ROOT MULTIPLICITIES FOR NICHOLS ALGEBRAS
OF DIAGONAL TYPE OF RANK TWO

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Abstract. We determine the multiplicities of a class of roots for Nichols algebras of diagonal type of rank two, and identify the corresponding root vectors. Our analysis is based on a precise description of the relations of the Nichols algebra in the corresponding degrees.

Keywords: Nichols algebra, super-letter, root vector, multiplicity

1. Introduction

Since the introduction of Nichols algebras in the late 1990-ies, the topic developed to an own-standing research field with many relationships to different (mainly algebraic or combinatorial) fields in mathematics. In particular, Nichols algebras are heavily used for the study of pointed Hopf algebras. Although Nichols algebras can be defined in any suitable braided monoidal category, a big part of the theory is dominated by Nichols algebras of diagonal type.

By now, a deep understanding of the structure of finite-dimensional Nichols algebras of diagonal type is available, based on the existence of a PBW basis [4] and the notion of roots [3]. In the general setting, one is constantly tempted to seek for relationships with Kac-Moody and Borcherds Lie (super) algebras. The latter seems to be very strong in the finite case because of the definitions of real roots in the two theories. However, the knowledge about imaginary roots and their multiplicities is little in the case of Kac-Moody algebras, and even poorer for Nichols algebras of diagonal type. For information on recent activities in the theory of Kac-Moody algebras we refer to [2]. With our results we make a small step towards a better understanding of the Nichols algebra theory in this respect.

In this paper, we concentrate on Nichols algebras of diagonal type of rank two. In order to clarify the context, we introduce the notion of root vector candidates and root vectors. We focus on the special

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roots \( m\alpha_1 + 2\alpha_2 \), where \( m \in \mathbb{N}_0 \) and \( \alpha_1, \alpha_2 \) is the standard basis of \( \mathbb{Z}^2 \).

(The root multiplicities of \( m\alpha_1 + k\alpha_2 \) with \( m \in \mathbb{N}_0, \ k \in \{0, 1\} \), have been known before.) We identify the family \((P_k)_{k \in \mathbb{N}_0}\) in the free algebra over a two-dimensional braided vector space \( V \) of diagonal type, and relate the relations in the Nichols algebra of \( V \) of degree \( m\alpha_1 + 2\alpha_2 \) to this family. We find two of our results particularly interesting. First, in Proposition 4.3 we prove that if a root vector candidate is a root vector, then any lexicographically larger root vector candidate of the same degree is a root vector, too. Second, in Theorem 4.16 we describe precisely when a root vector candidate is a root vector. To do so, we define a subset \( J \) of \( \mathbb{N}_0 \) depending on the given braiding, which measures the multiplicities of all roots of the form \( m\alpha_1 + 2\alpha_2 \) in a simple way. For the calculation of \( J \) one needs only elementary (and simple) calculations with Laurent polynomials in three indeterminates. Unfortunately, the proof of this theorem requires that we work over a field of characteristic 0.

The paper is organized as follows. In Section 2 we give some equations for Gaussian binomial coefficients, which will be needed later. In Section 3 we recall some fundamental definitions and results on which our work is based. In Section 4 we formulate and prove our main results mentioned above. We also conclude a non-trivial lower bound on root multiplicities.

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2. Quantum integers and Gaussian binomial coefficients

Throughout the paper let \( \mathbb{N} \) denote the set of positive integers and let \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). We write \( \mathbb{Z} \) for the set of integers.

For our study of Nichols algebras we will need some non-standard formulas for quantum integers and Gaussian binomial coefficients.

In the ring \( \mathbb{Z}[q] \), let \( (0)_q = 0 \) and for any \( m \in \mathbb{N} \), let

\[
(m)_q = 1 + q + q^2 + \cdots + q^{m-1}
\]

and \( (-m)_q = -(m)_q \). The polynomials \( (m)_q \) with \( m \in \mathbb{Z} \) are also known as quantum integers. Moreover, let \( (0)_q^1 = 1 \), and for any \( m \in \mathbb{N} \) let \( (m)_q^1 = \prod_{i=1}^{m}(i)_q \). For any \( i, m \in \mathbb{Z} \) with \( 0 \leq i \leq m \), the rational
function
\[
\binom{m}{i}_q = \frac{(m)_q^i}{(i)_q^i(m-i)_q^i} \in \mathbb{Q}(q)
\]
is in fact an element of \(\mathbb{Z}[q]\) and is called a Gaussian binomial coefficient. For \(m \in \mathbb{N}_0, i \in \mathbb{Z}\) with \(i < 0\) or \(i > m\) one defines \(\binom{m}{i}_q = 0\). The Gaussian binomial coefficients satisfy the following formulas:

(1) \[
\binom{m}{i}_q = \binom{m}{m-i}_p,
\]
(2) \[
\binom{m}{i}_q = q^i \binom{m-1}{i}_q + \binom{m-1}{i-1}_q,
\]
(3) \[
\binom{m}{i}_q = \binom{m-1}{i}_q + q^{m-i} \binom{m-1}{i-1}_q
\]
for any \(m \in \mathbb{N}, i \in \mathbb{Z}\).

**Lemma 2.1.** Let \(t \in \mathbb{N}_0\) and \(k \in \mathbb{Z}\) with \(k \geq -1\). Then

\[
\sum_{j=t}^{k} q^{-j(j+1)/2} q^{j-t}(j-t-1)/2 \binom{j}{t}_q = q^{-(t+1)(2k-t)/2} \binom{k+1}{t+1}_q.
\]

**Proof.** We proceed by induction on \(k\). For \(k < t\) the claim is trivial. Assume now that \(t, k \in \mathbb{N}_0\) and that the claim holds for \(t\) and \(k - 1\). The summand for \(j = k\) on the left hand side is \(q^{-(t+1)(2k-t)/2} \binom{k}{t}_q\). By subtracting this from both sides of the equation and using Equation (2), the claim follows from the induction hypothesis.

**Lemma 2.2.** Let \(m \in \mathbb{N}_0\) and \(n \in \mathbb{Z}\). Then in \(\mathbb{Z}[q, t]\) we have

\[
\sum_{i=0}^{m} \binom{m}{i}_q q^{i(i-1)/2} \prod_{j=0}^{i-1} (q^{j+n} t^2 - t) \prod_{j=1}^{m-i} (1 - q^{m+j-n} t^2) = \prod_{j=0}^{m-1} (1 - q^j t).
\]

**Proof.** We prove the claim by induction on \(m\).
For $m = 0$ the claim is trivial. Now assume that the claim holds for some $m \in \mathbb{N}_0$. By applying Equation (3) we obtain that

$$
\sum_{i=0}^{m+1} \binom{m+1}{i} q^{i(i-1)/2} \prod_{j=0}^{i-1} (q^{j+n+1} - t) \prod_{j=1}^{m+1} (1 - q^{m+n+1-j}t^2)
$$

$$
= \sum_{i=0}^{m} \binom{m}{i} q^{i(i-1)/2} \prod_{j=0}^{i-1} (q^{j+n+1} - t) \prod_{j=1}^{m+1} (1 - q^{m+n+1-j}t^2)
$$

$$
+ \sum_{i=1}^{m+1} \binom{m}{i-1} q^{m+1-i} q^{i(i-1)/2} \prod_{j=0}^{i-1} (q^{j+n+1} - t) \prod_{j=1}^{m+1} (1 - q^{m+n+1-j}t^2).
$$

Regarding the first term, note that

$$
\prod_{j=1}^{m+1} (1 - q^{m+n+1-j}t^2) = (1 - q^{m+n}t^2) \prod_{j=1}^{m-i} (1 - q^{m+n-j}t^2).
$$

Hence, by induction hypothesis, the first term is equal to

$$(1 - q^{m+n}t^2) \prod_{j=0}^{m-1} (1 - q^j t).
$$

Moreover, the second term is equal to

$$
\sum_{i=0}^{m} \binom{m}{i} q^{m-i} q^{i(i+1)/2} \prod_{j=0}^{i} (q^{j+n+1} - t) \prod_{j=1}^{m-i} (1 - q^{m+n+1-j}t^2).
$$

Now by using that

$$
q^{m-i} q^{i(i+1)/2} \prod_{j=0}^{i} (q^{j+n+1} - t) = q^m q^{i(i-1)/2} (q^n t^2 - t) \prod_{j=0}^{i-1} (q^{j+n+1} t^2 - t),
$$

induction hypothesis implies that the second term is equal to

$$
q^m (q^n t^2 - t) \prod_{j=0}^{m-1} (1 - q^j t).
$$

Since $1 - q^{m+n}t^2 + q^m (q^n t^2 - t) = 1 - q^m t$, the claim holds for $m+1$. \qed

**Lemma 2.3.** Let $k, m \in \mathbb{N}_0$ and

$$
Q_{1}^{k, m} = \sum_{i=0}^{m} \binom{m+1}{i} q^{i(2k+i-1)/2} \prod_{j=0}^{i-1} (q^{k+j} - r) \prod_{j=1}^{m-i} (1 - q^{2k+m-j}r^2)
$$

$$
Q_{2}^{k, m} = \frac{q^{(2k+m)(m+1)/2} (-r)^{m+1} - 1}{q^{2k+m} r^2 - 1} \prod_{i=0}^{m} (1 - q^{k+i} r)
$$
there exist Lyndon words \( w, v \) such that \( w < \text{lex} u \). Clearly, \( w, v \in Z[q, r] \).

