On the cohomology algebra of free loop spaces.

by

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Abstract. Let $X$ be a simply connected space and $\mathbb{K}$ be any field. The normalized singular cochains $N^*(X;\mathbb{K})$ admit a natural strongly homotopy commutative algebra structure, which induces a natural product on the Hochschild homology $HH_\ast N^* X$ of the space $X$. We prove that, endowed with this product, $HH_\ast N^* X$ is isomorphic to the cohomology algebra of the free loop space of $X$ with coefficients in $\mathbb{K}$. We also show how to construct a simpler Hochschild complex which allows direct computation.

Introduction.

The classical definition of the normalized Hochschild complex $\mathcal{C}_\ast A$ of an algebra $A$ extends naturally to a differential graded cochain algebra $(A,d_A)$ over a field $\mathbb{K}$ of characteristic $p \geq 0$ (see I-§2). Suppose $A$ is augmented and let $\rho : (A \otimes BA,D) \to (BA,\bar{D})$ denote the canonical projection on the reduced bar construction $BA$. The homology of the chain complex $\mathcal{C}_\ast A = (A \otimes BA,D)$ is the Hochschild homology (with coefficients in $A$) of $(A,d_A)$ and is denoted by $HH_\ast A$. If the graded algebra $A$ is commutative it is well-known that $H^* BA$ and $HH_\ast A$ are commutative graded algebras.

The purpose of the first part of this paper is to construct a product on $\mathcal{C}_\ast A$ when $A$ is not commutative. For this we first embed the category $DA$ of augmented differential graded algebras into the category $DASH$ of strongly homotopy graded algebras introduced by H.J. Munkholm in 1974, [27]. The objects of $DA$ and $DASH$ are the same but $DASH(A,A') = DC(BA,BA')$ where $DC$ denotes the category of coaugmented differential graded coalgebras. Then we consider the subcategory of shc-algebras (strongly homotopy commutative graded algebras) of which the objects are differential graded algebras $(A,d_A)$ with multiplication $m$ belonging to $DASH(A \otimes A,A)$ and satisfying some natural axioms (see I-§3). In particular, such algebras have graded commutative homology algebras $H(A,d_A)$ and it is well-known that there is a natural Hopf algebra structure on $BA$ such that $H^* BA$ is a commutative Hopf algebra. First we establish:

Theorem 1 Let $(A,d_A)$ be a shc-algebra. The shc-structure induces a natural graded commutative algebra structure on the Hochschild homology $HH_\ast A$. Moreover, the canonical projection $\rho : HH_\ast A \to H_\ast BA$.

The product structure of $HH_\ast A$ is given explicitly on the normalized Hochschild complex $\mathcal{C}_\ast A$. Nonetheless, complete computations are not tractable directly from $\mathcal{C}_\ast A$. To overcome this difficulty we introduce (§5) the notion of shc-equivalence between two shc-algebras. If $(A,d_A)$ and $(A',d_{A'})$ are shc-equivalent, then the algebra $HH_\ast A$ is isomorphic to $HH_\ast A'$. A particular case of interest is that of a differential graded algebra $(A,d_A)$ which is
*shc*-equivalent to a commutative differential graded algebra. This is the case, for instance, if \((A,d_A)\) is the algebra of normalized singular cochains of a space \(X\) with coefficients in \(\mathbb{K}\) whenever

a) \(X\) is a finite dimensional smooth manifold and \(\mathbb{K} = \mathbb{R}\), (example I-§3.4),

b) \(X\) is a connected topological space and \(\mathbb{K} = \mathbb{Q}\) (example I-§5.4),

c) \(X\) is an \(r\)-connected finite complex and \(\mathbb{K}\) is of characteristic \(p\), with \(p > \frac{\dim X}{r}\) \((r > 1)\) (example II-§4-3).

One more particular case is that of a \(shc\)-algebra \((A,d_A,\mu_A)\) which is \(shc\)-equivalent to the commutative graded algebra \(H(A,d_A)\) equipped with zero differential. This special case is far from being trivial, as illustrated by examples 1, 2, 3, 4, in section II-§4, or the computations made by K. Kuribayashi [23].

When \(H^0A = \mathbb{K}\), \(H^1A = 0\) and \(\dim H^iA < \infty\) for every \(i\), the \(shc\)-equivalence class of an \(shc\)-algebra \((A,d_A)\) can be represented by a \(shc\)-model. This is a free non-commutative model of the differential graded algebra \((A,d_A)\) which is a quotient of \(\Omega B(A,d_A)\), enriched by a structural map. In this case, a simpler Hochschild complex is constructed, (see I-§6). This is the main tool for the computations in examples 1, 4 and 5, in section II-§4.

Let \(X\) be a topological space and denote simply by \(N^*X\) the \(\mathbb{K}\)-algebra \(N^*(X;\mathbb{K})\) of normalized singular cochains of \(X\). By theorem 1, the natural \(shc\)-structure on \(N^*X\) allows one to define a natural commutative graded algebra structure on \(HH_*,N^*X\). Denote by \(X^{S^1}\) the free loop space of the topological space \(X\), that is the space of all continuous maps from the circle into \(X\). In 1987, J.D.S. Jones [22] constructed an isomorphism of graded vector spaces \(HH_*,N^*X \cong H^*(X^{S^1};\mathbb{K})\).

Using theorem 1 and the acyclic model theorem for cochain functors, we prove that Jones’ isomorphism is compatible with the product on \(HH_*,N^*X\) defined in theorem 1 and the usual cup product on \(H^*(X^{S^1};\mathbb{K})\). More precisely:

**Theorem 2** Let \(X\) be a simply connected space. There exists a natural equivalence of cochain complexes \(\mathfrak{e}^\bullet(N^*X) \rightarrow C^\bullet(X^{S^1})\) which induces a natural algebra isomorphism \(HH_*,N^*X \cong H^*(X^{S^1};\mathbb{K})\).

Furthermore the algebra map \(\rho_* : HH_*,N^*X \rightarrow H^*BN^*X\) identifies with \(j^* : H^*(X^{S^1};\mathbb{K}) \rightarrow H^*(\Omega X;\mathbb{K})\), where \(j : \Omega X \rightarrow X^{S^1}\) denotes the inclusion associated to some base point in \(X\).

This extends results obtained by M. Vigué-Poirrier, [30], N. Dupont-K. Hess [10] and by S. Halperin and M. Vigué, [18]. The algebraic techniques introduced in I-§6 allow one to make computations in the general case. For instance in §4 -examples 5 and 6 of part II we examine the cases \(X = \Sigma \mathbb{C}P^2\), \(\mathbb{K} = \mathbb{F}_2\) and \(X = G_2\), the exceptional Lie group, with \(\mathbb{K} = \mathbb{F}_5\).

Theorem 2 is somehow related to the formalism of path-integrals in supersymmetric quantum mechanics as explained below.

Let \(X\) be a finite dimensional smooth manifold and let \(LX\) be the space of smooth maps from \(S^1\) to \(X\). This space may be given the structure of an infinite dimensional manifold modelled on a Fréchet space. If \(X\) is paracompact, the natural inclusion \(LX \subset X^{S^1}\) is a homotopy equivalence. On the other hand, if \(A_{DR}(-)\) denotes the functor “differential
forms” then the “iterated integral map” defined by K.T. Chen, [8],
\[ \varepsilon_* A_{DR}(X) \to A_{DR}(LX) \]
is a homomorphism of differential graded algebras, which is a quasi-isomorphism when \( X \) is simply connected. Since \( N^*X \) is shc-equivalent to \( A_{DR}(X) \) there exists an isomorphism of graded algebras
\[ H_{DR}(LX) \cong H^*(LX; \mathbb{R}). \]
This is a de Rham theorem for the infinite dimensional manifold \( LX \).

In the light of the result of Gezler, Jones and Petrack, [16], one may ask: Is the natural equivalence \( C^*(N^*X) \to C^*(XS^1) \) a shm-map or a homomorphism of \( A_\infty \)-algebras in the sense of Stasheff, [29]?

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**Part I: Algebraic setting.**

Throughout the paper, the ground field \( \mathbb{k} \) of characteristic \( p \geq 0 \) is fixed. We use the Kronecker convention: an object with lower negative graduation has upper non-negative graduation.

1. **Review of the bar and cobar constructions**

1.1 Recall that \( DA \) (resp. \( DC \) resp. \( DM \)) denotes the category of augmented, differential graded algebras (resp. coaugmented differential graded coalgebras, resp. differential graded modules). An object \( A \in \text{Obj} \mathcal{D}A \) is a graded \( \mathbb{k} \)-vector space \( A = \{ A^k \}_{k \geq 0} \), equipped with structure
\[ d_A : A^k \to A^{k+1}, m : \oplus_{k+l=n} A^l \otimes A^k \to A^n, \eta_A : \mathbb{k} \to A, \epsilon_A : A \to \mathbb{k} \]
and an exact sequence \( 0 \to IA \xrightarrow{i} A \xrightarrow{q} \mathbb{k} \to 0 \). Similarly, an object \( C \in \text{Obj} \mathcal{D}C \) is a graded \( \mathbb{k} \)-vector space \( C = \{ C^k \}_{k \geq 0} \), with structure
\[ d_C : C^k \to C^{k+1}, \Delta : C^n \to \oplus_{k+l=n} C^l \otimes C^k, \epsilon_C : C \to \mathbb{k}, \eta_C : \mathbb{k} \to C \]
and an exact sequence \( 0 \to \mathbb{k} \xrightarrow{\eta_C} C \xrightarrow{\epsilon_C} JC \to 0 \).

