SPHERICAL SUPER VARIETIES

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Abstract. We give a definition of the notion of spherical varieties in the world of complex supervarieties with actions of algebraic supergroups. A characterization of affine spherical supervarieties is given which generalizes a characterization in the classical case. We also explain some general properties of the monoid of highest weights. Several examples are given that are interesting in their own right and highlight differences with the classical case, including the regular representation, symmetric supervarieties, and actions of split supergroups.

1. Introduction

Let $G$ be a complex algebraic supergroup with $G_0$ reductive, where $G_0$ is the even underlying algebraic group of $G$. We call such supergroups quasireductive. We would like to consider supervarieties with actions of such supergroups which have an especially large amount of symmetry; namely, we would like a hyperborel subsupergroup (see definition 4.8) to have an open orbit. For those familiar with Lie superalgebras, the notion of hyperborel subsuperalgebra agrees with the usual notion of Borel subsuperalgebra for many heavily studied cases, apart from queer superalgebras (see remark 4.3). We call such supervarieties spherical, generalizing the classical notion to the super world.

In the classical world, spherical varieties are a highly rich and well-studied class of varieties which simultaneously generalizes toric varieties, flag varieties, and symmetric spaces. They provide connections between representation theory, combinatorics, and algebraic geometry. Affine spherical varieties also have a close relationship with multiplicity-free spaces in symplectic geometry, and were used by F. Knop and I. Losev to prove Delzant’s conjecture ([Los09]). In work spanning several decades up to the mid-2010s, Bravi, Brion, Cupit-Foutou, Knop, Losev, Luna, Pezzini, Vust, and others completed the combinatorial classification of all spherical varieties.

It is interesting to ask how spherical varieties generalize to the super world. Classically, the first theorem giving a connection to representation theory states that an affine $G$-variety $X$ is spherical if and only if $\mathbb{C}[X]$ is multiplicity-free as a $G$-module. In the super case we do not have complete reducibility and thus such a statement is too much to hope for, a priori. Indeed it is seen in this article that $\mathbb{C}[X]$ may not be completely reducible for an affine spherical supervariety $X$; however the socle of $\mathbb{C}[X]$ must be multiplicity-free.

On the flip side, and perhaps more surprising superficially, there are situations in which a $G$-supervariety $X$ is affine, $\mathbb{C}[X]$ is completely reducible and multiplicity-free, but $X$ is not spherical. Thus this connection does not generalize nicely to the super world. However, we do find a characterization of sphericity in terms of the subsuperalgebra of $\mathbb{C}[X]$ generated by $B$-highest weight functions, where $B$ is a hyperborel subsupergroup (theorem 5.5). This characterization generalizes the classical fact that
an affine $G$-variety $X$ is spherical if and only if $X//U$ is a toric variety for a maximal torus $T$ of a Borel subgroup $B$, where $U$ is the unipotent radical of $B$.

The author began studying examples of spherical varieties in [She19]. In that work, indecomposable spherical representations were found for a large class of quasireductive groups, and the structure of the algebra of functions was determined. That paper and this one seek to understand affine $G$-supervarieties better, in particular in understanding how the geometry of the action is connected to the representation theory of the space of functions.

This work has been in progress for several years now by other authors within the study of symmetric superspaces. In [SS16] and [SSS18] the Capelli eigenvalue problem has been studied for supersymmetric pairs coming from simple Jordan superalgebras. In [All12] a generalization of the Harish-Chandra isomorphism theorem was given, and in [AS15] certain facts about the socle of the space of functions is proven, amongst other things. Further, in [SV17] the combinatorics of root systems gotten from supersymmetric pairs is used to construct integrable systems. We hope further insight can be gained through the more general lens of spherical supervarieties.

1.1. Structure of paper. We begin with definitions and explanations about notation for supervarieties in section 2. In section 3 we define actions by supergroups and state some lemmas about actions we plan to use later on, and in section 4 we define quasireductive supergroups and the notion of hyperborel subsupergroup. In section 5 we define spherical supervarieties and give the main characterization theorem and some consequences. Then in section 6 we discuss several examples the author has considered, with new results stated. The first appendix briefly looks at the notion of spherical actions of quasireductive Lie superalgebras on supervarieties by vector fields. Finally, the second appendix addresses some generalities about smoothness of supervarieties.

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2. Supergeometry

We are work in the algebraic setting. For the basic definitions on superschemes, see chapter 10 of [CCF11]. We work over the complex numbers, i.e. all superschemes are over $\mathbb{C}$, although one could just as well work over any algebraically closed field of characteristic zero.

2.1. Notation. For a super vector space $V$ we write $V = V_\bar{0} \oplus V_\bar{1}$ for its parity decomposition, and $\Pi V = C^{0|1} \otimes V$ for the parity shift, where $C^{0|1} = 0 \oplus \mathbb{C}$. Given a homogeneous element $v \in V$, we write $\overline{v} \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ for its parity.

For a superscheme $X$, write $|X|$ for the underlying topological space of $X$. Let $\mathcal{O}_X$ denote its structure sheaf and $\mathcal{O}_X = (\mathcal{O}_X)_\bar{0} \oplus (\mathcal{O}_X)_\bar{1}$ for the parity decomposition of this sheaf. For a point $x \in |X|$, we write $\mathcal{O}_{X,x}$ for the stalk of the sheaf $\mathcal{O}_X$ at $x$ which will be a local superalgebra, and $m_x$ for its unique maximal ideal. For a superalgebra $R$, we write $X(R)$ for the Spec $R$-points of $X$. In particular, we write $X(\mathbb{C})$ for the complex
points of $X$, which for the spaces we consider will be exactly the closed points. For an open subset $|V| \subseteq |X|$, we write $V$ for the superscheme obtained by restriction of $X = (|X|, \mathcal{O}_X)$ to $|V|$, and we call $V$ an open subsuperscheme of $X$.

We write $X_0$ for the even subvariety of a superscheme $X$, that is the space cut out by the ideal sheaf $\mathcal{J}_X$ generated by $(\mathcal{O}_X)^r$. Write $i_X : X_0 \to X$ for the corresponding closed embedding, or sometimes simply $i$ if the space is clear from context. Let $\mathcal{N}_X := \mathcal{J}_X/\mathcal{J}_X^2$ be the conormal sheaf, which is a quasi-coherent sheaf on $X_0$.

For a superscheme $X$ such that $|X|$ is Noetherian and irreducible, write $\mathbb{C}(X)$ for the stalk of $\mathcal{O}_X$ at the generic point of $|X|$. Then for any open subsuperscheme $V$ of $X$ we have a natural map $\Gamma(|V|, \mathcal{O}_X) \to \mathbb{C}(X)$. This map may not be injective (although for us it always will be), but if $f$ is a section over $|V|$ we will sometimes speak of it as an element of $\mathbb{C}(X)$ with the understanding that we are talking about its image under this restriction map.

**Definition 2.1.** We define a supervariety to be a superscheme $X$ over $\mathbb{C}$ such that the following conditions are satisfied:

1. $X_0$ is integral.
2. $X$ admits a finite cover by affine open subsuperschemes $\text{Spec} \, A$, where $A$ is a finitely-generated superalgebra over $\mathbb{C}$.
3. For any open subsuperscheme $V \subseteq X$, the map $\Gamma(|V|, \mathcal{O}_X) \to \mathbb{C}(X)$ is injective.
4. $X$ is locally split; i.e. for any $x \in |X|$ there exists an open subsuperscheme $V$ of $X$ with $x \in |V|$ such that $V \cong (|V|, \Lambda^*(\mathcal{N}_X)|_{|V|})$ (see definition 4.10).

**Remark 2.2.**
- If $X$ is a supervariety, then for all open sets $|V|, |V'| \subseteq |X|$ with $|V'| \subseteq |V|$, the restriction map $\Gamma(|V|, \mathcal{O}_X) \to \Gamma(|V'|, \mathcal{O}_X)$ is injective. This follows from functoriality of restriction.
- If $X$ is a supervariety, then $\mathcal{N}_X$ will be coherent sheaf on $X_0$. It follows that $\mathcal{N}_X$ is locally free on some dense open subset $|V|$ of $X$. Because $X$ is locally split, the open subsuperscheme $V$ will then be locally isomorphic to $\Lambda^* \mathcal{N}_X$. This implies $V$ is a smooth supervariety (see section 8 for a definition and discussion of smoothness). The superschemes we are interested in have dense open orbits under the action of a supergroup, thus have a dense smooth open subsupersvariety, so this is a natural property to assume.
- The property of being locally split is affine local; that is, if $X$ is an affine superscheme, $X$ is locally split if and only if it is split. This follows from the same cohomology argument given in [VMP90] that smooth affine superschemes are split.

**Definition 2.3.** For a supervariety $X$, define the tangent sheaf $\mathcal{T}_X$ as the unique sheaf defined on any affine open subsuperscheme $V = \text{Spec} \, A$ of $X$ by $\Gamma(|V|, \mathcal{T}_X) = \text{Der}(A)$, that is all (not necessarily even) $\mathbb{C}$-linear algebra derivations of $A$. In this way $\mathcal{T}_X$ is a coherent sheaf of Lie superalgebras on $X$, and $\Gamma(V, \mathcal{T}_X)$ acts by super derivations on $\Gamma(V, \mathcal{O}_X)$.

**Definition 2.4.** Given $x \in X(\mathbb{C})$, we define the tangent space at $x$ to be the super vector space $T_x X$ given by point derivations $\delta : \mathcal{O}_{X,x} \to \mathbb{C}$, i.e. maps of vector spaces.
such that $\delta(fg) = \delta(f)g(x) + (-1)^{\overline{f}}f(x)\delta(g)$. Note that the minus sign is not strictly necessary since if $\overline{f} = 1$ then $f(x) = 0$.

**Remark 2.5.** We have a natural identification $T_xX \cong (m_x/m_x^2)^\ast$. Further, there is a natural map of super vector spaces

$$T_{X,x} \to T_xX$$

given by $D \mapsto (f \mapsto D(f)(x))$. This map is not always surjective. We say that $X$ is smooth at $x$ if it is surjective. See section 8 for a discussion of smoothness of superschemes.

### 3. Supergroups and their Actions

#### 3.1. Supergroups.

See sections 8, 9, and 11 of [CCF11] for more on the foundations of (algebraic) supergroups and their actions.

**Definition 3.1.** An algebraic supergroup is a complex supervariety $G$ equipped with morphisms $m = m_G : G \times G \to G$, $s = s_G : G \to G$, and $e = e_G : \text{Spec } \mathbb{C} \to G$ satisfying the usual commutativity conditions:

$$m \circ (m \times \text{id}_G) = m \circ (\text{id}_G \times m),$$

$$m \circ (e \times \text{id}_G) = m \circ (\text{id}_G \times e) = \text{id}_G,$$

and

$$m \circ (\text{id}_G \times s) \circ \Delta_G = m \circ (s \times \text{id}_G) \circ \Delta_G = e.$$

where $\Delta_G : G \to G \times G$ is the diagonal embedding. In addition, we assume throughout this article that $G$ is linear, that is affine.

**Definition 3.2.** For $u_e \in T_eG$, construct a right-invariant vector field $u_L$ on $G$ via left infinitesimal translation by the equation

$$u_L(f) = -(u_e \otimes 1)(m^\ast(f))$$

Then the value of $u_L$ at $e$ as a tangent vector is $-u_e$. Write $\mathfrak{g} = \text{Lie } G$ for the Lie superalgebra of right-invariant vector fields on $G$. The restriction map $\mathfrak{g} \to T_eG$ is an isomorphism of super vector spaces, so we will freely identify $\mathfrak{g}$ with $T_eG$ when convenient. Given $u_e \in T_eG$ we may also construct a left-invariant vector field on $G$ via right infinitesimal translation given by

$$u_R(f) = (1 \otimes u_e)(m^\ast(f)).$$

The Lie superalgebra of left-invariant vector fields is canonically isomorphic to the lie superalgebra of right vector fields via $u_L \mapsto u_R$.

**Remark 3.3.** If $G$ is an algebraic supergroup, then $G_0$ is an algebraic group in the usual sense, and we have a canonical isomorphism $\overline{\mathfrak{g}_0} \cong \text{Lie}(G_0)$. 

3.2. Actions.

**Definition 3.4.** Let $X$ be a supervariety and $G$ an algebraic supergroup. An action of $G$ on $X$ is a morphism $a : G \times X \to X$ such that

$$a \circ (m_G \times \text{id}_X) = a \circ (\text{id}_G \times a)$$

and

$$a \circ (e \times \text{id}_X) = \text{id}_X$$

Given an action of $G$ on $X$, we obtain a homomorphism $\rho_a : g \to \Gamma(X, T_X)$ as follows.

For an open set $V \subseteq X$, choose an open subset $V' \subseteq G$ containing the identity such that $a$ sends $V' \times V$ into $V$. Let $f \in \Gamma(V, \mathcal{O}_X)$ and $u \in g$. Then define the action of $u$ on $f$ by

$$u(f) = -(u_e \otimes 1)(a^*(f)).$$

The map $\rho_a$ in fact determines an action of the Lie superalgebra $g$ on $X$, as discussed in section 7.

**Remark 3.5.** If a Lie supergroup $G$ acts on a supervariety $X$, then by functoriality $G_0$ acts on $X_0$ such that the following diagram commutes:

$$\begin{array}{ccc}
G \times X & \longrightarrow & X \\
\downarrow & & \downarrow \\
G_0 \times X_0 & \longrightarrow & X_0
\end{array}$$

We omit the proof of the following result. It can be proven by developing the notion of an action of a super Harish-Chandra pair, and showing it is equivalent to an action of the corresponding supergroup. The fact is stated for supermanifolds without proof in [DM99] and a full proof for supermanifolds is given in section 4.5 of [Bal11]. The author will provide a full proof for the algebraic case in their thesis.

