Homology for Operator Algebras I: Spectral homology for reflexive algebras

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Abstract

A stable homology theory is defined for completely distributive CSL algebras in terms of the point-neighbourhood homology of the partially ordered set of meet-irreducible elements of the invariant projection lattice. This specialises to the simplicial homology of the underlying simplicial complex in the case of a digraph algebra. These groups are computable and useful. In particular it is shown that if the first spectral homology group is trivial then Schur automorphisms are automatically quasispatial. This motivates the introduction of essential Hochschild cohomology which we define by using the point weak star closure of coboundaries in place of the usual coboundaries.

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1 Introduction

A leading theme in operator algebra is the analysis of automorphisms and derivations, as well as related Hochschild cohomology. In the present paper we are concerned largely with a certain tractable and well-known class of reflexive operator algebras on separable Hilbert space, namely those with completely distributive commutative invariant projection lattice. Starting with the partially ordered set of meet-irreducible projections in the lattice - the crucial ingredient for the intrinsic spectral representation theory of [22] and [23] - we define new homology groups, $H^{sp}_*(A)$, which we call the (integral) spectral homology of $A$. In contrast to Hochschild cohomology for Banach algebras, the spectral homology is often instantly computable. Furthermore we can obtain a Kunneth formula for the spectral homology of spatial tensor products, together with a natural suspension formula. These formulae follow fairly directly from the corresponding ones in simplicial homology.

What is less clear, except in finite-dimensions (cf. Proposition 3.7 below), is the relationship between $H^{sp}_1(A)$ and derivations, automorphisms and Hochschild cohomology. However the main point and result of this paper is that there is a useful connection. Specifically we show that if $H^{sp}_1(A)$ is trivial then every Schur automorphism $\alpha$ relative to a fixed masa of $A$ is quasispatial in the sense of Gilfeather and Moore [11]. Equivalently, $\alpha$ lies in the closure of the inner Schur automorphisms with respect to the point weak star topology. Alternatively put, we have the implication

$$H^{sp}_1(A) = 0 \Rightarrow \text{Hoch}^1_{ess}(A) = 0$$

where $\text{Hoch}^1_{ess}(A)$ is what we call the first essential Hochschild cohomology group. The essential Hochschild cohomology arises by replacing the usual space of coboundaries by their point weak star closure. The implica-
tion gives an appealing intuitive coordinate-theoretic understanding of why
specific algebras may have trivial first Hochschild cohomology (essential or
ordinary, because the two often coincide). Such examples include nest alge-ras (cf. Lance [21]), and tree algebras [5], as well as various algebras arising
from tensor products, and the cone and join constructions considered by Gil-
feather and Smith [12],[13]. We expect that the implication is also valid for
the higher order groups.

Essential Hochschild cohomology does seem to be, in many respects, the
most natural form of Hochschild cohomology for CSL algebras. In the case
of the tridiagonal algebras of the form

\[
\mathcal{A} = \begin{bmatrix}
\mathcal{L}(\mathcal{H}_1) & \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1) & 0 \\
0 & \mathcal{L}(\mathcal{H}_2) & 0 \\
0 & \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3) & \mathcal{L}(\mathcal{H}_3) \\
& \ast & \\
& \ast & \ast \\
& & \ddots
\end{bmatrix}
\]

Gilfeather, Hopenwasser and Larson [11] have computed the nonzero group
Hoch\(^1\)(\(\mathcal{A}\)). On the other hand all automorphisms of \(\mathcal{A}\) are quasispatial (see
[11]). For this algebra we have \(H^{op}_1(\mathcal{A}) = 0\), a fact which is immediately
apparent since this homology group coincides with the simplical homology
of the infinite complex

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From our perspective it is this vanishing of the first spectral homology group
that is the underlying reason for the quasispatiality of automorphisms in this
case.
An important ingredient of the proof of the main result is the representation of maps \( \Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \) which are bimodule maps relative to a maximal abelian self-adjoint subalgebra. Such maps are sums of elementary ones of the form \( A \to CAD \) with \( C \) and \( D \) in the masa. This type of representation goes back to Haagerup [14] and a convenient analysis of these issues is given by Smith in [27]. Although such representations play a crucial role in the completely bounded cohomology of Christensen, Effros and Sinclair [2], and in the cohomology calculations of Gilfeather and Smith, they play a different role in the proof of our main result. Using this representation, together with a local multicaitivity property for a Schur automorphism \( \alpha \), we obtain an elementary local representation \( \alpha(A) = CAD \) for all operators \( \mathcal{A} \) in the local space \( Q_1\mathcal{L}(\mathcal{H})Q_3 \) associated with a triple \( Q_1 \prec Q_2 \prec Q_3 \) of comparable intervals of \( \text{Lat} \mathcal{A} \). Here \( C \) and \( D \) are determined up to a nonzero scalar. The hypothesis of vanishing spectral homology is just the condition needed in order to extend the local representations to global ones on large enough subalgebras of \( \mathcal{A} \) to guarantee quasispatiality. To perform this last step it is necessary to make use of the intrinsic spectral representation of \( \text{Lat} \mathcal{A} \) obtained in [22] together with the additional propositions obtained below in section 2. The proof of the intrinsic representation theorem is quite lengthy and technical, and this combines to make the proof of the main result here a long one. Certainly it would be interesting to obtain a shorter proof. Nevertheless, this technical complexity should not obscure the usefulness of the ideas of the main result in practice. Indeed, in most concrete examples the spectral representation and the associated technicalities given in section 2 are completely apparent. See, for example, the algebras of section 3.10 below.

The domain of \( H^s_{sp} \) is the class of reflexive operator algebras with completely distributive commutative invariant projection lattice. We leave open the problem of how to extend the domain to larger classes of non-self-adjoint
operator algebras. The most obvious task here is to define spectral homology for general CSL algebras, that is, to drop the complete distributivity requirement, but we also mention other possibilities.

In [26] we develop a quite different homology theory for general non-self-adjoint operator algebras. This is based upon the homology of complexes arising from partial isometries in the associated stable algebra and is somewhat $K$-theoretic in nature. In the final section we comment on the fact that the spectral homology groups need not coincide with these partial isometry homology groups.

The notation and terminology used below is fairly standard and in the main is coherent with that given in the books of Davidson [3] and Power [25]. In particular, on occasion we write $A(G)$ for the digraph algebra (finite-dimensional CSL algebra) associated with the finite directed transitive graph $G$.

The results in this paper have been incubating and hibernating over a good few years. I would like to thank Vern Paulsen and Ken Davidson for their support in this project, and also David Larson and Roger Smith for excellent hospitality in January 1993 when this work was all but completed.

2 Spectral invariants

All operator algebras discussed below are assumed to act on a separable Hilbert space. A CSL algebra is a reflexive operator algebra $\mathcal{A}$ for which the lattice of invariant (self-adjoint) projections $\mathcal{L} = \text{Lat}\mathcal{A}$ is a set of commuting projections. A CDCSL algebra is a CSL algebra for which $\mathcal{L}$ is completely distributive as an abstract lattice. In many respects this subclass is the most tractable family of non-self-adjoint operator algebras. See, for example, the discussions in [1], [19], [23], [15], [4], [5], [6] and [7].
The spectral representation theorem for (lattices of) CDCSL algebras, which is given in Power [23] and Orr and Power [22], uses as underlying partially ordered space the set $M(\mathcal{L})$ of meet-irreducible elements of $\mathcal{L}$, excluding the identity operator. The partial ordering is the natural one: $L_1 \leq L_2$ if and only if $L_1 L_2 = L_1$. By way of illustration consider the 4-cycle digraph algebra $\mathcal{A}$ in $M_4(\mathbb{C})$ which is associated with the pattern

$$\begin{pmatrix}
* & 0 & * & *
\end{pmatrix}.
$$

In particular $\mathcal{A}$ contains the 4 minimal projections $e_{11}, e_{22}, e_{33}, e_{44}$. Consider the meet-irreducible projections $E_{ii} = \sup\{L : L \in \text{Lat}\mathcal{A}, L e_{ii} = 0\}$. Thus $E_{11} = e_{22}$, $E_{22} = e_{11}$, $E_{33} = e_{11} + e_{22} + e_{44}$, $E_{44} = e_{11} + e_{22} + e_{33}$. In fact $M(\mathcal{L}) = \{E_{ii} : 1 \leq i \leq 4\}$ and the partially ordered set $(M(\mathcal{L}), \leq)$ can be viewed as a copy of the 4-cycle digraph

This digraph (the convention is to omit the edges with a single vertex) is more usually associated with $\mathcal{A}$ in terms of the minimal interval projections of $\mathcal{L}$ (the atoms of $\mathcal{L}$) with the algebraic partial ordering: $e_{ii} \prec e_{jj}$ if and only if $e_{ii}Ae_{jj} = e_{ii}M_4e_{jj}$. However, for algebras that are not purely atomic, such as the spatial tensor product $\mathcal{A} \otimes \mathcal{T}$, where $\mathcal{T}$ is a continuous nest algebra, this kind of simple association is not available.

For many specific algebras, such as those given at the end of section 3, it is usually straightforward to identify the invariant $M(\mathcal{L})$ and its partial ordering. The following proposition is also useful.
Proposition 2.1 Let $\text{Alg}(L)$ be a completely distributive CSL algebra. Then

(i) $M(L)$ is totally ordered if and only if $\text{Alg}L$ is a nest algebra.

