Global Dynamics for the coupled Einstein-Maxwell system with pseudo-tensor of pressure on Bianchi spacetimes.

by

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Abstract

Global existence to the coupled Einstein-Maxwell system which rules the dynamics of a kind of charged matter with a pseudo-tensor of pressure is proved, in Bianchi I-VIII spacetimes. We study the geodesics completeness, the asymptotic behavior, the positivity conditions, and we prove that the problem is well-posed in the sense of Hadamard.

Keywords: global existence; local existence; pseudo-tensor of pressure; differential system; constraints; asymptotic behavior; geodesic completeness; positivity conditions.

1 Introduction

In relativistic kinetic theory, \textit{global dynamics} of several kinds of charged and uncharged matter remain an active research domain in General Relativity, in which cosmological constant plays a central role for astrophysical reasons. In the present paper, we consider the Einstein equations with in the sources, the pseudo-tensor of pressure due to A.LICHNEROWICZ [5] which is the general form of a relativistic fluid tensor. We prove the local and global existence and we study the asymptotic behavior. We study the positivity conditions and we prove that the problem is well-posed in the sense of Hadamard, which means that the global solution is a continuous function of the initial data. The work is organized as follows:

In section 2, we introduce the problem and we give the evolution and the constraints systems.

In section 3, we study the constraints, define the initial values problem and introduce the relative norms. In section 4, we construct the local solution by iterated method. In section 5, we prove the global existence theorem. In section 6, we study the asymptotic behavior and the geodesic completeness. In section 7, we study the positivity conditions. In section 8, we study the well-posedness in the sense of Hadamard.
2 The problem. Evolution and constraints systems

2.1 The problem

We are on Bianchi, time oriented spacetimes, which are \((M, \ 4g)\) spacetimes of type I-VIII, where \(4g\) is the metric with signature \((-, +, +, +)\) and \(M = \mathbb{R} \times G\) where \(G\) is a three dimensional connected Lie group. \(4g\) has the form:

\[
4g = -dt^2 + g_{ij}(t)e^i \otimes e^j
\]

where \((e_i)\) is a left invariant basis in \(G\) and \((e^i)\) the dual basis; \(g = (g_{ij})\) is a positive definite 3-dimensional metric on \(G\) and we adopt the Einstein summation convention \(A^\alpha B_\alpha = \sum_\alpha A^\alpha B_\alpha\). The Greek indexes \(\alpha, \beta, \ldots\) range from 0 to 3 and the Latin indexes \(i, j, \ldots\) from 1 to 3. The index 4 (as \(4g\)) is for quantities on \(M\). The vector \(n = \partial_t\) being orthogonal to \(G\), we complete the basis \((e_i)\) on \(G\) to obtain a basis \((n, e_i) = (\partial_0, e_i)\) on \(M\) with:

\[
4 e_0 = \partial_t; \quad 4 e_i = e_i; \quad 4 e_\alpha = 4 e_\lambda^\alpha \partial_\lambda
\]

where

\[
4 e_0^0 = 1; \quad 4 e_0^i = 0; \quad 4 e_i^0 = 0; \quad 4 e_i^j = e_j^i
\]

The structure constants of the Lie algebra \(G\) of the Lie group \(G\) are denoted \(C^k_{ij}\) and defined by:

\[
[e_i, e_j] = C^k_{ij} e_k
\]

where \([, , ]\) is the Lie brackets in \(G\). Due to the antisymmetry of \([, , ]\), we have:

\[
C^k_{ij} = -C^k_{ji}
\]

If \(4\nabla\) is the covariant derivative in \(4g\), the Ricci rotation coefficients \(4\gamma^\lambda_{\alpha\beta}\) are defined by:

\[
4\nabla_{4e_\alpha} 4e_\beta = 4\gamma^\lambda_{\alpha\beta} 4e_\lambda
\]

If \(4T^\alpha\) and \(4T_\alpha\) are tensors on \(M\), we have:

\[
\begin{align*}
4\nabla_\alpha 4T^\beta &= 4e_\alpha (4T^\beta) + 4\gamma^\beta_{\alpha\lambda} 4T^\lambda \\
4\nabla_\alpha 4T_\beta &= 4e_\alpha (4T_\beta) - 4\gamma^\lambda_{\alpha\beta} 4T_\lambda
\end{align*}
\]

We study a fluid, subject to the following system in which \(t\) is the only independent variable:
Dynamics of a plasma with pseudo-tensor of pressure

\[
\begin{align*}
4R_{\alpha\beta} - \frac{1}{2} 4R 4g_{\alpha\beta} + \Lambda 4g_{\alpha\beta} &= 8\pi (4\tau_{\alpha\beta} + 4T_{\alpha\beta}), \\
4\nabla_\alpha 4F^{\alpha\beta} &= e 4u^\beta, \\
4\nabla_\alpha 4F_{\beta\gamma} + 4\nabla_\beta 4F_{\gamma\alpha} + 4\nabla_\gamma 4F_{\alpha\beta} &= 0;
\end{align*}
\]  

(2.8) \hspace{1cm} (2.9) \hspace{1cm} (2.10)

where:

- (2.8) are the Einstein equations in \((4g_{\alpha\beta})\) with \(4R_{\alpha\beta}\) the Ricci tensor, \(4R = 4g^{\alpha\beta} 4R_{\alpha\beta}\) the scalar curvature, \(\Lambda\) a constant called the cosmological constant, \(4\tau_{\alpha\beta}\) the Maxwell tensor associated to the electromagnetic field \(4F = (4F_{\alpha\beta})\) and defined by:

\[
4\tau_{\alpha\beta} = -\frac{1}{4} 4g_{\alpha\beta} 4F^{\lambda\mu} 4F_{\lambda\mu} + 4F_{\alpha\lambda} 4F_{\beta}^\lambda
\]  

(2.11)

and \(4T_{\alpha\beta}\) the symmetric 2-tensor:

\[
4T_{\alpha\beta} = \frac{4}{3} \rho 4u_\alpha 4u_\beta + 4\Theta_{\alpha\beta}
\]  

(2.12)

where \(\rho > 0\) is an unknown function of \(t\), called the proper density, \(4u = (4u^\alpha)\) is the unknown material velocity of the fluid, \(4u\) is a unit vector oriented towards the future direction, and \(4\Theta_{\alpha\beta}\) is a symmetric 2-tensor called the pseudo-tensor of pressure, due to A.LICHNEROWICZ [5]. \(4\Theta_{\alpha\beta} = 0\) is the case corresponding to pure matter, and \(4\Theta_{\alpha\beta} = p 4g_{\alpha\beta}\) where \(p\) is a scalar function representing the pressure, corresponds to a perfect relativistic fluid. We adopt on \(4\Theta_{\alpha\beta}\) the assumptions:

\[
4\nabla_\alpha 4\Theta^{\alpha\beta} = \rho 4u^\beta \quad \text{and} \quad 4\Theta_{0\alpha} = 4z_0 4z_\alpha \left( \delta^0_\alpha + \frac{1}{2} \delta^{1,2,3}_\alpha \right)
\]  

(2.13)

where \((4z_\alpha)\) is a future pointing vector, \(\delta^0_\alpha\) is the Kroneker’s symbol and \(\delta^{1,2,3}_\alpha\) is a similar symbol such that \(\delta^{1,2,3}_i = 1\) and \(\delta^{1,2,3}_0 = 0\). The first formula in (2.13) is due to A.LICHNEROWICZ and the second formula which gives \(4\Theta_{00} = (4z_0)^2\) and hence, \(4T_{00} = \frac{4}{3} \rho (4u_0)^2 + 4\Theta_{00} \geq 0\) will be very helpful. We suppose that:

\[
4\Theta_{ij} = \frac{1}{3} \rho 4g_{ij}
\]  

(2.14)

from where, we have:

\[
4T_{ij} = \frac{4}{3} \rho 4u_i 4u_j + \frac{1}{3} \rho 4g_{ij}
\]

which is the stress-tensor of a perfect fluid of pure radiation.

- (2.9) and (2.10) are the first and second group of the Maxwell equations, for the electromagnetic field \(4F = (4F_0^i, 4F^i_j)\) which is an antisymmetric closed 2-form. \(4F_0^i\) and \(4F^i_j\) are
respectively, the electric and magnetic parts of $^4F$.

In equations (2.9), $e \geq 0$ is an unknown function called the Maxwell current created by the charged particles.

(2.10) expresses only the fact that $d^4F = 0$.

Let us recall that to solve the system (2.8)-(2.9)-(2.10) is to determine all the unknown functions $^4g_\alpha\beta$, $^4F_\alpha\beta$, $\rho$, $^4u^\alpha$, $^4\Theta_\alpha\beta$ and $e$, which depend only on the time variable $t$.

### 2.2 The equations

The unit vector $^4u$ satisfies the relation:

$$^4u^\alpha \cdot ^4u_\alpha = -1$$

(2.15)

which gives, since $^4u$ is future oriented and using (2.1):

$$^4u^0 = \sqrt{1 + g_{ij} ^4u^i ^4u^j}.$$  

(2.16)

We study the Einstein equations in 3+1 formulation, which means that, they are seen as giving the evolution of the triplet $(\Sigma_t, g_t, K_t)$, where $\Sigma_t = \{t\} \times G$, $g_t = (g_{ij}(t))$ is called the first fundamental form of $\Sigma_t$, and $K_t = (K_{ij}(t))$ is called the second fundamental form of $\Sigma_t$. In the present case, $K_{ij}$ is defined by:

$$K_{ij} = -\frac{1}{2} \partial_t g_{ij}.$$  

(2.17)

We now introduce a very useful quantity called the mean curvature and defined by:

$$H = g^{ij} K_{ij}.$$  

(2.18)

Let us give the expressions of $^4\gamma^\lambda_{\alpha\beta}$ as they are in [1]. They are:

\[
\begin{cases}
^4\gamma^\lambda_{00} = ^4\gamma^0_{00} = 0; & ^4\gamma^0_{i0} = 0; & ^4\gamma^0_{ij} = -K_{ij}; & ^4\gamma^j_{i0} = 0; & ^4\gamma^j_{00} = -K^j_i; \\
^4\gamma^l_{ij} := \gamma^l_{ij} = \frac{1}{2} g^{lk} \left[ -C^m_{jk} g_{im} + C^m_{ki} g_{jm} + C^m_{ij} g_{km} \right].
\end{cases}
\]

(2.19)

We deduce from (2.19) that:

$$\gamma^i_{ij} = C^i_{ij}; \quad \gamma^l_{ij} - \gamma^l_{ji} = C^l_{ij}.$$  

(2.20)
2.2.1 Equations in $\rho$, $\,^4u^\alpha$, $\,^4\Theta^{\alpha\beta}$ and $e$

We always have:

$$\,^4\nabla_\alpha \left( \,^4R^{\alpha\beta} - \frac{1}{2} \,^4R \,^4g^{\alpha\beta} + \Lambda \,^4g^{\alpha\beta} \right) = 0.$$  

Hence (2.8) implies the conservation conditions:

$$\,^4\nabla_\alpha \,^4\tau^{\alpha\beta} + \,^4\nabla_\alpha \,^4T^{\alpha\beta} = 0 \quad (2.21)$$

from where we deduce, using expression (2.12) of $\,^4T^{\alpha\beta}$ and the assumptions (2.13) on $\,^4\Theta^{\alpha\beta}$:

$$\,^4\nabla_\alpha \,^4\tau^{\alpha\beta} + \frac{4}{3} \,^4\nabla_\alpha (\rho \,^4u^\alpha \,^4u^\beta) + \rho \,^4u^\beta = 0. \quad (2.22)$$

Now a direct calculation gives:

$$\begin{cases} 
\,^4\nabla_\alpha \,^4\tau^{\alpha\beta} = \,^4F^{\beta}_{\lambda} \,^4\nabla_\alpha \,^4F^{\alpha\lambda} \quad (2.23) \\
\,^4\nabla_\alpha (\rho \,^4u^\alpha \,^4u^\beta) = \,^4u^\beta \,^4\nabla_\alpha (\rho \,^4u^\alpha) + (\rho \,^4u^\alpha) \,^4\nabla_\alpha (\,^4u^\beta). \quad (2.24)
\end{cases}$$

We deduce from (2.21), using, (2.23)-(2.24) and the Maxwell equation (2.9):

$$e \,^4F^{\beta}_{\lambda} \,^4u^\lambda + \frac{4}{3} \left[ \,^4u^\beta \,^4\nabla_\alpha (\rho \,^4u^\alpha) + (\rho \,^4u^\alpha) \,^4\nabla_\alpha (\,^4u^\beta) \right] + \rho \,^4u^\beta = 0. \quad (2.25)$$

The contracted multiplication of (2.25) by $\,^4u_\beta$ gives, using $\,^4F^{\beta}_{\lambda} \,^4u^\lambda \,^4u_\beta = \,^4F^{\beta}_{\lambda} \,^4u^\lambda \,^4u_\beta = 0$ (since $\,^4F$ is antisymmetric) and $\,^4u^\beta \,^4u_\beta = -1$ which implies $\,^4u_\beta \,^4\nabla_\alpha \,^4u^\beta = 0$:

$$\frac{4}{3} \,^4\nabla_\alpha (\rho \,^4u^\alpha) + \rho = 0 \quad (2.26)$$

If we return to (2.25), we obtain:

$$\frac{4}{3} (\rho \,^4u^\alpha) \,^4\nabla_\alpha \,^4u^\beta + e \,^4F^{\beta}_{\lambda} \,^4u^\lambda = 0 \quad (2.27)$$

We deduce from (2.26) that:

$$\begin{cases} 
\dot{\rho} = A(t) \rho \\
\text{with } A(t) = -\left( \frac{3}{4} \,^4u^0 + \frac{4}{3} \,^4u^0 \,^4u^0 - K^i + C_{ij}^k \,^4u^j \right) \quad (2.29)
\end{cases}$$

where the dot denotes the derivative with respect to $t$. Integrating (2.28), we obtain:

$$\rho = \rho(0) \exp \left( \int_0^t A(s)ds \right)$$
which shows that: $\rho = 0 \iff \rho(0) = 0$. In what follows, we suppose that:

$$\rho(0) > 0 \quad (2.30)$$

which implies $\rho > 0$. (2.27) gives:

$$4u^\alpha 4\nabla_\alpha (4u^\beta) = \frac{3e}{4\rho} 4F_\lambda^\beta 4u^\lambda$$

which is the differential system of current lines. It shows that in the vacuum ($4F = 0$), $4u$ satisfies the geodesics equation: $4u^\alpha 4\nabla_\alpha (4u^\beta) = 0$. Equation (2.31) gives, taking $\beta = 0$ and $\beta = i$, using (2.19) and (2.7):

$$\begin{cases}
4\dot{u}^0 = K_{ij} \frac{4u^i 4u^j}{4u^0} + \frac{3e}{4\rho} 4F_0^i \frac{4u^i}{4u^0} \\
4\dot{u}^i = 2K_i 4u^i - \gamma_{jk} 4u^j 4u^k \frac{4u^i}{4u^0} + \frac{3e}{4\rho} 4F_0^i + \frac{3e}{4\rho} 4F_j^i \frac{4u^i}{4u^0}
\end{cases}$$

and equation (2.26) in $\rho$ writes:

$$\dot{\rho} = -\left(\frac{3}{4} \frac{1}{4u^0} + \frac{3}{4} \frac{4u^i 4u^j}{4u^0} \right) \rho - \frac{3}{4} e \frac{4F_0^i 4u^i}{(4u^0)^2}. \quad (2.34)$$

Next, by (2.14) we have: $4\Theta^{ij} = \frac{2}{3} 4g^{ij}$; so, $4\Theta^{ij}$ is given by $\rho$ and $4g^{ij}$. We then only look for $4\Theta^{0\alpha}$. We deduce from (2.13), (2.7), that $4\Theta^{0\alpha}$ satisfy the system:

$$\begin{cases}
4\dot{\Theta}^{00} = K_i 4\Theta^{00} - C_{ij} 4\Theta^{0j} + \frac{1}{3} \rho g^{ij} K_{ij} + \rho 4u^0; \\
4\dot{\Theta}^{0i} = K_i 4\Theta^{0i} + 2K^k 4\Theta^{0k} - \frac{3e}{4\rho} (C_{kj} g^{ij} + \gamma_{jk} g^{ik}) + \rho 4u^i.
\end{cases}$$

Now the electromagnetic field always satisfies the identities $4\nabla_\beta 4\nabla_\alpha 4F^{\alpha\beta} = 0$. This implies, using the Maxwell equation (2.9):

$$4\nabla_\alpha (e 4u^\alpha) = 0. \quad (2.37)$$

(2.37) gives:

$$\dot{e} = -\left(\frac{1}{4u^0} 4\nabla_\alpha 4u^\alpha\right) e \quad (2.38)$$

which integrates to give:

$$e(t) = e(0) \exp \left( - \int_0^t 4\nabla_\alpha 4u^\alpha ds \right). \quad (2.39)$$

This shows that:

$$(e(0) \geq 0) \iff (e \geq 0).$$
In what follows we adopt:
\[ e_0 := e(0) \geq 0. \] (2.40)

Using the equations (2.32)-(2.33) in \( 4u^\alpha \), equation (2.38) in \( e \) can write:
\[ \dot{e} = -\left( \frac{K_{ij} u^i u^j}{(u^0)^2} + C_{ik} 4u^k \frac{4u^j}{u^0} - K_i^i \right) e + \frac{3}{4} g_{ij} \frac{4u^i 4F^{0j}}{(u^0)^2 \rho}. \] (2.41)

