On the Q-linear convergence of forward-backward splitting method and uniqueness of optimal solution to Lasso

J.Y. Bello-Cruz∗  G. Li †  T.T.A. Nghia‡

June 19, 2018

Abstract

In this paper, by using tools of second-order variational analysis, we study the popular forward-backward splitting method with Beck-Teboulle’s line-search for solving convex optimization problem where the objective function can be split into the sum of a differentiable function and a possible nonsmooth function. We first establish that this method exhibits global convergence to an optimal solution of the problem (if it exists) without the usual assumption that the gradient of the differentiable function involved is globally Lipschitz continuous. We also obtain the $o(k^{-1})$ complexity for the functional value sequence when this usual assumption is weaken from global Lipschitz continuity to local Lipschitz continuity; improving the existing $O(k^{-1})$ complexity result. We then derive the local and global Q-linear convergence of the method in terms of both the function value sequence and the iterative sequence, under a general metric subregularity assumption which is automatically satisfied for convex piecewise-linear-quadratic optimization problems. In particular, we provide verifiable sufficient conditions for metric subregularity assumptions, and so, local and global Q-linear convergence of the proposed method for broad structured optimization problems arise in machine learning and signal processing including Poisson linear inverse problem, the partly smooth optimization problems, as well as the $\ell_1$-regularized optimization problems. Our results complement the current literature by providing Q-linear convergence result to the forward-backward splitting method under weaker assumptions. Moreover, via this approach, we obtain several full characterizations for the uniqueness of optimal solution to Lasso problem, which covers some recent results in this direction.

Keywords: Iteration complexity; Nonsmooth and convex optimization problems; Forward-backward splitting method; Linear convergence; Uniqueness; Lasso; Metric subregularity; Variational Analysis.

Mathematics Subject Classification (2010): 65K05; 90C25; 90C30.

∗Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115, USA. E-mail: yunier-bello@niu.edu
†Department of Applied Mathematics, University of New South Wales, Sydney 2052, Australia. E-mail: g.li@unsw.edu.au
‡Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, USA. E-mail: nt-tran@oakland.edu
1 Introduction

In this paper we consider the following optimization problem

$$\min_{x \in \mathbb{R}^n} F(x) := f(x) + g(x),$$  \hspace{1cm} (1.1)

where $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ are proper, lower semi-continuous, and convex functions and $f$ is differentiable in its domain. Problems in this format have been appeared in many different fields of science and engineering including machine learning, compressed sensing, and image processing. A particular class of (1.1) known as $\ell_1$-regularized problem

$$\min_{x \in \mathbb{R}^n} F_1(x) := f(x) + \mu \|x\|_1,$$  \hspace{1cm} (1.2)

with constant $\mu > 0$ has been attracted huge attention and widely used in signal processing and statistics to derive sparse optimal solutions. One of the most popular cases of (1.2) is the Lasso problem \cite{38} (also known as $\ell_1$-regularized least square optimization problem) formulated by

$$\min_{x \in \mathbb{R}^n} F_2(x) := \frac{1}{2} \|Ax - b\|^2 + \mu \|x\|_1,$$  \hspace{1cm} (1.3)

where $A$ is an $m \times n$ matrix and $b$ is a vector in $\mathbb{R}^m$.

Among many methods of solving (1.1), the forward-backward splitting method (FBS in brief) \cite{5, 6, 11, 13, 14, 21, 33} is well-known due to its simplicity and efficiency as described below:

$$x^{k+1} = \text{prox}_{\alpha_k g}(x^k - \alpha_k \nabla f(x^k))$$  \hspace{1cm} (1.4)

with the proximal operator defined later in (3.1) and the stepsize $\alpha_k > 0$. The global convergence of FBS to an optimal solution of problem (1.1) and the complexity $O(k^{-1})$ of the functional $F(x^k)$ to the minimum value are usually proved under the assumption that $\nabla f$ is global Lipschitz continuous. By using some line searches motivated by the work of Tseng \cite{41}, Bello-Cruz and Nghia show that FBS indeed converges globally without the aforementioned Lipschitz condition, while the complexity of functional value is improved to $o(k^{-1})$ when the $\nabla f$ is only locally Lipschitz continuous. A recent work of Bauschke-Bolte-Teboulle \cite{4} also tackles the absence of Lipschitz continuous gradient on $f$ by introducing the so-called NoLips algorithm close to FBS with the involvement of Bregman distance. Their algorithm also shares the sublinear complexity $O(\frac{1}{k})$ of the functional sequence $(F(x^k))_{k \in \mathbb{N}}$ under some mild assumptions, and guarantees the global convergence of the solution sequence $(x^k)_{k \in \mathbb{N}}$ with an additional hypothesis on the closedness of the domain of the auxiliary Legendre function defined there. Unfortunately, the latter assumption is not satisfied for the Poisson inverse regularized problems with Kullback-Liebler divergence \cite{12, 43}, one of the main applications in \cite{4}. This situation is overcome in our paper by revisiting FBS with the line search of Beck-Teboulle \cite{11}. Under some minimal assumptions on initial data weaker than those in \cite{8, 37}, we show that the sequence $(x^k)_{k \in \mathbb{N}}$ in (1.4) is globally convergent to an optimal solution (if it exists) without any Lipschitz continuity on the gradient of $f$. Moreover, the sublinear rate $o(\frac{1}{k})$ is obtained when the gradient $\nabla f$ is locally Lipschitz continuous on its domain, which is automatic in the case of Kullback-Liebler divergence.

Our paper mainly devotes to the linear convergence of FBS. Despite of the popularity of FBS, the linear convergence of this method has been established recently throughout some error bound conditions \cite{17, 32, 45} with the base from \cite{27} or Kurdyka-Lojasiewicz inequality \cite{7, 26}. It is worth mentioning that those conditions are somehow equivalent; see, e.g., \cite{7, 17}. Our approach is close to
the recent work of Drusvyatskiy-Lewis [17], Bauschke-Phan-Noll [9], and Zhou-So [45] by using the so-called second-order growth condition and metric subregularity of the subdifferentials [1, 2, 18]; however, our proof of linear convergence is more direct without using the error bound [27] and reveals the $Q$-linear convergence rather than the $R$-one obtained in all the aforementioned works.

Local linear convergence of FBS iterative sequence to solve some structured optimization problems of (1.1) has been recently established in [6, 22, 23, 21] when the function $g$ is partly smooth relative to a manifold $M$ by using the idea of finite support identification. This notion introduced by Lewis [24] allows Liang-Fadili-Peyré [22, 23] to cover in their work many important problems such as the total variation semi-norm, the $\ell_1$-norm (1.2), the $\ell_\infty$-norm, and the nuclear norm problems. In their paper, a second-order condition was introduced to guarantee the $Q$-local linear convergence of FBS sequence generated by (1.4) under the non-degeneracy assumption [24]. When applying our results to this structured setting, we only need a weaker condition. Using the calculus in [25] is extremely helpful in computing the second-order limiting subdifferential [29] of a partly smooth function, but it technically sticks with the non-degeneracy assumption. When considering the $\ell_1$-regularized problem (1.2), we are able to avoid this assumption and introduce a new second-order condition by employing the recent result of Artacho-Geoffroy [3] who initiate a new characterization for the strong metrical subregularity of the subdifferential in term of graphical derivative [16]. This allows us to improve the well-known work of Hale-Yin-Zhang [21] in two aspects: (a) We completely ignore the aforementioned non-degeneracy assumption (b) Our second-order condition is strictly weaker than the one in [21, Theorem 4.10]. Our wider view is that when considering particular optimization problems listed in the spirit of [22, 23], the assumption of non-degeneracy may be not necessary. Furthermore, we revisit the iterative shrinkage thresholding algorithm (ISTA) [11, 15], which is indeed FBS for solving Lasso (1.3). It is well-known that the complexity of this algorithm is $O(k^{-1})$; however, the recent works [22, 42] shows the potential of local linear convergence. The stronger conclusion in this direction is obtained lately by Bolte-Nguyen-Peypouquet-Suter [7] that: ISTA is $R$-linearly convergent, but the rate may depend on the initial point. Inspired by this achievement, we provide two new information: (c) Both functional sequence $(F(x^k))_{k\in\mathbb{N}}$ and iterative sequence $(x^k)_{k\in\mathbb{N}}$ from ISTA are indeed globally $Q$-linearly convergent (d) They are eventually $Q$-linearly convergent to an optimal solution with a uniform rate that does not depend on the initial point. Another application of our work is solving Poisson inverse regularized problem [4, 12, 43] by using FBS. We show the linear convergence of this method in contrast to the sublinear complexity $O(1/k)$ obtained recently in [4, 37] by different methods.

Finally, we study the uniqueness of optimal solution to Lasso problem as one of the main applications from our approach of using second-order variational analysis. This property of optimal solution to (1.3) has been investigated vastly in the literature with immediate implementations to recovering sparse signals in compressed sensing; see, e.g., [19, 39, 40, 41, 40, 47] and the references therein. It is also used in [6, 42] to establish the linear convergence of ISTA. It seems to us that Fuchs [19] initializes this direction by introducing a simple sufficient condition for this property, which has been extended in other cited papers. Then Tibshirani in [39] shows that a sufficient condition closely related to Fuchs’ is also necessary for almost all $b$ in (1.3). The first full characterization for this property has been obtained recently in [46] by using results of strong duality in linear programming. This characterization, which is based on an existence of a vector satisfying a system of linear equations and inequalities, allows [46] to recover the aforementioned sufficient conditions and provide some situations in which these conditions turn necessary. As a direct application of our different approach, we also derive several new full characterizations. Our conditions in terms of positively linear independence and Slater type are well-recognized to be verifiable.

The outline of our paper is as follows. Section 2 briefly presents the relationship between the
metric subregularity of the subdifferential, quadratic growth condition, and Kurdyka-Łojasiewicz inequality. A second-order characterization for quadratic growth condition in term of graphical derivative is also recalled here. This section serves as the main tool for us to obtain the linear convergence of FBS. The reader could find further details about this topic in [12, 28, 18]. In Section 3, we provide the global convergence of FBS without the global Lipschitz condition on the gradient of $f$ and also the general complexity of $o(k^{-1})$. The study in this section is somewhat similar to the recent work [8]. However, we consider a different line search from those in [8] and our standing assumption is much weaker, which allows us to cover broader classes, e.g., the Poison inverse regularized problems studied in Section 5.1. The central part of our paper is Section 4, in which we show the Q-linear convergence of FBS under the metric subregularity of the subdifferential. Section 5 devotes to many applications of our work to structured optimization problems involving Poison inverse regularized, partial smoothness, $\ell_1$-regularized, and $\ell_1$-regularized least square optimization problems. In Section 6, we obtain several new full characterizations to the uniqueness of optimal solution to Lasso problem [1, 30]. The final Section 7 gives the conclusions and some potential future works in this direction.

2 Metric subregularity of the subdifferential and quadratic growth condition

Throughout the paper, $\mathbb{R}^n$ is the usual Euclidean space with dimension $n$ where $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ denote the corresponding Euclidean norm and inner product in $\mathbb{R}^n$. We use $\Gamma_0(\mathbb{R}^n)$ to denote the set of proper, lower semicontinuous, and convex functions on $\mathbb{R}^n$. Let $h \in \Gamma_0(\mathbb{R}^n)$, we write $\text{dom } h := \{ x \in \mathbb{R}^n \mid h(x) < +\infty \}$. The subdifferential of $h$ at $\bar{x} \in \text{dom } h$ is defined by

$$\partial h(\bar{x}) := \{ v \in \mathbb{R}^n \mid \langle v, x - \bar{x} \rangle \leq h(x) - h(\bar{x}), \ x \in \mathbb{R}^n \}. \quad (2.1)$$

Let $G : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a set-valued mapping. We define the domain and the graph of $G$, respectively as following:

$$\text{dom } G := \{ x \in \mathbb{R}^n \mid G(x) \neq \emptyset \} \quad \text{and} \quad \text{gph } G := \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in G(x) \}.$$

One of the key notions used in our paper is the so-called metric subregularity defined as follows; see [16] Section 3H and 3I.

**Definition 2.1** (Metric subregularity and strong metric subregularity). We say $G : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is metrically subregular at $\bar{x} \in \text{dom } G$ for $\bar{y} \in G(\bar{x})$ with modulus $\kappa > 0$ if there exists a neighborhood $U \subset \mathbb{R}^n$ of $\bar{x}$ such that

$$d(x; G^{-1}(\bar{y})) \leq \kappa d(y; G(x)) \quad \text{and} \quad x \in U, \quad (2.2)$$

where $d(x; \Omega)$ is the distance from $x \in \mathbb{R}^n$ to a set $\Omega \subset \mathbb{R}^n$ with the convention that $d(x; \emptyset) = \infty$. Furthermore, we say $G$ is strongly metrically subregular at $\bar{x}$ for $\bar{y} \in G(\bar{x})$ with modulus $\kappa > 0$ if $G$ is metrically subregular at $\bar{x}$ for $\bar{y}$ with modulus $\kappa$ and $\bar{x}$ is an isolated point of $G^{-1}(\bar{y})$.

Metric subregularity is automatic when $G$ is a piecewise polyhedral mapping, i.e., $\text{gph } G$ the union of finitely many convex polyhedral sets; see, e.g., [16] Proposition 3H.1 and [36] Example 9.57, where its roots comes from Robinson [35] and the landmark paper of Hoffman [20].

**Proposition 2.1** (Metric subregularity of polyhedral mappings). Let $G : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a piecewise polyhedral mapping. Then $G$ is metrically subregular at any $\bar{x} \in \text{dom } G$ for any $\bar{y} \in G(\bar{x})$ with a uniform modulus $\kappa > 0$ that does not depend on $(\bar{x}, \bar{y})$. 


Proof. Since $G : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ is piecewise polyhedral, its inverse is also piecewise polyhedral. Combining this with [16] Proposition 3H.1 and Proposition 3H.3 tells us that $G$ is metrically subregular at any $\bar{x}$ for any $\bar{y} \in G(\bar{x})$ with a uniform modulus $\kappa > 0$. \hfill \Box

From the above result, $G$ is strongly metric subregular at $\bar{x}$ for $\bar{y} \in G(\bar{x})$ when $G$ is piecewise polyhedral and $\bar{x}$ is an isolated point of $G^{-1}(\bar{y})$. Without polyhedrality, metric subregularity may be difficult to check; however, strong metric subregularity can be characterized and verified directly via the so-called graphical derivative (known also as contingent derivative); see [16] Section 4A.

**Definition 2.2** (Graphical derivative). Let $(\bar{x}, \bar{y}) \in \text{gph} G$. The graphical derivative of $G$ at $\bar{x}$ for $\bar{y}$ is the mapping $DG(\bar{x}|\bar{y}) : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ such that $v \in DG(\bar{x}|\bar{y})(u)$ if and only if there exist sequences $u^k \to u$, $v^k \to v$ and $t^k \downarrow 0$ such that $\bar{y} + t^k v^k \in G(\bar{x} + t^k u^k)$ for all $k \in \mathbb{N}$.

When $\text{gph} G$ is locally closed around $(\bar{x}, \bar{y}) \in \text{gph} G$, it is known from [16] Theorem 4C.1 that $G$ is strongly metrically subregular at $\bar{x}$ for $\bar{y}$ if and only if

$$DG(\bar{x}|\bar{y})^{-1}(0) = \{0\}. \tag{2.3}$$

Another important stability notion useful in our study is the so-called metric regularity [10] Sections 3G or [29] Definition 1.47.

**Definition 2.3** (Metric regularity). The set-valued mapping $G : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ is metrically regular at $\bar{x}$ for $\bar{y} \in G(\bar{x})$ with modulus $\kappa > 0$ if there exist neighborhoods $U \subset \mathbb{R}^n$ of $\bar{x}$ and $V \subset \mathbb{R}^m$ of $\bar{y}$ such that

$$d(x; G^{-1}(y)) \leq \kappa d(y; G(x)) \quad \text{for all} \quad (x, y) \in U \times V. \tag{2.4}$$

Furthermore, we say $G$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$ with modulus $\kappa > 0$ if there exist neighborhoods $U \subset \mathbb{R}^n$ of $\bar{x}$ and $V \subset \mathbb{R}^m$ of $\bar{y}$ such that (2.4) is satisfied and that the map $V \ni y \mapsto G^{-1}(y) \cap U$ is single-valued.

Metric regularity and strong metric regularity could be characterized fully in [29] Theorem 4.18 and [36] Theorem 9.40. However, we do not use these infinitesimal characterizations in the paper.

In later sections we mainly employ the metric subregular property of the subdifferential mapping $\partial h : \mathbb{R}^n \Rightarrow \mathbb{R}^n$ for $h \in \Gamma_0(\mathbb{R}^n)$ to derive the linear convergence of FBS method. We need the strict connection between metric subregularity of $\partial h$ and the growth condition established recently in [1] [2] [3] [18].

**Proposition 2.2** (Metric subregularity of the subdifferential and growth condition). Let $h \in \Gamma_0(\mathbb{R}^n)$ and $\bar{x}$ be an optimal solution to $h$, i.e., $0 \in \partial h(\bar{x})$. Consider the following assertions:

(i) $\partial h$ is metrically subregular at $\bar{x}$ for $0$ with modulus $\kappa > 0$.

(ii) $h$ satisfies the second-order growth condition at $\bar{x}$ in the sense that: there exist $c, \varepsilon > 0$ such that

$$h(x) \geq h(\bar{x}) + \frac{c}{2} d^2(x; (\partial h)^{-1}(0)) \quad \text{for all} \quad x \in B_\varepsilon(\bar{x}). \tag{2.5}$$

Then implication [(i) $\implies$ (ii)] holds, where $c$ can be chosen as $\kappa^{-1}$ in (2.5). Moreover, the converse implication [(ii) $\implies$ (i)] is also satisfied with $\kappa = \frac{2}{c}$. \hfill \Box

Proof. Applying [1] Theorem 6.2(a)] for convex functions clarifies the implication [(i) $\implies$ (ii)]. The converse implication follows from either [3] Theorem 3.3 or [1] Theorem 6.2(b)].
The equivalence between (i) and (ii) above was first established in [24, Theorem 3.3] in Hilbert spaces without studying the connection of \( \kappa \) and \( c \). The result has been improved and extended to Asplund spaces in [15] even for the case of nonconvex functions with further investigations on the modulus. The recent work [1, Theorem 6.2] has derived the tightest relationship between \( \kappa \) and \( c \) as described in Proposition 2.2. The quadratic growth condition (2.5) is also called \textit{quadratic functional growth} property in [32] when \( h \) is continuously differentiable over a closed convex set.

