Additivity of higher rho invariant for topological structure group from a differential point of view

Baojie Jiang\textsuperscript{1} and Hongzhi Liu\textsuperscript{2} \textsuperscript{†}

\textsuperscript{1}College of Mathematics and Statistics, Chongqing University, Chongqing 401331, P. R. China. e-Mail: jiangbaojie@gmail.com
\textsuperscript{2}School of Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, P. R. China. e-Mail: liu.hongzhi@mail.shufe.edu.cn
\textsuperscript{†}Corresponding author

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Abstract

In this paper, we adapt part of Weinberger, Xie and Yu’s breakthrough work, to define additive higher rho invariant for topological structure group by differential geometric version of signature operators, or in other words, unbounded Hilbert-Poincaré complexes.

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1 Introduction

In 2005, Higson and Roe \cite{4, 5, 6} investigate higher rho invariant for a homotopy equivalence between two closed smooth manifolds. And they proved that higher rho invariant establishes a natural set theoretic map from the smooth surgery exact sequence of a closed smooth manifold to the $K$-theory long exact sequence of geometric $C^*$-algebras (which is called the analytic surgery exact sequence). In particular, higher rho invariant for a homotopy equivalence induces a set theoretic map from smooth surgery set to $K$-theory of certain $C^*$-algebras.

In the same papers, Higson and Roe showed that this higher rho invariant can be described in 2 equivalent ways: the PL approach, giving rise to a theory of signature of bounded Hilbert-Poincaré complex, and the differential geometric approach, giving rise to a theory of signature of unbounded Hilbert-Poincaré complex.

Later, in \cite{12}, Piazza and Schick gave an alternative differential geometric construction of higher rho invariant for homotopy equivalence. They constructed the higher rho invariant for homotopy equivalence using Hilsum-Skandalis perturbation (\cite{7}). The technique Piazza and Schick used has also been exposed in \cite{11}. In \cite{18},...
Zenobi extended the work of Higson-Roe and Piazza-Schick to the case of topological manifold. Zenobi showed that the higher rho invariant is well-defined set theoretic map from the structure group of topological manifolds to $K$-theory of certain $C^*$-algebras. However, unlike smooth surgery set, the topological structure set actually carries a group structure. At the time, it was an open question whether the higher rho invariant defines a group homomorphism. In the breakthrough work of Weinberger, Xie and Yu ([16]), the question was answered positively in complete generality. There are two major novelties of their work:

1. it gives a new description of the topological structure group in terms of smooth manifolds with boundary. More specifically, the new description of structure group allows one to replace topological manifolds in the usual definition of structure group (of topological manifolds) by smooth manifolds with boundary. Such a description leads to a transparent group structure given by disjoint union;

2. it develops a theory of higher rho invariants in this new setting, in which higher rho invariants are easily seen to be additive.

Concerning the point (2) above, in [16], this part is carried out in terms of the PL version of signature operators, or in other words, bounded Hilbert-Poincaré complexes. In this paper, we adapt this part of [16] to work for the differential geometric version of signature operators, in other words, unbounded Hilbert-Poincaré complexes. Although it seems to be quite straightforward in principle, it is quite nontrivial work to carry out all the technical details, in particular, on the construction of higher rho invariant for homotopy equivalences between smooth manifolds with boundary.

This article is organized as follows. Some standard definitions are introduced in section 2. Signature operators and homotopy invariance of higher signature are introduced in section 3. Our results in section 3 rely on a detailed understanding of [7] [14]. The primary theme of section 4 and section 5 is to examine index map for $L_n(\pi_1 X)$ and local index map for $\mathcal{N}_n(X)$ respectively at great length. Roughly speaking, the infinitesimal controled homotopy equivalence represents zero element in $\mathcal{N}_n(X)$ (and $\mathcal{S}_n(X)$ also), while homotopy equivalence represents zero element in $L_n(\pi_1 X)$ ([16]). With these preparations, for closed oriented topological manifold of dimension $n \geq 5$, we define the mapping surgery to analysis in section 6. For the convenience of readers, we recall the reinterpretation of structure set, normal group, $L$-group given by Weinberger, Xie, and Yu in [16] in Appendix A, and Poincaré duality operators in Appendix B.

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2 Preliminary

In this section, we collect some basic notations and terminologies needed during the course of the paper. We refer the reader to [13, 17] for more details.
Let $X$ be a proper metric space. An $X$-module is a separable Hilbert space equipped with a $\ast$-representation of $C_0(X)$, where $C_0(X)$ is the $C^*$-algebra of all complex-valued continuous functions on $X$ which vanish at infinity. An $X$-module is called nondegenerate if the $\ast$-representation of $C_0(X)$ is nondegenerate. An $X$-module is said to be standard if no nonzero function in $C_0(X)$ acts as a compact operator.

**Definition 2.1.** Let $H_X$ be an $X$-module and $T$ a bounded linear operator acting on $H_X$.

1. The propagation of $T$ is defined to be $\sup\{d(x,y) \mid (x,y) \in \text{Supp}(T)\}$, where $\text{Supp}(T)$ is the complement (in $X \times X$) of the set of points $(x,y) \in X \times X$ for which there exists $f, g \in C_0(X)$ such that $gTf = 0$ and $f(x)g(y) \neq 0$;

2. $T$ is said to be locally compact if $fT$ and $Tf$ are compact for all $f \in C_0(X)$.

**Definition 2.2.** Let $H_X$ be a standard nondegenerate $X$-module and $B(H_X)$ the set of all bounded linear operators on $H_X$.

1. The Roe-algebra of $X$, denoted by $C^*(X)$, is the $C^*$-algebra generated by all locally compact and with finite propagation operators in $B(H_X)$;

2. Localization algebra $C^*_L(X)$ is the $C^*$-algebra generated by all bounded and uniformly-norm continuous functions $f : [1, \infty) \to C^*(X)$ such that

   \[
   \text{propagation of } f(t) \to 0 \text{ as } t \to \infty.
   \]

3. The kernel of the following evaluation map

   \[
   ev : C^*_L(X) \to C^*(X) \quad f \mapsto f(1)
   \]

   are defined to be the obstruction algebra $C^*_L,0(X)$. In particular, $C^*_L,0(X)$ is an ideal of $C^*_L(X)$;

4. If $Y$ is a subspace of $X$, then $C^*_L(Y;X)$ is defined to be a closed subalgebra of $C^*_L(X)$ generated by all elements $f$ such that there exist $c_t > 0$ satisfying $\lim_{t \to \infty} c_t = 0$, and $\text{Supp}(f(t)) \subset \{(x,y) \in X \times X \mid d((x,y),Y \times Y) \leq c_t\}$ for all $t$. Similarly, we can define $C^*_L,0(Y;X)$ which is a closed subalgebra of $C^*_L,0(X)$.

Let $G$ be a (countable) discrete group and acts properly on $X$ by isometries. Let $H_X$ be a $X$-module equipped with a covariant unitary representation of $G$. Let the representation of $C_0(X)$ be $\phi$ and let the action of $G$ be $\pi$, we call $(H_X,G,\phi,\pi)$ a covariant system is to say

\[
\pi(g)\phi(f) = \phi(g.f)\pi(g)
\]

where, $g.f(x) = f(g^{-1}x)$ for any $f \in C_0(X)$, $g \in G$.

**Definition 2.3.** A covariant system $(H_X,G,\phi,\pi)$ is called admissible if

1. The action of $G$ is proper and cocompact;

2. $H_X$ is a nondegenerate standard $X$-module;
3. For each \( x \in X \), the stabilizer group \( G_x \) acts on \( H_X \) regularly in the sense that the action is isomorphic to the obvious action of \( G_x \) on \( \ell^2(G_x) \otimes H \) for some infinite dimensional Hilbert space \( H \). Here \( G_x \) acts on \( \ell^2(G_x) \) by (left) translations and acts on \( H \) trivially.

We remark that for each locally compact metric space \( X \) with a proper and cocompact isometric action of \( G \), there exists an admissible covariant system \((H_X, G, \phi)\). We will denote an admissible covariant system \((H_X, G, \phi)\) by \( H_X \) and call it an admissible \((X, G)\)-module.

**Definition 2.4.** Let \( X \) be a locally compact metric space \( X \) with a proper and cocompact isometric action of \( G \). If \( H_X \) is an admissible \((X, G)\)-module, we denote by \( \mathbb{C}[X]^G \) to be \(*\)-algebra of all \( G \)-invariant locally compact operators with finite propagations in \( B(H_X) \). We define \( C^*(X)^G \) to be the completion of \( \mathbb{C}[X]^G \) in \( B(H_X) \). Similarly, we can define \( C^*_L(X)^G \), \( C^*_L(Y; X)^G \) and \( C^*_L(Y; X)^G \).

**Remark 2.5.** Up to isomorphism, \( C^*(X) \) does not depend on the choice of the standard nondegenerate \( X \)-module \( H_X \). The same holds for \( C^*_L(X), C^*_L(Y; X), C^*_L(Y; X)\) and their \( G \)-equivariant versions.

## 3 Homotopy invariance of higher signature

The purpose of this section is to review the literature on the homotopy invariance of the higher signatures for closed manifolds [4] [5] [7]. We start with the definition of signature operator, which is consistent with the one in [5, Section 5].

Let \( M \) be a closed smooth manifold of dimension \( n \) and let \( \tilde{M} \) be a regular \( G \)-covering space of \( M \), where \( G \) is a discrete group. Write \( \Lambda^p(M) \) for the smooth (compactly supported) differential \( p \)-forms on \( M \). Passing to \( L^2 \)-completions, we obtain Hilbert spaces \( \Lambda^p_{L^2}(M) \). We shall denote by \( d_{\tilde{M}} \) the operator-closure of the De Rham differential operator on \( \Lambda^p_{L^2}(\tilde{M}) \). For notation simplicity, we will use \( L^2(\Lambda(\tilde{M})) \) instead of \( \bigoplus \Lambda^p_{L^2}(\tilde{M}) \).

Let \( S_{\tilde{M}} \) be an operator obtained from attaching a proper coefficient to the usual Hodge-* operator, that is

\[
S_{\tilde{M}}(\omega) = i^{p(p-1)+[\frac{p}{2}]} * \omega, \quad \omega \in \Lambda^p(\tilde{M}),
\]

\( S_{\tilde{M}} \) is a self-adjoint operator with \( S_{\tilde{M}}^2 = 1 \).

For \( n \) even, the signature operator \( D_{\tilde{M}} \) on \( \tilde{M} \) is given by the de Rham operator \( d_{\tilde{M}} + d^*_{\tilde{M}} \), which anti-commutes with the grading operator provided by \( S_{\tilde{M}} \). For \( n \) odd, the de Rham operator \( d_{\tilde{M}} + d^*_{\tilde{M}} \) commutes with \( S_{\tilde{M}} \), the signature operator \( D_{\tilde{M}} \) on \( \tilde{M} \) is given by \( D_{\tilde{M}} := i(d_{\tilde{M}} + d^*_{\tilde{M}})S_{\tilde{M}} \) restricting to even forms. Following Higson and Roe [5, Section 5], we denote \( D := d_{\tilde{M}} + d^*_{\tilde{M}} \).

By [4, section 5], we know that \( D \pm S_{\tilde{M}} \) are invertible. Using this we can define our notion of higher index of the signature operator. We will denote it by \( \text{Ind}(D_{\tilde{M}}) \).

There are two cases:

For \( n \) even, \( K_0(C^*(\tilde{M})^G) \) is the receptacle for the higher index of the signature operator, and the higher index can be computed as

\[
[P_+(D + S_{\tilde{M}})] - [P_+(D - S_{\tilde{M}})] \in K_0(C^*(\tilde{M})^G)
\]
where, \( P_+(D + S_{\tilde{M}}) \) and \( P_+(D - S_{\tilde{M}}) \) denote the positive projections of \( D + S_{\tilde{M}} \) and \( D - S_{\tilde{M}} \) respectively.

For \( n \) odd, \( K_1(C^*(M)^G) \) is the receptacle for the higher index of the signature operator, and the higher index can be represented by the following invertible operator
\[
(D + S_{\tilde{M}})(D - S_{\tilde{M}})^{-1} : \Lambda_{\text{even}} \rightarrow \Lambda_{\text{even}}
\]

where, \( \Lambda_{\text{even}} = \bigoplus \Lambda_{L^2}^{2p}(\tilde{M}) \) denote the even forms.

Remark 3.1. Actually, for \( t > 0 \), we have
\[
[P_+(D + tS_{\tilde{M}})] - [P_+(D - tS_{\tilde{M}})] = [P_+(D + S_{\tilde{M}})] - [P_+(D - S_{\tilde{M}})] \in K_0(C^*(M)^G),
\]
and
\[
[(D + tS_{\tilde{M}})(D - tS_{\tilde{M}})^{-1}] = [(D + S_{\tilde{M}})(D - S_{\tilde{M}})^{-1}] \in K_1(C^*(M)^G).
\]

### 3.1 Hilsum-Skandalis submersion

In this subsection, we will concentrate on the most basic constructions exposed in [7]. Using this construction we will describe the homotopy invariance of higher signature in subsection 3.2.

Let \( M, N \) be closed oriented smooth manifolds of dimension \( n \) and let \( f : M \rightarrow N \) be a smooth map. Generally, the pull-back map \( f^* : L^2(\Lambda(N)) \rightarrow L^2(\Lambda(M)) \) does not induce a bounded operator from \( L^2(\Lambda(N)) \) to \( L^2(\Lambda(M)) \). However, following the line of [7], we can define a bounded operator \( T_f \) from \( L^2(\Lambda(N)) \) to \( L^2(\Lambda(M)) \).

Let \( I = (-1, 1) \) be an open interval. For \( k \in 8\mathbb{N} \) big enough, there exists a smooth map \( p : I^k \times M \rightarrow N \) which is a submersion such that \( p(x,0) = f(x) \). Let \( v \in \Lambda^k(I^k) \) be a real-valued \( k \)-form with \( \int_{I^k} v = 1 \). We define the map \( T_v(p) : L^2(\Lambda(N)) \rightarrow L^2(\Lambda(M)) \) by
\[
T_v(p)(\omega) = \int_{I^k} v \wedge p^* \omega.
\]

Note that any two submersions \( p \) and \( p' \) associated to \( f \) as above are (after possibly stabilizing) homotopic through a path of submersions \( p_t \) associated to \( f \). Thus we will denote \( T_v(p) \) by \( T_v(f) \) or \( T_f \).

Now, let \( f : M \rightarrow N \) be a smooth map and let \( G \rightarrow \tilde{N} \rightarrow N \) be a \( G \)-covering of \( N \), where \( G \) is a discrete group. Let \( \tilde{M} = f^*(\tilde{N}) \) be the pull back covering. Then \( G \rightarrow \tilde{M} \rightarrow M \) is a \( G \)-covering of \( M \), and \( f : M \rightarrow N \) can be lifted to a \( G \)-equivariant map \( \tilde{f} : \tilde{M} \rightarrow \tilde{N} \).

Similar to the compact case we can define \( T_{\tilde{f}} \). We list several properties which will be used in this paper

1. As an operator \( T_{\tilde{f}} : L^2(\Lambda(\tilde{N})) \rightarrow L^2(\Lambda(\tilde{M})) \) is uniformly bounded and \( G \)-equivariant;
2. It holds that \( T_{\tilde{f}}(\text{dom}(d_{\tilde{N}})) \subseteq \text{dom}(d_{\tilde{M}}) \) and \( d_{\tilde{M}}T_{\tilde{f}} = T_{\tilde{f}}d_{\tilde{N}} \), i.e. \( T_{\tilde{f}} \) induces a chain map from \( (L^2(\Lambda(\tilde{N})), d_{\tilde{N}}) \) to \( (L^2(\Lambda(\tilde{M})), d_{\tilde{M}}) \);
3. If \( f : M \rightarrow N \) is an orientation-preserving homotopy equivalence, then \( T_{\tilde{f}} \) induces an isomorphism from \( \text{Ker}(d_{\tilde{N}})/\text{Im}(d_{\tilde{N}}) \) to \( \text{Ker}(d_{\tilde{M}})/\text{Im}(d_{\tilde{M}}) \) and inverse is \( T_{\tilde{g}} \), where \( g : N \rightarrow M \) a homotopy inverse of \( f \);
4. Fix a metric on $\tilde{N}$ that agrees with the topology of $\tilde{N}$. Then as an operator between $G$-equivariant $\tilde{N}$-modules, one can make the propagation of $T_f$ as small as possible;

5. Let $v \in (L^2(\Lambda(\tilde{M})), d_{\tilde{M}})$, $w \in (L^2(\Lambda(\tilde{N})), d_{\tilde{N}})$. We define $T_f'$ by

$$\int_{\tilde{M}} v \wedge T_f' w = \int_{\tilde{N}} T_f' v \wedge w,$$

then there exists a bounded operator $y$ [7, Lemma 2.1], such that

$$1 - T_f'T_f = d_{\tilde{N}}y + yd_{\tilde{N}}.$$

For the properties listed above, more informations and proofs can be found for instance in [7, 14]. For notation simplicity, we will denote $T_f'$ by $T_f$.