(ii) $M(L)$ is path connected, when viewed as an infinite graph, if and only if the commutant of $A$ is trivial.

(iii) If $L_1 \otimes L_2$ is the spatial tensor product then $M(L_1 \otimes L_2) = M(L_1) \times M(L_2)$ with the product order.

Proof: (i) This follows from Theorem 2.2 below.

(ii) See Proposition 5.1 of [23].

(iii) The corresponding product structure for nonzero join-irreducible elements in a spatial tensor product has already been given in Lemmas 4.3 and 4.4 of [23]. A proof of (iii) follows on applying this to the complementary lattice of $L_1 \otimes L_2$. For completeness we give an independent proof below in the proof of the Kunneth formula.

The following construction of a CSL algebra in terms of a quadruple - the order-measure-multiplicity type - will feature in the fundamental representation theorem below.

Let $X$ be a partially ordered set with the order topology generated by the semi-open order intervals $[x, y)$, let $\mu$ be a Borel measure on $X$, let $\{E_k\}$ be a measurable partition of $X$ (by non-null sets), and let $(n_k)$ be a sequence of distinct numbers in $\mathbb{N} \cup \{\infty\}$. The range of the index $k$ may be finite. To the quadruple $\mathcal{X} = (X, \mu, \{E_k\}, (n_k))$ associate the CSL projection lattice $\mathcal{L}(\mathcal{X})$ on the Hilbert space

$$
\mathcal{H} = \sum_k \oplus (L^2(E_k, \mu) \otimes \mathbb{C}^{n_k})
$$

consisting of the projections

$$
\tilde{P}(B) = \sum_k \oplus (P(B) \otimes I_{n_k})|_{L^2(E_k, \mu) \otimes \mathbb{C}^{n_k}},
$$
where $B$ is a decreasing Borel subset of $X$, with associated projection $P(B)$ on $L^2(X, \mu)$. Of course the resulting lattice need not be completely distributive.

**Theorem 2.2** [22] Let $\mathcal{L}$ be a completely distributive separably acting commutative projection lattice. Let $X$ be the partially ordered set $\mathcal{M}(\mathcal{L})$ of meet-irreducible projections of $\mathcal{L}$, excepting the identity operator, carrying the order topology generated by the semi-intervals $[E, F)$. Then there is a Borel probability measure $\mu$ on $X$, and a multiplicity partition $\{E_k\}, (n_k)$, such that $\mathcal{L}$ is unitarily equivalent to the lattice $\mathcal{L}(X)$ associated with the quadruple $X = (X, \mu, \{E_k\}, (n_k))$.

The topology on $\mathcal{M}(\mathcal{L})$ plays a minor role in the theorem above, since it serves merely to locate the sigma algebra on which $\mu$ is defined. However, in what follows a topology on $\mathcal{M}(\mathcal{L})$ will be used in the definition of the spectral homology in terms of a point-neighbourhood homology and so we now give some attention to this.

The default topology on $\mathcal{M}(\mathcal{L})$ is defined to be that which is generated by the semi-intervals $[L, I)$ and their complements $[L, I)^c$, where $L$ ranges over all projections in the lattice $\mathcal{L}$. In particular, we can take a neighbourhood base for a projection $M$ in $\mathcal{M}(\mathcal{L})$ to be the family of subsets of $\mathcal{M}(\mathcal{L})$ of the form

$$[M, I) \cap [L_1, I)^c \cap \ldots \cap [L_n, I)^c$$

where $L_1, \ldots, L_n$ belong to $\mathcal{L}$ and $n$ is arbitrary. This topology is an algebra of sets, and in [22] it was shown that for a given separating vector $e$ for $\mathcal{L}$ there is a unique finitely additive measure $\mu$ on this algebra such that for all $L_1, \ldots, L_n$ in $\mathcal{L}$ we have
\[ \mu([L_1, I]^c \cap \ldots \cap [L_n, I]^c) = <L_1 L_2 \ldots L_n e, e> . \]

**Proposition 2.3** If \( B \subseteq M(\mathcal{L}) \) is open then \( \mu(B) > 0 \).

*Proof:* Let \( B = [M, I] \cap A \) where \( A = [L_1, I]^c \cap \ldots \cap [L_n, I]^c \). Then, for each \( k \), \( M^+ L_k \neq 0 \), and so \( M_k = M + M^+ L_k \) is strictly larger than \( M \). On the other hand if \( \mu(B) = 0 \) then \( \mu([M, I]^c \cap A) = \mu(A) \) and so from the definition of \( \mu \) and the fact that \( e \) is separating we have \( ML_1 L_2 \ldots L_n = L_1 L_2 \ldots L_n \). But now \( M = M_1 M_2 \ldots M_k \) contrary to the meet-irreducibility of \( M \). \( \square \)

**Proposition 2.4** Let \( A, B \) be Borel subsets of \( M(\mathcal{L}) \), with projections \( P(A), P(B) \) in \( \mathcal{L} \). Then \( \tilde{P}(A) \prec \tilde{P}(B) \) if and only if the set \( \{(x, y) : x \in A, y \in B, x \not\leq y\} \) has product measure zero.

*Proof:* Immediate from the definition of \( \tilde{P}(A) \) and \( \tilde{P}(B) \). \( \square \)

Let us write \( A \prec B \) if \( x \leq y \) for all \( x \) in \( A \) and \( y \) in \( B \). For the basic sets of the topology on \( M(\mathcal{L}) \) we have the following improvement of the last proposition.

**Proposition 2.5** Let \( A = [M, I] \cap C \) where \( C = [L_1, I]^c \cap \ldots \cap [L_n, I]^c \) and let \( B = [M', I] \cap C' \) with \( C' = [L'_1, I]^c \cap \ldots \cap [L'_n, I]^c \). Then \( \tilde{P}(A) \prec \tilde{P}(B) \) if and only if \( A \prec B \).
Proof: That \( A \prec B \) implies \( \hat{P}(A) \prec \hat{P}(B) \) is routine. For the converse direction notice that it is sufficient to prove that \( M \leq M' \). Indeed, if this is so then we deduce that \( M_1 \leq M' \) for all \( M_1 \) in \( A \) by noting that \([M_1, I) \cap C \prec B\). But if \( M \nleq M' \) then \( M' \in [M, I)^c \) and

\[
\hat{P}([M, I) \cap C)) \prec \hat{P}([M', I) \cap C' \cap [M, I)^c)
\]

where both these projections are nonzero, by Proposition 2.3. By Proposition 2.4 it follows that the set \( \{(x, y) : x \in [M, I), y \in [M, I)^c, x \leq y\} \) has positive product measure. But this is absurd since the set is empty. \( \square \)

The interpolation property of the following theorem is quite crucial to our proof of the main theorem of section 4 and it will also be used in the proof of the Kunneth formula. Note that if \( x, y \) belong to \( M(\mathcal{L}) \) and \( x \leq y \) then it is not necessarily true that there exist neighbourhoods \( U_x \) and \( U_y \) of \( x, y \) with \( U_x \prec U_y \). For this reason it seems that the proof of Theorem 2.6 must, inevitably, be somewhat nontrivial.

**Theorem 2.6** Let \( x, y \) be distinct points of \( M(\mathcal{L}) \) with neighbourhoods \( U_x, U_y \) with \( U_x \prec U_y \). Then there exist a point \( z \) in \( M(\mathcal{L}) \) and neighbourhoods \( V_x, V_y, V_z \) of \( x, y, z \) such that \( V_x \prec V_z \prec V_y \). Furthermore, \( z \) can be chosen in \( U_x \), and \( V_x \) and \( V_z \) can be chosen to be subsets of \( U_x \).

**Proof:** We may assume that \( U_x = [x, I) \cap C, U_y = [y, I) \cap C' \) with \( C, C' \) as in Proposition 2.4. If \( U_x \) is the singleton \( \{x\} \) then we may simply put \( z = x, V_x = V_z = U_x, V_y = U_y \).

Suppose that \( U_x \) is not a singleton and let \( \mathcal{L}_1 \) be the completely distributive projection lattice \( \hat{P}(U_x)\mathcal{L} \), with \( 0_1 \) and \( I_1 \) as the minimal and maximal projections. By the hypothesis, and since \( \mathcal{L} \) is completely...
distributive, there exists a projection $F \in \mathcal{L}_1$ such that $F_\prec \neq I_1$. The subset $[F_\prec, I_1)\subset M(\mathcal{L}_1)$ is nonvoid and so, by Theorem 2.2, there is a projection $M_1$ in $M(\mathcal{L}_1)$ with $F_\prec \leq M_1$. Note that from the definition of $F_\prec$ it follows that $[F, I_1)\prec \prec [F_\prec, I_1)$ and so $[F, I_1)\prec \prec [M_1, I_1)$. We now lift this comparability to $M(\mathcal{L})$ to get the desired new neighbourhoods of the points $x$ (the lift of $0_1$) and $z$ (the lift of $M_1$).

