INDUCTIVE LIMITS OF SEMIPROJECTIVE $C^*$-ALGEBRAS

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Abstract. We prove closure properties for the class of $C^*$-algebras that are inductive limits of semiprojective $C^*$-algebras. Most importantly, we show that this class is closed under shape domination, and so in particular under shape and homotopy equivalence. It follows that the considered class is quite large. It contains for instance the stable suspension of any nuclear $C^*$-algebra satisfying the UCT and with torsion-free $K_0$-group. In particular, the stabilized $C^*$-algebra of continuous functions on the pointed sphere is isomorphic to an inductive limit of semiprojectives.

1. Introduction

Shape theory is a tool to study global properties of metric spaces that have singularities. This is done by approximating the space under consideration by nicer spaces without singularities. Properties of the original space are encoded in the approximating system.

This idea has been transferred to the study of $C^*$-algebras by Blackadar in [Bla85]. The building blocks of this noncommutative shape theory are the semiprojective $C^*$-algebras; see Section 2 for definitions. One therefore seeks to approximate a given $C^*$-algebra $A$ by semiprojective $C^*$-algebras. This leads to the following fundamental question:

**Question 1.1** (Blackadar, [Bla85, 4.4]). Is every separable $C^*$-algebra isomorphic to a sequential inductive limit of separable, semiprojective $C^*$-algebras?

The analogue of this question for metric space has a positive answer: Every metric space is homeomorphic to an inverse limit of absolute neighborhood retracts.

Since the answer to **Question 1.1** is unknown, Blackadar developed a more general notion of approximation, called a shape system; see Section 2 for details. Every separable $C^*$-algebra is isomorphic to the inductive limit of a shape system.

In contrast, we say that a separable $C^*$-algebra has a strong shape system if it is isomorphic to a sequential inductive limit of separable, semiprojective $C^*$-algebras. Thus, **Question 1.1** asks if every separable $C^*$-algebra has a strong shape system. The main result of this paper is:

**Theorem (4.4).** Let $A$ and $B$ be separable $C^*$-algebras such that $A$ is shape dominated by $B$. Then, if $B$ has a strong shape system, so does $A$.

Consequently, if two separable $C^*$-algebras are shape equivalent (in particular, if they are homotopy equivalent), then one has a strong shape system if and only if the other does; see **Corollary 4.5**. In **Theorem 4.9** we summarize closure properties for the class of separable $C^*$-algebras that have a strong shape system.
We apply Theorem 4.4 to show that many nuclear $C^*$-algebras are inductive limits of semiprojective $C^*$-algebras.

**Theorem 5.3.** Let $A$ be a separable, stable, nuclear, homotopy symmetric $C^*$-algebra satisfying the UCT. Assume that $K_0(A)$ is torsion-free. Then $A$ has a strong shape system.

In particular, if $A$ is a separable, nuclear $C^*$-algebra satisfying the UCT, then the stable suspension $\Sigma A \otimes K$ has a strong shape system if $K_1(A)$ is torsion-free, and the stable second suspension $\Sigma^2 A \otimes K$ has a strong shape system if $K_0(A)$ is torsion-free; see Corollary 5.6.

Our results provide a partial answer to Question 1.1 by substantially increasing the class of $C^*$-algebras that are known to have a strong shape system. Moreover, we obtain many new concrete examples of $C^*$-algebras that are inductive limits of semiprojective $C^*$-algebras. For example, let $A$ be a UCT-Kirchberg algebra. Then $C([0,1]^n, A)$ has a strong shape system for every $n \in \mathbb{N}$; see Example 4.8. Further, if $X$ is a connected, compact, metrizable space and $x \in X$, then $C_0(X \setminus \{x\}) \otimes A \otimes K$ has a strong shape system whenever $K_0(C_0(X \setminus \{x\}) \otimes A)$ is torsion-free; see Example 5.7.

This paper proceeds as follows: In Section 2, we recall the basic notions of noncommutative shape theory. In Section 3, we generalize results of Loring and Shulman about cones of $C^*$-algebras [LS12, Section 7] to the setting of mapping cylinders. We show that the mapping cylinder has a strong shape system if the domain of the defining morphism is semiprojective; see Theorem 3.5.

In Section 4, we study closure properties of the class of $C^*$-algebras that have a strong shape system. We use the technical result about strong shape systems for mapping cylinders (Theorem 3.5) to prove the main result Theorem 4.4. In Section 5, we derive results about strong shape systems for nuclear $C^*$-algebras.

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## 2. Shape theory

In this section, we recall the basic notions of shape theory for separable $C^*$-algebras as developed by Blackadar in [Bl85].

We use the symbol $\simeq$ to denote homotopy equivalence. By $A, B, C, D$ we usually denote $C^*$-algebras. A morphism between $C^*$-algebras is understood to be a $^*$-homomorphism. By ideals in a $C^*$-algebra we always mean closed, two-sided ideals.

A morphism $\varphi: A \to B$ is said to be *projective* if for every $C^*$-algebra $C$, every ideal $J \trianglelefteq C$, and every morphism $\sigma: B \to C/J$, there exists a morphism $\psi: A \to C$ such that $\pi \circ \psi = \sigma \circ \varphi$, where $\pi: C \to C/J$ is the quotient morphism. This is indicated in the commutative diagram on the left below.

A morphism $\varphi: A \to B$ is said to be *semiprojective* if for every $C^*$-algebra $C$, every increasing sequence $J_0 \subseteq J_1 \subseteq \ldots$ of ideals in $C$, and for every morphism $\sigma: B \to C/\bigcup_n J_n$, there exist some $n \in \mathbb{N}$ and a morphism $\psi: A \to C/J_n$ such that $\pi_{\infty,n} \circ \psi = \sigma \circ \varphi$, where $\pi_{\infty,n}: C/J_n \to C/\bigcup_n J_n$ is the quotient morphism. This is indicated in the commutative diagram on the right below.

A $C^*$-algebra $A$ is *semiprojective* if the identity map $\text{id}_A: A \to A$ is.

By a *sequential inductive system* we mean a sequence $A_0, A_1, A_2, \ldots$ of $C^*$-algebras together with morphisms $\gamma_{n+1, n}: A_n \to A_{n+1}$ for each $n \in \mathbb{N}$. Given $n, m \in \mathbb{N}$ with $n < m$, we set $\gamma_{m, n} := \gamma_{m, m-1} \circ \ldots \circ \gamma_{n+2, n+1} \circ \gamma_{n+1, n}: A_n \to A_m$. We call $\gamma_{m, n}$ the connecting morphisms of the system. We use $\varprojlim A_n$ to denote the
relations defining $Z$. Let us assume that with the inclusion of $\phi$, of Theorem 3.5. Let $\phi$ of Theorem 3.5.

Let $A$ be a separable $C^*$-algebra. A shape system for $A$ is a sequential inductive system $(A_n, \gamma_{n+1,n})$ of separable $C^*$-algebras such that $A \cong \lim \limits_{\to} A_n$ and such that the connecting morphisms $\gamma_{n+1,n}$ are semiprojective. By [Bla85, Theorem 4.3], every separable $C^*$-algebra has a shape system.

