STRONG-COUPLED RELATIVITY WITHOUT RELATIVITY

Edward Anderson

Astronomy Unit, School of Mathematical Sciences, Queen Mary, Mile End Road, London E1 4NS, U.K.

Abstract

GR can be interpreted as a theory of evolving 3-geometries. A recent such formulation, the 3-space approach of Barbour, Foster and Ó Murchadha, also permits the construction of a limited number of other theories of evolving 3-geometries, including conformal gravity and strong gravity. In this paper, we use the 3-space approach to construct a 1-parameter family of theories which generalize strong gravity. The usual strong gravity is the strong-coupled limit of GR, which is appropriate near singularities and is one of very few regimes of GR which is amenable to quantization. Our new strong gravity theories are similar limits of scalar-tensor theories such as Brans–Dicke theory, and are likewise appropriate near singularities. They represent an extension of the regime amenable to quantization, which furthermore spans two qualitatively different types of inner product.

We find that these strong gravity theories permit coupling only to ultralocal matter fields and that they prevent gauge theory. Thus in the classical picture, gauge theory breaks down (rather than undergoing unification) as one approaches the GR initial singularity.

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Electronic address: e.anderson@qmul.ac.uk
1 Introduction

In the Arnowitt–Deser–Misner (ADM) [1] formulation of GR, the action is\(^1\)

\[
S = \int d\lambda \int d^3x (p^{ij} \dot{g}_{ij} - \xi^i \mathcal{H}_i - N \mathcal{H}),
\]

\[
\mathcal{H}_i \equiv -2\nabla_j p^j_i = 0,
\]

\[
\mathcal{H} \equiv G_{ijkl} p^{ij} p^{kl} - \sqrt{g} R = 0,
\]

where \(G_{ijkl} = \frac{1}{\sqrt{g}} \left( g_{ik} g_{jl} - \frac{1}{2} g_{ij} g_{kl} \right) \) is the DeWitt supermetric [2]. The purpose of this formulation is to treat GR as a dynamical system. The shift \(\xi^i\) and lapse \(N\) are merely auxiliary variables, variation with respect to which yields the momentum and Hamiltonian constraints [2] and [3]. GR has 2 true dynamical variables per space point. These are what is left of the 6 degrees of freedom of the 3-metric \(g_{ij}\), once one has taken the 4 constraints into account. Thus a configuration space for GR, that has a fourfold redundancy per space point, is Riem: the space of 3-metrics on a fixed topology (taken here to be compact without boundary). It is relatively straightforward to understand the restriction placed on this by the momentum constraint, which generates the 3-diffeomorphisms: the true dynamical variables are contained within the 3 degrees of freedom of the 3-geometries and not among the 3 coordinates painted onto these 3-geometries. Thus GR may be interpreted as a theory of evolving 3-geometries, or geometrodynamics [3]. The corresponding configuration space is the quotient space

\[
\{\text{Superspace}\} = \frac{\{\text{Riem}\}}{\{3\text{-diffeomorphisms}\}}
\]

which has a single redundancy per space point due to the still-remaining Hamiltonian constraint, \(\mathcal{H}\), which then plays a central role in geometrodynamics.

Possible classical and quantum interpretations of geometrodynamics have been discussed by Wheeler. In particular [3], he asked “If one did not know the Einstein–Hamilton–Jacobi equation, how might one hope to derive it straight off from plausible first principles without ever going through the formulation of the Einstein field equations themselves?” The first stage of answering this question involves finding a derivation for \(\mathcal{H}\), which is the fully classical analogue of this equation. Following Wheeler’s suggestion of presupposing embeddability into a spacetime to answer the question, Hojman, Kuchař and Teitelboim (HKT) [4] obtained the partial answer that the form [3] of \(\mathcal{H}\) (with an additional optional cosmological constant term) is required in order for the constraints to close as the Dirac Algebra, which is the condition to ensure this embeddability. More recently, Barbour, Foster and O’Murchadha (BFÓ)’s distinct 3-space approach [5] includes doing away with the presupposed embeddability by demanding mere closure, that is asking what self-consistent dynamical systems can describe evolving 3-geometries. Their first principles and method are as follows.

BFÓ’s first principles are 1) to consider prospective laws of physics that are based on relative quantities alone, and 2) that there should be no overall notion of time. Principle 1) is to be implemented by working with actions on the appropriate relative configuration space (which is done indirectly, by the method of ‘best matching’ [6] outlined below). In working with configuration spaces, the whole system is represented therein by a single point, and the evolution of the system is the curve traced out by these points. Because the whole system is represented thus, there is nothing external to the system with respect to which the parametrization time-label \(\lambda\) of the curve could be compared. Thus the theory should be invariant under an overall reparametrization of the time-label \(\lambda\). This is what is meant by principle 2) and its implementation by working with reparametrization-invariant actions.

To treat geometrodynamics, the relative configuration space in question is superspace. In this context, best matching is a method to implement 3-diffeomorphism invariance, by correcting the bare velocities of all fields \(\Psi\) present according to \(\dot{\Psi} \rightarrow \dot{\Psi} - \mathcal{L}_\xi \Psi\), where the dot is \(\dot{}\) and \(\mathcal{L}_\xi\) denotes the Lie derivative with respect to the vector field \(\xi^i\). This is an indirect implementation since nothing is done to eliminate any of the \(g_{ij}\) and furthermore \(\xi_i\) has been introduced, so that one’s action is on Riem \(\times \Xi\), where \(\Xi\) is the

\footnote{We use lower-case Latin letters for 3-space indices. Round brackets denote symmetrization of indices, and square brackets denote antisymmetrization. Indices unaffected by the (anti)symmetrization are set between vertical lines. \(g_{ij}\) is the spatial 3-metric with determinant \(g\) and conjugate momentum \(p^{ij}\). \(R\) is the spatial Ricci scalar and spatial covariant derivatives are denoted by a \(\nabla\) or a semicolon.}
space of the $\xi_i$. But variation with respect to $\xi_i$ gives rise to the momentum constraint. If one could solve this for $\xi_i$, then one could pass to superspace.

The form of the GR action that Baierlein, Sharp and Wheeler (BSW) \[7\] derived,

$$ S_{BSW} = \int d\lambda \int d^3x \sqrt{g} \sqrt{RT_g}, $$

$$ T_g = \frac{1}{\sqrt{g}} G^{abcd}(\dot{g}_{ab} - \mathcal{L}_{g_{ab}})(\dot{g}_{cd} - \mathcal{L}_{g_{cd}}) $$

may be taken to explicitly show that GR is indeed a theory of this form.

