SYMPELECTIC MAPPING CLASS GROUP RELATIONS
GENERALIZING THE CHAIN RELATION

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ABSTRACT. In this paper, we examine mapping class group relations of some symplectic manifolds. For each \( n \geq 1 \) and \( k \geq 1 \), we show that the \( 2n \)-dimensional Weinstein domain \( W = \{ f = \delta \} \cap B^{2n+2} \), determined by the degree \( k \) homogeneous polynomial \( f \in \mathbb{C}[z_0, \ldots, z_n] \), has a Boothby-Wang type boundary and a right-handed fibered Dehn twist along the boundary that is symplectically isotopic to a product of right-handed Dehn twists along Lagrangian spheres. We also present explicit descriptions of the symplectomorphisms in the case \( n = 2 \) recovering the classical chain relation for the torus with two boundary components.

1. INTRODUCTION

About fifteen years ago, Giroux [Gir02] described a correspondence between \((2n+1)\)-dimensional contact manifolds and symplectomorphisms of \( 2n \)-dimensional exact symplectic manifolds via the Thurston-Winkelnkemper construction [TW75]. Since then, there have been several achievements that allow us to understand this relation further. In dimension 3, considerable progress has been made in understanding this correspondence such as in [Wan14] as the symplectomorphism group of a surface is equal to its mapping class group, and hence can be studied entirely by Dehn twists along simple closed curves.

However, symplectomorphisms of symplectic manifolds of dimension greater than 2 are comparatively not well understood. There are many results that move us towards understanding symplectomorphism groups such as the results of Khovanov and Seidel [KS02] and also those of Seidel [Sei08] and [Sei98]. In [Sei98], Seidel examines the role of Dehn twists in the compactly supported symplectomorphism group of \( T^*S^n \). Moreover, one can construct Dehn twists for any symplectic manifold containing a Lagrangian sphere [Arn95], [Sei97] and [Sei98]. However, there exist exact symplectic manifolds which do not contain Lagrangian spheres (e.g., the \( 2n \)-dimensional disk). For such manifolds, there is no general way of...
constructing symplectomorphisms that are not symplectically isotopic to the identity.

To provide another means of constructing symplectomorphisms, Biran and Giroux [BG07] introduced the notion of a fibered Dehn twist which can be constructed if a symplectic manifold has a contact type boundary carrying a suitable circle action. This will be explained in detail in Section 3. Many such fibered Dehn twists have been shown not to be symplectically isotopic to the identity [BG07, CDvK12].

The main goal of this paper is to carry the effort to understand that relation one step further and examine symplectic mapping class group relations of some symplectic manifolds. We prove the following theorem which shows that in certain cases, fibered Dehn twists can be expressed as products of Dehn twists.

**Theorem 1.1.** Let \( f \in \mathbb{C}[z_0, \ldots, z_n] \) be a homogeneous polynomial of degree \( k \) with an isolated singularity at 0 for \( n \geq 1, \ k \geq 1 \). Denote by \((W, \beta)\), the \( 2n \)-dimensional Weinstein domain, where for \( \epsilon \geq 0 \) small and \( \delta = \delta(\epsilon) \) small

\[
W = \{ f(z_0, \ldots, z_n) = \delta \} \cap B_\epsilon^{2n+2} \subset \mathbb{C}^{n+1}, \beta = \frac{1}{2} \sum_{j=0}^{n} (x_j dy_j - y_j dx_j).
\]

Then \( W \) has Boothby-Wang type boundary and \( \partial W \) has a coherent open book \( OB(F, \Phi_{\partial}) \) such that a right-handed fibered Dehn twist \( \Phi_{\partial} \in \text{Symp}(F, d\beta, \partial F) \) along \( \partial W \) is boundary-relative symplectically isotopic to a product of \( k(k-1)^n \) right-handed Dehn twists \( \Phi_1 \cdots \Phi_{k(k-1)^n} \) along Lagrangian spheres.

**Corollary 1.2** (Roots of fibered Dehn twists). Let \( f \in \mathbb{C}[z_0, \ldots, z_n] \) be a homogeneous polynomial of degree \( k \) with an isolated singularity at 0 for \( n \geq 1, \ k \geq 1 \). Denote by \((W, \beta)\), the \( 2n \)-dimensional Weinstein domain, where for \( \epsilon \geq 0 \) small and \( \delta = \delta(\epsilon) \) small

\[
W = \{ f(z) = \sum_{j=0}^{n} z_j^k = \delta \} \cap B_\epsilon^{2n+2} \subset \mathbb{C}^{n+1}, \beta = \frac{1}{2} \sum_{j=0}^{n} (x_j dy_j - y_j dx_j).
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Details of the statements of the above theorem and corollary as well as the above discussion will be explained in Section 4.

In Section 5.2, we prove the following corollary which provides a visual representation of the case \( n = 2, \ k = 3 \) recovering the classical chain relation for a torus with two boundary components by using the star relation [Ger96] which will be explained in Section 5.3, the last section of the paper.
Corollary 1.3. Consider the genus 1 surface with 3 boundary components $S_{1,3}$ equipped with the embedded curves $\alpha_b, \alpha_g, \alpha_p, \alpha_r, b_1, b_2,$ and $b_3$ as depicted in Figure 1. Let $D_{\alpha_i}$ and $D_{b_j}$ represent Dehn twists along the curves $\alpha_i$ and $b_j$, respectively, where $i \in \{b, g, p, r\}$ and $j \in \{1, 2, 3\}$. Then

$$(D_{\alpha_r} \circ D_{\alpha_p} \circ D_{\alpha_b} \circ D_{\alpha_g})^3 = D_{b_1} \circ D_{b_2} \circ D_{b_3}$$

in the mapping class group of $S_{1,3}$.

Plan of the Paper.

- In Section 2, we give basic definitions.
- In Section 3, we describe the Boothby-Wang circle bundle and a right-handed fibered Dehn twist explicitly and present the conditions for a Boothby-Wang circle bundle to possess a supporting open book and a monodromy as a right-handed fibered Dehn twist.
- In Section 4, we give the proof of the main theorem by discussing the contact open book coming from the Lefschetz fibration with a product of Dehn twists as monodromy and the Boothby-Wang circle bundle with a right-handed fibered Dehn twist as monodromy. We also give the proof of Corollary 1.2 and the exact number of Dehn twists appeared in both Theorem 1.1 and Corollary 1.2.
- In Section 5, we discuss the explicit case $n = 2$ describing some mapping class group relations for surfaces.

