HOMOGENEOUS 2-NONDEGENERATE CR MANIFOLDS OF HYPERSURFACE TYPE IN LOW DIMENSIONS

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Abstract. In a recent paper, the author and I. Zelenko introduce the concept of modified CR symbols for organizing local invariants of 2-nondegenerate CR structures. In this paper, we consider homogeneous hypersurfaces in $\mathbb{C}^4$, a natural frontier in the CR hypersurface Erlangen program, and classify up to local equivalence the locally homogeneous 2-nondegenerate hypersurfaces in $\mathbb{C}^4$ whose symmetry group dimension is maximal among all such structures with the same local invariants encoded in their respective modified symbols. In the considered dimension, we show that among homogeneous structures with given modified CR symbols, the most symmetric structures (termed model structures) are unique. The classification is then achieved indirectly through classifying the modified symbols of homogeneous hypersurfaces in $\mathbb{C}^4$, obtaining (up to local equivalence) nine model structures. The methods used to obtain this classification are then applied to find homogeneous hypersurfaces in higher dimensional spaces. In total, $20$ locally non-equivalent maximally symmetric homogeneous 2-nondegenerate hypersurfaces are described in $\mathbb{C}^5$, and $40$ such hypersurfaces are described in $\mathbb{C}^6$, of which some have been described in other works while many are new. Lastly, two new sequences, indexed by $n$, of homogeneous 2-nondegenerate hypersurfaces in $\mathbb{C}^{n+1}$ are described. Notably, all examples from one of these latter sequences can be realized as left-invariant structures on nilpotent Lie groups.

1. Introduction

This article’s main result (Theorems 3.1 and 3.4) is a classification up to local equivalence of 2-nondegenerate real hypersurfaces in $\mathbb{C}^4$ that are locally equivalent to homogeneous CR manifolds whose symmetry groups have maximal dimension relative to the local invariants encoded in their modified CR symbols, which comprise sets that are local invariants introduced in [26]. In the sequel we refer to these most symmetric structures as modified symbol models (or shortly models). There are many more homogeneous structures than models (e.g., uncountably many homogeneous non-model structures are described by the formulas in [15, Theorem 1.3, with $n = 3$]), but these models occupy an important role in the general study of homogeneous structures due to a fundamental relationship given in Corollary 3.2 between general homogeneous 2-nondegenerate hypersurfaces in $\mathbb{C}^4$ and the model structures classified here. Broadly, homogeneous CR manifolds have been studied in several works, including [11, 2, 3, 8, 11, 18, 19, 20, 22, 23, 26], in part for their thematic role in classical differential geometric treatments of local equivalence problems, acting as prototypical structures of which general structures are described as generalizations or deformations, as in [6, 16] for example.

In particular, in [16], Kolář–Kossovskiy prescribe a complete normal form for 2-nondegenerate real hypersurfaces in $\mathbb{C}^3$, describing them as deformations of the thoroughly studied maximally symmetric 2-nondegenerate hypersurface in $\mathbb{C}^3$ (i.e., the tube over a future light cone). Due to dimension constraints there is just one modified CR symbol for hypersurfaces in $\mathbb{C}^3$, which corresponds to there being just one model structure in $\mathbb{C}^3$, but already in $\mathbb{C}^4$ there are many possible modified CR symbols with corresponding maximally symmetric structures. Seeking to generalize the Kolář–Kossovskiy
normal form to higher dimensional settings, one needs the appropriate notion of model structures to deform, and this paper’s classification indeed gives a natural choice for such structures. In total, there are nine CR structures in the present classification, enumerated as types I, II, III, IV.A, IV.B, V.A, V.B, VI, VII in Table 1, of which six (types IV through VII) have been described in other works \([13, 18, 22, 23]\), a seventh (type I) was given by the author in \([26, \text{Example 8.1}]\), and an eighth (type II) has the structure of a tube over a previously discovered affinely homogeneous hypersurface in \(\mathbb{R}^4\) [21, formula (1), Theorem 2]. For these structures that have been previously described in the aforementioned works, references to coordinate descriptions of the hypersurfaces (even given by defining equations) are in Table 1 and in the forthcoming text [14], we derive defining equation descriptions of the new hypersurfaces in the list (i.e., types I and III), which ultimately required considerable calculation and new techniques.

Hypersurfaces in \(\mathbb{C}^{n+1}\) that are locally equivalent to homogeneous CR manifolds have been classified up to local equivalence for \(n = 1\) in \([4, 5]\) and for \(n = 2\) in \([8, 11, 19]\), which naturally leads to our present study of structures in \(\mathbb{C}^4\). In more detail, for \(n = 1\), noting that the homogeneous CR hypersurfaces in \(\mathbb{C}^2\) that are not equivalent to a hyperplane are Levi-nondegenerate, Cartan was able to complete the classification by establishing the gap phenomenon that homogeneous Levi-nondegenerate hypersurfaces in \(\mathbb{C}^2\) with non-maximal symmetry group dimension are simply transitive (therefore having 3-dimensional symmetry groups), and Cartan then investigated the classification of these simply transitive structures using Bianchi’s classification of 3-dimensional Lie algebras. For \(n = 2\), homogeneous Levi-degenerate hypersurfaces in \(\mathbb{C}^3\) were classified in the major work of Fels–Kaup in [11], where they show that all such hypersurfaces are tubes over affinely homogeneous hypersurfaces in \(\mathbb{R}^3\) – a phenomenon that does not persist in higher dimensional settings; Fels–Kaup then obtain the CR classification through application of the earlier classifications \([7, 10]\) of affinely homogeneous hypersurfaces in \(\mathbb{R}^3\), where, notably, showing CR inequivalence of tubes over different affinely homogeneous hypersurfaces is crucial step in [11]. The complete classification of Levi-nondegenerate CR hypersurfaces in \(\mathbb{C}^3\) was obtained more recently in \([8, 19]\), building upon contributions from many research groups over the preceding decades, and we refer readers to \([8]\) for a historic outline of these developments \(\mathbb{C}^3\).

In higher dimensions the classification is complicated by features of Levi degeneracy. By limiting considerations to hypersurfaces that are locally equivalent to a homogeneous CR manifold whose symmetry group’s dimension is maximal relative to the signature of the structure’s Levi form, this limited classification for Levi-nondegenerate hypersurfaces is obtained in \([20]\) for all \(n\). The same classification problem is solved by the main results of \([22, 25]\) for the 2-nondegenerate hypersurfaces whose Levi form has a 1-dimensional kernel, that is, a classification of homogeneous structures whose symmetry algebras have maximal dimension relative to their Levi form for all \(n\). For Levi-degenerate structures with arbitrary Levi forms it is furthermore interesting to consider those with maximal symmetry groups relative to the local invariants encoded in maps referred to by Freeman in \([12]\) as generalized Levi forms – a term with several non-equivalent definitions, so we stress that the present usage refers to the definition in \([12]\).

For 2-nondegenerate structures, these latter local invariants are also encoded in the aforementioned modified CR symbols, so the classification that we obtain gives in particular all homogeneous CR structures on hypersurfaces in \(\mathbb{C}^4\) with a maximal symmetry algebra relative to the structure’s generalized Levi forms, of which there are 8 in total. The ninth structure in the classification obtained here is submaximal with respect to its generalized Levi forms despite being maximal with respect to its modified symbols, illustrating the general fact that modified CR symbols encode more local invariants than generalized Levi forms. We show that this submaximal model has a 9-dimensional symmetry algebra and the symmetry algebra of its associated dynamical Legendrian contact structure (introduced in \([26]\)) is the 14-dimensional exceptional Lie algebra \(\mathfrak{g}_2\), a case of special interest because this is the first known example for which a modified symbol model’s associated DLC structure has finite dimensional symmetry group and yet the CR structure’s symmetry group dimension is strictly
less than that of its DLC structure. Notably, in $\mathbb{C}^4$ the non-planar homogeneous Levi-degenerate hypersurfaces consist of 2-nondegenerate and 3-nondegenerate structures, and the recent [17] classifies the 3-nondegenerate homogeneous structures in $\mathbb{C}^3$, which further motivates our present study of the 2-nondegenerate class.

The secondary purpose of this text is to introduce the methods that were used to obtain the classification, which can, in principle, be applied in higher dimensional settings. Fundamentally, the approach consists of analyzing the algebraic properties satisfied by a homogeneous hypersurface’s modified CR symbols, which reduces to a problem of assessing consistency of a certain overdetermined algebraic system (given in (17)). Modified symbols for which this system can be solved admit reductions used to generate homogeneous CR manifolds that we refer to as flat CR manifolds (see Definition 2.9). These methods are effective for finding new examples of homogeneous 2-nondegenerate CR structures on hypersurfaces in higher dimensional spaces, namely flat structures. To illustrate this, in Sections 4 and 5 we describe 20 locally non-equivalent examples of 9-dimensional homogeneous hypersurfaces obtained via these methods, of which some have been studied previously while many are new. We emphasize the terms flat and modified symbol model have formally different definitions – the former being a structure uniquely defined by algebraic data, and the latter being a structure whose symmetry group dimension attains some upper bound – although they turn out to be equivalent properties in $\mathbb{C}^4$; we also caution that while our usage of flat is consistent with [26] and the context of the Tanaka theory therein, the term appears with several inequivalent meanings across related literature.

Of these 20 9-dimensional examples, several are obtained from the nine 7-dimensional modified symbol models (which are flat) via two constructions that we introduce in Section 4 called extending and linking abstract reduced modified symbols. By linking and extending the flat structures in $\mathbb{C}^4$, we obtain 14 locally homogeneous non-equivalent structures in $\mathbb{C}^5$ and 38 such structures in $\mathbb{C}^6$. For every $p, q, \in \mathbb{N}$ and 2-nondegenerate flat hypersurface $M$ in $\mathbb{C}^{n+1}$, we describe associated 2-nondegenerate hypersurface-type CR structures on $M \times \mathbb{C}^{(p+q)}$, that we call the $2(p+q)$-dimensional signature $(p, q)$ extensions of $M$. These extensions are themselves flat CR structures. Thus from every 2-nondegenerate flat hypersurface, we generate sequences of higher-dimensional homogeneous examples. Maximally symmetric homogeneous 2-nondegenerate hypersurfaces in $\mathbb{C}^{n+1}$ were found in [22] for arbitrary $n$, and, for $n > 4$, these structures are all extensions of the unique maximally symmetric model in $\mathbb{C}^4$ also described in [18, Example 1]. While all of these structures are described by Lie-theoretic means, we have developed techniques for deriving their hypersurface realizations described in terms of defining equations, which will appear in the forthcoming text [14].

Given the current paucity of known high-dimensional 2-nondegenerate homogeneous CR hypersurface examples, it is natural to ask if all sufficiently high-dimensional flat structures can be constructed from low-dimensional structures via combinations of extensions and linkings – that is, if their reduced modified symbols are indecomposable (in the sense of Definition 5.1). We address this in section 6 where, by applying the same methods used to obtain the aforementioned low-dimensional examples, we obtain two new sequences of homogeneous hypersurfaces in $\mathbb{C}^{n+1}$, indexed by their CR dimension $n$, that are different from any sequence generated by extensions and linkings. One of these two sequences has the property that each example in the sequence can be described as a left-invariant structure on a nilpotent Lie group. The other sequence is interesting in contrast to the first one because examples in both sequences share the same generalized Levi forms despite being locally non-equivalent.

2. Preliminaries

In this section we introduce definitions and precursory theorems necessary to derive this paper’s main results. We also introduce in Section 2.2 matrix representations of certain Lie algebras that will be of fundamental importance in the subsequent analysis. Introducing such representations is necessary because there is no established structure theory for the considered class of Lie algebras.
2.1. Definitions and precursory theorems. So that this text is self contained and because the concepts are rather new, we introduce here minimal working definitions of CR symbols, modified CR symbols, and reduced modified CR symbols of homogeneous 2-nondegenerate CR structures, along with some of their basic properties. For a more detailed exposition and study of these objects also defined in more general settings where homogeneity is not assumed, we refer the reader to [22, 26].

Throughout the sequel, let $M$ be a real 2-nondegenerate hypersurface in $\mathbb{C}^{n+1}$ that is locally equivalent to a homogeneous CR manifold. Let $H$ denote the tangential Cauchy–Riemann bundle of $M$ (i.e., the holomorphic part of the complexified maximal complex subbundle in $TM$), let $K \subset H$ be the Levi kernel, and let $r$ be the rank of $K$. The Levi form $\mathcal{L}_p : H_p \times H_p \to \mathbb{C}T_pM/(H_p \oplus \mathcal{P}_p)$ descends to a nondegenerate Hermitian form $\ell_p : H_p/K_p \times H_p/K_p \to \mathbb{C}T_pM/(H_p \oplus \mathcal{P}_p)$ at every point $p \in M$, and for each vector $v \in K_p$ there is an $\ell_p$-self-adjoint antilinear operator $ad_v : H/K \to H/K$ defined by

$$ad_v(X_p + K_p) := [V, X]_p \quad (\text{mod } \mathcal{P}_p \oplus K_p)$$

for all $p \in M$, $X \in \Gamma(X)$, and $V \in \Gamma(K)$ with $V_p = v$. One can show that this definition of $ad_v$ does not depend on the choices of $V$ and depends only on the value of $X$ at $p$. Note that uniform 2-nondegeneracy is equivalent to the map $v \mapsto ad_v$ being injective on $K_p$ for all $p \in M$. A consequence of uniform 2-nondegeneracy is that

$$\binom{n - r + 1}{2} \geq r.$$  

The space

$$\mathfrak{g}_-\,(p) := \mathfrak{g}_{-2,0}(p) \oplus \mathfrak{g}_{-1,-1}(p) \oplus \mathfrak{g}_{-1,1}(p)$$

with $\mathfrak{g}_{-2,0}(p) := \mathbb{C}T_pM/(H_p \oplus \mathcal{P}_p)$, $\mathfrak{g}_{-1,-1}(p) := \mathcal{P}_p/K_p$, and $\mathfrak{g}_{-1,1}(p) := H_p/K_p$, inherits the structure of a $(2n + 1 - 2r)$-dimensional complex Hiesenberg algebra from the Levi form by defining its nontrivial brackets via the formula

$$[v, \overline{w}] = 2i\ell_p(v, w) \quad \forall v, w \in \mathfrak{g}_{-1,1}(p).$$

Here, and throughout the sequel, we use the notation $\mathfrak{g}_j(p) := \bigoplus_k \mathfrak{g}_{j,k}(p)$, where the summation is over all $k$ for which $\mathfrak{g}_{j,k}$ has been defined. In a standard way, the Heisenberg algebra’s structure confers a conformal symplectic structure on $\mathfrak{g}_{-1}(p)$, and the conformal symplectic algebra $\mathfrak{csp}(\mathfrak{g}_{-1}(p))$ can be regarded both as an algebra of endomorphisms of $\mathfrak{g}_{-1}(p)$ and of derivations of $\mathfrak{g}_{-1}(p)$. We will switch freely between both interpretations of $\mathfrak{csp}(\mathfrak{g}_{-1}(p))$.

