Axisymmetric Multiple Contact Problem for a System of Annular Punches and Foundation with Nonuniform Coating of Variable Thickness

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Abstract. Contact problem for a system of rigid annular rough punches and foundation with nonuniform coating of variable thickness is considered. We assume that coating nonuniformity, it’s thickness, and forms of the punches can be described by rapidly changing functions. Mathematical model of the axisymmetric problem is compiled. It’s efficient solution is constructed by using special Manzhirov projection method. In this solution coating properties and shapes of the contacting surfaces are represented by separate terms and factors. Such a result can hardly be done by other known methods. Qualitative conclusions are presented.

1. Introduction

Contact problems for foundations with coatings have been studied in good number of papers (see, e.g., [1–15]. Various cases were considered: multiple and single contact, bodies with uniform and nonuniform coatings, various profiles of coatings and punches. This paper for the first time provide solution of axisymmetric problem for the common case of arbitrary system of annular rigid punches taking into account coating nonuniformity and shapes of the contacting surfaces.

2. Axisymmetric multiple contact problem

Layer of a thickness \( h_{\text{lower}} \) made of viscoelastic aging material at moment \( t_{\text{lower}} \) coated by elastic layer of variable thickness \( h(r) \) lies on a rigid undeformable basis. Modulus of elasticity and Poisson’s ratio depends of upper layer depends on the radial coordinate, i.e. \( E = E(r) \) and \( \nu = \nu(r) \). Such foundations sometimes are called as surface nonuniform foundations. The material nonuniformity usually results from the process of layer by layer manufacturing (additive manufacturing) of a coating (e.g., laser treatment, ion implantation, etc. [16, 17]) and its further processing. We assume that upper layer is soft, i.e. it’s modulus of elasticity is less than the modulus of elasticity of the lower layer or they are of the same order of magnitude. There is smooth contact or perfect contact between layers and between the lower layer and the rigid base.

At time \( t_0 \geq t_{\text{lower}} \), the forces \( P_i(t) \), applied along the \( z \)-axis, starts to indent system of \( n \) smooth annular rigid punches, which axes coincides with the \( z \)-axis, into the surface of such a foundation. The punch base forms describes by functions \( f_i(r) \). The contact regions are constant and bounded by the circles with radii \( r = a_i \) and \( r = b_i \), where \( a_i \) and \( b_i \) are inner and outer radii of \( i \)th punch, \( i = 1,2,\ldots,n \).
The coating is assumed to be thin compared with the contact areas, i.e., its thickness satisfies the condition \( h(r) \ll \max_{i=1,2,\ldots,n}(b_i - a_i) \). Scheme of contact interaction for 2 punches presented in figure 1.

To derive the mathematical model of the problem, we replace the system of punches by some normally distributed load acting on regions \( r \in [a_i, b_i], i=1,2,\ldots,n \) (under the punches), and equal to zero outside these regions. Then the vertical displacement of the upper face of the foundation described above on each region (under each punch) can be written as (see, for example, [18])

\[
\frac{q_i(r,t)h(r)}{R(r)} + \frac{2(1-\nu^2_{lower})}{h_{lower}} \sum_{j=1}^{n} \left\{ \frac{1}{E_{lower}(t-\tau_{lower})} \int_{\nu_j}^{\rho_j} k_{ax} \left( \frac{r}{h_{lower}}, \frac{\rho}{h_{lower}} \right) q_j(\rho, t) \rho d\rho \right. \\
- \int_{\nu_j}^{\rho_j} \frac{K(t-\tau_{lower}, \tau-\tau_{lower})}{E_{lower}(t-\tau_{lower})} \int_{\nu_j}^{\rho_j} k_{ax} \left( \frac{r}{h_{lower}}, \frac{\rho}{h_{lower}} \right) q_j(\rho, \tau) \rho d\rho = \delta_i(t) - g_i(r),
\]

