THE MOTION OF A SHOCK WAVE THROUGH A NON-UNIFORM ONE-DIMENSIONAL MEDIUM IN THE CASE OF ARBITRARY EQUATION OF STATE

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The derivation of the equation of one-dimensional movement of a solitary shock wave is given. This derivation shows, that the differential equation of movement of a solitary plane shock wave in the channel with variable area, is exact, if simplifying assumptions, made during derivation, are realized. But these assumptions in plane geometry it is possible to realize only approximately; situation with spherical and cylindrical shock waves is opposite.

1 Introduction

In 1957 Chisnell [1], being based on ideas of Chester [2], has deduced the equation of one-dimensional movement of a solitary shock wave on substance with polytropic equation of state. A little bit later the Whitham [3] has offered "... simple rule", which together with shock relations "... determines the motion of the shock wave". Actually Whitham was the first, who has considered the equation of one-dimensional movement of a solitary shock wave on substance with the arbitrary equation of state (see below the equation (14)), but as the approximate equation, from which for a special case of the polytropic equation of a state, one can derive the Chisnell equation.

2 The derivation of the equation

We shall consider one-dimensional movements of solitary shock waves. Movement of a shock wave is identified with movement of its front. One-dimensional movements of shock waves are understood as movements of spherically symmetric, spherically symmetric and plane shock waves. Shock waves are assumed solitary, i.e. it is not considered overtaking one shock wave by other shock wave etc. Naturally, the solitary shock wave is an abstraction. Be-
low, during the derivation of the equation of one-dimensional movement of a solitary plane shock wave the differential calculus will be used, how it was intuitively used in Newton time. After development of the nonstandard analysis by Robinson (see [4]) such operating with infinitesimal differentials may be regarded as quite correct.

Following Chisnell, we shall consider movement of a plane shock wave in the channel, having in some place infinitesimal jump of the sectional area \(dA\) – see figure (cp. it with Fig. 1 in Chisnell’s article [1], this figure can be also compared with Fig. 1, p. 193 in [5]). Let’s choose the system of coordinates connected to the channel, which axis \(X\) we shall direct along the channel. We shall assume, that walls of the channel are indefinitely rigid, so it is possible to neglect interaction of a shock wave with them, the plane front of a shock wave is perpendicular to walls of the channel etc. (i.e. it is supposed, that it is possible to neglect the three-dimensional effects arising at movement of a shock wave), the substance before front of a shock wave is homogeneous

![Figure 1: The shock wave, separating region 1, 2, is incident on small change in the area of a channel from \(A\) to \(A + dA\). The resulting transmitted shock separates regions 4, 5. Regions 3, 4 are separated by simple Riemann wave. The shape of the channel is shown on the top of the figure.](image)
and is motionless relatively the channel (at presence of the area change last assumption is implicitly containing in the assumption of homogeneity of substance before front of a shock wave). As it is usually accepted, by an index ”_0” we shall mark parameters of substance before front of a shock wave. As one-dimensional movement is considered, all vectors can be identified with their projections to axis X. Thus: \( u_0 = 0, \rho_0, p_0, \varepsilon_0, s_0 \) – respectively, speed (≡ projection of speed on axis X), density, pressure, specific (referred to a mass unit) internal energy, specific (referred to a mass unit) entropy of substance before front of a shock wave. As the parameter, defining strength of a shock wave, we shall take: 

\[ \sigma = \frac{\rho}{\rho_0}. \]  

– the ratio of density of substance behind front of a shock wave to density of substance before front of a shock wave. It is universal, dimensionless, finite parameter. Chisnell as the parameter, describing strength of a shock wave, have used the ratio of pressure: \( z = p/p_0 \), – that can not be used, for example, for some equations of a state of substance, when \( p_0 = 0 \). In one-dimensional case assignment of \( \sigma \) (or \( z \)), with known parameters of substance before the front of the shock wave \( u_0 = 0, \rho_0, p_0 \) completely determines parameters of substance behind front of a shock wave and speed of movement of shock wave \( D \) – see theorem 5.5 in [5].