The morphisms in \( DA \) (resp. \( DC \)) respect the whole structure and are called homomorphisms of \( DG \)-algebras (resp. homomorphisms of \( DG \)-coalgebras).

1.2 We denote by \( B : DA \to DC \), resp. \( (\Omega : DC \to DA) \) the bar construction, (resp. the cobar construction) and by \( B : DA \to DC \) resp. \( (\Omega : DC \to DA) \) the reduced bar construction (resp. the reduced cobar construction) [1], [26], [23], [27], [13]. The functors \( B \) and \( \Omega \) (resp. \( B \) and \( \Omega \)) are adjoint functors to each other. We denote by \( \omega : DA(\Omega C, A) \to DC(C, BA) \) the natural bijection. As \( \mathbb{k} \)-graded vector spaces
\[ BA = T(A), \quad BA = T(IA), \quad \Omega C = T(C) \quad \text{and} \quad \Omega C = T(JC) \]
and we denote by \([a_1|a_2|...|a_k]\) (resp. \((c_1|c_2|...|c_l)\)) the standard generators of \(B^k A\) or \(B^k \) (resp. \(\Omega^k C\) or \(\Omega^k C\)) of degree \(\sum_{i=1}^k \deg a_i - k\) (resp. \(\sum_{i=1}^l \deg a_i + l\)).

The natural inclusion \(BA \to \overline{BA}\) (resp. the quotient \(\overline{C} \to \Omega C\)) induces an isomorphism in (co)homology.

1.3 The homomorphism \(\alpha_A \in DA(\Omega BA, A)\) corresponding, by adjunction, to the identity \(id_{BA}\) is a natural homotopy equivalence \([27], [13]\). More precisely, \(\iota_A \in DM(A, \Omega BA)\), defined by

\[
\iota_A(a) = \begin{cases} 
\eta_A \epsilon_A(a) & \text{if } \epsilon_A(a) \neq 0 \\
\langle [a] \rangle & \text{if } \epsilon_A(a) = 0 
\end{cases}
\]

satisfies \(\alpha_A \circ \iota_A = id_A\) and \(id_{\Omega BA} - \iota_A \circ \alpha_A = d_{\Omega BA} \circ h + h \circ d_{\Omega BA}\), for some chain homotopy \(h : \Omega BA \to \Omega BA\) such that \(\alpha_A \circ h = 0\), \(h \circ \iota_A = 0\), \(h^2 = 0\).

1.4 Recall that \(f, g \in DA(A, A')\) are homotopic in \(DA\) if there exists a linear map \(h : A \to A'\) such that \(f - g = d_{A'} \circ h + h \circ d_A\) and \(h(xy) = h(x)g(y) + (-1)^{|x|} f(x)h(y)\) with \(x, y \in A\). Observe that \(\iota_A \circ \alpha_A \simeq id_{\Omega BA}\) in \(DM\) but not in \(DA\).

2. Hochschild homology of a differential graded cochain algebra.

2.1 Let \((A, d_A)\) be a cochain algebra:

\[A = \{A_i\}_{i \geq 0}, \quad d_A : A^k \to A^{k+1}, \quad x \in A^i, \quad |x| := i.\]

Let \(d_{BA}\) denote the differential of the bar construction \(BA\) or \(BA\). The tensor product \((A, d_A) \otimes (BA, d_{BA})\) (resp. \((A, d_A) \otimes (BA, d_{BA})\)) is then a differential module whose differential is denoted \(d_{A \otimes BA}\) in both cases. The Hochschild differential, denoted by \(D\), is defined by:

\[Da_0[a_1|...|a_n] = (d_0 - d_n)a_0[a_1|...|a_n] + d_{\otimes BA}a_0[a_1|...|a_n]\]

where \(d_0a_0[a_1|...|a_n] = (-1)^{|a_0|}a_0[a_1|...|a_n]\) and \(d_n0[a_1|...|a_n] = (-1)^{|a_n|+1}a_0[a_1|...|a_{n-1}+|a_n|]

\[a_n a_0 \otimes [a_1|...|a_n].\]

By definition,

\[\varepsilon_A := (A \otimes BA, D), \quad \varepsilon_A := (A \otimes BA, D)\]

is the \((normal\)\) Hochschild complex (resp. \((un-normalized)\) Hochschild complex) of \((A, d_A)\) and

\[HH_A = H \varepsilon_A = H \varepsilon_A\]

is the \(Hochschild homology\) of the cochain algebra \((A, d_A)\). One should notice here that \(\varepsilon_A\) is concentrated in non-negative total upper degrees. In particular \(HH_A\) is concentrated in non-negative upper degrees.

2.2 Let \(\Sigma_{n,m}\) be the set of \((n,m)\)-shuffles. Consider the shuffle map, \(\llbracket\rlbracket \ 4.2.1, sh : \varepsilon_A \otimes \varepsilon_A \to \varepsilon_A (A \otimes A)\) defined by:

\[
sh(a_0[a_1|a_2|...|a_n] \otimes b_0[b_1|b_2|...|b_m]) = (-1)^t \sum_{\sigma \in \Sigma_{n,m}} (-1)^{\epsilon(\sigma)} a_0 \otimes b_0[c_{\sigma(1)|...|c_{\sigma(n+m)}]
\]

where \(t = |b_0|[a_0|...|a_n]|b_0|[b_1|...|b_m]|\) and \(\epsilon(\sigma) = \sum((|c_{\sigma(i)}| - 1)(|c_{\sigma(m+j)}| - 1))\), summed over all pairs \((i, m+j)\) such that \(\sigma(m+j) <
σ(i). Clearly, sh induces a chain map \(sh : \mathfrak{e}_{\ast}A \otimes \mathfrak{e}_{\ast}A \to \mathfrak{e}_{\ast}(A \otimes A)\) and a map of differential coalgebras \(sh : BA \otimes BA \to B(A \otimes A)\).

3 \textbf{shc-algebras.}

3.1 A strongly homotopy commutative algebra (shc-algebra for short) is a triple \((A, d_{A}, \mu_{A})\) with \((A, d_{A}) \in \text{Obj}DA\) and \(\mu_{A} \in DA(\Omega B(A \otimes A), \Omega BA)\) satisfying

1. \(\alpha_{A} \circ \mu_{A} \circ \iota_{A \otimes A} = m_{A}\), where \(m_{A}\) is the product in \(A\);
2. \(\alpha_{A} \circ \mu_{A} \circ \Omega B(id_{A} \otimes \eta_{A}) \circ \iota_{A} = \alpha_{A} \circ \mu_{A} \circ \Omega B(id_{A} \otimes \eta_{A}) \circ \iota_{A} = id_{A}\);
3. \(\mu \circ \Omega B(\alpha_{A} \otimes id_{A}) \circ \Omega B(\mu \otimes id_{A}) \circ \chi_{(A \otimes A) \otimes A} \simeq \mu \circ \Omega B(id_{A} \otimes \alpha_{A}) \circ \Omega B(id_{A} \otimes \mu) \circ \chi_{A \otimes (A \otimes A)}\)
   in \(DA\);
4. \(\mu \circ \Omega BT \simeq \mu \) in \(DA\),

where \(T\) denotes the interchange map \(T(x \otimes y) = (-1)^{|x||y|} y \otimes x\). The two natural homomorphisms of DG-algebras, defined in [27]-2.2,

\[
\Omega B(\Omega B(A \otimes A) \otimes A) \xleftarrow{\chi_{(A \otimes A) \otimes A}} \Omega B(A \otimes A \otimes A) \xrightarrow{\chi_{A \otimes (A \otimes A)}} \Omega B(A \otimes \Omega B(A \otimes A))
\]

satisfy:

\[
\alpha_{(A \otimes A) \otimes A} \circ \chi_{(A \otimes A) \otimes A} = \alpha_{A \otimes (A \otimes A)} = \alpha_{A \otimes (A \otimes A)} \circ \chi_{A \otimes (A \otimes A)}.
\]

In particular, if \((A, d_{A}, \mu_{A})\) is a shc-algebra, then conditions 1 and 4 in the definition imply that \(H^{\ast}(A)\) is a commutative graded algebra.

3.2 Consider \(A\) and \(A'\) in \(\text{Obj}DA\). The linear map \(f \in DM(A, A')\) is said to be a \textbf{shc-map} from \((A, d_{A}, \mu_{A})\) to \((A', d'_{A}, \mu'_{A})\) if there exists \(\underline{f} \in DA(\Omega BA, \Omega BA')\) such that:

1. \(\alpha_{A} \circ \underline{f} \circ \iota_{A} = f\);
2. \(\alpha_{A'} \circ \underline{f} \circ \eta_{BA} = \eta_{A'}\);
3. \(\underline{f} \circ \mu_{A} \simeq \mu'_{A} \circ \underline{f} \otimes \underline{f}\) in \(DA\).

where \(\underline{f} \otimes \underline{f}\) is defined in [27]-2.2. Moreover, if \(f \in DA(A, A')\), then \(f\) is a strict shc-map. Observe that a shc-map is a shm-map in the sense of [27]. Obviously, if \(A\) and \(A'\) are commutative graded algebras, then any \(f \in DA(A, A')\) is a strict shc-map.

3.3 \textbf{Proposition.} Suppose that \((A, d_{A}, \mu_{A})\) is a shc-algebra, then \(BA\) is a Hopf algebra which is associative up to homotopy, and \(HBA\) is a commutative Hopf algebra.