A similar statement to Proposition 2.2 was also used recently in [17, Theorem 3.3] to prove the linear convergence of the classical forward-backward splitting method. Their quadratic growth condition is slightly different as follows:

\[
    h(x) \geq h(\bar{x}) + \frac{c}{2} d^2(x; (\partial h)^{-1}(0)) \quad \text{for all } x \in [h < h^* + \nu]
\]

for some constants \( c, \nu, \varepsilon > 0 \), where \([ h < h^* + \nu ] = \{ x \in \mathbb{R}^n \mid h(x) < h^* + \nu \} \) and \( h^* := \inf h(x) = h(\bar{x}) \).

It is easy to check that this growth condition implies (2.5). Indeed, suppose that (2.6) is satisfied for some \( c, \nu > 0 \) Define \( \eta := \sqrt{\frac{2\nu}{c}} \) and note that \( \bar{x} \in [h < h^* + \nu] \). Take any \( x \in B_\eta(\bar{x}) \), if \( x \in [h < h^* + \nu] \), inequality (2.5) is trivial. If \( x \notin [h < h^* + \nu] \), it follows that

\[
    h(x) \geq h(\bar{x}) + \nu = h(\bar{x}) + \frac{c}{2} \eta^2 \geq h(\bar{x}) + \frac{c}{2} \| x - \bar{x} \|^2 \geq h(\bar{x}) + \frac{c}{2} d^2(x; (\partial h)^{-1}(0)),
\]

which clearly verifies (2.5). Thus (2.5) is weaker than (2.6), but it is equivalent to the local version of (2.6) described as below:

\[
    h(x) \geq h(\bar{x}) + \frac{c}{2} d^2(x; (\partial h)^{-1}(0)) \quad \text{for all } x \in [h < h^* + \nu] \cap B_\varepsilon(\bar{x})
\]

for some constants \( \kappa, \nu, \varepsilon > 0 \). This property has been showed recently in [7, Theorem 5] to be equivalent to the fact that the convex function \( h \) satisfies the Kurdyka-Lojasiewicz inequality with order \( \frac{1}{2} \) recalled below.

\textbf{Definition 2.4 (Kurdyka-Lojasiewicz inequality with order \( \frac{1}{2} \)).} Let \( h \in \Gamma_0(\mathbb{R}^n) \) and \( \bar{x} \) be an optimal solution to \( h \), i.e., \( 0 \in \partial h(\bar{x}) \). We say \( h \) satisfies the Kurdyka-Lojasiewicz inequality with order \( \frac{1}{2} \) at \( \bar{x} \) if there exist some \( \nu, \varepsilon, c > 0 \) such that

\[
    d(0; \partial h(x)) \geq c |h(x) - h(\bar{x})|^{\frac{1}{2}} \quad \text{for all } x \in [h < h^* + \nu] \cap B_\varepsilon(\bar{x}).
\]

(2.7)

From the above discussion, \( h \) satisfies the Kurdyka-Lojasiewicz inequality with order \( \frac{1}{2} \) at \( \bar{x} \) if and only \( \partial h \) is metrically subregular at \( \bar{x} \) for 0. However, throughout the paper, we mainly use the metric subregularity of \( \partial h \) or the quadratic growth condition to reveal some new information for FBS (1.4) and the uniqueness of optimal solution to Lasso problem (1.3).

There are also similar characterizations for the strong metric subregularity of \( \partial h \) as discussed below, in which the positive-definiteness of \( D\partial h(\bar{x})|0 \) is introduced in [3].

\textbf{Proposition 2.3 (Strong metric subregularity of the subdifferential and growth condition).} Let \( h \in \Gamma_0(\mathbb{R}^n) \) and \( \bar{x} \) be an optimal solution, i.e., \( 0 \in \partial h(\bar{x}) \). Consider the following assertions:

(i) \( \partial h \) is strongly metrically subregular at \( \bar{x} \) for 0 with modulus \( \kappa > 0 \).

(ii) There exist \( c, \eta > 0 \) such that

\[
    h(x) \geq h(\bar{x}) + \frac{c}{2} \| x - \bar{x} \|^2 \quad \text{for all } x \in B_\eta(\bar{x}).
\]

(2.8)
D∂h(\bar{x}|0) is positive-definite with the modulus \( \ell > 0 \) in the sense that
\[
\langle v, u \rangle \geq \ell \|u\|^2 \quad \text{for all} \quad v \in D(\partial h)(\bar{x}|0)(u), u \in \mathbb{R}^n. \tag{2.9}
\]

Then the implication \([i] \implies (ii)\) holds with \( c \) in (2.8) chosen as \( \kappa^{-1} \). If (ii) is satisfied, (iii) is also valid with \( \ell = \frac{\kappa}{2} \). We also have the implication \([iii] \implies (i)\] with \( \kappa > \ell^{-1} \). Moreover, (iii) holds if and only if
\[
\langle v, u \rangle > 0 \quad \text{for all} \quad v \in D(\partial h)(\bar{x}|0)(u), u \in \mathbb{R}^n, u \neq 0. \tag{2.10}
\]

Consequently, if one of (i), (ii), and (iii) is fulfilled then \( \bar{x} \) is a unique optimal solution to \( h \).

Proof. The equivalences of (i), (ii), and (iii) follow from [3, Theorem 3.6 and Corollary 3.7]. Moreover, if (i) is satisfied, then \( c \) in (2.8) can be chosen as \( \kappa^{-1} \) by Proposition 2.2 and Definition 2.1; see also [1, Corollary 6.1]. When (iii) holds, it follows from [3, Corollary 3.7] that (i) is satisfied for any \( \kappa > \ell^{-1} \). To complete the first part of the proposition, we only need to verify that the validity of (2.10) implies the existence of \( \kappa \) in (2.9). Indeed, it follows from (2.10) that \( D(\partial f)(\bar{x}, 0)^{-1}(0) = \{0\} \). Thanks to (2.2), \( \partial h \) is strongly metrically subregular at \( \bar{x} \) for 0, which means (i) and obviously implies (iii).

Now suppose that one of (i), (ii), and (iii) is fulfilled. Thus \( \bar{x} \) is an isolated point of \( \partial h^{-1}(0) \) by Definition 2.1. Note also that \( (\partial h)^{-1}(0) \) is a convex set in \( \mathbb{R}^n \). It follows that \( (\partial h)^{-1}(0) = \{\bar{x}\} \). The proof is complete. \( \square \)

It is clear from the discussion after Definition 2.2, (2.3), and the above result that \( D(\partial h)(\bar{x}|0)^{-1}(0) = \{0\} \) is also equivalent to (2.9). Nevertheless, using (2.9) is somehow more convenient in some applications discussed later in Section 5.

Next let us discuss the metric regularity of the subdifferential with the connection with tilt stability introduced by Poliquin-Rockafellar [34].

**Proposition 2.4** (Metric regularity of the subdifferential and tilt stability). Let \( h \in \Gamma_0(\mathbb{R}^n) \) and \( \bar{x} \) be an optimal solution. The followings are equivalent:

(i) \( \partial h \) is metrically regular at \( \bar{x} \) for 0 with the modulus \( \kappa > 0 \).

(ii) \( \partial h \) is strongly metrically regular at \( \bar{x} \) for 0 with the modulus \( \kappa > 0 \).

(iii) \( \bar{x} \) is a tilt stable local minimizer with modulus \( \kappa > 0 \) to \( h \) in the sense that there exists \( \gamma > 0 \) such that the mapping

\[
M_\gamma : v \mapsto \text{argmin} \{ f(x) - \langle v, x \rangle \mid x \in B_\gamma(\bar{x}) \}
\]

is single-valued and Lipschitz continuous with constant \( \kappa \) on some neighborhood of 0 with \( M_\gamma(0) = \{\bar{x}\} \).

Proof. The equivalence of (i) and (ii) follows from [2 Proposition 3.8] or [16 Theorem 3G.5]. Moreover, the equivalence between (ii) and (iii) is derived from [30, Proposition 4.1]. \( \square \)

To complete this section, we recall a few important notions of linear convergence in our study. A sequence \( (x^k)_{k \in \mathbb{N}} \subset \mathbb{R}^n \) is called to be R-linearly convergent to \( x^* \) with rate \( \mu \in (0,1) \) if there exists \( M > 0 \) such that
\[
\|x^k - x^*\| \leq M \mu^k = \mathcal{O}(\mu^k) \quad \text{for all} \quad k \in \mathbb{N}.
\]
We say \((x^k)_{k \in \mathbb{N}}\) is \emph{Q-linearly convergent} to \(x^*\) with rate \(\mu \in (0, 1)\) if there exists \(K \in \mathbb{N}\) with
\[
\|x^{k+1} - x^*\| \leq \mu \|x^k - x^*\| \quad \text{for all} \quad k \geq K.
\]
Furthermore, \((x^k)_{k \in \mathbb{N}}\) is \emph{globally} Q-linearly convergent to \(x^*\) with rate \(\mu \in (0, 1)\) if
\[
\|x^{k+1} - x^*\| \leq \mu \|x^k - x^*\| \quad \text{for all} \quad k \in \mathbb{N}.
\]
It is obvious that the global Q-linear convergence implies the Q-linear convergence. Moreover, R-linear convergence holds under the validity of local Q-linear convergence with the same rate. On the other hand, R-linear convergence does not imply Q-linear convergence. A simple example is when \(a_k := \|x^k - x^*\|\) satisfies
\[
a_k = \begin{cases} 
\epsilon^k & \text{if} \quad k \text{ is odd}, \\
\epsilon^{2k} & \text{if} \quad k \text{ is even},
\end{cases}
\]
where \(\epsilon \in (0, 1)\). It is clear that \((x^k)_{k \in \mathbb{N}}\) is R-linearly convergent with rate \(\epsilon\); while it is not Q-linearly convergent.

3 Global convergence of forward-backward splitting methods

In this section, we recall the theory for forward-backward splitting methods (FBS) when the gradient \(\nabla f\) is not globally Lipschitz continuous. The results here are somewhat similar to \([8, 37]\) with slight relaxations on the standing assumptions. This section provides some facts used in our Section 4 and 5 to establish the Q-linear convergence of FBS.

Let us start with the standing assumptions on the initial data for problem (1.1) used throughout the paper:

\begin{enumerate}
\item[A1.] \(f, g \in \Gamma_0(\mathbb{R}^n)\) and \(\text{int}(\text{dom } f) \cap \text{dom } g \neq \emptyset\).
\item[A2.] For any \(x \in \text{int}(\text{dom } f) \cap \text{dom } g\), the sublevel set \(\{F \leq F(x)\}\) is contained in \(\text{int}(\text{dom } f) \cap \text{dom } g\) and \(f\) is continuously differentiable at any point in \(\{F \leq F(x)\}\) with \(F(\cdot) = (f + g)(\cdot)\).
\end{enumerate}

Our assumptions are certainly less restrictive than the standard ones broadly used in the theory of FBS \([13, 14, 37, 38]\):

\begin{enumerate}
\item[H1.] \(f, g \in \Gamma_0(\mathbb{R}^n)\) and \(\text{dom } g \subset \text{dom } f\).
\item[H2.] \(f \in C^{1,1}\), i.e., \(\nabla f\) is globally Lipschitz continuous.
\end{enumerate}

It is worth noting further that both assumptions A1 and A2 are valid whenever \(f\) is continuously differentiable on \(\text{int}(\text{dom } f) \cap \text{dom } g\) and \(\text{int}(\text{dom } f) \cap \text{dom } g = (\text{dom } f) \cap \text{dom } g \neq \emptyset\). In the Poisson inverse regularized problem discussed in Section 5.1, the latter condition is trivial, while both H1 and H2 are not satisfied. Our conditions are also strict relaxations of the following ones proposed recently in \([37]\) Section 4:

\begin{enumerate}
\item[H1'] \(f, g \in \Gamma_0(\mathbb{R}^n)\) are bounded from below with \(\text{int } (\text{dom } f) \cap \text{dom } g \neq \emptyset\).
\item[H2'] \(f\) is differentiable on \(\text{int}(\text{dom } f) \cap \text{dom } g\), \(\nabla f\) is uniformly continuous on any compact subset of \(\text{int}(\text{dom } f) \cap \text{dom } g\), and \(\nabla f\) is bounded on any sublevel sets of \(F\).
\end{enumerate}
**Lemma 3.1.** Under our standing assumptions, the following lemma is helpful in our proof of the finite termination of Beck–Teboulle’s line search.

Then

Moreover, \( f, g \) and \( g \) are indeed strictly weaker than \((H1′–H3′)\). To see this, consider \( p \in \{1, 2\} \),

\[
C_p := \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 \geq p, x_1 \geq 0, x_2 \geq 0 \},
\]

and \( f, g : \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{+\infty\} \) be defined as \( f(x) = -\log x_1 - \log x_2 \) if \( x \in \mathbb{R}^2_+ \) and +\( \infty \) otherwise, and \( g(x) = \delta_{C_1}(x) \) where \( \delta_{C_1} \) is the indicator function of the set \( C_1 \) defined above. It can be directly verified that the assumptions \((A1–A2)\) are satisfied. On the other hand, note that \( H1′ \) fails because \( f \) is unbounded on

\[
\text{int}(\text{dom } f) \cap \text{dom } g = \mathbb{R}^2_+ \cap C_1 = C_1 = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 \geq 1, x_1 \geq 0, x_2 \geq 0 \}.
\]

Moreover, \( (2, 1) \in \text{int}(\text{dom } f) \cap \text{dom } g \) and since \( F(2, 1) = (f + g)(2, 1) = -\log 2 \), we get

\[
\{ x \mid F(x) \leq -\log 2 \} = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 \geq 2, x_1 \geq 0, x_2 \geq 0 \} = C_2.
\]

So, clearly \( H2′ \) also fails because \( \nabla f(x_1, x_2) = (\frac{1}{x_1}, \frac{1}{x_2}) \) is unbounded on the above sublevel set of \( F \). Finally, observing that \( (\frac{2}{k}, k) \in C_2 \) and \( (0, k) \in \mathbb{R}^2 \setminus \text{int}(\text{dom } f) = \mathbb{R}^2 \setminus \mathbb{R}^2_+ \), we see that

\[
d(\{F \leq -\log 2\}; \mathbb{R}^2 \setminus \text{int}(\text{dom } f)) = d(C_2; \mathbb{R}^2 \setminus \mathbb{R}^2_+) \leq \lim_{k \rightarrow \infty} \|\frac{2}{k}, k\) - (0, k)\| = \lim_{k \rightarrow \infty} \frac{2}{k} = 0.
\]

This shows that \( H3′ \) fails in this case.

Next let us recall the proximal operator \( \text{prox}_g : \mathbb{R}^n \rightarrow \text{dom } g \) given by

\[
\text{prox}_g(z) := (\text{Id} + \partial g)^{-1}(z) \quad \text{for all } z \in \mathbb{R}^n,
\]

which is well-known to be a single-valued mapping with full domain. With \( \alpha > 0 \), it is easy to check that

\[
\frac{z - \text{prox}_{\alpha g}(z)}{\alpha} \in \partial g(\text{prox}_{\alpha g}(z)) \quad \text{for all } z \in \mathbb{R}^n.
\]

Let \( S^* \) be the optimal solution set to problem \( (1.1) \) and \( x^* \) be an element in \( \text{int}(\text{dom } f) \cap \text{dom } g \). Then \( x^* \in S^* \) if and only if

\[
0 \in \partial (f + g)(x^*) = \nabla f(x^*) + \partial g(x^*).
\]

The following lemma is helpful in our proof of the finite termination of Beck–Teboulle’s line search under our standing assumptions.

**Lemma 3.1.** Let \( g \in \Gamma_0(\mathbb{R}^n) \) and let \( \alpha > 0 \). Then, for every \( x \in \text{dom } g \), we have

\[
\text{prox}_{\alpha g}(x) \rightarrow x \quad \text{as } \alpha \rightarrow 0^+.
\]

**Proof.** Let \( x \in \text{dom } g \) and \( \alpha > 0 \). Define \( z = \text{prox}_{\alpha g}(x) \), we derive from \((3.2)\) that \( \frac{x - z}{\alpha} \in \partial g(z) \), which implies that

\[
g(x) - g(z) \geq \left\langle \frac{x - z}{\alpha}, x - z \right\rangle = \frac{1}{\alpha} \|x - z\|^2.
\]
Since \( g \) is proper, l.s.c. and convex, we have
\[
\infty > g(x) = g^{**}(x) = \sup_{u \in \mathbb{R}^n} \{ \langle u, x \rangle - g^*(u) \},
\]
which is the Fenchel biconjugate of \( g \) at \( x \). Hence, for any \( \varepsilon > 0 \), there exists \( u \in \mathbb{R}^n \) such that
\[
g(x) \leq \langle u, x \rangle - g^*(u) + \varepsilon \leq \langle u, x \rangle - g(z) + g(z) + \varepsilon.
\]
Combining this with (3.5) gives us that
\[
\alpha > g(\alpha g, x) \geq \langle u, x \rangle - \varepsilon \leq \langle u, x \rangle - (z - \alpha g(z)) + \varepsilon.
\]
which easily imply the following expression
\[
\alpha \geq \langle u, x \rangle - \varepsilon \leq \langle u, x \rangle - (z - \alpha g(z)) + \varepsilon,
\]
which implies that
\[
\|x - \text{prox}_{\alpha g}(x)\| = \|x - z\| \leq \frac{\alpha \|u\| + \sqrt{\alpha^2 \|u\|^2 + 4\alpha \varepsilon}}{2}.
\]
Since both \( u \) and \( \varepsilon \) do not depend on \( \alpha \), taking \( \alpha \to 0^+ \) from the latter inequality verifies (3.4). \( \square \)

Under our standing assumptions, we define the proximal forward-backward operator \( J : \text{int}(\text{dom } f) \cap \text{dom } g \times \mathbb{R}^+ \to \text{dom } g \) by
\[
J(x, \alpha) := \text{prox}_{\alpha g}(x - \alpha \nabla f(x)) \quad \text{for all } x \in \text{int}(\text{dom } f) \cap \text{dom } g, \alpha > 0. \tag{3.6}
\]
The following result is essentially from [8, Lemma 2.4]. Since the standing assumptions are different, we provide the proof for completeness.

