A POINCARÉ-BIRKHOFF-WITT THEOREM FOR
GENERALIZED LIE COLOR ALGEBRAS

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Abstract. A proof of Poincaré-Birkhoff-Witt theorem is given for a class of
generalized Lie algebras closely related to the Gurevich S-Lie algebras. As
concrete examples, we construct the positive (negative) parts of the quantized
universal enveloping algebras of type $A_n$ and $M_{p,q,\epsilon}(n,K)$, which is a non-
standard quantum deformation of $GL(n)$. In particular, we get, for both
algebras, a unified proof of the Poincaré-Birkhoff-Witt theorem and we show
that they are genuine universal enveloping algebras of certain generalized Lie
algebras.

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I. INTRODUCTION

In the paper \cite{1}, H. Yamane presented a proof of the Poincaré-Birkhoff-Witt
(PBW) theorem for some class of quantum groups: Drinfeld-Jimbo quantum groups
of type $A_n$. In his proof he did not use explicitly the Lie algebra theory concepts.

In this paper we show that Yamane used in an implicit manner some generalized
Lie algebra. Such a generalized Lie algebra will be called $T$-Lie algebra.

The $T$-Lie algebras satisfy not only generalized antisymmetry and Jacobi identity,
but aditional properties like multiplicativity, (also generalized, in the same
way as the Gurevich S-Lie algebras \cite{2}). Such $T$-Lie algebras arise in a natural
way embedded in the positive and negative parts of the Drinfeld-Jimbo quantum
groups $U_q(sl_{n+1})$ of type $A_n$.

Our $T$-Lie algebras share some properties with the S-Lie algebras. But they
are not equivalent, for example, $T$-Lie algebras satisfy a weaker multiplicativity
condition. In particular, there are some $T$-Lie algebras which are not $S$-Lie algebras.
However, classical Lie algebras \cite{3}, Lie superalgebras \cite{4} and Scheunert generalized
Lie algebras (Lie color algebras) \cite{5} are all $T$-Lie algebras.

These $T$-Lie algebras are related to the problem of finding the appropriate definition
of a quantum Lie algebra. There are already some generalized Lie algebras pro-
sposed to solve this problem: Majid braided Lie algebras \cite{6}, Delius-Gould quantum
Lie algebras \cite{7}, new generalized Lie algebras of Gurevich-Rubstov \cite{8}, generalized
Lie algebras due to Lyubashenko-Sudbery \cite{9}, among others. But the Delius-Gould
definition and the Gurevich-Rubstov also, depends on the associated universal en-
veloping algebra. This is not the case for the $T$-Lie algebras. Our axioms imply
the properties of the universal enveloping algebra. In particular we shall prove the
PBW theorem.

The generalized Lie algebras axioms of Lyubashenko-Sudbery are not enough in
order to obtain a PBW theorem (see example \cite{V.4}). While the main difference
with the braided Lie algebras of Majid is that the symmetry of our $T$-Lie algebras is not a braid morphism. Only a part of such symmetry is braided.

In particular, we get a $T$-Lie algebra $(\mathfrak{sl}_{n+1}^\pm)_q$ which is a deformation of the Lie subalgebra of upper (lower) triangular matrices. Such generalized Lie algebra meets almost all the requirements of a quantum Lie algebra in the sense of Lyubashenko-Sudbery [9], (only fails the point 7; actually the universal enveloping algebra of $(\mathfrak{sl}_{n+1}^\pm)_q$ has no a Hopf algebra structure, but it seems possible to define a braided Hopf algebra on it, however we do not try such matter in this paper). Moreover, the universal enveloping of $(\mathfrak{sl}_{n+1}^\pm)_q$ is $U_q^\pm(\mathfrak{sl}_{n+1})$ the positive part of the Drinfeld-Jimbo quantum group of type $A_n$, therefore the diagram in Figure 1 commutes.

This means that, relative to $U_q^\pm(\mathfrak{sl}_{n+1})$, the $T$-Lie algebra $(\mathfrak{sl}_{n+1}^\pm)_q$ satisfies, in some sense, the quantum Lie algebra condition of Delius [10].

Some possible physical applications of the formalism of generalized Lie algebras are in the affine Toda theories [11], quantum integrable systems [11], and gauge theory [9].

The paper is organized as follows. In Sec. II we shall define the $T$-Lie algebras. In Sec. III a list of classical and new Lie algebras is given. In Sec. IV we shall define the universal enveloping algebra of a $T$-Lie algebra and we shall prove that expecting an analogue at PBW theorem for any such universal enveloping algebra constructed by means of commutators is too much, we have to restrict our generalized Lie algebras in an adequate way. However, in Sec. V we persuit the classical idea to prove the PBW theorem [8] by constructing a representation of the universal enveloping algebra on the symmetric algebra (with modifications inspired by [1]). In Sec. VI the definition of a representation of $T$-Lie algebra is given. In Sec. VII we shall prove the PBW theorem for the universal enveloping of an adequate $T$-Lie algebra. Some remarks about braid morphisms are given in Sec. VIII. The Sec. IX is devoted to explain why we can apply the $T$-Lie algebras theory to a non-standard quantum deformation algebra $U_q^\pm(\mathfrak{sl}_{n+1})$ of $GL(n)$. Similar explanations are given in Sec. X but now dealing with $U_q^\pm(\mathfrak{sl}_{n+1})$ the positive (negative) parts of the Drinfeld-Jimbo quantum groups of type $A_n$. In particular, in Sec. XI we shall prove that $U_q^\pm(\mathfrak{sl}_{n+1})$ is a genuine universal enveloping algebra of certain $T$-Lie algebra.

II. THE NOTION OF $T$-LIE ALGEBRA

Let $k$ be a commutative unitary ring.

Definition II.1. A $k$-algebra $A$ is strictly graded if there exist $k$-submodules $(A_\eta)_{\eta \in \mathbb{N}}$ such that

$$A = \bigoplus_{\eta \in \mathbb{N}} A_\eta \quad \text{and} \quad A_{\eta_1} \cdot A_{\eta_2} \subseteq A_{\eta_1 + \eta_2 - 1}$$

for all $\eta_1, \eta_2 \in \mathbb{N}$. For $a \in A_\eta$, we shall put $\eta(a) = \eta$. 

\begin{figure}[h]
\centering
\begin{tikzcd}
\mathfrak{sl}_{n+1}^\pm \arrow{r} \arrow{d} & (\mathfrak{sl}_{n+1})_q \arrow{d} \\
U(\mathfrak{sl}_{n+1}^\pm) \arrow{r} & U^\pm_q(\mathfrak{sl}_{n+1}) = U_q(\mathfrak{sl}_{n+1})_q
\end{tikzcd}
\caption{The classical $\mathfrak{sl}_{n+1}^\pm$ and the quantum $(\mathfrak{sl}_{n+1})_q$}
\end{figure}
Remark II.1. For such graded algebras $A$ we can induce a filtration of $A \otimes_k A$ given by
\[
(A \otimes_k A)_\eta = \bigoplus_{n_1 + n_2 \leq \eta} A_{n_1} \otimes A_{n_2}
\]
Let $L$ be a free $k$-module with a given basis $B$ totally ordered.

