A GLOBAL BOUND FOR THE SINGULAR SET OF AREA-MINIMIZING HYPERSURFACES

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Abstract. We give an a priori bound on the \((n-7)\)-dimensional measure of the singular set for an area-minimizing \(n\)-dimensional hypersurface, in terms of the geometry of its boundary.

Area-minimizing surfaces in general will not be smooth, and a basic question in minimal surface theory is to understand the size and nature of the singular region. The cumulative works of many (Federer, De Giorgi, Allard, Simons, to name only a few) prove that for absolutely-area-minimizing \(n\)-dimensional hypersurfaces in \(\mathbb{R}^{n+1}\) ("codimension-one area-minimizing integral currents"), the interior singular set is at most \((n-7)\)-dimensional. This dimension bound is sharp, and is directly tied to the existence of low-dimensional, non-flat minimizing cones.

[HS79] proved that for such codimension-one area-minimizers, if the boundary is known to be \(C^{1,\alpha}\) and multiplicity-one, then in fact no singularities lie within a neighborhood of the boundary. Combined with interior regularity, this theorem gives a very nice structure of these minimizing hypersurfaces.

Recently [NV17], [NV15] quantified the interior partial regularity, by demonstrating effective local (interior) bounds on the \(\mathcal{H}^{n-7}\) measure of the singular set. Their methods also prove \((n-7)\)-rectifiability of the singular set, which was originally established through an entirely different approach by [Sim95].

In this short note, we obtain obtain a global, effective a priori estimate on the singular set of an area-minimizing hypersurface in terms of the boundary geometry. Our results are loosely analogous to the a priori bounds of [AL88] (see also the recent works [MMS18b], [MMS18a]).

We work in \(\mathbb{R}^{n+1}\), for \(n \geq 7\). Let us write \(\mathcal{I}_n(U)\) for the space of integral \(n\)-currents acting on forms supported in the open set \(U\). Given an \(n\)-dimensional, oriented manifold \(E\), write \([E]\) for the current induced by integration. Let \(\eta_\lambda(x) = \lambda x\), and \(\tau_y(x) = x + y\).

If \(T \in \mathcal{I}_n(U)\), we say \(T\) is area-minimizing if \(||T\|_W \leq ||T + S\|_W\) for every open \(W \subset U\), and every \(S \in \mathcal{I}_n(U)\) satisfying \(\partial S^* = 0\).
spt$S \subset W$. The regular set $\text{reg}T$ is the (open) set of points where $\text{spt}T$ is locally the union of embedded $C^{1,\alpha}$ manifolds. The singular set is $\text{sing}T = \text{spt}T \setminus \text{reg}T$. Write $||T||$ for the mass measure of $T$.

Given an $k$-manifold $S$, and $x \in S$, let $r_{1,\alpha}(S, x)$ be the largest radius $r$, so that $(S - x)/r$ is the graph of a $C^{1,\alpha}$ function $u$, with $|u|_{1,\alpha} \leq 1$. Define $r_{1,\alpha}(S) = \inf_{x \in S} r_{1,\alpha}(S, x)$.

Our main Theorem is the following.

**Theorem 0.1.** There is a constant $c = c(n, \alpha)$ so that the following holds. Let $T$ be a area-minimizing integral $n$-current in $\mathbb{R}^{n+1}$. Suppose $\partial T$ is a multiplicity-one, compact, oriented $C^{1,\alpha}$ manifold $S$, and assume that $S$ is contained in the boundary of some convex set. Then

$$H^{n-7}(\text{sing}T) \leq c(n, \alpha) \frac{||T||(\mathbb{R}^{n+1})}{r_{1,\alpha}(S)^7}.$$  

In particular, we have

$$H^{n-7}(\text{sing}T) \leq c'(n, \alpha) \frac{H^{n-1}(S)^{\frac{n}{n-7}}}{r_{1,\alpha}(S)^7}.$$  

I believe Theorem 0.1 should hold for more general $S$, but there are subtleties even in the idealized case when $S$ is a line. See the discussion below.