Letting $\iota$ denote the involution on $\mathfrak{g}_-(p)$ induced by the usual conjugation on $\mathbb{C}T_pM$, the map $\iota$ induces an involution on $\mathfrak{csp}(\mathfrak{g}_{-1}(p))$ given by the formula

$$\iota(\varphi)(v) := \iota \circ \varphi \circ \iota(v) \quad \forall \varphi \in \mathfrak{csp}(\mathfrak{g}_{-1}(p)), \, v \in \mathfrak{g}_-(p),$$

and hence $\iota$ extends uniquely to an antilinear involution on the Lie algebra $\mathfrak{g}_-(p) \times \mathfrak{csp}(\mathfrak{g}_{-1}(p))$ satisfying (3). One can show that, $\iota([v, w]) = [\iota(v), \iota(w)]$ for all $v, w \in \mathfrak{g}_-(p) \times \mathfrak{csp}(\mathfrak{g}_{-1}(p))$. For $v \in K_p$, let $ad_v \in \mathfrak{csp}(\mathfrak{g}_{-1}(p))$ be the endomorphism of $\mathfrak{g}_{-1}$ given by

$$\sim \quad \begin{cases} 
0 & \text{if } w \in \mathfrak{g}_{-1,1}(p) \\
\sim ad_v(\overline{w}) & \text{if } w \in \mathfrak{g}_{-1,1}(p) 
\end{cases}$$

and define

$$\mathfrak{g}_{0,2}(p) := \left\{ \sim ad_v \mid v \in K_p \right\} \quad \text{and} \quad \mathfrak{g}_{0,-2}(p) := \iota(\mathfrak{g}_{0,2}(p)).$$

Also define

$$\mathfrak{g}_{0,0}(p) := \left\{ v \in \mathfrak{csp}(\mathfrak{g}_{-1}(p)) \mid [v, \mathfrak{g}_{j,k}] \subset \mathfrak{g}_{j,k} \forall (j, k) \in \{-2, 0\}, \{-1, \pm 1\}, \{0, \pm 2\} \right\}.$$
Definition 2.1 (introduced in [22]). The CR symbol of the structure on $M$ at a point $p \in M$ is the bi-graded subspace $g^0(p)$ of $g_-(p) \times \mathfrak{csp}(g_{-\mathbf{1}}(p))$ given by

$$g^0(p) := g_-(p) \oplus g_{0,-2}(p) \oplus g_{0,0}(p) \oplus g_{0,2}(p).$$

The CR symbol is regular if it is a subalgebra of $g_-(p) \times \mathfrak{csp}(g_{-\mathbf{1}}(p))$.

CR symbols are basic local invariants of $2$-nondegenerate CR structures in one-to-one correspondence with generalized Levi forms. The symbol’s definition organizes data from the generalized Levi forms into an algebraic structure useful for applications of Tanaka prolongation and descriptions of the structures’ symmetry algebras.

Remark 2.2. After fixing a basis of $H_p/K_p$ and $CT_pM/(H_p \oplus \overline{H})$, we can represent $\ell_p$ with respect to those bases by a matrix $H_{\ell_p}$, and similarly we can represent each operator in $\{\text{ad}_v | v \in K_p\}$ by a matrix with respect to that same basis. If we then fix a basis $(v_1, \ldots, v_r)$ of $K_p$ and let $A_j$ be the matrix representing $\text{ad}_{v_j}$, then the CR symbol at $p$ is given by the matrix representation

$$(\text{span}_R\{H_{\ell_p}\}, \text{span}_C\{A_1, \ldots, A_r\}).$$

Definition 2.1 remains well posed if, instead of assuming that $(M, H)$ is locally homogeneous, we assume only that $(M, H)$ is uniformly 2-nondegenerate. Our assumption of homogeneity, however, implies that all CR symbols on $M$ are equivalent under the equivalence relation that, for $p, q \in M$, $g^0(p) \cong g^0(q)$ if there is a Lie algebra isomorphism between $g_-(p) \times \mathfrak{csp}(g_{-\mathbf{1}}(p))$ and $g_-(q) \times \mathfrak{csp}(g_{-\mathbf{1}}(q))$ that restricts to an isomorphism between the symbols $g^0(p)$ and $g^0(q)$ and commutes with the involution defined on $g^0(p)$ and $g^0(q)$. When this latter property is satisfied, we say that $M$ has a constant symbol of type $g^0$, where $g^0$ is any CR symbol such that $g^0 \cong g^0(p)$ for all $p \in M$.

Going forward let us fix a CR symbol $g^0$ such that $(M, H)$ has a constant symbol of type $g^0$. Such a symbol indeed exists because we are assuming that $M$ is locally homogeneous (e.g., one can take $g^0 = g^0(p)$ for some point $p \in M$). By definition, $g^0$ has an involution defined on it and has the same decomposition

$$g^0 = g_{-2} \oplus g_{-1} \oplus g_0 = g_{-2,0} \oplus g_{-1,1} \oplus g_{0,0} \oplus g_{0,2}$$

as in [2] and [11], with the Heisenberg component $g_- := g_{-2,0} \oplus g_{-1,1} \oplus g_{0,2}$.

To define the modified CR symbols of $(M, H)$, we first introduce the adapted (partial) frame bundle $pr : P^0 \to M$, which is the fiber bundle over $M$ whose fiber $P^0_p$ at a point $p \in M$ is given by

$$P^0_p := \left\{ \varphi |_{g_-} : g_- \to g_{-\mathbf{1}}(p) \right\}.$$  

The distribution $(K \oplus \overline{K}) \cap TM$ is involutive, and thus generates a foliation of $M$. We let $\mathcal{N}$ denote the space consisting of leaves of this foliation, and let $\pi : M \to \mathcal{N}$ denote the natural projection. Let us assume that $\mathcal{N}$ with its quotient topology has the structure of a smooth manifold, which can be achieved by shrinking $M$ (i.e., replacing $M$ by a sufficiently small neighborhood in $M$). Notice that the differential $\pi_* \circ \pi$ naturally identifies $g_{-\mathbf{1}}(p)$ with a subspace in $T_{\pi(p)}\mathcal{N}$ for all $p \in M$. For a fiber $P^0_{\pi(p)}$ of $\pi \circ pr : P^0 \to \mathcal{N}$ any fixed $\psi \in P^0_{\pi(p)}$, we have a local embedding of $\Phi_{\psi}$ into the conformal symplectic group $CSp(g_{-\mathbf{1}})$ given by

$$\Phi_{\psi}(\varphi) := (\pi_* \circ \psi)^{-1} \circ \pi_* \circ \varphi \in CSp(g_{-\mathbf{1}}),$$

and if we apply the (left) Maurer–Cartan of form of $CSp(g_{-\mathbf{1}})$ to the tangent space of the image of this embedding at the point $\Phi_{\psi}(\psi)$, the Maurer–Cartan form maps that tangent space to a subspace in $\mathfrak{csp}(g_{-\mathbf{1}})$ that we label as $g_{0,\text{mod}}(\psi)$. In terms of these subspaces $g_{0,\text{mod}}(\psi)$, we can now define the modified symbols.
Definition 2.3 (introduced in [26]). The modified CR symbol of the structure on $M$ at a point $\psi \in P^0$ is the subspace $\mathfrak{g}^{0,\text{mod}}(\psi)$ of $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$ of the form

$$\mathfrak{g}^{0,\text{mod}}(\psi) = \mathfrak{g}_- \oplus \mathfrak{g}_0^{\text{mod}}(\psi)$$

where $\mathfrak{g}_0^{\text{mod}}(\psi)$ is given through the above Maurer–Cartan form mediated construction.

The modified CR symbols $\mathfrak{g}^{0,\text{mod}}(\psi)$ can coincide with the CR symbol $\mathfrak{g}^0$, but in general the two symbols differ. There is, however, a weaker relationship between the two symbols that always holds, namely $\mathfrak{g}_{0,0} \subset \mathfrak{g}^{0,\text{mod}}(\psi)$ for all $\psi \in P^0$. The subgroup $G_{0,0}$ in $CSp(\mathfrak{g}_{-1})$ generated by $\mathfrak{g}_{0,0}$ acts on $P^0$ giving $P^0$ the structure of a principle bundle with structure group $G_{0,0}$. Modified symbols on the orbits of $G_{0,0}$ are related by the adjoint action of $G_{0,0}$ on $\mathfrak{csp}(\mathfrak{g}_{-1})$. Specifically, for $\psi \in P^0$ and $g \in G_{0,0}$, letting $g.\psi$ denote image of $\psi$ under the structure group action of $g$, we have

$$\mathfrak{g}_0^{\text{mod}}(g.\psi) = g^{-1} \circ \mathfrak{g}_0^{\text{mod}}(\psi) \circ g.$$

For a point $p \in M$, the set

$$\left\{ \mathfrak{g}^{0,\text{mod}}(g.\psi) \mid g \in G_{0,0} \right\} = \left\{ \mathfrak{g}_- \oplus \left( g^{-1} \circ \mathfrak{g}_0^{\text{mod}}(\psi) \circ g \right) \mid g \in G_{0,0} \right\},$$

given by choosing any $\psi \in P^0_p$, is a local invariant of the CR structure $H$ at $p$.

We are now going to describe a reduction procedure that produces subbundles of $P^0$, called reductions, from which we will obtain the aforementioned reduced modified CR symbols. Among these reductions are the level sets in $P^0$ of the mapping $\psi \mapsto \mathfrak{g}^{0,\text{mod}}(\psi)$. Homogeneity of $(M,H)$ implies each such level set will project surjectively onto $M$. and $P^{0,\text{red}}$ is a reduction of the $G_{0,0}$-principal bundle $P^0$, whose structure group we label $G^{\text{red}}_{0,0}$. We call the connected components of $P^{0,\text{red}}$ reductions of $P^0$. Note that this reduction depends on $\psi$ and that $P^0$ has many such reductions. Since we work only with connected components of the level sets, in the sequel we use $P^{0,\text{red}}$ to label any such connected component and let $G^{\text{red}}_{0,0}$ denote its connected structure group. We will now introduce analogous reductions of $P^{0,\text{red}}$ that we also simply call reductions of $P^0$.

Definition 2.4 (introduced in [26]). Let $P^{0,\text{red}} \subset P^0$ be a reduction of $P^0$. The reduced modified CR symbol of the structure on $M$ at a point $\psi \in P^{0,\text{red}}$ associated with the reduction $P^{0,\text{red}}$ is the subspace $\mathfrak{g}^{0,\text{red}}(\psi)$ of $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$ defined by exactly the same definition given for modified symbols but with $P^{0,\text{red}}$ used in place of $P^0$.

Just as the reduction $P^{0,\text{red}}$ of $P^0$ defined above was given as a level set of the mapping $\psi \mapsto \mathfrak{g}^{0,\text{mod}}(\psi)$, we can now define reductions of $P^{0,\text{red}}$ to be level sets of the mapping $\psi \mapsto \mathfrak{g}^{0,\text{red}}(\psi)$. As mentioned, given any reduction $P^{0,\text{red}}$ of $P^0$, we will call these subsequent reductions of $P^{0,\text{red}}$ of $P^0$ as well. For any such reduction, we define its associated reduced modified symbols using Definition 2.4. Thus we can apply an iterative process, where we first take a connected component of a level set of the mapping $\psi \mapsto \mathfrak{g}^{0,\text{mod}}(\psi)$ to obtain a reduction of $P^0$, then consider this reduction’s associated reduced modified symbols as defined in Definition 2.4, then take a connected component of a level set within this last reduction of the mapping $\psi \mapsto \mathfrak{g}^{0,\text{red}}(\psi)$ to obtain a new reduction of $P^0$, and finally repeat these last two steps any number of times. We call this iterative process the (geometric) reduction procedure for $P^0$, which we apply to obtain reductions of $P^0$ whose associated reduced modified symbols are in some ways easier to study than the original modified symbols, while, nevertheless encoding all of the information about $(M,H)$ that is encoded in the original modified symbols. These reductions are especially useful for the study of homogeneous hypersurfaces due to the following lemma.

Lemma 2.5. If $(M,H)$ is a homogeneous 2-nondegenerate CR manifold then there exists a reduction $P^{0,\text{red}}$ of $P^0$ whose associated reduced modified symbols are all the same and invariant under the previously defined involution $\iota$ of $\mathfrak{csp}(\mathfrak{g}_{-1})$, that is, the map $\psi \mapsto \mathfrak{g}^{0,\text{red}}(\psi)$ is constant on $P^{0,\text{red}}$ and
\( \iota(\g^{0,\text{red}}(\psi)) = \g^{0,\text{red}}(\psi) \). This reduction’s associated reduced modified symbol will, moreover, be a subalgebra of \( \g_- \rtimes \text{csp}(\g_{-1}) \).

Proof. From the homogeneity of \((M,H)\), at every step of the reduction procedure we obtain a reduction \( P^{0,\text{red}} \) of \( P^0 \) for which the set \( \{ \g^{0,\text{red}}(\psi) \mid \psi \in P^{0,\text{red}} \} \) of associated reduced modified symbols on the fiber \( P^{0,\text{red}}_p \) over a point \( p \in M \) does not depend on \( M \). Therefore any level set \( \widetilde{P}^{0,\text{red}} \) in \( P^{0,\text{red}} \) of the mapping \( \psi \mapsto \g^{0,\text{red}}(\psi) \) will project surjectively onto \( M \), and

\[
\dim(M) \leq \dim \left( \widetilde{P}^{0,\text{red}} \right) \leq \dim \left( P^{0,\text{red}} \right). 
\]

Since fibers of \( \widetilde{P}^{0,\text{red}} \) are orbits of a closed subgroup in the structure group \( G_{\text{red}} \) of \( P^{0,\text{red}} \) and \( G_{\text{red}} \) is connected, the upper bound in (5) is a strict inequality if this closed subgroup does not equal \( G_{0,0} \). Therefore, the upper bound in (5) is a strict inequality if \( P^{0,\text{red}} \) does not have a constant reduced modified symbol, and hence the reduction procedure reduces the dimension of the reduction obtained in every step unless it yields a reduction with constant reduced modified symbol in some step. The reductions obtained cannot, however, have dimension less than the lower bound in (5), so the procedure must at some step produce a reduction whose dimension is not less than the dimension of the reduction in the previous step, which implies that the reduction at that step has constant reduced modified symbol.

To ensure that the reduction procedure finally yields a reduction with a constant reduced modified symbol that is invariant under the involution \( \iota \), it suffices simply at every step of the procedure to take only level sets of reduced modified symbols that are themselves invariant under \( \iota \). This is indeed possible because it is shown in [26, Proposition 5.1 and Section 6] that if a reduction has constant reduced modified symbol then that reduced modified symbol is a subalgebra in \( \g_- \rtimes \text{csp}(\g_{-1}) \).

Dual to the geometric reduction procedure for \( P^0 \) described above, there is an algebraic reduction procedure that can be applied to any modified symbol \( \g^{0,\text{mod}}(\psi) \) associated to a point \( \psi \in P^0 \) that yields all of the possible modified symbols \( \g^{0,\text{red}}(\psi) \) that the geometric reduction procedure can produce. For our purposes, since we will ultimately care only about reductions of the type in Lemma 2.5, it will suffice to describe this procedure only for \( \psi \in \mathbb{R}P^0 \), which ensures that \( \g^{0,\text{red}}(\psi) \) is invariant under \( \iota \), so let us assume \( \psi \in \mathbb{R}P^0 \). Consider the filtration

\[
\g^{0,\text{mod}}(\psi) = V_0 \supset V_1 \supset \cdots \supset V_s
\]

for some number \( s \), where \( V_{j+1} \) is the maximal subspace in \( V_j \) such that \( [V_{j+1}, V_j] \subset V_j \),

\[
V_{j+1} := V_j \cap N_{\text{csp}(\g_{-1})}(V_j) \quad \forall j \in \mathbb{N} \cup \{0\},
\]

where \( N_{\text{csp}(\g_{-1})}(V_j) \) denotes the normalizer of \( V_j \) in \( \text{csp}(\g_{-1}) \). For homogeneous structures, spaces of the form \( V_s \) in (6) obtained for different choices of \( \psi \) are exactly the possible reduced modified symbols associated with reductions of \( P^0 \). Furthermore, if \( s \) is sufficiently large such that \( V_s = V_{s+1} \) then \( V_s \) will be a subalgebra of \( \text{csp}(\g_{-1}) \) corresponding to a reduction of the type in Lemma 2.5.

