The Vacuum Structure of $\mathcal{N}=2$ Super–QCD with Classical Gauge Groups

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**Abstract:** We determine the vacuum structure of $\mathcal{N}=2$ supersymmetric QCD with fundamental quarks for gauge groups $SO(n)$ and $Sp(2n)$, extending prior results for $SU(n)$. The solutions are all given in terms of families of hyperelliptic Riemann surfaces of genus equal to the rank of the gauge group. In the scale invariant cases, the solutions all have exact S-dualities which act on the couplings by subgroups of $PSL(2,\mathbb{Z})$ and on the masses by outer automorphisms of the flavor symmetry. They are shown to reproduce the complete pattern of symmetry breaking on the Coulomb branch and predict the correct weak–coupling monodromies. Simple breakings with squark vevs provide further consistency checks involving strong–coupling physics.

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1. Introduction and Summary

Seiberg and Witten [1,2] found the exact vacuum structure and spectrum of four-dimensional $\mathcal{N}=2$ supersymmetric $SU(2)$ QCD. Their analysis was extended to gauge group $SU(r+1)$ with matter in the fundamental representation [3,4,5,6]. In this paper we further extend this analysis to include the simple gauge groups $Sp(2r)$ with matter in the fundamental representation and $SO(n)$ with vector matter.

Following [5], we assume the solutions are uniformly described in terms of the moduli spaces of families of hyperelliptic Riemann surfaces. We then construct the unique solution consistent with this assumption by induction on $r$, the rank of the gauge group, using the patterns of symmetry breaking obtained by condensation of the complex scalar in the adjoint of the gauge group.

The resulting solutions satisfy a number of restrictive consistency requirements. First, they reproduce the complete pattern of symmetry breaking on the Coulomb branch (only a small subset of these breaking patterns are used in the induction argument). Second, there exist meromorphic one-forms on these curves whose periods generate the spectrum of low-energy excitations. These one-forms obey a set of differential equations and conditions on their residues which, for a generic curve, need have no solution. Third, these curves correctly reproduce all the weak-coupling monodromies in the $Sp(2r,\mathbb{Z})$ duality group. Finally, they reproduce the expected pattern of symmetry breaking on the Higgs branches. The latter property is a test of consistency at strong coupling.

The qualitative features of the solutions are similar to those of the $SU(r+1)$ theory: the generic vacuum is $U(1)^r$ $\mathcal{N}=2$ supersymmetric Abelian gauge theory; along special submanifolds of the Coulomb branch there is a rich spectrum of vacua, with massless electrically and magnetically charged states analogous to those studied in [7], new non-trivial fixed points like those studied in [8], as well as nonabelian Coulomb phases; and, in the cases where the beta function vanishes, the solutions exhibit exact scale invariance and strong-weak coupling duality. The duality acts in all cases as a subgroup of $SL(2,\mathbb{Z})$ on the couplings and on the masses by outer automorphisms of the flavor symmetry. The specific solutions follow; we first recall the $SU(r+1)$ solution with coupling $\tau$ [3].
\( SU(r+1) \) with \( N_f = 2r+2 \) fundamental hypermultiplets has the curve and one-form

\[
y^2 = \prod_{a=1}^{r+1} (x - \phi_a)^2 + 4h(h+1) \prod_{j=1}^{2r+2} (x - m_j - 2h\mu), \quad \sum \phi_a = 0,
\]

\[
\lambda = \frac{x - 2h\mu}{2\pi i} d \log \left( \frac{\prod (x - \phi_a) - y}{\prod (x - \phi_a) + y} \right), \quad h(\tau) = \frac{\vartheta_4^4}{\vartheta_2^4 - \vartheta_4^4}.
\] (1.1)

The masses \( m_j \) transform in the adjoint of the \( U(1) \times SU(N_f) \) flavor group, and \( \mu \equiv (1/N_f) \sum m_j \) is the flavor-singlet mass. For \( r=1 \) this curve is equivalent to the 4-flavor \( SU(2) \) curve [2]. For \( r>1 \) it is invariant under a \( \Gamma^0(2) \subset PSL(2, \mathbb{Z}) \) duality group generated by \( T^2: \tau \rightarrow \tau+2 \), and \( S: \tau \rightarrow -1/\tau, \ m_j \rightarrow m_j - 2\mu \). The solutions for the asymptotically free theories with \( N_f < 2r+2 \) flavors are obtained by taking \( 2r+2-N_f \) masses \( \sim M \rightarrow \infty \), while keeping \( \Lambda^{2r+2-N_f} = qM^{2r+2-N_f} \) finite. Another \( SU(3) \) curve has been presented in [3]; it differs from the above curve only by a (non-perturbative) redefinition of the coupling \( \tau \).

\( Sp(2r) \) with \( N_f = 2r+2 \) fundamental hypermultiplets has curve and one-form

\[
xy^2 = \left( x \prod_{a=1}^{r} (x - \phi_a^2) + g \prod_{j=1}^{2r+2} m_j \right)^2 - g^2 \prod_{j=1}^{2r+2} (x - m_j^2),
\]

\[
\lambda = \frac{\sqrt{x}}{2\pi i} d \log \left( \frac{x \prod (x - \phi_a^2) + g \prod m_j - \sqrt{xy}}{x \prod (x - \phi_a^2) + g \prod m_j + \sqrt{xy}} \right), \quad g(\tau) = \frac{\vartheta_4^4}{\vartheta_3^4 + \vartheta_4^4}.
\] (1.2)

(Note that the right side of the curve is divisible by \( x \), and that the \( \sqrt{x} \)'s in \( \lambda \) cancel upon expanding the derivative.) The masses transform in the adjoint of the \( SO(2N_f) \) flavor group. For \( r=1 \) this reduces to the 4-flavor \( SU(2) \) curve [4], which has a \( PSL(2, \mathbb{Z}) \) duality acting on the coupling and transforming the masses by the \( S_3 \) outer automorphisms of the \( SO(8) \) flavor symmetry. For \( r>1 \) this solution is invariant under a \( \Gamma_0(2) \subset PSL(2, \mathbb{Z}) \) duality group generated by \( T: \tau \rightarrow \tau+1, \ \prod m_j \rightarrow -\prod m_j \), and \( ST^2S: \tau \rightarrow \tau/(1-2\tau) \). The solutions for the asymptotically free theories with \( N_f < 2r+2 \) flavors are obtained by taking \( 2r+2-N_f \) masses \( \sim M \rightarrow \infty \), while keeping \( \Lambda^{2r+2-N_f} = qM^{2r+2-N_f} \) finite.
**SO(2r+1)** with \(N_f = 2r-1\) vector hypermultiplets has curve and one-form

\[
y^2 = x \prod_{a=1}^{r} (x - \phi_a^2)^2 + 4fx^2 \prod_{j=1}^{2r-1} (x - m_j^2),
\]

\[
\lambda = \frac{\sqrt{x}}{2\pi i} d \log \left( \frac{x \prod (x - \phi_a^2) - \sqrt{xy}}{x \prod (x - \phi_a^2) + \sqrt{xy}} \right),
\]

\[
f(\tau) = \frac{\vartheta_4^4 \vartheta_4^4}{(\vartheta_2^4 - \vartheta_4^4)^2}.
\]

The masses transform in the adjoint of the \(Sp(2N_f)\) flavor group. For \(r=1\) this reduces to the \(SU(2)\) curve with one adjoint hypermultiplet \([2]\), which has a \(PSL(2,\mathbb{Z})\) duality acting on the coupling. For \(r>1\) this solution is invariant under a \(\Gamma^0(2) \subset PSL(2,\mathbb{Z})\) duality group generated by \(T^2: \tau \rightarrow \tau + 2\), and \(S: \tau \rightarrow -1/\tau\). The solutions for the asymptotically free theories with \(N_f < 2r-1\) flavors are obtained by taking \(2r-1-N_f\) masses \(\sim M \rightarrow \infty\) keeping \(\Lambda^{4r-2-2N_f} = qM^{4r-2-2N_f}\) finite. In the Yang-Mills case \((N_f=0)\) this solution is equivalent to the solution of \([10]\).

**SO(2r)** with \(N_f = 2r-2\) vector hypermultiplets has the curve

\[
y^2 = x \prod_{a=1}^{r} (x - \phi_a^2)^2 + 4fx^3 \prod_{j=1}^{2r-2} (x - m_j^2),
\]

with one-form and \(f(\tau)\) the same as for the \(SO(2r+1)\) case (1.3), and the same duality group as well. In the Yang-Mills case this solution is equivalent to the solution of \([11]\).

