Evidence suggests that gamma-ray burst (GRB) ejecta are likely magnetized, although the degree of magnetization is unknown. When such magnetized ejecta are decelerated by the ambient medium, the characteristics of the reverse shock emission are strongly influenced by the degree of magnetization. We derive a rigorous analytical solution for the relativistic 90° shocks under the ideal MHD condition. The solution is reduced to the Blandford-McKee hydrodynamical solution when the magnetization parameter $\sigma$ approaches zero, and to the Kennel-Coroniti solution (which depends on $\sigma$ only) when the shock upstream and downstream are ultra-relativistic with each other. Our generalized solution can be used to treat the more general cases, e.g. when the shock upstream and downstream are mildly relativistic with each other. We find that the suppression factor of the shock in the strong magnetic field regime is only mild as long as the shock upstream is relativistic with respect to the downstream, and it saturates in the high-$\sigma$ regime. This indicates that generally strong relativistic shocks still exist in the high-$\sigma$ limit. This can effectively convert kinetic energy into heat. The overall efficiency of converting ejecta energy into heat, however, decreases with increasing $\sigma$, mainly because the fraction of the kinetic energy in the total energy decreases. We use the theory to study the reverse shock emission properties of arbitrarily magnetized ejecta in the GRB problem assuming a constant density of the circumburst medium. We study the shell-medium interaction in detail and categorize various critical radii for shell evolution. With typical GRB parameters, a reverse shock exists when $\sigma$ is less than a few tans or a few hundreds. The shell evolution can be still categorized into the thick and thin shell regimes, but the separation between the two regimes now depends on the $\sigma$ parameter and the thick shell regime greatly shrinks at high-$\sigma$. The thin shell regime can be also categorized into two sub-regions depending on whether the shell starts to spread during the first shock crossing. The early optical afterglow lightcurves are calculated for GRBs with a wide range of $\sigma$ value, with the main focus on the reverse shock component. We find that as $\sigma$ increases from below the reverse shock emission level increases steadily until reaching a peak at $\sigma<1$, then it decreases steadily when $\sigma>1$. At large $\sigma$ values, the reverse shock peak is broadened in the thin shell regime because of the separation of the shock crossing radius and the deceleration radius. This novel feature can be regarded as a signature of high $\sigma$. The early afterglow data of GRB 990123 and GRB 021211 could be understood within the theoretical framework developed in this paper, with the inferred $\sigma$ value $>0.1$. The case of GRB 021004 and GRB 030418 may be also interpreted with higher $\sigma$ values, although more detailed modeling is needed. Early tight optical upper limits could be interpreted as very high $\sigma$ cases, in which a reverse shock does not exist or very weak. Our model predictions could be further tested against future abundant early afterglow data collected by the Swift UV-optical telescope, so that the magnetic content of GRB fireballs can be diagnosed.
et al. 2002; Kumar & Panaitescu 2003). These findings suggest that magnetic fields may play a significant role in the GRB physics, as has been suggested by various authors previously (e.g. Usov 1994; Thompson 1994; Mészáros & Rees 1997b; Wheeler et al. 2000; Spruit, Daigne & Drenkhahn 2001; Blandford 2002). Within the framework of the currently favored collapsar progenitor model for GRBs (MacFadyen & Woosley 1999), the ejecta are found to be magnetized when MHD simulations are performed (Proga et al. 2003).

The degree of magnetization of the ejecta, however, is unknown. This is usually quantified by the parameter \( \sigma \) (see eq. \[4\] for a precise definition), the ratio of the electromagnetic energy flux to the kinetic energy flux. Current GRB models are focused on two extreme regimes. In the first regime, it is essentially assumed that the GRB fireball is purely hydrodynamical. Magnetic fields are introduced only through an equipartition parameter \( \epsilon_B \) (which is of the order of 0.001-0.1) for the purpose of calculating synchrotron radiation. This is the \( \sigma \to 0 \) regime. In this picture, the GRB prompt emission is produced from internal shocks (Rees & Mészáros 1994) or sometimes from external shocks (Mészáros & Rees 1993; Dermer & Mitman 1999). This is currently the standard scenario of GRB emission. The second is the \( \sigma \to \infty \) regime. This is the regime where a Poynting-flux dominated the flow, and GRB prompt emission is envisaged to be due to some less familiar magnetic dissipation processes (e.g. Usov 1994; Spruit et al. 2001; Blandford 2002; Lyutikov & Blandford 2003). In principle, a GRB event could include both a “hot component” as invoked in the \( \sigma = 0 \) model (e.g. due to neutrino annihilation) and a “cold component” as invoked in the \( \sigma = \infty \) model, the interplay between both components may result in a \( \sigma \) value varying in a wide range (Zhang & Mészáros 2002). It is an important but difficult task to pin down the degree of magnetization of GRB ejecta.

GRB early afterglow data (especially in the optical band) potentially contain essential information to diagnose the magnetic content of the fireball. The reason is that an early optical afterglow lightcurve is believed to include contributions from both the forward shock (which propagates into the ambient medium) and the reverse shock (which propagates into the ejecta itself). Since the magnetization degree of the ejecta influences the emission level of the reverse shock (or maybe even the level of the forward shock), by studying the interplay between the reverse shock and the forward shock emission components, one could potentially infer the degree of magnetization of the ejecta. In all the current analyses, the reverse shock emission is treated purely hydrodynamically (e.g. Mészáros & Rees 1997a; Sari & Piran 1999; Kobayashi 2000; Kobayashi & Zhang 2003a; ZKM03). When confronted with the available early afterglow data (four cases so far: GRB 990123, Akerlof et al. 1999; GRB 021004, Fox et al. 2003a; GRB 021211, Fox et al. 2003b, Li et al. 2003a; and GRB 030418, Rykoff et al. 2004), the model works reasonably well for two of them (GRB 990123 and GRB 021211), although a good fit requires that the magnetic field in the reverse shock region is much stronger than that in the forward shock region (ZKM03). For the other two, the lightcurves are not easy to explain with the simplest reverse shock model. On the other hand, GRB ejecta could in principle have an arbitrary \( \sigma \) value. When \( \sigma \) is large, the conventional hydrodynamical treatment is no longer a good approximation, and a full treatment involving MHD shock jump condition is desirable.

It is generally believed that a GRB involves a rapidly-rotating central engine. If the magnetic dissipation processes are not significant, field lines are essentially frozen in the expanding shells. The radial component of the magnetic field decays with radius as \( \propto R^{-2} \), while the toroidal magnetic field decays as \( \propto R^{-1} \). At the external shock radius, magnetic field lines are essentially frozen in the plane perpendicular to the moving direction. The MHD shock Rankine-Hugoniot relations are greatly simplified in such a 90° shock. Such relations have been studied extensively before both analytically and numerically. Kennel & Coroniti (1984) derived some simplified analytical expressions applicable for strong 90° degree shocks whose upstream and downstream are ultra-relativistic with each other. The model was used to treat the pulsar-wind nebula problem. In this regime, the strength of the shock is essentially characterized by only one parameter, i.e. the \( \sigma \) parameter. The conclusion was confirmed later by numerical simulations (e.g. Gallant et al. 1992).

Within the context of GRBs, since a GRB invokes a transient release of energy, the ejecta shell has a finite width (in contrast to the long-standing pulsar wind). Under some conditions, the reverse shock upstream and downstream could never become relativistic with each other when the reverse shock crosses the ejecta shell. In the \( \sigma = 0 \) limit, whether the reverse shock becomes relativistic depends on the comparison between the time scale \((T)\) of the central engine activity (essentially the duration of the burst) and the time scale \((t_r)\) when the mass of the ambient medium collected by the fireball reaches \( 1/\gamma_0 \) times the mass of the ejecta (e.g. Sari & Piran 1995; Kobayashi, Piran & Sari 1999). Both times are measured by the observer. The case of \( T > t_r \) is called the thick shell regime, and the reverse shock is relativistic. In many cases, however, one has \( T < t_r \), i.e. the thin shell regime\(^1\). The reverse shock is initially non-relativistic and only becomes mildly relativistic as the shock crosses the shell. For magnetized ejecta (e.g. a shell with a finite width but an arbitrary \( \sigma \) value), the separation between the thick and thin shell regimes becomes more complicated, but the non-relativistic reverse shock case (for the thin shell regime) is even more common (see §3B). The Kennel-Coroniti approximation cannot be directly used. The theory developed in this paper becomes essential to discuss the reverse shock physics in this parameter regime.

In this paper, we present a detailed treatment of reverse shock emission for an arbitrarily magnetized ejecta under the ideal MHD condition. The reverse shock emission in the mildly magnetized regime is also discussed by Fan, Wei & Wang (2004a) recently. Here we will develop a theoretical framework to include discussions for the reverse shock emission in a wider \( \sigma \) range, as well as for various ejecta-medium interaction parameter regimes. We first (§2) present a rigorous analytical solution for

\(^1\) If a GRB contains several well-separated emission episodes, the whole burst may be even separated into several discrete shells. In such cases, even a long-duration burst may be treated as the superposition of several thin shells rather than one single thick shell. See Zhang & Mészáros 2004 for more discussions.
the MHD 90° shock jump conditions, which is applicable for an arbitrary σ value and for an arbitrary Lorentz factor γ21 between the upstream and the downstream. The detailed derivation and the relevant equations are presented in the Appendix. We then (§3) discuss the ejecta-medium interaction within the context of GRB fireball deceleration and re-investigate the critical fireball radii and re-categorize the thick vs. thin shell regimes. This leads to a more complicated picture than in the pure hydrodynamical case (Sari & Piran 1995). In §4, we calculate the synchrotron emission from the shocks under the conventional assumptions about the particle acceleration in collisionless shocks, and present the predicted GRB early optical lightcurves for a wide range of σ value. We discuss how the early afterglow data may be used to diagnose the magnetic content of GRB ejecta. Our results are summarized in §5 with some discussions.

2. ANALYTICAL SOLUTION OF THE RELATIVISTIC 90° SHOCKS

We now consider a relativistic shock that propagates into a magnetized medium. In the following analysis, the unshocked region (upstream) is denoted as region 1, the shocked region (downstream) is denoted as region 2, and the shock itself is denoted as “s”2. Hereafter Qij denotes the value of the quantity Q in the region “i” in the rest frame of “j”, and Qs denotes the value of the quantity Q in the region “i” in its own rest frame. For example, γ12 is the relative Lorentz factor between the regions 1 and 2, βs is the relative velocity (in the unit of the speed of light c) between the region 1 and the shock, B2s is the magnetic field strength of the region 2 in the rest frame of the shock, while B1 is the comoving magnetic field strength in the region 1, etc. The relativistic 90° shock Rankine-Hugoniot relations could be written as (Kennel & Coroniti 1984)

\[ n_1 u_{1s} = n_2 u_{2s}, \] (1)

\[ \mathcal{E} = \beta_1 B_{1s} = \beta_2 B_{2s}, \] (2)

\[ \gamma_1 s \mu_1 + \frac{\mathcal{E} B_{1s}}{4\pi n_1 u_{1s}} - \gamma_2 s \mu_2 = \frac{\mathcal{E} B_{2s}}{4\pi n_2 u_{2s}}, \] (3)

\[ \mu_1 u_{1s} + \frac{p_1}{n_1 u_{1s}} + \frac{B_{1s}^2}{8\pi n_1 u_{1s}} = \mu_2 u_{2s} + \frac{p_2}{n_2 u_{2s}} + \frac{B_{2s}^2}{8\pi n_2 u_{2s}}, \]

where β denotes the dimensionless velocity, γ = (1 – β2)−1/2 denotes the Lorentz factor, and u = βγ denotes the radial four velocity, so that γ2 = 1 + u2. Hereafter, n, e, p, and \( \Gamma \) are the number density, internal energy and thermal pressure, respectively, and \( \Gamma \) is the adiabatic index. The enthalpy is \( nm_p c^2 + e + p \), and the specific enthalpy can be written as

\[ \mu = m_p c^2 + \frac{\Gamma}{\Gamma - 1} \left( \frac{p}{n} \right), \] (5)

where \( m_p \) is the proton mass and \( c \) is the speed of light. It is convenient to define a parameter

\[ \sigma = \frac{B_{1s}^2}{4\pi n_1 \mu_1} = \frac{B_{2s}^2}{4\pi n_2 \mu_1 \gamma_{1s}^2}, \] (6)

to denote the degree of magnetization in each region. The magnetization parameter in the upstream region (\( \sigma_1 \)) is a more fundamental parameter, since it characterizes the magnetization of the flow itself. We therefore define

\[ \sigma \equiv \sigma_1 = \frac{B_{1s}^2}{4\pi n_1 \mu_1 \gamma_{1s}^2}. \] (7)

In our problem, we are interested in a “cold” upstream flow, i.e., \( e_1 = p_1 = 0 \), so that \( \mu_1 = m_p c^2 \). This is the only assumption made in the derivation. After some algebra (see Appendix), we can finally write

\[ \frac{e_2}{n_2 m_p c^2} = (\gamma_2 - 1) \left[ 1 - \frac{\gamma_2 + 1}{2u_{1s}(\gamma_2, \sigma)u_{2s}(\gamma_2, \sigma)} \right]. \] (8)

Here \( u_{2s}(\gamma_2, \sigma) \) is a function of \( \gamma_2 \) and \( \sigma \), and can be solved once \( \gamma_2 \) and \( \sigma \) are known. After some analytical treatments of the relativistic Rankine-Hugoniot relations (eq. [4]), one can come up with an equation to solve \( u_{2s} \). Since it is complicated, we only present it in the Appendix (eq. [A10]). Once \( u_{2s} \) is solved, we can also solve \( u_{1s} \) (using eq. [A8]), i.e.,

\[ u_{1s}(\gamma_2, \sigma) = u_{2s}(\gamma_2, \sigma) \gamma_2 + [u_{2s}(\gamma_2, \sigma) + 1]^{1/2} (\gamma_2 - 1)^{1/2}. \] (9)

The compressive ratio can be derived directly from eq. (10), i.e.

\[ \frac{n_2}{n_1} = \frac{u_{1s}(\gamma_2, \sigma)}{u_{2s}(\gamma_2, \sigma)} = \gamma_2 + [u_{2s}(\gamma_2, \sigma) + 1]^{1/2} (\gamma_2 - 1)^{-1/2}. \] (10)

The main point here is that both \( e_2 / n_2 m_p c^2 \) and \( n_2 / n_1 \) can be determined by two unknown parameters, i.e. \( \gamma_2 \) and \( \sigma \), so that when they are given arbitrarily, the whole problem is solved.

