HYPERBOLIC METRIC ON THE STRIP AND THE SCHWARZ LEMMA FOR HQR MAPPINGS

MIODRAG MATELJEVIĆ & MAREK SVETLIK

Abstract. We give simple proofs of various versions of the Schwarz lemma for real valued harmonic functions and for holomorphic (more generally harmonic quasiregular, shortly HQR) mappings with the strip codomain. Along the way using the principle of subordination and the corresponding conformal mapping, depicted on the Figure 1 we get a simple proof of a new version of the Schwarz lemma for real valued harmonic functions (see Theorems 4 and 5) and Theorem 6 related to holomorphic mappings. Using the Schwarz-Pick lemma related to distortion for harmonic mappings and the elementary properties of the hyperbolic geometry of the strip we prove Lemma 8 which is a key ingredient in the proof of Theorem 7 which yields optimal estimates for modulus of HQR mappings.

1. Introduction and preliminaries

Motivated by the role of the Schwarz lemma in complex analysis and numerous fundamental results (see [1, 20, 8, 16] and references cited there and for some recent result which are in our research direction [2, 7, 8, 14, 22]), in 2016, cf. [21](a), the first author has posted the current research project “Schwarz lemma, the Carathéodory and Kobayashi Metrics and Applications in Complex Analysis” ∗. Various discussions regarding the subject can also be found in the Q&A section on ResearchGate under the question “What are the most recent versions of the Schwarz lemma ?”. [21](b)†. In this project and in [16], cf. also [7] we developed the method related to holomorphic mappings with strip codomain (we refer to this method as the approach via the Schwarz-Pick lemma for holomorphic maps from the unit disc into a strip). Note here that our use of terms the Schwarz lemma and the Schwarz-Pick lemma is refer to the corresponding versions for modulus and hyperbolic distances, respectively (we follow the terminology used in [3]).

In particular our work here is related to previous works [15, 13], and some recent results of Kalaj and Vuorinen [7] (shortly KV-results; see also D. Khavinson [10], G. Kresin and V. Maz’ya [11]). As we mentioned in [16], it seems that KV-results influenced further research by H. Chen [4], M. Marković [14], A. Khalfallah [9] and P. Melentijević [19].

Date: August 20, 2018.

2010 Mathematics Subject Classification. Primary 30C80; Secondary 31C05, 30C75.

Key words and phrases. The Schwarz lemma; the Schwarz-Pick lemma; harmonic functions; holomorphic and quasiregular maps; hyperbolic metric on the strip.

Research partially supported by Ministry of Education, Science and Technological Development of the Republic of Serbia, Grant No. 174 632.

* Motivated by S. G. Krantz paper [8].
†The subject has been presented at Belgrade analysis seminar [19].
One of the purpose of this paper, which is a relatively elementary contribution and continuation of these research, is to demonstrate our approach and make a common frame for previous works.

Throughout this paper by \( U \) we denote the unit disc \( \{ z \in \mathbb{C} : |z| < 1 \} \). By the Riemann mapping theorem simply connected plane domains different from \( \mathbb{C} \) (we call these domains hyperbolic) are conformally equivalent to \( U \). Accordingly if \( \Omega \) is a hyperbolic domain then by \( \rho_\Omega(z)dz \) we denote the hyperbolic metric of \( \Omega \). This metric induces a hyperbolic distance on \( \Omega \) in the following way

\[
d_\Omega(z_1, z_2) = \inf_\gamma \int_{\gamma} \rho_\Omega(z)|dz|,
\]

where the infimum is taken over all \( C^1 \) curves \( \gamma \) joining \( z_1 \) to \( z_2 \) in \( \Omega \).

It is well known that \( \rho_U(z)|dz| = \frac{2|dz|}{1 - |z|^2} \) and immediately follows that for all \( z_1, z_2 \in U \) it holds

\[
d_U(z_1, z_2) = \ln \left| 1 + \frac{\sigma_U(z_1, z_2)}{1 - \sigma_U(z_1, z_2)} \right| = 2 \text{artanh} \sigma_U(z_1, z_2),
\]

where the pseudo-hyperbolic distance \( \sigma_U \) is given by \( \sigma_U(z_1, z_2) = \frac{|z_1 - z_2|}{1 - |z_1z_2|} \).

If \( f \) is a conformal map from hyperbolic domain \( \Omega \) onto \( U \) then the hyperbolic metric \( \rho_\Omega(z)|dz| \) of \( \Omega \) is defined by \( \rho_\Omega(z)|dz| = \rho_U(f(z))|f'(z)| \). Hence, one can transfer the concept of the hyperbolic distance from \( U \) on hyperbolic domain \( \Omega \).

For more details related to hyperbolic domains, hyperbolic metric and distance, see, for example \([1, 3, 17]\).

In this paper, except the disc \( U \), of other hyperbolic domains we will mainly use the strip \( S = \{ z \in \mathbb{C} : -1 < \text{Re} \, z < 1 \} \).

Let \( D, G \) be domains in \( \mathbb{C} \). By \( \text{Hol}(D, G) \) (respectively \( \text{Har}(D, G) \)) we denote the set of all holomorphic (respectively harmonic) mappings \( f : D \to G \).

