A family of solutions of the Yang-Baxter equation

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Abstract

A new method to construct involutive non-degenerate set-theoretic solutions \((X^n, r^{(n)})\) of the Yang-Baxter equation from an initial solution \((X, r)\) is given. Furthermore, the permutation group \(G(X^n, r^{(n)})\) associated to the solution \((X^n, r^{(n)})\) is isomorphic to a subgroup of \(G(X, r)\), and in many cases \(G(X^n, r^{(n)}) \cong G(X, r)\).

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1 Introduction

The quantum Yang-Baxter equation is one of the basic equations in mathematical physics named after the authors of the two first works in which the equation arose: the solution of the delta function Fermi gas by C. N. Yang [15], and the solution of the 8-vertex model by R. J. Baxter [1]. It also lies at the foundation of the theory of quantum groups. One of the important open problems related to this equation is compute all its solutions. Those are linear maps \(R : V \otimes V \to V \otimes V\), with \(V\) a vector space, that satisfy

\[R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12},\]

where \(R_{ij}\) denotes the map \(R_{ij} : V \otimes V \otimes V \to V \otimes V \otimes V\) acting as \(R\) on the \((i, j)\) tensor factors and as the identity on the remaining factor.

Finding all the solutions of the Yang-Baxter equation is a difficult task far from being solved. Nevertheless, many solutions have been found during the last 20 years and the related algebraic structures (Hopf algebras) have been studied.

In [5], Drinfeld suggested the study of a simpler case: solutions induced by a linear extension of a mapping \(R : X \times X \to X \times X\), where \(X\) is a basis for \(V\). In this case, one says that \(R\) is a set-theoretic solution of the quantum Yang-Baxter equation. It is not difficult to see that, if \(\tau : X^2 \to X^2\) is the map defined by \(\tau(x, y) = (y, x)\), then the map \(R : X^2 \to X^2\) is a set-theoretic solution of the quantum Yang-Baxter equation if and only if the mapping \(r = \tau \circ R\) is a solution of the equation

\[r_{12} \circ r_{23} \circ r_{12} = r_{23} \circ r_{12} \circ r_{23},\]
where $r_{ij}$ is the map from $X^3$ to $X^3$ that acts as $r$ on the $(i,j)$ components and as the identity on the remaining component. In the sequel, we will always work with this last equivalent equation.

We study solutions with some additional conditions: involutively and non-degeneracy. A map

$$r : X \times X \rightarrow X \times X$$

is said to be involutive if $r \circ r = \text{id}_{X^2}$. Moreover, it is said to be left (resp. right) non-degenerate if each map $\sigma_x$ (respectively, $\gamma_y$) is bijective, and it is said to be non-degenerate if it is left and right non-degenerate. If $r$ is involutive and left non-degenerate, it can be checked that

$$r_{12} \circ r_{23} \circ r_{12} = r_{23} \circ r_{12} \circ r_{23}$$

if and only if it satisfies $\sigma_x \circ \sigma_x^{-1}(y) = \sigma_y \circ \sigma_y^{-1}(x)$ for all $x,y \in X$ (see the proof of [12, Theorem 9.3.10]).

In what follows, by a solution of the YBE we will mean a non-degenerate involutive set-theoretic solution of the Yang-Baxter equation.

In the last years, solutions of the YBE have received a lot of attention [3, 4, 6, 7, 8, 9, 10, 11, 13, 14]. In this case each solution $(X,r)$ of the YBE has an associated structure group, denoted by $G(X,r)$, and defined by

$$G(X,r) := \langle x \in X \mid xy = zt \text{ if and only if } r(x,y) = (z,t) \rangle.$$ 

When $X$ is finite, groups isomorphic to some $G(X,r)$ are called groups of $I$-type. There is another important group associated to every solution of the YBE, its permutation group $G(X,r)$, which is the subgroup of $\text{Sym}_X$ generated by the bijections $\sigma_x$, for all $x \in X$. It can be proved that $G(X,r)$ is a homomorphic image of $G(X,r)$. When $X$ is finite, groups isomorphic to some $G(X,r)$ are called IYB groups.

