WELL-POSEDNESS RESULTS FOR A CLASS OF SEMI-LINEAR SUPER-DIFFUSIVE EQUATIONS

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Abstract. In this paper we investigate the following fractional order in time Cauchy problem
\[
\begin{align*}
\frac{D^\alpha_t u(t)}{t} + Au(t) &= f(u(t)), \quad 1 < \alpha < 2, \\
u(0) &= u_0, \quad u'(0) = u_1.
\end{align*}
\]
The fractional in time derivative is taken in the classical Caputo sense. In the scientific literature such equations are sometimes dubbed as fractional-in time wave equations or super-diffusive equations. We obtain results on existence and regularity of local and global weak solutions assuming that \( A \) is a nonnegative self-adjoint operator with compact resolvent in a Hilbert space and with a nonlinearity \( f \in C^1(\mathbb{R}) \) that satisfies suitable growth conditions. Further theorems on the existence of strong solutions are also given in this general context.

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2010 Mathematics Subject Classification. 26A33, 35R11, 35G31, 34A12, 74G20, 74G25.

Key words and phrases. Fractional semi-linear wave equations, polynomial growth condition, weak and strong solutions, existence and uniqueness of local and global solutions.

The work of V. Keyantuo and M. Warma is partially supported by the Air Force Office of Scientific Research under the Award No: FA9550-18-1-0242.

The work E. Alvarez is partially supported by Colciencias under the Award No: 1215-569-33876.
1. Introduction

Of concern in the present paper is the existence of local and global solutions for a class of semi-linear super-diffusive fractional (wave) equations. More precisely, our aim is to investigate the following initial-value problem

$$\begin{cases}
\mathbb{D}_t^{\alpha} u(x, t) + Au(x, t) = f(u(x, t)), & \text{in } X \times (0, T), \\
u(\cdot, 0) = u_0, \quad \partial_t u(\cdot, 0) = u_1 & \text{in } X.
\end{cases} \tag{1.1}$$

In (1.1), $X$ is a (relatively) compact Hausdorff space, $T > 0$ and $1 < \alpha < 2$ are real numbers and $\mathbb{D}_t^{\alpha} u$ denotes the Caputo fractional derivative with respect to $t$, which is defined by

$$\mathbb{D}_t^{\alpha} u(x, t) := \frac{1}{\Gamma(2 - \alpha)} \int_0^t (t - s)^{1 - \alpha} \partial_s^2 u(x, s) ds, \quad (x, t) \in X \times (0, T). \tag{1.2}$$

When the function involved in (1.2) is sufficiently smooth, then (1.2) is equivalent to the following weaker form (see [33]; cf. also [7, 26, 25]):

$$\mathbb{D}_t^{\alpha} u(x, t) = \frac{1}{\Gamma(2 - \alpha)} \partial_t^2 \int_0^t (t - s)^{1 - \alpha} \left( u(x, s) - u(x, 0) - s \partial_s u(x, 0) \right) ds. \tag{1.3}$$

Notice that when $\alpha \to 2^-$, we have that $\mathbb{D}_t^{\alpha} \to \partial_t^2$, that is, the classical second order derivative. This can be easily seen from the formulas (1.2) or (1.3) with the interpretation that $\lim_{\beta \to 0^+} g_\beta(t) = \lim_{\beta \to 0^+} \frac{t^{\beta - 1}}{\Gamma(\beta)} = \delta_0$, the Dirac measure concentrated at the point 0. Finally, the nonlinearity $f \in C^1(\mathbb{R})$ satisfies some suitable growth conditions as $|u| \to \infty$.

The theory of fractional differential equations has found applications in many areas of science and technology and several monographs are devoted to their study. We mention [31, 33, 35] and the recently published book [20]. The interest in these equations lies in the fact that some nonlocal aspects of phenomena or systems that cannot be captured by the classical theory of partial differential equations fit well into the new models. Examples of this are phenomena with memory effects, anomalous diffusion, problems in rheology, material science and many other areas. The papers [15, 22, 21, 29, 30, 32] covers many of these applications. One important feature when dealing with evolution equations that are fractional in time is that several models of fractional derivatives are available. Historically, the most important ones are the Riemann-Liouville and Caputo fractional derivatives. We note that spatial models involving fractional derivatives are also being actively studied, due to their suitability for modeling concrete systems on the one hand, and due to the richness of the mathematical structure involved on the other hand. Here, stochastic models have received much attention as well. The above mentioned papers cover some of the important aspects. Both time and space fractional derivatives have a long history as can be seen in the above references.

As in the classical area of partial differential equations, linear and nonlinear models have been studied. The linear models are sometimes good approximations of the real problems

\footnote{Strictly speaking, the notion of Riemann-Liouville derivative requires a "lesser" degree of smoothness of the function involved.}
under consideration but as it is well known, their analysis also provides the mathematical tools needed to study nonlinear phenomena, especially for semi-linear and quasi-linear equations.

A theory of mild and classical solutions for the classical semi-linear wave equation ($\alpha = 2$) with a Lipschitz nonlinearity is developed in [19] and [38] by essentially adapting the techniques of Henry [24] exploited for semi-linear parabolic equations. In [37], the author surveys existence and regularity results for semi-linear wave equations with a polynomial like nonlinearity for the Laplacian in $\mathbb{R}^d$. The general existence and uniqueness theory for linear nonhomogeneous equations associated with (1.1) have been studied in [25] and [34]. In [27], a theory of integral solutions has been developed for the semi-linear problem (1.1) when $A$ is related to the classical Dirichlet Laplacian. In [28], the authors consider the operator $A = -\Delta_x$, $x \in \mathbb{R}^d$, in (1.1) and find a "critical" exponent in order to deduce the global existence of integral solutions for small data in low space dimension. The limiting case $\alpha = 2$ corresponding to the abstract wave equation (see, for example, [19, 38]) is used as an inspiration for the fractional case. However, the range of applicability of the results is much wider. In fact, well posedness for the linear wave equation fails in $L^p(\mathbb{R}^d)$ if $d \geq 2$ and $p \neq 2$. The classical theory of strongly continuous semigroups does not apply (see, e.g., [2, Chapter 8]). There is a substitute theory, namely that of integrated cosine functions (and for equations of the form (1.1), operator families suitable for such extensions have been introduced and treated in [25]) but we do not consider this approach in the present paper.

We note that a complete study of locally and/or globally defined weak and strong solutions and their fine regularities in the case $0 < \alpha \leq 1$ has been already performed in [18]. Here, the authors have established some precise and optimal conditions on the nonlinearity in order to have existence and the precise regularity of local and global weak and strong solutions. In particular, these results show that case $\alpha = 1$ can be recovered in a natural way. We refer the interested reader for an extensive comparison of our work in [18] with other investigations for the problem when $0 < \alpha \leq 1$, which lies outside the scope of the present paper. Outside the works of [27] and [28], we remark that not much seems to be known about semilinear fractional waves in general. Here we are interested in existence, uniqueness and regularity results for the semi-linear equation (1.1) when $1 < \alpha < 2$, under appropriate conditions on the data. In contrast to the works of [27], [28], which only consider the Laplacian for the operator and a general notion of integral solutions, the main novelties of the present paper are as follows:

- Our assumption on the operator $A$ is quite general. Indeed, we let $A$ be a self-adjoint operator in $L^2(X)$, that is associated with a bilinear symmetric and closed form $\mathcal{E}_A$, whose domain $D(\mathcal{E}_A)$ is compactly embedded into $L^2(X)$. We further assume that $(\mathcal{E}_A, D(\mathcal{E}_A))$ is a Dirichlet space in the broad sense of [16]. We refer to Section 2.1 for further details. Our framework contains in particular the realization in $L^2(\Omega)$ (where $\Omega \subset \mathbb{R}^d$ is an open set) of any second order elliptic operator in divergence form with Dirichlet, Neumann, Robin or Wentzell boundary conditions. It also contains any fractional powers of these operators and also the realization in $L^2(\Omega)$ of the fractional Laplace operator $(-\Delta)^s$ ($0 < s < 1$) with the zero Dirichlet exterior condition $u = 0$ in $\mathbb{R}^d \setminus \Omega$, and many other operators that are not explicitly stated in the paper (see
Section 5). We refer to \[36, 41, 40, 42\] and their references for more information on the operator \((-\Delta)\).

- We employ a notion of energy (weak) solution that is much stronger than the notion of integral solution devised by \[27, 28\]. Besides, in some cases we prove the existence of strong energy solutions that have the additional property that they also satisfy the fractional wave equation pointwise (see also Section 5.1 for further comments).

Our first main result (Theorem 4.4) states that if \(u_0 \in D(A_{1/2}^\alpha)\) (the domain of the fractional \(1/2\)-power), \(u_1 \in L^2(X)\) and \(f\) satisfies suitable growth assumptions (see Section 4), then our system has a unique weak solution on \((0, T^*)\) for some \(T^* > 0\). In most cases, these assumptions allow not only for nonlinearities of polynomial growth (at infinity) but essentially also for functions without a growth restriction. A critical value of \(\alpha\) will play here an essential role for finding classes of (unique) weak solutions that satisfy a certain energy identity and that exhibit finer properties of their solutions. The second main result (Theorem 4.5) shows that locally defined weak solutions can be always extended to a larger interval. Finally, Theorem 4.6 gives some results related to the existence of global weak solutions. In some cases, the existence of strong solutions (satisfying the equation pointwise on \(X \times (0, T)\)) can be also deduced in a certain range for \(\alpha\) (see Theorem 4.9).

The rest of the paper is structured as follows. In Section 2 firstly we introduce some notations and our general assumptions and secondly, we give the definition of the Mittag-Leffler functions and their properties that will be used throughout the paper. In Section 3 we study the linear problem where we have obtained some new regularity results of weak and strong solutions. In Section 4 we investigate the semi-linear system (1.1). We first introduce our notion of weak solutions of the considered system in noncritical and critical cases. We next give our general assumption on the nonlinearity. We conclude the section by stating the main results of the article regarding the semi-linear system. In Section 6 we give the proofs of our main results in the noncritical case. The proof of the critical case is contained in Section 7. Some examples of self-adjoint operators that fit our framework are given in Section 5.

2. Functional framework

We first introduce some background. Let \(Y, Z\) be two Banach spaces endowed with norms \(\|\cdot\|_Y\) and \(\|\cdot\|_Z\), respectively. We denote by \(Y \hookrightarrow Z\) if \(Y \subseteq Z\) and there exists a constant \(C > 0\) such that \(\|u\|_Z \leq C \|u\|_Y\), for \(u \in Y \subseteq Z\). This means that the injection of \(Y\) into \(Z\) is continuous. In addition, if the injection is also compact we shall denote it by \(Y \hookrightarrow_{cp} Z\).

By the dual \(Y^*\) of \(Y\), we think of \(Y^*\) as the set of all (continuous) linear functionals on \(Y\). When equipped with the operator norm \(\|\cdot\|_{Y^*}\), \(Y^*\) is also a Banach space.

2.1. Energy forms and Markovian semigroups. We introduce the notion of Dirichlet form on an \(L^2\)-type space (see [16, Chapter 1]). To this end, let \(X\) be a (relatively) compact metric space and \(m\) a Radon measure on \(X\) such that \(\text{supp}(m) = X\). Let \(L^2(X) = L^2(X, m)\) be the real Hilbert space with inner product \((\cdot, \cdot)\) and let \(\mathcal{E}_A\) with domain \(D(\mathcal{E}_A) =: V_{1/2}\) be a bilinear form on \(L^2(X)\). We consider \(L^p(X) = L^p(X, m)\) to be the corresponding Banach space for \(1 \leq p \leq \infty\), with norm \(\|\cdot\|_{L^p(X)}\). We notice that our assumption implies that \(m(X) < \infty\).
From now on, we shall refer to \( A \) Sobolev embedding theorem holds for \( V \) symmetric forms \( L \) form on 

\[ \text{Let } \] 

results in abstract form can be also found in the monographs \([13, 16]\). Then the following assertions hold.

**Remark 2.2.** We make the following important remarks.

- Clearly, \( D (A) = V_{1/2} \) is a real Hilbert space with inner product \( E_{A,\lambda} (\cdot, \cdot) \) for each \( \lambda > 0 \). We recall that a form \( E_A \) which satisfies (a)-(c) is closed and symmetric. If \( E_A \) also satisfies (d), then it is said to be a Markovian form.

- When \( E_A \) is closed, (d) is equivalent to the following more simple condition:

\[ (d) \quad \text{for each } \epsilon > 0 \text{ there exists a function } \phi_\epsilon : \mathbb{R} \to \mathbb{R}, \text{ such that } \phi_\epsilon \in C^\infty (\mathbb{R}), \phi_\epsilon (t) = t, \text{ for } t \in [0, 1], -\epsilon \leq \phi_\epsilon (t) \leq 1 + \epsilon, \text{ for all } t \in \mathbb{R}, 0 \leq \phi_\epsilon (t) - \phi_\epsilon (\tau) \leq t - \tau, \text{ whenever } \tau < t, \text{ such that } u \in V_{1/2} \text{ implies } \phi_\epsilon (u) \in V_{1/2} \text{ and } E_A (\phi_\epsilon (u), \phi_\epsilon (u)) \leq E_A (u, u). \]

We call \( v \in L^2 (X) \) a normal contraction of \( u \) implies \( v \in V_{1/2} \) and \( E_A (v, v) \leq E_A (u, u) \). We call \( v \) a normal contraction of some Borel version of \( u \) in \( L^2 (X) \), that is, \( |v (x)| \leq |u (x)| \), for all \( x \in X \), and

\[ |v (x) - v (y)| \leq |u (x) - u (y)|, \text{ for all } x, y \in X. \]

It is well-known that there is a one-to-one correspondence between the family of closed symmetric forms \( E_A \) on \( L^2 (X) \) and the family of non-negative (definite) self-adjoint operators \( A \) on \( L^2 (X) \) in the following sense:

\[
\begin{aligned}
\{ & V_{1/2} = D (A^{1/2}) , \ D (A) \hookrightarrow V_{1/2} , \text{ and} \\
& E_A (u, v) = (Au, v) , \ u \in D (A) , \ v \in V_{1/2} . \}
\end{aligned}
\]

From now on, we shall refer to \( (E_A, V_{1/2}) \) as a Dirichlet space whenever \( E_A \) is a Dirichlet form on \( L^2 (X) \) with \( D (E_A) = V_{1/2} \) in the sense of Definition 2.1.

Finally, any self-adjoint operator \( A \), that is in one-to-one correspondence with the Dirichlet form \( E_A \) (see (2.1)), turns out to possess a number of good properties provided a certain Sobolev embedding theorem holds for \( V_{1/2} \) (see, for instance, [17, Theorem 2.9]). Similar results in abstract form can be also found in the monographs [13, 16].

**Theorem 2.3.** Let \( A \) be the operator associated with the Dirichlet space \( (E_A, V_{1/2}) \). Assume \( V_{1/2} \hookrightarrow L^2 (X) \) and

\[ V_{1/2} \hookrightarrow L^{2q_A} (X) , \text{ for some } q_A > 1. \]

Then the following assertions hold.
(a) The operator $-A$ generates a submarkovian semigroup $(e^{-tA})_{t \geq 0}$ on $L^2(X)$. The semigroup can be extended to a contraction semigroup on $L^p(X)$ for every $p \in [1, \infty]$, and each semigroup is strongly continuous if $p \in [1, \infty)$ and bounded analytic if $p \in (1, \infty)$. Each such semigroup on $L^p(X)$ is compact for every $p \in [1, \infty]$.

(b) The operator $A$ has a compact resolvent, and hence has a discrete spectrum. The spectrum of $A$ is an increasing sequence of real numbers $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots$, that converges to $+\infty$.

(c) If $\varphi_n$ is an eigenfunction associated with $\lambda_n$, then $\varphi_n \in D(A) \cap L^\infty(X)$.

(d) For $\theta \in (0, 1]$, the embedding $D(A^\theta) \hookrightarrow L^\infty(X)$ holds provided that

$$\theta > \frac{qA}{2(qA - 1)} =: \theta_A.$$ 

(e) Moreover, when $\theta = \theta_A$ we have that $D(A^\theta) \hookrightarrow L^{2r_*}(X)$, for any $r_* \in (1, \infty)$. If $\theta < \theta_A$ then

$$D(A^\theta) \hookrightarrow L^{2r_*}(X), \text{ for } r_* = \frac{\theta_A}{\theta_A - \theta}.$$ 

In what follows, without loss of generality we can assume that $\lambda_1 > 0$; otherwise, one can replace the operator $A$ by $A + \varepsilon I$, $\varepsilon > 0$, to satisfy this assumption. In that case all the eigenvalues are of finite multiplicity.

For all $s \geq 0$, the operator $A^s$ also possesses the following representation:

$$A^s h = \sum_{n=1}^{\infty} (h, \varphi_n) \lambda_n^s \varphi_n, \quad h \in D(A^s) = \left\{ h \in L^2(X) : \sum_{n=1}^{\infty} |(h, \varphi_n)|^2 \lambda_n^{2s} < \infty \right\}.$$ \hspace{1cm} (2.3)

Consider on $D(A^s)$ the norm (recall that $\lambda_1 > 0$)

$$\|h\|_{D(A^s)} = \left( \sum_{n=1}^{\infty} |(h, \varphi_n)|^2 \lambda_n^{2s} \right)^{\frac{1}{2}}, \quad h \in D(A^s).$$

By duality, we can also set $D(A^{-s}) = (D(A^s))^*$ by identifying $(L^2(X))^* = L^2(X)$, and using the so called Gelfand triple (see e.g. [6]). Then $D(A^{-s})$ is a Hilbert space with the norm

$$\|h\|_{D(A^{-s})} = \left( \sum_{n=1}^{\infty} |(h, \varphi_n)|^2 \lambda_n^{-2s} \right)^{\frac{1}{2}},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket between $D(A^{-s})$ and $D(A)$. Since $D(A^{1/2}) = V_{1/2}$, we identify $V_{-1/2}$ with $D(A^{-1/2})$.

Throughout the remainder of the paper, without any mention we shall assume that $A$ satisfies the above assumptions. In addition, given any Banach space $Y$ and its dual $Y^*$, we shall denote by $\langle \cdot, \cdot \rangle_{Y^*,Y}$ their duality bracket.

