Interferometric detection of Chern numbers of Haldane model on bosonic optical lattices

JIAN XU

Science School, Guangdong Ocean University - Zhanjiang 524088, China and
Guangdong Provincial Key Laboratory of Quantum Engineering and Quantum Materials and School of Physics and Telecommunication Engineering, South China Normal University - Guangzhou 510006, China

received 14 August 2015; accepted in final form 9 November 2015
published online 30 November 2015

PACS 67.85.-d – Ultracold gases, trapped gases
PACS 03.65.Vf – Phases: geometric; dynamic or topological
PACS 03.75.Bg – Atom and neutron interferometry

Abstract – Topological states of matter emerge as a new type of quantum phases, which can be distinguished by their associated topological invariants, e.g., Chern numbers. Currently, there has been increasing interest toward the physical detection of the new predicted topological phases. Here, we propose an interferometric approach to directly measure the Chern number of the topological Haldane model in bosonic optical lattices via detecting the associated Zak phases. We show that this interferometric approach can distinguish Zak phases of $\pm 2\pi$ from 0 in the first Brillouin zone, and thus it provides a new tool to directly detect the Chern number of topological systems. In addition, we demonstrate that this method is feasible under realistic experimental conditions and may generalize to detect topological systems with higher Chern numbers.

Introduction. – Topology of quantum systems is a key concept in modern quantum many-body physics. For example, the quantum Hall effect has attracted much attention since it was discovered [1,2], of which the Hall conductance is proportional to the Chern number (CN) $C$ defined as the integral of Berry curvature over the first Brillouin zone (BZ) [3,4]. Specially, Haldane showed that the quantum Hall effect can exist without an external magnetic field and associated Landau levels [5]. Meanwhile, it has also been found important application in topological quantum computation [6–10]. On the other hand, the system with cold atoms in optical lattices is a powerful tool to simulate strongly correlated many-body models [11]. Recently, synthetic gauge fields and spin-orbit coupling have been experimentally realized in cold atomic gases [12]. These new developments allow one to study many exotic topological phases in condensed-matter physics using optical lattices. As remarkable examples, the implementation of the spin Hall effect are presented in cold atoms [13–15], the Haldane model in cold atom system has been theoretically proposed [16,17]. More recently, many remarkable progresses in the experimental realization of topological models have emerged. For example, the Haldane model has been experimentally realized in optical lattices and its topological band structure has also been characterised [18]. And then, the chiral currents [19] and chiral edge states [20] with ultracold atoms in optical ladders have been observed successively. Most recently the experimental realizations of the Thouless Quantum Pump in the lowest-energy [21] band and in the first-excited band [22] have been reported at about the same time.

The topological phases of many-body systems can be characterized by the CN, which can be manifested with the Hall conductance. Due to the absence of local orders, topological phases are typically hard to detect. Unlike condensed-matter systems where a routine measurement of the Hall conductance reveals the CN, the quantized Hall conductance is extremely difficult to be realized. Therefore, nowadays, measuring the Hall conductance is technically unrealistic for cold atoms in optical lattices. On the other hand, alternative strategies for detecting the CN include measuring the time evolution of the center of mass [23], the gapless edge modes [24–26], the pumped charge [27], the Landau-Zener-Stückelberg tunneling [28] and the bulk Chern number from Berry’s curvature over the BZ [29–31]. However, these methods either rely on the weak experimental signals or depend on complicated manipulations/measurements on the whole bulk band. For example, in ref. [31], the BZ is divided into many small
areas, and the Berry curvature of each area is measured separately. But, near the Dirac point, the nonadiabatic transition due to the small energy gap leads to experimental difficulties. Therefore, it is still challenging for the delicate cold atom systems [32]. Nevertheless, ref. [32] demonstrated that it can be inferred by measuring the energy spectrum at highly symmetric points of the BZ. Furthermore, it can be demonstrated that the method can distinguish the case of CN=1 and CN=0, but it cannot distinguish the case from CN=2 and CN=0. Although one cannot build a cold atom system with strict Hall conductance, a similar scheme detecting the CN by measuring the anomalous velocity has been experimentally realized [33]. It is well known that the CN is not only related to the Hall conductance but also the Zak phase [38], which can be realized by a combination of the gradient metric phase of the closed trajectory can be obtained by several gauge-invariant open trajectories, the total geometric phase of each area is measured. In this model, due to the fact that the geometric phase of the closed trajectory leads to experimental progress [18], we consider a spin-independent topological Haldane model. For bosons in optical lattices [35], which can be realized by a combination of the gradient magnetic field [36] and the accelerating optical lattice [37]. Based on the associated noncyclic Zak phases [38], we propose a interferometric scheme to detect the ±π phase when the trajectories inclose the Dirac points through measuring the geometric phase of half of a Dirac point. In our scheme, unlike previous ones [31], we can distinguish the ±π phases and avoid the problem of nonadiabatic transitions. In particular, based on this distinction, we provide a method to directly detect the CN in cold atom systems. In addition, our scheme can also realize a complex operation, e.g., one spin state is pinned and the other spin state is moved to obtain a Zak phase. Therefore, it provides a new possible idea to detect the \( \mathbb{Z}_2 \) topological invariants [39–41].

