On Hyperfine Splittings of Strange Baryons in the Skyrme Model

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Abstract

We calculate the complete order $1/N$ corrections to baryon masses in the rigid rotator approach to the 3-flavor Skyrme model.

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1 Introduction

After years of neglect, the Skyrme model [1] made a comeback in the 1980’s [4, 5] as a phenomenological low energy model of hadrons. In its simplest form, the Skyrme model is represented by a chiral lagrangian with the pseudoscalar mesons as the fundamental fields. The baryons are formed from meson solitons with non-zero winding number. Using the simplest chiral lagrangian which supports baryons of finite size and energy, the low energy properties of the nucleons and deltas can be derived from pion properties and agree with experiment to within 30 percent [4, 5, 6]. This is remarkable because the lagrangian is truncated to fourth order in derivatives.

Difficulties arise in trying to generalize the Skyrme model to include the strange quark. Because the strange quark mass is quite large, the SU(3) flavor symmetry is significantly broken. Although the baryon quantum numbers are predicted correctly by the SU(3) Skyrme model [7, 8, 9, 10, 11], early attempts at quantitative calculations yield baryon masses which are way off from experiment, even when the strange quark mass is included using first order perturbation theory [12, 13, 14].

It is hoped that this problem has more to do with the relatively large strange quark mass than with defects in the Skyrme model itself [12, 13, 14, 15, 16]. The Skyrme lagrangian may be treated under the rigid rotator approximation with the number of colors $N$ fixed to 3, but this does not work well. An alternative approach was proposed where hyperons are treated as kaon-soliton bound states [12, 13]. In this scheme, a $1/N$ expansion for hyperon properties is constructed where in principle the strange quark mass may be included exactly at each order. This approach has yielded good agreement with many observed properties of strange baryons [12, 13, 17, 18, 19].

In fact, the $1/N$ expansion [20] is an essential part of the Skyrme model because the connection between Skyrme model and QCD is based on the large $N$ limit [21, 3]. It is important that the spectrum of low strangeness, low isospin baryons makes sense in the large $N$ limit. In other words, the structure of the multiplets should not change for increasing values of $N$. For two flavors this is not a problem since the SU(2) multiplets for given spin contain the same number of baryons for arbitrary $N$. More precisely, for a given value of $N$, the allowed values of isospin are given by integral or half-odd-integral values, depending on whether $N$ is odd or even, ranging from 0 or $1/2$ up to $N/2$. Since spin and isospin are the same under the $J = I$ rule, the low-lying spin-flavor multiplets are independent of $N$. The situation is more complicated for three flavors, because the allowed spin-flavor multiplets grow in size as $N$ is increased. In fact, even the smallest multiplets contain baryons with strangeness up to $\sim N$. The strange quark mass breaks the SU(3) flavor symmetry and large strangeness baryons have much larger masses than low strangeness baryons. Thus, in the large $N$ limit the mass splittings within each multiplet are large, and in some sense the flavor symmetry is badly broken. It is convenient, therefore, to take the large $N$ limit for baryons of fixed strangeness. The low-lying spin-isospin quantum numbers of such baryons are independent of $N$, removing any conceptual problems. This is precisely the route taken
in the bound state approach.

In calculating the baryon masses using the bound state approach, the hamiltonian is expanded to order $1/N$ and compared with the quark model where phenomenological magnetic moment interactions are included. Calculations of this kind are carried out in [12, 13, 17, 18, 19], where the hamiltonian is treated exactly to order $N^0$ and some $\sim 1/N$ terms are taken into account. However, the strange-strange interactions, embodied in the terms in the chiral lagrangian that are quartic in the kaon field, were not included because of their complexity. Nevertheless, the model is remarkably successful because the calculated ratio of the strange-light to light-light interaction strengths, denoted by $c$, turns out to be close to the empirical value. It is also important to calculate the parameter $\bar{c}$, the ratio of the strange-strange to light-light interaction strengths. Unlike $c$, $\bar{c}$ is sensitive to the terms quartic in the kaon field, and our goal in this paper is to estimate their effect.

Inclusion of the quartic terms in the full bound state approach is a difficult task. For this reason, we will use the rigid rotator Skyrmion as a testing ground for our methods. Following ref. [16], we develop a $1/N$ expansion for the 3-flavor rigid rotator by treating the deviations into the strange directions as perturbations. This model bears a strong resemblance to the bound state approach, but is much simpler technically. Essentially, the dynamics of the kaon field is replaced by that of its most tightly bound mode. The price we pay is that the wave function of this mode is only an approximation to what it is in the full bound state approach, and this approximation becomes cruder with increasing kaon mass. Thus some numerical accuracy is lost, but the calculations become much more manageable and can be carried out analytically.

