Some Remarks on the Visual Angle Metric

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Abstract We show that the visual angle metric and the triangular ratio metric are comparable in convex domains. We also find the extremal points for the visual angle metric in the half space and in the ball by use of a construction based on hyperbolic geometry. Furthermore, we study distortion properties of quasiconformal maps with respect to the triangular ratio metric and the visual angle metric.

Keywords Triangular ratio metric · Visual angle metric · Quasiconformal maps

Mathematics Subject Classification 30C65 · 51M10

1 Introduction

Geometric function theory studies classes of mappings between subdomains of the Euclidean space $\mathbb{R}^n$, $n \geq 2$. These classes include both injective and non-injective mappings. In particular, Lipschitz, quasiconformal, and quasiregular mappings along with their generalizations such as maps with integrable dilatation are in focus. On the
other hand, this theory has also been extended to Banach spaces and even to metric spaces. What is common to these theories is that various metrics are extensively used as powerful tools, e.g., Väisälä’s theory of quasiconformality in Banach spaces [14] is based on the study of metrics: the norm metric, the quasihyperbolic metric and the distance ratio metric. In recent years, several authors have studied the geometries defined by these and other related metrics [7,8,10,12,13]. For a survey of these topics the reader is referred to [17].

The main purpose of this paper is to continue the study of some of these metrics. For a domain \( G \subset \mathbb{R}^n \) and \( x, y \in G \), the visual angle metric is defined by

\[
v_G(x, y) = \sup \{ \angle(x, z, y) : z \in \partial G \} \in [0, \pi],
\]

(1.1)

where \( \partial G \) is not a proper subset of a line. This metric was introduced and studied very recently in [11]. It is clear that a point \( z \in \partial G \) exists for which this supremum is attained, such a point \( z \) is called an extremal point for \( v_G(x, y) \). For a domain \( G \subset \mathbb{R}^n \) and \( x, y \in G \), the triangular ratio metric is defined by

\[
s_G(x, y) = \sup_{z \in \partial G} \frac{|x - y|}{|x - z| + |z - y|} \in [0, 1].
\]

(1.2)

Again, the existence of an extremal boundary point is obvious. This metric has been studied in [3,9]. The above two metrics are closely related, for instance, both depend on extremal boundary points. It is easy to see that both metrics are monotone with respect to domain. Thus, if \( D \) and \( G \) are domains in \( \mathbb{R}^n \) and \( D \subset G \) then for all \( x, y \in D \), we have

\[
 s_D(x, y) \geq s_G(x, y), \quad v_D(x, y) \geq v_G(x, y).
\]

On the other hand, we will see that these two metrics are not comparable in some domains.

This paper is organized into sections as below. In Sect. 2 some comparisons between the visual angle metric and the triangular ratio metric in convex domains are given. In Sect. 3 we find the extremal points for the visual angle metric in the half space and in the ball by use of a construction based on hyperbolic geometry. Our main results are given in Sect. 4, where uniform continuity of quasiconformal maps with respect to the triangular ratio metric and the visual angle metric is studied.

2 Notation and Preliminaries

In this section, we compare the visual angle metric and the triangular ratio metric in convex domains and we also show that these two metrics are not comparable in \( \mathbb{R}^2 \setminus \{0\} \).

Given two points \( x \) and \( y \) in \( \mathbb{R}^n \), the segment between them is denoted by

\[
[x, y] = \{(1 - t)x + ty : 0 \leq t \leq 1\}.
\]
Given three distinct points $x, y, z \in \mathbb{R}^n$, the notation $\angle (x, z, y)$ means the angle in the range $[0, \pi]$ between the segments $[x, z]$ and $[y, z]$. 

For a domain $G$ of $\mathbb{R}^n$, let $\text{Möb} (G)$ be the group of all Möbius transformations which map $G$ onto itself.

