THE NON-ORDINARY LOCUS OF THE TTV FAMILY OF CURVES

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Abstract. This article establishes a relation between the non-ordinary locus of the reduction mod $p$ of the TTV family of curves and the genus of certain triangular modular curves.

0. Introduction

Let $\Gamma_0(p) = \{ M \in \text{SL}_2(\mathbb{Z}) \mid M \equiv \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \pmod{p} \} \subseteq \text{SL}_2(\mathbb{Z})$ and $\mathcal{H}^* = \{ z \in \mathbb{C} \mid \Im(z) > 0 \} \cup \mathbb{Q}$. It is well-known that the modular curve $X_0(p) = \Gamma_0(p) \backslash \mathcal{H}^*$ admits an integral model for which the reduction modulo $p$ consists of two copies of $X_0(1)_{F_p} = \mathbb{P}^1_{F_p}$ crossing transversally at the points that represent supersingular elliptic curves.

This article’s motivating question is whether one can find a similar relation when the classical modular group $\text{SL}_2(\mathbb{Z})$ is replaced by the triangle group $\Gamma_{5,\infty,\infty}$ and elliptic curves are replaced by TTV curves (cf. Definition 2.1 below). The reason why TTV curves are chosen here is due to a connection between those curves and $\Gamma_{5,\infty,\infty}$ found by Henri Darmon via Frey representations (cf. Definition 1.1 and Theorem 1.10 in [Dar00] and Definition 8 in [Dar04]).

The definitions of $\Gamma_{5,\infty,\infty}$ and the TTV curves are given in sections 1 and 2. The main results lie in Section 4, where Theorem 4.7 states a relation between the genus of a version of $X_0(p)$ for $\Gamma_{5,\infty,\infty}$ and the number of non-ordinary TTV curves mod $p$, providing evidence that the answer to the motivating question might be positive.

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1. Triangle Groups

In this section, a particular triangle group and some important subgroups are defined. For more details and a more general description, cf. section 2 of [Tak12].

The triangle group \( \Gamma_{5,\infty,\infty} \) is defined to be the subgroup of \( SL_2(\mathbb{R}) \) generated by
\[
\gamma_1 = \begin{pmatrix} -2 \cos(\frac{\pi}{5}) & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.
\]

In what follows, whenever \( \Gamma \subseteq SL_2(\mathbb{R}) \), the group \( \Gamma \subseteq PSL_2(\mathbb{R}) \) will denote the image of \( \Gamma \) in \( PSL_2(\mathbb{R}) \).

It is a fact that \( \Gamma_{5,\infty,\infty} \) is a Fuchsian group and, moreover, \( \overline{\Gamma_{5,\infty,\infty}} \backslash \mathcal{H}^* \cong \mathbb{P}^1 \) where \( \mathcal{H}^* = \mathcal{H} \cup \{ \text{cusps of } \Gamma_{5,\infty,\infty} \} \).

Note that \( \Gamma_{5,\infty,\infty} \) is a subgroup of \( SL_2(\mathcal{O}) \), where \( \mathcal{O} = \mathbb{Z}[\zeta_{10} + \zeta_{10}^{-1}] \) and \( \zeta_{10} = e^{2\pi i/10} \).

**Definition 1.1.** Given a prime ideal \( p \) of \( \mathcal{O} \), the congruence subgroups of \( \Gamma_{5,\infty,\infty} \) with level \( p \) are defined to be
\[
\Gamma_{5,\infty,\infty}(p) = \left\{ M \in \Gamma_{5,\infty,\infty} \mid M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p} \right\}, \quad \text{and}
\]
\[
\Gamma_{5,\infty,\infty}^{(0)}(p) = \left\{ M \in \Gamma_{5,\infty,\infty} \mid M \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p} \right\}.
\]

2. The TTV Curves

**Definition 2.1.** The TTV curves are defined to be the following:
\[
C^-(t) : y^2 = x^5 - 5x^3 + 5x + 2 - 4t
\]
\[
C^+(t) : y^2 = (x + 2)(x^5 - 5x^3 + 5x + 2 - 4t).
\]

In [TTV91], W. Tautz, J. Top, and A. Verberkmoes studied the two families of hyperelliptic curves defined above. In particular, it was shown that their Jacobians have real multiplication by \( \mathcal{O}_L \), the ring of integers of \( L = \mathbb{Q}(\zeta_{10})^+ = \mathbb{Q}(\zeta_{10} + \zeta_{10}^{-1}) \).