### 3.2 Homotopy invariance of higher signature

In this subsection, we sketch Higson and Roe’s proof of homotopy invariance of the higher signatures for closed manifolds [4, 5].

Let $X$ be a closed oriented topological manifold of dimension $n$ and let $G \to \tilde{X} \to X$ be a $G$-covering of $X$, where $G$ is a discrete group. Moreover, we have the following maps

$$\begin{align*}
    M & \xrightarrow{f} N \\
    & \downarrow \phi \searrow \psi \\
    X & 
\end{align*}$$

(3.1)

where, $f : M \to N$ is a smooth orientation-preserving homotopy equivalence between two closed manifolds $M$ and $N$, $\phi$ and $\psi$ are continuous maps, such that $\phi = \psi \circ f$.

Let $\tilde{N} = \psi^*(\tilde{X})$ and $\tilde{M} = f^*(\tilde{N}) = \phi^*(\tilde{X})$ be the pull back coverings. We denote

$$d := \begin{pmatrix} d_{\tilde{M}} & 0 \\ 0 & d_{\tilde{N}} \end{pmatrix} \text{ and } S := \begin{pmatrix} S_{\tilde{M}} & 0 \\ 0 & -S_{\tilde{N}} \end{pmatrix},$$

which are operators acting on $L^2(\Lambda(\tilde{M})) \bigoplus L^2(\Lambda(\tilde{N}))$. By Definition B.1 $S$ is a Poincaré duality operator of $(L^2(\Lambda(\tilde{M})) \bigoplus L^2(\Lambda(\tilde{N})), d)$. Keep in mind that our definition of Poincaré duality operator is different from the ones in [4, 16].

In the following, if $S$ is a Poincaré duality operator of a complex $(\mathcal{H}, d)$ (c.f. Definition B.1), we suppress the complex $(\mathcal{H}, d)$ from the notation only say $T$ is a Poincaré duality operator of $d$.

We first introduce the following lemma, which is an analogy of [16, Lemma 4.36].

**Lemma 3.2.** Use the notations as above. Let $g : N \to M$ be a homotopy inverse of $f$, $S_{\tilde{N}}$ is chain homotopy equivalent to $T_f^* S_{\tilde{M}} T_f$ and $T_g S_{\tilde{M}} T_g^*$.

**Proof.** It is sufficient to prove that $S_{\tilde{N}}$ is chain homotopy equivalent to $T_f^* S_{\tilde{M}} T_f$. 

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Let \( v, w \in L^2(\Lambda(N)) \), then

\[
\int_N v \cdot S_N w - \int_N v \cdot T_f^* S_M T_f w = \int_N v \cdot S_N w - \int_N v \wedge T_f^* S_M T_f w
\]

\[
= i^{p(p-1)+[\frac{p}{2}]}(\int_N v \wedge w - \int_N T_f v \wedge T_f w)
\]

\[
= i^{p(p-1)+[\frac{p}{2}]}(\int_N v \wedge w - \int_N v \wedge T_f T_f^* w)
\]

\[
= i^{p(p-1)+[\frac{p}{2}]} \int_N v \wedge (d_N y + y d_N w)
\]

\[
= \int_N v \cdot S_N (d_N y + y d_N w).
\]

Thus, we have

\[
S_N - T_f^* S_M T_f = -d^*_N (S_N y) + S_N y d_N.
\]

We denote \( D := d + d^* \), and we have \( D \pm S \) are invertible [4, section 5]. Moreover, by Lemma 3.2, for \( t \in [0,1] \), we have the path

\[
\left( S_M^t \quad 0 \quad -(1-t) S_N - t T_f^* S_M T_f \right)
\]

are Poincaré duality operators (c.f. Definition B.1,[4, Lemma 3.4], and [4, Definition 3.1]) of \( d \).

Also, the path

\[
\left( \begin{array}{cc}
\cos(t \frac{\pi}{2}) S_M^t & \sin(t \frac{\pi}{2}) S_M^t T_f \\
\sin(t \frac{\pi}{2}) T_f S_M^t & -\cos(t \frac{\pi}{2}) T_f S_M^t T_f
\end{array} \right)
\]

(3.2)

are Poincaré duality operators of \( d \). Moreover, we can write the homotopy inverse of Eq.(3.2) as

\[
\left( \begin{array}{cc}
\cos(t \frac{\pi}{2}) S_M^t & \sin(t \frac{\pi}{2}) S_M^t T_f \\
\sin(t \frac{\pi}{2}) T_f S_M^t & -\cos(t \frac{\pi}{2}) T_f S_M^t T_f
\end{array} \right)^{-1}
\]

Notice that, \( t \in [0,1] \), using the path (Poincaré duality operators of \( d \))

\[
\left( \begin{array}{cc}
0 & e^{it\frac{\pi}{2}} S_M^t T_f \\
e^{-it\frac{\pi}{2}} T_f S_M^t & 0
\end{array} \right)
\]

we can be connected \( \left( \begin{array}{cc}0 & S_M^t T_f \\
T_f^* S_M & 0\end{array} \right) \) to \( \left( \begin{array}{cc}0 & -S_M^t T_f \\
-T_f^* S_M & 0\end{array} \right) \). After re-parameterize the \( t \) variable, we will denote the above path by \( S_t \).

Now, for any \( t \in [0,1] \), we have the following operators. For \( n \) even, we consider the operator

\[
P_+(D + \alpha S_t) \quad \text{and} \quad P_+(D - \alpha S_t).
\]

(3.3)

For \( n \) odd, we consider the invertible operator

\[
(D + S_t)(D - S_t)^{-1}
\]

(3.4)

which restricts to even forms.

We now prove Lemma 3.3 and Lemma 3.4, which shows that \( K \)-theory of the \( C^* \)-algebra \( C^*(\tilde{X})^G \) is the receptacle for the indices given by operators in Eq.(3.5) and Eq.(3.4).
Lemma 3.3 ([4, Lemma 5.7]). Let $i = \sqrt{-1}$, for any $t$, the operators $i \pm (D \pm S_t)$ are invertible, and $(i \pm (D \pm S_t))^{-1}$ belongs to $C^*(\hat{X})^G$.

Proof: Since $i \pm (D \pm S_t)$ is a self-adjoint operator and self-adjoint operator plus or minus $i$ are always invertible.

Hence, we have

\[(D + S_t + i) = (S_t(D + i)^{-1} + 1)(D + i)\]
\[= (D + i)((D + i)^{-1}S_t + 1).\]

Moreover, for any compact support $\phi$, we have

\[\phi(D + S_t + i)^{-1} = \phi((S_t(D + i)^{-1} + 1)(D + i))^{-1}\]
\[= \phi(D + i)^{-1}(S_t(D + i)^{-1} + 1)^{-1},\]
\[= (S_t(D + i)^{-1} + 1)\]
\[= \phi(D + i)^{-1}((D + i)^{-1} + 1)\]
\[= (S_t(D + i)^{-1} + 1)^{-1}(D + i)^{-1}\phi.\]

Operators $(S_t(D + i)^{-1} + 1)$ are certainly bounded, and $\phi(D + i)^{-1}, (D + i)^{-1}\phi$ are always compact. Thus, $(i \pm (D \pm S_t))^{-1}$ is locally compact.

Without loss of generality, we suppose $T_t$ have finite propagation, thus $(S_t(D + i)^{-1} + 1)$ can be approximated by finite propagation operators, and this is also true for the operator $(D + S_t + i)^{-1} = ((S_t(D + i)^{-1} + 1)(D + i))^{-1}$. \(\square\)

Lemma 3.4 ([4, Lemma 5.8]). For any bounded function $g : (-\infty, \infty) \rightarrow \mathbb{R}$, we have $g(D \pm S_t) - g(D)$ belongs to $C^*(\hat{X})^G$.

Proof: Using the integral

\[
\frac{x}{\sqrt{1 + x^2}} = \frac{1}{\pi} \int_1^\infty \frac{s}{\sqrt{s^2 - 1}} x(x^2 + s^2)^{-1} ds.
\]

Thus

\[
g(D) - g(D + S_t)
= \frac{1}{\pi} \int_1^\infty \frac{s}{\sqrt{s^2 - 1}} \left((D + is)^{-1}S_t(D + S_t + is)^{-1}\right) ds
+ \frac{1}{\pi} \int_1^\infty \frac{s}{\sqrt{s^2 - 1}} \left((D - is)^{-1}S_t(D + S_t - is)^{-1}\right) ds.
\]

\(\square\)

In summary, in the $K$-theory of the $C^*$-algebra $C^*(\hat{X})^G$, we have an explicit path connecting

\[
[P_+(D + \alpha S)] - [P_+(D - \alpha S)]
\]

and

\[
[(D + S)(D - S)^{-1}]
\]

to the trivial element respectively.

Now, we study the case that $X, M,$ and $N$ are not compact. The following is the definition of analytically controlled homotopy equivalence, introduced by Higson and Roe in [5].

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Definition 3.5 ([16, Section 4.5]). With the notation as above. And fix a metric on $X$ that agrees with the topology of $X$. Suppose $g : N \to M$ is a homotopy inverse of $f$. Let $\{h_s\}_{0 \leq s \leq 1}$ be a homotopy between $f \circ g$ and $\text{Id} : N \to N$. Similarly, let $\{h'_s\}_{0 \leq s \leq 1}$ be a homotopy between $g \circ f$ and $\text{Id} : M \to M$. Then $f : M \to N$ is analytically controlled homotopy equivalence over $X$ if all maps involved are have finite propagation.

Remark 3.6. Under the circumstances of Eq.(3.1), if we further suppose $G = \pi_1X$ is the fundamental group of $X$ and $\bar{X}$ is the universal cover of $X$. Let $\bar{N} = \psi^*(\bar{X})$ and $\bar{M} = f^*(\bar{N}) = \phi^*(\bar{X})$ be the pull back coverings. Then the lifting map $\tilde{f} : \bar{M} \to \bar{N}$ is automatically analytically controlled homotopy equivalence over $\bar{X}$.

With essentially the same argument for compact case, we have.

Proposition 3.7. With the notation as above. If $f : M \to N$ is an analytically controlled homotopy equivalence over $X$. Then

$$\phi_*(\text{Ind}(D_M)) = \psi_*(\text{Ind}(D_N)) \in K_*(C^*(X)),$$

where, $D_M$ and $D_N$ denote the signature operator on $M$ and $N$ respectively.

Proposition 3.7 has also been obtained by Kaminker-Miller [8], Kasparov [9], Hilsum-Skandalis [7]. However, we emphasize that the proof given in the current section follows form [4] and [5].

4 Index map for $L$-group

In this section we explore the index map for $L$-group. For each element in $L_n(\pi_1 X)$ (c.f. Appendix A or [16, Definition 3.7] for details), using the idea of relative index, we associate to it a $K$-theory class in $K_n(C^*(\bar{X})^G)$. We will show that this map is well defined and admire additivity.

4.1 Relative index

Before giving our construction, we would like to first introduce Lemma 4.1, cited from [7], and Lemma 4.2, which will play central role in our construction.

Lemma 4.1 ([7], Lemma 2.2). Let $\epsilon$ and $K$ be two real numbers such that $4\epsilon K^2 < 1$, $\nabla$ is a closed unbounded adjointable operator on Hilbert space $H$ such that $\text{ran}\nabla \subset \text{dom}\nabla$ and $\|\nabla^2\| \leq \epsilon^2$. Moreover, there exists $x, y$ in $B(H)$ such that $x\nabla + \nabla x = y$, where $y$ is invertible, $x\text{dom}\nabla \subset \text{dom}\nabla$, $\|x\| \leq K$ and $\|y^{-1}\| \leq K$, then $\nabla + \nabla^*$ is invertible.

With Lemma 4.1 in hand, we can show the following Lemma.

Lemma 4.2. For $i = 1, 2$, let $d_i$ be close unbounded operator on Hilbert space $H_i$. If there exist bounded operators $F, G, y, z$, and numbers $\epsilon \geq 0, K \geq 1$ satisfies the following properties:

1. The norm of $F, G, y$ and $z$ are all less than $K$;
2. For $i = 1, 2$, $\text{ran} \ d_i \subset \text{dom} \ d_i$ and $\|d_i^2\| \leq \epsilon^2$;
3. $F(\text{dom} \ d_2) \subset \text{dom} \ d_1$, $G(\text{dom} \ d_1) \subset \text{dom} \ d_2$, and $y(\text{dom} \ d_1) \subset \text{dom} \ d_1$, $z\text{dom} \ d_2 \subset \text{dom} \ d_2$;
is invertible, when \( \alpha \leq \frac{1}{2} K^{-2}(1 + K)^{-2} \) and \( \epsilon \leq \frac{1}{2} \sqrt{\frac{\alpha}{2\alpha + 1}} \).

**Proof:** For

\[
\begin{pmatrix} d_1 & \alpha F \\ 0 & -d_2 \end{pmatrix}, \quad \text{consider} \quad \begin{pmatrix} y & 0 \\ \alpha^{-1}G & -z \end{pmatrix}.
\]

Then we have

\[
\begin{pmatrix} y & 0 \\ \alpha^{-1}G & -z \end{pmatrix} \begin{pmatrix} d_1 & \alpha F \\ 0 & -d_2 \end{pmatrix} = \begin{pmatrix} yd_1 + d_1 y + FG & \alpha(yF - Fz) \\ \alpha^{-1}(Gd_1 - d_2 G) & GF + zd_2 + d_2 z \end{pmatrix}.
\]

If \( \alpha \leq \frac{1}{2} K^{-2}(1 + K)^{-2} \) and \( \epsilon \leq \frac{1}{2} \sqrt{\frac{\alpha}{2\alpha + 1}} \), we have the following estimate

\[
\left\| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} yd_1 + d_1 y + FG & \alpha(yF - Fz) \\ \alpha^{-1}(Gd_1 - d_2 G) & GF + zd_2 + d_2 z \end{pmatrix} \right\| \leq \frac{1}{2}.
\]

Thus

\[
\begin{pmatrix} y & 0 \\ \alpha^{-1}G & -z \end{pmatrix} \begin{pmatrix} d_1 & \alpha F \\ 0 & -d_2 \end{pmatrix} = \begin{pmatrix} yd_1 + d_1 y + FG & \alpha(yF - Fz) \\ \alpha^{-1}(Gd_1 - d_2 G) & GF + zd_2 + d_2 z \end{pmatrix}^{-1}
\]

is invertible, and

\[
\left\| \begin{pmatrix} y & 0 \\ \alpha^{-1}G & -z \end{pmatrix} \begin{pmatrix} d_1 & \alpha F \\ 0 & -d_2 \end{pmatrix} = \begin{pmatrix} yd_1 + d_1 y + FG & \alpha(yF - Fz) \\ \alpha^{-1}(Gd_1 - d_2 G) & GF + zd_2 + d_2 z \end{pmatrix}^{-1} \right\| < 2.
\]

Furthermore, we have

\[
\left\| \begin{pmatrix} y & 0 \\ \alpha^{-1}G & -z \end{pmatrix} \right\| \leq 2K + 2K^3(1 + K)^2.
\]

Then by Lemma 4.1, we see that

\[
\begin{pmatrix} d_1 + d_1^* & \alpha F \\ \alpha F^* & -d_2 - d_2^* \end{pmatrix}
\]

is invertible. \( \square \)

Suppose \( \theta = (M, \partial M, \phi, N, \partial N, \psi, f) \in L_n(\pi_1 X) \). We are now ready to construct the relative signature index of \( \theta \).

