REVISITING COSMOLOGICAL DIFFUSION MODELS IN UNIMODULAR GRAVITY AND THE $H_0$ TENSION

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ABSTRACT

Within the framework of Unimodular Gravity, we consider non–gravitational interactions between dark matter and dark energy. Particularly, we describe such interactions in the dark sector by considering diffusion models that couple the cold dark matter fluid with the dark energy component, where the latter has the form of a variable cosmological “constant”. For the first time, we solve the cosmological evolution for these models from the radiation domination era to the present day. We show how the diffusion processes take place by analyzing the cosmological evolution of the energy density parameters $\Omega_{\text{cdm}}$ and $\Omega_\Lambda$, as well as that of the Hubble parameter. Finally, we perform the statistical analysis, imposing constraints on the diffusion parameters, by using data from Planck 2018, SH0ES, Pantheon, and H0LICOW collaborations. We found that cosmological diffusion models in the framework of Unimodular Gravity can ease the current tension in the value of $H_0$.

Keywords Unimodular Gravity · Interacting dark sector · $H_0$ tension

1 Introduction

After more than 100 years, General Relativity (GR) remains as the theory successfully describing the gravitational interactions [1, 2]. In the cosmological context, GR offers a mathematical ground that has allowed to describe the evolution of the Universe, from the era when the first atomic nuclei form, to the current phase of accelerated expansion. Such description lies within the standard cosmological model $\Lambda$CDM, which demands, however, a new particle for the so-called Cold Dark Matter (CDM) component, responsible for the structure formation process, and which interacts mostly gravitationally with the rest of the known particles. The other ingredient of this model is the Cosmological Constant $\Lambda$, which is required to explain the current accelerated expansion of the Universe. Thus, GR is a successful gravitational framework to describe the evolution of the Universe as long as a new dark sector (dark matter particles as well as a cosmological constant) is added.

With the aim of gaining some theoretical understanding on what these new components of the Universe might be, other approaches have been proposed, such as modifying or including new contributions to the Einstein-Hilbert action. In particular, one of the main motivations to consider alternatives approaches to GR, is the observed current phase of accelerated expansion of the Universe [3, 4, 5, 6, 7], which is thought it is produced by the so-called Dark Energy (DE). Many models to explain DE have been proposed, such as fluids with variable equation of state [8, 9, 10, 11, 12, 13, 14, 15], scalar fields [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29], modified gravity [30, 31, 32, 33, 34, 35]. However, it seems that the most favored explanation according to several observations, is the cosmological constant term $\Lambda$ in the Einstein field equations,

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \] (1)
where $\kappa^2 = 8\pi G$. Historically, with the aim of having a quasi-static matter distribution in the Universe, Einstein introduced by-hand the term $\Lambda$ in his field equations [36]. After Edwin Hubble’s observations of how distant galaxies are receding away from us [37], and later with the discovery of the accelerating expansion of the Universe through supernovae observations [3, 4], the cosmological constant plays the role of the DE component responsible for such accelerated expansion that the Universe is currently experiencing. Its origin and physical interpretation have been debated and speculated since then, giving rise to the different approaches mentioned above.

One of the proposal for a possible origin of $\Lambda$ that is being currently studied with great interest, is related with the original formulation of the field equations of GR, where Einstein showed that it is always possible to consider a choice of coordinate such that the determinant of the metric tensor is fixed [38]. Specifically, when the determinant $g$ of the metric tensor $g_{\mu\nu}$ satisfies the unimodular condition $\sqrt{-g} = 1$, the Einstein tensor gets a simplified form. Later, in trying to understand the role of gravitational forces in the constitution of matter, Einstein showed that Einstein-Hilbert action through a Lagrange multiplier $\lambda$, where we have introduced the Einstein tensor $\frac{1}{2} R_{\mu\nu} + \lambda g_{\mu\nu} = 0$. (3)

Taking the trace of Eq. (3), the Lagrange multiplier is determined to be

$$\lambda = \frac{1}{4} R,$$

and therefore, Eq. (3) is written as

$$R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} = 0.$$

When considering matter content $S_M$ in the action (2), Eqs. (3) and (4) are given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad \lambda = \frac{1}{4} (R + \kappa^2 T),$$

where $T_{\mu\nu} = -2g^{-1/2}\delta S_M/\delta g^{\mu\nu}$ is the energy–momentum tensor, and $T$ its trace. After eliminating $\lambda$, we have

$$R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} = \kappa^2 \left( T_{\mu\nu} - \frac{1}{4} T g_{\mu\nu} \right),$$

which is the trace–free version of the Einstein field equations. This theory leading to the new set of equations (7) for the gravitational field has been dubbed Unimodular Gravity (UG). First works in UG are [40, 41, 42, 43, 44, 45], approaches of UG in the quantum regime have been explored by [46, 47, 48, 49, 50], and some recent cosmological applications have been studied in [51, 52, 53, 54, 55, 56, 57, 58, 59].

One of the main features of this theory can be seen as follows: notice that we can rewrite Eq. (7) in the following way,

$$G^\mu_{\nu} + \frac{1}{4} \left( R + \kappa^2 T \right) \delta^\mu_{\nu} = \kappa^2 T^\mu_{\nu},$$

where we have introduced the Einstein tensor $G^\mu_{\nu} \equiv R^\mu_{\nu} - \frac{1}{2} R \delta^\mu_{\nu}$. Then, applying the Bianchi identities we have

$$\nabla_{\mu} G^\mu_{\nu} + \frac{1}{4} \nabla_{\nu} \left( R + \kappa^2 T \right) = \kappa^2 \nabla_{\mu} T^\mu_{\nu}.$$
where $\Lambda$ is an integration constant, and $J_{\nu} \equiv \kappa^2 \nabla_{\mu} T_{\mu}^{\nu}$ is the energy–momentum current violation to be integrated on some arbitrary path $l$. Replacing this result into Eq. (6), we have

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \left( \Lambda + \int_{l} J(x) \right) g_{\mu\nu} = \kappa^2 T_{\mu\nu}. \quad (11)$$

The physical interpretation of the Lagrange multiplier $\lambda$ is now apparent: it plays the role of an effective cosmological "constant". In fact, in the particular case when the energy–momentum tensor is conserved ($J = 0$), the integration constant $\Lambda$ is identified as the cosmological constant term in the Einstein field equations (1). Thus, within the framework of UG, the cosmological constant $\Lambda$ is not a term introduced by–hand, but it arises naturally as an integration constant when considering the Einstein–Hilbert action with volume-preserving diffeomorphisms$^1$. However, there will be in general a non–null energy–momentum current violation. Thus, assuming that ordinary matter (photons, neutrinos, baryons) interact only gravitationally with the dark sector, the non-conservation of the energy–momentum tensor leads to a non-gravitational interaction between cold dark matter and the cosmological constant.

Interactions between the components of the dark sector have been studied broadly in the literature [62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78], and they have shown to be useful as alternative models to $\Lambda$CDM, to address the discrepancies found in the measurements of the current value of the Hubble parameter $H_0$, when it is inferred from early and late Universe observations [79, 80, 81, 82, 83, 84, 85, 86]. We will refer to such discrepancy as the $H_0$ tension (for recent discussions see [87, 88]). In Figure 1 it can be seen the current value of the Hubble parameter according different experiments. Whereas observations of the early Universe, based on the Cosmic Microwave Background (CMB), a value of $H_{0}^{\text{early}} \simeq 68 \text{km s}^{-1} \text{Mpc}^{-1}$ is inferred, the combined late Universe observations indicate that $H_{0}^{\text{late}} \simeq 73 \text{km s}^{-1} \text{Mpc}^{-1}$. This discrepancy may be due to systematic errors in the observations, but it can also be suggesting to look for extensions of the $\Lambda$CDM model with the aim of exploring new physics.

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**Figure 1**: Value of $H_0$ from different observations. Top: $H_0$ from CMB observations. Middle: $H_0$ from late time observations. Bottom: combined late time observations and the corresponding $H_0$ tension with early time measurements. Data from CMB [7, 89], BAO and BBN [90, 91], SNe Ia and Cepheids [92], SNe Ia and TRGB [93, 94], SNe Ia and Mira variables [95], lensed quasars [96], water megamasers [97], SBF and Cepheids [98]. This is an updated version of Figure 1 from [88]. Credits to Vivien Bonvin and Martin Millon [99].

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$^1$Theoretical analysis on the viability of UG for astrophysical and cosmological applications, as well as discussions about the integrability of $J_{\nu}$ are discussed in [60] and [61] respectively.
Non-gravitational interactions between dark matter and dark energy have been studied through diffusion models in the cosmological context within the framework of General Relativity [100, 101], as well as in the context of Two Measure Theories and dynamical spacetime theories [102, 103, 104]. We are particularly interested in diffusion models as those presented in [58, 59], where the authors explore cosmological diffusion processes in the framework of Unimodular Gravity. Specifically we will go further in the analysis by studying these models not only at late times, but from very early times deep within the radiation dominated era. This requires to consider the radiation component into account, which will be important in order to constraint the diffusion models we are interesting in with CMB data.

Thus, in this work we will study diffusion processes between the dark sector components within the framework of Unimodular Gravity. For the first time, these diffusion models are studied in a more realistic cosmological scenario, in which the component of radiation due to the presence of photons and ultra–relativistic neutrinos at early times, is taken into account. Since we will be focused on how these models can alleviate the $H_0$ tension, it is important to include the radiation component in order to use data from the CMB, and thus being able to test each diffusion model at the last scattering surface ($z_s \simeq 1100$). By performing a statistical analysis we will infer the most likely values of the diffusion models parameters in the light of current astrophysical and cosmological data. This will allow us to analyze the viability of such diffusion models as a possible solution to the $H_0$ tension. As we will show, the inferred value of $H_0$ at early times can be in agreement with that obtained from late time observations.

The outline of the present work is the following: in Section 2 we show the cosmological equations for the background evolution within the framework of UG. We analyze the cosmological evolution for each of the diffusion models of interest, studying the diffusion process in terms of the energy density parameters $\Omega_{\gamma}$, $\Omega_{\nu}$, $\Omega_{\cdm}$, and for $H_0$ are those determined by CMB observations (and thus $H_0 = H_0^{\text{early}}$ for $\Lambda$CDM), we show how the evolution of the Hubble parameter $H(z)$ for each diffusion model gives a current value $H_0$ consistent with the reported value from local observations, i.e., $H_0 = H_0^{\text{late}}$. In particular, when the parameters of the diffusion models are set to zero, we recover the $\Lambda$CDM results. We perform the statistical analysis in Section 3, where we: 1) consider only CMB observations from the Planck Compressed 2018 data, and then 2) we include into the analysis observations from the local Universe as those from Cepheids, Supernovae and lensed quasars. In the first analysis, the tension on the $H_0$ value is eased because the anticorrelation between $\Omega_{\cdm}$ and $H_0$ gets a broader range of values due to the presence of the new diffusion parameters. In the second analysis, we obtain that the local observations allow the diffusion models to ease the $H_0$ tension by shifting the mean value from $H_0^{\text{early}}$ to $H_0^{\text{late}}$, and thus, the diffusion models make CMB and late times observations to be in agreement in the value of the Hubble parameter at $z = 0$. Finally, in Section 4 we discuss our results and give some conclusions of our analysis.

