LOCAL SMOOTHING AND STRICHARTZ ESTIMATES FOR THE KLEIN-GORDON EQUATION WITH THE INVERSE-SQUARE POTENTIAL

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Abstract. We prove weighted $L^2$ estimates for the Klein-Gordon equation perturbed with singular potentials such as the inverse-square potential. We then deduce the well-posedness of the Cauchy problem for this equation with small perturbations, and go on to discuss local smoothing and Strichartz estimates which improve previously known ones.

1. Introduction. Consider the Cauchy problem for the Klein-Gordon equation with a potential $V$:

$$\begin{cases}
\frac{\partial^2}{\partial t^2}u - \Delta u + V(x)u + u = 0, \\
u(x,0) = f(x), \\
\frac{\partial}{\partial t}u(x,0) = g(x),
\end{cases}$$

where $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $\Delta$ is the $n$ dimensional Laplacian.

In [7], D’Ancona established the following smoothing estimate (also known as local energy decay) for the Klein-Gordon flow with $|V| \sim |x|^{-2}$

$$\left\| |x|^{-1}e^{it\sqrt{-\Delta + V + 1}}f \right\|_{L^2_{x,t}} \lesssim \|f\|_{H^{1/2}}$$

(2)

by extending Kato’s $H$-smoothing theory developed in [10, 11] to the flow. Recently, it was shown in [8] that the Klein-Gordon equation can have a solution with much smoothness locally as

$$\sup_{R>0} \frac{1}{R} \int_{|x|<R} \int_{-\infty}^{\infty} \| \nabla |x|^{1/2}e^{it\sqrt{-\Delta + V + 1}}f \|^2 dx dt \lesssim \|f\|^2_{H^{1/2}}$$

(3)

with $V$ such that for $\varepsilon > 0$ and $\delta > 0$

$$|V(x)| \sim \begin{cases}
|x|^{-2} \log |x|^{-(1+\delta)} & \text{near the origin}, \\
|x|^{-(2+\varepsilon)} \log |x|^{-(1+\delta)} & \text{at infinity}.
\end{cases}$$

(4)

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At this point, it is worth noting that this local smoothing estimate can be written in terms of a weighted $L^2$ norm as in (2). Indeed, if $\rho$ is any function such that $\sum_{j \in \mathbb{Z}} \|\rho\|_{L^\infty(|x|^{-2})}^2 < \infty$, one has

$$\|\rho|x|^{-1/2}v\|_{L^2_{x,t}}^2 \lesssim \sup_{R > 0} \frac{1}{R} \int_{|x| < R} \int_{-\infty}^{\infty} |v|^2 dt dx.$$ 

A typical example of such $\rho$ is given by

$$\rho = \sqrt{1 + (\log |x|)^2}^{-\frac{1}{4}(1+\varepsilon)} \varepsilon > 0,$$

as mentioned in [8], and then (3) implies a weaker estimate,

$$\|\rho|x|^{-1/2}|\nabla|^{1/2} e^{it\sqrt{-\Delta+V}+1}f\|_{L^2_{x,t}} \lesssim \|f\|_{H^{1/2}}.$$ 

Since the critical behavior for dispersion appears to be $|V| \sim |x|^{-2}$, recent studies for perturbed dispersive equations have intensively aimed to get as close as possible to this inverse-square potential. One of the main aims of this paper is to obtain the local smoothing estimate (3) allowing small perturbations with the inverse-square potential that improves (4). More generally, we consider Fefferman-Phong potentials $V \in \mathcal{F}^p$ with small $\|V\|_{\mathcal{F}^p}$. They are indeed defined for $1 \leq p \leq n/2$ by

$$\|V\|_{\mathcal{F}^p} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-n/p} \left( \int_{|y-x| < r} |V(y)|^p dy \right)^{1/p} < \infty.$$ 

It then follows that $L^{n/2} = \mathcal{F}^{n/2}$ and $|x|^{-2} \in L^{n/2,\infty} \subseteq \mathcal{F}^p$ if $1 \leq p \leq n/2$. Let us now denote by $L^2(w)$ a weighted $L^2$ space equipped with the norm $\|f\|_{L^2(w)} = \left( \int |f(x)|^2 w(x) dx \right)^{1/2}$. Our main result is then the following theorem in which (2) is generalized by (6) and (3) is improved by (7).

