THE GEODESIC MOTION ON GENERALIZED TAUB-NUT GRAVITATIONAL INSTANTONS

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Abstract

A class of generalized Taub-NUT gravitational instantons is reported in five - dimensional Einstein gravity coupled to a non-linear sigma model. The geodesic dynamics of a spinless particle of unit mass on these static gravitational instantons is studied. This is accomplished by finding a generalized Runge-Lenz vector. Unlike the Kepler problem, or, the dynamics of a spinless particle on the familiar Taub-NUT gravitational instantons, the orbits are not conic sections.

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1. INTRODUCTION

The classical equations of sourceless five-dimensional general relativity admit various stationary solutions among which the well-known Kaluza-Klein multi-monopoles [1]. The Kaluza-Klein monopole was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional theory, adding the time coordinate in a trivial way. Its line element is expressed as

\[ ds_5^2 = -dt^2 + ds_4^2 \]

\[ = -dt^2 + V^{-1}(r) \left[ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\varphi^2 \right] + V(r) \left[ dx^5 + A(r) \cdot dr \right]^2 \]  

(1.1)

where \( r \) denotes a three-vector \( r = (r, \theta, \varphi) \) and the gauge field \( A \) is that of a monopole

\[ A_r = A_\theta = 0, \quad A_\varphi = 4m(1 - \cos \theta) \]

\[ \mathbf{B} = \text{rot} \, \mathbf{A} = \frac{4m r}{r^3} . \]  

(1.2)

The function \( V(r) \) is

\[ V(r) = \left( 1 + \frac{4m}{r} \right)^{-1} \]  

(1.3)

and the so called NUT singularity is absent if \( x^5 \) is periodic with period \( 16\pi m \) [2].

Remarkably the same object has re-emerged in the study of monopole scattering. In the long distance limit, neglecting radiation, the relative motion of two monopoles is described by the geodesics of the space (1.1) [3,4]. Slow Bogomolny - Prasad - Sommer-field monopoles move along geodesics in a four-dimensional curved space with the line element \( ds_4^2 \). The dynamics of well-separated monopoles is completely soluble, but not trivial [3-8]. The problem of geodesic motion in this metric has therefore its own interest, independently of monopole scattering.

In this paper we use similar methods to investigate the geodesic motion in the space of a Kaluza-Klein monopole in the presence of a scalar sigma field coupled to the metric tensor field. It was noted several years ago by Omero and Percacci [9] and Gell-Mann and Zwiebach [10] that the nonlinear sigma-model can be used to induce the space-time compactification in Kaluza-Klein theory. This mechanism presents many interesting features among others being the absence of a cosmological term at the classical level.
The model we shall discuss consists of Einstein gravity in 5 dimensions coupled to a nonlinear sigma-model

$$S = -\frac{1}{16\pi G_k} \int_{M^5} d^5 x \sqrt{-g_5} \left[ R_5 - \frac{2}{\lambda^2} g^{AB} \frac{\partial \Phi}{\partial x^A} \frac{\partial \Phi}{\partial x^B} \right]$$

(1.4)

where $R_5$ is the five-dimensional scalar of the metric $g_{AB}$ of the manifold $M^5$ with the signature $-+++/$. Upper case Latin letters $A, B, C...$ denote five-dimensional indices $0, 1, 2, 3, 5$ and $\lambda^2$ is a constant giving the strength of the self-coupling of the scalar fields.

The corresponding Euler-Lagrange equations are the Einstein equations

$$R_{AB} = \frac{2}{\lambda^2} \frac{\partial \Phi}{\partial x^A} \frac{\partial \Phi}{\partial x^B}$$

(1.5)

and an additional one which prescribes that $\Phi$ is a harmonic map [9,11]. In [11], using the Einstein equations, we came to the conclusion that $\Phi$ is harmonic if the map is a differentiable submersion almost everywhere (see also [12]). Therefore a very general class of solutions of the model is given by submersions $\Phi : M^5 \to B$ satisfying Einstein equations (1.5) where $B$ is the manifold in which the scalar field $\Phi$ takes values [11,13,14].