Now, BFÓ systematically examined such reparametrization-invariant BSW-type actions

$$ S = \int d\lambda \int d^3x \sqrt{g} \mathcal{L}(g_{ij}, \dot{g}_{ij}; \xi_i) = \int d\lambda \int d^3x \sqrt{g} \sqrt{PT_W}, $$

where the kinetic term

$$ T_W = \frac{1}{\sqrt{g}} G_W^{abcd}(\dot{g}_{ab} - \mathcal{L}_{g_{ab}})(\dot{g}_{cd} - \mathcal{L}_{g_{cd}}). $$

is built using best matching and the most general ultralocal\(^2\) inverse supermetric,

$$ G_W^{ijkl} = \sqrt{g}(g^{ik}g^{jl} - W g^{ij}g^{kl}), $$

which is the inverse of the DeWitt supermetric when the free parameter $W$ takes the value 1. BFÓ chose a simple potential term, $P = \Lambda + sR$, for $\Lambda$ constant and without loss of generality $s \in \{1, 0, -1\}$; furthermore they showed that some more complicated potentials failed to be consistent.

BFÓ use Dirac’s generalized Hamiltonian method \[8\] exhaustively \[9\], as explained in Sec 2. In outline, in addition to the $H_{ij}$ constraint (relation between the momenta) obtained by $\xi_i$-variation, a relation $H_W$ between the momenta arises purely from the local square-root form of the Lagrangian in (7). Consistency and nontriviality then require $H_W$ and $H_i$ to be propagated by the equations of motion without the production of more constraints than the theory has remaining degrees of freedom. The remarkable consequences of this exhaustive interpretation include enforcing $H_W = H_{W=1} \equiv H$ in the Lorentzian ($s = 1$) case, as well as overruling the more complicated potentials, and giving further results on ‘adding on’ general bosonic matter, to which we now turn.

The reasons for ‘adding on’ matter in answering Wheeler’s question is that its context has changed, for his original hopes that vacuum geometrodynamics could be a unified theory (by extension of the Rainich–Misner–Wheeler ‘already-unified’ theory of gravity and electromagnetism \[10\] to include all the other fields of nature), have not been realized. Thus already in 1980 Teitelboim \[11\] extended HKT’s answer by ‘adding on’ matter and this is also the way in which BFÓ have treated matter. The BFÓ treatment appears to give some striking derivations of the classical laws of bosonic physics. In particular, rather than being presupposed, both gauge theory and the light-cone structure for bosonic theories are enforced, and share a common origin in the propagation of $H$. Also in BFÓ’s treatment and its extension by the author and Barbour \[9\], masslessness is enforced on fundamental vector fields; Maxwell and Yang–Mills theory are picked out.

The 3-space approach is not just a reformulation of GR. It is also a method by which a limited number of other theories of evolving 3-geometries may be constructed, which are very similar to the ADM formulation of GR. One such example is conformal gravity \[12\] \[13\], which has some promising features as a realistic physical theory.

Another example is the theory called strong gravity since it corresponds to the strong-coupled limit of GR, in which Newton’s gravitational constant $G \rightarrow \infty$, or equivalently $c \rightarrow 0 \[14\]$. This is the opposite of the more common Galilean limit $c \rightarrow \infty$ (see Fig. 1). In place of the Lorentz group and the light-cone structure, one has an ultralocal ‘Carroll group’ \[14\] structure in which each point is entirely isolated from the others. Strictly speaking, we must keep $\Lambda/G$ constant in evaluating this limit from GR. Otherwise the theory is dynamically trivial.

Strong gravity was first considered by Isham \[15\] as a new regime about which one might construct a perturbative theory of quantum gravity, akin to Klauder’s ultralocal field theory \[16\]. In this paper, we consider strong gravity as a dynamically-consistent theory on its own merit. Henneaux \[17\] showed that

\(^2\)By an expression being ultralocal, we mean that it does not contain spatial derivatives.
it has an unusual 4-geometry resulting from the degeneracy of the 4-metric; one consequence of this is
that, unlike in GR, the constraints and evolution equations do not combine to form a 4-tensor equation.
Moreover, conformal gravity again does not appear to admit a clean interpretation as a 4-tensor theory.
So strong gravity may provide some intuition as to what is possible in such theories.

Strong gravity approximates GR near the cosmological singularity, making it a worthwhile regime
to quantize [18]. Strong gravity gives an independent Kasner universe at each spatial point, which is
the conjectured behaviour of the general solution of GR. Belinskii, Khalatnikov and Lifshitz conjectured
mixmaster behaviour (a sequence of Kasner epochs) at each spatial point [19] whilst straightforward Kasner
behaviour at each spatial point can sometimes occur [20]. There is growing numerical evidence for these
behaviours [21]. The notion of strong gravity is related to the two most popular approaches to quantum
gravity as follows. It is analogous to the tensionless string [22], and it admits an Ashtekar variable
formulation [23, 24] (see Sec 2.3).

In Sec 2, we provide a further example: a 1-parameter family of theories of evolving 3-geometries. Their
discovery provides a different answer to Wheeler’s question from the uniqueness of BFÓ, in the case of the
strong-coupled limit $s = 0$: there is a consistent theory not only for the $W = 1$ DeWitt supermetric of the
usual strong gravity, but also for any ultralocal invertible ($W \neq \frac{1}{3}$) supermetric. Whereas in the GR case of the 3-space approach the presupposition or otherwise of best matching does not affect the final theory
that arises by consistency, strong gravity illustrates that this presupposition can alter the final theory. The spirit of the 3-space approach is to treat our strong gravities as consistent theories of evolving 3-geometries
in their own right, and hence akin to theories of gravity. Furthermore we discuss how they can be related
(as limits relevant to the very early universe) to the well-known scalar-tensor theories of gravity (Sec 2.1),
and used as toy models toward the study of conformal gravity (Sec 2.2).

GR offers only two regimes amenable to quantization: minisuperspace and strong gravity. This paper
provides an enlargement of this second arena, for which different ranges of $W$ give rise to considerable
mathematical differences [25]. In particular, for $W < \frac{1}{3}$ one has theories with positive-definite inner
products. The study of these could broaden the understanding of the inner product problem of quantum
gravity [26]. We furthermore discuss the possibility of the very early universe actually having a positive-
definite inner product. We finally show (Sec 2.3) that the Ashtekar variable formulation, of potentially
great use in quantization, is not readily available for $W \neq 1$.

