ON APPROXIMATION SPACES AND GREEDY-TYPE BASES

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ABSTRACT. The purpose of this paper is to introduce ω-Chebyshev-greedy and ω-partially greedy approximation classes and to study their relation with ω-approximation spaces, where the latter are a generalization of the classical approximation spaces. The relation gives us sufficient conditions of when certain continuous embeddings imply different greedy-type properties. Along the way, we generalize a result by P. Wojtaszczyk as well as characterize semi-greedy Schauder bases in quasi-Banach spaces, generalizing a previous result by the first author.

CONTENTS

1. Background and main results
2. Preliminary results
3. Semi-greedy Schauder bases in quasi-Banach spaces
4. Approximation classes and (semi-) greedy bases
5. Approximation classes and partially greedy bases
6. Examples
7. Annex: Lower and upper regularity properties
Acknowledgments
References

1. BACKGROUND AND MAIN RESULTS

Let \((X, \| \cdot \|)\) be a quasi-Banach space; that is, \(X\) is a complete vector space over \(\mathbb{F} = \mathbb{R} \) or \(\mathbb{C}\), with a quasi-norm \(\| \cdot \| : X \rightarrow [0, \infty)\) such that

(C1) \(\|x\| = 0\) if and only if \(x = 0\),
(C2) \(\|tx\| = |t|\|x\|\) for all \(t \in \mathbb{F}\) and all \(x \in X\), and
(C3) there is a constant \(\kappa \geq 1\) so that

\[
\|x + y\| \leq \kappa(\|x\| + \|y\|), \forall x, y \in X.
\]

Given \(0 < p \leq 1\), a \(p\)-norm is a map \(\| \cdot \| : X \rightarrow [0, \infty)\) satisfying (C1), (C2), and

(C4) \(\|x + y\|^p \leq \|x\|^p + \|y\|^p\) for all \(x, y \in X\).

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It follows that the condition \((C4)\) implies \((C3)\) with \(\kappa = 2^{1/p-1}\). A quasi-Banach space whose quasi-norm is a \(p\)-norm shall be called a \(p\)-Banach space. Thanks to the Aoki-Rolewicz’s Theorem (see \([2, 17]\)), we know that any quasi-Banach space \(X\) is \(p\)-convex for some \(0 < p \leq 1\), i.e., there is a constant \(C\) such that

\[
\left\| \sum_{j=1}^{n} x_j \right\|^p \leq C \sum_{j=1}^{n} \|x_j\|^p, \forall n \in \mathbb{N}, \forall x_j \in X.
\]

This way, a quasi-Banach space becomes \(p\)-Banach under a suitable renorming.

We say that \(B = \{e_n\}_{n=1}^{\infty}\) is a semi-normalized Markushevich basis (or an \(M\)-basis or simply a basis) of \(X\) if the following holds

\begin{enumerate}[a)]  
\item There exists a (unique) collection \(\{e_n\}_{n=1}^{\infty} \subset (X, \|\cdot\|_*)\), called the biorthogonal functionals, such that \(e_i^*(e_j) = \delta_{ij}\).
\item There exist \(c_1, c_2 > 0\) such that \(0 < c_1 \leq \{\|e_n\|, \|e_n^*\|_*\} \leq c_2 < \infty\) for all \(n \in \mathbb{N}\).
\item \(X = \text{span}\{e_n : n \in \mathbb{N}\}\).
\item If \(e_n^*(x) = 0\) for all \(n \in \mathbb{N}\), then \(x = 0\).
\end{enumerate}

Under these conditions, every \(x \in X\) is represented by \(x \sim \sum_{n=1}^{\infty} e_n^*(x)e_n\) and \(\lim_{n \to \infty} e_n^*(x) = 0\). Also, if we consider the algorithm of partial sums \(S_m(x) = \sum_{n=1}^{m} e_n^*(x)e_n\), we say that \(B\) is a Schauder basis if there is a positive constant \(C\) such that

\[
\|S_m(x)\| \leq C\|x\|, \forall m \in \mathbb{N}, \forall x \in X.
\]

We denote by \(K_\beta\) the smallest constant in \((1.1)\), which is called the basis constant.

One of the main objects studied in Approximation Theory is the nonlinear approximation spaces \(A^\alpha_q(X, B)\): for \(\alpha > 0\) and \(0 < q < \infty\),

\[
A^\alpha_q(B, X) = A^\alpha_q := \left\{ x \in X : \|x\|_{A^\alpha_q} := \|x\| \left[ \sum_{n=1}^{\infty} \left( n^\alpha \sigma_n(x) \right)^q \frac{1}{n} \right]^{1/q} \right\} < \infty
\]

and

\[
A^\alpha_\infty(B, X) = A^\alpha_\infty := \left\{ x \in X : \|x\|_{A^\alpha_\infty} := \|x\| + \sup_{n \geq 1} n^\alpha \sigma_n(x) < \infty \right\},
\]

where \(\sigma_m(x)\) is the \(m\)-term error of approximation:

\[
\sigma_m(B, x) = \sigma_m(x) := \inf \left\{ \left\| x - \sum_{n \in A} b_ne_n \right\| : |A| = m, b_n \in \mathbb{F} \right\}.
\]

There have been studies on embeddings among these approximation spaces and Lorentz spaces (see \([12, 14, 15, 18]\)). One of the recent results is given in \([13]\):

**Theorem 1.1.** \([13]\) Let \(X\) be a quasi-Banach space with an unconditional \(M\)-basis \(B\). Assume that \(h_1(m)\) is a doubling function. Then for \(\alpha > 0\) and \(q \in (0, \infty]\),
we have the following continuous embeddings
\[ \ell^q_{k_\omega h_i(k)}(B, X) \hookrightarrow \mathcal{A}_q(B, X) \hookrightarrow \ell^q_{k_\omega h_i(k)}(B, X) \]
where \( h_i(m) \) and \( h_r(m) \) are the so-called democracy functions.

Also in [13], the authors introduced a new class, called the greedy class, using the Thresholding Greedy Algorithm (TGA) [16]: a greedy sum of \( x \) of order \( m \) is given by
\[ G^\pi_m(x) = \sum_{n=1}^{m} e^*_\pi(n)(x)e_{\pi(n)}, \]
where \( \pi \) is a greedy ordering, i.e., \( \pi : \mathbb{N} \rightarrow \mathbb{N} \) is a permutation such that \( \text{supp}(x) \subseteq \pi(\mathbb{N}) \) and \( |e^*_\pi(i)(x)| \geq |e^*_\pi(j)(x)| \) whenever \( i \leq j \). Also, \( A = \{ \pi(n) : 1 \leq n \leq m \} \) is called a greedy set of \( x \) of order \( m \). The \( m \)th greedy error for \( x \in X \) is the quantity
\[ \gamma_m(x) := \sup_{\pi} \| x - G^\pi_m(x) \|. \quad (1.3) \]
The greedy class \( G^\alpha_q \) is defined as: for \( \alpha > 0 \) and \( q \in (0, \infty) \),
\[ G^\alpha_q(B, X) = G^\alpha_q := \left\{ x \in X : \| x \|_{G^\alpha_q} := \| x \| + \left[ \sum_{n=1}^{\infty} (n^\alpha \gamma_n(x))^{\frac{1}{q}} \right] \frac{1}{n} < \infty \right\}. \]
The class \( G^\alpha_q \) is defined in the same way as \( \mathcal{A}_\infty^\alpha \) with \( \sigma_n \) replaced by \( \gamma_n \). One of the results in [13] is that if an M-basis \( B \) in a quasi-Banach space is greedy, then \( \mathcal{A}_\infty^\alpha \approx G^\alpha_q \). The converse (in the context of Banach spaces) was given by Wojtaszczyk [19]:

**Theorem 1.2.** [19 Theorem 3.1 (restated)] Let \( B \) be an unconditional M-basis in a Banach space \( X \).

1. If \( B \) is greedy, then \( \mathcal{A}_\infty^\alpha \approx G^\alpha_q \) for all \( q \in (0, \infty) \) and \( \alpha > 0 \).
2. If \( \mathcal{A}_\infty^\alpha \approx G^\alpha_q \) for some \( q \in (0, \infty) \) and \( \alpha > 0 \), then \( B \) is greedy.

Statement (1) in Theorem 1.2 follows from definitions, while proving statement (2) is considerably more involved. The purpose of this paper is to study the above result in the more general context of weights and for different greedy-type bases. In particular, given \( q \in (0, \infty) \) and a weight \( \omega = (\omega(n))_{n=1}^{\infty} \) satisfying certain conditions (see definitions in Subsection 2.3), we define the following \( \omega \)-approximation spaces: for \( 0 < q < \infty \),
\[ \mathcal{A}_q^\omega := \left\{ x \in X : \| x \|_{\mathcal{A}_q^\omega} := \| x \| + \left[ \sum_{n=1}^{\infty} (\omega(n)\sigma_n(x))^{\frac{1}{q}} \right]\frac{1}{n} < \infty \right\}; \]
for \( q = \infty \),
\[ \mathcal{A}_\infty^\omega := \left\{ x \in X : \| x \|_{\mathcal{A}_\infty^\omega} := \| x \| + \sup_{n \geq 1} \omega(n)\sigma_n(x) < \infty \right\}. \]

These spaces were recently considered in [19]. When \( \omega(n) = n^\alpha \), we recover the classical approximation space \( \mathcal{A}_\infty^\alpha \). We shall introduce three different greedy approximation classes. The first one is a generalization of \( G^\alpha_q \): if \( 0 < q < \infty \) and \( \omega \) is a weight,
following continuous embeddings: for any \( q > 0 \) and for \( x \in X \) in a quasi-Banach space there exists an absolute constant \( A < \max \left\{ \omega(n) \right\} \).

Let \( \omega \) mean that \( \omega(n) \rightarrow 0 \) as \( n \rightarrow \infty \) is semi-greedy.

Using this error, we define the Chebyshev-greedy approximation class: for \( \omega \),
\[
CG_q^\omega := \left\{ x \in X : \|x\|_{CG_q^\omega} := \|x\| + \sum_{n=1}^\infty (\omega(n)\varphi_n(x))^q \right\}^{1/q} \quad < \infty
\]
and for \( q = \infty \),
\[
CG_\infty^\omega := \left\{ x \in X : \|x\|_{CG_\infty^\omega} := \|x\| + \sup_{n \geq 1} \omega(n)\varphi_n(x) < \infty \right\}.
\]

For \( \omega(n) = n^\alpha \), we recover the original greedy class introduced in [13].

In [11], Dilworth, Kalton, Kutzarova, and Temlyakov introduced a new algorithm that is an enhancement of the rate of convergence of the greedy algorithm. For \( x \in X \), the Thresholding Chebyshev Greedy Algorithm (TCGA) \( CG_m^\pi \) is defined as:

\[
\|x - CG_m^\pi(x)\| = \inf_{(a_n) \subset F} \left\{ x - \sum_{n=1}^m a_n e_{\pi(n)} \right\}.
\]

The \( m \)th Chebyshev-greedy error for \( x \in X \) is
\[
\vartheta_m(x) := \sup_{\pi} \|x - CG_m^\pi(x)\|.
\]
Using this error, we define the Chebyshev-greedy approximation class: for \( 0 < q < \infty \) and a weight \( \omega \),
\[
CG_q^\omega := \left\{ x \in X : \|x\|_{CG_q^\omega} := \|x\| + \sum_{n=1}^\infty (\omega(n)\varphi_n(x))^q \right\}^{1/q} \quad < \infty
\]
and for \( q = \infty \), we consider the usual modification as in \( CG_\infty^\omega \). Obviously, we have the following continuous embeddings: for any \( q > 0 \) and a weight \( \omega \),
\[
CG_q^\omega \hookrightarrow CG_\infty^\omega \hookrightarrow A_q^\omega.
\]

Finally, we introduce the partially greedy class \( PG_q^\omega \): for \( q \in (0, \infty) \) and a weight \( \omega \),
\[
PG_q^\omega := \left\{ x \in X : \|x\|_{PG_q^\omega} := \|x\| + \left( \sum_{n=1}^\infty (\omega(n)\phi_n(x))^q \right)^{1/q} \quad < \infty \right\},
\]
where \( \beta_m(x) := \|x - S_m(x)\| \).

For two functions \( f(a_1, a_2, \ldots) \) and \( g(a_1, a_2, \ldots) \), we write \( f \lesssim g \) to indicate that there exists an absolute constant \( C > 0 \) (independent of \( a_1, a_2, \ldots \)) such that \( f \leq C g \).
Similarly, \( f \gtrsim g \) means that \( Cf \geq g \) for some constant \( C \). Furthermore, \( f \asymp g \) means that \( f \lesssim g \) and \( f \gtrsim g \). For two sets \( A, B \subset \mathbb{N} \), we write \( A < B \) to mean that \( \max A < \min B \). We are ready to state the three main results of this paper.