(1)

where \( q_i(r,t) \) is unknown contact pressure, \( g_i(r) = f_i(r) - h(r) + h_{0i} (h_{0i} = -\min_{r \in [a_i,b_i]} f_i(r) - h(r)) \) describes distance between contact surfaces in nondeformable state and called backlash function; contact rigidity \( R(r) \) depend on the contact conditions between coating and lower layer (see, for example, [19]); in the case of a perfect coating-layer contact, we have \( R(r) = E(r)[1 - \nu(r)][1 + \nu(r)][1 - 2\nu(r)] \), and in the case of an smooth contact, \( R(r) = E(r)[1 - \nu^2(r)] \); \( E_{lower} \) and \( E_{lower}(t-\tau_{lower}) \) are the Young modulus and Poisson's ratio of the lower layer; \( \delta(t) \) is are the punch settlements; \( K(t, \tau) \) is tensile creep kernel \( (K(t, \tau) = E_{lower}(t-\tau_{lower}) \partial / \partial \tau) (1/E_{lower}(\tau) + C_{lower}(t, \tau) / \partial \tau, C_{lower}(t, \tau) \) is the tensile creep function; \( k_{ax}(r \rho_{lower}, h_{lower}) \) is known kernel of the plane contact problem, which has the form (see, for example, [19, 20]): \( k_{ax}(r, \rho) = \int_{\rho_0}^{\rho} L(u) J_\nu(r \rho) J_\nu(\rho u) du \), and, in the case of a smooth contact between the lower layer and the rigid base, \( L(u) = [\cosh(2u) - 1]/[\sinh(2u) + 2u] \) and in the case of a perfect contact, \( L(u) = 2\sinh(2u) - 4u]/[2\cosh(2u) + 4u^2 + 1 + \kappa'] \), \( \kappa = 3 - 4 \nu_{lower} \).

We supplement Eq. (1) with the conditions of the punches equilibrium on the foundation

\[
\int_{\rho_0}^{\rho} q_i(\rho, t) \rho d\rho = P_i(t), \quad i=1,2,\ldots,n.
\]

(2)
There exist several versions of mathematical statements for the axisymmetric contact problem for a system of punches. It is easy to show that there is one of two types of conditions on each punch: the load force is given and the punch settlement is given. Thus, there are three versions of mathematical statements of contact problem for a system of punches in axisymmetric case.

3. Model representation
Let us make the change of variables in (1) and (2) by the formulas

\[ (r')^2 = \frac{r^2 - a_i^2}{b_i^2 - a_i^2}, \quad (\rho')^2 = \frac{\rho^2 - a_j^2}{b_j^2 - a_j^2}, \quad t' = \frac{t}{\tau_0}, \quad \tau_{\text{lower}}' = \frac{\tau_{\text{lower}}}{\tau_0}, \quad \lambda = \frac{h_{\text{lower}}}{b_i - a_i}, \quad \eta' = \frac{a_j}{b_j - a_j}, \]

\[ (\varphi')^2 = \frac{b_j^2 - a_j^2}{(b_i - a_i)^2}, \quad \delta^v(t') = \frac{\delta_i(t)\varphi^j}{b_i - a_i}, \quad g^v(r') = \frac{g_i(r)\varphi^j}{b_i - a_i}, \quad m^v(r') = \frac{E_i h_i(r)}{2(1 - v_{\text{lower}}^2)R(r)(b_i - a_i)}, \]

\[ c'(t') = \frac{E_{\text{lower}}(t - \tau_{\text{lower}})}{E_i}, \quad q^v(r', t') = \frac{2(1 - v_{\text{lower}}^2)q_i(r,t)\varphi^j}{E_{\text{lower}}(t - \tau_{\text{lower}})}, \quad P^v(t') = \frac{4P_i(t)(1 - v_{\text{lower}}^2)\varphi^j}{\pi E_{\text{lower}}(t - \tau_{\text{lower}})(b_i - a_i)^2}, \]

\[ F^{\alpha\beta} f(r') = \int_0^1 k^{\beta}(r', \rho') f(\rho') \rho' d\rho', \quad k^{\beta}(r', \rho') = \frac{\varphi^i \varphi^j}{\lambda} s_{\beta} \left( \frac{r}{h_{\text{lower}}}, \frac{\rho}{h_{\text{lower}}} \right) \]

\[ V^v f(t') = \int_{\tau_0}^t K^v(t', \tau') f(\tau') d\tau', \quad K^v(t', \tau') = K(t - \tau_{\text{lower}}, t - \tau_{\text{lower}}) \tau_0, \quad i, j = 1, 2, \ldots, n. \]

Having omitted asterisks in the obtained relations, we arrive at the system of integral equations and additional conditions in the form \((r \in [0, 1], t \geq 1)\)

\[ c(t)m^i(r)q^i(r, t) + (I - V) \sum_{\beta=1}^n F^{\alpha\beta} q^\beta(r, t) = \delta^i(t) - g^i(r), \quad \int_0^1 g^i(\rho, t) \rho d\rho = P^i(t), \quad i = 1, 2, \ldots, n. \quad (4) \]

