Public key cryptography based on skew dihedral group rings

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Abstract

In this paper, we propose to use a skew dihedral group ring given by the group $D_{2n}$ and the finite field $\mathbb{F}_{q^2}$ for public-key cryptography. Using the ambient space $\mathbb{F}_{q^2}D_{2n}$ and a group homomorphism $\theta : D_{2n} \rightarrow \text{Aut}(\mathbb{F}_{q^2})$, we introduce a key exchange protocol and present an analysis of its security. Moreover, we explore the properties of the resulting skew group ring $\mathbb{F}_{q^2}D_{2n}$, exploiting them to enhance our key exchange protocol. We also introduce a probabilistic public-key scheme derived from our key exchange protocol and obtain a key encapsulation mechanism (KEM) by applying a well-known generic transformation to our public-key scheme. Finally, we present a proof-of-concept implementation of our cryptographic constructions. To the best of our knowledge, this is the first paper that proposes a skew dihedral group ring for public-key cryptography.

Keywords. Skew Dihedral Group Ring; Key Exchange Protocol; Encryption scheme.

MSC Classification 14G50, 94A60, 11T71, 16S35.

1 Introduction

The availability of quantum computers in the forthcoming future will make current public-key schemes insecure. Therefore, there is a need for devising quantum-secure cryptographic public-key primitives as a replacement for the current public-key algorithms. This need undoubtedly has propelled research towards creating quantum-secure public-key schemes.

There have been many proposed candidates so far, of which the most promising ones are classified into five families. These families are lattice-based cryptography, multivariate cryptography, hash-based cryptography, code-based cryptography, and supersingular elliptic curve isogeny cryptography. The third round of the post-quantum cryptography standardization process run by the National Institute of Standards and Technology (NIST) includes various candidates in each of the mentioned families [16].
However, recently a new promising family of cryptographic constructions, believed to be quantum-secure and based on variations of group rings [9, 5], has been introduced. In particular, the recent works [9, 5] exploit the structure of dihedral twisted group rings to introduce cryptographic constructions. The work [9] introduces a 2-cocycle $\beta$ in order to construct a dihedral twisted group algebra $F_q^\beta D_{2n}$. Over $F_q^\beta D_{2n}$, the authors build a key-exchange protocol à la Diffie-Hellman and a probabilistic public-key scheme. Following an alternative approach, the authors of [5] propose a key exchange protocol, a probabilistic public-key scheme, and a key encapsulation mechanism. They also introduce a 2-cocycle $\alpha \lambda$ to form the resulting twisted algebra $F_q^{\alpha \lambda} G$ non-equivalent to $F_q^\beta D_{2n}$ for a non-square $\lambda$ in the field $F_q$. They explore its properties and exploit them to enhance the introduced key exchange protocol.

In other related works, the authors in [4] investigate right ideals as codes in twisted group rings. In particular, they characterize all linear codes that are twisted group codes in terms of their automorphism group.

Our work takes an alternative path by introducing what we call a skew dihedral group ring which is the main tool for constructing a key exchange protocol, a probabilistic public-key scheme, and a derived key encapsulation mechanism. We first formally define the notion of a skew group ring and explore some of its properties. We then study skew dihedral group rings, and later construct a specific skew dihedral group ring by defining the group homomorphism $\theta: D_{2n} \to \text{Aut}(F_q^2)$ stated in Lemma 3.4. In particular, given the presentation $G = D_{2n} = \langle x, y : x^n = y^2 = 1, yxy^{-1} = x^{-1} \rangle$ of the dihedral group, the map $\theta(a) = a^q$ for all $a \in F_q^2$, for $g = x^i y$, $i \in \{0, \ldots, n-1\}$ and $\theta(a) = 1$otherwise is a group homomorphism. Over the resulting skew dihedral group ring $F_q^{\theta D_{2n}}$, we realize our cryptographic constructions and analyze their security. Finally, we present a proof-of-concept implementation of our key encapsulation mechanism.

The outline of the paper is as follows. In Section 2 we show the basic definitions and results we need, whereas in Section 3 we show the concrete presentation of the dihedral group ring we use. Section 4 presents the proposed key exchange protocol and analyzes its intractability assumptions. Section 5 presents a probabilistic public-key encryption scheme, and Section 6 introduces a key encapsulation mechanism using the ideas from the previous sections. Finally, Section 7 presents the pseudo-codes of a proof-of-concept Python implementation of our cryptographic constructions.

2 Preliminaries

Let $F_q$ be the finite field with $q = p^m$ elements where $p$ is a prime number and let $\text{Aut}(F_q)$ be the automorphism group of $F_q$. Recall that any automorphism $\Theta \in \text{Aut}(F_q)$ of the finite field $F_q$ is of the type $\Theta(x) = x^{p^\theta}$. Denote by $\text{Gal}(F_{q^k}, F_q)$ the Galois group of $F_{q^k}$ over $F_q$, i.e. the set of all automorphisms of $F_{q^k}$ that fix the subfield $F_q$. It holds that $\text{Aut}(F_{q^k}) = \text{Gal}(F_{q^k}, F_q) \times \text{Aut}(F_q)$, in particular $\text{Aut}(F_q) = \text{Gal}(F_q, F_p)$.

In the following paragraphs we summarise the definitions and properties we need on skew group rings.

**Definition 2.1.** Let $G$ be a finite multiplicative group and let $\theta : G \to \text{Aut}(F_q)$ be a
group homomorphism. The skew group ring $\mathbb{F}_q^\theta G$ is the set of all formal sums $\sum_{g \in G} a_g g$, where $a_g \in \mathbb{F}_q$, with the following skew multiplication

$$a_g g \cdot b_h h = a_g(\theta(g)(b_h))gh.$$ 

Note that as $\mathbb{F}_q$-vector space the skew group ring $\mathbb{F}_q^\theta G$ coincides with the group ring $\mathbb{F}_q G$. However, as rings not only may not coincide, but in general they are non-isomorphic. More precisely we have the following result.

