TRAVELING WAVE SOLUTION FOR A DIFFUSION SEIR EPIDEMIC MODEL WITH SELF-PROTECTION AND TREATMENT

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Abstract. A reaction-diffusion SEIR model, including the self-protection for susceptible individuals, treatments for infectious individuals and constant recruitment, is introduced. The existence of traveling wave solution, which is determined by the basic reproduction number $R_0$ and wave speed $c$, is firstly proved as $R_0 > 1$ and $c \geq c^*$ via the Schauder fixed point theorem, where $c^*$ is minimal wave speed. Asymptotic behavior of traveling wave solution at infinity is also proved by applying the Lyapunov functional. Furthermore, when $R_0 \leq 1$ or $R_0 > 1$ with $c \in (0, c^*)$, we derive the non-existence of traveling wave solution with utilizing two-sides Laplace transform. We take advantage of numerical simulations to indicate the existence of traveling wave, and show that self-protection and treatment can reduce the spread speed at last.

1. Introduction. Epidemics are widespread around the world, and always jeopardize public lives and health, such as COVID-19, malaria, severe acute respiratory syndrome (SARS), influenza and so on. The public has taken all kinds of measures to struggle against the different infectious epidemics. Various of mathematical models are established to research dynamics and influences of prevention and control for different epidemics [12, 23, 22, 10, 9, 15]. Recently, both the popularization of knowledge about infectious diseases and the improvement of treatment have built a solid foundation for epidemics’ prevention. This popularization can also lead to the strengthening in public self-protection. For instance, since Chinese had owned strongly self-protection and governments of China had taken efficient treatments, COVID-19 was first under effective control in China [11, 17, 13]. As a result, it is necessary to integrate self-protection and treatment into epidemic models. Because epidemics are always discovered at one location and then spread to other areas [21], reaction-diffusion models become essential to describe this spatial spread. Simultaneously, epidemic models with spatial spread usually can result in a development from a diseases-free state to an infective state, which can be predicted a wave for
this evolution of epidemics. Consequently, traveling wave solution becomes critical to study the spatial spread of epidemics [7, 8, 18, 20].

Many people have analyzed traveling wave solution and the asymptotic speed of propagation of classic compartmental epidemic models [16, 20, 1, 4, 30, 28, 32, 25]. Ducrot and Magal [3] studied the existence and the non-existence of traveling wave solution satisfying a diffusive epidemic model with age-structure and constant recruitment, and constructed a suitable Lyapunov functional to discuss their asymptotic behavior at infinity. Li et al. [14] found that the existence and the non-existence of traveling wave solution for the system which is a nonlocal dispersal delayed SIR model with constant recruitment and Holling-slowroman capii@ incidence rate are determined via both the minimal wave speed and the basic reproduction number $R_0$ defined by the corresponding reaction system. Zhang et al. [29] considered the existence of traveling wave solution satisfying an influenza model with treatment for infectious individuals. Zhao et al. [31] proved the existence and non-existence of traveling wave solution for a SIR model with multiple parallel infectious stages which own same diffusion coefficients. Then Zhao et al. [33] established a two-groups reaction-diffusion epidemic model to research the influence of host heterogeneity in spread of disease by applying the traveling wave solution. However, human self-protection for susceptible individuals and treatment for infectious individuals were not considered together in a SEIR diffusion epidemic model.

In this paper, we firstly integrate self-protection into the classical reaction-diffusion SEIR model with constant recruitment, so susceptible individuals are divided into two groups: susceptible individuals $S_1$ without self-protection and susceptible people $S_2$ with self-protection. Next, the treatment for infectious individuals is considered such that $I_1$ represents the infectious with treatment and $I_2$ means the infectious without it. Therefore, a reaction-diffusion SEIR model is constructed as system (1), where $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. In this model, $S_i(x, t)$ and $I_i(x, t)$ ($i = 1, 2$) represent the density of two groups of susceptible and infectious individuals above determined, respectively. $E(x, t)$ represents the density of individuals who are infected without infectivity. $R(x, t)$ represents the density of individuals who recover and own permanent immunity. $d_1$, $d_2$, $D_0$, $D_1$, $D_2$, and $D$ represent diffusion coefficients of $S_1$, $S_2$, $E$, $I_1$, $I_2$ and $R$, respectively. $\Lambda$ is constant recruitment. $\beta$ is bilinear incidence. $\mu$ is the sum of the mortality rate. $\alpha_i$ is the recovery rate of $I_i$ ($i = 1, 2$), respectively. $\gamma$ is the reciprocal of latency. $q_i$ is the proportion of individuals entering $I_i$ ($i = 1, 2$), respectively. $\rho$ is the migration rate from $S_1$ to $S_2$. $\sigma$ is the rate which susceptible individuals with self-protection reduce contacts with infections. $\theta$ represents the rate that the treatment for the infectious reduce their contacts with others.

$$\begin{align*}
\frac{\partial S_1(x, t)}{\partial t} & = d_1 \Delta S_1(x, t) + \Lambda - \beta S_1(x, t) \left[ \theta_1 I_1(x, t) + I_2(x, t) \right] - (\rho + \mu) S_1(x, t), \\
\frac{\partial S_2(x, t)}{\partial t} & = d_2 \Delta S_2(x, t) + \rho S_1(x, t) - \beta \sigma S_2(x, t) \left[ \theta_1 I_1(x, t) + I_2(x, t) \right] - \mu S_2(x, t), \\
\frac{\partial E(x, t)}{\partial t} & = D_0 \Delta E(x, t) + \beta S_1(x, t) + \sigma S_2(x, t) \left[ \theta_1 I_1(x, t) + I_2(x, t) \right] - (\mu + \gamma) E(x, t), \\
\frac{\partial I_1(x, t)}{\partial t} & = D_1 \Delta I_1(x, t) + q_1 \gamma E(x, t) - (\mu + \alpha_1) I_1(x, t), \\
\frac{\partial I_2(x, t)}{\partial t} & = D_2 \Delta I_2(x, t) + q_2 \gamma E(x, t) - (\mu + \alpha_2) I_2(x, t), \\
\frac{\partial R(x, t)}{\partial t} & = D \Delta R(x, t) + \alpha_1 I_1(x, t) + \alpha_2 I_2(x, t) - \mu R(x, t).
\end{align*}$$

Throughout this article, we make the following assumption:
In fact, for making a living, the susceptible has to move, and their diffusion is almost same without special cases. Especially, the susceptible with self-protection may take some methods, such as wearing masks, avoiding crowded public places and so on, to decrease the likelihood of connects with the infectious. The infectious could reduce going out owing to discomfort by disease. Thus, the assumption can better close to reality. In mathematical analysis of this model, the assumption (A) also can help us get an upper bound for the different susceptible and infected components to research the asymptotic behavior as $x \to +\infty$.

Since the equations $S_i(x, t)$, $E(x, t)$ and $I_i(x, t)$ ($i = 1, 2$) are fully decoupled from $R(x, t)$, we only need to study the sub-system:

\[
\begin{align*}
\frac{\partial S_1(x, t)}{\partial t} &= d \Delta S_1(x, t) + \Lambda - \beta S_1(x, t) [\theta I_1(x, t) + I_2(x, t)] \\
&\quad - (\rho + \mu) S_1(x, t), \\
\frac{\partial S_2(x, t)}{\partial t} &= d \Delta S_2(x, t) + \rho S_1(x, t) - \beta \sigma S_2(x, t) [\theta I_1(x, t) + I_2(x, t)] \\
&\quad - \mu S_2(x, t), \\
\frac{\partial E(x, t)}{\partial t} &= D_0 \Delta E(x, t) + \beta [S_1(x, t) + \sigma S_2(x, t)] [\theta I_1(x, t) + I_2(x, t)] \\
&\quad - (\mu + \gamma) E(x, t), \\
\frac{\partial I_1(x, t)}{\partial t} &= D_1 \Delta I_1(x, t) + q_1 \gamma E(x, t) - (\mu + \alpha_1) I_1(x, t), \\
\frac{\partial I_2(x, t)}{\partial t} &= D_2 \Delta I_2(x, t) + q_2 \gamma E(x, t) - (\mu + \alpha_2) I_2(x, t),
\end{align*}
\]

where $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

The structure of this article is as follows. In Section 2, we firstly obtain the basic reproduction number $R_0$ and two non-negative constant equilibriums for the corresponding reaction system, and these equilibriums are also two non-negative constant equilibriums for system (2). Then, we find a $c^* > 0$ called as the minimal spread speed. Next, as $R_0 > 1$ and the spread speed $c > c^*$, we prove that system (2) admits non-trivial and non-negative traveling wave solution by applying the Schauder fixed point theorem. Meanwhile, a suitable Lyapunov functional is used to prove the asymptotic behavior of traveling wave solution at $x \to +\infty$. At last, we obtain the existence of traveling wave solution connecting two non-negative constant equilibriums when $R_0 > 1$ and $c = c^*$. In Section 3, we firstly prove the non-existence of non-trivial and non-negative traveling wave solution connecting two non-negative constant equilibriums as $R_0 \leq 1$. In particular, when $R_0 > 1$ and $c \in (0, c^*)$, we also study the non-existence via the two-sided Laplace transform. In Section 4, we take advantage of numerical simulations to display the existence of traveling wave solution connecting two non-negative constant equilibriums. Simultaneously, we also conclude that self-protection and treatment can decrease the spread speed for an epidemic via numerical simulations.

2. Existence of traveling wave solution. In order to research traveling wave solution of system (2), the first step is to find what kind of constant equilibriums of system (2) exist. It is well known that there always exists an equilibrium
(S_1^0, S_2^0, 0, 0, 0) = (\frac{\Lambda}{\rho+\mu}, \frac{\rho \Lambda}{\mu(\rho+\mu)}, 0, 0, 0) named as the disease-free equilibrium of system (2). We can also find a positive equilibrium by the following system,

\[
\begin{align*}
\frac{dS_1(t)}{dt} &= \Lambda - \beta S_1(t) [\theta I_1(t) + I_2(t)] - (\rho + \mu) S_1(t), \\
\frac{dS_2(t)}{dt} &= \rho S_1(t) - \beta \sigma S_2(t) [\theta I_1(t) + I_2(t)] - \mu S_2(t), \\
\frac{dE(t)}{dt} &= \beta [S_1(t) + \sigma S_2(t)] [\theta I_1(t) + I_2(t)] - (\mu + \gamma) E(t), \\
\frac{dI_1(t)}{dt} &= q_1 \gamma E(t) - (\mu + \alpha_1) I_1(t), \\
\frac{dI_2(t)}{dt} &= q_2 \gamma E(t) - (\mu + \alpha_2) I_2(t). 
\end{align*}
\]

The point (S_1^0, S_2^0, 0, 0, 0) is also the disease-free equilibrium of system (3). Via the next generation matrix method formulated in [24], the basic reproduction number for system (3) denoted by R_0 can be expressed as

\[
R_0 = \left[ \frac{q_1 \gamma \theta \beta}{(\mu + \gamma)(\mu + \alpha_1)} + \frac{q_2 \gamma \beta}{(\mu + \gamma)(\mu + \alpha_2)} \right] (S_1^0 + \sigma S_2^0).
\]

Furthermore, taking advantage of the direct Lyapunov functional method, which is similar in [15], can claim the below theorem.

**Theorem 2.1.** If R_0 \leq 1, then there exists a unique constant equilibrium of system (3) which is the disease-free equilibrium (S_1^0, S_2^0, 0, 0, 0), and it is globally asymptotically stable. If R_0 > 1, system (3) admits a positive equilibrium which is the endemic equilibrium (S_1^*, S_2^*, E^*, I_1^*, I_2^*), and it is globally asymptotically stable.

In the rest of Section 2, we always assume R_0 > 1. Then system (2) exists two constant equilibriums: the disease-free equilibrium (S_1^0, S_2^0, 0, 0, 0) and the endemic equilibrium (S_1^*, S_2^*, E^*, I_1^*, I_2^*). So we research the existence of traveling wave solution satisfying system (2) and connecting (S_1^0, S_2^0, 0, 0, 0) and (S_1^*, S_2^*, E^*, I_1^*, I_2^*). Now, we show that traveling wave solution of system (2) is special solution with the form as

\[
(S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi)), \quad \xi = x + ct \in \mathbb{R}. \tag{4}
\]

Substitute formula (4) into system (2), and we can obtain the wave form equations as follows:

\[
\begin{align*}
\frac{dS_1' (\xi)}{d\xi} &- c S_1' (\xi) + \Lambda - \beta S_1(\xi) \theta I_1(\xi) + I_2(\xi)] - (\rho + \mu) S_1(\xi) = 0, \\
\frac{dS_2' (\xi)}{d\xi} &- c S_2' (\xi) + \rho S_1(\xi) - \beta \sigma S_2(\xi) \theta I_1(\xi) + I_2(\xi)] - \mu S_2(\xi) = 0, \\
D_0 E'' (\xi) &- c E' (\xi) + \beta \theta I_1(\xi) + I_2(\xi)] [S_1(\xi) + \sigma S_2(\xi)] - (\mu + \gamma) E(\xi) = 0, \\
D_1 I_1' (\xi) &- c I_1' (\xi) + q_1 \gamma E(\xi) - (\mu + \alpha_1) I_1(\xi) = 0, \\
D_2 I_2' (\xi) &- c I_2' (\xi) + q_2 \gamma E(\xi) - (\mu + \alpha_2) I_2(\xi) = 0,
\end{align*}
\]

for \( \xi \in \mathbb{R} \), where ’ and ” represent the first and the second derivative with respect to \( \xi \), respectively. Since we would like to find positive traveling wave solution connecting two equilibria, we need the positive solution (S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi)) of system (5) with following boundary conditions

\[
\begin{align*}
S_1(-\infty) &= S_1^0, \quad E(-\infty) = 0, \quad I_1(-\infty) = 0, \\
S_1(+\infty) &= S_1^*, \quad E(+\infty) = E^*, \quad I_1(+\infty) = I_1^*, \quad i = 1, 2.
\end{align*}
\]
Linearizing the $E$-th and $I_r$-th equations of (5) at the disease-free equilibrium $(S_1^0, S_2^0, 0, 0, 0)$ yields
\[
\begin{align*}
D_0 E''(\xi) - c E'(\xi) + \beta [\theta I_1(\xi) + I_2(\xi)] (S_1^0 + \sigma S_2^0) - (\mu + \gamma) E(\xi) &= 0, \\
D_1 I_1''(\xi) - c I_1'(\xi) + q_1 \gamma E(\xi) - (\mu + \alpha_1) I_1(\xi) &= 0, \\
D_2 I_2''(\xi) - c I_2'(\xi) + q_2 \gamma E(\xi) - (\mu + \alpha_2) I_2(\xi) &= 0.
\end{align*}
\] Set $(E(\xi), I_1(\xi), I_2(\xi)) = (\eta_0, \eta_1, \eta_2) e^{\lambda c}$ and plug it into above equations, then we can obtain characteristic equations
\[
\begin{align*}
D_0 \eta_0 \lambda^2 - c \eta_0 \lambda + \beta (\theta \eta_1 + \eta_2) (S_1^0 + \sigma S_2^0) - (\mu + \gamma) \eta_0 &= 0, \\
D_1 \eta_1 \lambda^2 - c \eta_1 \lambda + q_1 \gamma \eta_0 - (\mu + \alpha_1) \eta_1 &= 0, \\
D_2 \eta_2 \lambda^2 - c \eta_2 \lambda + q_2 \gamma \eta_0 - (\mu + \alpha_2) \eta_2 &= 0.
\end{align*}
\] (7)

Let
\[
\hat{A} = \begin{pmatrix} D_0 & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & D_2 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix},
\]
\[
\hat{V} = \begin{pmatrix} \mu + \gamma & 0 & 0 \\ 0 & \mu + \alpha_1 & 0 \\ 0 & 0 & \mu + \alpha_2 \end{pmatrix},
\]
\[
\hat{F} = \begin{pmatrix} 0 & \beta \theta (S_1^0 + \sigma S_2^0) & \beta (S_1^0 + \sigma S_2^0) \\ q_1 \gamma & 0 & 0 \\ q_2 \gamma & 0 & 0 \end{pmatrix}.
\]

Denote $\Theta(\lambda, c) = \lambda^2 \hat{A} - \mu \hat{B} - \hat{V} + \hat{F}$. Then system (7) can reduce to
\[
\Theta(\lambda, c) (\eta_0, \eta_1, \eta_2)^T = 0.
\]