In Secs 3.1 and 3.2, we couple to strong gravity scalar fields and many interacting vector fields respec-
tively. This enables fruitful comparison with matter coupling in the GR case (Secs 3.2 and 3.3), which
leads to better understanding of some of the GR 3-space approach results. First, we find that in BFÓ’s
approach, strong gravity theories impose an ultralocal structure rather than a Lorentz one, and that they
cause the breakdown of gauge theory, which reinforces BFÓ’s notion that the light-cone and gauge theory
have a common origin in the 3-space approach to GR coupled with bosonic fields. Second, we find that
massive vector fields are readily included coupled to strong gravity. This helps clarify the central role of
the differential Gauss laws of electromagnetism and Yang–Mills theory in the masslessness of vector fields
in the 3-space approach. Since the first of these results includes the usual GR strong-coupled limit, it
means that we have shown that gauge theory breaks down as one approaches the initial singularity of GR.

2 Strong Gravity and the 3-Space Approach

We are interested in finding consistent theories of evolving 3-geometries; we use BFÓ’s 3-space approach
to construct them. Consider then the following best-matched, reparametrization-invariant action,

$$S_{BSW} = \int d\lambda \int d^3 x \sqrt{g} \sqrt{sR + \Lambda} \sqrt{T_W}. \tag{9}$$

The supermetric whose inverse $G_{Xabcd}^{W}$ is used in the construction of $T_W$ is

$$G_{Xabcd} = \frac{1}{\sqrt{g}} \left( g_{ac} g_{bd} - \frac{X}{2} g_{ab} g_{cd} \right), \tag{10}$$

for

$$X = \frac{2W}{3W - 1}. \tag{11}$$
so that the DeWitt case $W = 1$ corresponds to $X = 1$, and $W = 0$ corresponds to $X = 0$. $W = \frac{1}{3}$ is excluded from the treatment on grounds of noninvertibility.

The canonical gravitational momenta are

$$p^{ij} \equiv \frac{\partial L}{\partial \dot{g}^{ij}} = G^{ijcd}_{\text{W}} \frac{1}{2N}(\dot{g}_{cd} - 2\nabla_{(c} \xi_{d)}),$$

(12)

where $2N = \sqrt{T_{W}/(sR + \Lambda)}$. We can invert (12) to find an expression for $\dot{g}_{ij}$.

The primary Hamiltonian constraint follows directly from the local square-root form of the Lagrangian. Since this statement is important and recurrent in our work, we provide here its derivation and interpretation:

$$G_{Xijkl}p^{ij}p^{kl} = G_{Xijkl}G^{ijcd}_{\text{W}} \frac{1}{2N}(\dot{g}_{cd} - 2\nabla_{(c} \xi_{d)})G^{klab}_{\text{W}} \frac{1}{2N}(\dot{g}_{ab} - 2\nabla_{(a} \xi_{b)}) = \sqrt{g} \frac{T_{W}}{(2N)^{2}} = \sqrt{g}(sR + \Lambda),$$

(13)

by (12) and (8). Hence the momenta are not independent, but rather there is a relation between them,

$$H \equiv \frac{1}{\sqrt{g}} \left( p^{ij}p_{ij} - \frac{X}{2}p^{2} \right) - \sqrt{g}(sR + \Lambda) = 0,$$

(14)

where $p$ is the trace of $p^{ij}$. Such a relation between the momenta is called a constraint; it is called primary to indicate that variation was not used in its derivation. Constraints arising from variation are called secondary; variation with respect to $\xi_{i}$ yields the secondary momentum constraint

$$H_{i} \equiv -2\nabla_{j}p^{ij} = 0.$$

(15)

If we can solve this as a differential equation for $\xi_{i}$ (which is the thin sandwich conjecture [24]), then one is actually working on superspace.

The Euler–Lagrange equations are

$$\frac{\partial p_{ij}}{\partial x} = \frac{\partial}{\partial g^{ij}} \left( \frac{1}{\sqrt{g}} \left( p^{ij}p_{ij} - \frac{X}{2}p^{2} \right) - \sqrt{g}(sR + \Lambda) \right) + \sqrt{g} sN p^{ij}$$

from which we evaluate $\mathcal{H}$:

$$\frac{\partial}{\partial x} \left[ \frac{1}{\sqrt{g}} \left( p^{ij}p_{ij} - \frac{X}{2}p^{2} \right) - \sqrt{g}(sR + \Lambda) \right] = \frac{Np(3X - 2)}{2\sqrt{g}} \left[ \frac{1}{\sqrt{g}} \left( p^{ij}p_{ij} - \frac{X}{2}p^{2} \right) - \sqrt{g}(sR + \Lambda) \right]$$

(16)

$$+ \frac{1}{2N}(N^{2}p_{ab},_{a} - 2s(N^{2}p_{ab})_{a} + 2s(1 - X)(N^{2}p_{ab})_{a}^{2}.$$

We demand that this vanishes weakly in the sense of Dirac [8], i.e. that it is zero modulo the constraints hitherto found, which we denote by the symbol $\approx$: $\mathcal{H} \approx 0$. Notice that expression (17) has been grouped so that the first two terms vanish by virtue of the Hamiltonian constraint and the third vanishes by virtue of the momentum constraint, which is thus an integrability condition for the Hamiltonian constraint in GR. Thus we are left with just the last term.

Supposing that this does not automatically vanish, then we would require new constraints. But we can apply exhaustive arguments as spelled out in general in [13], which rather cut down on how many further constraints can arise, since these quickly use up the theory’s degrees of freedom.

There are three possibilities for (17) to be consistent. First, $p_{a} = 0$, which gives a further constraint $p/\sqrt{g} = \text{const}$. Then we require furthermore that the new constraint propagates. This gives additionally the constant mean curvature (CMC) slicing equation

$$\frac{\partial}{\partial x} \left( \frac{p}{\sqrt{g}} \right) = C = 3\Lambda N + 2s(NR - \nabla^{2}N) + \frac{(3X - 2)Np^{2}}{2g},$$

(18)

(for $C$ a spatial constant) which nontrivially fixes the lapse $N$ provided that $s \neq 0$. This demonstrates that arbitrary geometries cannot be connected, which for $s \neq 0$ forces us to take the second possibility: that $X$ takes its DeWitt value $X = 1$. This is the main result, ‘Relativity Without Relativity’, of BFÒ’s paper, whereby GR is recovered. Giulini had also noted that $X = 1$ is mathematically special [25].