In the downstream region, the total pressure includes the contribution from the comoving thermal pressure \( p_2 = (\Gamma - 1)e_2 \) and the comoving magnetic pressure \( p_{B2} = B_{2s}^2 / 8\pi \). The ratio between the magnetic pressure to the thermal pressure is also a function of \( \sigma \) and \( \gamma_2 \):

\[ \frac{p_{B2}}{p_2} = \left( \frac{\beta_1}{\beta_2} \right)^2 \left( \frac{4\pi n_1 m_p c^2}{8\pi \gamma_2^2 (\Gamma - 1)e_2} \right) = \frac{1}{2(\Gamma - 1)} \left( \frac{u_{1s}}{u_{2s}} \right) \sigma \left( \frac{e_2}{n_2 m_p c^2} \right)^{-1}, \] (11)

where eqs. (2) and (4) have been used.

The correctness of the solution (eq. [A10]) are verified in two asymptotic regimes.

2.1. The \( \sigma = 0 \) regime

When \( \sigma = 0 \), the equation to solve \( u_{2s}(\gamma_2, \sigma) \) (eq. [4]) is greatly simplified (eq. [A23]). All quantities can be expressed as a function of \( \gamma_2 \). The solutions are

\[ u_{2s}^2 = \frac{(\gamma_2 - 1)(\Gamma - 1)^2}{\Gamma(2 - \Gamma)(\gamma_2 - 1) + 2} \] (12)

\footnote{Notice that such a notation system is only valid for \( \frac{3}{2} \) and the Appendix. When discussing the GRB problem, i.e. the ejecta-medium interaction (§7), we introduce different meanings for the subscript numbers.}

\footnote{Notice that the definition is slightly different from that in Kennel & Coroniti (1984). We find that our definition allows the parameters to be casted into an analytical form as a function of \( \sigma \) and \( \gamma_2 \).}
The equations (14-16) are just equations (3-5) of Blandford & McKee (1976), and they are the starting point for the hydrodynamical analysis of the reverse shock emission (e.g. Sari & Piran (1995)). Under the limit of $\gamma_21 \gg 1$ and $\hat{\Gamma} = 4/3$ (i.e. the downstream fluid is relativistic), the equations (15) and (16) are reduced to the familiar forms of $n_2/n_1 = 4\gamma_21 + 3$, $\gamma_1s \simeq \sqrt{2}\gamma_21$ and $\gamma_2s \simeq 3\sqrt{2}/4$ (or $u_{2s} \simeq \sqrt{2}/4$).

2.2. The $\gamma_21 \gg 1$ regime

In the $\gamma_21 \to \infty$ limit, the equation for $u_{2s}^2(\gamma_21, \sigma)$ (eq. [A16]) is also simplified (eq. [A26]). The solution of $u_{2s}^2$ is a function of $\sigma$ only, which reads

$$u_{2s}^2 = \frac{\hat{\Gamma}(1 - \frac{\hat{\Gamma}}{4})\sigma^2 + (\hat{\Gamma}^2 - 2\hat{\Gamma} + 2)\sigma + (\hat{\Gamma} - 1)^2 + \sqrt{X}}{2\hat{\Gamma}(2 - \hat{\Gamma})(\sigma + 1)}$$

(17)

where

$$X = \hat{\Gamma}^2(1 - \frac{\hat{\Gamma}}{4})^2 \sigma^4 + \hat{\Gamma}^2(\frac{\hat{\Gamma}^3}{2} - 3\hat{\Gamma}^2 + 7\hat{\Gamma} - 4)\sigma^3$$
$$+ (\frac{3}{2}\hat{\Gamma}^4 - 7\hat{\Gamma}^3 + \frac{31}{2}\hat{\Gamma}^2 - 14\hat{\Gamma} + 4)\sigma^2$$
$$+ 2(\hat{\Gamma} - 1)^2(\hat{\Gamma}^2 - 2\hat{\Gamma} + 2)\sigma + (\hat{\Gamma} - 1)^4.$$  

(18)

Notice that there are two solutions with the term $\pm \sqrt{X}$ in the numerator of eq. (17), but the minus term leads to negative pressure and is un-physical. For a relativistic downstream region, i.e. $\hat{\Gamma} = 4/3$, the solution is reduced to

$$u_{2s}^2 = \frac{8\sigma^2 + 10\sigma + 1 + \sqrt{64\sigma^2(\sigma + 1)^2 + 20\sigma(\sigma + 1) + 1}}{16(\sigma + 1)}$$

$$= \frac{8\sigma^2 + 10\sigma + 1 + (2\sigma + 1)\sqrt{16\sigma^2 + 16\sigma + 1}}{16(\sigma + 1)}.$$  

(19)

This is the eq.(4.11) of Kennel & Coroniti (1984). With $u_{2s}$, one can derive $u_{1s}$ using equation [12], which depends on $\gamma_21$ as well. The quantities $e_2/n_2m_p\gamma_21^2$ and $n_2/n_1$ can be also derived accordingly. In the $\sigma = 0$ limit, equation (19) is reduced to $u_{2s}^2 \simeq \sqrt{2}/4$, which is consistent with the asymptotic results in [2.1].

2.3. The general cases

For more general cases with arbitrary values of $\gamma_21$ and $\sigma$, $u_{2s}^2(\gamma_21, \sigma)$ has to be solved rigorously. The equation (eq. [A19]) is solved numerically, and the solutions indeed show deviations from both asymptotic regimes for arbitrary $\gamma_21$ and $\sigma$ values.

Figure 1 shows the variations of six parameters, i.e., $u_{2s}$, $\gamma_1s/\gamma_21$, $e_2/n_2m_p\gamma_21^2$, $n_2/n_1$, $p_{h2}/p_2$, and $F$, as a function of $\gamma_21$. The thick solid line indicates the case for $\sigma = 0$, which is the Blandford-McKee (1976) solution. For $p_{h2}/p_2$ (panel e), the $\sigma = 0$ line is at negative infinity. The dashed lines, starting from the one closest to the thick line, are for $\sigma = 0.01, 0.1, 1, 10, 100, 1000$, respectively. The parameter $e_2/n_2m_p\gamma_21^2$ (random Lorentz factor in the shocked, downstream region) is normalized to $(\gamma_21 - 1)$, and the parameter $n_2/n_1$ (compressive ratio) is normalized to $(4\gamma_21 + 3)$, both are the values expected in the $\sigma = 0$ case.

Figure 2.— The variations of six parameters, i.e., $u_{2s}$, $\gamma_1s/\gamma_21$, $e_2/n_2m_p\gamma_21^2$, $n_2/n_1$, $p_{h2}/p_2$, and $F$, as a function of $\gamma_21$. The thick solid line indicates the case for $\sigma = 0$, which is the Blandford-McKee (1976) solution. For $p_{h2}/p_2$ (panel e), the $\sigma = 0$ line is at negative infinity. The dashed lines, starting from the one closest to the thick line, are for $\sigma = 0.01, 0.1, 1, 10, 100, 1000$, respectively. The parameter $e_2/n_2m_p\gamma_21^2$ (random Lorentz factor in the shocked, downstream region) is normalized to $(\gamma_21 - 1)$, and the parameter $n_2/n_1$ (compressive ratio) is normalized to $(4\gamma_21 + 3)$, both are the values expected in the $\sigma = 0$ case.
mildly large (e.g. $> 3$), the ratio is essentially the function of $\sigma$ only. It starts from the conventional value $\sqrt{2}$ in the $\sigma \sim 0$ regime and increases quickly as $\sigma$ approaches unity, which means that the shock leads the fluid substantially (in the upstream rest frame) in the high $\sigma$ regime. Both $e_2/n_2 m_p c^2$ and $n_2/n_1$ are suppressed when $\sigma$ increases, but the suppression factor (with respect to the $\sigma = 0$ limit) is not large, especially when $\gamma_2 > 1$ is not too small. For example, for $\gamma_2 > 3$, the suppression factor for $e_2/n_2$ is larger than 0.6, while that for $n_2/n_1$ is larger than 0.4. Furthermore, the suppression factor reaches an asymptotic value when $\sigma$ approaches several. This result is very interesting, since conventionally it is believed that the shock is completely suppressed when $\sigma$ reaches larger values. Our results suggest that relativistic strong shocks still exist in the high-$\sigma$ regime. The suppression factor, which essentially does not depend on $\sigma$, is only mild as long as the shock is relativistic. The overall efficiency of converting the total energy (the kinetic energy plus the Poynting flux energy) to heat still decreases steadily with increasing $\sigma$. The reason is not that the shock (which converts the kinetic energy into heat) itself is less strong, but is that the fraction of the kinetic energy in the total energy, i.e., $(1+\sigma)^{-1}$, becomes smaller as $\sigma$ becomes larger.

3. EJECTA - MEDIUM INTERACTION

3.1. Basic equations

Now we consider an arbitrarily magnetized flow with magnetization parameter $\sigma$ and Lorentz factor $\gamma = \gamma_4$ being decelerated by an ambient medium with density $n = n_1$. A pair of shocks form when the shock forming condition is satisfied, i.e. the relative velocity between the two colliders exceeds the sound velocity in the medium and the magnetoacoustic wave velocity of the ejecta, and that the pressure in the shocked region exceeds the pressure in the unshocked region. In the GRB case (a relativistic ejecta), a forward shock always form, while a reverse shock may not always form if $\sigma$ is too large. In the high-$\sigma$ regime, the magnetoacoustic wave velocity is essentially the Alfvén velocity for a 90° shock. The first condition for the reverse shock formation is $\gamma_{41} > \gamma_A \sim \sqrt{1+\sigma}$, where $\gamma_A$ is the Alfvén Lorentz factor in the ejecta. For GRBs we have $\gamma_{41} \geq 100$ (to ensure that the observed gamma-ray spectrum is nonthermal), so that this condition is satisfied as long as $\sigma \leq 10^4$. The second condition is generally more stringent, which is expressed in eqs. (31) and (32) below. In any case, with reasonable parameters a reverse shock could be formed when $\sigma$ is less than hundreds or tens. When the reverse shock forms, we can then investigate a picture where two shocks and one contact discontinuity separate the ejecta and medium into four regions. Below we take the usual convention to define the four regions: (1) unshocked medium; (2) shocked medium; (3) shocked ejecta; (4) unshocked ejecta. Notice that after the numerical subscripts have different meanings from the ones used in §2 and Appendix, where the numbers “1” and “2” denote upstream and downstream, respectively. For the forward shock, both sets of notations coincide, while for the reverse shock, the previous “1” and “2” are replaced by “4” and “3”, respectively. We assume that the ambient medium is not magnetized so that $\sigma_1 = 0$, but will assign an arbitrary magnetization parameter

$$\sigma \equiv \sigma_4 = \frac{B_4^2}{4\pi n_4 m_p c^2} = \frac{B_{4s}^2}{4\pi n_4 m_p c^2 \gamma_{4s}^2}$$

for the ejecta. Since we are discussing the problem in the rest frame of the medium, we will drop out the subscript “1” whenever it means “in the rest frame of the region 1”. We can then write the following relations based on the shock jump conditions. Throughout the following discussions, $\Gamma = 4/3$ is adopted.

$$\frac{e_2}{n_2 m_p c^2} = (\gamma_2 - 1) \approx \gamma_2,$$  \hspace{1cm} (21)

$$\frac{n_2}{n_1} = 4\gamma_2 + 3 \approx 4\gamma_2,$$  \hspace{1cm} (22)

$$\frac{e_3}{n_3 m_p c^2} = (\gamma_3 - 1)f_a,$$  \hspace{1cm} (23)

$$\frac{n_3}{n_4} = (4\gamma_3 + 3)f_b,$$  \hspace{1cm} (24)

where

$$f_a = f_a(\sigma, \gamma_{34}) = 1 - \frac{\gamma_{34} + 1}{2[u_{3s}^2 \gamma_{34} + u_{3s}(u_{3s}^2 + 1)^{1/2}(\gamma_{34}^2 - 1)^{1/2}]}$$

and

$$f_b = f_b(\sigma, \gamma_{34}) = \frac{\gamma_{34} + 1}{4u_{3s}^2 + 3}$$

are the correction factors for $e_3/n_3 m_p c^2$ and $n_3/n_4$ with respect to the $\sigma = 0$ limit, and $u_{3s}$ is a function of $\gamma_{34}$ and $\sigma$, whose solution is found in the Appendix A (eq. (41)). The functions $f_a$ and $f_b$ are plotted as a function of $\sigma$ in Fig. (2c) and Fig. (2d), respectively, which shows $f_a \to 1$ and $f_b \to 1$ when $\sigma \to 0$. Constant speed across the contact discontinuity gives

$$\gamma_2 = \gamma_3,$$  \hspace{1cm} (27)

and constant pressure across the contact discontinuity gives

$$\frac{e_2}{3} = \frac{e_3}{3} + \frac{B_{4s}^2}{8\pi} = \frac{e_3}{3}(1 + \frac{p_{4s}}{p_3}),$$  \hspace{1cm} (28)

or

$$e_2 = e_3 f_c,$$  \hspace{1cm} (29)

where

$$f_c = f_c(\sigma, \gamma_{34}) = 1 + \frac{p_{4s}}{p_3}(\times \sigma)$$

The pressure ratio $p_{4s}/p_3$ is calculated according to eq. (11), whose dependence on $\sigma$ is plotted in Fig. (2e) which shows a $\times \sigma$ dependence in the $\sigma \gg 1$ asymptotic regime. Hereafter we denote the asymptotic behavior in the $\sigma \gg 1$ limit in a pair of parenthesis immediately following an equation.

