Abstract. We prove derived equivalence of Calabi–Yau threefolds constructed by Ito–Miura–Okawa–Ueda as an example of non-birational Calabi–Yau varieties whose difference in the Grothendieck ring of varieties is annihilated by the affine line.

In a recent paper [IMOU] there was constructed a pair of Calabi–Yau threefolds $X$ and $Y$ such that their classes $[X]$ and $[Y]$ in the Grothendieck group of varieties are different, but

$$([X] - [Y])[\mathbb{A}^1] = 0.$$ 

The goal of this short note is to show that these threefolds are derived equivalent

$$\mathcal{D}(X) \cong \mathcal{D}(Y).$$

In course of proof we will construct an explicit equivalence of the categories.

We denote by $k$ the base field. All the functors between triangulated categories are implicitly derived.

As explained in [IMOU] the threefolds $X$ and $Y$ are related by the following diagram

Here

- $F$ is the flag variety of the simple algebraic group of type $G_2$,
- $Q$ and $G$ are the Grassmannians of this group:
  - $Q$ is a 5-dimensional quadric in $\mathbb{P}(V)$, where $V$ is the 7-dimensional fundamental representation, and
  - $G = \text{Gr}(2, V) \cap \mathbb{P}(W)$, where $W \subset \wedge^2 V$ is the 14-dimensional adjoint representation (this intersection is not dimensionally transverse!),
- $\pi: F \to Q$ and $\rho: F \to G$ are Zariski locally trivial $\mathbb{P}^1$-fibrations,
- $M$ is a smooth half-anticanonical divisor in $F$,
- $\pi_M := \pi|_M: M \to Q$ is the blowup with center in the Calabi–Yau threefold $X$,
- $\rho_M := \rho|_M: M \to G$ is the blowup with center in the Calabi–Yau threefold $Y$,
- $D$ and $E$ are the exceptional divisors of the blowups,
- $p := \pi_D: D \to X$ and $q := \rho_E: E \to Y$ are the contractions.

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We denote by $h$ and $H$ the hyperplane classes of $Q$ and $G$, as well as their pullbacks to $F$ and $M$. Then $h$ and $H$ form a basis of $\text{Pic}(F)$ in which the canonical classes can be expressed as follows:

(1) $K_Q = -5h, \quad K_G = -3H, \quad K_F = -2H - 2h, \quad K_M = -H - h.$

The classes $h$ and $H$ are relative hyperplane classes for the $\mathbb{P}^1$-fibrations $\rho: F \to G$ and $\pi: F \to Q$ respectively. We define rank 2 vector bundles $\mathcal{X}$ and $\mathcal{U}$ on $Q$ and $G$ respectively by

(2) $\pi_* \mathcal{O}_F(H) \cong \mathcal{X}^\vee, \quad \rho_* \mathcal{O}_F(h) \cong \mathcal{U}^\vee.$

Then

$$\mathbb{P}_Q(\mathcal{X}) \cong F \cong \mathbb{P}_G(\mathcal{U}).$$

It follows from (2) that $X \subset Q$ is the zero locus of a section of the vector bundle $\mathcal{X}^\vee(h)$ on $Q$ and $Y \subset G$ is the zero locus of a section of the vector bundle $\mathcal{U}^\vee(H)$ on $G$.

Since $H$ and $h$ are relative hyperplane classes for $F = \mathbb{P}_Q(\mathcal{X})$ and $F = \mathbb{P}_G(\mathcal{U})$ respectively, we have on $F$ exact sequences

$$0 \to \omega_{F/Q} \to \mathcal{X}^\vee(-H) \to \mathcal{O}_F \to 0, \quad 0 \to \omega_{F/G} \to \mathcal{U}^\vee(-h) \to \mathcal{O}_F \to 0.$$  

By (1) we have $\omega_{F/Q} \cong \mathcal{O}_F(3h - 2H)$ and $\omega_{F/G} \cong \mathcal{O}_F(H - 2h)$. Taking the determinants of the above sequences and dualizing, we deduce