And define $A = \hat{V}^{-1} \hat{A}$, $B = \hat{V}^{-1} \hat{B}$, $F = \hat{V}^{-1} \hat{F}$, then we yield that
\[
(-A \lambda^2 + B \lambda + I)^{-1} F \eta = \eta,
\] (8)
and
\[
(-A \lambda^2 + B \lambda + I)^{-1} F = \begin{pmatrix} 0 & \frac{\beta \theta (S_1^0 + \sigma S_2^0)}{m_0(\lambda, c)} & \frac{\beta (S_1^0 + \sigma S_2^0)}{m_0(\lambda, c)} \\ \frac{q_1 \gamma}{m_1(\lambda, c)} & 0 & 0 \\ \frac{q_2 \gamma}{m_2(\lambda, c)} & 0 & 0 \end{pmatrix},
\]
where $\eta = (\eta_0, \eta_1, \eta_2)^T$, $m_0(\lambda, c) = -D_0 \lambda^2 + c \lambda + \mu + \gamma$, $m_1(\lambda, c) = -D_1 \lambda^2 + c \lambda + \mu + \alpha_1$, $m_2(\lambda, c) = -D_2 \lambda^2 + c \lambda + \mu + \alpha_2$. Let $M(\lambda, c) = (-A \lambda^2 + B \lambda + I)^{-1} F$, then equation (8) becomes
\[
M(\lambda, c) \eta = \eta.
\]

Let $\rho(\lambda, c)$ be the principal eigenvalue of $M(\lambda, c)$ and define
\[
\lambda_c = \min \left\{ \frac{c + \sqrt{c^2 - 4 D_0 (\mu + \gamma)}}{2 D_0}, \frac{c + \sqrt{c^2 - 4 D_1 (\mu + \alpha_1)}}{2 D_1}, \frac{c + \sqrt{c^2 - 4 D_2 (\mu + \alpha_2)}}{2 D_2} \right\}.
\]

For $c \geq 0$ and $\lambda \in [0, \lambda_c)$, a direct calculation indicates that
\[
\rho(\lambda, c) = \left[ \frac{\beta \theta q_1 \gamma}{m_0(\lambda, c) m_1(\lambda, c)} + \frac{\beta q_2 \gamma}{m_0(\lambda, c) m_2(\lambda, c)} \right] \frac{(S_1^0 + \sigma S_2^0)}{2}. \] (9)
Moreover, some properties of \( \rho(\lambda, c) \) can be described as the following lemma:

**Lemma 2.2.** Below statements hold:

(i) \( \lambda_c \) is strictly increasing in \( c \in [0, +\infty) \), and \( \lim_{c \to +\infty} \lambda_c = +\infty \),

(ii) \( \rho(0, c) = \sqrt{R_0} \) for any \( c \in [0, +\infty) \), \( \rho(\lambda, 0) \) is strictly increasing in \([0, \lambda_c)\),

and \( \rho(\lambda, c) \to +\infty \) as \( \lambda \to \lambda_c - 0 \) for any \( c \geq 0 \), where \( R_0 \) is the basic reproduction number for system (3).

(iii) \( \forall \lambda \in (0, \lambda_c), \frac{\partial}{\partial \lambda} \rho(\lambda, c) < 0 \).

**Proof.** It is well known that (i) is established, so we only prove (ii) and (iii). It follows from the definitions of both \( R_0 \) and \( \rho(\lambda, c) \) that \( \rho(0, c) = \sqrt{R_0} \) for any \( c \in [0, +\infty) \). Due to \( \rho(\lambda, 0) > 0 \) and \( m_i(\lambda, 0) > 0 \) for \( \lambda \in [0, \lambda_c) \) with \( i = 0, 1, 2 \), it is the fact that

\[
\frac{d}{d\lambda} \rho(\lambda, 0) = \rho^{-\frac{1}{2}}(\lambda, 0) \left( \beta q_1 \gamma(S_1^0 + \sigma S_2^0)(D_0 \lambda m_1(\lambda, 0) + D_2 \lambda m_0(\lambda, 0)) \right.
\]

\[
\left. + \beta q_2 \gamma(S_1^0 + \sigma S_2^0)(D_0 \lambda m_2(\lambda, 0) + D_2 \lambda m_0(\lambda, 0)) \right) > 0
\]

for \( \lambda \in [0, \lambda_c) \). In addition, we can indicate \( \lim_{\lambda \to \lambda_c - 0} m_i(\lambda, c) = 0 \) \((i = 0, 1, 2)\), respectively, so the principal eigenvalue \( \rho(\lambda, c) \) tends to \(+\infty\) as \( \lambda \to \lambda_c - 0 \). The direct derivative of the spread speed \( c \) can show that

\[
\frac{\partial}{\partial c} \rho(\lambda, c) = -\frac{1}{2} \lambda \rho^{-\frac{1}{2}}(\lambda, c) \left( \frac{\beta \theta q_1 \gamma(S_1^0 + \sigma S_2^0)}{m_0^2(\lambda, c)m_1^2(\lambda, c)} + \frac{\beta \theta q_2 \gamma(S_1^0 + \sigma S_2^0)}{m_0^2(\lambda, c)m_2^2(\lambda, c)} \right) < 0,
\]

for \( \forall \lambda \in [0, \lambda_c) \). The proof is complete. \( \square \)

According to Lemma 2.2, we define

\[
\hat{\lambda}(c) = \min_{\lambda \in [0, \lambda_c)} \rho(\lambda, c), \quad \forall c \geq 0.
\]

Then we conclude \( \hat{\lambda}(0) = \sqrt{R_0} \) and \( \lim_{c \to +\infty} \hat{\lambda}(c) = 0 \). Meanwhile, \( \hat{\lambda}(c) \) is continuous and strictly decreasing in \( c \geq 0 \). Assume that \( R_0 > 1 \), then there exists a \( c^* > 0 \) such that \( \hat{\lambda}(c^*) = 1 \), \( \hat{\lambda}(c) > 1 \) for \( c \in [0, c^*) \) and \( \hat{\lambda}(c) < 1 \) for \( c \in (c^*, +\infty) \). Let

\[
\lambda_* = \inf \{ \lambda \in [0, \lambda_c) : \rho(\lambda, c^*) = 1 \},
\]

it indicates that \( \rho(\lambda_*, c^*) = 1 \) and \( \rho(\lambda_*, c) < 1 \) for any \( c > c^* \). Define

\[
\lambda_1(c) = \sup \{ \lambda \in (0, \lambda_*) : \rho(\lambda, c) = 1, \rho(\lambda', c) \geq 1, \forall \lambda' \in (0, \lambda) \}.
\]

Because of \( \rho(\lambda_*, c) < 1 \) for \( c > c^* \), we claim that the following lemma is established.

**Lemma 2.3.** Assume that \( R_0 > 1 \), then there exist \( c^* > 0 \), named as the minimal spread speed, and \( \lambda_1 \in (0, \lambda_*) \) such that

(i) \( \rho(c, c) > 1 \) for any \( c \in [0, c^*) \) and \( \lambda \in (0, \lambda_*) \),

(ii) \( \rho(\lambda_*, c^*) = 1, \rho(\lambda, c^*) > 1 \) for any \( \lambda \in [0, c^*) \) and \( \rho(\lambda, c^*) \geq 1 \) for any \( \lambda \in (0, \lambda_*) \),

(iii) as \( c > c^* \), there exists \( \lambda_1(c) \in (0, \lambda_*) \) such that \( \rho(\lambda_1(c), c) = 1, \rho(\lambda, c) \geq 1 \) for \( \lambda \in (0, \lambda_1(c)) \) and \( \rho(\lambda_1(c) + \varepsilon_n(c), c) < 1 \) for some decreasing sequence \( \{\varepsilon_n(c)\} \) satisfying \( \lim_{n \to +\infty} \varepsilon_n = 0 \) and \( \lambda_1(c) + \varepsilon_n(c) < \lambda_* \) \((n \in \mathbb{Z})\). Especially, \( \lambda_1(c) \) is strictly decreasing in \( c > c^* \).

Since the \( \mathcal{M}(\lambda, c) \) is non-negative and irreducible for \( \lambda \in [0, \lambda_c) \), the below lemma holds with utilizing the Perron-Frobenius theorem.
Lemma 2.4. Assume that $R_0 > 1$, for $c > c^*$, there are positive unit vectors $\eta(c) = (\eta_0(c), \eta_1(c), \eta_2(c))^T$ and $\zeta^n(c) = (\zeta^n_0(c), \zeta^n_1(c), \zeta^n_2(c))^T$ for $n \in \mathbb{Z}$ such that

$$ \mathcal{M}(\lambda_1(c), c)\eta(c) = \eta(c), $$

$$ \mathcal{M}(\lambda_1(c) + \varepsilon_n(c), c)\zeta^n(c) = \rho(\lambda_1(c) + \varepsilon_n(c), c)\zeta^n(c). $$

Fix $c > c^*$ and let $\lambda_1(c), \eta(c) = (\eta_0(c), \eta_1(c), \eta_2(c))^T$ and $\zeta^n(c) = (\zeta^n_0(c), \zeta^n_1(c), \zeta^n_2(c))^T$ for $n \in \mathbb{Z}$ own same definitions in Lemma 2.3 and Lemma 2.4. For convenience, we use $\lambda_1, \varepsilon_n, \eta = (\eta_0, \eta_1, \eta_2)^T$ and $\zeta^n = (\zeta^n_0, \zeta^n_1, \zeta^n_2)^T (n \in \mathbb{Z})$ instead of them, respectively. It follows from Lemma 2.4 and $\rho(\lambda_1 + \varepsilon_n, c) < 1$ that following equations and inequalities are established for $n \in \mathbb{Z}$

$$ \begin{cases} 
- m_0(\lambda_1, c)\eta_0 + (S_1^0 + \sigma S_2^0)(\theta \eta_1 + \eta_2) = 0, \\
- m_1(\lambda_1, c)\eta_1 + q_1 \gamma \eta_0 = 0, \\
- m_2(\lambda_1, c)\eta_2 + q_2 \gamma \eta_0 = 0,
\end{cases} \quad (10) $$

and

$$ \begin{cases} 
- m_0(\lambda_1 + \varepsilon_n, c)\zeta^n_0 + (S_1^0 + \sigma S_2^0)(\theta \zeta^n_1 + \zeta^n_2) < 0, \\
- m_1(\lambda_1 + \varepsilon_n, c)\zeta^n_1 + q_1 \gamma \zeta^n_0 < 0, \\
- m_2(\lambda_1 + \varepsilon_n, c)\zeta^n_2 + q_2 \gamma \zeta^n_0 < 0.
\end{cases} \quad (11) $$

According to the above argument, we can construct suitable sub- and supersolutions which are defined in below lemmas. And then the local existence of solution for system (5) is proved via the Schauder fixed point theorem.

Lemma 2.5. Let the vector function $\mathcal{P}(\xi) = (p_0(\xi), p_1(\xi), p_2(\xi))^T$ with $p_i(\xi) = \eta_i(\xi)e^{\lambda_1 \xi}$ for $\xi \in \mathbb{R}$ and $i = 0, 1, 2$, and it satisfies

$$ \begin{cases} 
D_0p_0''(\xi) - cp_0'(\xi) + \beta(S_1^0 + \sigma S_2^0)(\theta p_1(\xi) + p_2(\xi)) - (\mu + \gamma)p_0(\xi) = 0, \\
D_1p_1''(\xi) - cp_1'(\xi) + q_1 \gamma p_0(\xi) - (\mu + \alpha_1)p_1(\xi) = 0, \\
D_2p_2''(\xi) - cp_2'(\xi) + q_2 \gamma p_0(\xi) - (\mu + \alpha_2)p_2(\xi) = 0.
\end{cases} \quad (i = 1, 2) $$

Lemma 2.6. For each $\omega > 0$ small enough with $\omega < \min\{\lambda_1, \frac{c}{\alpha}\}$ and $M > 1$ sufficiently large, the vector function $S^- = (S^-_1(\xi), S^-_2(\xi))^T$ defined by $S^-_i(\xi) = \max\{S_1^0(1 - M e^{\omega \xi}), 0\} (i = 1, 2)$ satisfies

$$ cS'_1(\xi) \leq dS''_1(\xi) + \Lambda - (\mu + \rho)S^-_1(\xi) - \beta S^-_1(\xi)(\theta p_1(\xi) + p_2(\xi)), $$

$$ cS'_2(\xi) \leq dS''_2(\xi) + \rho S^-_1(\xi) - \mu S^-_2(\xi) - \beta \sigma S^-_2(\xi)(\theta p_1(\xi) + p_2(\xi)), $$

for $\xi \in \mathbb{R}$ and $\xi \neq -\frac{1}{\omega} \ln M$.

Proof. When $\xi > -\frac{1}{\omega} \ln M$, $S^-_i(\xi) = 0$ for $i = 1, 2$, and inequalities (12)-(13) hold. When $\xi < -\frac{1}{\omega} \ln M$, it implies that $S^-_i(\xi) = S_1^0(1 - M e^{\omega \xi})$ and $p_i(\xi) = \eta_i e^{\lambda_1 \xi}$. Due to $\omega < \frac{c}{\alpha}$ and $e^{-\frac{\lambda_1 \xi}{\omega}} \ln M \to 0$ as $M \to +\infty$, it is shown that

$$ (-d\omega + c)S_1^0\omega M e^{\omega \xi} + \Lambda - (\mu + \rho)S_1^0(1 - M e^{\omega \xi}) - \beta(\theta \eta_1 + \eta_2)S_1^0(1 - M e^{\omega \xi})e^{\lambda_1 \xi} $$

$$ = (-d\omega + c)S_1^0\omega M e^{\omega \xi} + (\mu + \rho)S_1^0 Me^{\omega \xi} - \beta(\theta \eta_1 + \eta_2)S_1^0(1 - M e^{\omega \xi})e^{\lambda_1 \xi} $$

$$ \geq \left[(-d\omega + c)S_1^0\omega M + (\mu + \rho)S_1^0 M - \beta(\theta \eta_1 + \eta_2)S_1^0 e^{-\frac{\lambda_1 \xi}{\omega}} \ln M\right] e^{\omega \xi} $$

$$ \geq 0,$
so the inequality (12) is established. Furthermore,
\[
(-d\omega + c)S_0^2\omega M e^{\omega \xi} + \rho S_1^0(1 - M e^{\omega \xi}) - \beta \sigma(\theta \eta_1 + \eta_2)S_2^0(1 - M e^{\omega \xi})e^{\lambda_1 \xi} \\
- \mu S_0^0(1 - M e^{\omega \xi}) \\
= (-d\omega + c)S_0^2\omega M e^{\omega \xi} - \beta \sigma(\theta \eta_1 + \eta_2)S_2^0(1 - M e^{\omega \xi})e^{\lambda_1 \xi} \\
\geq \left[(-d\omega + c)S_1^0\omega M - \beta \sigma(\theta \eta_1 + \eta_2)S_1^0 e^{-\lambda_1 - \ln M}\right] e^{\omega \xi} \\
\geq 0,
\]
and the inequality (13) is set up. The lemma is completely proved.

