Distribution of the Number of Generations in Flux Compactifications

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Flux compactification of string theory generates an ensemble with a large number of vacua called the landscape. By using the statistics of various properties of low-energy effective theories in the string landscape, one can therefore hope to provide a scientific foundation to the notion of naturalness. This article discusses how to answer such questions of practical interest by using flux compactification of F-theory. It is found that the distribution is approximately in a factorized form given by the distribution of the choice of 7-brane gauge group, that of the number of generations \( N_{\text{gen}} \) and that of effective coupling constants. The distribution of \( N_{\text{gen}} \) is approximately Gaussian for the range \( |N_{\text{gen}}| \lesssim 10 \). The statistical cost of higher-rank gauge groups is also discussed.

1. Introduction:
String theory with compactified extra dimensions gives rise to a large number of vacua. The diversity of vacua originates from the choice of topology of compact internal space and flux configurations on it \cite{1}. String theory as understood in this way today therefore does not predict a unique low-energy effective theory. Despite this lack of prediction, there are still many ways in which we can take advantage of such an ensemble of string vacua for a better understanding of particle physics in the real world. Such an ensemble containing a large number of vacua provided by a fundamental theory is referred to as a landscape of vacua, or landscape for short.

At least two ideas have been proposed so far in how to take advantage of such landscape of string theory. One is in the context of understanding the non-vanishing (yet extremely small) dark energy. It has been realized that the very small value was predicted to be “natural” under the combination of three ansatzes, i) that the universe is occupied with many distinct areas facilitating effective theories with different values of the cosmological constant \cite{2}, ii) that the statistical distribution of dark energy is taken as a principle of arguing naturalness, and iii) that the observational factor (anthropics) is taken into account in the statistical distribution \cite{3}. The string landscape indeed give rise to such an ensemble of low-energy effective theories with different values of the cosmological constant \cite{7}, and eternal inflation has this ensemble of string vacua realized in the universe \cite{9}. Thus the string landscape provides a theoretical foundation for the attempt of understanding dark energy along the lines of i) ii) and iii).

The string landscape can also provide a scientific foundation for a notion of naturalness for various kinds of parameters of the standard model, not just for the value of dark energy. Naturalness has been exploited for decades as a guiding principle in the quest of models beyond the standard model. Arguments relying on naturalness, however, tend to depend on the class of vacua (an ensemble) one has in mind. Since string theory is able to provide a well-motivated ensemble of vacua on which naturalness arguments can be built, this is hence another place where string theory can contribute to progress of theoretical particle physics.

This article aims at making progress in the second direction above. It is known that flux compactification of Type IIB string theory / F-theory stabilizes not only complex structure moduli but also the brane configuration. This means that both gauge groups and coupling constants of the effective theory are determined once a topological flux configuration is given. Exploiting all of the theoretically possible topological flux configurations, an ensemble of low-energy effective theories with various gauge groups and coupling constants is generated in this framework, in principle. To get this done in practice, however, a clever approach is necessary. Low-energy effective theories are usually classified in terms of their algebraic information (such as gauge groups, matter representations and presence/absence of certain types of interactions) first, and then in terms of topological information (such as the number of generations of matter fields in a given representation). Effective theories with the same algebraic and topological information are then specified by the values of coupling constants. The string landscape will be of some use only when the statistical distribution of low-energy observables are presented and studied in compatibility with such a hierarchical classification of effective theories.

This can be done along the line described in \cite{12}, which is built on top of pioneering works \cite{8,10}. The study of \cite{12}, however, used K3 \( \times K3 \) compactification of F-theory, where analysis is a little easier, but we cannot even hope to obtain a semi-realistic model of low-energy physics. This article applies the method to more general Calabi–Yau fourfolds for F-theory compactification, derives answers to questions of practical interest, and exemplifies the potential power of the method.

Section 2 is devoted to a review of the method in \cite{8,10,12} along with new observations in \cite{13}. The method is then applied to a class of compactifications that lead to semi-realistic supersymmetric grand unification (GUT) models in section 3. We derive distribution of the number of generations in flux vacua of SU(5) GUT models, and also study how the number of flux vacua
scales when we require \( \text{SO}(10), \text{SU}(5) \) or no unification group on 7-branes, respectively. We find that the distribution is in a factorized form to a good approximation, independently of topological choice of geometry. Details and explanations omitted in this article are found in [13].

