STOCHASTIC SOLUTIONS OF A CLASS OF HIGHER ORDER CAUCHY PROBLEMS IN $\mathbb{R}^d$

ERKAN NANE

Abstract. We study solutions of a class of higher order partial differential equations in bounded domains. These partial differential equations appeared first time in the papers of Allouba and Zheng [4], Baeumer, Meerschaert and Nane [10], Meerschaert, Nane and Vellaisamy [37], and Nane [42]. We express the solutions by subordinating a killed Markov process by a hitting time of a stable subordinator of index $0 < \beta < 1$, or by the absolute value of a symmetric $\alpha$-stable process with $0 < \alpha \leq 2$, independent of the Markov process. In some special cases we represent the solutions by running composition of $k$ independent Brownian motions, called $k$-iterated Brownian motion for an integer $k \geq 2$. We make use of a connection between fractional-time diffusions and higher order partial differential equations established first by Allouba and Zheng [4] and later extended in several directions by Baeumer, Meerschaert and Nane [10].

1. Introduction and statement of main results

In recent years, there have been two lines of study of the stochastic solutions of partial differential equations (PDE’s): higher order Cauchy problems [3, 2, 1, 10, 37, 42] and time fractional Cauchy problems [9, 13, 36, 37]. We will use the equivalence of these two types of Cauchy problems on $\mathbb{R}^d$ and on bounded domains with Dirichlet boundary conditions to get classical as well as stochastic solutions of a class of higher order Cauchy problems that appeared in [10, 37, 42].

In this paper we suppose that Brownian motion has variance $2t$ (or that time clock is twice the speed of a standard Brownian motion). We express the solutions of these Cauchy problems using $k$-iterated Brownian motions for an integer $k \geq 2$. There are two ways to define $k$-iterated Brownian motions. The first one is just to let

$$I_k(t) = B_1(|B_2(|B_3(|\cdots(|B_k(t))\cdots))|)$$

where $B_j$’s are independent real-valued Brownian motions all started at 0. In $\mathbb{R}^d$, one takes $B_1$ to be an $\mathbb{R}^d$-valued Brownian motion with independent components. In this case we denote $k$-iterated Brownian motion by $I_k^d(t)$.

To define the second version of $k$-iterated Brownian motions, let $X^+(t)$, $X^-(t)$ be independent one-dimensional Brownian motions, all started at 0. Two-sided Brownian

Key words and phrases. Iterated Brownian motion of Burdzy, $k$-iterated Brownian motion, Brownian-time Brownian motion of Allouba and Zheng, exit time, bounded domain, heat equation, Caputo fractional derivative, fractional diffusion, higher of cauchy problems.
motion is defined to be
\[
X(t) = \begin{cases}
X^+(t), & t \geq 0 \\
X^-(t), & t < 0.
\end{cases}
\]

Then the second version of \( k \)-iterated Brownian motion is defined as
\[
J_k(t) = X_1(X_2(X_3(\cdots (X_k(t)) \cdots)))
\]
where \( X_j \)'s are independent real-valued two-sided Brownian motions all started at 0.

In \( \mathbb{R}^d \), one takes \( X_1 \) to be an \( \mathbb{R}^d \)-valued two-sided Brownian motion with independent components. In this case we denote \( k \)-iterated Brownian motion by \( J^d_k(t) \).

For \( k = 2 \), both of these processes \( I_2 \) and \( J_2 \) were called iterated Brownian motion (IBM) and they have been studied by several researchers; see, for example [3, 4, 16, 17, 28, 39, 40, 41, 44] and references therein.

Recently, Aurzada and Lifshits [8] studied the small ball probability for \( I_k, J_k \).

Arcones [6] studied the large deviation for \( k \)-iterated Brownian motions.

The classical well-known connection of a PDE and a stochastic process is the Brownian motion and heat equation connection. Let \( X(t) \in \mathbb{R}^d \) be Brownian motion started at \( x \). Then the function
\[
u(t,x) = \mathbb{E}_x[f(X(t))]
\]
is the unique solution of the Cauchy problem under mild conditions on \( f \)
\[
\frac{\partial}{\partial t} u(t,x) = \Delta u(t,x); \quad u(0,x) = f(x)
\]
for \( t > 0 \) and \( x \in \mathbb{R}^d \). There is a similar connection for any Markov process where we replace \( \Delta \) with the generator of the Markov process; see, for example, [7, 12, 19, 50].

Allouba and Zheng [11] and DeBlassie [21] obtained the PDE connection of 2-iterated Brownian motion. They showed that for \( Z(t) = I_2(t) \), or \( J_2(t) \)
\[
u(t,x) = \mathbb{E}_x[f(Z(t))]
\]
solves the Cauchy problem
\[
(1.3) \quad \frac{\partial}{\partial t} u(t,x) = \frac{\Delta f(x)}{\sqrt{\pi t}} + \Delta^2 u(t,x); \quad u(0,x) = f(x)
\]
for \( t > 0 \) and \( x \in \mathbb{R}^d \). The non-Markovian property of IBM is reflected by the appearance of the initial function \( f(x) \) in the PDE. The methods of Allouba and Zheng are more general and they allow one to replace \( \Delta \) with the generator of a Markov process. Another important characteristic of the Allouba-Zheng work is its setup. Let \( X^x \) be a continuous Markov process started at \( x \) and let \( B(t) \) be a Brownian motion independent of \( X \). They call \( X^x(t(|B(t)|)) \) a Brownian time process (BTP). Further more:
They introduced—for the first time—a large class of stochastic processes including kEBTPs that is obtained by taking at random one of the k copies of independent Markov process $X^x$ on each excursion interval of $|B_1(t)|$. By this method some important classes of processes are obtained including iterated Brownian motion of Burdzy [16] in the case $k = 2$, BTPs in the case of $k = 1$, Markov snake of Le Gall [30, 31, 32], and a process that is intermediate between iterated Brownian motion of Burdzy and Markov snake of Le Gall in the case of taking limit $k \to \infty$.

Their formulation and approach handle and link to fourth order PDEs not just IBMs, but a much larger class (in (1)) containing many other interesting new processes.

In the absence of the initial function in the PDE (1.3) this problem was studied in [23, 26, 14]. Nigmatullin [13] gave a Physical derivation of fractional diffusion

$$
\frac{\partial^\beta}{\partial t^\beta} u(t, x) = L_x u(t, x); \quad u(0, x) = f(x)
$$

for $t > 0$ and $x \in \mathbb{R}^d$, where $0 < \beta < 1$ and $L_x$ is the generator of some continuous Markov process $X_0(t)$ started at $x = 0$. Here $\frac{\partial^\beta g(t)}{\partial t^\beta}$ is the Caputo fractional derivative in time, which can be defined as the inverse Laplace transform of $s^\beta \tilde{g}(s) - s^{\beta-1}g(0)$, with $\tilde{g}(s) = \int_0^\infty e^{-st}g(t)dt$ the usual Laplace transform. Zaslavsky [55] used (1.4) to model Hamiltonian chaos.

Baeumer and Meerschaert [9] and Meerschaert and Scheffler [35] show that, in the case $p(t, x) = T(t)f(x)$ is a bounded continuous semigroup on a Banach space (with corresponding process $X(t)$, $E^\beta(t) = \inf\{u : D(u) > t\}$, $D(t)$ is a stable subordinator with index $0 < \beta < 1$), the formula

$$
u(t, x) = E_x(f(X(E^\beta(t)))) = \frac{t}{\beta} \int_0^\infty p(s, x)g_{\beta}(\frac{t}{s^{1/\beta}})s^{-1/\beta-1}ds$$

yields the unique solution to the fractional Cauchy problem (1.4). Here $g_{\beta}(t)$ is the smooth density of the stable subordinator $D(1)$, such that the Laplace transform $\tilde{g}_{\beta}(s) = \int_0^\infty e^{-st}g_{\beta}(t)dt = e^{-s^\beta}$.

Allouba and Zheng [1] were also the first to establish a connection between their class of BTPs and the time half-derivative through their BTP half-derivative generator (see their Theorem 0.5), in addition to connecting their BTPs to 4th order PDEs. Essentially, Allouba and Zheng [1] show that BTPs are also stochastic solution of (1.4) in the case $\beta = 1/2$ and $L_x$ is a second order elliptic differential operator of divergence form.

Later, Orsingher and Beghin [45, 46] show that $u(t, x) = E_x(f(I_{k+1}(t)))$ is the solution of

$$
\frac{\partial^{1/2k}}{\partial t^{1/2k}} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x); \quad u(0, x) = f(x)
$$
for \( t > 0 \) and \( x \in \mathbb{R} \).

We will denote the Laplace, Fourier, and Fourier-Laplace transforms (respectively) by:

\[
\tilde{u}(s, x) = \int_0^\infty e^{-st} u(t, x) dt;
\]

\[
\hat{u}(t, k) = \int_{\mathbb{R}^d} e^{-ik \cdot x} u(t, x) dx;
\]

\[
\bar{u}(s, k) = \int_{\mathbb{R}^d} e^{-ik \cdot x} \int_0^\infty e^{-st} u(t, x) dt dx.
\]

Using Fourier-Laplace transform method, Baeumer, Meerschaert, and Nane [10] showed the equivalence of a class of Higher order Cauchy problems and time fractional Cauchy problems: Suppose that \( X(t) = x + X_0(t) \) where \( X_0(t) \) is a Lévy process in \( \mathbb{R}^d \) starting at zero. If \( L_x \) is the generator of the semigroup \( p(t, x) = \mathbb{E}_x[(f(X(t)))] \), then for any \( f \in D(L_x) \), the domain of \( L_x \), and for any \( m = 2, 3, 4, \ldots \), both the Cauchy problem

\[
\frac{\partial u(t, x)}{\partial t} = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} L_x^j f(x) + L_x^m u(t, x); \quad u(0, x) = f(x)
\]

and the fractional Cauchy problem

\[
\frac{\partial^{1/m}}{\partial t^{1/m}} u(t, x) = L_x u(t, x); \quad u(0, x) = f(x),
\]

have the same unique solution given by

\[
u(t, x) = \mathbb{E}_x(f(X(E^{1/m}(t)))) = \int_0^\infty p((t/s)^{1/m}, x) g_1/m(s) ds.
\]

Considering (1.5), and the equivalence of (1.6) and (1.7), it is natural to expect the following Theorem. It establishes the PDE connection of \( k \)-iterated Brownian motion which extends PDE connection of 2-iterated Brownian motion (IBM) due to Allouba and Zheng [4] and DeBlassie [21].

