The automobile industry is experiencing a deep economic and technological change. It is moving from (i) a fossil-fueled, private-ownership based, manually-driven to (ii) an electric, sharing based, driver-less one\cite{1,2}. Understanding the supply-and-demand dynamics of automobile sales could help to achieve a smooth economic transition between (i) and (ii). Indeed, 88.1 million of both cars and light commercial vehicles were sold worldwide in 2016\cite{3}, involving billions of dollars per year. Unplanned circumstances or forecast failures could generate immense losses in this market and its ties, as previously seen in past crisis and economic transitions as that after 2007\cite{4}.

Thanks to the rise of both Big Data and digital tools we have today access to exhaustive databases that allow us to analyze socioeconomic systems using ideas of statistical physics\cite{5,6}. Examples range from allometric laws on city populations\cite{7,8}, evolution of firm sizes\cite{9}, transportation networks\cite{10,11} and human mobility\cite{12,13}, to popularity of digital products\cite{14,15}, the structure of the Internet\cite{16} or even diffusion of memes\cite{17}. In such an environment we analyze here the sales of new commercial vehicles in Spain from January 2007 to January 2017. The corresponding exhaustive database is published by the Spanish Directorate-General of Traffic (DGT) and contains registrations of more than 6,500 different configurations (model + body shape + engine) and more than 10 million sold cars\cite{18}. To analyze the evolution of car sales, we apply, in particular, the procedure used in Refs.\cite{19,20} on the aggregated data of monthly registrations (or sales) per each car configuration (model) as provided by Ref.\cite{21}. The procedure is summarized as follows:

1. We first identify the main dynamical variables. In our case, the variable is the total number of cars sold for the $i$-th automobile model at time $t$, $x_i(t)$.

2. We compare $x_i(t)$ versus its time derivative $\dot{x}_i(t)$ so as to find indications of any possible scaling rule in the underlying equation of motion, of the form

$$\dot{x}_i(t) = v_i(t)[x_i(t)]^q, \quad (1)$$

where $v_i(t)$ is the growth rate at time $t$ and $q$ is the exponent that parameterizes the dynamics. Due to the stochastic nature of the growth rates, it is convenient to compare $x_i$ with the variance of $\dot{x}_i$, obtained from all those $i$'s endowed with a similar value of $x$. Thus, we fit the variance to an expression of the form $\text{Var}[\dot{x}] = T_q x^{2q}$, where $T_q$ is used to measure the size of the fluctuations, that might be called a “temperature”\cite{22,23}. If one finds contributions of several independent components with the form of Eq. (1) with different exponents—as $\dot{x}_i(t) = \sum q v_{q,i}(t)[x_i(t)]^q$— we will define a temperature associated to each term from the fit $\text{Var}[\dot{x}] = \sum T_q x^{2q}$.

3. Next, we independently analyze the density distribution of total car sales per model $p(x)dx$. The principle of Maximum Entropy (MaxEnt) with dynamical information states that the natural variable for measuring the entropy is the one that linearizes the equation of motion, transforming the underlying symmetry into a translational one\cite{24}. For an equation of motion as Eq. (1), this is achieved via the Tsallis logarithm and its inverse function the Tsallis exponential\cite{25} defining the new variable $u = \log_q(x/x_0)$ —or $x = x_0 \exp_q(u)$— where $x_0$ is any value of reference (needed to keep the arguments dimensionless). After this transformation, the equation of motion becomes $\dot{u}_i(t) = v_i(t)$, which is indeed translationally invariant. If the system is in equilibrium, the most probable distribution is the one that maximizes the entropy under the system’s observable constraints. If the
form of the empirical distribution fits the form predicted by MaxEnt, we can perform an independent quantitative measure of $q$ directly from the equilibrium distribution.

4. If the value of $q$ one gets from the equation of motion, and the value of $q$ measured from the distribution-fit agree, we assume that our procedure is correct, proving that the system is in dynamical equilibrium and obeys the Maximum Entropy Principle.

5. As an additional test for our procedure, we also perform numerical experiments by defining microscopic equations of motion equivalent to those observed in the empirical system. We then analyze the evolution of the simulated system following the same steps as we traversed for the empirical one. If we obtain equivalent results, we would reconfirm that our procedure is indeed correct.

As stated in step 1, our relevant variable is the total of car sales $x_i$ for the $i$th automobile model. We display in Fig. 1 a sample of some trajectories for the ten years covered by our dataset, where a logistic profile is clearly visible (i.e., growth with saturation). Saturation occurs when the popularity of a model drops because of the availability of a newer version, or if a competitor advances on the same market niche. We also show in Fig. 1 the evolution of the number of different models sold per month during the ten years included in the dataset.

