Lattice-like subsets of Euclidean Jordan algebras *

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Abstract

While studying some properties of linear operators in a Euclidean Jordan algebra, Gowda, Szajder and Tao have introduced generalized lattice operations based on the projection onto the cone of squares. In two recent papers of the authors of the present paper it has been shown that these lattice-like operators and their generalizations are important tools in establishing the isotonicity of the metric projection onto some closed convex sets. The results of this kind are motivated by methods for proving the existence of solutions of variational inequalities and methods for finding these solutions in a recursive way. It turns out, that the closed convex sets admitting isotone projections are exactly the sets which are invariant with respect to these lattice-like operations, called lattice-like sets. In this paper it is shown that the Jordan subalgebras are lattice-like sets, but the converse in general is not true. In the case of simple Euclidean Jordan algebras of rank at least three the lattice-like property is rather restrictive, e.g., there are no lattice-like proper closed convex sets with interior points.

1. Introduction

By using and generalizing the extended lattice operations due to Gowda, Szajder and Tao [1], in [2] and [3] it has been shown that the projection onto a closed convex set is

*1991 A M S Subject Classification: Primary 90C33, Secondary 15A48; Key words and phrases: positive semidefinite cone, extended lattice operations, isotone projection onto a closed convex set, invariant sets with respect to extended lattice operations, variational inequalities.
isotone with respect to the order defined by a cone if and only if the set is invariant with respect to the extended lattice operations defined by the cone. We shall call such a set simply invariant with respect to the cone, or if there is no ambiguity, lattice-like, or shortly l-l. We also showed that the a closed convex set with interior points is l-l if and only if all of its tangent hyperplanes are l-l. These results were motivated by iterative methods for variational inequalities similar to the ones for complementarity problems in [4–7]. More specifically, a variational inequality defined by a closed convex set $C$ and a function $f$ can be equivalently written as the fixed point problem $x = P_C(x - f(x))$, where $P_C$ is the projection onto the closed convex set $C$. If the Picard iteration $x_{k+1} = P_C(x_k - f(x_k))$ is convergent and $f$ continuous, then the limit of $x_k$ is a solution of the variational inequality defined by $f$ and $C$. Therefore, it is important to give conditions under which the Picard iteration is convergent. This idea has been exploited in several papers, such as [8–18]. However, none of these papers used the monotonicity of the sequence $x_k$. If one can show that $x_k$ is monotone increasing (decreasing) and bounded from above (below) with respect to an order defined by a regular cone (that is, a cone for which all such sequences are convergent), then it is convergent and its limit is a solution of the variational inequality defined by $f$ and $C$. In [4–7] the convergence of the sequence $x_k$ was proved by using its monotonicity. Although they use non-iterative methods, we also mention the paper of H. Nishimura and E. A. Ok [19], where the isotonicity of the projection onto a closed convex set is used for studying the solvability of variational inequalities and related equilibrium problems. To further accentuate the importance of ordered vector structures let us also mention that recently they are getting more and more ground in studying various fixed point and related equilibrium problems (see the book [20] of S. Carl and S Heikkilä and the references therein). The case of a self-dual cone is of special importance because of the elegant examples for invariant sets with respect to the nonnegative orthant and the Lorentz cone [2]. Moreover, properties of self-dual cones are becoming increasingly important because of conic optimization and applications of the analysis on symmetric cones. Especially important self-dual cones in applications are the nonnegative orthant, the Lorentz cone and the positive semidefinite cone, however the class of self-dual cones is much larger [21]. The results of [2] and [3] extend the results of [22] and [19]. G. Isac showed in [22] that the projection onto a closed convex sublattice of the Euclidean space ordered by the nonnegative orthant is isotone. H. Nishimura and E. A. Ok proved an extension of this result and its converse to Hilbert spaces in [19]. The study of invariant sets with respect to the nonnegative orthant goes back to the results of D. M. Topkis [23] and A. F. Veinott Jr. [24], but it wasn’t until quite recently when all such invariant sets have been determined by M. Queyranne and F. Tardella [25]. The same results have been obtained in [2] in a more geometric way. Although [2] also determined the invariant sets with respect to the Lorentz cone, it left open the question of finding the invariant sets with respect to the cone $S^m_+$ of $n \times n$ positive semidefinite matrices, called the positive semidefinite cone.

As a particular case we show that if $n \geq 3$, then there is no proper closed convex l-l set with nonempty interior in the space $(S^n_+, S^n_+)$ (the space $S^n_+$ of $n \times n$ symmetric matrices ordered by the cone $S^n_+$ of symmetric positive semidefinite matrices). For this it is enough
to show that there are no invariant hyperplanes because the closed convex invariant sets with nonempty interior are the ones which have all tangent hyperplanes invariant.

All these problems can be handled in the unifying context of the Euclidean Jordan algebras. This way we can augment this field to an approach, where the order induced by the cone of squares (the basic notion of the Jordan algebra) becomes emphasized.

To shorten our exposition, we assume the knowledge of basic facts and results on Euclidean Jordan algebras. We strive to be in accordance with the terminology in [26]. A concise introduction of the used basic notions and facts in the field can be found in [1].

2. Preliminaries

Denote by $\mathbb{R}^m$ the $m$-dimensional Euclidean space endowed with the scalar product $\langle \cdot , \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, and the Euclidean norm $\| \cdot \|$ and topology this scalar product defines.

Throughout this note we shall use some standard terms and results from convex geometry (see e.g. [27] and [28]).

Let $K$ be a convex cone in $\mathbb{R}^m$, i.e., a nonempty set with (i) $K + K \subset K$ and (ii) $tK \subset K$, $\forall t \in \mathbb{R}_+ = [0, +\infty)$. The convex cone $K$ is called pointed, if $K \cap (-K) = \{0\}$.

The convex cone $K$ is generating if $K - K = \mathbb{R}^m$.

For any $x, y \in \mathbb{R}^m$, by the equivalence $x \leq_K y \iff y - x \in K$, the convex cone $K$ induces an order relation $\leq_K$ in $\mathbb{R}^m$, that is, a binary relation, which is reflexive and transitive. This order relation is translation invariant in the sense that $x \leq_K y$ implies $x + z \leq_K y + z$ for all $z \in \mathbb{R}^m$, and scale invariant in the sense that $x \leq_K y$ implies $tx \leq_K ty$ for any $t \in \mathbb{R}_+$. If $\leq$ is a translation invariant and scale invariant order relation on $\mathbb{R}^m$, then $\leq = \leq_K$, where $K = \{x \in \mathbb{R}^m : 0 \leq x\}$ is a convex cone. If $K$ is pointed, then $\leq_K$ is antisymmetric too, that is $x \leq_K y$ and $y \leq_K x$ imply that $x = y$. The elements $x$ and $y$ are called comparable if $x \leq_K y$ or $y \leq_K x$.