**Haldane model and its CN.** Based on the recently experimental progress [18], we consider a spin-independent topological Haldane model. For bosons in optical lattices with two sublattices, the two sublattices constitute the two pseudo-spin components of the Hamiltonian. In momentum space, the Hamiltonian of this model is described by

\[
H = H_0(k)I + H_x(k)\sigma_x + H_y(k)\sigma_y + H_z(k)\sigma_z, \tag{1}
\]

with

\[
H_0 = -2t' \cos \phi \sum_i \cos(k \cdot \bar{v}_i),
\]

\[
H_x = -2t' \sin \phi \sum_i \sin(k \cdot \bar{v}_i), \tag{2}
\]

\[
H_y = -t \sum_i \cos(k \cdot \bar{v}_i),
\]

\[
H_z = -t \sum_i \sin(k \cdot \bar{v}_i),
\]

where \( \bar{v}_1 = (0, a) \), \( \bar{v}_2 = (-\sqrt{3}a/2, -a/2) \) and \( \bar{v}_3 = (\sqrt{3}a/2, -a/2) \) are the lattice vectors, \( \bar{v}_1' = (\sqrt{3}a, 0) \), \( \bar{v}_2' = (-\sqrt{3}a/2, 3a/2) \) and \( \bar{v}_3' = (-\sqrt{3}a/2, -3a/2) \) are the vectors of next-nearest-neighbor hopping, \( a \) is the basis vector of lattices, \( t, t' \) and \( \phi \) are adjustable parameters. It is well known that the reciprocal lattice vectors of honeycomb lattice are \( \bar{b}_1 = (0, 4\pi/(3a)) \) and \( \bar{b}_2 = (2\pi/(\sqrt{3}a), -2\pi/(3a)) \). There are two types of Dirac points \( K \) and \( K' \) where \( K(K') = \pm(4\pi/(\sqrt{3}a), 0) \) in this model. Each Dirac point has ±π Berry’s phase, the sign of which, negative or positive, is determined by the system’s parameters. The definition of the CN is \( C = \frac{1}{2\pi} \int_{BZ} \Omega ds \) with \( \Omega = i\nabla \times \langle u(k)|\partial_k|u(k)\rangle \) being the Berry curvature. For the Haldane model, we find that the CN is

\[
C = \begin{cases} 
1, & \text{for } \phi \in (0, \pi), \\
0, & \text{for } \phi = 0, \\
-1, & \text{for } \phi \in (-\pi, 0). 
\end{cases} \tag{3}
\]

For the Haldane model, as the Berry curvatures only distribute within the area of the Dirac cones due to the power law distribution form [42], so an area that is bigger than that of the Dirac cone, which is generally tens of that of the first BZ, is enough to achieve an approximate integer quantization geometric phase. Thus, the definition of the CN can be approximated by \( C = \frac{1}{2\pi} \int_{S} \Omega ds \), where \( S \) is an area including all Dirac cones within the first BZ. We next link the CN to the geometric phase, i.e., the Zak phase [34], of an open trajectory in the first BZ. In this model, due to the fact that the geometric phase of the corresponding CN is 2πC, the interferometry cannot distinguish this phase in the case of a closed trajectory \( S \). However, if we divided the closed trajectory \( S \) into several gauge-invariant open trajectories, the total geometric phase of the closed trajectory can be obtained by adding those of open trajectories together. In addition, the geometric phase of an open trajectory can be measured through the atomic interferometry. Specially, the evolution trajectories of the wave function \( u(k) \) with quasi-momentum \( k \) and the straight trajectory between the end points \( k_b \) and \( k_c \) of the trajectories are the combination of the reciprocal lattice vectors \( \bar{b}_{1,2} \), as shown in fig. 1. The noncyclic Zak phase of an arbitrary trajectory \( \phi_{Zak}(C) \) can be approximated as [38]