Let $A = (A_n, \gamma_{n+1,n})$ and $B = (B_k, \theta_{k+1,k})$ be inductive systems. Then $A$ is said to be shape dominated by $B$, denoted $A \preceq B$, if there exist two strictly increasing sequences $n_0 < n_1 < \ldots$ and $k_0 < k_1 < \ldots$ in $\mathbb{N}$, and morphisms $\alpha_i : A_{n_i} \to B_{k_i}$ and $\beta_i : B_{k_i} \to A_{n_{i+1}}$ such that $\beta_i \circ \alpha_i \simeq \gamma_{n_{i+1},n_i}$ for $i \in \mathbb{N}$. If also $\alpha_{i+1} \circ \beta_i \simeq \theta_{k_{i+1},k_i}$ for all $i \in \mathbb{N}$, then $A$ and $B$ are said to be shape equivalent, denoted $A \sim B$. The situation is indicated in the following diagram:

$$
\begin{array}{ccccccc}
A_{n_0} & \xrightarrow{\gamma_{n_0,n_1}} & A_{n_1} & \xrightarrow{\gamma_{n_1,n_2}} & A_{n_2} & \cdots \\
B_{k_0} & \xrightarrow{\theta_{k_0,k_1}} & B_{k_1} & \xrightarrow{\theta_{k_1,k_2}} & B_{k_2} & \cdots \\
\end{array}
$$

The relation $\preceq$ for sequential inductive systems is transitive, and $\sim$ is an equivalence relation; see [Bla85, Definition 4.6].

By [Bla85, Corollary 4.9], any two shape systems of a $C^*$-algebra are shape equivalent. Therefore, the following definition makes sense: A $C^*$-algebra $A$ is said to be shape dominated by a $C^*$-algebra $B$, denoted $A \preceq_{\text{SH}} B$, if we have $A \preceq B$ for some, or equivalently every, shape system $A$ for $A$ and $B$ for $B$. Similarly, $A$ is said to be shape equivalent to $B$, denoted $A \sim_{\text{SH}} B$, if we have $A \sim B$ for shape systems $A$ for $A$ and $B$ for $B$.

Recall that $A$ is homotopy dominated by $B$ if there exist morphisms $\alpha : A \to B$ and $\beta : B \to A$ with $\beta \circ \alpha \simeq \text{id}_A$. If also $\alpha \circ \beta \simeq \text{id}_B$, then $A$ and $B$ are homotopy equivalent. By [Bla85, Corollary 4.11], shape is coarser than homotopy: If $A$ is homotopy dominated by (homotopy equivalent to) $B$, then $A \preceq_{\text{SH}} B$ ($A \sim_{\text{SH}} B$).

3. Mapping cylinders

In this section, we consider the mapping cylinder $Z_{\phi}$ associated to a morphism $\phi : A \to B$; see [Paragraph 3.2]. We let $\phi^+ : A \to B^+$ denote the composition of $\phi$ with the inclusion of $B$ into its forced unitization $B^+$. Given a presentation of $A$ and $B$ with self-adjoint generators and relations, we show how to present $Z_{\phi^+}$; see [Lemma 3.3]. Let us assume that $A$ is semiprojective. If we relax certain of the relations defining $Z_{\phi^+}$, we obtain a semiprojective $C^*$-algebra; see [Lemma 3.4]. We deduce that $Z_{\phi^+}$ is an inductive limit of semiprojective $C^*$-algebras; see the proof of Theorem 3.5.

To obtain the same conclusion for $Z_{\phi}$, we use that an ideal $J$ in a semiprojective $C^*$-algebra $B$ is semiprojective if the quotient $B/J$ is projective; see [End16].

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**Diagram:**

![Diagram](attachment:image.png)
Corollary 3.1.3]. We deduce the main result of this section: The mapping cylinder $Z_\varphi$ has a strong shape system; see Theorem 3.5.

3.1. The theory of universal $C^*$-algebras given by generators and relations is very rich and closely connected to noncommutative shape theory. Blackadar developed the basic theory in [Bla85, Section 1]. A comprehensive study of $C^*$-algebra relations was presented by Loring [Lor10]. We only recall the construction of a universal $C^*$-algebra given by self-adjoint generators subject to polynomial, order and norm relations.

Let $x = (x_0, x_1, x_2, \ldots)$ be a sequence of (self-adjoint) generators. We let $F(x)$ denote the free $*$-algebra on the self-adjoint generators $x$. Given a Hilbert space $H$, there is a natural bijective correspondence between sequences $X = (X_k)_{k \in \mathbb{N}}$ of self-adjoint elements in $B(H)$ and $*$-homomorphisms $F(x) \to B(H)$. Let $n = (c_k)_{k \in \mathbb{N}}$ be a sequence of numbers in $[0, \infty)$. We say that a sequence $X$ of self-adjoint elements in $B(H)$ satisfies the norm relations specified by $n$ if $\|X_k\| \leq c_k$ for every $k \in \mathbb{N}$.

A NC polynomial $p$ in $x$ is a polynomial in finitely many noncommuting variables from $x$ with coefficients in $\mathbb{C}$. We always assume that NC polynomials have vanishing constant term. An example of a NC polynomial is $x_1 x_2 x_3 - 3x_2 x_3$. Let $p = (p_l)_{l \in \mathbb{N}}$ be a sequence of NC polynomials in $x$. We say that a sequence $X$ of self-adjoint elements in $B(H)$ satisfies the polynomial relations specified by $p$ if $p_l(x) = 0$ for every $l \in \mathbb{N}$.

We also consider order-relations of the form $\alpha x_k \leq \beta x_l$, for some $\alpha, \beta \in \mathbb{R}$ and $k, l \in \mathbb{N}$. We formalize this by letting $o = ((\alpha_m, \beta_m, s_m, t_m))_{m \in \mathbb{N}}$ be a sequence of tuples with $\alpha_m, \beta_m \in \mathbb{R}$ and $s_m, t_m \in \mathbb{N}$. We say that a sequence $X$ of self-adjoint elements in $B(H)$ satisfies the order relations specified by $o$ if $\alpha_m X_{s_m} \leq \beta_m X_{t_m}$ for every $m \in \mathbb{N}$.

A representation of $(x|n, p, o)$ on a Hilbert space $H$ is a sequence $X$ of self-adjoint operators in $B(H)$ satisfying the norm relations $n$, the polynomial relations $p$ and the order relations $o$. Abusing notation, we identify a representation $\phi: F(x) \to B(H)$ of $(x|n, p, o)$ on $H$ with the $*$-homomorphism $\phi: F(x) \to B(H)$ satisfying $\phi(x) = X$. We obtain a universal $C^*$-seminorm on $F(x)$ given by

$$\|z\| := \sup \{\|\phi(z)\|: \phi: F(x) \to B(H) \text{ representation of } (x|n, p, o)\},$$

for $z \in F(x)$. The completion of $F(x)$ with respect to this $C^*$-seminorm is called the universal $C^*$-algebra given by self-adjoint generators $x$, subject to the relations specified by $n, p$, and $o$. Following [LS12] we denote it by

$$C^* \left\{ x = (x_0, x_1, \ldots) \left| \begin{array}{ll} -c_k \leq x_k \leq c_k, & k \in \mathbb{N} \\ p_l(x) = 0, & l \in \mathbb{N} \\ \alpha_m x_{s_m} \leq \beta_m x_{t_m}, & m \in \mathbb{N} \end{array} \right. \right\},$$

or just $C^*(x | n, p, o)$.

It was observed by Blackadar [Bla85 Example 1.3(b)] that every separable $C^*$-algebra has a presentation with countably many generators and corresponding norm relations and countably many polynomial relations. Loring and Shulman [LS12 Lemma 7.3] modified the construction of Blackadar to show that every separable $C^*$-algebra $A$ has such a presentation with self-adjoint generators. That is, given a separable $C^*$-algebra $A$, there is a sequence $x = (x_k)_k$ of self-adjoint generators, a sequence $(c_k)_k$ of positive real numbers, and a sequence of NC polynomials $(p_l)_l$ in $x$ such that

$$A \cong C^* \left\{ x = (x_0, x_1, \ldots) \left| \begin{array}{ll} -c_k \leq x_k \leq c_k, & k \in \mathbb{N} \\ p_l(x) = 0, & l \in \mathbb{N} \end{array} \right. \right\}.$$
Assume that \( A = C^*(x \mid n, p) \), for a sequence of self-adjoint generators \( x \), and for some norm conditions \( n \) and NC polynomials \( p \). We set
\[
C^*_n(x \mid n, p) := C^* \left\{ e, x_0, x_1, \ldots \mid -c_k \leq x_k \leq c_k, \quad k \in \mathbb{N}, \right. \\
p_l(x) = 0, \quad l \in \mathbb{N} \\
-1 \leq e \leq 1, e^2 = e \\
x_k e = x_k = e x_k, \quad k \in \mathbb{N} \right\}.
\]
Then \( C^*_n(x \mid n, p) \) is isomorphic to \( A^+ \), the forced unitization of \( A \); see [Blal6], II.1.2.1, p.53. For example, \( C^+ \cong \mathbb{C} \oplus \mathbb{C} \).

