Hardness Results on Curve/Point Set Matching with Fréchet Distance

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Abstract

Let $P$ be a polygonal curve in $\mathbb{R}^d$ of length $n$, and $S$ be a point-set of size $k$. We consider the problem of finding a polygonal curve $Q$ on $S$ such that all points in $S$ are visited and the Fréchet distance from $P$ is less than a given $\varepsilon$. We show that this problem is NP-complete, regardless of whether or not points from $S$ are allowed to be visited more than once. However, we also show that if the problem instance satisfies certain restrictions, the problem is polynomial-time solvable, and we briefly outline an algorithm that computes $Q$.

1 Introduction

Measuring the similarity between two geometric objects is a fundamental problem in many fields of science and engineering. However, to perform such comparisons, a good metric is required to formalize the intuitive concept of “similarity.” Among the many metrics that have been considered, Fréchet distance has emerged as a popular and powerful choice, especially when the geometric objects are curves. Shape matching with Fréchet distance has been applied in many different fields, including handwriting recognition [7], protein structure alignment [5], and vehicle tracking [3].

In this paper, we consider the basic problem of measuring the similarity of two polygonal curves. However, in our problem, the input is only partially defined. Instead of being given both curves, we are given only one polygonal curve $P$ as well as a point set $S$. Our problem is to complete this partial input by constructing a polygonal curve $Q$ that best matches the given curve, under the restriction that the constructed curve’s vertices are exactly $S$.

Since our metric of choice is Fréchet distance, we begin with an popular, intuitive description the concept. The metaphor of a person walking a dog is often used, with the dog walking along one curve and its owner walking along the other. The Fréchet distance between the two curves is the length of the smallest leash that would allow both the person and the dog to reach the end of their respective curves without ever backtracking or letting go of the leash. If a very short leash is sufficient, then the curves are very similar. But if a longer leash is required, then the curves are very different.

Suppose our dog owner wants to walk his dog while walking down a path in a given park. The extremely curious and territorial dog wants to sniff and/or mark every single tree along the path, running directly from tree to tree. Mindful of their dog’s proclivities, the owner goes shopping for a leash long enough to allow the dog to have its way without pulling the owner off the path. For a given path and set of trees, will a leash of a given length be sufficient? More formally, given a polygonal curve $P$, a point-set $S$, and a real number $\varepsilon > 0$, does there exist a polygonal curve on $S$ that visits every point in $S$ and has Fréchet distance less than $\varepsilon$ from $P$? Unfortunately for owners of territorial dogs, we show in this paper this problem is NP-complete.

2 Previous Work and New Results

The decision version of the Fréchet distance problem asks, given two geometric objects and a real number $\varepsilon > 0$, is the Fréchet distance $\delta_F$ between the two objects less than $\varepsilon$? Alt and Godau [1] showed that, when the objects in question are polygonal curves of length $n$ and $m$, this problem can be solved in $O(nm)$ time. They also showed that finding the exact Fréchet distance between the two curves can be done in $O(nm \log(nm))$ time.

Maheshwari et al. [6] examined the following variant of the Fréchet distance problem, which we refer to as the Curve/Point Set Matching (CPSM) problem. Given a polygonal curve $P$ of length $n$, a point set $S$ of size $k$, and a number $\varepsilon > 0$, determine whether there exists a
Table 1: Eight versions of the CPSM problem and their complexity classes. New results starred.

|                | Discrete | Continuous |
|----------------|----------|------------|
| Subset Unique  | NP-C     | Open       |
| Non-Unique     | P        | P          |
| All-Pts Unique | NP-C     | NP-C*      |
| Non-Unique     | P        | NP-C*      |

Table 1: Eight versions of the CPSM problem and their complexity classes. New results starred.

3 Preliminaries

Given two curves $P, Q : [0, 1] \rightarrow \mathbb{R}^d$, the Fréchet distance between $P$ and $Q$ is defined as $\delta_F(P, Q) = \inf_{\sigma, \tau} \max_{t \in [0, 1]} \| P(\sigma(t)), Q(\tau(t)) \|$, where $\sigma, \tau : [0, 1] \rightarrow [0, 1]$ range over all continuous non-decreasing surjective functions [4]. We make use of two commonly noted observations:

**Observation 1** Given four points $a, b, c, d \in \mathbb{R}^d$, if $\|ac\| \leq \varepsilon$ and $\|bd\| \leq \varepsilon$, then $\delta_F(ab, cd) \leq \varepsilon$.