\[
\phi_{Zak}(C) = \int_{S} \Omega ds - \int_{k_b}^{k_c} \langle u(k)|\partial_k|u(k)\rangle ds, \tag{4}
\]

where \( \int_{S} ds \) is the integral over the closed trajectory, constituted by \( u(k) \) and the straight trajectory connecting \( k_b \) and \( k_c \) points, and \( \int_{k_b}^{k_c} ds \) is the integral along the straight trajectory connecting the \( k_b \) and \( k_c \) points. Moreover, due to the mirror symmetry of this model with \( k_b = 0 \) being the axis of symmetry, one has \( \langle u(k_x, k_y) | \partial_{k_y} | u(k_x, k_y) \rangle = \langle u(k_x, -k_y) | \partial_{k_y} | u(k_x, k_y) \rangle = -\langle u(k_x, -k_y) | \partial_{k_y} | u(k_x, -k_y) \rangle \). Therefore, the second term
momentum space is described by [36]

$$\hbar \frac{dk}{dt} = \mp \mu_1, \mp \nabla B,$$

where $\mu_1 = \mu$ are the spin magnetic moments and $B$ is the magnetic field. On the other hand, if one properly modulates the lasers, the optical potential is accelerated in the stationary frame. Choosing the optical lattice as the reference frame, the optical potential is stationary and the atoms obtain an inertia force $F$. When a constant force $F$ is produced in the same direction for the two spin components, this result in momentum space is of the form [38]

$$\hbar \frac{dk}{dt} = -a_FM,$$

where $a_F$ is the acceleration and $M$ is the mass of the atom. Finally, a combination of the magnetic-field gradient and the accelerating optical lattice allows us to arbitrarily control over the evolution of two spin components in momentum space. Experimentally, the characteristic energies of the magnetic-field gradient and accelerating lattice are at least an order of magnitude smaller than the full bandwidth of the optical lattice, hence they do not transfer atoms from the lower band to the upper band.

To detect the CN, a bosonic gas with the spin state of $|\downarrow\rangle$ is initially prepared at the bottom of the energy band. Within the current technology, one can precisely control an atomic gas to move to anywhere over the BZ [43]. This detection needs one $\pi$-pulse and two $\pi/2$-pulses, where the $\pi$-pulse is equivalent to the $\sigma_z$ operation for the atomic spin states $|\uparrow\rangle$ and $|\downarrow\rangle$ and the $\pi/2$-pulse is an operation in the form of

$$U_{\pi/2,\phi} = \begin{pmatrix} 1 & e^{i\phi} \\ ie^{-i\phi} & 1 \end{pmatrix},$$

with $\phi$ being the controllable phase of the $\pi/2$-pulse.

The detection goes as follows. We first consider the detection of the contribution from the left Dirac point, as shown in fig. 1(a). A magnetic-field gradient in the $k_x$ direction is applied to create a constant force for the bosonic atoms. Such a constant force leads to a linear evolution of quasimomentum over time, i.e., the Bloch oscillation. When the spin state $|\downarrow\rangle$ moves to the site $(k_x = 2\pi/(3\sqrt{3}a), k_y = 0)$ of the BZ, one uses a $\pi/2$-pulse to bring the state $|\downarrow\rangle$ into a coherent superposition state of $|\downarrow\rangle$ and $|\uparrow\rangle$ using a $\pi/2$-pulse.

Detection of the CN. – For cold atom systems, to create an arbitrary interference trajectory in momentum space, one must control the relative movement between two parts of cold atoms. In contrast to the pseudo-spin of the Haldane model, we choose the hyperfine states of the bosonic atom as the manipulative spins $|\uparrow\rangle$ and $|\downarrow\rangle$ in this work. We start with a bosonic gas in the state $|\downarrow\rangle$ and bring it into a coherent superposition state $|\sqrt{2}(|\downarrow\rangle + |\uparrow\rangle)|$ using a $\pi/2$ Raman pulse. Then a magnetic-field gradient is applied to create a constant force in opposite directions for the two spin components. The moving of atoms in momentum space is described by [36]