A calculation of the hyperfine splittings, neglecting the strange-strange interaction terms, has been carried out in [16]. There it was found that the perturbative treatment of the strange quark mass breaks down, and that it has to be included exactly. In this paper we complete the calculation of the rigid rotator skyrmion masses to order $1/N$ by including the strange-strange interactions. Although the value of $c$ is unaffected by these additional terms, the value of $\bar{c}$ is seen to improve vastly over the partial calculations. The purpose of completing the rigid rotator calculation is twofold: (1) hopefully the improvement we observe in completing the rigid rotator calculation will carry over to the bound state approach, and (2) the complete rigid rotator calculation can be used to gain intuition about the complete bound state calculation.

This paper is organized as follows. In section 2, we introduce the Skyrme action. In section 3 we discuss the rigid rotator approximation and express the Skyrme action in terms of the rigid rotator excitations. In section 4, we carry out the $1/N$ expansion. Finally, in section 5 we quantize the resulting lagrangian and calculate $c$ and $\bar{c}$.

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1See, however, ref. [22] for some recent progress.
The Skyrme model

The Skyrme lagrangian is given by

\[ L = \frac{f^2}{16} \text{tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{32e^2} \text{tr}[\partial_\mu UU^\dagger, \partial_\nu UU^\dagger]^2 + \frac{f^2}{8} \text{tr} \mathcal{M}(U + U^\dagger - 2) \]

where \( U(\vec{x}, t) \in \text{SU}(3) \). \( \mathcal{M} \) is proportional to the quark mass matrix and, if we neglect the \( u \) and \( d \) masses, is given by

\[ \mathcal{M} = \begin{pmatrix} 0 & 0 \\ 0 & m_K^2 \end{pmatrix} \]

where, in general, we write \( 3 \times 3 \) matrices in the partitioned form

\[ \begin{pmatrix} 2 \times 2 & 2 \times 1 \\ 1 \times 2 & 1 \times 1 \end{pmatrix} \]

\( M_\mu \) is defined as

\( M_\mu := \partial_\mu UU^\dagger \)

If the quark masses are zero, the lagrangian is invariant under independent left and right global SU(3) rotations of \( U \)

\( U \mapsto AUB^\dagger \)

since under this transformation

\( M_\mu \mapsto AM_\mu A^\dagger \)

Eq. (1) is the simplest lagrangian invariant under this symmetry group for zero quark masses which also allows for classically stable baryons of finite radius and mass. Without the commutator term the baryon will collapse, but there is no a priori reason to exclude other higher derivative terms. With a non-zero strange quark mass the SU(3) flavor symmetry is broken to the SU(2) involving the massless \( u \) and \( d \) quarks.

There is another term which we must include in the action. It is called the Wess-Zumino term [23] and is written in non-local form as an integral over a five-dimensional disk with spacetime as its boundary:

\[ S_{WZ} = -\frac{iN}{240\pi^2} \int_D d^5\vec{x} \varepsilon^{\mu\nu\alpha\beta\gamma} \text{tr}(M_\mu M_\nu M_\alpha M_\beta M_\gamma) \]

where \( U(\vec{x}, t) \) is continuously extended to the disk (with our convention, \( \varepsilon^{01234} = 1 \)). In our treatment, the Wess-Zumino term is responsible for preventing anti-strange quarks from appearing in baryons. To obtain agreement with QCD, \( N \) in equation (2) is taken to be the number of colors [3].


3 Rigid rotator approximation

In this paper we will be concerned with the low-lying baryon states. We will construct their approximate description by quantizing the time-dependent rotations of the \( B = 1 \) Skyrme soliton

\[
U(\vec{x}, t) = A(t)U_0(\vec{x})A(t)^\dagger
\]

We choose the “hedgehog” ansatz

\[
U_0(\vec{x}) = \begin{pmatrix}
        e^{iF(\psi)\vec{r} \cdot \vec{\tau}} & 0 \\
        0^\dagger & 1
    \end{pmatrix}
\]

where \( \psi \) is defined by \( |\vec{x}| = e^\psi/(ef_\pi) \). The radial profile function \( F(\psi) \) satisfies the boundary conditions

\[
F(-\infty) = \pi \quad F(+\infty) = 0
\]

and is determined classically. Note that the static soliton \( U_0 \) lies entirely in the upper-left \( 2 \times 2 \) corner. This is reasonable because the \( N' \)'s and \( \Delta' \)'s are built out of light quarks.