**Observation 2** Let $P_1, P_2, Q_1,$ and $Q_2$ be four curves in $\mathbb{R}^d$ with $\delta_F(P_1, Q_1) \leq \varepsilon$ and $\delta_F(P_2, Q_2) \leq \varepsilon$. If the ending point of $P_1$ (resp. $Q_1$) is the same as the starting point of $P_2$ (resp. $Q_2$) then $\delta_F(P_1 + P_2, Q_1 + Q_2) \leq \varepsilon$, where $+$ denotes concatenation.

We now give a number of geometric definitions, some of which were used in [6]. For a given a point $p \in \mathbb{R}^d$ and a real number $\varepsilon > 0$, let $B(p, \varepsilon) \equiv \{ q \in \mathbb{R}^d : \|pq\| \leq \varepsilon \}$ denote the ball of radius $\varepsilon$ centered at $p$, where $\|\cdot\|$ denotes Euclidean distance. For a line segment $L \subset \mathbb{R}^d$, let $C(L, \varepsilon) \equiv \bigcup_{p \in L} B(p, \varepsilon)$ denote the cylinder of radius $\varepsilon$ around $L$. Note that a necessary condition for two polygonal curves $P$ and $Q$ to have Fréchet distance less than $\varepsilon$ is that the vertices of $Q$ must all lie within the cylinder of some segment of $P$.

Let the continuous function $F : [0, 1] \rightarrow \mathbb{R}^d$ represent a curve in $\mathbb{R}^d$. Given two points $u, v \in P$, we use the notation $u \prec v$ if $u$ occurs before $v$ on a traversal of $P$.

The relation $\succ$ is defined analogously.

4 Restricted Satisfiability Problem

Our NP-completeness result is obtained via reduction from a restricted version of the well-known 3SAT problem. The 3SAT problem takes as input a boolean formula with clauses of size 3, and asks whether there exists an assignment to the variables that makes the formula evaluate to TRUE. If we restrict the input to formulas in which each literal occurs exactly twice, the problem becomes the (3,B2)-SAT problem. This may seem to be a rather extreme restriction, and, indeed, formulas of this type with less than 20 clauses are always satisfiable. However, despite this restriction, the problem was shown to be NP-complete in [2], and an example of an unsatisfiable formula with 20 clauses was presented.

In order to simplify our reduction, we make the further restriction that no two clauses have two literals in common. In other words, we restrict the input to formulas in which the function that maps each literal to the pair of clauses it appears in is injective. For any formula that violates this assumption, an equivalent, compliant formula can easily be constructed using the “balanced enforcers” described in [2]. We therefore assume formulas to have this property for the remainder of the paper.

In the following sections, we first give a brief summary of the construction. We then describe the main gadget used, building it incrementally. Finally, we give the full construction in detail.

5 The Reduction

Let $\Phi$ be a formula given as input to the (3,B2)-SAT problem. We construct a polygonal curve $P$ and a point set $S$ such that $\Phi$ is satisfiable if and only if there exists a polygonal curve $Q$ whose vertices are exactly $S$ with Fréchet distance less than $\varepsilon$ from $P$.

First, we construct a gadget consisting of components of $P$ and $S$ that will force any algorithm to choose between two possible polygonal path constructions. The gadget is constructed in such a way that these two choices are the only possible polygonal paths along the gadget’s component of $S$ with Fréchet distance less than
5.1 Separation Gadget

We begin the description of our main gadget with a specific example, which we will proceed to generalize. Consider the problem instance shown in Figure 2a, with \( S = \{a, b, c, d\} \). It is clear that the answer to this problem instance is “no”: no polygonal curve on \( S \) with \( \delta_F(P, Q) \leq \varepsilon \) can visit both \( b \) and \( c \). However, suppose this \( P \) and \( S \) were part of a larger problem instance. Suppose further that other segments of \( P \) come within \( \varepsilon \) of \( b \) and \( c \). The answer to the problem instance is no longer so obvious. Even if both points cannot be reached the first time they are encountered, it is possible that whichever point was skipped could be covered in the future. This creates the fundamental difficulty that leads to our reduction.

