SYMMETRIES OF HANDLEBODIES AND THEIR FIXED POINTS: DIHEDRAL EXTENDED SCHOTTKY GROUPS

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Abstract. A Schottky structure on a handlebody $M$ of genus $g$ is provided by a Schottky group of rank $g$. A symmetry (an orientation-reversing involution) of $M$ is known to have at most $(g + 1)$ connected components of fixed points. Each of these components is either a point or a compact bordered surface (either orientable or not) whose boundary is contained in the border of $M$. In this paper, we derive sharp upper bounds for the total number of connected components of the sets of fixed points of given two or three symmetries of $M$. In order to obtain such an upper bound, we obtain a geometrical structure description of those extended Kleinian groups $K$ containing a Schottky group $\Gamma$ as finite index normal subgroup so that $K/\Gamma$ is a dihedral group (called dihedral Schottky groups). Our upper bounds turn out to be different from the corresponding ones at the level of closed Riemann surfaces. In contrast to the case of Riemann surfaces, we observe that $M$ cannot have two different maximal symmetries.

1. Introduction

A closed Riemann surface is symmetric if it has an anticonformal involution (called a symmetry). Under the well known equivalence between compact Riemann surfaces and smooth irreducible projective complex algebraic curves, the symmetric ones correspond to real algebraic curves, that is, curves which may be defined over the real numbers. Harnack’s theorem asserts that a symmetry of a closed Riemann surface of genus $g$ has at most $(g + 1)$ connected components of fixed points; each of these being a simple loop and called an oval (or mirror). Sharp upper bounds for the total number of ovals of given $k \geq 2$ non-conjugated symmetries of Riemann surfaces are well known (see, for instance, [2, 3, 4, 6, 24]). In [11] it was proved that given integers $k_1, k_2 \in \{0, 1, \ldots, g\}$, there is a symmetric Riemann surface admitting two symmetries $\tau_1$ and $\tau_2$ so that $\tau_1$ has $k_1$ ovals. In [14] there is derived a sharp upper bound on the number of ovals of two symmetries. Most of that study is consequence of the well known structure of non-Euclidian crystallographic (NEC) groups [15].

A lowest regular planar cover of a closed Riemann surface $S$ of genus $g \geq 1$ is known to be provided by a Schottky group $\Gamma$ of rank $g$ (i.e., a purely loxodromic Kleinian group, with non-empty region of discontinuity, isomorphic to a free group of rank $g$). If $\Omega$ is the region of discontinuity of $\Gamma$, then $M = (\mathbb{H}^3 \cup \Omega)/\Gamma$ is a compact manifold homeomorphic to a handlebody of genus $g$ (we say that $\Gamma$ induces a Schottky structure on $M$), its interior $M^0 = \mathbb{H}^3/\Gamma$ carries a natural complete hyperbolic structure (that is, a complete Riemannian structure with constant negative curvature) and $S = \Omega/\Gamma$ is its conformal boundary.

A symmetry of the handlebody $M$ is an orientation reversing self-homeomorphism $\tau$ of order two, whose restriction to $M^0$ is a hyperbolic isometry; we also say that $M$ is symmetric. By lifting $\tau$ to the universal cover, we obtain an extended Kleinian group $\Gamma$ whose orientation preserving half is $\Gamma$ (called an extended Schottky group). A geometrical structure of the extended Schottky groups, in terms of the Klein-Maskit combination theorems [16, 17], was obtained in [5]; it provides the type of the corresponding symmetry in a similar way as NEC groups do for the case of symmetries of Riemann surfaces. It follows from such a geometrical description (see also [12]) that the locus of fixed points of $\tau$ has at most $g + 1$ connected components and that each of such connected components is either an isolated point (in the interior $M^0$) or a 2-dimensional bordered compact surface (which may or not be orientable) whose border
is contained in $S$. The interior of the quotient orbifold $M/\langle \tau \rangle = (\mathbb{H}^3 \cup \Omega)/\hat{\Gamma}$ fails to be a manifold exactly at the projection of the isolated fixed points of $\tau$ (locally looks like a cone over the projective plane) and its conformal boundary is the compact Klein surface $S/\langle \tau \rangle = \Omega/\hat{\Gamma}$. At this point it is important to note that the presence of isolated fixed points of $\tau$ is not detected by its restriction to $S$; so the above upper bound is not a direct consequence of Harnack’s result for symmetric Riemann surfaces. We say that the symmetry is maximal (for $M$) if the number of connected components of its locus of fixed points is $g + 1$.

Let us note that a symmetry of $M$ induces a symmetry on its conformal boundary closed Riemann surface $S$, but a symmetry on $S$ may not be extended to a symmetry of $M$. Nevertheless, a symmetry $\eta$ of $S$ always keeps invariant a collection of pairwise disjoint simple loops which cut-off the surface into planar surfaces. It follows that there is a new handlebody $N$ (with a suitable Schottky structure) admitting $S$ as its conformal boundary so that $\eta$ is induced by a symmetry of $N$. We should remark that if $S$ has two different symmetries, then it is not clear the existence of a common handlebody $N$ admitting two symmetries, each one inducing the given ones on $S$; this makes the difference on the study of symmetries at the level of handlebodies with that on Riemann surfaces.

Let $\tau_1, \tau_2$ be two different symmetries of a handlebody $M$ with Schottky structure induced by the Schottky group $\Gamma$ (of rank $g \geq 2$). As said before, by lifting the symmetry $\tau_j$, for $j = 1, 2$, to the universal cover space, we obtain an extended Schottky groups $\hat{\Gamma}_j$ containing $\Gamma$ as its orientation preserving half. These two groups generate an extended Kleinian group $K$ (called a dihedral extended Schottky group). In Theorem 4.1 we provide a geometrical structural description, in terms of the Klein-Maskit combination theorems, of these dihedral extended Schottky groups. As a consequence of such a geometrical structure description we are able to obtain sharp upper bounds for the number of connected components of $k \in \{2, 3\}$ symmetries on a handlebody (Theorems 2.1 and 2.4). Such upper bounds are slightly different as those ones for Riemann surfaces. In particular, we obtain that a handlebody has at most one maximal symmetry (Corollary 2.2). Let us reobserve that at the level of Riemann surfaces, Natanzon [22] proved that if a hyperelliptic Riemann surfaces admits a maximal symmetry, then it admits two of them. In the same paper it was proved the uniqueness of maximal symmetry in the non-hyperelliptic case.

2. Upper bounds for components of fixed points of symmetries

Sharp upper bounds for the number of ovals of two symmetries on a symmetric Riemann surface of genus $g$ were provided in [13]: if $q$ is the order of the product of these two symmetries, then such an upper bound is $2(g - 1)/q + 4$ (if $q$ is odd) and $4g/q + 2$ (if $q$ is even). If moreover, $q \geq 3$ (i.e., the two symmetries do not commute) and $q$ does not divides $g - 1$, then in [14] it was obtained the upper bound

$$\left\lfloor \frac{2(g - 1)}{q} \right\rfloor + 3,$$

where $[ ]$ stands for the integer part and $q$, and, moreover, such an upper bound is sharp. Next result provides a sharp upper bound at the level of symmetric handlebodies.

**Theorem 2.1.** Let $M$ be a handlebody of genus $g \geq 2$, with a Schottky structure, and let $\tau_1$, and $\tau_2$ be two different symmetries with $m_1$ and $m_2$ connected components of their fixed points, respectively. If $q$ is the order of $\tau_1 \tau_2$, then

$$m_1 + m_2 \leq 2 \left\lfloor \frac{g - 1}{q} \right\rfloor + 4.$$

Moreover, for every positive integer $q \geq 2$, the above upper bound is sharp for infinitely many values of $g$.

The proof of the above is done in Section 6 and in Section 8 we provide examples to see the sharp part. Let us observe that, for instance, for $g \geq 2$ and $q = 2$, the sharp upper bound at the level of Riemann surfaces is $2g + 2$, but at the level of handlebodies this is $g + 3$, from which we obtain the following fact (already observed in [10] if both symmetries only have isolated fixed points).

**Corollary 2.2.** A handlebody of genus $g \geq 2$ admits at most one maximal symmetry.
Theorem 2.1 asserts that the upper bound \( m_1 + m_2 = g + 3 \) may only occur if \( g = 2 \), that is, when \( \langle \tau_1, \tau_2 \rangle = \mathbb{Z}_2^2 \) (explicit examples are provided in [10]); so the following fact holds.

**Corollary 2.3.** Let \( M \) be a handlebody of genus \( g \geq 2 \), with a Schottky structure, and let \( \tau_1 \) and \( \tau_2 \) be two symmetries with \( m_1 \) and \( m_2 \) connected components of their fixed points, respectively. If \( m_1 + m_2 = g + 3 \), then \( \langle \tau_1, \tau_2 \rangle = \mathbb{Z}_2^2 \).

For the case of three symmetries on a closed Riemann surface of genus \( g \), in [23] it was proved that number of ovals is bounded above by \( 2(g + 2) \) and that such upper bound is sharp. The following provides the corresponding situation for the case of handlebodies. We denote by \( D \), the dihedral group of order \( 2r \).

**Theorem 2.4.** Let \( M \) be a handlebody of genus \( g \geq 2 \) with a Schottky structure and let \( \tau_1, \tau_2 \) and \( \tau_3 \) be three different symmetries and set \( H = \langle \tau_1, \tau_2, \tau_3 \rangle \). If \( m_i \) is the number of connected components of fixed points of \( \tau_i \) then

\[
m_1 + m_2 + m_3 \leq \begin{cases} 5 & \text{if } g = 2, \\ 8 & \text{if } g = 3, \\ g + 5 & \text{if } g \geq 4 \text{ and } H \neq \mathbb{Z}_2 \times D_r \text{ for any } r, \\ (r + 1)g + 5r - 1 & \text{if } g \geq 4 \text{ and } H \cong \mathbb{Z}_2 \times D_r \text{ for some } r. \end{cases}
\]

Moreover, the above upper bounds are sharp for \( g = 2, 3 \) and for infinite many values of \( g \geq 4 \).

### 3. Preliminaries and previous results

In this section we briefly review several definitions and basic facts we will need in the rest of the paper. More details on these topics may be found, for instance, in [18 20].

#### 3.1. Extended Kleinian groups.

We denote by \( \tilde{M} \) the group of Möbius and extended Möbius transformations (the composition of a Möbius transformation with the complex conjugation) and by \( M \) its index two subgroup of Möbius transformations. The group \( \tilde{M} \) can also be viewed, by the Poincaré extension theorem, as the group of hyperbolic isometries of the hyperbolic space \( \mathbb{H}^3 \); in this case, \( M \) is the group of orientation-preserving ones. Möbius transformations are classified into parabolic, loxodromic (including hyperbolic) and elliptic. Similarly, extended Möbius transformations are classified into pseudo-parabolic (the square is parabolic), glide-reflections (the square is hyperbolic), pseudo-elliptic (the square is elliptic), reflections (of order two admitting a circle of fixed points on \( \mathbb{C} \)) and imaginary reflections (of order two and having no fixed points on \( \mathbb{C} \)). Each imaginary reflection in \( \tilde{M} \) has exactly one fixed point in \( \mathbb{H}^3 \) and this point determines such reflection uniquely. If \( K \) is a subgroup of \( \tilde{M} \) not contained in \( M \), then \( K^+ = K \cap M \) is its canonical orientation-preserving subgroup.