**Lemma 3.2.** For any \( x \in \text{int}(\text{dom } f) \cap \text{dom } g \), we have
\[
\frac{\alpha_2}{\alpha_1} \|x - J(x, \alpha_1)\| \geq \|x - J(x, \alpha_2)\| \geq \|x - J(x, \alpha_1)\| \quad \text{for all } \alpha_2 \geq \alpha_1 > 0. \tag{3.7}
\]

**Proof.** By using (3.2) and (3.6) with \( z = x - \alpha \nabla f(x) \), we have
\[
\frac{x - \alpha \nabla f(x) - J(x, \alpha)}{\alpha} \in \partial g(J(x, \alpha)) \tag{3.8}
\]
for all \( \alpha > 0 \). For any \( \alpha_2 \geq \alpha_1 > 0 \), it follows from the monotonicity of \( \partial g \) and (3.8) that
\[
0 \leq \left( \frac{x - \alpha_2 \nabla f(x) - J(x, \alpha_2)}{\alpha_2} - \frac{x - \alpha_1 \nabla f(x) - J(x, \alpha_1)}{\alpha_1}, J(x, \alpha_2) - J(x, \alpha_1) \right)
= \left( \frac{x - J(x, \alpha_2)}{\alpha_2} - \frac{x - J(x, \alpha_1)}{\alpha_1}, (x - J(x, \alpha_1)) - (x - J(x, \alpha_2)) \right)
= - \frac{\|x - J(x, \alpha_2)\|^2}{\alpha_2} - \frac{\|x - J(x, \alpha_1)\|^2}{\alpha_1} + \left( \frac{1}{\alpha_2} + \frac{1}{\alpha_1} \right) \langle x - J(x, \alpha_2), x - J(x, \alpha_1) \rangle
\leq - \frac{\|x - J(x, \alpha_2)\|^2}{\alpha_2} - \frac{\|x - J(x, \alpha_1)\|^2}{\alpha_1} + \left( \frac{1}{\alpha_2} + \frac{1}{\alpha_1} \right) \|x - J(x, \alpha_2)\| \cdot \|x - J(x, \alpha_1)\|,
\]
which easily imply the following expression
\[
\left( \|x - J(x, \alpha_2)\| - \|x - J(x, \alpha_1)\| \right) \cdot \left( \|x - J(x, \alpha_2)\| - \frac{\alpha_2}{\alpha_1} \|x - J(x, \alpha_1)\| \right) \leq 0.
\]
Since \( \frac{\alpha_2}{\alpha_1} \geq 1 \), we derive (3.7) and thus complete the proof of the lemma. \( \square \)
Next, let us present Beck-Teboulle’s backtracking line search \[\text{BT}\], which is specifically useful for forward-backward methods when the Lipschitz constant of \(\nabla f\) is not known or hard to estimate.

**Linesearch BT** (Beck–Teboulle’s line search)

Given \(x \in \text{int(dom } f) \cap \text{dom } g, \sigma > 0 \text{ and } \theta \in (0, 1)\).

**Input.** Set \(\alpha = \sigma \text{ and } J(x, \alpha) = \text{prox}_{\alpha g}(x - \alpha \nabla f(x))\) with \(x \in \text{dom } g\).

**While** \(f(J(x, \alpha)) > f(x) + \langle \nabla f(x), J(x, \alpha) - x \rangle + \frac{1}{2\alpha} \|x - J(x, \alpha)\|^2\), do

\[
\alpha = \theta \alpha.
\]

**End While**

**Output.** \(\alpha\).

The output \(\alpha\) in this line search will be denoted by \(LS(x, \sigma, \theta)\). Let us show the well-definedness and finite termination of this line search under the standing assumptions \(\mathbf{A1}\) and \(\mathbf{A2}\) below.

**Proposition 3.3** (Finite termination of Beck–Teboulle’s line search). Suppose that assumptions \(\mathbf{A1}\) and \(\mathbf{A2}\) hold. Then, for any \(x \in \text{int(dom } f) \cap \text{dom } g\), we have

(i) The above line search terminates after finitely many iterations with the positive output \(\bar{\alpha} = LS(x, \sigma, \theta)\).

(ii) \(\|x - u\|^2 - \|J(x, \bar{\alpha}) - u\|^2 \geq 2\bar{\alpha}[F(J(x, \bar{\alpha})) - F(u)]\) for any \(u \in \mathbb{R}^n\).

(iii) \(F(J(x, \bar{\alpha})) - F(x) \leq -\frac{1}{2\bar{\alpha}}\|J(x, \bar{\alpha}) - x\|^2 \leq 0\). Consequently, \(J(x, \bar{\alpha}) \in \text{int(dom } f) \cap \text{dom } g\) and \(f\) is continuously differential at \(J(x, \bar{\alpha})\).

**Proof.** Take any \(x \in \text{int(dom } f) \cap \text{dom } g\). Let us justify (i) first. Note that \(J(x, \alpha)\) is well-defined for any \(\alpha > 0\) because \(f\) is differentiable at \(x\) by assumption \(\mathbf{A2}\). If \(x \in S^*\), where \(S^*\) is the optimal solution set to problem (1.1), then \(x = J(x, \sigma)\) due to (3.3) and (3.2). Thus the line search stops with zero step and gives us the output \(\sigma\) and \(x = J(x, \sigma) \in \text{int(dom } f) \cap \text{dom } g\). If \(x \notin S^*\), suppose by contradiction that the line search does not terminate after finitely many steps. Hence, for all \(\alpha \in \mathcal{P} := \{\sigma, \sigma \theta, \sigma \theta^2, \ldots\}\) it follows that

\[
\langle \nabla f(x), J(x, \alpha) - x \rangle + \frac{1}{2\alpha} \|x - J(x, \alpha)\|^2 < f(J(x, \alpha)) - f(x).
\]

Since \(\text{prox}_{\alpha g}\) is non-expansive, we have

\[
\|J(x, \alpha) - x\| \leq \|\text{prox}_{\alpha g}(x - \alpha \nabla f(x)) - \text{prox}_{\alpha g}(x)\| + \|\text{prox}_{\alpha g}(x) - x\|
\]

\[
\leq \alpha \|\nabla f(x)\| + \|\text{prox}_{\alpha g}(x) - x\|.
\]

Due to the fact that \(\nabla f(x)\) and \(g(x)\) are finite, Lemma 3.1 tells us that

\[
\|J(x, \alpha) - x\| \rightarrow 0 \quad \text{as} \quad \alpha \rightarrow 0^+.
\]

Since \(x \in \text{int(dom } f)\), there exists \(\ell \in \mathbb{N}\) such that \(J(x, \alpha) \in \text{int(dom } f)\) for all \(\alpha \in \mathcal{P'} := \{\sigma \theta^\ell, \sigma \theta^{\ell+1}, \ldots\} \subseteq \mathcal{P}\). Thus \(J(x, \alpha) \in \text{int(dom } f) \cap \text{dom } g\) for all \(\alpha \in \mathcal{P'}\). Thanks to the convexity of \(f\), we have

\[
f(x) - f(J(x, \alpha)) \geq \langle \nabla f(J(x, \alpha)), x - J(x, \alpha) \rangle \quad \text{for} \quad \alpha \in \mathcal{P'}.
\]
This inequality together with (3.9) implies
\[
\frac{1}{2\alpha} \| J(x, \alpha) - x \|^2 < \langle \nabla f(J(x, \alpha)) - \nabla f(x), J(x, \alpha) - x \rangle \\
\leq \| \nabla f(J(x, \alpha)) - \nabla f(x) \| \cdot \| J(x, \alpha) - x \|, 
\]
which yields \( J(x, \alpha) \neq x \) and
\[
0 < \frac{\| J(x, \alpha) - x \|}{\alpha} < 2 \| \nabla f(J(x, \alpha)) - \nabla f(x) \| \text{ for all } \alpha \in \mathcal{P}'. \tag{3.13}
\]
Since \( \| x - J(x, \alpha) \| \to 0 \) as \( \alpha \to 0 \) by (3.11) and \( \nabla f \) is continuous on \( \text{int} (\text{dom } f) \cap \text{dom } g \) by Assumption A2, we obtain from (3.13) that
\[
\lim_{\alpha \to 0, \alpha \in \mathcal{P}'} \frac{\| x - J(x, \alpha) \|}{\alpha} = 0. \tag{3.14}
\]
Applying (3.2) with \( z = x - \alpha \nabla f(x) \) gives us that
\[
\frac{x - J(x, \alpha)}{\alpha} \in \nabla f(x) + \partial g(J(x, \alpha)).
\]
It follows from the convexity of \( g \) that
\[
\left\langle -\nabla f(x), y - J(x, \alpha) \right\rangle \leq g(y) - g(J(x, \alpha)) \text{ for all } y \in \mathbb{R}^n.
\]
Since the function \( g \) is lower semicontinuous, after taking \( \alpha \to 0, \alpha \in \mathcal{P}' \), we have
\[
\langle -\nabla f(x), y - x \rangle \leq g(y) - g(x) \text{ for all } y \in \mathbb{R}^n,
\]
which yields \(-\nabla f(x) \in \partial g(x)\), i.e., \( 0 \in \nabla f(x) + \partial g(x) \). This contradicts the hypothesis that \( x \notin S^* \) by (3.3). Hence, the line search terminates after finitely many steps with the output \( \bar{\alpha} \).

To proceed the proof of (ii), note that
\[
f(J(x, \bar{\alpha})) \leq f(x) + \langle \nabla f(x), J(x, \bar{\alpha}) - x \rangle + \frac{1}{2\bar{\alpha}} \| x - J(x, \bar{\alpha}) \|^2. \tag{3.15}
\]
Moreover, by (3.2), we have
\[
\frac{x - J(x, \bar{\alpha})}{\bar{\alpha}} - \nabla f(x) \in \partial g(J(x, \bar{\alpha})) = \partial g(J(x, \bar{\alpha})).
\]
Pick any \( u \in \mathbb{R}^n \), we get from the later that
\[
g(u) - g(J(x, \bar{\alpha})) \geq \left\langle \frac{x - J(x, \bar{\alpha})}{\bar{\alpha}} - \nabla f(x), u - J(x, \bar{\alpha}) \right\rangle. \tag{3.16}
\]
Observe further that
\[
f(u) - f(x) \geq \langle \nabla f(x), u - x \rangle. \tag{3.17}
\]
Adding (3.16) and (3.17) and using (3.15) give us that
\[
F(u) = (f + g)(u) \geq f(x) + g(J(x, \bar{\alpha})) + \left( \frac{x - J(x, \bar{\alpha})}{\bar{\alpha}} - \nabla f(x), u - J(x, \bar{\alpha}) \right) \tag{3.18}
\]
\[
= f(x) + g(J(x, \bar{\alpha})) + \frac{1}{\bar{\alpha}} \langle x - J(x, \bar{\alpha}), u - J(x, \bar{\alpha}) \rangle + \langle \nabla f(x), J(x, \bar{\alpha}) - x \rangle \\
\geq f(J(x, \bar{\alpha})) + g(J(x, \bar{\alpha})) + \frac{1}{\bar{\alpha}} \langle x - J(x, \bar{\alpha}), u - J(x, \bar{\alpha}) \rangle - \frac{1}{2\bar{\alpha}} \| J(x, \bar{\alpha}) - x \|^2.
\]
Hence, we have
\[
\langle x - J(x, \bar{a}), J(x, \bar{a}) - u \rangle \geq \bar{a}[F(J(x, \bar{a})) - F(x)] - \frac{1}{2}\|J(x, \bar{a}) - x\|^2.
\]
Since \(2(x - J(x, \bar{a}), J(x, \bar{a}) - u) = \|x - u\|^2 - \|J(x, \bar{a}) - x\|^2 - \|J(x, \bar{a}) - u\|^2\), the latter implies that
\[
\|x - u\|^2 - \|J(x, \bar{a}) - u\|^2 \geq 2\bar{a}[F(J(x, \bar{a})) - F(x)],
\]
which clearly ensures (ii).

Finally, (iii) is a direct consequence of (i) with \(x = u\). It follows that \(J(x, \bar{a})\) belongs to the sublevel set \(\{F \leq F(x)\}\). By \(A2\), \(J(x, \bar{a}) \in \text{int}(\text{dom } f) \cap \text{dom } g\) and \(f\) is continuously differential at \(J(x, \bar{a})\). The proof is complete. \(\square\)

Now we recall the forward-backward splitting method with line search proposed by \([11]\) as following.

**Forward-backward splitting method with backtracking line search (FBS method)**

**Step 0.** Take \(x^0 \in \text{int}(\text{dom } f) \cap \text{dom } g\), \(\sigma > 0\) and \(\theta \in (0, 1)\).

**Step k.** Set
\[
x^{k+1} := J(x^k, \alpha_k) = \text{prox}_{\alpha_k g}(x^k - \alpha_k \nabla f(x^k))
\]
with \(\alpha_{-1} := \sigma\) and
\[
\alpha_k := \text{LS}(x^k, \alpha_{k-1}, \theta).
\]

The following result which is a direct consequence of Proposition boundary-well plays the central role in our further study.

**Corollary 3.4 (Well-definedness of FBS method).** Let \(x^0 \in \text{int}(\text{dom } f) \cap \text{dom } g\). The sequence \((x^k)_{k \in \mathbb{N}}\) from FBS method is well-defined and \(f\) is differentiable at any \(x^k\). Moreover, for all \(k \in \mathbb{N}\) and \(x \in \mathbb{R}^n\), we have

(i) \(\|x^k - x\|^2 - \|x^{k+1} - x\|^2 \geq 2\alpha_k [F(x^{k+1}) - F(x)].\)

(ii) \(F(x^{k+1}) - F(x^k) \leq -\frac{1}{2\alpha_k} \|x^{k+1} - x^k\|^2.\)

**Proof.** Thanks to Proposition 3.3, \(x^k \in \text{int}(\text{dom } f) \cap \text{dom } g\) and \(f\) is differentiable at any \(x^k\) inductively. This verifies the well-definedness of \((x^k)_{k \in \mathbb{N}}\). Moreover, both (i) and (ii) are consequence of (ii) and (iii) from Proposition 3.3 by replacing \(u = x, x = x^k, \bar{a} = \alpha_k\), and \(J(x, \bar{a}) = J(x^k, \alpha_k) = x^{k+1}\). \(\square\)

The following result shows that the FBS method with backtracking line search is global convergent without assuming Lipschitz continuity on the gradient \(\nabla f\), and so, improves \([11]\) Theorem 1.2. A variant of this result for FBS method under different line searches was established in \([8\) Theorem 4.2\]. Here, the proof is also similar to that of \([8\) Theorem 4.2\] by using the well-definedness of \((x^k)_{k \in \mathbb{N}}\) and two properties in Corollary 3.4 and hence, we omit the details.
Theorem 3.5 (Global convergence of FBS method). Let \((x^k)_{k \in \mathbb{N}}\) be the sequence generated from FBS method. The following statements hold:

(i) If \(S^* \neq \emptyset\) then \((x^k)_{k \in \mathbb{N}}\) converges to a point in \(S^*\). Moreover,
\[
\lim_{k \to \infty} F(x^k) = \min_{x \in \mathbb{R}^n} F(x).
\] (3.20)

(ii) If \(S^* = \emptyset\) then we have
\[
\lim_{k \to \infty} \|x^k\| = +\infty \quad \text{and} \quad \lim_{k \to \infty} F(x^k) = \inf_{x \in \mathbb{R}^n} F(x).
\]

Proposition 3.6 (Boundedness from below for the step sizes). Let \((x^k)_{k \in \mathbb{N}}\) and \((\alpha_k)_{k \in \mathbb{N}}\) be the sequences generated from FBS method. Suppose that \(S^* \neq \emptyset\) and that the sequence \((x^k)_{k \in \mathbb{N}}\) is converging to some \(x^* \in S^*\). If \(\nabla f\) is locally Lipschitz continuous around \(x^*\) with modulus \(L\) then there exists some \(K \in \mathbb{N}\) such that
\[
\alpha_k \geq \min \left\{ \alpha_K, \frac{\theta}{L} \right\} > 0 \quad \text{for all} \quad k > K.
\] (3.21)

Furthermore, if \(\nabla f\) is globally Lipschitz continuous on \(\mathbf{int}(\text{dom } f) \cap \text{dom } g\) with uniform modulus \(L\) then \(\alpha_k \geq \min \{\sigma, \frac{\theta}{L}\} \) for any \(k \in \mathbb{N}\).