On Fig. 1 the situation is schematically represented, when the shock wave, moving from the left to the right in positive direction of axis X \( (D > 0) \), will pass through jump of area of the channel \( dA \). Arising gas-dynamic break splits into simple Riemann r-wave (according to definition in § 16, [5]) moving back from the front of the shock wave and shock wave with changed on \( d\sigma \) strength, moving to the right (see § 17 in [5]). For initial shock wave moving from right to left \( (D < 0) \) instead of r-wave we should have Riemann l-wave – theorem 16.2 in [5]. Situation, represented on Fig. 1 corresponds to supersonic motion behind front of the shock wave \( |u| > c \), where \( c \) – speed of a sound behind front of a shock wave. Following reasonings are correct without change and in a subsonic case \( |u| < c \).

Changes of speed \( u \) and pressure \( p \) behind front of the shock wave, corresponding to change of shock wave strength after passage of the area change, are ”compensated” by changes of speed \( d_Au \) and pressure \( d_Ap \) at adiabatic flowing of substance behind front of shock wave through the jump of the sectional area and by changes of speed and pressure in Riemann wave (for Riemann r-wave, when \( D > 0 \), index ”_+” will be used, for l-wave – index ”_-”). Thus, we have
the system of the equations:

$$\frac{du}{d\sigma} \cdot d\sigma = d_A u + d_\pm u,$$

(2)

$$\frac{dp}{d\sigma} \cdot d\sigma = d_A p + d_\pm p.$$  

(3)

At adiabatic flowing of substance behind front of the shock wave through the jump of the sectional area $dA$ following relations must be satisfied:

$$d_A (A\rho u) = 0,$$

(4)

$$d_A \left( \frac{u^2}{2} + \varepsilon + \frac{p}{\rho} \right) = 0,$$

(5)

$$d_A s = 0.$$  

(6)

First of them expresses law of conservation of mass, the second is Bernoulli equation, the last – adiabatic condition. The adiabatic condition implies equalities:

$$d_A p = c^2 \cdot d_A \rho,$$

(7)

$$d_A \varepsilon = \frac{p}{\rho^2} \cdot d_A \rho.$$  

(8)

Using (7) and (8), from (4)-(6) we receive system of the equations:

$$d_A u = -\frac{c^2}{u} \cdot \frac{d_A \rho}{\rho},$$

(9)

$$d_A p = c^2 \cdot d_A \rho,$$  

(10)

$$-\frac{dA}{A} = \frac{d_A \rho}{\rho} \cdot \left( 1 - \frac{c^2}{u^2} \right).$$  

(11)

The system of the equations (9)-(11) completely determines changes of parameters of substance at its motion through the jump of the sectional area of the channel.

In Riemann wave the changes of parameters of substance are in accord with relations:

$$d_\pm u \pm \frac{d_\pm p}{\rho c} = 0,$$

(12)

$$d_\pm p = c^2 \cdot d_\pm \rho.$$  

(13)

The equation (12) follows from definition of Riemann invariants and the equation (13) follows from adiabatic condition for movement in Riemann wave.
Using (9)-(11) and (12)-(13), from (2)-(3) we receive the equation:

$$-\frac{dA}{A} = \left(\frac{1}{u} \pm \frac{1}{c}\right) \cdot \left(\frac{du}{d\sigma} \pm \frac{1}{\rho c} \cdot \frac{dp}{d\sigma}\right) \cdot d\sigma. \tag{14}$$

For a shock wave, moving from left to right \((D > 0)\), in the right part (14) it is necessary to take signs ”+”. If \(D < 0\) (the shock wave moves from right to left), in the right part of the equation (14) it is necessary to take signs ”−”.

It is interesting to note, that the right part of the equation (14) is a product of a combination of speed of substance and speed of a sound on differential of appropriate Riemann invariant.

For the channel with arbitrary (smooth) dependence of the area of section on coordinate \(x - A(x)\), in the assumption of plane symmetry of considered movement, it is possible to pass from differential (14) to integrated equality:

$$-\int_{A_i}^{A} \frac{dA'}{A'} = \int_{\sigma_i}^{\sigma} \left(\frac{1}{u} \pm \frac{1}{c}\right) \cdot \left(\frac{du}{d\sigma'} \pm \frac{1}{\rho c} \cdot \frac{dp}{d\sigma'}\right) \cdot d\sigma', \tag{15}$$

– where \(A_i\) and \(\sigma_i\) – values \(A\) and \(\sigma\) at some moment of time \(t_i\).