The coupling dependence of each of these solutions is given in terms of the usual Jacobi theta functions defined by*

\[
\vartheta_2(\tau) = \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2}, \quad \vartheta_4^4 = 16q + \mathcal{O}(q^3),
\]

\[
\vartheta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad \vartheta_4^4 = 1 + 8q + \mathcal{O}(q^2),
\]

\[
\vartheta_4(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}, \quad \vartheta_4^4 = 1 - 8q + \mathcal{O}(q^2),
\]

* Note that these theta functions are labelled differently from the \(\theta_i\) used in \([3,5]\). The relation between the two is \(\theta_1 = \vartheta_2, \theta_2 = \vartheta_4, \theta_3 = \vartheta_3\).
which satisfy the Jacobi identity $\vartheta_2^4 - \vartheta_3^4 + \vartheta_4^4 = 0$. Here

$$q \equiv e^{i\pi \tau} \equiv e^{i\theta e^{-8\pi^2/g^2}}, \quad (1.6)$$

where $\theta$ is the theta angle and $g$ the gauge coupling. Under the modular transformations $S: \tau \rightarrow -1/\tau$ and $T: \tau \rightarrow \tau + 1$, the theta functions transform as

$$\vartheta_2^4(-1/\tau) = -\tau^2 \vartheta_2^4(\tau), \quad \vartheta_2^4(\tau + 1) = -\vartheta_2^4(\tau),$$

$$\vartheta_3^4(-1/\tau) = -\tau^2 \vartheta_3^4(\tau), \quad \vartheta_3^4(\tau + 1) = \vartheta_3^4(\tau),$$

$$\vartheta_4^4(-1/\tau) = -\tau^2 \vartheta_4^4(\tau), \quad \vartheta_4^4(\tau + 1) = \vartheta_4^4(\tau). \quad (1.7)$$

Two natural extensions of our work would be to apply similar techniques to other groups and other matter representations. Regarding the latter, we can write down solutions in a number of special cases, using the equivalences between low–rank groups. These are $SO(3)$ with adjoint matter, $SO(5)$, $SO(6)$, and $SO(8)$ with spinor matter, $Sp(4)$ with 5’s, $SU(4)$ with 6’s, and $SU(2) \times SU(2)$ with (2,2) matter. These examples should provide useful initial matching conditions for inductive generalizations to higher–rank groups.

Regarding the possibility of extending our results to include all Lie groups, it seems unlikely that the curves for the exceptional groups will be hyperelliptic. The hyperelliptic Ansatz is essentially the simplest assumption we can make about the form of the surface; although it works for the theories analyzed below, we do not know of any physical argument indicating it should be true of other theories as well. One of the main technical reasons it works for the classical groups is that the basis of holomorphic one-forms on a genus-$r$ hyperelliptic surface is $\omega_\ell = x^{r-\ell} dx/y$ for $\ell = 1, \ldots, r$. When $x$ and $y$ are assigned definite dimensions (or R-charges), the set of $\omega_\ell$ have evenly spaced dimensions. There is a natural one-form solving the differential equation $\partial \lambda/\partial s_\ell \propto \omega_\ell$ if the basis of polynomial invariants of the group $s_\ell$, $\ell = 1, \ldots, r$, also has evenly spaced dimensions. This is indeed the case for the classical groups.* Products of simple groups suffer from the same problem. Recent

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* Although $SO(2r)$ has an “extra” invariant $t=\sqrt{s_r}$, dividing by a global $Z_2$ symmetry corresponding to the outer automorphism of $SO(2r)$ that takes $t \rightarrow -t$ allows us to consider a curve depending only on $s_r$; presumably, a curve describing the $SO(2r)$ theory without dividing by this symmetry would not be hyperelliptic.
papers [12,13,14] on the Yang–Mills curves have proposed a uniform framework for all
gauge groups. It would be very interesting to understand the QCD solutions presented
here in terms of this framework, and thus dispense of the need for additional assumptions
on the form of the solution.

At least as interesting as the extension of the results of this paper to other groups
and representations would be the extraction of new physics from them. For example, there
is reason to believe that the $SO(n)$ theories possess a richer phenomenological structure
than the $SU(n)$ theories, based on the $\mathcal{N}=1$ results of [15]. A clearer understanding of
the origin of such $\mathcal{N}=1$ phenomena might be gained within the framework of our $\mathcal{N}=2$
solutions.

In the remainder of this paper we briefly review $U(1)^r$ duality in the low–energy
$\mathcal{N}=2$ supersymmetric effective theory, describe our basic Ansatz for the solution, and then
proceed to derive the curves and one-forms for the $Sp(2r)$, $SO(2r+1)$, and $SO(2r)$ theories
in turn. In each case the consistency requirements on the Coulomb branch mentioned above
are checked. Following our discussion of the various Coulomb branches, we check a Higgs
branch consistency requirement relating the different solutions.

2. $U(1)^r$ Duality and the Hyperelliptic Ansatz

$\mathcal{N}=2$ QCD is described in terms of $\mathcal{N}=1$ superfields by a chiral field strength multiplet
$W$ and a chiral multiplet $\Phi$ both in the adjoint of the gauge group, together with chiral
multiplets $Q^j$ in a representation $R$, and $\bar{Q}_j$ in the complex conjugate representation $\bar{R}$
of the gauge group. The flavor index $j$ runs from 1 to $N_f$. The Lagrangian contains $\mathcal{N}=1$
gauge–invariant kinetic terms for the fields with gauge coupling constant $\tau$ and superpoten-
tial $W = \sqrt{2}\bar{Q}_j\Phi Q^j$. Classically, the global symmetries are the $U(1)\times SU(2)$ chiral
R-symmetry, any outer automorphisms of the gauge group which leave the representation
$R\oplus\bar{R}$ invariant, and the flavor symmetry, a subgroup of $U(2N_f)$ determined by the super-
potential interaction. The $\mathcal{N}=2$ invariant quark mass matrix $M$ is a complex matrix in
the adjoint of the flavor group satisfying $[M, M^\dagger]=0$, implying that $M$ can be taken to be
in the Cartan subalgebra of the flavor group by a flavor rotation. In the quantum theory
the $U(1)$ R-symmetry is generally broken by anomalies to a discrete subgroup.
The theory has a rich vacuum structure consisting of Higgs, Coulomb, and mixed branches. We focus on the Coulomb branch since a nonrenormalization theorem \cite{16} implies that only the Coulomb branch can receive quantum corrections; the Higgs branch is determined by the classical equations of motion alone. We will use this fact in Section 7 to find relations between the solutions for various simple gauge groups. On the Coulomb branch the vevs of the lowest components of the chiral superfields satisfy \( q^j = \tilde{q}_j = 0 \) and \([\phi, \phi^\dagger] = 0\). This implies that \( \phi \) can be chosen by a color rotation to lie in the complexified Cartan subalgebra of the gauge group. \( \langle \phi \rangle \) generically breaks the gauge symmetry to \( U(1)^r \), where \( r \) is the rank of the gauge group, and gives all the quarks masses, so the low energy effective theory is an \( \mathcal{N}=2 \) supersymmetric \( U(1)^r \) Abelian gauge theory. Classically, for special values of \( \langle \phi \rangle \) and the quark masses \( m_j \), the unbroken gauge group will include nonabelian factors or massless quarks.

Assuming \( \mathcal{N}=2 \) supersymmetry is not dynamically broken, the Coulomb vacua are not lifted by quantum effects. At a generic point, the low energy effective Lagrangian can be written in terms of \( \mathcal{N}=2 \) \( U(1) \) gauge multiplets \( (A_\mu, W_\mu) \), where \( \mu, \nu = 1, \ldots, r \) and label quantities associated to each of the \( U(1) \) factors. We denote the scalar component of \( A_\mu \) by \( a_\mu \), which we will also take to stand for its vev. The effective Lagrangian is determined by an analytic prepotential \( \mathcal{F}(A_\mu) \),

\[
L_{\text{eff}} = \text{Im} \frac{1}{4\pi} \left[ \int d^4 \theta A_D^\mu \overline{A}_\mu + \frac{1}{2} \int d^2 \theta \tau^{\mu\nu} W_\mu W_\nu \right],
\]

with dual chiral fields \( A_D^\mu \equiv \partial \mathcal{F} / \partial A_\mu \), and effective couplings \( \tau^{\mu\nu} \equiv \partial^2 \mathcal{F} / \partial A_\mu \partial A_\nu \). Near submanifolds of moduli space where extra states become massless the range of validity of (2.1) shrinks to zero; on these singular submanifolds the effective Lagrangian must be replaced with one which includes the new massless degrees of freedom.

