ON THE GROUPS AND AUTOMORPHISM GROUPS
OF THE GROUPS OF ORDER 64p
WITHOUT A NORMAL SYLOW p-SUBGROUP

WALTER BECKER AND ELAINE W. BECKER

Abstract. The groups of order 64p without a normal sylow p-subgroup are listed, and their automorphism groups are also determined. As a by-product of our original effort to get these groups, we needed to determine the automorphism groups of those groups of order 64 with an odd-order automorphism. In view of the fact that we already had determined these groups and that these automorphism groups are not given explicitly in the literature, we have appended to this report these automorphism groups. In another project we were looking for new complete groups by following automorphism group towers up to completion when the computer memory allowed such followups. We did this for these groups of order 64. In another appendix we give the results of this work as applied to the groups of order 64.

1. Introduction

The calculations reported here were done in the mid 1990s in response to a then recent bibliographic survey by E. A. O’Brien and M. W. Short [1], in which it was stated that the number of groups of orders 192, 240, and 252 were the only orders below 256 for which the number of groups was still unknown. The groups of order 252, or more generally 36p, appear in the thesis of B. Malmrot [2] as a prelude to his determination of the groups of order 72p, for p > 3. In an effort to fill in these gaps we have looked at the groups and automorphism groups of the groups of these orders. The work on the groups of orders 240 and 36p (p > 3) will be given in later reports.

The number of groups of order 192 (or more generally 64p, p > 2) is very large, and from subsequent work of Bettina Eick and Hans Ulrich Besche [10] (done around the same time as the calculations reported below were done) we have the results reported below in Tables 1a and 1b for the groups of order 64p for various choices of the odd prime p. For comparison with other orders we give similar results for the groups of orders 16p and 32p from the earlier work of A. C. Lunn and J. K. Senior from the 1930s [3]. Also included in Tables 1a and 1b are other results taken from the work of Besche and Eick reported in [3].

One way to verify the numbers in the following Table 1a for the m = 3, 4, 5 and 6 cases with a normal sylow p-subgroup would be by just counting the number of different ways the quotient groups of C2, C4, . . . , C64 arise in the subgroup lattice diagrams for the groups of orders 16, 32 and 64 found in [3]. The C2 images (or equivalently the groups arising in the case of p ≡ 1 mod(2)) are called dimidiations in the older literature. If one just counts the number of normal subgroups of order
In the Hall-Senior lattice subgroup diagrams for the groups of order $2^m$, one should have the number of groups of order $2^m \ast p$ arising for each group of order $2^m$ for the case when $p \equiv 1 \mod(2)$ but $p \neq 1 \mod(4)$. This is the way in which we initially had an idea of just how many groups of order $64p$ one would encounter in this enumeration process. This report will deal with those groups of order $64p$ without a normal Sylow $p$-subgroup. We also give, in Tables 1c, 1c′ and 1d, some partial results for the number of groups in the orders $2^n p^2$ for $n \leq 8$. As one can see, the number of groups for $n \geq 5$ is very large, and a detailed study of the groups, order by order, is probably not feasible. If one is interested in certain subsets or classes of groups of these orders, this may or may not be feasible, e.g., by looking at groups of order $2^n p^2$ whose action of the 2-group on the $p$-group is by a $D_4$ or a $Q_2$ action. For $n = 5$, such a study was attempted in [12] with only partial results. For the case of $n = 6$, the number of groups even in this restricted subset increases very rapidly, yielding 374 cases for the $D_4$ action and 70 cases for the $Q_2$ action case.

Table 1a

| $m$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ | $n = 7$ | $n = 8$ |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|
| 3   | 5       | 7       | 2       | 1       |         |         |         |         |
| 4   | 14      | 28      | 9       | 2       | 1       |         |         |         |
| 5   | 51      | 144     | 40      | 9       | 2       | 1       |         |         |
| 6   | 267     | 1,120   | 243     | 42      | 9       | 2       | 1       |         |
| 7   | 2,328   | 16,996  | 2,180   | 262     | 42      | 9       | 2       | 1       |
| 8   | 56,092  | 1,027,380 | 32,836 | 2,339   | 263     | 42      | 9       | 2       |

Table 1b

| $n$ | $p = 3$ | $p = 5$ | $p = 7$ | $p = 17$ | $p = 31$ |
|-----|---------|---------|---------|----------|----------|
| 3   | (2,1)   | (2,0)   | (1,0)   | -        | -        |
| 4   | (6,4)   | (1,0)   | (1,0)   | -        | -        |
| 5   | (19,17) | (2,1)   | (2,0)   | -        | (1,0)    |
| 6   | (70,86) | (5,5)   | (9,0)   | -        | (1,0)    |
| 7   | (309,536) | (13,21) | (24,1)  | (0,0)    | (2,0)    |
| 8   | (1851,4912) | (49,104) | (77,4)  | (1,0)    | (5,0)    |

The listing in Table 1c′ is modeled on that for the groups of order $2^n p$. As one goes to higher and higher orders (e.g., $2^n p^2$ for $n \geq 5$) one needs a better way to list the results of the calculations. In these higher orders we also need to consider cases such as $p \equiv 7 \mod(8)$, etc. or equivalently $p \equiv -1 \mod(2^n)$. If in these higher orders one is only interested in the way in which the total number of groups varies with $p$, then the way these groups were listed in most of the early computations makes more sense and is shown in Table 1c. In the case of $n = 5$, for example, we...
would need to consider the cases when \( p \equiv 1, 3, 5, 7, 9, \) and \( 31 \mod(32) \), as well as \( 15 \) and \( 17 \mod(32) \). The most useful display, however, might be a combination of Tables 1c and 1c', showing the various quotient groups and how they act on the \( p \)-groups as a function of the prime \( p \). These tables, however, can become very large and cumbersome as the order of the 2-group increases past \( n = 4 \) or 5.

| \( p \) \mod(8) | \( p \equiv 1 \mod(8) \) | \( p \equiv 3 \mod(8) \) | \( p \equiv 5 \mod(8) \) | \( p \equiv 7 \mod(8) \) | \( p \equiv 9 \mod(16) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( p = 3 \)     | 50              | \( p = 3 \)     | 197             | \( p = 3 \)     | 197             |
| \( p = 5 \)     | 52              | \( p = 5 \)     | 221             | \( p = 5 \)     | 221             |
| \( p = 7 \)     | 44              | \( p = 7 \)     | 172             | \( p = 7 \)     | 172             |
| \( p = 1 \mod(8) \) | 60              | \( p = 3 \mod(16) \) | 257             | \( p = 1 \mod(16) \) | 257             |
| \( p = 3 \mod(8) \) | 42              | \( p = 3 \mod(8) \) | 167             | \( p = 3 \mod(8) \) | 167             |
| \( p = 5 \mod(8) \) | 52              | \( p = 5 \mod(8) \) | 219             | \( p = 5 \mod(8) \) | 219             |
| \( p = 7 \mod(8) \) | 42              | \( p = 7 \mod(8) \) | 169             | \( p = 9 \mod(16) \) | 243             |

In our previous work on the groups of order \( 32p \) we indicated how we estimated the number of groups of order 288 by using some counting arguments and the Hall-Senior Tables (or charts contained therein). The argument, there dealing with the \( C_2 \times C_2 \) type images, says to just count the number of lines hitting the set of \( C_2 \times C_2 \) images coming down from the normal subgroups of order 16, and this should give us the number of groups of order \( 32p^2 \) with a \( C_2 \times C_2 \) action on the \( p \)-group. When we went through and actually counted the number of such lines we did not get the exact same number as we got from determining the number of groups of order 288 which arise from the \( C_2 \times C_2 \) actions of the order 32 groups on the \( C_p \times C_p \) groups (for the cases of \( p = 3 \) or 5). The numbers were sufficiently close so that one might guess that the differences arose from errors in the Hall-Senior charts. We tried a similar counting argument here for groups of order \( 64p^2 \) for \( p = 3 \) and 5. Again some of the numbers are different, but again sufficiently close so that one might suspect the differences arise from omissions in the number of lines in the Hall-Senior charts. This might be an interesting question for the reader to look into. What we find is indicated in Table 9. There are many such cases, and we give the number of such extensions as a function of the isoclinic class. The explicit numbers expected for each 2-group are given in Tables A5 and A6. We have not checked the \( p = 3 \) cases other than to count the number of times the group \( QD_8 \) arises as a quotient group in the groups of order 64. The groups \( \Gamma_2b \) and \( \Gamma_2d \) of order 16, which appear in the Hall-Senior charts, only have a single line hitting them. Hence in these cases, according to our “little empirical rule”, we only get one new group for each occurrence of these order 16 actions (in order \( 64 \times 5^2 \)) on the \( p \)-group, whereas in cases such as \( C_2 \times C_2 \) we can have more than one group arising from that \( C_2 \times C_2 \) image. These multiple lines seem to be associated with quotient groups that are direct products, e.g., \( C_2 \times C_2, C_4 \times C_2, C_4 \times C_4, \) etc. Again this might be an interesting exercise for the reader to explain the reason for this apparent correspondence.
Table 1c'  

The case when the Sylow $p$-subgroup is normal in $G$

| $m$ | $p \equiv 1 \mod(2^n)$ | direct products | $C_2$ | $C_2 \times C_2$ | $C_4$ | $C_8$ | $(2,1)$ | $D_4$ | $Q_2$ | $G(16)$ | $C_{32}$ |
|-----|---------------------|----------------|------|----------------|------|------|--------|------|-----|--------|--------|
| 3   | $n = 1$             | 10             | 21   | 6              | 2    | 1    | 0      | 1    | 1   | 0      | 0      |
|     | $n = 2$             | 10             | 21   | 6              | 10   | 1    | 2      | 1    | 1   | 0      | 0      |
|     | $n = 3$             | 10             | 21   | 6              | 10   | 9    | 2      | 1    | 1   | 0      | 0      |
| 4   | $n = 1$             | 28             | 84   | 35             | 9    | 2    | 0      | 6    | 2   | 1      | 0      |
|     | $n = 2$             | 28             | 84   | 35             | 45   | 2    | 14     | 6    | 2   | 3      | 0      |
|     | $n = 3$             | 28             | 84   | 35             | 45   | 16   | 14     | 6    | 2   | 11     | 0      |
|     | $n = 4$             | 28             | 84   | 35             | 45   | 16   | 14     | 6    | 2   | 15     | 0      |
| 5   | $n = 1$             | 102            | 432  | 274            | 40   | 9    | 0      | 42   | 11  | 4      | 0      |
|     | $n = 2$             | 102            | 432  | 274            | 200  | 9    | 112    | 42   | 11  | 20*    | 1      |
|     | $n = 3$             | 102            | 432  | 274            | 80   | 112  | 42     | 11   |     |        |        |
| 6   | $n = 1$             | 534            | 3360 | 3362           | 241  | 42   | 0      | 374  | 70  | 36**   | 0      |
|     | $n = 2$             | 534            | 3360 | 3362           | 1213 | 42   | 1068   | 374  | 70  | 216†   | 8      |

$G(16)$ means a group of order 16. For $p \equiv 3 \mod(8)$ this group is $QD_8$ (in $16p^2$ case).

Note that this table does not explicitly contain the orders $p \equiv -1 \mod(8)$, but they just differ from the $p \equiv 3 \mod(8)$ case by only two groups in the $16p^2$ cases, i.e., delete $QD_8$ and add the three groups of order 16: $D_8$, $Q_4$, and $C_{16}$.

* These 20 groups break up into 6 $C_4 \times C_4$, 10 $C_4YQ_2$, and 4 $<2,2,2>$

** These 36 groups are all $QD_8$.

† The order 16 groups are: 63 $C_4 \times C_4$, 125 $\Gamma_2b$ cases and 28 $\Gamma_2d$ cases.

The number of groups for the case of $m = 4$ and $n = 4$ when the action is of order 16 is explicitly given in our 16$p^2$ article [6].

Another curiosity concerns the $C_4$ actions of the groups of order 64 on the group $C_p \times C_p$. In the case of $p = 3$, we have 241 cases, according to the GAP/Small Group Library runs. In previous cases we find that the number of groups arising when $p = 5$ is just five times the number arising in the $p = 3$ case. This can be understood quite simply; one new set of groups comes from the action of the $C_4$’s on the group $C_{25}$ whose automorphism group is $C_5 \times C_4$. The other four cases arise, as indicated in our $16p^2$ group discussion, by the set of actions:

$$a^5 = b^5 = (a, b) = a^x * a^2 = b^x * b^x = \cdots = 1,$$

where $x = 1, 2, 3, \text{ or } 4$. The case “coming up” from the $p = 3$ $C_4$ action corresponds to $x = 3$. According to our GAP/Small Group Library runs there are 241 such $C_4$ cases arising in the groups of order 576, but there are 1213 cases when we run the order 1600 groups. There are thus an additional eight groups apparently arising here. This difference also might be an interesting exercise for the reader to explain why these extra 8 groups arise in order 1600.

As a by-product of the work on the groups of order 64$p$ with a normal sylow 2-subgroup, we computed the automorphism groups for those groups of order 64 possessing odd-order automorphisms. A brief outline of this work is given in Appendix I and Table A1. This material was not contained in an earlier report [4]. In
several cases these automorphism groups have a very simple representation, e.g.,
direct products of other well-known groups, three cases yield complete groups, and
other cases have not yielded any simple interpretation. One of the side interests
in [4] was the finding of new complete groups. One way to do this is to find a
group with a trivial center and then follow its automorphism group tower up to
completion. In our attempts to come up with “reasonable” or “simple” represen-
tations for the automorphism groups of the groups of order 64, we encountered
several normal subgroups in these automorphism groups with a trivial center. In
many cases these groups led us, by means of their automorphism group towers, to
additional complete groups. In other cases, no such end, or termination point, was
found. This work is presented in the appendix. This paper contains frequent refer-
ences to groups of order 16, the automorphism groups of order 16p, and groups of
order 96. Convenient sources where these groups can be found are papers [6] and [7].

| n  | p = 3   | p = 5   | p = 7   | p = 17  | p = 31  |
|----|---------|---------|---------|---------|---------|
| 3  | (6,4)   | (0,0)   | (2,0)   | 0       | 0       |
| 4  | (13,17) | (4,0)   | (2,1)   | 0       | 0       |
| 5  | (41,90) | (4,4)   | (2,4)   | 0       | (2,0)   |
| 6  | (152,510)| (10,24)| (19,7)  | 0       | (2,≥1)  |
| 7  | (618,1072)* | (49,29)*| 0       | (4,≥2)  |
| 8  |         |         | (1,0)   |         | (10,≥7) |

The group $1^5@C_{31}$’s automorphism group is a complete
group of order 4950; hence the group $1^5@C_{31}$ only has
an outer automorphism of order 5. Hence there are no groups
of the form $(1^5@C_{31})@[2\text{-group}]$, but we can have groups
of the form $(1^5@C_{961})@[2\text{-group}]$ (number undetermined), and
of the form $1^5@C_{31} \times C_{31}@[2\text{-group}]$.

*For $2^7p$ when $p = 3$ we have (309,536);
for $2^7p^2$ when $p = 3$ we have at least (618,1072);
for $2^7p^2$ when $p = 7$ we have at least (49,29).
group. Using this method one should get all cases that will yield a group of order 64\(p\) with a normal subgroup of order 64. This method may yield some duplicates which will need to be weeded out by hand, but these are only a few cases to look at and present no great burden.

The automorphism groups for the groups of order 64\(p\) with a normal Sylow 2-subgroup yielded several new complete groups. In many other cases the automorphism groups of the groups of order 64\(p\) have a simple breakdown into direct products of other well-known groups. The tabulated results of these calculations are given in Table 2. The complete groups of orders 168, 384, 960, and 5760 are the same groups that arose in our other studies of groups of orders 8\(p\), 8\(p^2\), 16\(p\), and 16\(p^2\). The automorphism groups in many of the other cases involve well-known groups whose presentations are either well known (e.g., \(S_4\)) or which can be found in other published material (e.g., relations for 2-groups in [3] or [4]); others are explicitly written out here. Some of the groups appearing here as automorphism group factors are groups of order 192, without a normal Sylow 2-subgroup (e.g., \(\text{Aut}(C_4 \times C_2 \times C_2)\)), and explicit presentations for these groups are given below in the section devoted to these groups.

The automorphism groups for numbers 16 (= 192 #1506), 19 (= 192 #1507), and 63 (= 192 #1009) were rerun in GAP to check on the identity of their automorphism groups. These automorphism groups are in fact, as suspected, isomorphic with the 1152 order factor being group number 155,478 of order 1152 in the Small Group Library. Likewise the groups numbered 40 (= 192 #1508), 42 (= 192 #1509), 55 (=192 #1022) and 62 (= 192 #1024) were rerun for the same purpose. The results indicate that the order 576 groups in these automorphism groups are also, as suspected, isomorphic and are isomorphic to order 576 number 8654 in the Small Group Library.

In one case \((1^6 \oplus C_3)\) the order of the automorphism group is sufficiently large that very little other information about this group is available. The authors suspect that this group is a complete group.

In our initial listing of the groups of order 64\(p\) with a normal Sylow 2-subgroup we missed one of the groups for the case of \(p = 7\) (number 9 in Table 2b). The presentations for the groups of order 64\(p\) of the form \(1^6 \oplus C_7\) can be read from the following matrices:

\[
C_7 = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 1
\end{pmatrix}
\]

\[
C_7 = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}
\]

\[
C_7 = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

The middle \(C_7\) matrix generates the group with the automorphism group of order 677,376 [448,#1393]; the other \(6 \times 6\) matrix generates the one with an automorphism group of order 18,816 [448,#1394].
The other $p = 5$ and $p = 7$ actions are given either in Table 2b, or in previous papers, e.g., the $C_5$ actions in [6], and the $C_{31}$ case in [7].

The results reported here can easily be extended to yield the groups of order $64p^2$ with a normal sylow 2-subgroup. These groups can be broken up into the following classes:

1. The group of order 64 having only a single odd-order automorphism of order $p$.
   a. A group of order $64p \times C_p$.
   b. Take the group of order $64p$ and in its presentation change $C_3$ to $C_9$ (or more generally $C_p$ to $C_{p^2}$) without changing the actions on the order 64 group. This gives us groups of the type $[64] \oplus C_9$ (or $[64] \oplus C_{p^2}$).

2. The sylow $p$-subgroup having order $p^2$ or higher. These cases are shown in Table 3.

In these cases one can have groups with the structure $[64] \oplus (C_3 \times C_3)$ which may not be easily related to a group of order $64p$. Table 3 here shows that there are only a few cases of this type occurring here. The elementary abelian group of order 64 has sylow $p$-subgroups of orders $3^4$ and $7^2$, along with the ones of orders 5 and 31. The group $(2, 1^{14})$ has a sylow 3-subgroup of order $3^2$.

3. Groups of order 192 without a normal sylow subgroup

A systematic way of determining the groups of order 192 without a normal sylow 2-subgroup would be to look for 2-group extensions of groups, $G$, of orders $3 \times 2^n$ with $n < 6$, by 2-groups. This idea was tested out on the groups of orders 48 and 96, and this method yielded all of the groups without a normal sylow subgroup. If one recalls that there are 28 groups of order $16p$ produced via a $C_2$ action of the groups of order 16 on the $C_p$ group, it should be no surprise that there are 28 groups of the form:

$A_4 \oplus \text{[group of order 16]}$.

In fact, the automorphism groups of these groups are just

$S_4 \times \text{invariant factor appearing in Table 2a of [6]}$.

A similar remark can be made for finding the groups of order 192 coming from the groups of order 24, extended by the groups of order 8. Here the group in question is $\text{SL}(2, 3)$. Note that there are only two groups of order 8 with an automorphism of order 3: $C_2 \times C_2 \times C_2$ and $Q_2$. The first group extended by $C_3$ yields $A_4 \times C_2$, and $Q_2$ yields $\text{SL}(2, 3)$.

All extensions of $A_4 \times C_2$ by a group of order 8 will yield $\text{Aut}(G) \neq \text{Aut}(\text{SL}(2, 3)) \times \text{Invariant factor}$, where this invariant factor is the same as that arising in the groups of order $8p$ when we have a normal sylow $p$-subgroup. See Table 4b below.

Note, however, in this case, $\text{Aut}(G) \neq \text{Aut}(\text{SL}(2, 3)) \times \text{Invariant factor}$, where this invariant factor is the same as that arising in the groups of order $8p$ when we have a normal sylow $p$-subgroup. See Table 4b below.
duplicates of groups produced by the $A_4 \rtimes \{\text{groups of order } 16\}$. For the groups of order 16 there are four groups which have an automorphism of order 3, namely $(1^4)$, $(2, 1^2)$, $(2, 2)$ and $C_4 Y Q_2$. Of these groups, $(2, 1^2)$ will give a group of order 48 of the form $A_4 \times C_4$, which will again duplicate results from extensions of the form $A_4 \rtimes \{\text{group of order } 16\}$. The group $(1^4)$ has two extensions in order 48. They are $A_4 \times C_2 \times C_2$ and $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3$. The first case again duplicates results from the $A_4$ extensions. The second group will yield new groups in order 192.

Of the groups of order 48 that do not have a normal sylow subgroup, only $\langle 2, 3, 4 \rangle$ was not generated by a simple semi-direct product from a group of order 24 (or lower) with a normal sylow 2-subgroup. The group $\langle 2, 3, 4 \rangle$ can be obtained as a nonsplit extension of $SL(2, 3)$ and $C_2$, with the presentation:

\[
a^3 = a \ast b \ast a * ((b \ast a \ast b)^{-1}) = c^4 = c^2 \ast (a * (b^{-1}))^2
= a^c \ast a \ast b * (a^{-1}) = b^c \ast b = 1.
\]

The action of the element $[c]$ on $SL(2, 3)$ is the same as giving rise to $GL(2, 3)$\(^2\).

Table 4 gives those groups of order 192 without a normal sylow subgroup that arise as split extensions (i.e., semi-direct products) of a group with a normal sylow 2-subgroup and a 2-group. Some of the groups here arise from two or more different extensions. In many cases these “repeat cases” have been indicated for the reader.

For order 192 there are several cases of nonsplit extensions, which arise in exactly the same way as the group $\langle 2, 3, 4 \rangle$ does in order 48. The first two such cases, not involving $\langle 2, 3, 4 \rangle$, were pointed out to us by Dr. Antonio Vera Lopez [11]. In Dr. Lopez’s work classifying groups with 13 conjugacy classes he found four groups of order 192 with just 13 classes: the two that we found, arising as split extensions, namely

\[
\text{Aut}(C_4 \times C_2 \times C_2) \quad \text{and} \quad \text{Hol}(Q_2),
\]

and two others. The other two cases are described by Dr. Lopez as nonsplit extensions of $[Q_2 Y Q_2] \rtimes S_3$. In Table 5 we list those nonsplit extensions giving rise to groups of order 192 that have presentations analogous to that yielding the group $\langle 2, 3, 4 \rangle$ given above. The presentations given in Table 5 are modelled on those given in Table 4, in that the only modifications to those in Table 4 are finding the generator of the center of the kernel of the extension, and then mapping some power of the element coming from the quotient group of the extension onto the generating element of the center of the kernel. The only possible choices for the kernels in these cases are displayed in Table 6.

The group $SL(2, 3)$ presents an interesting case, and we show in Table 7 the relationship between the split and nonsplit extensions for the groups of order 192 arising from a normal $SL(2, 3)$. For the $C_8$, $D_4$ and $Q_2$ groups, the quotient group from which the $C_8$, $D_4$ or $Q_2$ image comes is given instead of the extension itself. The notation $C_2 [C_4]$ means that the generator of order 4 in the group acts as an

\(^2\)That is, if we set $c^4 \rightarrow c^2$ and omit the relation $c^2 * (a * b^{-1})^2$ what we get is a presentation for $GL(2, 3)$.\]
element of order 2 on SL(2, 3). The numbers in the split extension column and the last column refer to the number of the group in Tables 4 and 5.

In Table 7 we have tried to show some of the groups of order 192 that occur as a semi-direct product of SL(2, 3) and the groups $D_4$ and $Q_2$, and how they are related to some other groups that are not semi-direct products. In each case, the quotient group “$D_4$” or “$Q_2$” is modified or altered, and we show how this “alteration” is mapped into the center of SL(2, 3). The best way to explain this is just by showing the presentation used here. In the first case, i.e. #33, we have

$$a^3 = aba(bab)^{-1} = c^4 = d^c = a^caba^{-1} = b^c b$$

$$(a, d) = (b, d) = \begin{cases} d^2 = d^2(ab^{-1})^2 & \text{[33]} \\ d^4 = d^2(ab^{-1})^2 & \text{[72]} \end{cases} = 1.$$ 

One should also note that some of our normal semi-direct product cases can be obtained from some apparent nonsplit extensions as well. To see this, look at group #33 and group #59, or #34 and #64:

$$a^3 = aba(bab)^{-1} = d^2 = c^d = a^caba^{-1} = b^c b$$

$$(a, d) = (b, d) = \begin{cases} c^4 = c^4(ab^{-1})^2 & \text{[33]} \\ c^8 = c^4(ab^{-1})^2 & \text{[59]} \end{cases} = 1.$$ 

The nonsplit extensions using the form $c^8 = c^4(ab^{-1})^2$ all yield groups which were obtained by semi-direct products involving other groups of order 24, 48 or 96.

The only nonsplit extensions in which the actions were greater than order 2 are the $C_2 \times C_2$ cases listed in Tables 5 and 7. No nonsplit extensions were found here for any other cases. All of the actions for the higher-order groups with nonsplit extensions were also of order 2. In our initial search for nonsplit extensions our list was incomplete, in part because we did not have a systematic method to search for all of these groups in a manner analogous to that for the groups with a normal sylow 2-subgroup, and for those cases without a normal sylow subgroup that arise as a semi-direct product of lower-order groups. In this context see the comment below at the end of this section.

The reader should also note that some of the relations given in Table 4 for some of the groups of order 96 are not the same as those given in Tables 3a and 3b of [7]. In some of these cases they represent relations derived from a degree 8 permutation representation for that group which has fewer generators than those in Tables 3a and 3b of [7]. These relations and their permutation representations can be found in the work of J. Burns listed in Appendix 0 of [4].

The number of nonisomorphic groups listed in Tables 4 and 5 is 81. Comparing this with the number given in the Small Group Library of Besche and Eick, we find that they list 86 groups without a normal sylow subgroup. For the cases when the normal subgroup was either GL(2, 3) or the Coxeter group $<2, 3, 4>$ we got many duplicate cases. We show these cases in Tables A2 and A3 of the Appendix. In our attempts to identify which groups of order 192 we were missing we prepared Table A4, which shows the class/order structure for the groups we found,
and cross-referenced them with the groups of order 192 in the Small Group Library. The missing groups of order 192 are shown in Table 8. From this we were led to the five groups of order 192 that we missed. The ones missing from our list are #’s 949, 950, 954, 1489, and 1490 of order 192 in the Small Group Library [10].

Some of these groups (e.g., #950 in Table 8) were apparently missed because they possessed the same order structure and automorphism groups as others in Table 4. It is not clear why we missed the last one, #954 of order 192.

The groups numbered 1490 and 1489 in Table 8 have the structural form
\[ (Q_2 \times C_2 \times C_2) \rtimes C_3 \rtimes C_2. \]

For some reason in our initial working with the extensions of the order 96 group \((Q_2 \times C_2 \times C_2) \rtimes C_3\) it was believed that this order 96 group only gave rise to extensions with a normal sylow 2-subgroup. If one looks at the normal subgroups of these groups (#1489 and #1490) one finds that \((Q_2 \times C_2 \times C_2) \rtimes C_3\) does appear as a normal subgroup in both groups. Subsequently we obtained the following presentations for these last two groups:

\[
\begin{align*}
 a^4 &= b^4 = a^2 * b^{-2} = ab * a = e^2 = d^2 = (c, d) = (a, c) = (b, c) = (a, d) = (b, d) = \\
 e^3 &= a^4 * b^{-1} = b^5 * b^{-1} * a^{-1} = c^6 * d = d^6 * c * d = \\
 h^2 &= a^8 * a = b^h * b * a^{-1} = c^h * c = d^h * c * d = e^h * e = 1, \\
 h^4 &= a^h * h = b^h * b * a^{-1} = c^h * c = d^h * c * d = e^h * e = h^2 * b^{-2} = 1.
\end{align*}
\]

The first group is just a semi-direct product of the order 96 group with a \(C_2\). The second group, number 1489 in the Small Group Library, is a nontrivial central extension, obtained from group number 1490 by the replacement of \(C_2\) by a \(C_4\) and with the extra relation \(c^2 \rtimes (\text{center of 96 group})^{-1}\). It might also be of interest to point out that in the first case we have

\[
< a, b, e > \simeq \text{SL}(2, 3), \quad < a, b, e, h > \simeq \text{GL}(2, 3), \quad < c, d, e > \simeq A_4, \\
< c, d, e, h > \simeq S_4
\]

and in the second case,

\[
< a, b, e > \simeq \text{SL}(2, 3), \quad < a, b, e, h > \simeq < 2, 3, 4 >, \quad < c, d, e > \simeq A_4, \\
< c, d, e, h > \simeq A_4 \rtimes C_4.
\]

An interesting observation also is that if in the second form (i.e., \(h^4 = \cdots\)), if we replace \(h^2 * b^{-2}\) with \(h^2 * d^{-2}\) we get an alternate presentation for the first of these two groups.

In the case of the last missing group, #954 of order 192, we know that it has the structure \(\text{SL}(2, 3) @ D_4\). It has three normal subgroups of order 96, namely:

\[
\text{GL}(2, 3) \times C_2, \quad < 2, 3, 4 > \rtimes C_2, \quad \text{and} \quad \text{SL}(2, 3) \times C_4.
\]

In our attempts to get a “reasonable” presentation for this group we tried several permutations or modifications of the actions on the group \(\text{SL}(2, 3)\) given in Table 5. The extension forms we tried were based upon one or the other assumed structures:

\[
\text{SL}(2, 3) @ D_4, \quad (\text{SL}(2, 3) \times C_4) @ C_2.
\]
The only result from this exercise was our getting several different/alternate presentations of other groups of order 192 given in these tables. The Small Group Library gives rise to a presentation for this group on 7 generators with 19 relations. A slightly reduced set of relations can be obtained by eliminating the sixth and seventh generators, yielding the presentation:

\[
a^2 = b^4 = c^2 = d^3 = e^4 = (a, c) = (a * d^{-1})^2
= (b, d) = (c, d) = (b, e) = (c, e) = e^2 * b^{-1} * c * b^{-1}
= a * c * b * a * b^{-1} = (a * e^{-1} * d)^2 = (d^{-1} * e^{-1})^3 = 1.
\]

Acknowledgements

Almost all of the calculations reported here were done on the DEC computer in the Department of Cognitive and Linguistic Sciences at Brown University. We give special thanks to Dr. James Anderson for letting us use the computers. Margaret Doll helped with the installation of CAYLEY on the computers. Some of the calculations are in the Appendices. Thanks also go to Dr. John Cannon for giving us the CAYLEY program to use and for answering questions about using it.

We also used the programming system GAP for checking our earlier CAYLEY results. In this endeavor, Dr. Steve Linton was instrumental in helping us with the programming.

The bulk of this report was written by May 29, 1997. The comments related to the Small Group Library and other work by B. Eick et al. were added at a much later date (2006 to 2008).

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4. Appendix I

The Automorphism Groups of the Groups of Order 64 with an Odd-Order Automorphism

In the course of the work done on determining the groups of order 192 with a normal Sylow 2-subgroup we computed certain properties of the automorphism groups of the groups of order 64 which have an odd-order automorphism. The main purpose here was to determine the actions of the resulting operators of order 3 in the automorphism group on the group of order 64, thereby yielding the desired groups of order 192. In the course of this study we thought it would be interesting to determine presentations and other properties of these automorphism groups. A partial summary of this work is contained in Table A1. This table shows that in some cases these automorphism groups have a rather simple structure, while in other cases the orders are sufficiently large that direct computational procedures are not especially useful in determining some properties (e.g., their automorphism groups). In what follows let $U_{64}$ denote the set of automorphism groups of the groups of order 64 which possess odd-order automorphisms.

The two main areas of interest here were to obtain some information on the structure of the groups in $U_{64}$, and second to see if we could find any new complete groups related to groups in the set $U_{64}$. As the reader can see from Table A1 we have been successful in finding several new complete groups. We have not been as fortunate in arriving at some simple structural properties for many of the groups in $U_{64}$ as we would have liked.

The numbers appearing after the order of an automorphism group in Table A1 are to be interpreted as follows. For example, for group #3, $147,456(2)$ means that the automorphism group of the group number 3 of order 64 has order 147,456 and has center of order 2; in other cases, e.g., #15, the $(1^2)$ means that the center for this group is $C_2 \times C_2, \ldots$. The presentations for many of the automorphism groups of the groups of order 64 are given in Appendix II. Complete presentations for many of the groups in the automorphism tower sequences listed in Table A1 are given in Appendix III. Those automorphism towers leading to complete groups were originally planned for inclusion in a future paper devoted to the complete groups.
arising in our studies of automorphism groups. They are included here for now for the sake of completeness.

The groups in U64 for the most part do not have a normal Sylow subgroup, which means that the structure of these groups takes the schematic form:

\[ \text{Aut} = (T_1 @ C_3) @ T_2, \]

where \( T_1 \) and \( T_2 \) are 2-groups. The automorphism groups of the groups \( T_1 \) can be very large, but usually they have relatively small odd-order Sylow subgroups; typically these groups have orders such as \( 3 \times 2^n \), where \( n \) is often a number in the range 10 to 20. It is therefore very easy to obtain a presentation for the group \( T_1@C_3 \). But to complete the sequence one needs to know how \( T_2 \) acts on the group \( T_1@C_3 \), and these groups often have very large Sylow 2 components which make finding the relevant 2-group action on the required normal subgroup very difficult. In most cases the group \( T_1@C_3 \) has a trivial center, and we have followed the automorphism towers for these groups to obtain new complete groups.

There are three groups in U64 that have normal Sylow 2-subgroups. These are the automorphism groups for the following groups of order 64: #76, #93 and #153.