\textbf{Proof.} Take \(\omega\) as defined in 1.2. Observe that:

\[
\alpha_{A} \circ \mu_{A} \in DA(\Omega B(A \otimes A), A) \quad \text{and} \quad sh \in DC(BA \otimes BA, B(A \otimes A))
\]

hence \(\nu_{A} = \omega(\alpha_{A} \circ \mu_{A}) \in DC(BA \otimes A, BA)\) and the composite \(\nu_{A} \circ sh \in DC(BA \otimes BA, BA)\) defines a Hopf algebra structure on \(BA\). Moreover, \(\nu_{A} \circ T \simeq \nu_{A}\). Indeed \(\nu_{A} \circ T = \omega(\alpha_{A} \circ \mu_{A}) \circ BT = \omega(\alpha_{A} \circ \mu_{A} \circ T) \approx \omega(\alpha_{A} \circ \mu_{A}) = \nu_{A}\), since \(\omega\) preserves homotopy.

3.4 \textbf{Example.} Let \(X\) be a finite-dimensional smooth manifold and let \(U\) be an open cover of \(X\). We denote by \(A_{DR}(X)\) the de Rham complex of \(X\) and by \(N^{\ast}(X; \mathbb{R})\) the cochain complex of normalized singular cochains with coefficients in \(\mathbb{R}\). The natural homomorphism of cochain
complexes \( A_{DR}(X) \to N^*(X; \mathbb{R}) \), \( \omega \mapsto f \omega \), is a \( shc \)-map by [4]. Consider the Čech-de Rham bicomplex, \( N^{**}(U; A_{DR}(X)) = \bigoplus_{p,q \geq 0} N^p(U; A^q_{DR}(X)) \) with the usual differential and with product defined by \( (\alpha \cup \beta)|_{\cup_{i_0} \cup_{i_1} \ldots \cup_{i_{p+q}}} = (-1)^p \alpha|_{\cup_{i_0} \cup_{i_1} \ldots \cup_{i_p}} \cup \beta|_{\cup_{i_{p+1}} \cup_{i_{p+2}} \ldots \cup_{i_{p+q}}} \).

Now from [27] proposition 4.7 we see that \( \alpha \) is an identity in homology. It results from axiom 4 in the definition of a \( shc \)-structure and naturality of the constructions. The associativity property of \( \Phi \) follows directly from the definition of a \( shc \)-structure and naturality of the constructions.

**4 Proof of theorem 1**

4.1 Let \( (A, d_A, \epsilon_A) \) be an augmented cochain algebra. Since the cochain map \( \iota_A : A \to \Omega BA \), (1.3) is not a homomorphism of \( DG \)-algebras, it does not induce a homomorphism of Hochschild complexes. Nonetheless \( \epsilon_*(\alpha_A) : \epsilon_*(\Omega BA) \to \epsilon_*(A) \) is a surjective quasi-isomorphism. Therefore there exists a chain map

\[
s_A : \epsilon_* A \to \epsilon_* \Omega BA
\]

such that \( \epsilon_* \alpha_A \circ s_A = Id_{\epsilon_* A} \) and \( s_A \circ \epsilon_* \alpha_A \) is chain homotopic to \( id_{\epsilon_* (\Omega BA)} \).

Now suppose that \( (A, d_A, \mu_A) \) is a \( shc \)-algebra. We define a homomorphism of \( DG \)-modules

\[
\Phi : \epsilon_* A \otimes \epsilon_* A \to \epsilon_* A, \quad \Phi = \epsilon_* \alpha_A \circ \epsilon_* \mu \circ s_{A \otimes A} \circ sh.
\]

where \( s_{A \otimes A} \) denotes the linear section of \( \alpha_{A \otimes A} \), as defined above. Notice that \( \Phi \) induces a homomorphism of cochain complexes \( \epsilon_* A \otimes \epsilon_* A \to \epsilon_* A \), also denoted by \( \Phi \). Since \( HS_{A \otimes A} = (H \alpha_{A \otimes A})^{-1} \), the homomorphisms \( H \Phi \) does not depend on the choice of the section \( s_{A \otimes A} \). Actually, precomposing \( H_*(\Phi) \) by the \( K\text{"{u}nneth} \) isomorphism \( HH_* A \otimes HH_* A \to H_*(\epsilon_* A \otimes \epsilon_* A) \) yields a multiplication

\[
\Phi_* : HH_* A \otimes HH_* A \to HH_* A.
\]

The associativity property of \( \Phi_* \) is a direct consequence of the associativity of the shuffle map and of \( H \mu_A \) together with the fact that the morphisms \( \iota_- \) and \( \chi_- \) induce the identity in homology. It results from axiom 4 in the definition of a \( shc \)-algebra that \( H \eta_A : k \to HH_* A \) is a unit. Axiom 2 in the definition of a \( shc \)-structure and naturality imply the commutativity of \( HH_* A \).

**4.2 Lemma.** If \( \varphi : A \to B \) is a strict \( shc \)-map then \( HH_* \varphi \) is a homomorphism of graded algebras. Moreover, if \( \varphi \) is a quasi-isomorphism of \( DG \)-algebras then \( HH_* \varphi \) is an isomorphism.

**Proof.** If \( \varphi \) is a homomorphism of \( DG \)-algebras then \( HH_* \varphi \) is a well-defined linear map, which is an isomorphism when \( \varphi \) is a quasi-isomorphism. The fact that \( HH_* \varphi \) preserves multiplications follows directly from the definition of a \( shc \)-map together with the naturality of the constructions.
4.3 Finally, observe that the diagram

\[
\begin{array}{cccccc}
\mathfrak{c}_*A \otimes \mathfrak{c}_*A & \xrightarrow{sh} & \mathfrak{c}_*(A \otimes A) & \xrightarrow{\varepsilon_\Omega BA} & \mathfrak{c}_*\Omega BA & \xrightarrow{\varepsilon_\mathfrak{c}_*A} & \mathfrak{c}_*A \\
\downarrow \rho \otimes \rho & & \downarrow \rho & & \downarrow \rho & & \downarrow \rho \\
B A \otimes B A & \xrightarrow{sh} & B(A \otimes A) & \xleftarrow{B_\alpha A \otimes A} & B_\Omega B(A \otimes A) & \xrightarrow{B_\mu A} & B \Omega B A & \xrightarrow{B_\alpha A} & B A
\end{array}
\]

commutes. Now, using adjunction, we have that \( \nu_A \circ B_\alpha A \otimes A = B_\alpha A \circ B_\mu A \). This ends the proof of theorem 1.
5 shc-equivalence and shc-formality.

5.1 Two shc-algebras $A$ and $A'$ are **shc-equivalent** if there exists a sequence of strict shc-maps,

$$A \leftarrow A_1 \rightarrow A_2 \leftarrow \ldots \rightarrow A'$$

which are quasi-isomorphisms. This implies, in particular, that $HH_*A$ and $HH_*A'$ are isomorphic as graded algebras.

5.2 The shc-algebra $A$ is said to be **shc-commutative** if $(A, d_A, \mu_A)$ is shc-equivalent to a commutative differential graded algebra $(A', d_A)$.

The shc-algebra $A$ is said to be **shc-formal** if $A$ is shc-equivalent to the commutative differential graded algebra $H(A, d_A)$ equipped with zero differential; in this case $HH_*A \cong HH_*(H(A))$ can be computed using differential forms, as established by Hochschild, Kostant and Rosenberg, [19].

5.3 As proven in [7], any commutative graded algebra $A$ admits a free commutative model $\mathcal{M}_A$ which is of the form $(AX, d) \rightarrow A$, where $AX = E(X^{odd}) \otimes P(X^{even})$ and $E$ means exterior algebra and $P$ polynomial algebra. Therefore, if $(A, d_A)$ is shc-commutative, then $HH_*A = HH_*\mathcal{M}_A$ and one deduces easily from [8] and 25-§8-2.3 that there exists a commutative graded differential algebra of the form $(AX \otimes \Gamma sX, d')$, where $\Gamma$ stands for the free algebra of divided powers, such that

$$H^*(AX \otimes \Gamma sX, d') \cong HH_*A$$

and

$$H^*(\Gamma sX, d') \cong H^*BA$$

as graded algebras.

5.4 Example. Let $\mathbb{K}$ be a field of characteristic zero and let $X$ be a connected space. A simple generalization of example 3.4 with the polynomial de Rham forms, $A_{PL}(X)$, playing the role of $A_{DR}(X)$ and a simplicial Čech-de Rham bicomplex $N^{**}(X; A_{PL}(X))$ (see [7] or [13]-§10 for more details) proves that the shc-algebra $N^*(X; \mathbb{K})$ is shc-commutative. If there exist quasi-isomorphisms of DG-algebras $A_{PL}(X) \rightarrow (A, d_A) \leftarrow H^*(X, \mathbb{K})$ where $A$ is a commutative graded algebra, then the space $X$ is said to be $\mathbb{K}$-formal and $N^*(X; \mathbb{K})$ is shc-formal. This is no longer true for a field of positive characteristic, as shown by example II-4.4 or II-4.5. However, it follows from example 3.4 and [1] that, if $X$ is a compact Kähler manifold or a Riemannian symmetric space, and if $\mathbb{K} = \mathbb{R}$, then $HH_*N^*X = HH^*H_{DR}(X)$ as an algebra.