In order to have equation (26) satisfied, the condition should be that the thermal pressure generated in the forward shock is stronger than the magnetic pressure in the region 4. This gives $B_{4s}^2/8\pi < (4/3)\gamma_{2s}^2 n_1 m_p c^2$. Noticing equation (20), the condition for the existence of the reverse shock can be written as

$$\sigma < \frac{8}{3}\gamma_{2s}^2 n_1 n_4.$$  \hspace{1cm} (31)
When the reverse shock exists, using equations 24 and 29, one can finally get
\[ \frac{n_4}{n_1} F = \frac{(\gamma_2 - 1)(4\gamma_2 + 3)}{(\gamma_{34} - 1)(4\gamma_{34} + 3)}, \]  
where
\[ F = f_a f_b f_c, \]  
and \( f_a, f_b \), and \( f_c \) are defined in equations 25, 26 and 28, respectively. The parameter \( F \) has been calculated for different input parameters, and the results are shown in Fig.(1f) and Fig.(2f), respectively. We can see that \( F \) is very insensitive to \( \gamma_{34} \), and is essentially a function of \( \sigma \) only. The asymptotic behavior in the \( \sigma \gg 1 \) regime is \( F(\sigma) \propto \sigma \). We then write
\[ F(\gamma_{34}, \sigma) \simeq F(\sigma), (\propto \sigma). \]  
The equation 22 can be used to define whether the reverse shock upstream and downstream are relativistic with each other. Similar to the analysis of Sari & Piran (1995), we analyze the value of \((n_4/n_1)F/\gamma_2^2\) \( \gamma_{34} \). The relative Lorentz factor of the reverse shock upstream and downstream is
\[ \gamma_{34} \simeq \frac{1}{2} \left( \frac{\gamma_4}{\gamma_3} + \frac{\gamma_4}{\gamma_4} \right). \]  
For the relativistic case, i.e. \( \gamma_{34} \gg 1 \), we have \((n_4/n_1)F/\gamma_2^2 \sim \gamma_4^2(\gamma_4^2/\gamma_4^2) \sim \gamma_4 \). On the other hand, for a non-relativistic case, we have \( \gamma_{34} \sim 1 \), \( \gamma_4 \sim \gamma_3 \) and \( (\gamma_{34} - 1)(4\gamma_{34} + 3) = \epsilon \ll 1 \). This gives \((n_4/n_1)F/\gamma_2^2 \sim 1/\epsilon \gg 1 \). We thus conclude that the reverse shock upstream is relativistic with respect to the downstream when \( \gamma_2^2 \gg (n_4/n_1)F \), while it is non-relativistic when \( \gamma_2^2 \ll (n_4/n_1)F \). For \( \sigma = 0 \), we have \( F = 1 \), and the result is fully consistent with Sari & Piran (1995).

3.2. Critical radii

We consider an isotropic fireball with total energy \( E = E_K + E_P \), where \( E_K \) is the kinetic energy and \( E_P \) is the Poynting flux energy. The discussions are also valid for a collimated jet by regarding the various energy components as the “isotropic” values. With the definition of \( \sigma \) (eq. 4), we find \( E_P/E_K \sim \sigma \) (from eq. 24), so that \( E = E_K(1 + \sigma) \) or \( E_K = E/(1 + \sigma) \) (see also Zhang & Mészáros 2002). We follow the traditional convention to define the Sedov length \( l \sim (E/n_1m_pc^2)^{1/3} \), where the total energy is adopted. The shell baryon number density \( n_4 \) is however defined by \( E_K \). The density ratio is \( n_4/n_1 \sim 1/|\gamma_{34}^2\Delta R^2(1 + \sigma)| \), where \( \Delta = \max(\Delta_0, R/\gamma_{34}^2) \) is the thickness of the shell, \( \Delta_0 = cT \) is the initial width of the shell, and \( R \) is the fireball radius. This holds for both a non-sweeping shell (where \( \Delta = \Delta_0 \) is a constant) and for a spreading shell (where \( \Delta \sim R/\gamma^2 \)).

Below we revisit the four critical radii related to the reverse shock deceleration (Sari & Piran 1995). In our following discussion, we assume that a reverse shock exists. The asymptotic behaviors at \( \sigma \gg 1 \) for various correction factors (presented in parentheses) therefore are valid for the \( \sigma \) range in which the reverse shock forming condition is satisfied.

1. The fireball radius for the relative Lorentz factor between the reverse shock upstream and downstream (i.e. \( \gamma_{34} \)) to transform from the Newtonian regime to the relativistic regime can be estimated according to \( \gamma_4^2 \sim (n_4/n_1)F \), which gives
\[ R_N \sim \frac{l_{3/2}}{\Delta_{1/2}\gamma_{34}^2}C_N, \]  
where
\[ C_N(\gamma_{34}, \sigma) \simeq C_N(\sigma) = \left[ \frac{F(\sigma)}{1 + \sigma} \right]^{1/2} \sim 1, \left( \propto \sigma^0 \right) \]  
is the correction factor of \( R_N \) with respect to the \( \sigma = 0 \) case. Since both \( F(\sigma) \) and \( (1 + \sigma) \) have the same asymptotic behavior \((\propto \sigma) \) at high \( \sigma \), the final correction factor \( C_N \) is always of order unity throughout (see Fig.4), and we will neglect it in the following discussions.

2. The radius where the reverse shock crosses the shell is approximately (Sari & Piran 1995)
\[ R_3 = \frac{\Delta(\beta_2)(1 - \gamma_{34}/\gamma_3)(n_4/n_3)}, \]  
in the \( \sigma = 0 \) case, the factor \([1 - (\gamma_{34}/\gamma_3)(n_4/n_3)]\), which delineates the relative compression factor due to shock crossing, is a factor ranging from 1/2 to 6/7, which was neglected for order-of-magnitude estimates (Sari & Piran 1995). For arbitrary \( \sigma \) values, this parameter is \( \sigma \)-sensitive (through the dependence of \( f_4(\sigma) \), see eq. 26), i.e., becomes \( \leq 1 \) when \( \sigma \gg 1 \), so we can not drop it out. Following the similar procedure as in Sari & Piran (1995), and replacing \( n_4/n_1 \) (in the \( \sigma = 0 \) case) by \((n_4/n_1)F \), one finally has
\[ R_\Delta \sim \gamma_4^2 \left[ \frac{n_4}{n_1}F(\sigma) \right]^{1/2} \left[ 1 - \frac{\gamma_{34}n_4}{\gamma_3n_3} \right] \sim \gamma_4^4/\Delta^{1/4}/4C_\Delta, \]  
where
\[ C_\Delta(\gamma_{34}, \sigma) \simeq C_\Delta(\sigma) = \left[ \frac{F(\sigma)}{1 + \sigma} \right]^{1/4} \left[ 1 - \frac{\gamma_{34}n_4}{\gamma_3n_3} \right]^{1/2} \sim \left( 1 - \frac{\gamma_{34}n_4}{\gamma_3n_3} \right)^{1/2}, \left( \propto \sigma^{-1/2} \right) \]  
is the correction factor of \( R_\Delta \) with respect to the \( \sigma = 0 \) case. This correction factor suggests that the reverse shock crosses the shell faster when \( \sigma \) becomes larger.

3. The conventional “deceleration radius” (for the thin shell case) is modified in the high-\( \sigma \) regime. According to eq. 24, the ratio of the comoving magnetic fields in region 4 and 3 is \( B_4/B_3 = u_{4s}/u_{4s} \). The lab-frame ratio of the Poynting flux energy in both regions can be written as
\[ \frac{E_{P,4}}{E_{P,3}} = \frac{\gamma_{4}(B_{4}^2/n_4)}{\gamma_{3}(B_{3}^2/n_3)} = \frac{u_{4s}\gamma_{4}}{u_{4s}\gamma_{3}} \sim 1, \]  
where \( \sim \) applies in the \( \sigma \gg 1 \) limit so that \( u_{4s} \sim \gamma_{4s} \) and \( u_{4s} \sim \gamma_{3s} \). Since \( \gamma_{4s}/\gamma_{3s} = \gamma_{4}/\gamma_{3} \) (both have the same relation with \( \gamma_{34} \) (eq. 25)), eq. 40 is naturally derived. \footnote{The full presentation of eq. 4 should be \( \mu_{11}u_{11} + p_{1}/n_{1}u_{11} + B_{11}^2)/\mu_{11}u_{11} + B_{11}^2)/\mu_{11}u_{11} = \mu_{22}u_{22} + p_{2}/n_{2}u_{22} + B_{22}^2)/\mu_{22}u_{22} \). So strictly speaking, \( E_P/E_K = (\beta_1^2 + \beta_2^2)/2\beta_2^2 \). In the high-\( \sigma \) regime, we have \( \beta_2^2 \sim 1 \) and the factor \((1 + \beta_1^2)/2\beta_2^2 \sim 1 \). In the low \( \sigma \) regime, \( \beta_2^2 \ll 1 \), and the factor \((1 + \sigma(1 + \beta_1^2)/2\beta_2^2) \) is in any case \( \sim 1 \). We therefore neglect the \((1 + \beta_2^2)/2\beta_2^2 \) factor in the following discussion.}

\footnote{If one assumes that after shock crossing a shell with width \( \Delta \) becomes \( \Delta' \), this parameter is simply \((\Delta - \Delta')/\Delta \).}
We have calculated this ratio numerically and found that in the high-\(\sigma\) limit, the difference between this ratio and unity is a small quantity comparing \(\sigma^{-1}\). This manifests that shocks in the high-\(\sigma\) limit only effectively dissipate the kinetic energy (in the baryonic component) in the upstream, and the Poynting energy (in the lab frame) essentially remains the same. As a result, one should define the deceleration radius (for the thin shell case) using \(E_K\) alone, so that

\[
R_\gamma \sim \frac{l}{\gamma_4^{2/3} (1 + \sigma)^{1/3}} \sim \frac{l}{\gamma_4^{2/3} C_\gamma}, \tag{41}
\]

where

\[
C_\gamma(\sigma) = (1 + \sigma)^{-1/3} (\propto \sigma^{-1/3}). \tag{42}
\]

Notice that the radius \(R_\gamma\) is still the radius where the fireball collects \(1/\gamma_4\) of fireball rest mass\(^6\).

In the above discussion, we already assumed that a reverse shock exists. In order to satisfy the condition (31) at \(R_\gamma\), one can give a more explicit constraint on \(\sigma\), which reads

\[
\sigma < 100 \left(\frac{\gamma_4}{300}\right)^4 \left(\frac{T}{10 \text{ s}}\right)^{3/2} \left(\frac{E}{10^{32} \text{ ergs}}\right)^{-1/2}. \tag{43}
\]

We can see for typical GRB parameters, a reverse shock exists when \(\sigma\) is smaller than several tens to several hundreds.

It is worth noticing that at the deceleration radius, the Poynting energy is not transferred to the ISM yet. At the end shock crossing, the magnetic pressure behind the contact discontinuity balances the thermal pressure in the forward shock crossing. It is not until the reverse shock disappears during the deceleration phase when the bulk of magnetic energy in the ejecta transfers to the forward shock region. During the deceleration, the magnetic fields push the contact discontinuity from behind and transfer energy through \(\rho dV\) work. Eventually, the total energy will be transferred to the ISM, so that the late time afterglow level is still defined by the total energy of the fireball. The detailed energy transfer process is complicated and will be studied carefully in a future work (see Zhang & Kobayashi 2005 for a brief discussion). This point is relevant to the calculations of the forward shock emission level, and it will be further discussed in §4.4.

In Fig. 3 we numerically plot the functions \(C_N(\sigma)\), \(C_\Delta(\sigma)\) and \(C_\gamma(\sigma)\) for both the mildly relativistic case (\(\gamma_4 = 1.5\)) and the extremely relativistic case (\(\gamma_4 = 1000\)). We can see that \(C_\Delta\) is insensitive to both \(\gamma_4\) and \(\sigma\), and we will treat it as a constant of order unity. The correction factor \(C_\Delta\) is rather insensitive to \(\gamma_4\) and is essentially a function of \(\sigma\) only. In the \(\sigma \gg 1\) regime, we have \(C_\Delta \propto \sigma^{-1/2}\). By definition, \(C_\gamma\) is the function of \(\sigma\) only.