**Lemma 2.2.** Let $\theta$ and $\beta$ be homomorphisms of $G$ into $\text{Aut}(\mathbb{F}_q)$. There is a $\mathbb{F}_q$-isomorphism $\mathbb{F}_q^\theta G \cong \mathbb{F}_q^\beta G$ mapping $a_g g$ to $a_g \delta(g)$ for some $\delta(g) \in G$ if and only if $\theta(g) = \beta(\delta(g))$ for all $g \in G$.

**Proof.** Denote by $a_g g \cdot b_h h$ the product in $\mathbb{F}_q^\theta G$ and denote $a_g g \ast b_h h$ the product in $\mathbb{F}_q^\beta G$. The image of $a_g g \cdot b_h h = a_g \theta(g)(b_h)gh$ in $\mathbb{F}_q^\theta G$ is $a_g \theta(g)(b_h)\delta(g)$. The product in $\mathbb{F}_q^\beta G$ of the images of $a_g g$ and $b_h h$ is $a_g \beta(\delta(g))(b_h)\delta(g) = a_g \beta(\delta(g))(b_h)\delta(g)$. The two elements coincide if and only if $\theta(g) = \beta(\delta(g))$ for all $g \in G$. 

**Lemma 2.3.** The map $\varphi : \mathbb{F}_q^\theta G \rightarrow \mathbb{F}_q^\beta G$, $\sum_{g \in G} a_g g \mapsto \sum_{g \in G} \theta(g^{-1})(a_g)g^{-1}$, is a ring anti-isomorphism of $\mathbb{F}_q^\theta G$.

**Proof.** Let $a_g g, b_h h \in \mathbb{F}_q^\theta G$. Then we have

$$\varphi(a_g g \cdot b_h h) = \varphi(a_g \theta(g)(b_h)gh) = \theta(h^{-1}g^{-1})(a_g \theta(g)(b_h))h^{-1}g^{-1}$$

$$= \theta(h^{-1})(a_g \theta(g)(b_h))h^{-1}g^{-1} = \theta(h^{-1})(a_g \theta(h^{-1}))(b_h)h^{-1}g^{-1}$$

$$= \theta(h^{-1})(b_h)\theta(h^{-1})\theta(g^{-1})(a_g)h^{-1}g^{-1} = \varphi(b_h g)\varphi(a_g g).$$

**Definition 2.4.** For an element $a = \sum_{g \in G} a_g g \in \mathbb{F}_q^\theta G$ we define its adjunct as

$$\hat{a} := \varphi(a) = \sum_{g \in G} \theta(g^{-1})(a_g)g^{-1}.$$ 

## 3 A skew dihedral group ring

Let $G = D_{2n} = \langle x, y : x^n = y^2 = 1, yxy^{-1} = x^{-1} \rangle$ be a presentation of the dihedral group of order $2n$.

**Lemma 3.1.** Let $C_n = \langle x \rangle$ be the cyclic subgroup of $D_{2n}$ generated by $x$. Then we have

1. $\mathbb{F}_q^\theta D_{2n}$ is a free $\mathbb{F}_q^\theta C_n$-module with basis $\{1, y\}$. Therefore $\mathbb{F}_q^\theta D_{2n} = \mathbb{F}_q^\theta C_n \oplus \mathbb{F}_q^\theta C_n y$ as direct sum of $\mathbb{F}_q$-vector spaces.
2. $\mathbb{F}_q^\theta C_n y \cong \mathbb{F}_q^\theta C_n$ as $\mathbb{F}_q^\theta C_n$-modules.
3. For $a \in \mathbb{F}_q^\theta C_n y, ab \in \mathbb{F}_q^\theta C_n$ if $b \in \mathbb{F}_q^\theta C_n y$ or $ab \in \mathbb{F}_q^\theta C_n y$ if $b \in \mathbb{F}_q^\theta C_n$. 

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4. If \( a \in \mathbb{F}_q^\theta C_n \), then \( \hat{a} \in \mathbb{F}_q^\theta C_n \).

5. If \( a \in \mathbb{F}_q^\theta C_n y \), then \( \hat{a} \in \mathbb{F}_q^\theta C_n y \).

**Proof.** In what follows the symbol \([k]_n\) for \( k \in \mathbb{Z} \) denotes \( k \equiv [k]_n \mod n \).

1. Since \( \{1, y\} \) is a transversal for \( C_n \) in \( D_{2n} \), then \( D_{2n} = C_n \cup C_n y \) and the assertion follows.
2. Since \( x^i \cdot x^j = x^{[i+j]} \) and \( x^i \cdot y^j = x^{[i+j]} y \) for all \( i, j \in \{0, \ldots, n-1\} \), the assertions follow.
3. Since \( x^i y \cdot x^j y = x^{[i-j]} \) and \( x^i y \cdot x^j = x^{[i-j]} y \) for all \( i, j \in \{0, \ldots, n-1\} \), the assertions follow.
4. Since \( x^i \cdot x^{n-i} = 1 \) for all \( i \in \{0, \ldots, n-1\} \), then the assertion follows.
5. Since \( (x^i y)^2 = 1 \) for all \( i \in \{0, \ldots, n-1\} \), then it follows.

**Definition 3.2.**

1. We define the \( \theta \)-reversible subspace of \( \mathbb{F}_q^\theta C_n y \) as the vector subspace
\[
\Gamma_\theta = \{ a = \sum_{i=0}^{n-1} a_i x^i y \in \mathbb{F}_q^\theta C_n y \mid a_i = a_{n-i} \text{ for } i = 1, \ldots, n-1 \}.
\]

2. Given \( a = \sum_{i=0}^{n-1} a_i x^i y \in \mathbb{F}_q^\theta C_n y \) we define \( \Phi(a) = \sum_{i=0}^{n-1} a_i x^i \in \mathbb{F}_q^\theta C_n \).

