ON THE INDEPENDENCE OF A GENERALIZED STATEMENT OF EGOROFF’S THEOREM FROM ZFC, AFTER T. WEISS

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Abstract. We consider a generalized version (GES) of the well-known Severini–Egoroff theorem in real analysis, first shown to be undecidable in ZFC by Tomasz Weiss in [4]. This independence is easily derived from suitable hypotheses on some cardinal characteristics of the continuum like $b$ and $a$, the latter being the least cardinality of a subset of $[0, 1]$ having full outer measure.

In this paper we will consider the following Generalized Egoroff Statement, which is a version “without regularity assumptions” of the well-known Severini–Egoroff theorem from real analysis:

GES Given a sequence $(f_n : n \in \mathbb{N})$ of arbitrary functions $[0, 1] \rightarrow \mathbb{R}$ converging pointwise to 0, for each $\eta > 0$ there is a subset $A \subseteq [0, 1]$ of outer measure $\mu^*(A) > 1 - \eta$ such that $(f_n)$ converges uniformly on $A$.

This conjecture first emerged from some questions about the behaviour of bounded harmonic functions on the unit disc in $\mathbb{C}$; in particular, it has been used in [2] to show the independence from ZFC of a strong Littlewood-type statement about tangential approaches.

Notice that in GES it is necessary to consider Lebesgue outer measure to avoid simple counterexamples in ZFC:

Proposition 1. There is a decreasing sequence $(f_n : n \in \mathbb{N})$ of functions $[0, 1] \rightarrow \mathbb{R}$, converging pointwise to zero, such that every subset $A \subseteq [0, 1]$ on which $(f_n)$ converges uniformly has Lebesgue inner measure zero.

Proof. By a theorem of Lusin and Sierpiński there exists a partition of $[0, 1]$ into countably many (in fact, even continuum many) pieces

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\{B_n : n \in \mathbb{N}\}$ each having full outer measure. Consider then the sequence $(f_n)$ where, for every \( n \in \mathbb{N} \), \( f_n \) is the characteristic function of the subset \( B_{\geq n} = \bigcup_{k \geq n} B_k \) of the unit interval: clearly \( (f_n(x)) \) converges monotonically to zero on every point \( x \in [0, 1] \); if \( (f_n) \) converges uniformly on a subset \( A \), \( A \) has to be disjoint from \( B_{\geq \bar{n}} \) for some \( \bar{n} \in \mathbb{N} \), so \( \mu_*(A) \leq 1 - \mu^*(B_{\geq \bar{n}}) = 0. \)

Fix once and for all a decreasing vanishing sequence \( \varepsilon = (\varepsilon_n)_{n \in \mathbb{N}} \) of positive real numbers, e.g. \( \varepsilon_n = 2^{-n} \); consider the following function, mapping a sequence of reals to its \((\varepsilon)-\)order of convergence to zero:

\[
\text{oc} : \mathcal{C}_0 \rightarrow \mathbb{N}^\uparrow, \quad \text{defined on each } a = (a_n) \in \mathcal{C}_0 \text{ as }
(1) \quad (\text{oc } a)_n = \min \{ m : \forall l \geq m \ (|a_l| \leq \varepsilon_n) \},
\]

where \( \mathcal{C}_0 \) denotes the set of infinitesimal real-valued sequences and \( \mathbb{N}^\uparrow \subseteq \mathbb{N} \) is the set of nondecreasing sequences of natural numbers.

Using the natural identification of \( \mathbb{N}(\mathbb{R}) \) with \( \mathbb{X}(\mathbb{R}) \), we can view a sequence of real-valued functions \( X \rightarrow \mathbb{R} \) converging pointwise to zero as a single function \( F : X \rightarrow \mathcal{C}_0 \), and then study the associated order of convergence, \( \text{oc } F = \text{oc } \circ F : X \rightarrow \mathbb{N}^\uparrow \):

**Lemma 2.** \( F \) converges uniformly to zero if and only if the range of \( \text{oc } F \) is bounded in \( (\mathbb{N}, \leq ) \), where \( \leq \) is the partial order of everywhere domination: \( \alpha \leq \beta \iff \forall n \ (\alpha_n \leq \beta_n) \).