2. The Method:

A family of geometries over a restricted moduli space, \( \pi : \mathcal{Y} \rightarrow \mathcal{M}_* \), is a useful concept when one wants to focus on flux vacua with a given algebraic structure. A restricted family and its moduli space, \( \pi : \mathcal{Y} \rightarrow \mathcal{M}_*^{44} \) (resp. \( \mathcal{M}_*^{D5} \)), is specified for a topological choice of \( (B_3, [S]) \), where \([S]\) is a divisor class in \( B_3\). Each member of the family, \( \pi^{-1}(p) = Y_p \) for \( p \in \mathcal{M}_*^{44} \) (resp. \( \mathcal{M}_*^{D5} \)), is a smooth elliptically fibred Calabi–Yau fourfold \( \pi_Y : Y_p \rightarrow B_3 \) with a section, and the discriminant locus of the fibration \( \pi_Y \) contains an irreducible component in \([S]\) and the generic fibre over it is \( I_2^* \) (resp. \( I_1^* \)) in the Kodaira classification. Any one of such fourfolds in the family over \( \mathcal{M}_*^{44} \) (resp. \( \mathcal{M}_*^{D5} \)) can be used for F-theory compactification that results in a vacuum with an \( R = \text{SU}(5) \) (resp. \( R = \text{SO}(10) \)) unification group on 7-branes. \( \mathcal{M}_*^{44} \) or \( \mathcal{M}_*^{D5} \) parametrizes the complex structure of such geometries.\(^1\) Higher rank 7-brane gauge group implies a larger number of independent divisors

\[
[C_i] \in [H^2(Y_p; \mathbb{Z}) \otimes H^{1,1}(Y_p; \mathbb{R})], \quad i = 1, \ldots, \text{rank}(R),
\]

for a generic geometry \( \pi^{-1}(p) = Y_p \) in \( p \in \mathcal{M}_* \). An ensemble of F-theory flux vacua with a given algebraic and topological information is specified by a pair \( (H_{\text{scan}}, G_{\text{fix}}^{(4)}) \), where

\[
H_{\text{scan}} \subset [H^4(Y; \mathbb{Z})]_{\text{ker}}, \quad G_{\text{fix}}^{(4)} \in [H^4(Y; \mathbb{Z})]^\text{prim};
\]

the subscript “prim” implies \( J \wedge G = 0 \in H^6(Y; \mathbb{R}) \) (the D-term condition), and “ker” both \( J \wedge G = 0 \) and \( i_C^* (G) = 0 \in H^4(C; \mathbb{Q}) \). The last condition is to make sure that fluxes in \( H_{\text{scan}} \) do not introduce gauge symmetry breaking (cf [13]). An ensemble of 4-form flux

\[
\{G_{\text{tot}}^{(4)} = G_{\text{scan}}^{(4)} + G_{\text{fix}}^{(4)} | G_{\text{scan}}^{(4)} \in H_{\text{scan}}\}\)

determines an ensemble of vacua of complex structure through the superpotential \( W \propto \int_{Y_p} \Omega_{Y_p} \wedge G_{\text{tot}}^{(4)} \).

Statistics of such an ensemble of vacua can be presented as a distribution over the restricted moduli space \( \mathcal{M}_* \). The distribution was worked out analytically in a very robust way [8] for the vacuum index density \( d\mu_I \),

\[
d^{2m}z \sum_{G_{\text{scan}}^{(4)}_{\text{tot}}} \delta^{2m}(DW, \bar{DW}) \det \left[ \begin{array}{ccc} D^2W & \bar{DDW} \\ \bar{DDW} & D^2W \end{array} \right],
\]

to which each flux vacuum on \( \mathcal{M}_* \) contributes by a delta-function with coefficient \( \pm 1 \). Here, \( m := \dim \mathcal{M}_* = h^3(Y_\mathbb{C}) \), and the \( dz \)'s are local holomorphic coordinates on \( \mathcal{M} \). Derivatives in \( DW, \bar{DW} \) etc. are with respect to the fields corresponding to the complex structure moduli tangent to \( \mathcal{M}_* \).

