Abstract  Let $L_1$ be a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates and $L_2$ a second order divergence form elliptic operator with complex bounded measurable coefficients. A typical example of $L_1$ is the Schrödinger operator $-\Delta + V$, where $\Delta$ is the Laplace operator on $\mathbb{R}^n$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let $H^p_{L_1}(\mathbb{R}^n)$ be the Hardy space associated to $L_i$ for $i \in \{1, 2\}$. In this paper, the authors prove that the Riesz transform $D(L_1^{-1/2})$ is bounded from $H^p_{L_i}(\mathbb{R}^n)$ to the classical weak Hardy space $WH^p(\mathbb{R}^n)$ in the critical case that $p = n/(n+1)$. Recall that it is known that $D(L_1^{-1/2})$ is bounded from $H^p_{L_i}(\mathbb{R}^n)$ to the classical Hardy space $H^p(\mathbb{R}^n)$ when $p \in (n/(n+1), 1]$.

Keywords  Riesz transform · Davies-Gaffney estimate · Schrödinger operator · Second order elliptic operator · Hardy space · Weak Hardy space

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1 Introduction

The Hardy spaces, as a suitable substitute of Lebesgue spaces $L^p(\mathbb{R}^n)$ when $p \in (0, 1]$, play an important role in various fields of analysis and partial differential equations. For example, when $p \in (0, 1]$, the Riesz transform $\nabla(-\Delta)^{-1/2}$ is not bounded on $L^p(\mathbb{R}^n)$, but bounded on the Hardy space $H^p(\mathbb{R}^n)$, where $\Delta$ is the Laplacian operator $\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $\nabla$ is the gradient operator $(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$ on $\mathbb{R}^n$. It is well known that the classical Hardy spaces $H^p(\mathbb{R}^n)$ are essentially related to $\Delta$, which has been intensively studied in, for example, [7, 14, 30, 32, 33] and their references.

In recent years, the study of Hardy spaces associated to differential operators inspires great interests; see, for example, [2, 3, 4, 11, 12, 13, 16, 18, 19, 20, 9] and their references.

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In particular, Auscher, Duong and McIntosh [2] first introduced the Hardy space $H^1_{L^1}(\mathbb{R}^n)$ associated to $L$, where the heat kernel generated by $L$ satisfies a pointwise Poisson type upper bound. Later, Duong and Yan [10, 11] introduced its dual space $\text{BMO}_L(\mathbb{R}^n)$ and established the dual relation between $H^1_L(\mathbb{R}^n)$ and $\text{BMO}_{L^*}(\mathbb{R}^n)$, where $L^*$ denotes the adjoint operator of $L$ in $L^2(\mathbb{R}^n)$. Yan [35] further introduced the Hardy space $H^p_L(\mathbb{R}^n)$ for some $p \in (0, 1]$ but near to 1 and generalized these results to $H^p_L(\mathbb{R}^n)$ and their dual spaces. A theory of the Orlicz-Hardy space and its dual space associated to such $L$ were developed in [25, 22].

Moreover, for the Schrödinger operator $-\Delta + V$, Dziubański and Zienkiewicz [12, 13] first introduced the Hardy spaces $H^p_{-\Delta + V}(\mathbb{R}^n)$ with the nonnegative potential $V$ belonging to the reverse Hölder class $B_q(\mathbb{R}^n)$ for certain $q \in (1, \infty)$. As a special case, the Hardy space $H^p_{-\Delta + V}(\mathbb{R}^n)$ associated with $-\Delta + V$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $p \in (0, 1]$ but near to 1 was also studied in, for example, [11, 16, 25, 36, 37, 21, 8]. More generally, for nonnegative self-adjoint operators $L$ satisfying the Davies-Gaffney estimates, Hofmann et al. [16] introduced a new Hardy space $H^1_L(\mathbb{R}^n)$. In particular, when $L \equiv -\Delta + V$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$, Hofmann et al. originally showed that the Riesz transform $\nabla (L^{-1/2})$ is bounded from $H^1_L(\mathbb{R}^n)$ to the classical Hardy space $H^1(\mathbb{R}^n)$. These results in [16] were further extended to the Orlicz-Hardy space and its dual space in [21]. In particular, as a special case of [21, Theorem 6.3], it was proved that $\nabla (-\Delta + V)^{-1/2}$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ is bounded from the Hardy space $H^p_{-\Delta + V}(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ if $p \in \left(\frac{n}{n+1}, 1\right]$. Also, Auscher and Russ [4] studied the Hardy space $H^1_L$ on strongly Lipschitz domains associated with a second order divergence form elliptic operator $L$ whose heat kernels have the Gaussian upper bounds and certain regularity. Hofmann and Mayboroda [18, 19] and Hofmann et al. [20] introduced the Hardy and Sobolev spaces associated to a second order divergence form elliptic operator $L$ on $\mathbb{R}^n$ with complex bounded measurable coefficients. Notice that, for the second order divergence form elliptic operator $L$, the kernel of the heat semigroup may fail to satisfy the Gaussian upper bound estimate and, moreover, $L$ may not be nonnegative self-adjoint in $L^2(\mathbb{R}^n)$. Hofmann et al. [20] also proved that the associated Riesz transform $\nabla L^{-1/2}$ is bounded from $H^p_{L^1}(\mathbb{R}^n)$ to the classical Hardy space $H^p(\mathbb{R}^n)$ with $p \in (\frac{n}{n+1}, 1]$ which was also independently obtained by Jiang and Yang in [23, Theorem 7.4]. Moreover, a theory of the Orlicz-Hardy space and its dual space associated to $L$ were developed in [23, 24].

Recently, the Hardy space $H^1_{(-\Delta)^{2}+V^2}(\mathbb{R}^n)$ associated to the Schrödinger-type operators $(-\Delta)^{2}+V^2$ with $0 \leq V$ satisfying the reverse Hölder inequality was also studied in [5]. Moreover, the Hardy space $H^p_{T^1}(\mathbb{R}^n)$ associated to a one-to-one operator of type $\omega$ satisfying the $k$-Davies-Gaffney estimate and having a bounded $H_\infty$ functional calculus was introduced in [6], where $k \in \mathbb{N}$. Notice that when $k = 1$, the $k$-Davies-Gaffney estimate is just the Davies-Gaffney estimate. Typical examples of such operators include the $2k$-order divergence form homogeneous elliptic operator $T_1$ with complex bounded measurable coefficients and the $2k$-order Schrödinger-type operator $T_2 \equiv (-\Delta)^{k}+V^k$, where $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. It was further proved that the associated Riesz transform $\nabla^k T_i^{-1/2}$ for $i \in \{1, 2\}$ is bounded from $H^p_{T^1}(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ with $p \in \left(\frac{n}{n+k}, 1\right]$ in [6].