Note that the map \( \Phi : \mathbb{F}_q^\theta C_n y \to \mathbb{F}_q^\theta C_n \) is an \( \mathbb{F}_q \)-linear isomorphism.

**Lemma 3.3.** Let \( \theta : G = D_{2n} \to \text{Aut}(\mathbb{F}_q) \) be a group homomorphism. If \( \theta(x^i)(a) = a \) for all \( i \in \{0, \ldots, n-1\} \) and for all \( a \in \mathbb{F}_q \), then \( ab = b\hat{a} \) for \( a, b \in \Gamma_\theta \).

**Proof.** Let \( a = \sum_{i=0}^{n-1} a_i x^i y \in \Gamma_\theta \) and \( b = \sum_{j=0}^{n-1} b_j x^j y \in \Gamma_\theta \). Then
\[
a\hat{b} = \sum_{i=0}^{n-1} a_i x^i y \sum_{j=0}^{n-1} \theta(x^j)(b_j) x^j y = \sum_{j=0}^{n-1} \left( \sum_{i=0}^{n-1} a_i \theta(x^j)(b_{[i-j]}_n) \right) x^j
\]
and
\[
b\hat{a} = \sum_{i=0}^{n-1} b_i x^i y \sum_{j=0}^{n-1} \theta(x^j)(a_j) x^j y = \sum_{j=0}^{n-1} \left( \sum_{i=0}^{n-1} b_i \theta(x^j)(a_{[i-j]}_n) \right) x^j.
\]

Since \( a, b \in \Gamma_\theta \), then \( a_s = a_{[s]}_n \) and \( b_{[j-s]}_n = b_{[s-j]}_n \) for \( s \in \{0, \ldots, n-1\} \). Therefore, \( \theta(x^j)(a_{[s]}_n) = a_{[s]}_n \) and \( \theta(x^j)(b_{[s-j]}_n) = b_{[s-j]}_n = b_{[j-s]}_n \), which is equivalent to the \( s \)-th term in \( \sum_{i=0}^{n-1} a_i \theta(x^j)(b_{[i-j]}_n) \) coincides with the \([j-s]_n\)-th term of \( \sum_{i=0}^{n-1} b_i \theta(x^j)(a_{[i-j]}_n) \).

\[\square\]
From now on, we will consider a square extension of \( \mathbb{F}_q \), i.e., the skew group ring to consider will be \( \mathbb{F}_q^\theta D_{2n} \). We have the following result whose proof follows straightforwardly.

**Lemma 3.4.** Let \( \sigma \) be a generator element of the Galois group \( \text{Gal}(\mathbb{F}_{q^2}, \mathbb{F}_q) \). Then the map \( \theta_\sigma: G = D_{2n} \rightarrow \text{Gal}(\mathbb{F}_{q^2}, \mathbb{F}_q) \) defined by \( \theta_\sigma(g) = \sigma \) for \( g = x^i y, i \in \{0, \ldots, n-1\} \) and \( \theta_\sigma(g) = 1 \) otherwise is a group homomorphism.

**Proof.** This assertion can be checked straightforwardly.

\[ \Box \]

### 4 A key exchange protocol

This section presents a key exchange protocol based on two-sided multiplications over a skew dihedral group ring. We remark that other works have considered two-sided semi-group actions or matrices over group rings for key exchange \([8, 13, 14, 15]\). However, we follow an alternative approach. Recently, in \([5]\) the authors have proposed a key exchange protocol using two-sided multiplications over a dihedral twisted group ring. Following their construction, we introduce a similar key exchange protocol over \( \mathbb{F}_{q^2}^\theta D_{2n} \), with \( \theta \) being a suitable group homomorphism.

#### 4.1 The construction

We start by setting up our key exchange protocol’s public parameters.

1. Choose \( m, n \in \mathbb{N} \) and a prime number \( p \) such that \( p \) divides \( n \). We then set \( q = p^m \) and the finite field \( \mathbb{F}_{q^2} \).

2. Choose the map \( \theta_\sigma: D_{2n} \rightarrow \text{Aut}(\mathbb{F}_{q^2}) \) as it was defined in Lemma 3.4. In particular, for \( g = x^i y \) with \( i \in \{0, \ldots, n-1\} \), \( \theta_\sigma(g) = \sigma \), where \( \sigma(a) = a^\theta \) for all \( a \in \mathbb{F}_{q^2} \), and \( \theta_\sigma(g) = 1 \) otherwise.

3. Choose a random non-zero element \( h_1 \in \mathbb{F}_{q^2}^\theta C_n \) and a random non-zero element \( h_2 \in \mathbb{F}_{q^2}^\theta C_n y \). Set \( h = h_1 + h_2 \) and make \( h \) public.

We use the notation introduced in \([3]\). Let \( P_i \) and \( P_j \) be two parties and \( s \) be an identifier for a session. The key exchange protocol between \( P_i \) and \( P_j \) runs as shown by Protocol 1.

Note that if both \( P_i \) and \( P_j \) are uncorrupted during the exchange of the key and both complete the protocol for session-id \( s \), then they both establish the same key. Because of the choice of \( \theta_\sigma \), by Lemma 3.3 and Lemma 3.4

\[ k_i = a_ipk_j \hat{\gamma}_i = a_i a_j h \gamma_j \hat{\gamma}_i = a_j a_i h \gamma_i \hat{\gamma}_j = a_j pk_i \hat{\gamma}_j = k_j \]
Protocol 1 Our Key Exchange Protocol

1: The initiator $P_i$, on input $(P_i, P_j, s)$, chooses a secret pair $(a_i, \gamma_i) \leftarrow F_{\theta}^q C_n \times \Gamma_\theta$, and sends $(P_i, s, pk_i = a_i h^{\gamma_i})$ to $P_j$.