Making a continuous approximation [8] of the sum over flux configurations \( \sum G_{\text{scan}}^{(4)} \) in (4), the vacuum index density is cast into the following form

\[
d\mu_I = \frac{(2\pi L_s)^{K/2}}{(K/2)!} \rho_I, \quad K := \dim (H_{\text{scan}} \otimes \mathbb{R}).
\]

Here, \( L_s \) is the maximal D3-brane charge available. Although \( \rho_I \) depends on the choice of \( H_{\text{scan}} \), it is given by

\[
c_m(T, \mathcal{M}_* \otimes \mathcal{L}) = \det \left( -\frac{R}{2\pi i} + \frac{\omega}{2\pi} 1_{m \times m} \right)
\]

for the Kähler form \( \omega \) on \( \mathcal{M}_* \) whenever \( (H_{\text{scan}} \otimes \mathbb{R}) \) contains the real primary horizontal subspace \( H_{H_4}^4(Y; \mathbb{R}) \subset H^4(Y; \mathbb{R}) \) [8, 10, 12]. \( H_{H_4}^4(Y; \mathbb{R}) \) is the real part of the primary horizontal subspace in (4),

\[
\text{Span}_\mathbb{C} \{\Omega_{Y_p}, D\Omega_{Y_p}, D^2\Omega_{Y_p}, \cdots\} \subset H^4(Y; \mathbb{C}),
\]

which does not depend on the choice of \( p \in \mathcal{M}_* \).

In order to see how \( H_{\text{scan}} \) should be chosen to achieve the goal we have set in this article, note that the vector space \( H^4(Y; \mathbb{R}) \) is decomposed as follows:

\[
H_{H_4}^4(Y; \mathbb{R}) \oplus H^2_{R\mathbb{M}_*}^2(Y; \mathbb{R}) \oplus H_{V_2}^2(Y; \mathbb{R});
\]

the vertical component \( H^2_{V_2}^2(Y; \mathbb{R}) \) is the subspace generated by the wedge products of integral (1, 1)-forms on \( Y_p \), which defines a subspace of \( H^4(Y; \mathbb{R}) \) independent of \( p \in \mathcal{M}_* \). The remaining component \( H^2_{R\mathbb{M}_*}^2(Y; \mathbb{R}) \), thus, should not depend on \( p \in \mathcal{M}_* \).

We choose \( H_{\text{scan}} \otimes \mathbb{R} \) to be \( H_{H_4}^4(Y; \mathbb{R}) \) in this article. As discussed in more detail in [13], \( H_{H_4}^4(Y; \mathbb{R}) \) is contained in \( [H^4(Y; \mathbb{R})]_{\text{ker}} \). Thus, for this choice of \( H_{\text{scan}} \), all the vacua share the same symmetry group from 7-branes in the effective theories below the Kaluza–Klein scale. This argument does not exclude the option to take \( H_{\text{scan}} \otimes \mathbb{R} \) larger that \( H_{H_4}^4(Y; \mathbb{R}) \), but we should not take it to be as large as \( [H^2_{V_2}^2(Y; \mathbb{R})]^2 \). It is not hard to find families \( \pi : \mathcal{Y} \rightarrow \mathcal{M}_* \) with \( h^2(Y_p; \mathbb{Z}) = 0 \) where there are algebraic four-forms in \( H^2_{R\mathbb{M}_*}^2(Y; \mathbb{R}) \cap H^4(Y; \mathbb{Z}) \) that break the symmetry of the 7-brane gauge group [13]. A similar phenomenon has also been observed in [12], where \( Y = K3 \times K3 \) and the four-form in \( H^2_{R\mathbb{M}_*}^2(Y; \mathbb{R}) \) is not dual to an algebraic cycle.

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\(^1\) As in [12] and literatures therein, we only consider flux vacua for \((B_3, [S])\) of a given topology and for a given choice of Kähler form \( J \) on \( Y_p \) that is in \( H^2(Y_p; \mathbb{Z}) \otimes H^{1,1}(Y_p; \mathbb{R}) \) modulo multiplication of \( \mathbb{R}_+ \). The (restricted) moduli space \( \mathcal{M} \) or \( \mathcal{M}_* \) therefore refers only to that of complex structure. It is beyond the scope of this article or [13] to include the scanning over \((B_3, [S])\)'s of different topology or their Kähler moduli spaces.
3. The Results:

Let us apply the method described in the previous section to derive statistical distributions of observables of practical interest. The study in [12] used a family of $K^3 \times K^3$ for compactification of F-theory, where all the 7-branes are parallel, and there is no light matter fields except those in the adjoint representation of the 7-brane gauge groups. Ensembles of flux vacua in such a set-up do not include low-energy effective theories that look close to the (supersymmetric extensions of the) Standard Model. In this article, we therefore use a few other families of elliptically fibred Calabi–Yau fourfolds, for which the low-energy effective theories are at least semi-realistic. These effective theories have $SU(5)$ (resp. $SO(10)$) unification, and $N_{gen}$ generations of matter fields in the $10 + 5$ (resp. 16) representations. The task is to determine the value of $L_*$ and $K$ for such families and to study how those values depend on the choice of the unification group or the number of generations $N_{gen}$.