**Theorem 3.6.** Let $G$ be a Lie supergroup with $g = \text{Lie}(G)$, and suppose that $X$ is a supervariety. Suppose that $G_0$ acts on $X$ via $a_0 : G_0 \times X \to X$, and that we have a homomorphism of Lie superalgebras $\rho : g \to \Gamma(X, T_X)$ such that

1. $\rho|_{g_0}(u) = -(u \otimes 1) \circ a_0^*$ for all $u \in g_0$;
2. $\rho(\text{Ad}(g)(u))(u) = (a_0^g)^{-1} \circ \rho(u) \circ (a_0^g)^*$ for all $g \in G_0$ and $u \in g$, where $a_0^g = a_0 \circ (i_g \times \text{id}_X)$, where $i_g : \{g\} \to G_0$ is the natural inclusion.

Then there exists a unique action $a : G \times X \to X$ of $G$ on $X$ such that $a|_{G_0} = a_0$ and $\rho_a = \rho$.

We will often use this result in the form of the following corollary:

**Corollary 3.7.** Suppose that a Lie supergroup $G$ acts on a supervariety $X$, and that the open subset $|V| \subseteq |X|$ is stable under the action of $G_0$. Then the open subsupervariety $V$ is stable under the action of $G$, i.e. the action of $G$ on $X$ restricts to an action of $G$ on $V$.
3.3. Orbit maps and stabilizers. For \( x \in X(\mathbb{C}) \), we have an orbit map at \( x \), \( a_x : G \to X \), given by \( a \circ (\text{id}_G \times i_x) \), where \( i_x : \{x\} \to X \) is the natural inclusion. We refer to \( a_x^{-1}(x) \), the fiber of this morphism over \( x \), as the stabilizer \( \text{Stab}_G(x) \) of \( x \), a closed subsupergroup of \( G \) (see section 11.8 of [CCF11]). The following lemma is well-known (see e.g. Lemma 4 of [Vis11]).

**Lemma 3.8.** For \( x \in X(\mathbb{C}) \), the differential of the orbit map \( a_x \) at the identity of \( G \), \((da_x)_e : T_e G \to T_x X\), coincides with the natural evaluation map \( \rho_{a_x}(g) \to T_x X \).

The Lie superalgebra of \( \text{Stab}_G(x) \), which we write as \( \text{stab}_g(x) \), is then exactly the kernel of the restriction morphism \( \rho_{a_x}(g) \to T_x X \).

**Definition 3.9.** Suppose that \( G \) acts on \( X \). We say that the action is a submersion at a point \( x \in X(\mathbb{C}) \) if the map \( a_x : G \to X \) is a submersion at \( e_G \in G(\mathbb{C}) \) (or equivalently at any point of \( G \)). In this case, the locus of points where the map is a submersion will be an open subset of \( |X| \), and we refer to the open subsupervariety defined by this locus as an open orbit of \( G \). If all of \( X \) is an open orbit of \( G \), we say that \( X \) is a homogeneous \( G \)-supervariety.

**Remark 3.10.** Note that an action is a submersion at \( x \) if and only if the evaluation map \( g \to \Gamma(|X|, T_X) \to T_x X \) is surjective by lemma 3.8. Further, by proposition 8.2, an open orbit of \( G \) must be smooth.

The following two propositions are known to experts. We write out proofs for clarity.

**Proposition 3.11.** Let \( X \) be a supervariety, and let \( a : G \times X \to X \) be an action of \( G \) an algebraic supergroup on \( X \). Then for \( x \in X(\mathbb{C}) \), \( a_x \) is a submersion if and only if the pullback morphism of sheaves \( a_x^* : \mathcal{O}_X \to (a_x)_* \mathcal{O}_G \) is injective.

**Proof.** Let \( H \) be the stabilizer of \( x \), and write \( \pi : G \to G/H \) for the natural projection. Then the natural map of sheaves \( \mathcal{O}_{G/H} \to \pi_* \mathcal{O}_G \) is injective. There is an induced \( G \)-equivariant immersion \( b : G/H \to X \), i.e. the differential at every point is injective, and this map factors the orbit map \( a_x \). Therefore if \( a_x \) is a submersion, \( G/H \to X \) is too, and hence it induces an isomorphism of \( G/H \) onto an open subset of \( X \). By our assumption that restriction of functions is injective on supervarieties, the map

\[
\mathcal{O}_X \to b_* \mathcal{O}_{G/H} \to b_* \pi_* \mathcal{O}_G = (a_x)_* \mathcal{O}_G
\]

is injective.

If \( a_x \) is not a submersion, then first suppose that the underlying image of \( G/H \) in \( X \) is not open. Then we may choose a non-nilpotent function on \( X \) which vanishes on the underlying closed subscheme defined by its image, so that some power of this function will vanish under pullback, and \( a_x^* \) is not injective. Therefore assume \( G/H \) has an underlying open image, say \( |V| \subseteq |X| \). Then we may restrict to the open subsuperscheme \( V \) of \( X \), and there the morphism \( G/H \to V \) will be an isomorphism on closed points and an immersion, but not a submersion. One may then show that this map is a closed embedding, by considering the map on local rings and using Nakayama’s lemma. Hence the map on stalks is surjective, and so if it is also injective the map would be an isomorphism on this open set \( V \), contradicting the fact that \( a_x \) is not submersive. \( \square \)
Proposition 3.12. Suppose that a Lie supergroup $G$ acts on $X$. If $X$ is a homogeneous $G$-supervariety, then $\mathbb{C}[X]$ has no nontrivial $G$-invariant ideals. If $X$ is affine the converse also holds.

Proof. Because $\mathbb{C}[X]$ is a union of finite-dimensional $G$-submodules, any $G$-invariant ideal will also be $\mathfrak{g}$-invariant. Thus we may work with the action of the Lie algebra.

For the first statement, suppose that $I \subseteq \mathbb{C}[X]$ is a nonzero $\mathfrak{g}$-invariant ideal of $\mathbb{C}[X]$. If $I \neq \mathbb{C}[X]$, there exists a point $x \in X(\mathbb{C})$ such that $I \subseteq \mathfrak{m}_x$, where $\mathfrak{m}_x$ is the maximal ideal of functions vanishing at $x$. For each $f \in I$, we may find $n \in \mathbb{Z}_+$ such that $f \in \mathfrak{m}_x^n \setminus \mathfrak{m}_x^{n+1}$. Let $g \in I$ be such that it vanishes to the lowest degree at $x$ of all elements of $I$, say $m$, so that $g \in \mathfrak{m}_x^m \setminus \mathfrak{m}_x^{m+1}$. Then since $I \subseteq \mathfrak{m}_x$ we must have $m > 0$. If we lift a basis of $T^*_xX$ to $\mathfrak{m}_x$, giving $x_1, \ldots, x_r, \xi_1, \ldots, \xi_s$, we may find a homogeneous degree $m$ polynomial $p$ such that $g - p(x_1, \ldots, x_r, \xi_1, \ldots, \xi_s) \in \mathfrak{m}_x^{m+1}$. Since $\mathfrak{g} \rightarrow T^*_xX$ is a surjection, we may find $u \in \mathfrak{g}$ such $u(p) \in \mathfrak{m}_x^m \setminus \mathfrak{m}_x^{m+1}$. Therefore we also must have $u(g) \in \mathfrak{m}_x^{m-1} \setminus \mathfrak{m}_x^m$. But $u(g) \in I$. This contradicts the minimality of $m$, and thus $I = \mathbb{C}[X]$.

For the second statement, let $x \in X$, and consider that orbit map $a_x: G \rightarrow X$. It is a submersion if and only if $a_x^*: \mathbb{C}[X] \rightarrow \mathbb{C}[G]$ is injective by proposition 3.11 applied to the affine case. If it is not injective, its kernel will be a non-trivial $G$-invariant ideal. Therefore it must be injective and we are done. \hfill \Box

Remark 3.13. This proof shows that if a Lie superalgebra $\mathfrak{g}$ acts homogeneously on a supervariety $X$ (see section 7 for the meaning of this) then $\mathbb{C}[X]$ has no nontrivial $\mathfrak{g}$-invariant ideals.

3.4. Rational invariants. In the classical world, if an algebraic group $G$ acts on a space $X$, then it admits an open orbit if and only if $\mathbb{C}(X)^G = \mathbb{C}$. In the super world, this general principle no longer holds.

Example 3.14. Consider the action of $GL(0|n)$ on $X = \mathbb{C}^{0|n}$ by the standard representation of $GL(0|n)$. This supervariety has one point, and the orbit of that point is just itself, so there is not an open orbit. We have $\mathbb{C}(X) = \Lambda(\mathbb{C}^n)^*$, and this is a multiplicity-free representation of $\mathfrak{g} = \mathfrak{gl}(n)$, so in particular $\mathbb{C}(X)^\mathfrak{g} = \mathbb{C}$.

We do have the forward direction:

Proposition 3.15. If a Lie supergroup $G$ acts on a supervariety $X$ with an open orbit, we have $\mathbb{C}(X)^\mathfrak{g} = \mathbb{C}$.

Proof. Let $f \in \mathbb{C}(X)^\mathfrak{g}$ be non-zero, and choose an affine open subsupervariety $\text{Spec} A$ of $X$ contained in the open orbit of $G$ on which $f$ is regular. Then $A$ has no non-trivial $\mathfrak{g}$-stable ideals by proposition 3.12 and the remark following it. Therefore $(f) = A$, so $f$ is non-vanishing on $A$. However, if $x \in \text{Spec} A(\mathbb{C})$, then $f - f(x)$ is $\mathfrak{g}$-fixed and vanishes at $x$, i.e. is not invertible, so $(f - f(x))$ is a $\mathfrak{g}$-stable ideal not equal to $A$. Thus it must be trivial, i.e. $f = f(x)$, so $f$ is a constant function. \hfill \Box

As for a converse, we may state one for certain algebraic supergroups. First we need some notation. Suppose that $\mathfrak{b}$ is a solvable Lie superalgebra such that $[\mathfrak{b}_T, \mathfrak{b}_T] \subseteq [\mathfrak{b}_0, \mathfrak{b}_0]$. Then by lemma 1.37 of [CW12], every finite-dimensional irreducible representation of $\mathfrak{b}$ is one-dimensional.
If $V$ is a representation of $\mathfrak{b}$, we write $V^{(\mathfrak{b})}$ for the span of the $\mathfrak{b}$-eigenvectors of $V$, which will be a semisimple representation of $\mathfrak{b}$. Write $\Lambda_\mathfrak{b}(V)$ for the collection of characters $\Lambda$ of $\mathfrak{b}$ such that there is a $\mathfrak{b}$-eigenvector of weight $\Lambda$ in $V$. Finally, if $\mathfrak{b}$ acts by vector fields on the functions of a supervariety $X$, set

$$\Lambda^+_{\mathfrak{b}}(X) := \Lambda_\mathfrak{b}(\mathbb{C}[X]), \quad \Lambda_\mathfrak{b}(X) := \Lambda_\mathfrak{b}(\mathbb{C}(X)).$$

Observe that if $A$ is a superalgebra which $\mathfrak{b}$ acts on by derivations, then $A^{(\mathfrak{b})}$ is a subsuperalgebra of $A$.

**Proposition 3.16.** Suppose that $B$ is a solvable connected algebraic supergroup which acts on a supervariety $X$. Suppose that $[\mathfrak{b}_T, \mathfrak{b}_T] \subseteq [\mathfrak{b}_T, \mathfrak{b}_T]$ where $\mathfrak{b} = \text{Lie}(B)$. Then if $X_0$ is a normal variety and we have $\mathbb{C}(X)^{(\mathfrak{b})}$ is a multiplicity-free $\mathfrak{b}$-representation such that every non-zero $f \in \mathbb{C}(X)^{(\mathfrak{b})}$ is non-nilpotent, then $X$ has an open $B$-orbit.

**Proof.** Write $\Lambda$ for the character lattice of $B$, a finitely generated free abelian group. By our assumptions, $\mathbb{C}(X)^{(\mathfrak{b})}$ is isomorphic to the group algebra of a subgroup $\Lambda(X)$ of $\Lambda$, and hence $\Lambda(X)$ is free of some rank, say $n \in \mathbb{N}$. Choose rational functions $f_1, \ldots, f_n \in \mathbb{C}(X)^{(\mathfrak{b})}$ that are eigenfunctions of $B$ such that their weights form a $\mathbb{Z}$-basis of $\Lambda(X)$. Then by removing the divisors of zeroes and poles of $f_1, \ldots, f_n$, there exists a $B$-stable open subsupervariety $V$ of $X$ where $f_1, \ldots, f_n$ are regular and non-vanishing, and hence $\mathbb{C}(X)^{(\mathfrak{b})} \subseteq \mathbb{C}[V]$. By our assumption on $X$, we may now apply Sumihiro’s theorem (requiring normality of $X_0$) and corollary 3.7 to find a $B$-stable affine open subsupervariety $V'$ of $V$.

Now we claim that $V'$ is a homogeneous $B$-supervariety. Indeed, if $I \subseteq \mathbb{C}[V']$ is a nontrivial $B$-stable ideal, then it admits a $B$-eigenfunction $f \in I$. But then $f \in \mathbb{C}(X)^{(\mathfrak{b})}$, so by assumption $f$ is invertible on $V$, so $\mathbb{C}[V'] = (f) = I$. We conclude by proposition 3.12.\[\square\]

4. QUASIREDUCTIVE SUPERGROUPS AND HYPERBORELS

**Definition 4.1.** A supergroup $G$ is quasireductive if $G_0$ is reductive. We say a Lie superalgebra $\mathfrak{g}$ is quasireductive if it is the Lie superalgebra of a quasireductive supergroup.