Proof. To justify, suppose that \(S^* \neq \emptyset\), the sequence \((x^k)_{k \in \mathbb{N}}\) is converging to \(x^* \in S^*\), and that \(\nabla f\) is locally Lipschitz continuous around \(x^*\) with constant \(L > 0\). We find some \(\varepsilon > 0\) such that
\[
\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \quad \text{for all} \quad x, y \in \mathbb{B}_\varepsilon(x^*),
\] (3.22)
where \(\mathbb{B}_\varepsilon(x^*)\) is the closed ball in \(\mathbb{R}^n\) with center \(x^*\) and radius \(\varepsilon\). Since \((x^k)_{k \in \mathbb{N}}\) is converging to \(x^*\), there exists some \(K \in \mathbb{N}\) such that
\[
\|x^k - x^*\| \leq \frac{\theta \varepsilon}{2 + \theta} < \varepsilon \quad \text{for all} \quad k > K
\] (3.23)
with \(\theta \in (0, 1)\) defined in Linesearch BT. We claim that
\[
\alpha_k \geq \min \left\{ \alpha_{k-1}, \frac{\theta}{L} \right\} \quad \text{for any} \quad k > K.
\] (3.24)

Suppose by contradiction that \(\alpha_k < \min \{\alpha_{k-1}, \frac{\theta}{L}\}\). Then, \(\alpha_k < \alpha_{k-1}\), and so, the loop in Linesearch BT at \((x^k, \alpha_{k-1})\) needs more than one iteration. Define \(\hat{\alpha}_k := \frac{\alpha_k}{\theta} > 0\) and \(\hat{x}^k := J(x^k, \hat{\alpha}_k)\), Linesearch BT tells us that
\[
f(\hat{x}^k) > f(x^k) + \langle \nabla f(x^k), \hat{x}^k - x^k \rangle + \frac{1}{2\hat{\alpha}_k}\|x^k - \hat{x}^k\|^2.
\] (3.25)

Furthermore, it follows from Lemma 3.2 that
\[
\|x^k - \hat{x}^k\| = \|x^k - J(x^k, \hat{\alpha}_k)\| \leq \frac{\hat{\alpha}_k}{\alpha_k}\|x^k - J(x^k, \alpha_k)\| = \frac{1}{\theta}\|x^k - x^{k+1}\|,
\]
This together with (3.23) yields
\[
\|\hat{x}^k - x^*\| \leq \|\hat{x}^k - x^k\| + \|x^k - x^*\| \leq \frac{1}{\theta} \|x^k - x^{k+1}\| + \|x^k - x^*\| 
\leq \frac{1}{\theta} \frac{2\theta \epsilon}{2 + \theta} + \frac{\theta \epsilon}{2 + \theta} = \epsilon.
\]
(3.26)

Since $$x^k, \hat{x}^k \in B_\epsilon(x^*)$$ by (3.23) and (3.20), we get from (3.22) that
\[
f(\hat{x}^k) - f(x^k) - \langle \nabla f(x^k), \hat{x}^k - x^k \rangle = \int_0^1 \langle \nabla f(x^k + t(\hat{x}^k - x^k)) - \nabla f(x^k), \hat{x}^k - x^k \rangle dt 
\leq \int_0^1 t L \|\hat{x}^k - x^k\|^2 dt = \frac{L}{2} \|\hat{x}^k - x^k\|^2.
\]
Combining this with (3.25) yields $$\hat{\alpha}_k \geq \frac{1}{L}$$ and thus $$\alpha_k \geq \frac{\eta}{L}$$. This is a contradiction.

If there is some $$H > K$$ with $$H \in \mathbb{N}$$ such that $$\alpha_H > \frac{\eta}{L}$$, we get from (3.24) that $$\alpha_k \geq \frac{\eta}{L}$$ for all $$k \geq H$$. Otherwise, $$\alpha_k < \frac{\eta}{L}$$ for any $$k > K$$, which implies that $$\alpha_k = \alpha_{k-1} = \alpha_K$$ for all $$k > K$$ due to (3.24) and the nonincreasing property of $$(\alpha_k)_{k \in \mathbb{N}}$$. In both cases we have (3.21).

Finally suppose that $$\nabla f$$ is globally Lipschitz continuous with modulus $$L$$ on $$\text{int}(\text{dom } f) \cap \text{dom } g \subset \text{int}(\text{dom } f)$$. By using Proposition 3.3(iii), we can repeat the above proof without concerning $$\epsilon, K$$ and replace (3.24) by $$\alpha_k \geq \min\{\sigma, \frac{\eta}{L}\}$$. □

The following result showing the complexity $$o(k^{-1})$$ of FBS method when the function $$\nabla f$$ is locally Lipschitz continuous, improves [11] Theorem 1.1, which only obtains $$O(k^{-1})$$ of this method with the stronger assumption that $$\nabla f$$ is globally Lipschitz continuous. The proof of this theorem is quite similar to [8] Theorem 4.3 and Corollary 4.5 and so, we omit the details.

**Theorem 3.7** (Sublinear convergence of FBS method). Let $$(x^k)_{k \in \mathbb{N}}$$ be the sequence generated in FBS method. Suppose that $$S^* \neq \emptyset$$ and that $$\nabla f$$ is locally Lipschitz continuous around any point in $$S^*$$. Then we have
\[
\lim_{k \to \infty} k \left[ F(x^k) - \min_{x \in \mathbb{R}^n} F(x) \right] = 0.
\]
(3.27)

## 4 Local linear convergence of forward-backward splitting methods

In this section, we obtain the local Q-linear convergence for FBS method under a mild assumption of metric subregularity on $$\partial F$$ and local Lipschitz continuity of $$\nabla f$$, which is automatic in many problems including Lasso problem and Poisson linear inverse regularized problem. R-linear convergence of FBS method has been recently established under some different assumptions such as Kurdyka-Lojasiewicz inequality with order $$\frac{1}{2}$$ [7], and the quadratic growth condition [17], all of which are equivalent in the convex case; see [17] Corollary 3.6 and [7] Theorem 5]. Our results are close to [17] Theorem 3.2 and Corollary 3.7. However, we focus on the local linear convergence; our proof also suggests a direct way to obtain linear convergence of FBS from the quadratic growth condition (4.4) below without going through the error bound [17] Definition 3.1]. The first result is regarding the R-linear convergence of FBS method that will be improved later by Q-linear convergence in Theorem 1.2.

**Proposition 4.1** (R-linear convergence under metric subregularity). Let $$(x^k)_{k \in \mathbb{N}}$$ and $$(\alpha_k)_{k \in \mathbb{N}}$$ be the sequences generated from FBS method. Suppose that $$S^*$$ is not empty, $$(x^k)_{k \in \mathbb{N}}$$ converges
to some \( x^* \in S^* \) as in Theorem 3.6 and that \( \nabla f \) is locally Lipschitz continuous around \( x^* \) with constant \( L > 0 \). If \( \partial F = \nabla f + \partial g \) is metrically subregular at \( x^* \) for \( 0 \) with modulus \( \kappa^{-1} > 0 \), then there exists some \( K \in \mathbb{N} \) such that

\[
\begin{align*}
d(x^{k+1}; S^*) \leq \frac{1}{\sqrt{1 + \alpha k}} d(x^k; S^*) & \quad \text{for all } k > K, \\
\end{align*}
\]

where \( \alpha := \min \left\{ \frac{\alpha_k}{2}, \frac{\theta}{4} \right\} \). Consequently, we have

\[
\begin{align*}
F(x^{k+1}) - \min_{x \in \mathbb{R}^n} F(x) &= \mathcal{O}((1 + \alpha k)^{-k}), \\
\|x^{k+1} - x^*\| &= \mathcal{O}((1 + \alpha k)^{-\frac{k}{2}}).
\end{align*}
\]

If, in addition, \( \nabla f \) is globally Lipschitz continuous on \( \text{int}(\text{dom} \, f) \cap \text{dom} \, g \) with constant \( L, \alpha \) could be chosen as \( \min \left\{ \frac{\delta}{2}, \frac{\theta}{2} \right\} \), which is independent from \( K \).

Proof. When \( \partial F \) is metrically subregular at \( x^* \) for \( 0 \) with modulus \( \kappa^{-1} > 0 \), it follows from Proposition 2.2 that there exists \( \varepsilon > 0 \) such that

\[
F(x) - F(x^*) \geq \frac{K}{2} d^2(x; S^*) \quad \text{for all } x \in B_\varepsilon(x^*).
\]

Since \( (x^k)_{k \in \mathbb{N}} \) converges to \( x^* \) and \( \nabla f \) is locally Lipschitz continuous around \( x^* \), we find from Proposition 3.6 some constant \( K \in \mathbb{N} \) such that \( \alpha_k \geq 2\alpha \) and \( x^k \in B_\varepsilon(x^*) \) for any \( k > K \). Denote the projection of \( a \) onto the set \( S^* \) by \( \Pi_{S^*}(a) \). Combining (4.4) with Corollary 3.4(i) implies that

\[
\begin{align*}
d^2(x^k; S^*) - d^2(x^{k+1}; S^*) & \geq \|x^k - \Pi_{S^*}(x^k)\|^2 - \|x^{k+1} - \Pi_{S^*}(x^k)\|^2 \\
& \geq \alpha_k [F(x^{k+1}) - F(\Pi_{S^*}(x^k))] \\
& \geq 2\alpha [F(x^{k+1}) - F(x^*)] \geq \alpha \kappa d^2(x^{k+1}; S^*)
\end{align*}
\]

for all \( k > K \). This clearly verifies (4.1).

To justify (4.2), note from (4.1) that \( d(x^k; S^*) = \mathcal{O}((1 + \alpha k)^{-\frac{k}{2}}) \). This together with (4.5) allows us to find some \( M > 0 \) such that

\[
0 \leq F(x^{k+1}) - F(x^*) \leq \frac{1}{2\alpha} d^2(x^k; S^*) \leq M(1 + \alpha k)^{-k} \quad \text{for all } k \in \mathbb{N},
\]

which clearly ensures (4.2). To verify (4.3), we derive from Corollary 3.4(ii) that

\[
\|x^k - x^{k+1}\| \leq \sqrt{2\alpha_k [F(x^k) - F(x^{k+1})]} \leq \sqrt{2\sigma [F(x^k) - F(x^*)]} \leq \sqrt{2\sigma M(1 + \alpha k)^{-\frac{k}{2}}},
\]

Since \( (x^k)_{k \in \mathbb{N}} \) converges to \( x^* \), it follows from the latter inequality that

\[
\|x^{k+1} - x^*\| = \sum_{j=k+1}^\infty (\|x^j - x^*\| - \|x^{j+1} - x^*\|) \leq \sum_{j=k+1}^\infty \|x^j - x^{j+1}\| \\
\leq \sqrt{2\sigma M(1 + \alpha k)^{-\frac{k}{2}}} \sum_{j=0}^\infty (1 + \alpha k)^{-\frac{k}{2}} = \sqrt{2\sigma M(1 + \alpha k)^{-\frac{k}{2}}} \sum_{j=0}^\infty (1 - (1 + \alpha k)^{-\frac{k}{2}})^{-1},
\]

which verifies (4.3). To complete, we repeat the above proof with the note from Proposition 3.6 that \( \alpha_k \geq \min \{\sigma, \frac{\theta}{4} \} \) when \( \nabla f \) is globally Lipschitz continuous on \( \text{int}(\text{dom} \, f) \cap \text{dom} \, g \) with constant \( L \). \( \square \)
In the special case where \( g(x) = \delta_X(x) \), the indicator function to a closed convex set \( X \subset \mathbb{R}^n \), the obtained linear convergence of \( (d(x^k; S^*))_{k \in \mathbb{N}} \) in (4.1) is close to the [32] Theorem 12).

Next, we present the promised Q-linear convergence of the FBS method for both the objective value sequence \( (F(x^k))_{k \in \mathbb{N}} \) and the iterative sequence \( (x^k)_{k \in \mathbb{N}} \), under a general metric subregularity assumption. Easily verifiable sufficient conditions for this metric subregularity assumption will be provided in Corollary 4.4 and Section 5 later. We also point out that Q-linear convergence on the assumption. Easily verifiable sufficient conditions for this metric subregularity assumption will be provided in Corollary 4.4 and Section 5 later. We also point out that Q-linear convergence on the assumption.

**Theorem 4.2** (Q-linear convergence under metric subregularity). Let \( (x^k)_{k \in \mathbb{N}} \) and \( (\alpha_k)_{k \in \mathbb{N}} \) be the sequences generated from FBS method. Suppose that the solution set \( S^* \) is not empty, \( (x^k)_{k \in \mathbb{N}} \) converges to some \( x^* \in S \), and that \( \nabla f \) is locally Lipschitz continuous around \( x^* \) with constant \( L > 0 \). If \( \partial F = \nabla f + \partial g \) is metrically subregular at \( x^* \) for 0 with modulus \( \kappa^{-1} > 0 \), there exists \( K \in \mathbb{N} \) such that

\[
\|x^{k+1} - x^*\| \leq \frac{1}{\sqrt{1 + \frac{\alpha}{4\kappa}}} \|x^k - x^*\| \tag{4.6}
\]

\[
|F(x^{k+1}) - F(x^*)| \leq \frac{1}{2\sqrt{1 + \frac{\alpha}{4\kappa}}} |F(x^k) - F(x^*)| \tag{4.7}
\]

for any \( k > K \), where \( \alpha := \min \{ \frac{\alpha_k}{2}, \frac{\theta}{2\mathcal{L}} \} \).

If, in addition, \( \nabla f \) is globally Lipschitz continuous on \( \text{int}(\text{dom } f) \cap \text{dom } g \) with constant \( L > 0 \), \( \alpha \) could be chosen as \( \min \{ \frac{\alpha}{2\mathcal{L}}, \frac{\theta}{2\mathcal{L}} \} \).

**Proof.** Since \( \partial F = \nabla f + \partial g \) is metrically subregular at \( x^* \) for 0 with the modulus \( \kappa^{-1} > 0 \), we also have (4.1). This together with Corollary 3.4(i) gives us that

\[
\|x^k - x\|^2 - \|x^{k+1} - x\|^2 \geq 2\alpha [F(x^{k+1}) - F(x^*)] \geq \alpha \kappa d^2(x^k; S^*) \quad \text{for all } x \in S^*, \tag{4.8}
\]

when \( k > K \) for some large \( K \in \mathbb{N} \). Moreover, for any \( r > k > K \) we get that

\[
\|x^r - \Pi_{S^*}(x^k+1)\| \leq \|x^{k+1} - \Pi_{S^*}(x^{k+1})\| = d(x^{k+1}; S^*).
\]

Taking \( r \to \infty \) gives us that \( \|x^* - \Pi_{S^*}(x^{k+1})\| \leq d(x^{k+1}; S^*) \). It follows that

\[
\|x^{k+1} - x^*\| \leq \|x^{k+1} - \Pi_{S^*}(x^{k+1})\| + \|\Pi_{S^*}(x^{k+1}) - x^*\| \leq 2d(x^{k+1}; S^*).
\]

This together with (4.8) implies that

\[
\|x^k - x^*\|^2 \geq \|x^{k+1} - x^*\|^2 + \frac{\alpha_k}{4}\|x^{k+1} - x^*\|^2 = \left(1 + \frac{\alpha_k}{4}\right)\|x^{k+1} - x^*\|^2,
\]

which clearly verifies (4.6).

To see the second conclusion, we note from (4.6) that

\[
\|x^{k+1} - x^k\| \geq \|x^k - x^*\| - \|x^{k+1} - x^*\| \geq \beta(\|x^{k+1} - x^*\| + \|x^k - x^*\|) \tag{4.9}
\]

\[1\text{Proposition 4.1 was presented by the third author at ICCOPT 2016, when he was aware of [32] after attending the talk of I. Necoara.}\]
with \( \beta := \sqrt{1+\alpha\kappa/4} \) for \( k > K \) sufficiently large. We derive from this, Corollary 3.4(ii), and (4.9) that

\[
F(x^k) - F(x^{k+1}) \geq \frac{1}{2\alpha_k} ||x^{k+1} - x^k||^2 \geq \frac{1}{2\alpha_k} (||x^k - x^*|| - ||x^{k+1} - x^*||)^2 \\
\geq \frac{\beta}{2\alpha_k} (||x^k - x^*|| - ||x^{k+1} - x^*||)(||x^k - x^*|| + ||x^{k+1} - x^*||) \\
\geq \frac{\beta}{2\alpha_k}(||x^k - x^*||^2 - ||x^{k+1} - x^*||^2).
\]

Hence, we get from Corollary 3.4(i) that

\[
F(x^k) - F(x^{k+1}) \geq \beta[F(x^{k+1}) - F(x^*)].
\]

It follows that \( F(x^k) - F(x^*) \geq (1 + \beta)[F(x^{k+1}) - F(x^*)] \), which clarifies (4.7).

The last statement can be obtained similarly to the preceding proposition.

Next, we show that a sharper \( Q \)-linear convergence rate of \( (x^k)_{k \in \mathbb{N}} \) and \( (F(x^k))_{k \in \mathbb{N}} \) can be obtained under a stronger assumption: strong metric subregularity.

**Corollary 4.3** (Sharper \( Q \)-linear convergence rate under strong metric subregularity). Let \( (x^k)_{k \in \mathbb{N}} \) and \( (\alpha_k)_{k \in \mathbb{N}} \) be the sequences generated from FBS method. Suppose that the solution set \( S^* \) is not empty, \( (x^k)_{k \in \mathbb{N}} \) converges to some \( x^* \in S^* \), and that \( \nabla f \) is locally Lipschitz continuous around \( x^* \) with constant \( L > 0 \). If \( \partial F = \nabla f + \partial g \) is strongly metrically subregular at \( x^* \) for \( 0 \) with modulus \( \kappa^{-1} > 0 \), then \( x^* \) is the unique solution to problem (1.1). Moreover, there exists some \( K \in \mathbb{N} \) such that for any \( k > K \) we have

\[
||x^{k+1} - x^*|| \leq \frac{1}{\sqrt{1+\alpha\kappa}} ||x^k - x^*|| \tag{4.10}
\]

\[
|F(x^{k+1}) - F(x^*)| \leq \frac{\sqrt{1+\alpha\kappa} + 1}{2\sqrt{1+\alpha\kappa}} |F(x^k) - F(x^*)| \tag{4.11}
\]

with \( \alpha := \min \{ \frac{\alpha_k}{\alpha^2}, \frac{\theta_k}{2\alpha} \} \).