3. The Hasse-Witt and Cartier-Manin matrices

This section is mainly based on Chapters 9 and 10 of [Ser58] and [Yu78].
3.1. Hasse-Witt matrix. Let $k$ be a perfect field of characteristic $p > 2$ and $C$ a hyperelliptic curve of genus $g > 0$ defined over $k$. This notion can be defined in a more general context but we will focus on hyperelliptic curves.

**Definition 3.1.** Fix a basis of $H^1(C, \mathcal{O}_C)$. The Hasse-Witt matrix of $C$ is the matrix of the $p$-linear operator $F : H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_C)$, where $F$ is the Frobenius operator.

*Remark.* Notice that the Hasse-Witt matrix is dependent on the basis chosen. Because of the $p$-linearity of the Frobenius operator, if $H$ and $H'$ are Hasse-Witt matrices with respect to different bases, then there is a matrix $U$ such that

$$H' = U^{-1}HU^{(p)},$$

where $U^{(p)}$ is the matrix obtained from $U$ by raising all its entries to the $p$-th power.

There is another way to essentially define the Hasse-Witt matrix of a curve. This is done in terms of the so called Cartier operator, which is studied in the next section.

3.2. Cartier-Manin Matrix. Suppose $C$ is given by

$$(2) \quad y^2 = f(x)$$

where $f(x)$ is a polynomial over $k$ without multiple roots of degree $2g + 1$.

Every element of $\Omega^1_C$ can be written as

$$\omega = d\varphi + \eta^p x^{p-1}dx$$

for some $\varphi, \eta \in k(C)$.

**Definition 3.2.** The Cartier operator $\mathcal{C} : H^0(C, \Omega^1_C) \to H^0(C, \Omega^1_C)$ is defined by

$$\mathcal{C}(d\varphi + \eta^p x^{p-1}dx) = \eta dx.$$

**Definition 3.3.** The Cartier-Manin matrix is the matrix of the $1/p$-linear operator $\mathcal{C} : H^0(C, \Omega^1_C) \to H^0(C, \Omega^1_C)$.

*Remark.* Because of the $1/p$-linearity of the operator $\mathcal{C}$, if $M$ and $M'$ are Cartier-Manin matrices with respect to different bases, then there is a matrix $U$ such that

$$M' = U^{-1}MU^{(1/p)},$$

where $U^{(p)}$ is the matrix obtained from $U$ by raising all its entries to the $p$-th power.
Remark. The Cartier operator, as defined here, is called the modified Cartier operator in \cite{Yui78}. Moreover, the definition of the Cartier-Manin matrix given by N. Yui is slightly different (cf. page 381 of \cite{Yui78}).

The relation between the Hasse-Witt matrix and the Cartier-Manin matrix arises as follows. It is known that $H^0(C, \Omega^1_C)$ is the dual of $H^1(C, \mathcal{O}_C)$. Under this identification, the following result (cf. Prop. 9, Section 10 in \cite{Ser58}) holds.

**Proposition 3.4.** The map $\mathcal{C}: H^0(C, \Omega^1_C) \to H^0(C, \Omega^1_C)$ is the dual of $F: H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_C)$.

N. Yui (cf. pages 380-381 in \cite{Yui78}) gives a concrete way of computing the Cartier-Manin matrix of a curve:

**Proposition 3.5.** Let $C$ be given by (2). Then the Cartier-Manin matrix of $C$ with respect to the basis $\frac{dx}{y}, \frac{x\, dx}{y}, \ldots, \frac{x^{g-1}\, dx}{y}$ of $H^0(C, \Omega^1_C)$ is given by

$$N^{(1/p)},$$

where

$$N = (c_{ip-j}) = \begin{pmatrix} c_{p-1} & c_{p-2} & \cdots & c_{p-g} \\ c_{2p-1} & c_{2p-2} & \cdots & c_{2p-g} \\ \cdots & \cdots & \cdots & \cdots \\ c_{gp-1} & c_{gp-2} & \cdots & c_{gp-g} \end{pmatrix},$$

and

$$f(x)^{(p-1)/2} = \sum c_r x^r.$$

3.3. **Jacobian of $C$.** From now on, $k$ will be a finite field of characteristic $p > 2$.