Recently we shown [15,16] that in the model there exist static solutions of multi-monopole type. Inspired by the previous results [1,2,17] we used the static ansatz (1.1) for the five-dimensional metric. For a static configuration of $N$ monopoles located at $a_l$ we found that $V$ is a function of $r$ through the variable

$$\rho = 1 + \sum_{l=1}^{N} \frac{4m_l}{|r-a_l|}$$

(1.6)

satisfying the following equation

$$\frac{1}{V} \frac{d^2}{d\rho^2} \left( \frac{1}{V} \right) - \left[ \frac{d}{d\rho} \left( \frac{1}{V} \right) \right]^2 + 1 = 0 .$$

(1.7)

This equation can be solved explicitly and the general solutions are

$$V = \frac{\alpha}{\sin(\alpha \rho + \beta)}$$

(1.8)

$$V = \frac{\alpha}{\sinh(\alpha \rho + \beta)}$$

(1.9)

$$V = \frac{1}{\pm \rho + \beta}$$

(1.10)
where $\alpha$, $\beta$ are integration constants and the scalar field $\Phi$ can be determined from the differential equation:

$$\frac{d^2}{d\rho^2} \left( \frac{1}{V} \right) = -\frac{4}{\lambda^2} \frac{1}{V} \left( \frac{d}{d\rho} \Phi \right)^2$$

(1.11).

In the present paper we shall restrict ourselves to the monopole solution ($N = 1$) located at the origin of the system of coordinates ($a_1 = 0$). In the next section we shall investigate the geodesic motion in the proper space of the Kaluza-Klein monopole in the presence of a scalar field. Of course, for a vanishing scalar field (i.e. in the limit $\alpha \to 0, \beta \to 0, \lambda \to 0$ in eqs. (1.8)-(1.11) we recover the standard monopole solution (1.2), (1.3) with the dynamics described in papers [3-8].

In spite of the presence of the scalar field $\Phi$, there are many similarities with the dynamics of the standard monopoles in an euclidean Taub-NUT metric. However there are some notable differences and the most important one is a generalization of the unexpected Runge-Lenz vector which appears in the monopole dynamics. Unlike the Kepler problem, or, the dynamics of a spinless particle on the familiar Taub-NUT gravitational instantons, the orbits are not conic sections.

The paper ends with some concluding comments and with a perspective on various conceivable generalizations.

2. CLASSICAL DYNAMICS

As stated above we shall consider that the space-time compactification is induced by a scalar field in the form of a nonlinear sigma-model coupled to gravity as in eq. (1.4). We shall make the ansatz (1.1) for the metric with the function $V$ having one of the forms (1.8)-(1.10) with

$$\rho = 1 + \frac{4m}{r}$$

(2.1)

corresponding to the monopole configuration (1.2). It is convenient to make the coordinate transformation

$$4m (\psi + \varphi) = -x^5$$

(2.2)
with $0 \leq \psi < 4\pi$, which converts the four-dimensional line element $ds_4$ into
\[
ds_4^2 = V^{-1}(r) \left[ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \ d\varphi^2 \right] + 16m^2 V(r) \left[ d\psi + \cos \theta \ d\varphi \right]^2
\]
\[
= g_{\mu \nu} \, dx^\mu dx^\nu , \quad (\mu, \nu = 1, 2, 3, 5) \tag{2.3}
\]
Spaces with a metric of the form given above have an isometry group $SU(2) \times U(1)$. The four Killing vectors are
\[
\xi_1 = \sin \varphi \frac{\partial}{\partial \theta} + \cos \varphi \cot \theta \frac{\partial}{\partial \varphi} - \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \psi}
\]
\[
\xi_2 = -\cos \varphi \frac{\partial}{\partial \theta} + \sin \varphi \cot \theta \frac{\partial}{\partial \varphi} - \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \psi}
\]
\[
\xi_3 = \frac{\partial}{\partial \varphi}
\]
\[
\xi_5 = \frac{\partial}{\partial \psi} . \tag{2.4}
\]
\xi_5, which generates the $U(1)$ of $\psi$ translations, commutes with the other Killing vectors. In turn the remaining three vectors obey an $SU(2)$ algebra with
\[
\left[ \xi_1 , \xi_2 \right] = -\xi_3 , \text{ etc...} \tag{2.5}
\]
This can be contrasted with the Schwarzschild space-time where the isometry group at spacelike infinity is $SO(3) \times U(1)$. This illustrates the essential topological character of the magnetic monopole mass [18]. On the other hand, it stand to reason that the static metric (1.1) has $\xi_0 = \frac{\partial}{\partial t}$ as an additional Killing vector which generates the $U(1)$ of time translations and commute with the other four Killing vectors (2.4).