**Theorem 1.1.** Let $n \geq 3$ and $V \in \mathcal{F}^p$ with $\|V\|_{\mathcal{F}^p}$ small enough for $p > (n-1)/2$. Then there exists a unique solution $u \in L^2_{x,t}(|V|)$ to (1) with Cauchy data $(f, g) \in H^{1/2} \times H^{-1/2}$ such that

$$u \in C([0, \infty); H^{1/2}(\mathbb{R}^n)) \quad \text{and} \quad u_t \in C([0, \infty); H^{-1/2}(\mathbb{R}^n)).$$

Furthermore,

$$\|u\|_{L^2_{x,t}(|V|)} \lesssim \|V\|_{\mathcal{F}^p}^{1/2} (\|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}})$$

and

$$\sup_{x_0 \in \mathbb{R}^n, R > 0} \frac{1}{R} \int_{|x-x_0| < R} \int_{-\infty}^{\infty} |\nabla|^{1/2} u|^2 dt dx \lesssim \|f\|_{H^{1/2}}^2 + \|g\|_{H^{-1/2}}^2.$$ 

**Remark 1.** Our theorem particularly considers perturbations of the type $a/|x|^2$ with a small $|a| > 0$. We would like to mention that this smallness condition is a natural restriction when applying the quadratic form techniques to define the Schrödinger operator $-\Delta + a/|x|^2$ in some cases. See [12], p. 172.

**Remark 2.** The estimate (7) shows that the energy in a cylinder $\{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x-x_0| < R\}$ decays as the square root of the radius $R$. In this regard, it may be also seen as a local energy decay.

One of the key ingredients in the proof of Theorem 1.1 is weighted $L^2$ estimates for solutions of the inhomogeneous Klein-Gordon equation $\partial_t^2 u - \Delta u + u = F(x, t)$, which are obtained in Sections 2 and 3. Making use of these estimates, we also obtain the following Strichartz estimate for the perturbed Klein-Gordon equation (1).
Theorem 1.2. Let $n \geq 3$. Assume that $u$ is a solution of (1) with Cauchy data $(f, g) \in H^{1/2} \times H^{-1/2}$ and potential $V \in \mathcal{F}^p$ with small $\|V\|_{\mathcal{F}^p}$ for $p > (n-1)/2$. Then we have

$$
\|u\|_{L^q_t(L^2_R; H^\sigma_x(R^n))} \lesssim (1 + \|V\|_{\mathcal{F}^p})(\|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}})
$$

(8)

where the pairs $q > 2$ and $r \geq 2$ satisfy the admissible condition for $0 \leq \theta \leq 1$

$$
\frac{2}{q} + \frac{n-1+\theta}{r} \leq \frac{n-1+\theta}{2},
$$

(9)

and $\sigma \geq 0$ satisfies the gap condition

$$
\sigma = \frac{1}{q} \left( \frac{n+\theta}{r} - \frac{n-1+\theta}{2} \right). \tag{10}
$$

Remark 3. The condition (9) corresponds to the wave admissible pairs at $\theta = 0$ and the equality in (9) is the Schrödinger admissible pairs at $\theta = 1$. Note that the Klein-Gordon flow $e^{it \sqrt{1-\Delta}}$ behaves like the wave flow at high frequency and the Schrödinger flow at low frequency. In general, the Strichartz estimates for the Klein-Gordon equation are more complicated by the different scaling of $\sqrt{1-\Delta}$ for low and high frequencies.

We conclude with a short summary of earlier results. The Strichartz estimates for the Klein-Gordon equation have been studied for decades.

In the free case $V \equiv 0$, Strichartz [16] first established the following estimate in connection with the Fourier restriction theory in harmonic analysis:

$$
\|u\|_{L^q_t(L^{n+1})} \lesssim \|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}}
$$

(11)

for $2(n+2)/n \leq q \leq 2(n+1)/(n-1)$. Since then, there have been developments in extending (11) to mixed norm spaces $L^q_t(L^r_x(R^n))$ as follows (see e.g. [3, 17] and references therein):

$$
\|u\|_{L^q_t(L^r_x(R^n))} \lesssim \|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}}
$$

(12)

under the same conditions (9) and (10). Here the diagonal case $q = r$ with $\sigma = 0$ entirely recovers (11).

In the case of the potential perturbation, several works have recently treated (8). In [9] the potentials satisfy the decay assumption that $V(x)$ decays like $(|x|^{3/2-\varepsilon} + |x|^2)^{-1}$ at infinity only with the non-endpoint ($q \neq 2$) Schrödinger admissible pairs. See Remark 3. In [8] this is extended to the wave admissible pairs but with the stronger assumption (4) on the potential. Compared with these previous results, our theorem improves not only the perturbation by the inverse-square potential $c/|x|^2$ but also the pairs $(q, r)$ for which the estimate holds.

The rest of this paper is organized as follows. In Sections 2 and 3, we obtain some weighted-$L^2$ and local smoothing estimates concerning the free Klein-Gordon flow $e^{it \sqrt{1-\Delta}}$, which will be used in the next sections 4 and 5 for the proof of Theorems 1.1 and 1.2, respectively.

Throughout this paper, the letter $C$ stands for a positive constant which may be different at each occurrence. We also denote $A \lesssim B$ to mean $A \leq CB$ with unspecified constants $C > 0$. 

2. Estimates for the free flow. In this section we obtain some estimates for the free Klein-Gordon flow $e^{it\sqrt{1-\Delta}}$ which will be used in the next sections for the proof of Theorems 1.1 and 1.2. From now on, we shall use the notation $\langle \nabla \rangle = \sqrt{1-\Delta}$.