This lifting is obtained with the map $\Phi : M(\mathcal{L}_1) \to M(\mathcal{L})$ given by

$$\Phi(M) = \sup\{L \in \mathcal{L} : L\tilde{P}(U_x) = M\}.$$ 

Observe that the set $\Phi([F, I_1)\subset$ is equal to $[\Phi(F), I)^\subset \cap U_x$. To see this let $G \in [F, I_1)\subset$. Then the projection $K = G\tilde{P}$ is a nonzero projection in $\mathcal{L}_1$. If $L\tilde{P}(U_x) = G$ then $L\tilde{P}(U_x)\Phi(F) = L\tilde{P}(U_x)F = G\tilde{P} = K$, and so $\Phi(G)\tilde{P}(U_x)\Phi(F) = K$, and so $\Phi(G) \npreceq \Phi(F)$.

Thus we now have that the set $V_x = \Phi([F, I_1)\subset$ is an open set in $M(\mathcal{L})$ containing $x$, and is thus a neighbourhood of $x$. Since $\Phi$ is order preserving, $V_x \prec V_z$ where $V_z = \Phi([F_\prec, I_1))$ is a neighbourhood of $\Phi(M_1) = z$ say. Since $V_z \subseteq U_x$ we have $V_z \prec U_y$ and so, with $V_y = U_y$, the proof is complete. 

\[\square\]

3 Spectral Homology

First, we define a point-neighbourhood homology for partially ordered topological spaces.

Let $(X, \leq)$ be a separable topological space with an antisymmetric partial order. An edge of $(X, \leq)$, or topological edge, is an ordered pair $(x, y)$ of points of $X$ for which there exist neighbourhoods $U_x, U_y$ such that $s \leq t$ for all $s \in U_x$ and $t \in U_y$. If $F \subseteq X$ is a finite subset define the digraph $G(F)$
to have vertex set $F$ and directed edges $(x, y)$ where $x, y$ belong to $F$ and where $(x, y)$ is a topological edge of $(X, \leq)$. Associated with the undirected graph of $G(F)$ is a simplicial complex $\Delta(F)$ in which vertices correspond to 0-simplices, edges to 1-simplices, and where complete subgraphs on $t$ vertices correspond to $t - 1$ simplices. Define $H_n(\Delta(F))$ to be the usual integral simplicial homology of $\Delta(F)$ and note that if $F \subseteq G$ are finite sets then the inclusion $\Delta(F) \subseteq \Delta(G)$ gives a group homomorphism $H_t(\Delta(F)) \to H_t(\Delta(G))$ for each $t$.

**Definition 3.1** The point-neighbourhood homology of $(X, \leq)$ are the groups $H_t(X) = \varinjlim H_t(\Delta(F))$, $t = 0, 1, \ldots$, where the direct limit is taken over the net of finite subsets $F$ of $X$.

In other words, $H_t(X)$ is the integral simplicial homology of the infinite simplicial complex, $\Delta_{\text{top}}(X, \leq)$ say, arising from the topological edges of $(X, \leq)$. We refer to $\Delta_{\text{top}}(X, \leq)$ as the topological complex of $(X, \leq)$.

**Definition 3.2** The spectral homology of the reflexive operator algebra $\mathcal{A}$ associated with a completely distributive commutative projection lattice $\mathcal{L}$ is defined to be the point-neighbourhood homology $H_*(M(\mathcal{L}))$ of the partialy ordered set $M(\mathcal{L})$ of meet-irreducible projections (excluding the identity operator). The groups $H_t(M(\mathcal{L}))$ are also written as $H_t^{sp}(\mathcal{L})$ and $H_t^{sp}(\mathcal{A})$.

Write $H_t^{sp}(\mathcal{A}) = H_t^{sp}(\mathcal{L}) = H_t(M(\mathcal{L}))$, for $t \geq 0$.

We have chosen to define $H_*^{sp}(\mathcal{A})$ in terms of the topological complex $\Delta_{\text{top}}(M(\mathcal{L}), \leq)$ rather than the natural complex $\Delta(M(\mathcal{L}), \leq)$ of the partially ordered set $M(\mathcal{L})$. The reason for this is that this homology is adequate for
the proofs of Theorem 4.1 and 4.2, and it arises naturally in the arguments there involving comparability of interval projections. However, in all of the examples in this section the inclusion $\Delta_{\text{top}}(M(\mathcal{L}), \leq) \to \Delta(M(\mathcal{L}), \leq)$, when it is proper, nevertheless induces an isomorphism of homology, and so it would be interesting to know if this is a general phenomenon. (The proof of Theorem 3.9 provides a little affirmative evidence.)

**Proposition 3.3** $H_0^{sp}(\mathcal{A}) = \mathbb{Z}^d$ where $d$ is the linear space dimension of the commutant of $\mathcal{A}$.

**Proposition 3.4** If $\mathcal{A}$ is a nest algebra then $H_t^{sp}(\mathcal{A}) = 0$ for all $t > 0$.

**Proposition 3.5** Spectral homology is stable in the sense that

$$H_t^{sp}(\mathcal{A} \otimes \mathcal{L}(\mathcal{H})) = H_t^{sp}(\mathcal{A} \otimes M_k(\mathbb{C})) = H_t^{sp}(\mathcal{A})$$

for all $t \geq 0$.

**Proposition 3.6** If $\mathcal{A} = A(G)$ is a finite-dimensional digraph algebra, associated with the digraph $G$, then, for $t \geq 0$, $H_t^{sp}(\mathcal{A}) = H_t(\Delta(G_r))$, the simplicial homology of the simplicial complex of the reduced graph $G_r$ of $G$.

**Proposition 3.7** If $\mathcal{A}$ is an infinite tridiagonal algebra, as in the introduction, then $H_t^{sp}(\mathcal{A}) = 0$, for all $t > 0$.

**Proposition 3.8** Let $S\mathcal{A}$ denote the suspension of the CDCSL algebra $\mathcal{A}$ which is given by
where $B_1$ and $B_2$ are type I factors. Then $H_{t+1}^{sp}(SA) = H_t^{sp}(A)$ for $t \geq 1$.

Proofs: (3.3) This is a consequence of Proposition 2.1 (iii).

(3.4) By Proposition 2.1 (i) $M(\mathcal{L})$ is a totally ordered set when $\mathcal{L} = \text{Lat}A$ is a nest. Also the topology on $M(\mathcal{L})$ is the order topology and so the proposition follows.

(3.5) Immediate from the fact that $\text{Lat}(A \otimes \mathcal{L}(H))$ is isomorphic to $\text{Lat}A$.

(3.6) Let $\{e_{xy} : (x, y) \in E(G)\}$ be a (partial) system of matrix units for $A$, indexed by the edges of $G$. Define

$$L_x = \sup\{L \in \text{Lat}A : Le_{xx} = 0\}$$

and verify that $x \to L_x$ effects an isomorphism between $G_r = (V(G_r), E(G_r))$ and $(M(\mathcal{L}), \leq)$. (The reduced graph of $G$ is, roughly speaking, the antisymmetric graph obtained by collapsing each maximal complete subgraph of equivalent vertices to a single vertex.) In fact the proposition is also valid, with the same proof, for infinite digraph algebras - more usually referred to as purely atomic CSL algebras.

(3.7) $M(\text{Lat}A)$, as a graph, is an infinite (connected) chain whose consecutive proper edges have alternating direction. There is one end in the usual block staircase case, and no ends if the staircase is two-way infinite. Plainly such chains have trivial simplicial homology groups for $t > 0$. 

13
(3.8) Let \( L \in M(\text{Lat} \mathcal{A}) \). Then \( \tilde{L} = I \oplus I \oplus L \) belongs to \( M(\text{Lat}(\mathcal{S} \mathcal{A})) \).
Also the projections \( P = I \oplus 0 \oplus I \) and \( Q = 0 \oplus I \oplus I \) belong to \( M(\text{Lat} \mathcal{A}) \),
and \( M(\text{Lat}(\mathcal{S} \mathcal{A})) = \{P, Q\} \cup \{\tilde{L} : L \in M(\text{Lat} \mathcal{A})\} \). This partially ordered set
is isomorphic to the infinite digraph arising from the two point suspension \( S(M(\mathcal{L})) \) of the infinite digraph \( M(\mathcal{L}) \). Since \( H_{t+1}(S(G)) = H_t(G) \) for finite
digraphs (for \( t \geq 1 \)), by elementary simplicial homology, the conclusion follows readily. \( \square \)

Theorem 3.9 The spectral homology of the spatial tensor product \( \mathcal{A} \otimes \mathcal{A}' \)
of two CDCSL algebras is computable by the Kunneth formula:

\[
H_n^{sp}(\mathcal{A} \otimes \mathcal{A}') = \left( \sum_{p+q=n} (H_p^{sp}(\mathcal{A}) \otimes H_q^{sp}(\mathcal{A}')) \right) \oplus \left( \sum_{p+q=n-1} \oplus \text{Tor}(H_p^{sp}(\mathcal{A}), H_q^{sp}(\mathcal{A}')) \right).
\]

Proof: First, we identify the product structure of \( M(\mathcal{L}_1 \times \mathcal{L}_2) \), and we
begin by showing that if \( E \in \mathcal{L}_1 \otimes \mathcal{L}_2 = \mathcal{L} \) then \( E_+ \) has the form \( F_+ \otimes G_+ \)
for some \( F \) in \( \mathcal{L}_1 \) and \( G \) in \( \mathcal{L}_2 \). A lattice theoretic approach to this
product structure can be found in Fraser [8].