**Lemma 3.3.** Let \( A \) and \( B \) be separable \( C^* \)-algebras, and let \( \varphi : A \to B \) be a morphism. Assume that \( A \) is given via self-adjoint generators and relations as
\[
A = C^*(x = (x_0, x_1, \ldots) \mid (R) : -c_k \leq x_k \leq c_k, \quad q_k(x) = 0, \quad k \in \mathbb{N}).
\]
Set \( \bar{x}_k := \varphi(x_k) \) for each \( k \in \mathbb{N} \), and choose a sequence \( y = (y_j)_j \) of additional self-adjoint generators in \( B \), choose a sequence \( (d_j)_j \) of positive real numbers, and choose a sequence \( (q_l)_l \) of NC polynomials in \( x \) and \( y \) that each contain at least one term from \( y \) (that is, \( q_l \) is not just a NC polynomial in \( x \)) such that
\[
B \cong C^*
\begin{align*}
\bar{x} &= (\bar{x}_0, \bar{x}_1, \ldots) \\
y &= (y_0, y_1, \ldots) \\
(R) : -c_k \leq \bar{x}_k \leq c_k, &q_k(\bar{x}) = 0, \quad k \in \mathbb{N} \\
(S_{\text{norm}}) : -d_j \leq y_j \leq d_j, &j \in \mathbb{N} \\
(S_{\text{pol}}) : q_l(\bar{x}, y) = 0, &l \in \mathbb{N}.
\end{align*}
\]
Let \( \varphi^+ : A \to B^+ \) denote the composition of \( \varphi \) with the inclusion \( B \subseteq B^+ \). Then the mapping cylinder \( Z_{\varphi^+} \) has a presentation as
\[
Z_{\varphi^+} \cong C^*
\begin{align*}
x, y, h &\\
(R) : -c_k \leq x_k \leq c_k, &q_k(x) = 0, \quad k \in \mathbb{N} \\
(S_{\text{norm}}) : -d_j \leq y_j \leq d_j, &j \in \mathbb{N} \\
(S_{\text{pol}}) : q_l(\bar{x}, y, h) = 0, &l \in \mathbb{N} \\
(C) : 0 \leq h \leq 1, &h x_k = x_k h, \quad h y_k = y_k h, \quad k \in \mathbb{N}
\end{align*}
\]
where for each \( l \in \mathbb{N} \) the polynomial \( \tilde{q}_l(x, y, h) \) is obtained from \( q_l \) by ‘homogenizing’ on the left with \( h \) in the \( y \)-variables, that is, if \( q_l(x, y) = \sum_{d=0}^{N_l} q_{l,d}(x, y) \) with \( N_l \geq 1 \) and where each polynomial \( q_{l,d} \) is \( d \)-homogeneous in \( y \), then \( \tilde{q}_l(\bar{x}, y, h) := \sum_{d=0}^{N_l} h^{N_l-d} q_{l,d}(x, y) \).

**Proof.** The proof goes along the lines of [LS12] Lemma 7.1. Let \( U \) be the universal \( C^* \)-algebra that we want to show is isomorphic to \( Z_{\varphi^+} \). Recall that
\[
Z_{\varphi^+} = \{ (a, f) \in A \oplus C([0,1], B^+) : \varphi(a) = f(0) \}.
\]
To clarify the notation, we write \( a \oplus f \) for an element \( (a, f) \in Z_{\varphi^+} \). For \( k \in \mathbb{N} \), we let \( \varphi(x_k) \in C([0,1], B^+) \) denote the constant function with value \( \varphi(x_k) \), and we let \( t y_k \in C([0,1], B^+) \) denote the function \( [0,1] \to B^+ \) given by \( t \mapsto t y_k \). Define elements of \( Z_{\varphi^+} \) as
\[
\tilde{x}_k := x_k \oplus \varphi(x_k), \quad \tilde{y}_k := 0 \oplus t y_k, \quad \tilde{h} := 0 \oplus t 1_{B^+}.
\]
Define a map \( \omega: U \to Z_{\varphi^+} \) on the generators of \( U \) by

\[
x_k \mapsto \tilde{x}_k, \quad y_j \mapsto \tilde{y}_j, \quad h \mapsto \tilde{h}.
\]

To show that this assignment defines a morphism, we need to verify that \( \tilde{\omega} \) contains \( \varphi \). Since each \( \omega_r \) is \( \varphi \)-homogeneous, we need to show that \( \tilde{\omega} \) is \( \varphi \)-homogeneous. To verify \( \tilde{\omega} \), let \( j \in \mathbb{N} \).

To verify \( \tilde{\omega} \), let \( j \in \mathbb{N} \). We decompose \( q_l \) as \( q_l(x,y) = \sum_{d=0}^{\bar{n}_l} q_{l,d}(x,y) \) with \( \bar{n}_l \geq 1 \) and where each polynomial \( q_{l,d} \) is \( d \)-homogeneous in \( y \). Then

\[
\tilde{q}_l(x,y,h) = \sum_{d=0}^{\bar{n}_l} h^{\bar{n}_l-d} q_{l,d}(x,y).
\]

(For example, for \( q = y_0 + y_1 x_1 - x_0 \) we obtain \( \tilde{q} = y_0 + y_1 x_1 - h x_0 \), and for \( q = y_1^2 + y_0 - x_1 \) we obtain \( \tilde{q} = y_1^2 + h y_0 - h^2 x_1^2 \).) Using at the fourth step that \( q_{l,d}(\varphi(x),y) = t^d q_{l,d}(\varphi(x),y) \), and using at the last step that \( q_l(\varphi(x),y) = q_l(x,y) = 0 \), we deduce that

\[
\tilde{q}_l(\tilde{x},\tilde{y},\tilde{h}) = \sum_{d=0}^{\bar{n}_l} h^{\bar{n}_l-d} q_{l,d}(\tilde{x},\tilde{y})
\]

as desired. Thus, \( \omega: U \to Z_{\varphi^+} \) is a well-defined morphism. We proceed to show that \( \omega \) is bijective.

To show that \( \omega \) is surjective, note that every \( z = a \oplus f \in Z_{\varphi^+} \) can be written as \( z = (a \oplus \varphi(a)) + (0 \oplus (f - \varphi(a))) \).

We have \( f - \varphi(a) \in C_0([0,1],B^+) \). Thus, it is enough to show that the image of \( \omega \) contains \( a \oplus \varphi(a) \), for \( a \in A \), and \( 0 \oplus f \), for \( f \in C_0([0,1],B^+) \).

Let \( a \in A \). Since \( \tilde{x} \) generates \( A \), we can choose a sequence of NC polynomials \( (r_n)_n \) such that \( \lim_n d(a - r_n(x)) = 0 \). Then

\[
\lim_n \|(a \oplus \varphi(a)) - r_n(\tilde{x})\| = 0.
\]

Since each \( r_n(\tilde{x}) \) belongs to the image of \( \omega \), we obtain that \( a \oplus \varphi(a) \) belongs to the image of \( \omega \), as desired.