### 2.1 Hyperbolic Metric

The hyperbolic metric $\rho_{\mathbb{H}^n}$ and $\rho_{\mathbb{B}^n}$ of the upper half space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ and of the unit ball $\mathbb{B}^n = \{z \in \mathbb{R}^n : |z| < 1\}$ can be defined as weighted metrics with the weight functions $w_{\mathbb{H}^n}(x) = 1/x_n$ and $w_{\mathbb{B}^n}(x) = 2/(1 - |x|^2)$, respectively. This definition as such is rather abstract and for applications explicit formulas are needed. From [2, p. 35] we have

$$\text{ch} \rho_{\mathbb{H}^n} (x, y) = 1 + \frac{|x - y|^2}{2x_n y_n}, \quad (2.2)$$

for all $x, y \in \mathbb{H}^n$, and from [2, p. 40] we have

$$\text{sh} \frac{\rho_{\mathbb{B}^n} (x, y)}{2} = \frac{|x - y|}{\sqrt{(1 - |x|^2)(1 - |y|^2)}}, \quad (2.3)$$

and

$$\text{th} \frac{\rho_{\mathbb{B}^n} (x, y)}{2} = \frac{|x - y|}{\sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}}, \quad (2.4)$$

for all $x, y \in \mathbb{B}^n$.

Hyperbolic geodesic lines are arcs of circles which are orthogonal to the boundary of the domain.

### 2.2 Distance Ratio Metric

For a proper open subset $G$ of $\mathbb{R}^n$ and for all $x, y \in G$, the distance ratio metric $j_G$ is defined as

$$j_G(x, y) = \log \left( 1 + \frac{|x - y|}{\min \{d(x, \partial G), d(y, \partial G)\}} \right).$$

The distance ratio metric was introduced by Gehring and Palka [5] and in the above simplified form by Vuorinen [15]. Both definitions are frequently used in the study of hyperbolic-type metrics [8] and geometric theory of functions.

From [11, Thm. 3.8] and [1, Lem. 7.56],

$$v_{\mathbb{B}^n} (x, y) \leq \rho_{\mathbb{B}^n} (x, y) \leq 2 j_{\mathbb{B}^n} (x, y), \quad \text{for all } x, y \in \mathbb{B}^n. \quad (2.5)$$

The triangular ratio metric and the hyperbolic metric satisfy the following inequality in the unit ball [3, Lem. 3.4, Lem. 3.8] and [3, Thm. 3.22]:

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One of the main results in [11] is the following relation between the visual angle metric and the hyperbolic metric: for all \( x, y \in \mathbb{B}^n \),

\[
\arctan \left( \frac{\sinh \rho_{\mathbb{B}^n}(x, y)}{2} \right) \leq v_{\mathbb{B}^n}(x, y) \leq 2 \arctan \left( \frac{\sinh \rho_{\mathbb{B}^n}(x, y)}{2} \right),
\]

(2.7) see [11, Thm. 3.11].

If \( x, y \) are collinear or one of the two points \( x \) and \( y \) is 0, then from [11, Lem. 3.10] and (2.3),

\[
v_{\mathbb{B}^n}(x, y) = \arctan \left( \frac{|x - y|}{\sqrt{(1 - |x|^2)(1 - |y|^2)}} \right).
\]

(2.8)

If \( x, y \in \mathbb{H}^n \) are located on a line orthogonal to \( \partial \mathbb{H}^n \), then from [11, Lem. 3.18] and (2.2)

\[
v_{\mathbb{H}^n}(x, y) = \arctan \left( \frac{|x - y|}{2 \sqrt{x_n y_n}} \right).
\]

(2.9)

By the monotonicity of the function \( x \mapsto \arctan x / x \), it is easy to see that

\[
\frac{\pi}{4} x \leq \arctan x \leq x, \quad \forall x \in (0, 1).
\]

(2.10)

Our next goal is to prove Theorem 2.17. Our original argument gave the result with the weaker constant 8 in place of \( \pi \), for the unit ball. The present version is based on the suggestions of the referee, who also suggested the following two Lemmas. We would like to point out that (2.12) was already proved in our manuscript [6], which was written shortly after the first version of the present paper.