$$\hbar \frac{dk}{dt} = \mp \mu_1, \mp \nabla B,$$

where $\mu_1 = \mu$ are the spin magnetic moments and $B$ is the magnetic field. On the other hand, if one properly modulates the lasers, the optical potential is accelerated in the stationary frame. Choosing the optical lattice as the reference frame, the optical potential is stationary and the atoms obtain an inertia force $F$. When a constant force $F$ is produced in the same direction for the two spin components, this result in momentum space is of the form [38]

$$\hbar \frac{dk}{dt} = -a_FM,$$

where $a_F$ is the acceleration and $M$ is the mass of the atom. Finally, a combination of the magnetic-field gradient and the accelerating optical lattice allows us to arbitrarily control over the evolution of two spin components in momentum space. Experimentally, the characteristic energies of the magnetic-field gradient and accelerating lattice are at least an order of magnitude smaller than the full bandwidth of the optical lattice, hence they do not transfer atoms from the lower band to the upper band.

To detect the CN, a bosonic gas with the spin state of $|\downarrow\rangle$ is initially prepared at the bottom of the energy band. Within the current technology, one can precisely control an atomic gas to move to anywhere over the BZ [43]. This detection needs one $\pi$-pulse and two $\pi/2$-pulses, where the $\pi$-pulse is equivalent to the $\sigma_z$ operation for the atomic spin states $|\uparrow\rangle$ and $|\downarrow\rangle$ and the $\pi/2$-pulse is an operation in the form of

$$U_{\pi/2,\phi} = \begin{pmatrix} 1 & e^{i\phi} \\ ie^{-i\phi} & 1 \end{pmatrix},$$

with $\phi$ being the controllable phase of the $\pi/2$-pulse.

The detection goes as follows. We first consider the detection of the contribution from the left Dirac point, as shown in fig. 1(a). A magnetic-field gradient in the $k_x$ direction is applied to create a constant force for the bosonic atoms. Such a constant force leads to a linear evolution of quasimomentum over time, i.e., the Bloch oscillation. When the spin state $|\downarrow\rangle$ moves to the site $(k_x = 2\pi/(3\sqrt{3}a), k_y = 0)$ of the BZ, one uses a $\pi/2$-pulse to bring the state $|\downarrow\rangle$ into a coherent superposition state of $|\downarrow\rangle$ and $|\uparrow\rangle$ using a $\pi/2$-pulse.

Detection of the CN. – For cold atom systems, to create an arbitrary interference trajectory in momentum space, one must control the relative movement between two parts of cold atoms. In contrast to the pseudo-spin of the Haldane model, we choose the hyperfine states of the bosonic atom as the manipulative spins $|\uparrow\rangle$ and $|\downarrow\rangle$ in this work. We start with a bosonic gas in the state $|\downarrow\rangle$ and bring it into a coherent superposition state $|\sqrt{2}(|\downarrow\rangle + |\uparrow\rangle)|$ using a $\pi/2$ Raman pulse. Then a magnetic-field gradient is applied to create a constant force in opposite directions for the two spin components. The moving of atoms in
case of the right Dirac point, as shown in fig. 1(b). Atoms are now moved to the site \( (k_x = -2\pi/(3\sqrt{3}a), k_y = 0) \). A coherent superposition state of \( \{|\uparrow\rangle + i|\downarrow\rangle\}/\sqrt{2} \) is created again. After that, under the action of the accelerating optical lattice and the gradient magnetic field with \( |aM/(\mu vB)| = \sqrt{3} \) their directions are \( +x \) and \( -y \), respectively. In this process, the movements of \( |\uparrow\rangle \) and \( |\downarrow\rangle \) are exact \( b_1 + b_2 \) and \( b_1 \) and the superposition state evolves to the site \( (k_x,1,1) = 4\pi/(3\sqrt{3}a), k_y,1,1 = \pm 2\pi/(3a)) \).