Definition II.2. Denote by $L^n$ the $k$-submodule of $L^n \otimes$ generated by
\[
x_{i_1} \otimes \ldots \otimes x_{i_n}, \quad x_{i_1} < \ldots < x_{i_n}, \quad (x_{i_j} \in B),
\]
and by $^n L$ the $k$-submodule generated by
\[
x_{i_1} \otimes \ldots \otimes x_{i_n}, \quad x_{i_1} > \ldots > x_{i_n}, \quad (x_{i_j} \in B).
\]

Definition II.3. The module $L$ with $k$-morphisms
\[
S : L \otimes_k L \to L \otimes_k L, \text{ (presymmetry)} \tag{2.1}
\]
\[
T : L \otimes_k L \to L \otimes_k L, \text{ (symmetry)} \tag{2.2}
\]
\[
\langle, \rangle : L \otimes_k L \to L \otimes_k L, \text{ (pseudobracket)} \tag{2.3}
\]
\[
[, ] : L \otimes_k L \to L, \text{ (bracket)} \tag{2.4}
\]
is called $T$-Lie algebra with basis $B$ (or basic $T$-Lie algebra) if, for $S_{12} = S \otimes_k Id_L$, $S_{23} = Id_L \otimes_k S$, the following axioms are satisfied:

1. (a) $S^2 = Id$
   (b) $S(x \otimes y) = q_{x,y} y \otimes x$, for certain $q_{x,y} \in k$, $\forall x, y \in B$
   (c) (Multiplicativity)
   (i) $S([Id \otimes_k [, ]]|_{L^3}) = ([, ] \otimes_k Id)S_{23}S_{12}|_{L^3}$
   (ii) $S([[, ] \otimes_k Id]|_{L^3}) = (Id \otimes_k [, ])|S_{12}S_{23}|_{L^3}$

2. (Stability)
   (a) There exists a strict grading
   \[
   L = \bigoplus_{\eta} L_\eta
   \]
   of $L$ relative to $[,]$
   (b) $\langle L_{\eta_1} \otimes L_{\eta_2} \rangle \subseteq (L \otimes_k L)_{\eta_1 + \eta_2 - 1}$ for all $L_{\eta_1}, L_{\eta_2}$.

3. $T = S + \langle, \rangle$

4. (Antisymmetry)
   (a) $[,]T = -[,]$
   (b) $\langle, \rangle S = -\langle, \rangle$
   (c) $[,]\langle, \rangle = 0$

5. (Jacobi Identity)
\[
[,]((Id \otimes_k [, ])|S_{12}S_{23} - ([, ] \otimes_k Id)S_{23}S_{12} + (Id \otimes_k [, ])|S_{23}S_{12})|_{L} = 0
\]

Multiplicativity conditions are to control commutation relations in the universal enveloping algebra, whereas stability conditions are to obtain a good gradation in the corresponding symmetric algebra.

Definition II.4. Let $L_i$ be a basic $T$-Lie algebra with bracket $[,]_i$, pseudobracket $\langle, \rangle_i$ and presymmetry $S_i$, $i = 1, 2$. A $k$-morphism $f : L_1 \to L_2$ is called a $T$-Lie morphism if $f$ is a morphism of graded algebras relative to $[,]_i$, $i = 1, 2$ and the diagrams in the Figure 2 commute.
In order to obtain a graduation in the stability conditions it suffices to define a map \( \eta : B \to \mathbb{N} \) having properties (24) and (21) in the stability axiom. This remark will be used in the following examples.

### A. Some common Lie algebras.

#### Example III.1. Classical Lie algebras over fields are basic \( T \)-Lie algebras:

\[
\{,\} \text{ classical bracket}, \langle, \rangle = 0, T = S \text{ usual switch}, \eta = 1.
\]

#### Example III.2. Lie superalgebras over fields \( \mathbb{F} \) are basic \( T \)-Lie algebras:

Let \( L = L_0 \oplus L_1 \) be a Lie superalgebra with bracket \( [,] \) over a field \( k \). Let \( B_\alpha \) basis of \( L_\alpha \), \( \alpha = 0, 1 \). Define \( S : L \otimes_k L \to L \otimes_k L \) on the basis \( B = B_0 \cup B_1 \), by \( S(x \otimes y) = (-1)^{\alpha \beta} y \otimes x \) if \( x \in L_\alpha, y \in L_\beta \). Besides \( \langle, \rangle = 0, T = S, \eta = 1 \).

#### Example III.3 (Lie color algebras). Let \( k \) be a field of characteristic zero. Let

\[
L = \oplus_{\gamma \in \Gamma} L_\gamma
\]

be a \( \epsilon \) Lie algebra \( \mathbb{F} \), where \( \Gamma \) is an abelian group and \( \epsilon \) is a commutation factor on \( \Gamma \). Let \([,]\) be bracket of \( L \). Put \( B_\gamma \) basis of \( L_\gamma \) for each \( \gamma \in \Gamma \).

Define

\[
S(x \otimes y) = \epsilon(\alpha, \beta) y \otimes x, \text{ if } x \in B_\alpha, y \in B_\beta,
\]

besides \( \langle, \rangle = 0 \) and \( \eta = 1 \).

Multiplicativity conditions follow easily from the definition of commutation factor. We conclude that every \( \epsilon \) Lie algebra is a \( T \)-Lie algebra.

### B. Linear \( T \)-Lie algebras.

#### Example III.4. Let \( e_{ij}, 1 \leq i, j \leq n \) be standard basis of \( gl_n \) matrices \( n \times n \) over a field \( \mathbb{K} \). Let \([,]\) be the usual bracket in \( gl_n \), \( \text{sl}_n^+ \) the Lie subalgebra of upper triangular matrices having trace zero. We put \( x_i = e_{i,i+1}, i = 1, \ldots, n-1, x_n = [x_1, x_2], x_{n+1} = [x_2, x_3], \ldots, x_{n-3} = [x_{n-2}, x_{n-1}], x_{n-2} = [x_1, x_{n+1}] \ldots, x_{3n-6} = [x_{n-3}, x_{2n-3}], x_{3n-5} = [x_1, x_{2n-1}], x_{3n-4} = [x_2, x_{2n}], \ldots, x_m = [x_1, x_{m-1}] \) where \( m = n(n-1)/2 \), besides we define \( h_i = [x_i, x_i^t], i = 1, \ldots, m \) diagonal matrices in \( \text{sl}_n \). Further, \( q = exp(t) \in \mathbb{K}[t] \) formal series ring with indeterminate \( t \) and coefficients in \( \mathbb{K}, k = \mathbb{K}[q, q^{-1}], c_{i,j} \in \mathbb{Z} \) such that \( [h_i, x_j] = c_{i,j} x_j, 1 \leq i, j \leq m \).

Let \( (\text{sl}_n^+)_q \) be a free \( k \)-module with basis \( B = \{x_i | 1 \leq i \leq m\} \). We may define an order in \( B \) according to the Figure \( \frac{1}{2} \) from left to right and up to bottom. For example \( x_1 < x_n < x_2 < x_{2n} \). The first time that a diagram (Auslander-Reiten quiver of type \( A_{n-1} \)) of this type appears related to quantum groups, is in Ringel’s
work about the relationship between Poincaré-Birkhoff-Witt bases, quantum groups and Hall algebras [3].

Define:

\[
[x, y]_q = [x, y] \text{ if } x < y \in B
\]

\[
\langle e_{ij}, e_{uv} \rangle = \begin{cases} 
(q - q^{-1})e_{iv} \otimes e_{uj} & \text{if } i < u < j, u < j < v, \\
(q^{-1} - q)e_{uj} \otimes e_{iv} & \text{if } u < i < v, i < v < j \\
0, & \text{otherwise.}
\end{cases}
\]

\[
S(x_i \otimes x_j) = q^{c_{i,j}}x_j \otimes x_i, \text{ if } x_i < x_j,
\]

\[
T = S + \langle, \rangle
\]

Finally, we define \( \eta \) in such way that every basic element in the Figure 3 is in correspondence with a number belonging to the Figure 4, this yields, \( \eta(e_{ij}) = i(j - i), \forall i, j. \)

The multiplicativity condition follows from properties

\[
x_i < x_j < x_l \Rightarrow \begin{cases} 
[x_i, x_j] < x_l, \text{if } [x_i, x_j] \neq 0, \\
[x_i, x_l] < [x_j, x_l], \text{if } [x_j, x_l] \neq 0,
\end{cases}
\]

and

\[
[h_i, [x_j, x_l]] = (c_{i,j} + c_{j,i})[x_j, x_l]
\]

In the cases \( n = 2, 3, 4, 5 \), the Jacobi identity for \( [,]_q \) can be verified by straightforward calculations. We get that \((sl_+^n)_q\) with bracket \([,]_q\) is a basic \( T \)-Lie algebra, \( n = 2, 3, 4, 5 \).

Similarly, we can define \((sl_-^n)_q\).