We say that $\leq_K$ is a latticial order if for each pair of elements $x, y \in \mathbb{R}^m$ there exist the least upper bound $\sup\{x, y\}$ and the greatest lower bound $\inf\{x, y\}$ of the set $\{x, y\}$ with respect to the order relation $\leq_K$. In this case $K$ is said a latticial or simplicial cone, and $\mathbb{R}^m$ equipped with a latticial order is called an Euclidean vector lattice.

The dual of the convex cone $K$ is the set

$$K^* := \{y \in \mathbb{R}^m : \langle x, y \rangle \geq 0, \forall x \in K\},$$

with $\langle \cdot , \cdot \rangle$ the standard scalar product in $\mathbb{R}^m$.

The convex cone $K$ is called self-dual, if $K = K^*$. If $K$ is self-dual, then it is a generating pointed closed convex cone.

In all that follows we shall suppose that $\mathbb{R}^m$ is endowed with a Cartesian reference system with a basis $e_1, \ldots, e_m$. If $x \in \mathbb{R}^m$, then

$$x = x_1 e_1 + \ldots + x_m e_m$$

can be characterized by the ordered $m$-tuple of real numbers $x_1, \ldots, x_m$, called the coordinates of $x$ with respect to the given reference system, and we shall write $x = (x_1, \ldots, x_m)$. 

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With this notation we have $e_i = (0, ..., 0, 1, 0, ..., 0)$, with 1 in the $i$-th position and 0 elsewhere. Let $x, y \in \mathbb{R}^m$, $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m)$, where $x_i, y_i$ are the coordinates of $x$ and $y$, respectively with respect to the reference system. Then, the scalar product of $x$ and $y$ is the sum $\langle x, y \rangle = \sum_{i=1}^{m} x_i y_i$. It is easy to see that $e_1, \ldots, e_m$ is an orthonormal system of vectors with respect to this scalar product, in the sense that $\langle e_i, e_j \rangle = \delta^j_i$, where $\delta^j_i$ is the Kronecker symbol.

The set $\mathbb{R}_+^m = \{ x = (x_1, ..., x_m) \in \mathbb{R}^m : x_i \geq 0, i = 1, ..., m \}$ is called the nonnegative orthant of the above introduced Cartesian reference system. A direct verification shows that $\mathbb{R}_+^m$ is a self-dual cone.

The set $L_{m+1}^+ = \{ (x, x_{m+1}) \in \mathbb{R}^m \otimes \mathbb{R} = \mathbb{R}^{m+1} : \|x\| \leq x_{m+1} \}$, (1)
is a self-dual cone called the $m+1$-dimensional second order cone, or the $m+1$-dimensional Lorentz cone, or the $m+1$-dimensional ice-cream cone [1].

The nonnegative orthant $\mathbb{R}_+^m$ and the Lorentz cone $L$ defined above are the most important and commonly used self-dual cones in the Euclidean space. But the family of self-dual cones is rather rich [21].

3. Generalized lattice operations

A hyperplane through the origin, is a set of form

$$H(0, a) = \{ x \in \mathbb{R}^m : \langle a, x \rangle = 0 \}, \ a \neq 0. \quad (2)$$

For simplicity the hyperplanes through 0 will also be denoted by $H$. The nonzero vector $a$ in the above formula is called the normal of the hyperplane.

A hyperplane through $u \in \mathbb{R}^m$ with the normal $a$ is the set of the form

$$H(u, a) = \{ x \in \mathbb{R}^m : \langle a, x \rangle = \langle a, u \rangle, \ a \neq 0 \}. \quad (3)$$

A hyperplane $H(u, a)$ determines two closed halfspaces $H_-(u, a)$ and $H_+(u, a)$ of $\mathbb{R}^m$, defined by

$$H_-(u, a) = \{ x \in \mathbb{R}^m : \langle a, x \rangle \leq \langle a, u \rangle \},$$

and

$$H_+(u, a) = \{ x \in \mathbb{R}^m : \langle a, x \rangle \geq \langle a, u \rangle \}.$$}

Taking a Cartesian reference system in $\mathbb{R}^m$ and using the above introduced notations, the coordinate-wise order $\leq$ in $\mathbb{R}^m$ is defined by

$$x = (x_1, ..., x_m) \leq y = (y_1, ..., y_m) \iff x_i \leq y_i, \ i = 1, ..., m.$$ 

By using the notion of the order relation induced by a cone, defined in the preceding section, we see that $\leq = \leq_{\mathbb{R}_+^m}$.
With the above representation of \( x \) and \( y \), we define
\[
x \land y = (\min\{x_1, y_1\}, \ldots, \min\{x_m, y_m\}), \quad \text{and} \quad x \lor y = (\max\{x_1, y_1\}, \ldots, \max\{x_m, y_m\}).
\]

Then, \( x \land y \) is the greatest lower bound and \( x \lor y \) is the least upper bound of the set \( \{x, y\} \) with respect to the coordinate-wise order. Thus, \( \leq \) is a lattice order in \( \mathbb{R}^m \). The operations \( \land \) and \( \lor \) are called lattice operations.

A subset \( M \subset \mathbb{R}^m \) is called a sublattice of the coordinate-wise ordered Euclidean space \( \mathbb{R}^m \), if from \( x, y \in M \) it follows that \( x \land y, x \lor y \in M \).

Denote by \( P_D \) the projection mapping onto a nonempty closed convex set \( D \subset \mathbb{R}^m \), that is the mapping which associates to \( x \in \mathbb{R}^m \) the unique nearest point of \( x \) in \( D \):
\[
P_D x \in D, \quad \|x - P_D x\| = \inf\{\|x - y\| : y \in D\}.
\]
The nearest point \( P_D x \) can be characterized by
\[
P_D x \in D, \quad (P_D x - x, P_D x - y) \leq 0, \quad \forall y \in D.
\]
(4)

From the definition of the projection and the characterization (4) there follow immediately the relations:
\[
P_D(-x) = -P_D x, \tag{5}
\]
\[
P_{x+D} y = x + P_D(y - x) \tag{6}
\]
for any \( x, y \in \mathbb{R}^m \).