By \( d_e \) we denote Euclidean distance in \( \mathbb{C} \) and for \( z \in \mathbb{C} \) we define the functions \( e \) and \( R_e \), by \( e(z) = d_e(0, z) = |z| \) and \( R_e(z) = \text{Re} \, z \), respectively.

For completeness we first give the classical Schwarz lemma which is a direct corollary of maximum modulus principle.

**Theorem 1** (The classical Schwarz lemma - the Schwarz lemma for holomorphic maps from \( U \) into \( U \)). Let \( f \in \text{Hol}(U, U) \) and \( f(0) = 0 \). Then

\[
(1) \quad |f(z)| \leq |z|, \quad \text{for all} \quad z \in U,
\]

and

\[
(2) \quad |f'(0)| \leq 1.
\]

In \([1]\) the equality holds for one \( z \in U \setminus \{ 0 \} \) and in \([2]\) the equality holds if and only if \( f(z) = az \), where \( a \in \mathbb{C} \) such that \( |a| = 1 \).

The following theorem is known as the Schwarz lemma for harmonic maps from \( U \) into itself.

**Theorem 2** (The Schwarz lemma for harmonic maps from \( U \) into \( U \). \([6, 5, p. 77]\)). Let \( f \in \text{Har}(U, U) \) and \( f(0) = 0 \). Then

\[
(3) \quad |f(z)| \leq \frac{4}{\pi} \text{arctan} \, |z|, \quad \text{for all} \quad z \in U,
\]
and this inequality is sharp for each point \( z \in U \).

Using the concept of the hyperbolic metric and hyperbolic distance on hyperbolic domains one can derive the Schwarz-Pick lemma for simply connected domains as a corollary of the classical Schwarz lemma:

**Theorem 3** (The Schwarz-Pick lemma for simply connected domains, [3, Theorem 6.4.]). Let \( \Omega_1 \) and \( \Omega_2 \) be hyperbolic domains and \( f \in \text{Hol}(\Omega_1, \Omega_2) \). Then

\[
\rho_{\Omega_2}(f(z))|f'(z)| \leq \rho_{\Omega_1}(z), \quad \text{for all} \quad z \in \Omega_1.
\]

and

\[
d_{\Omega_2}(f(z_1), f(z_2)) \leq d_{\Omega_1}(z_1, z_2), \quad \text{for all} \quad z_1, z_2 \in \Omega_1.
\]

In (4) and (5) the equalities hold if and only if \( f \) is a conformal isomorphism from \( \Omega_1 \) into \( \Omega_2 \).

In this paper we will use only the special case of this result if the domain is the unit disc and the codomain is the strip.

In Example 1 we consider the conformal mapping \( \phi \) from the unit disc \( U \) onto the strip \( S \). In Lemmas 1 and 2 we explicitly find the maximum and minimum of the function \( R_e \) and the maximum of the function \( e \) on the closed hyperbolic disc in \( S \) which is obtained as image of the closed hyperbolic disc with center \( 0 \) in \( U \) by mapping \( \phi \). Note that the proof of Lemma 1 is elementary and it is based on the properties of mapping \( \phi \). The proof of Lemma 2 is based on Proposition 2 which gives us an interesting relation between the hyperbolic and Euclidean distance on \( S \). Theorem 3 follows directly from the formula (12) of Lemma 1 and the subordination principle. Further development of this method yields Theorem 4 (without hypothesis that \( 0 \) is mapped to \( 0 \)) which seems to be a new result (see also Example 3 and Lemma 3).

It seems here that it is right place to emphasize the following difference between holomorphic and harmonic maps.

If \( f \) is holomorphic mapping from \( U \) into itself such that \( f(0) = b \), where \( b \in U \), then using the mapping \( f^b = \varphi_b \circ f \), where \( \varphi_b \) is conformal automorphism of \( U \) (see Example 2 below), we reduce this situation to the case \( b = 0 \), since \( f^b(0) = 0 \). As far as we know the researchers have some difficulties to handle the case \( f(0) = b \) if \( f \) is harmonic mapping from \( U \) into \((-1, 1) \), since in that case the mapping \( f^b \) is not harmonic in general. Our method overcome this difficulty.

To get optimal estimate for modulus of holomorphic (more generally HQR) mappings we use the elementary properties of the hyperbolic geometry of the strip, see Lemmas 2 and 3 and Theorems 4 and 5.

In order to establish Theorem 7 (the Schwarz lemma for HQR maps from \( U \) into \( S \)) among other things we will use the following elementary considerations (see [16]):

(I) Suppose that \( f \in \text{Hol}(U, S) \). Then by Theorem 3 we have \( \rho_S(f(z))|f'(z)| \leq \rho_U(z) \), for all \( z \in U \).

(II) If \( f = u + iv \) is a complex valued harmonic and \( F = U + iV \) is a holomorphic function on a domain \( D \) such that \( \text{Re} f = \text{Re} F \) on \( D \) (in this setting we say that \( F \) is associated to \( f \) or to \( u \) ), then \( F' = U_x + iV_y = U_x - iU_y = u_x - iu_y \).

