$p$-Branes Electric-Magnetic Duality and Stueckelberg/Higgs Mechanism

--- A Path-Integral Approach ---

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(Received June 18, 1999)

We study the vacuum functional for a system of $p$-branes interacting with Maxwell fields of higher rank. This system represents a generalization of the usual electrodynamics of point particles, with one essential difference: the world-history of a $p$-brane, due to the spatial extension of the object, may possess a physical boundary. Thus, the objective of this study is twofold. First, we wish to exploit the breaking of gauge invariance due to the presence of a physical boundary in order to generate mass as an alternative to the Higgs mechanism. Second, we wish to investigate how the new mechanism of mass generation is affected by the duality transformation between electric and magnetic branes.

The entire analysis is performed using the path-integral method, as opposed to the more conventional canonical approach. The advantage of the path-integral formulation is that it enables us to Fourier transform the field strength (rather than the gauge potential) directly. To our knowledge, this field strength formulation represents a new application of the path-integral method, and it leads, in a straightforward way, to the dual representation of the vacuum functional. We find that the effect of the dual transformation is essentially that of exchanging the roles of the gauge fields defined respectively on the “bulk” and “boundary” of the $p$-brane history.

§1. Introduction

1.1. Synopsis and objectives

The electric-magnetic duality for closed $p$-branes embedded in a $D$-dimensional target space was established many years ago, leading to the general correspondence that the dual of a $p$-brane is a $\tilde{p}$-brane with $\tilde{p} = D - p - 4$. Table I illustrates this correspondence.

For instance, the brane solution of ($D = 11$) supergravity is the “magnetic dual” of an electric membrane, and there is a possibility that this type of solution represents the basic geometric elements of a unified theory of all fundamental interactions. Among the open questions that such a “final” theory must address, a preeminent one concerns the mechanism of mass generation in the universe. The celebrated
Table I. Closed $p$-brane duality in $D = 11$ spacetime dimensions.

| $p$ | $p$-brane | $\tilde{p} = 7 - p$ dual-brane |
|-----|-----------|-------------------------------|
| 0   | point particle | 7-brane                      |
| 1   | string      | 6-brane                       |
| 2   | membrane    | 5-brane                       |
| 3   | bag         | 4-brane                       |

Higgs mechanism was invented, and successfully applied, within the framework of local quantum field theory, i.e., a “low energy” framework dealing with the interactions of point particles in the Standard Model. Since point particles are currently thought of as low-energy manifestations of the underlying dynamics of strings and higher dimensional objects, it seems pertinent to ask two fundamentally related questions. First, what is the “engine” of mass production at the level of $p$-brane dynamics, say at the string scale of energy and beyond? Then, how is the new mechanism of mass generation affected by the duality transformation, which plays such a significant role in the theory of extended objects?

With the above questions in mind, we shall extend the notion of electric-magnetic duality to the case of open $p$-branes. Apart from its intrinsic interest as an example of a duality transformation, this extension of electric-magnetic duality will enable us to incorporate a new mechanism of mass generation which stems directly from the presence of a boundary and from a non-trivial interaction between gauge fields of different rank defined respectively on the bulk and boundary of the $p$-brane history. To our mind, this boundary effect, and concomitant mixing of gauge fields, represents a geometric realization of the Higgs mechanism never discussed in the physics of point particles for the simple reason that the world history of a point particle is usually assumed to have no boundary. On the other hand, the new mechanism of mass generation is precisely a boundary effect, and in order to fully appreciate its meaning, one must keep in mind the historical relationship between mass and gauge invariance. At first glance, these two concepts may seem contradictory, in the sense that mass explicitly breaks gauge invariance. On the other hand, mass is required for obvious experimental reasons, while gauge invariance is the essential prerequisite for the self-consistency (for instance, renormalizability) of any successful model of particle interactions. In the Standard Model, mass and gauge invariance are reconciled through the loophole of spontaneous symmetry breaking, and the ensuing Higgs mechanism. Upon fixing a gauge (say the unitary gauge), the role of the mass becomes manifest.

Thus, assuming that gauge invariance is just as important in the theory of extended objects as it is for point particles, what we wish to suggest in this paper is a geometric variation of the Higgs mechanism which is consistent with the requirement of gauge invariance in the theory of $p$-branes, both electric and magnetic, in interaction with higher rank gauge fields.

On the mathematical side, the originality of our approach stems from a new application of path-integral techniques to open $p$-branes. A self-contained discussion of this “sum over histories approach” is presented in Appendix A, which constitutes the mathematical backbone of the paper. This method is centered on the use of a functional representation of the Dirac-delta distribution as a tool to implement the duality transformation directly on the gauge field strength, rather than the gauge
potential. As we shall see, this new technique has the following desirable properties: i) it clearly separates the dynamics of the “bulk” from the dynamics of the “boundary” of the p-brane history; ii) it illustrates how bulk and boundary fields exchange their roles as a consequence of the duality transformation; and, iii) it is especially advantageous in handling the Bianchi identities in higher rank gauge theories that include both “electric” and “magnetic” objects.

1.2. Conventions and outline

In the following analysis, we assume that the dimensionality of the target space-time satisfies \( D > p + 1 \), in order to deal with dynamical extended objects. The interesting limiting case \( D = p + 1 \), in which there are no propagating degrees of freedom, requires a separate discussion that can be found in a recent publication. In order to elucidate the physical content of our discussion with a bare minimum of formalism, we have confined the bulk of our calculations to one self-contained appendix. However, we believe that those calculations represent, by themselves, a new and noteworthy contribution to the literature devoted to the case of open branes, especially in view of the increasing importance of extended objects, such as D-branes, in the formulation of non-perturbative string theory. Furthermore, in order to avoid a cumbersome proliferation of indices, in the main text we shall adopt the following index-free notation:

\[
A^{(p+1)} \equiv A_{\mu_1...\mu_{p+1}},
\]

\[
A_{(p+1)} \equiv A_{\mu_1...\mu_{p+1}},
\]

\[
A_{(p+1)} J^{(p+1)} \equiv \frac{1}{(p+1)!} A_{\mu_1...\mu_{p+1}} J^{\mu_1...\mu_{p+1}},
\]

\[
\partial J^{(p+1)} \equiv \partial_{\mu_1} J^{\mu_1...\mu_{p+1}},
\]

\[
dJ^{(p+1)} \equiv \partial^{[\nu_1} J_{\mu_1...\mu_{p+1}]},
\]

\[
dA^{(p+1)} \equiv \partial_{[\mu_1} A_{\mu_1...\mu_{p+1}]},
\]

\[
\frac{1}{!} J^{(p+1)}(x) \equiv \int dy G(x-y) J^{(p+1)}(y),
\]

\[
\ast K^{(p+1)} \equiv \frac{1}{(D-p-1)!} \epsilon^{\nu_1...\nu_{p+1}\nu_{p+2}...\nu_d} K_{\nu_p+2...\nu_d},
\]

\[
D_{(D-p-2)} \ast dI^{(p+1)} = (1)^{Dp} I^{(p+1)} \ast dD^{(D-p-2)},
\]

\[
N^{(p+1)} = (1)^{D(p+1)-(p+1)^2} \ast (N^{(p+1)}),
\]

(1.1)

For book-keeping purposes, in the above list, the upper or lower index on the left-hand side is a reminder of the actual number of indices carried by the corresponding tensor on the right-hand side. Finally, as a further notational simplification, we shall omit the volume of integration symbol, \( d^D x \), wherever a spacetime integration is performed.

