We consider the thermally activated escape of an overdamped Brownian particle over a potential barrier in the presence of periodic driving. A time-dependent path-integral formalism is developed which allows us to derive asymptotically exact weak-noise expressions for both the instantaneous and the time-averaged escape rate. Our results comprise a conceptually new, systematic treatment of the rate prefactor multiplying the exponentially leading Arrhenius factor. Moreover, an estimate for the deviations at finite noise-strengths is provided and a supersymmetry-type property of the time averaged escape rate is verified. For piecewise parabolic potentials, the rate-expression can be evaluated in closed analytical form, while in more general cases, as exemplified by a cubic potential, an action-integral remains to be minimized numerically. Our comparison with very accurate numerical results demonstrates an excellent agreement with the theoretical predictions over a wide range of driving strengths and driving frequencies.

PACS numbers: 05.40.-a, 82.20.Mj, 82.20.Pm

I. INTRODUCTION

The thermally activated escape over a potential barrier is a recurrent theme in a large variety of physical, chemical, and biological contexts \[1\]-\[3\]. In the case of foremost practical relevance, the characteristic strength of the thermal noise (the thermal energy \(k_B T\)) is much smaller than the potential barrier. As a consequence, successful barrier crossings constitute rare events and the escape statistics verifies with very high accuracy an exponential decay as a function of time. In other words, a meaningful escape-rate can be defined which completely characterizes the decay process. A seminal contribution to the theory of escape-rates represents the work by Kramers in 1940 \[4\], which subsequently has been refined, modified, and generalized in various important directions \[1\]-\[3\].

A particularly challenging novel direction are systems far away from thermal equilibrium, either due to non-thermal noise or external deterministic forces \[1\]. In such a case, the relevant probability distribution strongly deviates from the Boltzmann-form in the entire state space and its determination becomes a highly non-trivial problem. *Mutas mutandis*, this very same basic difficulty resurfaces again in all known theoretical methods of calculating escape rates in far from equilibrium systems \[2\]-\[4\].

The subject of our present paper is one of the simplest non-equilibrium descendants of Kramers’ original problem: namely the thermally activated escape of an overdamped Brownian particle over a potential barrier in the presence of a periodic driving (details are given in Sect. II). This is a prototypical setup in the sense that investigating the behavior of a system under the influence of a periodic forcing represents a particularly natural and straightforward experimental situation. Examples arise in the context of laser driven semiconductor heterostructures \[3\], stochastic resonance \[1\], directed transport in rocked Brownian motors \[4\]-\[7\], or periodically driven “resonant activation” processes \[8\]-\[10\] like AC driven biochemical reactions in protein membranes \[20\], to name only a few.

Despite its experimental importance, the theory of oscillating barrier crossing in the regime of weak thermal noise is still at its beginning. Previous quantitative, analytical investigations have been restricted to weak \[21\]-\[23\], slow \[24,25\], or fast \[21,24,26\] driving. In this article we continue our recent study \[27\] of the most challenging intermediate regime of *moderately strong* and *moderately fast* driving by means of path-integral methods. The general framework of this approach is derived from scratch in Sect. III, thus collecting, streamlining, and partially extending previously known material. The evaluation of the escape-rate is worked out in Sect. IV with the central results \[11\]-\[13\] for the time-averaged and \[10\] for the instantaneous escape rate. Especially, these results comprise a conceptually new, systematic treatment of the rate prefactor multiplying the exponentially leading Arrhenius factor. They become asymptotically exact for any finite amplitude and period of the driving as the noise strength tends to zero. On the other hand, for any fixed (small) noise strength, we have to exclude extremely small driving amplitudes and extremely long or short driving periods since this would lead us effectively back to an undriven escape problem, which is not covered by our present approach. Another situation which is excluded in our theory is the case of extremely strong driving such that escape events become possible even in the absence of the thermal noise \[22\]-\[24\]. Closest in spirit to our methodology is the recent work \[23\], which is restricted, however, to the linear response regime (weak driving) for the exponentially leading part (Arrhenius factor) and treats the prefactor by means of a matching procedure, involving the barrier region only. The approximation adopted in that work is complementary to ours in that it admits, for a fixed (small) noise strength, arbitrarily small driving amplitudes.

In Sect. V, our analytical predictions are verified for the case of sinusoidally rocked metastable potentials against very precise numerical results. A first example consists in a piecewise parabolic potential, for which our
general rate-expressions can be evaluated in closed analytical form. In more general cases, exemplified in Sect. V by a cubic potential, a few elementary numerically tasks remain before actual numbers can be obtained from our rate-expressions. The final conclusions are presented in Sect. VI.

II. THE ESCAPE PROBLEM

A. Model

We consider the following model for the one-dimensional Brownian motion of a particle with coordinate \( x(t) \) and mass \( m \) under the influence of a time-dependent force field \( F(x,t) \):

\[
m \ddot{x}(t) - F(x(t), t) = -\eta \dot{x}(t) + \sqrt{2D} \xi(t) .
\]

(1)

While the left hand side accounts for the dynamics of the isolated particle, the right hand side models the influence of its thermal environment with a viscous friction coefficient \( \eta \) and a randomly fluctuating force \( \xi(t) \), which is assumed to be unbiased Gaussian white noise with correlation

\[
\langle \xi(t) \xi(t') \rangle = \delta(t-t') .
\]

(2)

At thermal equilibrium, the intensity \( D \) of the noise is related to the temperature \( T \) according to the Einstein relation \( D = \eta k_B T \), where \( k_B \) is Boltzmann’s constant. Throughout this paper we will restrict ourselves to the overdamped motion such that inertia effects \( m \ddot{x}(t) \) in (1) are negligible. Choosing the time unit such that \( \eta = 1 \), the stochastic dynamics takes the form

\[
\dot{x}(t) = F(x(t), t) + \sqrt{2D} \xi(t) .
\]

(3)

The force field \( F(x,t) \) in (3) is assumed to derive from a metastable potential which undergoes an arbitrary periodic modulation in time with period \( T \):

\[
F(x, t+T) = F(x(t), t) .
\]

(4)

An example is a metastable static potential \( V(x) \) as cartooned in fig.1, supplemented by an additive sinusoidal driving

\[
F(x, t) = -V'(x) + A \sin(\Omega t) \quad \Omega = 2\pi/T
\]

(5)

(6)

Our next assumption is that the deterministic dynamics in (3) with \( D = 0 \) exhibits a stable periodic orbit \( x_s(t) \) and an unstable periodic orbit \( x_u(t) \), satisfying

\[
x_s(t + T) = x_s(t) , \quad x_u(t + T) = x_u(t) ,
\]

(7)

(8)

where ‘s, u’ means that the index may be either ‘s’ or ‘u’. Moreover, every deterministic trajectory is assumed to approach in the long-time limit either the attractor \( x_s(t) \) or to diverge towards \( x = \infty \), except if it starts exactly at the separatrix \( x_u(t) \) between those two basins of attraction. In other words, the metastable potential is required not to be rocked too violently such that particles cannot escape deterministically, i.e. without the assistance of the random fluctuations in (1). It is clear that \( x_s(t) \) and \( x_u(t) \) must be disjoint and by assuming a second “attractor” at \( x = \infty \) we have, without loss of generality, implicitly restricted ourselves to case that

\[
x_u(t) > x_s(t)
\]

(9)

for all \( t \). Note that the above requirements do not necessarily exclude the possibility that for certain times \( t \) the “instantaneous potential”, from which the force field \( F(x,t) \) derives, does no longer exhibit a potential barrier.

B. Escape rates

Next, we return to the stochastic dynamics (3) with a finite but very small noise-strength \( D \) such that a particle \( x(t) \) is able to leave the domain of attraction of the stable periodic orbit \( x_s(t) \) and subsequently disappear towards \( x = \infty \) but the typical waiting-time before such an event occurs is much longer than all characteristic time scales of the deterministic dynamics (separation of time scales (12)). For a quantitative characterization of such escape events, our starting point is the probability distribution \( p(x,t) \) of particles which is governed by the Fokker-Planck equation

\[
\frac{\partial}{\partial t} p(x,t) = \frac{\partial}{\partial x}\{-F(x,t) + D \frac{\partial}{\partial x}\} p(x,t) .
\]

(10)

Once \( p(x,t) \) is known, the population \( P_A(t) \) of the time-dependent basin of attraction \( A(t) := (-\infty, x_u(t)] \) of the stable periodic orbit \( x_s(t) \) follows as

\[
P_A(t) = \int_{-\infty}^{x_u(t)} p(x,t) dx .
\]

(11)

A suggestive definition of the “instantaneous rate” \( \Gamma(t) \) is then provided by the relative decrease of this population per time unit

\[
\Gamma(t) := -\frac{\dot{P}_A(t)}{P_A(t)} .
\]

(12)

We note that particles which leave the domain of attraction here give rise to a positive contribution to \( \Gamma(t) \). There is also a certain probability that particles from outside this domain recross the separatrix \( x_s(t) \), giving rise to a negative contribution to \( \Gamma(t) \). In other words, eq.(12) is the net flux of particles (outgoing flux minus back-flux) across the separatrix \( x_s(t) \) in units of the remaining population \( P_A(t) \). By exploiting the deterministic dynamics (3) for \( x_u(t) \), the Fokker-Planck equation (10) for \( p(x,t) \),
and the definition (11) of $P_A(t)$ we can rewrite the instantaneous rate (12) as

$$\Gamma(t) = -\frac{D}{P_A(t)} \frac{\partial p(x_u(t),t)}{\partial x_u(t)} .$$  \hspace{1cm} (13)

Inside the metastable state $x < x_u(t)$ the particle distribution is governed by intrawell relaxation processes. For small noise-strengths $D$, their characteristic time scales are well separated from the typical escape time itself. On this time scale of the intrawell relaxation, transients die out and the distribution $p(x,t)$ approaches a quasi-periodic dependence on time $t$. More precisely, $p(x,t)/\int_{-\infty}^{\infty} p(x, t) dx$ tends, at least for $x \leq x_u(t)$, towards a time-periodic function as $t$ grows. The same carries over to the escape probability (13) and thus the time-averaged escape rate

$$\bar{\Gamma} := \frac{1}{T} \int_{t}^{t+T} \Gamma(t') dt'$$  \hspace{1cm} (14)

becomes independent of the time $T$.

Our assumption of weak noise guarantees that the loss of population $P_A(t)$ is negligible on the time scale of the intrawell relaxation for any initial distribution $p(x, t_0)$ that is negligibly small in the vicinity and beyond the instantaneous separatrix $x_u(t_0)$. The denominator in (13) can thus be approximated by 1 for all times $t - t_0$ much smaller than the characteristic escape time $1/\bar{\Gamma}$ itself. Further, we can restrict ourselves to delta-distributed initial conditions of the form $p(x, t_0) = \delta(x - x_0)$ with $x_0$ inside the basin of attraction $A(t)$ of $x_u(t)$ such that the overwhelming majority of realizations will first relax towards a close neighborhood of the attractor $x_u(t)$ before they escape. The behavior of more general initial distributions then readily follows by way of linear superposition. Moreover, one expects that after transients (intrawell relaxation processes) have died out, the time dependent escape rate (13) will actually become independent of the initial conditions $x_0$ and $t_0$. Denoting by $p(x, t \mid x_0, t_0)$ the conditional probability associated with an initial delta peak at $x_0$ we thus can rewrite (13) as

$$\Gamma(t) = - D \frac{\partial p(x_u(t), t \mid x_0, t_0)}{\partial x_u(t)} .$$  \hspace{1cm} (15)

We recall that this expression is valid even if $t - t_0$ is not large, but then $\Gamma(t)$ still depends on $x_0$ and $t_0$. On the other hand, $t - t_0$ has been assumed to be much smaller than the typical escape time $1/\bar{\Gamma}$. However, on this time-scale the rate $\Gamma(t)$ has practically converged to its asymptotically periodic behavior and thus the extrapolation of $\Gamma(t)$ to arbitrarily large $t - t_0$ is trivial.

### C. Supersymmetry

Given a time-periodic force field $F(x, t)$, we define its supersymmetric partner field $\tilde{F}(x, t)$ according to

$$\tilde{F}(x, t) := F(-x, -t) .$$  \hspace{1cm} (16)

For instance, if the force field $F(x, t)$ derives from a periodically rocked potential $V(x)$ like in (6), then its supersymmetric partner is obtained by turning $V(x)$ upside down, followed by an inversion of the $x$-axis, i.e. $\tilde{V}(x) = -V(-x)$, see fig. 2, while the driving $A \sin(\Omega t)$ in (6) remains invariant (up to an irrelevant phase). In such a supersymmetric partner field $\tilde{F}(x, t)$, the stable and unstable periodic orbits exchange their roles, thus defining a new escape problem out of the basin of attraction of the new stable orbit $\tilde{x}_u(t) = -x_u(-t)$ across the new separatrix $\tilde{x}_u(t) = -x_u(-t)$.

It has been demonstrated in [22,34] that for force fields $F(x, t)$ of the form (6), the time-averaged rate (14) is invariant under the supersymmetry transformation (16). The same line of reasoning can be readily generalized to force fields of the form $F(x, t) = -V'(x) + y(t)$ with an arbitrary periodic driving $y(t)$. In our present paper we will show that for asymptotically weak noise $D$ the time averaged escape rate (14) is invariant under the general supersymmetry transformation (16) without any further restrictions on $F(x, t)$. Regarding the notion of supersymmetry and especially its connection with supersymmetric quantum mechanics, we refer to [22,34] and further references therein. We finally remark, that the standard definition of the supersymmetric partner force field is $-F(x, -t)$. For our present purposes, the definition (16) is equivalent but more convenient.

### III. PATH-INTEGRALS: GENERAL FRAMEWORK

In this section the general framework of a path-integral approach to the stochastic dynamics is outlined. Though these concepts are not new, we find it worthwhile to briefly review them here in order to make our paper self-contained. We also note that most of this section remains valid beyond the particular assumptions on the force field $F(x, t)$ from Sect. II.

#### A. Time-discretized path-integrals

Much like in quantum mechanics, also in the present context of stochastic processes, path-integral concepts have a tangible meaning only when considered as the limiting case of appropriate discrete-time approximations. Our first step is therefore a discretization in time of the
overdamped stochastic dynamics \( \text{(3)} \). Denoting the initial and final times by \( t_0 \) and \( t_f \), we introduce the definitions

\[
\Delta t := [t_f - t_0]/N \tag{17}
\]

\[
t_n := t_0 + n \Delta t \tag{18}
\]

\[
x_n := x(t_n) \tag{19}
\]

where \( n = 0, 1, ..., N \). The integer \( N \) is considered as large but finite and will ultimately be sent to infinity (continuous time limit). The discretized dynamics \( \text{(3)} \) then takes the form

\[
x_{n+1} - x_n = F(x_n, t_n) \Delta t + \sqrt{2D \Delta t} \xi_n \tag{20}
\]

where the \( \xi_n \) are independent, identically distributed Gaussian random numbers with probability distribution

\[
P(\xi_n) = (2\pi)^{-1/2} \exp(-\xi_n^2/2). \tag{21}
\]

As a side remark we notice that the so-called “prepoint discretization scheme” \( \text{(11)-(13)} \) (not to be confused with the Ito-scheme in the stochastic dynamics \( \text{(3)} \)) has been implicitly adopted in \( \text{(20)} \) for the sake of later convenience. Other “discretization schemes” \( \text{(40)-(42)} \) would give rise to a somewhat modified path-integral formalism but would – of course – lead to identical results as far as the actual stochastic dynamics \( \text{(3)} \) is concerned. In passing we further note that our treatment for Eq. \( \text{(3)} \) can be generalized to multiplicative noise \( g(x) \xi(t) \), with \( g(x) \neq 0 \), without encountering additional difficulties.