Recall the following definitions:

**Definition 3.6.** An abelian variety $A$ of dimension $g$ over $k$ is called

- *ordinary* if its $p$-rank is $g$, i.e., $#(A[p]) = p^g$;
- *supersingular* if $A$ is $k$-isogenous to a power of a supersingular elliptic curve.

Remark. As explained in Section 3.2 of \cite{Zhu00}, if $A$ is supersingular, then $A[p] = 0$. The converse holds if $g = 1$ or 2 but not necessarily if $g > 2$. 
Let \( J = J(C) \) be the Jacobian of the curve \( C \).

The results below show the relation between the Cartier-Manin matrix of \( C \) and \( J \).

**Proposition 3.7.** The \( p \)-rank of \( J \) is bounded above by the rank of the Cartier-Manin matrix, i.e., \( \sigma \leq \text{rk}(M) \), where \( \#(J[p]) = p^\sigma \) and \( M \) denotes the Cartier-Manin matrix.

**Proof.** This is a corollary of Proposition 10 in Section 11 of [Ser58]. □

**Proposition 3.8.** Let \( M \) be the Cartier-Manin matrix of \( C \) and \( N = M^{(p)} \). The following holds:

(a) \( \det(N) \neq 0 \) if and only if \( J \) is ordinary.

(b) \( N = 0 \) if and only if \( J \) is a product of supersingular elliptic curves.

(c) If the genus of \( C \) is 2, then \( N^{(p)} \) \( N = 0 \) if and only if \( J \) is supersingular.

**Proof.** Cf. [Yui78] (Theorems 3.1 and 4.1), [Nyg81] (Theorem 4.1) and [Man63] (p. 78). □

### 3.4. Curves with real multiplication.

Let \( L \) be a totally real number field such that \([L : \mathbb{Q}] = g\) and \( p \) a prime number that is unramified in \( L \). In this subsection \( C \) will denote a projective algebraic curve of genus \( g \) and \( J \) its Jacobian, which is assumed to have real multiplication by \( \mathcal{O}_L \), that is, with an embedding of rings

\[ \iota : \mathcal{O}_L \rightarrow \text{End}(J) \]

as explained in definition 2.2.1 of [Gor02].

In this section, the Cartier operator \( \mathcal{C} \) (hence, the Cartier-Manin matrix) is studied via the corresponding operator on the Jacobian of \( C \).

| \( C \) | \( J \) |
|---|---|
| action of \( \mathcal{C} \) on \( H^0(C, \Omega^1_C) \) | action of \( V \) on \( H^0(J, \Omega^1_J) \) |
| action of \( F \) on \( H^1(C, \mathcal{O}_C) \) | action of \( F \) on \( H^1(J, \mathcal{O}_J) \) |

The vector spaces on the left column are isomorphic to the ones on the right column. Furthermore, the semi-linear operators on the left column coincide (via that isomorphism) to the ones on the right column.

**Theorem 3.9.** As an \( \mathcal{O}_L \otimes k \)-module, the space \( H^1(J, \mathcal{O}_J) \) decomposes as

\[ H^1(J, \mathcal{O}_J) = \bigoplus_{\sigma \in B} W_\sigma, \]

where

\[ B = \{ \sigma : \mathcal{O}_L \rightarrow k \mid \sigma \text{ a ring homomorphism} \} \text{ and } \dim_k W_\sigma = 1. \]
Moreover, the action of $F$ commutes with the action of $\mathcal{O}_L \otimes k$ and satisfies the following

\[ F(W_\sigma) \subseteq W_{\text{Fr} \circ \sigma}. \]

**Proof.** Cf. Lemma 2.3.1 and Remark 2.2.8 in [GO00].

**Remark.** Being the dual of $F$, a similar statement holds for the action of $V$ on $H^0(J, \Omega^1_J)$.

**Remark.** Consider the factorization of $p$ in $\mathcal{O}_L$ given by

\[ p\mathcal{O}_L = p_1 p_2 \cdots p_r. \]

