Nematic Liquid Phase in a Frustrated Spin-1 System on the Square Lattice

Wen-Jun Hu\textsuperscript{1}, Shou-Shu Gong\textsuperscript{2,3,*}, Hsin-Hua Lai\textsuperscript{1}, Haoyu Hu\textsuperscript{1}, Qimiao Si\textsuperscript{1}, and Andriy H. Nevidomskyy\textsuperscript{1}

\textsuperscript{1}Department of Physics and Astronomy & Rice Center for Quantum Materials, Rice University, Houston, Texas 77005, USA
\textsuperscript{2}National High Magnetic Field Laboratory, Florida State University, Tallahassee, Florida 32310, USA
\textsuperscript{3}Department of Physics and International Research Institute of Multidisciplinary Science, Beihang University, Beijing 100191, China

Competing interactions in quantum spin systems promote a variety of unusual phases. An important example is the frustrated spin-1 model on the square lattice with the nearest-neighbor bilinear ($J_1$) and biquadratic ($K_1$) interactions, which has an enlarged SU(3) symmetry at $K_1 = J_1$. We find evidence for a nematic spin liquid phase for a range of $K_1/J_1$ near the SU(3)-symmetric point, based on the linear flavor-wave theory (LFWT) and the density matrix renormalization group (DMRG) method. This phase displays no spin dipolar or quadrupolar order, preserves translational symmetry but spontaneously breaks $C_4$ lattice rotational symmetry, and possesses fluctuations peaked at the wavevector $(\pi, 2\pi/3)$. The excitation spectrum appears gapless, and the nematic order is attributed to the dominant $(\pi, 2\pi/3)$ fluctuations. Our results provide a novel mechanism for electronic nematic order and, more generally, open up a new avenue to explore frustration-induced exotic ground states.

PACS numbers: 73.43.Nq, 75.10.Jm, 75.10.Kt

Introduction.— The spin-1/2 Heisenberg models represent a prototype for quantum magnetism. Such models can realize exotic phases of matter if the frustration, either due to geometry or competing interactions, becomes dominant. One of the intriguing phases in this system is the elusive quantum spin liquid (QSL) [1, 2]. Stimulated by experimental findings of spin liquid-like behaviors in antiferromagnetic materials, frustrated spin-1/2 Heisenberg models have been studied extensively and QSLs have been found in realistic models (see Ref. [2] and references therein).

Spin-1 systems are also of extensive interest in the search for novel quantum phases [3–12]. In one dimension, the celebrated Haldane phase exhibits a bulk energy gap and gapless excitations on the boundaries [3]. In two dimensions (2D), novel phases also emerge in frustrated spin-1 systems [13]. For systems with spin-1 (or higher), biquadratic interactions are also allowed for in addition to the bilinear (Heisenberg) ones. The competition between the two types of interactions represents an added form of frustration. This competition is particularly strong when the two interactions are comparable in strength. If they are exactly equal to each other, the system has an enlarged SU(3) symmetry [14, 15]. This symmetry gives rise to a large classical degeneracy between different ordered phases, which may lead to novel states when quantum fluctuations are incorporated [16]. One of the intriguing outstanding questions is whether proximity to an SU(3) symmetry promotes the spin-liquid phases.

Spin-1 models with bilinear-biquadratic interactions are also of considerable interest in a variety of strongly correlated systems. Models of this type on the square lattice have recently been analyzed in the context of nematic order for the iron-based superconductor FeSe [17–20]. They are also of interest to various antiferromagnetic materials [21–24] as well as to cold-atom systems with multiple flavors of fermions and a large Hubbard interaction [25–27]. Interestingly, in the spin-1 systems, spin liquid-like behavior has been reported in triangular (NiGa$_2$S$_4$ [21], Ba$_3$NiSb$_2$O$_9$ [22, 23]) and honeycomb (6HB-Ba$_3$NiSb$_2$O$_9$ [24]) antiferromagnets, and scenarios for spin liquid phase with a spinon Fermi surface have been theoretically proposed [23, 28]. To date, however, evidence for spin liquid state in spin-1 models has been scarce [19] and, in the SU(3) cases, model studies have only found conventional orders [10, 15, 29–33].

In this Letter, we study the quantum phases near the SU(3) point ($J_1 = K_1$) of the spin-1 bilinear-biquadratic Hamiltonian on the square lattice:

$$H = \sum_{\langle i,j \rangle} J_1 S_i \cdot S_j + K_1 (S_i \cdot S_j)^2. \quad (1)$$

Here, $S_i$ is the spin-1 operator at site $i$, and $J_1, K_1$ are the nearest-neighbor bilinear and biquadratic interactions, respectively. We have used LFWT to show that, over a range of parameters including the SU(3) point, a $(\pi, 2\pi/3)$ nematic and antiferroquadrupolar (AFQ23) order is energetically competitive, albeit ultimately unstable. We have also carried out DMRG studies on the large cylinder with circumference up to 9; this unrestricted geometry is chosen, because it does not bias the system towards three-sublattice orders and instead accommodates other types of ordering as well. To our surprise, we find that the ground state near the SU(3) point is a disordered phase without any spin dipolar, quadrupolar, or valence-bond crystal (VBC) order. This phase is found to possess a lattice nematic order with spontaneously broken $C_4$ lattice rotational symmetry. Finally, we have studied the low-lying excitations in several different ways. These analyses suggest the gapless nature of the excitation spectrum, and lead to a qualitative understanding for the dominating $(\pi, 2\pi/3)$ fluctuations and, by extension, the origin of the nematic order.

It should be noted that the square-lattice model defined by Eq. (1) contains an intriguing interplay between the three-flavor spin degrees of freedom and the bipartite nature of the lattice [15]. Previous semi-classical studies using a site-factorized wavefunction [14, 15] found the SU(3) point as the boundary between the Néel antiferromagnetic order (AFM2) and a “semiordered phase” with infinitely degenerate ground states. The inclusion of quantum fluctuations at the level of...
LFWT made the three-sublattice order at momentum $q = (2\pi/3, 2\pi/3)$ to have the lowest ground-state energy [15, 29]; such an order has also been suggested by numerical calculations [15, 29, 30]. Nonetheless, both the LFWT [29] and Schwinger boson mean-field [30] calculations had difficulties to determine the order parameter, which suggested that strong fluctuations may produce physics beyond such descriptions.

**Nematic order: indications from the linear flavor-wave theory.**— Using LFWT, we study the vicinity of the SU(3) point and consider different competing orders with finite stiffness. While we leave the details to Ref. [34], we note that, in addition to the previously studied three-sublattice antiferromagnetic (AFM3) and antiferroquadrupolar (AFQ3) orders at $q = (2\pi/3, 2\pi/3)$ [15, 29], the spin dipolar (AFM23) and quadrupolar (AFQ23) orders at $q = (\pi, 2\pi/3)/(2\pi/3, \pi)$ can also be locally stabilized. In particular, for $K_1/J_1 > 1$, the AFQ23 order is degenerate at the mean-field level with the AFQ3 phase, as is illustrated in Fig. 1; it has higher energy than the latter only as a result of the zero-point fluctuations. The AFQ23 order is of particular interest to us, because it is accompanied by a nematic order. Similar to the case of the AFQ3 phase at its SU(3) point [15], the corrections to the quadrupolar order parameters by the quantum fluctuations are divergent in the whole parameter regime including the SU(3) point. Nonetheless, the quantum correction to the nematic order is finite. What emerges is that the static orderings of both AFQ23 and AFQ3 are destroyed by the quantum fluctuations but, if the fluctuating order of the AFQ23 type remains important, as we will demonstrate using DMRG, a static nematic order can still be stabilized. To address this issue, we turn to the DMRG method, which is well-established to capture quantum fluctuations of highly frustrated spin systems.