Figure 2b shows an extension of the previous configuration, with more corners, all symmetrically the same as the first. Note how we have not increased the number of options; there are still only two possible paths to take. We can add as many of these corners as we like without breaking this property, as long as they all bend in the same direction.

The corner points must be placed very precisely to ensure the above properties hold. Because their position is so constrained, using them to represent elements of \( \Phi \) in our construction would be difficult. At each corner, the two path possibilities alternate between the boundary of the cylinders and the interior. As shown in Figure 2b, extra points in the cylinder interior are still only visible from the other interior points, and therefore we can add as many as we like without affecting the path possibilities. Thus, by extending the segments between the corners, we can create large regions which are only reachable along one of the two possibilities.

There is still a problem to be addressed; as more and more corners are added, more and more points are created that would be skipped by the chosen path. We would like to create a construction that forces a choice between only the points in the cylinder interiors, and ensure all the corner points will be visited regardless of which path is chosen. To accomplish this, after the last corner of the gadget, we can have \( P \) loop back around along the outer edge, covering all the corner points without covering any of the interior points. Figure 2c demonstrates this configuration.

Figure 4a shows an example usage of the gadget. The points in the cylinder interiors represent the clauses in which the variable appears. Only one set of clause points, either the clauses in which the positive literals appear or the clauses in which the negative literals appear, can be reached. However, all the corner points will always be covered. The full construction will include one of these gadgets for every variable, with each one passing through the points corresponding to clauses containing the variable’s positive and negative literals.
We now give the full specifications of the corner construction, granting them the flexibility to bend at an arbitrary angle $\alpha$. Each corner consists of two components of $P$ with four and three segments respectively, as well as four points of $S$.

Figure 3 shows the full $\alpha$-corner construction. The two components of $P$ are $A, B, C, D, E$, which we refer to as the forward path, and $F, G, H, I$, which we refer to as the return path. The four points of $S$ are $G, H, K$, and $L$. The line lengths are chosen so that the following properties hold:

- $\|BL\| = \|DK\| = \varepsilon$
- $AB$ (resp. $DE$) and the parallel line through $K$ (resp. $L$) are separated by exactly $\varepsilon$
- $G, H, K,$ and $L$ are all collinear
- The line through midpoints of $\overline{AJ}$ and $\overline{EF}$ passes through $C$.

The last property is enforced so that any points of $S$ in cylinders before $AB$ are not visible to any points in cylinders after $DE$, thus ensuring that external points cannot break the properties of the corner. The compactness of $\alpha$-corners can be shown using simple geometry; an infinite strip along $\overline{AB}$ of thickness $14\varepsilon$ is sufficient to contain all points of the structure for any $\alpha$.

### 5.2 $\alpha$-Corners

We now proceed to create $P$, adding more points to $S$ as needed. As we place components of $P$, we require that all joints and $\alpha$-corners be placed entirely outside all strips about all clause pairs, so as not to block future pieces. Furthermore, we require that new pieces of $P$ be placed so that their cylinders do not intersect the convex hull of all previously placed points in $S$. This ensures that previously placed pieces do not create unintended “shortcuts” that could break the properties of the new pieces. The exception to both these rules is the segment that passes directly through two clause points $i$ and $j$, which, of course, must pierce the convex hull. Its adjacent $\alpha$-corners will lie entirely inside the strip about $C_iC_j$, but must be placed outside all other clause strips. Note that this is always possible; beyond $4n^2\varepsilon$ units from the center of the clause ring, no clause strip intersects any other. Strips of different angles will grow further and further apart, creating regions of arbitrary size between them.

So long as the requirements in the preceding paragraph are met, the start point of $P$ can be placed arbitrarily. We then perform the following procedure for each variable $v_i$ in $\Phi$, building the construction incrementally. Let $x$ and $y$ be the clauses in which the positive literals of $v_i$ occur, and $z$ and $w$ be the clauses in which the negative literals occur. We begin by positioning an $\alpha$-corner so that the extension of the last segment of the forward path passes through $C_x$ and $C_y$. An extra point, which we refer to as a split point, is added on the boundary of the first forward path segment to induce the splitting of the two possible paths. From there, both the forward and return paths are extended through the clause ring, with the forward path crossing through $C_x$ and $C_y$.