Proof. Clearly, \( Q_1^{k,m} = Q_2^{k,m} \).

Since \( q^{k+m/2} - 1 \) divides \( \prod_{j=0}^{m} (1 - q^{k+j} r) \) and \( q^{k+m/2} - 1 \) divides \( -(q^{k+m/2} r)^{m+1} - 1 \) in \( Z[q, r] \), we conclude that \( Q_2^{k,m} \in Z[q, r] \). Moreover, \( Q_1^{k,m} = 0 \) and \( 1 \) divides \( Q_2^{k,m} \).

By Lemma 2.2 for \( m + 1, n = 0 \) and \( t = q^k r \), the first term is equal to \( \prod_{j=0}^{m} (1 - q^{k+j} r) \). From this it follows that \( Q_1^{k,m} = Q_2^{k,m} \).

3. Preliminaries on Nichols algebras

In the remaining part of the paper, let \( k \) be a field and let \( k^\times = k \setminus \{0\} \).

We start with collecting some information on Lyndon words, which is fairly standard.

Let \( A \) be a finite set (called the alphabet) and let \( A \) and \( A^\times \) denote the set of words and nonempty words, respectively, with letters in \( A \). For any \( s \in \mathbb{N}_0, a_1, a_2, \ldots, a_s \in A \) and \( u = a_1 \cdots a_s \in A \) we write \( |u| = s \) and call \( s \) the length of \( u \).

We fix a total ordering \( < \) on \( A \). It induces a total ordering \( \leq \) on \( A \) called the lexicographic ordering: Two elements \( u, v \in A \) satisfy \( u < v \) if and only if either \( v = uw \) for some \( w \in A^\times \), or there exist \( w, v' \in A \) and \( a, b \in A \) such that \( u = wau', v = wbv' \), and \( a < b \).

We say that a word \( u \in A^\times \) is a Lyndon word if for any decomposition \( u = vw, w, v \in A^\times \), the relation \( u \leq v \) holds.

A word \( u \in A^\times \) is a Lyndon word if and only if either \( u \in A \), or there exist Lyndon words \( w, v \in A^\times \) such that \( w \leq v \) and \( u = wv \).

Any Lyndon word \( u \) of length at least two has a unique decomposition into the product of two Lyndon words \( u = vw \), where \( |w| \) is minimal. It is called the Shirshow decomposition of \( u \).
The theory of Lyndon words is used in [4] to define PBW bases of Nichols algebras of diagonal type. (In fact, in [4] a much more general situation is considered.)

Let \( n \in \mathbb{N} \) and let \((V, c)\) be an \( n\)-dimensional braided vector space of diagonal type. Let \( I = \{1, \ldots, n\} \), and let \((q_{ij})_{i,j \in I} \in (\mathbb{k}^\times)^{n \times n}\) and \(x_1, \ldots, x_n\) be a basis of \( V \) such that\[
\forall i, j \in I, \quad c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i
\]
for any \( i, j \in I \). Let \( T(V) \) and \( \mathcal{B}(V) \) denote the tensor algebra and the Nichols algebra of \( V \), respectively. For the basics of the theory of Nichols algebras we refer to [1]. We write \( \pi : T(V) \to \mathcal{B}(V) \) for the canonical map.

Let \( \alpha_1, \ldots, \alpha_n \) be the standard basis of \( \mathbb{Z}^n \) and \( \chi : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{k}^\times \) be the bicharacter on \( \mathbb{Z}^n \) such that \( \chi(\alpha_i, \alpha_j) = q_{ij} \) for any \( i, j \in I \).

Let \( X = \{x_1, \ldots, x_n\} \), and fix the total ordering on \( X \) such that \( x_i < x_j \) whenever \( 1 \leq i < j \leq n \). Let \( X \) and \( X^\times \) denote the set of words and non-empty words over the alphabet \( X \), respectively. The elements of \( X \) can naturally be viewed as elements of (any quotient of) \( T(V) \), and as such they form a vector space basis of \( T(V) \). Both \( T(V) \) and \( \mathcal{B}(V) \) have a unique \( \mathbb{Z}^n \)-graded braided bialgebra structure such that \( \deg(x_i) = \alpha_i \) for any \( i \in I \). In particular, for any \( k \in \mathbb{N}_0 \) and \( l_1, \ldots, l_k \in I \) the degree of \( x_{l_1} \cdots x_{l_k} \) is \( \sum_{i=1}^k \alpha_{l_i} \). We write \( \deg(x) \) for the degree of any homogeneous element \( x \) of \( T(V) \) or \( \mathcal{B}(V) \).

For a Lyndon word \( u \in X^\times \), following [4] we define the super-letter \([u] \in \mathcal{B}(V)\) inductively as follows:

1. \([u] = u\), if \( u \in X \), and
2. \([u] = [v][w] - \chi(\deg(v), \deg(w))[w][v] \) if \( u \in X^\times \), \(|u| \geq 2 \), and \( u = vw \) is the Shirshov decomposition of \( u \).

Moreover, for any Lyndon word \( u \) and any integer \( k \geq 2 \) let \([u^k] = [u]^k\).

The total ordering on \( X \) induces a total ordering on the set of super-letters:

\([u] < [v] \iff u <_{\text{lex}} v\).

For any \( \alpha \in \mathbb{Z}^n \), let \( o_\alpha \in \mathbb{N} \cup \{\infty\} \) be the multiplicative order of \( \chi(\alpha, \alpha) \in \mathbb{k}^\times \). Moreover, let

\[ O_\alpha = \begin{cases} 
\{1, o_\alpha, \infty\} & \text{if } o_\alpha = \infty \text{ or } \text{char}(\mathbb{k}) = 0, \\
\{1, o_\alpha p^k, \infty \mid k \in \mathbb{N}_0\} & \text{if } o_\alpha < \infty, \ p = \text{char}(\mathbb{k}) > 0.
\]

Kharchenko proved the following fundamental result on Nichols algebras.
Theorem 3.1. There exists a set \( L \) of Lyndon words and a function \( h : L \to \mathbb{N} \cup \{\infty\} \), where \( h(v) \in O_{\deg v} \setminus \{1\} \) for any \( v \in L \), such that the elements
\[
[v_k]^{m_k} \cdots [v_1]^{m_1}, \quad k \in \mathbb{N}_0, \; v_1, \ldots, v_k \in L, \; v_1 \prec_{\text{lex}} v_2 \prec_{\text{lex}} \cdots \prec_{\text{lex}} v_k, \\
0 < m_i < h(v_i) \quad \text{for any } i,
\]
form a vector space basis of \( B(V) \).

In fact, the set \( L \) and the function \( h \) in the above theorem are uniquely determined.

In some situations it is more appropriate to work with a slightly different presentation of the above basis of \( B(V) \), in which the function \( h \) does not appear.

Definition 3.2. Let \( w \in \mathbb{X}^\times \). We say that \([w]\) is a root vector candidate if \( w = v^k \) for some Lyndon word \( v \) and \( k \in O_{\deg v} \setminus \{\infty\} \).

Definition 3.3. A root vector candidate \([w]\), where \( w \in \mathbb{X}^\times \), is called a root vector (of \( B(V) \)) if \([w] \in B(V)\) is not a linear combination of elements of the form \([v_k]^{m_k} \cdots [v_1]^{m_1}\), where \( k \in \mathbb{N}_0 \) and \([v_1], \ldots, [v_k]\) are root vector candidates with \( w <_{\text{lex}} v_1 <_{\text{lex}} \cdots <_{\text{lex}} v_k \).

Remark 3.4. By [4] Corollary 2, for any Lyndon word \( w \in \mathbb{X}^\times \) the root vector candidate \([w]\) is a root vector if and only if \([w] \in B(V)\) is not a linear combination of elements of the form \([v_k]^{m_k} \cdots [v_1]^{m_1}\), where \( k \in \mathbb{N}_0 \) and \([v_1], \ldots, [v_k]\) are root vector candidates with \( w <_{\text{lex}} v_1 <_{\text{lex}} \cdots <_{\text{lex}} v_k \).

Note that in Definition 3.3 it is not necessary to put assumptions on the degrees of the monomials, since \( B(V) \) is graded.