Choose \( F_\beta \in \mathcal{L}_1 \) and \( G_\beta \in \mathcal{L}_2 \), for \( \beta \) in some index set, so that \( E^\perp = \sup \{F^\perp_\beta \otimes G^\perp_\beta\} \). That this is possible follows from the fact that the
complementary lattice of \( \mathcal{L} \), the lattice of projections \( L^\perp \) with \( L \in \mathcal{L} \),
is the spatial tensor product of the complementary lattices of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \).

Set \( F = \inf \{F_\beta\} \), \( G = \inf \{G_\beta\} \). If \( H \in \mathcal{L}_1 \) and \( H \not\leq F \) then \( H \not\leq F_\beta \)
for some index \( \beta \) and so \( HF^\perp_\beta \not= 0 \). Thus \( (H \otimes I)(F^\perp_\beta \otimes G^\perp_\beta) \not= 0 \), and
so \( (H \otimes I)E^\perp \not= 0 \). That is, \( H \otimes I \not\leq E \). Taking the infimum over such
\( H \) obtain \( F_+ \otimes I \geq E_+ \). Similarly \( I \otimes G_+ \geq E_+ \), and so \( F_+ \otimes G_+ \geq E_+ \).

On the other hand, let \( K \not\leq E \) and express \( K^\perp \) as \( \sup \{L^\perp_\alpha \otimes M^\perp_\alpha\} \)
with \( \alpha \) ranging over some index set. Since \( K^\perp \not\leq E^\perp \) it follows that
\( F_\beta^\perp \otimes G_\beta^\perp \not\leq K^\perp \) for some index \( \beta \). Thus, for each index \( \alpha \) either
\( F_\beta^\perp \not\leq L^\perp_\alpha \) or \( G_\beta^\perp \not\leq M^\perp_\alpha \), that is, either \( L_\alpha \not\leq F_\beta \) or \( M_\alpha \not\leq G_\beta \) for each

14
Thus,
\[
K^\perp = ( \bigvee_{L_\alpha \not\leq F_\beta} L_\alpha^\perp \otimes M_\beta^\perp) \lor ( \bigvee_{M_\alpha \not\leq G_\beta} L_\alpha^\perp \otimes M_\beta^\perp)
\]
\[
\leq ( \bigvee_{L_\alpha \not\leq F_\beta} (L_\alpha \otimes I)^\perp) \lor ( \bigvee_{M_\alpha \not\leq G_\beta} (I \otimes M_\alpha)^\perp)
\]
\[
= ( \bigwedge_{L_\alpha \not\leq F_\beta} (L_\alpha \otimes I)^\perp) \lor ( \bigwedge_{M_\alpha \not\leq G_\beta} (I \otimes M_\alpha)^\perp)
\]
\[
\leq ((F_\beta)_+ \otimes I)^\perp \lor (I \otimes (G_\beta)_+)^\perp
\]

and so \( K \geq (F_\beta)_+ \otimes (G_\beta)_+ \geq F_+ \otimes G_+ \).

Now let \( L \in M(\mathcal{L}_1 \otimes \mathcal{L}_2) \) and set
\[
M_1 = \sup\{L_1 \in \mathcal{L}_1 : L_1 \otimes I \leq L\}
\]
\[
M_2 = \sup\{L_2 \in \mathcal{L}_2 : I \otimes L_2 \leq L\}
\]
so that, with tolerable abuse of notation, \( M_1 \lor M_2 \leq L \). Assume, by way of contradiction, that \( M_1 \lor M_2 \neq L \). Since \( \mathcal{L}_1 \otimes \mathcal{L}_2 \) is completely distributive there exists a projection \( E \) in \( \mathcal{L}_1 \otimes \mathcal{L}_2 \) such that \( E_+(M_1 \lor M_2)^\perp \neq 0 \) and \( E_+L_+ = 0 \). From the above, \( E_+ = F_+ \otimes G_+ \) for some \( F \in \mathcal{L}_1 \) and \( G \) in \( \mathcal{L}_2 \), and so \( 0 \neq (F_+ \otimes G_+)(M_1 \lor M_2)^\perp = (F_+ \otimes G_+)(M_1^\perp \otimes M_2^\perp) = F_+M_1^\perp \otimes G_+M_2^\perp \). Thus \( 0 \neq F_+M_1^\perp \) and \( 0 \neq G_+M_2^\perp \). But now \( L = (M_1 + F_+M_1^\perp)(M_2 + G_+M_2^\perp) \), and so \( L \) is not meet-irreducible, contrary to our assumption.

Finally, it is straightforward to show that if \( L_i \in M(\mathcal{L}_i) \), for \( i = 1, 2 \), then \( L_1 \lor L_2 \) is an element of \( M(\mathcal{L}_1 \otimes \mathcal{L}_2) \).

We may now identify \( M(\mathcal{L}_1 \otimes \mathcal{L}_2) \) and its ordering with the set \( M(\mathcal{L}_1) \times M(\mathcal{L}_2) \) with the product order. In particular the complex \( \Delta(M(L), \leq) \) coincides with the usual product complex \( \Delta(M(L_1), \leq) \times \Delta(M(L_2), \leq) \). On the other hand, because of the definition of the product topology, one sees that \( \Delta_{top}(M(L)) \) is naturally isomorphic to what we call the edge product complex.
\(\Delta_{\text{top}}(M(L_1)) \times_e \Delta_{\text{top}}(M(L_2))\). This is defined to be the complex which is determined by the digraph with the product vertex set \(M(L_1) \times M(L_2)\) and edges \(((x_1, y_1), (x_2, y_2))\) arising from the topological edges \((x_1, x_2), (y_1, y_2)\) of \(M(L_1)\) and \(M(L_2)\) respectively. Nevertheless we shall now show that the inclusion of \(\Delta_{\text{top}}(M(L_1)) \times_e \Delta_{\text{top}}(M(L_2))\) in \(\Delta_{\text{top}}(M(L_1)) \times \Delta_{\text{top}}(M(L_2))\) induces an isomorphism of homology. This will imply that

\[H^*_{\text{sp}}(A \otimes A') = H^*_{\text{sp}}(M(L)) = H_n(\Delta_{\text{top}}(M(L_1)) \times_e \Delta_{\text{top}}(M(L_2))) = H_n(\Delta_{\text{top}}(M(L_1)) \times \Delta_{\text{top}}(M(L_2))),\]

and the desired Kunneth formula then follows from the corresponding formula in simplicial homology.

To this end let \(F_i \subseteq L_i\) be finite sets determining the subcomplexes \(\Delta(F_i)\) of \(\Delta_{\text{top}}(M(L_i))\).

Each 1-simplex of \(\Delta(F_1) \times \Delta(F_2)\) which is not a 1-simplex of \(\Delta_{\text{top}}(M(L), \leq)\) comes from an edge of the form \(((x, y_1), (x, y_2))\) or \(((x_1, y), (x_2, y))\), where \(x\) (resp. \(y\)) is a point of \(F_1\) (resp. \(F_2\)) such that \(\{x\}\) (resp. \(\{y\}\)) is not a neighbourhood of \(x\) (resp. \(y\)). Refer to such edges, and their simplices, as the extremal edges and simplices of \(\Delta(F_1) \times \Delta(F_2)\). In the case of the extremal edge \(((x_1, y), (x_2, y))\), use Theorem 2.6 to choose a point \(y'\) in a neighbourhood of \(y\) and choose \(x'_2\) in a neighbourhood of \(x_2\) so that \((y, y')\) and \((x_2, x'_2)\) are topological edges. Then \(((x_1, y), (x'_2, y'))\) and \(((x_2, y), (x'_2, y'))\) are topological edges. Assume, moreover, that the point \((x_2, y)\) in \(F_1 \times F_2\) is maximal in the partial ordering, so that each 1-simplex of \(\Delta(F_1) \times \Delta(F_2)\), with \((x, y_2)\) as an endpoint, arises from an edge of the form above. By choosing \((x'_2, y')\) in a sufficiently small neighbourhood we can ensure that whenever \(((u, v), (x_2, y))\) is an edge corresponding to a 1-simplex of \(\Delta(F_1) \times \Delta(F_2)\) (even an extremal edge) then \(((u, v), (x'_2, y'))\) is a topological edge for \(M(L)\). The product complex \(\Delta(F_1) \times \Delta(F_2)\) is now chain homotopic to a subcomplex \(\Delta\).
of \( \Delta(F_1 \cup \{x_2\}) \times \Delta(F_2 \cup \{y\}) \) through the chain homotopy which moves \((x_2, y)\) to \((x'_2, y')\). The point of this homotopy is that the complex \( \Delta \) has fewer extremal edges. Note that maximal points of the type above always exist. Thus, repeating such homotopies a finite number of times, we find that there exist finite sets \( G_1 \) and \( G_2 \), containing \( F_1 \) and \( F_2 \), respectively, so that the product complex \( \Delta(F_1) \times \Delta(F_2) \) is homotopic in \( \Delta(G_1) \times \Delta(G_2) \) to a complex with no extremal edges. That is, \( \Delta(F_1) \times \Delta(F_2) \) is homotopic to a subcomplex of \( \Delta_{top}(M(\mathcal{L}), \leq) \), and the desired homology isomorphism follows. \( \square \)

**Examples 3.10 (a)** If \( \mathcal{A} \) is a tridiagonal algebra, as in the introduction, then \( H_1^{\text{sp}}(T\mathbb{R} \otimes \mathcal{A}) = 0 \).