2.2. Properties of Mittag-Leffler functions. The Mittag-Leffler function with two parameters is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \beta \in \mathbb{C}, \quad z \in \mathbb{C},$$
where Γ is the usual Gamma function. It is well-known that $E_{α,β}(z)$ is an entire function. The following estimate of the Mittag-Leffler function will be useful. Let $0 < α < 2$, $β ∈ ℝ$ and µ be such that $\frac{απ}{2} < µ < \min\{π, απ\}$. Then there is a constant $C = C(α, β, µ > 0$ such that

$$|E_{α,β}(z)| ≤ \frac{C}{1 + |z|^1}, \quad µ ≤ |\arg(z)| ≤ π.$$ (2.4)

In the literature, the notation $E_α = E_{α,1}$ is frequently used. It is well-know that, for $0 < α < 2$:

$$D_α^t E_{α,1}(zt^α) = zE_{α,1}(zt^α), \quad t > 0, z ∈ ℂ,$$ (2.5)

namely, for every $z ∈ ℂ$, the function $u(t) := E_{α,1}(zt^α)$ is a solution of the scalar valued ordinary differential equation

$$D_α^t u(t) = zu(t), \quad t > 0, 0 < α < 2.$$ 

Moreover, we have that for $α > 0$, $λ > 0$, $t > 0$ and $m ∈ ℤ$,

$$\frac{d^m}{dt^m}[E_{α,1}(−λt^α)] = −λt^{α−m}E_{α,α−m+1}(−λt^α),$$ (2.6)

$$\frac{d}{dt}[tE_{α,2}(−λt^α)] = E_{α,1}(−λt^α),$$ (2.7)

$$\frac{d}{dt}[t^{α−1}E_{α,α}(−λt^α)] = t^{α−2}E_{α,α−1}(−λt^α).$$ (2.8)

The proof of (2.4) is contained in [33, Theorem 1.6]. For (2.5) we refer to [7, Section 1.3] and the proofs of (2.6)-(2.8) are contained in [33, Section 1.2.3, Formula (1.83)]. For more details on the Mittag-Leffler functions we refer the reader to [1, 7, 22, 29, 30, 31, 33] and the references therein.

In what follows, we will also exploit the following estimates. They follow easily from (2.4) and some straightforward computations.

**Lemma 2.4.** Let $1 < α < 2$ and $α’ > 0$. Then the following assertions hold.

(a) Let $0 ≤ β ≤ 1$, $0 < γ < α$ and $λ > 0$. Then there is a constant $C > 0$ such that for every $t > 0$,

$$|λ^β t^γ E_{α,α'}(−λt^α)| ≤ Ct^{γ−αβ}.$$ (2.9)

(b) Let $0 ≤ γ ≤ 1$ and $λ > 0$. Then there is a constant $C > 0$ such that for every $t > 0$,

$$|λ^{1−γ} t^{α−2} E_{α,α'}(−λt^α)| ≤ Ct^{αγ−2}.$$ (2.10)

3. **THE LINEAR PROBLEM**

Recall that $X$ is a relatively compact metric space, $m$ is a Radon measure on $X$ and $A$ is the self-adjoint operator in $L^2(X)$ associated with a Dirichlet space $(E_A, V_{1/2})$ in the sense of (2.1). Throughout the remainder of the article, without any mention, by a.e. on $X$, we shall mean $m$ a.e. on $X$. Let $1 < α < 2$ and consider the following fractional in time wave equation

$$\begin{cases}
D_α^t u(x, t) + Au(x, t) = f(x, t) \quad \text{in } X × (0, T), \\
u(x, 0) = u_0, \quad ∂_t u(x, 0) = u_1 \quad \text{in } X,
\end{cases}$$ (3.1)

\[\text{See } [26, \text{Lemma 3.3}].\]
where \( u_0, u_1 \) and \( f \) are given functions. Our notion of weak solutions to the system \((3.1)\) is as follows.

**Definition 3.1.** Set \( \gamma := 1/\alpha \in (1/2, 1) \) and \( V_\gamma := D(A^{\gamma}) \). A function \( u \) is said to be a weak solution of \((3.1)\) on \((0, T)\), for some \( T > 0 \), if the following assertions hold.

- **Regularity:**
  
  \[
  \begin{aligned}
  u & \in C([0, T]; V_\gamma) \cap C^1([0, T]; L^2(X)), \\
  \mathcal{D}_t^\alpha u & \in C([0, T]; V_{-\gamma}).
  \end{aligned}
  \quad (3.2)
  
- **Initial conditions:**
  
  \[ u(\cdot, 0) = u_0, \quad \partial_\tau u(\cdot, 0) = u_1 \text{ a.e. in } X, \quad (3.3) \]
  and
  
  \[ \lim_{t \to 0^+} \|u(\cdot, t) - u_0\|_{V_\gamma} = 0, \quad \lim_{t \to 0^+} \|\partial_\tau u(\cdot, t) - u_1\|_{V_{-\gamma}} = 0, \quad (3.4) \]
  for some \( \frac{1}{\alpha} = \gamma \geq \sigma \geq 0 \) and \( 1 - \frac{1}{\alpha} \geq \beta > 0 \).

- **Variational identity:** for every \( \varphi \in V_\gamma \hookrightarrow V_{1/2} \) and a.e. \( t \in (0, T) \), we have
  
  \[ \langle \mathcal{D}_t^\alpha u(\cdot, t), \varphi \rangle_{V_{-\gamma}, V_{\gamma}} + \mathcal{E}_A(u(\cdot, t), \varphi) = (f(\cdot, t), \varphi). \quad (3.5) \]

We first prove well-posedness for the linear problem.

**Theorem 3.2.** Let \( u_0 \in V_\gamma, u_1 \in L^2(X) \) and

\[ f \in C([0, T]; V_{-\gamma}) \cap L^q((0, T); L^2(X)), \quad \text{for } \frac{1}{p} + \frac{1}{q} = 1, \text{ and } 1 \leq p < \frac{1}{2 - \alpha}. \]

Let the assumptions of Theorem 2.3 hold. Then the system \((3.1)\) has a unique weak solution \( u \) given by

\[ u(\cdot, t) = \sum_{n=1}^{\infty} (u_0, \varphi_n) E_{\alpha, 1}(-\lambda_n t^\alpha) \varphi_n + \sum_{n=1}^{\infty} (u_1, \varphi_n) t E_{\alpha, 2}(-\lambda_n t^\alpha) \varphi_n \quad (3.6) \]

\[ + \sum_{n=1}^{\infty} \left( \int_0^t (f(\cdot, \tau), \varphi_n)(t - \tau)^{\alpha - 1} E_{\alpha, \alpha}(-\lambda_n (t - \tau)^\alpha) d\tau \right) \varphi_n. \]

Moreover, there is a constant \( C > 0 \) such that for all \( t \in [0, T] \),

\[ \|u(\cdot, t)\|_{V_\gamma} + \|\partial_\tau u(\cdot, t)\|_{L^2(X)} \leq C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + t^{\frac{\alpha}{2} + \alpha - 2} \|f\|_{L^n([0, T]; L^2(X))} \right), \quad (3.7) \]

and

\[ \|\mathcal{D}_t^\alpha u(\cdot, t)\|_{V_{-\gamma}} \leq C \left( \|u_0\|_{V_\gamma} + t^{2 - \alpha} \|u_1\|_{L^2(X)} + t^{\frac{\alpha}{2}} \|f\|_{L^n([0, T]; L^2(X))} + \|f\|_{C([0, T]; V_{-\gamma})} \right). \quad (3.8) \]

**Proof.** We shall now use the notation

\[ (u_0, \varphi_n) = u_{0,n}, \quad (u_1, \varphi_n) = u_{1,n} \text{ and } (f(\cdot, t), \varphi_n) = f_n(t). \]

We prove the result in several steps.
Step 1. We show that \( u \in C([0, T]; V_{\gamma}) \). Let \( t \in [0, T] \) and set

\[
S_1(t)u_0 := \sum_{n=1}^{\infty} u_{0, n} E_{\alpha, 1}(-\lambda_n t^\alpha) \varphi_n, \quad S_2(t)u_1 := \sum_{n=1}^{\infty} u_{1, n} t E_{\alpha, 2}(-\lambda_n t^\alpha) \varphi_n,
\]

and

\[
S_3(t)f := \sum_{n=1}^{\infty} \left( \int_0^t f_n(\tau)(t - \tau)^{\alpha - 1} E_{\alpha, \alpha}(-\lambda_n (t - \tau)^\alpha) \, d\tau \right) \varphi_n
\]

so that

\[
u(t) = S_1(t)u_0 + S_2(t)u_1 + S_3(t)f.
\]

Using (2.4) we get that there is a constant \( C > 0 \) such that for every \( t \in [0, T] \),

\[
\|S_1(t)u_0\|_{V_{\gamma}}^2 \leq 4 \sum_{n=1}^{\infty} |u_{0, n} \lambda_n^\alpha E_{\alpha, 1}(-\lambda_n t^\alpha)|^2 \leq C \|u_0\|_{V_{\gamma}}^2.
\] (3.9)

Using (2.9) we obtain that there is a constant \( C > 0 \) such that for every \( t \in [0, T] \) (recall that \( \alpha \gamma = 1 \)),

\[
\|S_2(t)u_1\|_{V_{\gamma}}^2 \leq 4 \sum_{n=1}^{\infty} |u_{1, n} \lambda_n^\alpha t E_{\alpha, 2}(-\lambda_n t^\alpha)|^2 \leq Ct^{2(1 - \alpha \gamma)} \|u_1\|_{L^2(X)}^2 = C \|u_1\|_{L^2(X)}^2.
\] (3.10)

Using (2.9) again and the Hölder inequality, we get that there is a constant \( C > 0 \) such that for every \( t \in [0, T] \),

\[
\|S_3(t)f\|_{V_{\gamma}} \leq 2 \int_0^t \left( \sum_{n=1}^{\infty} |f_n(\tau) \lambda_n^\alpha (t - \tau)^{\alpha - 1} E_{\alpha, \alpha}(-\lambda_n (t - \tau)^\alpha)|^2 \right)^{\frac{1}{2}} \, d\tau
\]

\[
\leq C \int_0^t (t - \tau)^{\alpha - 1 - \alpha \gamma} \left( \sum_{n=1}^{\infty} |f_n(\tau)|^2 \right)^{\frac{1}{2}} \, d\tau
\]

\[
\leq C \int_0^t (t - \tau)^{\alpha - 2} \|f(\cdot, \tau)\|_{L^2(X)} \, d\tau
\]

\[
\leq Ct^{\frac{1}{2} + \alpha - 2} \|f\|_{L^2((0, T); L^2(X))}.
\] (3.11)

By the assumption on \( p, 1 + p (\alpha - 2) > 0 \). Since the series in (3.6) converges in \( V_{\gamma} \) uniformly for every \( t \in [0, T] \), we have shown that \( u \in C([0, T]; V_{\gamma}) \). It also follows from the estimates (3.9), (3.10) and (3.11) that there is a constant \( C > 0 \) such that for every \( t \in [0, T] \),

\[
\|u(\cdot, t)\|_{V_{\gamma}} \leq C_1 \left( \|u_0\|_{V_{\gamma}} + \|u_1\|_{L^2(X)} + t^{\frac{1}{2} + \alpha - 2} \|f\|_{L^2((0, T); L^2(X))} \right).
\] (3.12)
**Step 2.** Next, we show that \(u \in C^1([0,T];L^2(X))\). We notice that a simple calculation gives that for a.e. \(t \in (0,T),\)

\[
\partial_t u(\cdot,t) = \sum_{n=1}^{\infty} u_{0,n} \lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \varphi_n + \sum_{n=1}^{\infty} u_{1,n} E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n + \sum_{n=1}^{\infty} \int_0^t f_n(\tau)(t-\tau)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n(t-\tau)^\alpha) \, d\tau \varphi_n
\]

\[
=: S_1'(t)u_0 + S_2'(t)u_1 + S_3'(t)f.
\]

Proceeding as in Step 1, on account of (2.10), we get the following estimates:

\[
\|S_1'(t)u_0\|_{L^2(X)} \leq C\|u_0\|_{V_\gamma} \quad \text{and} \quad \|S_2'(t)u_1\|_{L^2(X)} \leq C\|u_1\|_{L^2(X)}, \quad \forall t \in [0,T],
\]

and

\[
\|S_3'(t)f\|_{L^2(X)} \leq C t^{\frac{1}{\gamma} + \alpha - 2} \|f\|_{L^\gamma((0,T);L^2(X))}, \quad \forall t \in [0,T].
\]

(3.14)

It thus follows from these estimates that there is a constant \(C > 0\) such that for every \(t \in [0,T],\)

\[
\|\partial_t u(\cdot,t)\|_{L^2(X)} \leq C_1 \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + t^{\frac{1}{\gamma} + \alpha - 2} \|f\|_{L^\gamma((0,T);L^2(X))} \right).
\]

(3.15)

Since the series (3.13) converges in \(L^2(X)\) uniformly for every \(t \in [0,T],\) it follows that \(u \in C^1([0,T];L^2(X)).\) In addition (3.7) follows from (3.12) and (3.15).

**Step 3.** Next, we prove that \(D_t^\alpha u \in C([0,T];V_{\gamma}).\) It follows from (3.6) that (using also some formulas of fractional derivatives of the Mittag-Leffler functions)

\[
D_t^\alpha u(\cdot,t) = -\sum_{n=1}^{\infty} u_{0,n} \lambda_n E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n - \sum_{n=1}^{\infty} u_{1,n} \lambda_n t E_{\alpha,2}(-\lambda_n t^\alpha) \varphi_n
\]

\[
- \sum_{n=1}^{\infty} \left( \int_0^t f_n(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha) \, d\tau \right) \varphi_n + f(\cdot,t)
\]

\[
= -Au(\cdot,t) + f(\cdot,t).
\]

Using (2.4) and (2.9), we get the following estimates (recall that \(\lambda_1 > 0\) and \(1 < 2\gamma = \frac{2}{\alpha} < 2\)):

\[
\left\| \sum_{n=1}^{\infty} u_{0,n} \lambda_n E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n \right\|_{V_{\gamma}} \leq \sum_{n=1}^{\infty} u_{0,n} \lambda_n^{1-\gamma} E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n \leq C \lambda_1^{1-2\gamma} \|u_0\|_{V_\gamma},
\]

(3.17)

and

\[
\left\| \sum_{n=1}^{\infty} u_{1,n} \lambda_n t E_{\alpha,2}(-\lambda_n t^\alpha) \varphi_n \right\|_{V_{\gamma}} \leq \sum_{n=1}^{\infty} u_{1,n} \lambda_n^{1-\gamma} t E_{\alpha,2}(-\lambda_n t^\alpha) \varphi_n \leq Ct^{2-\alpha} \|u_1\|_{L^2(X)}.
\]

(3.18)
Similarly on account of (2.9), we can deduce that
\[
\left\| \sum_{n=1}^{\infty} \left( \int_{0}^{t} f_n(\tau) \lambda_n(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha}) \, d\tau \right) \varphi_n \right\|_{V_{\gamma}} \leq C t^{\frac{1}{p}} \| f \|_{L^{q}([0,T];L^{2}(X))}. \tag{3.19}
\]
Since the series (3.16) converges in $V_{\gamma}$ uniformly in $[0,T]$, we can conclude that $D_{t}^{\alpha} u \in C([0,T];V_{\gamma})$ since $f \in C \left( [0,T];V_{\gamma} \right)$. The estimate (3.3) then follows from (3.17), (3.18), (3.19) on account of (3.16).

**Step 4.** Since $D_{t}^{\alpha} u(\cdot,t) \in V_{\gamma}$, $Au(\cdot,t) \in V_{\gamma}$ and $f(\cdot,t) \in L^{2}(X)$ for a.e. $t \in (0,T)$, then taking the duality product in (3.16) we get the variational identity (3.5).

**Step 5.** Using (3.6) and (3.13), we get that
\[
u(\cdot,0) = \sum_{n=1}^{\infty} u_{0,n} \varphi_n = u_0 \quad \text{and} \quad \partial_t u(\cdot,0) = \sum_{n=1}^{\infty} u_{1,n} \varphi_n = u_1,
\]
and we have shown (3.3). Now we check if (3.4) is also satisfied. By (3.6), for $t \ll 1$, we have
\[
u(\cdot,t) - u_0 = \sum_{n=1}^{\infty} u_{0,n} \left( E_{\alpha,1}(-\lambda_n t^{\alpha}) - 1 \right) \varphi_n + S_2(t) u_1 + S_3(t) f. \tag{3.20}
\]
The estimates (3.10) and (3.11) yield
\[
\| S_3(t) f \|_{V_{\sigma}} \leq \| S_3(t) f \|_{V_{\gamma}} \leq C t^{\frac{1}{p}+\alpha-2} \| f \|_{L^{q}([0,T];L^{2}(X))} \to 0, \tag{3.21}
\]
as $t \to 0^{+}$ since $V_{\gamma} \subseteq V_{\sigma}$ for $0 \leq \sigma < \gamma = 1/\alpha$, and
\[
\| S_2(t) u_1 \|_{V_{\sigma}} \leq 2 \left( \sum_{n=1}^{\infty} |u_{1,n} \lambda_{n}^{\sigma} t E_{\alpha,2}(-\lambda_n t^{\alpha})|^2 \right)^{\frac{1}{2}} \leq C t^{1-\alpha \sigma} \| u_1 \|_{L^{2}(X)} \to 0,
\]
as $t \to 0^{+}$. Notice that
\[
\left\| \sum_{n=1}^{\infty} u_{0,n} \left( E_{\alpha,1}(-\lambda_n t^{\alpha}) - 1 \right) \varphi_n \right\|_{V_{\sigma}} = \left( \sum_{n=1}^{\infty} |u_{0,n}|^2 \lambda_{n}^{2\sigma} \left( E_{\alpha,1}(-\lambda_n t^{\alpha}) - 1 \right)^2 \right)^{\frac{1}{2}},
\]
and $\lim_{t \to 0^{+}} \left( E_{\alpha,1}(-\lambda_n t^{\alpha}) - 1 \right) = 0$ for all $n \in N$ and using (2.4) we have that
\[
\sum_{n=1}^{\infty} |u_{0,n}|^2 \lambda_{n}^{2\sigma} \left( E_{\alpha,1}(-\lambda_n t^{\alpha}) - 1 \right)^2 \leq (C + 1)^2 \sum_{n=1}^{\infty} |u_{0,n}|^2 \lambda_{n}^{2\sigma} = (C + 1)^2 \| u_0 \|_{V_{\sigma}}^2 < \infty,
\]
for all $0 \leq t \leq T$. It follows from the Lebesgue Dominated Convergence Theorem that
\[
\lim_{t \to 0^{+}} \left\| \sum_{n=1}^{\infty} u_{0,n} \left( E_{\alpha,1}(-\lambda_n t^{\alpha}) - 1 \right) \varphi_n \right\|_{V_{\sigma}} = 0.
\]
Thus, the first statement of (3.21) is verified. On account of (3.13), we have
\[
\partial_t u(\cdot,t) - u_1 = S_1'(t) u_0 + \sum_{n=1}^{\infty} u_{1,n} \left( E_{\alpha,1}(-\lambda_n t^{\alpha}) - 1 \right) \varphi_n + S_3'(t) f. \tag{3.22}
\]
and once again, by (3.14), it is easy to see that
\[ \|S_2'(t)f\|_{L^2(X)} \to 0 \text{ as } t \to 0^+. \] (3.23)
A simple calculation for \(1 - \frac{1}{\alpha} \geq \beta > 0\) also yields from (2.9), that
\[ \|S_1'(t)u_0\|_{V_{-\beta}} = \left( \sum_{n=1}^{\infty} |u_{0,n}|^2 \lambda_n^{2\gamma} \lambda_n^{2(1-\beta)-2\gamma} |t^{\alpha-1}E_{\alpha,a}(-\lambda_n t^\alpha)|^2 \right)^{\frac{1}{2}} \] (3.24)
\[ \leq C t^{\alpha\beta} \|u_0\|_{V_{\gamma}} \to 0 \text{ as } t \to 0^+. \]

Recall that
\[ \left\| \sum_{n=1}^{\infty} u_{1,n}(E_{\alpha,1}(-\lambda_n t^\alpha) - 1)\varphi_n \right\|^2_{L^2(X)} = \sum_{n=1}^{\infty} \left| u_{1,n}(E_{\alpha,1}(-\lambda_n t^\alpha) - 1) \right|^2 \]
and \(\lim_{t \to 0^+} \left( E_{\alpha,1}(-\lambda_n t^\alpha) - 1 \right) = 0\) for all \(n \in \mathbb{N}\). It follows from (2.4) that
\[ \sum_{n=1}^{\infty} \left| u_{1,n}(E_{\alpha,1}(-\lambda_n t^\alpha) - 1) \right|^2 \leq (C + 1)^2 \|u_1\|^2_{L^2(X)}, \]
for all \(0 \leq t \leq T\). Using the Lebesgue Dominated Convergence Theorem again we get that
\[ \left\| \sum_{n=1}^{\infty} u_{1,n}(E_{\alpha,1}(-\lambda_n t^\alpha) - 1)\varphi_n \right\|_{L^2(X)} \to 0 \text{ as } t \to 0^+. \] (3.25)
Collecting the preceding estimates yields the second statement of (3.3).