A Kleinian group is a discrete subgroup of \( \tilde{M} \) and an extended Kleinian group is a discrete subgroup of \( \tilde{M} \) necessarily containing extended Möbius transformations. If \( K \) is a (extended) Kleinian group, then its region of discontinuity is the subset \( \Omega \) of \( \mathbb{C} \) composed by the points on which it acts discontinuously. Note that \( K \) is an extended Kleinian groups if and only if \( K^+ \) is a Kleinian group; both of them with the same region of discontinuity.

#### 3.2. Klein-Maskit’s combination theorems.

**Theorem 3.1** (Klein-Maskit’s combination theorem [16 17]).

1. (Free products) Let \( K_j \) be a (extended) Kleinian group with region of discontinuity \( \Omega_j \), for \( j = 1, 2 \). Let \( \mathcal{F}_j \) be a fundamental domain for \( K_j \) and assume that there is simple closed loop \( \Sigma \) contained in the interior of \( \mathcal{F}_1 \cap \mathcal{F}_2 \), bounding two discs \( D_1 \) and \( D_2 \), so that, for \( j \in \{1, 2\} \), \( \Sigma \cup D_j \subset \Omega_{3-j} \) is precisely invariant under the identity in \( K_{3-j} \). Then (i) \( K = \langle K_1, K_2 \rangle \) is a (extended) Kleinian group with fundamental domain \( \mathcal{F}_1 \cap \mathcal{F}_2 \) and \( K \) is the free product of \( K_1 \) and \( K_2 \) (ii) every finite order element in \( K \) is conjugated in \( K \) to a finite order element of either \( K_1 \) or \( K_2 \) and (iii) if both \( K_1 \) and \( K_2 \) are geometrically finite, then \( K \) is so.
2. (HNN-extensions) Let \( K_0 \) be a (extended) Kleinian group with region of discontinuity \( \Omega \), and let \( \mathcal{F} \) be a fundamental domain for \( K_0 \). Assume that there are two pairwise disjoint simple closed loops \( \Sigma_1 \) and \( \Sigma_2 \), both of them contained in the interior of \( \mathcal{F}_0 \), so that \( \Sigma_j \) bounds a disc \( D_j \) such that \( (\Sigma_1 \cup D_1) \cap (\Sigma_2 \cup D_2) = \emptyset \)
and that $\Sigma_j \cup D_j \subset \Omega$ is precisely invariant under the identity in $K_0$. Let $T$ be either a loxodromic transformation or a glide-reflection so that $T(\Sigma_j) = \Sigma_j$ and $T(D_j) \cap D_j = \emptyset$. Then (i) $K = (K_0, f)$ is a (extended) Kleinian group with fundamental domain $F_j \cap (D_1 \cup D_2)^c$ and $K$ is the HNN-extension of $K_0$ by the cyclic group $(T)$, (ii) every finite order element of $K$ is conjugated in $K$ to a finite order element of $K_0$ and (iii) if $K_0$ is geometrically finite, then $K$ is so.

3.3. Kleinian 3-manifolds and their automorphisms. If $K$ is a Kleinian group and $\Omega$ is its region of discontinuity, then associated to $K$ is a 3-dimensional orientable orbifold $M_K = (\mathbb{H}^3 \cup \Omega) / K$; its interior $M^0_K = \mathbb{H}^3 / K$ has a hyperbolic structure and its conformal border $S_K = \Omega / K$ has a natural conformal structure. If $K$ is torsion free, then $M_K$ and $M^0_K$ are orientable 3-manifolds and $S_K$ is a Riemann surface; we say that $M_K$ is a Kleinian 3-manifold and that $M_K$ and $S_K$ are uniformized by $K$.

Now, if $\hat{K}$ is an extended Kleinian group and $K = \hat{K}^+$, then the 3-orbifold $M_K$ admits the orientation-reversing homeomorphism $\tau : M_K \to M_K$ of order two induced by $\hat{K} - K$ and $M_K / \langle \tau \rangle = (\mathbb{H}^3 \cup \Omega) / \hat{K}$.

Let $M$ be a Kleinian 3-manifold, say $M = (\mathbb{H}^3 \cup \Omega) / \Gamma$, let $S = \Omega / \Gamma$ be its conformal boundary and let $M^0 = \mathbb{H}^3 / \Gamma$ be its interior hyperbolic 3-manifold. An automorphism of $M$ is a self-homeomorphism whose restriction to its interior $M^0$ is a hyperbolic isometry. An orientation-preserving automorphism is called a conformal automorphism and an orientation-reversing one an anticonformal automorphism. A symmetry of $M$ is an anticonformal involution. We denote by $\text{Aut}(M)$ the group of automorphisms of $M$ and by $\text{Aut}^+(M)$ the subgroup of conformal automorphisms. Let $\pi^0 : \mathbb{H}^3 \to M^0$ be the universal covering induced by $\Gamma$. Clearly, $\pi^0$ extends to a universal covering $\pi : \mathbb{H}^3 \cup \Omega \to M$ with $\Gamma$ as the group of Deck transformations. If $G \subset \text{Aut}(M)$ is a finite group and we lift it to the universal covering space $\mathbb{H}^3$ under $\pi^0$, then we obtain an (extended) Kleinian group $\hat{K}$ containing $\Gamma$ as a normal subgroup of finite index. The group $G$ contains orientation-reversing automorphisms if and only if $\hat{K}$ is an extended Kleinian group.

3.4. Schottky groups and their Kleinian manifolds. The Schottky group of rank 0 is just the trivial group. A Schottky group of rank $g \geq 1$ is a Kleinian group $\Gamma$ generated by loxodromic transformations $A_1, \ldots, A_g$, so that there are $2g$ disjoint simple loops, $C_1, C_1', \ldots, C_g, C_g'$, with a $2g$-connected outside $\mathcal{D} \subset \mathcal{G}$, where $A_i(C_i) = C_i'$, and $A_i(\mathcal{D}) \cap \mathcal{D} = \emptyset$, for $i = 1, \ldots, g$. The region of discontinuity $\Omega$ of $\Gamma$ is known to be connected and dense in $\mathcal{G}$, that $S = \Omega / \Gamma$ is a closed Riemann surface of genus $g$ and that the associated Kleinian manifold $M = (\mathbb{H}^3 \cup \Omega) / \Gamma$ is a handlebody of genus $g$. In this case, its interior $M^0 = \mathbb{H}^3 / \Gamma$ carries a geometrically finite complete hyperbolic Riemannian metric with injectivity radius bounded away from zero. Conversely, those geometrically finite hyperbolic structures for which the injectivity radius is bounded away from zero in the interior of a handlebody are provided by Schottky groups. As a consequence of the retrosection theorem \cite{11}, every closed Riemann surface can be uniformized by a Schottky group.

It is well known that a Schottky group of rank $g$ can be defined as a purely loxodromic Kleinian group of the second kind which is isomorphic to a free of rank $g$ \cite{19}. It also follows that every Kleinian structure on a handlebody is provided by a Schottky group. A Schottky group of rank $g$ can also be defined as a purely loxodromic geometrically finite Kleinian group which is isomorphic to a free of rank $g$ (essentially a consequence of the fact that a free group cannot be the fundamental group of a closed hyperbolic 3-manifold). It follows that every Kleinian structure on a handlebody is provided by a Schottky group. For this reason, we say that a Kleinian structure on a handlebody is a Schottky structure.

If $M$ is a handlebody of genus $g \geq 2$, with a Schottky structure, then it is known that $\text{Aut}(M)$ is finite; moreover, $\text{Aut}(M)$ has order at most $24(g - 1)$ and $\text{Aut}^+(M)$ has order at most $12(g - 1)$ \cite{23, 24}. Each conformal (respectively anticonformal) automorphism of $M$ induces a conformal (respectively anticonformal) automorphism of the conformal boundary $S$ and the later determines the former due to the Poincare extension theorem.

Let $M$ be a topological handlebody of genus $g$ and let $H$ be a finite group of homeomorphisms of $M$. It is well known that there are a (extended) Kleinian group $K$, containing as a normal subgroup a Schottky group $\Gamma$ of rank $g$, and an orientation preserving homeomorphism $f : M \to M_f$, where $M_f = (\mathbb{H}^3 \cup \Omega) / \Gamma$ is the handlebody uniformized by $\Gamma$, with $f \# f^{-1} = K / \Gamma$. This is in really consequence of the fact that a handlebody is a compression body (see also \cite{26}). In this way, to obtain examples
of handlebodies with groups of automorphisms, one may just work at topological constructions. These topological constructions may be produced by fattening up some symmetrical graphs; for instance Cayley graphs. The examples we produce in this paper are done in terms of Schottky groups, but they can be obtained as before by using (finite extensions of) dihedral groups and their Cayley graphs.

3.5. **Extended Schottky groups.** An extended Schottky group of rank \( g \) is an extended Kleinian group whose canonical orientation-preserving subgroup is a Schottky group of rank \( g \). Those extended Schottky groups containing no reflections are called **Klein-Schottky groups** (these were previously considered in \([10]\)) and the others are called **reflection Schottky groups**. As a consequence of the results in \([5, 12]\), it is possible to obtain a geometric structural description of extended Schottky groups in terms of the Klein-Maskit combination theorems as follows.

**Theorem 3.2** \([5]\). An extended Schottky group is the free product (in the Klein-Maskit combination theorem sense) of the following kind of groups:

(i) cyclic groups generated by reflections,
(ii) cyclic groups generated by imaginary reflections,
(iii) cyclic groups generated by glide-reflections,
(iv) cyclic groups generated by loxodromic transformations, and
(v) real Schottky groups (that is groups generated by a reflection and a Schottky group keeping invariant the corresponding circle of its fixed points).

Conversely any group of Möbius and extended Möbius transformations constructed using \( \alpha \) groups of type (i), \( \beta \) groups of type (ii), \( \gamma \) groups of type (iii), \( \delta \) groups of type (iv) and \( \epsilon \) groups of type (v), is an extended Schottky group if and only if \( \alpha + \beta + \gamma + \epsilon > 0 \). If, in addition, the real Schottky groups above have the ranks \( r_1, \ldots, r_\epsilon \geq 1 \), then \( \Gamma^+ \) is a Schottky group of the rank \( g = \alpha + \beta + 2(\gamma + \delta) + \epsilon - 1 + r_1 + \ldots + r_\epsilon \).

Let us consider a handlebody \( M \), with a Schottky structure induced by the Schottky group \( \Gamma \). If \( \tau : M \to M \) is a symmetry, then the lifting of \( \tau \) to the universal cover space produces an extended Schottky group \( \hat{\Gamma} \) containing \( \Gamma \) as its index two orientation preserving part and so that \( \hat{\Gamma}/\Gamma = \langle \tau \rangle \). Conversely, every extended Schottky group is obtained by the lifting of a symmetry of a handlebody with a Schottky structure. Direct consequence of Theorem 3.2 is then the following.

**Corollary 3.3.** Let \( \hat{\Gamma} \) be an extended Schottky group constructed, as in Theorem 3.2, using \( \alpha \) groups of type (i), \( \beta \) groups of type (ii), \( \gamma \) groups of type (iii), \( \delta \) groups of type (iv) and \( \epsilon \) groups of type (v). If \( \Gamma \) is the canonical orientation-preserving subgroup of \( \hat{\Gamma} \) and \( M = (\hat{\Gamma} \cup \Omega)/\Gamma \), then \( \hat{\Gamma} \) induces a symmetry of \( M \) whose connected components of fixed points consist of two dimensional closed discs, \( \beta \) isolated points, and \( \epsilon \) two dimensional non-simply connected compact surfaces. In particular, \( \hat{\Gamma} \) is a symmetry of a Kleinian manifold homeomorphic to a handlebody of genus \( g \), and \( n_0 \) is the number of isolated fixed points of \( \tau \), \( n_1 \) is the number of total ovals in the conformal boundary and \( n_2 \) is the number of two-dimensional connected components of the set of fixed points of \( \tau \), then \( n_0 + n_1, n_0 + n_2 \in \{0, 1, \ldots, g + 1\} \).