(3) $\det(\mathcal{X}) \cong \mathcal{O}_Q(-3h), \quad \det(\mathcal{U}) \cong \mathcal{O}_G(-H).$

Furthermore, twisting the sequences by $\mathcal{O}_F(H)$ and $\mathcal{O}_F(h)$ respectively, we obtain

(4) $0 \to \mathcal{O}_F(3h - H) \to \mathcal{X}^\vee \to \mathcal{O}_F(H) \to 0,$

and

(5) $0 \to \mathcal{O}_F(H - h) \to \mathcal{U}^\vee \to \mathcal{O}_F(h) \to 0.$

Derived categories of both $Q$ and $G$ are known to be generated by exceptional collections. In fact, for our purposes the most convenient collections are

(6) $\mathbf{D}(Q) = \langle \mathcal{O}_Q(-3h), \mathcal{O}_Q(-2h), \mathcal{O}_Q(-h), \mathcal{S}, \mathcal{O}_Q, \mathcal{O}_Q(h) \rangle,$

where $\mathcal{S}$ is the spinor vector bundle of rank 4, see [Kap], and

(7) $\mathbf{D}(G) = \langle \mathcal{O}_G(-H), \mathcal{U}, \mathcal{O}_G, \mathcal{U}^\vee, \mathcal{O}_G(H), \mathcal{U}^\vee(H) \rangle.$

This collection is obtained from the collection of [Kuz, Section 6.4] by a twist (note that $\mathcal{U} \cong \mathcal{U}^\vee(-H)$ by (3)). In fact, for the argument below one even does not need to know that this exceptional collection is full; on a contrary, one can use the argument to prove its fullness, see Remark [Kuz].

Using two blowup representations of $M$ and the corresponding semiorthogonal decompositions

(8) $\langle \pi_M^*(\mathbf{D}(Q)), i_* p^*(\mathbf{D}(X)) \rangle = \mathbf{D}(M) = \langle \rho_M^*(\mathbf{D}(G)), j_* q^*(\mathbf{D}(Y)) \rangle$

together with the above exceptional collections, we see that $\mathbf{D}(X)$ and $\mathbf{D}(Y)$ are the complements in $\mathbf{D}(M)$ of exceptional collections of length 6, so one can guess they are equivalent. Below we show that this is the case by constructing a sequence of mutations transforming one exceptional collection to the other.

We start with some cohomology computations:
Lemma 1. (i) Line bundles $\mathcal{O}_F(th-H)$ and $\mathcal{O}_F(tH-h)$ are acyclic for all $t \in \mathbb{Z}$.

(ii) Line bundles $\mathcal{O}_F(-2H)$ and $\mathcal{O}_F(2h-2H)$ are acyclic and

$$H^*(F, \mathcal{O}_F(3h-2H)) = k[-1].$$

(iii) Vector bundles $\mathcal{U}(-2H)$, $\mathcal{U}(-H)$, $\mathcal{U}(h-H)$, and $\mathcal{U} \otimes \mathcal{U}(-H)$ on $F$ are acyclic and

$$H^*(F, \mathcal{U}(h)) = k, \quad H^*(F, \mathcal{U} \otimes \mathcal{U}(h)) \cong k[-1].$$

Proof. Part (i) is easy since $\pi_*\mathcal{O}_F(-H) = 0$ and $\rho_*\mathcal{O}_F(-h) = 0$. For part (ii) we note that

$$(9) \quad \pi_*\mathcal{O}_F(-2H) \cong (\det \mathcal{X})[-1] \cong \mathcal{O}_Q(-3h)[-1],$$

so acyclicity of $\mathcal{O}_F(th-2H)$ for $-1 \leq t \leq 2$ and the formula for the cohomology of $\mathcal{O}_F(3h-2H)$ follow. For part (iii) we push forward the bundles $\mathcal{U}(-2H)$, $\mathcal{U}(-H)$, $\mathcal{U}(h-H)$, and $\mathcal{U} \otimes \mathcal{U}(-H)$ to $G$ and applying [2] we obtain

$$\mathcal{U}(-2H), \mathcal{U}(-H), \mathcal{U} \otimes \mathcal{U}(-H), \mathcal{U} \otimes \mathcal{U}(-H).$$

