Fuzzy sphere bimodule, ABS construction to the exact soliton solution

Bo-yu Hou\textsuperscript{a}, Bo-yuan Hou\textsuperscript{b} and Ruihong Yue\textsuperscript{a}

\textsuperscript{a}Institute of Modern Physics, Northwest University, Xi’an 710069, P. R. China
\textsuperscript{b}Graduate School, Chinese Academy of Science, Beijing 100039, P. R. China

Abstract

In this paper, we set up the bi-module of the algebra $\mathcal{A}$ on fuzzy sphere. Based on the differential operators in moving frame, we generalize the ABS construction into fuzzy sphere case. The applications of ABS construction are investigated in various physical systems.

PACS Numbers: 46.10.+z, 05.40.+j, 05.60.+w

1 Introduction

Non-commutative geometry is originally an old topics in mathematics [1]. Now, it becomes an interesting subject of quantum field theory since it was found that the non-commutative field theory appears naturally in string theory at low energy level in a constant NS B-field [2, 3]. The nontrivial B-field leads to the non-commutativity of the coordinates of string ends on the D-brane, which gives a non-commutative gauge field theory on the D-brane world volume.

Recently, the study of the non-perturbative dynamics of these fields has attracted much more attention [4-20]. Harvey, Kraus and Larsen set up a new method to investigate the soliton solution, the monopole solution and instanton solution in 3+1 dimension [24]. The Nielson-Olesen solution in abelian Higgs model was studied in [7] and [8]. All studies are carried out in the non-commutative Euclidean space. One natural question is to how to generalize such results into other non-commutative geometry with non-zero curvature. It is a challenging topics. Physically, the space near horizon in NS 5-brane is a sphere $S^3$. The
D-brane on such geometry can be described by a boundary WZW model on a sphere $S^2$ \cite{14}. Many works focused on the fuzzy sphere $S^2$ - a simplest space with non-trivial curvature \cite{10}. On the fuzzy sphere, one can set up a gauge filed theory \cite{17} directly or from the matrix theory \cite{21, 22}. The D-brane structure was discussed in \cite{13, 24}. The $CP(N)$ system on fuzzy sphere in ref. \cite{23}.

In the present paper, we will consider the gauge field theory on fuzzy sphere, which includes the soliton and Nielsen-Olesen solution. The main tool is ABS construction. In the standard ABS method, the quasi-unitarity operator has played an important role. However, on the fuzzy sphere, there does not exist such operator. We will generalize it and find a partial isometric operator which acts a same role as quasi-unitarity operator in usual one.

On the sphere, the spherical harmonic functions are convenient basis for studying other properties of the theory. But, most study about fuzzy sphere is based on the three-dimensional " coordinate" with a constrain. We hope that the correspondence of spherical function on fuzzy sphere gives a convenient basis and provides a clear physical picture. The best way to set up such correspondence is the coherent state technique. In ref \cite{18}, a coherent state is constructed with the help of the stereographic projection. It is not convenient for our purpose. In this paper, we take the standard method to define the coherent state \cite{19}.

\section{Fuzzy sphere and coherent state}

The algebra $\mathcal{A}$ of functions on the fuzzy sphere is defined as a finite dimensional algebra generated by "coordinate" $x_i, i = 1, 2, 3$ with relations

$$\left[x_i, x_j\right] = i\theta\epsilon_{ijk}x_k, \quad \sum_{i=1}^{3}x_i^2 = r^2$$

(1)

where the parameter $\theta$ stands for the non-commutativity and $r$ the radius of the fuzzy sphere. With appropriate value of $\theta$, this algebra has a finite number of basis. The basis may be represented by the "spherical harmonic functions $y^l_m, l \leq N$ with Moyer product (star product) \cite{14}

$$Y^I_i \ast Y^J_j = \sum_{K,k} \left[I \quad J \quad K\right]c_{IJK}^k Y^K_k$$

(2)

We has another metx forhod to realize it as an operator form. The matrixn form is \cite{13}

$$\left(T_{j,m}\right)_{m_1,m_2} = (-1)^{p-1-m_1} \sqrt{2j+1} \left(p-1 \atop \frac{p-1}{2} - m_1\right)_{j \atop m} \frac{p-1}{2} \left(m_m \atop m_2\right)$$

