In this article, we introduce a systematic and uniform construction of non-singular plane curves of odd degrees \( n \geq 5 \) which violate the local-global principle. Our construction works unconditionally for \( n \) divisible by \( p^2 \) for some odd prime number \( p \). Moreover, our construction also works for \( n \) divisible by some \( p \geq 5 \) which satisfies a conjecture on a \( p \)-adic property of the fundamental unit of the \( \mathbb{Q}(p^{1/3}) \). This conjecture is a natural cubic analogue of the classical Ankeny-Artin-Chowla-Mordell conjecture for \( \mathbb{Q}(p^{1/2}) \) and easily verified numerically.

1. Introduction

In the theory of Diophantine equations, the local-global principle for quadratic forms established by Minkowski and Hasse is one of the major culminations (cf. [26, Theorem 8, Ch. IV]).

In contrast, there exist many homogeneous forms of higher degrees which violate the local-global principle (i.e., counterexamples to the local-global principle). For example, Selmer [25] found that a non-singular plane cubic curve defined by

\[
3X^3 + 4Y^3 = 5Z^3
\]

has rational points over \( \mathbb{R} \) and \( \mathbb{Q}_p \) for every prime number \( p \) but not over \( \mathbb{Q} \). From eq. (1), we can easily construct reducible (especially singular) counterexamples of higher degrees.

After that, Fujiwara [11] found that a non-singular plane quintic curve defined by

\[
(X^3 + 5Z^3)(X^2 + XY + Y^2) = 17Z^5
\]

violates the local-global principle. More recently, Nguyen [21, 22] gave recipes for counterexamples of even degrees.

In [23], Poonen and Voloch made a qualitative conjecture on an old folklore that most hypersurfaces of degree \( n \geq d + 3 \) in the projective space \( \mathbb{P}^{d+1} \) violate the local-global principle. Probably based on this folklore and the Poonen-Voloch conjecture, there are many works for the existence and the proportion of counterexamples in certain classes [4, 6, 8, 10, 13, 23]. Among them, Dietmann and Marmon [10, Theorem 2] proved that,
under the abc conjecture \([19]\), 100% of non-singular plane curves of degree \(k\)

\[AX^k + BY^k = CZ^k \quad (A, B, C \in \mathbb{Z} \setminus \{0\})\]

violate the local-global principle for every \(k \in \mathbb{Z}_{\geq 6}\). However, their argument cannot give any specific (conjectural) counterexamples because the abc conjecture is ineffective to estimate the candidates of \(A, B, C\). It may be surprising that there seems to be no other concrete counterexamples to the local-global principle for non-singular plane curves of odd degrees \(\geq 5\) than \([11,12]\).

In this article, we exhibit how to construct such counterexamples of various odd degrees in a systematic and uniform manner. The following Theorem 1.1 is the main theorem of this article. We should emphasise that although it is unclear from the statement, in the proof, we shall exhibit how to generate parameters in the following equations, i.e., we can generate as many as we want explicit counterexamples as in eqs. (1) and (2).

**Theorem 1.1.** Let \(p\) be a prime number, and \(\epsilon = \alpha + \beta p^{1/3} + \gamma p^{2/3} \in \mathbb{R}_{>1}\) be the fundamental unit of \(\mathbb{Q}(p^{1/3})\) with \(\alpha, \beta, \gamma \in \frac{1}{2}\mathbb{Z}^{\oplus 3}\). Set

\[
\epsilon = \begin{cases} 
1 & \text{if } \beta \not\equiv 0 \pmod{p} \text{ or } \beta \equiv \gamma \equiv 0 \pmod{p} \\
2 & \text{if } \beta \equiv 0 \pmod{p} \text{ and } \gamma \not\equiv 0 \pmod{p} 
\end{cases}.
\]

Let \(n \in \mathbb{Z}_{>5}\) be an odd integer divisible by \(p^4\). Then, there exist infinitely many \(\frac{n-3}{3}\)-tuples of pairs of integers \((b_j, c_j)\) \((1 \leq j \leq \frac{n-3}{3})\) such that for each tuple there exist infinitely many \(L \in \mathbb{Z}\) such that the equation

\[
(X^3 + p\epsilon Y^3) \prod_{j=1}^{\frac{n-3}{3}} (b_j^2 X^2 + b_j c_j XY + c_j^2 Y^2) = LZ^n
\]

define non-singular plane curves which violate the local-global principle. Moreover, for each \(n\), there exist infinitely many such curves geometrically non-isomorphic to each other.

In particular, if \(\beta \equiv 0 \pmod{p}\) for a prime number \(p\), then we can generate infinite family of explicit counterexamples for every odd degree \(n \equiv 0 \pmod{p}\). The authors conjecture that this hypothesis is always true whenever \(p \not\equiv 3\).

**Conjecture 1.2.** Let \(p \not\equiv 3\) be a prime number, and \(\epsilon = \frac{1}{2}(\alpha + \beta p^{1/3} + \gamma p^{2/3}) \in \mathbb{R}_{>1}\) be the fundamental unit of \(\mathbb{Q}(p^{1/3})\) with \(\alpha, \beta, \gamma \in \mathbb{Z}\). Then, we have \(\beta \not\equiv 0 \pmod{p}\).

In fact, Conjecture 1.2 is a natural cubic field analogue of the following more classical conjecture for the real quadratic field \(\mathbb{Q}(p^{1/2})\),\(^2\) whose origin goes back to Ankeny-Artin-Chowla \([1]\) for \(p \equiv 1 \pmod{4}\) and Mordell \([20]\) for \(p \equiv 3 \pmod{4}\) respectively:

**Conjecture 1.3.** Let \(p \not\equiv 2\) be a prime number, and \(\epsilon = \frac{1}{2}(\alpha + \beta p^{1/2}) \in \mathbb{R}_{>1}\) be the fundamental unit of \(\mathbb{Q}(p^{1/2})\) with \(\alpha, \beta \in \mathbb{Z}\). Then, we have \(\beta \not\equiv 0 \pmod{p}\).

A key ingredient of our construction is the following theorem on the distribution of prime numbers represented by binary cubic polynomials:

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1 Here, we change the coefficients of \(\epsilon\) so that \(\alpha, \beta, \gamma \in \mathbb{Z}\) in order to emphasise the analogy. On the other hand, in the proof of Theorem 1.1, \(\alpha, \beta, \gamma \in \frac{1}{2}\mathbb{Z}\) is better because we can unify the three cases (i) \(p \equiv \pm 2, \pm 4 \pmod{9}\), (ii) \(p \equiv \pm 1 \pmod{9}\), and (iii) \(p = 3\). Note also that Conjecture 1.2 holds if and only if \(\mathbb{Q}(p^{1/3})\) has a unit \(a_0 + \beta_0 p^{1/3} + \gamma_0 p^{2/3}\) with \(a_0, \beta_0, \gamma_0 \in \frac{1}{2}\mathbb{Z}\) such that \(\beta_0 \not\equiv 0 \pmod{p}\). The authors verified Conjecture 1.2 for all prime numbers \(p < 10^5\) by Magma \([7]\), which cost just two hours. For the detail, see Appendix.

2 For numerical verification of Conjecture 1.3 see e.g. \([28,29]\).
In what follows, for each prime number $p$ let $\gamma_1, \gamma_2 \in \mathbb{Z}^{\oplus 2}$, and $\gamma_0$ be the greatest common divisor of the coefficients of $f_0(px + \gamma_1, py + \gamma_2)$. Set $f(x, y) := \gamma_0^{-1}f_0(px + \gamma_1, py + \gamma_2)$. Suppose that $\gcd(f(\mathbb{Z}^{\oplus 2})) = 1$. Then, the set $f(\mathbb{Z}^{\oplus 2})$ contains infinitely many prime numbers.

