JACOB’S LADDERS, NEW PROPERTIES OF THE FUNCTION 
arg\(\zeta\left(\frac{1}{2} + it\right)\) AND CORRESPONDING METAMORPHOSES

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Abstract. The notion of the Jacob’s ladders, reversely iterated integrals and the \(\zeta\)-factorization is used in this paper in order to obtain new results in study of the function \(\arg\zeta\left(\frac{1}{2} + it\right)\). Namely, we obtain new formulae for non-local and non-linear interaction of the functions \(|\zeta\left(\frac{1}{2} + it\right)|\) and \(\arg\zeta\left(\frac{1}{2} + it\right)\), and also a set of metamorphoses of the oscillating Q-system.

1. Introduction

1.1. Let us denote by \(N(T)\) the number of zeroes \(\beta + i\gamma\) of the \(\zeta(s)\)-function such that 
\[\beta \in (0, 1), \gamma \in (0, T).\]
We suppose that \(T\) is not equal to any \(\gamma\). Otherwise, we put 
\[N(T) = \frac{1}{2} \lim_{\epsilon \to 0^+} [N(T + \epsilon) + N(T - \epsilon)].\]
It us well-know that 
\[N(T) = \frac{1}{2\pi} T \ln \frac{T}{2\pi e} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right),\]
where
\[(1.1) S(T) = \frac{1}{\pi} \arg\zeta\left(\frac{1}{2} + it\right),\]
and the value of \(\arg\) is obtained by continuous variation along the straight lines joining the points 
\[2, 2 + iT, \frac{1}{2} + iT,\]
starting with the value zero. Next, we have the function
\[(1.2) S_1(T) = \int_0^T S(t)dt.\]

1.2. Further, let us remind the following facts 
\[\zeta\left(\frac{1}{2} + it\right) = \left|\zeta\left(\frac{1}{2} + it\right)\right| e^{i\arg\zeta\left(\frac{1}{2} + it\right)},\]
i.e. the functions
\[(1.3) \left|\zeta\left(\frac{1}{2} + it\right)\right|, \arg\zeta\left(\frac{1}{2} + it\right)\]

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are parts of the Riemann function
\[ \zeta \left( \frac{1}{2} + it \right). \]

The study of these functions have proceeded by isolated ways. Namely:

(a) the first one studied by Hardy-Littlewood
\[ \int_T^{T+U} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim U \ln T, \ldots \]

(b) the second one by Backlund, E. Landau, H. Bohr, Littlewood, Titchmarsh - to the fundamental Selberg’s results (see [3], [4]).

Let us mention two of the Selberg’s results:
\[ \int_T^{T+H} \{S_1(t)\}^2 dt = cH + O \left( \frac{H}{\ln T} \right), \]
(1.4)
\[ T^a \leq H \leq T; \quad \frac{1}{2} < a \leq 1, \quad l \in \mathbb{N}, \]
where \( l \) is arbitrary and fixed, (see [4], p. 130), and
\[ S_1(t) = \Omega_{\pm} \left\{ (\ln t)^{1/3} (\ln \ln t)^{-10/3} \right\}, \]
(1.5)
(see [4], p.150).

**Remark 1.** For our purpose it is sufficient to use the formula (1.4) in the minimal case
\[ H = T^{1/2+\epsilon}, \quad \epsilon > 0, \]
where \( \epsilon \) is sufficiently small (non-principal improvements of the exponent 1/2 are not relevant for our purpose).

1.3. To this date, there is no result in the theory of the Riemann zeta-function about the interaction of the functions (1.3), or the functions
\[ \left| \zeta \left( \frac{1}{2} + it \right) \right|, S_1(t). \]
That is, there is nothing known like
\[ F \left( \left| \zeta \left( \frac{1}{2} + it \right) \right|, |S_1(\tau)| \right) = 0 \]
for a set of values \( t, \tau \).

On the other hand, we have developed (see [1]) the method of \( \zeta \)-factorization that gives, for example, the following formula (see [1], (1.7))
\[ \frac{1}{\sqrt{\zeta \left( \frac{1}{2} + i\alpha_0 \right)}} \sim \frac{1}{\sqrt{\Lambda}} \prod_{\gamma=1}^{k} \zeta \left( \frac{1}{2} + i\alpha_r \right) \]
together with the infinite set of corresponding metamorphoses of the main multi-form.

