PEAK POSITIONS OF STRONGLY UNIMODAL SEQUENCES

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ABSTRACT. We study combinatorial and asymptotic properties of the rank of strongly unimodal integer sequences. We find a generating function for the rank enumeration function, and give a new combinatorial interpretation of the ospt-function introduced by Andrews, Chan, and Kim. We conjecture that the enumeration function for the number of unimodal sequences of a fixed size and varying rank is log-concave, and prove an asymptotic result in support of this conjecture. Finally, we determine the asymptotic behavior of the rank for strongly unimodal sequences, and prove that its values (when appropriately renormalized) are normally distributed with mean zero in the asymptotic limit.

1. INTRODUCTION AND STATEMENT OF RESULTS

A sequence of positive integers \( \{a_j\}_{j=1}^s \) is a strongly unimodal sequence of size \( n \) if it satisfies

\[
0 < a_1 < \cdots < a_{k-1} < a_k > a_{k+1} > \cdots > a_s > 0
\]

for some \( k \in \mathbb{N} \) and \( a_1 + \cdots + a_s = n \). If \( \sigma \) is a strongly unimodal sequence, then we denote its size by \( |\sigma| \), and for a given \( n \), let \( \mathcal{U}(n) \) be the set of all strongly unimodal sequences such that \( |\sigma| = n \). We also define the enumeration function for strongly unimodal sequences by \( u(n) := |\mathcal{U}(n)| \).

The generating function for \( u(n) \) is given by

\[
U(q) := \sum_{n \geq 1} u(n)q^n = \sum_{n \geq 0} (-q)_n^2 q^{n+1},
\]

where \((a)_n = (a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j)\). Andrews [2] previously studied the combinatorial properties of strongly unimodal sequences, where he used the terminology “strictly convex compositions” (with enumeration function \( X_d(n) = u(n) \)).

The fourth author [21] exploited a connection between \( U(q) \) and mixed mock modular forms using a technique developed by the first and the third author [10] to give a precise asymptotic formula for \( u(n) \). As a consequence one has for any \( N \in \mathbb{N}_{>0} \)

\[
u(n) = \frac{1}{8 \cdot 6^\frac{1}{2} n^\frac{3}{4}} e^{\pi \sqrt{\frac{2}{n}}} \left( 1 + \sum_{1 \leq r \leq N} \frac{\beta_r}{n^\frac{1}{2}} + O \left( n^{-\frac{N+1}{2}} \right) \right)
\]
for explicitly computable $\beta_r$. For example, $\beta_1 = \frac{-2\pi^2 + 9}{2\pi\sqrt{2k}}$.

The rank of a strongly unimodal sequence is the number of terms after the maximal term minus the number of terms that precede it, i.e., in the notation above, the rank is $s - 2k + 1$. By letting $w$ (resp. $w^{-1}$) keep track of the terms after (resp. before) a maximal term, we have that $u(m, n)$, the number of size $n$ and rank $m$ strongly unimodal sequences, satisfies (see [13, equation (1.1)])

$$U(w; q) := \sum_{m \in \mathbb{Z}, n \geq 0} u(m, n)w^m q^n = \sum_{n \geq 0} (-wq)_n (-w^{-1}q)_n q^{n+1}.$$  

In this paper we are interested in the the distribution of the unimodal rank statistic. By symmetry, it is clear that

$$u(m, n) = u(-m, n).$$  \hspace{1cm} (1.1)

For this reason, we only consider $m \geq 0$ throughout the article. The following table gives the first few values of $u(m, n)$:

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| 0   | 1 | 1 | 1 | 2 | 2 | 4 | 5 | 7 | 10 | 13 | 17 | 24 | 31 | 40 | 53 | 69 | 88 | 113 | 144 | 183 |
| 1   | 1 | 1 | 1 | 2 | 2 | 4 | 5 | 7 | 10 | 14 | 18 | 25 | 33 | 43 | 56 | 73 | 94 | 121 | 153 |    |
| 2   | 1 | 1 | 2 | 2 | 4 | 5 | 7 | 10 | 14 | 18 | 23 | 30 | 40 | 53 | 69 | 90 |    |    |    |    |
| 3   | 1 | 1 | 2 | 3 | 4 | 6 | 9 | 12 | 16 | 23 | 30 | 40 | 53 | 69 | 90 |    |    |    |    |    |
| 4   | 1 | 1 | 2 | 3 | 5 | 6 | 10 | 13 | 19 | 25 | 34 |    |    |    |    |    |    |    |    |    |

An interesting observation is that the first few non-zero values of $u(m, n)$ for fixed $m$ are equal to the values of the partition function. In particular, $u(m, \frac{1}{2}(m+2)(m+1)+n) = p(n)$ for $0 \leq n \leq m + 1$. The following theorem explains this phenomenon.

**Theorem 1.1.** We have the following generating function for $m \in \mathbb{N}_0$

$$U_m(q) := \sum_{n \geq 1} u(m, n)q^n = \frac{q^{m(n+1)}}{(q)_\infty} \sum_{n \geq 1} \frac{(-1)^n q^{\frac{n(n+1)}{2} + mn}}{1 - q^{n+m}} \left( q^{n+m} - 1 \right).$$  \hspace{1cm} (1.2)

This generating function is closely related to the generating function for partition ranks and cranks. The rank of a partition is its largest part minus the number of its parts. We denote by rank($\lambda$) the rank of a partition $\lambda$. To define the crank of a partition $\lambda$, let $o(\lambda)$ denote the number of ones in $\lambda$ and define $\mu(\lambda)$ to be the number of parts strictly larger than $o(\lambda)$. Then the crank of $\lambda$ is defined as

$$\text{crank}(\lambda) := \begin{cases} 
\text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\
\mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0.
\end{cases}$$

We let $|\lambda|$ denote the the size of a partition (the sum of its parts). We have the following corollary to Theorem 1.1.
Corollary 1.2. We have
\[
\sum_{n \geq 1} u(0, n)q^n = \sum_{n \geq 1} \text{ospt}(n)q^n,
\]
where
\[
\text{ospt}(n) := \sum_{|\lambda|=n \atop \text{crank}(\lambda)>0} \text{crank}(\lambda) - \sum_{|\lambda|=n \atop \text{rank}(\lambda)>0} \text{rank}(\lambda).
\]

The ospt-function was introduced by Andrews, Chan, and Kim [5]. There they gave a combinatorial interpretation of \(\text{ospt}(n)\) in terms of so-called even and odd strings in the partitions of \(n\). From their interpretation, it is clear that \(\text{ospt}(n) \geq 0\). Furthermore, the asymptotic behavior of the ospt-function was determined by the first and third author, who proved in [11, Theorem 1.4] that \(\text{ospt}(n) \sim \frac{p(n)}{4}\). In turn, Chan and Mao studied the combinatorial relationship between the ospt-function and partitions; one of their main results [14, equation (1.9)] proves that \(\text{ospt}(n) < \frac{p(n)}{2}\) for \(n \geq 3\). Furthermore, Chan and Mao showed several additional bounds, including [14, equation (1.8)]\]
\[
\text{ospt}(n) < \frac{p(n)}{4} + \frac{N(0, n)}{2} - \frac{M(0, n)}{4} + \frac{N(1, n)}{2} \quad \text{for } n \geq 7.
\]
Here \(N(m, n)\) (respectively \(M(m, n)\)) denotes the number of partitions of \(n\) with rank (resp. crank) \(m\). In particular, they commented that it might be possible to use their approach to prove \(p(n) \geq 2N(0, n) + 2N(1, n)\), which would give the refinement \(\text{ospt}(n) < \frac{p(n)}{2} - \frac{M(0, n)}{4}\).