(3)
The algebra $\mathcal{A}_N$ has a convenient realization by the $SU(2)$ Lie algebra. For a $N + 1$-dimensional irreducible representation, the generators of $SU(2)$ satisfy

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \quad \sum_{i=1}^{3} L_i^2 = \frac{N(N+2)}{4}. \quad (4)$$

The basis of $H$ can be chosen as $|N/2, m\rangle$ with relations

$$L_\pm |N/2, m\rangle = \sqrt{(N/2 \mp m)(N/2 \mp m + 1)}|N/2, m\rangle$$

$$L_3 |N/2, m\rangle = m|N/2, m\rangle, \quad m = N/2, N/2 - 1, \ldots, -N/2 \quad (5)$$

with $L_\pm = (L_1 \pm iL_2)/\sqrt{2}$.

It is well-known that there is an isomorphism between $\mathcal{A}_N$ and $SU(2)$ Lie algebra

$$x_i = \lambda_N L_i \quad (6)$$

On the $H_N$, acturally, we will show $L f = [x, f]$ on the function $f$. It gives a relation between the radius of fuzzy sphere and the Casimir

$$r^2 = \frac{\theta^2 N(N+2)}{4}. \quad (7)$$

For algebra $\mathcal{A}_N$, the basis can be written in terms of the spherical functions but with non-trivial multiplication Eq.(1). We will show it explicitly by using the generalized coherent states of $SU(2)$

$$|\omega\rangle = T(g)|v\rangle, \quad |v\rangle \in H_N \quad (8)$$

where $T(g)$ is an element of $SU(2)$ group

$$T(g) = e^{i\alpha L_3} e^{i\beta L_2} e^{i\gamma L_3} \quad (9)$$

These coherent states satisfy

$$\frac{N+1}{8\pi^2} \int d\Omega_3 |\omega\rangle(\omega)| = 1 \quad (10)$$

here $d\Omega_3$ stands for 3-volume form of the group manifold $\Omega(\alpha, \beta, \gamma)$. If taking a gauge $\alpha = \phi, \beta = \theta, \gamma = -\phi$ and $|v\rangle = |N/2, -N/2\rangle$, then this gives standard coherent state

$$|\omega(\theta, \phi)\rangle = \sum_{\mu = -N/2}^{N/2} D_{\mu, -N/2}^{N/2}(\phi, \theta, -\phi)|N/2, \mu\rangle \quad (11)$$

Here

$$D_{\mu, \nu}^{j}(\alpha, \beta, \gamma) = \sum_k \frac{(-1)^k \sqrt{(j + \mu)!(j - \mu)!(j + \nu)!(j - \nu)!}}{k!(j - \nu - k)!(j + k)!(k - \mu + \nu)!} e^{-i\alpha \mu} (\cos(\beta/2))^{2j - \nu + \mu - 2k} (\sin(\beta/2))^{2k - \mu + \nu} e^{-i\nu \gamma} \quad (12)$$

\[1\] The explicit form was given in Ref.[19], but no relation with D-function
For example $D_{0,m}^j(\phi, \theta, -\phi) = Y_m^j(\theta, \phi)$. In this Dirac gauge, the completeness condition changes into the integral on $S^2$.

$$\frac{N + 1}{4\pi} \int \sin(\theta) d\theta d\phi |\omega(\theta, \phi))(\omega(\theta, \phi)| = 1$$ (13)

First of all, let $f$ be a function of $(\theta, \phi)$ and define an operator $\hat{f}$ by

$$\hat{f} = \int d\Omega_2 f(\theta, \phi)|\omega(\theta, \phi))(\omega(\theta, \phi)|$$ (14)

For a basic function $y_m^l(\theta, \phi), 0 \leq l \leq N$, the corresponding operator $\hat{Y}_m^l$ is

$$\hat{Y}_m^l = \frac{N + 1}{4\pi} \int d\Omega_2 y_m^l(\theta, \phi)|\omega(\theta, \phi))(\omega(\theta, \phi)|.$$ (15)