We will now include the effect of rotation in flavor space. Calculation yields

\[
\partial_\mu U = A \left( \partial_\mu U_0 + [A^\dagger \partial_\mu A, U_0] \right) A^\dagger =: A d_\mu U_0 A^\dagger
\]

Since the action involves only traces of \( U \)'s and their derivatives, it follows that the rotation “gauges” the action in the following way,

\[
U \mapsto U_0 \\
\partial_\mu \mapsto d_\mu \\
\mathcal{M} \mapsto A^\dagger \mathcal{M} A
\]

The action is given by the sum of eqs. 1 and 2 with the redefinition above. Thus the effect of the time-dependent rotation is now hidden in the definition of \( \mathcal{M} \) and in the covariant derivative \( d_\mu \) given by

\[
d_\mu U = \partial_\mu U + [A^\dagger \partial_\mu A, U]
\]

Since \( A \) belongs to SU(3) it follows that \( A^\dagger \dot{A} \) is anti-hermitian and traceless, and so it can be expressed as a linear combination of \( i\lambda_a \):

\[
A^\dagger \dot{A} = (ef_\pi)i\nu^a \lambda_a = ief_\pi \left( \frac{\vec{v} \cdot \vec{r} + \nu^1}{V^\dagger} \left| \begin{array}{c} V \\ -2\nu \end{array} \right. \right)
\]

where

\[
\vec{v} = (v^1, v^2, v^3) \quad V = \begin{pmatrix} v^4 - iv^5 \\ v^6 - iv^7 \end{pmatrix} \quad \nu = v^8/\sqrt{3}
\]

Due to the significant SU(3) breaking, the bound state approach is, at the end, necessary to improve on this approximation.
We wish to express \( L := \int d^3 x \mathcal{L} \) in terms of \( F(\psi) \), \( \vec{v} \), \( V \) and \( \nu \). First we compute \( M_\mu \) which can be similarly expressed in terms of SU(3) generators

\[
M_\mu = (ef_\pi)i\omega_\mu^a \lambda_a = ief_\pi \left( \frac{\vec{w}_\mu \cdot \vec{\tau} + \omega_\mu^a 1}{W_\mu^\dagger} W_\mu - 2\omega_\mu \right)
\] (4)

Calculation yields

\[
d_r U = \partial_r U = ief_\pi e^{-\psi'} F'(\psi) \left( \begin{array}{cc} \vec{r} \cdot \vec{\tau} U & 0 \\ 0 & 0 \end{array} \right)
\]

\[
d_\theta U = \partial_\theta U = ief_\pi e^{-\psi} \sin F \left( \begin{array}{cc} \vec{\theta} \cdot \vec{\tau} & 0 \\ 0 & 0 \end{array} \right)
\]

\[
d_\phi U = \partial_\phi U = ief_\pi e^{-\psi} \sin F \left( \begin{array}{cc} \vec{\phi} \cdot \vec{\tau} & 0 \\ 0 & 0 \end{array} \right)
\]

\[
d_t U = [A^\dagger \dot{A}, U] = ief_\pi \left( \begin{array}{cc} 2 \sin F (\vec{r} \times \vec{v}) \cdot \vec{\tau} (1 - U)V & (1 - U)V \\ V^\dagger (U - 1) & 0 \end{array} \right)
\]

For an arbitrary vector \( \vec{n} \) perpendicular to \( \vec{r} \), we find that

\[
(\vec{n} \cdot \vec{\tau}) U^\dagger = (\vec{n} \cos F + \vec{n} \times \vec{r} \sin F) \cdot \vec{\tau} = \vec{n}_F \cdot \vec{\tau}
\]

where \( \vec{n}_F \) is \( \vec{n} \) rotated through angle \( F \) around \( \vec{r} \). With the help of this identity, further calculation yields

\[
\omega_\mu = 0
\]

\[
W_i = 0
\]

\[
W_t = (1 - U)V
\]

\[
\vec{w}_t = 2 \sin F (\vec{r} \times \vec{v})_F
\]

\[
\vec{w}_r = e^{-\psi} F'(\psi) \vec{r}
\]

\[
\vec{w}_\theta = e^{-\psi} \sin F \vec{\theta}_F
\]

\[
\vec{w}_\phi = e^{-\psi} \sin F \vec{\phi}_F
\]