2.2.2 Equations in \( 4F^{0i}, 4F_{ij} \) and constraints

The Maxwell equations (2.9) for \( \beta = i \), namely \( 4\nabla_\alpha 4F^{\alpha i} = 4e^i \), can write using (2.7) and (2.19), to give the equation for the electric part:
\[ 4\dot{F}^{0i} = K^j_i 4F^{0i} - C^j_{jk} 4F^{kj} - \frac{1}{2} C_{jk}^{ij} 4F^{jk} - C_{jk}^{0k} \frac{4u^i}{4u^0}. \] (2.42)

Now, the Maxwell equations (2.10) split into the equations:
\[ \begin{cases} 4\nabla_0 4F_{ij} + 4\nabla_i 4F_{0j} + 4\nabla_j 4F_{0i} = 0; \\ 4\nabla_i 4F_{jk} + 4\nabla_j 4F_{ki} + 4\nabla_k 4F_{ij} = 0. \end{cases} \] (2.43) (2.44)

(2.43) can write using (2.7), and (2.19), to give the equation for the magnetic part:
\[ 4\dot{F}_{ij} = C_{ij}^{kl} 4F^{0l}. \] (2.45)

Next, for \( \beta = 0 \), the Maxwell equations \( 4\nabla_\alpha 4F^{\alpha 0} = 4e^0 \) gives the constraints:
\[ C_{ik}^{0} 4F^{0k} + e 4u^0 = 0 \] (2.46)

and (2.44) gives the constraints:
\[ C_{ij}^{kl} 4F_{kl} + C_{jk}^{il} 4F_{il} + C_{ki}^{jl} 4F_{jl} \equiv C_{[ij]}^{kl} 4F_{kl} = 0. \] (2.47)

2.2.3 The Einstein equations in 3+1 formulation. Notations, System

The principle of the 3+1 formulation is to deduce from the Einstein equations which are a second order partial differential equations system, an equivalent first order system in \( g_{ij} \) and \( K_{ij} \). In the uncharged case, \( (4F = 0) \), the calculation is classical. We did the charged case \( (4F \neq 0) \) in [6] and it is adopted here without difficulty. So are the constraints. Expressing the tensor \( 4\tau_{\alpha\beta} \) we obtain:
\[ \begin{cases} 4\tau_{00} = \frac{1}{2} g_{ij} 4F^{0i} 4F^{0j} + \frac{1}{4} g^{ik} g^{jl} 4F_{kl} 4F_{ij} \\ 4\tau_{0j} = -\frac{3}{4} F^{0k} 4F_{jk} \\ 4\tau_{ij} = \left( \frac{1}{2} g_{ij} g_{kl} - g_{ik} g_{jl} \right) 4F^{0k} 4F^{0l} - \frac{1}{2} g_{ij} g^{km} g^{nl} 4F_{kl} 4F_{mn} + g^{kl} 4F_{ik} 4F_{jl} \end{cases} \] (2.48)
For $^4T_{\alpha\beta}$ we have:

$$^4T_{00} = \frac{4}{3}\rho (u_0)^2 + ^4\Theta_{00}, \quad ^4T_{ij} = \frac{4}{3}\rho u_i u_j + ^4\Theta_{ij}, \quad ^4T_{0j} = \frac{4}{3}\rho u_i u_j + \frac{1}{3}\rho g_{ij}.$$  \hspace{1cm} (2.49)

Since the indexes are now clearly specified, we will denote in what follows:

$$\left\{\begin{array}{l}
^4F^{0i} = E^i; \quad ^4F^{ij} = F^{ij}; \quad ^4T_{00} = T_{00}; \quad ^4T_{0i} = T_{0i}; \quad ^4T_{ij} = T_{ij} \\
^4\tau_{00} = \tau_{00}; \quad ^4\tau_{0i} = \tau_{0i}; \quad ^4\tau_{ij} = \tau_{ij}; \\
^4u^0 = u^0; \quad ^4u^i = u^i \\
^4\Theta^{00} = \Theta^{00}; \quad ^4\Theta^{0i} = \Theta^{0i}; \quad ^4\Theta^{ij} = \Theta^{ij} \\
^4g_{ij} = g_{ij}
\end{array}\right.$$  \hspace{1cm} (2.50)

and we will have for $F^{ki}$ and $F_i^0$, using (2.50):

$$\left\{\begin{array}{l}
F^{ki} = g^{kl} g^{im} F_{lm} \\
F_i^0 = 4 g_{i\lambda} F^{\lambda 0} = 4 g_{ij} F^{j0} = -g_{ij} E^j
\end{array}\right.$$  \hspace{1cm} (2.50)

Now using expression (2.48) of $^4\tau_{\alpha\beta}$, (2.49) of $^4T_{\alpha\beta}$, the notations (2.50), equations (2.33) of $^4u^i$, equation (2.34) of $\rho$, equations (2.35)-(2.36) of $^4\Theta^{0\alpha}$, equation (2.41) of $e$, equation (2.42) and (2.45) of $^4F^{0i}$ and $^4F^{ij}$, the constraints (2.46) and (2.47), reference [6], and the notations (2.50), we obtain the evolution system (S):

$$\left\{\begin{array}{l}
\dot{g}_{ij} = -2K_{ij}; \\
\dot{K}_{ij} = R_{ij} + HK_{ij} - 2K_{ij}^2 K_{il} - 8\pi (\tau_{ij} + T_{ij}) + 4\pi (-T_{00} + g^{lm} T_{lm}) g_{ij} - \Lambda g_{ij}; \\
\dot{F}_{ij} = C_{ij}^k g_{kl} E^l; \\
\dot{E}^i = H E^i - C_{jk}^i E^k \frac{u^i}{u^0} - C_{jk}^i g^{kl} g^{im} F_{lm} - \frac{1}{2} C_{jk}^i g^{j1} g^{km} F_{lm}; \\
\dot{u}^i = 2K_{ij}^i u^j - \gamma_{ij}^i \frac{u^j}{u^0} + \frac{3}{4} \rho u^0 C_{jk}^i E^k E^i - \frac{3}{4} \rho (u^0)^2 C_{jk}^i E^k g^{il} F_{ml} u^m; \\
\dot{\Theta}^{00} = H \Theta^{00} - C_{ij}^i \Theta^{0j} + \frac{1}{3} \rho H + \rho u^0; \\
\dot{\Theta}^{0i} = H \Theta^{0i} + 2K_{ij}^i \Theta^{0j} - \frac{2}{3} (C_{jk}^i g^{ij} + \gamma_{ij}^i g^{jk}) + \rho u^i; \\
\dot{\rho} = -(\frac{3}{4} u^0 + K_{ij}^i \frac{u^i}{(u^0)^2} - K_{ij}^i + C_{ij}^i \frac{u^i}{u^0}) \rho - \frac{3}{4} g_{ij} C_{jk}^i E^k E^i \frac{u^i}{(u^0)^2}; \\
\dot{e} = - \left( \frac{K_{ij}^i u^j}{(u^0)^2} + C_{jk}^i \frac{u^k}{u^0} - K_{ij}^i \right) e + \frac{3}{4} g_{ij} \frac{u^i E^j}{\rho} e^2.
\end{array}\right.$$  \hspace{1cm} (S)

Where in (2.52), $R_{ij}$ is the Ricci tensor associated to $g_{ij}$ and whose expression due to R.T. JANTZEN [3] is:

$$R_{ij} = \gamma^l_{lm} \gamma^m_{ij} - \gamma^l_{ji} \gamma^m_{mi} - C^l_{mj} \gamma^m_{li}.$$  \hspace{1cm} (2.60)
Next, we obtain the constraints:

\[
\begin{align*}
R - K_{ij}K^{ij} + H^2 &= 16\pi(\tau_{00} + T_{00}) + 2\Lambda, \quad (2.61) \\
\nabla^i K_{ij} &= -8\pi(\tau_{0j} + T_{0j}), \quad (2.62) \\
C^d_{ij}F_{kl} + C^d_{jk}F_{il} + C^d_{ki}F_{jl} &\equiv C^d_{[ij}F_{k]l} = 0, \quad (2.63) \\
C^d_{ik}F^{0k} + eu^0 &= 0, \quad (2.64)
\end{align*}
\]

where \( R = g^{ij}R_{ij} \) and \( \nabla \) is the Levi-Civita connection associated to \( g = (g_{ij}) \). The constraint (2.61) is called the Hamiltonian constraint. As we will see, this constraint is fundamental.

3 Study of constraints. The initial values problem. Relative Norms

3.1 Study of constraints

We prove that, if the constraints are satisfied by the solutions of the evolution system at \( t = 0 \), then the constraints are satisfied everywhere. Let us set:

\[
\begin{align*}
A &= R - K_{ij}K^{ij} + H^2 - 16\pi(\tau_{00} + T_{00}) - 2\Lambda, \\
A_j &= \nabla^i K_{ij} + 8\pi(\tau_{0j} + T_{0j}), \\
A_{ijk} &= C^d_{[ij}F_{k]l}, \\
B &= C^d_{ik}F^{0k} + eu^0, \\
W &= (A, A_j, A_{ijk}, B)^t.
\end{align*}
\]

Proposition 3.1. 1) The quantities \( A, A_j, A_{ijk}, B \) satisfy the relations:

\[
\begin{align*}
\dot{A} &= 2HA + 2g^{ij}\gamma^k_{ij}A_k, \quad (3.1) \\
\dot{A}_j &= HA_j, \quad (3.2) \\
\dot{A}_{ijk} &= 0, \quad (3.3) \\
\dot{B} &= HB. \quad (3.4)
\end{align*}
\]

2) The solutions of the evolution system satisfy the constraints everywhere, if and only if they satisfy the constraints at \( t = 0 \).
Proof. 1) See Appendix.

2) If we set $W = (A, A_j, A_{ijk}, B)^t$, then $W$ satisfies: $\dot{W} = LW$, where $L$ is a matrix. So we have:

$$W(t) = W(0) \exp \left( \int_0^t L(s) ds \right).$$

This shows that $(W = 0) \iff (W(0) = 0)$. This means that the constraints are satisfied everywhere if and only if they are satisfied at $t = 0$. In what follows, we consider that the constraints are properties of the solutions of the evolution system.

\[ \square \]

3.2 The initial values problem

Let: $g^0 = (g^0_{ij})$, $K^0 = (K^0_{ij})$, $F^0 = (F^0_{ij})$ be given $3 \times 3$ matrices with $g^0$ positive definite, $K^0$ symmetric and $F^0$ antisymmetric matrices.

Let: $U^0 = (U^{0,i})$, $E^0 = (E^{0,i})$, $\theta^0 = (\theta^{0,\alpha})$ be given vectors.

Let: $\rho^0 > 0$, and $e^0 \geq 0$ be given numbers.

We look for solutions: $g = (g_{ij})$ positive definite $3 \times 3$ matrix, $K = (K_{ij})$ symmetric $3 \times 3$ matrix, $F = (F_{ij})$ antisymmetric $3 \times 3$ matrix, $u = (u^i)$, $E = (E^i)$, $\Theta = (\Theta^{0,\alpha})$ vectors, $\rho > 0$, and $e \geq 0$ two functions, such that at $t = 0$, we have:

$$\begin{align*}
    g(0) &= g^0; & K(0) &= K^0; & F(0) &= F^0; & E(0) &= E^0; \\
    u(0) &= U^0; & \Theta(0) &= \theta^0; & \rho(0) &= \rho^0; & e(0) &= e^0;
\end{align*}$$

For $t \in [0, T]$, $T \leq +\infty$. We will set:

$$U^{0,0} = \sqrt{1 + g^0_{ij} U^{0,i} U^{0,j}}; \quad u^0 = \sqrt{1 + g_{ij} u^i u^j}$$

(3.5)

$g^0$, $K^0$, $F^0$, $E^0$, $U^0$, $\theta^0$, $\rho^0$ and $e^0$ are the initial data. In what follows, we suppose that they satisfy the constraints at $t = 0$. There are eight initial data for the four constraints. This means that there are four degrees of liberty to the initial data.

3.3 Relative norms

We now introduce the notion of relative norms due to RENDALL [7].
Lemma 3.1. Define the norm of the \( n \times n \) matrix \( A \) by:

\[
\|A\| = \sup \left\{ \frac{\|Ax\|}{\|x\|}, x \neq 0, x \in \mathbb{R}^n \right\}.
\]

If \( A_1 \) and \( A_2 \) are two \( n \times n \) matrices, \( A_1 \) positive definite, the norm of \( A_2 \) with respect to \( A_1 \) is:

\[
\|A_2\|_{A_1} = \sup \left\{ \frac{\|A_2x\|}{\|A_1x\|}, x \neq 0, x \in \mathbb{R}^n \right\}.
\]

From the definition we have:

\[
\|A_2\| \leq \|A_2\|_{A_1}\|A_1\| \tag{3.6}
\]

We also have:

\[
\|A_2\|_{A_1} \leq \left[ Tr(A_1^{-1}A_2A_1^{-1}A_2) \right]^\frac{1}{2} \tag{3.7}
\]

If \( A = (a_{ij}) \) is a \( n \times m \) matrix, one defines another norm by:

\[
|A| = \sup \{ |a_{ij}|; i = 1, 2, \ldots, n; j = 1, 2, \ldots, m \}. \tag{3.8}
\]

Lemma 3.2. Let \((u^a) = (u^0, u^i)\), where \( u^0 \) and \( u^i \) are linked by (3.5). Let \( F^0i \) be given. Then, there exists a constant \( C > 0 \) such that:

\[
\left| u^i \right| \leq C |g|^\frac{1}{2}; |F^0i| \leq (g_{rs}F^0rF^0s)^\frac{1}{2} |g|^\frac{1}{2}. \tag{3.9}
\]

Proof. Take in lemma 3.1, \( A_1 = (g^{ij}) \), \( A_2 = (a^{ij}) \) with \( a^{ii} = u^i \) and \( a^{ij} = 0 \) if \( i \neq j \). A direct calculation, using \( A_1^{-1} = (g_{ij}) \) gives:

\[
Tr(A_1^{-1}A_2A_1^{-1}A_2) = a^{ij}a_{ij} \tag{3.10}
\]

(3.6) and (3.7) then give, using (3.10):

\[
\|A_2\| \leq (a^{ii}a_{ii})^\frac{1}{2} \|g\|. \tag{3.11}
\]

But we have:

\[
(a^{ij}a_{ij})^\frac{1}{2} = (a^{ii}a_{ii})^\frac{1}{2} = (g_{ik}g_{il}a^{kl}a^{ii})^\frac{1}{2} = (g_{ik}^2a^{kk}a^{ii})^\frac{1}{2} \leq |g|^\frac{1}{2} \left( g_{ik}u^i u^k \right)^\frac{1}{2}.
\]

The first relation (3.9) is due to (3.11) using the inequalities \( |u^i| \leq |A_2| \leq C \|A_2\| \), since all the norms on a finite dimensional vector space are equivalent, the relation \( u^0 = (1 + g_{ik}u^i u^k)^\frac{1}{2} \) and the fact that \( |g| \) and \( \|g\| \) are equivalent. For the second relation (3.9), take \( a^{ii} = F^0i \) and \( a^{ij} = 0 \) if \( i \neq j \) and proceed as above. This ends the proof of lemma 3.2. \( \square \)
4 Local existence by iterated method

4.1 Construction of the iterated sequence

Consider the initial values in paragraph 3.2. Set:

\[
\begin{align*}
  g_0(t) &= g^0; \quad K_0(t) = K^0; \quad F_0(t) = F^0; \quad E_0(t) = E^0; \\
  u_0(t) &= U^0; \quad \Theta_0(t) = \theta^0; \quad \rho_0(t) = \rho_0; \quad \epsilon_0(t) = \epsilon_0;
\end{align*}
\]

We also set: \( V_0 = (g_0, K_0, F_0, E_0, u_0, \Theta_0, \rho_0, \epsilon_0) \). For \( n \in \mathbb{N} \), if \( V_n = (g_n, K_n, F_n, E_n, u_n, \Theta_n, \rho_n, \epsilon_n) \) is known, then define \( V_{n+1} \) by:

\[
V_{n+1} = V_0 + \int_0^t f(V_n(s))ds
\]

where \( f(V_n) \) is the right hand side of the evolution system (S) in which:

- \( u_n^0 = \sqrt{1 + g_{n,ij}u_n^i u_n^j} \geq 1 \);
- \( \gamma_{n,ij}^k \) is obtained by replacing in (2.19) \( g_{ij} \) by \( g_{n,ij} \);
- \( R_{n,ij}, \tau_{n,\alpha\beta}, T_{n,\alpha\beta}, \Theta_{n,ij} \) are defined by the same method.

We then obtain the iterated sequence \( (V_n) \) which is of class \( C^1 \) on a maximal interval \([0, T_n[\), \( T_n > 0 \) where \( g_n = (g_{n,ij}) \) is symmetric and positive definite, \( K_n = (K_{n,ij}) \) is symmetric, \( F_n = (F_{n,ij}) \) is antisymmetric, \( \rho_n > 0 \) and \( \epsilon_n \geq 0 \).

4.2 Estimation of the iterated sequence

**Proposition 4.1.** There exists a number \( T > 0 \), independent of \( n \), such that, the iterated sequence \( V_n = (g_n, K_n, F_n, E_n, u_n, \Theta_n, \rho_n, \epsilon_n) \) is defined and uniformly bounded on \([0, T[\).