4.1. Hyperborels. In order to discuss the notion of a spherical supervariety, it is necessary that we have a well-purposed generalization of Borel subgroup (subalgebra) to the super case. There are different notions of Borel subsuperalgebras used for quasireductive Lie superalgebras, although the most common one seems to coincide with the definition in section 9.3 of [Ser11]. We use a different notion that is closer to the definition given in the beginning of chapter 3 of [Mus12], and agrees with this definition when the Cartan subsuperalgebra of $\mathfrak{g}$ is purely even. In order to prevent confusion, we choose to call our subsuperalgebras hyperborels.

**Definition 4.2.** Let $\mathfrak{g}$ be quasireductive. A hyperborel subsuperalgebra of $\mathfrak{g}$ is a subsuperalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ such that

- $\mathfrak{b}_T$ is a Borel of $\mathfrak{g}_T$ in the usual sense;
- $[\mathfrak{b}_T, \mathfrak{b}_T] \subseteq [\mathfrak{b}_T, \mathfrak{b}_T]$; and,
- $\mathfrak{b}$ is maximal with this property.
We now give a brief discussion of this definition.

**Remark 4.3.**
- Given a hyperboreal subsuperalgebra $\mathfrak{b}$ and a choice of Cartan subsuperalgebra $\mathfrak{h}_\mathfrak{g} \subseteq \mathfrak{g}_\mathfrak{g}$, we have $\mathfrak{b} = \mathfrak{h}_\mathfrak{g} \ltimes \mathfrak{u}$ where $\mathfrak{u}$ is a nilpotent ideal. We call $\mathfrak{u}$ the unipotent radical of $\mathfrak{b}$.
- We may always conjugate a hyperboreal $\mathfrak{b}$ by an inner automorphism of $\mathfrak{g}$ so that $\mathfrak{b}_\mathfrak{g}$ is a chosen Borel of $\mathfrak{g}_\mathfrak{g}$.

**Remark 4.4.** By definition, hyperboles are solvable and all irreducible representations of them are one-dimensional (see lemma 1.37 of [CW12]). This property is the primary way in which the notion of hyperboreal subsuperalgebra is a generalization of Borel subalgebra for reductive Lie algebras. Further, it is this property that is of importance for us in the characterization of spherical supervarieties (proposition 5.3 and theorem 5.5).

Recall that for Lie superalgebras there is a notion of Cartan subsuperalgebras (see [Sch87] and [PS94]). For a quasireductive Lie superalgebra, a Cartan subsuperalgebra $\mathfrak{h}$ is given by a Cartan subalgebra $\mathfrak{h}_\mathfrak{g} \subseteq \mathfrak{g}_\mathfrak{g}$, and then $\mathfrak{h}$ is the centralizer of $\mathfrak{h}_\mathfrak{g}$ in $\mathfrak{g}$.

**Definition 4.5.** We say a quasireductive Lie superalgebra $\mathfrak{g}$ is Cartan-even if for a Cartan subsuperalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, $\mathfrak{h} = \mathfrak{h}_\mathfrak{g}$. We say a quasireductive supergroup $G$ is Cartan-even if $\text{Lie}(G)$ is.

The notion of hyperboreal is most natural for supergroups and superalgebras which are Cartan-even. If $\mathfrak{g}$ is Cartan-even then the notion of hyperboreal agrees with the definition of Borel given in [Mus12]. Further, the notion of hyperboreal and Borel (as defined in [Ser11]) coincide if $\mathfrak{g}$ is one of the following Cartan-even superalgebras: $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$ for $m \neq n$ and $(m, n) \neq (1, 1)$, $\mathfrak{psl}(n|n)$ or $\mathfrak{sl}(n|n)$ for $n \geq 3$, $\mathfrak{p}(n)$, $\mathfrak{osp}(m|2n)$, or is one of the exceptional basic simple Lie superalgebras. This is proven in Proposition 4.6.1 of [Mus12]. The case of $\mathfrak{p}(n)$ is not considered there, but one can show the notions agree for this superalgebra as well (although they do not agree for the derived subsuperalgebra of $\mathfrak{p}(n)$).

**Remark 4.6.** If $\mathfrak{g}$ is Cartan-even and $\mathfrak{b}$ is a Borel subsuperalgebra of $\mathfrak{g}$ (as defined in [Ser11]), then $\mathfrak{b}$ is contained in a hyperboreal subsuperalgebra. Indeed, a Borel subsuperalgebra satisfies all the conditions of being a hyperboreal but possibly maximality.

However if $\mathfrak{g}$ is not Cartan-even, for instance $\mathfrak{g}$ is the queer Lie superalgebra $\mathfrak{q}(n)$, then hyperboles greatly differ from Borels, as they do not contain a Cartan subsuperalgebra.  

**Remark 4.7.** If $\mathfrak{g}$ is quasireductive and $\mathfrak{b}$ a hyperboreal of $\mathfrak{g}$, then for a finite dimensional irreducible representation $V$ of $\mathfrak{g}$, $\dim V^{(b)} \geq 1$ by remark 4.4. However, it is possible that $\dim V^{(b)} > 1$, and thus we no longer have a bijective correspondence between certain characters of the Borel and finite dimensional irreducible representations.

Indeed even when $\mathfrak{g}$ is Cartan-even this phenomenon can occur; in [Ser18], a nontrivial central extension of the derived subsuperalgebra of $\mathfrak{p}(4)$ is considered, along with an irreducible representation $V_t$ deforming the standard representation of $\mathfrak{p}(4)$. If $t \neq 0$, it is shown that $V_t^{(b)}$ is irreducible. However there is a hyperboreal subsuperalgebra given by (in the notation of the paper) $\mathfrak{b} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ is a Borel subalgebra of $\mathfrak{g}_0$. One can check that $A^2 V_t^{(b)}$ is two-dimensional for any $t$.

However, if a hyperboreal subsuperalgebra $\mathfrak{b}$ contains a Borel subsuperalgebra then $\dim V^{(b)} = 1$ for an irreducible representation $V$ of $\mathfrak{g}$, by highest weight theory.
Definition 4.8. If $G$ is quasireductive, we call a subsupergroup $B$ a hyperborel subsupergroup if it is connected and gotten by integrating a hyperborel subsuperalgebra $b$ of $g$. If $u$ is the unipotent radical, we write $U$ for the connected subsupergroup of $B$ it integrates to in $G$ and call it the maximal unipotent subsupergroup of $B$. Finally, we write $T$ for the connected subgroup of $G$ that a chosen Cartan subsuperalgebra $b_\mathfrak{C} \subseteq b$ integrates to, which will be a maximal torus of $G_0$; we call $T$ a maximal torus of $B$.

Definition 4.9. If $X$ is a $G$-variety and $B$ a hyperborel of $G$ we set $\Lambda^+_B(X) := \Lambda^+_b(X)$ and $\Lambda_B(X) := \Lambda_b(X)$ (or simply $\Lambda^+(X)$, resp. $\Lambda(X)$ when there is no confusion).

There is a canonical identification of weights of $T$ with characters of $B$ via the composition of maps $T \to B \to B/U$. The algebra $\mathbb{C}[X]^U$ has a natural $T$-action, and $\Lambda_B^+(X)$ are the weights of this action under this identification. Also observe that neither $\Lambda_B^+(X)$ nor $\Lambda_B(X)$ are a monoid or group in general, due to the presence of nilpotent functions. For example, consider the action of an even torus on the functions of a purely odd representation of it.

4.2. $G_0$-equivariant splittings of supervarieties. Let $X_0$ be a variety and $\mathcal{N}$ a coherent sheaf on $X_0$. Then $X = (|X_0|, \Lambda^\bullet \mathcal{N})$ is a supervariety in a natural way.

Definition 4.10. We say that a supervariety $X$ is split if there exists a coherent sheaf $\mathcal{N}$ on $X_0$ and an isomorphism $X \cong (|X|, \Lambda^\bullet \mathcal{N})$. We call an isomorphism of $X$ with $(|X|, \Lambda^\bullet \mathcal{N})$ a splitting of $X$.

Remark 4.11. Observe that if $X$ is split and $X \cong (|X|, \Lambda^\bullet \mathcal{N})$, then $\mathcal{N}_X \cong \mathcal{N}$.

When $X$ is split, so that $X \cong (|X|, \Lambda^\bullet \mathcal{N}_X)$, its structure sheaf becomes endowed with a $\mathbb{Z}$ grading according to the exterior powers of the conormal sheaf, namely $(\Lambda^\bullet \mathcal{N}_X)_i = \Lambda^i \mathcal{N}_X$. However a split supervariety $X$ has, in general, many isomorphisms with $(|X|, \Lambda^\bullet \mathcal{N})$ (see for instance [Kos94]).

Definition 4.12. Let $G_0$ be an algebraic group. If $X$ is a $G_0$-supervariety, then we say it has a $G_0$-equivariant splitting if there exists a $G_0$-equivariant sheaf $\mathcal{M}$ on $X_0$ and a $G_0$-equivariant isomorphism $X \cong \Lambda^\bullet \mathcal{M}$.

Proposition 4.13. Let $G_0$ be a reductive group and $X$ a $G_0$-supervariety. If $X$ is split, then $X$ admits a $G_0$-equivariant splitting. In particular, if $G$ is quasireductive and $X$ is a $G$-supervariety which is split, then $X$ admits a $G_0$-equivariant splitting.

This question was considered by Rothstein in [Rot93] in the analytic setting. Adapting the proof ideas there to the algebraic setting one can prove the above proposition. The author will not include the proof here, however it will be written out carefully in their PhD dissertation.

5. Spherical Supervarieties

Let $G$ be quasireductive.

Definition 5.1. We say a $G$-supervariety $X$ is spherical if there exists a hyperborel $B$ of $G$ with an open orbit on $X$. If a hyperborel $B$ has an open orbit on $X$, we say that $X$ is $B$-spherical.
Remark 5.2.  
- If a $G$-supervariety $X$ is spherical, then the $G_0$ variety $X_0$ is also spherical.
- Note that a spherical supervariety need not be spherical with respect to every hyperborel; in fact if $\mathfrak{g}$ is basic classical this occurrence would be a degeneracy.

Proposition 5.3. Let $G$ be quasireductive, $B$ a hyperborel of $G$, and $X$ a supervariety such that $X_0$ is a normal variety. Then $X$ is $B$-spherical if and only if $\mathbb{C}(X)^{(b)}$ is a multiplicity-free $b$-module whose nonzero elements are non-nilpotent.

Proof. This follows immediately from proposition 3.16. □

5.1. Affine Spherical Supervarieties. In the classical case we have a characterization of affine spherical varieties by the fact the $\mathbb{C}[X]$ is a multiplicity-free representation. One might hope that this generalizes to the super case. Of course there is a first issue that for supergroups completely reducibility is a rare phenomenon to begin with. But one might hope that perhaps $\mathbb{C}[X]^U$ being multiplicity-free as a $T$-module is sufficient. This turns out to not be the case as the next examples demonstrate.

Example 5.4.  
- Consider the action of $GL(0|n)$ on $\mathbb{C}^{0|n}$ by the standard representation. The algebra of functions is $\Lambda^*(\mathbb{C}^n)^*$, which is completely reducible and multiplicity-free. However, there is only one point and the orbit of it under the whole group is itself, so this space is not spherical.
- An example which has a nontrivial even part is given by considering $G = OSP(1|2)$ and letting $X = OSP(1|2)/T$, where $T$ is a maximal torus of $G_0$. By the representation theory of $OSP(1|2)$ and Frobenius reciprocity, $\mathbb{C}[X] \cong \bigoplus_{n \geq 0} \Pi^a L(n)$, where $L(n)$ is the irreducible representation of highest weight $n$ with even highest weight vector. Hence $\mathbb{C}[X]$ is completely reducible and multiplicity-free. However, no hyperborel admits an open orbit since the odd dimension of $X$ is 2 while the odd dimension of any hyperborel is 1.

The next theorem demonstrates that the issue with the above two spaces is that some of the highest weight functions are nilpotent.

Theorem 5.5. Let $X$ be an affine $G$-supervariety, $B$ a hyperborel of $G$ with maximal unipotent subsupergroup $U$ and maximal torus $T$. Then the following are equivalent:

1. $X$ is spherical for $B$.
2. $X_0$ is spherical for $B_0$, and every nonzero $B$-highest weight function in $\mathbb{C}[X]$ is non-nilpotent.
3. Every nonzero $B$-highest weight function in $\mathbb{C}[X]$ is non-nilpotent, and $\dim \mathbb{C}[X]^U_\lambda \leq 1$ for all weights $\lambda$ of $T$.
4. $\mathbb{C}[X]^U$ is an even commutative algebra without nilpotents, and the natural $T$-action is multiplicity-free.

Proof. (1) $\implies$ (2): Let $x \in X(\mathbb{C})$ be such that $a_x : B \to X$ is a submersion, so that $a_x^*$ is injective. In $\mathbb{C}[B]$, all $B$-highest weight functions are non-nilpotent, and therefore the same must be true of the functions on $X$.

(2) $\implies$ (3): Since $X_0$ is spherical for $B_0$, we have $\dim \mathbb{C}[X_0]^U_\lambda \leq 1$ for all $\lambda$. Since the $B$-highest weight functions are non-nilpotent, the restriction map $\mathbb{C}[X]^{(b)} \to \mathbb{C}[X_0]^{(b)}$ is injective, and we are done.