Additionally, \( \nabla f \) is globally Lipschitz continuous on \( \text{int}(\text{dom} f) \cap \text{dom} g \) with constant \( L > 0 \), \( \alpha \) above could be chosen as \( \min \{ \frac{\alpha_k}{\alpha^2}, \frac{\theta_k}{2\alpha} \} \).

**Proof.** If \( \partial F = \nabla f + \partial g \) is strongly metrically subregular at \( x^* \) for \( 0 \) with modulus \( \kappa^{-1} > 0 \), \( x^* \) is an isolated point of \( \partial F^{-1}(0) = S^* \). Since \( S^* \) is a closed convex set, we have \( S^* = \{ x^* \} \). Thus (4.10) is a direct consequence of (4.11). To verify (4.11), we note from (4.10) that

\[
||x^{k+1} - x^k|| \geq ||x^k - x^*|| - ||x^{k+1} - x^*|| \geq \beta(||x^{k+1} - x^*|| + ||x^k - x^*||) \tag{4.12}
\]

with \( \beta = \frac{\sqrt{1+\alpha\kappa} + 1}{2\sqrt{1+\alpha\kappa}} \) for any \( k > K \) sufficiently large. The proof of \( Q \)-linear convergence of \( (F(x^k))_{k \in \mathbb{N}} \) in (4.11) can be obtained similarly as in Theorem 4.2 by using (4.12) instead of (4.9). 

The assumption that \( \partial F \) is metrically subregular in above results is automatic for a broad class of so-called **piecewise linear-quadratic functions** [36] Definition 10.20] defined below.
Definition 4.1 (convex piecewise linear-quadratic functions). A function \( h \in \Gamma_0(\mathbb{R}^n) \) is called convex piecewise linear-quadratic if \( \text{dom} \, h \) is a union of finitely many polyhedral sets, relative to each of which \( h(x) \) is given the expression of the form \( \frac{1}{2} \langle x, Ax \rangle + \langle b, x \rangle + c \) for some scalar \( c \in \mathbb{R} \), vector \( b \in \mathbb{R}^n \) and a symmetric positive semi-definite \( A \in \mathbb{R}^{n \times n} \).

If \( F = f + g \) is convex piecewise linear-quadratic function, it is known from [36, Proposition 12.30] that the set-valued mapping \( \partial F \) is polyhedral and thus is metrically subregular at any point \( \bar{x} \in \text{dom} \, \partial F \) for any \( \bar{v} \in \partial F(\bar{x}) \) by Proposition 2.1. This observation together with Theorem 4.1 tells us the local \( R \)-linear convergence of FBS for convex piecewise linear-quadratic functions. This fact has been obtained and discussed before in [17, 26, 45]. Our following result advances it with the \( Q \)-linear convergence and the uniform convergence rate.

Corollary 4.4 (Local linear convergence for piecewise linear-quadratic functions). Let \( (x^k)_{k \in \mathbb{N}} \) and \( (\alpha_k)_{k \in \mathbb{N}} \) be the sequences generated from FBS method. Suppose that \( F = f + g \) is a convex piecewise linear-quadratic function, the solution set \( S^* \) is nonempty, and that \( \nabla f \) is locally Lipschitz continuous around any point in \( S^* \). Then the sequences \( (x^k)_{k \in \mathbb{N}} \) and \( (F(x^k))_{k \in \mathbb{N}} \) are globally convergent to some optimal solution and optimal value respectively with local \( Q \)-linear rates.

Furthermore, if \( \nabla f \) is globally Lipschitz continuous on \( \text{int} \,(\text{dom} \, f) \cap \text{dom} \, g \), \( (x^k)_{k \in \mathbb{N}} \) and \( (F(x^k))_{k \in \mathbb{N}} \) are globally convergent to some optimal solution and optimal value, respectively, with uniform local linear rates that do not depend on the choice of the initial point \( x^0 \).

Proof. Suppose that the sequence \( (x^k)_{k \in \mathbb{N}} \) converges to some \( x^* \in S^* \) by Theorem 3.5. Since \( F \) is a convex piecewise linear-quadratic function, the graph of \( \partial F \) is polyhedral and thus it is metrically subregular at \( x^* \) for 0 with a uniform rate \( \kappa^{-1} > 0 \), which does not depend on the choice of \( (x^k)_{k \in \mathbb{N}} \) and \( x^* \) by Proposition 2.1. By Theorem 4.2 we have \( (x^k)_{k \in \mathbb{N}} \) and \( (F(x^k))_{k \in \mathbb{N}} \) are locally convergent to some \( x^* \) and the optimal value \( F(x^*) \), respectively, with \( Q \)-linear rate.

To complete the proof, suppose that \( \nabla f \) is globally Lipschitz continuous on \( \text{int} \,(\text{dom} \, f) \cap \text{dom} \, g \) with constant \( L \). It follows from the last part of Theorem 4.1 that \( \alpha \) could be chosen as \( \min \{\sigma, \frac{\theta}{2L} \} \). Since the metric subregularity modulus of \( \partial F \) is uniform as discussed above, the linear rate in Theorem 4.1 is independent from the choice of initial points.

Remark 4.1. It is worth noting that all the assumptions in Corollary 4.4 on initial data hold automatically in many important classes optimization problems in practice including the Tikhonov regularization, wavelet-based regularization, \( \ell_1 \) regularization, \( \ell_\infty \) regularization least square problems; see further discussions about using FBS method in these problems in [11, 21, 22, 32]. Let us discuss a bit here about Lasso problem (1.3). It is easy to see that \( F_2 \) in (1.3) is a convex piecewise linear-quadratic function. Moreover, the function \( f(x) = \frac{1}{2} \| Ax - b \|_2^2 \) has the gradient \( \nabla f(x) = A^T (Ax - b) \) that is globally Lipschitz continuous on \( \mathbb{R}^n \). FBS method for problem (1.3) is also called iterative shrinkage thresholding algorithm (ISTA) [11] via the shrinkage thresholding mapping \( \text{prox}_{\| \cdot \|_1} \). Recently, Tao-Boley-Zhang [42, Theorem 5.9] shows that the ISTA iteration eventually linearly convergent provided that (1.3) has a unique solution that satisfies a strict complementarity condition. Our Corollary 4.4 tells that not only ISTA iteration but also their functional iteration eventually reach the stage of linear convergence without adding any extra condition. Moreover, the linear rate is uniform and computable; see our Section 6.3 for computing this rate and also the global linear convergence of ISTA.
5 Linear convergence of forward-backward splitting method in some structured optimization problems

5.1 Poisson linear inverse problem

This subsection devotes to the study of the eventually linear convergence of FBS when solving the following standard Poisson regularized problem \[12, 43\]

\[
\min_{x \in \mathbb{R}^n_+} \sum_{i=1}^m b_i \log \frac{b_i}{(A x)_i} + (A x)_i - b_i, \tag{5.1}
\]

where \(A \in \mathbb{R}^{m \times n}_+\) is an \(m \times n\) matrix with nonnegative entries and nontrivial rows, and \(b \in \mathbb{R}^m_+\) is a positive vector. This problem is usually used to recover a signal \(x \in \mathbb{R}^n_+\) from the measurement \(b\) corrupted by Poisson noise satisfying \(A x \simeq b\). The problem (5.1) could be written in term of (1.1) in which

\[
f(x) := h(A x), \quad g(x) = \delta_{\mathbb{R}^n_+}(x), \quad \text{and} \quad F_3(x) := h(A x) + g(x), \tag{5.2}
\]

where \(h\) is the Kullback-Leibler divergence defined by

\[
h(y) = \begin{cases} 
\sum_{i=1}^m b_i \log \frac{b_i}{y_i} + y_i - b_i & \text{if } y \in \mathbb{R}^m_+ , \\
\infty & \text{if } y \in \mathbb{R}^m_+ \setminus \mathbb{R}^m_+ .
\end{cases} \tag{5.3}
\]

Note from (5.2) and (5.3) that \(\text{dom } f = A^{-1}(\mathbb{R}^m_+),\) which is an open set. Moreover, since \(A \in \mathbb{R}^{m \times n}_+\), we have \(\text{dom } f \cap \text{dom } g = \text{int}(\text{dom } f) \cap \text{dom } g = A^{-1}(\mathbb{R}^m_+) \cap \mathbb{R}^n_+ \neq \emptyset\) and \(f\) is continuously differentiable at any point on \(\text{dom } f \cap \text{dom } g\). The standing assumptions \(A1\) and \(A2\) are satisfied for Problem (5.1). Moreover, since the function \(F_3\) is bounded below and coercive, the optimal solution set to problem (5.1) is always nonempty.

It is worth noting further that \(\nabla f\) is locally Lipschitz continuous at any point \(\text{int}(\text{dom } f) \cap \text{dom } g\) but not globally Lipschitz continuous on \(\text{int}(\text{dom } f) \cap \text{dom } g\). Our Theorem 3.7 is applicable to solving (5.1) with global convergence rate \(o(1/k)\). In the recent work \([4]\), a new algorithm rather close to FBS was designed with applications to solving (5.1). However, the theory developed in \([4]\) could not guarantee the global convergence of their optimal sequence \((x^k)_{k \in \mathbb{N}}\) when solving (5.1), since one of their assumptions on the closedness of the domain of their auxiliary Legendre function in \([4, \text{Theorem } 2]\) is not satisfied. Our intent in this subsection is to reveal the Q-linear convergence of our method when solving (5.1) in the sense of Theorem 4.2. In order to do so, we need to verify the metric subregularity of \(\partial F_3\) at any optimal minimizer for 0, or the second-order growth condition of \(F_3\). Note further that the Kullback-Leibler divergence \(h\) is not strongly convex and \(\nabla f\) is not globally Lipschitz continuous; hence, standing assumptions in \([17]\) are not satisfied. Proving the metric subregularity of \(\partial F_3\) at an optimal solution via the approach of \([17]\) needs to be proceeded with caution.

**Lemma 5.1.** Let \(\bar{x}\) be an optimal solution to problem (5.1). Then for any \(R > 0\), we have

\[
F_3(x) - F_3(\bar{x}) \geq \nu d^2(x; S^*) \quad \text{for all } x \in B_R(\bar{x}) \tag{5.4}
\]

with some positive constant \(\nu\). Consequently, \(\partial F_3\) is metrically subregular with at \(\bar{x}\) for 0 with modulus \(\nu^{-1}\).
Proof. Pick any $R > 0$ and $x \in \mathcal{B}_R(\bar{x})$. We only need to prove (5.1) for the case that $x \in \text{dom } F_3 \cap \mathcal{B}_R(\bar{x})$, i.e., $x \in A^{-1}(\mathbb{R}_+^n) \cap \mathbb{R}_+^n \cap \mathcal{B}_R(\bar{x})$. Note that
\[
\nabla f(x) = \sum_{i=1}^{m} \left[1 - \frac{b_i}{\langle a_i, x \rangle}\right]a_i \quad \text{and} \quad \langle \nabla^2 f(x) \rangle^{ij} = \sum_{i=1}^{m} \frac{b_i}{\langle a_i, x \rangle^2} \quad \text{for all } d \in \mathbb{R}^n,
\]
where $a_i$ is the $i$-th row of $A$. Define $\bar{y} := A\bar{x}$, for any $x, u \in \mathcal{B}_R(\bar{x}) \cap \text{dom } f$ we have $[x, u] \subset \mathcal{B}_R(\bar{x}) \cap \text{dom } f$ and obtain from the mean-value theorem that
\[
f(x) - f(u) - \langle \nabla f(u), x - u \rangle = \frac{1}{2} \int_{0}^{1} \langle \nabla^2 f(x + t(x - u)) \rangle \, dt x - u, x - u \rangle \, dt
\]
\[
= \frac{1}{2} \int_{0}^{1} \sum_{i=1}^{m} b_i \frac{\langle a_i, x - u \rangle^2}{\langle a_i, u + t(x - u) \rangle^2} \, dt
\]
\[
\geq \frac{1}{2} \int_{0}^{1} \sum_{i=1}^{m} \frac{b_i}{\langle a_i, \bar{x} \rangle^2} \frac{\langle a_i, x - u \rangle^2}{\parallel a_i \parallel^2 \parallel u - \bar{x} - t(x - u) \parallel^2} \, dt
\]
\[
\geq \frac{1}{2} \sum_{i=1}^{m} \frac{b_i}{\langle a_i, \bar{x} \rangle^2 + 3 \parallel a_i \parallel R^2} \langle a_i, x - u \rangle^2.
\]
Similarly, we have
\[
f(u) - f(x) - \langle \nabla f(x), u - x \rangle \geq \frac{1}{2} \int_{0}^{1} \sum_{i=1}^{m} b_i \frac{\langle a_i, u - x \rangle^2}{\langle a_i, u + t(x - u) \rangle^2} \, dt\langle a_i, u - x \rangle^2 \quad \text{for } x, u \in \mathcal{B}_R(\bar{x}) \cap \text{dom } f. \tag{5.5}
\]

Adding the above two inequalities gives us that
\[
\langle \nabla f(x) - \nabla f(u), x - u \rangle \geq \sum_{i=1}^{m} \frac{b_i}{\langle a_i, \bar{x} \rangle^2 + 3 \parallel a_i \parallel R^2} \langle a_i, x - u \rangle^2 \quad \text{for all } x, u \in \mathcal{B}_R(\bar{x}) \cap \text{dom } f. \tag{5.6}
\]

We claim that the optimal solution set $S^*$ to problem (5.1) satisfies that
\[
S^* = A^{-1}(\bar{y}) \cap (\partial g)^{-1}(-\nabla f(\bar{x})) \quad \text{with } \bar{y} = A\bar{x}. \tag{5.7}
\]

Pick another optimal solution $\bar{u} \in S^*$, we have $\bar{u}_t := \bar{x} + t(\bar{x} - \bar{u}) \in S^* \subset \text{dom } f$ for any $t \in [0, 1]$ due to the convexity of $S^*$. By choosing $t$ sufficiently small, we have $\bar{u}_t \in \mathcal{B}_R(\bar{x}) \cap \text{dom } f$. Note further that $-\nabla f(\bar{u}_t) \in \partial g(\bar{u}_t)$ and $-\nabla f(\bar{x}) \in \partial g(\bar{x})$. Since $\partial g$ is a monotone operator, we obtain that
\[
0 \geq \langle \nabla f(\bar{x}), \bar{x} - \bar{u}_t \rangle.
\]

This together with (5.6) tells us that $\langle a_i, \bar{x} - \bar{u}_t \rangle = 0$ for all $i = 1, \ldots, m$. Hence $A\bar{x} = A\bar{u} = \bar{y}$ for any $\bar{u} \in S^*$, which also implies that
\[
\nabla f(\bar{u}) = A^T \nabla h(A\bar{u}) = A^T \nabla h(A\bar{x}) = \nabla f(\bar{x}). \tag{5.8}
\]

This verifies the inclusion “$\subset$” in (5.7). The opposite inclusion is trivial. Indeed, take any $u$ satisfying that $Au = \bar{y}$ and $-\nabla f(\bar{x}) \in \partial g(u)$, similarly to (5.8) we have $-\nabla f(u) = -\nabla f(\bar{x}) \in \partial g(u)$. This shows that $0 \in \nabla f(u) + \partial g(u)$, i.e., $u \in S^*$. The proof for equality (5.7) is completed.

Note from (5.7) that the optimal solution set $S^*$ is a polyhedral with the following format
\[
S^* = \{ u \in \mathbb{R}^n \mid Au = \bar{y} = A\bar{x}, \langle \nabla f(\bar{x}), u \rangle = 0, u \in \mathcal{B}_R(\bar{x}) \}.
\]
due to the fact that \((\partial g)^{-1}(-\nabla f(x)) = \{u \in \mathbb{R}^n_+ \mid \langle \nabla f(x), u \rangle = 0 = \langle \nabla f(x), \bar{x} \rangle\}. Thanks to the Hoffman’s lemma, there exists a constant \(\gamma > 0\) such that

\[
d(x; S^*) \leq \gamma(\|Ax - A\bar{x}\| + |\nabla f(\bar{x}), x - \bar{x}|) \quad \text{for all } x \in \mathbb{R}^n_+.
\] (5.9)

Moreover, for any \(x \in B_R(\bar{x}) \cap \mathbb{R}^n_+\), (5.9) tells us that

\[
f(x) - f(\bar{x}) - \langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq \frac{1}{2} \min_{1 \leq i \leq m} \left[ \frac{b_i}{\|a_i, \bar{x}\| + 3\|a_i\|R^2} \right] \|Ax - A\bar{x}\|^2.
\] (5.10)

This implies that

\[
f(x) - f(\bar{x}) \geq \frac{1}{2} \min_{1 \leq i \leq m} \left[ \frac{b_i}{\|a_i, \bar{x}\| + 3\|a_i\|R^2} \right] \|Ax - A\bar{x}\|^2 + \langle \nabla f(\bar{x}), x - \bar{x} \rangle
\]

\[
\geq \frac{1}{2} \min_{1 \leq i \leq m} \left[ \frac{b_i}{\|a_i, \bar{x}\| + 3\|a_i\|R^2} \right] \|Ax - A\bar{x}\|^2 + \frac{1}{\|\nabla f(\bar{x})\|} \langle Ax - A\bar{x}, x - \bar{x} \rangle^2
\]

\[
\geq \min \left\{ \frac{1}{2} \min_{1 \leq i \leq m} \left[ \frac{b_i}{\|a_i, \bar{x}\| + 3\|a_i\|R^2} \right] , \frac{1}{\|\nabla f(\bar{x})\|} \langle Ax - A\bar{x}, x - \bar{x} \rangle \right\} d^2(x; S^*),
\]

where the fourth inequality follows from the elementary inequality that \(\frac{(a+b)^2}{2} \leq a^2 + b^2\) with \(a, b \geq 0\), and the last inequality is from (5.9). This clearly ensures (5.4). The second part of the lemma is a consequence of Proposition 2.2.