Corollary 1.2 provides an alternative combinatorial interpretation of the ospt-function. By examining rank zero strongly unimodal sequences, we obtain the following improvement of Chan and Mao’s inequality.

Theorem 1.3. For \(n \geq 2\) we have
\[
\text{ospt}(n) \leq \frac{p(n) - M(0, n)}{2}.
\]
We note that \(M(0, n)\) is positive for \(n \geq 3\).

One of the striking features of the data in Table 1 is that \(m \mapsto u(m, n)\) appears to be unimodal. In fact, additional numerical data (checked by MAPLE for all \(n \leq 500\)) suggests that a stronger property holds. Recall that a sequence of positive real numbers \(\{a_m\}_{-M \leq m \leq M}\) is log-concave if \(a_m^2 - a_{m-1}a_{m+1} \geq 0\) for all \(-M + 1 \leq m \leq M - 1\). It is a straightforward fact that if \(\{a_m\}\) is symmetric \((a_{-m} = a_m)\) and log-concave, then it is unimodal with peak \(a_0\). We offer the following conjecture.

Conjecture 1.4. For \(n \geq \max(7, \lceil |m|/2 \rceil + 1)\) we have
\[
u(m, n)^2 > u(m - 1, n)u(m + 1, n).
\]

Remarks. 1. Conjecture 1.4 states that \(\{u(m, n)\}_m\) is strictly log-concave for \(n > 6\) (and hence strictly unimodal). The data in Table 1 shows that for \(n \leq 6\) the sequence is log-concave, but not necessarily strict; for example, \(u(1, 6)^2 - u(0, 6)u(2, 6) = 0\).
2. Since by Corollary 1.2 we have that \( u(0, n) = \text{ospt}(n) \), it is natural to ask if there are other combinatorial interpretations of \( u(m, n) \) for fixed \( m \geq 1 \). Such interpretations may give insight into Conjecture 1.4.

As further evidence for the unimodality/log-concavity of the values of \( u(m, n) \), we prove the following asymptotic version.

**Theorem 1.5.** For fixed \( m \in \mathbb{N} \), we have as \( n \to \infty \)

\[
\begin{align*}
    u(m, n) & \sim \frac{1}{16\sqrt{3n}} e^{\pi \sqrt{\frac{2n}{3}}} , \\
    u(m, n) - u(m + 1, n) & \sim \frac{\pi (2m + 1)}{96\sqrt{2n^3}} e^{\pi \sqrt{\frac{2n}{3}}} , \\
    u(m, n)^2 - u(m - 1, n)u(m + 1, n) & \sim \frac{\pi}{768\sqrt{6n^3}} e^{2\pi \sqrt{\frac{2n}{3}}} .
\end{align*}
\]

In particular, Conjecture 1.4 is true for sufficiently large \( n \).

**Remark.** We prove Theorem 1.5 using Wright’s Circle Method [23], which naturally gives asymptotic expansions of the form, for \( N \in \mathbb{N}_0 \),

\[
\begin{align*}
    u(m, n) = \frac{e^{\pi \sqrt{\frac{2n}{3}}}}{16\sqrt{3n}} \left( 1 + \sum_{1 \leq r \leq N} \frac{\alpha_r(m)}{n^{\frac{r}{2}}} + O\left( n^{-\frac{N+1}{2}} \right) \right) .
\end{align*}
\]

Here all of the \( \alpha_r(m) \) are explicitly computable, although the asymptotic terms stated in Theorem 1.5 only require the part of \( \alpha_1 \) that depends on the value of \( m \).

We now consider the values of the rank statistic among all \( \sigma \in U(n) \) for large \( n \). We calculate the moments of the rank, and then appeal to the probabilistic “Method of Moments” in order to describe the limiting distribution of the rank. For \( \ell \in \mathbb{N}_0 \), define

\[
    u_{2\ell}(n) := \sum_{m \in \mathbb{Z}} m^{2\ell} u(m, n) .
\]

Note that (1.1) implies that the analogous odd moments satisfy \( u_{2\ell+1}(n) = 0 \). The following theorem provides the asymptotic behavior of the even moments, where we use the double factorial notation \((2\ell - 1)!! := (2\ell - 1) \cdot (2\ell - 3) \cdots 3 \cdot 1\).

**Theorem 1.6.** For each \( \ell \in \mathbb{N}_0 \), we have

\[
    u_{2\ell}(n) \sim \frac{e^{\pi \sqrt{\frac{2n}{3}}}}{8 \cdot 6^{\frac{\ell}{2}} n^{\frac{\ell}{2}}} (2\ell - 1)!! \left( \frac{6n}{\pi^2} \right)^{\frac{\ell}{2}} .
\]

The shape of this result is explained by the close relationship between partitions into distinct parts and strongly unimodal sequences, as there is a map from pairs \( (\lambda, \mu) \) of such partitions to a strongly unimodal sequence given by

\[
(\lambda, \mu) \mapsto \{\lambda_\ell(\lambda), \ldots, \lambda_2, \lambda_1, \mu_1, \mu_2, \ldots, \mu_\ell(\mu)\} , \tag{1.3}
\]

where \( \ell(\lambda) \) is the number of parts in \( \lambda \). This map is only defined if the largest parts of \( \lambda \) and \( \mu \) are different, and is then in fact two-to-one onto the set of strongly unimodal sequences. Denoting the number of such pairs of total size \( n \) by \( q_2(n) \), with corresponding generating
function $\sum_{n \geq 0} q_2(n)q^n = (-q)^2$ it is known that $u(n) \sim \frac{q_2(n)}{2}$ (for example, this follows immediately from (3.3) and Theorem 4.1 below).

The rank of the unimodal sequence in (1.3) is $\ell(\mu) - \ell(\lambda) \pm 1$ (depending on which partition contributes the peak), and thus it is relevant to understand the typical number of parts in a partition into distinct parts. Let $Q(n)$ denote the set of partitions into distinct parts of size $n$, and let $q(n) := |Q(n)|$ be the enumeration function. Szekeres [22, Theorem 1] proved that for large $n$, if one picks $\lambda \in Q(n)$ uniformly at random, then $\ell(\lambda)$ is normally distributed, with mean $r_0 = \frac{2\sqrt{3} \log (2)}{\pi} \sqrt{n}$ and variance $s^2 = \frac{\sqrt{3}}{\pi} (1 - (\frac{2\sqrt{3} \log (2)}{\pi})^2) \sqrt{n}$.

As a rough estimate, we should therefore expect that for $\sigma \in U(n)$, we have $\text{rank}(\sigma) = \ell(\mu) - \ell(\lambda) \pm 1$ for some $\mu \in Q(n_1)$ and $\lambda \in Q(n_2)$ such that $n_1 \sim n_2 \sim \frac{n}{2}$. This follows from Hardy and Ramanujan’s famous asymptotic formula $\log (q(n)) \sim \pi \sqrt{n}$ (see [18, p. 109]), which implies that almost all $(\mu, \lambda)$ such that $|\mu| = |\lambda| = n$ satisfy $|\mu| \sim |\lambda| \sim \frac{n}{2}$.

