Polyakov loop and correlator of Polyakov loops at

next-to-next-to-leading order

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(Dated: October 5, 2010)

Abstract

We study the Polyakov loop and the correlator of two Polyakov loops at finite temperature in the weak-coupling regime. We calculate the Polyakov loop at order $g^4$. The calculation of the correlator of two Polyakov loops is performed at distances shorter than the inverse of the temperature and for electric screening masses larger than the Coulomb potential. In this regime, it is accurate up to order $g^6$. We also evaluate the Polyakov-loop correlator in an effective field theory framework that takes advantage of the hierarchy of energy scales in the problem and makes explicit the bound-state dynamics. In the effective field theory framework, we show that the Polyakov-loop correlator is at leading order in the multipole expansion the sum of a colour-singlet and a colour-octet quark-antiquark correlator, which are gauge invariant, and compute the corresponding colour-singlet and colour-octet free energies.

PACS numbers: 12.38.-t,12.38.Bx,12.38.Mh
I. INTRODUCTION

The Polyakov loop and the correlator of two Polyakov loops are the order parameters of the deconfinement phase transition in SU($N$) gauge theories [1, 2]. The phase transition is signaled by a non-vanishing expectation value of the Polyakov loop and a qualitative change in the large-distance behaviour of the correlation function (from confining to exponentially screened) [2]. In the deconfined phase, these quantities provide information about the electric screening and can be calculated at sufficiently high temperatures $T$ in perturbation theory. For the correlation function of Polyakov loops, the validity of the perturbative expansion is limited to distances $r$ smaller than the magnetic screening length $r \ll 1/(g^2 T)$ [3, 4].

From a phenomenological perspective, the Polyakov-loop correlator is interesting because it provides an insight into the in-medium modifications of the quark-antiquark interaction. Indeed, in-medium modified heavy-quark potentials, inspired also by the behaviour of the Polyakov-loop correlator, have been used since long time in potential models (see e.g. Ref. [5]). However, although the spectral decomposition of the Polyakov-loop correlator is known, its relation with the heavy-quark potential is still a matter of debate and in need of a clarifying analysis [6]. The issue has become particularly relevant since recently an in-medium modified heavy-quark potential has been derived rigorously from QCD [7–11]. One of the aims of the paper is to discuss, in the weak-coupling regime, the relation between the Polyakov-loop correlator and these recent findings.

The Polyakov-loop correlator is a gauge-invariant quantity, hence it is well suited for lattice calculations. In fact, the correlator of two Polyakov loops has be calculated on the lattice for the pure gauge theory [12–15] as well as for full QCD [16, 17] (for a review see Ref. [18]). Surprisingly, not much is known instead about the correlator in perturbation theory. The correlator is known at leading order (LO) since long time [2, 19]; beyond leading order, it was computed only for distances of the same order as the electric screening length in Ref. [20].

The purpose of the paper is to evaluate the (connected) Polyakov-loop correlator up to order $g^6$ at short distances, $rT \ll 1$. This corresponds to a next-to-next-to-leading order (NNLO) calculation, if we count the order $g^4$ as LO and the order $g^5$ as next-to-leading order (NLO). We also revisit the calculation of the expectation value of the Polyakov loop at order $g^4$, which corresponds also to a NNLO calculation, if we count 1 as the leading-order result.
and $g^3$ as the NLO one. We will find a result that differs from the long-time accepted result of Gava and Jengo [21]. Finally, we will add on the discussion about the relation between the Polyakov-loop correlator and the in-medium heavy-quark potential.

The paper is organized as follows. In the next section, we discuss the gluon propagator in static gauge at one-loop level. Section III contains the calculation of the Polyakov loop at NNLO, while in section IV we calculate the Polyakov-loop correlator. In Sec. V we rederive the Polyakov-loop correlator in an effective field theory language. There, we also define a singlet and an octet free energy that we compute. Finally, section VI contains the conclusion and outlook.

II. THE STATIC GAUGE AND THE SELF ENERGY

The Polyakov loop and the Polyakov-loop correlator are gauge-invariant quantities. We may exploit the gauge freedom by choosing the most suitable gauge. A convenient gauge choice is the static gauge [22], defined as

$$\partial_0 A^0(x) = 0.$$  \hspace{1cm} (1)

The reason for using the static gauge is that in this gauge the Polyakov line has a very simple form

$$L(x) = P \exp \left( ig \int_0^{1/T} d\tau A^0(x, \tau) \right) = \exp \left( \frac{ig A^0(x)}{T} \right),$$  \hspace{1cm} (2)

where $P$ stands for the path-ordering prescription. The spatial part of the gluon propagator reads

$$D_{ij}(\omega_n, k) = \frac{1}{k^2} \left( \delta_{ij} + \frac{k_i k_j}{\omega_n^2} \right) (1 - \delta_{n0}) + \frac{1}{k^2} \left( \delta_{ij} - (1 - \xi) \frac{k_i k_j}{k^2} \right) \delta_{n0},$$  \hspace{1cm} (3)

where $\omega_n = 2\pi T n$, $n \in \mathbb{Z}$, are the bosonic Matsubara frequencies and $k^2 = \omega_n^2 + k^2$. Throughout the paper italic letters will refer to Euclidean four-vectors and bold letters to the spatial components. The parameter $\xi$ is a residual gauge-fixing parameter. We call non-static modes those propagating with nonzero Matsubara frequencies and conversely we employ the term static mode for the zero mode. The first term in the r.h.s. of Eq. (3), proportional to $(1 - \delta_{n0})$, is then the non-static part, whereas the second, proportional to

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1 We will work in Euclidean space-time and $0$ will label the Euclidean-time component.
$\delta_{n0}$, is the static part. The temporal part of the gluon propagator reads

$$D_{00}(\omega_n, k) = \frac{\delta_{n0}}{k^2},$$

which is purely static. Note that the gauge-fixing parameter affects only the static part of the spatial gluon propagator. The complete set of Feynman rules in this gauge has been discussed in Refs. [22–24]. Feynman rules are listed in appendix A together with our Feynman diagram conventions. We will adopt the static gauge in all the calculations of the paper, if not otherwise specified.

A necessary ingredient for the calculation of the Polyakov-loop expectation value and the Polyakov-loop correlator at NNLO is the temporal component of the gluon self energy at LO. In the static gauge, due to the static nature of the temporal propagator in Eq. (4) only $\Pi_{00}(k) \equiv \Pi_{00}(0, k)$ enters. Furthermore, at LO static and non-static modes do not mix in $\Pi_{00}(k)$, which can thus be conveniently split into

$$\Pi_{00}(k) = \Pi_{00}^{NS}(k) + \Pi_{00}^{S}(k) + \Pi_{00}^{F}(k),$$

where the three terms correspond to the contribution of the non-static gluons, the static gluons and the fermion loops respectively.

![Diagram](image)

FIG. 1: Diagrams contributing to the non-static part of the gluon self-energy in the gluonic sector. Dashed lines are temporal gluons, curly lines are spatial non-static gluons.

1. $\Pi_{00}^{NS}(k)$

In the gluonic sector, the non-static part of the self-energy receives contributions only from the two diagrams shown in Fig. 1. Using the Feynman rules of appendix A, it can be written in terms of five dimensionally-regularized master sum integrals

$$\Pi_{00}^{NS}(k) = -2g^2C_A \left( \frac{d-1}{2}I_0 - (d-1)I_1 + I_2 + \frac{1}{2}I_3 + \frac{1}{4}I_4 \right),$$
where $C_A = N = 3$ is the number of colours, $d = 3 - 2\epsilon$ is the number of dimensions,

$$
I_0 = \int_p \frac{1}{p^2}, \quad I_1 = \int_p \frac{p^2}{p^2 q^2}, \quad I_2 = \int_p \frac{k^2}{p^2 q^2}, \quad I_3 = \int_p \frac{k^2}{p^2 q^2}, \quad I_4 = \int_p \frac{k^4}{p^2 q^2 \omega_n^2}.
$$

(7)

$q = k - p$, $\int_p'$ is a shorthand notation for the non-static, $n \neq 0$, sum integral:

$$
\int_p' \equiv T \sum_{n \neq 0} \mu^{2\epsilon} \int \left(\frac{2\pi}{d}\right)^d d^d p,
$$

(8)

and $\mu$ is the scale in dimensional regularization. The result (6) can be conveniently cast in a sum of a vacuum part, a matter part, a part made of the subtracted zero modes and a part that we may call singular, because it is singular for $T \to 0$; the singular part is a peculiar feature of the static gauge. We then have

$$
\Pi_{00}^{NS}(k) = \Pi_{00}^{NS}(k)_{\text{vac}} + \Pi_{00}^{NS}(k)_{\text{mat}} + \Pi_{00}^{NS}(k)_{\text{zero}} + \Pi_{00}^{NS}(k)_{\text{sing}},
$$

(9)

$$
\Pi_{00}^{NS}(k)_{\text{vac}} = -\frac{g^2 k^2}{(4\pi)^2} C_A \left[ \frac{11}{3} \left( \frac{1}{\epsilon} - \gamma_E + \ln(4\pi) - \ln \frac{k^2}{\mu^2} \right) + \frac{31}{9} \right],
$$

(10)

$$
\Pi_{00}^{NS}(k)_{\text{mat}} = g^2 C_A \left\{ \int_0^\infty \frac{d|p| n_B(|p|)}{\pi^2} \left[ 1 - \frac{k^2}{2p^2} \right]
+ \left( \frac{|p|}{|k|} - \frac{|k|}{2|p|} + \frac{|k|^3}{8|p|^3} \right) \ln \left( \frac{|k| + 2|p|}{|k| - 2|p|} \right) \right\},
$$

(11)

$$
\Pi_{00}^{NS}(k)_{\text{zero}} = g^2 C_A \frac{T|k|^{1-2\epsilon} \mu^{2\epsilon}}{4} \left[ 1 + \epsilon(-1 - \gamma_E + \ln(16\pi)) \right],
$$

(12)

$$
\Pi_{00}^{NS}(k)_{\text{sing}} = -g^2 C_A \frac{|k|^3}{192T},
$$

(13)

where $\gamma_E$ is the Euler constant and $n_B(k) = 1/(e^{k/T} - 1)$ is the Bose–Einstein distribution. We refer the reader to appendix B for details on the derivation of these equations. The vacuum part (10) agrees with the static gauge computation in [23]. Furthermore, the vacuum part and the matter part are identical to the $k^0 \to 0$ limit of their Coulomb gauge counterparts, computed respectively in [25, 26] and [27, 28]. $\Pi_{00}^{NS}(k)_{\text{zero}}$ consists of the subtracted zero modes. In the $\epsilon \to 0$ limit, it is $T|k|/4$; we have kept the order $\epsilon$ corrections, because, in the Polyakov-loop correlator calculation of Sec. [14] we will need to evaluate the Fourier transform of $|k|^{1-2\epsilon}/|k|^4$, coming from a self-energy insertion in a temporal-gluon propagator, which is divergent.
2. $\Pi_{00}^F(k)$

At leading order in the coupling, $\Pi_{00}^F(k)$ may be written in terms of three dimensionally-regularized master sum integrals \[28\]

$$\Pi_{00}^F(k) = 2g^2 n_f \left(-\bar{I}_0 + 2\bar{I}_1 + \frac{1}{2}\bar{I}_2\right), \quad (14)$$

where

$$\bar{I}_0 = T \sum_{n=-\infty}^{+\infty} \mu^{2\epsilon} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2}, \quad \bar{I}_1 = T \sum_{n=-\infty}^{+\infty} \mu^{2\epsilon} \int \frac{d^dp}{(2\pi)^d} \frac{\bar{\omega}_n^2}{p^2q^2},$$

$$\bar{I}_2 = T \sum_{n=-\infty}^{+\infty} \mu^{2\epsilon} \int \frac{d^dp}{(2\pi)^d} \frac{k^2}{p^2q^2}, \quad (15)$$

$q = p + k$ and $\bar{\omega}_n = (2n + 1)\pi T$ are the fermionic Matsubara frequencies and $n_f$ is the number of massless quarks contributing to the fermion loops. Since no fermionic Matsubara frequency vanishes, fermions are purely non-static. The fermionic contribution can be cast into a sum of a vacuum and a matter part: $\Pi_{00}^F(k) = \Pi_{00}^F(k)_{\text{vac}} + \Pi_{00}^F(k)_{\text{mat}}$.

After the Matsubara frequencies summation, the matter part can be read from \[29\]

$$\Pi_{00}^F(k)_{\text{mat}} = \frac{g^2}{2\pi^2} n_f \int_0^\infty d|p| |p| n_F(|p|) \left[2 + \frac{4|p^2 - k^2|}{2|p||k|} \ln \frac{|k| + 2|p|}{|k| - 2|p|} \right], \quad (16)$$

where $n_F(k) = 1/(e^{k/T} + 1)$ is the Fermi–Dirac distribution. The vacuum part is given by

$$\Pi_{00}^F(k)_{\text{vac}} = \frac{2}{3} \frac{g^2k^2}{(4\pi)^2} n_f \left[\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) - \ln \frac{k^2}{\mu^2} + \frac{5}{3}\right], \quad (17)$$

3. $\Pi_{00}^{NS}(k) + \Pi_{00}^F(k)$

Let us now consider the sum $\Pi_{00}^{NS}(k) + \Pi_{00}^F(k)$. The divergences in the vacuum parts \[10\] and \[17\] are of ultraviolet origin and are accounted for by the charge renormalization. In the $\overline{\text{MS}}$ scheme, the renormalized sum of vacuum parts reads

$$\Pi_{00}^{NS}(k)_{\text{vac}} + \Pi_{00}^F(k)_{\text{vac}} = -\frac{g^2k^2}{(4\pi)^2} \left[\beta_0 \ln \frac{\mu^2}{k^2} + \frac{31}{9} C_A - \frac{10}{9} n_f\right], \quad (18)$$

where $\beta_0 = 11C_A/3 - 2n_f/3$.

Simple analytical expressions can be obtained for the renormalized sum $\Pi_{00}^{NS}(k) + \Pi_{00}^F(k)$ in the two limiting cases $|k| \ll T$ and $|k| \gg T$. In the former case, we have

$$\left(\Pi_{00}^{NS} + \Pi_{00}^F\right)(|k| \ll T) = \frac{g^2T^2}{3} \left(C_A + \frac{n_f}{2}\right)$$
\[
- \frac{g^2 k^2}{(4\pi)^2} \left[ \frac{11}{3} C_A \left( - \ln \frac{(4\pi T)^2}{\mu^2} + 1 + 2\gamma_E \right) - \frac{2}{3} n_f \left( - \ln \frac{(4\pi T)^2}{\mu^2} - 1 + 2\gamma_E + 4 \ln 2 \right) \right] \\
\quad + g^2 k^2 \mathcal{O} \left( \frac{k^2 T^2}{T^2} \right),
\]

where the leading-order term is momentum independent and can be identified with the (square of the) Debye mass \( m_D \),

\[
m_D^2 \equiv \frac{g^2 T^2}{3} \left( N + \frac{n_f}{2} \right),
\]

which provides, in the weak-coupling regime, the inverse of an electric screening length.

We note that Eq. (19) presents a logarithm of the renormalization scale over the temperature rather than over the momentum: this happens because in the limit \( |k| \ll T \) the matter part produces a term proportional to \( k^2 \beta_0 \ln(T^2/k^2) \) that combines with the logarithm in the renormalized vacuum part (18) to cancel its momentum dependence.

In the opposite limit \( |k| \gg T \), we have

\[
\left( \Pi_{00}^{NS} + \Pi_{00}^F \right) (|k| \gg T) = \Pi_{00}^{NS}(k)_{\text{vac}} + \Pi_{00}^F(k)_{\text{vac}} + g^2 C_A \left( - \frac{T^2}{18} - \frac{|k|^3}{192 T} \right) \\
\quad + g^2 C_A T |k|^{1-2\epsilon} \mu^2 \frac{4}{T} \left[ 1 + \epsilon(-1 - \gamma_E + \ln(16\pi)) \right] \\
\quad + g^2 T^2 \mathcal{O} \left( \frac{T^2}{k^2} \right).
\]

We observe that, in this limit and at the considered order, fermions enter only through their contribution to the vacuum part. It should be also noted that, while the \(-g^2 C_A T^2/18\) term appears also in Coulomb gauge \([3, 27]\), the term proportional to \(|k|^3\) is instead a peculiar feature of the static gauge. The terms proportional to \( \epsilon T |k|^{1-2\epsilon} \), which appear in the second line, come from the subtracted zero modes and contribute only when plugged into divergent amplitudes. Details on the derivation of these expressions can be found in appendix C.

4. \( \Pi_{00}^S(k) \)

The diagrams contributing to the static part of the gluon self energy are shown in Fig. 2. They are not sensitive to the scale \( T \), since, by definition, static gluon propagators are just made of zero modes, however they are to the scale \( m_D \). Hence, when
evaluating the static contribution, it is important to keep in mind that, if the incoming momentum is of the order of the Debye mass, then insertions of gluon self-energies of the type of Eq. (19) into the temporal-gluon propagator need to be resummed modifying the temporal-gluon propagator into

$$D_{00}(\omega_n, k) = \frac{\delta_{n0}}{k^2 + m^2_D}. \tag{22}$$

The static part of the gluon self energy with resummed propagators reads, for all values of the gauge-fixing parameter $\xi$,

$$\Pi^{S}_{00}(k) = \frac{g^2 C_A T}{d^2 \pi^2} \int \frac{d^d p}{(2\pi)^d} \left( \frac{1}{p^2 + m^2_D} + \frac{d-2}{p^2} + \frac{2(m^2_D - k^2)}{p^2(q^2 + m^2_D)} + (\xi - 1)(k^2 + m^2_D) \frac{p^2 + 2p\cdot k}{p^4(q^2 + m^2_D)} \right), \tag{23}$$

where $q = k + p$. The result agrees with Ref. [3, 30]. Note that Eq. (23) applies for all gauges sharing the same static propagator, among which the static and the covariant gauges. The expression is finite in three dimensions and reads

$$\Pi^{S}_{00}(k) = \frac{g^2 C_A T}{4\pi} \left[ \frac{2m^2_D - k^2}{|k|} \arctan \frac{|k|}{m_D} - m_D + (\xi - 1)m_D \right]. \tag{24}$$

Finally for the static part $|k| \gg T$ implies $|k| \gg m_D$ and

$$\Pi^{S}_{00}(|k| \gg m_D) = -g^2 C_A \left\{ \frac{T|k|^{1-2\epsilon} \mu^{2\epsilon}}{4} \left[ 1 + \epsilon(-\gamma_E + \ln(16\pi)) \right] + O(m_D T) \right\}, \tag{25}$$

where again we have kept up to order $\epsilon$ terms proportional to $T|k|^{1-2\epsilon}$.