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2. Preliminaries

**Definition 2.1.** A contact manifold is a pair \((Y, \xi)\), where \(Y\) is a smooth, oriented, \((2n + 1)\)-dimensional manifold and \(\xi\) is a \(2n\)-dimensional hyperplane field on \(Y\) for which there exists a 1-form \(\alpha \in \Omega^1(Y)\) such that

1. \(\xi = \ker(\alpha)\),
2. \(\alpha \wedge (d\alpha)^n\) is a positive volume form, with respect to the prescribed orientation, on \(Y\).

We refer to any 1-form satisfying the above two properties as a contact form.

**Definition 2.2.** A Lefschetz fibration on a \(2n\)-dimensional manifold \(X\) is a mapping \(g : X \to S\), where \(S\) is a closed 2-manifold having the following properties:

1. The critical points of \(g\) are isolated.
2. If \(p \in X\) is a critical point of \(g\), then there are local coordinates \((z_1, \ldots, z_n)\) on \(X\) and \(z\) on \(S\) with \(p = (0, \ldots, 0)\) and such that in these coordinates, \(g\) is given by the complex map \(z = g(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2\).

Each critical point \(p\) of \(g\) is of Morse-type, i.e., non-degenerate. Moreover, for each \(z \in S \setminus \{c\}\), where \(c = g(p)\), we require that \(M_z = g^{-1}(z)\) is an exact symplectic submanifold of \(X\).

**Definition 2.3.** Let \(f \in \mathbb{C}[z_0, \ldots, z_n]\) be a hypersurface singularity germ at the origin in \(\mathbb{C}^n\), where \(n \geq 0\). Let \(S^{2n+1}_\epsilon = \partial B^{2n+2}_\epsilon\) be the boundary of the closed ball with radius \(\epsilon\) centered at the origin and let \(K = \{f = 0\} \cap S^{2n+1}_\epsilon\) be the corresponding link of singularity. Then the smooth, locally trivial fibration

\[ \phi : S^{2n+1}_\epsilon - K \to S^1, \quad \phi(z) = \frac{f(z)}{|f(z)|} \]

is called the Milnor fibration of the function germ \(f\). This is due to [Mil68].

When \(f\) has an isolated singularity at the origin, then the link \(K\) is a smooth manifold. Moreover, any fiber, namely Milnor fiber, \(F_t = \phi^{-1}(t)\) is a smooth open manifold. In particular, closure of the Milnor fiber is a manifold with boundary \(K\) [Mil68].

**Definition 2.4.** A Liouville manifold is a pair \((W, \beta)\), where \(W\) is a compact \(2n\)-dimensional manifold with boundary and a 1-form \(\beta \in \Omega^1(W)\) such that

1. \(d\beta\) is a symplectic form on \(W\).
2. \(\beta|_{\partial W}\) is a contact form.

**Definition 2.5.** Let \(\Sigma\) be a \(2n\)-dimensional compact smooth manifold with boundary. A Weinstein domain structure on \(\Sigma\) is a triple \((\omega, Z, \phi)\), where

- \(\omega\) is a symplectic form on \(\Sigma\),
- \(\phi : \Sigma \to \mathbb{R}\) is a generalized Morse function (i.e., \(f\) only has isolated critical points of Morse or birth-death type),
- \(Z\) is a complete Liouville vector field for \(\omega\) (i.e., there always exists such a vector field and the Lie derivative \(\mathcal{L}_Z\omega\) coincides with \(\omega\) and gradient-like for \(\phi\) which is constant on the boundary (i.e., \(d\phi(Z) > 0\) away from critical points). Furthermore, \(Z\) is transverse to the boundary of \(\Sigma\).
The quadruple $(\Sigma, \omega, Z, \phi)$ is then called a Weinstein domain.

3. OPEN BOOKS FOR BOOTHBY-WANG BUNDLES

**Definition 3.1.** An open book decomposition of a manifold $Y$ is a pair $(B, \pi)$, where

1. $B$ is a codimension 2 submanifold of $Y$ called the binding of the open book.
2. $\pi : Y - B \to S^1$ is a fiber bundle of the complement of $B$ such that each fiber $F_\theta := \pi^{-1}(\theta)$ in $Y$ and $\partial F_\theta = B$ for all $\theta \in S^1$. The fiber $F = F_\theta$, for any $\theta$, is called a page of the open book.

The holonomy of the fiber bundle $\pi$ determines a conjugacy class in $\text{Diff}^+(F, \partial F)$ which we call the monodromy.

**Definition 3.2.** The symplectomorphism group $\text{Symp}(F, \omega, \partial F) \subset \text{Diff}^+(F, \partial F)$ consists of all orientation preserving diffeomorphisms $\phi : F \to F$ such that $\phi$ preserves the symplectic form $\omega$, i.e., $\phi^*(\omega) = \omega$.

**Definition 3.3.** A contact structure $\xi$ on a manifold $Y$ is said to be supported by an open book $(B, \pi)$ of $Y$ if it is the kernel of a contact form $\lambda$ satisfying the following:

- $\lambda|_B > 0$ and
- $d\lambda > 0$ on every page.

If these two conditions hold, then the open book $(B, \pi)$ is called a supporting open book for the contact manifold $(Y, \xi)$ and the contact form $\lambda$ is said to be adapted to the open book $(B, \pi)$ or $\lambda$ is adapted by the fiber bundle $\pi$.

As mentioned in Section [1], Giroux [Gir02] and Thurston-Winkelnkemper [TW75] have shown that every open book decomposition gives rise to a supporting contact manifold. That is to say, we can associate to each triple $(W, \omega, \Phi)$ a contact manifold $(Y, \xi)$, where $(W, \omega)$ is a $2n$-dimensional Weinstein domain and $\Phi$ is a symplectomorphism of $(W, \omega)$ which restricts to the identity on some collar neighborhood of $\partial W$.

**Notation 3.1.** Let $(W, \omega)$ be a Weinstein domain with symplectic form $\omega$. Then as stated above, we can discuss its associated open book. Throughout, we will denote the symplectomorphism group of a page $F$ of the open book of the boundary of $W$ by $\text{Symp}(F, \omega, \partial F)$.