We also have a version of Theorem 0.1 in the case when $T$ has free-boundary. Given open sets $U$, $\Omega$, we say $T \in \mathcal{I}_n(U)$ is area-minimizing with free-boundary in $\Omega$ if: $\text{spt}T \subset \overline{\Omega}$, and $||T||(W) \leq ||S + T||(W)$ for all $W \subset\subset U$, and every $S \in \mathcal{I}_n(U)$ satisfying $\text{spt}S \subset \overline{\Omega} \cap W$ and $\text{spt} \partial S \subset \partial \Omega$. [Gru87] proved boundary singularities have dimension at most $n - 7$.

**Theorem 0.2.** Let $\Omega$ be a domain with $C^2$-boundary, and $\infty > r_{1,1}(\partial \Omega) > 0$. Let $T$ be a compactly supported, area-minimizing current with free-boundary in $\Omega$, with $\partial T \setminus \Omega = 0$. Then

$$H^{n-7}(\text{sing}T) \leq c(n) \frac{||T||(\Omega)}{r_{1,1}(\partial \Omega)^7}.$$  

The key to proving both Theorems is the observation that Naber-Valtorta's technique gives the following linear interior bound on the singular set: if $T$ is area-minimizing in $U \subset \mathbb{R}^{n+1}$, with $\partial T \setminus U = 0$, then for every $\epsilon > 0$, we have:

$$H^{n-7}(\text{sing}T \cap U \setminus B_\epsilon(\partial U)) \leq c(n) \epsilon^{-7} ||T||(U \setminus B_{\epsilon/2}(\partial U)).$$

For the Neumann problem (Theorem 0.2), we can adapt the techniques of [NV15] to prove a priori estimates on the singular set in a
neighborhood of the barrier. Unfortunately, it’s not clear that a good Dirichlet boundary version of Naber-Valtorta exists, in any more generality than is considered in Theorem 0.1. The problem is that there is not necessarily a good relationship between regularity and symmetry. If there exists a singular, minimizing hypersurface with Euclidean area growth and linear boundary, then by [HS79] any blow-down sequence would preclude an inclusion like $\text{sing} T \subset S_{n-7}^{n-7}$ (here $S_{n-7}^{n-7}$ being the $(n - 7, \epsilon)$-strata of [CN13]).

Instead, for Theorem 0.1 we can prove an effective version of [HS79], which says that the singular set is some uniform distance away from the boundary curve. It’s tempting to think an ineffective, quantitative version of [HS79] might hold for more general Dirichlet setups, but the problem is the same as above.

**Remark 0.3.** The following variant of Theorem 0.1 holds for almost-area-minimizers. Let $T \in \mathcal{I}_n(\mathbb{R}^{n+1})$ be almost-area-minimizing, in the sense that

$$||T||(B_r(x)) \leq ||T + S||(B_r(x)) + c_0 r^{n+2\alpha},$$

for any $S \in \mathcal{I}_n(\mathbb{R}^{n+1}), \partial S = 0, \text{spt} S \subset B_r(x)$, and some fixed $c_0$. Suppose $\partial T = [S]$ is an oriented, embedded, multiplicity-one $C^{1,\alpha}$-manifold $S$, and suppose there is a $C^{1,\alpha}$ domain $\Omega$ so that $\text{spt} T \subset \overline{\Omega}$, $S \subset \partial \Omega$. Then

$$\mathcal{H}^{n-7}(\text{sing} T) \leq c(n, \alpha) \max\{\frac{1}{2\alpha}, r_{1,\alpha}(\partial \Omega)^{-7}, r_{1,\alpha}(S)^{-7}\} ||T||(\mathbb{R}^{n+1}).$$

The same proof works, using [DS02], [Bom82] in place of [HS79], [All72], and a minor modification of [NV15].

The following examples illustrates some of the problems in extending our proof of Theorem 0.1 to more general settings.

**Example 0.4.** Both the half-helicoid and half of Enneper’s surface ([Whi96], [Per07]) are area-minimizing 2-dimensional currents in $\mathbb{R}^3$. (For the half-helicoid, just observe that by rotating the half-helicoid about the $z$-axis, one obtains a smooth foliation of $\mathbb{R}^3 \setminus z$-axis by oriented minimal surfaces). It would be interesting to know if there exists an example of a singular minimizing hypersurface bounding a multiplicity-one line.