5. $\Pi_{00}(k)$

$\Pi_{00}(k)$ is obtained by summing (10), (11), (12), (13), (16), (17) and (23) (or (24)). In
particular, the asymptotic expression for the gluon polarization at high momenta is

$$\Pi_{00}(|k| \gg T) = -\frac{g^2 k^2}{(4\pi)^2} \left( \beta_0 \ln \frac{k^2}{T^2} + \frac{31}{9} C_A - \frac{10}{9} n_f \right) + g^2 C_A \left( -\frac{T^2}{18} - \frac{|k|^2}{192T} \right) - \epsilon g^2 C_A \frac{T|k|^{1-2\epsilon}|\mu|^{2\epsilon}}{4} + \mathcal{O} \left( g^2 T^4, g^2 m_D T \right).$$  \hspace{1cm} (26)$$

Note that the term proportional to $T|k|^{1-2\epsilon}\epsilon^0$ in Eq. (25) has canceled against the term proportional to $T|k|^{1-2\epsilon}\epsilon^0$ in Eq. (21).

III. THE POLYAKOV LOOP

The quantity we are interested in computing is the trace of the Polyakov line $L \equiv L_R$ in a representation $R$ of dimension $d(R)$, where $R$ is either the fundamental representation ($R = F$, $d(F) = N$) or the adjoint representation ($R = A$, $d(A) = N^2 - 1$):

$$\langle L_R \rangle \equiv \langle \tilde{\text{Tr}} L_R \rangle, \quad \tilde{\text{Tr}} \equiv \frac{\text{Tr}}{d(R)}. \hspace{1cm} (27)$$

The brackets stand for the average in a thermal ensemble at a temperature $T$. Expanding the Polyakov line in the static gauge up to order $g^4$ yields

$$\langle L_R \rangle = 1 - \frac{g^2}{2!} \frac{\langle \tilde{\text{Tr}} A^2_0 \rangle}{T^2} - \frac{g^3}{3!} \frac{\langle \tilde{\text{Tr}} A^3_0 \rangle}{T^3} + \frac{g^4}{4!} \frac{\langle \tilde{\text{Tr}} A^4_0 \rangle}{T^4} + \ldots. \hspace{1cm} (28)$$

In computing Eq. (28) perturbatively, each diagram can receive contributions from both scales $T$ and $m_D$, for which we assume a weak-coupling hierarchy:

$$T \gg m_D. \hspace{1cm} (29)$$

In the weak-coupling regime, the calculation of $\langle L_R \rangle$ may be organized in an expansion in the coupling $g$; our aim is to compute $\langle L_R \rangle$ up to order $g^4$. Sometimes, we will find it useful to keep $m_D/T$ as a separate expansion parameter with respect to $g$, in order to identify more easily the origin of the various terms. We will call the $g^3$ term the NLO correction to the Polyakov loop and the $g^4$ term the NNLO correction. We will also identify the source of some higher-order corrections of order $g^5$ and $g^4 \times (m_D/T)^2$ that will play a role in Sec. [V]

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2 When discussing energy scales, we will consider $T$ and multiple of $\pi T$ to be parametrically of the same order.
A. The order $g^3$ contribution

Let us start examining $g^2 \langle \tilde{\text{Tr}} A_0^2 \rangle$. Diagrams contributing to $g^2 \langle \tilde{\text{Tr}} A_0^2 \rangle$ are shown in Fig. 3. Summing up all these diagrams, $g^2 \langle \tilde{\text{Tr}} A_0^2 \rangle$ can be written as

$$\delta \langle L_R \rangle = -\frac{g^2}{2!} \frac{\langle \tilde{\text{Tr}} A_0^2 \rangle}{T^2} = -\frac{g^2 C_R}{2T} \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + \Pi_{00}(k)},$$

where $C_R$ is the quadratic Casimir operator of the representation $R$ ($C_A = N$, $C_F = (N^2 - 1)/(2N)$). We observe that the integral receives contributions from the scales $T$ and $m_D$. We set out to separate the contributions from these two scales assuming the hierarchy (29).

1. Modes at the scale $T$

   We evaluate the integral (30) for $|k| \sim T \gg m_D$. In this momentum region, $\Pi_{00}(|k| \sim T \gg m_D) \ll k^2$ and we may expand the gluon propagator in $\Pi_{00}$. The LO term yields a scaleless integral

   $$\delta \langle L_R \rangle = -\frac{g^2}{2T} C_R \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = 0,$$

   whereas the following term gives

   $$\delta \langle L_R \rangle_T = \frac{g^2 C_R}{2T} \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{\Pi_{00}(|k| \sim T \gg m_D)}{k^2}.$$

   This term is of order $g^4$.

2. Modes at the scale $m_D$

   We evaluate now the contribution from the scale $m_D$. We recall from Eqs. (19) and (24) that, for $|k| \ll T$, $\Pi_{00}(k) = m_D^2 (1 + \mathcal{O}(g))$. We then rewrite the propagator in
Eq. (30) as $1/(k^2 + \Pi_{00}(|k| \ll T)) = 1/(k^2 + m_D^2 + (\Pi_{00}(|k| \ll T) - m_D^2))$ and expand in $\Pi_{00}(|k| \ll T) - m_D^2$. The LO term yields

$$
\delta \langle L_R \rangle_{\text{LO}} = \frac{g^2 C_R}{2T} \mu^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m_D^2} = \frac{C_R \alpha_s}{2} \frac{m_D}{T},
$$

(33)

whereas the following one gives

$$
\delta \langle L_R \rangle_{\text{NLO}} = \frac{g^2 C_R}{2T} \mu^2 \int \frac{d^d k}{(2\pi)^d} \Pi_{00}(|k| \sim m_D \ll T) - m_D^2,
$$

(34)

which is at least of order $g^4$.

Up to order $g^3$, we then have

$$
\langle L_R \rangle = 1 + \frac{C_R \alpha_s m_D}{2} \frac{1}{T} + O(g^4).
$$

(35)

FIG. 4: Diagram a) is the leading-order contribution to $\langle \tilde{\text{Tr}} A_0^3 \rangle$: it vanishes because of the three-gluon vertex involving only temporal gluons. Diagram b) is the LO term of $g^3 \langle \tilde{\text{Tr}} A_0^4 \rangle$: it vanishes because scaleless.

The LO contribution to the cubic term $g^3 \langle \tilde{\text{Tr}} A_0^3 \rangle$ is shown in Fig. 4 a). It vanishes due to the structure of the three-gluon vertex. This is just a LO manifestation of the charge conjugation symmetry; in fact, due to this symmetry, $g^3 \langle \tilde{\text{Tr}} A_0^3 \rangle$ vanishes to all orders. The quartic term $g^4 \langle \tilde{\text{Tr}} A_0^4 \rangle$ gets its LO contribution from the diagram shown in Fig. 4 b), which vanishes because scaleless. At higher order, a comparison with the analysis we have just performed for $\langle \tilde{\text{Tr}} A_0^3 \rangle$ makes it clear that $g^4 \langle \tilde{\text{Tr}} A_0^4 \rangle$ starts to contribute at order $g^4 \times (m_D/T)^2$, which is again beyond the accuracy of this analysis. We can therefore identify as the only contributions to the Polyakov loop at order $g^4$ the ones of Eqs. (32) and (34). In Sec. III B we will compute these contributions and, in Sec. III D we will analyze some sub-leading terms.
B. The order $g^4$ contribution

We now set out to compute Eqs. (32) and (34). Following the discussion in Sec. II, we separate the non-static from the static modes in $\Pi_{00}(k)$. We then have four sources of contributions: non-static modes at the scale $T$, non-static modes at the scale $m_D$, static modes at the scale $T$ and static modes at the scale $m_D$.

1. Non-static modes at the scale $T$

The non-static contribution to Eq. (32) reads

$$\delta\langle L_R \rangle_{NS,T} = g^4 C_R T \mu^{2\epsilon} \int \frac{d^4k}{(2\pi)^d} \frac{\Pi^{NS}_{00}(|k| \sim T) + \Pi^F_{00}(|k| \sim T)}{k^4},$$

where $\Pi^{NS}_{00}(|k| \sim T)$ is the full non-static contribution as defined in Eq. (9) and similarly $\Pi^F_{00}(|k| \sim T)$ is the full fermionic contribution as defined in Eqs. (16) and (17). We can rewrite Eq. (36) as

$$\delta\langle L_R \rangle_{NS,T} = \frac{g^4 C_R}{T} \left[ -C_A \left( \frac{d-1}{2} J_0 - (d-1) J_1 + J_2 + \frac{1}{2} J_3 + \frac{1}{4} J_4 \right) 
+n_f \left( -\tilde{J}_0 + 2\tilde{J}_1 + \frac{1}{2} \tilde{J}_2 \right) \right],$$

where we have defined the two-loop master sum-integrals $J_i$ and $\tilde{J}_i$ as

$$J_i = \mu^{2\epsilon} \int \frac{d^4k}{(2\pi)^d} \frac{1}{k^4} I_i, \quad \tilde{J}_i = \mu^{2\epsilon} \int \frac{d^4k}{(2\pi)^d} \frac{1}{k^4} \tilde{I}_i.$$

These integrals are evaluated in appendix D and their sum yields

$$\delta\langle L_R \rangle_{NS,T} = \frac{g^4 C_R}{2(4\pi)^2} \left[ C_A \left( \frac{1}{2\epsilon} - \ln \frac{4\pi^2}{\mu^2} + (1 - \gamma_E + \ln(4\pi)) - n_f \ln 2 \right) \right].$$

The divergence stems from the $J_2$ integral and is expected to cancel against an opposite divergence coming from the scale $m_D$.

2. Non-static modes at the scale $m_D$

The non-static contribution to Eq. (34) reads

$$\delta\langle L_R \rangle_{NS,m_D} = \frac{g^2 C_R}{2T} \mu^{2\epsilon} \int \frac{d^4k}{(2\pi)^d} \frac{\Pi^{NS}_{00}(|k| \sim m_D) + \Pi^F_{00}(|k| \sim m_D) - m_D^2}{(k^2 + m_D^2)^2}.$$  

For $|k|$ much smaller than the temperature, Eq. (19) applies and thus $\Pi^{NS}_{00}(k) + \Pi^F_{00}(k) = m_D^2 + \mathcal{O}(g^2 k^2)$. Therefore, the contribution of Eq. (40) is of order $g^4 \times (m_D/T) \sim g^5$. More explicitly, plugging Eq. (19) into Eq. (40) gives

$$\delta\langle L_R \rangle_{NS,m_D} = \frac{3g^4 C_R m_D}{4(4\pi)^3} \left[ \beta_0 \ln \left( \frac{\mu}{4\pi T} \right)^2 + 2\beta_0 \gamma_E + \frac{11}{3} C_A - \frac{2}{3} n_f (4 \ln 2 - 1) \right].$$
Although a term of order $g^5$ is beyond our accuracy, the contribution (41) is of interest because it fixes the renormalization scale of $g^3$ in the LO term (35) ($\alpha_s m_D/T \sim g^3$) to $\mu = 4\pi T$.

3. Static modes at the scale $T$

The static contribution at the scale $T$ to Eq. (32) reads

$$\delta \langle L_{R} \rangle_{ST} = \frac{g^2 C_R}{2T} \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{\Pi^S_{00}(|k| \sim T)}{k^4} = 0.$$ (42)

It vanishes because $\Pi^S_{00}(|k| \sim T \gg m_D) \sim g^2 T |k|$ (see Eq. (25)) and thus the resulting integration over $k$ is scaleless.

4. Static modes at the scale $m_D$

The static contribution to Eq. (34) is

$$\delta \langle L_{R} \rangle_{Sm_D} = \frac{g^4 C_R}{2 T} \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{\Pi^S_{00}(|k|)}{(k^2 + m_D^2)^2},$$ (43)

where $\Pi^S_{00}(|k|)$ is the full static contribution of Eq. (23). The computation is carried out in detail in appendix E; the result reads

$$\delta \langle L_{R} \rangle_{Sm_D} = \frac{g^4 C_R C_A}{2(4\pi)^2} \left( -\frac{1}{2\epsilon} - \ln \frac{\mu^2}{4m_D^2} - 1 + \gamma_E + \ln(4\pi) \right).$$ (44)

The divergence cancels against the one of Eq. (39) coming from non-static modes at the scale $T$.\(^3\) Note that the gauge-dependent part of Eq. (23) gives a vanishing integral, thus yielding the expected gauge-independent result.

Summing all contributions (static and the non-static) from the scales $T$ and $m_D$ up to order $g^4$ thus gives

$$\langle L_{R} \rangle = 1 + \frac{C_R \alpha_s m_D}{2 T} + \frac{C_R \alpha_s^2}{2} \left[ C_A \left( \ln \frac{m_D^2}{T^2} + \frac{1}{2} \right) - n_f \ln 2 \right] + O(g^5).$$ (45)

---

\(^3\) Both divergences in Eqs. (39) and (44) are of ultraviolet origin. This seems to contradict the expectation according to which infrared divergences from higher scales should cancel against ultraviolet divergences from lower scales. The contradiction is only apparent. The static modes at the scale $T$ develop both an ultraviolet and an infrared divergence that cancel against each other if regularized by the same cut off in dimensional regularization as assumed in Eq. (42). In general, however, the ultraviolet divergence of the static modes at the scale $T$ cancels against the ultraviolet divergence of the non-static modes, such that the sum of static and non-static modes at the scale $T$ ends up having only a residual infrared divergence. It is precisely this infrared divergence coming from the scale $T$, formally identical to the divergence in Eq. (39), that cancels against the ultraviolet divergence in (44) coming from the scale $m_D$. 

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C. Comparison with the literature

At order $g^4$, the Polyakov loop was first calculated in the pure gauge case ($n_f = 0$) and in Feynman gauge, by Gava and Jengo (GJ) [21], who find

$$\langle L_R \rangle_{\text{GJ}} = 1 + \frac{C_R \alpha_s m_D}{2T} + \frac{C_R C_A \alpha_s^2}{2} \left( \ln \frac{m_D^2}{T^2} - 2 \ln 2 + \frac{3}{2} \right) + \mathcal{O}(g^5). \quad (46)$$

Their result disagrees with ours, given in Eq. (45).

The disagreement may be traced back to an incorrect treatment of the static modes at the scale $m_D$ in [21]. In Feynman gauge, at order $g^4$, three terms contribute to the Polyakov loop: the non-static gluon self energy, whose dominant contribution comes from the scale $T$, the static gluon self energy, getting contributions from the scale $m_D$ only, and a third term coming from the fourth-order expansion of the Polyakov line. The computation of Gava and Jengo correctly reproduces the first and the third term. We show this with some detail in appendix F. However, in the evaluation of the static gluon self energy, the Debye mass is not resummed in the temporal gluons, leading to an inconsistent treatment of the scale $m_D$. Indeed, they have

$$\Pi_0^S(k)_{\text{GJ}} = g^2 C_A T \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \left( \frac{d - 1}{p^2} - \frac{2k^2}{p^2 q^2} \right), \quad (47)$$

which is the static self energy in Feynman gauge but without resumming the Debye mass in the internal propagators. If, instead, the Debye mass is resummed, the expression of the static self energy changes to Eq. (23) with $\xi = 1$. In this case, the calculation of the Polyakov loop in Feynman gauge leads to exactly the same result as in Eq. (15).

While the last part of this paper was being completed, Burnier, Laine and Vepsäläinen [31] published a perturbative analysis of the singlet quark-antiquark free energy. In the first part of their work, they consider also the Polyakov loop at order $g^4$ within a dimensionally reduced effective field theory framework in a covariant or Coulomb gauge. Our result (45) agrees with theirs.

\footnote{In [21], some contributions coming from the resummation of the Debye mass seem to have been included in $\delta W(0)$.}
D. Higher-order contributions

In Sec. III B, we obtained in Eq. (41) a term that is of order $g^4 \times (m_D/T) \sim g^5$. Other contributions of order $g^5$ can only come from $\langle \tilde{\text{Tr}}A_{0}^{2}\rangle$. Hence, they are encoded in the two-loop expression of the gluon self energy.

At order $g^6$, we can expect other contributions from the two-loop self energy and contributions coming from the diagram in Fig. 4 b). We explicitly calculate these last ones due to their relevance for Sec. V. The computation is carried out by evaluating the colour trace of the diagram in the representation $R$, whereas the loop integrations are easily obtained by comparison with Eq. (35). Thus we obtain

$$\delta\langle L_{R}\rangle = \left(3C_{R}^{2} - \frac{C_{R}C_{A}}{2}\right) \frac{\alpha_{s}^{2}}{24} \left(\frac{m_{D}}{T}\right)^{2}. \quad (48)$$

The colour structure of this quartic term is not linear in $C_{R}$, a fact that will play a role in Sec. V. We recall here that the linear dependence of $\ln\langle L_{R}\rangle$ on the Casimir operator $C_{R}$ is called Casimir scaling of the Polyakov loop. Equation (48) provides the leading perturbative correction that breaks the Casimir scaling. It is a tiny correction of order $g^6$, which may explain, at least in the weak-coupling regime, the approximate Casimir scaling observed in lattice calculations [32].

