The Background Field Method for $\mathcal{N} = 2$, $d3$ Super Chern-Simons-Matter Theories

I.L. Buchbinder *, N.G. Pletnev +

*Departamento de Fisica, UFJF, Juiz de Fora, MG, Brazil
and
*Department of Theoretical Physics, Tomsk State Pedagogical University, 634061 Tomsk, Russia1, email: joseph@tspu.edu.ru

+Department of Theoretical Physics, Institute of Mathematics, 630090 Novosibirsk, Russia
email: pletnev@math.nsc.ru

Abstract

We develop the superfield background field method and study the effective action in the $\mathcal{N} = 2$, $d3$ supersymmetric Chern-Simons-matter systems. The one-loop low-energy effective action for non-Abelian supersymmetric Chern-Simons theory is computed to order $F^4$ by use of $\mathcal{N} = 2$ superfield heat kernel techniques.

1 Introduction

During the last few years, quantum aspects of $d3$ supersymmetric theories at perturbative level attracted a considerable attention. This was inspired by the papers [1], [2], [3], [4], [5], where for an IR description of stacks of M2-branes a highly supersymmetric three dimensional conformal field theories was proposed in the same sense as maximally supersymmetric Yang-Mills theory provides an effective description of stacks of D-branes. Such models are referred to as the Bagger-Lambert-Gustavsson (BLG) and Aharony-Bergman-Jafferis-Maldacena (ABJM) theories. ABJM models defined as three-dimensional $\mathcal{N} = 6$ superconformal $U(N) \times U(N)$ Chern-Simons-matter theory with level $(k, -k)$. It is conjectured to describe $N$ M2-branes located at the fixed point of the $C^4/Z_k$ orbifold in the static gauge. It is also argued that the ABJM model is dual to M-theory on $AdS_4 \times S^7/Z_k$ at large $N$. For $SU(2) \times SU(2)$ gauge group, the $\mathcal{N} = 6$ supersymmetry is enhanced to $\mathcal{N} = 8$ and the ABJM model coincides with the BLG model. All these new superconformal field theories involve a non-dynamical gauge field, described by a Chern-Simons like term in the Lagrangian, which is coupled to matter fields, parameterizing the degrees of freedom transverse to the worldvolume of the M2-branes.

The interest to the extended $d3$ superconformal theories is connected with two points. First, these theories represent further evidence of the duality $AdS_4/CFT_3$ [6]. This duality opens interesting window that allows us to examine the various properties of condensed

1Permanent address
matter (for a review see e.g. [7]). Second, these duality also provide novel results about integrability of the free/planar sector of the $AdS/CFT$ pair of models and finiteness properties of these superconformal Chern-Simons-matter theories in the strong coupling regime (for a review see e.g. [8]).

It is known for a long time that the quantization of a membrane worldvolume theory is very challenging and one of difficulty is the nonlocality associated with the deformation of membrane without changing its volume. A quantum supermembrane theory faces a serious problem of quantum mechanical instability [9]. As a result, a single (quantum mechanical) super - membrane does not make sense and we get a multi-body problem in its nature, which can be regarded as the origin of the continuous spectrum. Therefore, from the field theory side, the action of M2-brane should go away from the infra-red fixed point to a nonperturbative Yang-Mills-Chern-Simons system.

One nontrivial test for the BLG and ABJM models as the dual field theory of M theory is the study of the membrane scattering amplitude. In the dual gravity description, it can be given by the effective action of the probe M2-brane due to the large number of source M2 branes. In the papers [10] the membrane scattering were examined in the context of the BLG and ABJM models for the first time. The analysis of dimensions of the target space from one loop effective $v^4$ potential shows that this is consistent with the calculation from D2-brane super Yang-Mills action. This phenomenon can be understood as a signal that the membranes propagating between two membranes always wrap on the one spacial direction which becomes the compactified direction. Such a result suggests that the open membrane, described as a perturbation from the background, always wraps the compactified direction, even when $k$ is finite and small, and therefore the BLG theory can probe only remaining ten dimensions. From the analysis of quantum correction [10] it was found out the complete agreement in membrane scattering dynamics between the results from the ABJM model and those from the dual supergravity on $AdS_4 \times S^7/Z_k$ in a specific gauge for worldvolume diffeomorphism. As a result, it was discovered the following: (i) there is no correction in the $v^2$ term, where $v$ is a probe brane velocity, (ii) $v^4$ term appears at one-loop, (iii) there should be a non-renormalization theorem, at least for $\mathcal{N} = 8$ supersymmetry. It would be very interesting and instructive to study the above problem using the manifest supersymmetric and gauge invariant formalism that requires, in its turn, the development of techniques for calculating the effective action in frame of superfield background field method.

On the other hand, it is known that multiple M2-branes in eleven dimensions are reduced to D2-branes in ten dimensions compactifying one of the transverse direction to the M2-brane. This procedure is performed by the novel Higgs mechanism proposed in [11], [12]. It has been shown in the frame of this mechanism that the BLG models and the ABJM-like theories are reduced to the super Yang-Mills theory describing $N$ coincident D2-branes. In this process the non-dynamical Chern-Simons gauge fields become dynamical. Since the super Yang-Mills action is the leading order approximation in string scale $\alpha'$ expansion of the non-Abelian Born-Infeld action, it is natural to expect that the BLG and ABJM theories directly gives rise to a perturbative expansion in terms of the inverse Yang-Mills coupling constant, or equivalently, in terms of the inverse vacuum expected
value of the Higgs field. As a result, arising new higher-order action is non-Abelian, plus a decoupled Abelian degree of freedom.

We want to draw attention to the fact that similar "Higgs" effect as quantum effect potentially occurs in all super Chern-Simons-matter models and even in pure non-Abelian Chern-Simons theory and its supersymmetric versions. Then the BLG and ABJM Lagrangians and supersymmetry transformations presented in [2], [3] can be thought as representing the leading order terms in Planck scale expansion of a (not yet determined) non-linear M2-brane theory. This circumstance is analogous to the fact that \( \mathcal{N} = 4 \), D4 super Yang-Mills theory represents the leading order terms of the Born-Infeld action, which is believed to describe the dynamics of coincident D3-branes. Therefore, it would be interesting to determine the full theory, in which the leading order terms are the BLG or ABJM Lagrangians. This ambitious program is similar to non-Abelian supersymmetric extension of the Born-Infeld-type action in the \( \mathcal{N} = 4 \), D4 super Yang-Mills quantum field theory (See as an example of just a few links [13], [14], [15], [16] from a large list of references).

The off-shell loop corrections in Chern-Simons-matter theory attracts much attention since they generate non-trivial quantum dynamics for classically non-dynamical gauge field (see e.g. [19]). The natural way to study these corrections is given by effective action which can be treated as a method to derive the new, higher order in strength, gauge invariant and supersymmetric functionals.

Gauge invariant and manifestly supersymmetric effective action is constructed on the base of the superfield background field method. Evaluation of the effective action within the background field method is often accompanied by use of proper time or heat kernel techniques. These techniques allow us to sum up efficiently an infinite set of Feynman diagrams with increasing number of insertions of the background fields and to develop a background field derivative expansion of the effective action in manifestly gauge covariant way. Precise determination of the effective action means an exact solution in an appropriate model of quantum field theory, that of course is impossible in general. Therefore, the various approximate approaches are used such as the expansion in the number of loops and an expansion in powers of derivatives. The coefficients in this series directly related to the local geometrical invariants, constructed from the background fields and their covariant derivatives. In supersymmetric theories such a procedure allows to find the gauge invariant and supersymmetric functionals.