**Lemma 2.11** Let \( D \subseteq \mathbb{R}^n \) be a domain. Then for \( x, y \in D \)

\[
\sin \frac{v_D(x, y)}{2} \leq s_D(x, y), \quad (2.12)
\]

\[
v_D(x, y) \leq \pi s_D(x, y). \quad (2.13)
\]

**Proof** Fix \( \theta \in (0, \pi) \) such that

\[
s_D(x, y) = \sin \frac{\theta}{2}. \quad (2.14)
\]

Then the ellipsoid

\[
E = \left\{ z \in \mathbb{R}^n : |z - x| + |z - y| < |x - y| / \sin \frac{\theta}{2} \right\}
\]
is contained in $D$. Hence, by domain monotonicity
\[ v_D(x, y) \leq v_E(x, y) = \theta, \tag{2.15} \]
and from (2.14) and (2.15) we see that \( \sin \frac{v_D(x, y)}{2} \leq s_D(x, y) \). Moreover,
\[ \frac{v_D(x, y)}{s_D(x, y)} \leq \frac{\theta}{\sin(\theta/2)} \leq \pi \]
where the last inequality follows from Jordan’s inequality. \(\square\)

**Lemma 2.16** If $D$ is a convex subdomain of $\mathbb{R}^n$, and $x, y \in D$, then
\[ s_D(x, y) \leq v_D(x, y). \]

**Proof** If $v_D(x, y) \geq \frac{\pi}{2}$, then $s_D(x, y) \leq 1 \leq v_D(x, y)$. Assume that $v_D(x, y) < \frac{\pi}{2}$. Let $z \in \partial D$ be such that $v_D(x, y) = \angle(x, z, y)$. Denote by $r$ the radius of the circle passing through $x, y, z$. Since $D$ is convex, the convex hull of the set \{ $z \in \mathbb{R}^n : \angle(x, z, y) > v_D(x, y)$\} is contained in $D$ and hence also the ellipsoid
\[ E = \{ z \in \mathbb{R}^n : |z - x| + |z - y| < 2r \} \]
is contained in the domain $D$, see Fig. 1. We easily conclude that $\sin v_D(x, y) = \frac{|x - y|}{2r}$. Moreover, by the domain monotonicity property of the $s$-metric,
\[ s_D(x, y) \leq s_E(x, y) = \frac{|x - y|}{2r} = \sin v_D(x, y) \leq v_D(x, y). \]
\(\square\)

**Theorem 2.17** Let $D \subset \mathbb{R}^n$ be a convex domain. Then for all $x, y \in D$ we have
\[ s_D(x, y) \leq v_D(x, y) \leq \pi s_D(x, y). \]

**Proof** The proof follows from Lemmas 2.11 and 2.16. \(\square\)

**Remark 2.18** The visual angle metric and the triangular ratio metric both highly depend on the boundary of the domain. If we replace the convex domain $D$ in Theorem 2.17 with $G = \mathbb{B}^2 \setminus \{0\}$, then the visual angle metric and the triangular ratio metric are not comparable in $G$. To this end, we consider two sequences of points $x_k = (1/k, 0)$, $y_k = (1/k^2, 0)$ ($k = 2, 3, \ldots$). Then
\[ s_G(x_k, y_k) = \frac{k - 1}{k + 1}. \]
From (2.8), we get
\[ v_G(x_k, y_k) = v_{\mathbb{B}^2}(x_k, y_k) = \arctan \left( \frac{k}{(k + 1)\sqrt{1 + k^2}} \right) < \arctan \left( \frac{1}{k + 1} \right). \]
Therefore,

\[
\frac{s_G(x_k, y_k)}{v_G(x_k, y_k)} \geq \frac{k - 1}{(k + 1) \arctan \left( \frac{1}{k+1} \right)} \to \infty, \quad \text{as } k \to \infty.
\]

### 3 The Extremal Points for the Visual Angle Metric

In this section, we aim to find the extremal points for the visual angle metric in the half space and in the ball by use of a construction based on hyperbolic geometry. Since the visual angle metric is similarity invariant we can consider it in the upper half space (Figs. 2, 3), and in the unit ball (Figs. 4, 5).

**Theorem 3.1** Given two distinct points \( x, y \in \mathbb{H}^n \), let \( J[x, y] \) be the hyperbolic segment joining \( x \) and \( y \). Let \( L_{xy} \) be the hyperbolic bisector of \( J[x, y] \) with two endpoints \( u \) and \( w \) in \( \partial \mathbb{H}^n \). Then one of the endpoints \( u \) and \( w \) is the extremal point for \( v_{\mathbb{H}^n}(x, y) \), specifically,

(i) if one of \( u \) and \( w \) is infinity, say \( w = \infty \), then \( v_{\mathbb{H}^n}(x, y) = \angle(x, u, y) \);

(ii) if none of \( u \) and \( w \) is infinity, then \( v_{\mathbb{H}^n}(x, y) = \max\{\angle(x, u, y), \angle(x, w, y)\} \).

