ASYMPTOTICS OF THE QUANTUM INVARIANTS FOR SURGERIES ON THE FIGURE 8 KNOT

JØRGEN ELLEGAARD ANDERSEN ∗†
Department of Mathematical Sciences
University of Aarhus, Building 530, Ny Munkegade, 8000 Aarhus, Denmark
andersen@imf.au.dk

SØREN KOLD HANSEN
Department of Mathematics, Kansas State University
138 Cardwell Hall, Manhattan, KS 66506, USA
hansen@math.ksu.edu

Received (Leave 1 inch blank space for publisher.)
Revised

ABSTRACT

We investigate the Reshetikhin–Turaev invariants associated to SU(2) for the 3–manifolds \( M \) obtained by doing any rational surgery along the figure 8 knot. In particular, we express these invariants in terms of certain complex double contour integrals. These integral formulae allow us to propose a formula for the leading asymptotics of the invariants in the limit of large quantum level. We analyze this expression using the saddle point method. We construct a certain surjection from the set of stationary points for the relevant phase functions onto the space of conjugacy classes of nonabelian \( SL(2, \mathbb{C}) \)-representations of the fundamental group of \( M \) and prove that the values of these phase functions at the relevant stationary points equals the classical Chern–Simons invariants of the corresponding flat SU(2)-connections. Our findings are in agreement with the asymptotic expansion conjecture. Moreover, we calculate the leading asymptotics of the colored Jones polynomial of the figure 8 knot following Kashaev [14]. This leads to a slightly finer asymptotic description of the invariant than predicted by the volume conjecture [24].

Keywords: Quantum invariants, surgeries on the figure 8 knot, asymptotic expansions, classical Chern–Simons theory

1. Introduction

In this paper we investigate the large level asymptotics of the Reshetikhin–Turaev invariants of the 3–manifolds obtained by doing surgery on the figure 8 knot with an arbitrary rational surgery coefficient. Let \( X \) be a closed oriented 3–manifold and
let $\tau_r(X)$ be the RT–invariant of $X$ associated to $\mathfrak{sl}_2(\mathbb{C})$ at level $r$, some integer $\geq 2$. The investigations of this paper are motivated by the following conjecture.

Conjecture 1 (Asymptotic expansion conjecture (AEC)). There exist constants (depending on $X$) $d_j \in \mathbb{Q}$ and $b_j \in \mathbb{C}$ for $j = 0, 1, \ldots, n$ and $a_j^l \in \mathbb{C}$ for $j = 0, 1, \ldots, n, \ l = 1, 2, \ldots$ such that the asymptotic expansion of $\tau_r(X)$ in the limit $r \to \infty$ is given by

$$\tau_r(X) \sim \sum_{j=0}^{n} e^{2\pi ir q_j r^d_j} b_j \left( 1 + \sum_{l=1}^{\infty} a^l_j r^{-l} \right),$$

where $q_0 = 0, q_1, \ldots, q_n$ are the finitely many different values of the Chern–Simons functional on the space of flat SU(2)–connections on $X$.

Here $\sim$ means asymptotic expansion in the Poincaré sense, which means the following: Let

$$d = \max\{d_0, \ldots, d_n\}.$$

Then, for any non-negative integer $L$, there is a $c_L \in \mathbb{R}$ such that

$$\left| \tau_r(X) - \sum_{j=0}^{n} e^{2\pi ir q_j r^d_j} b_j \left( 1 + \sum_{l=1}^{L} a^l_j r^{-l} \right) \right| \leq c_L r^{d-L-1}$$

for all levels $r$. Of course such a condition only puts limits on the large $r$ behaviour of $\tau_r(X)$.

Given an arbitrary complex function $f(r)$ defined on the positive integers a little simple argument shows that there at most exists one list of numbers $n \in \{0, 1, \ldots\}$, $q_0, q_1, \ldots, q_n \in \mathbb{R} \cap [0, 1]$, $d_j \in \mathbb{Q}$ and $b_j \in \mathbb{C}$ for $j = 0, 1, \ldots, n$ and $a_j^l \in \mathbb{C}$ for $j = 0, 1, \ldots, n, \ l = 1, 2, \ldots$ such that $q_0 < q_1 < \ldots < q_n, b_j \neq 0$ for $j > 0$ and such that the large $r$ asymptotic expansion of $f(r)$ is given by

$$f(r) \sim \sum_{j=0}^{n} e^{2\pi ir q_j r^d_j} b_j \left( 1 + \sum_{l=1}^{\infty} a^l_j r^{-l} \right).$$

This implies that if the function $r \mapsto \tau_r(X)$ has an asymptotic expansion of the form stated in Conjecture 1 then the $q_j$’s, $d_j$’s, $b_j$’s and the $a_j^l$’s are all uniquely determined by the set of invariants $\{\tau_r(X)\}_{r \geq 2}$, hence they are also topological invariants of $X$. As stated above the AEC already includes the claim that the $q_j$’s are the Chern–Simons invariants. There are also conjectured topological formulae for the $d_j$’s, and $b_j$’s (see e.g. [2] and the references given there).

In general, there should be expressions for each of the $a_j^l$’s in terms of sums over Feynman diagrams of certain contributions determined by the Feynman rules of the Chern–Simons theory. This has not yet been worked out in general, except in the case of an acyclic flat connection and the case of a smooth non-degenerate component of the moduli space of flat connections by Axelrod and Singer, cf. [4], [5], [3].
The AEC, Conjecture 1, however offers in a sense a converse point of view, where one seeks to derive the final output of perturbation theory after all cancellations have been made (i.e. collect all terms with the same Chern–Simons value). This seems actually rather reasonable in this case, since the exact invariant is known explicitly.

The AEC was proved by Andersen in [1] in the case of mapping tori of finite order diffeomorphisms of orientable surfaces of genus at least two using the gauge-theoretic approach to the quantum invariants. Later on the AEC was proved by Hansen in [9] for all Seifert manifolds with orientable base by supplementing the work of Rozansky [31] with the required analytic estimates. In [10] the AEC is further proved for the Seifert manifolds with nonorientable base of even genus.

Using the approach of Reshetikhin and Turaev to the quantum invariants, the AEC has not yet been proved for any hyperbolic 3–manifold. It is therefore particularly interesting to consider surgeries on the figure 8 knot. Let $M_{p/q}$ be the manifold obtained by (rational) Dehn surgery on the figure 8 knot with surgery coefficient $p/q$. Then $M_{p/q}$ has a hyperbolic structure if and only if $|p| > 4$ or $|q| > 1$, see e.g. [25, Theorem 10.5.10] or [33]. (It is well-known that for $|q| = 1$ and $|p| \in \{1, 2, 3\}$ $M_{p/q}$ is a Seifert manifold, and $M_{\pm 4}$ are obtained by gluing together the complement of the trefoil knot and the non-trivial $I$–bundle over the Klein bottle, see e.g. [15, p. 95]. The manifold $M_0$ is a mapping torus of a torus, see Appendix C.) We use here the convention of Rolfsen for surgery coefficients, cf. [30, Chap. 9]. In particular Dehn surgery on a knot $K$ in $S^3$ with surgery coefficient $f \in \mathbb{Z}$ is equal to the boundary of the compact 4–manifold obtained by attaching a 2–handle to the 4–ball using the knot $K$ with framing $f$, see [30, p. 261]. As usual $M_{p/q}$ is given the orientation induced by the standard right-handed orientation of $S^3$.

The advantage of working with surgeries on the figure 8 knot $K$ is that the (normalized) colored Jones polynomial $J_K'(\lambda)$ is known explicitly. In fact

$$J_K'(\lambda) = \sum_{m=0}^{\lambda-1} \xi^{-m\lambda} \prod_{t=1}^{\lambda} \left(1 - \xi^{\lambda-t}\right) \left(1 - \xi^{\lambda+t}\right),$$

where $\xi = \exp(2\pi i/r)$ (and the product is 1 for $m = 0$). The colors $\lambda$ are here dimensions of irreducible representations of the quantum group associated to $\mathfrak{sl}_2(\mathbb{C})$ and the root of unity $\xi$, so $\lambda = 1, 2, \ldots, r$. By the above expression for $J_K'(\lambda)$ we have an explicit formula for the quantum invariant $\tau_r(M_{p/q})$ (see formula (2.17)). Although this formula is completely explicit, it is not clear from it what the leading order large $r$ asymptotics of $\tau_r(M_{p/q})$ is. In order to study this asymptotics, we observe (generalizing from Kashaev’s work) that the product in the expression for the colored Jones polynomial can be expressed in terms of a quotient of two evaluations of Faddeev’s quantum dilogarithm $S_\gamma$ ($\gamma = \pi/r$):

$$J_K'(\lambda) = \sum_{m=0}^{\lambda-1} \frac{\xi^{-m\lambda}}{(1 - \xi^{\lambda})} \frac{S_\gamma(-\pi + 2\gamma(\lambda - m) - \gamma)}{S_\gamma(-\pi + 2\gamma(\lambda + m) + \gamma)},$$

(1.1)
This follows directly from the functional equation

\[(1 + e^{\sqrt{-1} \zeta}) S_\gamma(\zeta + \gamma) = S_\gamma(\zeta - \gamma)\]

which Faddeev’s $S_\gamma$ satisfies. Recall that for $\text{Re}(\zeta) < \pi + \gamma$ we have the expression

\[S_\gamma(\zeta) = \exp \left( \frac{1}{4} \int_{C_R} \frac{e^{\zeta z}}{\sinh(\pi z) \sinh(\gamma z)} \, dz \right),\]

which together with the functional equation determines $S_\gamma$ as a meromorphic function on $\mathbb{C}$. For the so-called top color, i.e. $\lambda = r$, we obtain the slightly simpler expression

\[J'_K(r) = r \sum_{m=0}^{r-1} \frac{S_\gamma(\pi - (2m + 1)\gamma)}{S_\gamma(-\pi + (2m + 1)\gamma)},\]

Then we simply use the residue formula to convert this expression into a contour integral

\[J'_K(r) = \int_{C_r} \tan(\pi r x) \tilde{g}_r(x) \, dx,\]

where $C_r$ is contained in the strip $\{ z \in \mathbb{C} \mid 0 < \text{Re}(z) < 1 \}$ and encloses $(m + \frac{1}{2})/r$ for $m = 0, \ldots, r - 1$, and $\tilde{g}_r$ is a holomorphic function in this strip expressed in terms of the above quotient of $S_\gamma$ functions (see formula (3.21)).

Similarly we get for the quantum invariant with the use of (1.1) and the residue theorem, now a double contour integral, since the quantum invariant also involves a sum over colors:

\[\tau_r(M_{p/q}) = \int_{C_r \times C_r} \cot(\pi r x) \tan(\pi r y) \tilde{f}_{p,q,r}(x,y) \, dx \, dy, \quad (1.2)\]

where $\tilde{f}_{p,q,r}$ is holomorphic on the double strip $\{ z \in \mathbb{C} \mid 0 < \text{Re}(z) < 1 \}^2$ and given by some expression involving quotients of evaluations of $S_\gamma$ functions, and where we furthermore require of $C_r$ that it also encloses $k/r$ for $k = 1, \ldots, r - 1$.

From this it is clear that we need to understand the small $\gamma$ asymptotics of $S_\gamma$.

We have that

\[S_\gamma(\zeta) = \exp \left( \frac{1}{2\sqrt{-1}\gamma} \text{Li}_2(-e^{\sqrt{-1}\zeta}) + I_\gamma(\zeta) \right),\]

where $\text{Li}_2$ is Euler’s dilogarithm function, and where we have certain analytic estimates on $I_\gamma(\zeta)$ (see Lemma 3).

Let us first explain how we use this to give a proof of the volume conjecture of Murakami and Murakami [24] for the figure 8 knot, namely that

\[\lim_{r \to \infty} \frac{2\pi \text{Log}(J'_K(r))}{r} = \text{Vol}(K),\]

where the right-hand side is the hyperbolic volume of the figure 8 knot, i.e. the hyperbolic volume of the complement $S^3 \setminus K$. The basic idea in analyzing the above
Asymptotics of the quantum invariants for the figure 8 knot

Contour integral expression for \( J'_K(r) \) is the following. In the upper half plane we approximate \( \tan \) by \( \sqrt{-1} \) and by \( -\sqrt{-1} \) in the lower half plane. Further we approximate \( S_\gamma \) by the above expression involving only the dilogarithm. In Appendix B we prove the required estimates which allows us to do these approximations and we end up with the following formula for the leading order large \( r \) asymptotics

\[
J'_K(r) \sim r^2 \int_{\epsilon}^{1-\epsilon} e^{r \Phi(x)} dx, \tag{1.3}
\]

where \( \epsilon < 1/(4r) \) is a small positive parameter and

\[
\Phi(x) = \frac{1}{2\pi \sqrt{-1}} \left( \text{Li}_2(e^{-2\pi \sqrt{-1} x}) - \text{Li}_2(e^{2\pi \sqrt{-1} x}) \right).
\]

Now we simply analyze the integral on the right-hand side of (1.3) by the saddle point method. This consists of finding the stationary points of \( \Phi \) and also the directions of steepest descent, see e.g. [6]. (In this paper we use critical point and stationary point interchangeably to mean a point in which the derivative is zero.)

This analysis leads to two interesting results. Firstly, the search for stationary points leads to the hyperbolicity equation for the complement of the figure 8 knot in the 3–sphere. Recall that this complement can be decomposed into two ideal hyperbolic tetrahedra each parametrized by a certain complex number. This decomposition then defines a hyperbolic structure on the complement exactly when the two parameters are equal and satisfy the hyperbolicity equation.

Secondly, we find that the value of the phase function in the relevant stationary point (there is only one such point in this case) is equal to the hyperbolic volume of the knot complement (divided by \( 2\pi \)), hence the leading asymptotics of \( J'_K(r) \) is determined by this volume.

These phenomena were first observed by Kashaev [14] and have been conjectured by Thurston [32] and Yokota [35] to be generally true for hyperbolic knots (see Remark 2).

Ultimately our asymptotic analysis leads to the following

**Theorem 1.** The leading order large \( r \) asymptotics of the colored Jones polynomial evaluated at the top color is given by

\[
J'_K(r) \sim 3^{-1/4} r^{3/2} \exp \left( \frac{r}{2\pi} \text{Vol}(K) \right).
\]

As a corollary we obtain the volume conjecture for the figure 8 knot. We note that none of the proofs so far given in the literature for the volume conjecture for the figure 8 knot have been able to see the finer details of the asymptotic behaviour, namely the polynomial part \( 3^{-1/4} r^{3/2} \).

After completion of this paper we have been informed that D. Zagier has computed the full asymptotic expansion of \( J'_K(r) \) by using the Euler–Maclaurin summation formula, however his techniques seem not applicable to the calculation of the large \( r \) asymptotics of \( \tau_r(M_{p/q}) \).
Asymptotics of the quantum invariants for the figure 8 knot

Let us now return to the study of the large $r$ asymptotics of the quantum invariant $\tau_r(M_{p/q})$. We expect that an analysis of the expression (1.2) paralleling our analysis of $J'_K(r)$ should be applicable. I.e. tan and cot should be approximated by $\pm\sqrt{-1}$ depending on the signs of $\text{Im}(y)$ and $\text{Im}(x)$ and $\tilde{f}_{p,q,r}(x,y)$ by an appropriate expression involving the dilogarithm for some deformation of the relevant part of $C_r \times C_r$. We have partial analytic results supporting this.

To be more specific we propose the following analog of (1.3) for the quantum invariant. Let $d$ be the inverse of $p$ mod $q$ and let $(a,b) \in \{0,1\}^2$ and $n \in \mathbb{Z}$. Define

$$\Phi_n(x,y) = -\frac{dn^2}{q} - \frac{p}{4q} x^2 + \frac{n}{q} x - xy + \frac{1}{4\pi^2} \left( \text{Li}_2(e^{2\pi i(x+y)}) - \text{Li}_2(e^{2\pi i(x-y)}) \right),$$

and

$$\Phi_n^{a,b}(x,y) = a(x + y) + b(x - y) + \Phi_n(x,y). \quad (1.4)$$

Finally, let

$$S = \{(x,y) \in \mathbb{R} \times \mathbb{C} \mid e^{2\pi i y} \notin ]-\infty,0]\}. \quad (1.5)$$

Thus $S$ is the union of infinitely many planes, namely the ones characterized by $x \in \mathbb{R}$ and $\text{Re}(y) \in \frac{1}{2} + \mathbb{Z}$.

**Conjecture 2.** There exist surfaces $\tilde{\Sigma}_{a,b}^{\mu,\nu,n} \subset \mathbb{C}^2$ for $(a,b) \in \{0,1\}^2$, $n \in \mathbb{Z}/|q|\mathbb{Z}$ and $(\mu,\nu) \in \{\pm 1\}^2$ such that the leading order large $r$ asymptotics of the quantum invariant is given by

$$\tilde{\tau}_r(M_{p/q}) \sim C r \sum_{n \in \mathbb{Z}/|q|\mathbb{Z}} \sum_{(a,b) \in \{0,1\}^2} \sum_{(\mu,\nu) \in \{\pm 1\}^2} \mu \nu \times \int_{\tilde{\Sigma}_{a,b}^{\mu,\nu,n}} \tilde{g}_n(x)e^{2\pi ir\Phi_n^{a,b}(x,y)} dx dy, \quad (1.6)$$

where $C$ is a constant only depending on $p$ and $q$ and $\tilde{g}_n$ is some simple $r$–independent function of $x \in \mathbb{C}$. If $p/q \neq 0$ the surfaces $\tilde{\Sigma}_{a,b}^{\mu,\nu,n} \subset \mathbb{C}^2$ can be chosen such that any critical point of $\Phi_n^{a,b}$ belonging to $\tilde{\Sigma}_{a,b}^{\mu,\nu,n}$ also belongs to the set $S$.

Please see Conjecture 3 for the more detailed version of this conjecture, including the precise formula for $\tilde{g}_n$. (We have here for sign-reasons switched to the complex conjugate invariant $\tilde{\tau}_r(M_{p/q}) = \tau_r(M_{p/q}) = \tau_r(M_{p/q})$. In case $p/q = 0$ our results show that we have to include critical points not belonging to $S$ in our surfaces $\tilde{\Sigma}_{a,b}^{\mu,\nu,n}$, see Appendix C.

We now proceed by making an asymptotic analysis of the right-hand side of (1.6) using the saddle point method like in our analysis of (1.3). Thus we need to analyze integrals of the form

$$I_{a,b}^{\mu,\nu,n} = \int_{\tilde{\Sigma}_{a,b}^{\mu,\nu,n}} \tilde{g}_n(x)e^{2\pi ir\Phi_n^{a,b}(x,y)} dx dy. \quad (1.8)$$

Again we have to determine the stationary points of $\Phi_n^{a,b}$ and the values of $\Phi_n^{a,b}$ in the relevant stationary points. The main idea behind the saddle point method is
to deform $\tilde{\Sigma}_{a,b,n}$ so that it contains certain stationary points of $\Phi^{a,b}_n$ satisfying that
the leading order large $r$ asymptotics of $I^{a,b,n}_{\mu,\nu}$ is given by integrating the integrant
of the integral (1.8) over small neighborhoods of these stationary points.

If we let $v = e^{\pi ix}$ and $w = e^{2\pi iy}$, then by exponentiating the two equations
for $(x,y)$ being a stationary point of $\Phi^{a,b}_n$ (see Theorem 2 below) we obtain the equations
\begin{align*}
v^{-p} &= \left(\frac{w - v^2}{1 - v^2 w}\right)^q, \\
v^2 w &= (1 - v^2 w)(w - v^2),
\end{align*}
which are independent of the integer parameters $a, b, n$. To link the asymptotics to
the flat connections (as proposed by the AEC) we then have to relate the relevant
stationary points of the phase functions $\Phi^{a,b}_n$ to the classical SU(2) Chern–Simons
theory on the manifolds $M_{p/q}$. Fortunately, this Chern–Simons theory has been
given a detailed description by Kirk and Klassen [18] using the work of Riley [28],
[29] on the SL(2, $\mathbb{C}$) representation variety of the knot group of the figure 8 knot.
According to Riley the conjugacy classes of the nonabelian elements of this variety
can be represented by a certain set of representations $\rho$ parametrized by $\rho = \rho(s,u)$,
where $(s, u) \in \mathbb{C}^\ast \times \mathbb{C}$ satisfies a certain polynomial equation. (To be precise $\rho(s_1, u_1)$
and $\rho(s_2, u_2)$ are conjugate if and only if $u_2 = u_1 \neq 0$ and $s_2 \in \{ s_1, s_1^{-1} \}$ or $u_2 =
u_1 = 0$ and $s_2 = s_1 \in A$, where $A$ is a certain subset of $\mathbb{C}^\ast$ consisting of 4 points.)

We show that $\rho(s, u)$ defines a SL(2, $\mathbb{C}$)–representation of $\pi_1(M_{p/q})$ if and only if $(v, w) = (s, u + 1)$ is a solution to (1.9) and $v^2 \neq 1$. By a result of Riley it is
known that this representation is conjugate to a SU(2)–representation if and
only if $(s, u) \in S^1 \times \mathbb{R}$. We relate the results of Kirk and Klassen [18] on the
Chern–Simons invariants of flat SU(2)–connections on the manifolds $M_{p/q}$ to the
asymptotic analysis of the quantum invariants $\tau_r(M_{p/q})$ via a detailed analysis of the
relevant critical values of the phase functions $\Phi^{a,b}_n$. Ultimately we arrive at

**Theorem 2.** The map
\[(x, y) \mapsto [\rho_{(e^{\pi ix}, e^{2\pi iy} - 1)}]\]
gives a surjection from the set of critical points $(x, y)$ of the functions $\Phi^{a,b}_n$ with
$x \notin \mathbb{Z}$ onto the set of conjugacy classes of nonabelian SL(2, $\mathbb{C}$)–representations
of $\pi_1(M_{p/q})$. Moreover, $(x, y) \in \mathbb{C}^2$ is a critical point of $\Phi^{a,b}_n$ if and only if
\begin{align*}
2a + \frac{n}{q} &= y + \left(\frac{p}{2q} + 1\right) x + \frac{i}{\pi} \log \left(1 - e^{2\pi i(x+y)}\right), \\
2b + \frac{n}{q} &= y + \left(\frac{p}{2q} - 1\right) x - \frac{i}{\pi} \log \left(1 - e^{2\pi i(x-y)}\right). 
\end{align*}

Furthermore, if $(x, y)$ is a critical point of $\Phi^{a,b}_n$ then $\rho_{(e^{\pi ix}, e^{2\pi iy} - 1)}$ is equivalent to a
SU(2)–representation $\tilde{\rho}$ of $\pi_1(M_{p/q})$ if and only if $(x, y) \in S$ (see (1.5)), and in that case
\[CS(\tilde{\rho}) = \Phi^{a,b}_n(x, y) \pmod{\mathbb{Z}},\]
where $CS$ is the Chern–Simons functional on the space of flat SU(2)–connections on $M_{p/q}$.

Continuing our asymptotic analysis of the integral (1.8) we point out that not all critical points have to give contributions to the asymptotics. The simplest case arise when all the relevant critical points are non-degenerate. Following Conjecture 2 we claim that the only critical points giving a contribution to the leading order large $r$ asymptotics of $\bar{\tau}_r(M_{p/q})$ are (some of) the ones belonging to the set $S$ in (1.5).

In Proposition 1 in Sec. 4 we prove that all these critical points are non-degenerate in case $|p/q| < \sqrt{20}$. In case $|p/q| > \sqrt{20}$ we have the following partial result:
The surjection described in the first part of Theorem 2 restricts to a surjection $\phi$ from the set of critical points belonging to $S$ onto the moduli space $M'_{p/q}$ of flat irreducible SU(2)–connections on $M_{p/q}$. If $p/q > \sqrt{20}$ we prove that all the critical points belonging to $S \setminus C_0$ are non-degenerate, where $C_0$ is a certain set of critical points. To be more precise the set $C_0$ is either empty or it is equal to the preimage of $\phi$ of one point if $p$ is odd and of two points if $p$ is even. We expect that the argument equation following from the first of the equations (1.9) together with the last statement in Proposition 1 should establish that $C_0$ is empty.

From Conjecture 2 we see that the only relevant phase functions $\Phi_{a,b}^{n,m}$ are the ones with $a, b \in \{0, 1\}$. This, however, does not imply that only a proper subset of the conjugacy classes of SL(2, $\mathbb{C}$)–representations of $\pi_1(M_{p/q})$ are in play. In fact, by using (1.10), we see that if $(x, y)$ is a critical point of $\Phi_{a,b}^{n,m}$ then $(x + 2k, y + 2l)$ is a critical point of $\Phi_{n+pk,a+b+l-k}^{a+l+k,b+l,k}$ for any $k, l \in \mathbb{Z}$. Moreover, the critical points $(x, y)$ and $(x + 2k, y + 2l)$ correspond to the same conjugacy class of representations by the surjection in Theorem 2. In particular all points of the moduli space of flat irreducible SU(2)–connections on $M_{p/q}$ could potentially give a contribution to the asymptotics of $\bar{\tau}_r(M_{p/q})$.

One is only likely to succeed in using the saddle point method to calculate the large $r$ asymptotics of an integral of the form (1.8) in case one can deform the surface $\Sigma_{a,b}^{n,m}$ so that it only contains critical point of a certain nice kind. Here we will only consider the case of non-degenerate critical points. Therefore, assume that $(x, y)$ is a non-degenerate critical point of $\Phi_{a,b}^{n,m}$ belonging to $\Sigma_{a,b}^{n,m}$. Then to be able to calculate that critical point’s contribution to the large $r$ asymptotics of the integral in (1.8) we assume that $(x, y)$ is positive definite, meaning that the imaginary part of a certain “twisted” Hessian of $\Phi_{a,b}^{n,m}$ in $(x, y)$ is positive definite, see Definition 1 for the precise definition. Ultimatively we arrive at the following result.

**Theorem 3.** Assume that $p/q \neq 0$. IF Conjecture 2 is true and if all the critical points of the phase functions $\Phi_{a,b}^{n,m}$ belonging to $S$ are non-degenerate and positive definite, then the leading order large $r$ asymptotics of the quantum invariant
\( \tau_r(M_{p/q}) \) is given by

\[
\bar{\tau}_r(M_{p/q}) \sim \frac{\text{sign}(q)}{4\sqrt{|q|}} \sum_{\bar{\rho} \in \mathcal{M}'_{p/q}} e^{2\pi i r CS(\bar{\rho})} b_{\bar{\rho}}, \tag{1.11}
\]

where \( \mathcal{M}'_{p/q} \) is the moduli space of flat irreducible SU(2)-connections on \( M_{p/q} \). For each \( \bar{\rho} \in \mathcal{M}'_{p/q} \) there are \((a, b) \in \{0, 1\}^2, n \in \mathbb{Z}\) and a stationary point \((x, y) \in S\) for the function \( \Phi_{n}^{a, b} \) given by (1.4), i.e. \((x, y)\) solves the equations (1.10), such that \( \bar{\rho} \) is equivalent to \( \rho(e^{\pi i x}, e^{2\pi i y - 1}) \). Moreover

\[ CS(\bar{\rho}) = \Phi_{n}^{a, b}(x, y) \text{ (mod } \mathbb{Z}), \]

and

\[ b_{\bar{\rho}} = m_{\bar{\rho}} e^{\frac{\pi a \sigma_{\bar{\rho}}}{2}} \sin \left( \frac{\pi}{q} (x - 2nd) \right) \left| 1 - 4 \cos(2\pi x) + \frac{p}{2q} \sinh(2\pi \text{Im}(y)) \right|^{-\frac{1}{2}}, \]

where \( m_{\bar{\rho}} \) and \( \sigma_{\bar{\rho}} \) are some integers.

The positive integers \( m_{\bar{\rho}} \) arise from the following two situations. First of all it can happen that two or more critical points of the relevant stationary phase functions correspond to the same irreducible flat SU(2)-connection on \( M_{p/q} \). Secondly, the same stationary point for the same phase function \( \Phi_{n}^{a, b} \) can belong to two or more of the surfaces \( \Sigma_{\mu, \nu, n}^{a, b} \). In fact, we expect that there is a one to one correspondence between the moduli space \( \mathcal{M}'_{p/q} \) of flat irreducible SU(2)-connections on \( M_{p/q} \) and the set of stationary points contributing to the leading large \( r \) asymptotics of \( \bar{\tau}_r(M_{p/q}) \). Moreover, we expect that a contributing stationary point for a relevant phase function \( \Phi_{n}^{a, b} \) belongs to (a deformation) of \( \Sigma_{\mu, \nu, n}^{a, b} \) for all \((\mu, \nu) \in \{\pm 1\}^2\) thus causing \( m_{\bar{\rho}} = 4 \) for all \( \bar{\rho} \in \mathcal{M}'_{p/q} \), see Remark 4. The above result, Theorem 3, coincides with the one found by the second author in [10] for the cases \(|p/q| \in \{1, 2, 3\}\) with \( m_{\bar{\rho}} = 4 \) for all \( \bar{\rho} \).

The invariant \( \tau_r(M_0) \) and its full asymptotic expansion have been calculated by Jeffrey [13]. We show (see Appendix C) that Jeffrey’s result is in agreement with the AEC. In this case the reducible flat SU(2)-connections contribute to the leading asymptotics. We find that the part of the leading order large \( r \) asymptotics of \( \tau_r(M_0) \) associated to the irreducible flat SU(2)-connections on \( M_0 \) is given by the right hand side of (1.11) with all \( m_{\bar{\rho}} = 4 \).

Acknowledgements

The first author thanks the Department of Mathematics and the Mathematical Sciences Research Institute, University of California, Berkeley for their hospitality during several visits, where part of this work was undertaken. The second author thanks the Université Louis Pasteur, Strasbourg, the University of Edinburgh, and the Max–Planck–Institut für Mathematik, Bonn for their hospitality during this
work. He was supported by the European Commission, the Danish Natural Science Research Council and the Max–Planck–Institut für Mathematik, Bonn.

2. The RT-invariant for surgeries on the figure 8 knot

This section is primarily intended to introduce notation. Moreover, we present some preliminary formulas for the colored Jones polynomial of the figure 8 knot and for the RT–invariants of the 3–manifolds $M_{p/q}$.

Let $t = \exp(2\pi\sqrt{-1}/(4r))$, $r$ an integer $\geq 2$, and let $U_t$ be the modular Hopf algebra considered in [27, Sec. 8], i.e. $U_t$ is a finite-dimensional factor of the quantum group $U_\xi(sl_2(C))$, $\xi = t^{4}$. (In [27], and in most literature on the subject, $\xi$ is denoted $q$, but we use in this paper $q$ to mean something different.) For an integer $k$ we let

$$[k] = \frac{t^{2k} - t^{-2k}}{t^2 - t^{-2}} = \frac{\sin(\pi k/r)}{\sin(\pi/r)},$$

sometimes called a quantum integer. For a knot $K$ in $S^3$ we denote by $K_0$ the knot $K$ considered as a framed knot with framing zero. The colored Jones polynomial associated to $U_t$ of a framed oriented knot $K$ with color $\lambda \in \{1, 2, \ldots, r\}$ is denoted $J_K(\lambda)$, and for an oriented knot $K$ in $S^3$ we let $J'_K(\lambda) = J_K(\lambda)/[\lambda]$. Here the colors are the dimensions (as complex vector spaces) of irreducible $U_t$–modules.