With the above remarks and conventions in mind, the plan of the paper is as follows. In §2, we discuss the Stueckelberg mechanism for open electric p-branes. This example clarifies how mass is tied up with the existence of a boundary and how gauge invariance is preserved, albeit in an extended form. In §3, we discuss
the quantization condition of electric and magnetic charges as a consequence of the arbitrariness in the choice of the “Dirac parent brane,” followed by a brief introduction to the duality symmetry among electric and magnetic \( p \)-branes. In \( \S 4 \), we perform a duality rotation and study the Stueckelberg mechanism in the “magnetic phase” of the model. Section 5 is devoted to summary and discussion of the results. Appendix A provides the details of the computations leading to the dual action for a system of interacting electric and magnetic \( p \)-branes. A set of accompanying Tables should help the reader to keep track of the definitions, and to correlate at a glance the essential components of the theory.

\section*{2. Electric Stueckelberg mechanism for open \( p \)-branes}

\subsection*{2.1. Background}

In order to place our work in the right perspective, we begin this section by briefly reviewing the manner in which the interaction of a \( p \)-brane with an antisymmetric tensor field can be described. This interaction is a higher-dimensional generalization of what is usually done in electromagnetic theory: there, we have a 0-dimensional object, namely a point particle, sweeping a 1-dimensional world-line. The most natural (i.e., “geometric”) way to couple a material particle to a field is through its tangent element. This is a vector field in the tangent bundle, and it gives rise to a current which, in turn, is coupled to a 1-form potential. At this point, one may contemplate a generalization of the electromagnetic scheme, and at least two possibilities come to mind. Historically, the first extension of electrodynamics was applied to non-abelian gauge fields in order to include internal symmetries with an eye on the weak and strong interactions, and it paved the way to a formulation of the phenomenologically successful theories embodied in the Standard Model of particle physics. With the advent of string and membrane theory as the only paradigm capable, at least in principle, of unifying gravity with the other fundamental interactions, the possibility of a new generalization of the electromagnetic scheme has emerged: in addition to the \( p = 0 \) case, why not consider arbitrary values of \( p \) and have a theory of \( p \)-dimensional objects endowed with \((p + 1)\)-dimensional tangent elements, interacting with \((p + 1)\)-differential forms? Following this universal blueprint, one might think that all formal properties of electromagnetism may be extrapolated to the extended case in a rather straightforward manner. For \textit{closed} \( p \)-\textit{branes}, this is indeed the case,\(^3\) as long as one keeps in mind that the physical properties of \( p \)-brane electrodynamics (namely, the number of degrees of freedom of the object, spin of the radiation field, etc.) depend on the dimensionality \( D \) of the target spacetime, and, for fixed \( D \), on the dimensionality of the \( p \)-brane.\(^4,5\) For instance, as mentioned above, in the limiting case of “bubble-dynamics,” \( D = p + 1 \), there is no radiation field, in contrast to the usual electrodynamics of point charges. On the other hand, in the case of \textit{open} \( p \)-\textit{branes}, one may reasonably expect some new formal, as well as physical properties that have no counterparts in the electrodynamics of point charges. Evidently, such new properties derive from boundary effects, regardless of the values of \( D \) and \( p \). The universality of such boundary effects is of primary
importance to us, since they are deeply intertwined with the gauge invariance of \( p \)-brane electrodynamics. In order to clarify this connection, let us call \( J^{(p+1)} \) the current associated with the \( p \)-brane, and \( A^{(p+1)} \) the corresponding \( p+1 \)-tensor gauge potential. If the \( p \)-brane is closed (that is, if its world-manifold — let us call it \( \mathcal{E} \) — has no boundary, \( \partial \mathcal{E} = \emptyset \)), then the current \( J \) is divergenceless,

\[
\partial J^{(p+1)} = 0, \tag{2.1}
\]

and the corresponding action is invariant under the tensor gauge transformation:

\[
\delta A^{(p+1)} = d\Lambda^{(p)}. \tag{2.2}
\]

It seems worth emphasizing that “charge conservation” is now associated with a topological property of the extended object, namely that it has no boundary. From here, one may infer two things concerning the general case. First, from a mathematical standpoint, one may expect that cohomology plays a central role in the theory, because one has to deal with differential forms of different orders defined on the bulk and boundary of the \( p \)-brane history. From a physical standpoint, on the other hand, one may anticipate that the presence of a boundary breaks gauge invariance, thereby violating the above conservation law. However, in the following subsection, we show that the symmetry can be restored by introducing a compensating field of the type originally suggested by Stueckelberg in the case of point particles. 6) This artifact brings into the theory a dimensional coupling constant, which ultimately leads to a gauge invariant generalized Higgs mechanism for generating mass.

2.2. Boundary effect, gauge invariance and mass

In this subsection, we go directly to the core of the problem and discuss in more detail the case of open \( p \)-branes endowed with electric charge only, in order to illustrate in the simplest case the difference between the open case and the closed case. As we have just seen, the source current of an open \( p \)-brane is not divergence-free, because there is a leakage of current through the boundary, and this breaks the gauge symmetry of the action

\[
S[A, J] = \int \left[ -\frac{1}{2} F^{(p+2)}(A)F_{(p+2)}(A) + e_p A^{(p+1)} J^{(p+1)} \right]. \tag{2.3}
\]

As a matter of fact, we now have

\[
\delta S[A, J]_{A \to A + dA} = e_p \int d\Lambda^{(p)} J^{(p+1)} \\
= e_p \int A^{(p)} \partial J^{(p+1)} \\
= e_p \int A^{(p)} j^{(p)} \neq 0. \tag{2.4}
\]

Furthermore, since the boundary of a \( p \)-brane is a world-manifold itself, it can also be described in terms of a current, say \( j^{(p)} \), which takes into account by how much the source of the \( p \)-brane fails to be conserved. Mathematically, this means

\[
j^{(p)} = \partial J_e^{(p+1)}. \]
However, we wish to show that the gauge invariance of the theory can be restored, along with the conservation law, by artificially “closing” the boundary of the $p$-brane. The artifact that is capable of doing this was discovered long ago by Stueckelberg in his attempt to construct a gauge-invariant theory of a massive vector field.\(^6\) With hindsight, the Goldstone boson, which is instrumental for the Higgs mechanism, is simply a Stueckelberg compensating field that provides the longitudinal degree of freedom necessary to turn a gauge field into a massive vector field without losing gauge invariance. This old recipe is the template that we wish to use “mutatis mutandis” in the case of open $p$-branes. Accordingly, we introduce a compensating antisymmetric tensor field coupled to the boundary of the extended object by modifying the action (2.3) as

\[
Z[N_e] = \frac{1}{Z[0]} \int [DA][DC] e^{-S[A,C,J_e]},
\]

\[
S[A,C,N_e] = \int \left[ -\frac{1}{2} F_{(p+2)}(A) F_{(p+2)}(A) + e_p \left( A_{(p+1)} - dC_{(p)} \right) J_{(p+1)}^{(p+1)} + \kappa \left( A_{(p+1)} - dC_{(p)} \right)^2 \right].
\]

(2.5)

The above action is gauge invariant, provided that the new field $C$ (the Stueckelberg compensating field), responds to the transformation of the gauge potential as

\[
\delta A_{(p+1)} = dA_p, \quad \delta C_p = A_p.
\]

(2.6)

(2.7)

Now, we wish to demonstrate that this new invariant action can be rearranged, to show how the additional interaction mediated by the Stueckelberg field can be traded off with a massive interaction among the $p$-brane elements. This can be seen as follows.