The \( U(1)^r \) theory has a lattice of allowed electric and magnetic charges, \( q^\mu \) and \( h_\mu \). Generically, the bare masses break the flavor symmetry to \( U(1)^{N_f} \), so states have associated quark number charges \( n^j \in \mathbb{Z} \). A BPS saturated \( \mathcal{N}=2 \) multiplet with quantum numbers \( q^\mu \), \( h_\mu \), and \( n^j \) has a mass given by \( M = |a_\mu q^\mu + a_D^\mu h_\mu + m_j n^j| \). The physics described by the \( U(1)^r \) effective theory is invariant under duality transformations \( (S, T) \in Sp(2r; \mathbb{Z}) \times \mathbb{Z}^{N_f} \) which act on the fields and charges as \( a \rightarrow S \cdot a + T \cdot m \), \( h \rightarrow S^{-1} \cdot h \), and \( n \rightarrow -T \cdot h + n \).
Here we have defined the column vectors $a ≡ (a_D^\mu, a_\nu)$, $m ≡ (m_j)$, $h ≡ (h_\mu, q_\nu)$, and $n ≡ (n^j)$. Encircling a singular submanifold in moduli space may produce a non-trivial duality transformation. The monodromy around a submanifold where one dyon with charges $(h, n)$ is massless is

$$S = 1 + h \otimes \delta (J \cdot h), \quad T = n \otimes \delta (J \cdot h), \quad (2.2)$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the symplectic metric. The action of $T$ on the periods corresponds to the freedom to shift the global quark–number current by a multiple of a $U(1)$ gauge current.

Our aim is to determine the analytic prepotential $F$ of the low–energy Abelian theory everywhere on moduli space. Let $\{s_\ell\}$ be good coordinates on the Coulomb branch. We will assume, following [1,2,3,4], that the effective coupling $\tau^{\mu\nu}(s_\ell)$ is the period matrix of a genus $r$ Riemann surface $\Sigma(s_\ell)$ varying holomorphically over moduli space, and the vevs of the scalar fields and their duals are given by $a_D^\mu = \oint a_\lambda^\mu \lambda$ and $a_\nu = \oint b_\lambda^\nu \lambda$, where $\lambda(s_\ell)$ is a meromorphic form on $\Sigma(s_\ell)$. Here $(\alpha^\mu, \beta_\nu)$ is a basis of $2r$ one-cycles on $\Sigma$ with the standard intersection form $\langle \alpha^\mu, \beta_\nu \rangle = \delta_\nu^\mu$, $\langle \alpha^\mu, \alpha^\nu \rangle = \langle \beta_\mu, \beta_\nu \rangle = 0$. The residues at the poles of $\lambda$ must be integral linear combinations of $m_j$, the bare quark masses, and the form must satisfy

$$\frac{\partial \lambda}{\partial s_\ell} = \omega_\ell + df_\ell, \quad (2.3)$$

with $\omega_\ell$ a basis of $r$ holomorphic one-forms on $\Sigma$, and $f_\ell$ arbitrary functions. Duality transformations $(S, T)$ have the effect of redefining the symplectic basis and shifting the winding numbers of the cycles around each of the poles according to (2.2). The condition on the residues of $\lambda$ guarantees that the correct action of $T$ on the vevs is realized.

We will further assume, as in [3,4,5], that $\Sigma$ is a hyperelliptic Riemann surface with polynomial dependence on the coordinates $s_\ell$ and the masses $m_j$. A curve $y^2 = \wp(x)$, where $\wp(x)$ is a polynomial in $x$ of degree $2r+2$, describes a hyperelliptic Riemann surface of genus $r$ as a double–sheeted cover of the $x$-plane branched over $2r+2$ points.
3. **Sp(2r)**

### 3.1. Symmetries

The unitary symplectic group $Sp(2r)$ has rank $r$ and dimension $2r^2 + 2$. The adjoint representation has index $2r + 2$, and the pseudoreal fundamental representation has dimension $2r$ and index $1$. Thus the $\mathcal{N}=2$ beta function for the theory with $N_f$ fundamental hypermultiplets is $i\pi\beta = N_f - 2r - 2$ and the instanton factor is $\Lambda^{2r+2-N_f}$ in the asymptotically free cases.

On the Coulomb branch, the adjoint chiral superfield $\Phi$ has expectation values that can be diagonalized as $\langle \phi \rangle = \text{diag}(\phi_1, \ldots, \phi_r, -\phi_1, \ldots, -\phi_r)$. The gauge–invariant combinations of the $\phi_a$’s are all the symmetric polynomials $s_\ell$ in $\phi_a^2$ up to degree $2r$. These are generated by

$$
\sum_{\ell=0}^{r} s_\ell x^{r-\ell} = \prod_{a=1}^{r} (x - \phi_a^2).
$$

(3.1)

The general (maximal) adjoint breaking is $Sp(2r) \rightarrow Sp(2r-2k) \times SU(k) \times U(1)$, which occurs via an expectation value $\langle \phi \rangle = (M, \ldots, M, 0, \ldots, 0)$ with $r-k$ $M$’s. The fundamental decomposes as $2r = (2r-2k, 1) \oplus (1, 2k) \oplus (1, 2k+1)$ under $Sp(2r-2k) \times SU(k)$.

The flavor symmetry of this theory is $O(2N_f)$, and an $\mathcal{N}=2$ supersymmetric mass term can be skew–diagonalized to masses $\pm m_j, j=1, \ldots, N_f$. The $O(2N_f)$ invariants in terms of these mass eigenvalues are all the symmetric polynomials in the $m_j^2$ up to degree $2N_f-2$, plus the product of the masses $\prod_{j=1}^{N_f} m_j$. To see the $O(2N_f)$ flavor symmetry, define the $2N_f$-component quark $X_a^j = (Q_a^j + i\tilde{Q}_a^j, Q_a^j - i\tilde{Q}_a^j)$ so the kinetic terms are $\mathcal{K} \propto X_a \cdot (X^+)^a$ and the superpotential is $W = X_a \cdot X_b \Phi^{ab}$ since $\Phi^{ab}$ is symmetric. (The adjoint of $Sp(2r)$ is the symmetric product of two fundamentals.) The global symmetry of $\mathcal{K}$ is $U(2N_f)$, while $W$ is left invariant by $O(2N_f, \mathbb{C})$. Their intersection is $O(2N_f)$.

$O(2N_f)$ differs from $SO(2N_f)$ by a global $\mathbb{Z}_2$ acting by interchanging $Q^1 \leftrightarrow \tilde{Q}^1$, leaving the other quarks invariant, which can also be thought of as the action of the outer automorphism of $SO(2N_f)$. This classical $\mathbb{Z}_2$ is anomalous.

### 3.2. Curve and One-Form

We now determine the form of the curve for the $Sp(2r)$ theory by imposing the fol-

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[8]
ollowing requirements: (1) that the form of the curve be uniform in \(r \geq 1\), and (2) that the curve have the form

\[ y^2 = P_{cl}(x, \phi_a) + Q_{qu}(x, \phi_a, m_i, \Lambda), \]  

(3.2)

where the “quantum piece” \(Q_{qu}\) vanishes when \(\Lambda \to 0\) or, in the scale invariant case, \(\tau \to +i\infty\). Only gauge- and flavor–invariant combinations of the \(\phi_a\) and \(m_j\) should appear in (3.2).

In the weak coupling limit \(\Lambda \to 0\) the branch points of this curve are at the zeros of \(P_{cl}\). When two (or more) of these coincide one or more cycles of the Riemann surface degenerate, corresponding to some charged state becoming massless. Such massless states appear at weak coupling whenever two of the \(\phi_a\) coincide (corresponding to an unbroken \(SU(2)\) gauge subgroup of \(Sp(2r)\)) or one of the \(\phi_a\) vanishes (corresponding to an unbroken \(Sp(2)\) gauge group). The symmetry in the \(\phi_a\) together with the fact that the whole curve must be singular as \(\Lambda \to 0\) implies \(P_{cl}\) must be of the form

\[ P_{cl} = x^\epsilon \prod_{a} (x - \phi_a^2)^2 \equiv x^\epsilon P^2, \quad \epsilon = 1 \text{ or } 2. \]  

(3.3)

This has the right degree to describe a curve of genus \(r\) if \(\epsilon=2\); if \(\epsilon=1\) it also describes a genus \(r\) surface, but with one of the branch points fixed at infinity on the \(x\)-plane. Note that \(x\) has dimension two. The known solution [1,2] for the \(SU(2)\approx Sp(2)\) case has one branch point fixed at infinity, so, in order to have a description uniform in \(r\), we choose \(\epsilon=1\).