(b) Let \( T_{[0,1]} \) be the Volterra nest algebra, that is, the nest algebra on \( L^2[0,1] \) associated with the nest of intervals \([0, t], 0 \leq t \leq 1\). Then \( H_1^{\text{sp}}(T_{[0,1]} \otimes A(D_4)) = \mathbb{Z} \). This a consequence of Propositions 3.6 and Theorem 3.9. However it is also possssible to obtain this by direct methods, as follows.

First, show directly that \((M(\mathcal{L}), \leq)\) is isomorphic to the product set \( X = [0, 1] \times \{1, 2, 3, 4\} \) with the natural product order: \((x, n) \leq (y, m)\) if and only if \( x \leq y \) and \((n, m)\) is an edge of the 4-cycle digraph \( D_4 \). Because of the nature of the product topology the topological edges are the pairs \(((x, n), (y, m))\) for which \( x < y \) and \((n, m)\) is an edge of \( D_4 \).

Next, let \( F = F_0 \times \{1, 2, 3, 4\} \) where \( F_0 \) is a finite subset of \([0, 1]\). It is an elemenary exercise in simplicial homology to show that the subcomplex \( \Delta(F) \) of the topological complex \( \Delta(X) \) has first simplicial homology group equal to \( \mathbb{Z} \). Furthermore the inclusion \( \Delta(F) \to \Delta(X) \) induces the identity map \( \mathbb{Z} \to \mathbb{Z} \). Since every finite subcomplex of \( \Delta(X) \) is contained in one of these special subcomplexes it follows that \( H_1^{\text{sp}}(\mathcal{L}) = \mathbb{Z} \).
Let $G$ be an antisymmetric digraph for which $|\Delta(G)|$, the geometric realisation of the complex of $G$, is the Klein bottle. Then $H_1(\mathcal{T}_{[0,1]} \otimes A(G)) = \mathbb{Z}_2$, and $H_1(\mathcal{T}_{[0,1]} \otimes A(D_4) \otimes A(G)) = \mathbb{Z}_2 \oplus \mathbb{Z}$.

Let $X = [-1, 1]^2$ be the closed square in $\mathbb{R}^2$ carrying the Borel measure $\mu$ which is the sum of Lebesgue area measure and arc length measure for the two diagonals. There is a partial ordering of $X$ which is suggested by the following diagram:

![Diagram](image)

That is, if points $P, Q$ belong to a triangle, say the triangle of $ABO$, then $Q \leq P$ if and only if the vector $PQ$ is equal to the vector $sAB + tBO$ for some $s, t \geq 0$. The triangles $CBO, CDO, ADO$ carry similar such orders, which are compatible on overlapping edges, and the partial ordering on $X$ is the union of these four partial orderings. The lattice of decreasing Borel sets for $(X, \leq, \mu)$ gives a completely distributive commutative lattice $\mathcal{L}$ of projections on $L^2(X, \mu)$, and an associated reflexive operator algebra $A$. It may seem curious, at first glance, that $H_{sp}^1(A) = \mathbb{Z}$, since there is no apparent hole to contribute 1-cycles that are not 1-boundaries. But the origin has measure zero, so it may be deleted, and the assertion now becomes plausible! In fact examination reveals that $(M(\mathcal{L}), \leq)$ is naturally isomorphic to the set

$$\{(-1, 1)^2 \setminus \{(0, 0)\} \cup \{(-1, 1), (1, 1), (-1, -1), (1, -1)\}$$

together with the relative ordering from $[-1, 1]^2$, already described, and the first point-neighbourhood homology group of this partially ordered set is $\mathbb{Z}$. 

18
Gilfeather and Smith [13] have obtained a Kunneth style formula for the Hochschild cohomology of the join of two operator algebras, $A \# B$. This is shown to be valid if one of the algebras acts on a finite-dimensional Hilbert space, and is shown to be false in general. For the spectral homology of CDCSL algebras the situation is much more straightforward. Also, we see that this context is considerably simpler than that of Theorem 3.9.

**Proposition 3.11** Let $A$ and $B$ be CDCSL algebras and let $A \# B$ be their join:

$$A \# B = \begin{bmatrix} B & 0 \\ \star & A \end{bmatrix}.$$  

Then

$$H_n(A \# B) = \bigoplus_{p+q=n-1} (H_p(A) \otimes H_q(B)).$$

**Proof:** It is elementary to verify that $M(\text{Lat}(A \# B))$ is the set of projections of the form $M \oplus I$ or $0 \oplus L$ where $L$ belongs to $M(\text{Lat}(A))$ and $M$ belongs to $M(\text{Lat}(B))$. It follows that the topological complex for the join algebra is precisely the simplicial complex join of the topological complexes for $A$ and $B$. Thus the desired formula follows from the corresponding formula in simplicial homology. \qed

We have seen that various constructions in simplicial homology, such as joins, suspensions and products, are also available for partially ordered measure spaces, and for CSL algebras. Similarly, the following somewhat unusual CSL algebra, a fibre sum of nest algebras, can be defined at the algebraic level, as a pull back, as below, or in terms of a fibre sum of the constituent partially ordered measure spaces. Such constructions can be used to create algebras, such as $(d)$ above, with interesting homology.
Example 3.12 Let \( T_\mu \) be the nest algebra on \( L^2([0,1],\mu) \) where \( \mu \) is equal to Lebesgue measure plus unit masses at 0 and at 1. Let \( T_\mu \oplus \mathbb{C} T_\mu \) be the fibre sum algebra associated with the summand maps \( T_\mu \to \mathbb{C} \) given by the compression maps for the atomic interval projections over 1. Let

\[
\mathcal{A} = (T_\mu \oplus \mathbb{C} T_\mu) \oplus \mathbb{C}^2 (T_\mu \oplus \mathbb{C} T_\mu)
\]

be the fibre sum algebra arising from summand maps

\[
T_\mu \oplus \mathbb{C} T_\mu \to \mathbb{C}^2
\]

corresponding to the compression maps for the atomic intervals associated with 0 in each copy of \([0,1] \). Then \( \mathcal{A} \) is a completely distributive CSL algebra for which \( M(\mathcal{L}) \), as a set, consists of four unit intervals joined at their endpoints to form a square. The partial ordering is that inherited from the four intervals and on consideration of the (infinite) topological complex of \( M(\mathcal{L}) \) it follows that \( H^1_{sp}(\mathcal{A}) = \mathbb{Z} \).

4 The Main Result

Let \( \mathcal{A} \) be a weak star closed operator algebra. Let \( Z^1(\mathcal{A}) \) denote the space of derivations that are weak star continuous and let \( B^1_{ess}(\mathcal{A}) \) denote the subspace of such derivations which are point weak star limits of inner derivations. Thus \( \delta \in B^1_{ess}(\mathcal{A}) \) if and only if for each weak star topology neighbourhood \( U \subseteq \mathcal{A} \) of zero and for each finite set \( a_1, \ldots, a_n \) in \( \mathcal{A} \) there exists an inner derivation \( \delta_0 \) such that \( \delta(a_i) - \delta_0(a_i) \in U \) for \( 1 \leq i \leq n \). We define the essential Hochschild cohomology group \( \text{Hoch}^1_{ess}(\mathcal{A}) \) to be the space \( Z^1(\mathcal{A})/B^1_{ess}(\mathcal{A}) \).

A Schur automorphism of \( \mathcal{A} \) is, by definition, an automorphism \( \alpha \) for which there exists a masa \( \mathcal{C} \) in \( \mathcal{A} \) which is fixed elementwise by \( \alpha \).
Theorem 4.1 Let $\mathcal{A}$ be a CDCSL algebra for which $H_{1}^{sp}(\mathcal{A}) = 0$. Then $Hoch_{\text{ess}}^{1}(\mathcal{A}) = 0$.

This theorem will be a consequence of the following closely related result.

Theorem 4.2 Let $\mathcal{A}$ be a CDCSL algebra with $H_{1}^{sp}(\mathcal{A}) = 0$ and let $\alpha$ be a Schur automorphism of $\mathcal{A}$ with respect to a masa $\mathcal{C}$. Then $\alpha$ is a point weak star limit of inner automorphisms of the form $A \to CAC^{-1}$ with $C \in \mathcal{C}$.

Fix $\mathcal{A}$ and $\mathcal{C}$ as in Theorem 4.2 and choose a spectral representation for $\mathcal{C}$ as follows. Identify the underlying Hilbert space $H$ with $L^{2}(m)$ for some finite measure space $(Y, \sigma, m)$ in such a way that $\mathcal{C}$ is identified with $L^{\infty}(m)$ by means of multiplication operators. The lattice $\mathcal{L} = \text{Lat}\mathcal{A}$ is a subset of $\mathcal{C}$, and we shall frequently consider partitions of the identity by the atoms $Q_{i}$ of a finite sublattice of $\mathcal{L}$. Such partitions correspond to particular finite measurable partitions of $Y$.

Lemma 4.3 If $Q_{1}$, $Q_{2}$, $Q_{3}$ are interval projections of $\mathcal{L}$ with $Q_{1} \prec Q_{2} \prec Q_{3}$ then there exist invertible operators $C_{1}$, $C_{3}$ in $\mathcal{C}$ such that

$$\alpha(Q_{1}AQ_{3}) = C_{1}Q_{1}AQ_{3}C_{3}$$

for all $A$ in $L(H)$.