**Lemma 2.7.** Let \(0 < \epsilon < \frac{\pi}{2}\) with \(\epsilon = \varepsilon_{n_0}\) for some \(n_0 \in \mathbb{Z}\), and the eigenvector \(\zeta^{n_0} = (\zeta_0^{n_0}, \zeta_1^{n_0}, \zeta_2^{n_0})^T\) is defined by \(\zeta = (\zeta_0, \zeta_1, \zeta_2)^T\). We define the vector map \(\mathcal{H}(\xi) = (h_0(\xi), h_1(\xi), h_2(\xi))^T\) with \(h_i(\xi) = \max\{\eta_i e^{\lambda_i \xi} - K \zeta_i e^{(\lambda_1 + \epsilon) \xi}, 0\}\), where \(\eta_i\) and \(\zeta_i\) are defined in Lemma 2.4 with \(i = 0, 1, 2\), respectively. For \(K > 0\) large enough such that
\[
\min\left\{\frac{1}{\epsilon} \ln \frac{K \zeta_0}{\eta_0}, \frac{1}{\epsilon} \ln \frac{K \zeta_1}{\eta_1}, \frac{1}{\epsilon} \ln \frac{K \zeta_2}{\eta_2}\right\} > \frac{1}{\omega} \ln M,
\]
then the vector map \(\mathcal{H}(\xi)\) satisfies
\[
ch_0^i(\xi) \leq D_0 h_{0''}(\xi) - (\mu + \gamma) h_0(\xi) + \beta (S_1^- (\xi) + \sigma S_2^- (\xi)) [\theta h_1(\xi) + h_2(\xi)], \\
\xi < \frac{1}{\epsilon} \ln \frac{\eta_0}{K \zeta_0}, \quad (14)
\]
\[
ch_1^i(\xi) \leq D_1 h_{1''}(\xi) + q_1 \gamma h_0(\xi) - (\mu + \alpha_1) h_1(\xi), \quad \xi < \frac{1}{\epsilon} \ln \frac{\eta_1}{K \zeta_1}, \quad (15)
\]
\[
ch_2^i(\xi) \leq D_2 h_{2''}(\xi) + q_2 \gamma h_0(\xi) - (\mu + \alpha_2) h_2(\xi), \quad \xi < \frac{1}{\epsilon} \ln \frac{\eta_2}{K \zeta_2}. \quad (16)
\]

**Proof.** Firstly, we prove the inequality (14). For \(\xi < \frac{1}{\epsilon} \ln \frac{\eta_0}{K \zeta_0}\), we yield \(h_0(\xi) = \eta_0 e^{\lambda_1 \xi} - K \zeta_0 e^{(\lambda_1 + \epsilon) \xi}, S_1^- (\xi) = S_1^0(1 - M e^{\omega \xi}), \) and \(S_2^- (\xi) = S_2^0(1 - M e^{\omega \xi})\). To prove the inequality (14), it is sufficient to show the following inequality:
\[
K \left[-D_0 \eta_0 (\lambda_1 + \epsilon)^2 + c \zeta_0 (\lambda_1 + \epsilon) + (\mu + \gamma) \zeta_0 \right] e^{(\lambda_1 + \epsilon) \xi} + \beta (S_1^- + \sigma S_2^-) [\theta h_1(\xi) \\
+ h_2(\xi)] + D_0 \eta_0 \lambda_1 e^{\lambda_1 \xi} - c \eta_0 \lambda_1 e^{\lambda_1 \xi} - (\mu + \gamma) \eta_0 e^{\lambda_1 \xi} \geq 0,
\]
which is
\[-K \zeta_0 m_0 (\lambda_1 + \epsilon, c) e^{(\lambda_1 + \epsilon) \xi} + \eta_0 m_0 (\lambda_1, c) e^{\lambda_1 \xi} - \beta (S_1^- + \sigma S_2^-) [\theta h_1(\xi) + h_2(\xi)] \leq 0.
\]
According to equations (10), the proof of the inequality (17) can be replaced by proving
\[
-K \zeta_0 m_0 (\lambda_1 + \epsilon, c) e^{(\lambda_1 + \epsilon) \xi} + \beta (S_1^0 + \sigma S_2^0)(\theta \eta_1 + \eta_2) e^{\lambda_1 \xi} - \beta (S_1^- + \sigma S_2^-) [\theta h_1(\xi) + h_2(\xi)] \leq 0. \quad (18)
\]
Because $S^0_i - S_i^- \leq S^0_i Me^{\omega \xi}$ and $\eta_i e^{\lambda_i \xi} - h_i(\xi) \leq K \zeta e^{(\lambda_i + \rho) \xi}$ for $i = 1, 2$, it is obtained that

$$
\begin{align*}
\beta(S^0_i + \sigma S^0_j)(\theta \eta_1 + \eta_2)e^{\lambda_i \xi} - \beta(S^-_i + \sigma S^-_j)[\theta h_1(\xi) + h_2(\xi)] \\
= \beta(S^0_i + \sigma S^0_j)(\theta \eta_1 + \eta_2)e^{\lambda_i \xi} - \beta(S_i^0 + \sigma S_j^0)[\theta h_1(\xi) + h_2(\xi)] \\
+ \beta(S_i^0 + \sigma S_j^0)[\theta h_1(\xi) + h_2(\xi)] - \beta(S_i^- + \sigma S_j^-)[\theta h_1(\xi) + h_2(\xi)] \\
= \beta(S_i^0 + \sigma S_j^0)[\theta(\eta_1 e^{\lambda_i \xi} - h_1(\xi)) + (\eta_2 e^{\lambda_i \xi}) - h_2(\xi)] \\
+ \beta[\theta h_1(\xi) + h_2(\xi)][(S_i^0 - S_i^-) + \sigma(S_j^0 - S_j^-)] \\
\leq \beta(S_i^0 + \sigma S_j^0)[\theta K \zeta e^{(\lambda_i + \rho) \xi} + K \zeta e^{(\lambda_i + \rho) \xi}] + \beta[\theta h_1(\xi) + h_2(\xi)](S_i^0 Me^{\omega \xi} + \sigma S_j^0 Me^{\omega \xi}).
\end{align*}
$$

Thus, for the proof of the inequality (18), we only need to prove

$$
K e^{(\lambda_i + \rho) \xi}[-m_0(\lambda_1 + \rho)\zeta_0 + \beta(S_i^0 + \sigma S_j^0)(\theta \zeta_1 + \zeta_2)] + M \beta(S_i^0 + \sigma S_j^0)[\theta \eta_1(\xi) + \eta_2(\xi)] e^{(\omega - \rho) \xi} \leq 0,
$$

which is

$$
K[-m_0(\lambda_1 + \rho)\zeta_0 + \beta(S_i^0 + \sigma S_j^0)(\theta \zeta_1 + \zeta_2)] + M \beta(S_i^0 + \sigma S_j^0)[\theta \eta_1(\xi) + \eta_2(\xi)] e^{(\omega - \rho) \xi} \leq 0. \tag{19}
$$

For $\xi < \frac{1}{\tau} \ln \frac{m_0}{K \zeta_0}$, it is the fact that $e^{(\omega - \rho) \xi} \to 0$ as $K \to +\infty$. Thus, combining inequalities (11), the inequality (19) is obtained, and the inequality (14) is established. Finally, we can prove inequalities (15) and (16) via the similar way in the proof of the inequality (14). The proof is complete. \hfill \Box

Now, we set $X > \max \left\{ \frac{1}{\tau} \ln \frac{m_0}{K \zeta_0}, \frac{1}{\tau} \ln \frac{m_1}{K \zeta_1}, \frac{1}{\tau} \ln \frac{m_2}{K \zeta_2} \right\}$. Define

$$
\Gamma_X = \begin{cases} 
\chi_i(\cdot), \chi_2(\cdot), \varphi_0(\cdot), \varphi_1(\cdot), \varphi_2(\cdot) & |\chi_i(\pm X) = S_i^- (\pm X), \\
\in C([-X, X], \mathbb{R}^5) & |\varphi_j(\pm X) = h_j(\pm X), \\
S_{-1}^- (\xi) \leq \chi_i(\xi) \leq S_i^0, & |S_i^- (\xi) \leq \chi_i(\xi) \leq S_i^0, \\
\end{cases}
$$

where $i = 1, 2$ and $j = 0, 1, 2$. And it is well known that $\Gamma_X$ is closed and convex. For any

$$
(\chi_1(\cdot), \chi_2(\cdot), \varphi_0(\cdot), \varphi_1(\cdot), \varphi_2(\cdot)) \in \Gamma_X,
$$

we consider the following boundary-value problem for $\xi \in (-X, X)$,

$$
\begin{cases} 
-dS_{1,X}''(\xi) + cS_{1,X}'(\xi) - \Lambda + (\mu + \rho)S_{1,X}(\xi) + \beta S_{1,X}(\xi)[\theta \varphi_1(\xi) + \varphi_2(\xi)] = 0, \\
-dS_{2,X}''(\xi) + cS_{2,X}'(\xi) - \rho \chi_1(\xi) + \mu S_{2,X}(\xi) + \beta \sigma S_{2,X}(\xi)[\theta \varphi_1(\xi) + \varphi_2(\xi)] = 0, \\
-D_0 E_{X}'(\xi) + cE_{X}'(\xi) - \beta [\chi_1(\xi) + \sigma \chi_2(\xi)][\theta \varphi_1(\xi) + \varphi_2(\xi)] + (\mu + \gamma)E_{X}(\xi) = 0, \\
-D_1 I_{1,X}'(\xi) + cI_{1,X}'(\xi) - q_{11} \gamma \varphi_0(\xi) + (\mu + \alpha_1)I_{1,X}(\xi) = 0, \\
-D_2 I_{2,X}'(\xi) + cI_{2,X}'(\xi) - q_{21} \gamma \varphi_0(\xi) + (\mu + \alpha_2)I_{2,X}(\xi) = 0,
\end{cases}
$$

satisfying the below boundary condition:

$$
S_{1,X}(\pm X) = S_i^- (\pm X), \quad S_{2,X}(\pm X) = S_i^- (\pm X), \\
E_{X}(\pm X) = h_0(\pm X), \quad I_{1,X}(\pm X) = h_1(\pm X), \quad I_{2,X}(\pm X) = h_2(\pm X). \tag{20}
$$

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Applying the Gilbarg and Trudinger’s Corollary 9.18 in [6], we can claim that there exists a unique solution

\[(S_{1,X}, S_{2,X}, E_X, I_{1,X}, I_{2,X}),\]

satisfying the problems (20)-(21), where

\[(S_{1,X}, S_{2,X}, E_X, I_{1,X}, I_{2,X}) \in W^{2,p}((-X, X), \mathbb{R}^5) \cap C([-X, X], \mathbb{R}^5),\]

for \(p > 1\). And it is shown that \((S_{1,X}, S_{2,X}, E_X, I_{1,X}, I_{2,X}) \in W^{2,p}((-X, X), \mathbb{R}^5) \hookrightarrow C^{1,\alpha}([-X, X], \mathbb{R}^5)\) for some \(\alpha \in (0, 1)\) via the embedding theorem (see Gilbarg and Trudinger’s Theorem 7.26 in [6]). Define the operator \(T = (T_1, T_2, T_3, T_4, T_5)\) on \(\Gamma_X\) such that

\[
\begin{align*}
S_{1,X} &= T_1(\chi_1, \chi_2, \varphi_0, \varphi_1, \varphi_2), \\
S_{2,X} &= T_2(\chi_1, \chi_2, \varphi_0, \varphi_1, \varphi_2), \\
E_X &= T_3(\chi_1, \chi_2, \varphi_0, \varphi_1, \varphi_2), \\
I_{1,X} &= T_4(\chi_1, \chi_2, \varphi_0, \varphi_1, \varphi_2), \\
I_{1,X} &= T_5(\chi_1, \chi_2, \varphi_0, \varphi_1, \varphi_2),
\end{align*}
\]

for any \((\chi_1, \chi_2, \varphi_0, \varphi_1, \varphi_2) \in \Gamma_X\).

**Lemma 2.8.** The operator \(T\) maps \(\Gamma_X\) into \(\Gamma_X\).

**Proof.** Firstly, we consider \(S_i(\xi)\) \((i = 1, 2)\). It is well known that 0 is the sub-solution of the \(S_i\)-th equation in equation (20), respectively. \(S_i^0\) and \(S_i^2\) are super-solutions of \(S_i\)-th equations in equation (20), respectively. According to \(0 < S_{1,X}(X) = S_1^- (X) < S_0^0\) and \(0 = S_{i,X}(-X) = S_i^- (-X) < S_0^0\), while combining the maximum principle in [19], it is the fact that \(0 \leq S_{i,X}(\xi) \leq S_0^0\) for \(\forall \xi \in [-X, X]\). Due to Lemma 2.6, it yields that \(S_i^- (\xi)\) satisfies

\[
0 \geq -dS_1^\omega (\xi) + cS_1^- (\xi) - \Lambda + (\mu + \rho)S_1^- (\xi) + \beta S_1^- (\xi)(\theta p_1(\xi) + p_2(\xi))
\]

\[
\geq -dS_1^\omega (\xi) + cS_1^- (\xi) - \Lambda + (\mu + \rho)S_1^- (\xi) + \beta S_1^- (\xi)(\theta \varphi_1(\xi) + \varphi_2(\xi)),
\]

and

\[
0 \geq -dS_2^\omega (\xi) + cS_2^- (\xi) - \rho S_1^- (\xi) + \mu S_2^- (\xi) + \beta \sigma S_2^- (\xi)(\theta p_1(\xi) + p_2(\xi))
\]

\[
\geq -dS_2^\omega (\xi) + cS_2^- (\xi) - \rho S_1^- (\xi) + \mu S_2^- (\xi) + \beta \sigma S_2^- (\xi)(\theta \varphi_1(\xi) + \varphi_2(\xi)).
\]

for any \(\xi \in [-X, X']\) with \(X' = -\frac{1}{\alpha} \ln M\). Because of \(S_{i,X}(X) = S_i^- (-X)\) and \(S_{i,X}(X') \geq S_i^- (X') = 0\), it is concluded that \(S_i^- (\xi) \leq S_{i,X}(\xi)\) for \(\xi \in [-X, X']\) by the maximum principle. Therefore, we claim that \(S_i^- (\xi) \leq S_{i,X}(\xi) \leq S_0^0\) for \(\xi \in [-X, X]\).

Secondly, we consider about \(E_X, I_{1,X}\) and \(I_{2,X}\). It follows from the maximum principle that \(E_X \geq 0, I_{1,X} \geq 0\) for any \(\xi \in [-X, X]\). According to Lemma 2.5, it is obtained that

\[
0 = -D_0 p_0''(\xi) + c p_0'(\xi) - \beta (S_0^0 + \sigma S_2^0)(\theta p_1(\xi) + p_2(\xi)) + (\mu + \gamma) p_0(\xi)
\]

\[
\leq -D_0 p_0''(\xi) + c p_0'(\xi) - \beta (S_0^0 + \sigma S_2^0)(\theta \varphi_1(\xi) + \varphi_2(\xi)) + (\mu + \gamma) p_0(\xi)
\]

\[
0 = -D_1 p_1''(\xi) + c p_1'(\xi) - q_1 \gamma p_0(\xi) + (\mu + \alpha_1) p_1(\xi)
\]

\[
\leq -D_1 p_1''(\xi) + c p_1'(\xi) - q_1 \gamma \varphi_0(\xi) + (\mu + \alpha_1) p_1(\xi),
\]
and

\[ 0 = -D_2 p_2''(\xi) + c p_2'(\xi) - q_2 \gamma p_0(\xi) + (\mu + \alpha_2) p_2(\xi) \]

\[ \leq -D_2 p_2''(\xi) + c p_2'(\xi) - q_2 \gamma \varphi_0(\xi) + (\mu + \alpha_2) p_2(\xi), \]

for any \( \xi \in [-X, X]. \) So the maximum principle shows that \( E_X(\xi) \leq p_0(\xi), \) \( I_{1,X}(\xi) \leq p_1(\xi) \) and \( I_{2,X}(\xi) \leq p_2(\xi). \) for \( \xi \in [-X, X]. \) Furthermore, Lemma 2.7 implies that

\[ 0 \geq -D_0 h_0''(\xi) + c h_0'(\xi) + (\mu + \alpha) h_0(\xi) - \beta(S_1^- (\xi) + \sigma S_2^- (\xi)) (\theta h_1(\xi) + h_2(\xi)) \]

\[ \geq -D_0 h_0''(\xi) + c h_0'(\xi) + (\mu + \alpha) h_0(\xi) - \beta(\chi_1^- (\xi) + \sigma \chi_2^- (\xi)) (\theta \varphi_1(\xi) + \varphi_2(\xi)), \]

for \( \xi \in [-X, X_0'] \) with \( X_0' = \frac{1}{2} \ln \frac{m + \mu}{m_0}, \)

\[ 0 \geq -D_1 h_1''(\xi) + c h_1'(\xi) - q_1 \gamma h_0(\xi) + (\mu + \alpha_1) h_1(\xi) \]

\[ \geq -D_1 h_1''(\xi) + c h_1'(\xi) - q_1 \gamma \varphi_0(\xi) + (\mu + \alpha_1) h_1(\xi), \]

for \( \xi \in [-X, X_1'] \) with \( X_1' = \frac{1}{2} \ln \frac{m}{m_0}, \) and

\[ 0 \geq -D_2 h_2''(\xi) + c h_2'(\xi) - q_2 \gamma h_0(\xi) + (\mu + \alpha_2) h_2(\xi) \]

\[ \geq -D_2 h_2''(\xi) + c h_2'(\xi) - q_2 \gamma \varphi_0(\xi) + (\mu + \alpha_2) h_2(\xi), \]

for \( \xi \in [-X, X_2'] \) with \( X_2' = \frac{1}{2} \ln \frac{m}{m_0}. \) Owing to \( E_X(-X) = h_0(-X), \) \( I_{1,X}(-X) = h_1(-X) \) and \( I_{2,X}(-X) = h_2(-X), \) while combining the fact \( E_X(X_0') \geq h_0(X_0') = 0, \) \( I_{1,X}(X_1') \geq h_1(X_1') = 0 \) and \( I_{2,X}(X_2') \geq h_2(X_2') = 0, \) we can implies that \( h_0(\xi) \leq E_X(\xi), \) \( h_1(\xi) \leq I_{1,X}(\xi) \) and \( h_2(\xi) \leq I_{2,X}(\xi) \) for \( \xi \in [-X, X'] \) \( (i = 0, 1, 2) \) by the maximum principle. Thus, one yields \( h_0(\xi) \leq E_X(\xi) \leq p_0(\xi), \) \( h_1(\xi) \leq I_{1,X}(\xi) \leq p_1(\xi) \) and \( h_2(\xi) \leq I_{2,X}(\xi) \leq p_2(\xi) \) for \( \xi \in [-X, X]. \) This completes the proof.