\textbf{Step 6.} Finally we show uniqueness. Let \(u, v\) be any two weak solutions of (3.1) with the same initial data \(u_0, u_1\) and source \(f = f(x,t)\) and then set \(w = u - v\). Then \(w\) is a weak solution of the system
\[
\begin{cases}
D_t^\alpha w = -Aw, & \text{in } X \times (0,T), \\
w(\cdot,0) = 0, \quad \partial_tw(\cdot,0) = 0 & \text{in } X.
\end{cases}
\] (3.26)
This system has only a unique solution that is the null solution \(w = 0\) (see e.g., [7, 26]). Hence \(u = v\). We include a brief argument for the sake of completeness. In the class of weak solutions in the sense of Definition 3.1 for (3.26), we may choose the test function \(\varphi = \varphi_n \in D(A) \hookrightarrow V_{\gamma}\) in the variational identity. Setting \(w_n(t) = \langle w(\cdot,t), \varphi_n \rangle_{V_{-\beta},V_{\gamma}}\) it then follows from (3.3) that
\[ D_t^\alpha w_n(t) = -\lambda_n w_n(t), \text{ for a.e. } t \in (0,T). \] (3.27)
Since \(w \in C([0,T];V_{\gamma})\) by the assumed regularity of the weak solutions \(u, v\) (see (3.2)), there actually holds \(w_n(t) = (w(\cdot,t), \varphi_n)\) as well as \(\lim_{t \to 0^+} \|w(\cdot,t)\|_{L^2(X)} = 0\) owing to the first of (3.3) and the fact that \(w(\cdot,0) = 0\). Therefore \(w_n(0) = 0\) and the uniqueness result (see e.g., [33]) for the ODE problem (3.27) implies that \(w_n(t) = (w(\cdot,t), \varphi_n) = 0\), for all \(n \geq 1\). Since the set \(\{\varphi_n\}_{n \in \mathbb{N}}\) is also an orthonormal basis in \(L^2(X)\), we finally obtain that \(w(\cdot,t) = 0\) as
Let the assumptions of Theorem 2.3 hold. Then the following assertions hold.

\[ (3.1) \] The proof of the theorem is finished.

We say that a function \( u \) is a strong solution of (3.1) on \((0, T)\), for some \( T > 0 \), if it is a weak solution in the sense of Definition 3.3 and it further has the regularity properties

\[ \mathcal{D}^{\alpha} u \in L^{1}((0, T); L^{2}(X)) \text{ and } u \in L^{1}((0, T); D(A)). \] (3.28)

In this case the first equation of (3.1) is satisfied pointwise a.e. in \( X \times (0, T) \).

We have the following existence result of strong solutions.

Theorem 3.4. Let \( \gamma = \frac{1}{\alpha} \) and

\[ f \in W^{1,q}((0, T); L^{2}(X)), \text{ for } \frac{1}{p} + \frac{1}{q} = 1, \text{ and } 1 \leq p < \frac{1}{2 - \alpha}. \]

Let the assumptions of Theorem 2.3 hold. Then the following assertions hold.

(a) If \( u_{0} \in V_{\gamma} \) and \( u_{1} \in L^{2}(X) \), then the system (3.1) has a unique strong solution in the sense of Definition 3.3 that is given exactly by (3.6). In particular there is a constant \( C > 0 \) such that for every \( t \in (0, T) \),

\[ \|\mathcal{D}^{\alpha} u(\cdot, t)\|_{L^{2}(X)} + \|Au(\cdot, t)\|_{L^{2}(X)} \leq C t^{1-\alpha} (\|u_{0}\|_{V_{\gamma}} + \|u_{1}\|_{L^{2}(X)}) \]

\[ + C \left( \|f(\cdot, t)\|_{L^{2}(X)} + \|f(\cdot, 0)\|_{L^{2}(X)} + C t^{\frac{1}{p}} \|\partial_{t} f\|_{L^{q}((0, T); L^{2}(X))} \right). \]

(b) If \( u_{0} \in V_{\sigma} \) for some \( \sigma > \frac{1}{\alpha} \) and \( u_{1} \in V_{\beta} \) for some \( \beta > \frac{\alpha - 1}{\alpha} \), then the strong solution of (3.1) also satisfies \( u \in W^{2,1}((0, T); L^{2}(X)) \) and there is a constant \( C > 0 \) such that

\[ \|\partial_{t}^{2} u\|_{L^{1}((0, T); L^{2}(X))} \leq C \left( T^{\alpha \sigma - 1} \|u_{0}\|_{V_{\sigma}} + T^{1-\alpha(1-\beta)} \|u_{1}\|_{V_{\beta}} \right) \]

\[ + T^{\frac{1}{p} + 1 - \alpha - 1} \|\partial_{t} f\|_{L^{q}((0, T); L^{2}(X))} + T^{\alpha - 1} \|f(\cdot, 0)\|_{L^{2}(X)}. \] (3.30)

In addition, the initial conditions are also satisfied in the following (strong) sense:

\[ \lim_{t \to 0^{+}} \|u(\cdot, t) - u_{0}\|_{V_{\gamma}} = 0, \lim_{t \to 0^{+}} \|\partial_{t} u(\cdot, t) - u_{1}\|_{L^{2}(X)} = 0. \] (3.31)

Proof. (a) First, since \( f \in W^{1,q}((0, T); L^{2}(X)) \), then \( f \in C([0, T]; L^{2}(X)) \hookrightarrow C([0, T]; V_{\gamma}) \). Second, let \( u_{0} \in V_{\gamma}, u_{1} \in L^{2}(X) \) and \( u \) the weak solution of (3.1). Recall that \( \mathcal{D}^{\alpha} u = -Au + f \) is given by (3.16). Using (2.6), we get the following estimates:

\[ \left\| \sum_{n=1}^{\infty} u_{0,n} \lambda_{n} E_{\alpha,1}(-\lambda_{n} t^{\alpha}) \varphi_{n} \right\|_{L^{2}(X)} \leq \left\| \sum_{n=1}^{\infty} \lambda_{n}^{\gamma} u_{0,n} \lambda_{n}^{1-\gamma} E_{\alpha,1}(-\lambda_{n} t^{\alpha}) \varphi_{n} \right\|_{L^{2}(X)} \]

\[ \leq C t^{1-\alpha} \|u_{0}\|_{V_{\gamma}}, \] (3.32)

and

\[ \left\| \sum_{n=1}^{\infty} u_{1,n} \lambda_{n} t E_{\alpha,2}(-\lambda_{n} t^{\alpha}) \varphi_{n} \right\|_{L^{2}(X)} \leq C t^{1-\alpha} \|u_{1}\|_{L^{2}(X)}. \] (3.33)
Using (2.6) and integrating by parts, we get that
\[
\int_0^t f_n(\tau)\lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha) \, d\tau \varphi_n
\]
\[
= \int_0^t f_n(\tau) \frac{d}{d\tau} \left( -E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha) \right) \, d\tau \varphi_n
\]
\[
= f_n(t)\varphi_n + f_n(0)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n + \int_0^t f_n'(\tau)E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha) \, d\tau \varphi_n.
\]
Therefore, proceeding as above we get that
\[
\left\| \sum_{n=1}^\infty \left( \int_0^t f_n(\tau)\lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha) \, d\tau \right) \varphi_n \right\|_{L^2(X)}
\]
\[
\leq \| f(\cdot, t) \|_{L^2(X)} + \| f(\cdot, 0) \|_{L^2(X)} + Ct^\frac{1}{\beta} \| \partial_t f \|_{L^2((0,T);L^2(X))}.
\]

It follows from (3.32), (3.33), (3.31) and the assumptions that \( \mathbb{D}^t \alpha u \in L^1((0,T);L^2(X)) \) since the function \( f \in W^{1,q}((0,T);L^2(X)) \). The estimate (3.29) then follows from (3.32), (3.33), (3.31) on account of (3.16). Since \( Au = -\mathbb{D}^t \alpha u + f \), we have also shown that \( u \in L^1((0,T);D(A)) \). Thus \( u \) is the unique strong solution of (3.1).

(b) Now assume that \( u_0 \in V_\sigma \) for some \( \sigma > \frac{1}{\alpha} \) and \( u_1 \in V_\beta \) for some \( \beta > \frac{\alpha-1}{\alpha} \), and let \( u \) be the strong solution of (3.1). Notice that it follows from the estimate (3.7) that \( u \in W^{1,1}((0,T);L^2(X)) \). From a simple calculation, for a.e. \( t \in (0,T) \), we have
\[
\partial_t^2 u(\cdot, t) = \sum_{n=1}^\infty u_{0,n}\lambda_n t^{-\alpha}\varphi_n - \sum_{n=1}^\infty u_{1,n}\lambda_n E_{\alpha,\alpha}(-\lambda_n t^\alpha)\varphi_n
\]
\[
+ \sum_{n=1}^\infty \int_0^t f_n'(\tau)(t-\tau)^{-\alpha}E_{\alpha,\alpha-1}(-\lambda_n(t-\tau)^\alpha) \, d\tau \varphi_n
\]
\[
+ \sum_{n=1}^\infty f_n(0)t^{-\alpha}E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) \varphi_n
\]
\[
=: S_1''(t)u_0 + S_2''(t)u_1 + S_3''(t)f + \sum_{n=1}^\infty f_n(0)t^{-\alpha}E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) \varphi_n.
\]

Exploiting (2.10), we have
\[
\| S_1''(t)u_0 \|_{L^2(X)} \leq 4 \left( \sum_{n=1}^\infty |u_{0,n}\lambda_n^\alpha|^2 |\lambda_n^{1-\sigma} t^{-\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)|^2 \right)^{\frac{1}{2}} \leq Ct^{\alpha\sigma-2} \| u_0 \|_{V_\sigma}.
\]
Since \( \sigma > \frac{1}{\alpha} \), we have that \( \alpha\sigma - 1 > 0 \) and this implies
\[
\int_0^T \| S_1''(t)u_0 \|_{L^2(X)} \, dt \leq CT^{\alpha\sigma-1} \| u_0 \|_{V_\sigma}.
\]
Since $\beta > \frac{\alpha-1}{\alpha}$, we have that $\alpha \beta - \alpha > -1$ and this also yields
\[
\int_0^T \|S''_2(t)u_1\|_{L^2(\mathcal{X})}dt \leq C T^{1-\alpha(1-\beta)} \|u_1\|_{V_\beta}.
\]
For the third term, using (2.10) and the Hölder inequality, we find
\[
\|S''_3(t)f\|_{L^2(\mathcal{X})} \leq C t^{\frac{1}{\alpha} - 2 + \alpha} \|\partial_t f\|_{L^q((0,T);L^2(\mathcal{X}))},
\]
which gives
\[
\int_0^T \|S''_3(t)f\|_{L^2(\mathcal{X})}dt \leq C T^{\frac{1}{\alpha} + \alpha - 1} \|\partial_t f\|_{L^q((0,T);L^2(\mathcal{X}))}.
\]
For the fourth term, using (2.4) we get that there is a constant $C > 0$ such that
\[
\left\| \sum_{n=1}^{\infty} f_n(0) t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) \varphi_n \right\|_{L^2(\mathcal{X})} \leq C t^{\alpha-2} \|f(\cdot,0)\|_{L^2(\mathcal{X})},
\]
and this estimate implies that
\[
\int_0^T \left\| \sum_{n=1}^{\infty} f_n(0) t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) \varphi_n \right\|_{L^2(\mathcal{X})} \, dt \leq C T^{\alpha-1} \|f(\cdot,0)\|_{L^2(\mathcal{X})}.
\]
It follows from these estimates together with the function $u \in W^{1,1}((0,T);L^2(\mathcal{X}))$, that $u \in W^{2,1}((0,T);L^2(\mathcal{X}))$ and one also has the estimate (3.30).

It remains to verify (3.31). For this argument, we shall exploit once again (3.20) and (3.22). Recall that $u_0 \in V_\sigma$ for $\sigma > \gamma$ and $u_1 \in V_\beta$ for $\beta > 1 - \gamma$. It first follows from (3.21) that $\|S_3(t)f\|_{V_\gamma} \to 0$ as $t \to 0^+$ and
\[
\|S_2(t)u_1\|_{V_\gamma} \leq 2 \left( \sum_{n=1}^{\infty} |u_{1,n}|^2 \lambda_n^{\gamma-\beta} t E_{\alpha,1}(-\lambda_n t^\alpha) \right)^{\frac{1}{2}} \leq C t^{\alpha \beta} \|u_1\|_{V_\beta} \to 0, \quad \text{as } t \to 0^+,
\]
whereas proceeding exactly as the proof of (3.25) we can deduce that
\[
\left\| \sum_{n=1}^{\infty} u_{0,n} \left( E_{\alpha,1}(-\lambda_n t^\alpha) - 1 \right) \varphi_n \right\|_{V_\gamma} = \left( \sum_{n=1}^{\infty} |u_{0,n}|^2 \lambda_n^{2\gamma} \left( E_{\alpha,1}(-\lambda_n t^\alpha) - 1 \right)^2 \right)^{\frac{1}{2}} \to 0,
\]
as $t \to 0^+$. Then the first of (3.31) is a simple corollary of these estimates and the identity (3.20). Analogously, for the second statement of (3.31) we may exploit the formula (3.22) together with (3.23), (3.25) and the fact that
\[
\|S'_1(t)u_0\|_{L^2(\mathcal{X})} = \left( \sum_{n=1}^{\infty} |u_{0,n}|^2 \lambda_n^{2\sigma} \lambda_n^{2-2\sigma} |t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 \right)^{1/2} \leq C t^{\alpha \sigma - 1} \|u_0\|_{V_\sigma},
\]
which goes to zero as $t \to 0^+$ since $\alpha \sigma - 1 > 0$. The proof of the theorem is now finished. □
4. The semi-linear problem

Once again, we recall that $X$ is a relatively compact metric space and $1 < \alpha < 2$. We focus our attention on the semi-linear problem

$$
\begin{align*}
\mathbb{D}_t^\alpha u(x,t) + Au(x,t) &= f(u(x,t)) \quad \text{in } X \times (0,T), \\
\hfill u(\cdot,0) = u_0, \quad \hfill \partial_t u(\cdot,0) = u_1 \quad \text{in } X.
\end{align*}
$$

(4.1)

Given an operator $A$ (see Theorem 2.3), we will define a critical value according to whether $q_A > 2$ or $1 < q_A \leq 2$ (see (2.2)). More precisely, in what follows we shall consider the following two important cases.

Case (i): If $q_A > 2$, we define

$$
\alpha_0 := \frac{1}{\theta_A} = \frac{2(q_A - 1)}{q_A} \in (1,2).
$$

(4.2)

With the assumptions of Theorem 2.3, the following embeddings then hold:

$$
V_{1/\alpha} \hookrightarrow L^\infty(X), \text{ for } \alpha \in (1,\alpha_0),
$$

(4.3)

$$
V_{1/\alpha_0} \hookrightarrow L^{2r_*}(X), \text{ for any } r_* \in (1,\infty),
$$

(4.4)

$$
V_{1/\alpha} \hookrightarrow L^{2r_*}(X), \text{ for } \alpha > \alpha_0 \text{ and } r_* = \frac{\alpha \theta_A}{\alpha \theta_A - 1} > 1.
$$

(4.5)

Case (ii): If $1 < q_A \leq 2$ (and so $\theta_A \geq 1$), the following embedding holds for all $\alpha \in (1,2)$:

$$
V_{1/\alpha} \hookrightarrow L^{2r_*}(X), \text{ for } r_* = \frac{\alpha \theta_A}{\alpha \theta_A - 1} > 1.
$$

(4.6)

Our notion of weak solution to the system (4.1) changes slightly according to these two cases. In what follows, let $u_0$, $u_1$ and $f$ be given functions and set once again $\gamma = 1/\alpha \in (1/2,1)$.

Definition 4.1 (The case (i) when $\alpha \in (1,\alpha_0)$). A function $u$ is said to be a (locally-defined) weak solution of (4.1) on $(0,T)$, for some $T > 0$, if the following assertions hold.