As it is known, the most powerful approach to study the quantum supersymmetric field theories is to make use of an unconstrained superfield formulation. Unfortunately, such a formulation for the \( \mathcal{N} = 6, 8 \) super Chern-Simons-matter theory is not known. The best what we know up to now is only \( \mathcal{N} = 3 \) off-shell formulation on \( \mathcal{N} = 3 \), d3 harmonic superspace [21]. In such a formulation three out of six or eight supersymmetries are realized off-shell while the other three or five are hidden and the supersymmetry algebra is closed only on shell. The corresponding superfield actions involve two hypermultiplet superfields in the bifundamental representations of the gauge groups and two Chern-Simons gauge superfields corresponding to the left and right gauge groups. The \( \mathcal{N} = 3 \) superconformal invariance allows only a minimal gauge interaction of the hypermultiplets. Therefore,
the $\mathcal{N} = 3, d3$ harmonic superspace methods should be helpful for these considerations. Alternatively, one can study the effective action in the $\mathcal{N} = 2$ superspace [22]. From the point of view of $\mathcal{N} = 2, d3$ supersymmetry, the $\mathcal{N} = 4, 6, 8$ Chern-Simons and super Yang-Mills theory describe coupling of the $\mathcal{N} = 2$ vector multiplet to the hypermultiplet $\Phi, \bar{\Phi}$ in the adjoint representation as well as one or another set of matter hypermultiplets $Q, \bar{Q}$ in the bifundamental representation.

The aim of this paper is to construct the background field method for $\mathcal{N} = 2$ super Chern-Simons theories, study the effective action in terms of unconstrained $\mathcal{N} = 2, d3$ superfields and calculate of the leading low-energy contributions to the effective action. Although the various classical and quantum aspects of $\mathcal{N} = 2, d3$ supersymmetric theories were extensively studied, the superfield background field method, allowing to develop manifestly gauge invariant and $\mathcal{N} = 2$ supersymmetric perturbation theory has not been formulated up to now. Just this problem is solved in the present paper. As the applications of background field method we show that in case of pure $\mathcal{N} = 2$ super Chern-Simons, $\eta$-invariant vanishes, but off-shell contributions to the effective action have a non-trivial complicated structure. For the computation of local gauge invariant and manifestly $\mathcal{N} = 2$ supersymmetric contributions we use the procedure previously proposed in [20], which we generalize to be applied for superfield theories, and the IR cutoff which is similar to that used in the work [15]. In our case, the scale of the IR cutoff will play the role of the Yang-Mills coupling constant and of course breaks the superconformal invariance. In the case of super Chern-Simons-matter models the role of the IR cutoff parameter is played by a vev of material fields.

The background field method and heat kernel techniques for $\mathcal{N} = 1, 2, d4$ super Yang-Mills theories were well-developed (see [23], [24], [25], [26] for reviews). In principle the $\mathcal{N} = 2, d3$ superfield formalism, is analogical, in some aspects, to $\mathcal{N} = 1, d4$ superfield formulation. Therefore, we pay basic attention here only to specific details of quantization, which are significant namely for $\mathcal{N} = 2, d3$ superfield theories.

This paper is organized as follows. In Section 2 we briefly discuss the formulation of the super Chern-Simons-matter theories in $\mathcal{N} = 2$ superfield. In Section 3 we formulate the background field method. In Section 4 we consider a structure of one-loop effective action and develop the superfield heat kernel technique for its computation. This technique is then used to compute the $F^2, F^3, (DF)^2, \ldots, F^4$ terms in the low-energy effective action in superfield form.

## 2 $\mathcal{N} = 2, d3$ superfield models

We start with a brief description of the $\mathcal{N} = 2, d3$ super Chern-Simons theory [27], [28], [29]. The constrained geometry of $\mathcal{N} = 2$ supergauge field is formulated in $R^{3|4}$ superspace with coordinates $z^M = \{x^m, \theta^\alpha, \bar{\theta}^\dot{\alpha}\}$\footnote{we use superspace conventions of [22].} in terms of the gauge covariant derivatives

$$D_M \equiv \{D_m, D_\alpha, D_{\dot{\alpha}}\} = D_M + i\Gamma_M^a T^a$$
The vector multiplet in three dimensions is built from one real scalar \( \phi \) and the gauge connection \( \Gamma^M \) where \( \Omega \) strained prepotential \([23],[24]\) is

\[
\text{It follows from (2.2) that the most general solution to this algebra in terms of the unconstrained prepotential}\ [23],[24]\ \text{is}
\]

\[
\mathcal{D}_\alpha = e^{-\Omega} D_\alpha e^{\Omega}, \quad \bar{\mathcal{D}}_\alpha = e^\Omega \bar{D}_\alpha e^{-\Omega}, \quad \Omega = \Omega^a T_a
\]

where \( \Omega^a \) is an arbitrary complex superfield. A covariantly chiral and antichiral superfields \( Q_c(z) = e^{\bar{\Omega}} Q, \bar{Q}_c(z) = \bar{Q} e^{\Omega} \) is defined to be annihilated by these operators,

\[
\mathcal{D}_\alpha Q_c(z) = 0, \quad \bar{\mathcal{D}}_\alpha \bar{Q}_c(z) = 0,
\]

Hermitian part \( \Omega \) defines a potential \( e^V = e^{\Omega} e^{\bar{\Omega}} \). In gauge chiral representation one gets \( \mathcal{D}_A = \{ \mathcal{D}_m, e^{-V} D_\alpha e^V, D_\alpha \} \) and \( Q_c = Q, \bar{Q}_c e^{-V} Q \).

The superfield strengths are the linear superfield \( G \) and chiral \( W_\alpha \) and antichiral \( \bar{W}_\alpha \) superfields satisfy the Bianchi identities

\[
\bar{W}_\alpha = D_\alpha G, \quad W_\alpha = \bar{D}_\alpha G
\]

\[
\mathcal{D}_\alpha \bar{W}_\beta = 0, \quad \bar{\mathcal{D}}_\alpha W_\beta = 0, \quad \mathcal{D}^\alpha \mathcal{D}_\alpha G = \bar{D}_\alpha \bar{D}^\alpha G = 0,
\]

\[
\mathcal{D}^\alpha W_\alpha + \bar{D}_\alpha W^\alpha = 0, \quad \mathcal{D}(\alpha W_\beta) + \bar{D}(\alpha \bar{W}_\beta) = -4i \mathcal{D}_\alpha G,
\]

\[
F_{\alpha\beta} = \frac{1}{8} \{ \mathcal{D}(\alpha W_\beta) - \bar{D}(\alpha \bar{W}_\beta) \},
\]

\[
\mathcal{D}_\alpha W_\beta = 2F_{\alpha\beta} - i \mathcal{D}_\alpha G + \frac{1}{2} \varepsilon_{\alpha\beta\rho} \mathcal{D}^\rho W_\rho, \quad \bar{D}_\alpha \bar{W}_\beta = -2F_{\alpha\beta} - i \bar{D}_\alpha G - \frac{1}{2} \varepsilon_{\alpha\beta\rho} \bar{D}_\rho \bar{W}^\rho.
\]

In \( \mathcal{N} = 2, d3 \) superspace, the gauge invariant Chern-Simons action reads in two equivalent forms \([27]\)

\[
S_{CS} = \frac{ik}{4\pi} \text{tr} \int_0^1 dt \int d^7 z D^\alpha \{ e^{-2V} D_\alpha e^{2V} \} e^{-2V} \partial_t e^{-2V}
\]

\[
= \frac{ik}{4\pi} \text{tr} \int_0^1 dt \int d^7 z D^\alpha \{ e^{2V} \bar{D}_\alpha e^{-2V} \} e^{2V} \partial_t e^{-2V}.
\]
Here the extra parameter $t$ satisfies the boundary conditions $V(t = 0) = 0$, $V(t = 1) = V$. After rescaling the potential as $V_{\text{new}} \equiv 2\sqrt{\pi} V$ we see that the coupling constant is $\sqrt{\pi} k$.