6 A Small Hochschild complex for a shc-algebra.

6.1 Let $(A, d_A, \mu_A)$ be a shc-algebra. The composite $BA \otimes BA \xrightarrow{sh} B(A \otimes A) \xrightarrow{\mu} BA$, where $\Omega_{sh} = \mu_A$, defines the product (denoted $*$) considered in 3.3. In particular, since $\Omega(BA \otimes BA) = (T((s^{-1}B^+A \otimes \mathbb{K}) \oplus (\mathbb{K} \otimes s^{-1}B^+A) \oplus s^{-1}(B^+A \otimes B^+A)), D)$, the composite

$$\mu_0 : (s^{-1}B^+A \otimes \mathbb{K}) \oplus (\mathbb{K} \otimes s^{-1}B^+A) \oplus s^{-1}(B^+A \otimes B^+A) \xrightarrow{s^{-1}sh} s^{-1}B^+(A \otimes A) \xrightarrow{\mu} s^{-1}B^+A$$

called the "linear part" of $\mu$ satisfies

$$\mu_0(s^{-1}[a_1][a_2] \otimes 1) = s^{-1}[a_1][a_2]$$

$$\mu_0(1 \otimes s^{-1}[a_1][a_2]) = s^{-1}[a_1][a_2]$$

$$\mu_0(s^{-1}[a_1][a_2] \otimes [b_1][b_2]) = s^{-1}([a_1][a_2] \star [b_1][b_2])$$.
6.2 Let \((A,d_A)\) be an augmented differential graded algebra and suppose we have an algebra quasi-isomorphism \((TU,D) \rightarrow (A,d_A)\). One example of this situation is \(\alpha_A : \Omega BA \rightarrow (A,d_A)\) where \(U = s^{-1}B^+A\), \(D = D_1 + D_2\) with \(D_1U \subset U\) while \(D_2U \subset T^{\geq 2}U\).

Here, \(D_1\) is, up to a shift of degrees, \(d_{BA}\) the differential of the bar construction, and \(D_2u = \sum_i(a_i|a'_i)\) where \(\Delta u = \sum_i a_i \otimes a'_i\) is the reduced coproduct in \(BA\).

6.3 Let \((A,d_A)\) be an augmented graded algebra and suppose that:

1) \(H^0(A,d_A) = \mathbb{k}\) and that \(H^1(A,d_A) = 0\)

2) There is a differential \(D\) on \(TU\) such that \(D = D_1 + D_2\) with \(D_1U \subset U\) and \(D_2U \subset T^{\geq 2}U\).

3) we have chosen a quasi-isomorphism \((TU,D) \rightarrow (A,d_A)\) of DG-algebras,

Set \(U = \ker D_1 \oplus S\), \(\ker D_1 = D_1S \oplus V\). Clearly, \(V \cong H^*(U,D_1)\). The decreasing filtration \(F^pTU = \oplus_{i \geq p} T^iU\) yields a first quadrant spectral sequence which converges to \(H^*TU \cong H^*A\). Let \(I\) denote the ideal generated by \(S\) and \(DS\) in \(TU\); the ideal \(I\) is acyclic and hence the projection \(p : (TU,D) \rightarrow (TU/I,D)\) is a quasi-isomorphism.

From the decomposition \(U = V \oplus S \oplus D_1S\), we deduce an isomorphism of graded algebras \(TU/I \cong TV\) which defines a differential \(d_V\) on \((TV,d_V)\) and a surjective quasi-isomorphism

\[p_V : (TU,D) \rightarrow (TV,d_V).\]

Moreover, there exists a homomorphism of DG-algebras \(\varphi_V : (TV,d_V) \rightarrow (TU,D)\) such that \(p_V \circ \varphi = id\). Therefore \(\varphi_V\) is a quasi-isomorphism. Observe that when \((TU,D) = \Omega BA\),

\[\Omega BA \xrightarrow{p_V} (TV,d_V) \xrightarrow{\varphi_V} \Omega BA,\]

and

a) \(V = \{V^i\}_{i \geq 2}\) and it is isomorphic to \(H(U,D_1) \cong s^{-1}H^+(BA)\),

b) the linear part of \(d_V\) is zero while its quadratic part is, up to a shift of degree, the comultiplication of \(H^*BA\).

The differential graded algebra \((TV,d_V)\) is called a free minimal model for \((A,d_A)\). Obviously, this model depends on the direct factor \(V\) in \(U\). It is easy to see that another choice produces a minimal model \(\varphi_{V'} : (TV',d_{V'}) \rightarrow (TU,D)\) and an isomorphism \(\varphi : (TV,d_V) \rightarrow (TV',d_{V'})\) such that \(\varphi_{V'} \circ \varphi \cong \varphi\). For this reason, one can speak of the minimal model of a space. (See also \([\mathbb{L}3]\)).

6.4 Let \((A,d_A,\mu_A)\) be an augmented shc-algebra and assume that \(H^0(A,d_A) = \mathbb{k}\) and that \(H^1(A,d_A) = 0\). Write \(\Omega (BA \otimes BA) = (TU,d_V)\). We denote by \(\mu_V : (TV,d_V) \rightarrow (TV,d_V)\) the composite

\[\mu_A \circ p_V : (TV,d_V) \rightarrow (TV,d_V)\]

The triple \((TV,d_V,\mu_V)\) is called a shc-minimal model for \((A,d_A,\mu_A)\).

6.5 In the remainder of this section we establish some of the main properties of the shc-minimal model of the shc-algebra \((A,d_A,\mu_A)\). First observe that, by 6.2, 6.3 and 6.4, we have a direct sum decomposition:

\[\hat{V} \cong s^{-1}(H^+BA \otimes \mathbb{k}) \oplus s^{-1}(\mathbb{k} \otimes H^+BA) \oplus s^{-1}(H^+BA \otimes H^+BA) \cong s^{-1}(sV \otimes \mathbb{k}) \oplus s^{-1}(\mathbb{k} \otimes sV) \oplus s^{-1}(sV \otimes sV)\].
Therefore we write: \( \hat{\nabla} := V \oplus W \oplus V \# W \), with \( V := s^{-1}(sV \otimes K) \), \( W := s^{-1}(K \otimes sV) \), \( V \# W := s^{-1}(sV \otimes sW) \) and \( v \# w := s^{-1}(sv \otimes sw) \).

Define \( \hat{\Psi}: \Omega(BA \otimes BA) = (T((s^{-1}B^+A \otimes K) \oplus (K \otimes s^{-1}B^+A)) \oplus s^{-1}(B^+A \otimes B^+A)), D) \rightarrow \Omega BA \otimes \Omega BA \) by: \( \hat{\Psi}(s^{-1}[a_1] \cdots [a_i] \otimes 1) = s^{-1}[a_1] \cdots [a_i] \otimes 1 \), \( \hat{\Psi}(1 \otimes s^{-1}[a_1] \cdots [a_i]) = 1 \otimes s^{-1}[a_1] \cdots [a_i] \), \( \hat{\Psi}(s^{-1}[a_1] \otimes [b_1] \cdots [b_j]) = 0 \). One can check easily that \( \hat{\Psi} \) commutes with the differentials. Moreover, the commutativity of the diagram

\[
\begin{array}{ccc}
\Omega B(A \otimes A) & \xrightarrow{\alpha_A \otimes \alpha_A} & A \otimes A \\
\Omega sh \downarrow & & \downarrow \alpha_A \otimes \alpha_A \\
\Omega(BA \otimes BA) & \xrightarrow{\hat{\Psi}} & \Omega BA \otimes \Omega BA
\end{array}
\]

implies that \( \hat{\Psi} \) is a surjective quasi-isomorphism. Set

\[
\Psi_V = (p_V \otimes p_V) \circ \hat{\Psi} \circ \varphi_{\hat{\nabla}}: (TV, d_{\hat{\nabla}}) \rightarrow (TV, d_V) \otimes (TV, d_V),
\]

hence \( \Psi_V \) is also a surjective quasi-isomorphism and since \( p_V \otimes \varphi_{\hat{\nabla}} = id \), \( \Psi_V \) satisfies:

\[
\Psi_V v = v \otimes 1, \quad \Psi_V w = 1 \otimes w, \quad \Psi_V v \# w = 0, \quad v \in V \text{ and } w \in W.
\]

**6.6 Proposition.** With the previous identification then

1) \( \mu_V \) identifies \( V \) and \( W \) with \( V \),

2) if \( \mu_0: V = V \oplus W \oplus V \# W \rightarrow V \) denotes the linear part of \( \mu_V \), then \( \mu_0(v \# w) = 0 \) if and only if \( sv \ast sw = 0 \) where \( \ast \) is the product in \( H^+BA \simeq sV \),

3) the differential \( d_{\hat{\nabla}} \) is completely determined by the fact that \( \Psi_V : (TV, d_{\hat{\nabla}}) \rightarrow (TV, d_V) \otimes (TV, d_V) \) is a surjective quasi-isomorphism,

4) if either \( d_Vv = 0 \) or \( d^Ww = 0 \), then \( d_V(v \# w) \) is determined by the Hirsch-type formulae described below.

**Proof** Parts 1 and 2 of the proposition are a direct consequence of the definitions. The rest of this section is devoted to the proof of parts 3 and 4.