**Proof.** Let \( N \in \mathbb{N}^\ast \). Suppose that for \( n \leq N - 1 \), we have the following inequalities:

\[
\begin{align*}
  |g_n - g^0| &\leq A_1; \quad |K_n - K^0| \leq A_2; \quad |F_n - F^0| \leq A_3; \\
  |E_n - E^0| &\leq A_4; \quad |u_n - U^0| \leq A_5; \quad |\Theta_n - \theta^0| \leq A_6; \\
  |\epsilon_n - \epsilon_0| &\leq A_7; \quad |\rho_n - \rho_0| \leq A_8; \quad (\det g_n)^{-1} \leq A_9
\end{align*}
\]

(4.1)
where $A_i, i = 1, 2, 3, 4, 5, 6, 7, 8, 9$, are strictly positive constants. We prove that we can choose these constants such that we still have the inequalities (4.1) for $n = N$, for $T$ sufficiently small.

Note that for $t$ sufficiently small, we have

$$|\rho_{N-1} - \rho^0| \leq \frac{\rho^0}{T}$$

from where we deduce that

$$\frac{1}{\rho_{N-1}} \leq \frac{2}{\rho^0}.$$ 

we take $A_8 = \frac{\rho^0}{T}$. Taking the inequalities (4.1) into account, the definition of the iterated sequence, the expressions of $H_{N-1}, R_{N-1,ij}, T_{N-1,ij}, \tau_{N-1,ij}$ and $u_{N-1}^0 = \sqrt{1 + g_{N-1,ij}u_{N-1}^{ij}u_{N-1}^{ij}} \geq 1$,

we can find constants $B_i, i = 1, 2, 3, 4, 5, 6, 7, 8$ strictly positive depending only on $A_i$ such that:

$$\begin{cases}
|\dot{g}_{N,ij}| \leq B_1; & |\dot{K}_{N,ij}| \leq B_2; & |\dot{F}_{N,ij}| \leq B_3; & |\dot{E}_{N}^{ij}| \leq B_4; \\
|\dot{u}_{N}^{ij}| \leq B_5; & |\dot{\Theta}_{N}^{ij}| \leq B_6; & |\dot{\rho}_{N}| \leq B_7; & |\dot{\rho}_{N}| \leq B_8.
\end{cases}$$ (4.2)

By integration of (4.2) we have:

$$\begin{cases}
|g_N - g^0| \leq B_1 t; & |K_N - K^0| \leq B_2 t; & |F_N - F^0| \leq B_3 t; & |E_N - E^0| \leq B_4 t; \\
|u_N - U^0| \leq B_5 t; & |\Theta_N - \theta^0| \leq B_6 t; & |e_N - e_0| \leq B_7 t; & |\rho_N - \rho_0| \leq B_8 t.
\end{cases}$$ (4.3)

Let us bound $(\det g_N)^{-1}$. By the definition of the iterated sequence we have:

$$\frac{d}{dt} g_{N,ij} = -2K_{N-1,ij}.$$ (4.4)

On one hand we have the equality:

$$\frac{d}{dt} \ln(\det g_N) = g_{N,ij} \frac{d}{dt} g_{N,ij}.$$ 

On the other hand we have:

$$\frac{d}{dt} \ln(\det g_N) = \frac{1}{(\det g_N)(\det g_N)^{-1}} \frac{d}{dt} (\det g_N)$$

$$= -\det g_N \frac{d}{dt} (\det g_N)^{-1}.$$ 

The relation (4.4) then implies:

$$\frac{d}{dt} (\det g_N)^{-1} = (2g_{N}^{ij}K_{N-1,ij})(\det g_N)^{-1}$$
which is a differential equation in \((\det g_N)^{-1}\) on \([0; t], t > 0\), whose solution is:

\[
(\det g_N)^{-1} = (\det g^0)^{-1} \exp \left( \int_0^t 2g_N^{i\bar{j}}K_{N-1;i\bar{j}}(s)ds \right).
\]  

(4.5)

The relation (4.4) which is similar to (2.17), shows that \(g_N\) and \(K_{N-1}\) are the first and second fundamental forms of a hypersurface. We must then have:

\[
g_N^{i\bar{j}}K_{N-1;i\bar{j}} = K_{N-1}^{i} = \text{Tr}(K_{N-1}).
\]

We then deduce from (4.1) and (4.5) that there exists a constant \(C > 0\) depending on \(A_i\), \(i = 1, 2, 3, 4, 5, 6, 7, 8\), and \(K^0\) such that:

\[
(\det g_N)^{-1} \leq (\det g^0)^{-1}\exp(Ct),
\]

(4.6)

Now if we take in (4.1):

\[(detg^0)A_9 > 1,
\]

we will have for \(t\) sufficiently small:

\[(det g^0)A_9 > \exp(Ct) > 1.
\]

This means, from (4.6) that, there exists \(t_1 > 0\), such that, for \(0 < t \leq t_1\), we have:

\[
(\det g_N)^{-1} \leq A_9.
\]

(4.7)

Then, using (4.3) and (4.7), we conclude that if \(T > 0\) is such that:

\[0 < T < t_1, \quad B_iT < A_i; i = 1, 2, 3, 4, 5, 6, 7, 8\]

then we still have the inequalities (4.1) for \(n = N\). Hence the iterated sequence \((V_n)\) is uniformly bounded on \([0, T]\). \(\Box\)

### 4.3 Local existence

**Proposition 4.2.** The initial values problem for the evolution system (2.51)...(2.59) has a unique local solution.

**Proof.** Let \([0, T[, T > 0\), be the interval obtained in proposition 4.1. We prove that the iterated sequence \(V_n = (g_n, K_n, F_n, E_n, u_n, \Theta_n, \rho_n, e_n)\) converges uniformly on every \([0, \delta \subset [0, T[, \delta > 0,\]

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towards a solution $V = (g, K, F, E, u, \Theta, \rho, c)$ of the evolution system.

We take the difference between two consecutive terms of $V_n$, we use the fact that they have the same initial data and since the sequences $\left( \frac{1}{\rho_n} \right)_n$, $(V_n)$ are uniformly bounded we have, with $C > 0$ a constant:

$$
|g_{n+1}(t) - g_n(t)| + |K_{n+1}(t) - K_n(t)| + |F_{n+1}(t) - F_n(t)|
+ |E_{n+1}(t) - E_n(t)| + |\Theta_{n+1}(t) - \Theta_n(t)|
+ |\rho_{n+1}(t) - \rho_n(t)| + |e_{n+1}(t) - e_n(t)|
\leq C \int_0^t |g_n(s) - g_{n-1}(s)|
+ |F_n(s) - F_{n-1}(s)| + |E_n(s) - E_{n-1}(s)|
+ |\Theta_n(s) - \Theta_{n-1}(s)| + |\rho_n(s) - \rho_{n-1}(s)| + |e_n(s) - e_{n-1}(s)| ds.
$$

For the same reasons we have:

$$
\left\| \frac{dg_{n+1}}{dt} - \frac{dg_n}{dt} \right\| + \left\| \frac{dK_{n+1}}{dt} - \frac{dK_n}{dt} \right\|
+ \left\| \frac{dF_{n+1}}{dt} - \frac{dF_n}{dt} \right\|
+ \left\| \frac{dE_{n+1}}{dt} - \frac{dE_n}{dt} \right\|
+ \left\| \frac{d\Theta_{n+1}}{dt} - \frac{d\Theta_n}{dt} \right\|
+ \left\| \frac{d\rho_{n+1}}{dt} - \frac{d\rho_n}{dt} \right\|
+ \left\| \frac{de_{n+1}}{dt} - \frac{de_n}{dt} \right\|
\leq C \int_0^t |g_n - g_{n-1}|
+ |F_n - F_{n-1}|
+ |E_n - E_{n-1}|
+ \left\| \Theta_n - \Theta_{n-1} \right\|
+ \left\| \rho_n - \rho_{n-1} \right\|
+ \left\| e_n - e_{n-1} \right\|.
$$

For $n \in \mathbb{N}$, we set:

$$
\alpha_n(t) = |g_{n+1}(t) - g_n(t)| + |K_{n+1}(t) - K_n(t)| + |F_{n+1}(t) - F_n(t)|
+ |E_{n+1}(t) - E_n(t)| + |\rho_{n+1}(t) - \rho_n(t)| + |e_{n+1}(t) - e_n(t)|
+ |\Theta_{n+1}(t) - \Theta_n(t)| + |\Theta_{n-1}(t) - \Theta_n(t)|.
$$

(4.8) and (4.10) give:

$$
\alpha_n(t) \leq C \int_0^t \alpha_{n-1}(s) ds.
$$

By induction on $n \geq 2$, we obtain, from (4.11):

$$
|\alpha_n(t)| \leq ||\alpha_2||_{\infty} \frac{(Ct)^{n-2}}{(n-2)!} \leq ||\alpha_2||_{\infty} \frac{(C\delta)^{n-2}}{(n-2)!},
$$

for $0 \leq t \leq \delta$ and $0 < \delta < T$.

But the series $\sum \frac{(C\delta)^n}{(n)!}$ converges. Hence we obtain from (4.12) that:

$$
\lim_{n \to +\infty} \sup_{0 \leq t \leq \delta} \alpha_n(t) = 0.
$$
Given the definition (4.10) of $\alpha_n$, we conclude that every sequence $(g_n), (K_n), (E_n), (u_n), (\Theta_n) (\rho_n)$ and $(e_n)$ converges uniformly on every interval $[0, \delta], 0 < \delta < T$ and we denote the different limits by $g, K, E, u, \Theta, \rho$ and $e$ which are continuous functions of $t$.

Now from the inequality (4.9), we conclude similarly that the sequences of derivatives $(\frac{dg_n}{dt}), (\frac{dK_n}{dt}), (\frac{dE_n}{dt}), (\frac{du_n}{dt}), (\frac{d\Theta_n}{dt})$ and $(\frac{de_n}{dt})$ converge uniformly on $[0, \delta], 0 < \delta < T$. In these conditions, the functions $g, K, E, u, \Theta, \rho$ and $e$ are of class $C^1$ on $[0, T]$. Hence $V = (g, K, F, E, u, \Theta, \rho, e)$ is a local solution of the evolution system (2.51)...(2.59).

We now prove that the solution is unique.

Consider two solutions $V_1$ and $V_2$ of the same initial values problem. Define $\alpha(t) = |V_1(t) - V_2(t)|$ with $\alpha(0) = 0$. Since the functions $g, K, F, E, u, \Theta, \rho$ and $e$ are bounded on $[0, \delta], 0 < \delta < T$, there exists a constant $C > 0$ such that:

$$\alpha(t) \leq C \int_0^t \alpha(s) ds.$$ 

By Gronwall Lemma, we obtain $\alpha(t) = 0$ since $\alpha(0) = 0$. So $V_1 = V_2$ and the local solution is unique. \hfill $\square$

5 Global existence theorem

We have to prove that, the solution $V = (g, K, F, E, u, \Theta, \rho, e)$ and the functions $\frac{1}{\rho}$ and $(\det g)^{-1}$ are bounded on every bounded interval. First of all, we prove the following important result on the mean curvature $H = g^{ij} K_{ij}$.

Proposition 5.1. Let the function $H = g^{ij} K_{ij}$ be bounded on $[0, T^*]$ where $T^* < +\infty$. Then the functions $g, K, F, E, u, \Theta, \rho, e, (\det g)^{-1}$ and $\frac{1}{\rho}$ are bounded on $[0, T^*]$.

Proof. We will use the following Lemma:

Lemma 5.1. The mean curvature $H = g^{ij} K_{ij}$ satisfies the following relation:

$$\frac{dH}{dt} = R + H^2 + 4\pi g^{ij} (\tau_{ij} + T_{ij}) - 12\pi (\tau_{00} + T_{00}) - 3\Lambda. \quad (5.1)$$

Proof of the Lemma

Since $H = g^{ij} K_{ij}$, the relation (2.17) gives: $\dot{g}^{ij} = 2K^{ij}$, from there we have:

$$\frac{dH}{dt} = 2K_{ij} K^{ij} + g^{ij} \dot{K}_{ij}$$
then obtain (5.1) from equation (2.52) by a direct calculation using $4g^{\alpha \beta}4\tau_{\alpha \beta} = 0$, which implies $g^{ij}\tau_{ij} = \tau_{00}$.

Proof of Proposition 5.1

• Boundedness of $|g|$ on $[0, T^*]$. By using the Hamiltonian constraint (2.61), (5.1) gives:

$$
\frac{dH}{dt} = K_{ij}K^{ij} - \Lambda + 4\pi g^{ij}(\tau_{ij} + T_{ij}) + 4\pi(\tau_{00} + T_{00}).
$$

(5.2)

But we have $\tau_{00} = g^{ij}\tau_{ij} \geq 0; T_{00} \geq 0; g^{ij}T_{ij} \geq 0$. We then deduce from (5.2):

$$
\frac{dH}{dt} \geq K_{ij}K^{ij} - \Lambda.
$$

(5.3)

Integrating (5.3) on $[0, t]$ for $0 \leq t \leq T^*$ gives:

$$
H(t) \geq H(0) - \Lambda t + \int_0^t K_{ij}K^{ij}ds.
$$

(5.4)

which shows, since $H$ is bounded on $[0, T^*]$ that we have:

$$
\int_0^{T^*} K_{ij}K^{ij}ds < +\infty.
$$

(5.5)

Now the integration of equation (2.51) on $[0; t], 0 \leq t \leq T^*$, gives:

$$
|g(t)| \leq |g(0)| + 2\int_0^t |K(s)|ds.
$$

(5.6)

(5.6) gives, using (3.6), the inequality:

$$
||g(t)|| \leq C \left[ ||g(0)|| + \int_0^t ||K(s)||_g(s)||g(s)||ds \right].
$$

(5.7)

(5.7) gives, using (3.7):

$$
||g(t)|| \leq C \left[ ||g(0)|| + \int_0^t (K_{ij}K^{ij})^{1/2}||g(s)||ds \right].
$$

By Gronwall Lemma, this implies:

$$
||g(t)|| \leq C_t||g(0)|| \exp \left( C \int_0^t (K_{ij}K^{ij})^{1/2}ds \right), 0 \leq t \leq T^*;
$$

This shows, using (5.5) that $||g||$, and therefore $|g|$, are bounded on $[0, T^*]$.

• Boundedness of $(\det g)^{-1}$ on $[0; T^*]$

The relation:

$$
\frac{d}{dt}\ln(\det g) = g^{ij}\frac{dg_{ij}}{dt}
$$
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gives, using equation (2.52) and \( H = g^{ij}K_{ij} \):

\[
\frac{d}{dt} \ln(\det g) = -2H.
\]

But \(|H|\) is bounded on \([0, T^*]\). Integrating on \([0, t], 0 \leq t < T^*\], there exists a constant \( C > 0 \) such that:

\[-C < \ln(\det g) < C\]

from where we have:

\[e^{-C} < \det g < e^C.\]

Then \( \det g \) and \((\det g)^{-1}\) are bounded on \([0, T^*]\).

**Boundedness of \(|K|\) on \([0, T^*]\)**

Since \(|g|\) and \((\det g)^{-1}\) are bounded on \([0, T^*]\), the expression (2.60) of \( R_{ij} \) shows that \( R = g^{ij}R_{ij} \) is bounded on \([0, T^*]\). We deduce from the Hamiltonian constraint (2.61), since \( \tau_{00} \geq 0, T_{00} \geq 0 \), that:

\[R + H^2 - 2\Lambda \geq K_{ij}K^{ij}.\]

\( R \) and \( H^2 \) being bounded, so is \( K_{ij}K^{ij} \). We then deduce from the inequality:

\[||K(s)|| \leq (K_{ij}K^{ij})^{\frac{1}{2}} ||g(s)||,\]

that \( ||K|| \) and hence \( |K| \) is bounded on \([0, T^*]\), since \( (K_{ij}K^{ij})^{\frac{1}{2}} \) and \( ||g|| \) are bounded.

**Boundedness of \( E \) on \([0, T^*]\)**

We use the Hamiltonian constraint (2.61) and \( \tau_{00} \geq 0, T_{00} \geq 0 \), to have:

\[0 \leq \max(16\pi\tau_{00}, 16\pi T_{00}) \leq 16\pi(\tau_{00} + T_{00}) = R + H^2 - K_{ij}K^{ij} - 2\Lambda.\]

This shows that \( \tau_{00} \) and \( T_{00} \) are bounded on \([0, T^*]\). We deduce from (2.11) that:

\[g^{ij}\tau_{ij} = \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}g_{ij}E^iE^j,\]

thus

\[0 \leq \frac{1}{2}g_{ij}E^iE^j \leq \tau_{00},\]

since \( F^{ij}F_{ij} \geq 0 \). We use (3.9) to conclude that \( E \) is bounded, since \( (g_{rs}F^{0r}F^{0s})^{\frac{1}{2}} \) and \( |g| \) are bounded.
• Boundedness of $F$ on $[0, T^\star]$  

Equation (2.53) gives, since $|g|$ and $|E|$ are bounded:

$$|\dot{F}_{ij}| \leq C, \forall t \in [0, T^\star].$$

where $C$ is a constant. Integrating on $[0, t]$, $0 \leq t \leq T^\star$ gives the result.