**Proof.** This is just a restatement of the definition of uniform convergence:

\[
\begin{align*}
F \text{ converges uniformly to } 0 & \iff \\
& \iff \forall n \exists m \ \forall x \in X \ \forall l \geq m \ (|F_l(x)| \leq \varepsilon_n) \iff \\
& \iff \exists (m_n) \in \mathbb{N}^\uparrow \ \forall n \in \mathbb{N} \ \forall x \in X \ ((\text{oc } F(x))_n \leq m_n). \quad \square
\end{align*}
\]

**Lemma 3.** For all \( \varphi : X \rightarrow \mathbb{N}^\uparrow \) there exists a sequence \( F \) of real-valued functions on \( X \) converging pointwise to zero with order \( \text{oc } F = \varphi \).

**Proof.** It is sufficient to prove the lemma pointwise: given a nondecreasing sequence of natural numbers \( \alpha = (\alpha_n) \in \mathbb{N}^\uparrow \), we construct a sequence \( a = (a_n) \in \mathcal{C}_0 \) converging to 0 with order \( \alpha \). For that, just let

\[
a = (a_n)_{n \in \mathbb{N}} \quad \text{where} \quad a_n = \inf \{ \varepsilon_k : \alpha_k \leq n \};
\]

it is straightforward to check that this works, i.e. \( \text{oc } a = \alpha. \) \quad \square

Let \( \mu^* \) be an upward continuous outer measure on a set \( X \), i.e. an outer measure \( \text{Pow } X \rightarrow [0, +\infty] \) satisfying

\[
A = \bigcup_{n \in \mathbb{N}} A_n \quad \Rightarrow \quad \mu^*(A) = \lim_{n \to \infty} \mu^* \left( \bigcup_{k < n} A_k \right).
\]

For every sequence \( F \) of real-valued functions on \( X \) converging pointwise to zero, consider the statement
GES\((X, \mu^*, F)\) for each \(M < \mu^*(X)\) there is a subset \(A \subseteq X\) such that \(\mu^*(A) > M\) and \(F\) converges uniformly on \(A\);

the Generalized Egoroff Statement relative to the space \((X, \mu^*)\) is the formula

\[
\text{GES}(X, \mu^*) = \forall F \ \text{GES}(X, \mu^*, F);
\]

clearly our original statement \text{GES} is just \text{GES}([0, 1], m^*), where \(m^*\) is Lebesgue outer measure on the unit interval \([0, 1] \subseteq \mathbb{R}\). Denote by \(\mathcal{K}_\sigma\) the \(\sigma\)-ideal generated by the bounded subsets of \((\mathbb{N}^\mathbb{N}, \leq)\); equivalently, \(\mathcal{K}_\sigma\) is the family of those subsets which are bounded with respect to the order \(\leq^*\) of eventual domination,

\[
\alpha \leq^* \beta \iff \forall n (\alpha_n \leq \beta_n) \iff \exists k \geq n (\alpha_k \leq \beta_k) \quad (\alpha, \beta \in \mathbb{N}^\mathbb{N}),
\]

and \(\mathcal{K}_\sigma\) is also the \(\sigma\)-ideal generated by the compact subsets of the Baire space \(\mathbb{N}^\mathbb{N}\) (see \[3\]).

**Lemma 4.** \text{GES}(X, \mu^*, F) holds iff there is a subset \(Y \subseteq X\) of full outer measure (i.e. \(\mu^*(Y) = \mu^*(X)\)) such that \(\text{oc } F[Y] \in \mathcal{K}_\sigma\).

**Proof.** Fix an increasing sequence of positive real numbers \((M_n)\) with limit \(\mu^*(X)\). Assume \text{GES}(X, \mu^*, F): by lemma \[2\] for every \(n \in \mathbb{N}\) there is a subset \(A_n \subseteq X\) such that \(\mu^*(A_n) > M_n\) and \(\text{oc } F[A_n]\) is bounded in \(\mathbb{N}^\mathbb{N}\); taking \(Y = \bigcup_{n \in \mathbb{N}} A_n\), \(Y\) has full outer measure and \(\text{oc } F[Y] = \bigcup_{n \in \mathbb{N}} \text{oc } F[A_n]\) is \(\sigma\)-bounded, as required. Conversely, suppose that \(\mu^*(Y) = \mu^*(X)\) and \(\text{oc } F[Y] \subseteq \bigcup_{n \in \mathbb{N}} B_n\), where each \(B_n\) is a bounded subset of \((\mathbb{N}^\mathbb{N}, \leq)\), and put

\[
A_n = (\text{oc } F)^{-1}[B_0 \cup \ldots \cup B_{n-1}]:
\]

since \(\text{oc } F[A_n]\) is bounded, \(F\) converges uniformly on every \(A_n\) (lemma \[2\]); moreover, as \(\mu^*\) is continuous and \(Y \subseteq \bigcup_{n \in \mathbb{N}} A_n\), for all \(m\) there is some \(n\) such that \(\mu^*(A_n) > M_m\), that is, \text{GES}(X, \mu^*, F) holds. \[\square\]

**Theorem 5.** \text{GES}(X, \mu^*) holds if and only if for all functions \(\varphi : X \to \mathbb{N}^\mathbb{N}\) there is a subset \(Y \subseteq X\) of full outer measure such that \(\varphi[Y] \in \mathcal{K}_\sigma\).