2: Upon receipt of $(P_i, s, pk_i)$, $P_j$ chooses a secret pair $(a_j, \gamma_j) \leftarrow F_{\theta}^q C_n \times \Gamma_\theta$ and sends $(P_j, s, pk_j = a_j h^{\gamma_j})$ to $P_i$, computes $k_j = a_j pk_i^{\gamma_j}$, erases $(a_j, \gamma_j)$ and outputs the key $k_j$ under the session-id $s$.

3: Upon receipt of $(P_j, s, pk_j)$, $P_i$, computes $k_i = a_i pk_j^{\gamma_i}$, erases $(a_i, \gamma_i)$ and outputs the key $k_i$ under the session-id $s$.

4.2 Intractability assumptions

With the notation above, let $h = h_1 + h_2$ be a public element in $F_{\theta}^q D_{2n}$, where $h_1$ is a random non-zero element from $F_{\theta}^q C_n$ and $h_2$ is a random non-zero element from $F_{\theta}^q C_n y$.

We now present attack games \cite{2, 18} for algebraic problems related to the security of our key exchange protocol.

**Game 4.1** (Skew Dihedral Product Decomposition). For a given adversary $A$, we define the following attack game:

- The challenger computes
  1: $(a, \gamma) \leftarrow F_{\theta}^q C_n \times \Gamma_\theta$;
  2: $pk \leftarrow ah^{\gamma}$;

  and gives the value of $pk$ to the adversary.

- The adversary outputs $(\tilde{a}, \tilde{\gamma}) \in F_{\theta}^q C_n \times \Gamma_\theta$.

We define $A$’s advantage in solving the Skew Dihedral Product Decomposition Problem for $F_{\theta}^q D_{2n}$, denoted $SDPDadv[A, F_{\theta}^q D_{2n}]$, as the probability that $\tilde{a}h^{\tilde{\gamma}} = ah^{\gamma}$.

**Definition 4.2** (Skew Dihedral Product Decomposition Assumption). We say that the Skew Dihedral Product Decomposition (SDPD) assumption holds for $F_{\theta}^q D_{2n}$ if for all efficient adversaries $A$ the quantity $SDPDadv[A, F_{\theta}^q D_{2n}]$ is negligible.

**Game 4.3** (Computational Skew Dihedral Product). For a given adversary $A$, we define the following attack game:

- The challenger computes
  1: $(a_1, \gamma_1) \leftarrow F_{\theta}^q C_n \times \Gamma_\theta$;
  2: $(a_2, \gamma_2) \leftarrow F_{\theta}^q C_n \times \Gamma_\theta$;
  3: $pk_1 \leftarrow a_1 h^{\gamma_1}$;
  4: $pk_2 \leftarrow a_2 h^{\gamma_2}$;
  5: $k \leftarrow a_2 pk_1^{\gamma_2}$;

  and gives the values of $pk_1$ and $pk_2$ to the adversary.
The adversary outputs some \( \tilde{k} \in \mathbb{F}_{q^2} D_{2n} \)

We define \( A \)'s advantage in solving the Computational Skew Dihedral Product (CSDP) Problem for \( \mathbb{F}_{q^2} D_{2n} \), denoted \( \text{CSDPAdv}[A, \mathbb{F}_{q^2} D_{2n}] \), as the probability that \( \tilde{k} = k \).

**Definition 4.4** (Computational Skew Dihedral Product Assumption). We say that the Computational Skew Dihedral Product (CSDP) assumption holds for \( \mathbb{F}_{q^2} D_{2n} \) if for all efficient adversaries \( A \) the quantity \( \text{CSDPAdv}[A, \mathbb{F}_{q^2} D_{2n}] \) is negligible.

**Lemma 4.5.** If the SDPD assumption does not hold for \( \mathbb{F}_{q^2} D_{2n} \), then CSDP assumption does not hold for \( \mathbb{F}_{q^2} D_{2n} \).

**Proof.** This assertion can be checked straightforwardly. \( \square \)

**Game 4.6** (Decisional Skew Dihedral Product). For a given adversary \( A \), we define two experiments:

**Experiment b**

- The challenger computes
  1. \((a_1, \gamma_1) \leftarrow R \mathbb{F}_{q^2} C_n \times \Gamma_{\theta};\)
  2. \((a_2, \gamma_2) \leftarrow R \mathbb{F}_{q^2} C_n \times \Gamma_{\theta};\)
  3. \((a_3, \gamma_3) \leftarrow R \mathbb{F}_{q^2} C_n \times \Gamma_{\theta};\)
  4. \(pk_1 \leftarrow a_1 h \gamma_1; \quad pk_2 \leftarrow a_2 h \gamma_2;\)
  5. \(k_0 \leftarrow a_2 pk_1 \hat{\gamma}_2; \quad k_1 \leftarrow a_3 h \gamma_3;\)

  and gives the triple \((pk_1, pk_2, k_b)\) to the adversary.

- The adversary outputs a bit \( \tilde{b} \in \{0, 1\} \)

Let \( W_b \) is the event that \( A \) outputs 1 in experiment \( b \). We define \( A \)'s advantage in solving the Decisional Skew Dihedral Product Problem for \( \mathbb{F}_{q^2} D_{2n} \) as

\[
\text{DSDPAdv}[A, \mathbb{F}_{q^2} D_{2n}] = |\Pr[W_0] - \Pr[W_1]|.
\]

**Definition 4.7** (Decisional Skew Dihedral Product Assumption). We say that the Decisional Skew Dihedral Product (DSDP) assumption holds for \( \mathbb{F}_{q^2} D_{2n} \) if for all efficient adversaries \( A \) the quantity \( \text{DSDPAdv}[A, \mathbb{F}_{q^2} D_{2n}] \) is negligible.