But clearly $s = 0$ gives a third possibility, for which any ultralocal supermetric is allowed. The theory traditionally called strong gravity has $W = 1$ because it is obtained as a truncation of GR. But our first result is that as far as dynamical consistency is concerned, we have now shown that there exists a family
of such theories parametrized by $W$. Because $W = 1/3$ is badly behaved, the family of dynamically-consistent theories naturally splits into $W > 1/3$ and $W < 1/3$ subfamilies. The $W < 1/3$ family should be qualitatively distinct from a quantum-mechanical perspective, as outlined at the end of the next subsection. A particularly simple example of such a dynamically-consistent theory is the $W = 0$ theory, for which the constraints are

$$p^{ab}p_{ab} = g\Lambda, \quad p^i_{\ j} = 0.$$  \hspace{1cm} (19)

$W = 0$ may be of particular relevance, in part because conformal gravity has been formulated i.e.o the $W > 1/3$ supermetric, and in part from string-theoretic considerations (see next subsection). Conformal gravity arises because, in a conformal generalization of the above working, the equivalent of the slicing equation (18) is independently guaranteed to hold. There is also a separate strong conformal theory [13].

Since for $s = 0$ (17) reduces to

$$\dot{\mathcal{H}} = \frac{(3X - 2)Np}{2\sqrt{g}} + \mathcal{L}_\xi \mathcal{H},$$  \hspace{1cm} (20)

the momentum constraint may no longer be seen to arise as an integrability condition. This fact was already noted by Henneaux [17]. Strong gravity thus provides a counterexample to the suggestion that all additional constraints need arise from the propagation of $\mathcal{H}$. However, all the other constraints can be interpreted as arising in this way in the standard approach to GR [28]: $\mathcal{H}_i$, the electromagnetism Gauss constraint $G$, the Yang–Mills Gauss constraint $G_J$ and the ‘locally Lorentz’ constraint $J_{AB}$ from working in some first order formalism.

It is also dynamically consistent [29] to have strong gravity without a momentum constraint $\mathcal{H}_i$ to start off with; the new $W \neq 1$ theories may be treated in this way too. In fact, it is this treatment that corresponds to strictly taking the $G \rightarrow \infty$ limit of GR (as opposed to Pilati’s approach [30] in which the momentum constraint is kept). This is because the GR momenta are proportional to $G^{-1}$ [23]. So there are distinct strong gravity theories with 5 and 2 degrees of freedom per space point respectively. Also, if one uses Ashtekar variables in place of the traditional ones, the analogues of $\mathcal{H}_i$ and $J_{AB}$ cease to be independent, so one is forced to have the theory with 5 degrees of freedom. [23].

Starting off without a momentum constraint ammounts to starting off with ‘bare velocities’ rather than best-matched ones, which corresponds to another 3-space scheme suggested in the GR case by Ó Murchadha [31]. Our generalization to arbitrary $s$ of this leads to

$$\dot{\mathcal{H}} = -\frac{2s}{N}(N^2[p^{ab}_i + (X - 1)p^a_i])_{,a} + \frac{(3X - 2)Np}{2\sqrt{g}} \mathcal{H}.$$  \hspace{1cm} (21)

For $s \neq 0$, $p^{ab}_i + (X - 1)p^a_i = 0$. But propagating this gives

$$\frac{\partial}{\partial x}[p^{ab}_i + (X - 1)p^a_i] = (X - 1) \left( \left[ 2s(NR - \nabla^2 N) + 3N\Lambda + \frac{N(3X - 2)p^2}{2g} \right]_{,a} - \frac{(3X - 2)Np}{2\sqrt{g}} \left( \eta^{a}_{\ b} \right)^{,a} \right)$$  \hspace{1cm} (22)

so if constraints alone arise (rather than conditions on $N$), we require $X = 1$ and we recover relativity: the constraint $\mathcal{H}_a \equiv -2p^{ab}_i = 0$ may be encoded into the bare action by the introduction of an auxiliary variable $\xi^a$. This encoding may be thought of as the content of best matching. Clearly, the $s = 0$ example is of value because it illustrates that it is possible to ‘miss out’ constraints if we interpret these as integrability conditions for $\mathcal{H}$. Thus although the ‘bare’ and ‘best-matched’ schemes are equivalent for pure GR, they are not in general equivalent.

### 2.1 Application to Scalar-Tensor Theories

The $X \neq 1$ departure from the DeWitt supermetric does not appear to affect Henneaux’s study of the geometry. Whereas these theories are no longer interpretable as truncations of GR, they do correspond to truncations of scalar-tensor theories (such as Brans–Dicke theory), in a region where the scalar field is a large constant. The relations between the Brans-Dicke parameter $\omega$ and our coefficients $W$ and $X$ are shown in Fig. 2. We now discuss the possibility that a positive-definite ($W < 1/3$ i.e. $\omega < 0$) inner product can occur in our universe. There is no point in considering Brans–Dicke theory since this has $\omega$ constant in space and time and we know from solar system tests that today $\omega \geq 3500$ [32], corresponding to $W$...
being very slightly larger than the GR value 1. However, general scalar-tensor theory permits \( \omega \) to vary with space and in particular time, so it could be that the very early universe had a very different value of \( \omega \) from that around us today, since the bounds on \( \omega \) from nucleosynthesis [35] permit \( \frac{\delta_{\text{nucleosynthesis}}}{\omega_{\text{today}}} \approx \frac{1}{25} \).

The bounds from [35] are less strict but presumably applicable to a wider range of theories since the origin of the departure from \( W \neq 1 \) (i.e. \( \omega \neq \infty \)) is there unspecified. Furthermore, \( \omega \) is attracted to the GR value at late times in scalar-tensor theories [30,37], so it need not have started off large. One would expect \( \omega \) of order unity in any fundamental scalar-tensor theory [36]. For example \( \omega = -1 \) arises in low-energy string theory [38].