**Theorem 1.3.** Let \( \omega \) be a doubling weight with \( i_\omega > 0 \). Let \( B \) be a quasi-greedy M-basis in a quasi-Banach space \( X \).

1. If \( B \) is semi-greedy, then \( A_q^\omega \approx CG_q^\omega \) for all \( q \in (0, \infty) \).
2. If \( B \) is Schauder with Property (W) and \( A_q^\omega \approx CG_q^\omega \) for some \( q \in (0, \infty) \), then \( B \) is semi-greedy.

The techniques used to prove the Theorem 1.3 allow us to generalize Theorem 1.2.
Theorem 1.4. Let $\omega$ be a doubling weight with $i_\omega > 0$. Let $\mathcal{B}$ be an unconditional $M$-basis in a quasi-Banach space $\mathbb{X}$.

(1) If $\mathcal{B}$ is greedy, then $A_\omega^q \approx G_\omega^q$ for all $q \in (0, \infty]$.

(2) If $A_\omega^q \approx G_\omega^q$ for some $q \in (0, \infty]$, then $\mathcal{B}$ is greedy.

Our final result is related to partially greedy bases, first introduced by Dilworth et al.

Theorem 1.5. Let $\omega$ be a doubling weight with $i_\omega > 0$. Let $\mathcal{B}$ be a quasi-greedy $M$-basis in a quasi-Banach space $\mathbb{X}$.

(1) If $\mathcal{B}$ is partially greedy, then $\mathcal{P}G_\omega^q \hookrightarrow G_\omega^q$ for all $q \in (0, \infty]$.

(2) If $\mathcal{B}$ is Schauder with Property (I) and Property (W$^*$) and $\mathcal{P}G_\omega^q \hookrightarrow G_\omega^q$ for some $q \in (0, \infty]$, then $\mathcal{B}$ is partially greedy.

The above-mentioned properties, the dilation index $i_\omega$, and other terminologies will be defined later.

Remark 1.6. Note that for no basis we have $G_\omega^q \hookrightarrow \mathcal{P}G_\omega^q$. Indeed, for $0 < q < \infty$, given $m \in \mathbb{N}$, we have

$$\|e_{m+1}\|_{G_\omega^q} = \|e_{m+1}\|,$$

but

$$\|e_{m+1}\|_{\mathcal{P}G_\omega^q} = \|e_{m+1}\| + \left( \sum_{n=1}^{m} \left( \omega(n) \|e_{m+1}\| \frac{1}{n} \right)^{\frac{q}{\omega}} \right)^{\frac{1}{q}} \geq \|e_{m+1}\| \omega(1) \left( \sum_{n=1}^{m} \frac{1}{n} \right)^{\frac{q}{\omega}}.$$

Hence,

$$\frac{\|e_{m+1}\|_{G_\omega^q}}{\|e_{m+1}\|_{\mathcal{P}G_\omega^q}} \leq \frac{1}{\omega(1) \left( \sum_{n=1}^{m} \frac{1}{n} \right)^{\frac{q}{\omega}}} \rightarrow 0 \quad m \rightarrow \infty.$$

A similar computation gives the result for $q = \infty$.

2. Preliminary results

2.1. Convexity. One of the arguably most important properties of Banach spaces is convexity. In the case of $p$-Banach spaces, we will use the following result that is an extension of classical results of convexity. Given $0 < p \leq 1$, we put

$$A_p = \frac{1}{(2p-1)^{1/p}}. \quad (2.1)$$

Proposition 2.1 ([1, Corollary 2.3]). Let $\mathbb{X}$ be a $p$-Banach space for some $0 < p \leq 1$. Let $(x_j)_{j \in J} \subset \mathbb{X}$ with $J$ finite, and $g \in \mathbb{X}$. Then

(1) For any scalars $(a_j)_{j \in J}$ with $0 \leq a_j \leq 1$, we have

$$\left\| g + \sum_{j \in J} a_j x_j \right\| \leq A_p \sup \left\{ \left\| g + \sum_{j \in A} x_j \right\| : A \subset J \right\}.$$
2.2. Greedy-type bases. In [16], Konyagin and Temlyakov introduced the TGA $(G^\pi_m)_{m=1}^\infty$ as we described in the previous section and defined greedy bases in Banach spaces.

**Definition 2.2.** An M-basis $B$ in a quasi-Banach space is greedy if $\gamma_m(x) \asymp \sigma_m(x)$.

Moreover, they characterized greedy bases in terms of unconditionality and democracy in the context of Banach spaces. In [1], the authors proved the same characterization of greediness for quasi-Banach spaces. Recall that an M-basis is $K$-unconditional if

$$K := \sup_{A \subset \mathbb{N}, |A| < \infty} \|P_A\| < \infty,$$

where $P_A(x) = \sum_{n \in A} e_n^*(x)e_n$ is the projection operator. To discuss democracy, we need the indicator sum

$$1_{\varepsilon, A} = 1_{\varepsilon, A}[B, X] := \sum_{n \in A} \varepsilon_n e_n,$$

where $\varepsilon = (\varepsilon_n)_{n \in A}$ with $|\varepsilon_n| = 1$ for all $n \in A$. We use the notation $|\varepsilon| = 1$.

**Definition 2.3.** We say that $B$ is a super-democratic basis in a quasi-Banach space if there is a positive constant $C$ such that

$$\|1_{\varepsilon, A}\| \leq C \|1_{\delta, B}\|,$$

for any $|A| \leq |B| < \infty$ and $|\varepsilon| = |\delta| = 1$. The smallest constant verifying (2.2) is denoted by $C_{sd}$ and we say that $B$ is $C_{sd}$-super-democratic. If $\varepsilon \equiv \delta \equiv 1$, we say that $B$ is $C_d$-democratic.

Equivalently, to define super-democracy, we can use the democracy functions: for each $m = 1, 2, \ldots$,

$$h_r(m) := \sup_{|A| \leq m, |\varepsilon| = 1} \|1_{\varepsilon, A}\| \quad \text{and} \quad h_l(m) := \inf_{|A| \geq m, |\varepsilon| = 1} \|1_{\varepsilon, A}\|.$$

Then $B$ is super-democratic if and only if

$$\sup_{m \geq 1} \frac{h_r(m)}{h_l(m)} < \infty.$$

**Remark 2.4.** For each $m \in \mathbb{N}$ in a $p$-Banach space,

$$2^{1-1/p} h_r(m) \leq h_r(m) := \sup_{|A| = m, |\varepsilon| = 1} \|1_{\varepsilon, A}\| \leq h_r(m).$$

Indeed, if $|A| \leq N$, take any $B \subset \mathbb{N}$ such that $A \subset B$ and $|B| = N$. Indeed, we have

$$\|1_{\varepsilon, A}\|^p = \left\|\frac{1}{2}(1_{\varepsilon, A} + 1_{B \setminus A}) + \frac{1}{2}(1_{\varepsilon, A} - 1_{B \setminus A})\right\|^p \leq \frac{1}{2^p}\|1_{\varepsilon, A} + 1_{B \setminus A}\|^p + \frac{1}{2^p}\|1_{\varepsilon, A} - 1_{B \setminus A}\|^p \leq 2^{1-p}(h_r(N))^p.$$
If $B$ is Schauder with basis constant $K_b$,

$$h_t(m) \leq h_t(m) := \inf_{|A|=m,|\varepsilon|=1} \|1_{\varepsilon A}\| \leq K_b h_t(m).$$

Moreover, Dilworth, Kalton, and Kutzarova [10] introduced the concept of semi-greedy bases.

**Definition 2.5.** We say that $B$ is a semi-greedy basis in a quasi-Banach space $X$ if there is a positive constant $C$ such that

$$\vartheta_m(x) \leq C \sigma_m(x), \forall x \in X, \forall m \in \mathbb{N}. \quad (2.3)$$

The smallest constant verifying $(2.3)$ is denoted by $C_{sg}$, and we say that $B$ is $C_{sg}$-semi-greedy.

Semi-greedy Schauder bases were first characterized in Banach spaces with finite cotype in terms of quasi-greediness and democracy [10], and later on, the first author of this paper [4] (see also [5]) removed the condition of finite cotype. The notion of quasi-greediness was introduced in [16]:

**Definition 2.6.** We say that $B$ is a quasi-greedy basis in a quasi-Banach space $X$ if there is a positive constant $C$ such that

$$\gamma_m(x) \leq C \|x\|, \forall x \in X, \forall m \in \mathbb{N}. \quad (2.4)$$

The smallest constant verifying $(2.4)$ is denoted by $C_q$, and we say that $B$ is $C_q$-quasi-greedy.

The characterization of semi-greediness in the context of quasi-Banach spaces is unknown and we shall prove that the same characterization of semi-greediness in Banach spaces holds for quasi-Banach spaces (see Section 3.) In the setting of Banach spaces, Dilworth, Kalton, Kutzarova, and Temlyakov [11] introduced partially greedy bases and characterized them as being quasi-greedy and conservative. Berná [3] showed that the same result holds for quasi-Banach spaces (under a stronger notion of partially greediness.)

**Definition 2.7.** We say that a basis $B$ in a quasi-Banach space is partially greedy if

$$\gamma_m(x) \lesssim \beta_m(x).$$

**Definition 2.8.** We say that $B$ is a super-conservative basis in a quasi-Banach space if there is a positive constant $C$ such that

$$\|1_{\varepsilon A}\| \leq C \|1_{\delta B}\|, \quad (2.5)$$

for any $|A| \leq |B| < \infty$, $\max A < \min B$, and $|\varepsilon| = 1, |\delta| = 1$. The smallest constant verifying $(2.5)$ is denoted by $C_{sc}$ and we say that $B$ is $C_{sc}$-super-conservative. If $\varepsilon \equiv \delta \equiv 1$, we say that $B$ is $C_{c}$-conservative.

In both Definitions 2.3 and 2.8, we require $|A| \leq |B|$. By $p$-convexity, this requirement can be replaced by $|A| = |B|$ (but the super-democratic (conservative) constants may differ.)

**Theorem 2.9** (Berná [3]). A Schauder basis $B$ of a quasi-Banach space is partially greedy if and only if it is quasi-greedy and is super-conservative.
Remark 2.10. In [3], it was actually proved that an $M$-basis of a quasi-Banach space is strongly partially greedy if and only if it is quasi-greedy and is super-conservative. In the context of Schauder bases, being strongly partially greedy is equivalent to being partially greedy (see [3, Remark 3.5].)

2.3. Weight classes. A weight is any sequence $\omega = (\omega(n))_{n=1}^{\infty}$ of nonnegative numbers with $\omega(1) > 0$. We use the following notation:

1. $\mathcal{W}$ for the set of nondecreasing and positive weights: $0 < \omega(1) \leq \omega(2) \leq \cdots$, and $\lim_{n \to \infty} \omega(n) = \infty$.
2. $\mathcal{W}_d$ is the set of doubling weights, i.e., $\omega \in \mathcal{W}$ and there is a positive constant $\theta \geq 1$ such that $\omega(2n) \leq \theta \cdot \omega(n)$ for all $n \in \mathbb{N}$. We say that $\omega$ is $\theta$-doubling.

Associated with a weight $\omega$, we consider the summing weight (see [6]): for each $m = 1, 2, \ldots$,

$$\tilde{\omega}(m) := \sum_{n=1}^{m} \frac{\omega(n)}{n}.$$

Remark 2.11. Thanks to [6, Proposition 2.4], we know that if $\omega \in \mathcal{W}_d$, then $\tilde{\omega} \in \mathcal{W}_d$. Specifically, if $\omega$ is $\theta$-doubling, then $\tilde{\omega}$ is $\frac{3\theta}{2}$-doubling.

A desired relation between a weight $\omega$ and its summing weight $\tilde{\omega}$ is that $\omega(n) \asymp \tilde{\omega}(n)$. To obtain the sufficient condition for this relation to hold, we need to define the so-called dilation indices.