For the conditional probability \( p_N(x_{n+1}, t_{n+1} | x_n, t_n) \) to reach the point \( x_{n+1} \) at time \( t_{n+1} \) when starting out from \( x_n \) at the previous time step \( t_n \) we find from the discretized dynamics \( \text{(20)} \) and the noise-distribution \( \text{(21)} \) that

\[
p_N(x_{n+1}, t_{n+1} | x_n, t_n) = \int \delta(x_{n+1} - x_n - F(x_n, t_n) \Delta t - \sqrt{2D \Delta t} \xi_n) P(\xi_n) d\xi_n = \frac{1}{(4\pi D \Delta t)^{N/2}} \exp \left\{ -\frac{[x_{n+1} - x_n - F(x_n, t_n) \Delta t]^2}{4D \Delta t} \right\}. \tag{22}
\]

Here and in the following, integrals over the entire real axis are written without the integration limits \( \pm \infty \). Further, the mutual independence of the random numbers \( \xi_n \) in \( \text{(20)} \) (Markov property) implies for the conditional probability the Chapman-Kolmogorov relation

\[
p_N(x_{n+2}, t_{n+2} | x_n, t_n) = \int p_N(x_{n+2}, t_{n+2} | x_{n+1}, t_{n+1}) p_N(x_{n+1}, t_{n+1} | x_n, t_n) \, dx_{n+1}. \tag{23}
\]

Upon iteration of this relation in combination with \( \text{(23)} \) one finds for the conditional probability the time-discretized path-integral representation

\[
p_N(x_f, t_f | x_0, t_0) = \int \frac{dx_1 \cdots dx_{N-1}}{(4\pi D \Delta t)^{N/2}} \exp \left\{ -\frac{S_N(x_0, ..., x_N)}{D} \right\}, \tag{24}
\]

where

\[
S_N(x_0, ..., x_N) := \sum_{n=0}^{N-1} \frac{\Delta t}{4} \left[ \frac{x_{n+1} - x_n - F(x_n, t_n)}{\Delta t} \right]^2
\]

is the discrete-time “action” or “Onsager-Machlup functional”. While \( x_1, ..., x_{N-1} \) are integration variables in \( \text{(23)} \), the initial- and end-points are fixed by the prescribed \( x_0 \) and by the additional constraint \( x_N = x_f \), see \( \text{(17)-(19)} \).

### B. Saddle-point approximation

For small noise-strengths \( D \) the path-integral \( \text{(23)} \) is dominated by the minima of the action \( S_N(x_0, ..., x_N) \). The existence of at least one (global) minimum can be readily inferred from the general structure of the action in \( \text{(23)} \). To keep things simple we assume for the moment that besides this global minimum no additional (local) minima play a role in \( \text{(23)} \). Denoting the global minimum by \( x^* := (x_0^*, ..., x_N^*) \) it follows that it satisfies the extremality conditions

\[
\frac{\partial S_N(x^*)}{\partial x_n^*} = 0 \tag{25}
\]

for \( n = 1, ..., N - 1 \), supplemented by the boundary conditions for \( n = 0, N \):

\[
x_0^* = x_0, \quad x_N^* = x_f. \tag{26}
\]

Under the assumption that the noise-strength \( D \) is small, the path-integral in \( \text{(23)} \) can be evaluated by means of a saddle point approximation about the minimizing path \( x^* \) with the result

\[
p_N(x_f, t_f | x_0, t_0) = Z_N(x^*) e^{-S_N(x^*)/D} \left[ 1 + O(D) \right]. \tag{27}
\]

where the prefactor \( Z_N(x^*) \) is given by a Gaussian integral of the form

\[
Z_N(x^*) := \int \frac{dy_1 \cdots dy_{N-1}}{(4\pi D \Delta t)^{N/2}} \exp \left\{ -\frac{1}{2D} \sum_{n,m=1}^{N-1} \frac{y_n}{\partial x_n^*} \partial x_m^* \frac{\partial^2 S(x^*)}{\partial x_n^* \partial x_m^*} y_m \right\}. \tag{28}
\]

and where in the order of magnitude expression \( O(D) \) only the dependence on the noise-strength \( D \) is being kept. A more detailed quantitative estimate of this correction \( O(D) \) is a difficult, and to our knowledge unsolved task.

The Gaussian integral in \( \text{(28)} \) is readily evaluated to yield
The fact that $x^*$ is a minimum of the action (24) guarantees that $Q_k^* > 0$. Here and in the following we use the abbreviations

$$ F'(x,t) := \frac{\partial F(x,t)}{\partial x}, \quad \tilde{F}(x,t) := \frac{\partial F(x,t)}{\partial t} $$

(33)

and bracket-saving expressions like $f(x)^2$ are understood as $|f(x)|^2$.

As we shall see later, we have to leave room for the possibility that even for small noise-strengths $D$ more than one (global or local) minimum of the action (24) notably contributes to the path-integral expression (23). We label those various non-negligible minima $x_k^*$ by the discrete index $k$ but leave for the moment the precise set of $k$-values unspecified. Each of the minimizing paths $x_k^*$ thus satisfies an extremality condition of the form (21). Under the assumption that those minima $x_k^*$ are well separated in the $N-1$-dimensional space of all paths $(x_0, ..., x_N)$ appearing in (23), the saddle point approximation (27) simply acquires an extra sum over $k$ with a corresponding extra index $k$ in (23). Combining (24) and (32) we thus arrive at

$$ p_N(x_f, t_f | x_0, t_0) = \sum_k \frac{e^{-S_N(x_k^*)/D}}{(4\pi D Q_{N,k}^*)^{\frac{N}{2}}} \left[ 1 + O(D) \right]. $$

(34)

### C. Continuous-time limit

Next we turn to the continuous-time limit $N \to \infty$, $\Delta t \to 0$ in (17). The continuous-time conditional probability $p(x_f, t_f | x_0, t_0)$ when $N \to \infty$ in (23) is symbolically indicated by the path-integral expression (36)

$$ p(x_f, t_f | x_0, t_0) = \int_{x(t_0)=x_0}^{x(t_f)=x_f} D x(t) \ e^{-S[x(t)]/D}, $$

(35)

where

$$ S[x(t)] := \int_{t_0}^{t_f} L(x(t), \dot{x}(t), t) \ dt $$

(36)

is the continuous-time limit of the action (24) with

$$ L(x, \dot{x}, t) := \frac{1}{4} (\dot{x} - F(x,t))^2 $$

(37)

as Lagrangian. The extremality conditions for the minimizing paths $x_k^*(t)$ in the continuous-time limit are obtained from (24) and (25) by letting $\Delta t \to 0$ as

$$ \dot{x}_k^*(t) = \tilde{F}(x_k^*(t), t) + F(x_k^*(t), t) F'(x_k^*(t), t), $$

(38)

with boundary conditions (cf. (26))

$$ x_k^*(t_0) = x_0, \quad x_k^*(t_f) = x_f. $$

(39)

The same result (38) can also be recovered as the Euler-Lagrange equation corresponding to the Lagrangian (37).

Equivalent to this Lagrangian dynamics is the following Hamiltonian counterpart:

$$ H(x, p, t) := p \ddot{x} - L = p^2 + p F(x,t) $$

(40)

$$ \dot{p}_k^*(t) = -p_k^*(t) F'(x_k^*(t), t) $$

(41)

$$ \dot{x}_k^*(t) = 2 p_k^*(t) + F(x_k^*(t), t). $$

(42)

The last equation (42) may also be considered as the definition of the momentum $p_k^*(t)$ in terms of $x_k^*(t)$ and $\dot{x}_k^*(t)$. With (37) and (12) the action (36) along a minimizing path $x_k^*(t)$ follows as

$$ \phi_k(x_f, t_f) := S[x_k^*(t)] = \int_{t_0}^{t_f} p_k^*(t)^2 \ dt, $$

(43)

where the dependence of the action $\phi_k(x_f, t_f)$ on the initial condition $x_0$ at time $t_0$ has been dropped. For later use we also recall the well-known result from classical mechanics that the derivative of the extremizing action with respect to its endpoint equals the canonical conjugate momentum, i.e.,

$$ \frac{\partial \phi_k(x_f, t_f)}{\partial x_f} = p_k^*(t_f). $$

(44)

Finally, the continuous-time limit for the conditional probability (24) in combination with (43) takes the form

$$ p(x_f, t_f | x_0, t_0) = \sum_k \frac{e^{-\phi_k(x_f, t_f)/D}}{(4\pi D Q_k^*)^{\frac{1}{2}}} \left[ 1 + O(D) \right], $$

(45)

where $Q_k^*(t)$ is governed by the second order homogeneous linear differential equation (27) that follows in the limit $\Delta t \to 0$ from (30) and (42):
\[ \frac{1}{2} \dot{Q}_k^*(t) - \frac{d}{dt}[Q_k^*(t) F'(x_k^*(t), t)] + Q_k^*(t) p_k^*(t) F''(x_k^*(t), t) = 0 . \] (46)

Similarly, the initial conditions (31) go over for \( \Delta t \to 0 \) into

\[ Q_k^*(t_0) = 0 \ , \ \dot{Q}_k^*(t_0) = 1 . \] (47)

We remark that according to (38) and (39) the minimizing paths \( x_k^*(t) \) are independent of the noise-strength \( D \). Consequently, neither \( \phi_k(x_f, t_f) \) from (38,39,43) nor \( Q_k^*(t) \) from (46,47) depend on the noise-strength, i.e., no implicit additional \( D \)-dependences are hidden in (45). We further note that by means of the substitution

\[ g_k^*(t) := \frac{\dot{Q}_k^*(t)}{2Q_k^*(t)} - F'(x_k^*(t), t) \] (48)

the linear homogeneous second order equation (40) goes over into the non-linear first order Riccati-equation

\[ g_k^*(t) + 2g_k^*(t)^2 + 2g_k^*(t) F'(x_k^*(t), t) = -p_k^*(t) F''(x_k^*(t), t) . \] (49)

Since (17) does not lead to a meaningful initial condition for \( g_k^*(t) \) in (48), the Riccati-equation (45) can only be used for times \( t > t_0 \). For this reason and also from the viewpoint of calculational efficiency we found that for practical purposes the linear second order equation (40) is often superior to the Riccati-equation (15).

To establish contact with previously known results we finally remark that one can identify

\[ \frac{\partial^2 \phi_k(x_f, t_f)}{\partial x_f^2} = g_k^*(t_f) . \] (50)

This relation (50) and the associated Riccati-equation (17) are usually derived by introducing a WKB-type ansatz into the Fokker-Planck-equation for the conditional probability distribution (cf. (17)) and then comparing powers of the noise-strength \( D \). Since a direct derivation by means of path-integral methods is not known to us, we have included such a derivation of the relation (50) in the Appendix B.

IV. PATH-INTEGRAL SOLUTION OF THE ESCAPE PROBLEM

By introducing the path-integral expression (15) for the conditional probability into the formula (15) for the instantaneous rate \( \Gamma(t) \) at time \( t = t_f \) and taking into account (44) we obtain our first main result (2), namely

\[ \Gamma(t_f) = \sum_k \frac{p_k^*(t_f) e^{-\phi_k(x_k^*(t_f), t_f)/D}}{[4 \pi D Q_k^*(t_f)]^{1/2}} [1 + O(D)] . \] (51)

In view of (15), the instantaneous rate (51) has the suggestive structure of “probability at the separatrix times velocity”.

As already mentioned in Sect. II, for sufficiently large times \( t_f - t_0 \) the instantaneous rate (51) is expected to become independent of the initial position \( x_0 \) as long as \( x_0 \) is located inside the domain of attraction of the stable periodic orbit \( x_s(t) \). A more detailed discussion of this point will be given in Subsect. IV.G. To keep things as simple as possible we focus in the following subsections on the particular case that \( x_0 \) is located at the stable periodic orbit, i.e.,

\[ x_0 = x_s(t_0) . \] (52)

A. Minimizing paths

Our next goal is the characterization of all the minimizing paths \( x_k^*(t) \) which significantly contribute to the sum in (51). Our first observation is that for any finite \( t_0 \) and \( t_f \) the action (20) exhibits in the generic case a unique global minimum respecting the boundary conditions

\[ x_k^*(t_0) = x_s(t_0) , \ \ x_k^*(t_f) = x_u(t_f) \] (53)

according to (20) and (51,52). To be specific, we denote this globally minimizing path as \( x_{k_0}^*(t) \). From the explicit form of the Lagrangian (37) we can infer that for large values of \( t_f - t_0 \) the minimal path \( x_{k_0}^*(t) \) follows most of the time rather closely a deterministic trajectory, i.e., \( \dot{x}_{k_0}^*(t) \sim F(x_{k_0}^*(t), t) \), in order not to accumulate a too large amount of action (60). In view of (4) and (53) it is thus suggestive that \( x_{k_0}^*(t) \) starts at \( t = t_0 \) and then continues to closely follow the stable periodic orbit \( x_s(t) \) for some time. At a certain moment, \( x_{k_0}^*(t) \) leaves this neighborhood and travels in a comparatively short time into the vicinity of the unstable periodic orbit \( x_u(t) \), where it remains for the rest of its time and ends at \( x_{k_0}^*(t_f) = x_u(t_f) \). Only during the crossover from the neighborhood of \( x_s(t) \) into that of \( x_u(t) \) does the path \( x_{k_0}^*(t) \) substantially deviate from a deterministic behavior and so gives rise to the main contribution to the action (60). We desist from a more rigorous derivation of these basic qualitative features since they are quite similar to the well-known barrier-crossing problem in a static potential (time-independent force-field) (3,3,4). Especially, the relatively short “crossover-segment” of \( x_{k_0}^*(t) \) between the long sojourns close to the stable and unstable orbits has to lead to the name “instanton” for such a path.