2 Background cosmological equations with diffusion

We start by considering a spatially-flat Friedmann–Robertson–Walker (FRW) line element, 
\[
\frac{dl^2}{ds^2} = -dt^2 + a^2(t) \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],
\]

where the scale factor $a(t)$ is function of the cosmic time $t$. When considering the FRW line element (12), we are assuming that at large scales the Cosmological Principle is valid, and then both homogeneity and isotropy imply that the effective cosmological constant (10) is now a function of the cosmic time only,
\[
\Lambda(t) = \Lambda + \int J(t) dt,
\]

where we have replaced the notation $\lambda(t) \rightarrow \Lambda(t)$. The Einstein field equations (11) for the background evolution are given by,
\[
\begin{align*}
H^2 &= \frac{\kappa^2}{3} \left( \rho_{\gamma} + \rho_{\nu} + \rho_b + \rho_{\cdm} + \rho_{\Lambda(t)} \right), \\
\dot{H} &= -\frac{\kappa^2}{2} \left[ (\rho_{\gamma} + p_{\gamma}) + (\rho_{\nu} + p_{\nu}) + (\rho_b + p_b) + (\rho_{\cdm} + p_{\cdm}) \right], \\
\dot{\rho}_{\gamma} &= -3H(\rho_{\gamma} + p_{\gamma}), \quad \dot{\rho}_{\nu} = -3H(\rho_{\nu} + p_{\nu}), \quad \dot{\rho}_b = -3H(\rho_b + p_b), \\
\dot{\rho}_{\cdm} &= -3H(\rho_{\cdm} + p_{\cdm}) - \frac{\dot{\Lambda}(t)}{\kappa^2}.
\end{align*}
\]

The dot denotes derivative with respect to cosmic time $t$, and $H = \dot{a}/a$ is the Hubble parameter. We will consider that baryons ($b$) and cold dark matter ($\cdm$) behave as dust, and then they have vanishing pressure $p_b = p_{\cdm} = 0$, whereas for photons ($\gamma$) and ultra–relativistic neutrinos ($\nu$) we have $p_{\gamma} = \rho_{\gamma}/3$ and $p_{\nu} = \rho_{\nu}/3.$
2.1 Diffusion models

Given a particular form of the cold dark matter energy density $\rho_{\text{cdm}}(t)$, Eq. (14d) can be integrated to find the function $\Lambda(t)$. With this approach, here we analyze two diffusion models previously studied in [58]: Sudden Transfer Model (STM) and Anomalous Decay of the Matter Density (ADMD). Both models are described in terms of two parameters, one regulating the amplitudes of $\rho_{\text{cdm}}$ and $\rho_\Lambda$ ($\alpha$ for STM and $\gamma$ for ADMD), and another one for the characteristic redshift $z^*$ at which the diffusion process takes place.

Another way to solve (14d) is by explicitly proposing a particular form for the diffusion function $Q(x)$. An explicit functional form for the diffusion function follows,

$$Q(x) = \frac{1}{\kappa^2} \int_1^x J(t) \, dt,$$

and thus, the integrated energy–momentum current violation is expressed in terms of the diffusion function $Q(x)$. Given homogeneity and isotropy, this function will depend only on the cosmic time $t$. Therefore, what we have denoted by $\Lambda(t)$ in Eq. (13) can now be written as

$$\Lambda(t) = \Lambda + \kappa^2 Q(t),$$

in whose case, Eq. (14d) reads

$$\dot{\rho}_{\text{cdm}} = -3H(\rho_{\text{cdm}} + p_{\text{cdm}}) - \dot{Q}(t).$$

An explicit functional form for the diffusion function $Q$ can be given, in order to find the CDM energy density by solving Eq. (17). This was considered by the authors in [59], where two phenomenological models are studied: Barotropic Model (BM) and Continuous Spontaneous Localization (CSL). Different from the previous two models, only one parameter characterizes the BM and CSL models ($x_{\text{cdm}}$ for BM and $\xi_{\text{CSL}}$ for CSL).

In this Section we will analyze the cosmological evolution of these four models mentioned above, from the radiation dominated era until the present day. To do so, we will solve the background equations (14) with the Boltzmann code CLASS [105].

2.1.1 Model 1: Sudden Transfer Model

The energy density proposed for the CDM component is given by

$$\rho_{\text{cdm}}(z) = \rho_{0,\text{cdm}}(1 + z)^3 \times \left\{ \begin{array}{ll} 1 & \text{if } z \geq z^*, \\ 1 - \alpha & \text{if } z < z^*, \end{array} \right. $$

where $\alpha$ is the dimensionless diffusion constant controlling the amplitude of the energy transfer, and $z^*$ is the characteristic redshift at which the sudden diffusion process takes place. After integration of Eq. (14d) (with $\rho_{\text{cdm}} = 0$), the effective cosmological constant is given by

$$\Lambda(z) = \left\{ \begin{array}{ll} \Lambda & \text{if } z \geq z^*, \\ \Lambda + 3H_0^2 \alpha (1 + z^*)^3 \Omega_{0,\text{cdm}} & \text{if } z < z^*, \end{array} \right. $$

and then, once $z < z^*$, the Friedmann equation (14a) can be written as

$$E(z) \equiv \frac{H(z)}{H_0} = \sqrt{\Omega_{0,r}(1 + z)^4 + \Omega_{0,b} h^2 (1 + z)^3 + \Omega_{\text{cdm}} (1 + z)^3 \left[ 1 - \alpha + \alpha \left( \frac{1 + z^*}{1 + z} \right)^3 \right] + \Omega_\Lambda}. $$

Thus, the diffusion process allows to have different present values of $\Omega_\Lambda$ and $\Omega_{\text{cdm}}$ for given values of $\alpha$ and $z^*$. This implies that the current Hubble parameter $H_0$ can also change its value since it is defined implicitly in the energy density parameters. It can be seen that the standard Friedmann equation is recovered when $\alpha = 0$.

In order to illustrate the diffusion process between CDM and $\Lambda$, we have considered the diffusion parameters given by $\{\alpha, z^*\} = \{0.8, 1\}$. As expected, the energy density for cold dark matter (yellow line) drops suddenly at $z^*$ (vertical dash–dotted black line), and the energy density for $\Lambda$ (red line) increases, as it is shown in Figure 2. The rest of the matter components evolve as usual. Only for comparison, we also have included the standard evolution of CDM and $\Lambda$ (black dotted and dashed lines respectively). The Friedman constraint is satisfied during all the cosmological evolution (horizontal gray line), this is, $1 = \sum_i \Omega_i(z)$.
With the aim of exploring the effects of the diffusion process between the dark sector components, different cosmological evolution with several values of the diffusion constant $\alpha$ are shown in Figure 3, where the sudden energy transfer between $\Omega_{\text{cdm}}$ and $\Omega_{\Lambda}$ can be observed. Particularly, it can be seen that given a characteristic redshift $z^*=1$ (vertical black line), the diffusion constant $\alpha$ will change the difference between the current values of the energy density parameters for the dark sector. This is expected since $\alpha$ regulates the amount of energy density that the CDM component transfers to the cosmological constant (see Eq. (18a) and (18b)). On the other hand, the effect of different characteristic redshifts $z^*$ on the diffusion process can be seen in Figure 4. We observe that higher values of $z^*$ will lead to lower (larger) values of the cold dark matter (cosmological constant) energy density parameter today.
Figure 4: Sudden diffusion between $\rho_{cdm}$ and $\Lambda$ (Model 1) at different characteristic redshifts: $z^* = 5$ (vertical dash–dotted purple line), and $z^* = 50$ (vertical dash–dotted cyan line). Black lines show the standard evolution without diffusion ($\alpha = 0$) for CDM (dotted line) and cosmological constant (dashed line)). For the diffusion processes between CDM and $\Lambda$ at $z^* = 5$ (blue and green lines), as well as $z^* = 50$ (yellow and red lines), the diffusion constant was settled to $\alpha = 0.15$.

2.1.2 Model 2: Anomalous Decay of the Matter Density

In this case, the mathematical model for the CDM energy density is proposed to be

$$\rho_{cdm}(z) = \rho_{0,cdm}(1 + z)^3 \times \begin{cases} 1 & \text{if } z \geq z^*, \\ \left(\frac{1 + z}{1 + z^*}\right)^\gamma & \text{if } z < z^*, \end{cases} \tag{19a}$$

where $\gamma$ is the dimensionless diffusion constant controlling the power of the energy transfer term, and $z^*$ is again the characteristic redshift, this time indicating when the anomalous decay of dark matter occurs. After integration of Eq. (14d), the energy density for $\Lambda$ is given by

$$\Lambda(z) = \begin{cases} \Lambda & \text{if } z \geq z^*, \\ \Lambda - \frac{3\gamma}{\gamma + 3} H_0^2 \left[\left(\frac{1 + z}{1 + z^*}\right)^\gamma (1 + z)^3 - (1 + z^*)^3\right] \Omega_{0,cdm} & \text{if } z < z^*. \end{cases} \tag{19b}$$

Once $z < z^*$, the Friedmann equation (14a) can be written as

$$E(z) \equiv \frac{H(z)}{H_0} = \sqrt{\Omega_{0,r}(1 + z)^4 + \Omega_b h^2 (1 + z)^3 + \Omega_{cdm}(1 + z)^3 \left[\frac{3}{3 + \gamma} \left(\frac{1 + z}{1 + z^*}\right)^\gamma + \frac{\gamma}{3 + \gamma} \left(\frac{1 + z^*}{1 + z}\right)^3\right]} + \Omega_\Lambda, \tag{19c}$$

where it can be seen that the $\Lambda$CDM scenario is recover when $\gamma = 0$. Figure 5 shows the evolution of all the matter components present in the Universe, considering a diffusion process described by the ADMD model with $\gamma = 0.2$ at a characteristic redshift $z^* = 1$.

In general, depending on the values of $\{\gamma, z^*\}$, the energy transfer between cold dark matter and $\Lambda$ will be smoother, or steeper than model 1. This can be seen in Figure 6, where we have settled $z^* = 1$ to explore the effect of $\gamma$. We observe that, at this redshift, the transition is smooth for different values of $\gamma$ (the reader can compare this to the case of sudden energy transfer of Model 1 in Figure 3). The cold dark matter fluid diffuses and the cosmological constant captures this energy, which allows to $\Omega_\Lambda$ to reach larger values in the present day in comparison with the $\Lambda$CDM.
Figure 5: ADMC model for the diffusion between $\rho_{cdm}$ and $\Lambda$ (Model 2). Photons (blue line), baryons (orange line) and neutrinos (green line) evolve as usual, whereas $\Omega_{cdm}$ (yellow line) and $\Omega_{\Lambda}$ (red line) evolve with a diffusion process with $\gamma = 0.2$. The characteristic redshift was set to $z^* = 1$ (vertical dashdotted line). The horizontal gray line indicates the Friedman constraint $\sum_i \Omega_i(z) = 1$.

