SATURATION GAMES FOR ODD CYCLES

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Abstract. Given a family of graphs \( \mathcal{F} \), we consider the \( \mathcal{F} \)-saturation game. In this game two players alternate adding edges to an initially empty graph on \( n \) vertices, with the only constraint being that neither player can add an edge that creates a subgraph that lies in \( \mathcal{F} \). The game ends when no more edges can be added to the graph. One of the players wishes to end the game as quickly as possible, while the other wishes to prolong the game. We let \( \text{sat}_g(\mathcal{F}; n) \) denote the number of edges that are in the final graph when both players play optimally.

The \( \{C_3\} \)-saturation game was the first saturation game to be considered, but as of now the order of magnitude of \( \text{sat}_g(\{C_3\}, n) \) remains unknown. We consider a generalization of this game.

Let \( \mathcal{C}_{2k+1} := \{C_3, C_5, \ldots, C_{2k+1}\} \). We prove that \( \text{sat}_g(\mathcal{C}_{2k+1}; n) \geq (\frac{1}{2} - \epsilon_k)n^2 + o(n^2) \) for all \( k \geq 2 \) and \( \text{sat}_g(\mathcal{C}_{2k+1}; n) \leq (\frac{1}{2} - \epsilon'_k)n^2 + o(n^2) \) for all \( k \geq 4 \), with \( \epsilon_k < \frac{1}{2} \) and \( \epsilon'_k > 0 \) constants tending to 0 as \( k \to \infty \). In addition to this we prove \( \text{sat}_g(\{C_{2k+1}\}; n) \leq \frac{27}{20}n^2 + o(n^2) \) for all \( k \geq 2 \), and \( \text{sat}_g(\mathcal{C}_\infty \setminus C_3; n) \leq 6n \), where \( \mathcal{C}_\infty \) denotes the set of all odd cycles.

1. Introduction

Hajnal proposed the following game. Initially \( G \) is an empty graph on \( n \) vertices. Two players alternate turns adding edges to \( G \), with the only restriction being that neither player is allowed to add an edge that would create a triangle. The last player to add an edge wins the game, and the central question is which player wins this game as a function of \( n \).

The answer to this problem is known only for small values of \( n \), the most recent result being \( n = 16 \) by Gordinowicz and Pra\'s [5]. A variation of this game was considered by F"uredi, Reimer, and Seress [4]. In the modified version of the game, there are two players, R and B, who alternate turns adding edges to an initially empty graph on \( n \) vertices with the same rules as in Hajnal’s original triangle-free game. The main difference is that once no more edges can be added to \( G \), R receives a point for every edge in the graph and B loses a point for every edge in the graph, with both players trying maximize the number of points they receive at the end. The question is now to figure out how many edges are at the end of the game when both players play optimally.

This game can be generalized. For a family of graphs \( \mathcal{F} \), we say that a graph \( G \) is \( \mathcal{F} \)-saturated if \( G \) contains no graph of \( \mathcal{F} \) as a subgraph, but adding any edge to \( G \) would create a subgraph of \( \mathcal{F} \). The \( \mathcal{F} \)-saturation game consists of two players, R and B, who alternate turns adding edges to an initially empty graph \( G \) on \( n \) vertices, with the only restriction being that \( G \) is never allowed to contain a subgraph that lies in \( \mathcal{F} \). The game ends when \( G \) is \( \mathcal{F} \)-saturated. The payoff for R is the number of edges in \( G \) when the game ends, and B’s payoff is the opposite of this. We let \( \text{sat}_g(\mathcal{F}; n) \) denote the number of edges that the graph in the \( \mathcal{F} \)-saturation game ends with when both players play optimally, and we call this quantity the \( \mathcal{F} \)-saturation number. We note that technically this game, and hence the value of \( \text{sat}_g(\mathcal{F}; n) \), depends on whether R or B makes the first move. This choice will not affect our asymptotic results, but for concreteness we will assume that R makes the first move.

Let \( C_k \) denote the cycle of length \( k \). The \( \{C_3\} \)-saturation game was the original saturation game studied in [4], where they proved what is still the best known lower bound of \( \frac{1}{2}n \log n + o(n \log n) \).
for $\text{sat}_g(\{C_3\}; n)$. Recently Biró, Horn, and Wildstrom [1] managed to prove the first non-trivial asymptotic upper bound of $\frac{26}{1771}n^2 + o(n^2)$ for $\text{sat}_g(\{C_3\}; n)$. A number of other results have been obtained for specific choices of $\mathcal{F}$, see for example [2], [3], and [8]. In addition to this, saturation games have recently been generalized to directed graphs [7], hypergraphs [9], and to avoiding more general graph properties such as $k$-connectivity [6].

1.1. Main Results.

Let $\mathcal{C}_{2k+1} := \{C_3, C_5, \ldots, C_{2k+1}\}$, and let $\mathcal{C}_\infty$ denote the set of all odd cycles. Most of this paper will be focused on studying the $\mathcal{C}_{2k+1}$-saturation games for $k \geq 2$. The key idea with these games is that by forbidding either player from making $C_5$'s, both players can utilize a strategy that keeps the graph essentially bipartite throughout the game. This makes it significantly easier to analyze the correctness of our proposed strategies and to bound the number of edges that are in the final graph. Our main result is the following upper and lower bounds for $\text{sat}_g(\mathcal{C}_{2k+1}; n)$ and most values of $k$.