In [3], in order to characterize the groups of $I$-type, it is suggested to follow two steps:

**Step 1:** Determine the finite groups that are IYB groups.

**Step 2:** Given an IYB group $G$, find all the solutions of the YBE $(Y,s)$ with $Y$ finite such that $G(Y,s) \cong G$.

Nowadays, these two problems remain unsolved. In the recent Ph.D. thesis of Nir Ben David [2], it is claimed that the result corresponding to [2, Corollary D] solves Step 2 in homological terms. In fact, this result reduces Step 2 to the following problem.

**Problem.** Let $G$ be an IYB group. Let $\pi : G \rightarrow A$ be a bijective 1-cocycle over a $G$-module $A$. Let $n$ be a positive integer. Find all the extensions of $G$-modules

$$0 \rightarrow \mathbb{Z}^n \rightarrow E \rightarrow A \rightarrow 0,$$

where $\mathbb{Z}^n$ is a trivial $G$-module, $E \cong \mathbb{Z}^n$ as abelian groups and there is a basis $Y$ of $E$, as free abelian group, which is invariant by the action of $G$ on $E$. 

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The aim of this work is to present some new results related to the resolution of Step 2. To this end, an algebraic structure called brace and introduced by Rump in [14] is very useful. Rump showed that this structure has deep connections with solutions of the YBE. We use some of this connections to prove our results.

This paper is organized as follows. In Section 2, we recall some results about braces. We only sketch there, without proofs, the theorems that we need later; for further reading, an introduction to braces and their connection to the Yang-Baxter equation can be found in [4]. Next, we devote Section 3 to present our main result about solutions of the YBE with fixed permutation group. We do not solve the general problem, but we generalize a result of [3] which gives a non-obvious construction of an infinite family of solutions with a fixed permutation group. Specifically, given an initial solution \((X, r)\) of the YBE, we state and prove a procedure to define, for each \(n \in \mathbb{N}\), a solution \((X^n, r^{(n)})\) such that \(\mathcal{G}(X^n, r^{(n)})\) is isomorphic to a subgroup of \(\mathcal{G}(X, r)\), and then we provide sufficient conditions for \(\mathcal{G}(X^n, r^{(n)})\) to be isomorphic to \(\mathcal{G}(X, r)\). One of the key steps of the proof of this result is the use of the properties of the brace structure.

# 2 Preliminaries about braces

We only present here very briefly the results about braces that we will need later. For a much more detailed account, see [4]. We begin recalling the definition of left brace.

**Definition 2.1** A left brace is a set \(G\) with two operations \(+\) and \(\cdot\) such that \((G, \cdot)\) is an abelian group, \((G, \cdot)\) is a group, and every \(a, b, c \in G\) satisfy

\[
a(b + c) + a = ab + ac.
\]

We will refer to this property as the brace property. We call \((G, \cdot)\) the additive group, and \((G, \cdot)\) the multiplicative group of the left brace. Right braces are defined similarly, changing the brace property by \((b + c)a + a = ba + ca\).

For any \(a \in G\), we define a map \(\lambda_a : G \to G\) by \(\lambda_a(b) = ab - a\). In the study of braces, these maps play an important role; here is a list of some of their properties.

**Lemma 2.2** Let \(G\) be a left brace. The following properties hold:

(i) \(\lambda_a\) is bijective and \(\lambda_a^{-1} = \lambda_a^{-1}\).

(ii) \(\lambda_a(x + y) = \lambda_a(x) + \lambda_a(y)\); that is, \(\lambda_a\) is an automorphism of the abelian group \((G, \cdot)\).

(iii) \(\lambda_a \lambda_b = \lambda_{ab}\); that is, the map \(\lambda : (G, \cdot) \to \text{Aut}(G, \cdot)\), defined by \(\lambda(a) = \lambda_a\) is a homomorphism of groups.
(iv) \( a + b = a \cdot \lambda_a^{-1}(b) \).
(v) \( a \cdot \lambda_a^{-1}(b) = b \cdot \lambda_b^{-1}(a) \).
(vi) \( \lambda_a \lambda_a^{-1}(b) = \lambda_b \lambda_b^{-1}(a) \).

(vii) The map \( r : G \times G \to G \times G \) defined by \( r(x, y) = \left( \lambda_x(y), \lambda_x^{-1}(x) \right) \) is a solution of the YBE. It is called solution associated to the left brace \( G \).