Here \(I\) is identity operator. Assuming that

\[ q(r, t) = q^i(r, t) i^i, \quad \delta(t) = \delta^i(t) i^i, \quad g(r) = g^i(r) i^i, \quad P(t) = P^i(t) i^i, \]

\[ k(r, \rho) = k^{\beta}(r, \rho) i^i i^j, \quad FF(r) = \int_0^1 k(r, \rho) \cdot f(\rho) \rho d\rho, \quad D(r) = D^{\beta}(r) i^i, \quad D^{\beta}(r) = \left\{ \begin{array}{ll} m^i(r) & i = j, \\ 0 & i \neq j. \end{array} \right. \]

we can represent system with additional conditions (4) as

\[ c(t)D(r)q(r, t) + (I - V)Fq(r, t) = \delta(t) - g(r), \quad \int_0^1 q(\rho, t) \rho d\rho = P(t). \quad r \in [0, 1], \quad t \geq 1. \quad (6) \]

Hereinafter, it will be the summation over repeated upper indices \(i\) and \(j\) from 1 to \(n\) if the left side of the formula is independent of the index. Note that vector functions \(D(r)\) and \(g(r)\) connect with coating rigidity, coating width, and punch base forms. These parameters can be described by a rapidly changing function.

In what follows, we construct the solution of the of two-dimensional operator equation with the vector auxiliary conditions (6), which contains integral operators with constant as well as variable limits of integration and 2\(n\) different rapidly changing functions.

4. Analytical solution

4.1. Special form of the contact pressure
We should see the solution for all versions of the problem in the form

\[ q(r, t) = \tilde{q}(r, t) - D^{\beta}(r) \cdot g(r) / c(t). \quad (7) \]

where \(D^{\beta}(r) = [D^{\beta}(r)]^{-1} i^i\) and \(\tilde{q}(r, t)\) is new unknown function. Then operator equation and auxiliary
conditions (6) can be reduced to the following equations:

\[ c(t)D(r)\tilde{q}(r,t) + (I - V)F\tilde{q}(r,t) = \delta(t) - \tilde{c}(t)\tilde{g}(r), \quad \int_0^\infty \tilde{q}(\rho,t)\rho\,d\rho = \tilde{P}(t), \quad r \in [0,1], \quad t \geq 1, \]  

(8)

where

\[ \tilde{g}(r) = \int_0^1 k(r,\rho) \cdot D^{-1}(\rho) \cdot g(\rho)\rho\,d\rho, \quad \tilde{c}(t) = -(I-V)c^{-1}(t), \quad \tilde{P}(t) = P(t) + c^{-1}(t) \int_0^1 D^{-1}(\rho) \cdot g(\rho)\rho\,d\rho. \]  

(9)

We obtain new mixed operator equation with \( n \) rapidly changing function \( m_i(r) \) \((i = 1,2,\ldots,n)\) supplemented by two vector conditions. Last term in right-hand side of operator equation (8) is “good”: its smoothness defined by kernel \( k(r,\rho) \).

We will construct the solution of equations (7) for the version when all forces are known.

4.2. Special basis and solution representation

By introducing in (8) notations \( \tilde{q}(r,t) = D^{-1/2}(r) \cdot Q(r,t), \quad k(r,\rho) = D^{1/2}(r) \cdot K(r,\rho) \cdot D^{1/2}(\rho), \quad Gf(r) = \int_0^1 K(r,\rho) \cdot f(\rho)\rho\,d\rho \) we obtain new operator equation and additional conditions

\[ c(t)Q(r,t) + (I - V)GQ(r,t) = D^{-1/2}(r) \cdot [\delta(t) - \tilde{c}(t)\tilde{g}(r)], \quad \int_0^1 D^{-1/2}(r)Q(\rho,t)\rho\,d\rho = \tilde{P}(t). \]  

(10)

Here \( D^{1/2}(r) = \sqrt{D^0(r)}i' i' \), \( D^{-1/2}(r) = \sqrt{1/D^0(r)}i' i' : Q(r,t) \) is new unknown function, and \( K(r,\rho) \) is new kernel.