4. Finally, the radius where the shell spreads is still defined by

\[
R_s \sim \frac{\gamma_4^2 \Delta_0}{\xi_4}. \tag{44}
\]

Taking the convention to define (Sari & Piran 1995)

\[
\xi \equiv (l/\Delta_{1/2})^{4/3}/\gamma_4, \tag{45}
\]

one has the following equation

\[
\frac{R_N}{\xi} = \frac{R_s}{C_\gamma} = \frac{\xi_4^{1/2}}{C_\Delta} R_\Delta = \frac{\xi_4^2 R_s}{\xi_4}, \tag{46}
\]

where

\[
\xi_4 = (l/\Delta_0)^{1/2}/\gamma_4 \sim \left(\frac{\gamma_4}{T}\right)^{1/2} \tag{47}
\]

is the \(\xi\) value for \(R \leq R_s\) (i.e. no spreading occurs), where

\[
\frac{R_s}{C_\gamma \gamma_4^2 c} \sim \left(\frac{l}{\gamma_4^4 c}\right)^{s} \tag{48}
\]

Notice that this notation is exactly the same as in the \(\sigma = 0\) case, and the total energy \(E\) is used (in defining \(l\)). This makes \(t_\gamma\) a constant not depending on \(\sigma\), which is convenient for the discussions of the various parameter regions in the next section.

3.3. Parameter regions: thick vs. thin shell regimes

Equating the four critical radii defines six critical lines in the \(\xi_4 - \sigma\) or the \(T/t_\gamma - \sigma\) space. This is justified by the fact that the spreading regime (which makes \(\xi\) deviating from \(\xi_4 = (t_\gamma/T)^{1/2}\)) always happens below the critical lines, and hence, does not influence the location of the critical lines. The six critical lines are

\[
R_\gamma \sim R_\Delta, \quad \frac{R_\gamma}{t_\gamma} \sim \left(\frac{C_\gamma}{C_\Delta}\right)^{4} \sim Q, (\propto \sigma^{2/3});
\]

\[
R_N \sim R_\Delta, \quad \frac{R_N}{t_\gamma} \sim C_\Delta^{-4/3} \sim Q, (\propto \sigma^{2/3});
\]

\[
R_N \sim R_\gamma, \quad \frac{R_N}{t_\gamma} \sim C_\gamma^{-2} \sim Q, (\propto \sigma^{2/3});
\]

\[
R_N \sim R_s, \quad \frac{R_N}{t_\gamma} \sim 1, (\propto \sigma^{0});
\]

\[
R_\gamma \sim R_s, \quad \frac{R_\gamma}{t_\gamma} \sim C_\gamma \sim Q^{-1/2}, (\propto \sigma^{-1/3});
\]

\[
R_\Delta \sim R_s, \quad \frac{R_\Delta}{t_\gamma} \sim C_\Delta^{1/3} \sim Q^{-1}, (\propto \sigma^{-2/3}). \tag{49}
\]
We can see that the first three lines have a same asymptotic behavior at high-$\sigma$, and calculations show that they essentially coincide with each other. Hereafter we define

$$Q(\sigma) \equiv [C_\gamma(\sigma)]^{-2} \sim C_{\Delta}^{4/3} \sim (C_\gamma/C_{\Delta})^4 \propto \sigma^{2/3}$$

so that $C_{\Delta} \sim Q^{-3/4}$ and $C_\gamma \sim Q^{-1/2}$, and eq. (50) can be re-written as

$$R_N^c = Q^{1/2} R_\gamma = \xi^{1/2} Q^{3/4} R_\Delta = \xi c R_s$$

In principle, changing the order between $R_N$ and $R_\gamma$ and between $R_\gamma$ and $R_s$ does not lead to essential modifications of the shock crossing and deceleration physics, so that the 4th and 5th lines in eq. (50) are not crucial. We therefore have two essentially lines that separate three physical regimes in the $T/t_\gamma - \sigma$ space (Fig. 4).

Region (I): the thick shell regime. This is the region where $T/t_\gamma > Q$ is satisfied. In this region, one has $R_N < R_\gamma < R_\Delta < R_s$. The downstream becomes relativistic with respect to the upstream before the reverse shock crosses the shell (Sari & Piran 1995). The full deceleration occurs at the end of shock crossing, i.e. at $R \sim R_\Delta$ (Kobayashi et al. 1999). Given a certain observed central engine activity time $T$, a total energy $E$, and an ambient density $n$, one can also define a critical Lorentz factor

$$\gamma_c \simeq 125 E_{52}^{1/8} n^{-1/8} T^{-3/8} Q^{3/8} \left(\frac{1 + z}{2}\right)^{3/8}$$

where some refined coefficients and the cosmological time dilution factor are explicitly taken into account in order to get the numerical value (here $z$ is the GRB redshift). For $\gamma_0 < \gamma_c$, one is in the thick shell regime, while for $\gamma_0 > \gamma_c$, one is in the thin shell regime. The function $Q(\sigma)$ is plotted as the top curve in Fig. 4. In the $\sigma \ll 1$ regime, $Q(\sigma) \sim 1$. For $\sigma \gg 1$, we have $Q(\sigma) \propto \sigma^{2/3}$, and hence, $\gamma_c \propto \sigma^{1/4}$. We can see that given same values for the other parameters, the parameter space for the thick shell regime is greatly reduced when $\sigma$ is high. More bursts are in the thin shell regime.

Region (II): the non-spreading thin-shell regime. This region is defined by $Q^{-1} < (T/t_\gamma) < Q$, in which $R_\Delta < R_\gamma < R_N$ and $R_\Delta < R_s$ are satisfied. The common features of this regime are that the reverse shock crosses the shell (at $R_\Delta$) before noticeable deceleration occurs (at $R_\gamma$, e.g. the Lorentz factor is reduced by roughly a factor of 2), that the relative speed between the upstream and the downstream never becomes relativistic, and that the shell does not spread during the first shock crossing. The separation between $R_\Delta$ and $R_s$ leads to some novel features for the reverse shock emission. During the first shock crossing the shell is heated so that electrons start to emit. However, after the first shock crossing, the shell is not decelerated significantly. It is difficult to delineate the detailed process at this stage, but a rough picture is that higher order shocks may form and bounce back and forth between the inner edge of the shell and the contact discontinuity. This happens until the shell reaches $R_\gamma$. It is likely that the shell remains heated by the multi-crossing of shocks and electrons keep radiating at a high level for an extended period of time. One then expects a broad reverse shock emission peak, which is a novel phenomenon in the high-$\sigma$ thin-shell regime. Notice that when $\sigma$ is very large, a reverse shock may not form at $R_\Delta$ at all. However, since $R_\Delta$ is the smallest in the problem, whenever a reverse shock forms, it quickly crosses the shell in a radius of $R_\Delta$, and the above discussion is still valid.

Region (III): the spreading thin-shell region. This is defined by $(T/t_\gamma) < Q^{-1}$, in which $R_s < R_\Delta < R_\gamma < R_N$ is satisfied. Since $R_\Delta < R_\gamma$, again multi-crossing of shocks are needed to slow down the ejecta, and the downstream never becomes relativistic with respect to the upstream. The novel feature in this region compared with the region (II) is that the shell starts to spread before shock crossing, so that the three radii have the relationship

$$C_{\gamma}^{2/3} R_N = R_\gamma = (C_\gamma/C_{\Delta})^2 R_\Delta$$

or

$$Q^{-1/3} R_N \simeq R_\gamma \simeq Q^{1/2} R_\Delta$$

We can see that the triple coincidence $R_N = R_\gamma = R_\Delta$ in the thin shell regime (Sari & Piran 1995) is only valid when $\sigma$ is small. According to Fig. 4, this practically happens when $\sigma \leq 0.01$.

3.4. Critical times

We finally derive the shock crossing time $t_x$ and the deceleration time $t_{dec}$ as measured by the observer. In the literature to study the $\sigma = 0$ regime, $t_{dec} = t_x \sim R_\Delta/\gamma^2 c \sim \max(T/t_\gamma, C_\gamma) \sim \max(T/t_\gamma)$ has been conventionally adopted (noticing $C_\gamma = 1$ when $\sigma = 0$). When an arbitrary $\sigma$ value is adopted, there are further complications to quantify these critical times. First, although in the thick shell regime (I) $t_{dec} = t_x$ is still valid, in the thin shell regimes (II and III) the deceleration radius $R_\Delta$ is larger than the shock crossing radius $R_\Delta$, so that $t_x < t_{dec}$. Second, the time scale $R_\Delta/\gamma^2 c$ only describes the delay time scales for the emission coming from the radius $R_\Delta$ with respect to the emission from the internal shock radius, for an infinitely thin shell. A more precise description of the reverse shock emission peak time should include the thickness of the radiation region. The real shock crossing time should correspond to the epoch when the emission from the end of the shell reaches the observer (see Fig. 5 for illustration). This gives

$$t_x \sim R_\Delta/\gamma^2 c + \frac{\Delta}{c}$$

Here $\Delta = \max(\Delta_0, R_s/\gamma^2)$, so that our definition is valid throughout the $(T/t_\gamma - \sigma$ space. In the $\sigma \ll 1$ limit,
we always have $R_\Delta / \gamma_4^2 c \sim \Delta / c$ so that to order of magnitude estimate, one can drop the latter term. In the $\sigma \gg 1$ limit, however, in certain regimes one could have $R_\Delta / \gamma_4^2 c \ll \Delta / c$. The correction factor introduced here is therefore essential to delineate the reverse shock behavior in the high-$\sigma$ regime. We notice that Nakar & Piran (2004) recently also noticed this correction within the context of $\sigma = 0$ regime, although their eq.(2) is slightly different from our definition.

In the thick shell regime (I) the reverse shock is relativistic at the crossing radius, and one has $\gamma_4^2 = \gamma_4((u_4/m_4)F)^{1/2}$. Using the definition of $R_\Delta$ (eq.(55)), one has $R_\Delta / \gamma_4^2 c \sim T C_\Delta^2 \sim TQ^{3/2}$. Since $\Delta / c = \Delta_0 / c = T$, with eq.(55), we have $t_{x}(I) = t_{dec}(I) \sim T(1 + Q^{-3/2})$. In the region (II), i.e., the non-spreading thick shell regime, one has $\gamma_3 \sim \gamma_4$ and $\Delta = \Delta_0$. Using eqs.(55) and (56), we get $t_{x}(II) \sim t_\gamma^3 T^{1/4} C_\Delta + T \sim \Delta_0 T^{1/4} Q^{-3/4} + T$. In the region (III), i.e., the spreading thin shell regime, one has $\gamma_2 \sim \gamma_4$ and $\Delta / c = R_\Delta / \gamma_4^2 c$. With eq.(55), one has $t_{x} \sim 2t_\gamma (C_\Delta^2 / C_\gamma) \sim 2t_\gamma Q^{-1}$. We define the deceleration time $t_{dec}$ as the epoch when the fireball is significantly decelerated. For the thick shell regime (I), this coincides with the shock crossing time. For the thin shell regimes (II, III), it is defined when the fireball reaches $R_\gamma$. Following the same argument to derive eqs.(56) (Fig.5), one can generally define

$$t_{dec} \sim \begin{cases} \frac{R_\gamma}{\gamma_4 c} + \frac{\Delta}{c} = t_x, & (I) \\ \frac{R_\gamma}{\gamma_4 c} + \frac{\Delta}{c} = t_\gamma C_\gamma + \Delta, & (II, III) \end{cases} \quad (56)$$

where we have interchanged $\gamma_2$ and $\gamma_4$ for the thin shell regimes. For the regime I, one has $t_{dec}(I) \sim t_\gamma Q^{-1/2} + T$, while for the regime III, one has $t_{dec}(III) \sim t_\gamma (Q^{-1/2} + Q^{-1})$.

For convenience, we collect the critical times as the following,

$$t_{x} \sim \begin{cases} T(1 + Q^{-3/2}), & (I) \\ L_2^{1/4} T^{1/4} Q^{-3/4} + T, & (II) \\ 2t_\gamma Q^{-1}, & (III) \end{cases} \quad (57)$$

and

$$t_{dec} \sim \begin{cases} T(1 + Q^{-3/2}), & (I) \\ t_\gamma Q^{-1/2} + T, & (II) \\ t_\gamma (Q^{-1/2} + Q^{-1}), & (III) \end{cases} \quad (58)$$

As demonstrated above, in the thin shell regime, the reverse shock emission should show a broad peak due to the separation between $t_\times$ and $t_{dec}$. The width of the peak of the reverse shock peak is defined as

$$t_{dec} - t_\times \sim \begin{cases} 0, & (I) \\ t_\gamma (Q^{-1/2} - (T/t_\gamma)^{1/4} Q^{-3/4}), & (II) \\ t_\gamma (Q^{-1/2} - Q^{-1}), & (III) \end{cases} \quad (59)$$

4. SYNCHROTRON EMISSION AND EARLY AFTERGLOW LIGHTCURVE

4.1. Particle acceleration

The hydrodynamical solution presented above is generic and lays a solid foundation for calculating synchrotron emission in the reverse shock and the early afterglow lightcurves. In this section, we will turn to the less certain aspects of the problem, i.e. particle acceleration and synchrotron emission from the reverse shock. In the conventional $\sigma = 0$ models, particles (ions and electrons) are assumed to be accelerated from the collision-less relativistic shocks through the first-order and probably also second-order stochastic Fermi acceleration mechanisms (Fermi 1949; Blanford & Eichler 1987). In the standard afterglow models, accelerated electrons are assumed to have a power law distribution with $dN_e / d\gamma_e \propto \gamma_e^{-p}$. Numerical simulations of particle acceleration in relativistic shocks confirm this simple treatment (e.g. Gallant et al. 1992; Achterberg et al. 2001) in the $\sigma = 0$ limit. For MHD shocks as discussed in this paper, more physical processes enter the problem. For example, in the 90° shock problem discussed in this paper, the existence of the electrostatic potential in the shock front plane and the influence of the Lorentz force exerted on the particles by the magnetic and electric fields in the upstream tend to trap ions in the shock plane. The influence of these effects on shock acceleration is unclear, although some investigations in this direction have started (e.g. Double et al. 2004; Spitkovsky & Arons 2004). Detailed treatments of particle acceleration in the high-$\sigma$ regime is beyond the scope of the current paper. Lacking a detailed model, here we simply extend the approach used in the $\sigma = 0$ regime to arbitrary $\sigma$ values, i.e. we assume a power law distribution of electron energy and assign the equipartition parameters $\epsilon_e$ and $\epsilon_B$ for the electrons and magnetic fields, respectively. Such an approximation is proven valid when $\sigma$ is small, but may progressively become not good enough as $\sigma$ increases, especially when $\sigma$ achieves very large values. The lightcurves we calculate below nonetheless provide a first-order picture on how the reverse shock emission level depends on $\sigma$. 

