An embedding from the Ringel-Hall algebra to the 
Bridgeland’s Ringel-Hall algebra associated to an algebra 
with global dimension at most two * 

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Abstract. For any finitely dimensional associative algebra with global dimension ≤ 2, we show that there is an embedding from the twisted Ringel-Hall algebra to the Bridgeland’s Ringel-Hall algebra. In particular, this result is true for tilted algebras and canonical algebras.

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1 Introduction

For any finite-dimensional semi-simple complex Lie algebra, there is a famous theorem saying that its positive roots correspond bijectively to the isomorphism classes of all (finitely dimensional) indecomposable modules of the corresponding hereditary algebra, which was proved by Gabriel [Gab] for ADE type and then extended by Dlab-Ringel [DR] for any type. This result gave a realization of the positive root system of the semi-simple complex Lie algebra from the modules of the hereditary algebra.

To realize the multiplication of the semi-simple complex Lie algebra from the module category of the hereditary algebra, Ringel [Ri2] introduced a Hall algebra from any abelian category with finite morphism spaces, which generalizes Hall’s definition from p-groups [Ha]. He [Ri3] showed that the positive part of the semi-simple complex Lie algebra can be realized via the Hall algebra from the module category of the hereditary algebra. Moreover, he [Ri4] proved that the positive part of the quantized enveloping algebra of the semi-simple complex Lie algebra is isomorphic to the Hall algebra in some twisted form from the module category of the hereditary algebra. Later, Green [Gr] extended this result to any symmetrizable Kac-Moody Lie algebra case by showing that there is a natural co-multiplication in the Hall algebra of any hereditary algebra.

So a natural question is how one can use the representation theory of hereditary algebras to realize whole but not only the positive part of any symmetrizable Kac-Moody Lie algebra

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and its quantized enveloping algebra. There have been many attempts to approach such a question. One way was to consider the Drinfeld double of the Hall algebra of any hereditary algebra constructed by Xiao in [X], and in this way he realized the full quantized enveloping algebra. Second way was given by Peng-Xiao in [PX1] and [PX2] who used the similar method as Ringel’s to construct the Ringel-Hall Lie algebra from any root category, which is the orbit category of all shift square orbits in the derived category of a hereditary algebra. In this way they realized the full symmetrizable Kac-Moody Lie algebra. Third way was introduced by Toën in [T] who constructed the derived Ringel-Hall algebra from the derived category. Recently, Bridgeland [Br] gave a more clever way to consider directly the \( \mathbb{Z}_2 \)-graded complexes (or 2-cycle complexes as called in [PX1], or 2-periodic complexes as in some literature) of projective modules of an algebra with finite global dimension to construct the Ringel-Hall algebra. He showed that the full quantized enveloping algebra can be also realized by his Ringel-Hall algebra. Later Yanagida [Y] proved that in hereditary algebra case the Bridgeland’s Ringel-Hall algebra is isomorphic to the Drinfeld double of the Hall algebra. Inspired by work of Bridgeland, Grosky recently in [Gro] constructed the semi-derived Ringel-Hall algebras from complex categories or \( \mathbb{Z}_2 \)-graded complex categories. In each way above there also were some further researches, see [Ka], [SVdB1], [SVdB2], [PT], [Cr], [Hu], [LP], [Sch], [XX], and so on.

In this paper we only consider Bridgeland’s Ringel-Hall algebras. For any hereditary algebra, Bridgeland [Br] has proven that the twisted Ringel-Hall algebra can be naturally embedded in the Bridgeland’s Ringel-Hall algebra. We extend this result to the case of any algebra with with global dimension \( \leq 2 \). Some special interesting consequences should be for tilted algebras and canonical algebras. Such two kinds of algebras are very important and they both have global dimension \( \leq 2 \).

**Notation.** We fix a field \( k = \mathbb{F}_q \) with \( q \) elements, and set \( t = \sqrt{q} \). We write \( \text{Iso}(\mathcal{A}) \) for the set of isomorphism classes of a small category \( \mathcal{A} \). The symbol \( |S| \) denotes the number of elements of a finite set \( S \).

## 2 Preliminaries

In this section, we give basic definitions and properties of Ringel-Hall algebra, following [Ri4], [Br], and [Sch].

In this section, \( \mathcal{A} \) denotes an abelian category and satisfy the following conditions:

(a) \( \mathcal{A} \) is essentially small, with finite morphism spaces,

(b) \( \mathcal{A} \) is linear over \( k = \mathbb{F}_q \),

(c) \( \mathcal{A} \) is of finite global dimension and has enough projectives.

Given objects \( A, B, C \in \mathcal{A} \), define \( \text{Ext}^1_{\mathcal{A}}(A, C)_B \subset \text{Ext}^1_{\mathcal{A}}(A, C) \) to be the subset parameterising extensions with middle term isomorphic to \( B \).

**Definition 2.1** The Ringel-Hall algebra \( \mathcal{H}(\mathcal{A}) \) is the vector space over \( \mathbb{C} \) with basis indexed by elements \( A \in \text{Iso}(\mathcal{A}) \), and with associative multiplication defined by

\[
[A] \circ [C] = \sum_{B \in \text{Iso}(\mathcal{A})} \frac{\text{Ext}^1_{\mathcal{A}}(A, C)_B}{|\text{Hom}_{\mathcal{A}}(A, C)|}[B].
\]

The unit is \([0]\).
Let $K(A)$ denote the Grothendieck group of $A$. We write $\hat{A} \in K(A)$ for the class of an object $A \in A$. Let $K_{\geq 0}(A) \subset K(A)$ be the subset consisting of these classes. For objects $A, B \in A$, define

$$\langle \hat{A}, \hat{B} \rangle = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \text{Ext}^i_A(A, B).$$

The sum is finite by our assumptions on $A$, and descends to give a bilinear form

$$\langle -, - \rangle : K(A) \times K(A) \to \mathbb{Z}$$

known as the Euler form. We also consider the symmetrized form

$$(-,-) : K(A) \times K(A) \to \mathbb{Z},$$

defined by $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$.