Remark III.1. We get

\[
(sl_+^n)_q|_{t=0} = sl_+^n,
\]

so in the cases \( n = 2, 3, 4, 5 \), \((sl_+^n)_q\) is a deformation of \( sl_+^n \) in the category of \( T \)-Lie algebras. Later, in the section [X], such property will be generalized for every \( n \).

Example III.5. Starting from \((sl_+^1)_q\) we are going to build a new basic \( T \)-Lie algebra, denoted \((sl_+^1)_q\). Its structure is:

\[
\tilde{[}, \tilde{]}_q = [,]_q, \quad \tilde{\langle}, \tilde{\rangle} = 0, \quad \tilde{S} = S, \quad \tilde{T} = S, \quad \tilde{\eta} = \eta.
\]

\[\begin{array}{cccccc}
\circ_1 & \circ_2 & \circ_3 & \ldots & \circ_{n-1} \\
\circ_n & \circ_{n+1} & \ldots & & \\
\circ_{2n} & & & & \\
& \ldots & \\
& & & \circ_m
\end{array}\]

Figure 3. The basic \( T \)-Lie algebra \((sl_+^{n+1})_q\)
Example III.6 (Non-standard quantum deformations of $GL(n)$). Let $p$, $q$ be units in a commutative unitary ring $k$ with $pq \neq 1$ and choose $n(n-1)/2$ discrete parameters $\epsilon_{ij}, \epsilon_{ji} = \pm 1, 1 \leq i < j \leq n, \epsilon_{ii} = 1, \epsilon_{ji} = \epsilon_{ij}$.

The $k$-module $L_{p,q,\epsilon}(n,k)$ is then defined to be the free $k$-module with basis

$$B = \{ Z_{ij} | 1 \leq i, j \leq n \}.$$ 

We ordered $B$ by putting $Z_{ij} > Z_{kl}$ if either $i > k$, or $i = k$ and $j > l$. Define $\eta$ by

$$\eta(Z_{ij}) = j3^{i-1}. \ (3.5)$$

To prove that $L_{p,q,\epsilon}(n,k)$ is a basic $T$-Lie algebra, since (3.5) it suffices to check the stability condition (2b) for $Z_{ij} > Z_{kl}$, such that $i > k$ and $l > v$:

$$(i - u)3^{l-1} > (i - u)3^{v-1}$$

then

$$i3^{l-1} + u3^{v-1} > i3^{v-1} + u3^{l-1}$$

but this equation has left side $\eta(Z_{ij}) + \eta(Z_{kl})$ whereas the right side is $\eta(Z_{kl}) + \eta(Z_{ij})$. This proves stability conditions.

IV. Universal Enveloping Algebras

A. Construction of $U(L)$.

Definition IV.1. Let $L$ be a $T$-Lie algebra with basis $B$, and $\otimes_k L$ the $k$-tensor algebra of the module $L$. The universal enveloping algebra $U(L)$ is the quotient

$$U(L) = \otimes_k L/J$$

where $J$ is the two sided ideal generated by

$$x \otimes y - T(x \otimes y) - [x, y], \ x, y \in B$$

Because the stability axiom (2b), the algebra $U(L)$ have a similar structure to a quadratic algebra with an ordering algorithm [4].
PBW FOR GENERALIZED LIE COLOR ALGEBRAS

B. Examples.

Example IV.2. Let \( \eta : \mathcal{A} = \mathcal{B} \rightarrow \mathcal{D} \) be a \( k \)-morphism on \( \mathcal{D} \) such that \( \eta(\Sigma) = \Sigma \) respectively and \( \mathcal{B} \). Examples.

Example IV.4. In \( \mathcal{D} \) the equation \( x_2x_6 = 0 \) holds. Then \( \mathcal{D} \) is a \( \mathcal{D} \)-symmetric algebra of \( \mathcal{D} \). For generalised Lie algebra \( \mathcal{D} \) and its PBW theorem does not hold. So, if we want to extend it to \( \mathcal{D} \)-Lie algebras.

Example IV.5. Let \( \mathcal{D} \) be a basic \( \mathcal{D} \)-Lie algebra. We are going to define a new \( \mathcal{D} \)-Lie algebra \( \mathcal{D}^0 : \mathcal{D} = \mathcal{D} \) in its structure of \( \mathcal{D} \)-module, \( \eta^0 = \eta, \eta = \eta \), and \( \mathcal{D} \) and define \( \mathcal{D} \) is a free \( \mathcal{D} \)-module with basis the monomials formed by the products \( \mathcal{D} \) and its universal enveloping algebra as such generalised Lie algebra is the same as \( \mathcal{D} \)-Lie algebra. Therefore, the generalised Lie algebra axioms of Lyubashenko-Sudbery are not enough in order to obtain a PBW theorem.

Example IV.6. In \( \mathcal{D} \) the equation \( x_2x_6 = 0 \) holds. Then \( \mathcal{D} \) is a \( \mathcal{D} \)-symmetric algebra of \( \mathcal{D} \). For generalised Lie algebra \( \mathcal{D} \) and its PBW theorem does not hold. So, if we want to extend it to \( \mathcal{D} \)-Lie algebras.

Example V.1 (A-B). Let \( \mathcal{D} \) be a \( \mathcal{D} \)-Lie algebra with basis \( \mathcal{D} \). \( \mathcal{D} = \mathcal{D} \) \( \mathcal{D} \)-symmetric algebra of \( \mathcal{D} \). For generalised Lie algebra \( \mathcal{D} \) and its PBW theorem does not hold. So, if we want to extend it to \( \mathcal{D} \)-Lie algebras.

\[
\mathcal{D} = \mathcal{D}^0 \quad \mathcal{D}^0 = \mathcal{D}^0 \quad \mathcal{D} = \mathcal{D}^0
\]

V. THE RELATIONSHIP BETWEEN UNIVERSAL ENVELOPING ALGEBRAS AND SYMMETRIC ALGEBRAS

Let \( \mathcal{D} \) be a \( \mathcal{D} \)-Lie algebra with basis \( \mathcal{D} \), \( \mathcal{D} = \mathcal{D} \) finite non-decreasing sequence of elements of \( \mathcal{D} \). We write \( \eta(\lambda) = \eta(z_\Sigma) = \eta(x_\lambda) \) such that \( x_\lambda \leq \Sigma \) if \( x_\lambda \leq x_\lambda \). Besides we put \( x_\lambda \leq \Sigma \) if \( x_\lambda \leq x_\lambda \).

**Lemma V.1 (A-B).** Let \( \mathcal{D} \) be a \( \mathcal{D} \)-Lie algebra with basis \( \mathcal{D} \). \( \mathcal{D} = \mathcal{D} \) \( \mathcal{D} \)-symmetric algebra, \( \mathcal{D} \), the \( \mathcal{D} \)-submodule generated by \( z_\Sigma \) such that \( \eta(\Sigma) \leq p \). There is a \( \mathcal{D} \)-morphism

\[
\mathcal{D} \otimes_k \mathcal{D} \rightarrow \mathcal{D}
\]
satisfying

(A) \( x_\lambda \cdot z_\Sigma = z_\lambda z_\Sigma \) for \( x_\lambda \leq \Sigma \);

(B) \( x_\lambda \cdot z_\Sigma - z_\lambda z_\Sigma \in \mathcal{P}_{\eta(\lambda)+\eta(\Sigma)-1} \).

**Proof.** By induction on \( \eta(\lambda) + \eta(\Sigma) \). If \( \eta(\lambda) + \eta(\Sigma) = 1 \) then \( \eta(\lambda) = 1 \) and \( \Sigma = 0 \); it follows \( z_\emptyset = 1 \). Then define

\[
x_\lambda \cdot 1 = z_\lambda
\]

so (A) and (B) holds. Assume the existence of \( x_\lambda \cdot z_\Sigma \) for \( \eta(\lambda') + \eta(\Sigma') < \eta(\lambda) + \eta(\Sigma) \) satisfying (A) and (B). We have to define \( x_\lambda \cdot z_\Sigma \).

There are two cases: \( \lambda \leq \Sigma \) or \( \lambda \not\subseteq \Sigma \).