First, from the action (2.5), we derive the field equations by means of variations with respect to the two gauge potentials, $A$ and $C$, respectively:

\[
\partial F_{(p+2)} + \kappa \left( A_{(p+1)} - dC_{(p)} \right) = e_p J_{(p+1)}^{(p+1)},
\]

(2.8)

\[
\kappa \partial \left( A_{(p+1)} - dC_{(p)} \right) = e_p j^{(p)}.
\]

(2.9)

Second, we solve these two equations in terms of currents and propagators. To this end, it is useful to split $A$ and $J$ into the sum of a divergenceless (hatted) part and a curl-free (tilded) part:

\[
J^{(p+1)} = \hat{J}^{(p+1)} + \tilde{J}^{(p+1)},
\]

(2.10)

\[
\partial \hat{J}^{(p+1)} = 0, \quad d\tilde{J}^{(p+1)} = 0,
\]

(2.11)

\[
\partial J^{(p+1)} = j^{(p)} \rightarrow \hat{J}^{(p+1)} = d \frac{1}{\Box} j^{(p)},
\]

(2.12)

\[
A^{(p+1)} = \hat{A}^{(p+1)} + \tilde{A}^{(p+1)},
\]

(2.13)

\[
\partial \hat{A}^{(p+1)} = 0, \quad d\tilde{A}^{(p+1)} = 0,
\]

(2.14)

\[
F_{(p+2)} = d\hat{A}^{(p+1)} \rightarrow \hat{A}^{(p+1)} = \partial \frac{1}{\Box} F_{(p+2)}.
\]

(2.15)
Then, from Eq. (2.9) we obtain
\[ \tilde{A}^{(p+1)} - dC^{(p)} = \frac{e_p}{\kappa} d \frac{1}{\Box} j^{(p)}, \quad (2.16) \]
while the Maxwell equation (2.8) can be written in terms of the field strength as
\[ \partial \left( \frac{\Box + \kappa}{\Box} F^{(p+2)} \right) = e_p j^{(p+1)} \rightarrow F^{(p+2)} = e_p d \frac{1}{\Box + \kappa} j^{(p+1)}. \quad (2.17) \]

Third, we substitute the solutions into the action (2.5) and obtain
\[ S[J,j] = \int \left[ -\frac{1}{2} F^{(p+2)} F^{(p+2)} - \frac{\kappa}{2} F^{(p+2)} j^{(p+1)} - e_p F^{(p+2)} j^{(p+1)} \right] \]
\[ = \int \left[ -\frac{e_p^2}{2} j^{(p+1)} - \frac{1}{\Box + \kappa} j^{(p+1)} + \frac{e_p^2}{2\kappa} j^{(p)} \right]. \quad (2.18) \]

Thanks to the extended gauge invariance (2.7) introduced by the Stueckelberg compensator, the action (2.18) is written in terms of divergence-free currents. The first term in the last line of Eq. (2.18) represents a short range, bulk interaction mediated by a massive field. On the other hand, the last term in Eq. (2.18) describes a long-range interaction confined to the boundary of the p-brane. The above effects represent the physical output of the Stueckelberg mechanism for electric p-branes. To sum up, one may trade the presence of the boundary, when concentrating on the gauge invariance properties of the theory, with an extra interaction mediated by the compensator field, and reinterpret it as the propagation of massive degrees of freedom on the p-dimensional extended object. With hindsight, one also recognizes the long-range interaction term in Eq. (2.18) as the residual trace of the interaction mediated by the massless Stueckelberg field $C^{(p)}$. This reminds us of a “Meissner-type effect,” in which the compensator field is “expelled” from the bulk and trapped on the boundary of the extended object. It has been suggested that in the limiting case $D = p + 1$, a “secret long-range force” can produce confinement in $D = 2$, and glueball formation in $D = 4$. Accordingly, we expect similar effects in higher dimensions.

In order to investigate the strong coupling dynamics of this mechanism, we have to switch to the “magnetic phase” of the model (2.3). Indeed, from previous results in dual models, one might expect that a strong electric coupling regime can be equivalently described in terms of a dual weak magnetic coupling phase. Thus, in the next section, we turn our attention to the duality procedure that will enable us to switch from the electric to the magnetic phase.

§3. Electric and magnetic branes

3.1. Extension of the Dirac formalism

In this section we are interested in objects carrying electric as well as magnetic charge. They are described by the following action, whose origin can be traced back
to the original work by Dirac on magnetic monopoles: 8)

\[
S[A,G;J] = \int \left[ -\frac{1}{2} \left( F_{(p+2)}(A) - G_{(p+2)}(x; n(\gamma)) \right)^2 + e_p A_{(p+1)} J^{(p+1)} \right] .
\]

(3.1)

Let us consider first the closed case as a testing ground for the subsequent discussion of the open case. In the expression (3.1) we have explicitly separated the usual electromagnetic contribution to the field strength, \( F \), from the magnetic field strength, \( G \), which is the singular part of the electromagnetic field due to the presence of magnetic charge. \((x = n(\gamma))\) represents the parent \((D - p - 3)\)-Dirac-brane. The electric field strength \( F = dA \) originates as the exterior derivative of the electromagnetic gauge potential \( A \), which in turn is coupled to the \((p + 1)\)-dimensional history of the extended object (an electric \( p \)-brane) through the source current \( J \). Since this brane represents a closed object, its source current \( J \) is conserved, and the full action is gauge invariant under the transformation (2.2). Moreover, since \( dF = 0 \), the Bianchi identities are satisfied, and \( F \) (as given by \( A \)) cannot describe magnetic sources. If we insist on having magnetic charges in addition to the electric ones, it is precisely the Bianchi identities that must be invalidated. In fact, this is the role of the \( G \) field, since the Bianchi identities that follow from the action (3.1) are not satisfied on \( M \), i.e., the world-history of the magnetic brane. Thus, to sum up the content of the action (3.1), we have a closed electric \( p \)-brane source, \( J \), coupled to the tensor potential \( A \) from which the \( F \) field originates, and a closed magnetic \((D - p - 4)\)-Dirac-brane, with source current \( \mathcal{J} \), responsible for the violation of the Bianchi identities for \( F \). It may also be useful to briefly discuss the dimensionality of the various components: The world manifold of a \( p \)-brane is \((p + 1)\)-dimensional, and its source current is a \((p + 1)\)-vector which couples to a \((p + 1)\)-form. Then the field is a \((p + 2)\)-form whose Hodge dual is a \((D - p - 2)\)-form. The divergence of the dual field is the magnetic current, which is thus a \((D - p - 3)\)-vector associated with the world-history of a \((D - p - 4)\)-brane. Moreover, note that the magnetic, in Dirac's formulation, is the boundary of a parent brane, \( \mathcal{N} \), and thus it is necessarily closed, because the boundary of a manifold does not have a boundary. The full action (3.1) possesses two distinct gauge symmetries: the original gauge symmetry (2.2), which reflects the fact that the electric brane is closed, and under which \( G \) is inert, and a new magnetic gauge invariance under the combined transformations

\[
G_{(p+2)}(x; U) \rightarrow G_{(p+2)}(x; V) + g_{D-p-4} d(\ast \Omega)^{(p+1)}(x; O),
\]

(3.2)

\[
A_{(p+1)} \rightarrow A_{(p+1)} + g_{D-p-4} (\ast \Omega)^{(p+1)}(x; O).
\]

(3.3)

This new symmetry reflects the freedom in the choice of the parent Dirac brane. More precisely, following Dirac's formulation, the freedom in the choice of a parent brane can be interpreted in terms of the gauge transformation given above, provided that the electric and magnetic charges satisfy the Dirac quantization condition

\[
e_p g_{D-p-4} = 2\pi n \quad \text{(in units for which } \hbar = c = 1),
\]

(3.4)

where \( n = 1, 2, 3, \ldots \). In the absence of the electric current, the action (3.1) is gauge invariant under (3.2) and (3.3). However, in the presence of the (\( AJ \))-interaction
term, the action $S[A,G; J]$ changes as

$$\delta \Omega S = (-1)^p e_p \int \left( \ast \Omega \right)_{(p+1)}(x; \Omega) J^{(p+1)}(x; E).$$

(3.5)

The target space integral of $\ast \Omega J$ is an integer that equals the number of times that the two branes intersect. Thus,

$$\delta \Omega S = (-1)^p e_p g_{D-p-4} \times \text{integer number}. \quad (3.6)$$

If the Dirac quantization condition holds, then

$$\delta \Omega S = \pm 2\pi n \times \text{integer number}. \quad (3.7)$$

Accordingly, the (Minkowskian) path-integral is unaffected by a phase shift $S \rightarrow S \pm 2\pi n$, and the gauge transformation given in (3.2) and (3.3) has no physical consequences for the interacting system (3.1).