**Definition 2.2** The twisted Ringel-Hall algebra $H_{tw}(A)$ is the same vector space as $H(A)$ but with twisted multiplication defined by

$$[A] * [C] = t^{\langle \hat{A}, \hat{C} \rangle} \cdot [A] \circ [C].$$

Let $C_2(A)$ be the abelian category of $\mathbb{Z}_2$-graded complexes in $A$. The objects of this category consist of diagrams

$$M_1 \xrightarrow{d_1} \xleftarrow{d_0} M_0,$$

with $d_{i+1} \circ d_i = 0$. Two morphisms $s_\bullet, t_\bullet : M_\bullet \rightarrow N_\bullet$ are said to be homotopic if there are morphisms $h_i : M_i \rightarrow N_{i+1}$ such that

$$t_i - s_i = d_{i+1} \circ h_i + h_{i+1} \circ d_i.$$

For an object $M_\bullet \in C_2(A)$, we define its class in the $K$-group by

$$\hat{M}_\bullet := \hat{M}_0 - \hat{M}_1 \in K(A).$$

Denote by $K_2(A)$ the homotopy category obtained from $C_2(A)$ by identifying homotopic morphisms. Let us also denote by $C_2(P) \subset C_2(A)$ the full subcategories whose objects are complexes of projectives in $A$. The shift functor [1] of complexes induces an involution

$C_2(A) \xrightarrow{\ast} C_2(A).$

This involution shifts the grading and changes the sign of the differential as follows:

$$M_\bullet = ( M_1 \xrightarrow{d_1} \xleftarrow{d_0} M_0 ) \xrightarrow{\ast} M_\bullet^* = ( M_0 \xrightarrow{-d_0} \xleftarrow{-d_1} M_1 ).$$
Lemma 2.6 (\([\mathcal{P}X1]\), \([\mathcal{B}r]\)) There is a fully faithful functor \(D : R_2(\mathcal{A}) \rightarrow \mathcal{K}_2(\mathcal{P})\) sending a \(\mathbb{Z}\)-graded complex of projectives \((P_i)_{i \in \mathbb{Z}}\) to the \(\mathbb{Z}_2\)-graded complex

\[
\bigoplus_{i \in \mathbb{Z}} P_{2i+1} \cong \bigoplus_{i \in \mathbb{Z}} P_{2i}.
\]

The functor \(D\) is an equivalent when \(\mathcal{A}\) is hereditary (see \([\mathcal{P}X1]\)).

Lemma 2.4 (\([\mathcal{B}r]\)) For \(M_* , N_* \in \mathcal{C}_2(\mathcal{P})\), we have

\[
\text{Ext}^1_{\mathcal{C}_2(\mathcal{A})}(N_*, M_*) \cong \text{Hom}_{\mathcal{K}_2(\mathcal{A})}(N_* , M_*^*).
\]

A complex \(M_* \in \mathcal{C}_2(\mathcal{A})\) is called acyclic if \(H_*(M_*) = 0\). To each object \(P \in \mathcal{P}\), we can attach acyclic complexes

\[
K_P = (P \xrightarrow{1} 0 \ P), \quad K_P^* = (P \xrightarrow{0 \ 1} P).
\]

Lemma 2.5 (\([\mathcal{B}r]\)) For each acyclic complex of projectives \(M_* \in \mathcal{C}_2(\mathcal{P})\), there are objects \(P, Q \in \mathcal{P}\), unique up to isomorphism, such that \(M_* \cong K_P \oplus K_Q^*\).

Let \(\mathcal{H}(\mathcal{C}_2(\mathcal{A}))\) be the Ringel-Hall algebra of the abelian category \(\mathcal{C}_2(\mathcal{A})\) defined in 2.1, \(\mathcal{H}(\mathcal{C}_2(\mathcal{P})) \subset \mathcal{H}(\mathcal{C}_2(\mathcal{A}))\) be the subspace spanned by complexes of projective objects. Define \(\mathcal{H}_{tw}(\mathcal{C}_2(\mathcal{P}))\) to be the same vector space as \(\mathcal{H}(\mathcal{C}_2(\mathcal{P}))\) with the twisted multiplication

\[
[M_*] * [N_*] := t^{(M_0,N_0)+(M_1,N_1)} \cdot [M_*] \circ [N_*],
\]

then \(\mathcal{H}_{tw}(\mathcal{C}_2(\mathcal{P}))\) is an associative algebra.

