Analytic exploration of safe basins in a benchmark problem of forced escape

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Abstract The paper presents an analytical approach to predicting the safe basins (SBs) in a plane of initial conditions (ICs) for escape of classical particle from the potential well under harmonic forcing. The solution is based on the approximation of isolated resonance, which reduces the dynamics to a conservative flow on a two-dimensional resonance manifold (RM). Such a reduction allows easy distinction between escaping and non-escaping ICs. As a benchmark potential, we choose a common parabolic-quartic well with truncation at varying energy levels. The method allows accurate predictions of the SB boundaries for relatively low forcing amplitudes. The derived SBs demonstrate an unexpected set of properties, including decomposition into two disjoint zones in the IC plane for a certain range of parameters. The latter peculiarity stems from two qualitatively different escape mechanisms on the RM. For higher forcing values, the accuracy of the analytic predictions decreases to some extent due to the inaccuracies of the basic isolated resonance approximation, but mainly due to the erosion of the SB boundaries caused by the secondary resonances. Nevertheless, even in this case the analytic approximation can serve as a viable initial guess for subsequent numeric estimation of the SB boundaries.

Keywords Escape · Safe basins · Global stability · Isolated resonance

Abbreviations
IC Initial condition
SB Safe basin
RM Resonance manifold
AIR Approximation of isolated resonance
AA Action angle
GIM Global integrity measure
SI Stochastic integrity
MM Maximum mechanism
SM Saddle mechanism
SMM Saddle maximum mechanism
SBMT Safe basin of maximum type
SBST Safe basin of saddle type
EC Excitation cycle
ESB Energy safe basin
DSB Displacement safe basin

1 Introduction and historical survey

Escape from a potential well is a tool for modeling a plethora of effects in many branches of science and engineering [1–6]. Examples of the escape phenomena include gravitational collapse in celestial mechanics [7], energy harvesting [6], absorption of particles [2, 8], responses of Josephson junctions [8], resonance
capture [9, 10], capsize of ships [3, 11–13], dynamic pull-in in microelectromechanical systems [14], collapses of arches [4], energy transport [15], capture of charged particles by an electric field [16], and buckling of engineering structures [17, 18], to mention a few.

One of the first seminal works on escape was written by Kramers almost 80 years ago [19]. While focusing on thermal activation of chemical reactions, it created a multitude of novel research areas. A particularly interesting phenomenon is the escape triggered by a stochastic resonance [20–22].

In recent years, a lot of attention was drawn to the dynamics of escape under the external harmonic forcing [23–28]. In [23], the escape from a squared hyperbolic secant potential was treated with the help of approximation of isolated 1:1 resonance (AIR). The AIR technique consists of two main steps: the action-angle (AA) transformation and the subsequent averaging over the fast phases. As a result, for the Hamiltonian system it yields an integral of motion describing a family of resonance manifolds (RMs), conveniently described by a two-dimensional phase portrait of the slow flow. Qualitative analysis of the slow-flow equation allows to identify the escaping trajectories, to describe the mechanisms of transition to escape, and to find the critical values of parameters.

Previous studies considered only special (primarily, zero) ICs; therefore, the question about the global dynamics remains open. This question can be formulated as follows: what is the set of all ICs corresponding to non-escape for a potential well? The aforementioned set of the ICs for the non-escaping trajectories is commonly referred to as a safe basin (SB) [29, 30]. In order to quantify the size of an SB, the notion of integrity measure (e.g., global integrity measure (GIM) which is the hyper-volume of the SB) is introduced [29–31]. Reduction of the integrity with the change of the system parameters is referred to as erosion, and the variation of integrity is called the erosion profile. In [29], the control was applied by alteration of the excitation shape leading to the shift of the erosion profile toward greater external forcing amplitudes by eliminating homo/heteroclinic bifurcations of hilltop saddles. Soliman and Thompson investigated stochastic integrity (SI) in the presence of a white noise [32], while Orlando et al. continued the study of SI for bistable elastic structures with harmonic and stochastic loading [33, 34].