3.6. **Lifting criteria.** Next we recall a simple criterion for lifting loops which we will need in the structure description of the dihedral extended Schottky groups. This is a direct consequence of the Equivariant Loop Theorem \([21]\), whose proof is based on minimal surfaces, that is, surfaces that minimize locally the area. In \([9]\) a proof which only uses arguments proper to (planar) Kleinian groups is provided. A function group is a pair \((G, \Delta)\), where \( G \) is a finitely generated Kleinian group and \( \Delta \) is a \( G \)-invariant connected component of its region of discontinuity.

**Theorem 3.4.** \([9, 21]\) Let \((G, \Delta)\) be a torsion free function group uniformizing a closed Riemann surface \( S \) of genus \( g \geq 2 \), that is, there is a regular covering \( P : \Delta \to S \) with \( G \) as the group of covering transformations, and let \( H \) be a group of automorphism of \( S \). Then, \( H \) lifts to the above regular planar covering if and only if there is a collection \( \mathcal{F} \) of pairwise disjoint simple loops on \( S \) such that:

(i) \( \mathcal{F} \) defines the regular planar covering \( P : \Delta \to S \); and
(ii) \( \mathcal{F} \) is invariant under the action of \( H \).
3.7. A counting formula. Let \( \hat{K} \) be an extended Kleinian group containing a Schottky group \( \Gamma \) of rank \( g \) as a normal subgroup of finite index (the last trivially holds if \( g \geq 1 \)). Let \( G = \hat{K}/\Gamma \) and let us denote by \( \theta : \hat{K} \to G \), the canonical projection. If there is some involution \( \tau \in G \), which is the \( \theta \)-image of an extended Möbius transformation, then \( \hat{\Gamma} = \theta^{-1}(\langle \tau \rangle) \) is an extended Schottky group whose canonical orientation-preserving subgroup is \( \Gamma \). As consequence of Theorem 3.2, the group \( \hat{\Gamma} \) is constructed using \( \alpha \) reflections, \( \beta \) imaginary reflections and \( \epsilon \) real Schottky groups. Theorem 3.5 below asserts that values \( \alpha, \beta \) and \( \epsilon \) can be obtained from \( \hat{K}, G \) and \( \theta \). First we provide some necessary definitions.

Let \( C = \{c_i : i \in I\} \) be a maximal collection of anticonformal involutions (i.e. reflections and imaginary reflections) in \( \hat{K} \) which are non-conjugate there. The group \( \hat{K} \) is geometrically finite, as it is a finite extension of a Schottky group, so one can prove that the set \( C \) is finite. However, due to the Theorem 3.2, we will not need it explicitly in the proof of the next theorem and so we do not go into details. We call the set \( C \) a complete set of symmetries of \( \hat{K} \) and we shall refer to its elements as to canonical symmetries. For each \( i \in I \) we set \( I(i) \subset I \) defined by those \( j \in I \) so that \( \theta(c_i) \) and \( \theta(c_j) \) are conjugate in \( G \) (in particular, \( i \in I(i) \)). Note at this point that it may happen that for \( j \in I(i) \), \( c_j \) can be imaginary reflection even if \( c_i \) is a reflection and viceversa. Such situations occur when \( \theta(c_i) \) is a symmetry containing both isolated fixed points and two-dimensional components of the set fixed points. We set by \( J(i) \) the subset of \( I(i) \) defined by those \( j \) for which \( c_j \) is an imaginary reflection. We also set by \( F(i) \subset I(i) - J(i) \) for those \( j \) for which \( c_j \) has a finite centralizer in \( \hat{K} \) and \( E(i) = I(i) - (J(i) \cup F(i)) \). Note that, as \( \hat{\Gamma} \) has a finite index in \( \hat{K} \), a reflection \( c \in \hat{K} \) has an infinite centralizer \( C(K, c) \) in \( \hat{K} \) if and only if it has an infinite centralizer in \( \hat{\Gamma} \).

**Theorem 3.5** ([3]). Let \( \hat{K} \) be an extended Kleinian group containing a Schottky group \( \Gamma \) as a finite index normal subgroup. Let \( G = \hat{K}/\Gamma \), let \( \theta : \hat{K} \to G \) be the canonical projection and let \( C = \{c_i : i \in I\} \) be a complete set of symmetries of \( \hat{K} \). Then \( \hat{\Gamma}_i = \theta^{-1}(\langle \theta(c_i) \rangle) = \langle \Gamma, c_i \rangle \) is an extended Schottky group, constructed using \( \alpha \) reflections, \( \beta \) imaginary reflections and \( \epsilon \) real Schottky groups as in Theorem 3.2.

where
\[
\alpha = \sum_{j \in F(i)}[C(G, \theta(c_j)) : \theta(C(\hat{K}, c_j))],
\beta = \sum_{j \in J(i)}[C(G, \theta(c_j)) : \theta(C(\hat{K}, c_j))],
\epsilon = \sum_{j \in E(i)}[C(G, \theta(c_j)) : \theta(C(\hat{K}, c_j))].
\]

4. Structural description of dihedral extended Schottky groups

In this section we provide the structural picture of dihedral extended Schottky groups in terms of the Klein-Maskit combination theorems.

**Theorem 4.1.** (1) A dihedral extended Schottky group is the free product (in the sense of the Klein-Maskit combination theorems) of the following groups (see Figure 1)

(i) \( \alpha \) cyclic groups generated by reflections;
(ii) \( \beta \) cyclic groups generated by imaginary reflections;
(iii) \( \gamma \) cyclic groups generated by loxodromic transformations;
(iv) \( \delta \) cyclic groups generated by glide-reflections;
(v) \( \varepsilon \) groups generated by a reflection and some finite number of elliptic transformations and reflections, each of them commuting with the previous reflection,

so that \( \alpha + \beta + \delta + \varepsilon > 0 \).

(2) Let \( K \) be any extended function group constructed by the previous groups. Then, \( K \) is a dihedral extended Schottky group if and only if there is a surjective homomorphism from
\[
\varphi : K \to D_p,
\]
with kernel a Schottky group so that \( \varphi(K^+) \) is the cyclic group of order \( p \), for some positive integer \( p > 1 \).

(3) Let \( K \) be an extended Kleinian group constructed as in (1), using \( \alpha \) groups of type (i), \( \beta \) groups of type (ii) \( \gamma \) groups of type (iii), \( \delta \) groups of type (iv) and \( \varepsilon \) groups of type (v) with \( \alpha + \beta + \delta + \varepsilon > 0 \). Let us assume that \( \Gamma_1, \ldots, \Gamma_\varepsilon \) are the groups of type (v) and assume \( \Gamma_i \) is constructed using the reflections
σ, σ₁, . . . , σₙ and elliptic transformations t₁, . . . , tₙ (the reflections σᵢ and the elliptics tᵢ each one commuting with σ₁). Let Ω be the region of discontinuity of K. Then,

(a) \( \hat{Ω}/K^+ \) has genus

\[
\hat{g} = α + β + 2(γ + δ) + ε - 1,
\]

with \( 2(m_1 + \cdots + m_ε) \) conical points of orders \( |t_{11}|, |t_{12}|, |t_{13}|, \ldots, |t_{r_i}|, |t_{r_j}| \), where \( |t| \) denotes the order of \( t \), and \( 2(n_1 + \cdots + n_ε) \) conical points of order 2.

(b) \( K \) is a dihedral extended Schottky group if and only if any of the followings hold.

(i) \( α + β + δ ≥ 2 \).

(ii) \( α + β + δ = 1 \) and \( ε > 0 \).

(iii) \( α + β + δ = 0 \) and \( ε ≥ 2 \).

(iv) \( α + β + δ = 0 \), \( ε = 1 \) and \( n_1 > 0 \).

(c) If \( K \) is a dihedral extended Schottky group and \( Γ \triangleleft K \) is a Schottky group of rank \( g \) which is a normal subgroup of \( K \) so that \( K/Γ \cong D_n \), then

\[
g = n(\hat{g} - 1) + 1 + n \sum_{i=1}^{ε} \sum_{k=1}^{m_i} \left(1 - \frac{1}{|t_{ik}|}\right) + \frac{n}{2} \sum_{i=1}^{ε} n_i,
\]

where \( \hat{g} \) is as in (a).

Remark 4.2. Let us note that Parts (3)(a) and (3)(c) are direct consequence of Riemann-Hurwitz formula and that Part (3)(b) is consequence of Part (2). Also, in Part (3), if \( t_{jk} \) is elliptic of order two, then \( \sigma_j t_{jk} \) is an imaginary reflection commuting with \( \sigma_j \). In this way, we only need to provide the proof of Parts (1) and (2) of the above theorem.

\[\begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram.png}
\end{array}\]

Figure 1. Structure of dihedral extended Schottky groups

5. Proof of Theorem 4.1 Parts (1) and (2)

Let us consider a dihedral extended Schottky group \( \hat{G} \), generated by two different extended Schottky groups \( G_1 \) and \( G_2 \), so that \( G_1^+ = G_2^+ = G \). Let us denote by \( \hat{G}^+ \) the index two orientation preserving half of \( \hat{G} \). It follows that \( G, G_1, G_2, \hat{G} \) and \( \hat{G}^+ \) all have the same limit set, and hence the same region of discontinuity, \( Ω \), on \( \hat{C} \). We set \( S^+ = \hat{Ω}/\hat{G} \); \( S_1 = \hat{Ω}/G_1 \); \( S_2 = \hat{Ω}/G_2 \); \( S_+ = \hat{Ω}/\hat{G}^+ \); and \( S = \hat{Ω}/\hat{G} \). Since \( \hat{G} \) is a finite extension of \( G \), and \( S^+ \) is a closed Riemann surface, \( S \) is a compact not necessarily orientable orbifold, with a finite number of orbifold points and possible non-empty boundary, and \( S_+ \) is a closed Riemann surface with some finite number of orbifold points. On \( S_+ \) we have an anticonformal involution, preserving the finite set of orbifold points, so that quotient of \( S_+ \) by it is \( S \).

Proposition 5.1. (i) The dihedral extended Schottky group \( \hat{G} \) contains no parabolic elements. (ii) If \( k \in \hat{G}^+ \) is an elliptic transformation with one fixed point in \( Ω \), then both fixed points of \( k \) belong to \( Ω \).
Proof. (i) follows at once from the fact that $[\hat{G} : G] < \infty$ and $G$ contains no parabolic elements. As $\hat{G}^+$ is a geometrically finite function group, (ii) follows from [8]. □

Since $G$ is a subgroup of index 2 in $G_i$, $i = 1, 2$, we have that each orientation reversing element $\eta_i \in G_i$ induces an orientation-reversing conformal automorphism $\tau_i$ of $S^+$, so that $S_i = S^+ / \langle \tau_i \rangle$. As $\eta_i^2 \in G$, we have that $\eta_i$ is either: (i) a reflection or (ii) an imaginary reflection or (iii) semi-hyperbolic.

**Remark 5.2.** In the case the extended Schottky group $G_j$ is a Klein-Schottky group, we have that $\eta_j$ is either an imaginary reflection or semi-hyperbolic.