For a closed convex cone \( K \) we define the following operations in \( \mathbb{R}^m \):
\[
x \cap_K y = P_{x-K} y, \quad \text{and} \quad x \cup_K y = P_{x+K} y
\]
(see [1]). Assume the operations \( \cup_K \) and \( \cap_K \) have precedence over the addition of vectors and multiplication of vectors by scalars.

A direct checking yields that if \( K = \mathbb{R}^m_+ \), then \( \cap_K = \land \), and \( \cup_K = \lor \). That is \( \cap_K \) and \( \cup_K \) are some generalized lattice operations. Moreover: \( \cap_K \) and \( \cup_K \) are lattice operations if and only if the self-dual cone used in their definitions is a nonnegative orthant of some Cartesian reference system. This suggest to call the operations \( \cap_K \) and \( \cup_K \) lattice-like operations, while a subset \( M \subset \mathbb{R}^m \) which is invariant with respect to \( \cap_K \) and \( \cup_K \) (i.e. if for any \( x, y \in M \) we have \( x \cap_K y, x \cup_K y \in M \)), a lattice-like or simply an l-l subset of \((\mathbb{R}^m, K)\).

The following assertions are direct consequences of the definition of lattice-like operations:

**Lemma 1** The following relations hold for any \( x, y \in (\mathbb{R}^m, K) \):
\[
x \cap_K y = x - P_K(x - y),
\]
\[
x \cup_K y = x + P_K(y - x).
\]
If $K$ is a nonzero closed convex cone, then the closed convex set $C \subset \mathbb{R}^m$ is called a $K$-isotone projection set or simply $K$-isotone if $x \leq_K y$ implies $P_C x \leq_K P_C y$. In this case we use equivalently the term $P_C$ is $K$-isotone.

We shall refer next often to the following theorems:

**Theorem 1** [3] Let $K \subset \mathbb{R}^m$ be a closed convex cone. Then, $C$ is a lattice-like set, if and only if $P_C$ is $K$-isotone.

**Theorem 2** [2] The closed convex set $C$ with nonempty interior in $(\mathbb{R}^m, K)$ is lattice-like, if and only if it is of form

$$C = \bigcap_{i \in \mathbb{N}} H_i(u_i, a_i),$$

where each hyperplane $H_i(u_i, a_i)$ through $u_i$ with the normal $a_i$ is tangent to $C$ and is lattice-like.

4. Characterization of the lattice-like subspaces of $(\mathbb{R}^m, K)$

Denote by $K$ a closed convex cone in $\mathbb{R}^m$ and by $(\mathbb{R}^m, K)$ the resulting ordered vector space.

The notation $G \subseteq H$ will mean $H$ and $G$ are subspaces of $\mathbb{R}^m$ and $G$ is a subspace of $H$. Let $H \subseteq \mathbb{R}^m$ and $L \subset H$ a closed convex cone. The notation $G \triangleleft L H$ will mean $G$ is an l-l subspace of $(H, L)$.

We gather some results from Theorem 1 [3] and Lemma 6 [2] and particularize them for subspaces:

**Corollary 1** Let $H$ a subspace in $(\mathbb{R}^m, K)$. the following assertions are equivalent:

1. $H \triangleleft K \mathbb{R}^m$,
2. $P_K H \subset H$,
3. $P_K K \subset K$.

**Proof.** The corollary is in fact a reformulation of Theorem 1 for the case of $D = H$ a subspace. Indeed, condition 2 is nothing else as the l-l property of of $H$ since if $x, y \in H$, then by Lemma 1 [4] one has

$$x \cap_K y = x - P_K(x - y) \in H,$$

since $x, x - y, P_K(x - y) \in H$.

Similarly, $x \cup_K y \in H$.

Condition 3 expresses, by the linearity of $P_H$ its $K$-isotonicity. \qed
Corollary 2 Let \( G \subseteq H \) and \( H \subseteq_K \mathbb{R}^m \). Then, \( G \subseteq_{K \cap H} H \iff G \subseteq_K \mathbb{R}^m \).

Proof. In our proof we shall use without further comments the equivalences in Corollary 1.

Let \( G \subseteq H \) and \( H \subseteq_K \mathbb{R}^m \).

First suppose that \( G \subseteq_K \mathbb{R}^m \), which is equivalent to \( P_GK \subset K \).

Hence, \( P_G(H \cap K) \subset P_GK \subset H \cap K \), since \( P_G(K) \subset G \subset H \). Thus, \( G \subseteq_{K \cap H} H \).

Conversely, assume that \( G \subseteq_{H \cap K} H \). Take \( x, y \in \mathbb{R}^m \) with \( x \leq_K y \). Then, from \( H \subseteq_K \mathbb{R}^m \) we have

\[
P_Hx \leq_K P_Hy, \text{ that is, } P_Hy - P_Hx \in H \cap K,
\]
hence

\[
P_Hx \leq_{H \cap K} P_Hy.
\]

From \( G \subseteq_{H \cap K} H \) it follows that

\[
P_GP_Hx \leq_{H \cap K} P_GP_Hy.
\]

From the property of orthogonal projections one has

\[
P_G = P_GP_H.
\]

Thus, the above relation writes as

\[
P_Gx \leq_{H \cap K} P_Gy,
\]
or \( P_Gy - P_Gx \in H \cap K \subset K \). That is,

\[
P_Gx \leq_K P_Gy,
\]
which shows that \( G \subseteq_K \mathbb{R}^m \).

\( \square \)

The following lemma is a direct consequence of Lemma 3 and Lemma 8 in [3]:

Lemma 2 Suppose that \( K \) is a closed convex cone in \( \mathbb{R}^m \). Let \( H(0, a) \subseteq \mathbb{R}^m \) be a hyperplane through the origin with unit normal vector \( a \in \mathbb{R}^m \). Then, the following assertions are equivalent:

(i) \( P_{H(0, a)} \) is \( K \)-isotone;
(ii) \( P_{H(b, a)} \) is \( K \)-isotone for any \( b \in \mathbb{R}^m \);
(iii) \[ \langle x, y \rangle \geq \langle a, x \rangle \langle a, y \rangle, \]
for any \( x, y \in K \).
5. Lattice-like subspaces of the Euclidean Jordan algebra

In the particular case of a self-dual cone $K \subset \mathbb{R}^m$, J. Moreau’s theorem \cite{29} reduces to the following lemma:

**Lemma 3** Let $K \subset \mathbb{R}^m$ be a self-dual cone. Then, for any $x \in \mathbb{R}^m$ the following two conditions are equivalent:

(i) $x = u - v$, $u, v \in K$, $(u, v) = 0$,

(ii) $u = P_K x$, $v = P_K (-x)$.

