A_2A_3 \), in order to derive from the inverse 3wFT problem for \( \triangle A_1A_2A_3 \) the inverse wFT problem for \( A_1A_2A_3A_4 \). The dynamic plasticity equations of Theorem 2 generalize the dynamic plasticity equations of Theorem 1 because \( A_0 \) could also lie on the side \( A_1A_3 \) of \( \triangle A_1A_2A_3 \) and the side \( A_2A_4 \) of \( \triangle A_1A_2A_4 \). These are the cases where the inverse 3wFT problem for \( \triangle A_1A_2A_3 \) and \( \triangle A_1A_2A_4 \) do not hold.

5. A universal Fermat-Torricelli minimal value of a family of absorbing generalized Gauss trees of degree three

Suppose that an absorbing weighted Fermat-Torricelli tree of degree four is derived as a limiting tree structure from an absorbing generalized Gauss-Menger tree of degree three regarding a fixed boundary quadrilateral \( A_1A_2A_3A_4 \) for a specific value \( \alpha_G \) of the generalized Gauss variable.

We need to consider the following lemma which gives \( \bar{B} \) as a linear function of \( B_4 \) regarding a fixed variable weighted Fermat-Torricelli tree of degree four in \( \mathbb{R}^2 \).

Lemma 6. [13] Corollary 4.5, p. 418 Let \( \sum_{1234} B := (\bar{B}_1)_{1234}(1 + \frac{\bar{B}_2}{B_1} + \frac{\bar{B}_3}{B_1} + \frac{\bar{B}_4}{B_1})_{1234} \).

If \( \sum_{1234} \bar{B} = \sum_{123} \bar{B} = \sum_{12} \bar{B} = \sum_{13} \bar{B} = \sum_{14} \bar{B} \), then

\[
(\bar{B}_1)_{1234} = x_i(\bar{B}_4)_{1234} + y_i, i = 1, 2, 3:
\]

\[
(x_1, y_1) = \left( \frac{\bar{B}_2}{B_1} \right)_{1234} + \left( \frac{\bar{B}_3}{B_1} \right)_{1234} - 1,
\]

\[
1 + \left( \frac{\bar{B}_2}{B_1} \right)_{1234} + \left( \frac{\bar{B}_3}{B_1} \right)_{1234},
\]

\[
(x_2, y_2) = (x_1)(\bar{B}_2)_{123} - \left( \frac{\bar{B}_1}{B_4} \right)_{1234}(\bar{B}_2)_{123}, (\bar{B}_4)_{1234}
\]

\[
(x_3, y_3) = (x_1)(\bar{B}_3)_{123} - \left( \frac{\bar{B}_1}{B_4} \right)_{1234}(\bar{B}_3)_{123}, (\bar{B}_4)_{1234}
\]

Theorem 6. A universal constant does not exist for a unique fixed (variable) weighted Fermat-Torricelli tree of degree four which is obtained as a limiting tree structure from a family of variable weighted Gauss-Menger trees (or full variable weighted Steiner trees) w.r. to a fixed boundary convex quadrilateral \( A_1A_2A_3A_4 \), which depend on the variable weights \( (\bar{B}_1)_{1234} \) which satisfy the dynamic plasticity equations

\[
(\bar{B}_2)_{1234} = (\bar{B}_2)_{1234}[1 - (\bar{B}_4)_{1234}(\bar{B}_1)_{1234}],
\]

\[
(\bar{B}_3)_{1234} = (\bar{B}_3)_{1234}[1 - (\bar{B}_4)_{1234}(\bar{B}_1)_{1234}],
\]

and

\[
(\bar{B}_1)_{1234} + (\bar{B}_2)_{1234} + (\bar{B}_3)_{1234} + (\bar{B}_4)_{1234} = cconstant.
\]
\[(\bar{B}_1)^2_{1234} + (\bar{B}_2)^2_{1234} + 2(\bar{B}_1)_{1234}(\bar{B}_2)_{1234} \cos \alpha_{102} = (\bar{B}_3)^2_{1234} + (\bar{B}_4)^2_{1234} + 2(\bar{B}_3)_{1234}(\bar{B}_4)_{1234} \cos \alpha_{304}, \tag{5.4}\]

\[(\bar{B}_1)^2_{1234} + (\bar{B}_4)^2_{1234} + 2(\bar{B}_1)_{1234}(\bar{B}_4)_{1234} \cos \alpha_{104} = (\bar{B}_2)^2_{1234} + (\bar{B}_3)^2_{1234} + 2(\bar{B}_2)_{1234}(\bar{B}_3)_{1234} \cos \alpha_{203}, \tag{5.5}\]

and

\[(\bar{B}_1)_{1234} + (\bar{B}_2)_{1234} + (\bar{B}_3)_{1234} + (\bar{B}_4)_{1234} = c_{const}. \tag{5.6}\]