3.A Number of Generations

We focus on a few choices of $(B_3, [S])$ for which the restricted moduli spaces and families for $SU(5)$ unification are constructed in the way stated at the beginning of the previous section as examples. We choose

$$B_3 = \mathbb{P}[O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(n)], \quad -3 \leq n \leq 3,$$
(9)

which is a $\mathbb{P}^1$-fibration over $\mathbb{P}^2$, and let $[S]$ be the “north-pole section” of the $\mathbb{P}^1$-fibration corresponding to the zero of a section of $O_{\mathbb{P}^2}$. The range of $n$ is set so that the 7-brane unification group at the $S$ can be as small as $SU(5)$, while the gauge group at the hidden sector (corresponding to the zero of a section of $O_{\mathbb{P}^2}(n)$) can be completely Higgsed away.

We set $(H_{\text{scan}} \otimes \mathbb{R}) = H^2_{\text{c}}(Y; \mathbb{R})$, and choose $G^{(4)}_{\text{fix}}$ to be the F-theory dual of the chirality-generating bundle twist in [5], parametrized by $\lambda_{F, \text{MW}} \in 1/2 + \mathbb{Z}$. This flux gives rise to the net chirality of the matter fields in the $10$ vs $\overline{10}$ (also $5$ vs $5$) representation of the $SU(5)$ unification group; $N_{gen} = -(18 - n)(3 - n)\lambda_{F, \text{MW}}$ [6]. Since the vanishing cycles for the chiral matter belong to $H^2_{\text{c}}(Y; \mathbb{Q})$, any flux vacua in the ensemble (3) have the same $N_{gen}$ [13]. In this way, we obtain an ensemble of F-theory flux vacua that share the same algebraic and topological ($N_{gen}$) information. The F-theory dual description of $G^{(4)}_{\text{fix}}$ has been determined in [11].

$$L_\ast = \chi(Y)/24 - (G^{(4)}_{\text{fix}})^2/2$$

is the upper bound on the net D3-brane charges from $G_{\text{scan}}$. It depends on $N_{gen}$ through $G^{(4)}_{\text{fix}}$ [12] and a straightforward computation reveals that [11, 13]

$$L_\ast = \frac{2163}{4} + \frac{125}{8} n (n + 7) - \frac{5 N_{gen}^2}{2(18 - n)(3 - n)}.$$
(10)

The maximal values of $L_{\ast \text{max}}$ range within $\sim 300$ to 800 for the families with $-3 \leq n \leq 3$ (Table I); details of the calculation are found in [13]. $L_\ast$ is always an upper convex quadratic function of $N_{gen}$ in F-theory, not just for the choice in (9).

The other number we need to use in (5) is $K = \dim_\mathbb{R} H^2_{\text{c}}(Y; \mathbb{R})$. This task boils down to the determination of the dimension of the horizontal component $H^2_{\text{c}}(Y; \mathbb{R})$, since $K = 2 + 2m + \dim_\mathbb{R} H^2_{\text{c}}(Y; \mathbb{R})$. A general recipe is to use mirror symmetry and determine the dimension of the vertical component of the mirror manifold of $Y$. The authors derived in [13] the formula for $h^{2,2}_V$, $h^{2,2}_H$, and $h^{2,2}_R$ that is valid for any Calabi–Yau hypersurface of a toric 5-dimensional ambient space.

We carried out the computation of $h^{2,2}_V$, $h^{2,2}_H$ and $K$ for the families for $SU(5)$ unification with $B_3$ in (9), it turns out that $h^{2,2}_R = 0$. Details of the computation are found in [13], and only the results are recorded in Table I. Certainly the results on $L_{\ast \text{max}}$ and $K$ in this table are only for a limited number of choices of $B_3$ and do not tell us whether they are typical among the results for all other choices of $B_3$ from 3-dimensional Fano varieties, or how much the values of $L_{\ast \text{max}}$ and $K$ can vary for different $B_3$. But the table at least provides the first example of such calculations.