If we ignore the mass term for now, we can compute \( \mathcal{L} \) in terms of \( W_t \) and \( \vec{w}_\mu \), which in turn can be expressed in terms of \( F \), \( \vec{v} \) and \( V \). The Wess-Zumino term can also be included (see, for instance, ref. [24]). After integration, we find

\[
L = -E_0 + 2(ef_\pi)^2 \Omega \vec{v}^2 + 2(ef_\pi)^2 \Phi V^\dagger V + (ef_\pi) N\nu
\] (5)

where the classical ground state energy \( E_0 \), and the two moments of inertia \( \Omega \) and \( \Phi \) are given in terms of \( F(\psi) \) as follows

\[
E_0 = \pi \frac{f_\pi}{e} \int_{-\infty}^{+\infty} e^{3\psi} d\psi \left( \frac{1}{2} e^{-2\psi} (F'^2 + 2 \sin^2 F) + 2 e^{-4\psi} \sin^2 F (2F'^2 + \sin^2 F) \right)
\]

5
\[
\Omega = \frac{1}{e^3 f_\pi} \int_{-\infty}^{+\infty} e^{3\psi} d\psi \sin^2 F \left(\frac{2}{3} + \frac{8}{3} e^{-2\psi} (F'^2 + \sin^2 F)\right)
\]
\[
\Phi = \frac{1}{e^3 f_\pi} \int_{-\infty}^{+\infty} e^{3\psi} d\psi \frac{1 - \cos F}{2} \left(1 + e^{-2\psi} (F'^2 + 2 \sin^2 F)\right)
\]

The profile function is determined classically by minimizing \(E_0\) with respect to \(F(\psi)\) with fixed boundary conditions. Numerical integration yields
\[
E_0 = 36.4 \frac{f_\pi}{e} \quad \Omega = 99.1 \frac{1}{e^3 f_\pi} \quad \Phi = 37.8 \frac{1}{e^3 f_\pi}
\]

If we write \(A(t)\) in local coordinates
\[
A(t) = A_0 e^{\frac{1}{2} i a^\ell \lambda_\ell} \approx A_0 (1 + \frac{1}{2} i a^\ell \lambda_\ell + \ldots)
\]
then \(A^\dagger \dot{A} \approx \frac{1}{2} i a^\ell \lambda_\ell\) from which we identify \(\dot{a}^\ell = 2e f_\pi v^\ell\). Then equation 5 can be reexpressed as
\[
L = -E_0 + \frac{1}{2} \Omega \sum_{j=1}^{3} \dot{a}^2_j + \frac{1}{2} \Phi \sum_{\ell=4}^{7} \dot{a}^2_\ell + \frac{N}{2\sqrt{3}} \dot{a}_8
\]

which is a convenient form for quantization.

4 The \(1/N\) expansion

So far the number of colors has yet to appear, except in the Wess-Zumino term where it appears explicitly. QCD without quark masses has but one coupling constant, which upon renormalization is converted into a mass scale \(\Lambda_{\text{QCD}}\) and disappears. Traditionally, the 3-flavor Skyrme model was handled by applying perturbation theory in \(m_s/\Lambda_{\text{QCD}}\), but it is shown in [8, 10, 11] that this does not work. Rather than do this, we introduce \(1/N\) as the expansion parameter [20, 21] and treat the deviations of the collective coordinate wave functions into the strange directions perturbatively [16].

To separate the SU(2) rotations from the deviations into strange directions, we write [16]
\[
A(t) = \begin{pmatrix} A(t) & 0 \\ 0^\dagger & 1 \end{pmatrix} S(t)
\]

where \(A(t) \in \text{SU}(2)\), and
\[
S(t) = \exp i \sum_{a=4}^{7} d^a \lambda_a = \exp iD
\]

where
\[
D = \begin{pmatrix} 0 & \sqrt{2}D \\ \sqrt{2}D^\dagger & 0 \end{pmatrix}, \quad D = \frac{1}{\sqrt{2}} \begin{pmatrix} d^4 - id^5 \\ d^6 - id^7 \end{pmatrix}
\]
It is convenient to introduce the angular velocity of the SU(2) rotation via

\[ A^\dagger \dot{A} = \frac{1}{2} i \dot{\alpha} \cdot \tau \]

Our goal will be to express the lagrangian in eq. 5 in terms of \( D \) and \( \dot{\alpha} \).