The groups #3 and #14 of order 64 have the structure:

\[ \text{Aut} = (T_1 @ (C_3 \times C_3)) @ T_2. \]

The automorphism group for #187 of order 64 is more complicated and appears to be representable as:

\[ \text{Aut} = ((1^4 @ C_3) \times 1^4) @ C_4. \]

The first five cases and numbers 14, 103, 104 and 105 in Table A1 have not been looked at for this breakdown. Numbers 1 and 2 appear to be too large to do this with CAYLEY.
5. Tables 2a to 9

Table 2a

| Group of order 192 | Automorphism Group |
|-------------------|-------------------|
| 1. $A_4 \times 4^1$ | $S_4 \times GL(4,2)$ |
| 2. $1^4 @ C_3 \times 1^2$ | $S_4 \times [5760]$ |
| 3. $1^6 @ C_3$ | order is [23,224,320] |
| 4. $A_4 \times (2,1^2)$ | $S_4 \times \text{Aut}(C_4 \times C_2 \times C_2)$ |
| 5. $1^4 @ C_3 \times C_4$ | $C_2 \times [5760]$ |
| 6. $[C_4 \times C_4] @ C_3 \times 1^2$ | $S_4 \times [384]$ complete |
| 7. $A_4 \times C_4 \times C_4$ | $S_4 \times \text{Aut}(C_4 \times C_4)$ |
| 8. $[2^2,1^2] @ C_3$ | $[73728]$ complete |
| 9. $A_4 \times C_8 \times C_2$ | $S_4 \times D_4 \times C_2$ |
| 10. $[2^2] @ C_3 \times C_4$ | $C_2 \times [384]$ complete |
| 11. $A_4 \times C_16$ | $S_4 \times C_4 \times C_2$ |
| 12. $[C_8 \times C_8] @ C_3$ | $[6144]$ complete |
| 13. $A_4 \times D_4 \times C_2$ | $S_4 \times \text{Hol}(C_4 \times C_2)$ |
| 14. $SL(2,3) \times 1^3$ | $S_4 \times \text{Hol}(C_2 \times C_2 \times C_2)$ |
| 15. $A_4 \times Q_2 \times C_2$ | $S_4 \times \text{Aut}(Q_2 \times C_2)$ |
| 16. $[Q_2 \times 1^3] @ C_3$ | $C_2 \times [1152]$, ncl = 17, z = 1 |
| C2× Table 3a in [7] # 11 | Aut(1152) = 2304 [complete] |
| 17. $A_4 \times [C_4 YQ_2]$ | $S_4 \times S_4 \times C_2$ |
| 18. $[C_4 YQ_2] @ C_3 \times C_2 \times C_2$ | $S_4 \times \text{Aut}(C_3 \times C_2 \times C_2)$ |
| 19. $[C_4 YQ_2 \times 1^2] @ C_3$ | $C_2 \times [1152]$, ncl = 17, z = 1 |
| C2× Table 3a in [7] # 12 | Aut(1152) = 2304 [complete] |
| 20. $A_4 \times (4,4[2,2)$ | $S_4 \times [32 # 33]$ |
| 21. $A_4 \times <2,2[4,2>$ | $S_4 \times [32 # 33]$ |
| 22. $A_4 \times <2,2[2> | $S_4 \times D_4 \times C_2$ |
| 23. $SL(2,3) \times C_4 \times C_2$ | $S_4 \times [32 # 33]$ |
| 24. $[C_8 YQ_2] @ C_3 \times C_2$ | $S_4 \times D_4 \times C_2$ |
| C2× Table 3a in [7] # 13 | $C_2 \times [384]$ complete |
| 25. $C_2 \times [96 # 13]$ | $C_2 \times [384]$ complete |
| 26. $[(C_4 YQ_2) \times C_3] @ C_3$ | $S_4 \times [32 # 33]$ |
| 27. $[C_4 YQ_2] @ C_3 \times C_4$ | $S_4 \times [32 # 33]$ |
| 28. $[C_4 YQ_2] @ C_3 \times C_4$ | $C_2 \times [384]$ complete |
| 29. $[36] @ C_3$ | $S_4 \times C_2 \times C_2$ |
| 30. $A_4 \times D_8$ | $S_4 \times C_4 \times C_2$ |
| 31. $A_4 \times < -2.4[2>$ | $S_4 \times D_4 \times C_2$ |
| 32. $A_4 \times Q_8$ | $S_4 \times \text{Hol}(C_8)$ |
| Group of order 192 | Automorphism Group |
|-------------------|--------------------|
| 33. \([32 \# 33] \times C_2 \times C_2\) | \(C_2 \times [384]\) complete |
| \(C_2 \times \text{Table 3a in [7]} \# 14\) | |
| 34. \(C_2 \times ([32 \# 34] \times C_3)\) | \(C_2 \times [384]\) complete |
| \(C_2 \times \text{Table 3a in [7]} \# 15\) | |
| 35. \(C_2 \times ([32 \# 41] \times C_3)\) | \(C_2 \times [64 \# 184]\) @ C_3 |
| \(C_2 \times \text{Table 3a in [7]} \# 16\) | |
| 36. \(# 81\) | \(C_2 \times [384]\) complete |
| 37. \(# 82\) | \(C_2 \times [384]\) complete |
| 38. \(# 93\) | \(C_2 \times [64 \# 184]\) @ C_3 |
| 39. \([64 \# 103] \times C_3\) | \(S_4 \times [192], \ ncl = 14, \ z = 1\) |
| \(C_2 \times \text{Table 3a in [7]} \# 17\) | |
| 40. \([64 \# 103] \times C_3\) | \(C_2 \times [576], \ ncl = 16, \ z = 1\) |
| \(C_2 \times \text{Table 3a in [7]} \# 18\) | |
| 41. \([64 \# 104] \times C_3\) | \(S_4 \times \text{Hol}(C_4 \times C_2)\) |
| \(C_2 \times \text{Table 3a in [7]} \# 19\) | |
| 42. \([64 \# 105] \times C_3\) | \(C_2 \times [576], \ ncl = 16, \ z = 1\) |
| 43. \([64 \# 105] \times C_3\) | \(S_4 \times S_4 \times C_2\) |
| 44. \(# 107\) | \(S_4 \times [32 \# 33]\) |
| 45. \(# 108\) | \(S_4 \times [32 \# 33]\) |
| 46. \(# 109\) | \(S_4 \times D_4 \times C_2\) |
| 47. \(# 144\) | \(C_2 \times [384]\) complete |
| 48. \(# 145\) | \(C_2 \times [384]\) complete |
| 49. \(# 147\) | \(C_2 \times [384]\) complete |
| 50. \(# 148\) | \(C_2 \times [384]\) complete |
| 51. \(# 150\) | \(C_2 \times [64 \# 184] \times C_3, \ ncl = 17, \ z = 1\) |
| 52. \(# 153\) | \(C_2 \times [64 \# 184] \times C_3, \ ncl = 17, \ z = 1\) |
| 53. \(\text{SL}(2,3) \times D_4\) | \(S_4 \times [32 \# 33]\) |
| 54. \(\text{SL}(2,3) \times Q_2\) | \(S_4 \times [1^4 C_3] @ C_2, \ ncl = 1\) |
| 55. \([Q_2 \times Q_2] @ C_3\) | \(\text{order 576, ncl=16}\) |
| 56. \(# 158\) | \(S_4 \times [32 \# 33]\) |
| 57. \(# 159\) | \(S_4 \times 1^4\) |
| 58. \(# 162\) | \(S_4 \times [32 \# 33]\) |
| 59. \(# 173\) | \([384]\) complete |
| 60. \(# 181\) | \([384]\) complete |
| #   | Group Description                                                                 |
|-----|----------------------------------------------------------------------------------|
| 61. | # 183 a. [36864] complete                                                        |
| 62. | # 183 b. [576], ncl = 16, z = 1                                                |
| 63. | # 183 c. [1152], ncl = 17, Aut(1152) = [2304] complete                          |
| 64. | # 184 [384] complete                                                             |
| 65. | # 187 [61440] complete                                                           |
| 66. | # 241 $S_4 \times \text{Hol}(C_8)$                                              |
| 67. | # 242 $S_4 \times D_4 \times C_2$                                               |
| 68. | # 243 $S_4 \times \text{Hol}(C_8)$                                              |
| 69. | # 259 $C_2 \times \text{Aut}(Q_2 \times C_2)$                                   |
| 70. | # 260 $C_2 \times \text{Aut}(Q_2 \times C_2)$                                   |

Note that the numbers in the group 192 column refer to the sylow 2-subgroup; e.g., #64 means that the order 192 group is #64 @ $C_3$. 
Notes for Table 2a

The complete group [2304] is the same as that obtained from Aut[Aut([Q_2 x C_2 x C_2]@C_3)].

Presentations for nondirect product cases.

| #27. | [64#30] | a^4 = b^4 = c^4 = (b, c) * a^4 = (a, b) = (a, c) = 
| #29. | [64#36] | b^2 = c^2 = (c, b) * a^2 = (a, b) = (a, c) = 
| #33. | See table for order 96 for 32 # 33 @ C_3 factor. | 
| #34. | See Table 3a in [7]. | 
| #35. | See Table 3a in [7]. | 
| #36. | [64#81] | a^2 = b^2 = c^4 = ((a, c), a) = ((a, c), b) = ((a, c), c) = 
| #37. | [64#82] | a^2 = b^2 = c^4 = (b, c) * a^2 = (a, c) = (a, b) = 
| #38. | [64#93] | a^2 = b^4 = c^2 = (a, c) * a^2 * b^2 = (b, c) * a^2 = (a, b) = 
| #39. | [64#103] | a^2 = b^4 = c^2 = (c, b) * a^2 * b^2 = (b, c) * a^2 = (a, b) = 
| #40. | | d^3 = (a, d) = b^3 * c * b = c^3 * b = 
| #41. | [64#104] | d^2 = a * (b^2) = a^2 * (c^2) = c^2 = (c, b) * a^2 = 
| #42. | [64#105] | b^2 = c^2 = d^2 = c^2 = (d, c) * a^2 = (c, b) * a^2 = 
| #43. | [64#105] | ..... = a^2 * ((a * d)^{-1}) = b^2 * ((b * c)^{-1}) = c^2 * b = 
| #44. | [64#107] | a^2 * (b^2) = a^2 * (c^2) = d^2 = (c, b) * a^2 = 
| #45. | [64#108] | a^2 = b^2 * (c^{-2}) = d^2 = (c, b) * b^2 = (a, c) = 
| #46. | [64#109] | a^2 = b^2 = c^2 = (c, b) * b^2 = (d, a) * a^2 = 

\[ \text{AUT}(G) \text{ FOR } |G| = 64 \]
### Notes for Table 2a (continued)

The complete group $[2304]$ is the same as that obtained from $\text{Aut}([Q_2 \times C_2 \times C_2] @ C_2]$. Presentations for non-direct product cases.

| #47. [64#144] | $a^2 = b^2 = c^2 = (a, b)^2 = (a, c)^2 = (b, c)^2 = ((a, b), c) = ((a, c), b) = $  
| #48. [64#145] | $d^3 = a^4 * c * b * c = b^d * b * c * b = c^d * b * c * a * c * b = 1$;  
| #49. [64#147] | $a^2 \cdot (b^{-1}) = c^3 = (b, a) * a^2 = (c, a) * (b, c) * c^2 = ((b, c), b) = ((b, c), c) = d^3 = a^d * (b^{-1}) = b^d * b * (a^{-1}) = c^d * (c^{-1}) * b = 1$;  
| #50. [64#148] | Sag Wamsley error in 64 # 147. An alternate set of relations for this 192 group is:  
| #51. [64#150] | $a^4 = b^4 = c^3 = (a, b) * a^2 = (a, c) * c^4 = (b, c) * b^2 = (a, c) * a = (a, c) = (d, e) = 1$;  
| #52. [64#153] | $a^2 \ast b^2 \ast (c^{-2}) \ast (b, a) \ast (b, c) \ast (a^{-2}) = $  
| #53. [64#158] | $c^4 = a^2 \ast (b^{-2}) = a^2 \ast (d^{-2}) = (b, a) * a^2 = (c, d) * c^2 = (a, c) = (a, d) = (b, c) = (b, d) = e^3 = a^c * (b^{-1}) = b^e * b * (a^{-1}) = (c, e) = (d, c) = 1$;  
| #54. [64#159] | $c^2 = a^2 \ast (b^{-2}) = a^2 \ast (d^{-2}) = (b, a) * a^2 = (c, d) * c^2 = (a, c) = (a, d) = (b, c) = (b, d) = e^3 = a^c * (b^{-1}) = b^e * b * (a^{-1}) = (c, e) = (d, c) = 1$;  
| #55. [64#162] | $a^2 \ast (b^{-2}) = a^2 \ast c^2 \ast (d^{-2}) = (b, a) * a^2 = (d, c) * c^2 = (a, c) = (a, d) = (b, c) = (b, d) = e^3 = a^c * (b^{-1}) = b^e * b * (a^{-1}) = (c, e) = (d, c) = 1$;  
| #56. [64#173] | $a^4 = b^4 = c^3 \ast (d^{-2}) = (d, c) * c^3 = (a, d) * (c, b) = (a, b) = (a, c) = (b, d) = e^3 = a^c * b * a = b^e * a = c^e * a * (d^{-1}) * (c^{-1}) * a = d^e * c = 1$;  
| #57. [64#181] | $a^4 \ast (b^{-4}) = (b, a)^2 * a^4 = ((a, b), a) = ((a, b), b) = c^3 = a^c * b * a^2 = b^e * ((a^3 * b)^{-1}) = 1$;  


Notes for Table 2a (continued)

The complete group $|G| = 64$ is the same as that obtained from $\text{Aut}[(Q_2 \times C_2 \times C_2)@C_3]$.

Presentations for nondirect product cases.

| #61. [64#183] | $c^2 = d^2 = (c, a) * a^2 = (a, d) * (b^{-1})*a^2 = (b, c)*b^2*(a^{-2}) = (d, b)*b^2 = (a, b) = (c, d) = e^3 = a^c * a * (b^{-1}) = b^e * a = e^c * ((a^2 * c * d)^{-1}) = d^{-1} * b * (c^{-1}) * (b^{-1}) = 1; $ |
|---|---|
| #62. [64#183] | $..a^c * c * (b^{-1}) = b^e * d * (b^{-1})*a^1 = e^c * d * c * (a^{-2}) = d^e * c * (a^{-2}) = 1; $ |
| #63. [64#183] | $..a^c * d * (a^{-1}) = b^e * d * c * (b^{-1}) = e^c * d * c * (a^{-2}) = d^e * c * (b^{-2}) = 1; $ |
| #64. [64#184] | $c^2 = d^2 = (c, a) * a^2 = (d, a) * b^2 = (c, b) * b^2 = (b, d) * b^2 * (a^{-2}) = (a, b) = (c, d) = e^3 = a^c * (b^{-1}) = b^e * b * a = e^c * c * (a^{-2}) = d^e * a * d * (a^{-1}) = 1; $ |
| #65. [64#187] | $a^2 * b^2 * (d^{-2}) = b^2 * (e^{-2}) = (c, a) * a^2 = (a, d) * (b^{-2})*a^2 = (b, c) * b^2 * (a^{-2}) = (d, b) * b^2 = (a, b) = (c, d) = e^3 = a^e * b = b^e * b * (a^{-1}) = e^d * (d^{-1}) * (a^{-2}) = d^e * c * (d^{-1}) = 1; $ |
| #66. [64#241] | $a^2 * (b^{-1}) = a^2 * (c^{-2}) = a^2 * (e^{-2}) = (d, a) * a^2 = (c, b) * a^2 = (a, b) = (c, d) = e^3 = a^e * c = b^e * (c^{-1}) * (b^{-1}) = e^c * b = (d, c) = 1; $ |
| #67. [64#242] | $a^2 * (b^{-1}) = a^2 * (c^{-2}) = a^2 * (e^{-2}) = (d, a) * a^2 = (c, b) * a^2 = (a, b) = (d, c) = e^3 = a^e * c = b^e * (c^{-1}) * (b^{-1}) = e^c * b = (d, c) = 1; $ |
| #68. [64#243] | $a^2 * (b^{-1}) = a^2 * (c^{-2}) = a^2 * (e^{-2}) = (d, a) * a^2 = (c, b) * a^2 = (a, b) = (d, c) = e^3 = a^e * c = b^e * (c^{-1}) * (b^{-1}) = e^c * b = (d, c) = 1; $ |
| #69. [64#259] | The relations for order 64 #259 as given by Sag and Wamsley appear to be incorrect. An alternate set of relations for this group is: $ (b, c) = (b, d) = (c, d) = (b, e) = (c, f) = (e, f) = (c, e) * a = (d, e) * b = (b, f) * a = (d, f) * c = (c, b) * a^4 = (a, b) = (c, d) = e^3 = (a, e) = b^e * (c^{-1}) * (b^{-1}) = c^e * b = (d, e) = 1; $ |
| #70. [64#260] | $b^2 = c^2 = b^2 = (b, c) = ((a, b), b) = ((a, c), c) = ((a, c), b) * (a^{-2}) = d^3 = (a, d) = b^d * c = c^d * b * c = 1; $ |
Table 2b

| Group of order 64 | Automorphism Group |
|------------------|-------------------|
| $p = 5$ (5 cases) |                   |
| 1. $[5]^1 @ C_5 \times 1^4$ | $S_3 \times [5760]$ complete |
| 2. $[5]^1 @ C_5 \times C_4$ | $C_2 \times [5760]$ complete |
| 3. # 104 | $C_2 \times [320]$ complete |
| 4. # 105 | $C_2 \times [320]$ complete |
| 5. # 187 | order 960 complete |
| $ncl = 13, z = 1$ |                   |
| actions for #104: | $f^o = a^f \cdot (b \cdot d \cdot e)^{-1} = b^f \cdot (a \cdot c \cdot d)^{-1}$ |
| $= c^f \cdot b \cdot e = d^f \cdot (d \cdot a)^{-1} = e, f = 1$ |                   |
| actions for #105: | $f^o = (a, f) = b^f \cdot e, c^f \cdot e \cdot d =$ |
| $= d^f \cdot c \cdot d \cdot c \cdot a = e, f = 1$ |                   |
| actions for #187: | $e^5 = a^e \cdot (a \cdot b \cdot d)^{-1} = b^e \cdot d \cdot c \cdot a^{-1}$ |
| $= e^c \cdot (a \cdot c)^{-1} = d^c \cdot (b \cdot d)^{-1} = 1$ |                   |
| $p = 7$ (9 cases) |                   |
| 1. $[56] \times 1^4$ | $GL(2,3) \times [168]$ |
| 2. $[6]^1 @ C_7$ | $[677376]$ complete |
| 3. $[56] \times C_4 \times C_2$ | $S_4 \times [168]$ complete |
| 4. $[56] \times C_8$ | $C_2 \times C_2 \times [168]$ complete |
| 5. $[2^3] @ C_7$ | $[10752]$ complete |
| 6. $[56] \times D_4$ | $S_4 \times [168]$ complete |
| 7. $[56] \times Q_2$ | $S_4 \times [168]$ complete |
| 8. # 153 | $[1344]$ complete |
| 9. $[1^6] @ C_7$ | $[18816]$ complete |
| actions for $2^3$: | $d^7 = a^d \cdot c \cdot (a \cdot b)^{-1} = b^d \cdot (a \cdot b^2)^{-1} =$ |
| $= e^d \cdot c \cdot a = 1$ |                   |
| actions for #153: | $d^7 = a^d \cdot b^{-1} = b^d \cdot c \cdot b \cdot a^{-1}$ |
| $e^d \cdot c^{-1} \cdot b^{-1} = 1$ |                   |
| $p = 31$ (1 case) |                   |
| 1. $1^6 @ C_{31} \times C_2$ | same as Aut$(1^6 @ C_{31})$ |

Table 3

| Group # | order of sylow $p$-subgroup of Aut$(G)$ | # of 64p groups |
|---------|----------------------------------------|-----------------|
| #1 GL(6, 2) | $3^4$ and $7^2,$ all others are $C_p$ | 3, 3 |
| #2 (2, 1^4) | $3^2$, all others are $C_p$ | 2 |
| (2^2, 1^2) | $3^2$ | 3 |
| $Q_8 \times 1^3$ | $3^2$ | 3 |
| $C_5 Y Q_2 \times 1^2$ | $3^2$ | 3 |
| # 103 | $3^2$ | 2 |
| #105 | $3^2$ | 2 |
| #183 | $3^2$ | 3 |
Table 4 April 11, 1996
Groups of order 192 without a normal sylow subgroup and their automorphism groups
split extensions

| A. Groups of the form $A_4 \rtimes \langle y \rangle$ (order 16 group) (28 cases) |
|---|
| $[A_4: x^2 = y^3 = (y \circ x)^3 = 1.$ |
| $C_2$ actions: $(g, x) = y^g \circ y = 1]$ |

There are 28 such cases and they take the same form as the groups of order $16p$. In these cases, however, instead of acting on a group $C_p$, the action is by the same element of order 2, and the automorphism group is $S_4 \times$ the invariant factor listed in Table 2a in [6].

In this set of groups the following direct products of groups of order 96 without a normal sylow 2-subgroup with $C_2$ can be found:

#’s 20 to 26 of Table 3b of [7].
B. Groups of the form \( \text{SL}(2, 3) \oplus \) [group of order 8] (13 cases)

| Actions: \( \text{SL}(2, 3) \): | \( a^3 = a \ast b \ast a \ast ((b \ast a \ast b)^{-1}) = 1 \); |
| --- | --- |
| \( C_2 \): | \( a^2 \ast a \ast b \ast (a^{-1}) = b^2 \ast b = 1 \); |
| \( C_3 \): | \( a^3 \ast b = b^3 \ast b \ast a \ast (b^{-1}) = 1 \); |
| \( C_2 \times C_2 \): | \( a^2 \ast a \ast (b^{-1}) \ast (a^{-1}) = b^2 \ast b \ast (a^{-1}) \ast (b^{-1}) = a^b \ast a \ast b \ast (a^{-1}) = b^b \ast b = 1; \) |
| \( D_4 \): | \( c^4 = d^2 = (d \ast (c^{-1}))^2 = a^c \ast b = b^c \ast b \ast a \ast (b^{-1}) = a^d \ast (b^{-1}) = b^d \ast (a^{-1}) = 1; \) |

There are seven cases in which the groups of order eight will yield groups of order 192, by means of a \( C_2 \) action on \( \text{SL}(2, 3) \). The action here is the same action as that giving \( \text{GL}(2, 3) \). There are additional groups in this area arising from actions of the forms \( \text{C}_2 \times \text{C}_2 \), \( \text{C}_4 \) and \( D_4 \). One should observe that the rule for getting the automorphism groups when the two-group acts as an operator of order 2 on \( C_j \) or \( A_4 \) does not work here; i.e., \( \text{Aut} \neq \text{Aut}(\text{SL}(2, 3)) \times \text{order} \ 8 \ \text{group} \ \text{invariant} \)

| #29. | \( \text{SL}(2, 3) \oplus C_8 \) | \( S_4 \times 1^4 \) |
| #30. | \( \text{SL}(2, 3) \oplus C_2 \times C_4 = \text{GL}(2, 3) \times C_4 \) | \( S_4 \times 1^4 \) |
| #31. | \( \text{SL}(2, 3) \oplus C_4 \times C_2 \) | \( \text{Table 3b in \#28} \times C_2 \) | \( S_4 \times \#33 \ \text{order} \ 32 \) |
| #32. | \( \text{SL}(2, 3) \oplus C_2 \times 1^2 = \text{GL}(2, 3) \times 1^2 \) | \( \text{Table 3b in \#27} \times C_2 \) | \( S_4 \times \#62 \ \text{order} \ 192 \) |
| #33. | \( \text{SL}(2, 3) \oplus D_4 \ [C_4 \ \text{acts as} \ C_2] \) | \( S_4 \times 1^4 \) |
| #34. | \( \text{SL}(2, 3) \oplus D_4 \ [C_2 \ \text{acts as} \ C_2] \) | \( S_4 \times \#33 \ \text{order} \ 32 \) |
| #35. | \( \text{SL}(2, 3) \oplus Q_2 \) | \( S_4 \times \#33 \ \text{order} \ 32 \) |
| #36. | \( \text{SL}(2, 3) \oplus C_4 \times C_2 \ [\text{full} \ C_4 \ \text{action}] \) | \( \text{Table 3b in \#29} \times C_2 \) | \( S_4 \times D_4 \times C_2 \) |
| #37. | \( \text{SL}(2, 3) \oplus C_4 \times C_2 \ [C_2 \times C_2 \ \text{action}] \) | \( \text{here} \ c^4 \ \text{acts as} \ [x] \ \text{above} \) | \( S_4 \times 1^4 \) |
| #38. | \( \text{SL}(2, 3) \oplus C_4 \times C_2 \ [C_2 \times C_2 \ \text{action}] \) | \( \text{here} \ c^4 \ \text{acts as} \ [y] \ \text{above} \) | \( S_4 \times \#33 \ \text{order} \ 32 \) |
| #39. | \( \text{SL}(2, 3) \oplus \text{II}^4 \ [C_2 \times C_2 \ \text{action}] \) | \( \text{same as a} \ C_1 \text{YQ}_2 \text{@} \ C_3 \text{case below} \) | \( S_4 \times \text{Hol}(C_4 \times C_2) \) |
| #40. | \( \text{SL}(2, 3) \oplus D_4 \ [C_2 \times C_2 \ \text{action}] \) | \( \text{here} \ \text{order} \ 4 \ \text{element} \ \text{acts as} \ [y] \) | \( S_4 \times 1^4 \) |
| #41. | \( \text{SL}(2, 3) \oplus D_4 \ [D_4 \ \text{action}] \) | | \( S_4 \times D_4 \times C_2 \) |
Table 4 (continued)

B. Groups of the form \( SL(2,3) \) @ [group of order 8] (13 cases)

\( SL(2,3) \): \( a^3 = a * b * a * ((b * a * b)^{-1}) = 1; \)

The other cases yield only groups with a normal sylow 2-subgroup.

|   |   |
|---|---|
| 1. | The \( C_2 \times C_2 \) image yields one of the missing cases: #950 (note added fall 2007). |
| 2. | One of the \( C_2 \times C_2 \) images from \( D_4 \) appears to give the same group as a \( C_2 \) case from \( D_4 \), #34 above. |
| 3. | The \( C_4 \) action from \( C_8 \) appears to give a duplication of the \( C_2 \) image case from \( C_8 \), i.e., #29 above. |

C. Groups from \( C_4 \times C_4 \) by a group of order 4 (4 cases)

\( a^4 = b^4 = (a, b) = c^3 = a^c * (b^{-1}) * (a^{-1}) = b^c * b^2 * (a^{-1}) = 1; \)

|   |   |
|---|---|
|   |   |

The following relations for the order 384 complete group, appearing as a factor in #’s 44 and 45 above are:

\( a^6 = c^2 = (b, c) = b^4 * c = (a^2 * b)^2 = a^2 * c * (a^{-1}) * c * (a^{-1}) * c = (a * (b^{-1}))^4 = (a * b)^2 * (a^{-1}) * (b^{-1}) * (a^{-1}) * b = a^2 * c * (b^{-1}) * a * (b^{-2}) * a * (b^{-1}) = 1; \)

This complete group of order 384 appears as a direct factor in, or as the automorphism group of many other groups of order 192.

This is group number (384, #5677) in the Small Group Library.
Table 4 (continued) June 21, 1994

D. Groups from $[C_4 Y Q_2] @ C_3$ (there are 6 cases here)

\[ a^3 = a \ast b \ast a \ast ((b \ast a \ast b)^{-1}) = c^2 = \]
\[ a^c \ast b \ast (a^{-1}) \ast (b^{-1}) = b^c \ast a \ast (b^{-1}) \ast (a^{-1}) = 1; \]

| Actions: |
|---------------------------------|
| $C_2$: |
| 1. $a^c \ast b \ast a \ast (b^{-1}) = b^c \ast b = c^2 \ast ((a \ast b \ast c \ast a)^{-1}) = 1$ |
| 2. same = same = $(a, c) = 1$ |
| $C_4$: |
| 1. $a^c \ast a \ast b \ast (a^{-1}) = b^c \ast b \ast a \ast (b^{-1}) = c^2 \ast b \ast (c^{-1}) \ast (a^{-1}) = 1$ |
| 2. same = same = $c^2 \ast (a^{-1}) \ast (c^{-1}) \ast a = 1$ |

| $C_2 \times C_2$: |
|---------------------------------|
| 1. $a^c \ast b \ast a \ast (b^{-1}) = b^c \ast b \ast a \ast (b^{-1}) = c^2 \ast ((a \ast b \ast c \ast a)^{-1}) = (a, y) = (b, y) = c^y \ast ((a \ast b \ast c \ast a)^{-1}) = 1$ |
| 2. $a^c \ast (b^{-1}) = b^c \ast (a^{-1}) = (c, x) = a^y \ast b = b^y \ast a = c^y \ast (a^{-1}) \ast (c^{-1}) \ast a = 1$ |
| 3. $a^c \ast b \ast a \ast (b^{-1}) = b^c \ast b = c^2 \ast ((a \ast b \ast c \ast a)^{-1}) = a^y \ast a = b^y \ast a \ast b \ast (a^{-1}) = [c^y \ast ((a \ast b \ast c \ast a)^{-1})] = 1$ |

| $C_2$ actions (4 cases) |
|---------------------------------|
| #46. $[C_4 Y Q_2] @ C_3 @ C_4$ | $C_2$ | action #1 | $S_4 \times [32#33]$ |
| $[(C_4 Y Q_2) @ C_3] @ C_4$ | $C_2$ | action #2 | $S_4 \times [32#33]$ |
| same as #37 | |

| #47. $[C_4 Y Q_2] @ C_3 @ C_2 \times C_2$ | $C_2$ | action #1 | $S_4 \times [32#33]$ |
| (table 3b in [2] # 32) \times C_2 | |
| $[(C_4 Y Q_2) @ C_3] @ C_2 \times C_2$ | $C_2 \times C_2$ | action #2 | $S_4 \times \text{Hol}(C_4 \times C_2)$ |
| same as $\text{SL}(2, 3) @ [C_2 \times C_2] \times C_2$ |
| # 39 above. | |

$C_4$ actions (2 cases)

| #48. $[(C_4 Y Q_2) @ C_3] @ C_4$ | $C_2$ | action #1 | $S_4 \times D_4 \times C_2$ |
| same as #36 above | |
| SL$(2, 3) @ C_2 \times C_2$ | |
| (Table 3b in [2] # 29) \times C_2 | |

$C_2 \times C_2$ actions (3 cases)

| #49. $[(C_4 Y Q_2) @ C_3] @ (C_2 \times C_2)$ | $C_2$ | action #1 | $S_4 \times D_4 \times C_2$ |
| #50. $[(C_4 Y Q_2) @ C_3] @ (C_2 \times C_2)$ | $C_2$ | action #2 | $S_4 \times S_4 \times C_2$ |
| #51. $[(C_4 Y Q_2) @ C_3] @ (C_2 \times C_2)$ | $C_2$ | action #3 | $S_4 \times D_4 \times C_2$ |
Table 4 (continued) June 21, 1994

E. Groups from \([C_2 \times C_2 \times C_2 \times C_2]@C_3\) (four cases here)

Relations for 48 group:

\[a^2 = b^2 = c^2 = d^2 = (a,b) = (a,c) = c^3 = a^e * b = b^e * a * b = e^d * d = e^f * c * d = 1;\]

\[a^f * c = b^f * c * d = c^f * a = d^f * a * b = e^f * e = 1;\]

or

\[(a, f) = b^f * b * a = (c, f) = d^f * d * a = e^f * e = 1;\]

\[C_2 \times C_2 \text{ actions:}\]

\[x^2 = y^2 = (x, y) =\]

1. \((a, x) = b^x * b * a = (c, x) = d^x * d * c = e^x * e = (a, y) = b^y * b * a = (c, y) = d^y * d * c = e^y * e = 1\]

2. \((a, x) = b^x * b * a = (c, x) = d^x * d * c = e^x * e = (a, y) = b^y * b * a = c^y * c * a = d^y * d * c * b * a = e^y * e = 1;\]

\[C_4 \text{ actions:}\]

1. \((a, f) = b^f * a * b = (c, f) = d^f * c * d = e^f * e * a = 1;\]

2. \((a, f) = b^f * a * b = c^f * d * b = d^f * c * a = e^f * e = 1;\]

| #52. | \([1^4@C_3]@C_2\) × \(C_2\) \(C_2\) action | \(C_2 \times [576]\) \text{Aut}(576)=S_4 \rtimes C_2\) |
| #53. | \([1^4@C_3]@C_4\) \(C_2\) action | \(C_2 \times [576]\) \text{Aut}(576)=S_4 \rtimes C_2\) |
| #54. | \([1^4@C_3]@((C_2 \times C_2) [1]) \text{ action}\) | 384 complete |
| #55. | \([1^4@C_3]@C_4\) \(C_4\) [1] action | \(C_2 \times [576]\) |
Table 4 (continued) June 21, 1994

F. Groups from the groups of order 96.

16 cases here, not all different.

| Order 96 group is [Table 3a in [7] # 15] | 384 complete |
|---|---|
| $C_4 \times Q_2 \wr C_3$ (four cases) | |
| Relations for the order 96 group. [Table 3a in [7] # 12] | |
| $b^3 = (a^{-3}) = b * a * b * ((a * b * a)^{-1}) = b * a^2 * b * a^5 = 1$; | |
| $c^2 = a * c^2 * (b * (b^{-1}) = b^c * a * (b^{-1}) * a = 1; S_4 \times D_4 \times C_2$ | |
| $c^2 = a^c * ((a * b^2 * a^{-2})^{-1}) = b^c * ((a^3 * b^2)^{-1} = 1; S_4 \times C_2 \times C_2 \times C_2$ | |
| $c^2 = a^c * ((a^5 * b * a)^{-1}) = b^c * ((a^2 * b * a^2)^{-1} = 1; S_4 \times C_2 \times C_2 \times C_2$ | |
| $c^2 = a^c * a * b * (a^{-1}) = b^c * b = 1; S_4 \times Hol(C_6)$ | |
| $([C_4 \times C_2] \wr C_3) \wr C_3$ (one case) | $C_2 \times 24 [192]$ |
| Order 96 group is [Table 3a in [7] # 13] | 384 complete |
| $a^3 = b^3 = c^3 = (b * (a^{-1}))^2 = c * b * (c^{-1}) * (a^{-1}) * c^{-1} * b = c * (a^{-1}) * b * (c^{-1}) * a * (b^{-1}) = d^2 = a^d * a * c * (a^{-1}) = b^d * b * c * (b^{-1}) = c^d * c * b * (a^{-1}) = 1$; | |
| $([C_2 \times C_2] \wr C_2) \wr C_3$ (one case) | 384 complete |
| Order 96 group is [Table 3a in [7] # 14] | |
| $a^2 = b^2 = c^2 = (a, b) = ((a, c), a) = ((b, c), b) = c^2 = (c * b) * a = c^e * a * c * a = d^e * b * d = 1$; An alternate presentation for this group is: | |
| $a^6 = b^6 = c^2 = (b * a)^2 = (b * (a^{-1}))^3 = b^2 * a * (b^{-1}) * (a^{-2}) * (b^{-1}) * a = a^c * ((a * b^3 * a)^{-1}) = b^c * ((b * a^3 * b)^{-1}) = 1; This group is Aut(Q_2 \times C_2).$ | |
| Same as # 54 above. | |
| $[Dih(C_4 \times C_3)] \wr C_3$ one case | 384 complete |
| Order 96 group is [Table 3a in [7] # 15] | |
| $a^4 = b^4 = c^2 = (a, b) = ((a, c), a) = (b, c) * b^2 = d^3 = a^d * b * (a^{-1}) = b^d * a * (b^{-2}) = (c, d) = c^2 = (a, e) = b^e * b * a = (c, e) = d^e * d = 1$; Same as # 44 above. | |
### Table 4. (continued) June 21, 1994

F. Groups from the groups of order 96.

16 cases here, not all different.

| # | Description | Presentation | Comments |
|---|---|---|---|
| #61. | order 96 group is that of $[\text{Table 3a in } \[7\] \# 18]$ (second 96 extension) | \[
\begin{align*}
\text{Relations for order 96 group.} \\
& f^2 = a^f * a = b^f * (c^{-1}) = c^f * (b^{-1}) = d^f * d * (a^{-1}) = e^f * e = 1; \\
& \text{same as # 41} \\
& \text{SL}(2,3)@D_4 \text{ (full } D_4 \text{ action) }
\end{align*}
\] | two cases |
| #62. | order 96 group is that of $[\text{Table 3a in } \[7\] \# 17]$ (first 96 extension) | \[
\begin{align*}
\text{Relations for order 96 group.} \\
& b^2 = a^b * (b^{-2}) = a^2 * c * (a^{-1}) * c = a * a^2 * c * (c^{-1}) = \\
& a * b * a * (b^{-1}) * a * (b^{-1}) = a * (a^{-1}) * b * c * (a^{-1}) * (b^{-1}) = \\
& b * c * (b^{-1}) * c * (b^{-1}) * c = 1.
\end{align*}
\] | $S_4 \times [576]$ same as in #52 and #53 |
| #63. | order 96 group is $[\text{Table 3a in } \[7\] \# 16]$ | \[
\begin{align*}
\text{Relations for order 96 group.} \\
& d^2 = a^d * ((b * a * c)^{-1}) = b^d * (a^{-1}) * c = \\
& e^d * ((a * b * c)^{-1}) = 1. \\
& \text{This group is the same as } \text{Aut}(C_4 \times C_2 \times C_2).
\end{align*}
\] | $C_2 \times [192]$ |
Table 4. (continued) June 21, 1994

F. Groups from the groups of order 96. (continued)

| Relations for order 96 group. | two cases |
|-------------------------------|----------|
| \(c^2 = c \ast b \ast (a^{-1}) \ast b \ast c \ast (a^{-1}) =\) |          |
| \(c \ast (a^{-1}) \ast b \ast c \ast (b^{-1}) \ast a = b^3 \ast (a^{-3}) =\) |          |
| \(b \ast a \ast b \ast ((a \ast b \ast a)^{-1}) = (b \ast a^2)^2 = 1;\) |          |
| \(d^2 = a^2 \ast a \ast b \ast (a^{-1}) = b^d \ast b =\) |          |
| \(c^d \ast ((a^2 \ast c \ast a)^{-1}) = 1;\) |          |
| \(d^d = a^2 \ast (c^{-1}) \ast a \ast (c^{-1}) =\) |          |
| \(b^d \ast b^2 \ast (a^{-1}) = (c, d) = 1;\) |          |
| looks like \#49. |          |
| \(d^2 = a^d \ast a^d \ast (b^{-1}) = b^d \ast (c^{-1}) \ast b \ast c =\) |          |
| \(c^d \ast a \ast (c^{-1}) \ast (a^{-1}) = 1;\) |          |

G. Groups coming from the groups \(GL(2, 3)\) and \(<2, 3, 4>\).

Note that most of the cases arising here are duplicates of previous listings. These alternate presentations, which might be of interest to the reader, are given in Tables A2 and A3.