![Fig. 5. — The space-time diagram of the GRB fireball evolution. Solid lines are the world lines of both ends of the shell, and the dashed lines are the world lines of light. The lowest dashed line is the first light (from the internal shock) that reaches the observer, and the other two dashed lines indicate the light emitted from both ends of the shell at the shock-crossing radius, $R_\Delta$. The observed shock crossing time is the sum of the delay time $R_\Delta / (2\gamma_4^2 c)$ and $\Delta / c$. To order of magnitude estimate, the compression of the shell after shock crossing is not taken into account.](image-url)
4.2. Magnetic fields

We follow our previous approach (ZKM03) to compare the flux level between the reverse shock peak and the forward shock peak. This is because, when studying the ratio of the peak flux levels, only the ratios of the microphysics parameters (e.g. $g(p) = (p-2)/(p-1)$, $\epsilon_e$, $\epsilon_B$, etc.) matter, and one does not need to invoke the absolute values of those parameters which are rather uncertain. A crucial parameter to study synchrotron spectrum is the comoving magnetic field strength in both shocked regions. For the forward shock region, since the medium is usually not magnetized, the downstream magnetic field is usually quantified by a fudge parameter $\epsilon_{B,f}$, which reflects the strength of the magnetic field presumably generated in situ due to a certain plasma instability (e.g. Medvedev & Loeb 1999). This magnetic field is randomly oriented, as has been supported by the observed weak-polarization level for the optical afterglow emission (e.g. Covino et al. 2003 for a review). The strength of this magnetic field component is low, with $\epsilon_{B,f} \sim (0.01 - 0.001)$, as inferred from broadband afterglow fits (Panaitescu & Kumar 2002; Yost et al. 2003). For the reverse shock, in the current model the magnetic field is predominantly due to the compression of the upstream magnetic field. Its level depends on the $\sigma$ value of the upstream, and the field is globally structured, so that the optical flash due to the reverse shock emission (such as the ones observed from GRB 990123 and GRB 021211) should have been strongly polarized (see also Granot & Königl 2003; Fan et al. 2004a; Sagiv, Waxman & Loeb 2004).

The forward shock comoving magnetic energy density is defined by

$$\frac{B_f^2}{8\pi} = \frac{B_0^2}{8\pi} = c_2 \epsilon_{B,f},$$

where $\epsilon_{B,f}$ is the conventional magnetic equipartition parameter in the afterglow theory which delineates the fraction of the total internal energy that is distributed to magnetic energy. In the reverse shock region, the comoving magnetic energy density is dominated by the shock-compressed upstream magnetic field, and can be denoted as

$$\frac{B_r^2}{8\pi} = \frac{B_\gamma^2}{8\pi} = (f_c - 1) \frac{\epsilon_3}{3},$$

where $f_c \equiv 1 + p_{\gamma}/p_3$ (eq.[46]) has been used. For easy comparison (with respect to the conventional definition of $\epsilon_{B,f}$), we can write (61) as

$$\frac{B_r^2}{8\pi} = c_2 \epsilon_{B,r},$$

where

$$\epsilon_{B,r} \equiv \frac{(f_c - 1)}{3f_c}$$

is an artificial parameter to simplify the discussions. With this definition, we can write

$$R_B \equiv \frac{B_r}{B_f} = \left(\frac{\epsilon_{B,r}}{\epsilon_{B,f}}\right)^{1/2}.$$

This is an important parameter which delineates the ratio of the magnetic field strength in the reverse shock and forward shock regions. This ratio has been found to be larger than unity in GRB 990123 and GRB 020121 (e.g. ZKM03), and discussion of this parameter is very essential to quantify the relative emission properties of both shocks.

In Fig[6] we plot $\epsilon_{B,r}$ as a function of $\sigma$ for both $\gamma_3 = 1000$ and $\gamma_4 = 1.5$. We can see that it increases with $\sigma$ initially, and saturates at a value $1/3$ as $\sigma \gg 1$. The asymptotic behavior is already obvious in eq.[63]. We note that the number $1/3$ is a pure artificial effect given the definition of $\epsilon_{B,r}$ (eq.[62]). The “real” magnetic equipartition factor in the region $3$ approaches unity when $\sigma \gg 1$. In the figure we also plotted the magnetic equipartition parameter in the forward shock region, i.e., $\epsilon_{B,f} \sim 0.001$. This level could also be regarded as the “bottom level” in the $\sigma \ll 1$ regime for the reverse shock region. The thick lines in Fig[6] are the “total” $\epsilon_B$ in the reverse shock region (which include both the amplified structured field component and the random field component), which saturates to $\epsilon_{B,r}$ in the low-$\sigma$ regime.

An interesting conclusion drawn from Fig[6] is that no matter what $\sigma$ or $\gamma_3$ values are taken, the ratio $\epsilon_B/\epsilon_{B,f}$ is at most $\sim 300$ for $\epsilon_{B,f} \sim 0.001$. Or effectively, the reverse-to-forward shock magnetic field ratio $R_B$ can not be significantly larger than 15. Since this value was inferred from the case of GRB 990123 (ZKM03), we tentatively conclude that the ejecta in GRB 990123 has $\sigma > 0.1$ where $\epsilon_{B,r}$ reaches its peak value. The absolute values of $\epsilon_{B,r}$ and $\epsilon_{B,f}$ are consistent with those obtained from the detailed modeling. For example, $\epsilon_{B,f} \sim 7.4 \times 10^{-4}$ was inferred by Panaitescu & Kumar (2002), while $\epsilon_{B,r}/\epsilon_{B,f} \sim 15^2 \sim 225$ was inferred by ZKM03, so that $\epsilon_{B,r} \sim 0.17$. This is close to the maximum $\epsilon_{B,r}$ we have calculated.

4.3. Lightcurve peak times and peak fluxes

Before describing the detailed process of calculating afterglow lightcurves, it is informative to define the so-called peak times and peak fluxes. An early optical lightcurve usually consists two peaks (ZKM03 and ref-
erences there in), i.e., a forward shock peak which corresponds to the epoch when the typical synchrotron frequency crosses the band (Sari et al. 1998; Kobayashi & Zhang 2003a), and a reverse shock peak at which the flux achieves the maximum and starts to decay thereafter. This corresponds to the time when no more shock heating is available and the shell starts to cool adiabatically.

For the $\sigma = 0$ case, the first shock crossing time and the shell deceleration time coincide, so that $t_{p,r} = t_x = t_{dec}$. This is still the case when $\sigma$ is larger as long as the shell is in the thick shell regime (I). For thin shell cases (II and III), however, this is no longer the case, and $t_{x}$ and $t_{dec}$ separate from each other (eqs. (50a, 58)). For easy discussion, hereafter we define the time $t_{dec}$ as the reverse shock peak time, and its corresponding afterglow flux as reverse shock peak flux, i.e.

$$t_{p,r} = t_{dec}$$

$$F_{p,r} = F_{\nu}(t_{dec})$$

For $t > t_{p,r}$, the shell cools adiabatically, and a decaying lightcurve results. The $t < t_{p,r}$ case is a little more complicated. For the thick shell case (I), since $t_x$ coincides with $t_{p,r}$, it is a rising lightcurve due to shock crossing. For the thin shell cases (II and III), as demonstrated above, the shell is heated at first shock crossing and remains heated until it is significantly decelerated. This results in a broad reverse shock peak which starts at $t_x$ and ends at $t_{p,r}$. Much more detailed studies are needed to reveal the physics during this stage, but to a first-order estimate, in this paper we assume that the heating level between $t_x$ and $t_{p,r}$ is roughly the same, so that the lightcurve shows a plateau during this period.

To calculate the synchrotron radiation flux, one needs to quantify the comoving random Lorentz factor of the leptons. We still take the convention to assume that the shock accelerated leptons have a single power-law distribution with the indices $p_f$ and $p_r$ for the forward and the reverse shocks, respectively, and that they occupy a fraction $\epsilon_{e,f}$ and $\epsilon_{e,r}$ of the total thermal energy in the forward shock region (which is $c_{2}$) and in the reverse shock region (which is $c_{3}$), respectively. For the reverse shock region, the lepton density may be enriched by the presence of pairs generated in the prompt emission phase (e.g. Li et al. 2003b). The pair-multiplicity parameter $y \equiv (N_b + N_{\pm})/N_b \geq 1$ (66) may be of order unity or mildly large in the low-$\sigma$ regime (depending on the compactness of the region when the prompt gamma-rays are emitted, e.g. Kobayashi, Ryde & MacFadyen 2002; Mészáros et al. 2002), and could be very large in the high-$\sigma$ regime (e.g. Zhang & Mészáros 2002). In this paper, we are mainly focusing on the novel features introduced by the $\sigma$ parameter, and will take $y \sim 1$ in the following calculations. The $y$-dependences are included in the expressions and their implication will be discussed in [43, 44].

The minimum comoving electron energy in the region “$i$” (2 or 3) is $\gamma_{e,m,i} = (\epsilon_{e,i}/\gamma_{e,i}/n_i/m_pc^2) g(p_i)(m_p/m_e)$, where $g(p) = (p - 2)/(p - 1)$ (assuming $p > 2$). Taking the values at the first shock crossing time $t_{x}$, one gets

$$\gamma_{e,m,r}(t_x) = \left( \frac{\epsilon_{e,r} g_{y,e,r}}{y_{e,f} g_f} \right) f_a \gamma_{34}(t_x) - 1 \sim R_e f_a \gamma_0 \gamma_y^{-1} \gamma_x^{-1},$$

where we have defined

$$R_e = \left( \frac{\epsilon_{e,r} g_{y_e}}{\epsilon_{e,f} g_f} \right),$$

used $\gamma_{34} \approx \gamma_4/\gamma_2(x)$ (which is valid for both thick and thin shells), and replaced $\gamma_4$ and $\gamma_2(x)$ by $\gamma_0$ (which means the initial Lorentz factor) and $\gamma_x$ (the fireball Lorentz factor at the shock crossing time), respectively. This allows the same notation system as in our previous work (ZKM03).

We are more interested in the behavior at the reverse shock peak time (i.e. the deceleration time), $t_{p,r}$. For the thick shell case, this is simply $t_x$. For the thin shell case, after the first shock crossing, the shell is kept heated by multi-crossing of successive shocks. To first order, we can take the approximation that the heating level in the ejecta during the time period between $t_x$ and $t_{p,r}$ is approximately constant, i.e., $c_{3}/\gamma_{34} \propto t_{p,r}$. During the same time period the random Lorentz factor in the shocked ISM region also remains constant (since $\gamma_{2} \approx \gamma_4$ before deceleration), we therefore also have

$$\gamma_{e,m,r}(t_{p,r}) \sim \gamma_{e,m,f}(t_x) \sim R_e f_a \gamma_0 \gamma_y^{-1} \gamma_x^{-1},$$

which is valid for both the thick and thin shell cases.

Let us denote the total electron numbers in the forward and reverse shocked region as $N_{e,f}$ and $N_{e,r}$, respectively. In the reverse shock region, the total lepton number (now includes pairs) is $N_{e,r} = N_b + N_{\pm} = y N_b$, where the definition in eq. (66) is used. According to eq. (31), at the deceleration radius, the total energy in the forward shock region is defined by $E_{K}$ alone. Although the bulk of the Poynting energy is expected to be transferred to the ISM eventually, shortly after the shock crossing and near the forward shock peak, this correction may not be significant. Below we will ignore this process in our calculations of the forward shock emission, but keeping in mind that the real forward shock emission level would increase with time, and may be much higher than our predicted level at later times. When we focus on early afterglow lightcurves, our calculations should be close to the real emission level (see more discussions in Zhang & Kobayashi 2005). A more careful treatment will be presented in a future work.

In our approximated treatment, one can write

$$E_{K} = \gamma_0 c^2(N_b m_p + N_{\pm} m_e) \sim \gamma_0 N_b m_p c^2 \sim N_{e,f} m_p c^2 \gamma_{dec}^2 \sim N_{e,f} m_p c^2 \gamma_y^2 \gamma_x^2,$$

where we have assumed $y \ll m_p/m_e$, so that the total pair mass $N_{\pm} m_e$

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7 When $\sigma$ is very large, additional heating for the shell may still happen if the post-shock-crossing energy transfer process time scale is short enough.