**Case** \( \lambda \leq \Sigma \): Because (A):

\[
x_\lambda \cdot z_\Sigma = z_\lambda z_\Sigma
\]

**Case** \( \lambda \not\subseteq \Sigma \): We may write \( \Sigma = (x_\mu, N) \) with \( x_\mu \leq N \) and \( x_\lambda > x_\mu \). Since \( \eta(N) < \eta(\Sigma) \) and because at the induction hypothesis \( x_\lambda \cdot z_N \) is already defined, and

\[
w = x_\lambda \cdot z_N - z_\lambda z_N \in \mathcal{P}_{\eta(\lambda)+\eta(N)-1}.
\]

Moreover, from \( \eta(\mu) + \eta(\lambda) + \eta(N) < 1 < \eta(\mu) + \eta(\lambda) + \eta(N) = \eta(\lambda) + \eta(\Sigma) \) it follows that \( x_\mu \cdot w \) is already defined.

We have

\[
T(x_\lambda \otimes x_\mu) = q_{\lambda \mu} x_\mu \otimes x_\lambda + \sum_i \xi_i x_{\mu_i} \otimes x_{\lambda_i} \quad (5.7)
\]

and because at (B) and the induction hypothesis \( x_\lambda \cdot z_N \in \mathcal{P}_{\eta(\lambda)+\eta(N)} \). As a consequence \( x_\mu \cdot (x_\lambda \cdot z_N) \) is already defined because \( \eta(\mu) + \eta(\lambda) + \eta(N) < \eta(\lambda) + \eta(\mu) + \eta(N) \) according to stability axiom.

We may define

\[
x_\lambda \cdot z_\Sigma = q_{\lambda \mu} z_\mu z_\lambda z_N + q_{\lambda \mu} x_\mu \cdot w + \sum_i \xi_i x_{\mu_i} \cdot (x_\lambda \cdot z_N) + [x_\lambda, x_\mu] \cdot z_N \quad (5.8)
\]

where \( w = x_\lambda \cdot z_N - z_\lambda z_N \); \( [x_\lambda, x_\mu] \cdot z_N \) is already defined because \( \eta([x_\lambda, x_\mu]) + \eta(N) < \eta(\lambda) + \eta(\mu) + \eta(N) = \eta(\lambda) + \eta(\Sigma) \).

Now only remains to prove (B). From \( z_\lambda z_\Sigma = q_{\lambda \mu} z_\mu z_\lambda z_N \) we obtain

\[
x_\lambda \cdot z_\Sigma - z_\lambda z_\Sigma = q_{\lambda \mu} x_\mu \cdot w + \sum_i \xi_i x_{\mu_i} \cdot x_\lambda \cdot z_N + [x_\lambda, x_\mu] \cdot z_N.
\]

Moreover

\[
x_\mu \cdot w \in \mathcal{P}_{\eta(\mu) + \eta(w)} = \mathcal{P}_{\eta(\mu) + \eta(\lambda) + \eta(N)} = \mathcal{P}_{\eta(\lambda) + \eta(\Sigma)}
\]

\[
x_{\mu_i} \cdot x_{\lambda_i} \cdot z_N \in \mathcal{P}_{\eta(\mu_i) + \eta(\lambda_i) + \eta(N)} = \mathcal{P}_{\eta(\mu) + \eta(\lambda) + \eta(N)} = \mathcal{P}_{\eta(\lambda) + \eta(\Sigma)}
\]

imply

\[
x_\lambda \cdot z_\Sigma - z_\lambda z_\Sigma \in \mathcal{P}_{\eta(\lambda) + \eta(\Sigma)}
\]

\[\square\]
Definition V.1. Let $L$ be a $T$-Lie algebra with basis $\mathcal{B}$. We call $L$ adequate if the morphism from the lemma (A-B) is such that the condition
\begin{equation}
(x_{\lambda'} \cdot x_{\mu'} \cdot z_M - T(x_{\lambda'} \otimes x_{\mu'}) \cdot z_M = [x_{\lambda'}, x_{\mu'}] \cdot z_M)
\end{equation}
for all $\eta(x_{\lambda'}) + \eta(x_{\mu'}) + \eta(M) \leq r$ implies
\begin{equation}
(x_{\lambda} \cdot x_{\mu} \cdot z_N - q_{\lambda \mu} [x_{\mu}, [x_{\lambda}, x_{\gamma}]] \cdot z_N = \\
q_{\mu \gamma} q_{\lambda \gamma} \langle x_{\gamma} [x_{\lambda}, x_{\mu}] \rangle \cdot z_N + q_{\mu \gamma} q_{\lambda \gamma} x_{\gamma} \cdot \langle x_{\lambda}, x_{\mu} \rangle \cdot z_N \\
+ q_{\mu \gamma} \langle x_{\lambda}, x_{\gamma} \rangle \cdot x_{\mu} \cdot z_N - q_{\lambda \mu} x_{\mu} \cdot \langle x_{\lambda}, x_{\gamma} \rangle \cdot z_N \\
+ q_{\mu \gamma} \langle x_{\lambda}, x_{\gamma} \rangle \cdot x_{\mu} \cdot z_N - q_{\lambda \mu} x_{\mu} \cdot [x_{\lambda}, x_{\gamma}] \cdot z_N \\
+ x_{\lambda} \cdot \langle x_{\mu}, x_{\gamma} \rangle \cdot z_N - q_{\lambda \mu} q_{\lambda \gamma} \langle x_{\mu}, x_{\gamma} \rangle \cdot x_{\lambda} \cdot z_N - q_{\lambda \gamma} q_{\lambda \mu} \langle [x_{\mu}, x_{\gamma}], x_{\lambda} \rangle \cdot z_N)
\end{equation}
for every $x_{\lambda} > x_{\mu} > x_{\gamma} \in \mathcal{B}$, $x_{\gamma} \leq z_N$ such that $\eta(x_{\lambda}) + \eta(x_{\mu}) + \eta(x_{\gamma}) + \eta(N) \leq r + 1$.

Lemma V.2 (C). Let $L$ be an adequate $T$-Lie algebra with basis $\mathcal{B}$, and $\mathcal{P}$ the related $q$-symmetric algebra. Then there exists a $k$-morphism $\_ \cdot \_ : L \otimes_k \mathcal{P} \to \mathcal{P}$ such that
\begin{enumerate}
\item $\mu \leq N$ or $\lambda \leq N$;
\item $\mu \not\leq N$ and $\lambda \not\leq N$;
\end{enumerate}

Assume $\mu \leq N$ and $\mu < \lambda$. Let $M = (\mu, N)$, then, by definition
\begin{equation}
x_{\lambda} \cdot x_{\mu} \cdot z_N = x_{\lambda} z_M \quad \text{where} \quad \lambda \not\leq M
\end{equation}
\begin{equation}
= z_{\lambda} \cdot z_M + q_{\lambda \mu} x_{\mu} \cdot w + \langle x_{\lambda}, x_{\mu} \rangle \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N
\end{equation}
On the other hand,
\begin{equation}
T(x_{\lambda} \otimes x_{\mu}) \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N
\end{equation}
\begin{equation}
= q_{\lambda \mu} x_{\mu} \cdot x_{\lambda} \cdot z_N + \langle x_{\lambda}, x_{\mu} \rangle \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N
\end{equation}
\begin{equation}
= q_{\lambda \mu} x_{\mu} \cdot (z_{\lambda} z_N) + q_{\lambda \mu} x_{\mu} \cdot w + \langle x_{\lambda}, x_{\mu} \rangle \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N
\end{equation}
since $\mu < \lambda$ and $\mu \leq N$ it holds $z_{\lambda} z_N = cz_N'$ where $\mu \leq N'$ and $c \in k$,
\begin{equation}
x_{\mu} \cdot (z_{\lambda} z_N) = c x_{\mu} \cdot z_{N'} = cz_{\mu} z_{N'} = z_{\mu} z_{\lambda} z_{N'},
\end{equation}
so
\begin{equation}
q_{\lambda \mu} x_{\mu} \cdot (z_{\lambda} z_N) = q_{\lambda \mu} z_{\lambda} z_{\lambda} z_{N} = z_{\lambda} z_{\mu} z_{N} = z_{\lambda} z_{M},
\end{equation}
therefore
\begin{equation}
T(x_{\lambda} \otimes x_{\mu}) \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N
\end{equation}
\begin{equation}
= z_{\lambda} z_M + q_{\lambda \mu} x_{\mu} \cdot w + \langle x_{\lambda}, x_{\mu} \rangle \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N
\end{equation}
\begin{equation}
= x_{\lambda} \cdot x_{\mu} \cdot z_N
\end{equation}(i.e. (C) holds for $\mu < \lambda$). It follows, multiplying by $-q_{\mu \lambda}$:
\begin{equation}
-q_{\mu \lambda} x_{\lambda} \cdot x_{\mu} \cdot z_N = -x_{\mu} \otimes x_{\lambda} \cdot z_N - q_{\mu \lambda} \langle x_{\lambda}, x_{\mu} \rangle \cdot z_N = q_{\mu \lambda} [x_{\lambda}, x_{\mu}] \cdot z_N.
\end{equation}
This implies, using antisymmetry,
\begin{equation}
x_{\mu} \cdot x_{\lambda} \cdot z_N - T(x_{\mu} \otimes x_{\lambda}) \cdot z_N = [x_{\mu}, x_{\lambda}] \cdot z_N
\end{equation}
and we conclude that (C) also holds for $\lambda < \mu$. 