We first consider manifolds \( \partial M \times R \) and \( \partial N \times R \). We will construct two bounded operators \( S_{\partial M \times R} \) and \( S_{\partial N \times R} \) in \( B(L^2(\Lambda(\partial M \times R))) \oplus L^2(\Lambda(\partial N \times R)) \).
Note that $\partial f : \partial M \to \partial N$ is a homotopy equivalent. And any form $\omega \in L^2(\Lambda(\partial M \times \mathbb{R})) \oplus L^2(\Lambda(\partial N \times \mathbb{R}))$ can be decomposed as

$$\omega_1 + \omega_2 \wedge dt,$$

where both $\omega_1, \omega_2$ are $L^2$ forms along $\partial M \sqcup \partial N$. Let $(x, t)$ be a point in $\partial M \times \mathbb{R} \sqcup \partial N \times \mathbb{R}$. Let $N_i (i = 0, 1, 2, 3)$ be positive integers, such that $\frac{1}{N_i - N_{i-1}}$ are sufficiently small for $i = 1, 2, 3$. We define $S_{\partial f \times \mathbb{R}}^\prime$ by setting $S_{\partial f \times \mathbb{R}}^\prime\omega(x, t)$ as follows:

1. when $-\infty < t \leq N_0$, set $S_{\partial f \times \mathbb{R}}^\prime\omega(x, t)$ as

$$\left[\left(\begin{array}{cc} 0 & -S_{\partial N} \\ S_{\partial M} & 0 \end{array}\right)\omega_1\right] \wedge dt + \left(\begin{array}{cc} 0 & S_{\partial N} \\ 0 & 0 \end{array}\right)\omega_2\left|_{(x, t)}\right. ;$$

2. when $N_0 < t \leq N_1$, set $S_{\partial f \times \mathbb{R}}^\prime\omega(x, t)$ as

$$\left[\left(\begin{array}{cc} 0 & -S_{\partial N} \\ S_{\partial M} & 0 \end{array}\right)\omega_1\right] \wedge dt + \left(\begin{array}{cc} 0 & S_{\partial N} \\ 0 & 0 \end{array}\right)\omega_2\left|_{(x, t)}\right. ;$$

3. when $N_1 < t \leq N_2$, set $S_{\partial f \times \mathbb{R}}^\prime\omega(x, t)$ as

$$\left[\left(\begin{array}{cc} \cos((\frac{t-N_1}{N_2-N_1} \pi) S_{\partial M} & \sin((\frac{t-N_1}{N_2-N_1} \pi) S_{\partial M} T_{\partial f} \\ \sin((\frac{t-N_1}{N_2-N_1} \pi) T_{\partial f} S_{\partial M} & -\cos((\frac{t-N_1}{N_2-N_1} \pi) T_{\partial f} S_{\partial M} T_{\partial f}\right)\omega_1\right] \wedge dt + \left(\begin{array}{cc} \cos((\frac{t-N_1}{N_2-N_1} \pi) S_{\partial M} & \sin((\frac{t-N_1}{N_2-N_1} \pi) S_{\partial M} T_{\partial f} \\ \sin((\frac{t-N_1}{N_2-N_1} \pi) T_{\partial f} S_{\partial M} & -\cos((\frac{t-N_1}{N_2-N_1} \pi) T_{\partial f} S_{\partial M} T_{\partial f}\right)\omega_2\left|_{(x, t)}\right. ;$$

4. when $N_2 < t < +\infty$, set $S_{\partial f \times \mathbb{R}}^\prime\omega(x, t)$ as

$$\left[\left(\begin{array}{cc} 0 & S_{\partial M} T_{\partial f} \\ T_{\partial f} S_{\partial M} & 0 \end{array}\right)\omega_1\right] \wedge dt + \left(\begin{array}{cc} 0 & S_{\partial M} T_{\partial f} \\ T_{\partial f} S_{\partial M} & 0 \end{array}\right)\omega_2\left|_{(x, t)}\right. .$$

In the meanwhile, we define $S_{\partial f \times \mathbb{R}}$ by setting $S_{\partial f \times \mathbb{R}}\omega(x, t)$ as follows:

1. when $-\infty < t \leq N_2$, set $S_{\partial f \times \mathbb{R}}\omega(x, t)$ as $S_{\partial f \times \mathbb{R}}^\prime\omega(x, t)$;

2. when $N_2 < t \leq N_3$, set $S_{\partial f \times \mathbb{R}}\omega(x, t)$ as

$$\left[\left(\begin{array}{cc} 0 & e^{i \frac{t-N_2}{N_3-N_2} \pi} S_{\partial M} T_{\partial f} \\ e^{-i \frac{t-N_2}{N_3-N_2} \pi} T_{\partial f} S_{\partial M} & 0 \end{array}\right)\omega_1\right] \wedge dt + \left(\begin{array}{cc} 0 & e^{i \frac{t-N_2}{N_3-N_2} \pi} S_{\partial M} T_{\partial f} \\ e^{-i \frac{t-N_2}{N_3-N_2} \pi} T_{\partial f} S_{\partial M} & 0 \end{array}\right)\omega_2\left|_{(x, t)}\right. ;$$

3. when $N_3 < t < +\infty$, set $S_{\partial f \times \mathbb{R}}\omega(x, t)$ as

$$\left[\left(\begin{array}{cc} 0 & -S_{\partial M} T_{\partial f} \\ -T_{\partial f} S_{\partial M} & 0 \end{array}\right)\omega_1\right] \wedge dt + \left(\begin{array}{cc} 0 & -S_{\partial M} T_{\partial f} \\ -T_{\partial f} S_{\partial M} & 0 \end{array}\right)\omega_2\left|_{(x, t)}\right. .$$
Remark 4.3.  1. In the above construction, positive integers $N_i$ ($i = 0, 1, 2, 3$) can be chosen arbitrarily.

2. We can replace all the functions parameterize by $t$ in the above construction by smooth functions.

3. Without loss of generality, we can assume $S_{\partial f \times \mathbb{R}}'$ and $S_{\partial f \times \mathbb{R}}$ are smoothly defined.

Let $M_\infty$ and $N_\infty$ be the manifolds with cylindrical ends associated to $M$ and $N$, i.e. $M_\infty = M \sqcup_{\partial M} \partial M \times [1, \infty)$ and $N_\infty = N \sqcup_{\partial N} \partial N \times [1, \infty)$.

We define operators $S_f'$ and $S_f$ on $L^2(\Lambda(M_\infty)) \bigoplus L^2(\Lambda(N_\infty))$ as

$$S_f' = \begin{pmatrix} S_{M_\infty} & 0 \\ 0 & -S_{N_\infty} \end{pmatrix} \begin{pmatrix} 1 - \chi_{\partial M \times [N_0, +\infty)} & 0 \\ 0 & 1 - \chi_{\partial N \times [N_0, +\infty)} \end{pmatrix}$$

$$S_f = \begin{pmatrix} S_{M_\infty} & 0 \\ 0 & -S_{N_\infty} \end{pmatrix} \begin{pmatrix} 1 - \chi_{\partial M \times [N_0, +\infty)} & 0 \\ 0 & 1 - \chi_{\partial N \times [N_0, +\infty)} \end{pmatrix} + S_{\partial f \times \mathbb{R}} \begin{pmatrix} \chi_{\partial M \times [N_0, +\infty)} & 0 \\ 0 & \chi_{\partial N \times [N_0, +\infty)} \end{pmatrix};$$

It is not hard to prove that $S_f$ and $S_f'$ are bounded operators on Hilbert space $L^2(\Lambda(M_\infty)) \bigoplus L^2(\Lambda(N_\infty))$.

Now, suppose $\theta = (M, \partial M, \phi, N, \partial N, \psi, f) \in L_n(\pi_1 X)$. Let $G = \pi_1 X$ be the fundamental group of $X$ and let $\tilde{X}$ be a $G$-covering space of $X$, where $\tilde{X}$ is the universal covering of $X$. Let $M_\infty$ and $N_\infty$ be the manifolds with cylindrical ends associated to $M$ and $N$, i.e. $M_\infty = M \sqcup_{\partial M} \partial M \times [1, \infty)$ and $N_\infty = N \sqcup_{\partial N} \partial N \times [1, \infty)$. Denote by $\tilde{M}_\infty$ (resp. $\tilde{N}_\infty$) the corresponding $G$-covering of $\Phi : M_\infty \to X \times [1, \infty)$ (resp. $\Psi : N_\infty \to X \times [1, \infty)$).

With these notations, we have $G$-equivariant versions of $S_f$ and $S_f'$, which are bounded operators on Hilbert space $L^2(\Lambda(\tilde{M}_\infty)) \bigoplus L^2(\Lambda(\tilde{N}_\infty))$. We will denote by $\tilde{S}_f$ and $\tilde{S}_f'$ as well.

The following Proposition is a corollary of Lemma 4.2.

**Proposition 4.4.** With the notation as above, for some proper chosen $\alpha > 0$, we have

$$\begin{pmatrix} d_{\tilde{M}_\infty} + d_{\tilde{M}_\infty}^* & 0 \\ 0 & d_{\tilde{N}_\infty} + d_{\tilde{N}_\infty}^* \end{pmatrix} \begin{pmatrix} \alpha S_f' \\ \alpha S_f \end{pmatrix},$$

and

$$\begin{pmatrix} d_{\tilde{M}_\infty} + d_{\tilde{M}_\infty}^* & 0 \\ 0 & d_{\tilde{N}_\infty} + d_{\tilde{N}_\infty}^* \end{pmatrix} \begin{pmatrix} \alpha S_f' \\ \alpha S_f \end{pmatrix},$$

are invertible.
Proof: We will prove the \( S_f \) case only. In our argument, \( S_f \) takes the role of \( F \) in Lemma 4.2. Thanks to Lemma 4.2, it is sufficient to show the existence of \( G, y, z \) and \( K \geq 1 \) which satisfy the conditions in Lemma 4.2.

We first prove the existence of \( G \). Recall that the restricting of \( f \) on the boundary \( \partial f : \partial M \to \partial N \) is homotopy equivalence. Suppose \( \partial g : \partial N \to \partial M \) to be the homotopy inverse of \( \partial f \). Similarly, we can define an operator \( S_{\partial g \times \mathbb{R}} \) in \( B(L^2(\Lambda(\partial M \times \mathbb{R})) \oplus L^2(\Lambda(\partial N \times \mathbb{R}))) \) as follows:

1. when \(-\infty < t \leq N_0 \), set \( S_{\partial g \times \mathbb{R}}^t \omega \mid_{(x,t)} \) as

\[
\left[ \left[ \left( \begin{array}{c}
S_{\partial M} \\
0
\end{array} \right) \omega_1 \right] \wedge dt + \left( \begin{array}{c}
0 \\
n_{\partial N}
\end{array} \right) \omega_2 \right] \mid_{(x,t)}.
\]

2. when \( N_0 < t \leq N_1 \), set \( S_{\partial g \times \mathbb{R}}^t \omega \mid_{(x,t)} \) as

\[
\left[ \begin{array}{c}
S_{\partial M} \\
n_{\partial N}
\end{array} \right] \omega_1 \wedge dt + \left( \begin{array}{c}
0 \\
n_{\partial N}
\end{array} \right) \omega_2 \mid_{(x,t)}.
\]

3. when \( N_1 < t \leq N_2 \), set \( S_{\partial g \times \mathbb{R}}^t \omega \mid_{(x,t)} \) as

\[
\left[ \begin{array}{c}
cos((t/N_2-N_1)/2)) S_{\partial M} \\
\sin((t/N_2-N_1)/2)) T_{\partial g} S_{\partial M}
\end{array} \right] \omega_1 \wedge dt + \left( \begin{array}{c}
\sin((t/N_2-N_1)/2)) S_{\partial M} \\
\cos((t/N_2-N_1)/2)) T_{\partial g} S_{\partial M}
\end{array} \right) \omega_2 \mid_{(x,t)}.
\]

4. when \( N_2 < t < +\infty \), set \( S_{\partial g \times \mathbb{R}}^t \omega \mid_{(x,t)} \) as

\[
\left[ \begin{array}{c}
0 \\
T_{\partial g} S_{\partial M} T_{\partial g}^* \\
0
\end{array} \right] \omega_1 \wedge dt + \left( \begin{array}{c}
0 \\
T_{\partial g} S_{\partial M} T_{\partial g}^* \\
0
\end{array} \right) \omega_2 \mid_{(x,t)}.
\]

Then we define an operator \( S_g \) in \( B(L^2(\Lambda(\tilde{M}_\infty))) \oplus L^2(\Lambda(\tilde{N}_\infty))) \) as:

\[
S_g = \left( \begin{array}{c}
S_{\tilde{M}_\infty} \\
-\chi_{\tilde{N}_\infty}
\end{array} \right) \left( 1 - \chi_{\partial M \times [N_0, +\infty]} \right) \left( 1 - \chi_{\partial N \times [N_0, +\infty]} \right) + S_{\partial g \times \mathbb{R}} \left( \begin{array}{c}
\chi_{\partial M \times [N_0, +\infty]} \\
\chi_{\partial N \times [N_0, +\infty]}
\end{array} \right).
\]

By choosing the smooth functions in Remark 4.3 so that their derivative functions all have small supremum-norm, say \( \varepsilon \), we have

\[
||d_{\infty} S_f + S_f d_{\infty}^*|| \leq \varepsilon;
\]

and

\[
|| - d_{\infty}^* S_g - S_g d_{\infty} || \leq \varepsilon,
\]

where \( d_{\infty} = \left( \begin{array}{c}
d_{\tilde{M}_\infty} \\
d_{\tilde{N}_\infty}
\end{array} \right) \).
We now prove the existence of \( y \) and \( z \). Again we define \( y \) by setting \( y(w)|_{(x,t)} \). We will show the definition of \( y(w)|_{(x,t)}, N_0 \leq t \leq N_1 \) in details only. The definition of \( y(w)|_{(x,t)} \) for other \( t \) are similar.

We consider \( \partial M \times \mathbb{R} \sqcup \partial \tilde{N} \times \mathbb{R} \) first. As shown in Lemma 3.2, \( \partial \tilde{N} \times \mathbb{R} \) is chain homotopy equivalent to \( T_{\alpha}^* \). By \( \partial \alpha \), it is easy to see that bounded operators \( T_{\alpha}^* \partial \tilde{N} \times \mathbb{R} \) and \( \partial \tilde{N} \times \mathbb{R} \alpha \) are all chain homotopy equivalent to \( 1 \). Let \( y_1, y_2 \) and \( y_3 \) be the connecting map respectively.

Then, for \( N_0 \leq t \leq N_1 \), we can define \( y(w)|_{(x,t)} \) by

\[
\left( \frac{(t-N_0)(N_1-t)}{(N_1-N_0)^2} y_1 + \frac{(t-N_0)(N_1-t)}{(N_1-N_0)^2} y_2 + \left( \frac{t-N_0}{N_1-N_0} \right)^2 y_3 \right) (w)_{(x,t)}.
\]

With a little modification, we can see that in \( B(L^2(\Lambda(\tilde{M}_\infty))) \bigoplus L^2(\Lambda(\tilde{N}_\infty))) \) there exist bounded operators \( y \) and \( z \) satisfying

\[
\| 1 - S_f \|_{\infty} - d_{\infty} y - y d_{\infty}^* \| \leq C \epsilon,
\]

and

\[
\| 1 - S_g \|_{\infty} - d_{\infty} z - z d_{\infty}^* \| \leq C \epsilon,
\]

where \( C \) is a constant. Now it is easy to see, choosing the smooth functions in Remark 4.3 so that their derivative functions all have small supremum-norm, \( G, y, z \) will satisfying the conditions in Lemma 4.2. On the other hand, we can choose constant \( K = \max\{ \| S_f \|, \| S_g \|, \| y \|, \| z \| \} \). Rather, change supremum-norm of the derivative functions do not increase \( K \). In summary, we have

\[
\begin{pmatrix}
\begin{pmatrix} d_{M_\infty}^* + d_{N_\infty}^* & 0 \\
0 & d_{N_\infty}^* + d_{N_\infty}^*
\end{pmatrix} & \alpha S_f \\
\alpha S_f & \begin{pmatrix} d_{M_\infty}^* + d_{M_\infty}^* & 0 \\
0 & d_{N_\infty}^* + d_{N_\infty}^*
\end{pmatrix}
\end{pmatrix}
\]

is invertible for some properly chosen \( \alpha \).

Corollary 4.5. For some proper chosen \( \alpha > 0 \), bounded operators

\[
\begin{pmatrix} d_{M_\infty}^* + d_{M_\infty}^* & 0 \\
0 & d_{N_\infty}^* + d_{N_\infty}^*
\end{pmatrix} \pm \alpha S_f',
\]

and

\[
\begin{pmatrix} d_{M_\infty}^* + d_{M_\infty}^* & 0 \\
0 & d_{N_\infty}^* + d_{N_\infty}^*
\end{pmatrix} \pm \alpha S_f
\]

are invertible.

Proof: Restricting

\[
\begin{pmatrix}
\begin{pmatrix} d_{M_\infty}^* + d_{M_\infty}^* & 0 \\
0 & d_{N_\infty}^* + d_{N_\infty}^*
\end{pmatrix} & \alpha S_f' \\
\alpha S_f & \begin{pmatrix} d_{M_\infty}^* + d_{M_\infty}^* & 0 \\
0 & d_{N_\infty}^* + d_{N_\infty}^*
\end{pmatrix}
\end{pmatrix}
\]

would give
to the subspace
\[ \bigvee \left\{ \begin{pmatrix} v \\ \overline{v} \end{pmatrix} \mid v \in L^2(\Lambda(\tilde{M}_\infty)) \oplus L^2(\Lambda(\tilde{N}_\infty)) \right\} \]
we have \( \begin{pmatrix} d_{\tilde{M}_\infty} + d^*_M & 0 \\ 0 & d_{\tilde{N}_\infty} + d^*_N \end{pmatrix} + \alpha S'_f \) is invertible.

And, restricting
\[ \begin{pmatrix} d_{\tilde{M}_\infty} + d^*_M & 0 \\ 0 & d_{\tilde{N}_\infty} + d^*_N \end{pmatrix} \alpha S'_f \]
to the subspace
\[ \bigvee \left\{ \begin{pmatrix} v \\ -\overline{v} \end{pmatrix} \mid v \in L^2(\Lambda(\tilde{M}_\infty)) \oplus L^2(\Lambda(\tilde{N}_\infty)) \right\} , \]
we have \( \begin{pmatrix} d_{\tilde{M}_\infty} + d^*_M & 0 \\ 0 & d_{\tilde{N}_\infty} + d^*_N \end{pmatrix} + \alpha S'_f \) is invertible.