Their acyclicity follows from orthogonality of $\mathcal{U}^\vee(H)$ to the collection $(\mathcal{O}_G(-H), \mathcal{U}, \mathcal{O}_Q, \mathcal{U}^\vee)$ in view of the exceptional collection [7]. Analogously, pushing forward $\mathcal{U}(h)$ to $G$ we obtain $\mathcal{U} \otimes \mathcal{U}^\vee$, and its cohomology is $k$ since $\mathcal{U}$ is exceptional. Finally, using [5] we see that $\mathcal{U} \otimes \mathcal{U}(h)$ has a filtration with factors $\mathcal{O}_F(-h)$, $\mathcal{O}_F(h-H)$, and $\mathcal{O}_F(3h-2H)$. The first two are acyclic by part (i) and the last one has cohomology $k[-1]$ by part (ii). It follows that the cohomology of $\mathcal{U} \otimes \mathcal{U}(h)$ is also $k[-1]$. \qed

Corollary 2. The following line and vector bundles are acyclic on $M$:

$$\mathcal{O}_M(h-H), \mathcal{O}_M(3h-H), \mathcal{U}(h-H).$$

Moreover,

$$H^*(M, \mathcal{U}(h)) = k, \quad H^*(M, \mathcal{U} \otimes \mathcal{U}(h)) = k[-1].$$

Proof. Since $M \subset F$ is a divisor with class $h+H$ we have a resolution

$$0 \to \mathcal{O}_F(-h-H) \to \mathcal{O}_F \to \mathcal{O}_M \to 0.$$

Tensoring it with the required bundles and using the Lemma [1] we obtain the required results. \qed

Proposition 3. We have an exact sequence on $F$ and $M$:

$$(10) \quad 0 \to \mathcal{U} \to \mathcal{I}' \to \mathcal{U}^\vee(-h) \to 0,$$

where $\mathcal{I}'$ is (the pullback to $F$ or $M$ of) a rank 4 vector bundle on $Q$.

Later we will identify the bundle $\mathcal{I}'$ constructed as extension [10] with the spinor bundle $\mathcal{I}$ on $Q$.

Proof. We will construct this exact sequence on $F$, and then restrict it to $M$. First, note that by Lemma [1] we have $\mathrm{Ext}^*(\mathcal{U}^\vee(-h), \mathcal{U}) \cong H^*(F, \mathcal{U} \otimes \mathcal{U}(h)) \cong k[-1]$, hence there is a canonical extension of $\mathcal{U}^\vee(-h)$ by $\mathcal{U}$. We denote by $\mathcal{I}'$ the extension, so that we have an exact sequence [10]. Obviously, $\mathcal{I}'$ is locally free of rank 4. We have to check that it is a pullback from $Q$.

Using exact sequences

$$0 \to \mathcal{O}_F(-h) \to \mathcal{U} \to \mathcal{O}_F(h-H) \to 0 \quad \text{and} \quad 0 \to \mathcal{O}_F(H-2h) \to \mathcal{U}^\vee(-h) \to \mathcal{O}_F \to 0$$