Putting between two vectors in $H_N$ will give the matrix element of $\hat{Y}_m^l$

$$\langle Y_m^l \rangle_{\mu, \nu} = \langle N/2, \mu | \hat{Y}_m^l | N/2, \nu \rangle$$

$$= a_{N,l} (-1)^{l+N/2+\mu} \sqrt{N+1} \begin{pmatrix} N/2 & l & N/2 \\ \nu & m & \mu \end{pmatrix}$$ (16)

with

$$a_{N,l} = (-1)^l \sqrt{N+1} \begin{pmatrix} N/2 & l & N/2 \\ -N/2 & 0 & N/2 \end{pmatrix}$$ (17)

In $(N+1)$-dimensional Hilbert space on which operator acts, one can define a symbol related to the operator through

$$\tilde{F}(\theta, \phi) = (\omega(\theta, \phi)|\hat{f} |\omega(\theta, \phi))$$ (18)

Since the coherent states are not orthogonal, the symbol $\tilde{F}$ is not equal to $f$. But there must some relations between them. In ref. [19], both $f$ and $F$ are called q-symbol and p-symbol respectively. Somehow it is related to the (anti)-normal order. To explore such relation, we put our attention on the symbol of typical basis $y_m^l$

$$\tilde{Y}_m^l(\theta, \phi) = (\omega(\theta, \phi)| \int d\Omega_2(\theta', \phi') y_m^l(\theta, \phi)|\omega(\theta', \phi'))(\omega(\theta', \phi')|\omega(\theta, \phi))$$

$$= \sum_{\mu, \nu, J} \langle N/2, -\mu, N/2, \nu | J, \nu - \mu \rangle \langle N/2, N/2, N/2, -N/2 | J, 0 \rangle a_{N,l} \frac{2l + 1}{4\pi} (-1)^{\mu+N/2} D_{\nu-\mu,0}^J(\theta, \phi)$$

$$= a_{N,l}^2 \hat{Y}_m^l(\theta, \phi)$$ (19)

With the help of symbol, we can define the "star product" (Moyer product) of two functions (symbols) to be the symbol of two operators, namely

$$\tilde{Y}_{m_1}^{j_1} \ast \tilde{Y}_{m_2}^{j_2} (\theta, \phi) = (\omega(\theta, \phi)| \hat{Y}_{m_1}^{j_1} \hat{Y}_{m_2}^{j_2} |\omega(\theta, \phi))$$
\[ \int d\Omega_2(\theta', \phi') (\omega(\theta, \phi) | \tilde{Y}_{m_1}^{j_1} \omega(\theta', \phi')) (\omega(\theta', \phi') | \tilde{Y}_{m_2}^{j_2} \omega(\theta, \phi)) = \sum_{\mu, \nu, J, m} (-1)^{\mu + N/2} a_{N,j_1} a_{N,j_2} \langle N/2, m, j_1, m_1 | N/2, \mu \rangle \langle N/2, \nu, j_2, m_2 | N/2, m \rangle \times \langle N/2, \nu, -\mu | J, \nu - \mu \rangle \langle N/2, -N/2, N/2, N/2 | J, 0 \rangle D_{\nu - \mu, 0}^J(\theta, \phi) \]

\[ = \sum_{j, m} \langle j, m, j_2, m_2 | J, m \rangle \sqrt{2J + 1} \left\{ \begin{array}{ccc} j_1 & j_2 & J \\ N/2 & N/2 & N/2 \end{array} \right\} \times a_{N,j_1} a_{N,j_2} a_{\tilde{a}_{N,J}} \tilde{Y}_m^J(\theta, \phi). \] (20)

where \( \cdots \) stands for 6j-symbol. Thus, the star product of two normalized functions

\[ Y_m^j = a_{\tilde{a}_{N,J}} \tilde{Y}_m^J(\theta, \phi) \]

is given by

\[ Y_{m_1}^{j_1} \star Y_{m_2}^{j_2}(\theta, \phi) = \sum_{j, m} (-1)^{j_2 - m - J(m + 1)} a_{N,j_1} a_{N,j_2} a_{\tilde{a}_{N,J}} \left( \begin{array}{ccc} j_1 & j_2 & J \\ m_1 & m_2 & -m \end{array} \right) \times \left\{ \begin{array}{ccc} j_1 & j_2 & J \\ N/2 & N/2 & N/2 \end{array} \right\} Y_m^J(\theta, \phi). \] (21)

This is nothing but the relation Eq.(2) appeared in ref.[14]. A little difference is due to the normalization of 6j-symbol. This provides a realization of algebra \( A_N \). It is clear that the symbol (14) is same as Eq.(3) given in Ref.[13] up to a normalization factor. Therefore, we have found a correspondence between function and operator realization. The integral in function space becomes the the trace of operators in Hilbert space, i.e.

\[ Tr \Rightarrow \frac{N + 1}{4\pi} \int d\Omega_2. \] (22)

Comparing with one in non-commutative plan, one can conclude \( \Theta = 2/(N + 1) \). This was obtained in Ref.[13] by taking the large \( N \) limit. Here it is valid for all value of \( N \).