Definition 2.12. The lower and upper dilation sequences associated with a positive sequence $\omega = (\omega(n))_{n=1}^{\infty}$ are given by

$$\varphi_\omega(M) := \inf_{k \geq 1} \frac{\omega(Mk)}{\omega(k)} \quad \text{and} \quad \Phi_\omega(M) := \sup_{k \geq 1} \frac{\omega(Mk)}{\omega(k)}, \quad M = 1, 2, 3, \ldots \quad (2.6)$$

Observe that $\varphi_\omega(M) \leq \frac{\omega(M)}{\omega(1)} \leq \Phi_\omega(M)$ and

$$\varphi_\omega(M_1) \varphi_\omega(M_2) \leq \varphi_\omega(M_1M_2) \leq \Phi_\omega(M_1M_2) \leq \Phi_\omega(M_1) \Phi_\omega(M_2), \quad (2.7)$$

that is, $\varphi_\omega$ is super-multiplicative and $\Phi_\omega$ is sub-multiplicative. If $\omega \in \mathcal{W}$, then $\varphi_\omega$ and $\Phi_\omega$ are nondecreasing and $\varphi_\omega(M) \geq 1$ for all $M \in \mathbb{N}$.

Proposition 2.13. Let $\omega \in \mathcal{W}$. The following statements are equivalent:

1. $\omega \in \mathcal{W}_d$.
2. $\Phi_\omega(M) < \infty$ for all $M \in \mathbb{N}$.
3. $\Phi_\omega(M_0) < \infty$ for some $M_0 \in \mathbb{N}_{\geq 2}$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. To prove (3) $\Rightarrow$ (2), fix $M \in \mathbb{N}$. Choose $k \in \mathbb{N}$ such that $M \leq M_0^k$. Since $\Phi_\omega$ is nondecreasing and sub-multiplicative, we have

$$\Phi_\omega(M) \leq \Phi_\omega(M_0^k) \leq \Phi_\omega(M_0)^k < \infty.$$

Now, we show that (2) implies (1). Taking into account that

$$\frac{\omega(2k)}{\omega(k)} \leq \Phi_\omega(2) < \infty,$$

we have the inequality $\omega(2k) \leq \Phi_\omega(2) \omega(k)$, so $\omega \in \mathcal{W}_d$ and the proof is done. \[\square\]
Definition 2.14. If \( \omega \in \mathbb{W} \), we define the associated lower and upper dilation indices, respectively, by

\[
i_{\omega} = \sup_{M > 1} \frac{\ln(\varphi_{\omega}(M))}{\ln M} \quad \text{and} \quad I_{\omega} = \inf_{M > 1} \frac{\ln(\Phi_{\omega}(M))}{\ln M}.
\]

(2.8)

Observe from (2.7) that

\[
(\varphi_{\omega}(M))^{n} \leq \varphi_{\omega}(M^{n}) \leq \Phi_{\omega}(M^{n}) \leq (\Phi_{\omega}(M))^{n},
\]

and therefore,

\[
\frac{\ln(\varphi_{\omega}(M))}{\ln M} \leq \frac{\ln(\varphi_{\omega}(M^{n}))}{\ln M^{n}} \leq \frac{\ln(\Phi_{\omega}(M^{n}))}{\ln M^{n}} \leq \frac{\ln(\Phi_{\omega}(M))}{\ln M}.
\]

(2.9)

Hence, one can replace “sup” and “inf” in (2.8) by “\( \lim \sup \)” and “\( \lim \inf \).”

Proposition 2.15. If \( \omega \in \mathbb{W} \), then

\[
0 \leq i_{\omega} \leq I_{\omega} \leq \infty,
\]

and

\[
i_{\omega} = \lim_{M \to \infty} \frac{\ln(\varphi_{\omega}(M))}{\ln M} \quad \text{and} \quad I_{\omega} = \lim_{M \to \infty} \frac{\ln(\Phi_{\omega}(M))}{\ln M}.
\]

(2.10)

Moreover, \( \omega \in \mathbb{W}_{d} \) if and only if \( I_{\omega} < \infty \).

Proof. Note that \( \omega \in \mathbb{W} \) implies that \( 1 \leq \varphi_{\omega} \leq \Phi_{\omega} \). Then (2.10) follows easily from (2.9). Now, we prove the last assertion. Assuming that \( I_{\omega} < \infty \), we have

\[
\frac{\ln(\Phi_{\omega}(M))}{\ln M} \leq I_{\omega}, \quad M = 1, 2, \ldots
\]

Thus, for every \( M, \Phi_{\omega}(M) < \infty \) and invoking Proposition 2.13 \( \omega \in \mathbb{W}_{d} \). For the converse, if \( \omega \in \mathbb{W}_{d} \), by Proposition 2.13 \( \Phi_{\omega}(M) < \infty \), hence \( \frac{\ln(\Phi_{\omega}(M))}{\ln M} \leq I_{\omega} \) for every natural \( M > 1 \). Hence, \( I_{\omega} < \infty \).

The last assertion is a direct consequence of Proposition 2.13.

\( \square \)

Lemma 2.16. If \( \omega \in \mathbb{W}_{d} \) with doubling constant \( \theta \) and \( i_{\omega} > 0 \), then for any \( q \in (0, \infty) \), \( \omega^{q} \in \mathbb{W}_{d} \) with doubling constant \( \theta^{q} \) and \( i_{\omega^{q}} > 0 \).

Proof. Trivially, \( i_{\omega^{q}} > 0 \). Also, since \( \omega(2n) \leq \theta \omega(n) \), it follows that \( \omega^{q}(2n) \leq \theta^{q} \omega^{q}(n) \) for every \( n = 1, 2, \ldots \).

\( \square \)

Proposition 2.17. \([6, \text{Proposition 2.4}] \) Let \( \omega \in \mathbb{W}_{d} \) with constant \( \theta \). Then

\[
\omega(N) \leq \frac{\theta}{\ln(2)} \tilde{\omega}(N), \quad N = 1, 2, \ldots
\]

Proposition 2.18. \([6, \text{Proposition 2.5}] \) Let \( \omega \in \mathbb{W} \). Then, \( \sup_{N \geq 1} \frac{\tilde{\omega}(N)}{\omega(N)} < \infty \) if and only if \( i_{\omega} > 0 \).

Proposition 2.19. Let \( \omega \in \mathbb{W}_{d} \) with \( i_{\omega} > 0 \). Then \( \omega(N) \approx \tilde{\omega}(N) \).

Proof. This follows immediately from Propositions 2.17 and 2.18.

Next, we prove an auxiliary result that shall be used in due course.

Lemma 2.20. Let \( \omega \in \mathbb{W}_{d} \) and \( \alpha > I_{\omega} \). There exists \( C_{\alpha} > 0 \) such that

\[
\omega(Mk) \leq C_{\alpha} M^{\alpha} \omega(k), \quad \forall M, k \geq 1.
\]
Proof. Let $\omega \in \mathcal{W}_d$ and $\alpha > I_{\omega}$. By (2.10), there exists $M_\alpha \geq 2$ such that
\[
\frac{\ln(\Phi_\omega(M))}{\ln(M)} \leq \alpha, \forall M \geq M_\alpha.
\]
Therefore, $\Phi_\omega(M) \leq M^\alpha$ for all $M \geq M_\alpha$. Now, by definition of $\Phi_\omega(M)$, one has $\omega(M) \leq M^\alpha \omega(k)$, $\forall M \geq M_\alpha$, $k \geq 1$.

For $M < M_\alpha$ and $k \geq 1$,
\[
\omega(Mk) \leq \omega(M_\alpha k) \leq M^\alpha \omega(k) \leq M^\alpha M^\alpha \omega(k).
\]
\hfill \square

Remark 2.21. In Section 7, we will give a relation between the dilation indices and the so-called Upper and Lower Regularity Properties. The relation is useful in proving our main theorems.

2.4. The truncation operator. For each $x \in X$ and each finite set $A \subset \mathbb{N}$, we define the restricted truncation operator and the truncation operator as follows:
\[
U(x, A) := \min_{n \in A} |e^*_n(x)| \sum_{n \in A} \text{sgn}(e^*_n(x)) e_n,
\]
\[
T(x, A) := U(x, A) + P_A(x).
\]
If $A$ is empty or is infinite, then we use the convention that $U(x, A) = 0$.

Similar operators were introduced in [10]. Write the quantity:
\[
\Gamma = \sup \{ \|U(x, A)\| : A \text{ is a greedy set of } x, \|x\| \leq 1 \},
\]
\[
\Upsilon = \sup \{ \|T(x, A)\| : A \text{ is a greedy set of } x, \|x\| \leq 1 \}.
\]

For quasi-Banach spaces, the authors in [1] proved the following result regarding the boundedness of these operators.

Theorem 2.22. [1, Theorem 4.12 and Theorem 4.13] Let $B$ be a $C_q$-quasi-greedy basis of a quasi-Banach space $X$. Then the restricted truncation operator is uniformly bounded, i.e., $\Gamma < \infty$. Also, if $X$ is $p$-Banach,
\[
\Gamma \leq C^2_s \eta_p(C_q), \text{ and } \Upsilon \leq (C_q^p + \Gamma^p)^{1/p},
\]
where, if $u > 0$,
\[
\eta_p(u) := \min_{0 < t < 1} (1 - t^p)^{-1/p} (1 - (1 + A_p^{-1} t^{-1})^{-p})^{-1/p}.
\]

3. SEMI-GREEDY SCHAUDEL BASES IN QUASI-BANACH SPACES

The first author [4] showed that a Schauder basis in a Banach space is semi-greedy if and only if it is quasi-greedy and super-democratic. We generalized the result to the setting of quasi-Banach spaces.

Theorem 3.1. Assume that $B$ is a Schauder basis in a quasi-Banach space. Then $B$ is quasi-greedy and super-democratic if and only if $B$ is semi-greedy. Moreover, if $X$ is a $p$-Banach space,
\[
C_{sd} \leq K_b(1 + K_b)C_{sg}^2,
\]
\[
C_q \leq K_b C_{sg}(1 + (1 + K_b)^p C_{sg}^p)^{1/p},
\]
\[ C_{sg} \leq (2(C_q^2 \eta_p(C_q))^p + (2A_pc_{sd}C_q^2 \eta(C_q))^p)^{1/p}. \]

**Proof.** Assume that \( B \) is \( C_q \)-quasi-greedy and \( C_{sd} \)-super-democratic. To show the semi-greediness, we proceed as in [10]. Take \( x \in X \) and \( m \in \mathbb{N} \). Choose \( z = \sum_{n \in B} b_n e_n \) with \( |B| = m \), \( A \) a greedy set of \( x \) with cardinality \( m \) and \( \varepsilon \equiv \text{sgn}(e_n(x - z)) \). Set 
\[ \alpha := \max_{n \in A} |e_n(x)| \text{ and } \Delta_\alpha := \{ n : |e_n(x - z)| > \alpha \}. \]
Thus, the set \( \Delta_\alpha \) is a greedy set for \( x - z \) and \( \Delta_\alpha \subset A \cup B \). Define 
\[ h := P_A(x) - P_A(T(x - z, \Delta_\alpha)). \]
It is easy to verify that 
\[ x - h = T(x - z, \Delta_\alpha) + P_{B \setminus A}(x - T(x - z, \Delta_\alpha)). \]
Let \( \Lambda \) be a greedy set of \( x - z \) with \( |\Lambda| = |B \setminus A| = |A \setminus B| \) and \( \min_{n \in \Lambda} |e_n(x - z)| \geq \min_{n \in A \setminus B} |e_n(x - z)| \geq \alpha \). On the one hand, Theorem 2.22 gives 
\[ \| T(x - z, \Delta_\alpha) \| \leq 2^{1/p} C_q^2 \eta_p(C_q)\|x - z\|. \] (3.1)
On the other hand, by Proposition 2.1 super-democracy, and Theorem 2.22 
\[ \| P_{B \setminus A}(x - T(x - z, \Delta_\alpha)) \| \leq 2A_p \alpha \sup_{|y|=1} \| 1_{\eta(B \setminus A)} \| \leq 2A_p C_{sd} \min_{n \in \Lambda} |e_n(x - z)| \| 1_{\varepsilon \Lambda} \| \leq 2A_p C_{sd} C_q^2 \eta_p(C_q)\|x - z\|. \] (3.2)
Hence, 
\[ \| x - h \| \leq [2(C_q^2 \eta_p(C_q))^p + (2A_p C_{sd} C_q^2 \eta(C_q))^p]^{1/p}. \]
Thus, since this works for any finite greedy set \( A \) of \( x \), the basis is \( C_{sg} \)-semi-greedy with 
\[ C_{sg} \leq (2(C_q^2 \eta_p(C_q))^p + (2A_p C_{sd} C_q^2 \eta(C_q))^p)^{1/p}. \]
Assume now that \( B \) is \( C_{sg} \)-semi-greedy. First, we prove that \( B \) is super-democratic. Take \( A, B \) such that \( |A| \leq |B|, \| \varepsilon \| = \| \delta \| = 1, C > (A \cup B) \text{ with } |C| = |A| \). Define the element \( y := 1_{\varepsilon A} + 1_C \). Hence, if \( m = |C| \), then \( C \) is a greedy set of \( y \) of order \( m \). Applying the Chebyshev Greedy Algorithm for a corresponding greedy ordering \( \pi \), 
\[ y - CG_m^\pi(y) = 1_{\varepsilon A} + \sum_{n \in C} c_n e_n, \]
for some scalars \( (c_n) \subset \mathbb{F} \). Hence, using the basis constant and semi-greediness, we have 
\[ \| 1_{\varepsilon A} \| \leq K_b \| y - CG_m^\pi(y) \| \leq K_b C_{sg} \sigma_m(y) \leq K_b C_{sg} \| 1_C \|. \]
To estimate \( \| 1_{\delta B} \| \), we define the element \( z := 1_C + 1_{\delta B} \). Arguing as before, if \( m = |B| \), then \( B \) is a greedy set corresponding to a greedy ordering \( \pi' \) and 
\[ \| 1_C \| \leq (1 + K_b) \| z - CG_m^\pi'(z) \| \leq (1 + K_b) C_{sg} \sigma_m(z) \leq (1 + K_b) C_{sg} \| 1_{\delta B} \|. \]
Thus, \( B \) is \( C_{sd} \)-super-democratic with 
\[ C_{sd} \leq K_b (1 + K_b) C_{sg}^2. \]
Next, we prove that \( B \) is quasi-greedy. Let \( x \in X \) with finite support, \( A \) be a finite greedy set of cardinality \( m \) of \( x \) and \( C > \text{supp}(x) \) such that \( |C| = m \). Define the
element \( z := (x - P_A(x)) + \alpha 1_C \), with \( \alpha = \max_{n \in A} |e_n^*(x)| \). Hence, \( C \) is a greedy set of \( z \), and, there is a greedy ordering \( \pi \) such that