We have the following simple relations satisfied by the acyclic complexes \(K_P\),

Lemma 2.6 (\([\mathcal{B}r]\)) For any object \(P \in \mathcal{P}\) and any complex \(M_* \in \mathcal{C}_2(\mathcal{P})\), we have the following relations in \(\mathcal{H}_{tw}(\mathcal{C}_2(\mathcal{P}))\) :

\[
[K_P] * [M_*] = t^{(P,M_*)} \cdot [K_P \oplus M_*], \quad [M_*] * [K_P] = t^{- (M_*, P)} \cdot [K_P \oplus M_*] \quad (1)
\]

\[
[K_P] * [M_*] = t^{(P,M_*)} \cdot [M_*] * [K_P], \quad [K_P^*] * [M_*] = t^{- (P,M_*)} \cdot [M_*] * [K_P] \quad (2)
\]

In particular, for \(P, Q \in \mathcal{P}\), we have

\[
[K_P] * [K_Q] = [K_P \oplus K_Q], \quad [K_P] * [K_Q^*] = [K_P \oplus K_Q^*]. \quad (3)
\]
Lemma 2.7
\[ [K_P^\ast] \ast [M_\ast] = t^{-\langle \hat{P}, \hat{M} \rangle} \cdot [K_P \oplus M_\ast], \quad [M_\ast] \ast [K_P^\ast] = t^{\langle \hat{M}, \hat{P} \rangle} \cdot [K_P \oplus M_\ast]. \] (4)

Proof: The proof is similar to the proof of the equations (1) (see [Br]). \[\square\]

Definition 2.8 The localized Ringel-Hall algebra \( \mathcal{DH}(A) \) is the localization of \( \mathcal{H}_{tw}(C_2(P)) \) with respect to the elements \([M_\ast]\) corresponding to acyclic complexes \( M_\ast \):
\[ \mathcal{DH}(A) := \mathcal{H}_{tw}(C_2(P))[\{M_\ast \mid H_s(M_\ast) = 0\}] \]
As explained in [Br], this is the same as localizing by the elements \([K_P]\) and \([K_P^\ast]\) for all objects \( P \in P \).

Write \( \alpha \in K(A) \) in the form \( \alpha = \hat{P} - \hat{Q} \) for objects \( P, Q \in P \), and set \( K_\alpha = [K_P] \ast [K_Q]^{-1}, K_\alpha^\ast = [K_P^\ast] \ast [K_Q^\ast]^{-1} \). Still as explained in [Br], the equations (2) continue to hold with the elements \([K_P]\) and \([K_P^\ast]\) replaced by \( K_\alpha \) and \( K_\alpha^\ast \) for arbitrary \( \alpha \in K(A) \).

Definition 2.9 The reduced localized Ringel-Hall algebra is given by setting \([M_\ast] = 1\) whenever \( M_\ast \) is an acyclic complex, invariant under the shift functor. In symbols
\[ \mathcal{DH}_{red}(A) = \mathcal{DH}(A)/\{[M_\ast]-1 : H_s(M_\ast) = 0, M_\ast \cong M_\ast^\ast\} \]
By Lemma 2.5, this is the same as setting \([K_P] \ast [K_P^\ast] = 1\) for all \( P \in P \).

3 Main results

In this section, let \( A = \text{mod} B \) be the category of finitely generated right \( B \)-modules, where \( B \) is a finite-dimensional algebra with global dimension \( \leq 2 \). Hence, every object \( A \in A \) has a projective resolution of the form
\[ 0 \rightarrow P_2 \xrightarrow{a_2} P_1 \xrightarrow{a_1} P_0 \xrightarrow{a_0} A \rightarrow 0. \] (5)

The following is well known.

Lemma 3.1 Any resolution (5) is isomorphic to a resolution of the form
\[ 0 \rightarrow P_2' \oplus R_1 \xrightarrow{\begin{pmatrix} a_2' & 0 \\ 0 & 0 \end{pmatrix}} P_1' \oplus R_0 \oplus R_1 \xrightarrow{\begin{pmatrix} a_1' & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} P_0' \oplus R_0 \xrightarrow{(a_0', 0)} A \rightarrow 0 \] (6)
for some objects \( R_0, R_1 \in P \), and some minimal projective resolution
\[ 0 \rightarrow P_2' \xrightarrow{a_2'} P_1' \xrightarrow{a_1'} P_0' \xrightarrow{a_0'} A \rightarrow 0. \] \[\square\]

Given an object \( A \in A \), take a minimal projective resolution (5) and consider the corresponding 2-periodic complex
\[ C_A := P_1 \xrightarrow{\begin{pmatrix} a_1 \\ 0 \end{pmatrix}} P_0 \oplus P_2 \] (7)
Note that any two minimal projective resolutions of \( A \) are isomorphic. So the complex \( C_A \) is well-defined up to isomorphism. In addition we have the homological groups \( H_0(C_A) = A, H_1(C_A) = 0 \), and in \( K(A) \) we have \( C_A = P_0 + P_2 - P_1 = A \).

**Proposition 3.2** Given objects \( A_1, A_2 \in A \), take minimal projective resolutions

\[
0 \to P_2 \xrightarrow{a_2} P_1 \xrightarrow{a_1} P_0 \xrightarrow{a_0} A_1 \to 0
\]

\[
0 \to Q_2 \xrightarrow{b_2} Q_1 \xrightarrow{b_1} Q_0 \xrightarrow{b_0} A_2 \to 0
\]

then there is an epimorphism \( \phi : \text{Hom}_{C_2(A)}(C_{A_1}, C_{A_2}) \to \text{Hom}_A(A_1, A_2) \) and

\[
|\ker \phi| = |\text{Hom}_A(P_2, Q_0)| \times |\text{Hom}_A(P_1, Q_2)| \times |\text{Hom}_A(P_0, Q_1)|/|\text{Hom}_A(P_0, Q_2)|.
\]

**Proof:** Let

\[
\bar{f} = \left( h , \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in \text{Hom}_{C_2(A)}(C_{A_1}, C_{A_2}).
\]