On the other hand, as in the proof of [LST1 Lemma 7.1], to show that \( 0 \oplus C_0([0,1],B^+) \) belongs to the image of \( \omega \), it is enough to verify that \( 0 \oplus f \) is in the image of \( \omega \) for \( f \) given by \( f(t) = t^s(ty_{j_1})(ty_{j_2}) \ldots (ty_{j_n}) \), for every \( s,n \in \mathbb{N} \) and \( j_1, \ldots, j_n \in \mathbb{N} \). This follows since

\[
0 \oplus f = (\tilde{h})^s \tilde{y}_{j_1} \tilde{y}_{j_2} \ldots \tilde{y}_{j_n}.
\]
To show that $\omega$ is injective, let $z \in U$ with $z \neq 0$. Choose an irreducible representation $\sigma: U \to \mathcal{B}(K)$ with $\sigma(z) \neq 0$. Set 

$$X := (\sigma(x_0), \sigma(x_1), \ldots), \text{ and } Y := (\sigma(y_0), \sigma(y_1), \ldots), \text{ and } H := \sigma(h).$$

The relation (C) tells us that $H$ is a positive contraction that commutes with all operators in the image of $\sigma$. Since $\sigma$ is irreducible, $H$ is a scalar multiple of the identity operator. Hence, $H = \lambda I$ for some $\lambda \in [0, 1]$. We distinguish the two cases $\lambda = 0$ and $\lambda > 0$.

Let $\pi: Z_{p+} \to A$ be given by $\pi(a \oplus f) := a$. Then $\pi$ is a surjective morphism. The kernel of $\pi$ is naturally identified with $C_0((0, 1], B^+)$. We have the following short exact sequence

$$0 \to C_0((0, 1], B^+) \xrightarrow{\lambda} Z_{p+} \xrightarrow{\pi} A \to 0.$$

**Case 1.** Assume that $\lambda = 0$. Then $H = 0$, and the relations $(\tilde{S}_{\text{norm}})$ imply that $Y = 0$. Then $X$ is a representation of $\langle x | (R) \rangle$. Let $\tau': A = C^*(x | (R)) \to \mathcal{B}(K)$ be the induced morphism. One checks that $\sigma$ agrees with $\tau \circ \sigma \circ \omega$ on each generator of $U$, which implies that $\sigma = \tau \circ \sigma \circ \omega$. It follows that $\omega(z) \neq 0$, as required.

**Case 2.** Assume that $\lambda > 0$. Then $H = \lambda$. Let us verify that $\langle X, \lambda^{-1} Y \rangle$ is a representation of $\langle x | (R) \rangle$. Let $\tau: A = C^*(x | (R)) \to \mathcal{B}(K)$ be the induced morphism. One checks that $\sigma$ agrees with $\tau \circ \sigma \circ \omega$ on each generator of $U$, which implies that $\sigma = \tau \circ \sigma \circ \omega$. It follows that $\omega(z) \neq 0$, as desired.

To verify $(S_{\text{pol}})$ for $\langle X, \lambda^{-1} Y \rangle$, let $l \in \mathbb{N}$. We decompose $q_l$ according to the degree of homogeneity in $Y$ as above and compute

$$q_l(X, \lambda^{-1} Y) = \sum_{d=0}^{N_l} q_{l,d}(X, \lambda^{-1} Y)$$

$$= \sum_{d=0}^{N_l} \lambda^{-d} q_{l,d}(X, Y)$$

$$= \lambda^{-N_l} \sum_{d=0}^{N_l} \lambda^{d} q_{l,d}(X, Y)$$

$$= \lambda^{-N_l} \tilde{q}_l(X, Y, H)$$

$$= 0.$$

Let $\tau: B = C^*(x, y | (R), (\tilde{S}_{\text{norm}}), (S_{\text{pol}})) \to \mathcal{B}(K)$ be the induced morphism, and let $\tau^+: B^+ \to \mathcal{B}(K)$ be the unique extension to a unital morphism. Let $\text{ev}_{\lambda}: Z_{p+} \to B^+$ be given by $\text{ev}_{\lambda}(a \oplus f) := f(\lambda)$. We claim that $\sigma = \tau^+ \circ \text{ev}_{\lambda} \circ \omega$. It is enough to verify this on the generators of $U$. We have

$$(\tau^+ \circ \text{ev}_{\lambda} \circ \omega)(h) = (\tau^+ \circ \text{ev}_{\lambda})(0 \oplus t) = \tau^+(\lambda) = \lambda = H = \sigma(h).$$

Further,

$$(\tau^+ \circ \text{ev}_{\lambda} \circ \omega)(y_j) = (\tau^+ \circ \text{ev}_{\lambda})(0 \oplus ty_j) = \tau^+(\lambda y_j) = \lambda \tau(y_j) = \lambda \lambda^{-1} Y_j = \sigma(y_j),$$

for each $j \in \mathbb{N}$. Moreover,

$$(\tau^+ \circ \text{ev}_{\lambda} \circ \omega)(x_k) = (\tau^+ \circ \text{ev}_{\lambda})(x_k \oplus \varphi(x_k)) = \tau^+(\varphi(x_k)) = \tau(\bar{x}_k) = X_k = \omega(x_k),$$

for each $k \in \mathbb{N}$. Thus, $\sigma = \tau^+ \circ \text{ev}_{\lambda} \circ \omega$. It follows that $\omega(z) \neq 0$, as desired. \qed

In the next result, we use $[x, y]$ to denote the commutator $[x, y] := xy - yx$. 
Lemma 3.4. We retain the notation from Lemma 3.3. Given $n \in \mathbb{N}$, set

$$Z^{(n)}_{\varphi^+} := C^* \left\langle \begin{array}{l} x \\ y \\ h \end{array} \right| \begin{array}{l} (R) : -c_k \leq x_k \leq c_k, \quad q_k(x) = 0, \quad k \in \mathbb{N} \\ 0 \leq h \leq 1 \end{array} \right\rangle.$$

Assume that $A = C^*\langle x \mid (R) \rangle$ is semiprojective. Then $Z^{(n)}_{\varphi^+}$ is semiprojective.

Proof. The proof goes along the lines of [LS12 Lemma 7.2]. Let $C$ be a $C^*$-algebra with an increasing sequence $J_0 \triangleleft J_1 \triangleleft \ldots \triangleleft C$ of ideals and set $J := \bigcup_{m} J_m$. Let $x = (x_0, x_1, \ldots)$, $y = (y_0, y_1, \ldots)$, and $h$ be (sequences) of self-adjoint elements in $C/J$ satisfying the relations defining $Z^{(n)}_{\varphi^+}$. We need to find $m \in \mathbb{N}$ and lifts $\tilde{x}$, $\tilde{y}$, and $\tilde{h}$ in $C/J_m$ that satisfy the same relations.

Since $C^*\langle x \mid (R) \rangle$ is semiprojective, there exist $m \in \mathbb{N}$ and a sequence $x = (\tilde{x}_0, \tilde{x}_1, \ldots)$ of self-adjoint elements in $C/J_m$ that satisfy $(R)$ and that lift $x$. We may also find a lift $h \in C/J_m$ of $h$ such that $0 \leq h \leq 1$. Applying Davidson’s order lifting theorem, [Dav91 Corollary 2.2], see also [Lor97 Corollary 8.2.3, p.63], we find a lift $\tilde{y}$ of $y$ in $C/J_m$ that satisfies $(\tilde{S}^n_{\text{norm}})$, that is, such that $-d_j \tilde{h} \leq \tilde{y}_j \leq d_j \tilde{h}$ for every $j \in \mathbb{N}$.