\[
z - CG_m(z) = (x - P_A(x)) + \sum_{n \in C} d_n e_n,
\]

for some \((d_n) \subset \mathbb{F}\). Thus,

\[
\|x - P_A(x)\|_p \leq K_b^p \|z - CG_m(z)\|_p \leq K_b^p C_{sg}^p \|x + \alpha 1_C\|_p ^p \leq K_b^p C_{sg}^p (\|x\|_p + \|\alpha 1_C\|_p ^p).
\]

If we consider the element \( y := x + \alpha 1_C \), a greedy set of \( y \) is \( A \). Hence, for some greedy ordering \( \pi' \), \( y - CG_m(y) = (x - P_A(x)) + \sum_{n \in A} a_n e_n + \alpha 1_C \). Then

\[
\|\alpha 1_C\| \leq (1 + K_b) \|y - CG_m(y)\| \leq (1 + K_b) C_{sg} \sigma_m(y) \leq (1 + K_b) C_{sg} \|x\|.
\]

Therefore, the basis is \( C_q \)-quasi-greedy with \( C_q \leq K_b C_{sg} (1 + (1 + K_b)^p C_{sg}^p)^{1/p} \). \( \Box \)

4. APPROXIMATION CLASSES AND (SEMI-) GREEDY BASES

Before proving Theorem \([13, \text{Proposition 7.1}]\) and \([19, \text{Lemma 3.3}]\),

**Proposition 4.1.** Let \( \omega \in \mathbb{W}_d \). Let \( B \) be a basis of a quasi-Banach space and \( f, g \) be two nondecreasing functions with \( f, g : \mathbb{N} \rightarrow (0, \infty) \), \( g \) doubling with constant \( d \), and

\[
\limsup_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty. \tag{4.1}
\]

Then there exist \( \eta_j \geq k_j \geq 1, j = 1, 2, \ldots \), such that

\[
\lim_{j \rightarrow \infty} \frac{\eta_j}{k_j} = \infty \text{ and } \frac{f(k_j)}{g(\eta_j)} \geq \frac{\omega(\eta_j)}{\omega(k_j)}.
\]

**Proof.** By (4.1), there exists an increasing sequence \((z_n)_{n=1}^\infty\) with \( z_n \rightarrow \infty \) such that

\[
\lim_{n \rightarrow \infty} \frac{f(z_n)}{g(z_n)} = \infty. \tag{4.2}
\]

Given \( z_n \), define \( r : \mathbb{N} \rightarrow \mathbb{N} \) such that \( 2^{r(n) - 1} \leq z_n < 2^{r(n)} \). Since \( g \) is doubling, for any \( n, M \in \mathbb{N} \),

\[
g(z_n M) \leq g(2^{r(n)} M) \leq d^{r(n)} g(M). \tag{4.3}
\]

Since \( \omega \in \mathbb{W}_d \), by Proposition \([2, \text{p. 15}] \), \( I_\omega < \infty \), and we can fix \( \alpha > I_\omega \). By (4.2), we take an increasing sequence \((k_j)_{j=1}^\infty\) where each \( k_j \) is some \( z_n \) such that

\[
\frac{f(k_j)}{g(k_j)} \geq d^{r(j)} C_\alpha z_j^\alpha, \tag{4.4}
\]

where \( C_\alpha \) is as in Lemma \([2, \text{p. 20}] \). Using Lemma \([2, \text{p. 20}] \) and (4.4), we obtain

\[
\frac{f(k_j)}{g(k_j)} \geq d^{r(j)} C_\alpha z_j^\alpha \geq \frac{d^{r(j)} \omega(z_j k_j)}{\omega(k_j)}. \tag{4.5}
\]
Define $\eta_j = z_j k_j$. Since $z_j \to \infty$,
\[ \lim_{j \to \infty} \frac{\eta_j}{k_j} = \infty, \]
so the first part of this proposition is proved. For the second,
\[ \frac{f(k_j)}{g(\eta_j)} = \frac{f(k_j)}{g(z_j k_j)} \geq \frac{f(k_j)}{\omega(z_j k_j)} \geq \frac{\omega(\eta_j)}{\omega(k_j)} = \omega(\eta_j) / \omega(k_j). \]
The proof is done. □

**Definition 4.2.** Let $B$ be a basis in a quasi-Banach space. We say that $B$ has Property (W) if there exists a positive constant $C$ such that for all $n \in \mathbb{N}$ and for all $m \geq n$, there exist $A \subset \mathbb{N}_{>m}$ with $|A| = n$ and $|\varepsilon| = 1$ such that
\[ h_r(n) \leq C \|1_{\varepsilon A}\|. \]  \tag{4.6}
The smallest constant in (4.6) is denoted by $K$, and we say that $B$ has the $K$-Property (W).

**Proposition 4.3.** If a basis $B$ in a $p$-Banach space is $C_{sc}$-super-conservative, then $B$ has $2^{1/p}C_{sc}$-Property (W).

**Proof.** Fix $n \in \mathbb{N}$ and $m \geq n$. Choose $B$ and $|\varepsilon| = 1$ such that $|B| = n$ and $h_r(n) \leq 2^{1/p} \|1_{\varepsilon B}\|$. Let $A = \{m + \max B + 1, \ldots, m + \max B + n\}$. Since $B$ is $C_{sc}$-super-conservative, we have
\[ h_r(n) \leq 2^{1/p} \|1_{\varepsilon B}\| \leq 2^{1/p} C_{sc} \|1_A\|. \]
Then, $B$ has $2^{1/p}C_{sc}$-Property (W). □

**Example 4.4.** All bases in Section 6 have Property (W). Subsection 6.4 gives an example of a non-conservative basis that has Property (W).

**Proposition 4.5.** Let $B$ be a Schauder basis with $K$-Property (W) in a $p$-Banach space $X$. Let $\omega \in \mathcal{W}_d$ with $i_\omega > 0$. Assume that there exist two sequences of integers $\eta_j \geq k_j \geq 1$, $j = 1, 2, \ldots$, such that
\[ \lim_{j \to \infty} \frac{\eta_j}{k_j} = \infty \quad \text{and} \quad \frac{h_r(k_j)}{h_l(\eta_j)} \geq \frac{\omega(\eta_j)}{\omega(k_j)}. \]  \tag{4.7}
Then $A_q^\omega \hookrightarrow CG_q^\omega$ does not hold for any $q \in (0, \infty]$.

**Proof.** Assume first that $0 < q < \infty$. For each $j \in \mathbb{N}$, choose $\Gamma_{i,j} \subset \mathbb{N}$ with $|\Gamma_{i,j}| = \eta_j$ and $|\varepsilon| = 1$ such that
\[ h_l(\eta_j) := \inf_{|A| = \eta_j, |\varepsilon| = 1} \|1_{\delta A}\| \geq \frac{\|1_{\varepsilon \Gamma_{i,j}}\|}{2}. \]
Since $h_l(\eta_j) \leq h_l(\eta_j) \leq K_l h_l(\eta_j)$,
\[ h_l(\eta_j) \geq \|1_{\varepsilon \Gamma_{i,j}}\|. \]  \tag{4.8}
By Property (W), there exist $|\delta| = 1$ and $\Gamma_{r,j}$ with $|\Gamma_{r,j}| = k_j \leq \eta_j$ such that $\Gamma_{r,j} \supset \Gamma_{l,j}$ and
\[ h_r(k_j) \leq K \|1_{\delta \Gamma_{r,j}}\|. \]  \tag{4.9}
Define the element $x_j := 2 \cdot 1_{\epsilon \Gamma_{l,j}} + 1_{\delta \Gamma_{r,j}}$. We have

$$
\|x_j\|^p \leq 2^p \|1_{\epsilon \Gamma_{l,j}}\|^p + \|1_{\delta \Gamma_{r,j}}\|^p \overset{4.13}{\leq} 2^{2p} K_b^p (h_t(\eta_j))^p + (h_r(k_j))^p. \tag{4.10}
$$

Since $\omega$ is nondecreasing and $\eta_j \geq k_j$, $\omega(\eta_j) \geq \omega(k_j)$. Hence, using our hypothesis,

$$
h_t(\eta_j) \leq h_r(k_j) \frac{\omega(k_j)}{\omega(\eta_j)} \leq h_r(k_j). \tag{4.11}
$$

By (4.10) and (4.11), we obtain

$$
\|x_j\| \lesssim h_r(k_j). \tag{4.12}
$$

Now, if $n \in \{1, \ldots, \eta_j\}$, a greedy set of $x_j$ of order $n$ is a subset of $\Gamma_{l,j}$; hence,

$$
\theta_n(x_j) \geq \frac{\|1_{\delta \Gamma_{r,j}}\|}{1 + K_b} \geq \frac{h_r(k_j)}{K(1 + K_b)}. \tag{4.13}
$$

Hence,

$$
\|x_j\|_{c_{\ell_q}^d} \overset{4.13}{\geq} \left( \sum_{n=1}^{\eta_j} \frac{1}{n} (\omega(n) h_r(k_j))^q \right)^{1/q} = h_r(k_j) (\tilde{\zeta}(\eta_j))^{1/q}, \tag{4.14}
$$

where $\zeta(j) = (\omega(j))^q$ and $\tilde{\zeta}$ is the summing weight corresponding to $\zeta$. Applying Lemma 2.16 and Corollary 2.19 we know that

$$
\zeta(n) \lesssim \tilde{\zeta}(n) \lesssim \zeta(n). \tag{4.15}
$$

Thus,

$$
\|x_j\|_{c_{\ell_q}^d} \gtrsim h_r(k_j) \omega(\eta_j). \tag{4.16}
$$

Regarding the error $\sigma_n(x_j)$, for any $1 \leq n \leq \eta_j + k_j$,

$$
\sigma_n(x_j) \leq \|x_j\| \overset{4.13}{\lesssim} h_r(k_j), \tag{4.17}
$$

but if $n > k_j$, we get

$$
\sigma_n(x_j) \leq 2 \|1_{\epsilon \Gamma_{l,j}}\| \overset{4.13}{\lesssim} h_t(\eta_j). \tag{4.18}
$$