3.2. Duality transformation for open electric and magnetic branes

Our specific purpose, now, is to extend the “dualization” procedure to a system of open $p$-branes coupled to higher rank gauge potentials. The purpose of this extension is to study how the dualization procedure, together with the formalism developed above, affects the mass generation mechanism that we have proposed in §2.

We believe that the full impact of the dualization procedure cannot be appreciated without keeping in mind some mathematical properties of the new formalism that we are proposing in this paper. Thus, as a preliminary step, we recall the results of the dualization procedure for closed objects. This case is discussed in detail in §A.1 of Appendix A using the path-integral method. The essential technical point of that approach is that the partition function of a system described, for instance, by the action (3.1) can be written in terms of the gauge invariant variable $F$ in place of the gauge potential $A$. To see this, we note that in the kinetic term the gauge potential $A$ appears only through its field strength $F(A)$, while the interaction term can be written as a constraint, $N \cdot F(A)$, after an integration by parts, where $N \equiv \partial J$ is an “electric parent current”:

$$S_{\text{INT}} = -e_p \int N^{(p+2)} F_{(p+2)}(A).$$

(3.8)

However, in switching to such a field strength formulation, one must exercise some extra care if extended magnetic objects are present. In that case, the field strength is the sum of the curl $dA$ and a “singular magnetic field strength” $G$:

$$F(A) \longrightarrow \bar{F} = dA - G,$$

(3.9)

where $G$ is chosen in such a way that the magnetic brane current $\bar{J}$ enters as a source in the “Bianchi Identities” for $\bar{F}$:

$$\ast d\bar{F} = g \bar{J}.$$  

(3.10)
The “new” Bianchi Identities (3.10) are encoded in the path integral through the following general relation:

\[
\int [DA] W[dA, J, \mathcal{J}] = \int [DF] [Dn] [D\pi] \delta [*dF - gJ] \delta [\partial N - J] \delta [\partial \pi - \mathcal{J}] W[F, N, \mathcal{J}],
\]

(3.11)

where \(n(\xi)\) and \(\pi(\sigma)\) are the embedding functions of the electric and magnetic parent branes, respectively. Note that, for clarity, we have suppressed all the non-essential labels in (3.11). Indices, coupling constants and numerical factors have been reinserted in (A.6).

The net result of the computations described in Appendix A.1 is the following expression for the dual action of (3.1):

\[
S[H, G, B, N, J] = - \int [dB]^{(D-p-3)} - i (-1)^{(p+1)} e_p (*N)^{(D-p-2)}
- i \int [gB^{(D-p-3)} J^{(D-p-3)} - e_p (*N)^{(p+2)} N^{(p+2)}].
\]

(3.12)

The first term in (3.12) is the kinetic term for the dual field strength. The regular part is the curl of the dual gauge potential, while the singular part is the dual of the electric brane current carrying the electric brane on its boundary. This term is the dual of the original kinetic term in (3.1).

The second term in (3.12) represents the coupling between the dual potential and the magnetic brane current. The strength of the coupling is represented by the magnetic charge \(gd_{p-4}\). It is worth observing that the strength of this coupling is the inverse of the original electric coupling, thanks to the Dirac quantization condition (3.7). Thus, an effect of the dualization procedure is to reverse the value of the coupling constant. Hence, a system of strongly coupled electric branes can be mapped onto a system of weakly coupled magnetic branes and vice versa. In current parlance, this is an example of the “S-duality” connecting the strong-weak coupling phases of a physical system.

Finally, the third term in (3.12) describes a contact interaction between “parent” branes. This term raises a potential problem: As we have seen, the Dirac brane is a “gauge artifact” in the sense that its motion can be compensated for by an appropriate gauge transformation. However, when the electric or magnetic brane coordinates are varied, extra contributions may enter the equation of motion, leading to physical effects. In order to avoid this inconsistency, we have to invoke the “Dirac veto”, namely, the condition that the Dirac brane world surface does not intersect the world surface of any other charged object. With this condition, no extra contribution enters the equation of motion. In this connection, we must emphasize that the physical object is the dual brane, coupled to the \(B\)-potential, and not the Dirac brane, which still maintains its pure gauge status.

Finally, note that the dual current \(J^{(D-p-3)}\) is divergenceless. Hence, it has support over the world-manifold of a closed \((D-p-4)\)-brane and acts as a conserved
source on the r.h.s. of the $B$-field Maxwell equation
\[
\partial \left[ d B^{(D-p-3)} - \iota (-1)^{p+1} e_p (\ast \mathcal{N})^{(D-p-2)} \right] = \iota g_{D-p-4} \mathcal{J}^{(D-p-3)}. \tag{3.13}
\]

In order to check the consistency of the above results, consider the two systems defined by Eqs. (3.1) and (3.12) in a vacuum, i.e., when the sources $J$ and $\mathcal{J}$ are switched off. In this case, we have two sets of field equations and Bianchi identities. The first set is given by
\[
\begin{align*}
\partial F^{(p+2)}(A) &= 0, \quad \text{(Maxwell equations)} \tag{3.14} \\
d F^{(p+2)}(A) &= 0, \quad \text{(Bianchi identities)} \tag{3.15}
\end{align*}
\]
while the second set involves the $B$-field
\[
\begin{align*}
\partial d B^{(D-p-3)} &= 0, \quad \text{(Maxwell equations)} \tag{3.16} \\
d d B^{(D-p-3)} &= 0. \quad \text{(Bianchi identities)} \tag{3.17}
\end{align*}
\]
Thus, we recover the familiar result that the (classical) “duality rotation,”
\[
F^{(p+2)} = \ast d B^{(D-p-3)}, \tag{3.18}
\]
exchanges the roles of the Maxwell equations and Bianchi identities:
\[
\begin{align*}
\text{A-Maxwell equations} &\leftrightarrow \text{B-Bianchi identities}, \tag{3.19} \\
\text{B-Maxwell equations} &\leftrightarrow \text{A-Bianchi identities}. \tag{3.20}
\end{align*}
\]
Of course, these relations also hold if we switch on both the electric and magnetic sources. In conclusion, by manipulating the path-integral representation of $Z[J]$, we have obtained the duality relation between relativistic extended objects of different dimensionality,
\[
\tilde{p} = D - p - 4, \tag{3.21}
\]
and the code of correspondence between dual quantities can be summarized as follows:
\[
\begin{align*}
A^{(p+1)} &\leftrightarrow B^{(D-p-3)}, \tag{3.22} \\
\text{Field equations (in vacuum)} &\leftrightarrow \text{Bianchi identities}, \tag{3.23} \\
J^{(p+1)} &\leftrightarrow (\mathcal{J})^{(D-p-3)}, \tag{3.24} \\
\text{closed $p$-brane} &\leftrightarrow \text{closed $(D-p-4)$-brane}, \tag{3.25} \\
Z[J] &\leftrightarrow Z[I]. \tag{3.26}
\end{align*}
\]
For the convenience of the reader, a “dictionary” of the various fields and currents in our model is given in Tables II and III.