To determine the “quantum” piece of the curve it is convenient to consider the most general possible form of the curve for \(Sp(2r+2k)\) with \(2r+2\) flavors, and to consider the resulting curve upon breaking to \(Sp(2r)\times SU(k)\times U(1)\). The classical curve will be corrected by powers of the instanton factor \(\Lambda^{2k}\) giving the general form

\[ y^2 = x\tilde{P}^2 + \Lambda^{2k}\tilde{Q}(x, \tilde{\phi}, \tilde{m}) + \Lambda^{4k}\tilde{R}(x, \tilde{\phi}, \tilde{m}) + O(\Lambda^{6k}) \]  

(3.4)

where \(\tilde{Q}\) and \(\tilde{R}\) are functions whose form is to be determined. Break to \(Sp(2r)\times SU(k)\times U(1)\) by setting \(\tilde{\phi}_a = \phi_a\) for \(a=1, \ldots, r\), and \(\tilde{\phi}_a = \phi'_a + M\) for \(a = r+1, \ldots, r+k\) with \(\sum \phi'_a = 0\). We also set \(\tilde{m}_j = m_j\) to ensure that the \(2r+2\) quarks remain light in the \(Sp(2r)\) factor. In
the limit \( M \gg (\phi_a, \phi'_a, m_j) \) the three factors decouple. The semiclassically scale invariant \( Sp(2r) \) factor will have finite coupling \( \tau \) if we send \( \Lambda \to \infty \) keeping \( g \equiv (\Lambda/M)^{2k} \) constant. At weak coupling, a one-loop renormalization group matching implies that \( g(q) \propto q \), which can be corrected by higher powers of \( q \) nonperturbatively.

In this limit the curve should factorize in a way corresponding to the decoupling low-energy sectors. The required factorization implies that the coefficients of positive powers of \( M \) must vanish, so the \( \mathcal{O}(\Lambda^{6k}) \) terms in (3.4) must vanish, and \( \tilde{R} = R(x, m_i) \) cannot depend on \( \tilde{\phi}_a \). Also, since \( \tilde{Q} \) is a symmetric function of the \( \tilde{\phi}_a^2 \), the only terms in \( \tilde{Q} \) which can survive the limit must have the form

\[
\tilde{Q} = \prod_{a=1}^{r+k} \left( Ax + B \sum m_i^2 - \tilde{\phi}_a^2 \right) \cdot Q(x, m_i) + \text{(lower order in } \tilde{\phi}^2),
\]

for some undetermined coefficients \( A, B \) and function \( Q \). The terms lower order in \( \tilde{\phi}^2 \) in general also give contributions surviving the factorization limit, but, fixing \( r \) and taking \( k \) arbitrarily large, fewer and fewer of these terms survive. Since the \( Sp(2r) \) curve should be independent of \( k \), the only contributions must come from the leading term shown in (3.5). Taking \( x \ll M \) and rescaling \( y \to M^{2k}y \) in the limit, the \( Sp(2r) \) curve thus becomes

\[
y^2 = xP^2 + gP' \cdot Q + g^2R, \quad P' \equiv \prod_{a=1}^{r} \left( Ax + B \sum m_i^2 - \phi_a^2 \right),
\]

and \( Q(x, m_i) \), \( R(x, m_i) \) are polynomials invariant under the Weyl group of the flavor symmetry \( SO(4r+4) \) of degrees \( r+1 \) and \( 2r+1 \) in \( x \), respectively.

Consider now the \( Sp(2) \) case \((r=1)\) with no flavors. This is the limit in which we take the four masses \( m_i=M \to \infty \) and the coupling \( \tau \to +i\infty \) such that \( qM^4 = \Lambda^4 \) is constant. The general form for \( Q \) in this case is \( Q = \alpha x^2 + \beta M^2 x + \gamma M^4 \) with some unknown constants \( \alpha, \beta, \gamma \). Then from (3.6) the \( R \) term drops out in the limit and the \( Sp(2) \) no-flavor curve is

\[
y^2 = x(x+u)^2 + \gamma(Ax+u)\Lambda^4 + \beta B x \Lambda^4 + \gamma B A^4 M^2,
\]

where we have defined \( u = -\phi_1^2 \), the \( Sp(2) \cong SU(2) \) gauge–invariant coordinate on the Coulomb branch. For this limit to be consistent, the last term must vanish (since \( M \to \infty \)), so either \( \gamma = 0 \) or \( B = 0 \). Comparing (3.7) to the \( SU(2) \) no flavor curve \( y^2 = \tilde{x}^2(\tilde{x} - u) + \Lambda^4 \tilde{x} \)
found in [2], we see that they are equivalent only if $B=0$, $A=1$, and we shift $x=\tilde{x}-u$. Thus we learn that $P'=P$ in (3.6).

Consider next the breaking of $Sp(2r)$ with $2r+2$ flavors to $Sp(2r-2)$ with $2r$ flavors. This is achieved by taking $\phi_r, m_{2r+1}, m_{2r+2} \sim M \to \infty$ and keeping the coupling, and therefore $g(q)$, finite. For this limit of (3.6) not to be singular, we must have $Q(x, m_i) = Cx^{r+1} + Dx^r \sum m_i^2 + E \prod m_i$, since all other $SO(2r+2)$ invariants would contribute higher powers of $M$ than $M^2$. But for $Q$ to preserve its form under this reduction, we must have $C=D=0$. Absorbing the constant $E$ into our definition of $g$, we have found, so far, that the $Sp(2r)$ curve with $2r+2$ flavors has the form

$$y^2 = xP^2 + 2gP \cdot Q + g^2 R, \quad P \equiv \prod_{a=1}^r (x - \phi_a^2), \quad Q \equiv \prod_{i=1}^{2r+2} m_i. \quad (3.8)$$

To further constrain the curve we construct the one-form $\lambda$. A basis of holomorphic one-forms on our hyperelliptic curve are $\omega_\ell = x^{r-\ell} dx/y$ for $\ell=1,\ldots,r$. From the definition of the $s_\ell$ (3.4), it follows that $\partial P/\partial s_\ell = x^{r-\ell}$ and $\partial y/\partial s_\ell = (1/y)(x^{r-\ell+1} P + x^{r-\ell} Q)$, so it is straightforward to integrate the differential equation (2.3) to find the solution, up to a total derivative,

$$\lambda = a \frac{dx}{2 \sqrt{x}} \log \left( \frac{xP + gQ + \sqrt{xy}}{xP + gQ - \sqrt{xy}} \right), \quad (3.9)$$

which has logarithmic singularities when

$$(xP + gQ + \sqrt{xy}) \cdot (xP + gQ - \sqrt{xy}) = g^2 (Q^2 - xR) = 0. \quad (3.10)$$

These logs can be converted into poles by adding the total derivative $d[a \sqrt{x} \log((xP+gQ-\sqrt{xy})/(xP+gQ+\sqrt{xy}))/\lambda]$ to $\lambda$. Denoting by $\epsilon^\pm_i$ the roots of the two factors in (3.10), the resulting form has poles $\pm a \sqrt{\epsilon^\pm_i} dx/(x - \epsilon^\pm_i)$. For the residues to equal the masses, we must have $\epsilon^\pm_i \propto m_i^2$. The flavor symmetry then implies $Q^2 - xR = \prod(x - m_i^2)$, where we have rescaled the masses and $x$ to fix the coefficients; this in turn implies $a=1/2\pi i$.

 Putting this all together gives the curve and one-form (1.2).

3.3. $\tau$-Dependence and S-Duality

It still remains to determine the coupling constant dependence of the coefficient $g(q)$. In principle $g$ could depend on $r$ as well as on $q$. We first determine its $r$-dependence by induction in $r$, then determine the $q$-dependence by matching to the $r=1$ case.
The induction proceeds by considering the breaking of $Sp(2r)$ with $N_f = 2r+2$ down to $Sp(2r-2) \times U(1)$ with $2r$ light hypermultiplets transforming as $(2r-2,0)$. Set

$$\phi_a = \begin{cases} 
\phi'_a/M & a = 1, \ldots, r-1, \\
\phi'_a & a = r,
\end{cases}$$

$$m_j = \begin{cases} 
h(q) m'_j & j = 1, \ldots, 2r, \\
k(q) M & j = 2r+1, 2r+2,
\end{cases}$$

(3.11)

where $h(q), k(q) = 1+O(q)$. Then the limit $M \to \infty$ keeping $\phi'_a$ and $m'_j$ fixed achieves the desired breaking. The matching conditions for the $\phi_a$ in (3.11) define the breaking we are considering. The function $h(q)$ in the matching for the masses can be absorbed in a redefinition of the masses, so can be defined to be $h(q)=1$. There is a single mass renormalization $h(q)$ for all the light masses since we are respecting the low-energy global flavor symmetry which is a simple group. We are free to choose the function $k(q)$ as well, as it defines the matching between the high- and low-energy scale theories. The simplest choice is $k(q)=1$. One then finds that (1.2) reduces to a curve of the same form with $r \to r-1$ and $g_{r-1}(q_{r-1}) = g_r(q_r)$. The one-loop renormalization group matching condition that $g_r(q_r) \propto q_r$ independent of $r$ implies that the bare couplings satisfy $\tau_{r-1} = \tau_r$ at weak coupling. Nonperturbatively this relation can be modified by positive powers of $q_r$ [17]: $q_{r-1} = q_r \ell(q_r)$ with $\ell(q) = 1+O(q)$. The choice of the function $\ell(q)$ is arbitrary; one can view it as a renormalization prescription defining what is meant by the coupling nonperturbatively. Our prescription will be to choose $\ell(q)=1$, in other words we choose $\tau_{r-1} = \tau_r$ nonperturbatively.