The superfield Lagrangian for $N_f$ matter chiral superfields $Q^i$ coupled to non-Abelian $\mathcal{N} = 2$ vector multiplet has the form

$$ S_{\text{matter}}[V, Q, \bar{Q}] = \text{tr} \int d^7 z \sum_{i=1}^{N_f} \bar{Q}^i e^V Q^i, \quad (2.10) $$

where the matter field $Q^i = \{f^i, \psi^i\}$, with global $U(N_f)$ flavor symmetry, is in an arbitrary representation $R$ of the gauge group. Such a $\mathcal{N} = 2$ theory can be formulated for any gauge group $G$ and chiral superfields in any representation, with arbitrary superpotential. The more extended supersymmetric theories can be formulated using some sets of $\mathcal{N} = 2$ superfields. For example, the $\mathcal{N} = 3$ superconformal theory on the $\mathcal{N} = 2$ language has $n$ pairs of chiral multiplets $Q^i, \bar{Q}^i$ ($i = 1, \ldots, n$), transforming in conjugate representations of the gauge group, and one chiral superfield $\Phi_a$. The action for the $\mathcal{N} = 3$ Chern-Simons-matter theory has the form

$$ S^{\mathcal{N}=3} = S_{\mathcal{N}=2}^{\mathcal{N}=3} + \int d^7 z (\bar{Q} e^V Q + \bar{Q} e^{-V} \bar{Q}) + \left\{ \int d^5 z \left( -\frac{k}{4\pi^2} \text{tr} \Phi^2 + \bar{Q}^i \Phi_a T^a_{ij} Q^j \right) + \text{c.c.} \right\}. \quad (2.11) $$

Here $\Phi$ is an auxiliary chiral superfield in the adjoint representation, combined with $V$ to give the $\mathcal{N} = 4$ vector multiplet. The scalar and the auxiliary components of $\Phi$ are combined with the corresponding components of $V$ to form a triplets under the $SU(2)_R$ symmetry. The R-symmetry also rotates as doublet the lowest component of the chiral superfield $Q$ and of its conjugate, antichiral superfield $\bar{Q}$. It should be noted that the most elegant presentation of a large class of classically marginal models Chern-Simons-matter with manifestly realized $\mathcal{N} = 3$ off-shell supersymmetry is provided in the $\mathcal{N} = 3$ harmonic superspace [21].

Maximally supersymmetric theories in $2 + 1$ dimensions with $SO(8)$ R-symmetry were constructed in [2]. These theories have an interesting property that the closure of the supersymmetry requires the particular combinations of the gauge group and the matter content, whereas there is no such restriction for $\mathcal{N} \leq 3$. The essential feature of these theories is that the matter fields $X^I X_d^I T^a$, $I = 1, \ldots, 4$ take the values in a metrized version of the Lie 3-algebra $A_n$:

$$ [T^a, T^b, T^c] = f^{abcd} T^d, \quad h^{ab} = \text{tr}(T^a, T^b), \quad (2.12) $$

where the structure constants $f^{abcd} = f^{abc} T^d$ are totally antisymmetric in upper indices and are subject to some basic identity. The gauge field takes values in the Lie algebra associated with the Lie 3-algebra $A_m = A_m^{abc} T^{ab}$, where the generators act in the fundamental representation as $(t^{ab})^c_d = f^{abc} T^d$. When $h^{ab}$ is positive definite, there is the only such $A_4$ 3-Lie algebra (with $f^{abcd} \propto \varepsilon^{abcd}, h^{ab} = \delta^{ab}$) which satisfies all reasonable physical requirements. On the associated Lie algebra there exist two invariant tensors which have the required structure of a Killing form, namely

$$ G^{ab,cd} = f^{abcd}, \quad g^{ab,cd} = f^{abe} f^{cdf} e. \quad (2.13) $$
Extending the BLG model to higher numbers of M2-branes by reducing the number of supersymmetries led to two generalizations of the notion of a 3-algebra: the generalized 3-Lie algebras and the Hermitian 3-algebras [4]. These algebras are used in the ABJM theories, with \( \mathcal{N} = 6 \) supersymmetry and \( U(N) \times U(N) \) gauge symmetry and in the ABJ theories [3] with \( \mathcal{N} = 6 \) supersymmetry and \( U(N) \times U(M) \) gauge symmetry, as well as with \( \mathcal{N} = 5 \) supersymmetry and \( Sp(2N) \times O(M) \) gauge symmetry. Similar theories were constructed in [30]. A classification of the possible \( \mathcal{N} = 6 \) theories of ABJM-type was presented in [31]. In all cases, the underlying 3-bracket is no longer required to be totally antisymmetric. So that in the case of theories with \( \mathcal{N} = 6 \) supersymmetry, the structure constants \( f^{abcd} = \text{tr}(\bar{T}^d[T^a,T^b];\bar{T}^c) \) must satisfy the relations:

\[
\begin{align*}
f^{ab\bar{c}\bar{d}} &= -f^{ba\bar{c}\bar{d}} \\
f^{ab\bar{c}\bar{d}} &= f^{\bar{c}\bar{d}ab}
\end{align*}
\]

The triple product is also required to satisfy the basic identity.

In order to get a compact form of the Feynman rules, it is convenient to use the capital Roman letters \( A, B, \ldots \) to denote the indices in associated gauge Lie algebra [2], [4], [5]. In terms of the gauge algebra indices, the invariant form is given by

\[
<X,Y> = X_{ab}Y_{cd}f^{abcd} = X_A Y_B G_{AB}.
\]

The structure constants \( F_{ABC} = F^{DA}_A G_{BC} \), where \( F^{DA}_A = C_{ab,cd}^{ef} \delta^{[f]}_d \), are totally antisymmetric due to ad-invariance of \( <\cdot,\cdot> \). Moreover, it is convenient to use the multi-indices \( ai \) combining flavor and 3-algebra indices for \( Q_{ai} = Q^I \). For example, we have for vertices \( <\bar{Q}^i, VQ_j> = Q^I V^A (T_A)^j_i \bar{Q}^j \).

By construction, all these models have at least \( \mathcal{N} = 2 \) supersymmetry. Higher supersymmetry depends on the underlying 3-algebra and the choices the superpotential. Therefore formally, the structure of the effective action in the sector of gauge fields (without violating the gauge symmetry) should have a universal form. The difference of effective actions of one model from another is stipulated by the choice of explicit 3-algebra representations and relations between various Casimir invariants for such Lie 3-algebras.

### 3 Background field quantization

We quantize the \( \mathcal{N} = 2 \) super Chern-Simons theory in the quantum-chiral but background vector representation. As a first step we split the initial superfields \( V, Q, \bar{Q} \) into background \( V, Q, \bar{Q} \) and quantum \( v, q, \bar{q} \) parts by the rule

\[
e^V \rightarrow e^V e^v, \quad Q \rightarrow Q + q.
\]

Since the background-quantum splitting for matter superfields is a simple sum, we will pay the basic attention only on gauge superfield.

The initial infinitesimal gauge transformations can be realized in two different ways:

(i) background transformations

\[
e^v \rightarrow e^{i\tau} e^v e^{-i\tau}, \quad \mathcal{D}_M \rightarrow e^{i\tau} \mathcal{D}_M e^{-i\tau},
\]

with a real parameter \( \tau = \bar{\tau} \)

(ii) quantum transformations

\[
e^v \rightarrow e^{i\bar{A}} e^v e^{-i\bar{A}}, \quad \mathcal{D}_M \rightarrow \mathcal{D}_M.
\]
with background covariantly chiral parameters, $\mathcal{D}_\alpha \Lambda = \mathcal{D}_\alpha \bar{\Lambda} = 0$. For infinitesimal $\Lambda$ we have (see [23])

$$\delta v = L_{\frac{1}{2}} \{ -i(\bar{\Lambda} + \Lambda) + \coth L_{\frac{1}{2}} i(\bar{\Lambda} - \Lambda) \} .$$

(3.17)

Both the superfields $Q$ ($\bar{Q}$) and $q$ ($\bar{q}$) are covariantly chiral (antichiral), $\bar{D}_\alpha Q = \bar{D}_\alpha \bar{q} = 0$, where the covariant spinor derivatives act in according with the representation of the gauge group. It is worth pointing out that the form of the background-quantum splitting (3.14) and the corresponding background and quantum transformations (3.15), (3.16) are analogous to the $\mathcal{N} = 1, d4$ non-Abelian super Yang-Mills model [23]. Our aim now is to construct an effective action as a gauge-invariant and $\mathcal{N} = 2$ supersymmetric functional of the background superfield $V$.

The presence of the parameter $t$ in (2.9) is very essential and the direct integration in (2.9) can be explicitly done only in the abelian case. However, the first (that noted in [27]) variation of (2.9) and second-order expansion in powers of quantum field $v$ contain no $t$ integration (modulo a total spinor derivative)

$$S \sim \int_0^1 dt \partial_t (v \bar{D}^\alpha \Gamma_\alpha) + \frac{1}{2} \int_0^1 dt \partial_t (v \bar{D}^\alpha D_\alpha v) + \mathcal{O}(v^3)$$

(3.18)

It is well known that the linear in $v$ term in (3.18) should be dropped when considering the effective action. The quadratic part $S_2$ of quantum action given in (3.18) depends on $V$ via the dependence of $\mathcal{D}_M$ on background superfield. Each term in the action (3.18) is manifestly invariant with respect to the background gauge transformations.