**Theorem 1.1.** For $k \geq 4$,

$$\left(\frac{1}{4} - \frac{1}{5k^2}\right)n^2 + o(n^2) \leq \text{sat}_g(\mathcal{C}_{2k+1}; n) \leq \left(\frac{1}{4} - \frac{1}{10^6(k+1)^2}\right)n^2 + o(n^2).$$

We can also obtain a quadratic lower bound for smaller values of $k$.

**Theorem 1.2.** For $k \geq 2$,

$$\text{sat}_g(\mathcal{C}_{2k+1}; n) \geq \frac{6}{25}n^2 + o(n^2).$$

We consider two more saturation games. The first is the game where only one odd cycle is forbidden.

**Theorem 1.3.** For $k \geq 2$,

$$\text{sat}_g(\{C_{2k+1}\}; n) \leq \frac{1}{12} \left(1 + \frac{1}{\ell}\right)^2 n^2 + o(n^2),$$

where $\ell = \max(3, \lfloor \sqrt{2k} \rfloor)$.

**Corollary 1.4.** For $k \geq 2$,

$$\text{sat}_g(\{C_{2k+1}\}; n) \leq \frac{4}{27}n^2 + o(n^2).$$

We also consider the “complement” of the $\{C_3\}$-saturation game where every odd cycle except $C_3$ is forbidden. It turns out that in this setting the game saturation number is linear.

**Theorem 1.5.**

$$\text{sat}_g(\mathcal{C}_\infty \setminus C_3; n) \leq 6n.$$

This result is in sharp contrast to the fact that $\text{sat}_g(\mathcal{C}_\infty; n) = \frac{1}{36}n^2$, see [2].

**Notation.** Throughout the paper we let $G^t$ denote the graph in the relevant saturation game after $t$ edges have been added, and we let $e^t$ denote the edge of $G^t$ that is not in $G^{t-1}$. We let $N^t(x)$ denote the neighborhood of $x$ in $G^t$, $d^t(x, y)$ the distance between $x$ and $y$ in $G^t$, and so on. We let $t = \infty$ correspond to the point in time when the graph has become $\mathcal{F}$-saturated. If $X^t$ is a real number depending on $t$, we let $\Delta(X^t) = X^t - X^{t-2}$.

**Organization.** In Section 2 we produce an algorithm for R that guarantees that the game ends with at least as many edges as stated in Theorem 1.2, which will work the same way for all $k \geq 2$. In Section 3 we modify this algorithm to take into account the choice of $k$, and from this we obtain the lower bound of Theorem 1.1. In Section 4 we produce an algorithm for B that guarantees that the
game ends with at most as many edges as the upper bound of Theorem 1.1. Theorem 1.3 is proven in Section 5. Theorem 1.5 is proven in Section 6. We end with some concluding remarks in Section 7.

2. Proof of Theorem 1.2

2.1. The Setup.

Let $uv$ be the edge of $G^1$. Let $1 < \gamma \leq 2$ and $\delta = \frac{1}{\gamma - 1}$. We say that $G^t$ is $\gamma$-good if it satisfies the following four conditions.

1. $(*)$ $G^t$ contains exactly one non-trivial connected component, and this component is bipartite with parts $U^t \ni u$ and $V^t \ni v$.

   Let $U_0^t = N^t(v)$ (the good vertices), and $U_1^t = U^t \setminus U_0^t$ (the bad vertices). Define an analogous partition for $V^t$.

2. $(2*)$ Every vertex of $U^t \cup V^t$ is adjacent to a vertex in $U_0^t \cup V_0^t$.

3. $(3*) b_U^t := |V_1^t| + (|U^t| - \gamma|V^t| - \delta) \leq 0$ and $b_V^t := |U_1^t| + (|V^t| - \gamma|U^t| - \delta) \leq 0$.

4. $(4*) b_U^t + b_V^t \leq -2$.

We note that $(2*)$ and $(4*)$ are trivially satisfied if $U_1^t = V_1^t = \emptyset$. We prove Theorem 1.2 by first proving the following result.

Theorem 2.1. There exists a strategy for $R$ in the $C_{2k+1}$-saturation game when $k \geq 2$ such that for all odd $t$, whenever $G^{t-1}$ contains an isolated vertex, $R$ can add an edge so that $G^t$ is $\frac{3}{2}$-good.

2.2. The Algorithm.

In this subsection we inductively construct the algorithm of Theorem 2.1 whenever $G^{t-1}$ has an isolated vertex. We will assume throughout this subsection that $t$ is odd and that $G^{t-1}$ contains an isolated vertex $z$. Let $e^{t-1} = xy$. We will say that $e^{t-1}$ is an $I$ (Internal) move if $x \in U^{t-2}$, $y \in V^{t-2}$, an $O$ (Outside) move if $x, y \notin U^{t-2} \cup V^{t-2}$, an $AU$ (Add to $U$) move if $x \in V^{t-2}$, $y \notin U^{t-2} \cup V^{t-2}$, and an $AV$ (Add to $V$) move if $x \in U^{t-2}$, $y \notin U^{t-2} \cup V^{t-2}$. Note that an $AU$ move causes $y$ to be added to $U^{t-1}$.