- Regularity:

$$
\begin{align*}
\left\{ \begin{array}{l}
\hfill u \in C([0,T];V_{\gamma}) \cap C^1([0,T];L^2(X)), \\
\hfill \mathbb{D}_t^\alpha u \in C([0,T];V_{-\gamma}).
\end{array} \right.
\end{align*}
$$

(4.7)

- Initial conditions:

$$
\begin{align*}
\hfill u(\cdot,0) = u_0, \quad \hfill \partial_t u(\cdot,0) = u_1 \quad \text{a.e. in } X,
\end{align*}
$$

(4.8)

and

$$
\begin{align*}
\lim_{t \to 0^+} \|u(\cdot,0) - u_0\|_{V_{\gamma}} = 0, \quad \lim_{t \to 0^+} \|\partial_t u(\cdot,0) - u_1\|_{V_{-\beta}} = 0,
\end{align*}
$$

(4.9)

for some

$$
\frac{1}{\alpha} > \sigma \geq 0 \text{ and } 1 - \frac{1}{\alpha} \geq \beta > 0.
$$

- Variational identity: for every $\varphi \in V_{\gamma}$ for a.e. $t \in (0,T)$, we have

$$
\langle \mathbb{D}_t^\alpha u(\cdot,t), \varphi \rangle_{V_{-\gamma},V_{\gamma}} + \mathcal{E}_A(u(\cdot,t),\varphi) = \langle f(u(\cdot,t)), \varphi \rangle.
$$

(4.10)
If the above properties hold for any \( T > 0 \), then we say that \( u \) is a global weak solution.

Definition 4.2 (The case (i) when \( \alpha \in [\alpha_0, 2) \) and case (ii) for all \( \alpha \in (1, 2) \)). Let

\[
1 \leq p < \frac{1}{2 - \alpha}
\]

and consider the dual conjugate \( q \) of \( p \),

\[
q = \frac{p}{p - 1} \in \left( \frac{1}{\alpha - 1}, \infty \right].
\]

A function \( u \) is said to be a (locally-defined) weak solution of \((4.1)\) on \((0, T)\), for some \( T > 0 \), if the following assertions hold.

- **Regularity:**
  \[
  \begin{cases}
    u \in C([0, T]; V_{\gamma}) \cap C^1([0, T]; L^2(X)) \cap L^{rq}((0, T); L^{2r}(X)) , \\
    \mathcal{D}^\alpha_t u \in C([0, T]; V_{\gamma} - \gamma) \oplus L^q((0, T); L^{2r}(X)).
  \end{cases}
  \tag{4.11}
  \]

  for some \( r > 1 \).

- **The initial conditions** \((4.8), (4.9)\) and variational identity \((4.10)\) are satisfied.

If the above properties hold for any \( T > 0 \), then we say that \( u \) is a global weak solution.

From now on, we shall also refer to any weak solution that satisfies the energy identity \((4.10)\) as an *energy solution*.

As far as the nonlinearity \( f \in C^1(\mathbb{R}) \) is concerned, we consider the following assumptions (not necessarily simultaneously).

(Hf1) There exist two constants \( C > 0 \) and \( r > 1 \) such that

\[
f(0) = 0 \quad \text{and} \quad \left| f'(s) \right| \leq C |s|^{r-1}, \quad \text{for all } s \in \mathbb{R}.
\]

We assume the following precise conditions on the exponent \( r \) under the conditions of Case (i):

(a) when \( \alpha = \alpha_0 \), \( r := r_* \) is any number in \((1, \infty)\),

(b) when \( \alpha > \alpha_0 \), \( r := r_* = \frac{\theta_A \alpha}{\theta_A \alpha - 1} > 1 \).

In Case (ii), we shall require that

\[
r := r_* = \frac{\theta_A \alpha}{\theta_A \alpha - 1} > 1 \quad \text{for all } \alpha \in (1, 2).
\]

(Hf2) There exist two monotone increasing (real-valued) functions \( Q_1, Q_2 \geq 0 \) such that

\[
f(0) = 0 \quad \text{and} \quad \left| f'(s) \right| \leq Q_1(|s|), \quad |f(s)| \leq Q_2(|s|), \quad \text{for all } s \in \mathbb{R}.
\]

Remark 4.3. Firstly, we notice that (Hf2) is not a growth condition. Secondly, we also mention that it can be replaced by the following simple condition:

\[
f \in C^1(\mathbb{R}) \quad \text{and} \quad f(0) = 0.
\]

Indeed, one can set in (Hf2),

\[
Q_1(\xi) = \sup_{0 \leq |s| \leq \xi} \left| f'(s) \right| + \varepsilon \xi \quad \text{and} \quad Q_2(\xi) = \sup_{0 \leq |s| \leq \xi} |f(s)| + \varepsilon \xi, \quad \xi \geq 0,
\]
for some $\varepsilon > 0$.

The following well-posedness result is our first main result.

**Theorem 4.4.** Let $\gamma = \frac{1}{\alpha}$, $u_0 \in V_\gamma$ and $u_1 \in L^2(X)$ and let the assumptions of Theorem 2.3 hold for the operator $A$. Consider the following cases:

(I) For **Case (i)** when $\alpha \in (1, \alpha_0)$, suppose that $f$ satisfies (Hf2). Then there is a time $T^* > 0$ (depending only on $(u_0, u_1)$) such that the system (4.1) has a unique weak solution on $(0, T^*)$ in the sense of Definition 4.1.

(II) For **Case (i)** when $\alpha \in [\alpha_0, 2)$ (provided that the interval is non-empty) and **Case (ii)**, respectively, suppose that $f$ satisfies (Hf1). Then there is a time $T^* > 0$ (depending only on $(u_0, u_1)$) such that the system (4.1) has a unique weak solution on $(0, T^*)$ in the sense of Definition 4.2.

In either case, the solution is given by

$$u(\cdot, t) = \sum_{n=1}^{\infty} u_{0,n} E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n + \sum_{n=1}^{\infty} u_{1,n} t E_{\alpha,2}(-\lambda_n t^\alpha) \varphi_n$$

$$+ \sum_{n=1}^{\infty} \left( \int_0^t f_n(u(\tau))(t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t - \tau)^\alpha) d\tau \right) \varphi_n,$$

where we have set $u_{0,n} = (u_0, \varphi_n)$, $u_{1,n} = (u_1, \varphi_n)$ and $f_n(u(t)) = (f(u(\cdot, t)), \varphi_n)$.

Our second main result shows that locally-defined weak solutions can be extended to a larger interval.

**Theorem 4.5.** Let the assumptions of Theorem 4.4 be satisfied. Then the unique weak solution on $(0, T^*)$ of (4.1) can be extended to the interval $[0, T^* + \tau]$, for some $\tau > 0$, so that, the extended function is the unique weak solution of (4.1) on $(0, T^* + \tau)$.

Our next result shows the precise conditions for which we have a global weak solution.

**Theorem 4.6.** Let the assumptions of Theorem 4.4 be satisfied. Then the system (4.1) has a unique global weak solution on $[0, \infty)$ or there exists a maximal time $T_{\max} \in (0, \infty)$ such that $u : X \times [0, T_{\max}) \to L^2(X)$ is a maximal locally-defined weak solution, and in that case, we have

$$\limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{V_\gamma} = \infty \quad \text{and} \quad \limsup_{t \to T_{\max}} \|\partial_t u(\cdot, t)\|_{L^2(X)} = \infty.$$  

(4.16)

**Remark 4.7.** The proofs of these statements contain an explicit dependence of the time of existence of solutions with respect to the initial data which allows for a longer time of existence for small initial data.

We now verify under what conditions a weak solution may become strong (in the sense that the fractional wave equation is satisfied pointwise a.e. on $X \times (0, T)$) for as long as the former exists.

**Theorem 4.8.** For **Case (i)** when $\alpha \in [\alpha_0, 2)$ (provided that the interval is non-empty) and **Case (ii)**, respectively, suppose that $f$ satisfies (Hf1). Let $u$ be a (maximally-defined) weak solution of problem (4.1) on $[0, T]$, for some $T \in (0, T_{\max})$ such that

$$\|u\|_{L^q(\tau^{-1}((0, T); L^\infty(X)))} < \infty.$$  

(4.17)
Then \( u \) is a strong solution on \([0, T]\), namely,
\[
\mathbb{D}_t^\alpha u \in L^1((0, T); L^2(X)) \quad \text{and} \quad u \in L^1((0, T); D(A)).
\]

**Proof.** The statement is an immediate consequence of (3.29) in Theorem 3.4 and (4.17). Indeed, it suffices to establish that \( \mathbb{D}_t^\alpha u \in L^1((0, T); L^2(X)) \) from (3.29). Notice that its validity depends upon on the fact that \( f(u) \in L^1((0, T); L^2(X)) \), \( u \in C^1([0, T]; L^2(X)) \), (4.18)
\[
\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \, dx \, dy < \infty,
\]
(4.20)
\[
v \in W^{2,1}((0, T); L^2(X))
\]
and the initial conditions are also satisfied in the sense of (3.31).

**Remark 4.10.** Notice that for the strong solutions of Theorem 4.9, the two notions of fractional derivative in (1.2) and (1.3) are equivalent. In particular, the Riemann-Liouville derivative (1.3) provides for another useful way to set up numerical schemes for the fractional wave equation (4.1), as well as to rigorously analyse the convergence of such numerical schemes.

5. Examples and concluding remarks

Before we give some examples, we introduce fractional order Sobolev spaces. For an arbitrary bounded open set \( \Omega \subset \mathbb{R}^d \) (\( d \geq 1 \)) and \( 0 < s < 1 \), we endow the Hilbert space
\[
W^{s,2}(\Omega) := \left\{ u \in L^2(\Omega), \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \, dx \, dy < \infty \right\},
\]
with the norm
\[
\|u\|_{W^{s,2}(\Omega)} := \left( \int_{\Omega} |u|^2 \, dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \, dxdy \right)^{\frac{1}{2}}.
\]
We also let
\[
W^{s,2}_0(\Omega) := \overline{D(\Omega)}^{W^{s,2}(\Omega)},
\]
and
\[
W^{s,2}_0(\Omega) := \left\{ u \in W^{s,2}(\mathbb{R}^d) : u = 0 \text{ in } \mathbb{R}^d \setminus \Omega \right\}.
\]
Since \( \Omega \) is assumed to be bounded we have the following continuous embeddings:
\[
W^{s,2}_0(\Omega), W^{s,2}_0(\overline{\Omega}) \hookrightarrow \begin{cases} L^{\frac{2d}{d-2s}}(\Omega) & \text{if } d > 2s, \\ L^p(\Omega), p \in (2, \infty) & \text{if } d = 2s, \\ C^{0,s-\frac{d}{2}}(\overline{\Omega}) & \text{if } d < 2s. \end{cases} \tag{5.1}
\]
For more details on fractional order Sobolev spaces we refer to [14, 23, 41] and their references.

5.1. The case of second order elliptic operators in divergence form. Let \( \Omega \) be a bounded domain of \( \mathbb{R}^d \) \((d \geq 1)\). In what follows, we define \( \mathcal{A} \) by the differential operator
\[
\mathcal{A}u(x) = -\sum_{i,j=1}^{d} \partial_{x_i} \left( a_{ij}(x) \partial_{x_j} u \right), \quad x \in \Omega,
\]
where \( a_{ij} = a_{ji} \in L^\infty(\Omega) \), satisfy the ellipticity condition
\[
\sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \geq c|\xi|^2, \quad x \in \Omega, \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d.
\]
In (11), we consider the Dirichlet operator\(\mathcal{D}\) which is also the case investigated in detail by [27] (assuming only that \( d \leq 3 \), and \( \Omega \) is of class \( C^2 \) and \( a_{ij} \in C^1(\overline{\Omega}) \)). Let \( \mathcal{A} \) be the realization in \( L^2(\Omega) \) of \( \mathcal{A} \) with the Dirichlet boundary condition \( u = 0 \) in \( \partial\Omega \). That is, \( \mathcal{A} \) is the self-adjoint operator in \( L^2(\Omega) \) associated with the Dirichlet form
\[
\mathcal{E}_A(u, v) = \sum_{i,j=1}^{d} \int_{\Omega} a_{ij}(x) \partial_{x_i} u \partial_{x_j} v \, dx, \quad u, v \in V_{1/2} = W^{1,2}_0(\Omega). \tag{5.2}
\]
We observe that
\[
V_{1/2} \hookrightarrow L^\infty(\Omega) \quad \text{when } d = 1, \quad \text{and so } q_A = \infty, \tag{5.3}
\]
\[
V_{1/2} \hookrightarrow L^{2q}(\Omega), \quad \text{when } d = 2, \quad \text{and so } q_A = q \in (1, \infty), \tag{5.4}
\]
\[
V_{1/2} \hookrightarrow L^{\frac{2d}{d-2}}(\Omega), \quad \text{when } d \geq 3, \quad \text{and so } q_A = \frac{d}{d-2}. \tag{5.5}
\]
The assumptions of Theorem 2.3 hold for the Dirichlet space \( (\mathcal{E}_A, W^{1,2}_0(\Omega)) \) (see e.g., [16]).

\(3\)Of course, the same differential operator \( \mathcal{A} \) subject to Neumann and/or Robin boundary conditions may be also allowed (see [18]). The results in this section remain also valid in these cases without any modifications to the main statements.
According to Section 4, we have in Case (i) whenever \( d < 4 \) (\( \Leftrightarrow q_A > 2 \)) as well as in Case (ii) when \( d \geq 4 \) (\( \Leftrightarrow 1 < q_A \leq 2 \)):

\[
\alpha_0 = \begin{cases} 
2, & \text{if } d = 1, \\
\frac{2(q-1)}{q}, & \text{for any } q \in (1, \infty) \text{ if } d = 2, \\
\frac{4}{3}, & \text{if } d = 3, \\
\text{no critical value}, & \text{if } d \geq 4.
\end{cases}
\]  

(5.6)

Recall that in all these cases \( \theta_A = \frac{q_A}{2(q_A-1)} \) (in particular, \( \theta_A = \frac{1}{2} \) in dimension \( d = 1 \)).

**Theorem 5.1.** All the statements of Theorems 4.4, 4.5, 4.6 and Theorem 4.9 are satisfied for the operator \( A \) associated with the Dirichlet space \( \mathcal{S} \).

When \( d \leq 3 \), Theorem 5.1 turns out to be a great improvement over the existence results of [27]. Recall that contrary to [27], we did not assume any regularity on the open set and the coefficients of the operator. For the class of energy solutions that we have considered in this article, our assumptions on the nonlinearity \( f \) turn out to be weaker than those enforced by [27]. Moreover, with our assumptions \((\text{Hf}1)-(\text{Hf}2)\) the results and corresponding estimates for the semi-linear problem (4.1) end up being more stable under perturbation especially as \( \alpha \to 1^+ \) (compare with [27] and [28]). Indeed, as \( \alpha \to 1^+ \) the class of weak solutions and estimates considered by [27] can be only recovered for a nonlinearity \( f(s) \) that is slightly super-linear at infinity (namely, \( f(s) \sim |s|^{1+\varepsilon} \), as \( |s| \to \infty \), for some \( 0 < \varepsilon = \varepsilon(\alpha) \) that converges toward the value \( \frac{1}{2} \) as \( \alpha \to 1^+ \) in all dimensions \( d \leq 3 \). We notice that under our assumptions \((\text{say when } d = 3)\), there are (unique) weak solutions that require no essential growth restrictions on the nonlinearity \((\text{see (Hf2)})\) in the range when \( \alpha \in (1, \frac{4}{3}) \) while for \( \alpha \in [\frac{4}{3}, 2) \), the nonlinearity \( f \) must obey the conditions \((\text{see (Hf1)})\):

\[
f(0) = 0 \text{ and } |f'(s)| \leq C |s|^{r-1}, \text{ for all } s \in \mathbb{R}.
\]  

(5.7)

Here \( r > 1 \) can be any number in \( (1, \infty) \), when \( \alpha_0 = \frac{4}{3} \) and \( r = \frac{3\alpha}{3\alpha-4} \) whenever \( \alpha_0 \in (\frac{4}{3}, 2) \). It is interesting to observe that as \( \alpha \to 2^- \), we recover in (5.7) a growth exponent \( r \to 3^+ \) that allows for nonlinearities of cubic growth exactly as in the case of the classical problem for \( \alpha = 2 \). Notice again that our notion of weak solutions is that of energy solutions which are well-known to be also suited in the classical case \( \alpha = 2 \). We emphasize that the class of weak solutions, considered by [27] Theorem 1.5, is a class of integral solutions that do not necessarily satisfy an energy identity like (4.10), so their notion is much weaker than ours. For this class of integral solutions, [27] Theorem 1.5] states additional estimates\(^4\) that allow an improvement of the growth exponent \( r > 3 \) to one that \( r \to 5^- \) as \( \alpha \to 2^- \) (in particular, this is known to recover a global existence result for the classical problem (4.1) with a quintic nonlinearity \( r = 5 \) when \( \alpha = 2 \)).

\(^4\)A similar discussion applies as well in the other remaining cases when \( d \neq 3 \). We have focused our attention to the case \( d = 3 \) only for the sake of simplicity.

\(^5\)These estimates reduce essentially to our estimates when \( \gamma = 1/\alpha \), \( r = 0 \) in [27] Theorem 1.5]. In this case, in the supercritical case \( \alpha \geq \alpha_0 \), their growth exponent \( (r =) b = d/(d-4\gamma) \) turns out to be exactly the same as ours and we did not use any Strichartz estimates.
Finally, we notice that no statements for the existence of strong solutions were given in [27] for any $\alpha \in (1,2)$. In the subcritical range for $\alpha \in (1,\alpha_0)$, by Theorem [19] every energy solution is also a strong solution.