In §2, we give a recipe which exhibits how to construct counterexamples to the local-global principle as in eq. (3) from certain Fermat type equations and prime numbers. These objects are generated in completely explicit manners via Theorem 1.4 in §3 and §4 respectively. In §4, the proofs of Theorem 1.4 is done by combining these arithmetic objects with a geometric argument (Lemma 4.1) on the non-isomorphy of complex algebraic curves defined by eq. (3). In §5, we demonstrate how our construction works for each given degree by exhibiting concrete examples of degree 7, 9, and 11. In Appendix, we explain how we can verify Conjecture 1.2 numerically for each given prime number $p$ by Magma [7].

It is fair to say that thanks to Theorem 1.4 (and Lemma 3.4), which is one of the culminations of highly sophisticated modern analytic number theory, our proof of Theorem 1.1 is relatively elementary and almost covered by a standard first course of algebraic number theory (as in e.g. [15, Ch. I – Ch. V], [17, Ch. I – Ch. V], and [24]). Moreover, after one admits Theorem 1.1, it is quite easy to generate as many as she/he wants explicit counterexamples to the local-global principle

We would like to conclude this introduction with a comment on the style of our proof. Each step of our proof (e.g. Lemmas 2.2 and 2.3 and Theorem 3.1) can be easily refined to more powerful forms. However, if we state them in a full generality and give their proofs, the amount of this article should get manyfold. In order to keep it as readable as possible, we state each proposition in a so restricted form that is sufficient to prove Theorem 1.1 with a complete recipe of how to generate parameters in eq. (3).

2. Construction from prime numbers and Fermat type equations

In this section, we prove the following proposition, which gives explicit counterexamples to the local-global principle of degree $n \equiv 0 \pmod{p}$ under the assumption that we have

- sufficiently many prime numbers of the form $p^ib_j^3 + c^3$ with $b, c \in \mathbb{Z}$ and
- integers $L$ such that the equation $x^3 + p^iy^3 = Lz^n$ has a specific property.

In what follows, for each prime number $l$, $v_l(n)$ denotes the additive $l$-adic valuation of $n \in \mathbb{Z}$.

Proposition 2.1. Let $p$ be a prime number, $n \in \mathbb{Z}_{\geq 5}$ be an odd integer divisible by $p$, and $b_1, \ldots, b_{n-3}, c_1, \ldots, c_{n-3}, L \in \mathbb{Z}$ such that

1. $p^ib_j^3 + c_j^3 \equiv 2 \pmod{3}$ is a prime number $\neq p$ for every $j$,
2. $L = \prod_{i \equiv 2 \pmod{3}} l_{v_i(L)}$ with $v_l(L) < n,$
3. $\gcd(L, b_jc_j) = 1$ for every $j$,
4. $L \equiv \prod_j b_j^3 \not\equiv 0 \pmod{3}$ and $\sum_j b_j^{-1}c_j \not\equiv 0 \pmod{3}$ if $p \not\equiv \pm 1 \pmod{9}$,
5. $L \equiv \prod_j b_j^3 \not\equiv 0 \pmod{3}$ and $\sum_j b_j^{-1}c_j \not\equiv 0 \pmod{3}$ if $p \equiv 2 \pmod{3}$, and
6. for every primitive triple $(x, y, z) \in \mathbb{Z}^{\oplus 3}$ satisfying $x^3 + p^iy^3 = Lz^n$, there exists a prime divisor $l$ of $L$ such that $x \equiv y \equiv 0 \pmod{l}$.

---

3 For example, if we make effort for 3-adic and $p$-adic solubility in the proof of Lemma 2.2 then we can relax the conditions on the coefficients and generalize our construction so that it includes the constructions of [11,12]. Moreover, the use of Theorem 1.4 in the proof of Theorem 3.1 is not essential. In fact, we can prove that a positive proportion of prime numbers $l$ can be used to generate the equations in Theorem 3.1 by using class field theory (cf. [17,27]).
Then, the equation
\[(X^3 + p'Y^3) \prod_{j=1}^{n-3} (b_j^2X^2 + b_jc_jXY + c_j^2Y^2) = LZ^n\]
violates the local-global principle.

This is a consequence of the following two lemmas.

**Lemma 2.2** (local solubility). Let \( p \) be a prime number, \( n \in \mathbb{Z}_{\geq 5} \) be an odd integer, \( \iota = 1 \) or 2, and \( b_1, \ldots, b_{n-3}, c_1, \ldots, c_{n-2}, L \in \mathbb{Z} \) such that

1. \( b_1, c_1 \neq 0 \),
2. \( L \equiv \prod_j b_j^2 \not\equiv 0 \pmod{3} \) and \( \sum_j b_j^{-1}c_j \neq 0 \pmod{3} \) if \( p \not\equiv \pm 1 \pmod{9} \), and
3. \( L \equiv \prod_j b_j^2 \not\equiv 0 \pmod{p} \) and \( \sum_j b_j^{-1}c_j \neq 0 \pmod{p} \) if \( p \equiv 2 \pmod{3} \).

Then, the equation
\[F(X,Y,Z) := (X^3 + p'Y^3) \prod_{j=1}^{n-3} (b_j^2X^2 + b_jc_jXY + c_j^2Y^2) - LZ^n = 0\]
has non-trivial solutions over \( \mathbb{R} \) and \( \mathbb{Q}_l \) for every prime number \( l \).

**Proof.** We prove this statement along Fujiwara’s argument in [11]: Since \( \iota = 1 \) or 2, and \( b_1, c_1 \neq 0 \), the minimal splitting field of \( (X^3 + p'Y^3)(b_1^2X^2 + b_1c_1XY + c_1^2Y^2) \) is a Galois extension over \( \mathbb{Q} \) whose Galois group is isomorphic to the symmetric group of degree 3. In particular, the residual degree at every prime number is 1, 2, or 3. Moreover, since this extension is unramified outside 3 and \( p \) (and \( \infty \)), the polynomial \( (X^3 + p'Y^3)(b_1^2X^2 + b_1c_1XY + c_1^2Y^2) \) has a linear factor over \( \mathbb{Q}_l \) for every prime number \( l \neq 3, p \). Therefore, the assertion follows if one notes the following facts:

1. \( X^3 + p'Y^3 \) is decomposed in \( \mathbb{R}[X,Y] \).
2. If \( p \equiv \pm 1 \pmod{9} \), then \( X^3 + p'Z^3 \) is decomposed in \( \mathbb{Z}_3[X,Y] \). On the other hand, if \( p \not\equiv \pm 1 \pmod{9} \), then since \( F(1,0,1) \equiv \prod_j b_j^2 - L \equiv 0 \pmod{3} \), and \( \frac{\partial F}{\partial Y}(1,0,1) \equiv (\prod_j b_j^2) \cdot (\sum_j b_j^{-1}c_j) \equiv 0 \pmod{3} \), we obtain a 3-adic lift of mod 3 solution \( (1,0,1) \) by Hensel’s lemma.
3. If \( p \equiv 1 \pmod{3} \), then \( b_1^2X^2 + b_1c_1XY + c_1^2Y^2 \) is decomposed in \( \mathbb{Z}_p[X,Y] \). On the other hand, if \( p \equiv 2 \pmod{3} \), Then, since \( F(1,0,1) \equiv \prod_j b_j^2 - L \equiv 0 \pmod{p} \) and \( \frac{\partial F}{\partial Y}(1,0,1) \equiv (\prod_j b_j^2) \cdot (\sum_j b_j^{-1}c_j) \equiv 0 \pmod{p} \), we obtain a \( p \)-adic lift of mod \( p \) solution \( (1,0,1) \) by Hensel’s lemma.