Hence, if \( \nabla u = (u_x, u_y) = u_x + iu_y \) then \( F' = \overline{\nabla u} \) and \( |F'| = |\overline{\nabla u}| = |\nabla u| \).
(III) Suppose that $D$ is a simply connected plane domain and $f : D \to \mathbb{S}$ is a complex valued harmonic function. Then it is known from the standard course of complex analysis that there is a holomorphic function $F$ on $D$ such that $\text{Re } f = \text{Re } F$ on $D$, and it is clear that $F : D \to \mathbb{S}$.

(IV) The hyperbolic density $\rho_S$ at point $z$ depends only on $\text{Re } z$.

By (I)-(IV) it is readable that we have

**Proposition 1** ([16, Proposition 2.4], [7, 4]). Let $u : U \to (-1, 1)$ be harmonic function and let $F$ be holomorphic function which is associated to $u$. Then

$$\rho_S(F(z)) |\nabla u(z)| \leq \rho_U(z) \quad \text{for all } z \in U.$$

Note the above described simple method basically based on the Schwarz-Pick lemma for holomorphic maps from $U$ into $\mathbb{S}$ yields a proof of the above proposition to which we refer as the Schwarz-Pick lemma related to distortion for harmonic maps from $U$ into $(-1, 1)$. By this proposition we control distortion of HQR mappings.

Note here that there is tightly connection between the subordination principle and the various versions of the Schwarz-Pick lemma for holomorphic maps from $U$ into $\mathbb{S}$. Namely in the proof of Theorem [11] and Theorem [8] (the Schwarz lemma for holomorphic maps from $U$ into $\mathbb{S}$) we have used a corollary of the subordination principle which can be stated in the form (see Definition [4] for notation):

(V) If $f \in \text{Hol}(U, \mathbb{S})$ and $a \in U$, then the image of the hyperbolic disc in $U$ with hyperbolic center $a$ and hyperbolic radius $\lambda$ under $f$ is in the hyperbolic disc in $\mathbb{S}$ with hyperbolic center at $f(a)$ and hyperbolic radius $\lambda$.

Note that (V) is the Schwarz-Pick lemma for holomorphic maps from $U$ into $\mathbb{S}$. Instead of subordination principle in the proof of Theorem [8] (the Schwarz lemma for HQR maps from $U$ into $\mathbb{S}$) we use Lemma [4] which can be consider as a generalization of (V):

(VI) If $f \in \text{HQR}_K(U, \mathbb{S})$ (see definition below) and $a \in U$, then the image the hyperbolic disc in $U$ with hyperbolic center $a$ and hyperbolic radius $\lambda$ under $f$ is in the hyperbolic disc in $\mathbb{S}$ with hyperbolic center $f(a)$ and hyperbolic radius $K\lambda$.

Note that proof of Lemma [4] is based on Proposition [11](the Schwarz-Pick lemma related to distortion for harmonic maps from $U$ into $(-1, 1)$).

2. Some useful examples

Here we consider mappings related to extremal mappings.

**Example 1.** Let $\varphi$ be the mapping defined by $\varphi(z) = \tan \left( \frac{\pi z}{4} \right)$. It is easy to check that the mapping $\varphi$ is holomorphic and injective on $\mathbb{S}$ and maps $\mathbb{S}$ onto $U$. Therefore the inverse mapping of the $\varphi$ maps $U$ onto $\mathbb{S}$. Denote that inverse mapping by $\phi$. It is of interest to consider in more detail the mapping $\phi$. Let

- $\phi_1(z) = iz$,
- $\phi_2(z) = \frac{1+z}{1-z}$,
- $\phi_3(z) = \ln z$, where $\ln$ is branch of logarithm defined on $\{z \in \mathbb{C} : \text{Re } z > 0\}$ and determined by $\ln 1 = 0$,
- $\phi_4(z) = -i \frac{\pi}{2} z$. 

It is easy to check that \( \phi = \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1 \). Also, \( \phi(0) = 0 \).

Let’s emphasize that throughout this text by \( \varphi \) and \( \phi \) we always denote the mappings defined in Example 1.

**Example 2.** For \( a \in U \), define \( \varphi_a(z) = \frac{a - z}{1 - \overline{a}z} \). It is well known that \( \varphi_a \) is a conformal automorphism of \( U \). Specially, for \( a \in (-1, 1) \), the mapping \( \varphi_a \) has the following properties:

i) it is decreasing on \((-1, 1)\) and maps \((-1, 1)\) onto itself;

ii) for \( r \in [0, 1) \) it holds \( \varphi_a([-r, r]) = [\varphi_a(r), \varphi_a(-r)] = \left[ \frac{a - r}{1 - ar}, \frac{a + r}{1 + ar} \right] \).

**Example 3.** Let \( b \in S \) be arbitrary and let \( \phi_b \) be a conformal isomorphism from \( U \) onto \( S \) such that \( \phi_b(0) = b \) and \( \phi_b'(0) > 0 \). It is straightforward to check that \( \phi_b = \phi \circ \varphi_a \), where \( a = \tan \frac{b\pi}{4} \) and \( \varphi_a \) is defined in Example 2. Specially, for \( b \in (-1, 1) \), the mapping \( \phi_b \) has the following properties:

i) it is decreasing on \((-1, 1)\) and maps \((-1, 1)\) onto itself;

ii) for \( r \in [0, 1) \) it holds \( \phi_b([-r, r]) = [m_b(r), M_b(r)] \), where

\[
m_b(r) = \phi_b(r) = \frac{4}{\pi} \arctan \frac{a - r}{1 - ar} \quad \text{and} \quad M_b(r) = \phi_b(-r) = \frac{4}{\pi} \arctan \frac{a + r}{1 + ar}.
\]

