Charge-voltage relation for a fractional capacitor

Author: Vikash Pandey

(Dated: July 13, 2020)

Abstract

Most capacitors of practical use deviate from the assumption of a constant capacitance. They exhibit memory and are often described by a time-varying capacitance. It is shown that a direct implementation of the classical relation, $Q(t) = CV(t)$, that relates the charge, $Q(t)$, with the constant capacitance, $C$, and the voltage, $V(t)$, is not applicable when the capacitance is time-varying. The resulting equivalent circuit that emerges from the substitution of, $C$, by, $C(t)$, is found to be inconsistent. Since, $C(t)$, leads to a time-variant system, the current, $\dot{Q}$, that is obtained from the product rule of the differentiation is not valid either. The search for a solution to this problem led to the expression for the charge, that is given by the convolution of the time-varying capacitance with the first-order derivative of the voltage, as, $Q(t) = C(t) \ast \dot{V}(t)$. Coincidentally, this equation also corresponds to the charge-voltage relation for a fractional-capacitor which is probably first reported in this Letter.

*vikashp@ifi.uio.no
It is well established that nearly all capacitors exhibit memory because of the time-dependent relaxation of the dielectric media that is sandwiched in between their plates [1–5]. An efficient way to represent a capacitor’s memory is to assume a time-varying capacitance, $C(t)$. Such an assumption has been used in the study of solid state devices [6, 7], time-varying storage components [8, 9], energy accumulation [10], brain microvascularity [11], and biomimetic membranes [12]. But most of these references use the equation,

$$Q_C(t) = C(t)V(t),$$  \hspace{1cm} (1)

that is motivated from the classical charge-voltage relation of a capacitor, where, $Q_C$ is the accumulated charge, and $V(t)$ is the applied voltage. The current is then obtained following the product rule of the differentiation as:

$$I_C(t) = \frac{d}{dt}Q_C(t) = C(t)\dot{V}(t) + V(t)\dot{C}(t).$$  \hspace{1cm} (2)

However the classical Eqs. (1) and (2), are not valid for a time-varying capacitance. This I show by assuming a capacitor with a capacitance, $C(t)$, that has a contribution from its constant geometric capacitance, $C_0$, and a time-varying capacitance, $C_{\phi}(t)$, due to the dielectric media present in the capacitor. Further assuming, $C_{\phi}(t) = \phi t$, such that $\phi$ is a real constant, and $\phi \neq 0$, implies, $C_{\phi}(t)$, could either increase or decrease linearly in time. So, I have,

$$C(t) = C_0 + C_{\phi}(t) = C_0 + \phi t.$$  \hspace{1cm} (3)

Since capacitances add in parallel circuits, the equivalent circuits are shown in Figs. (a) and (b). Now, assuming the validity of the Eqs. (1) and (2), I have the following relations for the capacitor with a capacitance, $C(t)$,

$$Q_C(t) = (C_0 + \phi t)V(t), \text{ and}$$

$$I_C(t) = C_0\dot{V} + \phi t\dot{V} + V\phi.$$  \hspace{1cm} (5)

On carefully observing the three current terms of the Eq. (5), I find that the first two corresponds to capacitor currents that flows through the capacitors of capacitances, $C_0$, and $C_{\phi}(t)$. But the third term is the Ohmic current that flows through a resistor of resistance, $R = 1/\phi$. Since currents add in parallel branches, Eq. (5), yields a parallel combination of the three elements as shown in Fig. (c). The resulting circuit is clearly not equivalent.
FIG. 1. (a) Circuit containing a time-varying capacitor, $C(t)$. (b) Equivalent model of (a), assuming a linearly time-varying capacitance, $C(t) = C_0 + \phi t$. (c) The incorrect circuit model with an additional resistor that emerges from the application of the classical relation, $Q_C(t) = C(t)V(t)$, to (b).