By taking advantage of the classic embedding theorem, \( \mathcal{T} \) is a compact operator from \( \Gamma_X \) to \( \Gamma_X. \) In fact, \( \mathcal{T} : \Gamma_X \to \Gamma_X \) is also a completely continuous operator (see [27]). Above all, the Schauder fixed point theorem implies that there exists a vector function \( (S_{1,X}, S_{2,X}, E_X, I_{1,X}, I_{2,X}) \in \Gamma_X \) satisfying

\[ (S_{1,X}, S_{2,X}, E_X, I_{1,X}, I_{2,X}) = \mathcal{T}(S_{1,X}, S_{2,X}, E_X, I_{1,X}, I_{2,X}), \]

for \( \xi \in [-X, X], \) which includes that \( (S_{1,X}, S_{2,X}, E_X, I_{1,X}, I_{2,X}) \) satisfies following equations for \( i = 1, 2, \)

\[
\begin{aligned}
-d S_{1,X}'(\xi) &+ c S_{1,X}'(\xi) - \Lambda + (\mu + \rho) S_{1,X}(\xi) + \beta S_{1,X}(\xi) [\theta I_{1,X}(\xi) + I_{2,X}(\xi)] = 0, \\
-d S_{2,X}'(\xi) &+ c S_{2,X}'(\xi) - \rho S_{1,X}(\xi) + \mu S_{2,X}(\xi) + \beta S_{2,X}(\xi) [\theta I_{1,X}(\xi) + I_{2,X}(\xi)] = 0, \\
-D_0 E_X''(\xi) &+ c E_X'(\xi) - \beta [S_{1,X}(\xi) + \sigma S_{2,X}(\xi)] [\theta I_{1,X}(\xi) + I_{2,X}(\xi)] \\
&+ (\mu + \gamma) E_X(\xi) = 0, \\
-D_1 I_{1,X}'(\xi) &+ c I_{1,X}'(\xi) - q_1 \gamma E_X(\xi) + (\mu + \alpha_1) I_{1,X}(\xi) = 0, \\
-D_2 I_{2,X}'(\xi) &+ c I_{2,X}'(\xi) - q_2 \gamma E_X(\xi) + (\mu + \alpha_2) I_{2,X}(\xi) = 0, \\
S_{1,X}(\pm X) & = S_{1}^- (\pm X), \quad E_X(\pm X) = h_0(\pm X), \quad I_{1,X}(\pm X) = h_1(\pm X), \\
I_{2,X}(\pm X) & = h_2(\pm X).
\end{aligned}
\]
We have proved the local existence of solution for system (5). In order to obtain the global existence, we need the following estimate.

**Lemma 2.9.** For a given $Y > 0$, there exist some positive constants $N_S(Y), N_E(Y)$ and $N_I(Y)$ such that

\[
\| S_{i,X} \|_{C^2[-Y,Y]} \leq N_S(Y), \quad \| E_X \|_{C^2[-Y,Y]} \leq N_E(Y), \quad \| I_{i,X} \|_{C^2[-Y,Y]} \leq N_I(Y),
\]

with $i = 1, 2, \text{ and these positive constants are independent of }$

\[
X > \max \left\{ Y, \frac{1}{\omega} \ln \frac{1}{M}, \frac{1}{\epsilon} \ln \frac{\eta_0}{K \zeta_0}, \frac{1}{\epsilon} \ln \frac{\eta_1}{K \zeta_1}, \frac{1}{\epsilon} \ln \frac{\eta_2}{K \zeta_2} \right\}.
\]

**Proof.** We always set $i = 1, 2$. The equation (22) implies that $S_{i,X}(\xi) \leq S^0_i$, $E_X(\xi) \leq \eta_0 e^{\lambda_1 Y} \leq \tilde{N}_E(Y)$, and $I_{i,X}(\xi) \leq \eta_1 e^{\lambda_1 Y} \leq \tilde{N}_I(Y)$ for any $\xi \in [-Y,Y]$. And applying $L^p (p \geq 2)$ estimates of linear elliptic differential equations to $S_{i,X}$, we can claim that

\[
\| S_{1,X} \|_{W^{2,p}(-Y,Y)} \leq \Omega_1 \left( A + \beta S^0_1[\theta \tilde{N}_I(Y) + \tilde{N}_E(Y)] + \| \phi_1 \|_{W^{2,p}(-Y,Y)} \right),
\]

and

\[
\| S_{2,X} \|_{W^{2,p}(-Y,Y)} \leq \Omega_2 \left( \rho S^0_1 + \beta S^0_1[\theta \tilde{N}_I(Y) + \tilde{N}_E(Y)] + \| \phi_2 \|_{W^{2,p}(-Y,Y)} \right),
\]

where $\Omega_1$ is a constant depending on $Y, \phi_i$ can be considered as a linear function connecting points $(-Y, S_{i,X}(-Y))$ and $(Y, S_{i,X}(Y))$, respectively. Therefore, there exists a constant $Q_{S_i}(Y)$ such that $\| S_{1,X} \|_{W^{2,p}(-Y,Y)} \leq Q_{S_i}(Y)$ for any $X > Y$, respectively. Owing to $W^{2,p}(-Y,Y) \rightarrow C^{1,\alpha}[-Y,Y]$ for $\alpha = 1 - \frac{1}{p}$, it is shown that $\| S_{i,X} \|_{C^{1,\alpha}[-Y,Y]} \leq \tilde{N}_S(Y)$ for $\tilde{N}_S(Y) = \tilde{Q}_{S_i}(Y)\tilde{N}_S(Y) > 0$. According to the first and second equations in equations (22), we also obtain $\| S_{i,X} \|_{C^{1,\alpha}[-Y,Y]} \leq \tilde{N}_S(Y)$ for $\tilde{N}_S(Y) > 0$, respectively. Via the similar way, we further yield that $\| E_X \|_{C^2[-Y,Y]} \leq \tilde{N}_E(Y)$ and $\| I_{i,X} \|_{C^2[-Y,Y]} \leq \tilde{N}_I(Y)$. Finally, the estimate (23) is established with differentiating two sides of the first five equations in equations (22). The proof is finished. \qed

Set a sequence of positive numbers $\{X_m \}_{m \geq 0}$ satisfying $X_m \rightarrow +\infty$ as $m \rightarrow +\infty$. Thus, by Lemma 2.9, there exists a solution $(S_1, S_2, E, I_1, I_2) \in C^2(\mathbb{R}, \mathbb{R}^5)$ of system (5) such that

\[
S_1(\xi) \leq S_1(\xi) \leq S^0_1, \quad S_2(\xi) \leq S_2(\xi) \leq S^0_2,
\]

\[
h_0(\xi) \leq E(\xi) \leq p_0(\xi), \quad h_1(\xi) \leq I_1(\xi) \leq p_1(\xi), \quad h_2(\xi) \leq I_2(\xi) \leq p_2(\xi)
\]

with $\forall \xi \in \mathbb{R}$. According to inequalities (24), we can gain

\[
\lim_{\xi \rightarrow -\infty} S_1(\xi) = S^0_1, \quad \lim_{\xi \rightarrow -\infty} S_2(\xi) = S^0_2, \quad \lim_{\xi \rightarrow -\infty} E(\xi) = 0,
\]

\[
\lim_{\xi \rightarrow -\infty} I_1(\xi) = 0, \quad \lim_{\xi \rightarrow -\infty} I_2(\xi) = 0.
\]

Now, we need to show some estimates about solution $(S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi))$ in order to research the asymptotic behavior as $\xi \rightarrow +\infty$. 
Lemma 2.10. Let \( r \leq \min\{\mu, \mu + \alpha_1, \mu + \alpha_2\} \), then we have

\[
0 \leq E(\xi) + I_1(\xi) + I_2(\xi) \leq \sqrt{\frac{d}{D_{\text{min}}}} \frac{\Lambda}{r}
\]

(25)

and

\[
\frac{\Lambda}{\mu + \rho + \beta \sqrt{\frac{d}{D_{\text{min}}}}} \leq S_1(\xi) \leq S_1^0,
\]

\[
\frac{\rho \Lambda}{\left(\mu + \rho + \beta \sqrt{\frac{d}{D_{\text{min}}}}\right) \left(\mu + \beta \sigma \sqrt{\frac{d}{D_{\text{min}}}}\right)} \leq S_2(\xi) \leq S_2^0,
\]

(26)

where \( D_{\text{min}} = \min\{D_0, D_1, D_2\} \) and \( \forall \xi \in \mathbb{R} \).

Proof. In this proof, we still set \( i = 1, 2 \). As \( E(\xi), I_1(\xi), I_2(\xi) \) are non-negative and not identically zero, the strong maximum principle implies that \( E(\xi), I_1(\xi), I_2(\xi) > 0 \) for any \( \xi \in \mathbb{R} \) and \( E(\xi) + I_1(\xi) + I_2(\xi) > 0 \). Define

\[
m_1(\xi) = \rho S_1(\xi) + \beta S_1(\xi) [\theta I_1(\xi) + I_2(\xi)],
\]

\[
m_2(\xi) = \rho S_1(\xi) - \beta \sigma S_2(\xi) [\theta I_1(\xi) + I_2(\xi)],
\]

and

\[
n_1(\xi) = q_1 \gamma E(\xi),
\]

\[
n_2(\xi) = q_1 \gamma E(\xi).
\]

Due to the define of \( r \), it then follows that

\[
\begin{cases}
-dS_1''(\xi) + cS_1'(\xi) + rS_1(\xi) \leq \Lambda - m_1(\xi), \\
-dS_2''(\xi) + cS_2'(\xi) + rS_2(\xi) \leq m_2(x_i), \\
-D_0E''(\xi) + cE'(\xi) + rE(\xi) \leq m_1(\xi) - m_2(\xi) - n_1(\xi) - n_2(\xi), \quad \forall \xi \in \mathbb{R}, \\
-D_1I_1''(\xi) + cI_1'(\xi) + rI_1(\xi) \leq n_1(\xi), \\
-D_2I_2''(\xi) + cI_2'(\xi) + rI_2(\xi) \leq n_2(\xi).
\end{cases}
\]

(27)

So we need to consider the following Cauchy problems

\[
\begin{align*}
\frac{\partial}{\partial t} u_1(t, \xi) - \frac{\partial}{\partial \xi^2} u_1(t, \xi) + c \frac{\partial}{\partial \xi} u_1(t, \xi) + ru_1(t, \xi) &= \Lambda - m_1(\xi), \\
u_1(0, \xi) &= S_1(\xi), \\
\frac{\partial}{\partial t} u_2(t, \xi) - \frac{\partial}{\partial \xi^2} u_2(t, \xi) + c \frac{\partial}{\partial \xi} u_2(t, \xi) + ru_2(t, \xi) &= m_2(\xi), \\
u_2(0, \xi) &= S_2(\xi), \\
\frac{\partial}{\partial t} v_0(t, \xi) - \frac{\partial}{\partial \xi^2} v_0(t, \xi) + c \frac{\partial}{\partial \xi} v_0(t, \xi) + rv_0(t, \xi) &= m_1(\xi) - m_2(\xi) - n_1(\xi) - n_2(\xi), \\
v_0(0, \xi) &= E(\xi), \\
\frac{\partial}{\partial t} v_i(t, \xi) - \frac{\partial}{\partial \xi^2} v_i(t, \xi) + c \frac{\partial}{\partial \xi} v_i(t, \xi) + rv_i(t, \xi) &= n_i(\xi), \\
v_i(0, \xi) &= I_i(\xi),
\end{align*}
\]
for \( \forall t > 0, \forall \xi \in \mathbb{R} \). Then, via the Theorem 12 and the Theorem 16 in [5], we can indicate

\[
\begin{align*}
u_1(t, \xi) &= e^{-rt} \int_0^t \frac{1}{\sqrt{4\pi dt}} e^{-\frac{(t-ct-y)^2}{4dt}} S_1(y)dy \\
&\quad + \int_0^t \int_0^t \frac{e^{-rs}}{\sqrt{4\pi ds}} e^{-\frac{(t-ct-y)^2}{4ds}} (\Lambda - m_1(y))dyds,
\end{align*}
\]

\[
\begin{align*}
u_2(t, \xi) &= e^{-rt} \int_0^t \frac{1}{\sqrt{4\pi dt}} e^{-\frac{(t-ct-y)^2}{4dt}} S_2(y)dy \\
&\quad + \int_0^t \int_0^t \frac{e^{-rs}}{\sqrt{4\pi ds}} e^{-\frac{(t-ct-y)^2}{4ds}} m_2(y)dyds,
\end{align*}
\]

\[
\begin{align*}
u_0(t, \xi) &= e^{-rt} \int \frac{1}{\sqrt{4\pi D_0}} e^{-\frac{(t-ct-y)^2}{4D_0}} E(y)dy \\
&\quad + \int \frac{e^{-rs}}{\sqrt{4\pi D_0}} e^{-\frac{(t-ct-y)^2}{4D_0}} (m_1(y) - m_2(y) - n(y) - n(y))dyds,
\end{align*}
\]

\[
\begin{align*}
u_i(t, \xi) &= e^{-rt} \int \frac{1}{\sqrt{4\pi D_i}} e^{-\frac{(t-ct-y)^2}{4D_i}} I_i(y)dy + \int \frac{e^{-rs}}{\sqrt{4\pi D_i}} e^{-\frac{(t-ct-y)^2}{4D_i}} n_i(y)dyds,
\end{align*}
\]

for \( \forall t > 0, \forall \xi \in \mathbb{R} \). Applying the comparison principle in [19] concludes

\[
S_1(\xi) \leq u_i(t, \xi), \quad E(\xi) \leq v_0(t, \xi), \quad I_i(\xi) \leq v_i(t, \xi), \quad \forall t > 0, \forall \xi \in \mathbb{R}.
\]

Let \( t \to +\infty \), then one yields

\[
S_1(\xi) \leq \frac{\Lambda}{4} - f_1(\xi), \quad S_2(\xi) \leq f_2(\xi), \quad E(\xi) \leq f_0(\xi) - g_0(\xi), \quad I_1(\xi) \leq g_1(\xi),
\]

for \( \forall t > 0, \forall \xi \in \mathbb{R} \), where

\[
\begin{align*}f_1(\xi) &= \int_0^{+\infty} e^{-\frac{r}{\sqrt{4D_1}}} \int_{-\infty}^{+\infty} m_1(\xi - y - ct)e^{-\frac{s^2}{4D_1}} dydt,
\end{align*}
\]

\[
\begin{align*}f_0(\xi) &= \int_0^{+\infty} e^{-\frac{r}{\sqrt{4D_0}}} \int_{-\infty}^{+\infty} [m_1(\xi - y - ct) - m_2(\xi - y - ct)]e^{-\frac{s^2}{4D_0}} dydt,
\end{align*}
\]

\[
\begin{align*}g_0(\xi) &= \int_0^{+\infty} e^{-\frac{r}{\sqrt{4D_0}}} \int_{-\infty}^{+\infty} [n_1(\xi - y - ct) + n_2(\xi - y - ct)]e^{-\frac{s^2}{4D_0}} dydt,
\end{align*}
\]

\[
\begin{align*}g_1(\xi) &= \int_0^{+\infty} e^{-\frac{r}{\sqrt{4D_1}}} \int_{-\infty}^{+\infty} n_1(\xi - y - ct)e^{-\frac{s^2}{4D_1}} dydt.
\end{align*}
\]

Owing to \( d \geq D_0 \geq D_i \), we yield

\[
\sqrt{D_0g_0(\xi)} \geq \sqrt{D_1g_1(\xi)} + \sqrt{D_2g_2(\xi)}
\]

and

\[
\sqrt{d}[f_1(\xi) - f_2(\xi)] \geq \sqrt{D_0f_0(\xi)}
\]

for any \( \xi \in \mathbb{R} \). We also have \( \sqrt{D_1I_i(\xi)} \leq \sqrt{D_2g_1(\xi)} \). Consequently, it is shown that

\[
E(\xi) + \sqrt{\frac{D_1}{D_0}I_1(\xi)} + \sqrt{\frac{D_2}{D_0}I_2(\xi)} \leq f_0(\xi) \leq \sqrt{\frac{d}{D_0}f_1(\xi)} - \sqrt{\frac{d}{D_0}f_2(\xi)}
\]

\[
\leq \sqrt{\frac{d}{D_0}} \left[ \frac{\Lambda}{r} - S_1(\xi) \right] - \sqrt{\frac{d}{D_0}} S_2(\xi)
\]

\[
\leq \sqrt{\frac{d}{D_0}} \frac{\Lambda}{r},
\]
which implies
\[
E(\xi) + I_1(\xi) + I_2(\xi) \leq \sqrt{\frac{D_0}{D_{\text{min}}}} E(\xi) + \sqrt{\frac{D_1}{D_{\text{min}}}} I_1(\xi) + \sqrt{\frac{D_2}{D_{\text{min}}}} I_2(\xi)
\]
\[
\leq \sqrt{\frac{d}{D_{\text{min}}}} \Lambda r,
\]
for \(D_{\text{min}} = \min\{D_0, D_1, D_2\}\) and any \(\xi \in \mathbb{R}\).