We next link this phase to a physical observable quantity of atomic spin states in our considered context. For ultracold atoms, bosons in the momentum space are usually considered to be point particles, \textit{i.e.}, the size of which is much smaller than that of Berry’s curvature of a Dirac point, and thus there is not a Zak phase difference between the bosons with same spin states. For this reason, eq. (4) for a single particle can be directly applied to the case of systems with many bosons as demonstrated in a recent experiment [43]. For the above-considered trajectory, the total phase difference between the atomic spin state \( |\downarrow\rangle \) and \( |\uparrow\rangle \) is \( \phi_{\text{tot}} = \phi_{\text{Zak}} + \phi_D \), where \( \phi_D = \int \delta E/\hbar dt \) is the dynamic phase with \( \delta E \) being the energy difference between two spin states and \( \phi_{\text{Zak}} \) is due to the Zeeman energy of the atoms in an external magnetic field. Due to the symmetry of the energy band and the trajectory, the dynamic energies of two spin states are always the same, and thus the dynamic phase is 0. In addition, \( \phi_{\text{Zak}} \) can be eliminated by a \( \pi \)-pulse, which flips the spin, in the middle of one straight trajectory while one keeps the direction of the accelerating optical lattice but changes that of the magnetic-field direction. Therefore, in this way, the Zak phase is the only contributed component for the total phase. Therefore, the wave function has evolved to \( |\psi(k_x)\rangle = e^{i\phi_{\text{Zak}}}|\psi(k_x)\rangle \). To detect the noncyclic Zak phase difference between two spin states, the second \( \pi/2 \)-pulse is applied to the atoms and the atom number in the two spin states \( N_{\uparrow,\downarrow} \) are found to be

\[
N_{\uparrow,\downarrow} = \frac{1}{2}[1 \pm \cos(\phi_{\text{Zak}} - \phi)]. \tag{10}
\]

When the atoms evolved to the specific sites I and II, the system obtains \( \pm \pi/2 \) or \( \pm 3\pi/2 \) noncycle Zak phases, and \( N_{\uparrow,\downarrow} = 1/2 \) without \( \phi \). The correspondence between the measured atomic population and the Zak phases for the two noncyclic trajectories is listed in table 1.

**Discussions.** — Although the atom numbers \( N_{\uparrow} \) without \( \phi \) in the different sites are the same \( 1/2 \), the detections should display different results in the presence of \( \phi \). If the geometrical phases of two Dirac points have different signs, which cancel each other, the total Zak phase is 0 and \( C = 0 \). In this case, the two Zak phases \( \phi_{\text{Zak}}^1 \) and \( \phi_{\text{Zak}}^\downarrow \) must be obtained as different values. Therefore, one just need to measure the Zak phase in these special sites for detecting the CN. In contrast to the case of \( C = 0 \), we next show that the detection display different properties when \( C = \pm 1 \). Because of the same geometrical phases of two Dirac points, the atoms should obtain the same Zak phases in these two special sites, that is, if the Zak phase \( \phi_{\text{Zak}}^1 \) is \( \pi/2 \), \( \phi_{\text{Zak}}^\downarrow \) must be \( \pi/2 \) for \( C = 1 \). Moreover, the absolute values of the Zak phase in the case of \( C = -1 \) are the same as that of the \( C = 1 \) case but with opposite sign. In other words, the different functions of \( \phi \) offer a signal of the topologically nontrivial phase characterized by a nonzero CN. Plus or minus of the CN can also be distinguished by comparing the different functions of \( \phi \) in the different sites. In short, this method can completely distinguish the CN for the considered model.

Experimentally, although one can move atoms to a precise place of the BZ using the current techniques, let us suppose that the atoms are not moved to the sites \( (k_x,1,1) = k_x,1,1 = \pm 4\pi/(3\sqrt{3}a), k_y,1,1 = -k_y,1,1) \), the changing trend of \( N_{\uparrow} \) with \( \phi \) is still distinguishable as long as the error of the Zak phase is within \( \pm \pi/4 \), \textit{i.e.}, the \( N_{\uparrow} \) without \( \phi \) is larger than 1/\( \sqrt{2} \). Specifically we suppose that there is an uncertainty in the determination of \( k_x \) and \( k_y \), which means \( \pm 4\pi/(3\sqrt{3}a) + \delta \) and \( \delta \) is the uncertainty in the \( k_x \) direction. With the data, we deduce what the value range of \( \delta \) is. For the sake of discussion and without loss of generality, let us suppose that the site of a Dirac point is the origin of coordinates and the sign of the Berry curvature is avoided. Hence the Berry curvature near the Dirac point has a simple form [42]:

\[
|\Omega| = \frac{2e^2\Delta}{4\epsilon^2(k_x^2 + k_y^2) + \Delta^{|3/2|}}.
\tag{11}
\]

with the slope of the Dirac cone \( c = 3at/2 \) and the energy gap \( \Delta = 2H_z \tau = 6\sqrt{3}\epsilon \sin \phi \) due to \( \partial H_z/\partial \phi \ll \partial H_x,y/\partial \phi \). Our method needs the error of the Zak phase to be smaller than \( \pi/4 \), \textit{i.e.}, \( \pi/4 < |\int_{-\infty}^{+\infty} \Omega dk_y dk_z| < 3\pi/4 \). \textit{Videlicet}, we can find the error \( \delta \) as

\[
|\delta| < \frac{2\sqrt{3}\epsilon}{at} \sin \phi. \tag{12}
\]