to the circuit shown in Fig. 1(b). This anomaly may also be verified on imposing the initial condition of, $t = 0$, in Eq. (5), that leads to, $I_C(t = 0) = C_0\ddot{V} + \phi_0$, instead of the expected, $I_C(t = 0) = I_{C_0} = C_0\dot{V}$. The root cause of this problem can be traced back to the classical equation, Eq. (1), which does not seem to be valid for a time-varying capacitance. The underlying reason behind the inequivalence of the circuit from Fig. 1(b) with the circuit from Fig. 1(c) is that the traditional charge-voltage relation assumes a linearly time-invariant system, i.e., $Q_C(t) = f(V(t))$. In contrast, a time-varying capacitance invokes a time-variant system, $Q_C(t) = f(V(t), t)$. The classical relation for the charge estimation leads to a term-by-term multiplication of $C(t)$ and $V(t)$ at any given instant of time, $t$. But, since both the variables, $C(t)$ and $V(t)$, are time-varying and also have a mutual effect on each other, a convolution operation turns out to be an appropriate option. The solution proposed for the charge estimation is,

$$Q_C(t) = C(t) \ast \dot{V}(t) = [C_0 + \phi t] \ast \dot{V}(t).$$

Further, the derivative property of the convolution leads to the expression for the current as:

$$I_C(t) = C(t) \ast \ddot{V}(t) = \left[ C_0 \ast \ddot{V}(t) \right] + \left[ \phi t \ast \ddot{V}(t) \right].$$

Both of the above equations are dimensionally consistent too. Interestingly, convolutions are also common in the field of fractional derivatives that provide a robust mathematical framework to study time-variant systems. The fractional framework has proven its versatility.
in describing systems that exhibit both spatial memory \cite{13} and temporal memory \cite{14}. The Caputo definition of a fractional derivative for a continuous, causal function, $f(t)$, is the convolution of a regular integer-order derivative with a power-law memory kernel, $\phi_\alpha(t)$, as \cite{15}:

$$\frac{d^\alpha}{dt^\alpha} f(t) \triangleq \dot{f}(t) * \phi_\alpha(t), \quad \phi_\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1,$$

where, $\alpha$ is the order, and $\Gamma(\cdot)$ is the Euler Gamma function that is used as a scaling factor. For negative values of $\alpha$, Eq. (8) corresponds to that of a fractional integral. The Fourier transform property, $\mathcal{F}\left[\frac{d^\alpha f(t)}{dt^\alpha}\right] = (i\omega)^\alpha f(\omega)$, of fractional derivatives, where $\omega$ is the angular frequency, confirm that they are a mere generalization of the regular integer-order derivatives. Furthermore, in appropriate limiting conditions, fractional derivatives asymptotically converge to the integer-order derivatives. In recent years, a connection between the fractional derivatives and the physics of complex media has also been established \cite{14, 16–18}. It is worth noting that the expressions for, $Q_C$ and $I_C$, in Eqs. (6) and (7), are actually motivated from the fractional derivatives. The current expression, Eq. (7), when seen in light of Eq. (8), gives,

$$I_C(t) = C_0 \dot{V}(t) + \phi V(t).$$

Also, at any instant of time, $V(t) \equiv t \dot{V}(t)$, which when substituted back in the Eq. (9), leads to,

$$I_C(t) = I_{C_0}(t) + I_{C_\phi}(t), \quad \text{where } I_{C_0}(t) = C_0 \dot{V}(t), \quad \text{and } I_{C_\phi} = C_\phi \dot{V}(t),$$

are the currents that flow through the respective branches of the circuit that contain the constant capacitance, $C_0$, and the time-varying capacitance, $C_\phi$. It should be emphasized that if the condition, $V(t) \equiv t \dot{V}(t)$, was applied on the last term of the Eq. (5), that would have led to a capacitor current identical to the second term of the same equation. The resulting circuit would then be again inequivalent to the circuit shown in Fig. 1(b). It is also possible to obtain Eq. (7) using the standard convolution integral. However if the time-varying capacitance is expressible in the form of a power-law, then fractional framework turns out to be a readily available tool for the study of those problems. It can be seen that Eq. (10) is equivalent to the current flowing in the circuit shown in Fig. 1(b). Thus the inequivalence that arose due to the conventional charge-voltage relation, Eq. (1), is restored through the convolution equation, Eq. (6). It can be inferred that the additional unwanted
term, $V \phi$, from Eq. (3), that had its origin from the term, $V(t) \dot{C}(t)$, of Eq. (2), vanishes. Therefore, if the last term of the Eq. (2) is neglected and a direct substitution of, $C(t)$, from Eq. (3), is made in the first term of the Eq. (2), then I get the same result as that from the Eq. (10). So, it can be concluded that the last term, $V(t) \dot{C}(t)$, in the Eq. (2), is really not required, and this has been experimentally verified as well [19]. This further consolidates the results presented in this manuscript.

Unfortunately, the inapplicability of the classical charge-voltage relation for capacitors with time-varying capacitance has been overlooked in the time-varying circuit theory [9], as well as in the basic circuit theory [20]. Even further the circuit simulation tools such as those from Matlab and Micro-Cap [8] also use the classical equations to model current through capacitors with time-varying capacitance, which seems incorrect. This makes those results doubtful that were affected by that ignorance.