The geodesic motion of a spinless particle of unit mass in (2.3) is described by the Lagrangian
\[
L = \frac{1}{2} g^{\mu \nu} \dot{x}_\mu \dot{x}_\nu
\]
\[
= \frac{1}{2} \left[ \frac{1}{V(r)} \dot{r}^2 + (4m)^2 V(r)(\dot{\psi} + \cos \theta \dot{\varphi})^2 \right] \tag{2.6}
\]
where the dot refers to differentiation with respect to proper time.

To the two cyclic variables $\psi$ and $t$ are associated the conserved quantities
\[
q = (4m)^2 V(r) (\dot{\psi} + \cos \theta \dot{\varphi}) \tag{2.7}
\]
\[
E = \frac{1}{2V(r)} \left[ \dot{r}^2 + \left( \frac{q}{4m} \right)^2 \right] = \frac{1}{2} \left[ V(r) \, p^2 + \frac{1}{V(r)} \left( \frac{q}{4m} \right)^2 \right] \tag{2.8}
\]

where
\[
p = \frac{1}{V(r)} \dot{r} \tag{2.9}
\]
is the "mechanical momentum" which is only part of the momentum canonically conjugate to \( r \). For the monopole scattering \( q \) and \( E \) are interpreted as "relative electric charge" and energy, respectively.

The equation of motion for \( p \) is
\[
p = \frac{1}{2} \nabla \left( \frac{1}{V(r)} \right) \left( \dot{r} \cdot \dot{r} \right) - \frac{1}{2} \left( \frac{q}{4m} \right)^2 \nabla \left( \frac{1}{V(r)} \right) - q \frac{\dot{r} \times r}{r^3} \tag{2.10}
\]
which is a complicated equation containing a velocity-squared dependent term, typical for the motion in curved space, plus a Coulomb term plus a Dirac-monopole term. This equation of motion was analyzed in literature [3-8] for the function \( V(r) \) given by eq. (1.3) which corresponds to the Kaluza-Klein monopole in the absence of a scalar field. The analysis of eq. (2.10) was facilitated by the existence of some additional constant of motion.

In what follows we shall show that eq. (2.10) is still tractable in spite of the complexity of the function \( V(r) \) which satisfies eq. (1.7) having one of the forms (1.8) - (1.10).

First of all we shall remark that the angular momentum
\[
j = \dot{r} \times P + q \frac{r}{r} \tag{2.11}
\]
is conserved as in the simplifying case (1.3). Here the first term is the orbital angular momentum and the second is the Poincaré contribution which occurs when the magnetic and electric charges are present [5]. Since these two terms are orthogonal, the magnitude of the orbital angular momentum
\[
l = |l| = |\dot{r} \times P| \tag{2.12}
\]
is also conserved. Eq. (2.11) implies that
\[
j \cdot \frac{r}{r} = q \tag{2.13}
\]
which fixes the relative motion to lie on a cone whose vertex is at the origin, and whose axis is \( j \).

Finally, there is a conserved vector analogous to the Runge-Lenz vector of the Coulomb problem. Its existence is rather surprising in view of the complexity of eq.(2.10). When the scalar field \( \Phi \) is omitted, therefore for \( V(r) \) given by eq. (1.3), this conserved vector is [5]

\[
K = p \times j + \left( \frac{q^2}{4m} - 4mE \right) \frac{r}{r^2} .
\]  

Unfortunately for a function \( V(r) \) as in eqs. (1.8)- (1.9) this simple form turns out to be inadequate. In general, for any central potential, it is possible to construct a constant of motion which generalizes the Runge-Lenz vector of the Coulomb (Kepler) problem. Motivated by the study of Peres [19] we shall construct a vector \( K \) as in eq. (2.14) with some arbitrary functions of \( r \) multiplying the vectors \( p \times j \) and \( r \). But we prefer to avoid this approach which implies some differential equations for these unknown functions and to express the vector \( K \) in terms of \( r, \theta \) and \( \varphi \) [20].