**Proof.** Suppose that we select four positive weights \((\bar{B}_i)_{1234}(0)\) which correspond the vertex \(A_i\) of the boundary convex quadrilateral \(A_1A_2A_3A_4\), such that the weighted floating inequalities \((5.2)\) of Theorem 1 hold, in order to locate the 4wFT point \(A_0\) at the interior of \(A_1A_2A_3A_4\) and particularly at the interior of \(\triangle A_1O_2\), where \(O\) is the intersection of the diagonals \(A_1A_3\) and \(A_2A_4\), closer to the vertex \(A_1\).

The location of the 3wFT points \(A_0\) and \(A_0'\), respectively, is given by the relations (Lemma 4):

\[
\cot \varphi = \frac{x_0 a_{12} + \bar{B}_4 a_{14} \cos(\alpha_{214} - \alpha_{400'}) + \bar{B}_3 a_{23} \cos(\alpha_{123} - \alpha_{300'})}{\bar{B}_4 a_{14} \sin(\alpha_{214} - \alpha_{400'}) - \bar{B}_3 a_{23} \sin(\alpha_{123} - \alpha_{300'})}, \tag{5.7}\]

\[
a_1 = \frac{a_{14} \sin(\alpha_{214} - \varphi - \alpha_{400'})}{\sin(\alpha_{100'} + \alpha_{400'})}, \tag{5.8}\]

and

\[
a_2 = \frac{a_{23} \sin(\alpha_{123} + \varphi - \alpha_{300'})}{\sin(\alpha_{200'} + \alpha_{300'})}, \tag{5.9}\]

where

\[
\alpha_{100'} = \arccos \left( \frac{\bar{B}_4^2 - \bar{B}_1^2 - x_G^2}{2B_1 x_G} \right), \tag{5.10}\]

\[
\alpha_{0'04} = \arccos \left( \frac{\bar{B}_1^2 - \bar{B}_4^2 - x_G^2}{2B_4 x_G} \right), \tag{5.11}\]

\[
\alpha_{104} = \arccos \left( \frac{x_G^2 - \bar{B}_1^2 - \bar{B}_4^2}{2B_1 B_4} \right), \tag{5.12}\]

\[
\alpha_{00'3} = \arccos \left( \frac{\bar{B}_2^2 - \bar{B}_3^2 - x_G^2}{2B_3 x_G} \right), \tag{5.13}\]

\[
\alpha_{00'2} = \arccos \left( \frac{\bar{B}_3^2 - x_G^2 - \bar{B}_2^2}{2x_G B_2} \right), \tag{5.14}\]

\[
\alpha_{20'3} = \arccos \left( \frac{x_G^2 - \bar{B}_2^2 - \bar{B}_3^2}{2B_2 B_3} \right), \tag{5.15}\]

and the variable weights \(\bar{B}_i \equiv (\bar{B}_i)_{1234}\) are taken from the system of equations \((5.1), \ (5.2), \ (5.3)\) or \((5.4), \ (5.5)\) and \((5.6)\). Taking into account Lemma 4, we express \((\bar{B}_i)_{1234}\) as a function of \(\bar{B}_4)_{1234}\).
By taking into account the distance of \( a_{12} \) from the line defined by \( A_0 \) and \( A'_0 \), and the distance of \( A_2 \) from the line which passes through \( A_1 \) and is parallel to the line defined by \( A_0 \) and \( A'_0 \), we express \( l \) as a function w.r. to \( B_4 \), \( x_G \) and the five Euclidean elements \( a_{12}, a_{13}, a_{23}, a_{34}, a_{24} \), for \( i = 1, 2, 3, 4 \).

\[
l = a_1 \cos(\alpha_{100'}) + a_2 \cos(\alpha_{200'}) + a_{12} \cos(\varphi) \tag{5.16}
\]

By letting \( l \equiv \epsilon \) a positive real number we derive a nonlinear equation which depends only on the absorbing rate of the generalized Gauss-Menger tree of degree three, where the 3wFT point \( A_0 \) remains the same because the weights \( (B_4)_{1234} \) satisfy the dynamic plasticity equations of the fixed variable weighted Fermat-Torricelli tree of degree four concerning the same boundary quadrilateral \( A_1A_2A_3A_4 \) where the position of \( A_0 \) remains invariant in \( \mathbb{R}^2 \).