On the other hand, the weak Hardy space $W^1_{H^1}(\mathbb{R}^n)$ was first introduced by Fefferman and Soria in [15]. Then, Liu [26] studied the weak $W^p_H(\mathbb{R}^n)$ space for $p \in (0, \infty)$ and
established a weak atomic decomposition for \( p \in (0, 1] \). Liu in [26] also showed that the \( \delta \)-Calderón-Zygmund operator is bounded from \( H^p(\mathbb{R}^n) \) to \( WH^p(\mathbb{R}^n) \) with \( p = n/(n + \delta) \), which was extended to the weighted weak Hardy spaces in [29].

Let \( L_1 \) be a nonnegative self-adjoint operator in \( L^2(\mathbb{R}^n) \) satisfying the Davies-Gaffney estimates and \( L_2 \) a second order divergence form elliptic operator with complex bounded measurable coefficients. A typical example of \( L_1 \) is the Schrödinger operator \(-\Delta + V\), where \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n) \). Let \( H^p_{L_i}(\mathbb{R}^n) \) be the Hardy space associated to \( L_i \) for \( i \in \{1, 2\} \).

In this paper, we prove that the Riesz transform \( D(L_1^{-1/2}) \) is bounded from \( H^p_{L_1}(\mathbb{R}^n) \) to the weak Hardy space \( WH^p(\mathbb{R}^n) \) in the critical case that \( p = n/(n + 1) \). To be precise, we have the following general result.

**Theorem 1.1.** Let \( p \equiv n/(n + 1) \), \( L_1 \) be a nonnegative self-adjoint operator in \( L^2(\mathbb{R}^n) \) satisfying the assumptions \((A_1)\) and \((A_2)\) as in Section 2 and \( D \) the operator satisfying the assumptions \((B_1)\), \((B_2)\) and \((B_3)\) as in Section 2. Then the operator \( D(L_1^{-1/2}) \) is bounded from \( H^p_{L_1}(\mathbb{R}^n) \) to the classical weak Hardy space \( WH^p(\mathbb{R}^n) \). Moreover, there exists a positive constant \( C \) such that for all \( f \in H^p_{L_1}(\mathbb{R}^n) \),

\[
\left\| D(L_1^{-1/2})f \right\|_{WH^p(\mathbb{R}^n)} \leq C \| f \|_{H^p_{L_1}(\mathbb{R}^n)}.
\]

As an application of Theorem 1.1, we obtain the boundedness of \( \nabla(-\Delta + V)^{-1/2} \) with \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n) \) from \( H^p_{-\Delta + V}(\mathbb{R}^n) \) to the classical weak Hardy space \( WH^p(\mathbb{R}^n) \) in the critical case that \( p = n/(n + 1) \) as follows.

**Corollary 1.1.** Let \( p \equiv n/(n + 1) \) and \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n) \). Then the Riesz transform \( \nabla(-\Delta + V)^{-1/2} \) is bounded from \( H^p_{-\Delta + V}(\mathbb{R}^n) \) to \( WH^p(\mathbb{R}^n) \). Moreover, there exists a positive constant \( C \) such that for all \( f \in H^p_{-\Delta + V}(\mathbb{R}^n) \),

\[
\left\| \nabla(-\Delta + V)^{-1/2}f \right\|_{WH^p(\mathbb{R}^n)} \leq C \| f \|_{H^p_{-\Delta + V}(\mathbb{R}^n)}.
\]

On the Riesz transform defined by the second order divergence form elliptic operator with complex bounded measurable coefficients, we also have the following endpoint boundedness in the critical case that \( p \equiv n/(n + 1) \).

**Theorem 1.2.** Let \( p \equiv n/(n + 1) \) and \( L_2 \) be the second order divergence form elliptic operator with complex bounded measurable coefficients. Then the Riesz transform \( \nabla(L_2^{-1/2}) \) is bounded from \( H^p_{L_2}(\mathbb{R}^n) \) to \( WH^p(\mathbb{R}^n) \). Moreover, there exists a positive constant \( C \) such that for all \( f \in H^p_{L_2}(\mathbb{R}^n) \),

\[
\left\| \nabla(L_2^{-1/2})f \right\|_{WH^p(\mathbb{R}^n)} \leq C \| f \|_{H^p_{L_2}(\mathbb{R}^n)}.
\]

Recall that the second order divergence form elliptic operator with complex bounded measurable coefficients may not be nonnegative self-adjoint operator in \( L^2(\mathbb{R}^n) \). Thus, we cannot deduce the conclusion of Theorem 1.2 from Theorem 1.1. However, if \( L \) is a second order divergence form elliptic operator with real symmetric bounded measurable coefficients, then \( L \) satisfies the assumptions of both Theorem 1.1 and Theorem 1.2.
We prove Theorems 1.1 and 1.2 by using the characterization of $WHP(p)$ in terms of the radial maximal function, namely, we need estimate the weak $L^p$ quasi-norm of the radial maximal function of the Riesz transform acting on the atoms or molecules of the Hardy spaces $H^p_{L_1}(\mathbb{R}^n)$. Unlike the proof of the endpoint boundedness of the classical Riesz transform $\nabla(-\Delta)^{-1/2}$, whose kernel has the pointwise size estimate and regularity, the strategy to show Theorems 1.1 and 1.2 is to divide the radial maximal function into two parts by the time $t$ based on the radius of the associated balls of atoms or molecules and then estimate each part via using $L^2$ off-diagonal estimates (see [17, 20] or Lemma 2.1 below).

This paper is organized as follows. In Section 2, we describe some assumptions on the operator $L_1$; then we recall some notion and properties concerning the Hardy spaces associated to $L_1$ and second order divergence form elliptic operator $L_2$ with complex bounded measurable coefficients. We also recall the definition of weak Hardy spaces and present some technical lemmas which are used later in the next section. Section 3 is devoted to the proof Theorem 1.1, Corollary 1.1, and Theorem 1.2. In Section 4, a similar result on the Riesz transforms defined by higher order divergence form homogeneous elliptic operators with complex bounded measurable coefficients or Schrödinger-type operators is also presented.

