Clifford Algebras and the Duflo Isomorphism

E. Meinrenken

Abstract

This article summarizes joint work with A. Alekseev (Geneva) on the Duflo isomorphism for quadratic Lie algebras. We describe a certain quantization map for Weil algebras, generalizing both the Duflo map and the quantization map for Clifford algebras. In this context, Duflo's theorem generalizes to a statement in equivariant cohomology.

2000 Mathematics Subject Classification: 17B, 22E60, 15A66, 55N91.

Keywords and Phrases: Clifford algebras, Quadratic Lie algebras, Duflo map, Equivariant cohomology.

1. Introduction

The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$ is the quotient of the tensor algebra $T(\mathfrak{g})$ by the relations, $\xi \xi' - \xi' \xi = [\xi, \xi']_\mathfrak{g}$. The inclusion of the symmetric algebra $S(\mathfrak{g})$ into $T(\mathfrak{g})$ as totally symmetric tensors, followed by the quotient map, gives an isomorphism of $\mathfrak{g}$-modules

$$\text{sym} : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$$

(1.1)

called the symmetrization map. The restriction of sym to $\mathfrak{g}$-invariants is a vector space isomorphism, but not an algebra isomorphism, from invariant polynomials to the center of the enveloping algebra. Let $J \in C^\infty(\mathfrak{g})$ be the function

$$J(\xi) = \det(j(ad_\xi)), \quad j(z) = \frac{\sinh(z/2)}{z/2},$$

and $J^{1/2}$ its square root (defined in a neighborhood of $\xi = 0$). Denote by $\hat{J}^{1/2}$ the infinite order differential operator on $S_{\mathfrak{g}} \subset C^\infty(\mathfrak{g}^*)$, obtained by replacing the...
variable $\xi \in g$ with a directional derivative $\frac{\partial}{\partial \mu}$, where $\mu$ is the dual variable on $g^*$. Duflo’s celebrated theorem says that the composition
\[
\text{sym} \circ \frac{J_1}{2} : Sg \to U(g)
\]
restricts to an algebra isomorphism, $(Sg)^0 \to \text{Cent}(U(g))$. In more geometric language, Duflo’s theorem gives an isomorphism between the algebra of invariant constant coefficient differential operators on $g$ and bi-invariant differential operators on the corresponding Lie group $G$.

The purpose of this note is to give a quick overview of joint work with A. Alekseev [1, 2], in which we obtained a new proof and a generalization of Duflo’s theorem for the special case of a quadratic Lie algebra. That is, we assume that $g$ comes equipped with an invariant, non-degenerate, symmetric bilinear form $B$.

Examples of quadratic Lie algebras include semi-simple Lie algebras, or the semi-direct product $g = s \ltimes s^*$ of a Lie algebra $s$ with its dual. Using $B$ we can define the Clifford algebra $\text{Cl}(g)$. Duflo’s factor $\frac{J_1}{2}(\xi)$ arises as the Berezin integral of $\exp(q(\lambda(\xi)))$, where $q : \wedge(g) \to \text{Cl}(g)$ is the quantization map, and $\lambda : g \to \wedge^2 g$ is the map dual to the Lie bracket.

2. Clifford algebras

Let $V$ be a finite-dimensional real vector space, equipped with a non-degenerate symmetric bilinear form $B$. Fix a basis $e_a \in V$ and let $e^a \in V$ be the dual basis. We denote by $\mathfrak{o}(V) \subset \text{End}(V)$ the space of endomorphisms $A$ of $V$ that are skew-symmetric with respect to $B$. For any $A \in \mathfrak{o}(V)$ we denote its components by $A_{ab} = B(e_a, Ae_b)$. Consider the function $S : \mathfrak{o}(V) \to \wedge(V)$ given by
\[
S(A) = \text{det}^{1/2}(j(A)) \exp_{\wedge(V)}\left(\frac{1}{2}f(A)_{ab}e^a \wedge e^b\right)
\]
(using summation convention), where
\[
f(z) = (\ln j)'(z) = \frac{1}{2}\coth(\frac{z}{2}) - \frac{1}{z}.
\]