3. **Some properties of the strip**

Since \( \varphi \) is a conformal isomorphism from \( S \) into \( U \) a simple computation gives

\[
\rho_S(z) = \rho_U(\varphi(z))|\varphi'(z)| = \frac{\pi}{2} \frac{1}{\cos \left( \frac{\pi}{2} \Re z \right)}, \quad \text{for all } z \in S.
\]

The following proposition gives us an interesting relation between the hyperbolic distance \( d_S \) and the Euclidean distance \( d_e \). It turns out that this relation is crucial for some of our investigation (see Lemma 2 and Theorems 6 and 7 below).

**Proposition 2.** Let \( z_1, z_2 \in S \). Then

\[
d_S(z_1, z_2) \geq \frac{\pi}{2} d_e(z_1, z_2).
\]

If \( z_1, z_2 \) are pure imaginary numbers then in (7) the equality holds.

**Proof.** Let \( \gamma \) be a \( C^1 \) curve such that joining \( z_1 \) to \( z_2 \) in \( S \). Since \( \rho_S(z) \geq \frac{\pi}{2} \) for all \( z \in S \), it follows that

\[
\int_{\gamma} \rho_S(z) |dz| \geq \frac{\pi}{2} \int_{\gamma} |dz|.
\]

In other words the hyperbolic length of the curve \( \gamma \) is great or equal to product of \( \frac{\pi}{2} \) and Euclidean length of the curve \( \gamma \). Since Euclidean length of the curve \( \gamma \) is great or equal to \( d_e(z_1, z_2) \) according to the inequality \( \text{8} \), we have

\[
\int_{\gamma} \rho_S(z) |dz| \geq \frac{\pi}{2} d_e(z_1, z_2).
\]

Take \( \frac{\pi}{2} \) infimum over all \( C^1 \) curves \( \gamma \) joining \( z_1 \) to \( z_2 \) in \( S \) we obtain

\[
d_S(z_1, z_2) \geq \frac{\pi}{2} d_e(z_1, z_2).
\]
On the other hand, let \( y_1, y_2 \in \mathbb{R} \) be arbitrary and let \( \hat{\gamma} : [0, 1] \to \mathbb{S} \) be defined by \( \hat{\gamma}(t) = iy_1 + i(y_2 - y_1)t \). An easy computation shows that

\[
\int_{\hat{\gamma}} \rho_S(z)|dz| = \frac{\pi}{2} |y_2 - y_1| = \frac{\pi}{2} d_e(y_1, y_2).
\]

\[\square\]

4. Euclidean properties of hyperbolic discs

**Definition 1.** Let \( \lambda > 0 \) be arbitrary. By \( D_\lambda(a) \) (respectively \( S_\lambda(b) \)) we denote the hyperbolic disc in \( \mathbb{U} \) (respectively in \( \mathbb{S} \)) with hyperbolic center \( a \in \mathbb{U} \) (respectively \( b \in \mathbb{S} \)) and hyperbolic radius \( \lambda \). More precisely \( D_\lambda(a) = \{ z \in \mathbb{U} : d_\mathbb{U}(z, a) < \lambda \} \) and \( S_\lambda(b) = \{ z \in \mathbb{S} : d_\mathbb{S}(z, b) < \lambda \} \). Also, \( \overline{D}_\lambda(a) = \{ z \in \mathbb{U} : d_\mathbb{U}(z, a) \leq \lambda \} \) and \( \overline{S}_\lambda(b) = \{ z \in \mathbb{S} : d_\mathbb{S}(z, b) \leq \lambda \} \) are corresponding closed discs. Specially, if \( a = 0 \) (respectively \( b = 0 \)) we omit \( a \) (respectively \( b \)) from the notations.

**Remark 1.** If \( f \) is a conformal isomorphism from \( \mathbb{U} \) onto \( \mathbb{S} \) such that \( f(a) = b \), then \( f(D_\lambda(a)) = S_\lambda(b) \) and \( f(\overline{D}_\lambda(a)) = \overline{S}_\lambda(b) \).

Let \( r \in (0, 1) \) be arbitrary. By \( U_r \) we denote Euclidean disc \( \{ z \in \mathbb{C} : |z| < r \} \) and by \( \overline{U}_r \) we denote the corresponding closed disc. Also, let

\[\lambda(r) = d_\mathbb{U}(r, 0) = \ln \frac{1 + r}{1 - r} = 2 \text{artanh } r.\]

Since \( d_\mathbb{U}(z, 0) = \ln \frac{1 + |z|}{1 - |z|} = 2 \text{artanh } |z| \) for all \( z \in \mathbb{U} \), we have

\[D_{\lambda(r)} = \{ z \in \mathbb{C} : 2 \text{artanh } |z| < 2 \text{artanh } r \} = \{ z \in \mathbb{C} : |z| < r \} = U_r,
\]

and similarly

\[\overline{D}_{\lambda(r)} = \overline{U}_r.\]