Indeed, this prediction is confirmed by Theorem 1.6 as we see that it may be equivalently written as

$$\frac{u_{2\ell}(n)}{u(n) \left(\frac{6n}{\pi^2}\right)^{\frac{1}{4}}} \sim (2\ell - 1)!!.$$ 

This matches the values of the even moments for the standard normal distribution (cf. Example 21.1 in [8]), and we therefore conclude that for large $n$, the rank is normally distributed around zero with variance $\frac{\sqrt{6n}}{\pi}$.

**Corollary 1.7.** For all $x \in \mathbb{R}$, we have

$$\lim_{n \to \infty} \frac{1}{u(n)} \left| \left\{ \sigma \in U(n) : \frac{\text{rank}(\sigma)}{\left(\frac{6n}{\pi^2}\right)^{\frac{1}{4}}} \leq x \right\} \right| = \Phi(x)$$

where $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$.

**Remark.** As a consequence of Corollary 1.7 we have that, for $a, b \in \mathbb{R}$ with $a \leq b$,

$$\lim_{n \to \infty} \frac{1}{u(n)} \left| \left\{ \sigma \in U(n) : a \leq \frac{\text{rank}(\sigma)}{\left(\frac{6n}{\pi^2}\right)^{\frac{1}{4}}} \leq b \right\} \right| = \Phi(b) - \Phi(a),$$

which tends to one as $b \to \infty, a \to -\infty$. This means that for any $\varepsilon > 0$, “almost all” strongly unimodal sequences $\sigma$ have $|\text{rank}(\sigma)| < n^{\frac{3}{2} + \varepsilon}$ (recall that the maximum value of the rank is roughly $\sqrt{|\sigma|}$).
We can use Corollary 1.7 to determine the asymptotic behavior of the absolute moments for the rank. For \( r \in \mathbb{N}_0 \), define the absolute moments
\[
u^r(n) := \sum_{m \in \mathbb{Z}} |m|^r u(m, n).
\]
Note that the even absolute moments are already described by Theorem 1.6, as \( u^2(n) = u_2(n) \).

**Corollary 1.8.** As \( n \to \infty \),
\[
\frac{u^r(n)}{u(n) \left( \frac{6n}{\pi^2} \right)^{\frac{r}{2}}} \sim \frac{2^r}{\sqrt{\pi}} \Gamma \left( \frac{r+1}{2} \right).
\]

**Remark.** Unlike Theorem 1.5 where we can obtain an asymptotic expansion in \( n^{\frac{3}{2}} \) with an arbitrary number of terms using Wright’s Circle Method, we do not have any control over the error terms in Corollary 1.8 due to the weaker notions of convergence used in the Method of Moments.

The paper is organized as follows. In Section 2 we consider the combinatorial properties of strongly unimodal sequences and related generating functions, and prove Theorem 1.1, Corollary 1.2, and Theorem 1.3. In Section 3 we establish Theorem 1.5 giving an asymptotic version of Conjecture 1.4. In Section 4 we determine the asymptotic behavior of the moments of the rank statistic, proving Theorem 1.6, Corollary 1.7, and Corollary 1.8. Finally, in Section 5 we discuss the modularity properties of the generating function for strongly unimodal sequences, and the relation to previously studied examples of mock modular and quantum modular forms.

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**2. PROOF OF THEOREM 1.1, COROLLARY 1.2, AND THEOREM 1.3**

In this section, we give some basic results for the generating function \( U(w; q) \). We prove Theorem 1.1, Corollary 1.2 and Theorem 1.3.

**Proof of Theorem 1.1.** From Entry 3.4.7 of [4] (or equivalently Theorem 4 of [15]) and Lemma 7.9 of [17], one can conclude the following identity,
\[
U(w; q) = \frac{1}{1 + w^{-1}} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1 + w q^n} - \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{w^{-n} q^{\frac{n(n+1)}{2}}}{1 + w q^n} \right) .
\]  
(2.1)

We let \([w^m] F(w; q)\) denote the coefficient of \( w^m \) in \( F(w; q) \), where \( F(w; q) \) is a series in \( w \) and \( q \). Our goal is to determine \([w^m] U(w; q)\). In order guarantee the absolute convergence of the various \( q \)-series that appear throughout the proof, we henceforth assume that \( |q| < |w| < 1 \).
Thus, for series, we obtain
\[
\frac{1}{1 + w^{-1}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^n q^{n(n+1)/2}}{1 + wq^n} = \frac{w}{1 + w} \left( \sum_{n \geq 1} \frac{(-1)^n q^{n(n+1)/2}}{1 + wq^n} + w^{-1} \sum_{n \geq 1} \frac{(-1)^n q^{n(n+1)/2}}{1 + w^{-1}q^n} \right)
\]
\[
= \sum_{j, \ell \geq 0, n \geq 1} (-1)^{n+j+\ell} w^j q^{n(n+1)/2} + n + j + \sum_{j, \ell \geq 0, n \geq 1} (-1)^{n+j+\ell} w^{-j+\ell} q^{n(n+1)/2} + n + j.
\]

Thus, for \( m \geq 0 \), we have
\[
[w^m] \frac{1}{1 + w^{-1}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^n q^{n(n+1)/2}}{1 + wq^n} = \sum_{n \geq 1} (-1)^{n+m} q^{n(n+1)/2} \left( - \sum_{0 \leq j \leq m-1} q^{nj} + \sum_{j \geq 0} q^{nj} \right)
\]
\[
= \sum_{n \geq 1} (-1)^{n+m} q^{n(n+1)/2} \sum_{j \geq m} \sum_{n \geq 1} \frac{(-1)^{n+m} q^{n(n+1)/2} + nm}{1 - q^n}.
\]

The second sum from (2.1) is expanded in a similar manner. Again using the geometric series, we obtain
\[
\frac{1}{1 + w^{-1}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{w^{-n} q^{n(n+1)/2}}{1 + wq^n} = \frac{w}{1 + w} \left( \sum_{n = 1}^{\infty} \frac{w^{-n} q^{n(n+1)/2}}{1 + wq^n} + w^{-1} \sum_{n = 1}^{\infty} \frac{w^n q^{n(n+1)/2}}{1 + w^{-1}q^n} \right)
\]
\[
= \sum_{j, \ell \geq 0, n \geq 1} (-1)^{j+\ell} w^{-n+j+\ell} q^{n(n+1)/2} + n + j + \sum_{j, \ell \geq 0, n \geq 1} (-1)^{j+\ell} w^{n+j-\ell} q^{n(n+1)/2} + n + j.
\]