3.1. Fibered Dehn Twist. Let $W$ be a symplectic manifold with contact type boundary $\partial W = P$ carrying a free $S^1$-action in the neighborhood $P \times [0, 1]$ that preserves the contact form $\lambda$ on $P$. Then we can define a right-handed fibered Dehn twist along the boundary $P$ as follows:

$$\tau : P \times [0, 1] \longrightarrow P \times [0, 1],$$

$$(x, t) \mapsto (x \cdot [f(t) \mod 2\pi], t),$$
where \( f : [0, 1] \rightarrow \mathbb{R} \) is a smooth function such that \( f(t) = 0 \) near \( t = 1 \) and \( f(t) = 2\pi \) near \( t = 0 \). It is easily verified that this is a symplectomorphism of \((P \times [0, 1], d(e^t \lambda))\) which is equal to identity near boundary whose isotopy class is independent of the choice of \( f \). See [CDvK12] for more details.

This notion was introduced by Biran and Giroux to emphasize that there are examples of symplectomorphisms such as fibered Dehn twists, which are not necessarily products of Dehn twists; see [BG07].

### 3.2. Boothby-Wang Circle Bundles or Prequantization Circle Bundles

Let \( M \) be a \( 2n \)-dimensional compact symplectic manifold with integral symplectic form \( \omega \), i.e., \([\omega] \in H^2(M; \mathbb{Z})\).

Let \( \Pi : P \rightarrow M \) be a principal circle bundle with first Chern class \( c_1(P) = [\omega] \). Then according to [BW58, Theorem 3], there exists a connection 1-form \( \lambda \) on \( P \) such that

1. \(-2\pi \Pi^*\omega = d\lambda \) and
2. \( \lambda \) is a contact 1-form on \( P \).

The contact form \( \lambda \) is called a Boothby-Wang form carried by the circle bundle \( P \xrightarrow{\Pi} M \). The fact that the contact form \( \lambda \) is a connection 1-form implies that the vector field, \( R_\lambda \), generating the circle action satisfies \( \mathcal{L}_{R_\lambda} \lambda \equiv 0 \) and \( \lambda(R_\lambda) \equiv 1 \). This implies that \( R_\lambda \) is the Reeb vector field for \( \lambda \). The circle bundle \((P \xrightarrow{\Pi} M, \lambda)\) is then called a Boothby-Wang circle bundle associated with \((M, \omega)\). Since the curvature 2-form \( \omega \) on \( M \) makes the base space \( M \) into a symplectic manifold, \((P, \lambda)\) is called prequantum circle bundle (or Boothby-Wang circle bundle) that provides a geometric prequantization of \((M, \omega)\). This is due to Boothby and Wang, 1958 [BW58].

### 3.3. Setup

Now we will provide a description of an adapted open book decomposition of a Boothby-Wang circle bundle over a symplectic manifold. This is due to Boothby and Wang [BW58] and Chiang, Ding and van Koert [CDvK12].

Let \( H \) be a compact symplectic hypersurface of a compact integral symplectic manifold \((M^{2n}, \omega)\) Poincaré dual to \([\omega]\), i.e., a codimension two symplectic submanifold whose homology class \([H]\) \( \in H_{2n-2}(M; \mathbb{Z}) \) is the Poincaré dual to \([\omega]\) \( \in H^2(M; \mathbb{Z}) \). Then \( F = M - \nu(H) \), the complement of an open tubular neighborhood \( \nu(H) \) of \( H \), carries a Weinstein domain structure; see [Don96] and [Gir02]. \( H \) is then called an adapted Donaldson hypersurface.

Now consider the Weinstein domain \((W, \beta)\) with a Boothby-Wang type boundary \((P, \lambda)\). Then it is possible to construct two contact manifolds out of the given data. Firstly, we can construct a Boothby-Wang circle bundle \((P \xrightarrow{\Pi} M, \lambda)\) over a symplectic manifold \( M \). We can also define a right-handed fibered Dehn twist \( \Phi_\partial \) along the boundary of \( W \), then define a contact open book with page \( F = M - H \)
and monodromy as $\Phi_\partial$. Thus, we can talk about the relation between the Boothby-Wang circle bundle and its associated open book. That relation is given by the following theorem.

**Theorem 3.1.** Let $(W, \beta)$ be a Weinstein domain with a Boothby-Wang type boundary ($\partial W = P, \lambda = \beta|_P$), $\Phi_\partial$ be a right-handed fibered Dehn twist on $W$ along the boundary $P$, $\omega$ be an integral symplectic form on $M$, $H$ be an adapted Donaldson hypersurface Poincaré dual to $[\omega]$ and $F = M - H$. Then

1. The Boothby-Wang circle bundle $(P \xrightarrow{\Pi} M, \lambda)$ associated with $(M, \omega)$ has an open book decomposition whose monodromy is a right-handed fibered Dehn twist.
2. The open book $(F, \Phi_\partial)$ with page $F$ and monodromy $\Phi_\partial$ is contactomorphic to the Boothby-Wang circle bundle $(P \xrightarrow{\Pi} M, \lambda)$ associated with $(M, \omega)$.

**Proof.** See Theorem 6.3 and Corollary 6.4 in [CDvK12]. □

**Definition 3.4.** An open book decomposition $(B, \pi)$ is **coherent** with a Boothby-Wang circle bundle $(P \xrightarrow{\Pi} M, \lambda)$ if

1. The contact form $\lambda$ is adapted by the fibration $\pi : P - \Pi^{-1}(H) \to S^1$, where $\Pi : P \to M$ is the Boothby-Wang circle bundle.
2. The binding $B = \Pi^{-1}(H)$ for some adapted Donaldson hypersurface $H \subset M$.
3. The monodromy $\Phi_\partial$ is a right-handed fibered Dehn twist.

**4. Proof of Theorem 1.1**

Throughout, we denote the coordinates on $\mathbb{C}^{n+1}$ by $z_j = x_j + iy_j, j = 0, \ldots, n$ and the open book with page $F$ and monodromy $\Phi$ by $OB(F, \Phi)$.

The proof is by comparing two points of view, namely the Lefschetz fibration point of view and the Boothby-Wang fibration point of view.

4.1. **Lefschetz Fibration Construction.** Let $z_n : W \to \mathbb{C}$ be a map with an isolated singularity at the origin. Then the restriction map, denoted by $h := z_n|_{\{|z_n| \leq \epsilon\}} : W \cap \{|z_n| \leq \epsilon\} \to \{|z_n| \leq \epsilon\}$ for $\epsilon$ small is a Lefschetz fibration and $F = W \cap \{z_n = \epsilon\}$ is the Lefschetz fiber of the Lefschetz fibration, i.e., the singular fiber (sphere) with one transverse self-intersection.