We now proceed to the quantization of the theory in a manifest $\mathcal{N} = 2$ supersymmetric form. To construct the effective action, we can use the Faddeev-Popov Ansatz. Within the framework of the background field method, we should fix only the quantum gauge transformations (3.16) keeping the invariance under the background gauge transformations. It is convenient to choose the gauge fixing functions in the form analogous to $\mathcal{N} = 1, d4$ theories: $\bar{f} = \bar{D}^2 v$, $f = D^2 v$. These functions are covariantly (anti)chiral and transform under the quantum gauge transformations (3.16). Therefore the ghost action is the same as in the four dimensional $\mathcal{N} = 1$ case [23], [24]:

$$S_{FP} = \text{tr} \int d^3 x d^3 \theta (b + \bar{b}) L_{\frac{1}{2}} [c + \bar{c} + \coth (L_{\frac{1}{2}}) (c - \bar{c})] = \text{tr} \int d^3 x d^4 \theta (\bar{b} c - b \bar{c}) + \mathcal{O}(v) .$$

(3.19)

where $c, \bar{c}, b, \bar{b}$ covariantly chiral and antichiral superfields. The effective action for pure Chern-Simons theory is given by the following functional integral

$$e^{i \Gamma_{CS}[V]} = e^{i S_{CS}[V]} \int D V D b D c \delta [f - \bar{D}^2 v] [\delta [\bar{f} - D^2 v] e^{i S_2[V,v]} + \mathcal{O}(v^3) + i S_{FP} .$$

(3.20)

Unlike in $\mathcal{N} = 1, d4$ case we average this expression with the following weight (see some details for $\mathcal{N} = 2, d3$ theory in [32])

$$1 = \int \mathcal{D} f \mathcal{D} \bar{f} \mathcal{D} \varphi \mathcal{D} \bar{\varphi} \exp \left\{ \frac{i}{2 \alpha} \int d^5 z f^2 + \frac{i}{2 \beta} \int d^5 \bar{z} \bar{f}^2 + i \int d^5 z \varphi^2 + i \int d^5 \bar{z} \bar{\varphi}^2 \right\} ,$$

(3.21)
where $\alpha, \beta$ are the gauge-fixing parameters and the anticommuting third ghost superfield $\varphi$ is background covariantly chiral. As a result, we see that the $\mathcal{N} = 2$ super Chern-Simons theory is described within the background field approach by three ghosts. However, the opposite of 4$d$ case, the Nielsen-Kallosh ghost gives no rise to the effective action even at one loop level.

Further we will study only one-loop effective action in gauge superfield sector. In this case it is sufficient to consider, under the functional integral for $\Gamma_{CS}[V]$, only the quadratic part of gauge fixed action for quantum fields. Then one gets

$$S_2 + S_{gf} = \frac{1}{2} \text{tr} \int d^7z v \frac{1}{4} (D_\alpha \bar{D}_\alpha + \bar{D}_\alpha D_\alpha + \frac{1}{\alpha} D^\alpha D_\alpha + \frac{1}{\beta} \bar{D}^\alpha \bar{D}_\alpha) v = \frac{1}{2} \text{tr} \int d^7z v \mathcal{H}_v v. \quad (3.22)$$

The operator $\mathcal{H}_v$ is defined by Eq. (3.22).

Now we should add the contribution of matter superfields. It is done by considering the integral over matter quantum fields $q, \bar{q}$ of $e^{iS_{\text{matter}}[V,q,\bar{q},v]}$, where $S_{\text{matter}}[V,q,\bar{q},v]$ is obtained from (2.10) by background-quantum splitting (3.14). For one-loop approximation it is sufficient to use $S_{\text{matter}}[V,q,\bar{q},v = 0] = S^{(2)}_{\text{hyper}}$. As a result, we get the following representation for the one-loop effective action in the gauge field sector

$$e^{i\Gamma^{(1)}[V]} = e^{iS_{CS}[V]} \int Dv Db Dc Dq D\bar{q} e^{iTr \int d^7z v \mathcal{H}_v + iS_{FP} + iS^{(2)}_{\text{hyper}}} \quad (3.23)$$

where

$$\mathcal{H}_{FP} = \left( \begin{array}{cc} 0 & \frac{1}{16} D_2 D^2 \\ -\frac{1}{16} \bar{D}_2 \bar{D}^2 & 0 \end{array} \right) \delta^{(7)}(z,z') \quad (3.24)$$

The matter superfield contributions to the effective action is

$$\Gamma^{(1)}_{\text{hyper}} = \frac{i}{2} \text{Tr} \ln \mathcal{H}_{\text{hyper}} = \frac{i}{2} \text{Tr} \ln \left( \begin{array}{cc} 0 & \frac{1}{4} D_2^2 \delta_+(z,z') \\ -\frac{1}{4} \bar{D}_2^2 \delta_-(z,z') & 0 \end{array} \right). \quad (3.25)$$

Note that it differs from the contributions of ghosts only by the sign and choice of the representation of a gauge group.

### 4 One-loop effective action

In this section we investigate the off-shell one-loop corrections to the action for $\mathcal{N} = 2$ super Chern-Simons quantum field theory. It is well known that the one loop effective action is given in terms of functional determinants of the differential operators in quadratic part of action for quantum fields. In the theory under consideration all these operators
are the generalized d’Alembertians acting on superfields. According to the previous section, there are three basic d’Alembertians which arise in the covariant supergraphs: (i) the vector d’Alembertian \( \Box \); (ii) the chiral d’Alembertian \( \Box_+ \); and (iii) the antichiral d’Alembertian \( \Box_- \). The vector d’Alembertian is defined by

\[
\Box_v = \mathcal{H}_v^2 = \frac{1}{16} [-D\partial D - \bar{D}\partial\bar{D} + \frac{1}{\alpha\beta} \{D^2, \bar{D}^2\} - 16G^2 - \frac{8i}{\alpha} \bar{W}^\alpha D_\alpha + \frac{8i}{\beta} W^\alpha \bar{\bar{D}}_\alpha] \tag{4.1}
\]

where

\[
\Box_{\text{cov}} = \frac{1}{2} d^{\alpha\beta} D_{\alpha\beta}
\]

and we have used the identities

\[
\frac{1}{8} D^\alpha D^2 D_\alpha = -\Box_{\text{cov}} + \frac{1}{16} \{D^2, \bar{D}^2\} + i\bar{W}^\alpha \bar{D}_\alpha + \frac{i}{2} (D^\alpha W_\alpha) - G^2, \tag{4.2}
\]

\[
\frac{1}{8} \bar{D}^\alpha \bar{D}^2 \bar{D}_\alpha = -\Box_{\text{cov}} + \frac{1}{16} \{D^2, \bar{D}^2\} - iW^\alpha \bar{D}_\alpha - \frac{i}{2} (\bar{D}^\alpha W_\alpha) - G^2.
\]

It is clear that the most convenient gauge choice is \( \alpha = \beta = 1 \).

The covariantly chiral d’Alembertian is defined by

\[
\Box_+ = \Box_{\text{cov}} + iW^\alpha \bar{D}_\alpha + \frac{i}{2} (D^\alpha W_\alpha) + G^2, \quad \Box_+ \Phi = \frac{1}{16} \bar{D}^2 \bar{D}^2 \Phi, \quad \bar{D}_\alpha \Phi = 0. \tag{4.3}
\]

The antichiral d’Alembertian is defined similarly,

\[
\Box_- = \Box_{\text{cov}} - i\bar{W}^\alpha \bar{D}_\alpha - \frac{i}{2} (\bar{D}^\alpha \bar{W}_\alpha) + G^2, \quad \Box_- \bar{\Phi} = \frac{1}{16} D^2 D^2 \bar{\Phi}, \quad \bar{D}_\alpha \bar{\Phi} = 0. \tag{4.4}
\]

Our aim is studying the low-energy effective action \( \Gamma[V] \) which is generated by integrating out the quantum fields \( v \), ghosts and matter fields and describes the quantum dynamics of \( \mathcal{N} = 2, d3 \) vector multiplet.