As we will see in more detail later, a meaningful limit of \( x_{k_0}^*(t) \) exists for \( t_0 \to -\infty \) and \( t_f \to \infty \) (henceforth abbreviated as \( t_f - t_0 \to \infty \)) in the sense that \( x_{k_0}^*(t) \) follows closer and closer the periodic orbits \( x_{s,u}(t) \) over longer and longer time intervals, while the “crossover-segment”
does practically not change its shape any more. Also the associated minimal action $S[x^*_k(t)]$ from (33) tends to a finite limit. In fact, one can readily show that the minimal action cannot increase upon increasing $t_f$ and/or decreasing $t_0$. Since it is furthermore bounded from below, the existence of the limit follows for the action as well as for the minimizing path itself. More importantly, from the time-periodicity of the force field \cite{44} one can infer that in the limit $t_f - t_0 \to \infty$ the action $S[x^*_k(t + n T)]$ has the same value for any integer $n$. In other words, for infinitely large $t_f - t_0$ the action no longer exhibits a unique absolute minimum, rather each path $x^*_k(t + n T)$ globally minimizes the action. However, these degenerate absolute minima are still well separated in the space of all paths $x(t)$ appearing in (33). This feature is the salient difference between our present problem and its time-independent counterpart \cite{32,33,34,36,37,38,44,45}, which exhibits a continuous degeneracy (Goldstone mode) in the limit $t_f - t_0 \to \infty$. Put differently, the time-periodic force field reduces the continuous time-translation symmetry into a discrete one. Since the rate-formula (51) assumes well separated minima $x^*_k(t)$ of the action, it is quite clear that the time-independent case must be excluded in the following.

We emphasize that the minimizing paths $x^*_k(t)$ remain well separated and thus the rate-formula (51) becomes asymptotically exact for any (arbitrary but fixed) finite values of the driving amplitude and period as the noise strength $D$ tends to zero. Apart from this fact that in the limit $D \to 0$ the $\mathcal{O}(D)$ correction in the saddle point approximation \cite{23} and thus in (33) vanishes, a more detailed quantitative statement seems difficult. On the other hand, for a given (small) noise strength $D$, we have to exclude extremely small driving amplitudes and extremely long or short driving periods since this would lead us effectively back to the static (undriven) escape problem, which requires a completely different treatment (especially of the (quasi-) Goldstone mode \cite{23,32,33,44,45,46}) than in \cite{23}. Put differently, in any of these three asymptotic regimes, the error $\mathcal{O}(D)$ from \cite{23} becomes very large.

For later reference we denote the minimizing path $x^*_k(t)$ when $t_f - t_0 \to \infty$ by $x^*_0(t)$, keeping in mind that we are still free to shift its time argument by an arbitrary multiple of $\mathcal{T}$. The corresponding action is

$$\phi_{\text{opt}} := S[x^*_0(t)] = \lim_{t_0 \to -\infty} \min_{t_f \to \infty} \min_{x(t), x(t_f) = x_u(t_f)} S[x(t)] , \quad (54)$$

where the second identity may also be considered as an implicit definition of $x^*_0(t)$. Similarly, any other quantity associated with $x^*_0(t)$ will be marked by an index "opt", for instance $p^*_0(t)$ (see \cite{12}), $Q^*_0(t)$ (see \cite{46}), and $q^*_0(t)$ (see \cite{18}).

In principle, besides the absolute minimum $x^*_0(t)$ of the action there may coexist further (absolute or relative) minima which cannot be identified with each other after a time-shift by an appropriate multiple of $\mathcal{T}$. While the coexistence of further absolute minima is non-generic, coexisting relative minima are irrelevant for sufficiently small noise-strengths $D$ as far as the sum in (33) is concerned. Though both cases could be easily taken into account in the following discussion, we will restrict ourselves to the simplest and most common case that $x^*_k(t + n T)$ are the only (relevant) minima of the action (33) in the limit $t_f - t_0 \to \infty$.

Returning to finite but large values of $t_f - t_0$, we expect $-\pi$ as a precursor of the $t_f - t_0 \to \infty$ limit – that besides the unique absolute minimum $x^*_0(t)$ there will coexist many additional relative minima $x^*_k(t)$ with an only slightly larger action. All those minima $x^*_k(t)$ possess a limit when $t_f - t_0 \to \infty$ in the same sense as for the case $k = k_0$ described above (quantitative details will be given later). Moreover, when $t_f - t_0 \to \infty$ then each $x^*_k(t)$ approaches $x^*_0(t + n(k) \mathcal{T})$ for a suitable choice of $n(k)$ and without loss of generality we can assume a (re-)labeling of the $x^*_k(t)$ such that $n(k) = k$. In other words, to each $x^*_k(t)$ belongs a very similarly looking "master-path" $x^*_0(t + k \mathcal{T})$, see fig.3. Since $t_f - t_0$ is finite, there is a finite number (of the order $(t_f - t_0)/\mathcal{T}$) of minimizing paths $x^*_k(t)$ and without loss of generality we can assume that the indices in the sum (33) start at $k = 0$ and run until a certain maximal value $K(t_f, t_0)$:

$$0 \leq k \leq K(t_f, t_0) = \mathcal{O}(t_f - t_0)/\mathcal{T} \quad (55)$$

Thus $x^*_0(t)$ is that minimizing path which closely follows $x_s(t)$ as long as possible and crosses over to the neighborhood of $x_u(t)$ at “the latest possible moment” (see fig.3), and similarly for $x^*_k(t_f, t_0)(t)$.

Note that all the general qualitative features discussed above are nicely illustrated by the explicit example in Sect. IV.A.

\section*{B. Neighborhood of periodic orbits}

Our final goal is to approximate the action $\phi_k(x_f, t_f)$ and the prefactor $p^*_k(t_f)/[Q^*_k(t_f)]^{1/2}$ for all minimizing paths $x^*_k(t)$ that play a non-negligible role in the sum (33) solely in terms of the “master-path” $x^*_0(t)$ and its descendents $p^*_0(t), Q^*_0(t), e \text{tc}$. To this end we first address the behavior of a path $x^*_k(t)$ within the neighborhood of either the stable periodic orbit $x_s(t)$ or of the unstable one $x_u(t)$. Within these regions, the time dependent force field can be approximately written as

$$F(x, t) = F(x_{\text{n}}, t) + (x - x_{\text{n}}(t)) F'(x_{\text{n}}(t), t) \quad (56)$$

Note that these approximations are valid not only if $x - x_{\text{n}}(t)$ is small but also if $F'(y, t)$ is small for all $y$ between $x$ and $x_{\text{n}}(t)$. An immediate consequence of (44) are the relations
\[ F'(x, t) = F'(x_{s,u}(t), t), \quad F''(x, t) = 0. \] (57)

As long as a minimizing path \( x_{s,u}^*(t) \) remains in a region where these approximations apply, the Hamiltonian equations (11, 12) take the form

\[ p^*_k(t) = -p^*_k(t) F'(x_{s,u}(t), t) \] (58)

\[ \Delta x^*_k(t) = 2 p^*_k(t) + \Delta x^*_k(t) F'(x_{s,u}(t), t), \] (59)

where we have introduced

\[ \Delta x^*_k(t) := x^*_k(t) - x_{s,u}(t). \] (60)

Their solutions are:

\[ p^*_k(t) = p^*_k(t_1) e^{-\Lambda_{s,u}(t, t_1)} \] (61)

\[ \Delta x^*_k(t) = \Delta x^*_k(t_1) e^{\Lambda_{s,u}(t, t_1)} + p^*_k(t) I_{s,u}(t, t_1), \] (62)

where \( t_1 \) is an arbitrary reference time (within our assumption that (50) applies for all the considered times \( t \)) and where

\[ \Lambda_{s,u}(t, t_1) := \int_{t_1}^{t} F'(x_{s,u}(t'), t') \, dt' \] (63)

\[ I_{s,u}(t, t_1) := 2 \int_{0}^{t_1} e^{2 \Lambda_{s,u}(t, t')} \, dt'. \] (64)

Obvious properties of the functions \( \Lambda_{s,u}(t, t_1) \) from (63) are:

\[ \Lambda_{s,u}(t, t_2) = \Lambda_{s,u}(t, t_1) + \Lambda_{s,u}(t_1, t_2) \] (65)

\[ \Lambda_{s,u}(t, t_1) = -\Lambda_{s,u}(t_1, t) \] (66)

\[ \Lambda_{s,u}(t + T, t_1 + T) = \Lambda_{s,u}(t, t_1). \] (67)

Further, one readily sees that the quantities

\[ \lambda_{s,u} := \Lambda_{s,u}(t + T, t)/T \] (68)

are indeed \( t \)-independent and characterize the stability/instability (“Lyapunov exponents”) of the periodic orbits, namely

\[ \lambda_s < 0 \quad \lambda_u > 0. \] (69)

One even expects that \( \Lambda_s(t, t_1) < 0 \) and \( \Lambda_u(t, t_1) > 0 \) not only for \( t - t_1 = T, 2T, \ldots \) (cf. (63)) but in fact for all \( t - t_1 > 0 \), however exceptions cannot be excluded for not too large \( t - t_1 \). From (66) and (68) it follows that \( \Lambda_{s,u}(t, t_1) \) can be written as the sum of a linear function \( \lambda_{s,u}(t - t_1) \) and a periodic function of \( t \). As a consequence, we obtain

\[ \Lambda_{s,u}(t, t_1) \sim \lambda_{s,u} \cdot (t - t_1) \] (70)

for asymptotically large positive and negative times \( t - t_1 \).

Turning to the discussion of \( I_{s,u}(t, t_1) \) from (24), we first note that

\[ I_s(t, t_1) = I_s(t) - e^{2 \Lambda_u(t, t_1)} I_s(t_1) \] (71)

\[ I_s(t) := \lim_{t_1 \to -\infty} I_s(t, t_1) = 2 \int_{-\infty}^{t} e^{2 \Lambda_s(t, t')} \, dt'. \] (72)

and similarly

\[ I_u(t, t_1) = -I_u(t) + e^{2 \Lambda_u(t, t_1)} I_u(t_1) \] (73)

\[ I_u(t) := -\lim_{t_1 \to -\infty} I_u(t, t_1) = 2 \int_{-\infty}^{t} e^{2 \Lambda_u(t, t')} \, dt'. \] (74)

Thus, \( I_{s,u}(t) \) are positive and finite for all \( t \) and obey

\[ I_{s,u}(t + T) = I_{s,u}(t). \] (75)

It follows that \( I_s(t, t_1) \) in (24) is given by a periodic function of \( t \) minus the product of another periodic function times an exponentially decreasing factor \( \exp\{-\lambda_s \cdot (t - t_1)\} \), and analogously for \( I_u(t, t_1) \) in (73).

Choosing as reference time \( t_1 = t_0 \) in (24) and taking into account that \( \Delta x^*_k(t_0) = 0 \) according to (21, 22), implies that in the neighborhood of \( x_s(t) \) we have

\[ \Delta x^*_k(t) = p^*_k(t) I_s(t, t_0). \] (76)

Dividing this result by the same identity evaluated at a different reference time \( t_s > t_0 \) and taking into account (21) we obtain

\[ \Delta x^*_k(t) = \Delta x^*_k(t_s) e^{-\Lambda_s(t, t_s)} \frac{I_s(t, t_0)}{I_s(t_s, t_0)} \] (77)

\[ p^*_k(t) = \Delta x^*_k(t_s) e^{-\Lambda_u(t, t_s)} / I_s(t_s, t_0). \] (78)

Both these expressions consist of an exponentially increasing factor \( \exp\{-\lambda_s \cdot (t - t_s)\} \) times some periodic function of \( t \). In (77) one has in addition a quickly decreasing correction. The corresponding behavior in the neighborhood of \( x_u(t) \) is given by

\[ \Delta x^*_k(t) = p^*_k(t) I_u(t, t_f) \] (79)

\[ \Delta x^*_k(t) = \Delta x^*_k(t_s) e^{-\Lambda_u(t, t_s)} \frac{I_u(t, t_f)}{I_u(t_s, t_f)} \] (80)

\[ p^*_k(t) = \Delta x^*_k(t_s) e^{-\Lambda_u(t, t_s)} / I_U(t_s, t_f), \] (81)

where \( t_u \) is some reference time with \( t_u < t_f \). As expected, (80) are now dominated by an exponentially decreasing behavior \( \exp\{-\lambda_u \cdot (t - t_u)\} \). We further remark that for the master-path \( x_{s,u}^*(t) \) we have \( t_0 \to -\infty \) and \( t_f \to \infty \), thus \( I_{s,u}(t_0, t_f) \) in (77) go over into \( I_{s,u}(t) \) according to (77, 78) and so all four equations (19-20, 21-22) are exactly given by \( \exp\{-\lambda_{s,u} \cdot (t - t_{s,u})\} \) times certain periodic functions of \( t \).

Within our above local analysis of the neighborhoods of \( x_{s,u}(t) \), the reference times \( I_{s,u}(t) \) are still arbitrary and the corresponding parameters \( \Delta x^*_k(t_{s,u}) \) remain undetermined. They can only be fixed through the global behavior of \( x_{s,u}^*(t) \). It is instructive to reconsider the same thing from a somewhat different viewpoint. From (71, 72, 76) we conclude that

\[ \frac{\Delta x^*_k(t_s)}{p^*_k(t_s)} = I_s(t_s) - 2 \int_{-\infty}^{t_0} e^{2 \Lambda_s(t, t')} \, dt'. \] (82)
and similarly
\[ \frac{\Delta x_k^*(t_u)}{p_k^*(t_u)} = -I_u(t_u) + 2 \int_{t_f}^{\infty} e^{2\lambda u(t_u')} \, dt'. \] (83)

Let us consider \( t_s > t_0 \) as fixed and such that the approximation \([56]\) is valid for all \( t \in [t_0, t_s] \). Within the same restriction, we now consider the quantity \( \Delta x_k^*(t_s) \) as a parameter. For any value of \( \Delta x_k^*(t_s) \), eq.\((82)\) thus fixes \( p_k^*(t_s) \). With these initial conditions for \( x_k^*(t) \) and \( p_k^*(t) \) at time \( t = t_s \), one may then propagate the Hamiltonian equations \([11][12]\) up to the time \( t = t_f \). It is clear that for a typical choice of \( \Delta x_k^*(t_s) \) such a “shooting procedure” does not lead to the desired end-result \( \Delta x_k^*(t_f) = 0 \). But we also know from the mere existence of the minimizing paths that there must be specific \( \Delta x_k^*(t_s) \)-values which do the job. Furthermore, eq.\((83)\) tells us that it is not necessary to proceed until \( t = t_f \); rather, it is sufficient to take any time \( t_u \) at which \( x_k^*(t) \) has reached the \( x_u(t) \)-neighborhood and then check whether \( \Delta x_k^*(t_u) \) is satisfied.

If \( t_s - t_0 \) and \( t_f - t_u \) are already large then the integrands in \([52][54]\) are extremely small. Thus, tiny changes of \( \Delta x_k^*(t_u) \) and \( p_k^*(t_u) \) will lead to huge changes of \( t_0 \) and \( t_f \). Especially, by letting \( t_0 \to -\infty \) and \( t_f \to \infty \) those integrals vanish and one recovers the master path \( x_{\text{opt}}^*(x + kT) \) associated with \( x_k^*(t) \). This confirms our conclusion from the previous subsection that a meaningful limit of \( x_k^*(t) \) for \( t_0 \to \infty \) and \( t_f \to \infty \) exists and that for finite large \( t_f - t_0 \) the difference between \( x_k^*(t) \) and the associated master path \( x_{\text{opt}}^*(t + kT) \) is extremely small for all \( t \in [t_0, t_f] \).