11
Definition 5.6. If a \( a \) exists weights \( \lambda_1, \ldots, \lambda_m \in S \) such that the monoid generated by \( S \) and \( -\lambda_1, \ldots, -\lambda_m \) is a group. Then if we invert \( f_{\lambda_1}, \ldots, f_{\lambda_m} \) in \( \mathbb{C}[X] \), all \( B \)-eigenfunctions in \( \mathbb{C}[V] \) will be invertible. Further, this open subsuperscheme \( V \) will be \( B \)-stable. Choose a point \( x \in V(\mathbb{C}) \), and consider the orbit map \( a_x : B \to X \). Since all \( f_{\lambda} \) become units on \( V \), they must not be in the kernel of \( a_x^* \). But if \( a_x^* \) is not injective, the kernel will contain a \( B \)-highest weight function, a contradiction. Therefore \( a_x \) must be a submersion, and so \( X \) is spherical. \( \square \)

Definition 5.7. If \( X \) is \( B \)-spherical of rank \( m \), there exist \( m \) \( B \)-highest weight functions \( f_{\lambda_1}, \ldots, f_{\lambda_m} \in \mathbb{C}[X] \) such that their common non-vanishing set is the open \( B \)-orbit.

Corollary 5.7. Suppose that an irreducible representation \( \mathbb{C}[X] \) shows up with multiplicity greater than 1. If \( B \) is a hyperborel for which \( X \) is \( B \)-spherical, there will be two \( B \)-eigenfunctions of the same weight in \( \mathbb{C}[X] \). This contradiction (3) of theorem 5.5. \( \square \)

Now suppose that \( X \) is an affine \( B \)-spherical supervariety and \( V \) is the open \( B \)-orbit. By the reasoning given in the proof of proposition 3.16, we know that all rational \( b \)-eigenfunctions will be regular (and in fact non-vanishing) on \( V \). Hence \( \mathbb{C}[V]^U = \mathbb{C}(X)^b \), and because these functions are all non-nilpotent we have

\[
\mathbb{C}(X)^b = \mathbb{C}[V]^U \cong \mathbb{C}[V_0^U] = \mathbb{C}(X_0)^{b_{\mathfrak{g}}}
\]

by restriction of functions. Further, these algebras are all isomorphic to group algebras on \( \Lambda_B(X) \), a finitely generated free abelian subgroup of the character lattice of \( T \).

Now on all of \( X \), restriction induces an injective map \( \mathbb{C}[X]^U \to \mathbb{C}[X_0]^U \), and hence an inclusion \( \Lambda^+_B(X) \subseteq \Lambda^+_B(X_0) \) and thus \( \Lambda^+_B(X) \) will be a submonoid of \( \Lambda^+_B(X_0) \). Note that \( \mathbb{C}[X]^U \) is the monoid algebra on \( \Lambda^+_B(X) \) and \( \mathbb{C}[X_0]^U \) is the monoid algebra on \( \Lambda^+_B(X_0) \).

It is a classical fact about spherical varieties that \( \mathbb{C}[X_0]^U \) is finitely generated, so choose generators \( g_1, \ldots, g_n \) which are \( b_{\mathfrak{g}} \)-eigenfunctions. Note that \( V_0 \) is precisely the non-vanishing locus of these functions. We may uniquely lift these to \( b \)-eigenfunctions \( f_1, \ldots, f_n \) on \( V \). Since \( X \) is affine it is split as a supervariety, and by proposition 4.13 we may choose a \( G_0 \)-equivariant splitting of \( X \). Thus we may write \( \mathbb{C}[X] = \Lambda^* M \), where \( M \) is a finitely generated \( G_0 \)-equivariant \( \mathbb{C}[X_0] \)-module. Let us assume the largest non-zero exterior power of \( M \) is \( k \). Then we may write

\[
f_i = g_i + m_{i1} + \cdots + m_{ik} \quad \text{where} \quad m_{ij} \in \Lambda^j M_{g_1 \cdots g_n}.
\]

Here \( M_{g_1 \cdots g_n} \) is the localization of \( M \) to the non-vanishing locus of \( g_1, \ldots, g_n \). We may do this because since \( f_i \) is a \( b \)-eigenvector, each \( m_{ij} \) must be a \( b_{\mathfrak{g}} \)-eigenvector and it
must be regular on the open $B_0$-orbit. Now the obstruction to regularity of $f_i$ is the poles of $m_{ij}$ along $g_1 \cdots g_n = 0$. For each $m_{ij}$, there exists a positive integer $k_{ij}$ such that $(g_1 \cdots g_k)^{k_{ij}} m_{ij} \in \mathcal{N} M$. By choosing an integer $N$ larger than $k + \max_{i,j} k_{ij}$, we now have:

**Proposition 5.9.** There exists an integer $N > 0$ such that $f_1^{k_1} \cdots f_n^{k_n}$ is regular whenever $k_1, \ldots, k_n \geq N$.

**Proof.** Expanding out the product, one sees that for any integer $N$ chosen as described in the paragraph before the proposition, the poles will be resolved.

**Corollary 5.10.** The set $\Lambda_B^+(X)$, which is a submonoid of $\Lambda_B^+(X_0)$, generates $\Lambda_B(X) = \Lambda_{B_0}(X_0)$ as a group. Further it is Zariski dense in the vector space spanned by its weights.

**Proof.** By proposition 5.9, $\Lambda_B^+(X)$ contains the lattice points of a translated orthant of $\mathbb{R} \otimes \mathbb{Z} \Lambda_B(X)$, and so the results follow.

Write $X//U := \text{Spec} \mathbb{C}[X]^U$. Then by (4) of theorem 5.5, $X//U$ is an even variety and admits a natural $T$-action such that $\mathbb{C}[X//U]$ is a multiplicity-free $T$-module. In particular, $X//U$ has an open $T$-orbit, hence is essentially a toric variety but that it need not be normal or Noetherian. Indeed, we observe it is isomorphic to the group algebra of $\Lambda_B^+(X)$, so being normal is equivalent to this monoid being saturated, and being Noetherian is equivalent to the monoid being finitely generated. We now present examples showing how these properties can fail.

**Example 5.11.** Consider the action of $G = GL(1|2)$ on $X = S^2 \mathbb{C}^{1|2}$ as the second symmetric power of the standard representation. This is a spherical supervariety as one can check (this was checked in [She19]), and is spherical exactly with respect to the hyperborels $B^+$ and $B^-$ of upper and lower triangular matrices, respectively. The coordinate ring $\mathbb{C}[X]$ is a supersymmetric polynomial algebra given by $S^\bullet (S^2(\mathbb{C}^{1|2})^\ast)$ as both an algebra and a $G$-module.

As a $G_0 = GL(1) \times GL(2)$-representation $X_0$ is a sum of two one-dimensional representations of distinct weights. Therefore the $B_0$-highest weight functions of $X_0$ are the monomials in two $G_0$-eigenfunctions $x, y$, where we let $x$ have weight $\lambda$ and $y$ have weight $\mu$. Let $\xi, \eta \in (S^2 \mathbb{C}^{1|2})^\ast_+$ be odd weight vectors of weights $\alpha, \beta$. Then $\mathbb{C}[X] = \mathbb{C}[x, y, \xi, \eta]$. One can show that $\xi \eta$ is a $G_0$-eigenvector of weight $\lambda + \mu$, and so one can show that for any hyperborel $B$ the rational $B$-eigenfunctions on $X$ are, up to scalar, all of the form:

$$f_{ij} = x^i y^j + c_{ij} x^i y^j \frac{\xi \eta}{xy}$$

where $i, j \in \mathbb{Z}$ and $c_{ij} \in \mathbb{C}$ is a coefficient in $\mathbb{C}$ to be determined depending on the choice of hyperborel. For the hyperborel $B^+$, we find that $c_{ij} = i$ and for $B^-$ we find that $c_{ij} = -j$. These values for $c_{ij}$ tell us which rational $B$-eigenfunctions are regular on all of $X$, or equivalently tell us what $\Lambda_{B^\pm}^+(X)$ are. We draw the two monoids below to visualize the result:

For comparison, the monoid $\Lambda_{B_0}^+(X)$ for any Borel subgroup $B_0$ of $G$ consists of all the lattice points that are a nonnegative linear combination of $\lambda$ and $\mu$. This example demonstrates that $\Lambda_B^+(X)$ need not be finitely generated as neither of the above monoids are finitely generated.
Example 5.12. Consider the action of $G = OSP(3|4) \times OSP(3|4)$ on $X = OSP(3|4)$ by left and right multiplication. The notion of Borel and hyperborel coincide for both $OSP(3|4)$ and $OSP(3|4) \times OSP(3|4)$. If $B$ is a Borel of $OSP(3|4)$, then $X$ is $B \times B^-$-spherical where $B^-$ is the opposite Borel of $B$. As we will see in the examples section, $\Lambda^+_{B \times B^-}(X)$ will be exactly the $B$-dominant weights of $OSP(3|4)$. Now if we choose the Borel determined by the simple roots $\delta_1 - \delta_2, \delta_2 - \epsilon_1, \epsilon_1$ as described in section 1.3.3 of [CW12], then by theorem 2.11 of [CW12] the weight $\lambda = \epsilon_1 + \epsilon_2 + \delta_1 + \delta_2$ is not dominant while $k \lambda$ is dominant for $k \geq 2$. Thus $\Lambda^+(X)$ is will not be saturated in this case.

6. Examples

We present some examples of spherical supervarieties.

6.1. Spherical Representations. Irreducible spherical representations of reductive algebraic groups were originally classified by Kac in [Kac80]. In [She19], the author classified all indecomposable spherical representations of the groups $GL(m|n)$, $OSP(m|2n)$, $P_{n|m}$, and the basic exceptional simple groups. The case of $Q(n)$ is also looked at, however a different notion of spherical was used for this supergroup there.

We found there are a few infinite families of irreducible representations, along with certain small exceptional cases. Below is a table of the infinite families; for the rest, we refer the reader to the paper. We write $GL_{m|n}$, $OSP_{m|2n}$, and $P_{n|m}$ respectively for the standard representations of $GL(m|n)$, $OSP(m|2n)$, and $P(n)$ respectively. We also state the dimension of the representation and whether the algebra of functions on it is completely reducible.
6.2. Symmetric Supervarieties. Let $\mathfrak{g}$ be quasireductive. Given an involution $\theta$ of $\mathfrak{g}$, we write $\mathfrak{t} = \mathfrak{g}^\theta$ for the fixed points of $\theta$, and call the pair $(\mathfrak{g}, \mathfrak{t})$ a supersymmetric pair. If $G$ is a Lie supergroup and $K$ a subsupergroup with $\text{Lie}(G) = \mathfrak{g}$ and $\text{Lie}(K) = \mathfrak{t}$, we call the coset space $G/K$ a symmetric supervariety.

In the classical world, symmetric varieties for reductive groups are always spherical by the Iwasawa decomposition. We recall how this decomposition works now, generalizing it to the super case. We keep the same notation, letting $\mathfrak{g}$ be quasireductive, $\theta$ an involution of $\mathfrak{g}$ with fixed points $\mathfrak{k}$ and $(-1)$-eigenspace $\mathfrak{p}$. Then let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal toral subalgebra of $\mathfrak{p}$, i.e. a maximal abelian subspace of $\mathfrak{p}_0$ with the property that the elements of $\mathfrak{a}$ are semisimple in $\mathfrak{g}_0$. Then we may decompose $\mathfrak{g}$ into weight spaces under the adjoint action of $\mathfrak{a}$. Write $\Sigma \subseteq \mathfrak{a}^*$ for the set of non-zero weights under this action. Choosing a generic hyperplane we obtain a subset $\Sigma^+ \subseteq \Sigma$ of positive weights, and we define

$$n = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha.$$ 

Write $C(\mathfrak{a})$ for the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Then we have $C(\mathfrak{a}) = C(\mathfrak{a}) \cap \mathfrak{t} \oplus C(\mathfrak{a}) \cap \mathfrak{p}$.

**Proposition 6.1.** The condition $C(\mathfrak{a}) \cap \mathfrak{p} = \mathfrak{a}$ is equivalent to the following decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus n.$$ 

We call such a decomposition an Iwasawa decomposition of the symmetric pair $(\mathfrak{g}, \mathfrak{t})$ (or of the involution $\theta$).

It is a well-known theorem that if $\mathfrak{g} = \mathfrak{g}_\mathfrak{T}$ is reductive then every symmetric pair has an Iwasawa decomposition (see for instance section 26.4 of [Tim11]). However in the super world this no longer remains true. In particular, it is possible for $C(\mathfrak{a}) \cap \mathfrak{p}_\mathfrak{T} \neq 0$. However, we do have the following:

**Theorem 6.2.** If $\mathfrak{g}$ is a basic classical simple Lie superalgebra and $\theta$ is an involution that preserves the invariant bilinear form on $\mathfrak{g}$, then either $\theta$ or $\delta \circ \theta$ has an Iwasawa decomposition, where $\delta \in \text{Aut}(\mathfrak{g})$ is the grading automorphism $\delta(x) = (-1)^{\mathfrak{T}x}$. 

| $V$          | $\dim^*V$ | $SV^*$ Completely Reducible? |
|--------------|-----------|-----------------------------|
| $GL_{m|n}$   | $(m|n)$   | Yes                         |
| $S^2GL_{m|n}$| $(\frac{n(n-1)}{2} + \frac{m(m+1)}{2})mn$ | Yes |
| $\Pi S^2GL_{n|n}$ | $(n^2|n^2)$ | Yes |
| $\Pi S^2GL_{n|n+1}$ | $(n(n+1)|n(n+1))$ | Yes |
| $OSP_{m|2n}$, $m \geq 2$ | $(m|2n)$ | If $m$ is odd or $m > 2n$ |
| $\Pi OSP_{m|2n}$ | $(2n|m)$ | Yes |
| $\Pi P_{n|n}$ | $(n|n)$ | No |
The author has proven this theorem using the framework of generalized root systems as developed by Serganova in [Ser96]. We do not write out the proof here.