We note that if we assume that $G^{t-2}$ satisfies $(1*)$, any vertex not in $U^{t-2} \cup V^{t-2}$ must be isolated. When $R$ plays it will always be obvious that $(1*)$ is maintained so we ignore this case in our analysis. We state our results in terms of general $\gamma$ whenever $\gamma = \frac{3}{2}$ isn’t required.

Lemma 2.2. If $G^{t-2}$ is $\gamma$-good and $e^{t-1}$ is an $I$ move, then $R$ can play so that $G^t$ is $\gamma$-good.

Proof. If there exists $u' \in U^{t-1}$, $v' \in V^{t-1}$ with $u'v' \notin G^{t-1}$, then $R$ adds the edge $u'v'$, and it’s not hard to see that in this case $G^t$ is $\gamma$-good. If no such pair of vertices exists, then $U^{t-1} \cup V^{t-1}$ is a complete bipartite graph with, say, $|U^{t-1}| \leq |V^{t-1}|$, in which case $R$ adds the edge $ze$. This gives $\Delta(|U^{t-1}|) = 1$ and $\Delta(|X^{t-1}|) = 0$ for every other set of interest. Since $U^{t-1} \cup V^{t-1}$ is a complete bipartite graph (and since $R$ added no vertex to $U_1^{t-1} \cup V_1^{t-1}$), $U_1^t = V_1^t = \emptyset$, so $(2*)$ and $(4*)$ hold. We have $\Delta(b_U^t) = -\gamma \leq 0$, and hence $b_U^t \leq 0$. If $|U^{t-1}| < \delta = \frac{1}{\gamma - 1}$, we automatically have $b_U^t \leq 0$. Otherwise $|V^{t-1}| \geq |U^{t-1}| \geq \frac{1}{\gamma - 1}$, which implies

$$|U^t| = |U^{t-1}| + 1 \leq |V^{t-1}| + (\gamma - 1) \frac{1}{\gamma - 1} \leq |V^{t-1}| + (\gamma - 1)|V^{t-1}| = \gamma|V^{t-1}| = \gamma|V^t|,$$

so $b_U^t \leq 0$ and $(3*)$ holds, so $G^t$ is $\gamma$-good. $\square$
Lemma 2.3. If $G^{t-2}$ is $\frac{3}{2}$-good and $e^{t-1}$ is an $O$ move, then $R$ can play so that $G^{t}$ is $\frac{3}{2}$-good.

Proof. Since $b_{V}^{t-2} + b_{V}^{t-2} \leq -2$, one of $b_{V}^{t-2}$ or $b_{V}^{t-2}$ must be at most $-1$, say $b_{V}^{t-2} \leq -1 \leq -\frac{1}{2}$. In this case, $R$ adds the edge $xy$ (otherwise $R$ adds the edge $xu$), which leads to $\Delta(|U^{t-1}|) = \Delta(|V^{t-1}|) = \Delta(|V_{1}^{t-1}|) = 1$, $\Delta(|U_{1}^{t-1}|) = 0$. $x$ and $y$ satisfy ($2^{*}$), so this continues to hold. We have $\Delta(b_{V}^{t-1}) = 1 - \gamma \leq 0$ and $\Delta(b_{V}^{t-1}) = 2 - \gamma = \frac{1}{2}$, so $b_{V}^{t-1} \leq 0$ since we assumed $b_{V}^{t-2} \leq -\frac{1}{2}$, and thus ($3^{*}$) holds. We have $\Delta(b_{V}^{t-1}) + \Delta(b_{V}^{t-1}) = 3 - 2\gamma = 0$, so ($4^{*}$) holds and $G^{t}$ is $\frac{3}{2}$-good. □

The situation becomes more complex if $e^{t-1}$ is an $AU$ or $AV$ move, in which case $R$ makes his move depending on the overall “State” of the game. We first make an observation.

Lemma 2.4. Assume that $G^{t-2}$ is $\gamma$-good.

If $|U^{t-1}| > \gamma|V^{t-1}| + \delta$, then $e^{t-1}$ is an $AU$ move, $V_{1}^{t-2} = V_{1}^{t-1} = \emptyset$, and $b_{V}^{t-2} \leq -1$.

If $|V^{t-1}| > \gamma|U^{t-1}| + \delta$, then $e^{t-1}$ is an $AV$ move., $U_{1}^{t-2} = U_{1}^{t-1} = \emptyset$, and $b_{V}^{t-2} \leq -1$.

Proof. Assume that $|U^{t-1}| > \gamma|V^{t-1}| + \delta$. Since we assumed that $b_{U}^{t-2} \leq 0$, and in particular that $|U^{t-2}| \leq \gamma|V^{t-2}| + \delta$ since $|V^{t-2}|$ is non-negative, we must have $e^{t-1}$ an $AU$ move, meaning $b_{U}^{t-1} = b_{U}^{t-2} + 1 \leq 1$. Thus $|V_{1}^{t-1}| = (-|U^{t-1}| + \gamma|V^{t-1}| + \delta) + b_{U}^{t-1} < 1$, which implies that $|V_{1}^{t-1}| = 0$ since $|V_{1}^{t-1}|$ is a non-negative integer, and thus $|V^{t-2}| = 0$ as nothing is removed from $V_{1}^{t-2}$ by an $AU$ move. Lastly, $b_{V}^{t-2} + b_{V}^{t-2} \leq -2$ by ($4^{*}$) and $b_{V}^{t-2} + b_{V}^{t-2} + 1 = b_{V}^{t-1} > 0$, so $b_{V}^{t-2} + b_{V}^{t-2} + b_{V}^{t-2} + 1 \leq -1$.