Proof: Consider the Hilbert space decomposition $H_{1} \oplus \ldots \oplus H_{4}$ where
\[ H_i = Q_i H, \text{ for } 1 \leq i \leq 3, \] and \[ H_4 = Q_4 H \] with \[ Q_4 = I - Q_1 - Q_2 - Q_3. \] Then

\[
\alpha : \begin{bmatrix}
0 & X_1 & X_3 & 0 \\
0 & 0 & X_2 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & \alpha_1(X_1) & \alpha_3(X_3) & 0 \\
0 & 0 & \alpha_2(X_2) & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

where \( \alpha_1, \alpha_2, \alpha_3 \) are bimodule maps with respect to the pairs of masas \((Q_1 C, Q_2 C), (Q_2 C, Q_3 C)\) and \((Q_1 C, Q_3 C)\) respectively. By a result of Haagerup [14] each \( \alpha_i \) is completely bounded and has the form

\[
\alpha_i : X \rightarrow \sum_k \phi_{k,i} X_i \psi_{k,i},
\]

where \( \phi_{k,i} \) and \( \psi_{k,i} \) belong to the appropriate restriction of \( C \) and

\[
\left( \sum_k ||\phi_{k,i}||^2 \right)^{\frac{1}{2}} \left( \sum_k ||\psi_{k,i}||^2 \right)^{\frac{1}{2}} \leq ||\alpha_i||_{cb}.
\]

View the operators \( \phi_{k,i}, \psi_{k,i} \) as elements in \( L^\infty(m) \). From the inequality we see that the function \( \Phi_i(x, y) = \sum_k \phi_{k,i}(x) \psi_{k,i}(y) \) defines an element in \( L^\infty(m \times m) \). In particular the restriction of \( \alpha_i \) to the subspace of Hilbert-Schmidt operators coincides with the Schur multiplier of the representing kernel functions induced by \( \Phi \).

Since \( \alpha \) is an automorphism we have \( \alpha_1(X_1) \alpha_2(X_2) = \alpha_3(X_1 X_2) \). From this it follows, by considering the cases when \( X_1, X_2 \) have rank one, for example, that for almost every pair \((x, z)\)

\[
\Phi_1(x, y) \Phi_2(y, z) = \Phi_3(x, z)
\]

for almost every \( y \). Thus we obtain a factorisation \( \Phi_3(x, y) = \theta(x) \eta(y) \) with \( \theta, \eta \) in \( L^\infty(m) \) and hence the representation \( \alpha_3(X_3) = C_1 X_3 C_3 \),
at least for Hilbert Schmidt operators. But $\alpha$ is weak star continuous, by Theorem 2.2 of [4], and so this equality holds generally. Since $\alpha_3$ is bounded below, and $C_1, C_3$ are bounded, it follows that $C_1, C_3$ are invertible.

Suppose now that $Q_i \mathcal{L}(H) Q_i \subseteq \mathcal{A}$, for $i = 2, 3$, and that there exist $C_2 \in Q_1 \mathcal{C}, D_2 \in Q_2 \mathcal{C}, C_3 \in Q_1 \mathcal{C}, D_3 \in Q_3 \mathcal{C}$, such that these operators are invertible in their respective spaces, and

$$\alpha(Q_1 X Q_2) = C_2 Q_1 X Q_2 D_2,$$
$$\alpha(Q_1 Y Q_3) = C_3 Q_1 Y Q_3 D_3,$$

for all $X, Y$ in $\mathcal{L}(H)$. Then we claim that $C_2 = \lambda C_3$ for some nonzero scalar $\lambda$. To see this we make use of the following well-known facts.

**Lemma 4.4** Let $\mathcal{B}$ be a CDCSL algebra with invariant projection lattice $\mathcal{L}$.

(i) If $L \in \mathcal{L}$ and if $R = x \otimes y^*$ is a nonzero rank one operator with range vector $x \in L \mathcal{H}$ and domain vector $y \in (I - L_-) \mathcal{H}$ then $R \in \mathcal{B}$.

(ii) $\sup\{L \in \mathcal{L} : I - L_\perp \neq 0\} = I$.

To establish the claim note that the algebra $\mathcal{B}_1 = Q_1 \mathcal{A} Q_1$ is a CDCSL algebra on $Q_1 \mathcal{H}$ and let $R = x \otimes y^*$ be a nonzero rank one operator in $\mathcal{B}_1$. If $X = Q_1 X Q_2$ and $Y = Q_1 Y Q_3$ then, since $\alpha$ is an automorphism, the following equations hold:

$$C_2 R X D_2 = \alpha(R) C_2 X D_2,$$
$$C_3 R Y D_3 = \alpha(R) C_3 Y D_3$$
From the first equation it follows that for all choices of \( X \) the range of \( \alpha(R)C_2XD_2 \) is \( \{ \lambda C_2x : \lambda \in \mathbb{C} \} \). Since \( C_2 \) and \( D_2 \) are invertible on their respective spaces \( \alpha(R) \) is necessarily of rank one. We remark at this point that an automorphism of a CDCSL algebra need not preserve rank - see [11]. Considering both equations, obtain \( C_2x = \lambda_x C_3x \) for some nonzero scalar \( \lambda_x \). The obvious manipulations show that if \( x_1 \) and \( x_2 \) are linearly independent then \( \lambda_{x_1} = \lambda_{x_2} \), and if \( y = \mu x \) then \( \lambda_y = \lambda_x \).

Thus, using the second part of the lemma, obtain \( C_2 = \lambda C_3 \) for some scalar \( \lambda \), and the claim is proven.

Suppose now that \( Q_1 \prec Q_2 \prec Q_3 \) and it is known that for any \( X \) of the form \( X = Q_1XQ_2 \) and for any \( Y \) of the form \( Q_2YQ_3 \) that \( \alpha(X) = C_1XD_1 \), \( \alpha(Y) = C_2YD_2 \) with \( C_1 \in Q_1\mathcal{C}, D_1, C_2 \in Q_2\mathcal{C} \) and \( D_2 \in Q_3\mathcal{C} \), where each operator is invertible in its respective space. Suppose further, that it is known that \( \alpha(Z) = C_3ZD_3 \) when \( Z = Q_1ZQ_3 \), with \( C_3 \in Q_1\mathcal{C}, D_3 \in Q_3\mathcal{C} \) invertible. Then since \( \alpha \) is an automorphism,

\[
C_3XYD_3 = C_1XD_1C_2YD_2.
\]

From this equality it follows readily that \( C_2 = \lambda D_1^{-1} \) for some nonzero scalar \( \lambda \).

Motivated by the relationships above between the various local implementing operators we now formulate a general lemma which is exactly suited to our needs.

**Lemma 4.5** Let \( Y = \{ x_1, \ldots, x_m \} \) be a finite set with an antisymmetric partial ordering. Let \( G_1, \ldots, G_m \) be abelian groups with \( \mathcal{C}_i \subseteq G_i \) for all \( i \) and suppose that for each pair \( x_i \prec x_j \) there is associated a pair \( (c_{ij}, d_{ij}) \) with \( c_{ij} \in G_i \) and \( d_{ij} \in G_j \) such that the following properties
hold:

(i) If \( x_i \prec x_j \) and \( x_i \prec x_k \) then \( c_{ij} = \lambda c_{ik} \) for some \( \lambda \in \mathbb{C}_* \).

(ii) If \( x_i \prec x_j \) and \( x_k \prec x_j \) then \( d_{ij} = \lambda d_{kj} \) for some \( \lambda \in \mathbb{C}_* \).

(iii) If \( x_i \prec x_j \prec x_k \) then \( d_{ij} = \lambda c_{jk} \) for some scalar \( \lambda \in \mathbb{C}_* \).

(iv) If \( x_i \prec x_j \prec x_k \) and the pair \( (c_{ij}, d_{ij}) \) is replaced by a scalar multiple so that in (iii) the scalar \( \lambda \) is unity, then, for some scalar \( \mu \in \mathbb{C}_* \) we have \( c_{ik} = \mu c_{jk} \) and \( d_{ik} = \mu d_{jk} \).

Suppose further, that \( X \subseteq Y \) is a subset such that the natural map \( H_1(\Delta(X)) \to H_1(\Delta(Y)) \) is the zero map. Then there exists a choice of elements \( g_i \) in \( G_i \), for all \( x_i \in X \), such that for each pair we have \( (c_{ij}, d_{ij}) = (\lambda_{ij}g_i, \lambda_{ij}g_j) \) for some \( \lambda_{ij} \in \mathbb{C}_* \).

**Proof:** Let \( (X, \prec) \) be viewed as a graph. We may assume that it is connected. Let \( \tau \) be a maximal tree in \( X \). Fix an edge \( (x_1, x_k) \) in \( \tau \) and define \( g_1 = d_{1,k} \). Using the edges of \( \tau \), and properties (i), (ii) and (iii), define \( g_i \) recursively for all the vertices \( x_i \) of \( X \). In the process of doing this, whenever property (iii) is used replace the ‘new’ pair by a scalar multiple so that the scalar \( \lambda \) of (iii) is unity.