8 Strictly speaking, the last factor $\gamma_{34}/\gamma_x^2$ in eq. (47) should be $\gamma_{0} - \gamma_{x})/\gamma_y^2$. The current approximation is valid as long as the reverse shock is mildly relativistic, say, $\gamma_{34} > 1.5$. For an even smaller $\gamma_{34}$ (which could be possible when $\sigma$ is large enough), there should be an additional correction factor (less than unity) in both eqs. (47) and (68).
is much smaller than the total baryon mass $N_v m_p$. This gives

$$N_{v,r}(t_{p,r}) \sim \frac{\gamma^2}{\gamma_0}.$$  

(70)

The characteristic synchrotron emission frequency is $\nu_m \propto \gamma B_n \gamma_m$, the cooling frequency is $\nu_c \propto \gamma^{-1} B^{-3} t^{-2}$, and the peak specific flux is $F_{\nu, m} \propto \gamma BN_c$, where $\gamma$ is the bulk Lorentz factor. For the thin shell case, we make another approximation that $B_3$ keeps constant from $t_x$ to $t_{dec}$ (so that $R_B(t_x) \simeq R_B(t_{dec}) = R_B$). Similar to Kobayashi & Zhang (2003a) and ZKM03, we can finally derive the following relations at $t_{dec}$:

$$\frac{\nu_{m,r}(t_{p,r})}{\nu_{m,f}(t_{p,r})} \sim \gamma^2 R_B^2 R_B f_a^2 \gamma^{-2},$$  

(71)

$$\frac{\nu_{c,r}(t_{p,r})}{\nu_{c,f}(t_{p,r})} \sim R_B^{-3},$$  

(72)

$$\frac{F_{\nu,m,r}(t_{p,r})}{F_{\nu,m,f}(t_{p,r})} \sim \frac{\gamma R_B}{y},$$  

(73)

where

$$\tilde{\gamma} \equiv \frac{\gamma^2}{\gamma_0} = \min \left( \gamma_0, \frac{\gamma^2}{\gamma_0} \right) \leq \gamma_c.$$  

(74)

Although there are in principle many cases of the reverse shock emission lightcurves (Kobayashi 2000), within the reasonable parameter regime the lightcurve behavior only has two variations depending on whether $R_\nu > 1$ or $R_\nu < 1$ (ZKM03) where

$$R_\nu \equiv \frac{R_B}{\nu_{m,r}(t_{p,r})}.$$  

(75)

In both cases, the ratio between the two peak-time fluxes

$$R_F \equiv \frac{F_{\nu,m,r}(t_{p,r})}{F_{\nu,m,f}(t_{p,r})}$$  

(76)

and the ratio between the two peak times

$$R_t \equiv \frac{t_{p,f}}{t_{p,r}}$$  

(77)

can be expressed in terms of $\tilde{\gamma}$, $R_B$ and $R_\nu$, respectively, for the $\sigma = 0$ case (ZKM03). Here the forward shock peak time $t_{p,f}$ corresponds to the epoch when $\nu_{m,f}$ crosses the band.

Below we repeat this process, but focus more on corrections introduced by the $\sigma$ factor. To further simplify the problem, we first estimate the numerical value of $R_\nu$. Due to the complication introduced by the $\sigma$ parameter, one can not close $R_\nu$ into a simple expression as in the $\sigma = 0$ case (e.g. eq.[24] in ZKM03). In any case, using eq.[21] and the standard expression for $\nu_{m,f}(t)$ (e.g. eq.[1] in Kobayashi & Zhang 2003a), one can write

$$R_\nu \sim 800 R_B^{-1} f_a^{-2} y^2 R_c^{-2} \left( \frac{k E_K}{10^{52} \text{erg}} \right)^{-1/2} \left( \frac{\tilde{\gamma}_2}{100} \right)^2 \times \left( \frac{t_{dec}}{0.001 \text{ s}} \right)^{-1/2} \left( \frac{c_{s,f}}{1.0} \right)^{-2} \left( \frac{g_f}{1/3} \right)^{-2} \times \left( \frac{t_{dec}}{100 \text{ s}} \right)^{3/2} \left( \frac{1 + z}{2} \right)^{-1/2}.$$  

(78)

9 When $\sigma$ is large enough, additional correction factor to $t_{p,f}$ is needed, but this factor is small if the energy injection time scale is long enough.

For $\tilde{\gamma} \leq \gamma_c \sim 125$ (eq.[22]), $R_c \sim 1$, $y \geq 1$, and $f_a < 1$, the above equation therefore essentially always gives $R_\nu > 1$. In the following discussions, we will not discuss the $R_\nu < 1$ case any further (which was also discussed in ZKM03).

The reverse shock emission lightcurve in the $R_\nu > 1$ case is simple. The lightcurve initially rises and reaches the peak at $t_x$. The flux level then keeps essentially constant until $t_{p,r}$ (for the thick shell case, both time scales coincident, so that there is no broadened peak), and starts to decay after $t_{p,r}$. The temporal indices of each segment of the lightcurve is also well-defined. In the rising part of the lightcurves, since all the correction factors introduced by the $\sigma$ parameter are essentially time-independent, the corrections essentially do not introduce extra time-dependence on the typical frequencies and the peak flux of the synchrotron radiation in the reverse shock. The rising lightcurves essentially remain unchanged as the $\sigma = 0$ case, as has been derived by Kobayashi (2000). This gives a $\sim 1/2$ temporal index for the thin shell case, and a $\sim 5$ temporal index for the thick shell case.

After the deceleration time, the shell cools. The optical band is typically in the regime of $\nu_{m,r}(t_x) < \nu_R < \nu_{c,r}(t_x)$. After the deceleration time, one has $\nu_{m,r} \propto t^{-3/2}$, $F_{\nu,m,r} \propto t^{-1}$ (Kobayashi 2000). Thus the temporal decay index (i.e. $F_{\nu} \propto t^{-\alpha}$) is

$$\alpha = \frac{3 p_r + 1}{4} \sim 2,$$  

(79)

where $p_r$ is the electron power-law index in the reverse shock region.

For $t > t_{p,r}$, one has $\nu_{m,f} \propto t^{-3/2}$, $F_{\nu,m,f} \propto t^0$ (Mészáros & Rees 1997a) and $\nu_{m,r} \propto t^{-3/2}$, $F_{\nu,m,r} \propto t^{-1}$ (Kobayashi 2000). Using the definitions of $R_F$ and $R_t$ (eqs.[78-77]) as well as eqs. (71) and (77), one can derive

$$R_F = \tilde{\gamma}^{4/3} R_B^{2/3} R_\nu^{-2/3} (R_c^{-4/3} y^{4/3} f_a^{-4/3}),$$  

(80)

$$R_F = \tilde{\gamma}^{1/3} R_B^{1/3} R_\nu^{-1/3} (y).$$  

(81)

These are valid for all the three parameter regions in Fig. These are compared with eqs.[12,13] in ZKM03, the extra correction factors are presented in parenthesis. Notice that the correction factors $R_c$ and $y$ should also exist in $\sigma = 0$ case, but we have previously assumed them to be unity. The extra $\sigma$-dependent correction factors are $f_a^{4/3}$ and $R_\nu$ (which is modified by the $\sigma$ parameter through many factors, e.g. $R_B$, $f_a$, $E_K$ and $t_{dec}$, see eq.[25]).

4.4. Sample lightcurves

We now calculate the typical early optical afterglow lightcurves for various parameter regimes. Equation 110 Detailed numerical calculations result in non-power law behavior in the rising lightcurve (Fan et al. 2004a).

11 A more detailed discussion such as that presented in Kobayashi & Sari (2000) leads to the similar conclusion.

12 Notice again that here we have assumed that the energy transfer time scale from a Poynting flux to the kinetic energy of the ISM long enough. This forward shock emission level should be regarded as a lower limit when the energy transfer process is taken into account.

13 In ZKM03, we have defined $R_F$ and $R_t$ at $t_x$, but in $\sigma = 0$ case one has $t_x = t_{dec}$ For the case of an arbitrary $\sigma$, the deceleration time $t_{dec}$ is more fundamental to define the problem.
states that the initial afterglow energy, which is essentially the kinetic part of the total energy, decreases with $\sigma$ given a constant total energy $E = E_K + E_P$. At high-$\sigma$, not only the reverse shock flux level drops, the forward shock flux level shortly after the shock crossing also decreases steadily. At later times, the forward shock level would increase due to magnetic energy injection. Since we are focusing on the early afterglow emission, this effect will be neglected in the following discussions. To explore the $\sigma$-effect, we fix the total energy of the fireball so that $E_K$ decreases with increasing $\sigma$. To simplify the calculations, we assume $R_e \sim 1$ and $y \sim 1$. The input parameters we adopt include $E_{52} = 1$, $\gamma_0 = 150$, $n = 1$, $\epsilon_{e,f} = 0.1$, $\epsilon_{B,f} = 0.001$, $p_f = 2.2$, and $z = 1$ (with the standard cosmological parameters $\Omega_m \sim 0.7$, $\Omega_m \sim 0.3$ and $H_0 \sim 70$ km s$^{-1}$ Mpc$^{-1}$). This gives the forward shock peak time and flux (Sari, Piran & Narayan 1998; Kobayashi & Zhang 2003a)

$$t_{p,f} \sim 1000 \text{ s}$$

$$F_{p,f} \sim 1.7(1 + \sigma)^{-1} \text{ mJy}$$

$$m_R \sim 15.6 + 2.5 \log(1 + \sigma)$$.

We also have $t_\gamma = [(3E/4\pi\gamma_0^2 n m_p c^2)^{1/3}/2\gamma_0 c](1 + z) \sim 60$ s. (For this calculation, we have added in all the precise coefficients previously neglected.) We then take two typical values of GRB durations. For the first case, we take $T = 100$ s. When $\sigma$ is small the burst is in the thick shell regime (region I). As $\sigma$ increases, the burst is in the non-superposing thin shell regime (region II). For the second case, we take $T = 20$ s. The burst is always in the thin shell regime for any $\sigma$ value, but transform from the spreading thin shell regime (region III) to the non-superposing thin shell regime (region II) when $\sigma$ is large enough.

For each $T$ value, we calculate both the reverse shock and the forward shock lightcurve for several values of $\sigma$, i.e. $\sigma = 0, 0.001, 0.01, 0.1, 1, 10, 100$ (Fig. 6), as long as the condition for the existence of the reverse shock (eq. 13) is satisfied. The procedure of our calculation is the following. First, with $t_\gamma$, $T$ and the assumed $\sigma$, one can judge which magnetic region the burst is in. With this information one can then calculate $t_{\text{dec}} = t_{p,r}$ and $R_k$ for the thick and thin shell regimes, as well as $t_\times$ for the thin shell case. Next, we calculate $R_B$ with the assumed $\sigma$ value (Fig. 6). For the thick shell case, we use the $\epsilon_B$ value for $\gamma_{34} \sim 1000$ since $\epsilon_B$ is insensitive to $\gamma_{34}$ when it is large. For the thin shell case we use the $\epsilon_{B_f}$ value for $\gamma_{34} \sim 1.5$, exclusively14. One can then solve $R_k$ (eq. 33), and then use the value of $R_k$ to calculate $R_F$ (eq. 31), and hence $F_{p,f,r}$. Since we know the temporal indices of the reverse shock lightcurve during the rising ($\sim 1/2$ for thick shell and $\sim 5$ for thin shell, Kobayashi 2000) and the decaying phase ($\sim -1.9$ for $p_r = 2.2$), the reverse shock lightcurve can be calculated once $t_{p,r} = t_{\text{dec}}$ and $F_{p,r}$ are known. For the thin shell case, with the current approximation, we roughly keep $F_{p,r}$ a constant between $t_\times$ to $t_{\text{dec}}$, both of which are known. For the forward shock emission, the temporal index is $3(1 - p_f)/4$ ($\sim 0.9$ for $p_f = 2.2$) after the peak time, and is $1/2$ before the peak time (but after the deceleration time)15. Given $F_{p,f}$ (which is dependent on $\sigma$ (eq. 25)), the forward shock lightcurve is also calculated.

Some sample R-band early afterglow lightcurves are presented in Fig. 7 with the contributions from both the reverse and the forward shocks superposed. For the forward shock emission component we assume that the time scale for the energy transfer from the Poynting energy to the afterglow energy is long enough, so that the forward shock level is defined by $E_K$ only, and its level decreases with $\sigma$. This approximation is good shortly after the reverse shock crossing. At later times, the real level could be progressively higher than this level and the calculation should be regarded as a lower limit. Lightcurves are calculated for different $\sigma$ values. Thick solid: $\sigma = 0$; thin dash-dotted: $\sigma = 0.001$; thin dotted: $\sigma = 0.1$; thin solid: $\sigma = 1$; thick dashed: $\sigma = 10$; thick dash-dotted: $\sigma = 100$. $\gamma_{34} = 1.5$ has been assumed for thin shell regimes. For high-$\sigma$ cases, $\gamma_{34}$ is closer to unity, and the reverse shock peak flux should be further suppressed. (a) $T = 100$ s case. According to eq. 25, the reverse shock exists when $\sigma < 200$. (b) $T = 20$ s case. The condition for the existence of the reverse shock is $\sigma < 20$.15

14 Notice that in reality, when $\sigma$ is very large, $\gamma_{34}$ could be much closer to unity. In such cases, the reverse shock peak flux should be further suppressed.