Thus, for \( m \geq 0 \), we have
\[
[w^m] \frac{1}{1 + w^{-1}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{w^{-n} q^{n(n+1)/2}}{1 + wq^n} = \sum_{n \geq 1} (-1)^{n+m} q^{n(n+1)/2} \left( - \sum_{0 \leq j \leq m+n-1} q^{nj} + \sum_{j \geq \max(0, n-m)} q^{nj} \right)
\]
\[
= - \sum_{n \geq 1} \frac{(-1)^{n+m} q^{n(n+1)/2} (1 - q^{n+m})}{1 - q^n} + \sum_{1 \leq n \leq m} \frac{(-1)^{n+m} q^{n(n+1)/2}}{1 - q^n}
\]
\[
+ \sum_{n \geq m+1} \frac{(-1)^{n+m} q^{n(n+1)/2} + n(n-m)}{1 - q^n}
\]
\[
= \sum_{n \geq 1} \frac{(-1)^{n+m} q^{n(n+1)/2} + nm}{1 - q^n} + \sum_{n \geq m+1} \frac{(-1)^{n+m} q^{n(n+1)/2} + (q^{n(n-m)} - 1)}{1 - q^n}.
\]
By equations (2.1), (2.2), and (2.3) we find that for \( m \geq 0 \),
\[
\sum_{n \geq 1} u(m, n) q^n = \left[ w^m \right] U(w; q) = \frac{1}{(q)\infty} \sum_{n \geq m+1} (-1)^{n+m} q^{\frac{n(n+1)}{2}} \frac{q^{n(n-m)} - 1}{1 - q^n} = \frac{q^{m(m+1)}}{(q)\infty} \sum_{n \geq 1} (-1)^{n} q^{\frac{n(n+1)}{2} + nm} \frac{q^{n(n+m)} - 1}{1 - q^{n+m}}. 
\]

We immediately obtain the relation between unimodal sequences and the ospt-function.

**Proof of Corollary 1.2** The proof follows directly from Theorem 1.1 using the identity
\[
\sum_{n \geq 0} \text{ospt}(n) q^n = \frac{1}{(q)\infty} \sum_{n \geq 1} \left( (-1)^{n+1} q^{\frac{n(n+1)}{2}} \frac{1}{1 - q^n} - (-1)^{n+1} q^{\frac{3(n+1)}{2}} \frac{1}{1 - q^n} \right),
\]
which is Theorem 1 of [5]. □

We conclude this section with the proof of Theorem 1.3.

**Proof of Theorem 1.3** Define a subset of pairs of partitions into distinct parts by \( S := \{ (\mu, \nu) : \ell(\mu) = \ell(\nu) + 1 \} \cup \emptyset \). There is a simple injection that maps a strongly unimodal sequence with rank zero to \( S \). In particular, suppose that \( \sigma = \{a_1, \ldots, a_k, \ldots, a_{2k-1}\} \) has peak \( a_k \), and define \( (\mu, \nu) \in S \) by
\[
\mu := (a_k, a_{k-1}, \ldots, a_1), \quad \nu := (a_{k+1}, \ldots, a_{2k-1}).
\]
This is an invertible injection, as its image consists of all \( (\mu, \nu) \in S \) such that the largest part in \( \mu \) is larger than all parts in \( \nu \). Consider the generating function
\[
S(q) := \sum_{n \geq 0} s(n) q^n = \sum_{(\mu, \nu) \in S} q^{\ell(\mu) + \ell(\nu)} = 1 + \sum_{n \geq 1} q^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}} \frac{1}{(q)_n (q)_{n-1}},
\]
and recall the following representation for the generating functions for partitions (equation (2.2.9) in [1]) and partitions with crank zero (Theorem 5 of [19]):
\[
\sum_{n \geq 1} p(n) q^n = \sum_{n \geq 1} \frac{q^n}{(q)_n^2}, \quad \sum_{n \geq 1} M(0, n) q^n = (1 - q) \sum_{n \geq 1} \frac{q^{n^2+2n}}{(q)_n^2}.
\]
Then
\[
\sum_{n \geq 1} p(n) q^n - 2 \sum_{n \geq 1} s(n) q^n = \sum_{n \geq 1} \frac{q^n}{(q)_n^2} (1 - 2(1 - q^n)) = - \sum_{n \geq 1} \frac{q^n}{(q)_n^2} + \sum_{n \geq 1} \frac{q^{n^2+2n}}{(q)_n^2} = -q + (1 - q) \sum_{n \geq 1} \frac{q^{n^2+2n}}{(q)_n^2} = -q + \sum_{n \geq 1} M(0, n) q^n.
\]
Thus, for $n \geq 2$,
\[ s(n) = \frac{p(n) - M(0, n)}{2}. \tag{2.4} \]
In particular, for $n \geq 2$ we have the inequality
\[ u(0, n) \leq s(n) = \frac{p(n) - M(0, n)}{2}. \]

Remark. We can also determine the asymptotic relationship between $s(n)$ and $\text{ospt}(n)$. Using (2.4), we find that $\text{ospt}(n) \sim \frac{1}{2} s(n) \sim \frac{1}{4} p(n)$, since is is known that $M(0, n) \sim \frac{\sqrt{n}}{\log n}$ \cite[Corollary 2.1]{20}. It is also not difficult to achieve minor improvements of our results by describing the image in $S$ more precisely; for example, by considering partitions in $S$ of the form $\mu = (j, 1)$ and $\nu = (n-j-1)$, for $2 \leq j \leq \lfloor \frac{n-1}{2} \rfloor$, we obtain $u(0, n) \leq s(n) - \lfloor \frac{n-1}{2} \rfloor + 1$ for $n \geq 4$. However, such special cases do not seem to lead to a qualitative improvement of the bound.

3. Proof of Theorem 1.5

Our primary goal in this section is to derive the first several terms in the asymptotic expansion for the coefficients of $U_m(q)$, which we achieve using Wright’s variant of the Hardy-Ramanujan Circle Method \cite{23, 24}. The proof begins with the determination of the first terms in the asymptotic expansion of the generating function in Section 3.1 and then proceeds by estimating its coefficients using a contour integral in Section 3.2. As before we only consider non-negative $m$ throughout.

3.1. Asymptotic expansions of generating functions. Recall Theorem 1.1 and define
\[ V_m(q) := (q)_\infty U_m(q). \]

The bulk of this section is devoted to determining the asymptotic behavior of $V_m(q)$. We recall a formula for the asymptotic expansion of a series that is a consequence of the Euler-MacLaurin summation formula (here $a \in \mathbb{R}^r$, $w \in \mathbb{C}$ with $\text{Re}(w) > 0$, and $F : \mathbb{C}^r \to \mathbb{C}$ is a $C^\infty$-function which, along with all of its derivatives, is of rapid decay)

\[
\sum_{n \in \mathbb{N}_0} F((n + a)w) \sim (-1)^r \sum_{n \in \mathbb{N}_0} F^{(n_1, \ldots, n_r)}(0) \prod_{j=1}^{r} B_{n_j+1}(a_j) \frac{w^{n_j}}{(n_j + 1)!}, \tag{3.1}
\]

\[ + \sum_{\mathcal{S} \subseteq \{1, \ldots, r\}} \frac{(-1)^{|\mathcal{S}|}}{w^{r-|\mathcal{S}|}} \sum_{n_j \in \mathbb{N}_0, j \in \mathcal{S}} \int_{[0, \infty)^{r-|\mathcal{S}|}} \left[ \prod_{j \in \mathcal{S}} \frac{\partial^{n_j}}{\partial x_j^{n_j}} F(x) \right]_{x_j = 0, k \notin \mathcal{S}} \prod_{j \notin \mathcal{S}} dx_k \prod_{j \notin \mathcal{S}} \frac{B_{n_j+1}(a_j)}{(n_j + 1)!} w^{n_j}, \]

where $B_n(x)$ denotes the $n$-th Bernoulli polynomial and throughout the paper we write vectors in bold letters and their components with subscripts. In particular, the one-dimensional case reduces to
\[
\sum_{n \in \mathbb{N}_0} F((n + a)t) \sim \frac{1}{t} \int_0^{\infty} F(x)dx - \sum_{n \geq 0} \frac{B_{n+1}(a)}{(n + 1)!} F^{(n)}(0)t^n.
\]
The following proposition gives the first few terms in the asymptotic expansion of \( V_m \).