Finally, it seems worth observing that the dual action (3.12) is gauge invariant under “magnetic gauge transformations”:
\[
\delta_A B^{(D-p-3)} = d \tilde{A}^{(D-p-4)}. \tag{3.27}
\]
Accordingly, $B$ is a massless field which is a solution of the field equation (3.16).
Table II. Closed $p$-brane gauge potentials and fields.

| Field        | Dimension in $\hbar = c = 1$ units | Rank | Physical Meaning                |
|--------------|------------------------------------|------|---------------------------------|
| $A_{\mu_1...\mu_{p+1}}$ | (length)$^{1-D/2}$                | $p + 1$ | “electric” potential            |
| $F_{\mu_1...\mu_{p+2}}$ | (length)$^{-D/2}$                | $p + 2$ | “electric” field strength       |
| $G_{\mu_1...\mu_{p+2}}$ | (length)$^{-D/2}$                | $p + 2$ | “magnetic” field strength       |
| $F_{\mu_1...\mu_{p+2}}$ | (length)$^{-D/2}$                | $p + 2$ | “singular” field strength       |
| $B_{\mu_1...\mu_{D-p-3}}$ | (length)$^{1-D/2}$                | $D - p - 3$ | dual gauge potential           |
| $H_{\mu_1...\mu_{D-p-2}}$ | (length)$^{-D/2}$                | $D - p - 2$ | dual field strength            |

Table III. Closed $p$-brane associated currents.

| Current Density | Dimension in $\hbar = c = 1$ units | Rank | Physical Meaning               |
|-----------------|------------------------------------|------|--------------------------------|
| $J_{\mu_1...\mu_{p+1}}$ | (length)$^{p+1-D}$                  | $p + 1$ | electric current              |
| $N_{\mu_1...\mu_{D-p-2}}$ | (length)$^{-D/2}$                | $D - p - 2$ | parent electric current      |
| $J_{\mu_1...\mu_{D-p-3}}$ | (length)$^{-p-3}$                | $D - p - 3$ | magnetic current             |
| $N_{\mu_1...\mu_{D-p-2}}$ | (length)$^{-D/2}$                | $D - p - 2$ | parent magnetic current       |
| $\Omega_{\mu_1...\mu_{D-p-1}}$ | (length)$^{-D/2}$                | $D - p - 1$ | gauge brane current          |

§4. Magnetic Stueckelberg mechanism

Given the formal apparatus outlined in the previous section, the discussion of §2, in which we applied the Stueckelberg mechanism to restore gauge invariance in a theory of open electric branes, can now be extended to the case in which there is also a magnetic charge present. In the latter case, we are dealing with the following objects:

1. an electric open $p$-brane, with an electric closed $(p - 1)$-boundary;
2. a magnetic open $(D - p - 3)$-brane, with a magnetic closed $(D - p - 4)$-boundary.

Therefore, the action for the full system is a natural generalization of the action (2.5):

$$S = \int \left[ -\frac{1}{2} \left( F_{(p+2)}(A) - G_{(p+2)} \right)^2 + \epsilon_p \left( A_{(p+1)} - dC_{(p)} \right) J_{(p+1)} \right] + \frac{\kappa}{2} \int \left( A_{(p+1)} - dC_{(p)} \right)^2. \quad (4.1)$$

Once again, for the interested reader, the dualization procedure for this action is described in detail in Appendix A.2. Here, we merely report the final result for the dual action,

$$Z[\mathcal{N}_e, \mathcal{J}_g] = \frac{1}{Z[0, 0]} \int [\mathcal{D}_F][\mathcal{D}_I] e^{-S[\mathcal{F}, \mathcal{J}_g]},$$

$$S[\mathcal{D}, \mathcal{B}, \mathcal{N}_e, \mathcal{J}_g] = -\frac{1}{2\kappa} \int \left[ dD^{(D-p-2)} - t (1)^{D(p+1)-(p+1)^2} \epsilon_p N_e^{(D-1)} \right]^2 + \frac{1}{2} \int \left[ D^{(D-p-2)} + (-1)^{Dp} dB^{(D-p-3)} \right]^2$$
\[ -i \int \left[ (-1)^{D_p} N^{(D-p-2)} D_{(D-p-2)} + g_{D-p-4} B_{(D-p-3)} J_g^{(D-p-3)} \right]. \]

(4.2)

Taking a closer look at the dual amplitude (4.2), its most evident feature is the exchange of roles of the compensator and gauge field with respect to the electric phase. Presently, the dual Stueckelberg strength tensor takes on the role of the gauge potential, while the dual gauge potential plays the role of the compensator. Together, these fields provide the new gauge symmetry

\[ \delta A_{(D-p-2)} = dA_{(D-p-3)}, \]

(4.3)

\[ \delta A_{(D-p-3)} = (-1)^{D_p} A_{(D-p-3)}. \]

(4.4)

Solving the field equations in the dual phase, one arrives at the effective action,

\[ S[\hat{N}, \hat{J}_g, J_e] = -\frac{\kappa}{2} \int \hat{N}^{(D-p-2)} \frac{1}{\Box + \kappa} \hat{N}^{(D-p-2)} + \int \left[ \frac{1}{2} g_{D-p-4} \hat{J}_g^{(D-p-3)} \frac{1}{\Box} \hat{J}_g^{(D-p-3)} + \frac{e_p^2}{2\kappa} j_e^{(p)} \frac{1}{\Box} j_e^{(p)} \right], \]

(4.5)

where the effect of the duality transformation becomes more transparent: The bulk interaction is short range, while the boundary interaction is still long range. On the basis of this result, we observe the following pattern of duality relations:

\[ A_{(p+1)} \leftrightarrow D^{(D-p-2)}, \]

(4.6)

\[ \hat{J}^{(p+1)} \leftrightarrow \hat{N}^{(D-p-2)}, \]

(4.7)

\[ C_{(p)} \leftrightarrow B^{(D-p-3)}, \]

(4.8)

\[ \frac{e_p}{\sqrt{\kappa}} j_e^{(p)} \leftrightarrow g_{D-p-4} \hat{J}_g^{(D-p-3)}. \]

(4.9)

In words, the above correspondence tells us that the gauge potential \( A_{(p+1)} \), which in the original phase is coupled to the electric \( p \)-brane current, transforms into \( D^{(D-p-2)} \) and interacts in a gauge invariant way with a \( D-p-3 \)-brane. The Stueckelberg mechanism induces a mass \( \sqrt{\kappa} \) for \( A_{(p+1)} \) and the same mass for \( D^{(D-p-2)} \). In other words, both the electric and magnetic gauge potentials are massive, due to the mixing between different rank tensors. This is the Stueckelberg mechanism operating on higher order tensors rather than vectors and scalars.

In the original action (4.1), the gauge compensator is \( C_{(p)} \), while in the dual action, the Stueckelberg field is the dual gauge potential \( B^{(D-p-3)} \). Overall, the dynamics of the model removes the compensator field from the bulk spectrum and confines it to the brane boundary, where it mediates a long-range interaction.

Finally, in the dual phase the electric parent brane disappears as a physical object; only \( \hat{N}^{(D-p-2)} \) and \( \hat{J}_g^{(D-p-3)} \) enter as conserved sources in the field equation. The only physical electric source is the divergence free current \( j_e^{(p)} \), providing
a boundary electric/magnetic duality symmetry. A similar phenomenon was discovered by Nambu to occur in the dual string model of mesons.\textsuperscript{10} Nambu showed that the mesonic open string acquires physical reality when propagating in the non-trivial Higgs vacuum because of the interaction between the end points (magnetic monopoles) and the charge of the scalar field. In our study, we have considered higher-dimensional open objects, and replaced the Higgs mechanism with the tensor mixing mechanism as the basic engine of mass production at the level of $p$-brane dynamics.