The closed discs \( \overline{D}_{\lambda(r)} \) and \( \overline{S}_{\lambda(r)} \) are shown on the Figure 1 and the following lemma to claim that disc \( \overline{S}_{\lambda(r)} \) be contained in a Euclidean rectangle.
Lemma 1. Let $r \in (0, 1)$ be arbitrary. Then
\begin{equation}
\overline{S}_{\lambda(r)} \subset \left[ -\frac{4}{\pi} \arctan r, \frac{4}{\pi} \arctan r \right] \times \left[ -\frac{2}{\pi} \lambda(r), \frac{2}{\pi} \lambda(r) \right].
\end{equation}
In particular,
\begin{equation}
Re(\overline{S}_{\lambda(r)}) = \left[ -\frac{4}{\pi} \arctan r, \frac{4}{\pi} \arctan r \right].
\end{equation}

Proof. Since $\overline{S}_{\lambda(r)} = \phi(U_r)$, where $\phi$ is defined in Example 1, it is sufficient to show that
\begin{equation}
\max \{|Re(\phi(z))| : z \in U_r\} = \frac{4}{\pi} \arctan r
\end{equation}
and
\begin{equation}
\max \{|Im(\phi(z))| : z \in U_r\} = \frac{2}{\pi} \lambda(r).
\end{equation}

Let $\phi_1, \phi_2, \phi_3$ and $\phi_4$ be as in Example 1. It is easy to check that $\phi_2 \circ \phi_1$ maps $\partial U_r$ onto $l_r$, where $l_r$ is circle with center $c = \frac{1 + r^2}{1 - r^2}$ and radius $R = \frac{2r}{1 - r^2}$. Also, for all $z \in l_r$ we have $Re(\phi_4(\phi_3(z))) = \frac{2}{\pi} \arg$ and $Im(\phi_4(\phi_3(z))) = -\frac{2}{\pi} \ln |z|$. Set $\theta_0 = \max \{ \arg z : z \in l_r \}$ and $L_0 = \max \{|\ln |z|| : z \in l_r\}$.

Let’s look at Figure 2. It is clear that line $y = (\tan \theta_0)x$ is a tangent from the point 0 on the circle $l_r$ and denote by $n_\theta_0$ the point of tangency. Also, note that $l_r$ intersect the $x-$axis at the points $c - R = \frac{1 - r}{1 + r}$ and $c + R = \frac{1 + r}{1 - r}$ which are

\[\text{Figure 1. } \overline{D}_{\lambda(r)} \text{ and } \overline{S}_{\lambda(r)}\]
reciprocal numbers. Thus, the power of the point 0 with respect to the circle $l_r$ is equal 1 and therefore $|n_{\theta_0}| = 1$. Now, it is obviously that $\tan \theta_0 = R$ and therefore

$$\theta_0 = \arctan R = \arctan \frac{2r}{1-r^2} = 2 \arctan r.$$  (15)

Further, since $\ln |n_{\theta_0}| = 1$ it is easy seen that $L_0 = \max\{-\ln(c - R), \ln(c + R)\}$. But, since $(c - R)(c + R) = 1$ it follows that $\ln(c + R) = -\ln(c - R)$ and therefore

$$L_0 = \ln(c + R) = -\ln(c - R) = \ln \frac{1+r}{1-r}. $$  (16)

From (15) and (16) we can get the equalities (13) and (14) (the details are left to the reader).

\[\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Angle $\theta_0$.}
\end{figure}\]

In order to appreciate the proof of the next lemma we give an example. For $s > 0$ set $R^s = [-1, 1] \times [-s, s]$ and let $\psi^s$ be conformal mapping of $U$ onto $R^s$ such that $\psi^s$ maps $(-i, i)$ onto $(-is, is)$ with $\psi^s(0) = 0$. Also, for $r \in (0, 1)$ set $E_{r,s} = \psi^s(U_r)$. We leave to the interested reader to check that for $r$ enough near to 1 the function $e$ on $E_{r,s}$ does not attain maximum at $\psi^s(ir)$.

**Lemma 2.** Let $\lambda > 0$ be arbitrary. Then

$$\max\{d_e(z, 0) : z \in \mathcal{S}_{\lambda}\} = \frac{2}{\pi} \lambda.$$  (17)

**Proof.** Let $z \in \mathcal{S}_{\lambda}$ be arbitrary. By Proposition 2 and since $z \in \mathcal{S}_{\lambda}$ we have

$$d_e(z, 0) \leq \frac{2}{\pi} d_{\mathcal{S}}(z, 0) \leq \frac{2}{\pi} \lambda.$$  (18)

It remains to show that there exists a $z_0 \in \mathcal{S}_{\lambda}$ such that $d_e(z_0, 0) = \frac{2}{\pi} \lambda$. Let $z_0 = i\frac{2}{\pi} \lambda$ or $z_0 = -i\frac{2}{\pi} \lambda$. Then it is clear that $d_e(z_0, 0) = \frac{2}{\pi} \lambda$ and by Proposition 2 we have $d_{\mathcal{S}}(z_0, 0) = \lambda$, i.e. $z_0 \in \mathcal{S}_{\lambda}$.

Recall that the above proof is based on the hyperbolic geometry of the strip. The reader can try to get a direct analytic proof without appeal to the geometry. \qed
Lemma 3. Let \( r \in (0, 1) \) and \( b \in (-1, 1) \) be arbitrary. Then
\[
Re(S_{\lambda(r)}(b)) = [m_b(r), M_b(r)],
\]
where \( m_b \) and \( M_b \) are defined in Example 3.