Despite all the advances in the study of SBs and their erosion, most of the related methods have significant limitations. Firstly, due to the complexity of the problem, the analytic treatment of SBs often stays beyond the consideration. Secondly, an integrity measure usually only quantifies the magnitude of an SB; however, it possesses little information about the shape. Recent developments, e.g., anisometric integrity measures, prove useful for capturing some shape-related properties of the SBs such as a relevance of each dimension (i.e., displacements and velocities) while neglecting unsafe fractal tongues [35, 36]. The boundary for possible placement of the attractors in a strongly nonlinear forced system can be assessed by semi-analytic procedure [37] and can be highly nontrivial.

Current work suggests an analytic approach for predicting the SB boundaries in a benchmark model system—a particle in truncated quartic potential—under harmonic forcing with constant amplitude. First, the AIR is applied to find the escape threshold for arbitrary initial conditions inside the well, under different levels of truncation. Then, the SB boundaries are identified and classified both on the RM and in the IC plane.

The paper is organized as follows: in Sect. 2, the model is presented, and the AIR is described. Those are followed by derivation of the critical forcing amplitude and classification of the safe basins. Section 3 presents the comparison of analytic results with numerical simulations and addresses the limitations of the suggested approach. Section 4 is devoted to discussion and conclusions.

2 Model description and analytic results

We consider a harmonically forced monostable Duffing oscillator with softening characteristics in the absence of damping. In the nondimensional form, the equation is written as:

\[
\ddot{q}(\tau) + q(\tau) - q(\tau)^3 = F \sin(\tau \Omega + \Psi),
\]

where \(q\) is the displacement, \(\tau\) stands for a nondimensional time, \(F\) is the amplitude of the external force, \(\Omega\) is the force frequency, and \(\Psi\) is the forcing phase.
Equation (1) defines the motion of a harmonically forced particle of the unit mass in a symmetric quartic potential:

\[ V(q) = \frac{q^2}{2} - \frac{q^4}{4}. \]  

(2)

Potential (2) serves an important example of a structural system prone to unstable symmetric bifurcation (buckling) [38]. It has a well (local minimum) with barriers (local maxima) at \( q = \pm 1 \). Assuming initially the particle is trapped inside the potential well, we are looking for the trajectories that correspond to the escape, i.e., particle leaving the well. Commonly, the escape is defined as \( \lim_{\tau \to \infty} |q(\tau)| > q_{\text{max}} \) where \( 0 < q_{\text{max}} \leq 1 \). However, this criterion is problematic for numerical implementation; therefore, we use an alternative definition of escape: \( \max_{\tau} |q(\tau)| > q_{\text{max}} \). Note that the two definitions are not equivalent, nonetheless, for the purpose of our problem, the latter definition is sufficient [23, 25].

In light of the range constraints due to the escape, one can generalize potential (2) by applying the truncation at a certain critical displacement inside the potential well:

\[ U(q; \xi_{\text{max}}) = \begin{cases} \frac{q^2}{2} - \frac{q^4}{4}, & |q| < q_{\text{max}}, \\ \xi_{\text{max}}, & |q| \geq q_{\text{max}} \end{cases} \]  

(3)

where \( q_{\text{max}} = \sqrt{1 - \sqrt{1 - 4\xi_{\text{max}}^2}}. \)  

(4)

At first glance, the truncated potential may appear far-fetched; however, in some applications it emerges naturally. For example, in the context of naval mechanics the escape from the full (non-truncated) potential corresponds to the ship capsizing [11–13], while the truncation corresponds to additional constraints imposed on the maximum allowed roll angle of the ship. Furthermore, truncated polynomial potentials can be used as a tool for approximating the escape problem in more intricate potentials [39].

It is convenient to represent Eq. (1) in the Hamiltonian form:

\[ \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \]  

(5)

where \( p = \dot{q} \) is the momentum of the particle and Hamiltonian \( H(p, q) \) is

\[ H(p, q) = H_0(p, q) - Fq \sin(\Omega \tau + \Psi). \]  

(6)

Here \( H_0 \) denotes the basic Hamiltonian

\[ H_0(p, q) = \frac{p^2}{2} + U(q; \xi_{\text{max}}), \quad -q_{\text{max}} < q < q_{\text{max}}, \]  

(7)

which describes the free motion of the particle in the truncated potential (3). The basic Hamiltonian \( H_0 \) provides yet another definition of escape: \( \max_{\tau} H_0(q(\tau), p(\tau)) > \xi_{\text{max}} \). The difference between the presented escape definitions is discussed in Appendix.