**Proof** It suffices to consider the two-dimensional case. We divide the proof into two cases.
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Fig. 2, 3 The Möbius transformation $\sigma \in \text{Möb}(\mathbb{H}^2)$ with $\sigma(w) = \infty$ maps Figure 2 onto Figure 3. Here $v_{\mathbb{H}^2}(x, y) = \angle(x, u, y)$

Case 1. $d(x, \partial \mathbb{H}^2) = d(y, \partial \mathbb{H}^2)$.
In this case, the hyperbolic bisector $L_{xy}$ of the hyperbolic segment $J[x, y]$ is also the bisector of the Euclidean segment $[x, y]$. We may assume that $u = L_{xy} \cap \partial \mathbb{H}^2$. Then by simple geometric observation, we see that

$$v_{\mathbb{H}^2}(x, y) = \angle(x, u, y).$$

Case 2. $d(x, \partial \mathbb{H}^2) \neq d(y, \partial \mathbb{H}^2)$.
Without loss of generality, we may assume that $d(x, \partial \mathbb{H}^2) < d(y, \partial \mathbb{H}^2)$ and that $u$ is located on the diameter of the circle containing $J[x, y]$. Then

$$\max\{\angle(x, u, y), \angle(x, w, y)\} = \angle(x, u, y).$$

Using a Möbius transformation $\sigma \in \text{Möb}(\mathbb{H}^2)$ with $\sigma(w) = \infty$, we see that $\sigma(L_{xy})$ is a Euclidean line orthogonal to $\partial \mathbb{H}^2$ and also the hyperbolic bisector of the hyperbolic segment $J[\sigma(x), \sigma(y)]$. Hence, $d(\sigma(x), \partial \mathbb{H}^2) = d(\sigma(y), \partial \mathbb{H}^2)$. By the argument used in Case 1, we have that

$$v_{\mathbb{H}^2}(\sigma(x), \sigma(y)) = \angle(\sigma(x), \sigma(u), \sigma(y)).$$

Therefore, it is clear that the circle $C$ through $\sigma(x), \sigma(u), \sigma(y)$ is tangent to $\partial \mathbb{H}^2$. Because Möbius transformations preserve circles, we conclude that the circle $\sigma^{-1}(C)$ through $x, u, y$ is also tangent to $\partial \mathbb{H}^2$, which implies that

$$v_{\mathbb{H}^2}(x, y) = \angle(x, u, y).$$

This completes the proof.

In a similar way, we have the following conclusion for the visual angle metric in the unit ball (Figs. 4, 5).

**Theorem 3.2** Given two distinct points $x, y \in \mathbb{B}^n$, let $J[x, y]$ be the hyperbolic segment joining $x$ and $y$. Let $L_{xy}$ be the hyperbolic bisector of $J[x, y]$ with two endpoints $u$ and $w$ in $\partial \mathbb{B}^n$. Then one of the endpoints $u$ and $w$ is the extremal point for $v_{\mathbb{B}^n}(x, y)$, specifically,

$$v_{\mathbb{B}^n}(x, y) = \angle(x, u, y).$$
The Möbius transformation \( T \in \text{M"{o}b}(B^2) \) with \( T(x) = -T(y) \) maps Figure 4 onto Figure 5.

Here \( v_{B^2}(x, y) = \angle(x, u, y) \)

(i) if \( x \) and \( y \) are symmetric with respect to the origin \( 0 \), then \( v_{B^2}(x, y) = \angle(x, u, y) = \angle(x, w, y) \);

(ii) if \( x \) and \( y \) are not symmetric with respect to the origin \( 0 \), then \( v_{B^2}(x, y) = \max\{\angle(x, u, y), \angle(x, w, y)\} \).

Proof

It suffices to consider the two-dimensional case. We divide the proof into two cases.

Case 1. The points \( x \) and \( y \) are symmetric with respect to the origin \( 0 \).