Let $J$ be the group of automorphisms of $S^+$ generated by $\tau_1$ and $\tau_2$; then $S = S^+ / J$. Since $S^+$ is closed, and $\tau_1 \neq \tau_2$, there is some smallest positive power $p > 1$ so that $(\tau_1 \circ \tau_2)^p = 1$, that is, $J$ is a dihedral group of order $2p$. It follows that every orientation-reversing element of $J$ is conjugate in $J$ to either $\tau_1$ or $\tau_2$.

**Remark 5.3.** (1) In the case that both extended Schottky groups $G_1$ and $G_2$ are Klein-Schottky groups, we have that all anticonformal involutions in $J$ are imaginary reflections. (2) If $p$ is odd, then $\tau_1$ and $\tau_2$ are conjugates in $J$ and, in particular, $G_1$ and $G_2$ are conjugates in $\hat{G}$.

Lifting to $\Omega$, we see that every orientation-reversing element of $\hat{G}$ either acts without fixed points on $\Omega$ or they are reflections.

**Proposition 5.4.** Every orientation-reversing element of $\hat{G}$ is either semi-hyperbolic or an imaginary reflection or a reflection.

Proof. Every orientation-reversing element of $\hat{G}$ is a lift of a conjugate of either $\tau_1$ or $\tau_2$, each one either a reflection or an imaginary reflection. It follows that if $a \in \hat{G}$ is orientation-reversing, then $a^2 \in G$. In this way $a^2 = 1$, in which case $a$ is either a reflection or an imaginary reflection, or $a^2$ is a loxodromic transformation, then $a$ is neither semi-parabolic nor semi-elliptic. □

**Proposition 5.5.** There is a canonical surjective homomorphism $\Phi : \hat{G} \to J$ with kernel $G$.

Proof. Denote the projection from $\Omega$ to $S$ by $\pi$, and denote the projection from $\Omega$ to $S^+$ by $\pi^+$. Choose any point $z \in \Omega$ and any element $a \in \hat{G}$. Then, since $\pi(z) = \pi(a(z))$, there is a $j \in J$ so that $\pi^+(a(z)) = j(\pi^+(z))$. It is clear that this map is continuous in $z$; hence $j$ depends only on $a$. Hence we have defined $\Phi(a) = j$. Now suppose that $\Phi(a_1) = j_1$, $i = 1, 2$. Then $\pi^+(a_i(z)) = \pi^+(a_i(\pi^+(z))) = \Phi(\pi^+(a_i(\pi^+(z))) = 1.

This shows that $\Phi$ is a homomorphism; it is obvious that its kernel is exactly $G$. It is also obvious that the image of $\Phi$ is contained in $J$. Since every element of $J$ lifts to $\Omega$ to yield an element of $\hat{G} - G$, $\Phi$ maps onto $J$.

The next result make explicit some properties of $\Phi$ we will need. First, we need some remark. The dihedral group $J$ has order $2p$. If $p$ is odd, then every involution on $J$ is orientation reversing; but if $p$ is even, then there are involutions in $J$ that preserve orientation. In the particular case that $p = 2$, we have that any two involutions in $J$ generate it.

**Proposition 5.6.** The homomorphism $\Phi : \hat{G} \to J$ given by Proposition 5.5 has the following properties.

1. No elliptic element of $\hat{G}$ lies in the kernel of $\Phi$.
2. Let $\alpha$ and $\beta$ be involutory generators (both orientation reversing if $p = 2$) of the dihedral group $J$; that is $J = \langle \alpha, \beta | \alpha^2 = \beta^2 = (\alpha \beta)^p = 1 \rangle$. Then every element of $\Phi^{-1}(\alpha)$, and every element of $\Phi^{-1}(\beta)$, reverses orientation.
3. If $\alpha$ and $\beta$ are as above, then $K_1 = \Phi^{-1}(\langle \alpha \rangle)$ and $K_2 = \Phi^{-1}(\langle \beta \rangle)$ are extended Schottky groups with $K_j^+ = G$ so that $\hat{G} = \langle K_1, K_2 \rangle$.

Proof. The first statement follows at once from the fact that the kernel of $\hat{G}$ is a Schottky group, which is free. The second statement follows at once from the fact that if $a_1, a_2 \in \Phi^{-1}(\alpha)$, then $a_1a_2 \in Ker(\Phi) = G$,
that is, both reverse orientation or both preserve it. As we have assumed that \( \alpha \) is orientation reversing, we must have some orientation reversing Möbius transformation in \( \Phi^{-1}(\alpha) \). The argument is similar for \( \beta \). The last statement is now easy to see.

\[ \square \]

**Remark 5.7.** Observe that \( \Phi(\hat{G}^+) \) is the cyclic group \( J^+ \) of order \( p \) generated by \( ab \) in Proposition 5.6. As \( G \triangleleft \hat{G}^+ \), \( \Phi \) restrict to a surjective homomorphism from \( \hat{G}^+ \) onto the cyclic group \( J^+ \) with kernel the Schottky group \( G \).

We need some topological information about the group \( \hat{G} \). We first need to generalize the decomposition of a function group into its structure subgroups to the case of an extended Kleinian group. Let us consider the regular planar Schottky covering \( (\hat{\Omega}, \hat{G}, P : \hat{\Omega} \to S^+) \). By construction, we have that the group \( J \) lifts under \( P \) and such a lifting is the group \( \hat{G} \). As a consequence of Theorem 3.4, we have the existence of a collection of pairwise disjoint loops \( F \) in \( S^+ \) so that it divides \( S^+ \) into genus zero surfaces (since we are dealing with a Schottky covering), each loop lifts to a simple loop on \( \hat{\Omega} \) and so that \( F \) is invariant under the action of \( J \).

Let \( A \) be a connected component of \( S - F \) and let \( J_A \) its stabilizer in \( J \). As \( A \) is a genus zero surface, we have that \( J_A \) is a finite subgroup of \( \hat{G} \). We also have that \( J_A \) is subgroup of a dihedral group of order \( 2p \), which restrict the possibilities for \( J_A \) as subgroup of \( \hat{G} \). If we set \( \sigma = \tau_1 \tau_2 \) the conformal automorphism of \( S^+ \) of order \( p \), then \( J_A \) is either:

(i) a cyclic conformal group generated by a power of \( \sigma \); or

(ii) a cyclic group of order two generated by an anticonformal involution conjugated to either \( \tau_1 \) or \( \tau_2 \); or

(iii) a dihedral subgroup of \( J \).

In either case (ii) or (iii) we have that \( A \) is invariant under an anticonformal involution \( \tau \in J \). If \( \tau \) is either (a) a reflection containing a loop of fixed points on \( A \) or (b) an imaginary reflection, then we may find a simple loop \( \beta \subset A \) which is invariant under \( \tau \); moreover, if \( \tau \) is reflection, then \( \beta \) is formed of only fixed points of it. We may add such a loop and its \( J \)-translated to \( F \) without destroying the conditions of Theorem 3.4 we still denoting such a collection by \( F \). In this way, we may assume that in either (ii) or (iii) the anticonformal involution is necessarily a reflection whose circle of fixed points is not completely contained in \( A \) (it intersects some boundary loops). Now we can delete loops from this new collection \( F \) so that we do not destroy all the above properties in order to make it minimal in the sense that eliminating from it any finite collection of loops will destroy any of the above properties. We have then, as a consequence of the above, the following existence fact.

**Proposition 5.8.** There is a collection of pairwise disjoint simple loops \( F \) in \( S^+ \) so that:

(i) each loop in \( F \) lifts to pairwise disjoint simple loops on \( \hat{\Omega} \);

(ii) \( F \) divides \( S^+ \) into genus zero surfaces;

(iii) \( F \) is invariant under the action of \( J \);

(iv) each connected component \( A \) of \( S^+ - F \) has stabilizer \( J_A \) being either:

(iv.1) trivial; or

(iv.2) a cyclic conformal group generated by a non-trivial power of \( \sigma = \tau_2 \tau_1 \); or

(iv.3) a cyclic group of order two generated by a reflection with no oval of fixed points completely contained inside \( A \); or

(iv.4) a dihedral group generated by a non-trivial power of \( \sigma \) and a reflection with no oval of fixed points completely contained inside \( A \).

(v) \( F \) is minimal in the sense that eliminating from it any finite collection of loops will destroy any of the above properties (i)-(iv).

**Remark 5.9.** In case both extended Schottky groups \( G_1 \) and \( G_2 \) are Klein-Schottky groups, as we know that in this case all anticonformal involutions in \( \hat{G} \) are imaginary reflections, Proposition 5.8 asserts that \( J_A \) is either trivial or a cyclic group generated by a non-trivial power of \( \sigma \).

Let us denote by \( \hat{\mathcal{G}} \) the loops on \( \hat{\Omega} \) obtained by the lifting of those in \( F \) (as in proposition 5.8). We call these simple loops the structure loops and the connected components of \( \hat{\Omega} - \hat{\mathcal{G}} \) the structure regions.
The loops in $\mathcal{F}$ are called the **base structure loops** and the connected components of $S^+ - \mathcal{F}$ are called the **base structure regions**. By the construction of the base structure loops, we may see that The set of structure loops $\mathcal{G}$ and the set of structure regions are invariant under $\hat{G}$.

If we consider a structure region $R$ so that $P(R) = A$, then we have that $P : R \to A$ is a conformal homeomorphism, in particular, the stabilizer in $\hat{G}$ of the structure region $R$, say $\hat{G}_R$ is isomorphic to $J_A$.

In particular, the following holds.

**Proposition 5.10.** The stabilizer in $\hat{G}$ of any structure region $R$ is either:

(i) trivial;
(ii) a finite cyclic conformal group;
(iii) a cyclic group of order two generated by a reflection whose circle of fixed points is not completely contained on $R$;
(iv) a dihedral group generated by a reflection $\tau$ with its circle of fixed points intersecting some boundary loop of $R$ and an elliptic element $\eta$ of finite order so that $(\eta \tau)^2 = 1$.

**Proposition 5.11.** If $R \subset \Omega$ is a structure region whose stabilizer is trivial, then the restriction to $R$ of the projection map from $\Omega$ to $S = S^+/J = \Omega/\hat{G}$ is a homeomorphism onto its image.

**Proof.** If $a \in \hat{G}$, then either $a \in \hat{G}_R = \{I\}$, in which case, $a(R) = R$, or $a \not\in \hat{G}_R$, in which case, $a(R) \cap R = \emptyset$. \qed

Let us now consider some structure loop $\beta \in \hat{G}$ and denote its stabilizer in $\hat{G}$ by $\hat{G}_\beta$.

As our group $J$ has no a dihedral subgroup inside the cyclic subgroup $J^*$, the orientation preserving half of $J$, we have the following fact.

**Proposition 5.12.** The orientation preserving half of $\hat{G}_\beta$ is either trivial or a finite cyclic group generated by some elliptic element.

We have from Proposition 5.4 that an orientation reversing transformation inside $\hat{G}$ is either an imaginary reflection or a reflection or a semi-hyperbolic transformation. Clearly, as a structure loop is contained in $\Omega$, we have that a semi-hyperbolic transformation cannot keep such a structure loop invariant. In particular, the only orientation reversing transformations in $\hat{G}$ that keep invariant some structure loop can be either an imaginary reflection or a reflection. Also, a structure loop can be stabilized by at most one imaginary reflection, for the product of two distinct imaginary reflections is always hyperbolic, and the entire structure loop must be contained in $\Omega$. A structure loop can be stabilized by a reflection in two different manners. One manner is that it fixes it point-wise, that is, the structure loop is the circle of fixed points of the reflection. The second manner is that the structure loop is not point-wise fixed by the reflection, in which case, there are exactly two fixed points of the reflection on the structure loop. These two points divide the loop into two arcs which are permuted by the reflection. All the above gives us the following possibilities.