2.1 Action-angle formalism and isolated resonance approximation

In order to obtain isolated resonance approximation, one needs to rewrite the Hamiltonian system (4) in AA variables and subsequently average it over the fast phases.

The canonical pair \((I, \theta)\) composed of action and angle is defined as follows [1]:

\[ I = \frac{1}{2\pi} \int_{\Gamma_E} p(q, E) dq, \quad \theta = \frac{\partial}{\partial I} \int_0^q p(x, I) dx, \]  

(8)

where \( \Gamma_E \) is a curve defined by a level set \( H_0 = E \). The canonical transformation (8) does not explicitly depend on time \( \tau \), and therefore, the Hamiltonian (6) can be written in AA variables:

\[ H(I, \theta) = H_0(I) - Fq(I, \theta) \sin(\Omega \tau + \Psi). \]  

(9)

Due to the \( 2\pi \)-periodicity of the angle variable, Hamiltonian (9) can be expanded in terms of the Fourier series:

\[ H(I, \theta) = H_0(I) - \sum_{m=-\infty}^{\infty} q_m(I) \left( e^{i(m\theta + \Omega \tau + \Psi)} - e^{i(m\theta - \Omega \tau - \Psi)} \right), \]  

\[ q_m = q_m^*, \]  

(10)

where symbol \(*\) denotes the complex conjugation. In order to treat 1:1 resonance, the phase \( \gamma = \theta - \Psi - \tau \Omega \) ...
is assumed to be slow, while all other combinations of phases must be considered fast \[15, 23, 25\]. After performing the averaging over the fast phases, one arrives at the following slow-flow equations:

\[
\dot{q} = -\frac{\partial C(\gamma, J)}{\partial \gamma}, \quad \dot{\gamma} = \frac{\partial C(\gamma, J)}{\partial J},
\]

with the first integral \[15, 23\]

\[
C(\gamma, J) = H_0(J) - \frac{i\epsilon}{2} (q_1(J)e^{i\gamma} - q_1(J)e^{-i\gamma}) - \Omega J = \text{const},
\]

where \(J = \langle I(\tau) \rangle\) denotes the averaged action. Non-trivial conservation law (12) defines a family of 1:1 RMs of the system, i.e., it approximates Hamiltonian (10) in the vicinity of the basic resonance. The existence of the first integral (12) is attributed to the fact that system (5) is Hamiltonian. The value of the constant is determined by the IC on the RM, i.e., it approximates Hamiltonian with the first integral \[15, 23\]

where \(K\) denotes the averaged action. Non-trivial conservation law (12) defines a family of 1:1 RMs of the system, i.e., it approximates Hamiltonian (10) in the vicinity of the basic resonance. The existence of the first integral (12) is attributed to the fact that system (5) is Hamiltonian. The value of the constant is determined by the IC on the RM, i.e., the values of the averaged action \(J\) and the slow phase \(\gamma\) at which the system is captured by the RM.

By applying transformation (8) to the basic Hamiltonian (7), one can express action-angle variables as functions of the energy \(E\) and the displacement \(q\) as follows (see \[25\] for the detailed derivation):

\[
I(E) = \frac{2\sqrt{2}}{3\pi} \sqrt{1 + \mu(E)} (\mathbb{E}(k(E)) - \mu(E) K(k(E))),
\]

\[
\theta(q, E) = \frac{\pi}{2K(k(E))} F\left(\sin^{-1}\left(\frac{q}{\sqrt{1 - \mu(E)}}\right) \mid k(E)\right),
\]

where \(\mu(E) = \sqrt{1 - 4E}\); functions \(K(k)\) and \(E(k)\) are complete elliptic integrals of the first and the second kind, respectively, with the modulus \(^1\) \(k = \sqrt{(1 - \mu)/(1 + \mu)}\). Finally, \(F(\phi, k)\) denotes the incomplete elliptic integral of the first kind.