This theorem provides a translation of \text{GES} into a purely set-theoretical statement.

**Proof.** The “if” direction follows directly from lemma \[4\] using \(\varphi = \text{oc } F\). For the converse, consider the function \(\Theta\) which maps a sequence \(\alpha = (\alpha_n)_{n \in \mathbb{N}}\) to the nondecreasing sequence \((\sum_{k \leq n} \alpha_k)_{n \in \mathbb{N}}\): it is a bijective order morphism \((\mathbb{N}^\mathbb{N}, \leq) \to (\mathbb{N}^\mathbb{N}, \leq)\) satisfying \(\alpha \leq \Theta(\alpha)\), therefore, for all \(Y \subseteq \mathbb{N}^\mathbb{N}\), \(\Theta[Y]\) is \((\sigma)\)-bounded iff \(Y\) is \((\sigma)\)-bounded. Assume \text{GES}(X, \mu^*) and let \(\varphi\) be a function \(X \to \mathbb{N}^\mathbb{N}\); by lemma \[3\] there exists a sequence \(F\) of real-valued functions converging pointwise to 0 with \(\text{oc } F = \Theta \circ \varphi\), so there is a set \(Y \subseteq X\) of full outer measure such that \(\Theta[\varphi[Y]] = \text{oc } F[Y] \in \mathcal{K}_\sigma\) (lemma \[4\]), i.e. \(\varphi[Y] \in \mathcal{K}_\sigma\) as desired. \[\square\]
Remark. Theorem \(5\) is still valid for measure spaces \((X, \mu)\) and the classical Egoroff Statement, provided that we only consider measurable maps \(\varphi\) and measurable subsets \(Y \subseteq X\). Thus theorem \(5\) entails the Severini–Egoroff theorem: if \(\mu\) is finite and \(\varphi : X \to \mathbb{N}\) is measurable, the image measure \(\varphi_* \mu\) is a finite Borel measure on \(\mathbb{N}\), hence it is regular and it is always supported by a \(\sigma\)-compact subset.

Recall that the \textit{bounding number} \(b = \text{non}(\mathcal{K}_\sigma)\) (see \([3]\)) is the smallest possible size of a subset of \(\mathbb{N}\) not belonging to \(\mathcal{K}_\sigma\). We also denote with \(o = o([0,1], m^*)\).

Corollary 6. Assuming \(o < b\), GES\((X, \mu^*)\) holds. In particular, \(o < b\) implies GES\(^2\).

Proof. Fix a subset \(Y \subseteq X\) of full outer measure with \(|Y| = o(X, \mu^*)\); then every function \(\varphi : X \to \mathbb{N}\) maps \(Y\) onto a set of cardinality less than \(b\), hence \(\varphi[Y] \in \mathcal{K}_\sigma\). □

We can also invoke theorem \(5\) to prove sufficient conditions for the failure of GES. Precisely, we infer \(\neg\)GES\((X, \mu^*)\) by constructing (under suitable hypotheses) a set \(Z \subseteq \mathbb{N}\) of cardinality \(|Z| \geq |X|\) such that all subsets of \(Z\) belonging to \(\mathcal{K}_\sigma\) have size less than \(o(X, \mu^*)\); once this is achieved, if \(\varphi\) is any injection \(X \to Z\), no subset \(Y \subseteq X\) of full measure can be mapped onto an element of \(\mathcal{K}_\sigma\), because \(|\varphi[Y]| = |Y| \geq o(X, \mu^*)\).

In order to state the next proposition, we recall that the \textit{dominating number} \(\mathfrak{d} \geq b\) is the least cardinality of a cofinal subset of \((\mathbb{N}, \leq^*)\) and that a \(\kappa\)-\textit{Lusin set} is a subset \(L \subseteq \mathbb{R}\) of cardinality \(\kappa\) whose meager (i.e. Baire first category) subsets have size less than \(\kappa\).

Proposition 7. Assume \(o(X, \mu^*) = |X| = \kappa\); then GES\((X, \mu^*)\) fails in each of the following cases:

1. \(\kappa = b\);
2. \(\kappa = \mathfrak{d}\);
3. there exists a \(\kappa\)-Lusin set.