Using the same arguments we can get
\[ \begin{pmatrix} d_{\tilde{M}_\infty} + d^*_M & 0 \\ 0 & d_{\tilde{N}_\infty} + d^*_N \end{pmatrix} \pm \alpha S_f \]
are invertible. \( \square \)

We denote
\[ \begin{pmatrix} d_{\tilde{M}_\infty} + d^*_M & 0 \\ 0 & d_{\tilde{N}_\infty} + d^*_N \end{pmatrix} \]
by \( D \). Following subsection 3.2. For \( n \) even, we consider the following well defined operator
\[ P_+(D + \alpha S_f) \quad \text{and} \quad P_+(D - \alpha S'_f) , \]
while \( \alpha \) is well chosen. For \( n \) odd, we consider the following invertible operator
\[ (D + \alpha S_f)(D - \alpha S'_f)^{-1} \]
restricts to even forms, while \( \alpha \) is well chosen. Now, our main task is to show that \([P_+(D + \alpha S_f)] - [P_+(D - \alpha S'_f)]\) and \((D + \alpha S_f)(D - \alpha S'_f)^{-1}\) actually defines an element in \( K_0(C^*(\tilde{X})^G) \) and \( K_1(C^*(\tilde{X})^G) \) respectively.

For \( n \) even, let \( \rho : M_\infty \sqcup N_\infty \to [0, 1] \) a smooth function which is constant 1 for \( x \leq N_3 \) and 0 when \( x \) is greater than \( N_3 + 1 \), constant along \( \partial M \sqcup \partial N \) and \( \rho' \) has
small supremum-norm. Then as in the proof of Lemma 3.4, we have

\[(D + \alpha S_f)\left(1 + (D + \alpha S_f)^2\right)^{-1/2} - (D - \alpha S'_f)\left(1 + (D - \alpha S'_f)^2\right)^{-1/2}\]

\[= \frac{1}{\pi} \int_{\pi}^{\infty} \frac{s}{\sqrt{s^2 - 1}} ((D + \alpha S_f + is)^{-1}(S'_f + S_f)\rho(D - \alpha S_f + is)^{-1}) \, ds\]

\[+ \frac{1}{\pi} \int_{\pi}^{\infty} \frac{s}{\sqrt{s^2 - 1}} ((D + \alpha S_f - is)^{-1}(S'_f + S_f)\rho(D - \alpha S_f - is)^{-1}) \, ds\]

\[= \frac{1}{\pi} \int_{\pi}^{\infty} \frac{s}{\sqrt{s^2 - 1}} ((D + \alpha S_f + is)^{-1}(S'_f + S_f)(D - \alpha S_f + is)^{-1}) \, ds \rho\]

\[+ \frac{1}{\pi} \int_{\pi}^{\infty} \frac{s}{\sqrt{s^2 - 1}} ((D + \alpha S_f - is)^{-1}(S'_f + S_f)(D - \alpha S_f - is)^{-1}) \, ds \rho + \epsilon\]

where \(\epsilon\) is a constant can be represent as following integral

\[\frac{1}{\pi} \int_{\pi}^{\infty} \frac{s}{\sqrt{s^2 - 1}} ((D + \alpha S_f + is)^{-1}(S'_f + S_f)(D - \alpha S_f + is)^{-1}[D, \rho]) \, ds\]

\[\frac{1}{\pi} \int_{\pi}^{\infty} \frac{s}{\sqrt{s^2 - 1}} ((D + \alpha S_f - is)^{-1}(S'_f + S_f)(D - \alpha S_f - is)^{-1}[D, \rho]) \, ds\]

from the above formula we see that \(\epsilon\) “only” dependents on \([D, \rho]\), so we can choose rho such that \(\epsilon\) is sufficiently small.

Above computation tell us that

\[[P_+(D + \alpha S_f)] - [P_+(D - \alpha S'_f)]\]

defines an class in \(K_0(C^*(\tilde{X} \times [1, N_3+1])^G),\) which is isomorphic to \(K_0(C^*(\tilde{X})^G)\). We still denote this \(K\)-theory class in \(K_0(C^*(\tilde{X})^G)\) by \([P_+(D + \alpha S_f)] - [P_+(D - \alpha S'_f)],\) and we think this abuse of notation would not cause any confusion.

The odd case are similar while simpler, in this case we consider the invertible operator

\[(D + \alpha S_f)(D - \alpha S'_f)^{-1}\]

from \(L^2(\Lambda^{even}(\tilde{M}_\infty) \oplus \Lambda^{even}(\tilde{N}_\infty))\) to \(L^2(\Lambda^{even}(\tilde{M}_\infty) \oplus \Lambda^{even}(\tilde{N}_\infty)).\)

In fact, let \(N_5 \geq N_4\) are two integers bigger than \(N_3.\) Let \(\rho : M_\infty \cup N_\infty \rightarrow [0, 1]\) a smooth function which is constant 1 for \(x \leq N_4\) and 0 when \(x\) is greater than \(N_5,\) always constant along \(\partial M \cup \partial N\) and \(\rho\) has small supremum-norm. Then we have

\[(D + \alpha S_f)(D - \alpha S'_f)^{-1} = 1 + \alpha(S'_f + S_f)(D - \alpha S'_f)^{-1}\rho,\]

since \(\alpha(S'_f + S_f)(D - \alpha S'_f)^{-1}\) belongs to \(C^*(\tilde{X} \times [1, \infty))^G,\) we can approximate it by an invertible element \(a\) in \(C^*(\tilde{X} \times [1, \infty))\) with finite propagation, we may suppose the propagation of \(a\) is bounded by \(p.\) Then obviously \(1 + a\rho\) defines an element in \(K_1(C^*(\tilde{X} \times [1, N_5 + p])^G).\) Thus natural isomorphism of \(K\)-groups

\(K_1(C^*(\tilde{X} \times [1, N]))^G \cong K_1(C^*(\tilde{X})^G)\)

give us an element in \(K_1(C^*(\tilde{X})^G).\) We also think it arise no confusion to omit \(a.\)

We denote the \(K\)-theory classes defined above in \(K_n(C^*(\tilde{X})^G)\) by \(\text{Ind} (\theta).\) Thus, we have a map \(\text{Ind} : L_n(\pi_1 X) \rightarrow K_n(C^*(\tilde{X})^G).\)

We conclude this subsection by the following simple consequence.

**Theorem 4.6.** Let \(\theta = (M, \partial M, \phi, N, \partial N, \psi, f) \in L_n(\pi_1 X)\) and suppose \(f : M \rightarrow N\) is a homotopy equivalent (not only restricting to the boundary). Then \(\text{Ind}(\theta)\) is trivial.
4.2 Additivity

In this subsection, we will continue our study of the index map \( \text{Ind} : L_n(\pi_1 X) \rightarrow K_n(C^*(X)) \). The theorem below plays a central role in our argument.

**Theorem 4.7.** Let \( \theta, \theta' \in L_n(\pi_1 X) \). We further suppose \( \theta = (M, \partial M, \phi, N, \partial N, \psi, f) \), \( \theta' = (M', \partial M', \phi', N', \partial N', \psi', f') \) and satisfies the following properties:

1. \( \partial M = \partial M' \), \( \partial N = \partial N' \), restricting to the boundary \( f = f' \), \( \phi = \phi' \) and \( \psi = \psi' \);
2. There exist two manifolds with boundary \( (W, \partial W) \) and \( (V, \partial V) \), continuous maps \( \Phi : W \rightarrow X \) (resp. \( \Psi : V \rightarrow X \)). Manifolds \( W \) and \( V \) both of dimension \( n + 1 \). Moreover, \( \partial W = M \sqcup_{\partial M} M' \) (resp. \( \partial V = N \sqcup_{\partial N} N' \)) and \( \Phi \) (resp. \( \Psi \)) restricts to \( \phi \sqcup \phi' \) (resp. \( \psi \sqcup \psi' \)) on \( M \sqcup_{\partial M} M' \) (resp. \( N \sqcup_{\partial N} N' \));
3. There exist a degree one normal map \( F : W \rightarrow V \) such that \( \Psi \circ F = \Phi \). Moreover, \( F \) restricts to \( f \sqcup f' \) on \( M \sqcup_{\partial M} M' \).

Then we have

\[
\text{Ind}(\theta) - \text{Ind}(\theta') = \text{Ind}(\theta \sqcup \theta') - \text{Ind}(\theta' \sqcup \theta').
\]  

**Proof:** In the following we go through details for odd case only. The even case are totally the same.

For real numbers \( T \) and \( T_1 \leq T_2 \), we denote \( M_T := M \sqcup_{\partial M} \partial M \times [0, T] \), \( M_{T_1} := \partial M \times [T, \infty) \), and \( M_{[T_1, T_2]} := \partial M \times [T_1, T_2] \). Similar, for \( M', N \) and \( N' \), we introduce the same notation.

Take a real number \( R \), we construct \( M_R \sqcup_{\partial M} M'_R \) by pasting along \( \partial M \times \{R\} \subset M \sqcup_{\partial M} \partial M \times [0, R] \) and \( \partial M' \times \{R\} \subset M' \sqcup_{\partial M'} \partial M' \times [0, R] \). Similar, for \( N' \), we introduce the same notation. Without loss of generality, we can identify \( M \sqcup_{\partial M} M' \) with \( M_R \sqcup_{\partial M \times R} M'_R \), and identify \( N \sqcup_{\partial N} N' \) with \( N_R \sqcup_{\partial N \times R} N'_R \).

For real number \( R \geq K \). We will introduce several Hilbert space

1. \( H_1 \) stands for \( L^2(\Lambda(M_K \sqcup N_K)) \) and \( H'_1 \) stands for \( L^2(\Lambda(M'_K \sqcup N'_K)) \);
2. \( H_2 \) stands for \( L^2(\Lambda(M_{[K,R]} \sqcup N_{[K,R]})) \) as well as \( L^2(\Lambda(M'_{[K,R]} \sqcup N'_{[K,R]})) \);
3. \( H_3 \) stands for \( L^2(\Lambda(M_{R} \sqcup N_{R})) \) as well as \( L^2(\Lambda(M'_{R} \sqcup N'_{R})) \).

With the above notation we have

1. \( L^2(\Lambda(M_\infty \sqcup N_{\infty})) = H_1 \oplus H_2 \oplus H_3 \);
2. \( L^2(\Lambda(M'_\infty \sqcup N'_{\infty})) = H'_1 \oplus H_2 \oplus H_3 \);
3. \( L^2(\Lambda((M_R \sqcup_{\partial M} M'_R) \sqcup (N_R \sqcup_{\partial N} N'_R))) = H_1 \oplus H_2 \oplus H_2 \oplus H'_1 \);
4. \( L^2(\Lambda((M'_R \sqcup_{\partial M'} M_R) \sqcup (N'_R \sqcup_{\partial N'} N_R))) = H'_1 \oplus H_2 \oplus H_2 \oplus H'_1 \).

Choose a represent of the \( K \)-theory class \( \text{Ind}(\theta) \), denote it by \( a \), with finite propagation. Moreover, by section 5.2 we can additional suppose \( [1 - a] \) is trivial on \( \tilde{X} \times [K, \infty) \). Similarly, for \( \text{Ind}(\theta') \) we have \( a' \).

Define operators on \( H_1 \oplus H_2 \oplus H_2 \oplus H'_1 \) and \( H'_1 \oplus H_2 \oplus H_2 \oplus H'_1 \) by

\[
A = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d' & 0 \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \\ H_2 \\ H'_1 \end{pmatrix}
\text{ and } A' = \begin{pmatrix} a' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a' \end{pmatrix} \begin{pmatrix} H'_1 \\ H_2 \\ H_2 \\ H'_1 \end{pmatrix}.
\]
Then it is obviously we have that
\[ \text{Ind}(\theta) - \text{Ind}(\theta') = [A] - [A'] . \]

However, we claim that
\[ [A] = \text{Ind}(\theta \sqcup \theta') . \]

Let \( R' = 2(R - K) \). In fact, we can assume that \( R \geq 2K \) and \( K \) is big enough, such that there is almost flat function \( \phi \) and \( \phi' \) satisfies
\[
1. \ \phi \text{ is constantly 1 on } M_K \sqcup M'_K, \text{ constantly zero on } M_{R'} \sqcup M'_{R'} ; \\
2. \ \phi' \text{ is constantly 1 on } N_K \sqcup N'_K, \text{ constantly zero on } N_{R'} \sqcup N'_{R'} ; \\
3. \ \text{Both } \phi \text{ and } \phi' \text{ are constant along boundary direction; } \\
4. \ \text{We have } \phi + \phi' = 1 \text{ on } H_1 \oplus H_2 \oplus H_2 \oplus H'_1 .
\]

Thus we have
\[
A = \begin{pmatrix}
    a & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
\end{pmatrix} \phi + \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & a' \\
\end{pmatrix} \phi' \\
= \sqrt{\phi} \begin{pmatrix}
    a & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
\end{pmatrix} \sqrt{\phi} + \sqrt{\phi'} \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & a' \\
\end{pmatrix} \sqrt{\phi'} + \epsilon
\]

here \( \epsilon \) stands for an operator with sufficient small norm. Actually, we can substitute the following matrixes
\[
\begin{pmatrix}
    a & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
\end{pmatrix} \text{ and } \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & a' \\
\end{pmatrix}
\]

by \( \tilde{a} \in B(L^2(M_\infty \sqcup N_\infty)) \) and \( \tilde{a}' \in B(L^2(M'_\infty \sqcup N'_\infty)) \) respectively.

Since
\[
\|(D + S_f)(D - S'_f)^{-1} - \tilde{a}\| \leq \eta, \ |(D' + S'_f)(D' - S'_f)^{-1} - \tilde{a}'| \leq \eta .
\]

Hence, we have that \([1 - A]\) and \( \sqrt{\phi}(D + S_f)(D - S'_f)^{-1}\sqrt{\phi} + \sqrt{\phi'}(D' + S'_f)(D - S'_f)^{-1}\sqrt{\phi'} \) give the same K-theory class.

Let \( D_\sqcup \) be the signature operator on \((M \sqcup M') \sqcup (N \sqcup N')\).

We can define \( S_{\sqcup,f} \) and \( S'_{\sqcup,f} \) as
\[
S_{\sqcup,f} = S_f \sqcup S'_f \text{ and } S'_{\sqcup,f} = S'_f \sqcup S'_f
\]

Now one can see that
\[
\sqrt{\phi}(D_\sqcup + S_{\sqcup,f})(D_\sqcup - S'_{\sqcup,f})^{-1}\sqrt{\phi} + \sqrt{\phi'}(D_\sqcup + S_{\sqcup,f})(D_\sqcup - S'_{\sqcup,f})^{-1}\sqrt{\phi'} \\
= (D + S_f)(D - S'_f)^{-1}\phi + (D' + S'_f)(D' - S'_f)^{-1}\phi' + \epsilon .
\]
where \( \epsilon \) is an operator with sufficiently small norm. Thus we have completed the proof. \( \square \)

Now, to show that map \( \text{Ind} : L_n(\pi_1 X) \to K_n(C^*(\tilde{X})^G) \) is well defined. It remains only to show that higher signature is bordism invariant. This is implied by the fact that \( K \)-homology class of the signature operator is a bordism invariant, which can be proven by relative \( K \)-homology exact sequence.

We briefly recite the argument here, for details the reader are referred to [1, 2, 3] and [10]. Let \( (M, \partial M) \) be a manifold with boundary and \( \phi : (M, \partial M) \to X \) be a continuous map. If the dimension of \( M \) is odd, one can show that the signature operator \( D_M \) defines a class, \( [D_M] \), in \( K_0(M, \partial M) \), the relative \( K \)-homology, and there is exact sequence

\[
K_1(M, \partial M) \xrightarrow{\partial} K_0(\partial M) \to K_0(M).
\]

Note that \( \partial[D_M] = [D_{\partial M}] \in K_0(\partial M) \), thus \( [D_{\partial M}] \) is zero in \( K_0(M) \), and consequently, is zero in \( K_0(X) \). For the \( G \)-equivariant case, the assemble map are factored through

\[
K^G_0(\partial \tilde{M}) \xrightarrow{\partial} K^G_0(\tilde{M}) \to K^G_0(\tilde{X}) \to K_0(C^*(\tilde{X})^G).
\]

For even case, use the same argument, we have

\[
\partial[D_M] = 2[D_{\partial M}] \in K_1(\partial M).
\]

Thus we can only show that twice the signature operator class and higher signature are bordism invariant.

Let \( \theta \) be zero element in \( L_n(\pi_1 X) \), and \( \theta' \) be an object from \( L_n(\pi_1 X) \) which is equivalent to \( \theta \) (Definition A.4). By Theorem 4.6, the second item in the left side of Eq.(4.1) is trivial in \( K_n(C^*(\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}] \). However, since higher signature is a bordism invariant, two items in right side of Eq.(4.1) are both trivial in \( K_n(C^*(\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}] \). Eq.(4.1) then says that

\[
\text{Ind} : L_n(\pi_1 X) \to K_n(C^*(\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}],
\]

maps zero element to zero element. Thus Ind is a well defined group homomorphism.