In this case, the hyperbolic bisector \( L_{xy} \) of the hyperbolic segment \( J[x, y] \) is also the bisector of the Euclidean segment \([x, y]\). Then by simple geometric observation, we see that

\[
v_{B^2}(x, y) = \angle(x, u, y) = \angle(x, w, y).
\]

Case 2. The two points \( x \) and \( y \) are not symmetric with respect to the origin \( 0 \).

The hyperbolic geodesic line through \( x \) and \( y \) divides \( \partial B^2 \) into two arcs. Without loss of generality, we may assume that \( u \) is in the minor arc or the semicircle (if \( x, 0, y \) are collinear) of \( \partial B^2 \). Then

\[
\max\{\angle(x, u, y), \angle(x, w, y)\} = \angle(x, u, y).
\]

Using a Möbius transformation \( T \in \text{M"{o}b}(B^2) \) with \( T(x) = -T(y) \), we see that \( T(L_{xy}) \) is the hyperbolic bisector of the hyperbolic segment \( J[T(x), T(y)] \). From the argument used in Case 1, we have that

\[
v_{B^2}(T(x), T(y)) = \angle(T(x), T(u), T(y)).
\]

Therefore, it is clear that the circle \( C \) through \( T(x), T(u), T(y) \) is tangent to \( \partial B^2 \). Because Möbius transformations preserve circles, we conclude that the circle \( T^{-1}(C) \) through \( x, u, y \) is also tangent to \( \partial B^2 \), which implies that

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\[ v_{\mathbb{B}^2}(x, y) = \angle(x, u, y). \]

This completes the proof. \(\square\)

Remark 3.3 In [18], the authors presented several methods of geometric construction to find the hyperbolic midpoint of a hyperbolic segment only based on Euclidean compass and ruler. These methods of construction can be used to find the extremal points for the visual angle metric in the upper half space and in the unit ball as the above theorems show.

4 Uniform Continuity of Quasiconformal Maps

In this section, we study the uniform continuity of quasiconformal maps with respect to the triangular ratio metric and the visual angle metric.

We use notation and terminology from [1] in the rest of this paper. We always take the boundary of a domain in \( \mathbb{R}^n \) with respect to \( \mathbb{R}^n \) in this section. Let \( \gamma_n \) and \( \tau_n \) be the conformal capacities of the \( n \)-dimensional Grötzsch ring and Teichmüller ring, respectively. Both of \( \gamma_n \) and \( \tau_n \) are continuous and strictly decreasing, see [1, (8.34), Thm. 8.37]. Let \( \lambda_n \in [4, 2e^{n-1}] \) be the Grötzsch ring constant, see [1, (8.38)].

For \( K > 0 \), the distortion function \( \varphi_{K,n}(r) \) is a self-homeomorphism of \((0, 1)\) defined by [1, (8.69)]

\[ \varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K \gamma_n(1/r))}. \]

For \( K \geq 1, n \geq 2, \) and \( r \in (0, 1), \)

\[ \varphi_{K,n}(r) \leq \lambda_n^{1-\alpha} r^\alpha, \quad \alpha = K^{1/(1-n)}, \quad (4.1) \]

see [16, Thm. 7.47].

For \( K \geq 1, n \geq 2, \) and \( t \in (0, 1), \) the function \( \Theta_{K,n}(t) \) in [4, (2.9)] is defined by:

\[ \Theta_{K,n}(t) = \frac{1}{\tau_n^{-1}(K \gamma_n(1/t))} = \frac{x^2}{1-x^2}, \quad x = \varphi^{-1}_{2^{n-1}K,n}(t). \]

Our proofs below make use of the modulus metric of a domain \( G \) whose boundary is of positive capacity, see [1, 8.80]. If \( D \) is a subdomain of \( G \), then \( \mu_G(x, y) \leq \mu_D(x, y) \) for all \( x, y \in D \), see [1, Rem. 8.83(2)]. A \( K \)-quasiconformal map \( f : G \to fG \) is \( K \)-bilipschitz in the \( \mu_G \) metric, see [1, (16.11)].