Then, it is not hard to see that, with the notation of Theorem 3.9, $B$ decomposes as

\[ B = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_r, \]

where $\#B_i = f = f(p_i/p) = [\mathcal{O}_L/p_i : \mathbb{F}_p]$. Furthermore, $\text{Fr}$ acts transitively on each $B_i$, i.e.,

\[ B_i = \{ \sigma_i \circ \text{Fr}^f \circ \sigma_i, \text{Fr} \circ \sigma_i, \ldots, \text{Fr}^{f-1} \circ \sigma_i \}. \]

4. **Studying $C^\pm(t)$**

In this section we return to the families of curves defined in (1). These curves have genus $g = 2$ and, as was mentioned in the previous chapter, they have real multiplication by $\mathcal{O}_L$ (where $L = \mathbb{Q}(\sqrt{5})$).

Consider a prime $p > 2$ that is unramified in $L$.

**Lemma 4.1.** There are only two possibilities for such a $p$:

- $p$ is a product of two primes in $\mathcal{O}_L$ (when $p \equiv 1, 4 \pmod{5}$); or
- $p$ is inert in $\mathcal{O}_L$ (when $p \equiv 2, 3 \pmod{5}$).

**Proof.** Cf. (1.1) in Chapter V of [FT93].

4.1. **The Cartier-Manin matrix of the curve $C^-$**. Example 3.5 (or the proof of the main result) in [TTV91] shows that the action of $\mathcal{O}_L$ on $H^0(C^-, \Omega^1)$ has two distinct eigenvectors, namely:

\[ \frac{dx}{y}, x \frac{dx}{y} \]

Thus, Lemma 4.1, Theorem 3.9 and the remarks that follow it yield the result below.
Theorem 4.2. The Cartier-Manin matrix of $C^-$ with respect to the basis $\{\frac{dx}{y}, x\frac{dx}{y}\}$ of $H^0(C^-, \Omega^1)$ is given by

$$M = \begin{cases} 
(\ast \ 0) , & \text{if } p \equiv 1, 4 \pmod{5} \\
0 \ast) , & \text{if } p \equiv 2, 3 \pmod{5}.
\end{cases}$$

where $\ast$ are elements of $\mathbb{F}_p[t]$.

Remark. A curious consequence of this fact is the following non-trivial result. Let $p$ be a prime number such that $p \neq 2, 5$, $f(x) = x^5 - 5x^3 + 5x + 2 - 4t \in \mathbb{Z}[t][x]$ and $f(x)^{(p-1)/2} = \sum c_r x^r$. If

- $p \equiv 1, 4 \pmod{5}$, then $c_{p-1} \equiv c_{2p-2} \equiv 0 \pmod{p}$
- $p \equiv 2, 3 \pmod{5}$, then $c_{p-2} \equiv c_{2p-1} \equiv 0 \pmod{p}$.

Proof. Follows from the previous result and Proposition 3.5.

Corollary 4.3. If $p \equiv 2$ or $3 \pmod{5}$, then the Jacobian of the curve $C^-$ is either supersingular or ordinary.

Proof. This is a direct consequence of the previous theorem and of Proposition 3.8.

4.2. The Cartier-Manin matrix of the curve $C^+$. Following the ideas of the proof of the main result of [TTV91], one can compute the action of $\mathcal{O}_L$ on $H^0(C^+, \Omega^1)$.

Proposition 4.4. The Jacobian of the curve $C^+$ has real multiplication by $\mathcal{O}_L$. Moreover, the action of $\mathcal{O}_L$ on $H^0(C^+, \Omega^1)$ has a basis of eigenvectors, namely:

$$dx/y , \ dx/y + ydx/y$$

Proof. Tautz-Top-Verberkmoes ([TTV91]) showed that $C^+$ is the quotient $D_t/\sigma$, where

$$D_t : y^2 = x^{10} + tx^5 + 1$$
and \( \sigma \in \text{End}(D_t) \) defined by

\[
\sigma : (x, y) \mapsto (1/x, y/x^5).
\]