**DMRG phase diagram.**— We establish quantum phases based on DMRG [35, 36] calculations on cylindrical geometry by keeping up to 4000 SU(2) states with truncation error below $1 \times 10^{-5}$ [34]. Two geometries are considered: a rectangular cylinder (RC) and a 45° tilted cylinder (TC) with the periodic boundary conditions in the $y$ direction and the open boundaries in the $x$ direction, which are denoted as RC/TC/L$_y$. (L$_y$ and L$_x$ are the number of sites in the two directions). We study the system with $L_y$ up to 9 and $L_x$ up to 36. We set $J_1 = 1$ as a unit energy scale.

The quantum phase diagram established by our DMRG calculation is shown in Fig. 2(a). With growing $K_1$, we find a Néel AFM2 phase for $K_1 \lesssim 0.75$ and a disordered phase for $0.75 \lesssim K_1 \lesssim 1.4$. First of all, we characterize the phases using spin and quadrupolar order parameters $m_S^2(q) = \sum_{i,j} S_i \cdot S_j e^{iq \cdot (r_i - r_j)}$ and $m_Q^2(q) = \sum_{i,j} Q_i \cdot Q_j e^{iq \cdot (r_i - r_j)}$ ($Q_i$ is the quadrupolar operator [37]), where the sites $i,j$ are chosen over the middle $N_e = L_y \times 2L_y$ sites in order to avoid the effects of open edge [38]. In Figs. 2(b) and 2(c), we show the peaks of both orders at four dominant momenta as a function of $K_1$. At $K_1 \approx 0.75$, the Néel order $m_S^2(\pi, \pi)$ drops sharply, suggesting a first-order transition from the Néel order to the disordered phase, which was also reported in a previous exact diagonalization study [15]. For $K_1 \gtrsim 1.4$, DMRG truncation error is enhanced [34] and our calculations are less converged. In this work, we will focus on the most intriguing disordered phase around the SU(3) point.

For the disordered regime, we find that the choice of the cylinder geometry is important: (1) For the RC cylinder, we find that the calculations on cylinders with $L_y = 3n$ ($n$ is an integer) have smaller truncation error and lower bulk energy than those on the other $L_y$ (see Supplemental Material [34]), which is consistent with the dominant structure factor peak at $q = (\pi, 2\pi/3)$ as shown in Figs. 2-3. (2) On the TC cylinder, this $(\pi, 2\pi/3)$ momentum is frustrated but the three-sublattice momentum $(2\pi/3, 2\pi/3)$ is allowed. Indeed, our DMRG calculations find the three-sublattice order on the TC cylinder [34]. Nevertheless, we note that DMRG truncation error is much larger on the TC cylinder, possibly because the dominant $(\pi, 2\pi/3)$ momentum is frustrated. (3) We also note that the shifted cylinders used in Ref. [30], which connect the top site of the $i$-th column to the bottom site of the $(i + 1)$-th column, also favor the three-sublattice momentum in DMRG. We compare the bulk energy on the RC and the shifted cylinders for the SU(3) model, showing that the RC cylinders with $L_y = 3n$ have the lowest energy on our studied system size. For example, the energy of the disordered phase on RC9 is

![FIG. 1. An illustration of the antiferroquadrupolar order with wavevector $(2\pi/3, \pi)$ (AFQ23).](image)
around $-0.652$ per site [34], which is close but $\sim 5\%$ lower than that of the three-sublattice order on the shifted RC8 cylinder ($-0.625$ from Ref. [30]). Therefore, we focus on the unrestricted RC cylinders with $L_y = 3, 6, 9$, which can harbor the dominant short-range order structure with a two-site period along the $x$-axis and a three-site period along the $y$-axis, as well as the three-sublattice structure.

Disordered nematic phase.—For $0.75 \lesssim K_1 \lesssim 1.4$, we have studied the finite-size scaling of the order parameters from calculations done with RC3, RC6, and RC9. We find that both spin and quadrupolar orders decay fast and properly extrapolate to vanish in the thermodynamic limit, as illustrated in Fig. 3(b) for the SU(3) point (see Supplemental Material for similar analyses at other values of $K_1$ [34]). Thus, we find no formation of either a spin or quadrupolar order in the disordered phase. We have also calculated the scalar spin chiral correlation function $\langle \chi_i \chi_j \rangle$ detecting time-reversal symmetry breaking, where $\chi_i = (S_{i,1} \times S_{i,2}) \cdot S_{i,3}$ is the scalar spin chirality associated with three spins at each plaquette. We find that this correlation function vanishes in the disordered phase.

Next, we examine the lattice translation symmetry by calculating the nearest-neighbor spin dipolar bond energy $\langle S_i \cdot S_j \rangle$ and spin quadrupolar bond energy $\langle Q_i \cdot Q_j \rangle$. In Figs. 4(a)-(b), we show the bond energy at the SU(3) point in the bulk of RC6. We find that the bond energy difference of the two bonds all decay quite fast from the open edge to the bulk, leading to the translationally uniform bond energy (the same for RC3 and RC9) in the bulk. The uniform bond energy indicates the preserved translational symmetry, precluding the possibility of a VBC order. Note that the apparent absence of the VBC on the square lattice is different from the kagome and honeycomb SU(3) models, in which the nonmagnetic ground states have been identified as breaking different lattice symmetries [31–33].

We now turn to analyzing the nematic order. In Figs. 4(a)-(b), one can see a strong anisotropy between the horizontal and vertical bond energies. We define the corresponding nematic order parameters $\sigma_1$ and $\sigma_Q$ as $\sigma_1 \equiv \frac{1}{N_m} \sum_i \langle S_i \cdot S_{i+\hat{x}} \rangle - \langle S_i \cdot S_{i+\hat{y}} \rangle$ [39] and $\sigma_Q \equiv \frac{1}{N_m} \sum_i \langle (Q_i \cdot Q_{i+\hat{x}}) - \langle Q_i \cdot Q_{i+\hat{y}} \rangle \rangle$ [17] ($\hat{x}$ and $\hat{y}$ denote the unit vectors along the two directions, $N_m$ is the number of sites of the two columns in the middle of cylinder). Both on RC6 and RC9, the results of $\sigma_1$ and $\sigma_Q$ versus $K_1$ are illustrated in Fig. 4(c). We remark that upon the transition from the Néel phase at $K_1 \approx 0.75$, both $\sigma_1$ and $\sigma_Q$ grow dramatically. Through the finite-size scaling shown in Fig. 4(d), we find that both $\sigma_1$ and $\sigma_Q$ remain nonzero in the thermodynamic limit. The finite-size scaling of the nematic order has been shown as an efficient method to identify the lattice $C_4$ symmetry breaking for different quantum phases in DMRG calculations [40]. Our results indicate that the $C_4$ symmetry breaking occurs spontaneously and is not the result of the cylindrical lattice geometry.

Singlet and triplet gaps.—In order to further characterize the nature of the disordered phase near the SU(3) point, we have also calculated the spin gaps with $\Delta S = 1, 2$ ($S$ is the total spin quantum number) as well as the spin-singlet gap in the bulk of a long cylinder by sweeping the excited states in the bulk [41]. The two spin gaps $\Delta S = 1, 2$ have the same values on our studied systems at the SU(3) point, as anticipated from the SU(3) symmetry. Limited by the system size on RC3 and RC6, we cannot perform finite-size scaling for the spectral gaps. However, we remark the fast drop of the gaps from RC3 to RC6, with the singlet gap reducing from 1.853 to 0.285 (more than 80%) and the triplet gap from 2.028 to 0.774 (more than 60%), which may suggest either tiny or vanishing gaps in the thermodynamic limit.