Proof. Following the plan outlined before stating the proposition, we try to build a “\(\kappa\)-Lusin set” \(Z\) for the ideal \(\mathcal{K}_\sigma\) instead of the ideal of meager sets. This is automatic under hypothesis (3): every (true) \(\kappa\)-Lusin set has the required properties, since all compact subsets of \(\mathbb{N}\) have empty interior and thus every \(\mathcal{K}_\sigma\) set is meager.

Assume \(\kappa = b\) and let \(\{\alpha_\xi\}_{\xi < b}\) be an unbounded family in \((\mathbb{N}, \leq^*)\). By transfinite recursion we build a wellordered unbounded chain \(Z = \{\beta_\xi\}_{\xi < b}\) of length \(b\): after the construction of all \(\beta^\eta\) for \(\eta < \xi\), pick \(\beta_\xi\).

\(^1\)We haven’t been able to find any specific name for this cardinal in the literature.
\(^2\)The latter fact has been pointed out by T. Weiss and I. Reclaw (see \([4]\)).
among the strict \( \leq^* \)-upper bounds of the set \( \{ \alpha^\xi \} \cup \{ \beta^n \}_{n<\xi} \) (which has size less than \( b \) and thus is \( \leq^* \)-bounded). It is clear that no \( \leq^* \)-bounded subset of \( Z \) can be cofinal in \( Z \), hence all \( K_\sigma \) subsets of \( Z \) have cardinality \( < b \).

Finally, suppose \( \kappa = d \) and let \( \{ \alpha^\xi \}_{\xi<d} \) be a cofinal family in \( (^N\mathbb{N}, \leq^*) \).

We build a set \( Z = \{ \beta^\xi \}_{\xi<d} \) of cardinality \( d \) by transfinite recursion as follows: after the construction of all \( \beta^n \) for \( \eta < \xi \), pick an element \( \beta^\xi \) which is not \( \leq^* \) any element of the set \( \{ \alpha^n \}_{n<\xi} \cup \{ \beta^n \}_{\eta<\xi} \) (which has size less than \( d \) and thus is not \( \leq^* \)-cofinal). \( Z \) has the desired properties: \( (\beta^\xi)_{\xi<d} \) is a sequence without repetitions, hence \( |Z| = d \), and moreover, if \( A \subseteq Z \) is in \( K_\sigma \), some \( \alpha^\xi \) has to eventually dominate all elements of \( A \), which implies that \( A \subseteq \{ \beta^n \}_{n<\xi} \) has cardinality less than \( d \). \( \square \)

**Corollary 8.** GES fails whenever at least one of the following hypotheses is satisfied:

1. \( o = d = c \) (the cardinality of the continuum);
2. there exists a \( c \)-Lusin set and \( o = c \);
3. there exists a \( c \)-Lusin set and \( c \) is a regular cardinal.

The last two conditions provide an affirmative answer (at least when \( c \) is regular or it coincides with \( o \)) to a question posed by T. Weiss; he also noticed that there are models of \( ZFC \) (e.g. the iterated Mathias real model, where \( o = d = c \)) which contain no \( c \)-Lusin sets but nevertheless satisfy \( \neg \)GES.

**Proof.** Assumptions (1) and (2) are just particular instances of cases (2) and (3) respectively of proposition 4. Moreover, hypothesis (3) is stronger than both (1) and (2): if \( \kappa \) is a regular cardinal and there is a \( \kappa \)-Lusin set, then \( \text{cov}(\mathcal{M}) \geq \kappa \) and thus \( o \geq \text{cov}(\mathcal{M}) \geq \kappa \) and \( o \geq \text{non}(\mathcal{N}) \geq \text{cov}(\mathcal{M}) \geq \kappa \) (see [1] for the relevant definitions of these cardinal characteristics associated to the \( \sigma \)-ideals \( \mathcal{M} \) of meager sets and \( \mathcal{N} \) of Lebesgue nullsets, as well as for the proofs in \( ZFC \) of the stated inequalities). \( \square \)

**Corollary 9 (T. Weiss).** GES is undecidable in \( ZFC \).

**Proof.** The hypothesis of corollary 8 and therefore GES, hold in the iterated Laver real model (see [1] and the proof of theorem 1 in [4]). On the other hand, \( o = d = c \) is certainly true (thus \( \neg \)GES holds) under the Continuum Hypothesis \( CH \) or just Martin’s Axiom \( MA \), which are consistent with \( ZFC \). \( \square \)

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