Comparing with the size of the first BZ, we can find that the requirement of accuracy control over the trajectories is not rigorous. On the other hand, analogues of the method may be applicable in systems with higher CN under the following conditions: 1) The Berry curvature distributes in a small region, \textit{e.g.}, within the Dirac cones; and 2) the geodesic of the needed open trajectories can be easily find out.

---

**Table 1:** The Zak phase \( \phi_{\text{Zak}} \) and the atom number of the spin up \( N_{\uparrow} \) in the different sites for different Chern numbers with \( \alpha_z = |1 - \cos(\pi/2 \pm \phi)|/2 \).

| \( \phi_{\text{Zak}} \) | \( N_{\uparrow} \) | \( \phi_{\text{Zak}} \) | \( N_{\uparrow} \) |
|---|---|---|---|
| 0 | \( \pi/2 \) | \( \alpha_- \) | \( -\pi/2 \) | \( \alpha_+ \) |
| 1 | \( -\pi/2 \) | \( \alpha_+ \) | \( \pi/2 \) | \( \alpha_- \) |
| -1 | \( -\pi/2 \) | \( \alpha_- \) | \( -\pi/2 \) | \( \alpha_+ \) |
Conclusion. — We propose to detect the CN of the topological Haldane model in bosonic optical lattices by measuring the atomic population with $\phi$ in some special sites of the BZ, which is connected with the Dirac points. We show the atomic population as a function of the phase of the final microwave $\pi/2$-pulse and how to distinguish the CN from these functions in the different sites. We further show that this method is experimentally realizable. Therefore, we provide a method to detect topological phases in the cold atom context and a new possible idea to detect the $\mathbb{Z}_2$ topological systems.

***

We thank Prof. Shi-Liang Zhu and Dr. Zheng-Yuan Xue for many helpful discussions.

REFERENCES

[1] V. Klitzing K., Dorda G. and Pepper M., Phys. Rev. Lett., 45 (1980) 494.

[2] Tsui D. C., Stormer H. L. and Gossard A. C., Phys. Rev. Lett., 48 (1982) 1559.

[3] Laughlin R. B., Phys. Rev. B, 23 (1981) 5632.

[4] Thouless D. J., Kohmoto M., Nightingale M. P. and den Nijs M., Phys. Rev. Lett., 49 (1982) 405.

[5] Haldane F. D. M., Phys. Rev. Lett., 61 (1988) 2015.

[6] Nayak C., Simon S. H., Stern A., Freedman M. and Sarma S. D., Rev. Mod. Phys., 80 (2008) 1083.

[7] Alicea J., Rep. Prog. Phys., 75 (2012) 076501.

[8] Xue Z.-Y., Shao L. B., Hu Y., Zhu S.-L. and Wang Z. D., Phys. Rev. A, 88 (2013) 024303.

[9] Xue Z.-Y., Eur. Phys. J. D, 67 (2013) 89.

[10] Xue Z.-Y., Gong M., Liu J., Hu Y., Zhu S.-L. and Wang Z. D., Sci. Rep., 5 (2015) 12233.

[11] Bloch I., Dalibard J. and Zwerger W., Rev. Mod. Phys., 80 (2008) 885.

[12] Dalibard J., Gerber F., Juzeliūnas G. and Ōhberg P., Rev. Mod. Phys., 83 (2011) 1523.

[13] Zhu S.-L., Fu H., Wu C.-J., Zhang S.-C. and Duan L.-M., Phys. Rev. Lett., 97 (2006) 240401.

[14] Beeler M. C., Williams R. A., Jiménez-García K., LeBlanc L. J., Perry A. R. and Spielman I. B., Nature, 498 (2013) 201.