**Theorem 1.1.** Suppose that \( X(t) = x + X_0(t) \in \mathbb{R}^d \) where \( X_0(t) \) is a Lévy process starting at zero. If \( L_x \) is the generator of the semigroup \( T(t)f(x) = \mathbb{E}_x[(f(X(t)))] \), then for any \( f \in D(L_x) \), domain of \( L_x \), \( u(t, x) = \mathbb{E}_x(f(X(|I_k(t)|))) \) is the unique solution of the Cauchy problems (1.6) and (1.7) with \( m = 2^k \). If \( Z(t) \) is a two-sided Lévy process with independent copies of \( X \) for positive and negative times then \( u(t, x) = \mathbb{E}_x(f(Z(J_k(t)))) \) is also the unique solution of the Cauchy problems (1.6) and (1.7) with \( m = 2^k \).

**Remark 1.1.** Only trivial extensions to special cases of the formulation in Allouba and Zheng [4] lead to the fact that the expressions with \( I_k(t) \) and \( J_k(t) \) in Theorem
are interchangeable in the solution expressions of the given PDEs (see Theorem 0.1 and its proof in Allouba and Zheng [4]).

Let $D$ be a domain in $\mathbb{R}^d$. We define the following spaces of functions.

- $C(D) = \{ u : D \to \mathbb{R} : u$ is continuous $\}$;
- $C(\bar{D}) = \{ u : \bar{D} \to \mathbb{R} : u$ is uniformly continuous $\}$;
- $C^j(D) = \{ u : D \to \mathbb{R} : u$ is $j$-times continuously differentiable $\}$;
- $C^j(\bar{D}) = \{ u \in C^j(D) : D^\gamma u$ is uniformly continuous for all $|\gamma| \leq j \}$.

Thus, if $u \in C^j(\bar{D})$, then $D^\gamma u$ continuously extends to $\bar{D}$ for each multi-index $\gamma$ with $|\gamma| \leq j$.

We define the spaces of functions $C^\infty(D) = \bigcap_{j=1}^{\infty} C^j(D)$ and $C^\infty(\bar{D}) = \bigcap_{j=1}^{\infty} C^j(\bar{D})$.

Also, let $C^{j,\alpha}(D)$ ($C^{j,\alpha}(\bar{D})$) be the subspace of $C^j(D)$ ($C^j(\bar{D})$) that consists of functions whose $j$-th order partial derivatives are uniformly Hölder continuous with exponent $\alpha$ in $D$. For simplicity, we will write $C^{0,\alpha}(D) = C^\alpha(D)$, $C^{0,\alpha}(\bar{D}) = C^\alpha(\bar{D})$ with the understanding that $0 < \alpha < 1$ whenever this notation is used, unless otherwise stated.

We use $C_c(D), C^j_c(D), C^{j,\alpha}_c(D)$ to denote those functions in $C(D), C^j(D), C^{j,\alpha}(D)$ with compact support.

A subset $D$ of $\mathbb{R}^d$ is an $l$-dimensional manifold with boundary if every point of $D$ possesses a neighborhood diffeomorphic to an open set in the space $H^l$, which is the upper half space in $\mathbb{R}^l$. Such a diffeomorphism is called a local parametrization of $D$. The boundary of $D$, denoted by $\partial D$, consists of those points that belong to the image of the boundary of $H^l$ under some local parametrization. If the diffeomorphism and its inverse are $C^{j,\alpha}$ functions, then we write $\partial D \in C^{j,\alpha}$.

Since we are working on a bounded domain, the Fourier transform methods in [36] are not useful. Instead we will employ Hilbert space methods used in [37]. Hence, given a complete orthonormal basis $\{ \psi_n(x) \}$ on $L^2(D)$, we will call

$$\tilde{u}(t, n) = \int_D \psi_n(x) u(t, x) dx;$$
$$\hat{u}(s, n) = \int_D \psi_n(x) \int_0^\infty e^{-st} u(t, x) dt dx = \int_D \psi_n(x) \tilde{u}(s, x) dx.$$ 

the $\psi_n$, and $\psi_n$-Laplace transforms, respectively. Since $\{ \psi_n \}$ is a complete orthonormal basis for $L^2(D)$, we can invert the $\psi_n$-transform

$$u(t, x) = \sum_n \tilde{u}(t, n) \psi_n(x)$$

for any $t > 0$, where the sum converges in the $L^2$ sense (e.g., see [48, Proposition 10.8.27]).
Mittag-Leffler function is defined by $E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}$, $0 < \beta < 1$; see, for example, [33].

Let $\beta \in (0, 1)$, $D_\infty = (0, \infty) \times D$ and define

$$\mathcal{H}_\Delta(D_\infty) \equiv \{ u : D_\infty \to \mathbb{R} : \frac{\partial}{\partial t^\beta} u, \frac{\partial}{\partial t} u, \Delta u \in C(D_\infty), \left| \frac{\partial}{\partial t} u(t, x) \right| \leq g(x) t^{\beta-1}, g \in L^\infty(D), t > 0 \}.$$ 

Let $\tau_D(X) = \inf\{ t \geq 0 : X(t) \notin D \}$ be the first exit time of the process $X$ from $D$. We denote by $\{ \phi_n, \lambda_n, n \geq 1 \}$ the set of eigenvalues $\lambda_n$ and corresponding eigenfunctions $\phi_n$ of the Laplacian $\Delta$ with Dirichlet boundary conditions:

$$\Delta \phi_n = -\lambda_n \phi_n \text{ in } D; \quad \phi_n |_{\partial D} = 0.$$ 

We will write $u \in C^k(\bar{D})$ to mean that for each fixed $t > 0$, $u(t, \cdot) \in C^k(\bar{D})$, and $u \in C^k_b(\bar{D}_\infty)$ to mean that $u \in C^k(\bar{D}_\infty)$ and is bounded.

Extending the Fourier-Laplace transform method to bounded domains, Meerschaert, Nane and Vellaisamy [37] gave a stochastic as well as an analytic solution to fractional Cauchy problem (1.4) in bounded domains: Let $0 < \gamma < 1$. Let $D$ be a bounded domain with $\partial D \in C^{1,\gamma}$, and $T_D(t)$ be the killed semigroup of Brownian motion $\{ X_t \}$ in $D$. Let $E^\beta(t)$ be the process inverse to a stable subordinator of index $\beta \in (0, 1)$ independent of $\{ X(t) \}$. Let $f \in D(\Delta) \cap C^1(\bar{D}) \cap C^2(D)$ for which the eigenfunction expansion (of $\Delta f$) with respect to the complete orthonormal basis $\{ \phi_n : n \in \mathbb{N} \}$ converges uniformly and absolutely. Then the unique (classical) solution of

$$u \in H_\Delta(D_\infty) \cap C^1(\bar{D});$$

is given by

$$\begin{align*}
\frac{\partial^\beta}{\partial t^\beta} u(t, x) &= \Delta u(t, x); \quad x \in D, t > 0; \\
u(t, x) &= 0, \quad x \in \partial D, t > 0; \\
u(0, x) &= f(x), \quad x \in D
\end{align*}$$

is given by

$$u(t, x) = \mathbb{E}_x[f(X(E^\beta(t)))I(\tau_D(X) > E^\beta(t))]$$

$$= \frac{t}{\beta} \int_0^\infty T_D(l) f(x) g_\beta(t l^{-1/\beta}) l^{-1/\beta-1} dl = \int_0^\infty T_D((t/l)^\beta) f(x) g_\beta(l) dl$$

$$= \sum_{n=1}^{\infty} \bar{f}(n) \phi_n(x) E_\beta(-\lambda_n t^\beta).$$

**Remark 1.2.** The analytic representation (1.9) of the solution is due to Agrawal [1] in the case $D = (0, M)$ is an interval in $\mathbb{R}$. For $\beta = 1$, the study of the Cauchy
problem \((1.8)\) boils down to studying heat equation in bounded domains with Dirichlet boundary conditions which has solution \((1.9)\). In this case \(E_1(-\lambda_n t) = e^{-\lambda_n t}\). This is valid under much less requirements on the initial function and the regularity of the boundary, see for example [12]. The solution to \((1.8)\) is also given by \((1.9)\) if we replace Brownian motion with a diffusion process in which case the Laplacian \(\Delta\) should be replaced with the diffusion operator, see Meerschaert, et. al. [37].

Let

\[
\mathcal{H}_{\Delta^m}(D_\infty) \equiv \bigg\{ u : D_\infty \to \mathbb{R} : \left| \frac{\partial}{\partial t} u(t, x) \right| \leq g(x)t^{1/m-1}, g \in L^\infty(D), \ t > 0, \frac{\partial}{\partial t} u, \Delta^k u \in C(D_\infty), k = 1, \cdots, m, \Delta^k u \in C^1(\bar{D}), k = 1, \cdots, m - 1 \bigg\}.
\]

Using the equivalence of fractional Cauchy problem \((1.8)\) with the Higher order Cauchy problems \((1.6)\) with the correct Dirichlet type Boundary conditions we obtain the second main result in this paper.