**Proposition 5.13.** Let $\beta \in \mathcal{G}$ a structure loop and $\hat{G}_\beta$ its stabilizer in $\hat{G}$. Then, we have the following possibilities:

(i) $\hat{G}_\beta$ is trivial;
(ii) $\hat{G}_\beta$ is a cyclic group generated by some elliptic element of finite order whose fixed points are separated by $\beta$;
(iii) $\hat{G}_\beta$ is a cyclic group of order two generated by some imaginary reflection;
(iv) $\hat{G}_\beta$ is a cyclic group of order two generated by a reflection and $\beta$ is the circle of fixed points of the reflection;
(v) $\hat{G}_\beta$ is a cyclic group of order two generated by some reflection and $\beta$ is not the circle of fixed points of the reflection. In this case, $\beta$ should contain exactly two fixed points of the reflection;
(vi) $\hat{G}_\beta$ is generated by a reflection that has exactly two fixed points on $\beta$ and an elliptic involution with same both fixed points. In this case, the composition of these two is an imaginary reflection and $\hat{G}_\beta$ is isomorphic to $Z_2^2$.\/
(viii) \( \hat{G}_\beta \) is generated by an elliptic involution whose fixed points are separated by \( \beta \) and a reflection that has \( \beta \) as its circle of fixed points on. In this case, \( \hat{G}_\beta \) is isomorphic to \( \mathbb{Z}_2^2 \).

(ix) \( \hat{G}_\beta \) is generated by an elliptic involution with both fixed points on \( \beta \) and a reflection for which \( \beta \) is its circle of fixed points. In this case, \( \hat{G}_\beta \) is isomorphic to \( \mathbb{Z}_2^2 \); and

(x) \( \hat{G}_\beta \) is generated by an elliptic involution with both fixed points on \( \beta \) and a reflection for which \( \beta \) intersects the circle of fixed points of the reflection at two points separating the fixed points of the conformal involution. In this case, \( \hat{G}_\beta \) is isomorphic to \( \mathbb{Z}_2^2 \); and

(xi) \( \hat{G}_\beta \) is generated by a reflection with exactly two fixed points on \( \beta \) and elliptic involution with both fixed points on the circle of fixed points of the reflection. In this case, \( \hat{G}_\beta \) is isomorphic to \( \mathbb{Z}_2^2 \).

The structure loops divide \( \Omega \) into structure regions; the stabilizer of every structure region is either (i) trivial or (ii) elliptic cyclic or (iii) a cyclic group of order two generated by a reflection whose circle of fixed points is not completely contained on the structure region or (iv) a dihedral group generated by an elliptic transformation and a reflection whose circle of fixed points is not completely contained on the structure region; hence every structure region has a finite number of structure loops on its boundary. Moreover, by Proposition 5.13, we also know all possibilities for the stabilizers of these structure loops. The following fact is easy to see.

**Proposition 5.14.** Let \( R \) and \( R' \) be any two different structure regions with a common boundary loop \( \beta \). Then, they are equivalent under \( \hat{G} \) if and only if \( i \) \( \hat{G}_\beta \) contains an element which does not belong to \( \hat{G}_R \) or \( ii \) there is another boundary loop \( \beta' \) of \( R \) and an element \( t \in \hat{G} - \hat{G}_R \) so that \( t(\beta') = \beta \).

Next result is related to those structure regions with stabilizer either trivial or a cyclic group generated by a reflection whose circle of fixed points is not completely contained inside the structure region.

**Proposition 5.15.** Let \( R \subset \Omega \) be a structure region with stabilizer \( \hat{G}_R \) either trivial or a cyclic group of order two generated by a reflection whose circle of fixed points is not completely contained on \( R \). If \( \beta \) is a boundary loop of \( R \), which is not invariant under a reflection in \( \hat{G}_R \), then there is a non-trivial element \( k \in \hat{G} - \hat{G}_R \) so that \( k(\beta) \) still a boundary loop of \( R \).

**Proof.** Let \( \beta \) be a structure loop on the boundary of the structure region \( R \). We assume that it is not invariant under a reflection that stabilizes \( R \). The hypothesis on \( \hat{G}_R \) and Proposition 5.13 gives us two general situations for \( \hat{G}_\beta \).

First case: The stabilizer \( \hat{G}_\beta \) of \( \beta \) in \( \hat{G} \) contains a cyclic group generated by either (i) a reflection whose circle of fixed points is \( \beta \) or (ii) an imaginary reflection or (ii) an elliptic element of order two with both fixed points on \( \beta \). In this case we may choose for \( k \) the generator of such a cyclic subgroup of \( \hat{G}_\beta \), which clearly does not belong to \( \hat{G}_R \).

Second case: The stabilizer \( \hat{G}_\beta \) is either: (i) trivial or (ii) a cyclic group generated by a reflection that does not fix it point-wise. In case (ii) we have that the reflection is the generator of the stabilizer of \( R \); by our hypothesis this case does not happen. Then we have that \( \hat{G}_\beta \) is trivial. In this case, the projection of \( \beta \) on \( S^+ \) is a simple loop \( \beta \), which has trivial stabilizer in \( J \). We have that \( \beta \) is free homotopic to the product of the other boundary loops of \( R \). If none of the other boundary loops of \( R \) is equivalent to \( \beta \) under \( \hat{G} \), then we may delete \( \beta \), and its \( J \)-translates from \( \mathcal{F} \), contradicting the minimality of \( \mathcal{F} \). \( \square \)

The other kind of structure regions \( R \) always contain an elliptic cyclic group \( H = \hat{G}_k \) inside its stabilizer \( \hat{G}_R \) (there are two cases, the stabilizer is either cyclic or a dihedral group). In the case that \( \hat{G}_R \) is a dihedral group, we know that the anticonformal involution in \( \hat{G}_R \) is a reflection whose circle of fixed points intersect some boundary loops of \( R \); that is, each boundary loop of \( R \) is either kept invariant (with exactly two fixed points of the reflection) or it is permuted with other boundary loop (in which case it has no fixed points of the reflection) by such a reflection. In any of the two possibilities for \( \hat{G}_R \), we have that \( R \) has either 0, 1 or 2 structure loops stabilized by \( H \) on its boundary; it has also some number of structure loops stabilized only by the identity in \( H \) on its boundary. It is clear that if no structure loop on the boundary of \( R \) is stabilized by \( H \), then both fixed points of \( H \) lie in \( R \). It is also clear that a structure
loop is stabilized by $H$ if and only if it separates the fixed points of $H$. Also, observe that if $\hat{G}_R$ is a dihedral group, as the reflection in $\hat{G}_R$ has no its circle of fixed points completely contained on $R$, then we must have that in the case that $R$ contains only one of the fixed points of $H$, such a reflection should fix that point, in particular, such a reflection should commute with a generator of $H$; this obligates to have $\hat{G}_R \cong \mathbb{Z}_2^2$.

**Proposition 5.16.** If $R \subset \Omega$ is a structure region with non-trivial conformal stabilizer $\hat{G}_R = H$ and there is a fixed point of $H$ in $R$, then both fixed points of $H$ lie in $R$.

**Proof.** Suppose there is only one fixed point of $H$ in $R$. Then there is a unique structure loop $W$ on the boundary of $R$ stabilized by $W$. Then every other structure loop on the boundary of $R$ is stabilized by the identity in $H$. If $\hat{G}_R = H$, then it follows that if were to fill in the discs bounded by the other structure loops on the boundary of $R$, then $W$ would be contractible; that is, if we delete the projection of $W$ and their $J$-translates from our list of base structure loops, this would leave unchanged the smallest normal subgroup containing the base structure loops raised to appropriate powers. Since we have chosen our base structure loops to be minimal, this cannot be. If $\hat{G}_R \neq H$, then we have a reflection $\tau \in \hat{G}_R$ whose circle of fixed points is not completely contained in $R$, and $\hat{G}_R = \langle H, \tau \rangle$ a dihedral group. In this case, we should have that the fixed point of $H$ in $R$ is also fixed by $\tau$; then, both fixed points of $H$ are fixed by $\tau$. As $\hat{G}_R \cong \mathbb{Z}_2^2$, we have $H \cong \mathbb{Z}_2$. In this case, we may also delete the projection of $W$ and its $J$-translates from $\mathcal{F}$ in order to get a contradiction to the minimality of $\mathcal{F}$. $\square$

**Proposition 5.17.** Assume we have a structure region $R \subset \Omega$ with stabilizer $\hat{G}_R$ containing a non-trivial conformal cyclic group, say generated by $k$. If $\beta_1, \beta_2$ are two different boundary loop of $R$ which are invariant under $k$, then there is a (non-trivial) element $k \in \hat{G}$ so that $k(\beta_1) = \beta_2$.

**Proof.** Assume this is not the case. We observe that $R/\hat{G}_R^+ \subset S^+$ is an annulus with some holes cut out of it. If we fill those holes with discs, then the projections of $\beta_1$ and $\beta_2$ become freely homotopic; the projection of $\beta_1$, raised to the appropriate power, is freely homotopic to a product of the projection of $\beta_2$, raised to the same power, and the projections of the other structure loops on the boundary of $R$. Hence we can delete the projection of $\beta_1$ on $S^+$ and its $J$-translates, which contradicts the minimality of this set. $\square$

**Structure regions with trivial stabilizers.** Let $\hat{G}$ be a dihedral extended Schottky group admitting a structure region $R$ with trivial stabilizer $\hat{G}_R = \{1\}$. As consequence of Proposition 5.15, every other structure region is equivalent under $\hat{G}$ to $R$ and $R$ is a fundamental domain for $\hat{G}$ and the boundary loops are paired by either (i) reflections, (ii) imaginary reflections, (iii) loxodromic transformations or (iv) semi-hyperbolic transformations (see Figure 1).

**Structure regions with non-trivial stabilizers.** Let us now consider the case our dihedral extended Schottky group $\hat{G}$ has not a structure region with trivial stabilizer.

Proposition 5.14 and the fact that $S$ is compact and connected, permits us to construct a connected set $\tilde{R}$ obtained as the union of a finite collection of non-equivalent structure regions (some of them may have non-trivial stabilizer) together their boundary loops which projects onto $S$. Some of the boundary loops of $\tilde{R}$ have a non-trivial stabilizer which sends $\tilde{R}$ at the other side of the boundary loop an the other boundary loops are equivalent in pairs under $\hat{G}$.

(1) Assume one of the structure regions $R \subset \tilde{R}$ has stabilizer a cyclic group of order two generated by a reflection $\tau$ we have the following. The circle of fixed points $C_\tau$ of $\tau$ intersects some structure loops in the boundary of $R$. Let us denote by $\gamma$ the intersection of $C_\tau$ with $R$. Let us denote by $R_1$ the union of one of the components of $R - \gamma$ with $\gamma$. Inside $R_1$ we have boundary loops, which, as a consequence of Proposition 5.15, are paired by either (i) reflections, (ii) imaginary reflections, (iii) loxodromic transformations or (iv) semi-hyperbolic transformations. The other boundary loops intersect $C_\tau$. Let $\beta$ be any structure loop in the boundary of $R$ which intersects $C_\tau$, necessarily at two points. If $\beta$ belongs to the boundary of $\tilde{R}$, then we already noted that either

(a) there is an involution $\kappa \in \hat{G}$ (conformal or anticonformal) with $\kappa(\beta) = \beta$ and $\kappa(\tilde{R}) \cap \tilde{R} = \beta$; or
(b) there is another boundary loop $\beta'$ of $\tilde{R}$ and an element $\sigma \in \tilde{G}$ so that $\sigma(\beta) = \beta'$ and $\sigma(\tilde{R}) \cap \tilde{R} = \beta'$.