Finally, we make some conventions on the notation. Throughout the whole paper, we always let $\mathbb{N} \equiv \{1, 2, \cdots \}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. We use $C$ to denote a positive constant, that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as $C_0$, do not change in different occurrences. If $f \leq Cg$, we then write $f \lesssim g$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$. For all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r) \equiv \{y \in \mathbb{R}^n : |x - y| < r\}$ and $\alpha B(x, r) \equiv B(x, \alpha r)$ for any $\alpha > 0$. Also, for any set $E \subset \mathbb{R}^n$, we use $E^c$ to denote the set $\mathbb{R}^n \setminus E$ and $\chi_E$ the characteristic function of $E$.

## 2 Preliminaries

We begin with recalling some known results on the Hardy spaces associated to operators and the weak Hardy spaces.

Let $L_1$ be a linear operator initially defined in $L^2(\mathbb{R}^n)$ satisfying the following assumptions:

(A1) $L_1$ is nonnegative self-adjoint;

(A2) The semigroup $\{e^{-tL_1}\}_{t>0}$ generated by $L_1$ is analytic on $L^2(\mathbb{R}^n)$ and satisfying the Davies-Gaffney estimates, namely, there exist positive constants $C_1$ and $C_2$ such that for all closed sets $E, F \subset \mathbb{R}^n$, $t \in (0, \infty)$ and $f \in L^2(\mathbb{R}^n)$ supported in $E$,

\begin{equation}
\|e^{-tL_1}f\|_{L^2(F)} \leq C_1 \exp \left\{ -\frac{[\text{dist}(E, F)]^2}{C_2 t} \right\} \|f\|_{L^2(E)},
\end{equation}

where and in what follows, $\text{dist}(E, F) \equiv \inf_{x \in E, y \in F} |x - y|$ is the distance between $E$ and $F$.

Typical examples of operators satisfying assumptions (A1) and (A2) include the second
order divergence form elliptic operator with real symmetric bounded measurable coefficients and the Schrödinger operator $-\Delta + V$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Let $\Gamma(x) \equiv \{(y, t) \in \mathbb{R}^n \times (0, \infty) : |x - y| < t\}$ be the cone with the vertex $x \in \mathbb{R}^n$. For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the $L_1$-adapted square function $S_{L_1}f(x)$ is defined by

$$S_{L_1}f(x) \equiv \left\{ \int_{\Gamma(x)} |t^2L_1 e^{-t^2L_1} f(y)|^2 \frac{dy \, dt}{t^{n+1}} \right\}^{1/2}.$$ 

As in [16, 21], we define the Hardy space $H^p_{L_1}(\mathbb{R}^n)$ associated to the operator $L_1$ as follows.

**Definition 2.1** ([16, 21]). Let $p \in (0, 1]$ and $L_1$ be an operator defined in $L^2(\mathbb{R}^n)$ satisfying the assumptions $(A_1)$ and $(A_2)$. A function $f \in L^2(\mathbb{R}^n)$ is said to be in $H^p_{L_1}(\mathbb{R}^n)$ if $S_{L_1}f \in L^p(\mathbb{R}^n)$; moreover, define $\|f\|_{H^p_{L_1}(\mathbb{R}^n)} \equiv \|S_{L_1}f\|_{L^p(\mathbb{R}^n)}$. The Hardy space $H^p_{L_1}(\mathbb{R}^n)$ is then defined to be the completion of $\mathbb{H}^p_{L_1}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^p_{L_1}(\mathbb{R}^n)}$.

For all $p \in (0, 1]$ and $M \in \mathbb{N}$, a function $a \in L^2(\mathbb{R}^n)$ is called a $(p, 2, M)_{L_1}$-atom if there exists a function $b \in D(L^M_1)$ and a ball $B \equiv B(x_B, r_B) \subset \mathbb{R}^n$ such that

(i) $a = L^M_1 b$;

(ii) for each $\ell \in \{0, 1, \cdots, M\}$, $\text{supp} L^\ell_1 b \subset B$;

(iii) for all $\ell \in \{0, 1, \cdots, M\}$,

$$\left\| \left( r_B^2 L_1 \right)^k b \right\|_{L^2(\mathbb{R}^n)} \leq r_B^{2M+n(\frac{1}{2} - \frac{1}{p})}.$$ 

We then have the following atomic decomposition of $H^p_{L_1}(\mathbb{R}^n)$.

**Theorem 2.1** ([16, 21]). Let $p \in (0, 1]$. Suppose that $M \in \mathbb{N}$ and $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$. Then for all $f \in L^2(\mathbb{R}^n) \cap H^p_{L_1}(\mathbb{R}^n)$, there exist a sequence $\{a_j\}_{j=0}^\infty$ of $(p, 2, M)_{L_1}$-atoms and a sequence $\{\lambda_j\}_{j=0}^\infty$ of numbers such that $f = \sum_{j=0}^\infty \lambda_j a_j$ in both $H^p_{L_1}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, and $\|f\|_{H^p_{L_1}(\mathbb{R}^n)} \sim \left( \sum_{j=0}^\infty |\lambda_j|^p \right)^{1/p}$.

For the second order divergence form operator, the associated Hardy space were studied in [18, 19, 20, 23]. More precisely, let $L_2 = -\text{div}(A\nabla)$ be a second order divergence form elliptic operator with complex bounded measurable coefficients. We say that $L_2$ is elliptic if the matrix $A \equiv \{a_{i,j}\}_{i,j=1}^n$ satisfying the elliptic condition, namely, there exist positive constants $0 < \lambda_0 \leq \Lambda < \infty$ such that $\lambda_0 |\xi|^2 \leq \Re(A\xi \cdot \xi)$ and $|A\xi \cdot \xi| \leq \Lambda |\xi|^2$, where for any $z \in \mathbb{C}$, $\Re z$ denotes the real part of $z$.