## 4.1 Vanishing the \( \eta \)-invariant

The operator \( \mathcal{H}_v \) (which is denoted for simplicity as \( \mathcal{H} \) in this subsection) has the "first order in power \( \partial^\alpha \). Therefore, we must worry about the phase of the functional determinant. Following the pioneering work [17], we define the phase of the path integral by means the superfield eta-invariant of Atiyah, Patodi and Singer as

\[
\eta_{\mathcal{H}}(s) = \frac{1}{2} \lim_{s \to 0} \sum_i \text{sign} \lambda_i |\lambda_i|^{-s} = \text{Tr}(\mathcal{H}(\mathcal{H})^{-\frac{s+1}{2}}) \tag{4.5}
\]

Here Tr is a functional trace of operator acting in superspace. It is evident that the \( \eta_{\mathcal{H}}(s) \) is a functional of background field \( V \). Then

\[
\frac{1}{\sqrt{\text{Det}[\mathcal{H}^\ast]}} e^{i\frac{\pi}{4} \eta_{\mathcal{H}}(0)} = \frac{1}{\sqrt{\text{Det}[\mathcal{H}^\ast]}} e^{i\frac{\pi}{4} \eta_{\mathcal{H}}(0)} \tag{4.6}
\]
In case of non-supersymmetric Chern-Simons theories the phase in Eq. (4.6) was discussed in [17], [18], [19].

Our aim is to compute the \( \eta_H(0) \) in the theory under consideration. To do that one use the identity
\[
\eta_H(s) = \frac{1}{\Gamma(s + \frac{1}{2})} \int_0^\infty dt t^{s-\frac{1}{2}} \text{Tr} e^{-tH^2}.
\] (4.7)
and then put \( s = 0 \). For evaluating the integral we, following [33], replace the background field \( V \) by the field \( gV \) with \( g \) be a real parameter. As a result ones get the operator \( H(g) \), such that \( H(1) = H \) and \( H(0) \) is background field independent. Differentiating Eq. (4.7) one obtains
\[
\delta_g \eta_H(g)(s) = \frac{1}{\Gamma(s + \frac{1}{2})} \int_0^\infty dt t^{s-\frac{1}{2}} \text{Tr} \left\{ \delta_g H(g) e^{-tH^2(g)} - 2t \delta_g H(g) H^2(g) e^{-tH^2(g)} \right\}.
\] (4.8)

Now we see that \( \delta_g \eta_H(g) \) is regular at \( s = 0 \) and its value is given by a local invariant
\[
\lim_{s \to 0} \delta_g \eta_H(g)(s) = \frac{2}{\sqrt{\pi}} \text{Tr} \left( \sqrt{t} \delta_g H(g) e^{-tH^2(g)} \right)|_{t=0} = -\frac{2}{\sqrt{\pi}} \text{Tr} \left( \sqrt{t} \delta_g H(g) e^{-tH^2(g)} \right)|_{t=0}.
\] (4.9)

We remind that for given an operator \( \hat{A} \) acting on the space of unconstrained superfields, its superkernel is determined to be biscalar \( A(z, z') \). Then \( \text{Tr} \hat{A} \) is given as
\[
\text{Tr} \hat{A} = \int d^3x d^4\theta A \delta^3(x - x') \delta^4(\theta - \theta')|_{x=x', \theta=\theta'}.
\] (4.10)

It is easy to see that the non-zero contribution in \( \delta_g \eta_H(g)(0) \) can be resulted only from zero and first terms of power series expansion of (4.9) in \( t \)
\[
\text{Tr} \delta_g H e^{-tH^2(g)} \sim \frac{1}{t^{3/2}} (h_0(z) + th_1(z) + ...)
\] (4.11)

Then, it is obvious that to obtain a nonzero result, we must put exactly four spinor derivatives on all Grassmann \( \delta \)-functions in Eq. (4.10). However, the operator \( \delta_g H(g) \) where \( \delta_g D_\alpha = [D_\alpha, e^{-gV} \delta_\alpha e^{gV}], \delta_g \bar{D}_\alpha = 0 \) has a combination of spinor derivatives \( D_\alpha, \bar{D}_\alpha \) of first-degree and the operator \( H^2(g) \) has also first-order spinor derivatives. Therefore, the both first terms \( h_0(z), h_1(z) \) in (4.11) vanish. Thus we see that \( \eta_H(g)(0) \) does not depend on \( g \) and hence it is background field independent. Therefore this quantity is no more then unessential constant and we can omit it in the (4.6). It is known that the background field dependent \( \eta_H(0) \) gives rise to a finite shift of coupling constant in non-Abelian Chern-Simons action. Our result means that such a shift is absent in the theory under consideration.
4.2 Low-energy contribution from vector multiplet

One-loop effective action, generated by vector multiplet, is given by the expression

\[ \Gamma^{(1)}[V] = \frac{i}{4} \text{Tr} \ln \mathcal{H}_v^2 = \frac{i}{4} \text{Tr} \int_0^\infty \frac{dt}{t} e^{-m^2 t} e^{-t \mathcal{H}_v^2} \]  

(4.12)

where \( m \) is infrared regulator. We will calculate the asymptotic expansion of the heat kernel in the integrand that takes the form of an expansion in the powers of covariant derivatives. Structure of such an expansion is defined by superfield DeWitt coefficient. At the component level, the non-trivial DeWitt coefficients, \( a_n \) for \( n \geq 4 \), contain in bosonic sector the field strength terms of the form \( F^n \). The first non-trivial coefficient, \( a_4 \), is well-known in \( d=4 \) [23]. In \( d=3 \) we also have analogous box diagram with factors \( i/2 W^\alpha(D + \bar{D})_\alpha \) at each vertex, and to get non-zero result one should keep terms with two \( D \)'s and two \( \bar{D} \)'s. Besides, we should treat the gauge strength as matrix in the adjoint representation \( W^\alpha_{ac} \equiv f_{abc} W^{ab} \). Then we get for a four-points contribution to the effective action:

\[ \Gamma^{(1)}_v = \frac{1}{256\pi^3 m^5} \int d^7 z g(a_1, a_2, a_3, a_4) (W^\alpha(a_1) W_{\alpha}(a_2) W^\beta(a_3) W_{\beta}(a_4)) \]  

(4.13)

where in \( \mathcal{N}=2, d=3 \) case \( W^\alpha \equiv (W - \bar{W})^\alpha \). Here we have used for colour structures the notation from work [15]:

\[ g(a_1, a_2, \ldots, a_n) = f_{b_1a_1b_2} f_{b_2a_2b_3} \cdots f_{b_na_ab_1} , \]

where \( f_{abc} \) are the structure constants for a gauge Lie-algebra or \( F^{ABC} \) for a gauge 3-algebra. Note that these terms do not have the Abelian analogue. They simply vanish in the Abelian case.

To obtain the component form and in particular, to get the \( \sim F^4 \) terms we should as usual to convert \( \int d^4 \theta \rightarrow \frac{1}{16} \bar{D}^2 D^2 \) and act with these four spinor derivatives on the \( W_\alpha = \bar{\theta}^\beta f_{\beta\alpha}(x_L) + \ldots \) and \( \bar{W}_\alpha = \theta^\beta f_{\beta\alpha}(x_R) + \ldots \), where dots stand for the terms with derivatives of the fields.

