DEPTH OF BOOLEAN ALGEBRAS

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Abstract. Suppose $D$ is an ultrafilter on $\kappa$ and $\lambda^\kappa = \lambda$. We prove that if $B_i$ is a Boolean algebra for every $i < \kappa$ and $\lambda$ bounds the Depth of every $B_i$, then the Depth of the ultraproduct mod $D$ is bounded by $\lambda^+$. We also show that for singular cardinals with small cofinality, there is no gap at all. This gives a full answer to this problem in the constructible universe.

2000 Mathematics Subject Classification. Primary: 06E05, 03G05. Secondary: 03E45.

Key words and phrases. Boolean algebras, Depth, Constructibility.

First typed: April 2007

Research supported by the United States-Israel Binational Science Foundation. Publication 911 of the second author.
0. INTRODUCTION

Let \( B \) be a Boolean Algebra. We define the depth of it as the supremum on the cardinalities of well-ordered subsets in \( B \). Now suppose that \( \langle B_i : i < \kappa \rangle \) is a sequence of Boolean algebras, and \( D \) is an ultrafilter on \( \kappa \). Define the ultra-product algebra \( B \) as \( \prod_{i<\kappa} B_i / D \). The question (raised also for other cardinal invariants, by Monk, in [3]) is about the relationship between \( \text{Depth}(B) \) and \( \prod_{i<\kappa} \text{Depth}(B_i) / D \).

Let us try to draw the picture:

\[
\begin{align*}
B &= \prod_{i<\kappa} B_i / D \\
\text{Depth}(B) &= \prod_{i<\kappa} \text{Depth}(B_i) / D \\
\langle B_i : i < \kappa \rangle, D &\rightarrow \langle \text{Depth}(B_i) : i < \kappa \rangle
\end{align*}
\]

As we can see from the picture, given a sequence of Boolean algebras (of length \( \kappa \)) and an ultrafilter on \( \kappa \), we have two alternating ways to produce a cardinal value. The left course creates, first, a new Boolean algebra namely the ultraproduct algebra \( B \). Then we compute the Depth of it. In the second way, first of all we get rid of the algebraic structure, producing a sequence of cardinals (namely \( \langle \text{Depth}(B_i) : i < \kappa \rangle \)). Then we compute the cardinality of its cartesian product divided by \( D \).

Shelah proved in [6] §5, under the assumption \( V = L \), that if \( \lambda = \lambda^\kappa \) and \( \kappa = \text{cf}(\kappa) < \lambda \), then you can build a sequence of Boolean algebras \( \langle B_i : i < \kappa \rangle \), such that \( \text{Depth}(B) > \prod_{i<\kappa} \text{Depth}(B_i) / D \) for every uniform ultrafilter \( D \). This result is based on the square principle, introduced and proved in \( L \) by Jensen.

A natural question is how far can this gap reach. We prove that if \( V = L \) then the gap is at most one cardinal. In other words, for every regular cardinal and for every singular cardinal with high cofinality we can create a gap (having the square for every infinite cardinal in \( L \)), but it is limited to one cardinal.

Observe that the assumption \( V = L \) is just to make sure that every ultrafilter is regular. We observe also that by [7], under some reasonable assumptions, there is no gap at all above a compact cardinal.

We can ask further what happens if \( \text{cf}(\lambda) < \lambda \), and \( \kappa \geq \text{cf}(\lambda) \). We prove here that if \( \lambda \) is singular with small cofinality, (i.e., all the cases which are not covered in the previous paragraph), then \( \prod_{i<\kappa} \text{Depth}(B_i) / D \geq \text{Depth}(B) \).

It is interesting to know that similar result holds above a compact cardinal for singular cardinals with countable cofinality. We suspect that it holds (for such cardinals) in ZFC.
The proof of those results is based on an improvement to the main Theorem in [2]. It says that under some assumptions we can dominate the gap between $\text{Depth}(B)$ and $\prod_{i<\kappa} \text{Depth}(B_i)/D$. In this paper we use weaker assumptions. We give here the full proof, so the paper is self-contained. We intend to shed light on the other side of the coin (i.e., under large cardinals assumptions) in a subsequent paper.
1. The main theorem

**Definition 1.1.** Depth.
Let $B$ be a Boolean Algebra.

$$\text{Depth}(B) := \sup \{ \theta : \exists \bar{b} = (b_\gamma : \gamma < \theta), \text{ increasing sequence in } B \}$$

We use also an important variant of the Depth:

**Definition 1.2.** $\text{Depth}^+(B)$.
Let $B$ be a Boolean Algebra.

$$\text{Depth}^+(B) := \sup \{ \theta^+ : \exists \bar{b} = (b_\gamma : \gamma < \theta), \text{ increasing sequence in } B \}$$

Through the paper, we use the following notation:

**Notation 1.3.**
(a) $\kappa, \lambda$ are infinite cardinals
(b) $D$ is a uniform ultrafilter on $\kappa$
(c) $B_i$ is a Boolean Algebra, for any $i < \kappa$
(d) $B = \prod_{i<\kappa} B_i / D$
(e) for $\kappa < \lambda$, $S_\kappa^\lambda = \{ \alpha < \lambda : \text{cf}(\alpha) = \kappa \}$.