It is clear that the above renormalization prescription is consistent; however, many other consistent possibilities exist. For instance, matching with $k \neq 1$ (and $\ell=1$) gives $g_{r-1} = k^2 g_r$. If the function $k$ is chosen to be singular enough, $g_{r-1}$ can have different modular properties than $g_r$. (In particular, $k(q)$ will need to have infinitely many poles in the $|q|<1$ disk, which accumulate on the boundary.) Which coupling dependence $g(q)$ of our curves is the right one? One cannot answer this question in the present framework since there we have no independent definition of what the coupling $\tau$ means (away from $\tau = +i\infty$). If one were directly computing the effective theory from a specific (e.g. lattice, string theory) regularization of the theory at high energies, then there would be a correct answer; however, this answer could depend on the regulator. With the less direct methods
we have at our disposal at present, we will be content to take the above, simplest, matching condition to determine the strong–coupling and modular behavior of our scale invariant solutions.

We determine the unknown function $g(q)$ by matching to Seiberg and Witten’s $SU(2) \simeq Sp(2)$ solution. Take the 4-flavor curve as given in Eq. (16.38) of [2], and make the following redefinitions, using their notation:

\[ y^2 \rightarrow \frac{c_1^2}{c_1^2 - c_2^2} y^2, \quad x \rightarrow x + \frac{c_1^2}{c_1^2 - c_2^2} u, \]
\[ u \rightarrow \frac{c_1}{c_1^2 - c_2^2} u, \quad m_i \rightarrow \frac{1}{\sqrt{c_1^2 - c_2^2}} m_i. \]

Then their curve becomes precisely the $r=1$ curve in (1.2) with $g = (c_2/c_1) = \vartheta_2^4 / (\vartheta_3^4 + \vartheta_4^4)$, in terms of the usual Jacobi theta functions (1.5).

It follows from (1.7) that $g$ is invariant under $T^2$ and $ST^2S$, while $T: g \rightarrow -g$. The curve is invariant under this sign change if, at the same time, the sign of a single mass is changed. Note that this sign change is not part of the nonanomalous $SO(4r+4)$ flavor symmetry (whose Weyl group contains only pairwise sign flips of the masses), but instead is the $\mathbb{Z}_2$ outer automorphism of the group. $T$ and $ST^2S$ generate a duality group $\Gamma_0 \subset PSL(2, \mathbb{Z})$ which can be characterized as the set of $SL(2, \mathbb{Z})$ matrices whose lower off–diagonal element is even. This should be contrasted with the $Sp(2)$ case [2], where the duality group is all of $PSL(2, \mathbb{Z})$, and is mixed with the $S_3$ outer automorphisms of the $SO(8)$ flavor group.

3.4. Checks

In the course of deriving the form of the $Sp(2r)$ curve above, we checked that the adjoint breaking $Sp(2r) \rightarrow Sp(2r-2) \times U(1)$ was consistently reproduced by our solution. We now check that the other adjoint breaking $Sp(2r) \rightarrow SU(r) \times U(1)$ is also reproduced. The semiclassical breaking of $Sp(2r)$ with $N_f = 2r+2$ down to $SU(r) \times U(1)$ with $2r$ light hypermultiplets transforming as $(r, 0)$ is achieved by tuning

\[ \phi_a = M + \tilde{\phi}_a \quad \quad \sum_{a=1}^{r} \tilde{\phi}_a = 0, \]
\[ m_j = \begin{cases} M + \tilde{m}_j + 2h(q) \tilde{\mu} & j = 2r+1, 2r+2, \\ 0 & \sum_{j=1}^{2r} \tilde{m}_j, \end{cases} \]

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in the limit $M \to \infty$ keeping $\tilde{\phi}_a$ and $\tilde{m}_j$ fixed, and where $h(q) \sim O(q)$. The relative renormalization $h(q)$ of the flavor–singlet mass reflects the fact that the global flavor symmetry of the $SU(r)$ theory is not a simple group. Substituting (3.13) into (1.2), shifting $x \to M^2 + 2M\tilde{x}$, $y \to Mr + \tilde{y}$, expanding around $|\tilde{x}| \ll |M|$, and tuning $h(q)$ appropriately, we indeed recover the $SU(r)$ curve (1.1). The other maximal adjoint breakings, $Sp(2r) \to Sp(2r−2k) \times SU(k) \times U(1)$, are equally easy to check.

The $Sp(2)$ curve should also be equivalent to the $SU(2)$ curve (1.1). If we write the latter curve as

$$y^2 = (x^2 - u)^2 + 4h(h+1) \prod_{j=1}^{4} \left(x - m_j - \frac{1}{2}h(\sum_k m_k)\right),$$

(3.14)

where $u = -\phi_1 \phi_2 = \phi_1^2$, and the $Sp(2)$ curve (1.2) as

$$xy^2 = \left(x(x - \tilde{u})^2 + g\prod_j \tilde{m}_j\right)^2 - g^2 \prod_{j=1}^{4} \left(x - \tilde{m}_j^2\right),$$

(3.15)

with $\tilde{u} = \phi_1^2$, then the discriminants of the two curves are the same if we relate the parameters by $\tilde{u} = \beta^2[4u - h(h+1)(\sum_j m_j)^2 + h \sum_j m_j^2]$, $\tilde{m}_j = 2\beta(1+\frac{3}{2}h)^{1/2}m_j$ with $\beta = (1+h)(1+\frac{3}{2}h)(1+2h)$. This reproduces the expected weak–coupling matching as $h \to 0$. The equality of the discriminants for these two tori imply that they are related by an $SL(2, \mathbb{C})$ transformation of $x$, and incidentally shows the equivalence of the $SU(2)$ solution (1.1) with the results of [2].

Another check on the validity of our solution is that it correctly reproduces the positions and monodromies of singularities at weak coupling. We will check two classes of such singularities: the gauge singularities which correspond classically to the restoration of a nonabelian gauge symmetry, and the quark singularities which correspond to hypermultiplets becoming massless.

For $Sp(2r)$ the gauge singularities occur whenever $\phi_a^2 = \phi_b^2$ or $\phi_a = 0$. Because the beta function vanishes, the semiclassical monodromies around the gauge singularities are actually the classical monodromies given by elements of the Weyl group of $Sp(2r)$, which act by permuting the $\phi_a$’s or flipping their signs. The breakings (3.11) and (3.13) imply that all the $Sp(2r−2)$ and $SU(r)$ singularities and associated monodromies are reproduced
by the $Sp(2r)$ curve, allowing us to check the gauge monodromies by induction in $r$. We need only compute for $Sp(2r)$ a generating monodromy not contained in the Weyl group of $Sp(2r-2)$. A convenient choice is a Coxeter element of the Weyl group corresponding to a cyclic permutation of the $\phi_a$ and a sign change of one element, which gives the monodromy $S = \begin{pmatrix} P^{-1} & 0 \\ 0 & P \end{pmatrix}$, where $P$ is the $r \times r$ matrix representation of the Coxeter element (i.e. its action on the $\phi_a$’s).

For weak coupling, $|q| \ll 1$, and vevs much larger than the bare masses $\phi_a \gg m_i$, the curve is approximately $y^2 = x \prod (x-\phi_a^2)^2 - q^2 x^{2r+1}$. Degenerations where two branch points collide occur whenever $\phi_a^2 = \phi_b^2$ or $\phi_a^2 = 0$, up to corrections of order $q$, corresponding to the semiclassical positions of the gauge singularities. The special monodromies can be conveniently measured by traversing a large circle in the $s_r$ complex plane, fixing the other $s_\ell = 0$, where the curve factorizes as $y^2 = x[(1 - q)x^{r} + s_r] \cdot [(1 + q)x^{r} + s_r]$. The branch points are arranged in $r$ pairs with a pair near each $r$th root of unity times $s_r^{1/r}$. As $s_r \rightarrow e^{2\pi i} s_r$, these pairs rotate into one another in a counterclockwise sense. In addition there is one branch cut extending from the origin to infinity. Choose cuts and a basis for the cycles as in the corresponding argument for the $SU(r)$ curve. As $s_r \rightarrow e^{2\pi i} s_r$, the cycles are cyclically permuted, and a pair of them pick up a minus sign as they pass through the extra cut extending from the origin, thus giving the the classical monodromy predicted above.