The significance of an Iwasawa decomposition for our purposes is that

**Theorem 6.3.** If a symmetric pair \((\mathfrak{g}, \mathfrak{t})\) admits an Iwasawa decomposition, then there exists a hyperborel \(\mathfrak{b}\) of \(\mathfrak{g}\) such that \(\mathfrak{b} + \mathfrak{t} = \mathfrak{g}\). In particular, a symmetric supervariety \(G/K\) constructed from this symmetric pair is spherical.

**Proof.** Write \(\mathfrak{g} = \mathfrak{t} + \mathfrak{a} + \mathfrak{n}\) for the Iwasawa decomposition. Write \(\Sigma^+ \subseteq \mathfrak{a}^*\) for the positive weights defining \(\mathfrak{n}\). Let \(\mathfrak{h}_0 \subseteq \mathfrak{g}_0\) be a Cartan subalgebra containing \(\mathfrak{a}\). Write \(\Delta \subseteq \mathfrak{h}^*\) for the roots of \(\mathfrak{g}\) with respect to \(\mathfrak{h}\). Then we have a natural projection map \(\mathfrak{h}^* \rightarrow \mathfrak{a}^*\) inducing a map \(\Delta \rightarrow \Sigma \cup \{0\}\). Choose a generic hyperplane in \(\Delta\) so that the image of \(\Delta^+\) under this projection lands in \(\Sigma^+ \cup \{0\}\). Then consider

\[
\mathfrak{b}' = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha
\]

Then \(\mathfrak{b}'\) satisfies all the properties of a hyperborel apart possibly from maximality, and thus is contained in a hyperborel subsuperalgebra \(\mathfrak{b}\) of \(\mathfrak{g}\). Further, \(\mathfrak{a} + \mathfrak{n} \subseteq \mathfrak{b}' \subseteq \mathfrak{b}\), and therefore \(\mathfrak{t} + \mathfrak{b} = \mathfrak{g}\) completing the proof. \(\square\)

Below we list all supersymmetric pairs (up to conjugacy) for the algebras \(\mathfrak{gl}(m|n)\), \(\mathfrak{osp}(m|2n)\), \(\mathfrak{p}(n)\), and the simple basic exceptional algebras. For each we state whether or not the pair is spherical as well as whether it admits an Iwasawa decomposition. Note that we only consider involutions of \(\mathfrak{gl}(m|n)\) that fix the center.

| Symmetric Pair                     | Spherical? | Iwasawa Decomposition? |
|-----------------------------------|------------|-------------------------|
| \((\mathfrak{g}, \mathfrak{g}^\Pi)\) | If \(\mathfrak{g} = \mathfrak{g}^\Pi\) | If \(\mathfrak{g} = \mathfrak{g}^\Pi\) |
| \((\mathfrak{gl}(m|n), \mathfrak{gl}(r|s) \times \mathfrak{gl}(m-r|n-s))\) | \(r \geq m - r\) and \(s \geq n - s\) or \(r \leq m - r\) and \(s \leq n - s\) | Same condition |
| \((\mathfrak{gl}(m|n), \mathfrak{osp}(m|2n))\) | Yes | Yes |
| \((\mathfrak{gl}(n|n), \mathfrak{p}(n))\) | Yes | No |
| \((\mathfrak{gl}(n|n), \mathfrak{q}(n))\) | Yes | Yes |
| \((\mathfrak{osp}(m|2n), \mathfrak{osp}(r|2s) \times \mathfrak{osp}(m-r, 2n-2s))\) | \(r \geq m - r\) and \(s \geq n - s\) or \(r \leq m - r\) and \(s \leq n - s\) | Same condition |
| \((\mathfrak{osp}(2m, 2n), \mathfrak{gl}(m|n))\) | Yes | Yes |
| \((\mathfrak{p}(n), \mathfrak{p}(r) \times \mathfrak{p}(n-r))\) | \(r = 1\) | No |
| \((\mathfrak{p}(n), \mathfrak{gl}(r|n-r))\) | \(n = 2, 3\) | No |
| \((D(1, 2; \alpha), \mathfrak{osp}(2|2) \times \mathfrak{so}(2))\) | Yes | Yes |

6.3. \(G\) as a spherical supervariety. Let \(G\) be a quasireductive supergroup. Then \(G \times G\) acts homogeneously on \(G\) by left and right translation, and this identifies \(G\) as a symmetric supervariety with respect to the involution \(\theta\) of \(G \times G\) which swaps the factors.

Some is already known about the structure of \(\mathbb{C}[G]\) as a representation. For instance, in [Ser11], the structure as a \(G\)-module under left translation was computed and was shown to be a sum of injective modules. In [LSZ12] a filtration of \(\mathbb{C}[GL(m|n)]\) as
a $G \times G$-module was constructed following the ideas of Donkin and Koppinen in the modular case, using the highest weight category structure of representations of $GL(m|n)$. Serganova’s result on the structure of $\mathbb{C}[G]$ under left translation also follows from Green’s work on coalgebras in [Gre76], generalized to the setting of supercoalgebras. We state some further results on $\mathbb{C}[G]$ looking at its structure as a $G \times G$-module that are straightforward extensions of results found in [Gre76], in particular on indecomposable block summands and the socle of $\mathbb{C}[G]$. Then we state a result that describes the Loewy layers of the socle filtration of $\mathbb{C}[G]$ (theorem 6.14) which the author has not found the literature. This description should hold in a rather general setting for coalgebras, and this will be addressed in a forthcoming paper.

**Theorem 6.4.** Let $\mathfrak{g}$ be a quasireductive Lie superalgebra and consider the supersymmetric pair $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$ defined by the involution $\theta$ of $\mathfrak{g} \times \mathfrak{g}$ which swaps the factors. Then this supersymmetric pair admits an Iwasawa decomposition if and only if $\mathfrak{g}$ is Cartan-even.

**Proof.** In this case a maximal toral subalgebra of the (-1)-eigenspace is given by $a = \{(h, -h) : h \in h_{\mathfrak{g}}\}$ where $h_{\mathfrak{g}} \subseteq g_{\mathfrak{g}}$ is a Cartan subalgebra of $g_{\mathfrak{g}}$. Therefore the centralizer of $a$ is just the centralizer of $h_{\mathfrak{g}} \times h_{\mathfrak{g}}$ in $\mathfrak{g} \times \mathfrak{g}$. This is equal to $h_{\mathfrak{g}} \times h_{\mathfrak{g}}$ if and only if $h_{\mathfrak{g}}$ is a Cartan subsuperalgebra of $g$, i.e. $g$ is Cartan-even. □

**Proposition 6.5.** If $G$ is Cartan-even, then the finite-dimensional irreducible representations of $G \times G$ are exactly those of the form $V_1 \boxtimes V_2$ for finite-dimensional irreducible representations $V_1, V_2$ of $G$.

**Proof.** A representation of this form is irreducible because $\text{End}_G(V_i) \cong \mathbb{C}$ for each $i$ and the Jacobson density theorem. Conversely, if $L$ is an irreducible representation of $G \times G$ then after choosing a Borel subsupergroup, it has a highest weight $\lambda_1 + \lambda_2$, where $\lambda_i$ is a weight of $i$th copy of $G$ in the direct product. Thus $L = L_B(\lambda_1) \boxtimes L_B(\lambda_2).$ □

**Definition 6.6.** Let $V$ be a finite-dimensional $G$-module corresponding to the coaction $V \to \mathbb{C}[G] \otimes V$. Define $\epsilon_V : V \otimes V^* \to \mathbb{C}[G]$ to be the canonical $G \times G$-equivariant map corresponding to the coaction. Notice that it is always nonzero if $V$ is nonzero. Equivalently, $\epsilon_V$ may be defined by Frobenius reciprocity; it is the unique element of $\text{Hom}_{G\times G}(V \boxtimes V^*, \mathbb{C}[G])$ that corresponds to the natural pairing $V \otimes V^* \to \mathbb{C}$ under the isomorphism $\text{Hom}_{G\times G}(V \boxtimes V^*, \mathbb{C}[G]) \cong \text{Hom}_G(V \otimes V, \mathbb{C})$

**Remark 6.7.** If $V$ is a finite-dimensional $G$-representation then there is a canonical isomorphism of $G \times G$-modules $V \boxtimes V^* \cong (V) \boxtimes (V)^*$, and this map factors $\epsilon_V$ through $\epsilon_{IV}$. In particular, $\text{Im} \epsilon_V = \text{Im} \epsilon_{IV}$.

For the rest of this section we will assume that $G$ is Cartan-even. Given an irreducible representation $V$ of $G$, the map $\epsilon_V : V \boxtimes V^* \to \mathbb{C}[G]$ is injective by irreducibility and the fact that $\epsilon_V$ is not the zero map. In this way we obtain a natural inclusion

$$\bigoplus_{V} V \boxtimes V^* \subseteq \text{soc}(\mathbb{C}[G]),$$

where the sum runs over all irreducible representations of $G$ up to parity. We now go about showing this is the entire socle.
Let $B'$ be a Borel subsupergroup of $G$ (as defined in [Ser11]) and $(B')^-$ its opposite Borel. Let $B$ be a hyperborel subsupergroup containing $B'$ and $B^-$ a hyperborel subsupergroup containing $(B')^-$. Then $B \times B^-$ is a hyperborel of $G \times G$, and $G$ is is $B \times B^-$-spherical. Further, $(B^-)_0$ is the Borel subgroup of $B_0$ in $G_0$.

**Lemma 6.8.** If $V$ is an irreducible representation of $G$, then $V^{(B)} = V^{(B')}$. 

*Proof.* Indeed $V^{(B)} \subseteq V^{(B')}$ but by remark 4.4 $1 \leq \dim V^{(B)} \leq \dim V^{(B')} = 1$. 

**Definition 6.9.** For a hyperborel subsupergroup $B$ of $G$, we say an integral weight $\lambda$ is $B$-dominant if there exists an irreducible representation $V$ of $G$ such that $\Lambda_B(V) = \{\lambda\}$.

Recall that (for instance by the Peter-Weyl theorem),

$$\Lambda^+_{B_0 \times (B^-)_0}(G_0) = \{(\lambda, -\lambda) : \lambda \text{ is a } B_0\text{-dominant weight}\}.$$

**Lemma 6.10.** We have

$$\Lambda^+_{B \times B^-}(G) = \{(\lambda, -\lambda) : \lambda \text{ is a } B\text{-dominant weight}\}.$$

*Proof.* By the inclusion $\Lambda^+_{B \times B^-}(G) \subseteq \Lambda^+_{B_0 \times (B^-)_0}(G_0)$ we know that $\Lambda^+_{B \times B^-}(G)$ must be contained in the RHS. However our socle computation above shows that $L(\lambda) \boxtimes L(\lambda)^* \subseteq \mathbb{C}[G]$ for all $B$-dominant weights $\lambda$, and this is exactly the $G \times G$ irreducible representation of highest weight $(\lambda, -\lambda)$. 

**Corollary 6.11.** $\text{soc}(\mathbb{C}[G]) \cong \bigoplus_V V \boxtimes V^*$, where the sum runs over all irreducible representations of $G$ up to parity.

We explain further the structure of $\mathbb{C}[G]$. Let $\text{Rep}(G)$ denote the category of finite-dimensional representations of $G$. Then we may decompose $\text{Rep}(G)$ into a sum of simple blocks, where a block $B$ is an abelian subcategory of $\text{Rep}(G)$ such that if $B'$ is another block distinct from $B$, then $\text{Ext}^i(V, W) = \text{Ext}^i(W, V) = 0$ for all $i$ and all objects $V$ of $B$ and $W$ of $B'$. A block $B$ is simple if it cannot be decomposed into a sum of smaller, nontrivial blocks. Notice that every block must contain an irreducible representation.

Given a block $B$ of $G$, we denote by $\Pi B$ the block consisting of all $G$-modules $\Pi V$ where $V$ is in $B$. If we write $\text{Bl}_G$ for the set of blocks of $G$, we want to consider the set $\text{Bl}_G/\sim$ where $\sim$ is the equivalence relation on blocks generated by $B \sim \Pi B$ for all blocks $B$. For $B \in \text{Bl}_G/\sim$, we write $\text{Irr}(B)$ for the set of irreducible representations that appear in $B$ up to parity. The following is an analogue of theorem (1.5g) part (ii) and theorem (1.6a) in [Gre76].

**Proposition 6.12.** We have as a $G \times G$-module

$$\mathbb{C}[G] = \bigoplus_{B \in \text{Bl}_G/\sim} M_B$$

where $M_B$ is an indecomposable $G \times G$-module given by

$$M_B = \sum_{V \in B} \text{Im } \epsilon_V.$$

Further,

$$\text{soc}(M_B) = \bigoplus_{V \in \text{Irr} B} V \boxtimes V^*.$$
Remark 6.13. It follows that the module $M_B$ is finite-dimensional if and only if $\text{Irr} \, B$ is finite. This example shows another phenomenon that may occur in the super case: given a spherical $G$-supervariety $X$, $\mathbb{C}[X]$ need not be a direct sum of finite-dimensional $G$-modules.