On the other hand, one has
\[
dS_1''(\xi) - cS_1'(\xi) + \Lambda - \beta S_1(\xi) \sqrt{\frac{d}{D_{\text{min}}}} \Lambda r - (\mu + \rho)S_1(\xi).
\]
for any \(\xi \in \mathbb{R}\). Then the maximum principle implies that
\[
\frac{\Lambda}{\mu + \rho + \beta \sqrt{\frac{d}{D_{\text{min}}}} \Lambda r} \leq S_1(\xi), \ \forall \xi \in \mathbb{R}.
\]
Via the similar argument on \(S_1(\xi)\), we can indicate
\[
\rho \Lambda \left( \frac{\mu + \rho + \beta \sqrt{\frac{d}{D_{\text{min}}}} \Lambda r}{\mu + \beta \sigma \sqrt{\frac{d}{D_{\text{min}}}} \Lambda} \right) \leq S_2(\xi), \ \forall \xi \in \mathbb{R}.
\]
This lemma is completely proved. \(\square\)

Since the system consisted by \(E\)-th and \(I_i\)-th equations in system (5) is cooperation and irreducible, the Theorem 2.2 in [2] and Lemma 2.10 can claim that there exists a positive constant \(M_1\) such that
\[
\max \left\{ \max_{[x-1,x+1]} E, \max_{[x-1,x+1]} I_1, \max_{[x-1,x+1]} I_2 \right\}
\]
\[
\leq M_1 \min \left\{ \min_{[x-1,x+1]} E, \min_{[x-1,x+1]} I_1, \min_{[x-1,x+1]} I_2 \right\}. \quad (28)
\]
Furthermore, there exists a constant \(\bar{M} > 0\) such that
\[
\left| \frac{E'(\xi)}{E(\xi)} + \frac{I_1'(\xi)}{I_1(\xi)} + \frac{I_2'(\xi)}{I_2(\xi)} \right| \leq \bar{M}, \quad (29)
\]
for \(\forall \xi \in \mathbb{R}\). Actually, \(L^p\) interior estimate shows there exists a positive constant \(M_2 > 0\) satisfying
\[
\max \left\{ \| E \|_{W^{2,p}(\xi-\frac{1}{2}, \xi+\frac{1}{2})}, \| I_1 \|_{W^{2,p}(\xi-\frac{1}{2}, \xi+\frac{1}{2})}, \| I_2 \|_{W^{2,p}(\xi-\frac{1}{2}, \xi+\frac{1}{2})} \right\}
\]
\[
\leq M_2 \left\{ \| E \|_{L^p(\xi-1, \xi+1)} + \| I_1 \|_{L^p(\xi-1, \xi+1)} + \| I_2 \|_{L^p(\xi-1, \xi+1)} \right\}
\]
\[
\leq 6M_2 \max \left\{ \max_{[\xi-1,\xi+1]} E, \max_{[\xi-1,\xi+1]} I_1, \max_{[\xi-1,\xi+1]} I_2 \right\},
\]
for any \(\xi \in \mathbb{R}\) and \(p > 1\). The above inequality, combining the embedding theorem, can conclude that there exists a positive constant \(M_3\) satisfying
\[
\max \left\{ \| E' \|_{C_{[-\frac{1}{2}, \frac{1}{2}]}} , \| I_1' \|_{C_{[-\frac{1}{2}, \frac{1}{2}]}} , \| I_2' \|_{C_{[-\frac{1}{2}, \frac{1}{2}]}} \right\}
\]
\[
\leq M_3 \max \left\{ \max_{[x-1,x+1]} E, \max_{[x-1,x+1]} I_1, \max_{[x-1,x+1]} I_2 \right\}. \quad \forall \xi \in \mathbb{R}. \quad (30)
\]
Set \(\bar{M} = M_1M_3\), then the inequality (29) is established via the estimate (30).
In order to take advantage of a suitable Lyapunov functional to research the asymptotic behavior of \((S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi))\) as \(\xi \to +\infty\), we define

\[
\hat{E} = \begin{cases} 
S_1(\cdot), S_2(\cdot), E(\cdot), I_1(\cdot), I_2(\cdot) \in C^1(\mathbb{R}, (0, +\infty)) \times \cdots \\
\times C^1(\mathbb{R}, (0, +\infty)), \\
S_1(\cdot) > 0, S_2(\cdot) > 0, E(\cdot) > 0, I_1(\cdot) > 0, I_2(\cdot) > 0, \\
\exists \hat{M} > 0, \left|\frac{E'(\xi)}{E(\xi)}\right| + \left|\frac{I_1'(\xi)}{I_1(\xi)}\right| + \left|\frac{I_2'(\xi)}{I_2(\xi)}\right| \leq \hat{M}.
\end{cases}
\]

Let \(g(x) = x - 1 - \ln x\) and define a suitable Lyapunov functional

\[
V(S_1, S_2, E, I_1, I_2)(\xi) = S_1^1 \left[ dS_1' \left( \frac{1}{S_1(\xi)} - \frac{1}{S_1^*} \right) + cg \left( \frac{S_1(\xi)}{S_1^*} \right) \right] \\
+ S_2^2 \left[ dS_2' \left( \frac{1}{S_2(\xi)} - \frac{1}{S_2^*} \right) + cg \left( \frac{S_2(\xi)}{S_2^*} \right) \right] \\
+ E^* \left[ D_0 E' \left( \frac{1}{E(\xi)} - \frac{1}{E^*} \right) + cg \left( \frac{E(\xi)}{E^*} \right) \right] \\
+ C_1 I_1^1 \left[ D_1 I_1' \left( \frac{1}{I_1(\xi)} - \frac{1}{I_1^*} \right) + cg \left( \frac{I_1(\xi)}{I_1^*} \right) \right] \\
+ C_2 I_2^2 \left[ D_2 I_2' \left( \frac{1}{I_2(\xi)} - \frac{1}{I_2^*} \right) + cg \left( \frac{I_2(\xi)}{I_2^*} \right) \right],
\]

where

\[
C_1 = \frac{\beta \theta \gamma I_1^1 (S_1^* + \sigma S_2^*)}{(1) q_1 \gamma E^*}, \quad C_2 = \frac{\beta I_2^2 (S_1^* + \sigma S_2^*)}{(2) q_2 \gamma E^*},
\]

for each \((S_1(\cdot), S_2(\cdot), E(\cdot), I_1(\cdot), I_2(\cdot)) \in \hat{E}\). Then we claim the below lemma.

**Lemma 2.11.** Let \((A)\) be satisfied and \((S_1(\cdot), S_2(\cdot), E(\cdot), I_1(\cdot), I_2(\cdot))\) be a positive solution of system (5) satisfying

\[
\frac{1}{N} \leq S_i(\xi) \leq S_i^*, \quad E(\xi) \leq N E^*, \\
I_i(\xi) \leq N I_i^*,
\]

and

\[
\left|\frac{E'(\xi)}{E(\xi)}\right| + \left|\frac{I_1'(\xi)}{I_1(\xi)}\right| + \left|\frac{I_2'(\xi)}{I_2(\xi)}\right| \leq N
\]

for any \(\xi \in \mathbb{R}\) and \(i = 1, 2\), where \(N\) is a positive constant. Then there exists a constant \(m > 0\), depending on \(N\), such that

\[
-m \leq V(\xi) < +\infty, \quad \forall \xi \in \mathbb{R},
\]

where the map \(V(\xi)\) is defined as formula (31). Moreover, the map \(V(\xi)\) is not increasing. In particular, \(S_1(\xi) = S_1^*, \quad E(\xi) = E^*, \quad I_i(\xi) = I_i^*\) as the map \(V(\xi)\) is a constant.
Proof. The previous description has shown $S_1$ and $S_2$ are bounded in $C^2(\mathbb{R})$. Via inequalities (32)-(35), we can conclude that for any $\xi \in \mathbb{R}$

\[
\left| \sum_{i=1}^{2} S_i^* S_i'(\xi) \left( \frac{1}{S_i(\xi)} - \frac{1}{S_i^*} \right) + D_0 E^* E'(\xi) \left( \frac{1}{E(\xi)} - \frac{1}{E^*} \right) \right|
\] 
\[
\quad + \sum_{i=1}^{2} C_i I_i^* D_i I_i'(x_i) \left( \frac{1}{I_i(\xi)} - \frac{1}{I_i^*} \right)
\] 
\[
\leq \sum_{i=1}^{2} S_i^* \| S_i \|_{\infty} (N + \frac{1}{S_i^*}) + \hat{D} \hat{C} N + \hat{D} \hat{C} N^2,
\]
\[
(37)
\]
where $\hat{D} = \max\{D_0 E^*, \ D_1 I_1^* \ D_2 I_2^*\}$, and $\hat{C} = \max\{1, \ C_1, \ C_2\}$. Let

\[
\Phi(\xi) = c g \left( \frac{S_2(\xi)}{S_2^*} \right) + c g \left( \frac{S_3(\xi)}{S_3^*} \right) + c g \left( \frac{E(\xi)}{E^*} \right) + c C_1 g \left( \frac{I_1(\xi)}{I_1^*} \right) + c C_2 g \left( \frac{I_2(\xi)}{I_2^*} \right).
\]
\[
(38)
\]
According to both the definition of $g(\cdot)$ and (32)-(34), it is shown that $0 \leq \Phi(\xi) < +\infty, \forall \xi \in \mathbb{R}$. Then, by combining the inequality (37), the inequality (36) holds.

Since a direct calculation with letting $I_1 = \frac{S_1(\xi)}{S_1^*}$, $I_2 = \frac{S_2(\xi)}{S_2^*}$, $x_3 = \frac{E(\xi)}{E^*}$, $x_4 = \frac{I_1(\xi)}{I_1^*}$, $x_5 = \frac{I_2(\xi)}{I_2^*}$ leads to

\[
\frac{dV(\xi)}{d\xi} = (\lambda - \beta \theta S_1^* I_1^* - \beta S_1^* I_2^* - \rho S_1^*) \left( 2 - x_1 - \frac{1}{x_1} \right)
\] 
\[
\quad + (\rho S_1^* - \beta \sigma S_2^* I_1^* - \beta \sigma S_2^* I_2^*) \left( 3 - \frac{1}{x_1} - x_2 - \frac{x_2}{x_2} \right)
\] 
\[
\quad + \beta \theta S_2^* I_1^* \left( 2 - x_1 - \frac{x_1 x_4}{x_3} - \frac{x_3}{x_4} \right) + \beta S_1^* I_2^* \left( 3 - \frac{1}{x_1} - x_1 x_5 - \frac{x_3}{x_5} \right)
\] 
\[
\quad + \beta S_1^* I_2^* \left( - \frac{1}{x_1} - \frac{x_1 x_4}{x_3} - \frac{x_3}{x_4} \right)
\] 
\[
\quad + \beta S_2^* I_1^* \left( 4 - \frac{1}{x_1} - \frac{x_1 x_4}{x_3} - \frac{x_3}{x_4} \right)
\] 
\[
\quad + \beta S_2^* I_2^* \left( 4 - \frac{1}{x_1} - \frac{x_1 x_4}{x_3} - \frac{x_3}{x_4} \right).
\]
Therefore, via the mean inequality, we conclude that

\[
\frac{dV(\xi)}{d\xi} \leq 0, \ \forall \xi \in \mathbb{R},
\]
which implies that the map $V(\xi)$ is non-increasing. Especially, when

\[
\frac{dV(\xi)}{d\xi} = 0, \ \forall \xi \in \mathbb{R},
\]
the map $V(\xi)$ is a constant, which indicates that

\[
S_1(\xi) = S_1^*, \ S_2(\xi) = S_2^*, \ E(\xi) = E^*, \ I_1(\xi) = I_1^*, \ I_2(\xi) = I_2^*, \ \forall \xi \in \mathbb{R}.
\]

This completes the proof. \(\square\)

Now, we gain the first theorem for existence of traveling wave solution for system (2) as below:
Theorem 2.12. If (A) and $R_0 > 1$ hold, system (2) admits a non-trivial and non-negative traveling wave solution $(S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi))$ satisfying the boundary condition (6) for each $c > c^*$, where $c^*$ is the minimal spread speed and $R_0$ is the basic reproduction number for system (3).

Proof. We still set $i = 1, 2$ in this proof. It follows from the previous argument that there exists a vector function

$$(S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi))$$

satisfying system (5), and

$$\lim_{\xi \to -\infty} S_1(\xi) = S_1^0, \quad \lim_{\xi \to -\infty} S_2(\xi) = S_2^0, \quad \lim_{\xi \to -\infty} E(\xi) = 0,$$

$$\lim_{\xi \to -\infty} I_1(\xi) = 0, \quad \lim_{\xi \to -\infty} I_2(\xi) = 0,$$

for any $\xi \in \mathbb{R}$. Therefore we only need to show that

$$\lim_{\xi \to +\infty} S_1(\xi) = S_1^*, \quad \lim_{\xi \to +\infty} S_2(\xi) = S_2^*, \quad \lim_{\xi \to +\infty} E(\xi) = E^*,$$

$$\lim_{\xi \to +\infty} I_1(\xi) = I_1^*, \quad \lim_{\xi \to +\infty} I_2(\xi) = I_2^*,$$

Take an arbitrary increasing sequence $\{\xi_m\}$ with $\xi_m > 0$ and $m \geq 0$ such that $\xi_m \to +\infty$ as $m \to +\infty$. Define

$$S_{i,m}(\xi) = S_i(\xi + \xi_m), \quad E_m(\xi) = E(\xi + \xi_m), \quad I_{i,m}(\xi) = I_i(\xi + \xi_m).$$

Via the elliptic estimate, it may assume that the sequence $(S_{1,m}, S_{2,m}, E_m, I_{1,m}, I_{2,m})$ converges towards $(S_{1,\infty}, S_{2,\infty}, E_{\infty}, I_{1,\infty}, I_{2,\infty})$ in $C^1_{loc}(\mathbb{R}) \times \cdots \times C^1_{loc}(\mathbb{R})$. As a result, $(S_{1,\infty}, S_{2,\infty}, E_{\infty}, I_{1,\infty}, I_{2,\infty})$ is also a solution of system (5). In addition, the map $V(\xi)$ defined as the formula (31) is not increasing, then we yield

$$V(S_{1,m}, S_{2,m}, E_m, I_{1,m}, I_{2,m})(\xi) \leq V(S_1, S_2, E_m, I_1, I_2)(\xi)$$

for any $\xi \in \mathbb{R}$. Since the map $V(\xi)$ is bounded via Lemma 2.11, there exists a constant $G \in \mathbb{R}$ such that

$$\lim_{m \to +\infty} V(S_{1,m}, S_{2,m}, E_m, I_{1,m}, I_{2,m})(\xi) = G, \quad \forall \xi \in \mathbb{R},$$

which implies that

$$V(S_{1,\infty}, S_{2,\infty}, E_{\infty}, I_{1,\infty}, I_{2,\infty})(\xi) \equiv \hat{G}$$

in $C^1_{loc}(\mathbb{R})$. Combining Lemma 2.11, we can claim that

$$S_{1,\infty} = S_1^*, \quad E_{\infty} = E^*, \quad I_{1,\infty} = I_1^*,$$

$$S_{1,\infty} = 0, \quad E_{\infty} = 0, \quad I_{1,\infty} = 0.$$

Via the arbitrariness of the sequence $\{\xi_m\}$, we finally indicate

$$\lim_{\xi \to +\infty} S_i(\xi) = S_i^*, \quad \lim_{\xi \to +\infty} E(\xi) = E^*, \quad \lim_{\xi \to +\infty} I_i(\xi) = I_i^*.$$

The proof is completed. \qed

Furthermore, the second theorem of existence for traveling wave solution for system (2) is stated as below:

Theorem 2.13. Assume that (A) is satisfied and $R_0 > 1$. Then for $c = c^*$, system (2) also admits a non-trivial and non-negative traveling wave solution $(S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi))$ satisfying the boundary condition (6), where $c^*$ is the minimal spread speed and $R_0$ is the basic reproduction number for system (3).
Proof. **Step 1.** Take a decreasing sequence \( \{c_m\} \in (c^*, c^* + 1) \) with \( \lim_{m \to +\infty} c_m = c^* \). It follows from Theorem 2.12 that there exists a solution \((S_{1,m}, S_{2,m}, E_m, I_{1,m}, I_{2,m})\) of system (5) for each \( c_m \) satisfying conditions (6), (25), (26), (28) and (29). Since \((S_{1,m}(\cdot + a), S_{2,m}(\cdot + a), E_m(\cdot + a), I_{1,m}(\cdot + a), I_{2,m}(\cdot + a))\) is also the solution of system (5) satisfying equations (6) for any \( a \in \mathbb{R} \), we can let

\[
S_{1,m}(0) = \frac{S_0^1 + S_1^*}{2}.
\]

