HIGHER-ORDER CHANGHEE NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper, we consider the higher-order Changhee numbers and polynomials which are derived from the fermionic $p$-adic integral on $\mathbb{Z}_p$ and give some relations between higher-order Changhee polynomials and special polynomials.

1. Introduction

As is well known, the Euler polynomials of order $\alpha (\in \mathbb{N})$ are defined by the generating function to be

$$\left( \frac{2}{e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}, \text{ (see [1-16]).}$$ (1.1)

When $x = 0$, $E_n^{(\alpha)} = E_n^{(\alpha)}(0)$ are called the Euler numbers of order $\alpha$.

The Stirling number of the first kind is defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n, l)x^l, \ (n \in \mathbb{Z}_{\geq 0}), \ (\text{see [5,6,7]}),$$ (1.2)

where $(x)_n = x(x-1) \cdots (x-n+1)$.

The Stirling number of the second kind is also defined by the generating function to be

$$(e^t - 1)^n = n! \sum_{l=0}^{\infty} S_2(l, n) \frac{t^l}{l!}, \ (n \in \mathbb{Z}_{\geq 0}).$$ (1.3)

Let $p$ be an odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm $| \cdot |_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \ (\text{see [9]}).$$ (1.4)

For $f_1(x) = f(x+1)$, we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0).$$ (1.5)

As is well-known, the Changhee polynomials are defined by the generating function to be

$$I_{-1}(f) = \sum_{n=0}^{\infty} C_n f_n(x), \ (n \in \mathbb{Z}_{\geq 0}).$$ (1.6)
When \( x = 0 \), \( Ch_n = Ch_n(0) \) are called the Changhee numbers. In this paper, we consider the higher-order Changhee numbers and polynomials which are derived from the multivariate fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) and give some relations between higher-order Changhee polynomials and special polynomials.

2. Higher-order Changhee polynomials

For \( k \in \mathbb{N} \), let us define the Changhee numbers of the first kind with order \( k \) as follows:

\[
Ch_n^{(k)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k),
\]

(2.1)

where \( n \) is a nonnegative integer.

From (2.1), we can derive the generating function of \( Ch_n^{(k)} \) as follows:

\[
\sum_{n=0}^{\infty} Ch_n^{(k)} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \sum_{n=0}^{\infty} \binom{x_1 + \cdots + x_k}{n} t^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \right)
\]

\[
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{x_1 + \cdots + x_k} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).
\]

(2.2)

By (1.5), we easily see that

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{x_1 + \cdots + x_k} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \left( \frac{2}{2 + t} \right)^k.
\]

(2.3)

From (2.2) and (2.3), we have

\[
\sum_{n=0}^{\infty} Ch_n^{(k)} \frac{t^n}{n!} = \left( \frac{2}{2 + t} \right)^k.
\]

(2.4)

It is easy to show that

\[
\left( \frac{2}{2 + t} \right)^k = \sum_{n=0}^{\infty} \left( \sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} Ch_{l_1} \cdots Ch_{l_k} \right) \frac{t^n}{n!}.
\]

(2.5)

Thus, by (2.4) and (2.5), we get

\[
Ch_n^{(k)} = \sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} Ch_{l_1} \cdots Ch_{l_k}.
\]

(2.6)

It is not difficult to show that

\[
\left( \frac{2}{2 + t} \right)^k = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \binom{k + n - 1}{n} \frac{t^n}{n!}.
\]

(2.7)
From (2.4) and (2.7), we have
\[ 2^n Ch_n^{(k)} = (-1)^n n! \binom{n+k-1}{n} = (-1)^n (k+n-1)_n \]
\[ = (-1)^n \sum_{l=0}^{n} S_1(n, l)(k+n-1)^l. \]  

(2.8)

Therefore, by (2.8), we obtain the following theorem.

**Theorem 2.1.** For \( n \geq 0 \), we have
\[ Ch_n^{(k)} = \left( -\frac{1}{2} \right)^n \sum_{l=0}^{n} S_1(n, l)(k+n-1)^l. \]

By (2.8), we get
\[ Ch_n^{(k)} = \int_{Z_p} \cdots \int_{Z_p} (x_1 + \cdots + x_k)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \]
\[ = \sum_{l=0}^{n} S_1(n, l) \int_{Z_p} \cdots \int_{Z_p} (x_1 + \cdots + x_k)^l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \]

(2.9)

Now, we observe that
\[ \int_{Z_p} \cdots \int_{Z_p} e^{(x_1 + \cdots + x_k)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \left( \frac{2}{e^t + 1} \right)^k = \sum_{n=0}^{\infty} E_n^{(k)} \frac{t^n}{n!}. \]

(2.10)

By (2.9) and (2.10), we get
\[ Ch_n^{(k)} = \sum_{l=0}^{n} S_1(n, l) E_l^{(k)}. \]

(2.11)

Therefore, by (2.2), we obtain the following theorem.