Let \( C_1(\Delta(Y)) \) be the group of 1-chains of the complex \( \Delta(Y) \). If \( (x_i, x_j) \) (with \( x_i \prec x_j \)) is a 1-simplex of \( \Delta(Y) \) then by (i), (ii) and (iii) we know that

\[
(c_{ij}, d_{ij}) = (\alpha g_i, \beta g_j)
\]

for some scalars \( \alpha, \beta \in \mathbb{C}_* \). Define the group homomorphism \( \Phi : C_1(\Delta(Y)) \to \mathbb{C}_* \) by taking the unique extension of the correspondences \( \Phi((x_i, x_j)) = \alpha \beta^{-1} \). By (iv) we have \( \Phi(\partial \sigma) = 1 \) if \( \sigma \) is a 2-simplex of \( \Delta(Y) \). It follows then that \( \Phi(w) = 1 \) whenever \( w \) is a 1-boundary.
Let \((x_i, x_j)\) be an edge of \((X, \prec)\) which is not an edge of \(\tau\). We are required to prove that \(\alpha = \beta\) in this case. But since \(\tau\) is a maximal tree, there is a 1-cycle \(w\) for \(\Delta(X)\) consisting of the simplex \((x_i, x_j)\) and (distinct) 1-simplexes from \(\tau\). Thus \(\Phi(w) = \Phi((x_i, x_j)) = \alpha\beta^{-1}\). By the hypothesis \(w\) is a boundary, and so, using the previous paragraph, \(\alpha = \beta\). \(\square\)

**The proof of Theorem 4.2**

Let \(L\) be represented as in Theorem 2.2. Let \(Q = \{U_1, \ldots, U_n\}\) be a partition of \(M(L)\) generated by basic clopen sets, and choose \(x_i \in U_i\) for each \(i\). Let \(Q_i = \tilde{P}(U_i)\) and associate with \(Q\) the subalgebra \(A(Q)\) of \(A\) given by

\[
A(Q) = \text{span} \{Q_i\mathcal{L}(H)Q_j : Q_i \ll Q_j\}
\]

where \(Q_i \ll Q_j\) if and only if there exists \(Q_k\) with \(Q_i \prec Q_k \prec Q_j\). By Lemma 4.3 for each such pair there exists an operator pair \((C_{ij}, D_{ij})\) with \(C_{ij} \in Q_i\mathcal{L}, D_{ij} \in Q_j\mathcal{L}\), such that \(\alpha(A) = C_{ij}AD_{ij}^{-1}\) for all operators \(A\) with \(A = Q_iAQ_j\).

We show that the restriction \(\alpha|A(Q)\) is inner. Let \((X, \ll)\) be the set \(X\) with the partial order inherited from \(Q\). View the associated complex \(\Delta(X, \ll)\) as a subcomplex of the topological complex \(\Delta_{\text{top}}(M(L), \leq)\). Then, by the spectral homology hypothesis the natural inclusion induced map

\[
H_1(\Delta(X, \ll)) \to H_1^{sp}(A)
\]

is the zero map and so there exists a finite subset \(Y_0\) of \(M(L)\) containing \(X\) such that the natural map
\[ H_1(\Delta(X, \triangleleft)) \to H_1(\Delta_{top}(Y_0, \leq)) \]

is the zero map. By Theorem 2.5 there is a finer partition associated with a finite set \( Y \) containing \( Y_0 \) such that \( \Delta_{top}(Y_0, \leq) \) is a subcomplex of \( (\Delta(Y, \triangleleft)) \). Thus, taking compositions, it follows that the natural map

\[ H_1(\Delta(X, \triangleleft)) \to H_1(\Delta(Y, \triangleleft)) \]

is the zero map.

Recall, from the discussion preceding Lemma 4.5 that the pairs of operators \((C_{ij}, D_{ij})\) that are associated with the partition satisfy the requirements (i), (ii), (iii), (iv) of Lemma 4.5. It follows from the lemma that there is a choice of non-zero scalars \( \lambda_{ij} \) and invertible operators \( C_i \) in the algebra \( Q_i C \), for \( i = 1, \ldots, m \), such that

\[(C_{ij}, D_{ij}) = (\lambda_{ij} C_i, \lambda_{ij} C_j)\]

for all \( Q_i \ll Q_j \). It now follows that \( \alpha|A(Q) \) is inner.

Finally we show that \( \alpha \) is a point weak star limit of inner automorphisms.

Consider a chain of partitions \( Q_1 \subseteq Q_2 \subseteq \ldots \) with associated subalgebra chain \( \{A(Q_k)\} \). Then the restriction of \( \alpha \) to \( A(Q_k) \) has the form \( A \to C_k A C_k^{-1} \), for some invertible operator \( C_k \) in \( \mathcal{C} \). Let

\[ P_k = \sup\{Q_i, Q_j : Q_i \ll Q_j, \ Q_i, Q_j \in Q_k\}. \]

Then we can arrange that \( C_{k+1} \) extends \( C_k \) in the sense that \( C_{k+1} P_k = C_k P_k \). To be more precise about this, let us restrict attention in the remainder
of the proof to the case of $\mathcal{A}$ irreducible. The general case then follows readily. By Proposition 5.1 of [23] $(X, \leq)$ is a connected binary relation. It follows that having chosen $C_1$ subsequent choices of the $C_k$ are naturally uniquely determined on the projection $P_k$. One way to see this is to note that Theorem 2.6 implies that projections $Q_i \in \mathcal{Q}_k$ and $Q_j \in \mathcal{Q}_1$ have subprojections $E$ and $F$, respectively, which are $\ll$-connected. Note that it follows that the bounded operator $C_k$ implements $\alpha$ on the space $P_k \mathcal{A} P_k$ as well as on the smaller subalgebra $\mathcal{A}(\mathcal{Q}_k)$.

Suppose for the moment that the union of the algebras $\mathcal{A}(\mathcal{Q}_k)$ is weak star dense in $\mathcal{A}$. We show that if $A \in \mathcal{A}$ and $\phi$ is a weak star continuous functional, then for $\epsilon > 0$ there exists an invertible operator $D$ in $\mathcal{C}$ such that

$$|\phi(\alpha(A)) - \phi(DAD^{-1})| < \epsilon.$$ 

In view of the hypothesised weak star density, the increasing projections $P_k$ converge to the identity operator in the strong operator topology. It follows that there is a densely defined (possibly) unbounded operator $C$ whose domain is the linear span of the ranges of the projections $P_k$, such that $CP_k = C_k P_k$. Let $E_{1k}, E_{2k}$ and $E_{3k}$ be the spectral projections for $|C|$ for the sets $[0, k^{-1})$, $[k, \infty)$ and $[k^{-1}, k)$ respectively. These are projections in $\mathcal{C}$ and the sequences $E_{1k}$ and $E_{2k}$ converge to zero in the strong operator topology. Consider the sequence of operators $D_k$ in $\mathcal{C}$ given by

$$D_k = k^{-1}E_{1k}^1 + kE_{2k}^2 + CE_{3k}^3.$$ 

Let $\phi$ be a weak star continuous linear linear functional on $\mathcal{A}$, so that $\phi(A) = \text{trace}(TA)$ for some trace class operator $T$. For fixed $A \in \mathcal{A}$ and $\epsilon > 0$ choose $k_0$ large enough so that

$$|\phi(\alpha(A)) - \phi(\alpha(E_{3k}^3 AE_{3k}^3))| < \epsilon/2.$$
for all \( k > k_0 \). (Recall that \( \alpha \) is automatically weak star continuous.) We shall show that by increasing \( k \), if necessary, we have

\[
|\phi(D_k A D_k^{-1}) - \phi(D_k E_k^3 A E_k^3 D_k^{-1})| < \epsilon/2,
\]

from which the desired conclusion follows, since \( D_k E_k^3 A E_k^3 D_k^{-1} = \alpha(E_k^3 A E_k^3) \).

It suffices to show that for large \( k \)

\[
|\text{trace}(XD_k E_k^3 A E_k^3 D_k^{-1})| < \epsilon/16
\]

for the eight pairs \((i, j)\) with \((i, j) \neq (3, 3)\). For \((i, j) = (1, 1)\) or \((2, 2)\) the quantity is simply \(|\text{trace}(X E_k^i A E_k^j)|\) and so these cases are clear. For the case \( i = 2 \) and \( j = 3 \) observe that

\[
D_k E_k^2 A E_k^3 D_k^{-1} = k E_k^2 A C^{-1} E_k^3 = (C_1 C E_k^2) E_k^2 A E_k^3 (C^{-1} E_k^3) = C_1 \alpha(E_k^2 A E_k^3)
\]

where \( C_1 \) is a contraction such that \( C_1 C E_k^2 = k E_k^2 \). Consequently this case is also clear, since \( X E_k^2 \rightarrow 0 \) in the trace class norm and the equality above shows that the operators \( D_k E_k^2 A E_k^3 D_k^{-1} \) are uniformly bounded by \( \|\alpha\| \|A\| \).

The case \((i, j) = (1, 3)\) is similar. The cases \((3, 2), (3, 1)\) and \((1, 2)\) are elementary, and so it remains to consider \( i = 2 \) and \( j = 1 \). In this case it can be seen that for \( k > \|\alpha\|^2 \) the operator \( E_k^2 A E_k^1 \) is zero. This need only be observed for rank one operators \( A \) in \( \mathcal{A} \), since their linear span is dense in the weak operator topology, and, in view of Lemma 4.4 (i), this verification is elementary.

To finish the proof we confirm the technical detail that a subalgebra chain \( \mathcal{A}(Q_k) \) can be found, with dense union.