Since all critical points of a Lefschetz fibration are of Morse-type, we need to perturb $z_n$ in a certain stable way so that the isolated degenerate singularity at 0 will split up into other isolated singularities which are non-degenerate. This deformation is called Morsification which is a small deformation of $z_n$ having $r$ distinct Morse critical points $p_1, \ldots, p_r$ and critical values $c_1, \ldots, c_r$. Here, $z_n + \epsilon g$ is a Morsification of $z_n$ if $g(z)$ is a linear function in general position and $\epsilon$ is very
Claim: There is finitely many such Morse-type critical points, say \( r \)-many, after Morsification of \( z_n \). Moreover, \( r = k(k - 1)^n \).

Proof: Assume to the contrary that there is an infinite number of Morse-type critical points, then there is a sequence of critical points which accumulates to a critical point. On the other hand, each Morse critical point has a neighborhood without other critical points. This contradicts with the fact that a sequence of critical points limits to a critical point. Therefore, there is finitely many Morse-type critical points. For the proof of the exact number of such critical points, see Section 4.5.

That is to say, we have finitely many Lagrangian spheres in the fiber coming from critical values. Moreover, it gives rise to associated monodromies. Denote the monodromy associated to critical values by \( \Phi_1 \cdot \ldots \cdot \Phi_k \cdot \Phi_{k-1} \cdot \ldots \cdot \Phi_1 \cdot \Phi_{k-1} \cdot \ldots \cdot \Phi_1 \). Each \( \Phi_i \in \text{Diff}^+ (F, \partial F) \), for \( i = 1, \ldots, k(k - 1)^n \), is a right-handed Dehn twist. Then we have the following open book decomposition associated to the boundary restriction of the Lefschetz fibration \( h \):

- Page: \( F = W \cap \{ z_n = \epsilon \} \).
- Binding: \( B = \partial W \cap \{ z_n = 0 \} \).
- Monodromy: \( \Phi_1 \cdot \ldots \cdot \Phi_k \cdot \Phi_{k-1} \cdot \ldots \cdot \Phi_1 \cdot \Phi_{k-1} \cdot \ldots \cdot \Phi_1 \).

Here, observe that \( F \) is a retracted page of the open book decomposition \( (B, \Phi_1 \cdot \ldots \cdot \Phi_k \cdot \Phi_{k-1} \cdot \ldots \cdot \Phi_1) \).

Let us consider the boundary of the Weinstein domain \( W \cap \{ |z_n| \leq \epsilon \} \). We know that an open book naturally arises as boundary restriction of a Lefschetz fibration. According to the Lefschetz fibration construction above, \( \partial(W \cap \{ |z_n| \leq \epsilon \}) \) is supported by the open book decomposition with page \( F \) and monodromy \( \Phi_1 \cdot \ldots \cdot \Phi_k \cdot \Phi_{k-1} \cdot \ldots \cdot \Phi_1 \). The binding of this open book is the intersection of the link associated to the homogeneous polynomial \( f \), i.e., \( \partial W = \{ f = 0 \} \cap S^{2n+1} \) with the hyperplane \( \{ z_n = 0 \} \).

The analysis above provides a description of the monodromy coming from the Lefschetz fibration point of view. In order to analyze the desired isotopy between the two monodromies mentioned in the main theorem, it suffices to describe the monodromy coming from the Boothby-Wang point of view, namely a right-handed fibered Dehn twist. In order to do that, we need the data given in Theorem 3.1.

4.2. Boothby-Wang Construction. Consider the Weinstein domain \( W \) with the boundary \( \partial W \) carrying a free \( S^1 \)-action. We can extend this action to the neighborhood \( \partial W \times [0, 1] \) so that contact form \( \lambda \) on \( \partial W \) is preserved. Let \( \Pi : \partial W \to M \) be a principal circle bundle. Then circle fibers are closed orbits of Reeb vector field \( R_\lambda \), i.e., Reeb foliates circle fibers. So we can view the boundary of the Weinstein
domain $W$ lying in $S^{2n+1}$ as a Boothby-Wang type boundary.

Now consider the fibration
\[
\pi : \partial W - \{z_n = 0\} \to S^1,
\]
\[
(z_1, \ldots, z_n) \mapsto \frac{z_n}{|z_n|}.
\]

Observe that $H = \{z_n = 0\}$ is an adapted Donaldson hypersurface in $M$ for which $\Pi^{-1}(\{z_n = 0\}) = \partial W \cap \{z_n = 0\}$ is the binding of the open book decomposition of $\partial W$ with page $F = \pi^{-1}(\theta_0)$ for some fixed angle $\theta_0 = arg(z_n)$ and monodromy a right-handed fibered Dehn twist. The Boothby-Wang circle bundle $(\partial W \xrightarrow{\Pi} M, \lambda)$ is then contactomorphic to a contact structure supported by the open book decomposition of $\partial W$ by Theorem 3.1.

Also, the open book decomposition discussed above is coherent with the Boothby-Wang circle bundle $(\partial W \xrightarrow{\Pi} M, \lambda)$ since the contact form on $\partial W$ is adapted to the open book $(B = \Pi^{-1}(\{z_n = 0\}), \pi)$ and the monodromy is a right-handed fibered Dehn twist.

So, the coherent open book decomposition is as follows:
- Page: $F = (\partial W - \{z_n = 0\}) \cap \{arg(z_n) = \theta_0\}$ for some fixed $\theta_0$.
- Binding: $B = \partial W \cap \{z_n = 0\}$.
- Monodromy: $\Phi_0$, a right-handed fibered Dehn twist.

4.3. Comparison of Two Views. Observe that the two open books discussed above are exactly the same up to symplectomorphism. They have the same binding and the same pages. To finish the proof, we have to check that the monodromy coming from the Lefschetz fibration point of view (product of Dehn twists along Lagrangian spheres) is symplectically isotopic to the monodromy coming from the Boothby-Wang fibration point of view (right-handed fibered Dehn twist) relative to the boundary. In order to show that, we will construct a symplectic isotopy so that we have the desired equivalence.