Due to (4.12), (4.17), and (4.18), we obtain

$$
\|x_j\|_{A_{\ell_q}^d} \lesssim h_r(k_j) + \left( \sum_{n=1}^{k_j} \frac{1}{n} (\omega(n) h_r(k_j))^q + \sum_{n=k_j+1}^{k_j+\eta_j} \frac{1}{n} (\omega(n) h_t(\eta_j))^q \right)^{1/q} \lesssim h_r(k_j) + \left( \tilde{\zeta}(k_j) (h_r(k_j))^q + \zeta(k_j+\eta_j) (h_t(\eta_j))^q \right)^{1/q} \overset{\text{hypothesis}}{\lesssim} h_r(k_j) (\omega(k_j))^q. \tag{4.19}
$$
Writing $\eta_j = s_j k_j$ and using our hypothesis, we have that $s_j \to \infty$, and applying Theorem \ref{thm:approximation} gives

$$\frac{\omega(\eta_j)}{\omega(k_j)} = \frac{\omega(s_j k_j)}{\omega(k_j)} \geq C_\alpha(s_j)^\alpha \to \infty,$$

where $\alpha$ is positive and strictly smaller than $i_\omega$. By \eqref{eq:4.16}, \eqref{eq:4.19}, and \eqref{eq:4.20},

$$\frac{\|x_j\|_{CG_q}}{\|x_j\|_{A_q}} \gtrsim \frac{h_r(k_j)\omega(\eta_j)}{h_r(k_j)\omega(k_j)} = \frac{\omega(\eta_j)}{\omega(k_j)} \to \infty \text{ as } j \to \infty.$$

Consider now $q = \infty$. By \eqref{eq:4.13},

$$\|x_j\|_{CG_\infty} \geq \sup_{1 \leq n \leq n_j} \omega(n)\vartheta_n(x_j) \gtrsim \omega(\eta_j) h_r(k_j).$$

Arguing as in \eqref{eq:4.19}, we obtain

$$\|x_j\|_{A_\infty} \gtrsim h_r(k_j) + \sup_{1 \leq n \leq n_j} \omega(n)\sigma_n(x_j) + \sup_{k_j < n \leq k_j + n_j} \omega(n)\sigma_n(x_j) \gtrsim h_r(k_j) + \omega(\eta_j) h_r(k_j).$$

Hypothesis

$$\|x_j\|_{A_\infty} \leq h_r(k_j) \omega(k_j).$$

We conclude that

$$\frac{\|x_j\|_{CG_\infty}}{\|x_j\|_{A_\infty}} \gtrsim \frac{h_r(k_j)\omega(\eta_j)}{h_r(k_j)\omega(k_j)} = \frac{\omega(\eta_j)}{\omega(k_j)} \to \infty \text{ as } j \to \infty.$$

This completes our proof.

\qed

**Proposition 4.6.** Let $B$ be a Schauder basis in a $p$-Banach space and $\omega \in \mathcal{W}_d$ with $i_\omega > 0$. If $A_\omega \subset CG_q$ for some $q \in (0, \infty)$, then $h_t$ is a doubling function.

**Proof.** Assume that $q \in (0, \infty)$ and that $h_t$ is not doubling.

Step 1: set up. Take sufficiently large $s > 1$. Then there exists $n_s \in \mathbb{N}$ such that

$$sh_t(n_s) \leq h_t(2n_s).$$

Choose a set $M_s$ with $|M_s| = n_s$ and $|\varepsilon| = 1$ such that

$$\|1_{\varepsilon M_s}\|^p - \frac{1}{sp} \leq h_t^p(n_s) \leq K_b h_t^p(n_s) \leq K_b^p \|1_{\varepsilon M_s}\|^p.$$

Let $D \subset \mathbb{N}$ such that $M_s < D$ and $|D| = n_s$. Then

$$s^p \left(\|1_{\varepsilon M_s}\|^p - \frac{1}{sp}\right) \leq (K_b s)^p h_t^p(n_s) \leq K_b^p h_t^p(2n_s) \leq K_b^p (\|1_{\varepsilon M_s}\|^p + \|1_D\|^p).$$

Hence,

$$s^p - K_b^p \|1_{\varepsilon M_s}\|^p - 1 \leq K_b^p \|1_D\|^p. \quad \text{(4.23)}$$

Partition $D = \bigcup_{i=1}^r V_i$ with $r = \lceil s^{n/2} \rceil$, and each set $V_i$ has cardinality $\lfloor n_s/r \rfloor$ or $\lceil n_s/r \rceil$. Let $V_s$ be a set in the partition such that $\|1_{V_s}\| = \max_{1 \leq i \leq r} \|1_{V_i}\|$. Dividing each term of \eqref{eq:4.23} by $r$, we obtain

$$\frac{(s^p - K_b^p)}{r} \|1_{\varepsilon M_s}\|^p \leq K_b^p \|1_{V_s}\|^p + \frac{1}{r}.$$
Thus,

$$\|1_{\varepsilon M_s}\|^p \leq \frac{1 + rK_b^p}{s^p - K_b^p}\|1_{V_s}\|^p.$$  \hfill (4.24)

For sufficiently large $s$, (4.24) implies that $\|1_{\varepsilon M_s}\| \leq \|1_{V_s}\|$. Define $x_s := 2 \cdot 1_{\varepsilon M_s} + 1_{V_s}$. Then

$$\|x_s\|^p \leq \|1_{V_s}\|^p + 2 \|1_{\varepsilon M_s}\|^p \leq (1 + 2^p) \|1_{V_s}\|^p.$$  \hfill (4.25)

Step 2: bound $\|x_s\|_{C^q_{\omega}}$. If $k \leq |M_s| = n_s$, a greedy set of $x_s$ is a subset $M_{1,s}$ of $M_s$ and so,

$$\|1_{V_s}\| \leq (1 + K_b) \left[1_{V_s} + 2 \cdot 1_{\varepsilon(M_s \setminus M_{1,s})} + \sum_{n \in M_{1,s}} a_n e_n\right] \leq (1 + K_b) \vartheta_k(x_s),$$  \hfill (4.26)

for some $(a_n) \subset F$. Let $\zeta(j) = (\omega(j))^q$ and $\tilde{\zeta}$ is the summing weight corresponding to $\zeta$. Applying lemma 2.16 and Proposition 2.19, we know that

$$\zeta(n) \lesssim \tilde{\zeta}(n) \lesssim \zeta(n).$$  \hfill (4.27)

Hence, for $0 < q < \infty$,

$$\|x\|_{C^q_{\omega}} \geq \left(\sum_{k=1}^{n_s} (\omega(k)\vartheta_k(x_s))^q 1_k\right)^{1/q} \geq \tilde{\zeta}(n_s)^{1/q} \|1_{V_s}\| \geq \omega(n_s) \|1_{V_s}\|.$$  \hfill (4.28)

Step 3: bound $\|x_s\|_{A^q_{\omega}}$. If $k \leq |V_s|$,\n
$$\sigma_k(x_s) \leq \|x_s\| \lesssim \|1_{V_s}\|.$$  \hfill (4.29)

If $k > |V_s|$,\n
$$\sigma_k(x_s) \leq 2\|1_{\varepsilon M_s}\| \lesssim \left(1 + rK_b^p\right)^{1/p} \|1_{V_s}\|.$$  \hfill (4.30)

Define $C(s,p) := \left(\frac{1 + rK_b^p}{s^p - K_b^p}\right)^{1/p}$. Since $i_\zeta > 0$, letting $\alpha = i_\zeta / 2$ and using Theorem 7.5 give

$$\frac{\zeta(n)}{\zeta(m)} \geq C_\alpha \left(\frac{n}{m}\right)^\alpha, \forall m \leq n.$$  \hfill (4.31)
We have
\[
\|x_s\|_{A_q^{\omega}} = \|x_s\| + \left(\sum_{k=1}^{\lceil V_s \rceil} (\omega(k) \sigma_k(x_s))^q \frac{1}{k} + \sum_{k=\lceil V_s \rceil + 1}^{2n_s} (\omega(k) \sigma_k(x_s))^q \frac{1}{k}\right)^{1/q}.
\]

Therefore,
\[
\|1_{V_s}\| \left(1 + \left(\tilde{\omega}(|V_s|) + C(s, p)\tilde{\omega}(2n_s)\right)^{1/q}\right).
\]

\[
\|1_{V_s}\| \left(1 + (\omega^q(|V_s|) + C(s, p)\omega^q(n_s))^{1/q}\right).
\]

\[
\|1_{V_s}\| \left(1 + (\omega^q(|V_s|) + C(s, p)\omega^q(n_s))^{1/q}\right).
\]

\[
\|x_s\|_{C^\omega_q} \left(\frac{1}{\omega(n_s)} + \left(\frac{1}{C_{\alpha}} \left(\frac{|V_s|}{n_s}\right)^{\alpha} + C(s, p)\right)^{1/q}\right).
\]

Therefore,
\[
\|x_s\|_{C^\omega_q} \left(\frac{1}{\omega(n_s)} + \left(\frac{1}{C_{\alpha}} \left(\frac{|V_s|}{n_s}\right)^{\alpha} + C(s, p)\right)^{1/q}\right)^{-1} \to \infty \text{ as } s \to \infty.
\]

The case when \( q = \infty \) is similar. We have that \( A_q^{\omega} \hookrightarrow C^\omega_q \) does not hold for any \( q \in (0, \infty] \).

**Proof of Theorem 1.3** Item (1) follows directly from the definitions of \( A_q^{\omega} \), \( C^\omega_q \), and semi-greediness. We prove item (2). By Theorem 3.1 it suffices to prove that \( B \) is super-democratic. Assume otherwise. By Proposition 4.6 \( h_l \) is doubling. Apply Proposition 4.1 with \( f = h_r \) and \( g = h_l \) to obtain sequences \( (k_j) \) and \( (n_j) \) satisfying (4.7). Now Proposition 4.3 implies that \( A_q^{\omega} \hookrightarrow C^\omega_q \) does not hold, which contradicts our hypothesis.

**Proof of Theorem 1.4** Item (1) follows directly from the definitions of \( A_q^{\omega} \), \( G_q^{\omega} \), and greediness. We prove item (2) using the exact argument as in the proof of Theorem 1.3. However, unlike Theorem 1.3 we do not need Property (W). In fact, we only have to consider the element \( x_j = 1_{G_{r, j} \setminus \Gamma_{r, j}} + 2 \cdot 1_{\Gamma_{r, j} \cap \Gamma_{r, j} \setminus x_{l, j}} \) defined in [13] Proposition 7.1 and apply unconditionality instead of the Property (W) in (4.13).

5. APPROXIMATION CLASSES AND PARTIALLY GREEDY BASES

Our goal of this section is to prove Theorem 1.5. Recall that bounding \( \sigma_m(x) \) effectively is crucial in the proof of Theorem 1.3 for greedy bases. However, for partially greedy bases, establishing an effective bound for \( \beta_m(x) \) is considerably more difficult. For example, if \( |\text{supp}(x)| = k \), then \( \sigma_m(x) = 0 \) for all \( m > k \), but the same conclusion does not necessarily hold for \( \beta_m(x) \). Furthermore, we have more freedom in choosing...
the vector $y$ in (1.2) to estimate $\sigma_m(x)$, while $S_m(x)$ in the definition of $\beta_m(x)$ is fixed. Hence, to have the equivalences as in Theorem 1.5, we require our bases to satisfy certain properties that allow us to estimate $\beta_m(x)$ more effectively. First, we need some definitions.

Set $\mathbb{D} = \{(m, u) \in \mathbb{N} \times \mathbb{N} : m \leq u\}$. Define the left and right restricted democracy functions as follows:

$$h_{R,l}(m, u) := \sup_{|A|=m, \max A \leq u} \|1_{\varepsilon A}\|$$ and $$h_{R,r}(m, u) := \inf_{|A|=m, \min A > u} \|1_{\varepsilon A}\|.$$ where $h_{R,r}(m, u)$ is defined on $\mathbb{N} \times \mathbb{N}$ and $h_{R,l}(m, u)$ is defined on $\mathbb{D}$.

**Proposition 5.1.**

1. For a Schauder basis $B$, it holds that

$$K_b h_{R,r}(m_1, u) \geq h_{R,r}(m_2, u), \forall m_1 \geq m_2. \quad (5.1)$$

2. For a $C_q$-quasi-greedy basis $B$, it holds that

$$C_q h_{R,l}(m_1, u) \geq h_{R,l}(m_2, u), \forall u \geq m_1 \geq m_2, \forall u \in \mathbb{N}. \quad (5.2)$$

3. There exists a non-quasi-greedy Schauder basis such that

$$\sup_{m_2 \leq m_1 \leq u} \frac{h_{R,l}(m_2, u)}{h_{R,l}(m_1, u)} = \infty.$$ 

**Proof.** (1) Let $A \subset \mathbb{N}, |A| = m_1, \min A > u$ and $|\varepsilon| = 1$. Choose $B$ to be the set of $m_2$ smallest numbers in $A$. We have

$$K_b \|1_{\varepsilon A}\| \geq \|1_{\varepsilon B}\| \geq h_{R,r}(m_2, u).$$

Taking the inf over all sets $A$ and $|\varepsilon| = 1$ gives the desired conclusion.