When we compare $x$ vs. $\text{Var}[\dot{x}]$ so as to fit Eq. (1), we find what looks like a three-component equation of motion, with an exponent i) for low sales of $q = 0.75 \pm 0.01$, ii) for mid-sales of $q = 0.63 \pm 0.01$, and iii) for high sales of $q = 0.33 \pm 0.01$. R2 adopts a value of 0.9997 (see Fig. 2). A naive interpretation might be that Eq. (1) is indeed the underlying equation of motion. The problem would be to explain the origin of the values for three exponents mentioned above. Even worse, an independent assessment of $q$ derived from the equilibrium distribution will not match these 3 values, as we will show later. However, paying attention to the high sales in Fig. 2 (arrow), we appreciate the existence of points with low variance. Indeed, here growth rates drop to such an extent that one approaches saturation and this would generate an under-estimation of the exponents. To get deeper insights into the actual underlying dynamics, we need to correct for each car model the effects of saturation. To this aim, we need to reconsider the form of the logistic equation

$$\dot{x}_i(t) = v_i(t)x_i(t)\left[1 - x_i(t)/X_i\right],$$  \hspace{1cm} (2)$$

where $X_i$ is the final number of total car sales. If we reconsider the trajectories by first defining $\dot{x}_i'(t) = \dot{x}_i(t)/[1 - x_i(t)/X_i]$, we should recover an expression of the form of Eq. (1) as $\dot{x}_i' = v_i x_i$ that corrects growth.
new result vouches for an equation of motion of the form \( \log(\text{an } R^2 \text{ coefficient equal to 0}) \) rates for the effects of saturation. Comparing in this way a 95\% with 0.1\% of the market (dashed blue lines and shaded 90\% of the total contribution with 0.03\% of monthly sales and the proportional term starts to be dominant, reaching around 2\% of the market (dashed blue lines and shaded areas). The average maximum market share for a single model is around 2\% (dashed gray line).

rates for the effects of saturation. Comparing in this way \( x \) vs. \( \text{Var}[\dot{x}'] \), we obtain the curve displayed in Fig. 2 which nicely describes a three-terms function: one has \( \text{Var}[\dot{x}'] = T_1 x^{q_1} + T_{1/2} x + T_0 \), with \( q = 1.016 \pm 0.002 \), and an \( R^2 \) coefficient equal to 0.9998 \( \log(T_1) = -3.41 \pm 0.05 \), \( \log(T_{1/2}) = 1.60 \pm 0.06 \), and \( \log(T_0) = 5.73 \pm 0.06 \). This new result vouches for an equation of motion of the form

\[
\dot{x}'(t) = v_{1,1}(t)x_i(t) + v_{1/2,1}(t)\sqrt{x_i(t)} + v_{0,1}(t). \tag{3}
\]

These three components have been already observed for firm sizes, city populations, and are well-understood. The first term with \( q = 1 \) is Gibrat’s law of proportional growth, which is expected for multiplicative processes. Such behavior indicates that the popularity of an automobile model grows as more cars are sold, following a rich-get-richer mechanism. The second term, with exponent \( q = 1/2 \), emerges from the proportional growth as a side effect of the central limit theorem (or finite-size effects), as shown in Ref. 27. It has been shown that this component is characterized by uncorrelated noise and no extra physics is expected. Finally, the last term in Eq. 3 correspond to linear forces \( (q = 0) \), independent of the value of \( x \). This last term becomes relevant only for small number of sales.

In our procedure we will focus only on the proportional regime, where the most interesting physics takes place. However, one may ask: which is the critical number of sold cars so as to consider that the pertinent trajectory would be accommodated by the proportional regime? A first estimation can be obtained via the overlap between the first and second terms of Eq. 3, obtaining \( x_0 = T_{1/2}/T_1 = 129 \) cars. At this critical value we have equal contributions from each of the two terms. It is then natural to ask when we can consider that the dynamics is fully governed by proportional or scalable growth. We can combine our two previous fits to compare the market share or relative number of sold cars per month \( \chi = \dot{x}/\sum_j \dot{x}_j \) with the relative contribution of the proportional term to the total growth, that we call scalability: \( \kappa(x) = \sqrt{T_1 x^q/\text{Var}[\dot{x}']} \). As seen in Fig. 3 an automobile model is \( \kappa = 95\% \) into the scalable growth regime when it reaches a \( \chi = 0.1\% \) of monthly relative sales or market-share.