The interior elliptic estimates, Arzela-Ascoli theorem and a diagonalization argument can indicate a subsequence of \( \{S_{1,m}, S_{2,m}, E_m, I_{1,m}, I_{2,m}\} \) defined again by \( \{(S_{1,m}, S_{2,m}, E_m, I_{1,m}, I_{2,m})\} \) satisfies

\[
(S_{1,m}, S_{2,m}, E_m, I_{1,m}, I_{2,m}) \to (S_1, S_2, E, I_1, I_2)
\]

as \( m \to +\infty \) in \( C^2_{loc}(\mathbb{R}, \mathbb{R}^5) \). And it is well known that \((S_1, S_2, E, I_1, I_2)\) satisfies system (5) and

\[
S_1(0) = \frac{S_0^1 + S_1^*}{2}, \tag{39}
\]

which implies that

\[
(S_1, S_2, E, I_1, I_2) \neq (S_0^1, S_2^0, 0, 0, 0).
\]

Moreover, we yield \( S_1 > 0, E > 0, I_i > 0 \). Above all, it is concluded that \((S_1, S_2, E, I_1, I_2)\) satisfies estimates (25), (26), (28) and (29). Via Lemma 2.11, one gains

\[
S_i(+\infty) = S_i^*, \quad E(+\infty) = E^*, \quad I_i(+\infty) = I_i^*, \quad i = 1, 2.
\]

Next, we need to prove what the solution converges to as \( \xi \to -\infty \). Since Lemma 2.11 implies \( V(\xi) \) defined as the formula (31) is non-increasing, then we can obtain either

\[
\lim_{\xi \to -\infty} V(\xi) = L < +\infty, \tag{40}
\]

or

\[
\lim_{\xi \to -\infty} V(\xi) = +\infty. \tag{41}
\]

If the formula (40) holds, via the similar way in Lemma 2.11, it can claim that

\[
S_i(-\infty) = S_i^*, \quad E(-\infty) = E^*, \quad I_i(-\infty) = I_i^*, \quad i = 1, 2,
\]

and then \( L = 0 \), which implies that \( V \equiv 0 \) for any \( \xi \in \mathbb{R} \). Consequently, applying Lemma 2.11 concludes that

\[
S_i(\xi) \equiv S_i^*, \quad E(\xi) \equiv E^*, \quad I_i(\xi) \equiv I_i^*, \quad \forall \xi \in \mathbb{R}, \quad i = 1, 2,
\]

which contradicts the equation (39). Thus, the equation (41) must be only workable. Due to the inequality (37), it is shown that

\[
\lim_{\xi \to -\infty} \Phi(\xi) = +\infty, \tag{42}
\]

where \( \Phi(\xi) \) is defined as the formula (38) in the proof of Lemma 2.11. Now, we firstly show that

\[
\lim_{\xi \to -\infty} \inf E(\xi) = 0. \tag{43}
\]

On the contrary, if \( E(\xi) > \delta \) in \( \xi \in \mathbb{R} \) for some \( \delta > 0 \). According to \( E(+\infty) = E^* \), then it follows from the estimate (28) that there exists a constant \( \hat{\delta} > 0 \) such that

\[
I_i(\xi) > \hat{\delta}, \quad \forall \xi \in \mathbb{R},
\]
which indicates that
\[
\lim_{\xi \to -\infty} \Phi(\xi) < +\infty,
\]
and contradicts the equation (42). Therefore, the equation (43) holds. Secondly, we prove that
\[
\lim_{\xi \to -\infty} E(\xi) = 0.
\]
If
\[
\lim_{\xi \to -\infty} \sup E(\xi) = \delta > 0,
\]
there exists a sequence \(\{\xi_j\}\) satisfying
\[
\lim_{j \to \infty} E(\xi_j) = \delta,
\]
Via the estimate (28), we can have
\[
I_i(\xi_j) \leq \frac{1}{2M_1} \delta, \quad i = 1, 2.
\]
Then, it is implied that
\[
\lim_{j \to \infty} \sup V(\xi_j) < +\infty,
\]
which contradicts the equation (41). Therefore,
\[
\lim_{\xi \to -\infty} E(\xi) = 0.
\]
In the similar way, we can also claim that
\[
\lim_{\xi \to -\infty} I_i(\xi) = 0.
\]
Finally, set \(i = 1, 2\) and we show that
\[
\lim_{\xi \to -\infty} S_i(\xi) = S^0_i,
\]
In order to hit the target, we firstly prove the existence of \(\lim_{\xi \to -\infty} S_i(\xi)\). On the contrary, we assume that one of \(\lim_{\xi \to -\infty} S_i(\xi)\) does not exist. Since \(S_i(\xi)\) satisfies (26), we obtain
\[
\lim_{\xi \to -\infty} \inf\{S_1(\xi) + S_2(\xi)\} < \lim_{\xi \to -\infty} \sup\{S_1(\xi) + S_2(\xi)\} \leq \frac{\Lambda}{\mu}.
\]
Take a sequence \(\{\xi_n\}\) satisfies \(\xi_n \to -\infty\) as \(n \to +\infty\) and
\[
\lim_{n \to +\infty} [S_1(\xi_n) + S_2(\xi_n)] = \lim_{\xi \to -\infty} \inf\{S_1(\xi) + S_2(\xi)\} < \frac{\Lambda}{\mu},
\]
\[
\frac{d}{d\xi_n^2}[S_1(\xi_n) + S_2(\xi_n)] = 0, \quad \frac{d^2}{d\xi_n^2}[S_1(\xi_n) + S_2(\xi_n)] \leq 0.
\]
Since \(\lim_{n \to +\infty} I_i(\xi_n) = 0\), adding the \(S_1\)-th equation and \(S_2\)-th equation implies that
\[
\lim_{n \to +\infty} [S_1(\xi_n) + S_2(\xi_n)] \geq \frac{\Lambda}{\mu},
\]
which leads to a contradiction to inequalities (44). Thus, \(\lim_{\xi \to -\infty} S_i(\xi)\) exists, respectively.

**Step 2.** This step is to prove \(\lim_{\xi \to -\infty} S_i(\xi) = S^0_i\). Let \(\lim_{\xi \to -\infty} S_i(\xi) = k_i\). Utilizing the equation
\[
-dS_i''(\xi) + cS_i'(\xi) + (\mu + \rho)S_i(\xi) = \Lambda - \beta S_i(\xi)[\theta I_1(\xi) + I_2(\xi)], \quad \xi \in \mathbb{R},
\]
we can obtain that
\[
S_1(\xi) = \frac{1}{z} \int_{-\infty}^{\xi} e^{A_1(\xi-x)} \left[ \lambda - \beta S_1(x)(\theta I_1(x) + I_2(x)) \right] dx
\]
\[
+ \frac{1}{z} \int_{\xi}^{+\infty} e^{A_2(\xi-x)} \left[ \lambda - \beta S_1(x)(\theta I_1(x) + I_2(x)) \right] dx
\]
\[
= \frac{1}{z} \int_{-\infty}^{0} e^{A_1x} \left[ \lambda - \beta S_1(\xi-x)(\theta I_1(\xi-x) + I_2(\xi-x)) \right] dx
\]
\[
+ \frac{1}{z} \int_{0}^{+\infty} e^{A_2x} \left[ \lambda - \beta S_1(\xi-x)(\theta I_1(\xi-x) + I_2(\xi-x)) \right] dx,
\]
where
\[
z = d(A_1 - A_2), \quad A_1 = \frac{c - \sqrt{c^2 + 4d(\mu + \rho)}}{2}, \quad A_2 = \frac{c - \sqrt{c^2 + 4d(\mu + \rho)}}{2}.
\]
As $\xi \to -\infty$, it follows from the Lebesgue dominated convergence theorem that
\[
k_1 = \frac{\Lambda}{z} \left( \int_{-\infty}^{0} s^{A_1x} dx + \int_{0}^{+\infty} s^{A_2x} dx \right),
\]
which leads to
\[
k_1 = \frac{\Lambda}{\mu + \rho} = S^0_1.
\]
Via the similar argument on $S_2(\xi)$, we can get
\[
k_2 = \frac{pk_1}{\mu} = S^0_2.
\]
This completes the proof.

3. Non-existence of traveling wave solution. The previous section has stated the existence of traveling wave solution connecting disease-free equilibrium and endemic equilibrium when $R_0 > 1$ and $c \geq c^*$. In this section, we aim to prove the non-existence of this solution in three cases: (1) $R_0 < 1$, (2) $R_0 = 1$, (3) $R_0 > 1$ and $c \in (0, c^*)$.

3.1. Case(1): $R_0 < 1$.

**Theorem 3.1.** If $R_0 < 1$, there does not exist a non-negative and non-trivial traveling wave solution of system (5) satisfying the boundary condition (6), where $R_0$ is the basic reproduction number for system (3).

**Proof.** On the contrary, assume that there exists a solution $(S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi))$ satisfying system (5) and the boundary condition (6). Let $E_{sup} = \sup_{\xi \in \mathbb{R}} E(\xi)$ and take $i = 1, 2$. We can yield from the $I_i$-th equation that
\[
D_i I_i''(\xi) - c I_i'(\xi) + q_i \gamma E_{sup} - (\mu + \alpha_i) I_i(\xi) \geq 0, \quad \forall \xi \in \mathbb{R}.
\]
It follows from the comparison principle that
\[
I_i(\xi) \leq \frac{q_i \gamma E_{sup}}{\mu + \alpha_i}, \quad \forall \xi \in \mathbb{R}.
\]
Next we research the following equation
\[
D_i E''(\xi) - c E'(\xi) + \Lambda \beta \left( \frac{q_1 \gamma E_{sup}}{\mu + \alpha_1} + \frac{q_2 \gamma E_{sup}}{\mu + \alpha_2} \right) - (\mu + \gamma) E(\xi) = 0,
\]
where $\tilde{\Lambda} = S_t^0 + \sigma S_{t}^0$. Via the comparison principle, we can show that

$$E(\xi) \leq \tilde{\Lambda} \left( \frac{q_1 \gamma}{\mu + \alpha_1} + \frac{q_2 \gamma}{\mu + \alpha_2} \right) E_{\text{sup}},$$

which indicates

$$\tilde{\Lambda} \left( \frac{q_1 \gamma}{\mu + \alpha_1} + \frac{q_2 \gamma}{\mu + \alpha_2} \right) \geq 1.$$ 

and contradicts the fact $R_0 < 1$. Above all, there exists no non-negative and non-trivial traveling wave solution as $R_0 < 1$. \hfill \Box

3.2. Case(2): $R_0 = 1$.

**Theorem 3.2.** As $R_0 = 1$, there does not exist a non-negative and non-trivial traveling wave solution of system (5) satisfying the boundary condition (6), where $R_0$ is the basic reproduction number for system (3).

**Proof.** Set a new sequence $\{\xi_m\} \subset \mathbb{R}$ such that

$$\lim_{m \to +\infty} E(\xi_m) = \hat{B} = \sup_{\xi \in \mathbb{R}} E(\xi).$$

Now, we need to show $\hat{B} = 0$ by contradiction. Assume $\hat{B} > 0$, and we consider the function sequence $(S_{1,m}(\xi), S_{2,m}(\xi), E_m(\xi), I_{1,m}(\xi), I_{2,m}(\xi)) = (S_1(\xi + \xi_m), S_2(\xi + \xi_m), E_m(\xi + \xi_m), I_1(\xi + \xi_m), I_2(\xi + \xi_m))$. We take a function subsequence denoted again by $(S_{1,m}(\xi), S_{2,m}(\xi), E_m(\xi), I_{1,m}(\xi), I_{2,m}(\xi))$, and utilizing elliptic estimates shows that

$$(S_{1,m}(\xi), S_{2,m}(\xi), E_m(\xi), I_{1,m}(\xi), I_{2,m}(\xi)) \to (\hat{S}_1(\xi), \hat{S}_2(\xi), \hat{E}(\xi), \hat{I}_1(\xi), \hat{I}_2(\xi)),$$

as $m \to +\infty$ in $C^2_{\text{loc}}(\mathbb{R})$ with $(\hat{S}_1(\xi), \hat{S}_2(\xi), \hat{E}(\xi), \hat{I}_1(\xi), \hat{I}_2(\xi))$ satisfying

\[
\begin{align*}
\begin{cases} 
 d\hat{S}_1^\prime(\xi) &= -\sigma \hat{S}_1(\xi) + \hat{S}_1(\xi) \left( \theta I_1(\xi) + I_2(\xi) \right) - (\rho + \mu) \hat{S}_1(\xi) = 0, \\
 d\hat{S}_2^\prime(\xi) &= -\sigma \hat{S}_2(\xi) + \hat{S}_2(\xi) \left( \theta I_1(\xi) + I_2(\xi) \right) - \mu \hat{S}_2(\xi) = 0, \\
 D_{0}\hat{E}^\prime(\xi) &= -\mu \hat{E}(\xi) - \beta \left( \theta I_1(\xi) + I_2(\xi) \right), \\
 D_1\hat{I}_1^\prime(\xi) &= -\hat{I}_1(\xi) + q_1 \gamma \hat{E}(\xi) - (\mu + \alpha_1) \hat{I}_1(\xi) = 0, \\
 D_2\hat{I}_2^\prime(\xi) &= -\hat{I}_2(\xi) + q_2 \gamma \hat{E}(\xi) - (\mu + \alpha_2) \hat{I}_2(\xi) = 0, \\
 \hat{E}(0) &= \hat{B}, \quad \hat{E}(\xi) \leq \hat{B}, \\
 0 &\leq \hat{S}_i(\xi) \leq S_t^0, \quad i = 1, 2, \quad \forall \xi \in \mathbb{R}.
\end{cases}
\end{align*}
\]

The maximum principle implies that

$$\hat{I}_i(\xi) \leq \frac{q_i \gamma \hat{B}}{\mu + \alpha_i}.$$ 

Moreover, we can yields

$$0 \leq D_{0}\hat{E}^\prime(0) + \beta [\hat{S}_1(0) + \sigma \hat{S}_1(0)] \left( \frac{q_1 \gamma}{\mu + \alpha_1} + \frac{q_2 \gamma}{\mu + \alpha_2} \right) \hat{B} - (\mu + \gamma) \hat{B}.$$

Owing to $\hat{E}^\prime(0) \leq 0$, $\beta [\hat{S}_1(0) + \sigma \hat{S}_1(0)] \left( \frac{q_1 \gamma}{\mu + \alpha_1} + \frac{q_2 \gamma}{\mu + \alpha_2} \right) \hat{B} - (\mu + \gamma) \hat{B} \geq 0$ and $\hat{S}_i \leq S_t^0$, we must claim $\beta [\hat{S}_1(0) + \sigma \hat{S}_1(0)] \left( \frac{q_1 \gamma}{\mu + \alpha_1} + \frac{q_2 \gamma}{\mu + \alpha_2} \right) \hat{B} - (\mu + \gamma) \hat{B} = 0$, which implies that $\hat{S}_i(0) = S_t^0$. The minimal principle with combining the $\hat{S}_i$-th equation...
can lead to \( \hat{S}_t(\xi) \equiv S^0_\xi, \hat{E}(\xi) \equiv 0, \hat{I}_i(\xi) \equiv 0, \forall \xi \in \mathbb{R} \). Consequently, we have \( \hat{B} = 0 \) that contradicts \( \hat{B} > 0 \). This case is completely proved. \( \square \)

3.3. Case(3): \( R_0 > 1 \) and \( c \in (0, c^*) \). In this case, we attempt to utilize the two-sides Laplace transform to gain the non-existence of traveling wave solution for system (2). Therefore, we firstly need to imply the exponential boundedness for traveling wave solution via next two lemmas.

**Lemma 3.3.** Assume that \( R_0 > 1 \), where \( R_0 \) is the basic reproduction number for system (3). For any \( c > 0 \), if system (5) admits a non-trivial and non-negative solution \((S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi))\) satisfying the boundary condition (6), then there exist two constants \( J > 0 \) and \( G > 0 \) large enough such that

\[
\int_{-\infty}^{\xi} E(x)dx < J, \quad \int_{-\infty}^{\xi} I_1(x)dx < J, \quad \int_{-\infty}^{\xi} I_2(x)dx < J,
\]

for \( \xi < -2G \).