Using that

\[
X^{2n} + 1 = X^n(X + X^{-1}) \cdot g(X^2 + X^{-2}) \in k[X, X^{-1}]
\]

for any odd \( n \), it follows that the map \( \varphi : D_t \to C^+ \) given by

\[
\varphi : (x, y) \mapsto (x + 1/x, y(x+1)/x^3)
\]

is well-defined and corresponds to the natural quotient map \( D_t \to D_t/\sigma \). Moreover, it makes the diagram below commutative

\[
\begin{array}{ccc}
D_t & \xrightarrow{\varphi} & C^+ \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & & \mathbb{P}^1
\end{array}
\]

where

\[
\begin{align*}
\mathbb{P}^1 & \to \mathbb{P}^1 \\
x & \mapsto x + 1/x
\end{align*}
\]

and the vertical maps are just

\[
(x, y) \mapsto x.
\]

The curve \( D_t \) has multiplication by \( \mathcal{O}_{\mathbb{Q}(\zeta_5)} \) coming from the map

\[
\zeta : (x, y) \mapsto (\zeta_5 x, y).
\]

To prove that the Jacobian of \( C^+ \) has multiplication by \( \mathcal{O}_L \), it is enough to show that the action of \( \zeta^* + (\zeta^{-1})^* \) preserves the space \((\Omega^1_{D_t})^\sigma\) of \( \sigma \)-invariant differentials of \( D_t \). One checks that a basis for \((\Omega^1_{D_t})^\sigma\) is given by

\[
\omega_1 = (x^2 - x)dx/y \ , \ \omega_2 = (x^3 - 1)dx/y.
\]

Now, by the definition of \( \zeta \), one computes that

\[
[\zeta^* + (\zeta^{-1})^*] \omega_1 = (\zeta_5^2 + \zeta_5^{-2}) \omega_1
\]

and

\[
[\zeta^* + (\zeta^{-1})^*] \omega_2 = (\zeta_5 + \zeta_5^{-1}) \omega_2.
\]

Now it remains only to show that

\[
dx/y \ , \ dx/y + ydx/y \in \Omega^1_C
\]

are eigenvectors for the action of \( \mathcal{O}_L \).

Notice that the action on \( \Omega^1_C \) comes from the action on \((\Omega^1_{D_t})^\sigma\) (these spaces are identified via \( \varphi \)). Now, the definition of \( \varphi \) yields

\[
\varphi^*(dx/y) = \omega_1
\]

and

\[
\varphi^*(dx/y + ydx/y) = \omega_2.
\]
From the previous computations, this finishes the proof. □

This result and Lemma 4.1 yield

**Theorem 4.5.** The Cartier-Manin matrix of $C^+$ with respect to the basis $\left\{ \frac{dx}{y}, x \frac{dx}{y} \right\}$ of $H^0(C^-, \Omega^1)$ is given by

$$M = \begin{cases} 
\begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix}, & \text{if } p \equiv 1, 4 \pmod{5} \\
\begin{pmatrix} a & b \\ a & -a \end{pmatrix}, & \text{if } p \equiv 2, 3 \pmod{5}.
\end{cases}$$

for some $a, b \in \mathbb{F}_p[t]$.

**Corollary 4.6.** If $p \equiv 2$ or $3 \pmod{5}$, then the Jacobian of the curve $C^+$ is either supersingular or ordinary.

**Proof.** This is a direct consequence of the previous theorem and of Proposition 3.8. The proof is similar to the proof of corollary 4.3. □

4.3. **A relation between $X^{(0)}_{5, \infty, \infty}(p)$ and the family $C^-$.** It is known that $X_0(p) = \Gamma_0(p) \backslash \mathcal{H}^*$ admits an integral model for which the reduction modulo $p$ consists of two copies of $X_0(1)_{\mathbb{F}_p} = \mathbb{P}^1_{\mathbb{F}_p}$ crossing transversally at the supersingular points as shown in figure 1 (cf. Theorem 6.9, page DeRa-144, in [DR73]). In particular, there is a relation between the genus of $X_0(p)$ and the number of supersingular elliptic curves modulo $p$.