Notation 4.1. Denote the page coming from the Lefschetz fibration construction by $\Sigma_0 = W \cap \{z_n = \epsilon\}$ and the slightly retracted page coming from the Boothby-Wang construction by $\Sigma_1 = (\partial W - \{z_n = t\epsilon|t \in [0,1]\}) \cap \{arg(z_n) = \theta_0\} \subset S^{2n+1}$.

Claim: There exists a symplectic isotopy from $\Sigma_0$ to $\Sigma_1$ relative to the endpoints, i.e., there exists a one parameter family of $(2n-2)$-dimensional symplectic submanifolds $(\Sigma_t, d\beta|_{\Sigma_t}), t \in [0,1]$.

Proof: Let $\beta$ be the usual Liouville 1-form on $\mathbb{C}^{n+1}$. Let $X = \frac{1}{2} \sum_{j=0}^{n} (x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j})$ be the radial vector field (i.e., the Liouville vector field of $\beta$ on all of
Consider the Liouville flow $gX$, where $g$ is some positive function on $\text{Int}(\Sigma_0)$ and vanishes along $\partial\Sigma_0$. Then we can obtain one parameter family of $(2n - 2)$-dimensional submanifolds $\Sigma_t$ by flowing from $\Sigma_0$ to $\Sigma_1$ along the radial vector field $gX$. Furthermore, $\Sigma_t$ is transverse to the vector fields $X$ and $Y$ since it is transverse to a multiple of $X$. Because the boundary of each $\Sigma_t$ is the same, the transversality condition still holds for each hypersurface $\Sigma_t$ at the boundary.

Let $v_1, \ldots, v_{2n-2}$ be a basis of $T\Sigma_t$ at some point $p$. We want to show that $d\beta = \sum_{j=0}^{n} dx_j \wedge dy_j$ is symplectic on each $\Sigma_t$, $t \in [0, 1]$.

We compute $(d\beta)^n$ as follows:

$$(d\beta)^n(X, Y, v_1, \ldots, v_{2n-2}) = n(dx_n \wedge dy_n) \wedge (dx_0 \wedge dy_0 + \cdots + dx_{n-1} \wedge dy_{n-1})^{n-1}(X, Y, v_1, \ldots, v_{2n-2})$$

$$= c(dx_0 \wedge dy_0 + \cdots + dx_{n-1} \wedge dy_{n-1})^{n-1}(v_1, \ldots, v_{2n-2}) > 0, \quad (*)$$

where $c > 0$.

The last inequality $(*)$ above is equivalent to the following condition: the projection of $\Sigma_t$ to the first $n$ coordinates (i.e., projection to the $z_0, \ldots, z_{n-1}$-plane $Z$) is symplectic. In other words, $pr : \Sigma_t \to pr(\Sigma_t)$ is a covering space over a symplectic submanifold of the plane $Z$. Also, notice that $\Sigma_0$ and $\Sigma_1$ satisfy $(*)$ above $\beta|_{\Sigma_0}$ and $\beta|_{\Sigma_1}$ are Liouville. So each $\Sigma_t$ is symplectic. Therefore, we have a symplectic isotopy relative to the boundary.

Thus, a right-handed fibered Dehn twist on $\Sigma_1$ is symplectically isotopic to a product of right-handed Dehn twists on $\Sigma_0$. This finishes the proof of the Theorem 1.1.

### 4.4. Proof of Corollary 1.2

With the Theorem 1.1 in hand, we are ready to complete the proof of Corollary 1.2.

**Proof:** Follows from the proof of the Theorem 1.1. In this case, the monodromy coming from the Lefschetz fibration is the $k$-th power of a product of right-handed Dehn twists. This follows from the fact that each fiber (page) $F_\theta$ has the following behavior:

$$F_\theta = W \cap \{z_n = e^{i\theta}\} = \{z_1^k + \cdots + z_{n-1}^k = -e^{i\theta k}\}.$$

In other words, the monodromy repeats itself $k$ times since it goes over the same $\Phi$ (a product of right-handed Dehn twists) $k$ times as $\theta$ varies. Observe also that the $\theta = 0$ case gives us the Lefschetz fiber together with the monodromy $\Phi$. Furthermore, when it sweeps out all of $\theta \in [0, 2\pi)$, one gets a right-handed fibered Dehn twist $\Phi_\theta$ along the boundary as monodromy which is equal to $\Phi^k$ up to symplectic isotopy.
One can Morsify $f$ by adding the linear function $\epsilon(z_0 + \cdots + z_n)$. Then it is easy to observe that degenerate isolated singularity at the origin splits into $(k-1)^n$ many non-degenerate critical points in the fiber. Therefore, $\Phi$ is a product of $(k-1)^n$ right-handed Dehn twists. So a right-handed fibered Dehn twists is then isotopic to a product of $k(k-1)^n$ right-handed Dehn twists since $\Phi_\partial = \Phi^k$. \hfill \Box

Note that, when $k = 1$, $W = \mathbb{C}^n$ and a fibered Dehn twist is symplectically isotopic to the identity.

In the case when $k = 2$, $W$ is the cotangent bundle to the $n$-sphere and $\Phi_\partial$ is the same as the square of a Dehn twist along the zero section.

4.5. **Proof of the Exact Number of Dehn Twists.** With the Corollary 1.2 in hand and [1,2], we show that there exists $k(k-1)^n$ right-handed Dehn twists for the full fibered Dehn twist when $f$ is any homogeneous polynomial of degree $k$ with an isolated singularity at the origin.

**Proof.** Let $g$ be a homogeneous polynomial of degree $k$ with an isolated singularity at the origin given by $g(z_0, \ldots, z_n) = z_0^k + \cdots + z_n^k$ and let $f$ be a homogeneous polynomial of degree $k$ with an isolated singularity at the origin. We want to show that there exists a 1-parameter family of homogeneous polynomials of degree $k$ with an isolated singularity at the origin, denoted by $p_t$ connecting $p_0$ and $p_1$ with the following properties:

- $p_0 = f$.
- $p_1 = g$.
- $p_t$ is a homogeneous polynomial of degree $k$ with an isolated singularity at the origin.
- The level set $p_t^{-1}(0) \cap S^{2n+1}$ is smooth for all $t \in I = [0, 1]$.

Firstly, if we can show that the complex codimension of the singular projective variety of homogeneous polynomials of degree $k$ in the projective variety of the space of all homogeneous polynomials of degree $k$ is $\geq 1$, then there is a path between any two polynomials ($f$ and $g$ in our case) in the nonsingular projective variety. Because one can always travel around the singular locus.