The action of differential operators \( L_a, a = +, -, 3 \) on symbol is given by

\[ (L_a Y_m^l)(\theta, \phi) = \sqrt{\frac{N(N + 1)(N + 2)}{12}} [Y_a^1, Y_m^l] \] (23)

Using Eq.(21), one can check

\[ r.h.s = \sum_j \sqrt{\frac{(2l + 1)N(N + 1)(N + 2)}{12}} \left\{ \begin{array}{ccc} 1 & l & j \\ N/2 & N/2 & N/2 \end{array} \right\} \times [1, a, l, m | j, a + m > - [ l, m, 1, a | j, a + m >] Y_a^j \]

\[ = \left\{ \begin{array}{c} \sqrt{(l \pm m)(l \pm m + 1)} Y_{m \pm 1}^l, \quad a = \pm m Y_m^l, \quad a = 3 \end{array} \right\} \] (24)

It is consistence with the left hand side of Eq.(23). This equation is same as one appeared in ref.[13] up to a factor. Notice that this extra factor on the right hand side of Eq.(23) comes from the non-commutative parameter \( \theta^2 = (N + 1)/2 \) and the radius.
3 Bimodule and Differential operator in moving frame

Up to now, we have set up the correspondence of the differential operators. In principle, we are ready to write down a quantum gauge field theory. However, these three differential operators are not independent. The fuzzy sphere, since the constraint $\sum x_i^2 = 1$, has two independent degree of freedom. Thus, the best way is to find two independent differential operators. On the usual sphere, there exist two tangent vector (in moving frame). this can be done by introducing the right acting operator on the basis. Somehow it is not necessary for usual case \[7\]. This idea can be generalized into fuzzy sphere. For present situation, the right acting operators are very important. On fuzzy sphere, the normal vector relates the spin of frame. Thus, the differential operators along two tangent and normal directions constitute a right-acting SU(2) group which commutative with the original left-acting SU(2). The rotation on the sphere is a subgroup. The Hilbert space $H_N$ corresponding to the left action of algebra $A_N$ also provides a $N + 1$-dimensional representation of such SU(2). Exactly, the basis of algebra $A_N$ are a bi-module. In this section, we will introduce right acting differential operators.

Let $J_a, a = +, -, 3$ the three right-acting operators in the moving frame acting on the symbol $D^l_{m, \mu}(\alpha, \beta, \gamma)$ as

$$ J_\pm D^l_{m, \mu}(\alpha, \beta, \gamma) = \sqrt{l \mp \mu}(l \pm \mu + 1)D^l_{m, \mu \pm 1}(\alpha, \beta, \gamma) $$

$$ J_\pm D^l_{m, \mu}(\alpha, \beta, \gamma) = \mu D^l_{m, \mu}(\alpha, \beta, \gamma) \quad (25) $$

Here we still write the right acting operators at the left of basis, but the action is different. They act on the second subscript of symbol $D^l_{m, \mu}$ in stead of the first index. Using the coherent state properties, we can show

$$ J_a D^l_{m, \mu} = \left[D^l_{0, \alpha}, D^l_{m, \mu}\right] \sqrt{\frac{N(N + 1)(N + 2)}{12}} \quad (26) $$

The proof is very similar with one in left multiplication case. In fact, this equation provides a method to define a local coordinates $\hat{x}_\pm$ by

$$ \hat{x}_\pm = \sqrt{\frac{N(N + 1)(N + 2)}{12}} D^l_{0, \alpha}, \quad (27) $$

then it becomes

$$ J_\pm D^l_{m, \mu} = \left[\hat{x}_\pm, D^l_{m, \mu}\right] \quad (28) $$

In Ref.\[23\], three operators $K_a$ of another SU(2) are found by using two sets of bosonic operators. They are commutative with the generators $L_a$. The $K_3$ must correspond to $J_3$ in
present paper describing a rotation freedom of the frame. The other two $K_\pm$ act like $J_\pm$ in our notation. They did not discuss it from the point of view of bimodule. The advantage of our method is to interpret the $L_a$ and $J_a$ as the left-right-acting operators on the basis of $A_N$. The counterpart of $J_a$ on function space is explicitly constructed. We find that this realization is natural and convenient.