As subspaces of \( \g_- \rtimes \text{csp}(\g_{-1}) \), reduced modified symbols share several properties, and we refer to all subspaces of \( \g_- \rtimes \text{csp}(\g_{-1}) \) having these properties as abstract reduced modified symbols (or ARMS), defined as follows.

**Definition 2.6.** An abstract reduced modified symbol (or ARMS) for 2-nondegenerate hypersurface-type CR structures is a fixed choice of antilinear involution \( \iota \) and decomposition of the Heisenberg algebra \( \g_- \) as in (2), together with a subspace \( \g^{0,\text{red}} \) of \( \g_- \rtimes \text{csp}(\g_{-1}) \) with the decomposition

\[
\g^{0,\text{red}} = \g_- \oplus \g_0^{\text{red}},
\]
where $\mathfrak{g}_0^{\text{red}}$ is a subspace in $\mathfrak{csp}(\mathfrak{g}_{-1})$ with a further decomposition

\begin{equation}
\mathfrak{g}_0^{\text{red}} = \mathfrak{g}_{0,-}^{\text{red}} \oplus \mathfrak{g}_{0,0}^{\text{red}} \oplus \mathfrak{g}_{0,+,}^{\text{red}}
\end{equation}

satisfying (1) $\iota(\mathfrak{g}_0^{\text{red}}) = \mathfrak{g}_0^{-1}$, (2) $\iota(\mathfrak{g}_0^{\text{red}}) = \mathfrak{g}_0^{+}$, (3) $[v, \mathfrak{g}_{-1,1}] \not\subset \mathfrak{g}_{-1,1}$ for all $v \in \mathfrak{g}_{0,-}^{\text{red}}$, (4) $[v, \mathfrak{g}_{-1,-1}] \subset \mathfrak{g}_{-1,-1}$ for all $v \in \mathfrak{g}_{0,-}^{\text{red}}$, and (5) $[v, \mathfrak{g}_{-1,i}] \subset \mathfrak{g}_{-1,i}$ for all $v \in \mathfrak{g}_{0,i}^{\text{red}}$, $i \in \{-1, 1\}$.

We stress that the decomposition in (8) is not canonical, whereas the decomposition in (7) is. The dimension $\dim(\mathfrak{g}_{0,+})$ is, however, independent of the splitting chosen in (8). Ultimately we will only care about ARMS that can be equivalent to the reduced modified symbols of the reduction in Lemma 2.6, so in the sequel we will only consider ARMS that are subalgebras of $\mathfrak{g}_{-} \times \mathfrak{csp}(\mathfrak{g}_{-1})$, motivating the following terminology.

**Definition 2.7.** We will say that an ARMS satisfies the subalgebra property if it is a subalgebra of $\mathfrak{g}_{-} \times \mathfrak{csp}(\mathfrak{g}_{-1})$.

An ARMS as in Definition 2.6 satisfying the subalgebra property can be used to construct a 2-nondegenerate CR manifold whose Levi kernel has rank $\dim(\mathfrak{g}_{0,+})$, which motivates the following definition.

**Definition 2.8.** An abstract reduced modified symbol with a decomposition as in Definition 2.6 has Levi kernel dimension $r$ if $\dim(\mathfrak{g}_{0,+}) = r$.

We will now describe homogeneous CR manifolds generated by ARMS satisfying the subalgebra property. Let $\mathfrak{g}_{0,}^{\text{red}}$ be an ARMS satisfying the subalgebra property, and let us fix a decomposition of $\mathfrak{g}_{0,}^{\text{red}}$ as in (7) and (8). Define

$$H := \mathfrak{g}_{-1,1} \oplus \mathfrak{g}_{0,+,}^{\text{red}} \oplus \mathfrak{g}_{0,0}^{\text{red}},$$

let $G$ be the connected simply-connected Lie group of $\mathfrak{g}_{0,}^{\text{red}}$, let $G_{0,0} \subset M^\mathbb{C}$ be the subgroup generated by $\mathfrak{g}_{0,0}^{\text{red}}$, let $\mathfrak{RG}$ be the subgroup generated by $\mathfrak{g}_{0,}^{\text{red}}$, let $\mathfrak{RG}$ be the subgroup generated by

$$\mathfrak{RG}(\mathfrak{g}_{0,}^{\text{red}}) := \{v \in \mathfrak{g}_{0,}^{\text{red}} | \iota(v) = v\},$$

and let $\pi : G \to G/G_{0,0}$ be the canonical projection to the left-coset space. Letting $\bar{H}$ denote the left invariant distribution on $G$ generated by $H$, the distribution $\pi_*(\bar{H})$ defines a tangential Cauchy–Riemann bundle on $\pi(\mathfrak{RG})$, defining a homogeneous CR manifold

\begin{equation}
\left(\pi(\mathfrak{RG}), \pi_*(\bar{H})\right).
\end{equation}

**Definition 2.9.** The flat structure (flat CR manifold or flat CR structure) generated by an ARMS $\mathfrak{g}_{0,}^{\text{red}}$ satisfying the subalgebra property (Definition 2.7) is the CR manifold (respectively CR manifold or CR structure) in (9).

**Remark 2.10.** If a CR symbol is regular as in Definition 2.7 then it satisfies the axioms of an ARMS given in Definition 2.6 as well as the subalgebra property (Definition 2.7), and thus generates a flat structure as described above.

The next two definitions address the circumstance that algebraically non-equivalent ARMS can generate the same CR structure, and we would like to sort the ARMS into equivalence classes based on the CR structures that they generate.

**Definition 2.11** (completion and base). Let $\mathfrak{g}_{0,}^{\text{red}}$ be an abstract reduced modified symbol with Levi kernel dimension $r$ and suppose furthermore that $\mathfrak{g}_{0,}^{\text{red}}$ is a subalgebra of $\mathfrak{g}_{-} \times \mathfrak{csp}(\mathfrak{g}_{-1})$. A completion (respectively base) of $\mathfrak{g}_{0,}^{\text{red}}$ is a maximal (respectively minimal) subalgebra of $\mathfrak{g}_{-} \times \mathfrak{csp}(\mathfrak{g}_{-1})$ containing $\mathfrak{g}_{0,}^{\text{red}}$ that is also an ARMS with Levi kernel dimension $r$.

We say that $\mathfrak{g}_{0,}^{\text{red}}$ is complete if it equals its own completion.
Definition 2.12. Two ARMS satisfying the subalgebra property (Definition 2.7) are equivalent if they have algebraically equivalent completions (see Definition 2.11), that is they have completions equipped with some decomposition as in (7) and (8) between which there is a Lie algebra isomorphism preserving the decompositions and the indexing of their respective components (including the bi-grading of the Heisenberg part) and commuting with the involution $\iota$.

It is easily seen that an ARMS completion and an ARMS base both generate the same flat structure as the ARMS does, so two ARMS are equivalent if and only if they generate equivalent flat CR structures.

The classification that we derive here will consist of flat structures generated by ARMS, and it will rely heavily on theorems of [22, 26]. Specifically, we will use the following theorems.

Theorem 2.13 (follows from [26, Theorem 5.2 and section 6]). If $(M, H)$ is a uniformly $2$-nondegenerate hypersurface-type CR manifold with a constant CR symbol $g^0$ and constant modified symbol on $P^0$ then its modified CR symbol coincides with $g^0$ and the CR symbol is regular. Conversely, if a CR symbol $g^0$ is regular then the flat structure generated by $g^0$ referred to in Remark 2.10 will have constant CR symbol and constant modified symbol on $P^0$ equivalent to $g^0$ itself.

In [22], a variation of Tanaka prolongation called a bi-graded prolongation is introduced and applied to obtain sharp upper bounds for the symmetry group dimension of uniformly $2$-nondegenerate hypersurface-type CR manifolds with regular CR symbols, which are referred to in the next theorem.

Theorem 2.14 (corollary of [22, Theorem 3.2]). If $(M, H)$ is a homogeneous $2$-nondegenerate hypersurface-type CR manifold having a regular CR symbol $g^0$ whose symmetry group’s dimension attains the upper bound given in [22, Theorem 3.1], then $(M, H)$ is locally equivalent to the flat structure generated by $g^0$ referred to in Remark 2.10.

In [26], a correspondence between CR structures and dynamical Legendrian contact structures is used to obtain sharp upper bounds for the symmetry group dimension of uniformly $2$-nondegenerate hypersurface-type CR manifolds referred to in [26] as being recoverable, a property meaning that the CR structure is uniquely determined by its corresponding dynamical Legendrian contact structure. We refer the reader to [26] for a thorough description and study of this property, but for the purposes of this text we need only to know the following lemma.

Lemma 2.15. If $(M, H)$ is a homogeneous $2$-nondegenerate hypersurface-type CR manifold with a rank 1 Levi kernel and the $g_{0,2}(p)$ component of the CR symbol (at an arbitrary point $p \in M$) is spanned by an operator in $\text{csp}(g_{-1}(p))$ whose rank is greater than 1, then $(M, H)$ is recoverable in the sense of [26]. In particular if, for such $(M, H)$, the CR symbol is non-regular then $(M, H)$ is recoverable.

These recoverable structures are the topic of the next theorem.

Theorem 2.16 (corollary of [26, Theorem 6.2]). Let $g^{0,\text{red}}$ be a fixed ARMS satisfying the subalgebra property (Definition 2.7). If $(M, H)$ is a homogeneous $2$-nondegenerate hypersurface-type recoverable CR manifold whose adapted (partial) frame bundle has a reduction with a constant reduced modified symbol equivalent to $g^{0,\text{red}}$ and whose symmetry group dimension attains the upper bound given in [26, Theorem 6.2], then $(M, H)$ is locally equivalent to the flat structure generated by $g^{0,\text{red}}$. Explicitly, the given upper bound is the complex dimension of the universal Tanaka prolongation of $g^{0,\text{red}}$, and this bound is sharp.

From Theorems 2.13 and 2.16 we see that ARMS generate the maximally symmetric homogeneous models for the structures that are either recoverable or have a constant modified symbol on $P^0$. We will see that all but one of the maximally symmetric $2$-nondegenerate hypersurfaces in $\mathbb{C}^4$ satisfy at least one of these last two properties, so it is from these theorems and analysis of ARMS that we obtain most of the classification in this paper’s main result Theorem 3.1.
Accordingly, applying (3), one obtains
\[
(11) \quad (e_0, e_1, \ldots, e_{2n-2r})
\]
of \( g \) satisfying
\[
g_{-2} = \text{span}\{e_0\} \quad \text{and} \quad g_{-1,-1} = \text{span}\{e_1, \ldots, e_{n-r}\} \quad \text{and} \quad \iota(e_0) = e_0,
\]
and \( \iota(e_j) = e_{j+n-r} \) for all \( j \in \{1, \ldots, n-r\} \). Let \( H_\ell \) be the Hermitian matrix satisfying \( [e_j, e_{n-r+k}] = i(H_\ell)_{j,k}e_0 \) for all \( j, k \in \{1, \ldots, n-r\} \), and fix a decomposition of \( g_{0, \text{red}} \) as in (3). There exists a set of \( (n-r) \times (n-r) \) matrices \( A_1, \ldots, A_r, \Omega_1, \ldots, \Omega_r \) such that, regarding \( g_{0, \text{red}} \) as a space of endomorphisms of \( g_{-1} \) represented by matrices with respect to the basis \( (e_1, \ldots, e_{2n-2r}) \), we can identify
\[
\begin{align*}
(12) \quad g_{0,+,0} &= \text{span}_C \left\{ \begin{pmatrix} \Omega_i & A_i \\ 0 & -H_\ell^{-1}\Omega_i^T H_\ell \end{pmatrix} \right\} \quad i \in \{1, \ldots, n-r\} \\
(13) \quad g_{0,-,0} &= \text{span}_C \left\{ \begin{pmatrix} -H_\ell^{-1}\Omega_i^T H_\ell & 0 \\ A_i & \Omega_i \end{pmatrix} \right\} \quad i \in \{1, \ldots, n-r\}.
\end{align*}
\]
Accordingly, applying (3), one obtains
\[
\begin{align*}
(14) \quad \alpha A_i H_\ell^{-1} + A_i H_\ell^{-1} \alpha^T &\in \text{span}\{A_j H_\ell^{-1}\}_{j=1}^{r_i} \quad \forall i \in \{1, \ldots, r\} \\
(15) \quad \alpha^T H_\ell^{-1}\Omega_i + H_\ell^{-1}\Omega_i^T \alpha &\in \text{span}\{H_\ell^{-1}\Omega_i\}_{j=1}^{r_i} \quad \forall i \in \{1, \ldots, r\},
\end{align*}
\]
respectively, and define the algebra \( \mathcal{A} \) to be their intersection, that is,
\[
\begin{align*}
(16) \quad \mathcal{A} := \left\{ \alpha \mid \alpha \text{ satisfies (14) and (15)} \right\}.
\end{align*}
\]
The space \( \mathcal{A} \) is indeed uniquely determined by this last property. To describe it more explicitly, consider the Lie algebras of \( (n-r) \times (n-r) \) matrices \( \alpha \) satisfying
\[
\begin{align*}
(17) \quad \mathcal{A} &= \text{span}\{A_j H_\ell^{-1}\}_{j=1}^{r_i} \quad \forall i \in \{1, \ldots, r\} \\
(18) \quad \mathcal{A} &= \text{span}\{H_\ell^{-1}\Omega_i\}_{j=1}^{r_i} \quad \forall i \in \{1, \ldots, r\},
\end{align*}
\]
respectively, and define the algebra \( \mathcal{A} \) to be their intersection, that is,
\[
\begin{align*}
(19) \quad \mathcal{A} := \left\{ \alpha \mid \alpha \text{ satisfies (17) and (18)} \right\}.
\end{align*}
\]
The space \( \mathcal{A} \) is a subspace of \( \mathcal{A} \). For our purposes, one can always assume that \( I \in \mathcal{A}_{0,0} \) because it does not change the flat CR structure generated by \( g_{0, \text{red}} \). We summarize this representation with the following lemma.

**Lemma 2.17.** Each ARMS (even without imposing the subalgebra property of Definition 2.7) is determined by a tuple \( (H_\ell, A_1, \ldots, A_r, \Omega_1, \ldots, \Omega_r, \mathcal{A}) \) consisting of \( 2r+1 (n-r) \times (n-r) \) matrices and a subspace of the vector space \( \mathcal{A} \) defined in (16).

This matrix representation of ARMS describes a general ARMS satisfying the axioms in Definition 2.7. A matrix representation of an ARMS satisfying the subalgebra property (Definition 2.7) has the additional properties described in the following lemma.

**Lemma 2.18** (compare to [20 Proposition 5.4]). The ARMS \( g_{0,\text{red}} \) is a Lie subalgebra of \( g_- \ltimes \mathfrak{csp}(g_{-1}) \) if and only if \( \mathcal{A} \) is a subalgebra of \( \mathcal{A} \) and there exist coefficients \( \eta_{ij}^k \in \mathbb{C} \) and \( \mu_{ij}^k \in \mathbb{C} \) indexed by
\[ \alpha \in \mathcal{A}_0 \text{ and } i, j, s \in \{1, \ldots, \text{rank } K\} \text{ such that the system of relations} \]

\[
\begin{align*}
(i) \quad & \alpha A_i H^{-1}_\ell + A_i H^{-1}_\ell \alpha^T = \sum_{s=1}^{r} \eta^s_{\alpha,i} A_s H^{-1}_\ell \\
(ii) \quad & [\alpha, \Omega_i] - \sum_{s=1}^{r} \eta^s_{\alpha,i} \Omega_s \in \mathcal{A}_0 \\
(iii) \quad & \Omega^T_j H_i A_i + H_i A_i \Omega_j = \sum_{s=1}^{r} \mu^s_{i,j} H_i A_i \\
(iv) \quad & \left[H^{-1}_\ell \Omega^T_i H, \Omega_j\right] + A_j A_i - \sum_{s=1}^{r} \left(\mu^s_{i,j} \Omega_s + \mu^s_{j,i} H^{-1}_\ell \Omega^T_s H\right) \in \mathcal{A}_0
\end{align*}
\]

holds for all \( \alpha \in \mathcal{A}_0 \) and \( i, j \in \{1, \ldots, \text{rank } K\} \).