By letting \( l \equiv \epsilon_i \) a decreasing sequence which converge to zero, \( A'_0 \) will approach the fixed position of \( A_0 \).

A universal constant \( u_c \) may occur if \( x_G(\epsilon_k) \to u_c \) for every positive real value of the variable weight \( (B_4)_{1234} \). Thus, a weighted Fermat-Torricelli degree four regarding the same boundary quadrilateral \( A_1A_2A_3A_4 \) would exist with a constant absorbing rate and could offer an analytical solution of the 4wFT problem in \( \mathbb{R}^2 \) which is contradictory with the result of Theorem 4.

The location of the 4wFT point for a convex quadrilateral or the position of a weighted Fermat-Torricelli tree of degree four is given by the following lemma which has been derived in [3] ((5.23) > (5.22) > (5.21) > (5.20) > (5.19) > (5.18) > (5.17)):

**Lemma 7.** [3] Formula (5),(6),(7),(8),p. 413 The following system of equations allow us to compute the position of the 4wFT point \( A_0 \) and provides a necessary condition to locate it at the interior of the convex quadrilateral \( A_1A_2A_3A_4 \):

\[
cot(\alpha_{013}) = \frac{\sin(\alpha_{213}) - \cos(\alpha_{213}) \cot(\alpha_{102}) - \frac{a_{14}^2}{a_{12}^2} \cot(\alpha_{304} + \alpha_{401})}{-\cos(\alpha_{213}) - \sin(\alpha_{213}) \cot(\alpha_{102}) + \frac{a_{14}^2}{a_{12}^2}}, \tag{5.17}
\]

\[
cot(\alpha_{013}) = \frac{\sin(\alpha_{314}) - \cos(\alpha_{314}) \cot(\alpha_{401}) + \frac{a_{24}^2}{a_{41}^2} \cot(\alpha_{304} + \alpha_{401})}{\cos(\alpha_{314}) + \sin(\alpha_{314}) \cot(\alpha_{401}) - \frac{a_{24}^2}{a_{41}^2}}, \tag{5.18}
\]

\[
(B_3)_{1234}^2 = (B_1)_{1234}^2 + (B_2)_{1234}^2 + (B_4)_{1234}^2 + 2(B_2)_{1234}(B_4)_{1234} \cos(\alpha_{401} + \alpha_{102}) + 2(B_1)_{1234}(B_2)_{1234} \cos(\alpha_{102}) + 2(B_1)_{1234}(B_4)_{1234} \cos(\alpha_{401}), \tag{5.19}
\]

\[
cot(\alpha_{304} + \alpha_{401}) = \frac{(B_1)_{1234} + (B_2)_{1234} \cos(\alpha_{102}) + (B_4)_{1234} \cos(\alpha_{401})}{(B_4)_{1234} \sin(\alpha_{401}) - (B_2)_{1234} \sin(\alpha_{102})}, \tag{5.20}
\]

\[
\alpha_{203} = 2\pi - \alpha_{102} - \alpha_{304} - \alpha_{401}, \tag{5.21}
\]

\[
a_{01} = a_{14} \frac{\sin(\alpha_{013} + \alpha_{314} + \alpha_{401})}{\sin \alpha_{401}} \tag{5.22}
\]

and

\[
a_{04} = a_{14} \frac{\sin(\alpha_{013} + \alpha_{314})}{\sin \alpha_{401}} \tag{5.23}
\]
Figure 2. A fixed weighted Fermat-Torricelli tree of degree four having a universal Fermat-Torricelli minimum value $u_{FT} = \min x_G = 3.8088826$ with respect to a boundary weighted rectangle taken from Example 2.