4 ABS construction

In soliton theory, the solution generating technique has been a significant tool. Recently, such approach was applied into non-commutative geometry. In ref. [24] the quasi-unitary operator was introduced and various solutions were obtained for several theories. The key point is the quasi-unitary operator $S$ which satisfies $S\bar{S} = 1 - P$, $\bar{S}S = 1$. This property is closely related to fact the infinite dimensional Hilbert space. For fuzzy sphere, however, the dimension of Hilbert space is just $N + 1$. The concept of quasi-unitary operator must be generalized if applying ABS method to generate new solutions in finite Hilbert space.

On fuzzy sphere, there are two independent degree of freedom. The method to choose such two coordinates labelled by $\hat{x}_\pm$ in the moving frame is given by Eq.(27). Define two operators (partial isometry) $T$ and $\bar{T}$ by

$$T = \frac{1}{\sqrt{x_- x_+}} \hat{x}_- , \quad \bar{T} = \frac{1}{\sqrt{x_+ x_-}} \hat{x}_+$$

(29)

which satisfy

$$TT = 1 - P_{N/2} , \quad T\bar{T} = 1 - P_{-N/2}$$

(30)

where $P_i$ is a projecting operator onto $i$-th dimension in Hilbert space. Since $\hat{x}_-$ ($\hat{x}_+$) has a kernel $|N/2, -N/2 > (|N/2, N/2 >)$, the $T$ ($\bar{T}$) has also same kernel as $\hat{x}_-$ ($\hat{x}_+$). This is ensured by choosing properly the order in the denominator of $T$ and $\bar{T}$. Such partial isometry operators will play an important role in constructing the new soliton-like solutions. In ref. [23], A similar result is obtained. But it was given in terms of operators. In present, all things are based on the partial isometry related to two tangent vectors. It can considered as the function realization. Using the same method, one can analyzes $\text{CP}(n)$ model on fuzzy sphere.

5 BPS solitons

Consider a complex scalar field theory on fuzzy sphere. The action reads

$$S = \int d\Omega_2 D_\alpha \Phi D^\alpha \Phi$$

(31)
where $D_a \Phi = J_a \Phi - i \Phi \ast A_a$. The equation of motion is

$$D_a \Phi = 0$$  \hspace{1cm} (32)

Since the gauge field $A_a$ has no kinetic term, we can get

$$A_a = -i \Phi^\dagger \ast J_a \Phi$$  \hspace{1cm} (33)

Thus the equation of motion can be written as

$$D_a \Phi = J_a \Phi - i \Phi \ast (-i \Phi^\dagger \ast J_a \Phi) = (1 - \Phi \ast \Phi^\dagger) \ast J_a \Phi = 0$$  \hspace{1cm} (34)

It is clear that the system has a trivial solution $\Phi = \text{const.}$ and $A_a = 0$. Now, we try to give other non-trivial solutions. Assume $\Phi = T^n, n \leq N$. Then one check that it is a new solution

$$D_a T^n = (1 - T^n \tilde{T}^n) \ast J_a T^n = P_{N,N-n} J_a T^n = 0$$  \hspace{1cm} (35)

where $P_{N,N-n} = P_N + P_{N-1} + \cdots + P_{N-n+1}$.

In the above discussion, we did not consider the affection of potential. It is quite easy to generalize into including potential case. Suppose the potential to be form $V(\Phi^\dagger \Phi - |\Phi_0|^2)$ which has an extrema at $\Phi = 0$ and a local minimum at $\Phi = \Phi_0$. Due to the appearance of potential, the equation of motion should be modified. Namely, the right hand side of Eq.(30) to be $V'(\Phi^\dagger \Phi - |\Phi_0|^2) \Phi^\dagger$ instead of zero. Putting $\Phi = T^n$ and suing the property of $T$, one can show the added term in equation of motion vanishing. So, it keeps the equation of motion unchanged.