Another relevant question regarding growth rates becomes legitimate at this point. It is crucial for predictability: Are car sales a non-Markovian process? In other words, are the sales of today correlated with those of yesterday, and thus predictable? Buying a car is an individual decision, undertaken according to a plethora of various considerations. We are looking at a complex decision-process that takes its time and relies on decisions taken previously by other individuals. As we did for cities’ populations in Refs. 11, 26, which unraveled the existence of a “memory” in cities, we have attempted to define here the putative system’s memory as the time correlation \( c(\tau) = \text{Cor}[\dot{x}(t), \dot{x}(t + \tau)] \), where \( \tau \) is the correlation time interval. We find an exponential-like decay, with slightly negative values after \( \tau = 5 \) years as shown in Fig. 4. This in turn establishes the limits of predictability: no accurate prediction on sales can be done for longer than that time-period.

We continue now with the step 3 outlined on page 1. For maximizing the entropy \( S \), it was shown in Ref. 15 that the only observable and objective constraints for systems with logistic growth are, due to the limitation in units of resources, the total available units (in our case, total number of sold cars \( X \)) and the number of elements sharing these units (for us, the number of available automobile models \( N \)). Considering only those car-models the sales of which exceeded \( x_0 = 130 \) units in the ten years (the limit of proportional growth as derived be-
fore) we have $X = 9957.537$ and $N = 3084$. Following Refs. 11, 23, 24 we write our thermodynamic potential as:

$$\Omega = -TS - \mu N + \Lambda X$$  \hfill (4)$$

where $T$, $\mu$, and $\Lambda$ are the concomitant Lagrange multipliers for the general variational problem of the thermodynamic potential $\Omega(S, N, X)$. For a general equation of motion with arbitrary $q$, we assume that an underlying probability density $p(u)$ governs our process, where $u = \log_q(x/x_0)$. $N = \int_0^\infty dup(u)$ and $X = \int_0^\infty dup(u) \exp_q(u)$. We explicitly cast the MaxEnt problem as a $p-$variational one:

$$0 = \delta_p \Omega = \delta_p \int_0^\infty dup(u) \left\{ T \log[p(u)/N] - \mu - \Lambda \exp_q(u) \right\}$$  \hfill (5)$$

where $\delta_p$ represent variations with respect to $p(u)$. We find the solution $p(u) = zN \exp[-\Lambda^* \exp_q(u)]$ which in terms of the variable $x$ is a power law with exponential cut-off:

$$p(x) = zN \exp(-\Lambda^* x/x_0) x^q$$  \hfill (6)$$

where $\Lambda^* = \Lambda/T$ and $z = \exp(\mu/T)$. The first constraint normalizes $p(u)$ to the number of car models $N$, and the second to the total car sales, yielding the equations of state:

$$z = (x_0 \Lambda^*)^{1-q} / \Gamma(1 - q, \Lambda^*)$$,$$
$$x_0 N / X = \Lambda^* \Gamma(1 - q, \Lambda^*) / \Gamma(2 - q, \Lambda^*)$$  \hfill (7a, b)$$

The cumulative distribution can be written as

$$P(x)/N = 1 - \frac{\Gamma(1 - q, \Lambda^* x/x_0)}{\Gamma(1 - q, \Lambda^*)}$$.  \hfill (8)$$

Passing now to step 4 of page 1, we show in Fig. 4 the comparison of the MaxEnt prediction with the empirical cumulative distribution of total car sales. We have considered only sizes larger than $x_0 = 130$ and fit Eq. (5) via $\log(x_0)$, $q$, and $\log(\Lambda^*)$ (logarithms are used for numerical stability). We find a remarkable fit with $\log(x_0) = 4.884 \pm 0.001$, $q = 1.048 \pm 0.001$, and $\log(\Lambda^*) = -4.560 \pm 0.003$. Our R2 value is 0.99999 — the relative difference with the empirical values $\log(130)$, 1, and $-4.8712$ (taken from the values of $q$, $X$ and $N$) are 0.3%, 5%, and 8.8%, respectively. Thus, we can safely claim that the dynamics of car sales are very close to equilibrium and the distribution of total sales obeys the Maximum Entropy Principle for scale-free systems.

We finally proceed to step 5 on page 2 by defining the equations needed for performing microscopic simulations of the empiric system. For writing these equations, we need the following considerations:

1. Geometric Brownian walkers are known to obey Gibrat’s law and reproduce states of dynamical equilibrium following entropic laws. However, Brownian walkers are Markovian. We will use instead a Langeving-like equation with a viscous term [27, 28], which is known to reproduce time correlations with exponential decay.

2. The constraints in the thermodynamic potential of Eq. (4) are introduced as forces in the Langevin equation, in analogy to what is done in molecular simulations.