§5. Discussion of results, concluding remarks and outlook

A central issue to be confronted by any theory involving relativistic extended objects is the search for an extension of the Higgs mechanism in order to account not only for the mass spectrum of ordinary matter in the form of elementary particles, but also for the existence of cosmic mass, such as dark matter. A second central issue in the theory of extended objects is to understand the role of “duality” in connecting seemingly different realizations of the underlying $M$-theory. In this paper we have addressed both issues using the relatively familiar testing ground provided by the “electrodynamics of $p$-branes.” The method of investigation that we have described in this paper is an original elaboration of the approach introduced in Refs. 11) and 12), in order to study the strong coupling phase of the Higgs model and its relation with the dual string model. Moreover, the derivation of Eq. (3-4) is an original extension of the method introduced in Ref. 13) for point-like charges and magnetic monopoles.

Our emphasis, throughout the paper, has been on the relationship between the dimensionality of electric and magnetic $p$-branes with an eye on the physical effects associated with the presence of a boundary for open $p$-branes. A compendium of our results is encoded in the dual action (4.2) and in Table V, where we have listed the relevant properties of fields, currents and coupling constants for the open case. In this connection, we observe that the current $K$ has support on the world-manifold of a $(D-p-2)$-brane. Hence, in contrast with the case of closed $p$-branes, we conclude that the spatial dimensionality of dual open branes is given by

$$\tilde{p} = D - p - 3.$$  

(5.1)

It seems worth elaborating slightly on the physical meaning of this formal relationship, since it reflects the central result of our discussion. Dual open objects in $D = 10$ spacetime are listed in Table IV. This list is consistent with the results reported in Ref. 14), that were obtained through a different approach. By comparing Table IV with Table I, one sees at a glance that the pattern of dual objects in $D = 10$ looks like that for closed objects in $D = 11$ supergravity. Far from being an accident, this similarity reflects the fact that a closed $p$-dimensional surface can always be considered as the boundary of an open $(p+1)$-dimensional volume. With hindsight, it is not surprising that the relation (5.1) can be obtained from (3-21) by replacing $p$ with $p+1$. However, the formal relationship (5.1) reflects a deeper physical phenomenon, namely, the generation of mass as a consequence of the presence of a physical boundary. To be
sure, the origin of a mass term is independent of the dualization procedure. As we have shown in §2, it can be traced back to the introduction of the Stueckelberg field, which is necessary to compensate for the leakage of symmetry through the boundary of the $p$-brane. A precursor of this mechanism was discussed, in the prehistory of string theory, by Kalb and Ramond for an open string with a quark-antiquark pair at the endpoints,\(^{15}\) and later extended to the case of an open membrane having a closed string as its boundary.\(^{16}\) Geometrically, the introduction of the compensating field is tantamount to “closing the surface,” thereby restoring gauge invariance, albeit in an extended form. The net result of the whole procedure is that the gauge field acquires a mass through the “mixing” of different gauge potentials. While the origin of mass is strictly a boundary effect, and therefore independent of duality, it seems pertinent to ask how the mixing mechanism, i.e., the relationship between mass, the Stueckelberg field and the gauge potential, is affected by the duality transformation. In order to answer this basic question, raised in the Introduction, in Appendix A we show how to implement the duality procedure in two distinct steps: The duality transformation is first applied to the Stueckelberg sector of the model, and then to the remaining gauge part of the partially dualized action. The final output is a massive, gauge invariant theory for higher rank tensor fields and currents, written in terms of a dual gauge potential and a dual Stueckelberg field. The main result of that laborious work is a pair of (semi-) classical effective actions describing, in a gauge invariant way, the interaction among electric and magnetic branes both in the original, “electric” phase, and in the dual, “magnetic” phase. In the two actions, bulk and boundary dynamics are clearly separated. While the bulk interaction is screened by the mass of the tensor gauge field, the boundary interaction is still long range. This is a somewhat unexpected result, which we interpret as a manifestation of the following holographic principle: Even when the Stueckelberg field is absorbed by the gauge potential (in analogy to the Goldstone boson in the Higgs model), and therefore disappears as a physical excitation in the bulk of the brane, an imprint of the associated long-range interaction is recorded on the boundary as a reminder that gauge invariance is rearranged, but not lost.

In conclusion, in view of the fact that the construction of a Higgs model for $p$-branes is at best tentative,\(^{17}\) with no obvious way of spontaneously breaking gauge symmetry, at present the only gauge invariant way to provide a tensor gauge field with a mass term is through the Stueckelberg mechanism discussed above. The cosmological implications of this new conversion mechanism that transforms the vacuum energy “stored” by a massless tensor gauge potential into massive particles, will be discussed in a forthcoming publication.\(^{18}\)

### Table IV. Open $p$-branes duality in $D = 10$ spacetime dimensions.

| $p$   | $p$-brane   | $\tilde{p} = 7 - p$ dual-brane |
|-------|-------------|--------------------------------|
| 0     | point-particle | 7-brane                        |
| 1     | string       | 6-brane                        |
| 2     | membrane     | 5-brane                        |
| 3     | bag          | 4-brane                        |
A.1. Closed branes

The precise meaning of the term “duality” is implicitly defined by the procedure employed in this subsection. It can be summarized as follows:

1. Exchanging the gauge potential $A$ in favor of the field strength $F$ by introducing the Bianchi identities as a constraint in the path-integral measure.
2. Introducing a “dual” gauge potential $B$ as the “Fourier conjugate” field to the Bianchi identities.
3. Integrating out $F$ and identifying the dual current as the object linearly coupled to $B$.

Step (1). In order to write the current-potential interaction in terms of $F$, we note that, since $J$ has vanishing divergence, it is a boundary current and can be written as the divergence of a “parent” electric current. In other words, there exists a $(p + 2)$-rank current $N^{\mu_1...\mu_{p+2}}(x;\pi)$ such that

$$\partial_\mu N^{\mu_1...\mu_{p+2}}(x;\pi) = J^{\mu_2...\mu_{p+2}}.$$  

(A.1)

$$\partial_\mu N^{\mu_1\mu_2...\mu_{p+2}} = J^{\mu_2...\mu_{p+2}}.$$  

(A.2)

To understand the role of these currents, it is useful to recall once again Dirac’s construction, in which a particle-antiparticle pair can be interpreted as the boundary of an “electric string” connecting them. In our case, which involves higher dimensions, the analogue of a particle-antiparticle pair is a closed $p$-brane, which we interpret as the boundary of an open, $(p + 1)$-dimensional parent brane. From this vantage point, the interaction term in the action can be written as

$$S_{\text{INT}} = -\frac{e_p}{(p + 2)!} \int N^{(p+2)} F^{(p+2)}(A).$$  

(A.3)

We note that one may employ the same procedure directly in the expression for the “non-magnetic” action (3.1) and write it only in terms of the field strength $F$ as follows:

$$S[A, G, N] = \int \left[ -\frac{1}{2} F^{(p+2)} F^{(p+2)} + e_p N^{(p+2)} \left( F^{(p+2)} + G^{(p+2)} \right) \right],$$  

(A.4)

$$F^{(p+2)}(A) \equiv F^{(p+2)}(A) - G^{(p+2)}.$$  

(A.5)

Proceeding with the dualization procedure, we further note that since the interaction term in the action (3.1) is written in the form (3.8), one may also use $F$ as the integration variable in the functional integral. One can do this, i.e., treat $F$ as an independent variable, by imposing the Bianchi identities as a constraint. The relationship between $N$ and $J$ must also be encoded in the path integral. This can be achieved by performing an integration over the parent brane coordinates $N = \ldots$
formed by means of the functional representation

\[ \int [DA] W[dA, G, J, \bar{J}] = \int [D\bar{F}][D\bar{N}][D\bar{\eta}] \delta \left[ d^*\bar{F}(p+2) - g_{D-p-4} \bar{J}^{(D-p-3)} \right] \times \delta \left[ \partial \bar{N}^{(D-p-2)} - g_{D-p-4} \bar{J}^{(D-p-2)} \right] W[F, N, \bar{N}]. \]  
\[ (A.6) \]

