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LIFTING OF COHOMOLOGY AND UNOBSERVEDNESS
OF CERTAIN HOLOMORPHIC MAPS

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Abstract. Let $f$ be a holomorphic mapping between compact complex manifolds. We give a criterion for $f$ to have unobstructed deformations, i.e. for the local moduli space of $f$ to be smooth: this says, roughly speaking, that the group of infinitesimal deformations of $f$, when viewed as a functor, itself satisfies a natural lifting property with respect to infinitesimal deformations. This lifting property is satisfied e.g. whenever the group in question admits a ‘topological’ or Hodge-theoretic interpretation, and we give a number of examples, mainly involving Calabi-Yau manifolds, where that is the case.

One of the most important objects associated to a compact complex manifold $X$ is its versal deformation or Kuranishi family

$$\pi: \mathcal{X} \to \text{Def}(X);$$

this is a holomorphic mapping onto a germ of an analytic space \(\text{Def}(X), 0\) (the Kuranishi space) with the universal property that $\pi^{-1}(0) = X$ and that any sufficiently small deformation of $X$ is induced by pullback from $\pi$ by a map unique to 1st order. In general, $\text{Def}(X)$ is singular and even nonreduced; in case $\text{Def}(X)$ is smooth, i.e. a germ of the origin in $\mathbb{C}^N$, we say that $X$ is unobstructed. In an analogous fashion, a holomorphic mapping

$$f: X \to Y$$

also possesses a versal deformation, which in this case is a diagram

$$\tilde{f}: \mathcal{X} \to \mathcal{Y} \xrightarrow{\text{Def}(f)}$$
with a similar universal property. Again we say that $f$ is unobstructed if $\text{Def}(f)$ is smooth.

Now in [R3], we gave a criterion which deduces the unobstructedness of a compact complex manifold $X$ from a lifting property (in particular, deformation invariance) of certain cohomology groups associated to $X$; this implies in particular the unobstructedness of Calabi-Yau manifolds, i.e. Kahler manifolds with trivial canonical bundle $K_X$ (theorem of Bogomolov-Tian-Todorov [B, Ti, To]), as well as that of certain manifolds with “big” anticanonical bundle $-K_X$. In this note we announce an extension of our criterion to the case of holomorphic maps of manifolds and discuss some applications, mainly to maps whose source is a Calabi-Yau manifold.

1. Generalities

Given a holomorphic map

$$f: X \to Y$$

of complex manifolds, we defined in [R1] certain groups $T^i_j$, $i \geq 0$, which are related to deformations of $f$; in particular, $T^1_j$ is the group of 1st-order deformations of $f$. For our present purposes, it will be necessary to consider the corresponding relative groups $T^i_{f/S}$, which are associated to a diagram

$$\tilde{f}: \mathcal{X} \to \mathcal{Y}$$

with $\mathcal{X}/S$, $\mathcal{Y}/S$ smooth (we call such a map $\tilde{f}$ an $S$-map, or a deformation of $f$).

In the notation of [R1, R2], we have

$$T^i_{f/S} = \text{Ext}^i(\delta_1, \delta_0)$$

where $\delta_0: f^*\mathcal{O}_Y \to \mathcal{O}_X$, $\delta_1: f^*\Omega_{Y/S} \to \Omega_{X/S}$ are the natural maps. As in [R1], we have an exact sequence

$$0 \to T^0_{f/S} \to T^0_{X/S} \oplus T^0_{Y/S} \to \text{Hom}_f(\Omega_{Y/S}, \mathcal{O}_X)$$

$$\to T^1_{f/S} \to T^1_{X/S} \oplus T^1_{Y/S} \to \text{Ext}^1(\Omega_{Y/S}, \mathcal{O}_X) \to \cdots$$

(1.1)

where $T^i_{X/S} = H^i(T_{X/S})$, $T^i_{Y/S}$ being the relative tangent bundle and similarly for $T^i_{Y/S}$, $\text{Hom}_f(\cdot, \cdot)$ and $\text{Ext}^i(\cdot, \cdot)$ are its derived functors.