Proof. Let's repeat that \( D_{\lambda(r)} = U_r \) and \( S_{\lambda(r)}(b) = \phi_b(D_{\lambda(r)}) = \phi_b(U_r) \). Further, one can show that
\[
S_{\lambda(r)}(b) \text{ is symmetric with respect to the } x\text{-axis.}
\]
Also, by [3, Theorem 7.11]
\[
S_{\lambda(r)}(b) \text{ is Euclidean convex.}
\]
Now, from (19), (20) and parts i) and ii) in Example 3 the lemma follows. \( \square \)

5. The Schwarz lemma for harmonic functions from \( \mathbb{U} \) into \( (-1, 1) \)

In this section we first give a simple proof of the classical Schwarz lemma for harmonic functions and then use the same method we prove a new version (Theorem 5).

Theorem 4 ([6], [5, p. 77]). Let \( u : \mathbb{U} \to (-1, 1) \) be harmonic function such that \( u(0) = 0 \). Then
\[
|u(z)| \leq \frac{4}{\pi} \arctan |z|, \quad \text{for all } z \in \mathbb{U},
\]
and this inequality is sharp for each point \( z \in \mathbb{U} \).

Proof. Let \( z \in \mathbb{U} \) be arbitrary and \( r = |z| \). Since \( \mathbb{U} \) is simply connected it is well known that there exists \( f \in Hol(\mathbb{U}, \mathbb{S}) \) such that \( u = Re f \) and \( f(0) = 0 \). By subordination principle we have \( f(U_r) \subset \phi(U_r) \), where \( \phi \) is mapping defined in Example 1. Now, since \( U_r = D_{\lambda(r)}(r) \) and since \( S_{\lambda(r)}(b) = \phi_b(D_{\lambda(r)}) \) by Lemma 1 we obtain \( u(U_r) \subset \left[ -\frac{4}{\pi} \arctan r, \frac{4}{\pi} \arctan r \right] \) and the inequality (21) follows.

If \( z = 0 \) it is clear that in (21) the equality holds. In order to show that the inequality (21) is sharp and for \( z \neq 0 \), we define function \( \tilde{u} : \mathbb{U} \to (-1, 1) \) on the following way \( \tilde{u}(\zeta) = (Re \phi)(e^{-i \arg z} \zeta) \), where \( \phi \) is mapping defined in Example 1. Note that function \( \tilde{u} \) depend on the point \( z \). It immediately follows that \( \tilde{u} \) is harmonic function and \( \tilde{u}(0) = 0 \). A simple computation gives
\[
|\tilde{u}(z)| = |(Re \phi)(e^{-i \arg z} z)| = |(Re \phi)(|z|)| = \frac{2}{\pi} \arctan \frac{2 \Re |z|}{1 - |z|^2} = \frac{4}{\pi} \arctan |z|.
\]
\( \square \)

We leave to the interested reader to elaborate proofs of Theorems 4 and 5 using the Schwarz-Pick lemma (as in Lemma 3) instead of the subordination principle.

Remark 2. Using the rotation Theorem 3 stated in the introduction, follows easily from Theorem 4. For details see [5, p. 77] cf. also [6].

\[\text{§} \]

\[\text{§} \] Here values of \( \arg \) belong to the interval \( [0, 2\pi) \).
Theorem 5. Let \( u: \mathbb{U} \to (-1, 1) \) be harmonic function such that \( u(0) = b \) and let \( m_b \) and \( M_b \) be defined in Example 3. Then
\[
(22) \quad m_b(|z|) \leq u(z) \leq M_b(|z|), \quad \text{for all } z \in \mathbb{U},
\]
and this inequality is sharp for each point \( z \in \mathbb{U} \).

Proof. The proof is analogously to the proof of Theorem 4. Whereby, instead of the mapping \( \phi \) defined in Example 1 and Lemma 1 should be used the mapping \( \phi_b \) defined in Example 3 and Lemma 3. \( \square \)

6. The Schwarz lemma for holomorphic maps from \( \mathbb{U} \) into \( \mathbb{S} \)

Theorem 2 is usually considered as harmonic version of Theorem 1. In analogy with Theorems 1 and 2 we prove the next results (Theorems 6 and 7). Whereby, the codomain \( \mathbb{U} \) and the function \( \arctan \) are replaced by the strip \( \mathbb{S} \) and the function \( \text{artanh} \), respectively. For \( K = 1 \) Theorem 7 is reduced to Theorem 6.

Theorem 6 (The Schwarz lemma for holomorphic maps from \( \mathbb{U} \) into \( \mathbb{S} \)). Let \( f \in \text{Hol}(\mathbb{U}, \mathbb{S}) \) and \( f(0) = 0 \). Then
\[
(23) \quad |f(z)| \leq \frac{4}{\pi} \text{artanh}|z|, \quad \text{for all } z \in \mathbb{U}.
\]
The inequality (23) is sharp for each point \( z \in \mathbb{U} \). Also,
\[
(24) \quad |f'(0)| \leq \frac{4}{\pi}.
\]
In (23) the equality holds if and only if \( f(z) = \phi(\alpha z) \), where \( \alpha \in \mathbb{C} \) such that \(|\alpha| = 1 \), and \( \phi \) is mapping defined in Example 4.