Only the new cases of groups without a normal sylow subgroup are given here.

| relations for \(GL(2, 3)\): |          |
|-------------------------------|----------|
| \(b \ast a \ast b \ast ((a \ast b \ast a)^{-1}) = (b^d \ast (a^{-1})^d = 1;\) |          |
| \(C_2\) actions |          |
| a. From \(C_2 \times C_2\), all direct products: |          |
| \([GL(2, 3) \ast C_2] \times C_2\) |          |
| action \(a^2 \ast a^3 = b^d \ast b^3 = 1\) |          |
| \(C_2 \times \) ([96] Table 3b in [7] \# 35) |          |
| relations for \(<2, 3, 4>\): |          |
| \(b \ast a \ast b((a \ast b \ast a)^{-1}) = b \ast a^2 \ast b \ast (a^{-2}) = 1;\) |          |

Direct Product Cases.

| S4 \#62[192] |          |
| <2, 3, 4> \times C_2 \times C_2 |          |
| S4 \times Hol(C_4 \times C_2) |          |
| C_2 \times (Table 3b in [7] \# 34) |          |
| S4 \times 1^4 |          |
| S4 \times 1^4 |          |
| b. From \(C_4\) case. |          |
| <2, 3, 4> \ast C_4 |          |
| action \(a^2 \ast a^3 = b^2 \ast b^3 = 1\) |          |
| #   | Group                           | Description                                                                 |
|-----|---------------------------------|-----------------------------------------------------------------------------|
| 71  | $\text{SL}(2,3) \rtimes C_4 \rtimes S_4$ | Replace $c^2$ in case #33 above with $e^{16} = e^8 * (a * (b^{-1}))^2 = 1$; |
|     | $\text{SL}(2,3) \rtimes (C_4 \times C_2) : C_2$ acting: | $S_4 \times 1^4$ replace $d^2$ in case number 30 with $d^4 = d^2 * (a * (b^{-1}))^2 = 1$; This is the group $< 2, 3, 4 > \times C_2$, same as #69 in Table 4. |
| 72  | $\text{SL}(2,3) \rtimes D_4$: we have five new cases here. | $1^4 \times S_4$ $c^4 = d^4 = c^d * c = d^2 * (a * (b^{-1}))^2 = 1$; $c$ acting as $C_2$ on $\text{SL}(2,3)$ |

| 73  | $(44|22)$ | $1^4 \times S_4$ $c^4 = d^2 = c^2 = (c, d) = (d, e) = c^c * (c^{-1}) * d = (a, d) = (a, e) = (b, d) = e * (a * b^{-1})^2 = ...$; $c$ acts as $C_2$ on $\text{SL}(2,3)$. |
| 74  | $e^8 = d^4 = c^d * (d^{-2}) = c^d * c = d^2 * (a * (b^{-1}))^2 = 1$; | $Q_4$ type with $c$ acting on $\text{SL}(2,3)$ |
| 75  | $Q_4$ type with $d$ acting on $\text{SL}(2,3)$ | $C_2 \times D_4 \times S_4$ $S_4 \times \text{Hol}(C_8)$ |
| 76  | $c^4 = d^4 = a^d * (a^{-3}) = d^2 * (a * (b^{-1}))^2 = 1$; | $C_2 \times D_4 \times S_4$ This can also be written as $[(D_4 \rtimes Q_2) @ C_3] @ C_2$. |

#71.
$\text{SL}(2,3) \rtimes C_8$: $C_2 \times C_4 \times S_4$

replace $c^2$ in case #35 above with $e^{16} = e^8 * (a * (b^{-1}))^2 = 1$;
### Table 5
Groups of order 192 without a normal Sylow subgroup and their automorphism groups:

#### nonsplit extensions

| #77 * | $\text{SL}(2, 3) @ Q_2$ | $1^4 \times S_4$ |
|-------|---------------------|-----------------|
| $c^4 = d^4 = e^4$ | $c^4 * c = e^4 * d^4 * (a * (b^{-1}))^2 = 1$; | both $c$ and $d$ act as $C_2$ on $\text{SL}(2, 3)$ |

**C₂ × C₂** type extensions:

| #78 | $\text{SL}(2, 3) @ (C_4 \times C_2)$ | $S_4 \times \#33[32]$ |
|------|---------------------|-----------------|
| $c^4 = d^4 = e^4$ | $c^4 (a * (b^{-1}))^2 = 1$; | $c$ acts like $x$ and $d$ like $y$. |

| #79 | $\text{SL}(2, 3) @ (C_4 \times C_2) \times C_2$ | $S_4 \times \text{Hol}(C_4 \times C_2)$ |
|------|---------------------|-----------------|
| $c^4 = d^4 = e^4 = (c, d) = (c, e) = (d, e) = c^2 * (a * (b^{-1}))^2 = 1$; | $c$ acts as $y$, $d$ acts as $x$. |

#### direct product

[2, 3, 4] @ $C_2 \times C_2$, same as #68 in Table 4.

### 2. Groups from the groups of order 96.

| #79 | ($C_4 \times C_2) @ C_3$ one case | $C_2 \times \text{Aut}(Q_2 \times C_2)$ |
|------|---------------------------------|-------------------------------------|
| Order 96 group is Table 3a in [7], # 13 | | |
| $a^3 = b^3 = c^3 = (b * (a^{-1}))^2$ | $c * b = (c^{-1}) * (a^{-1}) * (c^{-1}) * b = c * (a^{-1}) * b * (c^{-1}) * a * (b^{-1}) = d^2 =$ | $a^d * a * c * (a^{-1}) = b^d * b * c * (b^{-1}) =$ |
| $c^d * c * b * (a^{-1}) = 1$; | add $d^4 = d^2 * (a * (c^{-1}) * b * (c^{-1}) = 1$; |

*relations for group number 77

| $a^3 = a * b * a * ((b * a * b)^{-1}) = c^4 = d^4 = c^d * c$ | $a^e * a * b * (a^{-1}) = a^d * a * b * (a^{-1})$ | $b^d * b = c^2 * d^2 * (a * (b^{-1}))^2 = 1$ |

---

In this table, we list the groups of order 192 without a normal Sylow subgroup and their automorphism groups. The table includes both nonsplit extensions and $C_2 \times C_2$ type extensions. Each entry provides the group structure, its automorphism group, and any additional notes regarding the action of elements on the group. The table also includes groups derived from the groups of order 96, with specific relations and actions noted for clarity.
Table 5 (continued)

Groups of order 192 without a normal sylow subgroup and their automorphism groups:
nonsplit extensions

| Group | Relations for order 96 group. |
|-------|-------------------------------|
| \([Q_2YQ_2]@C_3\) [Table 3a in [7] # 18] | \(b^4 = a^3 * (b^{-2}) = a^2 * c * (a^{-1}) * c = a * c^2 * a * (c^{-1}) = a * b * a * (b^{-1}) * a * (b^{-1}) = b * c * (b^{-1}) * c * (b^{-1}) * c = 1\) |

\#80.

\(d^4 = d^2 * a^2\)

\(a^d * ((b * a * c)^{-1}) = b^d * (a^{-1}) * c = c^d * ((a * b * c)^{-1}) = 1\)

\(C_2 \times [576]\)

ncl = 16

\#81.

\(d^2 = a^d * ((c * a * b)^{-1}) = b^d * (a^{-1}) * (b^{-1}) * a = c^d * (b^{-1}) * c = 1;\)

\(C_2 \times \text{Aut}(Q_2 \times C_2)\)

\([Q_2YD_4]@C_3\) Table 3a in [7] # 19

| Relations for order 96 group. |
|-----------------------------|
| \(c^2 = c * b * (a^{-1}) * b * c * (a^{-1}) = c * (a^{-1}) * b * c * (b^{-1}) * a = b^3 * (a^{-3}) = b * a * b * ((a * b * a)^{-1}) = (b * a^2)^2 = 1;\) |

\(d^2 = a^d * a * b * (a^{-1}) = b^d * b = c^d * ((a^2 * c * a)^{-1}) = 1;\)

\(S_4 \times D_4 \times C_2\)

same as group \#76 above.

Table 6

Centers of groups of orders 24, 48, and 96 and nonsplit extensions

| order | group | generator for center of group | \(Z(G)\) | # of groups | group |
|-------|-------|-------------------------------|---------|------------|-------|
| 24    | SL(2, 3) | \(< (a * (b^{-1}))^2 >\) | \(C_2\) | 8          | \(a\) |
| 48    | \((C_4YQ_2)@C_3\) | \(< b * (a^{-1}) * c >\) | \(C_4\) | *          |       |
| 96    | \((Q_2 \times C_2 \times C_2)@C_3\) \(C_8YQ_2)@C_3\) \([32\#18]@C_3\) \((Q_2YQ_2)@C_3\) \((D_4YQ_2)@C_3\) | \(< a^2 \) or \(< b^2 \) \(< a^3 \) \(< a \) \(< a^2 \) \(< a \) \(< a \) \(< a \) \(< a \) \(< a \) \(< a \) \(< a \) \(< a \) \(< a \) \(< a \) \(< a \) \(< a \) | \(C_2\) \(C_8\) \(C_2\) \(C_2\) \(C_2\) \(C_2\) | 1489 | 79 | 80,81 |
Notes for Table 6

Note that some of the SL(2, 3) cases are just direct products, e.g., $<2, 3, 4> \times C_4$ and $<2, 3, 4> \times C_2 \times C_2$.

a. Note that all of the $<2, 3, 4>$ extensions could also be nonsplit SL(2, 3) extensions. These have not been counted here as SL(2, 3) cases. See table below.

* All of the groups found here appear to be duplicates of the SL(2, 3) extension by $C_8$.

** The four cases here have the same order structure and automorphism group as the split extension $A_4 \oplus C_16$.

*** This case also arises as an extension from SL(2, 3)$\oplus D_4$ ($<-2, 4|2>$ case).

Table 7

| 2-group | action | split extension | nonsplit extension | # |
|---------|--------|----------------|-------------------|---|
| $C_8$   | $C_2$  | #29            | $C_16$            | 71 |
| $C_4 \times C_2$ | $C_2[C_4]$ | #31          | $C_4 \times C_4$ | 69 |
|         | $C_4[C_4]$ | #36          |                   |   |
|         | $C_2[C_2]$ | #30 GL(2, 3) |                   |   |
|         | $\times C_4$ | #37          |                   |   |
|         | $C_2 \times C_2$ | #39          |                   |   |
|         | $\times C_2$ | #32 GL(2, 3) |                   |   |
| $C_2 \times C_2 \times C_2$ | $C_2$ | #32 GL(2, 3) | $<2, 3, 4> \times C_2 \times C_2$ | 67 |
|         | $C_2 \times C_2$ | #39          |                   |   |
| $D_4$   | $C_4[C_2]$ | #33          | $C_4 \oplus C_2[C_4$ | 72 |
|         | #33      |               | $(4, 4|2, 2)$      | 73 |
|         | #33      |               | $Q_4[c]$          | 74 |
|         | #34      |               | $Q_4[d]$          | 75 |
|         | #40      |               |                   |   |
|         | #41      |               | $<-2, 4|2>$        | 76 |
| $Q_2$   | $C_2$   | #35            | $(C_4 \oplus C_4)$| #949 M |
|         | $C_2 \times C_2$ | #950 M      |                   | 77 |
Table 8

| #   | GAP # | SGL # | Aut(g)                      |
|-----|-------|-------|-----------------------------|
| 35  | 945   | 3/3   | \((C_2 \times C_2) \perp C_2 \times S_4\) |
| -   | 949   | 3/3   | \((C_2 \times C_2) \perp C_2 \times S_4\) |
| 77  | 948   | 3/3   | \(1^4 \times S_4\)          |
| -   | 950   | 3/3   | \((C_2 \times C_2) \perp C_2 \times S_4\) |
| 60  | 181   | 32/1  | \(C_2 \times \#44[192] = 384[17949]\) |
| 79  | 180   | 32/1  | \(C_2 \times \#44[192] = 384[17949]\) |
| -   | 1490  | 32/1  | \(C_2 \times \text{Aut}(Q_2 \times C_2)\) |
| -   | 1498  | 32/1  | \(C_2 \times \text{Aut}(Q_2 \times C_2)\) |
| -   | 954   | 27/4  | \(1^4 \times S_4\)          |

Presentation for group number 949 in SGL:

- Just make the replacement in the presentation for our group #35 (SGL = #945)\(, c^2 \ast d^2 = 1 \rightarrow a \ast d \ast a^{-1} \ast d \ast b = 1\)

Presentation for group number 950 in SGL:

- Just make the replacement in the presentation for our group #77 (SGL = #948)\(, a^c \ast a \ast b \ast a^{-1} = 1 \rightarrow (a, c) = (b, c) = 1\)

This group can also be generated by the extension \(\text{SL}(2, 3) \ast Q_2\) with a \(C_2 \times C_2\) action; see Table 4B comment 1 at end of table.

Presentation for group number 79 above:

- Just make the replacement in the presentation for our group #60\(, d^2 = 1 \rightarrow d^4 = c \ast a^{-1} \ast c \ast b^{-1} = 1\)

Table 9

| Case of \(p = 3\) (See Table A6 for a listing of the individual groups.) |
|---|
| image \(\Gamma_3 a_2 \simeq QD_8\) | SGL # | Hall-Senior # | Hall-Senior Breakdown |
| \(C_4 \times C_2\) | 1068 | 757 | \(\Gamma_1[39], \Gamma_2[222], \Gamma_3[107], \Gamma_4[163],\) |
| \(\Gamma_2 b\) | 125 | 125 | \(\Gamma_2[11], \Gamma_3[4], \Gamma_4[35], \Gamma_6[4], \Gamma_9[16], \Gamma_{10}[15]\) |
| \(\Gamma_2 d\) | 28 | 28 | \(\Gamma_2[10], \Gamma_3[6], \Gamma_7[7], \Gamma_{12}[5]\) |

Case of \(p = 5\) (See Table A5 for a listing of the individual groups.)

| image \(\Gamma_5 a_2\) | SGL # | Hall-Senior # | Hall-Senior Breakdown |
|---|
| \(C_{16}\) | NA | 4 | \# 135,138,139,141 all in \(\Gamma_8\) |
| \(\Gamma_{3} a_1 \simeq D_8\) | NA | 43 | \(\Gamma_3[7], \Gamma_8[14], \Gamma_{14}[4], \Gamma_{15}[5], \Gamma_{17}[3], \Gamma_{19}[4],\) |
| \(\Gamma_3 a_3 \simeq Q_4\) | NA | 21 | \(\Gamma_3[7], \Gamma_8[3], \Gamma_{14}[4], \Gamma_{15}[4], \Gamma_{17}[3]\) |

NA means not available in the Small Group Library.
### Table A1

| number | Aut$(g)$ | Aut$^2(g)$ | Aut$^3(g)$ | Aut$^4(g)$ |
|--------|----------|------------|------------|------------|
| 1      | GL(2,6)  |            |            |            |
| 2      | 10,321,920 |            |            |            |
| 3      | 147,456(2)  |            |            | complete |
| 4ag    | $C_2 \times 21,504$ | 21,504 | 43,008 | 86,016 † |
| 5ag    | $C_2 \times 43,008$ | 86,016 | 2304 | 110,592(1) | 221,184 |
| 7g     | $C_2 \times 384$ | 2304 |            |            |
| 8      | 1536(4) | 24,576(8) [has 616 classes $z = 1^3$] |            |            |
| 12     | 688,128 = Hol$(C_4 \times C_2 \times C_2 \times C_2)$ |            |            |            |
| 13     | 2,064,384 |            |            |            |
| 14     | 73,728(2) | 294,912(1^2) | $2^{21} \times 3^3(2)$ | $2^{24} \times 3^3(1)$ |
| 15     | 49,152(1^2) | 196,608 not direct product $z = 1^2$ |            |            |
| 16     | 49,152(1^2) | 294,912(1) |            |            |
| 17g    | $C_2 \times 3072$ | $C_2 \times 3072$ | 294,912(2) |            |
| 19     | 12,288(1^2) | 2,359,296(2) |            |            |
| 21g    | $C_2 \times 768$ | $1^3 \times \text{Aut}(Q_2 \times C_2)$ |            |            |
| 22     | 12,288(2) | 196,608(2) has 741 classes |            |            |
| 27     | 3072(1^2) | 294,912(2) |            |            |
| 30bg   | $C_2 \times 6144$ | 24,576 | 49,152 | 98,304 † |
| 35k    | 96 [Table 3b in [7] # 36] $\times C_4 \times C_2$ |            |            |            |
| 36     | $C_2 \times C_4 \times S_4$ |            |            |            |
| 43e    | 12,288(2) | 98,304(1^3) 988 classes, not direct product |            |            |
| 44     | 6144(1^2) | 393,216(1^2) number of classes = 1566 |            |            |
| 45e    | 12,288(2) = Hol$(C_8 \times C_2 \times C_2)$ $\rightarrow$ 98,304(1^3), 988 classes $\rightarrow$ $2^{34} \times 3$ |            |            |            |
| 68f    | 12,288(2) |            |            |            |
| 69     | 49,152(1) = Hol$(C_4 \times C_4 \times C_2)$ |            |            |            |
| 76     | 6144(1^2) | 4,718,592(1) |            |            |
| 81bg   | $C_2 \times 6144$ | 24,576 | 49,152 | 98,304 † |
| 82     | 12,288(2) | 196,608(2) | $2^{29} \times 3^2(1^2)$ |            |
| 93bg   | $C_2 \times 3072$ | 294,912(1) complete |            |            |
| 103h   | 36,864 | 73,728 | 147,456 | 294,912 † |
| 104    | 61,440 complete |            |            |            |
| 105    | 23,040 complete |            |            |            |
| 107ik  | 96 [Table 3b in [7] # 36] $\times 32[33]$ |            |            |            |
| 108    | 12,288(2) | $1^2 \times 12288(2)$ | 196608, 864 classes $z = 1^3$ |            |
| 109    | 1536(2) | $C_4 \times D_4 \times \text{Aut}(Q_2 \times C_2)$ |            |            |
| 144cg  | $C_2 \times 1536$ | 1536 $\rightarrow$ 258048 = $C_2 \times C_2 \times 64,512$ complete |            |            |
| 145cg  | $C_2 \times 1536$ | 1536 $\rightarrow$ 258048 = $C_2 \times C_2 \times 64,512$ complete |            |            |
| 147cg  | $C_2 \times 1536$ | 1536 $\rightarrow$ 258048 = $C_2 \times C_2 \times 64,512$ complete |            |            |
| 148cg  | $C_2 \times 1536$ | 1536 $\rightarrow$ 258048 = $C_2 \times C_2 \times 64,512$ complete |            |            |

† These are complete groups.
Table A1 (continued)

| number | Aut(\(g\)) | Aut\(^2(\(g\))\) | Aut\(^3(\(g\))\) | Aut\(^4(\(g\))\) |
|--------|-------------|-----------------|----------------|-----------------|
| 150    | \(C_4 \times C_2 \times ([192] \# 3 \text{ in Table 2 above})\) | 64,512 complete |                |                 |
| 153d   | 10,752      |                 |                |                 |
| 155ik  | 96[\text{table 3b \# 36}] \times 32[33] | 64,512 complete |                |                 |
| 156k   | 96[\text{table 3b \# 36}] \rtimes C_2 | 110,592 complete |                |                 |
| 158ik  | 96[\text{table 3b \# 36}] \times 32[33] |                |                |                 |
| 159k   | 96[\text{table 3b \# 36}] \times C_2^4 |                |                |                 |
| 162ik  | 96[\text{table 3b \# 36}] \times 32[33] |                |                |                 |
| 173j   | 6144(1)     | 49,152          | 786,432(1)     | \(2^{19} + 3\) |
| 181    | 1536        | 6,144           | 12,288 complete |                 |
| 183    | 9216        | 18,432          | 36,864 complete |                 |
| 184c   | 1536        | 258,048 = C_2 \times C_2 \times 64,512 | | |
| 187    | 15,360 complete | | | |
| 241    | S_4 \rtimes \text{Hol}(C_8) | | | |
| 242    | C_2 \times D_4 \times S_4 | | | |
| 243    | S_4 \rtimes \text{Hol}(C_8) | | | |
| 259    | C_2 \times \text{Aut}(Q_2 \times C_2) | | | |
| 260    | C_2 \times \text{Aut}(Q_2 \times C_2) | | | |
Notes for Table A1

a. The complete groups of order 86,016 are not isomorphic. The one for #4 is the same as for order 32 #2.

b. These two sequences of groups are isomorphic.

c. These order 1536 groups are all isomorphic.

d. This complete group of order 64,512 is different from that in the other sequences (c type).

e. This automorphism group is isomorphic to Hol(C_8 \times C_2 \times C_2).

f. This group of order 64 is (C_2 \times C_2) \times C_2 \times C_2. The automorphism group for this group appears to be isomorphic to that for #22 above.

g. In these cases the automorphism group has the structure C_2 \times (large order group). The next step in the automorphism series is the automorphism group for this “large order” factor only, e.g., for group number 17 of order 64

\[ 64 \rightarrow C_2 \times 3072 \rightarrow 3072 \rightarrow 294,912. \]

h. These two complete groups of order 294,912 are not isomorphic: one has 115 (# 93) classes; the other 152 (# 103) classes.

i. The group labelled 32[33] is group number 33 of order 32.

j. The next two groups in this sequence have orders 2^{20} \times 3 and 2^{21} \times 3.

This last group is a complete group.

The 49,152 group is the automorphism group of the order 6144 group, and the 49,152 group has 16 normal subgroups with the same order structure as the previous 6144 group. These 16 normal subgroups are mapped into one another by the 49,152’s automorphism group. This type of behavior is frequently noted in automorphism towers leading to complete groups, i.e., in the sequence

\[ g \rightarrow \text{Aut}(g) \rightarrow \text{Aut}[\text{Aut}(g)] \rightarrow \cdots \text{Aut}^n(g) = \text{complete group} \]

with center of g of order one. In many cases the ratio of the multiplicity of g in two consecutive groups in the automorphism tower is equal to the ratio:

\[ \text{order[Aut}^{n+1}(g)] / \text{order[Aut}^n(g)]. \]

In many cases we have

\[ g \rightarrow \text{Aut}(g) \rightarrow \text{Aut}[\text{Aut}(g)] = \text{complete group} \]

with center of g of order one. Then the multiplicity of g in Aut(g) is equal to order [Aut[Aut(g)] ] / order(Aut(g)).

The fact that this happens does not mean that Aut[Aut(g)] is complete, but if Aut[Aut(g)] is complete, then this behavior is usually found. In most cases where this behavior is observed, the orders of adjacent members in the automorphism towers have orders differing by a factor of two. One should also note that the derived subgroup of the 786,432 group has order 49,152. This 49,152 group is not the same 49,152 group in Aut(49,152). They do not have the same number of conjugacy classes. We have not been able to determine whether this derived group has an automorphism group of order 786,432.

k. Table 3b refers to the table in [7]; this group is \( (1^4) \oplus C_3 \oplus C_2. \)
Table A2 June 21, 1994

Groups coming from the group GL(2, 3).

Note that most of the following cases are duplicates of previous listings as is noted below. These alternate presentations however might be of interest to the reader.

Relations for GL(2, 3):
\[ b^x a^x b^x ((a^x b^x a^x)^{-1}) = (b^x (a^x)^{-1})^2 = 1 \]

\[ C_2 \text{ actions:} \]
1. \[ a^x a^3 = b^y b^3 = 1 \]
2. \[ a^x a^3 = b^x b = 1 \]
3. \[ a^x b^x a^x (b^x)^{-1} = b^y b = 1 \]
4. \[ a^x (a^{-3}) = b^x a^x (b^{-1}) (a^{-1}) = 1 \]
5. \[ (a, x) = b^x (b^{-3}) = 1 \]

\[ C_2 \times C_2 \text{ actions:} \]
1. \[ a^x a^3 = b^y b^3 = \]
   \[ a^y a^3 = b^y b = 1 \]
2. \[ a^x a^3 = b^y b^3 = \]
   \[ a^y b^x a^x (b^x)^{-1} = b^y b = 1 \]
3. \[ a^x a^3 = b^x b = \]
   \[ a^y a = b^y b^3 = 1 \]
4. \[ a^x a^3 = b^x b = \]
   \[ a^y a = b^y b^x a^x (b^{-1}) = 1 \]
5. \[ (a, x) = b^x (b^{-3}) = \]
   \[ a^y (a^{-3}) = b^y (b^{-3}) = 1 \]

There are two other \( C_2 \times C_2 \) cases which appear to yield duplicate results.

These cases are:
2a. \[ a^x (a^{-3}) = b^y a^x (b^{-1}) (a^{-1}) = \]
   \[ a^y (a^{-3}) = b^y b^x (a^{-1}) (b^{-1}) = 1 \]
3a. \[ a^x b^x a^x (b^{-1}) = b^y b = \]
   \[ a^y a b^x (a^{-1}) = b^y b = 1 \]

\[ C_4 \text{ actions:} \]
1. \[ a^x a^3 = b^x a^x b^x (a^{-1}) = 1 \]
2. \[ (a, x) = b^x b^x (a^{-1}) (b^{-1}) = 1. \]

\[ C_2 \text{ actions a. From } C_2 \times C_2. \text{ All direct products:} \]

| #66 | \([\text{GL}(2, 3)@C_2] \times C_2 \text{ action [1]}\) | \(S_4 \times \#33 \text{ order 32}\) |
|-----|-------------------------------------------------|--------------------------------------|
| C_2 \times [96] Table 3b in [7] #35 | \([\text{GL}(2, 3)@C_2] \times C_2 \text{ action [2]}\) | \(S_4 \times \text{Hol}(C_2 \times C_2)\) |
| \(\text{same as } #39\) | \([\text{GL}(2, 3)@C_2] \times C_2 \text{ action [3]}\) | \(S_4 \times \text{Hol}(C_4 \times C_2)\) |
| \(\text{same as } #39\) | \([\text{GL}(2, 3)@C_2] \times C_2 \text{ action [4]}\) | \(S_4 \times \#62[192]\) |
| \(\text{same as } #32\) | | |
Table A2 (continued) June 21, 1994

| no new ones in this list | GL(2, 3) × 1^2 | S_4 × #33 order 32 |
|-------------------------|----------------|-------------------|
| GL(2, 3) × C_2 | S_4 × 1^4 | |

C_2 × C_2 actions:

| GL(2, 3)@C_4 action [1] | S_4 × S_4 × C_2 |
|-------------------------|-----------------|
| GL(2, 3)@C_4 action [2] | S_4 × D_4 × C_2 |
| GL(2, 3)@C_2 × C_2 | S_4 × D_4 × C_2 |
| GL(2, 3)@C_2 × C_2 | S_4 × S_4 × C_2 |
| GL(2, 3)@C_2 × C_2 | S_4 × S_4 × C_2 |
| GL(2, 3)@C_4 action [4] | S_4 × Hol(C_4 × C_2) |

C_4 actions:

| GL(2, 3)@C_4 action [1] | S_4 × 1^3 |
|-------------------------|-----------|
| GL(2, 3)@C_4 action [2] | S_4 × 1^3 |
G. Groups coming from the group \(<2,3,4>\).

Note that most of the following cases are duplicates of previous listings as is noted below. These alternate presentations however might be of interest to the reader.

Relations for \(<2,3,4>\):
\[ b \ast a \ast b((a \ast b \ast a)^{-1}) = b \ast a^2 \ast b \ast (a^{-2}) = 1 \]

| \(C_2\) actions: |
|------------------|
| 1. \(a^x \ast a^3 = b^x \ast b^3 = 1\) |
| 2. \(a^x \ast a^3 = b^x \ast (b^{-3}) = 1\) |
| 3. \(a^x \ast (a^{-1}) \ast b \ast (a^{-3}) = b^x \ast (b^{-3}) = 1\) |
| 4. \(a^x \ast a = b^x \ast a \ast (b^{-1}) \ast (a^{-1}) = 1\) |
| 5. \((a, x) = b^x \ast b = 1\) |

| \(C_2 \times C_2\) actions: |
|------------------|
| 1. \(a^x \ast a^3 = b^x \ast b^{-3} = a^y \ast a^3 = b^y \ast (b^{-3}) = 1\) |
| 2. \(a^x \ast a^3 = b^x \ast b^3 = a^y \ast (a^{-1}) \ast b \ast (a^{-3}) = b^y \ast (b^{-3}) = 1\) |
| 3. \(a^x \ast a^3 = b^x \ast (b^{-3}) = a^y \ast (a^{-3}) = b^y \ast b^3 = 1\) |
| 4. \((a, x) = b^x \ast b = a^y \ast a = b^y \ast b = 1\) |

There are three other \(C_2 \times C_2\) cases which appear to yield duplicate results.

These presentations are:

| \(C_4\) actions: |
|------------------|
| 1. \(a^x \ast a^3 = b^x \ast ((a^3 \ast b \ast a)^{-1}) = 1\) |
| 2. \((a, x) = b^x \ast a \ast b \ast (a^{-1}) = 1\) |

| Direct Product Cases. |
|-----------------------|
| \#67 \(<2,3,4> \times C_2 \times C_2\) | \(S_4 \times \#62[192]\) |
| \#68 \(<2,3,4> \times \@C_2 \times C_2\) | \(S_4 \times \text{Hol}(C_4 \times C_2)\) |
| \#69 \(<2,3,4> \times C_2\) | \(C_2\times\text{Table }3b\text{ in }[7]\#34\) |
| \(S_4 \times 1^4\) |

C₂ actions a. From \(C_2 \times C_2\). All direct products:

| \(<2,3,4> \times C_2\) action [1] |
|------------------|
| \(C_2 \times \text{Table }3b\text{ in }[7]\#35\) |
| \(\text{same as }\#66\) |
| \(S_4 \times \#33\text{ order }32\) |
Table A3 (continued) June 21, 1994

G. Groups coming from the group $<2,3,4>$. 

| Action Description | Other Information |
|--------------------|------------------|
| $<2,3,4>@C_2\times C_2$ action [2] | $S_4 \times \text{Hol}(C_4 \times C_2)$ |
| $C_2\times$ Table 3b in [7] # 34 | |
| $<2,3,4>@C_2\times C_2$ action [3] | $S_4 \times #33$ order 32 |
| $C_2\times$ Table 3b in [7] # 35 | |
| $<2,3,4>@C_2\times C_2$ action [4] | $S_4 \times #33$ order 32 |
| $C_2\times$ Table 3b in [7] # 32 | |
| $<2,3,4>@C_4$ action [1] | $S_4 \times 1^4$ |
| $<2,3,4>@C_4$ action [2] | $S_4 \times 1^4$ |
| same as # 70 | |
| $<2,3,4>@C_4$ action [3] | $S_4 \times 1^4$ |
| same as # 70 | |
| $<2,3,4>\times C_4$ action [4] | $S_4 \times 1^4$ |
| This shows there are awkward ways to write direct products! | |
| same as # 69 | |
| $<2,3,4>@C_4$ action [5] | $S_4 \times 1^4$ |
| same as # 69 | |
| $C_2\times C_2$ actions: | |
| $<2,3,4>@(C_2\times C_2)$ action [1] | $S_4 \times D_4 \times C_2$ |
| duplicate # 51 | |
| $<2,3,4>@(C_2\times C_2)$ action [1a] | $S_4 \times D_4 \times C_2$ |
| duplicate # 51 | |
| $<2,3,4>@(C_2\times C_2)$ action [2] | $S_4 \times D_4 \times C_2$ |
| duplicate # 49 | |
| $<2,3,4>@(C_2\times C_2)$ action [2a] | $S_4 \times D_4 \times C_2$ |
| duplicate # 49 | |
| $<2,3,4>@(C_2\times C_2)$ action [3] | $S_4 \times S_4 \times C_2$ |
| duplicate # 61 | |
| $<2,3,4>@(C_2\times C_2)$ action [3a] | $S_4 \times S_4 \times C_2$ |
| duplicate # 50 | |
| $<2,3,4>@(C_2\times C_2)$ action [4] | $S_4 \times S_4 \times C_2$ |
| duplicate # 51 | |
| $C_4$ actions: | |
| $<2,3,4>@C_4$ action [1] | $S_4 \times 1^3$ |
| duplicate # 57 | |
| $<2,3,4>@C_4$ action [2] | $S_4 \times 1^3$ |
| duplicate # 56 | |
Table A4
Class Structure for Groups of Order 192
Nonnormal Sylow Group Types

| ncl | group # | order structure of the group |
|-----|---------|-----------------------------|
|     | 4 / SGL | 2   | 3   | 4   | 6   | 8   | 12  | 16  | 24  |
| 11  | 55/184  | 19/3| 32/1| 60/3| 32/1| 48/2|
| 11  | 45/185  | 19/3| 32/1| 108/5| 32/1|
| 13  | 81/1492 | 19/3| 32/1| 60/5 | 32/1| 48/2|
| 13  | 80/1491 | 19/4| 32/1| 108/6| 32/1|
| 13  | 63/1494 | 43/4| 32/1| 36/4 | 32/1| 48/2|
| 13  | 62/1493 | 43/6| 32/1| 84/4 | 32/1|
| 14  | 44/956  | 43/5| 32/1| 36/4 | 32/1| 48/2|
| 14  | 54/955  | 43/6| 32/1| 84/5 | 32/1|
| 15  | 79/180  | 7/3 | 32/1| 72/5 | 32/1| 48/4|
| 15  | 60/181  | 31/4| 32/1| 48/4 | 32/1| 48/4|
| 17  | 76/989  | 11/3| 8/1 | 68/6 | 40/3| 48/2| 16/1|
| 17  | 74/987  | 19/3| 8/1 | 60/6 | 8/1 | 48/2| 48/3|
| 17  | 64/990  | 35/4| 8/1 | 44/5 | 40/3| 48/2| 16/1|
| 17  | 41/988  | 43/4| 8/1 | 36/5 | 8/1 | 48/2| 48/3|
| 19  | $Q_4$ a/975 | 7/3 | 8/1 | 72/6 | 8/1 | 48/4| 48/3|
| 19  | $QD_4$ a/973 | 23/5| 8/1 | 56/4 | 40/3| 48/4| 16/1|
| 19  | $QD_8$ ab/976 | 31/4| 8/1 | 48/5 | 8/1 | 48/4| 48/3|
| 19  | $D_8$ a/974 | 47/6| 8/1 | 32/3 | 40/3| 48/4| 16/1|
| 20  | 75/962  | 7/2 | 8/1 | 56/4 | 8/1 | 64/5| 16/2| 32/4|
| 20  | 42/182  | 7/3 | 32/1| 72/10| 32/1| 48/4|
| 20  | 56/965  | 31/3| 8/1 | 32/3 | 8/1 | 64/5| 16/2| 32/4|
| 20  | 43/944  | 31/5| 32/1| 48/8 | 32/1| 48/4|
| 20  | 53/1495 | 31/9| 32/1| 96/8 | 32/1|
| 20  | 59/966  | 55/4| 8/1 | 8/2  | 8/1 | 64/5| 16/2| 32/4|
| 20  | 52/1538 | 55/11| 32/1| 72/6 | 32/1|
| 22  | $Q_4$ a/957 | 7/3 | 8/1 | 104/6| 8/1 | 16/4| 16/2| 32/4|
| 22  | $QD_8$ b/960 | 31/4| 8/1 | 80/5 | 8/1 | 16/4| 16/2| 32/4|
| 22  | $D_8$ b/961 | 55/5| 8/1 | 56/4 | 8/1 | 16/4| 16/2| 32/4|
Table A4 (continued)
Class Structure for Groups of Order 192
Nonnormal Sylow Group Types