**Proof.** See [4, Lemmas 2.9 and 4.1].

So any left brace gives us a solution of the YBE. There are other relations between braces and solutions of the YBE, like the two next results, which are a characterization of groups of I-type and IYB groups through braces.

**Proposition 2.3** A group is isomorphic to \( G(X, r) \) for some solution of the YBE \( (X, r) \) if and only if it is the multiplicative group of a left brace.

In particular, a finite group \( G \) is an IYB group if and only if it is the multiplicative group of a finite left brace.

**Proof.** See [4, Corollary 4.6].

**Proposition 2.4** A group \( G \) is of I-type if and only if it is isomorphic to the multiplicative group of a left brace \( B \) such that the additive group of \( B \) is a free abelian group with a finite basis \( X \) such that \( \lambda_x(y) \in X \), for all \( x, y \in X \).

**Proof.** See [4, Proposition 5.2].

The next theorem is an essential tool for the proof of the main result of this paper. It allows us to embed any solution of the YBE \( (X, s) \) inside a left brace, and then we can use all the additional algebraic properties of this structure.

**Theorem 2.5** Let \( (X, s) \) be a solution of the YBE. Then \( G(X, s) \) is isomorphic to the multiplicative group of a left brace \( H \) such that, if \( (H, r) \) is the solution associated to it, then there exists a subset \( Y \) of \( H \) such that \( (Y, r') \), where \( r' \) is the restriction of \( r \) to \( Y^2 \), is a solution of the YBE isomorphic to \( (X, r) \). Furthermore, \( \lambda_h(y) \in Y \), for all \( y \in Y \) and all \( h \in H \).

**Proof.** See [4, Theorem 4.4] and its proof.
3 Solutions with fixed permutation group

In this section, we focus on the problem of the construction of solutions of the YBE with a fixed permutation group.

There is a simple way to produce solutions with the same permutation group. Note that, for any set $X$, the map $r(x, y) = (y, x)$, for all $x, y \in X$, is a solution of the YBE with trivial permutation group; we call $(X, r)$ the trivial solution on $X$. Note also that, if we have solutions $(X_i, r_i)$ of the YBE with permutation group $G_i = \mathcal{G}(X_i, r_i)$, for all $i = 1, \ldots, n$, and $X = \cup_{i=1}^{n} X_i$ is the disjoint union of the sets $X_i$, then $(X, r)$ is a solution of the YBE, where

$$r(x, y) = \begin{cases} r_i(x, y) & \text{if there exists an } i \text{ such that } x, y \in X_i, \\ (y, x) & \text{otherwise} \end{cases}$$

with permutation group isomorphic to $G_1 \times \cdots \times G_n$. Hence, since $\{\text{id}\} \times G \cong G$, we can associate to each solution $(X, r)$ of the YBE infinitely many solutions of the YBE with the same permutation group. However, this construction only increases the size of our set adding points that behave as the trivial solution, so we want to find “less trivial” constructions.

We will generalize [3, Lemma 5.2], which gives a non-obvious construction of an infinite family of solutions of the YBE associated to a fixed permutation group. This result is stated in [3] in terms of cycle sets. Translated to the language of solutions of the YBE equation, the result could be stated as follows.

**Proposition 3.1** Let $(X, r)$ be a solution of the YBE, with $r(x, y) = (\sigma_x(y), \gamma_y(x))$. Then, for $x_1, x_2 \in X$, the map $f_{(x_1, x_2)}: X^2 \to X^2$ defined by

$$f_{(x_1, x_2)}(y_1, y_2) = (\sigma_{x_1}(\sigma_{x_2}(y_1)), \sigma_{x_1, x_2}(y_1)\sigma_{x_2}(\gamma_{y_1}(y_2))),$$

for $y_1, y_2 \in X$, is bijective and $(X^2, r^{(2)})$, where $r^{(2)}: X^2 \times X^2 \to X^2 \times X^2$ is the map defined by

$$r^{(2)}((x_1, x_2), (y_1, y_2)) = (f_{(x_1, x_2)}(y_1, y_2), f_{f_{(x_1, x_2)}(y_1, y_2)}^{-1}(x_1, x_2)),$$

is a solution of the YBE. Moreover, if $\sigma_z = \text{id}$ for some $z \in X$, then $\mathcal{G}(X^2, r^{(2)}) \cong \mathcal{G}(X, r)$.