**Proposition 1.** Let $n \geq 3$ and $V \in \mathcal{F}^p$ for $p > (n-1)/2$. Then we have
\[
\|e^{it\sqrt{1-\Delta}} f\|_{L^2_t, L^2_x(\{|V|\})} \lesssim \|V\|_{\mathcal{F}^p}^{1/2} \|\langle \nabla \rangle^{1/2} f\|_{L^2}
\]
and
\[
\sup_{t \in \mathbb{R}} \left\|\langle \nabla \rangle^{-1/2} \int_0^t e^{i(t-s)\sqrt{1-\Delta}} F(s, \cdot) ds\right\|_{L^2_x} \lesssim \|V\|_{\mathcal{F}^p}^{1/2} \|F\|_{L^2_{t,s}(\{|V|\}^{-1})}.
\]

**Proof.** First we show the estimate (13). Using polar coordinates $\xi \to r\sigma$ and a change of variables $\sqrt{1+r^2} \to r$, we see
\[
e^{it\sqrt{1-\Delta}} f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it \sqrt{1+|\xi|^2}} \hat{f}(\xi) d\xi
\]
\[
= \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{ix \cdot r\sigma} e^{it \sqrt{1+r^2}} \hat{f}(r\sigma) d\sigma dr
\]
\[
= \int_1^\infty \int_{\mathbb{S}^{n-1}} e^{it \sqrt{r^2-1}} e^{it} \hat{f}(\sqrt{r^2-1}) \frac{r}{\sqrt{r^2-1}} d\sigma \frac{r}{\sqrt{r^2-1}} dr
\]
\[
= \int_{\infty}^{\infty} e^{itr} \chi_{(1,\infty)}(r) \frac{r}{\sqrt{r^2-1}} \langle \hat{f} d\sigma \rangle_{\sqrt{r^2-1}}(-x) dr.
\]
By applying Plancherel’s theorem in the $t$-variable and then using a change of variables $\sqrt{r^2-1} \to r$ again,
\[
\|e^{it\sqrt{1-\Delta}} f\|_{L^2_t, L^2_x(\{|V|\})}^2 = \int_{\mathbb{R}^n} \int_1^{\infty} \left| \frac{r}{\sqrt{r^2-1}} \langle \hat{f} d\sigma \rangle_{\sqrt{r^2-1}}(-x) \right|^2 |V(x)| dr dx
\]
\[
= \int_0^{\infty} \int_{\mathbb{R}^n} \left| \frac{r}{\sqrt{r^2-1}} \langle \hat{f} d\sigma \rangle_{\sqrt{r^2-1}}(-x) \right|^2 |V(x)| dx dr
\]
\[
= \int_0^{\infty} \frac{1+r^2}{r} \left\| \langle \hat{f} d\sigma \rangle_{\sqrt{r^2-1}}(-x) \right\|_{L^2_{t,s}(\{|V|\})}^2 dr.
\]
Now we make use of the following weighted $L^2$ restriction estimate for the Fourier transform:
\[
\|\langle \hat{f} d\sigma \rangle\|_{L^2(\{|V|\})} \lesssim \|V\|_{\mathcal{F}^p}^{1/2} \|f\|_{L^2(\mathbb{S}^{n-1})}
\]
for $V \in \mathcal{F}^p$ with $p > (n-1)/2$, $n \geq 3$, which can be found in [5, 4] (see also [2, 14]). Indeed, applying the re-scaled estimate of (16),
\[
\|\langle \hat{f} d\sigma \rangle\|_{L^2(\{|V|\})} \lesssim r^{1/2} \|V\|_{\mathcal{F}^p}^{1/2} \|f\|_{L^2(\mathbb{S}^{n-1})},
\]
to the right-hand side of (15), we get
\[
\|e^{it\sqrt{1-\Delta}} f\|_{L^2_t, L^2_x(\{|V|\})}^2 \lesssim \int_0^{\infty} \frac{1+r^2}{r} \|V\|_{\mathcal{F}^p} \int_{\mathbb{S}^{n-1}} |\hat{f}(r\sigma)|^2 d\sigma, dr
\]
\[
= \|V\|_{\mathcal{F}^p} \int_0^{\infty} \int_{\mathbb{S}^{n-1}} |(1+r^2)^{1/4} \hat{f}(r\sigma)|^2 d\sigma, dr
\]
\[
= \|V\|_{\mathcal{F}^p} \|\langle \nabla \rangle^{1/2} f\|_{L^2}^2
\]
as desired.
Lemma 2.1. Let

\[ \text{Theorem 2.1 in [1]}: \]

\[ \sup \left\| \nabla^{-1/2} \int_{-\infty}^{\infty} e^{-is\sqrt{1-\Delta}} F(\cdot, s) \, ds \right\|_{L^2_x} \lesssim \|V\|_{\mathcal{F}^p}^{1/2} \|F\|_{L^2_{x,t}(\mathbb{R}^n)} \]  