PBW FOR GENERALIZED LIE COLOR ALGEBRAS 9
Recall that $\eta(\lambda) + \eta(\mu) + \eta(N)$ where $\gamma \leq \lambda$, $\gamma < \mu$. We proceed by induction on $\eta(\lambda) + \eta(\mu) + \eta(N)$. Suppose that for each $\eta(\lambda') + \eta(\mu') + \eta(N') \leq r$ it holds (C).

Then, for $\eta(\lambda) + \eta(\mu) + \eta(N) \leq r + 1$ we have:

$$x_\mu \cdot z_N = x_\mu \cdot (x_\gamma \cdot z_Q) = T(x_\mu \otimes x_\gamma) \cdot z_Q + [x_\mu, x_\gamma] \cdot z_Q$$

(5.13)

because $\eta(\mu) + \eta(\gamma) + \eta(Q) = \eta(\mu) + \eta(N) \leq r$ and the induction hypothesis.

Now, $x_\mu \cdot z_Q = z_\mu z_Q + w$ where $w \in \mathcal{P}_{\eta(\mu)+\eta(Q)-1}$. We may apply (C) to $x_\lambda \cdot x_\gamma \cdot (z_\mu z_Q)$ since $z_\mu z_Q = cz_Q$, where $c \in k$ and $\gamma \leq Q'$ because $\gamma \leq Q$, $\gamma < \mu$ and case (A).

Also (C) applies to $x_\lambda \cdot x_\gamma \cdot w$ since

$$\eta(\lambda) + \eta(\gamma) + \eta(w) \leq \eta(\lambda) + \eta(\gamma) + \eta(\mu) + \eta(Q) - 1 = \eta(\lambda) + \eta(\gamma) + \eta(N) - 1 \leq r$$

and the induction hypothesis.

The preceding remarks show that (C) applies to

$$x_\lambda \cdot x_\gamma \cdot x_\mu \cdot z_Q = x_\lambda \cdot x_\gamma \cdot (z_\mu z_Q) + x_\lambda \cdot x_\gamma \cdot w$$

Using (5.13) and multiplying by $x_\lambda$,

$$x_\lambda \cdot x_\mu \cdot z_N = x_\lambda \cdot T(x_\mu \otimes x_\gamma) \cdot z_Q + x_\lambda \cdot [x_\mu, x_\gamma] \cdot z_Q$$

$$= q_{\mu\gamma}x_\lambda \cdot x_\mu \cdot x_\gamma \cdot z_Q + x_\lambda \cdot \langle x_\mu, x_\gamma \rangle \cdot z_Q + x_\lambda \cdot [x_\mu, x_\gamma] \cdot z_Q$$

$$= q_{\mu\gamma}q_{\lambda\gamma}x_\lambda \cdot x_\mu \cdot x_\gamma \cdot z_Q + q_{\mu\gamma}q_{\lambda\gamma} \cdot \langle x_\lambda, x_\gamma \rangle \cdot x_\mu \cdot z_Q$$

$$+ q_{\mu\gamma}q_{\lambda\gamma}x_\lambda \cdot x_\mu \cdot z_Q + x_\lambda \cdot \langle x_\mu, x_\gamma \rangle \cdot z_Q + x_\lambda \cdot [x_\mu, x_\gamma] \cdot z_Q.$$
Theorem VI.1. If \( x_\mu < x_\lambda \) then we can make use of multiplicity condition and since 
\[
\eta([x_\gamma, x_\mu]) + \eta(\lambda) + \eta(Q) < \eta(\lambda) + \eta(\mu) + \eta(\gamma) + \eta(Q) = \eta(\lambda) + \eta(\mu) + \eta(N)
\]
we obtain that (5.17) is equal to 
\[
q_{\mu\gamma}q_{\lambda\gamma}q_{\mu}\eta([x_\gamma, x_\mu], x_\lambda) \cdot zQ + q_{\mu\gamma}q_{\lambda\gamma}q_{\mu}\eta([x_\gamma, x_\mu], x_\lambda) \cdot zQ \\
= -q_{\lambda\gamma}q_{\mu\lambda}(x_\mu, x_\gamma), x_\lambda) \cdot zQ - q_{\lambda\gamma}q_{\mu\lambda}(x_\mu, x_\gamma), x_\lambda) \cdot zQ.
\]
(5.16)
Using multiplicity again and since \( \eta(x_\gamma) + \eta([x_\lambda, x_\mu]) + \eta(Q) < \eta(x_\lambda) + \eta(\mu) + \eta(\gamma) + \eta(Q) = \eta(x_\lambda) + \eta(x_\mu) + \eta(N) \) we may write 
\[
x_\gamma \cdot [x_\lambda, x_\mu] \cdot zQ = q_{\gamma\mu}q_{\lambda\gamma}[x_\lambda, x_\mu] \cdot x_\gamma \cdot zQ + [x_\gamma, [x_\lambda, x_\mu]] \cdot zQ + \langle x_\gamma, [x_\lambda, x_\mu] \rangle \cdot zQ
\]
(5.17)
Substitute (5.16) and (5.17) in (5.14), 
\[
x_\lambda \cdot x_\mu \cdot zN - q_{\mu\lambda}x_\mu \cdot x_\lambda \cdot zN = \\
[x_\lambda, x_\mu] \cdot x_\gamma \cdot zQ + q_{\mu\gamma}q_{\lambda\gamma}[x_\gamma, x_\lambda, x_\mu] \cdot zQ \\
+ q_{\mu\gamma}q_{\lambda\gamma}\langle x_\gamma, [x_\lambda, x_\mu] \rangle \cdot zQ + q_{\mu\gamma}q_{\lambda\gamma}x_\gamma \cdot \langle x_\lambda, x_\mu \rangle \cdot zQ \\
+ q_{\mu\gamma}\langle x_\lambda, x_\gamma \rangle \cdot x_\mu \cdot zQ + q_{\gamma\mu}q_{\lambda\gamma}\langle x_\lambda, x_\gamma \rangle \cdot x_\mu \cdot zQ \\
+ q_{\gamma\mu}\langle x_\lambda, x_\gamma \rangle \cdot x_\mu \cdot zQ + q_{\lambda\gamma}q_{\mu\lambda}(x_\mu, x_\gamma), x_\lambda) \cdot zQ \\
- q_{\lambda\gamma}q_{\mu\lambda}(x_\mu, x_\gamma), x_\lambda) \cdot zQ - q_{\lambda\gamma}q_{\mu\lambda}\langle x_\mu, x_\gamma \rangle, x_\lambda \rangle \cdot zQ
\]
since \( L \) is adequate, 
\[
= [x_\lambda, x_\mu] \cdot x_\gamma \cdot zQ + \langle x_\lambda, x_\mu \rangle \cdot x_\gamma \cdot zQ + \\
q_{\mu\gamma}q_{\lambda\gamma}\langle x_\gamma, [x_\lambda, x_\mu] \rangle \cdot zQ - q_{\lambda\gamma}q_{\mu\lambda}[x_\mu, x_\gamma], x_\lambda \rangle \cdot zQ - q_{\mu\lambda}[x_\mu, [x_\lambda, x_\gamma]] \cdot zQ.
\]
Thanks to Jacobi identity and (A) we get 
\[
x_\lambda \cdot x_\mu \cdot zN - q_{\lambda\mu}x_\mu \cdot x_\lambda \cdot zN - \langle x_\lambda, x_\mu \rangle \cdot zN = [x_\lambda, x_\mu] \cdot zN
\]
(5.18)
if \( x_\mu < x_\lambda \).
Multiplying both sides of (5.18) by \(-q_{\mu\lambda}\) and using antisymmetry, we get 
\[
x_\mu \cdot x_\lambda \cdot zN - q_{\lambda\mu}x_\mu \cdot x_\lambda \cdot zN - \langle x_\mu, x_\lambda \rangle \cdot zN = [x_\mu, x_\lambda] \cdot zN
\]
so (5.18) also holds if \( x_\lambda < x_\mu \).