We state our main result:

**Theorem 1.4.** Assume
(a) $\lambda = \text{cf}(\lambda)$
(b) $\lambda = \lambda^\kappa$
(c) $\text{Depth}^+(B_i) \leq \lambda$, for every $i < \kappa$.

Then $\text{Depth}^+(B) \leq \lambda^+$.

**Proof.**
Assume towards a contradiction that $\langle a_\alpha : \alpha < \lambda^+ \rangle$ is an increasing sequence in $B$. Let us write $a_\alpha$ as $\langle a_\alpha^i : i < \kappa \rangle / D$ for every $\alpha < \lambda^+$.

Let $\langle M_\alpha : \alpha < \lambda^+ \rangle$ be a continuous and increasing sequence of elementary submodels of $(\mathcal{H}(\chi), \in)$ for sufficiently large $\chi$ with the following properties $(\forall \alpha < \lambda^+)$: (*)

(a) $\| M_\alpha \| = \lambda$
(b) $\lambda + 1 \subseteq M_\alpha$
(c) $\langle M_\beta : \beta \leq \alpha \rangle \subseteq M_{\alpha + 1}$
(d) $\langle M_\alpha \rangle^\kappa \subseteq M_{\alpha + 1}$.

We may assume that $\langle a_\alpha^i : \alpha < \lambda^+, i < \kappa \rangle \in M_0$. We also assume that $B, \langle B_i : i < \kappa \rangle, D \in M_0$.

We will try to create a set $Z$, in the Lemma below, with the following properties:

($\ast)_2$

(a) $Z \subseteq \lambda^+, |Z| = \lambda$
(b) $\exists i_* \in \kappa$ such that for every $\alpha < \beta, \alpha, \beta \in Z$, we have $B_i_* \models a_\alpha^{i_*} < a_\beta^{i_*}$.
Since \(|Z| = \lambda\), we have an increasing sequence of length \(\lambda\) in \(B_i\), so \(\text{Depth}^+(B_i) \geq \lambda^+\), contradicting the assumptions of the Theorem.

\(\square\)

**Lemma 1.5.** There exists \(Z\) as above.

**Proof.**

For every \(\alpha < \beta < \lambda^+\), define:

\[ A_{\alpha,\beta} = \{ i < \kappa : B_i \models a^\alpha_i < a^\beta_i \} \]

By the assumption, \(A_{\alpha,\beta} \in D\) for all \(\alpha < \beta < \lambda^+\). Define \(C := \{ \gamma < \lambda^+ : \gamma = M_{\alpha} \cap \lambda^+ \}\), and \(S := C \cap S^\lambda_{\ast}\). Since \(C\) is a club subset of \(\lambda^+\), \(S\) is a stationary subset of \(\lambda^+\). Choose \(\delta^*\) as the \(\lambda\)-th member of \(S\). For every \(\alpha < \delta^*\), Let \(A_\alpha\) denote the set \(A_{\alpha,\delta^*}\).

Let \(u \subseteq \delta^*, |u| \leq \kappa\). Notice that \(u \in M_{\delta^*}\), by (d) of \((\ast)_1\) above. Define:

\[ S_u = \{ \beta < \delta^* : \beta > \sup(u) \text{ and } (\forall \alpha \in u)(A_{\alpha,\beta} = A_\alpha) \}. \]

Choose \(\delta_0 = 0\). Choose \(\delta_{\epsilon+1} \in S\) for every \(\epsilon < \lambda\) such that \(\epsilon < \zeta \Rightarrow \sup\{\delta_{\epsilon+1} : \epsilon < \zeta\} < \delta_{\zeta+1}\). Define \(\delta_\epsilon\) to be the limit of \(\delta_{\epsilon+1}\), when \(\gamma < \epsilon\), for every limit \(\epsilon < \lambda\). Since \(\text{otp}(S \cap \delta^*) = \lambda\), we have:

1. (a) \(\delta_\epsilon : \epsilon < \lambda\) is increasing and continuous
2. (b) \(\sup\{\delta_\epsilon : \epsilon < \lambda\} = \delta^*\)
3. (c) \(\delta_{\epsilon+1} \in S\), for every \(\epsilon < \lambda\)

Define, for every \(\epsilon < \lambda\), the following family:

\[ A_\epsilon = \{ S_u \cap \delta_{\epsilon+1} : u \subseteq [\delta_{\epsilon+1}]^{< \kappa} \}. \]

We get a family of non-empty sets, which is downward \(\kappa^+\)-directed. So, there is a \(\kappa^+\)-complete filter \(E_\epsilon\) on \(\delta_{\epsilon+1}\), with \(A_\epsilon \subseteq E_\epsilon\), for every \(\epsilon < \lambda\).