(17)

by duality. Substituting \( \chi_{[0,t]}(s)F(\cdot, s) \) for \( F(\cdot, s) \) and taking the supremum over \( t \), we then get

\[ \sup_{t \in \mathbb{R}} \left\| \nabla^{-1/2} \int_0^t e^{-i(t-s)\sqrt{1-\Delta}} F(\cdot, s) \, ds \right\|_{L^2_x} \lesssim \|V\|_{\mathcal{F}^p}^{1/2} \|F\|_{L^2_{x,t}(\mathbb{R}^n)} \]  

(18)

Combining this and the fact that \( e^{it\sqrt{1-\Delta}} \) is an isometry in \( L^2 \), we obtain the desired estimate,

\[ \sup_{t \in \mathbb{R}} \left\| \nabla^{-1/2} \int_0^t e^{-i(t-s)\sqrt{1-\Delta}} F(\cdot, s) \, ds \right\|_{L^2_x} = \sup_{t \in \mathbb{R}} \left\| \nabla^{-1/2} \int_0^t e^{-i(t-s)\sqrt{1-\Delta}} F(\cdot, s) \, ds \right\|_{L^2_x} \lesssim \|V\|_{\mathcal{F}^p}^{1/2} \|F\|_{L^2_{x,t}(\mathbb{R}^n)} \]  

Proof. Using polar coordinates \( \xi \to r\sigma \) and a change of variables \( \sqrt{1+r^2} \to r \) as before, we see

\[ |\nabla|^{1/2} e^{it\sqrt{1-\Delta}} f(x) = \int_0^\infty e^{it\sqrt{1+r^2}} r \int_{S^{n-1}} e^{i\sigma^\top x} \hat{f}(r\sigma) \, d\sigma \, dr \]

\[ = \int_0^\infty e^{itr} \chi_{(1,\infty)}(r) \frac{r}{(r^2-1)^{1/2}} \left( \hat{\tilde{f}}(r) \sqrt{r^2-1} \right)^{-}(-x) \, dr. \]

We then apply Plancherel’s theorem in the \( t \)-variable and use a change of variables \( \sqrt{r^2-1} \to r \) to obtain

\[ \sup_{x_0 \in \mathbb{R}^n, \mathcal{R} > 0} \frac{1}{\mathcal{R}} \int_{|x-x_0| < \mathcal{R}} \int_{-\infty}^{\infty} \left| \nabla |^{1/2} e^{it\sqrt{1-\Delta}} f(x) \right|^2 \, dt \, dx \]

\[ = \sup_{x_0, \mathcal{R}} \int_{|x-x_0| < \mathcal{R}} \int_{1}^{\infty} \left| \frac{r}{(r^2-1)^{1/4}} \left( \hat{\tilde{f}}(r) \sqrt{r^2-1} \right)^{-}(-x) \right|^2 \, dr \, dx \]

\[ = \sup_{x_0, \mathcal{R}} \int_{|x-x_0| < \mathcal{R}} \int_{0}^{\infty} \sqrt{1+r^2} \left| \hat{\tilde{f}}(r)(-x) \right|^2 \, dr \, dx. \]  

(19)

At this point we use an elementary result for Fourier transforms of \( L^2 \) densities (see Theorem 2.1 in [1]):

Lemma 2.1. Let \( K \) be a compact subset of a \( C^1 \) manifold \( M \) of codimension \( k \) in \( \mathbb{R}^n \), with the Euclidean surface area \( dS \). If the Fourier transform \( \hat{g} \) of a tempered distribution \( g \in S' \) is a square integrable density \( \hat{g} dS \) with support in \( K \), then

\[ \int_{|x| < \mathcal{R}} |g(x)|^2 \, dx \leq C R^k \int |\hat{g}(\xi)|^2 \, dS \]

(20)

where \( \xi = (\xi_1, \cdots, \xi_{n-k}) \in \mathbb{R}^{n-k} \) and the constant \( C \) is independent of \( g \) and \( R > 0 \).
Indeed, by changing the order of integration in the right side of (19) and then applying (20) with $k = 1$ and $g = \hat{f}d\sigma$, it follows that

$$\sup_{x_0, R} \frac{1}{R} \int_{|x-x_0|<R} \int_{-\infty}^{\infty} \left| \nabla \frac{1}{2} e^{it\sqrt{1-\Delta}} f \right|^2 dt \, dx \lesssim \sup_{x_0, R} \int_{0}^{\infty} \sqrt{1+r^2} \int_{S_{n-1}} |\hat{f}(r\sigma)|^2 d\sigma_r \, dr$$

$$\lesssim \int_{0}^{\infty} \int_{S_{n-1}} |(1+r^2)^{1/4} \hat{f}(r\sigma)|^2 d\sigma_r \, dr$$

$$= \|f\|_{H^{1/2}}^2$$

as desired. \(\square\)