Discussion on the nature of the disordered phase.—According to the Lieb-Schultz-Mattis-Hastings (LSMH) theo-
rem [42, 43], this spin-1 disordered phase can be either a gapped quantum paramagnet with a unique ground state such as the Haldane phase, a gapped topological spin liquid, or a gapless spin liquid. We have carefully studied these possibilities. First of all, we study this square model by continuously changing the interchain bilinear-biquadratic interactions. With equal interchain and intrachain couplings, the system is the studied 2D model. Without interchain coupling, the system reduces to decoupled spin-1 bilinear-biquadratic chain which is in the gapped Haldane phase for $K_1 < J_1$ [4]. Interestingly, we find an intermediate spin ordered phase between the weakly-coupled Haldane chains and the disordered phase with growing bilinear-biquadratic interchain couplings [34], which does not support the Haldane phase as a candidate for the disordered phase. Furthermore, in DRMG simulation of the spin-1/2 gapped spin liquid, one can obtain the spinon topological sector by removing or adding a spin-1/2 on each open edge of the cylinder [41]. Theoretically, three spinons (partons) are usually introduced to construct a spin-1 spin liquid state [28], which suggests that a spinon carries an integer spin. We follow the DRMG technique by adding or removing a spin-1 site on each open edge of the $L_y = 3, 6$ cylinders to detect the possible spinon sector in the disordered state. However, we do not find the near-degenerate spinon sector [34].

The numerical results and the discussions above motivate us to consider a gapless disordered phase. If the gapless spin liquid possesses a Fermi surface formed by the emergent or near-degenerate spinon sector [34]. With equal interchain and intrachain couplings, the system size, this issue certainly requires further scrutiny. Analyzing by alternative methods such as the variational Monte Carlo with Gutzwiller projected parton constructions are left for future studies.

In the frame of gapless spin liquid, the nematic nature can be understood based on the low-energy gapless fluctuations. For low energy description, the spin operator can be expressed as $S_I \sim e^{i\mathbf{q} \cdot \mathbf{r}} / \mathbf{S}_0 + H.c.$, where the vector $\mathbf{q}$ is the wavevector associated with the gapless excitations. The bond energy operator can be expressed as $\Delta S = S_I / S_{I+1} \sim e^{i\mathbf{q} \cdot \mathbf{S}_0 / \mathbf{S}_{-q} + H.c}$. The numerical results and the discussions above motivate us to consider a gapless disordered phase. If the gapless spin liquid possesses a Fermi surface formed by the emergent or near-degenerate spinon sector [34]. With equal interchain and intrachain couplings, the system size, this issue certainly requires further scrutiny. Analyzing by alternative methods such as the variational Monte Carlo with Gutzwiller projected parton constructions are left for future studies.

We stress that this mechanism for the nematic order operates not only for static order such as the $(2\pi/3, \pi)$ AFQ23 illustrated in Fig. 1, but also for fluctuations of such order. Our DMRG calculations have indeed found the $(\pi, 2\pi/3)/(2\pi/3, \pi)$ fluctuations at $L_y$-leg ladder near the SU(3) point, shown in Figs. 2-3. The precise origin of this wavevector $(\pi, 2\pi/3)/(2\pi/3, \pi)$ remains to be understood, but it is already hinted at by the local stability of such an order within the semiclassical analysis described earlier. In addition, the tendency towards dominant fluctuations of such a wavevector arises from a coupled-chain picture of the gapless spin liquid: $2\pi/3$ twice of the parton “Fermi-wavevector” for an SU(3)-chain, which is associated with the 1/3-filling in terms of the partons, and $\pi$ reflects the nearest-neighbor coupling being the interchain coupling that is relevant in the RG sense. Further details of this analysis are given in the Supplementary Material [34].

**Summary.**—We have studied the ground state of the spin-1 bilinear-biquadratic model on the square lattice using the semiclassical flavor-wave theory and the state-of-the-art DMRG on large cylinder geometry. We find a disordered state as the ground state near the SU(3) point by showing the vanishing of the spin and quadrupolar orders, as well as the absence of the lattice translational symmetry breaking. The bond energy texture indicates the spontaneous breaking of the $C_4$ lattice rotational symmetry. Based on our DMRG results, we suggest this disordered phase as a nematic spin liquid with excitations that are likely gapless. We also discuss the origin of the lattice nematic order and the dominant $(\pi, 2\pi/3)$ fluctuations in the framework of gapless spin liquid. Our results lead to a hitherto unexplored mechanism for quantum nematic spin liquid and, more generally, open up a new route towards novel phases of quantum spin systems.

**Acknowledgement.**—We thank Andreas Ludwig, Chao-Ming Jian, Cenke Xu, Donna Sheng, Federico Becca, Kun Yang, Meng Cheng, Matt Foster, Olexei I. Motrunich, Philippe Corboz, Rong Yu, Subir Sachdev, and Tim Hsieh for fruitful discussions. The work was supported in part by the NSF Grant No. DMR-1350237 (W.-J.H., H.-H.L. and A.H.N.), the NSF Grant No. DMR-1611392 and the Robert A. Welch Foundation Grant No. C-1411 (W.-J.H., H.-H.L., H. H., and Q.S.), a Smalley Postdoctoral Fellowship of the Rice Center for Quantum Materials (H.-H. L.), Cottrell Scholar Award from the Research Corporation for Science Advancement (W.-J.H. and A.H.N.), the National High Magnetic Field Labora-
tory through the NSF Grant No. DMR-1157490 and the State of Florida (S.-S.G.), and the Robert A. Welch Foundation Grant No. C-1818 (A.H.N.). The majority of the computational calculations have been performed on the Shared University Grid at Rice funded by NSF under Grant EIA-0216467, a partnership between Rice University, Sun Microsystems, and Sigma Solutions, Inc., the Big-Data Private-Cloud Research Cyberinfrastructure MRI-award funded by NSF under Grant No. CNS-1338099 and by Rice University, the Extreme Science and Engineering Discovery Environment (XSEDE) by NSF under Grant No. DMR160057.
Supplemental Material

Flavor wave theory calculations

Depending on the nature of the orders that we consider (either a magnetic or quadrupolar order), we choose either the time-reversal invariant basis of the SU(3) fundamental representation or the usual spin $S^z$ basis, respectively, which can be related by

$$|x\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}, \quad |y\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}, \quad |z\rangle = -i|0\rangle,$$

where we abbreviate $|S^z = \pm 1\rangle = |\pm 1\rangle$ ($|S^z = 0\rangle = |0\rangle$) and $|\pm 1\rangle \equiv |\pm 1\rangle$. The $|x, y, z\rangle$ are the time-reversal invariant basis and is more convenient for performing the flavor wave theory calculations for the quadrupolar orders, while the other basis in $S^z$ is more suitable for the magnetic orders. For the flavor wave theory, we associate 3 Schwinger-bosons at each site, $b_{i\alpha}$, to the states of Eq. (2), where $|\alpha\rangle$ with $|\alpha\rangle$ being the vacuum state of the Schwinger bosons and $\alpha = x, y, z$ or 0, 1, 1 depending on the nature of the orders that we consider. The bosons satisfy a local constraint $\sum_{\alpha} b^\dagger_{i\alpha} b_{i\alpha} = 1$. The model $J_1 - K_1$ bilinear-biquadratic Hamiltonian can be rewritten as

$$H = \sum_{i,\delta,\alpha,\beta} \left[ J_n b^\dagger_{i\alpha} b_{j\beta} + (K_n - J_n) b^\dagger_{i\alpha} b^\dagger_{j\beta} b_{i\delta} b_{j\beta} \right]$$

Following the usual procedure of the spin-wave theory calculations, we introduce different local rotations for each sublattice in different orders. For antiferroquadrupolar order (AFQ), we introduce

$$\begin{pmatrix} a_{ix} \\ a_{iy} \\ a_{iz} \end{pmatrix} = \begin{pmatrix} \cos \theta_i & \sin \theta_i & 0 \\ -\sin \theta_i & \cos \theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{ix} \\ b_{iy} \\ b_{iz} \end{pmatrix}$$

which still preserves the local constraint with $b_{i\alpha} \rightarrow a_{i\alpha}$. We assume that at each site only one flavor of bosons $a_{ix}$ condenses, and we replace $a_{ix}$ and $a_{iy}$ by $(M - a_{i\alpha}^a a_{i\alpha} - a_{i\alpha}^b a_{i+\hat{z}})^{1/2}$, with $M = 1$ in the present case. A 1/M expansion up to the quadratic order of the bosons $a_i$ and $a_x$ followed by an appropriate Holstein-Primakoff transformation transformation allows us to extract the ground state energy.