On the opposite side, outside the convex hull of all points in $S$ so far, another $\alpha$-corner is added, bending the path toward $C_zC_w$. More $\alpha$-corners, all bending in the same direction, are added as needed until one can be placed such that the forward path passes through $C_z$ and $C_w$. Note that there must be an odd number of $\alpha$-corners in order to ensure that $C_x, C_y, C_w, C_z$ are reachable on different curve possibilities. Once the paths have been extended through the clause ring and outside the convex hull, another split point is added on the boundary to collapse the curve possibilities. Finally, a path is added to $S$. At the end of the return path, more segments of $P$ are added, with each joint being added to $S$, in order to move to the next variable’s clause strips.

Once this process has been completed for all variables, the construction is complete. Note the units in our construction are all in terms of $\varepsilon$, so $\varepsilon$ can be chosen arbitrarily. Figure 4b shows a completed construction for
(a) A partial construction for formula with 12 clauses, showing the gadget for a single variable. The only two valid paths visit the either positive literal's clauses or the negative literal's clauses.

(b) A completed construction for the formula $\Phi = (x \lor y \lor z) \land (\overline{x} \lor y \lor z) \land (x \lor y \lor z)$ The upper right clause point represents the first clause, and the second, third, and fourth follow counterclockwise.

Figure 4: Example constructions

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Figure 4: Example constructions

6 Result

Lemma 3 There exists a polygonal path $Q$ on $S$ with $\delta_F(P, Q) \leq \varepsilon$ that visits every point in $S$ if and only if $\Phi$ is satisfiable.

Proof. For the forward direction, assume $\Phi$ has a satisfying assignment. It is easy to see that our construction always has a polygonal path $Q$ on $S$ with $\delta_F(P, Q) \leq \varepsilon$ that will visit every non-clause point; $\alpha$-corners are constructed specifically to ensure this. If $\Phi$ has a satisfying assignment, then one of the two path possibilities in each variable construct will cover the the clause points corresponding to the clauses satisfied by that variable, resulting in all clause points being visited as well.

For the backward direction, let $Q$ be a complete polygonal path $Q$ on $S$ with $\delta_F(P, Q) \leq \varepsilon$. By constructing each variable construct completely outside the convex hull of all previously placed points of $S$, we have ensured that any $Q$ with $\delta_F(P, Q) \leq \varepsilon$ must follow the path we have laid out. Each variable construct forces a choice between two paths, representing a true or false value for that variable. Since each $Q$ visits each clause point, the path taken in each variable construct represents an assignment to the variables that satisfies $\Phi$. □

It is straightforward to show that five $\alpha$-corners is sufficient to move between any two strips. Thus, the construction is clearly of polynomial size. This, together with the fact that the problem is in NP, leads to the final result.

Theorem 4 The Non-unique All-points Continuous CPSM Problem is NP-complete.

In the construction, the only points that occur more than once are the clause points and the inner $\alpha$-corner points. In all occurrences of both cases, the next point is always reachable from the previous point. Thus, for this class of problem instances, any solution to the Non-unique version of this problem can be converted to a solution to the Unique version by simply skipping the points that have already been visited. This shows that the same reduction applies to the Unique version.

Corollary 5 The Unique All-points Continuous CPSM Problem is NP-complete.

7 Restricted Problem

The hardness of the problem stems from the fact that $P$ could come within $\varepsilon$ of a point in $S$ multiple times.
Thus, it is natural to ask if the hardness remains if we make a restriction that prevents such a situation. In the following section, we show that the Non-unique All-Points Continous CPSM problem is polynomial-time solvable under the condition that, for all \( s \in S \), the set \( \{t \in [0, 1] \mid \|P(t), s\| \leq \varepsilon \} \) is connected.

### 7.1 Algorithm Outline

Let \( P_i \) be the \( i \)th segment of \( P \), and let \( C_i \) be the cylinder of radius \( \varepsilon \) around \( P_i \). For simplicity, let \( C_0 = B(P(0), \varepsilon) \) and \( C_{n+1} = B(P(1), \varepsilon) \). Let \( S_i = C_i \cap S \). For a point \( s \in S \), let \( l(s) \) be the earliest occurring point of \( P \) that is within \( \varepsilon \) of \( s \), and let \( r(s) \) be the latest. By the restriction imposed in the preceding section, \( l(s) \) and \( r(s) \) are uniquely defined.