In this paper we use this method to obtain a result of the new type
\[ |S_1(\alpha_0)| \sim \phi \left\{ \prod_{\gamma=1}^{k} \frac{\left| \zeta \left( \frac{1}{2} + i\alpha_r \right) \right|}{\left| \zeta \left( \frac{1}{2} + i\beta_r \right) \right|} \right\} \]
together with the infinite set of metamorphoses of the corresponding Q-system from [2].

Remark 2. A kind of nonlinear and nonlocal interaction of the functions (1.6) is expressed by the formula (1.7).

2. Theorem

2.1. We begin with the Selberg’s formula

\[
\int_{T}^{T+H} \{ S_{l}(t) \}^{2l} dt \sim c_{l} H, \ T \to \infty,
\]

\[ H = T^{1/2+\epsilon}, \ l \in \mathbb{N}, \ \epsilon > 0, \]

(comp. (1.4) and Remark 1), where \( l \) is arbitrary and fixed, \( \epsilon \) is sufficiently small. Now, if we use our method of transformation (see [2], (4.1)–(4.19)) in the case of the formula (2.1) then we obtain (see (1.1), (1.2)) the following

Theorem. Let

\[
[T, T + H] \longrightarrow \frac{1}{r}[T, T + H], \ldots, \frac{k}{k}[T, T + H],
\]

where

\[ [T, T + H], \ r = 1, \ldots, k, \ k \leq k_{0}, \ k_{0} \in \mathbb{N} \]

be the reversely iterated segment corresponding to the first segment in (2.2) and \( k_{0} \) be an arbitrary and fixed number. Then there is a sufficiently big

\[ T_{0} = T_{0}(l, \epsilon) > 0 \]

such that for every \( T > T_{0} \) and every admissible \( l, \epsilon, k \) there are the functions

\[
\alpha_r = \alpha_r(T, l; \epsilon, k), \ r = 0, 1, \ldots, k,
\]

\[
\beta_r = \beta_r(T; \epsilon, k), \ r = 1, \ldots, k,
\]

\[
\alpha_r, \beta_r \neq \gamma: \ \zeta \left( \frac{1}{2} + i\gamma \right) = 0
\]

such that

\[
\left| \int_{0}^{\alpha_0(T)} \arg \zeta \left( \frac{1}{2} + it \right) dt \right| \sim \left| \prod_{r=1}^{k} \left( \frac{z}{z + i\alpha_r(T, l)} \right) \right|^{-1}, \ T \to \infty.
\]

Moreover, the sequences

\[ \{ \alpha_r \}_{r=0}^{k}, \ \{ \beta_r \}_{r=1}^{k} \]

have the following properties

\[
T < \alpha_0 < \alpha_1 < \cdots < \alpha_k,
\]

\[
T < \beta_1 < \beta_2 < \cdots < \beta_k,
\]

\[
\alpha_0 \in (T, T + H), \ \alpha_r, \beta_r \in (T, T + H), \ r = 1, \ldots, k,
\]
\[
\alpha_{r+1} - \alpha_r \sim (1 - \epsilon) \pi(T), \ r = 0, 1, \ldots, k - 1,
\]
\[
\beta_{r+1} - \beta_r \sim (1 - \epsilon) \pi(T), \ r = 1, \ldots, k - 1,
\]
where
\[
\pi(T) \sim \frac{T}{\ln T}, \ T \to \infty
\]
is the prime-counting function and \(\epsilon\) is the Euler’s constant.

**Remark 3.** Let us notice that the asymptotic behavior of the sets
\[
\{\alpha_r\}_{r=0}^k, \ \{\beta_r\}_{r=1}^k
\]
is as follows: at \(T \to \infty\) the points of every set in (2.7) recede unboundedly each from other and all together recede to infinity. Hence, at \(T \to \infty\) each set in (2.7) looks like one-dimensional Friedmann-Hubble universe.

2.2. Let us denote the mean-value of the function
\[
\arg \zeta \left( \frac{1}{2} + it \right), \ t \in [0, T]
\]
by the symbol
\[
\langle \arg \zeta \left( \frac{1}{2} + it \right) \rangle_{[0,T]}.
\]
Let us mention that the function under consideration has an infinite set of first-order discontinuities. Since
\[
\int_0^\alpha_0(T) \arg \zeta \left( \frac{1}{2} + it \right) dt = \alpha_0(T) \langle \arg \zeta \left( \frac{1}{2} + it \right) \rangle_{[0,\alpha_0(T)]},
\]
then we obtain from (2.4) the following