IV. THE POLYAKOV-LOOP CORRELATOR AT ORDER $g^6$ FOR $rT \ll 1$

The spatial correlator of Polyakov loops in the fundamental representation is defined as [2]

$$\langle \tilde{\text{Tr}}L_{F}^{1}(0)\tilde{\text{Tr}}L_{F}(r)\rangle. \quad (49)$$

Following the notation of [20], we define $C_{PL}(r, T)$ as the connected part of the correlator

$$C_{PL}(r, T) \equiv \langle \tilde{\text{Tr}}L_{F}^{1}(0)\tilde{\text{Tr}}L_{F}(r)\rangle_{c} = \langle \tilde{\text{Tr}}L_{F}^{1}(0)\tilde{\text{Tr}}L_{F}(r)\rangle - \langle L_{F}\rangle^{2}. \quad (50)$$

Expanding Eq. (50) up to order $g^6$ yields^5

$$C_{PL}(r, T) = \frac{g^{4}}{(2!)^{2}} \left(\tilde{\text{Tr}}A_{0}^{2}(0)\tilde{\text{Tr}}A_{0}^{2}(r)\right)_{c} T^{4} + \frac{g^{6}}{(3!)^{2}} \left(\tilde{\text{Tr}}A_{0}^{3}(0)\tilde{\text{Tr}}A_{0}^{3}(r)\right)_{c} T^{6}.$$

^5 We adopt a slightly different definition of $C_{PL}(r, T)$ with respect to [20], in that we consider the zeroth-order term in the perturbative expansion, i.e 1, as part of $\langle L_{F}\rangle^{2}$ rather than of $C_{PL}$. 

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\[-\frac{2g^6}{2! \cdot 4!} \langle \tilde{\text{Tr}} A_0^4(0) \tilde{\text{Tr}} A_0^2(r) \rangle c + O(g^8). \tag{51}\]

Since the generators of SU($N$) are traceless, the first term in the expansion, which is $g^2 \langle \tilde{\text{Tr}} A_0(0) \tilde{\text{Tr}} A_0(r) \rangle c$, vanishes and thus the correlator starts in perturbation theory with a two-gluon exchange term. Terms with an odd number of gauge fields have been omitted from Eq. (51) since they vanish for charge-conjugation symmetry.

We will perform a complete calculation of the Polyakov-loop correlator for distances $rT \ll 1$. This situation corresponds to temperatures lower than the inverse distance of the quark-antiquark pair, hence it is the right one to make contact with known zero-temperature results. We assume the following hierarchy:

\[ \frac{1}{r} \gg T \gg m_D \gg \frac{g^2}{r}. \tag{52} \]

The scales $1/r$ and $g^2/r$ are the typical scales appearing in any perturbative static quark-antiquark correlator calculation \cite{33, 34}. The scales $T$ and $m_D$ are associated to the thermodynamics of the system. We assume that they are smaller than $1/r$, because we are interested in short distances. We assume that they are larger than $g^2/r$, because we would like to study a situation where both thermodynamical scales affect the quark-antiquark potential \cite{9}. In the weak-coupling regime, as discussed above, $T \gg m_D$, where $m_D$ is given by Eq. (20). Equation (52) amounts to having two largely unrelated small parameters, $g$ and $rT$, the hierarchy only requiring $rT \gg g$. Differently from the Polyakov-loop calculation where we had only $g$, the perturbative expansion of the Polyakov-loop correlator is, therefore, organized as a double expansion in $g$ and $rT$. We will stop the expansion for the Polyakov-loop correlator at order $g^6(rT)^0$, meaning that, given a term of order $g^k(rT)^n$, we will display it only if $k < 6$, for any (positive or negative) $n$, or if $k = 6$, for $n \leq 0$; we will not display it elsewhere. We should note here that, as in any double expansion whose expansion parameters are unrelated, undisplayed terms may, under some circumstances, turn out to be numerically as large as or larger than some of the displayed ones.\(^6\)

In \cite{20}, Nadkarni computed the Polyakov-loop correlator up to order $g^6$ using resummed temporal-gluon propagators throughout the computation, which amounts to calculating the Polyakov-loop correlator for distances $rm_D \sim 1$. Our calculation will differ from Nadkarni’s \(^6\)

\(^6\) A posteriori (see the final result in Eq. (57)), this may be avoided, in our case, by further requiring that $rT \gg \sqrt{g}$.\]
one in that we adopt the different hierarchy \((52)\). Nevertheless, some of our results can be obtained by expanding Nadkarni’s result for \(rm_D \ll 1\); we refer to Sec. \([V H]\) for a detailed comparison between the two results.

The calculation of the different contributions to Eq. \((51)\) will proceed similarly to the calculation of the Polyakov loop performed in the previous section. We will consider the different Feynman diagrams contributing to each of the terms in \((51)\), separate the contributions from the different energy scales and, in case, distinguish between static and non-static modes.

![Diagrams contributing to \(\langle \tilde{\text{Tr}} A_0^2(0) \tilde{\text{Tr}} A_0^2(r) \rangle_c\).](image)

**FIG. 5:** Diagrams contributing to \(\langle \tilde{\text{Tr}} A_0^2(0) \tilde{\text{Tr}} A_0^2(r) \rangle_c\).

### A. The leading-order contribution: diagram I

We start by evaluating the four-field correlation function: its leading-order contribution is given by diagram I in Fig. 5. It does not vanish only for momenta of order \(1/r\), giving

\[
\delta C_{\text{PL}}(r, T)_I = \frac{N^2}{8N^2} \frac{g^4}{T^2} \left( \mu^2 \int \frac{d^dk}{(2\pi)^d} e^{-ik\cdot r} \right)^2 = \frac{N^2 - 1}{8N^2} \frac{\alpha_s^2}{(rT)^2},
\]

\((53)\)

### B. The contribution from diagrams of type II

As we go beyond leading order, the first class of diagrams that we consider are those with gluon self-energy insertions in one temporal-gluon line, whose first example is diagram II in Fig. 5. They give

\[
\delta C_{\text{PL}}(r, T)_{II} = \frac{2N^2 - 1}{8N^2} \frac{g^4}{T^2} \frac{1}{4\pi r^2} \frac{1}{\mu^2} \int \frac{d^dk}{(2\pi)^d} e^{-ik\cdot r} \left( \frac{1}{k^2 + \Pi_{00}(k)} - \frac{1}{k^2} \right),
\]

\((54)\)
where the factor 2 comes from the symmetric diagrams and \( \Pi_{00} \) is the sum of bosonic and fermionic contributions to the gluon self energy, as in the Polyakov-loop case. This diagram receives contributions from all scales and depends on the gauge parameter \( \xi \). However it can be shown that the gauge dependence cancels with diagram IV \[20\], so, for simplicity, here we write our results in static Feynman gauge, \( \xi = 1 \).

1. **Contribution from the scale \( 1/r \)**

   We start by evaluating the contribution from the scale \( 1/r \) in the integral. If \( |k| \sim 1/r \gg T \), then we have

   \[
   \delta C_{PL}(r,T)_{II1/r} = -\frac{N^2 - 1}{4N^2} \frac{g^4}{T^2} \frac{1}{4\pi r} \mu^2 e^2 \int \frac{d^d k}{(2\pi)^d} e^{-ikr} \frac{\Pi_{00}(|k| \gg T)}{k^4} \left[ 1 + \mathcal{O} \left( \frac{g^2}{rT} \right) \right],
   \]

   where \( \Pi_{00}(|k| \gg T) \) is given by Eq. (26). The Fourier transform of the vacuum part corresponds to the one-loop static QCD potential and can be read from \[33, 36\]. Using that in dimensional regularization the Fourier transform of \( \frac{1}{|k|^n} \) becomes \[37\]

   \[
   \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ikr}}{|k|^n} = \frac{2^{-n} \pi^{-d/2} \Gamma(d/2 - n/2)}{\Gamma(n/2)},
   \]

   we have

   \[
   \delta C_{PL}(r,T)_{II1/r} = \frac{N^2 - 1}{8N^2} \frac{\alpha_s^3}{(rT)^2} \left\{ \frac{1}{2\pi} \left[ 2\beta_0 (\ln(\mu r) + \gamma_E) + \frac{31}{9} C_A - \frac{10}{9} n_f \right] + C_A \left( \frac{1}{12rT} - rT - \frac{2}{9} \pi (rT)^2 \right) \right\} + \mathcal{O} \left( g^6(rT)^2, g^7 \right).
   \]

   The term in the first line comes from the Fourier transform of the vacuum contribution, whereas the terms in the second line come respectively from the singular part, the (zero mode) order \( \epsilon \) term\(^7\) and the \( T^2 \) term in Eq. (26). Higher-order corrections to Eq. (26) contribute at order \( g^6(rT)^2 \) or \( g^7 \). Higher order radiative corrections to the gluon self energy contribute at order \( g^8 \). Note that the \( (\alpha_s^3/\pi)\beta_0 \ln(\mu r) \) term in Eq. (57) fixes the natural scale of \( \alpha_s^2 \) in the LO term \( \delta C_{PL}(r,T)_{I} \) to be \( 1/r \).

2. **Contributions from the scales \( T \) and \( m_D \)**

   We now consider the contributions from the thermal scales. For what concerns the

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\(^7\) The dimensionally-regularized Fourier transform of the order \( \epsilon \) term in Eq. (26) yields a \( 1/\epsilon \) pole, eventually leading to a finite contribution.
temperature, $|\mathbf{k}| \sim T$ translates into $r|\mathbf{k}| \ll 1$ and $m_D \ll |\mathbf{k}|$. Integrating out the temperature leads to the following contribution

$$
\delta C_{PL}(r,T)_{II_T} = -\frac{N^2 - 1}{4N^2} \frac{g^4}{T^2} \frac{1}{4\pi r} \mu^2 e^e \int \frac{d^d k}{(2\pi)^d} \left[ 1 + \mathcal{O}((\mathbf{k} \cdot \mathbf{r})^2) \right] \frac{1}{k^4} \Pi_{00}(|\mathbf{k}| \sim T) \times \left[ 1 + \mathcal{O}(g^2) \right], \tag{58}
$$

where we have implemented the condition $r|\mathbf{k}| \ll 1$ by expanding the Fourier exponent. Integrating out the Debye-mass scale leads to the following contribution

$$
\delta C_{PL}(r,T)_{II_mD} = \frac{N^2 - 1}{4N^2} \frac{g^4}{T^2} \frac{1}{4\pi r} \mu^2 e^e \int \frac{d^d k}{(2\pi)^d} \left[ 1 + \mathcal{O}((\mathbf{k} \cdot \mathbf{r})^2) \right] \left[ \frac{1}{k^2 + m_D^2 + \Pi_{00}(k)} - \frac{1}{k^2} \right] + \mathcal{O} \left( \frac{g^4}{m_D^2} \right). \tag{59}
$$

The integrals to be evaluated are the same needed to evaluate Eqs. (32), (33) and (34). Thus, summing the $T$ and $m_D$ contributions, we obtain

$$
\delta C_{PL}(r,T)_{II_T+mD} = -\frac{N^2 - 1}{4N^2} \frac{e^e}{r T} \left\{ \frac{m_D}{T} + \alpha_s \left[ C_A \left( \ln \frac{m_D^2}{T^2} + \frac{1}{2} \right) - n_f \ln 2 \right] \right\} + \mathcal{O} \left( \frac{g^7}{r T}, g^6(rT) \right). \tag{60}
$$

The term of order $g^5/(rT)$ comes from the first term in (59), the terms of order $g^6/(rT)$ come from the non-static modes in (58) and from the static ones in the second term of Eq. (59), the appearance of the logarithm $\ln m_D^2/T^2$ signals the cancellation between divergences at the scale $T$ and $m_D$, the suppressed term $g^7/(rT)$ comes from the non-static modes in the second term of Eq. (59) (see Eqs. (40) and (41) for the analogous case in the Polyakov-loop calculation), whereas the suppressed term $g^6(rT)$ comes from the $(\mathbf{k} \cdot \mathbf{r})^2$ term in Eq. (58).

**C. The contribution from diagrams of type III**

Diagram III in Fig. 3 is the first example of the class of diagrams with gluon self-energy insertions in both temporal-gluon lines. They may be evaluated from the diagrams of type II:

$$
\delta C_{PL}(r,T)_{III} = \frac{N^2 - 1}{8N^2} \frac{g^4}{T^2} \left[ \mu^2 e^e \int \frac{d^d k}{(2\pi)^d} e^{-i\mathbf{k} \cdot \mathbf{r}} \left( \frac{1}{k^2 + \Pi_{00}(\mathbf{k})} - \frac{1}{k^2} \right) \right]^2 = \frac{8N^2}{N^2 - 1} \frac{T^2}{g^4} \frac{\delta C_{PL}(r,T)_{II}}{2}. \tag{61}
$$
The leading-order term in (60) gives a $g^6$ contribution to $\delta C_{PL}(r,T)_{III}$, all other contributions being at least of order $g^7/(rT)^2$,

$$\delta C_{PL}(r,T)_{III} = \frac{N^2 - 1}{8N^2} \alpha_s^2 \frac{m_D^2}{T^2} + \mathcal{O}\left(\frac{g^7}{(rT)^2}\right). \quad (62)$$

**D. The contribution from diagrams of type IV**

The transverse static-gluon exchange between the two temporal-gluon lines (diagram IV and the diagrams derived from IV by inserting gluon self energies in each of the gluon lines) receives the following contributions.

1. **Contribution from the scale $1/r$**
   
   The contribution from the scale $1/r$ reads at leading order (with $\xi = 1$)
   
   $$\delta C_{PL}(r,T)_{IV 1/r} = \frac{g^6 N^2 - 1}{4T} C_A \mu^6 \int \frac{d^dk_1}{(2\pi)^d} \int \frac{d^dk_2}{(2\pi)^d} \int \frac{d^dp}{(2\pi)^d} e^{-i(k_1 - k_2) \cdot r}$$
   
   $$\times \frac{(2k_1 + p) \cdot (2k_2 + p)}{k_1^2 k_2^2 (k_1 + p)^2 (k_2 + p)^2 p^2}. \quad (63)$$
   
   Gluon self-energy insertions are suppressed by $g^2$.

2. **Contribution from the scale $T$**
   
   The contribution from the scale $T$ vanishes, because scaleless, if no self-energy insertions are considered. Hence, the leading contribution from the scale $T$ is of order $g^6/T \times g^2 T \sim g^8$.

3. **Contribution from the scale $m_D$**
   
   The contribution from the scale $m_D$ reads
   
   $$\delta C_{PL}(r,T)_{IV m_D} = \frac{g^6 N^2 - 1}{4T} C_A \mu^6 \int \frac{d^dk_1}{(2\pi)^d} \int \frac{d^dk_2}{(2\pi)^d} \int \frac{d^dp}{(2\pi)^d} \left[1 + \mathcal{O}\left((k_1 - k_2) \cdot r\right)^2\right]$$
   
   $$\times \frac{(2k_1 + p) \cdot (2k_2 + p)}{(k_1^2 + m_D^2)(k_2^2 + m_D^2)((k_1 + p)^2 + m_D^2)((k_2 + p)^2 + m_D^2)p^2} \left[1 + \mathcal{O}(g)\right]. \quad (64)$$
   
   This corresponds to a contribution of order $g^6(m_D/T) \sim g^7$, which is beyond our accuracy.
The leading contribution to $\delta C_{PL}(r, T)_{IV}$ comes, therefore, from $\delta C_{PL}(r, T)_{IV1/r}$, which can be computed in dimensional regularization with the help of Eq. (56). Our final result reads

$$\delta C_{PL}(r, T)_{IV} = \frac{N^2 - 1}{2N} \frac{\alpha_s^3}{rT} \left( 1 - \frac{\pi^2}{16} \right) + O(g^7).$$

(65)

The same result follows from [20] by expanding in $rm_D \ll 1$.

E. The contribution from diagrams of type V

Diagrams contributing to the correlators of six $A_0$ fields in Eq. (51) are shown in Fig. 6. The LO diagram contributing to $\langle \tilde{\text{Tr}} A_0^3(0) \tilde{\text{Tr}} A_0^3(r) \rangle_c$ is diagram V, which gives

$$\delta C_{PL}(r, T)_V = \frac{(N^2 - 4)(N^2 - 1)}{96N^3} \frac{\alpha_s^3}{(rT)^3}. \tag{66}$$

If we consider diagram V with gluon self-energy insertions in one of the temporal lines, in analogy to (59), then this starts contributing at order $g^7/(rT)^2$, which is beyond our accuracy.

F. The contribution from diagrams of type VI

Diagrams contributing to $\langle \tilde{\text{Tr}} A_0^4(0) \tilde{\text{Tr}} A_0^2(r) \rangle$ are like diagram VI in Fig. 6 and diagrams derived from VI by inserting gluon self energies and other radiative corrections. Colour factors aside, their leading contribution may be estimated by simply multiplying the contribution of the diagrams of Fig. 5 to the Polyakov-loop correlator with the contribution of
the diagrams of Fig. 3 to the Polyakov loop. Hence, diagrams of type VI contribute at LO to order $g^4/(rT)^2 \times g^2 m_D/T \sim g^7/(rT)^2$, which is beyond our accuracy.