If $\beta$ is in the interior of $\tilde{R}$, then we have another structure region $R' \subset \tilde{R}$ with $\beta$ as one of its border loop. In this case, as consequence of Proposition 5.14 we should have $\tilde{G}_H = \langle \tau \rangle$ and $\tau \in \tilde{G}_R$.

(2) Assume one of the structure regions $R \subset \tilde{R}$ has stabilizer a cyclic group generated by an elliptic transformation $\eta$. In this case we have two possibilities: either (i) both fixed points of $\eta$ belong to $R$ or (ii) there are two boundary loops $\beta_1$ and $\beta_2$ of $R$, each one invariant under $\eta$, and there is some $\kappa \in \tilde{G}$ with $\kappa(\beta_1) = \beta_2$, $\kappa(\tilde{R}) \cap \tilde{R} = \beta_2$. We have that each other boundary loop $\beta$ of $R$ is either (i) the boundary of another structure region inside $\tilde{R}$ (so it has trivial stabilizer in $\tilde{G}$) or (ii) it belongs to the boundary of $\tilde{R}$ (in which case we have either (a) or (b) above).

5.1. Geometrical Constructions: Part (1) of Theorem 4.1 All the above asserts that $\tilde{G}$ should be constructed, by using standard combination theorem techniques [18], by the groups in the first part of Theorem 4.1.

Remark 5.18.

(i) If both $G_1$ and $G_2$ are Klein-Schottky groups, then in $\tilde{G}$ we have no reflections, then in the above construction we only have to use, in the previous constructions, groups of type (ii), (iii) and (iv) [10]. In particular, $\tilde{G}^+$ is a Schottky group.

(ii) If $\tilde{G}$ is constructed with groups of types (i), (ii), (iii), (iv) and (v), then we have that $\tilde{G}^+$ is the free product of a Schottky group and a finite number of order two elliptic groups.

(iii) In the general situation we have that $\tilde{G}^+$ is a free product of a Schottky group and a finite number of elliptic groups of finite order; the orders of these elliptic elements should be divisors of some fixed positive integer $p > 1$.

5.2. Part (2) of Theorem 4.1 Let us consider an extended function group $K$ which is constructed by Klein-Maskit’s combination theorems as described in the first part of Theorem 4.1. In general, it may happens that $K$ is not a dihedral Schottky group. By hypothesis, there is a surjective homomorphism $\varphi: K \to D_p$ with kernel a Schottky group $\Gamma$ so that $\varphi(\Gamma^+)$ is the cyclic group of order $p$, for some positive integer $p > 1$. Choose $a, b \in D_p$, both of order 2, both generating $D_p$. As $K_1 = \varphi^{-1}(\langle a \rangle)$ and $K_2 = \varphi^{-1}(\langle b \rangle)$, $K_1$ and $K_2$ are extended Schottky groups (in both cases $\Gamma$ is its index two preserving half) and $K = \langle K_1, K_2 \rangle$, we obtain that $K$ is in fact a dihedral extended Schottky group. The other direction is clear from the definition of a dihedral Schottky group.

6. Proof of Theorem 4.1

Let $\Gamma$ be a Schottky group of rank $g \geq 2$ so that $M = (\mathbb{H}^3 \cup \Omega)/\Gamma$, where $\Omega$ denotes the region of discontinuity of $\Gamma$. By lifting both $\tau_1$ and $\tau_2$ to $\mathbb{H}^3$ we obtain an extended Kleinian group $\tilde{K}$ containing $\Gamma$ as a normal subgroup so that $\tilde{G} = \tilde{K}/\Gamma = \langle \tau_1, \tau_2 \rangle \cong D_q$, where $D_q$ denotes the dihedral group of order $2q$; that is, $\tilde{K}$ is a dihedral extended Schottky group. As before, we denote by $\theta: \tilde{K} \to \tilde{G}$ the canonical surjection and by $c_1, \ldots, c_r \in \tilde{K}$ a complete set of symmetries. By Theorem 4.1 $\tilde{K}$ is constructed using reflections $\xi_1, \ldots, \xi_a$, imaginary reflections $\eta_1, \ldots, \eta_{b_1}$, $\gamma$ cyclic groups generated by loxodromic transformations, $\delta$ cyclic groups generated by glide-reflections and $\varepsilon$ groups $K_1, \ldots, K_c$, so that each $K_i$ is generated by a reflection $\sigma_i$ and some other elliptic transformations $t_{i1}, \ldots, t_{im}$ (all of them of order at least 3 and commuting with $\sigma_i$) and some $f_i$ imaginary reflections and reflections (each of them commuting with $\sigma_i$); let us denote these involutions by $\sigma_{i1}, \ldots, \sigma_{if_i}$. By the same theorem, we have that $g \geq q(g - 1) + 1 + \frac{q}{2} \sum_{i=1}^p f_i$, where $\tilde{g} = g + \beta + \delta + \varepsilon - 1 \geq \alpha + \beta + \delta - 1$. So, it follows that

$$\frac{g - 1}{q} + 2 \geq \tilde{g} + 1 + \frac{1}{2} (f_1 + \cdots + f_e) \geq \alpha + \beta + \varepsilon + 1 \geq \frac{1}{2} (f_1 + \cdots + f_e).$$

Also, note that a complete set of symmetries for $\tilde{K}$ is given by $\xi_1, \ldots, \xi_a, \eta_1, \ldots, \eta_{b_1}, \sigma_1, \sigma_{i1}, \ldots, \sigma_{if_i}$, where $i = 1, \ldots, e$ and $k = 1, \ldots, f_j$. If $c$ denotes any of the above symmetries, then $\langle c \rangle \subset \langle \tilde{K}, c \rangle$ and so

$$\langle \theta(c) \rangle \subset \langle \theta(\tilde{K}, c) \rangle \subset \mathbb{C}(\tilde{G}, \theta(c)) = \begin{cases} \mathbb{Z}_2, & q \text{ even} \\ \mathbb{Z}_2, & q \text{ odd} \end{cases}$$
Now, it is easy to see the following
\[ \theta(C(\hat{K}; \xi_j)) = \theta(\xi_j) = \mathbb{Z}_2, \quad \theta(C(\hat{K}; \eta_j)) = \theta(\eta_j) = \mathbb{Z}_2, \quad \theta(C(\hat{K}; \sigma_{j,k})) = \theta(\sigma_{j,k}) = \mathbb{Z}_2^2, \]
\[ \theta(C(\hat{K}; \sigma_{j})) = \theta((\sigma_j, \sigma_{j,1}, \ldots, \sigma_{j,f}, \epsilon_{j1}, \ldots, \epsilon_{jm})) = \mathbb{Z}_2^2. \]
Finally, it follows from Theorem 2.3 that
\[ m_1 + m_2 \leq 2(\alpha + \beta) + \varepsilon + (f_1 + \cdots + f_{\varepsilon}) \leq 2(\alpha + \beta + \varepsilon) + (f_1 + \cdots + f_{\varepsilon}) \leq 2 \left( \frac{g - 1}{q} \right) + 4. \]

7. Proof of Theorem 2.4

Let \( q_{ij} \) be to denote the order of \( \tau_i \tau_j \), where \( i < j \). By a permutation of the indices, we may assume that \( 2 \leq q_{12} \leq q_{13} \leq q_{23} \). As consequence of Theorem 2.1 one has the inequality
\[ m_1 + m_2 + m_3 \leq \left[ \frac{g - 1}{q_{12}} \right] + \left[ \frac{g - 1}{q_{13}} \right] + \left[ \frac{g - 1}{q_{23}} \right] + 6. \]

7.1. Case \( g = 2 \). In this case, \( m_1 + m_2 + m_3 \leq 6 \). As \( m_i + m_j \leq 4 \), either \( m_1 + m_2 + m_3 \leq 5 \) or else \( m_1 = m_2 = m_3 = 2 \).

Claim 7.1. The case \( m_1 = m_2 = m_3 = 2 \) is not possible for \( g = 2 \).

Proof. Let \( \Gamma \) be a Schottky group of rank two, with region of discontinuity \( \Omega \), so that the handlebody \( M = (\mathbb{H}^3 \cup \Omega)/\Gamma \) admits three symmetries \( \tau_1, \tau_2, \tau_3 \), each of them having exactly two connected components of fixed points. Let \( \pi : \mathbb{H}^3 \cup \Omega \to M \) be the universal covering induced by \( \Gamma \), \( H = \langle \tau_1, \tau_2, \tau_3 \rangle \) and \( \hat{K} \) be the extended Kleinian group obtained by lifting \( H \) under \( \pi \). There is a surjective homomorphism \( \theta : \hat{K} \to H \) with \( \Gamma \) as its kernel. Let \( K_j = \theta^{-1}(\tau_j) \), where \( r \neq s \) and \( r, s \in \{1, 2, 3\} \). The extended Kleinian group \( K_j \) is a dihedral extended Schottky group and has finite index in \( \hat{K} \), so it has region of discontinuity \( \Omega \). As consequence of Theorem 2.1, \( K_j \) is constructed by using \( \alpha \) cyclic groups generated by reflections, \( \beta \) cyclic groups generated by imaginary reflections, \( \gamma \) cyclic groups generated by loxodromic transformations, \( \delta \) cyclic groups generated by glide-reflections and \( \varepsilon \) groups \( \Gamma_1, \ldots, \Gamma_\varepsilon \), as in (v) of that theorem, so that \( \alpha + \beta + \gamma + \delta + \varepsilon > 0 \). Assume that \( \Gamma_1, \ldots, \Gamma_\varepsilon \) are the groups of type (v) and assume \( \Gamma_i \) is constructed using the reflections \( \sigma_{i,1}, \ldots, \sigma_{i,m} \) and elliptic transformations \( t_{i,1}, \ldots, t_{i,m} \) (the reflections \( \sigma_{i,1} \) and the elliptics \( t_i \) each one commuting with \( \sigma_{i,1} \)). Then, \( \Omega/K_j^+ \) has genus \( \bar{g} = \alpha + \beta + 2(\gamma + \delta) + \varepsilon - 1 \), with \( 2(m_1 + \cdots + m_\varepsilon) \) conical points of orders \( |t_{1,1}|, |t_{1,2}|, |t_{1,3}|, \ldots, |t_{m_\varepsilon}|, |t_{m_\varepsilon}| \), where \( |t| \) denotes the order of \( t_i \) and \( 2(n_1 + \cdots + n_\varepsilon) \) conical points of order 2, and
\[ 2 = n(\bar{g} - 1) + 1 + n \sum_{i=1}^{\varepsilon} \sum_{k=1}^{m_i} \left( 1 - \frac{1}{|t_{ik}|} \right) + \sum_{i=1}^{\varepsilon} n_i. \]