VI. Representations

Definition VI.1. Let \( L \) be a basic \( T \)-Lie algebra and \( V \) a \( k \)-module. A \( \cdot \cdot \cdot : L \otimes_k V \to V \) is called representation of \( L \) if it satisfies 
\[
x \cdot y \cdot v - T(x \otimes y) \cdot v = [x, y] \cdot v, \forall x, y \in L, \forall v \in V
\]
where \((a \otimes b) \cdot v\) means \(a \cdot b \cdot v\).

Theorem VI.1. If \( L \) is an adequate basic \( T \)-Lie algebra then \( L \) has a natural representation on its q-symmetric algebra \( S(L) \).

Corollary VI.2. If \( L \) is an adequate basic \( T \)-Lie algebra then its universal enveloping algebra \( U(L) \) has a representation on the q-symmetric algebra \( S(L) \).
Suppose

\[ e_{ij} e_{jk} e_{kl} e_{il} \]

Example VI.3. Every basic T-Lie algebra of type \((sl_n^\pm)_q\) is adequate.

**Proof.** Suppose \(x_\lambda > x_\mu > x_\gamma \in B\), \(x_\alpha \leq z_N\) such that \(\eta(x_\lambda) + \eta(x_\mu) + \eta(x_\gamma) + \eta(N) \leq r + 1\). We have to prove that (5.10) holds.

Note that \((x_\lambda, x_\mu) = 0\) for any \(x_\lambda, x_\mu \in B\) except \(e_{jk}, e_{il}\), so each term in the equation (5.10) vanishes or \(e_{jk}, e_{il}\) appears. This means that the equation (5.10) holds trivially except in the following cases:

- **Case** \(e_{ij} < e_{jk} < e_{jl}\): The left side of (5.10) vanishes whereas the right side is:

  \[
  e_{jk} \cdot e_{il} \cdot z_N - q^2 e_{il} \cdot e_{jk} \cdot z_N + q(e_{ik}, e_{jl}) \cdot z_N \\
  = e_{jk} \cdot e_{il} \cdot z_N - q^2 e_{il} \cdot e_{jk} \cdot z_N + (q^2 - 1)e_{il} \cdot e_{jk} \cdot z_N = 0,
  \]

  because \(e_{il} \cdot e_{jk} \cdot z_N = e_{jk} \cdot e_{il} \cdot z_N\) since \(\eta(e_{ij}) + \eta(e_{jk}) < \eta(e_{ij}) + \eta(e_{ik}) + \eta(e_{jl})\) and supposition (5.9).

- **Case** \(e_{ij} < e_{ik} < e_{jl}\): Let be \(d = \eta(e_{ij}) + \eta(e_{ik}) + \eta(e_{jl})\). The left side of (5.10) is

  \[
  (q^{-1} - q)e_{il} \cdot e_{jk} \cdot e_{ij} \cdot z_N \\
  = (q^{-1} - q)(q e_{il} \cdot e_{ij} \cdot e_{jk} \cdot z_N - q e_{il} \cdot e_{ik} \cdot z_N) \\
  = (q^{-1} - q)(e_{ij} \cdot e_{il} \cdot e_{jk} \cdot z_N - q^{-1} - q) e_{il} \cdot e_{ik} \cdot z_N \\
  (\eta(e_{il}) + \eta(e_{ij}) + \eta(e_{jk}) < d \) and (5.3)
  \]

  \[
  = (q^{-1} - q)(e_{ij} \cdot e_{il} \cdot e_{jk} \cdot z_N - e_{il} \cdot e_{ik} \cdot z_N + q e_{ik} \cdot e_{il} \cdot z_N) \\
  (\eta(e_{ik}) + \eta(e_{il}) < d)\), and this is the right side of (5.10).

  The remaining cases are similar. \(\square\)

Example VI.4. Every basic T-Lie algebra of type \((sl_n^\pm)_q\) is adequate, \(n = 5, 6\).

**Proof.** By similar calculations as in the previous example. \(\square\)
Note that the symbol \( \cdot_z N \) is redundant in calculations at example VII.3. This remark leads to the following lemma.

Let \( \otimes_k L \) be the tensorial \( k \)-algebra and \( J_r \) the \( k \)-submodule generated by

\[
x_\alpha \otimes x_\beta \otimes x_\delta \sim T(x_\alpha \otimes x_\beta) \otimes x_\delta - [x_\alpha, x_\beta] \otimes x_\delta, \tag{6.19}
\]

\[
x_\alpha \otimes x_\beta \otimes x_\delta - x_\alpha \otimes T(x_\beta \otimes x_\delta) - x_\alpha \otimes [x_\beta, x_\delta]\tag{6.20}
\]

for \( \eta(\alpha) + \eta(\beta) + \eta(\delta) \leq r, \ \forall x_\alpha, x_\beta, x_\delta \in \mathcal{B} \).

**Lemma VI.2.** \( L \) is adequate if

\[
\langle x_\lambda, x_\mu \rangle \otimes x_\gamma - q_{\lambda\mu}[x_\mu, [x_\lambda, x_\gamma]] \equiv
\]

\[
q_{\mu\gamma}q_{\lambda\gamma}\langle x_\gamma[x_\lambda, x_\mu] \rangle + q_{\mu\gamma}q_{\lambda\gamma}x_\gamma \otimes \langle x_\lambda, x_\mu \rangle
\]

\[
+ q_{\mu\gamma}\langle x_\lambda, x_\gamma \rangle \otimes x_\mu - q_{\lambda\mu}x_\mu \otimes \langle x_\lambda, x_\gamma \rangle
\]

\[
+ q_{\mu\gamma}[x_\lambda, x_\gamma] \otimes x_\mu - q_{\lambda\mu}x_\mu \otimes [x_\lambda, x_\gamma]
\]

\[
+ x_\lambda \otimes \langle x_\mu, x_\gamma \rangle - q_{\lambda\mu}q_{\lambda\gamma}\langle x_\mu, x_\gamma \rangle \otimes x_\lambda - q_{\lambda\gamma}q_{\lambda\mu}(\langle x_\mu, x_\gamma \rangle, x_\lambda) \mod J_r \tag{6.21}
\]

for every \( x_\lambda > x_\mu > x_\gamma \in \mathcal{B} \) such that \( \eta(\lambda) + \eta(\mu) + \eta(\gamma) \leq r + 1 \).

VII. **POINCARE-BIRKHOFF-WITT THEOREM**

Let us define

\[
\mathcal{T}^n = \underbrace{L \otimes_k \ldots \otimes_k L}_{n \text{-times}}
\]

For \( u = x_{\lambda_1} \otimes \ldots \otimes x_{\lambda_n} \in \mathcal{T}^n \) define \( \delta(u) = \eta(x_{\lambda_1}) + \ldots + \eta(x_{\lambda_n}) \), \( D(u) = \# \{ (x_{\lambda_i}, x_{\lambda_j}) | x_{\lambda_i} > x_{\lambda_j} \text{ and } i < j \} \). If \( u \in \otimes_k L \) and \( u = \sum_i \xi_i u_i \) with \( u_i \in \mathcal{T}^i \), \( \xi_i \in k \), \( \forall i \), let us put

\[
D(u) = \max \{ D(u_i) | \xi_i \neq 0, i \} \tag{7.22}
\]

\[
\delta(u) = \max \{ \delta(u_i) | \xi_i \neq 0, i \} \tag{7.23}
\]

The number \( D(u) \) is called the **disorder** of \( u \).