When applying FBS to solving problem (5.1), we have

\[
x^{k+1} = \mathbb{P}_{\mathbb{R}^n_+} \left( x^k - \alpha_k \sum_{i=1}^m \left[ 1 - \frac{b_i}{\langle a_i, x^k \rangle} \right] a_i \right) \quad \text{with } x^0 \in A^{-1}(\mathbb{R}^n_+) \cap \mathbb{R}^n_+,
\] (5.11)

where \(\alpha_k\) is determined from the Beck-Teboulle’s line search and \(\mathbb{P}_{\mathbb{R}^n_+} (\cdot)\) is the projection mapping to \(\mathbb{R}^n_+\). Due to Corollary 3.4 all \(x^k\) are well-defined and \(F_3(x^k)\) are finite.

**Corollary 5.2.** (Q-linear convergence of method (5.11)) Let \((x^k)_{k \in \mathbb{N}}\) be the sequence generated from (5.11) with \(x^0 \in A^{-1}(\mathbb{R}^n_+) \cap \mathbb{R}^n_+\) for solving the Poisson regularized problem (5.1). Then the sequences \((x^k)_{k \in \mathbb{N}}\) and \((F_3(x^k))_{k \in \mathbb{N}}\) are Q-linearly convergent to an optimal solution and the optimal value to (5.1) respectively.

**Proof.** Since both functions \(f\) and \(g\) in problem (5.1) satisfy our standing assumptions A1 and A2, and problem (5.1) always has optimal solutions, the sequence \((x^k)_{k \in \mathbb{N}}\) converges to an optimal solution \(\bar{x}\) to problem (5.1) by Theorem 3.3. Moreover, it follows from Lemma 5.1 that \(\partial F_3\) is metrically subregular at \(\bar{x}\) for 0. Since \(\nabla f\) is locally Lipschitz continuous around \(\bar{x}\), the combination of Theorem 4.2 and Lemma 5.1 tells us that \((x^k)_{k \in \mathbb{N}}\) is Q-linearly convergent to \(\bar{x}\).

By using this approach, it is similar to show that quadratic growth condition in Lemma 5.1 is also valid for the following problem

\[
\min_{x \in \mathbb{R}^n_+} \sum_{i=1}^m b_i \log \frac{b_i}{(Ax)_i} + (Ax)_i - b_i + \mu k(x),
\] (5.12)
where \( k(x) \) is either the \( \ell_1 \) norm \( \| x \|_1 \) or the discrete total variation \( TV(x) \), while \( \mu > 0 \) is the penalty parameter (due to the polyhedral property of \( k(x) \)). In particular, when \( k(x) = \| x \|_1 \), the FBS method for solving (5.12) is practical by modifying the function \( f(x) \) in (5.2) to \( h(Ax) + \langle e, x \rangle \) with \( e = (1, 1, \ldots, 1) \in \mathbb{R}^n \). This together with Corollary 5.2 clearly shows that FBS (5.1) solves the Poisson inverse problem with sparse regularization [4] with linear rate.

### 5.2 Forward-backward splitting method under partial smoothness

This subsection is motivated from the recent work [22, 23] of Liang-Fadili-Peyré in which they study the local linear convergence of FBS method under additional assumptions of partial smoothness on the (possibly nonsmooth) function \( g \) that allows them to cover a wide range of important polyhedral/nonpolyhedral optimization problems. The main result of [22] is to obtain the linear convergence of FBS iteration under a nondegeneracy assumption and a local strong convexity one.

In this section, we revisit the problem in [22, 23] and obtain some improvements such as local Q-linear convergence of FBS iteration can be obtained under weaker conditions.

Let us proceed by providing some useful notions mainly used in this section. For any set \( \Omega \subset \mathbb{R}^n \), we denote \( \text{ri} \Omega, \text{aff} \Omega, \text{par} \Omega \) by the relative interior, the affine hull, and the subspace parallel to \( \Omega \), respectively. Given \( x \in \text{dom} g \), define \( T x := \text{par} (\partial g(\bar{x})) \perp \).

Next we recall the definition of partial smoothness of functions introduced by Lewis [24] with a slight modification for convex functions as in [22]. The class of partly smooth functions is broad, including, in particular, convex piecewise linear functions and spectral functions.

**Definition 5.1 (Partial smoothness).** The convex function \( g \in \Gamma_0(\mathbb{R}^n) \) is \( C^2 \)-partly smooth at \( \bar{x} \) relative to a set \( M \) containing \( \bar{x} \) if

1. (Smoothness) \( M \) is a \( C^2 \)-manifold around \( \bar{x} \) and \( g \) restricted to \( M \) is \( C^2 \) around \( \bar{x} \).
2. (Sharpness) The tangent space \( T_M(\bar{x}) \) is \( T_{\bar{x}} \).
3. (Continuity) The subgradient mapping \( \partial g \) is continuous at \( \bar{x} \) relative to \( M \).

The class of partly smooth and lower semi-continuous convex functions at \( \bar{x} \) relative to \( M \) defined above is denoted by \( \text{PS}_{\bar{x}}(M) \).

We also define here the so-called covariant Hessian of a partly smooth function [25, Definition 2.11] as follows. Its computation via the manifold \( M \) and the representation function of \( g \) on \( M \) can be found in [28].

**Definition 5.2 (Covariant Hessian).** Let \( g \in \Gamma_0(\mathbb{R}^n) \) be \( C^2 \)-partly smooth at \( \bar{x} \) relative to \( C^2 \) manifold \( M \) containing \( \bar{x} \). The covariant Hessian, \( \nabla^2_M g(\bar{x}) : T_M(\bar{x}) \times T_M(\bar{x}) \to \mathbb{R} \) is the unique self-adjoint and bilinear map satisfying

\[
\langle \nabla^2_M g(\bar{x}) u, u \rangle = \frac{d^2}{dt^2} g(\Pi_M(\bar{x} + tu)) \bigg|_{t=0}
\]

for all \( u \in T_M(\bar{x}) \), which is known to be well-defined.

The following result gives a characterization of strong metric subregularity for \( \partial F \). Its root can be found from the recent result of Lewis-Zhang [25, Theorem 6.3].
**Proposition 5.3** (Characterizations of strong metric subregularity: partial smoothness cases). Let $x^* \in S^*$ be an optimal solution to problem (1.1). Suppose that $f$ is $C^2$ around $x^*$ and $g$ is $C^2$-partly smooth at the point $x^*$ relative to the $C^2$ manifold $\mathcal{M}$. Suppose further that $-\nabla f(x^*) \in \ri \partial g(x^*)$. The following statements are equivalent:

(i) $\partial F$ is strongly metrically subregular at $x^*$ for 0.

(ii) $x^*$ is a tilt-stable local minimizer to $F$ in the sense of Proposition 2.4 (iii).

(iii) The following positive-definite condition holds:

$$\langle (\nabla^2 f(x^*) + \nabla^2 g(x^*))u, u \rangle > 0 \text{ for all } u \in T_{\mathcal{M}}(x^*) \neq \{0\}. \quad (5.13)$$

Moreover, if (iii) is fulfilled then $\partial F$ is strongly metrically subregular at $x^*$ for 0 with any modulus $\kappa > \mu^{-1}$, where $\mu$ is defined by

$$\mu := \min \left\{ \frac{\langle (\nabla^2 f(x^*) + \nabla^2 g(x^*))u, u \rangle}{\|u\|^2} \mid u \in T_{\mathcal{M}}(x^*) \right\} > 0 \quad (5.14)$$

with the convention $\frac{0}{0} = \infty$.

**Proof.** Let us start with the implication $[(i) \implies (ii)]$. Suppose that $\partial F$ is strongly metrically subregular at $x^*$ for 0 with modulus $\kappa^{-1} > 0$. It follows from Proposition 2.4 that there is some neighborhood $U$ of $x^*$ such that

$$F(x) \geq F(x^*) + \frac{1}{2\kappa} \|x - x^*\|^2 \text{ for all } x \in U. \quad (5.15)$$

Note that $0 \in \ri \partial F(x^*)$. This together (5.15) means that $x^*$ is a **strong critical point** of $F$ relative $\mathcal{M}$ in the sense of [25, Definition 3.4]. Since $F = f + g$ is $C^2$-partly smooth at the point $x^*$ relative to the $C^2$ manifold $\mathcal{M}$ due to [24, Corollary 4.7], we get from [25, Theorem 6.3] that $x^*$ is a tilt-stable local minimum of $f$. The converse implication $[(ii) \implies (i)]$ is trivial by Proposition 2.4.

Applying [25, Theorem 6.1 and Theorem 5.3] to the function $F$, we also have the equivalence of (ii) to the following condition

$$\langle \nabla^2\mathcal{M}F(x^*)u + v, u \rangle > 0 \text{ for all } u \in T_{\mathcal{M}}(x^*) \setminus \{0\}, v \in N_{\mathcal{M}}(x^*),$$

where $N_{\mathcal{M}}(x^*)$ is the normal cone to $\mathcal{M}$ at $x^*$, which is the orthogonal dual of the tangent cone $T_{\mathcal{M}}(x^*)$ in this case, since $T_{\mathcal{M}}(x^*)$ is indeed a subspace.

Since $f$ is $C^2$ around $x^*$, we have $\nabla^2\mathcal{M}F(x^*) = \nabla^2 f(x^*) + \nabla^2 g(x^*)$. Moreover, $\langle v, u \rangle = 0$ for all $u \in T_{\mathcal{M}}(x^*) \setminus \{0\}$ and $v \in N_{\mathcal{M}}(x^*)$, the latter is equivalent to (5.13). We derive the equivalence between (ii) and (iii).

Finally let us prove the connection between $\mu$ in (5.14) and the strong metric subregular modulus of $\partial F$ at $x^*$ for 0 in (i). Suppose that (5.13) holds. Then it follows from [31, Theorem 3.6] that $x^*$ is a tilt-stable local minimizer to $F$ with any modulus $\kappa > \mu^{-1}$. By Proposition 2.4 we have $\partial F$ is strongly metrically subregular at $x^*$ for 0 with such modulus $\kappa$. The proof is complete.

Recently, to prove the local linear convergence of FBS method when $g$ is partly smooth, [22] supposes the nondegeneracy condition $-\nabla f(x^*) \in \ri (\partial g(x^*))$ together with the following assumption

$$\langle \nabla^2 f(x^*)u, u \rangle \geq c\|u\|^2 \text{ for all } u \in T_{\mathcal{M}}(x^*) \quad (5.16)$$

for some $c > 0$. We show next that this condition is stronger than (5.13) and thus also guarantee the strong metric subregularity of $\partial F$ at $x^*$ for 0.
Corollary 5.4 (Sufficient condition for strong metric subregularity). Let \( x^* \in S^* \) be an optimal solution. Suppose that \( f \) is \( C^2 \) around \( x^* \) and \( g \) is \( C^2 \)-partly smooth at the point \( x^* \) relative to the \( C^2 \)-manifold \( M \). Suppose further that \( -\nabla f(x^*) \in \ri \partial g(x^*) \). If there exists some \( c > 0 \) such that \((5.16)\) is satisfied, then \( \partial F \) is strongly metrically subregular at \( x^* \) for 0 with any modulus \( \kappa > c^{-1} \).

Proof. Suppose that \((5.16)\) holds. Since \( g \) is a convex function, its subgradient mapping \( \partial g \) is maximal monotone. It follows from [34, Theorem 2.1] and [25, Theorem 5.3] that \( \mu \) in \((5.13)\) verifies (5.13) and that \( F \) is strongly metrically subregular at \( x^* \) for 0 with any modulus \( \kappa > c^{-1} \). This together with (5.16) verifies (5.13) and that is strongly metric subregular at \( \partial F \). If \( F \) is a convex function, its subgradient mapping \( \partial g \) is \( C^2 \)-partly smooth at the point \( x^* \) relative to the \( C^2 \)-manifold \( M \). Suppose further that \( \partial F \) is strongly metrically subregular at \( x^* \) for 0 with any radius \( \kappa > c^{-1} \). The proof is complete. \( \Box \)

Remark 5.1. Since \( f \) is convex, \( \nabla^2 f(x^*) \succeq 0 \). The condition \((5.16)\) is indeed equivalent to the following one

\[
(\nabla^2_{\partial g}(x^*))^T u, u \geq 0 \quad \text{for all} \quad u \in T_M(x^*).
\]

In some particular application, e.g., \( g(x) = \|x\|_1 \), the covariant Hessian \( \nabla_{\partial g}g \) is zero, and thus condition \((5.16)\) is strictly weaker than \((5.17)\). However, in general \( \nabla_{\partial g}g \) may be different from 0 and thus \((5.16)\) is set to be the same with \((5.17)\). This observation together with the above result and Proposition 2.3 tells us that \((5.17)\) guarantees the validity of second-order growth condition in \((2.8)\), which is part of [23, Proposition 4.1].

The following result motivated from [22, Theorem 3.1] also provides Q-linear convergence of FBS method under the partial smoothness and a positive definite condition. However, as discussed above, our condition \((5.13)\) is weaker.

Corollary 5.5 (Q-linear convergence under partial smoothness). Let \((x^k)_{k \in \mathbb{N}} \) and \((\alpha_k)_{k \in \mathbb{N}} \) be the sequences generated from FBS method. Suppose that the solution set \( S^* \) is not empty, \((x^k)_{k \in \mathbb{N}} \) is converging to some \( x^* \in S^* \), \( f \) is \( C^2 \) around \( x^* \), and that \( g \) is \( C^2 \)-partly smooth at the point \( x^* \) relative to the \( C^2 \) manifold \( M \). Suppose further that \( -\nabla f(x^*) \in \ri \partial g(x^*) \) and the condition \((5.13)\) holds for \( x^* \). Then there exists some \( k \in \mathbb{N} \) such that

\[
\|x^{k+1} - x^*\| \leq \frac{1}{\sqrt{1 + \alpha \kappa}} \|x^k - x^*\|
\]

\[
|F(x^{k+1}) - F(x^*)| \leq \frac{\sqrt{1 + \alpha \kappa} + 1}{2\sqrt{1 + \alpha \kappa}} |F(x^k) - F(x^*)|
\]

for any \( k > K \), where \( \alpha \) is any positive number bigger than \( \min \left\{ \frac{\alpha K}{2}, \frac{\theta}{2\lambda_{\max}(\nabla^2 f(x^*))} \right\} \), \( \lambda_{\max}(\nabla^2 f(x^*)) \) is the biggest eigenvalue of \((\nabla^2 f(x^*))\), and \( \kappa \) is any positive number smaller than \( \mu \) in \((5.14)\).

Proof. Since \( f \) is \( C^2 \) around \( x^* \), \( \nabla f \) is locally Lipschitz continuous around \( x^* \) with any constant \( L \) bigger than \( \lambda_{\max}(\nabla^2 f(x^*)) \). Note also from Proposition 5.3 that condition \((5.13)\) ensures that \( \partial F \) is strongly metrically subregular at \( x^* \) for 0 with any modulus bigger than \( \mu^{-1} \) from \((5.14)\). This together with Corollary 4.3 verifies all the conclusions of this result. \( \Box \)
5.3 Forward-backward splitting method for $\ell_1$-regularized problems

In this section we consider the $\ell_1$-regularized optimization problems in \((1.2)\). In this case the function $g(x) = \mu \|x\|_1$ belongs to the class of partial smooth functions discussed in Subsection 5.2. However, unlike the study there, we will avoid the nondegeneracy condition $-\nabla f(x^*) \in \partial g(x^*)$ (known also as the strict complementarity condition \([21]\)). To proceed, let us consider the following proposition computing the graphical derivative of $\partial \| \cdot \|_1$.

**Proposition 5.6** (Graphical derivative of $\partial \| \cdot \|_1$). Suppose that $\bar{s} \in \partial \| x^* \|_1$. Define $I := \{ j \in \{1, \ldots, n\} | |\bar{s}_j| = \mu \}$, $J := \{ j \in I | x^*_j \neq 0 \}$, $K := \{ j \in I | x^*_j = 0 \}$, and $H(x^*) := \{ u \in \mathbb{R}^n | u_j = 0, j \notin I \}$. Then $D\partial \| \cdot \|_1(x^*|\bar{s})$ is nonempty if and only if $u \in H(x^*)$. Furthermore, we have

$$D\partial \| \cdot \|_1(x^*|\bar{s})(u) = \left\{ v \in \mathbb{R}^n \left| \begin{array}{c} v_j = 0, j \in J \\ u_j v_j = 0, \bar{s}_j v_j \leq 0, j \in K \end{array} \right. \right\} \text{ for all } u \in H(x^*). \tag{5.18}$$

**Proof.** For any $x \in \mathbb{R}^n$, note that

$$\partial \| x \|_1 = \left\{ s \in \mathbb{R}^n \left| \begin{array}{c} \bar{s}_j = \mu \text{sgn}(x_j) \text{ if } x_j \neq 0 \\ \bar{s}_j \in [-\mu, \mu] \text{ if } x_j = 0 \end{array} \right. \right\}, \tag{5.19}$$

where $\text{sgn} : \mathbb{R} \to \{-1, 1\}$ is the sign function. Take any $v \in D\partial \| \cdot \|_1(x^*|\bar{s})(u)$, there exists sequence $t^k \downarrow 0$ and $(u^k, v^k) \to (u, v)$ such that $(x^*, \bar{s}) + t^k(u^k, v^k) \in \text{gph} \partial \| \cdot \|_1$. Let us consider three partitions of $J$ described below:

**Partition 1.1:** $j \notin I$, i.e., $|\bar{s}_j| < \mu$. It follows from \((5.19)\) that $x^*_j = 0$. For sufficiently large $k$, we have $|(\bar{s} + t^k v^k)_j| < \mu$ and thus $|(x^* + t^k u^k)_j| = 0$ by \((5.19)\) again. Hence $u^k_j = 0$, which implies that $u_j = 0$ for all $j \notin I$.

**Partition 1.2:** $j \in J$, i.e., $|\bar{s}_j| = \mu$ and $x^*_j \neq 0$. When $k$ is sufficiently large, we have $(x^* + t^k u^k)_j \neq 0$ and derive from \((5.19)\) that

$$(\bar{s} + t^k v^k)_j = \mu \text{sgn}(x^* + t^k u^k)_j = \mu \text{sgn} x^*_j = \bar{s}_j,$$

which implies that $v_j = 0$ for all $j \in J$.