3. Inhomogeneous estimates. In addition to the estimates in the previous section, we need to obtain the corresponding estimates for the inhomogeneous Klein-Gordon equation with zero initial data,

$$\begin{cases}
\partial_t^2 u - \Delta u + u = F(x, t), \\
u(x, 0) = 0, \\
\partial_t u(x, 0) = 0,
\end{cases} \quad (21)$$

as follows:

**Proposition 3.** Let $n \geq 3$ and $V \in \mathcal{F}^p$ for $p > (n-1)/2$. If $u$ is a solution of (21), then

$$\|u\|_{L^2_x, t(|V|)} \lesssim \|V\|_{\mathcal{F}^p} \|F\|_{L^2_x, t(|V|^{-1})} \quad (22)$$

and

$$\sup_{x_0 \in \mathbb{R}^n, R>0} \frac{1}{R} \int_{|x-x_0|<R} \int_{-\infty}^{\infty} \left| \nabla \frac{1}{2} u \right|^2 dt \, dx \lesssim \|V\|_{\mathcal{F}^p} \|F\|_{L^2_x, t(|V|^{-1})}^2 \quad (23)$$

The main ingredient in the proof of this proposition is the following estimates for the resolvent $R(z) := (-\Delta - z)^{-1}$ of the Laplacian. The first estimate (24) can be found in [4, 5] (see also [14]), and see Theorem 2 in [13] for the second estimate (25).

**Lemma 3.1.** Let $n \geq 3$ and $z \in \mathbb{C}$ with $\text{Im} \, z \neq 0$. If $V \in \mathcal{F}^p$ for $p > (n-1)/2$, then

$$\|R(z) f\|_{L^2(|V|)} \leq C \|V\|_{\mathcal{F}^p} \|f\|_{L^2(|V|^{-1})} \quad (24)$$

and

$$\sup_{x_0 \in \mathbb{R}^n, R>0} \frac{1}{R} \int_{|x-x_0|<R} \left| \nabla \frac{1}{2} R(z) f \right|^2 dx \leq C \|V\|_{\mathcal{F}^p} \|f\|_{L^2(|V|^{-1})}^2 \quad (25)$$

with constants $C > 0$ independent of $z$.

3.1. **Proof of (22).** We first decompose the solution $u$ of (21) as

$$u(x, t) = \tilde{u}(x, t) + R(x, t), \quad (26)$$

where

$$\tilde{u}(x, t) = \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} e^{i\xi \cdot x + i\tau} \frac{\hat{F}(\xi, \tau)}{1 + |\xi|^2 - (\tau + i\varepsilon)^2} \, d\xi d\tau$$
and
\[ R(x, t) = -\cos(t\sqrt{1-\Delta}) \int_{-\infty}^{\infty} \frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}} (\chi_{[0, \infty)}(t)F(\cdot, t)) dt - i\frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}} \int_{-\infty}^{\infty} \cos(t\sqrt{1-\Delta}) (\chi_{[0, \infty)}(t)F(\cdot, t)) dt. \]

This decomposition enables us to control the solution \( u \) by dividing it into two parts. We handle the main part \( \tilde{u} \) appealing to the resolvent estimate in Lemma 3.1 while applying the weighted \( L^2 \) estimates for the Klein-Gordon flow \( e^{it\sqrt{1-\Delta}} \) to the remainder part \( R \). Assuming for the moment this decomposition, we will show \( \|\tilde{u}\|_{L^2_{x,t}(|V|)} \lesssim \|V\|_{\mathcal{F}_2} \|F\|_{L^2_{x,t}(|V|^{-1})} \) (27) and
\[ \|R\|_{L^2_{x,t}(|V|)} \lesssim \|V\|_{\mathcal{F}_2} \|F\|_{L^2_{x,t}(|V|^{-1})} \] (28)
which implies immediately the desired estimate (22). The second estimate (28) follows easily by applying the weighted \( L^2 \) estimates (13) and (17). On the other hand, for the first estimate (27) we use Plancherel’s theorem in the \( t \)-variable and then change the order of integration to obtain
\[ \|\tilde{u}\|_{L^2_{x,t}(|V|)} \lesssim \lim_{\epsilon \to 0^+} \left( \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} e^{ix\cdot\xi + it\tau} \frac{\hat{F}(\xi, \tau)}{1 + |\xi|^2 - (\tau + i\epsilon)^2} d\xi d\tau \right)^{\frac{1}{2}} \]
(29)
Applying the resolvent estimate (24) and Plancherel’s theorem in the \( \tau \)-variable to (29), we have
\[
\|\tilde{u}\|_{L^2_{x,t}(|V|)} \lesssim \int_{-\infty}^{\infty} \|V\|_{\mathcal{F}_2} \left( \int_{\mathbb{R}^n} |\hat{F}(x, \cdot)(\tau)|^2 |V(x)|^{-1} dx d\tau \right)
= \|V\|_{\mathcal{F}_2} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} |F(x, t)|^2 |V(x)|^{-1} dt dx
= \|V\|_{\mathcal{F}_2} \|F\|_{L^2_{x,t}(|V|^{-1})}^2
\]
as desired.