Note that the finitely many polynomials defining $(\tilde{S}^n_{\text{pol}})$ and $(C^n)$ involve only finitely many variables and are homogeneous (of degree at least one) in the variables $y, h$ (but not necessarily in $x$). Let $\pi : C/J_m \to C/J$ denote the quotient morphism. We have

$$\|\tilde{q}_l(\pi(x), \pi(y), \pi(h))\| = \|\tilde{q}_l(x, y, h)\| \leq 1/n,$$

for $l = 0, \ldots, n$, and

$$\|\pi(h), \pi(\tilde{x}_k)\| = \|[h, x_k]\| \leq 1/n, \quad \text{and} \quad \|\pi(h), \pi(\tilde{y}_k)\| = \|\pi(h), \pi(\tilde{y}_k)\| \leq 1/n,$$

for $k = 0, \ldots, n$. By [LS12 Theorem 3.2] there exists $e \in J_{m+1}$ with $0 \leq e \leq 1$ such that $\tilde{h}' := e\tilde{h}e$ and the sequence $\tilde{y}' := e\tilde{y}e = (e\tilde{y}_0e, e\tilde{y}_1e, \ldots)$ lift $\tilde{h}$ and $\tilde{y}$, and such that $\tilde{h}'$, $\tilde{y}'$ and $\tilde{x}$ satisfy $(\tilde{S}^n_{\text{norm}})$ and $(C^n)$. Note that $\tilde{h}'$ and $\tilde{y}'$ also satisfy $(\tilde{s}^n_{\text{norm}})$ as

$$-d_j \tilde{h}' \leq -d_j e\tilde{h}e = e(-d_j \tilde{h})e \leq e\tilde{y}_je = \tilde{y}_j \leq e(-d_j \tilde{h})e = d_j \tilde{h},$$

for every $j \in \mathbb{N}$. This shows the semiprojectivity of $Z^{(n)}_{\varphi^+}$. \qed

The following is the main technical result of this paper. It shows that certain mapping cylinders are isomorphic to inductive limits of semiprojective $C^*$-algebras.

Theorem 3.5. Let $A$ and $B$ be separable $C^*$-algebras, and let $\varphi : A \to B$ be a morphism. If $A$ is semiprojective, then the mapping cylinder $Z_{\varphi}$ has a strong shape system.

Proof. The proof goes along the lines of [LS12 Theorem 7.4]. Let $\varphi^+ : A \to B^+$ denote the composition of $\varphi$ with $B \subseteq B^+$, as in Lemma 3.3. Recall that

$$Z_{\varphi^+} = \{(a, f) \in A \oplus C([0,1], B^+) : \varphi(a) = f(0)\}.$$

Let $\mathcal{g} : B^+ \to C$ be the natural quotient morphism, such that $B = \text{ker}(\mathcal{g})$. Given $(a, f) \in Z_{\varphi^+}$ we have $f(0) \in B$ and therefore $\mathcal{g}(f(0)) = 0$. We may therefore define $\pi : Z_{\varphi^+} \to C([0,1])$ by $\pi((a, f)) := \mathcal{g} \circ f$, for $a \in A$ and $f \in C([0,1], B^+)$. The natural inclusion $C([0,1], B) \to C([0,1], B^+)$ induces an injective morphism $\iota : Z_{\varphi} \to Z_{\varphi^+}$, which we use to identify $Z_{\varphi}$ with a sub-$C^*$-algebra of $Z_{\varphi^+}$. The
kernel of $\pi$ is $Z_\varphi$. Hence, we have a short exact sequence:

$$0 \to Z_\varphi \to Z_\varphi^+ \xrightarrow{\pi} C_0((0, 1]) \to 0.$$ 

For $n \in \mathbb{N}$, let $Z_{\varphi}^{(n)}$ be defined as in \[Lemma 3.4\] By construction, we have natural surjective morphisms $\gamma_{n+1,n}: Z_{\varphi}^{(n)} \to Z_{\varphi}^{(n+1)}$ such that the resulting inductive limit is isomorphic to $Z_\varphi^+$. Let $\gamma_{\infty,n}: Z_{\varphi}^{(n)} \to Z_\varphi^+$ be the surjective morphism to the inductive limit. For each $n \in \mathbb{N}$, set $D_n := \ker(\pi \circ \gamma_{\infty,n})$.

The morphism $\gamma_{n+1,n}$ induces a morphism $\psi_{n+1,n}: D_n \to D_{n+1}$. Then $Z_\varphi \cong \varinjlim D_n$. The situation is shown in the following commutative diagram, whose rows are short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \to & Z_\varphi & \xrightarrow{i} & Z_\varphi^+ & \xrightarrow{\pi} & C_0((0, 1]) & \to & 0 \\
\psi_{\infty,n} & & \gamma_{\infty,n} & & & & & & \\
0 & \to & D_{n+1} & \xrightarrow{\psi_{n+1,n}} & Z_{\varphi}^{(n+1)} & \xrightarrow{\gamma_{n+1,n}} & C_0((0, 1]) & \to & 0 \\
\psi_{n+1,n} & & \gamma_{n+1,n} & & & & & & \\
0 & \to & D_n & \xrightarrow{\psi_{n+1,n}} & Z_{\varphi}^{(n)} & \xrightarrow{\gamma_{n+1,n}} & C_0((0, 1]) & \to & 0 \\
\end{array}
$$

By \[Lemma 3.4\] each $Z_{\varphi}^{(n)}$ is semiprojective. An ideal $J$ in a semiprojective $C^*$-algebra $E$ is semiprojective if the quotient $E/J$ is projective; see \[End16, Corollary 3.1.3\], which generalizes \[LP98, Theorem 5.3\]. Since $C_0((0, 1])$ is projective, it follows that each $D_n$ is semiprojective. Thus, $Z_\varphi$ is isomorphic to an inductive limit of semiprojective $C^*$-algebras, as desired. \hfill $\square$

4. $C^*$-Algebras with Strong Shape System

In this section, we study closure properties of the class of separable $C^*$-algebras that have a strong shape system. Recall that a separable $C^*$-algebra is said to have a strong shape system if it is isomorphic to a sequential inductive limit of separable, semiprojective $C^*$-algebras; see \[Bla85, Definition 4.1\]. The main result of this section (and the whole paper) is \[Theorem 4.4\]. The class separable $C^*$-algebras that have a strong shape system is closed under shape domination. Hence, if two separable $C^*$-algebras are shape equivalent (in particular, if they are homotopy equivalent), then one has a strong shape system if and only if the other does; see \[Corollary 4.5\].

This provides many new examples of $C^*$-algebras with a strong shape system. For example, $C([0, 1]^n, A)$ has a strong shape system for every UCT-Kirchberg algebra $A$; see \[Example 4.8\].

**Proposition 4.1.** A separable $C^*$-algebra $A$ has a strong shape system if and only if its minimal unitization $\tilde{A}$ does.

**Proof.** If $A$ is unital, then there is nothing to show. So assume that $A$ is non-unital. Note that a $C^*$-algebra $\tilde{B}$ is semiprojective if and only if $\tilde{B}$ is; see \[Lor97, Theorem 14.1.7, p.108\].

To show the forward implication, assume that $A = \varinjlim A_n$, with each $A_n$ a semiprojective $C^*$-algebra. Then $\tilde{A} \cong \varinjlim \tilde{A}_n$, and each $\tilde{A}_n$ is semiprojective.