We have the following exchange diagram

\[
\begin{array}{ccc}
P_1 & \xrightarrow{(a_1, 0)} & P_0 \oplus P_2 \\
\downarrow h & & \downarrow \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \\
Q_1 & \xrightarrow{(b_1, 0)} & Q_0 \oplus Q_2
\end{array}
\]

i.e., we have

\[
\begin{cases}
a \circ a_1 = b_1 \circ h \\
c \circ a_1 = 0 \\
b_2 \circ c = 0 \\
b_2 \circ d = h \circ a_2
\end{cases}
\]

Because \( b_2 \) is a monomorphism, we have \( c = 0 \). And we have the following exchange graph

\[
\begin{array}{cccccc}
0 & \to & P_2 & \xrightarrow{a_2} & P_1 & \xrightarrow{a_1} & P_0 & \xrightarrow{a_0} & A_1 & \to & 0 \\
\downarrow d & & \downarrow h & & \downarrow a & & \downarrow \phi & & \downarrow f & & \downarrow 0 \\
0 & \to & Q_2 & \xrightarrow{b_2} & Q_1 & \xrightarrow{b_1} & Q_0 & \xrightarrow{b_0} & A_2 & \to & 0
\end{array}
\]

So there exists a unique \( f \in \text{Hom}_A(A_1, A_2) \) s.t. the graph

\[
\begin{array}{cccccc}
0 & \to & P_2 & \xrightarrow{a_2} & P_1 & \xrightarrow{a_1} & P_0 & \xrightarrow{a_0} & A_1 & \to & 0 \\
\downarrow d & & \downarrow h & & \downarrow a & & \downarrow f & & \downarrow 0 \\
0 & \to & Q_2 & \xrightarrow{b_2} & Q_1 & \xrightarrow{b_1} & Q_0 & \xrightarrow{b_0} & A_2 & \to & 0
\end{array}
\]

exchanged. We define \( \phi(\bar{f}) = f \).
On the other hand, for any \( f \in \text{Hom}_A(A_1, A_2) \), it is easy to find \( d, h, a, \) s.t.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & P_2 & \xrightarrow{a_2} & P_1 & \xrightarrow{a_1} & P_0 & \xrightarrow{a_0} & A_1 & \rightarrow & 0 \\
\downarrow{d} & & \downarrow{h} & & \downarrow{a} & & \downarrow{f} & & \\
0 & \rightarrow & Q_2 & \xrightarrow{b_2} & Q_1 & \xrightarrow{b_1} & Q_0 & \xrightarrow{b_0} & A_2 & \rightarrow & 0 \\
\end{array}
\]

exchanged. For any \( b \in \text{Hom}_A(P_2, Q_0) \), we have

\[
\tilde{f} = \left( h, \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) \in \text{Hom}_{C_2(A)}(C_{A_1}, C_{A_2}),
\]

and \( \phi(\tilde{f}) = f \) by the definition. Therefore, there is a surjection

\[
\phi : \text{Hom}_{C_2(A)}(C_{A_1}, C_{A_2}) \rightarrow \text{Hom}_A(A_1, A_2).
\]

It is easy to see that \( \phi \) is a morphism. So \( \phi \) is an epimorphism.

Let

\[
\tilde{f} = \left( h, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in \text{Hom}_{C_2(A)}(C_{A_1}, C_{A_2}),
\]

such that \( \phi(\tilde{f}) = 0 \). We have \( c = 0 \) and the following exchange graph

\[
\begin{array}{ccccccccc}
0 & \rightarrow & P_2 & \xrightarrow{a_2} & P_1 & \xrightarrow{a_1} & P_0 & \xrightarrow{a_0} & A_1 & \rightarrow & 0 \\
\downarrow{d} & & \downarrow{h} & & \downarrow{a} & & \downarrow{f} & & \\
0 & \rightarrow & Q_2 & \xrightarrow{b_2} & Q_1 & \xrightarrow{b_1} & Q_0 & \xrightarrow{b_0} & A_2 & \rightarrow & 0 \\
\end{array}
\]

So there exists only a morphism \( a' \) from \( P_0 \) to \( \ker(b_0) \). Conversely, consider the exact sequence

\[
0 \rightarrow \ker b_0 \xrightarrow{i_0} Q_0 \xrightarrow{b_0} A_2 \rightarrow 0
\]

and if there is a morphism \( a' \) from \( P_0 \) to \( \ker(b_0) \), we can let \( a = a' \circ i_0 \). So the number of \( a \in \text{Hom}_A(P_0, Q_0) \) with \( \phi(\tilde{f}) = 0 \) is equal to \( |\text{Hom}_A(P_0, \ker(b_0))| \). Since \( P_0 \) is a projective module, using the exact sequence

\[
0 \rightarrow Q_2 \xrightarrow{b_2} Q_1 \xrightarrow{\pi_1} \ker(b_0) \rightarrow 0
\]

we have \( \dim_k \text{Hom}_A(P_0, \ker(b_0)) = \dim_k \text{Hom}_A(P_0, Q_1) - \dim_k \text{Hom}_A(P_0, Q_2) \), i.e.

\[
|\text{Hom}_A(P_0, \ker(b_0))| = |\text{Hom}_A(P_0, Q_1)|/|\text{Hom}_A(P_0, Q_2)|.
\]

Now fix \( a \). Since \( d \) is determined uniquely by \( h \), we just need to decide \( h \). Note that \( h \) satisfies the following exchange graph

\[
\begin{array}{cccc}
P_1 & \xrightarrow{a_1} & P_0 & \\
\downarrow{h} & & \downarrow{a} & \\
Q_1 & \xrightarrow{b_1} & Q_0 & \\
\end{array}
\]

(8)
This reduces the following exchange graph

\[
\begin{array}{ccc}
P_1 & \xrightarrow{a_1} & P_0 \\
\downarrow{h} & & \downarrow{a'} \\
Q_1 & \xrightarrow{\pi_1} & \ker b_0
\end{array}
\]

Since \( P_1 \) is projective, using the exact sequence

\[
0 \rightarrow Q_2 \xrightarrow{b_2} Q_1 \xrightarrow{\pi_1} \ker b_0 \rightarrow 0,
\]

we get an exact sequence

\[
0 \rightarrow \Hom_A(P_1, Q_2) \rightarrow \Hom_A(P_1, Q_1) \rightarrow \Hom_A(P_1, \ker(b_0)) \rightarrow 0.
\]

Note that \( a' \circ a_1 \in \Hom(P_1, \ker(b_0)) \) is fixed. So the number of \( h \) satisfying the exchange graph (9) is equal to \( |\Hom_A(P_1, Q_2)| \).