In this subsection we investigate a similar property for the triangular modular curve $X^{(0)}_{5, \infty, \infty}(p)$: we show that in certain cases, the genus of the curve $X^{(0)}_{5, \infty, \infty}(p)$ is closely related to the number of non-ordinary elements of the family of curves $C^-$. More specifically, the following result holds:
Theorem 4.7. Let $p > 5$ be a prime number such that $p$ splits in $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}$ (i.e., $p \equiv 1$ or $4 \pmod{5}$) and take $p$ a prime ideal above $p$. Furthermore, let $g$ be the genus of $X_{5,\infty,\infty}^{(0)}(p)$ and $d(t) = \det(M^{(p)})$, where $M$ is the Cartier-Manin matrix of $C^{-}$ (as computed in Theorem 4.2). Then

$$g = \deg(d(t)) + \delta,$$

where

$$\delta = \begin{cases} -1, & \text{if } p \equiv 1 \pmod{5} \\ 1, & \text{if } p \equiv 4 \pmod{5}. \end{cases}$$

Proof. Since $p$ is assumed to be split, Proposition 4.5 in [Tak] implies that the genus of $X_{5,\infty,\infty}^{(0)}(p)$ is given by

$$g = 2n - 1,$$

where

$$p + 1 = 5n + m$$

with

$$m = \begin{cases} 0, & \text{if } p \equiv -1 \pmod{5} \\ 2, & \text{if } p \equiv 1 \pmod{5}. \end{cases}$$

Thus,

$$g = \frac{2}{5}(p + 1 - m) - 1.$$

It follows from Theorem 4.2 that the Cartier-Manin matrix is given by $(^0_0)$, it suffices to compute the degree of the entries of the main diagonal, which is done in the next lemma. □

Lemma 4.8. Let $p$ be as in the statement of the previous proposition and

$$\begin{pmatrix} a(t) & 0 \\ 0 & b(t) \end{pmatrix}$$

be the Cartier-Manin matrix of $C^{-}$ with respect to $p$. Then

$$\deg(a(t)) = \begin{cases} \frac{3}{2}k, & \text{if } p = 5k + 1 \\ \frac{3}{2}k - 1, & \text{if } p = 5k - 1 \end{cases}$$

and

$$\deg(b(t)) = \begin{cases} \frac{1}{2}k, & \text{if } p = 5k + 1 \\ \frac{1}{2}k - 1, & \text{if } p = 5k - 1 \end{cases}$$

Proof. By Proposition 3.5, $a(t)$ is the $(p-1)$-th coefficient of $f(x)^{(p-1)/2}$, where

$$f(x) = x^5 - 5x^3 + 5x + 2 - 4t.$$
Since this lemma is only concerned about the degree (with respect to $t$) of a certain coefficient, $f$ can be assumed to be
$$f(x) = x^5 - 5x^3 + 5x + t.$$

By the Multinomial Theorem,
$$f(x)^{(p-1)/2} = \sum_{a,b,c,d} (a,b,c,d)! \left( \frac{(p-1)}{2} \right)^b 5^{b+c} x^{5a+3b+c} t^d,$$
where the sum is taken over all integers $a, b, c, d \geq 0$ such that $a + b + c + d = (p - 1)/2$ and
$$(a,b,c,d)! = \frac{((p-1)/2)!}{a! b! c! d!}.$$

Therefore the $(p - 1)$-th coefficient is given by
$$a(t) = \sum_{5a+3b+c=p-1} (a,b,c,d)! \left( \frac{(p-1)}{2} \right)^b 5^{b+c} t^d.$$

This implies that $\text{deg}(a(t))$, at least over $\mathbb{Z}$, is given (possibly) by the largest $d$ such that
$$d = 4a + 2b - \frac{(p-1)}{2}$$
and
$$\left\{ \begin{array}{l}
5a + 3b \leq p - 1 \\
 a \geq 0 , b \geq 0.
\end{array} \right.$$ 

Assume now that that $p = 5k + 1$. One checks (using the graphical method of linear programming) that the solution is
$$d = \frac{3}{2}k$$
attained only once when
$$a = k \quad \text{and} \quad b = 0.$$

Since this is attained only once, $\text{deg}(a(t))$ over $\mathbb{Z}$ is actually $\frac{3}{2}k$. Using the fact that $p > 5$, it follows that the coefficient of the degree $\frac{3}{2}k$ term is not zero modulo $p$. Hence, $\text{deg}(a(t)) = \frac{3}{2}k$ over $\mathbb{F}_p$.