(obtained from [5] by the dualization and a twist) and the cohomology computations of Lemma [1] we see that extension [10] is induced by a class in $\mathrm{Ext}^1(\mathcal{O}_F(H-2h), \mathcal{O}_F(h-H)) \cong H^1(F, \mathcal{O}_F(3h-2H)) = k[-1]$. By [1] the corresponding extension is $\mathcal{I}^\vee(-2h)$. It follows that the sheaf $\mathcal{I}'$ has a 3-step filtration with factors being $\mathcal{O}_F(-h)$, $\mathcal{I}^\vee(-h)$, and $\mathcal{O}_F$. All these sheaves are pullbacks from $Q$, and since the subcategory $\pi^*(\mathcal{D}(Q)) \subset \mathcal{D}(F)$ is triangulated (because the functor $\pi^*$ is fully faithful), it follows that $\mathcal{I}'$ is also a pullback from $Q$. \qed
Now we are ready to explain the mutations. We start with a semiorthogonal decomposition
\begin{equation}
\mathbf{D}(M) = \langle \mathcal{O}_M(-H), \mathcal{U}, \mathcal{O}_M, \mathcal{U}^\vee, \mathcal{O}_M(H), \mathcal{U}^\vee(H), \Phi_0(\mathbf{D}(Y)) \rangle,
\end{equation}
\begin{equation}
\Phi_0 = j_* \circ q^*: \mathbf{D}(Y) \to \mathbf{D}(M),
\end{equation}
obtained by plugging (7) into the right hand side of (3). Now we apply a sequence of mutations, modifying the functor \(\Phi_0\).

First, we mutate \(\Phi_0(\mathbf{D}(Y))\) two steps to the left:
\begin{equation}
\mathbf{D}(M) = \langle \mathcal{O}_M(-H), \mathcal{U}, \mathcal{O}_M, \mathcal{U}^\vee, \Phi_1(\mathbf{D}(Y)), \mathcal{O}_M(H), \mathcal{U}^\vee(H) \rangle,
\end{equation}
\begin{equation}
\Phi_1 = \mathbf{L}(\mathcal{O}_M(H), \mathcal{U}^\vee(H)) \circ \Phi_0.
\end{equation}
Here \(\mathbf{L}\) denotes the left mutation functor.

Next, we mutate the last two terms to the far left (these objects got twisted by \(K_M = -h - H\))
\begin{equation}
\mathbf{D}(M) = \langle \mathcal{O}_M(-h), \mathcal{U}^\vee(-h), \mathcal{O}_M(-H), \mathcal{U}, \mathcal{O}_M, \mathcal{U}^\vee, \Phi_1(\mathbf{D}(Y)) \rangle.
\end{equation}
Next, we mutate \(\mathcal{O}_M(-h)\) and \(\mathcal{U}^\vee(-h)\) one step to the right. As \(\text{Ext}^\bullet(\mathcal{U}^\vee(-h), \mathcal{O}_M(-H)) \cong H^\bullet(M, \mathcal{U}(h - H)) = 0\), and \(\text{Ext}^\bullet(\mathcal{O}_M(-h), \mathcal{O}_M(-H)) \cong H^\bullet(M, \mathcal{O}_M(h - H)) = 0\) by Corollary 2 we obtain
\begin{equation}
\mathbf{D}(M) = \langle \mathcal{O}_M(-h), \mathcal{O}_M(-h), \mathcal{U}^\vee(-h), \mathcal{O}_M, \mathcal{U}^\vee, \mathcal{O}_M, \mathcal{U}^\vee, \Phi_1(\mathbf{D}(Y)) \rangle.
\end{equation}
Next, we mutate \(\mathcal{U}\) one step to the left. As \(\text{Ext}^\bullet(\mathcal{U}^\vee(-h), \mathcal{U}) \cong H^\bullet(\mathcal{U} \otimes \mathcal{U}(h)) \cong k[-1]\) by Corollary 2 the resulting mutation is an extension, which in view of (10) gives \(\mathcal{S}'\). Thus, we obtain
\begin{equation}
\mathbf{D}(M) = \langle \mathcal{O}_M(-h), \mathcal{O}_M(-h), \mathcal{S}', \mathcal{U}^\vee(-h), \mathcal{O}_M, \mathcal{U}^\vee, \mathcal{O}_M, \mathcal{U}^\vee, \Phi_1(\mathbf{D}(Y)) \rangle.
\end{equation}
Next, we mutate \(\mathcal{O}_M(-H)\) to the far right (this object got twisted by \(-K_M = h + H\)):
\begin{equation}
\mathbf{D}(M) = \langle \mathcal{O}_M(-h), \mathcal{S}', \mathcal{U}^\vee(-h), \mathcal{O}_M, \mathcal{U}^\vee, \Phi_1(\mathbf{D}(Y)), \mathcal{O}_M(h) \rangle.
\end{equation}
Next, we mutate \(\Phi_1(\mathbf{D}(Y))\) one step to the right:
\begin{equation}
\mathbf{D}(M) = \langle \mathcal{O}_M(-h), \mathcal{S}', \mathcal{U}^\vee(-h), \mathcal{O}_M, \mathcal{U}^\vee, \mathcal{O}_M(h), \mathcal{O}_M(h), \Phi_2(\mathbf{D}(Y)) \rangle,
\end{equation}
\begin{equation}
\Phi_2 = \mathbf{R}_{\mathcal{O}_M(h)} \circ \Phi_1.
\end{equation}
Here \(\mathbf{R}\) denotes the right mutation functor.