The analysis for the other case is similar. □

By Lemma 2.4, if we inductively assume that $G^{t-2}$ is $\gamma$-good, then after doing an $AU$ move the game must be in one of the following States.

- State N (Nice): $U_{1}^{t-1} = V_{1}^{t-1} = \emptyset$, $|U^{t-1}| \leq \gamma|V^{t-1}| + \delta$, and $|V^{t-1}| \leq \gamma|U^{t-1}| + \delta$.
- State OU (Overflow U): $|U^{t-1}| > \gamma|V^{t-1}| + \delta$ and $V_{1}^{t-1} = \emptyset$.
- State C (Clean-Up): $|U_{1}^{t-1} \cup V_{1}^{t-1}| \neq 0$, $|U^{t-1}| \leq \gamma|V^{t-1}| + \delta$, and $|V^{t-1}| \leq \gamma|U^{t-1}| + \delta$.

Lemma 2.5. If $G^{t-2}$ is $\gamma$-good and $e^{t-1}$ is an $AU$ move putting the game is State N, then $R$ can play so that $G^{t}$ is $\gamma$-good.

Proof. $R$ follows the same exact same strategy he did in response to $B$ playing an $I$ move, and the analysis remains the same. □

Lemma 2.6. If $G^{t-2}$ is $\frac{3}{2}$-good and $e^{t-1}$ is an $AU$ move putting the game in State OU, then $R$ can play so that $G^{t}$ is $\frac{3}{2}$-good.

Proof. $R$ adds the edge $zu$. This gives $\Delta(|U^{t-1}|) = \Delta(|V^{t-1}|) = 1$, $\Delta(|V_{1}^{t-1}|) = 0$, $\Delta(|U_{1}^{t-1}|) \leq 1$. Clearly $z$ satisfies condition ($2^{*}$), and $y$ does as well since $V_{1}^{t-1} = V_{1}^{t-1} = \emptyset$ by virtue of the game being in State OU, so ($2^{*}$) is maintained. We have $\Delta(b_{V}^{t-1}) = 1 - \gamma \leq 0$ and $\Delta(b_{V}^{t-1}) = 2 - \gamma \leq 1$, so $b_{V}^{t-1} \leq 0$ by Lemma 2.4, and thus ($3^{*}$) is maintained. Lastly, $\Delta(b_{V}^{t-1}) + \Delta(b_{V}^{t-1}) \leq 3 - 2\gamma = 0$, so ($4^{*}$) is maintained and $G^{t}$ is $\frac{3}{2}$-good. □

Lemma 2.7. If $G^{t-2}$ is $\gamma$-good and $e^{t-1}$ is an $AU$ move putting the game in State C, then $R$ can play so that $G^{t}$ is $\gamma$-good.
Proof. First assume that $U_t^{t-1} \neq \emptyset$. If $x \in U_t^{t-1}$, then $R$ adds the edge $xv$, otherwise $R$ picks an arbitrary $u' \in U_t^{t-1}$ and adds the edge $u'v$. After this we have $\Delta(V_t^t) = 1$, $\Delta(|U_t^t|) = 0$, $\Delta(|U_t^{t-1}|) = -1$, $\Delta(V_t^{t-1}) \leq 1$. $(2^*)$ is maintained since we made sure that $y$'s neighbor $x$ was in $U_0$. We have $\Delta(b_v') = 0$ and $\Delta(b_v'') \leq 1 - \gamma \leq 0$, so $(3^*)$ is maintained. $\Delta(b_v') + \Delta(b_v'') \leq 1 - \gamma \leq 0$, so $(4^*)$ is maintained and $G_t$ is $\gamma$-good.

Now assume $U_1^{t-1} = \emptyset$. In this case $R$ arbitrarily picks a $v' \in V_1^t$ and adds the edge $v'u$, giving $\Delta(V_t^t) = 1$, $\Delta(|U_t^t|) = \Delta(|U_t^{t-1}|) = 0$, $\Delta(V_t^{t-1}) \leq 0$. $(2^*)$ is maintained since $y$ has a neighbor in $U_t = U_0^t$. We automatically have $b_v' \leq 0$ by assumption of us not being in State OV and having $|U_t^t| = |U_t^{t-1}| = 0$, and $\Delta(b_v'') \leq -\gamma \leq 0$, so $(3^*)$ is maintained. Lastly, $\Delta(b_v') + \Delta(b_v'') \leq 1 - \gamma \leq 0$, so $(4^*)$ is maintained and $G_t$ is $\gamma$-good. \hfill $\Box$

The algorithm and analysis for $AV$ moves is analogous.

2.3. Proofs of Theorem 2.1 and Theorem 1.2.

Lemma 2.8. If $G_t$ satisfies $(1^*)$ and $(2^*)$, then $U_t^{t+1}$ and $V_t^{t+1}$ are both independent sets.