It is thus an open question whether \( \omega \) at early times could have passed through the value 0. This question is interesting for the quantum-mechanical reasons given in the next paragraph. We first wish to clarify the role of our strong gravities in such a study. Our strong gravity theories do not permit \( W \) (and hence \( \omega \)) to change with time, so we are not suggesting to use these to investigate whether such a transition through \( \omega = 0 \) is possible. But if such transitions are found to be possible, the very early universe could then be described by scalar-tensor theories which have \( \omega < 0 \). Then one of our strong gravity theories which behaves qualitatively differently from the usual \( W = 1 \) strong gravity corresponding to GR would be relevant as the approximation near the initial singularity. We propose to study the possibility of having such a transition using the full scalar-tensor theories. Unlike their strong gravity limits discussed in this paper, for which this transition involves passing through a noninvertible supermetric,\(^3\) the full scalar-tensor theories are not badly-behaved as \( \omega \to 0 \). This is because despite the degeneracy of the tensor (‘gravity’) supermetric for \( \omega = 1 \), what counts for the full theory is the larger\(^4\) scalar-tensor supermetric, which is usually well-behaved at \( \omega = 0 \) because of the presence of scalar-tensor cross-terms in the scalar-tensor supermetric, due to which the degeneracy of the \( 6 \times 6 \) block is not sufficient to cause the whole \( 7 \times 7 \) scalar-tensor supermetric to be degenerate. But in the approximation by which the theories in this paper arise from scalar-tensor theories the scalar momentum is negligible, so one is then left with only the ‘gravity’ supermetric.

Thus in principle there could be different possible early universe behaviours which admit 3 different sorts of strong gravity limits: \( \omega > 0 \) (indefinite), \( \omega = 0 \) (degenerate) and \( \omega < 0 \) (positive-definite). In each of these cases the natural inner product corresponding to the early-universe Wheeler–DeWitt equation (quantum Hamiltonian constraint) will be very different. This will cause the early universe’s quantum theory of gravity in the \( \omega < 0 \) (and \( \omega = 0 \)) cases to be substantially different from quantum general relativity, which belongs to the \( \omega > 0 \) case, by the following argument. Finding an inner product is crucial for the physical interpretation of a quantum system. The natural inner product potentially provides an easy solution to the problem of finding an inner product. But the natural inner product is only an acceptable solution to this problem if it admits a probabilistic interpretation [20].\(^5\) Now, the arguments for whether the natural inner product admits a probabilistic interpretation depend on the type of the natural inner product: given a positive-definite inner product, one must check whether it admits a Schrödinger-type probabilistic interpretation, which is a considerably different procedure from checking whether a given indefinite inner product admits a Klein–Gordon-type probabilistic interpretation.

The extension to \( W \neq 1 \) of Isham and Pilati’s perturbative quantization idea of expanding about strong gravity ought to be both tractable and of relevance to such a quantum study of scalar-tensor theories. The idea involves expanding about the strong gravity theory by introducing a comparatively small \( R \) term. In the GR case which alone has been investigated [38], one thus recovers a GR regime away from the singularity.

### 2.2 Application to Conformal Gravity

Another application of our \( W \leq \frac{1}{3} \) theories follows from conformal gravity being \( W \)-insensitive and thus expressible i.t.o a \( W = 0 < \frac{1}{3} \) supermetric. In fact, we can form a sequence of theories: \( W = 0 \) strong gravity, conformal strong gravity, conformal gravity. Thus we can isolate the study of some of the novel

\(^3\)In fact, this \( W = \frac{1}{3} \) supermetric is of the same form as the degenerate strong gravity 4-metric, so the pointwise geometry of superspace for \( W = \frac{1}{3} \) should be taken to be akin to Henneaux’s geometry of strong gravity spacetimes.

\(^4\)Just like the GR supermetric may be represented by a \( 6 \times 6 \) matrix [21], the scalar-tensor supermetric may be represented by a \( 7 \times 7 \) matrix where the new seventh index is due to the scalar field. For an account of this in Brans–Dicke theory, which also explains how other-signature supermetrics can occur elsewhere in full Brans–Dicke theory, see [34].

\(^5\)In particular, the indefinite natural inner product of GR does not admit a Klein–Gordon-type inner product. This is the inner product problem of quantum GR.
features of classical and quantum conformal gravity: \( W = 0 \) strong gravity has a positive-definite inner product, and in strong conformal gravity additionally \( \mathcal{H} \) begins to play a new role and there is a preferred slicing. Conformal gravity has in addition a nontrivial integro-differential lapse-fixing equation to solve (Whereas strong conformal gravity’s lapse-fixing equation merely integrates to give \( N = \) spatial constant), which represents an additional computational step before one can attempt to solve the evolution equations.

In answer to whether arguments from the Sec 2.1 are applicable to conformal gravity, we begin by noting that there is no ‘expansion term’ \( p \) in conformal gravity. Because this absence is due to \( p = 0 \) being seperately variationally imposed (rather than due to \( W = 0 \) occurring for the vacuum theory), the presence of non-minimally-coupled-scalars or the related use of conformal transformations are unable to reintroduce a \( p \) into conformal gravity. A consequence of this absence is that the usual notion of cosmology is not applicable to conformal gravity. It is not currently known whether any conventional or non-conventional cosmology can be recovered from conformal gravity by other means. Also, because \( p = 0 \) is seperately imposed, conformal gravity cannot be included among the ‘wider range of theories’ for which the less stringent bounds on \( W \) mentioned in 2.1 are applicable. Conformal gravity is a theory in which \( W \) plays no role at all. Presumably the classical and quantum study of conformal gravity on superspace with \( W < \frac{1}{3} \) and \( W < \frac{1}{4} \) are equivalent once projected down to conformal superspace. Working out how this happens may be interesting and instructive, at least from a theoretical point of view.