• Boundedness of $\rho$ on $[0, T^\star]$  

Since $|E|$, $|g|$, $|K|$, $\frac{u^i}{\rho}$, $\frac{1}{\rho}$ are bounded, equation (2.58) in $\rho$ shows that, there exists two constants $A > 0$, $B > 0$ such that:

$$\rho(t) \leq A \int_0^t \rho(s)ds + B, \ \forall t \in [0, T^\star].$$

By Gronwall Lemma, we have:

$$\rho(t) \leq Be^{At} \leq Be^{AT^\star}, \ \forall t \in [0, T^\star]$$

and $\rho$ is bounded.

• Boundedness of $u^0$, $\frac{1}{\rho}$ and $u^i$ on $[0, T^\star]$  

From (2.28) and (2.29), we have:

$$\dot{\rho} + \dot{u}^0 = -3 + \frac{1}{u^0} - K_i^i + C_{ij}^i u^j.$$

Since $\frac{1}{\rho^2}$, $H = K_i^i$ and $\frac{u^j}{\rho}$ are bounded on $[0; T^\star]$, there exists a constant $C > 0$ such that:

$$-C < -\int_0^t (3 + \frac{1}{u^0} - K_i^i + C_{ij}^i u^j)(s)ds < C,$$

for $0 < t < T^\star$. From where we deduce by integration:

$$e^{-C} \leq \rho u^0 \leq e^C \text{ on } [0, T^\star].$$

which proves that: $\rho u^0$ and $\frac{1}{\rho^2}$ are bounded on $[0, T^\star]$. Now write, using (3.9):

$$\left|\frac{u^j}{\rho^2}\right| = \frac{\rho u^j}{\rho u^0} \leq C |g|^\frac{3}{2}$$

from where we have:

$$\rho |u^i| \leq c(\rho u^0)|g|^\frac{3}{2}$$
and this prove that $\rho u^i$ is bounded. Now multiply equation (2.55) in $u^i$ by $\rho$, and use the fact that $\rho u^i$ is bounded to conclude that $\rho \dot{u}^i$ is also bounded. The relation

$$u^0 = \sqrt{1 + g_{ij}u^iu^j}$$

gives, derivating and using equation (2.51):

$$\frac{\dot{u}^0}{u^0} = -K_{ij} \frac{u^i}{u^0} \frac{u^j}{u^0} + g_{ij} \rho \dot{u}^i \frac{1}{\rho u^0} \frac{u^j}{u^0},$$

which shows, since $\frac{u^i}{u^0}$, $K$, $\rho u^0$, $g$, $\rho \dot{u}^i$ are bounded, that there exists a constant $A > 0$ such that:

$$\left| \frac{\dot{u}^0}{u^0} \right| < A.$$

Integrating on $[0, t]$, $0 \leq t \leq T^*$ we obtain:

$$1 \leq u^0(t) \leq u^0(0) \exp(AT^*).$$

Hence $u^0$ is bounded on $[0, T^*]$. Now write: $\frac{1}{\rho} = \frac{1}{\rho u^0} \times u^0$ and $|u^i| = \frac{|u^i|}{u^0} \times u^0$ to conclude that $\frac{1}{\rho}$ and $u^i$ are bounded on $[0, T^*]$.

- **Boundedness of $\Theta^{0\alpha}$ on $[0, T^*]$**

Since $|H|$, $\rho$, $u^0$, $|K|$, $(\det g)^{-1}$ and $u^i$ are bounded on $[0, T^*]$, by equations (2.56) and (2.57) in $\Theta^{0\alpha}$, there exists two constant $C_1$ and $C_2$: such that:

$$|\dot{\Theta}^{0\beta}| \leq C_1 \sum_{\alpha=0}^{3} |\Theta^{0\alpha}| + C_2.$$

Summing in $\alpha$ and integrating on $[0, t]$, $0 \leq t < T^*$, we obtain:

$$\sum_{\alpha=0}^{3} |\Theta^{0\alpha}|(t) \leq \sum_{\alpha=0}^{3} |\Theta^{0\alpha}(0)| + 4C_2T^* + 4C_1 \int_{0}^{t} \sum_{\alpha=0}^{3} |\Theta^{0\alpha}| ds;$$

and Gronwall Lemma gives:

$$\sum_{\alpha=0}^{3} |\Theta^{0\alpha}|(t) \leq \left( \sum_{\alpha=0}^{3} |\Theta^{0\alpha}(0)| + 4C_2T^* \right) \exp(4C_1T^*) \text{ on } [0, T^*].$$

which shows that each $\Theta^{0\alpha}$ is bounded on $[0, T^*]$.

- **Boundedness of $e$ on $[0, T^*]$**

From the constraint (2.64) we have:

$$e = -\frac{C_{ik} F^{0k}}{u^0};$$

which is bounded on $[0, T^*]$, since $\frac{1}{u^0}$ and $E$ are. This completes the proof of Proposition 5.1.
Théorème 5.1. Suppose that $\Lambda > 0$ and that the mean curvature satisfies $H(0) < 0$. Then the initial values problem for the coupled system of Einstein-Maxwell-pseudo tensor of pressure has a unique global solution.

Proof. From Proposition 5.1, we have to prove that the mean curvature $H$ is bounded on every interval $[0, T^*]$ such that $T^* < +\infty$. Consider the traceless tensor:

$$\sigma_{ij} = K_{ij} - \frac{1}{3} H g_{ij}. \tag{5.8}$$

By a direct calculation, we have:

$$\sigma_{ij} \sigma^{ij} = K_{ij} K^{ij} - \frac{1}{3} H^2. \tag{5.9}$$

Using the hamiltonian constraint (2.61) and (5.9), we obtain:

$$\frac{2}{3} H^2 - 2\Lambda = \sigma_{ij} \sigma^{ij} + 16\pi (\tau_{00} + T_{00}) - R \tag{5.10}$$

We have $\sigma_{ij} \sigma^{ij} \geq 0$, $\tau_{00} \geq 0$, $T_{00} \geq 0$. R.T. JANTZEN in [3] and R.WALD in [8] prove that, in the spacetimes considered here, we always have $R \leq 0$. We then deduce from (5.10):

$$\frac{2}{3} H^2 - 2\Lambda > 0,$$

this means:

$$H^2 > 3\Lambda.$$

From there we have:

$$H < -\sqrt{3\Lambda}, \text{ or } H > \sqrt{3\Lambda}$$

but $H$ is continuous and we have $H(0) < 0$, then we must have:

$$H < -\sqrt{3\Lambda}.$$

Now if we take (5.4) in which we have $K_{ij} K^{ij} \geq 0$, we conclude that:

$$H(t) \geq H(0) - \Lambda t.$$

Hence, on every interval $[0, T^*]$, where $T^* < +\infty$, we have:

$$H(0) - \Lambda T^* \leq H(t) \leq -\sqrt{3\Lambda}$$

which prove that $H$ is bounded on each interval $[0, T^*]$. This ends the proof of theorem 5.1. \qed
We end the paragraph by the following result:

**Proposition 5.2.** For $\Lambda < 0$, there exists no global solution to the Einstein-Maxwell-pseudo-tensor of pressure initial values problem.

**Proof.** Let $\Lambda < 0$ be given and suppose that the system has a global solution on the whole interval $[0, +\infty[$.

By (5.3) and (5.9), we have:

$$\frac{dH}{dt} \geq \sigma_{ij} \sigma^{ij} + \frac{1}{3} H^2 - \Lambda.$$  

But $\sigma_{ij} \sigma^{ij} \geq 0$, then:

$$\frac{dH}{dt} \geq \frac{1}{3} H^2 - \Lambda.$$ (5.11)

Since $\Lambda < 0$, we have $-\Lambda > 0$ and from (5.11) we deduce the inequalities:

$$\frac{dH}{dt} \geq \frac{1}{3} H^2,$$ (5.12)

$$\frac{dH}{dt} \geq -\Lambda.$$ (5.13)

(5.13) gives, by integration on $[0, t]$, $t \geq 0$:

$$H(t) \geq H(0) - \Lambda t.$$ (5.14)

(5.14) shows that $H(t) \to +\infty$ when $t \to +\infty$. So there exists $t_0 > 0$ such that: $H(t_0) > 0$. Now we have, given (5.12): $H(t) \geq W(t)$, for $t \geq t_0$, where $W$ is every function satisfying:

$$\begin{cases} 
\frac{dW}{dt} = \frac{1}{3} W^2, \\
W(t_0) = H(t_0) > 0.
\end{cases}$$ (5.15)

(5.16)

A solution $W$ of (5.15) on $[t_0, +\infty[$ is, given (5.16):

$$W(t) = \frac{3H(t_0)}{3 - H(t_0)(t - t_0)}.$$ (5.17)

(5.17) shows that: $W(t) \to +\infty$ when $t \to < t^* = t_0 + \frac{3}{H(t_0)} > t_0$. Hence $H(t) \to +\infty$ when $t \to < t^*$. This is impossible since the continuous function $H$ is bounded on the compact set $[t_0, t^*]$ and cannot tend to $+\infty$ when $t$ tends to $t^*$. So there can exist no global solution if $\Lambda < 0$. $\square$
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6 Asymptotic behavior and geodesic completeness

Here we extend the results of [4] to the charged case. We also study the asymptotic behavior of the charge \( e \), the matter density \( \rho \), the curvature tensor, the electromagnetic field and the pseudo-tensor of pressure.

6.1 Asymptotic behavior

Proposition 6.1. We have at late times:

\[
H(t) = -\sqrt{3\Lambda} + O(e^{-2\gamma t}) \quad (6.1)
\]
\[
\dot{H} = O(e^{-2\mu t}) \quad (6.2)
\]
\[
(\det g)^{-1} = O(e^{-6\gamma t}) \quad (6.3)
\]
\[
\det g = O(e^{6\gamma t}) \quad (6.4)
\]
\[
\sigma_{ij}\sigma^{ij} = O(e^{-2\gamma t}) \quad (6.5)
\]
\[
\sigma_{ij} = O(e^{\gamma t}) \quad (6.6)
\]
\[
\tau_{00}(t) = O(e^{-2\gamma t}) \quad (6.7)
\]
\[
T_{00}(t) = O(e^{-2\gamma t}) \quad (6.8)
\]
\[
R = O(e^{-2\gamma t}) \quad (6.9)
\]
\[
K_{ij}K^{ij} = \Lambda + O(e^{-2\gamma t}) \quad (6.10)
\]
\[
g_{ij}(t) = e^{2\gamma t}(g_{ij} + O(e^{-\gamma t})) \quad (6.11)
\]
\[
g^{ij}(t) = e^{-2\gamma t}(g^{ij} + O(e^{-\gamma t})) \quad (6.12)
\]
\[
\sigma^{ij}(t) = O(e^{-3\gamma t}) \quad (6.13)
\]
\[
E_iE^i(t) = O(e^{-2\gamma t}) \quad (6.14)
\]
\[
F_{ij}F^{ij}(t) = O(e^{-2\gamma t}) \quad (6.15)
\]
\[
E^i(t) = O(e^{2\gamma t}) \quad (6.16)
\]
F_{ij} = O(e^{4\gamma t}) \quad (6.17)

F^{ij} is bounded \quad (6.18)

K_{ij}(t) = O(e^{2\gamma t}) \quad (6.19)

\rho(u^0)^2 = O(e^{-2\mu t}) \quad (6.20)

\rho = O(e^{-2\mu t}) \quad (6.21)

e = O(e^{2\gamma t}) \quad (6.22)

\Theta_{ij}(t) = O(e^{2(\gamma-\mu)t}) \quad (6.23)

\tau_{ij}(t) = O(e^{8\gamma t}) \quad (6.24)

\gamma_{ij}^k is bounded \quad (6.25)

\Theta_{00}^0 = \Theta_{00} = O(e^{-2\gamma t}) \quad (6.26)

\tau_{0j}(t) = O(e^{6\gamma t}) \quad (6.27)

T_{0j} = O(e^{6\gamma t}) \quad (6.28)

\Theta_{0j} = O(e^{4\gamma t}) \quad (6.29)

T_{ij}(t) = O(e^{6(\gamma-2\mu)t}) \quad (6.30)

where \ \gamma^2 = \frac{\Lambda}{3}, \ \ 0 < \mu < \gamma, \ (G_{ij}) \ and \ (G^{ij}) \ are \ positive \ definite \ constant \ matrices.

\underline{proof \ of \ (6.1)}

We have by (5.3) and (5.9), since \ \sigma_{ij}\sigma^{ij} \geq 0:

\[ \frac{dH}{dt} \geq \frac{1}{3}H^2 - \Lambda > 0. \quad (6.31) \]

But: \ \frac{1}{3}H^2 - \Lambda = \frac{1}{3}(-H - \sqrt{3\Lambda})(-H + \sqrt{3\Lambda}). \ We \ are \ in \ the \ case \ of \ global \ existence, \ then: \ H < -\sqrt{3\Lambda}. \ So: -H + \sqrt{3\Lambda} > 2\sqrt{3\Lambda}. \ We \ then \ deduce \ from \ (6.31) \ that:

\[ \frac{dH}{dt} \geq \frac{1}{3}H^2 - \Lambda \geq \frac{2\sqrt{3\Lambda}}{3}(-H - \sqrt{3\Lambda}). \quad (6.32) \]

We can write (6.32) as:

\[ \frac{d(H + \sqrt{3\Lambda})}{dt} + 2\gamma(H + \sqrt{3\Lambda}) \geq 0; \quad (6.33) \]

where

\[ \gamma^2 = \frac{\Lambda}{3}. \quad (6.34) \]
Multiplying (6.33) by $e^{2\gamma t}$ and integrate over $[0, t]; \ t > 0,$ to obtain:

$$e^{2\gamma t}(H + \sqrt{3\Lambda}) \geq H(0) + \sqrt{3\Lambda}$$

which gives:

$$|H + \sqrt{3\Lambda}| \leq |H(0) + \sqrt{3\Lambda}|e^{-2\gamma t} \quad (6.35)$$

This proves (6.1) with $\gamma$ given by (6.34). □

**Proof of (6.2)**

We have, for $\mu \in ]0, \gamma[$:

$$e^{2\mu t} \frac{dH}{dt} = e^{2\mu t} \frac{d(H + \sqrt{3\Lambda})}{dt} = \frac{d}{dt}[e^{2\mu t}(H + \sqrt{3\Lambda})] - 2\mu e^{2\mu t}(H + \sqrt{3\Lambda}) \quad (6.36)$$

which proves, given (6.1) and (6.35) since $0 < \mu < \gamma,$ that $e^{2\mu t} \frac{dH}{dt}$ is integrable on $[0, +\infty[.$ We conclude that $e^{2\mu t} \frac{dH}{dt} \rightarrow 0$ as $t$ tends to $+\infty$ and we obtain (6.2). □

**Proof of (6.3) and (6.4)**

We have seen that:

$$\frac{d}{dt}[\ln (\det g)] = -2H \quad (6.37)$$

From $H < -\sqrt{3\Lambda}$ and (6.36) we have:

$$6\gamma \leq -2H \leq 6\gamma + 2|H(0) + \sqrt{3\Lambda}|e^{-2\gamma t}.$$ 

this means, given (6.37) that:

$$6\gamma \leq \frac{d}{dt}[\ln (\det g)] \leq 6\gamma + 2|H(0) + \sqrt{3\Lambda}|e^{-2\gamma t}. \quad (6.38)$$

Integrating over $[0, t], \ t > 0,$ we obtain:

$$(\det g^0)e^{6\gamma t} \leq \det g \leq (\det g^0) \exp \left( \frac{1}{\gamma} |H(0) + \sqrt{3\Lambda}| \right) e^{6\gamma t}.$$ 

which proves (6.3) and (6.4).

**Proof of (6.5)**

Since $\tau_{00} \geq 0, \ T_{00} \geq 0,$ and $-R \geq 0,$ (5.10) gives:

$$\frac{2}{3} H^2 - 2\Lambda \geq |\sigma_{ij}\sigma^{ij}| \geq 0. \quad (6.39)$$

But we have, since $H$ is increasing: $0 \leq -H + \sqrt{3\Lambda} \leq -H(0) + \sqrt{3\Lambda}$. Then:

$$0 \leq \frac{2}{3} H^2 - 2\Lambda \leq 2| - H(0) + \sqrt{3\Lambda}|.$$
(6.5) follows then from (6.39) and (6.1).

**Proof of (6.6)**

(6.6) is proven as in [4], using (6.5) and the notion of relative norms.

**Proof of (6.7), (6.8) and (6.9)**

(5.10) gives, using (6.5):

\[
16\pi (\tau_{00} + T_{00}) - R + \sigma_{ij}\sigma^{ij} = \frac{2}{3}H^2 - 2\Lambda = O(e^{-2\gamma t})
\]

But, \(T_{00} \geq 0, \tau_{00} \geq 0, R \leq 0\) and \(\sigma_{ij}\sigma^{ij} \geq 0\). Hence (6.7), (6.8) and (6.9) follow from (6.40).

**Proof of (6.10)**

(6.10) is a direct consequence of (5.9), (6.5) and (6.1).

**Proof of (6.11) and (6.12)**

The proof of (6.11) and (6.12) are given by [4] using (6.6).

**Proof of (6.13)**

We have: \(\sigma^{ij} = g^{ik}g^{jl}\sigma_{kl}\). (6.13) then follows from (6.12) and (6.6).