A obvious preprocessing step is to confirm that all points of \( S \) are in some cylinder \( C_i \). Another is to confirm that \( S_0 \) and \( S_{n+1} \) are nonempty. Since instances that do not satisfy these properties can be immediately ruled out, we assume them to be true for the remainder of this section.

The \( i \)th and \( j \)th segments of \( P \) are said to be connectable via \((s, t)\), where \( s \in S_i \) and \( t \in S_j \), when the following properties hold. Note that \( s \) and \( t \) could be the same point.

1. \( \bigcup_{1 < k < j} S_k - S_i - S_j = \emptyset \)
2. \( \delta_F(P', \mathcal{G}) \leq \varepsilon \), where \( P' \) is the subcurve of \( P \) from \( r(s) \) to \( l(t) \)
3. \( \forall v \in S_i \cup S_j, l(v) \leq r(s) \) or \( l(t) \leq r(v) \)

Let \( G \) be a directed graph whose vertices correspond to the segments of \( P \), with an edge between any two connectable segments of increasing index.

**Theorem 6** There is a path in \( G \) from \( P_1 \) to \( P_n \) if and only if there exists a polygonal curve \( Q \) whose vertices are exactly \( S \) with \( \delta_F(P, Q) \).

**Proof.** \((\Rightarrow)\) To show the forward direction of this statement, we will construct a curve with the requisite properties under the assumption that \( G \) has such a path, denoted by \( 1 = a_1 < a_2 < \ldots < a_m = n \). Let \( t_i \) and \( s_i \) be the points in \( S_{a_i} \) connecting \( P_{a_i-1} \) to \( P_{a_i} \), and \( P_{a_i+1} \), respectively. In other words, if \( C_{a_i-1} \) and \( C_{a_i} \) are connectable via \((x, y)\), and \( C_{a_i} \) and \( C_{a_i+1} \) are connectable via \((z, w)\), then \( t_i = y \) and \( s_i = z \). Let \( t_1 \) (resp. \( s_m \)) be an arbitrary point in \( S_0 \) (resp. \( S_{n+1} \)); these will be the first and last points of \( Q \).

Repeat the following for each \( i \) from 1 to \( m \). Add \( t_i \) to \( Q \). Then, visit every point in the set \( \{v \in S_{a_i} \mid l(t_i) \leq r(v) \} \) - \( \{s_i, t_i\} \), in order monotonic along the direction of \( P_{a_i} \). Finally, add \( s_i \) to \( Q \). Once this process has been performed for each \( i \), the path is complete.

We must show now that the constructed curve \((i)\) visits every point in \( S \) and \((ii)\) has Fréchet distance at most \( \varepsilon \) from \( P \).

We prove \((i)\) first. For a given iteration \( i \) of the above process, assume some point in \( v \in S_{a_i} \), where not did not satisfy the criteria \( l(t_i) \leq r(v) \) and \( l(v) \leq r(s_i) \). Without loss of generality, assume \( v \) did not satisfy the former of the two conditions. Then by Property 3, \( l(v) \leq r(s_{i-1}) \). If \( l(t_{i-1}) \leq r(v) \) as well, then \( v \) would have been added to \( Q \) in the \( i-1 \) iteration. If not, then \( l(v) \leq r(s_{i-2}) \), again by Property 3. Since \( l(t_1) \leq r(v) \) for all \( v \in S \), it follows that \( v \) must have been added to \( Q \) during some previous iteration. An identical argument can be used to show that, if \( v \) had failed the latter condition, \( v \) would definitely be added during some future iteration. Thus, the final constructed curve contains every point in \( \bigcup_{i \in [1,m]} S_{a_i} \), which by Property 1 above, equals \( S \).

Now we prove \((ii)\). To accomplish this, we need only show that \( \delta_F(P', Q') \leq \varepsilon \), where \( P' \) is the subcurve of \( P \) from \( l(t_i) \) to \( r(s_i) \) and \( Q' \) is the subcurve of \( Q \) consisting of all points added during the \( i \)th iteration. Combining this result with Property 2 will prove the statement by concatenation.