**Proposition 3.1.** Suppose that \( m \in \mathbb{N}_0 \). Then as \( \tau \to 0 \)

\[
V_m(q) = \frac{1}{4} + \left( \frac{m^2}{8} - \frac{1}{8} \right) 2\pi i \tau + O(|\tau|^2).
\]

**Proof.** Using finite geometric series, we deduce that

\[
V_m(q) = \sum_{n_1, n_2 \geq 0} (-1)^{n_1+n_2} q^{\frac{1}{2} \left( (n_1+m\frac{1}{2})^2 + \frac{1}{2} (n_2+\frac{1}{2})^2 + 2(n_1+m\frac{1}{2})(n_2+\frac{1}{2}) \right)}.
\]

We then write

\[
V_m(e^{2\pi i \tau}) = \sum_{\varepsilon_1, \varepsilon_2 \in \{0,1\}} (-1)^{\varepsilon_1+\varepsilon_2} \sum_{n_1, n_2 \geq 0} f \left( \sqrt{-2\pi i \tau} \left( n_1 + \frac{m}{2} + \frac{\varepsilon_1}{2}, n_2 + \frac{1}{4} + \frac{\varepsilon_2}{2} \right) \right),
\]

where \( f(x) := e^{-2x_1^2-6x_2^2-8x_1x_2} \). We now apply (3.1) to the sum on \( n_1, n_2 \). All terms except those corresponding to the first vanish due to the \((-1)^{\varepsilon_1+\varepsilon_2}\)-factor, and we are therefore left with

\[
V_m(e^{2\pi i \tau}) = \sum_{\varepsilon_1, \varepsilon_2 \in \{0,1\}} (-1)^{\varepsilon_1+\varepsilon_2} \sum_{n_1, n_2 \geq 0} B_{n_1+1} \left( \frac{m}{2} + \frac{1}{4} + \frac{\varepsilon_1}{2} \right) B_{n_2+1} \left( \frac{1}{4} + \frac{\varepsilon_2}{2} \right)
\]

\[
\times \frac{f(n_1,n_2)(0)}{(n_1+1)!(n_2+1)!} (-2\pi i \tau)^{n_1+n_2}.
\]

Using the facts that \( B_n(x) = (-1)^n B_n(1-x) \) and \( f(n_1,n_2)(0) = 0 \) unless \( n_1 \equiv n_2 \pmod{2} \), we obtain

\[
V_m(e^{2\pi i \tau}) = 2 \sum_{n_1, n_2 \geq 0} \frac{B_{2n_2+1} \left( \frac{1}{4} \right) (B_{2n_1+1} \left( \frac{m}{2} + \frac{1}{4} \right) - B_{2n_1+1} \left( \frac{m}{2} + \frac{3}{4} \right))}{(2n_2 + 1)!(2n_1 + 1)!} f(2n_1,2n_2)(0) (-2\pi i \tau)^{n_1+n_2}.
\]

Computing the first few terms yields the claim. \( \square \)

Proposition 3.1 enables us to determine the asymptotic behavior of \( U_m \) near \( q = 1 \).

**Proposition 3.2.** Assume that \( \tau = u + iv, v = \frac{1}{2\sqrt{6} \pi} \) and \(|u| \leq v\). As \( n \to \infty \) we have

\[
U_m(q) = \sqrt{-i\tau e^{\frac{\pi i}{12}}} \left( \frac{1}{4} + \pi i \left( \frac{m^2}{4} - \frac{11}{48} \right) \right) + O \left( n^{-\frac{5}{4}} e^{\pi \sqrt{v}} \right).
\]

**Proof.** From the well-known transformation law of the \( \eta \)-function (e.g., Theorem 3.1 in [2]), one directly concludes the asymptotic formula

\[
\frac{1}{(q)_{\infty}} = \sqrt{-i\tau e^{\frac{\pi i}{12} (\tau + \frac{1}{4})}} \left( 1 + O \left( e^{-2\pi \sqrt{v}} \right) \right).
\]

Expanding \( e^{\frac{\pi i}{12} \tau} \) gives the claim. \( \square \)

We next bound \( U_m \) away from the dominant pole \( q = 1 \).

**Proposition 3.3.** If \( v = \frac{1}{2\sqrt{6} \pi} \) and \( u \leq v \leq \frac{1}{2} \), then for some \( \delta < 1 \)

\[
|U_m(q)| \ll e^{\pi \delta \sqrt{v}}.
\]
Proof. We estimate
\[ |U_m(q)| \ll \frac{1}{|q|} \sum_{n \geq 1} n|q|^n. \]
The sum on \( n \) can be bounded against
\[ \sum_{n \geq 1} n|q|^n = \frac{|q|}{1 - |q|} \ll |q|^{-2}. \]

To estimate \( \frac{1}{|q|} \), we follow Wright’s argument from Lemma XVI of [23]. For the convenience of the reader we give the details. First, note that since \( |q| < 1 \), we have the logarithmic series expansion
\[ \log(\frac{1}{|q|}) = \sum_{m \geq 1} \frac{|q|^m}{m(1 - |q|^m)}. \]
The magnitude of this expression is bounded by
\[ \left| \log(\frac{1}{|q|}) \right| \leq \sum_{m \geq 1} \frac{|q|^m}{m(1 - |q|^m)} \leq \sum_{m \geq 1} \frac{|q|^m}{m(1 - |q|^m)} - \left( \frac{|q|}{1 - |q|} - \frac{|q|}{1 - |q|} \right). \]

By (3.3) the final sum in (3.4) has an asymptotic expansion given by
\[ \sum_{m \geq 1} \frac{|q|^m}{m(1 - |q|^m)} = \log \left( \frac{1}{|q|} \right) = \frac{\pi}{12v} + O(\log(v)). \]

To estimate the remaining terms in (3.4), we compute Taylor series to obtain
\[ 1 - |q| = 2\pi v (1 + O(v)), \quad |1 - q| = 2\sqrt{2}\pi v (1 + O(v)). \]
Indeed, the second identity holds since
\[ |1 - e^{2\pi i(u + iv)}|^2 = 1 - 2 \cos(2\pi u)e^{-2\pi v} + e^{-4\pi v} \geq 1 - 2 \cos(2\pi v)e^{-2\pi v} + e^{-4\pi v} \]
using the fact that \( v \leq |u| \leq \frac{1}{2} \). The claim now follows by the Taylor expansion
\[ 1 - 2 \cos(2\pi v)e^{-2\pi v} + e^{-4\pi v} = 8\pi^2 v^2 + O(v^3). \]

Plugging into the last two terms of (3.4) and combining with (3.5) implies that
\[ \log \left| \frac{1}{|q|} \right| \leq \log \left( \frac{1}{|q|} \right) - \frac{|q|}{1 - |q|} + \frac{|q|}{1 - e^{2\pi i(u + iv)}} \]
\[ = \frac{\pi}{12v} - \left( \frac{1}{2\pi v} - \frac{1}{2\sqrt{2}\pi v} \right) + O(\log(v)) \]
\[ = \frac{\pi}{12v} \left( 1 - \frac{6}{\pi^2} \left( 1 - \frac{1}{\sqrt{2}} \right) \right) + O(\log(v)). \]
Thus the claim holds for any \( 1 - \frac{6}{\pi^2} (1 - \frac{1}{\sqrt{2}}) = 0.8219 \ldots < \delta < 1 \).