Example 3.5. Assume that \( n \geq 2 \). Let \( k \in \mathbb{N}_0 \). The only Lyndon word of degree \( k\alpha_1 + \alpha_2 \) in \( \mathbb{X} \) is \( x_1^k x_2 \), and the only root vector candidate of degree \( k\alpha_1 + \alpha_2 \) is \( [x_1^k x_2] \). Since \( B(V) \) is \( \mathbb{N}_0^\times \)-graded, \([x_1^k x_2] \) is not a root vector if and only if \([x_1^k x_2] = 0 \) in \( B(V) \). In our setting, the latter can be characterized in terms of the matrix \((q_{ij})_{i,j \in \mathbb{I}}\) using Rossos Lemma [5 Lemma 14]: For any \( k \geq 0 \),
\[
[x_1^{k+1} x_2] = 0 \quad \iff \quad (k + 1)_{q_{11}} \prod_{i=0}^k (1 - q_{11}^i q_{12} q_{21}) = 0.
\]

The Lyndon words of degree \( k\alpha_1 + 2\alpha_2 \) in \( \mathbb{X} \) are the words \( x_1^{k_1} x_2 x_1^{k_2} x_2 \) with \( k_1, k_2 \in \mathbb{N}_0, \; k_1 + k_2 = k, \; k_1 > k_2 \). The elements \([x_1^{k_1} x_2 x_1^{k_2} x_2] \) are the only root vector candidates of degree \( k\alpha_1 + 2\alpha_2 \), except when \( k \) is even and \( q_{11}^{k^2/4} (q_{12} q_{21})^{k/2} q_{22} = -1 \). In the latter case, \([x_1^{k/2} x_2]^2 \) is the only
additional root vector candidate of degree $k\alpha_1 + 2\alpha_2$. The definition implies that the element $[x_1^{k_1}x_2^{k_2}]^i$ with $k_1 + k_2 = k$, $k_1 \geq k_2$, is not a root vector if and only if there exists a relation in $B(V)$ of the form

$$\sum_{i=k_2}^{k_1} \lambda_i [x_1^i x_2][x_1^{k-i} x_2] = 0$$

such that $\lambda_i \in k$ for any $k_2 \leq i \leq k_1$ and

$$\lambda_{k_1} = 1, \quad \lambda_{k_2} = -\chi(k_1\alpha_1 + \alpha_2, k_2\alpha_1 + \alpha_2).$$

(This is also true if $k_1 = k_2$!)

Note that the definitions of a root vector candidate and a root vector depend on the bicharacter $\chi$. Now Kharchenko’s theorem can be restated as follows.

**Theorem 3.6.** Let $L \subseteq X^\times$ such that $w \in L$ if and only if $[w]$ is a root vector. Then the elements $[v_k]^{m_k} \cdots [v_1]^{m_1}$, $k \in \mathbb{N}_0$, $v_1, \ldots, v_k \in L$, $v_1 <_{\text{lex}} v_2 <_{\text{lex}} \cdots <_{\text{lex}} v_k$, $0 < m_i < \min(\text{O}_{\deg(v_i}) \setminus \{1\})$ for any $i$, form a vector space basis of $B(V)$.

(Note that $\min(\text{O}_{\alpha} \setminus \{1\})$ for $\alpha \in \mathbb{Z}^n$ equals $o_\alpha$, except when $\alpha = 1$.) This reformulation of Kharchenko’s theorem allows to define the set

$$\Delta_+ = \{\deg(u) \mid u \in L\}$$

of positive roots of $B(V)$ and the root system $\Delta = \Delta_+ \cup -\Delta_+$ of $B(V)$, see [3]. It turns out that this definition is independent of choices. For any $\alpha \in \Delta_+$, the number of elements $u \in L$ with $\deg(u) = \alpha$ is called the multiplicity of $\alpha$.

One of the biggest open problems in the theory of Nichols algebras of diagonal type is to determine for any $V$ (in a suitable class) the set $L$ in Kharchenko’s theorem. In this paper we determine for $n = 2$ the subset of $L$ of elements of degree $m\alpha_1 + 2\alpha_2$, where $m \in \mathbb{N}$.

Next we recall standard tools for working with Nichols algebras of diagonal type.

For any $i \in \{1, \ldots, n\}$, there exists a unique skew-derivation $d_i$ of the tensor algebra $T(V)$ such that

$$d_i(x_j) = \delta_{ij}, \quad d_i(xy) = d_i(x)y + \chi(\alpha_i)xd_i(y)$$

for any $j \in I$ and $x, y \in T(V)$ with $\deg(x) = \alpha$. These skew-derivations induce skew-derivations of $B(V)$ which will be denoted by the same symbols.
Remark 3.7. An element $x \in \mathcal{B}(V)$ is constant if and only if $d_i(x) = 0$ in $\mathcal{B}(V)$ for any $i \in I$. In particular, a homogeneous element $x \in \mathcal{B}(V)$ of non-zero degree is zero if and only if $d_i(x) = 0$ in $\mathcal{B}(V)$ for any $i \in I$. Because of this, the skew-derivations $d_i$, $i \in I$, and their relatives belong to the main tools in the study of Nichols algebras of diagonal type.

Since $\mathcal{B}(V)$ is an $\mathbb{N}_0^n$-graded coalgebra, for any $x \in \mathcal{B}(V)$ and any $\beta, \gamma \in \mathbb{N}_0^n$ there exist uniquely determined elements

$$x_{\beta, \gamma} \in \mathcal{B}(V)(\beta) \otimes \mathcal{B}(V)(\gamma),$$

such that $\Delta(x) = \sum_{\beta, \gamma \in \mathbb{N}_0^n} x_{\beta, \gamma}$. For any $\beta, \gamma \in \mathbb{N}_0^n$, the map

$$\mathcal{B}(V) \to \mathcal{B}(V)(\beta) \otimes \mathcal{B}(V)(\gamma), \quad x \mapsto x_{\beta, \gamma},$$

is linear. The skew-derivations $d_i$ with $i \in I$ are closely related to these maps:

$$\Delta_{\alpha, \alpha - \alpha_i}(x) = x_i \otimes d_i(x)$$

for any $i \in I$, $\alpha \in \mathbb{N}_0^n$, and any homogeneous element $x \in \mathcal{B}(V)$ of degree $\alpha$.

Next we discuss reflections. Let $i \in I$. Assume that for any $j \in (I \setminus \{i\}$ there exists $k \in \mathbb{N}_0$ such that $(k + 1)q_{ij}(1 - q_i^k q_{ij}) = 0$. Following [3], we set $c_{ii} = 2$ and for any $j \in (I \setminus \{i\}$ we define

$$c_{ij} = -\min\{k \in \mathbb{N}_0 \mid (k + 1)q_{ij}(1 - q_i^k q_{ij}) = 0\}.$$

Let $s_i \in \text{GL}(\mathbb{Z}^n)$ be given by $s_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$ for any $j \in I$. The reflection of $V$ on the $i$-th vertex is the braided vector space $R_i(V)$ with basis $y_1, \ldots, y_n$ such that for all $j, k \in I$,

$$c(y_j \otimes y_k) = q'_{jk} y_k \otimes y_j,$$

where $q'_{jk} = \chi(s_i(\alpha_j), s_i(\alpha_k)) = q_{jk}^{-c_{ij}} q_{ji}^{-c_{ik}} q_{ii}^{-c_{ik}} q_{ii}^{-c_{ij}}$.

In the remaining part of the paper we restrict our attention to the case $n = 2$. Let $q = q_{11}$, $r = q_{12} q_{21}$, and $s = q_{22}$. We introduce special elements of $\mathcal{B}(V)$ and spell out some easy properties of them.

For all $k \in \mathbb{N}_0$ we define inductively $u_k \in T(V)$ by

$$u_0 = x_2, \quad u_k = x_1 u_{k-1} - q^{k-1} q_{12} u_{k-1} x_1$$

for $k \geq 1$. Note that then $u_k = [x_1^k x_2]$ for any $k \in \mathbb{N}_0$. Let ad denote the adjoint action of $T(V)$ on itself. Then

$$\text{ad} x_1(y) = x_1 y - \chi(\alpha_1, \text{deg}(y)) y x_1$$

for any homogeneous element $y \in T(V)$. In particular,

$$\text{ad} x_1(u_k) = u_{k+1}$$
for any \( k \in \mathbb{N}_0 \).

**Lemma 3.8.** Let \( m \in \mathbb{N}_0 \). Then
\[
d_1((\text{ad } x_1)^m(y)) = q^m(\text{ad } x_1)^m(d_1(y))
\]
\[
+ (m)_q (1 - q^{m-1} \chi(\alpha, \alpha)) (\text{ad } x_1)^{m-1}(y)
\]
for any homogeneous element \( y \in T(V) \) of degree \( \alpha \in \mathbb{N}_0 \).