For the magnetic order, we make local rotations at each site $i$ to align each spin at different sites along $\hat{1}$, and replace $a_{ix}$ and $a_{i\alpha}$ by $(M - a_{i\alpha}^a a_{i\alpha} - a_{i\alpha}^b a_{i+\hat{z}})^{1/2}$, and follow the same procedure above to perform a 1/M expansion up to the quadratic order of the bosons $a_0$ and $a_1$.

Within the flavor-wave theory calculations, for the quadrupolar orders, we consider the $(2\pi/3, 2\pi/3)$ AFQ (AFQ3), which is the three-sublattice order as shown in Fig. 5. In the figure, the red, green, and blue bars represent the sites occupied by only $b_{ix}$, $b_{iy}$, $b_{iz}$ bosons. We also consider the AFQ23, whose unit cell contains $2 \times 3$ sublattices, Fig. 6, which has been ignored previously. The consideration of the AFQ23 is motivated by the density matrix renormalization group (DMRG) analysis, which finds dominant ordering wavevector at $(2\pi/3, \pi)$, instead of $(2\pi/3, 2\pi/3)$, on the finite-size system near SU(3) point. We remind the readers that the order eventually is suppressed at the thermodynamic limit. For the magnetic order, we consider the ferromagnetic order (FM) and Néel antiferromagnetic order (AFM2).

We first give the dispersions associated with the gapless Goldstone modes together with the nematic order parameters in different orders. Two types of nematic order parameters are considered in this work, defined as

$$\sigma_1 \equiv \frac{1}{N_s} \sum_i \left[ \langle S_i \cdot S_{i+\hat{x}} \rangle - \langle S_i \cdot S_{i+\hat{y}} \rangle \right],$$

$$\sigma_Q \equiv \frac{1}{N_s} \sum_i \left[ \langle Q_i \cdot Q_{i+\hat{x}} \rangle - \langle Q_i \cdot Q_{i+\hat{y}} \rangle \right],$$

where $N_s$ represents the number of sites, $i$ is the site labelings, and the quadrupolar bond energy can be re-expressed in terms of the spin bond energy using the identity $Q_i \cdot Q_j = 2(S_i \cdot S_j)^2 + S_i \cdot S_j - 2S_i^2 S_j^2/3$.

(1) FM:

After the standard procedure, we find that the boson
FIG. 7. Nematic orders \( \sigma_1 \) and \( \sigma_Q \) in AFQ23 within the flavor-wave theory calculations. We can see that the nematic orders are finite indicating the broken \( \mathbb{C}_1 \) lattice rotation.

The boson Hamiltonian after performing Fourier transform is

\[
H = \sum_{k,\alpha = 0,1} \varepsilon_{k,\alpha} \left( a_{k,\alpha}^\dagger a_{k,\alpha} + \frac{1}{2} \right) + 8N_s J_1, \tag{7}
\]

where \( \varepsilon_{k,0} = 2J_1 \left[ \cos(k_x) + \cos(k_y) - 2 \right], \tag{8} \]
\( \varepsilon_{k,\bar{1}} = 2K_1 \left[ \cos(k_x) + \cos(k_y) \right] + 4 \left( K_1 - 2J_1 \right). \tag{9} \]

The nematic orders in FM are obviously zero \( \left( \sigma_1 = \sigma_Q = 0 \right) \) since there is no rational symmetry breaking in FM.

(2) AFM2 (Néel AFM):

The boson Hamiltonian after performing Fourier transform is

\[
H = \sum_{k,\alpha = 0,1} \varepsilon_{k,\alpha} \left( a_{k,\alpha}^\dagger a_{k,\alpha} + \frac{1}{2} \right) + N_s \left( \frac{16K}{3} - 8J \right), \tag{10}
\]

where \( N_s \) is the number of sites and the dispersions associated with \( a_{k,0} \) and \( a_{k,\bar{1}} \) are

\[
\varepsilon_{k,0} = \sqrt{A_{k,0}^2 - B_{k,0}^2}, \quad \varepsilon_{k,\bar{1}} = \sqrt{A_{k,\bar{1}}^2 - B_{k,\bar{1}}^2}, \tag{11}
\]

where we define

\[
A_{k,0} = 4(J - K), \tag{12} \]
\( A_{k,\bar{1}} = 4(2J - K), \tag{13} \]
\( B_{k,0} = 2(J - K) \left( \cos(k_x) + \cos(k_y) \right), \tag{14} \]
\( B_{k,\bar{1}} = 2K \left( \cos(k_x) + \cos(k_y) \right). \tag{15} \]

Since there is no rotational symmetry breaking in AFM2, the nematic orders are zero \( \left( \sigma_1 = \sigma_Q = 0 \right) \).

(3) FQ:

The boson Hamiltonian after performing Fourier transform is

\[
H = \sum_{k,\alpha \in \bar{0},\bar{1}} \varepsilon_{k,\alpha} \left( a_{k,\alpha}^\dagger a_{k,\alpha} + \frac{1}{2} \right) + 8N_s K_1, \tag{16}
\]

where \( N_s \) is again the number of sites and we find that the dispersions associated with \( a_{k,0} \) and \( a_{k,\bar{1}} \) are the same \( \varepsilon_{k,0} = \varepsilon_{k,\bar{1}} = \varepsilon_{k}^{FQ} \).

\[
\varepsilon_{k}^{FQ} = 2 \left[ \cos(k_x) + \cos(k_y) - 2K_1 \right]^2 - \\
\left[ (J_1 - K_1) \left( \cos(k_x) + \cos(k_y) \right) \right]^2 \frac{1}{2}. \tag{17}
\]

Since FQ does not break rotational symmetry, the nematic order are zero.

(4) AFQ3:

The bosonic Hamiltonian at the SU(3) time-reversal-invariant basis can be concise expressed as

\[
H = \sum_{k \in \text{BZ},\alpha = y,z} \varepsilon_{k,\alpha} \left( a_{k,\alpha}^\dagger a_{k,\alpha} + \frac{1}{2} \right), \tag{18}
\]

with

\[
\varepsilon_{k,\alpha = y,z} = \sqrt{A_{k,\alpha}^2 - B_{k,\alpha}^2}, \tag{19}
\]

where we define

\[
A_{k,y} = 2K_1 + (K_1 - J_1) \sqrt{2 \left( 1 + \cos(k_x + k_y) \right)}, \tag{20} \]
\( B_{k,y} = J_1 \sqrt{2 \left( 1 + \cos(k_x + k_y) \right)}, \tag{21} \]
\( A_{k,z} = 2K_1 - (K_1 - J_1) \sqrt{2 \left( 1 + \cos(k_x + k_y) \right)}, \tag{22} \]
\( B_{k,z} = J_1 \sqrt{2 \left( 1 + \cos(k_x + k_y) \right)}. \tag{23} \)

We note that the summation in momentum \( k \) is over the whole BZ for two bands. We can see that along the line with \( k_x + k_y = 0 \), the \( z \)-bosons have gapless lines along \( \varepsilon_{k,z} |_{k_x + k_y = 0} = 0 \). Due to this gapless line in the whole regime at \( K_1/J_1 \geq 1 \), the quantum corrections to the ordered moment extracted within the flavor-wave theory are divergent, which hints that the flavor-wave analysis may break down in this regime in this model. At the SU(3) point, the divergence of the quantum correction to the order moment within the flavor-wave picture was also reported previously [30]. The nematic orders vanish, \( \sigma_1 = \sigma_Q = 0 \).
In the AFQ23, a unit cell contains $2 \times 3$ sublattices as shown in Fig. 6. In our calculations, we choose the 2 sublattices in the first column to be a primitive cluster (sublattices $A_1$ and $B_2$) and perform local rotations on different sites in the unit cell to align with the states on the first column. Since the AFQ23 has never been considered previously, we provide more details of the calculations here. Focusing on the sublattices in the primitive clusters denoted as 1 for the original $A_1$ and 2 for the original $B_2$, we assume that at 1 only $a_x$-bosons condense and at 2 only $a_y$-bosons condense, and perform local rotations for $a_{r,\alpha=x,y,z}(\mu = 1, 2)$ to transform $a_{r,\alpha}(\mu) \rightarrow b_{r,\alpha}(\mu)$, and in the $b$-boson basis, only $b_z$ condense at each site. Explicitly, the local rotations give

$$\begin{align}
a_{r,x}(1) &= b_{r,x}(1) \\
a_{r,y}(1) &= b_{r,y}(1) \\
a_{r,z}(1) &= b_{r,z}(1) \\
a_{r,x}(2) &= b_{r,x}(2) \\
a_{r,y}(2) &= b_{r,y}(2) \\
a_{r,z}(2) &= b_{r,z}(2)
\end{align}$$