The equation (15) can be named (integrated) equation of movement of a solitary plane shock wave on homogeneous substance in the channel of variable area, as at known dependence \(A(x)\) it determines speed of shock wave \(D\) as implicit function of coordinate \(x\) and with trivial equation:

$$\frac{dx}{dt} = D(x), \tag{16}$$

– allows to describe movement of a shock wave completely. The equation (14) now, accordingly, can be named the differential equation of movement of a plane solitary shock wave on homogeneous substance in the channel of variable section.

The equation (14) can be trivial generalized, if substance before front of a shock wave is not homogeneous. For this purpose, in equations (9), (10) it is necessary to replace differentials \(\frac{du}{d\sigma} \cdot d\sigma\) and \(\frac{dp}{d\sigma} \cdot d\sigma\) with the more general expressions \(du\) and \(dp\), because using theorem 5.5 in [5] one can write:

$$du = \frac{\partial u}{\partial \sigma} \cdot d\sigma + \frac{\partial u}{\partial u_0} \cdot du_0 + \frac{\partial u}{\partial \rho_0} \cdot d\rho_0 + \frac{\partial u}{\partial p_0} \cdot dp_0, \tag{17}$$

$$dp = \frac{\partial p}{\partial \sigma} \cdot d\sigma + \frac{\partial p}{\partial u_0} \cdot du_0 + \frac{\partial p}{\partial \rho_0} \cdot d\rho_0 + \frac{\partial p}{\partial p_0} \cdot dp_0. \tag{18}$$
– instead of $\sigma$, $u_0$, $\rho_0$, $p_0$, other choice of parameters is possible also. After that, from (11)-(13), we can finally receive following differential equation of movement of a solitary shock wave:

$$- \frac{dA}{A} = \left( \frac{1}{u} \pm \frac{1}{c} \right) \cdot \left( du \pm \frac{1}{\rho c} \cdot dp \right). \quad (19)$$

It is supposed, that appearing in (17)-(19) functions $A(x)$, $u_0(x)$, $\rho_0(x)$, $p_0(x)$ are smooth. For the first time similar generalization was proposed by Vakhrameev – see [6], [7].

3 Chisnell’s solution

Chisnell in article [1] has derived analogue of the equation (14) for a special case of polytropic equation of state:

$$- \frac{1}{A} \cdot \frac{dA(z)}{dz} = \frac{1}{\gamma z} + \frac{1}{2(z-1)} - \frac{(\gamma+1)}{2((\gamma+1)z+(\gamma-1))} +$$

$$+ \left[ \frac{2}{\gamma z((\gamma-1)z+(\gamma+1))} \right] \times$$

$$\left[ 1 - \frac{(\gamma+1)(z-1)}{2((\gamma+1)z+(\gamma-1))} + \frac{(\gamma-1)z+(\gamma+1)}{2(z-1)} \right]. \quad (20)$$

Here $\gamma$ is adiabatic exponent for polytropic gas, $z = p/p_0$. The parameter $z$, describing force of a shock wave, is determined correctly, as for the polytropic equations of a state $\rho_0 \neq 0$ implies $p_0 \neq 0$, if only the temperature is not equal to absolute zero.

The indefinite integral of the equation (20) found by Chisnell, looks like:

$$Af(z) = \text{constant}. \quad (21)$$

Where

$$f(z) = z^{\frac{1}{\gamma}} (z - 1) \cdot \left( z + \frac{\gamma - 1}{\gamma + 1} \right)^{-\frac{1}{2}} \times$$

$$\times \left[ \frac{1 + \left( \frac{(\gamma+1)}{(\gamma-1)} \right)^{-\frac{1}{2}}} {1 - \left( \frac{(\gamma+1)}{(\gamma-1)} \right)^{-\frac{1}{2}}} \right] \times$$

$$\times \left[ 1 + \left( \frac{(\gamma+1)}{(\gamma-1)} \right)^{-\frac{1}{2}} \left( \frac{2}{\gamma-1} \right)^{\frac{1}{2}} \right] \times$$