**G. The Polyakov-loop correlator up to order $g^6$**

Summing up all contributions, we then have

$$C_{PL}(r, T) = \frac{N^2 - 1}{8N^2} \left\{ \frac{\alpha_s(1/r)^2}{(rT)^2} - 2 \frac{\alpha_s^2}{rT} \frac{m_D}{T} \right. \left. + \frac{\alpha_s^3}{(rT)^3} \frac{N^2 - 2}{6N} + \frac{1}{2\pi} \frac{\alpha_s^3}{(rT)^2} \left( \frac{31}{9} C_A - \frac{10}{9} n_f + 2\gamma_E/\beta_0 \right) + \frac{\alpha_s^3}{rT} \left[ C_A \left( -2 \ln \frac{m_D}{T^2} + 2 - \frac{\pi^2}{4} \right) + 2n_f \ln 2 \right] + \alpha_s^2 \frac{m_D^2}{T^2} - \frac{2}{9} \pi \alpha_s^2 C_A \right\} + \mathcal{O} \left( g^6/(rT)^2, \frac{g^7}{(rT)^2} \right),$$

(67)

where we have made explicit the scale dependence of $\alpha_s$ in the leading term. Note that the $r, T$ and $m_D$ independent term proportional to $-2\pi\alpha_s^3 C_A/9$ comes from Eq. (57), so it is actually a contribution from the scale $1/r$ that accounts for the matter part of the gluon self energy. The term proportional to $\alpha_s^3/(rT)^3$ comes from diagram V, Eq. (66), and from the singular part of the gluon self energy in the static gauge, Eq. (57).

**H. Comparison with the result of Nadkarni**

We compare here with Nadkarni’s (N) computation of the Polyakov-loop correlator [20]. The regime of validity of Nadkarni’s computation is $T \gg 1/r \sim m_D$, while ours is $1/r \gg T \gg m_D$. Therefore, we may only compare results obtained here that do not involve the hierarchy $rT \ll 1$, with Nadkarni’s results that do not involve the hierarchy $rT \gg 1$, expanded for $rm_D \ll 1$.

In [20], the tree-level expression of $g^4\langle \text{Tr}A_0^2(0)\text{Tr}A_0^2(r)\rangle_c/(4T^4)$ reads $(N^2 - 1)/(8N^2)\alpha_s^2$ \exp(−2$rm_D$)/(rT)$^2$, which expanded for $rm_D \ll 1$ gives $\delta C_{PL}(r, T)_{I}$, the LO of $\delta C_{PL}(r, T)_{II}$, (to be read from Eq. (60)) and $\delta C_{PL}(r, T)_{III}$. Also, the tree-level expression of $g^6\langle \text{Tr}A_0^3(0)\text{Tr}A_0^3(r)\rangle_c/(36T^6)$ in [20] agrees with $\delta C_{PL}(r, T)_{V}$ once expanded for $rm_D \ll 1$.

Diagram IV in Fig. 5 also contributes to Nadkarni’s calculation. The diagram does not involve gluon self-energy insertions and therefore its calculation does not rely on the
hierarchy between $1/r$ and $T$. As already remarked, $\delta C_{PL}(r, T)_{IV}$ agrees with Nadkarni’s result once expanded for $rm_D \ll 1$.

Let’s now consider the NLO contribution to $\delta C_{PL}(r, T)_{II m_D}$. This contribution is given by the static part of Eq. (59):

$$\delta C_{PL}(r, T)_{II m_D} = -\frac{N^2 - 1}{4N^2} \frac{g^4}{T^2} \frac{1}{4\pi r} \mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{\Pi_{00}^S(|k|)}{(k^2 + m_D^2)^2}. \tag{68}$$

The integral is divergent. In our case, i.e. assuming $1/r \gg T \gg m_D$, the divergence cancels against $\delta C_{PL}(r, T)_{II T}$, eventually leading to a finite result in $\delta C_{PL}(r, T)_{II T + m_D}$. The $\ln m_D/T$ term in Eq. (60) signals precisely that a divergence at the scale $m_D$ has canceled against a divergence at the scale $T$. In Nadkarni’s case, i.e. assuming $T \gg 1/r \gg m_D$, we get, along with $\delta C_{PL}(r, T)_{II m_D}$, a contribution from the scale $1/r$, which is

$$\delta C_{PL}(r, T)_{II 1/r} = -\frac{N^2 - 1}{4N^2} \frac{g^4}{T^2} \frac{1}{4\pi r} \mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} e^{-ik\cdot r} \Pi_{00}^S(|k| \gg m_D) \frac{\Pi_{00}^S(|k|)}{k^4}. \tag{69}$$

This is like Eq. (55), but involves only the static part of the self energy (25), since non-static modes have been already integrated out at the larger scale $T$. According to Eq. (25), we have $\Pi_{00}^S(|k| \gg m_D) \sim T|k|^{1-2\epsilon}$. The Fourier transform of $1/|k|^{3+2\epsilon}$ originates a $1/\epsilon$ pole. It is this divergence that in Nadkarni’s hierarchy cancels against the divergence in $\delta C_{PL}(r, T)_{II m_D}$, leading to the finite result

$$\delta C_{PL}(r, T)_{II m_D} + \delta C_{PL}(r, T)_{II 1/r} = -\frac{N^2 - 1}{2N} \frac{\alpha_s^3}{rT} \left[ \ln(2m_Dr) + \gamma_E - \frac{3}{4} + O(rm_D) \right], \tag{70}$$

which agrees with the result in [20]. In this case, the $\ln m_Dr$ term signals that a divergence at the scale $m_D$ has canceled against a divergence at the scale $r$.

V. THE POLYAKOV-LOOP CORRELATOR IN AN EFT LANGUAGE

The calculation of the Polyakov loop discussed in the previous section can be conveniently rephrased in an effective field theory (EFT) language that exploits at the Lagrangian level the hierarchy of energy scales in Eq. (52). The EFT framework has the advantage to allow more easily for systematic improvements of the calculation and to make more transparent its physical meaning.

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8 In Nadkarni’s paper this contribution is called $f_{II}$.

9 In Nadkarni’s paper this contribution is called $f_1$. 
Our starting point is QCD with a static quark and a static antiquark, denoted in the
following as static QCD. Its action in Euclidean space-time reads
\[
S_{\text{QCD}} = \int_0^{1/T} d\tau \int d^3x \left( \psi^\dagger D_0 \psi + \chi^\dagger D_0 \chi + \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_{l=1}^{n_f} \bar{q}_l D_l q_l \right),
\]
where \( D_0 = \partial_0 - igA_0 \), \( \psi \) is the Pauli spinor field that annihilates a static quark, \( \chi \) is the
Pauli spinor field that creates a static antiquark, and \( q_1, ..., q_{n_f} \) are the light quark fields,
which are assumed to be massless in this study.

The Polyakov-loop correlator may be expressed in static QCD as
\[
\langle \tilde{\text{Tr}} L_F^\dagger(0) \tilde{\text{Tr}} L_F(r) \rangle = \frac{1}{N^2} \frac{1}{\mathcal{N}} \langle \chi^\dagger_j(0, 1/T) \psi_i(r, 1/T) \psi^\dagger_i(0, r) \chi_j(0, 0) \rangle,
\]
where \( \mathcal{N} = [\delta^3(0)]^2 \) and we have written explicitly the colour indices. The thermal average
on the right-hand side reduces to the Polyakov-loop correlator on the left-hand side after
integrating out the fields \( \psi \) and \( \chi \). On general grounds, one also expects that \[38, 39\]
\[
\langle \tilde{\text{Tr}} L_F^\dagger(0) \tilde{\text{Tr}} L_F(r) \rangle = \frac{1}{N^2} \sum_n e^{-E_n/T},
\]
where \( E_n \) are the eigenvalues of the Hamiltonian associated to the static QCD Lagrangian.

### A. pNRQCD

Potential non-relativistic QCD (pNRQCD) is the EFT that follows from QCD by in-
tegrating out from the static quark-antiquark sector gluons of energy or momentum that
scale like the inverse of the distance \( r \) between the quark and the antiquark. Since \( 1/r \) is
the largest scale, the matching of the pNRQCD Lagrangian may be done by setting to zero
all other scales and, in particular, the thermal ones; as a consequence, the Lagrangian is
identical to the one derived at zero temperature \[34, 40–43\]. In Euclidean space-time, the
action reads
\[
S_{\text{pNRQCD}} = \int_0^{1/T} d\tau \int d^3x \int d^3r \text{Tr} \left\{ S^\dagger (\partial_0 + V_s) S + O^\dagger (D_0 + V_o) O \\
- iV_A \left( S^\dagger r \cdot gE O + O^\dagger r \cdot gE S \right) - \frac{i}{2} V_B \left( O^\dagger r \cdot gE O + O^\dagger O r \cdot gE \right) \\
+ \frac{i}{8} V_C \left( r^i r^j O^\dagger D^i gE^j O - r^i r^j O^\dagger O D^i gE^j \right) + \delta L_{\text{pNRQCD}} \right\}
\]
where the trace is over the colour indices, $S = 1_{N \times N} S/\sqrt{N}$ is a quark-antiquark field in a colour-singlet configuration, $O = \sqrt{T} a^{\alpha} O^\alpha$ is a quark-antiquark field in a colour-octet configuration, $D_0 O = \partial_0 - ig[A_0, O]$, $D = \nabla - igA$ and $E^i = F_{i0}$ is the chromoelectric field. The fields $S$ and $O^\alpha$ depend on the continuous parameter $r$ that labels the distance between the quark and the antiquark, the centre-of-mass coordinate $x$ and the Euclidean time $\tau$; the gluon fields have been multipole expanded and, therefore, depend on $x$ and $\tau$ only. The quantities $V_s$, $V_o$, $V_A$, $V_B$ and $V_C$ are the matching coefficients of the EFT. These are non-analytic functions of $r$. Since $V_A(r) = 1 + O(\alpha_s^2)$ and $V_C(r) = 1 + O(\alpha_s)$ it will suffice to our purposes to put $V_A(r) = V_B(r) = V_C(r) = 1$ from now on. $V_s$ and $V_o$ are the singlet and octet potentials in pNRQCD: $V_s$ is known up to three loops and $V_o$ is known up to two loops. For the purpose of obtaining the Polyakov-loop correlator at NNLO accuracy it is sufficient to know $V_s$ at one-loop accuracy and their difference at two-loop accuracy:

$$ + \int_0^{1/T} d\tau \int d^3x \left( \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \sum_{l=1}^{n_f} \hat{q}_l \hat{D} q_l \right),$$

(74)

Finally, $\delta L_{\text{pNRQCD}}$ includes all operators that are of order $r^3$ or smaller. At tree-level, they may be read from the multipole expansion of the quark and antiquark coupling to the temporal gluon in the static QCD Lagrangian, hence they just involve covariant derivatives acting on a chromoelectric field: the leading-order operator being $-i r^i r^j r^k \text{Tr} \{ O^i D^j D^k S + S^i D^j D^k g E^k O \}/24$. As we will argue in the next section, these terms are of order $g^4$, however, their contribution eventually cancels in the Polyakov-loop correlator up to order $g^6(rT)^0$. For this reason, we do not need to specify them further here.

Matching the connected Polyakov-loop correlator to pNRQCD gives

$$ C_{\text{PL}}(r, T) = \frac{1}{N^2} \left[ Z_s \frac{\langle S(r, 0, 1/T) S^\dagger(r, 0, 0) \rangle}{N} + Z_o \frac{\langle O^\alpha(r, 0, 1/T) O^{\alpha\dagger}(r, 0, 0) \rangle}{N} \right. $$

$$ + \mathcal{O}(\alpha_s^3(rT)^4) \left. \right] - \langle L_F \rangle^2.$$  

(78)
The right-hand side is the pNRQCD part of the matching. It contains the singlet and octet correlators, \( \langle S(r, 0, 1/T) S^\dagger(r, 0, 0) \rangle \) and \( \langle O^a(r, 0, 1/T) O^{a\dagger}(r, 0, 0) \rangle \), not surprisingly because in the \( r \to 0 \) limit the tensor fields \( \chi_j^\dagger(0, 1/T) \psi_i(r, 1/T) \) and \( \psi_i^\dagger(r, 0) \chi_j(0, 0) \), appearing in the right-hand side of Eq. (72), decompose into the direct sum of a colour-singlet and a colour-octet component. The colour-singlet and colour-octet correlators may be read from the Lagrangian (74):

\[
\frac{\langle S(r, 0, 1/T) S^\dagger(r, 0, 0) \rangle}{\mathcal{N}} = e^{-V_s(r)/T} (1 + \delta_s),
\]

(79)

\[
\frac{\langle O^a(r, 0, 1/T) O^{a\dagger}(r, 0, 0) \rangle}{\mathcal{N}} = e^{-V_o(r)/T} [(N^2 - 1) \langle L_A \rangle + \delta_o],
\]

(80)

where \( \delta_s \) and \( \delta_o \) stand for loop corrections to the singlet and octet correlators respectively. The factor \( \langle L_A \rangle \) comes from the covariant derivative \( D_0 \) acting on the octet field in (74).\footnote{The adjoint Polyakov loop \( \langle L_A \rangle \) factorizes the contribution coming from the gluons in the thermal bath that bind with the colour-octet quark-antiquark states to form part of the spectrum appearing in the right-hand side of Eq. (72). In pNRQCD at zero temperature, a similar expression factorizes the non-perturbative gluonic contribution to the gluelumps masses \cite{34}.}

Note that at finite temperature, for \( T \gtrsim g^2/r \), the octet correlator is not suppressed with respect to the singlet one, while in the opposite limit, \( T \ll g^2/r \), the Polyakov-loop correlator is dominated by the singlet contribution. Higher-dimensional operators have not been displayed, because they are negligible with respect to our present accuracy, which is of order \( g^6(rT)^0 \). The reason is that higher-dimensional operators involve the coupling with at least two field-strength tensors, hence the corresponding matrix elements are at least of order \( (rT)^4 \); moreover, as can be seen by adding two external gluons to diagram I of Fig. 5, the matrix element of an operator coupled with two external gluons is at least of order \( g^6 \). The normalization factors \( Z_s \) and \( Z_o \) have to be determined from the matching condition (78).

While \( V_s \) and \( V_o \) are the same at zero and finite temperature, the normalization factors are not for they depend on the boundary conditions.

In order to determine the normalization factors \( Z_s \) and \( Z_o \), let us consider in Eq. (78) only contributions coming from the scale \( 1/r \). In dimensional regularization, all loop corrections vanish in the pNRQCD part of the matching and the Polyakov loops \( \langle L_F \rangle \) and \( \langle L_A \rangle \) reduce to one; therefore, the matching condition reads

\[
C_{PL}(r, T)_{1/r} = \langle \tilde{\text{Tr}} L_F^\dagger(0) \tilde{\text{Tr}} L_F(r) \rangle_{1/r} - 1 = \frac{1}{N^2} [Z_s e^{-V_s(r)/T} + Z_o (N^2 - 1)e^{-V_o(r)/T}] - 1. \quad (81)
\]
We may now proceed in different ways. A way consists in matching with the spectral decomposition (73). By noting that at the scale $1/r$ the spectrum is just given by a singlet state of energy $V_s(r)$ and $N^2 - 1$ degenerate octet states of energy $V_o(r)$, the matching condition implies that $Z_s = Z_o = 1$. Another way consists in taking advantage of the Polyakov-loop correlator calculation done in Sec. IV and matching to it. $C_{PL}(r, T)_1/r$ is the sum of Eq. (53), Eq. (57) without the contribution from the matter part of the gluon self energy, Eq. (65) and Eq. (66); it reads

$$C_{PL}(r, T)_1/r = \frac{N^2 - 1}{8N^2} \left\{ \frac{\alpha_s(1/r)^2}{(rT)^2} + \frac{\alpha_s^3}{(rT)^3} \frac{N^2 - 2}{6N} + \frac{1}{2\pi (rT)^2} \left( \frac{31}{9} C_A - \frac{10}{9} n_f + 2\gamma_E \beta_0 \right) \right. \\
+ \frac{\alpha_s^3}{rT} C_A \left( 3 - \frac{\pi^2}{4} \right) + O \left( \frac{\alpha_s^4}{(rT)^4} \right) \}.$$  \hspace{1cm} (82)

A direct inspection shows that this expression satisfies

$$C_{PL}(r, T)_1/r = \frac{1}{N^2} \left[ e^{-V_s(r)/T} + (N^2 - 1) e^{-V_o(r)/T} \right] - 1,$$  \hspace{1cm} (83)

up to order $\alpha_s^3$, for $V_s(r)$ and $V_o(r)$ given by Eqs. (75)-(77). We note that Eqs. (82) and (83) are equivalent for $rT \gg g^2$, however, in Eq. (83), we resum some contributions that would become large for $rT \lesssim g^2$. Equation (83) is therefore valid also in that regime. Finally, we observe that the combination of the two procedures provides a non-trivial verification of Eq. (77), i.e. of the two-loop difference between the octet and the singlet potentials, known, so far, only from the direct calculation of the two-loop octet potential in a covariant gauge, done in Ref. [46].

Loop corrections to the singlet and octet correlators in Eqs. (79) and (80) get contributions from the scales $T, m_D$ and lower ones. We now proceed to evaluate these corrections, separating the contributions of the temperature from the ones of the Debye mass.