Note that \( h \) is chosen as \( h = h_1 + h_2 \), with \( h_1 \) being a random non-zero element from \( \mathbb{F}_{q^2} C_n \) and \( h_2 \) being a random non-zero element from \( \mathbb{F}_{q^2} C_n y \), to not let the attacker win the DSDP Game trivially. Indeed if \( h \) is chosen as \( h = h_1 + 0 \) with \( h_1 \in \mathbb{F}_{q^2} C_n \), then \( k_0 \in \mathbb{F}_{q^2} C_n \) and \( k_1 \in \mathbb{F}_{q^2} C_n y \) by Lemma 3.1. Similarly if \( h \) is chosen as \( h = 0 + h_2 \) with \( h_2 \in \mathbb{F}_{q^2} C_n y \), then \( k_0 \in \mathbb{F}_{q^2} C_n y \) and \( k_1 \in \mathbb{F}_{q^2} C_n \) by Lemma 3.1. Therefore the attacker can win the DSDP Game for both cases with non-negligible probability. Additionally, we have the following.
Lemma 4.8. If the CSDP assumption does not holds for \( \mathbb{F}_{q^2} D_{2n} \), then DSDP assumption does not holds for \( \mathbb{F}_{q^2} D_{2n} \).

Proof. This assertion can be checked straightforwardly. \(\Box\)

4.3 The Hardness of the SDPD Problem

The authors of [5] provide an algorithmic and algebraic analysis on the Dihedral Product Decomposition (DPD) Problem, which is the underlying problem associated with the security of their constructions. In particular, let \( \mathbb{F}_q^\alpha D_{2n} \) be a twisted dihedral group algebra, where the 2-cocycle \( \alpha : D_{2n} \times D_{2n} \to \mathbb{F}_q^* \) is defined by \( \alpha(g, h) = \lambda \) (a non-square in \( \mathbb{F}_q \)) for \( g = x^i y, h = x^j y \) with \( i, j \in \{0, \ldots, n-1\} \) and \( \alpha(g, h) = 1 \) otherwise. In [5], the authors demonstrate that \( \mathbb{F}_q^\alpha D_{2n} = \mathbb{F}_q^\alpha C_n \oplus \mathbb{F}_q^\alpha C_n y \) as direct sum of \( \mathbb{F} \)-vector spaces, and also define \( \Gamma_{\alpha} \subseteq \mathbb{F}_q^\alpha C_n y \) in a similar way.

The DPD attack game is defined as follows. Let \( h = h_1 + h_2 \) be a public element in \( \mathbb{F}_q^\alpha D_{2n} \), where \( h_1 \) is a random non-zero element from \( \mathbb{F}_q^\alpha C_n \) and \( h_2 \) is a random non-zero element from \( \mathbb{F}_q^\alpha C_n y \). For a given adversary \( A \),

- The challenger computes
  1: \((a, \gamma) \leftarrow \mathcal{R} \mathbb{F}_q^\alpha C_n \times \Gamma_{\alpha} ; \)
  2: \( pk \leftarrow ah\gamma ; \)

  and gives the value of \( pk \) to the adversary.

- The adversary outputs \((\bar{a}, \bar{\gamma}) \in \mathbb{F}_q^\alpha C_n \times \Gamma_{\alpha} \).

The \( A \)'s advantage in solving the Dihedral Product Decomposition Problem for \( \mathbb{F}_q^\alpha D_{2n} \) is defined as the probability that \( \bar{a}h\bar{\gamma} = ah\gamma \).

The authors of [5] analyze how an adversary, with access to a quantum computer, may leverage it to try to solve the DPD problem by exploiting quantum algorithms (e.g., Grover’s algorithm and Shor’s algorithm) [19]. Also, they analyze possible algebraic attacks on DPD problems and hence propose choosing their constructions’ public parameters to avoid that an adversary may leverage those algebraic techniques, such as [17], to solve the DPD problem. We remark that since the DPD problem and SDPD problem are very similar, such an algebraic and algorithmic analysis for the DPD problem presented in [5] may be adapted easily to the SDPD problem. However, we note that adjusting such an analysis to the SDPD problem does not mean that both DPD and SDPD problems are computationally equivalent. It is indeed an open question to prove whether these problems are computationally equivalent or not.

4.4 Security analysis in the authenticated-links adversarial model

This subsection is devoted to analysing further our key exchange protocol in a appropriate security model [12, 3]. In particular, we aim at proving that our protocol is session-key secure in the authenticated-links adversarial model (AM) of Canetti and Krawczyk [3], assuming the DSDP assumption holds for \( \mathbb{F}_q^\alpha \). We first recall the definition of session-key
security in the authenticated-links adversarial model of Canetti and Krawczyk [3], and
follow the description given in [5].

1. Let \( P = \{P_1, P_2, \ldots, P_n\} \) be a finite set of parties.

2. Let \( \mathcal{A} \) be an adversary that controls all communication between two parties, however

- \( \mathcal{A} \) is not allowed to inject or modify messages, except for messages sent by corrupted parties or sessions.
- \( \mathcal{A} \) may choose not to forward a message at all, but if \( \mathcal{A} \) chooses to forward a message \( m \), \( \mathcal{A} \) has to send it to the correct destination for \( m \), only once and without modifying \( m \).
- Parties give outgoing messages to \( \mathcal{A} \), who has control over their delivery via the \textbf{Send} query. \( \mathcal{A} \) can activate a party \( P_i \) by \text{Send} queries, i.e. the adversary has control over the creation of protocol sessions, which take place within each party. Two sessions \( s_1 \) and \( s_0 \) are matching if the outgoing messages of one are the incoming messages of the other, and vice versa. Additionally, \( \mathcal{A} \) is allowed to query the oracles \textbf{SessionStateReveal}, \textbf{SessionKeyReveal}, and \textbf{Corrupt}.
  - If \( \mathcal{A} \) query the \textbf{SessionStateReveal} oracle for a specified session-id \( s \) within some party \( P_i \), then \( \mathcal{A} \) obtains the contents of the specified session-id \( s \) within \( P_i \), including any secret information. This event is noted and hence produces no further output.
  - If \( \mathcal{A} \) query the \textbf{SessionKeyReveal} for a specified session-id \( s \), then \( \mathcal{A} \) obtains the session key for the specified session \( s \), assuming that \( s \) has an associated session.
  - If \( \mathcal{A} \) query the \textbf{Corrupt} oracle for a specified party \( P_i \), then \( \mathcal{A} \) takes over the party \( P_i \), i.e. \( \mathcal{A} \) has access to all information in \( P_i \)’s memory, including long-lived keys and any session-specific information still stored. A corrupted party produces no further output.
- Finally \( \mathcal{A} \) is given access to the \textbf{test} oracle, which can be queried once and at any stage to a completed, fresh, unexpired session-id \( s \). On input \( s \), the \textbf{test} oracle chooses \( b \leftarrow_R \{0, 1\} \), then it outputs the session key for the specified session-id \( s \) if \( b = 0 \). Otherwise, it returns a random value in the key space. Also, \( \mathcal{A} \) can issue subsequent queries as desired, with the exception that it cannot expose the test session. At any point, the adversary can try to guess \( b \). Let \( \text{Guess}[\mathcal{A}, \mathbb{F}_{q^2}^{\theta}D_{2n}] \) be the event that \( \mathcal{A} \) correctly guesses \( b \), and define the advantage \( \text{SKAdv}[\mathcal{A}, \mathbb{F}_{q^2}^{\theta}D_{2n}] = |\text{Guess}[\mathcal{A}, \mathbb{F}_{q^2}^{\theta}D_{2n}] - 1/2| \).