Figure 6: Anomalous decay of cold dark matter density $\rho_{cdm}$ into dark energy $\Lambda$ (Model 2). Black lines show the standard evolution without diffusion ($\gamma = 0$). Solid lines correspond to a diffusion process with $\gamma = 0.2$ (yellow and red for $\Omega_{cdm}$ and $\Omega_{\Lambda}$ respectively), and $\gamma = 0.5$ (blue and green for $\Omega_{cdm}$ and $\Omega_{\Lambda}$ respectively). The characteristic redshift was fixed to $z^* = 1$. 

The effects of this model on the dark sector energy density parameters due to the characteristic redshift $z_\star$ are shown in Figure 7. We can see that the diffusion process induces a steeper fall of the cold dark matter energy density when $z_\star = 5, 50$, in comparison to lower characteristic redshifts (for example, at $z_\star = 1$ in Figure 6). Besides, the present value of $\Omega_{cdm}$ ($\Omega_\Lambda$) decreases (increases) much more than the cases for lower redshifts.

![Figure 7: Anomalous decay of cold dark matter density $\rho_{cdm}$ into dark energy $\Lambda$ (Model 2) at different characteristic redshifts: $z_\star = 5$ (vertical dash–dotted purple line), and $z_\star = 50$ (vertical dash–dotted cyan line). Black lines show the standard evolution for $\Omega_{cdm}$ and $\Omega_\Lambda$ without diffusion ($\gamma = 0$). For the ADMD model, the diffusion constant was settled to $\gamma = 0.5$.](image)

### 2.1.3 Model 3: Barotropic Model

For the following two models we will solve Eq. (17), for which a diffusion function $Q$ has to be given. One of the forms of this function that has been explored in [59] is

$$ Q = x_i \rho_i, $$

this is, a constant barotropic equation of state $x_i$ relating the energy density of the different matter components $\rho_i$ ($i = b, \gamma, \nu, cdm, \Lambda$) and the diffusion function $Q$. In our case, the diffusion process will be only due to the CDM component, i.e., $i = cdm$, and thus we have from Eq. (17) that

$$ \rho_{cdm}(z) = \rho_{cdm}(1 + z)^{3(\omega_{cdm} + 1)} \frac{x_{cdm}}{x_{cdm} + 1}. $$

The normalized Friedmann equation for this model is given by,

$$ E(z) \equiv \frac{H(z)}{H_0} = \sqrt{\Omega_{0,\gamma} (1 + z)^4 + \Omega_{0,b} (1 + z)^3 + (1 + x_{cdm}) \Omega_{cdm}(1 + z)^{3(\omega_{cdm} + 1)} + \Omega_\Lambda}, $$

where we have considered the standard dust–like behavior for CDM ($\omega_{cdm} \approx 0$). The cosmological evolution for all matter components is shown in Figure 8, where it can be seen that the diffusion process driven by this model affects $\Omega_{cdm}$ from the moment when its amplitude starts to grow, and during most of the matter domination era. As in the previous models, for comparison we also show the standard $\Lambda$CDM case for $\Omega_{edm}$ (black dotted line) and $\Omega_\Lambda$ (black dashed line), but this time we also show the standard evolution of the baryon energy density $\Omega_b$ (black dash–dotted line) to show that in the case of the diffusion model (orange line) it increases in the right proportion to balance the CDM contribution (yellow line) such that the total budget of matter still satisfies the Friedmann constraint $\Omega_{tot} = 1$ (gray horizontal line) during all the cosmological evolution.

In Figure 9 we show the evolution of $\Omega_{cdm}$ and $\Omega_\Lambda$ for several values of $x_{cdm}$. Positive values of $x_{cdm}$ (red lines) lead to larger values of $\Omega_{cdm}$ at $z = 0$, whereas the current value of the cosmological constant energy density parameter $\Omega_{0,\Lambda}$ decreases (red dashed line). The opposite occurs when negative values of $x_{cdm}$ are considered (green lines).
Figure 8: Cosmological evolution of the energy density parameter of each matter component of the Universe considering the diffusion process of Model 3: Barotropic model. The diffusion between CDM and $\Lambda$ is mediated by the barotropic equation of state $Q = x_{\text{cdm}} \rho_{\text{cdm}}$.

Figure 9: Diffusion between CDM and $\Lambda$ mediated by the barotropic equation of state $Q = x_{\text{cdm}} \rho_{\text{cdm}}$. The effect of different values of $x_{\text{cdm}}$ can be summarized as an increment (reduction) on $\Omega_{\text{cdm}}$ for $x_{\text{cdm}} > 0$ ($x_{\text{cdm}} < 0$). Dotted and dashed black lines represent the standard evolution of $\Omega_{\text{cdm}}$ and $\Omega_{\Lambda}$ respectively.
We can see that, different from Models 1 and 2, this model presents diffusion during certain period of time of the cosmological evolution \((z \lesssim 10^6)\), and not only at a given characteristic redshift. In this sense, this model could have different implications in the cosmic history, as for example in the matter-radiation equality era \(z_{eq}\), as is shown in Figure 10. In fact, the most notorious effect on the cosmological parameters \(\Omega_{cdm}\) and \(\Omega_\Lambda\) are not at the present day, but approximately from \(z \approx 10^6\) to \(\sim 5\).

![Figure 10: Evolution of \(\Omega_m\) and \(\Omega_r\) for the \(\Lambda\) CDM model (black lines) and the barotropic model (orange lines). The redshift for the matter–radiation equality era \(z_{eq}\) depends on the diffusion parameter \(x_{cdm}\). For \(x_{cdm} = -0.05\), we have \(z_{eq} \approx 12.7 \times 10^3\) (vertical dash–dotted orange line), whereas for \(\Lambda\) CDM we have \(z_{eq} \approx 3.3 \times 10^3\) (vertical dash–dotted black line).](image)

### 2.1.4 Model 4: Continuous Spontaneous Localization model

An interesting model also studied in [59] is the Continuous Spontaneous Localization (CSL) model [106, 107, 108, 109, 110, 111, 112, 113, 61, 114, 115, 116]. This model arises as a proposal to explain the spontaneous collapse of the wave function in quantum mechanics, where the stochastic nature of the collapse is encoded in a new correction term in the Schrödinger equation. Such modification consists in a stochastic noise describing a diffusion process of the wavefunction in Hilbert space. The source of such noise can be of cosmological origin, for instance, due to the dark matter component in the Universe [117, 118]. In the case we are interested in, the predicted form of the diffusion function \(Q\) according to the CSL model is [59]

\[
\dot{Q} = -\xi_{CSL} \rho_{cdm},
\]

where \(\xi_{CSL}\) is the localization rate, which is interpreted as the frequency of the localization events. After integration of the above expression we have,

\[
Q(t) = Q_i - \xi_{CSL} \int_0^t \rho_{cdm}(t')dt',
\]

where \(Q_i = Q(t = 0)\) is an integration constant that will contribute to the total dark energy density, as we will show. The CDM energy density is then given by,

\[
\rho_{cdm}(z) = \rho_{cdm}(1 + z)^3 e^{\xi_{CSL}t},
\]

which lead to the following Friedmann equation,

\[
E(z) \equiv \frac{H(z)}{H_0} = \sqrt{\Omega_0, r(1 + z)^4 + \Omega_0 h^2 (1 + z)^3 + \Omega_{cdm} \left[ e^{\xi_{CSL}t}(1 + z)^3 - \xi_{CSL} \int_0^t e^{\xi_{CSL}t'}[1 + z(t')]^3 dt' \right] + \Omega_{\Lambda_{eff}}},
\]

### 2.1.4 Model 4: Continuous Spontaneous Localization model

An interesting model also studied in [59] is the Continuous Spontaneous Localization (CSL) model [106, 107, 108, 109, 110, 111, 112, 113, 61, 114, 115, 116]. This model arises as a proposal to explain the spontaneous collapse of the wave function in quantum mechanics, where the stochastic nature of the collapse is encoded in a new correction term in the Schrödinger equation. Such modification consists in a stochastic noise describing a diffusion process of the wavefunction in Hilbert space. The source of such noise can be of cosmological origin, for instance, due to the dark matter component in the Universe [117, 118]. In the case we are interested in, the predicted form of the diffusion function \(Q\) according to the CSL model is [59]

\[
\dot{Q} = -\xi_{CSL} \rho_{cdm},
\]

where \(\xi_{CSL}\) is the localization rate, which is interpreted as the frequency of the localization events. After integration of the above expression we have,

\[
Q(t) = Q_i - \xi_{CSL} \int_0^t \rho_{cdm}(t')dt',
\]

where \(Q_i = Q(t = 0)\) is an integration constant that will contribute to the total dark energy density, as we will show. The CDM energy density is then given by,

\[
\rho_{cdm}(z) = \rho_{cdm}(1 + z)^3 e^{\xi_{CSL}t},
\]

which lead to the following Friedmann equation,

\[
E(z) \equiv \frac{H(z)}{H_0} = \sqrt{\Omega_0, r(1 + z)^4 + \Omega_0 h^2 (1 + z)^3 + \Omega_{cdm} \left[ e^{\xi_{CSL}t}(1 + z)^3 - \xi_{CSL} \int_0^t e^{\xi_{CSL}t'}[1 + z(t')]^3 dt' \right] + \Omega_{\Lambda_{eff}}},
\]
where $\Omega_{\Lambda, eff} \equiv \Omega_\Lambda + (\kappa^2 Q_i / 3H^2_0)$. Once implemented a generalized form of the *lower incomplete Gamma function* to deal with the integral in the above expression, the Friedmann equation can be written as follows (see Appendix A)

\[
E(z) = \begin{cases} 
\sqrt{\Omega_r(z) + \Omega_b(z) + \Omega_{cdm}(z) \left[ e^{-u_0(1+z)^{-2}} + 2u_0(1+z)^{-2} \right] + \Omega_{\Lambda, eff}}, & \text{for Radiation domination}, \\
\sqrt{\Omega_r(z) + \Omega_b(z) + \Omega_{cdm}(z) \left[ e^{-u_0(1+z)^{-3/2}} - u_0(1+z)^{-3/2} \right] + \Omega_{\Lambda, eff}}, & \text{for Matter domination}.
\end{cases}
\]  

(21e)

The evolution of each $\Omega_i(z)$ is shown in Figure 11. Notice that we have introduced a dimensionless parameter $u_0 \equiv -\xi_{CSL}t_0$, where $t_0$ is the current age of the Universe. It seems that the condition $\xi_{CSL} \leq 0$ is required in order to have physically consistent cosmological evolution for this diffusion model during radiation domination era. We will consider, however, both positive and negative values of $u_0$ in our analysis. For more details see Appendix A.

---

**Figure 11:** Energy density parameter of each matter component of the Universe as function of the redshift considering the CSL diffusion process. As in previous Figures, the horizontal gray line stands for $\sum_i \Omega_i(z) = 1$.