Proof. Let \( z \in \mathbb{U} \) be arbitrary and \( r = |z| \). By subordination principle we have \( f(U_r) \subset \phi(U_r) \), where \( \phi \) is mapping defined in Example 4. Since \( U_r = \mathcal{D}_{\lambda(r)} \) and since \( \mathcal{S}_{\lambda(r)} = \phi(\mathcal{D}_{\lambda(r)}) \) we have \( f(U_r) \subset \mathcal{S}_{\lambda(r)} \). Hence, by Lemma 2 we obtain
\[
(25) \quad |f(z)| \leq \frac{2}{\pi} \lambda(|z|) = \frac{4}{\pi} \text{artanh}|z|.
\]
If \( z = 0 \) it is clear that in (23) the equality holds. In order to show that the inequality (23) is sharp and for \( z \neq 0 \), we define function \( \hat{f}: \mathbb{U} \to \mathbb{S} \) on the following way \( \hat{f}(\zeta) = \phi(ie^{-i\arg\zeta}) \), where \( \phi \) is defined in Example 4. Note that function \( \hat{f} \) depend on the point \( z \). It immediately follows that \( \hat{f} \in \text{Hol}(\mathbb{U}, \mathbb{S}) \) and \( \hat{f}(0) = 0 \). A simple computation gives
\[
|\hat{f}(z)| = |\phi(ie^{-i\arg\zeta})| = |\phi(i|z|)| = \left| -\frac{2}{\pi} \ln \frac{1 - |z|}{1 + |z|} \right| = \frac{4}{\pi} \text{artanh}|z|.
\]
Finally, by subordination principle we obtain \( |f'(0)| \leq |\phi'(0)| = \frac{4}{\pi} \) and theorem follows. \( \square \)

Here values of \( \arg \) belong to the interval \([0, 2\pi)\).
7. The Schwarz Lemma for Harmonic $K$-Quasiregular Maps from $U$ into $S$

Quasiregular maps are a class of continuous maps between Euclidean spaces $\mathbb{R}^n$ of the same dimension or, more generally, between Riemannian manifolds of the same dimension, which share some of the basic properties with holomorphic functions of one complex variable.

Let $D$ and $G$ be domains in $\mathbb{C}$. A $C^1$ mapping $f : D \to G$ we call sense-preserving $K$–quasiregular mapping if

a) $|f_z(z)| > |f_\bar{z}(z)|$ for all $z \in D$;

b) there exists $K \geq 1$ such that $\frac{|f_z(z)| + |f_\bar{z}(z)|}{|f_z(z)| - |f_\bar{z}(z)|} \leq K$ for all $z \in D$.

Thus the linear map $(df)_z = f_z(z)dz + f_\bar{z}(z)d\bar{z}$ maps circles with center at $z$ onto ellipses such that the ratio between the big axis and the small axis is uniformly bounded by $K$ with respect to $z \in D$.

Injective $K$–quasiregular mappings are called $K$–quasiconformal mappings. Quasiconformal maps play a crucial role in Teichmüller theory and complex dynamics.

The class of all harmonic sense-preserving $K$–quasiregular mapping $f : D \to G$ we denote by $\text{HQR}_K(D,G)$.

**Example 4.** Let $K \geq 1$ and let $A_K : S \to S$ be defined by $A_K(x, y) = (x, K y)$. It is clear that the mapping $A_K$ is sense-preserving $K$–quasiregular. Let $\psi_K = A_K \circ \phi$, where $\phi$ is the mapping defined in Example 1. It is easy to check that $\psi_K \in \text{HQR}_K(U, S)$.

**Lemma 4.** Let $K \geq 1$, $f \in \text{HQR}_K(U, S)$. Then

$$d_2(f(z_1), f(z_2)) \leq K d_2(z_1, z_2) \quad \text{for all} \quad z_1, z_2 \in U.$$  

**Proof.** Set $u = \text{Re} f$ and $\nabla u = (u_x, u_y)$. Since $f$ is $K$–quasiregular one can check that

$$|f_z(z)| + |f_\bar{z}(z)| \leq K |\nabla u(z)| \quad \text{for all} \quad z \in U.$$  

By Proposition 11 we have

$$|f_z(z)| |\nabla u(z)| \leq \rho_U(z) \quad \text{for all} \quad z \in U.$$  

From (24) and (28) it follows that

$$\rho_K(f(z)) \left( |f_z(z)| + |f_\bar{z}(z)| \right) \leq K \rho_U(z) \quad \text{for all} \quad z \in U.$$  

It is well known in general that the estimate of the gradient by means of the corresponding densities yields the corresponding estimate between the distances, see for example [16]. The detailed verification of it being left to the reader. In particular, we get (29).

Note that if codomain is $U$ the result of this type is proved in [13] and [12].

**Theorem 7** (The Schwarz lemma for HQR maps from $U$ into $S$). Let $K \geq 1$, $f \in \text{HQR}_K(U, S)$ and $f(0) = 0$. Then

$$|f(z)| \leq \frac{4}{\pi} K \text{artanh} |z|, \quad \text{for all} \quad z \in U,$$

and this inequality is sharp for each point $z \in U$. 