**Proof of (6.14) and (6.15)**

We have:

\[
\tau_{00} = g^{ij}\tau_{ij} = \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}g_{ij}E^iE^j \geq 0
\]

Hence,

\[
0 \leq \frac{1}{4}F^{ij}F_{ij} \leq \tau_{00} , \quad 0 \leq \frac{1}{2}g_{ij}E^iE^j \leq \tau_{00}.
\]

(6.14) and (6.15) are then consequences of (6.7).

**Proof of (6.16)**

The second relation (3.9) gives, using (6.11) and (6.14):

\[
|E^i| \leq (g_{rs}E^rE^s)^{\frac{1}{2}}|g|^{\frac{3}{2}} \leq (E_rE^r)^{\frac{1}{2}}|g|^{\frac{3}{2}} \leq ce^{-\gamma t}e^{3\gamma t} = ce^{2\gamma t}
\]

which proves (6.16).

**Proof of (6.17)**

The integration of equation (2.53) gives:

\[
F_{ij}(t) = F_{ij}(0) + C_{ij}^l \int_0^t g_{lm}E^m(s)ds
\]

(6.17) then follows from (6.11) and (6.16).

**Proof of (6.18)**
Use $F^{ij} = g^{ik} g^{jl} F_{kl}$, (6.12) and (6.17).

**proof of (6.19)**
The relation (5.8) gives:

$$|K_{ij}| \leq |\sigma_{ij}| + \frac{1}{3}|Hg_{ij}|.$$  

(6.19) then follows from (6.6), (6.1) and (6.11).

**proof of (6.20) and (6.21)**
The relation (5.1) gives, since $\tau_{00} = g^{ij} \tau_{ij}$:

$$4\pi e^{2\mu t} g^{ij} T_{ij} = e^{2\mu t} \left[ \frac{dH}{dt} + (3\Lambda - H^2) - R + 12\pi T_{00} + 8\pi \tau_{00} \right].$$  

The relations (6.1), (6.2), (6.7), (6.8), (6.9) and $0 < \mu < \gamma$, give:

$$g^{ij} T_{ij} = O(e^{-2\mu t}).$$

But we know that

$$T_{ij} = \frac{4}{3} \rho u_i u_j + \frac{1}{3} \rho g_{ij},$$

thus:

$$g^{ij} T_{ij} = \frac{4\rho}{3} g^{ij} u_i u_j + \rho \geq \rho(g^{ij} u_i u_j + 1) = \rho(u^0)^2 \geq \rho > 0.$$  

(6.41) gives (6.20) and (6.21).

**proof of (6.22)**
The constraint (2.64) *i.e.* $C^{i}_{ik} E^k + eu^0 = 0$, (6.16) and $u^0 \geq 1$ give (6.22).

**proof of (6.23)**

(6.23) is given by (2.14) *i.e.* $\Theta_{ij} = \frac{e}{3} g_{ij}$, (6.11) and (6.21).

**proof of (6.24)**
From (2.48) we have:

$$\tau_{ij} = \frac{1}{2} g_{ij} E^k E_k - g_{ik} g_{jl} E^k E^l \frac{1}{4} g_{ij} F^{lk} F_{lk} + g_{jl} F_{ik} F^{lk},$$

this proves (6.24) using (6.11), (6.14), (6.16), (6.15), (6.17) and (6.18).

**proof of (6.25)**
It is a consequence of (2.19), using (6.11) and (6.12).

**proof of (6.26)**
From (2.12) we obtain:

$$\Theta_{00} = T_{00} - \frac{4}{3} \rho (u^0)^2.$$
(6.26) is then a consequence of (6.8) and (6.20) since $0 < \mu < \gamma$.

**proof of (6.27)**

From (2.48) we have:

$$\tau_{0j} = -E^k F_{jk}$$

(6.27) is then a consequence of (6.16) and (6.17).

**proof of (6.28)**

We have, using the constraint (2.62):

$$4\pi T_{0j} = -8\pi \tau_{0j} + g^{ik} \gamma_k^l K_{lj} - g^{ik} \gamma_k^l K_{il}$$

(6.28) is then a consequence of (6.27), (6.19), (6.24) and (6.12).

**proof of (6.29)**

From (6.12) we obtain:

$$\Theta^{0j} = T^{0j} - \frac{4}{3} \rho u^0 u^j = -g^{ij} T_{0i} - \frac{4}{3} \rho u^0 u^j.$$  

(6.29) is then a consequence of (6.28), (6.12), (6.21) and (3.9).

**proof of (6.30)**

We have:

$$T_{ij} = \frac{4}{3} \rho u_i u_j + \frac{1}{3} \rho g_{ij}.$$  

From (3.9) we have:

$$|T_{ij}| \leq c \rho (u^0)^3 |g|^3 + \frac{1}{3} |\rho g_{ij}|.$$  

(6.30) then follows from (6.12), (6.20), (6.21), and (6.42)

This ends the proof of proposition 6.1.

### 6.2 Geodesic completeness

The geodesic equations for the metric (2.1) imply that along geodesics, the variables $t$, $u^0$, $u^i$ satisfy a differential system which contains, between others, the equation:

$$\frac{dt}{ds} = u^0, \quad (6.43)$$

where $s$ is an affine parameter. The space-time will be geodesically complete if the affine parameter $s$ tends to $+\infty$ as the time $t$ tends to $+\infty$. Since $(u^0)^2 = 1 + g_{ij} u^i u^j$, it will be enough if we prove that:

$$\frac{ds}{dt} = (1 + g_{ij} u^i u^j)^{-\frac{1}{2}} \geq c > 0.$$  

(6.44)
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since by integration we will have: \( s \geq ct + s_0 \), which proves that \( s \) tends to \(+\infty\) as \( t \) tends to \(+\infty\).

We begin by the following result:

**Proposition 6.2.** The quantity \( g^{ij}u_iu_j \) satisfies the equation:

\[
\frac{d}{dt} (g^{ij}u_iu_j) = \frac{2}{3} H g^{ij}u_iu_j + 2\sigma^{ij}u_iu_j - \frac{3\epsilon_0}{2\rho_0} \exp \left( \frac{3}{4} \int_0^t \frac{1}{u^0(s)} ds \right) E^iu_i. \tag{6.45}
\]

**Proof.** From the evolution equation (2.51) and since \( g^{ij} \) is symmetric, we have:

\[
\frac{d}{dt} (g^{ij}u_iu_j) = \frac{d}{dt} (g^{ij}u^iu^j) = -2K^{ij}u^iu^j + 2\dot{g}^{ij}u^iu^j. \tag{6.46}
\]

We can write this equation as:

\[
\frac{d}{dt} (g^{ij}u_iu_j) = -2K^{ij}u^iu^j + 2\dot{u}^iu^j. \tag{6.47}
\]

The last term is zero since \( F_{ml}u^mu^l = 0 \), and the constraint (2.64) gives: \( C^j_{jk}E^k = -\epsilon u^0 \).

Equation (6.47) then writes:

\[
2\dot{u}^iu_i = 4K^{ij}u_iu_j - \frac{2\gamma^i_{jk}u^ju^ku^i}{u^0} + \frac{3}{2} \frac{1}{\rho u^0} C^j_{jk}E^kE^iu_i - \frac{3}{2} \frac{1}{\rho(u^0)^2} C^j_{jk}E^kF_{ml}u^mu^l \tag{6.48}
\]

We have from (6.46), (6.47) and (6.48):

\[
\frac{d}{dt} (g^{ij}u_iu_j) = 2K^{ij}u_iu_j - \frac{3}{2} \frac{\epsilon}{\rho} E^iu_i - \frac{2\gamma^i_{jk}u^ju^ku^i}{u^0}. \tag{6.49}
\]

But we can write from expression (2.19) and \( \gamma^l_{ij} \):

\[
2\gamma^l_{ij}u^iu^ju_i = -g_{im}(C^m_{jk}u^ju^k)u^i + g_{jm}(C^m_{ki}u^iu^k)u^j + g_{km}(C^m_{ij}u^iu^k)u^k = 0
\]

since \( C^m_{ij} = -C^m_{ji} \). Then, use the traceless tensor \( \sigma^{ij} = K^{ij} - \frac{H}{2}g^{ij} \), (6.49) gives:

\[
\frac{d}{dt} (g^{ij}u_iu_j) = 2\sigma^{ij}u_iu_j + \frac{2}{3} H g^{ij}u_iu_j - \frac{3}{2} \frac{\epsilon}{\rho} E^iu_i. \tag{6.50}
\]
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Now we know by (6.26) that \( \rho \) satisfies:

\[
\dot{\rho} = -\left( \frac{4}{4u^0} \frac{\nabla_\alpha \frac{4}{4u^0} u^\alpha + 3}{4u^0} \right) \rho
\]

Integrating over \([0, t], t > 0\), we obtain:

\[
\rho(t) = \rho(0) \exp \left( - \int_0^t \left( \frac{4}{4u^0} \frac{\nabla_\alpha \frac{4}{4u^0} u^\alpha + 3}{4u^0} \right) ds \right).
\]

By (2.39) we have:

\[
e(t) = e(0) \exp \left( - \int_0^t \frac{4}{4u^0} \frac{\nabla_\alpha \frac{4}{4u^0} u^\alpha + 3}{4u^0} ds \right).
\]

Hence:

\[
\frac{e(t)}{\rho(t)} = \frac{e_0}{\rho_0} \exp \left( \frac{3}{4} \int_0^t \frac{1}{4u^0} ds \right). \quad (6.51)
\]

(6.45) is then a direct consequence of (6.50) and (6.51).

**Proposition 6.3.** If \( \sqrt{\frac{1}{3}} > \frac{3}{4} \), the space-time is geodesically complete.

**Proof.** Since by (6.13) \( \sigma^{ij} = O(e^{-3\gamma t}) \), \( e^{3\gamma t} \sigma^{ij} \) is bounded. The matrix \( G^{ij} \) in proposition 6.1 being constant and positive definite, we have:

\[
e^{3\gamma t} \sigma^{ij} u_i u_j \leq C G^{ij} u_i u_j \quad (6.52)
\]

where \( C \) is a constant. By (6.12) we can write:

\[
G^{ij} u_i u_j \leq C e^{2\gamma t} g^{ij} u_i u_j. \quad (6.53)
\]

Since \( g \) is a scalar product, we have:

\[
- E^i u_i \leq \left( E^i E_j \right)^{\frac{1}{2}} \left( g^{ij} u_i u_j \right)^{\frac{1}{2}}. \quad (6.54)
\]

So, (6.1), (6.45), (6.52), (6.53), (6.54) and (6.14) imply:

\[
\frac{d}{dt} (g^{ij} u_i u_j) \leq (-2\gamma + ce^{-3\gamma t}) g^{ij} u_i u_j + ce^{-\gamma t} g^{ij} u_i u_j + c \exp \left( -\gamma t + \frac{3}{4} \int_0^t \frac{1}{u^0(s)} ds \right) \left( g^{ij} u_i u_j \right)^{\frac{1}{2}}. \quad (6.55)
\]

From where, we deduce:

\[
\frac{d}{dt} (g^{ij} u_i u_j) \leq (-2\gamma + ce^{-\gamma t}) g^{ij} u_i u_j + c \exp \left( -\gamma t + \frac{3}{4} \int_0^t \frac{1}{u^0(s)} ds \right) \left( g^{ij} u_i u_j \right)^{\frac{1}{2}}. \quad (6.56)
\]

Sine \( u^0 \geq 1 \), we have \( \int_0^t \frac{ds}{u^0} \leq t \) and \( -\gamma t + \frac{3}{4} \int_0^t \frac{ds}{u^0} \leq t \left( -\gamma + \frac{3}{4} \right) \)

So if: \( -\gamma + \frac{3}{4} < 0 \) or, if we set: \( \omega = \gamma - \frac{3}{4} > 0 \), we deduce from (6.56):

\[
\frac{d}{dt} (g^{ij} u_i u_j) \leq (-2\omega + ce^{-\omega t}) g^{ij} u_i u_j + ce^{-\omega t} \left( g^{ij} u_i u_j \right)^{\frac{1}{2}} \quad (6.57)
\]
Let us set:

\[ Z = e^{\omega t} g^{ij} u_i u_j. \]

We have

\[ \frac{dZ}{dt} = \omega Z + e^{\omega t} \frac{d}{dt} (g^{ij} u_i u_j), \]

Hence, from (6.57) we obtain:

\[ \frac{dZ}{dt} \leq \omega Z + (-2\omega + ce^{-\omega t})Z + ce^{-\frac{\omega t}{2}}Z^{\frac{1}{2}}. \]

from where we have:

\[ \frac{dZ}{dt} \leq ce^{-\frac{\omega t}{2}} Z + ce^{-\frac{\omega t}{2}} Z^{\frac{1}{2}}. \] (6.58)

But

\[ e^{-\frac{\omega t}{2}} Z^{\frac{1}{2}} = e^{-\frac{\omega t}{2}} (e^{-\frac{\omega t}{2}} Z)^{\frac{1}{2}} \leq \frac{1}{2} [e^{-\frac{\omega t}{2}} + e^{-\frac{\omega t}{2}} Z]. \]

We then deduced, from (6.58) that:

\[ \frac{d(Z + 1)}{dt} \leq ce^{-\frac{\omega t}{2}} (Z + 1). \] (6.59)

(6.59) proves that \( Z = e^{\omega t} (g^{ij} u_i u_j) \) is bounded, and then, \( g^{ij} u_i u_j = g_{ij} u^i u^j \) is bounded. So there exists \( A > 1 \) such that

\[ 1 \leq u^0 \leq A. \]

We then have (6.44) and this ends the proof of proposition 6.3.

\[ \square \]

7 The positivity conditions

Let \( (4X^\alpha) \) be a future pointing vector. The quantity

\[ (4\tau_{\alpha\beta} + 4T_{\alpha\beta}) 4X^\alpha 4X^\beta \]

represents physically, the density of energy of a charged particle, measured by an observant whose speed is \( (4X^\alpha) \). Hence, this quantity should always be positive. Recall that a physical theory always has to satisfy at least one positivity condition [2]. There are three types of positivity conditions, they are, for \( (4X^\alpha) \), \( (4Y^\alpha) \) future pointing vectors:

a) The weak positivity condition which means:

\[ (4\tau_{\alpha\beta} + 4T_{\alpha\beta}) 4X^\alpha 4X^\beta \geq 0 \] (7.1)
b) The strong positivity condition which means:

\[ 4R_{\alpha\beta} 4X^\alpha 4X^\beta \geq 0; \quad (7.2) \]

c) The dominant energy condition which means:

\[ (4_{\alpha\beta} + 4T_{\alpha\beta}) 4X^\alpha 4Y^\beta \geq 0. \quad (7.3) \]

Obviously the dominant energy condition implies the weak energy condition, just setting

\[ 4Y^\alpha = 4X^\alpha. \]

We begin by proving:

**Proposition 7.1.** Let \( (4X^\alpha) \) and \( (4Y^\alpha) \) be two future pointing vectors. Then:

\[ 4X^\alpha 4Y_\alpha \leq 0. \quad (7.4) \]

**Proof.** Since \( 4X^\alpha 4Y_\alpha = 4g_{\alpha\beta} 4X^\alpha 4Y^\beta \), the definition (2.1) of the metric \( 4g \) implies that (7.4) is equivalent to:

\[ -4X^0 4Y^0 + g_{ij} 4X^i 4Y^j \leq 0. \quad (7.5) \]

But \( (4X^\alpha) \) and \( (4Y^\alpha) \) are future pointing, thus we have:

\[ 4X^\alpha 4X_\alpha \leq 0, \quad 4X^0 \geq 0, \quad 4Y^\alpha 4Y_\alpha \leq 0, \quad 4Y^0 \geq 0 \]

or equivalently:

\[ 0 \leq g_{ij} 4X^i 4X^j \leq (4X^0)^2; \quad 4X^0 \geq 0; \quad 0 \leq g_{ij} 4Y^i 4Y^j \leq (4Y^0)^2; \quad 4Y^0 \geq 0. \]

So, taking the square roots and the products:

\[ 0 \leq (g_{ij} 4X^i 4X^j)^{\frac{1}{2}} (g_{ij} 4Y^i 4Y^j)^{\frac{1}{2}} \leq 4X^0 4Y^0. \quad (7.6) \]

Since \( g = (g_{ij}) \) is a scalar product, we have the Schwartz inequality:

\[ |g_{ij} 4X^i 4X^j| \leq (g_{ij} 4X^i 4X^j)^{\frac{1}{2}} (g_{ij} 4Y^i 4Y^j)^{\frac{1}{2}}. \quad (7.7) \]

Hence, (7.4) is a direct consequence of (7.6) and (7.7). \( \square \)

We now prove an important result.