### B. The temperature scale

In the hierarchy (52), the next scale after the inverse distance is the temperature. Our aim is thus to compute the temperature contributions to loop corrections in pNRQCD. These loop corrections are the terms $\delta_s$ and $\delta_o$ that were introduced in Eqs. (79) and (80). We call

\[\text{More precisely, the matching to (82) fixes } Z_s = Z_o = 1 \text{ up to order } \alpha_s^2 \text{ and } Z_s + (N^2 - 1) Z_o = N^2 \text{ up to order } \alpha_s^3.\]
\( \delta_{s,T} \) and \( \delta_{o,T} \) the parts of \( \delta_s \) and \( \delta_o \) respectively that encode the contributions coming from the scale \( T \); they may be obtained by expanding \( \delta_s \) and \( \delta_o \) in \( m_D, V_s, V_o \) and in any lower energy scale. Similarly, \( \delta \langle L_R \rangle_T \) is the part of \( \langle L_R \rangle \) that encodes the contributions coming from the scale \( T \). Different terms contribute to \( \delta_{s,T}, \delta_{o,T} \) and \( \delta \langle L_R \rangle_T \); we examine them in the following.

![Feynman diagram](image.png)

FIG. 7: The pNRQCD Feynman diagram giving the leading-order correction to \( \delta_s \). The single continuous line stands for a singlet propagator, the double line for an octet propagator, the circle with a cross for the chromoelectric dipole vertex proportional to \( V_A \) in the Lagrangian (74) and the curly line connecting the two circles with a cross for a chromoelectric correlator.

1. **The singlet \( r^2 \) contributions**

   We start considering the one-loop, order \( r^2 \) in the multipole expansion, correction to the singlet correlator induced by the diagram shown in Fig. [7] it reads

   \[
   \delta_s^{O(r^2)} = \left( ig \sqrt{\frac{1}{2N}} \right)^2 r^i r^j T \sum_n \int \frac{d^4k}{(2\pi)^4} \int_{0}^{1/T} d\tau \int_{0}^{\tau} d\tau' e^{\tau V_s} e^{-(\tau-\tau')V_o} e^{-\tau' V_s} \times e^{-i(\tau-\tau')\omega_n} \langle E^i a U_{ab} E^j b \rangle(\omega_n, k). \tag{84}
   \]

   In the sum integral, we may distinguish between contributions coming from the non-zero modes and from the zero modes.

   For the contribution coming from the non-zero modes, only the leading-order chromoelectric correlator in momentum space \( \langle E^i a U_{ab} E^j b \rangle(\omega_n, k) \) (\( U_{ab} \) stands for a Wilson straight line in the adjoint representation connecting \( E^i a \) with \( E^j b \); at leading order \( U_{ab} = \delta_{ab} \)) is relevant at our accuracy:

   \[
   \langle E^i a U_{ab} E^j b \rangle(\omega_n, k) = (N^2 - 1) \left[ \frac{k^i k^j}{k^2} + (\delta_{ij} - \hat{k}^i \hat{k}^j) \frac{\omega_n^2}{\omega_n^2 + k^2} \right]. \tag{85}
   \]

   Loop corrections to the chromoelectric correlator contribute to the Polyakov-loop correlator at order \( g^6(rT) \) or smaller. Because of the hierarchy (52), we can expand the right-hand side of (84) in \( V_o - V_s \). The longitudinal part of the chromoelectric
correlator, i.e. the first term in square brackets, vanishes in dimensional regularization, whereas the transverse part is sensitive to the scale $T$ through the Matsubara frequencies. After performing the sum integral over the non-zero modes, we obtain

$$
\delta_{s,T}^{O(r^2)_{NS}} = -g^2C_F \frac{r^2T}{9}(V_o-V_s) + g^2C_F \frac{r^2}{36}(V_o-V_s)^2 + \mathcal{O} \left( g^6(rT), \frac{g^8}{rT} \right) \\
= -\frac{2}{9}\pi NCF\alpha_s^2 r T + \frac{\pi}{36}N^2 C_F\alpha_s^3 + \mathcal{O} \left( g^6(rT), \frac{g^8}{rT} \right).
$$

(86)

The contribution coming from the zero modes reads

$$
\delta_{s,T}^{O(r^2)_{S}} = \left( ig\sqrt{\frac{1}{2N}} \right)^2 \frac{r^i r^j}{2T} \int \frac{d^d k}{(2\pi)^d} \langle E^i a U_{ab} E^j b \rangle(0, k)|_{|k|\sim T} + \mathcal{O} \left( g^6(rT) \right). \quad \text{(87)}
$$

Here, the first non-vanishing contribution in dimensional regularization comes from the one-loop correction to the chromoelectric correlator. The integral with $\langle E^i a U_{ab} E^j b \rangle(0, k)$ at one loop has been calculated in [9]. Using that result we obtain

$$
\delta_{s,T}^{O(r^2)_{S}} = \frac{3}{2}\zeta(3)C_F \alpha_s (r m_D)^2 - \frac{2}{3}\zeta(3)NCF\alpha_s^2 (r T)^2 + \mathcal{O} \left( g^6(rT) \right). \quad \text{(88)}
$$

2. Higher multipole terms

Our aim is to calculate in the EFT the Polyakov-loop correlator at order $g^6$, neglecting terms of order $g^6(rT)$ or smaller. Contributions coming from the $\delta \mathcal{L}_{pNRQCD}$ part of the pNRQCD Lagrangian, which includes terms of order $r^3$ or smaller coming from the multipole expansion, share, at leading order, the same colour structure and the same order in $\alpha_s$ as Eqs. (86) and (88) but are suppressed by powers of $rT$. We may write these contributions as

$$
\delta_{s,T}^{\delta \mathcal{L}_{pNRQCD}} = \delta_{s,T}^{O(r^2)_{NS}} \sum_{n=0}^{\infty} c_n^{NS}(r T)^{2n+2} + \delta_{s,T}^{O(r^2)_{S}} \sum_{n=0}^{\infty} c_n^{S}(r T)^{2n+2} + \mathcal{O} \left( g^6(rT)^3 \right),
$$

12 The chromoelectric correlator is gauge invariant. In static gauge, thermal corrections arise from the non-static part of the spatial gluon propagator. Hence, at one loop, only gluon self-energy diagrams may provide thermal corrections; we have

$$
\langle E^i a U_{ab} E^j b \rangle(0, k)|_{|k|\sim T} = \langle \partial_i A_0^a \partial_j A_0^b \rangle(0, k)|_{|k|\sim T} = (N^2 - 1) \frac{k^i k^j}{k^2 + \Pi_{00}^{NS}(k)_{\text{mat}}},
$$

where $\Pi_{00}^{NS}(k)_{\text{mat}}$ is the matter part of the gluon self-energy’s temporal component calculated in static gauge, which can be read from Eq. (11). Finally, we recall that $\Pi_{00}^{NS}(k)_{\text{mat}}$ is the same in static gauge and in Coulomb gauge.
where the unknown coefficients $c_{n}^{NS}$ and $c_{n}^{S}$ are, as we will see, irrelevant for the purpose of calculating the Polyakov-loop correlator at order $g^{6}(rT)^{0}$.

3. The octet contributions

As in the singlet case, one loop-corrections to the octet correlator may be divided into order $r^{2}$ non-zero mode contributions ($\delta_{o,T}^{O(r^{2})NS}$), order $r^{2}$ zero-mode contributions ($\delta_{o,T}^{O(r^{2})S}$), and higher multipole terms ($\delta_{o,T}^{\delta L_{\text{NRQCD}}}$). It turns out that

$$\delta_{o,T}^{O(r^{2})NS} = \delta_{s,T}^{O(r^{2})NS}|_{V_{s}\leftrightarrow V_{o}},$$

and, up to order $g^{6}(rT)^{0}$,

$$\delta_{o,T}^{O(r^{2})S} = -\delta_{s,T}^{O(r^{2})S},$$

$$\delta_{o,T}^{\delta L_{\text{NRQCD}}} = -\delta_{s,T}^{\delta L_{\text{NRQCD}}}.$$

These equalities are proved in appendix G.

4. $\delta \langle L_{R} \rangle_{T}$

Finally, we need to calculate the contributions to the Polyakov loop coming from the scale $T$. The order $g^{4}$ contribution may be read from Eq. (39). Since we do not know the order $C_{R}g^{6}$ contribution, we write $\delta \langle L_{R} \rangle_{T}$ as

$$\delta \langle L_{R} \rangle_{T} = \frac{C_{R}\alpha_{s}^{2}}{2} \left[ C_{A} \left( \frac{1}{2\epsilon} - \ln \frac{4T^{2}}{\mu^{2}} + 1 - \gamma_{E} + \ln(4\pi) \right) - n_{f} \ln 2 + a\alpha_{s} \right] + O(\alpha_{s}^{4}),$$

where the explicit value of the coefficient $a$ does not matter. Instead, what matters here is that this coefficient is common to all colour representations. The first correction from the scale $T$ not of the type $C_{R}\alpha_{s}^{n}$ appears at order $\alpha_{s}^{4}$ and comes from diagram b) in Fig. 4 with two self-energy insertions, one in each temporal gluon. Note that Eq. (48) provides the first correction not of the type $C_{R}\alpha_{s}^{n}$ coming from the scale $m_{b}$.

In summary, we obtain the contribution of the scale $T$ to the singlet and octet correlators:

$$e^{-V_{s}(r)/T} \delta_{s,T} = e^{-V_{s}(r)/T} \left\{- \frac{2}{9\pi} NC_{F}\alpha_{s}^{2}rT \left[ 1 + \sum_{n=0}^{\infty} C_{n}^{NS}(rT)^{2n+2} \right] + \frac{\pi}{36} N^{2}C_{F}\alpha_{s}^{4} \right\}$$
\[ + \left( \frac{3}{2} \zeta(3) C_F \frac{\alpha_s}{\pi} (r m_D)^2 - \frac{2}{3} \zeta(3) N C_F \alpha_s^2 (r T)^2 \right) \left[ 1 + \sum_{n=0}^{\infty} C_n^S (r T)^{2n+2} \right] \]

\[ + O \left( g^6 (r T), \frac{g^8}{r T} \right) \}, \quad (94) \]

\[ e^{-V_0 (r)/T} \left[ (N^2 - 1) \delta \langle L_A \rangle_T + \delta_{o,T} \right] = \]

\[ (N^2 - 1) e^{-V_0 (r)/T} \left\{ \frac{C_A}{2} \frac{\alpha_s}{\pi} \left[ C_A \left( \frac{1}{2 \epsilon} - \ln \frac{4 T^2}{\mu^2} + 1 - \gamma_E + \ln(4 \pi) \right) - n_f \ln 2 + a \alpha_s \right] \right. \]

\[ + \frac{1}{9} \pi \frac{\alpha_s^2}{r T} \left[ 1 + \sum_{n=0}^{\infty} C_n^{NS} (r T)^{2n+2} \right] + \frac{\pi}{72} N \alpha_s^3 \]

\[ - \left( \frac{3}{4} \zeta(3) \frac{1}{N \pi} \frac{\alpha_s}{r m_D} \right)^2 - \frac{1}{3} \zeta(3) \alpha_s^2 (r T)^2 \right) \left[ 1 + \sum_{n=0}^{\infty} C_n^S (r T)^{2n+2} \right] \]

\[ + O \left( g^6 (r T), \frac{g^8}{r T} \right) \}. \quad (95) \]

Inserting Eqs. (93)-(95) into Eq. (78) and expanding, we obtain that the connected Polyakov-loop correlator is given by

\[ C_{PL}(r, T) = C_{PL}(r, T)_{1/r} \]

\[ - \frac{\pi}{18} C_F \alpha_s^3 + \frac{N^2 - 1}{8 N^2} \frac{\alpha_s^3}{r T} \left[ C_A \left( - \frac{1}{\epsilon} - 2 \ln \frac{\mu^2}{4 T^2} - 2 + 2 \gamma_E - 2 \ln(4 \pi) \right) \right. \]

\[ + 2 n_f \ln 2 \right] + O \left( g^6 (r T), \frac{g^8}{(r T)^4} \right) \]

\[ + \text{loop corrections at the scale } m_D \text{ or lower}, \quad (96) \]

where \( C_{PL}(r, T)_{1/r} \) may be read from Eq. (82). We observe that, in the connected Polyakov-loop correlator, terms proportional to the unknown coefficients \( c_n^{NS} \), \( c_n^S \) and \( a \) have canceled. The thermal corrections in (96) agree with those calculated in Sec. IV; in particular, they correspond to the sum of the gluon self-energy matter-part contribution in Eq. (57) with Eq. (58). The result in Eq. (96) has an infrared divergence that originates at the scale \( T \). This divergence shall cancel against an opposite ultraviolet one at the scale \( m_D \), which will be the subject of the next section.

C. The Debye mass scale

Here we compute the contributions to the singlet correlator, the octet correlator and the Polyakov loop coming from loop momenta sensitive to the Debye mass scale. We call these
contributions $\delta_{s,m_D}, \delta_{o,m_D}$ and $\delta\langle L_R \rangle_{m_D}$ respectively. They may be computed by evaluating the loop integrals in $\delta_s, \delta_o$ and $\delta\langle L_R \rangle$ over momenta of the order $m_D$ and expanding with respect to any other scale. The Debye mass scale is the lowest scale we need to consider here; contributions coming from scales lower than $m_D$ are beyond our accuracy. Different terms contribute to $\delta_{s,m_D}, \delta_{o,m_D}$ and $\delta\langle L_R \rangle_{m_D}$; we examine them in the following.

1. The singlet and octet contributions

The leading-order contribution to $\delta_{s,m_D}$ comes from the self-energy diagram shown in Fig. 7 when evaluated over loop momenta of order $m_D$. The contribution reads

$$\delta_{s,m_D} = \left(i g \sqrt{\frac{1}{2N}}\right)^2 r^i r^j T \sum_n \int \frac{d^d k}{(2\pi)^d} \int_0^{1/T} d\tau \int_0^\tau d\tau' e^{\tau V_s} e^{-(\tau-\tau')V_o} e^{-\tau' V_s} \times e^{-i(\tau-\tau')\omega_n} \langle E^i a U_{ab} E^j b \rangle (\omega_n, k) \left| k \right| \sim m_D. \quad (97)$$

The chromoelectric correlator evaluated over the region $\left| k \right| \sim m_D$ gives rise to scaleless momentum integrals unless for the temporal part of the zero mode, $n = 0$, which is at leading order $\langle E^i a U_{ab} E^j b \rangle (0, k) \left| k \right| \sim m_D = (N^2 - 1) k^i k^j / (k^2 + m_D^2)$. We obtain

$$\delta_{s,m_D} = -g^2 C_F r^i r^j T \int \frac{d^d k}{(2\pi)^d} \frac{k^i k^j}{k^2 + m_D^2} \left[ 1 + \mathcal{O} \left( \frac{g^2}{r T} \right) \right] = -C_F \frac{\alpha_s}{6} r^i r^j m_D^3 T + \mathcal{O} \left( g^7 (r T) \right). \quad (98)$$

The leading-order contribution to $\delta_{o,m_D}$ comes from the octet self-energy diagrams shown in Fig. 9 when evaluated over the region $\left| k \right| \sim m_D$. Also in this case, the only non-vanishing contribution comes from the zero mode of the temporal gluon propagator, which is $1/(k^2 + m_D^2)$ (see Eq. (22)). For the same argument developed in appendix C, we find that

$$\delta_{o,m_D} = -\delta_{s,m_D}. \quad (99)$$

Higher multipole terms are of order $\alpha_s r^i r^j m_D^3 \left( r m_D \right)^2 \sim g^7 (r T)^4$ or smaller and, therefore, beyond our accuracy.

2. $\delta\langle L_R \rangle_{m_D}$

We need to calculate the contribution to the Polyakov loop coming from the scale $m_D$. It may be read from Eqs. (33), (44) and (48). Since we do not know the order $C_R g^5$ and $C_R g^6$ contributions, we write $\langle L_R \rangle_{m_D}$ as

$$\delta\langle L_R \rangle_{m_D} = \frac{C_R \alpha_s m_D}{2} T.$$

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In terms of have canceled. The origin of the thermal corrections to the Polyakov-loop correlator in the
In summary, we obtain the contribution of the scale $m_D$ to the singlet and octet correlators:
\[
e^{-V_s(r)/T} \delta_{s, m_D} = e^{-V_s(r)/T} \left\{ - C_F \frac{\alpha_s}{6} r^2 \frac{m_D^3}{T} + O \left( g^7(rT) \right) \right\}
\]  
\[
e^{-V_o(r)/T} \left[ (N^2 - 1) \delta \langle L_A \rangle_{m_D} + \delta_{o, m_D} \right] = (N^2 - 1) e^{-V_o(r)/T} \left\{ C_A \frac{\alpha_s}{2} \frac{m_D}{T} + \frac{5}{48} C_A^2 \frac{\alpha_s^2}{24} \left( \frac{m_D}{T} \right)^2 \right. 
\]
\[
+ \frac{C_A \alpha_s^2}{2} \left[ C_A \left( - \frac{1}{2 \epsilon} - \ln \frac{\mu^2}{4m_D^2} - \frac{1}{2} + \gamma_E - \ln(4\pi) \right) + b_1 g + b_2 g^2 \right] 
\]
\[
+ \frac{1}{N} \frac{\alpha_s}{12} \frac{m_D^3}{T} + O \left( g^7 \right) \right\}.
\]  

Inserting Eqs. (100)-(102) into Eq. (96) and expanding, we obtain that the connected Polyakov-loop correlator is given by
\[
C_{PL}(r, T) = C_{PL}(r, T)_{1/r} 
\]
\[
- C_F \frac{\pi \alpha_s^3}{18} + \frac{N^2 - 1}{8N^2} \alpha_s^2 \left( \frac{m_D}{T} \right)^2 
\]
\[
+ \frac{N^2 - 1}{N^2} \frac{\alpha_s}{rT} \left\{ - \frac{\alpha_s m_D}{4T} - \frac{\alpha_s^2}{4} \left[ C_A \left( - \ln \frac{T^2}{m_D^2} + \frac{1}{2} \right) - n_f \ln 2 \right] \right\}
\]
\[
+ O \left( g^6(rT), \frac{g^7}{(rT)^2} \right),
\]  
where $C_{PL}(r, T)_{1/r}$ may be read from Eq. (52). We observe that, in the Polyakov-loop correlator, terms proportional to the unknown coefficients $b_1$ and $b_2$, as well as the divergences, have canceled. The origin of the thermal corrections to the Polyakov-loop correlator in the

\[\text{In terms of } \delta_{s, T}, \delta_{s, m_D}, \delta_{o, T}, \delta_{o, m_D}, \delta(L_F)_T, \delta(L_F)_m_D, \delta(L_A)_T \text{ and } \delta(L_A)_m_D, C_{PL}(r, T) \text{ reads}
\]
\[
C_{PL}(r, T) = \frac{1}{N^2} \left\{ e^{-V_s(r)/T} \left( 1 + \delta_{s, T} + \delta_{s, m_D} \right) 
\]
\[
+ e^{-V_o(r)/T} \left[ (N^2 - 1) \left( 1 + \delta(L_A)_T + \delta(L_A)_m_D + \delta_{o, T} + \delta_{o, m_D} \right) \right] 
\]
\[
- (1 + \delta(L_F)_T + \delta(L_F)_m_D)^2.
\]
situation $1/r \gg T \gg m_D \gg g^2/r$ is clear. The term $-C_F\pi\alpha_s^3/18$ arises from the dipole interaction contributions and from their interference with the zero-temperature potentials. The other thermal corrections arise from the interference of the adjoint Polyakov loop with the zero-temperature potentials.