**Corollary 1.**
\[
\left| \langle \arg \zeta \left( \frac{1}{2} + it \right) \rangle_{[0,\alpha_0(T)]} \right| \sim \frac{\pi(c_\ell)}{\alpha_0(T)} \frac{1}{2^k} \prod_{r=1}^k \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r(T, l) \right)}{\zeta \left( \frac{1}{2} + i\beta_r(T, l) \right)} \right|^{-\frac{1}{2}}
\]
\[
\alpha_0(T) \in (T, T + H), \ T \to \infty.
\]

Let us remind that the following Littlewood’s estimate (comp. [5], p. 189)
\[
S_1(t) = O(\ln t), \ t \to \infty
\]
holds true. Hence, we have (see (1.1), (1.2) the estimate
\[
\langle \arg \zeta \left( \frac{1}{2} + it \right) \rangle_{[0,T]} = O \left( \frac{\ln T}{T} \right), \ T \to \infty.
\]

**Remark 4.** Consequently, we have obtained in the direction of the estimate (2.9) the explicit asymptotic formula (2.8) for the mean-value
\[
\langle \arg \zeta \left( \frac{1}{2} + it \right) \rangle_{[0,T]}
\]
on the infinite subset
\[
\{\alpha_0(T)\}, \alpha_0(T) \in (T, T + T^{1/2+\epsilon}), \ T \to \infty.
\]
3. Reduction of the integral in (2.4)

3.1. Now, we use the Selberg’s Ω-theorem (1.5) to transform our formula (2.4). It follows from (1.5) that there are two sequences

\{a_n\}_{n=1}^\infty; \{b_n\}_{n=1}^\infty, \ a_n, b_n \to \infty

such that

\[ S_1(a_n) > A(ln a_n)^{1/3}(\ln \ln a_n)^{-10/3}, \]

\[ S_1(b_n) < -B(ln a_n)^{1/3}(\ln \ln a_n)^{-10/3}; \]

\[ A, B > 0. \]

(3.1)

Since

\[ S_1(t), \ t > 0 \]

is the continuous function then by (3.1) there is (Bolzano-Cauchy) the sequence

\{\mu_n\}_{n=1}^\infty: \ S_1(\mu_n) = 0, \ \mu_n \to \infty,

where \(\mu_n\) is the odd-order root of the equation

(3.2)

\[ S_1(t) = 0, \ t > 0. \]

Remark 5. We may suppose, of course, that the sequence (3.2) is complete one in the usual sense, the interval

\( (\mu_n, \mu_{n+1}) \)

does not contain any other odd-order root of the equation (3.3).

Remark 6. There is no need to discuss (for our purpose) the question about even-order roots of the equation (3.3).

Hence, we have: if

(3.4)

\[ \bar{k} = \bar{k}[\alpha_0(T)]: \ \mu_{\bar{k}} < \alpha_0(T) < \mu_{\bar{k}+1} \]

and (of course, see (2.4), (3.3))

\[ S_1[\alpha_0(T)] \neq 0, \]

then

(3.5)

\[ S_1[\alpha_0(T)] = \int_0^{\alpha_0(T)} S(t)dt = \int_0^{\mu_{\bar{k}}} + \int_{\mu_{\bar{k}}}^{\alpha_0(T)} = \int_{\mu_{\bar{k}}}^{\alpha_0(T)} S(t)dt. \]

Consequently, we have from (2.4) by (3.4), (3.5) the following

Corollary 2.

\[ \left| \int_{\mu_{\bar{k}}}^{\alpha_0(T)} \arg \zeta \left( \frac{1}{2} + it \right) dt \right| \sim \]

(3.6)

\[ \sim \pi(c_1)^{\frac{1}{2\pi}} \prod_{r=1}^{k} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r(T, l) \right)}{\zeta \left( \frac{1}{2} + i\beta_r(T) \right)} \right|^{-\frac{1}{2}}, \ T \to \infty. \]
3.2. Next, we obtain from \((3.6), \text{comp. (2.8)}\), the following