Now it remains to prove (26). Since \( \tilde{u} = u - R \) is a solution to the inhomogeneous equation via the space-time Fourier transform, the remainder term \( R \) is the solution of the homogeneous problem:
\[
\begin{align*}
\partial_x^2 R - \Delta R + R &= 0, \\
R(x, 0) &= -\tilde{u}(x, 0), \\
\partial_t R(x, 0) &= -\partial_t \tilde{u}(x, 0)
\end{align*}
\]
whose solution is given by
\[ R(x, t) = \cos(t\sqrt{1-\Delta})(-\tilde{u}(\cdot, 0)) + \frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}}(-\partial_t \tilde{u}(\cdot, 0)). \] (30)
To get \( R \) explicitly, we compute
\[ \tilde{u}(x, 0) = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\hat{F}(\xi, \tau)}{1 + |\xi|^2 - (\tau + i\epsilon)^2} d\xi d\tau \]
and note that
\[
\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} e^{-it\tau} \left( \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \frac{e^{-it\tau}}{1 + |\xi|^2 - (\tau + i\varepsilon)^2} d\tau \right) d\xi
\]

(see [15], p. 30). Therefore, we get
\[
\tilde{u}(x,0) = \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} e^{ix\xi} \frac{\sin (t\sqrt{1 + |\xi|^2})}{\sqrt{1 + |\xi|^2}} \hat{F}(\cdot,t)(\chi(0,\infty)(t)) d\xi dt
\]
\[
= \int_{-\infty}^{\infty} \sin \left( t \sqrt{1 - \Delta} \right) (\chi(0,\infty)(t) F(\cdot,t)) dt. \tag{31}
\]

Similar calculations give
\[
\partial_t \tilde{u}(x,0) = i \int_{-\infty}^{\infty} \cos \left( t \sqrt{1 - \Delta} \right) (\chi(0,\infty)(t) F(\cdot,t)) dt.
\]

Inserting this and (31) into (30), we get the decomposition (26).

3.2. Proof of (23). Next we prove (23). By the decomposition (26), it is enough to show that both \( \tilde{u} \) and \( R \) satisfy the estimate (23). For \( R \), it is easily shown by applying (18) and (17). On the other hand, for \( \tilde{u} \) we use Plancherel’s theorem in the \( t \)-variable and then change the order of integration to get

\[
\sup_{x_0, R} \frac{1}{R} \int_{|x - x_0| < R} \int_{-\infty}^{\infty} \left| \nabla \right|^{1/2} \tilde{u}(x,t) \right|^2 dx dt
\]
\[
= \sup_{x_0, R} \frac{1}{R} \int_{|x - x_0| < R} \int_{-\infty}^{\infty} \left| e^{ix\xi + it\tau} \right| \frac{|\xi|\sqrt{1 + |\xi|^2}}{1 + |\xi|^2 - (\tau + i\varepsilon)^2} d\xi d\tau \right|^2 dx dt
\]
\[
\lesssim \sup_{x_0, R} \frac{1}{R} \int_{|x - x_0| < R} \left| \int_{\mathbb{R}^n} e^{ix\xi} \frac{|\xi|\sqrt{1 + |\xi|^2}}{1 + |\xi|^2 - (\tau + i\varepsilon)^2} d\xi \right|^2 dx d\tau. \tag{32}
\]

Using (25) and then applying Plancherel’s theorem in the \( \tau \)-variable again, the right-hand side of (32) is bounded by

\[
\|V\|_{L^p} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |\hat{F}(x,\cdot)(\tau)|^2 |V(x)|^{-1} dx d\tau = \|V\|_{L^p} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} |F(x,t)|^2 |V(x)|^{-1} dt dx
\]
\[
= \|V\|_{L^p} \|F\|_{L_{x,t}^2(|V|^{-1})}^2.
\]

Hence we get

\[
\sup_{x_0, R} \frac{1}{R} \int_{|x - x_0| < R} \int_{-\infty}^{\infty} \left| \nabla \right|^{1/2} \tilde{u}(x,t) \right|^2 dx dt \lesssim \|V\|_{L^p} \|F\|_{L_{x,t}^2(|V|^{-1})}^2
\]
as desired.