Proof. If $v', v'' \in V_t$, let $u', u'' \in U_0^t$ be neighbors of $v'$ and $v''$ respectively, noting that such vertices exist by $(2^*)$. If $u'' = u''$, then $d_t(v', v'') = 2$, otherwise

$$d_t(v', v'') \leq d_t(v', u') + d_t(u', u'') + d_t(u'', v'') = 4.$$ 

Thus having $e_t+1 = v'v''$ would create either a $C_3$ or a $C_5$, which is forbidden in the $C_{2k+1}$-saturation game for $k \geq 2$. The analysis for $U_t$ is similar. \hfill $\Box$

Proof of Theorem 2.1. $G_1$ is $\frac{3}{2}$-good (in fact, it’s $\gamma$-good for any $1 < \gamma \leq 2$). Inductively assuming that $G_t$ was $\frac{3}{2}$-good, $e_t-1$ must be a move of type $I$, $O$, $AU$, or $AV$ by Lemma 2.8. The lemmas of the previous subsection show that $R$ can then play so that $G_t$ is $\frac{3}{2}$-good whenever $G_t$ contains an isolated vertex. \hfill $\Box$

Proposition 2.9. Let $1 < \gamma \leq 2$. Assume that there exists a strategy for $R$ in the $C_{2k+1}$-saturation game when $k \geq 2$ such that for all odd $t$, whenever $G_t$ contains an isolated vertex, $R$ can add an edge so that $G_t$ satisfies $(1^*)$, $(2^*)$, and $(3^*)$. Then

$$\text{sat}_g(C_{2k+1}; n) \geq \frac{\gamma}{(1 + \gamma)^2} n^2 + o(n^2).$$

Proof. Assume that $R$ uses such a strategy until there are no isolated vertices left. Let $T$ denote the smallest even number such that $G_T$ contains no isolated vertices. Let $S$ denote the set of vertices of $G_{T-1}$ that are isolated, noting that $|S| \leq 2$. We claim that $R$ can choose $e_{T+1}$ so that $G_{T+1}$ satisfies $(1^*)$ and $(2^*)$. Indeed, if $G_{T+1}$ doesn’t satisfy $(1^*)$ and $(2^*)$, then there must exist some $x \in S$ such that $G_T$ doesn’t contain either $xy$, $xu$, or $xv$. Since $G_T$ is bipartite by Lemma 2.8, $R$ can choose $e_{T+1}$ to be one of these edges, causing $G_{T+1}$ to satisfy $(1^*)$ and $(2^*)$. Thus $R$ can play so that $G_{T+1}$, and hence $G_\infty$, satisfies $(1^*)$ and $(2^*)$. After this $R$ plays arbitrarily.

By Lemma 2.8, $G_\infty$ contains all edges between $U_\infty$ and $V_\infty$. Since $|U_\infty| + |V_\infty| = n$, the product $|U_\infty||V_\infty|$ will be minimized when $|U_\infty| = |V_\infty|$. Thus we can assume that, say, $|U_\infty| = \gamma|V_\infty| + \delta + 2$ since $G_{T+1}$ satisfied $(3^*)$ (we omit floors and ceilings as they won’t affect the asymptotic result). Then

$$n = |U_\infty| + |V_\infty| = (1 + \gamma)|V_\infty| + \delta + 2 \implies |V_\infty| = \frac{n - 2 - \delta}{1 + \gamma},$$

and hence there are at least $|U_\infty||V_\infty| = \frac{\gamma}{(1 + \gamma)^2}(n - 2 - \delta)^2 + \delta|V_\infty|$ edges in $G_\infty$. \hfill $\Box$
Proof of Theorem 1.2. This is an immediate consequence of Theorem 2.1 and Proposition 2.9. □

3. The Lower Bound of Theorem 1.1

Let $\gamma' = \frac{4k^{-1} + \sqrt{16k^{-2} + 1}}{2}$.

Theorem 3.1. There exists a strategy for $R$ in the $C_{2k+1}$-saturation game when $k \geq 3$ such that for all odd $t$, whenever $G^{t-1}$ contains an isolated vertex, $R$ can add an edge so that $G^t$ is $\gamma'$-good.

The algorithm of Theorem 3.1 is essentially the same as that of Theorem 2.1, but with new definitions for the relevant parameters. Throughout this section we use the same notation as in the previous section unless stated otherwise, and we always assume that $k \geq 3$.

Let $\ell = k$ if $k$ is even and $\ell = k + 1$ if $k$ is odd. We redefine $U'_0 = \{u' \in V^t : d'(u', u) < \ell\}$ and define $U'_1 = U^t \setminus U'_0$. Order the vertices of $U^t$ in some way, say based on the order that they were added to the set. We will say that a vertex $x \in U^t$ is the representative for $u' \in U'_1$ if

1. $d'(x, u) = 4$.
2. $x$ lies along a shortest path from $u'$ to $u$.
3. $x$ is the minimal vertex (with respect to the ordering of $U^t$) satisfying these properties.

We note that since $k \geq 3$, $d'(x, u) \leq d'(u', u)$, so every $u' \in U'_1$ has a representative. Redefine $U'_1$ to be the set of vertices that are representatives for some vertex of $U_1$. Note that $|U'_1| = 0$ iff $|U_1| = 0$. We similarly define $V'_0$, $\tilde{V}_1$, and $V'_1$.