The sign of the diffusion parameter affects the behavior of both $\Omega_{cdm}$ and $\Omega_\Lambda$. This can be seen in Figure 12, where $u_0 > 0$ (blue lines) increases the energy density for the cosmological constant as CDM decreases. The opposite occurs for $u_0 < 0$ (orange lines).

---

**Figure 12:** $\Omega_{cdm}(z)$ and $\Omega_\Lambda(z)$ for the CSL model for $u_0 = -0.2, 0.2$. Black lines indicate the standard $\Lambda$CDM result.
As we mentioned before, all these changes in $\Omega_{cdm}$ and $\Omega_\Lambda$ due to the diffusion processes will affect the current value of the Hubble parameter. In particular, if the current values for each energy density parameter $\Omega_{0,i}$ ($i = b, \gamma, \nu, cdm, \Lambda$), as well as for $H_0$ are those obtained from CMB observations (and thus $H_0 = H_0^{early}$ for $\Lambda$CDM), it is possible to find values of the diffusion parameters such that $H_0$ shifts to the value inferred from late time observations, i.e., $H_0 = H_0^{late}$.

We show the evolution of the Hubble parameter in Figure 13. Considering particular values for the parameters of each of the diffusion models studied, we show that it is possible to obtain a cosmological solution for $H(z)$ consistent with the reported value of $H_0$ from local observations. Once the diffusion parameters are set to zero, the $\Lambda$CDM case is recovered and $H_0 = H_0^{early}$.

![Figure 13: Cosmological evolution of the Hubble parameter $H(z)$ for all the diffusion models: STM (yellow), ADMD (blue), BM (red), and CSL (green). Solid black line shows the standard evolution without diffusion, i.e., the diffusion parameters set to zero. Horizontal dashed lines indicate the value of $H_0$ inferred from CMB (purple) and local observations (gray), whose values are respectively given by $H_0^{early} \approx 68 \text{kms}^{-1}\text{Mpc}^{-1}$ and $H_0^{late} \approx 73 \text{kms}^{-1}\text{Mpc}^{-1}$.](image)

Therefore, from the numerical solutions that we have obtained, the four diffusion models studied seem to be good candidates to solve the tension in the value of $H_0$. Focused on an analysis for the late time Universe, this was already shown for model 1 (STM) and model 2 (ADMD) in [58], where the authors explore the parameter space in order to find the possible combinations of the diffusion parameters such that $H_0^{early} \rightarrow H_0^{late}$, although no direct tests with observations are made. On the other hand, the authors in [59] studied models 3 and 4 (BM and CSL respectively), at late times as well. Particularly, in their study they analyze by parts each model depending on the contribution of the dark energy component: for the BM, they consider the cases in which 1) $\Omega_{0,\Lambda} = 0$, 2) $\Omega_{0,\Lambda} = 1$, 3) $0 < \Omega_{0,\Lambda} < 1$, and 4) $\Omega_{0,\Lambda} < 0$. For the CSL model, the cases that the authors analyze are 1) $\Omega_{0,\Lambda} > \Omega_{0,\xi}$, 2) $\Omega_{0,\Lambda} < \Omega_{0,\xi}$, and 3) $\Omega_{0,\Lambda} = \Omega_{0,\xi}$, where $\Omega_{0,\xi} \equiv \xi^{2,CQG}/(9H_0^2)$. The authors include statistical analysis, but only considering local observations, such as Observational Hubble Data (OHD) used in [119], and supernovae (SNe Ia) [120].

We want to emphasise that it is crucial for any model trying to solve the $H_0$ tension to consider the radiation component of the Universe, since it plays an important role in the physics of the last scattering surface, specifically its contribution in the sound speed of the photon–baryon acoustic wave. In particular for models as those studied here, easing the $H_0$ tension should translate in the case in which CMB data are consistent with late time observations. This important part of the analysis is lacking in the previous works mentioned above, and it is the step further we are giving in this work in order to study the true viability of these models. We do not separate the analysis by cases, or restrict the diffusion models to low redshifts, but rather we contemplate all the cosmological evolution considering all the matter components present in the Universe.

Let us summarize the results obtained in this Section as follows: assuming that the values of the cosmological parameters are those inferred from CBM observations, in Figure 13 it is shown that, with the cosmological field equations obtained
from UG, it is possible to obtain a value of $H_0$ consistent with local observations by the means of diffusion processes between cold dark matter and dark energy in the form of a variable cosmological "constant". This is so, of course, for particular choices of the diffusion parameters. A statistical analysis have to be performed in order to infer the most likely values of such parameters in the light of current data taken from several astrophysical and cosmological observations.

3 Statistical analysis

In the previous Section, we have shown that diffusion processes in UG can ease the current tension on the $H_0$ parameter. Specifically, at the level of the numerical solutions for the background dynamics, the parameters of each diffusion model allow to have a consistent match between the $H_0$ inferred from CMB with that of local observations. It is crucial then, to analyze the viability of these models in the light of cosmological observations. Particularly, we want to constraint the diffusion parameters from models 1, 2, 3, and 4 with data from CMB observations. If these diffusion models ease the $H_0$ tension, then CMB data should allow values for $\gamma$ (for model 1), $\gamma$ and $z^*$ (for model 2), $x_{cdm}$ (for model 3), and $\xi_{CSL}$ (for model 4) such that the value of $H_0$ inferred by CMB coincides with that of local observations.

3.1 Constraints with CMB data set

Since we are focused on the background evolution, we will use the Planck Compressed 2018 (PC2018) data [121], instead of the full Planck 2018 likelihoods. Such compressed version of the CMB data have been probed to be as useful as the full version, to constraint not only the standard $\Lambda$CDM model, but also alternative models of dark energy, such as $\omega$CDM, CPL model, interacting dark energy, early dark energy, and all those models dubbed as smooth dark energy, which are models phenomenologically similar to a cosmological constant [122, 123, 124, 125, 126, 127, 128].

In our case, even when the gravitational theory is Unimodular Gravity, we have shown that the cosmological scenario is basically that of General Relativity with a non-gravitational interaction between the dark sector components. Particularly, we still have a cosmological "constant" which only changes its current value. Therefore, we can safely use PC2018 data to constraint the diffusion parameters.

The two main physical quantities to use in order to constraint cosmological models with PC2018 data are the acoustic scale $l_A$, which characterize the CMB temperature power spectrum in the transverse direction (and therefore leading to variations of the peak spacing), and the shift parameter $R$, which affects the CMB temperature spectrum in the line-of-sight direction (and therefore affecting the heights of the peaks),

$$ l_A = (1 + z_\star) \frac{\pi D_A(z_\star)}{r_s(z_\star)} ,$$

$$ R = (1 + z_\star) \frac{\Omega_{0,m} H_0}{c} D_A(z_\star) , \quad \text{(22)} $$

where $z_\star$ is the value of the redshift when photons decouple from baryons ($z_\star \approx 10^3$), $c$ is the speed of light, and $r_s$ and $D_A$ are the comoving sound horizon and the angular diameter distance respectively,

$$ r_s(z) = \frac{1}{H_0} \int_z^{\infty} \frac{c_s(z') dz'}{E(z')} , \quad \text{with} \quad c_s(z) = c \left[ 3 \left( 1 + \frac{3 \Omega_b}{4 \Omega_c (1 + z)} \right) \right]^{-1/2} , \quad \text{(23a)} $$

$$ D_A(z) = \frac{c}{H_0 (1 + z)} \int_0^z \frac{dz'}{E(z')} , \quad \text{for} \quad \Omega_b = 0 , \quad \text{(23b)} $$

where $c_s(z)$ is the sound speed of the photon-baryon acoustic wave. Additionally, the physical baryon energy density parameter $\Omega_b h^2$ with $h = H_0/100$, and the scalar spectral index $n_s$ are included in the PC2018 likelihood as well. It can be seen that the diffusion parameters enter through $E(z)$ given by Eq. (18c), (19c), (20c), (21e), for model 1, 2, 3, and 4 respectively, in the integrands shown in Eq. (23). Here it is evident the importance to have the radiation component in the analysis, since we have to evaluate integrals on $z_\star$ in order to compute $l_A$ and $R$ (see Eq. (22)).

The parameter space of interest for each diffusion model will be spanned by

$$ \Theta_1 = \{ \Omega_b h^2 , \Omega_{cdm} h^2 , H_0 , \alpha , z^* \} , \quad \text{(24a)} $$

$$ \Theta_2 = \{ \Omega_b h^2 , \Omega_{cdm} h^2 , H_0 , \gamma , z^* \} , \quad \text{(24b)} $$

$$ \Theta_3 = \{ \Omega_b h^2 , \Omega_{cdm} h^2 , H_0 , x_{cdm} \} , \quad \text{(24c)} $$

$$ \Theta_4 = \{ \Omega_b h^2 , \Omega_{cdm} h^2 , H_0 , \xi_{CSL} \} , \quad \text{(24d)} $$

This will explicitly show whether the diffusion parameters can take non–null values consistent with CMB data, and simultaneously solving the $H_0$ tension. The (logarithmic) likelihood function $\log L(\Theta)$ is given by

$$ \log L_{CMB}(\Theta) = -\frac{1}{2} \left[ \vec{\mu}_{CMB} - \vec{\mu}(\Theta) \right]^T C^{-1}_{CMB} \left[ \vec{\mu}_{CMB} - \vec{\mu}(\Theta) \right] , \quad \text{(25)} $$
where $C_{CMB}^{-1}$ is the inverse covariance matrix of the CMB observation, $\bar{\mu}_{CMB} = (R_{CMB}, l_A^{CMB}, (\Omega_b h^2)_{CMB}, n_s^{CMB})$ is the vector of CMB observations, and $\bar{\mu}(\Theta_i) = (R(\Theta_i), l_A(\Theta_i), \Omega_b h^2, n_s)$ is the vector of their corresponding theoretical values according to the diffusion models $i = 1, 2, 3, 4$.

To compute the posterior probabilities for each of the diffusion models, we use the software MONTE PYTHON [129]. We have considered flat priors, since they are the most conservatives to use when there is not previous knowledge about the parameters to analyze, as is the case for the diffusion models. As can be seen in Table 1, we have chosen the mean of the diffusion parameters to be those agreeing with $\Lambda$CDM, but with broad priors. For instance, the characteristic redshift $z_i^*(z_f^*)$ for Model 1 (Model 2) is such that, the diffusion process can take place at any moment between the last scattering surface ($z \simeq 1100$) and the present day ($z = 0$). The prior for $H_0$ is such that it contains the two preferred and different mean values reported by cosmological and local observations. On the other hand, the prior for the physical baryon density parameter $\Omega_b h^2$ was established such that $0 \leq \Omega_b \leq 1$, and according to the prior for $H_0$ mentioned above $0.65 \leq h \leq 0.75$. The prior for the CDM parameter $\Omega_{cdm} h^2$ was established in the same way.

| model | parameter | mean   | min prior | max prior   | Std. Dev. |
|-------|-----------|--------|-----------|-------------|-----------|
| ACMD  | $\Omega_b h^2$ | 0.0224 | 0         | 0.5625      | 0.015     |
|       | $\Omega_{cdm} h^2$ | 0.120  | 0         | 0.5625      | 0.0013    |
|       | $H_0$    | 70     | 65        | 75          | 0.01      |
| Model 1 | $\alpha$  | 0      | -1        | 10          | 0.005     |
| Model 2 | $z_1^*$  | 0      | 0         | 1100        | 0.005     |
| Model 3 | $\gamma$  | 0      | -10       | 10          | 0.005     |
| Model 4 | $\Omega_0$ | 0      | -0.01     | 0.05        | 0.001     |

Table 1: Input for the parameters of each diffusion model to generate the MCMC. Each diffusion model also has the same input for the $\Lambda$CDM parameters.