This completes the proof. \( \square \)

**Lemma 2.3** (global unsolvability). Let \( n \in \mathbb{Z}_{\geq 1} \) be an odd integer, \( a, b_1, \ldots, b_{n-3}, c_1, \ldots, c_{n-2}, L \in \mathbb{Z} \) such that

1. \( ab_j^2 + c_j^3 \equiv 2 \pmod{3} \) is a prime number prime to \( a \) for every \( j \),
2. \( L = \prod_{l \equiv 2 \pmod{3}} \nu_l(L) \) with \( \nu_l(L) < n \),
3. \( \gcd(L, b_jc_j) = 1 \) for every \( j \), and
4. for every primitive triple \( (x, y, z) \in \mathbb{Z}^{\geq 3} \) satisfying \( x^3 + ay^3 = Lz^n \), there exists a prime divisor \( l \) of \( L \) such that \( x \equiv y \equiv 0 \pmod{l} \).

Then, there is no triple \( (X, Y, Z) \in \mathbb{Z}^{\geq 3} \) satisfying

\[(X^3 + aY^3) \prod_{j=1}^{n-3} (b_j^2X^2 + b_jc_jXY + c_j^2Y^2) = LZ^n.\]

\footnote{In this article, we say that a triple \( (x, y, z) \in \mathbb{Z}^{\geq 3} \) is primitive if \( \gcd(x, y, z) = 1 \).}
Proof. We prove the assertion by contradiction. Let \((X, Y, Z) \in \mathbb{Z}^{3}\) be a triple satisfying eq. [4]. We may assume that it is primitive. It is sufficient to deduce that
\[
\gcd((X^3 + aY^3)L, b_j^2X^2 + b_jc_jXY + c_j^2Y^2) = 1 \quad \text{for every } j.
\]
Indeed, if eq. (5) holds, we have some divisor \(z\) of \(Z\) satisfying \(X^3 + aY^3 = Lz^n\). Hence, the fourth assumption implies that \(X \equiv Y \equiv 0 \pmod{l}\) for some prime divisor \(l\) of \(L\). However, since \(v_l(L) < n\), we have \(Z \equiv 0 \pmod{l}\), which contradicts \(\gcd(X, Y, Z) = 1\). In what follows, we deduce eq. (5).

First, suppose that a prime divisor \(q\) of \(X^3 + aY^3\) divides \(b_j^2X^2 + b_jc_jXY + c_j^2Y^2\) for some \(j\). Then, \(q\) also divides
\[
b_j^2(X^3 + aY^3) - (b_jX - c_jY)(b_j^2X^2 + b_jc_jXY + c_j^2Y^2) = (ab_j^3 + c_j^3)Y^3.
\]
Since \(\gcd(X, Y, Z) = 1\) and \(v_q(L) < n\), we see that \(Y \not\equiv 0 \pmod{q}\). Hence, by the first assumption, we have \(q = ab_j^3 + c_j^3 \equiv 2 \pmod{3}\). In particular, the polynomial \(b_j^2T^2 + b_jc_jT + c_j^2\) is irreducible in \(\mathbb{Z}_q[T]\). Since \(b_j^2X^2 + b_jc_jXY + c_j^2Y^2 \equiv 0 \pmod{q}\) and \(Y \not\equiv 0 \pmod{q}\), we have \(c_j \equiv 0 \pmod{q}\). However, \(q = ab_j^3 + c_j^3\) implies that \(a\) must be divisible by \(q\), a contradiction.

Secondly, suppose that a prime divisor \(l \equiv 2 \pmod{3}\) of \(L\) divides \(b_j^2X^2 + b_jc_jXY + c_j^2Y^2\) for some \(j\). Then, since \(T^2 + T + 1\) is irreducible in \(\mathbb{F}_l[T]\), we have \(b_jX \equiv c_jY \equiv 0 \pmod{l}\). On the other hand, since \(\gcd(X, Y, Z) = 1\) and \(v_l(L) < n\), we see that \(X \not\equiv 0 \pmod{l}\) or \(Y \not\equiv 0 \pmod{l}\). However, if \(X \not\equiv 0 \pmod{l}\) (resp. \(Y \not\equiv 0 \pmod{l}\)), then \(b_j \equiv 0 \pmod{l}\) (resp. \(c_j \equiv 0 \pmod{l}\)), which contradicts that \(\gcd(L, b_jc_j) = 1\).

This completes the proof.

\[\Box\]

3. Fermat type equations of the form \(X^3 + pY^3 = LZ^n\)

Let \(p \geq 3\) be a prime number, \(\pi = p^{1/3} \in \mathbb{R}\) be the cubic root of \(p\), \(K = \mathbb{Q}(\pi) \subset \mathbb{R}\), and \(\mathcal{O}_K\) denotes the ring of integers in \(K\). Let \(\epsilon = \alpha + \beta\pi + \gamma\pi^2 > 1\) be the fundamental unit of \(K\) with \(\alpha, \beta, \gamma \in \frac{1}{2} \mathbb{Z}\). Note that the Galois closure of \(K\) in \(\mathbb{C}\) is \(K(\zeta_3)\), where \(\zeta_3 \in \mathbb{C}\) is a primitive cubic root of unity. For basic properties of these objects, see [2] and its references.

Set
\[
\iota = \begin{cases} 
1 & \text{if } \beta \not\equiv 0 \pmod{p} \text{ or } \beta \equiv \gamma \equiv 0 \pmod{p}, \\
2 & \text{if } \beta \equiv 0 \pmod{p} \text{ and } \gamma \not\equiv 0 \pmod{p}.
\end{cases}
\]

For example, if \(p = 3\), then we have \((\alpha, \beta, \gamma) = (2, 0, -1)\), hence \(\iota = 2\). On the other hand, if Conjecture [1.2] holds for \(p \geq 5\), then we have \(\iota = 1\).

In this section, we prove the following theorem.

Theorem 3.1. Let \(p \geq 3\) be a prime number, and \(n \in \mathbb{Z}_{\geq 5}\) divisible by \(p^\iota\). Then, there exist infinitely many prime numbers \(l\) and an integer \(m \in \{1, 2, \ldots, p - 1\}\) such that

\begin{enumerate}
\item \(l \equiv 2 \pmod{3}\) and \(l^m \equiv 1 \pmod{3}\),
\item \(l^m \equiv 1 \pmod{p}\), and
\item every primitive solution of \(x^3 + p^\iota y^3 = l^m z^n\) satisfies \(x \equiv y \equiv 0 \pmod{l}\).
\end{enumerate}

In order to prove Theorem 3.1, we use Theorem 1.4. Suppose that \(p \neq 3\). Let \(f(A, C) = (3p^\iota A + 1)^3 + p^{2\iota}(3p^\iota C + 1)^3\). Then, since \(\gcd(f(0, 0), f(1, 0), f(-1, 0)) = 1\), we have \(\gcd(f(\mathbb{Z}^{\geq 2})) = 1\). Therefore, Theorem 1.4 implies that there exist infinitely many prime numbers \(l\) of the form
\[
l = a^3 + p^{2\iota}c^3 \quad \text{with} \quad (a, c) = (3p^\iota A + 1, 3p^\iota C + 1) \in \mathbb{Z}^{\geq 2}.
\]
On the other hand, if \( p = 3 \), we can use \( f(A, C) = (27A - 1)^3 + 81(27C + 1)^3 \). Thus, if one notes that \( 2^m \equiv 1 \pmod{3} \) if and only if \( m \) is even, the proof of Theorem 3.1 is reduced to prove the following proposition.