Lemma 4.3 [1, Lem. 8.86] Let \( G \) be a proper subdomain of \( \mathbb{R}^n \) such that \( \mathbb{R}^n \setminus G \) is a non-degenerate continuum. Then for \( x, y \in G, x \neq y, \)

\[ \mu_G(x, y) \geq \tau_n \left( \min \left( \frac{d(x, \partial G), d(y, \partial G)}{|x - y|} \right) \right) \geq \tau_n \left( \frac{d(x, \partial G)}{|x - y|} \right). \]
Lemma 4.4 [1, Ex. 8.85] Let $G$ be a proper subdomain of $\mathbb{R}^n$, let $x \in G$ and $B_x = B^n(x, d(x, \partial G))$. For $y \in B_x$, $x \neq y$,
\[ \mu_G(x, y) \leq \mu_{B_x}(x, y) = \gamma_n \left( \frac{d(x, \partial G)}{|x - y|} \right). \]

Lemma 4.5 For $K \geq 1$, $n \geq 2$, let $\alpha = K^{1/(1-n)}$.

(1) If $t_0 \in (0, 1)$ satisfies $\lambda_n^{2-\alpha} t_0^\alpha = 1/2$, then for $t \in (0, t_0]$,
\[ \Theta_{K, n}(t) \leq 2\lambda_n^{2-\alpha} t^\alpha. \]

(2) If $t_1 \in (0, t_0]$ satisfies $\lambda_n^{2-\alpha} t_1^\alpha = 1/4$, then for $t \in (0, t_1]$,
\[ \Theta_{K, n}(t) \leq \frac{1}{2}. \]

Proof (1) By (4.1) we have
\[ \varphi_{2n-1}^{2} \leq \lambda_n^{2-\alpha} t^\alpha, \]
and hence by (4.2) for $t \in (0, t_0]$,
\[ \Theta_{K, n}(t) \leq 2\lambda_n^{2-\alpha} t^\alpha. \]

(2) This claim follows immediately from part (1).

\[ \square \]

Theorem 4.6 Let $D, D'$ be two proper subdomains of $\mathbb{R}^n$ such that $\partial D$ is connected. Let $f : D \to D' = f D$ be a $K$-quasiconformal mapping. Then for all $x, y \in D$,
\[ s_{D'}(f(x), f(y)) \leq C_1 s_D(x, y)^\alpha, \quad \alpha = K^{1/(1-n)}, \]
where
\[ C_1 = \max \left\{ 2\lambda_n^{2-\alpha}(2 + t_0)^\alpha, \left( \frac{2 + t_0}{t_0} \right)^\alpha \right\}, \]
and $t_0$ is as in Lemma 4.5 (1).

Proof The result is trivial for $x = y$. Therefore, we only need to prove the theorem for $x \neq y$.

We first consider the case $|x - y| \leq t_0 d(x, \partial D)$ for $x \in D$. It is easy to see that $\partial D' = \partial(f D)$ is connected because $f$ is a homeomorphism and $\partial D$ is connected. Therefore, there exists an unbounded domain $G'$ such that $D' \subset G'$, $d(f(x), \partial G') = d(f(x), \partial D')$ and $\partial G'$ is connected. Then for $t \geq d(f(x), \partial G')$, we have
\[ S^{n-1}(f(x), t) \cap \partial G' \neq \emptyset, \]

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and hence $\mathbb{R}^n \setminus G'$ is a non-degenerate continuum. By Lemma 4.3, we have

$$\tau_n \left( \frac{d(f(x), \partial G')}{|f(x) - f(y)|} \right) \leq \mu_{G'}(f(x), f(y)) \leq \mu_{fD}(f(x), f(y)).$$

Let $B_x = B^n(x, d(x, \partial D))$. Then $y \in B_x$ and by Lemma 4.4,

$$\mu_{D}(x, y) \leq \mu_{B_x}(x, y) = \gamma_n \left( \frac{d(x, \partial D)}{|x - y|} \right).$$

Combining the above two inequalities, we get

$$\tau_n \left( \frac{d(f(x), \partial D')}{|f(x) - f(y)|} \right) \leq \mu_{fD}(f(x), f(y)) \leq K \mu_{D}(x, y) \leq K \gamma_n \left( \frac{d(x, \partial D)}{|x - y|} \right),$$

and hence by Lemma 4.5 (1),

$$s_{D'}(f(x), f(y)) \leq \frac{|f(x) - f(y)|}{d(f(x), \partial D')} \leq \Theta_{K,n} \left( \frac{|x - y|}{d(x, \partial D)} \right) \leq 2\lambda_n^{2-\alpha} \left( \frac{|x - y|}{d(x, \partial D)} \right)^\alpha.$$