Let $N_f$ be the 3–manifold obtained by surgery on $S^3$ along $K$ with surgery coefficient $f \in \mathbb{Z}$. By [16] or [27] the RT–invariant (at level $r - 2$) of $N_f$ is

$$\tau_r(N_f) = \alpha \sum_{k=1}^{r-1} \xi^{(k^2-1)f/4} [k]^2 J'_K(k),$$

(2.12)

where $K$ is given an arbitrary orientation. Here $\alpha = C^{\text{sign}(f)} D^{-2}$, where

$$D = \sqrt{\frac{r}{2}} \frac{1}{\sin(\pi/r)},$$

$$C = \exp\left(\frac{\sqrt{-1}\pi}{4} \frac{3(2 - r)}{r}\right).$$

We use here the normalization of [34]. This is $D^{-1}$ times the normalization of [16] and $C^{-b_1(N_f)} D^{-1}$ times the normalization of [27], where $b_1(N_f)$ is the first Betti number of $N_f$, see [8, Appendix A]. (In the notation of [34], $C = \Delta D^{-1}$.)

Let us next generalize to arbitrary rational surgery. Let $p, q$ be a pair of coprime integers with $q \neq 0$, and let $N_{p/q}$ be the 3–manifold obtained by surgery along $K$ with surgery coefficient $p/q$. Choose $c, d \in \mathbb{Z}$ such that $B = \begin{pmatrix} p & c \\ q & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$. Then (see e.g. [8, Theorem 5.1 and the proofs of Corollary 8.3 and Theorem 8.4]),

$$\tau_r(N_{p/q}) = \left( e^{\frac{i\pi}{2r}} \exp\left(\frac{-i\pi}{2r}\right) \Phi(B)^{-3\text{sign}(pq)} \right) \sqrt{\frac{2}{r}} \sin\left(\frac{\pi}{r}\right) \frac{1}{\sqrt{r}} \sum_{\lambda=1}^{r-1} [\lambda] J'_K(\lambda) \tilde{B}_{\lambda,1},$$
where $\Phi$ is the Rademacher Phi function, see [26], and $\tilde{}$ is the unitary representation of $\text{PSL}(2, \mathbb{Z})$ given by
\[
\tilde{B}_{j,k} = \sqrt{-1} \frac{\text{sign}(q)}{\sqrt{2r|q|}} e^{-\frac{\sqrt{-1}\pi}{2r|q|} \Phi(B)}
\times \sum_{\mu = \pm 1} \sum_{n \in \mathbb{Z}/|q|\mathbb{Z}} \mu \exp \left( \frac{\sqrt{-1}\pi}{2rq} [pj^2 - 2\mu j(k + 2rn\mu) + d(k + 2rn\mu)^2] \right).
\]

By evaluating the sum over $\mu$ we get
\[
\tau_r(N_{p/q}) = a(r) \sum_{n \in \mathbb{Z}/|q|\mathbb{Z}} \exp \left( \frac{2\pi i r}{q} \frac{dn^2}{q} \right)
\times \prod_{k=1}^{r-1} \sin \left( \frac{\pi}{q} \left[ 2nd - \frac{k}{r} \right] \right) \exp \left( \frac{\pi i r}{2q} \left[ p \left( \frac{k}{r} \right)^2 - 4n \frac{k}{r} \right] \right) [k] J'_K(k),
\]
where
\[
a(r) = -\frac{2\text{sign}(q)}{r\sqrt{|q|}} \sin \left( \frac{\pi}{r} \right) e^{\frac{\pi i}{4} \text{sign}(pq)} \exp \left( \frac{\pi i r}{2q} \left[ 3\text{sign}(pq) - \frac{p}{q} + S \left( \frac{p}{q} \right) \right] \right).
\]

Here $S$ is the Dedekind symbol, see e.g. [17]. We note that the quantum invariant $\tau_r$ is independent of the colored Jones polynomial $J'_K(k)$ for the top-color $k = r$.

In the remaining part of this paper $K$ will denote the figure 8 knot unless explicitly stated otherwise. Recall that $M_{p/q}$ denotes the 3–manifold obtained by surgery on $S^3$ along $K$ with surgery coefficient $p/q \in \mathbb{Q}$. By an $R$–matrix calculation (see e.g. [11]) we find that
\[
J'_K(\lambda) = \sum_{m=0}^{\lambda - 1} \xi^{-m\lambda} \prod_{l=1}^{m} (1 - \xi^{\lambda - l})(1 - \xi^{\lambda + l})
\]
for $\lambda = 1, 2, \ldots, r$, where $\prod_{l=1}^{m} (1 - \xi^{k-l})(1 - \xi^{k+l}) = 1$ for $m = 0$. Le and Habiro have obtained a similar formula, cf. [20].

**Remark 1.** Unitarity of the TQFT associated to $U_i$ implies that
\[
\tau_r(-M) = \tau_r(M)
\]
for any 3–manifold $M$, where $\overline{\ }$ means complex conjugation. This formula also follows directly from [16] and the remarks concerning normalization following (2.12).

Since the figure 8 knot is amphicheiral, $M_{-p/q}$ and $M_{p/q}$ are orientation reversing homeomorphic. By (2.15) we therefore have
\[
\overline{\tau_r(M_{p/q})} = \tau_r(M_{-p/q}).
\]
This formula also follows directly by (2.13) and the facts that $J'_K(\lambda)$ is real and $S(-p/q) = -S(p/q)$. That $J'_K(\lambda)$ is real follows by amphicheirality of $K$ but can
also be seen directly from (2.14) by

\[ J'_K(\lambda) = \sum_{m=0}^{\lambda-1} \prod_{l=1}^{m} \xi^{-\lambda/2} \xi^{l/2} (1 - \xi^{\lambda-l}) \xi^{-\lambda/2} \xi^{-l/2} (1 - \xi^{\lambda+l}) \]

\[ = \sum_{m=0}^{\lambda-1} \prod_{l=1}^{m} (\xi^{l/2} - \xi^{-(l/2)}) (\xi^{(\lambda+l)/2} - \xi^{-(\lambda+l)/2}) \]

\[ = \sum_{m=0}^{\lambda-1} (-4)^m m \prod_{l=1}^{m} \sin(\pi (\lambda - l)/r) \sin(\pi (\lambda + l)/r). \]

By (2.13) and (2.14) we get the following preliminary formula for the RT–invariants of the manifolds \( M_{p/q} \):

\[ \tau_r(M_{p/q}) = \frac{ia(r)}{2 \sin(\pi/r)} \sum_{n \in \mathbb{Z}/|q|\mathbb{Z}} \exp \left( \frac{2\pi i r}{q} \frac{dn^2}{q} \right) \]

\[ \times \sum_{k=1}^{r-1} \exp \left( \frac{\pi i r}{2q} \left[ p \left( \frac{k}{r} \right)^2 - 4n \frac{k}{r} \right] \right) \sin \left( \frac{\pi}{q} \frac{2nd - k}{r} \right) \]

\[ \times \sum_{m=0}^{r-1} \frac{\xi^{-(m+1)/2} k^{m}}{(1 - \xi^{k})} \prod_{l=0}^{m} (1 - \xi^{k-l})(1 - \xi^{k+l}). \] (2.17)

3. A complex double contour integral formula for \( \tau_r(M_{p/q}) \)

In this section we derive a complex double contour integral formula for the RT–invariants \( \tau_r(M_{p/q}) \) by using methods similar to Kashaev [14]. When we consider the expression for the summand in the multi sum (2.17), we see that the expression as it stands only makes sense for non-negative integers \( m \). In order to make sense of this expression for arbitrary complex values for \( m \), let us consider the quantum dilogarithm of Faddeev

\[ S_\gamma(\zeta) = \exp \left( \frac{1}{4} \int_{C_R} \frac{e^{\zeta z}}{\sinh(\pi z) \sinh(\gamma z)} dz \right) \]

defined on \( \Delta_\gamma = \{ \zeta \in \mathbb{C} \mid |\Re(\zeta)| < \pi + \gamma \} \), where \( \gamma \in [0,1] \) and \( C_R \) is the contour \( -\infty, -R \} + \mathbb{Y}_R + [R, \infty] \), where \( \mathbb{Y}_R(t) = Re^{\sqrt{-1}(\pi - t)}, t \in [0, \pi] \) and \( R \in [0,1] \).

The function \( S_\gamma : \Delta_\gamma \to \mathbb{C} \) is holomorphic and it satisfies the following well-known functional equation (see [7] or [14]).

**Lemma 1.** For \( \zeta \in \mathbb{C} \) with \( |\Re(\zeta)| < \pi \) we have

\[ (1 + e^{\sqrt{-1} \gamma}) S_\gamma(\zeta + \gamma) = S_\gamma(\zeta - \gamma). \]

For the sake of completeness we have given a proof in Appendix A. We use Lemma 1 to extend \( S_\gamma \) to a meromorphic function on the complex plane \( \mathbb{C} \).
From now on we fix $\gamma = \pi/r$ (so $r > 3$). By Lemma 1 we get that

$$S_\gamma(\zeta) = S_\gamma(\zeta + 2\pi) \prod_{j=0}^{r-1} \left(1 + e^{\sqrt{-1}(\zeta+(2j+1)\pi/r)}\right).$$

If we write $\zeta = -\pi + 2\pi x$ we get that

$$\prod_{j=0}^{r-1} \left(1 + e^{\sqrt{-1}(\zeta+(2j+1)\pi/r)}\right) = \prod_{j=0}^{r-1} \left(1 - w^j e^{2\pi\sqrt{-1}(x+1/j)}\right),$$

where $w = e^{2\pi\sqrt{-1}/r}$. Using $1 - z^r = \prod_{j=0}^{r-1} (1 - wz)$ we get that

$$S_\gamma(-\pi + 2\pi x) = \left(1 + e^{2\pi\sqrt{-1}x}\right) S_\gamma(-\pi + 2\pi(x + 1))$$

for $x \in \mathbb{C}$. Let

$$x_n = \frac{n}{r} + \frac{1}{2r}, \quad n \in \mathbb{Z}.$$ 

Then $x \mapsto S_\gamma(-\pi + 2\pi x)$ is analytic on $\mathbb{C} \setminus \{x_n | n = r, r + 1, \ldots\}$. If $m$ is a positive integer then $\{x_n | n = mr, mr + 1, \ldots, (m+1)r-1\}$ are poles of order $m$, while the points $\{x_n | n = -mr, -mr + 1, \ldots, -mr + r-1\}$ are zeros of order $m$.

Let us use the function $S_\gamma$ to give another expression for $\tau_r(M_{p/q})$. By Lemma 1 we have that

$$\prod_{l=0}^{m}(1 - \xi^{k+l}) = \prod_{l=0}^{m} \frac{S_\gamma(-\pi + 2\gamma(k + l) - \gamma)}{S_\gamma(-\pi + 2\gamma(k + l) + \gamma)}.$$

Therefore

$$\prod_{l=0}^{m}(1 - \xi^{k-l}) = \frac{S_\gamma(-\pi + 2\gamma(k - m) - \gamma)}{S_\gamma(-\pi + 2\gamma(k + \gamma))},$$

$$\prod_{l=0}^{m}(1 - \xi^{k+l}) = \frac{S_\gamma(-\pi + 2\gamma k - \gamma)}{S_\gamma(-\pi + 2\gamma(k + m) + \gamma)}.$$

So

$$\prod_{l=0}^{m}(1 - \xi^{k-l})(1 - \xi^{k+l}) = (1 - \xi^k) \frac{S_\gamma(-\pi + 2\gamma(k - m) - \gamma)}{S_\gamma(-\pi + 2\gamma(k + m) + \gamma)},$$

and then by (2.17)

$$\tau_r(M_{p/q}) = \beta(r) \sum_{n \in \mathbb{Z}/q\mathbb{Z}} \sum_{k=1}^{m} \sum_{m=0}^{r-1} f_{n,r} \left(\frac{k}{r}, \frac{m + 1/2}{r}\right),$$

where

$$f_{n,r}(x, y) = \sin \left(\frac{\pi}{q}(x - 2nd)\right) e^{2\pi ir\left(\frac{4\pi^2 + \frac{\pi}{q}x^2 - \frac{\pi}{q}x - xy}{q}\right)} \frac{S_\gamma(-\pi + 2\pi(x - y))}{S_\gamma(-\pi + 2\pi(x + y))}.$$
Asymptotics of the quantum invariants for the figure 8 knot

\[ \beta(r) = - \frac{ia(r)}{2 \sin(\pi/r)} \]

\[ = \frac{i \text{sign}(q)}{r \sqrt{|q|}} e^{-\frac{i\pi}{r}} \text{sign}(pq) \exp \left( \frac{\pi i}{2r} \left[ 3 \text{sign}(pq) - \frac{p}{q} + S \left( \frac{p}{q} \right) \right] \right). \tag{3.19} \]

Note that \( d \) is equal to the inverse of \( p \) mod \( q \) and that the functions \( f_{n,r} \) are independent of the choice of this inverse. By the remarks following Lemma 1 the functions \( f_{n,r} \) are holomorphic on \( \Omega_r \times \Omega_r \), where

\[ \Omega_s = \left\{ w \in \mathbb{C} \mid -\frac{1}{4s} < \text{Re}(w) < 1 + \frac{1}{4s} \right\} \tag{3.20} \]

for \( s \in ]0, \infty[ \). By the residue theorem we therefore end up with

**Lemma 2.** The quantum invariants of \( M_{p/q} \) are given by

\[ \tau_r(M_{p/q}) = \frac{\beta(r)^2}{4} \sum_{n \in \mathbb{Z}/|q|\mathbb{Z}} \int_{C_1^r} \cot(\pi r x) \left( \int_{C_2^r} \tan(\pi r y) f_{n,r}(x,y) dy \right) dx, \]

where \( \beta(r) \) is given by (3.19) and

\[ f_{n,r}(x, y) = \sin \left( \frac{\pi}{q} (x - 2nd) \right) e^{2\pi i r \left( \frac{\pi^2}{4} x^2 - \frac{\pi}{3} x - xy \right)} S_{\gamma}(\pi + 2\pi(x - y)) \]

and \( C_1^r \) is a closed curve in \( \Omega_r \) such that the poles \( \{ k/r \mid k = 1, 2, \ldots, r - 1 \} \) for \( x \mapsto \cot(\pi r x) \) lies inside \( C_1^r \) and the poles 0 and 1 lies outside \( C_1^r \), and \( C_2^r \) is a closed curve in \( \Omega_r \) such that the poles \( \{(m + 1/2)/r \mid m = 0, 1, \ldots, r - 1\} \) for \( y \mapsto \tan(\pi r y) \) lies inside \( C_2^r \). Both curves are oriented in the anti-clockwise direction.

Using the function \( S_{\gamma} \) we can also express \( J'_K(r) \) as a contour integral. By Lemma 1 we get that

\[ J'_K(r) = r^{r-1} \sum_{m=0}^{r-1} \frac{S_{\gamma}(\pi - (2m + 1)\gamma)}{S_{\gamma}(\pi + (2m + 1)\gamma)}. \]

We have here used that

\[ \frac{S_{\gamma}(-\pi + \gamma)}{S_{\gamma}(\pi - \gamma)} = \prod_{j=1}^{r-1} \frac{S_{\gamma}(\pi - (2j + 1)\gamma)}{S_{\gamma}(\pi + (2j + 1)\gamma)} = \prod_{j=1}^{r-1} \left( 1 - e^{-2\pi \sqrt{-1} \frac{j}{r}} \right) = r. \]

If we put

\[ g_r(x) = \frac{S_{\gamma}(\pi - 2\pi x)}{S_{\gamma}(\pi + 2\pi x)} \]
for \( x \in \Omega_{\frac{1}{2}} \) (see (3.20)) we get

\[
J'_K(r) = r \sum_{m=0}^{r-1} g_r \left( \frac{m + 1/2}{r} \right),
\]

and we can write this sum as the single contour integral

\[
J'_K(r) = \frac{\sqrt{-1r^2}}{2} \int_{C_2^r} \tan(\pi rx) g_r(x) dx,
\]

(3.21)

where \( C_2^r \) is given as in Lemma 2.

4. The large \( r \) asymptotics of \( J'_K(r) \) and \( \tau_r(M_{p/q}) \)

In this section we investigate the large \( r \) asymptotics of \( \tau_r(M_{p/q}) \) or more precisely the leading term of this asymptotics, using the saddle point method. We begin by calculating the large \( r \) asymptotics of \( J'_K(r) \) using the expression (3.21). This calculation will demonstrate the use of the saddle point method and will serve as a warm up for the more difficult considerations of the asymptotics of \( \tau_r(M_{p/q}) \) in the final part of this section.

4.1. Semiclassical asymptotics of the quantum dilogarithm

It is well-known that the semiclassical asymptotics, i.e. the small \( \gamma \) asymptotics of the quantum dilogarithm \( S_\gamma \), is given by Euler’s dilogarithm

\[
\text{Li}_2(z) = -\int_0^z \frac{\text{Log}(1-w)}{w} dw
\]

for \( z \in \mathbb{C} \setminus [1, \infty[. \) Here and elsewhere \( \text{Log} \) denotes the principal logarithm. For \( \zeta \in \mathbb{C} \) satisfying either \( \text{Re}(\zeta) = \pm \pi \) and \( \text{Im}(\zeta) \geq 0 \) or \( |\text{Re}(\zeta)| < \pi \) one can check (see Appendix A) that

\[
\frac{1}{2\sqrt{-1}\gamma} \text{Li}_2(-e^{\sqrt{-1}\zeta}) = \frac{1}{4} \int_{C_R} \frac{e^{cz}}{\sinh(\pi z)\gamma z^2} dz,
\]

hence we have that

\[
S_\gamma(\zeta) = \exp \left( \frac{1}{2\sqrt{-1}\gamma} \text{Li}_2(-e^{\sqrt{-1}\zeta}) + I_\gamma(\zeta) \right)
\]

(4.23)

for such \( \zeta \), where

\[
I_\gamma(\zeta) = \frac{1}{4} \int_{C_R} \frac{e^{cz}}{z \sinh(\pi z)} \left( \frac{1}{\sinh(\gamma z)} - \frac{1}{\gamma z} \right) dz.
\]

Lemma 3. If \( |\text{Re}(\zeta)| < \pi \) then

\[
|I_\gamma(\zeta)| \leq A \left( \frac{1}{\pi - \text{Re}(\zeta)} + \frac{1}{\pi + \text{Re}(\zeta)} \right) \gamma + B \left( 1 + e^{-\text{Im}(\zeta)R} \right) \gamma,
\]
and for $|\text{Re}(\zeta)| \leq \pi$ we have

$$|I_\gamma(\zeta)| \leq 2A + B \left(1 + e^{-\text{Im}(\zeta)R}\right) \gamma,$$

where $A$ and $B$ are positive constants only depending on $R$.

A proof is given in Appendix A. On the unit circle the imaginary part of the dilogarithm is given by Clausen’s function $\text{Cl}_2$, i.e.

$$\text{Im} (\text{Li}_2(e^{i\theta})) = \text{Cl}_2(\theta) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2} = - \int_0^\theta \log \left|2 \sin\left(\frac{t}{2}\right)\right| \, dt$$

for $\theta \in \mathbb{R}$. One sees that $\text{Cl}_2$ is increasing on $[0, \pi/3] \cup [5\pi/3, 2\pi]$ and decreasing on $[\pi/3, 5\pi/3]$. In particular, $\text{Cl}_2$ attains its maximum value at $\pi/3$ and its minimum value at $5\pi/3$. Moreover

$$-\text{Cl}_2\left(\frac{5\pi}{3}\right) = \text{Cl}_2\left(\frac{\pi}{3}\right) = -2 \int_{\pi/6}^\pi \log|2\sin(\phi)|d\phi = 2J\left(\frac{\pi}{6}\right) = \frac{1}{2}\text{Vol}(K),$$

where $J$ is Lobachevsky’s function and $\text{Vol}(K)$ is the hyperbolic volume of the complement of the figure 8 knot, see e.g. [25, Sec. 10.4].

### 4.2. The large $r$ asymptotics of $J'_K(r)$

We calculate the leading term of the large $r$ asymptotics of $J'_K(r)$, using the saddle point method like Kashaev [14]. Our calculation supplements the calculation of Kashaev with the required analytic error estimates. Let

$$C_\varepsilon^2 = C(\varepsilon) = \left\{ \begin{array} { c c c c c c } { \sqrt{-1} + \varepsilon, } & { - \sqrt{-1} + \varepsilon } & { - \sqrt{-1} + \varepsilon, } & { 1 - \varepsilon - \sqrt{-1} } \\ { 1 - \varepsilon - \sqrt{-1}, } & { 1 - \varepsilon + \sqrt{-1} } & { 1 - \varepsilon + \sqrt{-1}, } & { \varepsilon + \sqrt{-1}, } \end{array} \right\},$$

where $\varepsilon \in [0, \frac{1}{4}]$. We let $C_+(\varepsilon)$ be the part of the contour $C(\varepsilon)$ above the real axes and $C_-(\varepsilon)$ the part below the real axes. By (3.21) we have

$$J'_K(r) = \frac{\sqrt{-1}r^2}{2} \left( J_+(r, \varepsilon) + J_-(r, \varepsilon) \right),$$

where

$$J_\pm(r, \varepsilon) = \int_{C_\pm(\varepsilon)} \tan(\pi rx)g_\varepsilon(x)dx.$$ 

Away from the real axis, the factor $\tan(\pi rx)$ can be approximated by $\pm \sqrt{-1}$ depending on whether we are in the upper or lower half-plane. In fact we have

$$|\tan(\pi rx) - \sqrt{-1}| \leq \left\{ \begin{array} { c c c c c c } { 4e^{-2\pi r|\text{Im}(x)|}, } & { \text{Im}(x) \geq \frac{1}{\pi r}, } \\ { 2e^{-2\pi r|\text{Im}(x)|}, } & { \text{Re}(x) \in \mathbb{Z}, \text{Im}(x) \geq 0 } \end{array} \right\}$$

and

$$|\tan(\pi rx) + \sqrt{-1}| \leq \left\{ \begin{array} { c c c c c c } { 4e^{2\pi r|\text{Im}(x)|}, } & { \text{Im}(x) \leq -\frac{1}{\pi r}, } \\ { 2e^{2\pi r|\text{Im}(x)|}, } & { \text{Re}(x) \in \mathbb{Z}, \text{Im}(x) \leq 0 } \end{array} \right\}.$$
Therefore we write
\[ J_{\pm}(r, \varepsilon) = \pm \sqrt{-1} \int_{C_{\pm}(\varepsilon)} g_\varepsilon(x)dx + \int_{C_{\pm}(\varepsilon)} (\tan(\pi rx) \mp \sqrt{-1}) g_\varepsilon(x)dx. \]

The estimates on \(\tan(\pi rx) \pm \sqrt{-1}\) can be used (see Appendix B) to prove that
\[ \left| \sum_{\mu = \pm 1} \int_{C_{\mu}(\varepsilon)} (\tan(\pi rx) - \mu \sqrt{-1}) g_\varepsilon(x)dx \right| \leq K_1 \frac{1}{r}, \tag{4.28} \]
where \(K_1\) is a constant independent of \(r\) and \(\varepsilon\). Let now
\[ \Phi(x) = \frac{1}{2\pi \sqrt{-1}} \left( \text{Li}_2(e^{-2\pi \sqrt{-1}x}) - \text{Li}_2(e^{2\pi \sqrt{-1}x}) \right). \tag{4.29} \]

Note that \(\Phi\) is analytic on \(D = \mathbb{C} \setminus \{ x \in \mathbb{C} | \text{Re}(x) \in \mathbb{Z} \}\) but not in the points \(\mathbb{Z}\), so here we see the reason for using the small deformation parameter \(\varepsilon\). We have
\[ \int_{C_{\pm}(\varepsilon)} g_\varepsilon(x)dx = \int_{C_{\pm}(\varepsilon)} e^{r \Phi(x)}dx \]
\[ + \int_{C_{\pm}(\varepsilon)} (\exp(I\gamma(\pi - 2\pi x) - I\gamma(-\pi + 2\pi x)) - 1) e^{r \Phi(x)}dx. \]

However, as we will see in Appendix B, the estimate in Lemma 3 implies that
\[ \left| \int_{C_{\mu}(\varepsilon)} (\exp(I\gamma(\pi - 2\pi x) - I\gamma(-\pi + 2\pi x)) - 1) e^{r \Phi(x)}dx \right| \leq K_2 \frac{\log(r)}{r} e^{\frac{r}{2\pi} \text{Vol}(K)} \tag{4.30} \]
for \(\mu = \pm 1\), where \(K_2\) is a constant independent of \(r\) and \(\varepsilon\). We will see below that the estimates (4.28) and (4.30) imply that the leading order large \(r\) asymptotics of \(J'_{K}(r)\) is given by
\[ J'_{K}(r) \sim \frac{r^2}{2} \left( \int_{C_{-}(\varepsilon)} e^{r \Phi(x)}dx - \int_{C_{+}(\varepsilon)} e^{r \Phi(x)}dx \right), \tag{4.31} \]
to which we can apply the saddle point method, see e.g. [6, Chap. 5]. First we determine the stationary points of the phase function \(\Phi\). On \(D\) we have
\[ \Phi'(x) = \log \left( 1 - e^{2\pi \sqrt{-1}x} \right) + \log \left( 1 - e^{-2\pi \sqrt{-1}x} \right). \]

If we put \(z = e^{2\pi \sqrt{-1}x}\), then \(\Phi'(x) = 0\) implies that
\[ z^2 - z + 1 = 0. \tag{4.32} \]
The equation (4.32) has the solutions \(z_{\pm} = e^{\pm \sqrt{-1}\pi/3}\). We have \(1 - z_{\pm} = 1/2 \mp i\sqrt{3}/2\) which both have norm 1 and are each others conjugate, so
\[ \log (1 - z_+) + \log (1 - z_-) = 0. \]
We note that \( z_\pm = e^{\pm 2\pi \sqrt{-1} x} \) correspond to the \( x \)-points \( \pm 1/6 + \mathbb{Z} \). These points are non-degenerate critical points. In fact,

\[ \Phi''(x) = 2\pi \sqrt{1 - e^{2\pi \sqrt{-1} x} + 1 - e^{2\pi \sqrt{-1} x}} \]

on \( D \), so in particular \( \Phi''(x_\pm) = \pm 2\pi \sqrt{3} \) for \( x_\pm \in \pm 1/6 + \mathbb{Z} \). The imaginary part of \( \Phi(x) \) is zero for \( x \in \mathbb{R} \) and

\[ \Phi(x) = \frac{1}{2\pi} \text{Im} \left( \text{Li}_2(e^{-2\pi \sqrt{-1} x}) - \text{Li}_2(e^{2\pi \sqrt{-1} x}) \right) = -\frac{\pi}{\sqrt{3}} \text{Cl}_2(2\pi x), \]

by (4.24). Let \( x_\pm \in \pm 1/6 + \mathbb{Z} \). By (4.25) we have \( \text{Cl}_2(2\pi x_-) = -\text{Cl}_2(2\pi x_+) = -\text{Cl}_2(\pi/3) = -\text{Vol}(K)/2 \), i.e.

\[ \Phi(x_\pm) = \mp \frac{1}{2\pi} \text{Vol}(K). \]

By Cauchy’s theorem we have

\[
\int_{C_-(\varepsilon)} e^{r\Phi(x)} \, dx - \int_{C_+(\varepsilon)} e^{r\Phi(x)} \, dx = 2 \int_{C_-(\varepsilon)} e^{r\Phi(x)} \, dx = -2 \int_{C_+(\varepsilon)} e^{r\Phi(x)} \, dx.
\]

Deform \( C_-(\varepsilon) \) to \([\varepsilon, 1 - \varepsilon]\) keeping the end points fixed. This does not change the integral \( \int_{C_-(\varepsilon)} e^{r\Phi(x)} \, dx \). Let \( x_0 = 5/6 \). By terminology borrowed from [6, Sec. 5.4] the axis of the saddle point \( x_0 \) is the real axis (i.e. the directions of steepest descent are along the real axis). From the analysis of [6, Sec. 5.7] it follows that we can find a \( \delta > 0 \) (independent of \( r \) and \( \varepsilon \)) such that \( [x_0 - \delta, x_0 + \delta] \subseteq [1/(4r), 1 - 1/(4r)] \) and such that we have an asymptotic expansion

\[
\int_{-\delta}^{\delta} e^{r\Phi(x_0 + t)} \, dt \sim \frac{1}{3144\sqrt{r}} e^{\pi r \text{Vol}(K)} \left( 1 + \sum_{n=1}^{\infty} d_n r^{-n} \right)
\]

in the limit \( r \to \infty \), where the \( d_n \)'s are certain complex numbers. Finally we note that

\[
\left| \int_{\varepsilon}^{x_0 - \delta} e^{r\Phi(t)} \, dt + \int_{x_0 + \delta}^{1 - \varepsilon} e^{r\Phi(t)} \, dt \right| \leq e^{rc},
\]

where \( c = \max\{-\text{Cl}_2(2\pi(x_0 - \delta))/\pi, -\text{Cl}_2(2\pi(x_0 + \delta))/\pi\} < \text{Vol}(K)/2\pi \), see above (4.25). We have shown

**Lemma 4.** The leading order large \( r \) asymptotics of \( J'_K(r) \) is given by

\[
J'_K(r) \sim 3^{-1/4} r^{3/2} \exp \left( \frac{r}{2\pi} \text{Vol}(K) \right).
\]

(4.33)

In fact

\[
J'_K(r) = 3^{-1/4} r^{3/2} \exp \left( \frac{r}{2\pi} \text{Vol}(K) \right) + O \left( r \log(r) \exp \left( \frac{r}{2\pi} \text{Vol}(K) \right) \right)
\]

in the limit \( r \to \infty \). □
In particular
\[
\lim_{r \to \infty} \frac{2\pi \log(J'_K(r))}{r} = \text{Vol}(K)
\]
as predicted by the volume conjecture of Kashaev [14] and Murakami, Murakami [24] and as proven by Ekholm and others, see [22]. However, the arguments of Ekholm and others can’t see the finer details of the asymptotic behaviour (4.33), namely the polynomial part \(3^{-1/4}r^{3/2}\).

**Remark 2.** The complement \(S^3 \setminus K\) of the figure 8 knot can be decomposed into two ideal hyperbolic tetrahedra each parametrized by a certain complex number. This decomposition then defines a hyperbolic structure on this complement if and only if a certain set of conditions is satisfied. In fact, if the complex parameters for the two tetrahedra are respectively \(a\) and \(b\), then these conditions are equivalent to \(a = b\) and \(b^2 - b + 1 = 0\), which is equation (4.32) after substituting \(b\) for \(z\). We refer to [23, Sec. 3] for more details. This phenomenon, that one finds the hyperbolicity equation for the knot complement as the equation for the stationary points of the phase function, seems to be a general principal for hyperbolic knots as argued by Thurston and Yokota, cf. [32], [35]. However, there are major unsolved analytic difficulties in their approach. Basically they conjecture that one can carry out an asymptotic analysis similar to the one we carried out above for the figure 8 knot. To prove their conjecture one first has to show how to give an exact (multi-dimensional) contour integral formula for the Jones polynomial of a hyperbolic knot like our (3.21). A main part consists of choosing a correct (multi-dimensional) contour. Secondly, one has to carry out an asymptotic analysis similar to the one leading to (4.31). This analysis is relatively simple for the figure 8 knot due to the fact that we have a single (one-dimensional) contour in this case. In general one gets a contour of dimension \(> 1\) and the asymptotic analysis is expected to be harder (as also illustrated by the asymptotic analysis of the double-contour integral expression for the invariant \(\bar{\tau}_r(M_{p/q})\) in Lemma 2, see next section.)