In view of the separability of the underlying Hilbert space there is a countable family \( L_1, L_2, \ldots \) such that each projection \( L \) in \( \mathcal{L} \) is both
the supremum and infimum of projections in the family \( \{L_k\} \). Let \( Q_k \) be the partition of \( X \) generated by \( L_1, \ldots, L_k \). It will be enough to show that each rank one operator \( R \) is in the norm closure of the associated algebras \( \mathcal{A}(Q_k) \).

By Lemma 4.4, \( R = x \otimes y^* \) and there is a projection \( L \) in \( \mathcal{L} \) such that \( Lx = x \) and \( (I - L_\perp)y = y \). By norm approximation we may reduce to the case that both \( L \) and \( L_\perp \) belong to \( \{L_k\} \). We find a particular projection \( L_\epsilon \leq L \), with \( L_\epsilon \) also in \( \{L_k\} \), so that if \( x_\epsilon = L_\epsilon x \) then \( R_\epsilon = x_\epsilon \otimes y^* \) is a rank one operator close to \( R \) and lying in one of the algebras \( \mathcal{A}(Q_k) \). We do with the following argument borrowed from the proof of Lemma 2.7 of [22].

Since \( \mathcal{L} \) is completely distributive we have, by Lemma 2.3 of [22] for example,

\[
L = \sup\{G_+: L \not\leq G_+, G \in \mathcal{L}\}
\]

for all \( L \) in \( \mathcal{L} \) with \( L \neq 0 \). Readjusting our choice of \( \{L_k\} \) if necessary, we may assume that \( L \) is in fact the supremum of projections \( G_+ \) which belong to \( \{L_k\} \). Thus we may choose \( G_+^1, \ldots, G_+^n \) in \( \{L_k\} \) so that the projection \( L_\epsilon = \sup\{G_+^i : 1 \leq i \leq n\} \) determines the rank one operator \( R_\epsilon \) as above with \( \|R_\epsilon - R\| < \epsilon \). Note that \( [H, I]^- \prec [H_-, I] \) for any projection \( H \) and so

\[
[G_+^i, I]^- \prec [L, I]^- \cap [G_+^i, I]^- \prec [L_-, I]
\]

for each \( i \). Furthermore the middle sets here are nonempty. Thus by writing \( L_\epsilon x \) as a sum of vectors in the ranges of the projections \( G_+^i \) we see that \( R_\epsilon \) belongs to the algebra \( \mathcal{A}(Q) \) associated with the partition generated by \( [L_-, I] \) and \( [G_+^i, I]^- , 1 \leq i \leq n \).
The proof of Theorem 4.1

Let $\delta$ be a weak star continuous derivation of $\mathcal{A}$. We wish to show that $\delta \in B^1_{ess}(\mathcal{A})$. By a standard argument of Kadison and Ringrose [17] we may assume that $\delta(C) = 0$ if $C \in \mathcal{C}$ where $\mathcal{C}$ is a fixed masa in $\mathcal{A}$. Let $\alpha$ be the automorphism $\exp(\delta)$ of $\mathcal{A}$. Calculation shows that it is a Schur automorphism of $\mathcal{A}$ with respect to $\mathcal{C}$.

Let $\mathcal{A}(Q) \subseteq \mathcal{A}$ be a subalgebra associated with a finite partition as in the last proof. Then, by the arguments above, the restriction $\alpha_Q = \alpha|\mathcal{A}(Q)$ is inner and has the form $\alpha_C$ where $\alpha_C(A) = CAC^{-1}$ for all $A$ in $\mathcal{A}(Q)$, where $C$ is an invertible element of $\mathcal{C}$. Let $D$ be a logarithm for $C$ in $\mathcal{C}$ associated with the inner derivation $\delta_D$. Then $\exp(\delta_D) = \alpha_D = \alpha_C = \exp(\delta_Q)$, where $\delta_Q$ is the restriction $\delta|\mathcal{A}(Q)$. (Our assumption for $\delta$ implies that the weakly closed $\mathcal{C}$-bimodules in $\mathcal{A}$ are invariant for $\delta$.) Since $\delta_Q$ commutes with $\delta_D$ we deduce that $\exp(\delta_D - \delta_Q) = \text{id}$, and hence that $\delta_D = \delta_Q$. It now follows, as in the last proof, by the density of the algebras $\mathcal{A}(Q)$, that $\delta$ is a point weak star inner automorphism.

Remark 4.6 We conjecture that the higher order analogue of Theorem 4.1 also holds, that is,

$$H^sp_n(\mathcal{A}) = 0 \Rightarrow \text{Hoch}^n_{ess}(\mathcal{A}) = 0.$$ 

This is known to be true in the case of digraph algebras. In fact if $H^sp_n(A(G)) = 0$ then $H_n(\Delta(G_r)) = 0$, by Proposition 3.7. Thus $H^n(\Delta(G_r)) = 0$ by the duality for simplicial cohomology, and so Hoch$^n(A(G)) = 0$ by the cohomological identifications given in Gerstenhaber and Schack [7] and Kraus and Schack [18].

The analysis above complements some of the results of Gilfeather and Moore in [11]. They have shown, in particular, that if $\alpha$ is an automorphism
of any CDCSL algebra $\mathcal{A}$ then the following conditions are equivalent.

(i) $\text{rank}(\alpha(R)) = \text{rank}(R)$ for all finite rank operators in $\mathcal{A}$.

(ii) $\alpha$ is quasispatial in the sense that there is a closed injective linear transformation $T : \mathcal{H}_1 \to \mathcal{H}_2$, whose range and domain are dense, such that $\alpha(A)Ty = TAy$ for all $y$ in the domain of $T$.

Let $\alpha$ be a Schur automorphism, which is pointwise weakly inner in the sense of Theorem 4.2. Then $\alpha$ must preserve the rank of finite rank operators and so it follows from Theorem 4.2 and the Gilfeather-Moore result that if $H^sp_1(\mathcal{A}) = 0$ then all Schur automorphisms are quasispatial. In fact we can see this directly in the proof of Theorem 4.2 where the possibly unbounded implementing operator for the automorphism is constructed.

In the other direction it seems plausible that if $H^sp_1(\mathcal{A}) \neq 0$ then there exists a Schur automorphism of $\mathcal{A}$ which is not quasispatial. More generally, it would be interesting to determine whether there is a converse implication to the one conjectured above.

5 Final Remarks

It would be interesting to see to what extent it is possible to develop a homology theory for general CSL algebras which is based upon spectral invariants. One difficulty appears to be that the unitary invariants for general commutative projection lattices, as given by Arveson in [1] for example, do not have the explicit intrinsic nature as those of Theorem 2.2. Ideally one would wish to develop a general theory capable of calculations of the homology of even infinite tensor products. Another natural direction is, of course, to extend the spectral homology invariants to other classes of reflexive operator algebras. One can envisage that such a development is possible for classes of
projection lattices generated by CDCSL lattices together with ”amenable”
lattices, which are not necessarily commuting, through the natural opera-
tions - products, fibre sums, joins, and so forth - at the algebraic level. For
example, if \( \mathcal{A}_1 \) is a CDCSL algebra and \( ca_2 \) is the reflexive algebra asso-
ciated with two projections in generic position, (see Lambrou and Longstaff
[20] for example) then the vanishing of Hochschild cohomology of \( \mathcal{A}_1 \otimes \mathcal{A}_2 \)
should be a consequence of vanishing spectral homology.

In [26] we develop a stable homology theory for general non-self-adjoint
operator algebras which is based on partial isometries normalising a given
masa. Roughly speaking, if \( \mathcal{A} \) is an operator algebra with masa \( \mathcal{C} \) then
the stable homology group \( H_1(\mathcal{A};\mathcal{C}) \) is an abelian group associated with
cycles of partial isometries in the algebras \( M_n \otimes \mathcal{A} \) which normalise \( D_n \otimes \mathcal{C} \).
By hypothesis such partial isometries must form part of a complete matrix
unit system. The precise definition of \( H_n(\mathcal{A};\mathcal{C}) \) is analogous in spirit to the
definition of the \( K_0 \) -group. In fact if the inclusion \( \mathcal{C} \to C^*(\mathcal{A}) \) induces
a regular surjection \( K_0 \mathcal{C} \to K_0(C^*(\mathcal{A})) \) then \( H_0(\mathcal{A};\mathcal{C}) \) is often equal to
\( K_0(C^*(\mathcal{A})). \)

In a CDCSL algebra a masa is unique up to inner unitary equivalence,
and so in this case stable homology is an invariant for the algebra and we
may denote the groups simply as \( H_n(\mathcal{A}) \). The algebra \( \ell^\infty \) of diagonal
operators relative to an orthonormal basis is a CDCSL algebra and in this
case the first stable homology group \( H_0(\ell^\infty) \) is equal to \( K_0(\ell^\infty) \). On the
other hand \( H_n^{sp}(\ell^\infty) \) is simply the restricted direct product \( \mathbb{Z}^{\infty} \).

Another contrast, of a different and perhaps more significant nat-
ure, is that \( H_n^{sp}(\mathcal{A}) \) is computed purely in terms of the structure of \( \mathcal{L} = \text{Lat}\mathcal{A}, \)
and so spectral homology takes no account of the fact that \( cl \) may have
atoms of both finite and infinite rank. Stable homology on the other hand
is based on cycles of partial isometries of the same rank (in the case of
CSL algebras) and so from this point of view provides a more discriminating invariant.

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