On the other hand,

$$s_D(x, y) \geq \frac{|x - y|}{2 \min\{d(x, \partial D), d(y, \partial D)\} + |x - y|} \geq \frac{|x - y|}{(2 + t_0)d(x, \partial D)}.$$

Therefore, for $|x - y| \leq t_0 d(x, \partial D)$,

$$s_{D'}(f(x), f(y)) \leq 2\lambda_n^{2-\alpha} ((2 + t_0)s_D(x, y))^\alpha$$

$$= 2\lambda_n^{2-\alpha} (2 + t_0)^\alpha s_D(x, y)^\alpha.$$

Now it only remains to prove the case $|x - y| > t_0 d(x, \partial D)$ for $x, y \in D$. We easily see that

$$s_D(x, y) \geq \frac{|x - y|}{2|x - y|/t_0 + |x - y|} = \frac{t_0}{2 + t_0},$$

and hence

$$s_{D'}(f(x), f(y)) \leq 1 \leq \left( \frac{2 + t_0}{t_0} \right)^\alpha s_D(x, y)^\alpha.$$

Thus, we complete the proof by choosing the constant

$$C_1 = \max \left\{ 2\lambda_n^{2-\alpha} (2 + t_0)^\alpha, \left( \frac{2 + t_0}{t_0} \right)^\alpha \right\}.$$
Remark 4.7 In Theorem 4.6 the hypothesis of \( f \) being a quasiconformal map cannot be replaced with an analytic function. To see this, we consider the analytic function \( g : \mathbb{B}^2 \to \mathbb{B}^2 \setminus \{0\} = g\mathbb{B}^2 \) with \( g(z) = \exp(\frac{z+1}{z-1}) \). Let \( r_k = \frac{k-1}{k+1} \) (\( k = 1, 2, 3 \ldots \)). Then as \( k \to \infty \),

\[
 s_{\mathbb{B}^2}(r_k, r_{k+1}) = \frac{1}{3 + 2k} \to 0,
\]

while

\[
 s_{\mathbb{B}^2 \setminus \{0\}}(g(r_k), g(r_{k+1})) = \frac{e-1}{e+1}.
\]

Lemma 4.8 (1) Let the three points \( x, y, z \in \mathbb{H}^n \) be on the line orthogonal to \( \partial \mathbb{H}^n \) and \( x_n < y_n < z_n \). Then

\[
 v_{\mathbb{H}^n}(x, y) < v_{\mathbb{H}^n}(x, z).
\]

(2) Let \( \lambda \in (0, 1) \) and \( e_n = (0, 0, \ldots, 1) \in \mathbb{R}^n \). Let \( x \in \mathbb{H}^n \) and \( y \in S^{n-1}(x, \lambda x_n) \).

Then for \( y' = x + \lambda x_n e_n \),

\[
 v_{\mathbb{H}^n}(x, y) \geq v_{\mathbb{H}^n}(x, y') = \arctan \frac{\lambda}{2\sqrt{1+\lambda}} > \frac{\lambda}{4}.
\]

Proof (1) It is easy to see that the function \( t \mapsto \arctan \frac{t-a}{2\sqrt{a}} = \arctan \frac{1-a/t}{2\sqrt{a}} \) is increasing on \( (a, \infty) \) for \( a > 0 \). From (2.9),

\[
 v_{\mathbb{H}^n}(x, y) = \arctan \frac{y_n - x_n}{2\sqrt{x_n y_n}}.
\]

Since \( y_n < z_n \), we have that

\[
 v_{\mathbb{H}^n}(x, y) < v_{\mathbb{H}^n}(x, z).
\]

(2) By elementary geometry it is clear that the radius of the circle through \( x, y \) and tangent to \( \partial \mathbb{H}^n \) is a decreasing function of \( \theta = \angle(y', x, y) \in [0, \pi] \). Hence,

\[
 v_{\mathbb{H}^n}(x, y) \geq v_{\mathbb{H}^n}(x, y').
\]

