Construction of a genetic decomposition: theoretical analysis of the concept of the value theorem medium

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Abstract. In this study, from a cognitive perspective, we present the structures necessary for a student to construct the concept of the mean value theorem. The theoretical framework used for this research is based on the action-process-object-scheme theory. We propose, as a result, two genetic decompositions that describe two possible ways to construct the concept: a graphical path and an analytical path. Generic decompositions become a point of reference for the design of teaching strategies; in that sense, the results of this research are presented as a theoretical proposal for teaching and learning the model of the mean value theorem to undergraduate students of mathematics, physics, engineering or in whichever area this particular subject is studied, with the purpose of contributing to the strengthening of educational practices and student learning.

1. Introduction

Many researchers have written about the great difficulty students experience when trying to have a proper understanding of the derivative [1–5]. A derivative is a mathematical object, one where its other concepts are difficult to understand. Understanding the concept of the mean value theorem (MVT) is particularly problematic. Asiala [4] highlighted students’ problem with understanding the derivative of a function at a point as the slope of the tangent line to the curve, which represents the function, at that point. Therefore, we believe that the problem with understanding the concept of MVT may be due to a disconnect between the geometric and analytical interpretation of the conclusion of the MVT.

The mean value theorem is one of the most important results in mathematics; however, and despite the difficulty in understanding it, as far as we the authors know, there is very little research on MVT teaching and learning. A contribution presented in this research, in order to overcome the problem described above, is to study and analyze as an object of research the understanding of this mathematical concept, following the theoretical and methodological approach actions-processes-objects-scheme (APOE). Currently, the APOE theory is used as a frame of reference in multiple investigations within different areas of mathematics, for example, in linear algebra [6–9] and calculus [1–3, 10–12]. The APOE theory [13] has a research cycle based on three components: (i) theoretical analysis; (ii) design and application of instruments; (iii) data collection, analysis and verification.

In this investigation, we focus on developing the first component, proposing two genetic decompositions for the MVT concept as a result; the first one follows a graphical path and the other an analytical path. The results of this research will guide the development of future research on the following components of the cycle, the design and application of instruments and
the collection, analysis, and verification of data. Considering the above, the following research question is posed: How to characterize the cognitive mechanisms and structures that promote understanding of the concept of MVT? The answer to this question will be set forth in this research, based on the content of the following sections. In section 2, we present a theoretical framework whereby we establish the version of MVT, together with the APOE theoretical approach; in section 3, we present a brief description of the methodology and results of our study based on decomposition genetics, thus proposing two genetic decompositions of the MVT. Finally, in section 4, we present the conclusions of our investigation.

2. Theory foundation

The APOE theory was developed by Dubinsky in 1991, as an interpretation of constructivist theory, based mainly on the concept of reflective abstraction introduced by Piaget. It is used to describe the development of logical thinking in children, expanding this idea to more advanced mathematical notions [14]. The APOE theory defines actions, processes, objects, and schemes as the cognitive structures that a student may develop in order to construct a mathematical concept, with mechanisms of internalization, coordination, encapsulation and de-encapsulation as the means to achieve these structures.

In Figure 1 is a general representation of the understanding of a mathematical concept or in this case, of a theorem. It begins when a student acts upon previously constructed objects. When after repeating certain actions, the student reflects on them, ceasing to depend on external instructions by gaining internal control over whatever they do or imagine, we can say that the student has internalized such action into a process. The cognitive process of construction is defined by a student having reached the ability to imagine performing the steps to be followed in a mathematical activity, without necessarily having to carry out each one of them explicitly, even as to being able to apply transformations to a mathematical object in mind, without the need to go through each step, and proposing counterexamples if it is the case. On the other hand, two or more processes can be coordinated to build a new one, and also, this process can be reversed or generalized. Encapsulation of a process allows moving from a dynamic structure (process) to a static one (object), that is, reflection on cognitive processes can lead to the encapsulation of a process in an object, which implies that conceptual objects are entities in themselves and contain properties; thus, they can be deemed in their entirety and acted upon. Furthermore, if it is necessary to return from the object to the process where it was originated, it is said that the object in a process has been decapsulated. An outline of a mathematical concept is of a collection of actions, processes, objects and schemes pertaining to other concepts.