**Proposition 3.2.** Let \( p \geq 3 \) be a prime number, \( n \in \mathbb{Z}_{\geq 5} \) divisible by \( p' \), and \( l \neq p \) be a prime number such that

1. \( l \equiv 2 \pmod{3} \), and
2. there exist \( a, c \in \mathbb{Z} \) such that \( l = a^3 + p^2c^3 \), \( a \equiv 1 \pmod{p'} \), \( c \equiv 0 \pmod{p} \), and \( c \equiv 2 \pmod{3} \) if \( p = 3 \).

Then, there exists an even integer \( m \in \{1, 2, \ldots, p - 1\} \) such that every primitive solution of \( x^3 + p'y^3 = l^mz^n \) satisfies \( x \equiv y \equiv 0 \pmod{l} \).

First, we prove the following proposition as an intermediate step.

**Proposition 3.3.** Let \( p \geq 3 \) be a prime number, \( n \in \mathbb{Z}_{\geq 5} \) divisible by \( p' \), \( l \neq p \) be a prime number such that \( l \equiv 2 \pmod{3} \), and \( m \in \mathbb{Z}_{\geq 1} \). Assume that there exist \( a + b\pi^t + c\pi^{2t} \in \mathcal{O}_K \) with \( a, b, c \in \mathbb{Z} \) such that

1. \( l = a^3 + b^3p' + c^3p^{2t} - 3abc\pi^t \)
2. if we define \( A_k, B_k, C_k \in \frac{1}{3}\mathbb{Z} \) \((k \in \mathbb{Z})\) by

\[
A_k + B_k\pi^t + C_k\pi^{2t} = \pi^t(a + b\pi^t + c\pi^{2t})^m,
\]

then we have \( C_k \not\equiv 0 \pmod{p} \) for every \( k \in \mathbb{Z} \).

Then, every primitive solution of \( x^3 + p'y^3 = l^mz^n \) satisfies \( x \equiv y \equiv 0 \pmod{l} \).

We prove Proposition 3.3 along a classical idea as done in [11], where Fujiwara proved the insolubility of \( x^3 + 5y^3 = 17z^3 \). We use the following lemma, which is an immediate consequence of [3 Corollarie 2] and [9 Theorem 3].

**Lemma 3.4.** The class number of \( K \) is smaller than \( p \).

**Proof of Proposition 3.3.** We prove the assertion by contradiction. Suppose that there exists a primitive triple \((x, y, z) \in \mathbb{Z}^3\) such that \( x^3 + p'y^3 = l^mz^n \), and either \( x \) or \( y \) is prime to \( l \).

First, note that since either \( x \) or \( y \) is prime to \( l \), \( x^2 - xy\pi^t + y^2\pi^{2t} \) cannot be divisible by \( l \). Moreover, \( l \equiv 2 \pmod{3} \) splits to the product of two prime ideals \( p_1 \) and \( p_2 \) of degree 1 and 2 respectively. Suppose that \( x + y\pi^t \) is divisible by \( p_2 \). Then, the product of its conjugates \((x + 3\zeta_2\gamma\pi^t)(x + \zeta_2y\pi^t) = x^2 - xy\pi^t + y^2\pi^{2t} \) is divisible by \( l \), a contradiction (cf. the following argument for \( q \equiv 2 \pmod{3} \)). Therefore, \( x^2 - xy\pi^t + y^2\pi^{2t} \) is divisible by \( p_2^m \) but not divisible by \( p_1 \). Accordingly, \( x + y\pi^t \) is divisible by \( p_2^m \) but not divisible by \( p_2 \).

Next, suppose that \( x + y\pi^t \) is divisible by a prime ideal above a prime divisor \( q \) of \( z \). Then, \((x, y, z) \) is primitive, neither \( x + y\pi^t \) nor \( x^2 - xy\pi^t + y^2\pi^{2t} \) is divisible by \( q \) itself. Therefore, the possible decomposition types of \( q \) in \( K \) are as follows:

1. \((q) = p_{q,1}p_{q,2}p_{q,3}, \text{ i.e., } q \equiv 1 \pmod{3} \) and \( p (mod q) \in \mathbb{F}^{\times 3}_q \).
2. \((q) = p_{q}p_{q}, \text{ i.e., } q \equiv 2 \pmod{3} \) and \( q \neq p, \)
3. \((q) = p_{q,1}p_{q,2}, \text{ i.e., } q = 3 \) and \( p \equiv \pm 1 \pmod{9}, \)
4. \((q) = p_{q}^2, \text{ i.e., } q = p, \) or \( q = 3 \) and \( p \equiv \pm 1 \pmod{9}. \)

\[ \text{Since } l \equiv 2 \pmod{3}, \text{ } \mathcal{O}_K \text{ has prime ideals } p_1 \text{ and } p_2 \text{ of norms of degree } 1 \text{ and } 2 \text{ respectively. Therefore, the first condition holds (up to signature) if and only if } p_1 \text{ is generated by } a + b\pi + c\pi^2. \]

\[ \text{Indeed, we may check Lemma 3.3 for } p \leq 139 \text{ directly (e.g. by Magma [7]). If } p > 139, \text{ then by } \]

\[ \text{[3 Corollarie 2] and [9 Theorem 3], the class number of } K \text{ is bounded by} \]

\[ \frac{1}{4^p} \frac{2 \log p + \log 3}{2 \log p - 2 \log 3} \text{ or } \frac{3}{4^p} \frac{2 \log p + 3 \log 3}{2 \log p} \]

\[ \text{according to } p \equiv \pm 1 \pmod{9} \text{ or not.} \]
In each case, we have the following conclusion:

1. If \( x + y\pi^t \) is divisible by distinct two prime ideals above \( q \), say \( \mathfrak{p}_{q,1} \) and \( \mathfrak{p}_{q,2} \), then
   \( x^2 - x y\pi^t + y^2\pi^2t \) is divisible by \( (\mathfrak{p}_{q,1}\mathfrak{p}_{q,3}) \cdot (\mathfrak{p}_{q,2}\mathfrak{p}_{q,3}) \), hence by \( q \), a contradiction. Therefore, \( x + y\pi^t \) is divisible by \( \mathfrak{p}_{q,1}^{n_{\mathfrak{p}_{q,1}(z)}} \) but not by \( \mathfrak{p}_{q,2} \) nor \( \mathfrak{p}_{q,3} \) if we replace \( \mathfrak{p}_{q,1}, \mathfrak{p}_{q,2}, \mathfrak{p}_{q,3} \) to each other if necessary.