Next, we mutate simultaneously \(\mathcal{U}^\vee(-h)\) and \(\mathcal{U}^\vee\) one step to the right. As \(\text{Ext}^\bullet(\mathcal{U}^\vee(-h), \mathcal{S}') \cong \text{Ext}^\bullet(\mathcal{U}^\vee, \mathcal{O}_M(h)) = H^\bullet(M, \mathcal{U}(h)) = k\) by Corollary 2 the resulting mutation is the cone of a morphism, which in view of (3) and its twist by \(\mathcal{O}_M(-h)\) gives \(\mathcal{O}_M(H - 2h)\) and \(\mathcal{O}_M(H - h)\) respectively. Thus we obtain
\begin{equation}
\mathbf{D}(M) = \langle \mathcal{O}_M(-h), \mathcal{S}', \mathcal{O}_M, \mathcal{O}_M(H - 2h), \mathcal{O}_M(h), \mathcal{O}_M(H - h), \Phi_2(\mathbf{D}(Y)) \rangle.
\end{equation}
Next, we mutate \(\mathcal{O}_M(h)\) one step to the left. As \(\text{Ext}^\bullet(\mathcal{O}_M(H - 2h), \mathcal{O}_M(h)) \cong H^\bullet(M, \mathcal{O}_M(3h - H)) = 0\) by Corollary 2 we obtain
\begin{equation}
\mathbf{D}(M) = \langle \mathcal{O}_M(-h), \mathcal{S}', \mathcal{O}_M, \mathcal{O}_M(h), \mathcal{O}_M(H - 2h), \mathcal{O}_M(H - h), \Phi_2(\mathbf{D}(Y)) \rangle.
\end{equation}
Next, we mutate \(\Phi_2(\mathbf{D}(Y))\) two steps to the left:
\begin{equation}
\mathbf{D}(M) = \langle \mathcal{O}_M(-h), \mathcal{S}', \mathcal{O}_M, \mathcal{O}_M(h), \Phi_3(\mathbf{D}(Y)), \mathcal{O}_M(H - 2h), \mathcal{O}_M(H - h) \rangle,
\end{equation}
\begin{equation}
\Phi_3 = \mathbf{L}_{\mathcal{O}_M(H - 2h), \mathcal{O}_M(H - h)} \circ \Phi_2.
\end{equation}
Finally, we mutate \(\mathcal{O}_M(H - 2h)\) and \(\mathcal{O}_M(H - h)\) to the far left:
\begin{equation}
\mathbf{D}(M) = \langle \mathcal{O}_M(-3h), \mathcal{O}_M(-2h), \mathcal{O}_M(-h), \mathcal{S}', \mathcal{O}_M, \mathcal{O}_M(h), \Phi_3(\mathbf{D}(Y)) \rangle.
\end{equation}
Now we finished with mutations, and it remains to check that the resulting semiorthogonal decomposition provides an equivalence of categories. To do this, we first observe the following