From (2.9) and (2.10),

\[
 v_{\mathbb{H}^n}(x, y') = \arctan \frac{\lambda}{2\sqrt{1+\lambda}} \geq \frac{\lambda}{4} \frac{\pi}{2\sqrt{1+\lambda}} > \frac{\lambda}{4}.
\]

\[\square\]
Theorem 4.9 Let \( D, D' \) be two proper subdomains of \( \mathbb{R}^n \) such that \( D \) is convex. Let \( f: D \to D' = fD \) be a \( K \)-quasiconformal mapping. Then for all \( x, y \in D \),

\[
v_{D'}(f(x), f(y)) \leq C_2 v_D(x, y)^\alpha, \quad \alpha = K^{1/(1-n)},
\]

where

\[
C_2 = \max \left\{ 2^{3+2\alpha} \lambda_n^{2-\alpha}, \pi \left( \frac{4}{t_1} \right)^\alpha \right\},
\]

and \( t_1 \) is as in Lemma 4.5 (2).

Proof The result is trivial for \( x = y \). Therefore, we only need to prove the theorem for \( x \neq y \).

We first consider the case \( |x - y| \leq t_1 d(x, \partial D) \) for \( x \in D \). The boundary \( \partial D \) is connected because \( D \) is convex. From the proof of Theorem 4.6 and Lemma 4.5 (2), we have

\[
\frac{|f(x) - f(y)|}{d(f(x), \partial D')} \leq \Theta_{K,n} \left( \frac{|x - y|}{d(x, \partial D)} \right) \leq \frac{1}{2},
\]

and hence \( f(y) \in \mathbb{B}^n(f(x), d(f(x), \partial D')/2). \)

Without loss of generality, we may assume that \( f(x) = 0 \) and \( d(f(x), \partial D') = 1 \). Then from (2.5), we have

\[
v_{D'}(f(x), f(y)) \leq v_{\mathbb{B}^n}(f(x), f(y)) \leq \rho_{\mathbb{B}^n}(f(x), f(y)) \leq 2j_{\mathbb{B}^n}(f(x), f(y)).
\]

Because \( d(f(y), \partial D') \geq d(f(x), \partial D')/2 \) and \( \log(1 + a) \leq a \) for \( a \geq 0 \), we have

\[
j_{\mathbb{B}^n}(f(x), f(y)) \leq 2 \frac{|f(x) - f(y)|}{d(f(x), \partial D')}. \]

Therefore, the above three inequalities, combined with Lemma 4.5 (1), yield

\[
v_{D'}(f(x), f(y)) \leq 4 \Theta_{K,n} \left( \frac{|x - y|}{d(x, \partial D)} \right) \leq 8 \lambda_n^{2-\alpha} \left( \frac{|x - y|}{d(x, \partial D)} \right)^\alpha.
\]

Since \( y \in S^{n-1}(x, |x - y|) \), then by Lemma 4.8 (2), we have

\[
v_{D'}(f(x), f(y)) \leq 8 \lambda_n^{2-\alpha} \left( \frac{|x - y|}{4d(x, \partial D)} \right)^\alpha \leq 2^{3+2\alpha} \lambda_n^{2-\alpha} (v_H(x, y))^\alpha,
\]

where \( H \) is the half space which contains \( D \) such that \( d(x, \partial H) = d(x, \partial D) \). Therefore, for \( |x - y| \leq t_1 d(x, \partial D) \) we have

\[
v_{D'}(f(x), f(y)) \leq 2^{3+2\alpha} \lambda_n^{2-\alpha} (v_D(x, y))^\alpha.
\]
It remains to prove the case $|x - y| > t_1 d(x, \partial D)$ for $x, y \in D$. Again by Lemma 4.8, we have
\[ v_D(x, y) \geq v_H(x, y) \geq \frac{t_1}{4}, \]
and hence
\[ v_D'(f(x), f(y)) \leq \pi \leq \pi \left( \frac{4}{t_1} \right) ^\alpha v_D(x, y)^\alpha. \]

Thus, we complete the proof by choosing the constant
\[ C_2 = \max \left\{ 2^{3+2\alpha} \lambda_1 ^{2-\alpha}, \pi \left( \frac{4}{t_1} \right) ^\alpha \right\}. \]

\[ \square \]

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