It is worthy to point out that these solutions are the eigen-state of $J_0$ with value $(-2n)$. Similarly, one can also choose $\Phi = \tilde{T}^n$ which will give another kind of solutions with the eigen-value $(2n)$ of $J_0$. A similar results were also obtained in ref[23] for $CP(N)$ model on fuzzy sphere.

6  Flux-like solution

In this section, we will discuss another kind of solution in gauge field theory with a scalar field $\Phi$. The Lagrangian is taking the form

$$\mathcal{L} = -\frac{1}{g} \int d\Omega_2 \left( \frac{1}{4} F_{ab} F^{ab} + D_a \Phi D^a \Phi \right)$$  \hspace{1cm} (36)

where $\Phi$ takes the adjoint representation, i.e.

$$D_a = J_a \Phi + [A_a \ast \Phi]$$  \hspace{1cm} (37)
The equation of motion reads
\[
[D_a, [D^a, D^b]] + [\Phi, [\Phi, D^b]] = 0
\]
\[
[D_a, [D^a, \Phi]] = 0
\]  \hspace{1cm} (38)

It is easy to check that the Eq. (38) has a trivial solution \( \Phi = \text{const.} \) and \( A_a = 0 \). Let us consider another solution \( \Phi = T^n T^n = 1 - P_{N,N-n} \). First, we need show it to be a solution.

Using the explicit expression, one can find
\[
[D_a, \Phi] = T^n J_a \bar{T}^n T^n - T^n \bar{T}^n T^n J_a \bar{T}^n = 0
\]  \hspace{1cm} (39)

and the first equation of Eq. (38) is also valid. The gauge field strength is given by
\[
F_{+, -} = [D_+, D_-] - 2D_3 - ([J_+, J_-] - 2J_3) = n(N + 1)P_N
\]
\[
F_{3, \pm} = 0
\]  \hspace{1cm} (40)

### 7 D-brane on fuzzy sphere

Let us discuss the tachyon condensation on non-BPS D-brane on fuzzy sphere. As done in non-commutative \( R^n \) space, we chose the background field \( B \) which is not constant in our case. On any non-BPS D-brane there exists a tachyon field \( \phi \) (do not confuse with the partial isometry operator in above section) and a gauge field \( A_a \). The effective action of a non-BPS D2-brane can be written as the DBI form \( [25] \). In our case it reads
\[
S = \frac{\sqrt{2}}{\sqrt{\alpha' G_s}} \int dt Tr \left[ V(\phi) \sqrt{det (G + 2\pi \alpha'(F + B))} \right] + O(D_\phi, D_\phi F)
\]  \hspace{1cm} (41)

where \( D_\phi T, D_\phi F \) denote the covariant derivative of the tachyon and the gauge field strength on fuzzy sphere; the last term means including the higher derivative of \( T \) and \( F \). As argued in many cases, the tachyon condensation does not depend on the detailed form of the last term. This property is still valid in fuzzy sphere case. The Tachyon potential factor appeared in the front of DBI form is followed from Sen’s conjecture. It contains a local maximum \( T = 0 \) and local mimimum \( \phi = \phi_0 e^{i\theta} \).

On the non-BPS D-brane, both tachyon and gauge fields take the adjoint representation of the gauge group. Based on the correspondence between Moyer product and operator product, the derivative could be considered as the operator on fuzzy sphere. First, we want to solve the equation of motion of the tachyon field. The general form is
\[
D_a D^a \phi + \cdots = \frac{\partial V(T)}{\partial T}
\]  \hspace{1cm} (42)
Here \( \cdots \) stands for terms related to higher derivative of \( T \). The solutions of this equation can be represented by the projection operator on fuzzy sphere \( \phi = \phi_0(1 - P_{N/2,n}) \). The projection on fuzzy sphere consists of the partial isometry operator \( (88) \). Since \( 1 - P_{N/2,n} \) is also a projection, we have
\[
\left. \frac{\partial V(T)}{\partial T} \right|_{T=T_0(1-P_{N/2,n})} = \frac{\partial V(T_0)}{\partial T_0} (1 - P_{N/2,n}) = 0.
\]