In turns out that, despite the singularities of the exponential, $S$ is a global analytic function on all of $\mathfrak{o}(V)$. It has the following nice property. Let $\text{Cl}(V)$ denote the Clifford algebra of $V$, defined as a quotient of the tensor algebra $T(V)$ by the relations $vv' + v'v = B(v, v')$. The inclusion of $\wedge(V)$ into $T(V)$ as totally anti-symmetric tensors, followed by the quotient map to $\text{Cl}(V)$, gives a vector space isomorphism
\[
q : \wedge(V) \to \text{Cl}(V)
\]
known as the quantization map. Then $S(A)$ relates the exponentials of quadratic elements $1/2A_{ab}e^a \wedge e^b$ in the exterior algebra with the exponentials of the corresponding elements $1/2A_{ab}e^ae^b$ in the Clifford algebra:
\[
\exp_{\text{Cl}(V)}(1/2A_{ab}e^a e^b) = q\left(\iota(S(A)) \exp_{\wedge(V)}(1/2A_{ab}e^a \wedge e^b)\right).
\]
Here \( \iota : \wedge(V) \to \text{End}(V) \) is the contraction operator. In fact, one may add linear terms to the exponent: Let \( E \) be some vector space of “parameters”, and \( \phi^a \in E \). Then the following identity holds in the \( \mathbb{Z}_2 \)-graded tensor product \( \text{Cl}(V) \otimes \wedge(E) \):

\[
\exp(1/2A_{ab}e^ae^b + e_a \otimes \phi^a) = q\left(\iota(S(A)) \exp(1/2A_{ab}e^a \wedge e^b + e_a \otimes \phi^a)\right).
\]

### 3. Quadratic Lie algebras

Let us now consider the case \( V = g \) of a quadratic Lie algebra. Invariance of the bilinear form \( B \) means that the the adjoint representation \( \text{ad} : g \to \text{End}(g) \) takes values in \( \mathfrak{o}(g) \), or equivalently that the structure constants \( f_{abc} = B(e_a, [e_b, e_c]) \) are invariant under cyclic permutations of the indices \( a, b, c \). We specialize \( \mathbb{Z}_2 \) to \( A = \text{ad}_\xi \) for \( \xi \in g \), so that \( \lambda^g : g \to \wedge^2 g \),

\[
\lambda^g(\xi) = 1/2(\text{ad}_\xi)_{ab}e^a \wedge e^b
\]

is the map dual to the Lie bracket. Also, take \( E = T^*_\xi g \) and \( \phi^a = -d\xi^a \), where \( \xi^a = B(\xi, e^a) \) are the coordinate functions. Then our formula become the following identity in \( \text{Cl}(g) \otimes \Omega(g) \):

\[
\exp \left( q(\lambda^g) - e_a d\xi^a \right) = q\left(\iota(S^g) \exp(\lambda^g - e_a d\xi^a)\right), \tag{3.1}
\]

where \( S^g = S \circ \text{ad} : g \to \wedge g \). Consider now the following cubic element in the Clifford algebra,

\[
C = \frac{1}{6}f_{abc}e^a e^b e^c \in \text{Cl}(g).
\]

A beautiful observation of Kostant-Sternberg \[8\] says that \( C \) squares to a constant:

\[
C^2 = -\frac{1}{48}f_{abc}f^{abc}.
\]

It follows that the graded commutator \( d^{\text{Cl}g} := [C, \cdot] \) defines a differential on \( \text{Cl}(g) \). This Clifford differential is compatible with the filtration of \( \text{Cl}(g) \), and the induced differential \( d^{S^g} \) on the associated graded algebra \( \text{gr}(\text{Cl}(g)) = \wedge g \) is nothing but the Lie algebra differential. Let \( d^{\text{dR}} \) denote the exterior differential on the deRham complex \( \Omega(g) \).