**Proof.** Let \( y \in T(V) \) be a homogeneous element of degree \( \alpha \in \mathbb{N}_0 \). Then
\[
d_1(\text{ad } x_1(y)) = d_1(x_1y - \chi(\alpha, \alpha)y x_1)
\]
\[
= y + qx_1d_1(y) - \chi(\alpha, \alpha)(d_1(y)x_1 + \chi(\alpha, \alpha)y)
\]
\[
= (1 - \chi(\alpha, \alpha))y + q\text{ad } x_1(d_1(y)).
\]

Now we prove the lemma by induction on \( m \). For \( m = 0 \) the claim is trivial. Let now \( m \in \mathbb{N} \) and assume that the claim holds for \( m - 1 \). Let \( y \in T(V) \) be a homogeneous element of degree \( \alpha \in \mathbb{N}_0 \) and let \( \beta = \alpha + (m - 1)\alpha_1 \), \( q_\alpha = \chi(\alpha_1, \alpha) \chi(\alpha, \alpha_1) \). Then, by using the above formula, we conclude that
\[
d_1((\text{ad } x_1)^m(y)) = \text{qad } x_1((\text{ad } x_1)^{m-1}(y))
\]
\[
+ (1 - \chi(\alpha_1, \beta)) \chi(\beta, \alpha_1)(\text{ad } x_1)^{m-1}(y)
\]
\[
= \text{qad } x_1(q^{m-1}(\text{ad } x_1)^{m-1}(d_1(y)))
\]
\[
+ \text{qad } x_1((m - 1)_q (1 - q^{m-2} q_\alpha)(\text{ad } x_1)^{m-2}(y))
\]
\[
+ (1 - q^{2m-2} q_\alpha)(\text{ad } x_1)^{m-1}(y)
\]
because of the induction hypothesis. From this one obtains the claim for \( m \). \( \square \)

**Remark 3.9.** (1) It is well-known that
\[
\Delta(u_k) = u_k \otimes 1 + \sum_{i=0}^{k} \binom{k}{i} \prod_{j=k-i}^{k-1} (1 - q^j r) x_i^i \otimes u_{k-i}
\]
for any \( k \in \mathbb{N}_0 \). Hence from Equation (5) we conclude that
\[
d_1(u_k) = (k)_q (1 - q^{k-1} r) u_{k-1}, \quad d_2(u_k) = \delta_{k0} 1
\]
for any \( k \in \mathbb{N}_0 \).

(2) For all \( m \in \mathbb{N}_0 \) let
\[
b_m = \prod_{j=0}^{m-1} (1 - q^j r).
\]
In particular, \( b_0 = 1 \). Then Remark 3.7 implies that \( u_k = 0 \) in \( \mathcal{B}(V) \) if and only if \( (k)^i_q b_k = 0 \).

Later on, often a normalization of \( u_k \), \( k \in \mathbb{N}_0 \), will be very useful. For all \( k \in \mathbb{N}_0 \) let

\[
\hat{u}_k = \begin{cases} 
\frac{1}{(k)^i_q b_k} u_k & \text{if } (k)^i_q b_k \neq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

The following equations for \( k \in \mathbb{N}_0 \) with \( (k)^! q b_k \neq 0 \) follow directly from the analogous formulas for \( u_k \).

\[
\Delta(\hat{u}_k) = \hat{u}_k \otimes 1 + \sum_{i=0}^{k} \frac{x_i}{(i)^i_q} \otimes \hat{u}_{k-i},
\]

\[
d_1(\hat{u}_k) = \hat{u}_{k-1}, \quad d_2(\hat{u}_k) = \delta_{k0} 1,
\]

\[
ad x_1(\hat{u}_k) = (k + 1)_q (1 - q^k r) \hat{u}_{k+1},
\]

where \( \hat{u}_{-1} = 0 \) in (10).

We end the section with a lemma.

**Lemma 3.10.** Let \( k \in \mathbb{N}_0 \) such that \( (k)^i_q b_k \neq 0 \) and let \( \lambda_0, \ldots, \lambda_k \in k \). Let \( Z = \sum_{i=0}^{k} \lambda_i (-q^{21})^i \hat{u}_i \hat{u}_{k-i} \) in \( T(V) \). Then

\[
d_1(Z) = -q^{21} \sum_{i=0}^{k-1} (\lambda_{i+1} - q^i \lambda_i)(-q^{21})^i \hat{u}_i \hat{u}_{k-1-i},
\]

\[
d_2(Z) = (\lambda_0 + (-r)^k s \lambda_k) \hat{u}_k.
\]

**Proof.** This follows directly from Equations (4) and (10). \( \square \)

### 4. Multiplicities

We use the notation from the previous section. In this section we determine for all \( m \in \mathbb{N}_0 \) the set of root vectors of \( \mathcal{B}(V) \) of degree \( m \alpha_1 + 2 \alpha_2 \).

**Lemma 4.1.** Let \( m \in \mathbb{N} \) such that \( u_m \neq 0 \) in \( \mathcal{B}(V) \). Then \( u_m^2 = 0 \) in \( \mathcal{B}(V) \) if and only if \( q^{m^2} r^m s = -1 \) and \( u_{m+1} = 0 \) in \( \mathcal{B}(V) \).

**Proof.** Assume first that \( u_m^2 = 0 \). For all \( k \in \mathbb{N}_0 \) let \( \beta_k = k \alpha_1 + \alpha_2 \). Using Equation (7) we obtain that

\[
\Delta_{\beta_m, \beta_m}(u_m^2) = (1 + q^{m^2} r^m s) u_m \otimes u_m.
\]
Since $u_m \neq 0$, we conclude that $q^{m^2}r_m s = -1$. Moreover,
\[
\Delta_{\beta_{m+1},\beta_{m-1}}(u_m^2) = (m)q(1 - q^{m-1}r)(u_m x_1 + q^{m(m-1)}r^{m-1}s q_21 x_1 u_m) \otimes u_{m-1}
\]
\[
= (m)q(1 - q^{m-1}r)q^{m(m-1)}r^{m-1}s q_21 u_{m+1} \otimes u_{m-1}.
\]
Again, since $u_m \neq 0$, from Remark 3.9(2) it follows that $u_{m+1} = 0$.

Conversely, assume that $q^{m^2}r_m s = -1$ and that $u_{m+1} = 0$. Then
\[
x_1 u_m = q^{m} q_{12} u_m x_1,
\]
and hence
\[
\Delta_{\beta_m - \alpha_1, \alpha_1}(u_m^2) = 0,
\]
\[
\Delta_{\beta_{2m}, \alpha_2}(u_m^2) = b_m ((x_m^m \otimes u_0)(u_m \otimes 1) + u_m x_m^m \otimes u_0)
\]
\[
= b_m (q_m^m b q_{12}) u_m x_m^m \otimes u_0
\]
\[
= 0.
\]
Since $B(V)$ is a strictly graded coalgebra, it follows that $u_m = 0$.

\textbf{Lemma 4.2.} Let $k, l \in \mathbb{N}_0$ with $k > l$. Assume that $(k)_q^q b_k \neq 0$ and that $[x_1^k x_2 x_1^{l+1} x_2]$ is not a root vector. Then $[x_1^k x_2 x_1^{l+1} x_2]$ is not a root vector.

\textbf{Proof.} Lyndon words of degree $(k+l+1)\alpha_1 + 2\alpha_2$, which are larger than $x_1^k x_2 x_1^{l+1} x_2$, are of the form $x_1^m x_2 x_1^{k+l+1-m} x_2$ with $(k+l+1)/2 < m < k$. Hence the assumption implies that there exists $(\lambda_i)_{l+1 \leq i \leq k} \in k^{k-l}$ such that
\[
\lambda_k = 1 \text{ and } \sum_{i=l+1}^k \lambda_i \hat{u}_i \hat{u}_{k+i+1-l} = 0 \text{ in } B(V).
\]
Then
\[
d(\sum_{i=l+1}^k \lambda_i \hat{u}_i \hat{u}_{k+i+1-l}) = \sum_{i=l+1}^k \lambda_i (\hat{u}_{i-1} \hat{u}_{k+i+1-l} + q q_{21} \hat{u}_i \hat{u}_{k+i-l}) = 0.
\]
The coefficient of $\hat{u}_k \hat{u}_l$ in the last expression is $q^k q_{21}$ and hence the root vector candidate $[x_1^k x_2 x_1^{l+1} x_2]$ is not a root vector.

\textbf{Proposition 4.3.} Let $k, l \in \mathbb{N}$ with $k \geq l$. Assume that $[x_1^k x_2 x_1^{l+1} x_2]$ is a root vector candidate but not a root vector. Then $[x_1^k x_2 x_1^{l+1} x_2]$ is not a root vector.