We can say more about the socle filtration of $M_B$, and thus of $\mathbb{C}[G]$. Recall that for a finite-dimensional $G$-module $V$, the Loewy length of $V$, which we write as $\ell(V)$, is defined to be the length of a minimal semisimple filtration of $V$ (or equivalently the length of the socle or radical filtration of $V$). The first of the following results is an analogue of what was essentially known in [Gre76] for coalgebras. The author has not found the second result in the literature and will give a proof in a future paper.

Theorem 6.14. For each block $B \in \text{Bl}_G / \sim$ we have:

- $\text{soc}^k M_B = \sum_{V \in B, \, \ell(V) \leq k} \text{Im} \, \epsilon_V$

- For simple $G$-modules $L, L'$ which lie in a block of the equivalence class $B$, we have

$$[\text{soc}^k M_B / \text{soc}^{k-1} M_B : L' \otimes L^*] = [L' : \text{soc}^k I(L) / \text{soc}^{k-1} I(L)] = \dim \text{Hom}_G(P(L'), \text{soc}^k I(L) / \text{soc}^{k-1} I(L))$$

6.4. The case $G = GL(1|1)$. Let $G = GL(1|1)$, and $\mathfrak{g} = \text{Lie} \, G$. We give a very explicit description of the $\mathfrak{g} \times \mathfrak{g}$ action on $\mathbb{C}[G]$. In this case, there is only one block of $\text{Rep}(G)$ which is not semisimple, the principal block $B_0$, and it contains the irreducible representations where the center of $\mathfrak{gl}(1|1)$ acts trivially. We draw a picture depicting the local structure of $M_{B_0}$ below.

![Picture of weight vectors for GL(1|1)]

Note that $M_{B_0}$ is infinite-dimensional since there are infinitely many simple modules in $B_0$. Each dot in the picture represents a weight vector, with the bottom and top rows having even parity and the middle row having odd parity. We write $u, v$ for the action of the odd weight vectors of $\mathfrak{gl}(1|1)$ by left translation, and $\overline{u}, \overline{v}$ for the action of the odd weight vectors by right translation. One can see rather explicitly here that under left or right translation only this is just a sum of injective modules.

6.5. Split supergroups and actions. The following discussion follows closely the definitions and theorems of section 4 of [Vis11], except that we are working in the algebraic category and not the complex analytic category.

Introduce the category $SSV$ whose objects are supervarieties of the form $X = ([X_0], \Lambda^* \mathcal{N})$ where $X_0$ is a variety and $\mathcal{N}$ is a coherent sheaf on $X_0$. In other words the objects are split supervarieties with a given choice of splitting. This endows all objects of $SSV$
with a canonical \( \mathbb{Z} \)-grading on their structure sheaf. We then define morphisms in this category to be those morphisms of supervarieties that preserve the given \( \mathbb{Z} \)-gradings.

There is a natural functor \( \text{gr} \) from the category of supervarieties to \( \text{SSV} \). On objects it is given by

\[
\text{gr} \, X = (|X|, \bigoplus_{i \geq 0} J^i_X/J^{i+1}_X),
\]

so that in particular \(|X| = |\text{gr} \, X|\) and \(X(\mathbb{C}) = |\text{gr} \, X(\mathbb{C})|\). Note that the natural map \(\Lambda^i J^i_X/J^{i+1}_X \rightarrow J^i_X/J^{i+1}_X\) is an isomorphism because of our assumption that supervarieties are locally split. For a morphism \(\psi : X \rightarrow Y\) we let \(\psi : \text{gr} \, X \rightarrow \text{gr} \, Y\) be the same map of underlying topological spaces and set

\[
(\text{gr} \, \psi)^* : \bigoplus_{i \geq 0} J^i_Y/J^{i+1}_Y \rightarrow (\text{gr} \, \psi)^* \bigoplus_{i \geq 0} J^i_X/J^{i+1}_X
\]

to be

\[
(\text{gr} \, \psi)^* (f + J^i_Y) = \psi^*(f) + J^i_X
\]

where \(f \in J^i_Y\).

If \(X\) and \(Y\) are supervarieties, then \(X \times Y\) is a supervariety in a natural way, and \(J^i_{X \times Y} = p_X^* J^i_X + p_Y^* J^i_Y\), where \(p_X, p_Y\) are the natural projection maps. On the other hand, given two split supervarieties \(X' = (|X'|, \Lambda^i N_{X'}), Y' = (|Y'|, \Lambda^i N_{Y'})\), we define their direct product in \(\text{SSV}\) to be the direct product of supervarieties \(X' \times Y'\) with the natural splitting \(O_{X' \times Y'} = \Lambda^i (p^*_{X'}, N_{X'} + p^*_{Y'}, N_{Y'})\). Then there is a canonical isomorphism in \(\text{SSV}\) \(\text{gr}(X \times Y) \cong \text{gr} \, X \times \text{gr} \, Y\) coming from the fact that taking tensor product commutes with taking associated graded for filtered vector spaces with finite filtrations.

If \(G\) is a Lie supergroup, then using the canonical isomorphism \(\text{gr}(G \times G) \cong \text{gr} \, G \times \text{gr} \, G\) we have that \(\text{gr} \, G\) with the maps \(\text{gr} \, m_g, \text{gr} \, e_g\), and \(\text{gr} \, s_g\) forms a Lie supergroup. If \(g = \text{Lie} \, G\) we write \(g^\text{gr} := \text{Lie} \, \text{gr} \, G\). Further, if \(a : G \times X \rightarrow X\) is an action of a Lie supergroup on a supervariety \(X\), then \(\text{gr} \, a : \text{gr}(G \times X) \cong \text{gr} \, G \times \text{gr} \, X \rightarrow \text{gr} \, X\) defines an action of \(\text{gr} \, G\) on \(\text{gr} \, X\).

**Definition 6.15.** If \(G\) is a supergroup, we call \(\text{gr} \, G\) the split supergroup gotten from \(G\), and we say \(G\) is a split supergroup if \(G \cong \text{gr} \, G\) as supergroups. If \(a : G \times X \rightarrow X\) is an action of \(G\) on \(X\), we call \(\text{gr} \, a\) the split action of \(\text{gr} \, G\) on \(\text{gr} \, X\), and we say that \(a\) is a split action if it is isomorphic to \(\text{gr} \, a\) in the natural sense.

We give an explicit construction of \(\text{gr} \, G\). Being affine the supergroup \(G\) is split, so fix a splitting of \(G\) so that its structure sheaf is equipped with a \(\mathbb{Z}\)-grading. We call \(G\) with this chosen splitting \(\text{gr} \, G\), and we think of it as an object of \(\text{SSV}\). This choice of splitting determines a canonical splitting of \(G \times G\), and thus we may write

\[
(m_G)^* = \bigoplus_{k \geq 0} (m_G)^* k, \quad (s_G)^* = \bigoplus_{k \geq 0} (s_G)^* k
\]

where \((m_G)^*_i\), respectively \((s_G)^*_i\), increase the \(\mathbb{Z}\)-grading of an element by exactly \(i\). We set \(m^*_G = (m_G)^*_0\), \(s^*_G = (s_G)^*_0\), and \(e^*_G = e_G^*\), and these are all algebra homomorphisms. In this way, the induced maps on the supervariety \(G\) given by \(m_{\text{gr} \, G}, s_{\text{gr} \, G}, e_{\text{gr} \, G}\) become morphisms in \(\text{SSV}\) and define the structure of a supergroup on \(\text{gr} \, G\), and thus this supergroup is split. It follows in particular that we may identify \((\text{gr} \, G)_0\) and \(G_0\) as algebraic groups.
Now since we have constructed $\text{gr} G$ so that it is the same supervariety as $G$ (the only difference being that it has a chosen $\mathbb{Z}$-grading on its structure sheaf), we have an identification $T_e G = T_e \text{gr} G$. Thus we may canonically identify $\mathfrak{g} \cong \mathfrak{g}^{gr}$ as super vector spaces. Given $u_e \in T_e G$, we write $u_L$ (resp. $u_R$) for the corresponding $G$ right-invariant (resp. $G$ left-invariant) vector field on $G$, and $\text{gr} u_L$ (resp. $\text{gr} u_R$) for the corresponding $\mathfrak{g}$ right-invariant (resp. $\mathfrak{g}$ left-invariant) vector field on $G$. Using the $\mathbb{Z}$-grading on $\mathbb{C}[G]$ we may write $u_L = \sum_{i \in \mathbb{Z}} (u_L)_i$ (resp. $u_R = \sum_{i \in \mathbb{Z}} (u_R)_i$), where $(u_L)_i$ (resp. $(u_R)_i$) changes the $\mathbb{Z}$-grading by $i$.

**Lemma 6.16.** If $u_e$ is even then $\text{gr} u_L = (u_L)_0$ and $\text{gr} u_R = (u_R)_0$, and if $u_e$ is odd then $\text{gr} u_L = (u_L)_{-1}$ and $\text{gr} u_R = (u_R)_{-1}$.

**Proof.** We prove this for right-invariant vectors, with the case of left-invariant vector fields being similar. have

$$u_L = -(u_e \otimes 1) \circ (m_G^e) = \bigoplus_{i \geq 0} (-u_e \otimes 1) \circ (m_G^e)_i.$$ 

For $f \in \mathbb{C}[G]_k$, $(m_G^e)_i(f) \in \bigoplus_j \mathbb{C}[G]_j \otimes \mathbb{C}[G]_{k+i-j}$. If $u_e$ is even, then $u_e$ vanishes on $\mathbb{C}[G]_i$ for $i > 0$, so

$$-(u_e \otimes 1) \circ (m_G^e)_i = (u_L)_i,$$

so $\text{gr} u_L = (u_L)_0$. If $u_e$ is odd, then $u_e$ vanishes on $\mathbb{C}[G]_i$ for $i \neq 1$, so

$$-(u_e \otimes 1) \circ (m_G^e)_i = (u_L)_{i-1},$$

so $\text{gr} u_L = (u_L)_{-1}$. \hfill $\square$

**Corollary 6.17.** We have $[\mathfrak{g}_T^{gr}, \mathfrak{g}_T^{gr}] = 0$. In fact a supergroup $G$ is split if and only if $[\mathfrak{g}_T, \mathfrak{g}_T] = 0$, where $\mathfrak{g} = \text{Lie} G$.

**Proof.** For the first statement, the supercommutator of two degree (-1)-maps is of degree (-2) with respect to the $\mathbb{Z}$-grading. However there are no vector fields of degree (-2) on a split supervariety, thus the supercommutator must be zero. A proof of the second statement is given in proposition 4.4 of [Vis11]. \hfill $\square$

Now $G_0 \times G_0$ acts on $G$ by left and right translation. Using Koszul’s realization of $\mathbb{C}[G]$ as a coinduced algebra on $\mathbb{C}[G_0]$ (see [Kos82]), which gives a natural splitting of $G$, we obtain a natural $G_0 \times G_0$-equivariant splitting (this does not require that $G_0$ is reductive; if $G_0$ is reductive we could also use proposition 4.13 to find a $G_0 \times G_0$-equivariant splitting). Thus if we constructed $\text{gr} G$ as above using the $G_0 \times G_0$-equivariant splitting we would have that if $u_e$ is even, $u_L = (u_L)_0$ and $u_R = (u_R)_0$ since they will preserve the $\mathbb{Z}$-grading. Thus we have shown:

**Lemma 6.18.** If we construct $\text{gr} G$ by using a $G_0 \times G_0$-equivariant splitting of $G$, then for an even tangent vector $u_e \in T_e G$, $u_L = \text{gr} u_L$ and $u_R = \text{gr} u_R$. In particular $\mathfrak{g}_T = \mathfrak{g}_T^{gr}$ as Lie algebras of vector fields on $G$. Further, the natural isomorphism of super vector spaces $\mathfrak{g}_T \cong \mathfrak{g}_T^{gr}$ induced from this splitting is an isomorphism of $\mathfrak{g}_T$-modules.

**Proof.** It remains to show the second statement. For this, we observe that for $u \in \mathfrak{g}_T$, $v \in \mathfrak{g}_T$, $[u, v]_i = [u_i, v_i]$. Since $\text{gr} v = v_{-1}$, the statement follows. \hfill $\square$
We now move on to the study of split actions.

Lemma 6.19. Suppose $G$ is a supergroup which acts on a supervariety $X$, and consider the action of $\text{gr} \, G$ on $\text{gr} \, X$. Then for $u \in \mathfrak{g}_T^{gr}$, $u$ preserves the $\mathbb{Z}$-grading on $O_{gr} \, X$, and for $u \in \mathfrak{g}_T^{gr}$, $u$ acts by degree $-1$ on $O_{gr} \, X$.

Proof. For $f \in (O_{gr} \, X)_i$, we have
$$u(f) = -(u_e \otimes 1) \circ (\text{gr} \, a)^*(f).$$

Now since $\text{gr} \, a$ preserves the $\mathbb{Z}$-grading, we have $(\text{gr} \, a)^*(f) \in \bigoplus_{0 \leq j \leq i} (O_{gr} \, G)_j \otimes (O_{gr} \, X)_{i-j}$.

If $u \in \mathfrak{g}_T^{gr}$, then $u_e$ vanishes on $(O_G)_i$ for $i > 0$, and if $u \in \mathfrak{g}_T^{gr}$ then $u_e$ vanishes on $(O_G)_i$ for $i \neq 1$. The result follows.