[15] Kennedy C. J., Sivoglou G. A., Miyake H., Burton W. C. and Ketterle W., Phys. Rev. Lett., 111 (2013) 225301.

[16] Shao L.-B., Zhu S.-L., Sheng L., Xing D.-Y. and Wang Z.-D., Phys. Rev. Lett., 101 (2008) 246810.

[17] Struck J., Ölschläger C., Weinberg M., Hauke P., Simonet J., Eckardt A., Lewenstein M., Sengstock K. and Windpassinger P., Phys. Rev. Lett., 108 (2012) 225304.

[18] Jotzu G., Messer M., Desbuquois R., Lebrat M., Uehlinger T., Greif D. and Esslinger T., Nature, 515 (2014) 237.

[19] Atala M., Aidelsburger M., Lohse M., Barreiro J. T., Paredes B. and Bloch I., Nat. Phys., 10 (2014) 585.

[20] Mancini M., Pagano G., Cappellini G., Livi L., Rider L., Catani J., Sias C., Zoller P., Inguscio M., Dalmonte M. and Fallani L., arXiv:1502.02495 (2014).

[21] Nakajima S., Tomita T., Tae S., Ichinose T., Ozawa H., Wang L., Trover M. and Takahashi Y., arXiv:1507.02223 (2015).

[22] Lohse M., Schweizer C., Zilberberg O., Aidelsburger M. and Bloch I., arXiv:1507.02225 (2015).

[23] Dauphin A. and Goldman N., Phys. Rev. Lett., 111 (2013) 135302.

[24] Goldman N., Brugnon J. and Gerber F., Phys. Rev. Lett., 108 (2012) 255303.

[25] Buchhold M., Cocke D. and Hofstetter W., Phys. Rev. A, 85 (2012) 063614.

[26] Goldman N., Dalibard J., Dauphin A., Gerber F., Lewenstein M., Zoller P. and Spielman I. B., Proc. Natl. Acad. Sci. U.S.A., 110 (2013) 6736.

[27] Wang L., Soluyanov A. A. and Troyer M., Phys. Rev. Lett., 110 (2013) 166802.

[28] Lim L.-K., Fuchs J.-N. and Montambaux G., Phys. Rev. Lett., 112 (2014) 155302.

[29] Alba E., Fernandez-Gonzalvo X., Mur-Petit J., Pachos J. K. and Garcia-Ripoll J. J., Phys. Rev. Lett., 107 (2011) 235301.

[30] Price H. M. and Cooper N. R., Phys. Rev. A, 85 (2012) 033620.

[31] Abanin D. A., Kitaeva T., Bloch I. and Demler E., Phys. Rev. Lett., 110 (2013) 165304.

[32] Liu X.-J., Law K. T., Ng T. K. and Lee P. A., Phys. Rev. Lett., 111 (2013) 120402.

[33] Aidelsburger M., Lohse M., Schweizer C., Atala M., Barreiro J. T., Nasimbine S., Cooper N. R., Bloch I. and Goldman N., Nat. Phys., 11 (2015) 162.

[34] Zak J., Phys. Rev. Lett., 69 (1987) 2747.

[35] Duca L., Li T., Reitter M., Bloch I., Schleier-Smith M. and Schneider U., Science, 347 (2015) 288.

[36] Tarruell L., Greif D., Uehlinger T., Jotzu G. and Esslinger T., Nature, 483 (2012) 302.

[37] Madison K. W., Fischer M. C. and Raizen M. G., Phys. Rev. A, 60 (1999) R1767.

[38] Zak J., Europhys. Lett., 9 (1989) 615.

[39] de Lisle J., de S., Alba E., Bullivant A., Garcia-Ripoll J. J., Laitinen V. and Pachos J. K., New J. Phys., 16 (2014) 083022.

[40] Grusdt F., Abanin D. and Demler E., Phys. Rev. A, 89 (2014) 043621.

[41] Wang S.-T., Deng D.-L. and Duan L.-M., Phys. Rev. Lett., 113 (2014) 033002; Deng D.-L., Wang S.-T., Sun K. and Duan L.-M., Phys. Rev. B, 91 (2015) 094513.

[42] Xiao D., Yao W. and Ni Q., Phys. Rev. Lett., 99 (2007) 230609.

[43] Atala M., Aidelsburger M., Barreiro J. T., Abanin D., Kitaev T., Demler E. and Bloch I., Nat. Phys., 9 (2013) 795.