Denote by \( \mathcal{T}_p \) the \( k \)-submodule generated by \( u \in \otimes_k L \) such that \( \delta(u) \leq p \).

**Definition VII.1.** A sequence \( (x_{\lambda_1}, \ldots, x_{\lambda_n}) \) of elements in a basis of a basic \( T \)-Lie algebra is called non-decreasing if \( x_{\lambda_1} \leq \ldots \leq x_{\lambda_n} \) and \( x_{\lambda_i} = x_{\lambda_{i+1}} \) if and only if \( S(x_{\lambda_i} \otimes x_{\lambda_{i+1}}) = x_{\lambda_i} \otimes x_{\lambda_{i+1}} \).

**Theorem VII.1** (Poincaré-Birkhoff-Witt). Let \( L \) be an adequate \( T \)-Lie algebra with a basis \( \mathcal{B} \). The monomials formed by finite non-decreasing sequences of elements in \( \mathcal{B} \) constitute a free \( k \)-basis of the universal enveloping algebra \( U(L) \).

**Proof.** Let \( P : \otimes_k L \to U(L) \) be the canonical \( k \)-morphism, \( \mathcal{M} \) the \( k \)-submodule generated by the monomials described in the formulation of the theorem. We have to prove that \( U(L) = \mathcal{M} \). Note that

\[
U(L) = \sum_{p=1}^{\infty} P(\mathcal{T}_p)
\]

If \( p = 1 \) then \( \mathcal{T}_p \subseteq L \) it follows \( P(\mathcal{T}_p) \subseteq \mathcal{M} \). Suppose \( P(\mathcal{T}_r) \subseteq \mathcal{M} \). It suffices to show that \( P(\mathcal{T}_{r+1}) \subseteq \mathcal{M} \).

Define \( \mathcal{T}_u^0 \) as the \( k \)-submodule of \( \mathcal{T}_u \) generated by elements with disorder \( \leq u \), and proceed by a second induction on the disorder. We have \( P(\mathcal{T}_{r+1}^0) \subseteq \mathcal{M} \). Suppose
\(v = a \otimes x \otimes y \otimes b \in T_{r+1}^n\) where \(x > y \in B\), and \(a \in T^n, b \in T^m\) monomials form by basic elements in \(B\). Then
\[
P(v) = P(a \otimes q_{xy} x \otimes b) + P(a \otimes (x, y) \otimes b) + P(a \otimes [x, y] \otimes b)
\equiv P(a \otimes q_{xy} x \otimes b) \mod T_r
\]
but \(P(a \otimes q_{xy} x \otimes b) \in T_{r+1}^{n+1} \subseteq M\). Hence \(P(v) \in M\) it follows \(P(T_{r+1}) \subseteq M\).

It remains to prove linear independence. For a given sequence \(\Sigma = (x_{\lambda_1}, \ldots, x_{\lambda_n})\) of non-decreasing elements of \(B\), define \(x_\Sigma = x_{\lambda_1} \cdots x_{\lambda_n} \in U(L)\). Suppose
\[
\sum_i \xi_i x_{\Sigma_i} = 0
\]
where each \(\Sigma_i\) is a sequence non-decreasing and \(\xi_i \in k, \forall i\). Using the representation of \(U(L)\), we get from lemma (C)
\[
0 = \sum_i \xi_i x_{\Sigma_i} \cdot 1 = \sum_i \xi_i z_{\Sigma_i}
\]
and because the linear independence of the \(z_{\Sigma_i} \in S(L)\), it follows that \(\xi_i = 0, \forall i\).

**Corollary VII.2.** \(U(sl^\pm_n)_q\) has a basis of the Poincaré-Birkhoff-Witt type, \(n = 2, 3, 4, 5\).

**VIII. Braids**

**Proposition VIII.1.** In \(L = (sl^\pm_n)_q, n = 2, 3, 4\) it holds the braid equation:
\[
T_{12}T_{23}T_{12} \mid_{\gamma L} = T_{23}T_{12}T_{23} \mid_{\gamma L},
\]
where \(T_{i2} = T \otimes_k Id_L, T_{23} = Id_L \otimes_k T\).

**Proof.** By straightforward calculations on the basic elements. (Using Mathematica [E]).

**Proposition VIII.2.** The presymmetry \(S\) of a \(T\)-Lie algebras holds the braid equation
\[
S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23}
\]

**Proof.** Let \(x, y, z\) be basic elements. Then
\[
S_{12}S_{23}S_{12}(x \otimes y \otimes z) = q_{x,y}q_{x,z}q_{y,z}y \otimes x = S_{23}S_{12}S_{23}(x \otimes y \otimes z)
\]

**Remark VIII.1.** The symmetry of \(\tilde{sl}_4^+\) is a braid morphism, however we have no PBW theorem for \(U(\tilde{sl}_4^+)_q\). As a consequence the PBW theorem is independent from the braid equation.
IX. NON-STANDARD QUANTUM DEFORMATIONS OF $GL(n)$

Definition IX.1. Let $p, q$ be units in a commutative unitary ring $k$ with $pq \neq 1$ and choose $\alpha(\alpha - 1)/2$ discrete parameters $\epsilon_{ij}, \epsilon_{ji} = \pm 1, 1 \leq i < j \leq \alpha$, $\epsilon_{ii} = 1$. Let $m, n$ be positive integers such that $m, n \leq \alpha$.

The $k$-module $L_{p,q,\epsilon}(n, m, k)$ is then defined to be the free $k$-module with basis

$$B = \{Z^i_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Now define an order on $B$ and morphisms $S, T, \langle \cdot, \cdot \rangle, [\cdot, \cdot]$ copying the structure of $L(n, k)$ in example III.6.

Proposition IX.2. $L_{p,q,\epsilon}(n, m, k)$ has a structure of basic $T$-Lie algebra.

In a similar way to the algebras of type $(sl_n^+)_q$ (see section VI) we can define algebras of type $L_{p,q,\epsilon}(n, m, k)$.

Lemma IX.1. Every algebra of type $L_{p,q,\epsilon}(\lambda, \mu, k)$ is an adequate basic $T$-Lie algebra, where $\lambda, \mu \in \{2, 3\}$.

Lemma IX.2. If $Z^v_u > Z^j_i > Z^b_a$ then there exists $L$ being a $T$-Lie subalgebra of $L_{p,q,\epsilon}(n, m, k)$ and numbers $\lambda, \mu \in \{2, 3\}$ such that $L$ is of type $L_{p,q,\epsilon}(\lambda, \mu, k)$ and $\{Z^v_u, Z^j_i, Z^b_a\} \subset L$.

Proof. Let us put the basic elements in a matrix array (Figure 6 (a)).

$$\begin{array}{cccc}
Z_1^1 & Z_1^2 & \ldots & Z_1^n \\
\vdots & \vdots & \ddots & \vdots \\
Z_m^1 & Z_m^2 & \ldots & Z_m^n
\end{array}$$

$$\begin{array}{cccc}
\epsilon_i^j & \epsilon_j^i & \epsilon_i^{j+u} & \epsilon_j^{i+u} \\
\epsilon_i^{j+u} & \epsilon_j^{i+u}
\end{array}$$

Figure 6. (a) The basic $T$-Lie algebra $L_{p,q,\epsilon}(\lambda, \mu, k)$. (b) Diagonal relationship.

Note that for positive integers $u, v$ the elements appearing in the pseudobracket definition are in a diagonal relationship (Figure 7 (b)), and they form a free basis of a $T$-Lie algebra of type $L_{p,q,\epsilon'}(2, 2, k)$, where $\epsilon' = \{1, \epsilon_{i,i+u}, \epsilon_{i,j+u}\}$.