Now put $S_j = \text{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^j)$. Our main general result, which is an analogue for maps of a result given in [R3] for manifolds, is the following

**Theorem-Construction 1.1.** Suppose given $X_j/S_j$, $Y_j/S_j$ smooth and $f_j: X_j \to Y_j$ an $S_j$-map, for some $j \geq 2$, and let $X_{j-1}/S_{j-1}$, $Y_{j-1}/S_{j-1}$, $f_{j-1}: X_{j-1} \to Y_{j-1}$ be their respective restrictions via the natural inclusion $S_{j-1} \hookrightarrow S_j$. Then

(i) associated to $f_j$ is a canonical element $\alpha_{j-1} \in T^1_{f_{j-1}/S_{j-1}}$;

(ii) given any element $\alpha_j \in T^1_{f_j/S_j}$ which maps to $\alpha_{j-1}$ under the natural restriction map $T^1_{f_j/S_j} \to T^1_{f_{j-1}/S_{j-1}}$, there are canonically associated to $\alpha_j$ deformations $X_{j+1}/S_{j+1}$, $Y_{j+1}/S_{j+1}$ and an $S_{j+1}$-map $f_{j+1}: X_{j+1} \to Y_{j+1}$, extending $X_j/S_j$, $Y_j/S_j$ and $f_j: X_j \to Y_j$ respectively.

The proof is analogous to that of Theorem 1 in [R3] and will be presented elsewhere. In view of this theorem it makes sense to give the following
Definition 1.2. A map $f: X \to Y$ is said to satisfy the $T^1$-lifting property if for any deformation $f_j: X_j/S_j \to Y_j/S_j$ of $f$ and its restriction $f_{j-1}: X_{j-1}/S_{j-1} \to Y_{j-1}/S_{j-1}$, the natural map

$$T^1_{f_j/S_j} \to T^1_{f_{j-1}/S_{j-1}}$$

is surjective.

Abusing terminology somewhat, we will say that $T^1_f$ is deformation-invariant if the groups $T^1_{f_j/S_j}$ are always free $S_j$-modules and their formation commutes with base-change. Note, trivially, that whenever $T^1_f$ is deformation-invariant, $f$ satisfies the $T^1$-lifting property. As an easy consequence of Theorem 1.1, we have the following

Criterion 1.3. Suppose $f: X \to Y$ is a map of compact complex manifolds satisfying the $T^1$-lifting property (e.g. $T^1_f$ is deformation-invariant); then $f$ is unobstructed.

Remark 1.4. Various variants of this criterion are possible, e.g. for deformations of maps $f: X \to Y$ with fixed target $Y$. In the special case that $f$ is an embedding, with normal bundle $N$, we obtain that the Hilbert scheme of submanifolds of $Y$ is smooth at the point corresponding to $f(X)$ provided $H^0(N)$ satisfies the lifting property (e.g. is deformation-invariant). Also, the converse to Criterion 1.3 is trivially true, though we shall not need this.

2. Applications

Unless otherwise specified, all spaces $X, Y$ considered here are assumed smooth.

Theorem 2.1. Let $X$ be a Calabi-Yau manifold and $f: Y \hookrightarrow X$ the inclusion of a smooth divisor. Then $f$ is unobstructed and moreover the image and fibre of the natural map $\text{Def}(f) \to \text{Def}(X)$ are smooth.

Proof. In this case we may identify $T^1_f$ with $H^1(T')$ where $T'$ is defined by the exact sequence

$$(2.1) \quad 0 \to T' \to T_X \to N_{Y/X} \to 0,$$

and it will suffice to prove deformation invariance of $H^1(T')$. Now identifying $T_X \cong \Omega^{n-1}_X$, $N_{Y/X} \cong \Omega^{n-1}_Y$, $n = \dim X$, we may write the cohomology sequence of (2.1) as

$$0 \to H^{n-1,0}(Y) \to H^1(T') \to H^{n-1,1}(X) \xrightarrow{f^*} H^{n-1,1}(Y) \cdots.$$ 

As $H^{n-1,0}(Y)$ and $\ker(f^*)$ are both deformation-invariant, so is $H^1(T')$, hence $f$ is unobstructed, and since moreover the former groups are the respective tangent spaces to the fibre and image of $\text{Def}(f) \to \text{Def}(X)$, the latter are smooth. Q.E.D.

A similar argument can be used to reprove a recent theorem of C. Voisin [V] (see op. cit. for examples and further results):
Theorem 2.2 (Voisin). Let $X$ be a Kähler symplectic manifold, with (everywhere nondegenerate) symplectic form $\omega \in H^0(\Omega^2_X)$, and $f: Y \to X$ a Lagrangian embedding, i.e. $f^* \omega = 0$ and $\dim Y = \frac{1}{2} \dim X$. Then $f$ is unobstructed and the image and fibre of the natural map $\text{Def}(f) \to \text{Def}(X)$ are smooth.

Proof. In this case we may identify $T_X \cong \Omega_X$, $N_{Y/X} \cong \Omega_Y$, and we may argue as in the proof of Theorem 2.1 (note that this property of being Lagrangian is open).