![Figure 1. Graphic scheme of the structures and mechanisms proposed in the APOE theory [13].](image)

To practice this theory as a research framework, it is required for the design of a predictive model called genetic decomposition (GD). This hypothetical model describes in detail the constructs and cognitive mechanisms necessary for a student to learn a mathematical concept [13]. In the case of this particular research, it is interesting to design a GD that describes the construction of knowledge included in the MVT.
In literature, the MVT is defined in its different versions: real differentiable functions with a real variable [15], real integrable functions of a real variable [16], real functions on several variables and for vector functions [17]. For this research in particular we consider the version of the MVT for real differentiable functions with a real variable that Apostol presents in [15], and is stated below in Theorem 1.

Theorem 1. If \( f \) is a continuous function on the entire closed interval \([a, b]\) which has a derivative at each point of the open interval \((a, b)\), there is at least one point \( c \) inside \((a, b)\) for which \( f'(c) \) is expressed by Equation (1) [15].

\[
\frac{f(b) - f(a)}{b - a} = f'(c). 
\]

The concept of continuity of a function is required for the understanding of Equation (1) which is the first MVT studied analytically by university students.

3. Genetic decomposition
The APOE theory provides a research cycle made up of three components: theoretical analysis; design and application of instruments; and collection, analysis, and verification of data. Theoretical analysis, the component we are concerned with, consists of the in-depth study of the mathematical concepts involved in the MVT to determine the cognitive constructions necessary in learning the theorem through a hypothetical description of the cognitive constructions of the student. This is what we call a genetic decomposition (GD). GD is a hypothetical model that describes the structures and cognitive mechanisms that a student might need in order to learn a specific math concept [13].

The development of reported genetic decomposition of derivative concepts [4, 5, 18] and continuity [19], is a key input in our research for the construction of the genetic decomposition of the MVT, since all of these concepts are also hypotheses of the MVT. The derived process structure \( f'(c) \) of a function \( f \) at a point \( c \), described by Asiala, Dubinsky, Cottrill and Schwingendorf [4], like the slope \( m_l \) tangent line to the curve \( C \) (which is represented in the graph of the \( f \) function) on point \( c \) is fundamental for our study, since the coordination of said process with the process for the average rate of change of the function \( f \) [2], is then encapsulated in the pending object \( m_l \) of a straight \( l \). It allows for the construction of the MVT as we will describe further on. We assume that students come to study the MVT with the following prior knowledge.

3.1. Previous knowledge
3.1.1. Function. The student must have practical knowledge and a solid understanding of the concept of a function, and thus, a well-developed conceptual image of a function.

3.1.2. Continuity of a function. Given a function \( f : [a, b] \to \mathbb{R} \) the student must determine whether or not it is continuous at all points in the closed interval \([a, b]\). Brijlall and Maharaj [19] show that the continuity of a function at a point is the result of the coordination of the schemes previously built by the students on function and limit of functions, together with an appropriate notation (see Figure 2). Then, the student, in a single process, internalizes these actions as the points of the interval. The encapsulation of this process leads to the definition of continuity of the function throughout the interval.
3.1.3. Derived function. Given a function $f : (a, b) \rightarrow \mathbb{R}$, the student must determine the derived function on the interval $(a, b)$. The student internalizes the action of calculating the derivative at a point within the process of constructing the derived function, which takes as input a point $x$ and, in the output, generates the value of $f'(x)$ for any $x$ in the domain of the function if the limit exists. The encapsulation of this process generated the derived function as a new complex object (involving the process of object synthesis derived at a point) [5].

3.1.4. Parallel lines. The student must determine when two lines are parallel. The student encapsulates the process of matching the slopes of two lines on the parallel lines object.

3.2. Graphical and analytical paths to mean value theorem

In this work, we suggest that there are at least two related paths that can be taken when building a schematic GD for the MVT concept: a graphical path and an analytical path; these two roads are coordinated, even when not be entirely separate. The graphic path is described as follows:

(i) The student must carry out the act of connecting initial point $A = (a, f(a))$ and finally $B = (b, f(b))$ of the curve $C$ (graphical representation of the function $f$) through a straight line $l_{AB}$, Figure 3(a).