An example for which \( t_f - t_u \) is not large is the path \( x_0^*(t) \), i.e., the one which crosses over from the neighborhood of \( x_u(t) \) to that of \( x_u(t) \) at the latest possible moment, see fig.3. For this path \( x_0^*(t) \), the time \( t_u \) at which it enters the neighborhood of \( x_u(t) \) is already rather close to \( t_f \) and so the integral in \([53]\) is not any more small. As a consequence, the deviation of \( x_0^*(t) \) from \( x_{\text{opt}}(t) \) is no longer small as \( t \) approaches \( t_f \). In particular, for \( t = t_f \) it follows that \( x_0^*(t_f) - x_{\text{opt}}^*(t_f) = -\Delta x_{\text{opt}}^*(t_f) \) is no longer small and with \([79]\) we conclude that the same is true for the momentum \( p_{\text{opt}}(t_f) \), i.e.,
\[ p_{\text{opt}}^*(t_f) \text{ not small}. \] (84)

With increasing \( k \)-values, the deviations \( -\Delta x_{\text{opt}}^*(t + kT) \) between \( x_k^*(t) \) and the associated master path \( x_{\text{opt}}^*(t + kT) \) in the vicinity of \( t_f \) are rapidly decreasing, essentially like \( \exp \{-\lambda u(kT)\} \), see \([56][53]\). In the same way, for the largest possible \( k \)-values, \( k \approx (t_f, t_0) \) (see \([55]\)), corresponding to paths \( x_k^*(t) \) with only a very short initial time segment close to \( x_0(t) \), the deviations from \( x_{\text{opt}}(t + kT) \) are no longer small for \( t \) close to \( t_0 \). As we will see later, paths \( x_k^*(t) \) with such large \( k \)-values are negligible in the sum \([54]\). For this reason, we will henceforth neglect deviations between \( x_k^*(t) \) and \( x_{\text{opt}}^*(t + kT) \) and between \( p_k^*(t) \) and \( p_{\text{opt}}^*(t + kT) \) for times \( t \) near the starting point \( t_0 \). Formally, this approximation is equivalent to letting
\[ t_0 \to -\infty. \] (85)

**C. Approximations in terms of the master path**

Our next objective is to express the action \([13]\) of the path \( x_k^*(t) \) in terms of the associated master path \( x_{\text{opt}}(t + kT) \). We recall that while \( x_k^*(t) \) satisfies the boundary conditions \([13]\), those of \( x_{\text{opt}}^*(t) \) are \( x_{\text{opt}}^*(t) - x_u(t) \to 0 \) for \( t \to -\infty \) and \( x_{\text{opt}}^*(t) - x_u(t) \to 0 \) for \( t \to \infty \). We now modify the latter boundary condition and require instead that
\[ t_k \coloneqq t_f + kT \] (86)
is the final time and \( x_k \coloneqq x_{\text{opt}}(t_k) \) the final position. In other words, we simply truncate the master path \( x_{\text{opt}}^*(t) \) at the time \( t_k \), associated with the final time \( t_f \) of \( x_k^*(t) \). Since this “new” path \( x_{\text{opt}}^*(t) \) with \( t \in [\infty, t_k] \) obviously still satisfies the Hamiltonian equations \([11][12]\) it is again an extremizing path. The value of the action for this path follows like in \([13]\) as
\[ \phi_{\text{opt}}(x_k, t_k) \coloneqq \int_{t_k}^{t_f} p_{\text{opt}}^*(t)^2 \, dt \] (87)
and the relations \([44][50]\) take the form
\[ \frac{\partial \phi_{\text{opt}}(x_k, t_k)}{\partial x_k} = p_{\text{opt}}^*(t_k) \] (88)
\[ \frac{\partial^2 \phi_{\text{opt}}(x_k, t_k)}{\partial x_k^2} = g_{\text{opt}}(t_k). \] (89)

With \([13][54]\) we can rewrite \((87)\) as
\[ \phi_{\text{opt}}(x_k, t_k) = \phi_{\text{opt}} - \int_{t_k}^{\infty} p_{\text{opt}}^*(t)^2 \, dt. \] (90)

Next we express the action \([50]\) of the path \( x_k^*(t) \) by expanding the one belonging to the associated master path \( x_{\text{opt}}^*(t + kT) \) in powers of the difference \( -\Delta x_{\text{opt}}^*(t_k) \) between the end-points \( x_k^*(t_f) = x_u(t_f) \) and \( x_{\text{opt}}^*(t_f + kT) = x_{\text{opt}}^*(t_k) \):
\[ \phi_k(x_k^*(t_f), t_f) = \phi_{\text{opt}}(x_k, t_k) - \Delta x_{\text{opt}}^*(t_k) \frac{\partial \phi_{\text{opt}}(x_k, t_k)}{\partial x_k} + \frac{\Delta x_{\text{opt}}^*(t_k)^2}{2} \frac{\partial^2 \phi_{\text{opt}}(x_k, t_k)}{\partial x_k^2} + ... \] (91)

As justified above eq.\((53)\), the analogous contribution in powers of \( \Delta x_{\text{opt}}^*(t_0 + kT) \) is negligible on the right hand side of \([11]\). By exploiting \([53][97]\) and the counterparts of \([79][81]\) for \( x_{\text{opt}}^*(t + kT) \), one arrives after a short calculation at
\[
\phi_k(x_k(t_f), t_f) = \phi_{opt} + \int_{t_0}^{t_f} p_k^*(t)^2 \, dt \\
\times [1 + g_{opt}(t_k) I_u(t_k) + ...] .
\]

(92)

A similar expansion of \(p_{opt}^*(t_k)\) from (95) yields for \(p_k^*(t_f)\) the approximation

\[
p_k^*(t_f) = p_{opt}^*(t_k) + \Delta x_{opt}^* \frac{\partial^2 \phi_k(x_k,t_k)}{\partial x_k^2} + ...
= p_{opt}^*(t_k) [1 + g_{opt}(t_k) I_u(t_k) + ...] .
\]

(93)

We now turn to the prefactor-terms \(Q_k^*(t)\) in (51). Within the neighborhoods of \(x_u(t)\) for which the approximations (50) and thus (57) are valid, we can infer from the initial conditions (47) as

\[
\mu_s = \frac{Q_k^*(t_f)}{2 - Q_k^*(t)} = \text{const.} =: \mu_{s,u} .
\]

(94)

By comparison with (48) we further see that

\[
g_k^*(t) Q_k^*(t) = \mu_{s,u} .
\]

(95)

The constant \(\mu_s\), which is connected with the neighborhood of \(x_u(t)\) and is typically different from \(\mu_u\), follows from the initial conditions (47) as \(\mu_s = 1/2\). Hence, the solution of (94) takes the form

\[
Q_k^*(t) = I_s(t,t_0)/2 .
\]

(96)

As a by-product we find from (50), evaluated for an arbitrary final condition \(t_f = t\) and \(x_f = x\) in combination with (93) that

\[
\frac{\partial^2 \phi_k(x,t)}{\partial x^2} = \frac{1}{I_s(t,t_0)} .
\]

(97)

Within the linearization (50), closer inspection shows that only a single summand appears in the conditional probability (45) and one recovers the expected Gaussian result for \(x\) close to the stable periodic orbit \(x_s(x)\):

\[
p(x,t | x_s(t_0), t_0) = \left(\frac{1}{2 \pi D I_s(t,t_0)}\right)^{\frac{1}{2}} \exp \left\{ -\frac{(x - x_s(t))^2}{2 D I_s(t,t_0)} \right\} .
\]

(98)

Returning to (96), it is remarkable that besides the initial time \(t_0\) no further details of the path \(x_s^*(t)\) play a role. Especially, if \(x_s^*(t)\) remains for a long time in the neighborhood of \(x_u(t)\) where (93) is valid, then by the time it leaves this neighborhood, say \(t = t_s\), the quantity \(I_s(t,t_0)\) is practically equal to \(I_s(t)\) from (72) and thus \(Q_k^*(t)\) equal to the associated master-prefactor \(Q_{opt}^*(t + kT)\). Within our usual approximation (55) we thus have

\[
Q_k^*(t_s) = Q_{opt}^*(t_s + kT) = I_s(t_s)/2
\]

(99)

\[
Q_k^*(t_s) = Q_{opt}^*(t_s + kT) = I_s(t_s)/2 .
\]

(100)

These relations are then used as initial conditions in (10) in order to propagate \(Q_k^*(t_s)\) and \(Q_{opt}^*(t_s + kT)\) through the “crossover-segments” of the corresponding paths \(x_k^*(t_s)\) and \(x_{opt}^*(t + kT)\) up to a certain time-point, say \(t = t_u\), beyond which the linearization (50) about \(x_u(t)\) and thus (46) can be applied.

Once the neighborhood of \(x_u(t)\) is reached, i.e., for \(t \geq t_u\), the solution of (104,107) can be written with (64) as

\[
Q_k^*(t) = Q_{opt}^*(t_u) e^{\Delta (t,t_u)} [1 + g_{opt}(t_u) I_u(t_u,t)] .
\]

(101)

In view of (111) this yields furthermore

\[
Q_{opt}^*(t) |_{p_{opt}^*(t/u)}^2 = Q_{opt}^*(t_u) p_{opt}^*(t_u)^2 \\
\quad \times [1 - g_{opt}^*(t_u) I_u(t_u,t)] .
\]

(102)

Due to (73), the factor \(I_u(t_u,t)\) approaches \(-I_u(t_u)\) as \(t - t_u\) becomes large. It follows that the left hand side of (102) tends towards a finite limit as \(t \to \infty:\)

\[
g_{opt} := \lim_{t \to \infty} Q_{opt}^*(t) p_{opt}^*(t)^2 .
\]

(103)

Since \(t_u\) is an arbitrary reference-time in (102), we can first let \(t \to \infty\) and then rename \(t_u\) as \(t\) with the result

\[
Q_{opt}^*(t) = \frac{g_{opt}}{p_{opt}^*(t/u)} ,
\]

(104)

where the (finite) constant \(\mu_{opt}\) is defined analogously to (83) as

\[
\mu_{opt} := \lim_{t \to \infty} g_{opt}^*(t) Q_{opt}^*(t) .
\]

(105)

Exploiting once more (124) and (131), we can eliminate \(Q_{opt}^*(t)\) in (104) in favor of \(g_{opt}^*(t)\) with the result

\[
g_{opt}^*(t) = \frac{p_{opt}^*(t)^2}{g_{opt}/\mu_{opt} - p_{opt}^*(t)^2 I_u(t)} .
\]

(106)

As discussed below (83), the deviations of \(x_k^*(t)\) from the associated master path \(x_{opt}^*(t + kT)\) become smaller and smaller as \(k\) increases and in view of (99,100,101,105) we expect a similar convergence of \(Q_k^*(t)\) towards \(Q_{opt}^*(t + kT)\). In Appendix C, the following quantitative estimate for this convergence is established for all times \(t \in [t_u, t_f]\):

\[
Q_k^*(t) = Q_{opt}^*(t + kT) [1 + O(p_{opt}^*(t)^2)] ,
\]

(107)

where the order of magnitude is meant with respect to the dependence on \(k\).

From the technical viewpoint, Eq. (104) in combination with (107) is a central and highly non-trivial result of our present work. Since \(I_u(t)\) is periodic in \(t\) and since \(p_{opt}^*(t)\) decreases exponentially according to (81), we see from (104,107) that the prefactor \(Q_k^*(t)\) diverges exponentially with the time which the path \(x_k^*(t)\) spends in the neighborhood of \(x_u(t)\), in striking contrast to the behavior (85) close to \(x_u(t)\). The basic physical reason for this divergence of \(Q_k^*(t)\) is that the probability of a stochastic process (3) to permanently remain close to the unstable
periodic orbit \( x_u(t) \) decreases exponentially with increasing time. Since typically the process closely follows a deterministic trajectory, the action barely grows and thus it is the prefactor \( 1/\mathcal{Q}_r^1(t_f)^{1/2} \) in \( [15] \) which has to account for the exponential decrease in time!

Since \( p^*_o(t_k) \) decreases exponentially with \( k \) we see from \( [106] \) that \( g^*(t_k) \) tends to zero like \( p^*_o(t_k)^2 \). In view of \( [108] \) we therefore conjecture that also higher derivatives of \( \phi^*_o(x_k, t_k) \) continue to scale like the corresponding powers of \( p^*_o(t_k) \). The terms indicated by the dots in \( [101-103] \) are then indeed negligibly small.

### D. Evaluation and discussion of the rate

We are now in the position to evaluate the rate-formula \( [27] \) in terms of the master path \( x^*_o(t) \). To this end we approximate in \( [22] \) and \( [23] \) the square brackets by 1, neglect in \( [107] \) the term of order \( p^*_o(t_k)^2 \) and in \( [104] \) the last term (being also a correction of order \( p^*_o(t_k)^2 \)). By dropping the index of \( t_f \) we then can infer from \( [101-103] \) our central result for the \( \text{instantaneous rate} \):

\[
\Gamma(t) = \sqrt{D} \alpha \exp \frac{-\phi^*_o(D)}{D} \kappa(t, D) [1 + O(D^\gamma)]
\]

\[
\alpha := [4\pi T^2 \lim_{l \to -\infty} p^*_o(t^2) Q^*_o(t)]^{-1/2} \tag{109}
\]

\[
\kappa(t, D) := \sum_{k=0}^{K(t, t_0)} \frac{p^*_o(t_k)^2 C^k}{D} \times \exp \left\{ -\frac{1}{D} \int_t^{t+T} p^*_o(t'_k)^2 dt' \right\} \tag{110}
\]

The effect of our various approximations in deriving this result together with the corresponding “accuracy exponent” \( \gamma \to 0 \) in \([108]\) will be discussed in Sect. IV.E below. Next, we analyze in more detail the properties of \( \kappa(t, D) \). By means of \( [61-63, 74] \) we rewrite \( [110] \) as

\[
\kappa(t, D) = \sum_{k=0}^{K(t, t_0)} \frac{p^*_o(t_k)^2 C^k}{D} \times \exp \left\{ -\frac{1}{D} \int_t^{t+T} p^*_o(t'_k)^2 dt' \right\} \tag{111}
\]

\[
C := e^{-2\lambda_u T} \tag{112}
\]

Since \( 0 < C < 1 \) there is a competition in the sum \( [111] \) between the exponential terms which increase with \( k \) and the pre-exponential factors which decrease with \( k \). One readily sees that the dominant contribution to the sum stems from a few \( k \)-values around the real number \( \bar{k} \), implicitly defined via

\[
p^*_o(t_k)^2 C^\bar{k} I_u(t) = 2D \tag{113}
\]

Recalling that \( t \) stands here for \( t_f \) and since neither \( I_u(t_f) \) nor \( p^*_o(t_f) \) (cf. \( [74, 84] \)) are small quantities, it follows that \( \bar{k} \) is, for small noise-strengths \( D \), much larger than 0 but, for sufficiently large \( t_f \), according to \( [53] \) also much smaller than \( (t_f - t_0) \). Therefore the sum in \( [111] \) and thus in \( [110] \) can be extended to arbitrary integers \( k \) at the price of an error which is exponentially small in \( D \), i.e., without actually affecting the accuracy exponent \( \gamma \) in \( [108] \). As a further consequence of the fact that the dominant \( k \)-values are much smaller than the upper limit \( K(t, t_0) \) for large \( t - t_0 \), we see that our formal approximation \( [85] \) is indeed self-consistently satisfied.