It should be noted that though I have assumed a linearly time-varying capacitance, the proof that I have presented here can be generalized to all power-law forms of the time-varying capacitance using fractional derivatives. On replacing, $C(t)$, from Eq. (7), by, $C_0(\tau/t)^{\alpha-1}/\Gamma(2-\alpha)$, and then interpreting the resulting equation in light of Eq. (8), I have,

$$I_C(t) = C_0\tau^{\alpha-1}\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \ast \ddot{V}(t)\right] = C_f \frac{d^\alpha}{dt^\alpha} V(t),$$

(11)

where, $C_f = C_0\tau^{\alpha-1}$ is the pseudocapacitance and $\tau$ is the characteristic time constant. Coincidentally, Eq. (11), turns out to be the expression for the current of a fractional capacitor [21]. This infers that the relation expressed by Eq. (6), i.e., $Q_C(t) = C(t) \ast \dot{V}(t)$, corresponds to the charge-voltage relation of a fractional-capacitor. The relation is different than Eq. (5) from Ref. [21] which seems to have a dimensional inconsistency. The fractional capacitor has an interpolating behavior between a resistor and a capacitor for, $0 < \alpha < 1$. Besides, because of its constant phase angle, $|\alpha \pi/2|$, property, the fractional capacitor is also referred to as the constant phase element. However, most capacitors of daily applications exhibit, $\alpha \approx 1$, [2, 3], so fractional capacitors for other values of, $\alpha$, are fabricated in laboratories [22–24].

Since a power-law equals an infinite weighted sum of Debye relaxation responses, the fractional capacitor has also been modelled using an infinite ladder network of resistors and capacitors [25]. But that did not yield any physical interpretation of its parameters, $\tau$ and $\alpha$, which was however provided recently in Ref. [18]. In addition to that interpretation, the
finding of the charge-voltage relation for a fractional capacitor reported here, should further pave way in the emerging field of fractional-order circuits and systems [26]. The applications include modeling of, biological media [11, 12], supercapacitors [27, 28], electrochemical capacitors [29], and the design of the filters [30].

If the capacitance is assumed to be a constant, then results from Eqs. (6) and (7), reduce to the classical relations, \( Q_C(t) = CV(t) \) and \( I_C(t) = C\dot{V}(t) \), respectively, that is expected from a time-invariant system. This can be witnessed from the first term that appears on the right hand side of Eq. (10). The same also mirrors from the Eq. (11), for \( \alpha = 1 \). Therefore, the convolution equations, Eqs. (6) and (7), which correspond to the fractional capacitor, should be seen as relations that complete the bigger picture and yet retain the beauty of the classical relations.

[1] A. K. Jonscher. The “universal” dielectric response. Nature, 267:673–679, 1977.
[2] S. Westerlund. Dead matter has memory! Phys. Scripta, 43:174–179, 1991.
[3] S. Westerlund and L. Ekstam. Capacitor theory. IEEE T. Dielect. El. In., 1:826–839, 1994.
[4] M. Ershov, H. C. Liu, L. Li, M. Buchanan, and Z. R. Wasilewski. Unusual capacitance behavior of quantum well infrared photodetectors. Appl. Phys. Lett., 70:1828–1830, 1997.
[5] V. Uchaikin, R. Sibatov, and D. Uchaikin. Memory regeneration phenomenon in dielectrics: the fractional derivative approach. Phys. Scr., 2009:014002, 2009.
[6] M. Lee and K. Asada. Deep-submicrometer CMOS/SIMOX delay modeling by time-dependent capacitance model. IEEE Trans. Electron. Devices, 40:1897–1901, 1993.
[7] J. R. Brauer. Time-varying resistors, capacitors, and inductors in nonlinear transient finite element models. IEEE Trans. Magn., 34:3086–3089, 1998.
[8] D. Biolek, Z. Kolka, and V. Biolkova. Modeling time-varying storage components in PSpice. In Proceedings of the Electronic Devices and Systems IMAPS CS International Conference EDS, 2007:39–44, 2007.
[9] J. A. Richards. Analysis of periodically time-varying systems. Springer Science & Business Media, 2012.
[10] M. S. Mirmoosa, G. A. Ptitcyn, V. S. Asadchy, and S. A. Tretyakov. Time-varying reactive elements for extreme accumulation of electromagnetic energy. Phys. Rev. Applied, 11:014024,
[11] D. H. Jo, R. Lee, J. H. Kim, H. O. Jun, T. G. Lee, and J. H. Kim. Real-time estimation of paracellular permeability of cerebral endothelial cells by capacitance sensor array. *Sci. Rep.*, 5:1–9, 2015.