We seek the solution of Eq. (10) in the class of vector functions continuous in time \( t \) in the Hilbert space \( L_2([0,1],V) \) (see [21]). To this end, we at first construct an orthonormal system of vector functions in \( L_2([0,1],V) \) which contains the factors \( 1/\sqrt{m'(r)} \) and remaining basis functions can be written as the products of vector functions depending on \( r \) and weight functions \( 1/\sqrt{m'(r)} \). To this end we will orthonormal on \([0,1]\) the linearly independent system of vector-functions \( \{i'r^l/\sqrt{m'(r)}\}_{l=1,\ldots,n} \) by the formulas:

\[ \begin{align*}
& p_{1i}(r) = D^{-1/2}(r) \cdot p_{1i}(r), \quad p_{2i}(r) = p_{1i}(r)i', \quad d_{1i} = 1, \quad J_{k,i} = \int_0^1 D^{21/2}(\rho)\rho\,d\rho/\sqrt{m'(\rho)}, \\
& d_{k,i} = \begin{bmatrix}
J_{0,i} & \cdots & J_{k,i} \\
\vdots & \ddots & \vdots \\
J_{k,i} & \cdots & J_{2k,i}
\end{bmatrix}, \quad p_{k,i}(r) = \frac{1}{\sqrt{d_{k-1,i}d_{k,i}}} \begin{bmatrix}
J_{0,i} & \cdots & J_{k,i} \\
\vdots & \ddots & \vdots \\
J_{k-1,i} & \cdots & J_{2k-1,i}
\end{bmatrix} \begin{bmatrix}
J_{0,i} & \cdots & J_{k,i} \\
\vdots & \ddots & \vdots \\
J_{2k-2,i} & \cdots & J_{2k,i}
\end{bmatrix},
\end{align*} \]

(11)

Thus, \( p_{i}(r) \) \((i = 1,2,\ldots,n, k = 0,1,2,\ldots)\) is a basis in \( L_2([0,1],V) \).

4.3. Solution of the problem

If Hilbert space \( L_2([0,1],V) \) represents as the direct sum of the Euclidean space \( L_2^0([0,1],V) \) with the basis \( \{p^i_0(r)\}_{i=1,2,\ldots,n} \) and Hilbert space \( L_2^1([0,1],V) \) with the basis \( \{p^i_1(r),p^i_2(r),p^i_3(r),\ldots\}_{i=1,2,\ldots,n} \) then the integrand and the right-hand side of integral equation (10) can be presented as the algebraic sum of functions continuous in time \( t \) and ranging in \( L_2^0([0,1],V) \) and \( L_2^1([0,1],V) \), respectively, i.e.,

\[ \begin{align*}
& Q(r,t) = Q^0(r,t) + Q^1(r,t), \quad D^{-1/2}(r) \cdot [\delta(t) - \tilde{c}(t)\tilde{g}(r)] = \Lambda^0(r,t) + \Lambda^1(r,t), \\
& Q^0(r,t) = z^0_0(t)p^0_0(r), \quad \Lambda^0(r,t) = D^{-1/2}(r) \cdot [\delta(t) - \tilde{c}(t)\tilde{g}(r)] = \Lambda^0_0(t)p^0_0(r), \\
& \Lambda^0_i(t) = \int_0^1 \delta^i(t) - \tilde{g}_i^0\tilde{c}(t), \quad \Lambda^1_i(t) = -\tilde{g}_i^0\tilde{c}(t), \quad i = 1,2,\ldots,n, \quad m = 1,2,3,\ldots.
\end{align*} \]

(12)
here $\tilde{g}(r) = \tilde{g}_0(r) + \tilde{g}_1(r)$. $Q_l(r,t)$, $\Delta_l(r,t)$, and $\tilde{g}_l(r)$ belongs to $L_2^0([0,1], V)$ and $Q_l(r,t)$, $\Delta_l(r,t)$, $\tilde{g}_l(r)$ belongs to $L_2^0([0,1], V)$. It is easy to show that coefficients $\tilde{g}_m^i$ has a form

$$\tilde{g}_m^i = \sum_{k=0}^{\infty} K_{ml}^i \int_0^1 p_{m'}^i (\rho) g_{m'}^i (\rho) \rho d\rho, \quad i = 1, 2, \ldots, \quad m = 0, 1, 2, \ldots$$

where $K_{ml}^i$ are expansion coefficients of the kernel $K(r, \rho)$:

$$K_{ml}^i = \int_0^1 k_{ij}^i (r, \rho) p_{m'}^i (\rho) p_{m'}^i (\rho) r \rho d\rho, \quad i, j = 1, 2, \ldots, \quad m, l = 0, 1, 2, \ldots$$

The formula for $Q_l(r,t)$ contains known term $Q_l(r,t)$ which is determined by the auxiliary conditions (10): $z_l(t) = \tilde{P}_l(t) / \sqrt{J_{0,0}}$. Conversely, for the right-hand side, one should find $\Delta_l(r,t)$, while term $\Delta_l(r,t)$ is known and determined be the function $\tilde{g}_l(r)$. These peculiarities permit one to class the resulting problem as a specific case of the generalized projection problem stated in [21–23].