2. In this case, \( q \) is decomposed in \( K(\zeta_3) \) so that \( \mathfrak{p}_q = \mathfrak{p}_{q,1}\mathfrak{p}_{q,2}\mathfrak{p}_{q,3} \). Since the first assumption implies that \( \mathfrak{p}_q \) is generated by \( a + b\pi^t + c\pi^{2t} \), \( \mathfrak{p}_q^n \) is also a principal ideal. Moreover, Lemma 3.4 implies that \( \mathfrak{p}_q^n \) is also generated by a single element \( w_0 + w_1\pi + w_2\pi^2 \in \mathcal{O}_K \) with \( w_0, w_1, w_2 \in 1/3\mathbb{Z} \). Therefore, there exists \( k \in \mathbb{Z} \) such that
   \[
   x + y\pi^t = \epsilon^k(a + b\pi^t + c\pi^{2t})^m(w_0 + w_1\pi + w_2\pi^2)\eta^k
   \equiv A_kw_0^n + B_kw_0^n\pi^t + C_kw_0^na\pi^{2t} \pmod{\pi^{2t+1}}
   \]
   In particular, we have \( C_kw_0 \equiv 0 \pmod{p} \). On the other hand, since \( (p, \mathfrak{m}) = 1 \), we have \( w_0 \equiv 0 \pmod{p} \). Therefore, \( C_k \equiv 0 \pmod{p} \) for some \( k \), which contradicts the assumption. This completes the proof.

Now, we can prove Proposition 3.2. Let \( \rho(X, Y, Z) := \frac{Y}{2X} - \frac{Z}{X} \in \mathbb{Q}(X, Y, Z) \) and

\[
\delta(X, Z) := \begin{cases} \rho(\alpha, \beta, \gamma)^2 - 2 \cdot \frac{2}{X} & \text{if } \beta \neq 0 \pmod{p} \text{ or } \beta \equiv \gamma \equiv 0 \pmod{p} \\ \rho(\alpha, \gamma, \frac{\beta}{p})^2 - 2 \cdot \frac{2}{X} & \text{if } \beta \equiv 0 \pmod{p} \text{ and } \gamma \equiv 0 \pmod{p} \end{cases} \in \mathbb{Q}(X, Z).
\]

Lemma 3.5. Let \( a, c \in \mathbb{Z} \). Let \( (A_k, B_k, C_k) \in \mathbb{Z}^{1,3} \) \((k \in \mathbb{Z})\) such that

\[
A_k + B_k\pi^t + C_k\pi^{2t} \equiv \epsilon^k(a + c\pi^{2t}) \pmod{\pi^{3t}}.
\]

1. Suppose that \( \beta \equiv 0 \pmod{p} \), or \( \beta \equiv 0 \pmod{p} \) and \( \gamma \equiv 0 \pmod{p} \). Then, \( C_k \equiv 0 \pmod{p} \) for every \( k \) if and only if \( \delta(a, c) \) is not a quadratic residue modulo \( p \).

2. Suppose that \( \beta \equiv \gamma \equiv 0 \pmod{p} \). Then, \( C_k \equiv 0 \pmod{p} \) for every \( k \) if and only if \( c \not\equiv 0 \pmod{p} \).

Proof. First, by a simple induction on \( k \in \mathbb{Z} \), we have

\[
\frac{\epsilon^k}{\alpha^k} \equiv \begin{cases} 1 + k \cdot \frac{2}{\alpha} + k \cdot \left( \frac{k-1}{2} \cdot \frac{2}{\alpha} + \frac{\gamma}{\alpha} \right) \pi^2 \pmod{\pi^3} & \text{if } \ell = 1 \\ 1 + k \cdot \frac{2}{\alpha} + k \cdot \left( \frac{k-1}{2} \cdot \frac{2}{\alpha} + \frac{\gamma}{\alpha} \right) \pi^4 \pmod{\pi^6} & \text{if } \ell = 2 \end{cases}
\]
for every \( k \in \mathbb{Z} \). This implies that
\[
C_k \equiv \begin{cases} 
\frac{\beta^2}{2\pi^2}ak^2 - \left( \frac{\beta^2}{2\pi^2} - \frac{\gamma}{\alpha} \right) ak + c \pmod{p} & \text{if } \iota = 1 \\
\frac{\gamma^2}{2\pi^2}ak^2 - \left( \frac{\gamma^2}{2\pi^2} - \frac{\beta}{\alpha} \right) ak + c \pmod{p} & \text{if } \iota = 2
\end{cases}
\]
for every \( k \in \mathbb{Z} \). Therefore, \( C_k \not\equiv 0 \pmod{p} \) for every \( k \) if and only if the above quadratic polynomial of \( k \) has a zero in \( \mathbb{F}_p \). This implies the assertion. \( \square \)

**Proof of Proposition 3.3.** First, note that Lemma 3.5 shows that the assertion holds if \( \beta \equiv \gamma \equiv 0 \pmod{p} \). Suppose that \( \beta \not\equiv 0 \pmod{p} \), or \( \beta \equiv 0 \pmod{p} \) and \( \gamma \not\equiv 0 \pmod{p} \). For every \( m \in \mathbb{Z} \), define \( (a(m), b(m), c(m)) \in \mathbb{Z}_{\geq 2} \) by
\[
a(m) + b(m)\pi^t + c(m)\pi^{2t} = (a + c\pi^2)^m.
\]
Then, by the assumption, we have
\[
a(m) + b(m)\pi^t + c(m)\pi^{2t} \equiv (1 + c\pi^2)^m \equiv 1 + mc\pi^2 \not\equiv 0 \pmod{\pi^3},
\]
hence
\[
\delta(a(m), c(m)) \equiv \begin{cases} 
\rho(\alpha, \beta, \gamma)^2 - 2cm \pmod{p} & \text{if } \iota = 1 \\
\rho(\alpha, \gamma, \beta)^2 - 2cm \pmod{p} & \text{if } \iota = 2
\end{cases}
\]
for every \( m \in \{1, 2, \ldots, p-1\} \). Since \( \mathbb{F}_p \) has \( \frac{p-1}{2} \) quadratic residues and non-quadratic residues respectively, and \( 1, 4 \in \mathbb{F}_p^+ \) give distinct quadratic residues for every \( p \neq 3 \), there exists at least one even \( m \in \{1, 2, \ldots, p-1\} \) such that \( \delta(a(m), c(m)) \) is a non-quadratic residue modulo \( p \). \( \square \) Thus, the assertion follows from Proposition 3.3 and Lemma 3.5. \( \square \)

4. Proof of the main theorem

**Proof of Theorem 1.1.** First, we prove the assertion when \( p \equiv 1 \pmod{3} \) and \( n \not\equiv 0 \pmod{3} \). Let \( f(X, Y) = p^i(3X+1)^3 + (3Y+1)^3 \). Then, since \( \gcd(f(0, 0), f(0, 1), f(0, -1)) = 1 \), we have \( \gcd(f(x, y) \mid (x, y) \in \mathbb{Z}_{\geq 2}) = 1 \). Therefore, by Theorem 1.4 there exist infinitely many distinct prime numbers of the form \( q = f(B, C) \) with \( (B, C) \in \mathbb{Z}_{\geq 2} \). Among them, take distinct \( \frac{n-3}{2} \) prime numbers \( q_j = f(B_j, C_j) \) with \( (B_j, C_j) \in \mathbb{Z}_{\geq 2} \) \( (1 \leq j \leq \frac{n-3}{2}) \), and set \((b_j, c_j) := (3B_j + 1, 3C_j + 1)\).