Next we notice that under the sum in \( [110] \), the pre-exponential term is nothing else than \( \int_t^{t+T} \kappa(t', D) dt' \) for all \( t \) and all (small) \( D \). For the \( \text{time averaged rate} \)

\[
\Gamma = \sqrt{D} \alpha \exp \frac{-\phi^*_o(D)}{D} [1 + O(D^\gamma)] \tag{116}
\]

It consists of an Arrhenius-type exponentially leading part with an “effective potential barrier” \( \phi^*_o \) and a nontrivial pre-exponential \( D \)-dependence. The two quantities \( \alpha \) and \( \phi^*_o \) follow from the master-path \( x^*_o(t) \) according to \( [24] \) and \( [106] \).

Therefore they are independent of \( D \) but depend in a highly nontrivial way on various global properties of the deterministic force field \( F(x, t) \) in \( \{3\} \). In general, their explicit value can only be determined numerically or by means of approximations. An exactly analytically solvable special case will be presented in Sect. V.A.

We recall that for equilibrium systems, characterized by a time-independent force field \( F(x) = -V'(x) \) in \( \{3\} \), the escape rate exhibits an exponentially leading Arrhenius factor, which involves simply the barrier against escape of the static potential \( V(x) \), and a \( D \)-independent pre-exponential factor which depends only on local properties of the potential at the barrier and the well \( \{4\} \), see also \( [61] \) below. The different structure of \( [116] \) is thus a consequence of the far from equilibrium situation created by the time-dependence of the deterministic force field \( F(x, t) \).
As announced in Sect. II.C, the time-averaged escape rate for the periodic force field $F(x,t)$ can be identified with that of its supersymmetric partner force field (16) for asymptotically weak noise $D$ without any further restrictions on $F(x,t)$. The detailed proof of this highly non-trivial invariance property of (114) is carried out in Appendix D.

Returning to the instantaneous rate (108), we see that it exhibits in comparison with the time averaged rate (116) the additional time-dependent factor $\kappa_{\text{opt}}(t,D)$. The explicit evaluation of this factor requires the knowledge of one more global quantity, for instance of

$$\beta_{\text{opt}}(t) := \lim_{t \to \infty} p_{\text{opt}}^*(t) e^{A_u(t,t)}. \quad (117)$$

Note that due to relation (62) the $t$-dependency of this quantity is actually quite simple. According to (111), this definition (117) allows us to rewrite (111) -- with the range of $k$ extended to arbitrary integers -- as

$$\kappa_{\text{opt}}(t,D) = T \sum_{k=-\infty}^{\infty} \frac{\beta_{\text{opt}}(t)^2 C^k}{D} \times \exp \left\{ - \frac{\beta_{\text{opt}}(t)^2 C^k I_u(t)}{2D} \right\}. \quad (118)$$

Besides $\beta_{\text{opt}}(t)$ all other quantities in this expression are determined by local properties of the force field $F(x_u(t),t)$ along the unstable periodic orbit. By exploiting (103,112) it follows that

$$\kappa_{\text{opt}}(t+T,D) = \kappa_{\text{opt}}(t,D)$$
$$\kappa_{\text{opt}}(t,C,D) = \kappa_{\text{opt}}(t,D). \quad (119,120)$$

Together with (113) and the obvious property $0 < \kappa_{\text{opt}}(t,D) < \infty$ this completes our qualitative picture of the way in which $\Gamma(t)$ oscillates around its average value $\Gamma$.

E. The accuracy exponent $\gamma$

In the following, we come to the determination of the accuracy exponent $\gamma$ in (108,110). We will not elaborate here all the details of the rather involved calculations but restrict ourselves to the main steps.

First of all, we recall that a contribution $O(D)$ is inherited right away from formula (71). Next we have approximated the square brackets in (72) by 1. For those $k$-values which mainly contribute to the rate it can be inferred from (113) together with $t = t_f$ and (80) that

$$p_{\text{opt}}^*(t_k)^2 = O(D) \quad (121)$$

and hence with (106) that

$$g_{\text{opt}}(t_k) I_u(t_k) = O(D). \quad (122)$$

Since the integral in (122) is of the same order of magnitude as $p_{\text{opt}}^*(t_k)$ from (121) we conclude that the total error we committed in (122) is of the order $O(D^2)$, thus contributing once more a term of the order $O(D)$ in the rate-formulas (108,116). The same conclusion can be drawn with respect to our approximating the square brackets by 1 in (82) and neglecting the $O(p_{\text{opt}}^*(t_k)^2)$ -term in (107) as well as the last term in (104). In other words, the relative error induced by all our so far made approximations is of the order $O(D)$.

What remains is a closer inspection of the approximation (54) for $F(x,t)$ in the neighborhood of $x_{s,u}(t)$. One readily sees that actually only the approximation in the neighborhood of the unstable periodic orbit $x_u(t)$ matters in our quantitative evaluation of the rate; the basic reason for this is once more our assumption $t_0 \to -\infty$ in (83). In case that (64) happens to be an exact identity in this neighborhood of $x_u(t)$, then the total error committed in the rate-formulas (108,116) is thus of the order $O(D)$. Otherwise, a closer analysis of the relevant perturbative corrections shows that the error introduced via the approximation (54) amounts to corrections of the order $O(p_{\text{opt}}^*(t_k))$ in the rate-formula, i.e., of the order $O(\sqrt{D})$ according to (121). In other words, we can conclude that

$$\gamma = \begin{cases} 1 & \text{if } F''(x_u(t),t) \equiv 0 \\ 1/2 & \text{otherwise.} \end{cases} \quad (123)$$

In the case $\gamma = 1/2$ it is important that in the global quantities $\phi_{\text{opt}}$, $\alpha_{\text{opt}}$, $\beta_{\text{opt}}(t)$ from (54,109,117) the long-time limits are made and the exact master path is utilized without any further approximations. If instead in these definitions any finite reference-time in combination with relations based on the approximation (54) were used, then this would introduce a possibly very small but nevertheless $D$-independent error and so ruin the asymptotically exact predictions (108,116) in the weak noise limit $D \to 0$.

In cases for which (54) is not exactly satisfied in the neighborhood of the unstable periodic orbit $x_u(t)$ and hence $\gamma = 1/2$, it is in principle possible to calculate perturbatively the corresponding corrections such as to arrive again at a reduced relative error $O(D)$ in the so improved rate-formulas, though the actual calculations and the resulting expressions become very complicated. On the other hand, further reducing the $O(D)$-error is even in principle rather problematic since it would require to go beyond the saddle point approximation in the path-integral approach from Sect. III.

At this point it may also be worth to recall from Sect. IV.A that for any fixed (however small) $D$-value, the error $O(D)$ in (31), which is inherited by the final rate-formula, diverges as the amplitude of the time dependence of $F(x,t)$ tends to zero, but also if its period $T$ either tends to zero or to infinity. Thus, neither of these limits commutes with the limit $D \to 0$. 

12
F. The limits \( t \to \infty \) and \( D \to 0 \)

In the derivation of the rate-formula (108) we have assumed that all paths \( x^*_k(t) \) which notably contribute to (11) sojourn for a very long initial time-interval close to the stable periodic orbit \( x_s(t) \), see (13). On the other hand, eq. (113) tells us that the amount of time which those dominant paths spend in the neighborhood of the unstable periodic orbit \( x_u(t) \) is roughly speaking of the order \( O(\ln 1/D) \). Both these conditions are compatible only if \( t-t_0 \) substantially exceeds in order of magnitude \( \ln 1/D \). In the physically relevant case, the noise-strength \( D \) is small but finite and this condition is well satisfied after a comparatively short “transient” time-period. Thus, strictly speaking, in (108) and (116), with decreasing \( D \)-values also the lower limit of the admitted times \( t-t_0 \) is tacitly assumed to slowly increase in proportion to \( \ln 1/D \).

We remark that our result (108) obviously remains periodic in \( t \) for arbitrarily large \( t-t_0 \), see (13). Therefore, the restriction of the utilized basic formula (13) to values of \( t-t_0 \) much smaller than \( 1/D \) does no longer apply to the final result (108), see also the discussion below (13).

In the physically less relevant case that \( t-t_0 \) is kept at an arbitrary but fixed value and then \( D \) is made smaller and smaller, things become different. As pointed out in Sect. IV.A, for any finite initial and final times \( t_0 \) and \( t = t_f \), there exists generically a unique absolute minimum \( x^*_{k_0}(t) \) of the action. For sufficiently small \( D \) the \( k_0 \)-term will thus completely dominate the sum in (1), i.e.,

\[
\Gamma(t = t_f) = \frac{p^*_{k_0}(t_f)e^{-\phi_{k_0}(x_u(t_f),t_f)/D}}{[4 \pi D Q^*_{k_0}(t_f)]^{1/2}} . \tag{124}
\]

While most of the quantities on the right hand side of this result (including the index \( k_0 \)) still depend in a very complicated way on the time \( t = t_f \), no additional implicit \( D \)-dependence is hidden. The most striking feature is the \( 1/\sqrt{D} \) pre-exponential behavior in comparison with the \( \sqrt{D} \)-scaling in (108)!

Qualitatively, the crossover from (12) to (108), either as \( t \) increases or as \( D \) decreases, is clear: At some point the \( k \)-dependence of the pre-exponential factors in (12) is no longer negligible in comparison with the exponentially leading contributions and so the dominant \( k \)-value moves away from \( k_0 = k_0(t, t_0) \) towards smaller values \( k \simeq \tilde{k} \), cf. (113). At the same time, more that one term in the sum (12) starts to notably contribute. Quantitatively, a leading order approximation follows along the same line of reasoning as in the derivation of (108) from (51), except that in the approximation for the action (108), also contributions due to the deviations between \( x^*_k(t) \) and its associated master-path \( x^*_{k_0}(t+kT) \) at times close to \( t_0 \) to have been included, that is, the approximation (83) should be abandoned. The final result is again of the same form as in (108), but with a larger error than \( O(D^2) \) and instead of (108) with

\[
\kappa_{opt}(t, D) := T \sum_{k=0}^{K(t, t_0)} \frac{p^*_{opt}(t+kT)^2}{D} \times \exp \left\{ - \frac{1}{D} \left[ \int_{t}^{t+\infty} \int_{-\infty}^{t_0} p^*_{opt}(t'+kT)^2 \, dt' \right] \right\} . \tag{125}
\]

For moderate \( t-t_0 \) or extremely small \( D \) the exponential in (125) depends very strongly on \( k \) and therefore the sum is dominated by a single term \( k = k_0(t, t_0) \). Upon increasing \( t-t_0 \) or \( D \) this strong \( k \)-dependence of the exponential and hence the dominance of the \( k_0 \)-contribution is softened and the already discussed qualitative crossover-behavior is recovered.

G. More general seeds \( x_0 \)

So far, our rate-formula (108) is restricted to the case (12) that the initial position \( x_0 \) at time \( t_0 \) coincides with the stable periodic orbit \( x_s(t_0) \). As pointed out in Sect. II, one expects that for large enough times \( t-t_0 \) the initial position \( x_0 \) should not matter, provided it is chosen inside the domain of attraction of \( x_s(t) \). For sufficiently small noise-strengths \( D \) this is the case whenever

\[
x_0 < x_u(t_0) . \tag{126}
\]

In the following, we analyze this intuitive expectation in some more quantitative detail.

For arbitrary but fixed \( x_0 \) satisfying (126), the observation from Sect. IV.A remains true, namely that only minimizing paths \( x^*_k(t) \) play a role in the rate (11) which closely follow a deterministic behavior \( \delta^*_k(t) \simeq F(x^*_k(t), t) \) for most of the time. This requirement can be fulfilled in two basic ways and appropriate compromises thereof. The first possibility is that the path \( \delta^*_k(t) \) closely approximates a deterministic trajectory for a very long initial time-interval. During this time, \( x^*_k(t) \) approaches the periodic attractor \( x_s(t) \) very closely and does practically not accumulate any action (36,37). Consequently, one is back to the case (12) after an appropriate redefinition of the initial time \( t_0 \) by zero. With the initial conditions (17) one then recovers the same result as in (108) except that in (13) the function \( \Lambda_s(t, t_0) \) is now defined as \( \int_{0}^{t_0} F'(x^*_k(t'), t') \, dt' \).

Since \( x^*_k(t) \) practically agrees with \( x_u(t) \) during a very long time interval, one sees that also with respect to the prefactor \( Q^*_k(t) \) we are back to the case (12). As before, we may label such paths by low \( k \)-values and their contributions to the rate (11) are identical to those of the low-\( k \)-values in (108).

The second possibility is that the minimizing path \( x^*_k(t) \) travels from its starting point \( x_0 \) immediately into the neighborhood of the unstable periodic orbit \( x_u(t) \) and then very closely follows this deterministic trajectory \( x_u(t) \) for the rest of its time. If \( x_0 \) is already close to \( x_u(t_0) \), such paths lead to a very small value of the
action in [36,37] and thus will ultimately dominate the rate [51] if \( t - t_0 \) is kept fixed and \( D \) becomes asymptotically small. This puzzling observation has lead to some amount of confusion in the recent literature [15,48]. The resolution is that, much like in Sect. IV.F, things become very different for a small but fixed \( D \) in combination with larger and larger times \( t-t_0 \). The salient point is that the price to be paid for a long sojourn close to \( x_o(t) \) is a very small prefactor \( p^u_e(t)/\Omega_n^u(t_f)/2^2 \) in [54], as discussed below [107], namely of the order \( \exp\{-2\lambda_u \cdot (t - t_0)\} \). As a consequence, the paths with low \( k \)-values, as discussed in the preceding paragraph, will dominate in spite of their unfavorable action. Therefore, the rate-formula [108] applies for any \( x_o \) satisfying [126] on condition that

\[
t - t_0 \gg \phi_{opt}/(2D\lambda_u) .
\]

This condition characterizes the asymptotic time regime for which the rate-formula [108] is valid in the case of a general initial condition. Even for rather small \( D \), the preceding transient regime is typically confined to a few driving periods \( T \), as illustrated by the examples in Sect. V. Note that [127] comprises the condition from Sect. IV.F that \( t - t_0 \) has to substantially exceed in order of magnitude \( \ln 1/D \). In other words, for a generic initial condition, eq. [127] is the only restriction for the rate-formula [108], apart from the exclusion of vanishing driving amplitudes and vanishing or diverging periods \( T \).

H. Summary from the practical viewpoint

Given an arbitrary time-periodic force field \( F(x,t) \) that satisfies the condition in [34], what are the necessary practical (numerical or analytical) steps for an explicit quantitative evaluation of the rate [108]?