The momentum conjugate to \( \alpha \) is \( J_{ud} \), which can be interpreted as the angular momentum carried by the \( u \) and \( d \) quarks. The SU(2) rotator quantization yields its equality to the isospin, \( J_{ud} = I \). The low-lying states with isospin of order 1 correspond to angular velocities of order \( 1/N \), which is helpful in developing the \( 1/N \) expansion.

\( S(t) \) corresponds to deviations into the strange direction. The order 1 strangeness baryons correspond to \( D \) values of order \( 1/\sqrt{N} \). There are two effects which keep these deviations small. The strange quark mass is non-zero, so that the higher strangeness baryons will have a higher mass. But even if the strange quark mass were zero, the Wess-Zumino term acts as a magnetic field whose strength is order \( N \), thereby confining the wave functions of low strangeness baryons. Our intention is to calculate the baryon spectrum of these low lying states completely to order \( 1/N \). Since the lagrangian is order \( N \) to begin with, we need to retain the previously left out corrections of order \( D^4 \), which incorporate the strange-strange interactions.

We start with the systematic expansion of \( S = \exp iD \). In fact, \( S \) can be expanded to arbitrary order in \( D \) because \( D \) satisfies

\[ D^3 = d^2 D \quad d^2 := 2D^\dagger D \]

so that

\[ S = \sum_{n=0}^{\infty} \frac{i^n}{n!} D^n \]

\[ = 1 + i \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} d^{2n} \right) D - \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} d^{2n} \right) D^2 \]

\[ = 1 + iD \frac{\sin(d)}{d} - D^2 \frac{1 - \cos(d)}{d^2} \]

Thus, we have

\[ S = \begin{pmatrix} 1 - \frac{1 - \cos(d)}{d^2} 2DD^\dagger & i \frac{\sin(d)}{d} \sqrt{2}D \\ i \frac{\sin(d)}{d} \sqrt{2}D^\dagger & \cos(d) \end{pmatrix} \]

First, we take care of the mass term, \( \text{tr} \mathcal{M}(U + U^\dagger - 2) \). Straightforward calculation yields

\[ \text{tr} \mathcal{M}(U + U^\dagger - 2) = -2m_K^2(1 - \cos F) \sin^2(d) \]

Integrating over space and expanding to order \( D^4 \) we get

\[ L_M = -\Gamma m_K^2 (D^\dagger D - \frac{2}{3}(D^\dagger D)^2) + \ldots \]
where \( \Gamma \) is order \( N \) and is given by

\[
\Gamma := \frac{4\pi}{e^4f_\pi} \int_{-\infty}^{\infty} d\psi e^{-3\psi} \left( \frac{1 - \cos F}{2} \right) = 103.2 \frac{1}{e^4f_\pi}
\]

The other terms in the lagrangian, eq. \(3\), depend on \( \vec{v} \), \( V^\dagger V \) and \( \nu \), which are obtained by comparing eq. \(3\) with

\[
\mathcal{A}^\dagger \dot{\mathcal{A}} = S^\dagger A^\dagger (\dot{A}S + AS) = S^\dagger \left( \begin{array}{c} \frac{1}{2} i \vec{\alpha} \cdot \vec{\tau} \\ 0 \end{array} \right) S + S^\dagger \dot{S}
\]

A useful identity is

\[
\vec{v}^2 = \frac{1}{2} \text{tr}(\vec{v} \cdot \vec{\tau} + \nu 1)^2 - \nu^2
\]

In principle we could calculate \( \mathcal{A}^\dagger \dot{\mathcal{A}} \) systematically to all orders in \( \frac{1}{N} \), but it is much simpler to truncate to \( D^4 \) immediately. Since \( \vec{v} \cdot \vec{\tau} + \nu 1 \) starts off at order \( D^2 \), we only need it to order \( D^2 \) to compute \( \vec{v}^2 \) to order \( D^4 \). A straightforward but tedious calculation yields

\[
e f_\pi (\vec{v} \cdot \vec{\tau} + \nu 1) = \frac{1}{2} \dot{\vec{\alpha}} \cdot \vec{\tau} + i(\dot{D} D^\dagger - D \dot{D}^\dagger) + \ldots
\]