\textbf{Proof.} First assume that $k = l$. Then $[x_1^{k+1} x_2] = 0$ by Lemma 4.1. Therefore $[x_1^{k+1} x_2 x_1^{l+1} x_2]$ is not a root vector. Suppose now that $k > l$. Then $[x_1^k x_2 x_1^{l+1} x_2]$ is not a root vector by Lemma 4.2 and Remark 3.4 implies that $[x_1^{k+1} x_2 x_1^{l+1} x_2]$ is not a root vector.

For any $n \in \mathbb{N}_0$ let
\[
U_n = \bigoplus_{i=0}^n \mathbb{K} u_i u_{n-i} \subseteq T(V), \quad U'_n = \bigoplus_{i=0}^{n-1} \mathbb{K} u_i u_{n-i} \subseteq T(V).
\]
Lemma 4.4. The map $\text{ad} x_1 : U_m \to U_{m+1}$ is injective for any $m \in \mathbb{N}_0$.

Proof. Let $m \in \mathbb{N}_0$, $\lambda_0, \ldots, \lambda_m \in k$, and $v = \sum_{i=0}^{m} \lambda_i u_i u_{m-i}$. Then

$$\text{ad} x_1(v) = \sum_{i=0}^{m} \lambda_i(u_i u_{m-i} + q^i q_{12} u_i u_{m+1-i})$$

by Equation (6). Assume that $v \neq 0$. Let $0 \leq j \leq m$ such that $\lambda_j \neq 0$ and either $j = m$ or $\lambda_{j+1} = 0$. Then the coefficient of $u_{j+1} u_{m-j}$ in the above expression is $\lambda_j$, and hence $\text{ad} x_1(v) \neq 0$. \hfill $\Box$

Proposition 4.5. Let $k \in \mathbb{N}_0$ such that $(k)^{1}_q b_k \neq 0$. Let $v \in U'_k \cap \ker(\pi)$ and let $\mu_0, \ldots, \mu_{k-1} \in k$ such that

$$d_1(v) = \sum_{i=0}^{k-1} \mu_i(-q_{21})^i \hat{u}_i \hat{u}_{k-1-i}.$$ 

Then $\sum_{i=0}^{k-1} q^{-i(i+1)/2} \mu_i = 0$.

Proof. For any $\lambda = (\lambda_0, \ldots, \lambda_k) \in k^{k+1}$ let $\hat{\mu}(\lambda) = (\mu_i(\lambda))_{0 \leq i < k} \in k^k$ such that

$$\mu_i(\lambda) = \lambda_{i+1} - \lambda_i q^i$$

whenever $0 \leq i < k$. Let $W = \{ \lambda \in k^{k+1} | \lambda_0 = 0 \}$. Then the linear map $\hat{\mu} : W \to k^k$, $\lambda \mapsto \hat{\mu}(\lambda)$, is bijective. The inverse map is given by

$$\hat{\mu}^{-1}(\mu_0, \ldots, \mu_{k-1}) = (\lambda_i)_{0 \leq i < k}, \quad \lambda_i = \sum_{j=0}^{i-1} q^{(i-j)(i-j-1)/2} \mu_j.$$ 

Now let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{k-1}, 0) \in k^{k+1}$ such that

$$v = \sum_{i=0}^{k} \lambda_i(-q_{21})^i \hat{u}_i \hat{u}_{k-i},$$

Since $v \in \ker(\pi)$, it follows from Remark 3.7 that $d_2(v) \in \ker(\pi)$. Since $v \in U'_k$ and $u_k \neq 0$ in $B(V)$, Lemma 3.10 implies that $\lambda_0 = 0$, that is, $\lambda \in W$. Then we obtain from Equation (13) and from $\lambda_k = 0$ that $\sum_{j=0}^{k-1} q^{-j(j+1)/2} \mu_j(\lambda) = 0$. Moreover,

$$d_1(v) = -q_{21} \sum_{i=0}^{k-1} \mu_i(\lambda)(-q_{21})^i \hat{u}_i \hat{u}_{k-1-i}$$

by Lemma 3.10, and hence $\mu_j = -q_{21} \mu_j(\lambda)$ for any $0 \leq j < k$. This implies the claim. \hfill $\Box$
The following elements of $T(V)$ will play a fundamental role in Theorem 4.16.

**Definition 4.6.** For all $k \in \mathbb{N}_0$ with $(k)_q^i b_k \neq 0$, let

\[(14) \quad P_k = \sum_{i=0}^{k} (-q_{21})^i q^{i(i-1)/2} \hat{u}_i \hat{u}_{k-i} \in T(V).\]

**Lemma 4.7.** Let $k \in \mathbb{N}_0$ with $(k)_q^i b_k \neq 0$. Then $P_k = 0$ in $B(V)$ if and only if $q^{k(k-1)/2}(-r)^k S = -1$.

**Proof.** By Lemma 3.10

\[d_1(P_k) = 0, \quad d_2(P_k) = (1 + (-r)^k q^{k(k-1)/2}) \hat{u}_k.\]

Since $(k)_q^i b_k \neq 0$, the claim follows from this and from Remark 3.7. \hfill \Box

We also introduce a family of elements $S(k, t)$ of $T(V)$, which are related to the elements $P_k$ by Lemmas 4.9 and 4.10 below. Those lemmas themself are needed for Lemma 4.11, which is a crucial ingredient of the proof of Theorem 4.16.

**Definition 4.8.** For all $k, t \in \mathbb{N}_0$ with $0 \leq t \leq k$ and $(k)_q^i b_k \neq 0$ let

\[S(k, t) = \sum_{i=t}^{k} (-q_{21})^i q^{(i-t)(i-t-1)/2} \binom{i}{t}_q \hat{u}_i \hat{u}_{k-i} \in T(V).\]

In particular, $S(k, 0) = P_k$.

**Lemma 4.9.** Let $k, t \in \mathbb{N}_0$ with $0 \leq t \leq k$ such that $(k+1)_q^i b_{k+1} \neq 0$. Then

\[q_{12}^{-1} \text{ad} x_1(S(k, t)) = q^t(1 - q^{k-t} r)(k + 1 - t)_q S(k + 1, t) + r^{-1}(q^{2k-t} r^2 - 1)(t + 1)_q S(k + 1, t + 1).\]

**Proof.** First note that for $0 \leq i \leq k$ we get

\[\text{ad} x_1(\hat{u}_i \hat{u}_{k-i}) = (i + 1)_q (1 - q^i r) \hat{u}_{i+1} \hat{u}_{k-i} + q^i q_{12}(k + 1 - i)_q (1 - q^{k-i} r) \hat{u}_i \hat{u}_{k+1-i}.\]
by Equation (11). Moreover, \((k + 1)_q^i \neq 0\). Hence

\[
\text{ad } x_1(S(k, t))
\]

\[
= \sum_{i=t}^{k} (-q_{21})^i q^{(i-t)(i-t-1)/2}(i + 1)_q (1 - q^{-1}r) \left( \frac{i}{t} \right)_q \hat{u}_{i+1} \hat{u}_{k-i}
\]

\[
+ \sum_{i=t}^{k} (-q_{21})^i q^{(i-t)(i-t-1)/2} q^i q_{12} (k + 1 - i)_q (1 - q^{-1}r) \left( \frac{i}{t} \right)_q \hat{u}_i \hat{u}_{k+1-i}
\]

\[= -q_{12}^{-1} \sum_{i=t+1}^{k+1} (-q_{21})^i q^{(i-t-1)(i-t-2)/2}(i)_q (1 - q^{-1}r) \left( \frac{i}{t+1} \right)_q \hat{u}_i \hat{u}_{k+1-i}
\]

\[+ q_{12}^t \sum_{i=t}^{k+1} (-q_{21})^i q^{(i-t)(i-t+1)/2} (k + 1 - i)_q (1 - q^{-1}r) \left( \frac{i}{t} \right)_q \hat{u}_i \hat{u}_{k+1-i}.
\]

Now in the first term we replace \(i)_q \left( \frac{i}{t+1} \right)_q\) by \((t+1)_q \left( \frac{i}{t+1} \right)_q\) and \((i-t)_q \left( \frac{i}{t} \right)_q\), respectively. We then rewrite this first term as

\[(15) \quad -q_{12}^{-1} (t + 1)_q \sum_{i=t+1}^{k+1} (-q_{21})^i q^{(i-t-1)(i-t-2)/2}(i)_q \left( \frac{i}{t+1} \right)_q \hat{u}_i \hat{u}_{k+1-i}\]

\[(16) \quad + q_{12}^t \sum_{i=t}^{k+1} (-q_{21})^i q^{(i-t)(i-t+1)/2} (i)_q \left( \frac{i}{t} \right)_q \hat{u}_i \hat{u}_{k+1-i}.
\]

The second term of \(\text{ad } x_1(S(k, t))\) can be written as

\[(17) \quad q_{12}^t \sum_{i=t}^{k+1} (-q_{21})^i q^{(i-t)(i-t-1)/2} q^i q_{12} (k + 1 - i)_q \left( \frac{i}{t} \right)_q \hat{u}_i \hat{u}_{k+1-i}\]

\[(18) \quad -q_{12}^t r \sum_{i=t}^{k+1} (-q_{21})^i q^{(i-t)(i-t-1)/2} (k + 1 - i)_q \left( \frac{i}{t} \right)_q \hat{u}_i \hat{u}_{k+1-i}.
\]