**Proposition 7.2.** Let \( (4X^\alpha) \) and \( (4Y^\alpha) \) be two future pointing vectors. The Maxwell tensor \( 4\tau_{\alpha\beta} \) satisfies:

\[ 4\tau_{\alpha\beta} 4X^\alpha 4Y^\beta \geq 0. \quad (7.8) \]
Proof. It shall be enough if we prove the inequality (7.8) in a particular frame. Guided by the electromagnetic field itself, we choose a frame of four future pointing vectors \( l = (l^\alpha); n = (n^\alpha); \ x = (x^\alpha); \ y = (y^\alpha) \) such that:

\[
l_\alpha l^\alpha = n_\alpha n^\alpha = l_\alpha x^\alpha = l_\alpha y^\alpha = n_\alpha y^\alpha = n_\alpha x^\alpha = 0. \tag{7.9}
\]

The antisymmetric 2-form \( ^4F_{\alpha\beta} \) can be written in one of two forms:

\[
^4F_{\alpha\beta} = A \left( l_\alpha n^\beta - l_\beta n^\alpha \right) + B \left( x_\alpha y^\beta - x_\beta y^\alpha \right), \tag{7.10}
\]

or

\[
^4F_{\alpha\beta} = C \left( l_\alpha x^\beta - l_\beta x^\alpha \right). \tag{7.11}
\]

Where \( A, B, C \) are constants. It is important to choose the constants \( A, B, C \) such that:

\[
l_\alpha n^\alpha = -1; \quad x_\alpha x^\alpha = y_\alpha y^\alpha = 1; \quad x_\alpha y^\alpha = 0. \tag{7.12}
\]

Let us consider the Maxwell tensor:

\[
^4\tau_{\alpha\beta} = -\frac{1}{4} g_{\alpha\beta} \ ^4F^{\lambda\mu} \ ^4F_{\lambda\mu} + ^4F_{\alpha\lambda} \ ^4F_{\lambda\beta}. \tag{7.13}
\]

(7.11) gives, using (7.9) and (7.2):

\[
^4F^{\lambda\mu} \ ^4F_{\lambda\mu} = 0; \quad ^4F_{\alpha\lambda} \ ^4F_{\lambda\beta} = \frac{c^2}{4} l_\alpha l_\beta.
\]

(7.13) then gives:

\[
^4\tau_{\alpha\beta} = \frac{c^2}{4} l_\alpha l_\beta. \tag{7.14}
\]

Let \( (^4X^\alpha) \) and \( (^4Y^\alpha) \) be two future pointing vectors. We have:

\[
^4\tau_{\alpha\beta} \ ^4X^\alpha \ ^4Y^\beta = \frac{c^2}{4} (l_\alpha \ ^4X^\alpha)(l_\beta \ ^4Y^\beta)
\]

But since \( (l^\alpha), \ (^4X^\alpha), \ (^4Y^\alpha) \) are future pointing vectors, by (7.4) we have \( l_\alpha \ ^4X^\alpha \leq 0 \) and \( l_\beta \ ^4Y^\beta \leq 0 \). Then \( ^4\tau_{\alpha\beta} \ ^4X^\alpha \ ^4Y^\beta \geq 0 \) and this proves proposition 7.2

\[\square\]

\textbf{Théorème 7.1.} The global solution of the Einstein-Maxwell system with pseudo-tensor of pressure satisfies:

1°) The weak positivity condition;

2°) The strong positivity condition if \( \Lambda \geq \frac{(H(0))^2}{2} \).
Proof. -

1°) Let \( 4X^\alpha \) be a future pointing vector. By (7.8) we have:

\[
4\tau_{\alpha\beta} 4X^\alpha 4X^\beta \geq 0. \tag{7.15}
\]

Now by definition (2.12) of \( 4T_{\alpha\beta} \), we have:

\[
4T_{\alpha\beta} = \frac{4}{3} \rho 4u_\alpha 4u_\beta + 4\Theta_{\alpha\beta} \tag{7.16}
\]

If \( 4X^\alpha \) is a future pointing vector and since \( 4u^\alpha \) is by definition a future pointing vector, (7.4) implies, since \( \rho > 0 \):

\[
\left( \frac{4}{3} \rho 4u_\alpha 4u_\beta \right) 4X^\alpha 4X^\beta = \frac{4}{3} \rho (4u_\alpha 4X^\alpha) (4u_\beta 4X^\beta) \geq 0 \tag{7.17}
\]

Now we use the definition of the pseudo-tensor of pressure \( 4\Theta_{\alpha\beta} \). We have, using (2.13) and (2.14), for a future pointing vector:

\[
4\Theta_{\alpha\beta} 4X^\alpha 4X^\beta = 4\Theta_{00} (4X^0)^2 + 2 4\Theta_{0i} 4X^0 4X^i + 4\Theta_{ij} 4X^i 4X^j
\]

\[
= (4Z_0)^2 (4X^0)^2 + 2 \left( \frac{1}{2} 4Z_0 4Z_i \right) 4X^0 4X^i + \frac{1}{3} \rho g_{ij} 4X^i 4X^j
\]

\[
= (4Z_0)^2 (4X^0)^2 + (4Z_0 4X^0) 4Z_i 4X^i + \frac{1}{3} \rho g_{ij} 4X^i 4X^j
\]

\[
= (4Z_0)^2 (4X^0)^2 + (4Z_0 4X^0) (4Z_\alpha 4X^\alpha - 4Z_0 4X^0) + \frac{1}{3} \rho g_{ij} 4X^i 4X^j
\]

\[
4\Theta_{\alpha\beta} 4X^\alpha 4X^\beta = (4Z_0 4X^0) (4Z_\alpha 4X^\alpha) + \frac{1}{3} \rho g_{ij} 4X^i 4X^j \tag{7.18}
\]

But since \( 4Z^\alpha \) is future pointing, we have \( 4Z^0 \geq 0 \), then \( 4Z_0 = -4Z^0 \leq 0 \), thus, \( 4Z_0 4X^0 \leq 0 \) since \( 4X^0 \geq 0 \).

Using once more (7.4) we have \( 4Z_\alpha 4X^\alpha \leq 0 \), and finally: \( (4Z_0 4X^0) (4Z_\alpha 4X^\alpha) \geq 0 \). Now the metric \( (g_{ij}) \) is positive definite, thus \( g_{ij} 4X^i 4X^j \geq 0 \). In conclusion (7.18) implies

\[
4\Theta_{\alpha\beta} 4X^\alpha 4X^\beta \geq 0,
\]

and using (7.17) and (7.18) we obtain:

\[
4T_{\alpha\beta} 4X^\alpha 4X^\beta \geq 0. \tag{7.19}
\]

Finally we have by (7.15) and (7.19):

\[
4\tau_{\alpha\beta} 4X^\alpha 4X^\beta + 4T_{\alpha\beta} 4X^\alpha 4X^\beta = (4\tau_{\alpha\beta} + 4T_{\alpha\beta}) 4X^\alpha 4X^\beta \geq 0
\]

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and the weak positivity condition is proved.

Let \( 4X^\alpha \) be a future pointing vector.

The Einstein’s equations (2.8) imply:

\[
4R_{\alpha\beta} 4X^\alpha 4X^\beta = \left( \frac{4R}{2} - \Lambda \right) 4g_{\alpha\beta} 4X^\alpha 4X^\beta + 8\pi \left( 4\tau_{\alpha\beta} + 4T_{\alpha\beta} \right) 4X^\alpha 4X^\beta.
\] (7.20)

By the weak positivity condition we know that:

\[
8\pi \left( 4\tau_{\alpha\beta} + 4T_{\alpha\beta} \right) 4X^\alpha 4X^\beta \geq 0.
\] (7.21)

Let us see at what condition we will also have:

\[
\left( \frac{4R}{2} - \Lambda \right) 4g_{\alpha\beta} 4X^\alpha 4X^\beta = \left( \frac{4R}{2} - \Lambda \right) 4X^\alpha 4X_{\alpha} \geq 0
\]

By definition, we have:

\[
4R = 4g^{\alpha\beta} 4R_{\alpha\beta} = 4g^{00} 4R_{00} + g^{ij} 4R_{ij} = -4R_{00} + g^{ij} 4R_{ij}.
\] (7.22)

We know that \( 4R_{ij} \) and \( R_{ij} \) are linked by:

\[
4R_{ij} = R_{ij} - \partial_t K_{ij} + HK_{ij} - 2K_{ij}K_i^j.
\] (7.23)

Contracting with \( g^{ij} \) we obtain:

\[
g^{ij} 4R_{ij} = R - g^{ij} \partial_t K_{ij} + H^2 - 2K_{ij}K_i^j.
\] (7.24)

Now we have, using equation (2.51):

\[
g^{ij} \partial_t K_{ij} = \partial_t (g^{ij} K_{ij}) - K_{ij} \partial_t g^{ij} = \partial_t H - 2K_{ij}K_i^j
\] (7.25)

From (7.24) and (7.25) we obtain:

\[
g^{ij} 4R_{ij} = R - \partial_t H + H^2.
\] (7.26)

Now in the Einstein’s equations (2.8) take \( \alpha = \beta = 0 \) to obtain:

\[
4R_{00} = -\frac{4R}{2} + \Lambda + 8\pi (4\tau_{00} + 4T_{00}).
\] (7.27)

Then using (7.22), (7.26) and (7.27) we obtain:

\[
4R = \frac{4R}{2} - \Lambda - 8\pi (4\tau_{00} + 4T_{00}) + R - \partial_t H + H^2.
\]
From where we deduce, since \(4 \tau_{00} + T_{00} \geq 0, R \leq 0\) and \(\partial_t H \geq 0\) (see (6.31)):
\[
\frac{4R}{2} \leq -\Lambda + H^2
\]
But \(H\) is increasing and negative, then \(H^2 \leq H^2(0)\); hence:
\[
\frac{4R}{2} - \Lambda \leq -2\Lambda + (H(0))^2.
\]
So we will have \(\frac{4R}{2} - \Lambda \leq 0\) if \(-2\Lambda + (H(0))^2 \leq 0\) or \(\Lambda \geq \frac{(H(0))^2}{2}\). Since \(4X^\alpha\) is future pointing, we have by (7.4) \(4X^\alpha 4X_\alpha \leq 0\). In conclusion if \(\Lambda \geq \frac{(H(0))^2}{2}\), then:
\[
\left(\frac{4R}{2} - \Lambda\right) 4X^\alpha 4X_\alpha \geq 0
\]
(7.28) and (7.21) imply
\[
4R_{\alpha\beta} 4X^\alpha 4X^\beta \geq 0.
\]
This ends the proof of theorem (7.1)

8 Well-posedness

According to Hadamard, a mathematical problem is well-posed if its solution exists, if the solution is unique, and if the solution is a continuous function of the initial data.

We have only to prove the last point. In this paragraph we suppose \(e_0 > 0\). The initial data is denoted \(V_0\) where:
\[
V_0 = (g^0, K^0, F^0, E^0, U^{0,i}, \Theta^{0,\alpha}, \rho_0, e_0) \in \Omega := S_3(\mathbb{R}) \times S_3(\mathbb{R}) \times A_3(\mathbb{R}) \times \mathbb{R}^{10} \times \mathbb{R} \times [0, \infty[ \times [0, \infty[ \times \mathbb{R}_{>0},
\]
where \(S_3(\mathbb{R})\) and \(A_3(\mathbb{R})\) are respectively the sets of \(3 \times 3\) symmetric and antisymmetric matrices.

We suppose that the initial data satisfy the constraints (2.61), (2.62), (2.63), (2.64). The global solution:
\[
S = (g, K, F, E, u^i, \Theta^{\alpha}, \rho, e)
\]
of the evolution system is also in \(\Omega\) and is a function:
\[
S : \Omega \times [0, \infty[ \rightarrow \Omega
S(V_0, t) \mapsto S(V_0, t),
\]
We can also denote:
\[
S(V_0, t) : [0, \infty[ \rightarrow \Omega
S(V_0, t) \mapsto S(V_0, t),
\]
such that: \( S(V_0, 0) = V_0 \). Every component of \( S \) is of class \( C^1 \). Note that \( \Omega \) is an open set of \( \widetilde{\Omega} = S_3(\mathbb{R}) \times S_3(\mathbb{R}) \times A_3(\mathbb{R}) \times \mathbb{R}^{12} \). We can write:

\[
S : \Omega \rightarrow C^1([0, \infty[, \widetilde{\Omega})
\]

\( V_0 \mapsto S(V_0, \cdot) \),

If \( \Gamma \) is a compact set of \([0, \infty[\), we define the seminorm \( P_\Gamma \) on \( C^1([0, \infty[, \widetilde{\Omega}) \) by:

\[
P_\Gamma(\varphi) = \sup_{t \in \Gamma} (|\varphi|_1), \varphi \in C^1([0, \infty[, \widetilde{\Omega})
\]

Where \( |W|_1 = \sum_{i=1}^{n} |W_i| \) with \( W = (W_1, ..., W_n) \in \mathbb{R}^n \). \( C^1([0, \infty[, \widetilde{\Omega}) \) is a locally convex topological vector space, whose topology \( \tau \) is generated by the family of seminorms \( P_\Gamma \). We will prove that \( S \) is continuous from \( \Omega \) to \( C^1([0, \infty[, \widetilde{\Omega}) \), endowed with the topology \( \tau \). Let \( p \in \mathbb{N}^* \). Set:

\[
\Omega_p = B(0, 2^p) \times [2^{p}, 2^{p}[, 2^{p} \times [2^{p}, 2^{p}[, \mathbb{R}]
\]

where \( B(0, 2^p) \) is the ball of radius \( 2^p \) of \( S_3(\mathbb{R}) \times S_3(\mathbb{R}) \times A_3(\mathbb{R}) \times \mathbb{R}^{10} \). Notice that:

\[
\overline{\Omega_p} \subset \Omega_{p+1} \text{ and } \bigcup_{p \in \mathbb{N}} \Omega_p = \Omega
\]

The function \( S \) will be continuous on \( \Omega \), if its restriction to every \( \Omega_p \) is continuous.

In what follows, we associate to every initial data the iterated sequence defined in paragraph 4. Recall that the iterated sequence converges to a unique solution if \( \Lambda > 0 \) and \( H(0) < 0 \).

For every \( n \in \mathbb{N} \), the iterated sequence

\[
V_n = (g_n, K_n, F_n, E_n, u_n^i, \rho_n^0, \rho_n, e_n)
\]

is a function:

\[
V_n : \Omega \times [0, \infty[ \rightarrow \Omega
\]

\( (V_0, t) \mapsto V_n(V_0, t) \),

**Proposition 8.1.** \( V_n \) is continuous on \( \Omega \times [0, \infty[ \).

**Proof.** For \( V_0 \in \Omega, t \in [0, \infty[ \), we have by definition \( V_0(V_0, t) = V_0 \); hence \( V_0 \) is continuous on \( \Omega \times [0, \infty[ \). Suppose that for \( n \in \mathbb{N} \), \( V_0, V_1, ..., V_n \) are continuous, then the function:

\[
V_{n+1}(V_0, t) = \int_0^t f \circ V_n(V_0, s)ds + V_0
\]

where \( f \) defines the right hand side of the evolution system, is also continuous on \( \Omega \times [0, \infty[ \). Hence \( V_n \) is continuous for every \( n \in \mathbb{N} \). \( \square \)
Proposition 8.2. For \( p \in \mathbb{N}, 0 < T < +\infty \), the sequence \((V_n)_n\) is uniformly bounded on \( \Omega_p \times [0, T]\)

Proof. Let us recall that:

\[ H(0) = g^{0,ij}K^0_{ij} = H_0(V_0). \] (8.2)

(8.2) shows that \( H_0 \) is continuous on \( \overline{\Omega}_p \times [0, T] \), hence it is bounded on it. Under the hypothesis that the iterated sequence satisfy the constraints (2.61), (2.62), (2.63), (2.64), we must have:

\[ H_0(V_0) \leq H_n(V_0, t) \leq -\sqrt{3\Lambda}, \forall n \in \mathbb{N}, (V_0, t) \in \Omega \times [0, \infty[. \]

\( H_0 \) is bounded on \( \Omega_p \times [0, T] \) and it does not depend on \( t \). So there exists a constant \( C_p > 0 \) depending only on \( p \) such that:

\[-C_p \leq H_n(V_0, t) \leq -\sqrt{3\Lambda}, \forall n \in \mathbb{N}, (V_0, t) \in \Omega_p \times [0, T[.\]

This shows that the sequence \((H_n)_{n \in \mathbb{N}}\) is uniformly bounded on \( \Omega_p \times [0, T]\).