To see this, let $V_k$ denote the vector space of homogeneous polynomials of degree $k$ and $S_k$ denote the subset of $V_k$ consisting of those homogeneous polynomials of degree $k$ containing a singular locus. Note that, in this setting, a homogeneous polynomial of degree $\geq 1$ implies automatically that it has an isolated singularity at 0. Furthermore, we know that homogeneous polynomials of degree $k$ with an isolated singularity at the origin define smooth hypersurfaces in projective space and homogeneous polynomials of degree $k$ whose zero set forms a singular variety in projective space form themselves a singular variety in the space of all homogeneous polynomials of degree $k$. 
The goal is to show that $S_k$ is a variety of complex codimension at least 1 (real codimension 2). Then there exists a path between any two homogeneous polynomials with an isolated singularity at 0 in the complement of $S_k$. Rather than doing this directly, it is useful to consider the projectivizations $P(V_k)$ and $P(S_k)$ in the projective space. Observe that $P(S_k)$ is a projective variety lying strictly inside $P(V_k)$.

The goal is now equivalent to showing that $P(S_k)$ has complex codimension at least 1 in $P(V_k)$. But it is already known that the set of singular points of a projective variety has codimension at least 1, see [Sha13, pp. 127].

Once this is done, we get a path since one can locally travel around the singular polynomials. This proves the existence of a 1-parameter family of homogeneous polynomials of degree $k$ with an isolated singularity at the origin as desired.

Now consider $L(p_t) := \{ z \in \mathbb{C}^{n+1} \mid |z|^2 = 1, p_t(z) = 0 \}$, the link of singularity of the polynomial $p_t$ and check that the links $L(p_t)$ are all contactomorphic. If we can show that they are all contactomorphic, then there must be open books with the same properties. In other words, we have the same number of $(k(k - 1)^n$ many) right-handed Dehn twists for the full fibered Dehn twist in both Theorem 1.1 and Corollary 1.2.

Claim: Let $C := \{(t, z) \in I \times \mathbb{C}^{n+1} \mid |z|^2 = 1, z \in p_t^{-1}(0)\}$. Then the set $C$ is a topologically trivial cobordism between $L(p_0)$ and $L(p_1)$. In particular, $L(p_0)$ and $L(p_1)$ are diffeomorphic.

Proof: Consider the link $L(p_t) := p_t^{-1}(0) \cap S^{2n+1}$. Observe that it is smooth for all $t$. To see this, note that $L(p_t)$ is the zero set of the ideal generated by $|z|^2 = 1, p_t$ and $\bar{p}_t$. Its Jacobian is given by the following matrix:

$$
\begin{pmatrix}
\bar{z} & z \\
\frac{\partial p_t}{\partial z} & 0 \\
0 & \frac{\partial \bar{p}_t}{\partial z}
\end{pmatrix}
$$

It is clear that Jacobian has real rank 3 as $p_t$ has an isolated singularity at the origin. Therefore the link $L(p_t)$ is a smooth submanifold of $\mathbb{C}^{n+1}$ of real codimension 3.

Now observe that the boundary of $C$ consists of the links $L(p_0)$ and $L(p_1)$. It suffices to show that $C$ is a smooth manifold and hence a smooth cobordism between $L(p_0)$ and $L(p_1)$. To see this, define the map

$$P : I \times \mathbb{C}^{n+1} \longrightarrow \mathbb{R} \times \mathbb{C}, \quad (t, z) \longmapsto (|z|^2 - 1, p_t(z)).$$
Observe that $C = P^{-1}(0)$. The Jacobian of $P$ has real rank 3 which follows from the Jacobian of $L(p_t)$. So $C$ is a smooth submanifold of $I \times \mathbb{C}^{n+1}$ of codimension 3. Hence $C$ is a smooth cobordism between the links $L(p_0)$ and $L(p_1)$.

To show that this cobordism is topologically trivial, consider the function $h : C \to \mathbb{R}$,

$$(t, z) \mapsto t.$$

If we can show that $h$ has no critical points in $h^{-1}([0, 1])$, we then know that $C$ is a topologically trivial cobordism. To see this, take any point $(t_0, z_0) \in C$ and take a smooth curve $c(t) = (t, z(t))$ in $C$ through $(t_0, z_0)$ with $z(t)$ smooth: such a curve can be constructed in local coordinates and to construct it, one need to use the fact that the Jacobian of $p_t$ is invertible.

Then $h \circ c$ has non-vanishing derivative at $t = t_0$ due to the fact that our generic path lies in the nonsingular part of the projective space. So $dh$ does not vanish in $(t_0, z_0)$. So no point $(t_0, z_0) \in C$ is a critical point of $h$, i.e. $h$ has no critical points. Thus $C$ is a topologically trivial cobordism. Furthermore, $L(p_0)$ and $L(p_1)$ are diffeomorphic by Theorem 3.1 of [Mil63]. This concludes the proof of the diffeomorphism between the links $L(p_0)$ and $L(p_1)$.

Since the links are diffeomorphic, one can apply Gray’s stability theorem to see that they are all contactomorphic. As the resulting contact manifolds are contactomorphic, there must be open books with the same properties.

As there are $k(k - 1)^n$ right-handed Dehn twist for the full fibered Dehn twist in the case of Corollary 1.2, we have an open book with the same number of right-handed Dehn twists for the full fibered Dehn twist in the case of Theorem 1.1.

5. THE CASE $n = 2$

In this final section, we combine the Corollary 1.2 with the monodromy computations appearing in the literature to describe mapping class group relations for some surfaces.

5.1. The case of arbitrary $k \geq 2$.

**Lemma 5.1.** For each integer $k \geq 2$, consider the Weinstein domain

$$W = \{z_0^k + z_1^k + z_2^k = \delta\} \cap B^6.$$

Then the page of the open book of $\partial W$ (i.e., the Milnor fiber $F_{k,k}$ associated to the polynomial $z_0^k + z_1^k = 0$) is a surface with genus $g = \frac{1}{2}(k-1)(k-2)$ and $k$ boundary components. It is the ‘canonical’ Seifert surface for the $(k-1, k-1)$ torus link which can be drawn in $\mathbb{R}^3$ as shown in Figure 7 [HKP07]. The monodromy for the associated open book decomposition is isotopic to the product

$$\Phi = (D_{1,k-1} \circ \ldots D_{k-1,k-1}) \circ \ldots \circ (D_{1,1} \circ \ldots \circ D_{k-1,1}).$$
of right-handed Dehn twists $D_{i,j}$ about the curves $\alpha_{i,j} \subset F_{k,k}$ shown in the Figure 2.