Fix an iteration \( i \), and let the sequence \( t_i = v_1, v_2, \ldots, v_m = s_i \) denote the sequence of points added during the \( i \)th iteration. For \( j \in [2, m-1] \), let \( p_j \) denote the point on the segment \( P_i \cap P_{a_i} \) closest to \( v_j \). Let \( p_1 = l(t_i) \) and \( p_m = r(s_i) \). It follows from Property 3 and from the fact all points other than \( t_i \) and \( s_i \) are connectable to \( P_i \) that \( p_1 \geq p_2 \geq \ldots \geq p_m \). Thus, the concatenation of \( p_1 \) through \( p_m \) is exactly \( P' \). Since \( \|v_j, p_j\| \leq \varepsilon \) for all \( j \), either by definition or by virtue of being in \( C_{a_i} \), it follows by concatenation that \( \delta_F(P', Q') \leq \varepsilon \). \( \square \)

**Proof.** \((\Leftarrow)\) We now prove the backward direction of Theorem 6. Assume there exists a polygonal curve \( Q \) whose vertices are exactly \( S \) with \( \delta_F(P, Q) \leq \varepsilon \). Then there exists a sequence of edges in \( Q \) that correspond to a path in \( G \) from \( P_1 \) to \( P_n \).

Let \( q_1, \ldots, q_m \) be the sequence of vertices of \( Q \). From the definition of Fréchet distance, there exist continuous non-decreasing surjective functions \( \sigma \) and \( \tau \) such that \( \max_{t \in [0,1]} \|P(\sigma(t)), Q(\tau(t))\| \leq \varepsilon \). Let \( t_i \) be the minimum value such that \( Q(\tau(t_i)) = q_i \). As a special case, we impose that \( t_m = 1 \). Then, let \( a_i \) be the index of the segment on which \( P(\sigma(t_i)) \) lies. Note that the sequence \( A = a_1, \ldots, a_m \) is nondecreasing. We claim that, if \( a_i < a_{i+1} \), then \( P_{a_i} \) and \( P_{a_{i+1}} \) are connectable via \((q_i, q_{i+1})\). Thus, the maximal strictly increasing subsequence of \( A \) corresponds to a path in \( G \).

To show this, let \( i \) be such that \( a_i < a_{i+1} \). Since \( P(\sigma(t_i)) \in P_{a_i} \) and \( Q(\tau(t_i)) = q_i \) are within \( \varepsilon \) of each other, it must be that \( q_i \in S_{a_i} \), and likewise for \( i-1 \). We now show that the three properties of connectable seg-
ments are satisfied. The second property is immediate, as it is satisfied for all adjacent vertices of $Q$.

For the first property, assume the set $\bigcup_{a_i<k<a_i+1} S_k - S_{a_i} - S_{a_i+1}$ were nonempty, and let $q_j$ be a point in this set. Since $q_j$ is outside the cylinders $C_{a_i}$ and $C_{a_i+1}$, $q_j$ must be separated by more than $\varepsilon$ from either segment. Furthermore, by our restriction on the input, $q_j$ cannot be in any cylinder before $C_{a_i}$ or after $C_{a_i+1}$. Thus, it must be that $a_i < a_j < a_i+1$. This contradicts the fact that $A$ is nondecreasing.

For the third property, assume there was some point $q_j \in S_i \cup S_j$ such that $r(q_i) < l(q_j)$ and $r(q_j) < l(q_{i+1})$. The former condition implies that $q_j$ must be visited after $q_i$, and the latter condition implies that $q_j$ must be visited before $q_{i+1}$. This is a contradiction. □

With this property in hand, a polynomial time algorithm follows by simply finding the connecting edges, constructing the graph $G$, and then constructing $Q$ as in the proof above if $G$ has a valid path.

8 Conclusion and Open Problems

We have shown that both the Unique and Non-unique versions of the All-Points Continuous CPSM problem are NP-complete. Furthermore, for Non-unique case, we have shown that a modified version of the problem can be solved in polynomial time.

The Unique Subset Continuous version remains an open problem. However, given that the Discrete version is NP-complete, and that the Continuous versions tend to be harder, this version is almost certainly NP-complete as well.

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