For each \( \frac{n-3}{2} \)-tuple \(((b_j, c_j))_{1 \leq j \leq \frac{n-3}{2}} \) taken as above, by Theorem 3.1 there exist infinitely many prime numbers \( l \equiv 2 \pmod{3} \) and an integer \( m \in \{1, 2, \ldots, p-1\} \) such that \( l > \max\{p, b_j, c_j \mid j = 1, 2, \ldots, \frac{n-3}{2}\} \) and every primitive solution of \( x^3 + p^i y^3 = l^m z^n \) satisfies \( x \equiv y \equiv 0 \pmod{l} \). Therefore, Proposition 2.1 implies that the equation
\[
(X^3 + pY^3) \prod_{j=1}^{\frac{n-3}{2}} (b_j^2 X^2 + b_j c_j X Y + c_j^2 Y^2) = l^m Z^n
\]
violates the local-global principle. The non-singularity follows from the fact that \( q_j \) and \( q_k \) are distinct prime numbers, hence \([b_j : c_j] \neq [b_k : c_k] \) for any \( j \neq k \). The non-isomorphy follows from the following Lemma 4.1.

Note that when \( p \equiv 4, 7 \pmod{9} \) and \( n \equiv 0 \pmod{3} \), then we can combine \( f(X, Y) \) and \( g(X, Y) = p^i(3X-1)^3 + (3Y)^3 \) to generate \((b_j, c_j)\) so that \( \sum_j b_j^{-1} c_j \not\equiv 0 \pmod{3} \) as required in Proposition 2.1.

Next, suppose that \( p \equiv 2 \pmod{3} \). When \( p \equiv 8 \pmod{9} \) or \( n \equiv 0 \pmod{3} \), then, we can use \( f(X, Y) = p^i(3pX + (-1)^3 + (3Y+1)^3 \) to generate \((b_j, c_j)\), and the above construction gives desired counterexamples. When \( p \equiv 2, 5 \pmod{9} \) and \( n \equiv 0 \pmod{3} \),

\( \square \) If \( p = 3 \), then \( \delta(a(2), c(2)) \) is non-quadratic residue if (and only if) \( c \equiv 2 \pmod{3} \).
then we can combine \( f(X,Y) \) and \( g(X,Y) = p'(3pX + (-1)^{i-1})^3 + (3Y)^3 \) to generate \((b_j, c_j)\) such that \( \sum_j b_j^{-1} c_j \neq 0 \) (mod 3) and \( \sum_j b_j^{-1} c_j \neq 0 \) (mod \( p \)).

Finally, suppose that \( p = 3 \). Then, we can combine \( f_p(X,Y) = 9(27X \pm 1)^3 + (9Y - 1)^3 \) to generate \((b_j, c_j)\) so that we can apply Proposition 2.1 with a help of Theorem 3.1. This completes the proof.

**Lemma 4.1.** Let \( n \in \mathbb{Z}_{\geq 5} \) be an odd integer, \( a \in \mathbb{Z} \setminus \{0\} \). Let \( \mathcal{P} \subset \mathbb{P}^2 \) be an infinite set of 2-dimensional integral vectors \((b,c)\) such that \([b : c] \neq [b' : c']\) as a rational point of the projective line \( \mathbb{P}^1 \). For each \( \frac{n-3}{2} \)-tuple \( \mathbf{v} = (b_j,c_j)_{1 \leq j \leq \frac{n-3}{2}} \in \mathcal{P}^{\frac{n-3}{2}} \), let \( C(a, \mathbf{v}) \) be the plane curve defined by

\[
(X^3 + aY^3) \prod_{j=1}^{\frac{n-3}{2}} (b_j^2 X^2 + b_jc_j XY + c_j^2 Y^2) = Z^n.
\]

Then, the set

\[
\mathcal{C}_{a,n} := \left\{ C(a, \mathbf{v}) \mid \mathbf{v} \in \mathcal{P}^{\frac{n-3}{2}} \right\}
\]

contains infinitely many non-singular curves non-isomorphic to each other over \( \mathbb{C} \).

**Proof.** Let \( C \) be a non-singular curve in \( \mathcal{C}_{a,n} \). Then, since its genus \( \frac{(n-2)(n-1)}{2} > 1 \), Hurwitz’s automorphisms theorem [16] ensures that the automorphism group \( \text{Aut}(C) \) of \( C \) is a finite group. In particular, the set

\[
\text{Quot}(C) := \left\{ (C/\langle \varphi \rangle)^{\text{nb}} \mid \varphi \in \text{Aut}(C) \right\}
\]

contains finitely many \( n \)-punctured lines (i.e., \( \mathbb{P}^1 \) minus \( n \) points). Here, \( C/\langle \varphi \rangle \) denotes the quotient of \( C \) by the cyclic group \( \langle \varphi \rangle \) generated by \( \varphi \), and \( (C/\langle \varphi \rangle)^{\text{nb}} \) denotes its non-branched locus, i.e., the image of the points each of whose \( \varphi \)-orbit consists of exactly \( \#\langle \varphi \rangle \) distinct points.

On the other hand, for each \( \frac{n-3}{2} \)-tuple \( \mathbf{v} = (b_j,c_j)_{1 \leq j \leq \frac{n-3}{2}} \in \mathcal{P}^{\frac{n-3}{2}} \), let \( L(a, \mathbf{v}) \) be a punctured line defined by

\[
(X^3 + aY^3) \prod_{i=1}^{\frac{n-3}{2}} (b_i^2 X^2 + b_ic_i XY + c_i^2 Y^2) \neq 0.
\]

Then, we can prove that the set consisting of them

\[
\mathcal{L}_{a,n}^{\mathcal{C}} := \left\{ L(a, \mathbf{v}) \mid \mathbf{v} \in \mathcal{P}^{\frac{n-3}{2}} \right\}
\]

contains infinitely many non-isomorphic \( n \)-punctured lines. Indeed, for each \( \mathbf{v} \in \mathcal{P}^{\frac{n-3}{2}} \), there exist at most \( n(n - 1)(n - 2) \) tuples \( \mathbf{v}' \in \mathcal{P}^{\frac{n-3}{2}} \) such that \( L(a, \mathbf{v}) \) is isomorphic to \( L(a, \mathbf{v}') \) because such an isomorphism is extended to an element of \( \text{Aut}(\mathbb{P}^1) \), which is uniquely determined from the images of three points, say those satisfying \( X^3 + aY^3 = 0 \). As a consequence, we see that the set \( \mathcal{L}_{a,n}^{\mathcal{C}} / \sim_{\mathbb{C}} \) contains infinitely many isomorphism classes of \( n \)-punctured lines over \( \mathbb{C} \).

Finally, note that we have a natural injection

\[
\mathcal{L}_{a,n}^{\mathcal{C}} / \sim_{\mathbb{C}} \rightarrow \left( \bigcup_{C \in \mathcal{C}_{a,n}} \text{Quot}(C) \right) / \sim_{\mathbb{C}}
\]

induced by \( L(a, \mathbf{v}) \mapsto (C(a, \mathbf{v})/\langle \varphi \rangle)^{\text{nb}} \), where \( \varphi([X : Y : Z]) = [X : Y : \zeta_n Z] \) with a fixed primitive \( n \)-th root of unity \( \zeta_n \in \mathbb{C} \). Therefore, \( \mathcal{C}_{a,n} \) contains infinitely many isomorphism classes of non-singular curves over \( \mathbb{C} \) as claimed. \( \square \)
5. Examples

In this section, we demonstrate that the proof of Theorem 1.1 actually gives explicit counterexamples to the local-global principle of the form

\[(X^3 + pY^3) \prod_{j=1}^{n=3} (b_j^2X^2 + b_jc_jXY + c_j^2Y^2) = LZ^n.\]

5.1. degree 7

First of all, since the fundamental unit of \( \mathbb{Q}(7^{1/3}) \) is \( \epsilon = 4 + 2 \cdot 7^{1/3} + 7^{2/3} \), Conjecture 1.2 is verified for \( p = 7 \). Hence, we have \( \iota = 1 \), and so we can actually take \( n = 7 \). Our goal is to obtain the parameters \( (b_1, c_1; b_2, c_2; L) \) so that the equation

\[(X^3 + 7Y^3)(b_1^2X^2 + b_1c_1XY + c_1^2Y^2)(b_2^2X^2 + b_2c_2XY + c_2^2Y^2) = LZ^7\]

defines a non-singular plane curve which violates the local-global principle.