For $Z^v_u > Z^j_i > Z^b_a$ there are several cases. The cases given by Figure 7 (a), (b), (c), or they form a triangle which can be fitted, with vertices on the border, inside of the rectangle at Figure 7 (d).

In the case given by Figure 7 (a) we may complete each triangle to a square and obtain $L_{p,q,\epsilon_0}(2, 2, k)$. In the case given by Figure 7 (b), each triangle can be completed to a rectangle in the form of Figure 7 (c) and we get $L_{p,q,\epsilon_1}(3, 2, k)$. Similarly in the case given by Figure 7 (c) we get $L_{p,q,\epsilon_2}(2, 3, k)$. Finally, in the case given by Figure 7 (d), we obtain $L_{p,q,\epsilon_3}(3, 3, k)$.

Theorem IX.3. $L_{p,q,\epsilon}(n, m, k)$ is an adequate basic $T$-Lie algebra.

Corollary IX.3. The monomials formed by non-decreasing finite sequences of elements in

$$B = \{Z^i_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$$

constitute a free basis of the $k$-module $M_{p,q,\epsilon}(n, m, k) = U L_{p,q,\epsilon}(n, m, k)$. 


Lemma X.1. Let $e_{ab}, e_{uv}, e_{ij}$ be basic elements in $(sl_{n+1})_q$ and $[,]$ usual bracket in $sl_{n+1}$.

1. $e_{ab} < e_{uv}$ if and only if $a + b < i + j$ or $a + b = i + j$ and $b < j$.
2. If $S(e_{ab} \otimes e_{uv}) = q^{e_{ab,uv}} e_{uv} \otimes e_{ab}$ and $e_{ab} < e_{uv}$ then
   $$c_{ab,uv} = -\delta_{v,a} + \delta_{v,b} + \delta_{u,a} - \delta_{u,b}$$
3. If $e_{ab} < e_{uv} < e_{ij}$ then
   $$q^{e_{uv,ab}} [e_{uv}, [e_{ab}, e_{ij}]] = [e_{uv}, [e_{ab}, e_{ij}]]_q$$

Proof. 1. By the order definition.
2. It follows from the formula $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$ in the classical Lie algebra $sl_n$.
3. Since $e_{ab} < e_{ij}$ we can suppose $b = i$. We have to prove $q^{e_{uv,ab}} [e_{uv}, e_{aj}] = [e_{uv}, e_{aj}]_q$. There are two cases
   (a) $e_{uv} < e_{aj}$;
   (b) $e_{uv} > e_{aj}$.
   (a): If $[e_{uv}, e_{aj}] \neq 0$ then $v = a$ and $u < v = a < b$, it follows $e_{uv} < e_{ab}$ since $u + v < a + b$. A contradiction. Therefore $[e_{uv}, e_{aj}] = [e_{uv}, e_{aj}]_q = 0$.
   (b): We have to prove
   $$q^{e_{uv,ab}} [e_{aj}, e_{uv}] = q^{e_{uv,aj}} [e_{aj}, e_{uv}]$$
   Both sides are zero because if not then $j = u$ and $i < j = u < v$, these imply $i + j < v + u$, and then $e_{ij} < e_{uv}$. Again, we have a contradiction.

Theorem X.2. $(sl_{n+1})_q$ is an adequate basic $T$-Lie algebra.

Proof. Let $B$ be the canonical basis of $sl_{n+1}$, and write the basic elements of $B$ in the form $e_{ij}$. Now, we put this basic elements in an upper triangular array (Figure 8(a)). Note that, if $\langle e_{ij}, e_{(i+u)(j+v)} \rangle \neq 0$ then the elements appearing in the pseudobracket definition are in a diagonal relationship (Figure 8(b)), and if $\langle e_{ij}, e_{(i+u)(j+v)} \rangle = 0$ then $j = i + u$ and we get the Figure 8(c).

So, if we suppose $e_{ij} > e_{uv} > e_{ab}$ then the elements appearing in the formulation of Lemma VI.2 (brackets and pseudobrackets) can be fitted inside of a square of the form of Figure 8(d), and such square can be extended to an upper triangle (Figure 8(e)), but this triangle gives a strictly graded algebra of type $(sl_6^+)_q$. Since
Lemma X.3. Suppose since lemma X.1 and the Jacobi identity in these algebras satisfies the condition of lemma VI.2, in particular the elements $e_{ij}$, $e_{uv}$, $e_{ab}$ satisfy in several cases given by Figure 9, (in the first and third cases, since convergence, we may suppose $e_{ij}$, $e_{uv}$, $e_{ab}$ then, if $T$ in one, a triangle there is not arrow between 2 and $j$, whereas the equations (10.24) can be verified by straightforward calculations. So we may suppose $T$ triangles. The first one, a triangle consider the Figure 8(a). Such diagram can be thought as formed by two overlapping diagrams. By induction on $n$, the induction hypothesis there is not arrow between 12 and $j$, whereas there is not arrow between $j$ and $n+1$ because $e_{ij}$, $e_{uv}$, $e_{ab}$ are in a subalgebra isomorphic to $U_q^{+}(sl_2)$, $i = 1, 2$. Then, if $e_{ij}$ and $e_{uv}$ are both in $T_1$ or $T_2$, the equations (10.24) holds. As a consequence, we may suppose $i = 1$ and $b = n + 1$, and put $j \neq n + 1$ and $a \neq 1$.

At the Figure 8(a) join the node $rs$ with the node $uv$ if $[e_{rs}, e_{uv}]_q \neq 0$. We have several cases given by Figure 8 (in the first and third cases, since $e_{ij}$, $e_{uv}$, $e_{ab}$ are in $T_1$ and the induction hypothesis there is not arrow between 12 and $an$, whereas there is not arrow between $2j$ and $n(n+1)$ because $e_{ij}$, $e_{uv}$, $e_{ab}$ are in $T_2$).

At the first case we get a graph of type $A_4$, then $e_{1j} = [e_{12}, e_{2j}]_q$, $e_{a(n+1)} = [e_{an}, e_{n(n+1)}]_q$ are in a subalgebra isomorphic to $U_q^{+}(sl_2)$, it follows,

$$[e_{1j}, e_{j(n+1)}]_q = e_{1j}e_{j(n+1)} - q^{-1}e_{j(n+1)}e_{1j}$$
Theorem X.4. There exists an isomorphism

\[ U_q^+(s_{n+1}) \cong U(sl_{n+1})_q \]

of \( k \)-algebras.

Proof. Let us put \( c_{ab,cd} = c_{uv} \) where \( x_u = e_{ab}, x_v = e_{cd}, \) and \( x_u < x_v, \) \( 1 \leq a, b, c, d \leq n + 1. \) From

\[ [e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj} \]

it follows, if \( e_{ab} < e_{cd}, \)

\[ c_{ab,cd} = \begin{cases} 
1, & \text{if } a = c \text{ or } b = d, \\
0, & \text{if } a \neq c \text{ and } b \neq d, \\
-1, & \text{if } b = c, 
\end{cases} \]

Now use Lemma X.3 in order to obtain the following equations in \( U_q^+(s_{n+1}), \)

\[ e_{ab}e_{cd} - q^{c_{ab,cd}}e_{cd}e_{ab} = [e_{ab}, e_{cd}]_q + (e_{ab}, e_{cd}), \]
for all \(1 \leq a, b, c, d \leq n + 1\).

We conclude \(U_q^+(sl_{n+1}) \simeq U(sl_{n+1})_q\).

\[\square\]

**Corollary X.1.** 1. The monomials formed by non-decreasing finite sequences of elements in

\[B = \{e_{ij} \mid 1 \leq i < j \leq m\}\]

constitute a free basis of the \(k\)-module \(U_q^+(sl_{n+1})\), where \(m = n(n+1)/2\).

2. We have

\[\left.\left(\lfloor sl_{n+1}^+\right) \right|_{t=0} = sl_{n+1}^+\]

and \(\lfloor sl_{n+1}^+\rfloor_q\) is a deformation of \(sl_{n+1}^+\) in the category of \(T\)-Lie algebras.

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