2.3 Difficulty with Implementation of Ashtekar Variables

This paper proposes theories for which the inner product problem of quantum GR is altered (if not ameliorated). In the case of conformal gravity, the presence of a preferred foliation additionally represents an attempt to circumvent the problem of time of quantum GR. One must however recall that these problems of quantum gravity are always intertwined with other formidable problems, which include operator ordering and regularization [26]. At least in GR, Ashtekar variables [39] have nice properties as regards these last two problems: the constraints become polynomial functions (cutting down on the ordering ambiguities) and a natural regularization is provided. It therefore becomes of interest whether scalar-tensor theories or conformal gravity admit an analogue of Ashtekar variables. Indeed, how special is GR in admitting Ashtekar variables with their nice properties? The Ashtekar variables for GR [39] are an \( SU(2) \) connection \( A^A_B \) and its conjugate soldering form \( \tilde{\sigma}^a_{AB} \) (which is related to the 3-metric by \( g_{ab} = -tr(\tilde{\sigma}_a \tilde{\sigma}_b) \)).\(^6\) One then has the constraints

\[
D_a \tilde{\sigma}^a_{AB} \equiv \partial_a \tilde{\sigma}^a_{AB} + [A_a, \tilde{\sigma}^a_{AB}] = 0, \quad (23)
\]

\[
tr(\tilde{\sigma}^a [A_a, \tilde{\sigma}^a]) = 0, \quad (24)
\]

\[
tr(\tilde{\sigma}^a [A_a, \tilde{\sigma}^a]) = 0. \quad (25)
\]

\(^6\)The overline denotes that the soldering form is a densitized object: i.e it contains a factor of \( \sqrt{g} \). The capital indices in this section are spinorial internal \( SU(2) \) Yang–Mills indices. \( tr \) denotes the trace over these. \( D_a \) is the \( SU(2) \) covariant derivative as defined in the first equality of (23), and \( [\ , \ ] \) is the \( SU(2) \) commutator.

\(^7\)In Ashtekar variables, \( g \) is proportional to \( \epsilon_{a b c} \tilde{\sigma}^a \tilde{\sigma}^b \tilde{\sigma}^c \), so the cosmological constant term itself is also polynomial.
We can easily see, by the cyclic property of the trace and use of [26], that [27] is redundant as claimed on page 5. Furthermore, there is an equivalent form for the remaining constraints [26] and [28] [23], which we express as

\[ A_{[ab]} = 0, \quad A_{ab}A_{cd}G^{abcd} - \Lambda = 0 \]  

(for \( A_{ab} \equiv \text{tr}(A_\sigma \sigma_b) \)), to manifestly display the dependence on the (now overall undensitized inverse) DeWitt supermetric \( G^{abcd} = \frac{1}{\sqrt{g}} G^{abcd} \). We then investigate what happens when \( G^{abcd} \) is replaced by \( C^{abcd} \). Notice how then the Hamiltonian constraint no longer contains a truncation of the natural object \( F_{ab}^{AB} = 2\partial_\lambda A_{a\lambda} A_B^{\lambda} + [A_a, A_b]^{AB} \) of SU(2) Yang–Mills theory, in correspondence with \( W \neq 1 \) strong gravity not being a natural truncation of the GR Hamiltonian constraint. We find that whereas (29) and (30) still close for arbitrary \( W \), the closure of the original [26] and [28] requires the bracket in [28] to be antisymmetric, that is \( W = 1 \).

For full scalar-tensor theory, we do not think Ashtekar variable with \( W \neq 1 \) will work. One has there the option of making conformal transformations to put scalar-tensor theory into a \( W = 1 \) form, but the conformal factor required then causes the constraints to be non-polynomial [40]. As for conformal gravity, we could as well write the theory with conformal factor required then causes the constraints to be non-polynomial [40]. As for conformal gravity, we could as well write the theory with \( W = 1 \), but we see no way that conformal gravity’s analogue of (13) can be expressed polynomially. So for all these theories, we cannot so easily imitate the Ashtekar formulation means by which GR can be made to elude the operator ordering problem.

### 3 Coupling Matter to Strong Gravity

We now attempt to couple matter to this theory, following the procedure of BFÓ. This enables comparison with the GR case, and leads to a better understanding of how the 3-space approach works.

#### 3.1 Inclusion of Scalar Field

We include first a single scalar field by considering the action

\[ S_{\text{BSW}, \phi}^{(\text{strong})} = \int d\lambda \int d^3x \sqrt{g} \sqrt{\Lambda + U_\phi} \sqrt{T_W + T_\phi}, \]  

with the gravitationally best matched scalar kinetic term \( T_\phi = (\dot{\phi} - \mathcal{L}_\xi \phi)^2 \) and a scalar potential ansatz \( U_\phi = -(C/4) g^{ab} \phi_a \phi_b + V(\phi) \).

The conjugate momenta are given by the usual expression [12] and by

\[ \pi \equiv \frac{\partial L}{\partial \dot{\phi}} = \frac{\sqrt{g}}{2N}(\dot{\phi} - \mathcal{L}_\xi \phi), \]  

where now \( 2N = \sqrt{(T_W + T_\phi)/(\Lambda + U_\phi)} \). These can be inverted to obtain expressions for \( \dot{g}_{ij} \) and \( \dot{\phi} \). The local square root gives the primary Hamiltonian constraint

\[ \phi \mathcal{H} = \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{X}{2} p^2 + \pi^2 \right) - \sqrt{g}(\Lambda + U_\phi) = 0. \]  

Variation with respect to \( \xi_i \) gives the secondary momentum constraint

\[ \phi \mathcal{H}_i = -2p_{ij,j} + \pi \phi_i = 0. \]

The constraint \( \phi \mathcal{H} \) contains the canonical propagation speed \( \sqrt{C} \) of the scalar field. A priori, this is unrestricted. However, imposing \( \phi \mathcal{H} \approx 0 \) gives

\[ \frac{\partial}{\partial X} \left[ \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{X}{2} p^2 + \pi^2 \right) - \sqrt{g}(\Lambda + U_\phi) \right] = \frac{Np(3X-2)}{4g} \left[ \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{X}{2} p^2 + \pi^2 \right) - \sqrt{g}(\Lambda + U_\phi) \right] + \mathcal{L}_\xi \left[ \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{X}{2} p^2 + \pi^2 \right) - \sqrt{g}(\Lambda + U_\phi) \right] + \mathcal{L}_\xi \left[ \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{X}{2} p^2 + \pi^2 \right) - \sqrt{g}(\Lambda + U_\phi) \right] + \mathcal{L}_\xi \left[ \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{X}{2} p^2 + \pi^2 \right) - \sqrt{g}(\Lambda + U_\phi) \right] + \mathcal{L}_\xi \left[ \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{X}{2} p^2 + \pi^2 \right) - \sqrt{g}(\Lambda + U_\phi) \right]. \]  

\(^8\)This has been obtained using a new method that uses the Euler–Lagrange equations implicitly, so we feel no need to include these cumbersome expressions in this or the next section. This method (similar to that used in [13]) will be in the author’s thesis.
The theory has just one scalar degree of freedom, so if the cofactor of $C$ in the last term were zero, the scalar dynamics would be trivial. Thus one has derived that $C = 0$: the scalar field theory cannot have any spatial derivatives. So, strong gravity necessarily induces the Carroll group structure on scalar fields present, thereby forcing these to obey Klauder’s ultralocal field theory [16]. This is analogous to how relativity imposes the light-cone structure on scalar fields present in [5].