**Example 2.** Given a rectangle $A_1A_2A_3A_4$ in $\mathbb{R}^2$ such that: $A_1 = (0,0), A_2 = (7,0), A_3 = (7,4), A_4 = (0,4)$, with side lengths $a_{12} = a_{34} = 7, a_{23} = a_{41} = 4$ and initial positive weights $(B_1)_{1234} = 3, (B_2)_{1234} = 2.5, (B_3)_{1234} = 1.7, (B_4)_{1234} = 1.5$ which correspond to the vertices $A_1, A_2, A_3$ and $A_4$, respectively. By substituting these data in (5.17), (5.18), (5.19), (5.20), (5.21) of Lemma 7 we obtain the location of the 4wFT point $A_0 = (2.8274502, 1.2787811)$ which coincides with result obtained by the Weiszfeld algorithm ([11], [12]) with 7 digit precision and we derive that:

$$
\begin{align*}
\alpha_{102} &= 138.625^\circ, \alpha_{203} = 50.1502^\circ, \\
\alpha_{304} &= 102.986^\circ, \alpha_{401} = 68.2392^\circ.
\end{align*}
$$

(5.24)

By substituting (5.24) in (5.1), (5.2) and (5.3) or in (5.4), (5.5) and (5.6) we obtain the dynamic plasticity equations of $A_1A_2A_3A_4$:

$$
\begin{align*}
(B_1)_{1234} &= 4.2239621 - 0.8159745(B_4)_{1234}, \\
(B_2)_{1234} &= 0.8393665 + 1.1070888(B_4)_{1234}, \\
(B_3)_{1234} &= 3.6366712 - 1.2911143(B_4)_{1234}.
\end{align*}
$$

(5.25)

Then, we replace (5.24) for $(B_4)_{1234} = 1.5, (B_4)_{1234} = 1.2, (B_4)_{1234} = 1.7, (B_4)_{1234} = 1.7728955$, in the objective function (5.7) of a generalized Gauss tree of degree three, taking into account (5.4), (5.6) and (5.7) from Lemma 7 and by maximizing (5.1) w.r. to $x_G$, we derive the following table (Fig. 2).

| $(B_1)_{1234}$ | $(B_2)_{1234}$ | $(B_3)_{1234}$ | $(B_4)_{1234}$ | $x_G$ | $f$ |
|----------------|----------------|----------------|----------------|------|-----|
| 3              | 2.5            | 1.7            | 1.5            | 3.8192408 | 34.3746856 |
| 3.2447927     | 2.1678731     | 2.0873328     | 1.2            | 3.8541369  | 34.6371118 |
| 2.8368055     | 2.7214176     | 1.4477756     | 1.7            | 3.8096235  | 34.5330567 |
| 2.7773246     | 2.8021194     | 1.3476592     | 1.7728955      | 3.8088826  | 34.5178864 |
Figure 3. A fixed weighted Fermat-Torricelli tree of degree four having a universal Fermat-Torricelli minimum value $u_{FT} = \min x_G = 3.66326$ with respect to a boundary weighted rectangle taken from Example 3.

By substituting $l = 0.0000001$ and (5.25) in (5.16) and by minimizing (5.16) w.r. to the variables $x_G$ and $(B_4)_{1234}$, we obtain the universal minimum Fermat-Torricelli value $u_{FT} = 3.8088826$ for $(B_4)_{1234} = 1.7728955$ (see also the above Table, Fig. 2).

The universal absorbing rate is $\frac{u_{FT}}{\sum_{i=1}^{4} (B_i)_{1234}} = \frac{3.8088826}{8.7} = 0.4378025$.

Example 3. Given the same rectangle $A_1A_2A_3A_4$ in $\mathbb{R}^2$ with the one considered in Example 3 and initial positive weights $(B_1)_{1234} = 3.1$, $(B_2)_{1234} = 2.3$ $(B_3)_{1234} = 1.7$ $(B_4)_{1234} = 1.4$ which correspond to the vertices $A_1$, $A_2$, $A_3$ and $A_4$, respectively. By substituting these data in (5.17), (5.18), (5.20), (5.21) of Lemma 7 we obtain the location of the 4wFT point $A_0 = (2.381487, 1.1855484)$ which coincides with result obtained by the Weiszfeld algorithm ([11], [12]) with 7 digit precision and we derive that:

$$
\alpha_{102} = 139.138^\circ, \alpha_{203} = 45.754^\circ, \\
\alpha_{304} = 98.8792^\circ, \alpha_{401} = 76.2283^\circ.
$$