Equation (14) yields the displacement \(q\) as a function of the angle \(\theta\) and the energy \(E\):

\[
q(\theta, E) = \sqrt{1 - \mu(E)} \sin\left(\frac{2\theta K(k(E))}{\pi} K(k(E))\right),
\]

where \(\sin(\gamma, k)\) denotes the Jacobi elliptic sine function.

By taking the time derivative of (15) and noting that

\[
\dot{E}(E) = \frac{dE}{dt} = \frac{\pi \sqrt{1 + \mu(E)}}{2\sqrt{2K(k(E))}},
\]

we obtain the following expression for the momentum \(p\):

\[
p(\theta, E) = \sqrt{1 - \mu(E)^2} \operatorname{cn}\left(\frac{2\theta K(k(E))}{\pi} K(k(E))\right).
\]

In order to proceed with derivation of the conservation law (12), one needs to find \(H_0(I)\). Unfortunately, finding the inverse of (13) in a closed form is impossible; however, for the purpose of the escape problem it is not necessary. In fact, in order to parameterize the dynamics on the RM, one can use the averaged energy \(\xi = \langle E(t) \rangle\) that monotonously depends on the averaged action \(J\). With a slight abuse of notation, the conservation law for the slow flow in terms of \((\xi, \gamma)\) is written as follows:

\[
C(\xi, \gamma) = -F G(\xi) \cos(\gamma) - \Omega J(\xi) + \xi = \text{const},
\]

where

\[
J(\xi) = \frac{2\sqrt{2}}{3\pi} \sqrt{1 + \mu(\xi)} (\mathbb{E}(k(\xi)) - \mu(\xi) K(k(\xi))),
\]

\[
G(\xi) = \frac{\pi \sqrt{1 + \mu(\xi)}}{K(k(\xi))} \exp\left(-\frac{\pi K(\sqrt{1 - k(\xi)^2})}{2K(k(\xi))}\right).
\]

Note that in this case, the escape is defined in the simplest, most natural form: \(\max_\tau \xi(\tau) = \xi_{\text{max}}\).

2.2 Escape threshold on the control plane

We proceed with the qualitative analysis of the averaged dynamics of the system defined by the integral of motion (18). In order to determine whether a trajectory starting from a given IC \(q_{\text{ini}}, p_{\text{ini}}\) is escaping according to the maximum energy criterion,
one has to check whether the corresponding level curve:

\[ C(\gamma, \xi) = C(\gamma_{ini}, \xi_{ini}), \]  

(21)
crosses the circle \( \xi = \xi_{max} \). In [23, 25], two main scenarios of transition to escape were identified as a maximum mechanism (MM) and saddle mechanism (SM).

The maximum mechanism is a passage to escape described as follows. For a fixed frequency \( \omega \), the phase trajectory (LPT), i.e.,

\[ C(\gamma, \xi), \]  

is used for the quartic potential (2) and the limiting curve:

\[ \frac{\partial C}{\partial \gamma} | \gamma = \gamma^*, \xi = \xi_{max} = 0. \]  

(23)

It is easy to see that for any \( 0 < \xi_{max} < 1/4 \) the possible values of \( \gamma^* \) are 0 and \( \pi \). Therefore, the critical force \( F_{cr} \) is:

\[ F_{cr} = \pm \frac{C_0 + \Omega J(\xi_{max}) - \xi_{max}}{G(\xi_{max})}. \]  

(24)

In case of SM, the critical force \( F_{cr} \) is obtained by solving the system:

\[ \nabla C(\gamma, \xi) = 0, \quad \det H(C) < 0, \]  

(25)

\[ C(\gamma, \xi) = C(\gamma_{ini}, \xi_{ini}), \]  

(26)

where \( \nabla \) and \( H \) denote the gradient and the Hessian matrix, respectively. Note that solving the above equations analytically for \( \xi^\dagger \) is an impossible task. However, it is not necessary as one can use \( \xi^\dagger \) to parameterize the curve \( (\Omega(\xi^\dagger), F(\xi^\dagger)) \). Figures 1 and 2 show the behavior of the critical force curve depending on different values of \( \xi_{ini} \) and \( \gamma_{ini} \), respectively.

Generally, higher initial energies \( \xi_{ini} \) correspond to lower critical force amplitude required for the escape. However, our research shows that zero initial energy in the case of SM turns out to be an exception as its \( F_{cr}(\Omega) \) curve lies below analogous curves for some higher initial energies.