| ncl | group # | order structure of the group |
|-----|---------|-----------------------------|
|     | Tables  | 2   | 3   | 4   | 6   | 8   | 12  | 16  | 24  |
| 4 / SGL |        | 3/3 | 8/1 | 76/7 | 24/3 | 48/4 | 32/4 |
| 23x | 35/945  |     |     |      |      |      |      |    |
| 23x | 77/948  |     |     |      |      |      |      |    |
| 23  | 73/978  |     |     |      |      |      |      |    |
| 23  | 72/984  |     |     |      |      |      |      |    |
| 23  | 65/1486 |     |     |      |      |      |      |    |
| 23  | 61/1483 |     |     |      |      |      |      |    |
| 23  | 33/980  |     |     |      |      |      |      |    |
| 23  | 40/986  |     |     |      |      |      |      |    |
| 23  | 49/1485 |     |     |      |      |      |      |    |
| 23  | 34/952  |     |     |      |      |      |      |    |
| 23  | 50/1484 |     |     |      |      |      |      |    |
| 23M | /954    |     |     |      |      |      |      |    |
| 25  | $S_4 \times Q_2/66$ | 19/5 | 8/1 | 108/14 | 8/1 |      | 48/3 |
| 25** | $C_4Q_2$ ce/62 | 35/8 | 8/1 | 92/11 | 40/3 |      | 16/1 |
| 25** | $C_4Q_2$ cde/67 | 43/6 | 8/1 | 84/13 | 8/1 |      | 48/3 |
| 25  | $S_4 \times D_4/61$ | 59/11 | 8/1 | 68/8 | 40/3 |      | 16/1 |
| 26  | 70/947  |     |     |      |      |      |      |    |
| 26  | 78/979  |     |     |      |      |      |      |    |
| 26  | 48/982  |     |     |      |      |      |      |    |
| 26x | 38/985  |     |     |      |      |      |      |    |
| 26x | 68/1479 |     |     |      |      |      |      |    |
| 26  | 57/964  |     |     |      |      |      |      |    |
| 26  | 37/953  |     |     |      |      |      |      |    |
| 26  | 66/1476 |     |     |      |      |      |      |    |
| 26  | 51/1482 |     |     |      |      |      |      |    |
| 26  | 39/1481 |     |     |      |      |      |      |    |
| 28  | $< 2, 2/2 > a/968$ | 15/5 | 8/1 | 16/6 | 24/3 | 96/8 | 32/4 |
| 28x | $Q_2X \times C_2$ b | 15/7 | 8/1 | 112/12 | 24/3 |      | 32/4 |
| 28x | $C_4\@C_4$ a | 15/7 | 8/1 | 112/12 | 24/3 |      | 32/4 |
| 28x | $C_4\@C_4$ b | 15/7 | 8/1 | 112/12 | 24/3 |      | 32/4 |
| 28  | $< 2, 2/2 > b/959$ | 19/4 | 8/1 | 44/7 | 8/1 | 64/8 | 16/2 | 32/4 |
| 28* | (4,4\[2,2])b/991 | 31/11 | 8/1 | 96/8 | 56/7 |
| 28** | $C_4Q_2$ de/1471 | 39/7 | 8/1 | 88/12 | 24/3 |      | 32/4 |
| 28* | (4,4\[2,2])a/972 | 39/9 | 8/1 | 88/10 | 24/3 |      | 32/4 |
| 28  | $D_4 \times C_2$ a/1488 | 55/13 | 8/1 | 72/6 | 56/7 |      |
| 28  | $D_4 \times C_2$ b/1470 | 63/11 | 8/1 | 64/8 | 24/3 |      | 32/4 |
Table A4 (continued)
Class Structure for Groups of Order 192
Nonnormal Sylow Group Types

| ncl | group # | order structure of the group |
|-----|---------|-----------------------------|
|     | Tables | 2   | 3   | 4   | 6   | 8   | 12  | 16  | 24  |
| 32  | 29/183 | 3/3 | 8/1 | 28/8| 24/3| 96/12| 32/4|     |     |
| 32  | 69/946 | 3/3 | 8/1 | 76/12| 24/3| 48/8| 32/4|     |     |
| 32  | 71/187 | 7/2 | 8/1 | 8/3 | 8/1 | 16/6 | 16/2| 96/12| 32/4|
| 32x | 31/977 | 7/7 | 8/1 | 72/8| 56/7| 48/8|     |     |     |
| 32x | 67/1474| 7/7 | 8/1 | 72/8| 56/8| 48/8|     |     |     |
| 32  | 46/983 | 15/5| 8/1 | 64/10| 24/3| 48/8| 32/4|     |     |
| 32  | 36/981 | 15/5| 8/1 | 64/14| 24/3| 48/4| 32/4|     |     |
| 32  | 58/963 | 19/3| 8/1 | 44/8| 8/1 | 64/12| 16/2| 32/4|     |
| 32  | 30/951 | 27/5| 8/1 | 52/10| 24/3| 48/8| 32/4|     |     |
| 32  | 47/1480| 39/7| 8/1 | 40/8| 24/3| 48/8| 32/4|     |     |
| 32  | 32/1475| 55/11| 8/1 | 24/4| 56/7| 48/4|     |     |     |
| 40  | $A_4 \oplus C_{16}/186$ | 7/3 | 8/1 | 8/4 | 8/1 | 16/8 | 16/2 | 96/16| 32/4|
| 40  | $A_4 \oplus C_8 \times C_2/967$ | 15/7| 8/1 | 16/8  | 24/3 | 96/16 | 32/4|     |     |
| 40  | $A_4 \oplus C_4 \times C_4/969$ | 15/7| 8/1 | 112/24| 24/3| 32/4|     |     |     |
| 40  | $S_4 \times C_8/958$ | 19/5| 8/1 | 44/10| 8/1 | 64/16| 16/2 | 32/4|     |
| 40  | $A_4 \oplus C_2 \times 2^3/1487$ | 31/15| 8/1 | 96/16| 56/7 |     |     |     |     |
| 40  | $S_4 \times (2,1)/1469$ | 39/11| 8/1 | 88/20| 24/3| 32/4|     |     |     |
| 40  | $S_4 \times 1^3/1537$ | 79/23| 8/1 | 48/8 | 56/7|     |     |     |     |
Notes for Table A4.
ncl means the number of conjugacy classes in the group.
In the first row above, the number 19/3 means there are
19 elements of order 2 distributed in 3 classes,
and likewise for the other groups and element orders.

Notes: x in column 1 means groups having the same class/order structure but
having different automorphism groups; see Tables 4 and 5.

23M is a missing group in my list, GAP number given.

The groups of the form $A_4 \oplus$ (group of order 16) are not explicitly
written out in Table 4A, but the 2-group along with the generator
acting on $A_4$ as an element of order 2 is explicitly written out.
Thus one can reconstruct the group of order 192.

* The relations used here for $(4, 4|2, 2)$ were:

\[
a^2 = b^4 = (b, a) \ast (b^{-1}) \ast a \ast (b^{-1}) \ast a \ast b^2 = \\
(b^{-1}) \ast a \ast b \ast a \ast (b^{-1}) \ast a \ast b \ast a = 1
\]

** The relations used for $C_4YQ_2$ were:

\[
c^4 = d^2 = e^2 = (d, e) \ast c^2 = (c, d) = (c, e) = 1
\]
| image | Isoclinic Class | groups |
|-------|----------------|--------|
| $C_3 \times C_2$ | $\Gamma_1$ | $21^4[2], 2^44^2[4], 31^3[5], 2^{3}[2], 321[11]$ |
| 757   | $\Gamma_2$ | $c_1[4], c_2[4], d[13], e_1[6], e_2[4], f[8], g[13], h[4], i[11], j_1[11], j_2[11], k[11], l[6], m_1[10], m_2[10], n[4], o[14], p[17], q[18], r_1[11], r_2[5], s[5], t[4], u[4], v[5], w_1[3], w_2[3], x[3]$ |
| $C_4 \times C_4$ | $\Gamma_3$ | $c_1[4], c_2[4], d_1[4], d_2[4], e[11], f[11], g[6], h[6], i_1[4], i_2[6], i_3[4], j[11], k[4], l[6], m[4], n_1[3], n_2[3], o_1[3], o_2[3], p[3], q[3]$ |
| [54]  | $\Gamma_4$ | $g_1[2], g_2[2], h[6], i_1[6], i_2[4], i_3[4], i_4[4], i_5[4], j_1[4], j_2[4], k_1[6], k_2[6], l[2], m_1[11], m_2[11], n_1[11], n_2[7], o[18], p[18], p_2[11], p_3[11], q[10]$ |
| $\Gamma_5$ | $c_1[6], c_2[6], c_3[4], d[13]$ |
| $\Gamma_6$ | $c_1[4], c_2[6], c_3[4], d[4], e_1[4], e_2[6], e_3[4], f[11], g[4], h[11]$ |
| $\Gamma_7$ | $a_1[11], a_2[7], a_3[7], b_1[11], b_2[10], c_1[4], c_2[4], d[6], e_1[3], e_2[3], f[3]$ |
| $\Gamma_8$ | $c_1[2], c_2[2], d_1[2], d_2[2], e[3], f[3]$ |
| $\Gamma_{12}$ | $a_1[4], a_2[2], b[5]$ |
| $\Gamma_{17}$ | $a_1[2], a_2[2], a_3[2], b_1[3], b_2[3], c_1[3], c_2[2]$ |
| $\Gamma_{20}$ | $a[2]$ |
| $\Gamma_{21}$ | $a[1], a_2[3], a_3[2]$ |
| $\Gamma_{22}$ | $a_1[3], a_2[3]$ |
| $\Gamma_{23}$ | $a_1[3], a_2[3], a_3[3], a_4[3]$ |
| $\Gamma_2 b$ | $\Gamma_2$ | $b, c_1, c_2, f, l, m_1, m_2, n, o, r_1, r_2$ |
| [125] | $\Gamma_3$ | $i_1, i_2, i_3, j$ |
| $\Gamma_4$ | $b_1, b_2, c_1, c_2, c_3, d, e_1, e_2, e_3, f[3], h, i_1, i_2, i_3, i_4, i_5[2], j_1, j_2, k_1[4], k_2[4], l, o, p_1, p_2, p_3, q$ |
| $\Gamma_6$ | $e_1, e_2, e_3, h$ |
| $\Gamma_9$ | $a_1, a_2, b_1[2], b_2, b_3, b_4[2], c, d_1, d_2, e$ |
| $\Gamma_{10}$ | $b_1, b_2, b_3, b_4, b_5, b_6, c_1[2], c_2, c_3, c_4, c_5[2], c_6[2]$ |
| $\Gamma_{11}$ | $b_1, b_2, b_3, b_4, b_5$ |
| $\Gamma_{14}$ | $b_1, b_2, c_1, c_2, c_3, e$ |
| $\Gamma_{15}$ | $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, c_1, c_2, c_3, c_4, c_5[2], c_6[2]$ |
| $\Gamma_{16}$ | $b_1, b_2, b_3, c_1, c_2, c_3$ |
| $\Gamma_{24}$ | $a_1, a_2, a_3$ |
| $\Gamma_2 d$ | $\Gamma_2$ | $d, i, j_1, j_2, o, q, t, u, w_1, w_2$ |
| [28]  | $\Gamma_3$ | $n_1, n_2, o_1, o_2, p, q$ |
| $\Gamma_4$ | $m_1, m_2, o[2], p_1, p_2, p_3$ |
| $\Gamma_7$ | $e_1, e_2, f$ |
| $\Gamma_{12}$ | $b[2]$ |
Table A6. Details on Quotient Groups in Table 9, \( p = 3 \) and 7 cases

| image | Isoclinic Class | groups |
|-------|----------------|--------|
| \( QD_8 = \Gamma_3a_2 \) | \( \Gamma_3 \) | \( a_2, c_1, c_2, d_2, k, n_1, n_2, o_2 \) |
|        | \( \Gamma_8 \) | \( c_1, c_2, e \) |
|        | \( \Gamma_{14} \) | \( a_2, a_4, b_1, b_2, c_2 \) |
|        | \( \Gamma_{15} \) | \( a_2, a_3, b_2, b_5, b_6, c_2, c_3 \) |
|        | \( \Gamma_{17} \) | \( a_1, a_2(\ \text{twice}), a_3, b_1, b_2 \) |
|        | \( \Gamma_{20} \) | \( a(\ \text{twice}) \) |
|        | \( \Gamma_{21} \) | \( a_1, a_2, a_3 \) |
| \( p = 7 \) | \( C_{16} \) | \( \Gamma_1 \) | 411, 42, 51 (twice), and 6 |
|        | \( \Gamma_2 \) | \( w_1, w_2, x(\ \text{twice}) \) |
| \( \Gamma_3a_1 = D_8 \) | \( \Gamma_3 \) | \( a_1, c_1, d_1, t_1, k, n_1, o_1 \) |
|        | \( \Gamma_8 \) | \( a_1[2], a_2[2], a_3[2], b[2], c_1, c_2, d_1, d_2, e, f \) |
|        | \( \Gamma_{14} \) | \( a_1, a_3, b_1, c_1 \) |
|        | \( \Gamma_{15} \) | \( a_1, a_3, b_1, b_3, e_1? \) |
|        | \( \Gamma_{17} \) | \( a_1, a_2, b_1 \) |
|        | \( \Gamma_{19} \) | \( a_1[2], a_2[2] \) |
|        | \( \Gamma_{21} \) | \( a_1, a_2, a_3 \) |
|        | \( \Gamma_{27} \) | \( a_1, a_2, a_3 \) |
| \( \Gamma_3a_3 = Q_4 \) | \( \Gamma_3 \) | \( a_3, c_2, d_1, t_3, k, n_2, o_1 \) |
|        | \( \Gamma_8 \) | \( d_1, d_2, f \) |
|        | \( \Gamma_{14} \) | \( a_3, a_5, b_2, c_3 \) |
|        | \( \Gamma_{15} \) | \( a_4, b_7, b_8, c_4 \) |
|        | \( \Gamma_{17} \) | \( a_2, a_3, b_2 \) |

Table 7

| action | semi-direct case | nonsplit extensions | \( e^8 = e^4(ab^{-1})^2 \) | \( e^4 = d^4 = d^2(ab^{-1})^2 \) |
|--------|-----------------|---------------------|-----------------|-----------------|
| \( C_2[C_4] \) | 33 | 64 | 72 |
| \( C_2[C_2] \) | 34 | 50 | 74 |
| \( (C_2 \times C_2)(c, d) \) | [33] | [59] | 949 M |
| \( (C_2 \times C_2)(d, c) \) | 40 | [41] | 73 |
| \( D_4 \) | 41 | [40] | 76 |
7. Appendix II
Structure and relations for selected automorphism groups in Table A1

7.1. Group #3. [order 147,456]. The group has the structure:

\[(4096) @ (C_3 \times C_3) @ (C_2 \times C_2).\]

There are 13 possible \((C_3 \times C_3)\) actions on the 4096 group. CAYLEY generated the following two possible choices for a presentation of this group of order 36,864:

\[
\begin{align*}
T.1^2 &= T.2^2 = T.3^2 = T.4^2 = T.5^2 = T.6^2 = T.7^2 = T.8^2 = (T.1 \ast T.2)^2 = \\
(T.1 \ast T.4)^2 &= (T.1 \ast T.6)^2 = (T.2 \ast T.3)^2 = (T.2 \ast T.4)^2 = (T.2 \ast T.6)^2 = \\
(T.3 \ast T.5)^2 &= (T.3 \ast T.7)^2 = (T.3 \ast T.8)^2 = (T.4 \ast T.6)^2 = (T.5 \ast T.7)^2 = \\
(T.5 \ast T.8)^2 &= (T.6 \ast T.7)^2 = (T.7 \ast T.8)^2 = (T.1 \ast T.2 \ast T.5)^2 = \\
T.1 \ast T.3 \ast T.5 \ast T.5 \ast T.5 \ast T.3 &= (T.4 \ast T.6 \ast T.8)^2 = \\
T.4 \ast T.7 \ast T.8 \ast T.4 \ast T.8 \ast T.7 &= \\
T.1 \ast T.2 \ast T.7 \ast T.8 \ast T.2 \ast T.7 \ast T.1 \ast T.8 &= (T.1 \ast T.3 \ast T.4 \ast T.3)^2 = \\
T.1 \ast T.3 \ast T.4 \ast T.3 \ast T.4 \ast T.8 \ast T.1 \ast T.8 &= (T.1 \ast T.3 \ast T.6 \ast T.3)^2 = \\
T.1 \ast T.4 \ast T.5 \ast T.4 \ast T.5 \ast T.7 \ast T.1 \ast T.7 &= \\
T.2 \ast T.3 \ast T.6 \ast T.3 \ast T.6 \ast T.7 \ast T.2 \ast T.7 &= \\
T.2 \ast T.4 \ast T.5 \ast T.8 \ast T.2 \ast T.4 \ast T.8 \ast T.5 &= \\
T.3 \ast T.4 \ast T.5 \ast T.4 \ast T.6 \ast T.5 \ast T.3 \ast T.6 &= 1
\end{align*}
\]

with the actions:

\[
\begin{align*}
T.9^3 &= T.10^3 = (T.9, T.10) = \\
T.1^{T.9} \ast ((T.4 \ast T.7 \ast T.4 \ast T.8)^{-1}) &= \\
T.2^{T.9} \ast ((T.1 \ast T.2 \ast T.8 \ast T.1 \ast T.2)^{-1}) &= \\
T.3^{T.9} \ast ((T.1 \ast T.4 \ast T.6 \ast T.8 \ast T.1)^{-1}) &= \\
T.4^{T.9} \ast ((T.1 \ast T.3 \ast T.5 \ast T.7 \ast T.8 \ast T.1)^{-1}) &= \\
T.5^{T.9} \ast ((T.4 \ast T.5 \ast T.6 \ast T.5 \ast T.7 \ast T.4)^{-1}) &= \\
T.6^{T.9} \ast ((T.4 \ast T.7 \ast T.4 \ast T.5)^{-1}) &= \\
T.7^{T.9} \ast ((T.1 \ast T.7 \ast T.2 \ast T.5 \ast T.6)^{-1}) &= \\
T.8^{T.9} \ast ((T.2 \ast T.3 \ast T.4 \ast T.6 \ast T.8)^{-1}) &= 1
\end{align*}
\]
and

\[ T.1^{T.10} \ast ((T.1 \ast T.2 \ast T.4 \ast T.7 \ast T.2 \ast T.7)^{-1}) = \]
\[ T.2^{T.10} \ast ((T.4 \ast T.5 \ast T.2 \ast T.6 \ast T.5)^{-1}) = \]
\[ T.3^{T.10} \ast ((T.1 \ast T.2 \ast T.3 \ast T.8 \ast T.1 \ast T.2)^{-1}) = \]
\[ T.4^{T.10} \ast ((T.3 \ast T.1 \ast T.4 \ast T.3 \ast T.4)^{-1}) = \]
\[ T.5^{T.10} \ast ((T.1 \ast T.2 \ast T.7 \ast T.1 \ast T.5 \ast T.2)^{-1}) = \]
\[ T.6^{T.10} \ast ((T.1 \ast T.5 \ast T.2 \ast T.5)^{-1}) = \]
\[ T.7^{T.10} \ast ((T.4 \ast T.5 \ast T.4)^{-1}) = \]
\[ T.8^{T.10} \ast ((T.1 \ast T.2 \ast T.3 \ast T.1 \ast T.8 \ast T.2 \ast T.8)^{-1}) = 1 \]

or, alternatively, with this set of actions:

\[ T.9^3 = T.10^3 = (T.9, T.10) = \]
\[ T.1^{T.9} \ast ((T.4 \ast T.3 \ast T.4 \ast T.6 \ast T.5 \ast T.6)^{-1}) = \]
\[ T.2^{T.9} \ast ((T.4 \ast T.3 \ast T.4)^{-1}) = \]
\[ T.3^{T.9} \ast ((T.3 \ast T.8 \ast T.2 \ast T.8)^{-1}) = \]
\[ T.4^{T.9} \ast ((T.1 \ast T.4 \ast T.8 \ast T.1 \ast T.4 \ast T.7)^{-1}) = \]
\[ T.5^{T.9} \ast ((T.1 \ast T.7 \ast T.2 \ast T.5 \ast T.7)^{-1}) = \]
\[ T.6^{T.9} \ast ((T.2 \ast T.7 \ast T.2)^{-1}) = \]
\[ T.7^{T.9} \ast ((T.2 \ast T.6 \ast T.7 \ast T.2)^{-1}) = \]
\[ T.8^{T.9} \ast ((T.3 \ast T.6 \ast T.3 \ast T.8 \ast T.4)^{-1}) = 1 \]

and

\[ T.1^{T.10} \ast ((T.1 \ast T.5 \ast T.2 \ast T.6 \ast T.5 \ast T.6)^{-1}) = \]
\[ T.2^{T.10} \ast ((T.3 \ast T.7 \ast T.1 \ast T.3 \ast T.7)^{-1}) = \]
\[ T.3^{T.10} \ast ((T.3 \ast T.4 \ast T.7 \ast T.4 \ast T.5 \ast T.7)^{-1}) = \]
\[ T.4^{T.10} \ast ((T.3 \ast T.6 \ast T.3)^{-1}) = \]
\[ T.5^{T.10} \ast ((T.3)^{-1}) = \]
\[ T.6^{T.10} \ast ((T.1 \ast T.5 \ast T.1 \ast T.4 \ast T.5 \ast T.6)^{-1}) = \]
\[ T.7^{T.10} \ast ((T.1 \ast T.8 \ast T.1)^{-1}) = \]
\[ T.8^{T.10} \ast ((T.1 \ast T.4 \ast T.7 \ast T.4 \ast T.8 \ast T.1)^{-1}) = 1. \]

| order structure of \([(4096)@\langle C_5 \times C_3 \rangle]\) group |
|---|---|---|---|---|---|---|---|
| order of elements | 2 | 3 | 4 | 6 | 12 | order of center | 4 |
| number of elements | 927 | 2816 | 3168 | 20736 | 9216 | 12 | 4 |
| number of classes | 25 | 8 | 32 | 40 | 12 | | |
The automorphism group of this group of order 36,864 has order 221,184 = 2^{13} \times 3^3.

CAYLEY gives the following set of relations on three generators for the automorphism group (of order 147,456):

\[
a^6 = b^6 = c^2 = (a \ast (b^{-1}))^3 = a^2 \ast b \ast (a^{-1}) \ast (b^{-2}) \ast (a^{-1}) \ast b =
\]
\[
(a^2 \ast (b^{-2}))^2 = (a \ast (c^{-1}))^4 = (b \ast (c^{-1}))^4 =
\]
\[
a \ast b \ast (a^{-1}) \ast (b^{-1}) \ast c \ast (b^{-1}) \ast (a^{-1}) \ast b \ast a \ast c = (a^2 \ast (c^{-1}))^4 =
\]
\[
a^2 \ast (c^{-1}) \ast a \ast (b^{-1}) \ast (c^{-1}) \ast (b^{-2}) \ast (c^{-1}) \ast (b^{-1}) \ast a \ast (c^{-1}) =
\]
\[
a \ast b \ast (c^{-1}) \ast b \ast (a^{-1}) \ast (c^{-1}) \ast (a^{-1}) \ast (b^{-1}) \ast (c^{-1}) \ast (b^{-1}) \ast a \ast (c^{-1}) =
\]
\[
(a \ast (b^{-1}) \ast c \ast b \ast (a^{-1}) \ast c)^2 = (a \ast c \ast (a^{-1}) \ast b \ast c \ast (b^{-1}))^2 =
\]
\[
(a \ast c \ast (a^{-1}) \ast c)^3 = (b \ast c \ast (b^{-1}) \ast c)^3 =
\]
\[
a \ast b^2 \ast c \ast b^2 \ast c \ast (a^{-1}) \ast c \ast b \ast (a^{-1}) \ast c \ast (a^{-1}) \ast b =
\]
\[
a^2 \ast c \ast (b^{-2}) \ast (a^{-1}) \ast c \ast a \ast c \ast a \ast (b^{-1}) \ast (a^{-1}) \ast b \ast c \ast (a^{-1}) \ast b = 1.
\]

This automorphism group has only 19 normal subgroups.

The automorphism group of this order 147,456 group has order 294,912. It has 119 normal subgroups and has a center of $C_2 \times C_2$. GAP fails to get the automorphism group of this group of order 294,912.

7.2. **Group #4. [order 22,008]**. The automorphism group for the group $C_8 \times C_2 \times C_2 \times C_2$ is $C_2 \times [21,504]$. The presentation of the order 21,504 group is:

\[
a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = h^2 = (a, b) = (a, c) = (b, e) = (b, h)
\]
\[
= (c, d) = (c, f) = (d, f) = (e, h) =
\]
\[
a \ast b \ast c \ast b \ast c = a \ast b \ast e \ast a \ast e = a \ast b \ast h \ast a \ast h
\]
\[
= a \ast c \ast d \ast a \ast d = a \ast c \ast f \ast a \ast f
\]
\[
= (b \ast d)^3 = (b \ast f)^3 = (c \ast e)^3 = (c \ast h)^3 = (d \ast e)^4
\]
\[
= (d \ast e \ast d \ast h)^2 = (d \ast e \ast f \ast e)^2 = (e \ast f \ast h \ast f)^2 = 1.
\]

7.3. **Group #5. [order 86,016]**. The automorphism group of $C_4 \times C_4 \times C_4$ is $C_2 \times [43008]$. A presentation for the 43008 factor here is:

\[
a^2 = b^2 = c^2 = d^2 = (a \ast b)^2 = (a \ast c)^3 = (a \ast d)^3 =
\]
\[
(b \ast c)^3 = (c \ast d)^4 = (a \ast b \ast d \ast c \ast d)^2 =
\]
\[
a \ast b \ast d \ast a \ast b \ast d \ast a \ast b \ast d \ast b \ast a \ast d =
\]
\[
a \ast c \ast a \ast d \ast a \ast c \ast a \ast d \ast a \ast c \ast d \ast b \ast d \ast c \ast a \ast d =
\]
\[
a \ast b \ast c \ast a \ast d \ast a \ast c \ast d \ast c \ast b \ast d \ast c \ast b \ast d \ast c \ast b \ast d \ast c \ast b \ast d = 1.
\]

7.4. **Group #7. [order 768]**. The order 384 group [# 20095 in the small group library] here has the structure:

\[[(Q_2YQ_2)@C_3]@ (C_2 \times C_2).\]
This is a nonsplit extension via a $C_4 \times C_2$ quotient.

Relations for this case are:

\[
\begin{align*}
b^4 &= a^3 \cdot (b^{-2}) = a^2 \cdot c \cdot (a^{-1}) \cdot c = a \cdot c^2 \cdot a \cdot (c^{-1}) = \\
a \cdot b \cdot a \cdot (b^{-1}) \cdot a \cdot (b^{-1}) &= a \cdot (c^{-1}) \cdot b \cdot c \cdot (a^{-1}) \cdot (b^{-1}) = \\
b \cdot c \cdot (b^{-1}) \cdot c \cdot (b^{-1}) \cdot c = \\
d^4 &= c^2 = d^2 = (c, d) = \\
a^d \cdot ((a \cdot c \cdot b)^{-1}) &= b^d \cdot ((a \cdot b \cdot c)^{-1}) = c^d \cdot ((b \cdot a \cdot c)^{-1}) = \\
a^c \cdot ((b \cdot c \cdot a)^{-1}) &= b^c \cdot ((a \cdot c \cdot a)^{-1}) = c^c \cdot ((c \cdot a \cdot b)^{-1}) = 1.
\end{align*}
\]

7.5. **Group #8. [order 1,536]**. This group has the structure:

\[
[\text{order } 192 \times C_2 \times C_2] \oplus C_2.
\]

The order 192 group in this representation is group number 47 in Table 2a. As of now we have not been able to reconstruct a presentation for this automorphism group using this decomposition. Cayley returns the following presentation for this automorphism group:

\[
\begin{align*}
a^2 &= b^2 = c^2 = d^2 = (a \cdot b)^2 = (a \cdot c \cdot b \cdot c)^2 = \\
a \cdot c \cdot d \cdot c \cdot d \cdot c \cdot a \cdot d &= (a \cdot d)^4 = (a \cdot d \cdot b \cdot d)^2 = \\
(b \cdot c)^4 &= (b \cdot d)^4 = a \cdot b \cdot c \cdot a \cdot b \cdot c \cdot a \cdot c \cdot a \cdot c = \\
b \cdot c \cdot b \cdot d \cdot b \cdot d \cdot c \cdot d \cdot b \cdot d = 1.
\end{align*}
\]

7.6. **Group #14. [order 73,728]**. The group can be written as:

\[
[(2048) \oplus (C_3 \times C_3)] \oplus (C_2 \times C_2).
\]

The relations for this group of order 2048 are

\[
\begin{align*}
T \cdot 1^2 &= T \cdot 2^2 = T \cdot 3^2 = T \cdot 4^2 = T \cdot 5^2 = T \cdot 6^2 = T \cdot 7^2 = T \cdot 8^2 = \\
(T \cdot 1 \cdot T \cdot 2)^2 &= (T \cdot 1 \cdot T \cdot 3)^2 = (T \cdot 1 \cdot T \cdot 4)^2 = (T \cdot 1 \cdot T \cdot 5)^2 = \\
(T \cdot 1 \cdot T \cdot 6)^2 &= (T \cdot 1 \cdot T \cdot 7)^2 = (T \cdot 2 \cdot T \cdot 3)^2 = (T \cdot 2 \cdot T \cdot 4)^2 = \\
(T \cdot 2 \cdot T \cdot 5)^2 &= (T \cdot 2 \cdot T \cdot 6)^2 = (T \cdot 2 \cdot T \cdot 7)^2 = (T \cdot 3 \cdot T \cdot 4)^2 = \\
(T \cdot 3 \cdot T \cdot 5)^2 &= (T \cdot 3 \cdot T \cdot 6)^2 = (T \cdot 4 \cdot T \cdot 5)^2 = (T \cdot 4 \cdot T \cdot 6)^2 = \\
(T \cdot 5 \cdot T \cdot 6)^2 &= (T \cdot 7 \cdot T \cdot 8)^2 = (T \cdot 3 \cdot T \cdot 6 \cdot T \cdot 8)^2 = \\
(T \cdot A \cdot T \cdot 5 \cdot T \cdot 7)^2 &= T \cdot 1 \cdot T \cdot 2 \cdot T \cdot 7 \cdot T \cdot 2 \cdot T \cdot 7 \cdot T \cdot 8 \cdot T \cdot 1 \cdot T \cdot 8 = \\
T \cdot 3 \cdot T \cdot 4 \cdot T \cdot 5 \cdot T \cdot 8 \cdot T \cdot 3 \cdot T \cdot 5 \cdot T \cdot 4 \cdot T \cdot 8 = \\
T \cdot 3 \cdot T \cdot 4 \cdot T \cdot 6 \cdot T \cdot 7 \cdot T \cdot 3 \cdot T \cdot 6 \cdot T \cdot 4 \cdot T \cdot 7 = \\
T \cdot 3 \cdot T \cdot 4 \cdot T \cdot 7 \cdot T \cdot 3 \cdot T \cdot 8 \cdot T \cdot 4 \cdot T \cdot 8 \cdot T \cdot 7 = \\
T \cdot 3 \cdot T \cdot 4 \cdot T \cdot 7 \cdot T \cdot 4 \cdot T \cdot 7 \cdot T \cdot 8 \cdot T \cdot 3 \cdot T \cdot 8 = 1.
\end{align*}
\]
The automorphism group of this 2-group of order 2048 has order $2^{31} \ast 3^2 \ast 7 = 135,291,469,824$. The 3-group actions on the 2-group are:

$$T.9^3 = T.10^3 = (T.9, T.10) =$$

$$T.1^{T.9} \ast ((T.1 \ast T.3 \ast T.4 \ast T.8 \ast T.1 \ast T.5 \ast T.8)^{-1}) =$$

$$T.2^{T.9} \ast ((T.1 \ast T.4 \ast T.8 \ast T.1 \ast T.6 \ast T.8)^{-1}) =$$

$$T.3^{T.9} \ast ((T.2 \ast T.3 \ast T.4 \ast T.5 \ast T.7 \ast T.6 \ast T.7)^{-1}) =$$

$$T.4^{T.9} \ast ((T.3 \ast T.8 \ast T.1 \ast T.6 \ast T.8)^{-1}) =$$

$$T.5^{T.9} \ast ((T.1 \ast T.2 \ast T.3 \ast T.6)^{-1}) =$$

$$T.6^{T.9} \ast ((T.1 \ast T.3 \ast T.4 \ast T.5 \ast T.6)^{-1}) =$$

$$T.7^{T.9} \ast ((T.3 \ast T.8 \ast T.3)^{-1}) =$$

$$T.8^{T.9} \ast ((T.1 \ast T.3 \ast T.8 \ast T.1 \ast T.3 \ast T.7)^{-1}) =$$

$$T.1^{T.10} \ast ((T.6 \ast T.7 \ast T.3 \ast T.7)^{-1}) =$$

$$T.2^{T.10} \ast ((T.1 \ast T.3 \ast T.4 \ast T.8 \ast T.1 \ast T.5 \ast T.6 \ast T.8)^{-1}) =$$

$$T.3^{T.10} \ast ((T.1 \ast T.2 \ast T.3 \ast T.5 \ast T.8 \ast T.4 \ast T.8)^{-1}) =$$

$$T.4^{T.10} \ast ((T.3 \ast T.5 \ast T.6 \ast T.7 \ast T.2 \ast T.7)^{-1}) =$$

$$T.5^{T.10} \ast ((T.2 \ast T.4 \ast T.6)^{-1}) =$$

$$T.6^{T.10} \ast ((T.1 \ast T.3 \ast T.7 \ast T.2 \ast T.5 \ast T.6 \ast T.7)^{-1}) =$$

$$T.7^{T.10} \ast ((T.1 \ast T.4 \ast T.8 \ast T.1 \ast T.4)^{-1}) =$$

$$T.8^{T.10} \ast ((T.7 \ast T.8)^{-1}) = 1.$$ 

A possible alternate representation of this automorphism group is:

$$[1^6@C_3 \times 1^3] @(C_2 \times S_4),$$

where the order 192 group is (#3 in Table 2a). A set of relations for this form has not been obtained.

A set of relations for this automorphism group generated by CAYLEY is:

$$a^2 = b^2 = c^2 = d^2 = e^2 = f^4 = (a, b) = (a, d) = (a, e) = (a, f) =$$

$$(b, d) = (b, e) = (b, f) = (c, e) = (c, f) = (d, f) = (a \ast c)^3 =$$

$$(c \ast f)^3 = (b \ast c)^4 = (c \ast d)^4 = (d \ast e)^4 = (a \ast b \ast c)^2 \ast b \ast a \ast c =$$

$$d \ast e \ast f^2 \ast e \ast d \ast e \ast (f^{-2}) \ast e = a \ast b \ast c \ast d \ast c \ast a \ast d \ast b \ast c \ast d \ast c =$$

$$c \ast (d \ast c \ast e)^2 \ast d \ast e \ast c \ast d \ast e =$$

$$b \ast c \ast d \ast e \ast d \ast c \ast b \ast c \ast d \ast (f^{-1}) \ast e \ast f \ast e \ast (f^{-1}) \ast d \ast c = 1.$$ 

7.7. **Group #15. [order 49,152]**. This group’s structure takes the form:

$$[8192 @ C_3] @ C_2,$$
where the order 8192 relations are:

\[ a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = \]
\[ (a * b)^2 = (a * c)^2 = (a * f)^2 = (b * d)^2 = \]
\[ (b * e)^2 = (c * d)^2 = (c * e)^2 = a * b * e * f * e * b * f = \]
\[ (a * c)^4 = (a * c * a * d)^2 = (a * c * b * c)^2 = \]
\[ a * c * f * d * a * d * f * c = (b * c)^4 = (b * f)^4 = \]
\[ (b * f * d * f)^2 = (c * f)^4 = (c * f * d * f)^2 = \]
\[ (c * f * e * f)^2 = (d * e)^4 = \]
\[ (d * f)^4 = b * c * b * c * f * d * e * d * e * f = 1. \]

The order 8192 group can be generated with six generators, has 254 classes, center \( C_2 \times C_2 \) and an automorphism group of order \( 2^{26} \times 3 = 201,326,592 \).