(24) (25)

At Fourier space, we find that the Hamiltonian in terms of the bosons is

$$H = \sum_{k \in RBZ} \left\{ 3K_1 \left[ b_{k,y}^\dagger(1)b_{k,y}(1) + b_{k,z}^\dagger(2)b_{k,z}(2) \right] + K_1 \left[ b_{k,z}^\dagger(1)b_{k,z}(1) + b_{k,y}^\dagger(2)b_{k,y}(2) \right] + (K_1 - J_1) \left[ e^{ikx} \left( b_{k,y}^\dagger(1)b_{k,z}(1) + b_{k,y}(2)b_{k,z}(2) \right) + H.c. \right] + J_1 \left[ e^{ikx} \left( b_{k,y}^\dagger(1)b_{k,z}(1) + b_{k,y}(2)b_{k,z}(2) \right) + H.c. \right] + (K_1 - J_1) \left[ (1 + e^{ikx})b_{k,y}^\dagger(1)b_{k,z}(2) + H.c. \right] + J_1 \left[ (1 + e^{ikx})b_{k,y}^\dagger(1)b_{k,z}(2) + H.c. \right] + 4K_1 \right\}$$

(26)

Where $RBZ$ means the reduced Brillouin zone, which will be suppressed below for clarity. For diagnosing the Hamiltonian, we first simplify the Hamiltonian with new boson fields, defined as

$$\begin{align}b_{k,y}^\dagger(1) &= \frac{e^{-iky/2}}{\sqrt{2}}(c_k^\dagger(1) - c_k(2)) \\
b_{k,z}^\dagger(2) &= \frac{1}{\sqrt{2}}(c_k^\dagger(1) + c_k(2)) \\
b_{k,z}^\dagger(1) &= \frac{e^{ikz}}{\sqrt{2}}(d_k^\dagger(1) - d_k(2)) \\
b_{k,y}^\dagger(2) &= \frac{e^{-iky}}{\sqrt{2}}(d_k^\dagger(1) + d_k(2)),
\end{align}$$

followed by a generalized Bogoliubov transformation

$$\begin{align}c_k(1) &= u_{k,1}\gamma_{k,1} + u_{k,2}\gamma_{k,2} + v_{k,1}\gamma_{-k,1} + v_{k,2}\gamma_{-k,2} \\
d_k(1) &= u_{k,3}\gamma_{k,3} + u_{k,4}\gamma_{k,4} + v_{k,3}\gamma_{-k,3} + v_{k,4}\gamma_{-k,4} \\
c_k(2) &= s_{k,1}\gamma_{k,1} + s_{k,2}\gamma_{k,2} + t_{k,1}\gamma_{-k,1} + t_{k,2}\gamma_{-k,2} \\
d_k(2) &= s_{k,3}\gamma_{k,1} + s_{k,4}\gamma_{k,2} + t_{k,3}\gamma_{-k,1} + t_{k,4}\gamma_{-k,2},
\end{align}$$

(31) (32) (33) (34)

we can obtain the diagonalized bosonic Hamiltonian as

$$H = \sum_{k,\alpha=1,2,3,4} \varepsilon_{k,\alpha} \left( \gamma_{k,\alpha}^\dagger \gamma_{k,\alpha} + \frac{1}{2} \right),$$

(35)

where $\varepsilon_{k,\alpha}$ stands for the Goldstone bosons of band $\alpha$. The dispersions for the 4 bands are

$$\varepsilon_{k,1} = \left[ -A_{1,k} - (A_{1,k}^2 - 4B_{1,k})^{\frac{1}{2}} \right]^\frac{1}{2}$$

(36)

$$\varepsilon_{k,2} = \left[ -A_{1,k} + (A_{1,k}^2 - 4B_{1,k})^{\frac{1}{2}} \right]^\frac{1}{2}$$

(37)

$$\varepsilon_{k,3} = \left[ -A_{2,k} - (A_{2,k}^2 - 4B_{2,k})^{\frac{1}{2}} \right]^\frac{1}{2}$$

(38)

$$\varepsilon_{k,4} = \left[ -A_{2,k} + (A_{2,k}^2 - 4B_{2,k})^{\frac{1}{2}} \right]^\frac{1}{2},$$

(39)

where

$$A_{1/2,k} = 2K_1 (2J_1 - K_1) \left( \cos \left( \frac{k_y}{2} \right) \right)^2$$

(40)

$$B_{1/2,k} = K_1^3 (K_1 - 2J_1) \left( \cos \left( \frac{k_y}{2} \right) \right)^2$$

(41)
We note that the the dispersions in the AFQ23 are independent of \( k \) in this setup (If we choose the other completely degenerate order with \( q = (\pi, 2\pi/3) \) the dispersions would be independent of \( k \)), and there are a set of gapless lines for \( \epsilon_{k_z} = 0 \) due to the gapless lines, we are not able to extract the reduced order moment within the flavor-wave theory since the quantum corrections to the ordered moment are divergent in the whole regime at \( K_1/J_1 \geq 1 \).

For the nematic orders, we find that \( \sigma_1 \) and \( \sigma_Q \) are both finite indicating that the \( C_4 \) lattice rotation symmetry is broken. Explicitly, we find that the nematic orders can be expressed in terms of the coefficients for the Bogoliubov transformations, Eqs. (31)-(34), as \( \sigma_1 = \sum_k \xi_{k,1} \) and \( \sigma_Q = \sum_k \xi_{k,Q} \), with

\[
\xi_{k,1} = v_{k,1} u_{k,3} + v_{k,2} u_{k,4} - v_{k,1} v_{k,3} - v_{k,2} u_{k,4} + t_{k,1} s_{k,3} + t_{k,2} s_{k,4} - t_{k,1} t_{k,3} - t_{k,2} t_{k,4} - \frac{1}{2} f_k (v_{k,1} u_{k,1} + v_{k,2} u_{k,2} - v_{k,1}^2 - v_{k,2}^2) + \frac{1}{2} f_k (t_{k,1} s_{k,1} + t_{k,2} s_{k,2} - t_{k,1}^2 - t_{k,2}^2),
\]

and

\[
\xi_{k,Q} = v_{k,1} (u_{k,3} + v_{k,3} - v_{k,1}) + t_{k,1} (s_{k,3} + t_{k,3} - t_{k,1}) + t_{k,2} (s_{k,4} + t_{k,4} - t_{k,2}) + f_k \left[ t_{k,1} s_{k,1} + t_{k,2} s_{k,2} - v_{k,1} u_{k,1} - v_{k,2} u_{k,2} \right],
\]

where \( f_k = |e^{i k_x} + e^{-i k_x}| \). From these explicit expressions for the integrand, we find that the contributions from the gapless lines are finite. This is confirmed by numerical evaluations of these integrals: The results for \( \sigma_1 \) and \( \sigma_Q \) are illustrated in Fig. 7. Thus, the nematic orders are always finite indicating the broken \( C_4 \) lattice rotation symmetry in the AFQ23 order.