(5.26)

By substituting (5.26) in (5.1), (5.2) and (5.3) or in (5.4), (5.5) and (5.6) we obtain the dynamic plasticity equations of $A_1A_2A_3A_4$:

$$
(B_1)_{1234} = 4.1823652 - 0.7731178(B_4)_{1234}, \\
(B_2)_{1234} = 0.49794 + 1.2871855(B_4)_{1234}, \\
(B_3)_{1234} = 3.8196947 - 1.5140677(B_4)_{1234}.
$$

(5.27)

By substituting $l = 0.0000001$ and (5.25) in (5.16) and by minimizing (5.16) w.r. to the variables $x_G$ and $(B_4)_{1234}$, we obtain the universal minimum Fermat-Torricelli value $u_{FT} = 3.66326$ for $(B_4)_{1234} = 1.8199325$.

The universal absorbing rate is $\frac{u_{FT}}{\sum_{i=1}^{4} (B_i)_{1234}} = \frac{3.66326}{8.5} = 0.4309717$ (Fig. 3).

Definition 13. A universal absorbing Fermat-Torricelli set with respect to a fixed variable weighted Fermat-Torricelli tree of degree four is the set of values of the
Definition 14. We call a universal Fermat-Torricelli minimum value \( u_{FT} \) the minimum of the universal Fermat-Torricelli set regarding a fixed variable weighted Fermat-Torricelli tree of degree four.

Remark 3. The weighted Fermat-Torricelli tree of degree four taken from example 2 has greater universal Fermat-Torricelli minimum value \( u_{FT} \) and universal absorbing rate than the weighted Fermat-Torricelli tree of degree four w.r. to the same boundary quadrilateral \( A_1A_2A_3A_4 \) with different weights.

6. Steady trees and evolutionary trees for a boundary convex quadrilateral

Consider a universal Fermat-Torricelli set \( U \) and minimum value \( u_{FT} \) which corresponds to a fixed variable weighted Fermat-Torricelli tree of degree four regarding a boundary convex quadrilateral, which is derived as a limiting tree structure from a generalized Gauss tree of degree.

Definition 15. A steady tree of degree four is a weighted Fermat-Torricelli tree of degree four having as storage at the 4wFT point \( A_0 \) a positive quantity less than \( u_{FT} \).

Definition 16. An evolutionary tree of degree three is a generalized Gauss Menger tree of degree three which is derived by a weighted Fermat-Torricelli tree of degree four having as storage at the 4wFT point \( A_0 \) a positive quantity equal or greater than \( u_{FT} \) and then decreases by an absorbing rate \( 0 < a_G < u_{FT} \).

Example 4. Consider the same weighted Fermat-Torricelli tree four that we have obtained in Example 2 for the boundary rectangle \( A_1A_2A_3A_4 \). A steady tree is a weighted Fermat-Torricelli tree of degree four with storage level at \( A_0 \) less than \( u_{FT} = 3.8088826 \).

When the storage quantity at \( A_0 \) reaches \( u_{FT} = 3.8088826 \), it stimulates the steady tree which starts to evolve (Fig. 4). Thus, the universal minimum Fermat-Torricelli value unlocks the evolution of a generalized Gauss-Menger tree which could be derived by a weighted Fermat-Torricelli tree of degree four for the same boundary rectangle (or convex quadrilateral).

For instance, if the storage level reaches \( u = 3.8543169 > u_{FT} = 3.8088826 \) and spends \( a_G = 0.5 \), then we derive an evolutionary generalized Gauss tree having \( x_G = 3.3543169 \) (Fig. 5).