The first step is the determination of the stable and unstable periodic orbits \( x_o(t) \) and \( x_n(t) \). An efficient way to do this is by evolving the deterministic dynamics forward and backward over a long time, respectively, with a reasonably well chosen initial condition. Once \( x_{o,n}(t) \) is known, the functions \( \Lambda_{x,u}(t-t_1) \) and \( I_{x,n}(t-t_1) \) from [74,75] follow readily, with \( t_1 \) being an arbitrary reference time.

The next step is the determination of the master path \( x_{opt}(t) \). To this end, we chose an arbitrary but fixed time \( t_s \) and a very small but finite positive number \( \Delta x_{min} \), characterizing the neighborhood of \( x_o(t) \) within which we are willing to accept the errors introduced by the approximation [76]. We now consider the quantity \( \Delta x_{opt}(t_s) \) as a parameter that may take values in the interval \( [\Delta x_{min}, \Delta x_{min}e^{-\lambda_uT}] \). Each such parameter value \( \Delta x_{opt}(t_s) \) yields a set of initial conditions

\[
x_{opt}(t_s) = x_o(t_s) + \Delta x_{opt}(t_s) \quad (128)
\]

\[
p_{opt}(t_s) = \Delta x_{opt}(t_s)/I_o(t_s) , \quad (129)
\]

see [108,24]. With these initial conditions one then evolves \( x_{opt}(t) \) and \( p_{opt}(t) \) according to [44,42]. For a generic value of the parameter \( \Delta x_{opt}(t_s) \), the path \( x_{opt}(t) \) will either reach \( x_n(t) \) after a finite time and then proceed towards \( x = \infty \) or never reach \( x_o(t) \) and instead return into the vicinity of \( x_o(t) \) as \( t \) grows. By fine-tuning \( \Delta x_{opt}(t_s) \) one has to find a path \( x_{opt}(t) \) which remains close to \( x_o(t) \) as long as possible, say until \( t = t_{max} \). Upon varying \( \Delta x_{opt}(t_s) \) within \( [\Delta x_{min}, \Delta x_{min}e^{-\lambda_uT}] \), the existence of at least one such path is guaranteed by the theory. A second solution, corresponding to a saddle point instead of a minimum of the function, is also to be expected (see Sect. V.A, below eq. [158]). Further local extrema may coexist as well. Among them, the desired solution \( x_{opt}(t) \) is the one with the smallest value of the action

\[
\phi_{opt} = \frac{\Delta x_{opt}(t_s)^2}{2I_s(t_s)} + \int_{t_s}^{t_{max}} p_{opt}(t)^2 dt , \quad (130)
\]

see [13,54,61,74]. By approximating \( i \) in [117] by \( t_{max} \) we obtain

\[
\beta_{opt}(t) = p_{opt}^*(t) \cdot e^{\Lambda_u(t_{max},t)} , \quad (131)
\]

whence \( \kappa_{opt}(t,D) \) from [118] follows with \( C \) from [112]. Finally, one chooses the initial conditions

\[
Q_{opt}(t_s) = I_s(t_s)/2 , \quad \dot{Q}_{opt}(t_s) = \dot{I}_s(t_s)/2 , \quad (132)
\]

see [10,100], and then propagates \( Q_{opt}(t) \) according to [44] until \( t = t_{max} \), to obtain

\[
\alpha_{opt} = [4\pi T^2 p_{opt}^{max}/2Q_{opt}^{max}]^{-1/2} , \quad (133)
\]

see [109].

The accuracy of \( \phi_{opt}, \beta_{opt}(t), \alpha_{opt} \) from [130,131,133] can be estimated by observing how little these quantities change if \( t_{max} \) is varied and if \( \Delta x_{min} \) is changed by a factor \( e^{\lambda_uT} \).

We finally note that the association of \( x_{opt}(t+kT) \) with the specific path \( x_{opt}(t) \) as in the preceding subsections does not play a role any more in the above described practical procedure.

V. EXAMPLES

In general, the explicit quantitative evaluation of \( \phi_{opt}, \alpha_{opt}, \) and \( \kappa_{opt}(t,D) \) in the rate-formula [108] is not possible in closed analytical form. Exceptions are piecewise parabolic potentials \( V(x) \) in conjunction with an additive sinusoidal driving [5]. In Subsect. A, the simplest example [25] with two parabolic pieces will be worked out and compared with accurate numerical results and with the approximation for small driving amplitudes from [26]. In Subsect. B we elaborate as a second example the case of a force field [3] deriving from a cubic potential \( V(x) \) along the lines of the numerical recipe from Sect. IV.H.
A. Piecewise parabolic potential

We consider the force field from (5) with a piecewise parabolic potential of the form
\[
\begin{align*}
V(x \leq 0) &= \frac{\lambda_s}{2} [x_s^2 - (x - \bar{x}_s)^2] \\
V(x \geq 0) &= \frac{\lambda_u}{2} [x_u^2 - (x - \bar{x}_u)^2],
\end{align*}
\]  
(134)
where \(\bar{x}_s\) denotes the potential well (stable fixed point) and \(\bar{x}_u\) the saddle (unstable fixed point), with the properties
\[
\bar{x}_s < 0, \quad \bar{x}_u > 0.
\]  
(135)
The parameters
\[
\lambda_s < 0, \quad \lambda_u > 0
\]  
(136)
characterize the piecewise constant curvatures and thus the time scales (Lyapunov exponents) of the deterministic motion near the attractor \(\bar{x}_s\) and the repeller \(\bar{x}_u\), respectively. The force-field (3) then takes the explicit form
\[
\begin{align*}
F(x \leq 0, t) &= \lambda_s (x - \bar{x}_s) + A \sin(\Omega t) \\
F(x \geq 0, t) &= \lambda_u (x - \bar{x}_u) + A \sin(\Omega t).
\end{align*}
\]  
(137)
In particular, the quantities \(\lambda_s, \lambda_u\) in (134,137) are identical to those from (8).

Requiring continuity at \(x = 0\) we conclude from (137) that \(\lambda_s \bar{x}_s = \lambda_u \bar{x}_u\). Selecting as independent model parameters \(\lambda_s, \lambda_u\), and the static potential barrier
\[
\Delta V := V(\bar{x}_u) - V(\bar{x}_s)
\]  
(138)
the fixed points \(\bar{x}_{s,u}\) can be expressed through
\[
\lambda_s \bar{x}_s = \lambda_u \bar{x}_u = \sqrt{\frac{2 \Delta V |\lambda_s| \lambda_u}{|\lambda_s| + \lambda_u}}
\]  
(139)

Turning to the determination of the stable and unstable periodic orbits (8) it is convenient to make a somewhat stronger assumption than in (8) namely that both periodic orbits \(x_{s,u}(t)\) never cross the matching point \(x = 0\) of the two parabolic pieces of \(V(x)\), i.e. we require that
\[
x_s(t) < 0 < x_u(t)
\]  
(140)
for all times \(t\). One finds that this property is granted if and only if the conditions
\[
A^2 < (\lambda_{s,u}^2 + \Omega^2) \bar{x}_{s,u}^2
\]  
(141)
are satisfied for both the ‘s’- and the ‘u’-indices, and that the periodic orbits then take the explicit form
\[
x_{s,u}(t) = \bar{x}_{s,u} - \frac{A [\lambda_{s,u} \sin(\Omega t) + \Omega \cos(\Omega t)]}{\lambda_{s,u}^2 + \Omega^2}.
\]  
(142)
With the definitions \(\lambda_{s,u}(t, t_1) = \lambda_{s,u} \cdot (t - t_1)\) and \(I_{s,u}(t) = |\lambda_{s,u}|^{-1}\),
\[
\lambda_{s,u}(t_1) = \lambda_{s,u} \cdot (t - t_1)
\]  
(143)
(144)

Our next goal is the determination of the master path \(x_{s,u}(t)\). To simplify the analytical calculations we restrict ourselves to the case that the master path \(x_{s,u}(t)\) crosses the point \(x = 0\) exactly once, say at the time \(t = t_1\):
\[
x_{s,u}(t_1) = 0 \iff t = t_1
\]  
(145)
The self-consistency of this assumption with the final solution for \(x_{s,u}(t)\) remains to be checked later.

From (112) we see that both \(x_{s,u}(t)\) and \(p_{s,u}(t)\) are still continuous at \(t = t_1\). For all other times \(t\) the relation \(\Delta t\) and the hence following conclusions are not approximations but exact identities since the force-field \(F(x, t)\) in (135) is by construction piecewise linear. By introducing \(\lambda_{s,u}(t_{1,1})\) and \(\Delta t\) into (137,139) we obtain by letting \(t_0 \to \infty\) and \(t_f \to \infty\) for the master path the following two relations (one with index ‘s’ and one with ‘u’):
\[
x_{s,u}(t_1) = p_{s,u}(t_1)/\lambda_{s,u}.
\]  
(146)
These two equations for the two unknowns \(t_1\) and \(p_{s,u}(t_1)\) imply with (142) the result
\[
\tan(\Omega t_1) = \frac{1}{\Omega} \frac{\lambda_s - \Omega^2}{\lambda_s + \Omega u}
\]  
(147)
\[
p_{s,u}(t_1) = \lambda_u \bar{x}_u - \frac{A \lambda_s \lambda_u \cos(\Omega t_1)}{\Omega (\lambda_s + \lambda_u)}
\]  
(148)
We observe that the solutions \(t_1\) of (147) are independent of \(A\). Furthermore, there are obviously two solutions \(t_1\) within every time-period \(T = 2\pi/\Omega\). We anticipate, that only one of them corresponds to a minimum of the action, and thus to the master path. Hence we fix \(t_1\) uniquely (up to the usual degeneracy under \(t \mapsto t + T\)) by (147) in conjunction with
\[
\frac{A \Omega \cos(\Omega t_1)}{\lambda_s + \lambda_u} < 0,
\]  
(149)
and show later, that this condition singles out the right solution of (147). [The case \(\lambda_s + \lambda_u = 0\) has to be treated as limit \(\lambda_s + \lambda_u \to 0\).] Combining (147,148) it follows that
\[
p_{s,u}(t_1) = \lambda_u \bar{x}_u - |A| [\lambda_u |\lambda_u|/\nu^2 > 0,
\]  
(150)
where we have introduced the definition
\[
\nu^2 := [(\lambda_s^2 + \Omega^2)(\lambda_u^2 + \Omega^2)]^{1/2}.
\]  
(151)
Note that \(\lambda_u \bar{x}_u\) in (150) may be rewritten in various equivalent forms according to (133) and that the last relation \(p_{s,u}(t_1) > 0\) in (150) follows as a consequence of (136,140,146).
Given \( t_1 \) and \( p_{opt}(t_1) \), the entire time evolution of the master path \( x_{opt}(t) \) can be readily inferred from (143,144) with the result

\[
P_{opt}(t) = p_{opt}(t_1) e^{-\lambda_s u (t-t_1)}
\]

(152)

\[
x_{opt}(t) = x_{s,u}(t) - p_{opt}(t)/\lambda_s u
\]

(153)

where ‘s’ is associated with times \( t < t_1 \) and ‘u’ with \( t > t_1 \). All the general qualitative features discussed in Sect. IV.A are nicely illustrated by this explicit example (152,153).

Finally, we have to check the self-consistency of the solution (152,153) with our initial assumption (145), i.e., we have to verify that \( x_{opt}(t) \) is strictly positive for \( t > t_1 \) and negative for \( t < t_1 \). In general, in doing so, a transcendental equation arises which has to be evaluated numerically. Without going into the details of the proof we further mention that one can show analytically that \( A^2 < \lambda_s^2 u^2 \) is a sufficient but not necessary self-consistency criterion for (145). On the other hand, it is obvious that the assumption \( x_s(t) < 0 < x_u(t) \) in (140) is automatically covered by the stronger requirement (145). Thus, (140,141) is a necessary but not sufficient self-consistency criterion for (145).

Introducing the above relations (152,153) into (138,139), we obtain for the action of the master path

\[
\phi_{opt} = \Delta V \left[ 1 - \frac{A^2 \lambda_s \lambda_u (|\lambda_s| + |\lambda_u|) \beta Q_{opt}(t_1)}{2 \Delta V \nu^4} \right]^{1/2}.
\]

(154)

For \( A \to 0 \) or \( \Omega \to \infty \) we thus recover the static (undriven) potential barrier \( \Delta V \) from (138). The leading order corrections for small \( A \) decrease like \( |A|^2 \) (138). For any finite amplitude \( A \) and driving period \( \mathcal{T} = 2\pi/\Omega \) the “effective potential barrier” \( \phi_{opt} \) is smaller than the static barrier \( \Delta V \) and is monotonically decreasing both with increasing \( A \) and increasing \( \mathcal{T} \). Invoking the necessary but not sufficient self-consistency criterion (141) for (145), one can explicitly confirm that \( \phi_{opt} \) can never become zero (see (138,139)) by demonstrating that the argument in the square-brackets in (154) is always positive. If we had chosen the solution of (147) with the opposite inequality than in (149), then a plus instead of the minus sign in (152) would have been the consequence. Thus (149) is indeed the pertinent condition for singling out the solution which minimizes the action.

By using (143) and (152) in the definition (117) of \( \beta_{opt}(t) \) we obtain

\[
\beta_{opt}(t) = p_{opt}(t_1) e^{-\lambda_u (t-t_1)}.
\]

(155)

Turning to the prefactor \( Q_{opt}(t) \), we see from (140) and (144) that

\[
Q_{opt}(t) = \frac{1}{2 |\lambda_s|}
\]

(156)

for all times \( t < t_1 \). Since \( F''(x,t) = (\lambda_u - \lambda_s)\delta(x) \) according to (137), we can infer from (146) that the prefactor \( Q_{opt}(t) \) is continuous at \( t = t_1 \) while its derivative jumps from \( Q_{opt}^+(t_1) = 0 \) to the value

\[
\dot{Q}_{opt}(t_1) = \frac{|\lambda_s| + \lambda_u}{|\lambda_s|} \frac{\dot{x}_{opt}(t_1) - p_{opt}(t_1)}{\dot{x}_{opt}(t_1)}.
\]

(157)

where \( t_1^+ \) indicates the limit \( t \to t_1 \) from above and \( t_1^- \) from below. With these initial conditions, the solution of (144) in the domain \( t > t_1 \) is straightforward, yielding

\[
\lim_{t \to \infty} Q_{opt}(t) p_{opt}(t)^2 = \frac{p_{opt}(t_1)^2 Q_{opt}(t_1)}{2 \lambda_u}.
\]

(158)

Using (148,152,153) one can show that \( p_{opt}(t_1) - \dot{x}_{opt}(t_1) \) is identical to the expression on the left hand side of (149), so that (157) and thus (158) are positive quantities. With the opposite inequality in (149) they would be negative, confirming once more that the latter case corresponds to a saddle point rather than a minimum of the action. Collecting everything, we are finally in the position to evaluate eq. (109) with the result

\[
\alpha_{opt} = \left[ \frac{|A| (\Omega^2 + \lambda_s \lambda_u) + \sqrt{2 \Delta V \nu^4 (\lambda_s)^{-1} + \lambda_u^{-1}}} {16 \pi^3 |A| \phi_{opt}} \right]^{1/2}.
\]

(159)

Once again, the fact that the argument in the square root is positive can be explicitly verified by exploiting the necessary but not sufficient self-consistency criterion (141) for (143).