The result coincides with Eq. (67), obtained in Sec. IVV after a direct calculation. The differences in the way the two results were achieved illustrate well the typical differences between a direct computation and a computation in an EFT framework. In the EFT framework, some more conceptual work was necessary in order to identify the relevant contributions. Once this was done, we could take advantage of previously done calculations (in particular for $V_s(r)$ and $V_o(r)$) and reduce the calculation to essentially one diagram, shown in Fig. [7] evaluated in different momentum regions. In the EFT framework, we could also gain some new insight by reconstructing the spectral decomposition of the Polyakov-loop correlator and by providing two new quantities: the colour-singlet and the colour-octet quark-antiquark correlators.

### D. Singlet and octet free energies

Potential NRQCD at finite temperature allows to define a colour-singlet correlator, $\langle S(r,0,1/T)S^\dagger(r,0,0) \rangle$, and a colour-octet correlator, $\langle O^a(r,0,1/T)O^{a\dagger}(r,0,0) \rangle$, which are both gauge-invariant quantities. We may associate to them a colour-singlet free energy, $f_s(r,T,m_D)$, and a colour-octet free energy, $f_o(r,T,m_D)$, such that

\[
\langle S(r,0,1/T)S^\dagger(r,0,0) \rangle = e^{-V_s(r)/T} (1 + \delta_{s,T} + \delta_{s,m_D}) \\
\equiv e^{-f_s(r,T,m_D)/T},
\]

\[
\langle O^a(r,0,1/T)O^{a\dagger}(r,0,0) \rangle = e^{-V_o(r)/T} [(N^2 - 1) (1 + \delta(L_A)_T + \delta(L_A)_m_D) + \delta_{o,T} + \delta_{o,m_D}] \\
\equiv (N^2 - 1)e^{-f_o(r,T,m_D)/T}.
\]

Using the results of the previous sections, we have that

\[
f_s(r,T,m_D) = V_s(r) + \frac{2}{9}\pi NC_F\alpha_s^2 r^2 T^2 \left[ 1 + \sum_{n=0}^{\infty} c_n^{NS}(rT)^{2n+2} \right] - \frac{\pi}{36}N^2 C_F\alpha_s^3 T \\
- \left( \frac{3}{2} \zeta(3) C_F \alpha_s \frac{r m_D}{\pi} T^2 - \frac{2}{3} \zeta(3) NC_F\alpha_s^2 r^2 T^3 \right) \left[ 1 + \sum_{n=0}^{\infty} c_n^{S}(rT)^{2n+2} \right] \]

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\[ +C_F \frac{\alpha_s}{6} r^2 m_D^3 + T \mathcal{O} \left( g^6(rT), \frac{g^8}{rT} \right), \]  

(106)

and

\[
\begin{align*}
    f_o(r, T, m_D) &= V_o(r) \\
    &= \frac{C_A \alpha_s}{2} m_D + \frac{1}{48} C_A^2 \alpha_s^2 m_D^2 T \\
    &- \frac{C_A \alpha_s^2}{2} T \left[ C_A \left( -\ln \frac{T^2}{m_D^2} + \frac{1}{2} \right) - n_f \ln 2 + b_1 g + b_2 g^2 + a \alpha_s \right] \\
    &- \frac{\pi}{9} \alpha_s^2 r T^2 \left[ 1 + \sum_{n=0}^{\infty} c_n^{NS} (rT)^{2n+2} \right] - \frac{\pi}{72} N \alpha_s^3 T \\
    &+ \left( \frac{3}{4N} \zeta(3) \alpha_s \pi (r m_D)^2 T - \frac{1}{3} \zeta(3) \alpha_s^2 r^2 T^3 \right) \left[ 1 + \sum_{n=0}^{\infty} c_n^S (rT)^{2n+2} \right] \\
    &- \frac{1}{N} \frac{\alpha_s^2}{12} m_D^3 + T \mathcal{O} \left( g^6(rT), \frac{g^8}{rT} \right). 
\end{align*} \tag{107}
\]

We note that \( f_s(r, T, m_D) \) and \( f_o(r, T, m_D) \) are both finite and gauge invariant. They also do not depend on some special choice of Wilson lines connecting the initial and final quark and antiquark states.

In [9], the colour-singlet quark-antiquark potential was calculated in real-time formalism in the same thermodynamical situation considered here and specified by Eq. (52). The result may be found in Eq. (92) of Ref. [9]. Comparing terms of the same order, the real part of the real-time potential differs from \( f_s(r, T, m_D) \) by \( \frac{1}{9} \pi N C_F \alpha_s^2 r T^2 - \frac{\pi}{36} N^2 C_F \alpha_s^3 T \). The origin of the difference may be traced back to terms in Eq. (84) that would vanish for large real times. Indeed, performing the calculation of \( \langle S(r, 0, \tau) S^\dagger(r, 0, 0) \rangle \) for an imaginary time \( \tau \leq 1/T \), along the lines of Secs. [\ref{sec:vet}] and [\ref{sec:vcc}], and then continuing analytically \( \tau \) to large real times, one gets back exactly both the real and the imaginary parts of the real-time colour-singlet potential derived in [9]. The difference between the singlet free energy and the real part of the real-time colour-singlet potential appears to be a relevant finding to be considered when using free-energy lattice data for the quarkonium in medium phenomenology.

E. Comparison with the literature

An EFT approach for the calculation of the correlator of Polyakov loops was developed in [47] for the situation \( m_D \gg 1/r \) and in [20] for \( T \gg 1/r \). In neither of the two cases,
the scale $1/r$ was integrated out: the Polyakov-loop correlator was described in terms of dimensionally reduced effective field theories of QCD, while the complexity of the bound-state dynamics remained implicit in the correlator. The description developed in \cite{20,47} is valid for largely separated Polyakov loops. Under that condition, the correlator turns out to be screened either by the Debye mass, for $rm_D \sim 1$, or by the mass of the lowest-lying glueball, for $rm_D \gg 1$.

In \cite{38}, the spectral decomposition of the Polyakov-loop correlator was analyzed. It was concluded that the quark-antiquark component of an allowed intermediate state, i.e. a field $\varphi$ describing a quark located in $x_1$ and an antiquark located in $x_2$, should transform as $\varphi(x_1, x_2) \rightarrow g(x_1)\varphi(x_1, x_2)g^\dagger(x_2)$ under a gauge transformation $g$. Equation (78) is in accordance with that result for, in pNRQCD, both the singlet field $S$ and the octet field $O$ transform in that way \cite{40}. We remark, however, a difference in language: in our work, singlet and octet refer to the gauge transformation properties of the quark-antiquark fields, while, in \cite{38}, they refer to the gauge transformation properties of the physical states.

In \cite{31}, a weak-coupling calculation of the untraced Polyakov-loop correlator in Coulomb gauge and of the cyclic Wilson loop was performed up to order $g^4$. Each of these objects contributes to the correlator of two Polyakov loops through a Fierz transformation that also generates some octet counterparts. It is expected that large cancellations occur between those correlators and their octet counterparts in order to reproduce the Polyakov-loop correlator given in Eq. (67). Such large cancellations should occur at the level of the scales $1/r$, $T$ and $m_D$ as we have already experienced in this work. Note that in the case of the untraced Polyakov-loop correlator, the octet contribution shall also restore gauge invariance.

VI. SUMMARY AND OUTLOOK

In the weak-coupling regime, we have calculated the Polyakov loop up to order $g^4$ and the correlator of two Polyakov loops up to order $g^6(rT)^0$, assuming the hierarchy of scales $\frac{1}{r} \gg T \gg m_D \gg \frac{g^2}{r}$.

The Polyakov-loop calculation differs from the result of Gava and Jenko \cite{21} by a finite contribution at order $g^4$. We have analyzed in detail the origin of the difference and shown in an appendix that our result may be reproduced also performing the calculation in Feynman gauge. Our calculation agrees with the recent finding of Ref. \cite{31}.
The calculation of the Polyakov-loop correlator is new in the considered regime, although some partial results may be deduced from a previous work of Nadkarni, who studied distances \( r \sim 1/m_D \) \[20\]. We have performed the calculation in two different approaches: by a direct computation in static gauge and by calculating the Polyakov-loop correlator in a suitable EFT that exploits the hierarchy of scales in the problem. In this second approach, we have used pNRQCD at finite temperature and subsequently integrated out lower momentum regions. The advantages of this second approach are that the calculations do not rely on any specific choice of gauge and the systematics is clearer. Moreover, it makes explicit the quark-antiquark colour-singlet and colour-octet contributions to the Polyakov-loop correlator. In particular, we have shown that at leading order in the multipole expansion the Polyakov-loop correlator can be written as the colour average of a colour-singlet correlator, which defines a gauge-invariant colour-singlet free energy, and a colour-octet correlator, which defines a gauge-invariant colour-octet free energy. This is in line with some early intuitive arguments given in \[2, 19, 20\]. In general, however, such a decomposition does not hold.

In the weak-coupling regime, the degrees of freedom of pNRQCD are quark-antiquark colour-singlet fields, quark-antiquark colour-octet fields, gluons and light quarks. The obtained result for the Polyakov-loop correlator is consistent with its spectral decomposition. In the strong-coupling regime, the degrees of freedom are expected to change when the typical energy of the bound state is smaller than the confinement scale \( \Lambda_{\text{QCD}} \). In that situation, the bound state would become sensitive to confinement and give rise to a new spectrum of gluonic excitations (hybrids, glueballs). In the present work, we have not discussed this situation, which surely deserves investigation.

Possible further extensions of this work also include the study of the Polyakov-loop correlator in different scale hierarchies, in particular at temperatures of the same order as or higher than \( 1/r \), where the present analysis should smoothly go over the ones performed in \[20, 47\]. As mentioned above, also analyses that involve the strong-coupling scale should be addressed.

Finally, the present study should be completed by the study of correlators different from the Polyakov-loop one. Among these, the most studied in lattice gauge theories are the untraced Polyakov-loop correlator and the cyclic Wilson loop. Also the octet Wilson loop should be included for its role in the Polyakov-loop correlator. Since some partial perturbative results are already available for some of these correlators, it would be interesting to see
how they can be reproduced in the EFT framework introduced here and how they combine
to give back the Polyakov-loop correlator.

Acknowledgments

We thank Mikko Laine for correspondence and the authors of Ref. [31] for acknowledging
some of the results presented here prior to publication. N.B., J.G. and A.V. thank Owe
Philipsen for discussions. A part of this work was done at the Kavli Institute for Theoretical
Physics China (KITPC), CAS, Beijing. P.P. and J.G. thank the KITPC for hospitality and
support. Part of this work was also carried out during the CATHIE-INT mini program
“Quarkonia in hot matter: from QCD to experiment” held at the Institute for Nuclear
Theory (INT). N.B., P.P. and A.V. thank the Institute for Nuclear Theory at the University
of Washington for its hospitality and the Department of Energy for partial support. J.G.
thanks for hospitality Brookhaven National Laboratory, where this work was started. The
work of P.P. was supported by U.S. Department of Energy under Contract No. DE-AC02-
98CH10886. N.B., J.G. and A.V. acknowledge financial support from the RTN Flavianet
MRTN-CT-2006-035482 (EU) and from the DFG cluster of excellence “Origin and structure
of the universe” (www.universe-cluster.de).

Appendix A: Feynman rules in the static gauge

In the following, we list the Feynman rules in Euclidean space-time under the gauge
condition \( \partial_0 A^0 = 0 \). The temporal propagator reads (dropping colour indices)

\[
D_{00}(\omega_n, k) = \frac{\delta_{n0}}{k^2},
\]

(A1)

where, as usual, \( \omega_n = 2\pi n T \) and the Kronecker delta fixes \( n = 0 \), making this propagator
purely static. The spatial propagator can be divided into a non-static \( (n \neq 0) \) and a static
\( (n = 0) \) part. The former reads

\[
D_{ij}(\omega_n \neq 0, k) = \frac{1}{\omega_n^2 + k^2} \left( \delta_{ij} + \frac{k_i k_j}{\omega_n^2} \right) (1 - \delta_{n0}),
\]

(A2)

and thus mixes longitudinal and transverse components. The static part has a residual gauge
dependence on the parameter \( \xi \); it reads

\[
D_{ij}(\omega_n = 0, k) = \frac{1}{k^2} \left( \delta_{ij} - (1 - \xi) \frac{k_i k_j}{k^2} \right) \delta_{n0}.
\]

(A3)
Finally the ghost propagator reads

$$D_{\text{ghost}}(\omega_n, k) = \cdots \cdots = \frac{\delta_{n0}}{k^2},$$  \hspace{1cm} (A4)$$

and is thus purely static\(^{14}\). The interaction vertices (gluon-gluon and gluon-ghost) are the usual ones.

**Appendix B: The gluon self energy in the static gauge**

We proceed to the computation of the Matsubara sums in Eq. (7) in order to obtain Eqs. (10), (11), (12) and (13). We recall the two basic bosonic Matsubara sums \cite{29}

\[
T \sum_{n=-\infty}^{+\infty} \frac{1}{p^2 + \omega_n^2} = \frac{1 + 2n_B(|p|)}{2|p|},
\]

\[
T \sum_{n=-\infty}^{+\infty} \frac{1}{(p^2 + \omega_n^2)(q^2 + \omega_n^2)} = \frac{1}{2|p||q|} \left( \frac{1 + n_B(|p|) + n_B(|q|)}{|p| + |q|} + \frac{n_B(|q|) - n_B(|p|)}{|p| - |q|} \right),
\]

where \(n_B\) is the Bose–Einstein distribution. Since the sums include also the zero mode, in evaluating the master sum integrals defined in Eqs. (7) and (8) we will have to subtract it. Furthermore, we identify the temperature-independent part (the unity) in the numerators on the r.h.s of Eqs. (B1) and (B2) as the vacuum part and the part proportional to the thermal distributions as the matter part.

For \(I_0\), we have

\[
I_0 = \int' \frac{1}{p^2} = \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \left( \frac{1 + 2n_B(|p|)}{2|p|} - \frac{T}{p^2} \right) = \frac{T^2}{12};
\]

the subtracted zero mode along with the vacuum part vanish in dimensional regularization.

For \(I_1\), we have \((q = k - p)\)

\[
I_1 = \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \left[ \frac{|p|}{2|q|} \left( \frac{1 + n_B(|p|) + n_B(|q|)}{|p| + |q|} + \frac{n_B(|q|) - n_B(|p|)}{|p| - |q|} \right) - \frac{T}{q^2} \right]
\]

\[
= \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \left[ \frac{p^2}{2|p||q||(|p| + |q|)} + \frac{|p|n_B(|p|)}{2|q|} \left( \frac{-2|q|}{p^2 - q^2} \right) + \frac{|q'|n_B(|p|)}{2|p|} \left( \frac{-2|q'|}{p^2 - q'^2} \right) - \frac{T}{q^2} \right],
\]

\(^{14}\) The non-static ghost can be shown to decouple \cite{22}.\}
where we have operated a shift \( \mathbf{p} \to \mathbf{q}' = \mathbf{p} + \mathbf{k}, \mathbf{q} \to -\mathbf{p} \) in some terms of the matter part. The vacuum part can be brought into a more standard form by noting that
\[
\int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \frac{1}{(p^2 + p_0^2)(q^2 + p_0^2)} = \frac{1}{2|\mathbf{p}||\mathbf{q}|(|\mathbf{p}| + |\mathbf{q}|)}.
\]
(B4)

This allows to write the three-dimensional integral as a standard Euclidean four-dimensional integral, which can be computed with the formulas listed in appendix B1, setting \( d + 1 = 4 - 2\epsilon \). We thus have
\[
(I_1\,)_\text{vac} = \mu^{2\epsilon} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{p^\mu p^{\nu} (\delta_{\mu\nu} - \delta_{\mu0}\delta_{\nu0})}{p^2 q^2} = (\delta_{\mu\nu} - \delta_{\mu0}\delta_{\nu0})\mu^{2\epsilon} L_{d+1}^{\mu\nu}(k, 1, 1)|_{k^0=0}.
\]
(B5)

The zero-mode integral vanishes in dimensional regularization, whereas the remaining matter part is finite and gives
\[
(I_1\,)_\text{mat} = \frac{1}{2\pi^2} \int_0^{\infty} d|\mathbf{p}| |\mathbf{p}| n_B(|\mathbf{p}|) \left( 1 + \frac{|\mathbf{p}|}{2|\mathbf{k}|} \ln \frac{|\mathbf{k}| + |\mathbf{p}|}{|\mathbf{k}| - |\mathbf{p}|} \right).
\]
(B6)

Analogously, we have for \( I_2 \)
\[
I_2 = k^2 \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \left[ \frac{1}{2|\mathbf{p}||\mathbf{q}|} \left( \frac{1 + n_B(|\mathbf{p}|) + n_B(|\mathbf{q}|)}{|\mathbf{p}| + |\mathbf{q}|} + \frac{n_B(|\mathbf{q}|) - n_B(|\mathbf{p}|)}{|\mathbf{p}| - |\mathbf{q}|} \right) - \frac{T}{\mathbf{p}^2 \mathbf{q}^2} \right].
\]

The vacuum part is
\[
(I_2\,)_\text{vac} = \mu^{2\epsilon} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{k^2}{p^2 q^2} = k^2 \mu^{2\epsilon} L_{d+1}(k, 1, 1)|_{k^0=0},
\]
(B7)

the matter part is
\[
(I_2\,)_\text{mat} = \frac{1}{2\pi^2} \left( \int_0^{\infty} d|\mathbf{p}| n_B(|\mathbf{p}|) \frac{|\mathbf{k}|}{2|\mathbf{p}|} \ln \frac{|\mathbf{p}| + |\mathbf{k}|}{|\mathbf{p}| - |\mathbf{k}|} \right),
\]
(B8)

and the subtracted zero-mode part is
\[
(I_2\,)_\text{zero} = -\mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{T k^2}{\mathbf{p}^2 \mathbf{q}^2},
\]
(B9)

which has been kept in dimensional regularization.