### 4.3. The large \(r\) asymptotics of \(\tau_r(M_{p/q})\)

In Sec. 5 we will see that the signs of the phases in the asymptotics of \(\bar{\tau}_r(M_{p/q}) = \tau_r(M_{p/q})\) agrees with the Chern–Simons values, hence we work with this conjugate invariant. Because of (2.16) we can always obtain the asymptotic expansion of \(\tau_r(M_{p/q})\) by complex conjugation or by replacing either \(p\) by \(-p\) or \(q\) by \(-q\). By Lemma 2 we have

\[
\bar{\tau}_r(M_{p/q}) = \beta_1(r) \sum_{n \in \mathbb{Z}/|q|\mathbb{Z}} \int_{C_1^r \times C_2^r} \cot(\pi rx) \tan(\pi ry) \bar{f}_{n,r}(x, y) dx dy, \quad (4.34)
\]

where

\[
\beta_1(r) = \frac{i \text{sign}(q) r}{4 \sqrt{|q|}} e^{\frac{\pi i}{2} \text{sign}(pq)} \exp \left( -\frac{\pi i}{2r} \left[ 3 \text{sign}(pq) - \frac{p}{q} + S \left( \frac{p}{q} \right) \right] \right) \quad (4.35)
\]
and
\[ f_{n,r}(x, y) = \sin \left( \frac{\pi}{q} (x - 2nd) \right) e^{2\pi ir \left( -\frac{dn^2}{q} + \frac{p}{4q} x^2 + \frac{n}{q} xy \right)} \frac{S_\gamma(-\pi + 2\pi(x - y))}{S_\gamma(-\pi + 2\pi(x + y))}, \]
where \( \gamma = \pi/r \) as usual. Let for \( k, l \in \mathbb{Z} \)
\[ \Omega_{k,l} = \{(x, y) \in \mathbb{C}^2 \mid \text{Re}(x) + \text{Re}(y) \in [k, k+1], \text{Re}(x) - \text{Re}(y) \in [-l, -l+1]\}. \]
For \((x, y) \in \Omega_{k,l}\), we have by (3.18) and (4.23) that
\[ f_{n,r}(x, y) = \sin \left( \frac{\pi}{q} (x - 2nd) \right) \left( 1 + e^{2\pi i(x-y)r} \right)^k \left( 1 + e^{2\pi i(x+y)r} \right)^k \]
\[ \times \exp \left( I_{\gamma}(-\pi + 2\pi(x - y + l)) - I_{\gamma}(-\pi + 2\pi(x + y - k)) \right), \]
where
\[ \Phi_n(x, y) = -\frac{dn^2}{q} - \frac{p}{4q} x^2 + \frac{n}{q} xy + \frac{1}{4\pi^2} \left( \text{Li}_2(e^{2\pi i(x+y)}) - \text{Li}_2(e^{2\pi i(x-y)}) \right). \]
We note that this expression and the above expression for \( f_{n,r} \) are only valid for \((x, y) \in \mathbb{C}^2\) satisfying the two conditions
(i) \( \text{Re}(x) + \text{Re}(y) \notin \mathbb{Z} \lor \text{Im}(x) + \text{Im}(y) \geq 0, \)
(ii) \( \text{Re}(x) - \text{Re}(y) \notin \mathbb{Z} \lor \text{Im}(x) - \text{Im}(y) \geq 0. \)
Observe that for \( k, l \in \{0,1\} \), which corresponds to the four different \( \Omega_{k,l} \) intersecting \( C^1 \times C^2 \), we have that
\[ \left( 1 + e^{2\pi i(x-y)r} \right)^k \left( 1 + e^{2\pi i(x+y)r} \right)^k = \sum_{(a,b) \in F_{k,l}} e^{2\pi i(a(x+y)+b(x-y))r}, \]
where \( F_{k,l} = \{(a, b) \in \{0,1\}^2 \mid a \leq k, \ b \leq l\}. \) Hence, for such \( k, l \) we have
\[ \tilde{f}_{n,r}(x, y) = \sum_{(a,b) \in F_{k,l}} \sin \left( \frac{\pi}{q} (x - 2nd) \right) e^{2\pi ir \Phi_n^{a,b}(x, y)} \]
\[ \times \exp \left( I_{\gamma}(-\pi + 2\pi(x - y + l)) - I_{\gamma}(-\pi + 2\pi(x + y - k)) \right) \]
for \((x, y) \in \Omega_{k,l}, \) where
\[ \Phi_n^{a,b}(x, y) = a(x + y) + b(x - y) + \Phi_n(x, y). \]
(4.36)
Let
\[ \Omega_{k,l}^{\mu,\nu} = \{(x, y) \in \Omega_{k,l} \mid \mu \text{Im}(x) \geq 0, \ \nu \text{Im}(y) \geq 0\}. \]
Conjecture 3. There exists surfaces $\Sigma_{k,l,a,b}^{\mu,\nu,n} \subset \Omega_{k,l}^{\mu,\nu}$ for $(k,l) \in \{0,1\}^2$, $(a,b) \in F_{k,l}$, $(\mu,\nu) \in \{\pm 1\}^2$ and $n \in \mathbb{Z}/|q|\mathbb{Z}$ such that the leading order large $r$ asymptotics of the quantum invariant is given by

$$
\bar{\tau}_r(M_{p/q}) \sim \frac{i\text{sign}(q)r}{4|q|} \tau_{\pm\text{sign}(pq)} \sum_{n \in \mathbb{Z}/|q|\mathbb{Z}} \sum_{(k,l) \in \{0,1\}^2} \sum_{(a,b) \in F_{k,l}} \sum_{(\mu,\nu) \in \{\pm 1\}^2} \frac{\mu \nu}{(4\pi i)^n} \int_{\Sigma_{k,l,a,b}^{\mu,\nu,n}} \sin \left(\frac{\pi}{q}(x - 2nd)\right) e^{2\pi i r \Phi_n^a(x,y)} dx dy.
$$

(4.37)

Moreover, if $p/q \neq 0$ the surfaces $\Sigma_{k,l,a,b}^{\mu,\nu,n} \subset \Omega_{k,l}^{\mu,\nu}$ can be chosen such that any critical point of $\Phi_n^a$ belonging to $\Sigma_{k,l,a,b}^{\mu,\nu,n}$ also belongs to $\Sigma$ in (1.5).

The rational behind this conjecture is that we anticipate an analysis of the expression (4.34) paralleling our analysis of $J_K^r(r)$ should be applicable. I.e. tan and cot should be approximated by $\pm \sqrt{-1}$ depending on the signs of $\text{Im}(y)$ and $\text{Im}(x)$ and $\tilde{f}_{n,r}(x,y)$ by $\left(\sum_{(a,b) \in F_{k,l}}\frac{1}{q}\sin \left(\frac{\pi}{q}(x - 2nd)\right) e^{2\pi i r \Phi_n^a(x,y)}\right)$ for some deformation of the part of $C_1^r \times C_2^r$ which is contained in $\Omega_{k,l}$. We have partial analytic results supporting this conjecture. The factor in front of the sum is simply the leading term in the large $r$ asymptotics of $\beta_1(r)$, see (4.35). In case $p/q = 0$ our results in Appendix C show that we have to include critical points not belonging to $\Sigma$ in our surfaces $\Sigma_{k,l,a,b}^{\mu,\nu,n}$.

In the remaining part of this section we will compute the large $r$ asymptotics of the right-hand side of (4.37). The only task left is to calculate the leading term in the large $r$ asymptotic of the integrals

$$
I = \int_{\Sigma_{k,l,a,b}^{\mu,\nu,n}} \sin \left(\frac{\pi}{q}(x - 2nd)\right) e^{2\pi i r \Phi_n^a(x,y)} dx dy,
$$

(4.38)

where $k,l \in \{0,1\}$, $(a,b) \in F_{k,l}$, $\mu,\nu \in \{\pm 1\}$ and $n \in \mathbb{Z}$. By the properties of the surfaces $\Sigma_{k,l,a,b}^{\mu,\nu,n}$ postulated in Conjecture 3 this leading asymptotics should be calculable by the saddle point method. Let us give some details. Assume that $(x_0, y_0)$ is a critical point of $\Phi = \Phi_n^a$ belonging to the surface $\Sigma = \Sigma_{k,l,a,b}^{\mu,\nu,n}$ and assume that

$$
\text{Im} ((\Phi(x,y) - \Phi(x_0, y_0))) \geq 0
$$

for all $(x, y)$ in a small neighborhood of $(x_0, y_0)$ in $\Sigma$ with equality only in $(x_0, y_0)$. In this case we say that $\Sigma$ pass through the critical point $(x_0, y_0)$ in the directions of steepest descent. Moreover, assume this is satisfied for all critical points of $\Phi$ inside $\Sigma$. Then the main contribution to the integral $I$ in the large $r$ limit is given by integrating over small neighborhoods of the critical points in $\Sigma$.

Let us therefore begin by computing the stationary points of $\Phi_n^a$. To this end, it is more convenient to work with the functions

$$
\Psi_n^a(x,y) = ax + by + \Phi_n(x,y),
$$

(4.39)

$a, b, n \in \mathbb{Z}$, so $\Phi_n^a = \Psi_n^{a+b,a-b}$. 

Let $a, b, n \in \mathbb{Z}$ and put $\Psi = \Psi_{n}^{a,b}$. Let $z = e^{2\pi i x}$ and $w = e^{2\pi i y}$. Then

$$2\pi i \frac{\partial \Psi}{\partial x}(x, y) = 2\pi i (a - y) - \frac{p}{2q} 2\pi i x + \frac{2\pi in}{q} + \text{Log}(1 - zw) - \text{Log}(1 - zw^{-1}),$$

$$2\pi i \frac{\partial \Psi}{\partial y}(x, y) = 2\pi i (b - x) + \text{Log}(1 - zw) + \text{Log}(1 - zw^{-1}),$$

(4.40)

where we have to assume (which we will also assume in what follows) that both $zw$ and $zw^{-1}$ are different from 1.

We will need to specify a certain square root of $z$, namely let $v = e^{\pi i x}$. Then

$$\frac{\partial \Psi}{\partial x}(x, y) = 0 \text{ implies that } v^{-p} = \left( \frac{w - v^2}{1 - v^2 w} \right)^q,$$

(4.41)

and

$$\frac{\partial \Psi}{\partial y}(x, y) = 0 \text{ implies that } (1 - v^2 w)(w - v^2) = v^2 w.$$  

(4.42)

Both equations (4.41) and (4.42) are independent of $a, b,$ and $n$. We note that $(v, w) = (0, 0)$ is the only solution to (4.42) with $v$ or $w$ equal to zero. Note, moreover, that $(v, w)$ is a common solution to (4.41) and (4.42) if and only if $(\bar{v}, \bar{w})$ is a common solution to these two equations. By writing (4.41) as

$$(-1)^q v^{p+2q} = \left( \frac{1 - v^2 w}{1 - v^{-2} w} \right)^q$$

and (4.42) as

$$(1 - v^2 w)(1 - v^{-2} w) = -w$$

we see that $(v, w)$ is a nonzero solution to (4.41) and (4.42) if and only if $(v^{-1}, w)$ is such a solution. If $p$ is even, then $(v, w)$ is a solution to (4.41) and (4.42) if and only if $(-v, w)$ is such a solution.

Let us oppositely begin with a common solution $(v, w) \in \mathbb{C}^* \times \mathbb{C}^*$ to (4.41) and (4.42), where, as usual, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Write $v = e^{\pi i x}$ and $w = e^{2\pi iy}$ with $\text{Re}(x) \in ]-1, 1]$ and $\text{Re}(y) \in ]-1/2, 1/2]$, i.e. $x = \frac{1}{\pi i} \text{Log}(v)$ and $y = \frac{1}{2\pi i} \text{Log}(w)$. One now easily deduce from (4.42) that there exists a unique $b \in \mathbb{Z}$ such that

$$0 = 2\pi i (b - x) + \text{Log} \left( 1 - e^{2\pi i (x+y)} \right) + \text{Log} \left( 1 - e^{2\pi i (x-y)} \right),$$

(4.43)

and from (4.41) we deduce that there exists a unique $n \in \mathbb{Z}$ such that

$$0 = -2\pi iy - \frac{p}{q} \pi ix + \frac{2\pi in}{q} + \text{Log} \left( 1 - e^{2\pi i (x+y)} \right) - \text{Log} \left( 1 - e^{2\pi i (x-y)} \right).$$  

(4.44)

That is, there exists a unique pair of integers $b, n$ such that $(x, y)$ is a stationary point of $\Psi_{n}^{a,b}$. 
Remark 3. Let us make a slight digression by giving some general remarks about
the set of solutions to (4.42). Assume that \( v, w \in \mathbb{C}^* \) and let \( z = v^2 \). Then (4.42)
can be written in the following two ways

\[
\begin{align*}
  z^2 - \left( \frac{w + 1}{w} + 1 \right) z + 1 &= 0, \\
  w^2 - \left( z + \frac{1}{z} - 1 \right) w + 1 &= 0.
\end{align*}
\]

(4.45)

It is straightforward to see that if \((z, w)\) is a solution to (4.45) with
\( w \in \mathbb{R} \setminus \{0\} \), then \( z \) is real and positive if \( w > 0 \),
real and negative if \( w \in ]-\infty, -\frac{3 + \sqrt{5}}{2}] \cup [-\frac{3 - \sqrt{5}}{2}, 0[ \), and \( z \in S^1 \) if

\[-\frac{3 + \sqrt{5}}{2} \leq w \leq -\frac{3 - \sqrt{5}}{2}.
\]

If \((z, w)\) is a solution to (4.45) with \( z \in \mathbb{R} \setminus \{0\} \), then \( w \) is real and negative if \( z < 0 \),
real and positive if \( z \in ]0, \frac{3 - \sqrt{5}}{2}] \cup [\frac{3 + \sqrt{5}}{2}, \infty[ \), and \( w \in S^1 \) if

\[\frac{3 - \sqrt{5}}{2} \leq w \leq \frac{3 + \sqrt{5}}{2}.
\]

Later on we will be particularly interested in common solutions to (4.41) and
(4.42) with \( v \in S^1 \) and \( w \in \mathbb{R} \setminus \{0\} \). Assume that \((v, w)\) is such a solution, and
write \( v = e^{\pi i x} \), \( x \in ]-1, 1[ \). In that case the 2nd of the equations in (4.45) is

\[
  w^2 + (1 - 2\text{Re}(z)) w + 1 = 0.
\]

(4.46)

Since \( z \in S^1 \) we have \( 1 - 2\text{Re}(z) \in [-1, 3] \) and since \( w \) is real we also have \( |1 - 2\text{Re}(z)| \geq 2 \), so \( 1 - 2\text{Re}(z) \in [2, 3] \) or \( \cos(2\pi x) = \text{Re}(z) \in [-1, -1/2] \). By (4.46) we find that

\[
  w = w_\pm(x) = \cos(2\pi x) - \frac{1}{2} \pm \sqrt{\cos^2(2\pi x) - \cos(2\pi x) - \frac{3}{4}}.
\]

(4.47)

If \( w > 0 \) we have already seen that \( z \in ]0, \infty[ \). But then \( z = 1 \) so \( w^2 - w + 1 = 0 \) by
(4.45) contradicting the fact that \( w \) is real.

Taking absolute values we get from (4.41) that \( |1 - zw| = |w - z| \) and then from
(4.42) that \( |w| = |w - z|^2 = (w - z)(w - \bar{z}) = w^2 - 2\text{Re}(z)w + 1 \) (still assuming
that \((v, w) \in S^1 \times \mathbb{R} \)). Since \( w < 0 \) this equation is equivalent to (4.46). In other
words we have to consider the argument equation following from (4.41) to obtain
information from that equation not obtainable from (4.42).

Let us now turn to the second derivative of \( \Psi \) in a critical point \((x_0, y_0)\), i.e.
the Hessian \( H = H(x_0, y_0) \) of \( \Psi \) in \((x_0, y_0)\) (which is equal to the Hessian of \( \Phi_n \) in
\((x_0, y_0))\). Put \((v_0, w_0) = (e^{\pi i x_0}, e^{2\pi i y_0}) \) and \( z_0 = v_0^2 \). By a small computation, using
the fact that \((v_0, w_0)\) is a solution to (4.42), we find that
\[
H_{11} = \frac{\partial^2 \Psi}{\partial x^2}(x_0, y_0) = \frac{1}{w_0} - w_0 - \frac{p}{2q},
\]
\[
H_{12} = H_{21} = \frac{\partial^2 \Psi}{\partial x \partial y}(x_0, y_0) = z_0 - \frac{1}{z_0},
\]
\[
H_{22} = \frac{\partial^2 \Psi}{\partial y^2}(x_0, y_0) = \frac{1}{w_0} - w_0.
\]
(4.48)

Let us begin by establishing under which circumstances the critical point \((x_0, y_0)\) is non-degenerate. By (4.48)
\[
det(H) = \left( w_0 + \frac{1}{w_0} \right)^2 - \left( z_0 + \frac{1}{z_0} \right)^2 - \frac{p}{2q} \left( w_0 - \frac{1}{w_0} \right).
\]
By (4.45) we have
\[
det(H) = 1 - 2 \left( z_0 + \frac{1}{z_0} \right) + \frac{p}{2q} \left( w_0 - \frac{1}{w_0} \right).
\]
(4.49)

If we introduce \(y'_0 = iy_0\) we get the formula
\[
det(H) = 1 - 4 \cos(2\pi x_0) + \frac{p}{2q} \sinh(2\pi y_0).
\]
(4.50)

We are particularly interested in critical points where \(w_0 < 0\) and \(z_0 \in S^1\) (or equivalently \((x_0, y_0) \in S\), see (1.5)). By definition of \(z_0\) we have \(z_0 - 1/z_0 = 2i \sin(2\pi x_0)\) and by (4.48) we then get
\[
det(H) = \left( 1 - \frac{w_0^2}{w_0^2} - \frac{p}{2q} \right) \left( 1 - \frac{w_0^2}{w_0^2} \right) + 4 \sin^2(2\pi x_0).
\]
(4.51)

Thus if \(w_0 \in ]-1,0[\) and \(p/q \geq 0\) or if \(w_0 \in ]-\infty,-1[\) and \(p/q \leq 0\) we have \(\det(H) > 0\). If \(w_0 = -1\) we also have \(\det(H) > 0\) by (4.47). Since \(\Re(y_0) \in \frac{1}{2} + \mathbb{Z}\) we get by (4.49) the alternative formula
\[
det(H) = 1 - 4 \cos(2\pi x_0) + \frac{p}{2q} \sinh(2\pi \Im(y_0)).
\]
(4.52)

Finally we get by (4.47) and (4.49) that
\[
det(H) = 1 - 4 \cos(2\pi x_0) \pm \frac{p}{2q} \sqrt{\cos^2(2\pi x_0) - \cos(2\pi x_0)} - \frac{3}{4}.
\]
(4.53)

for \(w = w_\pm(x_0)\).

**Proposition 1.** Let the situation be as above and assume that \((x_0, y_0)\) is a critical point of \(\Psi\) belonging to the set \(S\) in (1.5). If \(|p/q| < \sqrt{20}\) then \((x_0, y_0)\) is non-degenerate. If \(|p/q| > \sqrt{20}\) then \((x_0, y_0)\) is non-degenerate except if
\[
\Re(z_0) = \cos(2\pi x_0) = \frac{1}{2} + \frac{4 - \frac{|p|}{q} \sqrt{\frac{p^2}{q^2} - 15}}{2 - \frac{|p|}{q} - 16}
\]
and \(w_0 = w_+(x_0)\) if \(p/q < 0\) and \(w_0 = w_-(x_0)\) if \(p/q > 0\).
Proof. Let \( z_0 = e^{2\pi ix_0} \) and \( w_0 = e^{2\pi iy_0} \) as above and put \( c = \cos(2\pi x_0) \). Assume that \((x_0, y_0)\) is degenerate, i.e. that \( \det(H) = 0 \). Then

\[
(16 - \frac{p^2}{q^2}) c^2 + \left(\frac{p^2}{q^2} - 8\right) c + 1 + \frac{3p^2}{4q^2} = 0
\]

by (4.53). If \( p^2/q^2 < 15 \) this equation does not have real solutions, so \( p^2/q^2 \geq 15 \). If \( p^2/q^2 = 16 \) we find that \( c = -13/8 \) giving a contradiction. Thus \( p^2/q^2 \in [15, 16]\cup[16, \infty] \) and

\[
c = \frac{8 - \frac{p^2}{q^2} \pm 2 |\frac{p}{q}| \sqrt{|\frac{p^2}{q^2} - 15|}}{2 \left(16 - \frac{p^2}{q^2}\right)}.
\]

(4.54)

By this and \( c \in [-1, -1/2] \), see below (4.46), we get that

\[
\frac{3p^2}{2q^2} - 20 \leq \pm \left|\frac{p}{q}\right| \sqrt{\frac{p^2}{q^2} - 15} \leq \frac{p^2}{q^2} - 12
\]

(4.55)

for \( p^2/q^2 \in [15, 16] \) with the opposite inequalities for \( p^2/q^2 \in ]16, \infty[ \). For \( p^2/q^2 \in [15, 16] \) we have \( \frac{3p^2}{2q^2} - 20 > 0 \) so we have a plus in front of \( |\frac{p}{q}| \sqrt{|\frac{p^2}{q^2} - 15|} \) in (4.55) and in that case the first inequality in (4.55) is equivalent to

\[
\frac{5}{4} \left(\frac{p}{q}\right)^4 - 45 \left(\frac{p}{q}\right)^2 + 400 \leq 0,
\]

(4.56)

which forces \( p^2/q^2 \in [16, 20] \) giving a contradiction.

Therefore \( p^2/q^2 > 16 \) and (4.54) and \( c \in [-1, -1/2] \) leads, as already stated, to (4.55) with the opposite inequalities. Since \( p^2/q^2 - 12 > 0 \) we again have a plus in front of \( |\frac{p}{q}| \sqrt{|\frac{p^2}{q^2} - 15|} \) and the second inequality is automatically satisfied for all \( p^2/q^2 \geq 16 \). This time the first inequality leads to (4.56) with the opposite inequalities, so we can conclude that \( p^2/q^2 \geq 20 \) and in fact therefore \( |p/q| > \sqrt{20} \).

The formula for \( \cos(2\pi x_0) \) is just (4.54) with the plus sign in front of \( |\frac{p}{q}| \sqrt{|\frac{p^2}{q^2} - 15|} \).

\[\Box\]

We note that the only other solutions \((z, w) \in S^1 \times ]-\infty, 0[\) to (4.46) satisfying that \( \text{Re}(z) \) is equal to the right-hand side of (4.54) are \((z_0, 1/w_0)\), \((\bar{z}_0, w_0)\) and \((\bar{z}_0, 1/w_0)\). By the remarks following (4.51) we get that among these three points only the point \((z_0, w_0)\) can actually satisfy that the right-hand side of (4.51) is zero. We conjecture that also in the case \(|p/q| > \sqrt{20}\) all critical points of \( \Psi \) which belong to the set \( S \) are non-degenerate. To confirm this one has probably to use the argument equation following from (4.41). By a direct check we have confirmed this conjecture in the cases \(|p/q| \in \{14/3, 5, 16/3, 6, 20/3, 22/3, 8, 26/3, 28/3, 10, 32/3\}\).

We are now ready to calculate the contribution to the leading order large \( r \) asymptotics of the integral (4.38) coming from a critical point \((x, y)\) of the phase function \( \Phi_{n,b}^{\mu, \sigma, n} \) belonging to \( S \cap \Sigma_{k,l,a,b}^{\mu, \sigma, n} \).
The integral $I$ in (4.38) is a double contour integral and as such is calculated by choosing a contour for each variable $x$ and $y$. We thus have an “inner” integral and an “outer” integral and in general the contour for the inner integral can depend on where we are on the outer contour. Here we will, however, only consider the very simple case where these two contours are independent of each other.

Therefore let $\alpha, \beta \in S^1$ and let $\gamma_{\alpha}, \gamma_{\beta} : \mathbb{I}_{\delta} \to \mathbb{C}$ be given by $\gamma_{\alpha}(s) = x_0 + \alpha s$, $\gamma_{\beta}(s) = y_0 + \beta s$, where $\mathbb{I}_{\delta} = [-\delta, \delta]$ for a sufficiently small $\delta > 0$. According to the saddle point method the main contributions to the integral (4.38) comes from integrating over a small neighborhood of $S \cap \Sigma$ in $\Sigma = \Sigma_{k,l,a,b}$. We are therefore lead to consider an integral of the form

$$K(x_0, y_0) = \int_{\gamma_\alpha} \left( \int_{\gamma_\beta} \sin \left( \frac{\pi}{q} (x - 2n\delta) \right) e^{2\pi i r \Psi(x,y)} dy \right) dx,$$

where $\Psi = \Psi_{a,b}^n$ as above. (If $(x_0, y_0)$ is a non-degenerate critical point on the boundary of $\Sigma$, then one or both of the integrals $\int_{-\delta}^\delta$ should be replaced by $\int_0^\delta$ (or $\int_{-\delta}^0$) and the contribution coming from that point in Theorem 4 should be multiplied by $\frac{1}{2}$ or $\frac{1}{4}$.) By a Taylor expansion we find that

$$\Psi(x_0 + \alpha s, y_0 + \beta t) = \Psi(x_0, y_0) + \frac{1}{2} \langle A \left( \begin{array}{c} s \\ t \end{array} \right), \left( \begin{array}{c} s \\ t \end{array} \right) \rangle + h(s, t),$$

where $A = \text{diag}(\alpha, \beta) H \text{diag}(\alpha, \beta)$ and

$$\langle \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right), \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right) \rangle = x_1 x_2 + y_1 y_2$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{C}^2$, and where $h(s, t)$ is a remainder term being a sum of terms of the form $c_{m,n}s^m t^n$, $m, n \in \{0, 1, 2, \ldots, \}$, $m + n \geq 3$, $c_{m,n} \in \mathbb{C}$. We search for $\alpha$ and $\beta$ such that there exists a $\delta > 0$ satisfying

$$\text{Re} \left( 2\pi i (\Psi(x_0 + \alpha s, y_0 + \beta t) - \Psi(x_0, y_0)) \right) < 0$$

for all $(s, t) \in \mathbb{I}_{\delta} \times \mathbb{I}_{\delta} \setminus \{(0, 0)\}$. This amounts to finding $\alpha$ and $\beta$ such that

$$\text{Im} \left( \langle A \left( \begin{array}{c} s \\ t \end{array} \right), \left( \begin{array}{c} s \\ t \end{array} \right) \rangle \right) > 0$$

for all $(s, t) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Since

$$\text{Im} \left( \langle A \left( \begin{array}{c} s \\ t \end{array} \right), \left( \begin{array}{c} s \\ t \end{array} \right) \rangle \right) = \langle \text{Im}(A) \left( \begin{array}{c} s \\ t \end{array} \right), \left( \begin{array}{c} s \\ t \end{array} \right) \rangle$$

this corresponds to finding $\alpha$ and $\beta$ such that $\text{Im}(A)$ is positive definite. It is essential for the method that such $\alpha$ and $\beta$ exist. (If not, the analysis becomes more
difficult since higher order terms in the Taylor expansion of $\Psi(x_0 + \alpha s, y_0 + \beta t)$ need to be involved, and it is likely that the analysis can not be carried through in this case.) For that reason we make the following definition.

**Definition 1.** A critical point $(x, y)$ of the phase function $\Psi$ is called positive definite if $(x, y)$ is non-degenerate and there exists $(\alpha, \beta) \in S^1 \times S^1$ such that $\text{Im}(A)$ is positive definite, where

$$A = \text{diag}(\alpha, \beta)H\text{diag}(\alpha, \beta),$$

where $H$ is the Hessian of $\Psi$ in $(x, y)$.

Assume that $(x_0, y_0)$ is positive definite and let $(\alpha, \beta) \in S^1 \times S^1$ such that $\text{Im}(A)$ is positive definite. Then the main contribution to the integral $K(x_0, y_0)$ in the limit of large $r$ is given by

$$K_{\text{main}}(x_0, y_0) = \alpha\beta\sin\left(\frac{\pi}{q}(x_0 - 2nd)\right)e^{2\pi ir\Psi(x_0, y_0)}$$

$$\int_{\mathbb{R}^2} e^{\pi ir(A\begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} s \\ t \end{pmatrix})} dsdt,$$

and this integral can be evaluated using the results of [12, Sec. 3.4]. In fact,

$$\int_{\mathbb{R}^2} e^{\pi ir(A\begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} s \\ t \end{pmatrix})} dsdt = \frac{1}{r} (\det(-iA))^{-1/2},$$

where $A$ is independent of $r$. Note here that the set $S$ of complex symmetric $2 \times 2$–matrices $B$ with $\text{Re}(B)$ positive definite is an open convex set in the 3–dimensional complex vector space of symmetric $2 \times 2$–matrices. It follows that there is a unique analytic branch of $B \mapsto (\det(B))^{1/2}$ on $S$ such that $(\det(B))^{1/2} > 0$ for $B$ real. We have used that branch in the above result. We note that $\det(-iA) = -((\alpha\beta)^2 \det(H)).$

In conclusion we can state the following result.

**Theorem 4.** Let $S_{\mu, \nu, n}^{k, l, a, b}$ be the set of critical points of $\Phi_{n}^{a, b}$ in $\Sigma_{k, l, a, b}^{\mu, \nu, n}$. IF Conjecture 3 is true and if all the critical points of $\Phi_{n}^{a, b}$ in $\Sigma_{k, l, a, b}^{\mu, \nu, n}$ are positive definite, $k, l = 0, 1, (a, b) \in F_{k, l}, \mu, \nu \in \{\pm 1\}$, $n \in \mathbb{Z}/|q|\mathbb{Z}$, then the leading order large $r$ asymptotics of the quantum invariant is given by

$$\tau_r(M_{p/q}) \sim \frac{i\text{sign}(q)}{4\sqrt{|q|}} e^{\frac{\pi i \text{sign}(pq)}{q}} \sum_{n \in \mathbb{Z}/|q|\mathbb{Z}} \sum_{(k, l) \in \{0, 1\}^2} \sum_{(a, b) \in F_{k, l}} \sum_{(\mu, \nu) \in \{\pm 1\}^2} \mu\nu$$

$$\times \sum_{(x, y) \in S_{k, l, a, b}^{\mu, \nu, n}} (-\det(H(x, y)\Phi_{n}^{a, b}))^{-1/2}$$

$$\times \sin\left(\frac{\pi}{q}(x - 2nd)\right)e^{2\pi ir\Phi_{n}^{a, b}(x, y)},$$

where $H(x, y)\Phi_{n}^{a, b}$ is the Hessian of $\Phi_{n}^{a, b}$ in $(x, y)$. 
According to the above theorem the growth rate of the quantum invariants of $M_{p/q}$ is $r^0$ if we can prove that the values of the phase functions $\Phi_{n}^{a,b}$ in the points $S_{k,l,a,b}^{\mu,\nu,n}$ are real. This we do in Sec. 5.3, see Corollary 5. This is in agreement with our computer studies of the quantum invariants of $M_{p/q}$. To obtain Theorem 3 from Theorem 4, hence a leading asymptotics as predicted by the AEC, we have to prove i) that the union of the sets of critical points $S_{k,l,a,b}^{\mu,\nu,n}$ corresponds to the flat (irreducible) SU(2)-connections on $M_{p/q}$ and ii) that the values of the relevant phase functions $\Phi_{n}^{a,b}$ in these critical points are equal to the Chern–Simons invariants of these flat connections. This we also do in Sec. 5.3. Note that we have used the expression (4.52) for the determinant of the Hessian $H(x,y)\Phi_{n}^{a,b}$ in Theorem 3.