We make the following observation.

Lemma 3.2. If (2*) holds, then $\tilde{U}_1 = \{u' : d(u, u') = \ell\}$, $\tilde{V}_1 = \{v' : d(v, v') = \ell\}$.

Proof. Let $v' \in V'_0$ be a neighbor of $u' \in \tilde{U}_1$, which exists by (2*).

$$d'(u', u) \leq d'(u', v') + d'(v', v) + d'(v, u) \leq 1 + (\ell - 2) + 1 = \ell,$$

with the $\ell - 2$ term coming from the fact that $d'(v', v) < \ell$ is even. Since $u' \in \tilde{U}_1$ implies $d'(u', u) \geq \ell$, the distance must be exactly $\ell$. The analysis for $\tilde{V}_1$ is similar. □

An application of Lemma 3.2 can be used to show that Lemma 2.8 continues to hold when we define (2*) in terms of these new sets. We omit the details.

The algorithm for Theorem 3.1 is the same algorithm as that of Theorem 2.1, except we now use these new definitions for $U'_0$, $U'_1$, $V'_0$, $V'_1$, and we make a slight change to how $R$ responds in State C. Namely, in the previous algorithm when $B$ added in the isolated vertex $y$, we checked to see if its neighbor $x$ was in, say, $U'_1$, in which case we added the edge $xy$. We now instead check if $x \in U'_1$, and if it is we add the edge $xz$ where $z$ is the representative for $x$. Adding this edge strictly decreases $d^{t-1}(x, u)$, so by Lemma 3.2 we will have $d'(x, u) < \ell$ and $y$ will have a neighbor in $U'_0$, so (2*) will still hold. One can check that outside of this specific subcase of State C, all of the previous analysis we did continues to hold with these new definitions of $U'_1$ and $V'_1$. It remains to address the two points in the algorithm where we required $\gamma = \frac{3}{2}$, namely Lemma 2.3 and Lemma 2.6.

Lemma 3.3. If $G^{t-2}$ is $\gamma$-good and $e^{t-1}$ is an O move, then $R$ can play so that $G^t$ is $\gamma$-good.
Proof. R reacts as he did in the previous algorithm. Observe that no vertices are added to $\bar{U}_{1}^{t-1}$ or $\bar{V}_{1}^{t-1}$. Indeed, the new vertices are within distance $2 < \ell$ of $u$ and $v$, so they’ll both be added to $\bar{U}_{0}^{t-1} \cup \bar{V}_{0}^{t-1}$. In particular, $\Delta(|U_{1}^{t}|) = \Delta(|V_{1}^{t}|) = 0$, and the remaining analysis is straightforward. □

**Lemma 3.4.** $|U_{1}^{t}| \leq 4k^{-1}|U^{t}|$ and $|V_{1}^{t}| \leq 4k^{-1}|V^{t}|$.

**Proof.** For each $x \in U_{1}^{t}$, let $u_{x}$ denote a vertex that $x$ is the representative for, and let $P_{x}$ denote the set of vertices that make up a shortest path from $x$ to $u_{x}$. We claim that $P_{x}$ and $P_{y}$ are disjoint if $x \neq y$. Indeed, let $w \in P_{x} \cap P_{y}$. If $d^{t}(w, x) < d^{t}(w, y)$, then

$$d^{t}(u_{y}, w) + d^{t}(w, y) + d^{t}(y, u) = d^{t}(u_{y}, u) + d^{t}(w, x) + d^{t}(x, u),$$

a contradiction since $d^{t}(y, u) = d^{t}(x, u) = 4$. By using a symmetric argument we see that we must have $d^{t}(w, x) = d^{t}(w, y)$. If we had, say, $x < y$ in the ordering of $U^{t}$, then $y$ couldn’t be the representative for $u_{y}$ since $x$ is less than $y$ and satisfies the relevant properties. A similar result occurs if $x > y$. We conclude that the only way $P_{x} \cap P_{y}$ could be non-empty is if $x = y$.

For each $x \in U_{1}^{t}$, we observe that the number of vertices in $P_{x} \cap U^{t}$ is

$$\frac{d^{t}(u_{x}, x)}{2} + 1 = \frac{\ell - 4}{2} + 1 = \frac{\ell}{2} - 1 \geq \frac{k}{4},$$

and none of these vertices appear in any other $P_{y}$ for $x \neq y \in U_{1}^{t}$. Since we can associate to each $x \in U_{1}^{t}$ a set of at least $k/4$ elements of $U^{t}$ without any element of $U^{t}$ appearing in more than one set, we must have $|U_{1}^{t}| \leq 4k^{-1}|U^{t}|$. The analysis for $|V_{1}^{t}|$ is analogous. □

**Lemma 3.5.** If $G^{t-2}$ is $\gamma'$-good and $e^{t-1}$ is an AU move putting the game in State OU, then R can play so that $G^{t}$ is $\gamma'$-good.