Figure 2 demonstrates critical force curves \( F_{cr}(\Omega) \) for various values of initial angle \( \gamma_{ini} \). As one can easily see that increasing the value of \( \gamma_{ini} \) lowers the \( F_{cr} \) curve for the SM, while raising the corresponding curve for the MM. Note that, the slow phase \( \gamma \) contains the phase \( \Psi \) of the external forcing. Its effect on the global dynamics is demonstrated in Sect. 3.

Figures 3 and 4 show curves \( F_{cr}(\Omega) \) for the zero IC and various values of the energy truncation level \( \xi_{max} \). In particular, Fig. 3 demonstrates three different escaping mechanisms: green dashed line corresponds to the MM through the tangent point \( (\gamma = \pi, \xi = 0.15) \), blue solid curve corresponds to the SM, and red dotted line—to the MM through the point \( (\gamma = 0, \xi = 0.15) \). The latter scenario occurs when the energy \( \xi^\dagger \) at the saddle point is above the truncation level \( \xi_{max} = 0.15 \), and therefore, we will reasonably call this particular case of the MM a saddle maximum mechanism (SMM).

Figure 4 presents the transformation of the critical forcing curve \( F_{cr}(\Omega) \) with increasing depth of the well. The shallower the well is, the lower the forcing amplitude is, while the resonance frequency (at the minima) increases. This behavior is attributed to the fact that a shallow well can be approximated with a quadratic potential (i.e., linear oscillator); hence, the resonance frequency is close to \( \Omega = 1 \).

2.3 Safe basins in the vicinity of the main resonance of the perturbed system

The escape of the particle corresponds to the phase curves that transverse the threshold energy level \( \xi = \ldots \)
max on the phase cylinder \( n \). On the contrary, the phase curves located below the circle \( n = n_{\text{max}} \) correspond to non-escaping trajectories of the particle. Therefore, in order to locate the SB, one needs to determine its boundary—the critical position of the non-escaping phase curves.

Similarly, to the available escape mechanisms, there are two types of the SBs depending on their boundaries: the maximum type and the saddle type. We say the basin is of maximum type (SBMT) if its boundary is a phase curve tangent to the circle \( \xi = \xi_{\text{max}} \). An SB is of the saddle type (SBST) if its boundary passes through the saddle point. Note that these two types are not mutually exclusive and can coexist.

The boundary of a SBMT is defined by the equation:

\[
C(\gamma, \xi) = C(\gamma^*, \xi^*),
\]

where \( \gamma^* \) is a solution to (23). It is clear that there are only two possible values of \( \gamma^* \): \( \gamma^* = 0 \) and \( \gamma^* = \pi \). Once the boundary is established, one needs to locate the safe region it is enclosing. There are two topologically non-equivalent kinds of SBMT, which we refer to simply as the first kind and the second kind. The main difference between the two is that the SBMT of the second kind wraps around the RM phase cylinder, whereas the first kind does not. The adopted notation is analogous to the well-established classification of the limit cycles in dynamical systems with cylindrical phase space [40]. For our purposes, the most important distinction between two kinds of SBMT is in the way we determine the safe region. For the SBMT of the first kind, the safe region is defined as all the points.
enclosed by the curve (27), whereas for the second kind the safe region comprises all the points bounded between the curve (27) and the circle \( \xi = 0 \). It is important to note that the latter type of SBMT contains points with all possible values of the angle \( \gamma \). A special case of an SBMT of the first kind has an LPT as its boundary. In this case, the safe region is defined the same way as for an SBMT of the second kind.

It is important to remark that there is a singularity at \( \xi = 1/4 \); thus, the tangent point \((\gamma^*, \xi^* = 1/4)\) is not defined.

After the boundary of SB in the space \((\gamma, \xi)\) is computed, one can use formulas (15), (17) to obtain the corresponding SB in the space of IC on the \((q_0, p_0)\) plane.

The SBMT of the first kind is illustrated in Fig. 5, while the examples of the SBMT of the second kind are shown in Figs. 6 and 7. The left panels show the level curves of the first integral (18), maximum energy level (blue dashed line), and the SB (grey region with red boundary). Panels on the right present the SB on the \((q_0, p_0)\) plane.