When running the chains, we have monitored the convergence with the Gelman–Rubin criterion [130], by considering $R - 1 < 0.05$. Figure 14 shows the posteriors for the parameters of Model 1 (top left, orange), 2 (top right, blue), 3 (bottom left, red), and 4 (bottom right, green). The posteriors for the $\Lambda$CDM parameters (gray) are shown for comparison.

As general features shared by all the diffusion models, we observe from the posteriors of the standard $\Lambda$CDM parameters that 1) the amount of baryonic matter $\Omega_b h^2$ remains unchanged, 2) the amount of cold dark matter $\Omega_{cdm} h^2$ gets a broader range of values, and since it is anticorrelated with the Hubble parameter, 3) $H_0$ is weakly constrained, and its posterior gets “stretched”. This is not the ideal way to ease the $H_0$ tension, in the sense that it would be desirable CMB data to be consistent with local observations by shifting the mean value of $H_0$ from $H_0^{early}$ to $H_0^{late}$.

In the case of the diffusion parameters from models 1 and 2 we have,

\[
\text{Model 1 (STM)} : \alpha = -0.0202^{+0.0317}_{-0.0152}, \quad z_1^* = 1.58^{+0.25}_{-1.58}. \tag{26a}
\]

\[
\text{Model 2 (ADMD)} : \gamma = -0.633^{+1.67}_{-0.728}, \quad z_2^* = 0.247^{+0.0269}_{-0.247}. \tag{26b}
\]

Whereas in [58] the diffusion parameters $\alpha$ and $\gamma$ are defined strictly positives, we are regarding their values to be constrained by observations, and in principle there is not restriction on the sign of these parameters. Moreover, if they have to be consistent with $\Lambda$CDM, it is convenient to consider a symmetric range around zero. This allow us to see that the most likely values for those parameters have negative mean values, with a standard deviation including $\alpha, \gamma$ equal to zero, where the $\Lambda$CDM model is recovered. On the other hand, the characteristic redshift is $z^* \geq 0$, since the energy transfer between CDM and $\Lambda$ occurs at some time between the last scattering surface and the present day. Both models present a transfer of energy taking place at $z_1^* = 1.58$ for model 1, and $z_2^* = 0.247$ for model 2, which is consistent with a late time diffusion process. The novelty here is that we were able to infer such values through CMB observations, which requires to calculate the angular diameter distance $D_A$ at $z_\ast \simeq 1100$ in order to compute the observables $l_A$ and $R$ (see Eq. (22) and Eq. (23b)),

\[
D_A(z_\ast) = \frac{c}{H_0(1+z_\ast)} \int_0^{z_\ast} \frac{dz'}{E(z')} = \frac{c}{H_0(1+z_\ast)} \left[ \int_0^{z_1^*} \frac{dz'}{E_i(z')} + \int_{z_1^*}^{z_\ast} \frac{dz'}{E_{\Lambda CDM}(z')} \right]. \tag{27}
\]

2While this anticorrelation is true in the $\Lambda$CDM case, strictly speaking, the anticorrelation between $\Omega_{cdm} h^2$ and $H_0$ occurs for the diffusion models 1, 2, and 4. In the case of model 3 the $\{\Omega_{cdm} h^2, H_0\}$--plane present a banana–shaped posterior, and the anticorrelation occurs only for the region corresponding with positive values of $x_{cdm}$. 


where $E_i(z)$ stands for the normalized Friedmann equations (18c), (19c) according to the models $i = 1, 2$ respectively, with $z_i^*$ the corresponding characteristic redshift, and $E_{\Lambda CDM}(z)$ for the standard $\Lambda$CDM case. The first integral within the square brackets quantify the modification induced by the diffusion process.

In the case of model 3, the diffusion parameter presents an upper bound at $x_{cdm} \simeq 0.03$. This allows to the CDM energy density parameter to have a broader range of values. However, the constraint on $H_0$ is not as weak as that imposed on such parameter by the other diffusion models. The parameter $\xi_{CSL}$ of model 4 presents a posterior with two peaks located at $\xi_{CSL} \simeq \left( -1.09 \times 10^{-19}, 5.26 \times 10^{-21} \right) s^{-1}$, which constitute the most likely values for such parameter. These peaks on $\xi_{CSL}$ induce a bimodal posterior in the CDM density parameter whose peaks are located at $\Omega_{cdm}h^2 \simeq \left( 0.105, 0.125 \right)$. While one of the peaks is approximately consistent with the amount of CDM according to the $\Lambda$CDM model, the second peak is located at less CDM contribution, which lead to higher values of $H_0$.

Therefore, whereas for $\Lambda$CDM we obtained the expected values inferred from CMB for $\Omega_b h^2, \Omega_{cdm} h^2$, and $H_0$ given by

$$
\Omega_b h^2 = 0.0223^{+0.000135}_{-0.000137}, \quad \Omega_{cdm} h^2 = 0.122^{+0.000968}_{-0.00095}, \quad H_0 = 67.1^{+0.427}_{-0.438} \text{ km s}^{-1} \text{ Mpc}^{-1},
$$

the diffusion parameters allow to increase the range of values that CDM energy density and $H_0$ usually have within the $\Lambda$CDM model according to CMB data. In particular, $H_0$ is weakly constrained and it can take values from $H_0^{\text{early}}$ to $H_0^{\text{late}}$ while being in agreement with CMB observations. However, and as we mentioned above, this is not the ideal way to ease the $H_0$ tension, in the sense that the mean value of $H_0$ should shift from $H_0^{\text{early}}$ to $H_0^{\text{late}}$, if it is the case.
that these models solve the tension successfully. The latter can be explored by adding new information to the analysis, new data sets that help to break some degeneracies between the parameters.

### 3.2 Constraints with CMB + Late time data set

So far we have obtained that the diffusion models presented here seem to ease the $H_0$ tension, but in such a way that, instead of an effective shift of the mean value, the constraint on $H_0$ is weaker than in the $\Lambda$CDM case due to the presence of the diffusion parameters. Therefore, let us now include observations from the late Universe to study whether the diffusion models are in agreement with $H_0 = H_0^{\text{late}}$ under the combined analysis CMB+SH0ES+H0LICOW+SNe Ia.

Besides the cosmological observation from the CMB ($z_\ast \sim 1100$), we now include some local observations to the analysis: Pantheon, light-curve from 1048 supernovae (SNe Ia) within the redshift range $0.01 < z < 2.26$ [120]. H0LICOW$^3$, time-delay distances of 6 lensed quasars at redshifts $z = 0.654, 1.394, 1.662, 1.693, 1.722, 1.789$ [132, 133, 134, 135, 136, 137, 138, 96]. SH0ES, $H_0$ measurement from Cepheids in the Large Magellanic Cloud ($d \sim 50$ kpc) [92].

The current constraints on $H_0$ from Planck, SH0ES, and H0LICOW can be seen in the next Figure$^4$, where it is also included the $H_0$ value for the combination of these two late Universe observations, as well as the corresponding tension $T_{H_0}$ calculated according to the estimator [139]

$$T_{H_0} = \frac{|\mu_{\text{early}} - \mu_{\text{late}}|}{\sqrt{\sigma_{\text{early}}^2 + \sigma_{\text{late}}^2}},$$

(29)

where $\mu_{\text{early}}$ ($\mu_{\text{late}}$) is the mean value of $H_0^{\text{early}}$ ($H_0^{\text{late}}$), and $\sigma$ their corresponding standard deviation.

![Figure 15](image)

Figure 15: Reduced version of Figure 1 considering the observations we will use. Constraints on $H_0$ from early (top) and late (middle) time observations. It is also shown the value of $H_0$ inferred from the combination of the two late Universe observations considered in this work (bottom). There is a tension of 5.2σ between the value of $H_0$ inferred from early Universe (Planck) and that obtained from late time observations (SH0ES+H0LICOW). Pantheon is not shown since it alone does not constraint $H_0$. See text for more details.

**Supernovae (SNe Ia)**

Observations from the luminosity of SNe Ia allow to estimate their distances. The model for the observed distance modulus $\mu_{\text{SNe}}$ is the following [120],

$$\mu_{\text{SNe}} = m^*_B - M, \quad \text{with} \quad m^*_B = m_B + \tilde{\alpha}X_1 - \beta C + \Delta_M + \Delta_B,$$

(30a)

$^3$For the MONTE PYTHON implementation of the H0LICOW likelihoods see here [131].

$^4$For an editable version of Figure 1, like the one presented in Figure 15 see here [99].
where $m^*_{L,i}$ corresponds to the corrected apparent peak magnitude, $X_1$ describes the time stretching of the light-curve, $C$ stands for the supernova color at maximum brightness, and $\alpha, \beta, M, \Delta_M, \Delta_B$ are nuisance parameters\(^5\) [140, 141, 142]. On the other hand, the distance modulus which relies on the cosmological model is

$$\mu(z) = 5 \log \left[ \frac{d_L(z)}{10 \text{pc}} \right] , \quad \text{with} \quad d_L(z) = \frac{c(1 + z)}{H_0} \int_0^z \frac{dz'}{E(z')} , \quad (30b)$$

where $d_L(z)$ is the luminosity distance. Then, the likelihood function will be,

$$\log \mathcal{L}_{\text{SNe}}(\Theta) = -\frac{1}{2} [\vec{\mu}_{\text{SNe}} - \vec{\mu}(\Theta)]^T C_{\text{SNe}}^{-1} [\vec{\mu}_{\text{SNe}} - \vec{\mu}(\Theta)] , \quad (30c)$$

where $\vec{\mu}_{\text{SNe}}$ and $\vec{\mu}(\Theta)$ are respectively given by Eq. (30a) and (30b) for each supernova, and $C_{\text{SNe}}$ is the covariance matrix.

It is important to mention that $H_0$ can not be constrained when using only SNe Ia because $H_0$ and the nuisance parameter $M$ are strongly degenerated [120]. Thus, data from SNe Ia should be combined with other observations in order to constraint $H_0$ properly.

**Quasars (QSR)**

Strong gravitational lenses allow to measure distances through the time delay between the multiple images of the lensed source. The $H_0$ Lenses in COSMOGRAIL’S Wellspring (H0LICOW) program have measured the current value of the Hubble parameter from a joint analysis of six gravitationally lensed quasars with measured time delay.