Now (15) is equal to \(-q_{12}^{-1} (t + 1)_q S(k + 1, t + 1)\), and the sum of (16) and (17) is equal to \(q_{12}^t (k + 1 - t)_q S(k + 1, t)\). Finally, in (18) we replace \((k + 1 - i)_q\) by \((k + 1 - t)_q - q_{21}^{k+1-i} (i-t)_q\) and \((i-t)_q \left( \frac{i}{t} \right)_q\) by \((t+1)_q \left( \frac{i}{t+1} \right)_q\). Thus (18) is equal to

\[-q_{12}^t r (k + 1 - t)_q S(k + 1, t) + q_{12}^t q^{2k-t-r} (t+1)_q S(k + 1, t + 1).
\]

This implies the lemma. \(\square\)
Lemma 4.10. Let $m, k \in \mathbb{N}_0$ such that $(k + m)^! q b_{k+m} \neq 0$. Then

$$q_{12}^{-m} (\text{ad } x_1)^m (P_k) = \sum_{i=0}^{m} \frac{(m)_q^i}{(m-i)_q} \lambda_{(m-i,k)} \beta_{(i,m,k)} S(k + m, i)$$

where for any $i, n, m' \in \mathbb{N}_0$,

$$\lambda_{(n,k)} = \prod_{j=1}^{n} (1 - q^{k-1+j} r) (k + j)_q, \quad \beta_{(i,m',k)} = \prod_{j=1}^{i} (q^{m'+2k-j} r - r^{-1}).$$

Proof. Note first that for any $i, n \in \mathbb{N}_0$,

$$\lambda_{(n+1,k)} = (1 - q^{k+n} r) (k + n + 1)_q \lambda_{(n,k)},$$
$$\beta_{(i,m,k)} = (q^{m+2k-i} r - r^{-1}) \beta_{(i-1,m,k)},$$
$$\beta_{(i,m+1,k)} = (q^{m+2k} r - r^{-1}) \beta_{(i-1,m,k)}.$$  

We prove the Lemma by induction on $m$.

For $m = 0$, both sides of the equation in the lemma are equal to $P_k$. Assume now that the formula in the Lemma holds for $m$ and that $(k + m + 1)_q b_{k+m+1} \neq 0$. Then

$$q_{12}^{-m-1} (\text{ad } x_1)^{m+1} (P_k)$$
$$= q_{12}^{-1} \text{ad } x_1 \left( q_{12}^{-m} (\text{ad } x_1)^m (P_k) \right)$$
$$= q_{12}^{-1} \text{ad } x_1 \left( \sum_{i=0}^{m} \frac{(m)_q^i}{(m-i)_q} \lambda_{(m-i,k)} \beta_{(i,m,k)} S(k + m, i) \right)$$

by induction hypothesis. Now apply Lemma 4.9 to obtain that

$$q_{12}^{-m-1} (\text{ad } x_1)^{m+1} (P_k)$$
$$= \sum_{i=0}^{m} \frac{(m)_q^i}{(m-i)_q} \lambda_{(m-i,k)} \beta_{(i,m,k)} \cdot$$
$$q^i (1 - q^{k+m-i} r) (k + m + 1 - i)_q S(k + m + 1, i)$$
$$+ \sum_{i=0}^{m} \frac{(m)_q^i}{(m-i)_q} \lambda_{(m-i,k)} \beta_{(i,m,k)} \cdot$$
$$(q^{2m+2k-i} r - r^{-1}) (i+1)_q S(k + m + 1, i + 1).$$
In the first term we use \([19]\), in the second we change the summation index. Then
\[
q_{12}^{-m-1}(\text{ad } x_1)^{m+1}(P_k)
= \sum_{i=0}^{m} \frac{(m)_q^i}{(m-i)_q^i} \lambda_{(m+1-i,k)} \beta_{(i,m,k)} q^i S(k + m + 1, i)
+ \sum_{i=1}^{m+1} \frac{(m)_q^i}{(m+1-i)_q^i} \lambda_{(m+1-i,k)} \beta_{(i-1,m,k)} 
\cdot (q^{2m+2k+1-i} r - r^{-1}) (i) q S(k + m + 1, i).
\]
Thus it remains to show that
\[
(m + 1 - i) q^i \beta_{(i,m,k)} + (q^{2m+2k+1-i} r - r^{-1}) (i) q \beta_{(i-1,m,k)}
= (m + 1) q \beta_{(i,m+1,k)}
\]
for any 0 \(\leq i \leq m + 1\). The latter is easily done by expressing \(\beta_{(i,m,k)}\) and \(\beta_{(i,m+1,k)}\) via \(\beta_{(i-1,m,k)}\) using Equations \([20]\) and \([21]\), respectively, and then comparing coefficients. This proves the claim for \(m + 1\).

Recall the definitions of \(Q_{1,k,m}^k, Q_{2,k,m}^k \in \mathbb{Z}[q, r]\) from Lemma \([2.3]\). In this section we view \(Q_{1,k,m}^k, Q_{2,k,m}^k\) as elements in \(\mathbf{k} = \mathbb{k} \otimes_{\mathbb{Z}[q,r]} \mathbb{Z}[q, r]\) by identifying \(q\) and \(r\) in \(\mathbb{Z}[q, r]\) with \(q\) and \(r\) in \(\mathbf{k}\), respectively.

**Lemma 4.11.** Let \(k, m \in \mathbb{N}_0\). Suppose that \((k+m+1)_{q} q_{k+m+1} \neq 0\) and that there exists \(v \in U'_{k+m+1} \cap \ker(\pi)\) such that \(d_1(v) = (\text{ad } x_1)^{m}(P_k)\) in \(T(V)\). Then \(Q_{2,k,m}^k = 0\).

**Proof.** Let \(v \in U'_{k+m+1} \cap \ker(\pi)\). Let \(\mu_0, \ldots, \mu_{k-1} \in \mathbf{k}\) such that
\[
d_1(v) = \sum_{j=0}^{k+m} \mu_j (-q_{21}^j) u_j u_{k+m-j}.
\]
Then \(\sum_{j=0}^{k+m} q^{-j(j+1)/2} \mu_j = 0\) by Proposition \([4.5]\). Assume now also that \(d_1(v) = (\text{ad } x_1)^{m}(P_k)\). Then from Lemma \([4.10]\) and Definition \([4.8]\) we obtain that
\[
q_{12}^m \sum_{i=0}^{k+m} \frac{(m)_q^i}{(m-i)_q^i} \lambda_{(m-i,k)} \beta_{(i,m,k)} \sum_{j=i}^{k+m} q^{(j-i)(j-i-1)/2} q^{-j(j+1)/2} \left(\frac{j}{i}\right)_q = 0
\]
in \(B(V)\). (We use the notation in Lemma \([4.10]\)). Then by Lemma \([2.1]\) it follows that
\[
\sum_{i=0}^{m} \frac{(m)_q^i}{(m-i)_q^i} \lambda_{(m-i,k)} \beta_{(i,m,k)} q^{-(i+1)(2k+2m-i)/2} \left(\frac{k+m+1}{i+1}\right)_q = 0
\]
Since
\[ \lambda(m-i,k) = \prod_{j=1}^{m-i} (1 - q^{k-1+j}r) \frac{(k+m-i)_{q}^!}{(k)_{q}^!}, \]
the latter implies that
\[ \sum_{i=0}^{m} \binom{m+1}{i+1} \prod_{j=1}^{m-i} (1 - q^{k-1+j}r) \beta(i,m,k) q^{-(i+1)(2k+2m-i)/2} = 0. \]

Now substitute \( i = m-l \). It follows that
\[ \sum_{l=0}^{m} \binom{m+1}{l+1} \prod_{j=0}^{l-1} (1 - q^{k+j}r) \prod_{j=1}^{m-l} (q^{2k+m-j}r - r^{-1}) q^{(2k+l-1)/2} = 0. \]
The latter is equal to \((-r)^{-m}Q_{1}^{k,m} \). Thus \( Q_{2}^{k,m} = 0 \) by Lemma 2.3. \( \square \)

Now we introduce the set \( \mathbb{J} \) which is crucial for Theorem 4.16 below.

**Definition 4.12.** Let \( \mathbb{J} = \mathbb{J}_{q,r,s} \subseteq \mathbb{N}_{0} \) be such that \( j \in \mathbb{J} \) if and only if
\[ q^{j(j-1)/2}(-r)^{j} s = -1 \]
and \( q^{j+n-1}r^{2} \neq 1 \) for any \( n \in \mathbb{J}, n < j \).