We consider now \( I_3 \):
\[
I_3 = k^2 \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \left[ \frac{1 + 2n_B(|\mathbf{p}|)}{2|\mathbf{p}|^3} - \frac{T}{\mathbf{p}^4} \right],
\]
(B10)
\[
(I_3\,)_\text{vac} = k^2 \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{1}{2|\mathbf{p}|^3} = 0,
\]
(B11)
\[
(I_3\,)_\text{mat} = \frac{1}{2\pi^2} \int_0^{\infty} d|\mathbf{p}| |\mathbf{p}| n_B(|\mathbf{p}|) \frac{k^2}{\mathbf{p}^2}.
\]
(B12)
In dimensional regularization the subtracted zero mode vanishes. The matter part is infrared divergent. Since this divergence will cancel against terms from $I_4$ in the sum (6), we present the result directly in the three-dimensional limit.

$I_4$ is given by

$$I_4 = I_4^a - I_4^b - I_4^c = \int_p \frac{k^4}{p^2 q^2 \omega_n^2} - \int_p \frac{k^4}{p^2 q^2 p^2} - \int_{p'} \frac{k^4}{p^2 q^2 q^2}. \quad (B13)$$

$I_4^a$ is

$$I_4^a = \frac{2T}{(2\pi)^2} \frac{\k^4}{8|k|^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{|k|^3}{96T}, \quad (B14)$$

which is a term peculiar to this gauge; it is singular in the $T \to 0$ limit and constitutes $\Pi_0^N(k)_{\text{sing}}$. $I_4^b$ is

$$I_4^b = k^4 \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \left[ \frac{1}{2|p^3|q^3} \left( \frac{1}{|p| + |q|} \right) + \frac{1}{2|p^3|q^3} \left( \frac{1}{|p| - |q|} \right) \right] \frac{1}{p^4 q^2}. \quad (B15)$$

The vacuum part can be brought into a more familiar form by adding and subtracting $1/(2|p^3|q^2)$

$$(I_4^b)^{\text{vac}} = k^4 \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \left[ \frac{1}{2|p^3|q^3} \left( \frac{1}{|p| + |q|} \right) - \frac{1}{2|p^3|q^3} \right] + k^4 \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{1}{2|p^3|q^3}$$

$$= -k^2 \frac{\ln 2}{2\pi^2} + \frac{k^4}{2|p^3|q^3} L_d(k, 3/2, 1). \quad (B16)$$

Although the matter part of $I_4$ is infrared divergent, its infrared divergence cancels against the matter part of $I_4$, i.e. Eq. (B12), in the sum (6). Hence, we may evaluate it directly in three dimensions. In contrast, we will keep regularized the subtracted zero modes. As discussed in the main text, these subtracted zero modes behave like $\epsilon|k|^{1-2\epsilon}$ and are going to contribute when evaluating the Fourier transform of $|k|^{1-2\epsilon}/|k|^4$ in the Polyakov-loop correlator calculation, like in Eq. (B5). Therefore, $(I_4^b)_{\text{mat}}$ and $(I_4^b)_{\text{zero}}$ read

$$(I_4^b)_{\text{mat}} = \frac{1}{2\pi^2} \int_0^\infty d|p| |p| n_B(|p|) \frac{|k|^3}{2|p|^3} \left[ \ln \left| \frac{|p| + |q|}{|k|} \right| + \ln \left| \frac{|p| - |q|}{|k|} \right| \right], \quad (B17)$$

$$(I_4^b)_{\text{zero}} = -\mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{T k^4}{p^4 q^2}. \quad (B18)$$

Similarly $I_4^c$ reads

$$I_4^c = k^4 \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \left[ \frac{1}{2p^2 q^3} - \frac{T}{p^2 q^4} \right], \quad (B19)$$
\[ (I_3^b)_{\text{vac}} = \kappa^4 \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{1}{2|p|^2|q|^3} = \frac{k^4}{2} \mu^{2\epsilon} L_d(k, 1, 3/2), \]  

\[ (I_4^b)_{\text{mat}} = \frac{1}{2\pi^2} \int_0^\infty \frac{d|p|}{|p|} \text{ln}(|p|) \frac{|k|^3}{2|p|^3} \ln \left( \frac{|k| + |p|}{|k| - |p|} \right), \]  

\[ (I_4^b)_{\text{zero}} = -\mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} T k^4 \frac{p^2 q^4}{p^2 q^4}. \]  

Notice that, as we anticipated, the sum \((I_3^b)_{\text{mat}}/2 - (I_4^b)_{\text{mat}}/4 - (I_4^b)_{\text{mat}}/4\), which is the combination appearing in \(\Pi_{00}^{\text{NS}}(k)\), is infrared finite. It is also worthwhile noticing that the vacuum parts \((I_3^b)_{\text{vac}}\) and \((I_4^b)_{\text{vac}}\) are infrared divergent, but that in the sum \((I_3^b)_{\text{vac}}/2 - (I_4^b)_{\text{vac}}/4 - (I_4^b)_{\text{vac}}/4\), these infrared divergences are canceled and replaced by an ultraviolet divergence eventually removed by renormalization. The canceling infrared divergence and the remaining ultraviolet one come from \((I_3)_{\text{vac}}\), which vanishes, like in Eq. (B11), if the two are set equal, as usually done in dimensional regularization.

Putting all pieces together in Eq. (6) and using

\[ \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{k^4}{p^2 q^2} = |k|^{1-2\epsilon} \mu^{2\epsilon} (4\pi)^{-3/2+\epsilon} \frac{\Gamma(3/2 + \epsilon) \Gamma(1/2 - \epsilon) \Gamma(-1/2 - \epsilon)}{\Gamma(-2\epsilon)} \]

\[ = \epsilon \frac{|k|^{1-2\epsilon} \mu^{2\epsilon}}{4} [1 + \mathcal{O}(\epsilon)], \]

\[ \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{k^2}{p^2 q^2} = |k|^{1-2\epsilon} \mu^{2\epsilon} (4\pi)^{-3/2+\epsilon} \frac{\Gamma(1/2 + \epsilon) \Gamma(1/2 - \epsilon)^2}{\Gamma(1-2\epsilon)} \]

\[ = \frac{|k|^{1-2\epsilon} \mu^{2\epsilon}}{8} \left[ 1 + \epsilon(-\gamma_E + \ln(16\pi)) + \mathcal{O}(\epsilon^2) \right], \]

we obtain Eqs. (11), (11), (12) and (13).

1. One-loop integrals

We list here the loop integrals \(L_d, L_d^\mu\) and \(L_d^{\mu\nu}\), obtained with the Gegenbauer polynomials technique [48]:

\[ L_d(k, r, s) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p + k)^2 r^{2s}} \]

\[ = \frac{k^{d-2(r+s)}}{(4\pi)^{d/2}} \frac{\Gamma(r + s - d/2)}{\Gamma(r) \Gamma(s)} \frac{\Gamma(d/2 - s) \Gamma(d/2 - r)}{\Gamma(d - s - r)}. \]  

\[ L_d^\mu(k, r, s) = \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu}{(p + k)^2 r^{2s}} \]

\[ = -k^\mu \frac{k^{d-2(r+s)}}{(4\pi)^{d/2}} \frac{\Gamma(r + s - d/2)}{\Gamma(r) \Gamma(s)} \frac{\Gamma(d/2 + 1 - s) \Gamma(d/2 - r)}{\Gamma(d + 1 - s - r)}, \]
\[ L_d^{\mu\nu}(k, r, s) = \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{(p + k)^{2r} p^{2s}} \]
\[ = k^{d-2(r+s)} \left[ \frac{k^2 \Gamma(r + s - 1 + d/2)}{(4\pi)^{d/2}} \left( \frac{\Gamma(d/2 + 1 - s) \Gamma(d/2 + 1 - r)}{\Gamma(r) \Gamma(s)} \delta^{\mu\nu} \right. \right. \]
\[ + \left. \left. \left( \frac{\Gamma(r - s - d/2) \Gamma(d/2 + 2 - s) \Gamma(d/2 - r)}{\Gamma(r) \Gamma(s) \Gamma(d + 2 - s - r)} \right) k^{\mu} k^{\nu} \right\} \right]. \quad (B25) \]

**Appendix C: Expansions**

In this appendix, we list the expansions of the gluon self energy for temperatures much greater or smaller than the momentum \( k \).

We start with \( T \gg |k| \). In the non-static sector, \( I_0 \) gives its exact result (B3) and \( I_3 \) reads in dimensional regularization
\[ I_3 = -\frac{2T k^2 \Gamma(1 - d/2)(2\pi T)^{d-4} \mu^{2\epsilon}}{(4\pi)^{d/2}} \zeta(4 - d). \] (C1)

For the other integrals, we first carry out the integral, then Taylor expand the result in \( k^2/\omega_n^2 \) and finally perform the sums with the zeta function, thus obtaining
\[ I_1 = \frac{T^2}{12} - \frac{\Gamma(2 - d/2) \mu^{2\epsilon} (\sqrt{\pi T})^d}{2\pi^2 T} \sum_{l=0}^{\infty} \frac{\Gamma(d/2 - 1) \Gamma(l + 1)}{\Gamma(d/2 - 1 - l) \Gamma(2l + 2)} \zeta(2l + 2 - d) \left( \frac{k}{2\pi T} \right)^{2l}, \] (C2)
\[ I_2 = \frac{k^2 \Gamma(2 - d/2) \mu^{2\epsilon} (\sqrt{\pi T})^d}{8\pi^4 T^3} \sum_{l=0}^{\infty} \frac{\Gamma(d/2 - 1) \Gamma(l + 1)}{\Gamma(d/2 - 1 - l) \Gamma(2l + 2)} \zeta(2l + 4 - d) \left( \frac{k}{2\pi T} \right)^{2l}, \] (C3)
\[ I_4 = \frac{k^4 \Gamma(2 - d/2) \mu^{2\epsilon} (\sqrt{\pi T})^d}{32\pi^6 T^5} \sum_{l=0}^{\infty} \frac{\Gamma(d/2 - 1) \Gamma(l + 1)}{\Gamma(d/2 - 1 - l) \Gamma(2l + 2)} \zeta(2l + 6 - d) \left( \frac{k}{2\pi T} \right)^{2l}. \] (C4)

In the fermionic, sector we have
\[ \tilde{I}_0 = -\frac{T^2}{24}, \] (C5)
and we can derive the expansions for \( \tilde{I}_1 \) and \( \tilde{I}_2 \) following the same procedure used for the bosonic integrals, but ending up with the Hurwitz zeta function as a result of the odd frequency sums. Thus we have
\[ \tilde{I}_1 = \frac{\Gamma(2 - d/2) \mu^{2\epsilon} (\sqrt{\pi T})^d}{2\pi^2 T} \sum_{l=0}^{\infty} \frac{\Gamma(d/2 - 1) \Gamma(l + 1)}{\Gamma(d/2 - 1 - l) \Gamma(2l + 2)} \zeta(2l + 2 - d, 1/2) \left( \frac{k}{2\pi T} \right)^{2l}, \] (C6)
\[ \tilde{I}_2 = \frac{k^2 \Gamma(2 - d/2) \mu^{2\epsilon} (\sqrt{\pi T})^d}{8\pi^4 T^3} \sum_{l=0}^{\infty} \frac{\Gamma(d/2 - 1) \Gamma(l + 1)}{\Gamma(d/2 - 1 - l) \Gamma(2l + 2)} \zeta(2l + 4 - d, 1/2) \left( \frac{k}{2\pi T} \right)^{2l}. \]
Plugging these expressions in Eqs. (6) and (14) we obtain the high-temperature expansion (19).

We consider now the low-temperature expansion. The vacuum part gives the order $k^2$ term in the expansion, whereas, for the matter part, the condition $|k| \gg T$ translates in Eq. (11) into $|k| \gg |p|$, since the internal momentum $|p|$ is of order $T$. Expanding this expression in $|p|/|k| \ll 1$ yields

$$\Pi_{00}^{NS}(|k| \gg T)_{\text{mat}} = -g^2 C_A T^2 \frac{T^2}{18} + g^2 T^2 O \left( \frac{T^2}{k^2} \right). \quad (C8)$$

The singular term ($\propto |k|^3/T$) and the subtracted zero-mode part also contribute in this region. The sum of Eq. (C8) with the vacuum, subtracted zero-mode and singular parts yields Eq. (21). For what concerns the static modes, the only scales are $|k|$ and $m_D$, thus the condition $|k| \gg T$ becomes $|k| \gg m_D$ and we end up with Eq. (25). Finally, the fermionic contribution is suppressed in this region, i.e. the first nonzero term in the expansion of Eq. (16) is of order $g^2 T^4/k^2$.

### Appendix D: Non-static two-loop sum-integrals

We set on the evaluation of the two-loop sum-integrals defined by Eq. (38). $J_0$ does not contribute in dimensional regularization because the integral over $k$ has no scale. $J_1$ can be rewritten as

$$J_1 = \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \int_p \frac{P^2}{k^4 p^2 q^2} = \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4} \int_p \frac{1}{q^2} - \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \int_p \frac{\omega_n^2}{k^4 p^2 q^2}. \quad (D1)$$

The first term vanishes in dimensional regularization, whereas the second one yields

$$J_1 = -\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \int_p \frac{\omega_n^2}{k^4 p^2 q^2} = -\frac{T}{8(4\pi)^2}. \quad (D2)$$

$J_2$ can be read from [49],

$$J_2 = \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \int_p \frac{1}{k^2 p^2 q^2} = \frac{T}{(4\pi)^2} \left( -\frac{1}{4\epsilon} + \ln \frac{2T}{\mu} - \frac{1}{2} + \frac{\gamma_E}{2} - \frac{\ln(4\pi)}{2} \right). \quad (D3)$$

15 A convenient way to proceed is by performing first the momentum integrations, by means of two Feynman parameters, and then the frequencies sum, which gives $\zeta(0) = -1/2$. 