**Theorem 1.2.** Let \(m = 2, 3, \cdots\), be an integer. Let \(D\) be a bounded domain with \(\partial D \in C^{1,\gamma}\), and \(T_D(t)\) be the killed semigroup of Brownian motion \(\{X_t\}\) in \(D\). Let \(\{E^{1/m}(t)\}\) be the process inverse to a stable subordinator of index \(1/m\) independent of \(\{X(t)\}\). Let \(f \in D(\Delta) \cap C^{2m-3}(\bar{D}) \cap C^{2m-2}(D) \cap L^2(D)\) be such that the eigenfunction expansion of \(\Delta^{m-1} f\) with respect to \(\{\phi_n : n \geq 1\}\) converges absolutely and uniformly. Then the (classical) solution of

\[
\begin{align*}
\frac{\partial u(t, x)}{\partial t} &= \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} \Delta^j f(x) + \Delta^m u(t, x), \ x \in D, \ t > 0; \\
u(t, x) &= \Delta^l u(t, x) = 0, \ t \geq 0, \ x \in \partial D, \ l = 1, \cdots, m - 1; \\
u(0, x) &= f(x), \ x \in D
\end{align*}
\]

is given by

\[
u(t, x) = \mathbb{E}_x[f(X(E^{1/m}(t)))I(\tau_D(X) > E^{1/m}(t))] = \mathbb{E}_x[f(X(E^{1/m}(t)))I(\tau_D(X(E^{1/m})) > t)] = tm \int_0^\infty T_D(l)f(x)g_{1/m}(hl^{-m})l^{-m-1}dl = \sum_{n=1}^\infty \bar{f}(n)\phi_n(x)E_{1/m}(-\lambda_n t^{1/m}).
\]

In the case \(m = 2^k\) for some integer \(k \geq 1\), the solution to \((1.10)\) is also given by

\[
u(t, x) = \mathbb{E}_x[f(X(|I_k(t)|))I(\tau_D(X) > |I_k(t)|)]. \text{ For } \beta = 1/2^k \text{ the solution to } (1.8) \text{ is also given by } u(t, x) = \mathbb{E}_x[f(X(|I_k(t)|))I(\tau_D(X) > |I_k(t)|)].
\]
Meerschaert, Nane and Vellaisamy \cite{37} proved this theorem in the case \( m = 2 \).

**Remark 1.3.** Theorem 1.2 also holds with the version of \( k \)-iterated Brownian motion \( J_k(t) \). Here, the outer process \( X(t) \) is a two-sided Brownian motion and \( J_k(t) \) is an independent \( k \)-iterated Brownian motion. In this case, using a simple conditioning argument, we can show that the function

\[
u(t, x) = \mathbb{E}_x[f(\sqrt{1+|\tau_D(X)|^2})]I(\tau_D(X^-) < J_k(t) < \tau_D(X^+))
\]

reduces to Equation (1.11) and hence is also a solution to both Cauchy problems (1.10) and (1.8) with \( m = 2^k \).

In \cite{42}, we studied the Cauchy problems that can be solved by running \( \alpha \)-time processes with \( 0 < \alpha \leq 2 \). An \( \alpha \)-time process is a Markov process in which the time parameter is replaced with the absolute value of an independent symmetric \( \alpha \)-stable process \( Y \) with \( 0 < \alpha \leq 2 \).

As a special case in Nane \cite{42, Theorem 2.1} we established: Let \( \{X(t)\} \) be a continuous Markov process with generator \( \mathcal{A} \), and let \( \{Y(t)\} \) be a Cauchy process independent of \( \{X(t)\} \). Let \( f \) be a bounded measurable function in the domain of \( \mathcal{A} \), with \( D_{ij} f \) bounded and Hölder continuous for all \( 1 \leq i, j \leq d \). Then \( u(t, x) = \mathbb{E}_x[f(X(|Y(t)|))] \) is a solution of

\[
\frac{\partial^2}{\partial t^2}u(t, x) = -\frac{2A f(x)}{\pi t} - \mathcal{A}^2 u(t, x); \quad u(0, x) = f(x)
\]

for \( t > 0, \ x \in \mathbb{R}^d \).

This reduces to nonhomogeneous wave equation in the case \( X \) is another Cauchy process independent of \( Y \), the generator \( \mathcal{A} = -(-\Delta)^{1/2} \), fractional Laplacian, i.e., \( u(t, x) = \mathbb{E}_x[f(X(|Y(t)|))] \) is a solution of

\[
\frac{\partial^2}{\partial t^2}u(t, x) = \frac{2(-\Delta)^{1/2} f(x)}{\pi t} + \Delta u(t, x); \quad u(0, x) = f(x)
\]

for \( t > 0, \ x \in \mathbb{R}^d \). This is one of the most interesting PDE connections of these iterated processes.

Let

\[
\mathcal{K}_{\Delta^2} = \{ u : D_{\infty} \to \mathbb{R} : \frac{\partial^2}{\partial t^2}u, \Delta u, \Delta^2 u \in C(D_{\infty}) \}
\]

In bounded domains we obtain the following

**Theorem 1.3.** Let \( D \) be a bounded domain with \( \partial D \in C^{1,\gamma}, \ 0 < \gamma < 1 \). Let \( \{X(t)\} \) be a Brownian motion in \( \mathbb{R}^d \) with independent components, and let \( \{Y(t)\} \) be a Cauchy process independent of \( \{X(t)\} \). Let \( f \in D(\Delta) \cap C^2(D) \) for which the eigenfunction expansion of \( \Delta f \) with respect to the complete orthonormal basis \( \{\phi_n : n \geq 1\} \) converges absolutely and uniformly. Then \( u(t, x) = \mathbb{E}_x[f(X(|Y(t)|))I(\tau_D(X) > |Y(t)|)] \) is
a solution of
\[ u \in \mathcal{K}_{\Delta^2} \cap C_b(D_\infty) \cap C^1(\bar{D}) \]
(1.12) \[ \frac{\partial^2}{\partial t^2} u(t, x) = -\frac{2\Delta f(x)}{\pi t} - \Delta^2 u(t, x), \quad t > 0, \quad x \in D \]
\[ u(0, x) = f(x), \quad x \in D, \]
\[ u(t, x) = \Delta u(t, x) = 0, \quad x \in \partial D, \quad t \geq 0. \]

Remark 1.4. In Theorem 1.2, the solution is expressed by subordinating a killed process by an increasing process \( E_{1/m}(t) \), inverse process to a stable subordinator of index \( 1/m \). And we see from Theorem 1.2 that killing the outer process and then subordinating, or subordinating and then killing gives the same solution of the Cauchy problem (1.10). The solution in the case \( m = 2^k \) is also given by subordinating the killed process by \( k \)-iterated Brownian motion. Is there an increasing process \( A(t) \) such that we get a similar relation for the solution of (1.12)?

Remark 1.5. We discuss how to extend Theorems 1.1, 1.2 and 1.3 to other continuous Markov processes in section 6.

This paper is organized as follows. In section 2 we give some preliminaries. Section 3 is devoted to the proof of Theorem 1.1. We prove Theorem 1.2 in section 4. Theorem 1.3 is proved in section 5. We also state and prove a theorem for Cauchy problems that can be solved by running a Brownian motion subordinated to the absolute value of a symmetric \( \alpha \)-stable process with \( \alpha \in (0, 2) \) a rational, and \( \alpha \neq 1 \). We discuss extensions of the theorems proved in this paper to other types of Markov process in section 6.

2. PRELIMINARIES

Let \( X_0(t) \) be a Lévy process started at zero and \( X(t) = x + X_0(t) \) for \( x \in \mathbb{R}^d \), the generator \( L_x \) of the semigroup \( T(t)f(x) = \mathbb{E}_x[f(X(t))] \) is a pseudo-differential operator \([5, 27, 52]\) that can be explicitly computed by inverting the Lévy representation. The Lévy process \( X_0(t) \) has characteristic function
\[ \mathbb{E}[\exp(ik \cdot X_0(t))] = \exp(t\psi(k)) \]
with
\[ \psi(k) = ik \cdot a - \frac{1}{2}k \cdot Qk + \int_{y \neq 0} \left( e^{ik \cdot y} - 1 - \frac{ik \cdot y}{1 + ||y||^2} \right) \nu(dy), \]
where \( a \in \mathbb{R}^d \), \( Q \) is a nonnegative definite matrix, and \( \nu \) is a \( \sigma \)-finite Borel measure on \( \mathbb{R}^d \) such that
\[ \int_{y \neq 0} \min\{1, ||y||^2\} \nu(dy) < \infty; \]
see for example [34, Theorem 3.1.11] and [5, Theorem 1.2.14]. Let
\[ \hat{f}(k) = \int_{\mathbb{R}^d} e^{-ik \cdot x} f(x) \, dx \]
denote the Fourier transform. Theorem 3.1 in [9] shows that \( L_x f(x) \) is the inverse Fourier transform of \( \psi(k) \hat{f}(k) \) for all \( f \in D(L_x) \), where \( \psi(k) \hat{f}(k) = \hat{h}(k) \) \( \exists h \in L^1(\mathbb{R}^d) \), and
\[
L_x f(x) = a \cdot \nabla f(x) + \frac{1}{2} \nabla \cdot Q \nabla f(x) + \int_{y \neq 0} \left( f(x + y) - f(x) - \nabla f(x) \cdot y \frac{1}{1 + y^2} \right) \nu(dy)
\]
for all \( f \in W^{2,1}(\mathbb{R}^d) \), the Sobolev space of \( L^1 \)-functions whose first and second partial derivatives are all \( L^1 \)-functions. This includes the special case where \( X_0(t) \) is an operator Lévy motion. We can also write \( L_x = \psi(-i\nabla) \) where \( \nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_d)' \).

We say that \( D \) satisfies an exterior cone condition at a fixed point \( x_0 \in \partial D \) if there exists a finite right circular cone \( V = V_{x_0} \) with vertex \( x_0 \) such that \( D \cap V_{x_0} = \{ x_0 \} \), and a uniform exterior cone condition if \( D \) satisfies an exterior cone condition at every point \( x_0 \in \partial D \) and the cones \( V_{x_0} \) are all congruent to some fixed cone \( V \).

The right condition for the existence of the solution to the Dirichlet problem turns out to be that every point of \( \partial D \) is regular for \( D^C \) (cf. [12, Section II.1]).

If a domain satisfies a uniform exterior cone condition, then every point of \( \partial D \) is regular for \( D^C \).