**Definition 2.2** ([18, 20, 23]). Let $p \in (0, 1]$ and $L_2$ be the second order divergence form elliptic operator with complex bounded measurable coefficients. A function $f \in L^2(\mathbb{R}^n)$ is said to be in $\mathbb{H}^p_{L_2}(\mathbb{R}^n)$ if $S_{L_2} f \in L^p(\mathbb{R}^n)$; moreover, define $\|f\|_{\mathbb{H}^p_{L_2}(\mathbb{R}^n)} \equiv \|S_{L_2} f\|_{L^p(\mathbb{R}^n)}$. The Hardy space $H^p_{L_2}(\mathbb{R}^n)$ is then defined to be the completion of $\mathbb{H}^p_{L_2}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^p_{L_2}(\mathbb{R}^n)}$. 

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Recall that in [20, 23], for all $p \in (0, 1]$, $\epsilon \in (0, \infty)$ and $M \in \mathbb{N}$, a function $A \in L^2(\mathbb{R}^n)$ is called an $(H^p_{L^2}, \epsilon, M)$-molecule if there exists a ball $B \equiv B(x_B, r_B) \subset \mathbb{R}^n$ such that

(i) for each $\ell \in \{1, \ldots, M\}$, $A$ belongs to the range of $L^\ell_2$ in $L^2(\mathbb{R}^n);$ 
(ii) for all $i \in \mathbb{Z}_+$ and $\ell \in \{0, 1, \ldots, M\}$, 

\[
\left\|\left(\frac{r_B^2}{L^\ell_2}\right)^{-\ell} A\right\|_{L^2(S_i(B))} \leq (2^i r_B)^{n(\frac{1}{2} - \frac{1}{p})} 2^{-\epsilon},
\]

where $S_0(B) \equiv B$ and $S_i(B) \equiv 2^i B \setminus 2^{i-1}B$ for all $i \in \mathbb{N}$.

Assume that $\{m_j\}_j$ is a sequence of $(H^p_{L^2}, \epsilon, M)$-molecules and $\{\lambda_j\}_j$ is a sequence of numbers satisfying $\sum_j |\lambda_j|^p < \infty$. For any $f \in L^2(\mathbb{R}^n)$, if $f = \sum_j \lambda_j m_j$ in $L^2(\mathbb{R}^n)$, then $\sum_j \lambda_j m_j$ is called a molecular $(H^p_{L^2}, 2, \epsilon, M)$-representation of $f$. The molecular Hardy space $H^p_{L^2, mol, M}(\mathbb{R}^n)$ is then defined to be the completion of the space 

\[
\mathbb{H}^p_{L^2, mol, M}(\mathbb{R}^n) \equiv \{f : f \text{ has a molecular } (H^p_{L^2}, 2, \epsilon, M)\text{-representation}\}
\]

with respect to the quasi-norm 

\[
\|f\|_{H^p_{L^2, mol, M}(\mathbb{R}^n)} \equiv \inf \left\{ \left( \sum_{j=0}^\infty |\lambda_j|^p \right)^{1/p} : f = \sum_{j=0}^\infty \lambda_j A_j \text{ is a molecular } (H^p_{L^2}, 2, \epsilon, M)\text{-representation} \right\},
\]

where the infimum is taken over all the molecular $(H^p_{L^2}, 2, \epsilon, M)$-representations of $f$ as above.

We have the following molecular characterization of $H^p_{L^2}(\mathbb{R}^n)$.

**Theorem 2.2** ([20, 23]). Let $p \in (0, 1]$. Suppose that $M > \frac{n}{2} \left(\frac{1}{2} - \frac{1}{p}\right)$ and $\epsilon > 0$. Then $H^p_{L^2}(\mathbb{R}^n) = H^p_{L^2, mol, M}(\mathbb{R}^n)$. Moreover, $\|f\|_{H^p_{L^2}(\mathbb{R}^n)} \sim \|f\|_{H^p_{L^2, mol, M}(\mathbb{R}^n)}$, where the implicit constants depend only on $M, n, p, \epsilon$ and the constants appearing in the ellipticity.

We now recall the definition of the weak Hardy space (see, for example, [15, 26, 27]). Let $p \in (0, 1]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with support in the unit ball $B(0, 1)$. The weak Hardy space $W H^p(\mathbb{R}^n)$ is defined to be the space

\[
\left\{ \varphi \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{W H^p(\mathbb{R}^n)} \equiv \sup_{\alpha > 0} \left( \alpha^p \left( \sup_{t > 0} |\varphi_t \ast f(x)| > \alpha \right) \right)^{1/p} < \infty \right\}.
\]

Let $L_1$ be a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the assumptions (A$_1$) and (A$_2$). Following [1], let the operator $D$ be a linear operator defined densely in $L^2(\mathbb{R}^n)$ and satisfy the following assumptions:

(B$_1$) $DL_1^{-1/2}$ is bounded on $L^2(\mathbb{R}^n);$ 
(B$_2$) the family of operators, $\{\sqrt{t}De^{-tL_1}\}_{t>0}$, satisfy the Davies-Gaffney estimates as in (2.1);
such that for all closed sets $E$ for example, we know that for all $t > 0$, \( \| \sum_{j \in \mathbb{N}} |\{x \in \mathbb{R}^n : |f_j| > \alpha\}| \leq C_2 \alpha^{-p} \). Then, for all $\alpha \in (0, \infty)$,

\[
\left\| \sum_{j \in \mathbb{N}} \lambda_j f_j(x) \right\|_{L^2(\mathbb{R}^n)} \leq \tilde{C} \frac{2 - p}{1 - p} \alpha^{-p} \sum_j |\lambda_j|^p.
\]

**Lemma 2.2** ([27, 31]). Let $p \in (0, 1)$ and \( \{f_j\}_j \) be a sequence of measurable functions. If \( \sum_j |\lambda_j|^p < \infty \) and there exists a positive constant $\tilde{C}$ such that for all \( \{f_j\}_j \) and $\alpha \in (0, \infty)$, 

\[
\left\| \sum_{j \in \mathbb{N}} \lambda_j f_j(x) \right\|_{L^2(\mathbb{R}^n)} \leq \tilde{C} \frac{2 - p}{1 - p} \alpha^{-p} \sum_j |\lambda_j|^p.
\]