With this preparation, we can derive the distribution of the number of generations $N_{gen}$ almost immediately. We have constructed ensembles of flux vacua that are labelled by $\lambda_{F, \text{MW}} \in 1/2 + \mathbb{Z}$. Each one of those ensembles consists of vacua that lead to effective theories with common algebraic information ($SU(5)$ GUT with chiral matter in the $10 + 5$ representation), but their topological information $N_{gen} \propto \lambda_{F, \text{MW}}$ varies from one ensemble to another. To compare the number of vacua that the individual ensembles contain, one simply needs to integrate $\rho_1$ over $M_\ast$, which is to ignore the difference in the value of the effective coupling constants of the vacua in a given ensemble. Since the integral of $\rho_1$ over $M_\ast$ usually yields a number of order unity (some region of $M_\ast$ may have to be excluded; cf the discussion of D-limits in [8]), we simply make an approximation $\int_{M_\ast} \rho_1 \approx 1$. Only the prefactor $(2\pi L_\ast)^{K/2}/\left([K/2]!ight)$ in (5) is then used as an estimate of the number of vacua in a given ensemble. We can use the value of $K$ in Table I, and the $N_{gen}$-dependence of $L_\ast$ has already been discussed in this article. The number of vacua depends on $N_{gen}$ in a way the volume of a $K$-dimensional sphere changes as the radius-square $L_\ast$.

### Table I: Various topological data of the families $Y \rightarrow M_{\text{av}}^{\text{SU(5)}}$ of Calabi–Yau fourfolds for $SU(5)$ unification, with $B_3$ given in (9).

| $n$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|-----|------|------|------|----|----|----|----|
| $h^{1,1}_\ast$ | 1249 | 1423 | 1723 | 2148 | 2698 | 3373 | 4173 |
| $h^{1,2}_\ast$ | 5057 | 5755 | 6955 | 8655 | 10855 | 13555 | 16756 |
| $h^{1,2}_\ast + 1$ | 9 | 9 | 9 | 9 | 9 | 8 | 8 |
| $L_{\ast \text{max}}$ | 237 | 297 | 387 | 507 | 657 | 837 | 1047 |
| $K$ | 7557 | 8603 | 10403 | 12953 | 16253 | 20303 | 25104 |
3.B Cost of Higher-Rank Gauge Groups

The distribution (5, 6, 11) can be used to derive the statistical cost of requiring a higher rank gauge group on 7-branes. This idea was pursued already in [12], using \( Y = K3 \times K3 \) for F-theory compactification; more examples are obtained in this article to estimate the systematics. The choices of \((B_3, [S])\) here are also more realistic than that of \(K3 \times K3\).

The SO(10) version of the family, \( \pi : \mathcal{Y} \longrightarrow \mathcal{M}^\text{SO} \), can also be constructed for the choice of \((B_3, [S])\) in (9) using toric geometry. Various topological data for the families over \(\mathcal{M}^\text{SO}, \mathcal{M}^\text{A4} \) and \(\mathcal{M}(\) no 7-brane gauge symmetry is required\) can be computed and the results are recorded in Table II for the \(n = 0\) case. For more details of computations, see [13].

One can see from (11) that the difference in the value of \(K\) for different choices of 7-brane symmetry \(R\) determines the relative number of the corresponding flux quanta (vacua). Ensembles with higher rank 7-brane gauge group have smaller dimension \(K\) (Table II), confirming the same observation in [12] based on the family \(Y = K3 \times K3\). This leads to the observation that the rank-4 gauge group on 7-branes is not as statistically “natural” as vacua without a gauge group on 7-branes. In the choice of \((B_3, [S])\) in (9) with \(n = 0\), for example, vacua with the rank-4 SU(5) unification constitutes only the fraction \(e^{-\Delta K/2} \approx e^{-3000}\) of the entire flux vacua (smaller than the fraction \(10^{-120}\) for the cosmological constant). The authors do not provide their interpretations for this inconvenient prediction; a popular attitude will be to hint at poor understanding of string theory, to count on cosmological factors that we did not study here, and/or to resort to anthropics.

4. Outlook:

This article only deals with the easiest applications of the method explained in section 2. Various ideas of using (11) and \(\rho_t\) to address questions of practical interest are described in detail in [13].

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