**Theorem 2.2.** For \( n \geq 0 \), we have
\[ Ch_n^{(k)} = \sum_{l=0}^{n} S_1(n, l) E_l^{(k)}. \]

Replacing \( t \) by \( e^t - 1 \) in (2.4), we get
\[ \sum_{n=0}^{\infty} Ch_n^{(k)} \frac{(e^t - 1)^n}{n!} = \left( \frac{2}{e^t + 1} \right)^k = \sum_{m=0}^{\infty} E_m^{(k)} \frac{t^m}{m!}. \]

(2.12)

and
\[ \sum_{n=0}^{\infty} Ch_n^{(k)} \frac{(e^t - 1)^n}{n!} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} Ch_n^{(k)} S_2(m, n) \right) \frac{t^m}{m!}. \]

(2.13)

Therefore, by (2.12) and (2.13), we obtain the following theorem.

**Theorem 2.3.** For \( n \geq 0 \), we have
\[ E_m^{(k)} = \sum_{n=0}^{m} Ch_n^{(k)} S_2(m, n). \]
Now, we consider the higher-order Changhee polynomials of the first kind as follows:

\[ Ch_n^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \] (2.14)

By (2.14), we get

\[ \sum_{n=0}^{\infty} Ch_n^{(k)}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{x_1 + \cdots + x_k + x} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \left( \frac{2}{2 + t} \right)^k (1 + t)^x. \] (2.15)

From (2.4), we have

\[ (2 + t)^k (1 + t)^x = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} Ch_{n-m}^{(k)} \right) \frac{t^n}{n!}. \] (2.16)

By (2.15) and (2.16), we get

\[ Ch_n^{(k)}(x) = \sum_{m=0}^{n} \binom{x}{m} \frac{n!}{(n-m)!} Ch_{n-m}^{(k)} = \sum_{m=0}^{n} \binom{x}{n-m} \frac{n!}{m!} Ch_m^{(k)}. \] (2.17)

From (2.14), we have

\[ Ch_n^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \sum_{l=0}^{n} S_1(n,l) E_l^{(k)}(x). \] (2.18)

Therefore, by (2.15), we obtain the following corollary.

**Corollary 2.4.** For \( n \geq 0 \), we have

\[ Ch_n^{(k)}(x) = \sum_{l=0}^{n} S_1(n,l) E_l^{(k)}(x). \]

In (2.15), by replacing \( t \) by \( e^t - 1 \), we get

\[ \sum_{n=0}^{\infty} Ch_n^{(k)}(x) \frac{(e^t - 1)^n}{n!} = \left( \frac{2}{e^t + 1} \right)^k e^{tx} = \sum_{m=0}^{\infty} E_m^{(k)}(x) \frac{t^m}{m!}; \] (2.19)

and

\[ \sum_{n=0}^{\infty} Ch_n^{(k)}(x) \frac{1}{n!} (e^t - 1)^n = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} Ch_n^{(k)}(x) S_2(m,n) \right) \frac{t^m}{m!}; \] (2.20)

Therefore, by (2.19) and (2.20), we obtain the following theorem.

**Theorem 2.5.** For \( m \geq 0 \), we have

\[ E_m^{(k)}(x) = \sum_{n=0}^{m} Ch_n^{(k)}(x) S_2(m,n). \]
The rising factorial is defined by
\[(x)^{(n)} = x(x+1)^{\cdots}(x+n-1) = (-1)^n(-x)^n.\] (2.21)

Here, we define the Changhee numbers of the second kind with order \(k \in \mathbb{N}\) as follows:
\[\hat{C}_n^{(k)} = \int_{\mathbb{Z}^k} \cdots \int_{\mathbb{Z}^k} (-x_1 - \cdots - x_k)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).\] (2.22)

Thus, by (2.22), we get
\[\hat{C}_n^{(k)} = \sum_{l=0}^{n} (-1)^l S_1(n, l) \int_{\mathbb{Z}^k} \cdots \int_{\mathbb{Z}^k} (x_1 + \cdots + x_k)^l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \sum_{l=0}^{n} (-1)^l S_1(n, l) E_l^{(k)}.\] (2.23)