**Remark 4.8.** In a word, we have to consider \( \mathbb{Z}[\frac{1}{2}] \) coefficient since the “boundary of signature operator” is not signature, but twice the signature when the manifold is even dimensional.

### 4.3 A product formula

To conclude this section, we present a product formula. The proof is postpone to appendix C. As our proof is essentially the same as [16, Appendix D], and we will go through details for even case only.

If \( \theta = (M, \partial M, \phi, N, \partial N, \psi, f) \in L_n(\pi_1 X) \), let \( \theta \times \mathbb{R} \) be the product of \( \theta \) and \( \mathbb{R} \), which defines an element in \( L_{n+1}(\pi_1 X) \). Here various undefined terms take the obvious meanings [16].

Note that the construction in previous subsection also applies to \( \theta \times \mathbb{R} \) and defines a \( K \)-theory class in \( K_{n+1}(C^*(\tilde{X} \times \mathbb{R})^G) \). And there is a natural homomorphism \( \alpha : C^*(\tilde{X})^G \otimes C^L(\mathbb{R}) \to C^*(\tilde{X} \times \mathbb{R})^G \).
Proposition 4.9. With the same notation as above, we have

\[ k_n \cdot \alpha_\ast (\text{Ind}(\theta) \otimes \text{Ind}_L(\mathbb{R})) = \text{Ind}(\theta \times \mathbb{R}) \]

in \( K_{n+1}(C^*(\tilde{X} \times \mathbb{R})^G) \), where \( \text{Ind}_L(\mathbb{R}) \) is the \( K \)-homology class of the signature operator on \( \mathbb{R} \) and \( k_n = 1 \) if \( n \) is even and \( 2 \) if \( n \) is odd.

Remark 4.10. We would like to remind readers that throughout this paper \( \text{Ind}_L(\mathbb{R}) \) is the \( K \)-homology class of the signature operator on \( \mathbb{R} \). Do NOT confuse with the map \( \text{Ind}_L \) which we will defined in subsection 5.2.

5 Local index map for \( \mathcal{N}_n(X) \)

In this section we address how to define the local index map \( \text{Ind}_L : \mathcal{N}_n(X) \to K_n(C^*_L(\tilde{X})^G) \) and show it is well defined and admire additivity.

Before we started, we need a fair idea of a certain hybrid \( C^* \)-algebra, which introduced in [16, Section 4.1]. For the convenience of readers, we briefly recall some facts about hybrid \( C^* \)-algebras in subsection 5.1.

In subsection 5.2, using infinitesimal controlled homotopy equivalence, we reduce the definition of local index map to the index map which we have carefully studied in section 4.

5.1 \( CX \) and hybrid \( C^* \)-algebra

Suppose \( G \) is a countable discrete group. Let \( X \) be proper metric space equipped with a proper \( G \)-action.

Definition 5.1 ([16, Definition 4.1]). We define \( C^*_c(X)^G \) to be the closed subalgebra of \( C^*(X)^G \) generated by all elements \( \alpha \) such that: for any \( \epsilon > 0 \), there exists \( G \)-invariant cocompact set \( K \subseteq Y \) such that the propagation of both \( \chi_{(X-K)} \alpha \) and \( \alpha \chi_{(X-K)} \) are less than \( \epsilon \).

Definition 5.2 ([16, Definition 4.2]). We define \( C^*_L(X)^G \) to be the closed subalgebra of \( C^*_L(X)^G \) generated by all elements \( \alpha \) such that: for any \( \epsilon > 0 \), there exists \( G \)-invariant cocompact set \( K \subseteq Y \) such that the propagation of both \( \chi_{(X-K)} \alpha \) and \( \alpha \chi_{(X-K)} \) are less than \( \epsilon \).

Let \( X \times [1, \infty) \) be the product space. Let \( \tilde{X} \) be the universal cover of \( X \) and let \( G = \pi_1X \) be the fundamental group of \( X \). It is obvious that for any \( r \geq 1 \), the \( C^* \) algebra \( C^*_L(\tilde{X} \times [1, r]; \tilde{X} \times [1, \infty)))^G \) is a two sided ideal of \( C^*_L(\tilde{X} \times [1, \infty))^G \).

Proposition 5.3. For \( i = 0, 1 \), we have

1. \( K_i(C^*_L(\tilde{X} \times [1, \infty))^G) \cong K_i(C^*_L(\tilde{X} \times [1, \infty))^G) \) [16, Proposition 4.4];
2. \( K_1(C^*_L(\tilde{X} \times [1, \infty))^G) = 0 \) [16, Lemma 4.6];
3. \( K_i(C^*_L(\tilde{X} \times [1, \infty))^G) \cong K_{i+1}(C^*_L(\tilde{X} \times [1, \infty))^G) \) [16, Corollary 4.7].

We now introduce the space \( CX \), which is defined as a rescaling of \( X \times [1, \infty) \) along \( X \) for any \( t \in [1, \infty) \) such that

1. For \( x \in X \) and \( t_1, t_2 \in [1, \infty) \), we have \( d((x, t_1), (x, t_2)) = |t_2 - t_1|; \)
2. For \( x_1, x_2 \in X \) and \( t \in [1, \infty) \), we have \( d((x_1, t), (x_2, t)) \) is no decreasingly tends to infinity as \( t \) tends to infinity.

**Remark 5.4.** If \( M \) is a closed Riemannian manifold, in \( M \times [1, \infty) \) we equipped with a (complete) Riemannian metric of the form

\[
ds^2 = dt^2 + t^2 g_{ij} dy^i dy^j
\]

where \( t \) is the coordinate on \([1, \infty)\), \( g_{ij} \) is a Riemannian metric on \( M \). This metric will satisfied above conditions.

Note that there is a proper continuous map

\[
\tau : CX \to X \times [1, \infty),
\]

define by \( \tau(x, t) = (x, t) \) which induces a \( C^* \)-algebra homomorphism

\[
\tau_* : C^*(CX) \to C^c_\ast(X \times [1, \infty)).
\]

Similarly, we have

\[
\tau_* : C^*_{\ell,0}(CX) \to C^*_{\ell,0,c}(X \times [1, \infty)).
\]

There are also obvious \( G \)-equivariant versions of theory of hybrid \( C^* \) algebra.

### 5.2 A construction on infinitesimal controlled homotopy equivalence

Consider \( \theta = (M, \partial M, \phi, N, \partial N, \psi, f) \in \mathcal{N}_n(X) \). \( CM, CN \) are obtained in the same way as \( CX \).

For each \( s \in [1, \infty) \) we denote \( X_s \) the sub-manifold \( X \times \{s\} \to CX \). Do not confuse it with \( X \) itself. And, for \( m \in \mathbb{N} \), we should also differ \( C^*(\sqcup_m X_m) \) from \( C^*({\sqcup_m X}_m) \), since \( \sqcup_m X_m \) is not coarsely equivalent to \( \sqcup_m X \). Similarly, we denote \( \tilde{\theta}_s \) the sub-manifold \( M \times \{s\} \to CM \) (resp. \( N \times \{s\} \to CN \)). We also have \( C^*({\sqcup_m M}_m) \) (resp. \( C^*({\sqcup_m N}_m) \)).

Let \( G = \pi_1 X \) be the fundamental group of \( X \) and \( G \to \tilde{X} \to X \) be a \( G \)-covering of \( X \), where \( \tilde{X} \) is the universal cover of \( X \). Denote by \( CM, CN \) the pull back \( G \)-covering induced by \( \phi : M \to X \) (resp. \( \psi : N \to X \)).

Hence, for each \( t \in [0, 1] \) we have \( \sqcup_m \tilde{M}_{m+t} \) and \( \sqcup_m \tilde{N}_{m+t} \). Let \( d_{\sqcup_m \tilde{M}_{m+t}} \) be the \((G\text{-equivariant})\ de\ Rham\ differential\ operator\ on\ \sqcup_m \tilde{M}_{m+t} \) and \( \partial_{\sqcup_m \tilde{N}_{m+t}} \) be the \((G\text{-equivariant})\ de\ Rham\ differential\ operator\ on\ \sqcup_m \tilde{N}_{m+t} \).

We introduce

\[
D_t := \begin{pmatrix} D_{\sqcup_m \tilde{M}_{m+t}} & 0 \\ 0 & D_{\sqcup_m \tilde{N}_{m+t}} \end{pmatrix} \quad \text{and} \quad S_t := \begin{pmatrix} S_{\sqcup_m \tilde{M}_{m+t}} & 0 \\ 0 & -S_{\sqcup_m \tilde{N}_{m+t}} \end{pmatrix}.
\]

By subsection 4.1, for any \( t \in [0, 1] \), we can define a relative index \( \text{Ind}(D_t) \) in \( K_n(C^*({\sqcup_m \tilde{X}_m+t})^G) \). Although we do not known whether there exist a path connected \( \text{Ind}(D_t) \) and the trivial element in \( K_n(C^*({\sqcup_m X}_m+t)^G) \), we can define a \( K \)-theory class in \( K_n(C^*_{G}(\tilde{X}))^G) \). We denote this \( K \)-theory class by \( \text{Ind}_L(\theta) \).
Remark 5.5. Yu has proved in [17] that $K_*(C^*_L(\tilde{X})^G)$ is isomorphic to $K$-homology of $X$, $K_*(X)$. Let $M$ and $N$ be two compact manifolds, and let $G$ be the fundamental group of $X$. Consider $\theta = (M, \emptyset, \phi, N, \emptyset, \psi, f) \in \mathcal{N}_n(X)$. Under the isomorphism constructed in [17], it is not hard to see that $\text{Ind}_L(\theta) = \phi_*([D_M]) - \psi_*([D_N])$, where $[D_M]$ (resp. $[D_N]$) is the $K$-homology class of signature operator on $M$ (resp. $N$).

Now, we will show that the map $\text{Ind}_L : \mathcal{N}_n(X) \to K_n(C^*_L(\tilde{X})^G)$ is well defined. Analogous to the Theorem 4.6 and Theorem 4.7 in subsection 4, we have the following two theorems.

**Theorem 5.6.** Suppose $\theta = (M, \partial M, \phi, N, \partial N, \psi, f) \in \mathcal{N}_n(X)$. And $f : M \to N$ is an infinitesimally controlled homotopy equivalence over $X$, then $\text{Ind}_L(\theta) = 0$.

**Theorem 5.7.** Let $\theta, \theta' \in \mathcal{N}_n(X)$. We further suppose $\theta = (M, \partial M, \phi, N, \partial N, \psi, f)$, $\theta' = (M', \partial M', \phi', N', \partial N', \psi', f')$ and satisfies the following conditions:

1. $\partial M = \partial M'$, $\partial N = \partial N'$, restricting to the boundary $f = f'$, $\phi = \phi'$ and $\psi = \psi'$;
2. There exist two manifolds with boundary $(W, \partial W)$ and $(V, \partial V)$, continuous maps $\Phi : W \to X$ (resp. $\Psi : V \to X$). Manifolds $W$ and $V$ both of dimension $n + 1$. Moreover, $\partial W = M \cup_{\partial M} M'$ (resp. $\partial V = N \cup_{\partial N} N'$) and $\Phi$ (resp. $\Psi$) restricts to $\phi \cup \phi'$ (resp. $\psi \cup \psi'$) on $M \cup_{\partial M} M'$ (resp. $N \cup_{\partial N} N'$);
3. There exist a degree one normal map $F : W \to V$ such that $\Psi \circ F = \Phi$.

Moreover, $F$ restricts to $f \cup f'$ on $M \cup_{\partial M} M'$.

we have

$$\text{Ind}_L(\theta) - \text{Ind}_L(\theta') = \text{Ind}_L(\theta \sqcup \theta') - \text{Ind}_L(\theta' \sqcup \theta').$$  \tag{5.1}$$

For the same reason as we mentioned in Remark 4.8, we have

$$\text{Ind}_L : \mathcal{N}_n(X) \to K_n(C^*_L(\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}],$$

is a well defined group homomorphism.

6 Mapping surgery to analysis

Let $X$ be a closed oriented topological manifold of dimension $n \geq 5$. With the technical and results of Section 4 and Section 5, we are now in a position to define “Mapping surgery to analysis” and to prove the additivity of higher rho map (Eq.(6.1) and Theorem 6.5).

For our purposes, we need to seek help from $L_{n+1}(\pi_1 X, X)$. Hence, before defining the higher rho map, it is helpful to investigate the index map $\hat{\rho} : L_{n+1}(\pi_1 X, X) \to K_{n+1}(C^*_L(\tilde{X} \times [1, \infty)^G)$ in the coming subsection.

6.1 Index of $L_{n}(\pi_1 X, X)$

For $\theta = (M, \partial_{\pm} M, \phi, N, \partial_{\pm} N, \psi, f) \in L_{n+1}(\pi_1 X, X)$, let $C_- M$ and $C_- N$ be $M \cup_{\partial_{-} M} C \partial_- M$ and $N \cup_{\partial_{-} N} C \partial_- N$ respectively.

$C_- M$ (resp. $C_- N$) is manifold with boundary $\partial C_- M = \partial_{+} M \cup_{\partial_{+} M} C \partial_{+} M$ (resp. $\partial C_- N = \partial_{+} N \cup_{\partial_{+} N} C \partial_{+} N$). Since $f : \partial_{\pm} M \to \partial_{\pm} N$ is an infinitesimally controlled homotopy equivalence over $X$, it induces an analytically controlled homotopy equivalence restricting to $\partial C_- M$. Hence, by subsection 4.1, this defines
a $K$-theory class in $K_{n+1}(C^*(C\check{X})^G)$. Then by rescaling, we actually obtain a class in $K_{n+1}(C^*_c(\check{X} \times [1, \infty))^G)$. We denote this class by $\hat{\rho}(\theta)$.

Analogous to the situation for index map for $L$-groups (Theorem 4.7), we have the following theorem. By this theorem we can see that $\hat{\rho} : L_{n+1}(\pi_1 X, X) \to K_{n+1}(C^*_c(\check{X} \times [1, \infty))^G)$ is well defined. The proof is essentially the same as Theorem 4.7, so we omit it.

**Theorem 6.1.** Let $\theta$ and $\theta'$ be two elements in $L_{n+1}(\pi_1 X, X)$. We further suppose $\theta = (M, \partial M, \phi, N, \partial N, \psi, f)$ and $\theta' = (M', \partial M', \phi', N', \partial N', \psi', f')$ such that $\partial_+ M = \partial_+ M'$, $\partial_+ N = \partial_+ N'$ and restricting to the $\partial_+$ boundary $f = f'$ and $\phi = \phi'$. Then we have

$$\hat{\rho}(\theta) - \hat{\rho}(\theta') = \hat{\rho}(\theta \sqcup \theta') - \hat{\rho}(\theta' \sqcup \theta').$$

Although we can not use bordism invariance to show that $\hat{\rho}(\theta \sqcup \theta')$ is zero, we can show this directly. Since $\hat{\rho}(\theta \sqcup \theta')$ lies in the image of the following map:

$$
\begin{array}{ccc}
K^G_*(C\check{X}) & \longrightarrow & K_*(C^*(C\check{X})^G) \\
\downarrow & & \downarrow \\
K_*(C^*_c(\check{X} \times [1, \infty))^G) & \longrightarrow & K_*(C^*_c(\check{X} \times [1, \infty))^G)
\end{array}
$$

where $K^G_*(C\check{X})$ is the $G$-equivariant $K$-homology of $C\check{X}$. By [16, Lemma 4.6], $K_*(C^*_c(\check{X} \times [1, \infty)^G)$ is always trivial. Thus $\hat{\rho}(\theta \sqcup \theta')$ is always zero in $K_*(C^*_c(\check{X} \times [1, \infty)^G)$.

Hence, we have

$$\hat{\rho} : L_*(\pi_1 X, X) \to K_*(C^*_c(\check{X} \times [1, \infty)^G)$$

is a well defined group homomorphism.

### 6.2 Higher rho map for $S_n(X)$

Suppose $\theta = (M, \partial M, \phi, N, \partial N, \psi, f) \in S_n(X)$. We use the same notations as in subsection 5. For $t \in [0, 1]$ and $m \in \mathbb{N}$, we have $\sqcup_m M_{m+t}$ and $\sqcup_m N_{m+t}$. And $\sqcup_m \hat{M}_{m+t}$ (resp. $\sqcup_m \hat{N}_{m+t}$), the Poincaré duality operator of $d_{\sqcup_m M_{m+t}}$ (resp. $d_{\sqcup_m N_{m+t}}$). As well as, $D_t$ and $S_t$. By Section 5, we can define an element $\alpha \in K_n(C^*_c(\check{X})^G) \otimes \mathbb{Z}[\frac{1}{2}]$.

When $t = 0$, $\hat{M} \times \{1\} \hookrightarrow \sqcup_m M_m$ (resp. $\hat{N} \times \{1\} \hookrightarrow \sqcup_m N_m$), $d_{\sqcup_m \hat{M}_m} = d_{\hat{M}}$ (resp. $d_{\sqcup_m \hat{N}_m} = d_{\hat{N}}$), and $S_{\sqcup_m \hat{M}_m} = S_{\hat{M}}$ (resp. $S_{\sqcup_m \hat{N}_m} = S_{\hat{N}}$).