Let \( \partial D \in C^1 \). Then at each point \( x \in \partial D \) there exists a unique outward pointing unit vector \( \theta(x) = (\theta_1(x), \ldots, \theta_d(x)) \).

Let \( u \in C^1(\bar{D}) \), the set of functions which have continuous extension of the first derivative up to the boundary. Let
\[ D_{\theta} u = \frac{\partial u}{\partial \theta} = \theta \cdot \nabla u, \]
denote the directional derivative, where \( \nabla u \) is the gradient vector of \( u \).

Now we recall Green’s first and second identities (see, for example, [24, Section 2.4]). Let \( u, v \in C^2(D) \cap C^1(\bar{D}) \). Then
\[
\int_D \frac{\partial u}{\partial x_i} v dx = \int_{\partial D} u v \theta_i ds - \int_D u \frac{\partial v}{\partial x_i} dx \quad \text{(integration by parts formula)},
\]
\[
\int_D \nabla v \cdot \nabla u dx = - \int_D u \Delta v dx + \int_{\partial D} \frac{\partial v}{\partial \theta} u ds \quad \text{(Green's first identity)},
\]
\[
\int_D [u \Delta v - v \Delta u] dx = \int_{\partial D} \left[ \frac{\partial v}{\partial \theta} - \frac{\partial u}{\partial \theta} \right] ds, \quad \text{(Green's second identity)}.
\]

Let \( D \) be bounded and every point of \( \partial D \) be regular for \( D^C \). Markov process corresponding to the Dirichlet problem is a killed Brownian motion. We denote the eigenvalues and the eigenfunctions of \( \Delta \) by \( \{\lambda_n, \phi_n\}_{n=1}^{\infty} \), where \( \phi_n \in C^\infty(D) \). The corresponding heat kernel is given by
\[
p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y).
\]
The series converges absolutely and uniformly on \([t_0, \infty) \times D \times D\) for all \( t_0 > 0 \). In this case, the semigroup given by
\[(2.2)\]
\[T_D(t) f(x) = E_x[f(X_t)I(t < \tau_D(X))] = \int_D p_D(t, x, y) f(y) dy = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) f(n)\]
solves the Heat equation in \( D \) with Dirichlet boundary conditions:
\[
\frac{\partial u(t, x)}{\partial t} = \Delta u(t, x), \quad x \in D, \quad t > 0,
\]
\[
u(t, x) = 0, \quad x \in \partial D,
\]
\[
u(0, x) = f(x), \quad x \in D.
\]

**Remark 2.1.** The eigenfunctions belong to \( L^\infty(D) \cap C^\infty(D) \), by [24] Corollary 8.11, Theorems 8.15 and 8.24. If \( D \) satisfies a uniform exterior cone condition all the eigenfunctions belong to \( C^\alpha(\bar{D}) \) by [24] Theorem 8.29. If \( \partial D \in C^{1,\alpha} \), then all the eigenfunctions belong to \( C^{1,\alpha}(\bar{D}) \) by [24] Corollary 8.36. If \( \partial D \in C^\infty \), then each eigenfunction of \( \Delta \) is in \( C^\infty(\bar{D}) \) by [24] Theorem 8.13.]
whose \( t \mapsto s \) Laplace transform \( s^{\beta-1}e^{-xs^\beta} \) can also be derived from the equation
\[
f_t(x) = \frac{d}{dx} P(D(x) \geq t) = \frac{d}{dx} \int_t^{\infty} x^{-1/\beta} g_\beta(x^{-1/\beta} u) \, du
\]
by taking Laplace transforms on both sides.

3. Cauchy problems

We prove Theorem 1.1 in this section. First we need the following Lemmas.

**Lemma 3.1.** Let \( E^{\beta_1}, E^{\beta_2} \) be two independent processes that are inverses to stable subordinators of index \( 0 < \beta_1, \beta_2 < 1 \). Then \( E^{\beta_1}(E^{\beta_2}(t)) \) is inverse to a stable subordinator of index \( \beta_1 \beta_2 \).

**Proof.** Since composition of stable subordinators gives another stable subordinator, see Bochner [15], the result follows as \( E^\beta \) is the inverse of a stable subordinator. \( \square \)

**Lemma 3.2.** For fixed \( t \geq 0, k \)-iterated Brownian motion
\[
|I_k(t)| = |B_1(|B_2(| \cdots (|B_k(t)| \cdots |)|)|)
\]
and \( E^{1/2k}(t) \) have the same one-dimensional distributions.

**Proof.** Since \( E^{1/2} \) and \( |B(t)| \) have the same 1-dimensional distributions, see, for example, proof of Theorem 3.1 in [10], the proof follows from Lemma 3.1 by induction on \( k \). Hence we have by composing \( k \) independent \( E^{1/2} \)s
\[
E_1^{1/2}(E_2^{1/2}( \cdots (E_k^{1/2}(t)))) = E^{1/2k}(t) \overset{(d)}{=} |I_k(t)|.
\]
\( \square \)

**Corollary 3.1.**
\[
I_{k+1}(t) = B_1(|B_2(| \cdots (B_{k+1}(t)) \cdots |)|) \overset{(d)}{=} B_1(E_t^{1/2k})
\]

**Proof of Theorem 1.1.** The proof is an adaptation of the proof of Theorem 3.1 in Baeumer et. al. [10], Using Lemma 3.2 we get that
\[
u(t,x) = \mathbb{E}_x(f(X(|I_k(t)|))) = \mathbb{E}(f(X(E_t^{1/2k})))
\]
is a solution to both the Higher order Cauchy problem (1.7) and the fractional Cauchy problem (1.6) for \( m = 2^k \). By a simple conditioning argument we also get that \( u(t,x) = \mathbb{E}_x(f(Z(J_k(t)))) \).
\( \square \)
4. CAUCHY PROBLEMS IN BOUNDED DOMAINS

The inverse stable subordinators with \( \beta = 1/2^k \) are related to Brownian subordinators by Lemma 3.2, this is well-known for the case \( k = 1 \), see, for example, [10]. Since Brownian subordinators are related to higher-order Cauchy problems by Theorem 1.1, this relationship can also be used to connect those higher-order Cauchy problems in bounded domains to their time-fractional analogues. In this section, we establish those connections for Cauchy problems on bounded domains in \( \mathbb{R}^d \). We extend this to establish an equivalence between a killed Markov process subordinated to an inverse stable subordinator with \( \beta = 1/2^k \), and the same process subject to a Brownian subordinator in section 6. Finally, we identify the boundary conditions that make the two formulations identical. This solves an open problem in [10]. This problem was solved in [37, Theorem 4.1] for \( k = 1 \).

**Lemma 4.1.** Let \( D \) be a bounded domain with \( \partial D \in C^{1,\alpha} \), \( 0 < \alpha < 1 \). Let \( \{\phi_n, \lambda_n, n \geq 1\} \) be the set of eigenvalues \( \lambda_n \) and corresponding eigenfunctions \( \phi_n \) of the Laplacian \( \Delta \). Let \( f \in C^j(D) \) for \( j = 1, \cdots, 2k \) and all the partial derivatives of \( f \) of order up to \( 2k - 1 \) vanish on the boundary (A simpler condition is to assume \( f \in C^2(D) \)). Then

\[
\int_D \phi_n(x) \Delta^j f(x) dx = (-\lambda_n)^j \int_D \phi_n(x) f(x) dx = (-\lambda_n)^j \tilde{f}(n), \quad j = 1, \cdots, k
\]

**Proof.** We use Green’s second identity and induction in \( j \)

\[
\int_D [\phi_n \Delta^j f - \Delta^{j-1} f \Delta \phi_n] dx = \int_{\partial D} \left[ \phi_n \frac{\partial \Delta^{j-1} f}{\partial \theta} - \Delta^{j-1} f \frac{\partial \phi_n}{\partial \theta} \right] ds,
\]

where we use the fact that \( \Delta^j f|_{\partial D} = 0 = \phi_n|_{\partial D} \), \( f \in C^j(D) \) for \( j = 1, \cdots, 2k \), and \( \phi_n \in C^{1,\gamma}(D) \) by Remark 2.1. Hence, by induction, the \( \phi_n \)-transform of \( \Delta^j u \) is \( (-\lambda_n)^j \tilde{f}(n) \), as \( \phi_n \) is the eigenfunction of the Laplacian corresponding to eigenvalue \( \lambda_n \).

**Proof of Theorem 1.2.** Suppose \( u \) is a solution to Equation (1.10). Taking the \( \phi_n \)-transform of (1.10) and using Lemma 4.1 we obtain

\[
\frac{\partial}{\partial t} \tilde{u}(t, n) = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} (-\lambda_n)^j \tilde{f}(n) + (-\lambda_n)^m \tilde{u}(t, n).
\]

Note that the time derivative commutes with the \( \phi_n \)-transform, as

\[
\left| \frac{\partial}{\partial t} u(t, x) \right| \leq g(x) t^{1/m-1}, \quad g \in L^\infty(D), \quad t > 0.
\]

Taking Laplace transforms on both sides and using the well-known Laplace transform formula

\[
\int_0^\infty \frac{t^{-\beta}}{\Gamma(1-\beta)} e^{-st} dt = s^{\beta-1}
\]
for $\beta < 1$, gives us

\begin{equation}
(4.4) \quad s \hat{u}(s, n) - \bar{u}(0, n) = \sum_{j=1}^{m-1} s^{-j/m}(-\lambda_n)^j \hat{f}(n) + (-\lambda_n)^m \hat{u}(s, n).
\end{equation}

Since $u$ is uniformly continuous on $C([0, e] \times \bar{D})$, it is also uniformly bounded on $[0, e] \times \bar{D}$. So, we have $\lim_{t\to 0} \int_D u(t, x)\phi_n(x)dx = \hat{f}(n)$. Hence, $\bar{u}(0, n) = \hat{f}(n)$. By collecting the like terms, we obtain

\begin{equation}
(4.5) \quad \hat{u}(s, n) = \frac{\hat{f}(n) \left(1 + \sum_{j=1}^{m-1} s^{-j/m}(-\lambda_n)^j\right)}{s - (-\lambda_n)^m}.
\end{equation}