$$\times \left[ 1 + \left( \frac{(\gamma+1)}{(\gamma-1)} \right)^{-\frac{1}{2}} \left( \frac{2}{\gamma-1} \right)^{\frac{1}{2}} \right] \times$$

$$\times \exp \left( \frac{2}{\gamma-1} \cdot \arctan \left( \frac{2}{(\gamma-1) \left( \frac{\gamma z}{z + \frac{\gamma - 1}{\gamma + 1}} \right) \right) \right). \quad (22)$$
Let’s look, how in case of polytropic equation of a state the equation (20) can be derived from the equation (14). For definiteness we shall consider shock wave, moving in a positive direction of axis $X$ ($D > 0$). Then we take signs “+” in (14):

$$
- \frac{1}{A} \cdot \frac{dA}{dz} = \left( \frac{1}{u} + \frac{1}{c} \right) \cdot \left( \frac{du}{dz} + \frac{1}{\rho c} \cdot \frac{dp}{dz} \right).
$$

Parameters of substance behind front of a shock wave are determined by Rankine-Hugoniot shock relations, which for polytropic equations of a state can be written in the following form:

$$
u (z) = (z - 1) \cdot \left[ \frac{2p_0}{\rho_0 \{ (\gamma + 1) z + (\gamma - 1) \}} \right]^{\frac{1}{2}},
$$

$$\rho (z) = \rho_0 \cdot \frac{(\gamma + 1) z + (\gamma - 1)}{(\gamma - 1) z + (\gamma + 1)},
$$

$$c (z) = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma z p_0}{\rho_0} \frac{(\gamma - 1) z + (\gamma + 1)}{(\gamma + 1) z + (\gamma - 1)}}.
$$

Differentiation of (24) gives:

$$
\frac{du}{dz} = \left[ \frac{2p_0}{\rho_0 \{ (\gamma + 1) z + (\gamma - 1) \}} \right]^{\frac{1}{2}} \cdot \left[ 1 - \frac{(z - 1) \cdot (\gamma + 1)}{2 \{ (\gamma + 1) z + (\gamma - 1) \}} \right].
$$

Substituting (24)–(27) in (23), we receive Chisnell equation (20).

### 4 Comparison of different solutions

One of the most important examples of one-dimensional movements of shock waves is movement on homogeneous substance solitary converging spherically symmetric shock wave. Choosing small enough solid angle with top in the centre of symmetry of a shock wave, we can see, that it is possible to consider movement of a shock wave inside such solid angle, as movement in the channel with the variable sectional area ($A = constant \cdot r^\alpha$, $\alpha = 2$, $r$ – radius of front of a shock wave in spherical system of coordinates with the beginning in the centre of symmetry of a shock wave), and all assumptions of section 2 are precisely carried out. Therefore movement of spherically symmetric shock wave in spherical system of coordinates, which beginning coincides with the
centre of symmetry of a shock wave, is described by the equation (see the

equation (15)):

\[ -\alpha \ln \frac{r}{r_i} = \int_{\sigma_i}^{\sigma} \left( \frac{1}{u} \pm \frac{1}{c} \right) \cdot \left( \frac{du}{d\sigma} \pm \frac{1}{\rho c} \cdot \frac{dp}{d\sigma} \right) \cdot d\sigma'. \]  

(28)

Similar reasonings show, that the equation (28) describes also movement of
solitary converging cylindrically symmetric shock waves (\( \alpha = 1 \), \( r \) – radius of
front of a shock wave in cylindrical system of coordinates with the beginning
on an axis of symmetry of a shock wave). Signs in the right part of the equation
(28) are taken according to agreements in Sec. 2).