4. **Proof of Theorem 1.1.** This section is devoted to proving Theorem 1.1. Let us first define the space $X$ and the operator $S$

\[ X = \{ F : \| F \|_{L^2_x([V])} < \infty \} \]

and

\[ SF = \int_0^t \frac{\sin((t-s)\sqrt{1-\Delta})}{\sqrt{1-\Delta}} (V(\cdot)F(\cdot, s))ds. \]

We then consider the potential term in (1) as a source term and thus write the solution of (1) as the sum of the solution to the free Klein-Gordon equation plus a Duhamel term, as follows:

\[ u = \cos(t\sqrt{1-\Delta})f + \frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}} g + Su. \quad (33) \]

For the existence of a unique solution in the space $X$, we will show that the first two terms in the right-hand side of (33) are in $X$, provided $(f, g) \in H^{1/2} \times H^{-1/2}$, and that the operator $S$ is a contraction in $X$ if $\|V\|_{\mathcal{F}^p}$ is small enough. The first part follows immediately from applying the estimate (13):

\[ \left\| \cos(t\sqrt{1-\Delta})f + \frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}} g \right\|_{L^2_x([V])} \lesssim \|V\|_{\mathcal{F}^p}^{1/2} (\|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}}). \quad (34) \]

For the second part, by the estimate (22) we see that

\[ \|SF\|_{L^2_x([V])} \lesssim \|V\|_{\mathcal{F}^p} \|VF\|_{L^2_x([V])} \lesssim \|V\|_{\mathcal{F}^p} \|F\|_{L^2_x([V])} \leq \frac{1}{2} \|F\|_{L^2_x([V])} \quad (35) \]

for $F \in X$, and thus the operator $S$ is a contraction in $X$ since we are assuming that $\|V\|_{\mathcal{F}^p}$ is small enough.

Next, we show (5). First note that

\[ \sup_{t \in \mathbb{R}} \left\| \cos(t\sqrt{1-\Delta})f + \frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}} g \right\|_{H^{1/2}} \lesssim \|\langle \nabla \rangle^{1/2} f\|_{L^2} + \|\langle \nabla \rangle^{-1/2} g\|_{L^2} \]

by Plancherel’s theorem, and applying (14) with $F = Vu$ gives

\[ \sup_{t \in \mathbb{R}} \|Su\|_{H^{1/2}} \lesssim \|V\|_{\mathcal{F}^p}^{1/2} \|u\|_{L^2_x([V])}. \]

It then follows easily that $u \in C([0, \infty); H^{1/2}(\mathbb{R}^n))$. Similarly as above, to show $u_t \in C([0, \infty); H^{-1/2}(\mathbb{R}^n))$, we first calculate

\[ \partial_t \left( \cos(t\sqrt{1-\Delta})f + \frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}} g \right) = -\langle \nabla \rangle \sin(t\sqrt{1-\Delta})f + \cos(t\sqrt{1-\Delta})g \]

and

\[ \frac{d}{dt} \int_0^t \frac{\sin((t-s)\sqrt{1-\Delta})}{\sqrt{1-\Delta}} F(\cdot, s)ds = \int_0^t \cos((t-s)\sqrt{1-\Delta})F(\cdot, s)ds. \]

Then we see

\[ \sup_{t \in \mathbb{R}} \left\| \partial_t \left( \cos(t\sqrt{1-\Delta})f + \frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}} g \right) \right\|_{H^{-1/2}} \lesssim \|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}} \].

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by Plancherel’s theorem, and see
\[
\sup_{t \in \mathbb{R}} \left\| \frac{d}{dt} \int_{0}^{t} \frac{\sin((t-s)\sqrt{1-\Delta})}{\sqrt{1-\Delta}} F(\cdot,s)ds \right\|_{H^{-1/2}} \lesssim \|V\|_{\mathcal{F}^{2p}}^{1/2} \|u\|_{L^{2,q}_{x,t}(\mathcal{V})}.
\]
by (14). It then follows from (33) that \(u_{t} \in C([0, \infty); H^{-1/2}(\mathbb{R}^{n}))\).
Combining (33), (34) and (35) directly yields the desired estimate (6). It remains
only to show the local smoothing estimate (7). But this follows immediately from
applying (33), (18), (23), and then (6).