**Remark 3.2** The assumption that $\sigma_x = \text{id}$, for some $x \in X$, is not true in general. However, increasing the size of our set, we can always construct a solution with this property preserving the same permutation group. Namely, consider the solution $(X \cup \{z\}, r')$ for the disjoint union of $X$ with a set of one point, defined as $r'(x, y) = r(x, y)$ if $x$ and $y$ belong to $X$ and $r'(x, y) = (y, x)$ if either $x = z$ or $y = z$ (note that it is the same trivial construction given at the beginning of this section).

Thus, beginning with an initial solution $(X, r)$ of the YBE and applying Proposition 3.1 to $X, X^2, (X^2)^2, \ldots$ we can construct infinitely many solutions of the YBE with the same associated permutation group $\mathcal{G}(X, r)$. 
In order to generalize this result we will need the following lemma, which is a generalization of [3, Lemma 5.1]. We use the notation $\pi = (x_1, \ldots , x_n)$ for the elements of $X^n$.

**Lemma 3.3** Let $X$ be a set and let $\sigma : X \rightarrow \text{Sym}_X$ be a map. Consider the map $\psi : \text{Sym}_X \rightarrow \text{Sym}_X$ defined by $\psi(\tau)(\pi) = (t_1^\tau(\pi), \ldots , t_n^\tau(\pi))$, for all $\tau \in \text{Sym}_X$ and all $\pi \in X^n$, where $t_j^\tau(\pi) = \tau(y_j)$, for $j \geq 1$.

Thus $\psi$ is a monomorphism.

**Proof.** Let $\tau \in \text{Sym}_X$. It is easy to check that $\psi(\tau)$ is a bijective map from $X^n$ to $X^n$ with inverse

$$(y_1, \ldots , y_n) \mapsto (T_1^\tau(\pi), \ldots , T_n^\tau(\pi)),$$

where $T_j^\tau(\pi) = \tau^{-1}(y_j)$,

$$T_{j+1}^\tau(\pi) = \sigma(T_j^\tau(\pi))^{-1} \cdots \gamma(T_1^\tau(\pi))^{-1} \gamma(\tau(\gamma(y_j))) \cdots \gamma(\gamma(\gamma(y_{j+1})))$$

Thus $\psi$ is well-defined.

To prove that $\psi$ is a morphism, we have to check $\psi(\gamma \circ \xi) = \psi(\gamma) \circ \psi(\xi)$ for all $\gamma, \xi \in \text{Sym}_X$. Component by component, this is equivalent to verify

$$t_j^{\gamma \circ \xi}(y_1, \ldots , y_n) = t_j^\gamma(t_j^\xi(y_1), \ldots , t_j^\xi(y_n)), \quad \forall j = 1, \ldots , n.$$  

This is done by induction. The first component is almost immediate; we write $t_j^\pi(\pi) = (t_1^\pi(\pi), \ldots , t_j^\pi(\pi))$ for short:

$$t_1^{\gamma \circ \xi}(\pi) = \tau(\pi(y_1)) = \tau(t_1^\xi(\pi)) = t_1^\gamma(t_1^\xi(\pi)).$$

Now, assume that we have checked it up to the $j$-th component, and we want to prove it for the $(j + 1)$-th component:

$$t_{j+1}^{\gamma \circ \xi}(\pi) = \sigma(t_j^\gamma(\pi))^{-1} \cdots \gamma(t_1^\gamma(\pi))^{-1} \gamma(\gamma(y_j)) \cdots \gamma(\gamma(y_{j+1}))$$

The second equality comes from the induction hypothesis, and at the end we use the definition of $t_j^\gamma(\pi)$.