**Theorem 4.9.** If the DSDP assumption holds for \( \mathbb{F}_{q^2}^{\theta}D_{2n} \), then our key exchange protocol is session-key secure in the the authenticated-links adversarial model, i.e. for any \( \mathcal{A} \) in the authenticated-links adversarial model (AM), then the following holds

1. The key-exchange protocol satisfies the property that if two uncorrupted parties complete matching sessions, then they both output the same key.
2. $\text{SKAdv}[\mathcal{A}, \mathbb{F}_q^D D_{2n}]$ is negligible.

Proof. The proof of this theorem is an adaptation of the proof, given in [5], for the key-exchange protocol over a twisted dihedral group algebra $\mathbb{F}_q^D D_{2n}$.

1. The proof of the first statement is given at the end of the Subsection 4.1.

2. To prove this statement, we proceed by contradiction. Let us suppose that there is an adversary $\mathcal{A}$ in the authentication-links model against our protocol that has a non-negligible advantage $\epsilon$ in guessing the bit $b$ chosen by the test oracle (when queried). Let $l$ be an upper bound on the number of sessions invoked by $\mathcal{A}$ in any interaction. We now present a distinguisher $\mathcal{D}$ for the DSDP problem.

1: Function $\mathcal{D}(h, \mathbb{F}_q^D D_{2n}, \text{pk}_1, \text{pk}_2, k)$
2: $r \leftarrow \{1, \ldots, l\}$;
3: Invoke $\mathcal{A}$ on a simulated interaction in the AM with parties $P_1, \ldots, P_n$, except for the $r$th session;
4: For the $r$-th session, let $P_i$ send $(P_i, s, \text{pk}_i = a_i h \gamma_i) \rightarrow P_j$ and let $P_j$ send $(P_j, s, \text{pk}_j = a_j h \gamma_j) \rightarrow P_i$;
5: if the $r$-th session is selected by $\mathcal{A}$ as the test session then
6: \hspace{1em} Give $k$ to $\mathcal{A}$ as the answer to his query;
7: \hspace{1em} $d \leftarrow \mathcal{A}(k)$;
8: else
9: \hspace{1em} $d \leftarrow \{0, 1\}$;
10: end if
11: return $d$
12: end Function

On the one hand, let us suppose that $\mathcal{A}$ picks the $r$-th as the test session, then $\mathcal{A}$ is provided with either $k_0$ or $k_1$, since the DSDP challenger gives either of the two keys to $\mathcal{D}$. Therefore, the probability that $\mathcal{A}$ correctly distinguishes is $1/2 + \epsilon$ with non-negligible $\epsilon$ (by assumption). On the other hand, assume that $\mathcal{A}$ does not choose the $r$-th as the test session, then $\mathcal{D}$ always returns a random bit, and hence the distinguishing probability for the input is $1/2$.

Note that the probability that the test session and the $r$-th session coincide is $1/l$. So these do not coincide with probability $1 - 1/l$. Hence the overall probability for $\mathcal{D}$ to win the DSDP Game is $1/(2l) + \epsilon/l + 1/2 - 1/(2l) = 1/2 + \epsilon/l$, which is non-negligible.

\[\square\]

5 Probabilistic Public Key Encryption

We now present a probabilistic public key encryption based on the key exchange protocol introduced in Section 4. Following the notation above, choose a random non-zero element
h₁ ∈ ℤ^θ_{q^2}C_n and a random non-zero element h₂ ∈ ℤ^θ_{q^2}C_n. Set h = h₁ + h₂ and make h public.

Let SK = ℤ^θ_{q^2}C_n × Γ_θ be the secret key space, PK = ℤ^θ_{q^2}D_{2n} be the public key space, M = ℤ^θ_{q^2}D_{2n} be the message space, and C = ℤ^θ_{q^2}D_{2n} the cipher-text space. We now define the public key encryption scheme E = (Gen, Enc, Dec).