The boundary of SBST is a phase curve, which passes through the saddle point, i.e., the curve defined by the equation:

\[ C(\gamma, \xi) = C\left(\gamma^\dagger, \xi^\dagger\right), \tag{28} \]

where \( \xi^\dagger \) and \( \gamma^\dagger \) satisfy Eq. (25). The safe region is bounded by the curve (28) and the circle \( \xi = 0 \).

An example of SBST is presented in Fig. 8. The case of coexisting regions is demonstrated in Fig. 9. We adopt the same notation as in Figs. 5, 6, and 7.

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**Fig. 4** Escape threshold versus excitation frequency for the zero IC for various values of the threshold energy level \( \xi_{\text{max}} \)

**Fig. 5** Grey region is SBMT of the first kind on the \((\gamma, \xi)\) cylinder (left) and on the \((q_0, p_0)\) plane (right). Thick dashed line corresponds to truncation level \( \xi_{\text{max}} = 0.24 \). The parameters of external forcing are \( F = 0.0876; \Omega = 0.92 \)
The amount and the location of the available critical points in Eq. (18) exhaust all the possibilities of escape scenarios. Therefore, for the problem in question the presented classification of SBs is complete.

3 Numeric verification

In order to validate theoretical predictions of the critical forcing curve $F_{cr}(\Omega)$ as well as the SBs, we compare the analytic results from Sects. 1 and 2 with numerical simulations. The averaged system (18) is expected to approximate the original system (1) in the vicinity of the primary 1:1 resonance for relatively

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**Fig. 6** SBMT of the second kind for the parameters $F = 0.0876$, $\Omega = 1.04$. Thick dashed line corresponds to truncation level $\xi_{\max} = 0.249$

**Fig. 7** SBMT of the second kind for the parameters $F = 0.0876$, $\Omega = 0.57$. Thick dashed line corresponds to truncation level $\xi_{\max} = 0.21$
small excitation amplitudes. However, for the larger forces we anticipate SB to undergo erosion due to the higher-order resonances, see [2, 29–34].

3.1 Critical escape force in the control plane

Figure 10 demonstrates the comparison of theoretical (solid) critical force curve with numerical simulations (dots). Every dot corresponds to a numerical estimation of $F_{cr}$ using the bisection method with an accuracy of $5 \times 10^{-5}$. Each simulation is performed for up to 3000 excitation cycles (EC), i.e., a trajectory is considered non-escaping if the escape does not occur within the time interval [0, 3000EC]. Three subplots of Fig. 10 represent different energy truncation levels.
Fig. 10  Escape threshold versus excitation frequency for different energy truncation levels $\xi_{\text{max}}$ and zero IC. Dots represent values obtained numerically, while solid curves correspond to the theoretical prediction.

Fig. 11  Escape threshold versus excitation frequency for various values of $\xi_{\text{ini}}$ and $\xi_{\text{max}} = 0.242$, $\gamma_{\text{ini}} = 0.25$, $\Psi = \pi$. The notation is the same as in Fig. 10.
Similarly, Figs. 11 and 12 show the corresponding curves $F_{cr}(\Omega)$ for nonzero ICs.

Comparison of numerical simulation results with these derived from a simplified model prediction clearly shows high-fidelity approximation in the vicinity of the main resonance, the dip, and weaker prediction quality on the distant frequencies, especially for the MM scenario, as the truncation level approaches the maximum. Physical reasons for this discrepancy are well-known appearance of the cascades of new steady-state solutions [2] and chaotic phenomena close to the separatrix.

3.2 The SB—analytic predictions versus numerical simulations

In order to verify the approximation of SB, we compare it with the numerical simulation results. We superimpose theoretic SB boundaries on the non-escaping regions obtained with numeric time integration of system (1). The saddle point $\left(\gamma^\dagger, \zeta^\dagger\right)$ required for determining the boundary of an SBST is approximated using the Newton–Raphson method. For all the numerical experiments, the ICs are drawn from a uniform $200 \times 200$ grid.