The action of the \( C_3 \) on this 2-group is

\[ h^3 = \]
\[ a^h * ((c * a * c * d * e * f * e * f * d)^{-1}) = \]
\[ b^h * ((f * e * f)^{-1}) = \]
\[ c^h * ((b * c * b * f * d * f)^{-1}) = \]
\[ d^h * ((a * f * c * a * f)^{-1}) = \]
\[ e^h * ((c * d * e * d * f * b * f * c)^{-1}) = \]
\[ (f, h) = 1, \]

which will yield the normal subgroup of order 24,576. This automorphism group does not have a normal sylow 2-subgroup. A CAYLEY generated set of relations for this automorphism group of order 49,152 is:

\[ a^6 = b^6 = c^2 = (a * (b^{-1}))^2 = a^2 * (b^{-2}) * a * b * (a^{-1}) * (b^{-1}) = \]
\[ (a * c)^4 = (b * c)^4 = a * (b^{-2}) * a * c * (a^{-1}) * b * a * (b^{-1}) * c = \]
\[ a^2 * b * a^2 * (b^{-2}) * (a^{-1}) * (b^{-2}) * (a^{-1}) * b = \]
\[ a^2 * c * a * (b^{-1}) * c * (b^{-2}) * c * (a^{-1}) * b * c = \]
\[ (a^2 * c)^4 = a^2 * c * b * a * c * b^2 * c * a * b * c = \]
\[ a * b * c * a * (b^{-1}) * c * (a^{-1}) * (b^{-1}) * c * (a^{-1}) * b * c = \]
\[ (a * c * (a^{-1}) * c)^3 = (a * c * (b^{-1}) * c)^3 = \]
\[ a^3 * (b^{-3}) * c * (b^{-2}) * (a^{-1}) * b * a^2 * c = 1. \]

7.8. Group #16. [order 49,152]. This group can be represented as:

\[ (8192\oplus C_3)\oplus C_2 = 8192\oplus S_3 \]

or

\[ (4096)\oplus C_3)\oplus (C_2 \times C_2) . \]
Relations for the 8192 group and its extensions by $C_3$:

Sylow 2-factor [order 8192 case]:

\[ a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = h^2 = \]
\[ (a, b) = (a, c) = (b, d) = (c, d) = (d, e) = (f, h) = \]
\[ (a * b * e)^2 = (a * b * f)^2 = (a * c * h)^2 = (b * c * e)^2 = \]
\[ (c * d * f)^2 = (d * e * h)^2 = \]
\[ a * b * c * b * c * d * a * d = (a * e)^4 = a * e * f * h * e * a * h * f = \]
\[ (a * f)^4 = (a * f * a * h)^2 = (a * f * e * f)^2 = \]
\[ (a * h)^4 = b * d * f * h * b * d * h * f = (b * h * c * h)^2 = 1. \]

\[ \text{Aut(8192 group)} \] has order $2^{27} * 3^2 = 1, 207, 959, 552$. The automorphism group of this group gives rise to the following four actions of order 3 on this 2-group. The group appearing as a normal subgroup of Aut(64# 16) is the above 2-group with the $C_3$ action in case four below.

Case 1:

\[ k^3 = \]
\[ a^k * ((c * f * d * e * b * f * e)^{-1}) = \]
\[ b^k * ((b * c * e * c * e)^{-1}) = \]
\[ c^k * ((a * b * f * c * e * a * f)^{-1}) = \]
\[ d^k * ((a * f * e * f * d)^{-1}) = \]
\[ e^k * ((f * e * h * c * h * e * f)^{-1}) = \]
\[ f^k * ((a * c * f * a * c)^{-1}) = \]
\[ (k, h) = 1. \]

This group can also be generated by $< a, b, d, e, h, k >$. It has 161 classes and center $= 1$. The automorphism group of this group has order $2^{17} * 3^2 = 1, 179, 648$.

Case 2:

\[ a^k * ((b * f * d * f * b)^{-1}) = \]
\[ b^k * ((h * e * b * e * h)^{-1}) = \]
\[ c^k * ((f * b * c * b * f)^{-1}) = \]
\[ d^k * ((a * c * h * e * b * h * d)^{-1}) = (e, k) = \]
\[ f^k * ((a * d * h * d * a * f)^{-1}) = \]
\[ h^k * ((c * e * f * e * c)^{-1}) = 1. \]

This group can also be generated by $< a, b, c, f, h, k >$. This group has 145 classes and center $= 1$. The automorphism group of this group has order $2^{20} * 3^2 = 9, 437, 184$. 
Case 3:

\[ a^k \ast ((a \ast b \ast c \ast e \ast f \ast e \ast f)^{-1}) = \]
\[ b^k \ast ((c \ast f \ast h \ast b \ast f \ast h \ast c)^{-1}) = \]
\[ c^k \ast ((b \ast f \ast e \ast f \ast h \ast c \ast h)^{-1}) = \]
\[ d^k \ast ((d \ast h \ast b \ast f \ast h \ast c \ast f)^{-1}) = \]
\[ e^k \ast ((f \ast e \ast h \ast c \ast h \ast e \ast f)^{-1}) = \]
\[ f^k \ast ((b \ast d \ast h \ast b \ast d)^{-1}) = \]
\[ h^k \ast ((a \ast b \ast f \ast h \ast a \ast b)^{-1}) = 1. \]

This group can also be generated by \( <a, b, d, f, h, k> \). This group has 145 classes and center = I. The automorphism group of this group has order \( 2^{20} \ast 3^2 = 9,437,184 \). This is the same as Case 2.

Case 4: This is the group action that will give the order 8192 \ast 3 group appearing as a normal subgroup of \( \text{Aut}(64 \# 16) \).

\[ k^3 = \]
\[ a^k \ast ((a \ast b \ast f \ast d \ast f \ast a \ast c)^{-1}) = \]
\[ b^k \ast ((a \ast f \ast h \ast b \ast f \ast h \ast a)^{-1}) = \]
\[ c^k \ast ((d \ast f \ast e \ast f \ast d)^{-1}) = \]
\[ d^k \ast ((a \ast b \ast d \ast f \ast h \ast c \ast f \ast h \ast b)^{-1}) = \]
\[ e^k \ast ((h \ast b \ast e \ast c \ast h)^{-1}) = \]
\[ f \ast k \ast ((a \ast b \ast h \ast a \ast b)^{-1}) = \]
\[ h^k \ast ((a \ast b \ast f \ast h \ast a \ast b)^{-1}) = 1. \]

This group can also be generated by \( <a, b, c, d, e, f, k> \). This group has 161 classes and a center of order 4. The automorphism group of this group has order \( 2^{13} \ast 3^2 = 73,728 \).

This group does not have a normal sylow 2-subgroup.
A presentation for the automorphism group of order 49,152 is:

\[
\begin{align*}
    a^5 &= b^6 = d^2 = (a * (b^{-1}))^3 = (a * (c^{-1}))^3 = (b * (c^{-1}))^2 = \\
    (b^2 * c)^2 &= (b * c^2)^2 = a^2 * b * a^2 * b^2 * a * b = \\
    a^2 * b * ((a * b^2 * a)^{-1}) * b &= a^2 * (b^{-1}) * c * (a^{-2}) * c * (b^{-1}) = \\
    (a * b)^2 * ((b * a)^{-1})^2 &= a * b * ((a * c)^{-1}) * c^2 * (b^{-2}) = \\
    a * (b^{-1}) * a * d * (b^{-1}) * a * (b^{-1}) * d = \\
    a * (b^{-1}) * d * c * (a^{-1}) * b * d * (c^{-1}) = \\
    (a * d * (b^{-1}) * d)^2 &= (a * d)^4 = (b * d)^4 = \\
    a^2 * b * (a^{-1}) * d * (a^{-1}) * b * a^2 * d = \\
    a * b * ((b * a)^{-1}) * d * ((a * b)^{-1}) * b * a * d = 1.
\end{align*}
\]

7.9. **Group #17. [order 6,144]**. This group has the structure:

\[
[(256 @ C_3) @ (C_2 \times C_2)].
\]

Four possible cases of a $C_3$ action on the order 256 group were found by CAYLEY. Only one generates the correct order 768 group.

2-group relations:

\[
\begin{align*}
    a^2 &= b^2 = c^2 = d^2 = e^2 = f^2 = (a, b) = (a, c) = (a, d) = (b, c) = (b, d) = \\
    (b, e) = (c, d) = (c, f) = (e, f) = (a * b * f)^2 &= a * e * f * a * f * e = (c * d * e)^2 = \\
    d * e * f * d * f * e = 1.
\end{align*}
\]

This 2-group is number 55805 of order 256 in the Small Group Library of Besche and Eick. This 2-group has an automorphism group of order 589,824 = 2^{16} \times 3^2.

$C_3$ actions:

**Case 1.** In this case the center equals I, and this action does not yield the correct group. According to GAP this is group number 1085112 of order 768 in the Small Group Library. This group’s automorphism group has order 18432.

\[
\begin{align*}
    a^h * e * a * e * d * c * a &= \\
    b^h * e * c * e &= \\
    c^h * e * a * e * d &= \\
    d^h * e * d * e * c * b &= \\
    e^h * f = f^h * f * e &= 1.
\end{align*}
\]

**Case 2.** This action generates a group that looks like case one, above, and is identified as [768 \# 1085110] in the Small Group Library. This group’s automorphism group has order 36864. The automorphism group arising in this case is a complete group with 56 classes. This complete group may be the same as the one
arising from Aut[64 # 183] @ $C_3$] below.

\[
\begin{align*}
    a^h * e * a * d * e &= \\
    b^h * e * d * e * c * b &= \\
    c^h * f * d * b * f * d * a &= \\
    d^h * e * c * e * c * a &= \\
    e^h * c * a * e * c * a &= \\
    f^h * a * f * a &= 1.
\end{align*}
\]

**Case 3.** According to GAP this action generates the same group as case one above, namely order 768 #1085112.

\[
\begin{align*}
    a^h * f * d * b * f * c &= \\
    b^h * c * b * a &= \\
    c^h * e * c * a * e * c &= \\
    d^h * b &= \\
    e^h * d * f * d &= \\
    f^h * f * c * a * e * c * a &= 1.
\end{align*}
\]

**Case 4.** This action looks like the one that generates the correct group of order 768, [768 # 1085111]. This group’s automorphism group has order 9216.

\[
\begin{align*}
    h^3 &= \\
    a^h * e * c * e * c * b &= \\
    b^h * e * c * e * c * b * a &= \\
    c^h * e * d * e &= \\
    d^h * e * d * e * c &= \\
    e^h * a * f * a &= f^h * f * c * e * c = 1.
\end{align*}
\]

The next step in the sequence: finding the $C_2 \times C_2$ actions on this 768 group has not been done. The subgroup lattice for this case may be rather hard to determine. An alternate presentation for this automorphism group’s factor of order 3072 is:

\[
\begin{align*}
    a^6 &= c^6 = d^2 = (a * (b^{-1}))^2 = (a * (c^{-1}))^2 = \\
    (a * (d^{-1}))^2 &= (b * (d^{-1}))^2 = (c * (d^{-1}))^2 = \\
    (a^2 * b)^2 &= (a * b^2)^2 = \\
    a^2 * (b^{-1}) * (c^{-1}) * b * (c^{-1}) * (a^{-1}) * b &= \\
    a^2 * c^{-2} * a * c * (a^{-1}) * (c^{-1}) &= \\
    a * (b^{-1}) * c^{-2} * (a^{-1}) * b * c^2 &= \\
    b * c * b * c^2 * (b^{-1}) * c^2 &= 1.
\end{align*}
\]
7.10. **Group #19. [order 12,288].** This group has the structure:

\[1536 \times C_2 \times C_2]_C.

The order 1536 group has as a normal subgroup:

\[1^6 \circ C_3\] with 24 classes and trivial center,

and the quotient group 1536/192 is the elementary abelian group of order 8. Cayley gives the following alternate description of this automorphism group of order 12,288:

\[a^2 = b^2 = c^2 = d^4 = e^4 = f^2 = (a, b) = (a, d) = (a, e) =
(b, d) = (b, e) = (c, d) = (c, e) = (d, f) =
\]

\[a \ast d^2 \ast f \ast a \ast f = d^2 \ast e \ast (d^{-2}) \ast e = d \ast e \ast d \ast (e^{-1}) \ast (d^{-1}) \ast (e^{-1}) =
\]

\[e^2 \ast f \ast (e^{-2}) \ast f = (a \ast c)^4 = (a \ast c \ast b \ast c)^2 =
(b \ast c)^4 = (b \ast c \ast b \ast f)^2 = (b \ast f)^4 =
\]

\[e \ast f \ast e \ast f \ast (e^{-1}) \ast f \ast (e^{-1}) \ast f =
\]

\[b \ast e \ast f \ast b \ast e \ast f \ast b \ast (e^{-1}) \ast f \ast (e^{-1}) \ast b \ast f =
\]

\[b \ast c \ast b \ast c \ast d \ast c \ast d \ast f \ast (e^{-1}) \ast f \ast (d^{-1}) \ast (e^{-1}) \ast f = 1.
\]

7.11. **Group #21. [order 1,536].** This group has the structure:

\[C_2 \times \text{[(order 96 group; #36 in Table 3b of [7])} \times C_2 \times C_2]_C.
\]

There are 5 possible actions here by the \(C_2\) quotient group, three of which yield a group of the form \(D_4 \times 96\). The \(C_2\) action on \(C_2 \times C_2\) is the same as in the case of a wreath product. The other two cases listed below appear to give the order 768 factor in the automorphism group for this group:

\[a^4 = b^4 = (a \ast b)^3 = (a \ast (b^{-1}))^3 = (a^2, b^2) = c^2 = d^2 =
\]

\[= (c \ast d)^4 = (a, c) = (b, c) =
\]

\[\begin{cases}
  a^d \ast a = b^d \ast ((a \ast b \ast a)^{-1}) = 1 \\
  \text{or}
  a^d \ast a = (b \ast a^2 \ast b)^{-1} = b^d \ast a \ast b \ast ((a \ast b \ast a)^{-1}) = 1.
\end{cases}
\]

The group in question here (of order 768) is [768, #1087581] in the Small Group Library. Its automorphism group has order 1536. The group generated by \(< a, b >\) is [96, #227] and has an automorphism group of order 576 [576, #8654].

7.12. **Group #22. [order 12,288].** This automorphism group and that for group #68 below appear to be isomorphic. This automorphism group has the structure:

\[[(\text{order 256 group} \times C_2 \times C_2) \circ C_3] \circ (C_2 \times C_2).
\]

The automorphism group of the order 3072 normal subgroup has order \(2^{20} \ast 3^3\).

There are 4 possible \(C_3\) actions on the order 256 group. The automorphism group of the 256 group is \(2^{19} \ast 3^2\). See group #68 below for details.
For this automorphism group, Cayley gives the following presentation:

\[ a^2 = b^4 = c^4 = d^2 = a \ast b \ast a \ast (b^{-1}) = (a \ast b \ast d)^2 = \\
  a \ast c^2 \ast a \ast (c^{-2}) = b^2 \ast c \ast (b^{-2}) \ast c = \\
  b \ast c^2 \ast b \ast (c^{-2}) = c^2 \ast d \ast (c^{-2}) \ast d = \\
  a \ast c \ast a \ast c \ast a \ast (c^{-1}) \ast a \ast (c^{-1}) = (a \ast d)^4 = \\
  c \ast d \ast c \ast d \ast (c^{-1}) \ast d \ast (c^{-1}) \ast d = \\
  a \ast c \ast b \ast c \ast a \ast (c^{-1}) \ast b \ast (c^{-1}) \ast (b^{-1}) \ast c = \\
  a \ast c \ast d \ast a \ast d \ast c \ast a \ast (c^{-1}) \ast d \ast a \ast d \ast (c^{-1}) = \\
  a \ast c \ast d \ast c \ast a \ast d \ast a \ast (c^{-1}) \ast d \ast (c^{-1}) \ast a \ast d = \\
  b \ast c \ast b \ast c \ast b \ast c \ast (b^{-1}) \ast (c^{-1}) \ast (b^{-1}) \ast (c^{-1}) = \\
  b \ast d \ast c \ast d \ast b \ast d \ast (c^{-1}) \ast d \ast (b^{-1}) \ast d \ast (c^{-1}) \ast d = \\
  a \ast b \ast c \ast d \ast b \ast c \ast b \ast c \ast a \ast d \ast (c^{-1}) \ast d \ast (c^{-1}) = 1. \\
\]

7.13. **Group #30. [order 12,288]**. This group has the structure:

\[ C_2 \times [1024 \oplus C_3] \oplus C_2, \]

where the order 6144 factor has 44 classes and a trivial center. This order 3072 group appears to be the same as the 3072 factor in the automorphism group 64 #93. See group 64 #173 below for an alternate presentation of the following groups of orders 1024 and 3072.

relations for order 1024 group:

\[ a^2 = b^2 = c^2 = d^2 = e^2 = (a \ast b)^2 = (a \ast c)^2 = (b \ast c)^2 = \\
  a \ast d \ast e \ast d \ast a \ast c = (a \ast d)^4 = (a \ast d \ast b \ast d)^2 = (a \ast d \ast c \ast d)^2 = \\
  (a \ast e)^4 = (a \ast e \ast b \ast e)^2 = (a \ast e \ast c \ast e)^2 = \\
  (b \ast d)^4 = \\
  (b \ast d \ast c \ast d)^2 = (b \ast e)^4 = (b \ast e \ast c \ast e)^2 = (c \ast d)^4 = (c \ast e)^4 = \\
  a \ast b \ast c \ast d \ast b \ast e \ast c \ast e \ast d = 1. \]

This 2-group has an automorphism group of order \(2^{20} \ast 3^2 = 9,437,184\), and center \(C_2\). Possible \(C_3\) actions:

The following two actions seem to generate the same group of order 3072 which appears in this decomposition:

\[ f^3 = \]

\[ \begin{cases} a^f \ast e \ast b \ast e \ast d \ast c \ast d \ast c \ast b = b^f \ast e \ast d \ast e \ast c \ast a \ast d \ast a = \\
     c^f \ast e \ast b \ast a \ast e \ast c \ast a = d^f \ast c \ast d \ast c \ast e \ast a = e^f \ast d = 1 \\
     \text{or} \\
     (a, f) = b^f \ast e \ast c \ast b \ast e = c^f \ast e \ast d \ast e \ast c \ast d \ast c \ast b = \\
     d^f \ast d \ast c \ast a \ast e \ast c = e^f \ast b \ast c \ast d \ast e \ast b = 1. \end{cases} \]
7.14. Group #44. [order 6,144]. The structure of this group is:

\[ [768@ (C_2 \times C_2 \times C_2)], \]

where the 768 group has 36 classes and a center of \( C_2 \times C_2 \). This 768 group appears to be the same as the one appearing in group #17 above and #108 below. 

Cayley gives the following presentation for this automorphism group on five generators:

\[
\begin{align*}
a^2 &= b^2 = d^4 = c^2 = e^2 = (a * b)^2 = a * d * a * (d^{-1}) = \\
(a * e)^2 &= b * d * b * (d^{-1}) = (b * e)^2 = (d * (e^{-1}))^2 = (a * c)^3 = \\
c * d^2 * c * (d^{-2}) &= (b * c)^4 = (b * c * d * c)^2 = \\
c * d * c * d * c * (d^{-1}) * c * (d^{-1}) = \\
(c * d * c * e)^2 &= (c * e)^4 = a * b * c * a * b * c * b * a * c = \\
b * c * b * e * c * b * c * b * c * c = 1.
\end{align*}
\]

7.15. Group #68. [order 12,288]. This automorphism group appears to be isomorphic to that for group #22 above.

This group does not have a normal sylow 2-subgroup. The structure looks like:

\[ [(256 \times C_2 \times C_2) \oplus C_3] \oplus (C_2 \times C_2), \]

where the order 256 2-group is generated by

\[
\begin{align*}
a^2 &= b^2 = c^2 = d^2 = (a * b)^2 = (c * d)^2 = (a * c)^4 = \\
(a * c * a * d)^2 &= (a * c * b * c)^2 = (a * d)^4 = \\
(a * d * b * d)^2 &= (b * c)^4 = (b * c * b * d)^2 = (b * d)^4 = 1.
\end{align*}
\]

This group is [256, #8935] in the Small Group Library. The order of the automorphism group of this 2-group of order 256 is \( 2^{10} * 3^2 = 4,718,592 \). CAYLEY gives the following four possible \( C_3 \) actions on this 2-group:

**Case 1.** This case has a trivial center. This action gives group 768 #1083695, with an automorphism group of order 36,864 and center I:

\[
a^e \ast ((a * d \ast a * b * d)^{-1}) = b^e \ast ((a * d \ast b * d)^{-1}) = \\
c^e \ast ((a * b * c * b * d * a * d)^{-1}) = d^e \ast ((a * b * c * a * c * d * b)^{-1}) = 1.
\]

**Case 2.** This case has center \( C_2 \times C_2 \) and is the desired 3072 group. This corresponds to 768 #1083694 and has an automorphism group of order 36,864 that is not isomorphic to that in Case 1:

\[
\begin{align*}
a^e \ast ((d * a * b * d)^{-1}) &= b^e \ast ((a * b * c * b * d)^{-1}) = \\
c^e \ast ((a * c * a * b * d * a * c * b * d)^{-1}) &= d^e \ast ((a * b * d * a * c * b * d)^{-1}) = 1.
\end{align*}
\]

**Case 3.** This action generates the same group as in Case 2:

\[
\begin{align*}
(a, c) &= b^e \ast ((a * c * a * b * c)^{-1}) = \\
e^e \ast (d^{-1}) &= d^e \ast (c * d)^{-1} = 1.
\end{align*}
\]
Case 4. This case is isomorphic to Case 2:

\[ a^c \times ((d \ast a \ast b \ast d)^{-1}) = b^c \times ((a \ast b \ast c \ast d \ast b \ast c \ast d)^{-1}) = c^c \times ((a \ast b \ast c \ast b \ast d \ast a \ast c)^{-1}) = d^c \times ((a \ast c \ast a \ast b \ast d \ast b)^{-1}) = 1, \]

with the presentation for the \([\langle 256 \rangle \times C_2 \times C_2 \rangle \mathfrak{g} C_3]\) subgroup given by:

\[
\begin{align*}
a^2 &= b^2 = c^2 = d^2 = (a \ast b)^2 = (c \ast d)^2 = (a \ast c)^2 = (a \ast c \ast b \ast c)^2 = (a \ast d)^4 = \\
(a \ast c \ast a \ast d)^2 &= (a \ast c \ast b \ast d)^2 = (b \ast c)^4 = (b \ast c \ast b \ast d)^2 = (b \ast d)^4 = \\
e^3 &= f^2 = h^2 = (f, h) = (a, f) = (b, f) = (c, f) = (d, f) = \\
(a, h) &= (b, h) = (c, h) = (d, h) = \\
a^c \times ((a \ast d \ast a \ast b \ast d)^{-1}) &= b^c \times ((a \ast d \ast b \ast d)^{-1}) = \\
c^c \times ((a \ast b \ast c \ast b \ast d \ast a \ast d)^{-1}) &= d^c \times ((a \ast b \ast c \ast a \ast b \ast c \ast d)^{-1} = \\
f^c \times h = h^c \times f \times h = 1.
\end{align*}
\]

The order of the automorphism group of the order 1024 group is \(2^{36} \ast 3^3\).

The automorphism group of the resulting order 3072 group has order \(2^{20} \ast 3^3 = 28,311,552\), with 223 conjugacy classes and a trivial center. The next step would be to find the \(C_2 \times C_2\) actions on the resulting order 3072 group. This, in view of the order of the automorphism group of this group, is difficult to do computationally.

Cayley gives the following presentation for this automorphism group of order 12,288 on five generators:

\[
\begin{align*}
a^2 &= b^2 = c^2 = d^2 = e^4 = (b \ast c)^2 = (b \ast d)^2 = \\
(b \ast e)^2 &= (c \ast d)^2 = a \ast e^2 \ast a \ast (e^{-2}) = (c \ast e)^3 = \\
d \ast e^2 \ast d \ast (e^{-2}) &= a \ast c \ast a \ast c \ast e \ast a \ast (e^{-1}) = \\
(a \ast b)^4 &= a \ast b \ast a \ast b \ast c \ast (e^{-2}) \ast c = (a \ast d)^4 = \\
d \ast e \ast d \ast e \ast d \ast (e^{-1}) \ast d \ast (e^{-1}) &= c \ast d \ast c \ast d \ast c \ast d \ast c \ast e = \\
a \ast c \ast d \ast a \ast c \ast d \ast a \ast d \ast c \ast a \ast d \ast c = \\
a \ast d \ast a \ast c \ast d \ast e \ast a \ast d \ast a \ast (e^{-1}) \ast d \ast (e^{-1}) = 1.
\end{align*}
\]

7.16. **Group #76.** [**order 6144**]. This group has a normal sylow 2-subgroup. The order of the automorphism group of the sylow 2-subgroup is \(2^{34} \ast 3^2 \ast 7\). A
presentation in terms of the sylow 2-subgroup and a $C_3$ quotient is given by:

$$\begin{align*}
a^2 &= b^2 = c^2 = d^2 = e^2 = (a * b)^2 = (a * c)^2 = \\
(b * c)^2 &= (d * e)^2 = (a * d)^4 = (a * d * a * e)^2 = \\
(a * d * b * d)^2 &= (a * d * c * d)^2 = (a * e)^4 = \\
(a * e * b * e)^2 &= (a * e * c * e)^2 = (b * d)^4 = \\
(b * d * b * e)^2 &= (b * d * c * d)^2 = (b * e)^4 = \\
(b * e * c * e)^2 &= (c * d)^4 = (c * d * c * e)^2 = (c * e)^4 = \\
f^3 &= \\
a^f * ((b * d * e * b * d * a * c * e)^{-1}) &= \\
b^f * ((a * c * d * e * a * b * d * e)^{-1}) &= \\
c^f * ((a * b * c * d * c * d * c * a * e)^{-1}) &= \\
d^f * ((a * b * d * b * e * a)^{-1}) &= \\
e^f * ((a * b * c * a * d * e * b)^{-1}) &= 1
\end{align*}$$

or

$$\begin{align*}
a^f * ((b * d * e * b * d * a * c * e)^{-1}) &= \\
b^f * ((c * e * b * e)^{-1}) &= \\
c^f * ((d * b * d)^{-1}) &= \\
d^f * ((a * b * c * d * b * c * e * a * d)^{-1}) &= \\
e^f * ((a * b * c * e * a * b * d * c)^{-1}) &= 1
\end{align*}$$

The automorphism group has order $4,718,592$ ($2^{19} * 3^2$) and a trivial center.

Cayley gives the following presentation for this automorphism group (Aut[64 $\#76$]) using only four generators:

$$\begin{align*}
a^2 &= b^2 = c^2 = d^6 = (b * c)^2 = a * b * a * d * b * (d^{-1}) = \\
a * d * a * d * a * (d^{-2}) &= (a * b)^4 = (a * b * a * c)^2 = (a * c)^4 = \\
(b * d * c * (d^{-1}))^2 &= (c * d * c * (d^{-1}))^2 = \\
c * d^3 * c * (d^{-3}) &= (a * c * (d^{-1}))^3 = \\
a * b * (d^{-1}) * b * d * a * (d^{-1}) * b * d * b &= \\
a * c * d * a * c * d * a * c * (d^{-2}) &= 1.
\end{align*}$$
Another alternate 4-generator version of Aut(#76) is
\[ a^2 = b^2 = c^2 = d^2 = (a, b) = (a, c) = (a, d) = (c, d) = 
\]
\[ a * c * e * a * c * e^{-1} = b * c * b * e * c * e^{-1} = 
\]
\[ e^6 = a * b * c * b * e^{-1} * c * e = b * e * b * e * b * e^{-2} = 
\]
\[ (b * c * b * d)^2 = (b * d)^4 = (d * e * d * e^{-1})^2 = 
\]
\[ d * e^3 * d * e^{-3} = (b * d * e^{-1})^3 = 
\]
\[ (b * d * e)^2 * b * d * e^{-2} = 1. 
\]

7.17. **Group #81. [order 12,288]**. See Group #30 above.

7.18. **Group #82. [order 12,288]**. This automorphism group has the structure $[512@C_3]@D_4$. The automorphism group of $512@C_3$ has order 589,824.

7.19. **Group #93. [order 6,144]**. The structure of this group is 
\[ C_2 \times [1024 @ C_3]. \]
The automorphism group of the 1024 group has order $2^{20} * 3^2$. See Group #30.

7.20. **Group #103. [order 36,864]**. A presentation for this automorphism group is:
\[ a^2 = b^4 = c^4 = d^2 = e^4 = a * b * a * (b^{-1}) = 
\]
\[ a * c * a * (c^{-1}) = a * e * a * (e^{-1}) = 
\]
\[ b * d * (b^{-1}) * d = a * b^2 * d * a * d = 
\]
\[ b^2 * c * (b^{-2}) * c = b^2 * e * (b^{-2}) * e = 
\]
\[ b * c^2 * (b^{-1}) * (c^{-2}) = 
\]
\[ b * e * b * (e^{-1}) * (b^{-1}) * (e^{-1}) = 
\]
\[ c^2 * d * (c^{-2}) * d = c^2 * e * (c^{-2}) * e = 
\]
\[ d * e^2 * d * (e^{-2}) = 
\]
\[ b * c * b * c * (b^{-1}) * (e^{-1}) * (b^{-1}) * (c^{-1}) = 
\]
\[ b * c * (d^{-1}) * c * (b^{-1}) * (c^{-1}) * (d^{-1}) * (c^{-1}) = 
\]
\[ b * c * e * c * (b^{-1}) * (c^{-1}) * (e^{-1}) * (c^{-1}) = 
\]
\[ c * d * c * d * (c^{-1}) * d * (c^{-1}) = 
\]
\[ c * d * (c^{-1}) * (e^{-1}) * (c^{-1}) * d * c * e = 
\]
\[ c * e * c * e * (c^{-1}) * (e^{-1}) * (c^{-1}) * e = 
\]
\[ d * e * d * e * d * (e^{-1}) * d * (e^{-1}) = 
\]
\[ b * d * e * b * d * (e^{-1}) * d * (b^{-1}) * (e^{-1}) = 1. 
\]
7.21. **Group #104.** [order 61,440]. A presentation on four generators for this automorphism group is:

\[
\begin{align*}
a^2 &= b^4 = c^4 = a \ast c \ast a \ast (c^{-1}) = \\
a \ast d \ast a \ast (d^{-1}) &= b^2 \ast (d^{-2}) = \\
b^2 \ast c \ast (b^{-2}) \ast c &= b \ast c^2 \ast (b^{-1}) \ast (c^{-2}) = \\
a \ast b \ast a \ast b \ast a \ast (b^{-1}) \ast a \ast (b^{-1}) = \\
a \ast b \ast a \ast b \ast d \ast (c^{-2}) \ast (d^{-1}) = \\
b \ast d \ast b \ast d \ast (b^{-1}) \ast (d^{-1}) \ast (b^{-1}) \ast (d^{-1}) = \\
c \ast d \ast c \ast (d^{-1}) \ast (c^{-1}) \ast (d^{-1}) \ast (c^{-1}) \ast (d^{-1}) = \\
a \ast b \ast a \ast b \ast c \ast b \ast a \ast (b^{-1}) \ast c \ast (b^{-1}) \ast (c^{-1}) = \\
b \ast c \ast b \ast c \ast d \ast c \ast b \ast c \ast d \ast (c^{-1}) \ast (d^{-1}) \ast (b^{-1}) \ast (c^{-1}) \ast (d^{-1}) = 1.
\end{align*}
\]

7.22. **Group #105.** [order 23,040]. A presentation for this complete group is:

\[
\begin{align*}
a^4 &= b^2 = c^4 = d^2 = a \ast b \ast (a^{-1}) \ast b = a^2 \ast c \ast (a^{-2}) \ast c = \\
a \ast c^2 \ast a \ast (c^{-2}) &= a^2 \ast b \ast d \ast (a^{-2}) \ast b \ast d = \\
a \ast d \ast a \ast d \ast (a^{-1}) \ast d \ast (a^{-1}) \ast d = a \ast d \ast c \ast d \ast a \ast d \ast (c^{-1}) \ast d = \\
(b \ast c)^4 = (b \ast c \ast b \ast (c^{-1}))^2 = (c \ast d \ast (c^{-1}) \ast d)^2 = \\
a \ast c^2 \ast (d^{-1}) \ast (b^{-1}) \ast (d^{-1}) \ast (a^{-1}) \ast (c^{-2}) \ast (b^{-1}) = \\
a \ast c \ast a \ast c \ast d \ast (c^{-2}) \ast b \ast d \ast (a^{-1}) \ast b \ast c = \\
a \ast c \ast b \ast c \ast d \ast a \ast d \ast (a^{-1}) \ast b \ast c \ast b \ast c = 1.
\end{align*}
\]

7.23. **Group #108.** [order 12,288]. This group has the structure

\[
768\cong(D_4 \times C_2).
\]

The order 768 group appears to be the same as appearing in groups #17 and #44 above. A set of relations for this form for the automorphism group has not yet been found.

An alternate presentation for this automorphism group is:

\[
\begin{align*}
a^2 &= b^4 = c^2 = a \ast b \ast a \ast b \ast a \ast (b^{-1}) \ast a \ast (b^{-1}) = \\
a \ast c \ast b^2 \ast c \ast a \ast c \ast (b^{-2}) \ast c &= \\
b^2 \ast c \ast (b^{-1}) \ast c \ast (b^{-2}) \ast c \ast (b^{-1}) \ast c = \\
a \ast b^2 \ast a \ast b \ast c \ast a \ast c \ast b \ast c \ast a \ast c &= \\
a \ast b^2 \ast c \ast b \ast c \ast a \ast c \ast (b^{-1}) \ast c \ast (b^{-2}) = \\
b \ast c \ast b \ast c \ast b \ast c \ast (b^{-1}) \ast c \ast (b^{-1}) \ast c \ast (b^{-1}) \ast c = \\
a \ast b \ast a \ast b^2 \ast c \ast a \ast c \ast a \ast (b^{-1}) \ast a \ast c \ast a \ast c &= \\
a \ast b \ast c \ast b \ast a \ast b \ast c \ast b \ast a \ast (b^{-1}) \ast c \ast (b^{-1}) \ast a \ast (b^{-1}) \ast c \ast (b^{-1}) = 1.
\end{align*}
\]
7.24. **Group #109. [order 1,536]**. This group has the structure

\[\text{[order 96, 11 classes]} @ (1^4 \text{group}),\]

where the automorphism group of the order 96 group has order 576. It appears that this extension is a nonsplit extension, since the straightforward application of the \((C_2 \times C_2 \times C_2 \times C_2)\) action on the order 96 group does not give the correct automorphism group.