To show the backward implication, assume that $\tilde{A} = \varinjlim B_n$, with each $B_n$ a semiprojective $C^*$-algebra. Let $\gamma_{n+1,n}: B_n \to B_{n+1}$ be the connecting morphisms. 

\[\tilde{A} = \varinjlim \tilde{B}_n, \text{ with each } \tilde{B}_n \text{ a semiprojective } C^*\text{-algebra.} \]
By [Thi11, Proposition 4.8], we may assume that each $\gamma_{n+1,n}$ is surjective. Let $\pi: \tilde{A} \to C$ be the quotient map such that $A = \ker(\pi)$. For each $n \in \mathbb{N}$, set $A_n := \ker(\pi \circ \gamma_{n,n+1})$. The morphism $\gamma_{n+1,n}$ induces a morphism $\psi_{n+1,n}: A_n \to A_{n+1}$. Then $A \cong \lim_{\to n} A_n$. The situation is shown in the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & A \\
\psi_{n+1,n} & \downarrow & \gamma_{n+1,n} \\
A_n & \to & B_n \\
\psi_{n+1,n} & \downarrow & \gamma_{n+1,n} \\
0 & \to & A_{n+1} \\
\end{array}
\]

For each $n$, the algebra $A_n$ is an ideal of finite codimension in the semiprojective $C^*$-algebra $B_n$. It follows from [End17, Theorem 3.2] that $A_n$ is semiprojective. Hence, $A$ has a strong shape system, as desired. \hfill \Box

By [Thi11, Corollary 4.5], every separable, contractible $C^*$-algebra is an inductive limit of projective $C^*$-algebras. Combining this result with [Proposition 4.1] we obtain:

**Corollary 4.2.** Let $A$ be a contractible $C^*$-algebra. Then $\tilde{A}$ has a strong shape system.

**Lemma 4.3.** Let $A$, $B$ and $C$ be $C^*$-algebras, and let $\gamma: A \to C$, $\alpha: A \to B$ and $\beta: B \to C$ be morphisms satisfying $\gamma \simeq \beta \circ \alpha$. Then there exist morphisms $\varphi: A \to Z_\beta$ and $\omega: Z_\beta \to C$ such that $\gamma = \omega \circ \varphi$.

**Proof.** The homotopy $\gamma \simeq \beta \circ \alpha$ is given by a morphism $\psi: A \to C([0,1],C)$ satisfying

$$
\beta \circ \alpha = ev_0 \circ \psi, \quad \text{and} \quad \gamma = ev_1 \circ \psi.
$$

The mapping cylinder $Z_\beta$ is the pullback of $B$ and $C([0,1],C)$ along $\beta$ and $ev_0$. Let $\delta: Z_\beta \to C([0,1],C)$ be the natural morphism from the pullback. By the universal property of pullbacks, the morphisms $\alpha$ and $\psi$ induce a morphism $\varphi: A \to Z_\beta$ such that $\delta \circ \varphi = \psi$. The situation is shown in the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & C \\
\downarrow{\alpha} & \swarrow{\psi} & \downarrow{ev_1} \\
\downarrow{\delta} & & \downarrow{ev_0} \\
B & \xrightarrow{\beta} & C \\
\end{array}
\]

Set $\omega := ev_1 \circ \delta$. Then

$$
\omega \circ \varphi = ev_1 \circ \delta \circ \varphi = ev_1 \circ \psi = \gamma,
$$

as desired. \hfill \Box

**Theorem 4.4.** Let $A$ and $B$ be separable $C^*$-algebras with $A \sim_{Sh} B$. Then, if $B$ has a strong shape system, so does $A$.

**Proof.** Let $(A_n, \gamma_{n+1,n})$ be a shape system for $A$, and let $(B_n, \theta_{n+1,n})$ be a strong shape system for $B$. Using that $A \sim_{Sh} B$, it follows from [Bla85, Theorem 4.8] that $(A_n, \gamma_{n+1,n})$ is shape dominated by $(B_n, \theta_{n+1,n})$. Thus, after reindexing, we may
assume that there are morphisms $\alpha_n : A_n \to B_n$ and $\beta_n : B_n \to A_{n+1}$ such that $\gamma_{n+1,n} \simeq \beta_n \circ \alpha_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, applying Lemma 4.3 we obtain morphisms $\varphi_n : A_n \to Z_{\beta_n}$ and $\omega_n : Z_{\beta_n} \to A_{n+1}$ such that $\gamma_{n+1,n} = \omega_n \circ \varphi_n$. Set $\psi_{n+1,n} := \varphi_{n+1} \circ \omega_n : Z_{\beta_n} \to Z_{\beta_{n+1}}$. Then the inductive systems $(A_n, \gamma_{n+1,n})$ and $(Z_{\beta_n}, \psi_{n+1,n})$ are intertwined, as shown in the following commutative diagram:

\[
\begin{array}{cccccc}
A_n & \xrightarrow{\gamma_{n+1,n}} & A_{n+1} & \xrightarrow{\gamma_{n+2,n+1}} & A_{n+2} & \cdots \\
\varphi_n & & \omega_n & & \varphi_{n+1} & \\
Z_{\beta_n} & \xrightarrow{\psi_{n+1,n}} & Z_{\beta_{n+1}} & & & \\
\end{array}
\]

It follows that $A \cong \varinjlim_n Z_{\beta_n}$. By Theorem 3.5 each $Z_{\beta_n}$ is an inductive limit of semiprojective $C^*$-algebras. Applying [Thi11] Theorem 3.12, it follows that $A$ is an inductive limit of semiprojective $C^*$-algebras, as desired. □

**Corollary 4.5.** Let $A$ and $B$ be shape equivalent, separable $C^*$-algebras. Then $A$ has a strong shape system if and only if $B$ does.

**Example 4.6.** Let $A$ be a $C^*$-algebra. We set $\Sigma := C_0(\mathbb{R})$. Further, $\Sigma A := C_0(\mathbb{R}) \otimes A$ denotes the suspension of $A$, and similarly $\Sigma^2 A := C_0(\mathbb{R}^2) \otimes A$ is the second suspension of $A$.

The $C^*$-algebra $qA$ was introduced by Cuntz in his study of $KK$-theory. It is defined as the kernel of the morphism $A \ast A \to A$ obtained from the universal property of the free product $A \ast A$ applied to the identity morphism on both factors. In particular, we have a short exact sequence

$$0 \to qA \to A \ast A \to A \to 0.$$ 

In [Shu05], Shulman showed that the $C^*$-algebras $qA \otimes \mathbb{K}$ and $\Sigma^2 A \otimes \mathbb{K}$ are asymptotically equivalent. Dadarlat showed in [Dad94] that the notion of asymptotic equivalence and shape equivalence agree. Thus, $qA \otimes \mathbb{K}$ and $\Sigma^2 A \otimes \mathbb{K}$ are shape equivalent. Hence, by Corollary 4.5 $qA \otimes \mathbb{K}$ has a strong shape system if and only if $\Sigma^2 A \otimes \mathbb{K}$ does.

Let us consider the case $A = \mathbb{C}$. It is known that $q\mathbb{C}$ is semiprojective; see [Lor97] Chapter 16. It follows that $q\mathbb{C} \otimes \mathbb{K}$, and consequently $\Sigma^2 \mathbb{C} \otimes \mathbb{K}$, has a strong shape system. Note that $\Sigma^2 \mathbb{C} \otimes \mathbb{K}$ is isomorphic to $C_0(S^2 \setminus \{\ast\}) \otimes \mathbb{K}$, the stabilized algebra of continuous functions on the pointed sphere. We do not know if the stabilization is necessary, that is, if $\Sigma^2$ has a strong shape system. By Proposition 4.1 this is equivalent to the following open question.

**Question 4.7.** Does the $C^*$-algebra $C(S^2)$ of continuous functions on the two-sphere have a strong shape system?

**Example 4.8.** Recall that a Kirchberg algebra is a separable, purely infinite, simple, nuclear $C^*$-algebra. To simplify, we call a Kirchberg algebra that satisfies the universal coefficient theorem (UCT) a UCT-Kirchberg algebra. We refer to [Bla98] Section 23, p.232ff for details on the UCT.

In [End15] Corollary 4.6, Enders solved a long-standing conjecture of Blackadar by showing that a UCT-Kirchberg algebra is semiprojective if and only if its $K$-groups are finitely generated. It follows from [Lor02] Proposition 8.4.13 that every UCT-Kirchberg algebra is isomorphic to an inductive limit of UCT-Kirchberg algebras with finitely generated $K$-groups. We deduce that every UCT-Kirchberg algebra has a strong shape system.