5.2. The case of fractional powers of elliptic operators. Assume that $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded Lipschitz domain. Denote by $L$ the self-adjoint operator considered in Section 5.1. For $0 < s < 1$, let $A := L^s$ be the fractional powers of $L$ as defined in (2.3). Letting $(\lambda_n)_{n \in \mathbb{N}}$ denote the eigenvalues of $L$ with associated eigenfunctions $(\varphi_n)_{n \in \mathbb{N}}$, it follows that $(\lambda_n^s)_{n \in \mathbb{N}}$ are the corresponding eigenvalues of $A = L^s$ with associated eigenfunctions $(\varphi_n)_{n \in \mathbb{N}}$. Let $H^s(\Omega) := D(L^s)$ where $D(L^s)$ is defined as in (2.3). It is well-known that

$$H^s(\Omega) = \begin{cases} W^{s,2}(\Omega) = W_0^{s,2}(\Omega) & \text{if } 0 < s < \frac{1}{2}, \\ W_0^{\frac{1}{2},2}(\Omega) & \text{if } s = \frac{1}{2}, \\ W_0^{s,2}(\Omega) & \text{if } \frac{1}{2} < s < 1, \end{cases}$$

(5.8)

where

$$W_0^{\frac{1}{2},2}(\Omega) := \left\{ u \in W^{\frac{1}{2},2}(\Omega), \int_\Omega \frac{|u(x)|^2}{(\text{dist}(x, \partial\Omega))^2} dx < \infty \right\}.$$

More precisely, we have

$$\begin{cases} H^s(\Omega) = W_0^{s,2}(\Omega) = [W_0^{s,2}(\Omega), L^2(\Omega)]_{1-s} & \text{if } s \in (0,1) \setminus \{1/2\}, \\ H^{1/2}(\Omega) = W_0^{\frac{1}{2},2}(\Omega) = [W_0^{\frac{1}{2},2}(\Omega), L^2(\Omega)]_{\frac{1}{2}}. \end{cases}$$

Here for $0 < \delta < 1$, $[\cdot, \cdot]_{\delta}$ denotes the complex interpolation space. Since $W_0^{\frac{1}{2},2}(\Omega) \hookrightarrow W_0^{s,2}(\Omega)$, it follows from (5.8) that the embedding (5.1) holds with $W_0^{s,2}(\Omega)$ replaced by $H^s(\Omega)$. The following integral representation of $A = L^s$ has been given in [12] Theorem 2.3. Let $u, v \in H^s(\Omega)$. Then

$$\langle Au, v \rangle_{(H^s(\Omega))^*, H^s(\Omega)} = \int_\Omega \int_\Omega \left( u(x) - u(y) \right) \left( v(x) - v(y) \right) K_s(x,y) dx dy$$

$$+ \int_\Omega \kappa_s(x) u(x) v(x) dx,$$

(5.9)

where

$$0 \leq K_s(x,y) := \frac{s}{\Gamma(1-s)} \int_0^\infty \frac{W_\Omega^D(t,x,y)}{t^{1+s}} dt, \quad x, y \in \Omega$$

and

$$0 \leq \kappa_s(x) := \frac{s}{\Gamma(1-s)} \int_0^\infty \left( 1 - e^{-tL} 1(x) \right) \frac{dt}{t^{1+s}}, \quad x \in \Omega.$$  

Here $W_\Omega^D$ denotes the heat kernel associated to the semigroup $(e^{-tL})_{t \geq 0}$, namely,

$$W_\Omega^D(t,x,y) = \sum_{n=1}^\infty e^{-t\lambda_n} \varphi_n(x) \varphi_n(y), \quad t > 0, \quad x, y \in \Omega.$$
We notice that it follows from (5.9) that \( A \) is associated with a closed, bilinear, symmetric, continuous and coercive form \( \mathcal{E}_A \), that is given by
\[
\mathcal{E}_A(u, v) = \int_{\Omega} \int_{\Omega} \left( u(x) - u(y) \right) \left( v(x) - v(y) \right) K_s(x, y) dx dy
+ \int_{\Omega} \kappa_s(x) u(x) v(x) dx, \quad D(\mathcal{E}_A) = V_{1/2, s} := \mathbb{H}^s(\Omega).
\]
(5.10)

Proceeding as in [41, Theorem 6.6], one has that \( (\mathcal{E}_A, V_{1/2, s}) \) is a Dirichlet space. This fact, together with (5.1) implies that \( A \) satisfies all the conditions in Theorem 2.3. It follows from (5.1) that
\[
V_{1/2, s} \hookrightarrow L^\infty(\Omega) \quad \text{when} \quad d < 2s, \quad \text{and so} \quad q_A = \infty, \quad (5.11)
\]
\[
V_{1/2, s} \hookrightarrow L^{2q}(\Omega), \quad \text{when} \quad d = 2s, \quad \text{and so} \quad q_A = q \in (1, \infty), \quad (5.12)
\]
\[
V_{1/2, s} \hookrightarrow L^{\frac{2d}{d-2s}}(\Omega), \quad \text{when} \quad d > 2s, \quad \text{and so} \quad q_A = \frac{d}{d-2s}. \quad (5.13)
\]
Notice that in (5.13), we have \( q_A > 2 \Leftrightarrow 2s < d < 4s \). According to Section 4 and taking into account the embeddings (5.11), (5.12) and (5.13), we have the following regarding the critical value \( \alpha_0 \):
\[
\begin{align*}
\alpha_0 &= 2 & \text{if} \quad d < 2s, \\
\alpha_0 &= \frac{2(q-1)}{q} & \text{for any} \quad q \in (1, \infty) \quad \text{if} \quad d = 2s, \\
\alpha_0 &= \frac{4s}{d} & \text{if} \quad 2s < d < 4s, \\
\text{no critical value} & \quad \text{if} \quad d \geq 4s.
\end{align*}
\]
(5.14)

We can then conclude with the following result.

**Theorem 5.2.** All the statements of Theorems 4.4, 4.5, 4.6 and Theorem 4.9 are satisfied for the operator \( A \) associated with the Dirichlet space given in (5.10).

5.3. **The case of the fractional Laplace operator.** Let \( 0 < s < 1 \) and the form \( \mathcal{E}_A \) with \( D(\mathcal{E}_A) := W^{s, 2}(\Omega) \) be defined by
\[
\mathcal{E}_A(u, v) := C_{d, s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+2s}} dx dy.
\]
Let \( A \) be the self-adjoint operator on \( L^2(\Omega) \) associated with \( \mathcal{E}_A \) in the sense of (2.1). An integration by parts argument gives that
\[
D(A) = \left\{ u \in W^{s, 2}(\Omega) : (-\Delta)^s u \in L^2(\Omega) \right\}, \quad Au = (-\Delta)^s u.
\]
The operator \( A \) is the realization in \( L^2(\Omega) \) of the fractional Laplace operator \( (-\Delta)^s \) with the Dirichlet exterior condition \( u = 0 \) in \( \mathbb{R}^d \setminus \Omega \). Here, \( (-\Delta)^s \) is given by the following singular integral
\[
(-\Delta)^s u(x) := C_{d, s} \text{P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy
= C_{d, s} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d \setminus |x-y| > \varepsilon} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy, \quad x \in \mathbb{R}^d,
\]
provided that the limit exists, where \( C_{d,s} \) is a normalization constant. We refer to [10, 11, 41] and their references for more information on the fractional Laplace operator. It has been shown in [36] that the operator \( A \) has a compact resolvent and its first eigenvalue \( \lambda_1 > 0 \). In addition, from [41, Theorem 6.6] and [18, Example 2.3.5] we can deduce that \((E_A, W^{s,2}_0(\Omega))\) is a Dirichlet space. From this and the embedding (5.13) we can conclude that the operator \( A \) satisfies all the assumptions in Theorem 2.3. We notice that even taking \( L \) of Section 5.2 to be the realization of the Laplacian \((-\Delta)\) with the Dirichlet boundary condition \( u = 0 \) on \( \partial\Omega \), the operator \( L^s \) and \( A \) are different in the sense that they have different eigenvalues and eigenfunctions. We refer to [9, 8, 36] for more details on this topic. As in Example 5.2, the critical value \( \alpha_0 \) is given exactly as in (5.14) and we can conclude once again with the following result.

**Theorem 5.3.** All the statements of Theorems 4.4, 4.5, 4.6 and Theorem 4.9 are satisfied for the operator \( A \) associated with the Dirichlet space \((E_A, W^{s,2}_0(\Omega))\).

5.4. **The case of the Laplace operator with Wentzell boundary conditions.** In all the above examples, we have that \( X = \Omega \subset \mathbb{R}^d \) is a bounded open set. In this section we give an example where \( X = \overline{\Omega} \), that is, the closure of a bounded open set \( \Omega \subset \mathbb{R}^d \).

Assume that \( \Omega \subset \mathbb{R}^d \) is a bounded domain with a Lipschitz continuous boundary. Let \( \beta \in L^\infty(\partial\Omega) \) be such that \( \beta(x) \geq \beta_0 > 0 \) for \( \sigma \)-a.e. on \( \partial\Omega \), where \( \beta_0 \) is a constant. Let \( \delta \in \{0, 1\} \) and

\[
W^{1,\delta,2}(\Omega) := \left\{ U = (u, u_{|\partial\Omega}) : u \in W^{1,2}(\Omega), \delta u_{|\partial\Omega} \in W^{1,2}(\partial\Omega) \right\},
\]

be endowed with the norm

\[
\|u\|_{W^{1,\delta,2}(\Omega)} = \begin{cases} 
\left( \|u\|^2_{W^{1,2}(\Omega)} + \|u\|^2_{W^{1,2}(\partial\Omega)} \right)^{\frac{1}{2}} & \text{if } \delta = 1 \\
\left( \|u\|^2_{W^{1,2}(\Omega)} + \|u\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} \right)^{\frac{1}{2}} & \text{if } \delta = 0.
\end{cases}
\]

Then

\[
W^{1,0,2}(\Omega) \hookrightarrow L^q(\Omega) \times L^q(\partial\Omega),
\]

with

\[
1 \leq q \leq \frac{2(d - 1)}{d - 2} \quad \text{if } d > 2 \quad \text{and} \quad 1 \leq q < \infty \quad \text{if } d \leq 2,
\]

and

\[
W^{1,1,2}(\Omega) \hookrightarrow L^q(\Omega) \times L^q(\partial\Omega),
\]

with

\[
1 \leq q \leq \frac{2d}{d - 2} \quad \text{if } d > 2 \quad \text{and} \quad 1 \leq q < \infty \quad \text{if } d \leq 2.
\]

Let \( E_{\delta,W} \) with \( D(E_{\delta,W}) := W^{1,\delta,2}(\Omega) \) be given by

\[
E_{\delta,W}(U, V) := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \delta \int_{\partial\Omega} \nabla u \cdot \nabla v \, d\sigma + \int_{\partial\Omega} \beta(x) uv \, d\sigma.
\]
Let $\Delta_{\delta,W}$ be the self-adjoint operator in $L^2(\Omega) \times L^2(\partial \Omega)$ associated with $E_{\delta,W}$ in the sense of (2.1). Then $\Delta_{\delta,W}$ is a realization in $L^2(\Omega) \times L^2(\partial \Omega)$ of $\left(-\Delta, -\delta \Delta_{\Gamma}\right)$ with the generalized Wentzell boundary conditions. More precisely, we have that

$$D(\Delta_{\delta,W}) = \left\{ U = (u, u|_{\Gamma}) \in W^{1,2}(\Omega), \Delta u \in L^2(\Omega), \right.$$  
and $-\delta \Delta_{\Gamma}(u|_{\partial \Omega}) + \partial_{\nu} u + \beta(u|_{\partial \Omega}) \in L^2(\partial \Omega) \right\},$

and

$$\Delta_{\delta,W} U = \left(-\Delta u, -\delta \Delta_{\Gamma}(u|_{\partial \Omega}) + \partial_{\nu} u + \beta(u|_{\partial \Omega})\right).$$

We notice that for $1 \leq q \leq \infty$, the space $L^q(\Omega) \times L^q(\partial \Omega)$ endowed with the norm

$$\|(f, g)\|_{L^q(\Omega) \times L^q(\partial \Omega)} = \begin{cases} \left(\|f\|_{L^q(\Omega)}^q + \|g\|_{L^q(\partial \Omega)}^q\right)^{1/q} & \text{if } 1 \leq q < \infty, \\
\max\{\|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}\} & \text{if } q = \infty, \end{cases}$$

can be identified with $L^q(\Omega, m)$ where the measure $m$ on $\overline{\Omega}$ is defined for a measurable set $M \subset \Omega$ by $m(M) = |\Omega \cap M| + \sigma(\partial \Omega \cap M)$. By [39], $(E_{0,W}, W^{1,2}(\Omega))$ and $(E_{1,W}, W^{1,2}(\Omega))$ are Dirichlet spaces on $L^2(\Omega, m)$. Hence, it follows from the coercivity of the form (5.19) and the embeddings (5.15) and (5.17) with $q$ given in (5.16) and (5.18) that the Dirichlet spaces $(E_{0,W}, W^{1,2}(\Omega))$ and $(E_{1,W}, W^{1,2}(\Omega))$ satisfy all the assumptions in Theorem 2.3. In the case $A = \Delta_{0,W}$, letting $V_{1/2} := W^{1,2}(\Omega)$ we have that

$$V_{1/2} \hookrightarrow L^\infty(\Omega, m) \text{ when } d = 1, \text{ and so } q_A = \infty, \quad (5.20)$$
$$V_{1/2} \hookrightarrow L^{2q}(\Omega, m), \text{ when } d = 2, \text{ and so } q_A = q \in (1, \infty), \quad (5.21)$$
$$V_{1/2} \hookrightarrow L^{2(d-1)/d}(\Omega, m), \text{ when } d > 2, \text{ and so } q_A = \frac{d-1}{d-2}. \quad (5.22)$$

In (5.22) we have $1 < q_A \leq 2$ for all $d > 2$. According to Section 4 and taking into account (5.20), (5.21) and (5.22), we have that

$$\begin{cases} \alpha_0 = 2 & \text{if } d = 1, \\
\alpha_0 = \frac{2(q-1)}{q} & \text{for any } q \in (1, \infty) \text{ if } d = 2, \\
\text{no critical value} & \text{if } d \geq 3. \end{cases} \quad (5.23)$$

If $A = \Delta_{1,W}$, then letting $V_{1/2} := W^{1,2}(\Omega)$ we have that

$$V_{1/2} \hookrightarrow L^\infty(\Omega, m) \text{ when } d = 1, \text{ and so } q_A = \infty, \quad (5.24)$$
$$V_{1/2} \hookrightarrow L^{2q}(\Omega, m), \text{ when } d = 2, \text{ and so } q_A = q \in (1, \infty), \quad (5.25)$$
$$V_{1/2} \hookrightarrow L^{2(d-1)/d}(\Omega, m), \text{ when } d > 2, \text{ and so } q_A = \frac{d}{d-2}. \quad (5.26)$$
In (5.26), \(q_A = 3 > 2 \Leftrightarrow d = 3\). Taking into account (5.24), (5.25) and (5.26), we have that

\[
\begin{cases}
\alpha_0 = \frac{2}{q-1} & \text{for any } q \in (1, \infty) \\
\alpha_0 = \frac{4}{3} & \text{if } d = 3,
\end{cases}
\]

The classical Dirichlet-to-Neumann map is the operator \(D\) now reads:

\[
D u(x) = \partial_\nu g(x),
\]

where \(g \in W^1(\partial \Omega)\) and given by

\[
\begin{align*}
\frac{\partial v}{\partial t} - \Delta v &= f_1(v(x,t)), \\
\frac{\partial w}{\partial t} - \delta \Delta w &= f_2(w(x,t)), \\
v(\cdot, 0) &= v_0, \\
w(\cdot, 0) &= w_0,
\end{align*}
\]

such that

\[
\partial_t v(\cdot, 0) = v_1 \text{ in } \Omega, \quad \partial_t w(\cdot, 0) = w_1 \text{ on } \partial \Omega.
\]

The nonlinear functions \(f_1, f_2 \in C^1\) satisfy the assumptions (Hf1)-(Hf2) (with the same growth exponent and functions \(Q_1, Q_2\)). Then setting

\[
u = (v, w), \quad f(u) = (f_1(v), f_2(w))
\]

and \(u_0 = (v_0, w_0), \ u_1 = (v_1, w_1)\), one can formally rewrite (5.28)-(5.29) as problem (4.1).

We can then conclude with the following result.

**Theorem 5.4.** All the statements of Theorems 4.4, 4.5, 4.6 and Theorem 4.9 are satisfied for the semilinear problem (5.28)-(5.29).

5.5. **The case of the Dirichlet-to-Neumann operator.** Here we give an example where the metric space \(X\) is given by the boundary of an open set. Let \(\Delta_D\) be the operator defined in Example 5.1 with \(L = -\Delta\). We denote its spectrum by \(\sigma(\Delta_D)\). Let \(\lambda \in \mathbb{R} \setminus \sigma(\Delta_D)\), \(g \in L^2(\partial \Omega)\) and let \(u \in W^{1,2}(\Omega)\) be the weak solution of the following Dirichlet problem

\[
-\Delta u = \lambda u \text{ in } \Omega, \quad u|_{\partial \Omega} = g.
\]

The classical Dirichlet-to-Neumann map is the operator \(D_{1,\lambda}\) on \(L^2(\partial \Omega)\) with domain

\[
D(D_{1,\lambda}) = \left\{ g \in L^2(\partial \Omega), \ \exists u \in W^{1,2}(\Omega) \text{ solution of } (5.30) \right\}
\]

and \(\partial_\nu u \) exists in \(L^2(\partial \Omega)\),

and given by

\[
D_{1,\lambda} g = \partial_\nu u.
\]

It is well known that one has the following orthogonal decomposition

\[
W^{1,2}(\Omega) = W^{1,2}_0(\Omega) \oplus \mathcal{H}^{1,\lambda}(\Omega),
\]

where

\[
\mathcal{H}^{1,\lambda}(\Omega) = \left\{ u \in W^{1,2}(\Omega), \ -\Delta u = \lambda u \right\},
\]
and by $-\Delta u = \lambda u$ we mean that
\[
\int_{\Omega} \nabla u \cdot \nabla u\,dx = \lambda \int_{\Omega} uv\,dx, \quad \forall \ v \in W^{1,2}_0(\Omega).
\]
Let
\[
W^{1,2}_{1/2}(\partial \Omega) := \left\{ u|_{\partial \Omega}, \ u \in W^{1,2}(\Omega) \right\}
\]
be the trace space. Since $\lambda \in \mathbb{R}\setminus\sigma(\Delta_D)$, we have that the trace operator restricted to $\mathcal{H}^{1,\lambda}(\Omega)$, that is, the mapping $u \in \mathcal{H}^{1,\lambda}(\Omega) \mapsto u|_{\partial \Omega} \in W^{1/2,2}(\partial \Omega)$, is linear and bijective. Letting
\[
\|u|_{\partial \Omega}\|_{W^{1/2,2}(\partial \Omega)} = \|u\|_{\mathcal{H}^{1,\lambda}(\Omega)},
\]
then $W^{1/2,2}(\partial \Omega)$ becomes a Hilbert space. By the closed graph theorem, different choice of $\lambda \in \mathbb{R}\setminus\sigma(\Delta_D)$ leads to an equivalent norm on $W^{1/2,2}(\partial \Omega)$. Moreover, we have the embedding $W^{1/2,2}(\partial \Omega) \hookrightarrow L^2(\partial \Omega)$. In addition we have the embedding
\[
W^{1/2,2}(\partial \Omega) \hookrightarrow L^q(\partial \Omega) \tag{5.31}
\]
with
\[
1 \leq q \leq \frac{2(d-1)}{d-2} \quad \text{if} \quad d > 2 \quad \text{and} \quad 1 \leq q < \infty \quad \text{if} \quad d \leq 2. \tag{5.32}
\]
It has been shown in [3] that $D_{1,\lambda}$ is the self-adjoint operator on $L^2(\partial \Omega)$ associated with the bilinear symmetric and continuous form $\mathcal{E}_{1,\lambda}$ with domain $W^{1/2,2}(\partial \Omega)$ given by
\[
\mathcal{E}_{1,\lambda}(\varphi, \psi) = \int_{\Omega} \nabla u \cdot \nabla v\,dx - \lambda \int_{\Omega} uv\,dx,
\]
where $\varphi, \psi \in W^{1/2,2}(\partial \Omega)$ and $u, v \in \mathcal{H}^{1,\lambda}(\Omega)$ are such that $u|_{\partial \Omega} = \varphi$ and $v|_{\partial \Omega} = \psi$. The operator $-D_{1,\lambda}$ generates a strongly continuous and analytic semigroup on $L^2(\partial \Omega)$ which is also submarkovian if $\lambda \leq 0$. If $\lambda < 0$ we also have that $\mathcal{E}_{1,\lambda}$ is coercive. For more information on the Dirichlet-to-Neumann map we refer to [3] [4] [5] and their references.