On the other hand, one can check that such tachyon solutions satisfy a simple equation \( D^+ \phi = D^+ \phi = 0 \). Thus this fact ensures the whole equation of motion even if we do not know the explicit form of second term in Eq.(11). Next we examine the gauge field. The equation of motion of gauge field is \( D^a F^{ab} = 0 \). This is automatically satisfied if the strength is proportional to a projection. For our case, the detail calculation shows the gauge field strength to be \( F^+ = n(N - n)P_{N/2} \). The mass of this excitation is
\[
M = \frac{\sqrt{2}}{\sqrt{\alpha'} G_s} Tr \left[ V(\phi_0)(1 - P_{N/2,n}) \sqrt{\det(G + 2\pi \alpha'(F + B))} \right] \quad (43)
\]

The last task of this section is to investigate the tachyon condensation on brane-antibrane systems. It is much more complicated that non-BPS D-brane. Of reasons is the tachyon becoming into complex and belonging to a bi-module. The actions of operators from the left and right of bi-module are different \[28, 26, 27\].

The effective action of a \( D2 - \bar{D}2 \) with two gauge fields has been computed through boundary string field theory \[28, 30\]. It was applied to non-commutative tori in \[29, 31\]. For our case the effective action takes
\[
S = \frac{1}{\sqrt{\alpha'} G_s} \int d^2r \left[ V^{(1)}(\phi \bar{\phi}) \sqrt{\det(G + 2\pi \alpha'(F^{(1)} + B))} \right] + O \left( D_a \phi, D_a F^+, D_a F^- \right) \quad (44)
\]

where \( O(x) \) denotes the derivative of tachyon and two gauge fields. The tachyon potentials \( V^{(i)} \) are assumed to be stationery at \( T_0 \). From the action, the equations of motion are
\[
D_a \phi = \bar{\phi} \frac{\partial V^{(1)}(x)}{\partial(x)} \bigg|_{x=\phi \bar{\phi}}
\]
\[
D_a \bar{\phi} = \phi \frac{\partial V^{(2)}(x)}{\partial(x)} \bigg|_{x=\phi \bar{\phi}}
\]
\[
D_a F^{(i)} = 0 \quad (45)
\]

Choose the following solution
\[
\phi = \phi_0 T^n \quad , \quad \bar{\phi} = \phi_0 \bar{T}^n \quad (46)
\]
one can check that they give the stationary points of the tachyon potential. A detailed calculation shows these solutions satisfying the equation of motion of tachyon. For gauge field, we choose

\[ A^{(1)}_a = 0 , \quad A^{(2)}_a = T^n \hat{j}_a \hat{T}^n - \hat{j}_a \]  

(47)

The proof of gauge field satisfying equation of motion is straightforward. The gauge field strength is \( F_{+}^{(2)} = n(N-n)P_{N/2} \).

8 Discussion

In this paper, we propose that the Hilbert space of algebra \( A \) is a bo-module. The operators acting on the bi-module are considered as the differential operators in both fixed frame and moving frame. Based on the two tangent vectors on fuzzy sphere, we carried out the ABS construction on fuzzy sphere and applied into the soliton and flux solutions of gauge field theory. The application to D-brane systems and the mass spectrum are discussed.

In Ref. [23], the bosonic realization of SU(2) algebra are used. Since the \( L_a \) is not a proper derivative to expose topologically nontrivial field configurations, Chan et al proposed another SU(2) \( K_a \). The operator differential equation looks like Eq.(26) (Eq.(2.24) in ref.[23]). The BPS solution obtained in ref.[23] must equal to Eq.(35). Since the BPS solitons are given in terms of partial isometry, it is not difficult to move away from the origin by shifting the parameters in symbols. These new solutions are similar to the \( W_{+}^k \) (Eq.(4.12) in ref.[23]). Thus, the results in section 5 partially recover those in ref.[?].

It would be desirable to extend the analysis to other fuzzy spheres such as \( S^3 \) and \( S^4 \). For \( S^3 \), it can be considered as a coset of \( SO(4)/SO(3) \). Using the similar method, one may construct the ABS operators and investigate the related noncommutative Yang-Mills theory.

Acknowledgment

We are grateful to Y.S. Wu for useful discussions and Miao Li for discussions and the hospitality during the Summer School on Strings from July 16-27. B.Y. Hou also likes to thank Martinec for discussions. We also acknowledge the NSFC for support.