\[
e f_\pi V = \frac{\sqrt{2}}{\sqrt{2}} \dot{D} + \frac{1}{2} i \dot{\vec{\alpha}} \cdot \vec{\tau} D - \frac{1}{3} (D^\dagger D) \dot{D} + \frac{1}{6} (D^\dagger \dot{D} + \dot{D}^\dagger D)D - \frac{1}{2} (D^\dagger \dot{D} - \dot{D}^\dagger D)D + \ldots
\]

\[
e f_\pi \nu = \frac{1}{2} i(D^\dagger \dot{D} - \dot{D}^\dagger D) - \frac{1}{2} D^\dagger \dot{\vec{\alpha}} \cdot \vec{\tau} D - \frac{1}{3} i(D^\dagger \dot{D} - \dot{D}^\dagger D)D^\dagger D + \ldots
\]

from which, after more calculation we conclude

\[
L = -E_0 + 4\Phi \dot{D}^\dagger \dot{D} + \frac{1}{2} i N \left( D^\dagger \dot{D} - \dot{D}^\dagger D \right) - \Gamma m_k^2 D^\dagger D
\]

\[
+ \frac{1}{2} \Omega \dot{\vec{\alpha}}^2 + i(\Omega - 2\Phi) \left( D^\dagger \dot{\vec{\alpha}} \cdot \vec{\tau} D - \dot{D}^\dagger \dot{\vec{\alpha}} \cdot \vec{\tau} D \right)
\]

\[
- \frac{1}{2} N D^\dagger \dot{\vec{\alpha}} \cdot \vec{\tau} D + 2 \left( \Omega - \frac{4}{3} \Phi \right) (D^\dagger D)(\dot{D}^\dagger \dot{D})
\]

\[
- \frac{1}{2} \left( \Omega - \frac{4}{3} \Phi \right) (\dot{D}^\dagger \dot{D} + \dot{D}^\dagger D)^2 + 2\Phi (D^\dagger \dot{D} - \dot{D}^\dagger D)^2
\]

\[
- \frac{1}{3} i N (D^\dagger \dot{D} - \dot{D}^\dagger D)D^\dagger D + \frac{2}{3} \Gamma m_k^2 (D^\dagger D)^2
\]

\[(6)\]

5 Quantizing the lagrangian

The complete expansion of the lagrangian to order \( 1/N \) is given by equation \(6\). The quantum variables are the local coordinates of the SU(2) rotation, \( \vec{\alpha} \), and the deviations into the strangeness directions \( d^a, a = 4, 5, 6, 7 \), arranged conveniently in a complex spinor

\[
D = \frac{1}{\sqrt{2}} \begin{pmatrix} d^4 - i d^5 \\ d^6 - i d^7 \end{pmatrix}
\]
The conjugate momenta are given by the $u$-$d$ quark angular momentum $\vec{J}_{ud}$ and $\pi_a, a = 4, 5, 6, 7$ which can also be arranged in a complex spinor

$$\Pi = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \pi^4 - i\pi^5 \\ \pi^6 - i\pi^7 \end{array} \right)$$

The momenta are given by the usual formula

$$(J_{ud})_i = \frac{\delta L}{\delta \dot{\alpha}_i} \quad \Pi^\gamma = \frac{\delta L}{\delta D^\dagger_i}$$

and satisfy the commutation relations

$$[(J_{ud})_i, \alpha^j] = \frac{1}{i} \delta_i^j \quad [\Pi^\gamma, D^\dagger_\beta] = [\Pi^\dagger_\beta, D^\gamma] = \frac{1}{i} \delta_\beta^\gamma$$

Calculation gives

$$\vec{J}_{ud} = \Omega \vec{\alpha} + i (\Omega - 2\Phi) \left( D^\dagger \vec{\tau} \dot{D} - \dot{D}^\dagger \vec{\tau} D \right) - \frac{1}{2} N D^\dagger \vec{\tau} D$$

$$\Pi = 4\Phi \dot{D} - \frac{1}{2} i N \dot{D} - i (\Omega - 2\Phi) \vec{\alpha} \cdot \vec{\tau} D$$

$$- \left( \Omega - \frac{4}{3} \Phi \right) (D^\dagger \dot{D} + \dot{D}^\dagger D) D - 4\Phi (D^\dagger \dot{D} - \dot{D}^\dagger D) D$$

$$+ 2 \left( \Omega - \frac{4}{3} \Phi \right) (D^\dagger D) \dot{D} + \frac{1}{3} N (D^\dagger D) D$$