4.3 The leading contribution of the ghost and matter superfields

Contribution to one-loop effective action from Faddeev-Popov ghosts is defined by the expression

\[ \text{Tr} \ln \mathcal{H}_{FP} = \text{Tr} \ln \frac{1}{16} \bar{D}^2 D^2 + \text{Tr} \ln \frac{1}{16} \bar{D}^2 D^2 = \text{Tr} \ln \Box_+ + \text{Tr} \ln \Box_+ \]

Contribution from matter superfields differs only by sign and choice of representation of gauge group. That allows to write the ghost contribution to effective action \( \Gamma^{(1)}_{gh} \) in the form of a integral over an auxiliary proper time \( t \)

\[ i\Gamma^{(1)}_{gh} = \int_0^\infty \frac{dt}{t} e^{-m^2 t} (K_+(t) + K_-(t)) . \]  

(4.14)
In this expression, \( m^2 \) is the infrared cutoff and \( K_+(t) \) and \( K_-(t) \) are the functional traces of the chiral and antichiral heat kernels respectively, which are defined by:

\[
K_\pm(t) = \text{tr}_\mathcal{R} \int d^5z_x (e^{-t\Box_x} - e^{-t\Box_0}) \delta_\pm(z, z')|_{z' = z} = \text{tr}_\mathcal{R} \int d^5z_x K_\pm(z, z'|t)|_{z' = z}. \tag{4.15}
\]

Here \( \text{tr}_\mathcal{R} \) denotes the trace over the representation \( \mathcal{R} \), \( dz_x \) is the integration measure over (anti)chiral subspace, \( \delta_\pm(z, z') \) is the (anti)chiral delta function, \( \delta_+ (z, z') = -\frac{i}{4} \mathcal{D}^2 \delta(z, z') \). It is well-known that \( K_+(t) = K_-(t) \). Therefore we discuss only the computation of the chiral kernel.

One of the procedures in computations of the heat kernel is to make use of the Fourier integral representation of delta-function as follows \([35]\):

\[
\delta^{(7)}(z - z')1 = \int \frac{d^5p}{(2\pi)^3} e^{ip\rho^m \rho_m} \zeta^2 \bar{\zeta}^2 I(z, z'), \tag{4.16}
\]

where

\[
\rho^m = (x - x')^m - i\zeta \gamma^m \bar{\theta}' + i\theta' \gamma^m \zeta, \quad \zeta = \theta - \theta', \quad \bar{\zeta} = \bar{\theta} - \bar{\theta}'.
\]

Here \( I(z, z') \) is the parallel displacement operator along the geodesic line connecting the point \( z' \) and \( z \), defined up to the gauge transformation \( I(z, z') \rightarrow e^{i\tau(z)} I(z, z') e^{-i\tau(z')} \). For our aims we need only the following properties of the \( I(z, z') \): \( I(z, z') I(z', z) = I(z, z) = 1 \) (as boundary condition) and the equation \( \zeta^A \mathcal{D}^I_i I(z, z') = \zeta^A \mathcal{D}^I_i I(z, z') = 0 \).

The heat kernel \( K_+(z, t) \) has an asymptotic Schwinger-DeWitt expansion, which is written as

\[
K_+(z, t) = \frac{i}{(4\pi t)^{3/2}} \sum_{n=0}^{\infty} t^n a_n(z), \quad a_0 = a_1 = 0. \tag{4.17}
\]

The \( a_n(z) \) are the DeWitt coefficients, which at the component level contain bosonic field strength terms of the form \( F^n \). From dimensional considerations and the requirement of gauge invariance, we can expect that the first non-trivial coefficient \( a_2 \) in the non-Abelian case is \( a_2 \sim \text{tr}_\mathcal{R} \int d^5z W^2 \sim \text{tr}_\mathcal{R} \int d^7z G^2 \). One can show that the \( a_n \) with \( n \geq 2 \) are obtained in form of \( \mathcal{D}^2 \) acting on superfield strengths and their covariant derivatives, and hence all terms in \( K_+(z, z'|t) \) can be written as the gauge-invariant superfunctionals on full superspace. By differentiating the kernel \( K_+(z, z'|t) \) with respect to \( t \), one observes that:

\[
\frac{dK_+(t)}{dt} = \text{tr}_\mathcal{R} \int d^7z \frac{1}{4} \mathcal{D}^2 e^{-t\Box} \delta_+(z, z')|_{z = z'}. \tag{4.18}
\]

It is convenient \([16]\) to introduce a new set of coefficients by writing

\[
\mathcal{D}^2 e^{-t\Box} \delta_+(z, z')|_{z = z'} = \frac{1}{(4\pi t)^{3/2}} \sum_{n=0}^{\infty} t^n c_n(z),
\]

as an asymptotic series. Here \( a_n(z) = \frac{1}{n-\frac{3}{2}} (-\frac{1}{4} \mathcal{D}^2) c_{n-1}(z) \). The effective action can then be written as

\[
\Gamma_{gh}^{(1)} = -\frac{1}{2\pi^{3/2}} \sum_{n=2}^{\infty} \frac{\Gamma(n - \frac{3}{2})}{(2n - 3)\pi^{2n-3}} \int d^7z \text{tr}_\mathcal{R} c_{n-1}. \tag{4.19}
\]
Here $R$ means adjoint representation. Matter contribution has the same form (4.19) with $R$ be a corresponding representation.

Our next goal is to discuss the computations of the superfield coefficients $c_1$, $c_2$ and $c_3$. We adopt and generalize for $\mathcal{N} = 2$, $d3$ case, the procedure developed in [36], [37] and modified for non-Abelian backgrounds in [16]. In some respects, this procedure is similar to that was used in [20] for constructing a gauge invariant derivative expansion of the effective action in the Yang-Mills theory.

Using the identities $\int d^2\eta \eta^2 = -4$, we present a $\zeta^2$ as $\zeta^2 = \int d^2\eta \eta^\alpha \zeta^\alpha$. Then $\frac{d}{dt} K_+(z, z', t)$ in the point coincidence limit becomes

$$K_+^\alpha = \int \frac{d^3p}{(2\pi)^3} \frac{1}{4} d^2\eta X^\alpha X_\alpha e^{-t\Delta} \cdot 1 .$$ (4.20)

The operator

$$\Delta = \frac{1}{2} X^{\alpha\beta} X_{\alpha\beta} + i X^\alpha W_\alpha + G^2,$$

is defined by

$$X_m = D_m + ip_m, \quad X^\alpha = D^\alpha + \eta^\alpha - p_{\alpha\beta} \bar{\zeta}^\beta$$

and the auxiliary field $-\frac{i}{2} D^\alpha W_\alpha$ may be omitted or included in the redefinition $G^2$. Note that the last term in $X^\alpha$ vanishes in the limit of point coincidence since the operator $\Box_+$ does not contain $D$. Expanding the exponential in powers of proper time around $e^{p^2 t}$ and integrating over $p$, we obtain the desired expansion, collecting together the coefficients at each degree of proper time $t$. Due to gauge invariance, these coefficients are actually expressed in terms of commutators of covariant derivatives. The pre-exponential factor $X^\alpha X_\alpha$ under the integral in (4.20) is important to write a contribution to effective action as integral over appropriate superspace. The following integrals are used:

$$\int \frac{d^3p}{(2\pi)^3} e^{ib^2} = \frac{i}{(4\pi t)^{3/2}}, \quad \int \frac{d^3p}{(2\pi)^3} P_m P_n e^{ib^2} = \frac{-1}{2} \delta_{mn} \frac{i}{(4\pi t)^{3/2}} ;$$

$$\int \frac{d^3p}{(2\pi)^3} P_m P_n P_k e^{-ib^2} = \frac{1}{4t^2} (\delta_{mn} \delta_{kl} + \delta_{mk} \delta_{nl} + \delta_{ml} \delta_{kn}) \frac{i}{(4\pi t)^{3/2}} .$$

Zeroth-order term does not depend on background fields. In the next terms of the expansion, we must take into account that $X^3 = 0$ and the integrals over odd powers of $p$ vanishes. Therefore in the first order of expansion of the (4.20) we have $-t(\Box + G^2)$, with a factor $i/(4\pi t)^{4}$ which is common in the expansion. In the next order of expansion after integration over $p$ we have exactly $+t\Box$ that cancels gauge non-invariant contribution and then one gets

$$c_1 = G^2$$ (4.21)

as mentioned above.\(^3\) As a result in the given order of the expansion of the heat kernel, we obtain the super Yang-Mills action as a leading low-energy contribution to effective

\(^3\)Such a cancelation of gauge non-invariant terms should take place in any order of heat kernel expansion.
action. The IR cutoff parameter plays a role of the dimensional coupling constant:

$$\Gamma^{(1)}_{gh} = -\frac{1}{4\pi} \text{tr}_{Ab} \int d^5z \frac{1}{m} W^\alpha W_\alpha. \quad (4.22)$$

Discuss some consequence of (4.21). First, we see that in the theory under consideration the Chern-Simons action is not induced by quantum corrections. This conclusion was also pointed out in the end of subsection 4.1 as a result of vanishing the $\eta$-invariant. Leading low-energy quantum correction to action is Yang-Mills and stipulated only by ghosts and matter, vector multiplet does not give rise. Second, since a contribution of matter to effective action has, up to a sign, the same form as ghost contribution one can conclude that for appropriate matter in adjoint representation a total contribution of ghost and matter to effective action vanishes. Third, one applies the above consideration to BLG model formulated in terms of $\mathcal{N} = 2, d3$ superfields. In this case the ghost and matter superfields take the values in a real 3-algebra. The induced Yang-Mills action contains a factor $F_{AC}^D F_{BD}^C - 2(T_A)_I \{ (T_B)_J \} = -2G_{AB}$ [4]. Therefore the leading low-energy correction to action is

$$\Gamma_{YM} = \frac{1}{2\pi} G_{AB} \int d^5z \frac{1}{m} W^{A\alpha} W^{B\alpha}. \quad (4.23)$$

The quantities $G_{AB}$ and $W^{A\alpha}$ are defined in the section 2.