1\(^0\) boundedness of \((g_n)_{n \in \mathbb{N}}\) on \( \Omega_p \times [0, T]\)

The sequence \((V_n)_n\) satisfy the Hamiltonian constraint (2.61), that is:

\[ H_n^2 = 16\pi(\tau_{n,00} + T_{n,00}) + K_{n,ij}K^0_{ij} - R_n. \] (8.3)

From the fact that: \( \Lambda \geq 0, \tau_{n,00} \geq 0, T_{n,00} \geq 0, K_{n,ij}K^0_{ij} \geq 0, -R_n \geq 0 \) and \((H_n)_n\) bounded on \( \Omega_p \times [0, T]\), (8.3) shows that the sequence \((K_{n,ij}K^0_{ij})_n\) of positive terms is bounded on \( \Omega_p \times [0, T]\). Using the definition of the iterated sequence, the integration of equation (2.61) on \([0, t], 0 \leq t < T\) gives:

\[ |g_{n+1}(V_0, t)| \leq |g^0| + 2\int_0^t |K_n(V_0, s)||g_n(V_0, s)|ds. \] (8.4)

Using the notion of relative norm introduced in paragraph 3.3, we deduce from (8.4) the inequality:

\[ |g_{n+1}(V_0, t)| \leq C\left[ |g^0| + \int_0^t ||g_n(V_0, s)||\|K_n(V_0, s)||g_n(V_0, s)||ds \right] \] (8.5)

where \( C > 0 \) is a constant. From (3.6) and (3.7), (8.5) implies:

\[ ||g_{n+1}(V_0, t)|| \leq C\left[ ||g^0|| + \int_0^t (K_{n,ij}K^0_{ij})^{\frac{1}{2}}||g_n(V_0, s)||ds \right]. \] (8.6)
But \((K_{n,ij}K_n^{ij})_n\) is uniformly bounded on \(\Omega_p \times [0, T[.\) Then (8.6) implies that there exists a constant \(C_p > 0\), depending only on \(p\), such that:

\[
||g_{n+1}(V_0, t)|| \leq C_p \left[ ||g^0|| + \int_0^t ||g_n(V_0, s)|| ds \right], \quad \forall (V_0, t) \in \Omega_p \times [0, T[.
\]

By induction on \(n\), we obtain:

\[
||g_{n+1}(V_0, t)|| \leq C_p ||g^0|| \left[ 1 + C_p t + \frac{(C_p t)^2}{2} + \ldots + \frac{(C_p t)^n}{n} + ||g^0|| \frac{(C_p t)^{n+1}}{n+1} \right], \quad \forall (V_0, t) \in \Omega_p \times [0, T[.
\]

Finally, we obtain:

\[
||g_{n+1}(V_0, t)|| \leq C_p ||g^0|| (1 + ||g^0||) \exp(C_p t), \quad \forall n \in \mathbb{N}, (V_0, t) \in \Omega_p \times [0, T[. \tag{8.7}
\]

(8.7) shows that the sequence \((g_n)_{n\in\mathbb{N}}\) is uniformly bounded on \(\Omega_p \times [0, T[.\)

2°) Boundedness of the sequence \((K_n)_{n\in\mathbb{N}}\) on \(\Omega_p \times [0, T[\)

Using once more the notion of relative norm, we have by (3.6):

\[
||K_n(V_0, t)|| \leq (K_{n,ij}K_n^{ij})^{\frac{1}{2}} ||g_n(V_0, t)||. \tag{8.8}
\]

But \((K_{n,ij}K_n^{ij})_n\) and \((g_n)_n\) are uniformly bounded on \(\Omega_p \times [0, T[.\) So is \((K_n)_n\).

3°) Boundedness of \((\det g_n)_{n\in\mathbb{N}}\), \((\det g_n)^{-1})_n\) and \((g_n^{ij})_n\) on \(\Omega_p \times [0, T[\)

The relation (6.37) is equivalent to:

\[
\det g = \det g^0 \exp(-2 \int_0^t H(V_0, s) ds).
\]

By definition of the iterated sequence, we must have:

\[
\det g_{n+1}(V_0, t) = \det g^0 \exp(-2 \int_0^t H_n(V_0, s) ds),
\]

From where, we deduce that, since \((H_n)_n\) is uniformly bounded on the compact \(\overline{\Omega_p} \times [0, T[\), there exist a constant \(C_{p,T} > 0\) depending only on \(p\) and \(T\) such that:

\[
\det g^0 \exp(-C_{p,T}) \leq \det g_n(V_0, t) \leq \det g^0 \exp(C_{p,T}), \quad \forall (V_0, t) \in \Omega_p \times [0, T[.
\]

This shows that \((\det g_n)_n\) and \((\det g_n)^{-1})_n\) are uniformly bounded on \(\Omega_p \times [0, T[.\) we can conclude that \((g_n^{ij})_n\) is uniformly bounded on \(\Omega_p \times [0, T[.\)
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4) Boundedness of \( (E_n)_{n \in \mathbb{N}} \) on \( \Omega_p \times [0, T] \)

From (8.3) we have:

\[
16\pi (\tau_{n,00} + T_{n,00}) = R_n + H_n^2 - K_n,ij K_n^{ij} - 2\Lambda \leq H_n^2 - 2\Lambda.
\]

Since \( K_n,ij K_n^{ij} \geq 0 \), \( R_n \leq 0 \). But \( (H_n)_n \) is uniformly bounded on \( \Omega_p \times [0, T] \). Then so are \( (\tau_{n,00})_n \) and \( (T_{n,00})_n \). We know that:

\[
\tau_{n,00} = g_n^{ij} \tau_{n,ij} = \frac{1}{4} F_n^{ij} F_{n,ij} + \frac{1}{2} g_n^{ij} E_n^{i} E_n^{j},
\]

then:

\[
0 \leq \frac{1}{2} g_n^{ij} E_n^{i} E_n^{j} \leq \tau_{n,ij},
\]

since \( F_n^{ij} F_{n,ij} \geq 0 \). \((g_n^{ij} E_n^{i} E_n^{j})_n\) is then uniformly bounded on \( \Omega_p \times [0, T] \). Applying (3.9) and the fact that \((g_n^{ij} E_n^{i} E_n^{j})_n\) and \((g_n)_n\) are uniformly bounded on \( \Omega_p \times [0, T] \), we conclude that so is \((E_n)_n\).

5) Boundedness of \( (F_n)_{n \in \mathbb{N}} \) on \( \Omega_p \times [0, T] \)

By the definition of the iterated sequence, equation (2.53) implies:

\[
F_{n,ij}(V_0, t) = F_{n,ij}^0 + \int_0^t C^{k}_{ij} g_{n,kl}(V_0, s) E_n^l(V_0, s) ds.
\]  (8.9)

But \((g_n)_n\) and \((E_n)_n\) are uniformly bounded on \( \Omega_p \times [0, T] \); by (8.9), \((F_n)_n\) is also uniformly bounded on \( \Omega_p \times [0, T] \).

6) Boundedness of \( (u^0_n)_n, (\frac{1}{\rho_n})_n, (\rho_n)_n \) and \((u_i^n)_n\) on \( \Omega_p \times [0, T] \)

We saw that (2.28) and (2.29) give:

\[
\frac{\dot{\rho}}{\rho} + \frac{\dot{u}^0}{u^0} = - \left( \frac{3}{4} \frac{1}{u^0} - H + C_n^{ji} \frac{u^j}{u^0} \right),
\]

and by integration on \([0, t], 0 < t < T\), we have:

\[
\rho u^0 = \rho_0 u^0(0) \exp \left[ - \int_0^t \left( \frac{3}{4} \frac{1}{u^0} - H + C_n^{ji} \frac{u^j}{u^0} \right) ds \right]
\]

By the definition of the iterated sequence, we must have:

\[
\rho_{n+1} u_{n+1}^0 = \rho_0 u^0(0) \exp \left[ - \int_0^t \left( \frac{3}{4} \frac{1}{u_n^0} - H + C_n^{ji} \frac{u_n^j}{u_n^0} \right) ds \right]  \]  (8.10)
By (3.9) we have $\frac{|u^i|}{u^0} \leq C |g_n|^\frac{3}{2}$, $(g_n)_n$ and $(H_n)_n$ are uniformly bounded on $\Omega_p \times [0, T]$, and $u^0_n \geq 1$. So, from (8.10) we deduce that there exist a constant $C_{p,T} > 0$ such that:

$$\rho_0 u^0(0) \exp(-C_{p,T}) \leq \rho_{n+1} u^0_{n+1} \leq \rho_0 u^0(0) \exp(C_{p,T}), \text{ on } \Omega_p \times [0, T].$$

So $(\rho_n u^0_n)_n$ and $(\frac{1}{\rho_n u^0_n})_n$ are uniformly bounded on $\Omega_p \times [0, T]$. But $u^0_n \geq 1$, then $(\rho_n)_n$ is also uniformly bounded on $\Omega_p \times [0, T]$. Now we have by (3.9):

$$\frac{|u^i|}{u^0} = \frac{|\rho_n u^i|}{\rho_n u^0} \leq C |g_n|^\frac{3}{2}$$

But $(\rho_n u^0_n)_n$ and $(g_n)_n$ are uniformly bounded on $\Omega_p \times [0, T]$. Hence so is $(\rho_n u^i_n)_n$. Now consider the equation in $u^i_{n+1}$ deduced from (2.55). We multiply this equation by $\rho_n$ and conclude that the sequence $(\rho_n u^i_{n+1})_n$ is uniformly bounded on $\Omega_p \times [0, T]$. Now derive the relation $u^0 = \sqrt{1 + g_{ij} u^i u^j}$ and obtain, using (2.51):

$$\frac{\dot{u}^0}{u^0} = -K_{ij} \frac{u^i u^j}{u^0 u^0} + g_{ij} \frac{\rho \dot{u}^i u^j}{\rho u^0 u^0}$$

From there, we deduce that for the iterated sequence, we must have:

$$\frac{\dot{u}^0_{n+1}}{u^0_{n+1}} = -K_{n,ij} \frac{u^i_n u^j_n}{u^0_n u^0_n} + g_{n,ij} \frac{(\rho_n u^i_n)}{(\rho_n u^0_n)} \frac{u^j_n}{u^0_n}$$

Since $(K_n)_n$, $(\frac{\dot{u}^i_n}{u^0_n})_n$, $(g_n)_n$, $(\rho_n u^i_n)_n$, $(\frac{1}{\rho_n u^0_n})_n$ are uniformly bounded on $\Omega_p \times [0, T]$ there exists a constant $C_{p,T} > 0$ such that:

$$\left| \frac{\dot{u}^0_{n+1}}{u^0_{n+1}} \right| \leq C_{p,T}$$

From there, we deduce that:

$$1 \leq u^0_{n+1} \leq U^{0,0} \exp(C_{p,T}), \text{ on } \Omega_p \times [0, T]$$

then $(u^0_{n+1})_n$ is uniformly bounded on $\Omega_p \times [0, T]$. Now $\frac{1}{\rho_n} = \frac{1}{\rho_n u^0_n} \times u^0_n$, thus $(\frac{1}{\rho_n})_n$ is uniformly bounded on $\Omega_p \times [0, T]$. We have $u^i_n = \frac{u^i_n}{u^0_n} \times u^0_n$, and $\frac{u^i_n}{u^0_n}$, $u^0_n$ are uniformly bounded on $\Omega_p \times [0, T]$. Then $(u^0_n)_n$ is uniformly bounded on $\Omega_p \times [0, T]$.

In conclusion, the sequences $(\rho_n)_n$, $(\frac{1}{\rho_n})_n$, $(u^0_n)_n$ and $(u^i_n)_n$ are uniformly bounded on $\Omega_p \times [0, T]$.

$\tau^n$) Boundedness of $(e_n)_{n \in \mathbb{N}}$ on $\Omega_p \times [0, T]$

The sequence $(e_n)_n$ from the iterated sequence satisfies constraint (2.64), hence:

$$e_n = -\frac{1}{u^0_n} C_{ik} E_n^k.$$ 

Since $u^0_n \geq 1$ and $(E_n)_n$ is uniformly bounded on $\Omega_p \times [0, T]$, so is $(e_n)_n$. 

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Boundedness of \((\theta^0_n)_{n \in \mathbb{N}}\) on \(\Omega_p \times [0, T]\)

By the equation defining the iterated sequence, since \((H_n)_{n}, (\rho_n)_{n}, (u^0_n)_{n}, (g^{ij}_n)_{n}, (K_{n,ij})_{n}, (\gamma^k_{n,ij})_{n}\) and \((u^k_n)_{n}\) are uniformly bounded on \(\Omega_p \times [0, T]\), we can conclude that, there exist two constants \(A_{p,T} > 0\), and \(B_{p,T} > 0\) such that:

\[
|\theta^0_{n+1}(V_0, t)| \leq |\theta^0(0, 0)| + A_{p,T} \int_0^t \sum_{\alpha=0}^3 |\theta^0_n(V_0, s)| \, ds + B_{p,T} \text{ on } \Omega_p \times [0, T]
\]

and

\[
|\theta^0_{n+1}(V_0, t)| \leq |\theta^0(0, 0)| + A_{p,T} \int_0^t \sum_{\alpha=0}^3 |\theta^0_n(V_0, s)| \, ds + B_{p,T} \text{ on } \Omega_p \times [0, T]
\]

From where we deduce that:

\[
\sum_{\alpha=0}^3 |\theta^0_{n+1}(V_0, t)| \leq 4A_{p,T} \int_0^t \sum_{\alpha=0}^3 |\theta^0_n(V_0, s)| \, ds + 4B_{p,T} + \sum_{\alpha=0}^3 |\theta^0(0, 0)| \text{ on } \Omega_p \times [0, T]
\]

By induction on \(n\), we conclude that:

\[
\sum_{\alpha=0}^3 |\theta^0_{n+1}(V_0, t)| \leq \left(4A_{p,T} + \sum_{\alpha=0}^3 |\theta^0_n(V_0, 0)| + 4B_{p,T}\right) \exp(4TC_{p,T}), \text{ on } \Omega_p \times [0, T].
\]

(8.11)

Where \(C_{p,T}\) is a constant. Now we know that

\[
\sum_{\alpha=0}^3 |\theta^0_{0}(V_0, 0)| \leq |V_0|_1;
\]

and \(|V_0|_1\) is continuous on the compact set \(\overline{\Omega}_p \times [0, T]\); then it is bounded on it. In conclusion, (8.11) shows that every sequence \((\theta^0_n)_{n}\) is uniformly bounded on \(\Omega_p \times [0, T]\). This ends the proof of proposition 8.2

\[\square\]

**Théorème 8.1.** *The Cauchy problem for the Einstein-Maxwell system with pseudo-tensor of pressure is well-posed in the sense of Hadamard.*

**Proof.** We prove that the function:

\[
S : \Omega \rightarrow C^1([0, \infty[, \overline{\Omega})
\]

\[
V_0 \mapsto S(V_0, .)
\]
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is continuous. We know that it will be enough if we prove that its restriction to every $\Omega_p$ is continuous.

Since $u_0^0 \geq 1$ and since the sequences $(V_n)_n$ and $(\frac{1}{\rho_n})_n$ are uniformly bounded on $\Omega_p \times [0, T]$, by the definition of the iterated sequence, there exist a constant $C_{p,T} > 0$ such that:

$$\left| \dot{V}_{n+1}(V_0, t) - \dot{V}_{n+1}({\tilde{V}}_0, t) \right|_1 \leq C_{p,T} \left| V_n(V_0, t) - V_n({\tilde{V}}_0, t) \right|_1, \forall V_0, {\tilde{V}}_0 \in \Omega_p, \ n \in \mathbb{N}, \ t \in [0, T].$$

We have:

$$V_n(V_0, 0) = V_0 ; \ V_n({\tilde{V}}_0, 0) = {\tilde{V}}_0.$$ 

Then, integrating this inequation on $[0, t], \ 0 < t < T$, we have:

$$|V_{n+1}(V_0, t) - V_{n+1}({\tilde{V}}_0, t)|_1 \leq C_{p,T} \int_0^t \left| V_n(V_0, s) - V_n({\tilde{V}}_0, s) \right|_1 ds + \left| V_0 - {\tilde{V}}_0 \right|_1.$$

By induction on $n$, we deduce from this inequation that:

$$|V_n(V_0, t) - V_n({\tilde{V}}_0, t)|_1 \leq |V_0 - {\tilde{V}}_0|_1 \sum_{i=0}^{n} \frac{(tC_{p,T})^i}{i!}$$

from where we deduce:

$$|V_{n+1}(V_0, t) - V_{n+1}({\tilde{V}}_0, t)|_1 \leq |V_0 - {\tilde{V}}_0|_1 \exp(TC_{p,T})$$

Taking the limit when $n \to +\infty$, we obtain:

$$|S(V_0, t) - S({\tilde{V}}_0, t)|_1 \leq |V_0 - {\tilde{V}}_0|_1 \exp(TC_{p,T}), \forall V_0, {\tilde{V}}_0 \in \Omega_p , t \in [0, T],$$

Using the seminorm $P_{[0,T]}$ we obtain:

$$P_{[0,T]}(S(V_0, .) - S({\tilde{V}}_0, .)) \leq |V_0 - {\tilde{V}}_0|_1 \exp(TC_{p,T}), \forall V_0, {\tilde{V}}_0 \in \Omega_p. \quad (8.12)$$

which proves that $V_0 \mapsto S(V_0, .)$ is continuous from $\Omega_p$ to $C^1([0, \infty[ , {\tilde{\Omega}})$. 