The half-helicoid structure could be seen locally for finite $S$, if one does not assume a priori area bounds on $S$. For example, one can imagine a connected boundary curve $S$, which is composed of line segment $L$, and a curve that wraps around $L$ many times. By taking the wrapping curve to go further and further out, one can arrange $S$ to satisfy $r_{1,\alpha}(S) \geq 1$, but take the separation along $L$ of the wrappings
to zero. The minimizing integral current $T$ spanning $S$ will look very much like a compressed half-helicoid near the line segment. We cannot decompose this $T$ near $L$ into pieces of uniformly bounded area.

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0.5. Proof of Theorem 0.1. The following quantifies Hardt-Simon’s boundary regularity.

Lemma 0.6. There is a constant $\epsilon_1(n, \alpha)$, so that the following holds. Let $T$ and $S$ be as in Theorem 0.1. Then for all $x \in \text{sing} T$, we have

$$\inf_{y \in S} \frac{|x - y|}{r_{1, \alpha}(S, y)} \geq \epsilon_1(n, \alpha).$$

Proof. Towards a contradiction, suppose we have minimizing currents $T_i$, with boundary curves $S_i$, each contained the boundary of the convex set $\Omega_i$, and $x_i \in \text{sing} T_i$, and $y_i \in S_i$, so that

$$\text{dist}(x_i, S_i) \leq r_{1, \alpha}(S, y_i)/i.$$

By the maximum principle, $\text{spt} T_i \subset \overline{\Omega_i}$, and by [HS79], $\text{dist}(x_i, S_i) > 0$. Since $r_{1, \alpha}(S, y) \leq r_{1, \alpha}(S, y_i)/2$ for $y \in B_{r_{1, \alpha}(S, y_i)/2}(y)$, there is no loss in generality in assuming that $y_i$ realizes the distance in $S$ to $x_i$.

After a rotation, dilation, translation, we can assume $y_i = 0$, $r_{1, \alpha}(S, y_i) = 2$, and $e_1$ is a choice of vector so that $\Omega_i \subset \{ x : x \cdot e_1 < 0 \}$. Moreover, we can take $S_i \cap B_1$ to be the graph of a function $u_i$, define on the line $L = \{ x_1 = x_{n+1} = 0 \}$, with $|u_i|_{1, \alpha} \leq 1$. Notice that $\text{dom}(u_i) \supset B_1 \cap L$, and that $x_i \rightarrow 0$. Let us assume $x_i \in B_1/2$ for all $i$.

Let $h(t, x) : [-1, 1] \times (L \cap B_1/2) \rightarrow \mathbb{R}^{n+1}$ be defined as

$$h(t, x) = \begin{cases} x + tu_i(x) & t \geq 0 \\ x - t\sqrt{1 - |x|^2}e_1 & t \leq 0 \end{cases},$$

and let $R_i = (h_i)_{\sharp}([-1, 1] \times [L \cap B_1/2])$. Then, as an element of $\mathcal{I}_n(B_1/2)$, $\partial R_i = [S_i] \cap B_1/2$. In particular, $T_i - R_i \in \mathcal{I}_n(B_1/2)$ has no boundary. By standard decomposition of codimension-one currents [Sim83], we can find open sets $E_{i,j} \subset E_{i,j+1} \subset \ldots B_1/2$, so that $[E_{i,j}] \in \mathcal{I}_n(B_1/2)$ satisfies:

$$T_i - R_i = \sum_j \partial [E_{i,j}], \quad ||T_i - R_i|| = \sum_i ||\partial [E_{i,j}]||.$$
Since \((T_i - R_i)_t \Omega^m = (-R_i)_t \Omega^m\), and the \(E_{i,j}\) are nested, we have spt\(\partial[E_{i,j}] \subset \overline{\Omega}\) for all but one \(j = j_i\). Therefore, we have
\[
||T_i|| = ||\partial[E_{i,j_i}] + R_i|| + \sum_{j \neq j_i} ||\partial[E_{i,j}]||,
\]
and hence \(\partial[E_{i,j_i}] + R_i\) and each \(\partial[E_{i,j}]\) (\(j \neq j_i\)) are area-minimizing. By volume comparison against balls, and the estimate \(||R_i||(B_r) \leq c(n)r^n\), we get that
\[
||\partial[E_{i,j}]||(B_r) \leq c(n)r^n \quad \forall r < 1/4.
\]

We break into two cases. First, assume that \(x_i \in \text{spt}(\partial[E_{i,j_i}] + R_i)\) for all \(i\). Let \(\lambda_i = |x_i|^{-1}\), and consider the dilates
\[
T'_i := \partial[(\eta_{\lambda_i})_t E_{i,j_i}] + (\eta_{\lambda_i})_t R_i.
\]
So that \(T'_i\) has a singularity at distance 1 from \(\lambda_i S_i\).