Figures 13 and 14 present the variations of the SBs with increasing energy truncation level $\xi_{max}$. Numerically obtained SBs are shaded according to the value of $\xi_{max}$. Lighter grey regions (smaller $\xi_{max}$) include all
the darker ones (larger $\varepsilon_{\text{max}}$). White color corresponds to the escape region common to all considered values of the parameters. Each curve in Figs. 13, 14, 15, 16, and 17 represents a theoretical prediction of the SB boundary for the corresponding parameter values (see legends for the detailed description of the patterns). While the demonstrated SBs in Fig. 13 have the same topology for all the cutoff energy levels, Fig. 14 shows the disjoint SBs for smaller values of $n_{\text{max}}$ (cf. Fig. 9). For the chosen parameters, the agreement between theoretically obtained SBs and their numerical counterparts is very good.

Distortion and rotation effects on SB boundary shapes caused by the phase shift of the forcing are demonstrated in Fig. 15 (cf. Fig. 14).

For relatively small force amplitudes (Fig. 16), the numeric results nicely conform to the theoretical predictions. Figure 17 presents a similar comparison for higher levels of forcing.

As one can see, for small values of the external forcing $F$ the analytically predicted boundaries perfectly match the numeric SBs, even when the SB is disjoint. However, when $F$ increases, the discrepancy between the theoretical prediction and the numeric SBs becomes more and more substantial. The reason for the reduction of the prediction quality can be attributed to the fact that the AIR method captures only the primary 1:1 resonance, while neglecting the secondary resonances, which cause the erosion of SBs [29–34]. Nevertheless, the obtained theoretical boundaries can serve as a reasonable initial approximation of the SB for subsequent possible numeric refining.
Note that due to the negative cubic nonlinearity, system (1) demonstrates a softening behavior, i.e., it exhibits coexisting stable solutions for the range below the natural frequency [41]. This fact is closely related to the presence of disjoint SB. Indeed, the Poincaré–Hopf index theorem implies the existence of at least one stable fixed point within each connected component of an SB. In particular, it means that existence of disjoint SB is impossible for the excitation frequency above the natural frequency of the well.

Figure 18 shows a comparison between the areas of numerically evaluated effective SB, i.e., part of the SB, which excludes the erosion zone, and its analytic approximation depends on the external forcing amplitude \( F \) for two different values of maximal energy level \( \bar{\varepsilon}_{\text{max}} \). The effective SB is obtained as follows: first, the boundary trajectory along with a handful of non-escaping trajectories inside the effective SB is located using the bisection method. As a result, we obtain a set of points belonging to the SB. Then, we find the \( \sim \)-shape of these points, which allows us to capture the effective SB region even if it is not convex [42]. The a priori knowledge of the approximate positions of the centers as well as the symmetry of the phase space provides us with the starting interval for the bisection method. One can easily see that the correspondence between the effective SB and its approximation, while improving toward the lower end of the parameter range, is more prominent for the smaller values of the maximal energy level \( \bar{\varepsilon}_{\text{max}} \). Besides, one observes that the analytic approximation always overestimates the SB area.

In order to further analyze the structure of SB for the larger values of external forcing, it is instructive to consider the stroboscopic (period) map

\[
(q(t), p(t)) \rightarrow (q(t + T), p(t + T)),
\]

with \( T = 2\pi/\Omega \). Figure 19 presents the phase portraits of 100 non-escaping trajectories of the map (29) starting from random ICs. We observe one big island of invariant tori centered around a fixed point, many smaller islands of invariant tori centered around various periodic solutions, and a stochastic layer in between. The structure of the phase portrait of the map (29) suggests the presence of the higher-order resonances contributing to the erosion of the SB. For example, invariant tori around a 6-periodic solution presented on the figure indicate the existence of 1:6 resonance. Dashed curve represents the theoretical

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**Fig. 18** Comparison of the areas of the effective SB (solid) and its approximation (dashed) depending on the external forcing amplitude \( F \) with \( \Omega = 0.91 \). Left: \( \xi_{\text{max}} = 0.249 \); right: \( \xi_{\text{max}} = 0.2 \)

**Fig. 19** Phase portrait of 100 non-escaping trajectories of the period map (29) for the following set of parameters: \( F = 0.0876, \psi = \pi, \Omega = 0.99, \xi_{\text{max}} = 0.24 \). Each trajectory runs for 3000 iterations. Dashed curve corresponds to the theoretically predicted boundary of the SB.
prediction of the SB boundary, and although it fails to capture the SB exactly, it still serves as a reasonable first guess for locating the SB.