Using the simple equality $(a - b)(a^{m-1} + a^{m-2}b + \cdots + ab^{m-1}) = a^m - b^m$ for any $m = 2, 3, \cdots$, for fixed $n$ and for large $s$, we get

\begin{equation}
(4.6) \quad \hat{u}(s, n) = \frac{s^{1/m-1} \hat{f}(n) \left(s^{1-1/m} + \sum_{j=1}^{m-1} s^{-j/m+1-1/m}(-\lambda_n)^j\right)}{s - (-\lambda_n)^m} = \frac{s^{1/m-1} \hat{f}(n) \left(s^{1-1/m} + \sum_{j=1}^{m-1} s^{-j/m+1-1/m}(-\lambda_n)^j\right)}{(s^{1/m} - (-\lambda_n))^m} = \frac{s^{1/m-1} \hat{f}(n) \left(s^{1-1/m} + \sum_{j=1}^{m-1} s^{-j/m+1-1/m}(-\lambda_n)^j\right)}{s^{1/m} + \lambda_n}.
\end{equation}

It follows from the proof of Theorem 3.1 and Corollary 3.2 in [37] that the inverse $\phi_n$-Laplace transform of $\hat{u}(s, n)$ is given by

\begin{equation}
(4.8) \quad u(t, x) = \sum_{n=1}^{\infty} \hat{f}(n) E_{1/m}(-\lambda_n t^{1/m})\phi_n(x) = \mathbb{E}_x[f(X(E^{1/m}(t)))I(\tau_D(X) > E^{1/m}(t))] = \mathbb{E}_x[f(X(E^{1/m}(t)))I(\tau_D(X(E^{1/m})) > t)],
\end{equation}

where $E^{1/m}(t)$ is the process inverse to stable subordinator of index $1/m$.

For any fixed $n \geq 1$, the two formulae (4.5) and (4.7) are well-defined and equal for all sufficiently large $s$. Since the inverse Laplace transform of $\hat{u}(s, n)$ in (4.7) is

\begin{equation}
(4.9) \quad \bar{u}(t, n) = \hat{f}(n) E_{1/m}(-\lambda_n t^{1/m}),
\end{equation}

see, for example, [33], we can see easily that $\bar{u}(t, n)$ is continuous in $t > 0$ for any $n \geq 1$. Hence, the uniqueness theorem for Laplace transforms [7, Theorem 1.7.3] shows that, for each $n \geq 1$, $\bar{u}(t, n)$ is the unique continuous function whose Laplace transform is given by (4.6). Since $x \mapsto u(t, x)$ is an element of $L^2(D)$ for every $t > 0$, and two elements of $L^2(D)$ with the same $\phi_n$-transform are equal $dx$-almost everywhere, we have (4.8) is the unique element of $L^2(D)$ and (4.9) is its $\phi_n$-transform.
Now, we show that the solution $u$ defined by (4.8) satisfies all the properties in (1.10). From the Proof of Theorem 3.1 in [37], we can get that the solution $u(t, x)$ defined by the series (4.8) belongs to $L^2(D)$, converges absolutely and uniformly for $t \geq t_0 > 0$ for some $t_0$.

Next we show that $\Delta^l u \in C(D_\infty)$ for $l = 1, \cdots, m$. To do this, we need only to show the absolute and uniform convergence of the series defining $\Delta^l u$ for $l = 1, \cdots, m$.

To apply $\Delta^l$ term-by-term to (4.8), we have to show that the series
\[
\sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)(-\lambda_n)^l E_{1/2}(-\lambda_n t^{1/m})
\]
is absolutely and uniformly convergent for $t > t_0 > 0$.

Note that by Lemma 4.1 the $\phi_n$-transform of $\Delta^j f$ is given by
\[
\int_D \phi_n(x) \Delta^j f(x) dx = (-\lambda_n)^j \bar{f}(n)
\]
and using [29, equation (13)], (4.10)
\[
0 \leq E_\beta(-\lambda_n t^\beta) \leq c/(1 + \lambda_n t^\beta),
\]
we get
\[
\sum_{n=1}^{\infty} |\bar{f}(n)||\phi_n(x)|(\lambda_n)^l E_{1/m}(-\lambda_n t^{1/m}) \leq \sum_{n=1}^{\infty} |\bar{f}(n)||\phi_n(x)|\lambda_n^l \frac{c}{1 + \lambda_n t^m}
\]
(4.11)
\[
\leq ct_0^{-\frac{1}{m}} \sum_{n=1}^{\infty} |\bar{f}(n)||\phi_n(x)|\lambda_n^{l-1} < \infty,
\]
where the last inequality follows from the absolute and uniform convergence of the eigenfunction expansion of $\Delta^{m-1} f$.

A similar argument using [29, Equation (17)]
\[
\left| \frac{dE_\beta(-\lambda_n t^\beta)}{dt} \right| \leq c \frac{\lambda_n t^{\beta-1}}{1 + \lambda_n t^\beta} \leq c\lambda_n t^{\beta-1},
\]
(4.12)
and the fact that the eigenfunction expansion of $\Delta^{m-1} f$ converges absolutely and uniformly allows us to differentiate the series (4.8) term by term with respect to $t$.

We next show that $\Delta^l u \in C(D)$ for $l = 1, \cdots, m - 1$: this follows from the bounds in [24, Theorem 8.33] and the absolute and uniform convergence of the series defining $\Delta^{m-1} f$.

(4.13)
\[
|\phi_n|_{1,\alpha;D} \leq C(1 + \lambda_n) \sup_D |\phi_n(x)|,
\]
where \( C = C(d, \partial D) \) is a finite constant. Here

\[
|u|_{k, \alpha; D} = \sup_{|\gamma| = k} |D^\gamma u|_{\alpha; D} + \sum_{j=0}^{k} \sup_{|\gamma| = j} \sup_{D} |D^\gamma u|, \quad k = 0, 1, 2, \ldots
\]

and

\[
[D^\gamma u]_{\alpha; D} = \sup_{x, y \in D, x \neq y} \frac{|D^\gamma u(x) - D^\gamma u(y)|}{|x - y|^\alpha}
\]

are norms on \( C^{k, \alpha}(\bar{D}) \). Hence for \( l = 0, 1, 2, \ldots, m - 1 \)

\[
|\Delta^l u(\cdot, t)|_{1, \alpha; D} \leq C \sum_{n=1}^{\infty} |\bar{f}(n)| E_{\beta}(\lambda_n^{-1}/m) (1 + \lambda_n) \sup_{D} |\phi_n(x)|
\]

\[
\leq C \sum_{n=1}^{\infty} |\bar{f}(n)| (\lambda_n^{-1}/m) \frac{1 + \lambda_n}{1 + \lambda_n} \sup_{D} |\phi_n(x)|
\]

\[
\leq Ct^{-\beta} \sum_{n=1}^{\infty} \sup_{D} |\phi_n(x)|(\lambda_n^{-1}/m) |\bar{f}(n)|
\]

\[
+ C \sum_{n=1}^{\infty} \sup_{D} |\phi_n(x)|(\lambda_n^{-1}/m) |\bar{f}(n)| < \infty.
\]

With this we established that \( u \in \mathcal{H}_{\Delta^m} \cap C_b(D_\infty) \cap C^1(\bar{D}) \).

Observe next that the Laplace transform of

\[
\frac{\partial}{\partial t} E_{\beta}(\lambda_n^{-1/m}) - \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} (\lambda_n^{-1}/m) (-\lambda_n)^j - (-\lambda_n)^m E_{\beta}(\lambda_n^{-1/m})
\]

is

\[
\frac{s^{1/m}}{s^{1/m} + \lambda_n} - 1 + \sum_{j=1}^{m-1} s^{-j/m} (-\lambda_n)^j - \frac{(-\lambda_n)^m s^{1/m-1}}{s^{1/m} + \lambda_n} = 0,
\]

since the Laplace transform of \( E_{1/m}(\lambda_n^{-1/m}) \) is \( \frac{s^{1/m-1}}{s^{1/m} + \lambda_n} \) and using (4.3). Hence, by the uniqueness of Laplace transforms, we get

\[
\frac{\partial}{\partial t} E_{\beta}(\lambda_n^{-1/m}) - \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} (\lambda_n^{-1}/m) (-\lambda_n)^j - (-\lambda_n)^m E_{\beta}(\lambda_n^{-1/m}) = 0.
\]
Now applying the time derivative and $\Delta^m$ to the series in (4.14) term by term gives
\[
\frac{\partial}{\partial t}u(t, x) - \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} \Delta^j f(x) - \Delta^m u(t, x)
\]
\[
= \sum_{n=1}^{\infty} \bar{f}(n) \left[ \frac{\partial}{\partial t} \frac{E_{1/m}}{-\lambda_n t^{1/m}} - \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} \Delta^j \phi_n(x) - E_{1/m}(-\lambda_n t^{1/m}) \Delta^m \phi_n(x) \right]
\]
\[
= \sum_{n=1}^{\infty} \bar{f}(n) \phi_n(x) \left[ \frac{\partial}{\partial t} \frac{E_{1/m}}{-\lambda_n t^{1/m}} + \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} (-\lambda_n)^j - (-\lambda_n)^m E_{1/m}(-\lambda_n t^{1/m}) \right]
\]
\[
= 0
\]
which shows that the PDE in (1.10) is satisfied. Thus, we conclude that $u$ defined by (4.8) is a classical solution to (1.10). This completes the first part of the proof.