Having introduced the orthoprojectors $P_0$ and $P_1$ mapping the space $L_2([0,1], V)$ onto subspaces $L_2^0([0,1], V)$ and $L_2^0([0,1], V)$, respectively, we will apply the orthoprojector $P_1$ to operator equation (10). As a result, we obtain the equation for determining $Q_l(r,t)$ with a known right-hand side

$$c(t) Q_l(r,t) + (I - V) P_1 G Q_l(r,t) = \Delta_l(r,t) - (I - V) P_1 G Q_l(r,t).$$

We will construct its solution in the form of an expansion in the eigenfunctions of the operator $P_1G$.

The spectral problem for this operator can be written in the form

$$P_1 G \varphi_k(r) = \gamma_k \varphi_k(r), \quad \varphi_k(r) = \sum_{m=1}^{\infty} \psi_{km} \psi_{km}^i (r), \quad k = 1, 2, 3, \ldots$$

This problem leads to the search for solutions of spectral problem about coefficients $\gamma_k$ and $\psi_{km}^i$ ($i = 1, 2, \ldots, n, k,m = 1, 2, 3, \ldots$): $\sum_{i=1}^{\infty} K_{mi}^i \psi_{ki}^i = \gamma_k \psi_{km}^i$. We expand the all functions in equation (15) with respect to the new basis functions $\varphi_k(r)$ ($k = 1, 2, 3, \ldots$) in $L_2^0([0,1], V)$, i.e.,

$$Q_l(r,t) = \sum_{k=1}^{\infty} z_k(t) \varphi_k(r), \quad P_1 G Q_l(r,t) = \sum_{k=1}^{\infty} \sigma_k(t) \varphi_k(r), \quad \Delta_l(r,t) = \sum_{k=1}^{\infty} \Delta_k(t) \varphi_k(r),$$

where functions $z_k(t)$ and $\varphi_k(t)$ has a form

$$\varphi_k(t) = \sum_{m=1}^{\infty} K_{mi}^i \psi_{km}^i z_m(t), \quad \Delta_k(t) = \sum_{m=1}^{\infty} \psi_{km}^i \Delta_m(t).$$

Substituting these representations into (11) we obtain formula for the unknown functions $z_k(t)$

$$z_k(t) = (I + W_k) - (I - V) \sigma_k(t) + \Delta_k(t) / c(t) \gamma_k, \quad W_k f(t) = \int R_k^*(t, \tau) f(\tau) d\tau, \quad k = 1, 2, 3, \ldots$$

where $R_k^*(t, \tau)$ is the resolvent of the kernel $K_k^*(t, \tau) = \gamma_k K(t, \tau) / [c(t) + \gamma_k]$.

Note that the solution has a form ($i = 1, 2, \ldots, n$)

$$q^i(r,t) = \frac{1}{m'(r)} \left[ z_0^i(t) p_{m'}^i (r) + \sum_{k=1}^{\infty} z_k(t) \sum_{m=1}^{\infty} \psi_{km}^i p_{m'}^i (r) \right] - \frac{g^i(r)}{c(t) m'(r)}, \quad r \in [0,1], \quad t \geq 1,$n

i.e., one can explicit write out the functions $m(r)$ and $g(r)$ in the solution. The coating nonuniformity and shapes of the contacting surfaces which can described by complicated and rapidly changing functions are related to these functions in the relations of change of variables. The formulas obtained permit obtaining efficient analytical solutions.

5. Conclusions

Axisymmetric multiple contact problem for foundation with nonuniform coating and arbitrary system of annular punches is posed. The corresponding mathematical model is given and analyzed. Possible variants of the problem statement are formulated. The analytical solution for the version of known forces is obtained using generalized projection method. In relation of contact stresses (19) coating
nonuniformity and shapes of the contacting surfaces are represented by separate terms and factors. It is allows one to perform effective computations for actual nonuniformities and punch shapes and analyze carefully the behavior of the punches on the layer, taking into account the complex properties and forms of the bodies and the mutual influence of the punches.

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