Next we consider deformations of fibre spaces $f: X^n \to Y^m$ with $X$ Calabi-Yau (i.e. $f$ is a flat map whose fibres are reduced and connected). Note that for a fibre space $f$, its general fibre is clearly a Calabi-Yau manifold. Also, it follows easily from the sequence (1.1) that $\text{Def}(f) \to \text{Def}(X)$ is an isomorphism by a theorem of Horikawa [H], hence in that case unobstructedness of $f$ follows from that of $X$. We will consider here two extreme cases: namely $m = n - 1$ and $m = 1$.

Theorem 2.3. Let $f: X \to Y$ be an elliptic fibre space (i.e. general fibre elliptic curve) with $X$ Calabi-Yau. Then $f$ is unobstructed.

Proof. Using the usual exact sequence (1.1) and Criterion 1.3, it suffices to prove the deformation invariance of

$$\ker(H^1(T_X) \to H^0(Y, R^1 f_* \mathcal{O}_X \otimes T_Y)).$$

Now by relative duality we have

$$R^1 f_* \mathcal{O}_X \cong \omega_{X/Y}^{-1} \cong \omega_Y,$$

hence we may identify $\alpha$ with the push-forward map (or “integration over the fibre”)

$$H^{n-1,1}(X) \to H^{n-2,0}(Y),$$

and in particular $\ker \alpha$ is deformation-invariant. (Note that we have $\text{Def}(f) \cong \text{Def}(X)$ whenever $\alpha = 0$, e.g. $H^{n-2,0}(Y) = 0$, which holds whenever $H^{n-2,0}(X) = 0$.)

Theorem 2.4. Let $f: X \to C$ be a fibre space from a Calabi-Yau manifold to a smooth curve. Then $f$ is unobstructed.

Proof. Note that for any fibre $Y$ of $f$ we have

$$h^0(\mathcal{O}_Y(Y)) = h^0(\mathcal{O}_Y) = 1,$$

and it follows that the scheme $\text{Div}^0(X)$ parametrizing reduced connected effective divisors of $X$ is smooth and 1-dimensional locally at the point corresponding to $Y$. Consequently if we denote by

$$p: Z \to \text{Div}^0(X)$$

the universal family and $q: Z \to X$ the natural map, then we have in fact a 1-1 correspondence between morphisms $f: X \to C$ as above and smooth compact connected 1-dimensional components $C \subset \text{Div}^0(X)$ such that $q|_{p^{-1}(C)}$ is an isomorphism. Now it follows from Theorem 2.1 and its proof that for any smooth fibre $Y$ of $f$, the locus $D' \subset \text{Def}(X)$ of deformations over which $Y$ extends is smooth and independent of $Y$. It follows that almost all, hence all, of $C$ as component of $\text{Div}^0(X)$ in fact extends over $D'$, hence so does $f$, so that $D' = \text{Def}(f)$, proving the theorem.

In the intermediate cases, we have only much weaker results:
Theorem 2.5. Let \( f : X \to Y \) be a smooth morphism and assume either

(i) \( K_X \) is trivial; or

(ii) \( K_{X/Y} \) is trivial.

Then \( \text{Def}(f) \to \text{Def}(Y) \) has smooth fibres.

Proof. We will prove (ii), as (i) is similar. It suffices to prove the deformation invariance of \( H^1(T_{X/Y}) \), where \( T_{X/Y} \) is the relative (vertical) tangent bundle. Now we have

\[
T_{X/Y} \cong \Omega^{n-1}_{X/Y} \otimes K_{X/Y}^{-1} \cong \Omega^{n-1}_{X/Y}, \quad n = \dim(X/Y).
\]

By relative Hodge theory, \( H^1(\Omega^{n-1}_{X/Y}) \) is a direct summand of \( H^n(f^{-1}\mathcal{O}_Y) \), and it will suffice to prove the deformation invariance of the latter. We have a Leray spectral sequence

\[
H^p(Y, R^qf_*f^{-1}\mathcal{O}_Y) \Rightarrow H^n(f^{-1}\mathcal{O}_Y).
\]

However \( H^p(Y, R^qf_*f^{-1}\mathcal{O}_Y) = H^{p,0}(Y, R^qf_*\mathbb{C}_X) \) is a direct summand of \( H^p(Y, R^qf_*\mathbb{C}_X) \), hence the degeneration of the Leray spectral sequence of \( \mathbb{C}_X \) implies that of (2.2), hence the deformation invariance of \( H^n(f^{-1}\mathcal{O}_Y) \).

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Added in proof

The above ideas are pursued further in the author’s preprints, Hodge theory and the Hilbert scheme (September 1990) and Hodge theory and deformations of maps (January 1991).

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