Algorithm 2 Key Generation Algorithm
1: Function Gen(h ∈ ℤ^θ_{q^2}D_{2n})
2: (a₁, γ₁) ← SK;
3: pk ← a₁h₁γ₁;
4: sk ← (a₁, γ₁);
5: return pk, sk;
6: end Function

Algorithm 3 Encryption Algorithm
1: Function Enc(m ∈ M, pk ∈ PK, r₂ ∈ SK, h ∈ ℤ^θ_{q^2}D_{2n})
2: (a₂, γ₂) ← r₂;
3: c₁ ← a₂h₂γ₂;
4: c₂ ← m + a₂pkγ₂;
5: c ← (c₁, c₂);
6: return c;
7: end Function

Algorithm 4 Decryption Algorithm
1: Function Dec(c ∈ C, sk ∈ SK)
2: (a₁, γ₁) ← sk;
3: (c₁, c₂) ← c;
4: k ← a₁c₁γ₁;
5: m ← c₂ - k;
6: return m;
7: end Function

Lemma 5.1 (Correctness). Let h be a public element in ℤ^θ_{q^2}D_{2n}. Consider the encryption scheme E constructed above. For any message m ∈ M, r₂ ← SK and (pk, sk) ← Gen(h), it holds that m ← Dec(Enc(m, pk, r₂, h), sk)

Proof. Since
(c₁ = a₂h₂γ₂, c₂ = m + a₂pkγ₂) ← Enc(m, pk, r₂, h)
and sk = (a₁, γ₁), then
k = a₁c₁γ₁ = a₁a₂h₂γ₁ = a₂a₁h₁γ₂ = a₂pkγ₂,
and therefore
\[ c_2 - k = m + a_2 p k \hat{\gamma}_2 - a_2 p k \hat{\gamma}_2 = m \]

\[ 1 \]

**Theorem 5.2.** If the DSDP assumption holds for $E_{q^2} D_{2n}$, then $E$ is semantically secure.

*Proof.* The proof of Theorem 5.2 in [5] can be easily adapted to this setting. \qed

### 6 A Key Encapsulation Mechanism

By applying a generic transformation of Hofheinz, Hövelmanns, and Kiltz [11] to $E$, we introduce a CCA-secure key encapsulation mechanism. Let $K = \{0, 1\}^l$ be the key space and $\text{rep}(x)$ be a function that simply returns the binary representation of $x$. Additionally, we construct the following two functions:

- $\mathcal{H}_1 : \{0, 1\}^* \rightarrow \mathcal{SK}$ is a hash function that takes in a bit-string, say $x$, and then uses cryptographic hash function, e.g. SHAKE256, to compute a key in the keyspace from it. Following the notation of [7], $\mathcal{H}_1(x) = \text{SHAKE256}(x, o)$, where $o = \lceil \log_2(p) \rceil 2m(n + \lceil \frac{n+1}{2} \rceil)$ is the bit length of the output. From this bit-string, the corresponding pair $(a, \gamma) \in \mathcal{SK}$ can be obtained easily.

- $\mathcal{H}_2 : \{0, 1\}^* \rightarrow \mathcal{K}$ is a hash function that applies a cryptographic hash function, e.g. SHAKE256, to the input. Specifically $\mathcal{H}_2(x) = \text{SHAKE256}(p_1 || x, l_1)$, where $p_1$ is a prepended fixed bit-string to make it different from $\mathcal{H}_1$.

Applying the generic transformation $U^L[T[E, \mathcal{H}_2], \mathcal{H}_1]$ from [11], we get KEM = (KeyGen, Encaps, Decaps).

**Algorithm 5** Key Generation Algorithm

1: Function **KeyGen**(h)
2: \( (pk, sk) \leftarrow \text{Gen}(h); \)
3: \( s \leftarrow M; \)
4: return \((s, sk, pk); \)
5: end Function

**Algorithm 6** Encapsulation Algorithm

1: Function **Encaps**(pk, h)
2: \( m \leftarrow M; \)
3: \( r \leftarrow \mathcal{H}_1(\text{rep}(m)||\text{rep}(pk)); \)
4: \( c \leftarrow \text{Enc}(m, pk, r, h); \)
5: \( K \leftarrow \mathcal{H}_2(\text{rep}(m)||\text{rep}(c)); \)
6: return \((c, K); \)
7: end Function
Algorithm 7 Decapsulation Algorithm

1: Function Decaps((s, pk, sk), c, h)
2: m ← Dec(c, sk);
3: r ← \( H_1(\text{rep}(m)||\text{rep}(pk)) \);
4: if c = Enc(m, pk, r, h) then
5: K ← \( H_2(\text{rep}(m)||\text{rep}(c)) \);
6: return K;
7: else
8: return \( H_2(\text{rep}(s)||\text{rep}(c)) \);
9: end if
10: end Function

7 Implementation of our cryptographic constructions

We implemented our proposed public-key encryption scheme and key encapsulation mechanism as a proof-of-concept in Python. The interested reader can see it on Google Colaboratory [6].

7.1 Dihedral Group

To implement a dihedral group of order \( 2n \), we simply represent a dihedral group element \( g = x^{i_1}y^{j_1} \) as the integer \( k_1 = j_1 \cdot n + i_1 \). Also, we compute a \( 2n \times 2n \) integer array \( \text{table} \) such that the row \( \text{table}[k_1] \), \( 0 \leq k_1 < 2n \), stores a \( 2n \) array with the integer representations of \( g, gx, gx^2, \ldots, gx^{n-1}, gy, \ldots, gx^{n-1}y \). To compute the operation of two given group elements \( g = x^{i_1}y^{j_1} \) and \( h = x^{i_2}y^{j_2} \), we simply return \( \text{table}[k_1][k_2] \), where \( k_1 = j_1 \cdot n + i_1 \) and \( k_2 = j_2 \cdot n + i_2 \). To compute the multiplicative inverse of a given group element \( g = x^{i_1}y^{j_1} \), the function \( \text{inverse}(k_1) \) returns 0 if \( k_1 = 0 \), or \( n - k_1 \) if \( 1 \leq k_1 < n \), or \( k_1 \) if \( n \leq k_1 < 2n \).

7.2 Homomorphism \( \theta \)

The homomorphism \( \theta \) is implemented as described next. Given \( k_1 \) and \( k_2 \), two representations of two group elements, then the function \( \text{homomorphism}(k_1, k_2) \) returns a pointer to the function \( \sigma \) if \( n \leq k_1 < 2n \) and \( n \leq k_2 < 2n \). Otherwise, it returns a pointer to the function identity \( I \). We will next describe the inner working of each function.