The time delay of an image $i$ in comparison with the no lensing case is [143],

$$t(\theta_i, \beta) = \frac{D_{\Delta t}}{c} \phi(\theta_i, \beta) , \quad (31a)$$

where $\theta_i$ is the position of the lensed image of $i$, $\beta$ is the source position, $D_{\Delta t}$ is the so-called time-delay distance, $\phi$ is the Fermat potential, and $c$ is the speed of light. For a lens at redshift $z_d$ and a source at redshift $z_s$, the time-delay distance is given by,

$$D_{\Delta t} = (1 + z_d) \frac{D_d D_s}{D_{ds}} , \quad (31b)$$

where $D_d$ and $D_s$ are the angular diameter distances to the lens and to the source respectively, whereas $D_{ds}$ is the angular diameter distance between the lens and the source (see Eq. (23b)). By measuring the time delay between two images $i$ and $j$, we have

$$\Delta t_{ij} = t(\theta_i, \beta) - t(\theta_j, \beta) = \frac{D_{\Delta t}}{c} \Delta \phi_{ij} . \quad (31c)$$

Once determined the Fermat potential (by modeling the lens mass distribution), and with $\Delta t$ measured, it is possible to infer the value of the time-delay distance $D_{\Delta t}$. Thus, the likelihood function is,

$$\log \mathcal{L}_{\text{QSR}}(\Theta) = -\frac{1}{2} [\vec{\mu}_{\text{QSR}} - \vec{\mu}(\Theta)]^T C_{\text{QSR}}^{-1} [\vec{\mu}_{\text{QSR}} - \vec{\mu}(\Theta)] , \quad (31d)$$

where $\vec{\mu}(\Theta)$ and $\vec{\mu}_{\text{QSR}}$ are respectively given by Eq. (31b) and (31c) for each lensed quasar, and $C_{\text{QSR}}$ is the covariance matrix.

**Cepheids (CPH)**

Cepheids provide a way to measure distances through the period-luminosity relation characterizing them. Particularly, from 70 long-period Cepheids observed by the Hubble Space Telescope (HST) in the Large Magellanic Cloud, the Supernova $H_0$ for the Equation of State (SH0ES) collaboration have determined the current value of the Hubble parameter $H_0^{\text{CPH}} = 74.03 \pm 1.42 \text{km/s/Mpc}$ [92]. The likelihood function is then given by,

$$\log \mathcal{L}_{\text{CPH}}(H_0) = -\frac{1}{2} \left[ \frac{H_0^{\text{CPH}} - H_0}{\sigma_{H_0}^{\text{CPH}}} \right]^2 . \quad (32)$$

The likelihood function associated to these late Universe observations is given by,

$$\log \mathcal{L}_{\text{Late}}(\Theta) = \log \mathcal{L}_{\text{SNe}}(\Theta) + \log \mathcal{L}_{\text{QSR}}(\Theta) + \log \mathcal{L}_{\text{CPH}}(H_0) , \quad (33)$$

\(^5\)We have used a tilde on the nuisance parameter $\alpha$ to avoid confusions with the diffusion parameter $\alpha$ from Model 1 STM.
and finally, the total likelihood function will be the following.

\[
\log L_{\text{tot}}(\Theta) = \log L_{\text{CMB}}(\Theta) + \log L_{\text{Late}}(\Theta). \tag{34}
\]

Considering the same initial mean values, priors, and standard deviations shown in Table 1, Figure 16 shows the posteriors for each diffusion model as well as those for the \(\Lambda \)CDM model. It can be observed that the posteriors for the \(\Lambda \)CDM parameters (gray) do not change considerably with respect to the posteriors obtained in the previous analysis when only CMB data were considered (see Eq. (28)). This time we have,

\[
\Omega_b h^2 = 0.0225^{+0.000134}_{-0.000132}, \quad \Omega_{cdm} h^2 = 0.121^{+0.000877}_{-0.000887}, \quad H_0 = 67.9^{+0.399}_{-0.403} \text{ km s}^{-1}\text{Mpc}^{-1}. \tag{35}
\]

Particularly, the mean of \(H_0\) slightly increases its value from 67.1 to 67.9 km s\(^{-1}\) Mpc\(^{-1}\) and, as expected from the \(\Lambda \)CDM model, the tension persists when this result is compared with the mean value of \(H_0\) inferred from late time observations.

Figure 16: Posterior probability distributions considering CMB+SNe+QSR+CPH data for all diffusion models under study: Sudden Transfer model (top right, orange), Anomalous Decay of the Matter Density model (top right, blue), Barotropic model (bottom left, red), and Continuous Spontaneous Localization model (bottom right, green). We have included the posteriors for each diffusion model as well (gray).

With the aim of quantify how much the tension is eased in the case of the diffusion models, let us propose a similar estimator to that of Eq. (29),

\[
\tau_{H_0} = \left| \mu_{\text{comb}} - \mu_{\text{late}} \right| / \sqrt{\sigma_{\text{comb}}^2 + \sigma_{\text{late}}^2}, \tag{36}
\]

where \(\mu_{\text{comb}}\) and \(\sigma_{\text{comb}}\) are respectively the mean and standard deviation of \(H_0\) according to the combined analysis CMB+SNe+QSR+CPH, whereas \(\mu_{\text{late}}\) and \(\sigma_{\text{late}}\) are the mean and standard deviation of \(H_0\) from the SH0ES and
H0LICOW combined observations given by $H_0 = 73.7^{+1.1}_{-1.0}$ km s$^{-1}$ Mpc$^{-1}$ (see bottom panel of Figure 15$^6$). In particular, we define $\sigma_{comb}$ as the following mean value $\sigma_{comb} \equiv (\sigma_{comb}^+ + \sigma_{comb}^-)/2$ to incorporate the case of an asymmetric standard deviation.

It is important to recall that when only CMB data are considered, the posteriors of $H_0$ for the diffusion models are weakly constrained, and there is not a mean value with an associated standard deviation that can be used to compare with local observations. As discussed in the previous Section, the presence of new parameters allows a broad range of values for the current Hubble parameter. On the other hand, since the value of $H_0$ does not change considerably between the CMB analysis and that of CMB+SNe+QSR+CPH, the results from the latter give approximately the same tension as that obtained from the former. This motivate us to propose Eq. (36). Table 2 shows the mean and 1–$\sigma$ confidence level for the $\Lambda$CDM and diffusion parameters. The tension $T_{H_0}$ for each model is shown as well.

| model | parameter | mean$^{\pm \sigma}$ | $T_{H_0}$ |
|-------|-----------|----------------------|-----------|
| $\Lambda$CDM | $\Omega_{cdm} h^2$ | 0.121$^{+0.000887}_{-0.000877}$ | ... |
| $\Lambda$CDM | $H_0$ | 67.9$^{+0.399}_{-0.403}$ | 5.0$\sigma$ |
| Model 1 | $\Omega_{c0} h^2$ | 0.2023$^{+0.000135}_{-0.000135}$ | ... |
| Model 1 | $\Omega_{c0} h^2$ | 0.109$^{+0.000233}_{-0.000233}$ | ... |
| Model 1 | $H_0$ | 73.4$^{+0.88}_{-0.88}$ | 0.2$\sigma$ |
| Model 1 | $\alpha$ | $-0.0582^{+0.0134}_{-0.0182}$ | ... |
| Model 1 | $z_b^*$ | 0.842$^{+0.318}_{-0.547}$ | ... |
| Model 2 | $\Omega_{c0} h^2$ | 0.109$^{+0.000233}_{-0.000233}$ | ... |
| Model 2 | $H_0$ | 73.2$^{+0.88}_{-0.88}$ | 0.3$\sigma$ |
| Model 2 | $\gamma$ | $-0.089^{+0.0672}_{-0.0212}$ | ... |
| Model 2 | $z_b^*$ | 1.41$^{+0.414}_{-1.13}$ | ... |
| Model 3 | $\Omega_{c0} h^2$ | 0.094$^{+0.000784}_{-0.000784}$ | ... |
| Model 3 | $H_0$ | 70$^{+0.919}_{-1.177}$ | 2.4$\sigma$ |
| Model 3 | $x_{cdm}$ | 0.0135$^{+0.00475}_{-0.00436}$ | ... |
| Model 4 | $\Omega_{c0} h^2$ | 0.106$^{+0.00284}_{-0.00301}$ | ... |
| Model 4 | $H_0$ | 73.9$^{+0.951}_{-0.969}$ | 1.1$\sigma$ |
| Model 4 | $\xi_{CSL}$ | $(-8.4^{+1.08}_{-1.85}) \times 10^{-20}$ | ... |

Table 2: Mean values and corresponding $1\sigma$ confidence level for the parameters of each diffusion model, as well as for the $\Lambda$CDM parameters. The units of $H_0$ and $\xi_{CSL}$ are given respectively by km s$^{-1}$ Mpc$^{-1}$ and $s^{-1}$. The last column indicates the tension $T_{H_0}$ according to Eq. (36). See the text for more details.

We want to end this Section by highlighting the following: having included local observations in the analysis led to specific and well–defined values of the diffusion parameters. In particular, the amount of CDM and $H_0$ get precise values, opposite to the case in which only CMB data was used. The amount of total matter ($\Omega_{o} + \Omega_{c0}$) $h^2$ predicted by the diffusion models is less than the $\Lambda$CDM case, which translates in a higher value for $H_0$. This indicates that the diffusion models are, effectively, good candidates to alleviate the $H_0$ tension. In this sense, our statistical analysis verifies what we have obtained numerically in the previous Section: within the framework of Unimodal Gravity, the non–gravitational interaction between the dark sector components through diffusion processes, leads to an inferred value of $H_0$ consistent with local observation when constraining the diffusion parameters with CMB data and CMB+local observations.

$^6$The presence of SNe Ia in the combined analysis we have made does not affect the results obtained from the estimator (36). Since SH0ES and H0LICOW impose the late time constraint on $H_0$, once combined, SNe Ia data are in agreement with such constraints. Thus, we do not expect a substantial change on the combined SH0ES+H0LICOW value $H_0^{\text{late}} = 73.7^{+1.1}_{-1.0}$ km s$^{-1}$ Mpc$^{-1}$ due to SNe Ia.
4 Discussion and final remarks

Whereas the main ingredient responsible of the dynamics of the late Universe remains unknown, the best model to explain the current accelerated expansion is given by the cosmological constant in the Einstein field equations. Nonetheless, the origin of such constant is still unknown, although there are several theoretical proposal trying to understand its nature. Unimodular Gravity offers an explanation of the origin of the cosmological constant: it arises as an integration constant once volume–preserving diffeomorphisms are considered in the Einstein–Hilbert action.

The dynamics of the spacetime metric is governed by the trace–free version of the Einstein field equations and, as a direct consequence, the energy–momentum tensor is not conserved once the Bianchi identities are applied. The non–conservation of the energy–momentum tensor allows matter components to interact with each other non–gravitationally. Since the physics of the standard model of particles is very well understood, we have assumed the non–gravitational interaction to be possible only between the dark sector components, which are given by a cold dark matter fluid ρ_{cdm} and a effective cosmological constant with dependence on the cosmic time Λ(t). To describe the interaction between ρ_{cdm} and Λ(t), we have considered phenomenological models previously studied in [58, 59], where the authors propose diffusion processes as the mechanism for the energy transfer from one dark component to the other.