3. In addition to the proportional growth $q = 1$, the model should include linear forces ($q = 0$) and finite size fluctuations ($q = 1/2$). We include them as independent thermal baths with no inertia.

4. The number of sold cars can grow, but does not decrease (un-selling cars is not allowed). Thus, a Maxwell demon [28] is introduced to select only positive values from the fluctuations.

After these considerations, we write the full set of microscopic equations as:

$$\dot{v}_{1,i}(t) = \eta_i(t) - \Lambda x_i(t) - \gamma v_{1,i}(t),$$

$$\dot{x}_i(t) = v_{1,i}(t) x_i(t) + v_{1/2,i}(t) \sqrt{x_i(t)} + v_{0,i}(t) \sqrt{\gamma},$$

where $\gamma$ is the dumping or viscosity which controls the inertia, and the thermal bath obeys $\langle \eta_i(t) \eta_j(t') \rangle = 2T_0 \delta(t - t')$ with $T_0$ defining its temperature. The other two sources of noise are described as White Noise with $\langle v_{1/2,i}(t) v_{1/2,j}(t') \rangle = 2T_{1/2} / \Delta t \delta(t - t')$, and $\langle v_{0,i}(t) v_{0,j}(t') \rangle = 2T_0 / \Delta t \delta(t - t')$ with $T_{1/2}$ and $T_0$ their respective temperatures. Here, $\delta$ is the Dirac’s Delta and $\Delta t$ is the discretized interval of time used to numerically solve our equations. Finally, the symbol $\lor$ indicates choosing the maximum between the quantity on the left and on the right, preventing negative values of the growth $x_i$.

Since our aim is to prove that the entropic procedure properly predicts the equilibrium distribution in the proportional regime, we have reduced Eqs. (9) for the sake...
of simplicity by removing the finite size term \( q = 1/2 \) (as done in Ref. [11] and considering over-dumping dynamics by setting \( \gamma = 1/\Delta t \). (For the interested reader, a full exhaustive exercise solving and exploring Eqs. [6] will be published elsewhere.)

We have solved the simplified version of Eqs. [6] for the range of values \( 0.1 \leq T_\eta \leq 500 \) and \( 10^{-4} \leq \Lambda \leq 0.5 \) with \( N = 3000 \), \( T_0 = 100 \), and intervals of times in a range of \( 10^{-3} \leq \Delta t \leq 10^{-2} \), starting at an initial configuration where every walker has \( x(0) = 1 \). Eventually, the growth rate of the walker becomes zero due to the force induced by the \( \Lambda \) term, reaching saturation. So as to keep the number of active walkers constant, we add a new one at \( x = 1 \) every time an older walker saturates. After some time-intervals have elapsed, the system reaches a thermodynamical equilibrium.

We study the scalable growth by selecting only walkers with values \( x_i(t) > 100 \) and measuring the effective temperature as the variance of the growth rates \( T_1 = \text{Var}[x(x)] \) at equilibrium. We find that the system’s temperature and that of the thermal bath obey \( \log(T_1) = (0.52 \pm 0.06) + (0.65 \pm 0.02) \log(T_\eta) \). The equilibrium distribution \( p(x)dx \) fits the form of the MaxEnt prediction in Eq. [8]: when fitting via \( q = 1 \) and \( \Lambda^* \), we find \( q = 1.01 \pm 0.01 \) and \( \Lambda^* = (3.0 \pm 0.2) + \log(T_\eta/T_0) \) —independently of the other parameters \( \Delta t \) and \( T_0 \), see Fig. 5— confirming the correctness of our entropic procedure. We show in Fig. 5 the cumulative distribution for a simulation using the empirical temperature \( T_1 = \exp(-3.41) \ (T_\eta = 0.0024) \) and \( \Lambda = 1.35 \times 10^{-5} \), obtaining \( \Lambda^* = 0.01 \).

Summing up, we have shown that the dynamics underlying car sales corresponds to a non-Markovian logistic growth with scale symmetry, and the dynamical equilibrium obeys the maximum entropy principle. We have also found that the sales become scalable up to a 95% when a model reaches a 0.1% of the monthly market share. Numerical experiments reproduce indeed the macroscopic features of the empirical system and confirm the theoretical procedure proposed here. In view of such results, our macroscopic theoretical framework should be recommendable in any simulation of car sales that aims to be predictive. Today, the industry uses algorithms based on, for example, deep learning on time series that are very useful to fit internal correlations between every involved variable. Our approach can be used to constrain the degrees of freedom in such methods, which should further improve their efficiency. We hope that accurate forecasts will help to describe controlled evolution of the market and to reduce the economic risk inherent to profound technological transitions, such as the one that the automobile market is nowadays experiencing.

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