**Proof.** The computation of the genus and number of boundary components of $F_{k,k}$ follows immediately from the degree-genus formula for complex curves in $\mathbb{CP}^2$. The drawing of the surface in $\mathbb{R}^3$ follows from Example 6.3.10 in [GS99]. The Dehn twist presentation of the monodromy follows from the proof of Proposition 3.2 in [HKP07].

Combining the above lemma with Corollary 1.2 provides the following:

**Corollary 5.2.** The surface $F_{k,k}$ described in Figure 7 ([HKP07]) is such that

$$((D_{1,k-1} \circ \ldots \circ D_{k-1,k-1}) \circ \ldots \circ (D_{1,1} \circ \ldots \circ D_{k-1,1}))^k$$

is isotopic to a product of right-handed fibered Dehn twists about each of the boundary components of $F_{k,k}$.
5.2. **The case** \( k = 3 \). Now we describe the case \( n = 2, k = 3 \) in greater detail, proving Corollary 1.3. For simplicity, we will henceforth refer to \( F_{3,3} \) as \( S \).

To complete the proof, we consider arcs \( a_g, a_b, a_p, \) and \( a_r \) in \( S \) as depicted in the Figure 3. We also label the boundary components \( b_1, b_2, \) and \( b_3 \) of \( S \) as shown. The orientations of \( a \)-arcs are indicated with arrows.

![Figure 3](image)

**Figure 3.** We label the green, blue, purple, and red arcs as \( a_g, a_b, a_p, \) and \( a_r \), respectively.

If we cut the surface \( S \) along the arcs \( a_g, a_b, a_p, \) and \( a_r \), all that will remain is a 16-gon, which is topologically just a disk as shown in the Figure 4. In the Figure 4 the segments of the boundary are colored according to the identifying colors for the \( a \)-arcs. The small arrows indicate where the boundary orientation of the 16-gon agrees and disagrees with the orientations of the arcs.

By regluing along the cut arcs, we obtain an easier-to-visualize depiction of \( S \) shown in the Figure 5. Call the diffeomorphism from the surface in Figure 2 to the glued up surface in the Figure 5 induced by the cutting and gluing of the \( a \)-arcs, by \( \Psi \).

To complete the proof of Corollary 1.3 one can easily check that \( \Psi \) maps the curves

\[
\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \alpha_{2,2}
\]

of the Figure 2 to the curves

\[
\alpha_b, \alpha_g, \alpha_r, \alpha_p
\]
of the Figure[1] respectively. To see this, note that for each color-label $i \in \{g, b, p, r\}$, the simple-closed curve $\alpha_i$ intersects $a_i$ in a single point, and is disjoint from each $a_j$ for $j \neq i$. These intersection conditions completely determine the isotopy classes of the $\alpha$-curves. This completes the proof of Corollary 1.3.

5.3. From Monodromy to Mapping Class Group Relations. In this section, we study explicit descriptions of the symplectomorphisms in the case $n = 2$, which provides a visual representation of the case $n = 2$, $k = 3$ recovering the classical chain relation for a torus with two boundary components.
**Definition 5.1.** Consider the arcs \(c_1, c_2, c_3, c, b_1, b_2, \) and \(b_3\) in \(S_{1,3}\) as depicted in the Figure 6. If \(c_1, c_2, c_3, c, d_1, d_2, \) and \(d_3\) are the isotopy classes of simple closed curves in \(S_{1,3}\) then we have the following relation:

\[
(D_{c_1} \circ D_{c_2} \circ D_{c_3} \circ D_{c})^3 = D_{b_1} \circ D_{b_2} \circ D_{b_3}
\]

![Figure 6](image.png)

**Figure 6.** We label the boundary components as \(b_1, b_2, b_3\) and other components of the isotopy classes of simple closed curves in \(S_{1,3}\) as \(c_1, c_2, c_3, \) and \(c.\)

This relation is called the *star relation* \([Ger96]\).

Consider a chain of circles \(\alpha_1, \ldots, \alpha_k\) in a compact, connected, orientable surface \(S.\) Denote isotopy classes of boundary curves by \(b\) in the even case and by \(b_1\) and \(b_2\) in the odd case. Then each chain induces one of the following relations in the mapping class group of \(S.\) In each case, the relation is called a *chain relation* or a \(k\)-chain relation.

\[
(D_{\alpha_1} \cdots D_{\alpha_m})^{2m+2} = D_b \quad \text{if } m \text{ is even},
\]

\[
(D_{\alpha_1} \cdots D_{\alpha_m})^{m+1} = D_{b_1} D_{b_2} \quad \text{if } m \text{ is odd}.
\]

There is another version of the chain relation described in \([FM12]\) which will be useful in our case:

\[
(D_{\alpha_1}D_{\alpha_2} \cdots D_{\alpha_m})^{2m} = D_b \quad \text{if } m \text{ is even},
\]

\[
(D_{\alpha_1}D_{\alpha_2} \cdots D_{\alpha_m})^{m} = D_{b_1} D_{b_2} \quad \text{if } m \text{ is odd}.
\]

Notice how the Dehn twist along the boundary differs in each case.

Now consider any two arcs, \(\alpha\) and \(\beta\) on \(S\) with geometric intersection number 1 (i.e., the minimum number of intersections of curves \(\alpha'\) and \(\beta'\) isotopic to \(\alpha\) and \(\beta\), respectively). Then the following relation is called the *braid relation*:

\[
D_\alpha D_\beta D_\alpha = D_\beta D_\alpha D_\beta
\]
It is known that the star relation implies the 2-chain relation \((D_{\alpha_1} D_{\alpha_2})^6 = D_b\) by using the braid relation. If one can show that the relation
\[
(D_{\alpha_r} \circ D_{\alpha_p} \circ D_{\alpha_b} \circ D_{\alpha_g})^3 = D_{b_1} \circ D_{b_2} \circ D_{b_3}
\]
in the mapping class group of \(S_{1,3}\) given in Corollary 1.3 is isotopic to the star relation, then it satisfies the 2-chain relation. To see this, we will use the following argument explained by Dan Margalit [Mar14].