In all what follows we will consider that the ordered Euclidean space is $(V, Q)$, the Euclidean Jordan algebra $V$ of unit $e$ ordered by the cone $Q$ of squares in $V$. All the terms concerning $V$ will be equally used for $(V, Q)$.

Since the hyperplanes in Theorem 2 play an important role, and since the l-l property is invariant with respect to translations (Lemma 3, \cite{2}), it is natural to study the l-l subspaces in $V$ which are naturally connected with the algebraic structure of this space.

**Theorem 3** Any Jordan subalgebra of $(V, Q)$ is a lattice-like subspace.

**Proof.**

Take a Jordan subalgebra $L$ in $V$ and denote by $Q_0$ its cone of squares. We have

$$Q_0 = \{x^2 : x \in L\} \subset \{x^2 : x \in V\} = Q. \quad (7)$$

We shall prove that

$$x \in L \Rightarrow P_Q x = P_{Q_0} x \in L. \quad (8)$$

Indeed, we have, by Lemma 3 applied in the ordered vector space $(L, Q_0)$, that

$$x = P_{Q_0} x - P_{Q_0} (-x), \quad (P_{Q_0} x, P_{Q_0} (-x)) = 0. \quad (9)$$

By (7)

$$P_{Q_0} x, \quad P_{Q_0} (-x) \in Q_0 \subset Q,$$

which, by equations (8) and Lemma 3 yield $P_{Q_0} x = P_Q x$, or equivalently (8).

Accordingly $P_Q L \subset L$, which by Corollary 1 translates into $L \subseteq Q V$.

6. The Pierce decomposition of the Euclidean Jordan algebra and its lattice-like subspaces

Let $r$ be the rank of $V$ and \{c_1, \ldots, c_r\} be an arbitrary Jordan frame in $V$, that is, $c_k$ are primitive idempotents such that

$$c_i c_j = 0, \quad \text{if } i \neq j, \quad c_i^2 = c_i,$$
With the notation
\[ V_{ii} = V(c_i, 1) = \mathbb{R}c_i, \]
\[ V_{ij} = V \left( c_i, \frac{1}{2} \right) \cap V \left( c_j, \frac{1}{2} \right), \]
(where for \( \lambda \in \mathbb{R} \), \( V(c_i, \lambda) = \{ x \in V : c_i x = \lambda x \} \)), we have by Theorem IV.2.1. [26] the following orthogonal decomposition (the so-called Pierce decomposition) of \( V \):
\[ V = \bigoplus_{i \leq j} V_{ij}, \quad \text{(10)} \]
where
\[ V_{ij} V_{ij} \subset V_{ii} + V_{jj}; \quad V_{ij} V_{jk} \subset V_{ik}, \quad \text{if} \quad i \neq k; \quad V_{ij} V_{kl} = \{ 0 \}, \quad \text{if} \quad \{ i, j \} \cap \{ k, l \} = \emptyset. \quad \text{(11)} \]
Taking for \( 1 \leq k < r \)
\[ V^{(k)} = \bigoplus_{i \leq j \leq k} V_{ij} \quad \text{(12)} \]
is a Jordan algebra with the unit
\[ e_k = c_1 + \cdots + c_k. \]
Indeed, relations (11) imply the invariance of \( V^{(k)} \) with respect to the Jordan product. The same relations and the definitions imply \( e_k x_{ii} = c_i x_{ii} = x_{ii} \), for any \( x_{ii} \in V_{ii} \) and \( i \leq k; \ c_i V_{ij} = \{ 0 \} \) if \( l \notin \{ i, j \}; \ e_k x_{ij} = (c_i + c_j) x_{ij} = x_{ij} \), for any \( x_{ij} \in V_{ij} \) and \( i, j \leq k, \ i \neq j \). Hence \( e_k \) is the unity of \( V^{(k)} \). These relations also imply that
\[ V^{(k)} = V(e_k, 1) = \{ x \in V : e_k x = x \}. \quad \text{(13)} \]
Thus, \( V(e_k, 1) \) is a subalgebra (this follows also by Proposition IV.1.1 in [26] since \( e_k \) is idempotent). Hence by Theorem 3, \( V(e_k, 1) \) is an l-l subspace in \( (V, Q) \).

A Jordan algebra is said simple if it contains no nontrivial ideal. A consequence of the above cited theorem and the content of paragraph IV.2. of [26] is that \( V \) is simple if and only if \( V_{ij} \neq \{ 0 \} \) for any \( V_{ij} \) in (10). By the same conclusion \( V^{(k)} \) given by (12) is simple too, and by Corollary IV.2.6. in [26] the spaces \( V_{ij}, \ i \neq j \) have the common dimension \( d \), hence by (12)
\[ \dim V^{(k)} = k + \frac{d}{2} k (k - 1). \]

The subcone \( F \subset Q \) is called a face of \( Q \) if whenever \( 0 \leq Q x \leq Q y \) and \( y \in F \) it follows that \( x \in F \).

It is well known that for an arbitrary face \( F \) of \( Q \) one has \( P_{\text{span} F} Q \subset Q \) (see e.g. Proposition II.1.3 in [30]). Hence by Corollary 4 it follows the assertion:
Corollary 3 Each subspace generated by some face of $Q$ is a lattice-like subspace in $(V, Q)$.

We give an independent proof of this.

**Proof.** Let $\{c_1, ..., c_r\}$ be a Jordan frame in $V$, $k \leq r$. If

$$e_k = c_1 + \cdots + c_k, \quad 0 \leq k \leq r,$$

then by Theorem 3.1 in [31]

$$F = V(e_k, 1) \cap Q = \{x \in Q : e_k x = x\}$$

is a face of $Q$ and each face of $Q$ can be represented in this form for some Jordan frame.

The cone $F = V(e_k, 1) \cap Q$ is the cone of squares in the subalgebra $V(e_k, 1)$, hence its relative interior is non-empty, accordingly

$$V(e_k, 1) = \text{span } F = F - F.$$

Since $V(e_k, 1)$ is a subalgebra, by Theorem [3] it is an l-l subspace. \qed

7. The subalgebras and the lattice-like subspaces of the space spanned by a Jordan frame

Suppose that the dimension of the Euclidean Jordan algebra $V$ is at least 2. Let $\{c_1, ..., c_r\}$ be a Jordan frame in $V$. Then,

$$V_r := \text{span}\{c_1, ..., c_r\}$$

is a Jordan subalgebra of $V$. Obviously, $V_r = V_{11} \oplus \cdots \oplus V_{rr}$. If $x, y \in V_r$, then

$$xy = (x_1 y_1, ..., x_r y_r),$$

where $x_i$ and $y_i$ are the coordinates of $x$, respectively $y$ with respect to the above Jordan frame.