Since \( g = 2 \), Riemann–Hurwitz formula asserts that \( \bar{g} \in \{0, 1\} \). As \( \alpha + \beta + \gamma + \delta + \varepsilon > 0 \), it must be that \( \gamma = \delta = 0 \). If \( \varepsilon = 0 \), the same asserts that \( K_j \) will be elementary group containing the non-elementary group \( \Gamma \), a contradiction. So \( \varepsilon \geq 1 \), and as \( \bar{g} \geq 0 \), in fact \( \bar{g} = 1 \) and \( \bar{g} = \alpha + \beta \). It follows that
\[ (\alpha, \beta, \gamma, \delta, \varepsilon) = \{(0, 0, 0, 0, 1), (1, 0, 0, 0, 1), (0, 1, 0, 0, 1)\}. \]

In the case \( (\alpha, \beta, \gamma, \delta, \varepsilon) = (0, 0, 0, 0, 1) \), we have \( \bar{g} = 0 \) and
\[ 2 = 1 - n + \sum_{k=1}^{m_1} \left( 1 - \frac{1}{|t_{ik}|} \right) + \frac{nm_1}{2}. \]

As each \( |t_{ik}| \geq 2 \), the above ensures that \( 2 \geq n(n_1 + m_1 - 2) \). Now, if \( n_1 + m_1 \geq 3 \), then \( n = 2 \) (as required) and, necessarily, \( n_1 + m_1 = 3 \). If \( n_1 + m_2 \leq 2 \), then \( K_j \) will be elementary, a contradiction. In the case \( (\alpha, \beta, \gamma, \delta, \varepsilon) = (0, 0, 0, 0, 1), (0, 1, 0, 0, 1) \), we have \( \bar{g} = 1 \) and
\[ 2 = 1 + \sum_{k=1}^{m_1} \left( 1 - \frac{1}{|t_{ik}|} \right) + \frac{nm_1}{2}. \]

Again, as \( |t_{ik}| \geq 2 \), one obtains that \( 2 \geq n(m_1 + m_1) \). Then \( n = 2 \) and \( n_1 + m_1 = 1 \), but again this makes \( K_j \) to be elementary, a contradiction.
As a consequence of the above, we see that $K_j$ is generated by a reflection $\rho_{j1}$ and three other involutions $\rho_{j2}, \rho_{j3}, \rho_{j4}$, each one commuting with $\rho_{j1}$ (some of the involutions may be reflections and others may be imaginary reflections). Note also that $K_j$ keeps invariant the circle of fixed points of $\rho_{j1}$, that is, the limit set of $K_j = \mathbb{Z}_2 \oplus (\mathbb{Z}_2 \ast \mathbb{Z}_2)$ is contained in the circle of fixed points of $\rho_{j1}$. As the limit set of $K$ is infinite, such a circle is uniquely determined by it. As a consequence, $\rho_{11} = \rho_{21} = \rho_{31} = \rho$. This asserts that $\theta(\rho) \in \theta(K_1) \cap \theta(K_2) \cap \theta(K_3)$. As, by the definition of the groups $K_j$, $\tau_j \notin \theta(K_1) \cap \theta(K_2) \cap \theta(K_3)$, $\theta(\rho)$ is a symmetry and the symmetries in $H$ are exactly $\tau_1, \tau_2, \tau_3$ and $\tau_1 \tau_2 \tau_3$, the only possibility is to have $\theta(\rho) = \tau_1 \tau_2 \tau_3$, from which, for instance, $\tau_1 \tau_2 \tau_3 \in \langle \tau_1, \tau_2 \rangle$; obligating to have that $\tau_3 \in \langle \tau_1, \tau_2 \rangle$, a contradiction.

7.2. Case $g = 3$. If $q_{23} \geq 3$, then $m_2 + m_3 \leq 4$. Since $m_1 \leq g + 1 = 4$, we get in this case that $m_1 + m_2 + m_3 \leq 8$. Let us now assume $q_{12} = q_{13} = q_{23} = 2$, in which case $\langle \tau_1, \tau_2, \tau_3 \rangle \cong \mathbb{Z}_2^2$. We may reorder again the indices to assume that $m_1 \leq m_2 \leq m_3 \leq 4$. If $m_3 = 4$, then inequality $\mathbb{I}$ asserts that $m_1 + m_2 \leq 2$, so $m_1 + m_2 + m_3 \leq 8$. Now, the only case with $m_3 \leq 3$ for which we do not have $m_1 + m_2 + m_3 \leq 8$ is when $m_1 = m_2 = m_3 = 3$.

Claim 7.2. The case $m_1 = m_2 = m_3 = 3$ is not possible for $g = 3$.

Proof. The Proof follows the same ideas as for the Claim[7.1] Let $\Gamma$ be a Schottky group of rank $g = 3$, with region of discontinuity $\Omega$, so that the handlebody $M = (\mathbb{H}^3 \cup \Omega)/\Gamma$ admits three symmetries $\tau_1, \tau_2$ and $\tau_3$, each of them having exactly three connected components of fixed points, and with $H = \langle \tau_1, \tau_2, \tau_3 \rangle = \mathbb{Z}_2^2$. Let $\pi : \mathbb{H}^3 \cup \Omega \rightarrow M$ be the universal covering induced by $\Gamma$ and let $\tilde{K}$ be the extended Kleinian group obtained by lifting $H$ under $\pi$. There is a surjective homomorphism $\theta : \tilde{K} \rightarrow H$ with $\Gamma$ as its kernel. Let $K_j = \theta^{-1}(\tau_r, \tau_s)$, where $r \neq s$ and $r, s \in \{1, 2, 3\} - \{j\}$; so $\theta(K_j) = \mathbb{Z}_2^2$. The extended Kleinian group $K_j$ is a dihedral extended Schottky group and has index two in $\tilde{K}$, so it has region of discontinuity $\Omega$. As consequence of Theorem[4.4] $K_j$ is constructed by using a cyclic groups generated by reflections, $\beta$ cyclic groups generated by imaginary reflections, $\gamma$ cyclic groups generated by loxodromic transformations, $\delta$ cyclic groups generated by glide-reflections and $\epsilon$ groups $\Gamma_1, \Gamma_\epsilon$, as in (v) of that theorem, so that $\alpha + \beta + \delta + \epsilon > 0$. Assume that $\Gamma_1, \ldots, \Gamma_\epsilon$ are the groups of type (v) and assume $\Gamma_1$ is constructed using the reflections $\sigma_1, \sigma_2, \ldots, \sigma_m$, and elliptic transformations of order two $i_1, \ldots, i_m$ (the reflections $\sigma_{ik}$ and the elliptics $i_k$ each one commuting with $\sigma_i$). Then, $\Omega/K_j^+$ has genus $\bar{g} = \alpha + \beta + 2(\gamma + \delta) + \epsilon - 1$, with $2(m_1 + \cdots + m_\epsilon + n_1 + \cdots + n_\epsilon)$ conical points of order 2, and so that

$$2 = 2(g - 1) + \sum_{i=1}^\epsilon (m_i + n_i).$$

Since $g = 3$ and $H^+ = \mathbb{Z}_2^2$, Riemann-Hurwitz formula asserts that $\bar{g} \in \{0, 1\}$. As $\alpha + \beta + \epsilon > 0$, then $\epsilon = 0$, the same asserts that $K_j$ will be elementary group containing the non-elementary group $\Gamma$, a contradiction, so $\epsilon \geq 1$. If $\epsilon \geq 3$, then $\alpha + \beta < 0$, a contradiction. If $\epsilon = 2$, then $\alpha + \beta = \bar{g} - 1$, so the only possible case is to have $g = 1$ and $\alpha = \beta = 0$. In this way,

$$(\alpha, \beta, \gamma, \delta, \epsilon; \bar{g}) \in \{(0, 0, 0, 0, 1; 0), (1, 0, 0, 0, 1; 1), (0, 1, 0, 0, 1; 1), (0, 0, 0, 0, 2; 1)\}.$$
\[ k = 1, 2, 3, 4. \] So, in particular, \( \theta(\eta_1) = \theta(\eta_2) = \theta(\eta_3) = \theta(\eta_4) \). In this case, a complete set of symmetries of \( K_j \) is given by \( \langle \rho, \eta_1, \eta_2, \eta_3, \eta_4 \rangle \) and

\[
C(K_j, \rho) = K_j, \quad C(K_j, \eta_k) = \langle \rho, \eta_k \rangle = \mathbb{Z}_2^2
\]

and it follows, from Theorem 3.5, that \( \theta(\rho) \) has at most 1 connected components of fixed points, a contradiction to the assumption the symmetries have 3 connected components.

Let us consider the case \( (\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \rho) = (1, 0, 0, 0, 1; 0) \) (similar arguments for the case \( (\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \rho) = (0, 1, 0, 0, 1; 0) \)). In this case \( K_j \) is a free product of a cyclic group generated by a reflection \( \zeta \) and a group \( K^0_j = \langle \rho, \eta_1, \eta_2, \eta_3, \eta_4 \rangle \), where \( \rho \) is a reflection and each of the \( \eta_k \) is either a reflection or an imaginary reflection commuting with \( \rho \) (each element of \( K^0_j \) keeps invariant the circle of fixed points of \( \rho \)). Again, as before, \( \theta(\rho) \neq \theta(\eta_k) \) for each \( k = 1, 2, 3, 4 \). So, in particular, \( \theta(\eta_1) = \theta(\eta_2) = \theta(\eta_3) = \theta(\eta_4) \). In this case, a complete set of symmetries of \( K_j \) is given by \( \langle \zeta, \rho, \eta_1, \eta_2, \eta_3, \eta_4 \rangle \)

\[
C(K_j, \zeta) = \langle \zeta \rangle = \mathbb{Z}_2, \quad C(K_j, \rho) = K^0_j, \quad C(K_j, \eta_k) = \langle \rho, \eta_k \rangle = \mathbb{Z}_2^2.
\]

The only way for both symmetries in \( \theta(K_j) \) to have exactly 3 connected components of fixed points is to have that \( \theta(\zeta) = \theta(\rho) \). But in this case we should have that \( \Gamma \) is the Schottky group generated by the loxodromic transformations \( \rho, \zeta, \eta_1, \eta_2, \eta_3, \eta_4 \), which is of rank 4, a contradiction.