Consider the yellow curve \(\alpha_y\) as depicted in Figure 7 which intersects \(\alpha_g\) once and is disjoint from \(\alpha_b\) and \(\alpha_p\). Then we have:

\[
D_{\alpha_r} = D_{\alpha_g}^{-1} \circ D_{\alpha_y} \circ D_{\alpha_g}
\]

![Figure 7](image)

Figure 7. We label the blue, green, purple, red, and yellow curves as \(\alpha_b, \alpha_g, \alpha_p, \alpha_r, \) and \(\alpha_y\), respectively. The boundary components are labeled as \(b_1, b_2, \) and \(b_3.\)

After plugging \(D_{\alpha_r} = D_{\alpha_g}^{-1} \circ D_{\alpha_y} \circ D_{\alpha_g}\) into the relation given in Corollary 1.3 we get the following relation:

\[
((D_{\alpha_g}^{-1} \circ D_{\alpha_y} \circ D_{\alpha_g}) \circ D_{\alpha_p} \circ D_{\alpha_b} \circ D_{\alpha_g})^3 = D_{b_1} \circ D_{b_2} \circ D_{b_3}
\]

After multiplying out \(D_{\alpha_g}^{-1} \circ D_{\alpha_y} \circ D_{\alpha_g} \circ D_{\alpha_p} \circ D_{\alpha_b} \circ D_{\alpha_g}\) with itself three times and canceling out the terms, we obtain:

\[
D_{\alpha_g}^{-1} \circ D_{\alpha_y} \circ D_{\alpha_g} \circ D_{\alpha_p} \circ D_{\alpha_b} \circ D_{\alpha_g} \circ D_{\alpha_y} \circ D_{\alpha_g} \circ D_{\alpha_p} \circ D_{\alpha_b} \circ D_{\alpha_g} = D_{b_1} \circ D_{b_2} \circ D_{b_3}
\]

Using the fact that \(D_{b_1} \circ D_{b_2} \circ D_{b_3}\) is central, we can conjugate both sides by \(D_{\alpha_g}\) and get:

\[
D_{\alpha_y} \circ D_{\alpha_g} \circ D_{\alpha_p} \circ D_{\alpha_b} \circ D_{\alpha_y} \circ D_{\alpha_g} \circ D_{\alpha_p} \circ D_{\alpha_b} \circ D_{\alpha_y} \circ D_{\alpha_g} = D_{b_1} \circ D_{b_2} \circ D_{b_3}
\]

We can also apply conjugation by \(D_{\alpha_y}\). After applying conjugation, we have:

\[
(D_{\alpha_g} \circ D_{\alpha_p} \circ D_{\alpha_b} \circ D_{\alpha_y}) \circ (D_{\alpha_g} \circ D_{\alpha_p} \circ D_{\alpha_b} \circ D_{\alpha_y}) \circ (D_{\alpha_g} \circ D_{\alpha_p} \circ D_{\alpha_b} \circ D_{\alpha_y}) = D_{b_1} \circ D_{b_2} \circ D_{b_3}
\]
So we have the following:

\[(D_{\alpha g} \circ D_{\alpha p} \circ D_{\alpha b} \circ D_{\alpha y})^3 = D_{b_1} \circ D_{b_2} \circ D_{b_3}\]

which is the star relation.

The star relation and the braid relation imply the 2-chain relation. If you cap off one boundary component, then two curves in the star relation become the same. Then it is an easy application of the braid relation to get the 2-chain relation

\[(D_{\alpha_1}^2 D_{\alpha_2} D_{\alpha_3})^3 = D_{b_1} D_{b_2}.\]

So \(n = 2\) and \(k = 3\) case of Corollary 1.2 recovers the classical chain relation for a torus with 2 boundary components.

**REFERENCES**

[Arn95] V. I. Arnol’d, *Some remarks on symplectic monodromy of Milnor fibrations*, Progr. Math., vol. 133, Birkhäuser Basel, 1995.

[BG07] P. Biran and E. Giroux, *Symplectic mapping classes and fillings*, unpublished manuscript 2005-6, slides of a talk (2007).

[BW58] W. M. Boothby and H. C. Wang, *On contact manifolds*, Annals of Math, Vol. 68 (1958), no. 3, 721–734.

[CDvK12] R. Chiang, F. Ding, and O. van Koert, *Open books for Boothby-Wang bundles, fibered Dehn twists and the mean Euler characteristic*, Preprint, arXiv:1211.0201 (2012).

[Don96] S. K. Donaldson, *Symplectic submanifolds and almost-complex geometry*, Journal of Differential Geometry 44 (1996), 666–705.

[FM12] B. Farb and D. Margalit, *A primer on mapping class groups*, vol. 49, Princeton University Press, 2012.

[Ger96] S. Gervais, *Presentation and central extensions of mapping class groups*, American Mathematical Society 348 (1996), no. 8, 3097–3132.

[Gir02] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 405–414.

[GS99] R. E. Gompf and A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. 20, 1999.

[HKP07] S. Harvey, K. Kawamuro, and O. Plamenevskaya, *On transverse knots and branched covers*, Preprint, arXiv:0712.1557 (2007).

[KS02] M. Khovanov and P. Seidel, *Quivers, Floer cohomology and braid group actions*, Journal of the American Mathematical Society, 15(1) (2002), 203–271 (electronic).

[Mar14] Dan Margalit, *Personal communication*, November 2014.

[Mil63] J. Milnor, *Morse theory*, Annals of Mathematics Studies, Princeton University Press, 1963.

[Mil68] _____, *Singular points of the complex hypersurfaces*, Annals of Mathematics Studies, no. 61, Princeton University Press, 1968.

[Sei97] P. Seidel, *Floer homology and the symplectic isotopy problem*, Ph.D. Thesis, Oxford University (1997).

[Sei98] _____, *Symplectic automorphisms of cotangent bundle of sphere*, Preprint, arXiv:9803084 (1998).

[Sei08] _____, *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Vol. VII, Zürich (2008).

[Sha13] Igor R. Shafarevich, *Basic algebraic geometry 1: Varieties in projective space*, Springer-Verlag, 2013.
[TW75] W. P. Thurston and H. E. Winkelnkemper, *On the existence of contact forms*, Proceedings of AMS, Vol. 52 (1975), no. 1, 345–347.

[vK14] Otto van Koert, *Personal communication*, October 2014.

[Wan14] A. Wand, *Tightness is preserved by Legendrian surgery*, Preprint, arXiv:1404.1705 (2014).

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