By using the notations of the above section, denote $Q_r = Q \cap V_r$ and let us show that

$$Q_r = \text{cone}\{c_1, ..., c_r\} := \left\{ \sum_{i=1}^{r} \lambda_i c_i : \lambda_i \geq 0, \forall 1 \leq i \leq r \right\}.$$

The inclusion $\text{cone}\{c_1, ..., c_r\} \subset Q_r$ is obvious. Next, we show that $Q_r \subset \text{cone}\{c_1, ..., c_r\}$. Suppose to the contrary, that there exists $x \in Q_r \setminus \text{cone}\{c_1, ..., c_r\}$. It follows that $(c_k, x) < 0$ for some $k \in \{1, \ldots, r\}$. Since $Q$ is selfdual, this implies $x \notin Q$, which is a contradiction.
The ordered vector space \((V_r, Q_r)\) can be considered an \(r\)-dimensional Euclidean vector space ordered with the positive orthant \(Q_r\) engendered by the Jordan frame.

Let \(H_{r-1}\) be an \(l\)-l hyperplane in \((V_r, Q_r)\), with the unit normal \(a \in V_r\). Thus, the results in [2] and [3] applies, hence if

\[
a = (a_1, ..., a_r),
\]

then we must have

\[
a_ia_j \leq 0, \quad \text{if} \quad i \neq j.
\]  

(14)

(15)

Then, there are two possibilities:

**Case 1.** There exists an \(i\) such that \(a_i = 1\) and \(a_j = 0\) for \(j \neq i\).

**Case 2.** There are only two nonzero coordinates, say \(a_k\) and \(a_l\) with \(a_ka_l < 0\).

**Ad 1.** In the **Case 1**

\[
H_{r-1} = \text{span}\{c_1, ..., c_{i-1}, c_{i+1}, ..., c_r\}
\]

and \(H_{r-1}\) is obviously a Jordan algebra.

**Ad 2.** In the **Case 2**

\[
H_{r-1} = \{x \in V_r : a_kx_k + a_lx_l = 0\}.
\]

We know from the above cited result, that \(H_{r-1}\) is an \(l\)-l subspace in \((V_r, Q_r)\) and since \(V_r\) is a subalgebra of \(V\), by Theorem 3 \(V_r \subseteq Q V\). By using Corollary 2 we have, for the \(l\)-l subspace \(H_{r-1} \subseteq Q V\), that

\[
H_{r-1} \subseteq Q V.
\]

In the case **Ad 1** the \(l\)-l hyperplane \(H_{r-1}\) is also a Jordan algebra.

Suppose that **Ad 2** holds. We would like to see under which condition the \(l\)-l hyperplane \(H_{r-1}\) is a Jordan algebra.

Let us suppose that \(H_{r-1}\) is a Jordan algebra, and take \(x \in H_{r-1}\), \(x = (x_1, ..., x_r)\). Then, \(x^2 = (x_1^2, ..., x_r^2) \in H_{r-1}\). Take \(x\) with \(x_l = a_k\) and \(x_k = -a_l\). Then, \(x \in H_{r-1}\) and we must have \(x^2 \in H_{r-1}\). Hence

\[
a_ka_l^2 + a_la_k^2 = a_ka_l(a_l + a_k) = 0,
\]

and since \(a_ka_l \neq 0\), we must have

\[
a_k = \frac{\sqrt{2}}{2}, \quad a_l = -\frac{\sqrt{2}}{2},
\]

or conversely. In this case

\[
H_{r-1} = \{x : x_k = x_l\}
\]

(16)

is obviously a subalgebra.
Remark 1 For any hyperplane \( H_{r-1} \) in \( V_r \) with the unit normal \( \mathbf{a} \) having only two nonzero components with opposite signs and different absolute values \( H_{r-1} \) is an l-l subspace, but not a Jordan subalgebra.

If \( \mathbf{a} \in V \) is arbitrary, then there exists a Jordan frame \( \{\mathbf{c}_1, \ldots, \mathbf{c}_r\} \) such that \( \mathbf{a} \) can be represented in the form (14) (Theorem III.1.2 in [26]). We will call such a Jordan frame as being attached to \( \mathbf{a} \).

Corollary 4 Let \( H \) be a lattice-like hyperplane in \( (V, Q) \) with the normal \( \mathbf{a} \) and \( \{\mathbf{c}_1, \ldots, \mathbf{c}_r\} \) be a Jordan frame attached to it. If \( \mathbf{a} \) is represented by (14), then the coordinates \( a_i, i = 1, \ldots, r \) of \( \mathbf{a} \) satisfy the relations (15).

Proof. If \( V_r = \text{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_r\} \), then \( H_{r-1} = H \cap V_r \) is an l-l hyperplane in \( (V_r, Q_r) \) with the normal \( \mathbf{a} \) because it is the intersection of two l-l sets: \( V_r \) (a subalgebra) and \( H \). Thus, we can apply the characterization of l-l hyperplanes in \( (V_r, Q_r) \) described above in this section. \( \square \)

Denote by \( \mathcal{F}(Q) \) the family of faces of \( Q \), by \( \mathcal{A} \) the family of subalgebras of \( V \) and by \( \mathcal{L} \) the family of the l-l subspaces in \( V \). Then, by the above reasonings, we conclude the

Corollary 5 We have the following strict inclusions:

\[ \{\text{span} F : F \in \mathcal{F}(Q)\} \subset \mathcal{A} \subset \mathcal{L}. \]

Proof. The second strict inclusion follows from Remark 1. The first inclusion is strict since for instance the subspaces in (16) are subalgebras which are not generated by faces of \( Q \). Indeed, take in \( V_r \) the reference system engendered by \( \mathbf{c}_1, \ldots, \mathbf{c}_r \) and let

\[ H_{r-1} = \{(t, t, x_{r+3}, \ldots, x_r) \in V_r : t, x_j \in \mathbb{R}\}. \]

Take \( \mathbf{y} = (1, 1, 0, \ldots, 0) \) and \( \mathbf{x} = (1, 0, 0, \ldots, 0) \) in \( Q_r = Q \cap V_r \). Since in \( V_r \), \( \leq Q = \leq Q_r \) and the latter is coordinate-vise ordering,

\[ 0 \leq Q \mathbf{x} \leq Q \mathbf{y}, \]

and we have \( \mathbf{y} \in H_{r-1} \cap Q \), but \( \mathbf{x} \notin H_{r-1} \cap Q \), which shows that \( H_{r-1} \cap Q \) is not a face. \( \square \)

8. The inexistence of lattice-like hyperplanes in simple Euclidean Jordan algebras of rank \( r \geq 3 \).

Theorem 4 Suppose that \( V \) is a simple Euclidean Jordan algebra of rank \( r \geq 3 \). Then, \( V \) does not contain lattice-like hyperplanes.
Proof. Assume the contrary: $H$ is an l-l hyperplane through 0 in $V$ with the unit normal $a$.