**Proof.** Set \( i = 1, 2 \). Let \( c > 0 \) and \((S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi))\) be traveling wave solution of system (5) satisfying the boundary condition (6). Thus, we can claim there exist \( G > 0 \) large enough and small \( \varepsilon \in (0, 1) \) such that

\[
S_1(\xi) > S^0_\xi(1 - \varepsilon), \quad \xi \in (-\infty, -2G).
\]

For any \( \xi < -2G \), we can yield

\[
\beta(S_1(\xi) + \sigma S_2(\xi))\theta I_1(\xi) + I_2(\xi)) - (\mu + \gamma)E(\xi) \geq \beta(S^0_\xi + \sigma S^0_\xi)(1 - \varepsilon)\theta I_1(\xi) + I_2(\xi)) - (\mu + \gamma)E(\xi).
\]

For \( y < \xi < -2G \), let

\[
\tilde{J}_0(\xi, y) = \int_{y}^{\xi} E(x)dx, \quad \tilde{J}_1(\xi, y) = \int_{y}^{\xi} I_1(x)dx, \quad \tilde{J}_2(\xi, y) = \int_{y}^{\xi} I_2(x)dx.
\]

Integrate both sides of inequalities (45) from \( y \) to \( \xi \), with \( y < \xi < -2G \), and we can yield

\[
\beta(S^0_\xi + \sigma S^0_\xi)(1 - \varepsilon)\theta \tilde{J}_1(\xi, y) + \tilde{J}_2(\xi, y)) - (\mu + \gamma)\tilde{J}_0(\xi, y)
\]

\[
\leq \int_{y}^{\xi} [\beta(S_1(x) + \sigma S_2(x))\theta I_1(x) + I_2(x)) - (\mu + \gamma)E(x)]dx.
\]

It follows from Lemma 2.10 and inequality (29) that

\[
\| E(\cdot) \|_{C^2(\mathbb{R})} \leq \hat{P}, \quad \| I_1(\cdot) \|_{C^2(\mathbb{R})} \leq \hat{P},
\]

and

\[
\lim_{\xi \to -\infty} \hat{E}'(\xi) = \lim_{\xi \to -\infty} \hat{I}'_1(\xi) = 0,
\]

(48)
where \( \hat{P} \) is a positive constant. Due to inequalities (47)-(48), we have
\[
\int_{-\infty}^{\xi} [\beta(S_1(x) + \sigma S_2(x)) (\theta I_1(x) + I_2(x)) - (\mu + \gamma) E(x)] dx = \lim_{y \to -\infty} \int_{y}^{\xi} [-D_0 E''(x) + c E'(x)] dx = -D_0 E'(\xi) + c E(\xi),
\]
\[
\int_{-\infty}^{\xi} [q_1 \gamma E(x) - (\mu + \alpha_1) I_1(x)] dx = \lim_{y \to -\infty} \int_{y}^{\xi} [-D_1 I_1''(x) + c I_1'(x)] dx = -D_1 I_1'(\xi) + c I_1(\xi).
\]
In the following, we aim to prove that there exists a constant \( \mathcal{J} > 0 \) such that
\[
\int_{-\infty}^{\xi} E(x) dx < \mathcal{J}, \quad \int_{-\infty}^{\xi} I_1(x) dx < \mathcal{J}, \quad \forall \xi < -2G.
\]
Let
\[
A = \begin{pmatrix}
\mu + \gamma & \beta \theta (S_1^0 + \sigma S_2^0)(1 - \varepsilon) & \beta (S_1^0 + \sigma S_2^0)(1 - \varepsilon) \\
q_1 \gamma & \mu + \alpha_1 & 0 \\
q_2 \gamma & \mu + \alpha_2 & 0
\end{pmatrix},
\]
and we conclude \( |A| < 0 \) for \( \varepsilon > 0 \) small enough on the basis of assumption \( R_0 > 1 \).
Add
\[
\hat{J}(\xi, y) = - (\mu + \alpha_2) \beta \theta (S_1^0 + \sigma S_2^0)(1 - \varepsilon)(q_1 \gamma \hat{J}_0(\xi, y) - (\mu + \alpha_1) \hat{J}_1(\xi, y))
\]
\[
- (\mu + \alpha_1) \beta (S_1^0 + \sigma S_2^0)(1 - \varepsilon)(q_2 \gamma \hat{J}_0(\xi, y) - (\mu + \alpha_2) \hat{J}_2(\xi, y))
\]
to both sides of the inequality (46) multiplied by \(- (\mu + \alpha_1)(\mu + \alpha_2)\), and it is obtained that
\[
-|A| \hat{J}_0(\xi, y) \leq (\mu + \alpha_1)(\mu + \alpha_2) \int_{y}^{\xi} [\beta(S_1(x) + \sigma S_2(x))(\theta I_1(x) + I_2(x))
\]
\[
- (\mu + \gamma) E(x)] dx - \hat{J}(\xi, y),
\]
for \( y < \xi < -2G \). As \( y \to -\infty \), the inequality (51), combining equations (49) and (50), leads to
\[
\int_{-\infty}^{\xi} E(x) dx < \mathcal{J}, \quad \forall \xi < -2G.
\]
In the similar way, we also can claim that
\[
\int_{-\infty}^{\xi} I_1(x) dx < \mathcal{J}, \quad \forall \xi < -2G.
\]
The proof is completed.

**Lemma 3.4.** Assume that \( R_0 > 1 \), where \( R_0 \) is the basic reproduction number for system (3). For any \( c > 0 \), if there exists a non-trivial and non-negative traveling wave solution \((S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi))\) of system (5) satisfying the boundary condition (6), a positive constant \( \mu_0 \) can be found and satisfies
\[
\sup_{\xi \in \mathbb{R}} E(\xi)e^{-\mu_0 \xi} < \infty, \quad \sup_{\xi \in \mathbb{R}} |E'(\xi)|e^{-\mu_0 \xi} < \infty, \quad \sup_{\xi \in \mathbb{R}} |E''(\xi)|e^{-\mu_0 \xi} < \infty,
\]
\[
\sup_{\xi \in \mathbb{R}} I_1(\xi)e^{-\mu_0 \xi} < \infty, \quad \sup_{\xi \in \mathbb{R}} |I_1'(\xi)|e^{-\mu_0 \xi} < \infty, \quad \sup_{\xi \in \mathbb{R}} |I_1''(\xi)|e^{-\mu_0 \xi} < \infty,
\]
\[
\sup_{\xi \in \mathbb{R}} I_2(\xi)e^{-\mu_0 \xi} < \infty, \quad \sup_{\xi \in \mathbb{R}} |I_2'(\xi)|e^{-\mu_0 \xi} < \infty, \quad \sup_{\xi \in \mathbb{R}} |I_2''(\xi)|e^{-\mu_0 \xi} < \infty.
\]
Proof. Fix $c > 0$. According to $\varepsilon > 0$ and $G > 0$ defined in Lemma 3.3, it is also shown that

$$S_1(\xi) > S_1^0(1 - \varepsilon), \quad S_2(\xi) > S_2^0(1 - \varepsilon), \quad \forall \xi < -2G.$$ 

And $R_0 > 1$ implies that

$$\frac{[q_1\gamma (\mu + \alpha_2) + q_2\gamma (\mu + \alpha_1)](S_1^0 + \sigma S_2^0)(1 - \varepsilon)}{(\mu + \gamma)(\mu^2 + \alpha_2)} > 1. \quad (53)$$

Thus, for any $\xi < -2G$, we yield that

$$cE'(\xi) \geq D_0 E''(\xi) + \beta(S_1^0 + \sigma S_2^0)(1 - \varepsilon)\theta J_1(\xi) + J_2(\xi) - (\mu + \gamma)E(\xi), \quad (54)$$

$$cI_1'(\xi) = D_1 I_1''(\xi) + q_2\gamma E(\xi) - (\mu + \alpha_1)J_1(\xi), \quad (55)$$

$$cI_2'(\xi) = D_2 I_2''(\xi) + q_2\gamma E(\xi) - (\mu + \alpha_2)J_2(\xi). \quad (56)$$

Set $i = 1, 2$ and remain in what follows. According to Lemma 3.3, it is indicated that

$$\bar{J}_0(\xi) = \int_{-\infty}^{\xi} E(x)dx < J, \quad \bar{J}_i(\xi) = \int_{-\infty}^{\xi} I_i(x)dx < J, \quad \forall \xi < -2G.$$ 

for any $\xi < -2G$. Moreover, we integrate both sides of the inequality (54) from $-\infty$ to $\xi$ and can obtain that

$$cE(\xi) \geq D_0 E'(\xi) + \beta(S_1^0 + \sigma S_2^0)(1 - \varepsilon)\theta \bar{J}_1(\xi) + \bar{J}_2(\xi) - (\mu + \gamma)\bar{J}_0(\xi). \quad (57)$$

Integrating two sides of the inequality (57) from $-\infty$ to $\xi$ with $\xi < -2G$ indicates that

$$\beta(S_1^0 + \sigma S_2^0)(1 - \varepsilon)\theta \int_{-\infty}^{\xi} \bar{J}_1(x)dx + \int_{-\infty}^{\xi} \bar{J}_2(x)dx - (\mu + \gamma) \int_{-\infty}^{\xi} \bar{J}_0(x)dx \quad (58)$$

$$+ D_0 E(\xi) \geq c\bar{J}_0(\xi).$$

Via the similar way in equations (55) and (56), we can yield

$$q_1\gamma \int_{-\infty}^{\xi} \bar{J}_0(x)dx - (\mu + \alpha_1) \int_{-\infty}^{\xi} \bar{J}_1(x)dx + D_1 I_1(\xi) = c\bar{J}_1(\xi), \quad (59)$$

and

$$q_2\gamma \int_{-\infty}^{\xi} \bar{J}_0(x)dx - (\mu + \alpha_2) \int_{-\infty}^{\xi} \bar{J}_2(x)dx + D_2 I_2(\xi) = c\bar{J}_2(\xi), \quad (60)$$

which reduces to

$$\int_{-\infty}^{\xi} \bar{J}_1(x)dx = \frac{1}{\mu + \alpha_1} \left( q_1\gamma \int_{-\infty}^{\xi} \bar{J}_0(x)dx + D_1 I_1(\xi) - c\bar{J}_1(\xi) \right), \quad (61)$$

and

$$\int_{-\infty}^{\xi} \bar{J}_2(x)dx = \frac{1}{\mu + \alpha_2} \left( q_2\gamma \int_{-\infty}^{\xi} \bar{J}_0(x)dx + D_2 I_2(\xi) - c\bar{J}_2(\xi) \right). \quad (62)$$

Next, we need to claim there exist two positive constants $a$ and $b$ such that

$$a \sum_{j=0}^{2} \int_{-\infty}^{\xi} \bar{J}_j(x)dx \leq b \sum_{j=0}^{2} \bar{J}_j(\xi)dx. \quad (63)$$
Substituting equations (61) and (62) into the equation (59) has
\[
c\tilde{J}_0(\xi) \geq \beta(S^0_1 + \sigma S^0_2)(1-\varepsilon) \left[ \frac{\theta}{\mu + \alpha_1} \left( q_1 \gamma \int_{-\infty}^{\xi} \tilde{J}_0(x)dx + D_1 I_1(\xi) - c\tilde{J}_1(\xi) \right) + \frac{1}{\mu + \alpha_2} \left( q_2 \gamma \int_{-\infty}^{\xi} \tilde{J}_0(x)dx + D_2 I_2(\xi) - c\tilde{J}_2(\xi) \right) \right] \\
+ D_0 E(\xi) - (\mu + \gamma) \int_{-\infty}^{\xi} \tilde{J}_0(x)dx
\]
which implies
\[
c \left[ \tilde{J}_0(\xi) + \beta(S^0_1 + \sigma S^0_2)(1-\varepsilon) \left( \frac{\theta}{\mu + \alpha_1} \tilde{J}_1(\xi) + \frac{1}{\mu + \alpha_2} \tilde{J}_2(\xi) \right) \right] \geq \beta(S^0_1 + \sigma S^0_2)(1-\varepsilon) \left( \frac{q_1 \gamma \theta}{\mu + \alpha_1} + \frac{q_2 \gamma}{\mu + \alpha_2} \right) - (\mu + \gamma) \int_{-\infty}^{\xi} \tilde{J}_0(x)dx
\]
Because $E(\xi)$ and $I_i(\xi)$ are non-negative, the above inequality reduces to
\[
\beta(S^0_1 + \sigma S^0_2)(1-\varepsilon) \left( \frac{q_1 \gamma \theta}{\mu + \alpha_1} + \frac{q_2 \gamma}{\mu + \alpha_2} \right) - (\mu + \gamma) \int_{-\infty}^{\xi} \tilde{J}_0(x)dx \leq c \left[ \tilde{J}_0(\xi) + \beta(S^0_1 + \sigma S^0_2)(1-\varepsilon) \left( \frac{\theta}{\mu + \alpha_1} \tilde{J}_1(\xi) + \frac{1}{\mu + \alpha_2} \tilde{J}_2(\xi) \right) \right].
\]
According to the inequality (53), we can claim that there exist two positive constants $\tilde{a}_0$ and $\tilde{b}_0$ such that
\[
\tilde{a}_0 \int_{-\infty}^{\xi} \tilde{J}_0(x)dx \leq \tilde{b}_0 (\tilde{J}_0(\xi) + \tilde{J}_1(\xi) + \tilde{J}_2(\xi)). \quad (64)
\]
Plug the inequality (64) into the inequality (58), and it is shown that there are two constants $\tilde{a}$ and $\tilde{b}$ satisfying
\[
\tilde{a} \left( \int_{-\infty}^{\xi} \tilde{J}_1(x)dx + \int_{-\infty}^{\xi} \tilde{J}_2(x)dx \right) \leq \tilde{b} (\tilde{J}_0(\xi) + \tilde{J}_1(\xi) + \tilde{J}_2(\xi)). \quad (65)
\]
Hence, for any $\xi < -2G$, the inequality (3.3) is established by adding the inequality (64) and the inequality (65). Let
\[
J(\xi) = \tilde{J}_0(\xi) + \tilde{J}_1(\xi) + \tilde{J}_2(\xi),
\]
then we gain that
\[
a \int_{-\infty}^{\xi} J(x)dx \leq b J(\xi), \quad \forall \xi < -2G,
\]
that is
\[ a \int_{-\infty}^{0} J(x + \xi)dx \leq bJ(\xi), \quad \forall \xi < -2G. \]

Because \( J(\cdot) \) is increasing, it is implied that \( a\xi J(\xi - \eta) \leq bJ(\xi) \) for any \( \xi < -2G \) and \( \eta > 0 \). Therefore, there exist a large enough \( \xi_0 > 0 \) and a small \( \omega_0 \in (0, 1) \) such that

\[ J(\xi - \xi_0) \leq \omega_0 J(\xi) \quad \forall \xi < -2G. \]

Let \( \Psi(\xi) = J(\xi)e^{-\mu_0 \xi} \) with \( 0 < \mu_0 = \frac{1}{\xi_0} \ln \frac{1}{\omega_0} < \lambda_1 \), and then it is concluded that

\[ \Psi(\xi - \xi_0) = J(\xi - \xi_0)e^{-\mu_0 (\xi - \xi_0)} \leq \omega_0 J(\xi)e^{-\mu_0 (\xi - \xi_0)} = \Psi(\xi), \]

for any \( \xi < -2G \). According to \( J(\xi) < \infty \) for any \( \xi < -2G \), we can find a constant \( k_0 > 0 \) such that \( \Psi(\xi) \leq k_0 \) for \( \forall \xi < -2G \), that is \( J(\xi) \leq k_0 e^{\mu_0 \xi} \) for any \( \xi < -2G \). Therefore, there exists a constant \( q_0 > 0 \) such that \( \int_{-\infty}^{0} J(x)dx \leq q_0 e^{\mu_0 \xi} \) in \( \xi \in (-\infty, -2G) \). Combining equations (59)-(60), we can also find a constant \( p_0 > 0 \) satisfying

\[ E(\xi) \leq p_0 e^{\mu_0 \xi}, \quad I_i(\xi) \leq p_0 e^{\mu_0 \xi}, \quad \forall \xi < -2G. \]

Since \( E(\xi) \) and \( I_i(\xi) \) are bounded in \( \mathbb{R} \), we obtain that

\[ E(\xi) \leq p_0 e^{\mu_0 \xi}, \quad I_i(\xi) \leq p_0 e^{\mu_0 \xi}, \quad \forall \xi \in \mathbb{R}. \]

According to estimates (29) and (57), it yields

\[ \sup_{\xi \in \mathbb{R}} |E^{(i)}(\xi)| e^{-\mu_0 \xi} < \infty. \]

Via the \( E \)-th equation in system (5) and the inequality (54), we can conclude that

\[ \sup_{\xi \in \mathbb{R}} |E^{(i)}(\xi)| e^{-\mu_0 \xi} < \infty. \]

Finally, applying the similar argument on \( I_i(\xi) \) shows that

\[ \sup_{\xi \in \mathbb{R}} I_i(\xi) e^{-\mu_0 \xi} < \infty, \quad \sup_{\xi \in \mathbb{R}} |I_i(\xi)| e^{-\mu_0 \xi} < \infty, \quad \sup_{\xi \in \mathbb{R}} |I_i^{(i)}(\xi)| e^{-\mu_0 \xi} < \infty. \]

This completes the proof. \( \square \)

According to above two theorems, we can obtain the following non-existence theorem.