For this purpose let us choose the \( z \) axis along \( j \) so that the motion of the particle may be conveniently described in terms of the polar coordinates

\[
r = r \, e(\theta, \varphi)
\]  

with

\[
e = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)
\]  

and from eq. (2.13)

\[
j \cdot \frac{r}{r} = j \cos \theta = q .
\]  

Having in mind that \( j \) and \( q \) are constants in time, eq. (2.17) implies that \( \theta \) is also constant. For the radial variable \( r \) we have from eqs. (2.7), (2.8), (2.11) and (2.12)

\[
\dot{r}(r) = \frac{dr}{dt} = \left[ 2EV(r) - \frac{l^2V^2(r)}{r^2} - \frac{q^2}{16m^2} \right]^{\frac{1}{2}} .
\]  

The turning points are the roots of the equation \( \dot{r}(r) = 0 \). Assuming that at \( t = 0 \) the particle starts from the turning point \( r_1 \), the time dependence of the motion is given by

\[
t(r) = \int_{r_1}^{r} \frac{d\rho}{\dot{r}(\rho)} .
\]
For the function $\varphi(r)$ it is convenient to evaluate $l^2$ and $j^2$ from eqs.(2.11) and (2.12)

\[
l^2 = \frac{r^4}{V^2(r)} \left( 1 - \frac{q^2}{j^2} \right) \dot{\varphi}^2 \tag{2.20}
\]

\[
j^2 = l^2 + q^2 . \tag{2.21}
\]

These equations have two solutions:

a) The angular momentum vanishes

\[
l = 0 , \quad j = q , \quad \theta = 0 . \tag{2.22}
\]

In this case the energy is

\[
E = \frac{1}{V(r)} \left( \dot{r}^2 + \frac{q^2}{16m^2} \right) , \tag{2.23}
\]

and the motion is restricted to the $j$ axis and the angle $\varphi$ is irrelevant.

b) The angular momentum is nonvanishing

\[
j^2 \sin^2 \theta = l^2 , \quad \theta \neq 0
\]

\[
\dot{j} = \frac{r^2 \dot{\varphi}}{V(r)} \tag{2.24}
\]

and consequently

\[
\varphi(r) = j \int_{r_1}^{r} \frac{V(r)}{\rho^2 \dot{r}(\rho)} d\rho . \tag{2.25}
\]

Finally, the velocity vector can be express as

\[
\dot{\mathbf{r}} = \dot{\mathbf{r}} \mathbf{e} + r \dot{\varphi} \mathbf{e}' = \dot{\mathbf{r}} \mathbf{e} + \frac{jV(r)}{r} \mathbf{e}' \tag{2.26}
\]

where

\[
\mathbf{e}' = \frac{d\mathbf{e}}{d\varphi} = (-\sin \theta \sin \varphi , \sin \theta \cos \varphi , 0) . \tag{2.27}
\]

With these preparatives the generalized Runge-Lenz vector can be written in a local rotating basis ($\mathbf{e}(\varphi)$, $\mathbf{e}'(\varphi)$, $\mathbf{j}$) as

\[
\mathbf{K} = X_1 \left( \mathbf{e} - \cos \theta \frac{\mathbf{j}}{j} \right) + X_2 \mathbf{e}' . \tag{2.28}
\]

$\mathbf{K}$ will remain constant in the laboratory frame if it will rotate in the opposite direction with respect to its local basis [20]
\[ X_1 = X_0 \cos(\varphi - \varphi_0) \]
\[ X_2 = -X_0 \sin(\varphi - \varphi_0) \]  \hspace{1cm} (2.29)

where \( X_0, \varphi_0 \) are constants which can be chosen as one wishes.