(2) Let $A \subset \mathbb{N}, |A| = m_2, \max A \leq u$ and $|\varepsilon| = 1$. Choose $B$ to be a subset of $\{1, \ldots, u\}$ such that $B \cap A = \emptyset$ and $|B| = m_1 - m_2$. Consider $x = 1_{\varepsilon A} + 1_B$. Since $B$ is a greedy set of $x$, we obtain

$$\|1_{\varepsilon A}\| = \|x - 1_B\| \leq C_q \|x\| \leq C_q h_{R,l}(m_1, u).$$

Taking the sup over all sets $A$ and $|\varepsilon| = 1$ finishes the proof.

(3) See Subsection 6.2. \qed

**Proposition 5.2.** A basis $B$ is super-conservative if and only if

$$\sup_{(m, u) \in \mathbb{D}} \frac{h_{R,l}(m, u)}{h_{R,r}(m, u)} < \infty.$$ 

**Proof.** Assume that $B$ is $C_{sc}$-super-conservative. Fix $(m, u) \in \mathbb{D}$. Choose $A, B \subset \mathbb{N}$ with $|A| = |B| = m$ and $\max A \leq u < \min B$. For any $|\varepsilon| = |\delta| = 1$, we have

$$\frac{\|1_{\varepsilon A}\|}{\|1_{\delta B}\|} \leq C_{sc},$$

so taking the sup over all $A, |\varepsilon| = 1$ and the inf over all $B, |\delta| = 1$ gives

$$\frac{h_{R,l}(m, u)}{h_{R,r}(m, u)} \leq C_{sc}.$$ 

Taking the sup over $(m, u) \in \mathbb{D}$ completes the proof.
Now assume that
\[
\sup_{(m,u) \in D} \frac{h_{R,l}(m,u)}{h_{R,r}(m,u)} < C,
\]
for some constant $C$. Choose $|\varepsilon| = |\delta| = 1$ and $A, B \subset \mathbb{N}$ with $|A| = |B|$, $\max A < \min B$. We have
\[
\|1_{\varepsilon A}\| \leq h_{R,l}(|A|, \max A) \quad \text{and} \quad \|1_{\delta B}\| \geq h_{R,r}(|B|, \max A).
\]
Hence,
\[
\frac{\|1_{\varepsilon A}\|}{\|1_{\delta B}\|} \leq \frac{h_{R,l}(|A|, \max A)}{h_{R,r}(|B|, \max A)} < C.
\]
Hence, $B$ is super-conservative. \qed

The next definition expedites our introduction of Property (I).

**Definition 5.3.** A function $\psi : \mathbb{N} \to \mathbb{N}$ is called a characteristic function of $h_{R,l}$ if $(n, \psi(n)) \in D$ for all $n \in \mathbb{N}$ and
\[
\sup_{u} h_{R,l}(m, u) \lesssim h_{R,l}(m, \psi(m)).
\]

Clearly, such a function $\psi$ exists but is not unique.

**Example 5.4.** A characteristic function of $h_{R,l}$ is found as follows: let $\psi(m)$ be the smallest integer at least $m$ such that $h_r(m) \leq 2 h_{R,l}(m, \psi(m))$.

**Definition 5.5.** A basis $B$ is said to have Property (I) if
\begin{enumerate}
    \item we have
    \[
    \sup_{\varepsilon \in \mathbb{N} \cup \{0\}} \frac{h_{R,l}(2^{\varepsilon+1}u, u)}{h_{R,r}(2^{\varepsilon}u, u)} < \infty, \quad \text{and}
    \]
    \item there is a characteristic function $\psi$ of $h_{R,l}$ such that $h_{R,r}(u, u) \lesssim h_{R,l}(m, u)$ whenever $u \leq \psi(m)$.
\end{enumerate}

**Definition 5.6.** A basis $B$ is said to have Property (W$^*$) if there is constants $C_1, C_2 > 0$ such that for every $m \in \mathbb{N}$, there exist $A \subset \mathbb{N} \leq C_1 m$ and $|\varepsilon| = 1$ such that $|A| = m$ and $\|1_{\varepsilon A}\| \leq C_2 h_l(m)$. We say $B$ has $(C_1, C_2)$-Property (W$^*$).

**Example 5.7.** All bases in Section 6 have Property (I) and Property (W$^*$). In particular, Subsection 6.3 gives an unconditional and conservative basis with Property (I) and Property (W$^*$), but the basis is not democratic. Subsection 6.3 gives an unconditional basis with Property (I) and Property (W$^*$) but is not conservative.

**Proposition 5.8.** Let $w \in \mathcal{W}_d$. If a Schauder basis $B$ has property (I) and is not conservative, then there exist $\eta_j \geq u_j \geq k_j \geq 1$, $j = 1, 2, \ldots$, such that
\[
\lim_{j \to \infty} \frac{\eta_j}{u_j} = \infty \quad \text{and} \quad \frac{h_{R,l}(k_j, u_j)}{h_{R,r}(\eta_j, u_j)} \gtrsim \frac{\omega(\eta_j)}{\omega(u_j)}.
\]

**Proof.** Since $B$ is not conservative, Proposition 5.2 gives
\[
\sup_{(m,u) \in D} \frac{h_{R,l}(m,u)}{h_{R,r}(m,u)} = \infty.
\]
Let \((z_n, v'_n) \in \mathbb{D}\) be chosen such that
\[
\lim_{n \to \infty} \frac{h_{R,l}(z_n, v'_n)}{h_{R,r}(z_n, v'_n)} = \infty. \tag{5.2}
\]
Let \(\psi\) be a characteristic function of \(h_{R,l}\) that is also in the definition of Property (I). By the definition of characteristic functions, we have
\[
h_{R,l}(z_n, v'_n) \lesssim h_{R,l}(z_n, \psi(z_n)). \tag{5.3}
\]
We now build a sequence \((v_n)\):
\[
v_n = \begin{cases} 
\psi(z_n) & \text{if } v'_n > \psi(z_n), \\
v'_n & \text{if } v'_n \leq \psi(z_n).
\end{cases}
\]
Note that if \(v'_n > \psi(z_n)\), we have
\[
\frac{h_{R,l}(z_n, v'_n)}{h_{R,r}(z_n, v'_n)} = \frac{h_{R,l}(z_n, \psi(z_n))}{h_{R,r}(z_n, \psi(z_n))} \lesssim \frac{h_{R,l}(z_n, v'_n)}{h_{R,r}(z_n, v'_n)},
\]
which, along with (5.2), implies that
\[
\lim_{n \to \infty} \frac{h_{R,l}(z_n, v'_n)}{h_{R,r}(z_n, v'_n)} = \infty \text{ and } v_n \leq \psi(z_n). \tag{5.4}
\]
Observe that \(h_{R,r}(z_n, v_n) \geq K_\beta^{-1} \inf_n \|e_n\|\) and \(h_{R,l}(z_n, v_n) \leq z_n^{1/p} \sup_n \|e_n\|\), so we can assume that \(z_n, v_n \to \infty\). Let \(r : \mathbb{N} \to \mathbb{N}\) be such that \(2^{r(n)-1} \leq z_n \leq 2^{r(n)}\). Since \(\omega \in \mathbb{W}_d\), Proposition 2.15 gives \(I_\omega < \infty\). Choose \(\alpha > I_\omega\) and \(C_\alpha\) as in Lemma 2.20. Choose \((k_j, u_j)\) to be some \((z_n, v_n)\) such that
\[
\frac{h_{R,l}(k_j, u_j)}{h_{R,r}(k_j, u_j)} \geq K_\delta d^{r(j)} C_\alpha z_j^\alpha \geq K_\delta d^{r(j)} \frac{\omega(u_j z_j)}{\omega(u_j)}, \tag{5.5}
\]
where \(d\) is equal to the sup in item (1) of Property (I). Set \(\eta_j = u_j z_j\). We obtain
\[
h_{R,r}(\eta_j, u_j) = h_{R,r}(u_j z_j, u_j) \overset{\text{(5.1)}}{=} K_b h_{R,r}(2^{r(j)} u_j, u_j) \leq K_d d^{r(j)} h_{R,r}(u_j, u_j) \overset{\text{(5.6)}}{=} K_d d^{r(j)} h_{R,r}(k_j, u_j),
\]
where the last two inequalities are due to Property (I) and the fact that \(u_j \leq \psi(k_j)\). Clearly, \(\eta_j / u_j \to \infty\). Furthermore,
\[
\frac{h_{R,l}(k_j, u_j)}{h_{R,r}(\eta_j, u_j)} \overset{\text{(5.6)}}{=} K_b^{-1} d^{-r(j)} \frac{h_{R,l}(k_j, u_j)}{h_{R,r}(k_j, u_j)} \overset{\text{(5.5)}}{=} \frac{\omega(\eta_j)}{\omega(u_j)}.
\]

If our basis is quasi-greedy, we have an immediate corollary.

**Corollary 5.9.** Let \(w \in \mathbb{W}_d\). If a quasi-greedy Schauder basis \(B\) has property (I) and is not conservative, then there exist \(\eta_j \geq u_j \geq 1, j = 1, 2, \ldots\), such that
\[
\lim_{j \to \infty} \frac{\eta_j}{u_j} \to \infty \text{ and } \frac{h_{R,l}(u_j, u_j)}{h_{R,r}(\eta_j, u_j)} \overset{\text{(5.5)}}{=} \frac{\omega(\eta_j)}{\omega(u_j)}.
\]

**Proof.** Use Propositions 5.1 item (2) and 5.3. \(\square\)
**Proposition 5.10.** Let $B$ be a Schauder basis with Property (I) and Property (W$^*$) of a $p$-Banach space. Let $\omega \in \mathbb{W}_q$ with $i_\omega > 0$. If $B$ is not conservative, then $\mathcal{P}G_q^\omega \hookrightarrow G_q^\omega$ does not hold for any $q \in (0, \infty]$.

**Proof.** We assume that $q \in (0, \infty)$. (The case $q = \infty$ is similar.)

Step 1: set up. By Proposition 5.8 there exist $\eta_j \geq u_j \geq k_j \geq 1$, $j = 1, 2, \ldots$, such that

$$\lim_{j \to \infty} \frac{\eta_j}{u_j} \to \infty \quad \text{and} \quad \frac{h_{R,l}(k_j, u_j)}{h_{R,r}(\eta_j, u_j)} \gtrsim \frac{\omega(\eta_j)}{\omega(u_j)}. \quad (5.7)$$

Assume that $B$ has $(C_1, C_2)$-Property (W$^*$). Choose $|\varepsilon| = 1$, $\Gamma_{r,j}^\varepsilon \subset \mathbb{N} \leq C_1 \eta_j$ with $|\Gamma_{r,j}^\varepsilon| = \eta_j$ and

$$\|1_{\varepsilon \Gamma_{r,j}^\varepsilon}\| \leq C_2 h_l(\eta_j). \quad (5.8)$$

Define $\Gamma_{r,j} = \Gamma_{r,j}^\varepsilon \cap [u_j + 1, \infty)$. Then

$$\|1_{\varepsilon \Gamma_{r,j}}\| \leq (K_b + 1) \|1_{\varepsilon \Gamma_{r,j}^\varepsilon}\| \leq C_2 (K_b + 1) h_{R,r}(\eta_j, u_j). \quad (5.9)$$

Choose $|\delta| = 1$, $\Gamma_{l,j} \subset \mathbb{N} \leq u_j$ with $|\Gamma_{l,j}| = k_j$ and

$$\|1_{\delta \Gamma_{l,j}}\| = h_{R,l}(k_j, u_j). \quad (5.10)$$

Set $x_j := 1_{\delta \Gamma_{l,j}} + 2 \cdot 1_{\varepsilon \Gamma_{r,j}}$. We have

$$\|x_j\|^p \leq \|1_{\delta \Gamma_{l,j}}\|^p + 2^p\|1_{\varepsilon \Gamma_{r,j}}\|^p \quad (5.9, 5.10) \lesssim (h_{R,l}(k_j, u_j))^p + (h_{R,r}(\eta_j, u_j))^p \quad (5.7) \lesssim (h_{R,l}(k_j, u_j))^p, \quad (5.11)$$

which gives

$$\|x_j\| \lesssim h_{R,l}(k_j, u_j). \quad (5.12)$$

For $n \in \{1, \ldots, \eta_j - u_j\}$, a greedy set of order $n$ of $x_j$ is a subset of $\Gamma_{r,j}$. Therefore,

$$\|\gamma_n(x_j)\| \geq \frac{1}{K_b} \|1_{\delta \Gamma_{l,j}}\| = \frac{1}{K_b} h_{R,l}(k_j, u_j). \quad (5.13)$$