**Remark 4.** The manifolds $M_{p/q}$, $|p/q| \in \{1, 2, 3\}$, are Seifert manifolds and by [10] (1.11) holds for these manifolds if we put $m_{\bar{\rho}} = 4$ for all $\bar{\rho} \in \mathcal{M}^{r}_{p/q}$. Moreover, the results in Appendix C show that the part of the leading order large $r$ asymptotics of $\tau_r(M_0)$ associated to the irreducible SU(2)-connections on $M_0$ is equal to the right-hand side of (1.11) if we put $m_{\bar{\rho}} = 4$ for both points $\bar{\rho}$ in $\mathcal{M}^{r}_{p/q}$. We actually expect that $m_{\bar{\rho}} = 4$ for all $\bar{\rho} \in \mathcal{M}^{r}_{p/q}$ for all rational $p/q$. Recall here that when we calculated the leading order large $r$ asymptotics of $J_{K}^{r}(r)$ in Sec. 4.2 we had to part the contour $C(\varepsilon)$ into two parts $C_{\pm}(\varepsilon)$ because of the tangent factor in the contour integral (3.21). The relevant stationary point was in that case on the real axes, namely it was $x_0 = 5/6$, and to obtain a contour passing through that point we had to deform $C_{-}(\varepsilon)$ to $[\varepsilon, 1 - \varepsilon]$ and $C_{+}(\varepsilon)$ to $[1 - \varepsilon, \varepsilon]$. In that way the stationary point $x_0$ gave two equal contributions. (In fact we used Cauchy’s theorem so that we only had to work with either $C_{-}(\varepsilon)$ or $C_{+}(\varepsilon)$, but that of course also produced the factor 2.) In the case with the quantum invariants $\tau_r(M_{p/q})$ we expect a similar phenomenon with the difference that each relevant stationary point contribute with 4 instead of 2 equal contributions. We expect that the relevant critical points $(x, y)$ belong to $[1/3, 2/3] \times \{y \in \mathbb{C}|\text{Re}(y) = 1/2\}$. Because of the cotangent and tangent factors we had to separate our double contour integral expression for $\tau_r(M_{p/q})$ in (4.34) into 4 parts causing the sum $\sum_{(\mu, \nu) \in \{\pm 1\}^2} \mu \nu$ in the asymptotic formula for $\tau_r(M_{p/q})$. The contour $C_{\varepsilon}^1$ was thus separated into the part with $\text{Im}(x) \leq 0$ and the part with $\text{Im}(x) \geq 0$. The first of these parts can be deformed to $[0, 1]$ while the second can be deformed to $[1, 0]$. The contour $C_{\varepsilon}^2$ was separated into the part with $\text{Im}(y) \geq 0$ and the part with $\text{Im}(y) \leq 0$. Both parts can be deformed to a contour containing the part of the line $\text{Re}(y) = 1/2$ containing the $y$’s in the relevant stationary points. In this way we obtain as claimed that each relevant critical point contribute 4 times. Like in the case of the the knot invariant $J_{K}^{r}(r)$ we find that the signs $\mu$ and $\nu$ are cancelled by taking care of the orientations of the different involved contours.

Let us end this section by examine to what extend the critical points $S_{k,l,a,b}^{\mu,\nu,n}$ are positive definite, see Definition 1. Since $A$ is symmetric we have $\text{Im}(A)_{ij} = \text{Im}(A_{ij})$. Here

$$A_{11} = \alpha^2 H_{11},$$
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\[ A_{12} = A_{21} = \alpha \beta H_{12}, \]
\[ A_{22} = \beta^2 H_{22}, \]
where \( H \) is the Hessian in (4.48). We have \( z_0 = e^{2\pi i x_0} \) and therefore \( H_{12} = H_{21} = 2i \sin(2\pi x_0) \). Moreover, \( w_0 \in \{w_\pm(x_0)\} \) using notation from (4.47) and \( H_{11} = H_{22} - \frac{4}{x_0^2} \), where

\[ H_{22} = \frac{1}{w_\pm(x_0)} - w_\pm(x_0) = \mp 2 \sqrt{\cos(2\pi x_0) - \cos(2\pi x_0) - \frac{3}{4}}. \]

If \( M = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \) is a real symmetric matrix, then \( M \) has real eigenvalues, and \( M \) is positive definite if and only if both of these eigenvalues are positive, i.e. if and only if \( a \) and \( b \) are both positive and \( ab > c^2 \). Now assume that \( M = \text{Im}(A) \). Since \( \text{Re}(z_0) \in [-1, -1/2] \) we have that \( H_{12} \) is zero if and only if \( z_0 = -1 \). In that case \( H_{22} = \mp \sqrt{5} \) which is irrational. Therefore \( H_{11} \neq 0 \) and we can always choose \( \alpha \) and \( \beta \) so \( a > 0 \) and \( b > 0 \) and hence \( ab > c^2 \). Therefore assume in what follows that \( H_{12} \neq 0 \). If \( H_{11} \) and \( H_{22} \) are both positive (meaning that \( w_0 = w_-^\prime(x_0) \)) we can let \( \alpha = \beta = e^{i\pi/4} \) giving \( a = H_{11}, \ b = H_{22} \) and \( c = 0 \). If \( H_{11} \) and \( H_{22} \) are both negative (meaning that \( w_0 = w_+^\prime(x_0) \)) then we can let \( \alpha = \beta = e^{-i\pi/4} \) giving \( a = -H_{11}, \ b = -H_{22} \) and \( c = 0 \). If \( \mu = \text{sign}(H_{11}) \neq \text{sign}(H_{22}) \) we can let \( \alpha = e^{i\mu\pi/4} \) and \( \beta = e^{-i\mu\pi/4} \) giving \( a = |H_{11}|, \ b = |H_{22}| \) and \( c = D \), writing \( H_{12} = iD, \ D \in \mathbb{R} \). Thus \( \det(M) = -\det(H) \), so if \( \det(H) < 0 \) this choice of \( \alpha \) and \( \beta \) works. The case \( \text{sign}(H_{11}) \neq \text{sign}(H_{22}) \) and \( \det(H) > 0 \) is more problematic, and the above analysis is actually too simplified as the examples \( |p/q| \in \{4/3, 10/3, 16/3, 22/3\} \) show.

One can in an even simpler way see that the above approach is too simplified, namely by considering the cases where \( H_{11} = 0 \) or \( H_{22} = 0 \). In that case \( a = 0 \) or \( b = 0 \) so \( \text{Im}(A) \) is not positive definite. We have \( H_{22} = 0 \) if and only if \( \cos(2\pi x_0) = -1/2 \) and that case occurs if and only if \( p = 6m + 3 \) for some integer \( m \).

Since \( \cos(2\pi x_0) \in [-1, -1/2] \) the entry \( H_{11} \) can not be zero if \( |p/q| > \sqrt{20} \). For \( |p/q| < \sqrt{20}, \ H_{11} = 0 \) implies that \( \cos(2\pi x_0) = \frac{1}{2} - \frac{x_0^2}{16} \). Finally note that if \( H_{11} = H_{22} = 0 \) then \( p/q = 0 \), but in that case \( H_{22} \neq 0 \) for critical points \( (x_0, y_0) \in \mathcal{S} \), see Appendix C.

The solution to these problems is to let the contour for the “inner” integral depend on the contour for the “outer” integral. We will not give more details here, but refer to a later paper. Note that the expression for the matrix \( A \) in Definition 1 in this more general approach is different from the one stated in that definition.

5. Classical Chern–Simons theory on \( M_{p/q} \)

In this section we will describe the classical theory, that is the classical Chern–Simons theory on the manifolds \( M_{p/q} \). The SU(2) Chern–Simons functional is a map with values in \( \mathbb{R}/\mathbb{Z} \) defined on the set \( \mathcal{A} \) of gauge equivalence classes of connections in a principal SU(2) bundle on \( M_{p/q} \) (all such bundles being trivializable). Inside
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\( \mathcal{A} \) sits the moduli space \( \mathcal{M}_{p/q} \) of flat SU(2)–connections on \( M_{p/q} \), that is the set of classical solutions to the SU(2) Chern–Simons field theory. Recall that

\[
\mathcal{M}_{p/q} = \text{Hom}(\pi_1(M_{p/q}), \text{SU}(2))/\text{SU}(2)
\]

from which it is clear that \( \mathcal{M}_{p/q} \) is a compact space. The Chern–Simons functional is constant on the connected components of \( \mathcal{M}_{p/q} \), thus there are only finitely many different values on flat connections. Let \( \mathcal{M}'_{p/q} \) be the subset of \( \mathcal{M}_{p/q} \) consisting of nonabelian representations. Recall that these representations correspond to the irreducible connections, while the abelian representations correspond to the reducible connections.

The main results in this section are Theorem 7 and Theorem 8 which tie up the Chern–Simons theory to the large \( r \) asymptotics of \( \bar{\tau}_r(\mathcal{M}_{p/q}) \) by showing that a certain subset of the critical points of the phase functions \( \Phi_{a,b}^n \) corresponds to \( \mathcal{M}'_{p/q} \). Under this correspondence, the Chern–Simons functional is taken to the phase functions \( \Phi_{a,b}^n \).

We begin by giving a description of \( \mathcal{M}_{p/q} \) following Riley [28], [29] and Kirk & Klassen [18].

5.1. The moduli space of flat SU(2)–connections on \( M_{p/q} \)

In the following \( \pi = \pi_1(S^3 \setminus \text{nbd}(K)) \) denotes the knot group of the figure 8 knot. We have a presentation

\[
\pi = \langle x, y | wx = yw \rangle,
\]

where \( w = [x^{-1}, y] \), and where \( \mu = x \) and \( \lambda = yx^{-1}y^{-1}x^2y^{-1}x^{-1}y \) are the elements of \( \pi \) corresponding to the meridian and the preferred longitude of \( K \). The SL(2, \( \mathbb{C} \)) representation variety of \( \pi \) was analyzed by Riley [28], [29] relevant to our work. Consider a group \( G \) given by a presentation

\[
G = \langle x, y | wx = yw \rangle,
\]

where \( w = x^{\varepsilon_1}y^{\varepsilon_2}x^{\varepsilon_3} \cdots y^{\varepsilon_{\alpha-1}} \), where \( \alpha \) is odd and \( \varepsilon_j = \varepsilon_{\alpha-j} = \pm 1, j = 1, 2, \ldots, \alpha-1 \). Such groups are denoted 2–bridge knot groups by Riley since they generalize the 2–bridge knot groups. Following Riley we say that a representation \( \psi : G \to \text{SL}(2, \mathbb{C}) \) is affine when the image of \( \psi \) fixes exactly one point in \( \mathbb{C}P^1 \) and not affine when this image has no fixed points. We note that if \( \psi \) is nonabelian then \( \psi \) is affine if and only if \( \psi(x) \) and \( \psi(y) \) have a common eigenvector, and \( \psi \) is not affine if these two matrices have no common eigenvector. Let \( H \) be some subgroup of SL(2, \( \mathbb{C} \)). Then we will say that two representations \( \psi_1, \psi_2 : G \to \text{SL}(2, \mathbb{C}) \) are \( H \)–equivalent if they are conjugate to each other by a matrix in \( H \), i.e. if there exists a matrix \( U \in H \) such that \( \psi_2(\gamma) = U\psi_1(\gamma)U^{-1} \) for all \( \gamma \in G \). In particular, we will say that \( \psi_1 \) and \( \psi_2 \) are equivalent if they are \( \text{SL}(2, \mathbb{C}) \)–equivalent. For \( (t, u) \in \mathbb{C}^* \times \mathbb{C} \) we put

\[
C_0(t) = \begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix}, \quad D_0(t, u) = \begin{pmatrix} t & 0 \\ -tu & 1 \end{pmatrix},
\]
We note that \( C_\nu(t) \) and \( D_\nu(t, u) \) are elements of \( \text{SL}(2, \mathbb{C}) \), \( \nu = 1, 2 \). If \( s \) is a square root of \( t \) and \( V(s) = \text{diag}(s, s^{-1}) \) then
\[
V(s)C_2(t)V(s)^{-1} = C_1(t), \quad V(s)D_2(t, u)V(s)^{-1} = D_1(t, u),
\]
and
\[
C_0(t) = sC_2(s), \quad D_0(t, u) = sD_2(s, u).
\]
Let \( W_\nu(t, u) \) denote the matrix obtained by replacing \( x \) and \( y \) by respectively \( C_\nu(t) \) and \( D_\nu(t, u) \) in the expression for \( w \), and let
\[
\phi(t, u) = W_{11} + (1 - t)W_{12},
\]
where \( W = W(t, u) = W_0(t, u) \). We let \( \rho_{(t,u)} \) be the assignment \( x \mapsto C_2(t) \), \( y \mapsto D_2(t, u) \). We have the following \( \text{SL}(2, \mathbb{C}) \) version of [28, Theorem 1].

**Theorem 5.** Let \( (s, u) \in \mathbb{C}^* \times \mathbb{C} \). None of the assignments \( \rho_{(s, u)} \) extend to an abelian \( \text{SL}(2, \mathbb{C}) \)-representation of \( G \). The assignment \( \rho_{(s, u)} \) extends to a nonabelian representation \( \rho_{(s, u)} : G \rightarrow \text{SL}(2, \mathbb{C}) \) if and only if
\[
\phi(s^2, u) = 0.
\]
Conversely, if \( \psi : G \rightarrow \text{SL}(2, \mathbb{C}) \) is a nonabelian representation, then there exists a pair \( (s, u) \in \mathbb{C}^* \times \mathbb{C} \) satisfying (5.60) such that \( \psi \) and \( \rho_{(s, u)} \) are equivalent. When \( \psi \) is affine this pair is unique, and when \( \psi \) is not affine the pair \( (s, u) \) can only be replaced by \( (s^{-1}, u) \).

**Proof.** The theorem follows by results of [28], [29]. The assignment \( \rho_{(s, u)} \) extends to a \( \text{SL}(2, \mathbb{C}) \)-representation of \( G \) if and only if \( W_2(s, u)C_2(s, u) = D_2(s, u)W_2(s, u) \). The matrices \( C_2(s) \) and \( D_2(s, u) \) commute if and only if \( s = \pm 1 \) and \( u = 0 \), and since \( C_2(\pm 1) = \pm C_2(1) \) is different from \( D_2(\pm 1, 0) = \pm D_2(1, 0) \) we have that the assignment \( \rho_{(\pm 1, 0)} \) does not extend to a \( \text{SL}(2, \mathbb{C}) \)-representation of \( G \). (Note that if \( W = W(1, 0) \) then \( W_{21} = 0 \) since the matrices \( C_0(1), D_0(1, 0) = I \) and their inverses are upper triangular. But we also have that \( \phi(1, 0) = 0 \) would imply that \( W_{11} = 0 \) contradicting the fact that \( W \) is invertible. Therefore \( \phi(1, 0) \neq 0 \).)

Let \( \sigma = \sum_{j=1}^{n-1} \xi_j \). Then \( s^\sigma W_2(s, u) = W(s^2, u) \). By the proof of [28, Theorem 1] we have \( W(s^2, u)C_0(s^2) = D_0(s^2, u)W(s^2, u) \) if and only if \( \phi(s^2, u) = 0 \), so by (5.59) we find that \( \rho_{(s, u)} \) extends to a (necessarily nonabelian) \( \text{SL}(2, \mathbb{C}) \)-representation of \( G \) if and only if \( \phi(s^2, u) = 0 \).

If \( \psi : G \rightarrow \text{SL}(2, \mathbb{C}) \) is an arbitrary nonabelian representation it follows by [29, Lemma 7] and (5.58) that there exists a pair \( (s, u) \in \mathbb{C}^* \times \mathbb{C} \) (necessarily satisfying (5.60)) such that \( \psi \) and \( \rho_{(s, u)} \) are equivalent. By the above any such pair is different from \( (\pm 1, 0) \) and by [29, Lemma 8] and (5.58) we then get the final statement of the theorem. \( \square \)
If \((s, u) \in \mathbb{C}^* \times \mathbb{C}\) with \(\phi(s^2, u) = 0\) then \(\rho_{(s,u)}\) is affine if and only if \(C_2(s)\) and \(D_2(s, u)\) have a common eigenvector. But this happens exactly when \(u = 0\) or \(u = (s - s^{-1})^2\).

Let us now restrict to the case where \(G\) is the figure 8 knot group. Then

\[
\phi(t, u) = u^2 + (3 - (t + t^{-1})) (u + 1) \tag{5.61}
\]

so in particular \(\phi(s^2, 0) = 3 - s^2 - s^{-2} = 0\) if and only if \(s^4 - 3s^2 + 1 = 0\) i.e. if and only if \(s = \mu_1 \sqrt{(3 + \mu_2 \sqrt{5})}/2\) for some \(\mu_1, \mu_2 \in \{\pm 1\}\). If \(u = (s - s^{-1})^2 = s^2 + s^{-2} - 2\) then \(\phi(s^2, u) = u^2 + (3 - u - 2)(u + 1) = u^2 + 1 - u^2 = 1\), so we conclude that \(\rho_{(s,u)}\) is affine if and only if \(u = 0\) and \(s^4 - 3s^2 + 1 = 0\). Let

\[
\mathcal{N} = \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})
\]

be the space of conjugacy classes of \(\text{SL}(2, \mathbb{C})\)--representations of \(\pi\) and let \(\mathcal{N}_{\text{nab}}\) be the subset consisting of classes represented by nonabelian \(\text{SL}(2, \mathbb{C})\)--representations. Moreover, let

\[
\tilde{\mathcal{N}} = \{ (s, u) \in \mathbb{C}^* \times \mathbb{C} \mid \phi(s^2, u) = 0 \}, \tag{5.62}
\]

and let \(\Phi : \tilde{\mathcal{N}} \to \mathcal{N}\) be the map which maps \((s, u)\) to the class represented by \(\rho_{(s,u)}\).

We have shown

**Corollary 1.** The image of \(\Phi\) is \(\mathcal{N}_{\text{nab}}\). If we let

\[
\tilde{\mathcal{N}}_0 = \{ (\mu_1 \sqrt{(3 + \mu_2 \sqrt{5})}/2, 0) \mid \mu_1, \mu_2 \in \{\pm 1\} \}
\]

then \(\Phi|_{\tilde{\mathcal{N}}_0} : \tilde{\mathcal{N}}_0 \to \mathcal{N}\) is injective and \(\Phi^{-1}(\Phi(s, u)) = \{(s, u), (s^{-1}, u)\}\) for any \((s, u) \in \tilde{\mathcal{N}} \setminus \tilde{\mathcal{N}}_0\).

Recall that \(M_{p/q}\) denotes the closed oriented 3--manifold obtained by surgery on \(S^3\) along the figure 8 knot with rational surgery coefficient \(p/q\). The representation \(\rho_{(s,u)}, (s, u) \in \tilde{\mathcal{N}}\), extends to a \(\text{SL}(2, \mathbb{C})\)--representation of \(\pi_1(M_{p/q})\) if and only if

\[
\rho_{(s,u)}(\mu)^p \rho_{(s,u)}(\lambda)^q = 1.
\]

To analyze this criterion it is an advantage to diagonalize \(\rho_{(s,u)}(\lambda)\). Assume in the following that \((s, u) \in \tilde{\mathcal{N}}\). By a rather long but completely elementary and straightforward calculation we find that

\[
\rho_{(s,u)}(\lambda) = \begin{pmatrix} \lambda_{11}(s, u) & \lambda_{12}(s, u) \\ 0 & \lambda_{11}(s^{-1}, u) \end{pmatrix},
\]

where

\[
\lambda_{11}(s, u) = -1 + s^{-2} - 2s^2 + s^4 + u(s^{-2} - s^2)
\]

and \(\lambda_{12}(s, u) = 2u(1 - u)\) if \(s^2 = 1\) and

\[
\lambda_{12}(s, u) = \frac{\lambda_{11}(s, u) - \lambda_{11}(s^{-1}, u)}{s^2 - 1}
\]
We conclude that if $s \neq 1$. We note that $\lambda_{11} s^{-1}, u = \lambda_{11} (s, u)^{-1}$.

In general, if $A = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \in \text{SL}(2, \mathbb{C})$, then $A$ can be diagonalized if and only if $A$ is not parabolic, i.e. if and only if $\text{tr}(A) \neq \pm 2$ or equivalently if and only if $\alpha \neq \pm 1$ (except, of course, if $\beta = 0$). If $\alpha \neq \pm 1$ then $(1, 0)$ is an eigenvector with eigenvalue $\alpha$ and $(-\beta/(\alpha - \alpha^{-1}), 1)$ is an eigenvector with eigenvalue $\alpha^{-1}$. If $s^2 = 1$ then $\lambda_{11} (s, u) = -1$ and $\lambda_{12} (s, u) = \pm 2\sqrt{3}$. If $s^2 \neq 1$ then $\lambda_{11} (s, u) = \pm 1$ if and only if $\lambda_{12} (s, u) = 0$. Therefore $\rho_{(s, u)} (\lambda)$ is diagonalizable (or diagonal) if and only if $s^2 \neq 1$. In case $s^2 \neq 1$ and $\lambda_{12} (s, u) \neq 0$ we have

$$\frac{\lambda_{12} (s, u)}{\lambda_{11} (s, u) - \lambda_{11} (s, u)^{-1}} = \frac{1}{s^2 - 1} = \frac{s^{-1}}{s - s^{-1}}.$$  

We conclude that if $s^2 \neq 1$, then $\mathbb{C}^2$ has a basis consisting of a set of common eigenvectors for the matrices $\rho_{(s, u)} (\mu)$ and $\rho_{(s, u)} (\lambda)$, namely $u_1 = (1, 0)$ and $u_2 = (-1)/(s^2 - 1), 1)$. If we let $\tilde{\rho}_{(s, u)} : \pi \to \text{SL}(2, \mathbb{C})$ be the representation $\tilde{\rho}_{(s, u)} (\gamma) = U^{-1} \rho_{(s, u)} (\gamma) U$, where $U \in \text{SL}(2, \mathbb{C})$ with $j$th column $u_j$, we therefore have

$$\tilde{\rho}_{(s, u)} (x) = \text{diag} (s, s^{-1}), \quad \tilde{\rho}_{(s, u)} (\lambda) = \text{diag} (\lambda_{11} (s, u), \lambda_{11} (s, u)^{-1}). \quad (5.63)$$

In particular, $\rho_{(s, u)} : \pi \to \text{SL}(2, \mathbb{C})$, $s^2 \neq 1$, extends to a representation of $\pi_1 (M_{p/q})$ if and only if

$$s^{-p} = \lambda_{11} (s, u)^q. \quad (5.64)$$

Recall here that $\rho_{(s, u)}$ and $\rho_{(s^{-1}, u)}$ are equivalent for $(s, u) \in \tilde{N} \setminus \tilde{N}_0$, cf. Corollary 1. But as noted above $\lambda_{11} (s^{-1}, u) = \lambda_{11} (s, u)^{-1}$ in accordance with (5.64). For $(s, u) \in \tilde{N}_0$ we have $\lambda_{11} (s, u) = 1$ and $|s| \neq 1$, so $\rho_{(s, u)}$ extends to a representation of $\pi_1 (M_{p/q})$ if and only if $p = 0$.

A direct check shows that if $s^2 = 1$ then $\rho_{(s, u)}$ does not extend to a representation of $\pi_1 (M_{p/q})$ for any rational number $p/q$. In fact, if $s = \pm 1$, then

$$\rho_{(s, u)} (\mu)^p = (\pm 1)^{|p|} \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix},$$

and

$$\rho_{(s, u)} (\lambda)^q = \left( - \begin{pmatrix} 1 & \varepsilon i 2\sqrt{3} \\ 0 & 1 \end{pmatrix} \right)^q = (-1)^{|q|} \begin{pmatrix} 1 & \varepsilon i 2q\sqrt{3} \\ 0 & 1 \end{pmatrix}$$

for $\varepsilon \in \{-1, 1\}$. On the other hand, since $\lambda_{11} (s, u) = -1$, we have that (5.64) is satisfied if $s = 1$ and $q$ is even or $s = -1$ and both $p$ and $q$ are odd. For the following we note that if $s^2 = 1$ then $u^2 + u + 1 = \phi(1, u) = 0$ so $u$ is not real.

We are mostly interested in the SU(2)–representations of $\pi$. Let in the following $\mathcal{M}_{\text{lab}}$ be the set of conjugacy classes of nonabelian SU(2)–representations of $\pi$.

**Proposition 2.** Let $(s, u) \in \tilde{N}$. The representation $\rho_{(s, u)} : \pi \to \text{SL}(2, \mathbb{C})$ is SL(2, $\mathbb{C}$)–equivalent to a representation $\pi \to \text{SU}(2)$ if and only if $|s| = 1$ and $u$
is real. If we write \( s = e^{2\pi i \theta} \), \( \theta \in ]-1/2, 1/2[ \), then \( u \in \{ u_{\pm} \} \), where \( u_{\pm} = u_{\pm}(\theta) \) are the two solutions to \( \phi(e^{4\pi i \theta}, u) = 0 \), i.e.

\[
u_{\pm}(\theta) = \cos(4\pi \theta) - \frac{3}{2} \pm \sqrt{\cos^2(4\pi \theta) - \cos(4\pi \theta) - \frac{3}{4}}.
\]

Since \( u \) is real we have \( \theta \in [-1/3, -1/6]\cup[1/6, 1/3] \). The representation \( \rho(e^{2\pi i \theta}, u_{\pm}) \), \( \theta \in [-1/3, -1/6]\cup[1/6, 1/3] \) and \( \varepsilon \in \{ \pm \} \), is \( \text{SL}(2, \mathbb{C}) \)-equivalent to a \( \text{SU}(2) \)-representation \( \tilde{\rho}_{\theta, \varepsilon} \) which satisfies

\[
\tilde{\rho}_{\theta, \varepsilon}(\mu) = \text{diag}(e^{2\pi i \theta}, e^{-2\pi i \theta}),
\]

\[
\tilde{\rho}_{\theta, \varepsilon}(\lambda) = \text{diag}(L_{\varepsilon}, L_{\varepsilon}^{-1}),
\]

where \( \mu \) and \( \lambda \) are the elements of \( \pi \) corresponding to the meridian and the preferred longitude of \( K \), and

\[
L_{\pm} = L_{\pm}(\theta) = \lambda_{11} \left( e^{2\pi i \theta}, u_{\pm} \right) = -1 + e^{-4\pi i \theta} - 2e^{4\pi i \theta} + e^{8\pi i \theta} + u_{\pm} \left( e^{-4\pi i \theta} - e^{4\pi i \theta} \right).
\]

We note that \( \tilde{\rho}_{-\theta, \varepsilon} \) and \( \tilde{\rho}_{\theta, \varepsilon} \) are \( \text{SU}(2) \)-equivalent. In particular, the space \( \mathcal{M}_{\text{nab}} \) can be parametrized by the two arcs \( (e^{2\pi i \theta}, u_{\pm}(\theta)), \theta \in [1/6, 1/3] \), and \( (e^{2\pi i \theta}, u_{\pm}(\theta)), \theta \in [1/6, 1/3] \). These two arcs only coincide at the endpoints, so topologically \( \mathcal{M}_{\text{nab}} \) is a circle.

This proposition follows from [28, Proposition 4], see also [18, Proposition 5.4]. For a more geometric argument determining the topological type of \( \mathcal{M}_{\text{nab}} \), see [19].

For \( \theta \in [-1/3, -1/6]\cup[1/6, 1/3] \) and \( \varepsilon \in \{ \pm \} \), the representation \( \tilde{\rho}_{\theta, \varepsilon} \) extends to a representation of \( \pi_1(M_{p/q}) \) if and only if

\[
\tilde{\rho}_{\theta, \varepsilon}(\mu)^p \tilde{\rho}_{\theta, \varepsilon}(\lambda)^q = 1,
\]

i.e. if and only if

\[
e^{-2\pi ip\theta} = L_{\varepsilon}(\theta)^q.
\]

From this (use e.g. (5.69)) we see that

**Corollary 2.** Let \( p/q \in \mathbb{Q} \) be arbitrary. The moduli space of irreducible flat \( \text{SU}(2) \)-connections on \( M_{p/q} \) is a finite set.

Let us end this section by finding the abelian \( \text{SU}(2) \)-representations of \( \pi_1(M_{p/q}) \) (up to equivalence). Therefore, let \( \theta \in [-1/2, 1/2] \) and let \( \rho_{\theta} \) be the assignment

\[
\rho_{\theta}(\mu) = \text{diag}(e^{2\pi i \theta}, e^{-2\pi i \theta}).
\]

By (5.57) this assignment extends to an abelian \( \text{SU}(2) \)-representation of \( \pi \) for any \( \theta \in [-1/2, 1/2] \) by letting \( \rho_{\theta}(y) = \rho_{\theta}(x) \). Moreover, any abelian \( \text{SU}(2) \)-representation of \( \pi \) is \( \text{SU}(2) \)-equivalent to \( \rho_{\theta} \) for some \( \theta \in [-1/2, 1/2] \). For any \( \theta \in [-1/2, 1/2] \) we have \( \rho_{\theta}(\lambda) = 1 \), and \( \rho_{\theta} \) extends to a representation of \( \pi_1(M_{p/q}) \) if and only if \( \rho_{\theta}(\mu)^p = 1 \), i.e. if and only if \( p \theta \in \mathbb{Z} \). If \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) then

\[
A\rho_{\theta}(\mu)A^{-1} = \rho_{-\theta}(\mu) \text{ so we can assume that } \theta \in [0, 1/2].
\]

Note, moreover, that two matrices \( \text{diag}(e^{i\phi}, e^{-i\phi}) \) and \( \text{diag}(e^{i\psi}, e^{-i\psi}) \), \( \phi, \psi \in [-\pi, \pi] \), are conjugate in \( \text{SU}(2) \) if and only if \( \phi = \psi \) or \( \phi = -\psi \). We conclude
Proposition 3. For \( p \neq 0 \) the set of conjugacy classes of abelian SU(2)–represen-
tations of \( \pi_1(M_{p/q}) \) is given by

\[
\left\{ \left[ \rho_j/p \right] \mid j = 0, 1, \ldots, \left[\frac{|p|}{2}\right] \right\},
\]

where \( \left[|p|/2\right] \) is the integer part of \(|p|/2\). For \( p = 0 \) the set of conjugacy classes of
abelian SU(2)–representations of \( \pi_1(M_{p/q}) \) is given by

\[
\left\{ \left[ \rho_0 \right] \mid \theta \in [0, \frac{1}{2}] \right\},
\]

so topologically this set is a closed interval.

5.2. Chern–Simons invariants

We begin by recalling formulas from [18] for the Chern–Simons invarian-
ts of the flat SU(2)–connections on \( M_{p/q} \). The basic tool will be Theorem 6 below due to P.
A. Kirk and E. P. Klassen.

Let \( M \) be a closed oriented 3–manifold with a knot \( K \) in its interior and let
\( X \) be the complement of a tubular neighborhood of \( K \) in \( M \). Moreover, let \( \mu \) be
a meridian of \( K \) and \( \lambda \) a longitude, both in \( \partial X \). Let \( G \) be SU(2) or SL(2, \mathbb{C}). If
\( \rho : \pi_1(X) \to G \) is a representation, then \( \rho \) extends to a \( G \)–representation of \( \pi_1(M) \)
if and only if \( \rho(\mu) = 1 \). Now assume that \( \rho_t : \pi_1(X) \to G \) is a piecewise smooth
path of representations, \( t \in I = [0, 1] \). Choose a piecewise smooth path \( g : I \to G \)
such that

\[
g_t \rho_t(\mu) g_t^{-1} = \text{diag} \left( e^{2\pi i \alpha(t)}, e^{-2\pi i \alpha(t)} \right),
\]

\[
g_t \rho_t(\lambda) g_t^{-1} = \text{diag} \left( e^{2\pi i \beta(t)}, e^{-2\pi i \beta(t)} \right)
\]

(5.66)

for some piecewise smooth curves \( \alpha, \beta \). If \( G = \text{SU}(2) \) this is always possible by [18,
Lemma 3.1], and in that case \( \alpha \) and \( \beta \) are real-valued. If \( G = \text{SL}(2, \mathbb{C}) \) the above
is possible if the path \( \rho \) avoids the parabolic representations (i.e. upper triangular
with 1s or \(-1\)s on the diagonal), cf. [18, Remark p. 354]. (See the text in connection
to (5.63) for the case of the figure 8 knot.) In that case the curves \( \alpha \) and \( \beta \) are
complex-valued. We then have

Theorem 6 ([18, Theorem 4.2] ). Assume that \( \rho_0(\mu) = \rho_1(\mu) = 1 \). Thinking
of \( \rho_0 \) and \( \rho_1 \) as flat \( G \)–connections on \( M \), we have

\[
\text{CS}(\rho_1) - \text{CS}(\rho_0) = -2 \int_0^1 \beta(t) \alpha'(t) dt \pmod{\mathbb{Z}},
\]

where \( \text{CS} \) is the Chern–Simons functional associated to \( G \).