**Remark 2.19.** The matrices \( A_1, \ldots, A_r, H_\ell \) determine the CR symbol of the flat structure generated by the ARMS \( g^{0,\text{red}} \). We therefore say that \( g^{0,\text{red}} \) corresponds to the CR symbol of its flat structure.

We have the following useful characterizations of regular symbols.

**Lemma 2.20 (\[26\] Remark 5.3 and part of Lemma \[25\] Lemma 4.3).** The flat structure generated by the ARMS given by

\[ (H_\ell, A_1, \ldots, A_r, \Omega_1, \ldots, \Omega_r, \mathcal{A}_0) \]

has a regular CR symbol if and only if

\[ A_i A_j A_k + A_k A_j A_i \in \text{span}\{A_1, \ldots, A_{n-r}\} \quad \forall i, j, k. \]

Furthermore, if an ARMS satisfying the subalgebra property (Definition \[22\]) having Levi kernel dimension 1 is encoded by \((H_\ell, A_1, \Omega_1, \mathcal{A}_0)\) with \( \Omega_1 \in \mathcal{A} \) then this ARMS corresponds to a regular CR symbol (in the sense of Remark \[27\]).

3. The classification of modified symbol models in \( \mathbb{C}^4 \)

In this section we classify all ARMS that are reductions of modified CR symbols associated with homogeneous models on 7-dimensional manifolds and show that the modified symbol models in \( \mathbb{C}^4 \) (i.e., homogeneous 2-nondegenerate hypersurfaces in \( \mathbb{C}^4 \) that are maximally symmetric relative to their modified symbols) are locally equivalent to the flat structures in \( \mathbb{C}^4 \) generated by such ARMS. Hence the classification of modified symbol models follows from the classification of ARMS generating flat structures. Since we set \( \text{dim}(M) = 2n+1 \) and \( \text{rank } K = r \), in this section we have \( n = 3 \). Note that by \( \Box \) with \( n = 3 \), we get that \( r = 1 \), which corresponds to the fact that all uniformly 2-nondegenerate 7-dimensional CR manifolds have a rank 1 Levi kernel. Accordingly, each of the reduced modified symbols that we are classifying in this section is determined by a tuple \( \{H_\ell, A_1, \Omega_1, \mathcal{A}_0\} \) satisfying the system in \((17)\) with \( r = 1 \). Here we have written \( A_1 \) and \( \Omega_1 \) to match the notation of Section \[22\] but for convenience let us omit the subscripts because they are unnecessary in this case with \( r = 1 \).

Since \( n - r = 2 \), \( H_\ell \) and \( A \) are \( 2 \times 2 \) matrices representing a nondegenerate Hermitian form \( \ell \) and an \( \ell \)-selfadjoint antilinear operator, and, by changing a basis to bring such a pair to the canonical form of \[24\] Theorem 2.2, we can assume (after possibly rescaling \( H_\ell \) and \( A \) by different real coefficients) that they have one of the forms

\[ H_\ell = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for some } \epsilon = \pm 1, \]

\[ H_\ell = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \quad \text{and} \quad A = I \quad \text{for some } \epsilon = \pm 1, \]
H_ℓ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (20)

H_ℓ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (21)

H_ℓ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \quad (22)

H_ℓ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & e^{iθ} \\ 1 & 0 \end{pmatrix} \text{ for some } θ \in \left(0, \frac{π}{2}\right) \cup \left(\frac{π}{2}, π\right), \quad (23)

H_ℓ = \begin{pmatrix} 1 & 0 \\ 0 & ε \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & λ \end{pmatrix} \text{ for some } ε = \pm 1, λ > 1, \quad (24)

H_ℓ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (25)

These possible forms for the pair \((H_ℓ, A)\) are ordered above to highlight a few key patterns. By Lemma 2.20, the CR symbols corresponding to (18), (19), (20), and (21) are regular and thus of the type classified in [22]. Therefore, as noted in Remark 2.10, (18), (19), (20), and (21) are all associated with homogeneous models, and it remains for us to determine which ARMS satisfying the subalgebra property (Definition 2.7) exist corresponding to these four cases. We furthermore need to determine which ARMS if any satisfy the subalgebra property and correspond to (22), (23), (24), or (25). Each such ARMS will generate a flat structure, and we will show that this flat structure is the unique modified symbol model having the same modified symbols. The conclusions of this analysis are summarized below in Theorems 3.1 and 3.4.

**Theorem 3.1.** Up to local equivalence, there are nine 7-dimensional 2-nondegenerate flat structures generated by ARMS (as described in Section 2). They are respectively generated by each of the nine ARMS described in Table 1. In particular:

1. There exist three equivalence classes of ARMS satisfying the subalgebra property (Definition 2.7) corresponding to (18), one for \(ε = 1\) and two for \(ε = -1\). These are represented by types III, IV.A, and IV.B in Table 1.

2. There exist two equivalence classes of ARMS satisfying the subalgebra property (Definition 2.7) corresponding to (19), one for each parameter setting of \(ε\). These are represented by types V.A and V.B in Table 1.

3. There exists one equivalence class of ARMS satisfying the subalgebra property (Definition 2.7) corresponding to each of the four cases (20), (21), (22), and (25). These are represented by types I, II, VI, and VII in Table 1.

4. No ARMS satisfying the subalgebra property (Definition 2.7) correspond to any of the cases in (23) and (24).

The symmetry groups of these nine flat structures have the respective dimensions indicated in Table 1.

**Corollary 3.2.** The adapted (partial) frame bundle \(P^0\) of a 7-dimensional homogeneous 2-nondegenerate hypersurface-type CR manifold admits a reduction having the structure of a principal bundle over the complexified Levi leaf space (defined locally in [26]) whose structure group has as its Lie algebra the degree zero component of one of the nine ARMS in Table 1. Thus every such CR manifold is canonically assigned one of the nine types labeled in Table 1, and, moreover, this type is determined by any one of its modified CR symbols.

**Proof.** This is an immediate corollary of Lemma 3.3 and Theorem 3.1. \(\square\)
Lemma 3.3. If a 7-dimensional homogeneous 2-nondegenerate hypersurface-type CR manifold \((M, H)\) whose \(P^0\) bundle admits a reduction \(P^0,\text{red}\) with constant reduced modified symbol \(g^{0,\text{red}}\) of type III (as enumerated in Table 1) has a 9-dimensional symmetry group then \((M, H)\) is locally equivalent to the flat structure generated by \(g^{0,\text{red}}\).

We defer the proof of Lemma 3.3 to Section 3.4.

Theorem 3.4. The symmetry group \(\text{Aut}(M)\) of a 7-dimensional homogeneous 2-nondegenerate hypersurface-type CR manifold \(M\) has dimension bounded by that of the flat structure having the same type as \(M\) (as described in Corollary 3.2). This bound is given in Table 1 and if \(\dim \text{Aut}(M)\) attains its bound then the CR structure on \(M\) is flat (as defined in Definition 2.9). In particular the flat structures are in one-to-one correspondence with modified symbol models (as defined in Section 1).

Proof. This theorem is a corollary of Lemma 3.3, Theorem 2.16, and the main results of [22]. Indeed, for types I, II, V, and VI it follows immediately from Theorem 2.16, whereas for types IV, V, VI, and VII it is a special case of the results in [22]. See Section 3.3 for more detail on how these previous theorems are applied.

The remaining case, type III, is addressed by Lemma 3.3.

We prove Theorem 3.4 and Lemma 3.3 and expand on the proof of Theorem 3.4 in Sections 3.1 through 3.4, presenting the proofs with the following outline, partitioned into two steps.

Step 1 (sections 3.1, 3.2): We give a constructive proof of the existence statements in Theorem 3.4 explicitly describing the equivalence classes of reduced modified CR symbols referred to in Theorem 3.1. Since our classification goal reduces to describing the tuples \((H_\ell, A, \Omega, \mathcal{A}_0)\) for which the system (17) is consistent, we will suppose that \(\Omega\) and \(\mathcal{A}_0\) are fixed such that (17) is satisfied. We also let \(g^0\) be the CR symbol encoded by the pair \((H_\ell, A)\), as described in Remark 2.19. Depending on the value of \((H_\ell, A)\) we will either describe this pair \((\Omega, \mathcal{A}_0)\) in more detail, deriving the corresponding formulas in Table 1, or derive a contradiction from the assumption that such a pair exists. Doing this for all \((H_\ell, A)\) in the normal forms of (18) through (25) completes the proof of the existence statements in Theorem 3.1.

Step 2 (sections 3.3, 3.4): Establishing the symmetry group bounds of Theorems 3.1 and 3.4 and the local uniqueness statements of Theorem 3.4 requires different arguments for the different types in Table 1 which we present in Sections 3.3 and 3.4. As noted in the proof of Theorem 3.4, types I, II, V.A, V.B, and VI can be treated as immediate applications of [26, Theorem 6.2], which we explain further in Section 3.3 whereas Theorem 3.4 is already proven in [22] for types IV.A, IV.B, V.A, V.B, VI, and VII.

3.1. Symbols corresponding to formulas (18) through (21). Suppose that \(H_\ell\) and \(A\) are as in (18), (19), (20), or (21). Since, by Lemma 2.20, \(g^0\) is itself a reduced modified CR symbol corresponding to the pair \((H_\ell, A)\), as noted in Remark 2.10. This reduced modified symbol is described by taking \(\mathcal{A}_0 = \mathcal{A}\) and taking \(\Omega\) to be any matrix in \(\mathcal{A}\). Hence, all that remains for us to do is determine whether or not there exist reduced modified symbols for which \(\Omega\) is not in \(\mathcal{A}\).

If \((H_\ell, A)\) is as in (18), then there turns out to be exactly one equivalence class of solutions with \(\Omega\) not in \(\mathcal{A}\) provided that \(\epsilon = -1\), whereas there is no such solution if \(\epsilon = 1\). So we record this as a lemma.

Lemma 3.5. Suppose \((H_\ell, A)\) is as in (18). If \(\epsilon = -1\) then (up to the equivalence in Definition 2.12) there exists exactly one ARMS \(g^{0,\text{red}}\) satisfying the subalgebra property (Definition 2.7) such that the matrix \(\Omega\) is not in \(\mathcal{A}\), and if \(\epsilon = 1\) then there is no such ARMS. In the former case, this equivalence class of reduced modified symbols is represented by any one of the ARMS described by (18) and

\[
\Omega = e^{i\theta} \begin{pmatrix} 0 & 0 \\ \sqrt{3} & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{A}_0 = \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \right\} \text{ for some } \theta \in \mathbb{R}.
\]
Accordingly, with \( \alpha \) in item (iv) of (17) being equal to zero yields

\[
0 = -\frac{1}{\sqrt{\frac{\Omega}{2}}}\quad \text{and is therefore diagonal. Since we are searching for a solution with } \Omega \text{ not in } \mathcal{A}, \text{ we can assume that } \Omega_{2,1} \neq 0, \text{ and hence setting the off-diagonal entries in (28) equal to zero yields}
\]

\[
\Omega_{2,2} = 3\Omega_{1,1}.
\]

Accordingly,

\[
\alpha = \begin{pmatrix}
\epsilon|\Omega_{2,1}|^2 - 4|\Omega_{1,1}|^2 + 1 & \frac{\epsilon|\Omega_{2,1}|}{\Omega_{2,1}(\Omega_{2,2} - 3\Omega_{1,1})}
\end{pmatrix},
\]

where

\[
a = -\frac{1}{\sqrt{\frac{\Omega}{2}}} + 2(\Omega_{1,1}\Omega_{2,2} + \Omega_{1,1}\Omega_{2,2}) .
\]

By item (iv) of (17), \( \alpha \) belongs to \( \mathcal{A} \), and is therefore diagonal. Since we are searching for a solution with \( \Omega \) not in \( \mathcal{A} \), we can assume that \( \Omega_{2,1} \neq 0 \), and hence setting the off-diagonal entries in (28) equal to zero yields

\[
\Omega_{2,2} = 3\Omega_{1,1}.
\]

Table 1. Flat structures of Theorem 3.1 described in the notation of Lemma 2.17. The letters \( a, b, \) and \( c \) denote complex variables. The last column gives references to known coordinate descriptions of the respective flat structures. Some of these references describe real hypersurfaces in \( \mathbb{R}^4 \), and it is rather the tube over this real hypersurface that has the relevant CR structure.

Proof. Notice that \( \mathcal{A} \) is the space of all \( 2 \times 2 \) diagonal matrices, and item (iii) in (17) implies \( \Omega_{1,2} = 0 \). With \( \Omega_{1,2} = 0 \), we get

\[
(27) \quad \left[ H^{-1}_\ell \Omega^T H_\ell, \Omega \right] = \begin{pmatrix}
\epsilon|\Omega_{2,1}|^2 & \epsilon\Omega_{2,1}(\Omega_{2,2} - \Omega_{1,1}) \\
\Omega_{2,1}(\Omega_{2,2} - \Omega_{1,1}) & -\epsilon|\Omega_{2,1}|^2
\end{pmatrix}.
\]

The coefficient \( \mu_{1,1} \) in item (iii) of (17) is equal to \( 2\Omega_{1,1} \), and hence, by (27), labeling the matrix \( \left[ H^{-1}_\ell \Omega^T H_\ell, \Omega \right] + A\Omega - \left( \mu_{1,1} \Omega + \mu_{1,1} H^{-1}_\ell \Omega^T H_\ell \right) \) in item (iv) of (17) \( \alpha \), we have

\[
(28) \quad \alpha = \begin{pmatrix}
\epsilon|\Omega_{2,1}|^2 - 4|\Omega_{1,1}|^2 + 1 & \frac{\epsilon|\Omega_{2,1}|}{\Omega_{2,1}(\Omega_{2,2} - 3\Omega_{1,1})}
\end{pmatrix},
\]

where

\[
a = -\frac{1}{\sqrt{\frac{\Omega}{2}}} - 2(\Omega_{1,1}\Omega_{2,2} + \Omega_{1,1}\Omega_{2,2}) .
\]

By item (iv) of (17), \( \alpha \) belongs to \( \mathcal{A} \), and is therefore diagonal. Since we are searching for a solution with \( \Omega \) not in \( \mathcal{A} \), we can assume that \( \Omega_{2,1} \neq 0 \), and hence setting the off-diagonal entries in (28) equal to zero yields

\[
\Omega_{2,2} = 3\Omega_{1,1}.
\]

Accordingly,

\[
(29) \quad \alpha = \begin{pmatrix}
\epsilon|\Omega_{2,1}|^2 - 4|\Omega_{1,1}|^2 + 1 & 0 \\
0 & -\epsilon|\Omega_{2,1}|^2 - 12|\Omega_{1,1}|^2
\end{pmatrix}.
\]
Evaluating item (i) in (17) with $\alpha$ given by (29), we obtain $\eta_{0,1}^1 = 2\epsilon|\Omega_{2,1}|^2 - 8|\Omega_{1,1}|^2 + 2$, and since, noting $\Omega_{1,2} = 0$,

$$[\alpha, \Omega] = \begin{pmatrix} 0 & 0 \\ (-2\epsilon|\Omega_{2,1}|^2 - 8|\Omega_{1,1}|^2 - 1) \Omega_{2,1} & 0 \end{pmatrix},$$

the $(2, 1)$ entry of $[\alpha, \Omega] - \eta_{0,1}^1 \Omega$ is equal to

$$\begin{pmatrix} [\alpha, \Omega] - \eta_{0,1}^1 \Omega \end{pmatrix}_{2,1} = - (4\epsilon|\Omega_{2,1}|^2 + 3) \Omega_{2,1}. \quad (30)$$

By item (ii) in (17), $[\alpha, \Omega] - \eta_{0,1}^1 \Omega$ belongs to $\mathcal{A}$, and hence $([\alpha, \Omega] - \eta_{0,1}^1 \Omega)_{2,1} = 0$. If $\epsilon = 1$ then we have obtained a contradiction because then the value in (30) is nonzero. Accordingly, if $\Omega \not\in \mathcal{A}$ then $\epsilon = -1$. Setting (30) equal to zero with $\epsilon = -1$, we get

$$|\Omega_{2,1}|^2 = \frac{3}{4}. \quad (31)$$

By (29) and (31)

$$\alpha = \begin{pmatrix} -4|\Omega_{1,1}|^2 - \frac{1}{4} & 0 \\ 0 & 3(-4|\Omega_{1,1}|^2 - \frac{1}{4}) \end{pmatrix},$$

and hence

$$\text{span} \left\{ \frac{\alpha}{-4|\Omega_{1,1}|^2 - \frac{1}{4}} \right\} = \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \right\} \subset \mathcal{A}_0. \quad (32)$$

Notice that $I$ is not in $\mathcal{A}_0$ because then items (i) and (ii) of (17) would imply that $\Omega$ is in $\mathcal{A}_0$, so equality actually holds in (32), which together with (31) implies that (18) and (26) indeed give a solution to the system (17).