5. **Proof of Theorem 1.2.** In this final section we prove Theorem 1.2 by making
use of the weighted \(L^{2}\) estimates obtained in Sections 2 and 3. Recalling (33), we
first write the solution \(u\) of (1) as
\[
u = \cos(t\sqrt{1-\Delta})f + \frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}}g + Su.
\] (36)
Applying the following Strichartz estimate for the free case (see (12)),
\[
\|e^{it\sqrt{1-\Delta}}f\|_{L^{q}_{t}H^{r}_{x}} \lesssim \|f\|_{H^{1/2}},
\] (37)
to the first two terms of (36), we then have
\[
\left\| \cos(t\sqrt{1-\Delta})f + \frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}}g \right\|_{L^{q}_{t}H^{r}_{x}} \lesssim \|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}}
\]
under the same conditions on \((q, r)\) and \(\sigma\) as in Theorem 1.2. Now it remains
to show that
\[
\|Su\|_{L^{q}_{t}H^{r}_{x}} = \left\| \int_{0}^{t} \frac{\sin((t-s)\sqrt{1-\Delta})}{\sqrt{1-\Delta}} (V(\cdot)u(\cdot, s))ds \right\|_{L^{q}_{t}H^{r}_{x}} \lesssim \|V\|_{\mathcal{F}^{2p}}(\|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}})
\]
for the same \((q, r)\) and \(\sigma\). By duality, it suffices to show that
\[
\left\langle \langle \nabla \rangle^{\sigma} \int_{0}^{t} \frac{\sin((t-s)\sqrt{1-\Delta})}{\sqrt{1-\Delta}} (V(\cdot)u(\cdot, s))ds, G \right\rangle_{x,t} \lesssim \|V\|_{\mathcal{F}^{2p}}(\|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}}) \|G\|_{L^{q}_{t}L^{r}_{y}}.
\] (38)
The left-hand side of (38) is equivalent to
\[
\int_{-\infty}^{\infty} \int_{0}^{t} \left\langle \langle \nabla \rangle^{\sigma-1} \sin((t-s)\sqrt{1-\Delta}) (V(\cdot)u(\cdot, s)), G \right\rangle_{x} duds
t = \int_{-\infty}^{\infty} \int_{0}^{t} \langle Vu, \langle \nabla \rangle^{\sigma-1} \sin((t-s)\sqrt{1-\Delta})G \rangle_{x} duds
 = \left\langle V^{1/2}u, V^{1/2} \langle \nabla \rangle^{\sigma-1} \int_{s}^{\infty} \sin((t-s)\sqrt{1-\Delta})G d\right\rangle_{x,s}.
\] (39)
Using Hölder’s inequality, (39) is bounded by
\[
\|u\|_{L^{2,q}_{x,t}(\mathcal{V})} \left\| \langle \nabla \rangle^{\sigma-1} \int_{s}^{\infty} \sin((t-s)\sqrt{1-\Delta})G d\right\|_{L^{2,q}_{x,t}(\mathcal{V})}.
\]
We will show that
\[
\|u\|_{L^{2,q}_{x,t}(\mathcal{V})} \lesssim \|V\|_{\mathcal{F}^{2p}}^{1/2}(\|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}})
\] (40)
and
\[ \left\| \langle \nabla \rangle^{-1} \int_t^\infty \sin \left( (t-s)\sqrt{1-\Delta} \right) G \, ds \right\|_{L^2_x,\nu(\mathbb{R}^d)} \lesssim \| V \|^{1/2}_{\mathcal{F}^p} \left\| L^q_x L^{q'}_t \right\| . \] (41)

Then the desired estimate (38) is proved.

To show the first estimate (40), we apply (13) and (22) to (36) to get
\[ \| u \|_{L^2_x,\nu(\mathbb{R}^d)} \lesssim \| V \|^{1/2}_{\mathcal{F}^p} \left( \| f \|_{H^{1/2}} + \| g \|_{H^{-1/2}} \right) + \| V \|_{\mathcal{F}^p} \| u \|_{L^2_x,\nu(\mathbb{R}^d)}. \] (42)

Since we are assuming that \( \| V \|_{\mathcal{F}^p} \) is small enough, the last term on the right-hand side of (42) can be absorbed into the left-hand side. Hence, we get (40). For the second estimate (41), we first note that
\[ \left\| \langle \nabla \rangle^{-1} \int_{-\infty}^\infty \sin \left( (t-s)\sqrt{1-\Delta} \right) G \, ds \right\|_{L^2_x,\nu(\mathbb{R}^d)} \]
\[ \lesssim \left\| \langle \nabla \rangle^{-1} e^{it\sqrt{1-\Delta}} \int_{-\infty}^\infty e^{-is\sqrt{1-\Delta}} G \, ds \right\|_{L^2_x,\nu(\mathbb{R}^d)} \]
\[ \lesssim \| V \|^{1/2}_{\mathcal{F}^p} \left\| \langle \nabla \rangle^{-1/2} \int_{-\infty}^\infty e^{-is\sqrt{1-\Delta}} G \, ds \right\|_{L^2} \]
\[ \lesssim \| V \|^{1/2}_{\mathcal{F}^p} \left\| L^q_x L^{q'}_t \right\|. \]

using (13) and the dual estimate of (37). We then use the following Christ-Kiselev lemma ([6]) to conclude
\[ \left\| \langle \nabla \rangle^{-1} \int_{-\infty}^t \sin \left( (t-s)\sqrt{1-\Delta} \right) G \, ds \right\|_{L^2_x,\nu(\mathbb{R}^d)} \lesssim \| V \|^{1/2}_{\mathcal{F}^p} \left\| L^q_x L^{q'}_t \right\| \]
for \( 2 > q' \), which yields (41) simply by changing some variables.

**Lemma 5.1** (Christ-Kiselev lemma). Let \( X \) and \( Y \) be two Banach spaces and let \( T \) be a bounded linear operator from \( L^p(\mathbb{R}; X) \) to \( L^q(\mathbb{R}; Y) \) such that
\[ Tf(t) = \int_{-\infty}^\infty K(t,s) f(s) \, ds. \]

Then the operator
\[ \tilde{T} f(t) = \int_{-\infty}^t K(t,s) f(s) \, ds \]
has the same boundedness when \( \beta > \alpha \), and \( \| \tilde{T} \| \lesssim \| T \|. \)

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