A similar argument proves all the other cases. The only exception is the last case ($\text{deg}(b(t))$ when $p = 5k - 1$), where the maximum $d$ is attained twice. But in this case a straight forward computation shows that the coefficient is still non-zero modulo $p$. \hfill $\square$

Remark. Theorem 4.7 presents an interesting relation between the genus of $X_{5,\infty,\infty}^{(0)}(p)$ and the number of non-ordinary elements in the family $C^{-}$ modulo $p$ when $p$ is split. Unfortunately when $p$ is inert, the same does not hold. The example below shows that the difference between
the degree of \(d(t)\) and the genus of \(X_{5,\infty,\infty}(p)\) grows with \(p\) when \(p\) is inert.

It would be interesting to understand why there is this discrepancy between primes that are split and primes that are inert.

**Example.** Contrary to the split case, the difference between the genus of \(X_{5,\infty,\infty}^{(0)}(p)\) and the degree of \(d(t)\) is not \(\pm 1\) when \(p\) is inert. Here are the first few inert primes and their corresponding data as calculated using the computer algebra system SAGE ([S+12]):

| \(p\) | genus of \(X_{5,\infty,\infty}^{(0)}(p)\) | degree of \(d(t)\) | genus - degree |
|---|---|---|---|
| 7  | 13 | 2 | 11 |
| 13 | 55 | 4 | 51 |
| 17 | 99 | 6 | 93 |
| 23 | 189| 8 | 181|
| 37 | 511| 14| 497|
| 43 | 697| 16| 681|
| 47 | 837| 18| 819|
| 53 | 1071| 20| 1051|
| 67 | 1729| 26| 1703|
| 73 | 2059| 28| 2031|
| 83 | 2673| 32| 2641|
| 97 | 3667| 38| 3629|
| 103| 4141| 40| 4101|

**Remark.** Note that Theorem 4.7 actually describes a relation between the genus of \(X_{5,\infty,\infty}^{(0)}(p)\) and the degree of \(d(t)\), which is not exactly the same as the number of non-ordinary elements in the \(C^-\) family. In our computations, summarized in the table below, the difference between the degree and the exact number of non-ordinary elements seems to be reasonably small. It would be interesting to understand how the difference grows and check whether it is bounded or not.

| \(p\) | degree of \(d(t)\) | \# of non-ordinary curves | difference |
|---|---|---|---|
| 11 | 4 | 3 | 1 |
| 19 | 6 | 5 | 1 |
| 29 | 10| 10| 0 |
| 31 | 12| 11| 1 |
| 41 | 16| 16| 0 |
| 59 | 22| 21| 1 |
| 61 | 24| 22| 2 |
| 71 | 28| 25| 3 |
|    |    |    |    |
|----|----|----|----|
| 79 | 30 | 27 | 3  |
| 89 | 34 | 32 | 2  |
| 101| 40 | 38 | 2  |
| 109| 42 | 42 | 0  |
| 131| 52 | 45 | 7  |
| 139| 54 | 53 | 1  |
| 149| 58 | 54 | 4  |
| 151| 60 | 57 | 3  |
| 179| 70 | 69 | 1  |
| 181| 72 | 68 | 4  |
| 191| 76 | 75 | 1  |
| 199| 78 | 75 | 3  |
| 211| 84 | 79 | 5  |
| 229| 90 | 90 | 0  |
| 239| 94 | 91 | 3  |
| 241| 96 | 92 | 4  |
| 251| 100| 95 | 5  |
| 269| 106| 106| 0  |
| 271| 108| 105| 3  |
| 281| 112| 110| 2  |
| 311| 124| 123| 1  |
| 331| 132| 129| 3  |
| 349| 138| 134| 4  |
| 359| 142| 139| 3  |
| 379| 150| 147| 3  |
| 389| 154| 154| 0  |
| 401| 160| 158| 2  |
| 409| 162| 156| 6  |
| 419| 166| 165| 1  |
| 421| 168| 162| 6  |
| 431| 172| 165| 7  |
| 439| 174| 169| 5  |

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