In order to generate the coefficients \( (b_1, c_1; b_2, c_2) \), we can use the cubic polynomial \( f(X,Y) = 7(3X + 1)^3 + (3Y + 1)^3 \) as in the proof of Theorem 1.1. Indeed, by Theorem 1.4 the set \( f(\mathbb{Z}^{\oplus 2}) \) contains infinitely many prime numbers, for example,

\[
\begin{align*}
71 & = 7 \cdot (3 \cdot 0 + 1)^3 + (3 \cdot 1 + 1)^3, \\
449 & = 7 \cdot (3 \cdot 1 + 1)^3 + (3 \cdot 0 + 1)^3, \\
503 & = 7 \cdot (3 \cdot 11 + 1)^3 + (3 \cdot (-22) + 1)^3, \\
& \vdots \text{ etc.}
\end{align*}
\]

Among such \( (b_j, c_j) = (3X + 1, 3Y + 1) \), we can take, for example, \( (b_1, c_1) = (1, 4) \) and \( (b_2, c_2) = (4, 1) \).

For each choice of the above coefficients, we can take \( L = \ell^m \) with a prime number \( \ell > \max\{p, b_1, c_1, b_2, c_2\}(= 7) \) and even integer \( m \in \mathbb{Z}_{\geq 2} \) so that every primitive solution of \( x^3 + 7y^3 = \ell^mz^7 \) satisfies \( x \equiv y \equiv 0 \pmod{7} \) (cf. condition (6) in Proposition 2.1). In fact, as in the proof of Theorem 3.1 we can generate such \( \ell \) as integral values of another cubic polynomial \( g(A, C) = (21A + 1)^3 + 49(21C + 1)^3 \). For example,

\[
\begin{align*}
262193 & = (21 \cdot 3 + 1)^3 + 49(21 \cdot 0 + 1)^3, \\
452831 & = (21 \cdot (-2) + 1)^3 + 49(21 \cdot 1 + 1)^3, \\
521753 & = (21 \cdot 0 + 1)^3 + 49(21 \cdot 1 + 1)^3, \\
& \vdots \text{ etc.}
\end{align*}
\]

Here, we take \( \ell = 262193 \) with \( (a, c) := (21A + 1, 21C + 1) = (64, 1) \).

Finally, to generate the exponent \( m \), we use Proposition 3.3 and Lemma 3.5. They ensure that every primitive solution of \( x^3 + 7y^3 = 262193^mz^7 \) satisfies \( x \equiv y \equiv 0 \pmod{7} \) whenever \( \delta(a, mc) \equiv \delta(1, m) \equiv 4 - 2m \pmod{7} \) is a non-quadratic residue. Thus, we can take \( m = 4 \).

As a consequence, we obtain an explicit counterexample to the local-global principle:

\[
(X^3 + 7Y^3)(X^2 + 4XY + 16Y^2)(16X^2 + 4XY + Y^2) = 262193^4Z^7.
\]

5.2. degree 9

In this case, we can take \( p = 3 \). First of all, since the fundamental unit of \( \mathbb{Q}(3^{1/3}) \) is \( \epsilon = 4 + 3 \cdot 3^{1/3} + 2 \cdot 3^{2/3} \), we have \( \iota = 2 \). Our goal is to obtain the parameters...
We ensure that every primitive solution of the equation
\[(X^3 + 7Y^3)(b_1^2 X^2 + b_1 c_1 XY + c_1^2 Y^2)(b_2^2 X^2 + b_2 c_2 XY + c_2^2 Y^2)(b_3^2 X^2 + b_3 c_3 XY + c_3^2 Y^2) = L Z^9\]
defines a non-singular plane curve which violates the local-global principle.

In order to generate the coefficients \((b_1, c_1; b_2, c_2; b_3, c_3)\), we can combine the cubic polynomials \(f(X, Y) = 9(27X \pm 1)^3 + (9Y - 1)^3\) as in the proof of Theorem 1.1. Indeed, by Theorem 1.4, the sets \(f(L, \mathbb{Z}^{>0})\) contain infinitely many prime numbers, for example,
\[
\begin{align*}
521 &= 9 \cdot (27 \cdot 0 + 1)^3 + (9 \cdot 1 - 1)^3, \\
45179 &= 9 \cdot (27 \cdot 1 + 1)^3 + (9 \cdot (-4) - 1)^3, \\
85193 &= 9 \cdot (27 \cdot 0 + 1)^3 + (9 \cdot 5 - 1)^3, \\
&\vdots \quad \text{etc}
\end{align*}
\]
and
\[
\begin{align*}
503 &= 9 \cdot (27 \cdot 0 - 1)^3 + (9 \cdot 1 - 1)^3, \\
21347 &= 9 \cdot (27 \cdot 1 - 1)^3 + (9 \cdot (-4) - 1)^3, \\
65141 &= 9 \cdot (27 \cdot 1 - 1)^3 + (9 \cdot (-2) - 1)^3, \\
&\vdots \quad \text{etc}
\end{align*}
\]
respectively. Among such \((b_j, c_j) = (3X + 1, 3Y + 1)\), we can take them so that \(\sum_j b_j^{-1} c_j \neq 0 \mod 3\) (cf. condition (4) in Proposition 2.1). For example, we can take \((b_1, c_1) = (1, 8)\), \((b_2, c_2) = (-1, 8)\), and \((b_3, c_3) = (1, 44)\).

For each choice of the above coefficients, we can take \(L = l^m\) with a prime number \(l > \max\{p, b_1, c_1, b_2, c_2, b_3, c_3\} = 44\) and even integer \(m \in \mathbb{Z}_{>2}\) so that every primitive solution of \(x^3 + 9y^3 = l^m z^9\) satisfies \(x \equiv y \equiv 0 \mod 0\) (cf. condition (6) in Proposition 2.1). In fact, as in the proof of Theorem 3.1, we can generate such \(l\) as integral values of another cubic polynomial \(g(A, C) = (27A - 1)^3 + 81(27C + 1)^3\). For example,
\[
\begin{align*}
17657 &= (27 \cdot 1 - 1)^3 + 81(27 \cdot 0 + 1)^3, \\
1611737 &= (27 \cdot (-2) - 1)^3 + 81(27 \cdot 1 + 1)^3, \\
1778111 &= (27 \cdot 0 - 1)^3 + 81(27 \cdot 1 + 1)^3, \\
&\vdots \quad \text{etc.}
\end{align*}
\]
Here, we take \(l = 17657\) with \((a, c) := (27A - 1, 27C + 1) = (26, 1)\).

Finally, to generate the exponent \(m\), we use Proposition 3.3 and Lemma 3.5. They ensure that every primitive solution of \(x^3 + 9y^3 = 17657^m z^9\) satisfies \(x \equiv y \equiv 0 \mod 0\) whenever \(m = 2\).