Assuming that $\lambda < 0$, using (5.31) and (5.32), we get that $(\mathcal{E}_{1,\lambda}, W^{1/2,2}(\partial \Omega))$ is a Dirichlet space and it satisfies all the conditions in Theorem 2.3 with $X = \partial \Omega$, $m$ is the Lebesgue surface measure and $A = D_{1,\lambda}$. Letting $V_{1/2} := W^{1/2,2}(\partial \Omega)$, we have that
\[
V_{1/2} \hookrightarrow L^\infty(\partial \Omega) \quad \text{when} \quad d = 1, \quad \text{and so} \quad q_A = \infty, \tag{5.33}
\]
\[
V_{1/2} \hookrightarrow L^q(\partial \Omega), \quad \text{when} \quad d = 2, \quad \text{and so} \quad q_A = q \in (1, \infty), \tag{5.34}
\]
\[
V_{1/2} \hookrightarrow L^{2(d-1)/(d-2)}(\partial \Omega), \quad \text{when} \quad d > 2, \quad \text{and so} \quad q_A = \frac{d-1}{d-2}. \tag{5.35}
\]
In (5.33), $1 < q_A \leq 2$ for all $d > 2$. According to Section 4 and taking into account (5.33), (5.34) and (5.35), we can deduce that
\[
\begin{align*}
\alpha_0 &= 2 & \text{if} \quad d = 1, \\
\alpha_0 &= \frac{2(q-1)}{q} & \text{for any} \quad q \in (1, \infty) \quad \text{if} \quad d = 2, \\
\text{no critical value} & & \text{if} \quad d \geq 3.
\end{align*} \tag{5.36}
\]
We can once again conclude with the following result for (4.1).

**Theorem 5.5.** All the statements of Theorems 4.4, 4.5, 4.6 and Theorem 4.9 are satisfied for the operator $A$ associated with the Dirichlet space $(E_{1,\lambda}, W^{\frac{1}{2}, 2}(\partial \Omega))$.

5.6. **Final remarks.** In this contribution, we have developed a theory for well-posedness of weak solutions that further generalizes the theory of integral solutions developed in [27, Theorem 1.5]. In particular, our theory allows us to include many interesting examples of self-adjoint operators that can be currently found in the scientific literature. However, we emphasize once again that our assumptions that we employ on $A$ and/or $(X, m)$ are of general character, and as a result do not require a specific form as suggested by the examples of Section 5; this abstraction allows (1.1) to represent a much larger family of super-diffusive equations, that have not been explicitly studied anywhere in detail, including fractional wave equations associated with operators on (compact) Riemannian manifolds with or without boundary. Other examples of operators that satisfy the assumptions of Theorem 2.3 can be also found in [18]. Among them one can find other non-standard operators of “fractional” type subject to appropriate boundary conditions.

We finally remark that in contrast to the classical case, the following is the only integration by parts formula available for fractional derivatives defined in the sense of (1.2):

$$\int_0^T v(t) D_t^\alpha u(t) dt = \int_0^T u(t) D_{t,T}^{\beta} v(t) dt + [u'(t) I_{t,T}^{2-\alpha} v(t) + u(t) D_{t,T}^{\alpha-1} v(t)]_{t=0}^{t=T},$$

provided that the left and right-hand side expressions make sense. Here, $D_{t,T}^{\beta}$ and $I_{t,T}^{\beta}$ denote the right Riemann-Liouville fractional derivative and fractional integral of order $\beta > 0$, respectively (see e.g. [26]). Therefore, at present it is not clear how to verify the conditions of Theorem 4.6 in order to deduce global-in-time existence results for problem (1.1) with arbitrary initial data.

6. **Proofs in the case (1) when $1 < \alpha < \alpha_0$**

In this section we give the proofs of the main results stated in Section 4 in the sub-critical case $1 < \alpha < \alpha_0$ for Case (i).

**Proof of Theorem 4.4.** Fix $0 < T^* \leq T$. Consider the space

$$X := \left\{ u \in C([0, T^*]; V_\gamma) \cap C^1([0, T^*]; L^2(X)) : u(\cdot, 0) = u_0, \partial_t u(\cdot, 0) = u_1 \text{ and} \right.$$ 

$$\|u(\cdot, t)\|_{V_\gamma} + \|\partial_t u(\cdot, t)\|_{L^2(X)} \leq R^* \quad \forall \ t \in [0, T^*] \right\},$$

for some $R^* > 0$, and define the mapping $\Phi$ on $X$ by

$$\Phi(u)(t) = \sum_{n=1}^{\infty} u_{0,n} E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n + \sum_{n=1}^{\infty} u_{1,n} t E_{\alpha,2}(-\lambda_n t^\alpha) \varphi_n$$

$$+ \sum_{n=1}^{\infty} \left( \int_0^t f_n(u(\tau))(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha) d\tau \right) \varphi_n.$$

(6.1)
Firstly, it is clear that
\[
\|u\|_X := \sup_{t \in [0,T^*)} (\|u(\cdot,t)\|_{V_\gamma} + \|\partial_t u(\cdot,t)\|_{L^2(X)})
\]  
(6.2)
defines a norm on $X$. Secondly, it follows from (4.3) that there is a constant $C > 0$ such that
\[
\|w\|_{L^\infty(X)} \leq C\|w\|_{V_\gamma}, \quad \forall \ w \in V_\gamma.
\]
Note that $X$ when endowed with the norm in (6.2) is a closed subspace of the Banach space $C([0,T^*]; V_\gamma) \cap C^1([0,T^*]; L^2(X))$. We prove the existence of a locally defined solution of (4.1) by a fixed point argument.

**Step 1.** Since $f$ is continuously differentiable, we have that $\Phi(u)(t)$ is continuously differentiable on $[0,T^*)$. We will show that by an appropriate choice of $T^*, R^* > 0$, $\Phi : X \to X$ is a contraction with respect to the metric induced by the norm of $C([0,T^*]; V_\gamma) \cap C^1([0,T^*]; L^2(X))$. The appropriate choice of $T^*, R^* > 0$ will be specified below. We first show that $\Phi$ maps $X$ into $X$. Indeed, let $u \in X$. Then
\[
\Phi(u)'(t) = \sum_{n=1}^{\infty} u_{0,n} \lambda_n t^{\alpha - 1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \varphi_n + \sum_{n=1}^{\infty} u_{1,n} E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n
\]  
(6.3)
\[
+ \sum_{n=1}^{\infty} \left( \int_0^t f_n(u(\tau))(t-\tau)^{\alpha - 2} E_{\alpha,\alpha-1}(-\lambda_n (t-\tau)^\alpha) d\tau \right) \varphi_n.
\]
Next, by assumption (Hf2), for every $t \in [0,T^*)$,
\[
\|f(u(\cdot,t))\|_{L^2(X)} \leq C Q_2 \left( \|u(\cdot,t)\|_{V_\gamma} \right),
\]  
(6.4)
for some $C > 0$. Proceeding as the proof of Theorem 3.2 and using the estimate (6.4) we get that there is a constant $C > 0$ such that for every $t \in [0,T^*)$,
\[
\|\Phi(u)(t)\|_{V_\gamma} \leq C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + t^{\alpha - 1} \|f(u)\|_{L^\infty([0,T^*];L^2(X))} \right)
\]  
(6.5)
\[
\leq C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + (T^*)^{\alpha - 1} Q_2 \left( \|u\|_{C([0,T^*];V_\gamma)} \right) \right)
\leq C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + (T^*)^{\alpha - 1} Q_2 R^* \right).
\]
Thus $\Phi(u) \in C([0,T^*]; V_\gamma)$ where we have also used the fact that the series in (6.1) converges in $V_\gamma$ uniformly for $t \in [0,T^*)$. Similarly, we have that there is a constant $C > 0$ such that for every $t \in [0,T^*)$,
\[
\|\Phi(u)'(t)\|_{L^2(X)} \leq C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + (T^*)^{\alpha - 1} Q_2 \left( \|u\|_{C([0,T^*];V_\gamma)} \right) \right)
\]  
(6.6)
\[
\leq C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + (T^*)^{\alpha - 1} Q_2 R^* \right).
\]
Since the series in (6.3) converges in $L^2(X)$ uniformly for every $t \in [0,T^*)$, we also have that $\Phi(u) \in C^1([0,T^*]; L^2(X))$. It also follows from (6.5) and (6.6) that
\[
\|\Phi(u)(t)\|_{V_\gamma} + \|\Phi(u)'(t)\|_{L^2(X)} \leq C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + (T^*)^{\alpha - 1} Q_2 R^* \right).
\]
Letting
\[
R^* \geq 2C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} \right),
\]
we can find a sufficiently small time $T^* > 0$ such that
\[
2C(T^*)^{\alpha - 1} Q_2 R^* \leq R^*,
\]  
(6.7)
in which case, it follows that $\Phi(u) \in X$ for all $u \in X$.

**Step 2.** Next, we show that by choosing a possibly smaller $T^* > 0$, $\Phi : X \to X$ is a contraction. Indeed, let $u, v \in X$. Using the assumption (Hf2), the Hölder inequality, we have that there is a constant $C > 0$ (albeit possibly with a different value in each line) such that for every $t \in [0, T^*]$, \[
\|f(u(\cdot, t)) - f(v(\cdot, t))\|_{L^2(X)} \leq C\|u(\cdot, t) - v(\cdot, t)\|_{L^\infty(X)} \left(Q_1(\|u\|_{C([0,T^*];V_\gamma)}) + Q_1(\|v\|_{C([0,T^*];V_\gamma)})\right) \leq C\|u - v\|_{C([0,T^*];V_\gamma)}Q_1(R^*).
\]
It follows from (6.5), (6.6) and (6.8) that there is a constant $C > 0$ such that for every $t \in [0, T^*]$, \[
\|\Phi(u)(t) - \Phi(v)(t)\|_{V_\gamma} + \|\Phi(u)'(t) - \Phi(v)'(t)\|_{L^2(X)} \leq C(T^*)^{\alpha-1}Q_1(R^*)\|u - v\|_X,
\]
and this implies that \[
\|\Phi(u) - \Phi(v)\|_X \leq C(T^*)^{\alpha-1}Q_1(R^*)\|u - v\|_X.
\]
Choosing $T^*$ smaller than the one determined by (6.7) such that $C(T^*)^{\alpha-1}Q_1(R^*) < 1$, it follows that the mapping $\Phi$ is a contraction on $X$. Therefore, owing to the contraction mapping principle, we can conclude that the mapping $\Phi$ has a unique fixed point $u$ in $X$.

**Step 3.** Finally we show that $u$ has the regularity specified in (4.7) and also satisfies the variational identity. For the regularity part, it remains to show that $D_t^\alpha u \in C([0, T^*]; V_{-\gamma})$. In fact, it follows from (4.15) that \[
D_t^\alpha u(\cdot, t) = -\sum_{n=1}^{\infty} u_{0,n}\lambda_nE_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n - \sum_{n=1}^{\infty} u_{1,n}\lambda_n t E_{\alpha,2}(-\lambda_n t^\alpha)\varphi_n - \sum_{n=1}^{\infty} \left(\int_0^t f_n(u(\tau))\lambda_n(t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t - \tau)^\alpha) d\tau\right) \varphi_n + f(u(\cdot, t)) = -Au(\cdot, t) + f(u(\cdot, t)).
\]
Proceeding as in (3.17) and (3.18) we get that there is a constant $C > 0$ such that for every $t \in [0, T^*]$, \[
\left\|\sum_{n=1}^{\infty} u_{0,n}\lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n\right\|_{V_{-\gamma}} \leq C\|u_0\|_{V_{-\gamma}}, \tag{6.10}
\]
and \[
\left\|\sum_{n=1}^{\infty} u_{1,n}\lambda_n t E_{\alpha,2}(-\lambda_n t^\alpha)\varphi_n\right\|_{V_{-\gamma}} \leq Ct^{2-\alpha}\|u_1\|_{L^2(X)}. \tag{6.11}
\]
Similarly, using the assumptions on \( f \), we have that for every \( t \in [0, T^*] \),
\[
\left\| \sum_{n=1}^{\infty} \left( \int_0^t f_n(u(\tau))\lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha}) \, d\tau \right) \varphi_n \right\|_{V_{-\gamma}} \leq C t \| f(u) \|_{L^\infty([0, T^*]; L^2(X))} \leq C t Q_2 \left( \sup_{t \in [0, T^*]} \| u(\cdot, t) \|_{V_{-\gamma}} \right).
\]
Since the series in (6.9) converges in \( V_{-\gamma} \) uniformly in \([0, T^*]\), we have shown that \( \mathbb{D}_t^\alpha u \in C([0, T^*]; V_{-\gamma}) \) owing also to the fact that
\[
f(u) \in C \left([0, T^*]; L^2(X)\right) \hookrightarrow C \left([0, T^*]; V_{-\gamma}\right).
\]
Since \( \mathbb{D}_t^\alpha u(\cdot, t) \in V_{-\gamma} \), \( Au(\cdot, t) \in V^{-1/2} \hookrightarrow V_{-\gamma} \) and \( f(u(\cdot, t)) \in L^2(X) \) for all \( t \in (0, T^*) \), then taking the duality product in (6.9) we immediately get the variational identity (3.5).

It is clear that \( \Phi(u)(0) = u_0 \) and that \( \Phi(u)'(0) = u_1 \) and we have shown (6.8). The initial conditions are satisfied in the sense of (4.9) on account of the regularity property (6.12) and Step 5 of the proof of Theorem 3.2. We have shown that the function \( u \) given by (4.15) is the unique weak solution of (4.11) on \([0, T^*]\). The proof is finished.

**Proof of Theorem 4.5**

Let
\[
S_1(t)u_0 := \sum_{n=1}^{\infty} u_{0,n} E_{\alpha,1}(-\lambda_n t^{\alpha}) \varphi_n \quad \text{and} \quad S_2(t)u_1 := \sum_{n=1}^{\infty} u_{1,n} E_{\alpha,2}(-\lambda_n t^{\alpha}) \varphi_n,
\]
and
\[
S_3(t)f := \sum_{n=1}^{\infty} \left( \int_0^t f_n(u(s))(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha}) \, ds \right) \varphi_n
\]
so that
\[
u(t) = S_1(t)u_0 + S_2(t)u_1 + S_3(t)f.
\]
Let also
\[
S_1'(t)u_0 := \sum_{n=1}^{\infty} u_{0,n} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) \varphi_n \quad \text{and} \quad S_2'(t)u_1 := \sum_{n=1}^{\infty} u_{1,n} t E_{\alpha,1}(-\lambda_n t^{\alpha}) \varphi_n,
\]
and
\[
S_3'(t)f := \sum_{n=1}^{\infty} \left( \int_0^t f_n(u(s))(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(t-s)^{\alpha}) \, ds \right) \varphi_n
\]
so that
\[
\partial_t u(\cdot, t) = S_1'(t)u_0 + S_2'(t)u_1 + S_3'(t)f.
\]
Let \( T^* \) be the time from Theorem 4.4. Fix \( \tau > 0 \) and consider the space
\[
\mathbb{K} := \left\{ v \in C([0, T^* + \tau]; V_{-\gamma}) \cap C^1([0, T^* + \tau]; L^2(X)) : \right. \\
\left. v(\cdot, t) = u(\cdot, t) \quad \forall \, t \in [0, T^*], \right. \\
\left. \| v(\cdot, t) - u(\cdot, T^*) \|_{V_{-\gamma}} + \| \partial_t v(\cdot, t) - \partial_t u(\cdot, T^*) \|_{L^2(X)} \leq R, \; \forall \, t \in [T^*, T^* + \tau] \right\}.
\]
Define the mapping $\Phi$ on $K$ by

$$
\Phi(v)(t) = \sum_{n=1}^{\infty} u_{0,n} E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n + \sum_{n=1}^{\infty} u_{1,n} t E_{\alpha,2}(-\lambda_n t^\alpha) \varphi_n + \sum_{n=1}^{\infty} \left( \int_0^t f_n(v(s))(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) \, ds \right) \varphi_n.
$$

(6.17)

Note that $K$ when endowed with the norm of $C([0,T^*+\tau];V_\gamma) \cap C^1([0,T^*+\tau];L^2(X))$ is a closed subspace of $C([0,T^*+\tau];V_\gamma) \cap C^1([0,T^*+\tau];L^2(X))$. We show that $\Phi$ has a fixed point in $K$.

**Step 1.** Since $f$ is continuously differentiable, we have that the mapping $t \mapsto \Phi(v)(t)$ is continuously differentiable on $[0,T^*+\tau]$. We will show that by properly choosing $\tau, R > 0$, $\Phi : K \to K$ is a contraction mapping with respect to the metric induced by the norm of $C([0,T^*+\tau];V_\gamma) \cap C^1([0,T^*+\tau];L^2(X))$. The appropriate choice of $\tau, R > 0$ will be specified below. First, We show that $\Phi$ maps $K$ into $K$. Indeed, let $v \in K$.