Lastly, we need to show that changing the parameter $\theta$ in (26) does not change the equivalence class represented by the corresponding reduced modified CR symbol. To see this last observation, consider the 1-parameter subgroup

$$\left\{ \begin{pmatrix} e^{it}I & 0 \\ 0 & e^{-it}I \end{pmatrix} : t \in \mathbb{R} \right\}$$

of $CSp(\mathfrak{g}_{-1})$. This subgroup belongs to the group $\Re G_{0,0} = G_{0,0} \cap \Re G$ (where $G$ and $G_{0,0}$ are as in Section 2) and it acts transitively (via the natural adjoint action) on the set of reduced modified symbols parameterized by $\theta$ described by (18) and (26), giving isomorphisms between these ARMS establishing their equivalence in the sense of Definition 2.12. \hfill \square

**Lemma 3.6.** If $(H_\ell, A)$ is as in (19), (20) or (21) then (up to the equivalence in Definition 2.12) the only corresponding ARMS satisfying the subalgebra property (Definition 2.7) is the one described by taking $\mathcal{A}_0 = \mathcal{A}$ and taking $\Omega$ to be any matrix in $\mathcal{A}_0$. \hfill \square

**Proof.** If $(H_\ell, A)$ is as in (19), it is easily checked that item (iii) in (17) implies that both of the set inclusion conditions in (16) are satisfied by setting $\alpha = \Omega$. In other words, if $(H_\ell, A)$ is as in (19) then $\Omega$ is in $\mathcal{A}$.

This also happens if $(H_\ell, A)$ is as in (21) instead by exactly the same calculation, which is clear because if $(H_\ell, A)$ is as in (21) then $AH_\ell^{-1}$ and $H_\ell A$ are the same in this case as they are in the case where (19) holds.

Similarly, if $(H_\ell, A)$ is as in (20) then $\mathcal{A}$ is the space of $2 \times 2$ upper-triangular matrices, and item (iii) in (17) implies $\Omega_{2,1} = 0$. In other words, if $(H_\ell, A)$ is as in (20) then again we get that $\Omega$ is in $\mathcal{A}$.
3.2. Symbols corresponding to formulas (22) through (25). In each of these cases (i.e., in (22) through (25)), $\mathcal{A}$ is spanned by the identity matrix $I$. If $I$ is in $\mathcal{A}_0$ then items (i) and (ii) of (17) imply that $\Omega$ is in $\mathcal{A}_0$, which contradicts Lemma 2.20 so

\[
\mathcal{A}_0 = 0.
\]

Proof. Suppose first that $H_\ell$ and $A$ are as in (22) and (23). To treat both cases with common formulas, for the case where $H_\ell$ and $A$ are as in (22), we set $\theta = \frac{\pi}{2}$ so that $A$ is described by the same formula as in (23). Item (iii) in (17) implies

\[
\Omega_{1,1} = \Omega_{2,2}, \quad \Omega_{1,2} = -e^{-i\theta}\Omega_{2,1}, \quad \text{and} \quad \mu_{1,1} = 2\Omega_{1,1}.
\]

Labeling the matrix $H_\ell + \Omega^2 H_\ell, \Omega$ and applying (35), we obtain

\[
\alpha_{1,2} = 4R\left(e^{i\theta}\Omega_{1,1}\Omega_{2,1}\right), \quad \alpha_{2,1} = -4R\left(\Omega_{1,1}\Omega_{2,1}\right),
\]

and

\[
\alpha_{1,1} = e^{i\theta} - 4|\Omega_{1,1}|^2 + (e^{-i\theta} - e^{i\theta})|\Omega_{2,1}|^2.
\]

Since item (iv) of (17) gives that $\alpha$ belongs to $\mathcal{A}_0$, by (34), the values in (36) and (37) are equal to zero. Since $0 < \theta < \pi$, setting the values in (36) equal to zero implies $\Omega_{1,1}\Omega_{2,1}$, whereas setting (37) equal to zero implies $\Omega_{2,1} \neq 0$. Therefore, $\Omega_{1,1} = 0$, and, by (37), the equation $\alpha_{1,1} = 0$ simplifies to

\[
\Omega_{1,1} = 0 \quad \text{and} \quad |\Omega_{2,1}|^2 = \frac{e^{i\theta}}{e^{i\theta} - e^{-i\theta}}.
\]

Since $0 < \theta < \pi$ and $0 \leq |\Omega_{2,1}|$, (38) implies that $\theta = \frac{\pi}{2}$, and hence the system (17) is inconsistent if $(H_\ell, A)$ is as in (23), which yields the following result.

**Lemma 3.7.** There are no ARMS satisfying the subalgebra property (Definition 2.7) corresponding to any of the cases in (23).

**Lemma 3.8.** There exists exactly one equivalence class of ARMS $\mathfrak{g}^{0,\text{red}}$ satisfying the subalgebra property (in the sense of Definition 2.12) corresponding to the case where $(H_\ell, A)$ is as in (22). This equivalence class of ARMS is represented by any one of the symbols described by (22) and

\[
\Omega = e^{i\theta} \left( \begin{array}{cc} 0 & i\frac{\sqrt{2}}{2} \\ i\frac{\sqrt{2}}{2} & 0 \end{array} \right) \quad \text{and} \quad \mathcal{A}_0 = 0 \quad \text{for some} \ \theta \in \mathbb{R}.
\]

**Proof.** By (34), (35), and (38), if $(H_\ell, A, \Omega, \mathcal{A}_0)$ satisfies (17) with $(H_\ell, A)$ as in (22) then, indeed (39) holds. Conversely, if $(H_\ell, A, \Omega, \mathcal{A}_0)$ is as in (22) and (39) then it is straightforward to check that the system (17) is consistent.

We finish this proof using the same conclusion as in the proof of Lemma 3.5. That is, the 1-parameter subgroup of $CSp(\mathbb{Q}_{-1})$ given in (33) acts transitively on the set of reduced modified symbols parameterized by $\theta$ described by (22) and (39), providing the Lie algebra isomorphisms that show as $\theta$ varies in (39), the corresponding reduced modified CR symbols belong to the same equivalence class (in the sense of Definition 2.12).

The following lemmas address the cases in (24) and (25).

**Lemma 3.9.** There are no ARMS satisfying the subalgebra property (Definition 2.7) corresponding to either of the cases (i.e., $\epsilon = 1$ and $\epsilon = -1$) in (24).

**Proof.** Item (iii) in (17) implies

\[
\Omega_{1,1} = \Omega_{2,2}, \quad \Omega_{1,2} = -e\lambda\Omega_{2,1}, \quad \text{and} \quad \mu_{1,1} = 2\Omega_{1,1}.
\]
Labeling the matrix \( \left[ H_{\ell}^{-1}\Omega^T H_\ell, \Omega \right] + A\bar{A} - \left( \mu_{1,1}^1\Omega + \mu_{1,1}^1H_{\ell}^{-1}\Omega^T H_\ell \right) \) in item (iv) of (17) \( \alpha \) and applying (40) to simplify \( \alpha \), we obtain
\[
\alpha_{1,2} = 2\epsilon(\lambda\Omega_{1,1}\Omega_{2,1} - \Omega_{1,1}\Omega_{2,1}) \quad \text{and} \quad \alpha_{2,1} = -2(\Omega_{1,1}\Omega_{2,1} - \lambda\Omega_{1,1}\Omega_{2,1})
\]
and
\[
\alpha_{1,1} = 1 - 4|\Omega|_{1,1}^2 + \epsilon(1 - \lambda^2)|\Omega_{2,1}|^2 \quad \text{and} \quad \alpha_{2,2} = \lambda^2 - 4|\Omega|_{1,1}^2 - \epsilon(1 - \lambda^2)|\Omega_{2,1}|^2.
\]
Since item (iv) of (17) gives that \( \alpha \) belongs to \( \mathcal{A}_0 \), by (33), the values in (41) and (42) are equal to zero. Accordingly,
\[
\epsilon(\lambda^2 - 1)|\Omega|_{1,1}\Omega_{2,1} = \frac{\alpha_{1,2} + \epsilon\lambda\alpha_{2,1}}{2} = 0,
\]
which implies that either \( \Omega_{1,1} = 0 \) or \( \Omega_{2,1} = 0 \) because \( \lambda^2 \neq 1 \). If \( \Omega_{2,1} = 0 \) then \( \Omega \) is a multiple of the identity, which implies that \( \Omega \in \mathcal{A} \), contradicting Lemma 2.20. Therefore, \( \Omega_{1,1} = 0 \). Yet if \( \Omega_{1,1} = 0 \), since \( \alpha_{1,1} = \alpha_{2,2} = 0 \), the two equations in (42) respectively imply
\[
1 = -\epsilon(1 - \lambda^2)|\Omega_{2,1}|^2 \quad \text{and} \quad \lambda^2 = \epsilon(1 - \lambda^2)|\Omega_{2,1}|^2,
\]
implying \( \lambda^2 = -1 \), contradicting the assumption in (24) that \( \lambda > 1 \).

\[\square\]

**Lemma 3.10.** There exists exactly one equivalence class of ARMS \( \mathfrak{g}^{0,\text{red}} \) satisfying the subalgebra property (in the sense of Definition 2.14) corresponding to the case where \((H_\ell,A)\) is as in (25). This equivalence class of reduced modified symbols is represented by any one of the symbols described by (25) and
\[
\Omega = e^{i\theta} \left( \begin{array}{cc} 1 & \frac{1}{2} \\ 0 & 0 \end{array} \right) \quad \text{and} \quad \mathcal{A}_0 = 0 \quad \text{for some} \ \theta \in \mathbb{R}.
\]

**Proof.** Item (iii) in (17) implies
\[
\Omega_{1,1} = 2\Omega_{1,2} + \Omega_{2,2}, \quad \Omega_{2,1} = 0, \quad \text{and} \quad \mu_{1,1}^1 = 2(\Omega_{1,2} + \Omega_{2,2}).
\]
Labeling the matrix \( \left[ H_{\ell}^{-1}\Omega^T H_\ell, \Omega \right] + A\bar{A} - \left( \mu_{1,1}^1\Omega + \mu_{1,1}^1H_{\ell}^{-1}\Omega^T H_\ell \right) \) in item (iv) of (17) \( \alpha \) and applying (40) to simplify \( \alpha \), we obtain
\[
\alpha_{1,2} = -2\left( \Omega_{1,2} \left( \Omega_{1,2} + \Omega_{2,2} \right) + \Omega_{1,2} \left( 3\Omega_{1,2} + \Omega_{2,2} \right) - 1 \right)
\]
and
\[
\alpha_{1,1} = 1 - 2\Omega_{2,2}^2(\Omega_{1,2} + \Omega_{2,2}) - 2(\Omega_{1,2} + \Omega_{2,2})^2(2\Omega_{1,2} + \Omega_{2,2}) + \left( 1 - 4|\Omega_{1,2}|^2 \right) - 2(\Omega_{2,2}\Omega_{1,2} + \Omega_{2,2}\Omega_{1,2}) - 4|\Omega_{2,2}|^2 - 4\Omega_{2,2}\Omega_{1,2}.
\]
Since item (iv) of (17) gives that \( \alpha \) belongs to \( \mathcal{A}_0 \), by (34), \( \alpha = 0 \). Setting the value in (45) equal to zero is equivalent to
\[
\Omega_{2,2}\Omega_{1,2} + \Omega_{2,2}\Omega_{1,2} = 1 - 4|\Omega_{1,2}|^2.
\]
Setting \( \alpha_{1,1} = 0 \) and applying (47) to simplify (46), we obtain
\[
0 = \left( 1 - 4|\Omega_{1,2}|^2 \right) - 2(\Omega_{2,2}\Omega_{1,2} + \Omega_{2,2}\Omega_{1,2}) - 2|\Omega_{2,2}|^2 - 4\Omega_{2,2}\Omega_{1,2}
\]
\[
= \left( 1 - 4|\Omega_{1,2}|^2 \right) - 2|\Omega_{2,2}|^2 - 4\Omega_{2,2}\Omega_{1,2}.
\]
Therefore, \( \Omega_{2,2}\Omega_{1,2} \) is a real number and (47) implies
\[
\Omega_{2,2} = \frac{1 - 4|\Omega_{1,2}|^2}{2\Omega_{1,2}}.
\]
Together (48) and (49) imply
\[
\frac{(1 - 4 |\Omega_{1,2}|^2)^2}{2 |\Omega_{1,2}|^2} = 2 |\Omega_{2,2}|^2 = -(1 - 4 |\Omega_{1,2}|^2)^2 - 4\Omega_{2,2}^2 \Omega_{1,2} = -3 \left(1 - 4 |\Omega_{1,2}|^2\right),
\]
which is equivalent to
\[
(50) \quad 0 = \left(1 - 4 |\Omega_{1,2}|^2\right)^2 + 6 \left(1 - 4 |\Omega_{1,2}|^2\right) |\Omega_{1,2}|^2 = (1 - 4 |\Omega_{1,2}|^2)(2 |\Omega_{1,2}|^2 + 1).
\]
By (49) and (50),
\[
|\Omega_{1,2}|^2 = \frac{1}{4} \quad \text{and} \quad \Omega_{2,2} = 0.
\]

Therefore, noting (15) and (51), if \((H_\ell, \Omega, \mathcal{A}_0)\) satisfies (17) with \((H_\ell, A)\) as in (25) then (43) holds. Conversely, if \((H_\ell, A, \Omega, \mathcal{A}_0)\) is as in (25) and (43) then it is straightforward to check that the system (17) is consistent.