The hamiltonian is calculated to order $1/N$ by Legendre-transforming the lagrangian with the aid of a computer. The result is

$$H = \vec{\alpha} \cdot \vec{J}_{ud} + \Pi^\dagger D + D^\dagger \Pi - L$$

$$= E_0 + \frac{1}{4\Phi} \Pi^\dagger D - i \frac{N}{8\Phi} (D^\dagger D) - \left( \Gamma m_K^2 + \frac{N^2}{16\Phi} \right) D^\dagger D$$

$$+ \frac{1}{2\Omega} \vec{J}_{ud}^2 + i \left( \frac{1}{2\Omega} - \frac{1}{4\Phi} \right) \left( D^\dagger \vec{J}_{ud} \cdot \vec{\tau} D - \Pi^\dagger \vec{J}_{ud} \cdot \vec{\tau} D \right)$$

$$+ \frac{N}{4\Phi} D^\dagger \vec{J}_{ud} \cdot \vec{\tau} D + \left( \frac{1}{2\Omega} - \frac{1}{3\Phi} \right) (D^\dagger D)(\Pi^\dagger D)$$

$$+ \left( \frac{1}{12\Phi} - \frac{1}{8\Omega} \right) (D^\dagger D + \Pi^\dagger D)^2 - \frac{1}{8\Phi} (D^\dagger D - \Pi^\dagger D)^2$$

$$- i \frac{N}{8\Phi} (D^\dagger D - \Pi^\dagger D) D^\dagger D + \left( \frac{N^2}{12\Phi} - \frac{2}{3} \Gamma m_K^2 \right) (D^\dagger D)^2$$

The order $N$ piece of the hamiltonian is simply the classical ground state energy $E_0$. The order 1 piece includes the terms quadratic in $D$ and $\Pi$, and thus may be diagonalized exactly using creation and annihilation operators

$$D^\gamma = \frac{1}{\sqrt{N\mu}} (a^\gamma + (b^\dagger)^\gamma) \quad \Pi^\gamma = \frac{\sqrt{N\mu}}{2i} (a^\gamma - (b^\dagger)^\gamma)$$
where
\[ \mu = \sqrt{1 + \left(\frac{m_K}{M_0}\right)^2}, \quad M_0 = \frac{N}{4\sqrt{4\Phi}} \]

The operators \( a^\dagger (b^\dagger) \) may be thought of as creation operators for constituent strange quarks (anti-quarks). Strangeness and the angular momentum of the strange quarks are given respectively by
\[ S = b^\dagger b - a^\dagger a, \quad \vec{J}_s = \frac{1}{2}(a^\dagger \vec{\tau}a - b\vec{\tau}b^\dagger) \]

In terms of the creation and annihilation operators, the normal-ordered hamiltonian to order 1 is given by
\[ H = E_0 + \omega a^\dagger a + \bar{\omega} b^\dagger b; \quad \omega = \frac{N}{8\Phi}(\mu - 1), \quad \bar{\omega} = \frac{N}{8\Phi}(\mu + 1). \]

Thus, replacing a light quark with a strange quark (anti-quark) costs energy \( \omega (\bar{\omega}) \). Note that for vanishing \( m_K \), \( \omega \) also vanishes, thereby restoring the original SU(3) symmetry (it costs no energy to replace a \( u \) or \( d \) quark with an \( s \) quark) but that \( \bar{\omega} \) tends to a rather large value, \( N/4\Phi \). Indeed, baryons containing strange anti-quarks are exotic states that have not been observed in nature. The Wess-Zumino term, which acts as magnetic field in the \( D - D^\dagger \) plane, breaks the \( s \leftrightarrow \bar{s} \) symmetry.

From now on, we consider those states for which \( n_s = 0 \). \( b \) annihilates these states, and since we plan to normal-order the hamiltonian, we may simply drop the contributions of the \( b \)-oscillators. Upon doing so, we find that the hamiltonian to order \( 1/N \) is
\[ H = E_0 + \frac{N}{8\Phi}(\mu - 1)a^\dagger a + \frac{1}{2\Omega}\vec{J}_{ud}^2 + \left(\frac{1}{2\Omega} - \frac{1}{4\Phi \mu}(\mu - 1)\right)a^\dagger \vec{J}_{ud} \cdot \vec{\tau}a \]
\[ + \left(\frac{1}{8\Omega} - \frac{1}{8\Phi \mu^2}(\mu - 1)\right)(a^\dagger a)^2 \quad (10) \]