- If $t \in [0,T^*]$, then $v(\cdot, t) = u(\cdot, t)$. Hence $\Phi(v)(t) = \Phi(u)(t) = u(\cdot, t)$ and there is nothing to prove.
- If $t \in [T^*,T^*+\tau]$, then

$$
\|\Phi(v)(t) - u(\cdot, T^*)\|_{V_\gamma}
\leq \|S_1(t)u_0 - S_1(T^*)u_0\|_{V_\gamma} + \|S_2(t)u_1 - S_2(T^*)u_1\|_{V_\gamma}
+ \sum_{n=1}^{\infty} \left( \int_0^t f_n(v(s))(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) \, ds \right) \varphi_n
- \sum_{n=1}^{\infty} \left( \int_0^T f_n(u(s))(T^*-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T^*-s)^\alpha) \, ds \right) \varphi_n \bigg|_{V_\gamma}
\leq \|S_1(t)u_0 - S_1(T^*)u_0\|_{V_\gamma} + \|S_2(t)u_1 - S_2(T^*)u_1\|_{V_\gamma}
+ \int_0^T \left[ \sum_{n=1}^{\infty} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) f_n(v(s)) \varphi_n \right]_{V_\gamma} \, ds
+ \int_0^T \left[ \sum_{n=1}^{\infty} [(t-s)^{\alpha-1} - (T^*-s)^{\alpha-1}] E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) f_n(u(s)) \varphi_n \right]_{V_\gamma} \, ds
+ \int_0^T \left[ \sum_{n=1}^{\infty} (T^*-s)^{\alpha-1} [E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) - E_{\alpha,\alpha}(-\lambda_n (T^*-s)^\alpha)] f_n(u(s)) \varphi_n \right]_{V_\gamma} \, ds
= N_1 + N_2 + N_3 + N_4.
$$

Since for every $T \geq 0$, the mappings $t \mapsto S_1(t)u_0$ and $t \mapsto S_2(t)u_1$ belong to $C([0,T],V_\gamma)$, we can choose $\tau > 0$ small such that for $t \in [T^*,T^*+\tau]$, we have

$$
N_1 := \|S_1(t)u_0 - S_1(T^*)u_0\|_{V_\gamma} + \|S_2(t)u_1 - S_2(T^*)u_1\|_{V_\gamma} \leq \frac{R}{4}.
$$

(6.18)
Proceeding as the proof of Theorem 1.4 we can choose $\tau > 0$ small such that for $t \in [T^*, T^* + \tau]$, we have

$$N_2 := \int_{T^*}^{T^*} \left\| \sum_{n=1}^{\infty} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha}) f_n(v(s)) \varphi_n \right\|_{V_\gamma} ds \quad (6.19)$$

$$\leq C \tau^{\alpha-1} Q_2 \left( \|v(\cdot, t)\|_{V_\gamma} \right) \leq 2C \tau^{\alpha-1} Q_2(R^*) \leq \frac{R}{4}.$$ 

For the third norm we have that

$$N_3 := \int_{0}^{T^*} \left\| \sum_{n=1}^{\infty} \left[ (t-s)^{\alpha-1} - (T^* - s)^{\alpha-1} \right] E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha}) f_n(u(s)) \varphi_n \right\|_{V_\gamma} ds. \quad (6.20)$$

Note that the series in (6.20) converges in $V_\gamma$ uniformly for $t \in [T^*, T^* + \tau]$. Moreover,

$$\left\| \sum_{n=1}^{\infty} \left[ (t-s)^{\alpha-1} - (T^* - s)^{\alpha-1} \right] E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha}) f_n(u(s)) \varphi_n \right\|_{V_\gamma} \to 0 \quad \text{as} \quad t \to T^*,$n}

and there is a constant $C > 0$ such that

$$\left\| \sum_{n=1}^{\infty} \left[ (t-s)^{\alpha-1} - (T^* - s)^{\alpha-1} \right] E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha}) f_n(u(s)) \varphi_n \right\|_{V_\gamma} \leq C(t-s)^{\alpha-2} \|f(u(\cdot, s))\|_{L^2(\gamma)} \leq C(T^* - s)^{\alpha-2} \|f(u(\cdot, s))\|_{L^2(\gamma)}.$$

Thus by the Lebesgue Dominated Convergence Theorem, we can choose $\tau > 0$ small such that for $t \in [T^*, T^* + \tau]$,

$$N_3 = \int_{0}^{T^*} \left\| \sum_{n=1}^{\infty} \left[ (t-s)^{\alpha-1} - (T^* - s)^{\alpha-1} \right] E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha}) f_n(u(s)) \varphi_n \right\|_{V_\gamma} ds \leq \frac{R}{4} \quad (6.21)$$

With the same argument as for $N_3$, we can choose $\tau > 0$ small such that for every $t \in [T^*, T^* + \tau]$ we have

$$N_4 := \int_{0}^{T^*} \left\| \sum_{n=1}^{\infty} (T^* - s)^{\alpha-1} \left[ E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha}) - E_{\alpha,\alpha}(-\lambda_n(T^*-s)^{\alpha}) \right] f_n(u(s)) \varphi_n \right\|_{V_\gamma} ds \leq \frac{R}{8}. \quad (6.22)$$
For the time derivative, proceeding as above, we also have that
\[
\|\Phi(v)'(t) - \partial_t u(\cdot, T^*)\|_{L^2(X)} \\
\leq \|S_1'(t)u_0 - S_1'(T^*)u_0\|_{V_\gamma} + \|S_2'(t)u_1 - S_2'(T^*)u_1\|_{L^2(X)} \\
+ \int_{T^*}^t \left\| \sum_{n=1}^{\infty} (t - s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n(t - s)^{\alpha}) f_n(v(s)) \varphi_n \right\|_{L^2(X)} ds \\
+ \int_0^{T^*} \left\| \sum_{n=1}^{\infty} [(t - s)^{\alpha-2} - (T^* - s)^{\alpha-2}] E_{\alpha,\alpha-2}(-\lambda_n(t - s)^{\alpha}) f_n(u(s)) \varphi_n \right\|_{L^2(X)} ds \\
+ \int_0^{T^*} \left\| \sum_{n=1}^{\infty} (T^* - s)^{\alpha-2} [E_{\alpha,\alpha-1}(-\lambda_n(t - s)^{\alpha}) - E_{\alpha,\alpha-1}(-\lambda_n(T^* - s)^{\alpha})] f_n(u(s)) \varphi_n \right\|_{V_\gamma} ds \\
= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4.
\]

Using the same argument as the corresponding terms above, we can choose \(\tau > 0\) small such that for every \(t \in [T^*, T^* + \tau]\),
\[
\mathcal{M}_1 := \|S_1'(t)u_0 - S_1'(T^*)u_0\|_{L^2(X)} + \|S_2'(t)u_1 - S_2'(T^*)u_1\|_{L^2(X)} \leq \frac{R}{8}, \quad (6.23)
\]
and
\[
\mathcal{M}_2 := \int_{T^*}^t \left\| \sum_{n=1}^{\infty} (t - s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n(t - s)^{\alpha}) f_n(v(s)) \varphi_n \right\|_{L^2(X)} ds \leq \frac{R}{8}, \quad (6.24)
\]
and
\[
\mathcal{M}_3 := \int_0^{T^*} \left\| \sum_{n=1}^{\infty} [(t - s)^{\alpha-2} - (T^* - s)^{\alpha-2}] E_{\alpha,\alpha-2}(-\lambda_n(t - s)^{\alpha}) f_n(u(s)) \varphi_n \right\|_{L^2(X)} ds \leq \frac{R}{8}, \quad (6.25)
\]
and
\[
\mathcal{M}_4 := \int_0^{T^*} \left\| \sum_{n=1}^{\infty} (T^* - s)^{\alpha-2} [E_{\alpha,\alpha-1}(-\lambda_n(t - s)^{\alpha}) - E_{\alpha,\alpha-1}(-\lambda_n(T^* - s)^{\alpha})] f_n(u(s)) \varphi_n \right\|_{V_\gamma} ds \\
\leq \frac{R}{8}, \quad (6.26)
\]

It follows from (6.18), (6.22), (6.23), and (6.26) that there exists \(\tau > 0\) small such that for every \(t \in [T^*, T^* + \tau]\),
\[
\|\Phi(v)(t) - u(\cdot, T^*)\|_{V_\gamma} + \|\Phi(v)'(t) - \partial_t u(\cdot, T^*)\|_{L^2(X)} \leq R.
\]

We have shown that \(\Phi\) maps \(K\) into \(K\).

**Step 2.** We show that \(\Phi\) is a contraction on \(K\). Let \(v, w \in K\). Then
\[
\Phi(v)(t) - \Phi(w)(t) = \sum_{n=1}^{\infty} \left( \int_0^t (f_n(v(s)) - f_n(w(s)))(t - s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t - s)^{\alpha}) ds \right) \varphi_n.
\]
• If \( t \in [0, T^*] \), then it follows from the proof of Theorem 4.4 that
\[
\| \Phi(u)(t) - \Phi(v)(t) \|_{V_\gamma} + \| \Phi(u)'(t) - \Phi(v)'(t) \|_{L^2(X)} \\
\leq C(T^*)^{\alpha - 1} Q_1(R^*) \|u - v\|_{C([0,T^*];V_\gamma)}.
\]

• If \( t \in [T^*, T^* + \tau] \), then proceeding as in (6.8) and (6.19) we get that there is a constant \( C > 0 \) such that
\[
\| \Phi(v)(t) - \Phi(w)(t) \|_{V_\gamma} \\
= \left\| \sum_{n=1}^{\infty} \left( \int_{T^*}^{t} (f_n(v(s)) - f_n(w(s))(t-s)^{\alpha - 1} E_{\alpha,\alpha} \{ -\lambda_n(t-s)^{\alpha} \} \right) \varphi_n \right\|_{V_\gamma}
\leq C \tau^{\alpha - 1} \| f(v) - f(w) \|_{L^\infty([T^*,T^*];L^2(X))}
\leq C \tau^{\alpha - 1} Q_1(R) \| v - w \|_{C([T^*,T^*+\tau];V_\gamma)}.
\]

In a similar way we have that there is a constant \( C > 0 \) such that
\[
\| \Phi(v)'(t) - \Phi(w)'(t) \|_{L^2(X)} \\
= \left\| \sum_{n=1}^{\infty} \left( \int_{T^*}^{t} (f_n(v(s)) - f_n(w(s))(t-s)^{\alpha - 2} E_{\alpha,\alpha - 1} \{ -\lambda_n(t-s)^{\alpha} \} \right) \varphi_n \right\|_{L^2(X)}
\leq C \tau^{\alpha - 1} \| f(v) - f(w) \|_{L^\infty([T^*,T^*];L^2(X))}
\leq C \tau^{\alpha - 1} Q_1(R) \| v - w \|_{C([T^*,T^*+\tau];V_\gamma)}.
\]

It follows from (6.27) and (6.28) that there is a constant \( C > 0 \) such that
\[
\| \Phi(v) - \Phi(w) \|_K \leq C \tau^{\alpha - 1} Q_1(R) \| v - w \|_K.
\]

Then choosing \( \tau > 0 \) even smaller again (if necessary) so that \( C \tau^{\alpha - 1} Q_1(R) < 1 \), we deduce once again that \( \Phi \) is a contraction on \( K \). Hence, \( \Phi \) has a unique fixed point \( v \) on \( K \).

**Step 3** We show that the function \( v \) given by the right hand side of (6.17) has the regularity specified in (4.7). In fact we need to show that \( \mathbb{D}^\alpha v \in C([0,T^*+\tau];V_{\gamma - \epsilon}) \). The proof follows the lines of the corresponding result in the proof of Theorem 4.4. The proof is finished.

To complete the proof of Theorem 4.6 we need the following lemma.

**Lemma 6.1.** Let \( T \in (0, \infty) \) and \( u : X \times [0, T) \rightarrow L^2(X) \) be such that \( u(x, \cdot) \) is continuously differentiable for a.e. \( x \in X \) with
\[
\sup_{t \in [0,T)} \left( \| u(\cdot,t) \|_{V_\gamma} + \| \partial_t u(\cdot,t) \|_{L^2(X)} \right) < \infty.
\]

Let
\[
\mathbb{E}_k(t) := t^{\alpha - 1} E_{\alpha,\alpha} \{ -\lambda_k t^\alpha \} \quad \text{and} \quad \mathbb{E}'_k(t) := t^{\alpha - 2} E_{\alpha,\alpha} \{ -\lambda_k t^\alpha \}.
\]
Let \( t_n \in [0, T) \) be a sequence such that \( \lim_{n \to \infty} t_n = T \). Then

\[
\lim_{n \to \infty} \int_0^{t_n} \left\| \sum_{k=1}^{\infty} \left[ \mathbb{E}_k(t - \tau) - \mathbb{E}_k(T - \tau) \right] f_k(u(\tau)) \varphi_k \right\|_{V_{\gamma}} \ d\tau = 0, \quad (6.30)
\]

and

\[
\lim_{n \to \infty} \int_0^{t_n} \left\| \sum_{k=1}^{\infty} \left[ \mathbb{E}'_k(t - \tau) - \mathbb{E}'_k(T - \tau) \right] f_k(u(\tau)) \varphi_k \right\|_{L^2(X)} \ d\tau = 0. \quad (6.31)
\]

**Proof.** Recall that \( \gamma = \frac{1}{\alpha} \). Let us prove the first claim (6.30). Set

\[
K := \sup_{s \in [0, T)} \| f(u(\cdot, s)) \|_{L^2(X)} < \infty.
\]

Given \( \epsilon > 0 \), fix \( \delta \in (0, T) \) such that

\[
\frac{CK}{\alpha} (T - \delta)^\alpha < \frac{\epsilon}{4}.
\]

Using (2.8) and (2.4) we get that there is a constant \( C > 0 \) such that for every \( 0 < T < t \),

\[
\left| \lambda_k^\alpha \left[ (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k(t - \tau)^\alpha) - (T - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k(T - \tau)^\alpha) \right] \right|
\]

\[
= \left| \int_{T+\tau}^{t+\tau} \lambda_k^\gamma s^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_k s^\alpha) \ ds \right|
\]

\[
\leq C \int_{T+\tau}^{t+\tau} s^{\alpha-2} \lambda^{\gamma-1} s^{-\alpha} \ ds \leq C \int_{T+\tau}^{t+\tau} s^{-2} \ ds = C \left( \frac{1}{T+\tau} - \frac{1}{t+\tau} \right).
\]

This estimate implies that

\[
\left\| \sum_{k=1}^{\infty} \left[ \mathbb{E}_k(t - \tau) - \mathbb{E}_k(T - \tau) \right] f_k(u(\tau)) \varphi_k \right\|_{V_{\gamma}} \leq CK \left( \frac{1}{T+\tau} - \frac{1}{t+\tau} \right). \quad (6.32)
\]

Thus

\[
\lim_{t \to T^+} \left\| \sum_{k=1}^{\infty} \left[ \mathbb{E}_k(t - \tau) - \mathbb{E}_k(T - \tau) \right] f_k(u(\tau)) \varphi_k \right\|_{V_{\gamma}} = 0.
\]

It also follows from (6.32) that

\[
\left\| \sum_{k=1}^{\infty} \left[ \mathbb{E}_k(t - \tau) - \mathbb{E}_k(T - \tau) \right] f_k(u(\tau)) \varphi_k \right\|_{V_{\gamma}} \leq CK \frac{1}{T},
\]

and the right hand-side belongs to \( L^1((0, t)) \). Therefore, by the Lebesgue Dominated Convergence Theorem, we can choose \( N \in \mathbb{N} \) such that \( t_n > \delta \) and

\[
\int_0^{\delta} \left\| \sum_{k=1}^{\infty} \left[ \mathbb{E}_k(t - \tau) - \mathbb{E}_k(T - \tau) \right] f_k(u(\tau)) \varphi_k \right\|_{V_{\gamma}} \ d\tau < \frac{\epsilon}{2}.
\]
for all $n \geq N$. Therefore, for all $n \geq N$,
\[
\int_0^{t_n} \left\| \sum_{k=1}^\infty [E_k(t-\tau) - E_k(T-\tau)] f_k(u(\tau)) \varphi_k \right\|_{L^2(\Omega)} d\tau
\leq \int_0^{t_n} \left\| \sum_{k=1}^\infty [E_k(t-\tau) - E_k(T-\tau)] f_k(u(\tau)) \varphi_k \right\|_{L^2(\Omega)} d\tau
+ C \int_\delta^{t_n} \left[ (t_n - \tau)^{\alpha-1} + (T - \tau)^{\alpha-1} \right] \|f(u(\cdot, \tau))\|_{L^2(\Omega)} d\tau
\leq \frac{\epsilon}{2} + \frac{CK}{\alpha-1} ((t_n - \delta)^{\alpha-1} + (T - \delta)^{\alpha-1}) < \frac{\epsilon}{2} + \frac{2CK}{\alpha-1} (T - \delta)^{\alpha-1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]
and we have shown (6.30).

Analogously, given $\epsilon > 0$, fix $\delta \in (0, T)$ such that
\[
\frac{CK}{\alpha-1} (T - \delta)^{\alpha-1} < \frac{\epsilon}{4}.
\]
Proceeding as above, we can choose $N_0 \in \mathbb{N}$ such that $t_n > \delta$ and
\[
\int_0^{t_n} \left\| \sum_{k=1}^\infty \left[ E_k'(t-\tau) - E_k'(T-\tau) \right] f_k(u(\tau)) \varphi_k \right\|_{L^2(\Omega)} d\tau < \frac{\epsilon}{2},
\]
for all $n \geq N_0$. Using this estimate, we get that for all $n \geq N_0$,
\[
\int_0^{t_n} \left\| \sum_{k=1}^\infty \left[ E_k'(t-\tau) - E_k'(T-\tau) \right] f_k(u(\tau)) \varphi_k \right\|_{L^2(\Omega)} d\tau
\leq \int_0^{\delta} \left\| \sum_{k=1}^\infty \left[ E_k'(t-\tau) - E_k'(T-\tau) \right] f_k(u(\tau)) \varphi_k \right\|_{L^2(\Omega)} d\tau
+ C \int_\delta^{t_n} \left[ (t_n - \tau)^{\alpha-2} + (T - \tau)^{\alpha-2} \right] \|f(u(\cdot, \tau))\|_{L^2(\Omega)} d\tau
\leq \frac{\epsilon}{2} + \frac{CK}{\alpha-1} ((t_n - \delta)^{\alpha-1} + (T - \delta)^{\alpha-1}) < \frac{\epsilon}{2} + \frac{2CK}{\alpha-1} (T - \delta)^{\alpha-1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
We have shown (6.31) and the proof is finished. \qed

Now we are ready to give the proof of our last main result.