We can pass to a subsequence (also denoted \(i\)), so that \((\eta_{\lambda_i})_t E_{i,j_i} \to [E]\), for some open set \(E\). Since \(\lambda_i S_i \to L\) in \(C^1,\alpha\), we have \((\eta_{\lambda_i})_t R_i \to [H]\), where \(H = \{x_{n+1} = 0, x_1 > 0\}\) and \([H]\) is endowed with the orientation so that \(\partial[H] = L\).

In particular, we have \(T'_i \to T = \partial[E] + [H]\), where \(\partial T = [L]\). Since each \(T'_i\) is minimizing, \(T\) is minimizing also, and \(T'_i\) converge as both currents and measures. By construction, \(T\) has a singularity at distance 1 from \(L\), \(T\) has Euclidean volume growth, and spt\(T \subset \{x : x \cdot e_1 \leq 0\}\).

Since \(T\) is minimizing with Euclidean volume growth, we can take a tangent cone \(C\) at infinity (as both currents and varifolds). \(C\) satisfies \(\partial[C] = [L]\), and so by [HS79] \(C\) is planar. Since we can write \(C = \partial[F] + [H]\) for some open set \(F\), and spt\(C \subset \{x : x \cdot e_1 \leq 0\}\), in fact \(C\) must be a multiplicity-one half-plane. By monotonicity we must have that \(T\) is a multiplicity-one half-plane also, and hence \(T\) is regular. This is a contradiction.

We are left with the second case: for all \(i\), \(x_i \in \text{spt}\partial[E_{i,j}]\) for some \(j \neq j_i\). Write \(E_i = E_{i,j}\) for the open set, for which \(x_i \in \text{spt}\partial E_i\). Consider the dilates \(E'_i = \lambda_i E_i\). Then we can pass to a subsequence, to get convergence as currents \([E'_i] \to [E]\), convergence as currents and measures \(\partial[E'_i] \to \partial[E]\), for \(0 \neq [E] \in \mathcal{I}_{n+1}(\mathbb{R}^{n+1})\) satisfying: a) \(\partial[E]\) is minimizing; b) \(\partial[E]\) has a singularity at distance 1 from the origin; and c) \(E \subset \{x : x \cdot e_1 \leq 0\}\).

Properties a), c) imply that any tangent cone at infinity of \(\partial[E]\) is a multiplicity-one plane, and hence \(\partial[E]\) is a multiplicity-one plane. This contradicts property b), and therefore completes the proof of Lemma [L.6] \(\Box\).
Lemma 0.7. Let $T \in \mathcal{I}_n(B_1)$ be area-minimizing, with $\partial T = 0$. Then we have

$$\mathcal{H}^{n-7}(\text{sing} T \cap B_{1/2}) \leq c(n) ||T|| \cdot (B_1).$$

Proof. We can decompose $T = \sum_i \partial [E_i]$, for $E_i \subset E_{i+1} \subset \ldots \subset B_1$, so that $||T|| = \sum_i ||\partial [E_i]||$, and hence each $\partial [E_i] \in \mathcal{I}_n(B_1)$ is minimizing

Since $||\partial [E_i]|| \cdot (B_1) \leq ||\partial [E_i \cup B_{3/4}]|| \cdot (B_1)$, we have $||\partial [E_i]|| \cdot (B_{3/4}) \leq c(n)$. On the other hand, by monotonicity, if $\text{spt} \partial [E_i] \cap B_{1/2} \neq \emptyset$, then $||\partial [E_i]|| \cdot (B_1) \geq 1/c(n)$. From the estimates of [NV15], we have

$$\mathcal{H}^{n-7}(\text{sing} \partial [E_i] \cap B_{1/2}) \leq c(n) \leq c(n) ||\partial [E_i]|| \cdot (B_1).$$

We can sum up contributions:

$$\mathcal{H}^{n-7}(\text{sing} T \cap B_{1/2}) \leq \sum_i \mathcal{H}^{n-7}(\text{sing} \partial [E_i] \cap B_{1/2})$$

$$\leq c(n) \sum_i ||\partial [E_i]|| \cdot (B_1)$$

$$= c(n) ||T|| \cdot (B_1).$$

Proof of Theorem 0.1. By scaling, there is no loss in assuming $r_{1, \alpha}(S) = 1$. Lemma 0.6 implies that $B_\epsilon(S) \cap \text{sing} T = \emptyset$, where $\epsilon = \epsilon_1(n, \alpha)$.