We finally note that these results are unaffected by whether one chooses to use the ‘bare’ instead of the ‘best-matched’ formulation.

3.2 Inclusion of $K$ interacting vector fields

We consider a BSW-type action containing the a priori unrestricted vector fields $A^I_a$, $I = 1$ to $K$;\footnote{Capital Latin letters are used in this section for tensorial Yang–Mills-type internal indices; we attach no importance to whether these internal indices are raised or lowered, but their order is important in the GR case.}

$$S_{BSW_A}^{(strong)} = \int d\lambda \int d^3x \sqrt{g} (g_{ij}, \dot{g}^{ij}, A^I, \dot{A}_I, N, N^i) = \int d\lambda \int d^3x \sqrt{g} \sqrt{\Lambda + U_A} \sqrt{T_W + T_{A_I}}. \quad (36)$$

We use the most general homogeneous quadratic best matched kinetic term $T_{A_I}$, and a general ansatz for the potential term $U_A$. $T_{A_I}$ is unambiguously

$$T_{A_I} = P_{I,IJ} g^{ad} (A^I_a - \mathcal{L}_\xi A^i_a) (\dot{A}_d^I - \mathcal{L}_\xi A_d^i) \quad (37)$$

for $P_{I,J}$ without loss of generality a symmetric constant matrix. We further assume that $P_{I,J}$ is positive-definite so that the quantum theory of $A^I_a$ has a well-behaved inner product. In this case, we can take $P_{I,J} = \delta_{I,J}$ by rescaling the vector fields.

The $U_A$ considered here is the most general up to first derivatives of $A^I_a$, and up to four spatial index contractions. In the GR case, the latter was a good assumption because it is equivalent to the necessary naive power-counting requirement for the renormalizability of any emergent four-dimensional quantum field theory for $A_{1a}$ [9].\footnote{We shall however see that for strong gravity these ‘quantum’ conditions may be dropped.} Then $U_A$ has the form

$$U_A = O_{IK} \tilde{C}^{abcd} A^I_a A^K_c + B^I_{JK} C^{abcd} A_{ab} A^I_c A^K_d + I_{JKLM} \tilde{C}^{abcd} A^I_a A^K_b A^L_c A^M_d + \frac{1}{\sqrt{g}} e^{abc} (D_{IK} A^I_{ab} A^K_c + E_{JK} A^I_c A^K_d) + F_{I,J} g^{ab} A^I_a A^K_d + M_{JK} g^{ab} A^I_a A^K_d,$$  

$$\quad (38)$$

where $C^{abcd} = C_1 g^{ac} g^{bd} + C_2 g^{ad} g^{bc} + C_3 g^{ab} g^{cd}$ is a generalized supermetric, and similarly for $\tilde{C}$ and $\tilde{C}$ with distinct coefficients. $O_{I,J}$, $B_{IJK}$, $I_{IJKL}$, $D_{IJK}$, $E_{IJK}$, $F_{I,J}$, $M_{IJ}$ are constant arbitrary arrays. Without loss of generality $O_{I,J}$, $M_{IJ}$ are symmetric and $E_{IJK}$ is totally antisymmetric.

The conjugate momenta are given by [12] and

$$\pi^i_I \equiv \frac{\partial L}{\partial \dot{A}_I^i} = \frac{\sqrt{g}}{2N} (\dot{A}_I^i - \mathcal{L}_\xi A^i_a), \quad (39)$$

where now $2N = \sqrt{(T_W + T_{A_I})/(\Lambda + U_{A_I})}$. These can be inverted to give expressions for $\dot{g}_{ij}$ and $\dot{A}_I^i$. The local square root gives the primary Hamiltonian constraint,

$$A^I_i \mathcal{H} = \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{X}{2} p^2 + \pi^i_I \pi^i_I \right) - \sqrt{g} (\Lambda + U_A) = 0. \quad (40)$$

We get the second momentum constraint by varying with respect to $\xi_i$:

$$A^I_i \mathcal{H}_i = -2p_{ij} \pi^{ij} + \pi^c_I (A_{Ic;i} - A_{1i;c}) - \pi^c_{I,i} A^I_a = 0. \quad (41)$$
The evolution of the Hamiltonian constraint is then

\[
\begin{align*}
\frac{(3X-2)^N}{2\sqrt{g}} & \frac{\partial}{\partial x} \left[ \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{X}{2} p^2 + \pi^I \pi^I_1 \right) - \sqrt{g}(\Lambda + U_N) \right] = \\
& = \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{X}{2} p^2 + \pi^I \pi^I_1 \right) - \sqrt{g}(\Lambda + U_N) + \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{X}{2} p^2 + \pi^I \pi^I_1 \right) - \sqrt{g}(\Lambda + U_N) \\
& \quad - \frac{4}{N} O^{IK} \left( C_1(N^2 \pi^a_A A_{Ka}^b)_b + C_2(N^2 \pi^a_A A^b_{Ka,a})_b + C_3(N^2 \pi^a_A A^b_{K,a:b})_a \right) \\
& \quad - \frac{4}{N} C^{abcd} B^{J}_K (N^2 \pi^a_A A^a_A^b)_b + \frac{2}{N} C^{abc} D^{JK} (N^2 \pi^a_A A^b_{K,a})_b + \frac{2}{N} F^{I} (N^2 \pi^a_A)_i^j. \\
& \quad + \frac{1}{N} B^{I}_K \left( N^2 (p_{ij} - \frac{X}{2} g_{ij}) A^a_A^b A^b_{a} (2 A^a_i C^{aib} - A^a_i C^{aib}) \right)^a_i^a + \frac{1}{N} F^{I} \left( N^2 (p_{ij} - \frac{X}{2} g_{ij}) (2 A^a_i C^{aib} - A^a_i C^{aib}) \right)^a_i^a .
\end{align*}
\]

(42)

We demand that \( \mathcal{H} \) vanishes weakly. The first two terms vanish weakly by the Hamiltonian constraint, leaving us with nine extra terms. Because we have less than 3K vector degrees of freedom to use up, nontriviality dictates that most of these extra terms can only vanish strongly, that is by fixing coefficients in the potential ansatz. Furthermore, we notice that all contributions to (42) are terms in \( N^a \) or are partnered by such terms. Since further constraints are independent of \( N \), these terms in \( N^a \) are of the form \( (N^a V_{Ja}) S^I \), and nontriviality dictates that it must be the scalar factors \( S \) that vanish. We proceed in three steps.