Moreover, since $f : M \to N$ is a homotopy equivalence over $X$. Thus we can connect $\alpha(1)$ to the trivial element in $K_n(C^*(\check{X})^G)$. Hence, we have the following map

$$\rho : S_n(X) \to K_n(C^*_c(\check{X})^G).$$

We want to show the map

$$k_n \cdot \rho : S_n(X) \to K_n(C^*_c(\check{X})^G) \otimes \mathbb{Z}[\frac{1}{2}], \quad (6.1)$$

is a well defined group homomorphism, where $k_n = 1$ if $n$ is even and 2 if $n$ is odd. This is a corollary of the following theorem. Recall that we have $S_n(X) \cong L_{n+1}(\pi_1 X, X)$ (c.f [16, Section 3.3] or see appendix A).
Theorem 6.2 ([16, Theorem 6.9]). Let \( X \) be a closed oriented topological manifold of dimension \( \geq 5 \). We have the following commutative diagram of \( K \)-groups

\[
\begin{array}{ccc}
L_{n+1}(\pi_1 X, X) & \xrightarrow{\hat{\rho}} & K_{n+1}(C_c^* (\tilde{X} \times [1, \infty))^G) \otimes \mathbb{Z}[\frac{1}{2}] \\
\downarrow \varepsilon_{*} & & \downarrow \partial_{*} \\
S_{n}(X) & \xrightarrow{k_{n,\rho}} & K_n(C_{L,0,c}^* (\tilde{X} \times [1, \infty))^G) \otimes \mathbb{Z}[\frac{1}{2}] = K_n(C_{L,0}^* (\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}]
\end{array}
\]

where \( \partial_{*} \) is the connecting map in the \( K \)-theory long exact sequence associated to the following short exact sequence of \( C^* \)-algebras

\[
0 \to C_{L,0,c}^* (\tilde{X} \times [1, \infty))^G \to C_{L,c}^* (\tilde{X} \times [1, \infty))^G \to C_c^* (\tilde{X} \times [1, \infty))^G \to 0,
\]

and \( k_n = 1 \) if \( n \) is even and \( 2 \) if \( n \) is odd.

The strategy to prove this theorem is to compute the map \( \partial_{*} \) in a Mayer-Vietoris sequence.

Before we go into details, we specify which Mayer-Vietoris sequence we are going to use. Let \( \mathcal{A} := C_{L,0}^* (\tilde{X} \times \mathbb{R})^G \). Define a bunch of \( C^* \) algebras as

\[
\begin{align*}
\mathcal{A}_- &= \bigcup_{n \in \mathbb{N}} C_{L,0}^*(\tilde{X} \times [-\infty, n]; \tilde{X} \times \mathbb{R})^G \\
\mathcal{A}_+ &= \bigcup_{n \in \mathbb{N}} C_{L,0}^*(\tilde{X} \times [-n, \infty]; \tilde{X} \times \mathbb{R})^G \\
\mathcal{A}_\cap &= \bigcup_{n \in \mathbb{N}} C_{L,0}^*(\tilde{X} \times [-n, n]; \tilde{X} \times \mathbb{R})^G,
\end{align*}
\]

all of which are two sided ideal of \( \mathcal{A} \). Also, we have

\[
\mathcal{A}_- + \mathcal{A}_+ = \mathcal{A}, \mathcal{A}_- \cap \mathcal{A}_+ = \mathcal{A}_\cap.
\]

Thus there is Mayer-Vietoris sequence

\[
\begin{array}{cccc}
K_0(\mathcal{A}_\cap) & \xrightarrow{\partial_{M\text{V}}} & K_0(\mathcal{A}_+) \otimes K_0(\mathcal{A}_-) & \xrightarrow{\partial_{M\text{V}}} & K_0(\mathcal{A}) \\
\uparrow \varepsilon_{M\text{V}} & & \uparrow \varepsilon_{M\text{V}} & & \uparrow \varepsilon_{M\text{V}} \\
K_1(\mathcal{A}) & \xleftarrow{\partial_{M\text{V}}} & K_1(\mathcal{A}_+) \otimes K_0(\mathcal{A}_-) & \xleftarrow{\partial_{M\text{V}}} & K_0(\mathcal{A}_\cap).
\end{array}
\]

Similarly, let \( \mathcal{B} := C_{L}^*(\mathbb{R}) \), we define

\[
\begin{align*}
\mathcal{B}_- &= \bigcup_{n \in \mathbb{N}} C_{L}^*([-\infty, n]; \mathbb{R}) \\
\mathcal{B}_+ &= \bigcup_{n \in \mathbb{N}} C_{L}^*([-n, \infty]; \mathbb{R}) \\
\mathcal{B}_\cap &= \bigcup_{n \in \mathbb{N}} C_{L}^*([-n, n]; \mathbb{R}),
\end{align*}
\]

and again, we have Mayer-Vietoris sequence

\[
\begin{array}{cccc}
K_0(\mathcal{B}_\cap) & \xrightarrow{\partial_{M\text{V}}} & K_0(\mathcal{B}_+) \otimes K_0(\mathcal{B}_-) & \xrightarrow{\partial_{M\text{V}}} & K_0(\mathcal{B}) \\
\uparrow \varepsilon_{M\text{V}} & & \uparrow \varepsilon_{M\text{V}} & & \uparrow \varepsilon_{M\text{V}} \\
K_1(\mathcal{B}) & \xleftarrow{\partial_{M\text{V}}} & K_1(\mathcal{B}_+) \otimes K_0(\mathcal{B}_-) & \xleftarrow{\partial_{M\text{V}}} & K_0(\mathcal{B}_\cap).
\end{array}
\]
Proof: To compute $\partial_\ast(\hat{\rho}(c_\ast(\theta)))$, we explicitly construct the lifting of $\hat{\rho}(c_\ast(\theta))$ in $C_{1,\ast}^\ast(\hat{X} \times [1, \infty])^G$.

Let $a_0(n)$ be $\chi_n\hat{\rho}(c_\ast(\theta))\chi_n$, where $\chi_n$ the characteristic function on $\hat{X} \times [n, \infty]$. Then $a_0(t) = (n + 1 - t)a_0(n) + (t - n)a_0(n + 1) \in C_{1,\ast}^\ast(\hat{X} \times [1, \infty])^G$ is a lift of $\hat{\rho}(c_\ast(\theta))$.

On the other hand, one can see $a_0$ is also a lift of $\rho'(\theta \times \mathbb{R})$ under the homomorphism

$$\partial_{MV}: K_{n+1}(C_{1,\ast}^\ast(\hat{X} \times \mathbb{R}))^G \to K_n(C_{1,\ast}^\ast(\hat{X}))^G.$$ 

Thus, we have

$$\partial_\ast(\hat{\rho}(c_\ast(\theta))) = \partial_{MV}(\rho'(\theta \times \mathbb{R})).$$

However, the right side one can be compute by the following formalism argument.

Note that there is a natural homomorphism $\alpha: C_{1,\ast}^\ast(\hat{X}) \times \mathbb{B} \to \mathcal{A}$, which restrict to homomorphisms

$$\alpha: C_{1,\ast}^\ast(\hat{X})^G \times \mathbb{B}_\pm \to \mathcal{A}_\pm, \quad \alpha: C_{1,\ast}^\ast(\hat{X})^G \times \mathbb{B}_\gamma \to \mathcal{A}_\gamma.$$ 

And we have the following commutative diagram

$$\begin{array}{ccc}
K_n(C_{1,\ast}^\ast(\hat{X})^G) \otimes K_1(\mathbb{B}) & \to & K_{n+1}(C_{1,\ast}^\ast(\hat{X})^G \otimes \mathbb{B}) \\
\downarrow{1 \times \partial_{MV}} & \longrightarrow & \downarrow{\partial_{MV}} \\
K_n(C_{1,\ast}^\ast(\hat{X})^G) \otimes K_0(\mathcal{B}_\gamma) & \to & K_n(C_{1,\ast}^\ast(\hat{X})^G \otimes \mathcal{B}_\gamma) \to K_n(\mathcal{A}_\gamma) = K_n(C_{1,\ast}^\ast(\hat{X})^G). \end{array}$$

Using the main result of [17, Theorem 3.2], we know that the local index map from $K_i(\mathbb{R})$ to $K_i(C_1^\ast(\mathbb{R}))$ is an isomorphism, where $K_i(\mathbb{R})$ denote the $K$-homology of $\mathbb{R}$ [17].

Now the proof is completed by the a product formula(Proposition 6.3 or [16, Theorem 6.8]).

Given $\theta = (M, \partial M, \phi, N, \partial N, \psi, f) \in S_n(X)$, let $\theta \times \mathbb{R}$ be the product of $\theta$ and $\mathbb{R}$, which defines an element in $S_{n+1}(X \times \mathbb{R})$. Here various undefined terms take the obvious meanings [16].

Note that $\theta \times \mathbb{R}$ defines a $K$-theory class in $K_{n+1}(C_{1,\ast}^\ast(\hat{X} \times \mathbb{R}))^G$. Also there is a natural homomorphism $\alpha: C_{1,\ast}^\ast(\hat{X})^G \otimes C_1^\ast(\mathbb{R}) \to C_{1,\ast}^\ast(\hat{X} \times \mathbb{R})^G$. Now we have the following product formula, which essentially the same as [16, Theorem 6.8].

Proposition 6.3 ([16, Theorem 6.8]). With the notation as above, we have

$$k_n \cdot \alpha_\ast(\rho(\theta) \otimes \text{Ind}_L(\mathbb{R})) = \rho(\theta \otimes \mathbb{R})$$

in $K_{n+1}(C_{1,\ast}^\ast(\hat{X} \times \mathbb{R}))^G$, where $\text{Ind}_L(\mathbb{R})$ is the $K$-homology class of the signature operator on $\mathbb{R}$ and $k_n = 1$ if $n$ is even and 2 if $n$ is odd.

Proof: Use the same notation as before. We need to consider $\theta \times \mathbb{R} \in S_{n+1}(X \times \mathbb{R})$. Note that, for $s \in [1, \infty]$, $(\tilde{M} \times \mathbb{R})_s \neq \tilde{M} \times \mathbb{R}$ (resp. $(\tilde{N} \times \mathbb{R})_s \neq \tilde{N} \times \mathbb{R}$). As $(\tilde{M} \times \mathbb{R})_s$ (resp. $(\tilde{N} \times \mathbb{R})_s$) is obtained from $\tilde{M} \times \mathbb{R}$ (resp. $\tilde{N} \times \mathbb{R}$) by rescaling $\tilde{M} \times \mathbb{R}$ (resp. $\tilde{N} \times \mathbb{R}$) according to $s$. Moreover, we use product metric on $\tilde{M} \times \mathbb{R}$ (resp. $\tilde{N} \times \mathbb{R}$).

Nevertheless, we can choose the $K$-homology class of the signature operator on $\mathbb{R}$ with arbitrary small propagation [17].

In a word, in order to compute $\rho(\theta \otimes \mathbb{R})$, we can replace $\sqcup_m(\tilde{M} \times \mathbb{R})_{m+t}$ (resp. $\sqcup_m(\tilde{N} \times \mathbb{R})_{m+t}$) by $(\sqcup_m \tilde{M}_{m+t}) \times \mathbb{R}$ (resp. $(\sqcup_m \tilde{N}_{m+t}) \times \mathbb{R}$).
For \( t \in [0, 1] \), the K-theory class in \( K_n(C^*(\sqcup_{m+t}\tilde{X}_{m+t})^G) \), which we get from \( \sqcup_m\tilde{M}_{m+t} \) and \( \sqcup_m\tilde{N}_{m+t} \), is the relative index. So satisfied the product formula of Proposition 4.9.

Moreover, by the proof of Proposition 4.9 we have a path connect \( \alpha(1) \) and trivial element in \( K_n(C^*(\tilde{X})^G) \). It is no have to see this path also satisfied the product formula of Proposition 4.9. \( \square \)

Remark 6.4. The \( k_n \) appears here and in the argument of bordism invariance of signature operator is the reason why we need to consider K-theory with coefficient \( \mathbb{Z}\left[\frac{1}{2}\right] \) coefficient.

6.3 Commutative diagram

In this subsection, we prove the following theorem.

**Theorem 6.5.** Let \( X \) be a closed oriented topological manifold of dimension \( \geq 5 \). We have commutative diagram of abelian groups

\[
\begin{array}{cccc}
N_{n+1}(X) & \xrightarrow{Ind_L} & L_{n+1}(\pi_1 X) & \xrightarrow{Ind} & S_n(X) & \xrightarrow{k_n \cdot \rho} & N_n(X) \\
K_{n+1}(C^*_L(\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}] & \xrightarrow{\iota^*} & K_{n+1}(C^*(\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}] & \xrightarrow{\iota^*} & K_n(C^*_L,0(\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}] & \xrightarrow{\iota^*} & K_n(C^*_L(\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}]
\end{array}
\]

where, \( G = \pi_1 X \) is the fundamental group of \( X \), \( \tilde{X} \) is the universal covering of \( X \), and \( k_n = 1 \) if \( n \) is even and \( 2 \) if \( n \) is odd.

**Proof:** The commutativity of the left and the right square follows immediately from definition. Now, we focus on the middle square

\[
\begin{array}{cccc}
L_{n+1}(\pi_1 X) & \xrightarrow{Ind} & S_n(X) & \\
K_{n+1}(C^*(\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}] & \xrightarrow{k_n \cdot \rho} & K_n(C^*_L,0(\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}].
\end{array}
\]

The commutativity of the above square can implied by following diagram

\[
\begin{array}{cccc}
L_{n+1}(\pi_1 X) & \xrightarrow{j_*} & L_{n+1}(\pi_1 X, X) & \\
K_{n+1}(C^*(\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}] & \xrightarrow{\iota^*} & K_{n+1}(C^*_L,0(\tilde{X} \times [1, \infty))^G) \otimes \mathbb{Z}[\frac{1}{2}] & \xrightarrow{\iota^*} K_n(C^*_L,0(\tilde{X})^G) \otimes \mathbb{Z}[\frac{1}{2}].
\end{array}
\]

**Remark 6.6.** We actually can eliminate the \( \mathbb{Z}[\frac{1}{2}] \) coefficient, by the isomorphism of de Rham complex and simplicial complex. However, we do not know how to eliminate it with a pure de Rham argument until now, so we will leave the \( \mathbb{Z}[\frac{1}{2}] \) alone.

In the last we mention that our definition coincide with the one in [16]. This is actually immediate since both of them can be defined by hybrid \( C^* \)-algebra, and these hybrid algebra definition coincide with each other obviously.
A Surgery long exact sequence

In [16], Weinberger, Xie, and Yu give a new description of the topological structure group in terms of smooth manifolds with boundary. This description has analytic advantages.

For the convenience of readers, in this appendix we list the definitions and main results of [16, Section 3]. Throughout this paper, manifolds are suppose to be oriented and maps are suppose to be oriented-preserving. For simplicity, we will suppress the terminology orientation in the context.

Let $X$ be a topological manifold. Fix a metric on $X$ which is compatible with the topology of $X$. If $X$ is a manifold with boundary, we denote the boundary of $X$ by $\partial X$. We begin with infinitesimal controlled homotopy equivalence.

**Definition A.1.** Let $M$ and $N$ be two compact Hausdorff spaces equipped with continuous maps $\phi: M \to X$ and $\psi: N \to X$. A continuous map $f: M \to N$ is said to be an infinitesimally controlled homotopy equivalence over $X$, if there exist proper continuous maps $M \times [1, \infty) \to N \times [1, \infty)$ and $N \times [1, \infty) \to M \times [1, \infty)$ such that

1. $\Psi \circ F = \Phi$;
2. $F|_{M \times \{1\}} = f$, $\Phi|_{M \times \{1\}} = \phi$, $\Psi|_{N \times \{1\}} = \psi$;
3. There exists a proper continuous homotopy $\{H_s\}_{0 \leq s \leq 1}$ between $H_0 = F \circ G$ and $H_1 = Id: N \times [1, \infty) \to N \times [1, \infty)$ such that the diameter of the set $\{\Phi(H_s(z,t)) | 0 \leq s \leq 1\}$ goes uniformly (i.e. independent of $z \in N$) to zero, as $t \to \infty$;
4. There exists a proper continuous homotopy $\{H'_s\}_{0 \leq s \leq 1}$ between $H'_0 = G \circ F$ and $H'_1 = Id: M \times [1, \infty) \to M \times [1, \infty)$ such that the diameter of the set $\{\Psi(H'_s(y,t)) | 0 \leq s \leq 1\}$ goes uniformly (i.e. independent of $y \in M$) to zero, as $t \to \infty$.

We will also need the following notion of restrictions of homotopy equivalences gaining infinitesimal control on parts of spaces.