On the other hand, to prove that $\psi$ is injective, suppose that $\psi(\gamma) = \psi(\xi)$ for some $\gamma, \xi \in \text{Sym}_X$. Then, $\psi(\gamma)(\pi) = \psi(\xi)(\pi)$ for all $\pi \in X^n$. Looking at the first component, we get $\gamma(y_1) = \xi(y_1)$ for all $y_1 \in X$, so $\gamma = \xi$. }

The next two results give the announced generalization of [3, Lemma 5.2].
Theorem 3.4 Let \((X,r)\) be a solution of the YBE, with \(r(x,y) = (\sigma_x(y), \gamma_y(x))\). Let \(n\) be an integer greater that 1. For \(\varpi \in X^n\), consider the map \(f_{\varpi}: X^n \rightarrow X^n\)
defined by
\[
f_{\varpi}(\vareta) = (h_1(\varpi, \vareta), h_2(\varpi, \vareta), \ldots, h_n(\varpi, \vareta)),
\]
for \(\varpi \in X^n\), where the \(h_j\) is defined recursively by
\[
h_1(\varpi, \varrho) = \sigma_{x_1} \cdots \sigma_{x_n}(y_1),
\]
and
\[
h_j(\varpi, \varrho) = \sigma_{h_{j-1}(\varpi, \varrho)}^{-1} \cdots \sigma_{h_1(\varpi, \varrho)}^{-1} \sigma_{x_1} \cdots \sigma_{x_n} \sigma_{y_1} \cdots \sigma_{y_{j-1}}(y_j),
\]
for \(j = 2, \ldots, n\). Then \(f_{\varpi}\) is bijective and \((X^n, r^{(n)})\), where \(r^{(n)} : X^n \times X^n \rightarrow X^n \times X^n\) is the map defined by
\[
r^{(n)}(\varpi, \varrho) = (f_{\varpi}(\varrho), f_{\varpi}^{-1}(\varpi))
\]
is a solution of the YBE.

Proof. Let \(\sigma: X \rightarrow \text{Sym}_X\) be the map defined by \(\sigma(x) = \sigma_x\), for all \(x \in X\). Consider the map \(\psi: \text{Sym}_X \rightarrow \text{Sym}_{X^n}\) defined as in Lemma 3.3. It is clear
that \(f_{\varpi} = \psi(\sigma_{x_1} \cdots \sigma_{x_n})\). Hence \(f_{\varpi}\) is bijective.

By Theorem 2.5, we may assume that \(X\) is a subset of a left brace \(H\) and
that \(\sigma_x\) is the restriction of \(\lambda_x\) to \(X\), for all \(x \in X\). Recall that \(\lambda_a(b) = ab - a\),
for all \(a, b \in H\). Therefore, by Lemma 2.2(iii),
\[
h_1(\varpi, \varrho) = \sigma_{x_1} \cdots \sigma_{x_n}(y_1) = \lambda_{x_1 \cdots x_n}(y_1).
\]
We claim that
\[
\lambda_{x_1 \cdots x_n}(y_1 \cdots y_j) = h_1(\varpi, \varrho) \cdots h_j(\varpi, \varrho), \tag{1}
\]
for all \(1 \leq j \leq n\) and \(x_1, \ldots, x_n, y_1, \ldots, y_n \in X\).

We prove this claim by induction on \(j\). We know that it is true for \(j = 1\).
Suppose that \(j > 1\) and the claim is true for \(j - 1\). We have
\[
\lambda_{x_1 \cdots x_n}(y_1 \cdots y_j) = \lambda_{x_1 \cdots x_n}(y_1 \cdots y_{j-1} + \lambda_{y_1 \cdots y_{j-1}}(y_j)) = \\
\lambda_{x_1 \cdots x_n}(y_1 \cdots y_{j-1}) + \lambda_{x_1 \cdots x_n} \lambda_{y_1 \cdots y_{j-1}}(y_j) \quad \text{(by Lemma 2.2(ii))} = \\
h_1(\varpi, \varrho) \cdots h_{j-1}(\varpi, \varrho) + \lambda_{x_1 \cdots x_n} \lambda_{y_1 \cdots y_{j-1}}(y_j) \quad \text{(by induction hypothesis)} = \\
h_1(\varpi, \varrho) \cdots h_{j-1}(\varpi, \varrho) \lambda_{h_{j-1}(\varpi, \varrho)}^{-1} \cdots \lambda_{h_1(\varpi, \varrho)}^{-1} \lambda_{x_1 \cdots x_n} \lambda_{y_1 \cdots y_{j-1}}(y_j) \quad \text{(by Lemma 2.2(iv))}
\]
By Lemma 2.2(iii),
\[
\lambda_{h_{j-1}(\varpi, \varrho)}^{-1} \cdots \lambda_{h_1(\varpi, \varrho)}^{-1} \lambda_{x_1 \cdots x_n} \lambda_{y_1 \cdots y_{j-1}}(y_j) = h_j(\varpi, \varrho).
\]
Hence the claim follows. By (1), we have that
\[ h_j(\overline{x}, \overline{y}) = \lambda_{x_1 \cdots x_n}(y_1 \cdots y_{j-1})^{-1} \lambda_{x_1 \cdots x_n}(y_1 \cdots y_j), \tag{2} \]
for \(2 \leq j \leq n\). Let \( H_1(\overline{x}, \overline{y}) = \lambda_{x_1 \cdots x_n}^{-1}(y_1) \) and
\[ H_j(\overline{x}, \overline{y}) = \lambda_{x_1 \cdots x_n}^{-1}(y_1 \cdots y_{j-1})^{-1} \lambda_{x_1 \cdots x_n}^{-1}(y_1 \cdots y_j), \]
for \(2 \leq j \leq n\). It is easy to check that the map \( X^n \rightarrow X^n \) defined by \( \overline{y} \mapsto (H_1(\overline{x}, \overline{y}), \ldots, H_n(\overline{x}, \overline{y})) \) is \( f_{\overline{x}}^{-1} \).