Using the second and third terms in expansion in proper time under the integral (4.20) one finds the coefficient $c_2$ in the form

$$c_2 = \{ \frac{1}{2} G^4 + \frac{1}{12} [D^m, D^n][D_m, D_n] + \frac{1}{6} [D^m, [D_m, G^2]] - \frac{1}{2} [D^\alpha, G^2] iW_\alpha \}
+ \frac{1}{6} [D^m, D^n][iW_\alpha] + \frac{1}{3} [D^m, D^n][D_m, iW_\alpha]. \quad (4.24)$$

In principle all commutators can be expressed in terms of strengths and their covariant derivatives. The third term in (4.24) vanishes since the transfer of the total derivative on the operator of parallel transport in the limit of coincidence gives $\int \ldots \{ D_m I (z, z') \} \{ z = z' = 0 \} = 0$. To find a component structure of the coefficient $c_2$ we use the definition (2.2) and Bianchi identities. As a result one gets in bosonic sector the terms of the form $\sim f^3, (Df)^2$ where $f_{mn}$ is bosonic strength. Note that for Abelian constant background the coefficient $c_2$ vanishes, since $\text{tr}(G\tilde{W}^\alpha W_\alpha + \tilde{W}^\alpha G W_\alpha) = \text{tr}G[\tilde{W}^\alpha, W_\alpha].$

The next $c_3$ coefficient in expansion (4.19) has a complicated and cumbersome enough structure. To compute it we should use the expansion of exponential in (4.20) in proper time from third to six order. The final result for $c_3$ is written as a sum of two kinds of terms. First, the terms of the form:

$$-\frac{1}{180} \text{tr}[ -6\mathcal{O}_1 + \mathcal{O}_2 + 4\mathcal{O}_3 + 3\mathcal{O}_4 + 3\mathcal{O}_5 ], \quad (4.25)$$

where

$$\mathcal{O}_1 = [D_m, D_n][D_n, D_l][D_l, D_m], \quad \mathcal{O}_2 = [D_m, [D_m, D_n]][D_l, [D_l, D_n]]. \quad (4.26)$$
\[ \mathcal{O}_3 = [\mathcal{D}_m, [\mathcal{D}_n, \mathcal{D}_l]][\mathcal{D}_m, [\mathcal{D}_n, \mathcal{D}_l]], \quad \mathcal{O}_4 = [\mathcal{D}_m, [\mathcal{D}_n, [\mathcal{D}_l, \mathcal{D}_l]]][\mathcal{D}_n, \mathcal{D}_l], \]
\[ \mathcal{O}_5 = [\mathcal{D}_n, \mathcal{D}_l][\mathcal{D}_m, [\mathcal{D}_n, [\mathcal{D}_l, \mathcal{D}_l]]]. \]

Under the sign of the matrix trace and the integral we have \( \text{tr} \int \mathcal{O}_3 = - \text{tr} \int \mathcal{O}_4 = - \text{tr} \int \mathcal{O}_5 \)
and \( \text{tr} \int \mathcal{O}_3 = \text{tr} \int (2\mathcal{O}_2 - 4\mathcal{O}_4) \) and then these terms are written as follows:

\[-\frac{1}{180} \text{tr}(2F^3 - 3(DF)^2).\]

Analogous contribution to effective action was first constructed in [20] for pure Yang-Mills theory.

Second kind of the terms it is convenient to group according to the degree \( W \). They have the form:

\[-\frac{1}{6} G^6 \] (4.27)

\[+ \frac{1}{6}[G^2, \mathcal{D}^\alpha] \{ W_\alpha, \mathcal{D}^\beta \} W_\beta - \frac{1}{6} \{ [G^2, \mathcal{D}^\alpha], \mathcal{D}^\beta \} W_\alpha W_\beta \] (4.28)

\[+ \frac{1}{12} [\mathcal{D}^\beta, \mathcal{D}_m] \{ W_\beta, \mathcal{D}^\alpha \} [W_\alpha, \mathcal{D}_m] + \frac{1}{12} [\mathcal{D}^\beta, \mathcal{D}_m] W_\beta [\mathcal{D}^\alpha, \mathcal{D}_m] W_\alpha \] (4.29)

\[+ \frac{i}{6} \{ 3[\mathcal{D}^\alpha, G^2] W_\alpha G^2 - [\mathcal{D}^\alpha, G^2][W_\alpha, G^2] \} \]

\[-\frac{i}{60} \{ -[\mathcal{D}_m, [\mathcal{D}_m, \mathcal{D}^\alpha]] [\mathcal{D}_n, [\mathcal{D}_n, W_\alpha]] + [\mathcal{D}^\alpha, [\mathcal{D}_m, [\mathcal{D}_n, \mathcal{D}_n]]][\mathcal{D}_m, W_\alpha] \} \] (4.30)

\[-2[\mathcal{D}^\alpha, \mathcal{D}_m][\mathcal{D}_m, \mathcal{D}_n] W_\alpha - 3[\mathcal{D}_m, \mathcal{D}_n][\mathcal{D}^\alpha, \mathcal{D}_n] W_\alpha \}

\[+ \frac{i}{12} \{ 2[\mathcal{D}_m, \mathcal{D}_n][\mathcal{D}_m, W_\alpha] G^2 - [\mathcal{D}^\alpha, G^2][\mathcal{D}_m, [\mathcal{D}_m, W_\alpha]] \} \] (4.31)

\[+ \frac{1}{12} [\mathcal{D}_m, G^2][\mathcal{D}_m, G^2] \] (4.32)

\[-\frac{1}{60} \{ [\mathcal{D}_m, [\mathcal{D}_m, [\mathcal{D}_n, G^2]]] + [\mathcal{D}_m, G^2][\mathcal{D}_n, [\mathcal{D}_m, \mathcal{D}_n]] - [\mathcal{D}_n, [\mathcal{D}_m, \mathcal{D}_n]][\mathcal{D}_m, G^2] \] (4.33)

\[-2G^2[\mathcal{D}_m, \mathcal{D}_n][\mathcal{D}_m, \mathcal{D}_n] + 2[\mathcal{D}_m, \mathcal{D}_n][\mathcal{D}_m, \mathcal{D}_n] G^2 + [\mathcal{D}_m, \mathcal{D}_n] G^2[\mathcal{D}_m, \mathcal{D}_n] \]

Total coefficient \( c_3 \) is given by sum of the terms (4.25)-(4.33). In leading bosonic component sector this coefficient gives us the terms of dimension 8 like \((f_{mn})^4\) and the products of some power of \( f_{mn} \) and some powers of covariant derivatives \( \mathcal{D}_m f_{mn} \) with total dimension 8.

In principle the coefficient \( c_3 \) can be transformed to the form which is expressed completely in terms of superfield strengths and their supercovariant derivatives, however such an expression will be extremely tedious and we are not going to write down it there. Note, that for specific goals and approximations only some certain terms in \( c_3 \) can be essential.
For example, let us consider the coefficient $c_3$ for constant Abelian background. Then one can show that this coefficient is reduced to the following form
\[
c_3 = \frac{1}{8} \bar{W}^\alpha \bar{W}_\alpha W^\beta W_\beta.
\]
(4.34)

As mentioned above, although we considered the heat kernel for the ghost, the case of the matter chiral superfields can also be treated along these lines. The results differ from ghost ones only by sign and choice of representation of gauge group for matter.