[12] J. S. Najem, M. S. Hasan, R. S. Williams, R. J. Weiss, G. S. Rose, G. J. Taylor, S. A. Sarles, and C. P. Collier. Dynamical nonlinear memory capacitance in biomimetic membranes. *Nat. Commun.*, 10, 2019.

[13] V. Pandey, S. P. Näsholm, and S. Holm. Spatial dispersion of elastic waves in a bar characterized by tempered nonlocal elasticity. *Fract. Calc. Appl. Anal.*, 19:498–51, 2016.

[14] V. Pandey and S. Holm. Linking the fractional derivative and the Lomnitz creep law to non-newtonian time-varying viscosity. *Phys. Rev. E*, 94:032606, 2016.

[15] F. Mainardi. *Fractional calculus and waves in linear viscoelasticity: An Introduction to mathematical models*. Imperial College Press, London, 2010.

[16] V. Pandey and S. Holm. Connecting the grain-shearing mechanism of wave propagation in marine sediments to fractional order wave equations. *J. Acoust. Soc. Am.*, 140:4225–4236, 2016.

[17] V. Pandey and S. Holm. Connecting the viscous grain-shearing mechanism of wave propagation in marine sediments to fractional calculus. In *78th EAGE Conference and Exhibition*, pages 1–5. European Association of Geoscientists & Engineers, 2016.

[18] V. Pandey. Origin of the curie-von schweidler law and the fractional capacitor from time-varying capacitance. *arXiv:2006.06073*, 2020.

[19] U. Jadli, F. Mohd-Yasin, H. A. Moghadam, J. R. Nicholls, P. Pande, and S. Dimitrijev. The correct equation for the current through voltage-dependent capacitors. *IEEE Access*, 8:98038–98043, 2020.

[20] C. A. Desoer and E. S. Kuh. *Basic circuit theory*. Tata McGraw-Hill Education, 2010.

[21] M. E. Fouda, A. Allagui, A. S. Elwakil, S. Das, C. Psychalinos, and A. G. Radwan. Nonlinear charge-voltage relationship in constant phase element. *Int. J. Electron. Commun.*, 117:153104, 2020.

[22] A. M. Elshurafa, M. N. Almadhoun, K. N. Salama, and H. N. Alshareef. Microscale electrostatic fractional capacitors using reduced graphene oxide percolated polymer composites. *Appl. Phys. Lett.*, 102:232901, 2013.
[23] G. Tsirimokou, C. Psychalinos, A. S. Elwakil, and K. N. Salama. Experimental verification of on-chip CMOS fractional-order capacitor emulators. *Electron. Lett.*, 52:1298–1300, 2016.

[24] D. A. John, S. Banerjee, G. W. Bohannan, and K. Biswas. Solid-state fractional capacitor using MWCNT-epoxy nanocomposite. *Appl. Phys. Lett.*, 2017.

[25] I. S. Jesus and J. A. Tenreiro Machado. Development of fractional order capacitors based on electrolyte processes. *Nonlinear Dyn.*, 56:45–55, 2009.

[26] A. S. Elwakil. Fractional-order circuits and systems: An emerging interdisciplinary research area. *IEEE Circuits Syst. Mag.*, 10:40–50, 2010.

[27] A. Allagui, A. S. Elwakil, M. E. Fouda, and A. G. Radwan. Capacitive behavior and stored energy in supercapacitors at power line frequencies. *J. Power Sources*, 390:142–147, 2018.

[28] A. Allagui, T. J. Freeborn, A. S. Elwakil, M. E. Fouda, B. J. Maundy, A. G. Radwan, Z. Saida, and M. A. Abdelkareem. Review of fractional-order electrical characterization of supercapacitors. *J. Power Sources*, 400:457–467, 2018.

[29] V. Martynyuk and M. Ortigueira and M. Fedula and O. Savenko. Methodology of electrochemical capacitor quality control with fractional order model. *Int. J. Electron. Commun.*, 91:118–124, 2018.

[30] M. C. Tripathy, K. Biswas, and S. Sen. A design example of a fractional-order KerwinHuelsmanNewcomb biquad filter with two fractional capacitors of different order. *Circuits Syst. Signal Process.*, 32:1523–1536, 2013.