Step 2: bound $\|x_j\|_{G_q^\omega}$. Let $\zeta = \omega^q$ and $\tilde{\zeta}(m) = \sum_{n=1}^{m} \zeta(n)/n$, $m = 1, 2, \ldots$ Then

$$\|x_j\|_{G_q^\omega} \gtrsim \left(\frac{\eta_j - u_j}{n} \frac{1}{\omega(n)} \|\gamma_n(x_j)\|\right)^{1/q} \quad (5.14)$$

$$\gtrsim \left(\frac{\eta_j - u_j}{n} \frac{1}{\omega(n) h_{R,l}(k_j, u_j)}\right)^{1/q} = \left(\tilde{\zeta}(\eta_j - u_j)\right)^{1/q} h_{R,l}(k_j, u_j) \quad (5.14)$$

$$\gtrsim \omega(\eta_j - u_j) h_{R,l}(k_j, u_j). \quad (5.14)$$

Step 3: bound $\|x_j\|_{\mathcal{P}G_q^\omega}$. For $n \leq u_j$,

$$\|\beta_n(x_j)\| \leq (1 + K_b) \|x_j\| \lesssim h_{R,l}(k_j, u_j), \quad (5.15)$$
and for \( u_j < n \leq C_1 \eta_j \),
\[
\| \beta_n(x_j) \| \leq (1 + K_b) \| 1_{e_{r,j}} \| \lesssim h_{R,r}(\eta_j, u_j).
\]  

(5.16)

We have
\[
\| x_j \|_{\mathcal{P}G^\omega_q} \leq \| x_j \| + \left( \sum_{n=1}^{u_j} \frac{(\omega(n)\beta_n(x_j))^q}{n} + \sum_{n=u_j+1}^{[C_1 \eta_j]} \frac{(\omega(n)\beta_n(x_j))^q}{n} \right)^{1/q} \lesssim \| x_j \| + \left( \sum_{n=1}^{u_j} \frac{(\omega(n)h_{R,l}(k_j, u_j))^q}{n} + \sum_{n=u_j+1}^{[C_1 \eta_j]} \frac{(\omega(n)h_{R,r}(\eta_j, u_j))^q}{n} \right)^{1/q} \lesssim h_{R,l}(k_j, u_j) + \left( (h_{R,l}(k_j, u_j))^q (\eta_j - u_j) + (h_{R,r}(\eta_j, u_j))^q [C_1 \eta_j] \right)^{1/q} \lesssim \omega(u_j)h_{R,l}(k_j, u_j).
\]  

(5.17)

By (5.14) and (5.17), we obtain
\[
\frac{\| x_j \|_{\mathcal{P}G^\omega_q}}{\| x_j \|_{G^\omega_q}} \gtrsim \frac{\omega(\eta_j - u_j)}{\omega(u_j)} \rightarrow \infty \text{ due to Theorem 7.5 and } \frac{\eta_j - u_j}{u_j} \rightarrow \infty.
\]

Hence, \( \mathcal{P}G^\omega_q \hookrightarrow G^\omega_q \) does not hold.

\[ \square \]

**Proof of Theorem 7.5** Item (1) follows from the definitions of \( \mathcal{P}G^\omega_q, G^\omega_q \), and partial greediness. We prove item (2). Since \( B \) is quasi-greedy and Schauder, by Theorem 2.9 it suffices to show that \( B \) is conservative, which is clearly true due to Proposition 5.10.

\[ \square \]

### 6. EXAMPLES

All examples we consider are Banach spaces over real scalars having a Schauder basis.

#### 6.1. The summing basis of \( c_0 \)

Let \( B = (e_n) \) be the canonical basis in \( c_0 \) and consider the collection \( (x_n) = (\sum_{i=1}^{n} e_i), n = 1, 2, \ldots, \) which is a conditional Schauder basis of \( c_0 \). We have
\[
\left\| \sum_{n=1}^{\infty} a_n x_n \right\| = \sup_{N \geq 1} \left| \sum_{n=1}^{N} a_n \right|.
\]

6.1.1. **Calculating democracy functions.** It is easy to see that \( h_l(N) = 1 \) and \( h_r(N) = N, \forall N \in \mathbb{N} \). Similarly, \( h_{R,l}(m, u) = m, \forall (m, u) \in \mathbb{D} \) and \( h_{R,r}(m, u) = 1, \forall (m, u) \in \mathbb{N} \times \mathbb{N} \). Therefore, \( B \) is not conservative.
6.1.2. Properties. We verify each desired property below.

1. Property (W): for every \( m, n \in \mathbb{N} \) with \( m \geq n \), we let \( A = \{ m + 1, m + 2, \ldots, m + n \} \). Clearly, \( \|1_A\| = n = h_r(n) \).

2. Property (W\(^\star\)): for every \( m \in \mathbb{N} \), let \( A = \{ 1, 2, \ldots, m \} \) and \( \varepsilon = ((-1)^n)^{m}_{n=1} \) to have \( \|\varepsilon_A\| = 1 = h_l(m) \).

3. Property (I) is due to the fact that \( h_{R,r} \equiv 1 \).

6.2. The difference basis in \( \ell_1 \). Consider \( B = (e_n) \), the canonical basis in \( \ell_1(\mathbb{N}) \) and consider the following vectors:

\[
x_1 = e_1, \quad x_n = e_n - e_{n-1}, \quad n = 2, 3, \ldots
\]

The collection \( (x_n)^\infty_{n=1} \) is a monotone conditional Schauder basis in \( \ell_1 \). For \( (a_n)^N_{n=1} \subset \mathbb{R} \),

\[
\left\| \sum_{n=1}^{N} a_n x_n \right\| = \sum_{n=1}^{N-1} |a_n - a_{n+1}| + |a_N|.
\]

6.2.1. Calculating democracy functions. We have

\[
h_{R,l}(2N, 2N) = \left\| (1, 1, \ldots, 1, \ldots) \right\| = 1 \quad \text{and} \quad \text{length } 2N
\]

\[
h_{R,l}(N, 2N) \geq \left\| (0, 1, 0 \ldots, 0, 1, 0, \ldots) \right\| = 2N \quad \text{length } 2N
\]

Hence, \( h_{R,l}(2N, 2N) = o(h_{R,l}(N, 2N)) \), which illustrates item (3) in Proposition 5.1.

Due to [6, Lemma 8.1], we know that

\[
h_r(N) = 2N \quad \text{and} \quad h_l(N) = 1.
\]

Proposition 6.1. For \( (x_n) \) as above, we have

\[
h_{R,l}(m, u) = \begin{cases} 2m & \text{if } u \geq 2m, \\ 2u - 2m + 1 & \text{if } m \leq u < 2m, \end{cases}
\]

and

\[
h_{R,r}(m, u) = 2.
\]

Proof. If \( u \geq 2m \), we have

\[
h_{R,l}(m, u) \geq \left\| (0, 1, 0 \ldots, 0, 1, 0, \ldots) \right\| = 2m \quad \text{length } 2m
\]

On the other hand, \( h_{R,l}(m, u) \leq h_r(m) = 2m \). Hence, if \( u \geq 2m \), \( h_{R,l}(m, u) = 2m \).

If \( m \leq u < 2m \), assume that \( u \) is even. The case for odd \( u \) is similar. Consider the problem of distributing \( m \) 1’s among the first \( u \) spots to maximize the norm. We distribute \( u/2 \) 1’s as follows:

\[
\left\| (0, 1, 0 \ldots, 0, 1, 0, \ldots) \right\| = u \quad \text{length } u
\]

There are \( (m - u/2) \) 1’s remaining to be distributed. Note that putting an 1 into the first spot reduces the norm by 1, while putting an 1 into other spots reduces the norm by 2.
Calculating democracy functions.

6.3.1. Corollary 6.2. The difference basis $B$ is not conservative.

6.2.2. Properties. We verify each desired property:

1. Property (W): for every $m, n \in \mathbb{N}$ with $m \geq n$, we let $A = \{m + 1, m + 3, \ldots, m + 2n - 1\}$. Clearly, $\|1_A\| = 2n = h_r(n)$.

2. Property (W*): for every $m \in \mathbb{N}$, let $A = \{1, 2, \ldots, m\}$ to have $\|1_A\| = 1 = h_l(m)$.

3. Property (I) is due to the fact that $h_{R,r} \equiv 2$.

6.3. A modification of the Schreier space. We shall use the same example as in [8, Proposition 6.10], which is a modification of the Schreier space. Let $\mathcal{S}$ be the completion of $c_{00}$ under the following norm:

$$\|(x_1, x_2, x_3, \ldots)\|_\mathcal{S} = \sup_{F \in \mathcal{F}} \sum_{i \in F} |x_i|,$$

where $\mathcal{F} = \{F \subseteq \mathbb{N} : \sqrt{\min F} \geq |F|\}$. It is easy to check that the canonical basis $\mathcal{B}$ is an 1-unconditional and 1-conservative monotone Schauder basis of $\mathcal{S}$.

6.3.1. Calculating democracy functions. Let $N \in \mathbb{N}$. By the definition of $\|\cdot\|_\mathcal{S}$, we have

$$h_r(N) = N \text{ and } h_l(N) = \|(1, \ldots, 1, 0, \ldots)\|_\mathcal{S}.$$

For $M \in \mathbb{N}$ and $N \in \mathbb{N} \cup \{0\}$, set $x_{N,M} := \sum_{n=N+1}^{N+M} e_n$ and write $x_{N,M} = (x_1, x_2, x_3, \ldots)$.

Proposition 6.3. We have

$$\|x_{N,M}\|_\mathcal{S} \lesssim \sqrt{N + M}.$$

Proof. Let $F \in \mathcal{F}$ with $\min F = N + j_0$ for some $1 \leq j_0 \leq M$.

Case 1: $\min F + [\sqrt{\min F}] - 1 \leq N + M$; equivalently, $j_0 + [\sqrt{N + j_0}] \leq M + 1$, which implies that

$$j_0 \leq M + \frac{5}{2} - \sqrt{M + N + 9/4} =: f(M, N). \quad (6.1)$$

If $f(M, N) < 1$, then Case 1 cannot happen. If $f(M, N) \geq 1$, then

$$\sum_{i \in F} |x_i| \leq [\sqrt{\min F}] \leq \sqrt{N + j_0} \lesssim \sqrt{N + M}. \quad (6.1)$$

Case 2: $\min F + [\sqrt{\min F}] - 1 \geq N + M$; equivalently, $j_0 + [\sqrt{N + j_0}] \geq M + 1$, which implies that

$$j_0 \geq M + \frac{3}{2} - \sqrt{M + N + 5/4} =: g(M, N). \quad (6.2)$$
Hence,
\[
\sum_{i \in F} |x_i| \leq (N + M) - \min F + 1 = M - j_0 + 1 \leq M - g(M, N) + 1 \lesssim \sqrt{M + N}.
\]

\[\Box\]

**Proposition 6.4.** For \( M \in \mathbb{N} \) and \( N \in \mathbb{N} \cup \{0\} \),
\[
\|x_{N,M}\|_S \gtrsim \begin{cases} 
M & \text{if } N \geq M^2 - 1, \\
\sqrt{N + M} & \text{if } N \leq M^2 - 1.
\end{cases}
\]

**Proof.** Let \( F = \{N + j_0, \ldots, N + M\} \), where \( 1 \leq j_0 \leq M \) is the smallest such that \( \sqrt{N + j_0} \geq M - j_0 + 1 \), i.e., \( F \in \mathcal{F} \). Equivalently,
\[
j_0 \geq M + 3/2 - \sqrt{M + N + 5/4} =: g(M, N).
\]

Case 1: If \( g(M, N) \leq 1 \); equivalently, \( N \geq M^2 - 1 \), we choose \( j_0 = 1 \) and obtain \( \|x_{N,M}\|_S \geq M \).

Case 2: If \( g(M, N) \geq 1 \); equivalently, \( N \leq M^2 - 1 \), we choose \( j_0 = \lceil g(M, N) \rceil \). Then
\[
\|x_{N,M}\|_S \geq M - j_0 + 1 \geq M - g(M, N) \gtrsim \sqrt{M + N}.
\]
\[\Box\]

The following corollaries are immediate from Propositions 6.3 and 6.4.

**Corollary 6.5.** We have
\[
h_l(N) = \|\underbrace{1, \ldots, 1}_N, 0, \ldots\|_S \asymp \sqrt{N},
\]
and so \( \mathcal{B} \) is not democratic.