**Lemma 2.3** ([1, 17]). Let $L_1$ be a nonnegative self-adjoint operator satisfying the assumptions $(A_1)$ and $(A_2)$ and $D$ the operator satisfying the assumptions $(B_1)$, $(B_2)$ and $(B_3)$. Let \( M \in \mathbb{N} \). Then there exists a positive constant $C$, depending on $M$, such that for all closed sets $E$, $F$ in $\mathbb{R}^n$ with \( \text{dist}(E, F) > 0 \), \( f \in L^2(\mathbb{R}^n) \) supported in $E$ and $t \in (0, \infty)$,

\[
\left\| DL_1^{-1/2} (I - e^{-tL_1})^M f \right\|_{L^2(F)} \leq C \left( \frac{t}{[\text{dist}(E, F)]^2} \right)^{M} \| f \|_{L^2(E)}
\]
and

\[(2.5) \quad \left\| DL_1^{-1/2} \left( tL_1 e^{-tL_1} \right)^M f \right\|_{L^2(F)} \leq C \left( \frac{t}{\text{dist}(E, F)^2} \right)^M \| f \|_{L^2(E)}. \]

Moreover, if \( L_2 \) is a second order divergence form elliptic operator with complex bounded measurable coefficients, then (2.4) and (2.5) still hold when \( D \) and \( L_1 \) are replaced, respectively, by the gradient operator \( \nabla \) and \( L_2 \).

3 Proofs of main results

In this section, we show Theorem 1.1, Corollary 1.1 and Theorem 1.2.

**Proof of Theorem 1.1.** Let \( p \equiv \frac{n}{n+1} \). By the density of \( H^p_{L_1} (\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) in \( H^p_{L_1} (\mathbb{R}^n) \), we only need consider \( f \in H^p_{L_1} (\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \). Let \( M \in \mathbb{N} \) and \( M > \max \{ \frac{1}{2} + \frac{n}{4}, 1 \} \). By Theorem 2.1, we know that there exist a sequence \( \{ a_j \}_j \) of \( (p, 2, M)_{L_1} \)-atoms and a sequence \( \{ \lambda_j \}_j \) of numbers such that

\[(3.1) \quad f = \sum_j \lambda_j a_j \]

in \( L^2(\mathbb{R}^n) \) and \( \| f \|_{H^p_{L_1} (\mathbb{R}^n)} \sim \{ \sum_j |\lambda_j|^p \}_j^{1/p} \). To show Theorem 1.1, by (3.1) and the definition of \( WH^p(\mathbb{R}^n) \), we see that it suffices to prove that for all \( \alpha \in (0, \infty) \),

\[(3.2) \quad \left\{ x \in \mathbb{R}^n : \sup_{0 < t < \infty} \varphi_t * \left( \sum_j \lambda_j DL_1^{-1/2} a_j \right)(x) > \alpha \right\} \leq \frac{1}{\alpha^p} \sum_j |\lambda_j|^p, \]

where \( \varphi \in C_c^\infty (\mathbb{R}^n) \) satisfies \( \text{supp} \varphi \subset B(0, 1) \), and for all \( x \in \mathbb{R}^n \) and \( t \in (0, \infty) \), \( \varphi(t) \equiv \frac{1}{t} \varphi \left( \frac{x}{t} \right) \). In order to prove (3.2), by Lemma 2.2, it suffices to show that for any \( (p, 2, M)_{L_1} \)-atom \( a \) associated with the ball \( B \equiv B(x_B, r_B) \) and \( \alpha \in (0, \infty) \),

\[
\left\{ x \in \mathbb{R}^n : \sup_{0 < t < \infty} \varphi_t * \left( DL_1^{-1/2} a \right)(x) > \alpha \right\} \lesssim \frac{1}{\alpha^p}.
\]

Let \( \mathcal{M} \) be the *Hardy-Littlewood maximal function*. It is easy to see that

\[
\sup_{0 < t < \infty} \left| \varphi_t * (DL_1^{-1/2} a) \right| \lesssim \mathcal{M}(DL_1^{-1/2} a).
\]

Then by Chebyshev’s inequality, Hölder’s inequality, the \( L^2(\mathbb{R}^n) \)-boundedness of \( \mathcal{M} \), the \( L^2(\mathbb{R}^n) \)-boundedness of \( DL_1^{-1/2} \) via \( (B_1) \), and (2.2), we know that

\[
\left\{ x \in 16B : \sup_{0 < t < \infty} \left| \varphi_t * \left( DL_1^{-1/2} a \right)(x) \right| > \alpha \right\} \lesssim \frac{1}{\alpha^p} \left\| \sup_{0 < t < \infty} \left| \varphi_t * \left( DL_1^{-1/2} a \right) \right| \right\|_{L^p(16B)}^p \lesssim \frac{1}{\alpha^p} \left\| \mathcal{M} \left( DL_1^{-1/2} a \right) \right\|_{L^p(16B)}^p.
\]
By Chebyshev's inequality, H"older's inequality, the $L^i$ which is a desired estimate for $I$.

On the other hand, we have

$$|I| \lesssim \frac{1}{\alpha^p} \left\| M \left( DL_1^{-1/2} a \right) \right\|_{L^2(\mathbb{R}^n)}^p |B|^{-\frac{p}{2}} \lesssim \frac{1}{\alpha^p} \left\| a \right\|_{L^2(\mathbb{R}^n)}^p |B|^{-\frac{p}{2}} \lesssim \frac{1}{\alpha^p}. $$

To estimate $I$, let $S_i(B) \equiv 2^i B \setminus 2^{i-1} B$ and $\overline{S}_i(B) \equiv 2^{i+1} B \setminus 2^{i-2} B$ with $i \in \mathbb{N}$. For all $i \geq 5$, $x \in S_i(B)$ and $y \in B(x, r_B)$, from $\sup \varphi (\cdot) \in L^\infty$, it follows that $y \in \overline{S}_i(B)$. For $i \geq 5$, let

$$I_i \equiv \left\{ x \in S_i(B) : \sup_{0 < t < r_B} |\varphi_t * (DL_1^{-1/2} a)(x)| > \alpha/2 \right\}. $$