The degree \(-1\) identification \( V \# W \leftrightarrow s^{-1}(sV \otimes sW) \), \( v \# w \leftrightarrow s^{-1}(sv \otimes sw) \) defines a linear map of degree \(-1\): \( V \otimes W \rightarrow T(V \oplus W \oplus V \# W) \) which we wish to extend to \( T(V) \otimes T(W) \). Set

\[
a) \quad (v_1 \cdots v_k) \# w = \sum_{i=1}^{k} (-1)^{a_i} v_{i-1} \cdots v_1 (v_i \# w) v_{i+1} \cdots v_k \quad v_i \in V, w \in W, a_i = |v_i| + \cdots + |v_{i-1}| + |w||v_{i+1}| + \cdots + |v_k|)
\]

\[
b) \quad v \# (w_1 \cdots w_l) = \sum_{j=1}^{l} (-1)^{b_j} w_{j-1} (v \# w_j) w_{j+1} \cdots w_l \quad v \in V, w_j \in W, b_j = |w_1| + \cdots + |w_{j-1}|
\]

\[
c) \quad (v_1 \cdots v_k) \# w_1 \cdots w_l = \sum_{j=1}^{k} (-1)^{c_j} w_1 \cdots w_{j-1} (v_1 \cdots v_k \# w) w_{j+1} \cdots w_l \quad v \in V, w_j \in W, c_j = (|v_1| + \cdots + |v_{i+1}| + \cdots + |v_k| + 1)(|w_1| + \cdots + |w_{j-1}|)
\]

Formula c) means that we first expand \# with respect to the first argument and then with respect to the second argument. If one reverses this order, i.e., we expand first with respect to the second argument and then with respect to the first argument, one obtains the formula

\[
d) \quad (v_1 \cdots v_k) \# w_1 \cdots w_l := \sum_{j=1}^{k} (-1)^{d_i} v_{i-1} (v_i \# w_1 \cdots w_l) v_{i+1} \cdots v_k \quad v_i \in V, w_j \in W, d_i = (|v_1| + \cdots + |v_i-1| + |v_{i+1}| + \cdots + |v_k|)(|w_1| + \cdots + |w_l|)
\]
Observe that, in general, \((v_1...v_k)^\#w_1...w_l \neq (v_1...v_k)^\#'w_1...w_l\) and that formula a) can be read as \((v_1...v_k)^\#'w\). To avoid confusion we keep the notation used in formula a).

We define a derivation of degree 1, \(D_0 : T(V \oplus W \oplus V^\#W) \to T(V \oplus W \oplus V^\#W)\) by:

\[
D_0v = d_Vv, \quad D_0w = d_Ww, \quad D_0(v^\#w) = v \cdot w - (-1)^{|v||w|}w \cdot v - D_0v^\#w - (-1)^{|v|}v^\#D_0w
\]

where \(v \in V, w \in W\).

6.7 Lemma For \(v \in V, w \in W, a \in TV, b \in TW\),

1. \(D_0(a^\#')w = a \cdot w - (-1)^{|a||w|}w \cdot a - (D_0)a^\#'w - (-1)^{|a|}a^\#D_0w\),
2. \(D_0(v^\#b) = vb - (-1)^{|v||b|}bv - D_0v^\#b - (-1)^{|v|}v^\#D_0b\).

Proof. We proceed by the induction on word-length. First observe that equalities 1 and 2 hold when \(a \in V\) and \(b \in W\). Suppose the first equality true when \(a \in T^pV\) and consider \(a = a_0a_1 \in T^{p+1}V\) with \(a_0 \in T^pV\) and \(a_1 \in V\). Then by the induction hypothesis,

\[
D_0(a^\#w) = (-1)^{|a_0|}D_0a_0(a_1^\#w) + a_0(a_1w - (-1)^{|a_1||w|}wa_1) - a_0(D_0a_1^\#w - (-1)^{|a_1|}a_1^\#D_0w)
\]

so that \(D_0(a^\#w) = aw - (-1)^{|a||w|}wa - D_0a^\#w - (-1)^{|a|}a^\#D_0w\).

The second equality is obtained in the same way.

6.8 End of the proof of proposition 6.5. Recall that \(\Psi_V\) is a surjective quasi-isomorphism. The end of the proof of part 3 follows directly from the uniqueness, up to isomorphism, of the minimal model (6.3). Moreover, we deduce from lemma 6.7 that

\[
D_0^2(v^\#w) = D_0v^\#D_0w - D_0v^\#D_0w,
\]

which is not zero in general. Thus \(d_V\) does not coincide with \(D_0\). Nonetheless we have \(d_V(v^\#w) = D_0(v^\#w)\), if \(v\) or \(w\) is a cocycle in \(V\).

6.9 Proposition. Let \(\mathcal{M}_A := (TV, d_V, \mu_V)\) be a shc-model of the shc-algebra \((A, d_A, \mu_A)\). There exists a product on the Hochschild homology \(HH_*\mathcal{M}_A\) which coincides, up to an isomorphism, with the product defined on \(HH_*A\).

Proof Consider the following diagram

\[
\begin{array}{ccccc}
\mathcal{e}_*A \otimes \mathcal{e}_*A & \xrightarrow{\text{sh}} & \mathcal{e}_*(A \otimes A) & \xleftarrow{\mathcal{e}_*\alpha_\otimes A} & \mathcal{e}_*\Omega B(A \otimes A) & \xrightarrow{\mathcal{e}_*\mu_\otimes A} & \mathcal{e}_*A \\
\mathcal{e}_*TV \otimes \mathcal{e}_*TV & \xrightarrow{\text{sh}} & \mathcal{e}_*(TV \otimes TV) & \xleftarrow{\mathcal{e}_*\psi} & \mathcal{e}_*\hat{T}V & \xrightarrow{\mathcal{e}_*\mu_V} & \mathcal{e}_*TV
\end{array}
\]

where the vertical arrows are respectively from left, \(\mathcal{e}_*(\alpha_A \circ \varphi_V) \otimes \mathcal{e}_*(\alpha_A \circ \varphi_V)\), \(\mathcal{e}_*(\alpha_A \otimes \alpha_A \circ \varphi_V \circ \varphi_V)\), \(\mathcal{e}_*(\varphi_V)\) and \(\mathcal{e}_*(\alpha_A \circ \varphi_V)\). The two left hand squares commute, while the right hand square commutes up to homotopy.
If \( s_\Psi \) denotes a linear section of \( \xi_\Psi \), one defines the product \( \Phi_V \) on \( \xi_*TV \) by
\[
\Phi_V = \xi_*\mu_V \circ s_\Psi \circ sh.
\]

**Part II: From algebra to topology.**

1. Hochschild homology of a space.

1.1 Let \( X \) be a topological space. We denote by \( C^*(X) \) (resp. \( N^*(X) \)) the algebra of un-normalized singular cochains (resp. of normalized singular cochains) on \( X \). Since the inclusion \( N^*(X) \to C^*(X) \) is an algebra map and induces an isomorphism in (co)homology, we define the Hochschild homology of \( X \) as the graded vector space
\[
HH_*N^*(X) \cong HH_*C^*(X).
\]

From [27] 1.2, we know that there exists a commutative natural diagram in \( DM \)
\[
\begin{array}{ccc}
\Omega B(N^*X \otimes N^*X) & \xrightarrow{AW} & \Omega BN^*(X \times X) \\
\alpha_{N^*X \otimes N^*X} & & \alpha_{N^*(X \times X)} \\
N^*X \otimes N^*X & \xrightarrow{AW} & N^*(X \times X)
\end{array}
\]
where \( AW \) denotes the normalized Alexander-Whitney map.

1.2 Proposition. Let \( \Delta \) be the topological diagonal map. If \( X \) is path connected, the natural shc-structural map of \( N^*(X) \) defined by \( \mu_X = \Omega BN^*\Delta \circ AW \) induces a natural graded commutative algebra structure on \( HH_*X \).

1.3 Example. A space \( X \) is called \( \mathbb{K}\text{-shc-formal} \) if \( N^*(X) \) is shc-formal. Spheres and complex projective spaces are shc-formal for any field. If \( X \) is a simply connected space which is \( \mathbb{K}\text{-shc-formal} \), then the multiplicative structure on \( H^*(X;\mathbb{K}) \) is completely determined by the graded algebra \( H^*(X;\mathbb{K}) \). See [21] and examples 4.1, 4.2 below.

2. The acyclic model theorem for cochain functors

The proof of theorem 2 relies heavily on the acyclic model theorem for cochains functors. For the convenience of the reader we recall some definitions here.

2.1 Let us denote the category of non-negatively graded vector spaces by \( DM^* \). Let \( \mathcal{A} \) be a category with models \( \mathcal{M} \). Recall that a functor \( F : \mathcal{A} \to DM^* \)
   a) admits a unit if, for each object \( A \) in \( \mathcal{A} \), there exists a linear map \( \eta_A : \mathbb{K} \to FA \) such that \( d \circ \eta_A = 0 \);
   b) is acyclic on the models if, for any object \( M \) in \( \mathcal{M} \), there exists a linear map \( \epsilon_M : FM \to \mathbb{K} \) such that \( \epsilon_M \circ \eta_M \simeq Id_{\mathbb{K}} \) and \( \eta_M \circ \epsilon_M \simeq Id_{FM} \).
   c) is corepresentable on the models if there exists a natural transformation \( \kappa : \hat{F} \to F \) such that \( \kappa \circ \xi = Id_F \), where : \( \hat{F} \) denotes the contravariant functor
\[
\hat{F} : \mathcal{A} \to DM^*, \quad \hat{F}(A) = \prod_{M \in \mathcal{M}} (F(M) \times \mathcal{A}(M, A)).
\]
and \( \xi \) the natural transformation: \( \xi_A : F(A) \to \hat{F}(A), \ a \mapsto \{F(f)(a)\}_{M \in \mathcal{M}, f \in \mathcal{A}(M, A)}, a \in A \).
For instance, the functor \( X \mapsto C^*X \) is corepresentable on the standard simplexes \( \Delta^n \) in \( \text{Top} \).