**B. Comparison of analytical and numerical results**

We have compared the above analytical predictions for the instantaneous rate (103) with very accurate numerical results in fig.4. To this end, we have computed the solution of the Fokker-Planck-equation (10) and then evaluated the rate according to (4), starting with an narrow Gaussian initial distribution \( \rho(x,t_0) \) about the potential well \( x_s \) and then waiting until transients have died out, i.e. until \( \Gamma(t) \) has reached its \( \mathcal{T} \)-periodic asymptotic behavior. In order to numerically evolve the one-dimensional time-dependent Fokker-Planck-equation (11) one can employ standard parabolic partial differential equation solving procedures in one spatial variable. We have adopted a Chebyshev collocation method (49) to reduce the problem to a coupled system of ordinary differential equations, which is then solved by standard numerical methods. By changing the various parameters of the numerical procedure, the typical relative errors of the numerical rates \( \Gamma(t) \) in our figures are estimated to be at most of the order of \( 10^{-2} \) for rates down to about \( 10^{-100} \) and of the order of \( 10^{-3} \) for rates down to about \( 10^{-200} \).

The results in fig.4 confirm for a representative set of parameter values that the agreement between the analytical predictions and the practically exact numerical
results for the instantaneous rate $\Gamma(t)$ indeed improves with decreasing noise-strength $D$. While the absolute values of $\Gamma(t)$ and the location of the extrema strongly depend on $D$, the overall shape changes very little and does not develop singularities as $D \to 0$.

The corresponding time averages $\bar{\Gamma}$ are depicted in fig.5, exhibiting excellent agreement between theory and numerics even for relatively large $D$. The inset of fig. 5 confirms our analytical prediction that the relative error in (116) decreases asymptotically like $D^3$.

Finally, fig.6 illustrates the dependence of the time averaged rate $\bar{\Gamma}$ upon the amplitude $A$ of the periodic driving force. As expected, our theoretical prediction compares very well with the (numerically) exact rate, except for very small driving amplitudes $A$. The latter discrepancy is in accordance with our discussion in Sect. IV.A and Sect. IV.E.

We have furthermore included in fig.6 a comparison with the analytical approximation for the time averaged rate $\bar{\Gamma}$ from Ref. [23]. By way of a matching procedure, involving the barrier region only, it is predicted [23] that

$$\bar{\Gamma} = \Gamma_0 \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-s(\phi)/D},$$

where $\Gamma_0$ is the well-known Kramers-Smoluchowsky-rate in the absence of the periodic driving force [1].

The leading order effect of an additive sinusoidal driving force, such that the associated periodic modulations of the potential barrier are small in comparison with the unperturbed barrier $\Delta V$, but not necessarily in comparison with the noise-strength $D$, are captured by the function $s(\phi)$ in (161). It can be written for a general metastable potential $V(x)$ in (3) under the form [23]

$$s(\phi) = A \left( S \sin \phi - C \cos \phi \right)$$

$$S = S(x_1) := \int_{\bar{x}_u}^{\bar{x}_s} dx u \left( \Omega \int_{x_1}^{x} dy V'(y) \right)$$

$$C = C(x_1) := \int_{\bar{x}_u}^{\bar{x}_s} dx \cos \left( \Omega \int_{x_1}^{x} dy V'(y) \right)$$

with an arbitrary reference point $x_1 \in (\bar{x}_s, \bar{x}_u)$. Fig.6 confirms that this approximation from [23] is indeed complementary to ours in that it is very accurate for small driving amplitudes $A$ but develops considerable deviations with increasing $A$. Those approximations have been omitted in figs.4 and 5 since they are not valid in this parameter regime and indeed are way off.

### C. Cubic potential

As a second example we consider a force field (5) with a cubic metastable potential

$$V(x) = -\frac{a}{3} x^3 + \frac{b}{2} x^2, \quad a, b > 0.$$  

The stable and unstable fixed points $\bar{x}_{s,u}$ of this potential are given by

$$\bar{x}_s = 0, \quad \bar{x}_u = \frac{b}{a},$$

with curvatures at those fixed points

$$V''(\bar{x}_s) = b, \quad V''(\bar{x}_u) = -b,$$

and a static potential barrier height

$$\Delta V := V(\bar{x}_u) - V(\bar{x}_s) = \frac{b^3}{6a^2}.$$  

The time-dependent force field (3) takes the following form:

$$F(x, t) = a x^2 - b x + A \sin(\Omega t).$$

Since already the analytical evaluation of such a force field’s periodic orbits is impossible, one has to recourse to numerical methods for the calculation of the quantities $\phi_{opt}$, $\kappa_{opt}$, and $\kappa_{opt}(t, D)$ appearing in the rate expressions (108-116). A convenient numerical strategy for doing so has been discussed in detail already in Sect. IV.H. The so obtained predictions for the time averaged rate (116) are compared in fig.7 against precise numerical results for a representative set of parameter values. Note that these parameter values are quantitatively very similar to those in fig.6, hence also the rates as a function of the noise strength $D$ are very similar. The agreement between the theoretical prediction and the practically exact numerical results is again excellent even for relatively large noise strengths $D$. However, in contrast to the piecewise parabolic case, the numerically accessible $D$-values are still not small enough in order to check the validity of our prediction (116) for the behavior of the relative error in the analytical approximation (110).

### VI. CONCLUSIONS

In our present work we have scrutinized by means of path integral methods the thermally activated escape of an overdamped Brownian particle over a periodically oscillating potential barrier in the most challenging regime of weak thermal noise in combination with moderately strong and moderately fast driving.

A first major result of our novel path-integral approach is the expression (51) for the instantaneous escape rate, which displays the suggestive general structure of “probability at the separatrix times velocity”. The summation appearing in this expression reflects the fact that several local minima of the relevant action in the path-integral formulation of the escape problem notably contribute
to the rate. In contrast to the undriven escape problem, giving rise to a (quasi-) Goldstone mode due to the (quasi-) time-translation symmetry, in our present case the paths corresponding to the local minima of the action are well separated and therefore admit a standard saddle point approximation of the path-integral for small noise strengths $D$. Pictorially speaking, by switching on the periodic driving, the continuous time-translation symmetry of the escape problem is reduced to a time-discrete one.

Our present explorations indicate that from the practical (technical) viewpoint, a path-integral approach which keeps an entire sum of possibly relevant contributions to the rate may be easier to handle than WKB-type or quasipotential-type methods [3,12], which operate with the concept of a single exponentially dominating weak-noise contribution and a single pre-exponential factor, both of them typically of a non-analytic nature.

The central result of our present paper represents the formula (108) for the instantaneous rate, supplemented by the result (123) for the “accuracy exponent” $\gamma$. The above discussed summation over the relevant local minima of the action resurfaces in all the equivalent alternative expressions (110,111,118) for $\kappa_{\text{opt}}(t,D)$ but drops out (can be performed) in the time averaged rate (116) due to the miraculous identity (113).

The rate expressions (108,116) share the general Arrhenius-type structure of the exponentially leading weak-noise contribution with the typical form of an equilibrium (undriven) rate (161). However, both, the Arrhenius factor and the pre-exponential contribution to the rates (108,116) now depend in a very complicated way on global features of the periodically oscillating potential (in contrast to the purely local properties $\Delta V = V(\bar{x}_u) - V(\bar{x}_s)$ and $V''(\bar{x}_s,u)$ governing (161)). Moreover, a non-trivial $\sqrt{D}$-dependence of the pre-exponential factor on the noise strength $D$ arises.

For the time averaged rate (116) we have shown in Appendix D that for asymptotically weak noise $D$ an invariance property holds under the supersymmetry transformation (16) without any further restrictions on the force field $F(x,t)$. Such an invariance property can be established rigorously on very general grounds [3,4] for force fields of the form $F(x,t) = -V'(x) + y(t)$ and arbitrary noise-strengths $D$.

The time averaged rate (116) displays a remarkable structural similarity with the rate-expressions obtained in [24] for one-dimensional discrete-time systems in the presence of weak Gaussian white noise. While a general qualitative connection between these two different types of escape problems via some kind of stroboscopic mapping is quite suggestive, the quantitative details are not so simple. Especially, the Gaussianity of the resulting noise after the stroboscopic mapping is crucial [51] but is far from obvious [52] for the rare but strong fluctuations (large deviations) which govern the escape events.

The conditions for the validity of our rate-formulas are (127) and that for a fixed (small) noise strength $D$, extremely weak, fast, and slow periodic driving forces should be excluded. Especially, the weak noise limit $D \to 0$ displays a rather intriguing non-interchangeability with the long-time limit $t \to \infty$ (see Sect. IV.F), and with the limits of asymptotically weak, fast, or slow driving.

In general, an action-integral remains to be minimized numerically and an ordinary linear differential equation of second order for the prefactor to be solved (Sect. IV.H) before actual numbers can be extracted from our rate-formulas. However, for the special case of a sinusoidally driven, piecewise parabolic metastable potential this entire program can be executed in closed analytical from. This example retains all the typical features of more general setups and exhibits excellent agreement with high-precision numerical results (Sect. V).

Conceptionally, our new path-integral approach should be of considerable interest for many related problems. Generalizations for higher dimensional systems and for non-periodic driving forces are currently under investigation [30]. Also the proper handling of the tantalizing weak, fast, and slow driving limits within a consistent path-integral formalism remains as an open problem for future research.

Finally, the complicated dependence of the rate on the global details of the oscillating potential poses an challenging inverse problem, namely to reconstruct the underlying force field from a given (e.g. measured) behavior of the escape rates.

Acknowledgement

This work has been supported by the DFG-Sachbeihilfe HA1517/13-2 (P.H., P.R.), the Sonderforschungsbereich 486 (P.H., J.L.), the Studienstiftung des deutschen Volkes (J.L.), and the Graduiertenkolleg GRK283 (P.H., P.R.).

APPENDIX A

To prove the equivalence of (24) and (30-32) one first needs the Hessian of the discrete-time action $S_N(x_0,\ldots,x_N)$ in (24), which is given by the $(N-1) \times (N-1)$-matrix

$$
\frac{\partial^2 S(x^*)}{\partial x_n^* \partial x_m^*} =
\begin{pmatrix}
2\Delta & -b_1 \\
-a_1 & -b_1 & a_2 \\
& -b_1 & \ddots & \ddots \\
& & \ddots & \ddots & -b_{N-2} & a_{N-1} \\
& & & -b_{N-2} & a_{N-1}
\end{pmatrix}
$$

(169)

where
\[a_n := 2 + \left[ 2F'(x_n, t_n) - (x_{n+1}^* - x_n^*) \right] F''(x_n^*, t_n) \Delta t
\]
\[+ \left[ (F'(x_n, t_n))^2 + F(x_n^*, t_n) F''(x_n^*, t_n) \right] \Delta t^2 \quad (170)\]
\[b_n := 1 + F'(x_n^*, t_n) \Delta t \quad (171)\]

For the prefactor \(Z_N(x^*)\) in \([28]\), the determinant of the matrix on the right-hand side of \([169]\) has to be evaluated. This can be done by a standard procedure (cf. \([39]\)). The result is a linear two-step recursion relation for the principal minor \(\hat{Q}_n^*\), consisting of the first \(n\) columns and rows of \([169]\), of the form
\[\hat{Q}_{n+1}^* = a_n \hat{Q}_n^* - b_{n-1}^2 \hat{Q}_{n-1}^* \quad (172)\]
for \(2 \leq n \leq N\), with initial conditions
\[\hat{Q}_1^* = 1, \quad \hat{Q}_2^* = a_1 \quad (173)\]

Comparing \([29, 32]\) with \([169]\), we observe that \(Q_N = \Delta t \hat{Q}_N^*\), and due to the linearity of \([172]\) we can conclude that \(Q_n^* := \Delta t \hat{Q}_n^*\), also obeys these equations. Therefore, using the definitions \([170, 171]\), it is readily shown that \(Q_n^*\) satisfies the required recursion relations \([30]\) together with the initial conditions \([31]\).

As a by-product, needed in Appendix B, we notice that by defining for \(1 \leq n < N\)
\[\mu_n := \frac{\hat{Q}_{n+1}^*}{\hat{Q}_n^*} = \frac{Q_{n+1}^*}{Q_n^*} \quad (174)\]
the linear two-step recursion relation \([172]\) can be rewritten as an equivalent non-linear one-step recursion
\[\mu_{n+1} = a_{n+1} - \frac{b_n^2}{\mu_n}, \quad \mu_1 = a_1 \quad (175)\]

The actual quantity of interest \(\hat{Q}_N^*\) then follows as
\[\hat{Q}_N^* = \prod_{n=1}^{N-1} \mu_n \quad (176)\]

APPENDIX B

In the following we derive Eq. \([50]\) by showing that
\[\frac{\partial^2 \phi_k(x_f, t_f)}{\partial x_f^2} = \frac{\hat{Q}_k(t_f)}{2 \hat{Q}_k(t_f)} - F'(x_f, t_f) \quad (177)\]
which gives together with \([37, 48]\) for \(t = t_f\) the desired result.

We work with the time-discrete version of the quantities in \([177]\) and consider in a first step the dependency of a minimizing path \(x^* = x^*(x_f)\) on the end-point \(x_f\) for fixed \(t_0, t_f,\) and \(x_0\). In order to complicate the notation as little as possible, we have left out the index \(k\) labeling the different paths \(x_k^*\).