We finish this proof using the same conclusion as in the proof of Lemma 3.5. That is, the 1-parameter subgroup of \(CSp(g_{-1})\) given in (33) acts transitively on the set of reduced modified symbols parameterized by \(\theta\) described by (25) and (39), providing the Lie algebra isomorphisms that show as \(\theta\) varies in (43) the corresponding reduced modified CR symbols belong to the same equivalence class (in the sense of Definition 2.12).

\[\square\]

3.3. Local uniqueness and symmetry group dimensions. For structures not of type III, we can establish the symmetry bounds and local uniqueness statements in Theorems 3.1 and 3.4 by applying Theorem 2.14 (a corollary of [22, Theorem 3.1]) and Theorem 2.16 (a corollary of [26, Theorem 6.2]). For structures of type III, however, neither of these previous theorems apply. Indeed Theorem 2.14 does not apply because the local uniqueness results of [22] apply only to structures that are maximally symmetric relative to their CR symbol, which type III structures are not. And Theorem 2.16 applies only to recoverable structures, which again type III structures are not.

In more detail, for an ARMS \(g^{0,\text{red}}\) of type I, II, V, or VI, its universal Tanaka prolongation has dimension equal to the symmetry group bound indicated in Table 1, a fact that is easily checked by direct calculation, as these prolongations can, for example, even be calculated using a computer algebra system such as Maple. By Lemma 2.15, structures of each of these types are recoverable, and hence Theorem 2.16 (a corollary of [26, Theorem 6.2]) indeed establishes the symmetry bounds and local uniqueness statements in Theorems 3.1 and 3.4 for structures of types I, II, V, and VI.

For types IV, V, VI, and VII the symmetry bounds of Theorem 3.1 where calculated in [22], using the bi-graded Tanaka prolongation method introduced therein. The local uniqueness statements in Theorem 3.4 for structures of types I, II, V, and VI follow from the main result in [22] and a weaker version of this main result sufficient for our present application is stated above in Theorem 2.14.

Symmetry bounds and local uniqueness of maximally symmetric type III structures are addressed by Lemma 3.3, proven in the next section.

3.4. Proof of Lemma 3.3. Throughout section 3.4, let \((M, H)\) be as in Lemma 3.3, so, in particular, \((M, H)\) is a homogeneous structure of type III (as described in Corollary 3.2).

Notice that type III and type IV.B structures have the same CR symbols. They are, however, distinguished by having non-equivalent modified CR symbols, and are therefore not locally equivalent as CR structures. It is shown in [22] that the flat structure of type IV.B is the unique structure with its CR symbol whose symmetry group is at least 10-dimensional, and hence the symmetry group of a type III structure is at most 9-dimensional. On the other hand the flat structure of type III is generated by a 9-dimensional ARMS, so its symmetry group is at least 9-dimensional. Therefore 9 is the sharp upper bound for the symmetry group dimension of a type III structure.
Let $g^{0,\text{red}}$ be the maximal ARMS of type III given in Table 11. We can assign it a basis $(e_0, \ldots, e_8)$ to this 9-dimensional Lie algebra with respect to which it has the matrix representation $\rho : g^{0,\text{red}} \to \mathfrak{gl}_6(\mathbb{C})$ given by

$$
\rho \left( \sum_{j=0}^{8} t_j e_j \right) = \begin{pmatrix}
2t_7 & -t_3 & 0 & 0 & 0 & 0 \\
0 & t_7 + t_8 & \frac{\sqrt{3}}{2} t_6 & 0 & 0 & 0 \\
0 & 0 & t_7 + 3t_8 & 0 & 0 & 0 \\
0 & 0 & 0 & t_7 - t_8 & \frac{\sqrt{3}}{2} t_5 & 0 \\
0 & 0 & 0 & 0 & t_7 - 3t_8 & -t_4 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

To describe a decomposition of this ARMS as in (11) and (13), note that $(e_0, \ldots, e_5)$ forms a basis of the ARMS Heisenberg component as in (10), and one can take $g^{\text{red}}_{0,+}$ and $g^{\text{red}}_{0,-}$ to be the subspaces spanned by $e_5$ and $e_6$ respectively. Consequently, $g^{0,\text{red}}_0$ is spanned by $e_7$ and $e_8$. Notice that $(2e_5, 2e_6, e_8)$ is a standard $\mathfrak{sl}_2$ triple, and hence $g^{0,\text{red}}$ is not solvable.

Let $\mathfrak{a}$ denote the standard universal Tanaka prolongation of $g^{0,\text{red}}$, calculated with respect to the graded decomposition $g^{0,\text{red}} = g_{-2} \oplus g_{-1} \oplus g^{0,\text{red}}_0$ with $g^{0,\text{red}}_0$ regarded as the degree zero component. Calculating $\mathfrak{a}$ explicitly, one finds from its Killing form and Cartan’s criterion that $\mathfrak{a}$ is a semisimple, 14-dimensional, rank 2, complex Lie algebra, which from the well known classification of semisimple Lie algebras implies that $\mathfrak{a}$ is isomorphic to the Lie algebra of the exceptional complex Lie group $G_2$.

To relate the well know structure of $\text{Lie}(G_2)$ to our considered subalgebra $g^{0,\text{red}}$, consider the $\mathbb{Z}$-graded decomposition

$$
u = u_{-2} \oplus u_{-1} \oplus u_0 \oplus u_1 \oplus u_2
$$

with $u_{-2} = g_{-2}$, $u_{-1} = g_{-1}$, and $u_0 = g^{0,\text{red}}_0$. In our chosen basis of $g^{0,\text{red}}_0$, the Cartan subalgebra of $\mathfrak{a}$ is $u_{0,0} := \langle e_7, e_8 \rangle$. We describe the root space decomposition of $\mathfrak{a}$ by letting $u_{j,k}$ denote the intersection of the eigenspace of $\text{ad}(-e_7)$ with eigenvalue $j$ and the eigenspace of $\text{ad}(e_8)$ with eigenvalue $k$, and have the following root space decomposition diagram of $\mathfrak{a}$ labeled with these components.

Here $u_{-2,0} = \langle e_0 \rangle$, $u_{-1,-3} = \langle e_4 \rangle$, $u_{-1,-1} = \langle e_3 \rangle$, $u_{-1,1} = \langle e_1 \rangle$, $u_{-1,3} = \langle e_2 \rangle$, $u_{0,-2} = \langle e_6 \rangle$, and $u_{0,2} = \langle e_5 \rangle$.

We now need several basic facts about the Tanaka-theoretic prolongation constructions in [26]. We recall these facts here, and refer the reader to [26] for a full description of the Tanaka theory. Geometric Tanaka prolongations $P^1$ and $P^2$ of $P^0,\text{red}$ constructed in [26] Section 9] are special fiber bundles fitting into a sequence of fiber bundles

$$\mathcal{CN} \leftarrow \pi \mathcal{CM} \xleftarrow{\mathcal{P}^{0,\text{red}}} P^0,\text{red} \xrightarrow{\mathcal{P}^1} P^1 \xrightarrow{\mathcal{P}^2} P^2,$$

where $\mathcal{CM}$ and $\mathcal{CN}$ denote the complexified CR manifold and Levi leaf space, defined locally in [26] essentially by just replacing real local coordinates with complex ones. The complex structure on $\mathcal{CM}$ induces an antilinear involution on $\mathfrak{a}$ that extends the involution already defined on $g^{0,\text{red}}$. A structure on the Levi leaf space $\mathcal{CN}$ called a dynamical Legendrian contact structure (DLC structure) is introduced in [26], and this structure is induced by the CR structure on $\mathcal{CM}$. Symmetries of the DLC structure on $\mathcal{CN}$ have a naturally induced action on each bundle $P^0,\text{red}$, $P^1$, and $P^2$. The prolongation procedure also yields an absolute parallelism on $P^2$ identifying each tangent space in $P^2$.
with $u$, and the symmetry group of $(CM, H)$ is embedded in the symmetry group of the DLC structure on $CN$ which in turn is embedded in the symmetries of this parallelism via the aforementioned natural action on $P^2$.

Let us fix a set of points $q \in N$, $\psi_0 \in P^{0,\text{red}}$, and $\psi_1 \in P^1$ satisfying 

$$\text{pr}_1(\psi_1) = \psi_0 \quad \text{and} \quad \text{pr}_1(\psi_0) = q,$$

and let $G$ denote the symmetry group of the complexified CR manifold $(CM, H)$ with Lie algebra $g$. Let $G_q$ be the subgroup of $G$ whose induced action on $CN$ fixes the point $q$; let $G_0$ be the subgroup of $G_q$ whose induced action on $P^{0,\text{red}}$ fixes the point $\psi_0$; and let $G_1$ be the subgroup of $G_0$ whose induced action on $P^1$ fixes the point $\psi_1$. The Tanaka prolongation procedure naturally induces injective linear maps

$$\text{gr}(g)_- := g/\text{Lie}(G_q) \hookrightarrow u_{-2} \oplus u_{-1}, \quad \text{gr}(g)_0 := \text{Lie}(G_q)/\text{Lie}(G_0) \hookrightarrow u_0,$$

$$\text{gr}(g)_1 := \text{Lie}(G_0)/\text{Lie}(G_1) \hookrightarrow u_1, \quad \text{and} \quad \text{gr}(g)_2 := \text{Lie}(G_1) \hookrightarrow u_2$$

such that the graded Lie algebra 

$$\text{gr}(g) := \text{gr}(g)_- \oplus \text{gr}(g)_0 \oplus \text{gr}(g)_1 \oplus \text{gr}(g)_2$$

obtained from the filtration $g \supset \text{Lie}(G_q) \supset \text{Lie}(G_0) \supset \text{Lie}(G_1)$ is mapped homomorphically onto a subalgebra of $u$. For notational convenience, let us simply identify $\text{gr}(g)$ with that subalgebra. The last fact that we will take for granted here, which follows from the construction of the parallelism on $P^2$, is that infinitesimal symmetries on $(CM, H)$ at a point $o \in CM$ whose values at $o$ are nonzero are identified (modulo symmetries in $\text{Lie}(G_o)$) in one-to-one correspondence with elements in $u_{-2} \oplus u_{-1} \oplus u_{0,-2} \oplus u_{0,2}$, where informally this identification is given by lifting the infinitesimal symmetry to $P^o$ and taking its value at a point in the fiber above $o$.

This last point implies $u_{-2} \oplus u_{-1} \oplus u_{0,-2} \oplus u_{0,2} \subset \text{gr}(g)$. Yet the only 9-dimensional (complex) subalgebra of $u$ having a grading compatible with [32] containing $u_{-2} \oplus u_{-1} \oplus u_{0,-2} \oplus u_{0,2}$ is $u_{-2} \oplus u_{-1} \oplus u_{0}$, and hence

$$\text{gr}(g) = u_{-2} \oplus u_{-1} \oplus u_0 = g^{0,\text{red}}.$$

By [3] Lemma 3], $g \cong \text{gr}(g)$, and hence $g \cong g^{0,\text{red}}$. For any point $o \in P^{0,\text{red}}$, the isotropy subgroup $G_o$ of $G$ fixing $o$ acts freely and transitively on the fiber $P^{0,\text{red}}_o$, and hence the Lie algebra of $G_o$ is exactly the Lie algebra $g^{0,\text{red}}_o$ of the $P^{0,\text{red}}$ bundle's structure group. The symmetry group $RG$ and isotropy subgroup $RG_o$ of $(M, H)$ are generated by the real forms $g^{0,\text{red}}$ and $g^{0,\text{red}}_o$ defined as fixed points in $g^{0,\text{red}}$ and $g^{0,\text{red}}_o$ of the involution on $g^{0,\text{red}}$ induced by the complex structure on $CM$. Since $M = RG/RG_0$ and its CR structure descends from the left invariant distribution $g_{-1,1} \oplus u_{0,0} \oplus u_{0,2}$, this shows that $(M, H)$ is locally equivalent to the flat structure generated by $g^{0,\text{red}}$.

4. Extending and linking ARMS

In this section we describe two processes, extending ARMS and linking ARMS, by which we can construct new ARMS satisfying the subalgebra property (Definition [2.7]) from others. We describe all flat structures of dimension at most 11 that are generated by combinations of ARMS extensions and ARMS links built from the classification in Section [3] of ARMS that generate 7-dimensional flat structures and the 5-dimensional flat structure generated by the ARMS encoded by $(H_1, A, \Omega, \omega_0)$ with $1 \times 1$ matrices $H_1 = A = I$, $\Omega = 0$, and $\omega_0 = \mathbb{C}$ – which happens to be the only ARMS that generates a model of dimension less than 7. While this does not account for all ARMS that generate flat structures of dimension at most 11, it does give a large set of new flat structure examples resulting almost immediately from the ARMS of Theorem [3.1].

For an ARMS \{ $H_1, A_1, \ldots, A_r, \Omega_1, \ldots, \Omega_r, \omega_0$ \} satisfying the subalgebra property (Definition [2.7]), there we two new ARMS called the **positive 2-dimensional extension** and the **negative 2-dimensional extension**.
extension; these two new ARMS are of the form \(
\{ \tilde{H}_\ell, \tilde{A}_1, \ldots, \tilde{A}_r, \tilde{\Omega}_1, \ldots, \tilde{\Omega}_r, \tilde{\mathcal{A}}_0 \}\) with

\[
(53) \quad \tilde{H}_\ell = \begin{pmatrix} H_\ell & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad \tilde{A}_j = \begin{pmatrix} A_j & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
(54) \quad \tilde{\Omega}_j = \begin{pmatrix} \Omega_j & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\mathcal{A}}_0 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} : \alpha \in \mathcal{A}_0 \right\}
\]

for all \(j \in \{1, \ldots, r\}\), where \(\varepsilon = \pm 1\). If \(\varepsilon = 1\) then \((53)\) and \((54)\) gives the positive 2-dimensional extension, whereas \(\varepsilon = -1\) gives the negative 2-dimensional extension. It is easily checked that these extensions also satisfy the subalgebra property (Definition 2.7), thereby generating flat structures themselves, and that, up to a change of sign (positive or negative), these extensions do not depend on the specific matrix representation of the ARMS represented by \(\{H_\ell, A_1, \ldots, A_r, \Omega_1, \ldots, \Omega_r, \mathcal{A}_0\}\).

We call them 2-dimensional extensions because they generate flat structures whose dimension is two greater than that generated by \(\{H_\ell, A_1, \ldots, A_r, \Omega_1, \ldots, \Omega_r, \mathcal{A}_0\}\). Repeatedly constructing these 2-dimensional extensions yields higher dimensional extensions.

**Definition 4.1.** For two non-negative integers \(p, q \in \mathbb{Z}\) and an ARMS \(\mathfrak{g}^{0,\text{red}}\) satisfying the subalgebra property (Definition 2.7) represented by \(\{H_\ell, A_1, \ldots, A_r, \Omega_1, \ldots, \Omega_r, \mathcal{A}_0\}\), the \((2p + q)\)-dimensional signature \((p, q)\) extensions of \(\mathfrak{g}^{0,\text{red}}\) are the ARMS obtained by applying the positive 2-dimensional extension to \(\mathfrak{g}^{0,\text{red}}\) in sequence \(m\) times for \(m \in \{p, q\}\) followed by applying the negative 2-dimensional extension to \(\mathfrak{g}^{0,\text{red}}\) in sequence \(p + q - m\) times.

The order that the 2-dimensional extensions are applied in Definition 4.1 does not actually matter, because changing this order yields an equivalent ARMS in the sense of Definition 2.12. There can be two non-equivalent \((2p + q)\)-dimensional signature \((p, q)\) extensions, which correspond to taking \(m = p\) or \(m = q\) in Definition 4.1 but it can also happen that either choice \(m = p\) or \(m = q\) yields the same ARMS.