The evolutionary Gauss-Menger tree is obtained by substituting \( x_G = 3.3543169 \), \( (B_1)_{1234} = 3.2447927 \), \( (B_2)_{1234} = 2.1678731 \), \( (B_3)_{1234} = 2.0873328 \), \( (B_4)_{1234} = 1.2 \) in (3.4), (3.6) and (3.5) from Lemma 4. Thus, we get:

\[
\begin{align*}
a_1 &= 1.6642065, \quad a_2 = 2.7738702, \quad a_3 = 3.6321319, \\
a_4 &= 3.4873166, \quad l = 3.3543169.
\end{align*}
\] (6.1)

Suppose that the storage level reaches \( u = 3.82 > u_{FT} = 3.8088826 \) and spends \( a_G = 0.2 \), then we derive another evolutionary weighted Fermat-Torricelli tree...
Figure 4. An evolutionary tree of degree four having an initial storage level 3.8543169 and absorbing rate $a_G = 0$ with respect to a boundary weighted rectangle taken from Example 2 for $(B_1)_{1234} = 3.2447927$, $(B_2)_{1234} = 2.1678731$, $(B_3)_{1234} = 2.0873328$, $(B_4)_{1234} = 1.2$.

Figure 5. An evolutionary tree of degree three having an initial storage level 3.8543169 and absorbing rate $a_G = 0.5$ with respect to a boundary weighted rectangle taken from Example 2 for $(B_1)_{1234} = 3.2447927$, $(B_2)_{1234} = 2.1678731$, $(B_3)_{1234} = 2.0873328$, $(B_4)_{1234} = 1.2$.

of degree four and a generalized Gauss tree of degree three having $x_G = 3.62$ (Figs. 6, 7, 8).

By substituting $u = x_G = 3.82$ and the dynamic plasticity equations (5.25) taken from Example 2 in (3.1), the maximum of (3.1) w.r. to $(B_4)_{1234}$ yields $(B_4)_{1234} = 1.4901507$ or $(B_4)_{1234} = 2.0556426$. By letting $(B_4)_{1234} = 1.4901507$ in (5.25), we get: $(B_1)_{1234} = 3.0080371$, $(B_2)_{1234} = 2.4890958$, $(B_3)_{1234} = 1.7127149$,

and by letting $(B_4)_{1234} = 2.0556426$ in (5.25), we get: $(B_1)_{1234} = 2.5466101$, $(B_2)_{1234} = 3.1151456$, $(B_3)_{1234} = 0.9826002$.

The first evolutionary Gauss-Menger tree is obtained by substituting $x_G = 3.62$, $(B_1)_{1234} = 3.0080371$, $(B_2)_{1234} = 2.4890958$, $(B_3)_{1234} = 1.7127149$, $(B_4)_{1234} =$
Figure 6. An evolutionary tree of degree four having an initial storage level 3.82 and absorbing rate $a_G = 0$ with respect to a boundary weighted rectangle taken from Example 2 for $(B_1)_{1234} = 3.0080371$, $(B_2)_{1234} = 2.4890958$, $(B_3)_{1234} = 1.7127149$, $(B_4)_{1234} = 1.4901507$.

Figure 7. An evolutionary tree of degree three having an initial storage level 3.82 and absorbing rate $a_G = 0.2$ with respect to a boundary weighted rectangle taken from Example 2 $(B_1)_{1234} = 3.0080371$, $(B_2)_{1234} = 2.4890958$, $(B_3)_{1234} = 1.7127149$, $(B_4)_{1234} = 1.4901507$.

1.4901507 in (3.4), (3.6) and (3.5) from Lemma 4 (Fig. 7). Thus, we get:

$$a_1 = 2.5638686, a_2 = 3.4255328, a_3 = 4.2080591,$$

$$a_4 = 3.6397828, l = 1.5309344.$$  \hspace{1cm} (6.2)

The second evolutionary Gauss-Menger tree is obtained by substituting $x_G = 3.62$, $(B_1)_{1234} = 2.5466101$, $(B_2)_{1234} = 3.1151456$, $(B_3)_{1234} = 0.9826002$, $(B_4)_{1234} = 2.0556426$. in (3.4), (3.6) and (3.5) from Lemma 4 (Fig. 8). Thus, we get:

$$a_1 = 2.6836315, a_2 = 3.0204233, a_3 = 4.0857226,$$

$$a_4 = 3.6424502, l = 1.8001622.$$  \hspace{1cm} (6.3)
Figure 8. An evolutionary tree of degree three having an initial storage level 3.82 and absorbing rate $a_G = 0.2$ with respect to a boundary weighted rectangle taken from Example 2 $(B_1)_{1234} = 2.5466101$, $(B_2)_{1234} = 3.1151456$, $(B_3)_{1234} = 0.9826002$. $(B_4)_{1234} = 2.0556426$.

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