It is easily verified that \( \lambda^g - e_a d\xi^a \in \wedge g \otimes \Omega(g) \) is closed for the differential \( d^{\wedge g} + d^{\text{dR}} \), while \( q(\lambda^g) - e_a d\xi^a \in \text{Cl}(g) \otimes \Omega(g) \) is closed under \( d^{\text{Cl}(g)} + d^{\text{dR}} \). Together with \( E \), this leads to a number of consistency conditions for the function \( S^g \). One of these conditions gives a solution of the classical dynamical Yang-Baxter equation (CDYBE): Let \( \tau : g \to \mathfrak{o}(g) \) be the meromorphic function \( \tau^g(\xi) = f(\text{ad}_\xi) \) appearing in the exponential factor of \( S^g \). Then

\[
cycl_{abc} \left( \frac{\partial \tau_{ab}}{\partial \xi^c} - \tau_{ak} f^{kl}_{b} \tau_{lc} \right) = -\frac{1}{4} f_{abc} \tag{3.2}
\]

where \( \text{cycl}_{abc} \) denotes the sum over cyclic permutations of \( a, b, c \). This solution of the CDYBE was obtained by Etingof-Varchenko \[5\] and in \[1\] by different methods.
In Etingof-Schiffmann [6], it is shown that $r_{ab}$ is in fact the unique solution of this particular CDYBE, up to gauge transformation. More general CDYBE’s are associated to a pair $\mathfrak{h} \subset \mathfrak{g}$ of Lie algebras, here $\mathfrak{h} = \mathfrak{g}$. The proof sketched above can be modified to produce some of these more general solutions.

4. The non-commutative Weil algebra

Using $B$ to identify the Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$, the Weil algebra of $\mathfrak{g}$ is the $\mathbb{Z}$-graded $\mathfrak{g}$-module given as a tensor product

$$W_\mathfrak{g} = S\mathfrak{g} \otimes \wedge \mathfrak{g},$$

where generators of $S\mathfrak{g}$ are assigned degree 2. Let $L^\mathfrak{g}_\xi$ for $\xi \in \mathfrak{g}$ denote the generators for the $\mathfrak{g}$-action on $W_\mathfrak{g}$, and $\iota^\mathfrak{g}_\xi = 1 \otimes \iota_\xi$ the contraction operators. The Weil differential $d^\mathfrak{g}$ is a derivation of degree 1, uniquely characterized by its properties $d^\mathfrak{g} \circ d^\mathfrak{g} = 0$ and $d^\mathfrak{g} (1 \otimes \xi) = \xi \otimes 1$ for $\xi \in \mathfrak{g}$. The Weil algebra $W_\mathfrak{g}$ with these three types of derivations is an example of a $\mathfrak{g}$-differential algebra: That is, $L^\mathfrak{g}_\xi, \iota^\mathfrak{g}_\xi, d^\mathfrak{g}$ satisfy relations similar to contraction operators, Lie derivatives, and de Rham differential for a manifold with group action.

In [1], we introduced the following non-commutative version of the Weil algebra,

$$W_\mathfrak{g} = U\mathfrak{g} \otimes \text{Cl}(\mathfrak{g}).$$

It carries a $\mathbb{Z}$-filtration, where generators of $U\mathfrak{g}$ are assigned filtration degree 2, with associated graded algebra $\text{gr}(W_\mathfrak{g}) = W_\mathfrak{g}$. Moreover, it carries a $\mathbb{Z}_2$-grading, compatible with the $\mathbb{Z}$-filtration in the sense of [8]. Define contraction operators as $\mathbb{Z}_2$-graded commutators $\iota^W_\xi = [1 \otimes \xi, \cdot]$, let $L^W_\xi$ be the generators for the natural $\mathfrak{g}$-module structure, and set $d^W = [D, \cdot]$ where

$$D = e_a \otimes e^a - 1 \otimes C \in W_\mathfrak{g}$$

is the cubic Dirac operator [7]. Its square

$$D^2 = \frac{1}{2} e_a e^a \otimes 1 - \frac{1}{48} f_{abc} f^{abc}$$

is in the center of $W_\mathfrak{g}$, hence $d^W$ is a differential. As it turns out, $W_\mathfrak{g}$ is again a $\mathfrak{g}$-differential algebra. The derivations $d^W, \iota^W_\xi, L^W_\xi$ respect the $\mathbb{Z}$-filtration, and the induced derivations on the associated graded algebra are just the standard derivations for the Weil algebra $W_\mathfrak{g}$.