7.3. Case \( g \geq 4 \). In this case, inequality (1) asserts

\[
m_1 + m_2 + m_3 \leq \left( \frac{1}{q_{12}} + \frac{1}{q_{13}} + \frac{1}{q_{23}} \right) (g - 1) + 6.
\]

If \( q_{12}^{-1} + q_{13}^{-1} + q_{23}^{-1} \leq 1 \), then the above ensures that \( m_1 + m_2 + m_3 \leq g + 5 \). If \( q_{12}^{-1} + q_{13}^{-1} + q_{23}^{-1} > 1 \), then we have the following cases:

\[
\begin{array}{cccc}
q_{12} & q_{13} & q_{23} & H = \langle \tau_1, \tau_2, \tau_3 \rangle \\
2 & 2 & r \geq 2 & \mathbb{Z}_2 \times D_r \\
2 & 3 & 3 & \mathbb{Z}_2 \rtimes \mathcal{A}_4 \\
2 & 3 & 4 & \mathbb{Z}_2 \rtimes \mathcal{S}_4 \\
2 & 3 & 5 & \mathbb{Z}_2 \rtimes \mathcal{A}_5 \\
\end{array}
\]

where \( \mathbb{Z}_2 = \langle \tau_1 \rangle, r = \langle \tau_2, \tau_3 \rangle \) and \( \mathcal{A}_4, \mathcal{S}_4 \) and \( \mathcal{A}_5 \) are generated by \( \langle \tau_1, \tau_2, \tau_3 \rangle \) in each case. In the cases \( q_{13} = 3 \) one has that \( \langle \tau_1, \tau_3 \rangle \cong D_3 \), so \( \tau_1 \) and \( \tau_3 \) are conjugated, that is, \( m_1 = m_3 \). It follows that

\[
2m_1 = 2m_3 = m_1 + m_3 \leq 2 \left[ \frac{g - 1}{3} \right] + 4,
\]

so \( m_1 \leq m_3 \leq \left[ \frac{g - 1}{3} \right] + 2 \). As \( q_{23} \geq 3 \),

\[
m_1 + (m_2 + m_3) \leq \left( \left[ \frac{g - 1}{3} \right] + 2 \right) + \left( 2 \left[ \frac{g - 1}{q_{23}} \right] + 4 \right) \leq 3 \left[ \frac{g - 1}{3} \right] + 6 \leq g + 5.
\]

Now, for the case \( q_{12} = q_{13} = 2 \) and \( q_{23} = r \geq 2 \), inequality (1) asserts

\[
m_1 + m_2 + m_3 \leq 2 \left[ \frac{g - 1}{2} \right] + \left[ \frac{g - 1}{r} \right] + 6 \leq 2 \left( \frac{g - 1}{2} \right) + \frac{g - 1}{r} + 6 = \frac{(r + 1)g + 5r - 1}{r}.
\]
8. Examples

Example 8.1 (Sharp upper bound in Theorem 2.1). Let \( q \geq 2 \) and let us consider an extended Schottky group \( \tilde{K} \), constructed by Theorem 3.2 using exactly \( r + 1 \) reflections \( E_1, \ldots, E_{r+1} \). The orbifold uniformized by \( \tilde{K} \) is a planar surface bounded by exactly \( r + 1 \) boundary loops. Let us consider the surjective homomorphism
\[
\theta : \tilde{K} \to D_q = \langle x, y : x^2 = y^2 = (yx)^q = 1 \rangle : \theta(E_1) = x, \theta(E_2) = \cdots = \theta(E_{r+1}) = y.
\]
Let \( \Gamma = \ker \theta \). If we set \( L = E_2E_1 \) and \( C_j = E_{r+1}E_{j+1} \), for \( j = 1, \ldots, r - 1 \), then it is not difficult to see that
\[
\Gamma = \langle L^q, C_1, \ldots, C_{r-1}, LC_1L^{-1}, \ldots, LC_{r-1}L^{-1}, \ldots, L^{q-1}C_1L^{q-1}, \ldots, L^{q-1}C_{r-1}L^{q-1} \rangle
\]
is a Schottky group of rank \( g = (r - 1)q + 1 \). Let us consider the extended Schottky groups
\[
\Gamma_1 = \theta^{-1}(\langle x \rangle) = \langle E_1, \Gamma \rangle, \quad \Gamma_2 = \theta^{-1}(\langle y \rangle) = \langle E_2, \Gamma \rangle.
\]
As the group \( \tilde{K} \) contains no imaginary reflections nor real Schottky groups, it follows that \( \Gamma_j \) is constructed, by Theorem 3.2 using \( \alpha_i \) reflections and \( \beta_i \) loxodromic and glide-reflection transformations. The handlebody \( M = (\mathbb{H}^3 \cup \Omega)/\Gamma \), where \( \Omega \) denotes the region of discontinuity of \( \tilde{K} \), admits two symmetries, say \( \tau_1 \) and \( \tau_2 \) induced by \( \Gamma_1 \) and \( \Gamma_2 \). The number of connected components of fixed points of \( \tau_i \) is exactly \( \alpha_i \). As consequence of Theorem 2.1 we should have \( \alpha_1 + \alpha_2 \leq 2(r + 1) \). Next, we proceed to see that in fact we have an equality, showing that the upper bound in Theorem 2.1 is sharp. In order to achieve the above, we use Theorem 3.5. A complete set of symmetries in \( \tilde{K} \) is provided by \( E_1, \ldots, E_{r+1} \). We also should note that \( C(E, E) = \langle E \rangle \), for every \( j \), and that \( J(\tilde{i}) = \emptyset \) and \( I(\tilde{i}) = F(\tilde{i}) \).

Case \( q \) odd. In this case, \( I(1) = \{1, 2, \ldots, r + 1\} \) and \( C(D_q, x) = \langle x \rangle \). It follows, from Theorem 3.5 that \( \alpha_1 = \alpha_2 = r + 1 \) and we are done.

Case \( q \) even. In this case, \( I(1) = \{1\}, I(2) = \{2, \ldots, r + 1\}, C(D_q, x) = \langle x, (yx)^{q/2} \rangle \cong \mathbb{Z}_2^2 \) and \( C(D_q, y) = \langle y, (yx)^{q/2} \rangle \cong \mathbb{Z}_2^2 \). It follows, from Theorem 3.5 that \( \alpha_1 = 2, \alpha_2 = 2r \) and we are done.

Example 8.2 (Sharp upper bounds in Theorem 2.4 for \( g = 2 \)). Let us consider the extended Kleinian group \( \tilde{K} \) generated by four reflections, say \( \eta_1, \ldots, \eta_4 \), where \( \eta_1(z) = \overline{z}, \eta_2(z) = -z, \eta_2(z) = 1/\overline{z} \) and \( \eta_3 \) is the reflection on a circle \( \Sigma \) which is orthogonal to the unit circle and disjoint from the real and imaginary axis. One may see that
\[
\tilde{K} = \langle \eta_1 \rangle \times ((\eta_1, \eta_2) * (\eta_3)) \cong \mathbb{Z}_2 \times (\mathbb{D}_2 * \mathbb{Z}_2).
\]
Let \( \Gamma = \langle A = \eta_1\eta_3, B = \eta_1\eta_2\eta_3\eta_2 \rangle \). It is not difficult to see that \( \Gamma \) is a Schottky group of rank 2 with a fundamental domain bounded by the 4 circles \( C_1 = \Sigma, C_1' = \eta_1(\Sigma), C_2 = \eta_2(\Sigma) \) and \( C_2' = \eta_2\eta_1(\Sigma) \) so that \( A(C_1) = C_1' \) and \( B(C_2) = C_2' \). As \( \eta_1\eta_3 \eta_1 \eta_3 = A^{-1}, \eta_2\eta_3 \eta_2 \eta_3 = A^{-1} \), \( \eta_2\eta_3 \eta_2 \eta_3 = A \), \( \eta_1\eta_3 \eta_1 \eta_3 = B^{-1} \) and \( \eta_1\eta_2 \eta_1 \eta_2 = A^{-1}B^{-1}A \), it follows that \( \Gamma \) is a normal subgroup of \( \tilde{K} \). Moreover, \( \tilde{K}/\Gamma \cong \mathbb{Z}_2^2 \times \mathbb{D}_2 \). On the handlebody \( M \) by \( \Gamma \) we see that \( \eta_1 \) and \( \eta_3 \) both induce the same symmetry \( \tau_1 \) with exactly 3 connected components of fixed points (each of them a disc), \( \eta_2 \) induces a symmetry \( \tau_2 \) with exactly one connected component of fixed points (a dividing disc) and \( \eta_4 \) induces a symmetry \( \tau_3 \) with exactly one connected component of fixed points (this being an sphere with three borders). It follows that the three induced symmetries are non-conjugated.

Example 8.3 (Sharp upper bounds in Theorem 2.4 for \( g = 3 \)). Let us consider the extended Kleinian group \( \tilde{K} \) generated by three reflections, \( \eta_1(z) = \overline{z}, \eta_2(z) = -z \) and \( \eta_3 \) the reflection on a circle \( \Sigma \) disjoint from the real and imaginary lines. One has that
\[
\tilde{K} = \langle \eta_1, \eta_2, \eta_3 \rangle \cong \mathbb{D}_2 \times \mathbb{Z}_2.
\]
Let \( \Gamma = \langle A_1 = (\eta_1\eta_3)^2, A_2 = (\eta_3\eta_2)^2, A_3 = \eta_1\eta_2\eta_3\eta_1\eta_2\eta_3 \rangle \). It is not difficult to see that \( \Gamma \) is a Schottky group of rank 3 with a fundamental domain bounded by the 6 circles \( C_1 = \eta_1(\Sigma), C_1' = \eta_1(\Sigma), C_2 = \eta_2(\Sigma), C_2' = \eta_2(\Sigma), C_3 = \eta_3(\Sigma) \) and \( C_3' = \eta_3(\Sigma) \), so that \( A_1(C_1) = C_1', A_2(C_2) = C_2' \) and \( A_3(C_3) = C_3' \). Similarly to the previous case, one may check that \( \Gamma \) is a normal subgroup of \( \tilde{K} \) and \( \tilde{K}/\Gamma \cong \mathbb{Z}_2^3 \). The reflection \( \eta_j \) induces a symmetry \( \tau_j \) (for each \( j = 1, 2, 3 \)) on the handlebody \( M \) uniformized by \( \Gamma \). In this
case (either by direct inspection or by using Theorem 3.3) one sees that each of \( \tau_1 \) and \( \tau_2 \) has exactly 2 connected components of fixed points, and \( \tau_1 \) has 4 connect components of fixed points. In some cases the handlebody \( M \) will have extra automorphisms conjugating \( \tau_1 \) with \( \tau_2 \) (for instance, when \( \mathbb{L} \) is orthogonal to the line \( \mathcal{L} = \{ \text{Re}(z) = \text{Im}(z) \} \)); but in the generic case this will not happen (that is, the three of them will be non-conjugated).

**Example 8.4** (Sharp upper bounds in Theorem 2.4 for \( g \geq 4 \) and \( H \not\cong \mathbb{Z}_2 \times \mathbb{Z}_4 \)). Let \( \widetilde{K} \) be an extended Schottky group constructed by using \( 2n + 3 \) reflections (or imaginary reflections or combination of them), say \( \eta_1, \ldots, \eta_{2n+3} \). Consider the surjective homomorphism

\[
\theta : \widetilde{K} \to D_3 = \langle a, b : a^2 = b^2 = (ab)^3 = 1 \rangle : \theta(\eta_1) = \cdots = \theta(\eta_{2n+2}) = a, \ \theta(\eta_{2n+3}) = b.
\]

In this case \( \Gamma = \ker \theta \) is a Schottky group of rank \( g = 6n + 4 \). The handlebody produced by \( \Gamma \) admits the symmetries \( \tau_1 = a, \tau_2 = b \) and \( \tau_3 = bab \). It is clear (again by direct inspection or by using Theorem 3.3) that \( m_1 = m_2 = m_3 = 2n + 3 \).

**Example 8.5** (Sharp upper bounds in Theorem 2.4 for \( g \geq 4 \) and \( H \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \)). Let \( \widetilde{K} \) be an extended Schottky group constructed by using \( 3 \) reflections (or imaginary reflections or combination of them), say \( \eta_1, \eta_2 \) and \( \eta_3 \). Consider the surjective homomorphism

\[
\theta : \widetilde{K} \to \mathbb{Z}_2 \times D_3 = \langle c \rangle \times \langle a, b : a^2 = b^2 = (ab)^3 = 1 \rangle : \theta(\eta_1) = c, \ \theta(\eta_2) = a, \ \theta(\eta_3) = b.
\]

In this case \( \Gamma = \ker \theta \) is a Schottky group of rank \( g = 2r + 1 \). The handlebody produced by \( \Gamma \) admits the symmetries \( \tau_1 = c, \tau_2 = a \) and \( \tau_3 = b \). It is clear (again by direct inspection or by using Theorem 3.3) that \( m_1 = 2r, m_2 = m_3 = 4 \).

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