From the above discussions and noting that there is no restriction to the morphism \( b \) from \( P_2 \) to \( Q_0 \), we have

\[
|\ker(\phi)| = |\Hom_A(P_2, Q_0)| \times |\Hom_A(P_1, Q_2)| \times |\Hom_A(P_0, Q_1)| / |\Hom_A(P_0, Q_2)|.
\]

By Lemma 2.3 and Lemma 2.4, we have the following relations:

**Lemma 3.3** \( \Ext_{C_2(A)}(C_{A_1}, C_{A_2}) \cong \Ext_A(A_1, A_2) \).

**Proof**:

\[
\Ext_{C_2(A)}(C_{A_1}, C_{A_2}) \cong \Hom_{K_2(A)}(C_{A_1}, C_{A_2}^*) \\
\cong \Hom_{R_2(A)}(A_1, A_2[1]) \\
\cong \bigoplus_{i \in \mathbb{Z}} \Hom_{D^b(A)}(A_1, A_2[2i + 1]) \\
= \bigoplus_{i \geq 0} \Hom_{D^b(A)}(A_1, A_2[2i + 1]) \\
= \bigoplus_{i \geq 0} \Ext^i_A(A_1, A_2)
\]

Since the global dimension of \( A \) is at most two, we have

\[
\Ext_{C_2(A)}(C_{A_1}, C_{A_2}) \cong \bigoplus_{i \geq 0} \Ext^{2i+1}_A(A_1, A_2) = \Ext_A(A_1, A_2).
\]

Given \( A \in \mathcal{A} \), take a minimal projective resolution (5), define

\[
E_A = t(\hat{\mathcal{P}}_1 - \hat{\mathcal{P}}_2, \hat{A}) \cdot K_{-\hat{\mathcal{P}}_1} * K_{\hat{\mathcal{P}}_2} * K_{-\hat{\mathcal{P}}_2} * [C_A] \in \mathcal{D}_H(A).
\]

From the following proposition, one can see that \( E_A \) is not depend on whether the projective resolution is minimal or not.
Proposition 3.4 Suppose we take a different, not necessarily minimal, projective resolution (6), consider the corresponding complex (7) in $C_2(P)$, i.e. let

$$C'_A := R_0 \oplus P_1 \oplus R_1 \rightarrow R_0 \oplus P_0 \oplus R_1 \oplus P_2;$$

and

$$E'_A = t(P_1 + R_0 + R_1 - 2P_2 + R_1) \cdot K_{-\hat{p}_1 - R_0 - R_1} * K_{\hat{p}_2 + R_1} * K_{-\hat{p}_2 - R_1} * [C'_A].$$

Then we have

$$E'_A = E_A.$$

Proof: Note that $[C'_A] = [K_{R_1}^* + K_{R_0} + C_A]$. By Lemma 2.6 and Lemma 2.7, we have

$$[C'_A] = [K_{R_1}^* + K_{R_0} + C_A] = t(P_1, A) \cdot K_{R_1}^* * K_{R_0} * [C_A].$$

Hence,

$$E'_A = t(P_1 + R_0 + R_1 - 2(\hat{p}_2 + R_1)) \cdot K_{-\hat{p}_1 - R_0 - R_1} * K_{\hat{p}_2 + R_1} * K_{-\hat{p}_2 - R_1} * [C'_A]$$

$$= t(P_1 + R_0 + R_1 - 2(\hat{p}_2 + R_1)) \cdot K_{-\hat{p}_1 - R_0 - R_1} * K_{\hat{p}_2 + R_1} * K_{-\hat{p}_2 - R_1} * [C_A]$$

$$= t(P_1 + R_0 + R_1 - 2(\hat{p}_2 + R_1)) \cdot (R_1, A) \cdot K_{-\hat{p}_1 - R_0 - R_1} * K_{\hat{p}_2 + R_1} * K_{-\hat{p}_2 - R_1} * [C_A]$$

$$= t(P_1 - 2\hat{p}_2, A) \cdot K_{-\hat{p}_1} * K_{\hat{p}_2} * K_{-\hat{p}_2} * [C_A]$$

$$= E_A \quad \Box$$

Proposition 3.5 For $B_i \in A$, $1 \leq i \leq s$, assume that $\{[B_i]|1 \leq i \leq s\}$ is linearly independent in $\mathcal{H}_{tw}(A)$. Then for any $P_i, Q_i, R_i, S_i(1 \leq i \leq s) \in P$, $\{K_{P_i}^* \cdot K_{Q_i}^* \cdot K_{R_i} \cdot K_{S_i}^* \cdot [B_i]|1 \leq i \leq s\}$ is linearly independent in $\mathcal{D}H(A)$.