### 2.2 Theorem ([1])

Let \( \mathcal{A} \) be a category with models \( \mathcal{M} \) and \( F_1, F_2 : \mathcal{A} \to \text{DM}^* \) two contravariant functors with units. If \( F_1 \) is acyclic and \( F_2 \) is corepresentable on the models, then:

1. there exists a natural transformation \( \tau : F_1 \to F_2 \) which preserves the units,
2. any two such natural transformations are naturally homotopic.

### 3. Proof of theorem 2.

#### 3.1

Consider the simplicial set \( K \) defined as follows: \( K(n) = \mathbb{Z}/(n+1)\mathbb{Z} \), and if \( \overline{k}^n \) denotes an element in \( \mathbb{Z}/n\mathbb{Z} \), the face maps \( d_i : K(n) \to K(n-1) \) with \( 0 \leq i \leq n-1 \) and the degeneracy maps \( s_j : K(n) \to K(n+1) \) with \( 0 \leq i \leq n \) are:

\[
d_i \overline{k}^{n+1} = \begin{cases} 
\overline{k}^n & \text{if } k \leq i \\
\overline{k}^{-1}^n & \text{if } k > i
\end{cases}, \quad
s_j \overline{k}^{n+1} = \begin{cases} 
\overline{k}^{n+2} & \text{if } k \leq i \\
\overline{k}^{-1}^{n+2} & \text{if } k > i.
\end{cases}
\]

and \( d_n \overline{k}^{n+1} = \overline{1}^n \). Consider also the simplicial set \( P \) defined by \( P(n) = \overline{0}^{n+1} \in \mathbb{Z}/(n+1)\mathbb{Z} \) with obvious face and degeneracy maps. If \( \Sigma \) is a simplicial set, we denote as usual by \( |\Sigma| \) its geometric realization, [26]. By [1] (proposition 1.4), \( |K| \) is homeomorphic to the circle \( S^1 \) while \( |P| \) is a point.

#### 3.2

To any topological space \( X \) one can associate the cosimplicial topological spaces \( \underline{X} \), \( \underline{Y} \) and \( \underline{P} \) defined by:

\[
\underline{X}(n) = \text{Map}(K(n), X) = X \times \cdots \times X, \quad \underline{P}(n) = \text{Map}(P(n), X) = X, \quad n \geq 0,
\]

and if \( * \in X \), \( \underline{Y}(n) = \{*\} \times X \times \cdots \times X \), \( n \geq 0 \).

Observe that \( Y(n) = \{f \in \text{Map}(K(n), X)\text{ such that } f(\overline{0}^{n+1}) = *\} \).

The coface and codegeneracy maps for the cosimplicial spaces \( \underline{X} \) and the sub-cosimplicial space \( \underline{Y} \subset \underline{X} \) are:

\[
d_i(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, x_i, x_{i+1}, \ldots, x_n), \quad 0 \leq i \leq n
\]

\[
d_{n+1}(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, x_n, x_0)
\]

\[
s_j(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, x_j, x_{j+1}, \ldots, x_n), \quad 0 \leq j \leq n.
\]

We have a sequence of obvious cosimplicial maps \( \underline{Y} \to \underline{X} \to \underline{P} \).

#### 3.3

Write \( \text{Top} \) (resp. \( \text{Costop} \)) for the category of topological spaces (resp. of cosimplicial topological spaces). There is a covariant functor \( ||\cdot|| : \text{Costop} \to \text{Top} \) called the geometric realization

\[
||Z|| = \text{Costop}(\Delta, Z) \subset \prod_{n \geq 0} \text{Top}(\Delta^n, Z(n)),
\]

where \( ||Z|| \) is equipped with the topology induced by this inclusion. Here \( \Delta \) denotes the cosimplicial space defined by \( \underline{\Delta}(n) = \Delta^n \) with the usual coface and codegeneracy maps \( \delta_i \) and \( \sigma_j \) respectively.
If $\Sigma$ is a simplicial set and $T$ a topological space then $Z = Map(\Sigma, T)$ is a cosimplicial topological space. We recall the following duality result due to Bott and Segal, \cite{Segal} (proposition 5.1).

**3.4 Proposition** There is a homeomorphism:

\[
||T^\Sigma|| = ||Z|| = \text{Costop}(\Delta, Z) \cong \text{Top}(|\Sigma|, T) = T^{||\Sigma||}.
\]

With this notation introduced in 3.2 this yields that

\[
||X|| = \text{Top}(|\Sigma|, X) = X^{S^1} \quad \text{and} \quad ||P|| \cong X
\]

where $X^{S^1}$ is the free loop space of $X$.

**3.5 Proposition** $||Y|| \cong \Omega X$.

**Proof** From 3.4 one has $||X|| \xrightarrow{\Omega} \text{Top}(|K|, X) \xrightarrow{F} ||X||$ where $F$ and $G$ are inverse homeomorphisms. We denote by $*_{0}$ the $0$-simplex of $\Delta_0$ and by $[\overline{0}, *_{0}]$ the base point of $|K| \cong S^1$. If $f \in ||Y|| = \text{Costop}(\Delta, Y)$, then $f = \{f_n\}_{n \geq 0}$ with $f_n : \Delta^n \rightarrow Y(n)$ compatible with the coface and codegeneracy maps and satisfying $f_n(s^n_0 \overline{1}) = *$, $n \geq 0$ and where $s^n_0 = id$. By the definition of $G$, $G(f)([\overline{1}, *_{0}]) = f(x_0)(\overline{1}) = *$. That is, $G(f) \in \text{Top}(|K|, X)$ preserves the base point. Let $f = \{f_n\}_{n \geq 0} \in \text{Top}(|K|, X)$ preserving the base point, that is $f([\overline{1}, *_{0}]) = *$. We have $F(f) = \{g_n\}_{n \geq 0}$, with $g_n : \Delta^n \rightarrow X(n)$ compatible with the coface and codegeneracy maps. By the definition of $F$, $g_n(t)(x) = f_n([t, x])$, $t \in \Delta^n$ and $x \in K(n)$. For any $t \in \Delta^n$, $(t, s^n_0 \overline{1})$ is equivalent to $(\sigma^*_0 t, \overline{1}) = (x_0, \overline{1})$ where the $\sigma_j$ are the codegeneracy maps of $\Delta^n$. Thus $g_n(t, s^n_0 \overline{1}) = F(f)([t, s^n_0 \overline{1}]) = *$ and $F(f) \in \text{Costop}(\Delta, Y)$.

**3.6** The naturality of the above constructions yields the commutative diagram

\[
\begin{array}{ccc}
||Y|| & \rightarrow & ||X|| \\
\downarrow & & \downarrow \\
\Omega X & \rightarrow & X^{S^1} \\
\end{array}
\]

in which the vertical arrows are homeomorphisms. In particular the top line is a Serre fibration.

If $Z$ is any cosimplicial topological space, then $C^*Z$ is a simplicial cochain complex. We define $\text{Tot}C^*Z$ by $(\text{Tot}C^*Z)_n = \bigoplus_{p-q=n} C^qZ(p)$ with differential $Dx = \sum_{i=1}^p (-1)^iC^i(d_i) + (-1)^p \delta x$, where $x \in C^*Z(p)$, the $d_i$ are the coface operators and $\delta$ is the internal differential of $C^*Z(p)$. Observe that $Z \rightarrow \text{Tot}C^*Z$ and $Z \rightarrow C^*||Z||$ are contravariant functors from $\text{Costop}$ to $\text{DA}$.

There exists a natural transformation, \cite{Segal} (corollary 5.3) and \cite{DA} (proof of theorem 4.1 and lemma 6.3):

\[
\psi_Z : \text{Tot}C^*Z \rightarrow C^*||Z||
\]

such that $\psi_Z$ induces an isomorphism in (co)homology when $X$ is simply connected. Since it is obvious that $\psi_Z : C^*X \rightarrow C^*X$ is the identity map, we have the commutative diagram
in DA

\[
\begin{array}{ccc}
C^*X & \xrightarrow{id} & C^*X \\
\downarrow & & \downarrow \\
\text{Tot}C^*X & \xrightarrow{\psi_X} & C^*|X| \\
\downarrow & & \downarrow \\
\text{Tot}C^*Y & \xrightarrow{\psi_Y} & C^*|Y|.
\end{array}
\]

As pointed out by J. Jones in [22] (6. proofs of theorems A and B), the iteration of the Alexander-Whitney natural transformation \(AW : C^*X \otimes C^*X \to C^*(X \times X)\) yields a natural transformation

\[\theta_X : \mathfrak{e},C^*X \to \text{Tot}C^*X\]

inducing an isomorphism in (co)homology.

It is straightforward to check that:
- \(\theta_X\) restricts to the identity on \(C^*X\),
- \(\theta_X\) induces \(\theta_X' : BC^*X \to \text{Tot}C^*X\)
- there is a commutative diagram

\[\begin{array}{ccc}
N^*X & \xrightarrow{i_X} & C^*X \\
\downarrow & & \downarrow \\
\mathfrak{e},N^*X & \xrightarrow{\theta_X \circ i_X} & \text{Tot}C^*X \\
\downarrow & & \downarrow \\
BN^*X & \xrightarrow{\theta_X' \circ i'_X} & \text{Tot}C^*X \\
\end{array}\]

(\(i_X : N^*X \to C^*X\) is the inclusion inducing \(j_X : \mathfrak{e},N^*X \to \mathfrak{e},C^*X\) and \(j'_X : BN^*X \to BC^*X\) respectively. The maps \(i_X\) and \(\Theta_X = \psi_X \circ \theta_X \circ j_X\) induce isomorphisms in (co)homology.