\textbf{Corollary 3.}

\[
\begin{align*}
\left| \langle \arg \zeta \left( \frac{1}{2} + it \right) \rangle_{[\mu_k, \alpha_0(T)]} \right| & \sim \\
& \sim \frac{\pi(c_1)}{\alpha_0(T) - \mu_k} \prod_{r=1}^{k} \left| \zeta \left( \frac{1}{2} + i\alpha_r(T, t) \right) \right|^{-\frac{1}{2}},
\end{align*}
\]

and, of course, \((2.8), (3.7)\)

\[
\left| \langle \arg \zeta \left( \frac{1}{2} + it \right) \rangle_{[\mu_k, \alpha_0(T)]} \right| \sim \frac{\alpha_0(T)}{\alpha_0(T) - \mu_k} \left| \langle \arg \zeta \left( \frac{1}{2} + it \right) \rangle_{[0, \alpha_0(T)]} \right|, \ T \to \infty.
\]

4. On infinite set of metamorphoses of the Q-system that is generated by the factorization formula \((2.4)\)

4.1. Let us remind the Riemann-Siegel formula

\[
Z(t) = 2 \sum_{n \leq \tau(t)} \frac{1}{\sqrt{n}} \cos \{ \vartheta(t) - t \ln n \} + O(t^{-1/4}),
\]

where

\[
\begin{align*}
Z(t) &= e^{i\vartheta(t)} \zeta \left( \frac{1}{2} + it \right), \ \tau(t) = \sqrt{\frac{t}{2\pi}}, \\
\vartheta(t) &= -\frac{t}{2} \ln \pi + \text{Im} \ln \Gamma \left( \frac{1}{4} + i\frac{t}{2} \right),
\end{align*}
\]

(see [5], pp. 79, 239). Next, we have introduced (see [2], (2.1)) the following oscillatory Q-system (based exactly on the Riemann-Siegel formula (4.1))

\[
G(x_1, \ldots, x_k; y_1, \ldots, y_k) = \prod_{r=1}^{k} \left| \frac{Z(x_r)}{Z(y_r)} \right|,
\]

(4.2)

\[
= \prod_{r=1}^{k} \left| \sum_{n \leq \tau(x_r)} \frac{1}{\sqrt{n}} \cos \{ \vartheta(x_r) - x_r \ln n \} + R(x_r) \right| \left| \sum_{n \leq \tau(y_r)} \frac{1}{\sqrt{n}} \cos \{ \vartheta(y_r) - y_r \ln n \} + R(y_r) \right|,
\]

\((x_1, \ldots, x_k) \in M_1^k, (y_1, \ldots, y_k) \in M_2^k, R(t) = O(t^{-1/4}), k \leq k_0 \in \mathbb{N},\)

where

\[
M_1^k = \{(x_1, \ldots, x_k) \in (T_0, +\infty)^k, \ T_0 < x_1 < \cdots < x_k\},
\]

\[
M_2^k = \{(y_1, \ldots, y_k) \in (T_0, +\infty)^k, \ T_0 < y_1 < \cdots < y_k\},
\]

\(x_r, y_r \neq \gamma : \zeta \left( \frac{1}{2} + i\gamma \right) = 0, \ r = 1, \ldots, k.\)
Next, we have obtained (see [2], (3.1)) the following spectral formula

\[ Z(t) = 2 \sum_{n \leq \tau(x_r)} \frac{1}{\sqrt{n}} \cos \left( t \ln \frac{\tau(x_r)}{n} - \frac{x_r}{2} - \frac{\pi}{8} \right) + \]

\[ + O(x_r^{-1/4}), \quad \tau(x_r) = \sqrt{\frac{x_r}{2\pi}}, \]

\[ t \in [x_r, x_r + V], \quad V \in (0, \sqrt{x_r}], \]

(and similarly for \( x_r \to y_r \)), where

\[ T_0 < x_r, y_r, \quad r = 1, \ldots, k. \]

**Remark 7.** The spectral formula (4.4) is, of course, a variant of the Riemann-Siegel formula (4.1).

**Remark 8.** We call the expressions

\[ \frac{2}{\sqrt{n}} \cos \left\{ t\omega_n(x_r) - \frac{x_r}{2} - \frac{\pi}{8} \right\} \ldots \]

as the local Riemann’s oscillators with:

(a) the amplitudes

\[ \frac{2}{\sqrt{n}}, \]

(b) the incoherent local phase constants

\[ \left\{ -\frac{x_r}{2} - \frac{\pi}{8} \right\}, \left\{ -\frac{y_r}{2} - \frac{\pi}{8} \right\}, \]

(c) the non-synchronized local times

\[ t = t(x_r) \in [x_r, x_r + V], \ldots \]

(d) the local spectrum of the cyclic frequencies

\[ \{\omega_n(x_r)\}_{n \leq \tau(x_r)}, \quad \omega_n(x_r) = \ln \frac{\tau(x_r)}{n}, \]

\[ \{\omega_n(y_r)\}_{n \leq \tau(y_r)}, \quad \omega_n(y_r) = \ln \frac{\tau(y_r)}{n}. \]

**Remark 9.** The Q-system \( (4.2) \) represents a complicated oscillating process generated by oscillations of big number of the local Riemann’s oscillators \( 4.4 \).