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3 vanishes in dimensional regularization because the $k$ integral has no scale and finally $J_4$ yields

$$J_4 = \mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \int \frac{T}{p^2 q^2 \omega_n^2} = T \sum_{n \neq 0} \frac{1}{\omega_n^2} \left( \frac{\omega_n}{4\pi} \right)^2 = -\frac{T}{(4\pi)^2}. \quad (D4)$$

We consider now the fermionic integrals. $\tilde{J}_0$ vanishes because it has a scaleless $k$ integration, whereas $\tilde{J}_1$ can be computed along the lines of its bosonic counterpart, performing the sum over odd frequencies by means of the the generalized (Hurwitz) zeta function,

$$\tilde{J}_1 = \mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} T \sum_n \mu^{2\epsilon} \int \frac{dp}{(2\pi)^d} \tilde{\omega}_n^2 = -\frac{T\zeta(0,1/2)}{4(4\pi)^2} = 0. \quad (D5)$$

$\tilde{J}_2$ can be read from [50],

$$\tilde{J}_2 = \mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} T \sum_n \mu^{2\epsilon} \int \frac{dp}{(2\pi)^d} \frac{1}{k^2 p^2 q^2} = -\frac{T}{(4\pi)^2} \ln 2. \quad (D6)$$

### Appendix E: Static-modes contribution to the Polyakov loop

In this appendix, we evaluate the 6-dimensional two-loop integral entering Eq. (43). We will perform the calculation modifying the magnetostatic propagator in Eq. (3) into

$$\frac{1}{k^2} \left( \delta_{ij} - (1 - \xi) \frac{k_i k_j}{k^2} \right) \delta_{n0} \rightarrow \frac{1}{k^2 + m_m^2} \left( \delta_{ij} - (1 - \xi) \frac{k_i k_j}{k^2} \right) \delta_{n0}, \quad (E1)$$

where $m_m$ may be interpreted as a small magnetic mass to be put to zero at the end of the calculation. The magnetic mass modifies the static gluon self-energy expression with resummed gluon propagators from Eq. (23) to

$$\Pi^S_{00}(k) = g^2 C_A T \mu^{2\epsilon} \int \frac{d^dp}{(2\pi)^d} \left[ d - (1 - \xi) \frac{(k + q)^2 - (1 - \xi) (k + q) \cdot p}{p^2 + m_m^2} \frac{p^2}{(p^2 + m_m^2)(q^2 + m_D^2)} \right], \quad (E2)$$

where $q = k - p$.\textsuperscript{16}

In Eq. (43), the integral over the first term in Eq. (E2), i.e. the tadpole contribution, gives

$$-\frac{d - (1 - \xi)}{4\pi} \frac{g^4 C_RC_A m_m}{2} \mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + m_D^2)^2} = -\frac{d - (1 - \xi)}{4(4\pi)^2} \frac{g^4 C_RC_A m_m}{m_D}. \quad (E3)$$

\textsuperscript{16} We use here a different parameterization of the integrand with respect to Eq. (24).
For the second term, we start by considering the term proportional to \((k + q)^2\). We rewrite

\[(k + q)^2 = 2(k^2 + m_D^2) + 2(q^2 + m_D^2) - (p^2 + m_m^2) + (m_m^2 - 4m_D^2), \tag{E4}\]

and consider the contributions given by each of the four terms in brackets. The first one gives

\[
2\mu^4 \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m_D^2)(p^2 + m_m^2)} = \frac{2}{(4\pi)^2} \left[ \frac{1}{4\epsilon} + \ln \frac{\mu}{2m_D + m_m} + \frac{1}{2} - \frac{\gamma_E}{2} + \frac{\ln(4\pi)}{2} + \mathcal{O}(\epsilon) \right], \tag{E5}\]

the second one gives

\[
2\mu^4 \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m_D^2)^2(p^2 + m_m^2)} = -\frac{1}{(4\pi)^2 m_D}, \tag{E6}\]

the third one gives

\[-\mu^4 \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m_D^2)^2(q^2 + m_D^2)} = \frac{1}{2(4\pi)^2}, \tag{E7}\]

and the last one

\[
\frac{(m_m^2 - 4m_D^2)\mu^2}{(2m_D^2)(2m_D + m_m)} = \frac{m_m^2 - 4m_D^2}{-2m} \left. \frac{\partial}{\partial m} \mu^2 \frac{1}{(2\pi)^d} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(k^2 + m_m^2)^2(p^2 + m_m^2)(q^2 + m_D^2)} \right|_{m = m_D} = \frac{1}{(4\pi)^2 \mu^2}.
\]

Finally, we consider the term proportional to \(\frac{(k + q) \cdot p}{p^2}\) in Eq. (E2). We rewrite the numerator as

\[
(1 - \xi) \frac{(k + q) \cdot p}{p^2} = \frac{1 - \xi}{p^2} [(k^2 + m_D^2)^2 + (q^2 + m_D^2)^2 - 2(k^2 + m_D^2)(q^2 + m_D^2)]. \tag{E8}\]

The first term gives

\[
\mu^4 \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{p^2(p^2 + m_m^2)(q^2 + m_D^2)} = -\frac{1}{(4\pi)^2 m_m}, \tag{E9}\]

the third term is \(-2\) times this one and the second term gives

\[
\mu^2 \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2(p^2 + m_m^2)} \mu^2 \int \frac{d^d k}{(2\pi)^d} \frac{k^2 + p^2 - 2p \cdot k + m_D^2}{(k^2 + m_m^2)^2} = \mu^2 \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2(p^2 + m_m^2)} \left[ \frac{p^2}{8\pi m_D} - \frac{m_D}{4\pi} \right] = \frac{1}{(4\pi)^2} \left[ \frac{m_D}{m_m} + \frac{m_m}{2m_D} \right].
\]
The static contribution is thus
\[
\delta(L_R)_{S_{mD}} = \frac{g^4 C_A C_R}{2(4\pi^2)} \left[ -\frac{1}{2\epsilon} - \ln \left( \frac{\mu^2}{2m_D + m_m^2} \right) + \gamma_E - \ln(4\pi) + \frac{2 - d m_m}{2 m_D} \right. \\
- \left. \frac{3}{2} - \frac{m_m^2 - 4m_D^2}{2m_D(2m_D + m_m)} \right].
\] (E10)

The final result is independent of the gauge parameter \(\xi\). The expression is well behaved for \(m_m \to 0\) and yields Eq. (44).

Appendix F: The Polyakov loop in Feynman gauge

In this section, we sketch the computation of the vacuum expectation value of the Polyakov loop in Feynman gauge. We restrict ourselves to the fundamental representation \((L \equiv L_F)\). Since the fermionic contribution, evaluated in Sec. III, is to that order gauge-invariant, we do not need to compute it here again.

The perturbative expansion of the Polyakov line through the Baker–Campbell–Hausdorff formula is, following [24] and up to order \(g^4\),
\[
\langle \tilde{\text{Tr}} L \rangle = \frac{1}{N} \left\langle \text{Tr} \exp \left( i g \int_0^{1/T} d\tau A^0(\tau, x) \right) \right\rangle = \frac{1}{N} \left\langle \text{Tr} \left( 1 + \frac{g^2}{2} (H_2^2 + g^2 H_1^2) + 2g H_0 H_1 + 2g^2 H_0 H_2 + \frac{1}{3!} g^3 (H_0^3 + 3g H_0^2 H_1) + \frac{1}{4!} g^4 H_0^4 \right) \right\rangle + \ldots, \quad (F1)
\]
where
\[
H_0 = i \int_0^{1/T} d\tau A^0(\tau),
\]
\[
H_1 = -\frac{1}{2} \int_0^{1/T} d\tau_1 \int_0^{\tau_1} d\tau_2 \left[ A^0(\tau_2), A^0(\tau_1) \right],
\]
\[
H_2 = -\frac{1}{6} \left[ H_0, H_1 \right] - \frac{i}{6} \int_0^{1/T} d\tau_1 \int_0^{\tau_1} d\tau_2 \left[ A^0(\tau_2), \left[ A^0(\tau_2), A^0(\tau_1) \right] \right] \]
\[-\frac{i}{3} \int_0^{1/T} d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \left[ A^0(\tau_3), \left[ A^0(\tau_2), A^0(\tau_1) \right] \right], \quad (F2)
\]
and \(A^0(\tau, x) \equiv A^0(\tau)\). We recall that
\[
D_{00}(\tau) \equiv \theta(\tau) A_0(\tau) A_0(0) + \theta(-\tau) A_0(0) A_0(\tau) = T \sum_n e^{i\omega_n \tau} \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} D_{00}(\omega_n, k),
\]
where in Feynman gauge the free temporal-gluon propagator is
\[
D_{00}^{(0)}(\omega_n, k) = \frac{1}{\omega_n^2 + k^2}. \quad (F3)
\]
FIG. 8: Diagrams contributing to the Polyakov loop up to order $g^4$ in Feynman gauge. The blob stands for the one-loop gluon self energy, the solid line for the Polyakov line and the dots at its beginning/end represent the points $(0, x)$ and $(1/T, x)$, which are compactified by the periodic boundary conditions. When integrating over loop momenta of order $m_D$, the dashed lines stand for resummed temporal propagators, elsewhere for free ones.

We can now start working on the different terms in Eq. (F1). The first one gives

$$\frac{1}{N} \langle \text{Tr} \frac{g^2}{2} H_0^2 \rangle = -\frac{1}{2} g^2 C_F \mu^{2\epsilon} \int \frac{d^4k}{(2\pi)^d} D_{00}(0,k).$$  \hspace{1cm} (F4)$$

Following the same approach as in Sec. III, at order $g^4$, the relevant diagrams contributing to (F4) are shown in Fig. 8a) and b). At leading order, the Debye mass is gauge invariant, whereas the full one-loop gluon self-energy is not. It is convenient to separate non-static from static modes. The former yield [49]

$$\Pi_{00}^{\text{NS}}(0,k) = -2g^2 C_A \left( \frac{d-1}{2} I_0 - (d-1) I_1 + I_2 \right),$$  \hspace{1cm} (F5)$$

where the master integrals $I_j$ are those defined in Eq. (7), hence Eq. (F5) equals the first three terms of the static-gauge expression (3). The static mode contribution to the self energy is common to all gauges that share the same static propagator as the static gauge and the Feynman gauge do. Therefore, the static part of the self energy in Feynman gauge is just Eq. (23) with $\xi = 1$. We then have, separating the contributions coming from the scale $T$ from those coming from the scale $m_D$,

$$\mu^{2\epsilon} \int \frac{d^4k}{(2\pi)^d} D_{00}(0,k) = \mu^{2\epsilon} \int \frac{d^4k}{(2\pi)^d} \left[ \frac{1}{k^2 + m_D^2} - \frac{\Pi_{00}^{\text{NS}}(\vert k \vert \sim T)}{k^4} - \frac{\Pi_{00}^{\text{NS}}(\vert k \vert)}{(k^2 + m_D^2)^2} \right] + \ldots,$$  \hspace{1cm} (F6)$$

where the dots stand for higher orders in the perturbative expansion. We have omitted the non-static contribution at the scale $m_D$ (cf. Eq. (40)) since it can be shown that also in
Feynman gauge $\Pi_{00}^{NS}(|k| \sim m_D) - m_D^2 = \mathcal{O}(g^2k^2)$, leading to a higher-order contribution, whereas the contribution of the static modes at the scale $T$ leads to a scaleless integral. Plugging Eq. (F6) into Eq. (F4) and using the results of appendices D and E we obtain most of the final, order $g^4$, result, except for the contribution of $J_4$ in Eq. (37).

We then consider the other terms in the Baker–Campbell–Hausdorff expansion, starting from $H_2^2$:

$$
\frac{1}{N} \left< \text{Tr} \frac{g^4}{2} H_1^2 \right> = \frac{C_F C_A}{8} g^4 \int_0^{1/T} d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{1/T} d\tau_3 \int_0^{\tau_3} d\tau_4 \left[ D_{00}(\tau_2 - \tau_3) D_{00}(\tau_1 - \tau_4) - D_{00}(\tau_2 - \tau_4) D_{00}(\tau_1 - \tau_3) \right]
$$

where we have used free propagators and the dots stand for higher orders. This result corresponds exactly to the contribution of $J_4$ in the static gauge. The contribution can be traced back to diagram c) in Fig. 8 and it corresponds to the term called $L_4$ in Eq. (4) of [21].

We now need to show that the sum of the remaining terms yields zero at order $g^4$. $\langle \text{Tr} 2gH_0H_1 \rangle$ vanishes because it involves a three temporal-gluon vertex. $\langle \text{Tr} 2g^2H_0H_2 \rangle$ is a more complicated object, however one can show that, working with free propagators [24],

$$
\frac{1}{N} \langle \text{Tr} g^4H_0H_2 \rangle = 0 + \mathcal{O}(g^5, g^4 \times (m_D/T)).
$$

The $H_0^3$ term vanishes, again due to the three temporal-gluon vertex and the $H_0^2H_1$ term can be easily shown to be zero after performing the colour trace. The $H_0^4$ term gives

$$
\frac{1}{4!N} \langle \text{Tr} g^4H_0^4 \rangle = \frac{g^4}{4!} \left( 3C_F^2 - \frac{C_F C_A}{2} \right) \frac{1}{T^2} \left( \mu^2 \int \frac{d^d k}{(2\pi)^d} D_{00}(0,k) \right)^2,
$$

which is at least of order $g^4 \times (m_D/T)^2$. This, finally, shows that the Feynman-gauge computation of the Polyakov loop agrees with the static-gauge computation that led to Eq. (45).

**Appendix G: Octet contributions**

In this appendix, we want to prove that, up to order $g^6(rT)^0$, $\delta_{O_T}^{\mathcal{O}(r^2)^{NS}} = \delta_{s_T}^{\mathcal{O}(r^2)^{NS}}|_{V_s \leftrightarrow V_o}$, $\delta_{O_T}^{(r^2)^{S}} = -\delta_{s_T}^{(r^2)^{S}}$, and $\delta_{o_T}^{\mathcal{L}_{\text{NRQCD}}} = -\delta_{s_T}^{\mathcal{L}_{\text{NRQCD}}}$, where the left- and right-hand sides of the equalities encode non-zero modes, zero-modes and higher-multipole one-loop corrections
to the pNRQCD octet and singlet propagators respectively induced by interaction vertices of the type $S^+ r^{i_1} ... r^{i_n} \partial_{i_1} ... \partial_{i_{n-1}} E^{i_n} O +$ Hermitian conjugate or $O^+ r^{i_1} ... r^{i_n} \partial_{i_1} ... \partial_{i_{n-1}} E^{i_n} O +$ charge conjugate.

![Feynman diagrams](attachment:image.png)

FIG. 9: The pNRQCD Feynman diagrams giving the leading-order correction to $\delta_o$. The single continuous line stands for a singlet propagator, the double line for an octet propagator, the circle with a cross for the chromoelectric dipole vertex proportional to $V_A$ in the Lagrangian (74), the square with a cross for the chromoelectric dipole vertex proportional to $V_B$ in the Lagrangian (74), the circle with a dot for the chromoelectric dipole vertex proportional to $V_C$ in the Lagrangian (74), the curly line for a chromoelectric correlator and the dashed line for a temporal-gluon propagator.

The general argument goes as follows. Let’s first consider contributions coming from the non-zero modes of the loop integral, Fig. 7 providing the leading-order contribution to the singlet propagator and diagram a) in Fig. 9 providing the leading-order contribution to the octet propagator. As the leading-order example shows, there is a one to one correspondence between diagrams in the singlet and in the octet channel, to each singlet diagram corresponds an octet diagram whose contribution is equal to the singlet diagram contribution with $V_s$ replaced by $V_o$ and viceversa. We note that, since at order $g^4$ these contributions are linear in $V_o - V_s$, they are at that order one the opposite of the other.

Let’s now consider contributions coming from the zero modes of the loop integral. In order to see how things work, we consider, first, the order $r^2$ contribution. In the singlet channel, only one diagram, Fig. 7 contributes; that contribution has been written in Eq. (87) and evaluated in Eq. (88). In the octet channel, four diagrams contribute, which are
shown in Fig. 9. Diagram a) gives the same contribution as the singlet channel:

\[
\delta_{o,T}^{a)S} = -g^2 \frac{1}{2N} \frac{r^i r^j}{2T} \int \frac{d^d k}{(2\pi)^d} \langle E^{i a} U_{ab} E^{j b} \rangle (0, k)|_{k \sim T} + \mathcal{O} \left( g^6 (r T) \right). \tag{G1}
\]

Diagram b) is like diagram a) with the colour factor \(1/(2N)\) replaced by \(d^{abc} f^{abc}/[4(N^2 - 1)]\):

\[
\delta_{o,T}^{b)S} = -g^2 \frac{N^2 - 4}{4N} \frac{r^i r^j}{2T} \int \frac{d^d k}{(2\pi)^d} \langle E^{i a} U_{ab} E^{j b} \rangle (0, k)|_{k \sim T} + \mathcal{O} \left( g^b (r T) \right). \tag{G2}
\]

Finally, diagrams c) and d) are like diagram a) with the colour factor \(1/(2N)\) replaced by \(f^{abc} f^{abc}/[8(N^2 - 1)]\):

\[
\delta_{o,T}^{c)+d)S} = g^2 \frac{N}{4} \frac{r^i r^j}{2T} \int \frac{d^d k}{(2\pi)^d} \langle E^{i a} U_{ab} E^{j b} \rangle (0, k)|_{k \sim T} + \mathcal{O} \left( g^6 (r T) \right) , \tag{G3}
\]

where the positive sign comes from moving a derivative acting on the chromoelectric field in one vertex to the temporal gluon in the other one (see also footnote 12). Summing Eqs. (G1)-(G3) we obtain the opposite of the singlet contribution in Eq. (87).

This argument may be easily generalized to any order in the multipole expansion. Let’s consider diagrams contributing to order \(2n\) in the multipole expansion. The singlet contribution is proportional to

\[
\delta_{s,T}^{O(r^{2n})} \propto r^{2n} \frac{1}{2N} \frac{1}{2} \sum_{\ell=0}^{n-1} \frac{1}{(2\ell + 1)!} \frac{1}{(2n - (2\ell + 1))!}. \tag{G4}
\]

Again there are three classes of octet contributions that correspond to the three classes discussed at order \(r^2\). Except for the first class, each one has a different colour factor with respect to the singlet contribution, but for the rest they are equal:

\[
\delta_{o,T}^{a)S} \propto -r^{2n} \frac{1}{2N} \frac{1}{2} \sum_{\ell=0}^{n-1} \frac{1}{(2\ell + 1)!} \frac{1}{(2n - (2\ell + 1))!}, \tag{G5}
\]

\[
\delta_{o,T}^{b)S} \propto -r^{2n} \frac{N^2 - 4}{4N} \frac{1}{2} \sum_{\ell=0}^{n-1} \frac{1}{(2\ell + 1)!} \frac{1}{(2n - (2\ell + 1))!}, \tag{G6}
\]

\[
\delta_{o,T}^{c)+d)S} \propto r^{2n} \frac{N}{4} \frac{1}{2} \sum_{\ell=0}^{n} \frac{1}{(2\ell)!} \frac{1}{(2n - 2\ell)!}. \tag{G7}
\]

where the positive sign in the last expression comes from moving an odd number of derivatives acting on the field in one vertex to the field in the other one. Since \(\sum_{\ell=0}^{n} \frac{1}{(2\ell)!} \frac{1}{(2n - 2\ell)!} = \)
\[
\sum_{\ell=0}^{n-1} \frac{1}{(2\ell + 1)!} \frac{1}{(2n - (2\ell + 1))!}, \text{ the sum of all octet contributions is just the opposite of the singlet contribution.}
\]
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