15 Before the deceleration time, the forward shock lightcurve should have different temporal slopes. During the shock crossing, we have $\gamma_2 \propto t^0$ for thin shells and $\gamma_2 \propto t^{-1/4}$ for thick shells. Using the standard synchrotron radiation analysis (e.g. Sari et al. 1998), the forward shock emission temporal slope is 3 and 4/3 for the thin and thick shell cases, respectively. Between the shock crossing time and the deceleration time in the high-$\sigma$ thin shell case, the temporal slope is flat.

**Fig. 7.— Sample early afterglow lightcurves for GRBs with an arbitrary magnetization parameter $\sigma$. Following parameters are adopted. $E_{52} = 1$, $\gamma_0 = 150$, $n = 1$, $\epsilon_{e,f} = 0.1$, $\epsilon_{B,f} = 0.001$, $p_f = 2.2$, and $z = 1$. We also take $R_e \sim 1$ and $y \sim 1$. Both the forward shock and the reverse shock emission components are calculated and they are superposed to get the final lightcurve. For the forward shock emission component we assumed that the time scale for the energy transfer from the Poynting energy to the afterglow energy is long enough, so that the forward shock level is defined by $E_K$ only, and its level decreases with $\sigma$. This approximation is good shortly after the reverse shock crossing. At later times, the real level could be progressively higher than this level and the calculation should be regarded as a lower limit. Lightcurves are calculated for different $\sigma$ values. Thick solid: $\sigma = 0$; thin dash-dotted: $\sigma = 0.001$; thin dotted: $\sigma = 0.1$; thin solid: $\sigma = 1$; thick dashed: $\sigma = 10$; thick dash-dotted: $\sigma = 100$. $\gamma_{34} = 1.5$ has been assumed for thin shell regimes. For high-$\sigma$ cases, $\gamma_{34}$ is closer to unity, and the reverse shock peak flux should be further suppressed. (a) $T = 100$ s case. According to eq. 25, the reverse shock exists when $\sigma < 200$. (b) $T = 20$ s case. The condition for the existence of the reverse shock is $\sigma < 20$.**
tion should be regarded as a lower limit (see Zhang & Kobayashi 2005 for more explanations). In Fig. 7, the cases for \( T = 100 \) s are calculated. According to eq. (1), a reverse shock exists as long as \( \sigma < 200 \). We therefore calculate the lightcurves up to \( \sigma = 100 \). We can see that when \( \sigma \leq 1 \), the parameters are in the thick shell (relativistic reverse shock) regime. When \( \sigma \) increases from below, the contrast between the reverse and forward shock peak fluxes (i.e. \( R_F \)) increases steadily. Since the forward shock emission level does not change much when \( \sigma < 1 \), the reverse shock peak flux increases steadily with \( \sigma \). At even higher \( \sigma \) values, the reverse shock peak flux drops steadily with \( \sigma \). In the mean time, the burst enters the thin shell regime so that the separation between the peak fluxes of the forward and reverse shock increases (i.e. \( R_F \)) increases steadily with \( \sigma \). When \( \sigma \) decreases, the reverse shock emission has a broader peak. In Fig. 7, the cases for \( T = 20 \) s are calculated. According to eq. (6), a reverse shock exists as long as \( \sigma < 20 \), and we calculate the lightcurves up to \( \sigma = 10 \). The shell is in the thin shell regime throughout the whole \( \sigma \) range calculated. The transition from spreading to non-spreading thin shell regime does not bring any noticeable signature in the lightcurves. Again, the reverse shock peak flux increases with \( \sigma \) initially (when \( \sigma \leq 1 \)), and starts to decrease when \( \sigma \geq 0.1 \). The reverse shock peak is broad, but the separation gradually shrinks due to the decrease of the \( (Q^{-1/2} - Q^{-1}) \) parameter (eq. (59)). Throughout our calculations, \( R_\nu \) remains larger than 25 (up to \( \approx 1000 \) for \( \sigma = 0 \) in the \( T = 100 \) s case), so that our treatment by neglecting \( R_\nu < 1 \) regime is justified.

We notice several interesting features from our results. First, the reverse shock component is still noticeable even with \( \sigma \geq 1 \) (until reaching several tens or even hundreds when the condition (63) is no longer satisfied). The absolute reverse shock peak flux increases with \( \sigma \) initially, but drops steadily when \( \sigma > 1 \). Second, the forward shock emission level right after shock crossing also drops with \( \sigma \). This is because only the kinetic energy of the baryonic component \( (E_K) \) defines the afterglow level after the shock crossing time. The forward shock level will increase later due to the transfer of the remaining magnetic energy into the medium. One then expects an initially dim early afterglow for a high-\( \sigma \) flow, which would be brightened at later times. If GRB prompt emission is due to magnetic dissipation (e.g. Drenkhahn & Spruit 2002), and if \( \alpha \) is still high in the afterglow phase (e.g. \( \approx 10 \)), one may account for the very large apparent GRB efficiencies inferred from some GRBs (e.g. Lloyd-Ronning & Zhang 2004). Such a picture may be also relevant to the recent December 27 giant flare afterglow from the soft gamma-ray repeater 1806-20, for which a very high gamma-ray efficiency is inferred (Wang et al. 2005 and references therein). Third, the broad reverse shock peak is a novel feature identified in the high-\( \sigma \) model, it can be used to diagnose the existence of a Poynting-flux-dominated flow. The physical origin of the broad peak is that a high-\( \sigma \) value leads the decoupling of the shock crossing radius \( R_s \) and the deceleration radius \( R_d \), so that multi-crossing of a series of successive shocks leads to continuous heating of the ejecta shell before cooling starts.

In the above calculations, \( y = 1 \) has been adopted (i.e. we assume that the pair fraction is negligible in the ejecta). In some cases, especially in the high-\( \sigma \) regime, \( y \) could be much larger than unity. It would be essential to investigate the \( y \)-dependence of the current analysis. Solve \( R_\nu \) from eq. (59) and submit it to eq. (64), we find \( R_F \propto y^{(7/4 - \alpha)} \), which is \( \propto y^{-1/3} \) for \( \alpha = 2 \). We can see that a larger \( y \) will lower the reverse-to-forward shock peak flux contrast, although the dependence is mild. The \( R_F \) factor is more sensitive to \( R_\nu \) (i.e. \( \propto R_\nu^{(\alpha - 1/3)} \)), but assuming a similar shock acceleration mechanism, \( R_\nu \) may not deviate too much from unity.

Our results can be directly compared with the early afterglow data of the four bursts whose such information is available so far. The case of GRB 990123 (Akerlof et al. 1999) is consistent with a flow with \( 0.1 < \sigma < 1 \) in which regime \( R_F \) is large and the reverse shock peak is not broadened. The observed bright afterglow also argues against a higher-\( \sigma \) flow. The case of GRB 021211 (Fox et al. 2002b; Li et al. 2002a) also shows a large \( R_F \), which also suggests that \( \sigma > 0.1 \). For GRB 021004, Kobayashi & Zhang (2003a) attempted to fit the data with the \( \sigma = 0 \), \( R_B = 1 \) model. Another data point at an earlier epoch after the burst trigger reported by Fox et al. (2003a) makes that model more difficult to fit, and it has been attributed to a continuously energy injection (Fox et al. 2003a) or to the emission from a wind-type medium (Li & Chevalier 2003). However, using the theory developed in this paper, the data may be consistent with a high-\( \sigma \) flow (e.g. \( \sigma > 10 \)), so that the extended early afterglow emission could be interpreted as the combination of the broad reverse shock peak and the gradual transfer of the Poynting energy into the afterglow energy. In the high-\( \sigma \) regime, the \( (\gamma_{33} - 1) \) is quite small during the shock crossing, which will lower the reverse shock peak flux and \( R_F \). Similarly, a broad early afterglow bump was identified in GRB 030418 (Rykoff et al. 2004) which challenges the conventional reverse shock model but may be consistent with a high-\( \sigma \) flow (weak or no reverse shock component). Although detailed modeling is needed (we plan to do it in a future work), we tentatively conclude that all the current early optical afterglow data may be understood within the theoretical framework developed in this paper, if \( \sigma \) is allowed to vary for different GRB fireballs.

5. CONCLUSIONS AND DISCUSSION

We have derived a rigorous analytical solution for the relativistic 90° shocks under the ideal MHD condition (eq. (60)). Generally, the solution depends both on the magnetization \( \sigma \) parameter and the Lorentz factor of the shock, \( \gamma_{12} \). The solution can be reduced to the Blandford-McKee hydrodynamical solution when \( \gamma = 0 \), and to the Kennel-Coronti solution (which depends on \( \sigma \) only) when the \( \gamma_{21} \gg 1 \). Our generalized solution can be used to treat the more general cases, e.g. when the reverse shock upstream and downstream are mildly relativistic with each other. Since GRBs invoke a shell with finite width, this latter possibility is common (e.g. the parameter space for thin shell greatly increases in the high-\( \sigma \) regime), so that our generalized solution is essential to deal with the GRB reverse shock problem.

Several interesting conclusions emerge from our analysis. (1) Strong shocks still exist in the high-\( \sigma \) regime, as long as the shock is relativistic. Figs. 2-4 indicate that as \( \sigma \) increases, both the downstream “temperature” \( e_2/n_2 \)
and the “shock compression factor” \( n_2/n_1 \) decreases with respect to the \( \sigma = 0 \) values. However, the suppression factors in both cases are only mild (a factor of \( \sim 0.5 \)), and they saturate when \( \sigma \gg 1 \). In the relativistic shock regime, the results are actually consistent with Kennel & Coroniti (1984). However, these authors did not calculate the suppression factor with respect to the \( \sigma = 0 \) case, and did not explore further into the high-\( \sigma \) regime, so that their results leave the impression to the readers that the shock is completely suppressed when \( \sigma \) reaches higher values. For typical GRB parameters, we found that a reverse shock still exists when \( \sigma \) is as high as several tens or even hundreds. When the reverse shock exists, its emission level decreases when \( \sigma \) gets higher. This is not only because the reverse shock becomes weaker since \( \gamma_{34} \) gets close to unity in the high-\( \sigma \) regime, but also because the total kinetic energy in the flow (which is the energy reservoir for shock dissipation) gets smaller given a same total energy. (2) When discussing ejecta-medium interaction, somewhat surprisingly, some important parameters, such as \( F \) and \( C_\Delta \), are very insensitive to the reverse shock Lorentz factor, \( \gamma_{34} \), and can be regarded as the function of \( \sigma \) only (§2.3). This greatly simplify the problem, and is essential to characterize the parameter regimes. (3) The triple coincidence of the first three critical lines in eq.(49) is very crucial for a self-consistent description of the problem (§2.3).

Comparing with the conventional hydrodynamical treatment, we reveal several novel features for the early lightcurves. First, as \( \sigma \) increases, the reverse shock peak flux increases rapidly initially, reaching a peak around \( \sigma \sim (0.1-1) \), and starts to decrease when \( \sigma \geq 1 \). Second, due to the inability of tapping the Poynting flux energy during the shock crossing process, the fireball deceleration radius for the thin shell case decreases as \( \sigma \) increases \([\propto (1+\sigma)^{-1/3}]\). The forward shock emission level is also lower right after shock crossing. Third, in the high-\( \sigma \) thin shell regime, the reverse shock peak is broadened due to the separation of the shock crossing radius and the deceleration radius. This is a signature for a high-\( \sigma \) flow, which can be used to diagnose the magnetic content of the fireball. Fourth, as \( \sigma \) becomes large enough (larger than several tens or several hundreds), the condition for forming a reverse shock is no longer satisfied, and there should be no reverse shock component in the early afterglow lightcurves. This could be consistent with very early tight optical upper limits for some GRBs, such as the recent Swift dark burst GRB 050319a (Roming et al. 2005). In summary, the above new features allows the current theory to potentially interpret known GRB early afterglow cases collected so far as well as the case of the dark bursts, if one allows \( \sigma \) to vary in a wide enough range (say, from 0.01 to 100). The Swift GRB mission, launched on November 20, 2004, is expected to detect many early optical afterglow lightcurves with the UV-optical telescope on board. We expect to further test our theoretical predictions against the abundant Swift data, and to systematically diagnose the magnetic content of GRB fireballs.

If the GRB ejecta is indeed magnetized, as inferred from the early afterglow data, the internal shocks should also be corrected by the magnetic suppression factor. This aspect has been investigated by Fan, Wei & Zhang (2004b)

Throughout the paper, we have treated the problem under the ideal MHD limit. In the high-\( \sigma \) case, strong magnetic dissipation may occur. The magnetic dissipation effect has been included in the internal shock study of Fan et al. (2004b). Our treatment in this paper presents a first order picture to the early afterglow problem (see also Fan et al. 2004a, whose treatment in the mildly magnetized regime is consistent with ours), and further considerations are needed to fully delineate the physics involved. Also our whole discussion is relevant when a reverse shock is present. It does not apply to the regime for an even higher \( \sigma \) value (e.g., Lyutikov & Blandford 2003). Finally, as discussed in §4.1, further studies on particle acceleration in MHD shocks are essential to give a more accurate calculation on the reverse shock emission in the high-\( \sigma \) regime.

We only discussed one type of the medium, i.e., assuming a constant medium density, typically for the interstellar medium. In principle, the medium density can vary with distance from the central engine. In particular, a wind-type medium, characterized by the \( n \propto R^{-2} \) profile, has been widely discussed. Our MHD shock theory could be straightforwardly used to the wind case to study the reverse shock emission in combination with the previous pure hydrodynamical treatments (Chevalier & Li 2000; Wu et al. 2003; Kobayashi & Zhang 2003b; Kobayashi, Mészáros & Zhang 2004, see Fan et al. 2004a for a preliminary treatment).