Set \(\Theta_X' = \psi_Y \circ \theta_X' \circ j'_X\), then

\[3.7 \text{ Proposition} \quad \text{The map } \Theta_X' \text{ induces an isomorphism in (co)homology.}\]

\[\text{Proof} \quad \text{The map } \rho : \mathfrak{e},N^*X \to BN^*X \text{ factors through the diagram}\]

\[\begin{array}{ccc}
\mathfrak{e},N^*X & \xrightarrow{\rho} & BN^*X \\
\rho_1 \downarrow & & \downarrow \rho_3 \\
\mathfrak{e},N^*X \otimes_{N^*X} B(N^*X,N^*X) & \xrightarrow{\rho_2} & BN^*X \otimes B(N^*X,N^*X)
\end{array}\]

where \(B(N^*X,N^*X)\) is the bar construction with coefficients, [20], \(\rho_1\) is the inclusion, \(\rho_2\) is the usual isomorphism, \(\rho_3\) is a quasi-isomorphism since \(B(N^*X,N^*X)\) is a semi-free resolution of the field \(\mathbb{k}\) as \(N^*X\)-module, [14] (lemma 4.3). Therefore, \(H^*(\mathfrak{e},N^*X \otimes_{N^*X} B(N^*X,N^*X)) = Tor_{N^*X}(\mathfrak{e},N^*X,\mathbb{k}) \cong H^*(BN^*X)\).

In view of diagram (\(*\)), we also obtain, [12], that

\[H^*(C^*|X||) \simeq Tor_{C^*X}(C^*|X||,\mathbb{k}),\]

and that the linear map \(H^*(\Theta_X') : H^*BN^*X \to H^*(C^*|Y||)\) coincides with \(Tor_{i_X}(\Theta_X; \mathbb{k}) : Tor_{N^*X}(\mathfrak{e},N^*X,\mathbb{k}) \to Tor_{C^*X}(C^*|X||,\mathbb{k})\). The latter is an isomorphism, since \(i_X\) and \(\Theta_X\) are quasi-isomorphisms, [14] (proposition 2.3).
Most of the rest of this section is devoted to the proof of:

3.8 Proposition If $X$ is simply connected, the natural chain equivalence of cochain complexes

$$\Theta_X : \varepsilon_* N^*X \to C^*||X||$$

induces an isomorphism of graded algebras in (co)homology.

We could not prove each assertion completely without going into too much detail, so we only prove the main one i.e. the commutativity of the following diagram

\[
\begin{array}{ccc}
HH_*X \otimes HH_*X & \xrightarrow{H(\Theta_X) \otimes H(\Theta_X)} & HC^*||X|| \otimes HC^*||X|| \\
\Phi_* & \downarrow & \downarrow \\
HH_*X & \xrightarrow{H(\Theta_X)} & HC^*||X||
\end{array}
\]  

where $\Phi_*$ is the product considered in 1.4-(a) and $\cup$ is the usual cup product.

In order to establish the commutativity of the diagram $(\ast)$, consider the natural Alexander-Whitney transformation, [26], Chap. VIII-8.1 and 8.6:

\[
AW : \varepsilon_* (N^*X \otimes N^*X) \to \varepsilon_* N^*X \otimes \varepsilon_* N^*X
\]

which satisfies $AW \circ sh = id$ and $H(sh \circ AW) = id$.

From this fact and the definition of $s_{N^*X \otimes N^*X}$, the commutativity of the above diagram is a consequence of the commutativity up to homotopy of the following diagram:

\[
\begin{array}{ccc}
\varepsilon_* \Omega B(N^*X \otimes N^*X) & \xrightarrow{\varepsilon_* \alpha_{N^*X \otimes N^*X}} & \varepsilon_* (N^*X \otimes N^*X) \\
\downarrow & & \downarrow \\
\varepsilon_* (\alpha_{N^*X} \circ \mu_X) & & C^*||X|| \otimes C^*||X|| \\
\downarrow & & \downarrow \\
\varepsilon_* N^*X & \xrightarrow{\Theta_X} & C^*||X||
\end{array}
\]

To establish the commutativity up to homotopy, of this diagram, we consider the contravariant functors $F_1, F_2 : \text{Top} \to \text{DM}^*$,

$$F_1(X) = \varepsilon_* \Omega B(N^*X \otimes N^*X), \quad F_2(X) = C^*||X||.$$

It results from lemma 3.9 below and from theorem 2.2 that the two natural transformations $\beta_X = \cup \circ (\Theta_X \otimes \Theta_X) \circ AW \circ \varepsilon_* \alpha_{N^*X \otimes N^*X}$ and $\gamma_X = \Theta_X \circ \varepsilon_* (\alpha_{N^*X} \circ \mu_X)$ are chain homotopic and thus the above diagram is commutative up to homotopy.

Indeed, the functor $F_1$ admits a natural unit (i.e. a linear map $\eta_X : \mathbb{K} \to F_1(X)$ such that $d \circ \eta = 0$). The composition of the natural inclusions $N^0||X|| \to C^0||X|| \to C^*||X||$ defines a unit on $F_2$. By a tedious verification, one checks that the natural transformations $\beta$ and $\gamma$ respect these units.

3.9 Lemma. The functor $F_1$ (resp. $F_2$) is acyclic (resp. corepresentable) on a set of models in $\text{Top}$.

Proof. Let $\vee^p Z$ be the wedge of $(p + 1)$ copies of $Z$. Consider the topological spaces:
$O^n = \vee_{p \geq 0}(\vee_{i=0}^{p+1}(\triangle^n \times \triangle^p))$ topologized with the weak topology,

$T^n_X = \{ f = \{ f_p \}_{p \geq 0} | f_p : \triangle^n \times \triangle^p \to X(p) \text{ such that } f_p(\ast_p) = (f_0(\ast_0), \ldots, f_0(\ast_0)) \}$

$\triangle^n$ (resp. $\triangle^n \times \triangle^p$) pointed by $\ast$ (resp. $\ast_p$).

Since $\triangle^n \times \triangle^p$ is contractible for $n, p \geq 0$, so is $O^n$ for $n \geq 0$ and $F_1$ is acyclic on the models $O^n$. To prove the corepresentability of $F_2$ on these models, we establish the corepresentability of $F(X) = C^n|X|$, for each $n \geq 0$. Consider

$$\hat{F}(X) = \prod_{m \geq 0} \prod_{f \in \text{Top}(O^m, X)} C^n||O^m|| \times \{ f \}.$$  

The natural transformation $\xi : F \to \hat{F}$ is defined by:

$$\xi(w) = (C^n|\int w|)_f, \quad w \in C^n|X|, \quad f \in \text{Top}(O^m, X), m \geq 0$$

where $\int$ denotes the cosimplicial map associated to $f$.

Let $\sigma \in C_n||X||$ be a singular simplex. Recall that $||X|| = \text{Costop}(\triangle, X) \subset \prod_{p \geq 0} \text{Top}(\triangle^p, X(p))$.

Observe that there is an obvious bijection

$$\vartheta : \text{Top}(\triangle^n, ||X||) \to \text{Costop}(\triangle^n \times \triangle, X)$$

where $\triangle^n \times \triangle$ is the cosimplicial space defined by $(\triangle^n \times \triangle)(p) = \triangle^n \times \triangle^p$. Moreover $\text{Costop}(\triangle^n \times \triangle, X)$ is a subspace of $T^n_X$. Indeed, if $\tau \in \text{Costop}(\triangle^n \times \triangle, X)$, then $\tau = \{ \tau_p \}_{p \geq 0}$ with each $\tau_p : \triangle^n \times \triangle^p \to X(p)$ compatible with the first coface map $d_0$

For any $n \geq 0$, there is a natural bijection

$$\tau^n : T^n_X \to \text{Top}(O^n, X).$$

defined by $\tau^n(g) = \vee_{p \geq 0}(\vee_{i=0}^{p+1}g_{p,i})$.

Thus $\sigma$ determines a unique element $\tilde{\sigma} = \tau^n \circ \vartheta(\sigma)$ in $\text{Top}(O^n, X)$.

Finally observe that, if $id$ is the identity map of $O^n$, then $(\tau^n)^{-1}(id) \in \text{Costop}(\triangle^n \times \triangle, O^n)$. Set $\iota_n = (\vartheta)^{-1} \circ (\tau^n)^{-1}(id) \in C_n||O^n||$.

Let $(a_{m,f}) \in \hat{F}(X)$ and $\sigma \in C_n||X||$. The formula $(\kappa_X(a_{m,f}))(\sigma) = a_{n,\tilde{\sigma}}(\iota_n)$ defines a natural transformation $\kappa : \hat{F} \to F$ such that $\kappa \circ \xi = id$, as can be seen from the diagram:

$$\begin{array}{ccc}
\text{Top}(O^n, O^n) & \xrightarrow{\tilde{\sigma}} & \text{Top}(O^n, X) \\
\tau^n \uparrow & & \tau^n \uparrow \\
T^n_O & \xrightarrow{\hat{\iota}_n} & T^n_X \\
\uparrow & & \uparrow \\
\text{Costop}(\triangle^n \times \triangle, O^n) & \xrightarrow{\vartheta} & \text{Costop}(\triangle^n \times \triangle, X) \\
\vartheta \uparrow & & \vartheta \uparrow \\
\text{Top}(\triangle^n, ||O^n||) & \xrightarrow{\tilde{C}_n(\vartheta)} & \text{Top}(\triangle^n, ||X||)
\end{array}$$

3.10 End of the proof of theorem 2. It remains to show that $\Theta_X : BN^X \to C^*||Y|| \cong C^*\Omega X$ induces an isomorphism of graded algebras in (co)homology. Since it has already been proved that $\Theta_X$ induces a linear isomorphism in (co)homology (proposition 3.7), we have only to show that this isomorphism is compatible with the products.
This is proved as in lemma 3.7, by setting:
\[ F'_i X = BN^* X \otimes BN^* X, \quad F'_i X = C^* |Y|, \]
\[ \beta'_X = \Theta'_X \circ \nu \quad \text{and} \quad \gamma'_X = \cup \circ \Theta'_X \otimes \Theta'_X \]
where \( \nu \) is the product defined in I-3.3 and \( \cup \) is the cup product in \( C^* |Y| \) and considering the models \( \vee_{p \geq 0} (\Delta^n \times \Delta^p) \) instead of \( O^n \).