Define, for any \(i < \kappa\) and \(\epsilon < \lambda\), the sets \(W_{\epsilon,i} \subseteq [\delta_\epsilon, \delta_{\epsilon+1}]\) and \(B_\epsilon \subseteq \kappa\), by:

\[ W_{\epsilon,i} := \{ \beta : \delta_\epsilon \leq \beta < \delta_{\epsilon+1} \text{ and } i \in A_{\beta,\delta_{\epsilon+1}} \} \]

\[ B_\epsilon := \{ i < \kappa : W_{\epsilon,i} \in E_\epsilon^+ \}. \]

Finally, take a look at \(W_\epsilon := \cap\{[\delta_\epsilon, \delta_{\epsilon+1}] \setminus W_{\epsilon,i} : i \in \kappa \setminus B_\epsilon\}\). For every \(\epsilon < \lambda\), \(W_\epsilon \in E_\epsilon\), since \(E_\epsilon\) is \(\kappa^+\)-complete, so clearly \(W_\epsilon \neq \emptyset\).

Choose \(\beta = \beta_\epsilon \in W_\epsilon\). If \(i \in A_{\beta,\delta_{\epsilon+1}}\), then \(W_{\epsilon,i} \in E_\epsilon^+\), so \(A_{\beta,\delta_{\epsilon+1}} \subseteq B_\epsilon\) (by the definition of \(B_\epsilon\)). But, \(A_{\beta,\delta_{\epsilon+1}} \subseteq D\), so \(B_\epsilon \subseteq D\). For every \(\epsilon < \lambda\), \(A_{\delta_{\epsilon+1},\delta^*}\) (which is \(A_{\delta_{\epsilon+1},\delta^*}\)) belongs to \(D\), so \(B_\epsilon \cap A_{\delta_{\epsilon+1}} \subseteq D\).

Choose \(i_\epsilon \in B_\epsilon \cap A_{\delta_{\epsilon+1}}\), for every \(\epsilon < \lambda\). You have chose \(\lambda\) \(i_\epsilon\)-s from \(\kappa\), so we can arrange a fixed \(i_\epsilon\) in \(\kappa\) such that the set \(Y = \{ \epsilon < \lambda : \epsilon\text{ is an even ordinal, and } i_\epsilon = i_\ast \}\) has cardinality \(\lambda\).

The last step will be as follows:

define \(Z = \{ \delta_{\epsilon+1} : \epsilon \in Y \}\). Clearly, \(Z \in [\delta^*]^\lambda \subseteq [\lambda^+]^\lambda\). We will show that for \(\alpha < \beta\) from \(Z\) we get \(B_{i_\ast} \models a^\alpha_{i_\ast} < a^\beta_{i_\ast}\). The idea is that if \(\alpha < \beta\) and \(\alpha, \beta \in Z\), then \(i_\ast \in A_{\alpha,\beta}\).
Why? Recall that $\alpha = \delta_{\epsilon+1}$ and $\beta = \delta_{\zeta+1}$, for some $\epsilon < \zeta < \lambda$ (that’s the form of the members of $Z$). Define:

$U_1 := S_{\delta_{\epsilon+1}} \cap [\delta_{\zeta}, \delta_{\zeta+1}) \in \mathfrak{A}_\zeta \subseteq E_{\zeta}$.

$U_2 := \{ \gamma : \delta_{\zeta} \leq \gamma < \delta_{\zeta+1} \text{ and } i_s \in A_{\gamma, \delta_{\zeta+1}} \} \in E_{\zeta}^+$.

So, $U_1 \cap U_2 \neq \emptyset$.

Choose $\iota \in U_1 \cap U_2$.

Now the following statements hold:

(a) $B_{i_\iota} \models a_{i_\iota}^\alpha < a_{i_\iota}^\iota$

[Why? Well, $\iota \in U_1$, so $A_{\delta_{\iota+1}, \iota} = A_{\delta_{\iota+1}}$. But, $i_s \in B_{\iota} \cap A_{\delta_{\iota+1}} \subseteq A_{\delta_{\iota+1}}$, so $i_s \in A_{\delta_{\iota+1}, \iota}$, which means that $B_{i_\iota} \models a_{i_\iota}^{\delta_{\iota+1}} (= a_{i_\iota}^\alpha) < a_{i_\iota}^\iota$.]