An alternate presentation for this automorphism group is:

\[
a^2 = b^2 = c^2 = d^2 = e^4 = (a * b)^2 = (a * d)^2 = a * e * a * (e^{-1}) = \\
(b * c)^2 = b * e * b * (e^{-1}) = (d * e)^3 = (a * c)^4 = \\
a * c * a * c * d * (e^{-2}) * d = (b * d)^4 = b * d * b * e * d * (e^{-1}) * d * e = \\
c * e * c * e * c * (e^{-1}) * c * (e^{-1}) = c * e * d * (e^{-1}) * c * (e^{-1}) * d * e = 1.\]

7.25. **Group #153. [order 10,752]**. This automorphism group has the structure

\[(1^9 @ C_7) @ C_3.\]

The action of the group \(C_7 @ C_3\) on the elementary abelian group of order 512 is obtained by considering the \(C_3\) and the \(C_7\) to act on each of the three triplets \(C_2 \times C_2 \times C_2\) as follows:

\[
a^2 = b^2 = c^2 = (a, b) = (a, c) = (b, c) = \\
h^7 = k^3 = h^5 = \\
a^h * b * c = b^h * a = c^h * b = \\
(a, k) = b^k * c = c^k * a * b * c = 1.\]

A Cayley-generated presentation for this automorphism group is:

\[
a^3 = b^3 = c^3 = (a * (b^{-1}))^2 = \\
a * b * a * c * b * (a^{-1}) * c * (a^{-1}) * b * c = \\
a * c * a * c * (a^{-1}) * (c^{-1}) * (a^{-1}) * c * (a^{-1}) * (c^{-1}) = \\
a * b * (c^{-1}) * b * c * a * c * (b^{-1}) * (c^{-1}) * (a^{-1}) * c = \\
a * c * (a^{-1}) * (b^{-1}) * (c^{-1}) * (a^{-1}) * (c^{-1}) * b * a * c * b * (c^{-1}) = 1.\]

7.26. **Group #173. [order 6,144]**. This automorphism group does not have a normal sylow 2-subgroup. It does however have a normal subgroup of order 3072 that does have a normal sylow 2-subgroup. The automorphism group thus can be represented as

\[[1024 @ C_3] @ C_2.\]

There are three different possible groups of order 3072 that can be used to express the automorphism group of the form here. The automorphism groups of these groups have the orders 294,912 (2 cases) or 589,824. An alternate presentation on
four generators for the 1024 group is:

\[
\begin{align*}
a^2 &= b^2 = c^2 = d^2 = (a \ast b)^2 = (a \ast c)^4 = (a \ast c \ast b \ast c)^2 = \\
(a \ast d)^4 &= (a \ast d \ast b \ast d)^2 = (b \ast c)^4 = (b \ast d)^4 = (c \ast d)^4 = \\
a \ast c \ast d \ast c \ast d \ast a \ast d \ast c = b \ast c \ast d \ast c \ast d \ast b \ast d \ast c \ast d \ast c = \\
a \ast b \ast c \ast a \ast b \ast d \ast a \ast b \ast c \ast b \ast a \ast d = \\
a \ast c \ast d \ast a \ast c \ast d \ast a \ast d \ast c \ast a \ast d \ast c = 1.
\end{align*}
\]

The corresponding $C_3$ action to yield the order 3072 group is:

\[
\begin{align*}
e^3 &= a^c \ast b = b^c \ast b \ast a = c^c \ast d \ast a \ast c \ast a \ast d = \\
d^c \ast b \ast c \ast a \ast d \ast a \ast c \ast b = 1
\end{align*}
\]

and yields a group that has 46 conjugacy classes and an automorphism group of order 589,824.

A presentation for the full automorphism group on four generators is:

\[
\begin{align*}
a^4 &= b^4 = c^4 = d^2 = a^2 \ast b \ast (a^{-2}) \ast b = a^2 \ast c \ast (b^{-2}) \ast (c^{-1}) = \\
a^2 \ast (c^{-1}) \ast (b^{-2}) \ast c &= a^2 \ast d \ast (a^{-2}) \ast d = \\
a \ast b^2 \ast (a^{-1}) \ast (b^{-2}) = b^2 \ast d \ast (b^{-2}) \ast d = \\
(b \ast (c^{-1}))^3 &= c^2 \ast d \ast (c^{-2}) \ast d = \\
a \ast b \ast a \ast b \ast (a^{-1}) \ast (b^{-1}) \ast (a^{-1}) \ast b = \\
a \ast b \ast c \ast b \ast (c^{-1}) \ast (a^{-1}) \ast b \ast c = \\
a \ast c^2 \ast a \ast (c^{-2}) \ast (b^{-2}) = a \ast d \ast a \ast d \ast (a^{-1}) \ast d \ast (a^{-1}) \ast d = \\
b \ast c \ast (b^{-1}) \ast d \ast (b^{-1}) \ast (c^{-1}) \ast b \ast d = \\
b \ast d \ast b \ast d \ast (b^{-1}) \ast d \ast (b^{-1}) \ast d = \\
c \ast d \ast c \ast d \ast (c^{-1}) \ast d \ast (c^{-1}) \ast d = \\
a^2 \ast b \ast c \ast d \ast (c^{-1}) \ast d \ast b \ast d = \\
a \ast b \ast d \ast a \ast (c^{-1}) \ast a \ast c \ast d \ast (a^{-1}) \ast b = \\
a \ast b \ast d \ast b \ast (a^{-1}) \ast d \ast (a^{-1}) \ast (b^{-1}) \ast d \ast (b^{-1}) \ast a \ast d = 1.
\end{align*}
\]

7.27. **Group #181. [order 1,536].** This automorphism group does not have a normal sylow 2-subgroup. The largest normal sylow 2-subgroup has order 256 (#55999). Of the three normal subgroups of order 768, only one has a normal sylow 2-subgroup. A possible representation of this automorphism group is therefore

\[
[(256 \ast C_3) \ast @ C_2].
\]
The order 768 group has the following presentation:

\[
\begin{align*}
    a^2 &= b^2 = c^2 = d^2 = e^2 = f^2 = (a \cdot b)^2 = (a \cdot c)^2 = \\
    (a \cdot d)^2 &= (a \cdot e)^2 = (b \cdot c)^2 = (b \cdot e)^2 = (b \cdot f)^2 = \\
    (c \cdot d)^2 &= (c \cdot f)^2 = (d \cdot e)^2 = (d \cdot f)^2 = (e \cdot f)^2 = \\
    (a \cdot f)^4 &= (b \cdot d)^4 = (c \cdot e)^4 = \\
    a \cdot b \cdot c \cdot d \cdot b \cdot d \cdot c \cdot f \cdot a \cdot f \cdot c \cdot e = \\
    h^3 &= \\
    a^h \cdot f \cdot a \cdot f \cdot c \cdot a &= b^h \cdot f \cdot a \cdot f = \\
    c^h \cdot f \cdot a \cdot f \cdot b \cdot a = \\
    d^h \cdot f = \\
    e^h \cdot b \cdot d \cdot b = \\
    f^h \cdot c \cdot c \cdot c = 1.
\end{align*}
\]

This group of order 768 (\#1085205) appears to be a characteristic subgroup of the complete group of order 12,288 below.

The smallest order normal subgroup with a factor of three in its order is 192. This order 192 group is \#61 in Table 2a, which is \#1023 in the Small Group Library. The quotient group \text{aut}(\#181)/(normal subgroup of order 192) is \(D_4\). (This order 192 group has 9 classes and is \#183 \@ \(C_3\).)

A presentation for this automorphism group of order 1536 is:

\[
\begin{align*}
    b^4 &= c^4 = d^2 = b \cdot d \cdot (b^{-1}) \cdot d = (c \cdot d)^2 = \\
    a^3 \cdot d \cdot (a^{-1}) \cdot d &= a^2 \cdot b \cdot a^2 \cdot (b^{-1}) = \\
    (a \cdot (c^{-1}))^3 &= b \cdot c^2 \cdot b \cdot c^2 = \\
    a \cdot b \cdot c \cdot b \cdot (c^{-1}) \cdot (a^{-1}) \cdot (b^{-1}) = \\
    a \cdot c \cdot a \cdot c \cdot b^2 \cdot a \cdot (c^{-1}) &= 1.
\end{align*}
\]

This group has 33 classes and the order structure:

| order of element | number of elements | number of classes |
|------------------|--------------------|-------------------|
| 2                | 159                | 10                |
| 3                | 128                | 1                 |
| 4                | 480                | 13                |
| 6                | 384                | 3                 |
| 8                | 384                | 5                 |

The group presentation below gives rise to the automorphism group of this order 1536 group and has order 6144. This is the presentation used as the starting
point for the automorphism tower here (see Appendix III):

\[
b^4 = c^4 = d^4 = b * d * b * (d^{-1}) = (c * (d^{-1}))^2 = \\
a^4 * (d^{-2}) = a^2 * b * a^2 * (b^{-1}) = a * b * d * a * d * (b^{-1}) = \\
a * b * (d^{-1}) * a * (d^{-1}) * (b^{-1}) = \\
a * (b^{-1}) * d * a * (b^{-1}) * (d^{-1}) = \\
(a * (c^{-1}))^3 = b^2 * d * (c^{-1}) * (d^{-1}) * (c^{-1}) = \\
b * c^2 * b * (c^{-2}) = a * b * c * b * (c^{-1}) * (a^{-1}) * (b^{-1}) = \\
a * c * a * c * (b^{-2}) * a * (c^{-1}) = 1.
\]

7.28. **Group #183. [order 9,216]**. The automorphism group does not have a normal sylow 2-subgroup. The largest-order normal 2-group is the elementary abelian group of order 256. A group of order 2304 has a normal sylow 2-subgroup. This group has the structural form:

\[
[(256)@((C_3 \times C_3))@ (C_2 \times C_2)].
\]

Possible presentations for this group of order 2304 are:

\[
a^3 = b^3 = c^3 = (a * (b^{-1}))^2 = (a * (c^{-1}))^2 = (b * (c^{-1}))^2 = \]

\[
(a * c * b)^2 = \\
(a, a1) = (b, a1) = (c, a1) = \\
a1^2 = b1^2 = c1^2 = d1^2 = \\
(a1, b1) = (a1, c1) = (a1, d1) = (b1, c1) = (b1, d1) = (c1, d1) = \\
da^3 = \\
1^4 * b1 = b1^4 * a1 * b1 = c1^4 * d1 = d1^4 * c1 * d1 = \\
(a, b1) = (b, b1) = (c, b1) = \\
(a, c1) = (b, c1) = (c, c1) = \\
(a, d1) = (b, d1) = (c, d1) =
\]
coupled with one of the following actions of the order-three element \(d\) on the generators \(a\), \(b\), and \(c\). Cases 1, 2 and 3 all seem to generate this order 2304 group. The automorphism group of this group has order \(2^{12} * 3^4 * 5\). Case 4 has a different number of classes (64) and order structure [255 elements of order 2 in 35 classes, 1088 elements of order 3 in 8 classes, 960 elements of order 6 in 20 classes, and 255 elements of order 2 in 35 classes]. Case 4’s automorphism group has order \(2^{215} * 3^4 * 5^2\).

Case 1.

\[
a^d * c * b = b^d * ((b * c)^{-1}) * a = (c, d) = 1.
\]

Case 2.

\[
a^d * b * ((c * a)^{-1}) = b^d * b * a = c^d * a * b = 1.
\]

Case 3.

\[
(a, d) = b^d * c * a = c^d * c * ((a * b)^{-1}) = 1.
\]

Case 4.

\[
a^d * b * ((c * a)^{-1}) = b^d * (c^{-1}) * b * (a^{-1}) = c^d * c * ((a * b)^{-1}) = 1.
\]
A three-generator set of relations for this automorphism group is:

\[ a^2 = b^2 = c^2 = (a * b * a * (b^{-1}))^2 = a * (b^{-2}) * c * a * c * b^2 = \]
\[ (a * c)^4 = a * b^2 * a * c * a * c * (b^{-2}) = (a * b * a * (b^{-1}) * c)^2 = \]
\[ a * b * c * a * b * c * a * c * (b^{-1}) * c * (b^{-1}) = a * b * a * b * c * b^4 * c * b^2 = \]
\[ (b^2 * (c^{-1}))^4 = (b * (c^{-1}))^6 = \]
\[ a * b * c * (b^{-1}) * a * c * b * c * (b^{-1}) * c * b * c * (b^{-1}) * c = 1. \]

7.29. **Group #187. [order 15,360]**. The structure of this automorphism group, which is a complete group, has the form:

\[ ((1^4 @ C_3) \times 1^4) @ C_5 \times C_4. \]

A presentation for the \[ [(1^4 @ C_3) \times 1^4] @ C_5 \] normal subgroup of order 3840 is:

\[ a^3 = b^3 = c^3 = (a * (b^{-1}))^2 = (a * (c^{-1}))^2 = (b * (c^{-1}))^2 = \]
\[ (a * c * b)^2 = \]
\[ a^2 = b^2 = c^2 = d^2 = (a1, b1) = (a1, c1) = (a1, d1) = \]
\[ (b1, c1) = (b1, d1) = (c1, d1) = \]
\[ (a, a1) = (b, a1) = (c, a1) = (a, b1) = (b, b1) = (c, b1) = \]
\[ (a, c1) = (b, c1) = (c, c1) = \]
\[ d^2 = \]
\[ a^2 d^2 * a * c1 * d1 = b1 d^2 * a1 = c1 d^2 * a1 * b1 * c1 * d1 = d1 d^2 * a1 * d1 = 1. \]

The group \( < a, b, c > \) is \( 1^4 @ C_3 \) and is the group of order 48 whose automorphism group has order 5760. The actions of \( C_5 \) on each of the two pairs of \( 1^4 \) groups should be as in the group of order 80 with automorphism group of order 960.

In the representation above we have the order-5 automorphism acting on the group of order 48 directly. We have not yet found a consistent representation with the form

\[ [1^4 \times (1^4 @ C_3) @ C_5] @ C_4. \]

The \( C_4 \) acts as an element of order 2 on the \( C_3 \) and as an element of order 4 on the \( C_5 \). Relations based upon this normal subgroup structure have not yet been found.
A presentation of this automorphism group using only three generators is:

\[(a * (b^{-1}))^2 = a^3 * (b^{-3}) * (a^{-1}) * b = \]
\[a * b * a * b * (a^{-1}) * (b^{-1}) * (a^{-1}) * (b^{-1}) = \]
\[a * b * a * c * (b^{-1}) * c * b * (c^{-1}) = \]
\[a * b * c * b * (c^{-2}) * b * c = \]
\[b^2 * c * (b^{-1}) * (c^{-2}) * b * (c^{-1}) = \]
\[a^2 * (c^{-1}) * (a^{-1}) * (b^{-1}) * a * c^2 * (a^{-1}) * (c^{-1}) = \]
\[a * c * (a^{-1}) * (c^{-2}) * (a^{-1}) * b * c * a * (c^{-1}) = 1. \]

7.30. Relations for the group of order 1536 appearing in the automorphism groups of numbers 144, 145, 147, 148 and 184.

The following 2-groups arising in the above discussions seem to be isomorphic:

- order 64,512 appearing in #144,... above.
- order 192 group as follows:
  - \(g^3 = a^g * b = b^g * a * b = c^g * d = f^g * e * f = \)
  - \(h^2 = (a, h) = (b, h) = (c, h) = (d, h) = (e, h) = (f, h) = (g, h) = \)
  - \(k^2 = a^k * b = c^k * d = e^k * f = (h * k)^4 = g^k * g = 1. \)

The structure of this group is:

\[ [(1^6 \times C_3) \times C_2 \times C_2] @ C_2 \]

with the action of \(C_2\) on the order 192 group as follows:

- \(C_2\) acts on pairs of \(C_2 \times C_2\) as in \(C_2 \times C_2\).
- \(C_2\) inverts the generator of \(C_3\).
- The action of \(C_2\) on the other \(C_2 \times C_2\) is as a wreath product.

The group \((1^6 @ C_3) @ C_2\) has for its automorphism group the complete group of order 64,512 appearing in #144,... above.

7.31. Some common 2-groups arising in the above automorphism groups.

The following 2-groups arising in the above discussions seem to be isomorphic:

- the 1024 groups in cases 64#'s 30, 81, 93, 173 (yes, Dr. Newman confirmed these). The 3072 groups \((1024 @ C_3)\) in cases # 30, 81 and 91 appear isomorphic but #173 is different. The automorphism groups for #30 and #81 have the form \((3072) @ C_2\) and may be isomorphic.

Also,

- the 256 groups and the 768 groups that occur in the automorphism groups of #'s 17, 44, and 108 of order 64 are all isomorphic. These automorphism groups have the structures:
  - \([(256) @ C_3] @ (C_2 \times C_2) \times C_2,\]
  - \([(256) @ C_3] @ (C_2 \times C_2 \times C_2),\) and
  - \([(256) @ C_3] @ (D_4 \times C_2)\)
with the same \((256 \oplus C_3)\) normal subgroup. In other cases, such as \#’s 104, 105, 153, and 187, among others, presentations on three or four generators are available.
8. Appendix III
Automorphism Towers for Groups in Table A1

The presentations for these automorphism towers were originally computed by CAYLEY runs in the mid 1990s. When we started revising this report only a paper copy of these runs was on hand. In order to check these presentations in their current published form, the following was done. This set of presentations was scanned in from a computer transfer listing, but the scanning was not perfect; hence a fair amount of hand editing was involved in getting this listing for these presentations. The listings given previously for the automorphism groups of the groups of order 64 were taken directly from computer readable files, and hence are more reliable than the ones given here. In those cases (the automorphism groups for the groups of order 64) one also did some transfers from an original text editor output from CAYLEY to an MS Word format and then to the LaTeX form, so even there some minor editing was done. These presentations were run recently in GAP in order to verify that these presentations gave the correct orders and in many cases the correct class structures for these previously obtained results. One word of caution should be noted here. Namely, in the original CAYLEY runs, the presentations from one member of the automorphism tower were used as the input for the next group assuring the continuity of the automorphism tower structures. With GAP this was not done for the following reason. In CAYLEY the output was reproducible while in GAP this is not the case. The way things were done with CAYLEY was the following. One ran the initial group in CAYLEY and asked for, among other things, the automorphism group of the input group. In many cases these CAYLEY generated automorphism groups had large orders and many generators. We then asked CAYLEY to select a subset of this initial generating set that would also give this automorphism group, e.g., say generators numbers 1, 3, 6, 8 and 10. We then ran CAYLEY a second time and asked CAYLEY to get a presentation for the automorphism group using this specific generating set. Then further calculations on the automorphism group were carried out using this presentation. In the automorphism towers this process was then repeated for each subsequent group in the series until either the tower terminated or became too large for CAYLEY to continue up the sequence. In some cases these runs took several hours, depending upon the degree of the automorphism group and the number of generators needed to specify the group, so doing things without these “built-in breaks” would be rather impractical. One really does need to know just how many generators one needs to specify the various groups, and this seems to be the best way to do that. Furthermore the use of permutations is not as efficient as the use of a presentation input. From our experience with CAYLEY, many calculations using permutations become very time-consuming or impossible, whereas with a presentation input many of these calculations are doable and are considerably faster.

In order to check these presentations, the following was done. We scanned the printed file into the computer. We then took the edited and hopefully corrected scanned presentations and made the following substitutions: $a \rightarrow f.1$, $b \rightarrow f.2$, etc., and $=$ in the presentation was replaced by a comma, and we rewrote $a^{-2} \rightarrow f.1 \land -2$, etc. to get a GAP type input format. We then ran these edited presentations in

\footnote{Subsequently the computer file which contained computer transfers of these presentations was located. Even so we still checked these presentations with runs using GAP.}
GAP. Some errors were found in these initial runs, and hopefully the corrections have been incorporated back into the original LaTeX manuscript. A comment or two is in order here as well. In the CAYLEY runs, the orders of the centers and the order/class structure were obtained reasonably quickly. With GAP the determinations of the centers of the groups (mostly of orders 1, 2, 4, or 8) were very slow. This is probably a built-in problem with GAP since GAP converts everything to permutations before doing any calculations. In using CAYLEY it was found that using presentations rather than permutations as input resulted in much faster run times, and in fact some calculations probably could not have been done using a permutation input for many groups.\footnote{It was suggested to us that we should check these runs using GAP, possibly to see if there were any errors in our original presentations. As a matter of general information we encountered many more errors (i.e., typos) due to the rewriting of these presentations in GAP in the process of checking them than were found in the original listings.}

Some of the presentations could have been simplified, e.g., words such as
\[a * b * d * a * b * d * a * b * d * b * a * d = (a * b * d)^3 * b * a * d\]
and many others below. These words were left intact to avoid any additional typing/editing errors here.

We now proceed to give the presentations for the groups in these automorphism group tower sequences.

8.1. **Group #5. Automorphism tower for** $C_4 \times C_4 \times C_4$. a. The automorphism group of $C_4 \times C_4 \times C_4$ is
\[
a^2 = b^2 = c^2 = d^2 = (a * b)^2 = (a * c)^2 = (a * d)^2 = (a * b * c)^3 = (a * b * d)^3 = \\
(b * c)^3 = (c * d)^4 = (a * b * d * c * d)^2 = \\
a * b * d * a * b * d * a * b * d * b * a * d = \\
a * c * a * d * a * c * a * d * a * c * d * b * d * c * a * d = \\
a * b * c * a * d * a * c * d * c * b * d * b * c * b * d * c * b * d = 1.
\]
b. The second group in this sequence is complete:
\[
a^2 = b^2 = c^2 = d^2 = e^2 = (a * b)^2 = (a * e)^2 = (a * c)^3 = (a * b * c * a * e * d * e)^3 = (b * c)^3 = (b * e)^4 = \\
(c * d)^4 = a * b * d * a * c * e * b * d * c * d = \\
a * b * d * a * b * d * a * b * d * b * a * d = \\
a * c * a * d * a * c * a * d * a * c * d * b * d * c * a * d = \\
a * b * c * a * b * d * c * e * c * b * d * c * b * d * b * c * d * c * e = 1.
\]

8.2. **Group #7.** Automorphism sequences for this group start out with $C_2 \times (\text{order } 384 \text{ group})$. 
a. Aut(64 \# 7) is $C_2$ cross the following order 384 (# 20095) group:

\[
\begin{align*}
d^2 &= (a \ast b^{-1})^2 = (a \ast c^{-1})^2 = (b \ast c^{-1})^2 = \\
a^2 \ast (b^{-2}) \ast (a^{-1}) \ast b &= \\
a^2 \ast (c^{-2}) \ast (a^{-1}) \ast c &= \\
a^2 \ast (d^{-1}) \ast c \ast a \ast (d^{-1}) &= (a \ast c \ast b)^2 = \\
(a \ast (d^{-1}) \ast (b^{-1}))^2 &= \\
a \ast (c^{-1}) \ast (b^{-1}) \ast (d^{-1}) \ast (a^{-1}) \ast (b^{-1}) \ast a \ast (d^{-1}) = 1.
\end{align*}
\]

b. The second entry in the sequence is Aut(384) with order $2^8 \ast 3^2 = 2304$:

\[
\begin{align*}
a^2 &= d^2 = e^2 = a \ast c \ast a \ast (c^{-1}) = (a \ast e)^2 = \\
b^2 \ast (c^{-2}) &= b \ast c \ast (b^{-1}) \ast (c^{-1}) = b \ast e \ast (b^{-1}) \ast e = \\
c \ast e \ast (c^{-1}) \ast e &= (d \ast e)^2 = (a \ast b)^3 = \\
a \ast b \ast d \ast b \ast a \ast (b^{-1}) \ast d \ast (b^{-1}) &= (a \ast d)^4 = \\
b^2 \ast c \ast d \ast c^{-1} \ast b^{-2} \ast d &= (b \ast d^{-1})^4 = \\
(c \ast (d^{-1}))^4 &= a \ast b \ast a \ast d \ast (b^{-1}) \ast d \ast a \ast d \ast (b^{-1}) \ast d = 1.
\end{align*}
\]

c. The third entry in the sequence has order $2^{12} \ast 3^3 = 110,592$:

\[
\begin{align*}
c^2 &= d^2 = a^2 \ast b^{-2} = a \ast b \ast a^{-1} \ast b^{-1} = \\
a \ast d \ast a^{-1} \ast d &= a \ast e \ast a^{-1} \ast e^{-1} = (c \ast d)^2 = d \ast e \ast d \ast e^{-1} = \\
(b \ast d^{-1})^3 &= a^2 \ast b \ast c \ast b^{-1} \ast a^{-2} \ast c = \\
b \ast e \ast b \ast e \ast (b^{-1}) \ast (e^{-1}) \ast (b^{-1}) \ast (e^{-1}) &= \\
b \ast d \ast b \ast e \ast b \ast e \ast c \ast d \ast (b^{-1}) \ast (e^{-1}) = \\
a \ast c \ast b \ast (e^{-1}) \ast c \ast (a^{-1}) \ast c \ast e \ast (b^{-1}) \ast c &= \\
a^2 \ast e \ast c \ast e \ast c \ast e \ast (b^{-1}) \ast c \ast b \ast e \ast c &= \\
a \ast c \ast a \ast c \ast (a^{-1}) \ast c \ast a \ast c \ast (b^{-1}) \ast c &= \\
b \ast c \ast e \ast (b^{-1}) \ast c \ast e \ast c \ast (b^{-1}) \ast (e^{-1}) \ast (b^{-1}) \ast c \ast e &= \\
(b \ast (c^{-1}))^6 &= \\
a^2 \ast c \ast a \ast c \ast (a^{-1}) \ast c \ast (b^{-1}) \ast c \ast (a^{-1}) \ast b \ast c \ast a \ast c = 1.
\end{align*}
\]
d. The fourth entry has order $2^{13} \cdot 3^3 = 221,184$:

\[
\begin{align*}
d^2 &= e^2 = f^2 = a^2 \ast (b^{-2}) = a^2 \ast (c^{-2}) = \\
a \ast b \ast (a^{-1}) \ast (b^{-1}) &= a \ast c \ast (a^{-1}) \ast (c^{-1}) = \\
a \ast e \ast (a^{-1}) \ast e &= a \ast f \ast (a^{-1}) \ast f = \\
b \ast c \ast (b^{-1}) \ast (c^{-1}) &= c \ast f \ast (c^{-1}) \ast f = (d \ast e)^2 = \\
(d \ast f)^2 &= (c \ast (e^{-1}))^3 = a^2 \ast c \ast d \ast (c^{-1}) \ast (a^{-2}) \ast d = \\
b \ast (d^{-1}) \ast (f^{-1}) \ast b \ast (f^{-1}) \ast (c^{-1}) \ast (d^{-1}) \ast (c^{-1}) = \\
b \ast e \ast b \ast c \ast (b^{-1}) \ast e \ast (b^{-1}) \ast e &= \\
b \ast e \ast (b^{-1}) \ast f \ast (b^{-1}) \ast e \ast b \ast f &= \\
(b \ast f \ast (b^{-1}) \ast d)^2 &= (e \ast f)^4 = \\
b \ast c \ast e \ast b \ast c \ast e \ast c \ast b \ast e &= \\
b \ast (e^{-1}) \ast (f^{-1}) \ast (e^{-1}) \ast (f^{-1}) \ast (b^{-1}) \ast (f^{-1}) \ast (c^{-1}) \ast (f^{-1}) \ast (e^{-1}) &= \\
(a \ast d \ast a \ast d \ast (a^{-1}) \ast d \ast c \ast d \ast (c^{-1}) \ast d &= \\
b \ast (d^{-1}) \ast b \ast (d^{-1}) \ast b \ast (d^{-1}) \ast b \ast (d^{-1}) \ast c \ast \\
(d^{-1}) \ast c \ast (d^{-1}) &= \\
a \ast (b^{-1}) \ast (d^{-1}) \ast b \ast (d^{-1}) \ast (b^{-1}) \ast (d^{-1}) \ast (a^{-1}) \ast \\
d^{-1} \ast a^{-1} \ast d^{-1} \ast b^{-1} \ast a \ast d^{-1} &= 1.
\end{align*}
\]
e. The last entry in this sequence is complete and has order $= 2^{14} \cdot 3^3 = 442,368$:

$$d^2 = f^2 = h^2 = a^2 \ast (b^{-2}) = a^2 \ast (c^{-2}) =$$

$$a^2 \ast (e^{-2}) = a \ast b \ast (a^{-1}) \ast (b^{-1}) =$$

$$a \ast c \ast a^{-1} \ast (c^{-1}) = a \ast d \ast (a^{-1}) \ast d =$$

$$a \ast e \ast a \ast (f^{-1}) = a \ast e \ast (a^{-1}) \ast (e^{-1}) =$$

$$a \ast f \ast (a^{-1}) \ast f = a \ast k \ast (a^{-1}) \ast (k^{-1}) =$$

$$b \ast c \ast (b^{-1}) \ast (c^{-1}) = b \ast d \ast (b^{-1}) \ast d =$$

$$b \ast e \ast (b^{-1}) \ast (e^{-1}) = b \ast f \ast (b^{-1}) \ast f =$$

$$c \ast d \ast (c^{-1}) \ast d = c \ast e \ast (c^{-1}) \ast (e^{-1}) =$$

$$c \ast f \ast (c^{-1}) \ast f = (d \ast h)^2 = e \ast f \ast (e^{-1}) \ast f =$$

$$(f \ast h)^2 = b \ast (c^{-1}) \ast (k^{-1}) \ast d \ast k =$$

$$b \ast k \ast d \ast (k^{-1}) \ast (e^{-1}) =$$

$$a \ast (k^{-1}) \ast h \ast k \ast (a^{-1}) \ast h =$$

$$b \ast e \ast (d^{-1}) \ast (e^{-1}) \ast (d^{-1}) \ast (c^{-1}) =$$

$$b \ast (e^{-1}) \ast h \ast (e^{-1}) \ast e \ast h =$$

$$b \ast (k^{-2}) \ast (c^{-1}) \ast k^2 = e \ast k^2 \ast (e^{-1}) \ast (k^{-2}) =$$

$$e \ast (k^{-4}) \ast (f^{-1}) =$$

$$b \ast k \ast b \ast (k^{-1}) \ast (b^{-1}) \ast (k^{-1}) \ast k =$$

$$b \ast e \ast k \ast b \ast k \ast e \ast (k^{-1}) \ast (b^{-1}) \ast (k^{-1}) =$$

$$a \ast h \ast a \ast h \ast (a^{-1}) \ast h \ast (a^{-1}) \ast h \ast c \ast h \ast (b^{-1}) \ast h =$$

$$b \ast (h^{-1}) \ast b \ast (h^{-1}) \ast b \ast (h^{-1}) \ast b \ast (h^{-1}) \ast c \ast (h^{-1}) =$$

$$e \ast k \ast e \ast k \ast e \ast k \ast f \ast (k^{-1}) \ast (e^{-1}) \ast (k^{-1}) \ast (e^{-1}) \ast k =$$

$$a \ast h \ast (a^{-1}) \ast h \ast e^{-1} \ast h \ast a^{-1} \ast$$

$$b \ast f \ast h \ast a \ast h \ast b^{-1} \ast h = 1.$$
8.3. **Group #8. $C_8 \times C_8$ case.** For the first step, see table of $\text{aut}(64)$ groups. The next group in the sequence is a group of order $24,576 = 2^{13} \times 3$ with the presentation:

\[
\begin{align*}
a^2 &= b^2 = c^2 = d^2 = e^2 = (a \ast c)^2 = \\
(a \ast d)^2 &= (a \ast e)^2 = (a \ast f)^2 = (c \ast d)^2 = \\
(c \ast e)^2 &= (c \ast f)^2 = (d \ast e)^2 = f^4 = \\
b \ast c \ast d \ast b \ast d \ast c &= b \ast c \ast e \ast b \ast e \ast c = \\
b \ast c \ast (f^{-1}) \ast b \ast f \ast c &= d \ast f^2 \ast d \ast (f^{-1}) = \\
e \ast f^2 \ast e \ast f^{-2} &= d \ast f \ast d \ast f \ast d^{-1} \ast d \ast f^{-1} = \\
d \ast f^2 \ast e \ast (f^{-1}) \ast d \ast (f^{-1}) \ast e \ast f = \\
e \ast f \ast e \ast f \ast e \ast (f^{-1}) \ast e \ast (f^{-1}) = \\
a \ast b \ast f^2 \ast b \ast a \ast b \ast (f^{-1}) \ast b = \\
a \ast b \ast a \ast b \ast a \ast b \ast (f^{-1}) \ast c \ast b \ast (f^{-1}) \ast c = \\
(a \ast b \ast c \ast b \ast c \ast b)^2 = (b \ast c)^6 = 1.
\end{align*}
\]

This order 24,576 group has for its automorphism group one of order $2^{28} \ast 3^2 \ast 7 = 16,911,433,728$ with a center $C_2$.

8.4. **Group #14.** a. For the automorphism group, see table of $\text{aut}(g)$ for order 64 groups. This is the order 294,912 group. The relations for the 294,912 group are

\[
\begin{align*}
a^2 &= b^2 = c^2 = d^2 = e^2 = f^2 = h^2 = (a \ast b)^2 = \\
(a \ast d)^2 &= (a \ast e)^2 = (a \ast f)^2 = (a \ast h)^2 = (a \ast k)^2 = \\
(b \ast d)^2 &= (b \ast e)^2 = (b \ast f)^2 = (c \ast e)^2 = (c \ast f)^2 = \\
(d \ast f)^2 &= (d \ast h)^2 = d \ast k \ast d \ast (k^{-1}) = (e \ast h)^2 = (f \ast h)^2 = \\
f \ast k \ast f \ast (k^{-1}) &= k^4 = b \ast d \ast k \ast b \ast k = d \ast h \ast k \ast h \ast (k^{-1}) = \\
(a \ast c)^3 &= a \ast (c \ast a \ast h \ast c \ast h = a \ast c \ast a \ast k \ast c \ast (k^{-1}) = \\
(b \ast c)^4 &= (b \ast h)^4 = (d \ast e)^4 = (d \ast e \ast f \ast e)^2 = \\
(e \ast f)^4 &= a \ast b \ast c \ast a \ast b \ast c \ast b \ast a \ast c = d \ast e \ast f \ast e \ast k \ast e \ast f \ast e \ast k = \\
d \ast e \ast d \ast f \ast k^2 \ast e \ast (k^{-2}) \ast f = (b \ast c \ast h \ast b \ast h \ast c)^2 = \\
b \ast d \ast e \ast h \ast b \ast k \ast e \ast h \ast d \ast e \ast (k^{-1}) \ast e = \\
c \ast d \ast e \ast c \ast d \ast c \ast e \ast d \ast e \ast c \ast d \ast e \ast c = \\
e \ast k \ast e \ast k \ast e \ast k \ast (k^{-1}) \ast e \ast (k^{-1}) \ast e \ast (k^{-1}) = 1.
\end{align*}
\]

The center of this group is generated by $< d \ast f \ast k, c \ast d \ast e \ast (k^{-1}) >$. The next three groups in the automorphism tower sequence have orders $2^{21} \ast 3^3 = 56,623,104$, $2^{24} \ast 3^3 = 452,984,832$, and $2^{25} \ast 3^3 = 905,969,664$. 

b. A presentation for the quotient group \(294,912/Z\) is:

\[
\begin{align*}
a^2 &= b^2 = c^2 = d^2 = e^2 = (a * b)^2 = (a * d)^2 = \\
(a * e)^2 &= (a * f)^2 = (b * d)^2 = (c * d)^2 = (d * e)^2 = \\
f^4 &= (a * c)^3 = a * c * a * e * c * e = a * c * a * f * c * (f^{-1}) = \\
b * e * f * e * b * f = b * f^2 * b * (f^{-2}) = (b * c)^4 = (b * e)^4 = \\
b * f * b * f * b * (f^{-1}) * b * (f^{-1}) = a * b * c * a * b * c * b * a * c = \\
(b * c * e * b * c * e)^2 = \\
b * (f^{-1}) * d * f * d * f * b * (f^{-1}) * d * (f^{-1}) * d * f = \\
d * f * d * f * d * f * d * (f^{-2}) * d * (f^{-1}) * d * (f^{-1}) = \\
(d * f * d * (f^{-1}))^3 = \\
d * f^2 * d * f^2 * d * (f^{-2}) * d * (f^{-2}) = \\
a * b * c * d * e * (f^{-1}) * d * f * d * f^{-1} * c * b * e = \\
a * c * f * d * f^2 * c * d * f^2 * d * f^{-1} * c * f * d * f = 1.
\end{align*}
\]

The automorphism group of this group is a complete group of order 147,456. A presentation for this complete group is:

\[
\begin{align*}
a^2 &= b^2 = c^2 = d^2 = e^2 = (a * b)^2 = (a * d)^2 = \\
(a * e)^2 &= (a * f)^2 = (b * d)^2 = (c * d)^2 = (d * e)^2 = \\
(d * e)^2 &= d * h * d * (h^{-1}) = f^4 = (a * c)^3 = \\
a * c * a * e * c * e &= a * c * a * f * c * (f^{-1}) = \\
b * e * f * e * b * f &= b * f^2 * b * (f^{-2}) = h^6 = \\
a * b * e * c * b * h * c * a * h^{-1} &= (a * h * c * (h^{-1}))^2 = \\
(a * h * e * h)^2 &= (a * h^{-1}) * b * c)^2 = (b * c)^4 = \\
b * c * (h^{-2}) * b * h^2 * c &= (b * e)^4 = \\
b * f * b * f * b * (f^{-1}) * b * (f^{-1}) &= (b * h * b * (h^{-1}))^2 = \\
(b * h * f * h) &= (c * (h^{-1}) * c * h)^2 = \\
a * b * c * a * b * c * b * a * c &= a * b * c * h * b * c * h * b * e = \\
a * b * h * f * (h^{-1}) * c * h * (f^{-1}) * (h^{-1}) &= \\
a * d * f * h * e * f * (h^{-1}) * e * f &= \\
b * h * f * h * f^2 * h * (f^{-1}) * h &= 1.
\end{align*}
\]
This group can be generated by \(<e, f, h>\) with the following presentation:
\[
\begin{align*}
\sigma^2 &= b^4 = a \ast b^2 \ast a \ast b^2 = c^5 = a \ast b \ast a \ast b \ast a \ast (b^{-1}) \ast a \ast (b^{-1}) = \\
\ast b \ast a \ast b \ast c^2 \ast a \ast (b^{-1}) \ast a \ast (b^{-1}) \ast (c^{-2}) = \\
\ast b \ast a \ast c \ast a \ast c \ast b^2 \ast (c^{-1}) \ast a \ast (c^{-1}) \ast b^2 \ast c \ast a \ast c = \\
\ast b \ast (b^{-1}) \ast c \ast a \ast (c^{-1}) \ast a \ast c^2 \ast (b^{-1}) \ast (c^{-1}) \ast a \ast c = \\
\ast (b^{-1}) \ast c \ast b^2 \ast c \ast a \ast (b^{-1}) \ast (c^{-1}) \ast b^2 \ast (c^{-1}) = \\
(a \ast c^2)^4 = b^2 \ast c \ast b^2 \ast (c^{-1}) \ast b^2 \ast c \ast b^2 \ast (c^{-1}) = \\
(b \ast c^2 \ast (b^{-1}) \ast c^2)^2 = \\
\ast b \ast a \ast c^2 \ast b^2 \ast c^2 \ast b^2 \ast c^2 \ast (b^{-1}) = \\
\ast b \ast a \ast c^2 \ast (b^{-1}) \ast c^2 \ast (b^{-1}) \ast c^2 = \\
\ast b^2 \ast c \ast a \ast b \ast (c^{-1}) \ast a \ast b^2 \ast c \ast a \ast b \ast (c^{-1}) = \\
\ast b^2 \ast c \ast b \ast c \ast (b^{-1}) \ast c^2 \ast a \ast b \ast (c^{-1}) \ast (b^{-1}) \ast c = \\
a \ast b \ast c \ast b \ast (c^{-1}) \ast b \ast (c^{-1}) \ast b \ast a \ast c \ast (b^{-1}) \ast \\
c \ast (b^{-1}) \ast (c^{-1}) = 1.
\end{align*}
\]

This complete group has the following order structure:

| order of element | number of elements | number of classes |
|------------------|--------------------|------------------|
| 2                | 2847               | 35               |
| 3                | 1664               | 3                |
| 4                | 45792              | 64               |
| 6                | 41856              | 23               |
| 8                | 16896              | 6                |
| 12               | 32256              | 9                |
| 24               | 6144               | 1                |

8.5. **Group #15.** The order 196,608 group in this sequence has the presentation:
\[
\begin{align*}
\sigma^2 &= a^2 = a^2 \ast b^2 = b \ast d \ast (b^{-1}) \ast d = (c \ast d)^2 = \\
\ast b \ast a \ast b \ast (a^{-1}) \ast (b^{-1}) \ast (a^{-1}) \ast (b^{-1}) = \\
a \ast b \ast (a^{-1}) \ast d \ast (a^{-1}) \ast (b^{-1}) \ast a \ast d = \\
(a \ast (b^{-1}) \ast c \ast (b^{-1}))^2 = (a \ast c)^4 = \\
\ast d \ast a \ast d \ast (a^{-1}) \ast d \ast a^{-1} \ast d = \\
\ast c \ast a \ast c \ast (b^{-2}) \ast c \ast (b^{-2}) \ast c \ast (b^{-2}) \ast c = \\
a \ast b \ast c \ast b \ast a \ast c \ast (a^{-1}) \ast (b^{-1}) \ast c \ast (b^{-1}) \ast (a^{-1}) \ast c = \\
(a \ast c \ast (a^{-1}) \ast c)^2 = (b \ast c \ast (b^{-1}) \ast c)^2 = \\
a \ast (b^{-1}) \ast c \ast (a^{-1}) \ast c \ast (b^{-1}) \ast c \ast (b^{-1}) \ast c \ast (a^{-1}) \ast b \ast c \ast a \\
\ast c \ast b \ast c \ast b \ast c = 1.
\end{align*}
\]
The center of this group is generated by \(< a \ast d \ast (a^{-1}) \ast d, (a \ast c \ast b \ast b \ast c)^2 >\). 