By Chebyshev's inequality, H"older's inequality, the $L^2(\mathbb{R}^n)$-boundedness of $M$, Lemma 2.3 and (2.2), we conclude that

$$|I_i| \lesssim \alpha^{-p} \int_{S_i(B)} \left[ \sup_{0 < t < r_B} \int_{\overline{S}_i(B)} t^n \varphi \left( \frac{x - y}{t} \right) \left| \chi_{\overline{S}_i(B)}(y) DL_1^{-1/2} a(y) \right| dy \right]^p dx $$

$$\lesssim \alpha^{-p} \int_{S_i(B)} \left[ M \left( \chi_{\overline{S}_i(B)} DL_1^{-1/2} a \right)(x) \right]^p dx $$

$$\lesssim \alpha^{-p} |S_i(B)|^{1-p/2} \left\| DL_1^{-1/2} a \right\|_{L^2(\overline{S}_i(B))}^p $$

$$\lesssim \alpha^{-p} |S_i(B)|^{1-p/2} \left\| DL_1^{-1/2} \left( I - e^{-r_B^2 L_1} \right) M a \right\|_{L^2(\overline{S}_i(B))}^p $$

$$+ \sum_{k=1}^M \left\| DL_1^{-1/2} \left( r_B^2 L_1 e^{-\frac{k}{\pi} r_B^2 L_1} \right) M r_B^{-2M} b \right\|_{L^2(\overline{S}_i(B))}^p $$

$$\lesssim \alpha^{-p} |S_i(B)|^{1-p/2} \left[ \frac{r_B^2}{(2r_B)^2} \right]^M |B|^p/2 - 1 \sim 2^{-i[2M - n(1-p/2)]} \alpha^{-p}. $$

From this, the definition of $I_2$, $p = \frac{n}{n+1}$ and $M > \frac{1}{2} + \frac{2}{n}$, we deduce that $|I| \lesssim \sum_{i=1}^{\infty} |I_i| \lesssim \frac{1}{\alpha^p}$, which is a desired estimate for $I$.

To estimate $I_2$, by the assumption that $\int_{\mathbb{R}^n} DL_1^{-1/2} a(y) dy = 0$ via (B3), we know that

$$|J_2| \lesssim \left\{ x \in (16B)^c : \right\} \sum_{i=0}^{\infty} \sup_{r_B < t < \infty} \left[ \int_{S_i(B)} \frac{1}{t^n} \left| \varphi \left( \frac{x - y}{t} \right) - \varphi \left( \frac{x - x_B}{t} \right) \right| DL_1^{-1/2} a(y) dy \right] \cdot \alpha/2 \right\}. $$
Let $F_i(x) \equiv \sup_{r_B < t < \infty} \left| \int_{S_i(B)} \frac{1}{|t|^{n/2}} [\varphi(x - y) - \varphi(x - \frac{t}{r_B}y)] DL_i^{-\frac{1}{2}} a(y) dy \right|$ and

$$J_i \equiv \left\{ x \in (16B)^c : F_i(x) > \alpha/2 \right\}.$$ 

To obtain a desired estimate for $J$, by Lemma 2.2, it suffices to show that there exists a positive constant $C_0$ such that

$$(3.3) \quad |J_i| \lesssim \frac{2^{-C_0^i}}{\alpha^p}.$$ 

From the mean value theorem, Hölder’s inequality, supp $\varphi \subset B(0, 1)$, Lemma 2.3 and (2.2), we infer that

$$F_i(x) \lesssim \sup_{j \in \mathbb{Z}^+} \chi(2^{i+1}+2^{j+1})B(x) \sup_{2^jr_B \leq t < 2^{j+1}r_B} 2^{-j(n+1)}B|^{-1} \sup_{2^jr_B \leq t < 2^{j+1}r_B} \left| DL_i^{-\frac{1}{2}} a(y) \right| dy$$

$$\lesssim \sup_{j \in \mathbb{Z}^+} \chi(2^{i+1}+2^{j+1})B(x) \sup_{2^jr_B \leq t < 2^{j+1}r_B} 2^{-j(n+1)}B|^{-1} \left[ \frac{r_B^2}{(2^r)^{2j+1}} \right] |B|^{-1/p}$$

$$\equiv C_3 \sup_{j \in \mathbb{Z}^+} \chi(2^{i+1}+2^{j+1})B(x) \sup_{2^jr_B \leq t < 2^{j+1}r_B} 2^{-j(n+1)}2^{-i(2M-n-2)}|B|^{-1/p}.$$ 

Let $j_0 \equiv \max \left\{ j \in \mathbb{Z}_+ : C_3 2^{-j(n+1)}2^{-i(2M-n-2)}|B|^{-1/p} > \alpha/2 \right\}.$

For all $x \in \left( 2^{i+1} + 2^{j_0+1} \right)B^c$, we see that

$$F_i(x) \lesssim C_3 \sup_{j \geq j_0} \chi(2^{i+1}+2^{j+1})B(x) \sup_{2^jr_B \leq t < 2^{j+1}r_B} 2^{-j(n+1)}2^{-i(2M-n-2)}|B|^{-1/p} \leq \alpha/2,$$

which implies that $x \in J_i^c$. Thus, $J_i \subset \left( 2^{i+1} + 2^{j_0+1} \right)B$. From this and Chebyshev’s inequality, we then deduce that

$$|J_i| \lesssim \alpha^{-p} \int_{\left( 2^{i+1} + 2^{j_0+1} \right)B} 2^{-p}2^{-i(2M-n-1)}|B|^{-1} dx \lesssim 2^{-i[(2M-1)p-n(1-p)]} \alpha^{-p},$$

which implies that (3.3) holds with $C_0 \equiv (2M-1)p-n(1-p)$. Observe that $C_0 > 0$, since $M > 1$ and $p = \frac{n}{n+1}$. Thus, combining the estimate of I and J, we then complete the proof of Theorem 1.1.

**Proof of Corollary 1.1.** From Lemma 2.1, we deduce that the Schrödinger operator $-\Delta + V$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies the assumptions (A1) and (A2) as in Section 2, and both $-\Delta + V$ and the gradient operator $\nabla$ satisfy the assumptions (B1), (B2) and (B3) as in Section 2. Thus, from Theorem 1.1, we deduce that the Riesz transform $\nabla(-\Delta + V)^{-1/2}$ is bounded from $H^p_{-\Delta + V}(\mathbb{R}^n)$ to the classical weak Hardy space $WH^p(\mathbb{R}^n)$ in the critical case that $p = n/(n+1)$, which completes the proof of Corollary 1.1.
Proof of Theorem 1.2. Let \( p = \frac{n}{n+1} \) and \( M \in \mathbb{N} \) satisfy \( M > \frac{3}{2} + \frac{1}{2} \). To prove Theorem 1.2, similar to the proof of Theorem 1.1, by Theorem 2.2 and Lemma 2.2, for each \( (B^p_\ell, \epsilon, M) \)-molecule \( A \) associated to the ball \( B(x_B, r_B) \), \( m \in \mathbb{Z}_+ \) and \( \alpha \in (0, \infty) \), we only need estimate the measure of the following sets:

\[
\tilde{I} \equiv \left\{ x \in (16B)^C : \sup_{0 < t < r_B} \left| \varphi_t \ast (\nabla L_2^{-1/2} A)(x) \right| > \alpha/2 \right\}
\]

and

\[
\tilde{J} \equiv \left\{ x \in (16B)^C : \sup_{r_B \leq t < \infty} \left| \varphi_t \ast \left( \nabla L_2^{-1/2} A \right)(x) \right| > \alpha/2 \right\}.
\]