In general, models with non–gravitational interactions in the dark sector have been studied with emphasis in the H₀ tension, this is, the current discrepancy between the value of the Hubble parameter at present day inferred from early Universe, and that inferred from local observations. The authors in [58, 59] have also analyzed the possibility of ease such tension through diffusion processes between ρ_{cdm} and Λ(t) in UG. However, their analysis have been restricted to the dynamics of the late Universe which, although some hints about the viability of these models can be obtained, a more suitable analysis must involves physics of the early Universe, in particular the radiation component has to be included in the cosmological evolution. This is important because any model addressing the H₀ tension must be tested not only with late time observations, but with CMB data as well. For the latter it is mandatory to evaluate the model at high redshifts, in particular at the moment of photon–baryon decoupling occurring at zₙ ≃ 1100. Thus, we have implemented each of the diffusion models studied in the works mentioned before, in a more realistic cosmological scenario, where the background evolution includes the presence of photons and ultra–relativistic neutrinos as the components of radiation in the Universe.

Considering a spatially–flat FRW line element, we have solved the UG field equations, from very deep in the radiation domination era (initial redshift z(tᵢ) = 10^{14}) to the present day (z(t₀) = 0), with special interest in the cosmological evolution of the energy density parameters Ωᵦ where i = b, γ, ν, cdm, Λ stands for baryons, photons, neutrinos, cold dark matter, and cosmological constant respectively. We explicitly showed how the diffusion process takes place between Ω_{cdm} and Ω_{Λ} for each of the four diffusion models studied: Sudden Transfer Model (STM) and Anomalous Decay of the Matter Density (ADMD) from [58], and Barotropic Model (BM) and Continuous Spontaneous Localization (CSL) from [59], for which we have obtained the corresponding modified Friedmann equation (18c), (19c), (20c), and (21e). For the BM and CSL model, we have implemented a different approach to that of the authors in [59]: instead of separating the analysis by cases depending on the contribution of the cosmological constant energy density, we have solved the full set of background equations for all the cosmological evolution considering the modification on H(z) due to the diffusion models. In particular, this was not direct to do for the CSL model, which expression for the Friedmann equation involves an integro–differential equation for z(t).

It was crucial to have the expression of the modified Friedmann equation for each diffusion model, since later in the statistical analysis, H(z) enters in the angular diameter distance Dₘ that is needed to compute the observables from CMB (Eq. (22)) and Quasars (Eq. (31b)), as well as in the distance modulus μ through the luminosity distance dₗ for SNe Ia (Eq. (30b)).

We explored the influence of the diffusion parameters on the dynamics of the dark sector, as well as how these parameters modify the current amount of the dark components energy densities. The magnitude and sign of the diffusion parameters set the contribution of both, Ω_{cdm} and Ω_{Λ}, and thus, the Hubble parameter at present day change its value as well. In particular, when considering that the values of each Ωᵦ and H₀ are those reported by CMB observations (and then H₀ = H₀^{early} ≃ 67 km s^{-1} Mpc^{-1}), we have shown that it is possible to find values of the diffusion parameters such that the diffusion models lead to a Hubble parameter at z = 0 consistent with the value obtained from data of the local Universe (i.e., H₀ = H₀^{late} ≃ 73 km s^{-1} Mpc^{-1}). Therefore, at the level of the numerical solutions, all the diffusion models ease the H₀ tension. Nonetheless, to truly test these models and to analyze properly their viability as compelling candidates to solve the discrepancy on the measurement of the current value of the Hubble parameter, it is mandatory a statistical analysis. While such analysis was not performed in [58], the authors in [59] constrain the BM and CSL model by using data from late time observations (Observational Hubble Data (OHD) and supernovae (SNe Ia)). As we mentioned before, to establish a proper comparison between the predictions of the diffusion models in what the H₀
tension is concerned, not only data set from local observations have to be considered, but from the early Universe as well. We have done this by using the Planck Compressed 2018 data.

Since the diffusion models predict less amount of matter in order to rise the value of \(H_0\), we have let to vary both the physical energy density for baryons \(\Omega_b h^2\) and CDM \(\Omega_{cdm} h^2\), considering the broadest priors possible. Besides, the prior for \(H_0\) included the observed mean values from both, early and late time Universe. In the case of the diffusion parameters, we also considered the broadest priors possible. In fact, by setting the more broadest priors have costed to expend more time for the chains to converge when exploring the parameter space. This is so because the new parameters from the diffusion models induce a longer anticorrelation between \(\Omega_{cdm} h^2\) and \(H_0\), and more possible combinations of such parameters are allowed by the observations in comparison with the \(ΛCDM\) case. When considering only CMB observations, the additional parameters from the diffusion models allow to ease the tension on \(H_0\), for which weaker constraints are imposed. In fact, the CDM energy density parameter \(\Omega_{cdm}\) is weakly constrained as well, and given that the diffusion parameters lead to lower values of \(\Omega_{cdm}\), this translates to higher values of \(H_0\). Thus, the posteriors for \(H_0\) get a broader range of values, which includes those inferred from both early and late time Universe (see Figure 14). It is when local observations are included into the analysis that an actual shift on the mean value of \(H_0\) occurs (see Figure 16).

In the case of the model 1 (STM), negative values of the diffusion parameter \(\alpha\) are preferred by observations. This is not the case in [58] where such parameter is defined such that \(0 < \alpha < 1\), and the parameter space is restricted for positive \(\alpha\) only. However, our result is not in contradiction with the prediction of the model: the contribution of matter today will be less than the predicted by \(ΛCDM\). In fact, this is a prediction of all the diffusion models studied, as can be seen in Figure 16 where both \(\Omega_b h^2\) and \(\Omega_{cdm} h^2\) for the STM (red), ADMD (blue), BM (red), and CSL (green) are shifted to the left in comparison with the \(ΛCDM\) case (gray). Since the amount of baryons is a well–measured quantity, the shift is less than 1%, whereas for CDM the differences are approximately 9.9% for STM and ADMD, 22.3% for BM, and 12.4% for CSL. For the diffusion parameter \(\gamma\) from model 2 (ADMD), we observe that a negative value is preferred as well. The characteristic redshift \(z^*\) for models 1 and 2 are constrained to low redshifts, this is, the diffusion processes described by these models occur in the late Universe. Such constraints were obtained under the assumption that the diffusion process could take place at any time between the last scattering surface and the present day, which was important to consider since, according to the analysis of the parameter space carried out in [58], larger values of \(\alpha\) lead to values of \(\alpha\) and \(\gamma\) such that the \(ΛCDM\) case is recovered, i.e., for \(z^* \gg 1\), \(\alpha, \gamma \to 0\) (see Figures 3 and 6 from [58]).

The constraint we have obtained on the diffusion parameter \(x_{cdm}\) from model 3 (BM) is in agreement with that reported in [59] for the case in which the authors infer the value of \(x_{cdm}\) from OHD and SNe Ia independently. The authors have rewritten the supernovae likelihood only as a function of the redshift and the cosmological parameters, and no nuisance parameters are present (see Eq.55 in [59]). This gives a value of \(H_0 = 73.2^{+1.7}_{-1.5}\), which is larger than the inferred value in our analysis. Nonetheless, as can be compared from the reported values in [59] shown above and our results, our uncertainties for \(x_{cdm}\) are smaller (see Model 3 in Table 2). In the case of \(H_0\), our result is consistent with that of the authors, at least when compared with their analysis for OHD and OHD+SNe Ia. As we mentioned in Section 3.2, the Pantheon data set alone can not be used to constrain the \(H_0\) since it is degenerated with \(M\) [120]. In order to have a \(H_0\) measurement only from SNe Ia, we have noticed that in [59] the authors have rewritten the supernovae likelihood only as a function of the redshift and the cosmological parameters, and no nuisance parameters are present (see Eq.55 in [59]). This gives a value of \(H_0 = 73.2^{+1.7}_{-1.5}\), which is larger than the inferred value of \(H_0\) in our analysis for model 3. In the case of model 4 (CSL), our inferred value for \(\xi_{CSL}\) is incompatible with the constraints found in [61, 144, 59], where positive values for such parameter are inferred, whereas from our analysis we have obtained that \(\xi_{CSL} < 0\). Nonetheless, let us to pinpoint out the following aspects to be regarded: in the works mentioned above, the non–gravitational interaction includes ordinary matter, since the authors consider energy transfer between the cosmological constant and the total matter content \(\rho_m\). It has to be recalled that in our case, the violation of the energy–momentum tensor is produced only through the components of the dark sector. On the other hand, as discussed in [61] the sign of \(\xi_{CSL}\) can be negative, which implies an endothermic evolution. The latter scenario can be obtained, for instance, from approaches to quantum gravity such as causal set [145, 146].

With the aim of quantify how much these diffusion models ease the tension on \(H_0\), we proposed an estimator \(\bar{T}_H\) to compute the differences between our combined analysis CMB+SNe+QSR+CPH and local observations from SHOES and HOLICOW (see Eq. (36)). The model 1 (STM) is the one reducing more the tension with a difference of \(0.2\sigma\), whereas model 3 (BM) gives the larger difference by easing the tension at \(2.4\sigma\). These results indicate that diffusion models in UG are viable theoretical proposals to solve the \(H_0\) tension.

It will be interesting to explore the cosmological perturbations for the diffusion models studied. Previous works have made some contributions in this direction, although imposing by hand the conservation of the energy–momentum tensor, which as we have shown it is not the most general way to solve the UG field equations. Even so, in [147] the authors
show that temperature fluctuations of the Cosmic Microwave Background (CMB) radiation, and specifically the Sachs–Wolfe effect [148] has a correction term given by a scalar metric perturbation that is demanded to be non–vanishing in UG. Assuming adiabatic fluctuations, the authors of the mentioned work found that the main difference between the GR and UG prediction is only a dipole–like term which is suppressed at large scale, and thus, the effect induced by UG is negligible. Moreover, in terms of gauge–invariant quantities, it was shown in [149] that the GR and UG cosmological perturbations are identical, and then, CMB photons will not distinguish between the two theories. An analysis including the non–conservation of the energy–momentum tensor has to be made in order to explore possible deviations of GR in cosmological observables such as CMB anisotropies and large scale structures.

While the discrepancies between early and late Universe in the $H_0$ measurements may be due to still unaccounted systematic errors in the observations, there exists the possibility that new physics is needed in order to unravel this cosmological conundrum. A deeper comprehension of the physical mechanisms driving the diffusion processes is needed in order to have a complete description of the non–gravitational interaction between the dark sector components. Nonetheless, the phenomenological models studied here might constitute a compelling variation to the ΛCDM paradigm. In this sense, Unimodular Gravity offers not only a explanation to the origin of the cosmological constant, but also the non–conservation of the energy–momentum tensor that naturally arises in this gravitational framework allows to address the $H_0$ tension successfully.