**Lemma 4.** The bundle $\mathcal{S}'$ is isomorphic to the spinor bundle $\mathcal{S}$ on $Q$.

**Proof.** The first six objects in (17) are pullbacks from $Q$ by $\pi_M$. Since $\pi_M^*$ is fully faithful, the corresponding objects on $Q$ are also semiorthogonal. In particular, the bundle $\mathcal{S}'$ on $Q$ is right orthogonal to $\mathcal{O}_Q$ and $\mathcal{O}_Q(h)$ and left orthogonal to $\mathcal{O}_Q(-3h)$, $\mathcal{O}_Q(-2h)$, and $\mathcal{O}_Q(-h)$. By (3) the intersection of these orthogonals is generated by the spinor bundle $\mathcal{S}$. Therefore, $\mathcal{S}'$ is a multiple of the spinor bundle $\mathcal{S}$. Since the ranks of both $\mathcal{S}$ and $\mathcal{S}'$ are 4, the multiplicity is 1, so $\mathcal{S}' \cong \mathcal{S}$. 

Thus the first six objects of (17) generate $\pi_M^*(\mathcal{D}(Q))$. Comparing (17) with (5) and (8), we conclude that the last component $\Phi_3(\mathcal{D}(Y))$ coincides with $i_*q^*(\mathcal{D}(X))$. Altogether, this proves the following

**Theorem 5.** The functor

$$
\Phi_3 = \mathbf{L}(\mathcal{O}(H-2h),\mathcal{O}(H-h)) \circ \mathbf{R}(\mathcal{O}(h)) \circ \mathbf{L}(\mathcal{O}(H),\mathcal{H}(H)) \circ j_* \circ q^*: \mathcal{D}(Y) \to \mathcal{D}(M)
$$

is an equivalence of $\mathcal{D}(Y)$ onto the triangulated subcategory of $\mathcal{D}(M)$ equivalent to $\mathcal{D}(X)$ via the embedding $i_* \circ p^*: \mathcal{D}(X) \to \mathcal{D}(M)$. In particular, the functor

$$
\Psi = p_* \circ i^* \circ \mathbf{L}(\mathcal{O}(H-2h),\mathcal{O}(H-h)) \circ \mathbf{R}(\mathcal{O}(h)) \circ \mathbf{L}(\mathcal{O}(H),\mathcal{H}(H)) \circ j_* \circ q^*: \mathcal{D}(Y) \to \mathcal{D}(X)
$$

is an equivalence of categories.

**Remark 6.** Let us sketch how the arguments above can be also used to prove fullness of (7). Denote by $\mathcal{C}$ the orthogonal to the collection (7) in $\mathcal{D}(G)$. Then we still have a semiorthogonal decomposition (14), with $\Phi_0(\mathcal{D}(Y))$ replaced by $\langle \mathcal{C}', \mathcal{D}(Y) \rangle$. We can perform the same sequence of mutation, keeping the subcategory $\mathcal{C}$ together with $\mathcal{D}(Y)$. For instance, in (13) we write $\mathbf{L}(\mathcal{O}(H,H),\mathcal{H}(H)) \circ \mathbf{L}(\mathcal{O}(H),\mathcal{H}(H)) \circ j_* \circ q^*$ instead of just $\Phi_1(\mathcal{D}(Y))$ and so on. In the end, we arrive at (17) with $\Phi_3(\mathcal{D}(Y))$ replaced by $\langle \mathcal{C}', \mathcal{C}' \rangle$ with $\mathcal{C}'$ equivalent to $\mathcal{C}$. Comparing it with (6) and (8), we deduce that $\mathcal{D}(X)$ has a semiorthogonal decomposition with two components equivalent to $\mathcal{C}$ and $\mathcal{D}(Y)$. But $X$ is a Calabi–Yau variety, hence its derived category has no nontrivial semiorthogonal decompositions by [Bri]. Therefore $\mathcal{C} = 0$ and so exceptional collection (7) is full.

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