**Proof.** R reacts as he did in the previous algorithm. By definition of State OU, we have $V_{1}^{t} = \emptyset$ and $|U^{t-1}| > \gamma'|V^{t-1}| + \delta$. The latter implies that

$$|V^{t}| = |V^{t-1}| + 1 < \frac{1}{\gamma'}|U^{t-1}| - \frac{1}{\gamma'}\delta + 1 = \frac{1}{\gamma'}|U^{t}| - \frac{1}{\gamma'}\delta + 1.$$

Combining these observations with Lemma 3.4 gives

$$b_{U}^{t} = b_{U}^{t-1} = |U_{1}^{t}| + (1 - \gamma')(|U^{t}| + |V^{t}|) - 2\delta \leq 4k^{-1}|U^{t}| + (1 - \gamma')(|U^{t}| + \frac{1}{\gamma'}|U^{t}| - \frac{1}{\gamma'}\delta + 1) - 2\delta = (-\gamma' + 4k^{-1} + \frac{1}{\gamma'})|U^{t}| - (1 + \frac{1}{\gamma'})\delta + (1 - \gamma') = -(1 + \frac{1}{\gamma'})\delta + (1 - \gamma'),$$

with the last equality coming from the fact that $\gamma'$ is a root of $-x^{2} + 4k^{-1}x + 1$. It’s not too hard to see that the remaining value is at most $-2$. □

**Proof of Theorem 3.1.** The proof is essentially the same as that of Theorem 2.1, except here we use Lemma 3.3 and Lemma 3.5 instead of Lemma 2.3 and Lemma 2.6. □

**Proof of the lower bound of Theorem 1.1.** Note that

$$\frac{\gamma'}{(1 + \gamma')^{2}} = \frac{k}{8}(\sqrt{k^{2} + 4} - k) \leq \left(\frac{1}{4} - \frac{1}{5k^{2}}\right)$$

when $k \geq 4$. The result is then a consequence of Theorem 3.1 and Proposition 2.9. □
4. The Upper Bound of Theorem 1.1

Throughout this section we will consider the $C_{2k-1}$-saturation game, so that the smallest odd cycle that can be made is a $C_{2k+1}$, and we will always have $k \geq 5$. We again let $e^1 = uv$.

The algorithm needed to prove the upper bound of Theorem 1.1 will take some time to develop. The main idea is that B will maintain a number of long, disjoint paths in $G^t$, and then eventually either B will be able to join these paths together and create many odd cycles, or the graph will look like a bipartite graph with one side much larger than the other.

4.1. Paths.

We wish to define a special set of paths $P^t$ in $G^t$, with each path having $v$ as one of its endpoints. We start with $P^1 = \{uv\}$, and inductively we define $P^t$ based on the following procedure (regardless of the parity of $t$).

Step 0. Set $P^0 := P^{t-1}$.

Step 1. Consider the case that B added the edge $e^1 = xw$ where $x$ is either an isolated vertex or a vertex in an isolated edge $xy$ (i.e. $d(x) = d(y) = 1$) in $G^{t-1}$. If there exists some $p \in P^{t-1}$ with $p = w \cdots v$, then $P^t := (P^t \setminus \{p\}) \cup \{xp\}$. If $w = v$, $P^t := P^t \cup \{xv\}$.

Step 2. If $w, w'$ are vertices of $p, p'$ with $p, p' \in P^t$, $p \neq p'$, $w, w' \neq v$, and if $w$ and $w'$ lie in the same connected component of $G^t - \{v\}$, then $P^t := P^t \setminus \{p, p'\}$.

Step 3. Let $d_P^t(x, y)$ denote the distance between $x$ and $y$ in the graph induced by the vertices of $P^t$. If $w$ is a vertex of $p \in P^t$ with $d_P^t(w, v) < d_P^t(w, v)$, then $P^t := P^t \setminus \{p\}$.

Observe that by these Steps, each vertex besides $v$ is in at most one path of $P^t$. Let $D^t_{\ell}$ denote the set of vertices that are the endpoint of a path of length $\ell$ in $P^t$ and that aren’t $v$.

**Lemma 4.1.** $|D^t_{\ell}| - |D^{t-1}_{\ell}| \geq -2$, and this number is 0 whenever R adds an edge to $G^{t-1}$ that involved an isolated vertex of $G^{t-1}$.

**Proof.** $e^t$ can involve at most two components of $G^{t-1} - \{v\}$, and each component contains at most one path of $P^{t-1}$ by Step 2. Any path not in these components won’t be modified or deleted by the Steps. Hence $|D^t_{\ell}| - |D^{t-1}_{\ell}| \geq -2$. If R makes an edge involving an isolated vertex, then none of the Steps for modifying $P^t$ apply and we have $D^t_{\ell} = D^{t-1}_{\ell}$. □

**Lemma 4.2.** If $w_1 \neq w_2$ are two vertices of $D^t_{\ell}$, then choosing $e^{t+1} = w_1w_2$ is a legal move.

**Proof.** Let $p_i$ denote the path for which $w_i$ is an endpoint for, noting that $p_1 \neq p_2$ since $w_1 \neq w_2$ and neither are equal to $v$. Since $p_1 \neq p_2$, $w_1$ and $w_2$ lie in different components of $G^t - \{v\}$ by Step 2. Thus if the edge $w_1w_2$ created a forbidden cycle it would have to involve the vertex $v$. Since $d^t(w_i, v) = d^t_p(w_i, v) = k$ for $i = 1, 2$ by Step 3, the smallest cycle that could be formed is a $C_{2k+1}$, which is allowed. □