We next deal with the case $m = 2^k$. Observe that $E^{1/2^k}(t)$, process inverse to stable subordinator of index $1/2^k$ and $|I_k(t)|$ have the same density, as they have the same one dimensional distribution by Lemma 3.2. Let $p(t, l)$ denote the common density of $|I_k(t)|$ and $E^{1/2^k}(t)$.

\begin{equation}
(4.14) \quad u(t, x) = \sum_{n=1}^{\infty} \bar{f}(n) E_{1/2^k}(-\lambda_n t^{1/2^k}) \phi_n(x)
\end{equation}

\[
= \int_0^{\infty} \sum_{n=1}^{\infty} \bar{f}(n) e^{-\lambda_n t} \phi_n(x) \left[ 2^k t g_{1/2^k}(tl^{-2^k}) l^{-(1+2^k)} \right] dl
\]

\begin{equation}
(4.15) \quad = \int_0^{\infty} \sum_{n=1}^{\infty} \bar{f}(n) e^{-\lambda_n t} \phi_n(x) \left[ T_D(l) f(x) p(t, l) \right] dl
\end{equation}

\begin{equation}
(4.16) \quad = \mathbb{E}_x [f(X(\tau_D(X))) I(\sigma_D(X) > |I_k(t)|)].
\end{equation}

Note that Equation (4.16) follows by a conditioning argument.

Finally we prove uniqueness. Let $u_1, u_2$ be two solutions of (1.11) with initial data $u(0, x) = f(x)$ and Dirichlet boundary condition $u(t, x) = 0$ for $x \in \partial D$. Then $U = u_1 - u_2$ is a solution of (1.11) with zero initial data and zero boundary value. Taking $\phi_n$-transform on both sides of (1.11) we get
\[
\frac{\partial}{\partial t} \tilde{U}(t, n) = (-\lambda_n)^m \tilde{U}(t, n), \quad \tilde{U}(0, n) = 0,
\]
and then $\tilde{U}(t, n) = 0$ for all $t > 0$ and all $n \geq 1$. This implies that $U(t, x) = 0$ in the sense of $L^2$ functions, since $\{\phi_n : n \geq 1\}$ forms a complete orthonormal basis for $L^2(D)$. Hence, $U(t, x) = 0$ for all $t > 0$ and almost all $x \in D$. Since $U$ is a
continuous function on $D$, we have $U(t, x) = 0$ for all $(t, x) \in [0, \infty) \times D$, thereby proving uniqueness.

**Corollary 4.1.** Let $f \in C^{2k}_c(D)$ be a $2k$-times continuously differentiable function of compact support in $D$. If $k > m - 1 + 3d/4$, then the Equation (1.10) has a strong solution. In particular, if $f \in C^\infty_c(D)$, then the classical solution of Equation (1.10) is in $C^\infty(D)$.

**Proof.** By Example 2.1.8 of [19], $|\phi_n(x)| \leq (\lambda_n)^{d/4}$. Also, from Corollary 6.2.2 of [20], we have $\lambda_n \sim n^{2/d}$.

By Lemma 4.1 we get

\[(4.17) \Delta^k f(n) = (-\lambda_n)^k \bar{f}(n).\]

Using Cauchy-Schwartz inequality and the fact $f \in C^{2k}_c(D)$, we get

\[
\Delta^k f(n) \leq \left[ \int_D (\Delta^k f(x))^2 \, dx \right]^{1/2} \left[ \int_D (\phi_n(x))^2 \, dx \right]^{1/2} = \left[ \int_D (f^{(2k)}(x))^2 \, dx \right]^{1/2} = c_k,
\]

where $c_k$ is a constant independent of $n$.

This and Equation (4.17) give $|\bar{f}(n)| \leq c_k (\lambda_n)^{-k}$.

Since

\[
\Delta^{m-1} f(x) = \sum_{n=1}^{\infty} (-\lambda_n)^{m-1} \bar{f}(n) \phi_n(x),
\]

to get the absolute and uniform convergence of the series defining $\Delta^{m-1} f$, we consider

\[
\sum_{n=1}^{\infty} (\lambda_n)^{m-1} |\phi_n(x)||\bar{f}(n)| \leq \sum_{n=1}^{\infty} (\lambda_n)^{d/4 + m - 1} c_k (\lambda_n)^{-k}
\]

\[
= c_n \sum_{n=1}^{\infty} (n^{2/d})^{d/4 + m - 1 - k} = c_k \sum_{n=1}^{\infty} n^{1/2 + 2(m-1)/d - 2k/d}
\]

which is finite if $-2(m-1)/d - 1/2 + 2k/d > 1$, i.e. $k > m - 1 + 3d/4$. \qed

5. **Other higher order Cauchy Problems**

The $d$-dimensional symmetric $\alpha$-stable process $X(t)$ with $\alpha \in (0, 2]$ is the process with stationary independent increments whose transition density

\[
p^\alpha_t(x, y) = p^\alpha(t, x - y), \quad (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d,
\]

is characterized by

\[
\int_{\mathbb{R}^d} e^{i\xi \cdot y} p^\alpha(t, y) \, dy = \exp(-t|\xi|^{\alpha}), \quad t > 0, \xi \in \mathbb{R}^d.
\]

The process has right continuous paths, it is rotation and translation invariant. For $\alpha = 2$, this is Brownian motion running twice the speed of standard Brownian motion.
Since the Laplace transform method does not apply by the appearance of $t^{-1}$ in the PDE (1.12), we use a direct method to prove Theorem 1.3.

**Proof of Theorem 1.3.** By a simple conditioning argument and using the series representation of the killed semigroup $T_D(t)$ in (2.2), we can express $u(t, x)$ as

$$u(t, x) = 2 \int_0^\infty \left( \sum_{n=1}^{\infty} e^{-\lambda_n s} \bar{f}(n) \phi_n(x) \right) p^1(t, s) ds$$

(5.1)

We use Fubini-Tonelli theorem, the simple inequality

$$\int_0^\infty e^{-\lambda_n s} \frac{t}{\pi(t^2 + s^2)} ds \leq \frac{1}{\pi t \lambda_n},$$

(5.2)

and the fact that the series defining $\Delta f$ is absolutely and uniformly convergent to show that

$$u(t, x) = 2 \sum_{n=1}^{\infty} \bar{f}(n) \phi_n(x) \int_0^\infty e^{-\lambda_n s} \frac{t}{\pi(t^2 + s^2)} ds.$$

Using this, and (5.2) we can show that we can apply $\Delta^2$ to the series (5.1) term by term. Hence it follows that $u, \Delta u, \Delta^2 u \in C(D_\infty)$.

We next show that each term in the series (5.1) satisfy the PDE (1.3). We use the fact that $p^1(t, s)$ satisfy

$$\left( \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} \right) p^1(t, s) = 0,$$

dominated convergence theorem, and integration by parts twice to get

$$\frac{\partial^2}{\partial t^2} \left( \bar{f}(n) \phi_n(x) \int_0^\infty e^{-\lambda_n s} p^1(t, s) ds \right) = \bar{f}(n) \phi_n(x) \int_0^\infty e^{-\lambda_n s} \frac{\partial^2}{\partial t^2} p^1(t, s) ds$$

$$= -\bar{f}(n) \phi_n(x) \int_0^\infty e^{-\lambda_n s} \frac{\partial^2}{\partial s^2} p^1(t, s) ds$$

$$= \frac{1}{\pi t} \lambda_n \phi_n(x) \bar{f}(n)$$

$$+ \lambda_n^2 \phi_n(x) \bar{f}(n) \int_0^\infty e^{-\lambda_n s} p^1(t, s) ds$$

$$= -\frac{1}{\pi t} \Delta \phi_n(x) \bar{f}(n)$$

$$+ -\Delta^2 \phi_n(x) \bar{f}(n) \int_0^\infty e^{-\lambda_n s} p^1(t, s) ds.$$

From this we get that the time derivative can be applied term by term to the series (5.1) since the series for $\Delta f$ and $\Delta^2 u$ converge absolutely and uniformly. Hence
applying the time derivative and the $\Delta^2$ term by term to the series \((5.1)\) we obtain

\[
\frac{\partial^2}{\partial t^2} u(t, x) + \frac{2\Delta f(x)}{\pi t} + \Delta^2 u(t, x) = \sum_{n=1}^{\infty} \bar{f}(n) \left( \phi_n(x) \int_0^{\infty} e^{-\lambda_ns} \frac{\partial^2}{\partial t^2} p^1(t, s) ds + \frac{1}{\pi t} \Delta \phi_n(x) \right)
\]

\[
+ \Delta^2 \phi_n(x) \int_0^{\infty} e^{-\lambda_ns} p^1(t, s) ds = 0.
\]

(5.3)

\[\square\]

For a rational $\alpha \neq 1$ the PDE is more complicated since kernels of symmetric $\alpha$-stable processes satisfy a higher order PDE.

**Theorem 5.1** (Nane [42]). Let $\alpha \in (0, 2)$ be rational $\alpha = l/m$, where $l$ and $m$ are relatively prime. Let $T(s) f(x) = \mathbb{E}_x[f(X(s))]$ be the semigroup of Brownian motion $X(t)$ and let $\Delta$ be its generator. Let $f$ be a bounded measurable function in the domain of $\Delta$, with $D\gamma f$ bounded and Hölder continuous for all multi index $\gamma$ such that $|\gamma| = 2l$. Then $u(t, x) = \mathbb{E}_x[f(X(|Y(t)|))]$ is a solution of

\[
(-1)^{l+1} \frac{\partial^{2m}}{\partial t^{2m}} u(t, x) = -2 \sum_{i=1}^{l} \left( \frac{\partial^{2l-2i}}{\partial s^{2l-2i}} p^0(t, s) \big|_{s=0} \right) \Delta^{2i-1} f(x)
\]

\[
- \Delta^{2l} u(t, x), \quad t > 0, \quad x \in \mathbb{R}^d
\]

\[
u(0, x) = f(x), \quad x \in \mathbb{R}^d.
\]

Let

\[
\mathcal{K}_{\Delta^{2l}} \equiv \left\{ u : D_{\infty} \rightarrow \mathbb{R} : \frac{\partial^2}{\partial t^2} u, \Delta^j u, \in C(D_{\infty}) \text{ for } j = 1, \cdots, 2l,
\]

\[
\Delta^j u, \in C(\bar{D}) \text{ for } j = 1, \cdots, 2l - 1 \right\}
\]