This group has 354 conjugacy classes and the order structure:

| order of element | number of elements | number of classes |
|------------------|-------------------|------------------|
| 2                | 6143              | 106              |
| 3                | 512               | 1                |
| 4                | 81920             | 181              |
| 6                | 36352             | 36               |
| 8                | 43008             | 22               |
| 12               | 28672             | 7                |

The automorphism group of this order 196,608 group has order 100,663,296, and a center of order 8 \((C_2 \times C_2 \times C_2)\). The quotient group here has a center of order 32.

8.6. **Group #16.** The group of order 294,912 has the following presentation on five generators:

\[
\begin{align*}
  a^2 &= b^2 = c^2 = d^4 = e^4 = (a \ast b)^2 = (b \ast c)^2 = \\
  (b \ast d)^2 &= a \ast c \ast d \ast a \ast (d^{-1}) \ast c = a \ast c \ast e \ast a \ast (c^{-1}) \ast c = \\
  (c \ast d)^3 &= c \ast e^2 \ast c \ast (e^2) = d^2 \ast e \ast (d^2) \ast (e^{-1}) = \\
  (b \ast e)^4 &= (b \ast c \ast b \ast (c^{-1}))^2 = (b \ast c \ast e \ast e^2)^2 = \\
  b \ast e \ast (d^{-1}) \ast e \ast b \ast (e^{-1}) \ast (d^{-1}) \ast (c^{-1}) = \\
  c \ast e \ast c \ast (e^{-1}) \ast c \ast (e^{-1}) = \\
  d \ast e \ast d \ast e \ast (d^{-1}) \ast e \ast (d^{-1}) = \\
  d \ast e \ast d \ast (e^{-1}) \ast (d^{-1}) \ast e \ast d \ast (e^{-1}) = \\
  (a \ast c \ast a \ast c \ast (e^{-1}))^2 = \\
  a \ast e \ast b \ast (e^{-1}) \ast a \ast e^2 \ast b \ast (e^2) = \\
  a \ast e \ast d \ast e^2 \ast a \ast e \ast (d^{-1}) \ast (e^2) = \\
  c \ast d^2 \ast c \ast e \ast c \ast (d^2) \ast c \ast (e^{-1}) = (a \ast c)^6 = \\
  c \ast d \ast e \ast d \ast c \ast e \ast c \ast d \ast e \ast (d^{-1}) \ast c \ast e = \\
  a \ast c \ast a \ast d \ast c \ast e \ast c \ast d \ast e \ast (d^{-1}) \ast (e^{-1}) \ast a \ast e \ast c \ast (e^{-1}) = 1.
\end{align*}
\]

This group has the following order structure:

| order of element | number of elements | number of classes |
|------------------|-------------------|------------------|
| 2                | 6463              | 79               |
| 3                | 5120              | 3                |
| 4                | 106176            | 156              |
| 6                | 78848             | 20               |
| 8                | 18432             | 4                |
| 12               | 79872             | 11               |
8.7. **Group #17.** The following is the second order 3072 group in this automorphism group sequence. The relations are:

\[
\begin{align*}
8.7. & \quad a^2 = b^2 = c^2 = d^2 = f^2 = e^3 = (a \ast b)^2 = \\
     & = a \ast e \ast a \ast (e^{-1}) = (b \ast c)^2 = b \ast e \ast b \ast (e^{-1}) = \\
     & = (c \ast d)^2 = (c \ast f)^2 = (d \ast f)^2 = (a \ast b \ast d)^2 = \\
     & = (a \ast b \ast f)^2 = a \ast c \ast d \ast a \ast d \ast c = a \ast c \ast f \ast a \ast f \ast c = \\
     & = (c \ast e)^3 = (d \ast e)^3 = d \ast e \ast d \ast f \ast e \ast f = \\
     & = a \ast c \ast a \ast e \ast c \ast a \ast c \ast (e^{-1}) \ast c \ast e \ast a \ast c \ast (e^{-1}) = 1.
\end{align*}
\]

The next group has order 294,912 = \(2^{15} \ast 3^2\) and has a center of order 2. Its presentation on six generators and class structure are:

\[
\begin{align*}
8.7. & \quad a^3 = b^2 = c^2 = d^2 = e^4 = f^2 = (b \ast d)^2 = \\
     & = d \ast e \ast d \ast (e^{-1}) = a \ast d \ast e \ast (a^{-1}) \ast (e^{-1}) \ast d = \\
     & = a \ast d \ast f \ast (a^{-1}) \ast f \ast d = a \ast e \ast (a^{-1}) \ast b \ast (e^{-1}) \ast b = \\
     & = (b \ast c \ast (e^{-1}))^2 = (e \ast f)^3 = \\
     & = a \ast b \ast c \ast (e^{-1}) \ast c \ast (e^{-1}) \ast (a^{-1}) \ast b = \\
     & = a \ast b \ast e \ast c \ast e \ast c \ast (a^{-1}) \ast b = (a \ast c)^4 = \\
     & = b \ast c \ast b \ast d \ast e \ast c \ast (e^{-1}) \ast d = (b \ast f)^4 = \\
     & = (c \ast d \ast f \ast d)^2 = (c \ast f)^4 = (d \ast f)^4 = (b \ast e \ast f)^3 = \\
     & = a \ast b \ast (a^{-1}) \ast c \ast a \ast e \ast b \ast (e^{-1}) \ast (a^{-1}) \ast c = \\
     & = a \ast f \ast c \ast f \ast b \ast f \ast c \ast f \ast (a^{-1}) \ast b = \\
     & = c \ast d \ast c \ast f \ast c \ast f \ast d \ast f \ast c \ast f = \\
     & = a \ast b \ast a \ast f \ast c \ast b \ast (a^{-1}) \ast c \ast b \ast f \ast b = \\
     & = a \ast d \ast a \ast (e^{-1}) \ast f \ast (e^{-1}) \ast d \ast f \ast (e^{-1}) \ast d \ast a \ast f = 1.
\end{align*}
\]

This order 294,912 group has an automorphism group of order 1,179,648 and a center \(C_2 \times C_2\).

| order of element | number of elements | number of classes |
|------------------|-------------------|-------------------|
| 2                | 4351              | 39                |
| 3                | 6272              | 3                 |
| 4                | 89856             | 84                |
| 6                | 83840             | 22                |
| 8                | 36864             | 11                |
| 12               | 73728             | 14                |
8.8. **Group #19.** This tower gives an order of $2^{17} \ast 3^2 = 2,359,296$ for the next group in the tower.

8.9. **Group #22.** The order 196,608 = $2^{16} \ast 3$ in this sequence has the following presentation:

\[ a^4 = b^4 = c^2 = d^2 = e^2 = f^2 = (a \ast d)^2 = \]
\[ b \ast c \ast (b^{-1}) \ast c = (c \ast e)^2 = (c \ast f)^2 = (d \ast e)^2 = \]
\[ (d \ast f)^2 = (c \ast f)^2 = a^2 \ast c \ast (a^{-2}) \ast c = \]
\[ a \ast d \ast c \ast d \ast (a^{-1}) \ast c = b \ast e \ast f \ast (b^{-1}) \ast f \ast e = \]
\[ a^2 \ast b^2 \ast (a^{-2}) \ast (b^{-2}) = a \ast c \ast e \ast a \ast e \ast c \ast (b^{-2}) = \]
\[ a \ast d \ast (b^{-2}) \ast d \ast a^{-1} \ast b^2 = a \ast e \ast b^{-2} \ast e \ast a^{-1} \ast b^2 = \]
\[ (a \ast f \ast (a^{-1}) \ast e)^2 = (a \ast f \ast (a^{-1}) \ast f)^2 = \]
\[ a \ast b^2 \ast (a^{-1}) \ast b \ast a \ast (b^{-2}) \ast (a^{-1}) \ast b = \]
\[ a \ast e \ast f \ast (a^{-1}) \ast (b^{-1}) \ast a \ast f \ast e \ast (a^{-1}) \ast b = \]
\[ a^2 \ast b \ast a^2 \ast b \ast (a^{-2}) \ast (b^{-1}) \ast (a^{-2}) \ast (b^{-1}) = \]
\[ a^2 \ast b \ast c \ast d \ast (b^{-1}) \ast (a^{-2}) \ast (b^{-1}) \ast d \ast (a^{-1}) \ast b = \]
\[ a^2 \ast b \ast (a^{-1}) \ast f \ast (b^{-1}) \ast (a^{-2}) \ast (b^{-1}) \ast (a^{-1}) \ast f \ast (b^{-1}) = \]
\[ a^2 \ast b^{-1} \ast a^{-1} \ast b^{-1} \ast f \ast a^{-1} \ast f \ast d \ast b^{-1} \ast d \ast b^{-1} = \]
\[ a^2 \ast e \ast b \ast d \ast a \ast (b^{-1}) \ast e \ast (b^{-1}) \ast (a^{-1}) \ast d \ast (b^{-1}) = \]
\[ a \ast b \ast a \ast b \ast d \ast (b^{-1}) \ast d \ast (b^{-1}) \ast (a^{-1}) \ast (b^{-1}) \ast (a^{-1}) \ast b = \]
\[ a \ast b \ast a \ast (b^{-1}) \ast (a^{-1}) \ast d \ast c \ast (b^{-1}) \ast a \ast b \ast c \ast d = \]
\[ a \ast b \ast a \ast (a^{-1}) \ast (b^{-1}) \ast (a^{-1}) \ast (b^{-1}) \ast a \ast c \ast (a^{-1}) \ast b = \]
\[ a \ast b \ast (a^{-1}) \ast b \ast c \ast b \ast a \ast (b^{-1}) \ast (a^{-1}) \ast (b^{-1}) \ast e \ast (b^{-1}) = \]
\[ a \ast d \ast b \ast a \ast d \ast (b^{-1}) \ast d \ast (a^{-1}) \ast (b^{-1}) \ast d \ast a^{-1} \ast b = \]
\[ a \ast e \ast b \ast a \ast e \ast b \ast c \ast (a^{-1}) \ast (b^{-1}) \ast e \ast (a^{-1}) \ast b = 1. \]

This group has 741 conjugacy classes and the order structure:

| order of element | number of elements | number of classes |
|------------------|--------------------|------------------|
| 2                | 7487               | 187              |
| 3                | 512                | 1                |
| 4                | 86720              | 425              |
| 6                | 48640              | 63               |
| 8                | 36864              | 48               |
| 12               | 16384              | 16               |
8.10. **Group #27.** This is the second term in the automorphism tower. The presentation for this group of order $294,912 = 2^{15} \cdot 3^2$ is

\[
\begin{align*}
a^2 &= b^2 = c^2 = e^2 = f^2 = g^2 = h^3 = (a \ast c)^2 = \\
a \ast d \ast a \ast d^{-1} &= (a \ast e)^2 = (a \ast f)^2 = (a \ast g)^2 = \\
a \ast h \ast a \ast (h^{-1}) &= b \ast d \ast b \ast (d^{-1}) = (b \ast c)^2 = (c \ast e)^2 = \\
(c \ast f)^2 &= (c \ast g)^2 = c \ast h \ast c \ast (h^{-1}) = d^4 = \\
d \ast e \ast (d^{-1}) \ast e &= d \ast f \ast (d^{-1}) \ast f = d \ast g \ast (d^{-1}) \ast g = \\
d \ast h \ast (d^{-1}) \ast (h^{-1}) &= (e \ast f)^2 = (e \ast g)^2 = (g \ast h)^2 = \\
a \ast b \ast a \ast b \ast (d^{-2}) &= a \ast b \ast a \ast f \ast b \ast f = b \ast g \ast (h^{-1}) \ast b \ast h \ast g = \\
(c \ast d)^3 &= (e \ast h)^3 = (b \ast c)^4 = (b \ast c \ast b \ast g)^2 = \\
(b \ast g)^4 &= (e \ast (h^{-1}) \ast f \ast h)^2 = (f \ast g)^4 = \\
(f \ast h \ast f \ast (h^{-1}))^2 &= f \ast g \ast f \ast h \ast f \ast h \ast g \ast f \ast (h^{-1}) = \\
a \ast b \ast a \ast g \ast f \ast g \ast b \ast g \ast f \ast g = \\
a \ast b \ast a \ast (h^{-1}) \ast f \ast h \ast b \ast (h^{-1}) \ast f \ast h = \\
b \ast c \ast d^2 &= \ast c \ast b \ast c \ast (d^{-2}) \ast c = \\
b \ast c \ast d \ast c \ast h \ast b \ast c \ast h \ast b \ast h \ast (d^{-1}) \ast c = 1.
\end{align*}
\]

There are 1008 conjugacy classes in this group and the order structure of this group is

| order of element | number of elements | number of classes |
|------------------|--------------------|------------------|
| 2                | 13375              | 370              |
| 3                | 4256               | 3                |
| 4                | 117696             | 445              |
| 6                | 70496              | 139              |
| 12               | 89088              | 50               |
8.11. **Group #43.** The automorphism group tower is:

a. \( \text{aut}(g) \): order 12,288. This version of \( \text{Hol}(C_5 \times C_2 \times C_2) \) is the presentation given in the holomorph article. [arXiv:math/0609571](http://arxiv.org/abs/math/0609571)

\[
a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = \\
(a * b)^2 = (a * c)^2 = (a * d)^2 = \\
(a * e)^2 = (a * f)^2 = (b * c)^2 = \\
(b * d)^2 = (b * e)^2 = (b * f)^2 = \\
(c * d)^2 = (c * e)^2 = (c * f)^2 = \\
(d * e)^2 = (d * f)^2 = (e * f)^2 = \\
g^4 = h^4 = g^2 * h * g^{-2} * h = g * h * g * ((h * g * h)^{-1}) = \\
a^{-1} * g^{-1} * a * g = b^9 * a * b = c^9 * a * b * c = \\
a^{-1} * h^{-1} * a * h = b^6 * a * b * c = c^5 * a * c = \\
d^{-1} * g^{-1} * d * g = e^9 * d * e = f^9 * d * e * f = \\
d^{-1} * h^{-1} * d * h = e^h * d * e * f = f^h * d * f = \\
j^8 = j^4 * a = \\
a^{-1} * j^{-1} * a * j = b^{-1} * j^{-1} * b * j = c^{-1} * j^{-1} * c * j = \\
d^j * a * d = e^j * b * e = f^j * c * f = \\
g^{-1} * j^{-1} * g * j = h^{-1} * j^{-1} * h * j = \\
k^2 = j^k * j = \\
a^{-1} * k^{-1} * a * k = b^{-1} * k^{-1} * b * k = c^{-1} * k^{-1} * c * k = \\
d^{-1} * k^{-1} * d * k = e^{-1} * k^{-1} * e * k = f^{-1} * k^{-1} * f * k = \\
g^{-1} * k^{-1} * g * k = h^{-1} * k^{-1} * h * k = 1.
\]

A 5-generator presentation for this group obtained with CAYLEY is:

\[
a^2 = b^2 = c^2 = a * d * a * (d^{-1}) = a * e * a * (e^{-1}) = \\
b * d * b * d^{-1} = c * d * c * d^{-1} = (d * e)^2 = (d * e^{-1})^2 = \\
e^4 = (a * b)^3 = a * b * c * b * c * a * c = \\
b * c * b * c * b * c * (e^{-1}) = (b * e)^4 = \\
(b * e * b * (e^{-1}))^2 = (c * e)^4 = (c * e * c * (e^{-1}))^2 = \\
d^8 = b * c * e * c * (e^{-1}) * b * e * c * (e^{-1}) * c = \\
c * d^4 * e^2 * c * (e^{-2}) = \\
a * b * a * e^2 * b * a * (e^{-2}) * b * (e^{-2}) = 1.
\]
b. Order 98,304 case:
\[ a^2 = b^2 = c^2 = f^2 = h^2 = x^2 = a \ast d \ast a \ast (d^{-1}) = \]
\[ a \ast e \ast a \ast (e^{-1}) = (a \ast f)^2 = a \ast h \ast a \ast (h^{-1}) = (a \ast k)^2 = \]
\[ (a \ast x)^2 = b \ast d \ast b \ast (d^{-1}) = (b \ast f)^2 = b \ast h \ast b \ast (h^{-1}) = \]
\[ (b \ast k)^2 = c \ast d \ast c \ast (d^{-1}) = (c \ast f)^2 = c \ast h \ast c \ast (h^{-1}) = \]
\[ d^4 = (d \ast (e^{-1}))^2 = d \ast f \ast (d^{-1}) \ast f = d \ast h \ast (d^{-1}) \ast h = \]
\[ d \ast k \ast d^{-1} \ast k = d \ast x \ast d^{-1} \ast x = e^4 = (e \ast f^{-1})^2 = \]
\[ e \ast h \ast e^{-1} \ast h^{-1} = e \ast k \ast e^{-1} \ast k = e \ast x \ast e^{-1} \ast x = \]
\[ (f \ast h)^2 = (f \ast k)^2 = (f \ast x)^2 = h^4 = h \ast k \ast (h^{-1}) \ast k = \]
\[ h \ast x \ast (h^{-1}) \ast x = (k \ast x)^2 = (a \ast b)^2 = a \ast b \ast a \ast x \ast b \ast x = \]
\[ a \ast c \ast a \ast x \ast c \ast x = c \ast e^2 \ast c \ast (e^{-2}) = a \ast b \ast c \ast b \ast c \ast a \ast c = \]
\[ b \ast c \ast b \ast e \ast b \ast c \ast b \ast (e^{-1}) = b \ast c \ast k \ast c \ast b \ast k \ast (e^{-2}) = (b \ast e)^4 = \]
\[ c \ast d^2 \ast e \ast c \ast (h^{-2}) \ast (e^{-1}) = 1. \]

8.12. **Group #44.** The first two groups in this automorphism tower have the presentations:

a. \( \text{aut}(g) \):

\[ a^2 = b^2 = c^2 = (a \ast b)^2 = a \ast d \ast a \ast (d^{-1}) = (a \ast e)^2 = \]
\[ b \ast d \ast b \ast (d^{-1}) = (b \ast e)^2 = d^4 = (d \ast (e^{-1}))^2 = (a \ast c)^3 = \]
\[ c \ast d^2 \ast c \ast (d^{-2}) = (b \ast c)^4 = (b \ast c \ast d \ast c)^2 = \]
\[ c \ast d \ast c \ast d \ast c \ast (d^{-1}) \ast c \ast (d^{-1}) = (c \ast d \ast c \ast e)^2 = \]
\[ (c \ast e)^4 = a \ast b \ast c \ast a \ast b \ast c \ast b \ast a \ast c = \]
\[ b \ast c \ast b \ast e \ast c \ast b \ast e \ast c \ast e \ast b \ast c \ast e = 1. \]
b. Second member. This group has order $393,216 = 2^{17} \times 3$ and center $C_2 \times C_2$.

\[
a^2 = b^2 = c^2 = d^2 = (a \ast b)^2 = a \ast c \ast a \ast (c^{-1}) = \\
(a \ast e)^2 = (a \ast f)^2 = (a \ast h)^2 = b \ast c \ast b \ast (c^{-1}) = (b \ast e)^2 = \\
(b \ast f)^2 = c^4 = c \ast e \ast (c^{-1}) \ast e = d^4 = d \ast f \ast (d^{-1}) \ast f = \\
(f \ast h)^2 = b \ast d^2 \ast b \ast (d^{-2}) = c^2 \ast d \ast (c^{-2}) \ast (d^{-1}) = \\
c^2 \ast f \ast (c^{-2}) \ast f = c \ast e \ast (d^{-1}) \ast e \ast (c^{-1}) \ast d = \\
c \ast e \ast h \ast (c^{-1}) \ast e \ast h = b \ast d \ast h \ast d^2 \ast h \ast (d^{-1}) = \\
a \ast b \ast d \ast a \ast h \ast (d^{-1}) \ast b \ast h = (a \ast b \ast h \ast d)^2 = \\
a \ast (d^{-1}) \ast c \ast d \ast a \ast (d^{-1}) \ast (c^{-1}) \ast d = \\
c \ast d \ast c \ast d \ast (c^{-1}) \ast d \ast (c^{-1}) \ast d = \\
c \ast d \ast c \ast (d^{-1}) \ast (c^{-1}) \ast d \ast (c^{-1}) \ast (d^{-1}) = \\
c \ast d \ast c^{-1} \ast f^{-1} \ast c^{-1} \ast d^{-1} \ast c \ast f^{-1} = \\
c \ast f \ast c \ast f \ast (c^{-1}) \ast f \ast (c^{-1}) \ast f = \\
c \ast f \ast (c^{-1}) \ast h \ast (c^{-1}) \ast f \ast c \ast h = (c \ast f \ast e \ast f)^2 = \\
c \ast h \ast c \ast h \ast (c^{-1}) \ast h \ast (c^{-1}) \ast h = (c \ast f)^4 = \\
c \ast d \ast (c^{-1}) \ast (h^{-1}) \ast (c^{-1}) \ast (d^{-1}) \ast c \ast d \ast (h^{-1}) \ast (d^{-1}) = \\
a \ast d \ast a \ast d \ast a \ast (d^{-1}) \ast a \ast (d^{-1}) \ast a \ast (d^{-1}) = 1.
\]

8.13. Group #81. The group of order 6144 in this tower has the following presentation:

\[
a^2 = b^2 = c^2 = d^2 = (a \ast d)^2 = (a \ast b)^3 = (b \ast c)^3 = \\
(b \ast d)^3 = (a \ast c)^4 = (a \ast c \ast d \ast c)^2 = (c \ast d)^4 = \\
(a \ast b \ast a \ast c)^3 = a \ast b \ast a \ast c \ast a \ast c \ast b \ast a \ast d \ast b \ast c \ast a \ast c \ast a \ast b \ast d = \\
a \ast b \ast a \ast c \ast b \ast d \ast c \ast a \ast d \ast b \ast c \ast a \ast b \ast d \ast c \ast d = \\
a \ast b \ast a \ast c \ast d \ast c \ast b \ast a \ast d \ast b \ast c \ast d \ast c \ast a \ast b \ast d = 1.
\]

The second group in this sequence has order 24,576 = $2^{13} \times 3$ and the presentation:

\[
a^2 = b^2 = c^2 = d^2 = (a \ast d)^2 = a \ast e \ast a \ast (e^{-1}) = \\
b \ast e \ast b \ast (e^{-1}) = c \ast e \ast c \ast (e^{-1}) = e^4 = (a \ast b)^3 = \\
(b \ast c)^3 = (b \ast d)^3 = (a \ast c)^4 = (a \ast c \ast d \ast c)^2 = \\
(c \ast d)^4 = (d \ast e)^4 = (d \ast e \ast d \ast (e^{-1}))^2 = \\
a \ast c \ast a \ast c \ast d \ast e^2 \ast d \ast (e^{-2}) = \\
a \ast e \ast a \ast d \ast e \ast d \ast c \ast d \ast (e^{-1}) \ast d = (a \ast b \ast a \ast c)^3 = \\
b \ast c \ast b \ast d \ast c \ast b \ast (e^{-1}) \ast d \ast e \ast b \ast c \ast d = \\
a \ast b \ast a \ast c \ast b \ast d \ast c \ast a \ast d \ast b \ast c \ast a \ast b \ast d \ast c \ast d = 1.
\]
The next factor has order 49, 152 = $2^{14} \cdot 3$ and presentation:

\[
a^2 = b^2 = c^2 = d^2 = f^2 = (a * d)^2 = a * e * a * (e^{-1}) = 1 \\
b * c * b * (e^{-1}) = c * e * c * (e^{-1}) = e^4 = e * f * (e^{-1}) * f = (a * b)^3 = 1 \\
(a * b * f)^2 = a * c * a * e * c * (e^{-1}) = (a * c * f)^2 = 1 \\
a * e^2 * f * a * f = (b * c) = (b * d)^3 = (a * c)^4 = 1 \\
(a * c * d * c)^2 = (c * d)^4 = c * d * e * d * c * d * (e^{-1}) * c * (e^{-1}) = 1 \\
(c * d * e^2 * d * (e^{-1}) * c * (e^{-1}) = (d * e)^4 = (d * f)^4 = 1 \\
a * c * d * f * d * c * e * d * a * f * d = b * c * e * c * b * e * d * e * f * d * f = 1.
\]

The last group in this sequence is a complete group of order 98, 304 = $2^{15} \cdot 3$ and presentation:

\[
a^2 = b^2 = c^2 = d^2 = (a * d)^2 = a * e * a * (e^{-1}) = 1 \\
b * e * b * (e^{-1}) = c * c * e * (e^{-1}) = e^4 = e * f * (e^{-1}) = 1 \\
(a * b)^3 = a * b * f * a * b * (f^{-1}) = a * c * f * c * a * (f^{-1}) = 1 \\
(a * c)^4 = (a * c * d * c)^2 = (c * d)^4 = (d * e)^4 = 1 \\
(b * d * e * d * b * (f^{-1}) * d * (e^{-1}) * (f^{-1}) = 1 \\
(c * d * f * d * c * e * d * f * d * (e^{-1}) = 1 \\
(b * c * b * d * c * b * d * b * f * c * d * (f^{-1}) = 1.
\]

8.14. **Group #82.** A presentation for group number 82’s automorphism group is:

\[
a^2 = b^2 = c^2 = d^3 = b * c * b * c = e^4 = a * e^{-1} * a * e^{-1} = 1 \\
(a * c * a * c = c * e^2 * c * e^{-2} = d^{-1} * c * d^{-1} * c * d^{-1} * c = 1 \\
(d * e^{-1} * d^{-1} * b * e * b * e = b * a * d^{-1} * a * b * a * d^{-1} * a = 1 \\
(c * d^{-1} * a * d * c * e * d^{-1} * a * d = d^{-1} * c * d^{-1} * c * e^{-1} * c * e^{-1} * c = 1 \\
(a * d * e * d^{-1} * a * d * e^{-1} * d^{-1} = b * a * b * a * b * a * b = 1 \\
(c * e^{-1} * c * d^{-1} * c * e^{-1} * d^{-1} = c * d^{-1} * e^2 * d * c * d^{-1} * e^2 * d = 1 \\
(c * c * c * d^{-1} * c * e * c * d^{-1} * c * e^{-1} * d^{-1} = b * d * a * d^{-1} * a * b * d * a * d^{-1} * a = 1 \\
(e * d^{-1} * a * d * c * e * d^{-1} * a * d * e^{-1} = 1 \\
(d^{-1} * c * b * d * c * b * a * d^{-1} * a * c = 1 \\
(c * d * a * d * c * d^{-1} * c * d * a * d = 1 \\
(d * a * d^{-1} * a * e^{-2} * d * a * d^{-1} * a * e^{-2} = 1 \\
(b * a * b * d^{-1} * c * a * d * b * a * b * d^{-1} * c * a * d = 1 \\
a * a * d * e^2 * d * a * d * e^{-2} * d = 1.
\]
This automorphism group has the structure $[512@C_3]@D_4$. All but five of the normal subgroups of this automorphism group are 2-groups. Those that are not have orders 1536 (one case), 3072 (one case), and three cases with orders 6144. The quotient groups $\text{aut}(g)/(\text{order 512 groups})$ are either small group $(24,12)$, i.e., $S_4$ (seven cases), or $(24,8) = (4,6|2,2)$ or $C_3@D_4$ (one case). A presentation for the order 1536 group is:

$$a^2 = g^2 = f^3 = c^4 = b^{-1} * e * b * e = c^{-2} * d^2 = b^4 = c * d * c * d^{-1} = e^2 * b^2 =$$

$$a * c * a * c^{-1} = b * c * b^{-1} * c^{-1} = a * b * a * b^{-1} = a * e * a * e^{-1} =$$

$$a * d * a * d^{-1} = b^{-1} * d^{-1} * b^{-1} * d^{-1} =$$

$$d^{-1} * e^{-1} * d^{-1} * e^{-1} = a * f * a * f^{-1} = c * f * b^{-1} * f^{-1} =$$

$$c^{-1} * e^{-1} * c^{-1} * e^{-1} = a * g * a * g = b * g * b^{-1} * g = c * a * g * c^{-1} * g =$$

$$d * a * g * d^{-1} * g = e * a * g * e^{-1} * g = c * b * f * c * f^{-1} =$$

$$f^{-1} * a * d^{-1} * f * e^{-1} * b^{-1} = b * f * d * f^{-1} * d * e =$$

$$f^{-1} * g * f^{-1} * g * f^{-1} * g = 1.$$  

The sylow 2-subgroup of this order 1536 group, according to GAP, is generated by the following presentation:

$$a^4 = b^4 = c^2 = d^2 = e^2 = f^2 = g^2 = (a, b) =$$

$$b^{-1} * d)^2 = (c * d)^2 = a^{-1} * e * a * e =$$

$$b^{-1} * e * b * e = (c * e)^2 = (d * e)^2 =$$

$$a^{-1} * f * a * f = (c * f)^2 = (d * f)^2 =$$

$$(e * f)^2 = (c * g)^2 = (d * g)^2 = (e * g)^2 =$$

$$(f * g)^2 = a * c * a^{-1} * b^2 * c =$$

$$a^{-1} * d * a^{-1} * b^2 * d = g * e * a^{-1} * g * a =$$

$$c * b^{-1} * a^2 * c * b = f * e * b^{-1} * f * b =$$

$$g * e * b^{-1} * g * b = 1.$$  

The automorphism group of this order 1536 group is 589,824 = $2^{16} * 3^2$. The next one in the automorphism tower sequence has order 1,179,648 and is a complete group. We were unable to generate presentations for these last two groups using GAP. The automorphism group orders were computed with MAGMA.
8.15. **Group #93.** Below are the presentations for the two members of this sequence.

a. order 3072:

\[
\begin{align*}
    a^3 &= b^3 = c^3 = d^3 = (a * (c^{-1}))^2 = (b * (c^{-1}))^2 = \\
    (b * d^{-1})^2 &= (c * d^{-1})^2 = (a * b)^3 = (b * d * c)^2 = \\
    (a * (b^{-1}) * a * (d^{-1}))^2 &= \\
    a * (b^{-1}) * c * a * (b^{-1}) * (c^{-1}) * (a^{-1}) * b &= \\
    (a * d^{-1})^4 &= a * b * c * d^{-1} * a * d * a * d^{-1} * b^{-1} = 1.
\end{align*}
\]

b. order 294, 912 = 2^{15} * 3^2:

\[
\begin{align*}
    a^4 &= b^3 = c^6 = d^3 = e^4 = a * b * (a^{-1}) * (b^{-1}) = \\
    a * c * a * (a^{-1}) * (c^{-1}) &= a^2 * c * (a^{-2}) * (c^{-1}) = \\
    a * c * a * (a^{-1}) * (c^{-1}) * c &= a * d * (a^{-1}) * (c^{-1}) * d * c * d = \\
    a * c * a * (c^{-1}) * (a^{-1}) * c * a * (c^{-1}) &= \\
    a * c * e * c * a * (c^{-1}) * (e^{-1}) * (c^{-1}) &= \\
    a * c * (e^{-1}) * c * a * (c^{-1}) * e * (c^{-1}) &= \\
    (a * d * a * d^{-1})^2 &= (a * d * a^{-1} * d^{-1})^2 = \\
    a * e^2 * c^{-1} * e * c * e * c^{-1} &= b * c^3 * b^{-1} * c^{-3} = \\
    b * (c^{-1}) * (d^{-1}) * (e^{-1}) * (b^{-1}) * (e^{-1}) * c * (d^{-1}) &= \\
    b * d * b * (d^{-1}) * (b^{-1}) * (c^{-1}) * (d^{-1}) * c &= \\
    b * d * (b^{-1}) * e * (d^{-1}) * b * d * (e^{-1}) &= \\
    b * d * (b^{-1}) * (e^{-1}) * (d^{-1}) * b * d * e &= \\
    (b * (d^{-1}))^4 &= \\
    b * (d^{-1}) * (e^{-1}) * e * (b^{-1}) * e * (d^{-1}) * c &= \\
    b * e^2 * b^{-1} * d^{-1} * e^{-2} * e &= (b * e^{-1})^4 = \\
    c * e * d * (e^{-1}) * (e^{-1}) * (d^{-1}) * e &= \\
    c * e^{-1} * d * e * c^{-1} * e * d^{-1} * e^{-1} &= \\
    (d * (e^{-1}))^4 &= 1.
\end{align*}
\]
8.16. **Group #103.** The following are the higher-order presentations for the groups in this automorphism tower.