**Partition 1.3:** $j \in K$, i.e., $|\bar{s}_j| = \mu$ and $x^*_j = 0$. If there is a subsequence of $(x^*, \bar{s}) + t^k(u^k, v^k)_j$ (without relabeling) such that $|(\bar{s} + t^k v^k)_j| < \mu = |\bar{s}_j|$, we have $\bar{s}_j v_j < 0$ and $(x^* + t^k u^k)_j = 0$ by \((5.19)\). It follows that $u^k_j = 0$. Letting $k \to \infty$, we have $u_j = 0$ and $\bar{s}_j v_j \leq 0$. Otherwise, we find some $K > 0$ such that $|(\bar{s} + t^k v^k)_j| = \mu = |\bar{s}_j|$ for all $k > K$, which yields $v^k_j = 0$. Taking $k \to \infty$ gives us that $v_j = 0$. In both situations, we have $u_j v_j = 0$ and $\bar{s}_j v_j \leq 0$.

Combining the conclusions in three cases above gives us that $u \in H(x^*)$ and also verifies the inclusion “$\subseteq$” in \((5.18)\). To justify the converse inclusion “$\supseteq$”, take $u \in H(x^*)$ and any $v \in \mathbb{R}^n$ with $v_j = 0$ for $j \in J$ and $u_j v_j = 0, \bar{s}_j v_j \leq 0$ for $j \in K$. For any $t^k \downarrow 0$, we prove that $(x^*, \bar{s}) + t^k(u, v) \in \text{gph} \partial \| \cdot \|_1$ and thus verify that $v \in D\partial \| \cdot \|_1(x^*|\bar{s})(u)$. For any $t \in \mathbb{R}$, define the set-valued mapping:

$$\text{SGN}(t) := \partial |t| = \left\{ \begin{array}{ll} \text{sgn}(t) & \text{if } t \neq 0 \\ [-1, 1] & \text{if } t = 0. \end{array} \right.$$
Partition 2.1: \( j \neq I, \) i.e., \(|s_j| < \mu\). Since \( u \in H(x^*)\), we have \( u_j = 0\). Note also that \( x_j^* = 0\). Hence we get \((x^* + tku)_j = 0\) and \((\bar{s} + tkv)_j \in [-\mu, \mu]\) when \( k \) is sufficiently large, which means \((\bar{s} + tkv)_j \in \mu \text{SGN}(x^* + tku)_j\).

Partition 2.2: \( j \in J, \) i.e., \(|s_j| = \mu\) and \( x_j^* \neq 0\). Since \( v_j = 0\), we have
\[
\text{sgn}(\bar{s} + tkv)_j = \text{sgn} \bar{s}_j = \text{sgn}(x_j^*) = \text{sgn}(x^* + tku)_j
\]
and \((x^* + tku)_j \neq 0\) when \( k \) is large. It follows that \((\bar{s} + tkv)_j \in \mu \text{SGN}(x^* + tku)_j\).

Partition 2.3: \( j \in K, \) i.e., \(|s_j| = \mu\) and \( x_j^* = 0\). If \( u_j = 0\), we have \((x^* + tku)_j = 0\) and \(|(\bar{s} + tkv)_j| \leq |s_j| \leq \mu\) for sufficiently large \( k\), since \( s_jv_j \leq 0\). If \( u_j \neq 0\), we have \( v_j = 0\) and
\[
(\bar{s} + tkv)_j = \bar{s}_j = \text{sgn}(u_j) = \text{sgn}(x^* + tku)_j
\]
when \( k \) is large, since \( u_j\bar{s}_j \geq 0\). In both cases, we have \((\bar{s} + tkv)_j \in \mu \text{SGN}(x^* + tku)_j\).

From those cases, we always have \((x^*, \bar{s}) + tk(u, v) \in gph \partial \mu \|\cdot\|_1\) and thus \( v \in D\partial \mu \|\cdot\|_1(x^*)\langle s\rangle(u)\).

As a consequence, we establish a characterization of strong metric subregularity for \( \partial F_1 \).

**Theorem 5.7** (Characterization of strong metric subregularity for \( \partial F_1 \)). Let \( x^* \) be an optimal solution to problem \( (1.2) \). Suppose that \( \nabla f \) is differentiable at \( x^* \). Define \( E := \{j \in \{1, \ldots, n\} \mid |(\nabla f(x^*))_j| = \mu\}, K := \{j \in E \mid x_j^* = 0\}, U := \{u \in \mathbb{R}^E \mid u_j(\nabla f(x^*))_j \leq 0, j \in K\}, \) and \( H_E(x^*) := \{\nabla^2 f(x^*)_i, i, j \in E\}. \) Then the following statements are equivalent:

(i) The subdifferential mapping \( \partial F_1 \) is strongly metrically regular at \( x^* \) for 0

(ii) \( H_E(x^*) \) is positive definite over \( U \) in the sense that
\[
\langle H_E(x^*)u, u \rangle > 0 \quad \text{for all} \quad u \in U \setminus \{0\}, \quad (5.20)
\]

(iii) \( H_E(x^*) \) is nonsingular over \( U \) in the sense that
\[
\ker H_E(x^*) \cap U = \{0\}. \quad (5.21)
\]

Moreover, if \( (5.20) \) is satisfied then \( \partial F_1 \) is strongly metrically regular at \( x^* \) for 0 with any modulus \( \kappa > c^{-1} \), where
\[
c := \min \left\{ \frac{\langle H_E(x^*)u, u \rangle}{\|u\|^2} \mid u \in U \right\} \quad (5.22)
\]

with the convention \( \frac{0}{0} = \infty \).

**Proof.** First let us verify the equivalence between (i) and (ii). Suppose that (i) is valid, i.e., \( \partial F_1 \) is strongly metrically subregular at \( x^* \) for 0. It follows from Proposition 2.3 that there is some \( c_1 > 0 \) satisfying that
\[
\langle w, u \rangle \geq c_1\|u\|^2 \quad \text{for all} \quad w \in D(\nabla f + \partial \mu \|\cdot\|_1)(x^*)(0)(u). \quad (5.23)
\]

Due to the sum rule of graphical derivative [16] Proposition 4A.2, we have
\[
D(\nabla f + \partial \mu \|\cdot\|_1)(x^*)(0)(u) = \nabla^2 f(x^*)u + D\partial \mu \|\cdot\|_1(x^*) - \nabla f(x^*))u.
\]
Thus (5.24) is equivalent to
\[ (\nabla^2 f(x^*) u, u) + \langle v, u \rangle \geq c_1 \| u \|^2 \quad \text{for all} \quad v \in D\partial\mu \cdot \|_1 (x^*) - \nabla f(x^*) (u). \quad (5.24) \]

Define \( \mathcal{V} := \{ u \in \mathbb{R}^n \mid u_j = 0, j \notin \mathcal{E}, u_j (\nabla f (x^*))_j \leq 0, j \in K \} \). Thanks to Proposition 5.6 we have
\[ \langle v, u \rangle = 0 \quad \text{for all} \quad v \in D\partial\mu \cdot \|_1 (x^*) - \nabla f(x^*) (u), u \in \mathcal{V}. \quad (5.25) \]

This together with (5.24) clearly verifies (5.20).

Conversely, suppose that (5.20) is satisfied meaning that \( c \) in (5.22) is positive. Hence we have
\[ \langle \mathcal{H}_\mathcal{E} (x^*) u, u \rangle \geq c \| u \|^2 \quad \text{for all} \quad u \in \mathcal{U}. \quad (5.26) \]

It follows from Proposition 5.6 that \( D\partial\mu \cdot \|_1 (x^*) - \nabla f(x^*) (u) \neq 0 \) if and only if \( u \in \mathcal{V} \). For any \( u \in \mathcal{V} \) and \( v \in D\partial\mu \cdot \|_1 (x^*) - \nabla f(x^*) (u) \), we write \( u = \{ 0_{c^c} \} \times \{ u_{c^c} \} \in \mathbb{R}^{c^c} \times \mathbb{R}^{c^c} \) with \( \mathcal{E}^c = \{ 1, \ldots, n \} \setminus \mathcal{E} \). Obtain from (5.20) and (5.25) that
\[ \langle \nabla^2 f(x^*) u, u \rangle + \langle v, u \rangle = \langle \mathcal{H}_\mathcal{E} (x^*) u_{c^c}, u_{c^c} \rangle + \langle v, u \rangle \geq c \| u_{c^c} \|^2 + 0 = c \| u \|^2, \]

which verifies (5.24) and thus (5.24) with \( c_1 = c \). By Proposition 2.3 \( \partial F_1 \) is strongly metrically subregular with any modulus \( \kappa > c^{-1} \). This clarifies the equivalence between (i) and (ii) and the last statement of the theorem. Moreover, the equivalence between (ii) and (iii) is trivial due to the fact that \( f \) is convex and thus \( \mathcal{H}_\mathcal{E} (x^*) \) is positive semi-definite.

**Corollary 5.8** (Linear convergence of FBS method for \( \ell_1 \)-regularized problems). Let \( (x^k)_{k \in \mathbb{N}} \) and \( (\alpha_k)_{k \in \mathbb{N}} \) be the sequences generated from FBS method for problem (1.2). Suppose that the solution set \( S^* \) is not empty, \( (x^k)_{k \in \mathbb{N}} \) is converging to some \( x^* \in S^* \), and that \( f \) is \( C^2 \) around \( x^* \). If condition (5.20) holds, then \( (x^k)_{k \in \mathbb{N}} \) and \( (F_1(x^k))_{k \in \mathbb{N}} \) are \( Q \)-linearly convergent to \( x^* \) and \( F_1(x^*) \) respectively with rates determined in Corollary 5.6, where \( \kappa \) is any positive number smaller than \( c \) in (5.22).

**Proof.** The result follows from Corollary 4.3, Proposition 5.6, and the proof of Corollary 5.5.

**Remark 5.2.** It is worth noting that condition (5.21) is strictly weaker than the assumption used in [21] that \( \mathcal{H}_\mathcal{E} \) has full rank to obtain the linear convergence of FBS for (1.2). Indeed, let us take into account the case \( n = 2, \mu = 1, f(x_1, x_2) = \frac{1}{2}(x_1 + x_2)^2 + x_1 + x_2 \). Note that \( x^* = (0, 0) \) is an optimal solution to problem (1.2). Moreover, direct computation gives us that \( \nabla f(x^*) = (1, 1), \mathcal{E} = \{ 1, 2 \}, \mathcal{V} = \mathbb{R}_- \times \mathbb{R}_-, \) and \( \mathcal{H}_\mathcal{E}(x^*) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \). It is clear that \( \mathcal{H}_\mathcal{E}(x^*) \) does not have full rank, but condition (5.20) and its equivalence (5.21) hold.

### 5.4 Global \( Q \)-linear convergence of ISTA on Lasso problem

In this section we study the linear convergence of ISTA for Lasso problem (1.3). The following lemma taken from [7, Lemma 10] plays an important role in our proof.

**Lemma 5.9** (Global error bound). Fix any \( R > \frac{\| b \|^2}{2\mu} \). Suppose that \( x^* \) is an optimal solution to problem (1.3). Then we have
\[ F_2(x) - F_2(x^*) \geq \frac{\gamma R}{2} d^2(x; S^*) \quad \text{for all} \quad \| x \|_1 \leq R, \quad (5.27) \]
where
\[
\gamma_R := \nu^2 \left( 1 + \frac{\sqrt{5}}{2} \mu R + (R\|A\| + \|b\|)(4R\|A\| + \|b\|) \right)^{-1}
\] (5.28)
while \( \nu \) is the Hoffman constant defined in [7, Definition 1] only depending on the initial data \( A, b, \mu \).

**Theorem 5.10** (Global Q-linear convergence of ISTA). Let \((x^k)_{k \in \mathbb{N}}\) be the sequence generated by ISTA for problem (1.3) that converges to an optimal solution \(x^* \in S^*\). Then \((x^k)_{k \in \mathbb{N}}\) and \((F_2(x^k))_{k \in \mathbb{N}}\) are globally Q-linearly convergent to \(x^*\) and \(F_2(x^*)\) respectively:
\[
\|x^{k+1} - x^*\| \leq \frac{1}{\sqrt{1 + \frac{\alpha \gamma_R}{4}}} \|x^k - x^*\| \quad (5.29)
\]
\[
|F_2(x^{k+1}) - F_2(x^*)| \leq \frac{2 \sqrt{\frac{1 + \alpha \gamma_R}{4}}}{\sqrt{1 + \frac{\alpha \gamma_R}{4}} + 1} |F_2(x^k) - F_2(x^*)| \quad (5.30)
\]
for all \( k \in \mathbb{N} \), where \( R \) is any number bigger than \( \|x^0\| + \frac{\|b\|^2}{\mu} \) and \( \gamma_R \) is given as in (5.28) while \( \alpha := \frac{1}{2} \min \left\{ \sigma, \frac{\theta}{\lambda_{\max}(A^T A)} \right\} \).

**Proof.** Note that Lasso always has optimal solutions. With \( x^* \in S^* \), we have
\[
F_2(0) = \frac{1}{2} \|b\|^2 \geq F_2(x^*) \geq \mu \|x^*\|_1,
\]
which implies that \( \|x^*\|_1 \leq \frac{1}{2\mu} \|b\|^2 \). It follows from Corollary 3.4(i) that
\[
\|x^k\| \leq \|x^k - x^*\| + \|x^*\| \leq \|x^0 - x^*\| + \|x^*\| \leq \|x^0\| + 2\|x^*\| \leq \|x^0\| + \frac{\|b\|^2}{\mu} < R
\]
for all \( k \in \mathbb{N} \). Thanks to Lemma 5.9 Corollary 3.4(i), and Proposition 3.6 we have
\[
\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 \geq \alpha \gamma_R d^2(x^{k+1}; S^*)
\] (5.31)
with \( \alpha = \frac{1}{2} \min \left\{ \sigma, \frac{\theta}{\lambda_{\max}(A^T A)} \right\} \) and the note that \( \lambda_{\max}(A^T A) \) is the global Lipschitz constant of \( \frac{1}{2}\|Ax - b\|^2 \). The proof of (5.29) and (5.30) are quite similar to the one of (4.6) and (4.7) in Theorem 4.2 by using (5.31) instead of (4.8) there. \( \square \)

By using Lemma 5.9 and Theorem 4.1 we can prove that indeed \((x^k)_{k \in \mathbb{N}}\) and \((F_2(x^k))_{k \in \mathbb{N}}\) are converging globally R-linearly to \(x^*\) and \(F_2(x^*)\) with better rates \((1 + \alpha \gamma_R)^{-\frac{1}{2}}\) and \((1 + \alpha \gamma_R)^{-1}\), respectively. A very similar argument has been obtained recently from [7] with a different approach via KL-inequality. Here we prove the global Q-linear convergence. Observe further that the linear rates in Theorem 5.10 depends on the initial point \(x^0\). However, the local linear rates around optimal solutions are uniform and independent from the choice of \(x^0\) as mentioned in Remark 4.1.

**Corollary 5.11** (Local Q-linear convergence of ISTA with uniform rate). Let \((x^k)_{k \in \mathbb{N}}\) be the sequence generated by ISTA for problem (1.3) that converges to an optimal solution \(x^* \in S^*\). Then (5.29) and (5.30) are satisfied when \( k \) is sufficiently large, where \( \alpha = \min \left\{ \frac{\theta}{2}, \frac{\theta}{2\lambda_{\max}(A^T A)} \right\} \) and \( R \) is any number bigger than \( \frac{\|b\|^2}{2\mu} \).
Proof. Since $x^k$ is converging to $x^* \in S^*$. It follows from the proof of Theorem 5.10 that $\|x^*\| \leq \frac{\|x^k\|}{2\mu} < R$. Hence there exists $K \in \mathbb{N}$ such that $\|x^k\| < R$ for any $k > K$. By using Lemma 5.9 and Corollary 3.4(i), we also obtain (5.31) for all $k > K$. Following the same arguments as in Theorem 5.10 justifies the corollary.

6 Uniqueness of optimal solution to $\ell_1$-regularized least square optimization problems

As discussed in Section 1, the linear convergence of ISTA for Lasso was sometimes obtained by imposing an additional assumption that Lasso has a unique optimal solution $x^*$; see, e.g., [2]. Since $\partial F_2$ is always metrically subregular at $x^*$ for 0 from Remark 4.1, the uniqueness of $x^*$ is equivalent to the strong metric subregularity of $\partial F_2$ at $x^*$ for 0. This observation together with Theorem 5.7 allows us to characterize the uniqueness of optimal solution to Lasso in the below theorem. A different characterization for this property could be found in [40, Theorem 2.1]. Suppose that $x^*$ is an optimal solution, which means $-A^T(Ax^* - b) \in \mu \partial \|x^*\|_1$. In the spirit of Proposition 5.7, with $f(x) = \frac{1}{2} \|Ax - b\|^2$, define

$$E := \{j \in \{1, \ldots, n\} \mid (A^T(Ax^* - b))_j = \mu\}, \quad K := \{j \in E \mid x^*_j = 0\}, \quad J := E \setminus K.$$  \hspace{1cm} (6.1)

Since $-A^T(Ax^* - b) \in \partial \mu \|x^*\|_1$, if $x^*_j \neq 0$ then $(A^T(Ax^* - b))_j = -\mu \text{sign}(x^*_j)$. This tells us that $J = \{j \in \{1, \ldots, n\} \mid x^*_j \neq 0\} = \text{supp}(x^*)$. Furthermore, given an index set $I \subset \{1, \ldots, n\}$, we denote $A_I$ by the submatrix of $A$ formed by its columns $A_i$, $i \in I$ and $x_I$ by the subvector of $x \in \mathbb{R}^n$ formed by $x_i$, $i \in I$. For any $x \in \mathbb{R}^n$, we also define $\text{sign}(x) := (\text{sign}(x_1), \ldots, \text{sign}(x_n))^T$ and $\text{Diag}(x)$ by the square diagonal matrix with the main entries $x_1, x_2, \ldots, x_n$.