For \( J_1, K_1 > 0 \), the phase diagram based on the energetics obtained by the linear flavor wave theory is illustrated in the upper panel in Fig. 8. The result based on the energetics is qualitatively consistent with the previous exact diagonalization result with a small system size (up to 20 sites) [15, 29]. However, we remark that if we calculate the order parameters for the different orders, i.e. on-site magnetization \( \langle S \rangle \) for AFM2 and the \( z \)-boson density \( \langle a_1^\dagger a_z \rangle = M - \sum_{x=\pm\pi/3} \langle a_x^\dagger a_x \rangle \) for AFQ3 that dictates the reduction of the ordered moments due to the quantum fluctuations. As illustrated in the bottom panel in Fig. 8, for the AFM2 we find that the order parameter is gradually reduced toward the SU(3) point. For \( K_1/J_1 \geq 1 \), we find that for both AFQ3 and AFQ23 the reduction of the order moment is always divergent due to the presence of gapless lines in the boson dispersions. The divergence of the reduction of the order moment was also found previously at the SU(3) point within the flavor-wave theory calculations [30], and here we illustrate that the divergence for AFQ3 not only occurs at SU(3) but in the whole regime at \( K_1/J_1 \geq 1 \).

For a general case of \( J_1, K_1 \) taking both signs, we obtain the phase diagram in Fig. 9. We remark that the phase diagrams in Figs. 8 and 9 solely based on the energetics within the flavor-wave theory are qualitatively consistent with the previous results [15].

**DMRG results**

*Spin dipolar and quadrupolar orders on RC9.*— Figure 10 shows the spin dipolar (\( m_2^z \)) and quadrupolar (\( m_2^z \)) order par-
rameters, as well as nematic order parameters ($\sigma_1$ and $\sigma_Q$), for $K_1 \geq 0.8$ on the RC9 cylinder. As shown in Fig. 10(a), $m_\Sigma^2$ decreases smoothly as $K_1$ increases, indicating the absent spin dipolar order. Instead, a sharp increase of $m_\Sigma^2$ at around $K_1 = 1.6$ suggests a phase transition from the disordered phase to the AFQ3 phase. Correspondingly, the nematic order parameters ($\sigma_1$ and $\sigma_Q$) have a big drop around $K_1 = 1.6$, which could be consistent with an emergent AFQ3 order.

Finite-size scaling of the order parameters.— We have shown the decay of magnetic and quadrupolar order parameters at the SU(3) point via finite-size scaling in Fig. 3(b) in the main text. Here, we complement these findings by showing the finite-size scaling of the order parameters for two different values of $K_1 = 0.9$ and 1.1, both inside the disordered phase proposed in the present work. In Figs. 11(a)-(b), we find the spin and quadrupolar orders also scaling to zero, which are consistent with our results for the SU(3) point in the main text. In Figs. 11(c)-(d), we further show the scaling of the nematic order parameters that are defined as $\sigma_1 \equiv \frac{1}{N_m} \sum_i \langle \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_{i+\hat{x}} \rangle - \langle \hat{\mathbf{S}}_i \rangle \cdot \langle \hat{\mathbf{S}}_{i+\hat{x}} \rangle$ and $\sigma_Q \equiv \frac{1}{N_m} \sum_i \langle \langle \hat{Q}_i \cdot \hat{Q}_{i+\hat{y}} \rangle - \langle \hat{Q}_i \rangle \cdot \langle \hat{Q}_{i+\hat{y}} \rangle \rangle$ in the bulk of cylinder ($\hat{x}$ and $\hat{y}$ denote the unit vectors along the two directions, $N_m$ is the number of sites of the two columns in the middle of cylinder). Consistent with the findings at the SU(3) point in Fig. 4(d) of the main text, we find that the nematic order parameters extrapolate to finite values in the thermodynamic limit, indicating a spontaneous lattice rotational symmetry breaking in this disordered phase.

Spin gaps and ground-state energy.— In Fig. 12, we show the spin gaps between the ground state in the total spin $S = 0$ sector and the lowest-energy states in the total spin $S = 1$ and $S = 2$ sectors, respectively; which are denoted as $\Delta_T(S = 1)$ and $\Delta_T(S = 2)$. The gaps are obtained on the RC6 cylinder by sweeping the ground state first and then targeting the total spin $S = 1$ and $S = 2$ sectors to find the lowest-energy state. In the Néel AFM phase for $K_1 \lesssim 0.75$, both spin gaps are quite small, consistent with the broken spin rotational symmetry. Interestingly, at the transition $K_1 \approx 0.75$, the spin gap $\Delta_T(S = 2)$ undergoes a sharp increase on the RC6, while the spin-triplet gap $\Delta_T(S = 1)$ grows gradually. For large $K_1$,
the AFQ order is dominant, which agrees with the smaller spin gap $\Delta_T(S = 2)$ we find for $K_1 > 1$. For the SU(3) model, we find that the two spin gaps have the same values on our studied systems (see also Table I). In Table I, we list the ground-state bulk energy and spin gaps for the SU(3) model on different geometries. The ground-state energy for RC9 is obtained by extrapolating the energy versus DMRG truncation error as shown in Fig. 13.

**Anisotropic model.**—We study the bilinear-biquadratic model by continuously changing the interchain interactions as shown in the inset of Fig. 14. We set $J_{1x} = 1.0$ as energy scale and we fix $K_{1x}/J_{1x} = K_{1y}/J_{1y}$. Then we can change two parameters $K_{1x}$ and $J_{1y}$. According to previous studies [10], 1D spin-1 bilinear-biquadratic chain is in the gapped Haldane phase for $K_{1x} < J_{1x}$. For $K_{1x}/J_{1x} > 0.75$, we increase the interchain couplings $J_{1y}, K_{1y}$. For $J_{1y} = J_{1x}$ and $K_{1y} = K_{1x}$, the model reduces to the isotropic model that we have studied. With growing interchain couplings, we find that the spin correlations clearly show an emergent antiferromagnetic order with slowly decaying correlations as shown in Fig. 14. Therefore, we find that the disordered phase in the isotropic model is not connected to the Haldane phase, which suggests that the disordered phase is not the extension of the 1D Haldane phase.

**Cylinder system with edge pinning.**—In DMRG simulation for a topological gapped spin liquid, the spinon sector usually can be found by removing or adding a spinon on each open edge of cylinder. On a finite-size system, the normal sector and the spinon sector have an energy difference that decays exponentially with increasing system width. This edge pinning technique has been successfully used for studying gapped spin liquids in spin-1/2 systems [41]. For spin-1 gapped spin liquid, a spinon may carry spin-1 spin. Thus, we have performed the pinning technique for the nematic disordered phase by removing a spin-1 on each open edge of cylinder on the $L_y = 3, 6$ cylinders. We find that for $L_y = 3, 6$, the ground states with and without removing spin-1 sites have the same bulk energy (see Fig. 15), which seems not to support a

| $e$ | $M_{SU(2)}$ | $\varepsilon$ | $\Delta_T(S = 1)$ | $\Delta_T(S = 2)$ | $\Delta_S$ |
|-----|-------------|---------------|------------------|------------------|-----------|
| RC3 | –0.8652     | 2000          | $4 \times 10^{-14}$ | 2.028            | 2.028     | 1.853     |
| RC4 | –0.6360     | 4000          | $4 \times 10^{-6}$  |                  |           |           |
| RC5 | –0.6459     | 4000          | $6 \times 10^{-6}$  |                  |           |           |
| RC6 | –0.6764     | 4000          | $2 \times 10^{-7}$  | 0.774            | 0.777     | 0.285     |
| RC7 | –0.6392     | 4000          | $6 \times 10^{-6}$  |                  |           |           |
| RC8 | –0.6405     | 4000          | $4 \times 10^{-5}$  |                  |           |           |
| RC9 | –0.6518(12) |               |                  |                  |           |           |
| TC3 | –0.6825     | 2000          | $4 \times 10^{-14}$ | 2.028            | 2.028     | 1.853     |
| TC4 | –0.6612     | 4000          | $2 \times 10^{-6}$  |                  |           |           |

**TABLE I.** The bulk energy per site $e$, the spin gaps $\Delta_T(S = 1, 2)$, and the spin singlet gap $\Delta_S$ of the SU(3) model on the RC and TC cylinders. $M_{SU(2)}$ and $\varepsilon$ are the kept SU(2) DMRG states and the DMRG truncation error, respectively. The bulk energy on the RC9 is obtained by the quadratic fitting of the energy as a function of the truncation error as shown in Fig. 13. We keep the SU(2) states up to 4000 with the truncation error $\varepsilon \sim 1 \times 10^{-5}$. For the RC cylinder, one can see that the RC3, RC6, and RC9 cylinders have the lower energy and the smaller truncation error than the other cylinders.
Entanglement Entropy

0.5
1.0
1.5

Bulk Energy

–0.6764
–0.6762
–0.6760
–0.6758
–0.6756

ε

Bulk Energy

Ly=6, without removing one site
Ly=6, removing one site

ε

Entanglement Entropy

Ly=3
Ly=6

l_x

von Neumann Entanglement Entropy.— Figure 16 shows the von Neumann entanglement entropy versus the column number $l_x$ along the $x$-direction for the SU(3) model on the RC3-24 and RC6-24 cylinders. We find that the entanglement entropy versus $l_x$ is quite flat, indicating no emergent large Fermi surface.