Let $\{x_j\}_j$ be a maximal ($\epsilon/2$)-net in $\text{spt} T \setminus B_\epsilon(S)$. Then the balls $\{B_{\epsilon/2}(x_j)\}_j$ cover $\text{spt} T \setminus B_\epsilon(S)$, and the balls $\{B_{\epsilon}(x_j)\}_j$ have overlap bounded by $c(n)$. For each $j$, $\partial T \cap B_{\epsilon}(x_j) = 0$, and so by Lemma 0.7 we have

$$\mathcal{H}^{n-7}(\text{sing} T \cap B_{\epsilon/2}(x_j)) \leq \frac{c(n)}{\epsilon^7} ||T|| \cdot (B_\epsilon(x_j)).$$

Using bounded overlap of the $\{B_\epsilon(x_j)\}_j$, and the isoperimetric inequality due to [FJA86], we deduce that

$$\mathcal{H}^{n-7}(\text{sing} T) = \mathcal{H}^{n-7}(\text{sing} T \setminus B_\epsilon(S))$$

$$\leq \sum_j \mathcal{H}^{n-7}(\text{sing} T \cap B_{\epsilon/2}(x_j))$$

$$\leq c(n, \alpha) \sum_j ||T|| \cdot (B_\epsilon(x_j))$$

$$\leq c(n, \alpha) ||T|| \cdot (\mathbb{R}^{n+1})$$

$$\leq c(n, \alpha) \mathcal{H}^{n-1}(S)^{n/(n-1)}. \quad \square$$
0.8. **Proof of Theorem 0.2.** We will show that the arguments of [NV15], [Gru87], and [GJ86] prove the following: there is an $\epsilon = \epsilon(n)$, so that for $x \in \text{spt}T \cap \partial\Omega$, and $r = r_{1,1}(\partial\Omega)$, we have

$$\mathcal{H}^{n-7}(\text{sing}T \cap B_{\epsilon r/2}(x)) \leq c(n)||T|||B_{\epsilon r}(x))$$

Given this estimate, the bound of Theorem 0.2 follows by a straightforward covering argument as in the proof of Theorem 0.1.

By scaling, we can and shall assume that $r_{1,1}(\partial\Omega) = 1/\Gamma$, for $\Gamma \leq \epsilon_2(n)$ chosen sufficiently small so that in $B_1(\partial\Omega)$ the nearest-point projection $\xi(x)$ to $\partial\Omega$ is well-defined and satisfies $|\xi|_{C^1} \leq 1$. Define the reflection function $\sigma(x) = 2\xi(x) - x$, and the linear reflection $i_\sigma$ about $T_\xi(x) \partial\Omega$.

Take $T \in \mathcal{I}_n(B_2)$ area-minimizing with free-boundary in $\Omega$. Define $T' = T - \sigma_\tau T$, so that $\partial T' = 0$. Then we can decompose $T'$ as

$$T' = \sum_i \partial[E_i], \quad ||T'|| = \sum_i ||\partial[E_i]||,$$

for nested open sets $E_i \subset E_{i+1}$. Moreover, since $T' \cup \Omega = T$ we can write

$$T = \sum_i \partial[E_i] \cup \Omega, \quad ||T|| = \sum_i ||\partial[E_i] \cup \Omega||.$$

From [1], we get that each $\partial[E_i] \cup \Omega$ is area-minimizing, with free-boundary in $\Omega$. By comparison against $\partial[E_i \cup B_r(x) \cup \Omega$, we have the a priori mass bounds

$$||\partial[E_i] \cup \Omega||(B_r(x)) \leq c(n)r^n \quad \forall B_r(x) \subset B_2.$$

Additionally, [Gru87] showed $T'$ admits a certain almost-minimizing property, in the following sense:

$$||T'||(B_r(x)) \leq ||T'| + S||(B_r(x)) + c(n)r||T'||(B_r(x)).$$

for every $S \in \mathcal{I}_n(B_2)$ with $\partial S = 0$, spt$S \subset B_r(x)$, and every $B_r(x) \subset B_2$ with $x \in \partial\Omega$.