By applying 2-dimensional and 4 dimensional extensions to the ARMS of Theorem 3.1 we obtain eleven 9-dimensional flat structures, represented by vertices in the middle ring of the graph in Figure 1B and 19 11-dimensional models, represented vertices in the outer ring of the graph in Figure 1B. Note also that the two structures of types IV.A and IV.B are the 2-dimensional ARMS extensions of the unique ARMS that generates a 5 dimensional flat 2-nondegenerate structure.

Let us now define linking of ARMS with Levi-kernel dimension 1. To prepare this we need to introduce compatibility criteria under which two ARMS can be linked.

Consider the sesquilinear maps of the form

\[
\Psi \circ \Phi : \mathbb{C}^2 \rightarrow \mathbb{C}
\]

for which \(A\) is the matrix from a tuple

\[
(55) \quad \{H_\ell, A, \Omega, \mathcal{A}_0\} \quad \text{(with } A = A_1 \text{ and } \Omega = \Omega_1)\]

representing an ARMS \(\mathfrak{g}^{0,\text{red}}\) satisfying the subalgebra property (Definition 2.7), and with \(\Psi : \mathcal{A}_0 \rightarrow \mathbb{C}\) and \(\Phi : \mathbb{C}^2 \rightarrow \mathcal{A}_0\) defined as follows. For a representation as in (55) of an ARMS satisfying the subalgebra property, define

\[
(56) \quad \Psi(\alpha) := \eta^{1}_{\alpha,1} \quad \forall \alpha \in \mathcal{A}_0,
\]

where \(\eta^{1}_{\alpha,1}\) is defined by item (i) in (17), and let \(\Phi\) be the sesquilinear map given by

\[
(57) \quad \Phi(a, b) := \overline{ab} \left( \frac{H_\ell^{-1} \Omega^T H_\ell}{H_\ell \Omega} + A \overline{\alpha} - \left( \mu^{1}_{1,1} \Omega + \mu^{1}_{1,1} H_\ell^{-1} \Omega^T H_\ell \right) \right)
\]

where the \(\mu^{1}_{1,1}\) is the coefficient defined by item (iii) in (17). Notice that \(\Phi(1, 1)\) is exactly the matrix of item (iv) in (17). To see that \(\Phi(a, b)\) also belongs to \(\mathcal{A}_0\) in general, consider the index 1 matrix
in (12) rescaled by $\mathbf{\sigma}$ and the index 1 matrix in (11) rescaled by $b$ and compute the Lie bracket of these two matrices. By the subalgebra property (Definition 2.7), this Lie bracket belongs to $\mathfrak{g}_0^{\text{red}}$, and taking its $\mathfrak{g}_0^{\text{red}}$ part with respect to (5), the upper left $(n-1) \times (n-1)$ block of this matrix is $\Phi(a,b)$, implying by (13) that $\Phi(a,b) \in \mathcal{A}_0$.

The system (17) can be simplified significantly in our present setting because $r = 1$.

**Lemma 4.2.** Let $\mathfrak{g}_0^{\text{red}}$ be an ARMS of Levi kernel dimension 1 with a representation $\{H_\ell, A, \Omega, \mathcal{A}_0\}$ as in Lemma 2.17. The matrices $A$ and $\Omega$ can be rescaled such that the coefficient $\mu_{1,1}^1$ in (17) is either 1 or 0. Furthermore, if there exists $\alpha \in \mathcal{A}_0$ for which the $\eta_{\alpha,1}^1$ coefficient in (17) is nonzero then $\Omega$ can be chosen so that $\mu_{1,1}^1 = 0$.

Lastly, if $\{H_\ell, A, \Omega, \mathcal{A}_0\}$ can be chosen such that $\mu_{1,1}^1 = 0$ then $\{H_\ell, A, \Omega, \mathcal{A}_0\}$ can be chosen so that both $\mu_{1,1}^1 = 0$ and $\Psi \circ \Phi(1,1) \in \{-1,0,1\}$, where, furthermore, this normalized value of $\Psi \circ \Phi(1,1)$ is uniquely determined by $\mathfrak{g}_0^{\text{red}}$.

**Proof.** Suppose $\mu_{1,1}^1 \neq 0$. By replacing $\{H_\ell, A, \Omega, \mathcal{A}_0\}$ with $\left\{H_\ell, \frac{1}{\mu_{1,1}^1} A, \frac{1}{\mu_{1,1}^1} \Omega, \mathcal{A}_0\right\}$, item (iii) in (17) implies that the new $\mu_{1,1}^1$ coefficient for this new matrix representation of $\mathfrak{g}_0^{\text{red}}$ is equal to 1.

Suppose instead that there exists $\alpha \in \mathcal{A}_0$ for which the $\eta_{\alpha,1}^1$ coefficient in (17) is nonzero. Recall that the decomposition in (5) is not unique. Rather, choosing $\Omega$ and $A$ determines this splitting via (11). The different splittings are in 1-to-1 correspondence representations of $\mathfrak{g}_0^{\text{red}}$ of the form $\{H_\ell, A, \Omega + \alpha, \mathcal{A}_0\}$ for $\alpha \in \mathcal{A}_0$. In particular, if $\eta_{\alpha,1}^1 \neq 0$ then

$$\left\{H_\ell, A, \Omega - \mu_{1,1}^1 \left(\frac{1}{\eta_{\alpha,1}^1}\right)^{-1} \overline{\mathbf{P}_\ell \alpha}^* \mathbf{P}_\ell^{-1}, \mathcal{A}_0\right\}$$

represents the same ARMS as $\{H_\ell, A, \Omega, \mathcal{A}_0\}$ because $\overline{\mathbf{P}_\ell \alpha}^* \mathbf{P}_\ell^{-1}$ also belongs $\mathcal{A}_0$. After replacing $\{H_\ell, A, \Omega, \mathcal{A}_0\}$ by (58), the new $\mu_{1,1}^1$ coefficient appearing in item (iii) of (17) is zero.

Lastly, suppose now that $\{H_\ell, A, \Omega, \mathcal{A}_0\}$ is chosen such that $\mu_{1,1}^1 = 0$. The map $\Psi \circ \Phi$ depends only on the structure coefficients of the Lie algebra $\mathfrak{g}_0^{\text{red}}$ and the splitting in (5), but not the basis of $\mathfrak{g}_- \mathfrak{g}_-$ with respect to which we obtain its matrix representations. So we need to consider all possible representations $\{\tilde{H}_\ell, \tilde{A}, \tilde{\Omega}, \tilde{\mathcal{A}}_0\}$ of $\mathfrak{g}_0^{\text{red}}$ as in Lemma 2.17 for which $\mu_{1,1}^1 = 0$ given with respect to any fixed basis of the form in (10). In general, these representations have the form

$$\tilde{H}_\ell = a H_\ell, \quad \tilde{A} = b A, \quad \tilde{\Omega} = b \Omega + \alpha, \quad \tilde{\mathcal{A}}_0 = \mathcal{A}_0,$$

where $a \in \mathbb{R}$, $b \in \mathbb{C}$, and $\alpha \in \mathcal{A}_0$ has the coefficient $\eta_{\alpha,1}^1 = 0$ in the system (17) calculated with respect to $\{H_\ell, A, \Omega, \mathcal{A}_0\}$; indeed $\alpha$ must satisfy $\eta_{\alpha,1}^1 = 0$ in (59) because otherwise the coefficient $\mu_{1,1}^1$ in the system (17) calculated with respect to $\{\tilde{H}_\ell, \tilde{A}, \tilde{\Omega}, \tilde{\mathcal{A}}_0\}$ would be nonzero.

Letting $\tilde{\Psi} \circ \tilde{\Phi}$ be the maps defined by (56) and (57) with respect to $\{\tilde{H}_\ell, \tilde{A}, \tilde{\Omega}, \tilde{\mathcal{A}}_0\}$ as in (57), we have $\tilde{\Psi} \circ \tilde{\Phi}(1,1) = \Psi \circ \Phi(b,b) = |b|^2 \Psi \circ \Phi(1,1)$. Therefore, by changing the matrix representation of $\mathfrak{g}_0^{\text{red}}$ while keeping $\mu_{1,1}^1 = 0$ we can only change $\tilde{\Psi} \circ \tilde{\Phi}(1,1)$ by a positive coefficient, but any positive coefficient can be achieved. Since $\Psi \circ \Phi$ is sesquilinear, $\Psi \circ \Phi(1,1) \in \mathbb{R}$, and hence the possible transformations of the matrix representations in (59) allow us to normalize $\Psi \circ \Phi(1,1)$ to equal one of the values in $\{-1,0,1\}$ uniquely determined by the original sign of $\Psi \circ \Phi(1,1)$. □

Noting Lemma 4.2 we adopt the following normalization convention.

**Definition 4.3** (normalization condition). A tuple $\{H_\ell, A_1, \Omega_1, \mathcal{A}_0\}$ representing an ARMS $\mathfrak{g}_0^{\text{red}}$ satisfying the subalgebra property (Definition 2.7) with Levi kernel dimension 1 is normalized if either

- (case 1) $\mu_{1,1}^1 = 0$ and $\Psi \circ \Phi(1,1) \in \{-1,0,1\}$, or
- (case 2) $\mu_{1,1}^1 = 1$, or
where in both cases $\mu_{1,1}^1$ is the coefficient appearing in (17).

By Lemma 4.2, the value of $\mu_{1,1}^1$ in this normalization is uniquely determined by $g^{0,\text{red}}$, and every ARMS $g^{0,\text{red}}$ indeed has a normalized representation $\{H_\ell,A_\ell,\Omega,\alpha\}$. Using the normalization convention, we characterize compatibility of ARMS.

**Definition 4.4** (compatibility criteria). Two ARMS satisfying the subalgebra property (Definition 2.7) with Levi kernel dimension 1 are compatible if they have normalized representations such that

- the normalized representations have the same respective values of $\mu_{1,1}^1$ and $\Psi \circ \Phi(1,1)$, and
- the image of the linear map $\Psi : \mathcal{A}_0 \rightarrow \mathbb{C}$ defined in (56) with respect to each of the normalized representations is the same.

We can now define the linking of compatible ARMS with Levi kernel dimension 1.

**Definition 4.5.** Let $g^{0,\text{red}}$ and $g^{0,\text{red}}$ be compatible ARMS satisfying the subalgebra property (Definition 2.7) with Levi kernel dimension 1 and with respective normalized representations $\{H_\ell,A_\ell,\Omega,\alpha\}$ and $\{\tilde{H}_\ell,\tilde{A}_\ell,\tilde{\Omega},\tilde{\alpha}\}$. A link of $g^{0,\text{red}}$ and $g^{0,\text{red}}$ is an ARMS $g^{0,\text{red}}$ of the form $\{\tilde{H}_\ell,\tilde{A}_\ell,\tilde{\Omega},\tilde{\alpha}\}$ with

$$\tilde{H}_\ell = \left( \begin{array}{cc} H_\ell & 0 \\ 0 & \epsilon H_\ell \end{array} \right), \quad \tilde{A}_\ell = \left( \begin{array}{cc} A & 0 \\ 0 & A \end{array} \right), \quad \tilde{\Omega} = \left( \begin{array}{cc} 0 & 0 \\ 0 & \Omega \end{array} \right)$$

and

$$\tilde{\alpha} := \left\{ \left( \begin{array}{c} \alpha \\ 0 \\ \frac{1}{\alpha} \end{array} \right) \bigg| \alpha \in \alpha_0, \tilde{\alpha} \in \tilde{\alpha}_0, \text{ and } \eta_{\alpha,1}^1 = \eta_{\tilde{\alpha},1}^1 \right\},$$

where $\epsilon = \pm 1$, and $\eta_{\alpha,1}^1$ and $\eta_{\tilde{\alpha},1}^1$ are the coefficients of item (i) in (17) calculated with respect to $\{H_\ell,A_\ell,\Omega,\alpha\}$ and $\{\tilde{H}_\ell,\tilde{A}_\ell,\tilde{\Omega},\tilde{\alpha}\}$ respectively.

By construction, the linked ARMS of Definition 4.5 satisfy the subalgebra property, and therefore generate homogeneous 2-nondegenerate hypersurface-type CR manifolds. As with the extension construction of Definition 4.1, linking a pair of compatible ARMS can yield two non-equivalent links, which correspond to taking $\epsilon = 1$ or $\epsilon = -1$ in Definition 4.5 but it can also happen that either choice of $\epsilon$ yields the same link.

By linking pairs of ARMS from among the ARMS of Theorem 3.1 as well as the unique ARMS that generates a 5-dimensional homogeneous hypersurface, we obtain six 9-dimensional models, represented by the dashed edges in Figure 1a and 24 11-dimensional models, represented by the solid edges in Figure 1b.

In total, using the classification of ARMS in Theorem 3.1 and simple constructions of linking and extending ARMS, we obtain 14 homogeneous locally non-equivalent structures in $\mathbb{C}^5$ and 38 homogeneous locally non-equivalent structures in $\mathbb{C}^6$. Notice that 9 and 11-dimensional structures obtained by linking pairs of ARMS among those of types IV.A and IV.B and the 5-dimensional flat structure’s ARMS are equivalent to those obtained from 2 and 4-dimensional extensions of the ARMS of types V.A and V.B, which accounts for three 9-dimensional and five 11-dimensional structures appearing in both graphs of Figure 1. All other 9 and 11-dimensional structures described by Figure 1b do not also appear in the graph of Figure 1b.

These totals are obtained purely by linking pairs of ARMS and applying extensions without combining these two operations in sequence. It is natural therefore to check if other structures arise from linking more than two ARMS or by combining the extending and linking constructions, but, in fact, no additional structures arise from such combinations. This last observation readily follows from noticing that all constructions of extending and linking ARMS commute, and the structures of types IV and V are themselves obtained as extensions and linkings of the ARMS generating the 5-dimensional structure. Specifically, all of these combinations of linking and extending the ARMS of 5 and 7-dimensional structures can be reduced to combinations of linking and extending the ARMS.
HOMOGENEOUS 2-NONDEGENERATE HYPERSURFACES

Figure 1. These graphs show extensions and links of ARMS generating 9 and 11-dimensional flat structures. The vertex label marks the ARMS of 7-d. structures obtained as links or extensions of the ARMS that generates the unique flat 5-d. structure in Figure 1a labeled by • (i.e., without using the ARMS labeled by ⊙). All possible combinations written in this reduced form are indeed accounted for in the graphs of Figure 1.

5. EXAMPLES IN $\mathbb{C}^5$

Having introduced links and extensions of ARMS, the following definition is very natural.

**Definition 5.1.** An ARMS is **indecomposable** if it is not equivalent to a combination of links and extensions of other ARMS, and it is **decomposable** otherwise.

In addition to the 14 ARMS found in Section 4 that generate flat structures in $\mathbb{C}^5$, six other such non-equivalent ARMS with Levi-kernel dimension 1 are known to the author and they are listed here, given by their matrix representations $\{H_\ell, A, \Omega, \mathcal{A}_0\}$. All six are indecomposable, as they do not appear in Section 4.

5.1. Example 5.1. Let

\[
A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H_\ell = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
\Omega = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad \mathcal{A}_0 = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & -a & 0 \end{pmatrix} \middle| a \in \mathbb{C} \right\}.
\]
By calculating the complete ARMS represented by \((H_\ell, A, \Omega, \mathcal{A}_0)\) and applying [26, Theorem 6.2] and [25, Theorem 3.8], we find that this structure’s symmetry algebra is 11-dimensional.

5.2. Example 5.2. Let

\[
A = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & \sqrt{3}
\end{pmatrix}, \quad H_\ell = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
\Omega = \frac{1}{\sqrt{3}} \begin{pmatrix}
\frac{1}{2} & -1 & -\sqrt{3} \\
-1 & \frac{1}{2} & \sqrt{3} \\
1 & 1 & \frac{1}{2}
\end{pmatrix}
\]

and \(\mathcal{A}_0 = 0\).