**Theorem 3.5.** If \( R_0 > 1 \), for \( c \in (0, c^*) \), there exists no non-trivial and non-negative traveling wave solution \( (S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi)) \) of system (5) satisfying the boundary condition (6), where \( c^* \) is the minimal spread speed and \( R_0 \) is the basic reproduction number for system (3).

**Proof.** We intend to prove this theorem by contradiction. Fix \( c \in (0, c^*) \). On the contrary, we assume there exists a non-trivial and non-negative \( (S_1(\xi), S_2(\xi), E(\xi), I_1(\xi), I_2(\xi)) \) of system (5) satisfying the boundary condition (6). Via Lemma 3.4, then there exists a positive constant \( \mu_0 \) defined in the proof of Lemma 3.4 such that inequalities (52) are established. Set \( K_1(\xi) = S_1(\xi) - S_1(0) \) and \( K_2(\xi) = S_2(\xi) - S_2(0) \) in \( \mathbb{R} \). Plugging \( K_1(\xi) \) and \( K_2(\xi) \) into the 1-st and 2-nd equations yields

\[
\begin{align*}
    cK_1'(\xi) - dK_1''(\xi) + (\mu + \rho)K_1'(\xi) - \beta S_1(\xi)\theta I_1(\xi) + I_2(\xi) &= 0, \\
    cK_2'(\xi) - dK_2''(\xi) - \rho K_2'(\xi) + \mu K_2'(\xi) - \beta S_2(\xi)\theta I_1(\xi) + I_2(\xi) &= 0.
\end{align*}
\]  

(66)  

(67)
According to the inequality
\[ \| K_i' \|_{C^\infty([0,0],[R])} \leq 2 \| K_i'' \|_{C^\infty([0,0],[R])} \| K_i \|_{C^\infty([0,0],[R])}, \]
and the fact that
\[ \lim_{\xi \to -\infty} K_i(\xi) = 0, \]
we can claim
\[ \lim_{\xi \to -\infty} K_i'(\xi) = 0, \]
for \( i = 1, 2 \). In addition, since \( K_i(\xi) \) is bounded and satisfies equations (68)-(69), integrating both sides of the sum of equations (66) and (67) from \(-\infty\) to \( \xi \leq 0 \) can show that
\[ c[K_1(\xi) + K_2(\xi)] - d[K_1'(\xi) + K_2'(\xi)] + \mu \int_{-\infty}^{\xi} [K_1(x) + K_2(x)]dx \]
\[ - \int_{-\infty}^{\xi} \beta[S_1(x) + \sigma S_2(x)]\theta I_1(x) + I_2(x)]dx = 0. \]
Let \( f(\xi) = \int_{-\infty}^{\xi} \beta[S_1(x) + \sigma S_2(x)]\theta I_1(x) + I_2(x)]dx \), and \( K_{\mu}(\xi) = \mu \int_{-\infty}^{\xi} [K_1(x) + K_2(x)]dx \) for any \( \xi \leq 0 \). It follows from Lemma 3.4 that \( f(\xi) \leq C_f e^{\mu_0 \xi} \) on \( \xi \in \mathbb{R} \), where \( C_f \) is a positive constant. Hence, a direct calculation indicates that
\[ K_1(\xi) + K_2(\xi) = [K_1(0) + K_2(0)]e^{\xi \xi} + \frac{1}{d} e^{\xi \xi} \int_{\xi}^{0} e^{-\xi x} [-K_{\mu}(x) + f(x)]dx \]
\[ \leq [K_1(0) + K_2(0)]e^{\xi \xi} + \frac{1}{d} e^{\xi \xi} \int_{\xi}^{0} e^{-\xi x} f(x)dx, \]
for \( \xi \leq 0 \). Owing to \( f(\xi) = O(e^{\mu_0 \xi}) \) as \( \xi \to -\infty \), we can claim that \( K_1(\xi) + K_2(\xi) = O(e^{\mu_0 \xi}) \) as \( \xi \to -\infty \), where \( \mu_0 = \min \{ \mu_0, \frac{\mu}{d} \} \). Combining \( 0 \leq K_1(\xi) + K_2(\xi) \leq \frac{\mu}{d} \), we can conclude that
\[ \sup_{\xi \in \mathbb{R}} [K_1(\xi) + K_2(\xi)]e^{-\mu_0 \xi} < \infty, \]
which implies
\[ \sup_{\xi \in \mathbb{R}} K_1(\xi)e^{-\mu_0 \xi} < \infty, \quad \sup_{\xi \in \mathbb{R}} K_2(\xi)e^{-\mu_0 \xi} < \infty. \]

On the basis of the above discussion, we define the one-sided Laplace transforms for \( E(\xi), I_1(\xi) \) and \( I_2(\xi) \) by
\[ L_0(\lambda) = \int_{-\infty}^{0} e^{-\lambda \xi} E(\xi)d\xi, \quad L_1(\lambda) = \int_{-\infty}^{0} e^{-\lambda \xi} I_1(\xi)d\xi, \quad L_2(\lambda) = \int_{-\infty}^{0} e^{-\lambda \xi} I_2(\xi)d\xi. \]
(71)
Next, we only consider \( \lambda \in \mathbb{R}^+ \). Since \( E(\xi) > 0, I_1(\xi) > 0, I_2(\xi) > 0 \) for any \( \xi \in \mathbb{R} \), and \( L_i(\cdot) \) is increasing for \( \lambda \in \mathbb{R}^+ \), respectively, there exist two possibilities: (i) a positive constant \( v_i > \mu_0 \) can be found such that \( L_i(\cdot) < \infty \) for any \( 0 \leq \lambda < v_i \) and \( \lim_{\lambda \to v_i} L_i(\lambda) = \infty \). (ii) \( L_i(\cdot) < \infty \) for any \( \lambda \geq 0 \), where \( i = 0, 1, 2 \). Therefore, we define two-sided Laplace transforms for \( E(\cdot), I_1(\cdot) \) and \( I_2(\cdot) \) by
\[ L_0(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda \xi} E(\xi)d\xi, \quad L_1(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda \xi} I_1(\xi)d\xi, \quad L_2(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda \xi} I_2(\xi)d\xi, \]
and we also only research $\hat{\lambda} > 0$. Because $E(\cdot)$, $I_1(\cdot)$ and $I_2(\cdot)$ are bounded in $\mathbb{R}$, respectively, it is the fact that
\[
\int_0^{+\infty} e^{-\hat{\lambda} \xi} E(\xi) d\xi < \infty, \quad \int_0^{+\infty} e^{-\hat{\lambda} \xi} I_1(\xi) d\xi < \infty, \quad \int_0^{+\infty} e^{-\hat{\lambda} \xi} I_2(\xi) d\xi < \infty,
\]
for any $\hat{\lambda} > 0$, which implies $L_i(\cdot)$ owns the same property as $L_i(\cdot)$ in $\mathbb{R}_+$ with $i = 0, 1, 2$, respectively.

In the following, we indicate that there indeed exists a $v_i = +\infty$ such that $L_i(\hat{\lambda}) < \infty$ for any $\hat{\lambda} \geq 0$ with $i = 0, 1, 2$, respectively. The first step is to prove $v_0 = v_1 = v_2$. According to
\[
D_0 E''(\xi) - cE'(\xi) + \beta(S_1^0 + \sigma S_2^0)[\theta I_1(\xi) + I_2(\xi)] - (\mu + \gamma) E(\xi) = \beta[S_1^0 - S_1(\xi) + \sigma S_2^1 - S_2(\xi)][\theta I_1(\xi) + I_2(\xi)], \quad \forall \xi \in \mathbb{R},
\]
it is implied that
\[
[D_0 \hat{\lambda}^2 - c\hat{\lambda} - (\mu + \gamma)]L_0(\hat{\lambda}) + \beta(S_1^0 + \sigma S_2^0)[\theta L_1(\hat{\lambda}) + L_2(\hat{\lambda})]
= \beta \int_{-\infty}^{+\infty} [S_1^0 - S_1(\xi) + \sigma S_2^1 - S_2(\xi)][\theta I_1(\xi) + I_2(\xi)] e^{-\hat{\lambda} \xi} d\xi, \quad \forall \xi \in \mathbb{R}. \tag{72}
\]
In the same way, we can obtain
\[
[D_1 \hat{\lambda}^2 - c\hat{\lambda} - (\mu + \alpha_1)]L_1(\hat{\lambda}) + q_1 \gamma L_0(\hat{\lambda}) = 0, \tag{73}
\]
and
\[
[D_2 \hat{\lambda}^2 - c\hat{\lambda} - (\mu + \alpha_2)]L_2(\hat{\lambda}) + q_2 \gamma L_0(\hat{\lambda}) = 0. \tag{74}
\]
Thus, it follows from equations (73) and (74) that $v_0 = v_1 = v_2$. Let $v = v_0 = v_1 = v_2$. The second step is to prove $v = +\infty$ via a contradiction. We assume that $v < +\infty$ and $L_i(\hat{\lambda}) < \infty (i = 0, 1, 2)$ for any $\hat{\lambda} \in [0, v)$ and $\lim_{\hat{\lambda} \to v^-} L_i(\hat{\lambda}) = \infty$.

Equations (72)-(74) can show that $-D_1 v^2 + cv + (\mu + \alpha_1) > 0$ and $-D_2 v^2 + cv + (\mu + \alpha_2) > 0$. If $D_0 v^2 - cv - (\mu + \gamma) \geq 0$ and let $\hat{\lambda} \to v - 0$ in both sides of (72), the right side shows
\[
\beta \int_{-\infty}^{+\infty} [S_1^0 - S_1(\xi) + \sigma S_2^1 - S_2(\xi)][\theta I_1(\xi) + I_2(\xi)] e^{-\hat{\lambda} \xi} d\xi
\leq \beta \sup_{\xi \in \mathbb{R}} \left\{ [S_1^0 - S_1(\xi) + \sigma S_2^1 - S_2(\xi)] e^{-\mu_0 \xi} \right\} \left[ \int_{-\infty}^{+\infty} \theta I_1(\xi) e^{-(\nu - \mu_0) \xi} d\xi + \int_{-\infty}^{+\infty} I_2(\xi) e^{-(\nu - \mu_0) \xi} d\xi \right] \leq \infty.
\]
However, the left side tends to $\infty$, which leads to a contraction. If $D_0 v^2 - cv - (\mu + \gamma) < 0$, plugging equations (73) and (74) into the equation (72) can obtain that
\[
\hat{m}_0(\hat{\lambda}, c) L_0(\hat{\lambda}) [\rho^2(\hat{\lambda}, c) - 1] = \beta \int_{-\infty}^{+\infty} [S_1^0 - S_1(\xi) + \sigma S_2^1 - S_2(\xi)][\theta I_1(\xi) + I_2(\xi)] e^{-\hat{\lambda} \xi} d\xi, \tag{75}
\]
and $0 < v < \lambda_c$. As $\hat{\lambda} \to v - 0$, the right side is bounded, but the left side tends to $\infty$. As a result, the case (1) is impossible.
If the case (ii) is established, similarly, let \( \hat{\lambda} \to +\infty \) in both sides of the equation (72). The right side is bounded but the left side tends to \( \infty \) owing to \( D_0 \lambda^2 - c \hat{\lambda} - (\mu + \gamma) \to \infty \), which also leads to a contraction. Above all, the non-existence is completely proved.

4. Numerical simulations and discussion. Section 2 and Section 3 have proved the existence and non-existence of traveling wave solution for system (2) satisfying the boundary condition (6). In this section, we aim to visually display the existence of traveling wave solution for system (2) connecting disease-free equilibrium and endemic equilibrium. Now, we firstly take a set of parameters for system (2) as follows:

\[
\begin{align*}
    d &= 0.02, \quad D_0 = 0.015, \quad D_1 = 0.005, \quad D_2 = 0.01, \quad D = 0.02, \\
    \Lambda &= 8, \quad \mu = 0.3, \quad \beta = 0.375, \quad \rho = 0.7, \\
    \sigma &= 0.5, \quad \gamma = 0.75, \quad \alpha_1 = 0.65, \quad \alpha_2 = 0.5, \\
    \theta &= 0.5, \quad q_1 = 0.6, \quad q_2 = 0.4.
\end{align*}
\]

As a result, we can obtain the disease-free equilibrium \((S_1^0, S_2^0, E^0, I_1^0, I_2^0, R^0) = (8, 18.6667, 0, 0, 0, 0)\), the endemic equilibrium \((S_1^*, S_2^*, E^*, I_1^*, I_2^*, R^*) \approx (3.3762, 2.3999, 5.9687, 2.8274, 2.2383, 9.8565)\) and the basic reproduction number \(R_0 \approx 3.2821 > 1\). For simulations, we further intercept \([-200, 200]\) from spatial domain and \([0, 200]\) from time domain. Moreover, we take the Neumann boundary condition and the below piecewise functions as initial conditions for system (2):

\[
\begin{align*}
    S_i(x, t) &= \begin{cases} 
        S_i^*, & 0 < x \leq 200, \quad t = 0, \quad i = 1, 2, \\
        S_i^0, & -200 \leq x \leq 0, \quad t = 0, \quad i = 1, 2,
    \end{cases} \\
    E(x, t) &= \begin{cases} 
        E^*, & 0 < x \leq 200, \quad t = 0, \\
        0, & -200 \leq x \leq 0, \quad t = 0,
    \end{cases} \\
    I_i(x, t) &= \begin{cases} 
        I_i^*, & 0 < x \leq 200, \quad t = 0, \quad i = 1, 2, \\
        0, & -200 \leq x \leq 0, \quad t = 0, \quad i = 1, 2,
    \end{cases} \\
    R(x, t) &= \begin{cases} 
        R^*, & 0 < x \leq 200, \quad t = 0, \\
        0, & -200 \leq x \leq 0, \quad t = 0.
    \end{cases}
\end{align*}
\]

The figure 1, simulations with applying above conditions, indicate that there exists a traveling wave solution of system (2) connecting disease-free equilibrium and endemic equilibrium. Meanwhile, we cross section curves of traveling wave solution in figure 1 as \( t = 200\) (see figure 2). And we can find that traveling waves for system (2) are not monotonic in figure 2.

Since we pay more attention to influences of self-protection and treatment in the spatial spread for an epidemic, it is critical to research the change of the minimal spread speed \( c^* \) while self-protection \( \sigma \) and treatment \( \theta \) changing. The direct derivations of \( \rho(\lambda, c) \) with respect to \( \sigma \) and \( \theta \) show that

\[
\frac{\partial \rho}{\partial \theta} > 0, \quad \frac{\partial \rho}{\partial \sigma} > 0,
\]

respectively, where \( \rho(\lambda, c) \) is defined as the formula (9). The inequality (76) means that the numerical increase in \( \sigma \) and \( \theta \) could lead to increasing of the minimal spread speed \( c^* \) with applying Lemma 2.3. The figures (a) and (b) in figure 3 also display that the minimal spread speed \( c^* \) is increasing with respect to \( \sigma \) and \( \theta \) for \( \sigma, \theta \in [0.05, 1] \), respectively. In fact, measures about the enhancement of
self-protection and treatment can lead to numerical reductions about both $\sigma$ and $\theta$, which implies the minimal spread speed $c^*$ will decrease via the inequality (76) and figure 3. Above all, the effective self-protection and treatment can reduce the spread speed for an epidemic.

In this paper, we mainly construct a non-monotonic reaction diffusion SEIR model with effects of self-protection and treatment in incident rate, and determine the existence and non-existence of traveling wave solution connecting disease-free equilibrium and endemic equilibrium. We prove the existence as $R_0 > 1$ and $c \geq c^*$. And when $R_0 \leq 1$ or $R_0 > 1$ with $c \in (0, c^*)$, there exists no non-trivial and non-negative traveling wave solution satisfying the boundary condition (6). Finally, the numerical simulations show the existence and indicate that self-protection and treatment can reduce the spread speed of an epidemic. However, self-protection and treatment may affect the movement of different individuals in various times and spaces. Some individuals may even choose or be forcibly quarantined. These factors are worthy being researched in the future.

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Figure 2. Cross section curve of traveling wave solution for system (2) as $t = 200$.

Figure 3. Show the effects of self-protection $\sigma$ and treatment $\theta$ on minimal spread speed $c^*$, where $\sigma$ and $\theta$ are taken from 0.05 to 1.
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