Remark. The proof of Proposition 3.3 also corrects the proof of Corollary 3.4 in the published version of [11].
3.2. Asymptotic behavior of coefficients. Here we use a variant of the Circle Method due to Wright [24]. By Cauchy’s Theorem, we obtain 

\[ u(m, n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{U_m(q)}{q^{n+1}} dq = \int_{-\frac{1}{2}}^{\frac{1}{2}} U_m \left( e^{-\frac{\pi}{\sqrt{6}m}+2\pi iu} \right) e^{\pi \sqrt{\frac{6}{n}}-2\pi i nu} du, \]

where \( \mathcal{C} \) denotes the circle with radius \( e^{-\frac{\pi}{\sqrt{6}m}} \) surrounding the origin counterclockwise.

We then split 

\[ u(m, n) = I'(n) + I''(n) \]

with

\[ I'(n) := \int_{|u| \leq \frac{1}{2\sqrt{6}m}} U_m \left( e^{-\frac{\pi}{\sqrt{6}m}+2\pi iu} \right) e^{\pi \sqrt{\frac{6}{n}}-2\pi i nu} du, \]

\[ I''(n) := \int_{\frac{1}{2\sqrt{6}m} \leq |u| \leq \frac{1}{2}} U_m \left( e^{-\frac{\pi}{\sqrt{6}m}+2\pi iu} \right) e^{\pi \sqrt{\frac{6}{n}}-2\pi i nu} du. \]

It turns out that \( I'(n) \) contributes the asymptotic main term, whereas \( I''(n) \) is part of the asymptotic error term. To see this, we rewrite 

\[ I'(n) = \frac{1}{2\sqrt{6}m} \int_{-1}^{1} U_m \left( e^{\frac{i \pi}{\sqrt{6}m}(-1+iu)} \right) e^{\pi \sqrt{\frac{6}{n}}(1-\pi iu)} du. \]

We next approximate \( I'(n) \) by a Bessel function. For this define for \( s \in \mathbb{R} \)

\[ P_s(n) := \frac{1}{2\pi i} \int_{1-i}^{1+i} w^s e^{\pi \sqrt{\frac{6}{n}}(w+\frac{1}{2})} dw. \]

We then may write, using Proposition 3.2

\[ I'(n) = \frac{\pi}{8 \cdot 2^\frac{1}{4} \cdot 3^\frac{1}{4} \cdot n^\frac{3}{4}} P_{\frac{1}{2}}(n) - \frac{\pi^2 \left( \frac{m^2}{4} - \frac{11}{48} \right)}{12 \cdot 2^\frac{1}{4} \cdot 3^\frac{1}{4} \cdot n^\frac{5}{4}} P_{\frac{3}{2}}(n) + O \left( n^{-\frac{7}{4}} e^{\pi \sqrt{2n}} \right) \]

We have (see Section 5 of [24]) the following approximation

\[ P_s(n) - I_{-s-1} \left( \pi \sqrt{\frac{2n}{3}} \right) \ll e^{\frac{3\pi}{4} \sqrt{\frac{6}{n}}}, \quad \text{as} \quad n \to \infty. \]

We next turn to bounding \( I''(n) \). Using Proposition 3.3 gives

\[ |I''(n)| \ll \int_{\frac{1}{2\sqrt{6}m} \leq |u| \leq \frac{1}{2}} \left| U_m \left( e^{-\frac{\pi}{\sqrt{6}m}+2\pi iu} \right) \right| e^{\pi \sqrt{\frac{6}{n}}} du \ll e^{(1+\delta) \pi \sqrt{\frac{6}{n}}}, \]

the important feature of this bound is that it is exponentially smaller than the initial terms in the asymptotic expansion.

Thus we find that

\[ u(m, n) = \frac{\pi}{8 \cdot 2^\frac{1}{4} \cdot 3^\frac{1}{4} \cdot n^\frac{3}{4}} I_{\frac{1}{2}}(n) - \frac{\pi^2 \left( \frac{m^2}{4} - \frac{11}{48} \right)}{12 \cdot 2^\frac{1}{4} \cdot 3^\frac{1}{4} \cdot n^\frac{5}{4}} I_{\frac{3}{2}}(n) + O \left( n^{-\frac{7}{4}} e^{\pi \sqrt{2n}} \right). \]
To finish the proof, we use the asymptotic expansion of the Bessel function \[3, (4.12.7)\]

\[I_\ell(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left(1 - \frac{4\ell^2 - 1}{8x} + O\left(\frac{1}{x^2}\right)\right) \quad \text{as } x \to \infty.\]

Plugging in, we find that

\[u(m, n) = \frac{e^{\pi\sqrt{\frac{3}{n}+\frac{1}{n}}} \left(1 - \frac{1}{\sqrt{n}} \left(\frac{\pi m^2}{2\sqrt{6}} + \nu\right) + O\left(\frac{1}{n}\right)\right)}{16\sqrt{3n}}\]

where \(\nu\) is an explicit constant that does not depend on \(m\) or \(n\) (the value of \(\nu\) is not needed to conclude the formulas in Theorem 1.5, but for the sake of the interested reader we note that \(\nu = \frac{\sqrt{3}}{\sqrt{2\pi}} - \frac{11\pi}{21\sqrt{6}}\)).

### 4. Proof of Theorem 1.6, Corollary 1.7, and Corollary 1.8

We use Ingham’s Tauberian theorem to obtain the asymptotic main term of the rank moments.

**Theorem 4.1.** Let \(f(q) = \sum_{n \geq 0} a(n) q^n\) be a power series with weakly increasing non-negative coefficients and radius of convergence equal to one. If there exist constants \(A > 0\) and \(\lambda, \alpha \in \mathbb{R}\) such that as \(t \to 0^+\) we have

\[f(e^{-t}) \sim \lambda t^{\alpha} e^{\frac{A}{t}},\]

then, as \(n \to \infty\),

\[a(n) \sim \frac{\lambda}{2\sqrt{\pi}} \frac{A^{\frac{\alpha}{2} + \frac{1}{2}}}{n^{\frac{\alpha}{2} + \frac{1}{2}}} e^{2\sqrt{An}}.\]

In order to apply Theorem 4.1, we need to know that the moments of the unimodal rank are monotonic.