We note that in case \( G = \text{SL}(2, \mathbb{C}) \) the Chern–Simons functional takes values in
\( \mathbb{C}/\mathbb{Z} \).
Next consider a knot $K$ in $S^3$ and let $X$ be the knot complement. Let $p/q$ be a rational number and let $N_{p/q}$ be the closed oriented 3–manifold obtained by $p/q$ surgery on $S^3$ along $K$. Let $\mu$ and $\lambda$ be classes in $\pi_1(\partial X)$ represented by respectively a meridian and the preferred longitude of $K$. Choose integers $c, d \in \mathbb{Z}$ such that $pd - qc = 1$. Let $V$ be a tubular neighborhood of $K$ considered as a subspace of $N_{p/q}$. We note that $\mu' = p\mu + q\lambda$ and $\lambda' = c\mu + d\lambda$ are represented by respectively a meridian of $V$ and a longitude of $V$. Assume that $\rho_t : \pi_1(X) \to G$, $t \in I$, is a piecewise smooth curve of representations from the trivial representation to a representation, which extends to a representation of $\pi_1(N_{p/q})$, i.e. $\rho_t(\mu') = 1$. Assume, moreover, that $\rho_t$ avoids the parabolic representations in case $G = \text{SL}(2, \mathbb{C})$, and choose curves $\alpha, \beta$ as in (5.66) with $\alpha(0) = \beta(0) = 0$. By Theorem 6 we have

$$\text{CS}(\rho_t) = -2 \int_0^1 (ca(t) + db(t))(pa'(t) + q\beta'(t))dt$$

$$= -2 \int_0^1 \beta(t)a'(t)dt - cpa^2(1) - dq\beta^2(1) - 2cqa(1)\beta(1) \pmod{\mathbb{Z}}.$$  \hspace{1cm} (5.67)

(We have corrected a sign error in [18, Formula (*) p. 361].) We note that this expression is independent of the choice of $c, d$. The condition $\rho_t(\mu') = 1$ is equivalent to

$$pa(1) + q\beta(1) \in \mathbb{Z}. \hspace{1cm} (5.68)$$

Now let $K$ be the figure 8 knot and let $\bar{\rho}_{\theta, c}$ be a SU(2)–representation of $\pi_1(M_{p/q})$, i.e. (5.65) is satisfied. Following [18] we determine a formula for $\text{CS}(\bar{\rho}_{\theta, c})$. For later we will here pay special attention to the branches of the logarithm. By Proposition 2 we have

$$\text{Re} \left( L_- (\theta) \right) = \text{Re} \left( L_+ (\theta) \right) = 2 \cos^2(4\pi\theta) - \cos(4\pi\theta) - 2,$$

$$\text{Im} \left( L_{\pm} (\theta) \right) = \mp 2 \sin(4\pi\theta) \sqrt{\cos^2(4\pi\theta) - \cos(4\pi\theta) - \frac{3}{4}} \hspace{1cm} (5.69)$$

for all $\theta \in [-1/3, -1/6] \cup [1/6, 1/3]$. From these identities we see that

$$L_{\pm}(1/3) = L_{\pm}(1/6) = -1, \quad L_{\pm}(1/4) = 1, \hspace{1cm} (5.70)$$

and that $\text{Im}(L_+) < 0$ and $\text{Im}(L_-) > 0$ on $[1/6, 1/4]$, with the opposite signs on $[1/4, 1/3]$. We conclude that $L_+(\theta)$ and $L_- (\theta)$ run through $S^1$ both beginning and ending in $-1$, $L_+$ in the anti-clockwise and $L_-$ in the clockwise direction, as $\theta$ runs through $[1/6, 1/3]$. We can therefore use the principal logarithm $\text{Log}$ to define continuous curves $\beta_{\pm} : [1/6, 1/3] \to \mathbb{R}$ by

$$\beta_{\pm}(\theta) = \frac{1}{2\pi i} \text{Log}(L_{\pm}(\theta)) + f_{\pm}(\theta),$$

where

$$f_{\pm}(\theta) = \begin{cases} 0, & \theta = \frac{1}{6}, \\ 1, & \theta \in \left[ \frac{1}{6}, \frac{1}{3} \right], \end{cases} \hspace{1cm} (5.71)$$

$$f_{\pm}(\theta) = \begin{cases} 0, & \theta = \frac{1}{6}, \\ 1, & \theta \in \left[ \frac{1}{6}, \frac{1}{3} \right], \end{cases} \hspace{1cm} (5.72)$$
and
\[ f_-(\theta) = \begin{cases} 0, & \theta \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ -1, & \theta = \frac{2}{3}. \end{cases} \] (5.73)

By (5.65) and the definition of \( \beta_\pm \) the representation \( \bar{\rho}_{\theta, \varepsilon} \) extends to a representation of \( \pi_1(M_{p/q}) \) if and only if
\[ p\theta + q\beta_\varepsilon(\theta) \in \mathbb{Z} \] (5.74)

which is nothing but (5.68) (reparametrized). We note that \( \beta_\pm \) are smooth on \( \left[\frac{1}{3}, \frac{2}{3}\right] \) but not in the end points 1/6 and 1/3. The terms \( f_\pm \) have been chosen so that \( \beta_{1/6} = 1/2 \). This is needed for the proof of

**Proposition 4** ([18, p. 362]). Let \( \theta \in [1/6, 1/3] \) and let \( \bar{\rho}_{\theta, \pm} \) be as in Proposition 2. If \( \bar{\rho}_{\theta, \varepsilon} \) extends to a representation of \( \pi_1(M_{p/q}) \) for a \( \varepsilon \in \{\pm 1\} \) then
\[ \text{CS}(\bar{\rho}_{\theta, \varepsilon}) = -\frac{1}{6} - cp\theta^2 - dq\beta_\varepsilon^2(\theta) - 2cp\theta\beta_\varepsilon(\theta) - 2\int_{1/6}^{\theta} \beta_\varepsilon(t)dt \pmod{\mathbb{Z}}, \]

where \( \beta_\pm \) are the curves defined by (5.71). \( \square \)

Kirk and Klassen prove the above result by explicitly constructing a piecewise smooth path \( \rho : [0, 1] \to \text{Hom}(\pi, \text{SL}(2, \mathbb{C})) \) from the trivial representation to \( \bar{\rho}_{\pm} = \bar{\rho}_{\bar{\rho}} \) with piecewise smooth functions \( \alpha, \beta : [0, 1] \to \mathbb{C} \) as in (5.66) satisfying \( \alpha(1) = \frac{1}{3} \) and \( \beta(1) = \frac{1}{2} \). Moreover, they use that \( \int_0^1 \beta(t)\alpha'(t)dt = \frac{1}{12} \) and the fact that \( y \to \bar{\rho}_{y, \varepsilon} : [1/6, \theta] \to \text{Hom}(\pi, \text{SU}(2)) \) is a path from \( \bar{\rho}_{\theta, \varepsilon} \) to \( \bar{\rho}_{\theta, \varepsilon} \) with associated functions \( \alpha(y) = y \) and \( \beta(y) = \beta_\varepsilon(y) \). By the choice of the functions \( f_\pm \), these \( \alpha \)- and \( \beta \)-functions are continuations of the ones used for the path \( \rho \) from the trivial representation to \( \bar{\rho}_{\pm} \).

Kirk and Klassen argue that \( \int_0^1 \beta(t)\alpha'(t)dt = \frac{1}{12} \) using a comparison between a computer calculation and the Chern–Simons invariants of flat SU(2)–connection on the Seifert manifold \( M_{-3} \). By following Kirk and Klassen’s argument [18, pp. 361–362] it is actually not hard to give an explicit calculation of this integral. Let us give some details. The path \( \rho \) from the trivial connection to \( \bar{\rho}_{\pm} \) consists of three parts, where \( \beta \) is identical zero along the first two parts. We can therefore concentrate on the third part. Let \( a = (3+\sqrt{5})/2 \), and let \( \alpha : [0, 1] \to \mathbb{C} \) be a piecewise smooth function from \( \frac{1}{2\pi} \log(\sqrt{a}) \) to 1/6. Let \( t(s) = e^{2\pi ia(s)} \) and choose a piecewise smooth solution \( u(s) \) to \( \phi(t(s), u(s)) = 0 \) where \( \phi \) is given by (5.61). Then \( s \mapsto \rho(t(e^{2\pi ia(s)}), u(s)) = \eta_s \) is the third piece of our curve \( \rho \) (reparametrized). Assuming \( \alpha(s) \notin \frac{1}{2}\mathbb{Z} \) (so as to avoid the parabolic representations) we can diagonalize exactly as demonstrated after Corollary 1 and get
\[ \hat{\eta}_s(\mu) = \text{diag}(T, T^{-1}), \quad \hat{\eta}_s(\lambda) = \text{diag}(\lambda_{11}(T, u), \lambda_{11}(T, u)^{-1}), \]
where \( T = T(s) = e^{2\pi ia(s)}, \ u = u(s), \ \lambda_{11} \) is as below Corollary 1, and where
\[ \hat{\eta}_s(\gamma) = U^{-1}\eta_s(\gamma)U \quad \text{for} \ \gamma \in \pi, \ \text{where} \ U = U(s) = \begin{pmatrix} 1 & 0 \\ -1/(T^2) & 1 \end{pmatrix}. \] This
Asymptotics of the quantum invariants for the figure 8 knot shows that \( \alpha(s) \) indeed plays the role as the \( \alpha \)-curve for our path \( \eta_s \). The \( \beta \)-curve should be a piecewise smooth curve \( \beta(s) \) starting at zero such that

\[
e^{2\pi i \beta(s)} = \lambda_{11}(e^{2\pi i \alpha(s)}, u(s))
\]  \hspace{1cm} (5.75)

for \( s \in [0, 1] \). We note that \( \lambda_{11}(e^{2\pi i \alpha(1)}, u(1)) = -1 \) so we must have \( \beta(1) \in \frac{1}{2} + \mathbb{Z} \).

Since \( \bar{\rho}_{1/6, \pm} \) extends to a SU(2)-representation of \( \pi_1(M_{-3}) \) we get from (5.67) that

\[
-2 \int_0^1 \beta(s) \alpha'(s) ds - \frac{1}{12} + \frac{1}{3} \beta(1) \pmod{\mathbb{Z}}
\]

is a Chern–Simons value of a flat SU(2)-connection on \( M_{-3} \), so \( \int_0^1 \beta(s) \alpha'(s) ds \) is real.

We now only have to choose a nice \( \alpha \)-curve. Let \( \delta \) be a small positive parameter less than \( |\alpha(0)| < 1/6 \), and let \( \alpha = \alpha_1 + \alpha_2 + \alpha_3 \), where \( \alpha_1 \) is the line segment \( [\alpha(0), -i\delta] \), \( \alpha_2 \) is the part of the circle with centre zero and radius \( \delta \) running from \( -i\delta \) to \( \delta \), and \( \alpha_3 = [\delta, 1/6] \). Here the parameter \( \delta \) is introduced to avoid passing through zero (so as to avoid parabolic representations). With this choice it is not hard to show that there is a unique continuous solution \( (u(s), \beta(s)) \) to \( \phi(t(s), u(s)) = 0 \) and (5.75) with \( \beta(0) = 0 \) and \( \beta(1) = \frac{1}{2} \) and to show that \( \int_0^1 \beta(s) \alpha'(s) ds = \frac{1}{12} \) for this solution. (Use that the integral is real and independent of \( \delta \) and calculate its \( \delta \to 0 \) limit).

Proposition 4 gives a formula for the Chern–Simons invariants of the irreducible flat SU(2)-connections on \( M_{p/q} \). Let us also determine the Chern–Simons invariants of the reducible flat SU(2)-connections on \( M_{p/q} \). Therefore let \( \rho_{j/|p|} \) be as in Proposition 3, \( p \neq 0 \), and let \( \alpha(t) = j t / |p| \) and \( \beta(t) = 0 \), \( t \in [0, 1] \), and get by (5.67) that

\[
\text{CS}(\rho_{j/|p|}) = -c q j^2 / p^2 = -c j^2 / p \pmod{\mathbb{Z}},
\]  \hspace{1cm} (5.76)

where \( c \) is the inverse of \( -q \mod p \).

In case \( p = 0 \) the moduli space of flat reducible SU(2)-connections on \( M_0 \) can be identified with a closed interval, cf. Proposition 3. Since the Chern–Simons functional is constant on each of the connected components of the moduli space of flat connections, we conclude that the Chern–Simons invariant of any of the reducible SU(2)-connections is equal to the Chern–Simons invariant of the trivial connection, i.e. it is equal to zero. This of course also follows from Kirk and Klassen’s result. In fact, if \( \rho_0 \) denotes the abelian representation from Proposition 3, then we can put \( \alpha(t) = \theta t \) and \( \beta(t) = 0 \), \( t \in [0, 1] \), and get by (5.67) that

\[
\text{CS}(\rho_0) = 0 \pmod{\mathbb{Z}}.
\]  \hspace{1cm} (5.77)
5.3. A comparison between Chern–Simons invariants and critical values of the phase functions \( \Phi_n^{a,b} \)

The purpose of this section is to combine the Chern–Simons theory described in the previous two sections with the asymptotic analysis in Sec. 4.3. First we will describe a correspondence between the critical points of the phase functions \( \Phi_n^{a,b} \) in (4.36) and the nonabelian SL(2, C)–representations of \( \pi_1(M_{p/q}) \). Thereafter we will show that the set of Chern–Simons invariants of flat irreducible SU(2)–connections on \( M_{p/q} \) is a certain subset of the critical values of the functions \( \Phi_n^{a,b} \). Like in Sec. 4.3 we will most of the time work with the shifted phase functions \( \Psi_n^{a,b} \) in (4.39). We refer to Remark 5 below for a comparison between the phase functions \( \Phi_n^{a,b} \) and \( \Psi_n^{a,b} \).

Recall the set \( \tilde{N} \) given by (5.62).

**Theorem 7.** The map \( (x, y) \mapsto \rho_{(e^{\pi i x}, e^{2\pi i y} - 1)} \) gives a surjection \( \varphi \) from the set of critical points \( (x, y) \) of the functions \( \Psi_n^{a,b}, a, b, n \in \mathbb{Z}, \) with \( x \notin \mathbb{Z} \) onto the set of representations \( \rho(s, u) : \pi \to \text{SL}(2, \mathbb{C}), (s, u) \in \tilde{N}, \) which extend to SL(2, C)–representations of \( \pi_1(M_{p/q}) \).

If \( (x, y) \) is a critical point, let us say of \( \Psi_n^{a,b} \), then \( (x + 2k, y + l) \) is a critical point of \( \Psi_{n+l,b+2k} \) for any \( k, l \in \mathbb{Z} \), so \( \varphi^{-1}(\varphi(x, y)) = \{ (x + 2k, y + l) | k, l \in \mathbb{Z} \} \) if \( x \notin \mathbb{Z} \).

**Proof.** The last statement follows immediately from (4.40), so let us concentrate on the first part. Let \( \psi : \mathbb{C}^2 \to \mathbb{C} \) be given by

\[
\psi(z, w) = (1 - zw)(w - z) - zw,
\]

so \( \psi(v^2, w) = 0 \) if and only if \( (v, w) \) is a solution to (4.42). Then

\[
\psi(z, u + 1) = -z\phi(z, u)
\]

for \( (z, u) \in \mathbb{C}^* \times \mathbb{C}, \) where \( \phi \) is the function (5.61). It follows that \( (v, w) \) is a solution to (4.42) if and only if either \( (v, w) = (0, 0) \) or \( (v, w - 1) \in \tilde{N} \).

Next assume that \( (v, w) \in \tilde{N} \) and put \( (z, w) = (v^2, u + 1) \). Since \( \psi(z, w) = 0 \) and \( z \neq 0 \) we have that \( w \neq 0 \) and \( w - z \neq 0 \). Therefore \( 1 - zw = wz/(w - z) \) and \( (v, w) \) is a solution to (4.41) if and only if

\[
v^{-p} = \left( \frac{(w - z)^2}{wz} \right)^q = (wz^{-1} - 2 + zw^{-1})^q = (\lambda_{11}(v, w - 1) - 1 - z^2 + zw^{-1} + wz + z)^q.
\]

But \( \psi(z, w) = 0 \) and \( w \neq 0 \) implies that \( -1 - z^2 + zw^{-1} + wz + z = 0 \). Thus \( (v, w) \) is a solution to (4.41) if and only if \( (s, u) = (v, u) \) is a solution to (5.64).

The above shows together with the discussion around (5.64) that there is a one to one correspondence between common nonzero solutions \( (v, w) \) to (4.41), (4.42) with \( v^2 \neq 1 \) and representations \( \rho_{(v, w - 1)} : \pi \to \text{SL}(2, \mathbb{C}), (v, w - 1) \in \tilde{N}, \) which extend to SL(2, C)–representations of \( \pi_1(M_{p/q}) \). By the remarks following (4.40) this proves the theorem.

\[\square\]
Recall the set $\mathcal{S}$ in (1.5). Moreover, recall that if $(x, y)$ is a critical point of $\Psi_{a,b}^n$, $a, b, n \in \mathbb{Z}$, with $(x, y) \in \mathcal{S}$ then $x \notin \mathbb{Z}$ is automatically satisfied. By Proposition 2 and the above theorem we therefore have

**Corollary 3.** The surjection $\varphi$ in Theorem 7 restricts to a surjection from the set of critical points $(x, y)$ of $\Psi_{a,b}^n$, $a, b, n \in \mathbb{Z}$, with $(x, y) \in \mathcal{S}$ onto the set of representations $\rho_{(s, u)} : \pi_1 (\mathcal{M}_{p/q}) \rightarrow \text{SL}(2, \mathbb{C})$, $(s, u) \in \mathcal{N}$, which are equivalent to SU(2)-representations of $\pi_1 (\mathcal{M}_{p/q})$.

**Theorem 8.** Let $(x, y) \in \mathbb{C}^2$ such that $\rho(e^{\pi i x}, e^{2\pi i y - 1})$ is equivalent to a nonabelian SU(2)-representation of $\pi_1 (\mathcal{M}_{p/q})$ and choose according to Theorem 7 integers $a, b, n$ such that $(x, y)$ is a critical point of $\Psi_{a,b}^n$. If $a', b', n'$ is another such set of integers, then $b' = b$ and $a' + n' / q = a + n / q$. In fact we have

$$a + \frac{n}{q} = y + \frac{p}{2q} x + \frac{i}{2\pi} \left( \log \left( 1 - e^{2\pi i (x+y)} \right) - \log \left( 1 - e^{2\pi i (x-y)} \right) \right),$$

$$b = x + \frac{i}{2\pi} \left( \log \left( 1 - e^{2\pi i (x+y)} \right) + \log \left( 1 - e^{2\pi i (x-y)} \right) \right).$$

Moreover, for any such set of integers $a, b, n$ we have

$$\text{CS}(\bar{\rho}_{\theta, \varepsilon}) = \Psi_{a,b}^n (x, y) \pmod{\mathbb{Z}},$$

where $\theta \in [-1/3, -1/6] \cup [1/6, 1/3]$ and $\varepsilon \in \{\pm\}$ with $e^{2\pi i \theta} = e^{\pi i x}$ and $1 + \nu_{\varepsilon} (\theta) = e^{2\pi i y}$, and $\bar{\rho}_{\theta, \varepsilon}$ is a SU(2)-representation of $\pi_1 (\mathcal{M}_{p/q})$ equivalent to $\rho(e^{\pi i x}, e^{2\pi i y - 1})$ and given by Proposition 2.

**Remark 5.** i) In general we have $\Phi_{a,b}^n = \Psi_{a+b, a-b}^n$, so there is a one to one correspondence between the phase functions $\Phi_{a,b}^n$ and the phase functions $\Psi_{a,b}^n$ with $a' + b'$ even. If $(x, y)$ is a critical point of $\Psi_{a,b}^n$ with $a' + b'$ odd, then $(x, y)$ is also a critical point of $\Psi_{a'+1, b'}^n$ by Theorem 8. In that way we see that any critical point of one of the functions $\Psi_{a,b}^n$ is also a critical point of one of the functions $\Phi_{a,b}^n$.

ii) Assume that $|p/q| > \sqrt{20}$ and assume that $(x_j, y_j)$, $j = 1, 2$, are degenerate critical points of phase functions $\Phi_{a,b}^n$ belonging to $\mathcal{S}$. Let $v_j = e^{\pi i x_j}$ and $w_j = e^{2\pi i y_j}$. By Proposition 1 we have $\cos(2\pi x_1) = \cos(2\pi x_2)$ and $w_j = w_{-j}(x_j)$, where $\varepsilon = \text{sign}(p/q)$, so $w_1 = w_2$. Moreover, $x_1, x_2 \in [-2/3, -1/3] \cup [1/3, 2/3] + 2\mathbb{Z}$. We are interested in comparing the equivalence classes of the representations $\rho_j = \rho(v_j, w_j)$. For that purpose we can without loss of generality assume that $x_1, x_2 \in [1/3, 2/3]$. Then $x_2 = x_1$ or $x_2 = 1 - x_1$. In the second case $v_2 = -v_1$. Note that $v_1 = v_1^{-1}$ if and only if $v_1 \in \{\pm i\}$. But in that case $\cos(2\pi x_1) = -1$ and by (4.53) we get that $|p/q| \in \{\pm 2/\sqrt{5}\}$ which is impossible. Thus $\rho_1$ and $\rho_2$ are not equivalent. If $p$ is odd we must have $x_2 = x_1$ since only one of the points $(v_1, w_1)$, $(-v_1, w_1)$ can be a solution to (4.41). If $p$ is even both $(v_1, w_1)$ and $(-v_1, w_1)$ are solutions to both of the equations (4.41) and (4.42). Thus Proposition 1 shows that the set of degenerate critical points in $\mathcal{S}$ of the phase functions $\Phi_{a,b}^n$, $a, b, n \in \mathbb{Z}$, is either empty or corresponds under the surjection in Theorem 7 to one point in $\mathcal{M}_{p/q}$ in case $p$ is odd and to two points in the moduli space in case $p$ is even.
The phase functions $\Psi_{n}^{0,0}, \Psi_{n}^{1,-1}, \Psi_{n}^{1,1}$ and $\Psi_{n}^{2,0}$ are the ones entering Conjecture 3. By Theorem 8 the parameter $b$ is fixed by a critical point of $\Psi_{n}^{a,b}$, while $a$ and $n$ can be varied as long as we keep fixed $a + n/q$. Because of this (see also Lemma 5 below) we only need to consider $\Psi_{n}^{0,0}, \Psi_{n}^{0,-1}$, and $\Psi_{n}^{0,1}$. By Theorem 8 (and also the proof of this theorem and Proposition 2) we get

**Corollary 4.** Let $(x, y) \in \mathbb{C}^2$ such that $\rho_{(e^{\pi i x}, e^{2\pi i y})}$ is equivalent to a nonabelian SU(2)-representation of $\pi_1(M_{p/q})$. Let $(x', y') \in \mathbb{C}^2$ such that $(e^{\pi i x'}, e^{2\pi i y'}) = (e^{\pi i x}, e^{2\pi i y})$ and $\text{Re}(x') \in [-1, 1]$ and $\text{Re}(y') \in [-1/2, 1/2]$. Moreover, let

$$
\begin{align*}
    n &= q \left( y' + \frac{p}{2q} x' + \frac{i}{2\pi} \left( \log \left( 1 - e^{2\pi i(x+y)} \right) - \log \left( 1 - e^{2\pi i(x-y)} \right) \right) \right), \\
    b &= x' + \frac{i}{2\pi} \left( \log \left( 1 - e^{2\pi i(x+y)} \right) + \log \left( 1 - e^{2\pi i(x-y)} \right) \right).
\end{align*}
$$

Then $\theta := x'/2 \in [-1/3, -1/6] \cup [1/6, 1/3]$ and

$$
\begin{align*}
    b &= \begin{cases} 
        -1, & \theta \in [-1/3, -1/4] \\
        0, & \theta \in [-1/4, -1/6] \cup [1/6, 1/4] \\
        1, & \theta \in [1/4, 1/3].
    \end{cases}
\end{align*}
$$

Moreover, $n \in \mathbb{Z}$ and $(x', y')$ is a critical point of $\Psi_{n}^{0,b}$ and

$$
CS(\tilde{\rho}_{\theta, \varepsilon}) = \Psi_{n}^{0,b}(x', y') \pmod{\mathbb{Z}},
$$

where $\varepsilon \in \{\pm\}$ with $1+u_{\varepsilon}(\theta) = e^{2\pi i y}$, and $\tilde{\rho}_{\theta, \varepsilon}$ is a SU(2)-representation of $\pi_1(M_{p/q})$ equivalent to $\rho_{(e^{\pi i x}, e^{2\pi i y})}$ and given by Proposition 2.

We note that the first part of Theorem 8 is an immediate consequence of (4.40). To prove the second part we start by observing that if $a, a', n, n', b$ are integers such that $a' + n'/q = a + n/q$ then $\Psi_{n}^{a,b}(x, y) - \Psi_{n'}^{a',b'}(x, y)$ is an integer independent of $(x, y) \in \mathbb{C}^2$. The remaining part of the proof will consists of a series of lemmas. We start by

**Lemma 5.** Let $a, b, n \in \mathbb{Z}$ and assume that $(x, y)$ is a critical point of $\Psi_{n}^{a,b}$. Let $l, k \in \mathbb{Z}$ and put $a' = a + l, b' = b + 2k$, and $n' = n + pk$. Then $(x + 2k, y + l)$ is a critical point of $\Psi_{n'}^{a',b'}$ and

$$
\Psi_{n'}^{a',b'}(x + 2k, y + l) - \Psi_{n}^{a,b}(x, y) \equiv 0 \pmod{\mathbb{Z}}.
$$

**Proof.** That $(x + 2k, y + l)$ is a critical point of $\Psi_{n'}^{a',b'}$ is an immediate consequence of (4.40) and was already observed in Theorem 7. Put $x' = x + 2k$ and $y' = y + l$. Since $e^{2\pi i x'} = e^{2\pi i x}$ and $e^{2\pi i y'} = e^{2\pi i y}$ we get

$$
\begin{align*}
    \Psi_{n'}^{a',b'}(x', y') - \Psi_{n}^{a,b}(x, y) &= a'x' + b'y' - x'y' + \frac{n'}{q} x' - \frac{p}{4q} x'^2 - \frac{d}{q} n'^2 \\
    &\quad - ax - by + xy - \frac{n}{q} x + \frac{p}{4q} x^2 + \frac{d}{q} n^2 \\
    &= 2(a + l)k + 2\frac{n}{q}(1 - pd)k + \frac{p}{q}(1 - pd)k^2.
\end{align*}
$$
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Here \( pd - cq = 1 \) for an integer \( c \) so

\[
\Psi_{n,a,b}'(x',y') - \Psi_n^{a,b}(x,y) = 2(a + l - nc)k - pck^2.
\]

By Proposition 2 and the above lemma we are thus left with the following to prove: Let \( \theta \in [-1/3, -1/6] \cup [1/6, 1/3] \) and assume that \( \tilde{\rho}_{\theta, \varepsilon} \) extends to a representation of \( \pi_1(M_{p/q}) \) for \( \varepsilon \in \{ \pm \} \). Let \( b = b(\theta) \) and \( n = n(\theta) \) be the unique integers such that \( (2\theta, y) \) is a stationary point of \( \Psi_{n,a,b}^0 \), where \( y = \frac{1}{2\pi} \Log(1 + u_{\varepsilon}(\theta)) \) (so \( \Re(y) = \frac{1}{2} \)) (see around (4.43) and (4.44)). Then

\[
\text{CS}(\tilde{\rho}_{\theta, \varepsilon}) = \Psi_{n,a,b}^0(2\theta, y) \pmod{\mathbb{Z}}. \tag{5.78}
\]

In the course of the proof of this identity we will prove that

\[
\text{CS}(\tilde{\rho}_{\theta, \varepsilon}) = -\frac{1}{6} - \frac{p}{q} \theta^2 + \frac{m}{q} 2\theta - \frac{d}{q} m^2 - 2 \int_{1/6}^{\theta} \beta_\varepsilon(t) dt \pmod{\mathbb{Z}}, \tag{5.79}
\]

for \( \theta \in [1/6, 1/3] \), where \( m = p\theta + q\beta_\varepsilon(\theta) \in \mathbb{Z} \) by (5.74). Unfortunately the proof of (5.78) is rather technical, but we have tried to emphasize the main ideas of the proof here and defer many technicalities to the appendices C and D.

The Chern–Simons invariants of SU(2)–connections are real, so we begin by investigating to what extent a general phase function \( \Psi_{n,a,b}^0, a, b, n \in \mathbb{Z} \), is real in its critical points. Assume that \( (x, y) \) is such a critical point and write \( z = e^{2\pi ix} \) and \( w = e^{2\pi iy} \) as usual. From (4.40) we find that

\[
a - \Re(y) = \frac{p}{2q} \Re(x) - \frac{n}{q} - \frac{1}{2\pi} \Im \left( \Log(1 - zw) - \Log(1 - zw^{-1}) \right),
\]

and

\[
b - \Re(x) = -\frac{1}{2\pi} \Im \left( \Log(1 - zw) + \Log(1 - zw^{-1}) \right).
\]

Moreover, we have \( \Im(x) = -\frac{1}{2\pi} \Log|z| \) and \( \Im(y) = -\frac{1}{2\pi} \Log|w| \). By (4.39) we therefore get

\[
\Im(\Psi_{n,a,b}^0(x,y)) = (a - \Re(y)) \Im(x) + (b - \Re(x)) \Im(y) - \frac{p}{2q} \Im(x) \Re(x)
\]

\[
+ \frac{n}{q} \Im(x) + \frac{1}{4\pi^2} \Im \left( \Li_2(zw) - \Li_2(zw^{-1}) \right).
\]

This together with the above expressions for \( a - \Re(y) \) and \( b - \Re(x) \) leads to the formula

\[
\Im(\Psi_{n,a,b}^0(x,y)) = \frac{1}{4\pi^2} \left( \Im(\Log(1 - zw) - \Log(1 - zw^{-1})) \Log|z| 
\right.
\]

\[
+ \Im(\Log(1 - zw) + \Log(1 - zw^{-1})) \Log|w|
\]

\[
+ \Im \left( \Li_2(zw) - \Li_2(zw^{-1}) \right).
\]
By introducing the Bloch–Wigner dilogarithm function
\[ D(z) = \text{Im} (\text{Li}_2(z)) + \text{Arg}(1 - z) \log|z| \]
we obtain

**Lemma 6.** Let \( a, b, n \in \mathbb{Z} \) and let \((x, y)\) be a critical point of \( \Psi_n^{a,b} \). Then
\[
\text{Im} \left( \Psi_n^{a,b}(x, y) \right) = \frac{1}{4\pi^2} \left( D\left(e^{2\pi i(x+y)}\right) - D\left(e^{2\pi i(x-y)}\right) \right).
\]

We note that \( D \) is analytic on \( \mathbb{C} \setminus \{0, 1\} \) and continuous on \( \mathbb{C} \). Moreover, \( D \) satisfies the identities
\[
D(z) + D(\bar{z}) = 0, \quad D(z) + D(1/z) = 0 \quad (5.80)
\]
for all \( z \in \mathbb{C} \setminus \{0\} \). From this we have

**Corollary 5.** Let \( a, b, n \in \mathbb{Z} \) and let \((x, y)\) be a critical point of \( \Psi_n^{a,b} \) with \( e^{2\pi ix} \in S^1 \) and \( e^{2\pi iy} \in \mathbb{R} \). Then
\[
\text{Im} \left( \Psi_n^{a,b}(x, y) \right) = 0.
\]

**Remark 6.** Let \((v, w)\) be a non-zero solution to (4.41) and (4.42). Then \((\bar{v}, \bar{w})\) is also a solution to these two equations as already observed. Let \( z = e^\theta \) and write \((z, w) = (e^{2\pi ix}, e^{2\pi iy})\). Then \((\bar{z}, \bar{w}) = (e^{-2\pi ix}, e^{-2\pi iy})\). If \((x, y)\) is a critical point of \( \Psi_n^{a,b} \), then \((-\bar{x}, -\bar{y})\) is a critical point of \( \Psi_n^{-a,-b} \). By Lemma 6 and (5.80) we have
\[
\text{Im} \left( \Psi_n^{a,b}(x, y) \right) = -\text{Im} \left( \Psi_n^{-a,-b}(-\bar{x}, -\bar{y}) \right).
\]

Hence, if this value is different from zero, then either \( \exp(2\pi i \Psi_n^{-a,-b}(-\bar{x}, -\bar{y})) \) or \( \exp(2\pi i \Psi_n^{a,b}(x, y)) \) grows exponentially. We note that by Conjecture 3 and Corollary 5 stationary points leading to such exponential growth do not contribute to the large \( r \) asymptotics of \( \bar{\tau}_r(M_{p/q}) \), see also Theorem 4.