This proves that the Cauchy problem for the Einstein-Maxwell system with pseudo-tensor of pressure is well-posed. \qed
Conclusion and future investigation

We have proved global existence of solutions to the evolution system $(S)$ when the cosmological constant $\Lambda$ is strictly positive and $H(0) < 0$. We have also proved the geodesic completeness (when $\sqrt{\frac{4}{3} \geq \frac{\Lambda}{4}}$), and determined the asymptotic behavior (of the spacetimes) in the neighborhood of $+\infty$ in the case of global existence. On the other hand, we have finally proved that the problem is well posed in the sense of Hadamard.
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Point 1 of Proposition 3.1

1) Evolution of \( A = R - K_{ij}K^{ij} + H^2 - 16\pi(\tau_{00} + T_{00}) - 2\Lambda \)

1.1) Evolution of \( H \)

From definition (2.18) of \( H \), we have:

\[
\partial_t H = \partial_t (g^{ij}K_{ij}) = 2K^{ij}K_{ij} + g^{ij}\partial_t K_{ij}
\]

But equation (2.52) gives:

\[
g^{ij}\partial_t K_{ij} = g^{ij} \left[ R_{ij} + HK_{ij} - 2K^i_jK_l - 8\pi(\tau_{ij} + T_{ij}) + 4\pi(-T_{00} + g^{lm}T_{lm})g_{ij} - \Lambda g_{ij} \right]
\]

\[
= R + H^2 - 2K^i_lK_l - 8\pi g^{ij}\tau_{ij} - 8\pi g^{ij}T_{ij} + 12\pi(-T_{00} + g^{lm}T_{lm}) - 3\Lambda.
\]

From the two relations, we have the evolution of \( H \):

\[
\partial_t H = R + H^2 + 4\pi g^{ij}T_{ij} - 8\pi\tau_{00} - 12\pi T_{00} - 3\Lambda. \tag{A.1}
\]

1.2) Evolution of \( \tau_{00} + T_{00} \)

We use the index \((^4\)) to write the conservation laws:

\[
4\nabla_\alpha (4\tau^{\alpha\beta} + 4T^{\alpha\beta}) = 4g^{\alpha\lambda}4g^{\mu\beta}4\nabla_\alpha (4\tau^{\lambda\mu} + 4T^{\lambda\mu}) = 0.
\]

Now, using (2.7) we have:

\[
4g^{\alpha\lambda}4g^{\mu\beta} \left[ 4\epsilon_\alpha (4\tau^{\lambda\mu} + 4T^{\lambda\mu}) - 4\gamma^\theta_{\alpha\lambda} (4\tau^{\theta\mu} + 4T^{\theta\mu}) - 4\gamma^\theta_{\alpha\mu} (4\tau^{\lambda\theta} + 4T^{\lambda\theta}) \right] = 0.
\]

Then use (2.2), (2.3) and (2.19) to obtain the evolution of \( \tau_{00} + T_{00} \):

\[
\frac{d}{dt}(\tau_{00} + T_{00}) = H(\tau_{00} + T_{00}) - g^{ij}\gamma_i^k(\tau_{0k} + T_{0k}) + K^{ij}(\tau_{ij} + T_{ij}). \tag{A.2}
\]

1.3) Evolution of \( R \)

We have by definition:

\[
\begin{align*}
R^{ij} &= g^{ij} R_{ij} \\
4R^\alpha_{\beta\delta} &= 4\epsilon_\beta (4\gamma^{\alpha\delta}) - 4\epsilon_\delta (4\gamma^{\alpha\beta}) + 4\gamma^{\lambda\alpha} 4\gamma^{\mu\beta} - 4\gamma^{\lambda\beta} 4\gamma^{\mu\alpha} - 4C^{\mu}_{\beta\delta} 4\gamma^{\lambda}. \tag{A.3}
\end{align*}
\]
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Where $C_{\beta \delta}^{\mu}$ are defined by:

$$[\ 4e_\alpha, \ 4e_\beta] = 4C_{\alpha \beta}^{\lambda} \ 4e_\lambda \quad (A.4)$$

with:

$$4C_{0 \beta}^{\alpha} = 4C_{0 \alpha}^{\lambda} = 0; \ 4C_{ij}^k = C_{ij}^k \quad (A.5)$$

In particular, we deduce from (A.3):

$$4R_{j,i0}^l = 4e_i(4e_j^l) - 4e_0(4e_j^l) + 4\gamma_0^l 4\gamma_{ij}^l - 4\gamma_0^l 4\gamma_{ij}^l - 4C_{00}^l 4\gamma_{ij}^l$$

$$= - \frac{d}{dt}(\gamma_{ij}^l) - (K_{j}^{k} \gamma_{ik}^l - K_{k}^{l} \gamma_{ij}^k).$$

Hence:

$$4R_{j,i0}^l = - \frac{d}{dt}(\gamma_{ij}^l) - \nabla_i K_j^l \quad (A.6)$$

Now we have from the Codazzi relation:

$$4R_{\lambda i,jl}^\lambda = - \nabla_l K_{ij} + \nabla_j K_{il}.$$

But here we have $n = \partial_t$; this implies:

$$4R_{0i,jl} = - \nabla_l K_{ij} + \nabla_j K_{il}$$

or:

$$4R_{j,i0}^l = - \nabla_l K_{ij} + \nabla_j K_i^l; \quad (A.7)$$

(A.6) and (A.7) give:

$$\frac{d}{dt}(\gamma_{ij}^l) = \nabla_l K_{ij} - \nabla_j K_i^l - \nabla_i K_j^l \quad (A.8).$$

Now a direct calculation gives:

$$\nabla_l \left( \frac{d}{dt} \gamma_{ij}^l \right) = \frac{d}{dt} R_{ij}.$$  

We deduce from (A.8) that:

$$\frac{d}{dt} R_{ij} = \nabla_l \nabla_i K_{ij} - \nabla_l \nabla_j K_i^l - \nabla_i \nabla_j K_l^j.$$
We deduce:

\[ g^{ij} \frac{d}{dt} R_{ij} = -2g^{lm} \nabla_i \nabla^m K_{mn} \]

\[ = -2g^{lm} \left[ \gamma^m_{il} \nabla^l K_{mn} - \gamma^l_{im} \nabla^m K_{ln} - \gamma^i_{lm} \nabla^m K_{mn} \right] \]

\[ = -2 \left[ \gamma^m_{li} \nabla^l K_{im} - \gamma^l_{im} \nabla^m K^i_l - g^{lm} \gamma^i_{ln} \nabla^m K_{ni} \right] \]

and:

\[ g^{ij} \frac{d}{dt} R_{ij} = 2g^{ij} \gamma^k_{ij} \nabla^l K_{lk}. \]

But:

\[ R = g^{ij} R_{ij} \quad \Rightarrow \quad \partial_t R = 2K^{ij} R_{ij} + g^{ij} \frac{d}{dt} R_{ij} \]

and the evolution of \( R \):

\[ \partial_t R = 2K^{ij} R_{ij} + 2g^{ij} \gamma^k_{ij} \nabla^l K_{lk}. \quad (A.9) \]

1.4) Evolution of \( K_{ij} K^{ij} \)

We have:

\[ \partial_t (K_{ij} K^{ij}) = K^{ij} K_{ij} + K_{ij} K^{ij} \]

\[ \quad = K^{ij} K_{ij} + K_{ij} \left[ g^{ik} g^{jl} K_{kl} \right] \]

\[ \quad = K^{ij} K_{ij} + K_{ij} \left[ 4K^{ik} g^{jl} K_{kl} + g^{ik} g^{j l} K^{kl} \right]. \]

Hence:

\[ \partial_t (K_{ij} K^{ij}) = 2K^{ij} K_{ij} + 4K_{ij} K^{il} K^j_l. \]

Then equation (2.52) gives the evolution of \( K_{ij} K^{ij} \):

\[ \partial_t (K_{ij} K^{ij}) = 2K^{ij} R_{ij} + 2HK_{ij} K^{ij} - 16\pi K^{ij}(\tau_{ij} + T_{ij}) + 8\pi H(-T_{00} + g^{lm} T_{lm}) - 2HA. \quad (A.10) \]

1.5) Evolution of \( A \)

We have:

\[ \partial_t A = \partial_t R - \partial_t (K_{ij} K^{ij}) + 2H \partial_t H - 16\pi \partial_t (\tau_{00} + T_{00}) \]

and using (A.9), (A.10), (A.1) and (A.2), we obtain the evolution of \( A \):

\[ \partial_t A = 2HA + 2g^{ij} \gamma^k_{ij} A_k. \quad (A.11) \]
2) Evolution of $A_j = \nabla^i K_{ij} + 8\pi (\tau_{0j} + T_{0j})$

We have to compute:

$$\frac{d}{dt}(\nabla^i k_{ij}), \quad \text{and} \quad \frac{d}{dt}(\tau_{0j} + T_{0j}).$$

If we take $\beta = j$ in the conservation laws (2.21), we obtain:

$$4\nabla_{\alpha}(4\tau^{\alpha j} + 4T^{\alpha j}) = 4\nabla_0(4\tau^{0j} + 4T^{0j}) + 4\nabla_i(4\tau^{ij} + 4T^{ij}) = 0,$$

or, using (2.7):

$$4\epsilon_0(4\tau^{0j} + 4T^{0j}) + 4\epsilon_{0\alpha}(4\tau^{\lambda j} + 4T^{\lambda j}) + 4\epsilon_{0\lambda}(4\tau^{0j} + 4T^{0j}) + 4\gamma_{ij}(4\tau^{ij} + 4T^{ij}) + 4\gamma_{ji}(4\tau^{ij} + 4T^{ij}) = 0.$$

From there we deduce, using (2.19) that:

$$\frac{d}{dt}(\tau^{0j} + T^{0j}) = H(\tau^{0j} + T^{0j}) - \nabla^i(\tau^{ij} + T^{ij}) + 2K^j_i(\tau^{0i} + T^{0i}). \quad (A.12)$$

Now:

$$\tau_{0n} + T_{0n} = 4g_{0\alpha}4g_{n\beta}(4\tau^{\alpha\beta} + 4T^{\alpha\beta}) = -g_{nj}(\tau^{0j} + T^{0j})$$

Then, using $\frac{d}{dt}g_{nj} = -2K_{nj}$ given by equation (2.51), we obtain:

$$\frac{d}{dt}(\tau_{0n} + T_{0n}) = 2K_{nj}(\tau^{0j} + T^{0j}) - g_{nj}\frac{d}{dt}(\tau^{0j} + T^{0j}).$$

We then use (A.12) to obtain:

$$\frac{d}{dt}(\tau_{0j} + T_{0j}) = H(\tau_{0j} + T_{0j}) + \nabla^i(\tau_{ij} + T_{ij}). \quad (A.13)$$

Now we have by Bianchi identities:

$$4\nabla_{\alpha}4R^{\alpha\beta} - \frac{1}{2}4\nabla^\beta 4R = 0.$$ But using (2.19), with $\beta = j$, we have:

$$4\nabla^\beta 4R = 4\nabla^\beta 4R^\alpha_\alpha = g^{ij}4\nabla_i 4R^\alpha_i = 4g^{ij}[4\gamma_{j\lambda}^i 4R^\lambda_\alpha - 4\gamma_{j\alpha}^i 4R^\lambda_\alpha] = 0.$$
Then we have:

\[ 4 \nabla_\alpha 4 R^{\alpha j} = 0. \]

But:

\[ (4 \nabla_\alpha 4 R^{\alpha j} = 0) \implies (4 \nabla_0 4 R^{0j} + 4 \nabla_i 4 R^{ij} = 0), \]

and developing, we obtain:

\[ \frac{d}{dt} (4 R^{0j}) - K^j_i 4 R^{0i} + 4 \nabla_i 4 R^{ij} = 0. \quad (A.14) \]

This means that we have to compute

\[ 4 R^{0i} \text{ and } 4 \nabla_i 4 R^{ij}. \]

We know that:

\[ 4 R_{\alpha\beta} = 4 e_\lambda \left( 4 \gamma^\lambda_{\alpha\beta} \right) - 4 e_\beta \left( 4 \gamma^\lambda_{\beta\alpha} \right) + 4 \gamma^\lambda_{\mu\alpha} 4 \gamma^\mu_{\beta\alpha} - 4 \gamma^\lambda_{\mu\beta} 4 \gamma^\mu_{\alpha\lambda} - C^\mu_{\lambda\beta} 4 \gamma^\lambda_{\mu\alpha}. \quad (A.15) \]

We then obtain, using once more (2.19):

\[ 4 R^{0j} = 4 g^{0\alpha} 4 g^{j\beta} 4 R_{\alpha\beta} = -g^{ij} 4 R^{0i} = g^{ij} \left[ K^k_i 4 \gamma^k_{lj} - K^l_i 4 \gamma^k_{lj} \right] = g^{ij} \nabla_l K^l_i. \]

Thus:

\[ 4 R^{0j} = \nabla_l K^l_i. \quad (A.16) \]

If we consider formula (A.15), we deduce using (2.19) that:

\[ 4 R^{ij} = 4 g^{ik} 4 g^{jl} 4 R_{kl} = -g^{ik} g^{jl} \partial_l K_{kl} + HK^{ij} + g^{ik} g^{jl} \left[ 4 \gamma^p_{lj} 4 \gamma^q_{lk} - 4 \gamma^p_{lk} 4 \gamma^q_{lj} - C^p_{q} 4 \gamma^q_{lk} \right] - 2K^l_j K^{pl}. \]

i.e., using (2.60):

\[ 4 R^{ij} = R^{ij} + HK^{ij} - 2K^l_j K^{pl} - g^{ik} g^{jl} \partial_l K_{kl}. \quad (A.17) \]
We then deduce, using (A.16):

\[ 4\nabla_i 4R^{ij} = 4\gamma_i^{i\lambda} 4R^{j\lambda} + 4\gamma_j^{i\lambda} 4R^{ij} = -H 4R^{ij} - K_i^{i\lambda} 4R^{j0\lambda} + 4\gamma_i^{ij} 4R^{kj} + 4\gamma_j^{ij} 4R^{ik} \]

and we deduce from (A.17) and (2.19):

\[ 4\nabla_i 4R^{ij} = -K_j^{ij} \nabla_k K^{ki} + \nabla_i R^{ij} - 2\nabla_i (K_p^{i} K^{pj}) - g^{ip} g^{jq} \nabla_i (\partial_i K_{pq}). \]  

We obtain from (A.14), (A.16) and (A.18):

\[ d_{\partial} (4\nabla_i K^{ij}) - H \nabla_i K^{ij} - 8\pi \nabla_i (\tau_{ij} + T_{ij}) + 4\pi (-T_{00} + g^{lm} T_{lm}) g_{pq} - A g_{pq} = 0 \]

We now use equation (2.52) to express \( \partial_i K_{pq} \) and we obtain:

\[ \frac{d}{d\tau} (g^{jk} \nabla_i K^{ik}) - 2K_j^{ij} \nabla_k K^{ik} + \nabla_i R^{ij} - 2\nabla_i (K_p^{i} K^{pj}) - g^{ip} g^{jq} \nabla_i (\partial_i K_{pq}) = 0 \]

We finally obtain, using (A.13) and (A.19):

\[ \frac{d}{d\tau} A_j = H A_j \]
3) Evolution of $A_{ij k} = C^l_{ij} F_{kl} + C^l_{j k} F_{il} + C^l_{k i} F_{jl}$

Using equation (2.53) in $F_{ij}$, we have:

$$\partial_t (A_{ij k}) = C^l_{ij} \partial_t (F_{kl}) + C^l_{j k} \partial_t (F_{il}) + C^l_{k i} \partial_t (F_{jl})$$

$$= C^l_{ij} \left( C^m_{kl} g_{ml} E^l \right) + C^l_{j k} \left( C^m_{il} g_{ml} E^l \right) + C^l_{k i} \left( C^m_{j k} g_{ml} E^l \right)$$

$$= \left( C^l_{ij} C^m_{kl} + C^l_{j k} C^m_{il} + C^l_{k i} C^m_{j k} \right) g_{ml} E^l$$

$$= 0,$$

from the equality of Jacobi. Hence:

$$\partial_t A_{ij k} = 0.$$

4) Evolution of $B = C^0_{ik} F^{0k} + e u^0$

Set

$$B^\alpha = C^i_{ik} F^{\alpha k} + e u^\alpha.$$

We have $B^0 = B$. Setting $J^k = e u^k$, we have from equation (2.9):

$$4 \nabla_\alpha B^\alpha = C^i_{ik} 4 \nabla_\alpha 4 F^{\alpha k}$$

$$= C^i_{ik} J^k.$$

Thus:

$$4 \nabla_0 B^0 + 4 \nabla_i B^i = C^i_{ik} J^k.$$

or:

$$\frac{d}{dt} B + 4 \gamma^i_{i\lambda} B^\lambda = C^i_{ik} J^k,$$

i.e.

$$\frac{d}{dt} B + 4 \gamma^i_{i0} B + 4 \gamma^i_{ij} B^j = C^i_{ik} J^k;$$

Use (2.20) to obtain:

$$\frac{d}{dt} B - HB + C^i_{ij} \left[ C^l_{lk} F^{jk} + e u^j \right] = C^i_{ik} e u^k.$$
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or:

\[
\frac{d}{dt} B - HB + C^i_{ij} C^h_{lk} F^{jk} + C^i_{ij} eu^j = C^i_{ik} eu^k.
\]

But

\[
C^i_{ij} C^h_{lk} F^{jk} = 0.
\]

Hence:

\[
\frac{d}{dt} B - HB = 0
\]

and the evolution of B:

\[
\frac{d}{dt} B = HB.
\]

This ends the appendix. \(\square\)
References

[1] CHOQUET-BRUHAT, DE WITT-MORETTE, C, DILLARD-BLEICK, M., Analysis, manifolds and Physics, north-holland publishing Company, Amsterdam-Newyork-Oxford, 1977.

[2] HAWKING S.W and ELLIS F.R,1973, the large structure of space-time (Cambridge, Monographs on Maths Phys. )
(Cambridge: Cambridge University press)

[3] JANTZEN R.T., Cosmology of the Early Universe
L.Z. Fang and R. Ruffini Eds, World Scientific, Singapore, 1984.

[4] LEE HAYOUNG, 2004, Asymptotic behaviour of the Einstein-Vlasov system with positive cosmological constant, Math.Proc. Comb. Phil. Soc. 137, 495-509.

[5] LICHNEROWICZ, A., Théories Relativistes de la Gravitation et de l’Electromagnétisme, MASSON, 1955.

[6] NOUTCHEGUEME N. and TETSADJIO E.M.
Global Dynamics for a relativistic charged plasma in Bianchi spacetimes; Class Quantum Grav. 26 (2009) 195001 (16 pp).

[7] RENDAL, A.D., Cosmic Censorship for some spatially homogeneous models
Ann. Phys. 233, 82-96 (1994)

[8] WALD, R., Asymptotic behaviour of homogeneous cosmological models in the presence of positive cosmological constant.
Phys. Rev. D 28, 2118-2120
1983