Classically, quark singularities occur whenever $\phi_a = \pm m_i / \sqrt{2}$, corresponding to the $q_i^a$, $\tilde{q}_i^a$ hypermultiplets becoming massless. In the effective theory, the massless quark can be taken to have electric charge one with respect to a single $U(1)$ factor and to carry quark numbers $n_j = \delta_j^1$. The semiclassical monodromy around the quark singularity can be read off from (2.2).

Consider the curve near a classical quark singularity, say $\phi_1 \sim m_1 / \sqrt{2}$. At weak coupling and for $x \sim \phi_1^2$ the curve is approximately $y^2 = [C_1(x-\phi_1^2)-qC_2]^2 - q^2 C_3(x-m_1^2)$, where the $C_i$ are slowly-varying functions of $x$ and $s_\ell$. This has a double zero at $x = \frac{1}{2} m_1^2 + \frac{1}{2} q^2 (C_2/C_1)$ for $\phi_1^2 = \frac{1}{2} m_1^2 - q(C_2/C_1) - \frac{1}{2} q^2 (C_3/C_1^2)$, which is indeed near the classical quark singularity for small $q$. Define the period $a_1$ by a contour enclosing the pole at $m_1^2$ (recall that changing which poles are enclosed by a given contour corresponds
to a physically unobservable redefinition of the quark number charges). One then finds that as $\phi_1$ winds around the singular point the two branch points are interchanged. The monodromy which follows from this is nontrivial only in a $2 \times 2$ block of $[2,2]$, for which we find $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, in agreement with the semiclassical prediction.

4. SO($2r+1$)

The arguments in this case are essentially the same as in the $Sp(2r)$ case, so we will run through them more quickly. Also, there is some simplification compared to the $Sp(2r)$ case due to the fact that the flavor symmetry in the $SO(n)$ case does not admit the “extra” invariant $\prod m_j$.

4.1. Symmetries

$SO(2r+1)$ has rank $r$ and dimension $r(2r+1)$, its adjoint representation has index $4r-2$, and the $2r+1$-dimensional vector representation is real and has index $2$. The beta function for the theory with $N_f$ vector hypermultiplets is then $i\pi \beta = 2N_f - 4r + 2$, and the instanton factor is $\Lambda^{4r-2-2N_f}$.

On the Coulomb branch, the adjoint chiral superfield $\Phi$ expectation value can be skew–diagonalized as

$$\langle \phi \rangle = \begin{pmatrix} 0 & \phi_1 \\ -\phi_1 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \phi_r \\ -\phi_r & 0 \\ 0 \end{pmatrix}.$$ (4.1)

The gauge invariant combinations of the $\phi_a$’s are all the symmetric polynomials in $\phi_a^2$ up to degree $2r$. The generating polynomial for these invariants is $\prod_a (x - \phi_a^2) \equiv \sum_{\ell=0}^r s_\ell x^{r-\ell}$. The general (maximal) adjoint breaking is $SO(2r+1) \rightarrow SO(2k+1) \times SU(r-k) \times U(1)$, which occurs via an expectation $\langle \phi \rangle = (M, \ldots, M, 0, \ldots, 0)$ with $r-k$ $M$’s. The vector decomposes as $2r+1 = (r-k, 1) \oplus (r-k, 1) \oplus (1, 2k+1)$ under $SU(r-k) \times SO(2k+1)$.

The flavor symmetry is $Sp(2N_f)$ and the $N=2$ invariant masses transform in the adjoint representation, and can be diagonalized to $\pm m_i$, $i=1, \ldots, N_f$. The $Sp(2N_f)$ invariants are all the symmetric polynomials in the $m_i^2$ up to degree $2N_f$. To see the $Sp(2N_f)$
flavor symmetry, define \( X = (Q, \tilde{Q}) \), so \( W = X_a \cdot J^i X_b \Phi^{ab} \) where \( J^i_r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes 1_{N_f} \), the symplectic metric, since the adjoint \( \Phi^{ab} \) is antisymmetric. So, the global symmetry is \( U(2N_f) \cap Sp(2N_f, \mathbb{C}) \equiv USp(2N_f) \), the unitary symplectic group.

4.2. Curve and One-Form

As in the \( Sp(2r) \) case, we impose that the curve be hyperelliptic of the form (3.2). The same argument gives the “classical” piece as (3.3). In the scale invariant case with \( 2r-1 \) masses there will be an overall factor of \( x^\epsilon \) in the curve. \( \epsilon = 2 \) would make the curve singular everywhere, so we must take \( \epsilon = 1 \). The “quantum” piece is determined by considering the most general curve for \( SO(4r+1) \) with \( 2r-1 \) flavors and breaking to \( SU(r) \times SO(2r+1) \times U(1) \) at a large scale \( M \), by letting \( \phi_a \rightarrow M \) for \( a = 1, \ldots, r \). In the limit \( M \gg (\phi_a, m_j) \) the three factors decouple. To obtain the \( SO(2r+1) \) factor with \( 2r-1 \) flavors at finite coupling \( \tau \), we should send \( \Lambda \rightarrow \infty \) such that \( (\Lambda/M)^{4r} f(q) \sim O(q) \), by a one-loop renormalization group matching. Taking the limit \( M \rightarrow \infty \), the classical piece factorizes into a piece with \( 2r+1 \) zeros near \( x = 0 \) (relative to the scale \( M \)) and another piece with zeros all of order \( M \). The whole curve should factorize in this way to correspond to the decoupling low-energy sectors. This implies that the scale invariant curve must have the form \( y^2 = xP^2 + 4fQ \) where \( Q = Q(x, m_j^2) \) is symmetric in the \( m_j^2 \) and homogeneous of degree \( 2r \). \( Q \) is fixed by integrating (2.3) to find

\[
\lambda \propto \left( \frac{dx}{\sqrt{x}} \right) \log\left[ \frac{(xP - \sqrt{xy})/(xP + \sqrt{xy})}{(xP - \sqrt{xy})/(xP + \sqrt{xy})} \right].
\]

The logarithmic singularities at \( x^2 = \epsilon_j \), the zeros of \( Q \), are converted into poles by integrating by parts. The residues of \( \lambda \) are linear in the quark masses if \( \epsilon_j \propto m_j^2 \), and the only flavor-symmetric \( Q \) with this property is \( Q = x^2 \prod_j (x - m_j^2) \). Putting this all together gives the \( SO(2r+1) \) curve and one-form (1.3).

4.3. \( \tau \)-Dependence and S-Duality

We still need to determine \( f(q) \) in (1.3). In principle, \( f \) could depend on \( r \) as well as \( q \). We determine its \( r \)-dependence by considering the breaking of \( SO(2r+1) \) with \( N_f = 2r-1 \) down to \( SO(2r-1) \times U(1) \) with \( 2r-3 \) light hypermultiplets transforming as \( (2r-1, 0) \). To this end, set the parameters as in (3.11). In the limit \( M \rightarrow \infty \) one finds that the \( SO(2r+1) \) curve reduces to a curve of the same form with \( r \rightarrow r-1 \) and \( f_{r-1} = f_r \). The one-loop

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renormalization group matching condition that \( f(q) \propto q \) independent of \( r \) implies that the bare couplings satisfy \( \tau_r = \tau_{r-1} \) at weak coupling. We choose this as our renormalization prescription at strong coupling as well.

We determine the unknown function \( f(q) \) by matching to the solution [2] of the SU(2) theory with a massless adjoint hypermultiplet. It will be easiest to match to the SU(2) curve in the form (1.1). According to [2], the SU(2) solution with a single adjoint hypermultiplet of mass \( \tilde{m} \) is given by the 4-fundamental-flavor solution with masses \( (\tilde{m}, \tilde{m}, 0, 0) \). With these masses, (1.1) becomes

\[
y^2 = (x^2 - u)^2 + 4h(h+1)(x - (h+1)\tilde{m})^2(x - h\tilde{m})^2, \tag{4.2}
\]

where we have defined the gauge–invariant coordinate on the Coulomb branch as \( u = -\phi_1 \phi_2 = \phi_1^2 \). The discriminant of (4.2) is

\[
\Delta_{su} = 4096h^2(h+1)^2(2h+1)^2(u - h^2\tilde{m}^2)^2(u - h(h+1)\tilde{m}^2)^2(u - (h+1)^2\tilde{m}^2)^2. \tag{4.3}
\]

Our SO(3) curve, on the other hand, is

\[
y^2 = x(x - v)^2 + 4fx^2(x - m^2), \tag{4.4}
\]

where \( v = \phi_1^2 \) is the gauge–invariant coordinate, and its discriminant is

\[
\Delta_{so} = -16fv^4(v^2 - m^2v - fm^4). \tag{4.5}
\]

The most general relation between the coordinates and masses allowed by dimensional considerations and agreeing with the weak–coupling limit is

\[
v = A(q)u + B(q)\tilde{m}^2, \quad m^2 = C(q)\tilde{m}^2, \tag{4.6}
\]

where \( A(q), C(q) = 1 + O(q) \) and \( B(q) \sim O(q) \).