RC cylinder with $L_y$ not a multiple of 3.— On the RC cylinder with $L_y$ not a multiple of 3, we find the ground state with a spontaneous lattice translational symmetry breaking in the bulk of the cylinder. In Fig. 17, we show the bulk bond energy $\langle S_i \cdot S_j \rangle$ and $\langle Q_i \cdot Q_j \rangle$ on the RC4 and RC5 cylinders. In all the cases, the bond energy is not uniform and shows a period of 3 along the $x$ direction, suggesting the breaking of translational symmetry. We note that on these geometries, the bulk energy is higher than those on the RC cylinder with $L_y = 3n$ such as RC3, RC6, and RC9, which may be owing to that the TC lattice frustrates the short-range $(\pi, 2\pi/3)$ order pattern.

Three-sublattice order on the tilted cylinders.— On the $\pi/4$ tilted cylinder (TC), the lattice geometry frustrates the momentum at $(\pi, 2\pi/3)$ but is compatible with $(2\pi/3, 2\pi/3)$. Accordingly, we find the three-sublattice order on the TC cylinders, as shown in Fig. 18. DMRG calculations on the RC cylinder are harder to converge than the calculations on the RC cylinder. Here, we only show the convergent results on TC3 and TC4.

Spinon-Gauge Theory for Weakly-coupled SU(3) chains

As far as the origin of the $(\pi, 2\pi/3)/(2\pi/3, \pi)$ orders at $L_y$-leg ladder near the SU(3) point in Figs. 2-3, which van-
ish at thermodynamic limit, we can also rely on the picture of the gapless spin liquid. Focusing on the SU(3) point, where the gapless disordered phase is present, we can introduce a 8-component operator \( \Omega^\mu \), with \( \mu = 1 \sim 8 \), consisting of 3 components of spin (\( S^{x}\equiv x,y,z \)) and 5 components of the quadrupolar moment (\( Q^{i}=1\sim 5 \)). The operator can be concisely expressed in terms of 3-flavor partons, \( f_{a=x,y,z} \) which couple to a U(1) gauge field to gauge out the total charge mode, as \( \Omega^\mu = f^\dagger_{a}(\lambda^\mu_{a})f_{b} \), where \( \lambda^\mu \) are 3 x 3 matrices constructed based on Gell-Mann matrices [34]. For a single SU(3) chain, we can expand the operator in terms of continuum field \( \Omega^\mu \sim \sum_{P=R/L} Q^\mu_{P} + \sum_{q=\pm 2k_F} Q^\mu_{q} e^{iqy} \), where \( k_F = \pi/3 \). Using Bosonization method [46–48], we find that the scaling dimensions for each are \( \Delta[Q^\mu_{P=R/L}] = 1 \) and \( \Delta[Q^\mu_{q=2\pi/3}] = 2/3 \) [34], which leads to the power-law behaviors in the real-space correlation function, \( \langle \delta(x) \cdot \delta(y) \rangle \sim r^{-2} + r^{-4/3} \cos(2k_F x) \), consistent with the previous results [49]. We can clearly see that the wavevector \( 2\pi/3 \) indeed appears at a single SU(3) chain limit. We then consider the multiple SU(3) chains coupled by weak nearest-neighbor inter-chain interactions. Based on weak-coupling analysis, we find that the inter-chain interaction can be written as \( H' = \sum_{y} \int dxF' \), where

\[
H' = g_1 Q^\mu_{q}\bar{Q}^\mu_{q+1} + \lambda_b Q^\mu_{R,y} Q^\mu_{L,y} + g_2 Q^\mu_{P,y} Q^\mu_{P,y+1},
\]

where repeated indices means summation and \( \bar{q} = -q = \pm 2k_F \). Based on the scaling dimension analysis, we find that only the coupling \( g_1 \) is relevant, which suggests that the oscillating components \( Q^\mu_{q} \) at y-leg and \( Q^\mu_{q+1} \) sitting at \( (y+1) \)-leg tend to “anti-align” with each other leading to a \( \pi \) period along y direction. The above weak-coupling analysis for the SU(3) ladder system gives the tendency toward the formation of \( (2\pi/3, \pi) / (\pi, 2\pi/3) \) order, which is frustrated in the thermodynamic limit.

**Further details.**—We now present further details to make the above points more explicit. In two dimensions (2D) the usual approach is to decompose the 3-component spin (\( S^{x,y,z} \)) and 5-component quadrupole (\( Q^{i}=1\sim 5 \)) operators in terms of 3-flavor spinor, the fermionic partons. At SU(3) point, we can concisely construct a 8-component operator \( \Omega \equiv f^\dagger_{a} M^\mu_{a} f_{b} \), with

\[
\begin{align*}
\Omega^{\mu=1\sim 3} &= S^\alpha = -ie^{\alpha\beta\gamma} f^\dagger_{\beta} f_{\gamma}, \\
\Omega^{\mu=4\sim 8} &= Q^{i=1\sim 5}, \\
f^\dagger_{a} f_{a} &= 1
\end{align*}
\]

where repeated indices mean summation, and we can identify \( Q^1 = Q^{x^2-y^2} \), \( Q^2 = Q^{3z^2-r^2} \), \( Q^3 = Q^{3x^2} \), \( Q^4 = Q^{yz} \), and \( Q^5 = Q^{zx} \) in the usual convention for the definition of the quadrupolar operator, and the 8-component matrices \( M \) can be related to Gell-Mann matrices \( \lambda^\mu_{a=1\sim 8} \) as \( M^1 = \lambda_\gamma, \ M^2 = -\lambda_5, \ M^3 = \lambda_2, \ M^4 = -\lambda_3, \ M^5 = \lambda_8, \ M^6 = -\lambda_1, \ M^7 = -\lambda_6, \) and \( \lambda^8 = -\lambda_4 \). The Gell-Mann matrices are

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{align*}
\]

In the mean-field approach, one assumes that the partons are non-interacting and hopping freely on the lattice [44]. This artificially enlarges the Hilbert space, since the non-interacting parton hopping Hamiltonian allows for unoccupied and doubly-occupied sites, which are strictly forbidden in the present quantum spin model. One route to project the enlarged Hilbert space into a physical one is to perform Gutzwiller projection to project the enlarged Hilbert space at the mean field level back into the physical Hilbert space for the quantum spin model restricting the partons to single occupancy. The alternate approach for implementing the constraint of the single occupancy is by introducing a gauge field, for which the simplest case is the U(1) gauge field, minimally coupled to the partons in the hopping Hamiltonian. By doing this, the theory becomes a strongly-coupled lattice gauge field theory, which is hard to be solved analytically. Fortunately, on the chain limit, we can employ Bosonization to analyze the quasi-1D gauge theory, which can capture universal low energy properties of the ground state in the spin Hamiltonian [44, 45].