The finding of the analytical solution of the equation (28) for the concrete
equation of state of substance can be not trivial problem – see Chisnell’s
solution (21)-(22). However, with the help of asymptotic analysis of functions
of real variable (see, for example, chapter 5 in [8]) it is often easily to receive
asymptotic solution (at \( r \to 0 \)) of the equation (28), that is also interesting
enough. Thus received asymptotic solution of the equation (28), generally
speaking, are logarithmically equivalent to the true solution – see definition 5,
§ 1, chapters 5 in [8]. It is dictated by structure of the right part of the equation
(28). Analytical Chisnell solution for polytropic equations of state allows to
receive strongly equivalent solution of the equation (28) (see definition 4,
§ 1, chapters 5 in [8]):

\[ \frac{p}{p_i} \sim \left( \frac{r}{r_i} \right)^{-\alpha \nu}. \]  

(29)

Here \( p_i \) – pressure behind front of the shock wave, which is taking place at
some moment of time \( t_i \) on radius \( r_i \) (it is supposed, that \( p_i > p_0 \)); \( \alpha \) – is
determined above, \( \nu \) is expressed by the formula:

\[ \nu = \left[ \frac{\gamma + 2}{2\gamma} + \frac{1}{2} \cdot \sqrt{\frac{2\gamma}{\gamma - 1}} \right]^{-1}. \]  

(30)

In the table exponent \( \alpha \nu \) are given for \( \gamma = 5/3 \), 7/5, 6/5 together with the
corresponding values from self-similar solutions: Guderley, Butler – see [11], [9],
[10], Landau and Stanukovich – see § 64 in [11], see also [12].

It is visible, that the accordance is very good. Small distinction of values
could be tried to explain, how it tried to make Chisnell [11], that at derivation
of the equation (14) possible change \( d^2\sigma \) of strength of shock wave, caused
by reflection of the simple Riemann wave, which has arisen at jump of the
sectional area of the channel \( dA_2 \), from earlier arisen distortion (at jump of
Table 1: Values of the module of exponent according to calculations of different authors.

|                  | Cylindrical wave ($\alpha = 1$) |         |         |         |         |
|------------------|----------------------------------|---------|---------|---------|---------|
|                  | Chisnell                         | Butler  | Guderley| Landau,| Stanukovich|
| $\gamma = \frac{6}{5}$ | 0,326223                        | 0,322441|         |         |         |
| $\gamma = \frac{7}{5}$ | 0,394141                        | 0,394589| 0,396   | 0,398   |         |
| $\gamma = \frac{8}{5}$ | 0,450850                        | 0,452108|         |         |         |

|                  | Spherical wave ($\alpha = 2$) |         |         |         |         |
|------------------|--------------------------------|---------|---------|---------|---------|
|                  | Chisnell                         | Butler  | Guderley| Landau,| Stanukovich|
| $\gamma = \frac{6}{5}$ | 0,652447                        | 0,641513|         |         |         |
| $\gamma = \frac{7}{5}$ | 0,788283                        | 0,788728|         | 0,789   |         |
| $\gamma = \frac{8}{5}$ | 0,901699                        | 0,905385|         |         |         |

the sectional area of the channel $dA_1$) behind front of a shock wave. But $d^2\sigma$ must have second infinitesimal order, because $d^2\sigma$ is bilinear function of $dA_1$ and $dA_2 - d^2\sigma \approx dA_1 \cdot dA_2$ ($d^2\sigma$ can have first infinitesimal order for divergent shock waves).

To understand the true reason of discrepancy of solutions we shall return to derivation of the equation (14). The derivation of the equation (14) is unusual. It is not local in the sense, that changes of the values relating to various points of space are considered. Therefore it would be possible to expect, that the type of symmetry of task will somehow show itself. Equation (12) determining linear connection of infinitesimal changes of speed of substance and pressure in simple Riemann wave in case of spherical (cylindrical) symmetry is incorrect. It would be necessary to replace it with more general relation (see, for example, [13], § 2.7 of chapter 2):

$$D_\pm u \pm \frac{1}{\rho c} D_\pm p \pm \frac{\alpha cu}{r} = 0.$$  (31)
Where
\[ D_{\pm} = \frac{\partial}{\partial t} + (u \pm c) \cdot \frac{\partial}{\partial r}, \]  
(32)

\( \alpha = 0, 1, 2 \) – correspondingly, in case of plane, cylindrical and spherical symmetry. Thus, the equation (14) for spherical (cylindrical) sound waves should be considered as approximated.

5 Conclusions

Given derivation of equation of movement of a solitary shock wave in the channel of variable area shows, why this equation may be only approximate for cylindrical and spherical convergent waves. But such approximation has high precision and may be used for estimations.

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