4. Examples.

4.1 \( X = S^{2n}, \ n \geq 1 \). For the convenience of the reader, we give some details on the computation of \( HH^*(X; \mathbb{F}_p) \) when \( X = S^{2n}, \ n \geq 1 \) (compare with [11] and [24]). First observe that as graded vector spaces, \( H^* \Omega X \cong H^*_s \Omega X = T(u), \ |u| = 2n-1, \ n \geq 1 \). Thus the minimal model of \( X = S^{2n} \) is of the form \( (TV, d_V) \), as seen in I-§6.1-a, with \( V \cong s^{-1}T^+(u) \).

More precisely, \( V = \oplus_{k \geq 1} v_k \mathbb{F}_p \) with \( v_k \) identified to \( s^{-1}u^k \) and \( dv_k = \sum_{i+j=k} v_i v_j \). The map \( \mu_V : (T(V' \oplus V'' \oplus V''')V'''), (TV, d) \) identifies \( V' \) and \( V'' \) with \( V \) and satisfies \( \mu_V(v'_k \# v''_l) = \binom{k+l}{l} v_{k+l} \). For degree reasons, \( X = S^{2n} \) is \( \mathbb{F}_p \)-shc-formal. On the other hand generators of \( \mathfrak{c}_s H^* X \) are of the form \( a_k = 1 \{x]\ldots[x], \ k \geq 1 \) or \( b_l = x \{x]\ldots[x], \ l \geq 0 \).

when \( H^{2n} X = x \mathbb{F}_p \). The differential is \( db_l = 0, \ da_k = \begin{cases} 0, & \text{if } k \text{ is odd} \\ 2b_{k-1}, & \text{if } k \text{ is even} \end{cases} \).

The product in \( \mathfrak{c}_s H^* X \) is the composite \( \mathfrak{c}_s H^* X \otimes \mathfrak{c}_s H^* X \xrightarrow{\mu} \mathfrak{c}_s(H^* X \otimes H^* X) \xrightarrow{\mathfrak{c}_s m} \mathfrak{c}_s H^* X \) where \( m \) is the product in \( H^* X \).

Thus \( a_k a_l = \binom{k+l}{l} a_{k+l}, \ b_k b_l = 0, \ a_k b_l = \binom{k+l}{l} b_{k+l} = b_l a_k \)

if either \( k \) or \( l \) is even and the other products are trivial. We recover that \( H(X; \mathbb{F}_p) \) is an \( s \)-acyclic \( \mathbb{F}_p \)-algebra.

4.2 \( X = \mathbb{C}P(n) \). The space \( X \) is shc-formal for every \( \mathbb{K} = \mathbb{F}_p, [22] \). The graded algebra \( H^*(X; \mathbb{F}_p) \) admits a free commutative minimal model of the form \( (\Lambda(x, y), d) \) with \( dx = 0, \ dy = x^{n+1}, \ |x| = 2, \ |y| = 2n+1 \). Thus, we deduce from I-§5.3:

\[ H^*(\Omega X; \mathbb{F}_p) \cong \Lambda x \otimes \Gamma y \]
\[ H^*(X; \mathbb{F}_p) \cong \begin{cases} \mathbb{K}[x]/x^{n+1} \otimes \Lambda x \otimes \Gamma y & \text{if } n+1 = 0 \mod p \\ \mathbb{K}[x]/x^{n+1} \otimes \Lambda x \otimes \Gamma y & \text{if } n+1 \neq 0 \mod p \end{cases} \]

Observe in particular that the fibration \( \Omega X \rightarrow X^{S^1} \rightarrow X \) is not T.N.C.Z., if \( n+1 \neq 0 \mod p \). Compare with [21] or [4].

4.3 Finite complexes. D. Anick proved in [3] that, if \( X \) is a \( r \)-connected CW-complex of dimension \( n \), then, for any prime \( p \geq \frac{n}{r}, \) the mod \( p \) Adams-Hilton model is isomorphic, as a differential Hopf algebra, to the universal enveloping algebra of some free differential graded Lie algebra \( L \). Therefore, I-5.2 or 5.3 apply.

4.4 \( X = \Sigma \mathbb{C}P(2) \) and \( \mathbb{K} = \mathbb{F}_2 \). From the Bott-Samelson theorem, [20] (Appendix 2), it is known that \( H_s(\Omega X; \mathbb{F}_2) = T(a_2 b_4) \) with \( \Delta b_4 = b_4 \otimes 1 + a_2 \otimes a_2 + 1 \otimes b_4 \). As graded
vector spaces, \( H^*\Omega X \) is the dual of \( H_*\Omega X \) and the product in \( H^*\Omega X \) is the dual of the coproduct in \( H_*\Omega X \). Thus the element \( a_0 \) of degree 2 in \( H^*\Omega X \) satisfies \( a_0^2 = b_4 \). The minimal model of \( X \) is \((TV, d_V)\) where \( V = s^{-1}T^+(a_2, b_4) \). Set \( x_3 = s^{-1}a_2', \ x_5 = s^{-1}b_4' \).

The map \( \mu_V : (T(V' \oplus V'' \oplus V'\#V''), D) \to (TV, d) \) identifies \( V' \) and \( V'' \) with \( V \). By 2 of proposition I-§6.6, \( \mu_0(x_3'\#x_5'') \neq 0 \). (Notice in passing that the space \( S^3 \vee S^5 \) admits the same minimal model as \( X \) but not the same \( shc \)-structure). We also have \( sh(1[x_3], 1[x_3]) = [x_3 \otimes 1 + 1 \otimes x_3 \otimes 1] \).

If \( s_V \) is any section of \( \mathfrak{c}_*(\Psi) \), a straightforward computation shows that \( s_V(1[x_3 \otimes 1|x_3]) = 1[x_3'\#x_3''] \) and \( s_V(1 \otimes 1|x_3 \otimes x_3 \otimes 1) = 1[x_3'\#x_3''] + 1[x_5', 1[x_3'\#x_3'']] \), so that \( \mathfrak{c}_*\mu_V(\text{sh}(1[x_3], 1[x_3])) = 1[x_5] \).

Thus, the cohomology class \( [1[x_3]]^2 = [1[x_5]] \) is not trivial in \( H^*\left((\Sigma\mathbb{C}P(p))^3; \mathbb{F}_p\right) \). More generally, for any prime \( p \), if \( [1[x_3]] \) is a generator of \( H^2\left((\Sigma\mathbb{C}P(p))^3; \mathbb{F}_p\right) \), then \( [1[x_3]]^p \neq 0 \).

### 4.5 \( X = G_2 \) and \( \mathbb{K} = \mathbb{F}_5 \). From [28] (5.9 and 5.18), we know that if \( X \) is the exceptional Lie group \( G_2 \) then \( H^*[G_2; \mathbb{F}_5] = \Lambda(v_3, v_{11}) \) is the free commutative algebra on two generators of degree 3 and 11 respectively, and the first Steenrod operation satisfies: \( \mathcal{P}_3 v_3 \neq 0 \). The Kudo transgression theorem implies that there exists \( y_2 \in H^*(\Omega G_2; \mathbb{F}_5) \) such that \( y_2^5 \neq 0 \). Since the multiplicative fibration \( \Omega G_2 \to G_2^3 \to G_2 \) admits a section, \( G_2^3 \cong G_2 \times \Omega G_2 \). Thus there is a cohomology class \( \alpha \in H^2(G_2^3; \mathbb{F}_5) \) such that \( \alpha^5 \neq 0 \). We recover this result with our construction.

The minimal model of \( G_2 \) is given by \((TV, d_V)\) where \( V = s^{-1}H^*(\Omega G_2; \mathbb{F}_5) \). More precisely, this minimal model is \((T(x_3, x_5, x_7, x_9, x_{11}, ...), d)\) with \( dx_3 = dy_{11} = 0 \), \( dx_5 = x_3^2 \), \( dx_7 = [x_3, x_3] \), \( dx_9 = [x_3, x_7] + x_5^2 \), and \( dx_{11} = [x_3, x_9] + [x_5, x_7] \).

Consider the map \( \mu_V : (T(V' \oplus V'' \oplus V'\#V''), D) \to (TV, d) \). We have \( \mu_V(x_3'\#x_5'') = 2x_5 \), \( \mu_V(x_3'\#x_5'') = 3x_7 \), \( \mu_V(x_3'\#x_5'') = 4x_9 \), and \( \mu_V(x_3'\#x_5'') = \epsilon y_{11} \) with \( \epsilon \neq 0 \), by 2 of proposition I-§6.6. A section \( s_V \) of \( \mathfrak{c}_*(\Psi) \) can be determined in low degrees and a tedious but straightforward computation shows that the cocycle \( (1[x_3])^5 \) is cohomologous to \( 4e(1[y_{11}]) \) and thus non trivial. By contrast, the cohomology class of \( (1[x_3])^{25} \) is zero as can be shown by a straightforward computation.

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