Suppose $(M, \partial M)$ and $(N, \partial N)$ are two manifolds with boundary equipped with continuous maps $\phi: M \to X$ and $\psi: N \to X$. Let $f: (M, \partial M) \to (N, \partial N)$ be a homotopy equivalence such that $\psi \circ f = \phi$. Let $g: (N, \partial N) \to (M, \partial M)$ be a homotopy inverse of $f$. Note that $\phi \circ g \neq \psi$ in general. Let $\{h_s\}_{0 \leq s \leq 1}$ be a homotopy between $f \circ g$ and $Id: (N, \partial N) \to (N, \partial N)$. Similarly, let $\{h'_s\}_{0 \leq s \leq 1}$ be a homotopy between $g \circ f$ and $Id: (M, \partial M) \to (M, \partial M)$.

Since we have use $CM$ for other purpose, we denote $M \sqcup_{\partial M} \partial M \times [1, \infty)$ by $M_\infty$ instead.
Definition A.2 ([16, Definition 3.3]). With the above notations, we say that on the boundary $f$ restricts to an *infinitesimally controlled homotopy equivalence* $f|_{\partial M} : \partial M \to \partial N$ over $X$, if there exist proper continuous maps

\[
\begin{array}{ccc}
M_\infty & \xrightarrow{F} & N_\infty \\
\Phi \downarrow & & \downarrow \Psi \\
X \times [1, \infty) & \xrightarrow{G} & \partial N
\end{array}
\]

such that

1. $\Psi \circ F = \Phi$;
2. $F|_M = f$, $\Phi|_M = \phi$, $\Psi|_N = \psi$;
3. There exists a proper continuous homotopy $\{H_s\}_{0 \leq s \leq 1}$ between

\[
H_0 = F \circ G \quad \text{and} \quad H_1 = \text{Id} : N_\infty \to N_\infty
\]

such that the diameter of the set $\{\Phi(H_s(z,t)) \mid 0 \leq s \leq 1\}$ goes uniformly (i.e. independent of $z \in \partial N$) to zero, as $t \to \infty$;
4. There exists a proper continuous homotopy $\{H'_s\}_{0 \leq s \leq 1}$ between

\[
H'_0 = G \circ F \quad \text{and} \quad H'_1 = \text{Id} : M_\infty \to M_\infty
\]

such that the diameter of the set $\{\Psi(H'_s(y,t)) \mid 0 \leq s \leq 1\}$ goes uniformly (i.e. independent of $y \in \partial M$) to zero, as $t \to \infty$.

The following geometric definition of $L$-groups due to Wall [15].

Definition A.3 (Objects for definition of $L_n(\pi_1 X)$, [16]). An object

\[
\theta = (M, \partial M, \phi, N, \partial N, \psi, f)
\]

of $L_n(\pi_1 X)$ consists of the following data:

1. Two manifolds with boundary $M$ and $N$ both of dimension $n$;
2. Continuous maps $\phi : M \to X$ and $\psi : N \to X$;
3. A degree one normal map $f : (M, \partial M) \to (N, \partial N)$ such that $\psi \circ f = \phi$.

Moreover, on the boundary $f|_{\partial M} : \partial M \to \partial N$ is a homotopy equivalence.

If $\theta = (M, \partial M, \phi, N, \partial N, \psi, f)$ is an element, then we denote by $-\theta$ to be the same object except that the fundamental classes of $M$ and $N$ switch sign. For two objects $\theta_1$ and $\theta_2$, we write $\theta_1 + \theta_2$ to be the disjoint union of $\theta_1$ and $\theta_2$. This sum operation is clearly commutative and associative, and admits a zero element: the element with $M$ (hence $N$) empty. We denote the zero element by $0$.

Definition A.4 (Equivalence relation for definition $L_n(\pi_1 X)$, [16]). Let

\[
\theta = (M, \partial M, \phi, N, \partial N, \psi, f)
\]

be an object from $L_n(\pi_1 X)$. We write $\theta \sim 0$ if the following conditions are satisfied:
1. There exists a manifold with boundary $W$ and a continuous map $\Phi : W \to X$. The dimension of manifold $W$ is $n + 1$. Moreover, $\partial W = M \sqcup_{\partial M} \partial_2 W$, in particular $\partial M = \partial \partial_2 W$ (We can decompose $\partial W$ into two pieces $\partial_1 W$ and $\partial_2 W$, each piece is a manifold with boundary. Moreover, $\partial W = \partial_1 W \sqcup \partial_2 W$ in particular $\partial \partial_1 W = \partial \partial_2 W$);

2. Similarly, there exists a manifold with boundary $V$ and a continuous map $\Psi : V \to X$. The dimension of manifold $V$ is $n + 1$. Moreover, $\partial V = N \sqcup_{\partial N} \partial_2 V$, in particular $\partial N = \partial \partial_2 V$;

3. There is a degree one normal map $F : (W, \partial W) \to (V, \partial V)$ such that $\Psi \circ F = \Phi$. Moreover, $F$ restricts to $f$ on $M$, and $F|_{\partial_2 W} : \partial_2 W \to \partial_2 V$ is a homotopy equivalence over $X$.

A little bit of abuse our notation. We denote by $\mathcal{N}_n(X)$ the set of equivalence classes from Definition A.5 under equivalence relation from Definition A.6. Note that $\mathcal{N}_n(X)$ is an abelian group under disjoint union. It is a theorem of Wall that the above definition of $L$-groups is equivalent to the algebraic definition of $L$-groups [15, Chapter 9].

**Definition A.5** (Objects for definition of $\mathcal{N}_n(X)$, [16]). An object

$$\theta = (M, \partial M, \phi, N, \partial N, \psi, f)$$

of $\mathcal{N}_n(X)$ consists of the following data:

1. Two manifolds with boundary $M$ and $N$ both of dimensional $n$;
2. Continuous maps $\phi : M \to X$ and $\psi : N \to X$;
3. A degree one normal map $f : (M, \partial M) \to (N, \partial N)$ such that $\psi \circ f = \phi$. Moreover, on the boundary $f|_{\partial M} : \partial M \to \partial N$ is an infinitesimally controlled homotopy equivalence over $X$.

**Definition A.6** (Equivalence relation for definition of $\mathcal{N}_n(X)$, [16]). Let

$$\theta = (M, \partial M, \phi, N, \partial N, \psi, f)$$

be an object from $\mathcal{N}_n(X)$. We write $\theta \sim_0$ if the following conditions are satisfied.

1. There exists a manifold with boundary $W$ and a continuous map $\Phi : W \to X$. The dimension of manifold $W$ is $n + 1$. Moreover, $\partial W = M \sqcup_{\partial M} \partial_2 W$, in particular $\partial M = \partial \partial_2 W$;
2. Similarly, there exists a manifold with boundary $V$ and a continuous map $\Psi : V \to X$. The dimension of manifold $V$ is $n + 1$. Moreover, $\partial V = N \sqcup_{\partial N} \partial_2 V$, in particular $\partial N = \partial \partial_2 V$;
3. There is a degree one normal map $F : (W, \partial W) \to (V, \partial V)$ such that $\Psi \circ F = \Phi$. Moreover, $F$ restricts to $f$ on $M$, and $F|_{\partial_2 W} : \partial_2 W \to \partial_2 V$ is an infinitesimally controlled homotopy equivalence over $X$.

We denote by $\mathcal{N}_n(X)$ the set of equivalence classes from Definition A.5 under equivalence relation from Definition A.6. Note that $\mathcal{N}_n(X)$ is an abelian group with the sum operation being disjoint union. $\mathcal{N}_n(X)$ is the usual Normal group in surgery theory. Now we recall the relative $L$-group $L_n(\pi_1 X, X)$. 

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Definition A.7 (Objects for definition of \( L_n(\pi_1 X, X) \), [16]). An object
\[
\theta = (M, \partial_{\pm} M, \phi, N, \partial_{\pm} N, \psi, f)
\]
of \( L_n(\pi_1 X, X) \) consists of the following data:

1. Two manifolds with boundary \( M \) and \( N \) both of dimensional \( n \). Moreover, \( \partial M = \partial_{+} M \sqcup \partial_{-} M \) (resp. \( \partial N = \partial_{+} N \sqcup \partial_{-} N \)) where \( \partial M \) (resp. \( \partial N \)) is the boundary of \( M \) (resp. \( N \)). In particular, \( \partial_{\pm} M = \partial \partial_{\pm} M \) and \( \partial \partial_{+} N = \partial \partial_{-} N \);
2. Continuous maps \( \phi : M \to X \) and \( \psi : N \to X \);
3. A degree one normal map \( \theta : (M, \partial M) \to (N, \partial N) \) such that \( \phi \circ f = \psi \);
4. The restriction \( f|_{\partial_{+} M} : \partial_{+} M \to \partial_{+} N \) is a homotopy equivalence over \( X \);
5. The restriction \( f|_{\partial_{-} M} : \partial_{-} M \to \partial_{-} N \) is a degree one normal map over \( X \);
6. The homotopy equivalence \( f|_{\partial_{+} M} \) restricts to an infinitesimally controlled homotopy equivalence \( f|_{\partial_{-} M} : \partial_{\pm} M \to \partial_{\pm} N \) over \( X \).

Definition A.8 (Equivalence relation for definition of \( L_n(\pi_1 X, X) \), [16]). Let
\[
\theta = (M, \partial_{\pm} M, \phi, N, \partial_{\pm} N, \psi, f)
\]
be an object from \( L_n(\pi_1 X, X) \). We write \( \theta \sim 0 \) if the following conditions are satisfied.

1. There exists a manifold with boundary \( W \) and a continuous map \( \Phi : W \to X \). The dimension of manifold \( W \) is \( n + 1 \). Moreover, \( \partial W = M \sqcup \partial_{2} W \sqcup \partial_{3} W \) and we have decompositions \( \partial M = \partial_{+} M \sqcup \partial_{-} M \), \( \partial_{2} W = \partial \partial_{2} W \sqcup \partial \partial_{2} - W \) and \( \partial \partial_{3} W = \partial \partial_{3} + W \sqcup \partial \partial_{3} - W \) such that
\[
\partial_{+} M = \partial \partial_{2} + W, \quad \partial_{-} M = \partial \partial_{3} - W, \quad \text{and} \quad \partial \partial_{2} - W = \partial \partial_{3} + W.
\]
Furthermore, we have
\[
\partial_{+} M \cap \partial_{-} M = \partial \partial_{2} + W \cap \partial \partial_{2} - W = \partial \partial_{3} + W \cap \partial \partial_{3} - W;
\]
2. Similarly, there exists a manifold with boundary \( V \) and a continuous map \( \Psi : V \to X \). The dimension of manifold \( V \) is \( n + 1 \). Moreover, \( \partial V = N \sqcup \partial_{2} V \sqcup \partial_{3} V \) satisfies similar conditions as \( W \);
3. There is a degree one normal map \( F : (V, \partial V) \to (W, \partial W) \) such that \( \Psi \circ F = \Phi \). Moreover, \( F \) restricts to \( f \) on \( N \);
4. \( F|_{\partial_{2} W} : \partial_{2} W \to \partial_{2} V \) is a homotopy equivalence over \( X \);
5. \( F|_{\partial_{2} W} \) restricts to an infinitesimally controlled homotopy equivalence \( F|_{\partial \partial_{2} - W} : \partial \partial_{2} - W \to \partial \partial_{2} - V \) over \( X \).

We denote by \( L_n(\pi_1 X, X) \) the set of equivalence classes from Definition A.7 under equivalence relation from Definition A.8. Note that \( L_n(\pi_1 X, X) \) is an abelian group with the sum operation being disjoint union.

Remark A.9. The equivalence relation can also be stated as follows, there exists \( \theta' = (M', \partial_{\pm} M', \phi', N', \partial_{\pm} N', \psi', f') \in L_n(\pi_1 X, X) \), such that

1. \( f' \) is a homotopy equivalence over \( X \);
2. \( \partial_{-} M' = \partial_{+} M, \partial_{-} N' = \partial_{+} N, f'|_{\partial_{+} M'} = f|_{\partial_{+} M}, \psi'|_{\partial_{+} N'} = \psi|_{\partial_{+} N} \);
3. $f'_{\partial M}$ restricts to an infinitesimally controlled homotopy equivalence $f'_{\partial M} : \partial M \rightarrow \partial N$ over $X$.

In fact, one can take $M \sqcup_{\partial M} M'$ as $\partial_3 W$, and $N \sqcup_{\partial N} N'$ as $\partial_3 V$ in definition A.8. Note that both

$$(M \sqcup_{\partial M} M') \sqcup_{\partial_{M}\sqcup_{\partial M} M'} (M \sqcup_{\partial M} M')$$

and

$$(N \sqcup_{\partial N} N') \sqcup_{\partial_{N}\sqcup_{\partial N} N'} (N \sqcup_{\partial N} N')$$

are boundary of some manifold with boundary.

**Definition A.10** (Objects for definition of $S_n(X)$, [16]). An object

$$\theta = (M, \partial M, \phi, N, \partial N, \psi, f)$$

of $S_n(X)$ consists of the following data:

1. Two manifolds with boundary $M$ and $N$ both of dimensional $n$;
2. Continuous maps $\phi : M \rightarrow X$ and $\psi : N \rightarrow X$;
3. Homotopy equivalence $f : (M, \partial M) \rightarrow (N, \partial N)$ such that $\psi \circ f = \phi$. Moreover, on the boundary $f|_{\partial M} : \partial M \rightarrow \partial N$ is an infinitesimally controlled homotopy equivalence over $X$.

**Definition A.11** (Equivalence relation for definition of $S_n(X)$, [16]). Let

$$\theta = (M, \partial M, \phi, N, \partial N, \psi, f)$$

be an object from $S_n(X)$. We write $\theta \sim 0$ if the following conditions are satisfied.

1. There exists a manifold with boundary $W$ and a continuous map $\Phi : W \rightarrow X$.
   The dimension of manifold $W$ is $n + 1$. Moreover, $\partial W = M \sqcup_{\partial M} \partial_2 W$, in particular $\partial M = \partial \partial_2 W$;
2. Similarly, there exists a manifold with boundary $V$ and a continuous map $\Psi : V \rightarrow X$. The dimension of manifold $V$ is $n + 1$. Moreover, $\partial V = N \sqcup_{\partial N} \partial_2 V$, in particular $\partial N = \partial \partial_2 V$;
3. There is a homotopy equivalence $F : (W, \partial W) \rightarrow (V, \partial V)$ such that $\Psi \circ F = \Phi$. Moreover, on the boundary $f|_{\partial M} : \partial M \rightarrow \partial N$ is an infinitesimally controlled homotopy equivalence over $X$.

We denote by $S_n(X)$ the set of equivalence classes from Definition A.10 under equivalence relation from Definition A.11. Note that $S_n(X)$ is an abelian group with the sum operation being disjoint union.

Following [16], we begin to give a description of the surgery long exact sequence based on ideas of Wall. There is natural group homomorphism

$$i_* : N_n(X) \rightarrow L_n(\pi_1 X)$$

by forgetting control. Moreover, every element

$$\theta = (M, \partial M, \phi, N, \partial N, \psi, f) \in L_n(\pi_1 X)$$

naturally defines an element in $L_n(\pi_1 X, X)$ by letting $\partial_\cdot M = \emptyset$. 

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We denote the corresponding natural homomorphism by

\[ j_\ast : L_n(\pi_1 X) \to L_n(\pi_1 X, X). \]

Moreover, for \( \theta = (M, \partial M, \phi, N, \partial N, \psi, f) \in L_n(\pi_1 X, X) \), we see that \( \theta_- = (\partial M, \partial^\pm \partial M, \phi, \partial^\pm N, \partial\partial^\pm N, \psi, f) \) defines an element in \( N_{n-1}(X) \).

We denote the corresponding natural homomorphism by

\[ \partial_\ast : L_n(\pi_1 X, X) \to N_{n-1}(X). \]

**Theorem A.12** ([16, Theorem 3.14]). We have the following long exact sequence

\[ \cdots \to N_n(X) \overset{i}{\to} L_n(\pi_1 X) \overset{j}{\to} L_n(\pi_1 X, X) \overset{\partial}{\to} N_{n-1}(X) \to \cdots. \]

We shall identify the groups \( S_n(X) \) and \( L_{n+1}(\pi_1 X, X) \). There is a natural group homomorphism

\[ c_\ast : S_n(X) \to L_{n+1}(\pi_1 X, X) \]

by mapping

\[ \theta = (M, \partial M, \phi, N, \partial N, \psi, f) \mapsto \theta \times I \]

where \( \theta \times I \) consists of the following data:

1. Manifold \( (M \times I, \partial \pm (M \times I)) \) with \( \partial_+(M \times I) = M \times \{0\} \) and \( \partial_-(M \times I) = \partial M \times I \sqcup M \times \{1\} \). In particular, we have \( \partial \partial_+(M \times I) = \partial M = \partial \partial_-(M \times I) \). Similarly, for \( N \) we have manifold \( (N \times I, \partial \pm (N \times I)) \);

2. Continuous maps \( \tilde{\phi} := \phi \circ p : M \times I \overset{p}{\to} M \overset{\phi}{\to} X \) and \( \tilde{\psi} := \psi \circ p : N \times I \overset{p}{\to} N \overset{\psi}{\to} X \), where \( p \) is the canonical projection map from \( M \times I \to M \) or from \( N \times I \to N \);

3. A degree one normal map \( \tilde{f} := f \times Id : (M \times I, \partial \pm (M \times I)) \to (N \times I, \partial \pm (N \times I)) \) such that \( \tilde{\psi} \circ \tilde{f} = \tilde{\phi} \);

4. The restriction \( \tilde{f}|_{\partial_+(M \times I)} : \partial_+(M \times I) \to \partial_+(N \times I) \) is a homotopy equivalence over \( X \);

5. The restriction \( \tilde{f}|_{\partial_-(M \times I)} : \partial_-(M \times I) \to \partial_-(N \times I) \) is a degree one normal map over \( X \);

6. The homotopy equivalence \( \tilde{f}|_{\partial_+(M \times I)} : \partial_+(M \times I) \to \partial_+(N \times I) \) over \( X \) restricts to an infinitesimally controlled homotopy equivalence \( \tilde{f}|_{\partial \partial_+(M \times I)} : \partial \partial_+(M \times I) \to \partial \partial_+(N \times I) \) over \( X \).