Consider a Jordan frame $\{c_1, ..., c_r\}$ attached to $a$.

The set $H_{r-1} = H \cap V_r$ is obviously a hyperplane through 0 in $V_r$.

Since by hypothesis $H \subset Q$, by Corollary 2, $H_{r-1} \subset Q_r V_r$, where $Q_r = Q \cap V_r$.

If $a = (a_1, ..., a_r)$ is the representation of $a$ in the reference system engendered by the Jordan frame, then using Corollary 4, the l-l property of $H_{r-1}$ in $(V_r, Q_r)$ implies that one of the following cases must hold:

Case 1. For some $i$ $a_i = 1$ and $a_j = 0$ for $j \neq i$.

Case 2. There are only two nonzero coordinates, say $a_i$ and $a_j$ with $a_i a_j < 0$.

Suppose that $i = 1$, $j = 2$.

Since $V$ is simple, $V_{12} \neq \{0\}$ (by Proposition IV.2.3 [26]), hence we can take $x \in V_{12}$ with $\|x\|^2 = 2$. Then, by Exercise IV. 7 in [26], we have that

$$u = \frac{1}{2} c_1 + \frac{1}{2} c_2 + \frac{1}{2} x, \quad \text{and} \quad v = \frac{1}{2} c_1 + \frac{1}{2} c_2 - \frac{1}{2} x$$

are idempotent elements, hence $u, v \in Q$. We further have

$$uv = \left(\frac{1}{2} c_1 + \frac{1}{2} c_2\right)^2 - \frac{1}{4} x^2,$$

whereby, by using Proposition IV.1.4 in [26], we have

$$x^2 = \frac{1}{2} \|x\|^2 (c_1 + c_2) = c_1 + c_2,$$

and after raising to the second power and substitution

$$uv = \frac{1}{4} c_1 + \frac{1}{4} c_2 - \frac{1}{4} (c_1 + c_2) = 0.$$

Hence

$$\langle u, v \rangle = 0.$$

Since $H_{r-1}$ is l-l, we have by Lemma 2

$$0 = \langle u, v \rangle \geq \langle a, u \rangle \langle a, v \rangle.$$

If $a_1 = 1$, and $a_j = 0$ for $j \neq 1$, the above relation becomes $0 \geq \frac{1}{4} ||c_i||^4$, which is impossible.

Assume $a_1 a_2 < 0$ and $a_j = 0$ for $j > 2$. 

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Take now
\[ w = \frac{1}{2} c_1 + \frac{1}{2} c_3 + \frac{1}{2} y \]
\[ z = \frac{1}{2} c_1 + \frac{1}{2} c_3 - \frac{1}{2} y \]
with \( y \in V_{13}, \| y \|^2 = 2 \). Then, \( w, z \in Q \) (similarly to \( u, v \in Q \)) and, by using the mutual orthogonality of the elements \( c_1, c_2, c_3, y \) and Lemma 2, it follows that
\[ 0 = \langle w, z \rangle \geq \langle a, w \rangle \langle a, z \rangle = \langle a_1 c_1 + a_2 c_2, \frac{1}{2} c_1 + \frac{1}{2} c_3 + \frac{1}{2} y \rangle \langle a_1 c_1 + a_2 c_2, \frac{1}{2} c_1 + \frac{1}{2} c_3 - \frac{1}{2} y \rangle = \frac{1}{4} a_1^2 \| c_1 \|^2, \]
which is a contradiction.

This theorem conferred with Theorem 2 and Lemma 3 in [2] yields the

**Corollary 6** *In the ordered Euclidean Jordan algebra \((V, Q)\) of rank at least 3 there are no proper closed convex lattice-like set with nonempty interior. In particular, for \( n \geq 3 \) the ordered space \((S^n, S^n_+)\) contains no proper, closed, convex lattice-like set with nonempty interior.*

9. **The case of the simple Euclidean Jordan algebras of rank 2**

A simple Euclidean Jordan algebra of rank 2 is isomorphic to an algebra associated with a positive definite bilinear form (Corollary IV.1.5 [26]). This is in fact a Jordan algebra associated with the Lorentz cone. Hence the problem of the existence of 1-l hyperplanes in this case is answered positively in [2] and [3]. In this section we use the formalism developed in the preceding sections to this case too.

**Lemma 4** *Suppose that \( a \) is the unit normal to a lattice-like hyperplane \( H \) through 0 in the simple Euclidean Jordan algebra \( V \) of rank 2. Let \( \{ c_1, c_2 \} \) be the Jordan frame attached to \( a \) and \( a = a_1 c_1 + a_2 c_2 \). Then, supposing \( a_1 > 0 \), we obtain*
\[ a = \frac{\sqrt{2}}{2} c_1 - \frac{\sqrt{2}}{2} c_2. \]  

**Proof.** Take \( u \) and \( v \) as in the formula (17). Then, \( u, v \in Q \) and using Lemma 2 we obtain
\[ 0 = \langle u, v \rangle \geq \langle a, u \rangle \langle a, v \rangle = \langle a_1 c_1 + a_2 c_2, \frac{1}{2} c_1 + \frac{1}{2} c_2 + \frac{1}{2} y \rangle \langle a_1 c_1 + a_2 c_2, \frac{1}{2} c_1 + \frac{1}{2} c_2 - \frac{1}{2} y \rangle = \frac{1}{4} (a_1 + a_2)^2, \]
whereby our assumption follows.

\textbf{Theorem 5} Let $V$ be a simple Euclidean Jordan algebra of rank 2 and $H$ be a hyperplane through 0 with unit normal $a$ in $V$. Then, $H$ is lattice-like if and only if $a = \sqrt{2}/2 c_1 - \sqrt{2}/2 c_2$ in its Jordan frame representation. In this case $H$ is a subalgebra.