References

[1] A. Connes, ”Noncommutative geometry”, Academic Press 1994

[2] A.Connes, M.R. Douglas and A. Schwarz, ”Noncommutative geometry and Matrix theory: Compactification on tori”, JHEP 9802(1998)003; hep-th/9711162
[3] N. Seiberg and E. Witten, "String theory and Noncommutative geometry", JHEP 9909 (1999) 032; hep-th/9908142

[4] R. Gopakumar, S. Minwalla and A. Strominger, "Noncommutative solitons", JHEP 0005 (2000) 048; hep-th/0003160

[5] J. Harver, P. Kraus, F. Larsen and E. Martinec, "D branes and strings as noncommutative solitons", hep-th/0005031

[6] K. Dasgupta, S. Mukhi and G. Rajesh, "Noncommutative tachyons", hep-th/0005006

[7] D. Jatkar, G. Mandal and S. Wadia, "Nielson-Olesen vortices in noncommutative abelian Higgs model", hep-th/0007078

[8] D. Bak, "Exact solitons of multi-vortices and false vacuum bubbles in noncommutative abelian Higgs model", hep-th/0008204

[9] D. Bak, K. Lee and J. Park, "Noncommutative vortex solitons", hep-th/0011099

[10] M. Li, "Note on noncommutative tachyon in matrix models", hep-th/0010058

[11] B. Lee, K. Lee and H. Yang, "CP(N) models on noncommutative plan", hep-th/0007140

[12] M. Hamanaka and S. Terashima, "On exact noncommutative solitons" hep-th/0010221

[13] Y. Hikida, M. Nozaki and T. Takayanagi, Nucl. Phys. B595 (2000) 319; hep-th/0008023

[14] A. Alekseev, A. Recknagel and V. Schomerus, "Noncommutative world-volume geometries: Branes on SU(2) and fuzzy spheres", JHEP 9909 (1999) 023; hep-th/9908040

[15] A. Alekseev, A. Recknagel and V. Schomerus, " Brane dynamics in background fluxes and Noncommutative geometry", JHEP 0005 (2000) 010; hep-th/0003187

[16] H. Grosse, P. Klimčík and P. Prešnajder, Commun. Math. Phys. 178 (1996) 507; Lett. Math. Phys. 46 (1998) 61; Commun. Math. Phys. 180 (1996) 429

[17] U. Carrow-Watamura and S. Watamura, Commun. Math. Phys. 212 (2000) 413

[18] G. Alexanian, A. Pinzul and A. Stern, " Generalized coherent state approach to star product and application to the fuzzy sphere", hep-th/0010187

[19] Peremolov, "Generalized coherent state"
[20] Y. Hikida, M. Nozaki and Y. Sugawara,” Formation of spherical D2-brane from multiple D0-brane”, hep-th/0101211

[21] Y. Kimura,” Noncommutative gauge theories on fuzzy sphere and fuzzy torus from matrix model”, hep-th/0103492

[22] S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki,” Noncommutative gauge theory on fuzzy sphere from matrix model”, hep-th/0

[23] C. Chan, C. Chen, F. Lin and H. Yang,” CP(N) model on fuzzy sphere”, hep-th/0105087

[24] J. Harver, P. Kraus, and F. Larsen, ”Exact noncommutative solitons”, hep-th/0010060

[25] M. R. Garousi, ”Tachyon couplings on non-BPS D-brane and Dirac-Born-Infeld action”, Nucl. Phys. B584(2000)284, hep-th/0003122

[26] D. Kutasov, M. Marino and G. Moore,”Some exact results on tachyon condensation in string field theory” JHEP 0010(2000)045, hep-th/0009148

[27] D. Kutasov, M. Marino and G. Moore,”Remarks on tachyon condensation in superstring field theory”, hep-th/0010108

[28] T. Takayanagi, S. Terashima and T. Uesugi,”Brane-antibrane action from boundary string field theory”, hep-th/0012210

[29] I.Bars, H. Kajiura, Y. Matsuo and T. Takayanagi,” Tachyon condensation on noncommutative torus”, hep-th/0010101

[30] H. Kajiura, Y. Matsuo and T. Takayanagi,”Exact tachyon condensation on noncommutative torus”, hep-th/0104143

[31] E. Martinec,”Noncommutative soliton on orbifold”, hep-th/0101199