**Proposition 4.2.** If \(\ell, n \in \mathbb{N}_0\), then

\[u_{2\ell}(n+1) \geq u_{2\ell}(n).\]

**Proof.** Recall that the rank moments are defined as

\[u_{2\ell}(n) = \sum_{|\sigma| = n} \text{rank}(\sigma)^{2\ell} = \sum_{|\sigma| = n} |\text{rank}(\sigma)|^{2\ell}.\]

There is a natural injection which we denote by \(\phi\) that sends unimodal sequences of size \(n\) to unimodal sequences of size \(n + 1\) and preserves the rank. In particular, suppose that \(\sigma\) is the unimodal sequence \(\{a_1, \ldots, a_k, \ldots a_s\}\) with \(|\sigma| = n\) and peak \(a_k\). Then set

\[\phi(\sigma) := \{a_1, \ldots, a_k + 1, \ldots a_s\}.\]

It is clear that \(\text{rank}(\phi(\sigma)) = \text{rank}(\sigma)\), and that \(\phi\) is an injection (whose image contains all strongly unimodal sequences of \(n + 1\) whose peak is at least two larger than any part). The moments therefore satisfy

\[u_{2\ell}(n) = \sum_{|\sigma| = n} |\text{rank}(\sigma)|^{2\ell} = \sum_{|\sigma| = n} |\text{rank}(\phi(\sigma))|^{2\ell} \leq \sum_{|\sigma| = n+1} |\text{rank}(\sigma)|^{2\ell} = u_{2\ell}(n+1).\]

The inequality holds because every term in the sum is non-negative. \(\square\)
Remark. Proposition 4.2 can be modified so that it applies to the case of moments for the partition rank and crank statistics. If the rank of a partition is positive, then the injection is defined by increasing the largest part by one, and otherwise, a part of size one is added; the definition for the crank is identical. In all cases the magnitude of the rank or crank statistics do not decrease (in fact, the statistic is only preserved by the injection in the case that the crank is positive and the partition contains ones; in all other cases the statistic changes by at least one). This would allow one to similarly use Ingham’s Tauberian theorem in order to prove the main asymptotic terms in [11]. However, this is not enough to prove the asymptotic inequality for rank and crank moments that is the main result of that paper, as it requires a more detailed asymptotic expansion.

Theorem 1.6 follows from the asymptotic behavior of the moment generating function. For this set

\[ U_{2\ell}(q) := \sum_{n \geq 0} u_{2\ell}(n)q^n. \]

**Theorem 4.3.** As \( t \to 0^+ \), we have

\[ U_{2\ell}(e^{-t}) \sim \frac{(2\ell - 1)!!}{4} t^{-\ell} e^{x^2}. \]

**Proof.** For the proof we use the three-dimensional Euler-Maclaurin summation formula (see (3.1)). We start by writing

\[ U_{2\ell}(q) = \frac{1}{(q)_{\infty}} \left( \delta_{\ell,0} V_0(q) + 2 \sum_{m \geq 1} m^{2\ell} V_m(q) \right), \]

where \( \delta_{\ell,0} \) equals zero unless \( \ell = 0 \), in which case we have one. By Proposition 3.1 we have

\[ V_0(e^{-t}) \sim \frac{1}{4}. \]

Next we write

\[ \sum_{m \geq 1} m^{2\ell} V_m(e^{-t}) = t^{-\ell} \sum_{\delta_1, \delta_2 \in \{0,1\}} (-1)^{\delta_1+\delta_2} \sum_{n \in \mathbb{N}_0^3} F \left( \sqrt{t} \left( n_1 + \frac{\delta_1}{2}, n_2 + \frac{\delta_2}{2}, n_3 + 1 \right) \right), \]

where \( F(x) := x_3^2 e^{-2(x_1 + \frac{\delta_1}{2})^2 - 6x_2^2 - 8(x_1 + \frac{\delta_1}{2})x_2} \). We now apply (3.1) in dimension three. Because of the weighting factor \( (-1)^{\delta_1+\delta_2} \), any term in (3.1) that does not depend on both \( \delta_1 \) and \( \delta_2 \) vanishes, leaving just two sums to consider.

The term corresponding to \( S = \{1, 2\} \) is

\[ t^{-\ell - \frac{1}{2}} \sum_{\delta_1, \delta_2 \in \{0,1\}} (-1)^{\delta_1+\delta_2} \sum_{n_1, n_2 \geq 0} B_{n_1+1} \left( \frac{1}{4} + \frac{\delta_1}{2} \right) B_{n_2+1} \left( \frac{1}{4} + \frac{\delta_2}{2} \right) n_1^{n_1+1} n_2^{n_2+1} t^{-\frac{n_1+n_2}{2}} \int_0^{\infty} F(n_1, n_2, 0)(0, 0, x_3)dx_3. \]

The sum on \( \delta_1, \delta_2 \) evaluates as

\[ \sum_{\delta_1, \delta_2 \in \{0,1\}} (-1)^{\delta_1+\delta_2} B_{n_1+1} \left( \frac{1}{4} + \frac{\delta_1}{2} \right) B_{n_2+1} \left( \frac{1}{4} + \frac{\delta_2}{2} \right) \]

\[ (4.1) \]
The dominant term from (4.1) comes from $n_1 = n_2 = 0$ and contributes
\[
\frac{4}{\sqrt{t}} t^{-\ell} B_1 \left( \frac{1}{4} \right)^2 \int_0^\infty F(0, 0, x^3)dx = \frac{(2\ell - 1)!!}{4\sqrt{2}} t^{-\ell - \frac{3}{2}}.
\]

The first term in (3.1) is of higher order. Thus we get
\[
U_{2\ell} (e^{-t}) \sim \frac{1}{(e^{-t})_\infty} \left( \delta_{t, 0} \frac{1}{4} + 2^{\ell - \frac{3}{2}} \Gamma \left( \ell + \frac{1}{2} \right) t^{-\ell - \frac{3}{2}} \right) \sim \frac{2^{\ell - \frac{3}{2}} \Gamma \left( \ell + \frac{1}{2} \right)}{(e^{-t})_\infty t^{\ell + \frac{1}{2}}}.\]

Now
\[
\frac{1}{(e^{-t})_\infty} \sim \sqrt{\frac{t}{2\pi}} e^\frac{\pi^2}{12t}.
\]
Combining gives the claim.  □

Theorem 1.6 now follows from Theorem 4.1, Proposition 4.2, and Theorem 4.3. The subsequent corollaries are then a straightforward consequence of the “Method of Moments”, which uses the limiting behavior of the moments of a sequence of random variables to determine the limiting distribution. In the following key result $X$ (respectively $X_n$) is a random variable with distribution $\mu$ (resp. $\mu_n$), so that $\mu([a, b]) := P_n \{ \sigma \in U(n) : X_n(\sigma) \leq x \}$.