[GJ86] define the following monotonicity. For $x \in B_1$, and $r < 1 - |x|$, let

$$\tilde{\theta}_T(x, r) = r^{-n}||T||(B_r(x)) + r^{-n}||T'||(\{y : |\sigma(y) - x| < r\}).$$

Notice that when $\Omega$ is a half-space, then $\tilde{\theta}_T(x, r) = \theta_T(x, r)$, and in general we have $\tilde{\theta}_T(x, r) = \theta_T(x, r)$ when $r < \text{dist}(x, \partial\Omega)$. Here $\theta_T(x, r) = r^{-n}||T||(B_r(x))$ for the usual Euclidean density ratio, and $\theta_T(x) = \lim_{r \to 0} \theta_T(x, r)$ whenever it exists.
For $0 < s < r < 1 - |x|$, [GJ86] prove

\begin{equation}
\int_{B_r(x) \setminus B_s(x)} |y - x|^{-n-2} \left( |(y - x)^{\perp}|^2 + |i(y - x)^{\perp}|^2 \right) d||T||(|y|) \leq \tilde{\theta}_T(x, r) - \tilde{\theta}_T(x, s) + c(n) \Gamma r \tilde{\theta}_T(n, r).
\end{equation}

Monotonicity (6) implies that the density $\tilde{\theta}_T(x) = \lim_{r \to 0} \tilde{\theta}_T(x, r)$ is a well-defined, upper-semi-continuous function on $B_1$, which is $\geq 1$ on spt$T$.

The above discussion, and the works of [Gru87], [GJ86], give:

**Lemma 0.9.** Let $\Omega_i$ be a sequence of $C^2$ domains, with $r_{1,1} (\partial \Omega_i \cap B_2) \to \infty$, and $T_i \in T_\ast (B_2)$ a sequence of area-minimizing currents with free-boundary in $\Omega_i$. Suppose $T_i \to T$. Then

1. $T$ is area-minimizing, with free-boundary in a half-space, and $||T_i|| \to ||T||$.
2. $T_i' \to T'$ as currents and measures, and $\tilde{\theta}_{T_i}(x, r) \to \theta_{T'}(x, r)$ for all $x \in B_2$, and a.e. $0 < r < 2 - |x|$. Here $T_i' = T_i - (\sigma_i)_{2} T_i$, where $\sigma_i$ is the reflection function associated to $\Omega_i$.
3. If $x_i \to x \in B_2$, and $r_i \to 0$, then $\limsup_i \tilde{\theta}_{T_i}(x_i, r_i) \leq \tilde{\theta}_T(x) = \theta_{T'}(x)$.
4. If $T'$ is regular, then the $T_i' \cap B_1$ are regular for $i$ sufficiently large.

**Proof.** Since $\sigma_i \to \sigma$ in $C^1$, we have $T_i' \to T'$. The convergence of measures $||T_i'|| \to ||T'||$ is a standard argument using the almost-minimizing property (6). Convergence $||T_i|| \to ||T||$ then follows from the fact that $T_i' \cap \overline{\Omega_i} = T_i$.

Convergence of the $\tilde{\theta}_T$ follows because we can estimate

\[
||T_i|| \left( \{ y : |\sigma_i(y) - x| < r \} \right) - ||T_i|| \left( \sigma(B_r(x)) \right) \\
\leq c ||T_i|| \left( B_{(1+\kappa_i)} r(x) \setminus B_{(1-\kappa_i)} r(x) \right),
\]

where $\kappa_i \to 0$ as $i \to \infty$, and because $||T||(\partial B_r(x)) = 0$ for a.e. $r$. Upper-semi-continuity follows by convergence of $\tilde{\theta}_T(x, r)$, and monotonicity.

The last property (4) is a direct consequence of the decomposition (4) and the Allard-type regularity theory of [GJ86]. \qed

We show the following variant of [NV15] (recall that $r_{1,1} (\partial \Omega) = 1/\Gamma$).