Now if $K$ is a closed subgroup of $G$ via the inclusion $\phi : K \to G$, then the $\mathbb{Z}$-gradings induced on $\mathbb{C}[G]$ and $\mathbb{C}[K]$ from Koszul’s realization make the natural pullback surjection $\phi^* : \mathbb{C}[G] \to \mathbb{C}[K]$ into a graded map. Thus the kernel of this map, $I_K \subseteq \mathbb{C}[G]$, becomes a graded ideal. Further, if we consider the split supergroups structure on $K$ and $G$ from these gradings, $\phi$ will be a homomorphism of supergroups $\text{gr} \, K \to \text{gr} \, G$. Thus $\text{gr} \, \phi = \phi$, and so $I_K = I_{gr} \, K$.

Lemma 6.20. If $X$ is a supervariety and $x \in X(\mathbb{C})$, $\text{Stab}_{\text{gr} \, G}(x) = \text{Stab}_G(x)$ as closed subvarieties of $G$.

Proof. Write $K = \text{Stab}_G(x)$, $\mathfrak{m}_x$ for the maximal ideal sheaf of $x \in X(\mathbb{C})$ and $\mathfrak{m}_x^{gr}$ for the maximal ideal sheaf of $x \in \text{gr} \, X(\mathbb{C})$. Then by assumption we have $(a_x)^*(\mathfrak{m}_x) = I_K$.

But with respect to the $\mathbb{Z}$-grading from Koszul’s realization, $I_K$ is a graded ideal and thus $(\text{gr} \, a_x)^*(\mathfrak{m}_x^{gr}) = I_K = I_{gr} \, K$, and we are done.

Corollary 6.21. If $X$ is a homogeneous $G$-supervariety isomorphic to $G/K$, then $\text{gr} \, X$ is a homogeneous $\text{gr} \, G$-variety isomorphic to $\text{gr} \, G/\text{gr} \, K$.

6.6. $G$ a quasireductive split supergroup. Let $G$ be a quasireductive supergroup, and write $\mathfrak{g} = \text{Lie} \, G$ as always.

Lemma 6.22. If $\mathfrak{l} \subseteq \mathfrak{g}_T$ is an abelian ideal of $\mathfrak{g}$, then $\mathfrak{l}$ is contained in every hyperborel subsuperalgebra of $\mathfrak{g}$.

Proof. If $\mathfrak{b}$ is a hyperborel subsuperalgebra, then $\mathfrak{b} + \mathfrak{l}$ is a subsuperalgebra that still satisfies the first two properties of being a hyperborel, and thus by maximality $\mathfrak{b} = \mathfrak{b} + \mathfrak{l}$.

Corollary 6.23. Let $G$ be a split quasireductive group. Then every hyperborel of $\mathfrak{g}$ is of the form $\mathfrak{b}_T \oplus \mathfrak{g}_T$, where $\mathfrak{b}_T$ is a Borel subalgebra of $\mathfrak{g}_T$. In particular $G$ has only one hyperborel up to conjugacy.

Proof. In this case $\mathfrak{g}_T$ is an abelian ideal of $\mathfrak{g}$, so we use lemma 6.22 to get that every hyperborel must contain $\mathfrak{g}_T$, and thus they are all of this form. If $\mathfrak{b}, \mathfrak{b}'$ are two hyperborels, then conjugating $\mathfrak{b}_T$ to $\mathfrak{b}'_T$ will conjugate $\mathfrak{b}$ to $\mathfrak{b}'$.

The following lemma now follows easily from what we have shown so far.

Lemma 6.24. If $G$ is a quasireductive supergroup, and $B$ is a hyperborel subsupergroup of $G$, then $\text{gr} \, G$ is quasireductive and $\text{gr} \, B$ is a subsupergroup of a hyperborel of $\text{gr} \, G$. 


We now prove that the functor $\text{gr}$ preserves sphericity.

**Corollary 6.25.** Suppose that $G$ is quasireductive and $X$ is a spherical $G$-supervariety. Then $\text{gr} \ X$ is a spherical $\text{gr} \ G$-supervariety under the split action.

*Proof.* Let $B$ be a hyperborel of $G$ with an open orbit on $X$. Then by corollary 6.21, $\text{gr} \ B$ has an open orbit on the same underlying open subset of $|X|$. By lemma 6.24, $\text{gr} \ B$ is contained in a hyperborel of $\text{gr} \ G$, and the hyperborel of $\text{gr} \ G$ containing $\text{gr} \ B$ has an open orbit at $x$. Thus $\text{gr} \ X$ is spherical. \qed

For the rest of this section we assume that $G$ is a split quasireductive supergroup.

**Proposition 6.26.** Suppose that $X$ is a spherical $G$-supervariety. Then $\text{soc} \ C[X]$ is a subsuperalgebra of $C[X]$. Further the restriction of $i_X$ to $\text{soc} \ C[X]$ is injective. In particular, $\text{soc} \ C[X]$ is an even subalgera of $C[X]$ without nilpotents.

*Proof.* A semisimple representation of $G$ is exactly the pullback of a semisimple representation of $G_0$ under the natural surjection $G \rightarrow G_0$. Therefore $\text{soc} \ C[X]$ can be thought of as a sum of simple $G_0$-representations, and thus the tensor product of two subrepresentations of $\text{soc} \ C[X]$ is again a semisimple $G_0$-representation. Since multiplication is $G$-equivariant, it follows that $\text{soc} \ C[X]$ is a subsuperalgebra of $C[X]$.

Recall that $i_X$ is a $G_0$-equivariant map of algebras. If $\text{soc} C[X] \cap \ker i_X \neq 0$, then it must contain a simple subrepresentation $V$. Let $f \in V$ be the $B$-highest weight vector for some hyperborel $B$ of $G$. Then by proposition 5.3, $f$ is non-nilpotent and thus $i_X(f) \neq 0$, a contradiction. This completes the proof. \qed

**Corollary 6.27.** If $X$ is an affine spherical $G$-supervariety, then $C[X]$ is completely reducible if and only if $X = X_0$.

*Proof.* If $X = X_0$ then $G$ acts via the quotient to $G_0$ so $X_0$ is a spherical variety in the classical sense, and thus $C[X]$ is completely reducible.

On the other hand, the condition that $C[X]$ is completely reducible is equivalent to $C[X] = \text{soc} \ C[X]$. By proposition 6.26, this condition implies that $i_X$ is an isomorphism, so $X = X_0$. \qed

We now focus on the case of homogeneous spherical supervarieties for $G$.

**Lemma 6.28.** If $X$ is a homogeneous $G$-supervariety, then $X$ is split, and the action $a : G \times X \rightarrow X$ is isomorphic to the split action $\text{gr} \ a$.

*Proof.* This follows directly from corollary 6.21. \qed

**Proposition 6.29.** If $X$ is a homogeneous $G$-supervariety, then $X$ is spherical if and only if $X_0$ is a spherical $G_0$-variety.

*Proof.* If $X = G/K$, then we want to determine when $\mathfrak{h} = \text{Lie} \ K$ has a complimentary hyperborel in $G$. By corollary 6.23, the hyperborels of $\mathfrak{g} = \text{Lie} \ G$ are all of the form $\mathfrak{b}_0 \oplus \mathfrak{b}_\tau$ for a Borel subalgebra $\mathfrak{b}_\tau$ of $\mathfrak{g}_\tau$. Thus it is equivalent to find a Borel subalgebra $\mathfrak{b}_\tau$ complimentary to $\mathfrak{h}_\tau$ in $\mathfrak{g}_\tau$. Since $X_0 = G_0/K_0$, this completes the proof. \qed

**Proposition 6.30.** If $X$ is a homogeneous spherical $G$-supervariety, then there exists a splitting of $X$ for which $C[X]_0 = \text{soc} \ C[X]$. In particular, if $B$ is a hyperborel of $G$, then $\Lambda^+_B(X) = \Lambda^+_{\mathfrak{b}_0}(X_0)$. 23
Proof. By lemma 6.28, there exists a splitting of $X$ for which the action of $G$ is split. With respect to this action, $\mathfrak{g}^\tau$ acts by degree $-1$ derivations on $O_X$. Thus $\mathbb{C}[X]_0 \subseteq \mathbb{C}[X]^{\mathfrak{g}^\tau} = \text{soc} \mathbb{C}[X]$. On the other hand, by proposition 6.26, $i_X : \text{soc} \mathbb{C}[X] \rightarrow \mathbb{C}[X_0]$ is injective. Since $i_X : \mathbb{C}[X]_0 \rightarrow \mathbb{C}[X_0]$ is an isomorphism we must have $\mathbb{C}[X]_0 = \text{soc} \mathbb{C}[X]$. \hfill $\square$

In the case of homogeneous affine spaces, we have the following strengthening of corollary 6.27. Note that a homogeneous space $G/K$ is affine if and only if $K_0$ is reductive, i.e. $K$ is quasireductive.

**Proposition 6.31.** If $X = G/K$ is a homogeneous affine $G$-space, then the following are equivalent.

1. $X = X_0$.
2. $\mathbb{C}[X]$ is completely reducible.
3. $\mathbb{C}$ splits of from $\mathbb{C}[X]$ as a $G$-module.

Before proving this, we first state a lemma.

**Lemma 6.32.** Suppose that $G$ is quasireductive and that $\mathfrak{g} = \text{Lie}(G)$ has an odd abelian ideal $\mathfrak{l} \subseteq \mathfrak{g}^\tau$. Then if $K \subseteq G$ is a quasireductive subsupergroup, $\mathbb{C}$ splits off from $\mathbb{C}[G/K]$ only if $\mathfrak{l} \subseteq \mathfrak{k} = \text{Lie}(K)$.

**Proof.** Suppose that $\mathfrak{l}$ is not contained in $\mathfrak{k}$. Let $\mathfrak{m} = \mathfrak{k} \cap \mathfrak{l}$, and let $\mathfrak{n}$ be a $\mathfrak{g}^\tau$-invariant complement to $\mathfrak{m}$ in $\mathfrak{l}$, where we are using that $K_0$ is reductive. Write $L,M,$ and $N$ for the purely even vector spaces with $L^\tau = L,M^\tau = \mathfrak{m}^\tau,$ and $N^\tau = \mathfrak{n}^\tau$. We may naturally view $L$ as a $\mathfrak{g}^\tau$-module according to the restriction of the adjoint action of $\mathfrak{g}^\tau$ to $\mathfrak{l}$, using that $\mathfrak{l}$ is an ideal of $\mathfrak{g}$.

Now consider the following $\mathfrak{g}$-module $V$. As a $\mathfrak{g}^\tau$-module, $V = L \otimes L^* \oplus PL^*$. Choose a $\mathfrak{g}^\tau$-invariant complement $\mathfrak{l}'$ to $\mathfrak{l}$ in $\mathfrak{g}^\tau$. Then we say that for $u \in \mathfrak{l}'$, $u$ acts by $0$ on $V$, and for $u \in \mathfrak{l}$, $u$ acts by $0$ on $V_0 = L \otimes L^*$, while for $\varphi \in V_\mathfrak{l}' = PL^*$, we set $u \cdot \varphi := u \otimes \varphi \in V_\mathfrak{l}'$. Then this defines a representation of $\mathfrak{g}$ on $V$. Further, the span of the element $v_L \in V_\mathfrak{l}' = L \otimes L^*$ which correspond to the identity map on $L$ defines an even trivial subrepresentation $\mathbb{C} \langle v_L \rangle$ of $V$. This subrepresentation does not split off of $V$, as we see that if $u_1, \ldots, u_n$ is a basis of $L$ and $\varphi_1, \ldots, \varphi_n$ is a dual basis of $L^*$, then we have the following equation in $V$:

$$\sum_{i=1}^n u_i \cdot \varphi_i = \sum_{i=1}^n u_i \otimes \varphi_i = v_L$$

Consider the element $\psi \in V^*$ corresponding to the trace form on $N \otimes N^* \subseteq L \otimes L^*$. Then as an element of $V^*$, $\psi$ is $\mathfrak{g}^\tau$-invariant since $N$ is a $\mathfrak{g}^\tau$-submodule. If $u \in \mathfrak{g}^\tau$ and $\varphi \in V_\mathfrak{g}^\tau$, then $u \cdot \varphi = u \otimes \varphi \in M \otimes L^*$, and thus $\psi(u \otimes \varphi) = 0$. It follows that $\psi \in (V^*)^\tau$, i.e. it defines an even coinvariant of $V$, so by Frobenius reciprocity it defines a $G$-module morphism $\Psi : V \rightarrow \mathbb{C}[G/K]$. Further, since $\psi(v_L) \neq 0$ and $v_L$ is $G$-fixed, $\Psi(v_L)$ is a non-zero constant function on $G/K$. We see that

$$\sum_{i=1}^n u_i \cdot \Psi(\varphi_i) = \Psi \left( \sum_{i=1}^n u_i \cdot \varphi_i \right) = \Psi(v_L).$$

It follows that $\mathbb{C}$ does not split off from $\mathbb{C}[G/K]$, and we are done. \hfill $\square$
Now we prove proposition 6.31.

*Proof.* Since \( g \) if an odd abelian ideal of \( g \), if \( K \subseteq G \) is a quasireductive subsupergroup, \( \mathbb{C} \) splits off from \( \mathbb{C}[G/K] \) only if \( g_\mathfrak{m} \subseteq \mathfrak{k} \) by lemma 6.32, and in this case \( G/K \) is a purely even variety. This shows \((3) \implies (1)\). Both \((1) \implies (2)\) and \((2) \implies (3)\) are obvious. \( \square \)

### 7. Appendix: Action of Lie Superalgebras

Here we define the notion of an action of a Lie superalgebra on a supervariety, so that we may slightly extend our results on spherical supervarieties.