Proof: Assume there exist $l_i \in \mathbb{C}$, $1 \leq i \leq s$, such that in $\mathcal{D}H(A)$

$$\sum_{i=1}^{s} l_i K_{P_i}^* \cdot K_{Q_i}^* \cdot K_{-R_i} \cdot K_{-S_i}^* \cdot [C_{B_i}] = 0.$$

Then multiplying suitable $K_{P_i}, K_{Q_i}$ for some $P, Q \in P$ we can get

$$\sum_{i=1}^{s} l_i K_{X_i}^* \cdot K_{Y_i}^* \cdot [C_{B_i}] = 0.$$
in $\mathcal{H}_{tw}(C_2(P))$ for some $X_i, Y_i \in \mathcal{P}$, $1 \leq i \leq s$. By Lemma 2.6 and Lemma 2.7, we have
\[
\sum_{i=1}^{s} l_i \cdot t(\hat{x}_i - \hat{y}_i, \hat{b}_i) \cdot [K_{X_i} \oplus K_{Y_i}^* \oplus C_{B_i}] = 0
\]
in $\mathcal{H}_{tw}(C_2(P))$. If there is $u (1 \leq u \leq s)$ such that $l_u \neq 0$, there must be $v (1 \leq v \leq s)$ such that $v \neq u, l_v \neq 0$ and
\[
K_{X_u} \oplus K_{Y_u}^* \oplus C_{B_u} \cong K_{X_v} \oplus K_{Y_v}^* \oplus C_{B_v}.
\]
Then
\[
H_0(K_{X_u} \oplus K_{Y_u}^* \oplus C_{B_u}) \cong H_0(K_{X_v} \oplus K_{Y_v}^* \oplus C_{B_v}).
\]
Since for any $P \in \mathcal{P}$, $H_0(K_P) = H_1(K_P) = H_0(K_P^*) = H_1(K_P^*) = 0$, and $H_0(C_{B_i}) = B_i (1 \leq i \leq s)$, we have $B_u \cong B_v$, a contradiction with $\{[B_i]|1 \leq i \leq s\}$ linearly independent in $\mathcal{H}_{tw}(A)$. So for any $u (1 \leq u \leq s)$, we have $l_u = 0$. Hence, $\{K_{\hat{P}_i}^* * K_{\hat{Q}_i}^* * K_{\hat{R}_i} * K_{\hat{S}_i} * [C_{B_i}]|1 \leq i \leq s\}$ is linearly independent in $\mathcal{D}\mathcal{H}(A)$. $\square$

From the above proposition, it is easy to get the following result:

**Corollary 3.6** For $B_i \in \mathcal{A}$, $1 \leq i \leq s$, assume that $\{[B_i]|1 \leq i \leq s\}$ is linearly independent in $\mathcal{H}_{tw}(A)$. Then $\{[C_{B_i}]|1 \leq i \leq s\}$ is linearly independent in $\mathcal{D}\mathcal{H}(A)$, and $\{[E_{B_i}]|1 \leq i \leq s\}$ are also linearly independent in $\mathcal{D}\mathcal{H}(A)$. $\square$

**Theorem 3.7** There is an injective algebra homomorphism
\[
I_+: \mathcal{H}_{tw}(A) \hookrightarrow \mathcal{D}\mathcal{H}(A), \quad [A] \mapsto E_A.
\]

**Proof** : Given objects $A_1, A_2 \in \mathcal{A}$ in $\mathcal{H}_{tw}(A)$ we have
\[
[A_1] * [A_2] = t(\hat{A}_1, \hat{A}_2) \sum_{[A_3]} |\text{Ext}_A(A_1, A_2, A_3)| / |\text{Hom}_A(A_1, A_2)| \cdot [A_3].
\]
Take minimal projective resolutions
\[
0 \rightarrow P_2 \xrightarrow{a_2} P_1 \xrightarrow{a_1} P_0 \xrightarrow{a_0} A_1 \rightarrow 0
\]
\[
0 \rightarrow Q_2 \xrightarrow{b_2} Q_1 \xrightarrow{b_1} Q_0 \xrightarrow{b_0} A_2 \rightarrow 0
\]
It is easy to get one projective resolution of $A_3$,
\[
0 \rightarrow P_2 \oplus Q_2 \rightarrow P_1 \oplus Q_1 \rightarrow P_0 \oplus Q_0 \rightarrow A_3 \rightarrow 0.
\]
By Lemma 3.3, $\text{Ext}_{C_2(A)}(C_{A_1}, C_{A_2}) = \text{Hom}_{K_2(A)}(C_{A_1}, C_{A_2}^*) = \text{Ext}_A(A_1, A_2)$. Then it is easy to see that any extension of $A_1$ by $A_2$ is the complex $C_{A_3}$ defined by the corresponding extension $A_3$ of $A_1$ by $A_2$. Using Lemma 2.6 and Lemma 2.7, we have
\[
I_+(A_1) * I_+(A_2)
\]
\[
= t(\hat{P}_1 - 2\hat{P}_2, A_1) + (\hat{Q}_1 - 2\hat{Q}_2, A_1) \cdot K_{-\hat{P}_1} * K_{\hat{P}_2} * K_{-\hat{P}_2} * [C_{A_1}] + K_{\hat{Q}_1} * K_{-\hat{Q}_2} * [C_{A_1}]
\]
\[
= t(\hat{P}_1 - 2\hat{P}_2, A_1) + (\hat{Q}_1 - 2\hat{Q}_2, A_2) + (\hat{Q}_1 - A_1) - 2(\hat{Q}_2, A_1) \cdot K_{-\hat{P}_1} * K_{\hat{P}_2} * K_{-\hat{P}_2} * K_{\hat{Q}_1} * K_{-\hat{Q}_2} * [C_{A_1}] * [C_{A_2}]
\]
\[
= t(\hat{P}_1 - 2\hat{P}_2, A_1) + (\hat{Q}_1 - 2\hat{Q}_2, A_2) + (\hat{Q}_1 - A_1) - 2(\hat{Q}_2, A_1) + (\hat{P}_1 - \hat{Q}_1 + \hat{Q}_2, A_1) + (\hat{P}_0 + \hat{P}_2, A_0 + \hat{Q}_2) - (\hat{P}_1 - \hat{Q}_1 - 2\hat{P}_2 - 2\hat{Q}_2, A_3)
\]
\[
\sum_{[A_3]} |\text{Ext}_A(A_1, A_2, A_3)| / |\text{Hom}_{C_2(A)}(C_{A_1}, C_{A_2})| * E_{A_3}.
\]
From Proposition 3.2, we know