Now, if the two tori (4.2) and (4.4) are equivalent by an SL(2, \( \mathbb{C} \)) transformation of \( x \), then their discriminants should be equal for some \( f(q) \) after a suitable change of variables (4.6). However, it is clear that this is impossible, since (4.3) has three double zeros in \( u \), while (4.5) has one quartic and two single zeros in \( v \). Nevertheless, the two curves are
equivalent, being related by an isogeny: for fixed values of the parameters, each is a double
cover of the other. (An example of isogenous descriptions of the same physics appeared in
[2].) More explicitly, the double cover of the $SO(3)$ torus (4.4) is given by
\[ y^2 = (z^2 - v)^2 + 4fz^2(z^2 - m^2), \] (4.7)
which is obtained from (4.4) by dropping the overall factor of $x$ and then replacing $x \rightarrow z^2$.
The fact that (4.4) is cubic in $x$ and has an overall factor of $x$ implies that two of its four
branch points are at $\infty$ and 0 on the $x$-plane, independent of the values of the parameters.
(By an $SL(2, \mathbb{C})$ transformation any torus can be brought to this form.) Moving to a double
cover of the $x$-plane by $x = z^2$ then effectively removes the branch points at $x = 0, \infty$,
leaving the isogenous curve (4.7). Its discriminant is
\[ \Delta_{so} = 4096f^2(1+4f)v^2(v^2 - m^2v - fm^4)^2. \] (4.8)
This indeed becomes (1.3) under a change of variables (4.6) with $A = C = 1$, $B = h(h+1)$,
and $f = h(h+1) = \vartheta_2 \vartheta_4^4/(\vartheta_2^4 - \vartheta_4^4)^2$.

The modular transformation properties of the theta functions (1.7) imply that $f$, and
thus the $SO(2r+1)$ curve and one-form, are invariant under $T^2$ and $S$, which generate a
duality group $\Gamma^0(2) \subset PSL(2, \mathbb{Z})$ characterized as the set of $SL(2, \mathbb{Z})$ matrices with even
upper off–diagonal element. For $r=1$, the curve is in fact invariant under all of $PSL(2, \mathbb{Z})$,
though this is not manifest in the form given in (1.3). One way of seeing this is to note that
(by an argument similar to the one given above) the $SO(3)$ curve is isogenous to the $Sp(2)$
curve with masses $(m, m, 0, 0)$, and this can be rewritten in a manifestly $SL(2, \mathbb{Z})$-invariant
form using the change of variables (3.12).

4.4. Checks

It is easy to check that the curve (1.3) reproduces the maximal adjoint breaking
patterns $SO(2r+1) \rightarrow SO(2r-2k+1) \times SU(k) \times U(1)$ in a manner similar to the analogous
check for the $Sp(2r)$ curves in section 3.4. The positions and monodromies of gauge and
quark singularities also match with perturbation theory at weak coupling. Here the gauge
singularity monodromy is precisely the same as in the $Sp(2r)$ case, since both curves have
the same limit at weak coupling and small bare quark masses, and because their Weyl
groups are the same. Checking the quark singularities also involves a calculation very
similar to (though slightly simpler than) the $Sp(2r)$ case, which we will not repeat.

Finally, in the limit that we take all the bare quark masses large (and $q \to 0$ appropriately), our curve should describe pure $SO(2r+1)$ Yang–Mills theory. The curve is, in this limit,

$$y^2 = x \prod_{a=1}^{r} (x - \phi_a^2)^2 + x^2 \Lambda^{4r-2}.$$  

(4.9)

A seemingly different $SO(2r+1)$ Yang–Mills curve,

$$y^2 = \prod_{a=1}^{r} (z^2 - \phi_a^2)^2 + z^2 \Lambda^{4r-2},$$  

(4.10)

was proposed in [10]. In fact, the two curves are equivalent, the second being a double cover
of the first. Indeed, one can transform the first curve into the second by the same “isogeny”
change of variables as we used to go from (4.4) to (4.7). In general, this transformation
takes us from a genus-$r$ curve to a genus $2r-1$ curve with a $Z_2$ symmetry. In order for the
higher–genus curve to reproduce the periods (and hence the physics) of the lower–genus
curve, we must divide by the $Z_2$ symmetry just as was done in [10].

5. $SO(2r)$

The argument and results for $SO(2r)$ closely parallel those of $SO(2r+1)$. Since $SO(2r)$
can not be obtained from $SO(2r+1)$ by adjoint symmetry breaking, we need to give a new
induction and matching argument. In section 6, we will see how to obtain the $SO(2r)$
curve directly from $SO(2r+1)$ by giving an expectation value to a squark in the vector
representation.

The adjoint representation has dimension $r(2r-1)$ and index $4r-4$; the vector repre-
sentation has dimension $2r$ and index $2$. Thus the $\mathcal{N}=2$ beta function is $i\pi \beta = 2N_f - 4r + 4$
and the instanton factor is $\Lambda^{4r-4-2N_f}$. As we did for $SO(2r+1)$, let $\phi_1, \ldots, \phi_r$ be the skew–
diagonal entries of the $2r \times 2r$ matrix $\langle \phi \rangle$. The Weyl group is generated by permutations
and by simultaneous sign changes of pairs of the $\phi_a$, so the symmetric polynomials $s_\ell$ of the
\( \phi_a^2 \), generated by \( \sum_{\ell} s_{\ell} x^{r-\ell} = \prod_a (x-\phi_a^2) \), are gauge-invariant. In addition to the \( s_{\ell} \), there is one “extra” Weyl invariant \( t = \phi_1 \phi_2 \cdots \phi_r \), which might be expected to appear in the curve. However, \( SO(2r) \) also possesses an outer automorphism corresponding to reflection of the Dynkin diagram about its principal axis, which interchanges the two spinor roots and takes \( t \to -t \). This additional global symmetry means that our curve can be taken to depend only on \( s_r = t^2 \). As was the case for \( SO(2r+1) \), the flavor symmetry is \( Sp(2N_f) \); hence the flavor–invariant mass combinations are again the symmetric polynomials in the masses, up to degree \( 2N_f \).

By following the argument of the previous section, we can deduce that the scale invariant curve takes the form \( y^2 = x P^2 - g^2 Q \) with \( P \) as before and \( Q = Q(x, m_j^2) \) a symmetric function of the \( m_j \) of degree \( 2r+1 \) in \( x \). The differential form \( \lambda \) is the same as in the \( SO(2r+1) \) case, implying that \( Q \) has zeroes at \( x = m_j^2 \) and therefore that \( Q = x^3 \prod_{j=1}^{2r-2} (x-m_j^2) \). We thus obtain a curve of the form (1.4). One unexpected feature of this solution is that the “classical” piece of the curve \( xP = x \prod (x-\phi_a) \) has singularities whenever any one \( \phi_a = 0 \), in addition to singularities whenever \( \phi_a = \phi_b \). The latter corresponds to an enhanced gauge group classically, but there is no such enhanced symmetry when \( \phi_a = 0 \). Therefore, for this curve to be correct there must be no monodromy \( i.e. \), only a trivial monodromy) around such singularities. In terms of gauge invariant parameters, \( \phi_a = 0 \) means that \( t = 0 \); for \( t \sim 0 \) the curve near \( x \sim 0 \) is approximately \( y^2 = x(A_1 x-t^2)(A_2 x-t^2) \) with nonzero constants \( A_i \). As \( t \to e^{2\pi i t} \), the branch points at \( x = t^2/A_i \) wind twice around the origin, and a simple contour deformation argument shows that the resulting monodromy is, in fact, trivial.