**Lemma 4.13.** For any \( j \in \mathbb{J} \), the integers \( j+1 \) and \( j+2 \) are not in \( \mathbb{J} \). In particular, for any \( m \in \mathbb{N}_{0} \),
\[ |\mathbb{J} \cap [0,m]| \leq \frac{m}{3} + 1. \]

**Proof.** Let \( j \in \mathbb{N}_{0} \) and \( t \in \mathbb{N} \). Assume that \( j, j+t \in \mathbb{J} \). Then
\[ q^{j(j-1)/2}(-r)^{j} s = -1, \quad q^{j(t)(j+t-1)/2}(-r)^{j+t} s = -1, \]
and \( q^{2j+t-1}r^{2} \neq 1 \). Hence \( q^{t(2j+t-1)/2}(-r)^{t} = 1 \). This gives a contradiction both for \( t = 1 \) and for \( t = 2 \). \( \square \)

**Example 4.14.** By the definition of \( \mathbb{J} \) and by Lemma 4.13 the following hold.

1. \( 0 \in \mathbb{J} \) if and only if \( s = -1 \).
2. \( 1 \in \mathbb{J} \) if and only if \( rs = 1 \) and \( s \neq -1 \).
3. \( 2 \in \mathbb{J} \) if and only if \( qr^{2} s = -1 \), \( s \neq -1 \), and \( rs \neq 1 \).

For the proof of the next theorem we will need a technicality.

**Lemma 4.15.** Assume that \( \text{char}(\mathbb{k}) = 0 \). Let \( k, m \in \mathbb{N}_{0} \), and assume that \( b_{k+m+1} \neq 0 \) and \( q^{2k+m}r^{2} = 1 \). Then \( Q_{2}^{k,m} \neq 0 \) in \( \mathbb{k} \).
Proof. Assume first that \( m \) is odd and that \( q^{2k+m}r^2 = 1 \). Then
\[
Q^{k,m}_2 = \sum_{i=0}^{(m-1)/2} (q^{2k+m}r^2)^i \prod_{i=0}^{m} (1 - q^{k+i}r) = \frac{m + 1}{2} \prod_{i=0}^{m} (1 - q^{k+i}r).
\]
Since \( b_{k+m+1} \neq 0 \) and \( \text{char}(k) = 0 \), we conclude that \( Q^{k,m}_2 \neq 0 \) in \( k \).

Assume now that \( q^{2k+m}r^2 = 1 \) and that \( m \) is even. Let \( n = m/2 \).
Since \( b_{k+m+1} \neq 0 \), it follows that \( q^{k+n}r = -1 \). Hence
\[
Q^{k,m}_2 = \sum_{i=0}^{m} (-q^{k+n}r)^i \prod_{i=0}^{n-1} (1 - q^{k+i}r) \prod_{i=n+1}^{m} (1 - q^{k+i}r).
\]
Thus we again obtain that \( Q^{k,m}_2 \neq 0 \) in \( k \). \( \square \)

Theorem 4.16. Assume that \( \text{char}(k) = 0 \). Let \( m \in \mathbb{N}_0 \) such that \( (m)_q^l b_m \neq 0 \). Then the elements \( (\text{ad} x_1)^{m-j}(P_j) \) with \( j \in \mathbb{J} \cap [0,m] \) form a basis of \( \ker(\pi) \cap U_m \).

Proof. First note that \( P_j \in \ker(\pi) \cap U_j \) for any \( j \in \mathbb{J} \) because of Lemma 4.7. Hence \( (\text{ad} x_1)^{m-j}(P_j) \in \ker(\pi) \cap U_m \) for any \( j \in \mathbb{J} \cap [0,m] \).

Now we prove by induction on \( m \) that the elements \( (\text{ad} x_1)^{m-j}(P_j) \) with \( j \in \mathbb{J} \cap [0,m] \) are linearly independent. This is clear for \( m = 0 \).

Assume now that \( m > 0 \), and for any \( j \in \mathbb{J} \cap [0,m] \) let \( \lambda_j \in k \) such that
\[
\sum_{j \in \mathbb{J} \cap [0,m]} \lambda_j (\text{ad} x_1)^{m-j}(P_j) = 0.
\]
If \( m \notin \mathbb{J} \), then \( \sum_{j \in \mathbb{J} \cap [0,m]} \lambda_j (\text{ad} x_1)^{m-1-j}(P_j) = 0 \) by Lemma 4.4. Hence \( \lambda_j = 0 \) for all \( j \in \mathbb{J} \cap [0,m] \) by induction hypothesis.

Assume now that \( m \in \mathbb{J} \). Lemma 3.10 implies that \( d_1(P_n) = 0 \) for any \( n \in \mathbb{N}_0 \), and hence
\[
\sum_{j \in \mathbb{J} \cap [0,m-1]} \lambda_j (\text{ad} x_1)^{m-j}(P_j) = 0.
\]
From Lemma 3.8 then it follows that
\[
\sum_{j \in \mathbb{J} \cap [0,m-1]} \lambda_j (m-j)_q (1 - q^{m-j-1}q^{2j}r^2)(\text{ad} x_1)^{m-1-j}(P_j) = 0.
\]
Note that \( q^{m+j-1}r^2 \neq 1 \) for all \( j \in \mathbb{J} \cap [0,m-1] \) because of \( m \in \mathbb{J} \). Moreover, \( (m)_q^l \neq 0 \) by assumption. Therefore induction hypothesis implies that \( \lambda_j = 0 \) for all \( j \in \mathbb{J} \cap [0,m-1] \). Then clearly \( \lambda_m = 0 \) holds, too.

It remains to show that
\[
\text{(22)} \quad \dim \left( \ker(\pi) \cap U_m \right) = |\mathbb{J} \cap [0,m]|.
\]

Again we proceed by induction on $m$. Note that $P_0 = u_0 \in \ker(\pi)$ if and only if $s = -1$, that is, $0 \in \mathcal{J}$, according to Lemma 4.7. Thus the claim holds for $m = 0$.

Let now $m \in \mathbb{N}$. Induction hypothesis and the first part of the proof of the Theorem imply that the elements $(\text{ad } x_1)^{m-1-j}(P_j)$, where $j \in \mathcal{J} \cap [0, m-1]$, form a basis of $\ker(\pi) \cap U_{m-1}$. Since ad $x_1$ is injective by Lemma 4.4 and since ad $x_1(\ker(\pi)) \subseteq \ker(\pi)$, we further obtain that

$$\dim \left( \ker(\pi) \cap U_m \right) \geq \dim \left( \ker(\pi) \cap U_{m-1} \right).$$

Assume first that $\dim \left( \ker(\pi) \cap U_m \right) = \dim \left( \ker(\pi) \cap U_{m-1} \right)$. Then the elements $(\text{ad } x_1)^{m-j}(P_j)$, where $j \in \mathcal{J} \cap [0, m-1]$, form a basis of $\ker(\pi) \cap U_m$. Moreover, the linear independence of the elements $(\text{ad } x_1)^{m-j}(P_j)$, $j \in \mathcal{J} \cap [0, m]$, implies that $m \notin \mathcal{J}$. This proves (22).

Assume now that $\dim \left( \ker(\pi) \cap U_m \right) > \dim \left( \ker(\pi) \cap U_{m-1} \right)$. Since

$$d_1(\ker(\pi) \cap U_m) \subseteq \ker(\pi) \cap U_{m-1},$$

we conclude that $\ker(\pi) \cap U_m \cap \ker(d_1) \neq 0$. Since $(m)^0_j b_m \neq 0$, Lemma 3.10 implies that $\ker(d_1|U_m) = \mathbb{k}P_m$. Hence $P_m \in \ker(\pi) \cap U_m$,

$$\dim \left( \ker(\pi) \cap U_m \right) = 1 + \dim \left( \ker(\pi) \cap U_{m-1} \right),$$

and for any $j \in \mathcal{J}$ there exists $v_j \in \ker(\pi) \cap U_m$ such that

$$d_1(v_j) = (\text{ad } x_1)^{m-1-j}(P_j).$$

Since $P_m \in \ker(\pi) \cap U_m$, we obtain from Lemma 4.7 that

$$q^{m(m-1)/2}(-r)^m s = -1.$$

Further, we may assume that $v_j \in \ker(\pi) \cap U'_m$ for any $j \in \mathcal{J} \cap [0, m-1]$. Hence $Q_2^{m-1-j} = 0$ for any $j \in \mathcal{J} \cap [0, m-1]$ by Lemma 4.11. Since $\text{char}(\mathbb{k}) = 0$, from Lemma 4.15 we conclude that $q^{m+j-1/2} \neq 1$ for any $j \in \mathcal{J} \cap [0, m-1]$. Thus $m \notin \mathcal{J}$. Then Equation (22) follows from (23) and from induction hypothesis.