**Corollary 6.6.** For \((m,u) \in \mathbb{D}\), we have
\[
h_{R,l}(m,u) \lesssim \sqrt{u} \text{ and } h_{R,l}(m,u) \gtrsim \begin{cases} 
m & \text{if } u \geq m^2 + m - 1 \\
\sqrt{u} & \text{if } u \leq m^2 + m - 1.
\end{cases}
\]

**Proof.** Observe that \( h_{R,l}(m,u) = \|x_{u-m,m}\| \) and apply Propositions 6.3 and 6.4 \(\Box\)

**Corollary 6.7.** For \((m,u) \in \mathbb{N} \times \mathbb{N}\), we have
\[
h_{R,r}(m,u) \lesssim \sqrt{u + m} \text{ and } h_{R,r}(m,u) \gtrsim \begin{cases} 
m & \text{if } u \geq m^2 - 1 \\
\sqrt{u + m} & \text{if } u \leq m^2 - 1.
\end{cases}
\]

**Proof.** Observe that \( h_{R,r}(m,u) = \|x_{u,m}\| \) and apply Propositions 6.3 and 6.4 \(\Box\)
6.3.2. Properties.

1. Property (W): for every \(m, n \in \mathbb{N}\) with \(m \geq n\), we let \(A = \{m^2 + 1, m^2 + 2, \ldots, m^2 + n\}\). Clearly, \(\|1_A\| = n = h_r(n)\).

2. Property (W*) for every \(m \in \mathbb{N}\), let \(A = \{1, 2, \ldots, m\}\) to have \(\|1_A\| \approx \sqrt{m} \approx h_l(m)\).

3. Property (I): fix \(u \in \mathbb{N}\) and \(\ell \in \mathbb{N} \cup \{0\}\). We can assume that \(u \leq 2\ell u^2 - 1\) since the only case that the inequality fails is when \(u = 1\) and \(\ell = 0\). We have

\[
 h_{R,l}(2\ell + 1, u) \lesssim \sqrt{2\ell + 1 + u} \leq \sqrt{2}\sqrt{2\ell u + u} \lesssim h_{R,r}(2\ell u, u),
\]

where the last inequality is due to (6.4) and the fact that \(u \leq 2\ell u^2 - 1\). Next, we choose a suitable characteristic function of \(h_{R,l}\). The function \(\psi_m = (m^2 - 1)\lor 1\) is a candidate because

\[
 \sup_u h_{R,l}(m, u) \leq m \lesssim h_{R,l}(m, (m^2 - 1) \lor 1).
\]

We check that \(h_{R,r}(u, u) \lesssim h_{R,l}(m, u)\) whenever \(u \leq \psi^m\): for \(m \geq 2\), \(\psi^m = m^2 - 1\),

\[
 h_{R,r}(u, u) \lesssim \sqrt{u} < \sqrt{u + m} \lesssim h_{R,r}(m, u).
\]

Therefore, the canonical basis has Property (I).

6.4. An unconditional basis with Property (I) and Property (W*) but is not conservative. We shall consider the example of a basis given in [7, Subsection 5.4]. Let \(s_n = \sum_{i=1}^{n} \frac{1}{i}\), and \(\Pi\) be the set of all permutations of \(\mathbb{N}\). Let \(X\) be the completion of \(c_{00}\) with respect to the following norm:

\[
 \|(x_1, x_2, x_3, \ldots)\| = \max \left\{ \sup_n \frac{1}{\sqrt{s_n}} \sup_{\pi \in \Pi} \sum_{i=1}^{n} \frac{|x_{\pi(i)}|}{i^{1/2}}, \left( \sum_i |x_{2i}|^2 \right)^{1/2} \right\}.
\]

The canonical basis \(B\) is an 1-unconditional and non-conservative Schauder basis. Indeed, letting \(A_N = \{2, 4, 6, \ldots, 2N\}\) and \(B_N = \{2N + 1, 2N + 3, \ldots, 4N - 1\}\), we have

\[
 \frac{\|1_{A_N}\|}{\|1_{B_N}\|} \gtrsim \frac{\sqrt{N}}{\sqrt{N} / \sqrt{\ln(N + 1)}} = \sqrt{\ln(N + 1)} \to \infty.
\]

Hence, \(B\) is not conservative.

6.4.1. Calculating democracy functions. For \(N \in \mathbb{N}\), it is obvious that

\[
 h_r(N) = \sqrt{N} \text{ and } h_{\ell}(N) \approx \frac{\sqrt{N}}{\sqrt{\ln(N + 1)}}.
\]

Similarly, for \((m, u) \in \mathbb{D}\),

\[
 h_{R,l}(m, u) \approx \sqrt{m};
\]

for \((m, u) \in \mathbb{N} \times \mathbb{N}\),

\[
 h_{R,r}(m, u) \approx \frac{\sqrt{m}}{\sqrt{\ln(m + 1)}}.
\]
6.4.2. Properties. We verify each desired property below.

(1) Property (W): for every \( m, n \in \mathbb{N} \) with \( m \geq n \), we let \( A = \{ j, j + 2, \ldots, j + 2(n - 1) \} \), where \( j \) is the smallest even integer greater than \( m \). Clearly, \( \norm{1_A} = \sqrt{n} = h_r(n) \).

(2) Property (W\(^*\)): for every \( n \in \mathbb{N} \), let \( A = \{ 1, 3, \ldots, 2n - 1 \} \) to have \( \norm{1_A} \asymp h_l(n) \).

(3) Property (I): Pick \( u \in \mathbb{N} \) and \( \ell \in \mathbb{N} \cup \{0\} \). Then
\[
\frac{h_{R,r}(2^{\ell+1}u, u)}{h_{R,r}(2^{\ell}u, u)} \asymp \frac{\sqrt{2^{\ell+1}u/\ln(2^{\ell+1}u)}}{\sqrt{2^{\ell}u/\ln(2^{\ell}u)}} = O(1).
\]
Furthermore, choose the characteristic function \( \psi(m) = m \). For all \( u \leq m = \psi(m) \), we have
\[
\frac{h_R, r(u, u)}{\sqrt{\ln(u + 1)}} \leq \frac{\sqrt{m}}{\sqrt{\ln(m + 1)}} \asymp h_R, r(m, u).
\]

Corollary 6.8. For the basis \( B \) above, we have
\[
\mathcal{P} \mathcal{G}_q^\omega \not\supseteq \mathcal{G}_q^\omega, \forall \omega \in \mathcal{W}_d \text{ with } i_\omega > 0, \forall q \in (0, \infty].
\]

7. Annex: Lower and upper regularity properties

Duality for greedy-type bases has been studied in [11]. Some of their results depend on two properties: the upper regularity property (URP) and lower regularity property (LRP). The purpose of this section is to show that URP and LRP are equivalent to the upper dilation index of the parameter been smaller than 1 and the lower one being greater than zero, respectively.

Definition 7.1. A positive sequence \( \omega = (\omega(n))_{n=1}^{\infty} \) has the lower regularity property, denoted by \( \omega \in LRP \), if there exist \( \alpha > 0 \) and \( C_\alpha > 0 \) such that
\[
\omega(N) \geq C_\alpha \left( \frac{N}{k} \right)^\alpha \omega(k), \forall N \geq k. \tag{7.1}
\]

A positive sequence \( \omega \) has the upper regularity property, denoted by \( \omega \in URP \), if there exist \( \beta < 1 \) and \( C_\beta \geq 1 \) such that
\[
\omega(N) \leq C_\beta \left( \frac{N}{k} \right)^\beta \omega(k), \forall N \geq k. \tag{7.2}
\]

We write \( \omega \in LRP(\alpha) \) and \( \omega \in URP(\beta) \) when (7.1) and (7.2) hold, respectively.

Proposition 7.2. Let \( \omega \in \mathcal{W} \). Then

(1) \( \omega \in LRP(\alpha) \) if and only if there exists \( c > 0 \) such that
\[
\varphi_\omega(M) \geq cM^\alpha, \forall M \geq 1.
\]

(2) \( \omega \in URP(\beta) \) if and only if there exists \( c > 0 \) such that
\[
\Phi_\omega(M) \leq cM^\beta, \forall M \geq 1.
\]
Proof. We prove (1). (The proof of (2) is similar.) If \( \omega \in LRP(\alpha) \), then
\[
\omega(Mk) \geq C_\alpha M^\alpha \omega(k), \forall M, k \in \mathbb{N}.
\]
Therefore, \( \varphi_\omega(M) \geq C_\alpha M^\alpha \) for all \( M \).

Conversely, suppose that \( c' := \inf_{M \geq 1} \varphi_\omega(M)/M^\alpha > 0 \). Let \( N > k \) and choose \( M \geq 1 \) such that \( Mk < N \leq (M + 1)k \). Then
\[
\omega(N) \geq \omega(Mk) \geq c'M^\alpha \omega(k) \geq c' \left( \frac{M + 1}{2} \right)^\alpha \omega(k) \geq c'2^{-\alpha}N/k^\alpha \omega(k).
\]
Therefore, \( \omega \in LRP(\alpha) \) with constant \( c'/2^\alpha \). \( \square \)

**Proposition 7.3.** For \( \omega \in \mathbb{W} \), the following are equivalent:

1. \( \omega \in LRP \).
2. \( \lim_{M \to \infty} \varphi_\omega(M) = \infty \).
3. There exists \( M_0 \geq 2 \) such that \( \varphi_\omega(M_0) > 1 \).

Proof. (1) \( \Rightarrow \) (2) follows from Proposition 7.2 and (2) \( \Rightarrow \) (3) is trivial. We show that (3) \( \Rightarrow \) (1). Let \( \lambda = \varphi_\omega(M_0) > 1 \) and \( \alpha = \ln \lambda/\ln M_0 \). Then
\[
\omega(M_0k) \geq \lambda \omega(k) = M_0^\alpha \omega(k), \forall k \in \mathbb{N}.
\]
Hence, if \( k < N \), we can find \( j \in \mathbb{N}_0 \) such that \( M_0^j \leq N/k < M_0^{j+1} \) to have
\[
\omega(N) \geq \omega(M_0^j) \geq M_0^\alpha \omega(k) \geq M_0^{-\alpha}(N/k)^\alpha \omega(k).
\]
Thus, we have shown that \( \omega \in LPR(\alpha) \). \( \square \)

Similarly, one can prove the following.

**Proposition 7.4.** For \( \omega \in \mathbb{W} \), the following are equivalent:

1. \( \omega \in URP \)
2. \( \omega \in \mathbb{W}_d \) and \( \lim_{M \to \infty} \Phi_\omega(M)/M = 0 \)
3. There exists \( M_0 \geq 2 \) such that \( \Phi_\omega(M_0)/M_0 < 1 \).

We now deduce the main result in this section.

**Theorem 7.5.** Let \( \omega \in \mathbb{W} \). Then

1. \( \omega \in LRP \) if and only if \( i_\omega > 0 \). Moreover,
\[
i_\omega = \sup \{ \alpha > 0 : \omega \in LRP(\alpha) \}.
\] (7.3)
2. \( \omega \in URP \) if and only if \( I_\omega < 1 \). Moreover,
\[
I_\omega = \inf \{ \beta < 1 : \omega \in URP(\beta) \}.
\] (7.4)

Proof. We prove (1). (The proof of (2) is similar.) If \( \omega \in LRP(\alpha) \), then Proposition 7.2 gives \( i_\omega \geq \alpha \). Hence, \( \omega \in LRP \) implies \( i_\omega > 0 \). Conversely, if \( i_\omega > 0 \), by 2.8, there exists \( M_0 \geq 2 \) such that \( \varphi_\omega(M_0) > 1 \). So \( \omega \in LRP \) by Proposition 7.3. Finally, it remains to prove “\( \leq \)” in (7.3). That is, we must show that if \( 0 < \alpha < i_\omega \) then \( \omega \in LRP(\alpha) \). Indeed, by definition of \( i_\omega \), there exists \( M_\alpha \geq 2 \) such that \( \ln(\varphi_\omega(M))/\ln M \geq \alpha \) for all \( M \geq M_\alpha \). Then we conclude using Proposition 7.2 item (1). \( \square \)
Remark 7.6. The sequence $\omega(n) = \sqrt{n} \ln(n + 1)$, $\gamma \in \mathbb{R}$, shows that, in general, it may not hold that $\omega \in LRP(i_{\omega})$ or $\omega \in URP(I_{\omega})$. Indeed, in this case $i_{\omega} = I_{\omega} = 1/2$, but

$$\varphi_{\omega}(M) \asymp \sqrt{M} / (\ln(M + 1))^{-\gamma} \text{ and } \Phi_{\omega}(M) \asymp \sqrt{M} (\ln(M + 1))^{\gamma + 1}.$$ 

Hence, by Proposition 7.2, $\omega \notin LRP(1/2)$ if $\gamma < 0$, and $\omega \notin URP(1/2)$ if $\gamma > 0$.

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