This theorem takes the following form in bounded domains

**Theorem 5.2.** Let $\alpha \in (0, 2)$ be rational $\alpha = l/m$, where $l$ and $m$ are relatively prime. Let $D$ be a bounded domain with $\partial D \in C^{1, \gamma}$, $0 < \gamma < 1$. Let $\{X(t)\}$ be a Brownian motion, and let $\{Y(t)\}$ be symmetric $\alpha$-stable process independent of $\{X(t)\}$. Let $f \in D(\Delta) \cap C^{4l-2}(D)$ for which the eigenfunction expansion of $\Delta^{2l-1} f$ with respect to the complete orthonormal basis $\{\phi_n : n \geq 1\}$ converges absolutely and uniformly.
Then $u(t,x) = \mathbb{E}_x[f(X(\tau_D(X)))] I(\tau_D(X) > |Y(t)|)$ is a classical solution of
$u \in \mathcal{K}_{\Delta 2^l} \cap C_b(D_\infty) \cap C^1(\bar{D})$

$$(5.4) (-1)^{l+1} \frac{\partial^{2m}}{\partial t^{2m}} u(t,x) = -2 \sum_{i=1}^{l} \left( \frac{\partial^{2i-2} f(t,s)}{\partial s^{2i-2}} \right) \Delta^{2i-1} f(x)$$

$$- \Delta^{2l} u(t,x), \quad t > 0, \ x \in D$$

$u(0,x) = f(x), \quad x \in D.$

$u(t,x) = \Delta^j u(t,x) = 0, \ x \in \partial D, \ t \geq 0, \ j = 1, \cdots, 2l - 1.$

**Proof.** Let $T_D(t)$ be the killed semigroup of Brownian motion in $D$. We use the representation

$$u(t,x) = \mathbb{E}_x[f(X(\tau_D(X)))] I(\tau_D(X) > |Y(t)|) = 2 \int_0^\infty p^\alpha(t,s) T_D(s) f(x) ds$$

$$(5.5) = 2 \int_0^\infty \left( \sum_{n=1}^{\infty} e^{-\lambda_n s} \tilde{f}(n) \phi_n(x) \right) p^\alpha(t,s) ds$$

and the fact that the transition density $p^\alpha(t,s)$ of the process $Y(t)$ satisfies

$$\left( \frac{\partial^2}{\partial s^2} \right)^l + (-1)^{l+1} \frac{\partial^{2m}}{\partial t^{2m}} p^\alpha(t,s) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R},$$

from Lemma 3.2 in [12].

We also use the well-known fact that for $\alpha \in (0,2)$

$$p^\alpha(t,x) = t^{-d/\alpha} p(1,t^{-1/\alpha} x) \leq t^{-d/\alpha} p(1,0) = t^{-d/\alpha} M_{d,\alpha}, t > 0, x \in \mathbb{R}^d,$$

where

$$M_{d,\alpha} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|x|^\alpha} dx.$$ 

Hence

$$\int_0^\infty e^{-\lambda_n s} p^\alpha(t,s) ds \leq \frac{M_{d,\alpha}}{\lambda_n t^{d/\alpha}}.$$ 

This allows us to deduce the fact that $\Delta^j u \in C(D_\infty)$ for $j = 0, 1, \cdots, 2l$ and also that $\Delta^j u \in C(\bar{D})$ for $j = 0, 1, \cdots, 2l - 1.$

To get that $\frac{\partial^{2m}}{\partial t^{2m}} u(t,x) \in C(D_\infty)$ we use the fact that the series defining $\Delta^{2l} u$ converges absolutely and uniformly as well as the series defining $\Delta^{2l-1} f$ converges absolutely and uniformly and the fact that the terms in the series defining $u(t,x)$ in (5.5) satisfy the PDE (5.4).

Hence we can interchange the sum and powers of the Laplacian in the series (5.5) term by term to show that PDE in (5.4) is satisfied. \qed
6. Extensions and Discussion

A uniformly elliptic operator of divergence form is defined on \( C^2 \) functions by

\[
Lu = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right)
\]

with \( a_{ij}(x) = a_{ji}(x) \) and, for some \( \lambda > 0 \),

\[
\lambda \sum_{i=1}^{n} y_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \leq \lambda^{-1} \sum_{i=1}^{n} y_i^2, \quad \forall y \in \mathbb{R}^d.
\]

The operator \( L \) acts on the Hilbert space \( L^2(D) \). We define the initial domain \( C^\infty_0(\bar{D}) \) of the operator as follows. We say that \( f \) is in \( C^\infty_0(\bar{D}) \), if \( f \in C^\infty(\bar{D}) \) and \( f(x) = 0 \) for all \( x \in \partial D \). This condition incorporates the notion of Dirichlet boundary conditions.

From [20, Corollary 6.1], we have that the associated quadratic form

\[
Q(f, g) = \int_D \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} dx
\]

is closable on the domain \( C^\infty_0(\bar{D}) \) and the domain of the closure is independent of the particular coefficients \( (a_{ij}) \) chosen. In particular, for \( f, g \in C^\infty_0(\bar{D}) \) by integration by parts

\[
\int_D g(x)Lf(x)dx = Q(f, g) = \int_D f(x)Lg(x)dx,
\]

which shows that \( L \) is symmetric.

From now on, we will use the symbol \( L_D \) if we particularly want to emphasize the choice of Dirichlet boundary conditions, to refer to the self-adjoint operator associated with the closure of the quadratic form above by the use of [20, Theorem 4.4.5]. Thus, \( L_D \) is the Friedrichs extension of the operator defined initially on \( C^\infty_0(\bar{D}) \).

If the coefficients \( a_{ij}(x) \) are smooth \( (a_{ij}(x) \in C^1(D)) \), then \( L_D u \) takes the form

\[
L_D u = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \left( \sum_{j=1}^{d} \frac{\partial a_{ij}(x)}{\partial x_j} \right) \frac{\partial u}{\partial x_i} = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i}.
\]

If \( X_t \) is a solution to

\[
dX_t = \sigma(X_t)dW_t + b(X_t)dt, \quad X_0 = x_0,
\]

where \( \sigma \) is a \( d \times d \) matrix, and \( W_t \) is a Brownian motion, then \( X_t \) is associated with the operator \( L_D \) with \( a = \sigma^T \sigma \), (see Chapters 1 and 5 of Bass [12]). Define the first exit time as \( \tau_D(X) = \inf\{t \geq 0 : X_t \notin D\} \). The semigroup defined by \( T(t)f(x) = E_x[f(X_t)1(\tau_D(X) > t)] \) has generator \( L_D \), which follows by an application of the Itô formula.
Let $D$ be a bounded domain in $\mathbb{R}^d$. Suppose $L$ is a uniformly elliptic operator of divergence form with Dirichlet boundary conditions on $D$, and that there exists a constant $\Lambda$ such that for all $x \in D$,

$$
\sum_{i,j=1}^{d} |a_{ij}(x)| \leq \Lambda.
$$

Let $T_D(t)$ be the corresponding semigroup. Then $T_D(t)$ is an ultracontractive semigroup (even an intrinsically ultracontractive semigroup), see Corollary 3.2.8, Theorem 2.1.4, Theorem 4.2.4, and Note 4.6.10 in [19]. Every ultracontractive semigroup has a kernel for the killed semigroup on a bounded domain which can be represented as a series expansion of the eigenvalues and the eigenfunctions of $L_D$ (cf. [19, Theorems 2.1.4 and 2.3.6] and [24, Theorems 8.37 and 8.38]): There exist eigenvalues $0 < \mu_1 < \mu_2 \leq \mu_3 \cdots$, such that $\mu_n \to \infty$, as $n \to \infty$, with the corresponding complete orthonormal set (in $H^2_0$) of eigenfunctions $\psi_n$ of the operator $L_D$ satisfying

$$
L_D \psi_n(x) = -\mu_n \psi_n(x), \; x \in D : \psi_n|_{\partial D} = 0.
$$

In this case,

$$
p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\mu_n t} \psi_n(x) \psi_n(y)
$$

is the heat kernel of the killed semigroup $T_D$. The series converges absolutely and uniformly on $[t_0, \infty) \times D \times D$ for all $t_0 > 0$.

Suppose $D$ satisfies a uniform exterior cone condition. Let $\{X_t\}$ be a Markov process in $\mathbb{R}^d$ with generator $L_D$, and $f$ be continuous on $\bar{D}$. Then the semigroup

$$
T_D(t)f(x) = E_x[f(X_t)I(t < \tau_D(X))]
$$

$$
= \int_D p_D(t, x, y)f(y)dy
$$

$$
= \sum_{n=1}^{\infty} e^{-\mu_n t} \psi_n(x) \bar{f}(n)
$$

solves the Dirichlet initial-boundary value problem in $D$:

$$
\frac{\partial u(t,x)}{\partial t} = L_Du(t,x), \; x \in D, \; t > 0,
$$

$$
u(t, x) = 0, \; x \in \partial D,
$$

$$
u(0, x) = f(x), \; x \in D.
$$

Remark 6.1. The eigenfunctions belong to $L^\infty(D) \cap C^\alpha(D)$ for some $\alpha > 0$, by [24, Theorems 8.15 and 8.24]. If $D$ satisfies a uniform exterior cone condition all the eigenfunctions belong to $C^\alpha(\bar{D})$ by [24, Theorem 8.29]. If $a_{ij} \in C^\alpha(\bar{D})$ and $\partial D \in C^{1,\alpha}$, then all the eigenfunctions belong to $C^{1,\alpha}(\bar{D})$ by [24, Corollary 8.36]. If $a_{ij} \in C^\infty(D)$
then each eigenfunction of $L$ is in $C^\infty(D)$ by [24, Corollary 8.11]. If $a_{ij} \in C^\infty(\bar{D})$ and $\partial D \in C^\infty$, then each eigenfunction of $L$ is in $C^\infty(\bar{D})$ by [24, Theorem 8.13].

**Remark 6.2.** Theorems 1.1, 1.2 and 1.3 are valid if we replace the outer Brownian motion with a Diffusion process. This can be verified by considering (4.13), (6.4), (6.5), Remark 6.1 and Theorem 3.1 in [37].