1: Function \( \sigma(a \in \mathbb{F}_q^*) \)
2: \( [b_s, b_{s-1}, \ldots, b_0] \leftarrow \text{getBinaryReprepresation}(q) \);
3: \( r \leftarrow \text{getOneFromQuadraticField}() \);
4: for \( i \leftarrow s \) to 0 do
5: \( r \leftarrow r \cdot r \)
6: if \( b_i = 1 \) then
7: \( r \leftarrow r \cdot a \)
8: end if
9: end for
easily implemented as shown next.

A field element array of $2n$ field elements $a = [a_0, a_1, a_2, \ldots, a_{2n-1}]$, where $a_i$ is the representation of the field element $a_i \in \mathbb{F}_q^D$. Therefore the addition and product of two elements of this ring is easily implemented as shown next.

### 7.2.1 The skew dihedral group ring $\mathbb{F}_q^D D_{2n}$

An element $a = \sum_{i=0}^{n-1} a_i x^i + \sum_{i=0}^{n-1} a_{n+i} x^i y$ in the group ring $\mathbb{F}_q^D D_{2n}$ is represented as an array of $2n$ field elements $a = [a_0, a_1, a_2, \ldots, a_{2n-1}]$, where $a_i$ is the representation of the field element $a_i \in \mathbb{F}_q^D$. Therefore the addition and product of two elements of this ring is easily implemented as shown next.

```plaintext
1: Function ADDITION(a, b)
2:    c ← [0, \ldots, 0]
3:    for (i ← 0; i < 2n; i ← i + 1) do
4:        c[i] ← a[i] + b[i];
5:    end for
6:    return c
7: end Function
```

```plaintext
1: Function PRODUCT(a, b)
2:    c ← [0, \ldots, 0]
3:    for (i ← 0; i < 2n; i ← i + 1) do
4:        for (j ← 0; j < 2n; j ← j + 1) do
5:            k ← table[i, j];
6:            fe ← a[i] \cdot (\text{homomorphism}(i)(b[j]));
7:            c[k] ← c[k] + fe;
8:        end for
9:    end for
10: return c
11: end Function
```

On the one hand, the addition function has a cost of $2n$ field additions to compute an ring element $c$. On the other hand, the product function has a cost of $4n^2$ field additions and $4n^2 \cdot (1 + f)$ field multiplications, where $f$ is the number of field multiplication to compute $\text{homomorphism}(i)(b[j])$. In addition to these functions, we implement the function $\text{adjunct}$, which computes the adjunct of a ring element and its cost is $2n \cdot f$ multiplications. Also, functions for computing a random element in $\Gamma_\theta$ ($\mathbb{F}_q^D D_{2n}$, $\mathbb{F}_q^D C_n$ and $\mathbb{F}_q^D C_n y$) are described next.

```plaintext
1: Function ADJUNCT(a)
2:    c ← [0, \ldots, 0]
3:    for (i ← 0; i < 2n; i ← i + 1) do
4:        j ← inverse(i)
5:        c[j] ← \text{homomorphism}(j)(a[i])
6:    end for
7:    return c
8: end Function
```

```plaintext
1: Function GETRANDOMFRONT()
2:    c ← [0, \ldots, 0]
3:    c[n] ← \text{getRandomFieldElement()}
4:    n_1 ← n/2
5:    for (i ← 1; i ≤ n_1; i ← i + 1) do
6:        c[i+n] ← \text{getRandomFieldElement()}
7:        c[n + (n - i) \mod n] ← c[i + n]
8:    end for
9:    return c
10: end Function
```

On the one hand, the addition function has a cost of $2n$ field additions to compute an ring element $c$. On the other hand, the product function has a cost of $4n^2$ field additions and $4n^2 \cdot (1 + f)$ field multiplications, where $f$ is the number of field multiplication to compute $\text{homomorphism}(i)(b[j])$. In addition to these functions, we implement the function $\text{adjunct}$, which computes the adjunct of a ring element and its cost is $2n \cdot f$ multiplications. Also, functions for computing a random element in $\Gamma_\theta$ ($\mathbb{F}_q^D D_{2n}$, $\mathbb{F}_q^D C_n$ and $\mathbb{F}_q^D C_n y$) are described next.
We also implement the following function to compute a random public element $h$.

1: Function getRandomFD2n()
2: \( c \leftarrow [0, \ldots, 0] \)
3: for \((i \leftarrow 0; i < 2n; i \leftarrow i + 1)\) do
4: \( c[i] \leftarrow \text{getRandomFieldElement()} \)
5: end for
6: return \( c \)
7: end Function

1: Function getRandomFCn()
2: \( c \leftarrow [0, \ldots, 0] \)
3: for \((i \leftarrow 0; i < n; i \leftarrow i + 1)\) do
4: \( c[i] \leftarrow \text{getRandomFieldElement()} \)
5: end for
6: return \( c \)
7: end Function

1: Function getRandomFCny()
2: \( c \leftarrow [0, \ldots, 0] \)
3: for \((i \leftarrow n; i < 2n; i \leftarrow i + 1)\) do
4: \( c[i] \leftarrow \text{getRandomFieldElement()} \)
5: end for
6: return \( c \)
7: end Function

7.2.2 Parameters choice

For our KEM, we propose to use the parameters shown by Table 1, which provide varying degrees of security.
Table 1: Proposed parameters

|   |   |   |    |    |
|---|---|---|----|----|
|   |   |   |    |    |
| 19| 1 | 19| {128, 192, 256} | 124 |
| 23| 1 | 23| {128, 192, 256} | 149 |
| 31| 1 | 31| {128, 192, 256} | 200 |
| 41| 1 | 41| {128, 192, 256} | 264 |

Table 1 shows four sets of parameters providing various degrees of security, where \( l_1 \in \{128, 192, 256\} \) refers to the length of the output key. The values in the level of security column were calculated as proposed in [5]. To see the code of our implementation, please see [6].

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