1') The first, second, third, fifth and sixth non-weakly-vanishing terms have nontrivial scalar factors, so we are forced to have \( O^{IK} = g^{IK}, C_1 = C_2 = C_3 = 0, D_{IK} = 0 \) and \( F_I = 0 \). The conditions on the \( C \)'s correspond to the vector fields obeying the local Carroll structure.

2') This automatically implies that the seventh, eighth and ninth terms also vanish.

3') The only nontrivial possibility for the vanishing of the fourth term is if \( B_{IJK} = 0 \), in which case the constraint algebra has been closed.

It is enlightening to contrast these (primed) steps with their (unprimed) counterparts from the GR case [5].

1) is the same as 1') except that \( C_1 = -C_2 = -1/4 \), which corresponds to the vector fields obeying the local Lorentz light-cone structure

2) is the same as 2') except that instead of the automatic vanishing of the eighth term, one is forced to take \( B_{IK} = B_{IJK} \), which is the start of the imposition of an algebraic structure on the hitherto unknown arrays.

3) One is now left with \( K \) new nontrivial scalar constraints, which happen to form the Yang–Mills Gauss constraint,

\[
\mathcal{G}_I \equiv \pi_{Ja}^a - g B_{IJK} \pi_{Ja}^I A^K_a \approx 0,
\]

(43)

where \( g = -4 \bar{c}_1 = 4 \bar{c}_2 \) will become the coupling constant. So the algebra is not yet closed; we have the following further steps.

Propagation of the new constraints requires that the \( M_{IJ} \) and \( E_{IJK} \) terms are killed off; the first of these conditions means that the vector fields are massless. Furthermore the propagation forces \( I_{JKLM} = B_{IJK} B_{LM} \) and the Jacobi identity \( B^{I}_K B_{IJK} + B^{J}_L B_{IMJ} + B^{J}_M B_{IJL} = 0 \). So in this case, one obtains a gauge theory for which \( B_{IJK} \) are structure constants. Finally, one is forced to have \( B_{IJK} = B_{IJK} \), which allows one to restrict the algebra associated with the gauge group to being the direct sum of compact simple and \( U(1) \) subalgebras by use of the standard Gell-Mann–Glashow result [14], albeit in a slightly different way from its use in the flat spacetime derivation of Yang–Mills theory. The new constraint may now be encoded as the variation in a further auxiliary variable introduced according to the best matching procedure corresponding to the gauge symmetry. Thus one arrives at the \((3 + 1)\) decomposition of Einstein–Yang–Mills theory.

3.3 Discussion of Matter-Coupling Results

The results of this section help clarify some aspects of the 3-space approach results for GR. First, notice also how now that a family of supermetrics is allowed the matter dynamics is insensitive to a possible change of supermetric, which is encouraging for the coupling of conformal gravity to matter fields.

Second, we can take further the view that local causal structure and gauge theory are manifestations of the same thing. By the BFΩ procedure, in the GR case the universal light-cone and gauge theory come
together from the $\dot{R}$ term in $\dot{\mathcal{H}}$, whilst the absence of this in strong gravity ensures that the collapse of the light-cone to the Carrollian line is accompanied by the breakdown of gauge theory: there is neither gauge symmetry nor a Gauss law. In the GR case, the quantum-mechanics-inspired positive-definiteness assumed of the vector field kinetic matrix $P_{IJ}$ then turns out to be necessary in the restriction of the choice of gauge group, so there would be a price to pay if one insisted instead on entirely classical assumptions. In the strong gravity case, the absence of emergent gauge structure means that there is no such price to pay for using classical assumptions alone. Provided that $P_{IJ}$ is invertible, the outcome of steps 1') to 3') is unaltered.

We see through what happens in the absence of the Gauss law in this paper that it is specifically this characteristic of the 3-vector theory that kills off the vector field mass terms in the 3-space approach, rather than some underlying principle for general matter. This is a useful first insight into the status of mass in the 3-space approach to GR. It is also easy to demonstrate that the general derivative-free potential term built out of vector fields persists coupled to strong gravity.

We emphasize that our result concerning the breakdown of gauge theory is in particular a result about GR, although it clearly occurs for all our theories and the theories they approximate. As $G \to \infty$ such as in the vicinity of the initial singularity, dynamical consistency dictates that gauge theory breaks down in GR. Gauge interactions become impossible as one approaches such a regime. This appears not be in accord with the view that gauge interactions persist in extreme regimes to form part of a unified theory with gravity, such as in string theory. However, little is known about physics in such regimes, so this classical GR intuition might not hold. If string theory can tame such singularities, the circumstances under which gauge theory breaks down according to GR might not occur. However, it could even be that string theory breaks down in such a regime, since according to one interpretation, stringy matter could be a phase of some larger theory which breaks down in a high-energy phase transition. Also, Carrollian regimes might arise in string theory under other circumstances, and exhibit different behaviour from the strong-coupled limit of GR coupled to gauge theory, as suggested by the recent Born–Infeld study.

4 Conclusion

We have found by the 3-space approach a 1-parameter family of consistent theories of evolving 3-geometries. Whereas for the parameter value $W = 1$ the corresponding theory is well-known to be the strong-coupled limit of GR, we have interpreted our other theories as similar limits of scalar-tensor theories, and argued that such scalar-tensor theories may describe our universe.

For the parameter value $W < \frac{1}{3}$, our theories exhibit qualitatively different behaviour from GR, but similar to the behaviour of conformal gravity. This qualitative difference shows up in the quantum-mechanical regime by offering a different resolution to the inner product problem.

However, we have argued that this feature is unlikely to be simply combinable with the use of Ashtekar variables.

On coupling matter, our theories enforce that this matter is ultralocal, just like BFÓ’s treatment of matter enforces a local Lorentz light-cone structure on the matter. Whereas BFÓ’s treatment enforces gauge theory, the collapse of the light-cone structure in the strong-coupled limit is accompanied by the breakdown of gauge theory.

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