The estimate of \( \tilde{I} \) is similar to that of \( I \) in the proof of Theorem 1.1. We omit the details. Now we estimate \( \tilde{J} \). Since

\[
|\tilde{J}| \lesssim \left\{ x \in (16B)^C : \sum_{i=0}^{\infty} \sup_{r_B \leq t < \infty} \left| \int_{S_i(\bar{B})} \frac{1}{t^n} \left[ \varphi \left( \frac{x - y}{t} \right) - \varphi \left( \frac{x - x_B}{t} \right) \right] \right. \right. \nabla L_2^{-\frac{1}{2}} \left( I - e^{-r_B^2 L_2} \right)^M A(y) dy \right| > \alpha/2 \right\} + \left\{ x \in (16B)^C : \sum_{i=1}^{M} \sum_{k=1}^{M} \sup_{r_B \leq t < \infty} \left| \int_{S_i(\bar{B})} \frac{1}{t^n} \left[ \varphi \left( \frac{x - y}{t} \right) - \varphi \left( \frac{x - x_B}{t} \right) \right] \right. \right. \nabla L_2^{-\frac{1}{2}} \left( r_B^2 L_2 - \frac{1}{2} r_B^2 L_2 \right)^M (r_B^2 L_2)^{-M} A(y) dy \right| > \alpha/2 \right\}.
\]

Let \( \tilde{F}_{1,i}(x) \equiv \sup_{r_B \leq t < \infty} \left| \int_{S_i(\bar{B})} \frac{1}{t^n} \left[ \varphi \left( \frac{x - y}{t} \right) - \varphi \left( \frac{x - x_B}{t} \right) \right] \nabla L_2^{-\frac{1}{2}} \left( I - e^{-r_B^2 L_2} \right)^M A(y) dy \right| \),

\[
\tilde{F}_{2,i}(x) \equiv \sup_{r_B \leq t < \infty} \left| \int_{S_i(\bar{B})} \frac{1}{t^n} \left[ \varphi \left( \frac{x - y}{t} \right) - \varphi \left( \frac{x - x_B}{t} \right) \right] \nabla L_2^{-\frac{1}{2}} \left( r_B^2 L_2 - \frac{1}{2} r_B^2 L_2 \right)^M (r_B^2 L_2)^{-M} A(y) dy \right|,
\]

\( \tilde{J}_{1,k} \equiv \{ x \in (16B)^C : \tilde{F}_{1,i}(x) > \alpha/2 \} \) and \( \tilde{J}_{2,k} \equiv \{ x \in (16B)^C : \tilde{F}_{2,i}(x) > \alpha/2 \} \). By Lemma 2.2, it suffices to show that there exist positive constants \( C_4 \) and \( C_5 \) such that for all \( \alpha \in (0, \infty) \), \( |\tilde{J}_{1,k}| \lesssim \frac{2^{-C_4}}{\alpha^p} \) and \( |\tilde{J}_{2,k}| \lesssim \frac{2^{-C_5}}{\alpha^p} \). We only prove the first inequality, the proof of the second inequality is similar. Take \( \epsilon \in (n + 1 - 1/(n+1), \infty) \). By the mean value theorem, Hölder’s inequality, Lemma 2.3, (2.3) and \( \text{supp} \varphi \subset B(0, 1) \), we conclude that

\[
\tilde{F}_{1,i}(x) \lesssim \sup_{j \in \mathbb{Z}_+} \chi_{(2j^2+1)^2 j^2 B}(x) \sup_{2j^2 r_B \leq t < 2j^2 + 1} \int_{S_i(\bar{B})} \frac{1}{t^n} \nabla \varphi \| \nabla \varphi \|_{L^\infty(\mathbb{R}^n)} \left| \frac{y - x_B}{t} \right| \nabla L_2^{-\frac{1}{2}} \left( I - e^{-r_B^2 L_2} \right)^M A(y) dy
\]

\[ \times \left| \nabla L_2^{-\frac{1}{2}} \left( r_B^2 L_2 - \frac{1}{2} r_B^2 L_2 \right)^M (r_B^2 L_2)^{-M} A(y) \right| dy. \]
\[ \lesssim \sup_{j \in \mathbb{Z}_+} \chi(2^{j+1} + 2j^{+1})B(x) \sup_{2j^{+1}_B \leq t < 2^{j+1}_B} \int_{S_i(B)} \frac{1}{t^n} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \left| \frac{y - x_B}{t} \right| dy \times \left| \nabla L_2^{-\frac{1}{2}} \left( I - e^{-\frac{1}{2}L^2} \right) \chi(\overline{S}_i(B)) A(y) \right| dy \]

where \( S_i(B) \) and \( \overline{S}_i(B) \) are as in the proof of Theorem 1.1. The rest of the proof is similar to that of Theorem 1.1; we omit the details. This finishes the proof of Theorem 1.2. \( \square \)

4 Further remarks

In this section, we establish a variant of Theorems 1.1 and 1.2 for the higher order divergence form elliptic operators with complex bounded measurable coefficients and the higher order Schrödinger-type operators.