It follows from Theorem 4.4 that every separable $C^*$-algebra that is shape dominated by a UCT-Kirchberg algebra has a strong shape system. For example,
$C([0,1]^n,A)$ has a strong shape system for every UCT-Kirchberg algebra $A$ and every $n \geq 1$.

The next result records permanence properties of the class of $C^*$-algebras that are isomorphic to inductive limits of semiprojective $C^*$-algebras.

**Theorem 4.9.** The class of separable $C^*$-algebras that have a strong shape system is closed under:

1. countable direct sums;
2. sequential inductive limits;
3. approximation by sub-$C^*$-algebras in the sense of [Thi11 Paragraph 3.1];
4. shape domination, in particular shape equivalence, homotopy domination and homotopy equivalence;
5. passing to matrix algebras or stabilization.

**Proof.** To prove statement (1), let $(A_n)_{n \in \mathbb{N}}$ be a sequence of separable $C^*$-algebras with strong shape systems. For each $n \in \mathbb{N}$, let $(A_n,k; \varphi_{k+1,n,k})$ be a strong shape system for $A_n$. Set $B := \bigoplus_{n \in \mathbb{N}} A_n$ and set $B_k := \bigoplus_{k \in \mathbb{N}} A_{n,k}$ for each $k \in \mathbb{N}$. Define $\psi_{k+1,n,k} : B_k \to B_{k+1}$ by mapping the summand $A_{n,k}$ in $B_k$ to the summand $A_{n,k+1}$ in $B_{k+1}$ by the map $\varphi_{k+1,n,k}^{(n)}$, for each $n = 0, \ldots, k$. Then each $B_k$ is semiprojective, and $B \cong \lim_{\to k} B_k$, as desired.

Statements (2) and (3) follow from Theorem 3.12 and Theorem 3.9 in [Thi11], respectively. Statement (4) follows from [Theorem 4.4].

To prove statement (5), let $A$ be a separable $C^*$-algebra with $A \cong \lim_{\to n} A_n$ for semiprojective $C^*$-algebras $A_n$. Given $k \in \mathbb{N}$, we have $A \otimes M_k \cong \lim_{\to n} A_n \otimes M_k$.

Further, we have $A \otimes K \cong \lim_{\to n} A_n \otimes M_n$, with connecting maps $A_n \otimes M_n \to A_{n+1} \otimes M_{n+1}$ given by the amplification of the connecting map $A_n \to A_{n+1}$ to $n \times n$-matrices, followed by the upper left corner embedding $A_{n+1} \otimes M_n \to A_{n+1} \otimes M_n$.

By [Lor97 Theorem 14.2.2, p.110], a matrix algebra of a separable, semiprojective $C^*$-algebra is again semiprojective. This shows that $A \otimes M_k$ and $A \otimes K$ have strong shape systems.

5. **Nuclear $C^*$-algebras with strong shape system**

In this section, we show that every separable, stable, nuclear, homotopy symmetric $C^*$-algebra satisfying the universal coefficient theorem (UCT) in $KK$-theory and with torsion-free $K_0$-group has a strong shape system; [Theorem 5.4]. Hence, if $A$ is a separable, nuclear $C^*$-algebra satisfying the UCT, then the stable suspension $\Sigma A \otimes K$ has a strong shape system if $K_1(A)$ is torsion-free, and the stable second suspension $\Sigma^2 A \otimes K$ has a strong shape system if $K_0(A)$ is torsion-free; see Corollary 5.6.

The notion of ‘homotopy symmetry’ was introduced by Dadarlat and Loring in [DL94 Section 5], to which we refer the reader for the definition. For homotopy symmetric $C^*$-algebras, shape theory and $E$-theory are closely related: By combining results in [DL94] and [Dad94], we obtain that two separable, stable, homotopy symmetric $C^*$-algebras are shape equivalent if and only if they are $E$-equivalent; see [Lemma 5.3]. We refer to [Bla98] for details on $KK$-theory, $E$-theory and the UCT.

To prove [Theorem 5.4] we construct for every given pair $(G_0, G_1)$ of countable, abelian groups with $G_0$ torsion-free a model $C^*$-algebra $A$ that is stable, nuclear, homotopy symmetric, satisfies the UCT, has a strong shape system, and such that $K_*(A) \cong (G_0, G_1)$; see [Lemma 5.2]. It is not clear if such model $C^*$-algebras exist with torsion in $K_0$; see Question 5.8 and Remark 5.9.
5.1. We set \( \Sigma := C_0(\mathbb{R}) \). Given \( n \in \mathbb{N} \) with \( n \geq 2 \), we define
\[
I_n := \{ f \in C((0,1], M_n) : f(1) \in C1_{M_n} \}.
\]
The \( C^* \)-algebra \( I_n \) is called the (nonunital) dimension-drop algebra of order \( n \). We have
\[
K_n(\Sigma) \cong (0, \mathbb{Z}), \quad \text{and} \quad K_n(I_n) \cong (0, \mathbb{Z}_n).
\]
The \( C^* \)-algebras \( \Sigma \) and \( I_n \) are homotopy symmetric; see Propositions 5.3 and 6.1 in [DL94].

**Lemma 5.2.** Let \( G_0 \) and \( G_1 \) be countable, abelian groups. Assume that \( G_0 \) is torsion-free. Then there exists a stable, nuclear, homotopy symmetric \( C^* \)-algebra \( A \) satisfying the UCT, with a strong shape system, and such that \( K_n(A) \cong (G_0, G_1) \).

**Proof.** We will construct stable, nuclear, homotopy symmetric \( A \) that satisfy the UCT, that have strong shape systems, and that satisfy \( K_n(A_0) \cong (G_0, 0) \) and \( K_n(A_1) \cong (0, G_1) \). Since the properties of being stable, nuclear, homotopy symmetric, satisfying the UCT, and having a strong shape system are preserved by direct sums, it will follow that \( A := A_0 \oplus A_1 \) has the desired properties.

To construct \( A_0 \), we use that every countable, torsion-free, abelian group arises as the \( K_0 \)-group of a separable AF-algebra; see for example [Blad98, Corollary 23.10.4, p.241]. Thus, we may choose a separable AF-algebra \( B \) with \( K_0(B) \cong G_0 \). Set \( A_0 := \Sigma^2 B \otimes \mathbb{K} \). Then \( A_0 \) is clearly nuclear and satisfies \( K_0(A_0) \cong (G_0, 0) \). By [DL94, Lemma 5.1], the tensor product of a (nuclear) homotopy symmetric \( C^* \)-algebra with any other \( C^* \)-algebra is again homotopy symmetric. Thus, since \( \Sigma \) is homotopy symmetric, so is \( A_0 \).

Let \( F_n \) be finite-dimensional \( C^* \)-algebras such that \( B \cong \lim_{\rightarrow n} F_n \). Then
\[
\Sigma^2 F_n \otimes \mathbb{K} \cong \lim_{\rightarrow n}(\Sigma^2 F_n \otimes \mathbb{K}).
\]
Thus, by [Theorem 4.9(2)], it is enough to verify that each \( \Sigma^2 F_n \otimes \mathbb{K} \) has a strong shape system. Since \( F_n \) is a direct sum of matrix algebras, applying [Theorem 4.9(1)], it is even enough to show that \( \Sigma^2 M_{k} \otimes \mathbb{K} \) has a strong shape system. This follows from [Example 4.6] since \( \Sigma^2 M_{k} \otimes \mathbb{K} \cong \Sigma^2 C \otimes \mathbb{K} \).