\textbf{Proof.} Suppose that $H = \ker a$, $\|a\| = 1$ is l-l, and that the Jordan frame attached to $a$ is $\{c_1, c_2\}$.

Then, by Lemma 4, it follows that $a$ is of form (19).

Suppose that the Jordan frame representation of $a$ is of form (19). Then, equations (10) and (11) imply that

$$\ker a = \{t(c_1 + c_2) + x = te + x : t \in \mathbb{R}, x \in V_{12}\}.$$ 

Then, for two arbitrary elements $u, v \in \ker a$, we have the representations:

$$u = t_1 e + x; \quad v = t_2 e + y; \quad x, y \in V_{12}; \quad t_i \in \mathbb{R}, \quad i = 1, 2.$$ 

Then,

$$uv = t_1 t_2 e + t_1 y + t_2 x + xy.$$ 

Since $xy = (1/4)((x + y)^2 - (x - y)^2)$, by using Proposition IV.1.4 in [26], we conclude that $xy = q(c_1 + c_2) = qe$ with $q \in \mathbb{R}$. Hence

$$uv = (t_1 t_2 + q)e + t_1 y + t_2 x \in \ker a.$$ 

This shows that $H = \ker a$ is a subalgebra, and hence an l-l set. \qed

\textbf{Remark 2} With the notations in the above proof we have that $\text{span}\{c_1, c_2\}$ is a subalgebra of dimension 2 in $V$.

Similarly to Remark 1, it follows that there exist l-l subspaces of dimension 1 in $\text{span}\{c_1, c_2\}$ which are not subalgebras.

Collating Theorem 5 and Theorem 2 it follows the result:

\textbf{Corollary 7} The closed convex set with nonempty interior $M \subset V$ is a lattice-like set if and only if it is of the form:

$$M = \bigcap_{i \in \mathbb{N}} H_-(u_i, a_i),$$

with the $a_i$ normal unit vectors represented in their Jordan frame $c_1^i, c_2^i$ by

$$a_i = \varepsilon_i \left( \frac{\sqrt{2}}{2} c_1^i - \frac{\sqrt{2}}{2} c_2^i \right), \quad \varepsilon_i = 1 \text{ or } -1.$$ (20)

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Example 1 Write the elements of $\mathbb{R}^{m+1}$ in the form $(x, x_{m+1})$ with $x \in \mathbb{R}^m$ and $x_{m+1} \in \mathbb{R}$. The Jordan product in $\mathbb{R}^{m+1}$ is defined by

$$(x, x_{m+1}) \circ (y, y_{m+1}) = (y_{m+1}x + x_{m+1}y, \langle x, y \rangle + x_{m+1}y_{m+1}),$$

where $\langle x, y \rangle$ is the usual scalar product in $\mathbb{R}^m$. The space $\mathbb{R}^{m+1}$ equipped with the usual scalar product and the operation $\circ$ just defined becomes an Euclidean Jordan algebra of rank 2, denoted by $\mathcal{L}^{m+1}$, with the cone of squares $Q = \mathcal{L}^{m+1}$, the Lorentz cone defined by (1).

The unit element in $\mathcal{L}^{m+1}$ is $(0, 1)$, where $0$ is the zero vector in $\mathbb{R}^m$.

The Jordan frame attached to $(x, x_{m+1}) \in \mathcal{L}^{m+1}$ with $x \neq 0$ is

$$c_1 = \frac{1}{2} \left( \frac{x}{\|x\|}, 1 \right), \quad c_2 = \frac{1}{2} \left( -\frac{x}{\|x\|}, 1 \right).$$

The unit normal $a$ from Lemma 4 will be then parallel with $(b, 0)$ with some $b \in \mathbb{R}^m$, $b \neq 0$. This means, that the hyperplanes $H(u_i, a_i)$ in Corollary 7 are parallel with the $m+1$-th axis, and the closed convex set in the corollary is in fact of form

$$(C \times \mathbb{R}),$$

with $C$ closed convex set with nonempty interior in $\mathbb{R}^m$.

This is exactly the result in Example 1 of [3].

10. The general case

For a general Euclidean Jordan algebra $V$, gathering the results of Proposition III.4.4, Proposition III.4.5, and Theorem V.3.7, of [26], in Theorem 5 of [1] the following result is stated:

Theorem 6 Any Euclidean Jordan algebra $V$ is, in unique way, a direct sum

$$V = \bigoplus_{i=1}^k V_i$$

of simple Euclidean Jordan algebras $V_i$, $i = 1, \ldots, k$. Moreover the cone of squares $Q$ in $V$ is, in a unique way, a direct sum

$$Q = \bigoplus_{i=1}^k Q_i$$

of the cones of squares $Q_i$ in $V_i$, $i = 1, \ldots, k$.

(Here the direct sum, (by a difference to that in the Pierce decomposition), means Jordan-algebraic and hence also orthogonal direct sum.)

Let $C \subset V$ a closed convex set. From the results in Theorem 6 it follows easily (using the notations in the theorem), that

$$P_C = \sum_{i=1}^k P_{C_i},$$

with $C_i = C \cap V_i$, $i = 1, \ldots, k$.

Collating these results with Corollary 2 we have the following:
Corollary 8 With the notations in Theorem 6, for the subspace $M \in V$ we have the equivalence:
\[
M \subset Q \iff M \cap V_i \subset Q_i, \ i = 1, \ldots, k.
\] (24)
For the closed convex set $C$ the projection $P_C$ is $Q$-isotone if and only if $P_{C \cap V_i}$ is $Q_i$-isotone in $(V_i, Q_i)$, $i = 1, \ldots, k$.

Corollary 9 If $H$ is a lattice-like hyperplane in $V$ represented as (27) in Theorem 6, then $V_i \in H$ for each simple subalgebra in (27) of rank at least 3.