**Proof of Theorem 4.6.** Let
\[
\mathcal{T} := \left\{ T \in [0, \infty) : \exists u : X \times [0, T] \to L^2(\Omega) \text{ unique local solution to (4.1) in } (0, T) \right\},
\]
and set $T_{\max} := \sup \mathcal{T}$. Then we have a continuously differentiable function (in the second variable) $u : X \times [0, T_{\max}) \to L^2(\Omega)$ which is the local solution of (4.1) on $[0, T_{\max})$. If $T_{\max} = \infty$, then $u$ is a global solution. Now if $T_{\max} < \infty$ we shall show that we have (4.16). Assume that there exists $K_0 < \infty$ such that
\[
\|u(\cdot, t)\|_{V_{\gamma}} + \|\partial_t u(\cdot, t)\|_{L^2(\Omega)} \leq K_0, \quad \forall t \in [0, T_{\max}). \tag{6.33}
\]
Let \((t_n)_{n \in \mathbb{N}} \subset [0, T_{\text{max}})\) be a sequence that converges to \(T_{\text{max}}\). Let \(t_n > t_m\) and

\[
K := \sup_{t \in [0, T_{\text{max}})} \| f(u(\cdot, t)) \|_{L^2(X)} < \infty.
\]

Then using the assumption \(6.33\), we get from Lemma \(6.1\) that

\[
\left\| \int_{t_m}^{t_n} \sum_{k=1}^{\infty} \mathbb{E}_k(T_{\text{max}} - s)f_k(u(s))\varphi_k \, ds \right\|_{V_{\gamma}} \\
\leq \int_{t_m}^{t_n} \sum_{k=1}^{\infty} \left\| \mathbb{E}_k(T_{\text{max}} - s)f_k(u(s))\varphi_k \right\|_{V_{\gamma}} \, ds \\
\leq C \int_{t_m}^{t_n} (T_{\text{max}} - s)^{\alpha - 1}\| f(u(s)) \|_{L^2(X)} \, ds \\
\leq CK \int_{t_m}^{t_n} (T_{\text{max}} - s)^{\alpha - 1} \, ds \\
= \frac{CK}{\alpha} \left[ (T_{\text{max}} - t_n)^{\alpha} - (T_{\text{max}} - t_m)^{\alpha} \right] \to 0,
\]

as \(n, m \to \infty\).

We use the notations of \(S_j\) and \(S_j'\) \((j = 1, 2, 3)\) given in \(6.13\)–\(6.16\). Then

\[
\| u(\cdot, t_n) - u(\cdot, t_m) \|_{V_{\gamma}} \\
\leq \| S_1(t_n)u_0 - S_1(t_m)u_0 \|_{V_{\gamma}} + \| S_2(t_n)u_1 - S_2(t_m)u_1 \|_{V_{\gamma}} + \| S_3(t_n)f - S_3(t_m)f \|_{V_{\gamma}} \\
\leq \| S_1(t_n)u_0 - S_1(t_m)u_0 \|_{V_{\gamma}} + \| S_2(t_n)u_1 - S_2(t_m)u_1 \|_{V_{\gamma}} \\
+ \left\| \int_{0}^{t_n} \sum_{k=1}^{\infty} \left[ \mathbb{E}_k(t_n - s) - \mathbb{E}_k(T_{\text{max}} - s) \right] f_k(u(s))\varphi_k \, ds \right\|_{V_{\gamma}} \\
+ \left\| \int_{0}^{t_m} \sum_{k=1}^{\infty} \left[ \mathbb{E}_k(t_m - s) - \mathbb{E}_k(T_{\text{max}} - s) \right] f_k(u(s))\varphi_k \, ds \right\|_{V_{\gamma}} \\
+ \left\| \int_{t_m}^{t_n} \sum_{k=1}^{\infty} \mathbb{E}_k(T_{\text{max}} - s)f_k(u(s))\varphi_k \, ds \right\|_{V_{\gamma}} \to 0 \quad \text{as} \; n, m \to \infty,
\]
where we have used Lemma 6.1. Analogously, for $t_n > t_m$ we have that

\[
\left\| \int_{t_m}^{t_n} \sum_{k=1}^{\infty} \mathcal{E}'_k(T_{\max} - s) f_k(u(s)) \varphi_k \, ds \right\|_{L^2(X)} \\
\leq \int_{t_m}^{t_n} \sum_{k=1}^{\infty} \left\| \mathcal{E}'_k(T_{\max} - s) f_k(u(s)) \varphi_k \right\|_{L^2(X)} \, ds \\
\leq C \int_{t_m}^{t_n} (T_{\max} - s)^{\alpha - 2} \|f(u(s))\|_{L^2(X)} \, ds \\
\leq CK \int_{t_m}^{t_n} (T_{\max} - s)^{\alpha - 2} \, ds \\
= \frac{C}{\alpha - 1} \left[ (T_{\max} - t_n)^{\alpha - 1} - (T_{\max} - t_m)^{\alpha - 1} \right] \to 0,
\]

as $n, m \to \infty$. Thus, by Lemma 6.1 again we obtain that

\[
\|\partial_t u(\cdot, t_n) - \partial_t u(\cdot, t_m)\|_{L^2(X)} \\
\leq \|S'_1(t_n)u_0 - S'_1(t_m)u_0\|_{L^2(X)} + \|S'_2(t_n)u_1 - S'_2(t_m)u_1\|_{L^2(X)} + \|S'_3(t_n)f - S'_3(t_m)f\|_{L^2(X)} \\
\leq \|S'_1(t_n)u_0 - S'_1(t_m)u_0\|_{L^2(X)} + \|S'_2(t_n)u_1 - S'_2(t_m)u_1\|_{L^2(X)} \\
+ \left\| \int_{0}^{t_n} \sum_{k=1}^{\infty} \left[ \mathcal{E}'_k(t_n - s) - \mathcal{E}'_k(T_{\max} - s) \right] f_k(u(s)) \varphi_k \, ds \right\|_{L^2(X)} \\
+ \left\| \int_{0}^{t_m} \sum_{k=1}^{\infty} \left[ \mathcal{E}'_k(t_m - s) - \mathcal{E}'_k(T_{\max} - s) \right] f_k(u(s)) \varphi_k \, ds \right\|_{L^2(X)} \\
+ \left\| \int_{t_m}^{t_n} \sum_{k=1}^{\infty} \mathcal{E}'_k(T_{\max} - s) f_k(u(s)) \varphi_k \, ds \right\|_{L^2(X)} \to 0,
\]

as $n, m \to \infty$. It follows that $(u(\cdot, t_n))_{n \in \mathbb{N}}$ and $(\partial_t u(\cdot, t_n))_{n \in \mathbb{N}}$ are Cauchy sequences and therefore have limits $u_{T_{\max}}(\cdot, t)$ and $\partial_t u_{T_{\max}}(\cdot, t)$ such that $u_{T_{\max}}(\cdot, t) \in V_\gamma$ and $\partial_t u_{T_{\max}}(\cdot, t) \in L^2(X)$. Then, we can extend $u$ over $[0, T_{\max}]$ to obtain the equality

\[
u(\cdot, t) = S_1(t)u_0 + S_2(t)u_1 + S_3(t)f,
\]

for all $t \in [0, T_{\max}]$. By Theorem 4.4 we can extend the solution to some larger interval. This is a contradiction with the definition of $T_{\max} > 0$. The proof is finished. \qed

7. Proofs in the case (i) when $2 > \alpha \geq \alpha_0$ and case (ii)

In this section we briefly discuss the proofs of the results stated in Section 4 in the super-critical case $\alpha_0 \leq \alpha < 2$ (for Case (i)) and in the Case (ii), respectively.

**Proof of Theorem 4.4** Fix $0 < T^* \leq T$. Let

\[
\mathcal{Y} := C([0, T^*]; V_\gamma) \cap C^1([0, T^*]; L^2(X)) \cap L^{r_q}(0, T^*; L^{2r}(X))
\]
and consider the space
\[ \mathbb{Y}_{T^*} = \left\{ u \in \mathbb{Y} : u(t, 0) = u_0, \partial_t u(t, 0) = u_1 \text{ and } \|u\|_{\mathbb{Y}_{T^*}} \leq R^* \right\}, \]
for some \( R^* > 0 \), with norm
\[ \|u\|_{\mathbb{Y}_{T^*}} := \sup_{t \in [0, T^*]} \left( \|u(t)\|_{V_\gamma} + \|\partial_t u(t)\|_{L^2(X)} + \|u\|_{L^p((0, T^*); L^{2r}(X))} \right). \]
Next, define the same mapping \( \Phi \) on \( \mathbb{Y}_{T^*} \) by (6.1). Note that \( \mathbb{Y}_{T^*} \) when endowed with the previous norm is a closed subspace of the Banach space \( \mathbb{Y} \). We prove the existence of a locally defined solution of (1.1) by a fixed point argument. Furthermore, we recall that \( \Phi(u)'(t) \) can be also defined as the mapping from (6.3) since \( f \) is \( C^1 \).

**Step 1.** As in the proof in the previous sections, we must first check that \( \Phi \) maps \( \mathbb{Y}_{T^*} \) into \( \mathbb{Y}_{T^*} \). To this end, by assumption (Hf1), for every \( t \in [0, T^*] \), one can check for every \( u, v \in \mathbb{Y}_{T^*} \) that
\[ \|f(u) - f(v)\|_{L^q((0, T^*); L^{2r}(X))} \leq \|u - v\|_{L^q((0, T^*); L^{2r}(X))} \leq \|u\|_{L^q((0, T^*); L^{2r}(X))} + \|v\|_{L^q((0, T^*); L^{2r}(X))} \]
as well as
\[ \|f(u)\|_{L^q((0, T^*); L^{2r}(X))} \leq \|u\|_{L^q((0, T^*); L^{2r}(X))}. \]
We can proceed as in the proof of Theorem 3.2. Instead, exploiting estimate (7.2), we can find a constant \( C > 0 \) such that for every \( t \in [0, T^*] \),
\[ \|\Phi(u)(t)\|_{V_\gamma} \leq C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + \frac{1}{p} \|f(u)\|_{L^q((0, T^*); L^{2r}(X))} \right) \]
\[ \leq C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + (T^*)^{\frac{1}{p} + \alpha - 2} \|u\|_{L^q((0, T^*); L^{2r}(X))} \right) \]
\[ \leq C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + (T^*)^{\frac{1}{p} + \alpha - 2} (R^*)^r \right), \]
since \( 1 + p(\alpha - 2) > 0 \). Thus \( \Phi(u) \in C([0, T^*]; V_\gamma) \) where we have also used the fact that the series in (6.1) converges in \( V_\gamma \) uniformly for \( t \in [0, T^*] \). Similarly, we have that there is a constant \( C > 0 \) such that for every \( t \in [0, T^*] \),
\[ \|\Phi(u)'(t)\|_{L^2(X)} \leq C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + (T^*)^{\frac{1}{p} + \alpha - 2} (R^*)^r \right). \]
Since the series in (6.3) converges in \( L^2(X) \) uniformly for every \( t \in [0, T^*] \), we also have that \( \Phi(u) \in C^1([0, T^*]; L^2(X)) \). It also follows from (7.3) and (7.4) that
\[ \|\Phi(u)(t)\|_{V_\gamma} + \|\Phi(u)'(t)\|_{L^2(X)} \leq C \left( \|u_0\|_{V_\gamma} + \|u_1\|_{L^2(X)} + (T^*)^{\frac{1}{p} + \alpha - 2} (R^*)^r \right). \]
Finally, since \( V_\gamma \hookrightarrow L^{2r}(X) \) it holds
\[ \|u\|_{L^q((0, T^*); L^{2r}(X))} \leq (T^*)^{\frac{1}{rp}} \|u\|_{C([0, T^*]; V_\gamma)} \]
and therefore by (7.5), it follows that
\[
\|\Phi(u)\|_{Y_{T^*}} \leq C \left( \|u_0\|_{V_{\gamma}} + \|u_1\|_{L^2(X)} + (T^*)^{\frac{1}{p} + \alpha - 2} (R^*)^r \right) \\
+ C (T^*)^{\frac{1}{p}} \left( \|u_0\|_{V_{\gamma}} + \|u_1\|_{L^2(X)} \right) + C(T^*)^{\frac{1}{p} + \alpha - 2 + \frac{1}{r}} (R^*)^r.
\]

Letting now
\[ R^* \geq 2C (\|u_0\|_{V_{\gamma}} + \|u_1\|_{L^2(X)}) , \]
we can find a sufficiently small time \( T^* > 0 \) such that
\[ C (T^*)^{\frac{1}{p}} \left( \|u_0\|_{V_{\gamma}} + \|u_1\|_{L^2(X)} \right) + C(T^*)^{\frac{1}{p} + \alpha - 2 + \frac{1}{r}} (R^*)^r \leq \frac{R^*}{2}, \]
in which case it follows that \( \Phi(u) \in Y_{T^*} \) for all \( u \in Y_{T^*} \).

**Step 2.** Next, we show that by choosing a possibly smaller \( T^* > 0 \), \( \Phi : Y_{T^*} \to Y_{T^*} \) is a contraction. Similarly to the foregoing estimates, we can exploit (7.1) such that for every \( t \in [0, T^*] \),
\[
\|\Phi(u)(t) - \Phi(v)(t)\|_{V_{\gamma}} + \|\Phi(u)'(t) - \Phi(v)'(t)\|_{L^2(X)} \leq (T^*)^{\frac{1}{p} + \alpha - 2} (R^*)^r \|u - v\|_{Y_{T^*}},
\]
as well as
\[
\|\Phi(u) - \Phi(v)\|_{L^q((0,T^*);L^2(X))} \leq T^{\frac{1}{q}} \|\Phi(u) - \Phi(v)\|_{C([0,T^*];V_{\gamma})}.
\]
Choosing \( T^* \leq 1 \) smaller than the one determined by (7.7) such that \( (T^*)^{1+\rho(\alpha-2)} (R^*)^r < 1 \) it follows that the mapping \( \Phi \) is a contraction on \( Y_{T^*} \). Therefore, owing to the contraction mapping principle, we can conclude that the mapping \( \Phi \) has a unique fixed point \( u \in Y_{T^*} \).

**Step 3.** Finally we show that \( u \) has the regularity specified in (4.11) and also satisfies the variational identity. For the regularity part, it remains to show that
\[
\mathbb{D}_t^\alpha u = \theta_1 + \theta_2,
\]
with \( \theta_1 \in C([0, T^*]; V_{-\gamma}) \) and \( \theta_2 \in L^q((0, T^*); L^2(X)) \). As before, by (1.13) we have
\[
\mathbb{D}_t^\alpha u(\cdot, t) = -\sum_{n=1}^\infty u_{0,n} \lambda_n E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n - \sum_{n=1}^\infty u_{1,n} \lambda_n t E_{\alpha,2}(-\lambda_n t^\alpha) \varphi_n - \sum_{n=1}^\infty \left( \int_0^t f_n(u(\tau)) \lambda_n (t - \tau)^{-\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t - \tau)^\alpha) d\tau \right) \varphi_n + f(u(\cdot, t))
\]
\[ = -Au(\cdot, t) + f(u(\cdot, t)). \]

Proceeding as in (3.17) and (3.18) we deduce the estimates (6.10) and (6.11) for the first two summands in (7.8). For the third summand, we exploit (3.19) to easily conclude that
\[
\left\| \sum_{n=1}^\infty \left( \int_0^t f_n(u(\tau)) \lambda_n (t - \tau)^{-\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t - \tau)^\alpha) d\tau \right) \varphi_n \right\|_{V_{-\gamma}} \leq C t^{\frac{1}{p}} \|f(u)\|_{L^q((0, T^*); L^2(X))} \leq t^{\frac{1}{p}} \|u\|_{L^q((0, T^*); L^2(X))},
\]
for all \( t \in [0, T^*] \), owing also to (7.2). Since all series in (7.8) converge in \( V_{-\gamma} \) uniformly in \([0, T^*]\), we can let \( \theta_1 \) to be the sum of the first three summands in (7.8) and observe that \( \theta_1 \in C([0, T^*]; V_{-\gamma}) \); setting

\[
\theta_2 = f(u) \in L^q \left( (0, T^*); L^2(X) \right) \hookrightarrow L^q \left( (0, T^*); V_{-\gamma} \right),
\]

we immediately deduce the claim about the regularity of \( \mathbb{D}_t \theta \). Finally, since \( \mathbb{D}_t \theta \in V_{-\gamma} \), \( Au(\cdot, t) \in V_{-\gamma} \) and \( f(u(\cdot, t)) \in L^2(X) \) for a.e. \( t \in (0, T^*) \), then taking the duality product in (7.8) we immediately get the variational identity (3.5). The initial conditions are satisfied in the sense of (4.9) on account of the regularity property (7.10) and Step 5 of the proof of Theorem 3.2. We have shown that the function \( u \) given by (4.15) is the unique weak solution of (4.11) on \((0, T^*)\). The proof is finished. \( \square \)

**Proofs of Theorem 4.5 and Theorem 4.6** One argues almost verbatim (with some minor modifications) as in the proofs provided in Section 6. Indeed, let \( T^* > 0 \) be the time from the preceding proof. Fix \( T > 0 \) and consider the space

\[
\mathbb{K} := \left\{ v \in \mathbb{V}_{T^* + \tau} : v(\cdot, t) = u(\cdot, t) \ \forall t \in [0, T^*], \ |v - u|_{z_{T^*}} \leq R \right\},
\]

where

\[
|v - u|_{z_{T^*}} := \sup_{T^* \leq t \leq T^* + \tau} \left[ \|v(\cdot, t) - u(\cdot, T^*)\|_{V_{-\gamma}} + \|\partial_t v(\cdot, t) - \partial_t u(\cdot, T^*)\|_{L^2(X)} \right] + \|v - u\|_{L^q((T^*; T^* + \tau); L^2(X))},
\]

for \( t \in [T^*, T^* + \tau] \). With the same mapping \( \Phi \) as in (6.1), and arguing in a similar fashion as in the preceding proof by taking advantage of the basic estimates (7.1)–(7.2), we can show once again that \( \Phi : \mathbb{K} \to \mathbb{K} \) is a contraction mapping with respect to the metric induced by (7.11). In addition, it follows that \( \mathbb{D}_t^\alpha v = v_1 + v_2 \in C([0, T^* + \tau]; V_{-\gamma}) \oplus L^q \left( (0, T^* + \tau); L^{2r}(X) \right) \).

For the proof of Theorem 4.6, one may argue as in the subcritical case \( 1 < \alpha < \alpha_0 \) (see Section 6) with some (albeit) minor modifications. In particular, one updates the value \( K > 0 \) from the proof of the crucial Lemma 6.1 to

\[
K := \|f(u)\|_{L^q((0, T_{\max}); L^2(X))} < \infty.
\]

We leave the obvious details to the interested reader. \( \square \)

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