**Theorem 0.10** (compare from [NV15]). There is an $\epsilon_3 = \epsilon_3(n, \Lambda)$, so that if $T \in T_\ast (B_2)$ is area-minimizing, with free-boundary in $\Omega$, and
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\[ ||T|| \leq \Lambda, \text{ and } \Gamma \leq \epsilon_3, \text{ then} \]

\[ \mathcal{H}^{n-7}(\text{sing} T \cap B_1) \leq c(n, \Lambda). \]

When \( T = \partial[E] \), we get \( \Lambda = c(n) \), and then using the decomposition (4) in an identical argument to Lemma 0.7 we deduce the required (3).

The argument of [NV15] requires only the monotonicity formula (6), and the following two theorems, which are essentially Lemmas 7.2, 7.3 and Theorem 6.1 in [NV15] (or Lemma 3.1, Theorem 5.1 in [EE17]).

The rest of [NV15] is entirely general (see e.g. [Edc]).

**Theorem 0.11.** There is an \( \eta_0 = \eta_0(n, \alpha, \Lambda, \gamma, \rho) \), so that the following holds. Take \( B_{6r}(x) \subset B_2 \). Let \( T \in \mathcal{I}_n(B_{6r}(x)) \) be an area-minimizer with free-boundary in \( \Omega \), and take \( \eta \leq \eta_0 \). Suppose

\[ \tilde{\theta}_T(x, 6r) \leq \Lambda, \quad \Gamma \leq \eta, \quad \sup_{B_{3r}(x)} \tilde{\theta}_T(z, 3r) \leq E, \]

then at least one of the following occurs:

1. we have

\[ \text{sing} T \cap B_r(x) \subset \{ z \in B_r(x) : \tilde{\theta}_T(z, \gamma \rho r) \geq E - \gamma \}, \text{ or} \]

2. there is an affine \((n - 8)\)-space \( p + L^{n-8} \), so that

\[ \{ z \in B_r(x) : \tilde{\theta}_T(z, 3\eta r) \geq E - \eta \} \subset B_{\rho r}(p + L). \]

**Theorem 0.12.** There is a \( \delta(n, \alpha, \Lambda) \) so that the following holds. Take \( B_{10r}(x) \subset B_2 \). Let \( T \in \mathcal{I}_n(B_{10r}(x)) \) be an area-minimizer with free-boundary in \( \Omega \), and \( \mu \) a finite Borel measure. Suppose that

\[ \tilde{\theta}_T(x, 10r) \leq \Lambda, \quad \Gamma \leq \delta, \quad \tilde{\theta}_T(x, 8r) - \tilde{\theta}_T(x, \delta r) < \delta, \quad x \in \text{sing}(T). \]

Then we have

\[ \inf_{p + L^{n-7}} \frac{1}{r^{n-5}} \int_{B_{r}(x)} \text{dist}(z, p + L)^2 d\mu(z) \leq \frac{c(n, \alpha, \Lambda)}{r^{n-7}} \int_{B_{r}(x)} \tilde{\theta}_T(z, 8r) - \tilde{\theta}_T(z, r) + c(n) \Gamma d||T|||z|, \]

where the infimum is over affine \((n - 7)\)-planes \( p + L^{n-7} \).

**Proof of Theorem 0.11** The proof consists of two contradiction arguments, verbatim to Theorem 5.1 in [EE17]. In place of the \( \epsilon \)-strata, we use the following consequence of Lemma 0.9. Suppose \( T_i \in \mathcal{I}_n(B_6) \) is a sequence of area-minimizers with free-boundary in \( \Omega_i \), so that \( r_{1,1}(\partial \Omega_i \cap B_6) \to \infty \) and \( T_i \to T \). If \( T'_\perp B_2 \) coincides with a cone, that is invariant along an \((n - 6)\)-space, then \( T_i \perp B_1 \) is regular for sufficiently large \( i \). \( \square \)
Proof of Theorem 0.12. The proof divides into two parts, which are verbatim to Lemma 6.2 and Proposition 6.6 in [NV15] (or Theorem 5.1 in [EE17]). The first part is a direct consequence of the monotonicity formula (6). The second part is a straightforward contradiction argument. The proof in [NV15] uses varifold convergence. For integral currents, one can use the fact for any \((n+1)\)-form \(\omega\) and any vector \(v\), we have
\[
< \bar{T}, \omega, v > = < v \wedge \bar{T}, \omega > \quad \text{and} \quad \| v \wedge \bar{T} \| = |\pi_{T^\bot}(v)|. \tag*{\blacksquare}
\]

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