(b) $B_{i_\iota} \models a_{i_\iota}^\iota < a_{i_\iota}^\beta$

[Why? Well, $\iota \in U_2$, so $i_s \in A_{\iota, \delta_{\zeta+1}}$, which means that $B_{i_\iota} \models a_{i_\iota}^\iota < a_{i_\iota}^{\delta_{\iota+1}} (= a_{i_\iota}^\beta)$.]

(c) $B_{i_\iota} \models a_{i_\iota}^\alpha < a_{i_\iota}^\beta$

[Why? By (a)+(b)].

So, we are done.
2. Depth in $L$

As a consequence of the main result from the previous section we have, under the constructibility axiom, as follows:

**Theorem 2.1.** (GCH)
Assume

(a) $\kappa < \lambda$
(b) $\text{Depth}(B_i) \leq \lambda$, for every $i < \kappa$.
(c) $\lambda = \lim D(\langle \text{Depth}(B_i) : i < \kappa \rangle)$

Then $\text{Depth}(B) \leq \lambda^+$.

Proof.
For every successor cardinal $\lambda^+$ we have (under the GCH)

$$(\lambda^+)^\kappa = (2^\lambda)^\kappa = 2^{\lambda^\kappa} = 2^\lambda = \lambda^+$$

Clearly, $\lambda^+$ is a regular cardinal, and by (b) we know that $\text{Depth}^+(B_i) \leq \lambda^+$ for every $i < \kappa$. Now apply Theorem 1.4 and conclude that $\text{Depth}^+(B) \leq \lambda^{+2}$, so $\text{Depth}(B) \leq \lambda^+$ as required.

**Remark 2.2.** In $L$ equality holds. The proof is similar to the proof in Theorem 2.3 below.

So if $\lambda$ is regular and $\kappa < \lambda$, or even $\lambda > \text{cf}(\lambda) > \kappa$, we can build in $L$ an example for $\text{Depth}(B) > \prod_{i<\kappa} \text{Depth}(B_i)/D$, but the discrepancy is just one cardinal. We can ask what happens if $\lambda$ is singular with small cofinality. The following Theorem gives an answer. Notice that this answers problem No. 12 from [4], for the case of singular cardinals with countable cofinality.

**Theorem 2.3.** ($V = L$)
Assume

(a) $\text{cf}(\lambda) < \lambda$
(b) $\kappa \geq \text{cf}(\lambda)$
(c) $\text{Depth}(B_i) \leq \lambda$, for every $i < \kappa$
(d) $\lambda = \lim D(\langle \text{Depth}(B_i) : i < \kappa \rangle)$.

Then $\text{Depth}(B) = \prod_{i<\kappa} \text{Depth}(B_i)/D$.

Proof.
First we claim that $\prod_{i<\kappa} \text{Depth}(B_i)/D = \lambda^+$. The basic idea is that in $L$ we know that $D$ is regular (by the fundamental result of Donder, from $\Pi$), so $\prod_{i<\kappa} \text{Depth}(B_i)/D = \lambda^\kappa = \lambda^+$ (recall that $\text{cf}(\lambda) \leq \kappa$).

Now $\text{Depth}(B) \geq \prod_{i<\kappa} \text{Depth}(B_i)/D = \lambda^+$, by Theorem 4.14 from [3] (since $L \models \text{GCH}$). On the other hand, Theorem 2.1 makes sure that $\text{Depth}(B) \leq \lambda^+$ (by (c) of the present Theorem). So $\prod_{i<\kappa} \text{Depth}(B_i)/D = \lambda^+ = \text{Depth}(B)$, and we are done.
We know that if $\kappa$ is less than the first measurable cardinal, then every uniform ultrafilter on $\kappa$ is $\aleph_0$-regular. It gives us the result of Theorem 2.3 for singular cardinals with countable cofinality, if the length of the sequence (i.e., $\kappa$) is below the first measurable. We have good evidence that something similar holds for singular cardinals with countable cofinality above a compact cardinal. Moreover, if $\text{cf}(\lambda) = \aleph_0$ then $\kappa \geq \text{cf}(\lambda)$ for every infinite cardinal $\kappa$. It means that it is consistent with ZFC not to have a counterexample in this case. So the following conjecture does make sense:

**Conjecture 2.4. (ZFC)**

Assume

(a) $\aleph_0 = \text{cf}(\lambda) < \lambda$, and $2^{\aleph_0} < \lambda$
(b) $\kappa < \lambda$
(c) $\text{Depth}(B_i) \leq \lambda$, for every $i < \kappa$
(d) $\lambda = \lim_D(\text{Depth}(B_i) : i < \kappa)$.

Then $\text{Depth}(B) \leq \prod_{i<\kappa} \text{Depth}(B_i)/D$.

Notice that by [5] we know that this question is independent when $2^{\aleph_0} > \lambda$, as follows from Theorem 3.2 there.

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