Thus we have constructed the vector \( \mathbf{K} \) which is constant in time, for the motion corresponding to the lagrangian (2.6) with an arbitrary function \( V(r) \) satisfying eq.(1.7). The function \( \varphi(r) \) occurring in its expression is given explicitly in terms of \( V(r) \) by eqs. (2.25) and (2.18).

Of course it is possible to express the generalized Runge-Lenz vector (2.27) in a form similar to eq.(2.14) with some functions depending of \( r \) as coefficients. To put into practice this task proves to be quite involved and by no means illuminating.

More important it is to note that the property of being constant in time is not sufficient for \( \mathbf{K} \) to be an integral of the motion. It must also be a one-valued function of the state of the particle [20,21]. Suppose that for a set of constants \( q, l, e \) in eq. (2.18), \( r \) varies between the endpoints \( r_1, r_2 \) which are the roots of the equation \( \dot{r}(r) = 0 \). Let us define the interval \( \tau \) as time during which \( r \) varies from \( r_1 \) to \( r_2 \) and back. During time \( \tau \) the phase \( \varphi \) increases with \( \Delta \varphi \). From eq.(2.29) the sufficient condition for \( \mathbf{K} \) to be a one-valued function is

\[ \Delta \varphi = 2\pi k , \quad k = 1, 2, 3, ... \]  \hspace{1cm} (2.30)

The orbit of the motion satisfying the above condition form a closed path and the period \( T \) of the motion is just the interval \( \tau \). Closed paths are possible not only under the above condition, but also in the more general case

\[ \Delta \varphi = 2\pi \frac{k}{n} \]  \hspace{1cm} (2.31)

with natural numbers \( k \) and \( n \). In this case the period of the motion is \( T = n\tau \). In general, the conditions (2.30), (2.31), with natural \( k \) and \( n \) may be met for orbits with some particular values of \( E, j, l \). Therefore, contrary to the Kepler (Coulomb) potential [21], or pure Kaluza-Klein monopole [1] only in some particular cases the orbits are closed paths and the vector \( \mathbf{K} \) (2.28) may serve as an integral of motion.
3. CONCLUDING REMARKS

The nonlinear sigma-model in curved space proved to be an interesting mechanism for space-time compactification in Kaluza-Klein theories. The compactified space becomes isomorphic to the manifold in which the scalar fields take values and the four-dimensional space has no cosmological term at the classical level.

The presence of the nonlinear sigma-model coupled to gravity does not hamper the existence of static solutions of multi-monopole type. However, the dynamics in this case is more involved with some notable differences. The most important one refers to the conserved vector $K$ analogous to the Runge-Lenz vector of the Coulomb problem. In contrast with the pure Kaluza-Klein theory, without sources, the generalized Runge-Lenz vector is not in general a one-valued function of the state of the particle. For the vector $K$ given by eq. (2.14), we have

$$\left[ K + \frac{4m}{q} \left( E - \frac{q^2}{16m^2} \right) j \right] \cdot \frac{r}{r} = j^2 - q^2$$

and consequently the relative motion is in a fixed plane. Combining this result with the fact that the relative motion is on a cone whose axis is $j$ one gets that the orbits are conic sections. When the sigma fields are present, an equation similar to (3.1), with a constant in the right hand side, cannot be established. The relative motion remains on a cone, but not restricted to a plane. Moreover, in general, the orbits are not closed paths.

The quantum dynamics of the monopoles in sourceless five-dimensional general relativity was studied using various methods [5,6,8,22,23]. Even in the pure Kaluza-Klein model the quantization of the Runge-Lenz vector $K$ is a hard task [6,24]. We expect that the quantization of the monopole system in the presence of a sigma-model to be more involved. The classical trajectories are not periodic in general and new delicate problems arise [25].

An interesting problem is to consider the motion of spinning particles in curved space-time. We intend to extend the spaces considered previously with additional fermionic dimensions, parametrised by vectorial Grassmann co-ordinate $\Psi^\mu$ [26]. The relation between symmetries of the graded manifolds and constant of motion for spinning particles is more complicated than in the case of scalar point particles.

These generalizations will be presented elsewhere [27].
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