The first Dirac delta distribution takes into account the presence of the magnetic brane as the singular surface where the Bianchi identities are violated. The second and third delta functions encode the relationship between the boundary currents \( J \) and \( \bar{J} \) and their respective bulk counterparts \( N \) and \( \bar{N} \). It may be worth emphasizing that the electric and magnetic parent branes enter the path integral as “dummy variables” to be summed over. In other words, the physical sources are the boundary currents \( J \) and \( \bar{J} \) alone. After performing the above operations, the generating functional, written in terms of the total electric-magnetic field strength \( F \), takes the form

\[ \begin{align*}
Z[J, \bar{J}] &= \frac{1}{Z[0]} \int [D\bar{F}][D\bar{N}][D\bar{\eta}] \delta \left[ d^*\bar{F}(p+2) - \bar{J}^{(D-p-3)} \right] \\
&\quad \times \delta \left[ \partial \bar{N}^{(p+2)} - J^{(p+2)} \right] \delta \left[ \partial \bar{N}^{(D-p-2)} - g_{D-p-4} \bar{J}^{(D-p-2)} \right] e^{-S[F, G, \bar{N}]}.
\end{align*} \]
\[ (A.7) \]

Step (2). The “Bianchi identities Dirac delta distribution” can be Fourier transformed by means of the functional representation

\[ \delta \left( d^*\bar{F}(p+2) - g_{D-p-4} \bar{J}^{(D-p-3)} \right) = \int [DB] \exp \left( i \int L(B, F, \bar{J}) \right), \]  
\[ (A.8) \]

where

\[ L(B, F, \bar{J}) = B^{(D-p-3)} \left[ d^*\bar{F}(p+2) - g_{D-p-4} \bar{J}^{(p+3)} \right]. \]  
\[ (A.9) \]

The net result is a “field strength formulation” of the model (A.7), (3.1) in terms of a dynamical field \( F \) and a Lagrange multiplier \( B \):

\[ \begin{align*}
Z[J, \bar{J}] &= \frac{1}{Z[0]} \int [D\bar{N}][D\bar{\eta}][D\bar{F}][DB] \delta \left[ \partial \bar{N}^{(D-p-2)} - g_{D-p-4} \bar{J}^{(D-p-3)} \right] \\
&\quad \times \delta \left[ \partial \bar{N}^{(p+2)} - J^{(p+2)} \right] e^{-S[F, G, \bar{N}]},
\end{align*} \]
\[ (A.10) \]

where

\[ S[F, G, B, \bar{N}, \bar{J}] = - \int \left[ \frac{1}{2} F^{(p+2)}(p+2) + e_p \left( F^{(p+2)}(p+2) + G^{(p+2)}(p+2) \right) \bar{N}^{(p+2)} \\
- \iota (-1)^{D(p+1)} * H^{(p+2)}(B) \bar{F}^{(p+2)}(p+2) - \iota g_{D-p-4} B^{(D-p-3)} \bar{J}^{(D-p-3)} \right], \]  
\[ (A.11) \]

and \( *H \) represents the dual field strength

\[ H^{(D-p-2)}(B) \equiv dB^{(D-p-3)}. \]  
\[ (A.12) \]
Step (3). Finally, we are ready to switch to the dual description of the model by integrating away the field strength $F$. Since the path integral is Gaussian in $F$, the integration can be carried out in closed form:

$$Z[J, \bar{J}] = \frac{1}{Z[0]} \int [D\bar{N}][DN][DB] \delta \left[ \partial N - g_{D-p-4} \bar{J} \right] \delta \left[ \partial \bar{N} - J \right] e^{-S[G,B,N,J]} ,$$

(A.13)

$$S[G, B, \bar{N}, J] = -\frac{1}{2} \int \left[ i(-1)^{(p+1)} (H)^{(p+2)} + e_p N^{(p+2)} \right]^2$$

$$- \int \left[ e_p G_{(p+2)} N^{(p+2)} + i g_{D-p-4} B_{(D-p-3)} J^{(D-p-3)} \right]$$

$$= -\frac{1}{2} \int \left[ dB_{(D-p-3)} - i (-1)^{(p+1)} e_p (N^{(p+2)})^2 \right]^2$$

$$+ 1 \int \left[ g_{D-p-4} B_{(p+3)} J^{(p+3)} + e_p (N^{(p+2)})^2 \right] .$$

(A.14)

A.2. Open branes

Equipped with the formalism developed in the previous subsection, we now wish to consider the extension of the path-integral method to the case of open $p$-branes. As emphasized throughout the paper, the main difference stems from the fact that a gauge invariant action for an open $p$-brane requires the introduction of new gauge fields to compensate for the gauge symmetry “leakage” through the boundary. 4)

Thus, we replace the system (2.3) with

$$Z[N_e] = \frac{1}{Z[0]} \int [DA][DC] e^{-S[A,C,J_e]}$$

$$S[A, C, N_e] = \int \left[ -\frac{1}{2} F^{(p+2)} F_{(p+2)} + e_p \left( A_{(p+1)} - dC_{(p)} \right) N_e^{(p+1)} \right.$$

$$\left. - \kappa \left( dC_{(p)} - A_{(p+1)} \right)^2 \right] ,$$

(A.15)

where $C^{(p)}$ has canonical dimensions $L^{2-D/2}$, and $\kappa$ is a constant introduced for dimensional reasons.

In this case, the divergence of the $p$-brane current is no longer vanishing, but is equal to the current $J_e^{(p)}$ associated with the free boundary of the world-manifold. In other words,

$$\partial N_e^{(p+1)} = J_e^{(p)}. \quad (A.16)$$

However, the action (2.5) is still invariant under the extended gauge transformation:

$$\delta A_{(p+1)} = dA_{(p)} , \quad (A.17)$$

$$\delta A_{(p)} = A_{(p)} . \quad (A.18)$$

Indeed, the role of the $C^{(p)}$-field, which is a Stueckelberg compensating field, is to restore the gauge invariance broken by the boundary of the $p$-brane. Perhaps it
is worth emphasizing that $S[A,C,N_e]$ depends on $C(p)$ only through its covariant curl $dC(p) \equiv \Theta(p+1)$, which is not gauge invariant, but transforms as

$$\delta \Theta(p+1) = d\Lambda(p).$$  \hfill (A.19)

The advantage of the path-integral method for constructing the dual action becomes evident at this point, since we can eliminate the Stueckelberg potential $C(p)$ in favor of its curl $\Theta(p+1)$ by introducing the dual Stueckelberg potential $D(D-p-2)(x)$, as we did in Eq. (A.8):

$$\delta \left[ *d\Theta(p+1) \right] = \int [DD] e^{iS[D,\Theta]}, \hfill (A.20)$$

$$S[D,\Theta] = \int D^{(p+2)} \left( *d\Theta(p+1) \right), \hfill (A.21)$$

$$(\ast K(D))_{(p+1)} = (*dD_{D-p-2}). \hfill (A.22)$$

The resulting vacuum amplitude is

$$Z[N_e] = \frac{1}{Z[0]} \int [DA][DD][DI] e^{-S[A,D,N_e]}, \hfill (A.23)$$

where

$$S[A,D,\Theta,N_e] = \int \left[ -\frac{1}{2} F_{(p+2)}^{(p+2)} + e_p \left( A_{(p+1)} - \Theta_{(p+1)} \right) N_{e(p+1)}^\ast \right.$$

$$\left. + \frac{\kappa}{2} \left( \Theta_{(p+1)} - A_{(p+1)} \right)^2 + \iota(-1)^D_p \Theta_{(p+1)} \left( *K(D) \right)_{(p+1)} \right] \hfill (A.24)$$

represents the “Stueckelberg dual action,” in the sense that the dualization procedure was applied to the field $C(p)$ only.