We now embark upon proving (5.78). We start by reducing to the case \( \theta \in [1/6, 1/3] \). By Proposition 2 we know that the representations \( \rho_{\theta, z} \) and \( \bar{\rho}_{-\theta, z} \) are SU(2)-equivalent, so in particular they have the same Chern–Simons invariant.

**Lemma 7.** Let the situation be as in (5.78). Then
\[
\Psi_n^{0,b(\theta)}(2\theta, y) - \Psi_n^{0,b(-\theta)}(-2\theta, y) \equiv 0 \pmod{\mathbb{Z}}.
\]

To prove this and (5.78) we need the technical Lemma 8 below keeping track of branches of the logarithm in certain expressions. Let \( I = [-1/3, -1/6] \cup [1/6, 1/3] \) and put
\[
Q_1^\pm(\theta) = 1 - (1 + u_\pm(\theta))^{-1} e^{4\pi i \theta},
Q_2^\pm(\theta) = 1 - (1 + u_\pm(\theta)) e^{4\pi i \theta}
\]
for \( \theta \in I \), where \( u_\pm \) are defined in Proposition 2. We also put \( Q_3^\pm(\theta) = 1 + u_\pm(\theta) \).
Lemma 8. We have
\[
\log \left( Q_1^+ (\theta) \right) + \log \left( Q_2^+ (\theta) \right) = \log \left( Q_1^+ (\theta) Q_2^+ (\theta) \right)
\]
and
\[
Q_1^+ (\theta) Q_2^+ (\theta) = e^{4\pi i \theta}
\]
for all \( \theta \in I \). Moreover
\[
\log \left( Q_1^+ (\theta) \right) + \log \left( Q_3^+ (\theta) \right) - \log \left( Q_2^+ (\theta) \right) = \log \left( \frac{Q_1^+ (\theta) Q_3^+ (\theta)}{Q_2^+ (\theta)} \right) + e_{\pm} (\theta) 2\pi i
\]
for all \( \theta \in I \), where
\[
e_{+}(\theta) = \begin{cases} 1, & \theta \in [1/6, 1/4], \\ 0, & \theta \in [1/4, 1/3], \end{cases}
\]
and \( e_{-}(\theta) = 1 - e_{+}(\theta) \) for \( \theta \in [1/6, 1/3] \), \( e_{+}(-1/4) = e_{+}(1/4) \), \( e_{-}(\theta) = 1 - e_{+}(-\theta) \) for \( \theta \in I \setminus \{\pm 1/6, \pm 1/4, \pm 1/3\} \), and \( e_{\pm}(\theta) = 0 \) for \( \theta \in \{\pm 1/6, \pm 1/3\} \). Finally
\[
Q_1^+ (\theta) Q_3^+ (\theta) Q_2^+ (\theta) = L_{\pm} (\theta)
\]
for all \( \theta \in I \), where \( L_{\pm} \) are given by Proposition 2.

The proof of this lemma is given in Appendix D.

Proof of Lemma 7 By (4.43) and (4.44) we have
\[
b(\theta) = 2\theta - \frac{1}{2\pi i} \left( \log \left( Q_1^+ (\theta) \right) + \log \left( Q_2^+ (\theta) \right) \right)
\]
and
\[
n(\theta) = p\theta + qy + \frac{1}{2\pi i} \left( \log \left( Q_1^+ (\theta) \right) - \log \left( Q_2^+ (\theta) \right) \right).
\]
By Lemma 8
\[
b(\theta) = 2\theta - \frac{1}{2\pi i} \log \left( e^{4\pi i \theta} \right). \quad (5.86)
\]
Taking the real part of the expression for \( n(\theta) \) we get
\[
n(\theta) = p\theta + \frac{1}{2} q + \frac{q}{2\pi} \left( \text{Arg} (Q_1^+ (\theta)) - \text{Arg} (Q_2^+ (\theta)) \right). \quad (5.87)
\]
By (5.86) we get that
\[
b(-\theta) = -b(\theta), \quad \theta \in [-1/3, -1/6] \cup [1/6, 1/3] \setminus \{\pm 1/4\},
\]
and
\[
b(-1/4) = -1, \quad b(1/4) = 0.
\]
By (5.87), (D.5), and (D.6) we get
\[
n(-\theta) = q - n(\theta), \quad \theta \in [-1/3, -1/6] \cup [1/6, 1/3] \setminus \{\pm 1/4\},
\]
and
\[ n(-1/4) = n(1/4) - \frac{p}{2}. \]

Let \( b = b(\theta) \) and \( n = n(\theta) \). Since \( \Psi_{n(\theta)}^{0,b(\theta)}(2\theta, y) \) is real by Corollary 5 and since \( \text{Re} (\text{Li}_2(z)) = \text{Re} (\text{Li}_2(z)) \) for \( z \in \mathbb{C}[\mathbb{I}, \infty] \) we find that
\[
\Psi_{n(\theta)}^{0,b(\theta)}(2\theta, y) - \Psi_{n(-\theta)}^{0,b(-\theta)}(-2\theta, y) = (b - b(-\theta)) \text{Re}(y) - 4\theta \text{Re}(y) + \frac{(n + n(-\theta)2\theta}{q} - \frac{d(n^2 - n(-\theta)^2)}{q}.
\]

(If \( \theta \in \{ \pm 1/4 \} \), use that \( \text{Li}_2(e^{2\pi i (2\theta + y)}) - \text{Li}_2(e^{2\pi i (2\theta - y)}) \) is the same for \( \theta = 1/4 \) and \( \theta = -1/4 \).) Assume first that \( \theta \neq \pm 1/4 \). Then, since \( \text{Re}(y) = 1/2 \), we get
\[
\Psi_{n(\theta)}^{0,b(\theta)}(2\theta, y) - \Psi_{n(-\theta)}^{0,b(-\theta)}(-2\theta, y) = b + dq - 2dn \in \mathbb{Z}.
\]

Next assume that \( \theta = 1/4 \). Then one finds
\[
\Psi_{n(\theta)}^{0,b(\theta)}(2\theta, y) - \Psi_{n(-\theta)}^{0,b(-\theta)}(-2\theta, y) = \frac{n}{q}(1 - pd) - \frac{p}{4q}(1 - pd).
\]

But \( 1 - pd = -qc \) for an integer \( c \) so
\[
\Psi_{n(\theta)}^{0,b(\theta)}(2\theta, y) - \Psi_{n(-\theta)}^{0,b(-\theta)}(-2\theta, y) = \left( \frac{p}{4} - n \right) c.
\]

By (5.74) \( p/4 + q\beta(1/4) \in \mathbb{Z} \). But \( L_{1/4}(1/4) = 1 \) so \( \beta(1/4) = f_e(1/4) \in \mathbb{Z} \), so \( p \) is divisible by 4.

By using that \( y = \frac{1}{2\pi i} \log (Q_3^e(\theta)) \) together with Lemma 8 we obtain the alternative formula
\[
n(\theta) = p\theta + q\beta(\theta) + \frac{q}{2\pi i} \log (L_e(\theta)).
\]

This and (5.71) immediately leads to
\[
n(\theta) = p\theta + q\beta(\theta) + q(e_\theta(\theta) - f_\theta(\theta)) \tag{5.88}
\]
for \( \theta \in [1/6, 1/3] \). In the proof of (5.78) we need certain symmetries for the functions \( L_{\pm} \). First we note that
\[
L_{\pm} \left( \frac{1}{2} - \theta \right) = L_{\mp}(\theta) \tag{5.89}
\]
for \( \theta \in [1/6, 1/3] \) by (5.69). Second, by the next lemma and Proposition 2, we have
\[
L_{\mp}(\theta) = L_{\pm}(\theta)^{-1} \tag{5.90}
\]
for \( \theta \in [-1/3, -1/6] \cup [1/6, 1/3] \). (In particular, \( \widetilde{\rho}_{\theta, \pm} \) extends to a representation of \( \pi_1(M_{p/q}) \) if and only if \( \widetilde{\rho}_{\theta, \mp} \) extends to a representation of \( \pi_1(M_{-p/q}) \).)
Lemma 9. Let $s \in \mathbb{C}^*$ and let $u_\pm$ be the two solutions to $\phi(s^2, u) = 0$. Then

$$(1 + u_+)(1 + u_-) = 1,$$

and

$$\lambda_{11}(s, u_+)\lambda_{11}(s, u_-) = 1.$$ 

Proof. We have

$$1 + u_\pm = \frac{1}{2}(s^2 + s^{-2} - 1) \pm \frac{1}{2}\sqrt{s^4 + s^{-4} - 2(s^2 + s^{-2}) - 1},$$

so

$$(1 + u_+)(1 + u_-) = \frac{1}{4}(s^2 + s^{-2} - 1)^2 - \frac{1}{4}(s^4 + s^{-4} - 2(s^2 + s^{-2}) - 1) = 1.$$ 

Moreover,

$$\lambda_{11}(s, u_\pm) = \frac{1}{2}(s^4 + s^{-4}) - \frac{1}{2}(s^2 + s^{-2}) - 1 \pm \frac{1}{2}(s^2 - s^{-2})\sqrt{s^4 + s^{-4} - 2(s^2 + s^{-2}) - 1},$$

so

$$\lambda_{11}(s, u_+)\lambda_{11}(s, u_-) = \left(\frac{1}{2}(s^4 + s^{-4}) - \frac{1}{2}(s^2 + s^{-2}) - 1\right)^2$$

$$- \frac{1}{4}(s^2 - s^{-2})^2 (s^4 + s^{-4} - 2(s^2 + s^{-2}) - 1).$$

By simple reductions one gets the result. 

We are now ready to finalize the proof of (5.78) and thereby the proof of Theorem 8.

Proof of (5.78) By Lemma 7 we can assume that $\theta \in [1/6, 1/3]$. Write $\theta_0$ for $\theta$ in the following. Let us first observe that formula (5.79) is an immediate consequence of Proposition 4. In fact, by letting $c, d$ be integers as in Proposition 4 we get

$$-2cq\theta_0\beta_\epsilon(\theta_0) = 2cp\theta_0^2 - 2cm\theta_0,$$

$$-dq\beta_\epsilon^2(\theta_0) = -\frac{d}{q}(m^2 - 2pm\theta_0 + p^2\theta_0^2),$$

and therefore

$$-cp\theta_0^2 - dq\beta_\epsilon^2(\theta_0) - 2cq\theta_0\beta_\epsilon(\theta_0) = \left(c - \frac{dp}{q}\right)p\theta_0^2 - 2\left(c - \frac{dp}{q}\right)m\theta_0 - \frac{d}{q}m^2.$$ 

But $c - dp/q = -1/q$, hence (5.79) follows. On the other hand we have

$$\Psi_{n}^{0, b}(2\theta_0, y) = (b - 2\theta_0)y + \frac{2}{q}2\theta_0 - \frac{p}{q}2\theta_0 - \frac{d}{q}n^2 + \frac{1}{4\pi^2} \left(\text{Li}_2(z_0 w_\epsilon) - \text{Li}_2(z_0 w^{-1}_\epsilon)\right),$$
where \( z_0 = e^{4\pi i\theta_0} \) and \( w_\varepsilon = 1 + u_\varepsilon(\theta_0) \). By (5.86) and (5.88) we have \( b - 2\theta_0 = -\frac{1}{2\pi i} \log(z_0) \) and \( n = m + q(e_\varepsilon(\theta_0) - f_\varepsilon(\theta_0)) \) so
\[
\Psi_{n}^{0,b}(2\theta_0, y) = -\frac{p}{q}\theta_0^2 + \frac{m}{q}2\theta_0 - \frac{d}{q}m^2 + 2(e_\varepsilon(\theta_0) - f_\varepsilon(\theta_0))\theta_0 - l_\varepsilon(\theta_0) + \frac{1}{4\pi^2} \left( \log(z_0) \log(w_\varepsilon) + \operatorname{Li}_2(z_0 w_\varepsilon) - \operatorname{Li}_2(z_0 w_\varepsilon^{-1}) \right),
\]
where \( l_\varepsilon(\theta_0) = 2m(e_\varepsilon(\theta_0) - f_\varepsilon(\theta_0)) + dq(e_\varepsilon(\theta_0) - f_\varepsilon(\theta_0))^2 \in \mathbb{Z} \). We therefore get that
\[
\Psi_{n}^{0,b}(2\theta_0, y) - \text{CS}(\rho_{\theta_0,\varepsilon}) = \frac{1}{6} + 2(e_\varepsilon(\theta_0) - f_\varepsilon(\theta_0))\theta_0 + \frac{1}{4\pi^2} \left( \log(z_0) \log(w_\varepsilon) + \operatorname{Li}_2(z_0 w_\varepsilon) - \operatorname{Li}_2(z_0 w_\varepsilon^{-1}) \right) + 2\int_{1/6}^{\theta_0} \beta_\varepsilon(t) dt \quad \text{(mod Z)}.
\]

For \( \theta_0 \in [1/6, 1/4] \) we note that \( e_\varepsilon(\theta_0) = f_\varepsilon(\theta_0) \). We will consider the special cases \( \theta_0 \in \{1/6, 1/4, 1/3\} \) first and then handle the other cases afterwards.

**The cases** \( \theta_0 \in \{1/6, 1/3\} \). In these cases we have \( w_\varepsilon = -1 \) so
\[
\Psi_{n}^{0,b}(2\theta_0, y) - \text{CS}(\rho_{\theta_0,\varepsilon}) = \frac{1}{6} + \frac{i}{4\pi} \log(z_0) + 2(e_\varepsilon(\theta_0) - f_\varepsilon(\theta_0))\theta_0 + 2\int_{1/6}^{\theta_0} \beta_\varepsilon(t) dt \quad \text{(mod Z)}.
\]

If \( \theta_0 = 1/6 \) we immediately get that this is zero. If \( \theta_0 = 1/3 \) we get
\[
\Psi_{n}^{0,b}(2\theta_0, y) - \text{CS}(\rho_{\theta_0,\varepsilon}) = \frac{1}{6} + \frac{i}{4\pi} (4\pi i/3 - 2\pi i) + \frac{2}{3} (e_\varepsilon(\theta_0) - f_\varepsilon(\theta_0)) + 2f_\varepsilon(1/4)(1/3 - 1/6) + \frac{1}{\pi i} \int_{1/6}^{1/3} \log(L_\varepsilon(t)) dt \quad \text{(mod Z)}.
\]

But if \( \theta_0 \in [1/4, 1/3] \) then
\[
\int_{1/4}^{\theta_0} \log(L_\pm(\theta)) d\theta = \int_{1/4}^{1/2-\theta_0} \log(L_\mp(t)) dt = -\int_{1/2-\theta_0}^{1/4} \log(L_\pm(t)) dt,
\]
by (5.89) and (5.90) so
\[
\int_{1/2-\theta_0}^{\theta_0} \log(L_\pm(\theta)) d\theta = 0. \quad (5.91)
\]

In particular,
\[
\Psi_{n}^{0,b}(2\theta_0, y) - \text{CS}(\rho_{\theta_0,\varepsilon}) = \frac{1}{3} - \frac{2}{3} f_\varepsilon(1/3) + \frac{1}{3} f_\varepsilon(1/4) \quad \text{(mod Z)},
\]
where we also use that \( e_\pm(1/3) = 0 \) by Lemma 8. By (5.72) and (5.73) this is zero.

**The case** \( \theta_0 = 1/4 \). In this case we have \( z_0 = -1 \) so

\[
\Psi_n^{0,b}(2\theta_0, y) - CS(\tilde{\rho}_{\theta_0, \varepsilon}) = \frac{1}{6} + \frac{1}{4\pi^2} \left( \pi \log(w_\varepsilon) + \text{Li}_2(\varepsilon|w_\varepsilon|) - \text{Li}_2(\varepsilon|w_\varepsilon|^{-1}) \right) + 2 \int_{1/6}^{1/4} \beta_\varepsilon(t) dt \pmod{\mathbb{Z}}.
\]

Since \((v_0, w_\varepsilon) = (e^{2\pi i \theta_0}, w_\varepsilon)\) is a solution to (4.42) we have \( w_\varepsilon \in \{(-3 - \sqrt{5})/2, (-3 + \sqrt{5})/2\} \), and by (D.2) we then conclude that \( w_\varepsilon = \frac{\sqrt{2}}{2} \) if \( \varepsilon = -1 \) and \( w_\varepsilon = -\sqrt{2} \) if \( \varepsilon = +1 \). By (C.10) we then get

\[
\Psi_n^{0,b}(2\theta_0, y) - CS(\tilde{\rho}_{\theta_0, \varepsilon}) = \frac{1}{6} + \frac{1}{4\pi^2} \left( -\pi^2 - \frac{\pi^2}{9} \right) + 2 \int_{1/6}^{1/4} \beta_\varepsilon(t) dt
\]

\[
= \frac{1}{6} + \frac{1}{4} \left( -1 - \frac{\pi}{3} \right) + \frac{1}{6} f_\varepsilon(1/4)
\]

\[
+ \frac{1}{\pi} \int_{1/6}^{1/4} \text{Arg}(L_\varepsilon(t)) dt \pmod{\mathbb{Z}},
\]

and this is zero by (5.72), (5.73), and (C.5).

**The case** \( \theta \in [1/6, 1/3] \setminus \{1/4\} \). We have

\[
\Psi_n^{0,b}(2\theta_0, y) - CS(\tilde{\rho}_{\theta_0, \varepsilon}) = \frac{1}{6} + \frac{1}{4\pi^2} \log(z_0) \log(w_\varepsilon) + 2 \left( e_\varepsilon(\theta_0) - f_\varepsilon(\theta_0) \right) \theta_0
\]

\[
+ 2 \int_{1/6}^{\theta_0} \beta_\varepsilon(t) dt + R(2\theta_0, y) \pmod{\mathbb{Z}},
\]

where

\[
R(x, y) = \frac{1}{4\pi^2} \left( \text{Li}_2(e^{2\pi i(x+y)}) - \text{Li}_2(e^{2\pi i((x-y)}) \right).
\]

Let us write \( u \) for \( u_\varepsilon \) in the following. By definition of the dilogarithm we have

\[
4\pi^2 R(2\theta_0, y) = \int_0^{(1+u)^{-1} e^{4\pi i \theta_0}} \frac{\log(1-t)}{t} dt - \int_0^{(1+u) e^{4\pi i \theta_0}} \frac{\log(1-t)}{t} dt.
\]

By (D.2) we have \( 1 + u_{-}(\theta) \leq -1 \leq u_{+}(\theta) < 0 \) for all \( \theta \in [1/6, 1/3] \) and \( 1 + u_{\pm}(\theta) = -1 \) if and only if \( \theta \in \{1/6, 1/4\} \). Let \( \theta_1 = 1/6 \) if \( \theta_0 \in [1/6, 1/4] \) and \( \theta_1 = 1/3 \) if \( \theta_0 \in (1/4, 1/3] \). We note that \( z \mapsto \log(1-z)/z \) is analytic on \( \mathbb{C} \setminus [1, \infty[ \) so by Cauchy’s theorem

\[
4\pi^2 R(2\theta_0, y) = \int_0^{-(1+u)^{-1} e^{4\pi i \theta_1}} \frac{\log(1-t)}{t} dt + \int_{-(1+u) e^{4\pi i \theta_0}}^{(1+u)^{-1} e^{4\pi i \theta_0}} \frac{\log(1-t)}{t} dt - \int_{-(1+u) e^{4\pi i \theta_0}}^{(1+u) e^{4\pi i \theta_0}} \frac{\log(1-t)}{t} dt
\]

\[
\int_{-e^{4\pi i \theta_1}}^{(1+u)^{-1} e^{4\pi i \theta_1}} \frac{\log(1-t)}{t} dt - \int_{-e^{4\pi i \theta_1}}^{(1+u) e^{4\pi i \theta_0}} \frac{\log(1-t)}{t} dt.
\]
The curves $\gamma_{\pm}(\theta) = (1 + u(\theta))^{\pm 1} e^{\pm i\theta}$ are smooth on $[1/6, 1/3]$ so

$$4\pi^2 R(2\theta_0, y) = \lim_{\eta \to 0^+} \left( \int_{\theta_1 + \mu \eta}^{\theta_0} \log(1 - \gamma_-(\theta)) \frac{\gamma'_-(\theta)}{\gamma_-(\theta)} d\theta ight.$$ 

$$- \int_{\theta_1 + \mu \eta}^{\theta_0} \log(1 - \gamma_+(\theta)) \frac{\gamma'_+(\theta)}{\gamma_+(\theta)} d\theta \right),$$

where $\mu = 1$ if $\theta_1 = 1/6$ and $\mu = -1$ if $\theta_1 = 1/3$. (The parameter $\eta$ is necessary because $u$ is not differentiable in $1/6$ and $1/3$.) It follows that

$$4\pi^2 R(2\theta_0, y) = \lim_{\eta \to 0^+} (4\pi i R_1(\theta_0, \eta) - R_2(\theta_0, \eta)), $$

where

$$R_1(\theta_0, \eta) = \int_{\theta_1 + \mu \eta}^{\theta_0} \log(Q_1^+(\theta)) - \log(Q_2^+(\theta)) d\theta,$$

$$R_2(\theta_0, \eta) = \int_{\theta_1 + \mu \eta}^{\theta_0} \left\{ \log(Q_1^-(\theta)) + \log(Q_2^-(\theta)) \right\} \frac{u'(-\theta)}{1 + u(-\theta)} d\theta,$$

where the functions $Q_i^\pm$ are defined above Lemma 8. By Lemma 8

$$R_2(\theta_0, \eta) = \int_{\theta_1 + \mu \eta}^{\theta_0} \log(e^{4\pi i \theta}) \frac{u'}{1 + u} d\theta$$

$$= 4\pi i \int_{\theta_1 + \mu \eta}^{\theta_0} \theta \frac{u'(-\theta)}{1 + u(-\theta)} d\theta - 2\pi i b(\theta_0) \int_{\theta_1 + \mu \eta}^{\theta_0} u'(-\theta) \frac{d\theta}{1 + u(-\theta)}.$$

Since $u'(-\theta)/(1 + u(-\theta)) = \frac{d}{d\theta} \log(1 + u(-\theta))$ for any branch log of the logarithm defined on an open section of $\mathbb{C}^*$ containing $[-\infty, 0]$ we have

$$\int_{\theta_1 + \mu \eta}^{\theta_0} \theta \frac{u'(-\theta)}{1 + u(-\theta)} d\theta = [\theta \log(1 + u(-\theta))]_{\theta_1 + \mu \eta}^{\theta_0} - \int_{\theta_1 + \mu \eta}^{\theta_0} \log(1 + u(-\theta)) d\theta,$$

and

$$\int_{\theta_1 + \mu \eta}^{\theta_0} \frac{u'(-\theta)}{1 + u(-\theta)} d\theta = [\log(1 + u(-\theta))]_{\theta_1 + \mu \eta}^{\theta_0}.$$

We therefore get

$$R(2\theta_0, y) = \frac{1}{4\pi^2} \lim_{\eta \to 0^+} (4\pi i R_1(\theta_0, \eta) - R_2(\theta_0, \eta))$$

$$= \frac{b(\theta_0)}{2} - \theta_1 - \frac{1}{4\pi^2} \log(z_0) \log(w\varepsilon)$$

$$+ \frac{i}{\pi} \lim_{\eta \to 0^+} \left( R_3(\theta_0, \eta) + \int_{\theta_1 + \mu \eta}^{\theta_0} \log(Q_3^\pm(\theta)) d\theta \right),$$
50  Asymptotics of the quantum invariants for the figure 8 knot

where we use that $1 + w_ε(θ) = Q_3^ε(θ) = w_ε$ and (5.86). By Lemma 8 we get

$$R_1(θ_0, η) + \int_{θ_1 + μη}^{θ_0} \log (Q_3^ε(θ)) dθ = \int_{θ_1 + μη}^{θ_0} \log (L_ε(θ)) + 2πiε(θ)dθ,$$

so

$$\lim_{η→0⁺} \left( R_1(θ_0, η) + \int_{θ_1 + μη}^{θ_0} \log (Q_3^ε(θ)) dθ \right) = ε(θ_0)2πi(θ_0 - θ_1) + \int_{θ_1}^{θ_0} \log (L_ε(θ))dθ.$$

But then

$$\Psi_{n,b}^0(2θ_0, y) - \text{CS}(\bar{ρ}_{b,ε}) = \frac{1}{6} + 2ε(θ_0)θ_1 - 2f_ε(θ_0)θ_0 + \frac{b(θ_0)}{2} - θ_1$$

$$+ 2 \int_{1/6}^{θ_0} β_ε(t)dt - \frac{1}{πi} \int_{θ_1}^{θ_0} \text{Log}(L_ε(θ))dθ \quad (\text{mod } Z).$$

Here

$$2 \int_{1/6}^{θ_0} β_ε(t)dt = 2f_ε(θ_0) \left( θ_0 - \frac{1}{6} \right) + \frac{1}{πi} \int_{θ_1}^{θ_0} \text{Log}(L_ε(θ))dθ,$$

so

$$\Psi_{n,b}^0(2θ_0, y) - \text{CS}(\bar{ρ}_{b,ε}) = \frac{1}{πi} \left( \int_{1/6}^{θ_0} \text{Log}(L_ε(θ))dθ - \int_{θ_1}^{θ_0} \text{Log}(L_ε(θ))dθ \right)$$

$$+ \frac{1}{6} + 2ε(θ_0)θ_1 - \frac{1}{3}f_ε(θ_0) + \frac{b(θ_0)}{2} - θ_1 \quad (\text{mod } Z).$$

The subcase $θ_0 \in [1/6, 1/4[$. Here we have $θ_1 = 1/6$ and $b(θ_0) = 0$ so the result follows by the fact that $ε(θ) = f_ε(θ)$ for $θ \in [1/6, 1/4[$.

The subcase $θ_0 \in [1/4, 1/3]$. In this case we have $b(θ_0) = 1$ and $θ_1 = 1/3$ so

$$\Psi_{n,b}^0(2θ_0, y) - \text{CS}(\bar{ρ}_{b,ε}) = \frac{1}{πi} \left( \int_{1/6}^{θ_0} \text{Log}(L_ε(θ))dθ - \int_{1/3}^{θ_0} \text{Log}(L_ε(θ))dθ \right)$$

$$+ \frac{1}{3} + \frac{2}{3}ε(θ_0) - \frac{1}{3}f_ε(θ_0) \quad (\text{mod } Z).$$

By (5.91) we get

$$\int_{1/3}^{θ_0} \text{Log}(L_ε(t))dt = \int_{1/3}^{θ_0} \text{Log}(L_ε(t))dt = - \int_{1/6}^{θ_0} \text{Log}(L_ε(t))dt.$$

By (5.89) and (5.90) we then have

$$\int_{1/3}^{θ_0} \text{Log}(L_ε(t))dt = \int_{1/6}^{θ_0} \text{Log}(L_ε(t))dt,$$

so

$$\Psi_{n,b}^0(2θ_0, y) - \text{CS}(\bar{ρ}_{b,ε}) = \frac{1}{3} + \frac{2}{3}ε(θ_0) - \frac{1}{3}f_ε(θ_0) \quad (\text{mod } Z),$$

and this is zero by Lemma 8, (5.72) and (5.73).
Remark 7. If \( u_{\pm} = u_{\pm}(v) \) are the two solutions to \( \phi(v^2, u) = 0 \) for \( v \in \mathbb{C}^* \) fixed, then \( \lambda_{11}(v, u_{\pm}) = \lambda_{11}(v, u_{\pm})^{-1} \) by Lemma 9. By the proof of Theorem 7 we therefore conclude that \((v, 1 + u_{\pm}), \varepsilon \in \{\pm\}\) is a solution to (4.41) and (4.42) if and only if \((v, 1 + u_{\pm})\) is a solution to (4.42) and

\[
v^p = \left( \frac{w - v^2}{1 - v^2 w} \right)^q.
\] (5.92)

If we work with the invariants \( \tau_r \) instead of the invariants \( \bar{\tau}_r \), then by (2.16) we have to change \( p/q \) to \(-p/q\) everywhere in the above. The equation (4.42) will be the same but (4.41) will change to (5.92). If \((v, u) \in \tilde{N}\) then we find as in the proof of Theorem 7 that \((v, w) = (v, u + 1)\) is a solution to (5.92) if and only if \((v, u)\) is a solution to

\[
v^p = \lambda_{11}(v, u)^q,
\]

which is the “wrong” equation. This is one of the main reasons for working with \( \bar{\tau}_r \) instead of \( \tau_r \). Another reason is that one would get a minus sign in (5.78).

Let us end this section by some further results on the Chern–Simons invariants of flat irreducible SU(2)–connections on \( M_{p/q} \). By (5.90) \( \bar{\rho}_{\theta, \varepsilon} \) extends to a SU(2)–representation of \( \pi_1(M_{p/q}) \) if and only if \( \bar{\rho}_{\theta, -\varepsilon} \) extends to a SU(2)–representation of \( \pi_1(M_{-p/q}) \). In that case we find that

\[
CS(\bar{\rho}_{\theta, -\varepsilon}) = -CS(\bar{\rho}_{\theta, \varepsilon})
\]

by using (5.74).

If \( p \) is even then \( e^{2\pi ip(1/2 - \theta)} = e^{2\pi ip\theta} \). Since \( L_\varepsilon(1/2 - \theta) = L_\varepsilon(\theta)^{-1} \) by (5.89) and (5.90) and since \( q \) is odd it follows that \( \bar{\rho}_{\theta, \varepsilon} \) extends to a SU(2)–representation of \( \pi_1(M_{p/q}) \) if and only if \( \bar{\rho}_{1/2 - \theta, \varepsilon} \) extends to such a representation, see (5.65). Assuming that \( \bar{\rho}_{\theta, \varepsilon} \) extends to such a representation we get

\[
CS(\bar{\rho}_{1/2 - \theta, \varepsilon}) = CS(\bar{\rho}_{\theta, \varepsilon}) - \frac{1}{2} \frac{p}{2} \text{ (mod } \mathbb{Z})
\]

by using (5.91) and (5.74).