Proof. Let \( z \in \mathbb{U} \) be arbitrary. By the Lemma 4 we have
\[
d_S(f(z), 0) \leq Kd_U(z, 0).
\]
Since \( d_U(z, 0) = \lambda(|z|) \), from (31) it follows that \( f(z) \) belongs to the closed hyperbolic disc with hyperbolic center 0 and hyperbolic radius \( K\lambda(|z|) \), i.e. \( f(z) \in S_{K\lambda(|z|)} \).
Hence, by Lemma 2 we get
\[
|f(z)| \leq \frac{2}{\pi}K\lambda(|z|) = \frac{4}{\pi}K \text{artanh} |z|.
\]
As in the proof of Theorem 6, one can show that the inequality (30) is sharp. In this case, instead of mapping \( \phi \) defined in Example 1 the mapping \( \psi_K \) defined in Example 4 should be used.

\[\Box\]

Acknowledgement. The authors discussed the results of these types with members of Belgrade analysis seminar, B. Karapetrović, N. Mutavdić and B. Jevtić, who also obtained some results related to this subject. We are indebted to the above mention colleagues for useful discussions, and plan in forthcoming paper (hopefully with them) to discuss further progress.

References

[1] L. V. Ahlfors, Conformal Invariants, McGraw-Hill, New York, 1973.
[2] T. A. Azeroglu and B. N. Ornek, A refined Schwarz inequality on the boundary, Complex Var. Elliptic Equ. 58 (2013), no. 4, 571-577.
[3] A.F. Beardon and D. Minda, The Hyperbolic Metric and Geometric Function Theory, Proceedings of the International Workshop on Quasiconformal Mappings and their Applications (New Delhi, India, 2007), Narosa Publishing House, pp. 10-56.
[4] H. Chen, The Schwarz-Pick lemma and Julia lemma for real planar harmonic mappings, Sci. China Math. November 2013, Volume 56, Issue 11, pp 2327-2334.
[5] P. Duren, Harmonic mappings in the plane, Cambridge University Press, 2004.
[6] E. Heinz, On one-to-one harmonic mappings, Pacific J. Math. 9 (1959), 101-105.
[7] D. Kalaj and M. Vuorinen, On harmonic functions and the Schwarz lemma, Proc. Amer. Math. Soc. 140 (2012), no. 1, 161-165.
[8] S. G. Krantz, The Carathéodory and Kobayashi Metrics and Applications in Complex Analysis, arXiv:math/0608772v1 [math.CV] 31 Aug 2006.
[9] A. Khalfallah, Old and new invariant pseudo-distances defined by pluriharmonic functions, Complex Anal. Oper. Theory (2015) 9:113-119.
[10] D. Khavinson, An extremal problem for harmonic functions in the ball, Canadian Math. Bulletin 35(2) (1992), 218-220.
[11] G. Kresin and V. Mazya, Maximum Principles and Sharp Constants for Solutions of Elliptic and Parabolic Systems, Mathematical Surveys and Monographs, vol. 183, American Mathematical Society, Providence, RI, 2012.
[12] M. Knežević, A Note on the Harmonic Quasiconformal Diffeomorphisms of the Unit Disc, Filomat 29:2 (2015), 335-341.
[13] M. Knežević, M. Mateljević, On the quasi-isometries of harmonic quasi-conformal mappings, J. Math. Anal. Appl. 2007, 334(1), 404-413.
[14] M. Marković, On harmonic functions and the hyperbolic metric, Indag. Math. 26 (2015) 19-23.
[15] M. Mateljević, Ahlfors-Schwarz lemma and curvature, Kragujevac J. Math. 25 (2003) 155-164.
[16] M. Mateljević, Schwarz lemma and Kobayashi metrics for harmonic and holomorphic functions, J. Math. Anal. Appl. 464 (2018) 78-100.
[17] M. Mateljević, Topics in Conformal, Quasiconformal and Harmonic Maps, Zavod za udžbenike, Beograd, 2012.
[18] M. Mateljević, Communications at Belgrade analysis seminar, University of Belgrade, 2017 and 2018.
[19] P. Melentijević, Invariant gradient in refinements of Schwarz lemma and Harnack inequalities, Ann. Acad. Sci. Fenn. Math. 43 (2018) 391-399.
[20] R. Osserman, From Schwarz to Pick to Ahlfors and beyond, Notices Amer. Math. Soc. 46 (1999) 868-873.
[21] (a) https://www.researchgate.net/project/Schwarz-lemma-the-Caratheodory-and-Kobayashi-Metrics-and-Applications-in-Complex-Analysis
(b) https://www.researchgate.net/post/What_are_the_most_recent_versions_of_The_Schwarz_Lemma
(c) https://www.researchgate.net/publication/325430073_Miodrag_Mateljevic_Rigidity_of_holomorphic_mappings_Schwarz_and_Jack_Lemma
[22] Jian-Feng Zhu, Schwarz lemma and boundary Schwarz lemma for pluriharmonic mappings, manuscript December 2017, to appear in Filomat.

Faculty of mathematics, University of Belgrade, Studentski Trg 16, Belgrade, Republic of Serbia
E-mail address: miodrag@matf.bg.ac.rs
E-mail address: svetlik@matf.bg.ac.rs