To this end, we first recall some notion and notations. For \( \theta \in [0, \pi) \), the \textit{closed sector}, \( S_\theta \), of angle \( \theta \) in the complex plane \( \mathbb{C} \) is defined by \( S_\theta \equiv \{ z \in \mathbb{C} \setminus \{ 0 \} : |\arg z| \leq \theta \} \cup \{ 0 \} \). Let \( \omega \in [0, \pi) \). A closed operator \( T \) in \( L^2(\mathbb{R}^n) \) is called of \textit{type} \( \omega \) (see, for example, [28]), if its spectrum, \( \sigma(T) \), is contained in \( S_\omega \), and for each \( \theta \in (\omega, \pi) \), there exists a nonnegative constant \( C \) such that for all \( z \in \mathbb{C} \setminus S_\theta \), \( \|(I - zI)^{-1}\|_{L^2(\mathbb{R}^n)} \leq C|z|^{-1} \), where and in what follows, \( \|S\|_{L^2(\mathcal{H})} \) denotes the operator norm of the linear operator \( S \) on the normed linear space \( \mathcal{H} \). Let \( T \) be a one-to-one operator of type \( \omega \), with \( \omega \in [0, \pi) \) and \( \mu \in (\omega, \pi) \), and \( f \in H_\infty(S^0_\mu) \equiv \{ f \text{ is holomorphic on } S^0_\mu : \|f\|_{L^\infty(S^0_\mu)} < \infty \} \), where \( S^0_\mu \) denotes the \textit{interior} of \( S_\mu \). By the \( H_\infty \) functional calculus, the function of the operator \( T \), \( f(T) \) is well defined. The operator \( T \) is said to have a \textit{bounded} \( H_\infty \) functional calculus in the Hilbert space \( \mathcal{H} \), if there exist \( \mu \in (0, \pi) \) and positive constant \( C \) such that for all \( \psi \in H_\infty(S^0_\mu) \), \( \|\psi(T)\|_{L^2(\mathcal{H})} \leq C\|\psi\|_{L^\infty(S^0_\mu)} \).

As in [6], let \( T \) be an operator defined in \( L^2(\mathbb{R}^n) \) which satisfies the following assumptions:

(E1) The operator \( T \) is a one-to-one operator of type \( \omega \) in \( L^2(\mathbb{R}^n) \) with \( \omega \in [0, \pi/2) \);

(E2) The operator \( T \) has a bounded \( H_\infty \) functional calculus in \( L^2(\mathbb{R}^n) \);

(E3) Let \( k \in \mathbb{N} \). The operator \( T \) generates a holomorphic semigroup \( \{e^{-tT}\}_{t \geq 0} \) which satisfies the \textit{k-Davies-Gaffney estimate}, namely, there exist positive constants \( C_6 \) and \( C_7 \) such that for all closed sets \( E \) and \( F \) in \( \mathbb{R}^n \), \( t \in (0, \infty) \) and \( f \in L^2(\mathbb{R}^n) \) supported in \( E \),

\[ \|e^{-tT}f\|_{L^2(F)} \leq C_6 \exp \left\{ - \frac{\text{dist}(E, F)}{C_7 t^{1/(2k-1)}} \right\} \|f\|_{L^2(E)}, \]
When $k = 1$, the $k$-Davies-Gaffney estimate is just (2.1).

Let $k \in \mathbb{N}$. Typical examples of operators, satisfying the above assumptions $(E_1)$, $(E_2)$ and $(E_3)$, include the following 2$k$-order divergence form homogeneous elliptic operator

$$T_1 \equiv (-1)^k \sum_{|\alpha|=|\beta|=k} \partial^{\alpha}(a_{\alpha,\beta}\partial^{\beta})$$

with complex bounded measurable coefficients $\{a_{\alpha,\beta}\} \Rightarrow |\alpha|=|\beta|=k$, and the following 2$k$-order Schrödinger-type operator

$$T_2 \equiv (-\Delta)^k + V^k$$

with $0 \leq V \in L^k_{loc}(\mathbb{R}^n)$.

For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the $T$-adapted square function $S_T f(x)$ by

$$S_T f(x) = \left\{ \int \int_{\Gamma(x)} |t^{2k}T e^{-\ell^{2k}T} f(y)|^2 \frac{dy dt}{t^{n+1}} \right\}^{1/2}.$$ 

Using the $T$-adapted square function $S_T f$, Cao and Yang [6] introduced the following Hardy space $H_T^p(\mathbb{R}^n)$ associated to $T$.

**Definition 4.1** ([6]). Let $p \in (0, 1]$ and $T$ satisfy the assumptions $(E_1)$, $(E_2)$ and $(E_3)$. A function $f \in L^2(\mathbb{R}^n)$ is said to be in $H_T^p(\mathbb{R}^n)$ if $S_T f \in L^p(\mathbb{R}^n)$; moreover, define $\|f\|_{H_T^p(\mathbb{R}^n)} \equiv \|S_T f\|_{L^p(\mathbb{R}^n)}$. The Hardy space $H_T^p(\mathbb{R}^n)$ is then defined to be the completion of $H_T^p(\mathbb{R}^n)$ with respect to the quasi-norm $\| \cdot \|_{H_T^p(\mathbb{R}^n)}$.

Let $i \in \{1, 2\}$. By first establishing the molecular characterization of $H_{T_1}^p(\mathbb{R}^n)$, Cao and Yang [6] then obtain the following boundedness of the Riesz transform $\nabla^k(T_i^{-1/2})$ from $H_{T_i}^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ when $p \in (n/(n+k), 1]$.

**Theorem 4.1** ([6]). Let $k \in \mathbb{N}$, $p \in (n/(n+k), 1]$, $T_1$ be the 2$k$-order divergence form homogeneous elliptic operator with complex bounded measurable coefficients as in (4.1), and $T_2$ the 2$k$-order Schrödinger-type operator as in (4.2). Then, for $i \in \{1, 2\}$, the Riesz transform $\nabla^k(T_i^{-1/2})$ is bounded from $H_{T_i}^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$.

Again, for $i \in \{1, 2\}$, applying the molecular characterization of $H_{T_i}^p(\mathbb{R}^n)$ from [6], by an argument similar to that used in the proof of Theorem 1.2, we obtain the endpoint boundedness of $\nabla^k(T_i^{-1/2})$ in the critical case that $p = n/(n+k)$. We omit the details by similarity.

**Theorem 4.2.** Let $k \in \mathbb{N}$, $p \equiv n/(n+k)$, $T_1$ be the 2$k$-order divergence form homogeneous elliptic operator with complex bounded measurable coefficients as in (4.1), and $T_2$ the 2$k$-order Schrödinger-type operator as in (4.2). Then, for $i \in \{1, 2\}$, the Riesz transform $\nabla^k(T_i^{-1/2})$ is bounded from $H_{T_i}^p(\mathbb{R}^n)$ to $WH^p(\mathbb{R}^n)$.

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