The vector space isomorphism $\text{sym} \otimes q : W_\mathfrak{g} \rightarrow W_\mathfrak{g}$ intertwines the contraction operators and Lie derivatives, but not the differentials. There does exist, however, a better “quantization map” $Q : W_\mathfrak{g} \rightarrow W_\mathfrak{g}$ that is also a chain map. Using our function $S^\mathfrak{g} \in C^\infty(\mathfrak{g}) \otimes \wedge \mathfrak{g}$, let $\iota(S^\mathfrak{g})$ denote the operator on $W_\mathfrak{g}$, where the $\wedge \mathfrak{g}$-factor acts by contraction on $\wedge \mathfrak{g}$ and the $C^\infty(\mathfrak{g})$-factor as an infinite order differential operator.
Theorem. \[\text{The quantization map}\]
\[
\mathcal{Q} := (\text{sym} \otimes q) \circ \iota(\widehat{S^g}) : Wg \to Wg
\]

intertwines the contraction operators, Lie derivatives, and differentials on \(Wg\) and on \(Wg\).

The fact that \(\mathcal{Q}\) intertwines the two differentials \(d^W, d^W\) relies on a number of special properties of the function \(S^g\), including the CDYBE.

Put differently, the quantization map \(\mathcal{Q}\) defines a new, graded non-commutative ring structure on the Weil algebra \(Wg\), in such a way that the derivations \(\iota_W^g, L^W, d^W\) are still derivations for the new ring structure, and in fact become inner derivations. Notice that \(\mathcal{Q}\) restricts to the quantization map for Clifford algebras \(q : \wedge g \to \text{Cl}(g)\) on the second factor and to the Duflo map on the first factor, but is not just the product of these two maps.

5. Equivariant cohomology

H. Cartan in \[\text{[3]}\] introduced the Weil algebra \(Wg\) as an algebraic model for the algebra of differential forms on the classifying bundle \(EG\), at least in the case \(G\) compact.

In particular, it can be used to compute the equivariant cohomology \(H_G(M)\) (with real coefficients) for any \(G\)-manifold \(M\). Let \(\iota^R, L^R, d^R\) denote the contraction operators, Lie derivatives, and differential on the de Rham complex \(\Omega(M)\) of differential forms. Let

\[
H_g(M) = H((Wg \otimes \Omega(M))_{\text{basic}}, d^W + d^R)
\]

where \((Wg \otimes \Omega(M))_{\text{basic}}\) is the subspace annihilated by all Lie derivatives \(L^W\) and all contraction operators \(\iota^W\). Cartan’s result says that \(H_g(M) = H_G(M, \mathbb{R})\) provided \(G\) is compact.

More generally, we can define \(H_g(A)\) for any \(g\)-differential algebra \(A\). Let \(H_g(A)\) be defined by replacing \(Wg\) with \(Wg\). The quantization map \(\mathcal{Q} : Wg \to Wg\) induces a map \(\mathcal{Q} : H_g(A) \to H_g(A)\).

Theorem. \[\text{For any } g\text{-differential algebra } A, \text{ the vector space isomorphism } \mathcal{Q} : H_g(A) \to H_g(A) \text{ is in fact an algebra isomorphism.}\]

Our proof is by construction of an explicit chain homotopy between the two maps \(Wg \otimes Wg \to Wg\) given by “quantization followed by multiplication” and “multiplication followed by quantization”, respectively. Taking \(A\) to be the trivial \(g\)-differential algebra (i.e. \(A = \Omega(\text{point})\)), the statement specializes to Duflo’s theorem for quadratic \(g\).

References

[1] A. Alekseev & E. Meinrenken, The non-commutative Weil algebra, Invent. Math. 139 (2000), 135–172.
[2] A. Alekseev & E. Meinrenken, Clifford algebras and the classical dynamical Yang-Baxter equation (in preparation).

[3] H. Cartan, Notions d’algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie., Colloque de topologie (espaces fibrés), Bruxelles, (1950).

[4] M. Duflo, Opérateurs différentiels bi-invariants sur un groupe de Lie, Ann. Sci. École Norm. Sup. 10 (1977), 265–288.

[5] P. Etingof & A. Varchenko, Geometry and classification of solutions of the classical dynamical Yang-Baxter equation, Comm. Math. Phys., 192 (1988), 77-120.

[6] P. Etingof & O. Schiffmann, On the moduli space of classical dynamical $r$-matrices, Math. Res. Lett. 8 (2001), 157–170.

[7] B. Kostant, A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups, Duke Math. J. 100 (1999), no. 3, 447–501.

[8] B. Kostant & S. Sternberg, Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras, Ann. Physics 176 (1987), no. 1, 49–113.