We now start by using Bosonization to analyze the gauge theory. We assume a mean field state in which the partons are hopping on the chain with nearest-neighbor hopping strengths denoted \( t_1 \). The dispersion of each flavor of parton is

\[
\xi(k) = -2t_1 \cos(k).
\]

At the mean-field level, each flavor of parton is \( 1/3 \) filled, which gives one set of Fermi crossings at wave vectors \( \pm k_F = \pm \pi/3 \). The parton operators are expanded in terms of continuum fields,

\[
f_{a}(x) = \sum_{P} e^{iP_{k}x} f_{P,a},
\]

with \( \alpha = x,y,z \) denoting the flavor, and \( P = R/L = \pm \) denoting the right and left moving fermions. We now use Bosonization re-expressing these low energy parton operators with Bosonic fields,

\[
f_{P,a} = \eta_{a} e^{i(\varphi_{a}+P\theta_{a})},
\]

with canonically conjugate boson fields:

\[
[\varphi_{a}(x), \varphi_{\beta}(x')] = \left[ \theta_{a}(x), \theta_{\beta}(x') \right] = 0,
\]

\[
[\varphi_{a}(x), \theta_{\beta}(x')] = i\pi \delta_{a\beta} Q(x-x'),
\]
where \( \Theta(x) \) is the Heaviside step function, and \( \eta_\alpha \) are the Klein factors, the Majorana fermions \( \{ \eta_\alpha, \eta^\beta \} \), for assuring the anti-commutation between partons with different flavors. Under bosonization, the slowly varying fermionic densities are simply \( \int P_\alpha f_\alpha = \partial_x (P \phi_\alpha + \theta_\alpha) / (2\pi) \).

In the present 1+1D continuum theory, we work under the gauge constraint that eliminate spatial components of the gauge field. In the imaginary-time formalism, the bosonized Lagrangian density is,

\[
\mathcal{L} = \frac{1}{2\pi} \sum_{\alpha = x, y, z} \left[ \frac{1}{v_\alpha} \left( \partial_\tau \phi_\alpha \right)^2 + v_\alpha \left( \partial_x \phi_\alpha \right)^2 \right] + \mathcal{L}_A. \tag{54}
\]

Here \( \mathcal{L}_A \) encodes the coupling to the slowly varying 1D (scalar) potential field \( A(x) \),

\[
\mathcal{L}_A = \frac{1}{m} \left( \frac{\partial_x A}{\pi} \right)^2 + i \rho_A A, \tag{55}
\]

where \( \rho_A \) denotes the total gauge charge density,

\[
\rho_A = \frac{1}{\pi} \sum_\alpha \partial_\tau \phi_\alpha. \tag{56}
\]

In the present SU(3) chain, it is useful to define fields as

\[
\theta_\rho = \frac{1}{\sqrt{3}} \sum_\alpha \theta_\alpha, \tag{57}
\]

\[
\theta_1 = \frac{1}{\sqrt{2}} (\theta_x - \theta_y), \tag{58}
\]

\[
\theta_2 = \frac{1}{\sqrt{6}} (\theta_x + \theta_y - 2\theta_z), \tag{59}
\]

and similar expressions for \( \varphi_\rho, \varphi_1, \varphi_2 \) fields, which leads to the Lagrangian density in the same form as before,

\[
\mathcal{L} = \frac{1}{2\pi} \sum_{\alpha = \rho, 1, 2} \left[ \frac{1}{v_\alpha} \left( \partial_\tau \phi_\alpha \right)^2 + v_\alpha \left( \partial_x \phi_\alpha \right)^2 \right] + \mathcal{L}_A. \tag{60}
\]

Integration over the gauge potential generate a mass term,

\[
\mathcal{L}_A \sim m \left( \theta_\rho - \theta_\rho^{(o)} \right)^2, \tag{61}
\]

for the field \( \theta_\rho = \sum_\alpha \theta_\alpha / 2 \). Due to the presence of the mass term for the total charge mode, \( \theta_\rho \), the \( \theta_\rho \) becomes gapped and can be ignored essentially.

The spin and quadrupolar operators can also be re-expressed in terms of the bosonic fields, and their corresponding correlation functions can be determined based on the bosonic Lagrangian above, Eqs. (60)-(61). Let’s take the spin operator as an illustration. We find that the spin operator at the low-energy theory description consists of a uniform and a oscillating parts with wave vectors \( q = \pm 2k_F = \pm 2\pi / 3 \),

\[
S^\alpha(x) \simeq S^\alpha_{\text{uni}}(x) + \sum_{q = \pm 2k_F} S^\alpha_q(x)e^{iqx}, \tag{62}
\]

with

\[
S^\alpha_{\text{uni}}(x) \simeq \frac{1}{2\pi} \left( \partial_x \varphi_{R\alpha} - \partial_x \varphi_{L\alpha} \right) = \frac{1}{\pi} \partial_\tau \theta_\alpha, \tag{63}
\]

\[
S^\alpha_q(x) \simeq -ie^{i\theta_\rho^{(o)} + i\theta_{\rho}} \eta_\rho e^{-i(\theta_\alpha + \theta_\beta)} \cos (\varphi_\beta - \varphi_\gamma). \tag{64}
\]

Based on the scaling dimension analysis, we find the scaling dimensions for the uniform and oscillating parts to be

\[
\Delta[S^\alpha_{\text{uni}}] = 1, \quad \Delta[S^\alpha_q] = 2/3, \tag{65}
\]

which leads to the conclusion that a spin correlation function at such a SU(3) spin chain shows the power-law behavior as

\[
\left\langle \hat{S}(x) \cdot \hat{S}(0) \right\rangle \simeq r^{-2\Delta[S^\alpha_{\text{uni}}]} + r^{-2\Delta[S^\alpha_q]} \cos(2k_F x), \tag{66}
\]

\[
r = r^{-2} + r^{-4/3} \cos(2k_F x), \tag{67}
\]

which is consistent with the previous studies [49]. Similarly, we can find that the quadrupolar operator has the similar behaviors as

\[
Q^i(x) \simeq Q^i_{\text{uni}}(x) + \sum_{q = \pm 2k_F} Q^i_q e^{iqx}, \tag{68}
\]

and their scaling dimensions are

\[
\Delta[Q^i_{\text{uni}}] = 1, \quad \Delta[Q^i_q] = 2/3. \tag{69}
\]

We can clearly see the 2\( \pi / 3 \) wave vector.

Now in order to see the \( \pi \) wave vector found in the DMRG calculations, we proceed to consider weak couplings between each SU(3) chains. Introducing a more compact 8-component operator \( \Omega \) consisting both spin and quadrupolar operators, we can then expand the 8-component in terms of low-energy fields as

\[
\Omega^{\mu=1\sim 8} \simeq Q^\mu_{\text{uni}} + \sum_{q = \pm 2k_F} Q^\mu_q e^{iqx}. \tag{70}
\]

At weak-coupling regime, we find that the nearest-neighbor couplings between \( y \)-leg and \( (y+1) \)-leg chains can be written down at the low-energy description as \( H' = \sum_y \int dx H' \) with

\[
H' = g_1 Q^\mu_{q,y} Q^\mu_{q,y+1} + \lambda_{3k} Q^\mu_{R,y} Q^\mu_{L,y} + g_2 Q^\mu_{P,y} Q^\mu_{P,y+1}, \tag{71}
\]

where the repeated indices mean summations. We define \( Q^\mu_q \equiv \sum_{\mu} Q^\mu_{\mu} \), and \( q = -q \). At tree-level renormalization group (RG) analysis, we find the scaling dimensions for the couplings as \( \Delta[g_1] = 2/3 \), where \( \lambda_{3k} \) and \( g_2 \) remain marginal. Therefore, we can see if \( g_1 > 0 \), under RG the \( Q^\mu_q \) and \( Q^\mu_{q,y+1} \) tend to “anti-align” with each other leading to a staggered pattern along \( y \)-direction with a period of \( \pi \).

We can then see based on the weak-coupling analysis of multiple SU(3) chains weakly coupled by weak inter-chain interactions, the wave vector (2\( \pi / 3 \), \( \pi / 3 \)) can naturally arise, which is consistent with the DMRG results.