By (5.70) and (5.65) we find that \( \bar{\rho}_{1/6, -} = \bar{\rho}_{1/6, +} \) is a SU(2)–representation of \( \pi_1(M_{p/q}) \) if and only if \( p = 6m + 3, m \in \mathbb{Z} \), and \( q \) is odd. In that case

\[
CS(\bar{\rho}_{1/6, \varepsilon}) = -\frac{cp}{36} - \frac{dq}{4} - \frac{1}{2}d \text{ (mod } \mathbb{Z}).
\]

By (5.70) and (5.65) we also have that \( \bar{\rho}_{1/3, -} = \bar{\rho}_{1/3, +} \) is a SU(2)–representation of \( \pi_1(M_{p/q}) \) if and only if \( p = 6m + 3, m \in \mathbb{Z} \), and \( q \) is even. In that case

\[
CS(\bar{\rho}_{1/3, \varepsilon}) = \frac{1}{2} \frac{cp}{9} - \frac{dq}{4} \text{ (mod } \mathbb{Z}).
\]
Asymptotics of the quantum invariants for the figure 8 knot

Finally, by (5.70) and (5.65), $\rho_{1/4,-}$ and $\rho_{1/4,+}$ are SU(2)–representations of $\pi_1(M_{p/q})$ if and only if 4 divides $p$. In that case we find that

\[
\begin{align*}
\text{CS}(\bar{\rho}_{1/4,+}) &= \frac{1}{5} - \frac{cp}{16} \pmod\mathbb{Z}, \\
\text{CS}(\bar{\rho}_{1/4,-}) &= -\frac{1}{5} - \frac{cp}{16} \pmod\mathbb{Z}.
\end{align*}
\]

This follows by using (C.5).

We have in the following appendices collected material of a technical nature.

**Appendix A Proofs of Lemma 1, Lemma 3 and (4.23)**

**Proof of Lemma 1** Let $a > 0$. Let $\varepsilon = 1$ if $\text{Im}(\zeta) \geq 0$ and let $\varepsilon = -1$ otherwise. Put $\delta^+ = [-a, -\sqrt{-1}a]$ and $\delta^- = [\sqrt{-1}a, a]$. (Here, as usual, $[z_1, z_2]$ denotes the line segment in $\mathbb{C}$ beginning at $z_1$ and ending at $z_2$.) We have

\[
\begin{align*}
\frac{S_2(\zeta - \gamma)}{S_2(\zeta + \gamma)} &= \exp\left(-\frac{1}{2} \int_{C_{\gamma}} \frac{e^{\zeta z}}{\sinh(\pi z) z} \, dz \right).
\end{align*}
\]

By an elementary argument one finds that the integrals $\int_{\delta^\pm} \frac{e^{\zeta z}}{\sinh(\pi z) z} \, dz$ converge to zero as $a \to \infty$. Therefore

\[
\int_{C_{\gamma}} \frac{e^{\zeta z}}{\sinh(\pi z) z} \, dz = \varepsilon 2\pi \sqrt{-1} \left( b_1 + \sum_{n=1}^{\infty} \text{Res}_{z=\varepsilon \sqrt{-1}n} \left\{ \frac{e^{\zeta z}}{\sinh(\pi z) z} \right\} \right),
\]

where $b_1 = 0$ and $b_{-1} = \text{Res}_{z=0} \left\{ \frac{e^{\zeta z}}{\sinh(\pi z) z} \right\} = \frac{\zeta}{\pi}$. For $n \in \mathbb{Z} \setminus \{0\}$ we have

\[
\text{Res}_{z=\sqrt{-1}n} \left\{ \frac{e^{\zeta z}}{\sinh(\pi z) z} \right\} = \frac{(-1)^n e^{\sqrt{-1} \zeta n}}{\pi \sqrt{-1} n},
\]

so

\[
\int_{C_{\gamma}} \frac{e^{\zeta z}}{\sinh(\pi z) z} \, dz = -(1-\varepsilon)\sqrt{-1}\zeta - 2\text{Log} \left( 1 + e^{\sqrt{-1}\zeta} \right)
\]

giving the result.

To prove the identity (4.23) we use the power series expansion

\[
\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}
\]

for the dilogarithm, valid for $|z| \leq 1$. In the course of the following proof we will establish the identity

\[
\frac{\zeta^2}{2} - \frac{\pi^2}{6} - \text{Li}_2(-e^{-\sqrt{-1}\zeta}) = \text{Li}_2(-e^{\sqrt{-1}\zeta})
\]

valid for $\zeta = \pm \pi$ and all $\zeta \in \mathbb{C}$ with $|\text{Re}(\zeta)| < \pi$. 

\[
(A.1)
\]
Proof of (4.23) Note first that the integral $A_\gamma(\zeta) := \frac{1}{4\pi i} \int_{C_R} \frac{e^{\zeta t}}{\sinh(\pi t)^2} \, dz$ is convergent for all $\zeta \in \mathbb{C}$ with $|\Re(\zeta)| \leq \pi$ since

$$\int_{-\infty}^{-R} \frac{e^{\zeta t}}{\sinh(\pi t)^2} \, dt = -\int_{R}^{\infty} \frac{e^{-\zeta t}}{\sinh(\pi t)^2} \, dt$$

and

$$\left| \int_{R}^{\infty} \frac{e^{\zeta t}}{\sinh(\pi t)^2} \, dt \right| \leq \frac{2}{1 - e^{-2\pi R}} \int_{R}^{\infty} e^{(\Re(\zeta) - \pi)t} \frac{1}{t^2} \, dt.$$ 

Let $b = \text{sign}(\Im(\zeta))$, where sign(0) can be put to both 1 and $-1$ in the following. Moreover, let $h$ be a positive parameter and let $\delta_h^-(t) = (1 + ib)ht + ibh$ for $t \in [-1, 0]$ and let $\delta_h^+(t) = (1 - ib)ht + ibh$ for $t \in [0, 1]$. It is elementary to show that the integrals $\int_{\delta_h^\pm} \frac{e^{\zeta t}}{\sinh(\pi t)^2} \, dz$ converge to zero as $h$ converges to infinity for $|\Re(\zeta)| \leq \pi$.

By the residue theorem we conclude that

$$A_\gamma(\zeta) = \frac{1}{4\pi i} 2\pi i \sum_{n=1}^{\infty} \text{Res}_{z=\pm i n} \left\{ \frac{e^{\zeta z}}{\sinh(\pi z)^2} \right\}$$

for $\Im(\zeta) \geq 0$ and $|\Re(\zeta)| \leq \pi$ and

$$A_\gamma(\zeta) = -\frac{1}{4\pi i} 2\pi i \sum_{n=1}^{\infty} \text{Res}_{z=-\pm i n} \left\{ \frac{e^{\zeta z}}{\sinh(\pi z)^2} \right\}$$

for $\Im(\zeta) \leq 0$ and $|\Re(\zeta)| \leq \pi$. Using (A.1) this leads directly to

$$A_\gamma(\zeta) = \begin{cases} \frac{1}{2\sqrt{1 - \gamma^2}} \text{Li}_2(-e^{\sqrt{-\gamma^2}}) & \text{if } \Im(\zeta) \geq 0, \\ \frac{1}{2\sqrt{1 - \gamma^2}} \left[ \frac{\zeta^2}{2} - \frac{\pi^2}{6} - \text{Li}_2(-e^{-\sqrt{-\gamma^2}}) \right] & \text{if } \Im(\zeta) \leq 0. \end{cases}$$

for $|\Re(\zeta)| \leq \pi$. Left is to prove the identity (A.2). To this end, let

$$g(\zeta) = \frac{\zeta^2}{2} - \frac{\pi^2}{6} - \text{Li}_2(-e^{i\gamma}) - \text{Li}_2(-e^{i\gamma})$$

for $\zeta \in \Omega := \{ \zeta \in \mathbb{C} \mid |\Re(\zeta)| < \pi \}$. By (4.22) we have

$$g'(\zeta) = \zeta + i \left( \text{Log}(1 + e^{i\gamma}) - \text{Log}(1 + e^{-i\gamma}) \right)$$

and therefore $e^{ig'(\zeta)} = 1$. Since $\Omega$ is connected, $g$ is $C^1$ and $g'(0) = 0$ we get that $g'$ is identically zero on $\Omega$ so $g$ is constant on $\Omega$. Now $g(0) = -\frac{\pi^2}{6} - 2\text{Li}_2(-1)$ and

$$\text{Li}_2(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

so $g(0) = 0$. Finally we note that $g$ is well-defined and continuous on $\Omega \cup \{ \pm \pi \}$ so $g(\pm \pi) = 0$ by continuity.
Note that the function $g$ in the above proof is a well-defined analytic function on $W = \{ \zeta \in \mathbb{C} \mid \Re(\zeta) \notin \pi + 2\pi \mathbb{Z} \}$ and that $g$ is continuous on $W \cup \mathbb{R}$. As in the proof above we find that $g$ is constant on each connected component of $W$. Moreover, we can on each of these connected components choose a branch of the dilogarithm such that $g$ extends to a continuous (and hence a constant) function on the connected set $W \cup \mathbb{R}$.

**Proof of Lemma 3** The function $1/\sinh(w)$ has a simple pole at $w = 0$ with principal part $1/w$, i.e.

$$\phi(w) = \frac{1}{\sinh(w)} - \frac{1}{w}$$

is holomorphic in a neighborhood of zero, in fact on the open disk $D(0, \pi)$ with centre 0 and radius $\pi$. Let $a \geq R$, let $C_{R,a} = [-a, -R] \cup \Upsilon_R \cup [a, R]$, and write $I_\gamma(\zeta) = J_\gamma(\zeta) + K_\gamma(\zeta)$, where

$$4J_\gamma(\zeta) = \int_{C_{R,a}} \frac{e^{\zeta z}}{\sinh(\pi z) \phi(\gamma z)} \, dz,$$

$$4K_\gamma(\zeta) = \int_{-\infty}^{-a} \frac{e^{\zeta z}}{\sinh(\pi z) \phi(\gamma z)} \, dz + \int_{a}^{\infty} \frac{e^{\zeta z}}{\sinh(\pi z) \phi(\gamma z)} \, dz.$$

To estimate $K_\gamma(\zeta)$ we simply use that

$$|\phi(\gamma t)| = \frac{1}{\sinh(\gamma t)} + \frac{1}{\gamma t} \leq \frac{2}{\gamma t}$$

for $t > 0$ leading to the bound

$$|K_\gamma(\zeta)| \leq \frac{1}{2\gamma} \int_{a}^{\infty} \frac{e^{\Re(\zeta)t} + e^{-\Re(\zeta)t}}{\sinh(\pi t) \frac{1}{t^2}} \, dt$$

$$\leq \frac{1}{\gamma(1 - e^{-2\pi a})} \int_{a}^{\infty} \left( e^{-(\pi - \Re(\zeta))t} + e^{-(\pi + \Re(\zeta))t} \right) \frac{1}{t^2} \, dt.$$

For $|\Re(\zeta)| \leq \pi$ we therefore get

$$|K_\gamma(\zeta)| \leq \frac{2}{\gamma(1 - e^{-2\pi a})} \int_{a}^{\infty} \frac{1}{t^2} \, dt = \frac{2}{a \gamma(1 - e^{-2\pi a})}.$$

If $|\Re(\zeta)| < \pi$ we find that

$$|K_\gamma(\zeta)| \leq \frac{1}{\gamma a^2(1 - e^{-2\pi a})} \int_{a}^{\infty} (e^{-(\pi - \Re(\zeta))t} + e^{-(\pi + \Re(\zeta))t}) \, dt$$

$$= \frac{1}{\gamma a^2(1 - e^{-2\pi a})} \left( \frac{e^{-(\pi - \Re(\zeta))a}}{\pi - \Re(\zeta)} + \frac{e^{-(\pi + \Re(\zeta))a}}{\pi + \Re(\zeta)} \right).$$

Next let us estimate $J_\gamma(\zeta)$. First we use the standard estimate

$$\left| \int_{\tau_R} e^{\zeta z} \frac{\phi(\gamma z)}{\sinh(\pi z)} \, dz \right| \leq \pi \rho M(\zeta, R),$$
where \( M(\zeta, R) = \max_{z \in \Upsilon_R} \left| \frac{e^{\zeta z}}{\sinh(\pi z)} \phi(\gamma z) \right| \). We have
\[
|\phi(w)| = \frac{\sinh(w) - w}{w \sinh(w)}.
\]
Here \( \sinh(w) - w = w^3 h(w) \) and \( w \sinh(w) = w^2 k(w) \), where \( h \) and \( k \) are entire functions. Note that \( k \) is different from zero on \( D(0, \pi) \). Since \( \gamma \in [0, 1] \) we get
\[
M(\zeta, R) = 2\gamma L(R) N(\zeta, R),
\]
where \( L(R) = \max_{|z| \leq R} |h(z)/k(z)| \) and \( N(\zeta, R) = \max_{z \in \Upsilon_R} \left| \frac{e^{\zeta z}}{\sinh(\pi z)} \right| \). We note that
\[
N(\zeta, R) \leq Q_{\pm}(R) \max_{z \in \Upsilon_R} \left| e^{\langle\zeta \pm \pi\rangle z} \right|,
\]
where \( Q_{\pm}(R) = \max_{z \in \Upsilon_R} \frac{1}{|1 - e^{\langle\pm \pi\rangle z}|} \). Put \( Q(R) = Q_{-}(R) + Q_{+}(R) \) and get
\[
N(\zeta, R) \leq Q(R) \min_{\mu = \pm 1} \left( \max_{z \in \Upsilon_R} \left| e^{\langle\zeta + \mu \pi\rangle z} \right| \right).
\]
Since \( \Re(z) \in [-R, R] \) and \( \Im(z) \in [0, R] \) for \( z \in \Upsilon_R \) we finally get
\[
N(\zeta, R) \leq Q(R) e^{2\pi R} \left( 1 + e^{-\Im(\zeta) R} \right).
\]
We have thus obtained the estimate
\[
\left| \frac{1}{4} \int_{\Upsilon_R} \frac{e^{\zeta z}}{\sinh(\pi z)} \phi(\gamma z) dz \right| \leq \gamma B \left( 1 + e^{-\Im(\zeta) R} \right),
\]
where \( B = \frac{\pi}{2} RL(R) Q(R) e^{2\pi R} \).

Finally we have to estimate \( \int_R^a \frac{e^{\zeta t} - e^{-\zeta t}}{\sinh(\pi t)} \phi(\gamma t) dt \). First observe that
\[
h(y) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n + 3)!}
\]
and
\[
k(y) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n + 1)!}
\]
so \( h(y) \leq k(y)/6 \) for \( y \in \mathbb{R} \). Therefore
\[
\left| \int_R^a \frac{e^{\zeta t} - e^{-\zeta t}}{\sinh(\pi t)} \phi(\gamma t) dt \right| \leq \frac{\gamma}{3} \int_R^a \frac{e^{\Re(\zeta)t} + e^{-\Re(\zeta)t}}{e^{\pi t} - e^{-\pi t}} dt
\]
\[
\leq \frac{\gamma}{3(1 - e^{-2\pi R})} \int_R^a \left( e^{-\pi t} - e^{\pi t} \right) dt.
\]
For \( |\Re(\zeta)| \leq \pi \) we get
\[
\left| \int_R^a \frac{e^{\zeta t} - e^{-\zeta t}}{\sinh(\pi t)} \phi(\gamma t) dt \right| \leq \frac{2\pi \gamma}{3(1 - e^{-2\pi R})}.
\]
If $|\text{Re}(\zeta)| < \pi$ we get
\[
\left| \int_R^a e^{\zeta t} - e^{-\zeta t} \phi(\gamma(t)) dt \right| \leq \frac{\gamma}{3(1 - e^{-2\pi R})} \int_0^a \left( e^{-(\pi - \text{Re}(\zeta))t} + e^{-(\pi + \text{Re}(\zeta))t} \right) dt \]
\[
= \frac{\gamma}{3(1 - e^{-2\pi R})} \left( \frac{1 - e^{-(\pi - \text{Re}(\zeta))a}}{\pi - \text{Re}(\zeta)} + \frac{1 - e^{-(\pi + \text{Re}(\zeta))a}}{\pi + \text{Re}(\zeta)} \right).
\]
The lemma now follows by putting $a = 1/\gamma > 1 > R$ and $A = \frac{13}{12(1 - e^{-2\pi R})}$. \hfill \Box

Appendix B Proofs of the estimates (4.28) and (4.30)

Let
\[
J'_\pm(r,\varepsilon) = \int_{C_\pm(\varepsilon)} (\exp(I_\gamma(\pi - 2\pi x) - I\gamma(-\pi + 2\pi x)) - 1) e^{r\Phi(x)} dx,
\]
\[
J''_\pm(r,\varepsilon) = \int_{C_\pm(\varepsilon)} (\tan(\pi r x) \mp \sqrt{-1}) g_r(x) dx,
\]
where $\Phi$ is given by (4.29). Note first that we are free to deform the contour $C_\pm(\varepsilon)$ as long as we stay inside the domain of analyticity of the integrands. For the integrals $J'_\pm(r,\varepsilon)$ we will deform $C_\pm(\varepsilon)$ to $[\varepsilon, 1 - \varepsilon]$. Since the integrands of the integrals $J'_\pm(r,\varepsilon)$ are analytic on $\Omega_{1r} \setminus \{(m + 1/2)/r \mid m = 0, 1, \ldots, r - 1\}$, we can deform $C_\pm(\varepsilon)$ to $C_\pm(0)$ in these integrals without changing their sum, i.e.
\[
J''_-(r,\varepsilon) + J''_+(r,\varepsilon) = J''_-(r) + J''_+(r),
\]
where $J''_+(r) = J''_+(r,0)$. In the following calculations we will need the identity
\[
\text{Re}(\Phi(x)) = 2\pi \text{Re}(x) \text{Im}(x) - \pi \text{Im}(x) - \frac{1}{\pi} \text{Im} \left( \text{Li}_2 \left( e^{-2\pi \text{Im}(x)} e^{2\pi i \text{Re}(x)} \right) \right) \quad (B.1)
\]
valid for $x \in \Omega_{\infty}$ (see (3.20)). This identity is an immediate consequence of (A.2).

Proof of (4.28) Let us first estimate $J''_+(r)$. We partition $C_+(0)$ into the three pieces $C^1_+ = [-1, 0]$, $C^2_+ = [1 + \sqrt{-1}, \sqrt{-1}]$, and $C^3_+ = [1, 1 + \sqrt{-1}]$. Put
\[
I^+_+(r) = \int_{C^+_+(r)} (\tan(\pi r x) - \sqrt{-1}) g_r(x) dx.
\]
By (4.26) we immediately get
\[
|I^+_+(r)| \leq 2 \int_0^1 e^{-2\pi rt} |g_r(\sqrt{-1}t)| dt.
\]
To be able to use (4.23) we introduce the positive parameter $\varepsilon$ again. In fact, we have by (4.23), Lemma 3 and Lebesgue's dominated convergence theorem that
\[
|I^+_+(r)| \leq 2 \exp(2A + 2B \pi/r) \lim_{\varepsilon \to 0^+} \int_0^1 e^{-2\pi rt} \left| e^{r\Phi(\varepsilon + \sqrt{-1}t)} \right| dt.
\]
By (B.1) we immediately get that $|e^{r\Phi(e^{\sqrt{-1}t})}| = \exp\left(r\left[2\pi e t - \pi t - \frac{1}{\pi} \text{Im} \left( \text{Li}_2 \left(e^{2\pi\sqrt{-1}e^{-2\pi t}} \right) \right) \right] \right)$. Now by continuity of $(t, \epsilon) \mapsto \text{Li}_2 \left(e^{2\pi\sqrt{-1}\epsilon e^{-2\pi t}} \right)$ on $[0,1] \times [0,1]$ we can remove the parameter $\epsilon$ again by using Lebesgue’s dominated convergence theorem once more. This gives us

$$
\lim_{\epsilon \to 0^+} \int_0^1 e^{-2\pi rt} |e^{r\Phi(e^{\sqrt{-1}t})}| \, dt = \int_0^1 e^{-3\pi rt} \, dt
$$

leading to the estimate

$$
|I_1^t(r)| \leq \frac{2}{3\pi r} \exp(2A + 2B\pi/r) \left(1 - e^{-3\pi r} \right).
$$

Next we estimate $I_2^t$. By (4.26) we get

$$
|I_2^t(r)| \leq 4e^{-2\pi r} \int_0^1 |g_r(\sqrt{-1} + t)| \, dt.
$$

Similarly to the analysis of $I_1^t$ we introduce the parameter $\epsilon$ and get

$$
|I_2^t(r)| \leq 4 \exp(2A + 2B\pi/r) e^{-2\pi r} \lim_{\epsilon \to 0^+} \int_\epsilon^{1-\epsilon} |e^{r\Phi(\sqrt{-1}t)}| \, dt.
$$

Here

$$
|e^{r\Phi(\sqrt{-1}t)}| = e^{-\pi r(1-2it)} \exp\left(-\frac{r}{\pi} \text{Im} \left( \text{Li}_2 \left(e^{-2\pi e^{-2\pi \sqrt{-1}t}} \right) \right) \right)
$$

by (B.1), so by Lebesgue’s dominated convergence theorem we get

$$
|I_2^t(r)| \leq 4 \exp(2A + 2B\pi/r) e^{-3\pi r} \int_0^1 e^{2\pi rt} \exp\left(-\frac{r}{\pi} \text{Im} \left( \text{Li}_2 \left(e^{-2\pi e^{2\pi \sqrt{-1}t}} \right) \right) \right) \, dt.
$$

By definition of the dilogarithm we have

$$
\text{Im} \left( \text{Li}_2 \left(e^{-2\pi e^{2\pi \sqrt{-1}t}} \right) \right) = -\int_0^1 \text{Arg} \left( \frac{1 - se^{-2\pi e^{2\pi \sqrt{-1}t}}}{s} \right) \, ds,
$$

which is non-negative for $t \in [0, 1/2]$. For $t \in [1/2, 1]$ we use that

$$
\text{Im} \left( \text{Li}_2 \left(e^{-2\pi e^{2\pi \sqrt{-1}t}} \right) \right) = \text{Im} \left( \text{Li}_2 \left(e^{2\pi \sqrt{-1}t} \right) \right) + \int_{1-2\pi}^1 \frac{\text{Arg} \left( 1 - se^{2\pi \sqrt{-1}t} \right)}{s} \, ds,
$$

where the last integral is positive. The first term is bounded from below by $-\sum_{n=1}^{\infty} \frac{1}{n^2} = -\pi^2/6$ by (4.24). We therefore end up with

$$
|I_2^t(r)| \leq 4 \exp(2A + 2B\pi/r) e^{-(3-1/6)\pi r} \int_0^1 e^{2\pi rt} \, dt
$$

$$
= \frac{2}{\pi r} \exp(2A + 2B\pi/r) e^{-(3-1/6)\pi r} \left(e^{2\pi r} - 1 \right).
$$
Finally, we estimate $I_3^3$. Similarly to the other cases we get

$$|I_3^3(r)| \leq 2 \exp(2A + 2B \pi / r) \lim_{\varepsilon \to 0} \int_0^1 e^{-2\pi rt} \left| e^{r \Phi(1 - \varepsilon + \sqrt{-1} t)} \right| dt.$$ 

By (B.1) we have

$$\left| e^{r \Phi(1 - \varepsilon + \sqrt{-1} t)} \right| = \exp \left( r \left[ \pi (1 - 2\varepsilon) t - \frac{1}{\pi} \text{Im} \left( \text{Li}_2(e^{-2\pi t} e^{-2\pi \sqrt{-1} t}) \right) \right] \right)$$

leading to

$$|I_3^3(r)| \leq 2 \exp(2A + 2B \pi / r) \int_0^1 e^{-\pi rt} dt = \frac{2}{\pi r} \exp(2A + 2B \pi / r) \left( 1 - e^{-\pi r} \right).$$

By letting $C_1 = [-\sqrt{-1}, 0], C_2 = [1 - \sqrt{-1}, -\sqrt{-1}], C_3 = [1, 1 - \sqrt{-1}]$ and

$$I_i^+(r) = \int_{C_i} (\tan(\pi rx) + \sqrt{-1}) g_r(x) dx$$

we find (now by the use of (4.27)) an upper bound for $|I_i^+(r)|$ identical with the upper bound for $|I_i^+(r)|$, $i = 1, 2, 3$, with the exception that $\exp(2A + 2B \pi / r)$ should be replaced by $\exp(2A + 4B \pi / r)$ in these bounds. To conclude we have shown that there exists a constant $K_1$ independent of $r$ and $\varepsilon$ such that

$$|J''(r, \varepsilon) + J''(r, \varepsilon)| \leq \sum_{i=1}^3 (|I^+_i| + |I^-_i|) \leq \frac{K_1}{r}$$

for all $r \in \mathbb{Z}_{\geq 2}$.

**Proof of (4.30)** By the remarks prior to the proof of (4.28) we have

$$J'_\pm(r, \varepsilon) = \mp \int_{\varepsilon}^0 e^{r \Phi(t)} h_r(t) dt,$$

where

$$h_r(t) = \exp(I_r(\pi - 2\pi t) - I_r(-\pi + 2\pi t)) - 1.$$ 

The integrand is continuous on $[0, 1]$. Therefore

$$\left| J'_\pm(r, \varepsilon) \right| \leq \int_0^1 \exp(r \text{Re}(\Phi(t))) |h_r(t)| dt$$

for all $\varepsilon \in [0, 1/2]$. By (4.24) and (4.25) and the remarks prior to (4.25) we have

$$\text{Re}(\Phi(t)) = \frac{1}{2\pi} \text{Im} \left( \text{Li}_2(e^{-2\pi \sqrt{-1} t}) - \text{Li}_2(e^{2\pi \sqrt{-1} t}) \right) = -\frac{1}{\pi} \text{Cl}_2(2\pi t) \leq \frac{1}{2\pi} \text{Vol}(K).$$

Therefore

$$\left| J'_\pm(r, \varepsilon) \right| \leq \exp \left( \frac{r}{2\pi} \text{Vol}(K) \right) \int_0^1 |h_r(t)| dt.$$
By definition we have
\[ h_\gamma(t) = \sum_{n=1}^{\infty} \frac{1}{n!} (I_\gamma(\pi - 2\pi t) - I_\gamma(-\pi + 2\pi t))^n. \]

From Lemma 3 we get
\[ |I_\gamma(\pi - 2\pi t) - I_\gamma(-\pi + 2\pi t)| \leq (C f(t) + D) \gamma \]
for \( t \in [0, 1] \), where \( C \) and \( D \) are positive constants independent of \( \gamma \) and \( t \), and
\[ f(t) = \frac{1}{t} + \frac{1}{1-t}. \]

Since \( f : [0, 1] \to \mathbb{R} \) is bigger than or equal to 4 we can choose \( C \) so big that
\[ |I_\gamma(\pi - 2\pi t) - I_\gamma(-\pi + 2\pi t)| \leq C f(t) \gamma \]
for \( t \in [0, 1] \). From Lemma 3 we also have
\[ |h_\gamma(t)| \leq \exp(|I_\gamma(\pi - 2\pi t) - I_\gamma(-\pi + 2\pi t)|) \leq \exp(4A + 4B \pi/r) \]
for \( t \in [0, 1] \), where \( A \) and \( B \) are as in Lemma 3, so \( \pi r \exp(4A + 4B \pi/r) \) is an upper bound for both of the integrals \( \int_{0}^{1} |h_\gamma(t)| dt \) and \( \int_{1-\gamma}^{1} |h_\gamma(t)| dt \) (for \( r \geq 4 \)). Left is to evaluate (for \( r \geq 7 \))
\[ \int_{\gamma}^{1-\gamma} |h_\gamma(t)| dt \leq \sum_{n=1}^{\infty} \frac{C^n \gamma^n}{n!} \int_{\gamma}^{1-\gamma} f(t)^n dt. \]

By using that
\[ f(t)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{t^k} \frac{1}{(1-t)^n-k} \]
we get
\[ \int_{\gamma}^{1-\gamma} f(t)^n dt = \sum_{k=0}^{n} \binom{n}{k} \left( \int_{\gamma}^{1-\gamma} \frac{1}{t^k} \frac{1}{(1-t)^n-k} dt \right) \]
\[ \leq \sum_{k=0}^{n} \binom{n}{k} \left( 2^{n-k} \int_{\gamma}^{1/2} \frac{1}{t^k} dt + 2^k \int_{1/2}^{1-\gamma} \frac{1}{(1-t)^n-k} dt \right) \]
\[ \leq 2^n \sum_{k=0}^{n} \binom{n}{k} \left( \int_{\gamma}^{1/2} \frac{1}{t^k} dt + \int_{1/2}^{1} \frac{1}{t^{n-k}} dt \right) \]
\[ \leq 2^{n+1} \left( \sum_{k=0}^{n} \binom{n}{k} \right) \int_{\gamma}^{1/2} \frac{1}{t^n} dt = 2^{2n+1} \int_{\gamma}^{1/2} \frac{1}{t^n} dt. \]

Here
\[ \int_{\gamma}^{1/2} \frac{1}{t} dt = -\log(2) - \log(\gamma) \leq \log(r) \]
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\[ \int_{\gamma}^{1/2} \frac{dt}{t^n} = \frac{1}{n-1} \left( \frac{1}{\gamma^{n-1}} - 2^{n-1} \right) \leq \frac{1}{n-1} \frac{1}{\gamma^{n-1}} \]

for \( n \geq 2 \). Therefore

\[ \int_{\gamma}^{1} \frac{dt}{\gamma^{n-1}} \leq \frac{1}{n-1} \left( \frac{1}{\gamma^{n-1}} - 1 \right) \]

We conclude that there exists a constant \( K_2 \) independent of \( r \) and \( \varepsilon \) such that

\[ |J'_+(r, \varepsilon) + J'_-(r, \varepsilon)| \leq K_2 \frac{\log(r)}{r} \exp \left( \frac{r}{2\pi} \text{Vol}(K) \right), \]

for all \( r \in \mathbb{Z}_{\geq 2} \).

\[ \square \]

Appendix C The case \( M_0 \)

The manifold \( M_0 \) is the mapping torus of a torus with monodromy matrix

\[ \left( \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right), \]

see [18, p. 366]. The invariant \( \tau_r(M_0) \) has been calculated by Jeffrey [13, Theorem 4.1]. This theorem gives the large \( r \) asymptotics of the invariant as well. In fact, we have

\[ \tau_r(M_0) = \frac{1}{2} - \frac{1}{2\sqrt{5}} - \frac{1}{\sqrt{5}} \left( \exp \left( 2\pi \sqrt{-1}r \left( -\frac{1}{5} \right) \right) + \exp \left( 2\pi \sqrt{-1}r \frac{1}{5} \right) \right) \]

which at the same time can be taken as the large \( r \) asymptotics of the invariant. Let us relate this result to our contour integral formula for the invariant \( \tau_r(M_0) = \tau_r(M_0) \). By Lemma 2

\[ \tau_r(M_0) = \frac{ri}{4} \int_{C_1^r} \cot(\pi x) \left( \int_{C_2^r} \tan(\pi y) f_{0,r}(x,y) dy \right) dx, \]

where

\[ f_{0,r}(x,y) = \sin(\pi x) e^{-2\pi i xy} \frac{S_{\pi/2}(-\pi + 2\pi(x-y))}{S_{\pi/2}(-\pi + 2\pi(x+y))}. \]

Following the discussion in Sec. 4.3 the relevant (shifted) phase functions to consider are given by

\[ \Psi^a_{0,b}(x,y) = ax + by - xy + \frac{1}{4\pi^2} \left( \text{Li}_2(e^{2\pi i(x+y)}) - \text{Li}_2(e^{2\pi i(x-y)}) \right), \]
where \(a, b\) are certain integers. If we put \(\Psi = \Psi_{a,b}^0\), then by (4.40) \((x, y)\) is a critical point of \(\Psi\) if and only if
\[
0 = 2\pi i(a - y) + \log(1 - zw) - \log(1 - zw^{-1}), \\
0 = 2\pi i(b - x) + \log(1 - zw) + \log(1 - zw^{-1}),
\]
where \(z = e^{2\pi ix}\) and \(w = e^{2\pi iy}\) as usual, and this set of equations implies that
\[
w - z = 1 - zw, \\
(1 - zw)(w - z) = zw,
\]
(C.2)
compare with (4.41) and (4.42). We note that the first of these equations is equivalent to
\[
w - 1 = z(1 - w)
\]
so \(w = 1\) or \(z = -1\). For \(z = -1\) we get \(w^2 + 3w + 1 = 0\) so \(w = \frac{-3 \pm \sqrt{5}}{2}\). For \(w = 1\) we find that \(z^2 - 3z + 1 = 0\) so \(z = \frac{3 - \sqrt{5}}{2}\). However, only the point \((z, w) = (\frac{3 - \sqrt{5}}{2}, 1)\) satisfies that \(zw, zw^{-1} \notin [1, \infty]\). For this point we find that \(y \in \mathbb{Z}\) and from (C.2) we get that \(y = a\). But then
\[
\Psi(x, y) = ax + by - xy + \frac{1}{4\pi^2} (\text{Li}_2(z) - \text{Li}_2(z)) = yx + ba - xy = ba \in \mathbb{Z}.
\]
Let \(\bar{\rho}_{\theta, \varepsilon}\) be the nonabelian SU(2)--representations of \(\pi\) from Proposition 2, where \(\theta \in [-1/3, -1/6] \cup [1/6, 1/3]\) and \(\varepsilon \in \{\pm\}\). Here \(\bar{\rho}_{\theta, \varepsilon}\) and \(\bar{\rho}_{-\theta, \varepsilon}\) are conjugate. By (5.65) \(\bar{\rho}_{\theta, \varepsilon}\) extends to a representation of \(\pi_1(M_0)\) if and only if
\[L_\varepsilon(\theta) = 1.
\]
But this happens if and only if \(\theta = \pm 1/4\) for both \(\varepsilon = +\) and \(\varepsilon = -\), i.e. the set of conjugacy classes of nonabelian representations of \(\pi_1(M_0)\) into SU(2) is \(\{[\bar{\rho}_{1/4, -}], [\bar{\rho}_{1/4, +}]\}\).