Translational invariance of the functional integration measure enables us to shift the $I$-field and introduce the gauge invariant field strength $\bar{\Theta}$,

$$\bar{\Theta}^{(p+1)} \equiv \Theta^{(p+1)} - A^{(p+1)}, \hfill (A.25)$$

as a new integration variable instead of $\Theta$.

Once $\bar{\Theta}$ has been integrated out, we find

$$Z[N_e] = \frac{1}{Z[0]} \int [DA][DD] e^{-S[A,D,N_e]}, \hfill (A.26)$$

with

$$S[A,D,N_e] = \int \left[ -\frac{1}{2} F_{(p+2)}^{(p+2)} + \iota(-1)^{D-p} \left( *D \right)^{(p+2)} \left( F_{(p+2)}^{(p+2)} + G_{(p+2)} \right) \right.$$

$$\left. + \frac{1}{2\kappa} \left( e_p N_{e(p+1)}^\ast - \iota(-1)^D_p \left( *K \right)^{(p+1)} \right)^2 \right]. \hfill (A.27)$$
By recognizing that the first line in Eq. (A.27) is the same as (A.4), once \( \ast D \) is identified with \( \mathcal{N} \), i.e.,

\[
e_{p} \mathcal{N}_{e}^{(p+2)} \rightarrow i(-1)^{(D-p)p} \ast(D_{(D-p-2)}), \tag{A.28}
\]

we can write the “complete dual action” without repeating all the previous calculations:

\[
Z[\mathcal{N}_{e}, J_{g}] = \frac{1}{Z[0,0]} \int [DB][DD][Dw] \delta \left[ \partial \mathcal{N}^{D-p-2} - g \mathcal{J}^{D-p-3} \right] e^{-S[B, D, \mathcal{N}_{e}]},
\]

\[
S[G, B, \mathcal{N}_{e}, J_{g}] = \frac{1}{2\kappa} \int \left[ dD^{(D-p-2)} + i(-1)^{(D-p+1)^{2}e_{p}(\ast \mathcal{N}_{e})^{(D-p-1)}} \right]^{2}
- \frac{1}{2} \int \left[ D^{(D-p-2)} - \mathcal{D}^{(D-p-3)} \right]^{2}
- i \int \left[ (-1)^{D} p \mathcal{N}_{(D-p-2)} D_{(D-p-2)} + gD_{(D-p-4)} B_{(D-p-3)} \mathcal{J}_{g}^{(D-p-3)} \right]. \tag{A.29}
\]

From the dual action (A.29) we obtain the field equations

\[
\partial \left[ D^{(D-p-2)} - \mathcal{D}^{(D-p-1)} \right] = -i(-1)^{D} p g D_{(D-p-4)} \mathcal{J}_{g}^{(D-p-3)}, \tag{A.30}
\]

\[
\partial \left[ dD^{(D-p-2)} + i(-1)^{(D-p+1)^{2}e_{p}(\ast \mathcal{N}_{e})^{(D-p-1)}} \right] + \kappa D^{(D-p-2)}
= -i \kappa(-1)^{D} p \mathcal{N}^{(D-p-2)}. \tag{A.31}
\]

Solving Eq. (A.30), we find

\[
D^{(D-p-2)} - \mathcal{D}^{(D-p-1)} = \mathcal{D}^{(D-p-2)} - i(-1)^{D} p g D_{(D-p-4)} d\frac{1}{\Box} \mathcal{J}_{g}^{(D-p-3)},
\]

\[
\nabla \mathcal{D}^{(D-p-2)} = 0. \tag{A.32}
\]

After decomposing the magnetic parent current as follows:

\[
\mathcal{N}_{(D-p-2)} = \mathcal{N}_{(D-p-2)} + gD_{(D-p-4)} d\frac{1}{\Box} \mathcal{J}_{g}^{(D-p-3)}, \quad \nabla \mathcal{N}_{(D-p-2)} = 0, \tag{A.33}
\]

Eq. (A.31) becomes

\[
\partial d \mathcal{D}^{(D-p-2)} + \kappa \mathcal{D}^{(D-p-2)} = i \kappa(-1)^{D} p \mathcal{N}^{(D-p-2)}. \tag{A.34}
\]

and gives for \( \mathcal{D}^{(D-p-2)} \) the solution

\[
\mathcal{D}^{(D-p-2)} = -i \kappa(-1)^{D} p \frac{1}{\Box + \kappa} \mathcal{N}^{(D-p-2)}, \tag{A.35}
\]

\[
d \mathcal{D}^{(D-p-2)} = -i \kappa(-1)^{D} p d \frac{1}{\Box + \kappa} \mathcal{N}^{(D-p-2)}. \tag{A.36}
\]

When Eqs. (A.34) and (A.35) are inserted back into Eq. (A.29) we find the following expression:

\[
S[\mathcal{N}, \mathcal{J}_{g}, J_{e}] = -\frac{\kappa}{2} \int \mathcal{N}^{(D-p-2)} \frac{1}{\Box + \kappa} \mathcal{N}^{(D-p-2)}
\]
+ \int \left[ \frac{g_2^2}{2} \mathcal{J}^{D-p-3} \sqrt{g} \mathcal{J}^{D-p-3} + \frac{\epsilon_j^2}{2\kappa} \mathcal{J}_j \mathcal{J}_j \right],

(A-37)

in which the short-range bulk interaction and the long-range boundary interaction are clearly displayed.

Table V. Open $p$-brane gauge potentials and fields.

| Field | Dimension in $\hbar = c = 1$ units | Rank | Physical Meaning |
|-------|-----------------------------------|------|-------------------|
| $A_{\mu_1 \cdots \mu_{p+1}}$ | $(\text{length})^{1-D/2}$ | $p + 1$ | “electric” potential |
| $F_{\mu_1 \cdots \mu_{p+2}}$ | $(\text{length})^{-D/2}$ | $p + 2$ | “electric” field strength |
| $C_{\mu_1 \cdots \mu_p}$ | $(\text{length})^{2-D/2}$ | $p$ | Stueckelberg gauge potential |
| $K_{\mu_1 \cdots \mu_{p+1}}$ | $(\text{length})^{1-D/2}$ | $p + 1$ | Stueckelberg field strength |
| $D_{\mu_1 \cdots \mu_{D-p}}$ | $(\text{length})^{-D/2}$ | $D - p - 2$ | dual Stueckelberg potential |
| $I_{\mu_1 \cdots \mu_{p+1}}$ | $(\text{length})^{1-D/2}$ | $p + 1$ | dual Stueckelberg strength |

Table VI. Open $p$-brane associated currents.

| Current Density | Dimension in $\hbar = c = 1$ units | Rank | Physical Meaning |
|----------------|-----------------------------------|------|-------------------|
| $N_{\nu_1 \cdots \nu_{D-p-2}}$ | $(\text{length})^{p+1-D}$ | $p + 1$ | electric parent current |
| $J_{\nu_1 \cdots \nu_{p}}$ | $(\text{length})^{-D}$ | $p$ | boundary current |
| $N_{\nu_1 \cdots \nu_{D-p-2}}$ | $(\text{length})^{-D/2}$ | $D - p - 2$ | parent magnetic current |
| $J_{\nu_1 \cdots \nu_{p-3}}$ | $(\text{length})^{-D/2}$ | $D - p - 3$ | magnetic current |

Table VII. Dimension of couplings.

| Coupling Constants | Dimension in $\hbar = c = 1$ units |
|--------------------|-----------------------------------|
| $\epsilon$         | $(\text{length})^{D/2-p-2}$        |
| $g$                | $(\text{length})^{p+2-D/2}$        |
| $\epsilon g$       | $1$                                |
| $\kappa$           | $(\text{length})^{-2}$             |

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Note added in proof: After this paper was accepted for publication we were informed of the
existence of some additional references on p-brane electric/magnetic duality and generalized Higgs
mechanism. Reference 19) contains a selected list of those articles.