Proof. Assume the contrary. Then, $H \cap V_i$ is an $1$-l hyperplane in $V_i$, which contradicts Theorem 4. \hfill \Box

Gathering the results in Theorem 2, Section 7, Corollary 7 and Corollary 9 we have

Theorem 7 Suppose that $V$ is an Euclidean Jordan algebra of form (27) with $V_i$ simple subalgebras. Let us write this sum as
\[
V = W_1 \oplus W_2 \oplus W_3
\] (25)
where
\[
W_1 = \oplus_{i \in I_1} V_i, \quad W_2 = \oplus_{i \in I_2} V_i, \quad W_3 = \oplus_{i \in I_3} V_i
\] (26)
such that $V_i$ for $i \in I_1$ are the subalgebras of rank 1, for $i \in I_2$, the subalgebras of rank 2, and for $i \in I_3$ the subalgebras of rank at least 3. Then, $C \subset V$ is a closed convex lattice-like subset with nonempty interior if and only if the following conditions hold:
\[
C = \bigcap_{i \in \mathbb{N}} H_{-}(u_i, a_i)
\] (27)
where each hyperplane $H(u_i, a_i)$ through $u_i$ and with the unit normal $a_i$ is tangent to $C$ and is lattice-like. Let $\{c_1^i, \ldots, c_r^i\}$ be a Jordan frame attached to $a_i$. The last conditions hold if and only if
\[
a_i = a_1^i c_1^i + \ldots + a_{r_1}^i c_{r_1}^i + a_{r_1+1}^i c_{r_1+1}^i + \ldots + a_{r_2}^i c_{r_2}^i + a_{r_2+1}^i c_{r_2+1}^i + \ldots + a_r^i c_r^i
\] (28)
with $c_1^i, \ldots, c_{r_1}^i \in W_1$, $c_{r_1+1}^i, \ldots, c_{r_2}^i \in W_2$, and $c_{r_2+1}^i, \ldots, c_r^i \in W_3$, and exactly one of the following two cases hold:

(i) There exists a $k \in \{1, \ldots, r_1\}$ with $a_k^i \neq 0$ and exactly one of the following two statements is true

(i)’ The equality $a_j^i = 0$ holds for $j \neq k$,

(i)” There exists an $l \in \{1, \ldots, r_1\}$ such that $a_l^i a_k^i < 0$ and $a_j^i = 0$, for $j \notin \{k, l\}$,

(ii) There exists $k, l \in \{r_1+1, \ldots, r_2\}$ and $p \in I_2$ such that $c_k^i, c_l^i \in V_p$, $a_k^i = \sqrt{2}/2$, $a_l^i = -\sqrt{2}/2$ and $a_j^i = 0$, for $j \notin \{k, l\}$. 

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Proof. Observe first that using the representation (28) of \( a_i \) and the partition (26) of \( V \), we have the following relations:

\[
    r_1 = |I_1|, \quad r_2 - r_1 = 2|I_2|, \quad r - r_2 \geq 3|I_3|.
\]

The representation (27) follows from Theorem 2. Let us see first that the alternative (i), respectively (ii) is sufficient for \( H(u_i, a_i) \) to be an \( l \)-l set.

If (i) holds, then the hyperplane \( H_{r_1-1} \) through 0 with the normal \( a_i^{ij} u = a_i^j c^i_1 + \ldots + a_i^j c^i_{r_1} \) is by Section 7 an \( l \)-l set in \( \text{span}\{c^i_1, ..., c^i_{r_1}\} \) ordered by the orthant engendered by \( c^i_1, ..., c^i_{r_1} \). Hence \( H(u_i, a_i) = (H_{r_1-1} + u_i) \bigoplus W_2 \bigoplus W_3 \) is \( l \)-l in \( V \) (by Theorem 1 and Lemma 2).

If (ii) holds, then the hyperplane \( H' \) through 0 with the normal \( a_i^j = a_i^j c^i_k + \ldots + \sqrt{2}/2(c^i_k - c^i_l) \) in \( V_p \) is \( l \)-l (by Theorem 3, Theorem 1 and Lemma 2), hence \( H(u_i, a_i) = (H' + u_i) \bigoplus (\bigoplus_{j \neq p} V_j) \) is \( l \)-l in \( V \).

To complete the proof we have to show the necessity of the alternatives (i) and (ii). Observe first that if \( H(u_i, a_i) \) is \( l \)-l, then in the representation (28) of \( a_i \), by Corollary 9 we must have \( a_i^j = 0 \) whenever \( j > r_2 \). Thus, if \( a_i^j \neq 0 \), then \( j \leq r_2 \).

Suppose that \( a_i^j \neq 0 \) for some \( c^i_k \in W_2 \). Then, there exists an \( a_i^j \neq 0 \) and \( c^i_k, c^i_l \in V_p \), for some \( V_p \) in the representation of \( W_2 \). Indeed, in the case \( a_i^j \neq 0 \) it follows that \( a_i^j c^i_k \in V_p \setminus \{0\} \) for some \( V_p \subset W_2 \), hence \( H(u_i, a_i) \cap V_p \) is a hyperplane in \( V_p \) and our assertion follows from Lemma 4 (and in particular one of \( a_i^j \) and \( a_i^l \) is \( \sqrt{2}/2 \) the other is \( -\sqrt{2}/2 \)). From Corollary 4 it follows then that \( a_i^j = 0 \) for \( j \notin \{k, l\} \). Thus, the alternative (ii) must hold.

Suppose now that \( a_i^j \neq 0 \) for some \( j \leq r_1 \). Then from the reasoning of the above paragraph and Corollary 4 we must have \( a_i^j = 0 \) if \( k > r_1 \). In this case two situations are possible: (i)' \( a_i^j = 0 \) and \( a_i^k = 0 \) for \( j \neq l \), and (i)" there exists \( a_i^j \neq 0 \), \( (l \leq r_1) \) with \( a_i^j a_i^k < 0 \) and \( a_i^k = 0 \) for \( k \notin \{j, l\} \). Thus, the alternative (i) must hold.

\[ \square \]

Example 2 Let \( V \) be a simple Euclidean Jordan algebra with the Pierce decomposition given by (10) and (11), and \( d \) the common dimension of \( V_{ij} \), \( i \neq j \) (see Corollary IV.2.6 [26]). Denote

\[
    W_{k,l} = \bigoplus_{k \leq i \leq j \leq l} V_{ij}.
\]

Then, \( W_{k,l} \) is a subalgebra, hence an \( l \)-l subspace. The sum

\[
    W_{1,k} \bigoplus W_{k+1,r}, \quad k < r
\]

is a subalgebra too, and hence an \( l \)-l subspace. Suppose that \( r \geq 4 \) and \( 2 \leq k \leq r-2 \). Let \( H_0 \) be an \( l \)-l hyperplane in \( W_{k+1,r} \) which is not its subalgebra. Then,

\[
    W_{1,k} + H_0
\]

is an \( l \)-l subspace in \( V \) of dimension \( k + (d/2)k(k-1) + r - k - 1 \) which is not an algebra.

Question: Is every \( l \)-l subspace of \( V \) which is not a subalgebra of this type?
Acknowledgement

The authors express their gratitude to Roman Sznajder for his helpful comments and information on Euclidean Jordan algebras.

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