By calculating the complete ARMS represented by \((H_\ell, A, \Omega, \mathcal{A}_0)\) and applying [26, Theorem 6.2] and [25, Theorem 3.8], we find that this structure’s symmetry algebra is 10-dimensional.

5.3. Example 5.3. Let

\[
A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}, \quad H_\ell = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad \Omega = \begin{pmatrix}
-\frac{3}{2} & -1 & 0 \\
0 & -\frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}
\]

and \(\mathcal{A}_0 = 0\). By calculating the complete ARMS represented by \((H_\ell, A, \Omega, \mathcal{A}_0)\) and applying [26, Theorem 6.2] and [25, Theorem 3.8], we find that this structure’s symmetry algebra is 10-dimensional.

5.4. Example 5.4. Let

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad H_\ell = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
\Omega = \begin{pmatrix}
-\frac{3}{2} & -1 & 0 \\
0 & -\frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}
\]

and \(\mathcal{A}_0 = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a \in \mathbb{C} \right\}\).

By calculating the complete ARMS represented by \((H_\ell, A, \Omega, \mathcal{A}_0)\) and applying [26, Theorem 6.2] and [25, Theorem 3.8], we find that this structure’s symmetry algebra is 11-dimensional.

5.5. Example 5.5. Let

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

and \(H_\ell = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}\).

The ARMS represented by this pair \((H_\ell, A)\) and

\[
\Omega = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

and \(\mathcal{A}_0 = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a \in \mathbb{C} \right\}\),

generates a structure that was studied in [22], whereas the ARMS given by \((H_\ell, A)\) and

\[
\Omega = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

and \(\mathcal{A}_0 = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}\),

generates a structure that was studied in [26]. These two structures are compared in [26] Examples 8.2 and 8.3. Their respective symmetry algebras have dimension 16 and 14.
6. Sequences of flat structures in higher dimensions

In [22], for every integer $n \geq 3$, several $(2n+1)$-dimensional homogenous 2-nondegenerate hypersurface-type CR manifolds are explicitly described as flat structures generated by regular CR symbols. In terms of extensions and linkings of ARMS, for the structures with $n > 4$ and Levi kernel rank 1, all examples in [22] are flat structures generated by combinations of linking and extending the ARMS equal to regular CR symbols whose flat structures are hypersurfaces in $\mathbb{C}^3$, $\mathbb{C}^4$, or $\mathbb{C}^5$. In this section, we introduce two sequences of examples of indecomposable ARMS, showing empirically that indecomposable ARMS generating hypersurfaces in $\mathbb{C}^{n+1}$ exist for all $n \geq 2$.

Lemmas 6.1 and 6.2 each give a sequence of indecomposable ARMS whose CR symbols are the same. The ARMS of 6.1 have nilpotent bases, whereas the ARMS of 6.2 do not have completions with a nilpotent base, which shows that Lemmas 6.1 and 6.2 indeed describe non-equivalent structures.

In what follows we let $T_k$ denote the $k \times k$ nilpotent matrix in Jordan normal form whose eigenspace is 1-dimensional, and let $\text{Diag}(a_1, \ldots, a_k)$ denote the diagonal matrix whose $(j, j)$ entry is $a_k$, i.e.,

$$T_k = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 1 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix} \quad \text{and} \quad \text{Diag}(a_1, \ldots, a_k) = \begin{pmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & a_k
\end{pmatrix}.
$$

The following two lemmas give non-equivalent examples of $(2n+1)$-dimensional flat structures generated by ARMS of the form $(H_\ell, A, \Omega, \omega_0)$ with

$$(H_\ell)_{i,j} = \delta_{i+j,n} \quad \text{and} \quad A = T_{n-1}.
$$

**Lemma 6.1.** For each $n \geq 3$, there exists a sequence $(\omega_1, \ldots, \omega_{n-1})$ such that the ARMS encoded by (60) and

$$(\Omega = \text{Diag}(\omega_1, \ldots, \omega_{n-1})T_{n-1}) \quad \text{and} \quad \omega_0 = 0
$$

is a nilpotent subalgebra of $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_-)$, and hence the flat structure generated by this ARMS has the CR symbol encoded by (60). A defining formula for $(\omega_1, \ldots, \omega_{n-1})$ is given in (63).

**Proof.** For $n = 3$, the lemma is satisfied by the type VII structure of Table 1. For $n = 4$, the lemma is satisfied by Example 5.5. For $n > 4$, we define such sequences $(\omega_1, \ldots, \omega_{n-1})$ as follows.

Let $(\lambda_i)$ be the sequence indexed by $\mathbb{Z}$ given by

$$\lambda_1 = \frac{1}{\sqrt{3}}, \quad \lambda_2 = \frac{2}{\sqrt{3}}, \quad \lambda_3 = \frac{1}{\sqrt{3}}, \quad \text{and} \quad \lambda_i = -\lambda_{i+3} \quad \forall i \in \mathbb{Z}.
$$

Consider now the six vectors $v_i^r$ of some length $n - 1$ given by

$$v_i^r = (\lambda_{r+i}, \ldots, \lambda_{r+i+n-1}) \quad \forall i \in \{0, \ldots, 5\}
$$

for some integer $r$, and notice that

$$(v_1^r) \odot (v_2^r) - (v_2^r) \odot (v_1^r) = -(1, \ldots, 1),
$$

independently of $r$, where $v_i \odot v_j$ denotes the Hadamard product. Define

$$(\omega_1, \ldots, \omega_{n-1}) := \left\{ \begin{array}{ll}
\left[ \begin{array}{c} 0 \\
\frac{\bar{n}}{2} \end{array} \right]_6 & \text{if } n \text{ is odd} \\
\left( \frac{n}{2} - 1, \frac{n}{2} - 2, \ldots, 1 - \frac{n}{2} \right) & \text{if } n \text{ is even}
\end{array} \right.
$$

where $\left[ \frac{\bar{n}}{2} \right]_6$ denotes the equivalence class of $\frac{\bar{n}}{2}$ reduced modulo 6. It is straightforward now to check that the ARMS given by (61) and (63) indeed defines a subalgebra of $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_-)$. 

We will show now that indeed this choice of \( \omega_i \) works. Letting \( k \) denote the length of the vectors \( v^r_i \), define the permutation \( R: \{v^r_0, \ldots, v^r_5\} \to \{v^r_0, \ldots, v^r_5\} \) by
\[
R(\kappa_r, \ldots, \kappa_{r+n-1}) := (\kappa_{r+n-2}, \kappa_{r+n-3}, \ldots, \kappa_{r-1}).
\]
In other words,
\[
R(v^r_i) = v^r_{[5-n-i-2r]_0} \quad \forall k \in \mathbb{N}, \ r \in \mathbb{Z}, \ i \in \{0, \ldots, 5\}
\]
where the bracket notation denotes the integer \( 3 - k - i - 2r \) reduced modulo 6.

Applying this last formula, we get
\[
- \left[ \text{Diag}(v^r_i) T_{n-1}, \text{Diag}(R(v^r_i)) T_{n-1} \right] =
\]
\[
= \text{Diag} \left( v^r_i \odot v^r_{[3-n-2r-i]_0} - v^r_{[3-n-2r-i]_0} \odot v^r_{i+1} \right) T_{n-1}^2
\]
and hence, by (62), if \( n \) is odd then
\[
(64) \quad - \left[ \text{Diag}(v^r_i) T_{n-1}, \text{Diag} \left( R \left( v^r_{[3-n]_0} \right) \right) T_{n-1} \right] = - T_{n-1}^2.
\]
Similarly, if \( n \) is even then it is straightforward to check that
\[
(65) \quad \left[ \text{Diag} \left( \frac{n}{2} - 1, \frac{n}{2} - 2, \ldots, 1 - \frac{n}{2} \right) T_{n-1}, \text{Diag} \left( \frac{n}{2} - 2, \frac{n}{2} - 3, \ldots, - \frac{n}{2} \right) T_{n-1} \right] = - T_{n-1}^2.
\]

With \( \Omega, \ H_\ell, \) and \( A \) as in (60), (61), and (62) we get that \( [\Omega, -H_\ell^{-1} \Omega^* H_\ell] \) equals the left side of (64) or (65) depending on the parity of \( n \). It then follows readily from (63) or (65) that
\[
(66) \quad \left[ \begin{pmatrix} \Omega & A \\ 0 & -H_\ell^{-1} \Omega^* H_\ell \end{pmatrix} \right] = 0.
\]
Noting the matrix representation of this ARMS given in Section 2.2, it follows that \( [g_0^{\text{red}}, g_0^{\text{red}}] = 0 \), so \( g_0^{\text{red}} \) is indeed a subalgebra of \( \mathfrak{sp}(g_{-1}) \). Lastly, nilpotency of \( g^{0, \text{red}} \) follows from \( [g_0^{\text{red}}, g_0^{\text{red}}] = 0. \]

**Lemma 6.2.** There exists a unique sequence \( (\omega_1, \omega_2, \ldots) \) such that for any odd integer \( n > 3 \) the ARMS encoded by the \( (n-1) \times (n-1) \) matrices (60),
\[
\Omega = \text{Diag}(n-2, n-4, \ldots, 2-n) + \sum_{k=1}^{(n-1)/2} \text{Diag} \left( \frac{n-2k-1}{2} \omega_k, \frac{n-2k-1}{2} - 1 \omega_k, \ldots, \frac{n-2k-1}{2} - n - 1 \omega_k \right) T_{n-1}^{2k},
\]
and
\[
\mathcal{A}_0 = \text{span} \left\{ \text{Diag}(a, -a, a, -a, \ldots, (-1)^n a) T_{n-1}^{2k+1} \mid a \in \mathbb{C}, k \in \mathbb{N} \right\}
\]
is a subalgebra of \( \mathfrak{g}_{-} \times \mathfrak{sp}(g_{-1}) \), and hence the flat structure generated by this ARMS has the CR symbol encoded by (60). The sequence \( \{\omega_s\} \) begins
\[
\omega_1 = \frac{1}{2}, \quad \omega_2 = \frac{1}{12}, \quad \omega_3 = \frac{1}{40}, \quad \omega_4 = \frac{23}{2520}, \quad \omega_5 = \frac{67}{18144}, \quad \ldots
\]
and is, in general, defined by the recursive formula
\[
(68) \quad \omega_s := \frac{s}{2 - 4s} \sum_{k=1}^{s-1} \omega_{s-k} \omega_k \quad \forall s > 1.
\]
Proof. This proof is a matter of direct calculation, which we outline here in sufficient detail to reproduce. Some explicit formulas are, however, omitted.

Apply induction on \( n \). For the base case of induction, first assume \( n = 5 \), and set \( \omega_1 = \frac{1}{2} \). In this case, by direct calculation with (60) and (67) we get

\[
\begin{pmatrix}
\Omega & A \\
0 & -H^{-1}_{\ell}\Omega^T H_{\ell}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
-\Omega & A \\
\Omega^+ & 0
\end{pmatrix}
\end{pmatrix}
= -2
\begin{pmatrix}
\Omega & A \\
0 & -H^{-1}_{\ell}\Omega^T H_{\ell}
\end{pmatrix}
\begin{pmatrix}
0 & -H^{-1}_{\ell}\Omega^T H_{\ell}
\end{pmatrix}
+ 2
\begin{pmatrix}
0 & -H^{-1}_{\ell}\Omega^T H_{\ell}
\end{pmatrix}
\begin{pmatrix}
0 & -H^{-1}_{\ell}\Omega^T H_{\ell}
\end{pmatrix}
\]

(69)

and

\[
\begin{pmatrix}
\alpha & 0 \\
0 & -H^{-1}_{\ell}\alpha^T H_{\ell}
\end{pmatrix}
\begin{pmatrix}
\Omega & A \\
0 & -H^{-1}_{\ell}\Omega^T H_{\ell}
\end{pmatrix}
= \begin{pmatrix}
\beta & 0 \\
0 & -H^{-1}_{\ell}\beta^T H_{\ell}
\end{pmatrix}
\]

(70)

where

\[
\alpha = \begin{pmatrix}
0 & a_1 & 0 & a_2 \\
0 & 0 & -a_1 & 0 \\
0 & 0 & 0 & a_1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\beta = \begin{pmatrix}
0 & -2a_1 & 0 & 6a_2 + a_1/2 \\
0 & 0 & 2a_1 & 0 \\
0 & 0 & 0 & -2a_1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

It follows that \([\mathfrak{g}_0^{\text{red}}, \mathfrak{g}_0^{\text{red}}] \subset \mathfrak{g}_0^{\text{red}}\), and hence \(\mathfrak{g}_0^{\text{red}}\) is indeed a subalgebra of \(\mathfrak{g}_- \times \mathfrak{sp}(\mathfrak{g}_-)\) when \( n = 5 \).

Proceeding, suppose that for some odd integer \( k > 5 \), we have chosen \( \omega_1, \ldots, \omega_{(k-5)/2} \) such that the lemma holds for all odd \( n < k \), and now let us assume \( n = k \). We will see that this assumption implies (69), but, to begin, let us furthermore assume that our choice satisfies (69) for all \( n < k \). We will choose \( \omega_{(n-3)/2} \) so that (69) holds with \( n = k \), and need to show that such a choice exists.

By direct calculation, one finds that the upper right and lower left \( n \times n \) blocks in the matrix equation (69) holds for any choice of \( \omega_{(n-3)/2} \). Using the assumption that (69) holds for all \( n < k \) with our choice of \( \omega_1, \ldots, \omega_{(k-5)/2} \), it is also straightforward to calculate that with \( n = k \) the \((i, j)\) scalar component of (69) holds for each \((i, j) \notin \{(1, n-2), (2, n-1), (n, n-2), (n, n-1)\}\), that is for every scalar component in which \( \omega_{(n-3)/2} \) does not appear on either side of (69). Lastly, each \((i, j)\) scalar component of (69) for \((i, j) \in \{(1, n-2), (2, n-1), (n, n-2), (n, n-1)\}\) gives the same defining equation for \( \omega_{(n-3)/2} \), namely, (69) with \( s = (n-3)/2 \). In this way, \( \omega_{(n-3)/2} \) is uniquely determined. With this \( \omega_{(n-3)/2} \) set, another direct calculation of the Lie bracket in (70) with \( \alpha \) taken as an arbitrary matrix in \( \mathfrak{a}_0 \) that the ARMS is indeed closed under Lie brackets.

By linking and extending the ARMS of Lemma 6.1 we obtain the following theorem.

**Theorem 6.3.** Every CR symbol encoded by \((H_{\ell}, A)\) with \( A \) nilpotent can be obtained from a homogeneous 2-nildegenerate hypersurface.

**Proof.** If \( A \) has a 1-dimensional eigenspace, then Lemma 6.1 gives an example of an ARMS whose flat structures has the corresponding CR symbol \((H_{\ell}, A)\). For an arbitrary nilpotent \( A \), by [24 Theorem 2.2], we can assume without loss of generality that \( A \) is in Jordan normal form, and thus represents the CR symbol of an ARMS obtained by linking and extending ARMS of the type in Lemma 6.1. For this last point, one needs that indeed all ARMS in Lemma 6.1 are compatible (in the sense of Definition 4.3), and indeed the needed compatibility criteria of Definition 4.4 follow readily from (69). □

**Remark 6.4.** In the very recent work [15], for every CR symbol of the form in Theorem 6.3, we obtain local coordinate descriptions given as defining equations of large classes of homogeneous 2-nildegenerate CR hypersurfaces with the given symbol.
Acknowledgments

The author would like to thank Igor Zelenko for many helpful discussions on this topic. The author was supported by the GACR grant GA21-09220S.

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