**Remark 6.3.** It might be an interesting project to consider the PDEs treated in this paper with the Neumann boundary conditions. Probably the solutions will be obtained by running reflected diffusions subordinated by $k$-iterated Brownian motions. We will treat this problem elsewhere.

**Acknowledgments.** I would like to thank Mark Meerschaert for encouragement on working on the problems in this paper. I also would like to thank anonymous referee for his or her helpful comments on the comments on the historical priority on the equivalence of fractional Cauchy problems and higher order PDEs which improved the accuracy of the results leading to the present paper.

**References**

[1] Agrawal, O.P. (2002). Solution for a fractional diffusion-wave equation defined in a bounded domain. Fractional order calculus and its applications. *Nonlinear Dynam.* 29 145–155.

[2] Allouba, H., A linearized Kuramoto-Sivashinsky PDE via an imaginary-Brownian-time-Brownian-angle process, C.R. Acad. Sci. Paris Ser. 1336 (2003), 309-314.

[3] Allouba, H., Brownian-time processes: The pde connection and the corresponding Feynman-Kac formula, Trans. Amer. Math. Soc. 354 (2002), no.11 4627 - 4637.

[4] Allouba, H. and Zheng, W. (2001). Brownian-time processes: The PDE connection and the half-derivative generator, *Ann. Prob.* 29 1780-1795.

[5] Applebaum, D. (2004). *Lévy Processes and Stochastic Calculus*. Cambridge studies in advanced mathematics.

[6] Arcones, M. A. The large deviation principle for stochastic processes. I. Teor. Veroyatnost. i Primenen. 47 (2002), no. 4, 727–746; translation in Theory Probab. Appl. 47 (2003), no. 4, 567–583.

[7] Arendt, W., Batt, C., Hieber, M. and Neubrander, F. (2001). *Vector-valued Laplace transforms and Cauchy problems*. Monographs in Mathematics, Birkhäuser-Verlag, Berlin.

[8] Aurzada, F. and Lifshits, M. (2009). On the Small Deviation Problem for Some Iterated Processes. Electronic. J. Probab. Vol. 14, Paper no. 68, pages 1992010.

[9] Baeumer, B. and Meerschaert, M.M. (2001). Stochastic solutions for fractional Cauchy problems, *Fractional Calculus Appl. Anal.* 4 481–500.

[10] Baeumer, B., Meerschaert, M.M. and Nane, E. Brownian subordinators and fractional Cauchy problems, *Trans. Amer. Math. Soc.* 361 (2009), 3915-3930.

[11] Bass, R.F. (1995). *Probabilistic Techniques in Analysis*. Springer-Verlag, New York.

[12] Bass, R. F. (1998). *Diffusions and Elliptic Operators*. Springer-Verlag, New York.

[13] Becker-Kern, P., Meerschaert, M.M. and Scheffler, H.P. (2004). Limit theorem for continuous time random walks with two time scales. *J. Applied Probab.* 41 No. 2, 455–466.

[14] Benachour, S.; Roynette, B.; Vallois, P. *Explicit solutions of some fourth order partial differential equations via iterated Brownian motion*. Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1996), 39–61, Progr. Probab., 45, Birkhäuser, Basel, 1999.
[15] Bochner, S., *Harmonic Analysis and the Theory of Probability*, Univ. California press, Berkeley, CA (1955).

[16] Burdzy, K. (1993). Some path properties of iterated Brownian motion. *In Seminar on Stochastic Processes* (E. Çinlar, K.L. Chung and M.J. Sharpe, eds.) 67-87. Birkhäuser, Boston.

[17] Burdzy, K. (1994). Variation of iterated Brownian motion. *In Workshops and Conference on Measure-valued Processes, Stochastic Partial Differential Equations and Interacting Particle Systems* (D.A. Dawson, ed.) 35-53. Amer. Math. Soc. Providence, RI.

[18] Caputo, M. (1967). Linear models of dissipation whose Q is almost frequency independent, Part II. *Geophys. J. R. Astr. Soc.* 13 529-539.

[19] Davies, E.B. (1989). *Heat Kernels and Spectral Theory*. Cambridge Univ. Press, Cambridge.

[20] Davies, E.B. (1996). *Spectral Theory and Differential Operators*. Cambridge Univ. Press, Cambridge.

[21] DeBlassie, R.D. (2004). Iterated Brownian motion in an open set. *Ann. Appl. Probab.* 14 1529–1558.

[22] DiBenedetto, E. (1995). *Partial Differential Equations*. Birkhäuser, Boston.

[23] Funaki, T. *A probabilistic construction of the solution of some higher order parabolic differential equations*, Proc. Japan Acad. Ser. A. Math. Sci. (5) 55 (1979), 176 - 179.

[24] Gilbarg, D. and N.S. Trudinger, N.S. (2001) *Elliptic Partial Differential Equations of Second Order*. Reprint of the 1998 ed.- New York Springer.

[25] Gorenflo, R. and Mainardi, F. (2003). Fractional diffusion processes: Probability distribution and continuous time random walk. *Lecture Notes in Physics* 621 148–166.

[26] Hochberg, K. J. and Orsingher, E. (1996). Composition of stochastic processes governed by higher-order parabolic and hyperbolic equations. *J. Theoret. Probab.* 9 511532.

[27] Jacob, N. (1996) *Pseudo-Differential Operators and Markov Processes*. Berlin : Akad. Verl.

[28] Khoshnevisan, D. and Lewis, T.M., *Chung’s law of the iterated logarithm for iterated Brownian motion*, *Ann. Inst. H. Poincaré Probab. Statist.* 32 (1996), no. 3, 349-359.

[29] Krägeloh, A.M. (2003). Two families of functions related to the fractional powers of generators of strongly continuous contraction semigroups, *J. Math. Anal. Appl.* 283 459-467.

[30] Le Gall, J.-F. (1993). Solutions positives de $\Delta u = u^2$ dans le disque unité. *C.R. Acad. Sci. Paris Sér. I* 317 873-878.

[31] Le Gall, J.-F. (1994). A path-valued Markov process and its connections with partial differential equations. In *European Congress of Mathematics* 2 185-212. Birkhäuser, Boston.

[32] Le Gall, J.-F. (1995). The Brownian snake and solutions of $\Delta u = u^2$ in a domain. *Probab. Theory Related Fields* 102 393-432.

[33] Mainardi, F. and Gorenflo, R. (2000). On Mittag-Leffler-type functions in fractional evolution processes, *J. Comput. Appl. Math.* 118 283–299.

[34] Meerschaert, M.M. and Scheffler, H.P. (2001) *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*. Wiley Interscience, New York.

[35] Meerschaert, M.M. and Scheffler, H.P. (2004). Limit theorems for continuous time random walks with infinite mean waiting times. *J. Applied Probab.* 41 No. 3, 623–638.

[36] Meerschaert, M.M., Benson, D.A., Scheffler, H.P. and Baeumer, B. (2002). Stochastic solution of space-time fractional diffusion equations. *Phys. Rev. E* 65, 1103–1106.

[37] Meerschaert, M.M., Nane, E. and Vellaisamy, P. (2009). *Fractional Cauchy problems in bounded domains*, *Ann. Prob. Vol. 37, No. 3, 979-1007*

[38] Meerschaert, M.M., Nane, E. and Xiao, Y., *Correlated continuous time random walks*, *Stat. & Probab. Lett.* Volume 79, Issue 9, 1 May 2009, Pages 1194-1202

[39] Nane, E., *Iterated Brownian motion in parabola-shaped domains*, Potential Analysis, 24 (2006), 105-123.
Nane, E., *Iterated Brownian motion in bounded domains in \( \mathbb{R}^n \)*, Stochastic Processes and Their Applications, 116 (2006), 905-916.

Nane, E., *Lifetime asymptotics of iterated Brownian motion in \( \mathbb{R}^n \):* ESAIM: P&S, March 2007, Vol. 11, pp. 147-160.

Nane, E., *Higher order PDE’s and iterated processes*, Trans. Amer. Math. Soc. 360 (2008), 2681-2692.

Nigmatullin, R.R. (1986). The realization of the generalized transfer in a medium with fractal geometry. *Phys. Status Solidi B* 133 425-430.

Nourdin, I. and Peccati, G., Weighted power variations of iterated Brownian motion. Electronic J. Probab. Vol. 13 (2008), 1229-1256.

Orsingher, E. and Beghin, L., (2004) Time-fractional telegraph equations and telegraph processes with Brownian time. *Prob. Theory Rel. Fields* 128, 141–160.

Orsingher, E. and Beghin, L., (2008) Fractional diffusion equations and processes with randomly varying time, *Ann. Probab.* Volume 37, Number 1 (2009), 206-249.

Podlubny, I. (1999). *Fractional Differential Equations*, Academic Press, San Diego.

Royden H.L. (1968). *Real Analysis*. 2nd Edition, MacMillan, New York.

Samko, S., Kilbas, A. and Marichev, O. (1993). *Fractional Integrals and derivatives: Theory and Applications*. Gordon and Breach, London.

Sato, K.I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

Scalas, E. (2004). Five years of Continuous-Time Random Walks in Econophysics. *Proceedings of WEHIA 2004*, A. Namatame (ed.), Kyoto.

Schilling, R.L., Growth and Hölder conditions for sample paths of Feller processes. *Probability Theory and Related Fields* 112, 565–611 (1998)

Schneider, W. R. and Wyss, W. (1989). Fractional diffusion and wave equations. *J. Math. Phys.* 30, 134-144.

Song, R. and Vondraček, Z. (2003). Potential theory of subordinate killed Brownian motion in a domain. *Probab. Theory Relat. Fields* 125 578-592.

Zaslavsky, G. (1994). Fractional kinetic equation for Hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion. *Phys. D* 76 110-122.

**Erkan Nane, Department of Mathematics and Statistics, 221 Parker Hall, Auburn University, Auburn, AL 36849**

*E-mail address: nane@auburn.edu*