For

\[ \theta = (M, \partial_\pm M, \phi, N, \partial_\pm N, \psi, f) \in L_{n+1}(\pi_1 X, X), \]

we see that

\[ \theta_+ = (\partial_\pm M, \partial \partial_\pm M, \phi, \partial_\pm N, \partial \partial_\pm N, \psi, f) \]

defines an element in \( S(X) \).

We denote the natural group homomorphism by

\[ r_\ast : L_{n+1}(\pi_1 X, X) \to S_n(X). \]

It follows from definition that the homomorphisms \( c_\ast \) and \( r_\ast \) are well defined. By [16, Proposition 3.16] the homomorphisms \( c_\ast \) and \( r_\ast \) are inverses of each other. In particular, we have \( S_n(X) \cong L_{n+1}(\pi_1 X, X) \). Hence, we have the following theorem.
Theorem A.13 ([16]). We have the following long exact sequence:
\[ \cdots \to N_{n+1}(X) \to L_{n+1}(\pi_1 X) \to S_n(X) \to N_n(X) \to \cdots. \]

Remark A.14. When \( X \) is of dimension \( n \geq 5 \), then by [16, Proposition 3.18] and [16, Proposition 3.19] \( S_n(X) \) is isomorphic to \( S^{TOP}(X) \).

Our paper is based on the Proposition A.16. The point here is that in the definition of \( S_{\infty}^C(X) \) we use smooth manifolds with boundary. Hence, our constructions in previous sections works. We first introduce \( S_{\infty}^C(X) \).

Definition A.15 ([16, Definition 3.21]). An element \( \theta \in S_{\infty}^C(X) \) consists of the following data:

1. \( \theta \) is an element of \( S_n(X) \);
2. \( M \) and \( N \) are smooth manifolds with boundary, and the map \( f : M \to N \) is smooth.

Proposition A.16 ([16, Proposition 3.22]). For \( n \geq 5 \), we have natural isomorphisms
\[ S_{\infty}^C(X) \cong S_n(X). \]
In particular, every element in \( S_n(X) \) has a smooth representative.

B Poincaré duality operators

In this appendix we recall the concept of Poincaré duality operator. Our definition is slightly different from the one in [4, 16]. We adapt this definition for sake of simplicity.

We shall begin with a complex of Hilbert spaces
\[
\mathcal{H}_0 \xrightarrow{d_1} \mathcal{H}_1 \xrightarrow{d_2} \cdots \xrightarrow{d_n} \mathcal{H}_n
\]
where the spaces \( \mathcal{H}_i \) are Hilbert spaces, and the operators \( d_i \) are closable, unbounded operators. We shall denote by \( \mathcal{H} := \oplus \mathcal{H}_i \) the direct sum of \( \mathcal{H}_i \). Denote by \( d := \oplus d_i \) the direct sum of \( d_i \), acting on \( \mathcal{H} \). It is a closable operator, and \( d^2 = 0 \).

Definition B.1. Let \( (\mathcal{H}, d) \) be a complex of Hilbert spaces. We shall call bounded operator \( S : \mathcal{H} \to \mathcal{H} \) a Poincaré duality operator of \( (\mathcal{H}, d) \) if

1. \( S \) is self-adjoint, and \( S : \mathcal{H}_p \to \mathcal{H}_{n-p} \);
2. \( S \) maps the domain of \( d \) into the domain of \( d^* \), and \( Sd + d^* S = 0 \);
3. \( S \) induces an isomorphism from the homology of the dual complex
\[
\mathcal{H}_n \xrightarrow{d_n^*} \mathcal{H}_{n-1} \xrightarrow{d_{n-1}^*} \cdots \xrightarrow{d_{1}^*} \mathcal{H}_0
\]
to the homology of the complex \( (\mathcal{H}, d) \).

Denote \( d + d^* \) by \( D \), then by [4, section 5] we know that \( D \pm S \) are invertible.
Example B.2. Let $M$ be a closed smooth manifold of dimension $n$ and let $G \to \tilde{M} \to M$ be a $G$-covering of $M$, where $G$ is a discrete group. We have de Rham complex

$$
\Lambda^0(\tilde{M}) \xrightarrow{d_{\tilde{M}}} \Lambda^1(\tilde{M}) \xrightarrow{d_{\tilde{M}}} \cdots \xrightarrow{d_{\tilde{M}}} \Lambda^n(\tilde{M}),
$$

of smooth (compactly supported) differential forms on $\tilde{M}$. Passing to $L^2$-completions, we obtain Hilbert spaces complex

$$
\Lambda^0_{L^2}(\tilde{M}) \xrightarrow{d_{\tilde{M}}} \Lambda^1_{L^2}(\tilde{M}) \xrightarrow{d_{\tilde{M}}} \cdots \xrightarrow{d_{\tilde{M}}} \Lambda^n_{L^2}(\tilde{M}).
$$

Let $\ast$ be the usual Hodge-$\ast$ operator. Then

$$
S_{\tilde{M}}(\omega) = i^{p(p-1)+|\tilde{G}|} \ast \omega, \omega \in \Lambda^p(\tilde{M}).
$$

is a Poincaré duality operator of $(L^2(\Lambda(\tilde{M})),d)$.

Example B.3. Let $X$ be a closed oriented topological manifold of dimension $n$ and let $G \to \tilde{X} \to X$ be a $G$-covering of $X$, where $G$ is a discrete group. Moreover, we have the following maps

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\phi & \downarrow & \psi \\
X & \downarrow & \\
\end{array}
$$

where, $f : M \to N$ is a smooth orientation-preserving homotopy equivalence between two closed manifolds $M$ and $N$, $\phi$ and $\psi$ are continuous maps, such that $\phi = \psi \circ f$. Let $\tilde{N} = \psi^*(\tilde{X})$ and $\tilde{M} = f^*(\tilde{N}) = \phi^*(\tilde{X})$ be the pull back coverings. We have the following complex of Hilbert spaces

$$
\Lambda^0_{L^2}(\tilde{M}) \oplus \Lambda^0_{L^2}(\tilde{N}) \xrightarrow{d} \Lambda^1_{L^2}(\tilde{M}) \oplus \Lambda^1_{L^2}(\tilde{N}) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n_{L^2}(\tilde{M}) \oplus \Lambda^n_{L^2}(\tilde{N}),
$$

where, $d := \begin{pmatrix} d_{\tilde{M}} & 0 \\ 0 & -d_{\tilde{N}} \end{pmatrix}$.

We denote $S := \begin{pmatrix} S_{\tilde{M}} & 0 \\ 0 & -S_{\tilde{N}} \end{pmatrix}$. Then $S$ is a Poincaré duality operator of $(L^2(\Lambda(\tilde{M}))) \oplus L^2(\Lambda(\tilde{N})),d)$. Moreover, by item 5 at the end of subsection 3.1 and Lemma 3.2, for $t \in [0,1]$, operators

$$
\begin{pmatrix}
S_{\tilde{M}} & 0 \\
0 & (1-t)S_{\tilde{N}} - tT_{\tilde{f}}^*S_{\tilde{M}}T_{\tilde{f}} \\
\end{pmatrix}, \begin{pmatrix}
\cos(t\tilde{\pi})S_{\tilde{M}} & \sin(t\tilde{\pi})S_{\tilde{M}}T_{\tilde{f}} \\
\sin(t\tilde{\pi})T_{\tilde{f}}^*S_{\tilde{M}} & -\cos(t\tilde{\pi})T_{\tilde{f}}^*S_{\tilde{M}}T_{\tilde{f}} \\
\end{pmatrix},
$$

are all Poincaré duality operators of $(L^2(\Lambda(\tilde{M}))) \oplus L^2(\Lambda(\tilde{N})),d)$.

Remark B.4. If $X, M, N$ are not assumed to be compact, then “homotopy equivalence” $f$ needs to be replaced by analytically controlled one instead.
C  Proof of product formula

Now, we will prove the even case of the product formula in subsection 4.3. If \( \theta = (M, \partial M, \phi, N, \partial N, \psi, f) \in L_0(\pi_1 X) \), let \( \theta \times \mathbb{R} \) be the product of \( \theta \) and \( \mathbb{R} \), which defines an element in \( L_{n+1}(\pi_1 X) \). Here various undefined terms take the obvious meanings [16].

Note that the construction in subsection 4.1 also applies to \( \theta \times \mathbb{R} \) and defines a \( K \)-theory class in \( K_{n+1}(C^*(X \times \mathbb{R})^G) \). And there is a natural homomorphism \( \alpha : C^*(X)^G \otimes C^*_L(\mathbb{R}) \to C^*(X \times \mathbb{R})^G \).

Proposition C.1. With the same notation as above, we have

\[
  k_n \cdot \alpha_* (\text{Ind}(\theta) \otimes \text{Ind}_L(\mathbb{R})) = \text{Ind}(\theta \times \mathbb{R})
\]

in \( K_{n+1}(C^*(X \times \mathbb{R})^G) \), where \( \text{Ind}_L(\mathbb{R}) \) is the \( K \)-homology class of the signature operator on \( \mathbb{R} \) and \( k_n = 1 \) if \( n \) is even and \( 2 \) if \( n \) is odd.

Let \( Z \) and \( Z' \) be (smooth) manifolds of dimensional \( n \) and \( n' \) respectively. Then \( L^2(\Lambda(Z \times Z')) = L^2(\Lambda(Z)) \otimes L^2(\Lambda(Z')) \). \( d_{Z \times Z'} = d_Z \otimes 1 + 1 \otimes d_{Z'} \), where \( \otimes \) stands for graded tensor product. And \( S_{Z \times Z'}(x \otimes y) = (-1)^{(n-p)q} S_Z(x) \otimes S_{Z'}(y) \), where \( x \otimes y \in \Lambda^p(Z) \otimes \Lambda^q(Z') \).

Proof: With the notations as above, and borrow the notation from subsection 4.1. We use \( \Lambda \) denote \( \Lambda(\hat{M}_\infty) \oplus \Lambda(\hat{N}_\infty) \), and \( \Lambda_{x \otimes \mathbb{R}} \) denote \( \Lambda(\hat{M}_\infty \times \mathbb{R}) \oplus \Lambda(\hat{N}_\infty \times \mathbb{R}) \).

By \( h dt \mapsto h \) we can identify \( \Lambda^1(\mathbb{R}) \) with \( \Lambda^0(\mathbb{R}) \). With this identification, we have \( d^*_R = -d_R \) and \( d_R \) is skew-adjoint.

Then we have

\[
\Lambda^0_{x \otimes \mathbb{R}} = (\Lambda^{\text{even}} \otimes \Lambda^1(\mathbb{R})) \oplus (\Lambda^{\text{odd}} \otimes \Lambda^0(\mathbb{R})) \\
\cong (\Lambda^{\text{even}} \oplus \Lambda^{\text{odd}}) \otimes \Lambda^0(\mathbb{R}) = \Lambda \otimes \Lambda^0(\mathbb{R}); \\
\Lambda^0_{x \otimes \mathbb{R}} = (\Lambda^{\text{even}} \otimes \Lambda^0(\mathbb{R})) \oplus (\Lambda^{\text{odd}} \otimes \Lambda^1(\mathbb{R})) \\
\cong (\Lambda^{\text{even}} \oplus \Lambda^{\text{odd}}) \otimes \Lambda^0(\mathbb{R}) = \Lambda \otimes \Lambda^0(\mathbb{R}).
\]

If we denote

\[
d_\infty := \begin{pmatrix} d_{\hat{M}_\infty} & 0 \\ 0 & d_{\hat{N}_\infty} \end{pmatrix}, \quad d^*_{\infty} := \begin{pmatrix} d_{\hat{M}_{x \otimes \mathbb{R}}} & 0 \\ 0 & d_{\hat{N}_{x \otimes \mathbb{R}}} \end{pmatrix}
\]

and \( D_R := id_R \).

Then as a map from \( \Lambda \otimes \Lambda^0(\mathbb{R}) \) to \( \Lambda \otimes \Lambda^0(\mathbb{R}) \). We have

\[
d_\infty + d^*_{\infty} = (d_\infty + d^*_\infty) \otimes 1 - 1 \otimes iD_R; \\
S_{(\hat{M}_{x \otimes \mathbb{R}}) \times \mathbb{R}} = S_{\hat{M}_{x \otimes \hat{N}_{\infty}}} \otimes 1.
\]

We now define the “Poincaré duality operators”. As the equation above we can take the “Poincaré duality operators” as \( S_f \otimes 1 \) and \( S'_f \otimes 1 \). And obviously, everything involved here is analytically controlled over \( X \times \mathbb{R} \). By Proposition 4.4. We know that \( \text{Ind}(\theta \times \mathbb{R}) \) can be represented by invertible operator

\[
(D \otimes 1 - 1 \otimes iD_R + \alpha S_f \otimes 1)(D \otimes 1 - 1 \otimes iD_R - \alpha S'_f \otimes 1)^{-1}
\]

in \( B(L^2(\Lambda^0_{x \otimes \mathbb{R}})) \).
Recall that \((D + \alpha S_f)\) and \((D - \alpha S'_f)\) is a self-adjoint invertible operator. Therefore, \((D + \alpha S_f)\) and \((D - \alpha S'_f)\) is homotopic to \(P_+(D + \alpha S_f) - P_-(D + \alpha S_f)\) and \(P_+(D - \alpha S'_f) - P_-(D - \alpha S'_f)\) through a path of invertible elements, where \(P_\pm(D + \alpha S_f)\) and \(P_\pm(D - \alpha S'_f)\) is the positive/negative projection of \((D + \alpha S_f)\) and \((D - \alpha S'_f)\).

For notational simplicity, we use \(P_\pm\) and \(P'_\pm\) denote \(P_\pm(D + \alpha S_f)\) and \(P_\pm(D - \alpha S'_f)\) respectively.

We see that
\[
D \otimes 1 - 1 \otimes iD_R + \alpha S_f \otimes 1 = (D + \alpha S_f) \otimes 1 - 1 \otimes iD_R
\]
is homotopic to
\[
(P_+ - P_-) \otimes 1 - (P_+ + P_-) \otimes iD_R = P_+ \otimes (1 - iD_R) - P_- \otimes (1 + iD_R).
\]
Similarly,
\[
D \otimes 1 - 1 \otimes iD_R - \alpha S'_f \otimes 1 = (D - \alpha S'_f) \otimes 1 - 1 \otimes iD_R
\]
is homotopic to
\[
(P'_+ - P'_-) \otimes 1 - (P'_+ + P'_-) \otimes iD_R = P'_+ \otimes (1 - iD_R) - P'_- \otimes (1 + iD_R).
\]

A routine calculation shows that
\[
(P'_+ \otimes (1 - iD_R) - P'_- \otimes (1 + iD_R))^{-1} = P'_+ \otimes (1 - iD_R)^{-1} - P'_- \otimes (1 + iD_R)^{-1}.
\]

It follows that,
\[
[\left(P_+ \otimes (1 - iD_R) - P_- \otimes (1 + iD_R)\right)\left(P'_+ \otimes (1 - iD_R)^{-1} - P'_- \otimes (1 + iD_R)^{-1}\right)] \\
= [P_+ P'_+ \otimes 1 - P_+ P'_- \otimes (1 - iD_R)) (1 + iD_R)^{-1} \\
+ P_- P'_+ \otimes (1 + iD_R)(iD_R - 1)^{-1} \\
+ P_- P'_- \otimes 1] \\
= [(1 - P_+) \otimes 1)(P'_+ \otimes (1 + iD_R)(iD_R - 1)^{-1} + (1 - P'_+) \otimes 1) \\
+ (P_+ \otimes (iD_R - 1)(1 + iD_R)^{-1}] \\
= ([P_+] - [P'_+]) \otimes [(D_R + i)(D_R - i)^{-1}]
\]

the last term is precisely \(\text{Ind}(\theta) \otimes \text{Ind}_L(\mathbb{R})\).

To summarize, when \(n\) is even, we have proved that
\[
\alpha_\ast (\text{Ind}(\theta) \otimes \text{Ind}_L(\mathbb{R})) = \text{Ind}(\theta \times \mathbb{R}).
\]

\[
\square
\]

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