**Theorem 4.4** (Theorem 30.2 in [8]). Suppose that the distribution of $X$ is determined by its moments, that moments of all orders exist for each $\{X_n\}_{n \geq 1}$, and that $\lim_{n \to \infty} E[X_n^r] = E[X^r]$ for $r \geq 1$. Then $X_n$ converges in distribution to $X$; i.e., if $f$ is bounded and continuous, then
\[
\lim_{n \to \infty} \int_a^b f(x)d\mu_n(x) = \int_a^b f(x)d\mu(x).
\]

**Proof of Corollary 1.7** For each $n$, let $P_n$ denote the uniform probability distribution on $U(n)$, so that each unimodal sequence of size $n$ is chosen with probability $\frac{1}{u(n)}$. Now define a random variable $X_n$ on $U(n)$ by taking the normalized rank; in particular, if the outcome of the random selection is $\sigma \in U(n)$, then
\[
X_n = X_n(\sigma) := \frac{\text{rank}(\sigma)}{(\frac{6n}{\pi})^{\frac{1}{4}}}.\]
Denote the corresponding distribution $X_n$ by $\mu_n$, and distribution function by $F_n$, so
\[
F_n(x) = \mu_n((\infty, x]) := P_n \{ \sigma \in U(n) : X_n(\sigma) \leq x \} = \frac{1}{u(n)} \sum_{m \leq \left(\frac{6n}{\pi}\right)^{\frac{1}{4}} x} u(m, n).
\]

Theorem 1.6 implies that
\[
\lim_{n \to \infty} E[X_n^{2\ell}] = \lim_{n \to \infty} \frac{1}{u(n)} \sum_{\sigma \in U(n)} X_n^{2\ell} = \lim_{n \to \infty} \frac{u_2(n)}{u(n) \left(\frac{6n}{\pi}\right)^{\frac{1}{2}}} = (2\ell - 1)!!.
\]
and we also know by symmetry that \( E[X_{2\ell+1}^2] = 0 \) for \( \ell, n \in \mathbb{N}_0 \). As mentioned in the introduction, these limiting values are the moments for the standard normal random variable \( Z \), which has the well-known distribution \( \Phi \). We now apply Theorem 4.4 to conclude that \( X_n \) converges in distribution to \( Z \). In particular, setting \( f(x) = 1 \) gives that

\[
\lim_{n \to \infty} \frac{1}{u(n)} \sum_{m \leq \left( \frac{6n}{\pi^2} \right)^{\frac{3}{2}}} u(m, n) = \lim_{n \to \infty} \int_{-\infty}^{x} d\mu_n(x) = \Phi(x),
\]

which is precisely the statement of the corollary. □

**Proof of Corollary 1.8.** It is known [8, Problem 21.2] that the absolute moments of the standard normal distribution \( Z \) are given by

\[
E[|Z|^r] = \frac{2^r}{\sqrt{\pi}} \Gamma\left( \frac{r+1}{2} \right).
\]

On the other hand, we have

\[
E[|X_n|^r] = \frac{1}{\left( \frac{6n}{\pi^2} \right)^{\frac{3}{2}}} \sum_{m \in \mathbb{Z}} |m|^r \frac{u(m, n)}{u(n)} = \frac{u^+(n)}{u(n) \left( \frac{6n}{\pi^2} \right)^{\frac{3}{2}}}.
\]

Theorem 4.4 now implies that

\[
\lim_{n \to \infty} E[|X_n|^r] = E[|Z|^r],
\]

which is the claimed result. □

5. (GENERALIZED) QUANTUM MODULAR PROPERTIES

Due to equation (2.1), the function \( U(w; q) \) can be recognized as essentially a mock Jacobi form [12]. Furthermore, Bryson, Ono, Pitman, and the fourth author [13] found \( U(-1; q) \) to be a so-called quantum modular form (the definition of which is given below). It is then reasonable to ask if the functions \( U_m(q) \) enjoy any modular properties. While it is likely too much to ask for them to be mock modular forms, some sort of quantum modular properties are not an unreasonable expectation.

To explain our questions, we first consider a classical example. Andrews, Dyson, and Hickerson [6] defined

\[
\sigma(q) := \sum_{n \geq 0} q^{n(n+1)/2} (-q)_n, \quad \sigma^*(q) := 2 \sum_{n \geq 1} \frac{(-1)^n q^{n^2}}{(q; q^2)_n}.
\]

Note [6, 26] that \( \sigma \) and \( \sigma^* \) can be written as indefinite theta functions

\[
q^{\frac{1}{24}} \sigma(q) = \left( \sum_{n+j \geq 0} + \sum_{n+j < 0} \right) (-1)^{n+j} q^{\frac{3}{2}(n+\frac{1}{6})^2 - j^2},
\]

\[
q^{-\frac{1}{24}} \sigma^*(q) = \left( \sum_{2j+3n \geq 0} + \sum_{2j+3n < 0} \right) (-1)^{n+j} q^{-\frac{3}{2}(n+\frac{1}{6})^2 + j^2}.
\]
Cohen [16] then viewed these functions in the framework of Maass forms. To recall his results, define the coefficients $T(n)$ by

$$\sum_{n \equiv 1 \mod 24} T(n)q^{\frac{n}{24}} := q^{\frac{-1}{24}}\sigma(q) + q^{\frac{-1}{24}}\sigma^*(q)$$

and set, $\tau = u + iv$,

$$\varphi_0(\tau) := v^{\frac{1}{2}} \sum_{n \in \mathbb{Z}\setminus\{0\}} T(n)K_0\left(\frac{2\pi|n|v}{24}\right)e^{\frac{2\pi inu}{24}},$$

where $K_0$ is the $K$-Bessel function of weight zero of the second kind. Cohen then proved that $\varphi_0$ is a Maass form of weight zero on $\Gamma_0(2)$ (with some multiplier) and eigenvalue $\frac{1}{4}$. Maass forms transform like modular forms. However, instead of being meromorphic they are eigenfunctions under the Laplace operator

$$\Delta := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right).$$

The connection to Maass forms directly gives that $\sigma$ is a quantum modular form. Roughly speaking Zagier [25] defined quantum modular forms to be functions $f : \mathbb{Q} \to \mathbb{C}$ ($\mathbb{Q} \subset \mathbb{Q}$) such that for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, some subgroup of $\text{SL}_2(\mathbb{Z})$, $\chi$ a certain multiplier,

$$f(\tau) - \chi(M)^{-1}(c\tau + d)^{-k}f(M\tau)$$

can be extended to an open subset of $\mathbb{R}$ and is real-analytic there.

Zwegers [26] considered general indefinite theta series of the shape satisfied by $\sigma$ and $\sigma^*$ and associated functions similar to that of $\phi_0$. For convenience, suppose that $\Phi^+$ and $\Phi^-$ are the indefinite theta series and the associated function is $\Phi$. By construction, $\Phi$ is harmonic, but $\Phi$ may not have any modular properties. However, Zwegers was able to “complete” $\Phi$ to a function $\hat{\Phi}$ that satisfies a modular transformation, but may no longer be harmonic. In the case that $\Phi = \hat{\Phi}$, the functions $\Phi^+$ and $\Phi^-$ are quantum modular forms due to $\Phi$ being a Maass form (see Theorem 2.8 of [9] for a precise statement).

From equation (3.2), we have an indefinite theta representation of $V_m(q)$ and so we can apply Zwegers’ machinery. However, in doing so we find that we are in the case that the associated harmonic object is not equal to its modular completion. For this reason, we suspect that our functions are not quantum modular forms in the sense above.

We pose two problems. First, determine any generalized quantum modular properties of $U_m(q)$. Second, more generally, in the case of Zwegers’ construction when $\Phi \neq \hat{\Phi}$, determine any generalized quantum modular properties of $\Phi^+$ and $\Phi^-$. 

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