Since the initial point \(x_0\) is kept fixed, we have that \(dx_0^*/dx_f = 0\). Further, we know from \([27]\) that for all \(n = 1, \ldots, N - 1\)
\[\frac{\partial S_N}{\partial x_n} \bigg|_{x^*(x_f)} = 0 \quad (178)\]
for any \(x_f\)-value, which implies after taking the derivative with respect to \(x_f\) that
\[\sum_{m=1}^{N-1} \frac{d x_m^*}{d x_f} \frac{\partial^2 S_N}{\partial x_m \partial x_n} \bigg|_{x^*(x_f)} = 0 \quad (179)\]

Using \([169]\), we thus get:
\[a_n \frac{d x_n^*}{d x_f} - b_{n-1}^2 \frac{d x_{n-1}^*}{d x_f} - b_n \frac{d x_{n+1}^*}{d x_f} = 0 \quad (180)\]

Introducing the new quantities \(\eta_n\) by
\[\eta_n := b_n \frac{d x_{n+1}^*}{d x_n^*} \quad (181)\]
one obtains from \([180]\) the one-step recursion relation
\[\eta_{n+1} = a_{n+1} - \frac{b_n^2}{\eta_n} \quad (182)\]
The corresponding initial condition follows from \([181]\) for \(n = 1\) by taking into account the above mentioned fact that \(dx_0^*/dx_f = 0:\)
\[\eta_1 = a_1 \quad (183)\]

Comparing \([176]\) with \([182, 183]\) yields \(\eta_n = \mu_n\) for \(n = 1, \ldots, N - 1\) and therefore, using the definitions \([171, 174, 181]\), for \(n = N - 1\)
\[\frac{Q_N^*}{Q_{N-1}^*} = \left[ 1 + F'(x_{N-1}, t_{N-1}) \right] \frac{dx_N^*}{dx_{N-1}^*} \quad (184)\]

In a next step an explicit expression for \(dx_N^*/dx_{N-1}^*\) in terms of well-known quantities has to be found. This can be achieved by taking the second derivative of the discrete-time action \(S_N(x^*(x_f))\) of the same minimizing path \(x^*\) as above with respect to the end-point \(x_f\). With \([178, 179]\), and the boundary conditions \([26]\) we find
\[\frac{d^2 S_N(x^*(x_f))}{d x_f^2} = \sum_{n=1}^{N} \frac{d x_n^*}{d x_f} \frac{\partial^2 S_N}{\partial x_n \partial x_N} \bigg|_{x^*(x_f)} \quad (185)\]
and using \([24]\) we can conclude that
\[\frac{dx_N^*}{dx_{N-1}^*} = \frac{1 + \Delta t \frac{d x_f}{d x_f} S_N(x^*(x_f))}{1 - 2 \Delta t \frac{d x_f}{d x_f} S_N(x^*(x_f))} \quad (186)\]
Together with \([184]\) we thus arrive at
\[\frac{Q_N^*}{Q_{N-1}^*} = \left[ 1 + \Delta t F'(x_{N-1}, t_{N-1}) \right] \frac{dx_N^*}{dx_{N-1}^*} \quad (187)\]
which with \([43]\) yields in the continuous-time limit the searched for relation \([177]\).
APPENDIX C

In this appendix we derive (107) for \( t \in [u, t_f] \), where the order of magnitude refers to the asymptotics with respect to \( k \).

As discussed below (83), the differences

\[
\delta x_k^*(t) := x_k^*(t) - x_{\text{opt}}^*(t + kT)
\]
\[
\delta p_k^*(t) := p_k^*(t) - p_{\text{opt}}^*(t + kT)
\]

(188)

(189)

rapidly decrease with increasing index \( k \) uniformly on the entire time-interval \([t_0, t_f]\). Our first conclusion is that the time-point \( t_u \) at which \( x_u^*(t) \) enters the neighborhood of \( x_u(t) \) depends itself on the index \( k \), basically it decreases like \(-kT\), see fig.3. On the other hand, the distance \( \Delta x_k^*(t_u) \) at which the path enters this neighborhood is, by definition, \( k \)-independent. The corresponding momentum \( p_k^*(t_u) \) is not strictly \( k \)-independent, but approaches an asymptotic \( k \)-independence for large \( k \). Similar conclusions apply for the time \( t_u \) at which \( x_u^*(t) \) leaves the neighborhood of the stable periodic orbit.

Next we can conclude from (71,76) that

\[\delta x_k^*(t) = \delta p_k^*(t) I_u(t) - \delta p_k^*(t) e^{2\Lambda_2(t,t_0)} I_s(t_u) .\]

(190)

Within the approximation (85) it follows that \( \delta p_k^*(t) = \delta x_k^*(t_u)/I_u(t_u) \). With these initial conditions at \( t = t_u \), the small perturbations \( \delta x_k^*(t) \) and \( \delta p_k^*(t) \) are then propagated according to (41,42) until \( t = t_u \). In linear order of these small perturbations it follows that \( \delta p_k^*(t_u)/\delta x_k^*(t_u) \) is an asymptotically \( k \)-independent constant, which, however, depends on all the details of the force field \( F(x,t) \) along the crossover segment of the master path \( x_{\text{opt}}^*(t) \).

The counterpart of (190) in the neighborhood of \( x_u(t) \) follows along the same line of reasoning, reading

\[\delta x_k^*(t) = -\delta p_k^*(t) I_u(t) + \delta p_k^*(t) e^{-2\Lambda_2(t,t_0)} I_s(t_u) .\]

(191)

Replacing on the right hand side \( e^{-2\Lambda_2(t,t_f)} \) by \( p_{\text{opt}}^*(t_f + kT)^2/p_{\text{opt}}^*(t + kT)^2 \) according to (51,52), choosing \( t = t_u \), and making use of (86,189) we can infer that

\[\delta x_k^*(t_u) + \delta p_k^*(t_u) I_u(t_u) = \frac{p_{\text{opt}}^*(t_u) I_u(t_f)}{[p_k^*(t_u) - \delta p_k^*(t_u)]^2} p_{\text{opt}}^*(t_k)^2 .\]

(192)

As we have just pointed out, the quantity \( \delta x_k^*(t_u) \) is proportional to \( \delta p_k^*(t_u) \) with an asymptotically \( k \)-independent proportionality constant that depends on the details of the force field \( F(x,t) \) along the crossover segment of the master path \( x_{\text{opt}}^*(t) \). In the generic case, this proportionality constant is thus not expected to coincide with \(-I_u(t_u)\) since the latter depends on the behavior of \( F(x,t) \) along the unstable periodic orbit \( x_u(t) \) only. Consequently, both \( \delta x_k^*(t_u) \) and \( \delta p_k^*(t_u) \) on the left hand side of (192) are, with respect to their \( k \)-dependence, of the same order of magnitude as the right hand side. Since \( p_k^*(t_u) \) is asymptotically \( k \)-independent and \( \delta p_k^*(t_u) \) tends to zero, we can infer from (192) that

\[\delta x_k^*(t_u) = O(p_{\text{opt}}^*(t_k)^2) , \quad \delta p_k^*(t_u) = O(p_{\text{opt}}^*(t_k)^2) .\]

(193)

With the initial conditions (83) it follows from (46) that the relative difference between \( Q_k^*(t_u) \) and \( Q_{\text{opt}}^*(t_u + kT) \) scales as a function of \( k \) like \( \delta x_k^*(t_u) \) and \( \delta p_k^*(t_u) \). With (193) this implies that

\[Q_k^*(t_u) = Q_{\text{opt}}^*(t_u + kT) [1 + O(p_{\text{opt}}^*(t_k)^2)] .\]

(194)

A similar relation follows for \( \dot{Q}_k(t_u) \) and thus for \( g_k^*(t_u) \) (cf. (48)), namely

\[g_k^*(t_u) = g_{\text{opt}}^*(t_u + kT) [1 + O(p_{\text{opt}}^*(t_k)^2)] .\]

(195)

Finally we conclude from (101) that

\[\frac{Q_k^*(t)}{Q_{\text{opt}}^*(t + kT)} = \frac{Q_k^*(t_u)}{Q_{\text{opt}}^*(t_u + kT)} \times \frac{1 - g_k^*(t_u) I_u(t_u,t)}{1 - g_{\text{opt}}^*(t_u + kT) I_u(t_u,t)} .\]

(196)

Like for \( \Delta x_k^*(t_u) \) and \( p_k^*(t_u) \) (see below (189)) one can convince oneself that also \( g_k^*(t_u) \) is asymptotically \( k \)-independent. With (194,195) the result (107) then follows from (190).

APPENDIX D

The purpose of this appendix is to verify that our expression (116) for the time-averaged rate is invariant with respect to the supersymmetry transformation (10). To this end, we first note that the path defined via

\[\tilde{x}_{\text{opt}}^*(t) := -x_{\text{opt}}^*(t) , \quad \tilde{p}_{\text{opt}}^*(t) := p_{\text{opt}}^*(t) \]

(197)

(198)

satisfies the Hamilton equations (11,12) for the supersymmetric partner field \( \tilde{F}(x,t) \) from eq. (10). Since the periodic orbits of this new force field are given by \( \tilde{x}(t) = -x_u(-t) \) and \( \tilde{x}_u(t) = -x_u(-t) \) (see Sect. I.C) one can readily see that \( \tilde{x}_{\text{opt}}^*(t) \) from (197) also obeys the boundary conditions (53) in the relevant limit \( t_f - t_0 \to \infty \). Hence we have found (up to the usual degeneracy with respect to time shifts by arbitrary multiples of \( T \)) the unique solution of the supersymmetric partner variational problem (54). Inserting \( \tilde{p}_{\text{opt}}^*(t) \) from (198) into the definitions (53) and (54) then leads to the following result:

\[\tilde{\phi}_{\text{opt}} = \phi_{\text{opt}} .\]

(199)

Somewhat more elaborate considerations are necessary in order to establish a corresponding identity for the prefactor \( \alpha_{\text{opt}} \) in (116). To this end we first consider two arbitrary but linear independent solutions \( Q_i(t) \) \((i = 1,2)\) of
the prefactor equation \( (198) \) for \( Q^*_{\text{opt}}(t) \). One can then easily verify that the prefactor \( Q^*_{\text{opt}}(t) \), which moreover has to fulfill the initial conditions \( (47) \) in the limit \( t_0 \to -\infty \), is given by

\[
Q^*_{\text{opt}}(t) = \lim_{t_0 \to -\infty} \frac{Q_1(t) Q_2(t_0) - Q_1(t_0) Q_2(t)}{W(t_0)}, \tag{200}
\]

with the Wronskian

\[
W(t) := Q_1(t) Q_2(t) - Q_1(t) Q_2(t). \tag{201}
\]

Due to \( (198) \) one can infer that

\[
\dot{W}(t) = 2W(t) F'(x^*_{\text{opt}}(t), t). \tag{202}
\]

With help of the Hamilton equation \( (41) \) it follows that

\[
p^*_{\text{opt}}(t)^2 W(t) = \text{const}. \tag{203}
\]

Turning now to the supersymmetric partner problem, it is readily seen that one obtains via \( \tilde{Q}_1(t) := Q_1(-t) \) two linear independent solutions of the prefactor equation \( (198) \) for the supersymmetric partner field \( (197) \) and the path given by \( (197) \). Thus we can use \( (198) \) and

\[
(200) \tag{200}
\]

(with tildes) to establish the identity

\[
\lim_{t \to -\infty} \tilde{Q}^*_{\text{opt}}(t) = \lim_{t_0 \to -\infty} \frac{p^*_{\text{opt}}(t)^2}{W(-t_0)} \times [Q_1(-t) Q_2(t_0) - Q_1(t_0) Q_2(-t)]. \tag{204}
\]

According to \( (203) \) we can now rewrite \( p^*_{\text{opt}}(-t)^2/W(-t_0) \) as \( p^*_{\text{opt}}(-t_0)^2/W(-t) \). Replacing \( t \to -t_0 \) and vice versa one can then conclude with help of \( (200) \) that

\[
\lim_{t \to -\infty} \tilde{p}^*_{\text{opt}}(t)^2 \tilde{Q}^*_{\text{opt}}(t) = \lim_{t \to -\infty} p^*_{\text{opt}}(t)^2 Q^*_{\text{opt}}(t). \tag{205}
\]

Hence we finally obtain

\[
\alpha_{\text{opt}} = \tilde{\alpha}_{\text{opt}}. \tag{206}
\]

This, in combination with \( (192) \), proves our proposition that the time-averaged rate \( (114) \) is invariant with respect to the supersymmetric transformation \( (14) \).

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FIG. 1. Sketch of a typical metastable potential $V(x)$ in eq. [3]. Plotted is the piecewise parabolic example [34] with parameters $\Delta V = 0.9$, $\lambda_s = -0.6$, and $\lambda_u = 0.3$ in arbitrary, dimensionless units.

FIG. 2. The supersymmetric partner potential $\tilde{V}(x) := -V(-x)$ of the potential $V(x)$ from fig.1.

FIG. 3. Solid: The paths $x^*_k(t)$, $k = 0, \ldots, K(t_f, t_0) = 3$ which minimize the action (36,37) with boundary conditions (53). Dashed: The associated “master paths” $x^*_\text{opt}(t + k T)$, implicitly defined via (54). Dotted: Stable and unstable periodic orbits $x_s(t)$ and $x_u(t)$ from (7). In this plot, $t_f - t_0$ has been chosen rather small. As $t_f - t_0$ increases, more and more intermediate paths $x^*_k(t)$ appear which better and better agree with their associated master paths $x^*_\text{opt}(t + k T)$. The depicted curves have been obtained for the additively driven piecewise parabolic potential [34] with parameters $A = 0.5$, $\Omega = 1$, $\lambda_s = -1$, $\lambda_u = 1$, $\Delta V = 1$, $t_0 = -12$, $t_f = 7.5$ (dimensionless units).
FIG. 4. Instantaneous rate $\Gamma(t)$ versus time $t$ for the piecewise linear force field \( (\ref{137}) \) with parameters $x_s = \lambda_s = -1$, $x_u = \lambda_u = 1$, $\Omega = 1$, and $A = 0.5$, corresponding to a static ($A = 0$) potential barrier $\Delta V = 1$ in \( (\ref{138},\ref{33}) \). Solid line: Analytical prediction \( (\ref{108}),\ref{118},\ref{150},\ref{151},\ref{154},\ref{155},\ref{159} ) \) by neglecting the $O(D^2)$-term in \( (\ref{108}) \). Dashed line: High-precision numerical results, obtained as described in Sect. V.B.

FIG. 5. (a) Arrhenius plot of the time-averaged rate $\bar{\Gamma}$ for the piecewise linear force field \( (\ref{137}) \) with the same parameters as in fig.4. Solid line: Analytical prediction \( (\ref{116}),\ref{154},\ref{159} ) \) by neglecting the $O(D^2)$-term in \( (\ref{116}) \). Crosses: High-precision numerical results, obtained as described in Sect. V.B. (b) Relative difference between analytical ($\bar{\Gamma}$) and numerical ($\bar{\Gamma}_{\text{num}}$) rate.

FIG. 6. Time-averaged rate $\bar{\Gamma}$ versus driving amplitude $A$ for the piecewise linear force field \( (\ref{137}) \) with parameters $x_s = \lambda_s = -1$, $x_u = \lambda_u = 1$, $\Omega = 1$, and $D = 0.05$. Solid line: Analytical prediction \( (\ref{116}),\ref{157},\ref{159} ) \) by neglecting the $O(D^2)$-term in \( (\ref{117}) \). Dotted line: Theoretical approximation \( (\ref{160})-(\ref{163}) \) according to Ref. \[23\] Crosses: High-precision numerical results, obtained as described in Sect. V.B. Inset: Magnification of the small-$A$ regime.

FIG. 7. Arrhenius plot of the time-averaged rate $\bar{\Gamma}$ for the cubic potential \( (\ref{164}) \) with parameters $a = 1/\sqrt{6}$, $b = 1$, $A = 0.5$, and $\Omega = 1$, corresponding to a static ($A = 0$) potential barrier $\Delta V = 1$ in \( (\ref{167}) \) and curvatures $|V''(\bar{x}_{s,u})| = 1$ in \( (\ref{166}) \). Solid line: Analytical prediction from \( (\ref{116}) \) without the $O(D^3)$-term by adopting the calculational procedure from Sect. IV.H. Crosses: High-precision numerical results, obtained as described in Sect. V.B.