The same induction argument as for \( SO(2r+1) \) implies that we can take \( f_r(q) = f_{r-1}(q) \); to find \( f(q) \), we study the breaking of \( SO(2r) \) to \( SU(k) \) and match to eq. (1.1). Consider the breaking of \( SO(2r+2k) \) with the critical number of flavors \( 2r+2k-2 \), to \( SU(k) \times SO(2r) \) with \( 2k \) and \( 2r+2 \) flavors, respectively. The breaking is achieved by taking \( \tilde{\phi}_a = \phi_a \) for \( a = 1, \ldots, k \) and \( \tilde{\phi}_b = M + \phi'_b \) for \( b = k+1, \ldots, k+r \), with \( \sum_{r+1}^{r+k} \tilde{\phi}_b = 0 \). In order for the resulting \( SU(k) \) and \( SO(2r) \) limits to be critical, we must also shift the masses \( \tilde{m}_i = m_i \) for \( i = 1, \ldots, 2r-2 \) and \( \tilde{m}_j = M + m_j + r(q) \mu \) for \( j = 2r-1, \ldots, 2r+2k-2 \). The resulting curve
By expanding near $x \sim 0$, we readily recover an $SO(2r)$ curve of the same form as the $SO(2r+2k)$ curve. To obtain $SU(k)$, we expand around $M^2$ using $x = M^2 + 2M\tilde{x}$. In the large-$M$ limit, the overall $M$-dependence factors out, leaving just

$$y^2 = \prod_{1}^{r} (x - \phi_b^2)^2 \prod_{1}^{k} (x - (\phi_b + M)^2) + fx^3 \prod_{1}^{2r-2} (x - m_i^2) \prod_{1}^{2k} (x - (M + m_j + r(q)\mu)^2) \quad (5.1)$$

This is exactly the curve (1.1), with $f = 4h(h+1)$ and $r(q) = 2h(q)$. We have thus found the complete form of the curve for $SO(2r)$. This result will be confirmed in the next section.

It is easily checked that the positions and monodromies of the semiclassical singularities of this curve match perturbation theory, by an argument similar to that given for $Sp(2r)$. The only subtlety which arises is that the Coxeter monodromy is generated by traversing a large circle in the $t$-plane, which corresponds to traversing a large circle twice in the $s_r$-plane. The resulting monodromy corresponds to a cyclic permutation of the $\alpha$-cycles times a sign flip of two of them—precisely the Coxeter element of the $SO(2r)$ Weyl group which only includes pair-wise sign flips. Finally, the $SO(2r)$ Yang–Mills (no flavors) curve found in [11] is simply a double cover of the one derived from (1.4) by sending the bare quark masses to infinity at weak coupling.

6. Higgs Breaking

We now perform another check on the curves (1.1)-(1.4), this time coming from physics on the Higgs branches of these theories. This argument depends on a nonrenormalization theorem [16], which states that the prepotential $F(A)$ determining the low–energy effective action can have no dependence on the vev of any hypermultiplet. The theorem is proved by considering the form of the most general low–energy effective action for $\mathcal{N}=2$ vector and hypermultiplets [19]. It implies in particular that the low–energy theory along any
Coulomb or mixed Higgs–Coulomb branch can not depend on the squark vevs. We will use this theorem to extend a solution valid for large squarks (and weak coupling) to arbitrary squark expectation values, including those that correspond to strong coupling.

6.1. SU(n)

We first write down the $F$- and $D$-term equations which describe the classical moduli space of the SU(n) theory. Denote, as usual, the hypermultiplet vevs by $q^i_a$ and $\tilde{q}^a_i$, and the vector multiplet vev by the traceless $\phi_{ab}$, where $a=1,\ldots,n$ is a color index and $i=1,\ldots,N_f$ is the flavor index. The $F$- and $D$-terms are then $[\phi,\phi^\dagger] = 0$ determining the Coulomb branch (where $q^i_a = \tilde{q}^a_i = 0$), $q^i_a \tilde{q}^a_i = q^i_a (q^\dagger)^b_i - (\tilde{q}^\dagger)_a^i \tilde{q}^a_i \propto \delta^b_a$ determining the Higgs branch (when $\phi=0$), and $q^i_a m^i_j + \phi^b_a q^i_b = m^i_j q^a_j + \tilde{q}^a_i \phi^a_i = 0$ governing the mixed Higgs–Coulomb branch. Here $m^i_j$ denotes the quark mass matrix which is in the adjoint of the $U(N_f)$ flavor group.

We are interested in the simplest nontrivial solution to the Higgs equations, namely $q^i_a = \langle q \rangle \delta^i_a \delta^a_N, \tilde{q}^a_i = \langle q \rangle \delta^N_i \delta^a$. Along this direction on the Higgs branch only two flavors of squark get a vev, breaking the gauge group to $SO(n-1)$. For the mixed Higgs–Coulomb equations to be satisfied, the masses of these two squarks must vanish $m_{N_f-1} = m_{N_f} = 0$. It is then clear that the mixed Higgs–Coulomb equations admit solutions with nonzero $\phi^a_b$ satisfying the Coulomb equation and the condition $\phi^a_a = \phi^b_b = 0$ for all $a, b$. This condition simply reduces the rank of $\phi$ so that it describes the Coulomb branch of the SU(n−1) unbroken by $\langle q \rangle$.

Physically, we have shown that there is a classical flat direction along which two quarks get a vev $\langle q \rangle$, Higgsing SU(n)→SU(n−1) and reducing the number of light flavors from $N_f$ to $N_f-2$. We expect this picture to be quantum–mechanically accurate only in the limit $\langle q \rangle \to \infty$, where the physics on the Higgs branch becomes arbitrarily weakly coupled. Thus, in this limit, when two of the bare masses $m_{N_f-1} = m_{N_f} = 0$, the Coulomb branch for SU(n−1) with $N_f-2$ flavors emanates from the Higgs branch. By the nonrenormalization theorem stated above, this SU(n−1) Coulomb branch cannot depend on the value of $\langle q \rangle$. So we are free to take the limit $\langle q \rangle \to 0$, which identifies the SU(n−1) Coulomb branch as the “root” of the mixed branch where it intersects the SU(n) Coulomb branch (see Fig. 1). This intersection is determined from the curve (1.1) for SU(n) as the submanifold where
the renormalized mass of two quarks is zero (i.e. the submanifold of points from which a Higgs branch can emanate).

\[ y^2 = \prod_{a=1}^{n} (x - \phi_a)^2 + 4h(h+1)(x-2h\mu)^2 \prod_{j=1}^{2n-2} (x - m_j - 2h\mu), \]  

(6.1)

where \( \sum \phi_a = 0 \) and \( \mu \equiv (1/2n) \sum m_j \). There is a quark singularity precisely when \( \phi_n = 2h\mu \), in which case the curve becomes the singular piece \((x-2h\mu)^2\) times the curve

\[ y^2 = \prod_{a=1}^{n-1} (x - \phi_a)^2 + 4h(h+1) \prod_{j=1}^{2n-2} (x - m_j - 2h\mu), \]  

(6.2)

which by the above argument we should identify as the \( SU(n-1) \) curve with \( 2n-2 \) flavors. Redefining \( x = \tilde{x} - 2h\mu/(n-1) \), \( \phi_a = \tilde{\phi}_a - 2h\mu/(n-1) \), and \( \mu = n\tilde{\mu}/(n-1) \), we see that (5.2) indeed becomes precisely the \( SU(n-1) \) curve.
6.2. SO(n)

The SO(n) case is simpler. Again assembling the squark vevs into $2N_f$-component vectors $X_i^a$, the Higgs–branch equations are $\overline{X}_i^a X_i^b = X_i^a J_{ij} X_j^b = 0$ where $J$ is the symplectic metric. The moduli space has flat directions for $X_i^a$ having a single non-zero entry with the associated bare mass zero. Such a vev parameterizes a mixed Higgs–Coulomb branch, along which $SO(n)$ is broken to $SO(n-1)$ and the Higgs mechanism lifts one of the flavors. By the nonrenormalization theorem, we can identify the $SO(n-1)$ curve from the $SO(n)$ curve with one bare mass set to zero, by looking at the intersection of the $SO(n)$ Coulomb branch with the mixed branch.

When breaking $SO(2r) \rightarrow SO(2r-1)$, we need to tune $\phi_r = 0$ in (1.4) to find this intersection (where a single quark is massless). Factoring out the $x^2$ singularity indeed gives the $SO(2r-1)$ curve (1.3). When breaking $SO(2r+1) \rightarrow SO(2r)$ we do not need to tune the $\phi_a$’s at all since the two groups have the same rank. Thus we learn that the whole $SO(2r+1)$ Coulomb branch with one bare mass set to zero should be identified with the $SO(2r)$ Coulomb branch. This is immediate from the curves (1.3) and (1.4).

6.3. Sp(2n)

The corresponding argument for the $Sp(2n)$ curve is less powerful, since along the flat direction where a single fundamental squark has a vev, the Higgs mechanism breaking $Sp(2n) \rightarrow Sp(2n-2)$ only gives one flavor a mass. Thus, starting from the scale invariant theory we flow to a non-asymptotically-free theory at weak coupling on the Higgs branch. In order to recover the $Sp(2n-2)$ scale invariant theory we must tune the bare coupling $q \rightarrow 0$ and another bare mass $m \rightarrow \infty$ appropriately. We thus lose any information concerning the strong–coupling dependence of the original curve. This does not rule out the possibility that there might be other, more complicated flat directions that would produce the required $Sp(2n-2)$ curve directly.

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