5 \hspace{1em} Summary and Conclusion

We have developed the background method for constructing the gauge invariant effective action in non-Abelian $\mathcal{N} = 2, d = 3$ supersymmetric Chern-Simons theory coupled to matter. Using the background field method we have studied a structure of one-loop effective action for the theory under consideration. One-loop effective action was formulated in terms of superfield determinants of the differential operators on superspace. To evaluate the determinants we have developed the $\mathcal{N} = 2, d = 3$ superfield proper time technique and formulated a procedure for computing the low-energy one-loop effective action. It was shown that the leading quantum correction is defined by ghost and matter superfields. As a result, the leading contribution in the case of adjoint matter is $\mathcal{N} = 2, d = 3$ Yang-Mills action. A few sub-leading higher derivative corrections are also calculated.

Background field method opens the possibilities for studying the effective action in various extended supersymmetric $d = 3$ models, which can be formulated in terms of $\mathcal{N} = 2$ superfields. From our point of view, a most important application of the methods developed in the paper is investigation of the effective action in BLG and ABJM models. These models are the $\mathcal{N} = 8$ and $\mathcal{N} = 6$ supersymmetric Chern-Simons theories coupled to specific matter supermultiplets. Both models are formulated in terms of $\mathcal{N} = 2$ superfields and hence their quantum aspects can be studied on the base of the background field method and proper time technique developed the given paper. However in this case we should consider the backgrounds containing not only vector multiplet superfield but also the matter superfields.

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References

[1] J.H. Schwarz, JHEP 0411 (2004) 078.

[2] J. Bagger, N. Lambert, Phys. Rev. D75 (2007) 045020; Phys. Rev. D77 (2008) 065008; JHEP 02 (2008) 105; A. Gustavsson, Nucl. Phys. B811 (2009) 66.

[3] O. Aharony, O. Bergman, D.L. Jafferis, J. Maldacena, JHEP 0810 (2008) 091; O. Aharony, O. Bergman, D.L. Jafferis, JHEP 0811 (2008) 043.

[4] J. Bagger, N. Lambert, Phys. Rev. D79 (2009) 025002; S. Cherkis and C. Saemann, Phys. Rev. D 78 (2008) 066019; P. de Medeiros, J. Figueroa-OFarrill, E. Mendez-Escobar, and P. Ritter, Commun. Math. Phys. 290 (2009) 871.

[5] A. Gustavsson, Nucl. Phys. B807 (2009) 315; N. Akerblom, C. Saemann, M. Wolf, Nucl. Phys. B826 (2010) 456.

[6] J.M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231.

[7] S.A. Hartnoll, Class. Quant. Grav. 26 (2009) 224002; C.P. Herzog, J. Phys. A42 (2009) 343001; J. McGreevy, Adv. High Energy Phys. 2010 (2010) 723105.

[8] Beisert et. al., Review of AdS/CFT Integrability: An Overview, arXiv:1012.3982 [hep-th].

[9] B. de Wit, J. Hoppe, H. Nicolai, Nucl. Phys. B305 (1988) 545; B. de Wit, M. Lusher, H. Nicolai, Nucl. Phys. B320 (1989) 135; W. Taylor, Rev. Mod. Phys. 73 (2001) 419.

[10] H. D. Berenstein, D. Trancanelli, Phys. Rev. D78 (2008) 106009; J.-H. Baek, S. Hyun, W. Jang, S.-H. Yi, Membrane Dynamics in Three dimensional N=6 Supersymmetric Chern-Simons Theory, arXiv:0812.1772 [hep-th]; T. Hirayama, D. Tomino, JHEP 0908 (2009) 071.

[11] S. Mukhi and C. Papageorgakis, JHEP 07 (2008) 085; B. Ezhuthachan, S. Mukhi and C. Papageouragis, JHEP 04 (2009) 101; T. Li, Y. Liu and D. Xie, Int. Journ. Mod. Phys. A24 (2009) 3039.

[12] S.V. Ketov, S. Kobayashi, Phys. Rev. D83 (2011) 045003.

[13] I. Chepelev, A.A. Tseytlin, Nucl. Phys. B515 (1998) 73; A. A. Tseytlin, Born-Infeld action, supersymmetry and string theory, arXiv: hep-th/9908105.

[14] P. Koerber, Fortsch. Phys. 52 (2004) 871.

[15] A. Refolli, A. Santambrogio, N. Terzi, D. Zanon, Nucl. Phys. B613 (2001) 64, Erratum-ibid. B648 (2003) 453; A. Bilal, Nucl. Phys. B618 (2001) 21.

[16] D.T. Grasso, JHEP 0211 (2002) 012, JHEP 0409 (2004) 054.
[17] E. Witten, Commun. Math. Phys. 121 (1989) 351.
[18] D. Birmingham, M. Blau, M. Rakowski, G. Thompson, Phys. Rept. 209 (1991) 129.
[19] D.G.C. McKeon, Annals Phys. 218 (1992) 325; F.A. Dilkes, L.C. Martin, D.G.C. McKeon, T.N. Sherry, Int. J. Mod. Phys. A14 (1999) 463; W. Chen, G.W. Semenoff, Y.-S. Wu, Phys. Rev. D46 (1992) 5521.
[20] D. Diakonov, V.Yu. Petrov, A.V. Yung, Phys. Lett. B130 (1983) 385; Yad. Fiz.39 (1984) 240; Sov. J. Nucl. Phys. 39 (1984) 150.
[21] I.L. Buchbinder, E.A. Ivanov, O. Lechtenfeld, N.G. Pletnev, I.B. Samsonov, B.M. Zupnik, JHEP 0903 (2009) 096, JHEP 0910 (2009) 075.
[22] I.L. Buchbinder, N.G. Pletnev, I.B. Samsonov, JHEP 04 (2010) 124, JHEP 1101 (2011) 121.
[23] S.J. Gates, M.T. Grisaru, M. Roček, W. Siegel, Superspace or one thousand and one lessons in supersymmetry, Benjamin/Cummings, Reading, U.S.A. (1983).
[24] I.L. Buchbinder, S.M. Kuzenko, Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace, IOP Publishing, Bristol and Philadelphia, 1998.
[25] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev, Harmonic Super- space, Campridge, UK: Univ. Press. 2001.
[26] E.I. Buchbinder, I.L. Buchbinder, E.A. Ivanov, S.M. Kuzenko, B.A. Ovrut, Phys. Part. Nucl.32 (2001) 641, Fiz.Elem.Chast.Atom.Yadra 32 (2001) 1222.
[27] E.A. Ivanov, Phys. Lett. B268 (1991) 203.
[28] B.M. Zupnik, D.G. Pak, Teor. Mat. Fiz. 77 (1988) 97; Theor. Math. Phys. 77 (1988) 1070; B.M. Zupnik, D.G. Pak, Sov. Phys. J. 31 (1988) 962.
[29] S.J. Gates, Jr., H. Nishino, Phys. Lett. B281 (1992) 72; H. Nishino, S.J. Gates, Jr., Int. J. Mod. Phys. A8 (1993) 3371; R. Brooks, S.J. Gates, Jr., Nucl. Phys. B432 (1994) 205; H. Nishino, S.J. Gates, Jr. Nucl. Phys. B480 (1996) 573.
[30] K. Hosomichi, K. M. Lee, S. Lee, S. Lee and J. Park, JHEP 0809 (2008) 002.
[31] M. Schnabl, Y. Tachikawa, JHEP 1009 (2010) 103.
[32] M.S. Bianchi, S. Penati, M. Siani, JHEP 1005 (2010) 106.
[33] P. Gilkey, Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem, CRC Press, 1995.
[34] H.C. Kao, K.M. Lee, T. Lee, Phys. Lett. B373 (1996) 94.
[35] S.M. Kuzenko, I.N. McArthur, JHEP 0305 (2003) 015.

[36] I.N. McArthur, T.D. Gargett, Nucl. Phys. B497 (1997) 525; T.D. Gargett, I.N. McArthur, J. Math. Phys. 39 (1998) 4430.

[37] N.G. Pletnev, A.T. Banin, Phys. Rev. D60 (1999) 105017.