\[ |\text{Hom}_{\mathcal{C}(\mathcal{A})}(C_{A_1}, C_{A_2})| = |\text{Hom}_{\mathcal{A}}(A_1, A_2)| \cdot |\text{Hom}_{\mathcal{A}}(P_2, Q_0)| \cdot |\text{Hom}_{\mathcal{A}}(P_1, Q_2)| \cdot |\text{Hom}_{\mathcal{A}}(P_0, Q_1)|/|\text{Hom}_{\mathcal{A}}(P_0, Q_2)| \]

Because

\[ I_+([A_1] * [A_2]) = \iota^{(A_1, A_2)} \cdot \sum_{[A_3]} \frac{|\text{Ext}_{\mathcal{A}}(A_1, A_2) A_3|}{|\text{Hom}_{\mathcal{A}}(A_1, A_2)|} * E_{A_3}, \tag{11} \]

comparing (10) and (11), since \( |\text{Hom}_{\mathcal{A}}(P, Q)| = q(P, Q) = \iota^2(P, Q) \) for any projective modules \( P, Q \), we have \( I_+([A_1] * [A_2]) = I_+([A_1]) * I_+([A_2]) \) if and only if

\[
\iota^{(\hat{P}_1 - 2\hat{P}_2, \hat{A}_1)} + (\hat{Q}_1 - 2\hat{Q}_2, \hat{A}_2) = 2(\hat{Q}_2, \hat{A}_1) + (\hat{P}_1, \hat{Q}_1) + (\hat{P}_0 + \hat{Q}_0 + \hat{Q}_2) - (\hat{P}_1 + \hat{Q}_1 - 2\hat{P}_2 - 2\hat{Q}_2, \hat{A}_3)
\]

\[ = \iota^{(A_1, A_2) + 2(\hat{P}_2, \hat{Q}_0) + 2(\hat{P}_1, \hat{Q}_2) + 2(\hat{P}_0, \hat{Q}_1) - 2(\hat{P}_1, \hat{Q}_1)} \quad \tag{12} \]

Using \( \tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2, \tilde{A}_1 = \tilde{P}_0 + \tilde{P}_2 - \tilde{P}_1 \) and \( \tilde{A}_2 = \tilde{Q}_0 + \tilde{Q}_2 - \tilde{Q}_1 \), it is easy to verify that (12) is established.

By Corollary 3.6, if \( \{[B_i]|1 \leq i \leq s\} \) is linearly independent in \( \mathcal{H}_{tw}(\mathcal{A}) \) for any \( B_i(1 \leq i \leq s) \in \mathcal{A} \), then \( \{[E_{B_i}]|1 \leq i \leq s\} \) is also linearly independent in \( \mathcal{D}(\mathcal{A}) \). Therefore, \( I_+ \) is injective. \( \square \)

Let \( \pi \) be the natural homomorphism from \( \mathcal{D}(\mathcal{A}) \) to \( \mathcal{D}(\mathcal{A}) \), we have

**Theorem 3.8** There is an embedding of algebras \( \tilde{I}_+ = \pi \circ I_+: \mathcal{H}_{tw}(A) \hookrightarrow \mathcal{D}(\mathcal{A}) \).

**Proof**: Clearly \( \tilde{I}_+ \) is an algebra homomorphism. So we only need to prove that \( \tilde{I}_+ \) is injective. Let \( J \) be the ideal in \( \mathcal{D}(\mathcal{A}) \) generated by \( K_{\hat{P}} - K_{\hat{P}}^* \) for all \( P \in \mathcal{P} \). Then by above theorem and \( \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A})/J \), we just need to prove \( \text{im}I_+ \cap J = 0 \).

It is easy to see that all \( K_{\hat{P}}K_{\mathcal{A}}^*K_{\mathcal{A}}K_{\mathcal{A}}^* \cdots K_{\mathcal{A}}^*[M] \) span \( \mathcal{D}(\mathcal{A}) \), where \( P, Q, R, S \in \mathcal{P} \) and \( M \in C_2(\mathcal{P}) \) has no non-zero direct summand of acyclic complexes. Therefore any element in \( \text{im}I_+ \cap J \) has the form

\[
\sum_{i=1}^s l_i E_{B_i} = \sum_{j=1}^n h_j (K_{-X_j} - K_{X_j}^*) * K_{\hat{P}_j} * K_{\hat{W}_j} * K_{-Y_j} * K_{-Z_j} * [(M_j)\cdot] \]