The generating function of \(\hat{C}_n^{(k)}\) is given by
\[\sum_{n=0}^{\infty} \hat{C}_n^{(k)} t^n/n! = \int_{\mathbb{Z}^k} \cdots \int_{\mathbb{Z}^k} (1 + t)^{-x_1 - \cdots - x_k} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \left(\frac{2}{2+t}\right)^k (1 + t)^k.\] (2.24)

Now, we observe that
\[\left(\frac{2}{2+t}\right)^k (1 + t)^k = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{k}{m} \frac{n!}{(n-m)!} \hat{C}_{n-m}^{(k)} \frac{t^n}{n!}.\] (2.25)

Thus, by (2.24) and (2.25), we get
\[\hat{C}_n^{(k)} = \sum_{m=0}^{n} m! \binom{k}{m} \left(\frac{n}{m}\right) \hat{C}_{n-m}^{(k)}.\] (2.26)

Therefore, by (2.26), we obtain the following theorem.

**Theorem 2.6.** For \(n \geq 0\), we have
\[\hat{C}_n^{(k)} = \sum_{m=0}^{n} m! \binom{k}{m} \left(\frac{n}{m}\right) C_{n-m}^{(k)}.\]

In (2.24), by replacing \(t\) by \(e^t - 1\), we get
\[\sum_{n=0}^{\infty} \hat{C}_n^{(k)} \frac{(e^t - 1)^n}{n!} = \left(\frac{2}{e^t + 1}\right)^k e^{tk} = \sum_{m=0}^{\infty} E_m^{(k)} \frac{t^m}{m!}.\] (2.27)

and
\[\sum_{n=0}^{\infty} \hat{C}_n^{(k)} \frac{(e^t - 1)^n}{n!} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \hat{C}_n^{(k)} S_2(m, n) \frac{t^m}{m!}.\] (2.28)

Therefore, by (2.27) and (2.28), we obtain the following theorem.
**Theorem 2.7.** For $m \geq 0$, we have

$$E_m^{(k)}(k) = \sum_{n=0}^{m} \tilde{C}_n^{(k)} S_2(m, n).$$

Now, we consider the Changhee polynomials of the second kind with order $k \in \mathbb{N}$ as follows:

$$\tilde{C}_n^{(k)}(x) = \int_{z_p} \cdots \int_{z_p} (-x_1 - \cdots - x_k + x)_n d\mu_1(x_1) \cdots d\mu_1(x_k). \quad (2.29)$$

From (2.24) and (2.29), we have

$$\sum_{n=0}^{\infty} \tilde{C}_n^{(k)}(x) \frac{t^n}{n!} = \int_{z_p} \cdots \int_{z_p} (1 + t)^{-x_1 - \cdots - x_k + x} d\mu_1(x_1) \cdots d\mu_1(x_k) = (1 + t)^{x+k} \left( \frac{2}{2+t} \right)^k. \quad (2.30)$$

We observe that

$$\left( \frac{2}{2+t} \right)^k (1 + t)^{x+k} = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \frac{n!}{m!} \binom{x}{m} \binom{n}{m} \tilde{C}_{n-m}^{(k)} \right) \frac{t^n}{n!}. \quad (2.31)$$

Thus, by (2.30) and (2.31), we obtain the following theorem.

**Theorem 2.8.** For $m \geq 0$, we have

$$\tilde{C}_n^{(k)}(x) = \sum_{m=0}^{n} m! \binom{x}{m} \binom{n}{m} \tilde{C}_{n-m}^{(k)}. \quad (2.32)$$

From (2.24), we have

$$\tilde{C}_n^{(k)}(x) = \sum_{l=0}^{n} S_1(n, l)(-1)^l \int_{z_p} \cdots \int_{z_p} (x_1 + \cdots + x_k - x)^l d\mu_1(x_1) \cdots d\mu_1(x_k)$$

$$= \sum_{l=0}^{n} S_1(n, l)(-1)^l E_l^{(k)}(-x). \quad (2.32)$$

In (2.30), by replacing $t$ by $e^t - 1$, we get

$$\sum_{n=0}^{\infty} \tilde{C}_n^{(k)}(x) \left( \frac{e^t - 1}{t} \right)^n = e^{(x+k)t} \left( \frac{2}{e^t + 1} \right)^k = \sum_{m=0}^{\infty} E_m^{(k)}(x + k) \frac{t^m}{m!}, \quad (2.33)$$

and

$$\sum_{n=0}^{\infty} \tilde{C}_n^{(k)}(x) \frac{1}{n!} (e^t - 1)^n = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \tilde{C}_n^{(k)} S_2(n, m) \frac{t^m}{m!}. \quad (2.34)$$

Therefore, by (2.33) and (2.34), we obtain the following theorem.