Using the identity
\[ \vec{J}_s^2 = \frac{1}{4}(a^\dagger a)^2 + \frac{1}{2}a^\dagger a \]
we may rewrite the hamiltonian as
\[ H = E_0 + \omega a^\dagger a + \frac{1}{2\Omega}(\vec{J}_{ud}^2 + 2c\vec{J}_{ud} \cdot \vec{J}_s + \bar{c}\vec{J}_s^2) \]

Here \( \omega \) differs from \( \frac{N}{8\Phi}(\mu - 1) \) by a subleading term of order \( 1/N \), which we neglect. The explicit formulae for \( c \) and \( \bar{c} \) are
\[ c = 1 - \frac{\Omega}{2\mu \Phi}(\mu - 1), \quad \bar{c} = 1 - \frac{\Omega}{\mu^2 \Phi}(\mu - 1). \]

Let us note that the structure of the hyperfine (order \( 1/N \)) splittings is the same as in the quark model with phenomenological magnetic moment interactions,
\[ H_{hf} = \frac{1}{\Omega}\left(\sum_{i<k} \vec{J}_i \cdot \vec{J}_k + c \sum_{i,K} \vec{J}_i \cdot \vec{J}_K + \bar{c} \sum_{i<K} \vec{J}_i \cdot \vec{J}_K \right) \]
Table 1: Values of $c$ and $\bar{c}$

| Source                        | $c$  | $\bar{c}$ |
|-------------------------------|------|-----------|
| Experiment                    | .67  | .27       |
| Rigid rotator, partial        | .28  | .08       |
| Rigid rotator, complete       | .28  | .35       |
| Bound state, partial          | .60  | .36       |
| Bound state, complete         | .60  | ?         |

(the small indices refer to light quarks, while the capital – to strange quarks). It is remarkable that in the Skyrme model such interactions are derived rather than postulated, and that the values of the parameters are explicitly calculable. Using the fact that $J^P_{ud} = I(I + 1)$, the hyperfine splittings can be expressed as

$$\delta M = \frac{1}{2\Omega} \left\{ cJ(J + 1) + (1 - c)(I(I + 1) - \frac{Y^2}{4}) + (1 + \bar{c} - 2c)\frac{Y^2}{4} \right\}.$$  

We use the parameters

$$f_\pi = 129 \text{ MeV} \quad e = 5.45$$

from the standard fit to $N$ and $\Delta$. Thus numerical calculation gives us

$$E_0 = 862 \text{ MeV} \quad \Omega^{-1} = 211 \text{ MeV} \quad \Phi^{-1} = 552 \text{ MeV} \quad \Gamma^{-1} = 202 \text{ MeV}$$

which leads to

$$M_0 = 250 \text{ MeV} \quad \mu = 2.22$$

We finally conclude that

$$c = .28 \quad \bar{c} = .35$$

At this point, we can see why first order perturbation theory in $m_K^2$ fails so badly: the expansion parameter turns out to be $m_K^2/M_0^2$ which is almost as large as 4! Note that the unexpectedly low mass scale $M_0$ is peculiar to baryons and does not appear in purely mesonic physics [16]. Thus, the Skyrme model analysis suggests that chiral perturbation theory for strange baryons is not reliable. We regard this is an interesting and non-trivial prediction. Although $M_0$ is model-dependent, we believe that our qualitative conclusions are sound.

## 6 Conclusions

A summary of previous calculations of $c$ and $\bar{c}$ is given in Table 1. In this paper we include all terms up to order $1/N$, including the strange-strange interactions. These interactions
affect the value of $\bar{c}$, although they do not affect $c$. In the quark model one typically uses the relation $\bar{c} = c^2$, which is equivalent to expressing the hamiltonian in the form

$$H = E_0 + \omega a^\dagger a + \frac{1}{2\Omega}(\vec{J}_{ud} + c\vec{J}_s)^2$$

The same hamiltonian follows from the bound state approach with the quartic terms in the kaon field neglected. We have explicitly shown here that the strange-strange interactions break this relation, thereby improving the value of $\bar{c}$.

Inclusion of the quartic terms in the bound state approach, and calculation of their effect on $\bar{c}$, is a laborious calculation which we leave for the future. Judging from the results presented here, we do not expect the shift in $\bar{c}$ to be too large. As suggested by the recent results in ref. [22], the validity of the bound state approach will undoubtedly be preserved.

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