(ii) The student must perform the act of graphing tangent lines $t_{c_i}$ to the curve $C$ at arbitrary points on the curve ($c_i, f(c_i)$), Figure 3(b).

(iii) Once the previous actions have been carried out, the student internalizes them in a single process as you only identify those lines $t_{c_i}$ parallel to the line $l_{AB}$, Figure 3(c).

(iv) The student encapsulates the above process to generate the MVT as the existence of points $c_i$ in the domain of the function $f$, where lines $t_{c_i}$ and $l_{AB}$ are parallel. Finally, the student obtains the conclusion of the MVT decapsulating the object parallel lines in the process of matching the slopes $m_{t_{c_i}}$ and $m_{l_{AB}}$ of the lines $t_{c_i}$ y $l_{AB}$ respectively, that is $m_{t_{c_i}} = m_{l_{AB}}$.

Figure 2. Diagram of the concept of continuity of a function which relates the function scheme and the limit scheme that have been reached in previous processes.

Figure 3. (a) Graphical representation of the function through a straight line; (b) graph of the line tangent to the curve at the point; (c) graphical representation of the parallel line process.
Accordingly, the analytical path is described as follows:

(i) The student performs the act of calculating the average rate of change of the function \( f : [a, b] \rightarrow \mathbb{R} \). Calculating the following quotient with Equation (2).

\[
\frac{f(b) - f(a)}{b - a}.
\]

(ii) The student internalizes the previous action in the process of calculating the slope \( m_{l_{AB}} \) of the straight \( l_{AB} \), that joins the starting point \( A = (a, f(a)) \) and finally \( B = (b, f(b)) \) of the curve \( C \) which graphically represents the function \( f \), this is Equation (3).

\[
m_{l_{AB}} = \frac{f(b) - f(a)}{b - a}.
\]

(iii) The student performs the act of calculating the derived function \( f' \) at an arbitrary point \( c \) of the domain of the function, that is \( f'(c) \).

(iv) The student internalizes the previous action in the process of a function that takes as an input arbitrary points \( c_i \) and generates output values \( f'(c_i) \) like the slopes \( m_{t_{c_i}} \) of tangent lines \( t_{c_i} \) to the curve \( C \), in the points \( c_i \), Equation (4).

\[
m_{t_{c_i}} = f'(c_i).
\]

(v) Point processes ii y iv are coordinated through matching slopes \( m_{l_{AB}} \) y \( m_{t_{c_i}} \), that is to say, the Equation (5).

\[
m_{l_{AB}} = m_{t_{c_i}}.
\]

(vi) The student encapsulates the new process above to produce the MVT this is that is to say, the Equation (6).

\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]

When assimilating the solution of this equation, gives us the points \( c_i \) of the domain of the function \( f \), where the slope of the line \( m_{l_{AB}} \) is the same as the slope of the line \( m_{t_{c_i}} \), Figure 4.

**Figure 4.** Graphic scheme that represents the coordination between the process of calculating the slope and the process of calculating the derivative of a function to originate the MVT as an object.
4. Conclusions

In this study, we have attempted to contribute to knowledge on how a student can learn the concept of the mean value theorem. The theoretical framework of action, process, object, scheme, or APOE, was a very useful tool to build the two genetic decompositions, whereby we describe the possible cognitive structures and mechanisms that a student might need in order to understand the concept of the mean value theorem. Our theoretical proposal of these two genetic decompositions was generated out of literature that exposes problems with understanding the derivative, along with the authors’ experience as teachers for undergraduate students.

The results obtained in this research show the cognitive constructions and mechanisms that model the learning of the mean value theorem. For this purposes, we propose two genetic decompositions, one graphical and the other analytical, both of which are clearly coordinated and suggest that, for a better understanding of the theorem, rather than memorizing the result and solving the equation to achieve the theorem’s conclusion, students should build this equation by interpreting the theorem as the search for points inside the domain of the function, where a tangent line passes to the curve, representing said function parallel to the line that passes through the initial and final points of said curve.

Our theoretical proposal is meant for a community interested in learning about these topics as a possible model for teaching and learning about the MVT, and as a starting point for future research on the two components of the APOE theory: design and application of instruments and data collection, as well as analysis and verification.

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