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APPENDIX

DERIVATION OF THE SOLUTION OF THE RELATIVISTIC 90° SHOCK JUMP CONDITIONS

Lorentz transformations

Given the definitions of \( \gamma_{ij}, \beta_{ij} \) and \( u_{ij} \), the following Lorentz transformations are frequently used in the derivations.

\[
\beta_{2s} = \frac{\beta_{1s} - \beta_{21}}{1 - \beta_{1s} a_{21}} \tag{A1}
\]

We only discussed one type of the medium, i.e., assuming a constant medium density, typically for the interstellar medium. In principle, the medium density can vary with distance from the central engine. In particular, a wind-type medium, characterized by the \( n \propto R^{-2} \) profile, has been widely discussed. Our MHD shock theory could be straightforwardly used to the wind case to study the reverse shock emission in combination with the previous pure hydrodynamical treatments (Chevalier & Li 2000; Wu et al. 2003; Kobayashi & Zhang 2003b; Kobayashi, Mészáros & Zhang 2004, see Fan et al. 2004a for a preliminary treatment).

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The equation (8) in the text can be then derived straightforwardly.

Let us define (Kennel & Coroniti 1984)
\[ Y = \frac{B_{2s}}{B_{1s}} = \frac{\gamma_{1s}u_{1s}}{\gamma_{1s}u_{2s}} = \frac{\beta_{1s}}{\beta_{2s}}, \]  
(A11)

the equations (B) and (H) can be re-written as
\[ \gamma_{1s}\mu_{1}[1 + (1 - Y)\sigma] = \gamma_{2s}\mu_{2} \]  
(A12)
\[ u_{1s}\mu_{1}\left[1 + \frac{\sigma}{2\beta_{1s}}(1 - Y^2)\right] = u_{2s}\mu_{2} + \frac{p_{2}}{n_{2}u_{2s}}. \]  
(A13)

Multiply eq. (A12) by \( \beta_{1s} \) and substitute the resultant formula into eq. (A13), one can derive
\[ \frac{n_{2}m_{p}c^2 + e_{2}}{n_{2}} = \left(\gamma_{1s}\gamma_{2s}[1 + (1 - Y)\sigma] - u_{1s}u_{2s}[1 + \frac{\sigma}{2}(\beta_{1s}^{-2} - \beta_{2s}^{-2})]\right)\mu_{1} \]
\[ = \left(\gamma_{21} - \frac{u_{21}^{2}}{2u_{1s}u_{2s}}\sigma\right)\mu_{1}. \]  
(A14)

The equation (B) in the text can be then derived straightforwardly.

Solving \( u_{2s}^{2} \)

Combining eq. (A12) and the definition of \( \mu_{2} \) (eq. (B)), one can derive
\[ \frac{\gamma_{1s}}{\gamma_{2s}}[1 + (1 - Y)\sigma] = 1 + \hat{\Gamma}(\gamma_{21} - 1) - \frac{\hat{\Gamma}u_{21}^{2}}{2u_{1s}u_{2s}}\sigma. \]  
(A15)

This turns out to be a three-order equation of \( x \equiv u_{2s}^{2} \), i.e.
\[ Ax^{3} + Bx^{2} + Cx + D = 0, \]  
(A16)

where
\[ A = \hat{\Gamma}(2 - \hat{\Gamma})(\gamma_{21} - 1) + 2 \]  
(A17)
\[ B = -(\gamma_{21} + 1)\left[2 - \hat{\Gamma}(\gamma_{21}^{2} + 1) + \hat{\Gamma}(\gamma_{21} - 1)\gamma_{21}\right]\sigma \]
\[ - (\gamma_{21} - 1)\left[\hat{\Gamma}(2 - \hat{\Gamma})(\gamma_{21}^{2} - 2) + (2\gamma_{21} + 3)\right] \]  
(A18)
\[ C = (\gamma_{21} + 1)\left[\hat{\Gamma}(1 - \frac{\hat{\Gamma}}{4})(\gamma_{21}^{2} - 1) + 1\right]\sigma^{2} \]
\[ + (\gamma_{21}^{2} - 1)[2\gamma_{21} - (2 - \hat{\Gamma})(\gamma_{21} - 1)]\sigma \]
\[ + (\gamma_{21} + 1)(\gamma_{21} - 1)(\hat{\Gamma} - 1)^{2} \]  
(A19)
\[ D = -(\gamma_{21} - 1)(\gamma_{21} + 1)^{2}(2 - \hat{\Gamma})^{2}\sigma^{2} \]  
(A20)

For \( \hat{\Gamma} = 4/3 \), the four coefficients could be written equivalently as
\[ A = 8\gamma_{21} + 10 \]  
(A21)
\[ B = -(\gamma_{21} + 1)(8\gamma_{21}^{2} + 4\gamma_{21} + 6)\sigma - (\gamma_{21} - 1)(8\gamma_{21}^{2} + 18\gamma_{21} + 11) \]  
(A22)
\[ C = (\gamma_{21} + 1)(8\gamma_{21}^{2} + 1)\sigma^{2} + (\gamma_{21}^{2} - 1)(10\gamma_{21} + 6)\sigma + (\gamma_{21} + 1)(\gamma_{21} - 1)^{2} \]  
(A23)
\[ D = -(\gamma_{21} - 1)(\gamma_{21} + 1)^{2}\sigma^{2} \]  
(A24)
When $\sigma = 0$, the equation (A10) is reduced to

$$x \left[ x - (\gamma_{21}^2 - 1) \right] \left\{ \hat{\Gamma}(2 - \hat{\Gamma})(\gamma_{21} - 1) + 2x - (\gamma_{21} - 1)(\hat{\Gamma} - 1)^2 \right\} = 0,$$  \hspace{1cm} (A25)

which gives equation (12) besides the other two non-physical solutions $u_{2s} = 0$ and $u_{2s} = u_{21}$.

When $\gamma_{21} \gg 1$, the $x^3$ term is a small quantity and is negligible. The equation (A10) is reduced to

$$\hat{\Gamma}(2 - \hat{\Gamma})(\sigma + 1)x^2 - \left[ \hat{\Gamma}(1 - \frac{\hat{\Gamma}}{2})\sigma^2 + (\hat{\Gamma}^2 - 2\hat{\Gamma} + 2)\sigma + (\hat{\Gamma} - 1)^2 \right] x + (2 - \hat{\Gamma})^2 \frac{\sigma^2}{4} = 0.$$  \hspace{1cm} (A26)

This gives the solution (17) in the text (when the non-physical solution is neglected).

**NOTATION LIST**

- subscript 1 upstream (§2 and Appendix A), or unshocked medium (§3 and §4)
- subscript 2 downstream (§2 and Appendix A), or shocked medium (§3 and §4)
- subscript 3 shocked ejecta
- subscript 4 unshocked ejecta
- $c$ speed of light
- $e_i$ internal energy density in region $i (= 1, 2, 3, 4)$
- $f_a$ correction factor of $(e_2/n_2m_p c^2)$ normalized to the $\sigma = 0$ value
- $f_b$ correction factor of $(n_2/n_1)$ normalized to the $\sigma = 0$ value
- $f_c$ magnetic-to-thermal pressure ratio plus 1
- $g_f$ $(p_f - 2)/(p_f - 1)$
- $g_r$ $(p_r - 2)/(p_r - 1)$
- $l$ Sedov length
- $m_e$ electron rest mass
- $m_p$ proton rest mass
- $n_i$ baryon number density in region $i (= 1, 2, 3, 4)$
- $p_i$ thermal pressure in region $i (= 1, 2, 3, 4)$
- $p_{b,i}$ magnetic pressure in region $i (= 1, 2, 3, 4)$
- $p_f$ electron power law index in the forward shock
- $p_r$ electron power law index in the reverse shock
- $t_{\text{dec}}$ deceleration time measured by the observer
- $t_\gamma$ $R_\gamma / C_\gamma \gamma_2^2 c$ (eq. 18)
- $t_{p,f}$ emission peak time of the forward shock component
- $t_{p,r}$ emission peak time of the reverse shock component
- $t_x$ shock crossing time measured by the observer
- $u_{ij}$ four speed in the region $i (= 1, 2, 3, 4)$ in the rest frame of $j (= 1, 2, 3, 4, s)$
- $x$ $u_{2s}^2$
- $y$ pair multiplicity parameter
- $z$ redshift
$A, B, C, D$ coefficients to solve the equation for $u^2_{2s}$

$B_i$ comoving magnetic field in the region $i(= 1, 2, 3, 4)$

$B_{1s}$ magnetic field in the region $i(= 1, 2, 3, 4)$ in the rest frame of the shock

$B_f$ comoving magnetic field in the forward shocked region

$B_r$ comoving magnetic field in the reverse shocked region

$C_N$ correction factor to $R_N$ with respect to the $\sigma = 0$ case

$C_\Delta$ correction factor to $R_\Delta$ with respect to the $\sigma = 0$ case

$C_\gamma$ correction factor to $R_\gamma$ with respect to the $\sigma = 0$ case

$\mathcal{E}$ shock frame electric field

$E$ isotropic total energy of the fireball

$E_K$ isotropic kinetic energy of the fireball

$E_P$ isotropic Poynting-flux energy of the fireball

$F$ the product of $f_a$, $f_b$ and $f_c$

$F_\nu$ specific flux at the frequency $\nu$

$F_{\nu,m,f}$ maximum synchrotron emission specific flux in the forward shock

$F_{\nu,m,r}$ maximum synchrotron emission specific flux in the reverse shock

$F_{\nu,p,f}$ peak flux for the forward shock emission component in certain (e.g. R) band

$F_{\nu,p,r}$ peak flux for the reverse shock emission component in certain (e.g. R) band

$H_0$ Hubble constant

$M_0$ mass in the ejecta

$M_{\text{ISM}}$ mass of the interstellar medium collected by the shock

$N_b$ total baryon number in the shell

$N_\pm$ total electron-positron pair number in the shell

$N_{e,f}$ lepton (electron) number in the forward shock

$N_{e,r}$ lepton (electron and pairs) number in the reverse shock

$Q$ a parameter introduced to categorize the parameter regimes (defined by eq. [50])

$R$ radius from the central engine

$R_B$ reverse-to-forward comoving magnetic field ratio

$R_\epsilon$ reverse-to-forward ratio of the $\epsilon_{e,g}$ parameter

$R_F$ reverse-to-forward peak flux ratio

$R_k$ forward-to-reverse peak time ratio

$R_{\nu}$ the ratio between $\nu_R$ and $\nu_{m,r}(t_{\text{dec}})$

$R_N$ radius where the reverse shock becomes relativistic

$R_s$ radius where the ejecta shell starts to spread

$R_\Delta$ radius where the reverse shock crosses the ejecta shell

$R_\gamma$ radius where the fireball collects $1/\gamma_0$ rest mass of the fireball

$T$ central engine activity time scale

$U_{B,0}$ initial comoving magnetic energy

$U_B$ comoving magnetic energy after shock crossing

$X$ an intermediate parameter introduced in eq. [17]

$Y$ ratio between $B_{2s}$ and $B_{1s}$
**GRB early afterglows**

- $\Gamma$: adiabatic index
- $\Delta$: shell width in the lab frame
- $\Delta_0$: initial shell width in the lab frame
- $\Omega_m$: cosmology mass density parameter
- $\Omega_\Lambda$: cosmology $\Lambda$ density parameter

- $\alpha$: temporal decay index of the reverse shock emission component after peak time
- $\beta_{ij}$: dimensionless velocity of region $i(=1,2,3,4)$ in the rest frame of $j(=1,2,3,4,s)$
- $\tilde{\gamma}$: an equivalent Lorentz factor defined in eq. (74)
- $\gamma_{ij}$: Lorentz factor of region $i(=1,2,3,4)$ in the rest frame of $j(=1,2,3,4,s)$
- $\gamma_i$: Lorentz factor of region $i(=2,3,4)$ in the rest frame of the circumburst medium
- $\gamma_0$: initial Lorentz factor of the fireball, $\gamma_0 \equiv \gamma_4$
- $\gamma_c$: critical initial Lorentz factor that separates thick vs. thin shell regimes (eq. [52])
- $\gamma_x$: fireball Lorentz factor at the shock crossing time

- $\gamma_{e,m,f}$: electron minimum Lorentz factor in the forward shock
- $\gamma_{e,m,r}$: electron minimum Lorentz factor in the reverse shock
- $\epsilon_{e,f}$: electron energy equipartition parameter in the forward shock
- $\epsilon_{e,r}$: electron energy equipartition parameter in the reverse shock
- $\epsilon_{B,f}$: magnetic energy equipartition parameter in the forward shock
- $\epsilon_{B,r}$: equivalent magnetic energy equipartition parameter in the reverse shock
- $\mu_i$: specific enthalpy in region $i(=1,2,3,4)$
- $\nu_{c,f}$: forward shock synchrotron cooling frequency
- $\nu_{c,r}$: reverse shock synchrotron cooling frequency
- $\nu_{m,f}$: forward shock synchrotron typical frequency
- $\nu_{m,r}$: reverse shock synchrotron typical frequency
- $\nu_R$: R-band frequency
- $\xi$: a parameter defined in eq. (48)
- $\xi_0$: the $\xi$ value when $\Delta = \Delta_0$
- $\sigma$: magnetization parameter as defined in eq. (7)
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