**Definition 7.1.** Let \( g \) be a Lie superalgebra and let \( X \) be a complex supervariety. We say that \( g \) acts on \( X \) if there is homomorphism of Lie superalgebras \( \rho : g \to \Gamma(X, T_X) \).

**Remark 7.2.**
- If a supergroup \( G \) acts on a supervariety \( X \), then the map \( \rho_g \) defined after definition 3.4 defines an action of \( \text{Lie } G \) on \( X \).
- If \( g \) acts on a supervariety \( X \), then it naturally acts on any open subsupervariety of \( X \) by restriction of vector fields.

**Definition 7.3.** If \( g \) acts on \( X \), then we say \( g \) has an open orbit on \( X \) if there exists a point \( x \in X(\mathbb{C}) \) such that the natural restriction map \( g \to T_x X \) is a surjection. In this case, the locus of points where \( g \to T_x X \) is surjective is open, and we call this open set an open orbit of \( g \). We say \( X \) is a homogeneous \( g \)-space if all of \( X \) is an open orbit.

An open orbit of \( g \) will be smooth by proposition 8.2. Also observe that an open subsupervariety of a homogeneous supervariety is still homogeneous for the natural restricted action.

**Proposition 7.4.** If \( g \) acts on \( X \) and is homogeneous for this action, then \( \mathbb{C}[X] \) has no \( g \)-invariant ideals.

*Proof.* This is proven in the same way as the first part of the proof of of proposition 3.12. \( \square \)

Now assume that \( g \) is quasireductive.

**Definition 7.5.** A supervariety \( X \) with a \( g \)-action is said to be spherical if there exists a hyperborel subsuperalgebra \( b \) in \( g \) such that \( b \) has an open orbit on \( X \). In this case we say that \( X \) is \( b \)-spherical.

**Remark 7.6.** If \( G \) is quasireductive and acts on a variety \( X \), and \( B \) is a hyperborel subsupergroup of \( G \), then \( X \) is \( B \)-spherical if and only if \( X \) is \( b \)-spherical for the induced action of \( g \) on \( X \).

**Theorem 7.7.** Let \( X \) be a \( g \)-supervariety, \( b \) a hyperborel of \( g \) and \( h_\mathfrak{m} \subseteq b \) a Cartan subalgebra of \( g_\mathfrak{m} \). Then \( X \) is \( b \)-spherical only if \( \mathbb{C}[X](b) \) is a multiplicity-free \( h_\mathfrak{m} \)-module and consists only of non-nilpotent functions.

*Proof.* Suppose that \( f \in \mathbb{C}[X] \) is a non-zero weight vector of \( b \). Then \((f)\) is a \( b \)-invariant ideal. If we localize \((f)\) to the open orbit \( V \) of \( b \), by proposition 7.4 we must get all of \( \mathbb{C}[V] \) since \((f)\) cannot become the zero ideal (since restriction of functions is injective by assumption). This implies the restriction of \( f \) to \( V \) is a unit, and thus \( f \) is non-nilpotent.
Now if \( f_1, f_2 \) are non-zero weight vectors for \( \mathfrak{b} \) of the same weight, then \( g = f_1/f_2 \) is a rational \( \mathfrak{b} \)-invariant function. Since \( f_2 \) is a unit on \( V \), \( g \) is regular on \( V \). Then since \( V \) is \( \mathfrak{b} \)-homogeneous, \( \mathbb{C}[V] \) has no nontrivial \( \mathfrak{b} \)-invariant ideals. However for \( x \in V \), \( (g - g(x)) \) will be an invariant ideal which is not equal to \( \mathbb{C}[V] \) since it is contained in \( \mathfrak{m}_x \). Therefore \( g - g(x) = 0 \), so \( g \) is constant, and thus \( f_1 \) and \( f_2 \) are proportional. This completes the proof. \( \square \)

8. Appendix: Smoothness

Let \( X \) be a complex supervariety and let \( x \in X(\mathbb{C}) \). We say that \( X \) is smooth at \( x \) if the natural evaluation map \( T_{X,x} \to T_x X \) is surjective (see remark 2.5). We seek to give a list of conditions that are equivalent to this, so as to clarify the existing literature on smoothness of superschemes. For a supervariety \( X \), write \( \Omega_X \) for its sheaf of differentials, which can be defined as the conormal sheaf to \( X \) under the diagonal embedding \( X \to X \times X \).

First, we need a lemma.

**Lemma 8.1.** Let \( K \) be a finitely generated field over \( \mathbb{C} \) of transcendence degree \( m \), and let \( K = K[\xi_1,\ldots,\xi_n] \) for odd variables \( \xi_1,\ldots,\xi_n \). Then \( \Omega_{K/\mathbb{C}} \) is a free \( K \)-module of rank \( (m|n) \).

**Proof.** We have the short exact sequence

\[
K \otimes_K \Omega_{K/\mathbb{C}} \to \Omega_{K/\mathbb{C}} \to \Omega_{K/K} \to 0
\]

Since \( \Omega_{K/K} \) is a free \( K \)-module of rank \((0|n)\) with generators \( d\xi_1,\ldots,d\xi_n \), the last map splits which implies that \( d\xi_1,\ldots,d\xi_n \) generate a free summand of \( \Omega_{K/\mathbb{C}} \) of rank \((0|n)\). We know that \( \Omega_{K/K} \) is a free \( K \)-module of rank \((m|0)\) with generators \( dt_1,\ldots,dt_m \), where \( t_1,\ldots,t_m \) form a transcendence basis of \( K \) over \( \mathbb{C} \). Hence \( \Omega_{K/K} \) is generated by \( dt_1,\ldots,dt_m, \) \( d\xi_1,\ldots, d\xi_n \), and it suffices to show that \( dt_1,\ldots,dt_m \) are \( K \)-linearly independent.

However if we compute \( \text{Hom}_K(\Omega_{K/\mathbb{C}},K) \) we get \( \mathbb{C} \)-linear derivations of \( K \), which contains a free submodule of rank \((m|0)\) generated by \( \partial t_1,\ldots, \partial t_m \). These may be used to show that \( dt_1,\ldots,dt_m \) are \( K \)-linearly independent, and we are done. \( \square \)

To state our characterization of smoothness, we need to introduce a few notions, most of which should be familiar.

- For \( x \in X(\mathbb{C}) \) we may view \( T_x X \) as the affine superspace \( \text{Spec} S^*(\mathfrak{m}_x/\mathfrak{m}_x^2) \). Define the tangent cone at \( x \), \( TC_x X \), to be the closed conical subsupervariety of \( T_x X \) given by

\[
TC_x X = \text{Spec} \left( \bigoplus_{n \geq 0} \frac{\mathfrak{m}_x^n}{\mathfrak{m}_x^{n+1}} \right)
\]

The derivations in \( T_{X,x} \) act on both \( \mathbb{C}[T_x X] \) and \( \mathbb{C}[TC_x] \) by derivations of degree -1, and the action is equivariant with respect to the above closed embedding.

- For a local superalgebra \( A \) with unique maximal ideal \( \mathfrak{m} \), we write \( \hat{A} \) for the completion of \( A \) with respect to the \( \mathfrak{m} \)-adic topology.
Following [Sch89], given a superalgebra $A$ we say that an even element $t \in A_r$ is $A$-regular if the multiplication map $A \to A$ is injective. We say an odd element $\xi \in A_r$ is $A$-regular if the cohomology of the multiplication map by $\xi$ is trivial. Finally, if $(r_1, \ldots, r_k)$ is a sequence of homogeneous elements of $A$, we say the sequence is $A$-regular if $r_i$ is regular in $A/(r_1, \ldots, r_{i-1})$. Now we will say that a local superalgebra $A$ is regular iff the unique maximal ideal $m$ is generated by an $A$-regular sequence.

Proposition 8.2. For a supervariety $X$, and closed point $x \in X(\mathbb{C})$, let $A = O_{X,x}$ with maximal ideal $m = m_x$. Let $t_1, \ldots, t_m, \xi_1, \ldots, \xi_n \in m$ project to a homogeneous basis of $m/m^2$, where $\overline{t}_i = 0$ and $\overline{\xi}_i = 1$. Then the following are equivalent.

1. $\hat{A} \cong \mathbb{C}[t_1, \ldots, t_m, \xi_1, \ldots, \xi_n]$.
2. $\text{Gr}_mA \cong \mathbb{C}[\overline{t}_1, \ldots, \overline{t}_m, \overline{\xi}_1, \ldots, \overline{\xi}_n]$, where $(\cdot) : m \to m/m^2$ is the natural projection.
3. $\Omega_{X,x} = \Omega_{A/C}$ is free over $A$.
4. $\text{Spec } A \to \mathbb{C}$ is a formally smooth morphism.
5. $\hat{A} = A/\hat{(A_r)}$ is a regular local ring, and $A \cong \hat{A}[\xi_1, \ldots, \xi_n]$.
6. There exists an affine neighborhood $V = \text{Spec } B$ of $x$ such that $B = B/\hat{(B_r)}$ is regular and $B \cong \Lambda[\xi_1, \ldots, \xi_n]^{\text{even}}$.
7. We have $T_xX = TC_xX$.
8. The natural map $T_{X,x} \to T_xX$ is surjective.
9. $A$ is a regular local superalgebra.

Proof. The equivalence (1) $\iff$ (2) is proven in [Fio08], (2) $\iff$ (7) is clear, (3) $\iff$ (4) is proven in [KV11], and (5) $\iff$ (9) is proven in [Sch89].

For (1) $\implies$ (3), we have that $m/m^2 \cong \hat{m}/\hat{m}^2$ is $(m|n)$-dimensional, so by Nakayama’s lemma $\Omega_{A/C}$ is generated by $(m|n)$ elements. Localizing $A$ to the generic point, we obtain a superalgebra $K$ which by our assumption is isomorphic to $\mathbb{K}[\xi_1, \ldots, \xi_n]$ by the Cohen structure theorem, where $\mathbb{K}$ is the fraction field of $\hat{A}$. Hence by lemma 8.1 $\Omega_{K/C}$, which is the localization of $\Omega_{A/C}$ at the generic point, is free of rank $(m|n)$. It follows that $\Omega_{A/C}$ must itself be free of rank $(m|n)$.

For (3) $\implies$ (8), we have $dt_1, \ldots, dt_m, d\xi_1, \ldots, d\xi_n$ form a basis of $\Omega_{A/C}$. Then $T_{X,x} = \text{Hom}_A(\Omega_{A/C}, A)$ will be free with basis $\partial_{t_1}, \ldots, \partial_{t_m}, \partial_{\xi_1}, \ldots, \partial_{\xi_n}$ and these derivations map to a basis of $T_xX$, namely the dual basis of $t_1, \ldots, t_m, \xi_1, \ldots, \xi_n \in m/m^2$.

(8) $\implies$ (7): If $TC_xX \neq T_xX$, then the vanishing ideal of $TC_xX$ must be preserved by all derivations from $T_{X,x}$. By our assumption, we get all coordinate derivations from the derivations of $T_{X,x}$, so no such non-trivial ideals exist.

For (5) $\iff$ (6), the backward direction follows from localizing. For the forward direction, the isomorphism $O_{X,x} \to \hat{A}[\xi_1, \ldots, \xi_n]$ may be extended to a morphism of sheaves $O_X \to \text{gr }X$ on a small enough affine open of $x$ which is an isomorphism of stalks at $x$, and so using Noetherian and coherent properties, we get an isomorphism in an open neighborhood of $x$.

The implication (5) $\implies$ (1) is clear.

Now we assume (1), and use (3) (which we have so far shown is equivalent to 1) to prove (5). First, (1) implies that $\hat{A}$ is regular. As noted previously, by (3) we know that $A$ has derivations $\partial_{t_1}, \ldots, \partial_{t_m}, \partial_{\xi_1}, \ldots, \partial_{\xi_n}$. These derivations extend canonically to $\hat{A}$ as the usual coordinate derivations, and these derivations preserve $A$ as a subalgebra. We
have the following diagram:

$$
\begin{array}{ccl}
\mathbb{A} & \longrightarrow & \mathbb{C}[t_1, \ldots, t_m, \xi_1, \ldots, \xi_n] \\
\pi \downarrow & & \downarrow \tilde{\pi} \\
\mathbb{A} & \longrightarrow & \mathbb{C}[t_1, \ldots, t_m]
\end{array}
$$

where $\pi$ is the natural quotient map. To construct a splitting $\mathbb{A} \to \mathbb{A}$, we observe that $\tilde{\pi}$ has a natural splitting $\tilde{s}$ sending $t_i$ to $\xi_i$. We would like to show that $\tilde{s}(\mathbb{A})$ lies in the image of $\mathbb{A}$ in the completion.

Let $f \in \mathbb{A}$, thought of as a power series. Then we may lift $f$ to $\tilde{f} \in \mathbb{A}_\mathbb{F}$. The power series expansion of $\tilde{f}$ will then be

$$\tilde{f} = f + \sum_{I \neq \emptyset} f_I \xi_I \in \mathbb{A}$$

where $\xi_I = \xi_{i_1} \cdots \xi_{i_k}$ if $I = \{i_1, \ldots, i_k\}$, and $f_I \in \mathbb{C}[t_1, \ldots, t_m]$. Using the derivations $\partial_{\xi_i}$ for varying $I$, we may show that each function $f_I$ lies in $\mathbb{A}$, and so $\tilde{f}$ itself lies in $\mathbb{A}$. Therefore we have our splitting, and now it follows that $\mathbb{A} \cong \mathbb{A}[\xi_1, \ldots, \xi_n]$. [CW12]. □

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