**4.3.** Now, in connection with the oscillating Q-system \( (4.2) \), the following corollary follows from our Theorem

**Corollary 4.**

\[ \prod_{r=1}^{k} \left| \sum_{n \leq \tau(\alpha_r)} \frac{2}{\sqrt{n}} \cos \left\{ \vartheta(\alpha_r) - \alpha_r \ln n \right\} + R(\alpha_r) \right| \]

\[ \left| \sum_{n \leq \tau(\beta_r)} \frac{2}{\sqrt{n}} \cos \left\{ \vartheta(\beta_r) - \beta_r \ln n \right\} + R(\beta_r) \right| \sim \]

\[ \sim n' \sqrt{c_1} \left| \int_{0}^{\alpha(7)} \arg \zeta \left( \frac{1}{2} + it \right) \right|^{-t}, \quad T \to \infty. \]
Remark 10. Hence, we have two resp. one parametric sets of control functions (=Golem's shem) for admissible and fixed \( \epsilon, k \), (see (4.3)),

\[
\{ \alpha_0(T, l), \alpha_1(T, l), \ldots, \alpha_k(T, l) \},
\]

\[
\{ \beta_1(T), \ldots, \beta_k(T) \},
\]

\( T \in (T_0, +\infty), \ l \in \mathbb{N}, \)

of the metamorphoses (4.6), (comp. [1], [2]).

Remark 11. The mechanism of the metamorphosis is as follows. Let (comp. (4.3), (4.7))

\[
M^3_k = \{ \alpha_1(T, l), \ldots, \alpha_k(T, l) \},
\]

\[
M^4_k = \{ \beta_1(T), \ldots, \beta_k(T) \},
\]

where, of course,

\[
M^3_k \subset M^4_k \subset (T_0, +\infty)^k,
\]

\[
M^4_k \subset M^2_k \subset (T_0, +\infty)^k.
\]

Now, if we obtain after random sampling of the points

\[
(x_1, \ldots, x_k), (y_1, \ldots, y_k)
\]

(see the conditions (4.3)) such that

\[
(x_1, \ldots, x_k) = (\alpha_1(T, l), \ldots, \alpha_k(T, l)) \in M^3_k,
\]

\[
(y_1, \ldots, y_k) = (\beta_1(T), \ldots, \beta_k(T)) \in M^4_k,
\]

(see (4.8), (4.9)), then - at the points (4.10) - the Q-system (4.2) changes its old form (=chrysalis) to the new one (=butterfly), and the last ist controlled by the function \( \alpha_0(T) \).

4.4. Now, we rewrite the formula (4.6), (comp. (4.6)), as follows:

\[
\left| \int_{\mu_k}^{\alpha_0(T)} \arg \zeta \left( \frac{1}{2} + it \right) dt \right| \sim \pi(c_l)^k \prod_{r=1}^{k} \left| \sum_{n \leq \tau(\alpha_r)} \frac{2}{\sqrt{n}} \cos \left( \theta(\alpha_r) - \alpha_r \ln n \right) + R(\alpha_r) \right|^{-\frac{1}{2}}
\]

Remark 12. The formula (4.11) expresses the metamorphosis in the reverse direction. We describe the mechanism of this as follows: we begin with the integral

\[
\left| \int_{0}^{w} \arg \zeta \left( \frac{1}{2} + it \right) dt \right|
\]

that is the Aaron staff,

\[
\rightarrow \left| \int_{\mu_k}^{\alpha_0(T)} \arg \zeta \left( \frac{1}{2} + it \right) dt \right|
\]

that is the bud of the Aaron staff corresponding to \( w = \alpha_0(T) \),

\[
\sim \pi(c_l)^k \prod_{r=1}^{k} \left| \sum_{n \leq \tau(\alpha_r)} \frac{2}{\sqrt{n}} \cos \left( \theta(\alpha_r) - \alpha_r \ln n \right) + R(\alpha_r) \right|^{-\frac{1}{2}}
\]
already metamorphosed one into almonds ripened, (motivation: Chumash, Bamidbar, 17:23).

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