**Corollary 4.17.** Assume that $\text{char}(\mathbb{k}) = 0$. Let $k, l \in \mathbb{N}_0$ with $k \geq l$.

Suppose that $(k+l)^0_j b_{k+l} \neq 0$, and that $q^{k^2} r^k s = -1$ if $k = l$. Then the following are equivalent.

1. $[x^k_1 x^k_2 x^l_1 x^l_2] \text{ is a root vector},$
2. $|\mathcal{J} \cap [0, k+l]| \leq l.$

**Proof.** By assumption, $[x^k_1 x^k_2 x^l_1 x^l_2]$ is a root vector candidate. Proposition 4.3 and Example 3.5 imply that $[x^k_1 x^k_2 x^l_1 x^l_2]$ is a root vector if and only if any root vector candidate of degree $(k+l)\alpha_1 + 2\alpha_2$, which is not a root vector, is of the form $[x^k_1 x^k_2 x^l_1 x^l_2]$ with $k_1 + k_2 = k + l,$
$0 \leq k_2 < l$. This just means that $\dim(\ker(\pi) \cap U_{k+l}) \leq l$. According to Theorem 4.16, the latter is equivalent to $|J \cap [0, k + l]| \leq l$. □

**Corollary 4.18.** Assume that $\text{char}(k) = 0$. Let $m \in \mathbb{N}_0$ such that $(m)_q! b_m \neq 0$. Then the multiplicity of $m\alpha_1 + 2\alpha_2$ is

$$m' = |J \cap [0, m]|,$$

where

$$m' = \begin{cases} (m + 1)/2 & \text{if } m \text{ is odd, } \\ m/2 & \text{if } m \text{ is even and } q^{m^2/4_r^m/2_s} \neq -1, \\ m/2 + 1 & \text{if } m \text{ is even and } q^{m^2/4_r^m/2_s} = -1. \end{cases}$$

**Proof.** By Example 3.8, $m'$ is just the number of root vector candidates of degree $m\alpha_1 + 2\alpha_2$. Corollary 4.17 implies that $|J \cap [0, m]|$ is the number of root vector candidates of degree $m\alpha_1 + 2\alpha_2$ which are not root vectors. This implies the claim. □

The following proposition treats the question in Corollary 4.18 if the assumption on $m$ is not satisfied. Recall that $R_1(V)$ is the reflection of $V$ on the first vertex.

**Proposition 4.19.** Assume that $\text{char}(k) = 0$. Let $k, m \in \mathbb{N}_0$ such that $(k)_q! b_k \neq 0$, $(k + 1)_q(1 - q^k r) = 0$, and $m \geq k$. Then the multiplicity of $m\alpha_1 + 2\alpha_2$ is the same as the multiplicity of $(2k - m)\alpha_1 + 2\alpha_2$ of $B(R_1(V))$.

**Proof.** The claim is a very special case of the invariance of multiplicities under reflections, which was proved in [3]. □

**Remark 4.20.** According to the explanations in Section 3 in Proposition 4.19 we have $c_{12} = -k$. Hence the braiding matrix $(q'_{ij})_{i,j \in \{1,2\}}$ of $R_1(V)$ satisfies

$$q'_{11} = q, \quad q'_{12}q'_{21} = r, \quad q'_{22} = s$$

whenever $q^k r = 1$, and

$$q'_{11} = q, \quad q'_{12}q'_{21} = q^{-2}r^{-1}, \quad q'_{22} = qrks$$

whenever $q^k r \neq 1$ (and then $(k + 1)_q = 0$). Since $2k - m \leq k$, the multiplicity of $(2k - m)\alpha_1 + 2\alpha_2$ of $B(R_1(V))$ can be obtained using Corollary 4.18 with the set $J$ for $(q'_{ij})_{i,j \in \{1,2\}}$.

Finally, we discuss the multiplicity of roots in some special cases.

**Corollary 4.21.** Assume that $\text{char}(k) = 0$. Let $m \in \mathbb{N}_0$. 

(1) Assume that \( m \in \{1,2,3,4,6\} \) and that \((m)_q^{1}b_m \neq 0\). Then \( m\alpha_1 + 2\alpha_2 \) is not a root if and only if \( q, r, s \) satisfy the conditions given in Table A.

(2) Assume that \( m = 2k + 1 \geq 5 \) is odd and that \((k + 3)_q^{1}b_{k+3} \neq 0\). Then \( m\alpha_1 + 2\alpha_2 \) is a root of \( B(V) \).

(3) Assume that \( m = 2k \geq 8 \) and that \((k + 4)_q^{1}b_{k+4} \neq 0\). Then \( m\alpha_1 + 2\alpha_2 \) is a root of \( B(V) \).

Proof. (1) We apply Corollary 4.18 case by case.
Assume that \( m = 1 \). Then \( m' = 1 \). Hence \( \alpha_1 + 2\alpha_2 \) is not a root if and only if \( \lfloor \mathbb{J} \cap [0, 1] \rfloor = 1 \). According to Example 4.14, this is equivalent to \((1 + s)(1 - rs) = 0\).

Assume that \( m = 2 \). Then \( \lfloor \mathbb{J} \cap [0, 2] \rfloor \leq 1 \) by Example 4.14, and equality holds if and only if \((1 + s)(1 - rs)(1 + qr^2s) = 0\). Hence, if \( qr \neq -1 \), then \( m' = 1 \) and the claim is proven. On the other hand, if \( qr = -1 \), then \( m' = 2 \) and hence \( 2\alpha_1 + 2\alpha_2 \) is a root. Note that in this case \((1 + s)(1 - rs)(1 + qr^2s) \neq 0 \) since
\[
(m)_q^{1}b_m = (2)_q(1 - r)(1 - qr) \neq 0.
\]
Thus the claim is valid also in this case.

Assume that \( m = 3 \). Then \( m' = 2 \). Hence \( 3\alpha_1 + 2\alpha_2 \) is not a root if and only if \( \lfloor \mathbb{J} \cap [0, 3] \rfloor = 2 \). Due to Lemma 4.13, the latter is only possible if \( \mathbb{J} \cap [0, 3] = \{0, 3\} \). This means that \( s = -1 \), \( q^3r^3s = 1 \), and \( q^2r^2 \neq 1 \). Because of \((3)_q^{1}b_3 \neq 0 \) we can rewrite this condition to \( s = -1 \), \( (3)_{-qr} = 0 \).

The conditions for \( m = 4 \) and \( m = 6 \) can be obtained similarly.

(2) Assume first that \((m)_q^{1}b_m \neq 0\). By Corollary 4.18, the multiplicity of \( m\alpha_1 + 2\alpha_2 \) is \( k + 1 - \lfloor \mathbb{J} \cap [0, m] \rfloor \). By Lemma 4.13, \( \lfloor \mathbb{J} \cap [0, m] \rfloor \leq m/3 + 1 \). Since \( 3k - m = k - 1 > 0 \), we conclude that \( m\alpha_1 + 2\alpha_2 \) is a root of \( B(V) \).

Assume now that \((m)_q^{1}b_m = 0\). Since \((k + 3)_q^{1}b_{k+3} \neq 0 \) by assumption, for the Cartan matrix entry \( c_{12} \) we obtain that \( k + 3 \leq -c_{12} < m \). Moreover,
\[
s_1(m\alpha_1 + 2\alpha_2) = (-2c_{12} - m)\alpha_1 + 2\alpha_2
\]
and \(-2c_{12} - m \) is odd and lesser than \(-c_{12} \). Moreover,
\[
-2c_{12} - m - 5 = -2c_{12} - 2k - 6 \geq 0
\]
and hence Proposition 4.19 and the previous paragraph for \( R_1(V) \) imply that \( m\alpha_1 + 2\alpha_2 \) is a root of \( B(V) \).

(3) Similar to the proof of (2). Note that \( 2k\alpha_1 + 2\alpha_2 \) is always a root if \( q^{k^2}r^ks = -1 \) and \((k + 1)_q^{1}b_{k+1} \neq 0 \) because of Lemma 4.1. Hence only the case where \( q^{k^2}r^ks \neq -1 \) has to be considered in detail. \( \square \)
\[
\begin{array}{c|c}
 m\alpha_1 + 2\alpha_2 & \text{non-root conditions} \\
 \alpha_1 + 2\alpha_2 & (1+s)(1-rs) = 0 \\
 2\alpha_1 + 2\alpha_2 & (1+s)(1-rs)(1+qr^2s) = 0 \\
 3\alpha_1 + 2\alpha_2 & s = -1, (3)_{-qr} = 0 \\
 4\alpha_1 + 2\alpha_2 & s = -1, (3)_{-qr} = 0 \text{ or } s = -1, q^3r^2 = -1 \text{ or } rs = 1, (3)_{-q^2r} = 0 \\
 6\alpha_1 + 2\alpha_2 & q = 1, s = -1, (3)_{-r} = 0
\end{array}
\]

Table 1. Table for Corollary 4.21

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