To construct \( A_1 \), choose finitely generated, abelian groups \( H_n \) and group homomorphisms \( \varphi_{n+1,n} : H_n \to H_{n+1} \), for \( n \in \mathbb{N} \), such that \( G_1 \cong \lim_{\rightarrow n} H_n \). Every finitely generated, abelian group is a direct sum of cyclic groups. Thus, for each \( n \) there exist \( e_n \in \mathbb{N} \) and cyclic groups \( H_{n,0}, \ldots, H_{n,e_n} \), such that
\[
H_n \cong H_{n,0} \oplus \ldots \oplus H_{n,e_n}.
\]
Using theses decompositions of \( H_n \) and \( H_{n+1} \), the morphism \( \varphi_{n+1,n} \) corresponds to a tuple \((\varphi_{n+1,n}^{(s,t)})_{s,t}\) of morphisms \( \varphi_{n+1,n}^{(s,t)} : H_{n,s} \to H_{n+1,t} \), for \( s = 0, \ldots, e_n \) and \( t = 0, \ldots, e_{n+1} \). Given \( n \in \mathbb{N} \) and \( k \in \{0, \ldots, e_n\} \), set \( B_{n,k} := \Sigma \) if \( H_{n,k} \cong \mathbb{Z}_n \) and set \( B_{n,k} := I_m \) if \( H_{n,k} \cong \mathbb{Z}_m \). We further define
\[
B_n := B_{n,0} \oplus \ldots \oplus B_{n,e_n}.
\]
Then \( K_n(B_n) \cong (0, H_n) \).

Let \( n \in \mathbb{N} \), \( s \in \{0, \ldots, e_n\} \) and \( t \in \{0, \ldots, e_{n+1}\} \). It follows from [RLL00, Lemma 13.2.1, p.222] that there exists a morphism \( \psi_{n,1,n}^{(s,t)} : B_{n,s} \otimes \mathbb{K} \to B_{n+1,t} \otimes \mathbb{K} \) such that \( K_1(\psi_{n+1,n}^{(s,t)}) = K_1(\varphi_{n+1,n}^{(s,t)}) \). This induces a morphism \( \psi_{n+1,n} : B_n \otimes \mathbb{K} \to B_{n+1} \otimes \mathbb{K} \) with \( K_1(\psi_{n+1,n}) = \varphi_{n+1,n} \).

Let \( A_1 \) be the inductive limit of the system \((B_n, \psi_{n+1,n})\). Then
\[
K_1(A_1) \cong \lim_{\rightarrow n} K_1(B_n) \cong \lim_{\rightarrow n} H_n \cong G_1.
\]
Further, \( K_0(B_n) \cong 0 \) for each \( n \), and therefore \( K_0(A_1) \cong 0 \).

Each \( B_n \) is stable, nuclear and satisfies the UCT, whence \( A_1 \) has the same properties. Moreover, each \( B_n \) is homotopy symmetric. By [Dad93, Theorem 3], homotopy symmetry passes to sequential inductive limits of separable \( C^* \)-algebras. Moreover, \( A_1 \) has a strong shape system since each \( B_n \) is semiprojective. \( \square \)

**Lemma 5.3.** Let \( A \) and \( B \) be separable, stable, homotopy symmetric \( C^* \)-algebras. Then \( A \) and \( B \) are shape equivalent if and only if they are \( E \)-equivalent.

**Proof.** By [Dad94, Theorem 3.9], two separable \( C^* \)-algebras are shape equivalent if and only if they are equivalent in the asymptotic homotopy category \( A \) studied in [Dad94]. By [DL94, Theorem 4.3], two stable, homotopy symmetric \( C^* \)-algebras are equivalent in \( A \) if and only if they are \( E \)-equivalent. \( \square \)

**Theorem 5.4.** Let \( A \) be a separable, stable, nuclear, homotopy symmetric \( C^* \)-algebra satisfying the UCT. Assume that \( K_0(A) \) is torsion-free. Then \( A \) has a strong shape system.

**Proof.** Apply [Lemma 5.2] to obtain a stable, nuclear, homotopy symmetric \( C^* \)-algebra \( B \) satisfying the UCT, with a strong shape system, and such that \( K_*(A) \cong K_*(B) \). We will show the following implications:

\[
K_*(A) \cong K_*(B) \Rightarrow A \sim_{KK} B \Rightarrow A \sim_{E} B \Rightarrow A \sim_{sh} B,
\]

where \( \sim_{KK}, \sim_{E} \) and \( \sim_{sh} \) mean \( KK \)-equivalence, \( E \)-equivalence and shape equivalence, respectively.

The first implication follows from [Bla98, Corollary 23.10.2, p.241], which shows that two \( C^* \)-algebras satisfying the UCT are \( KK \)-equivalent if (and only if) they have isomorphic \( K \)-groups. The second implication follows from [Bla98, Theorem 25.6.3, p.278], which shows that \( E \)-theory and \( KK \)-theory agree for separable, nuclear \( C^* \)-algebras. The third implication follows from [Lemma 5.3].

Hence, \( A \) and \( B \) are shape equivalent. Since \( B \) has a strong shape system, it follows from [Theorem 4.4] that \( A \) does as well. \( \square \)

**Remarks 5.5.** (1) [Theorem 5.4] still holds if the assumption that \( A \) is nuclear and satisfies the UCT is replaced by the assumption that \( A \) satisfies the UCT for \( E \)-theory as considered for example in [Bla98, 25.7.5, p.281].

(2) It is easy to construct \( C^* \)-algebras to which [Theorem 5.4] applies. Let \( B \) be a nuclear, stable, homotopy symmetric \( C^* \)-algebra satisfying the UCT. By [Bla98, 22.3.5(f), p.229], the class of nuclear \( C^* \)-algebras satisfying the UCT is closed under tensor products. Hence, for any nuclear \( C^* \)-algebra \( A \) satisfying the UCT, the tensor product \( A \otimes B \) is nuclear, stable, homotopy symmetric and satisfies the UCT.

Examples of homotopy symmetric \( C^* \)-algebras include \( C_0(X \setminus \{ * \}) \) for any connected, compact, metrizable space \( X \); see [Dad93] and [DL94, Proposition 5.3].

**Corollary 5.6.** Let \( A \) be a separable, nuclear \( C^* \)-algebra satisfying the UCT. If \( K_0(A) \) is torsion-free, then \( \Sigma^2 A \otimes K \) has a strong shape system. If \( K_1(A) \) is torsion-free, then \( \Sigma A \otimes K \) has a strong shape system.

**Example 5.7.** Recall from [Example 4.8] that every UCT-Kirchberg algebra has a strong shape system. Let \( A \) be a UCT-Kirchberg algebra, let \( X \) be a connected, compact, metrizable space, and let \( x \in X \). If \( K_0(C_0(X \setminus \{ x \}) \otimes A) \) is torsion-free, then \( C_0(X \setminus \{ x \}) \otimes A \otimes K \) has a strong shape system.

The restriction on the \( K \)-theory in [Lemma 5.2] and [Theorem 5.4] arises since we do not know if there exist suitable building blocks realizing torsion in \( K_0 \). In particular, we do not know the answer to the following question.

**Question 5.8.** Does \( \Sigma I_n \otimes K \) have a strong shape system, for \( n \geq 2 \)?
Remark 5.9. Let $n \geq 2$. Question 5.8 is closely related to [DL94, Question 7.2], where Dadarlat and Loring ask if there exists a separable, nuclear, homotopy symmetric, semiprojective $C^*$-algebra with $K$-theory $(\mathbb{Z}_n, 0)$.

Note that $\Sigma I_n \otimes \mathbb{K}$ is separable, stable, nuclear, homotopy symmetric and satisfies the UCT. Hence, the argument in the proof of Theorem 5.4 shows that Question 5.8 is equivalent to the following: Does there exist any separable, stable, nuclear, homotopy symmetric $C^*$-algebra $D$ satisfying the UCT, with strong shape system, and such that $K_n(D) \cong (\mathbb{Z}_n, 0)$?

In particular, a positive answer to the question of Dadarlat and Loring (additionally assumed to satisfy the UCT) implies a positive answer to Question 5.8.

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