**Theorem 6.1** (Uniqueness of optimal solution to Lasso problem). Let $x^*$ be an optimal solution to problem (1.3). The following statements are equivalent:

(i) $x^*$ is the unique optimal solution to Lasso (1.3).

(ii) The system $A_J x_J - A_K Q_K x_K = 0$ and $x_K \in \mathbb{R}_+^K$ has a unique solution $(x_J, x_K) = (0_J, 0_K) \in \mathbb{R}^J \times \mathbb{R}^K$, where $Q_K := \text{Diag} \left[ \text{sign}(A_K^T(A_J x_J^* - b)) \right]$.

(iii) The submatrix $A_J$ has full column rank and the columns of $A_J A_J^T A_K Q_K - A_K Q_K$ are positively linearly independent in the sense that

$$\text{Ker} \left( A_J A_J^T A_K Q_K - A_K Q_K \right) \cap \mathbb{R}_+^K = \{0_K\},$$  \hspace{1cm} (6.2)

where $A_J^T := (A_J^T A_J)^{-1} A_J^T$ is the Moore-Penrose pseudoinverse of $A_J$.

(iv) The submatrix $A_J$ has full column rank and there exists a Slater point $y \in \mathbb{R}^n$ such that

$$\langle Q_K A_K^T A_J A_J^T - Q_K A_K^T \rangle y < 0.$$  \hspace{1cm} (6.3)

**Proof.** Since $\partial F_2$ is always metrically subregular at $x^*$ for 0 from Remark 4.1(i) means that $\partial F_2$ is strongly metrically subregular at $x^*$ for 0. Thus, by Theorem 5.7 (i) is equivalent to

$$\langle \mathcal{H}_E u, u \rangle > 0 \quad \text{for all} \quad u \in \mathcal{U} \setminus \{0\}$$  \hspace{1cm} (6.4)
with \( f(x) = \frac{1}{2} \|Ax - b\|^2 \) and \( \mathcal{U} = \{u \in \mathbb{R}^\mathcal{E} \mid u_j(\nabla f(x^*))_j \leq 0, j \in K\} \). Note that \( \mathcal{H}_\mathcal{E} = [\nabla^2 f(x^*)]_{i,j} \in \mathcal{E} = [(A^TA)_{i,j}]_{i,j} \in \mathcal{E} = A^T \mathcal{E} A \). Hence (6.4) means the system

\[
0 = A^T \mathcal{E} u = A_J u_J + A_K u_K \quad \text{and} \quad u_K \in \mathcal{U}_K
\]

has a unique solution \( u = (u_J, u_K) = (0, 0)_K \in \mathbb{R}^J \times \mathbb{R}^K \), where \( \mathcal{U}_K \) is defined by

\[
\mathcal{U}_K := \{u \in \mathbb{R}^K \mid u_k(A^T(Ax^* - b))_k \leq 0, k \in K\}.
\]

As observed after (6.1), \( J = \text{supp}(x^*) \), for each \( k \in K \) we have

\[
(A^T(Ax^* - b))_k = (A^T(A_J x^*_J - b))_k = (A^T_K(A_J x^*_J - b))_k.
\]

It follows that \( \mathcal{U}_K = -Q_K(\mathbb{R}^K_+) \) and \( Q_K \) is a nonsingular diagonal square matrix (each diagonal entry is either 1 or \(-1\)). Uniqueness of system (6.5) is equivalent to (ii). This verifies the equivalence between (i) and (ii).

Let us justify the equivalence between (ii) and (iii). To proceed, suppose that (ii) is valid, i.e., the system

\[
A_J x_J - A_K Q_K x_K = 0 \quad \text{with} \quad (x_J, x_K) \in \mathbb{R}^J \times \mathbb{R}^K_+.
\]

has a unique solution \((0, 0)_K \in \mathbb{R}^J \times \mathbb{R}^K_+\). Choose \( x_K = 0 \), the latter tells us that equation \( A_J x_J = 0 \) has a unique solution \( x_J = 0 \), i.e., \( A_J \) has full column rank. Thus \( A^T_J A_J \) is nonsingular. Furthermore, it follows from (6.6) that \( A^T_J A_J x_J = A^T_J A_J Q_K x_K \), which means

\[
x_J = (A^T_J A_J)^{-1} A^T_J A_J Q_K x_K = A^T_J A_J Q_K x_K.
\]

This together with (6.6) tells us that the system

\[
A_J A^T_J A_J Q_K x_K - A_K Q_K x_K = (A_J A^T_J A_J Q_K - A_K Q_K) x_K = 0, x_K \in \mathbb{R}^K_+
\]

has a unique solution \( x_K = 0 \in \mathbb{R}^K_+ \), which clearly verifies (6.2) and thus (iii).

To justify the converse implication, suppose that (iii) is valid. Consider the equation (6.6) in (ii), since \( A_J \) has the full rank column, we also have (6.7). It is similar to the above justification that \( x_K \) satisfies equation (6.8). Thanks to (6.2) in (iii), we get from (6.8) that \( x_K = 0 \) and thus \( x_J = 0 \) by (6.7). This verifies that the equation (6.6) in (ii) has a unique solution \((x_J, x_K) = (0, 0)_K\).

Finally, the equivalence between (iii) and (iv) follows from the well-known Gordan’s lemma and the fact that the matrix \( A_J A^T_J \) is symmetric. \( \square \)

Next let us discuss some known conditions relating the uniqueness of optimal solution to Lasso. In [19], Fuchs introduced a sufficient condition for the above property:

\[
A^T_J (A_J x^*_J - b) = -\mu \text{sign}(x^*_J), \quad (6.9)
\]

\[
\|A^T_J (A_J x^*_J - b)\|_\infty < \mu, \quad (6.10)
\]

\[
A_J \text{ has full column rank}. \quad (6.11)
\]

The first equality (6.9) indeed tells us that \( x^* \) is an optimal solution to Lasso problem. Inequality (6.10) means that \( \mathcal{E} = J \), i.e., \( K = \emptyset \) in Theorem 6.1. (6.11) is also present in our characterizations. Hence Fuchs’ condition implies (iii) in Theorem 6.1 and is clearly not a necessary condition for the uniqueness of optimal solution to Lasso problem, since in many situations the set \( K \) is not empty.
Furthermore, in the recent work [39] Tibshirani shows that the optimal solution $x^*$ to problem (1.3) is unique when the matrix $A_E$ has full column rank. This condition is sufficient for our (ii) in Theorem 6.1. Indeed, if $(x_J, x_K)$ satisfies system (6.6) in (ii), we have $A_E[x_J - Q_Kx_K]^T = 0$, which implies that $x_J = 0$ and $Q_Kx_K = 0$ when $\ker A_E = 0$. Since $Q_K$ is invertible, the latter tells us that $x_J = 0$ and $x_K = 0$, which clearly verifies (ii). Tibshirani’s condition is also necessary for the uniqueness of optimal solution to Lasso problem for almost all $b$ in (1.3), but it is not for any $b$; a concrete example could be found in [46].

In the recent works [46, 47], the following useful characterization of unique solution to Lasso has been established under mild assumptions:

There exists $y \in \mathbb{R}^m$ satisfying $A_J^T y = \text{sign} (x_J^*)$ and $\|A_K^T y\|_\infty < 1$, \hspace{1cm} (6.12)

$A_J$ has full column rank.

It is still open to us to connect directly this condition to those ones in Theorem 6.1 although they must be logically equivalent under the assumptions required in [46, 47]. However, our approach via second-order variational analysis is completely different and also provides several new characterizations for the uniqueness of optimal solution to Lasso. It is also worth mentioning here that the standing assumption in [46] that $A$ has full row rank is relaxed in our study.

To end this section, we note that the procedure in this section could be extended to investigate the same property for other structured optimization problem in [47] as well as the well-known nuclear norm regularized least square optimization problem

$$\min_{X \in \mathbb{R}^{p \times q}} h(X) := \|AX - b\|^2 + \mu\|X\|_*, \hspace{1cm} (6.13)$$

where $A : \mathbb{R}^{p \times q} \to \mathbb{R}^m$ is a linear operator and $\|X\|_*$ is the trace norm (known as well the nuclear norm) of $X$. The recent main result in [45] could imply that $\partial h$ is locally metrically regular at any point on its graph under a mild assumption. So $X^*$ is a unique optimal solution to (6.13) if and only if it is an optimal solution and $\partial h$ is strongly metrically regular at $X^*$ for 0 under a mild assumption. Then the same approach to problem (6.13) will lead us to several new complete characterizations for the uniqueness of optimal solution to this problem provided that the graphical derivative $D\partial \| \cdot \|_*$ is fully calculated.

7 Conclusion

In this paper we analyze the Q-linear convergence of the forward-backward splitting method for solving nonsmooth convex optimization problems and the uniqueness of optimal solution to Lasso. Our work recovers several recent results in [7, 17, 19, 21, 22, 23, 32, 39, 42] and reveals many new information. (Strong) Metric subregularity on the subdifferential and second-order growth condition play significant roles in our analysis. It is well-recognized that KL-inequality with order $\frac{1}{2}$, which is equivalent to both latter properties in convex frameworks, is a very useful tool to guarantee the convergence of many proximal-type algorithms even for nonconvex optimization problems. In future research we intend to study the connection of metric subregularity of subdifferential and second-order growth condition with KL inequality and their effects to the convergence of proximal algorithms in nonconvex settings. Extending the approach in Section 6 to investigate the uniqueness of optimal solution to $\ell_0$-optimization problem is also a potential project that we are working on.
References

[1] D. Azé and J.-N. Corvellec: Nonlinear local error bounds via a change of metric, *J. Fixed Point Theory Appl.* 16 (2014), 251–372.

[2] F. J. Aragón Artacho and M. H. Geoffroy, Characterizations of metric regularity of subdifferentials, *J. Convex Anal.* 15 (2008), 365–380.

[3] F. J. Aragón Artacho and M. H. Geoffroy: Metric subregularity of the convex subdifferential in Banach spaces, *J. Nonlinear Convex Anal.* 15 (2014), 35–47.

[4] H.H. Bauschke, J. Bolte, and M. Teboulle: A descent lemma beyond Lipschitz gradient continuity: first-order methods revisited and applications, *Math. Oper. Res.* 42 (2017), 330–348.

[5] H. H. Bauschke and P. L. Combettes: *Convex Analysis and Monotone Operator Theory in Hilbert Spaces.* Springer, New York (2011).

[6] K. Bredies and D. A. Lorenz: Linear convergence of iterative soft-thresholding. *Journal of Fourier Analysis and Applications* 14 (2008), 813–837.

[7] J. Bolte, T.P. Nguyen, J. Peypouquet, and B. W. Suter: From error bounds to the complexity of first-order descent methods for convex functions, *Mathematical Programming*, (2016). doi:10.1007/s10107-016-1091-6.

[8] J. Y. Bello Cruz and T. T. A. Nghia: On the convergence of the proximal forward-backward splitting method with linesearches, *Optim. Method Softw.* 31 (2016), 1209–1238.

[9] H. H. Bauschke, H. M. Phan, and D. Noll: Linear and strong convergence of algorithms involving averaged nonexpansive operators, *J. Math. Anal. Appl.*, 421 (2015), 1–20.

[10] S. Bonettini and V. Ruggiero, On the convergence of primal-dual hybrid gradient algorithms for total variation image restoration. *J. Math. Imaging Vision*, 44 (2012) 236–253.

[11] A. Beck and M. Teboulle: *Gradient-Based Algorithms with Applications to Signal Recovery Problems.* in *Convex Optimization in Signal Processing and Communications*, (D. Palomar and Y. Eldar, eds.) 42–88 University Press, Cambribge (2010).

[12] I. Csiszár: Why least squares and maximum entropy? An axiomatic approach to inference for linear inverse problems, *Ann. Statist.* 19 (1991), 2032–2066.

[13] P. L. Combettes and J.-C. Pesquet: Proximal splitting methods in signal processing. in *Fixed-Point Algorithms for Inverse Problems. Science and Engineering. Springer Optimization and Its Applications* 49 (2011), 185–212 Springer, New York.

[14] P. L. Combettes and V. R. Wajs: Signal recovery by proximal forward-backward splitting, *Multiscale Model. Simul.* 4 (2005), 1168–1200.

[15] I. Daubechies, M. Defrise, and D. De Mol: An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Comm. Pure Appl. Math.* 57 (2004), 1413–1457.

[16] A. L. Dontchev and R. T. Rockafellar: *Implicit Functions and Solution Mappings. A View from Variational Analysis*, Springer, Dordrecht, 2009.

[17] D. Drusvyatskiy and A. Lewis: Error bounds, quadratic growth, and linear convergence of proximal methods, *Math. Oper. Res. doi.org/10.1287/moor.2017.0889*

[18] D. Drusvyatskiy, B. S. Mordukhovich and T. T. A. Nghia, Second-order growth, tilt stability, and metric regularity of the subdifferential, *J. Convex Anal.* 21 (2014), 1165–1192.
[19] J.-J. Fuchs: On sparse representations in arbitrary redundant bases. *IEEE Trans. Inform. Theory*, **50** (2004), 1341–1344.

[20] A. J. Hoffman: On approximate solutions of systems of linear inequalities. *J. Res. Nat. Bur. Standards*, **49** (1952), 263–265.

[21] E. T. Hale, W. Yin, and Y. Zhang: Fixed-point continuation for $\ell_1$-minimization: methodology and convergence. *SIAM J. Optim.* **19** (2008), 1107–1130.

[22] J. Liang, J. Fadili, and G. Peyré: Local linear convergence of forward-backward under partial smoothness, *Adv. Neural Inf. Process Syst.* (2014).

[23] J. Liang, J. Fadili, and G. Peyré: Activity identification and local linear convergence of forward–backward type methods, *SIAM J. Optim.* **27** (2017), 408–437.

[24] A. S. Lewis: Active sets, nonsmoothness, and sensitivity, *SIAM J. Optim.* **23** (2002), 702–725.

[25] A. S. Lewis and S. Zhang: Partial smoothness, tilt stability, and generalized Hessians, *SIAM J. Optim.* **23** (2013), 74–94.

[26] G. Li and T.K. Pong: Calculus of the exponent of Kurdyka-Łojasiewicz inequality and its applications to linear convergence of first-order methods, *Found. Comp. Math.* (2018) doi.org/10.1007/s10208-017-9366-8

[27] Z.-Q. Luo and P. Tseng: Error bounds and convergence analysis of feasible descent methods: a general approach, *Ann. Oper. Res.* **46** (1993), 157–178.

[28] S. A. Miller and J. Malick: Newton methods for nonsmooth convex minimization: connections among U-Lagrangian, Riemannian Newton and SQP methods, *Math. Program.* **104** (2005), 609–633.

[29] B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications*, Springer, Berlin (2006).

[30] B. S. Mordukhovich and T. T. A. Nghia: Second-order variational analysis and characterizations of tilt-stable optimal solutions in infinite-dimensional spaces, *Nonlinear Anal.* **86** (2013), 159–180.

[31] B. S. Mordukhovich and T. T. A. Nghia: Second-order characterizations of tilt stability with applications to nonlinear programming, *Math. Program.* **149** (2015), 83–104.

[32] I. Necoara, Yu. Nesterov, and F. Glineur: Linear convergence of first order methods for non-strongly convex optimization, *Math. Program.* (2018) doi.org/10.1007/s10107-018-1232-1.

[33] P. Neal and S. Boyd: Proximal Algorithms, *Foundations and Trends in Optimization* **1** (2014), 127–239.

[34] R. A. Poliquin and R. T. Rockafellar: Tilt stability of a local minimum, *SIAM J. Optim.* **8** (1998), 287–299.

[35] S. M. Robinson: Some continuity properties of polyhedral multifunctions, *Math. Program. Study* **14** (1981), 206–214.

[36] R. T. Rockafellar and R. J-B. Wets: *Variational Analysis*, Springer, Berlin, 1998.

[37] S. Salzo: The variable metric forward-backward splitting algorithm under mild differentiability assumptions, *SIAM J. Optim.*, **27** (2017), 2153–2181.

[38] R. Tibshirani: Regression shrinkage and selection via the Lasso, *J. R. Stat. Soc.* **58** (1996), 267–288

[39] R. J. Tibshirani: The Lasso problem and uniqueness, *Electron. J. Stat.* **7** (2013), 1456–1490.
[40] J. Tropp: Just relax: Convex programming methods for identifying sparse signals in noise, *IEEE Trans. Inform. Theory*, **52** (2006), 1030–1051.

[41] P. Tseng: A modified forward-backward splitting method for maximal monotone mappings, *SIAM J. Control Optim.*, **38** (2000), 431–446.

[42] S. Tao, D. Boley, and S. Zhang: Local linear convergence of ISTA and FISTA on the Lasso problem, *SIAM J. Optim.*, **26** (2016), 313–336.

[43] Y. Vardi, L.A. Shepp, L. Kaufman: A statistical model for positron emission tomography, *J. Amer. Statist. Assoc.*, **80** (1985), 8–37.

[44] M. J. Wainwright: Sharp thresholds for high-dimensional and noisy sparsity recovery using $\ell_1$-constrained quadratic programming (lasso), *IEEE Trans. Inform. Theory*, **55** (2009), 2183–2202.

[45] Z. Zhou and A. M-C. So: A unified approach to error bounds for structured convex optimization, *Math. Program.*, **165** (2017), 689–728.

[46] H. Zhang, W. Yin, L. Cheng: Necessary and sufficient conditions of solution uniqueness in 1-norm minimization, *J. Optim. Theory Appl.*, **164** (2015), 109–122.

[47] H. Zhang, M. Yan, W. Yin: One condition for solution uniqueness and robustness of both $\ell_1$-synthesis and $\ell_1$-analysis minimizations, *Adv. Comput. Math.*, **42** (2016), 1381–1399.