Group of order $73,728 = 2^{13} \ast 3^2$:

\[
\begin{align*}
  a^4 &= b^4 = c^4 = d^2 = b^2 \ast (e^{-2}) = a \ast d \ast (a^{-1}) \ast d = \\
  a \ast c \ast (a^{-1}) \ast (e^{-1}) &= c \ast d \ast (c^{-1}) \ast d = a^2 \ast b \ast a^2 \ast b = \\
  a^2 \ast c \ast a^2 \ast c &= a^2 \ast (e^{-1}) \ast d \ast (e^{-1}) \ast d = \\
  a \ast c \ast a \ast (a^{-1}) \ast (a^{-1}) \ast (e^{-1}) &= \\
  a \ast c \ast c \ast (a^{-1}) \ast (a^{-1}) \ast (c^{-1}) &= b^2 \ast d \ast b^2 \ast d = \\
  a \ast b \ast a \ast b \ast (a^{-1}) \ast (b^{-1}) \ast (a^{-1}) \ast (b^{-1}) = \\
  a \ast b \ast c \ast b \ast a^{-1} \ast b^{-1} \ast c^{-1} \ast b^{-1} = \\
  a \ast b \ast d \ast b \ast (a^{-1}) \ast (b^{-1}) \ast d \ast (b^{-1}) = \\
  a \ast b \ast c \ast b \ast (a^{-1}) \ast (b^{-1}) \ast (e^{-1}) \ast (b^{-1}) = \\
  b \ast c \ast b \ast c \ast (b^{-1}) \ast (c^{-1}) \ast (b^{-1}) \ast c = \\
  b \ast d \ast b \ast d \ast (b^{-1}) \ast d \ast (b^{-1}) \ast d = \\
  b \ast d \ast (b^{-1}) \ast (e^{-1}) \ast (b^{-1}) \ast d \ast b \ast c = \\
  b \ast e \ast b \ast e \ast (b^{-1}) \ast (e^{-1}) \ast (b^{-1}) \ast (e^{-1}) = \\
  a \ast (b^{-1}) \ast (e^{-1}) \ast (b^{-1}) \ast (c^{-1}) \ast (a^{-1}) \ast b \ast e \ast b \ast c = \\
  b \ast c^2 \ast (b^{-1}) \ast d \ast (b^{-1}) \ast c^2 \ast b \ast d = \\
  a \ast b \ast d \ast b \ast c \ast a \ast b \ast d \ast (b^{-1}) \ast c \ast a \ast (b^{-1}) \ast d \ast (b^{-1}) \ast (c^{-1}) &= 1.
\end{align*}
\]

The number of classes in this group is 119 and its order structure is:

| order of element | number of elements | number of classes |
|------------------|--------------------|------------------|
| 2                | 2127               | 28               |
| 3                | 1088               | 2                |
| 4                | 22704              | 56               |
| 6                | 14784              | 16               |
| 8                | 16128              | 5                |
| 12               | 16896              | 11               |
The next group in this tower is a group of order 147,456 = 2^{14} \cdot 3^2 with the presentation:

\begin{align*}
a^4 &= b^4 = c^2 = d^4 = b^2 \cdot e^{-4} = a \cdot c \cdot a^{-1} \cdot c = \\
a^2 \cdot b \cdot a^{-2} \cdot b &= a^2 \cdot d \cdot a^{-2} \cdot d = a \cdot b^2 \cdot a^{-1} \cdot b^{-2} = \\
(a \cdot (b^{-1}) \cdot (e^{-1}))^2 &= \\
a \cdot d \cdot a \cdot (d^{-1}) \cdot (a^{-1}) \cdot (d^{-1}) &= \\
(a \cdot c \cdot b^{-1})^2 &= a \cdot e^2 \cdot a^{-1} \cdot e^{-2} = \\
a \cdot (e^{-1}) \cdot b \cdot c \cdot (a^{-1}) \cdot (b^{-1}) &= b^2 \cdot c \cdot b^2 \cdot c = \\
b^2 \cdot d \cdot b^2 \cdot d &= b^2 \cdot c \cdot b^2 \cdot (e^{-1}) = c \cdot d^2 \cdot c \cdot d^2 = \\
a \cdot b \cdot a \cdot b \cdot (a^{-1}) \cdot (b^{-1}) \cdot (a^{-1}) \cdot (b^{-1}) &= \\
a \cdot b \cdot c \cdot b \cdot (a^{-1}) \cdot (b^{-1}) \cdot c \cdot (b^{-1}) &= \\
a \cdot b \cdot d \cdot b \cdot a^{-1} \cdot b^{-1} \cdot d^{-1} \cdot b^{-1} &= \\
a \cdot (d^{-1}) \cdot b \cdot c \cdot (d^{-1}) \cdot (b^{-1}) \cdot a \cdot (e^{-1}) &= \\
b \cdot c \cdot b \cdot c \cdot (b^{-1}) \cdot c \cdot (b^{-1}) \cdot c &= \\
b \cdot c \cdot (b^{-1}) \cdot (d^{-1}) \cdot (b^{-1}) \cdot c \cdot b \cdot d &= \\
b \cdot d \cdot b \cdot d \cdot (b^{-1}) \cdot (d^{-1}) \cdot (b^{-1}) \cdot d &= \\
b \cdot e \cdot c \cdot d^2 \cdot (e^{-1}) \cdot (b^{-1}) \cdot c &= \\
c \cdot d \cdot c \cdot d \cdot c \cdot (d^{-1}) \cdot c \cdot (d^{-1}) &= \\
a \cdot c \cdot d \cdot a \cdot c \cdot (d^{-1}) \cdot c \cdot (a^{-1}) \cdot (d^{-1}) &= 1.
\end{align*}

The number of classes in this group is 106 and it has the following order structure:

| order of element | number of elements | number of classes |
|------------------|--------------------|------------------|
| 2                | 2383               | 20               |
| 3                | 1088               | 2                |
| 4                | 37296              | 41               |
| 6                | 26048              | 18               |
| 8                | 42240              | 10               |
| 12               | 32256              | 12               |
| 24               | 6144               | 2                |
The last group in this tower has order 294, 912 = $2^{15} \cdot 3^2$ and presentation:

\[
\begin{align*}
b^2 &= e^2 = a^4 = a \cdot b \cdot a^{-1} \cdot b = a \cdot d \cdot a^{-1} \cdot d^{-1} = \\
a \cdot c \cdot a^{-1} \cdot c &= a \cdot h \cdot a^{-1} \cdot h^{-1} = b \cdot c \cdot e \cdot c^{-1} = \\
b \cdot c^{-1} \cdot e \cdot c &= (b \cdot e)^2 = b \cdot g \cdot b \cdot g^{-1} = e \cdot h \cdot e \cdot h^{-1} = \\
g^4 &= a^2 \cdot c \cdot a^{-2} \cdot c = a^2 \cdot d^{-1} \cdot b \cdot d^{-1} \cdot b^{-1} = \\
a^2 \cdot g \cdot a^{-2} \cdot g &= a \cdot d^{-1} \cdot g^{-1} \cdot a^{-1} \cdot d \cdot g = \\
a \cdot g \cdot a \cdot g^{-1} \cdot a^{-1} \cdot g^{-1} &= a \cdot g \cdot c \cdot h \cdot e \cdot c^{-1} = \\
a \cdot h \cdot g \cdot h^{-1} \cdot a^{-1} \cdot g^{-1} &= a \cdot g \cdot e \cdot g^{-2} = (a \cdot c)^2 \cdot (a^{-1} \cdot c^{-1})^2 = \\
a \cdot c \cdot h^{-1} \cdot c^{-2} \cdot h^{-1} \cdot c^{-1} \cdot g^{-1} = \\
b \cdot h \cdot c^{-1} \cdot g \cdot c \cdot g \cdot c \cdot h^{-1} &= (d \cdot h)^2 \cdot (d^{-1} \cdot h^{-1})^2 = \\
(e \cdot g)^2 \cdot (e \cdot g^{-1})^2 &= a \cdot c \cdot b \cdot h^{-1} \cdot b \cdot h^{-1} \cdot c \cdot e \cdot g = \\
a \cdot c \cdot h^{-1} \cdot d^{-1} \cdot h^{-2} \cdot d^{-1} \cdot h^{-1} \cdot c \cdot a^{-1} = \\
a \cdot c^{-1} \cdot h \cdot c \cdot d \cdot a^{-1} \cdot c^{-1} \cdot h^{-1} \cdot d^{-1} \cdot c^{-1} = \\
c \cdot d \cdot h^{-1} \cdot c^{-1} \cdot h^{-1} \cdot c^{-1} \cdot d \cdot h \cdot c^{-1} \cdot h = \\
a \cdot c \cdot d \cdot h \cdot d \cdot g \cdot h \cdot c^{-1} \cdot h^{-1} \cdot d^{-1} \cdot h = 1.
\end{align*}
\]

The number of classes in this group is 152, and it has the order structure:

| order of element | number of elements | number of classes |
|------------------|--------------------|-------------------|
| 2                | 3087               | 21                |
| 3                | 1088               | 2                 |
| 4                | 60144              | 50                |
| 6                | 40896              | 33                |
| 8                | 82176              | 17                |
| 12               | 76800              | 23                |
| 16               | 18432              | 1                 |
| 24               | 12288              | 4                 |
8.17. **Group #108.** For this automorphism sequence we have found the following three groups with the following presentations:

a. order 12,288, and center $C_2$:

$$a^2 = b^4 = c^2 = d^2 = e^2 = f^2 = (a \ast b)^2 = (a \ast e)^2 =$$

$$(a \ast f)^2 = b \ast d \ast (b^{-1}) \ast d = b \ast e \ast (b^{-1}) \ast e =$$

$$b \ast f \ast (b^{-1}) \ast f = (c \ast f)^2 = (d \ast e)^2 = (d \ast f)^2 =$$

$$(c \ast f)^2 = a \ast b^2 \ast a \ast b = a \ast b \ast a \ast b \ast b \ast b^{-1} \ast a \ast b^{-1} =$$

$$a \ast c \ast b \ast c \ast a \ast c \ast (b^{-1}) \ast c = (b \ast c \ast e \ast c)^2 = (c \ast e)^4 =$$

$$a \ast c \ast d \ast b^2 \ast c \ast a \ast c \ast d \ast c = a \ast c \ast d \ast c \ast a \ast c \ast d \ast c \ast e =$$

$$a \ast c \ast e \ast b^2 \ast c \ast a \ast c \ast e \ast c =$$

$$b^2 \ast c \ast (b^{-1}) \ast c \ast b^2 \ast c \ast (b^{-1}) \ast c =$$

$$b^2 \ast c \ast d \ast c \ast b^2 \ast c \ast d \ast c = (a \ast b \ast a \ast c \ast a \ast c)^2 =$$

$$a \ast c \ast a \ast d \ast c \ast a \ast d \ast c \ast d \ast a \ast c \ast d =$$

$$b \ast c \ast b \ast c \ast b \ast c \ast (b^{-1}) \ast c \ast (b^{-1}) \ast c \ast (b^{-1}) \ast c =$$

$$a \ast b \ast a \ast b \ast c \ast a \ast b \ast c \ast d \ast c \ast d \ast c =$$

$$a \ast b \ast a \ast c \ast b \ast a \ast c \ast d \ast c \ast b \ast c \ast c \ast (b^{-1}) \ast c \ast d \ast c = 1.$$
This group has 464 classes with the order structure:

| order of element | number of elements | number of classes |
|------------------|--------------------|-------------------|
| 2                | 3583               | 163               |
| 3                | 128                | 1                 |
| 4                | 23040              | 228               |
| 6                | 10112              | 43                |
| 8                | 144                | 16                |
| 12               | 6144               | 12                |

The automorphism group of the above order 49,152 group has order \( 2^{27} \times 3^2 = 1,207,959,552 \). GAP did not return/find the center of this order \( 2^{27} \times 3^2 \) group.

b. A representation for the order 12,288 group that appears in the above 49,152 order of element

\[
\begin{align*}
a^2 &= b^2 = c^2 = d^2 = e^2 = f^2 = (a,b) = (a,d) = (b,d) = 1, \\
(c,e) &= (c,f) = (a*b*f)^2 = (a*e*f)^2 = 1, \\
a* (e^{-1})*f*a*f*e &= b*e*f*b*f*(e^{-1}) = 1, \\
(d*e)^3 &= d*e*f*d*f*(e^{-1}) = (a*e)^4 = 1, \\
(a*c*b*c)^2 &= (a*c*d*c)^2 = (b*c)^4 = (c*d)^4 = 1, \\
a*b*c*d*c*b*c*d*(e^{-1})*c*a*e &= 1, \\
a*c*a*d*c*e*d*c*f*d*f*c*d*(e^{-1}) &= 1.
\end{align*}
\]

c. The automorphism group of the order 12,288 group in the order 49,152 group has order \( 2^{16} \times 3 = 196,608 \) and has for a center the elementary abelian group of order 8. A presentation for this group is

\[
\begin{align*}
a^2 &= b^2 = c^2 = d^3 = e^2 = f^2 = g^2 = h^2 = (a*c)^2 = 1, \\
(a*e)^2 &= b*d*b*(d^{-1}) = (b*e)^2 = (b*f)^2 = (b*g)^2 = 1, \\
(b*h)^2 &= (c*h)^2 = (e*g)^2 = a*b*g*a*g*b = 1, \\
a*d*a*e*(d^{-1})*e &= a*d*a*g*(d^{-1})*g = 1, \\
a*d*a*h*(d^{-1})*h &= a*d*f*a*f*(d^{-1}) = 1, \\
a*(d^{-1})*f^2 &= a*f*a*g*f*g = a*f*a*h*f*h = 1, \\
b*c*b*g*c*g &= (c*d)^3 = c*d*f*c*f*(d^{-1}) = 1, \\
c*d^{-1}e*d*c*e &= (a*b)^4 = (a*b*a*h)^2 = 1, \\
a*b*e*f*a*e*f*b &= (a*h)^4 = (a*h*e*h)^2 = 1, \\
(b*c)^4 &= (e*h)^4 = (e*h*g*h)^2 = (g*h)^4 = 1, \\
a*b*h*g*h*a*h*g*h*b &= 1, \\
b*c*b*d*c*b*f*c*f*b*c*d^{-1} &= 1.
\end{align*}
\]

This group has 864 classes and an order structure of:
| order of element | number of elements | number of classes |
|------------------|--------------------|------------------|
| 2                | 8575               | 242              |
| 3                | 128                | 1                |
| 4                | 97920              | 465              |
| 6                | 28544              | 79               |
| 8                | 24576              | 40               |
| 12               | 36864              | 36               |

8.18. **Group #144.** The complete group factor in this tower.

\[
\begin{align*}
  a^2 &= b^2 = c^2 = d^2 = (a * b)^3 = (a * d)^3 = (b * c)^4 = \\
  (a * b * d * b * d)^2 &= (a * b * a * d)^3 = (a * b * c)^4 = (c * d)^6 = \\
  a * b * c * a * b * c * d * b * a * c * b * a * c * d &= \\
  a * b * c * a * b * d * b * a * c * b * a * d * c * d &= \\
  (a * b * c * a * d * a * c)^2 &= (a * d * b * c)^4 = 1.
\end{align*}
\]

8.19. **Group #153.** The first group in this automorphism tower has order 10,752 and presentation:

\[
\begin{align*}
  a^3 &= b^3 = c^3 = (a * (b^{-1}))^2 = \\
  a * b * a * c * b * (a^{-1}) * c * (a^{-1}) * b * c &= \\
  a * c * a * c * (a^{-1}) * (c^{-1}) * (a^{-1}) * c * (a^{-1}) * (c^{-1}) &= \\
  a * b * (c^{-1}) * b * c * a * c * (b^{-1}) * (c^{-1}) * (a^{-1}) * c &= \\
  a * c * a^{-1} * b^{-1} * c^{-1} * a^{-1} * c^{-1} * b * a * c * b * c^{-1} &= 1.
\end{align*}
\]

This group has 48 classes and the order structure:

| order of element | number of elements | number of classes |
|------------------|--------------------|------------------|
| 2                | 511                | 29               |
| 3                | 896                | 2                |
| 6                | 6272               | 14               |
| 7                | 3072               | 2                |

The next group in this tower has order 64,512 and presentation:

\[
\begin{align*}
  a^4 &= b^3 = c^6 = (a * b * a * (b^{-1}))^2 = \\
  (a * b * (a^{-1}) * (b^{-1}))^2 &= \\
  a * c * a * c * (a^{-1}) * (c^{-1}) * (a^{-1}) * (c^{-1}) &= \\
  a * (a^{-1}) * c * a * (c^{-1}) * a * (c^{-1}) &= \\
  a * c^3 * (a^{-1}) * c^3 &= a^2 * b * a^2 * b * a^2 * b &= \\
  a * b * c * (b^{-1}) * c * (b^{-1}) * (a^{-1}) * b^{-1} * c^{-1} * b &= \\
  a * (b^{-1}) * a * c * (a^{-1}) * (c^{-1}) * b * c * (a^{-1}) * c^{-1} &= \\
  a^2 * b * c * b * c * (a^{-1}) * (b^{-1}) * (a^{-1}) * (b^{-1}) * c &= 1.
\end{align*}
\]
This group has 48 classes and the order structure:

| order of element | number of elements | number of classes |
|------------------|--------------------|------------------|
| 2                | 703                | 11               |
| 3                | 2816               | 5                |
| 4                | 1344               | 4                |
| 6                | 30464              | 17               |
| 7                | 3072               | 2                |
| 12               | 10752              | 4                |
| 14               | 9216               | 2                |
| 21               | 6144               | 2                |

8.20. **Group #156.** A permutation representation and a presentation for the group [96 number 2301] \( G_2 \) is

\[
A = (1, 2, 3, 4)(5, 7) \\
B = (1, 2)(5, 8, 6, 7) \\
C = (1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(8, 16) \\
A^4 = B^4 = C^2 = (A * B)^3 = (A * (B^{-1})^3 = \\
A^2 * B^2 * (A^{-2}) * (B^{-2}) = \\
A * C * A * C * (A^{-1}) * C * (A^{-1}) * C = \\
A * C * B * C * (A^{-1}) * C * (B^{-1}) * C = \\
B * C * B * C * (B^{-1}) * C * (B^{-1}) * C = 1.
\]

The automorphism group of this group has order 110, 592 = \(2^{12} * 3^3\) and a presentation on four generators:

\[
a^4 = b^2 = c^4 = d^6 = a * c^2 * a^{-1} * c^2 = \\
a * d * b * (d^{-1}) * (a^{-1}) * b = a * d^2 * (a^{-1}) * (d^{-2}) = \\
(a * (d^{-1}) * b^{-1})^2 = b * c * (d^{-1}) * b * d * (c^{-1}) = \\
a * b * a * b * (a^{-1}) * b * (a^{-1}) * b = a * b * c^2 * b * (c^{-1}) * (a^{-1}) * c = \\
a * b * d * c^2 * (a^{-1}) * b * d = a * c * a * c * a * (c^{-1}) * a * (c^{-1}) = \\
a * d^{-1} * c * d * a^{-1} * d^{-1} * c^{-1} * d = \\
a^2 * d^{-1} * c^{-1} * d * c * a^{-1} * b * d^{-1} * c^{-1} * a^{-1} * b = 1.
\]
8.21. **Group #173.** This automorphism tower has had the following members’ presentations determined:

a. Group of order 6144:

\[
\begin{align*}
    a^4 &= b^4 = c^4 = d^2 = a^2 * b * a^2 * b = a^2 * c * b^2 * c^{-1} = \\
    a^2 * (c^{-1}) * (b^2) * c &= a^2 * d * (a^2) * d = a * b^2 * (a^{-1}) * (b^2) = \\
    b^2 * d * b^2 * d &= (b * c^{-1})^3 = c^2 * d * c^2 * d = \\
    a * b * a * b * (a^{-1}) * (b^{-1}) * (a^{-1}) * b = \\
    a * b * c * b * (c^{-1}) * (a^{-1}) * b * c &= a * c^2 * a * (c^2) * (b^2) = \\
    a * d * a * d * (a^{-1}) * d * (a^{-1}) * d = \\
    b * c * (b^{-1}) * (d^{-1}) * (b^{-1}) * (c^{-1}) * b * (d^{-1}) = \\
    b * d * b * d * (b^{-1}) * d * (b^{-1}) * d &= c * d * c * d * (c^{-1}) * d * (c^{-1}) * d = \\
    a^2 * b * c * d * (c^{-1}) * d * b * d = \\
    a * b * d * a * (c^{-1}) * a * c * d * (a^{-1}) * b = \\
    a * b * (d^{-1}) * b * (a^{-1}) * (d^{-1}) * (a^{-1}) * (b^{-1}) * (d^{-1}) * (b^{-1}) * a * (d^{-1}) = 1.
\end{align*}
\]

This group has 44 classes, a trivial center, and the following order structure:

| order of element | number of elements | number of classes |
|------------------|--------------------|-------------------|
| 2                | 399                | 20                |
| 3                | 512                | 1                 |
| 4                | 2544               | 16                |
| 6                | 1536               | 3                 |
| 8                | 1152               | 3                 |
b. The order 49,152 group has the following presentation:
\[
a^4 = b^4 = c^2 = e^4 = f^2 = g^2 = 2 = \\
a \ast b \ast (a^{-1}) \ast (b^{-1}) = a \ast d \ast (a^{-1}) \ast (d^{-1}) = \\
a \ast g \ast (a^{-1}) \ast g = b \ast g \ast (b^{-1}) \ast g = \\
c^2 \ast (d^2) = c \ast d \ast (c^{-1}) \ast (d^{-1}) = c \ast g \ast (c^{-1}) \ast g = \\
d \ast g \ast (d^{-1}) \ast g = e \ast g \ast (e^{-1}) \ast g = a^2 \ast c \ast (a^2) \ast c = \\
a^2 \ast (e^{-1}) \ast (c^2) \ast e = a^2 \ast f \ast (a^2) \ast f = \\
a^2 \ast (g^{-1}) \ast (f^{-1}) \ast (g^{-1}) \ast (f^{-1}) = \\
a \ast e \ast (a^{-1}) \ast b^2 \ast e = b^2 \ast c \ast (b^2) \ast (c^{-1}) = \\
b^2 \ast d \ast (b^2) \ast d = (b \ast (d^{-1}) \ast f)^2 = \\
b \ast f \ast (b^{-1}) \ast e^2 \ast f = c^2 \ast f \ast (c^2) \ast f = (c \ast (e^{-1}))^3 = \\
a \ast c \ast e \ast c \ast e^{-1} \ast a^{-1} \ast c \ast e = \\
a \ast (c^{-1}) \ast b \ast (e^2) \ast (a^{-1}) \ast b \ast c = \\
a \ast (f^{-1}) \ast a \ast (f^{-1}) \ast (d^{-1}) \ast (b^{-1}) \ast (d^{-1}) \ast (b^{-1}) = \\
b \ast c \ast (b^{-1}) \ast (e^{-1}) \ast (b^{-1}) \ast (c^{-1}) \ast b \ast (e^{-1}) = \\
b \ast (c^{-1}) \ast e \ast c \ast b \ast (c^{-1}) \ast (e^{-1}) \ast (c^{-1}) = \\
b \ast e \ast c \ast d \ast b \ast e \ast (d^{-1}) \ast (c^{-1}) = \\
b \ast e \ast (f^{-1}) \ast e \ast (b^{-1}) \ast c \ast (f^{-1}) \ast (c^{-1}) = \\
c \ast e \ast (c^{-1}) \ast (f^{-1}) \ast (c^{-1}) \ast (e^{-1}) \ast c \ast (f^{-1}) = \\
c \ast f \ast c \ast f \ast (e^{-1}) \ast f \ast (c^{-1}) \ast f = \\
a^2 \ast c \ast e \ast f \ast (e^{-1}) \ast f \ast c \ast f = 1.
\]
c. The next member in this tower has order $2^{18} \times 3 = 786,432$. The derived group of this group has order 49,152, but is not isomorphic to the previous order 49,152 group. A presentation for this derived group is:

\[ a^4 = b^4 = a^2 * (e^{-2}) = a \ast b \ast (a^{-1}) \ast (b^{-1}) = \]
\[ (a, e) = (b \ast e^{-1})^2 = \]
\[ a^2 \ast (c^{-1}) \ast b^2 \ast c = a^2 \ast (d^{-1}) \ast b^2 \ast d = \]
\[ a^2 \ast c \ast b^2 \ast (c^{-1}) \ast b^2 = a^2 \ast d \ast b^2 \ast (d^{-1}) \ast b^2 = \]
\[ (a \ast c \ast (a^{-1}) \ast (c^{-1}))^2 = \]
\[ a \ast c \ast d^2 \ast a \ast (d^{-2}) \ast (c^{-1}) = \]
\[ a \ast c \ast (d^{-1}) \ast b \ast c \ast (a^{-1}) \ast (d^{-1}) \ast (b^{-1}) = \]
\[ (a \ast c \ast (e^{-1}) \ast (e^{-1}))^2 = \]
\[ a \ast (c^{-1}) \ast b \ast c \ast a \ast (c^{-1}) \ast (b^{-1}) \ast c = \]
\[ a \ast (c^{-1}) \ast d \ast (b^{-1}) \ast (d^{-1}) \ast (a^{-1}) \ast c \ast (b^{-1}) = \]
\[ (a \ast d \ast (a^{-1}) \ast (d^{-1}))^2 = \]
\[ a \ast (d^{-1}) \ast c \ast (e^{-1}) \ast (d^{-1}) \ast c \ast (a^{-1}) \ast c = \]
\[ b \ast c \ast b \ast (c^{-1}) \ast (b^{-1}) \ast c \ast b \ast (e^{-1}) = \]
\[ (b \ast (c^{-1}) \ast (d^{-1}) \ast (e^{-1}))^2 = \]
\[ b \ast d \ast (c^{-1}) \ast (e^{-1}) \ast (b^{-1}) \ast c \ast e \ast (d^{-1}) = \]
\[ b \ast d^2 \ast c \ast d^2 \ast (b^{-1}) \ast c = (b \ast (d^{-3}))^2 = \]
\[ b \ast e \ast c^2 \ast (e^{-1}) \ast (b^{-1}) \ast (c^{-2}) = (c^2 \ast (d^{-2}))^2 = \]
\[ (c \ast d \ast (c^{-1}) \ast (d^{-1}))^2 = \]
\[ a \ast (c^{-2}) \ast (d^{-1}) \ast (a^{-1}) \ast b^2 \ast d \ast c^2 = 1. \]

This group has 244 classes with the following order structure:

| order of element | number of elements | number of classes |
|------------------|--------------------|------------------|
| 2                | 1343               | 51               |
| 3                | 512                | 2                |
| 4                | 11968              | 104              |
| 6                | 24064              | 62               |
| 8                | 3072               | 8                |
| 12               | 8192               | 16               |
Several attempts to find the automorphism group of this group have failed. Running times were over 2 days without completion.

8.22. **Group #181.** The automorphism tower for this group starts out with the following group of order 1536:

$$b^4 = c^4 = d^2 = b * d * (b^{-1}) * d = (c * d)^2 =$$

$$a^3 * d * (a^{-1}) * d = a^2 * b * a^2 * (b^{-1}) =$$

$$(a * (c^{-1}))^3 = b * c^2 * b * c^2 =$$

$$a * b * c * b * (c^{-1}) * (a^{-1}) * (b^{-1}) =$$

$$a * c * a * c * b^2 * a * (c^{-1}) = 1.$$ 

This group has 33 classes and the order structure:

| order of element | number of elements | number of classes |
|------------------|--------------------|------------------|
| 2                | 159                | 10               |
| 3                | 128                | 1                |
| 4                | 480                | 13               |
| 6                | 384                | 3                |
| 8                | 384                | 5                |

An alternate presentation for this order 1536 group is:

$$b^4 = c^4 = d^4 = b * d * b * (d^{-1}) = (c * (d^{-1}))^2 =$$

$$a^4 * (d^{-2}) = a^2 * b * a^2 * (b^{-1}) = a * b * d * a * d * (b^{-1}) =$$

$$a * b * (d^{-1}) * a * (d^{-1}) * (b^{-1}) =$$

$$a * (b^{-1}) * d * a * (b^{-1}) * (d^{-1}) =$$

$$(a * (c^{-1}))^3 = b^2 * d * (c^{-1}) * (d^{-1}) * (c^{-1}) =$$

$$b * c^2 * b * (c^{-2}) = a * b * c * b * (c^{-1}) * (a^{-1}) * (b^{-1}) =$$

$$a * c * a * c * (b^{-2}) * a * (c^{-1}) = 1.$$ 

The second member of this tower is a group of order 6144 = 2^{13} * 3.

$$a^2 = b^6 = c^4 = d^2 = a * b * a * (b^{-1}) = (a * c)^2 =$$

$$a * c^2 * d * a * c^{-2} * d = (a * d)^4 = (a * d * (c^{-1}) * d)^2 =$$

$$b^3 * c * (b^{-3}) * (c^{-1}) = (b * c * b * (c^{-1}))^2 =$$

$$b * c * d * (c^{-1}) * b * (c^{-1}) * d * (c^{-1}) =$$

$$c * d * c * d * (c^{-1}) * d * (c^{-1}) * d =$$

$$a * b * (c^{-1}) * (b^{-1}) * d * (b^{-1}) * (c^{-1}) * a * b * d =$$

$$b * (c^{-1}) * d * (b^{-1}) * d * (b^{-1}) * c * d * b * d = 1.$$ 

This group has 72 classes and the order structure:
The final, complete group, in this tower is a group of order $12,288 = 2^{12} \times 3$ and has the presentation:

\[
b^4 = c^4 = a^2 \cdot d^2 = a \cdot d \cdot (a^{-1}) \cdot (d^{-1}) =
\]
\[
a^2 \cdot (d^{-1}) = b^2 \cdot c \cdot b^2 \cdot c = b \cdot c^2 \cdot (b^{-1}) \cdot c^2 =
\]
\[
a^3 \cdot c \cdot a^{-3} \cdot c^{-1} = (a \cdot b \cdot a \cdot b^{-1})^2 =
\]
\[
(a \cdot b \cdot d \cdot c^{-1})^2 = (a \cdot b \cdot d^{-1} \cdot b^{-1})^2 =
\]
\[
a \cdot (b^{-1}) \cdot d \cdot c \cdot a \cdot (b^{-1}) \cdot d \cdot (c^{-1}) = (a \cdot c \cdot a \cdot (c^{-1}))^2 =
\]
\[
a \cdot c \cdot d^{-1} \cdot c^{-1} \cdot a \cdot c^{-1} \cdot d^{-1} \cdot c =
\]
\[
a \cdot d \cdot b \cdot c \cdot (d^{-1}) \cdot (a^{-1}) \cdot b = (c \cdot d)^4 =
\]
\[
(c \cdot (d^{-1}))^4 = a^2 \cdot c \cdot a \cdot b \cdot (a^{-1}) \cdot (c^{-1}) \cdot a \cdot (b^{-1}) =
\]
\[
a \cdot b \cdot c \cdot b \cdot (c^{-1}) \cdot (a^{-1}) \cdot b \cdot (a^{-1}) \cdot b = 1.
\]

This group has 78 classes and an order structure:

| order of element | number of elements | number of classes |
|-----------------|-------------------|------------------|
| 2               | 559               | 18               |
| 3               | 128               | 1                |
| 4               | 4560              | 34               |
| 6               | 2432              | 10               |
| 8               | 3072              | 11               |
| 12              | 1536              | 3                |

8.23. **Group #183.** The automorphism tower for this group has three steps with the following presentations and order structure:

a. **Group of order $9216 = 2^{10} \times 3^2$:**

\[
a^2 = b^{12} = c^2 = (a \cdot b \cdot a \cdot (b^{-1}))^2 = a \cdot (b^{-2}) \cdot c \cdot a \cdot c \cdot b^2 =
\]
\[
(a \cdot c)^4 = a \cdot b^2 \cdot a \cdot c \cdot a \cdot c \cdot (b^{-2}) = (a \cdot b \cdot a \cdot (b^{-1}) \cdot c)^2 =
\]
\[
(a \cdot b \cdot c \cdot a \cdot b \cdot c \cdot a \cdot c \cdot (b^{-1}) \cdot c \cdot (b^{-1}) =
\]
\[
a \cdot b \cdot a \cdot b \cdot c \cdot b^4 \cdot c \cdot b^2 = (b^2 \cdot c)^4 = (b \cdot c)^6 =
\]
\[
a \cdot b \cdot c \cdot (b^{-1}) \cdot a \cdot c \cdot b \cdot c \cdot (b^{-1}) \cdot c \cdot b \cdot c \cdot (b^{-1}) \cdot c = 1.
\]
This group has 50 classes and the order structure:

| order of element | number of elements | number of classes |
|------------------|-------------------|------------------|
| 2                | 495               | 18               |
| 3                | 800               | 3                |
| 4                | 3600              | 18               |
| 6                | 3168              | 7                |
| 12               | 1152              | 3                |

b. The next group has order 18,432 and presentation:
\[
a^2 = c^2 = d^2 = e^2 = (a * c)^2 = (a * e)^2 = b * c * (b^{-1}) * c = \\
(c * c)^2 = a * b * a * e * b * e = b^6 * c = b * d * c * e * d * (b^{-1}) * e = \\
(a * b * a * (b^{-1}))^2 = a * (b^{-2}) * d * a * d * b^2 = (a * d)^4 = \\
b^2 * e * d * e * (b^{-2}) * d = (c * d)^4 = a * b^2 * a * d * a * d * (b^{-2}) = \\
(b * d^{-1})^6 = a * b * d * (b^{-1}) * a * d * b * d * (b^{-1}) * d = 1.
\]

This group has 88 classes and the order structure:

| order of element | number of elements | number of classes |
|------------------|-------------------|------------------|
| 2                | 943               | 26               |
| 3                | 800               | 3                |
| 4                | 7248              | 41               |
| 6                | 6752              | 12               |
| 12               | 2688              | 5                |

c. The final group is complete of order 36,864 and has a presentation:
\[
a^2 = c^2 = d^2 = e^2 = f^4 = (a * c)^2 = (a * e)^2 = \\
a * f * a * (f^{-1}) = b * c * (b^{-1}) * c = (c * e)^2 = \\
a * b * a * e * b * e = c * d * f * c * (f^{-1}) * d = \\
c * d * (f^{-1}) * c * f * d = b * f * (b^{-1}) * e * d * b * (f^{-1}) = \\
(a * b * a * (b^{-1}))^2 = a * (b^{-2}) * d * a * d * b^2 = \\
a * c * f * (b^{-1}) * f^2 * b * f = a * c * f * e * f^2 * e * f = \\
(a * d)^4 = b^2 * e * d * e * (b^{-2}) * d = (b * e * (b^{-1}) * d)^2 = \\
e * f * e * (f^{-1})^2 = a * b * a * d * c * (f^{-1}) * b * d * f = \\
a * b^2 * a * d * a * d * (b^{-2}) = a * c * d * a * e * f * d * e * f = 1.
\]
This complete group has 98 classes and the order structure:

| order of element | number of elements | number of classes |
|------------------|--------------------|------------------|
| 2                | 1167               | 23               |
| 3                | 800                | 3                |
| 4                | 12144              | 41               |
| 6                | 13152              | 19               |
| 8                | 3072               | 5                |
| 12               | 4992               | 5                |
| 24               | 1536               | 1                |

266 Brian Drive, Warwick, Rhode Island 02886
E-mail address: w.becker@hotmail.com

266 Brian Drive, Warwick, Rhode Island 02886
E-mail address: elainewbecker@gmail.com