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## A Incomplete Gamma Function and the CSL model

The Continuous Spontaneous Localization (CSL) model leads to an expression for the CDM energy density that involves the following integral

$$I(t) \equiv \int_0^t e^{\xi_{CSL}t'} [1 + z(t')]^3 dt'. \tag{37}$$

We need to solve the background evolution to be able to give the explicit function $z(t)$, and thus, the Friedmann equation (21d) is an integro–differential equation,

$$\frac{1}{H_0^2} \left[ \frac{dz(t)}{dt} \right]^2 = \Omega_r[z(t)] + \Omega_b[z(t)] + \Omega_{cdm}[z(t)]e^{\xi_{CSL}t} - \Omega_{cdm}\xi_{CSL} \int_0^t e^{\xi_{CSL}t'} [1 + z(t')]^3 dt' + \Omega_{\Lambda_{eff}}. \tag{38}$$

This will be the case even if we try to integrate $I(t)$ by parts: let be $f(t) \equiv [1 + z(t)]^3$, then

$$I(t) = \left. \frac{f(t')e^{\xi_{CSL}t'}}{\xi_{CSL}} \right|_0^t - \int_0^t \frac{e^{\xi_{CSL}t'}}{\xi_{CSL}} f(t') dt'$$

$$= \frac{[1 + z(t)]^3 e^{\xi_{CSL}t}}{\xi_{CSL}} - \frac{1}{\xi_{CSL}} + \frac{3}{\xi_{CSL}} \int_0^t e^{\xi_{CSL}t'} [1 + z(t')]^3 H(t') dt'. \tag{39}$$

A way to handle this situation, is by proposing an ansatz on the temporal dependence of the scale factor $a(t)$, or equivalently, the redshift $z(t)$. This is in fact usually done in the framework of ΛCDM model, where it is well–known that a power law relates the scale factor with the cosmic time as $a(t) = (t/t_0)^p$, with $p = 1/2$ ($p = 2/3$) for radiation (matter) domination era. Thus, considering that the new term arising from the CSL diffusion process will not change drastically such power law relation, let us propose

$$a(t) = \frac{1}{1 + z(t)} \equiv \left( \frac{t}{t_0} \right)^p, \tag{40}$$

and then the integral (37) is then written as

$$I(t) = \int_0^t e^{\xi_{CSL}t'} \left( \frac{t'}{t_0} \right)^{-3p} dt'. \tag{41}$$

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If we now introduce the following change of variable \( u \equiv -\xi_{CSL} t \), we have

\[
I_s(u) \equiv \frac{t_0}{u_0} \int_0^u e^{-u'} u'^{s-1} du',
\]

where we have defined \(-3p \equiv s - 1\), and \(u_0 \equiv -\xi_{CSL} t_0\). The integral in the last expression looks like the lower incomplete Gamma function \( \gamma(s, u) \) [150, 151, 152, 153], which has the following series representation [154, 155, 156]

\[
\gamma(s, u) = \int_0^u e^{-u'} u'^{s-1} du' = \sum_{k=0}^{\infty} \frac{(-1)^k u^{k+s}}{k!(k+s)} , \quad \text{for } s > 0 \text{ and } u \geq 0 .
\]

Thus, it is mandatory to verify whether \( \{ u, s \} \) satisfy such conditions. Given the change of variable \( u \equiv -\xi_{CSL} t \), we observe that \( \xi_{CSL} \leq 0 \) is needed to have \( u \geq 0 \). On the other hand, \( s > 0 \) implies \( 1 - 3p > 0 \). Considering values of the diffusion constant such that \( | \xi_{CSL} | < 1 \), we expect that the power law given by the ansatz (40) does not differ too much from the \( \Lambda \)CDM result, and then \( s = -1/2 \) (\( s = -1 \)) for radiation (matter) domination era. Moreover, it can be seen that the constant term out of the integral (42) will be imaginary in radiation domination if \( \xi_{CSL} > 0 \).

Before going further, let us summarize the above discussion as follows: the only case that will lead to an integral with physical meaning \( I_s(u) \in \mathbb{R} \), is when \( \xi_{CSL} \leq 0 \Rightarrow u \geq 0 \), with \( s < 0 \). Since the lower incomplete Gamma function \( \gamma(s, u) \) does not admits negative values of the so-called shape parameter \( s \), some sort of extension or generalization has to be implemented. This have been possible by using the tools of Neutrix Calculus [157, 158, 159], when studying the asymptotic behavior of divergent integrals. The main use of neutrix calculus and neutrix limit, is to extract the finite part of divergent quantities. This have been used, for example, in Quantum Field Theory to obtain finite renormalizations in loop calculations [160, 161]. Within this context, different generalizations to the standard and well–known lower incomplete Gamma function have been developed [162, 163, 164, 165, 166, 167, 168, 169]. Particularly in [163, 165, 168] have developed an extension to consider negative integers given by

\[
\gamma(-s, u) = \frac{(-1)^s}{s!} \ln u + \sum_{k=0}^{\infty} \frac{(-1)^k u^{k-s}}{k!(k-s)} , \quad \text{for } s \in \mathbb{N} \text{ and } u > 0 .
\]

Since \( s \in \mathbb{N} \), the above equation will be useful for the matter domination era, where we are interested in \( \gamma(-1, u) \).

On the other hand, another way to write the lower incomplete Gamma function is given by its suitably normalized form [170, 171]

\[
\gamma^*(s, u) = \frac{u^{-s}}{\Gamma(s)} \gamma(s, u) .
\]

Written like this, \( \gamma^*(s, u) \) is a real–valued function for \( s, u \in \mathbb{R} \) and such that \( s, u > 0 \). An extension of this formula was given in [172, 173] to consider negative and real values of \( s \) with \( u > 0 \), where particularly for \( s \geq -1/2 \) we have

\[
\gamma^*(s, u) = \frac{1}{\Gamma(s+1)} \sum_{k=0}^{\infty} \frac{s(-u)^k}{k!(k+s)} .
\]

Therefore, using Eq. (45) we have that

\[
\gamma(s, u) = \frac{\Gamma(s) u^s}{\Gamma(s+1)} \sum_{k=0}^{\infty} \frac{s(-u)^k}{k!(k+s)} , \quad \text{for } s \geq -1/2 \text{ and } u > 0 .
\]

The last equation can then be used during the radiation domination era, where \( s = -1/2 \). Thus, the integral in Eq. (42) can be now formally identified with the Generalized lower incomplete Gamma function \( \gamma^G(s, u) \) as follows,

\[
I_s(u) = \frac{t_0}{u_0} \gamma^G(s, u) , \quad \text{for } u > 0 ,
\]

where, using the results of Eq. (44) and Eq. (47), we have

\[
\gamma^G(s, u) = \begin{cases} 
\gamma(-1, u) = - \ln u + \sum_{k=0}^{\infty} \frac{(-1)^k u^{k-1}}{k!(k-1)} , & \text{for } s = -1 , \\
\gamma(-1/2, u) = - \frac{1}{u^{1/2}} \sum_{k=0}^{\infty} \frac{(-u)^k}{k!(k-1/2)} , & \text{for } s = -1/2 ,
\end{cases}
\]
where we have used that $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(-1/2) = -2\sqrt{\pi}$. The Gamma function $\Gamma(s)$ is classically defined for positive integers, but it can be extended to both, real and imaginary numbers as well (see for instance [174]). Now, let us work on the series shown in Eq. (49).

A.1 $s = -1$

For matter domination era we have

$$\gamma^G(-1, u) = -\ln u + \sum_{k=0}^{\infty} \frac{(-1)^k u^{k-1}}{k!(k - 1)} \simeq -\ln u - \frac{1}{u} + \frac{u}{2} - \frac{u^2}{12} + \ldots ,$$

(50)

which as function of the redshift is written as

$$\gamma^G(-1, z) \simeq -\ln \left[ \frac{u_0}{(1 + z)^{3/2}} \right] - \frac{(1 + z)^{3/2}}{u_0} + \frac{u_0}{2(1 + z)^{3/2}} - \frac{u_0^2}{12(1 + z)^3} + \ldots .$$

(51)

Then, Eq. (48) is given by

$$I_{-1}(z) \simeq -t_0 u_0 \ln \left[ \frac{u_0}{(1 + z)^{3/2}} \right] - t_0 (1 + z)^{3/2} + \frac{t_0 u_0^2}{2(1 + z)^{3/2}} - \frac{t_0 u_0^3}{12(1 + z)^3} + \ldots .$$

(52)

Recalling that this integral has a multiplicative factor of $\xi_{CSL}$ (see Eq. (38)), and considering the dimensionless variable $u_0 = -\xi_{CSL}t_0$ as the parameter with which the expansion is developed,

$$\xi_{CSL}I_{-1}(z) \simeq u_0(1 + z)^{3/2} + u_0^2 \ln \left[ \frac{u_0}{(1 + z)^{3/2}} \right] - \frac{u_0^3}{2(1 + z)^{3/2}} + \frac{u_0^4}{12(1 + z)^3} + \ldots , \text{ for } u_0 < 1 .$$

(53)

Therefore, if we neglect terms of order equal and higher than $O(u_0^2)$, we have that in the matter domination era, the modified Friedmann equation is written as

$$E(z) = \sqrt{\Omega_r(z) + \Omega_b(z) + \Omega_{cdm}(z) \left[ e^{-u_0(1+z)^{-3/2}} - u_0(1+z)^{-3/2} \right] + \Omega_{\Lambda_{eff}} .}$$

(54)

A.2 $s = -1/2$

In the case of the radiation domination era, Eq. (49) reads as follows,

$$\gamma^G(-1/2, u) = -\frac{1}{u^{1/2}} \sum_{k=0}^{\infty} \frac{(-u)^k}{k!(k - 1/2)} \simeq \frac{2}{u^{1/2}} + 2u^{1/2} - \frac{u^{3/2}}{3} + \frac{u^{5/2}}{15} + \ldots .$$

(55)

When the above expression is written in terms of the redshift we have

$$\gamma^G(-1/2, z) \simeq \frac{2(1 + z)}{u_0^{1/2}} + \frac{2u_0^{1/2}}{1 + z} - \frac{u_0^{3/2}}{3(1 + z)^2} + \frac{u_0^{5/2}}{15(1 + z)^3} + \ldots ,$$

(56)

which leads to the integral (48)

$$\xi_{CSL}I_{-1/2}(z) \simeq -2u_0(1 + z) - \frac{2u_0^2}{1 + z} + \frac{u_0^3}{3(1 + z)^2} - \frac{u_0^4}{15(1 + z)^3} + \ldots .$$

(57)

where we have already included the multiplicative factor $\xi_{CSL}$. Again, neglecting terms of order $O(u_0^2)$ and higher, the Friedmann equation is written as

$$E(z) = \sqrt{\Omega_r(z) + \Omega_b(z) + \Omega_{cdm}(z) \left[ e^{-u_0(1+z)^{-1/2}} + 2u_0(1+z)^{-1/2} \right] + \Omega_{\Lambda_{eff}} .}$$

(58)

Therefore, the Friedmann equation for the CSL model, can be written as Eq. (54) and (58) for matter domination and radiation domination respectively, as is shown in Eq. (21e).
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