As a consequence, we obtain an explicit counterexample to the local-global principle:
\[
(X^3 + 9Y^3)(X^2 + 8XY + 64Y^2)(X^2 - 8XY + 64Y^2)(X^2 + 44XY + 1936Y^2) = 17657^2 Z^9.
\]

### 5.3. degree 11

First of all, since the fundamental unit of \(\mathbb{Q}(11^{1/3})\) is \(\epsilon = 1 + 4 \cdot 11^{1/3} - 2 \cdot 11^{2/3}\), Conjecture 1.2 is verified for \(p = 11\). Hence, we have \(\epsilon = 1\), and so we can actually take \(n = 11\). Our goal is to obtain the parameters \((b_1, c_1; \ldots; b_4, c_4; L)\) so that the equation
\[
(X^3 + 7Y^3)(b_1^2 X^2 + b_1 c_1 XY + c_1^2 Y^2) \cdots (b_4^2 X^2 + b_4 c_4 XY + c_4^2 Y^2) = L Z^{11}
\]
defines a non-singular plane curve which violates the local-global principle.

In order to generate the coefficients \((b_1, c_1; \ldots; b_4, c_4)\), we can combine the cubic polynomials \(f(X, Y) = 11(33X - 1)^3 + (3Y + 1)^3\) and \(g(X, Y) = 11(33X + 1)^3 + (3Y)^3\) as in
the proof of Theorem 1.1. Indeed, by Theorem 1.4, the set \( f(\mathbb{Z}^2) \) contain infinitely many prime numbers, for example,

\[
39293 = 11 \cdot (33 \cdot 0 - 1)^3 + (3 \cdot 1 + 1)^3,
1265903 = 11 \cdot (33 \cdot (-2) - 1)^3 + (3 \cdot 5 + 1)^3,
3060179 = 11 \cdot (33 \cdot 2 - 1)^3 + (3 \cdot 1 + 1)^3,
\vdots \text{ etc}
\]

and

\[
227 = 11 \cdot (33 \cdot 0 + 1)^3 + (3 \cdot 2)^3,
5843 = 11 \cdot (33 \cdot 0 + 1)^3 + (3 \cdot 6)^3,
27011 = 11 \cdot (33 \cdot 0 + 1)^3 + (3 \cdot 10)^3,
\vdots \text{ etc}
\]

respectively. Among such \((b_j, c_j) = (3X^j + 1, 3Y^j + 1)\), we can take them so that

\[
\sum_j b_j^{-1} c_j \not\equiv 0 \pmod{3} \quad \text{and} \quad \sum_j b_j^{-1} c_j \not\equiv 0 \pmod{p} \quad \text{(cf. conditions (4) and (5) in Proposition 2.1)}.
\]

For example, we can take \((b_1, c_1) = (-1, 1), (b_2, c_2) = (1, 6), (b_3, c_3) = (1, 18), \) and \((b_4, c_4) = (1, 30)\).

For each choice of the above coefficients, we can take \(L = l^m\) with a prime number \(l > \max\{p, b_1, c_1, \ldots, b_4, c_4\}(= 30)\) and even integer \(m \in \mathbb{Z} \geq 2\) so that every primitive solution of \(x^3 + 11y^3 = l^m z^{11}\) satisfies \(x \equiv y \equiv 0 \pmod{0}\) (cf. condition (6) in Proposition 2.1). In fact, as in the proof of Theorem 3.1, we can generate such \(l\) as integral values of another cubic polynomial \(h(A, C) = (33A + 1)^3 + 121(33C + 1)^3\). For example,

\[
1000121 = (33 \cdot 3 + 1)^3 + 121(33 \cdot 0 + 1)^3,
2507693 = (33 \cdot (-4) + 1)^3 + 121(33 \cdot 1 + 1)^3,
4574417 = (33 \cdot 5 + 1)^3 + 121(33 \cdot 0 + 1)^3,
\vdots \text{ etc.}
\]

Here, we take \(l = 1000121\) with \((a, c) := (33A + 1, 33C + 1) = (100, 1)\).

Finally, to generate the exponent \(m\), we use Proposition 3.3 and Lemma 3.5. They ensure that every primitive solution of \(x^3 + 11y^3 = 1000121^{m^6} z^{11}\) satisfies \(x \equiv y \equiv 0 \pmod{0}\) whenever \(\delta(a, mc) \equiv \delta(1, m) \equiv 8 - 2m \pmod{11}\) is a non-quadratic residue. Thus, we can take \(m = 6\). As a consequence, we obtain an explicit counterexample to the local-global principle:

\[
(X^3 + 11Y^3)(X^2 - XY + Y^2)(X^2 + 6XY + 36Y^2) \\
\times (X^2 + 18XY + 324Y^2)(X^2 + 30XY + 900Y^2) = 1000121^6 Z^{11}.
\]

Appendix : Numerical verification of Conjecture 1.2

The following is the command of Magma which we used for numerical verification of Conjecture 1.2 for prime numbers \(p < 10^5\). Here, recall that Conjecture 1.2 holds for \(p\) if the order \(\mathbb{Z}[p^{1/3}] \subset \mathcal{O}_K\) has a unit \(\alpha + \beta p^{1/3} + \gamma p^{2/3}\) with \(\alpha, \beta, \gamma \in \mathbb{Z}\) such that \(\beta \neq 0 \pmod{p}\).

```
> Z<x> := PolynomialRing(Integers());
> for p in [1..10^5] do;
    > if IsPrime(p) then
```
Note that the return $p = 3$ is the conjectural unique exception.

Moreover, the above numerical experiment implies that there exist non-singular plane curves of degree $n$ which violate the local-global principle for “most” odd integers $n$ in the sense of natural density: Let $N := \mathbb{Z}_{\geq 1}$, and $\mathbb{N}^{\text{odd}}$ be the set of positive odd integers. Set

\[
P := \{ p : \text{prime number} \},
\]
\[
BP := \{ p \in P \mid p < 10^5 \},
\]
\[
M := \{ n \in \mathbb{N} \mid n \not\equiv 0 \pmod{p} \text{ for all } p \in BP \text{ and } n \not\equiv 0 \pmod{p^2} \text{ for all } p \in P \},
\]
\[
N := \mathbb{N} \setminus (M \cup \{1\}),
\]
\[
\mathbb{N}^{\text{odd}} := \{ n \in \mathbb{N} \mid n \text{ is an odd integer} \},
\]
\[
\mathbb{N}^{\text{odd}} := N \cap \mathbb{N}^{\text{odd}},
\]

then Theorem 1.1 and the above numerical verification of Conjecture 1.2 (with Selmer’s construction for $n = 3$) ensures that we can construct infinitely many explicit non-singular plane curves of degree $n$ which violates the local-global principle for each $n \in \mathbb{N}^{\text{odd}}$. Moreover, if we denote the natural density of $S \subset \mathbb{N}$ by $d(S)$ (if it exists), then we have

\[
d(M) = \prod_{p \in BP} (1 - p^{-1}) \times \prod_{p \in P \setminus BP} (1 - p^{-2}) = \prod_{p \in BP} (1 + p^{-1})^{-1} \times \zeta(2)^{-1}
\]

\[
\approx 0.080195 \times 0.60793 \approx 0.048753
\]

and

\[
d(\mathbb{N}^{\text{odd}}) = \frac{1}{2},
\]

hence

\[
\frac{d(\mathbb{N}^{\text{odd}})}{d(\mathbb{N}^{\text{odd}})} = 1 - \frac{d(M)}{d(\mathbb{N}^{\text{odd}})} \approx 0.90249.
\]

Therefore, the around 90% of odd integers are expected to lie in $\mathbb{N}^{\text{odd}}$.

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