By the definition of \( r^{(n)} \), it is straightforward to check that \( r^{(n)} \circ r^{(n)} = \text{id} \).

Thus, in order to prove that \((X^n, r^{(n)})\) is an involutive non-degenerate set-theoretic solution of the Yang-Baxter equation, we should show that

(a) \( f_{\overline{x}} f_{\overline{x}}^{-1}(\overline{y}) = f_{\overline{x}} f_{\overline{x}}^{-1}(\overline{z}) \), for all \( \overline{x}, \overline{y}, \overline{z} \in X^n \), and

(b) the map \( \gamma_{\overline{x}} : X^n \rightarrow X^n \) defined by \( \gamma_{\overline{x}}(\overline{y}) = f_{\overline{x}}^{-1}(\overline{y}) \) is bijective.

By (1) and the definition of \( H_j(\overline{x}, \overline{y}) \), the first component of \( f_{\overline{x}} f_{\overline{x}}^{-1}(\overline{y}) = f_{\overline{x}} f_{\overline{x}}^{-1}(\overline{y}) \) is
\[ \lambda_{x_1 \cdots x_n} H_1(\overline{x}, \overline{y}) H_2(\overline{x}, \overline{y}) (z_1) = \lambda_{x_1 \cdots x_n} \lambda_{x_1 \cdots x_n}^{-1}(y_1 \cdots y_n) (z_1) = \lambda_{y_1 \cdots y_n} \lambda_{y_1 \cdots y_n}^{-1}(x_1 \cdots x_n) (z_1), \]
where the last equality follows from Lemma \((2.24)\) (vi). For \( j > 1 \), the \( j \)-th component of \( f_{\overline{x}} f_{\overline{x}}^{-1}(\overline{y}) = f_{\overline{x}} f_{\overline{x}}^{-1}(\overline{y}) \) is
\[ (\lambda_{x_1 \cdots x_n} H_1(\overline{x}, \overline{y}) H_2(\overline{x}, \overline{y}) (z_1 \cdots z_j-1))^{-1} \cdot (\lambda_{x_1 \cdots x_n} \lambda_{x_1 \cdots x_n}^{-1}(y_1 \cdots y_n) (z_1 \cdots z_j-1)) \]
\[ = (\lambda_{x_1 \cdots x_n} \lambda_{x_1 \cdots x_n}^{-1}(y_1 \cdots y_n) (z_1 \cdots z_j-1))^{-1} \cdot (\lambda_{x_1 \cdots x_n} \lambda_{x_1 \cdots x_n}^{-1}(y_1 \cdots y_n) (z_1 \cdots z_j)) \]
and, by Lemma \((2.24)\) (vi), we can interchange the \( x \)'s and the \( y \)'s, and (a) follows.