for some \( l_i, h_j \in \mathbb{C}, 1 \leq i \leq s, 1 \leq j \leq n \), and some \( B_i \in \mathcal{A}, 1 \leq i \leq s \), where \( \{[B_i]|1 \leq i \leq s\} \) is linearly independent in \( \mathcal{H}_{tw}(\mathcal{A}) \), and some \( T_j, W_j, X_j, Y_j, Z_j \in \mathcal{P} \) and \( (M_j)\cdot \in C_2(\mathcal{P}) \) for \( 1 \leq j \leq n \) so that each \( (M_j)\cdot \) has no non-zero direct summand of acyclic complexes. Multiplying \( K_{\hat{P}}K_{\mathcal{A}}^* \) on the both sides for some suitable \( P, Q \in \mathcal{P} \), we can get

\[
\sum_{i=1}^s l_i t_i K_{\hat{P}_i} * K_{\mathcal{A}}^* [C_{B_i}] = \sum_{j=1}^n h_j (1 - K_{X_j}^* K_{X_j}^*) * K_{\hat{P}_j} * K_{\mathcal{A}}^* [M] \]

in \( \mathcal{H}_{tw}(C_2(\mathcal{P})) \) for some \( P_i, Q_i, R_j, S_j \in \mathcal{P}, 1 \leq i \leq s, 1 \leq j \leq n \), where all \( t_i \in \mathbb{C} \) are not zero.

Suppose some \( l_i \neq 0 \) and take \( (M_1)\cdot, (M_2)\cdot, \cdots, (M_{im})\cdot \in \{ (M_1)\cdot, (M_2)\cdot, \cdots, (M_n)\cdot \} \) to be all the \( \mathbb{Z}_2 \)-graded complexes such that each of them is isomorphic to \( C_{B_i} \). Note that

\[ \{ K_{\hat{P}}K_{\mathcal{A}}^*[M] \mid P, Q \in \mathcal{P}, M \in C_2(\mathcal{P}) \} \]
is also a basis in $\mathcal{H}_{tw}(C_2(\mathcal{P}))$. We have
\[ l_i t_i K_{F_i}^* * K_{Q_i}^* * [C_{B_i}] = \sum_{j=1}^{m_i} h_{ij}(1 - K_{X_{ij}}^* * K_{R_{ij}}^* * K_{S_{ij}}^* *[M_{ij}]) \]  \tag{13}

in $\mathcal{H}_{tw}(C_2(\mathcal{P}))$. Under the above basis, since the sum of the coefficients of the equation (13) on the right is zero, we have $l_i t_i = 0$ as the coefficient on the left and so $l_i = 0$, which is a contradiction with $l_i \neq 0$. Hence for any $1 \leq i \leq s$, we have $l_i = 0$. Therefore, $\text{im}I_+ \cap J = 0$. \( \square \)

**Remarks 3.9** If we set $F_A = E_A^*$, then there is also an injective algebra homomorphism
\[ I_- : \mathcal{H}_{tw}(A) \hookrightarrow \mathcal{D}(A), \quad [A] \mapsto F_A \]
and this embedding can induce an embedding form $\mathcal{H}_{tw}(A)$ to $\mathcal{D}_{red}(A)$. \( \square \)

In the following we will get some interesting consequences.

Let us recall the definitions of tilted algebras and canonical algebras.

**Definition 3.10** ([HR]) Let $H$ be a finite-dimensional hereditary algebra. An $H$-module $T$ is a tilting module if it satisfies:

1. $\text{proj.dim}_HT \leq 1$;
2. $\text{Ext}^1_H(T, T) = 0$;
3. there exists a short exact sequence $0 \rightarrow H \rightarrow T_0 \rightarrow T_1 \rightarrow 0$, where $T_0, T_1$ are direct summands of a finite direct sum of copies of $T$ respectively.

The endomorphism algebra $B = \text{End}_HT$ is called a tilted algebra.

**Definition 3.11** ([Ri1]) Let $p = (p_0, p_1, \ldots, p_n)$ be a $(n+1)$-tuple of integers $p_i \geq 1$, $\lambda = (\lambda_2, \ldots, \lambda_n)$ be pairwise distinct elements of $k$. A canonical algebra $B$ of type $(p, \lambda)$ is the following quiver

with relations given by
\[ X_i^{p_i} = X_0^{p_0} - \lambda_i X_1^{p_1} \quad \text{for} \quad i = 2, 3, \ldots, n, \]

By [GL], $\mathcal{D}^b(B)$ is equivalent to $\mathcal{D}^b(\text{coh}(\mathcal{X}))$, where $\text{coh}(\mathcal{X})$ is the category of coherent sheaves on a weighted projective line $\mathcal{X}$ of type $(p, \lambda)$. 

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By [HR] tilted algebras have global dimension at most two and it is obvious that canonical algebras have global dimension \( \leq 2 \). As an application, we have

**Corollary 3.12** Assume that \( \mathcal{A} = \text{mod} B \) is the category of finitely generated right \( B \)-modules where \( B \) is a tilted algebra or a canonical algebra. Then there is an embedding of algebras from \( \mathcal{H}_{\text{tw}}(\mathcal{A}) \) to \( \mathcal{D}\mathcal{H}(\mathcal{A}) \) and \( \mathcal{D}\mathcal{H}_{\text{red}}(\mathcal{A}) \).

**Remarks 3.13** It should be also interesting to apply to some other algebras which are neither tilted algebras nor tubular algebras but with global dimension \( \leq 2 \). For example, the algebra is given respectively by the following quivers with relations

\[
\begin{align*}
\alpha &\quad \beta \\
\beta &\quad \alpha
\end{align*}
\]

\( \alpha \beta = 0 \), and

\[
\begin{align*}
\alpha & \quad \beta \\
\gamma & \quad \alpha
\end{align*}
\]

\( \beta \alpha = 0 \)

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