By (5.77) the flat reducible SU(2)--connections on \(M_0\) all have a Chern--Simons invariant equal to zero, so we conclude that the image set of the SU(2) Chern--Simons functional on \(M_0\) has at most three elements. By [18, Theorem 5.6 and precedent text] we can therefore conclude that the set of Chern--Simons invariants of flat SU(2)--connections on \(M_0\) is
\[
\left\{0 \mod \mathbb{Z}, -\frac{1}{5} \mod \mathbb{Z}, \frac{1}{5} \mod \mathbb{Z}\right\}.
\]
(C.4)
In particular, the set of Chern--Simons invariants of \([\bar{\rho}_{1/4, +}]\) and \([\bar{\rho}_{1/4, -}]\) is equal to \(\{-\frac{1}{5} \mod \mathbb{Z}, \frac{1}{5} \mod \mathbb{Z}\}\). We note that (C.1) and (C.4) prove the AEC for the invariants \(\tau_\varepsilon(M_0)\).

By (5.70) and (5.71) we have \(\beta_\varepsilon(1/4) = f_\varepsilon\), where \(f_- = 0\) and \(f_+ = 1\). By Proposition 4 and (5.71) we find
\[
\text{CS}(\bar{\rho}_{1/4, \varepsilon}) = \frac{\pm 1}{6} - \frac{1}{\pi} \int_{1/6}^{1/4} \text{Arg}(L_\varepsilon(t))dt \pmod{\mathbb{Z}},
\]
where we use that $L_{\pm}(t) \in S^1$. By (5.90) we have
\[
\int_{1/6}^{1/4} \text{Arg}(L_-(t))dt = -\int_{1/6}^{1/4} \text{Arg}(L_+(t))dt
\]
so we finally get
\[
\text{CS}(\hat{\rho}_{1/4, \pm}) = \pm \left( \frac{1}{6} - \frac{1}{\pi} \int_{1/6}^{1/4} \text{Arg}(L_+(t))dt \right) \quad \text{(mod Z)}.
\]
Note that $\text{Im}(L_+(t)) < 0$ on $]1/6, 1/4[$ so
\[
-\frac{1}{12} = \frac{1}{\pi}(-\pi)(1/4 - 1/6) < \frac{1}{\pi} \int_{1/6}^{1/4} \text{Arg}(L_+(t))dt < 0.
\]
Therefore
\[
1/6 - \frac{1}{\pi} \int_{1/6}^{1/4} \text{Arg}(L_+(t))dt \in ]1/6, 1/4[.
\]
Since this value mod Z belongs to the set $\{\pm 1/5 \text{ mod Z}\}$ we conclude that it is equal to 1/5 so
\[
\frac{1}{\pi} \int_{1/6}^{1/4} \text{Arg}(L_-(t))dt = -\frac{1}{\pi} \int_{1/6}^{1/4} \text{Arg}(L_+(t))dt = \frac{1}{30}.
\]
(C.5)

Let us finally calculate the value of $\Psi = \Psi_{0}^{a,b}$ in the critical points corresponding to the solutions $(z, w) = (3 + \sqrt{5})/2, 1$ and $(z, w) = (-1, -\frac{3\pm \sqrt{5}}{2})$ to (C.3). Since the set of solutions to (C.3) is in one to one correspondence with the set of conjugacy classes of nonabelian SL(2, C)–representations of $\pi_1(M_0)$ by the proof of Theorem 7 and since the subset of solutions $(z, w)$ with $z \in S^1$ and $w \in ]-\infty, 0[$ corresponds to the set of conjugacy classes of SU(2)–representations we see that the points $(z, w) = (-1, -\frac{3\pm \sqrt{5}}{2})$ correspond to nonabelian SU(2)–representations of $\pi_1(M_0)$ while the points $(z, w) = (\frac{3\pm \sqrt{5}}{2}, 1)$ correspond to nonabelian SL(2, C)–representations of $\pi_1(M_0)$ which are not equivalent to SU(2)–representations.

For $(z, w) = (e^{2\pi ix}, e^{2\pi iy}) = ((3 + \sqrt{5})/2, 1)$ we find again that $y = a$ and then $\Psi(x, y) = ab$ (independent of the choice of branch of the dilogarithm along $]1, \infty[$).

Finally, let $(z, w) = (e^{2\pi ix}, e^{2\pi iy}) = (-1, -\frac{3\pm \sqrt{5}}{2})$. The real values of the right-hand sides of (C.2) do not depend on $a$ and $b$. Taking the imaginary values of these equations we get
\[
0 = 2\pi (a - \text{Re}(y)) + \text{Im} \left( \text{Log}(1 + w) - \text{Log}(1 + w^{-1}) \right),
\]
\[
0 = 2\pi (b - x) + \text{Im} \left( \text{Log}(1 + w) + \text{Log}(1 + w^{-1}) \right).
\]
The second of these equations is equivalent to
\[
x - b = \frac{1}{2\pi} \left( \text{Arg}(1 + w) + \text{Arg}(1 + w^{-1}) \right) = \frac{1}{2}
\]
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for both \( w = \frac{-3 + \sqrt{5}}{2} \), and the first is equivalent to

\[
a - \text{Re}(y) = \frac{1}{2\pi} \left( \text{Arg}(1 + w^{-1}) - \text{Arg}(1 + w) \right) = \begin{cases} \frac{1}{2}, & w = \frac{-3 + \sqrt{5}}{2} \\ -\frac{1}{2}, & w = \frac{-3 - \sqrt{5}}{2} \end{cases}
\]

We have

\[
\Psi(x, y) = ax + by - xy + \frac{1}{4\pi^2} \left( \text{Li}_2(|w|) - \text{Li}_2(1/|w|) \right),
\]

and since this is real by Corollary 5, we get

\[
\Psi(x, y) = ax + (b - x)\text{Re}(y) + \frac{1}{4\pi^2} \text{Re} \left( \text{Li}_2(|w|) - \text{Li}_2(1/|w|) \right).
\]

Here

\[
ax + (b - x)\text{Re}(y) = (b - x)(\text{Re}(y) - a) + ab = ab + \frac{1}{2} (a - \text{Re}(y))
\]

\[
= ab + \begin{cases} \frac{1}{4}, & w = \frac{-3 + \sqrt{5}}{2} \\ -\frac{1}{4}, & w = \frac{-3 - \sqrt{5}}{2} \end{cases}
\]

For \( z \in \mathbb{C} \setminus [0, \infty[ \) we have

\[
\text{Li}_2(z) + \text{Li}_2(1/z) = -\frac{\pi^2}{6} - \frac{1}{2} \log^2(-z).
\]

This identity e.g. follows by differentiating the difference of the two sides in the identity, showing that this difference is constant on \( \mathbb{C} \setminus [0, \infty[ \), and then evaluating in \( z = -1 \) using that \( \text{Li}_2(-1) = -\pi^2/12 \). Therefore

\[
\text{Li}_2(t) = -\frac{\pi^2}{6} - \frac{1}{2} \log^2(-t) - \text{Li}_2(t^{-1}) \tag{C.6}
\]

for \( t > 1 \) for a branch of \( \text{Li}_2 \) continuously extended across \( ]1, \infty[ \). Let \( w_\pm = \frac{-3 \pm \sqrt{5}}{2} \).

We note that

\[
w_+w_- = 1. \tag{C.7}
\]

Moreover, by [21, Formula (1.20) p. 7],

\[
\text{Li}_2 \left( \frac{3 - \sqrt{5}}{2} \right) = \frac{\pi^2}{15} - \text{Log}^2 \left( \frac{1 + \sqrt{5}}{2} \right). \tag{C.8}
\]

Note also that

\[
\left( \frac{1 + \sqrt{5}}{2} \right)^2 = 3 + \sqrt{5}. \tag{C.9}
\]

Assume that \( w = w_- = \frac{-3 - \sqrt{5}}{2} \). Then, by (C.6),

\[
\text{Li}_2(|w|) = -\frac{\pi^2}{6} - \frac{1}{2} \log^2(-|w|) - \text{Li}_2(|w|^{-1}),
\]
where \( \log(-|w|) = \log(|w|) + i\pi \), so

\[
\text{Li}_2(|w|) = \frac{\pi^2}{2} - \frac{\pi^2}{6} - \frac{1}{2} \log^2(|w|) - \text{Li}_2(|w|^{-1}) - i\pi \log(|w|).
\]

But then

\[
\text{Li}_2(|w|) - \text{Li}_2(|w|^{-1}) = \frac{\pi^2}{2} - \frac{\pi^2}{6} - \frac{1}{2} \log^2(|w|) - 2\text{Li}_2(|w|^{-1}) - i\pi \log(|w|).
\]

Here \(|w|^{-1} = \frac{3 + \sqrt{5}}{2}\) by (C.7) so by (C.8) and (C.9) we get

\[
\text{Li}_2(|w|) - \text{Li}_2(|w|^{-1}) = \frac{\pi^2}{2} - \frac{\pi^2}{6} - \frac{2\pi^2}{15} - i\pi \log(|w|) = \frac{\pi^2}{5} - i\pi \log(|w|).
\]

By (C.7) we conclude that

\[
\text{Li}_2(|w|) - \text{Li}_2(|w|^{-1}) = \mp \frac{\pi^2}{5} - i\pi \log(|w|).
\]

For \( w = w_\pm \) we therefore get

\[
\Psi(x, y) = ab \pm \frac{1}{4} \mp \frac{1}{20} = ab \pm \frac{1}{5}.
\]

Note here that we use the identities (C.5) and (C.10) in the proof of Theorem 8 to handle the cases \( \theta = \pm 1/4 \), so we can not here refer to this theorem.

We have above shown that the set of values \( \text{mod}Z \) of \( \Psi^a_b \) in its critical points is identical with the set (C.4) of Chern–Simons invariants of flat SU(2)–connections on \( M_0 \) for arbitrary \( a, b \in Z \). The determinant of the Hessian \( H \) of \( \Psi^a_b \) in a critical point \((x, y)\) is given by

\[
\det(H) = 1 - 2 \left( z + \frac{1}{z} \right)
\]

according to (4.49). Thus all the critical points are non-degenerate. It is interesting to note that to obtain the leading order large \( r \) asymptotics of \( \tau_r(M_0) \) using the saddle point method we have to use critical points corresponding to SL(2,\(C\))–representations of \( \pi_1(M_0) \) which are not equivalent to SU(2)–representations in order to get the part of that leading asymptotics being associated to the reducible flat SU(2)–connections on \( M_0 \). By Conjecture 3 this phenomenon should only occur for \( p/q = 0 \). Recall here that the case \( p/q = 0 \) is in any case special; only in that case the moduli space of reducible flat SU(2)–connections on \( M_{p/q} \) is not discrete.

For \((z, w) = \left(-1, -\frac{3 + \sqrt{5}}{2}\right)\) we see that \( \det(H) = 5 \). Moreover, \( z = e^{2\pi i x} = -1 \) implies that \( x \in \frac{1}{2} + Z \), so \( \sin(\pi x) = \pm 1 \). Thus we see that the right-hand side of the formula (1.11) gives the part of the leading order large \( r \) asymptotics of \( \tau_r(M_0) \) corresponding to the irreducible flat SU(2)–connections on \( M_0 \) if we let \( m_\rho = 4 \) for both points in \( M'_0 \).
The identity (C.10) was proved by using the explicit value of the dilogarithm in $(3 - \sqrt{5})/2$, cf. (C.8). We note that only very few explicit values of the dilogarithm are known, see [21, Chap 1].

Appendix D Proof of Lemma 8

Let us begin by showing the identities (5.83) and (5.85). Let $\theta \in I$ and let $u = u_\pm(\theta)$, $Q_i = Q_i^\pm(\theta)$ and $t = e^{4\pi i \theta}$ and get

$$Q_1Q_2 = (1 - (1 + u)^{-1} t)(1 - (1 + u)t) = 1 - t (1 + u + (1 + u)^{-1}) + t^2$$

$$= -t(1 + u)^{-1} ((1 + u)^2 - (t + t^{-1})(u + 1)) = t,$$

where the last equality follows by the fact that $\phi(t, u) = 0$, where $\phi$ is given by (5.61).

To show (5.85) we observe that

$$Q_1Q_3 = Q_3Q_2 = t^{-1}Q_3Q_2^2$$

$$= t^{-1}(1 + u) (1 - t(1 + u)^{-1})^2 = t^{-1}(1 + u) + t(1 + u)^{-1} - 2$$

by (5.83). Now $\phi(t, u) = 0$ implies that

$$t(1 + u)^{-1} = t^2 + 1 - t(u + 1)$$

leading to the identity

$$\frac{Q_1Q_3}{Q_2} = -1 + t^{-1} - 2t + t^2 + u(t^{-1} - t) = L_\pm(\theta).$$

To prove the identities (5.82) and (5.84) it is necessary to examine the arguments of the functions $Q_i^\pm(\theta)$, $i = 1, 2$. By Lemma 9 we note that

$$Q_1^\pm(\theta) = Q_2^\mp(\theta)$$

for $\theta \in I$.

Assume first that $\theta \in [1/6, 1/3]$. An elementary calculation shows that

$$1 + u_+(\theta) \in [-1, (\sqrt{5} - 3)/2],$$
$$1 + u_-(\theta) \in [-3 + \sqrt{5])/2, -1]$$

for $\theta \in [1/6, 1/3]$ and $1 + u_+(\theta) = -1$ if and only if $\theta = 1/6, 1/3$. In particular, $1 + u_\pm(\theta)$ is negative. We therefore get

$$\text{Im} (Q_1) = |1 + u|^{-1} \sin(4\pi \theta) \begin{cases} 
> 0, & \theta \in [1/6, 1/4[, \\
= 0, & \theta = 1/4, \\
< 0, & \theta \in [1/4, 1/3].
\end{cases}$$
By (5.81) $\Re(Q_1) = 1 - (1 + u)^{-1} \cos(4\pi \theta)$. Since $1 + u$ is negative we see that $\Re(Q_1)$ have the same sign as $\cos(4\pi \theta) - (1 + u)$, where $\text{sign}(0) = 0$ as usual. By (D.2) we conclude that

$$\Re \left( Q_1^{-} (\theta) \right) > 0$$

for all $\theta \in [1/6, 1/3]$. Let us next consider $Q_1^{+}$. First note that

$$\cos(4\pi \theta) - u_+ (\theta) - 1 = \frac{1}{2} - \sqrt{\cos^2(4\pi \theta) - \cos(4\pi \theta) - \frac{3}{4}}.$$

We therefore get that $\Re \left( Q_1^{+} (\theta) \right)$ has the opposite sign as $\cos^2(4\pi \theta) - \cos(4\pi \theta) - 1$. By the assumption on $\theta$ we have $\cos(4\pi \theta) \in [-1, -1/2]$, and we therefore get

$$\Re \left( Q_1^{+} (\theta) \right) \begin{cases} > 0, & \theta \in [1/6, \theta_0 \cup 1/2 - \theta_0, 1/3], \\ = 0, & \theta \in \{\theta_0, 1/2 - \theta_0\}, \\ < 0, & \theta \in [\theta_0, 1/2 - \theta_0], \end{cases}$$

where $\theta_0 \in [1/6, 1/4]$ is the unique element such that $\cos(4\pi \theta_0) = (1 - \sqrt{5})/2 \in [-1, -1/2]$ is the negative solution to $t^2 - t - 1 = 0$.

Let $\psi_1^\pm (\theta) \in [-\pi, \pi]$ be the principal argument of $Q_1^\pm (\theta)$. Then the above analysis shows, also using (D.1), that

$$\psi_2^- (\theta) = \psi_1^+ (\theta) \begin{cases} \in [0, \frac{\pi}{2}], & \theta \in [\frac{\pi}{4}, \theta_0], \\ = \frac{\pi}{2}, & \theta = \theta_0, \\ \in [\frac{\pi}{2}, \pi], & \theta \in \left( \theta_0, \frac{\pi}{4} \right], \\ = \pi, & \theta = \frac{\pi}{4}, \\ \in [\pi, -\frac{\pi}{2}], & \theta \in \left[ \frac{\pi}{2}, \frac{1}{3} - \theta_0 \right], \\ = -\frac{\pi}{2}, & \theta = \frac{1}{3} - \theta_0, \\ \in [-\frac{\pi}{2}, 0], & \theta \in \left[ \frac{1}{3} - \theta_0, \frac{1}{4} \right]. \end{cases}$$

and

$$\psi_2^+ (\theta) = \psi_1^- (\theta) \begin{cases} \in [0, \frac{\pi}{2}], & \theta \in [\frac{1}{3}, \frac{1}{4}], \\ = 0, & \theta = \frac{1}{4}, \\ \in [-\frac{\pi}{2}, 0], & \theta \in \left[ \frac{1}{4}, \frac{1}{3} \right], \end{cases}$$

so

$$\psi_1^+ (\theta) + \psi_2^+ (\theta) \in \begin{cases} [0, \pi], & \theta \in \left[ \frac{1}{3}, \theta_0 \right], \\ \left[ \frac{\pi}{2}, \pi \right], & \theta \in \left[ \theta_0, \frac{1}{3} \right], \\ \left[ \frac{\pi}{2}, \pi \right], & \theta = \frac{1}{3}, \\ \left[ -\frac{\pi}{2}, -\pi \right], & \theta \in \left[ \frac{1}{3}, \frac{1}{2} - \theta_0 \right], \\ -\pi, 0, & \theta \in \left[ \frac{1}{2} - \theta_0, \frac{1}{4} \right]. \end{cases}$$

By (5.83) we have

$$\psi_1^+ (\theta) + \psi_2^+ (\theta) \in 4\pi \theta + 2\pi \mathbb{Z},$$

so we conclude that

$$\psi_1^+ (\theta) + \psi_2^+ (\theta) = 4\pi \theta$$

for all $\theta \in [1/6, 1/3]$ proving (5.82) for these $\theta$.
Next assume that $\theta \in [-1/3, -1/6]$. First observe that
\[
Q_1^\pm(-\theta) = Q_1^\pm(\theta), \quad \theta \in I \setminus \{\pm1/4\},
\]
and
\[
Q_i^\pm(-1/4) = Q_i^\pm(1/4)
\]
for $i = 1, 2$. By (D.6) we immediately get that (5.82) holds for $\theta = -1/4$. For $\theta \in [-1/3, -1/6] \setminus \{-1/4\}$, (5.82) follows by (D.5) and the fact that $\Log(p) = \Log(p)$ for $p \in \mathbb{C} \setminus (-\infty, 0]$. Note that (5.84) is true if we choose $e_\pm(\theta) \in \mathbb{Z}$ such that
\[
\psi_1^\pm(\theta) + \pi - \psi_2^\pm(\theta) - e_\pm(\theta)2\pi \in [-\pi, \pi].
\]
By (D.3) and (D.4) we have that $\psi_1^\pm(\theta) - \psi_2^\pm(\theta) \in [-\pi, \pi]$, and we conclude that we have to put $e_\pm(\theta) = 0$ if and only if $\psi_1^\pm(\theta) \leq \psi_2^\pm(\theta)$ and $e_\pm(\theta) = 1$ elsewhere. By (D.3) and (D.4) we conclude that $e_-(1/4) = 0$ and $e_+(1/4) = 1$.

Assume that $\theta \in [1/6, 1/3] \setminus \{1/4\}$. Then $\psi_1^\pm(\theta)$ and $\psi_2^\pm(\theta)$ both belong to either $[-\pi, 0]$ or to $[0, \pi]$. We use this fact together with the fact that $\cot: \mathbb{R} \to \mathbb{R}$ is strictly decreasing for any $m \in \mathbb{Z}$. In fact, $\cot(\psi_1) = \frac{\Re(Q_1)}{\Im(Q_1)}$ so
\[
\cot(\psi_1) = \cot(4\pi \theta) + \frac{\Re Q_1}{\Im Q_1} - 1 + u|1 + u|^{-1} \sin(4\pi \theta),
\]
\[
\cot(\psi_2) = \cot(4\pi \theta) + \frac{1 + u}{\sin(4\pi \theta)}.
\]
By this we find that
\[
\text{sign}(\psi_1^\pm(\theta) - \psi_2^\pm(\theta)) = \text{sign}\left(\sin(4\pi \theta)\left(1 - (1 + u_\pm(\theta))^2\right)\right).
\]
Since $(1 + u_+(\theta))^2 \leq 1$ and $(1 + u_-(\theta))^2 \geq 1$ with equalities if and only if $\theta \in \{1/6, 1/3\}$, we get $\psi_1^\pm(\theta) = \psi_2^\pm(\theta)$ for $\theta \in \{1/6, 1/3\}$ and
\[
\psi_1^\pm(\theta) \leq \psi_2^\pm(\theta)
\]
for $\theta \in [1/6, 1/4]$ and $\varepsilon = -$ or for $\theta \in [1/4, 1/3]$ and $\varepsilon = +$. Moreover,
\[
\psi_1^\pm(\theta) > \psi_2^\pm(\theta)
\]
for $\theta \in [1/6, 1/4]$ and $\varepsilon = +$ or for $\theta \in [1/4, 1/3]$ and $\varepsilon = -$. We therefore get that (5.84) is true if we put $e_\pm(\theta) = 0$ for $\theta \in \{1/6, 1/3\}$ and
\[
e_+(\theta) = \begin{cases} 1, & \theta \in [1/6, 1/4], \\ 0, & \theta \in [1/4, 1/3], \end{cases}
\]
and let $e_-(\theta) = 1 - e_+(\theta)$ for $\theta \in [1/6, 1/3] \setminus \{1/4\}$. 

\[\text{Asymptotics of the quantum invariants for the figure } 8 \text{ knot} \quad 67\]
Let us finally consider the case \( \theta \in [-1/3, -1/6] \). First observe that

\[
Q^\pm_3(-\theta) = Q^\pm_3(\theta)
\]

for all \( \theta \in I \). By this and (D.6) we conclude that \( e_\pm(-1/4) = e_\pm(1/4) \). For \( \theta \in [-1/3, -1/6] \setminus \{-1/4\} \) we get by (D.5) and (D.7) that

\[
\begin{align*}
\log (Q^\pm_1(\theta)) + \log (Q^\pm_3(\theta)) - \log (Q^\pm_2(\theta)) &= \log (Q^\pm_3(\theta)) - \log (Q^\pm_3(\theta)) \\
+ \log (Q^\pm_1(-\theta)) + \log (Q^\pm_3(-\theta)) - \log (Q^\pm_2(-\theta)) &= 2 \text{Im} \log (Q^\pm_3(\theta)) + \left( \log \left( \frac{Q^\pm_1(-\theta)Q^\pm_3(-\theta)}{Q^\pm_2(-\theta)} \right) + e_\pm(-\theta)2\pi i \right).
\end{align*}
\]

By using that \( Q_3 \) is negative and that \( L_\pm(\theta) \in \mathbb{C} \setminus (-\infty, 0] \) for \( \theta \in ]1/6, 1/3[ \) we get for \( \theta \in ]-1/3, -1/6[ \setminus \{-1/4\} \) that

\[
\log (Q^\pm_1(\theta)) + \log (Q^\pm_3(\theta)) - \log (Q^\pm_2(\theta)) = \log \left( \frac{Q^\pm_1(\theta)}{Q^\pm_2(\theta)} \right) + \left( 1 - e_\pm(-\theta) \right)2\pi i,
\]

so we conclude that \( e_\pm(\theta) = 1 - e_\pm(-\theta) \) for these \( \theta \). Finally we get for \( \theta \in \{-1/3, -1/6\} \) that \( Q^\pm_3(\theta) = Q^\pm_2(\theta) \) so (5.84) is satisfied for \( e_\pm(\theta) = 0 \) for these \( \theta \).

[1] J. E. Andersen, The Witten invariant of finite order mapping tori I, to appear in Journal für die Reine und Angewante Mathematik. Preprint version available at http://home.imf.au.dk/andersen/papers

[2] J. E. Andersen, The asymptotic expansion conjecture, Sec. 7.2 in Problems on invariants of knots and 3--manifolds, ed. T. Ohtsuki, in Invariants of knots and 3--manifolds (Kyoto 2001), Geometry and Topology Monographs 4 (2002), 377--572. Available at http://www.maths.warwick.ac.uk/gt/GTMon4/paper24.abs.html

[3] S. Axelrod, Overview and warmup example for perturbation theory with instantons, in Geometry and physics (Aarhus 1995), Lecture Notes in Pure and Appl. Math. 184, Marcel Dekker, Inc. (1997), 321--338.

[4] S. Axelrod, I. M. Singer, Chern--Simons perturbation theory, Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2, World Sci. Publishing (1992), 3--45.

[5] S. Axelrod, I. M. Singer, Chern--Simons perturbation theory. II, J. Differential Geom. 39 (1994), no. 1, 173--213.

[6] N. G. de Bruijn, Asymptotic methods in analysis, North--Holland Publishing Co. (1961).

[7] L. D. Faddeev, Discrete Heisenberg--Weyl group and modular group, Lett. Math. Phys. 34 (1995), 249--254.

[8] S. K. Hansen, Reshetikhin--Turaev invariants of Seifert 3--manifolds and a rational surgery formula, Algebr. Geom. Topol. 1 (2001), 627--686. Available at http://www.maths.warwick.ac.uk/agt/AGTVol1/agt-1-32.abs.html

[9] S. K. Hansen, Reshetikhin--Turaev invariants of Seifert 3--manifolds, and their asymptotic expansions, D. Phil. thesis, The University of Aarhus (Sept. 1999).

[10] S. K. Hansen, Analytic asymptotic expansions of the Reshetikhin--Turaev invariants of Seifert 3--manifolds for SU(2), in preparation. Draft version available at http://www.math.ksu.edu/~hansen
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[11] S. K. Hansen, The colored Jones polynomial of the trefoil and the figure 8, unpublished. Available at http://www.math.ksu.edu/~hansen

[12] L. Hörmander, The analysis of linear partial differential operators I, Springer–Verlag (1983).

[13] L. C. Jeffrey, Chern–Simons–Witten invariants of lens spaces and torus bundles, and the semiclassical approximation, Comm. Math. Phys. 147 (1992), no. 3, 563–604.

[14] R. M. Kashaev, The hyperbolic volume of knots from the quantum dilogarithm, Letters in Math. Physics 39 (1997), 269–275. Preprint version available at arXiv:q-alg/9601025

[15] R. C. Kirby, Problems in low-dimensional topology. Edited by Rob Kirby. AMS/IP Stud. Adv. Math. 2, in Geometric topology (Athens, GA, 1993), 35–473, Amer. Math. Soc., Providence, RI (1997).

[16] R. C. Kirby, P. M. Melvin, On the 3–manifold invariants of Witten and Reshetikhin–Turaev for $sl_2(C)$, Invent. Math. 105 (1991), no. 3, 473–545.

[17] R. C. Kirby, P. M. Melvin, Dedekind sums, $\mu$–invariants and the signature cocycle, Math. Ann. 299 (1994), 231–267.

[18] P. A. Kirk, E. P. Klassen, Chern–Simons invariants of 3–manifolds and representation spaces of knot groups, Math. Ann. 287 (1990), 343–367.

[19] E. P. Klassen, Representations of knot groups in $SU(2)$, Trans. A.M.S. 326 (1991), no. 2, 795–828.

[20] Thang T. Q. Le, Quantum invariants of 3–manifolds: integrality, splitting, and perturbative expansion, preprint math.QA/0004099

[21] L. Lewin, Polylogarithms and associated functions, North-Holland Publishing Co. (1981).

[22] H. Murakami, The asymptotic behaviour of the colored Jones function of a knot and its volume, preprint arXiv:math.GT/0004036

[23] H. Murakami, Kashaev’s invariant and the volume of a hyperbolic knot after Y. Yokota, preprint arXiv:math.GT/0008027

[24] H. Murakami, J. Murakami, The colored Jones polynomials and the simplicial volume of a knot, Acta Math. 186 (2001), 85–104

[25] J. G. Ratcliffe, Foundations of hyperbolic manifolds, Grad. Texts in Math. 149, Springer–Verlag (1994).

[26] H. Rademacher, E. Grosswald, Dedekind sums, The Carus Mathematical Monographs 16, The Mathematical Association of America (1972).

[27] N. Reshetikhin, V. G. Turaev, Invariants of 3–manifolds via link polynomials and quantum groups, Invent. Math. 103 (1991), no. 3, 547–597.

[28] R. Riley, Nonabelian representations of 2-bridge knot groups, Quart. J. Math. Oxf. 35 (1984), 191–208.

[29] R. Riley, Holomorphic parameterized families of subgroups of $SL(2, C)$, Mathematika 32 (1985), 248–264.

[30] D. Rolfsen, Knots and links, 2nd printing with corrections, Mathematics Lecture Series 7, Publish or Perish, Inc. (1990).

[31] L. Rozansky, Residue formulas for the large $k$ asymptotics of Witten’s invariants of Seifert manifolds. The case of $SU(2)$, Comm. Math. Phys. 178 (1996), no. 1, 27–60. Preprint version available at arXiv:hep-th/9412075

[32] D. Thurston, Hyperbolic volume and the Jones polynomial, Lecture notes, École d’été de Mathématiques 'Invariants de noeuds et de variétés de dimension 3', Institut Fourier - UMR 5582 du CNRS et de l’UJF Grenoble (France) du 21 juin au 9 juillet 1999.

[33] W. P. Thurston, The geometry and topology of 3–manifolds, Lecture notes, Princeton University (1978).

[34] V. G. Turaev, Quantum invariants of knots and 3–manifolds, de Gruyter Stud. Math.
18. Walter de Gruyter (1994).

[35] Y. Yokota, *On the volume conjecture for hyperbolic knots*, preprint arXiv:math.QA/0009165