To prove (b), first we shall see that \( \gamma_{\overline{x}} \) is injective. Let \( \overline{x}, \overline{z} \in X^n \) be elements such that \( \gamma_{\overline{x}}(\overline{y}) = \gamma_{\overline{z}}(\overline{y}) \). Hence \( H_j((h_1(\overline{x}, \overline{y}), \ldots, h_n(\overline{x}, \overline{y})), \overline{y}) = H_j((h_1(\overline{z}, \overline{y}), \ldots, h_n(\overline{z}, \overline{y})), \overline{x}) \), for all \( j = 1, \ldots, n \).

That is
\[ \lambda_{h_1(\overline{x}, \overline{y}) \cdots h_n(\overline{x}, \overline{y})} (x_1) = \lambda_{h_1(\overline{y}) \cdots h_n(\overline{y})} (z_1) \]
and
\[ \lambda_{h_1(\overline{x}, \overline{y}) \cdots h_n(\overline{x}, \overline{y})} (x_1 \cdots x_j-1) = \lambda_{h_1(\overline{y}) \cdots h_n(\overline{y})} (z_1 \cdots z_j-1), \]
for all \( j = 2, \ldots, n \). Therefore
\[ \lambda_{h_1(\overline{x}, \overline{y}) \cdots h_n(\overline{x}, \overline{y})} (x_1 \cdots x_j) = \lambda_{h_1(\overline{y}) \cdots h_n(\overline{y})} (z_1 \cdots z_j), \]
for all \( j = 1, \ldots, n \). By (1),
\[ \lambda_{x_1 \cdots x_n} (y_1 \cdots y_n) (x_1 \cdots x_j) = \lambda_{x_1 \cdots x_n} (y_1 \cdots y_n) (z_1 \cdots z_j), \tag{3} \]
for all $j = 1, \ldots, n$. By Lemma 2.2(vi), since
\[
\lambda^{-1}_{\lambda a_n(y_1 \cdots y_n)}(x_1 \cdots x_n) = \lambda^{-1}_{\lambda a_n(y_1 \cdots y_n)}(z_1 \cdots z_n),
\]
we have that $x_1 \cdots x_n = z_1 \cdots z_n$. Thus, by (3), $x_1 \cdots x_j = z_1 \cdots z_j$, for all $j = 1, \ldots, n$. Hence $x_j = z_j$, for all $j = 1, \ldots, n$. Therefore $\gamma_\sigma$ is injective.

We shall see that $\gamma_\sigma$ is surjective. Let $\sigma = (z_1, \ldots, z_n) \in X^n$. By Lemma 2.2(vii), there exists $a_n \in H$ such that $\lambda^{-1}_{\lambda a_n(y_1 \cdots y_n)}(a_n) = z_1 \cdots z_n$. Let
\[
a_j = \lambda_{a_n(y_1 \cdots y_n)}(z_1 \cdots z_j),
\]
for all $j = 1, \ldots, n - 1$. By Theorem 2.5, $a_1 \in X$. Let $x_1 = a_1$ and $x_i = a_i^{-1}a_i$, for $1 < i \leq n$. We shall prove that $x_i \in X$, for all $i$, and $\gamma_\sigma(\sigma) = \sigma$. Suppose that $i > 1$ and $x_1, \ldots, x_{i-1} \in X$. We have that
\[
a_i = \lambda_{a_n(y_1 \cdots y_n)}(z_1 \cdots z_i) = \lambda_{a_n(y_1 \cdots y_n)}(z_1 \cdots z_{i-1} + \lambda_{z_1 \cdots z_{i-1}}(z_i)) = \lambda_{a_n(y_1 \cdots y_n)}(z_1 \cdots z_{i-1}) + \lambda_{a_n(y_1 \cdots y_n)}(\lambda_{z_1 \cdots z_{i-1}}(z_i)) \quad \text{(by Lemma 2.2(ii))}
\[
= a_{i-1} + \lambda_{a_n(y_1 \cdots y_n)}(z_1 \cdots z_{i-1}) \quad \text{(by Lemma 2.2(iii))}
\[
= a_{i-1} \lambda_{a_n(y_1 \cdots y_n)}(z_1 \cdots z_{i-1}) \quad \text{(by Lemma 2.2(iv)).}
\]
Hence $x_i = a_{i-1}^{-1}a_i = \lambda_{a_n(y_1 \cdots y_n)}(z_1 \cdots z_{i-1}) \in X$, by Theorem 2.5. Thus, by induction, $x_1, \ldots, x_n \in X$. Now we have
\[
\gamma_\sigma(\sigma) = (H_1((h_1(\sigma, \sigma), \ldots, h_n(\sigma, \sigma)), \sigma), \ldots, H_n((h_1(\sigma, \sigma), \ldots, h_n(\sigma, \sigma)), \sigma)).
\]
By (1) and the definition of $H_j$,
\[
\gamma_\sigma(\sigma) = (\lambda^{-1}_{\lambda a_1 \cdots a_n(y_1 \cdots y_n)}(x_1), \lambda^{-1}_{\lambda a_1 \cdots a_n(y_1 \cdots y_n)}(x_1) \cdots \lambda^{-1}_{\lambda a_1 \cdots a_n(y_1 \cdots y_n)}(x_1 x_2), \ldots
\[
\lambda^{-1}_{\lambda a_1 \cdots a_n(y_1 \cdots y_n)}(x_1 \cdots x_n) = (\lambda^{-1}_{\lambda a_n(y_1 \cdots y_n)}(a_1), \lambda^{-1}_{\lambda a_n(y_1 \cdots y_n)}(a_1) \cdots \lambda^{-1}_{\lambda a_n(y_1 \cdots y_n)}(a_n)) = (z_1, z_2, \ldots, z_n).
\]
Therefore $\gamma_\sigma$ is bijective and (b) follows. This finishes the proof.