4 Discussion and conclusions

The results obtained above indicate that the approximation of isolated resonance enables analytic prediction of the safe boundaries in the problem of forced escape. For relatively small forcing amplitudes, these predictions are very accurate. The reason is that the forcing term is in fact a perturbation of integrable basic Hamiltonian describing the conservative motion in the well. This approach reveals some new possible properties of the SB, e.g., complicated, or even disjoint geometric shape.

For higher forcing amplitudes, the quality of the prediction deteriorates due to intrinsic inaccuracies of the AIR and, more significantly, due to secondary resonances that are not considered in the AIR and cause the SB erosion. Still, the analytic prediction can serve as a viable initial guess for the SB with further numeric tightening.

The analysis presented above is substantially based on the ability to perform the action-angle transformation in the benchmark model with quartic potential. Application of the developed method to more generic models, where this transformation is not possible, may be based on appropriate approximation techniques and requires further exploration.

Finally, all the analysis in the present work was performed under the assumption that the damping can be neglected. In particular, this assumption leads to the conservation law (18)—the basic element for identifying the boundaries of the SB. However, even in the presence of damping it is possible to apply the AA formalism and then to use perturbation techniques to obtain the critical forcing curve and the boundaries of SB. This idea requires a lot of additional research and will be a subject of future work. Here, we demonstrate that the damping indeed strongly affects the SB. Figure 20 shows the SB of system (1) with an addition of a linear damping term \(0.005q\), superimposed with the theoretical SB boundary of the corresponding system without damping. As one can expect, the Hamiltonian analysis underestimates the SB for the damped case and can serve as a conservative approximation.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest or competing interests.

Appendix: Comparison of the escape criteria

As it was mentioned before, the two escape criteria—maximum energy and maximum displacement—are not equivalent. It is obvious that the energy criterion immediately follows from the displacement criterion, and therefore, the SB based on the energy criterion (ESB) is always included in the corresponding SB defined through the maximum displacement criterion (DSB). However, the other way is not necessarily true.

In order to compare two criteria, the following numeric test is performed. For each value of the forcing amplitude \(F\), we pick 10,000 random ICs from the square \([-1, 1] \times [-1, 1]\). Then, we perform numerical time integration of system (1) starting from the
chosen set selecting only the non-escaped trajectories according to the ESB. Next, we run the simulations again for the rest of ICs but with the displacement criterion to obtain the set belonging to the DSB but not to the ESB. Let $A_q$ and $A_E$ denote the number of points that belong to the DSB and the ESB, respectively. The difference between $A_q$ and $A_E$ relative to $A_E$ exhibits the discrepancy between the two criteria.

The experiment is repeated five times for the values of the external forcing amplitude $F$ ranging from 0.001 to 0.071 with a step $\Delta F = 0.005$. The results are presented in Fig. 21. As one can see, for the small values of the external forcing amplitude $F$, SBs for both criteria are almost identical. However, with increasing $F$ the difference becomes substantial, see Fig. 21.

Furthermore, our findings reveal that the choice of the escape criterion also strongly affects the shape of the SBs. The maximum energy escape criterion yields SBs with smoother boundary.

Figure 22 presents a comparison of SBs in the $(q_0, p_0)$ plane for two superimposed escape criteria for a short evaluation time $t_{\text{eval}}$ (left) and a long $t_{\text{eval}}$ (right). Colors grey and black correspond to the displacement and the energy criteria, respectively.

For a short evaluation time $t_{\text{eval}}$, the boundaries of SBs for both criteria look irregular representing the transient behavior. For a long time $t_{\text{eval}}$ after the transient dynamics fades out, the SB boundary becomes clearer and smoother. The full view of SB for the same values of parameters as in Fig. 22 is shown in Fig. 23.

Figure 24 presents the change of the SB area with time. As one can see, the area undergoes a rapid shrinking for about 2000EC, followed by almost no change after. Each point on the graph is a natural
logarithm of the SB area computed numerically at a given evaluation time.

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