**Theorem 2.9.** For $m \geq 0$, we have

$$E_m^{(k)}(x + k) = \sum_{n=0}^{m} \tilde{C}_n^{(k)} S_2(m, n).$$
Now, we observe that

\[
(-1)^n \frac{\overline{C}_n^{(k)}(x)}{n!} = (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{- (x_1 + \cdots + x_k) + x}{n} \right) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{x_1 + \cdots + x_k - x + n - 1}{n} \right) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
= \sum_{m=0}^{n} \frac{(n-1)}{n} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{x_1 + \cdots + x_k - x}{m} \right) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
= \sum_{m=0}^{n} \frac{(n-1)}{m!} \overline{C}_m^{(k)}(-x) = \sum_{m=1}^{n} \frac{(n-1)}{m!} \overline{C}_m^{(k)}(-x).
\]

Therefore, by (2.35), we obtain the following theorem.

**Theorem 2.10.** For \( n \in \mathbb{N} \), we have

\[
(-1)^n \frac{\overline{C}_n^{(k)}(x)}{n!} = \sum_{m=1}^{n} \frac{(n-1)}{m!} \overline{C}_m^{(k)}(-x).
\]

By (2.14), we get

\[
(-1)^n \frac{C_n^{(k)}(x)}{n!} = (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{x_1 + \cdots + x_k + x}{n} \right) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{-x_1 - x_2 - \cdots - x_k - x + n - 1}{n} \right) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
= \sum_{m=0}^{n} \frac{(n-1)}{m!} \overline{C}_m^{(k)}(-x) = \sum_{m=1}^{n} \frac{(n-1)}{m!} \overline{C}_m^{(k)}(-x).
\]

Therefore, by (2.36), we obtain the following theorem.

**Theorem 2.11.** For \( n \in \mathbb{N} \), we have

\[
(-1)^n \frac{\overline{C}_n^{(k)}(x)}{n!} = \sum_{m=1}^{n} \frac{(n-1)}{m!} \overline{C}_m^{(k)}(-x).
\]

**References**

[1] S. Araci, M. Acikgoz, A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials, Adv. Stud. Contemp. Math., 22 (2012), no.3, 399-406.
[2] A. Bayad, T. Kim, Identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials, Adv. Stud. Contemp. Math. 20 (2010), no. 2, 247-253.
[3] M. Cenkci, The \( p \)-adic generalized twisted \((h, q)\)-Euler-t-function and its applications, Adv. Stud. Contemp. Math. 15(2007), no.1, 37-47.
[4] K.-W. Hwang, D. V. Dolgy, D. S. Kim, T. Kim, S. H. Lee, Some theorems on Bernoulli and Euler numbers, Ars Combin. 109 (2013), 285-297.
[5] D. S. Kim, T. Kim, A study on the integral of the product of several Bernoulli polynomials, Rocky Mountain J. Math.(2013), Forthcoming Article.
[6] D. S. Kim, T. Kim, J. J. Seo, A note on Changhee polynomials and numbers, Adv. Studies Theor. Phys. 7(2013), 1-10.
[7] D. S. Kim, T. Kim, S.-H. Lee, A note on poly-Bernoulli polynomials arising from umbral calculus, Adv. Studies Theor. Phys. 7(2013), no. 15, 731-744.
[8] T. Kim, p-adic q-integrals associated with the Changhee-Barnes’ q-Bernoulli polynomials, Integral Transforms Spec. Funct. 15 (2004), no. 5, 415-420.
[9] T. Kim, Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials, J. Difference Equ. Appl. 14(2008), no.12, 1267-1277.
[10] H. Ozden, p-adic distribution of the unification of the Bernoulli, Euler and Genocchi polynomials, Appl. Math. Comput. 218(2011), no. 3, 970-973.
[11] B. Kurt, Y. Simsek, On the generalized Apostol-type Frobenius-Euler polynomials, Adv. Difference Equ. 2013, 2013:1, 9 pp.
[12] S.-H. Rim, J. Jeong, On the modified q-Euler numbers of higher order with weight, Adv. Stud. Contemp. Math. 22(2012), no. 1, 93-98.

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