The next proposition describes the permutation group of the solutions $(X^n, r(n))$ as a certain subgroup of the permutation group of $(X, r)$. After that, we give two cases in which $G(X^n, r(n)) \cong G(X, r)$.

**Proposition 3.5** With the above notation, $G(X^n, r(n))$ is isomorphic to the subgroup of $G(X, r)$ generated by all the permutations $\sigma_{x_1} \cdots \sigma_{x_n}$, for all $x_i \in X$. In particular,

1. If $\sigma_z = id$ for some $z \in X$, then $G(X^n, r(n)) \cong G(X, r)$ for all $n$.

2. If $X$ is a finite set and $|G(X, r)| = m < +\infty$, then, for all $n$ such that $\gcd(m, n) = 1$, we have $G(X^n, r(n)) \cong G(X, r)$. 

9
Proof. Recall that \( r(x,y) = (\sigma_x(y), \gamma_y(x)) \). Let \( \sigma : X \to \text{Sym}_X \) be the map defined by \( \sigma(x) = \sigma_x \), for all \( x \in X \). Consider the map \( \psi : \text{Sym}_X \to \text{Sym}_{X^n} \) defined as in Lemma 3.3. We have that

\[
G(X^n, r^{(n)}) = \langle f_x : x \in X^n \rangle = \langle \psi(\sigma_{x_1} \cdots \sigma_{x_n}) : x_i \in X \rangle
= \psi(\langle \sigma_{x_1} \cdots \sigma_{x_n} : x_i \in X \rangle) \cong \langle \sigma_{x_1} \cdots \sigma_{x_n} : x_i \in X \rangle,
\]

using in the last isomorphism that \( \psi \) is a monomorphism.

In particular,

1. If \( \sigma_z = \text{id} \) for some \( z \in X \), then any \( \sigma_x \) can be written as a product of \( n \) permutations using \( \sigma_x = \sigma_z^{-1} \sigma_x \), so \( \langle \sigma_{x_1} \cdots \sigma_{x_n} : x_i \in X \rangle = \langle \sigma_x : x \in X \rangle \), and

\[
G(X^n, r^{(n)}) \cong \langle \sigma_{x_1} \cdots \sigma_{x_n} : x_i \in X \rangle = \langle \sigma_x : x \in X \rangle = G(X,r).
\]

2. Suppose that \( X \) is finite, and that \( n \) is a positive integer coprime with \( m = |G(X,r)| \). We can find a positive integer \( k \) such that \( nk \equiv 1 \pmod{m} \). Then, \( \sigma_x = \sigma_x^{nk} \) for all \( x \in X \), which implies \( \langle \sigma_{x_1} \cdots \sigma_{x_n} : x_i \in X \rangle = \langle \sigma_x : x \in X \rangle \), and the conclusion follows as in the previous case.

\[
\]

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