4.2. Phases and Phase Transitions.

For the rest of the section we assume that $t$ is even. We wish to describe each $G^t$ as belonging to a certain “Phase.” To do this formally we’ll need some definitions.
Set $U^0 := \emptyset$, $V^0 := \{v\}$. Let $e^i = xy$. If $x \in V^{i-1}$ and $y$ is an isolated vertex, then $U^t := U^{t-1} \cup \{y\}$, $V^t := V^{t-1}$. If $x \in V^{i-1}$ and $y$ is in an isolated edge $yz$ in $G^{i-1}$, then $U^t := U^{t-1} \cup \{y\}$, $V^t := V^{t-1} \cup \{z\}$. If $x \in U^{t-1}$ we define $U^t$ and $V^t$ analogously. For any other case of $e^i$, $U^t := U^{t-1}$, $V^t := V^{t-1}$. Let $i^t$ denote the number of isolated vertices in $G^t$. Let $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{3}$, $c_k = c_{-1} = (1000k^2)^{-1}$.

We will say that $G^t$ is in Phase $\ell$ for some $-1 \leq \ell \leq k$ based on the following set of rules.

- $G^0$ is in Phase 0.
- If $G^{t-2}$ is in Phase 0, $|D^i| \geq c_2 n + 9c_{-1} n$, $|U^t| - |V^t| < c_{-1} n$, and either $i^t = \lfloor c_1 n \rfloor$ or $i^t = \lfloor c_1 n \rfloor + 1$, then $G^t$ is in Phase 1.
- If $G^{t-2}$ is in Phase 1, $|D^i| \geq c_2 n$, $|U^t| - |V^t| < c_{-1} n$, and $i^t \geq \lfloor c_1 n \rfloor - 2c_2 n - 8c_{-1} n$, then $G^t$ is in Phase 2.
- For $2 \leq \ell < k$, if $G^{t-2}$ is in Phase $\ell$, $|D^i_{\ell+1}| \geq 4c_{\ell} n + 9(k - \ell - 1)c_{-1} n$, $|U^t| - |V^t| < c_{-1} n$, and $i^t \geq 8(k - \ell - 1)c_{\ell} n + \sum_{j=\ell+1}^{k} 27(k - j)c_{-1} n$, then $G^t$ is in Phase $\ell + 1$.
- If $G^{t-2}$ is in Phase $\ell$ with $\ell < k$ and $|U^t| - |V^t| \geq c_{-1} n$, then $G^t$ is in Phase $-1$.
- If $G^{t-2}$ is in Phase $\ell$ and if $G^t$ satisfies none of the above situations, then $G^t$ is in Phase $\ell$.

We note that our exact choices of $c_1$, $c_2$, $c_k$, and $c_{-1}$ aren’t important. The only thing that matters is that they satisfy the following set of inequalities for $k \geq 5$ and $n$ sufficiently large, which isn’t too difficult to verify for the values we’ve chosen (in particular one can use that $\sum_{j=2}^{k} (k - j) \leq k^2$).

\[
1 > c_1, \quad c_2, \quad c_k, \quad c_{-1} > 0 \quad (1)
\]

\[
\frac{1}{4}(1 - c_1 - \frac{1}{n}) - 2c_{-1} \geq c_2 + 9c_{-1} \quad (2)
\]

\[
c_2 \geq 4c_k + 9(k - 2)c_{-1} \quad (3)
\]

\[
c_1 - 2c_2 - 8c_{-1} \geq 8(k - 2)c_k + \sum_{j=2}^{k} 27(k - j)c_{-1} \quad (4)
\]

\[
(k - 2)c_k - k^2 c_k^2 \geq c_{-1}^2 \quad (5)
\]

\[
c_2 \geq \frac{1}{4}(c_1 + \frac{1}{n} - c_2) + \frac{1}{n} \quad (6)
\]

4.3. The Beginning of the Game.

We will say that a path in $U^t \cup V^t$ is alternating if the vertices in the path alternate being in $U^t$ and $V^t$, and we define $d^t_{\alpha}(x,y)$ for $x, y \in U^t \cup V^t$ to be the length of the shortest alternating path in $U^t \cup V^t$ from $x$ to $y$. Note that $d^t_{\alpha}(x,v) = d^t(x,v)$ if $x$ is a vertex in some path of $P^t$, and that $d^t_{\alpha}(x,v)$ is even if $x \in V^t$ and odd if $x \in U^t$.

**Theorem 4.3.** For $n$ sufficiently large there exists a strategy for $B$ in the $C_{2k-1}$-saturation game for $k \geq 5$ such that, for every even $t$ with $G^t$ in Phase $\ell$ for $0 \leq \ell < k$, the following conditions hold.

1. $G^t$ contains exactly one non-trivial connected component whose vertices are $U^t \cup V^t$.
2. $d^t_{\alpha}(x,v) \leq \ell + 2$ for all $x \in U^t \cup V^t$.
3. $i^t \geq 3$, and if $\ell \neq 0$ then $D^t_{\ell} \geq 3$. 

(1') $G^t$ contains exactly one non-trivial connected component whose vertices are $U^t \cup V^t$.

(2') $d^t_{\alpha}(x,v) \leq \ell + 2$ for all $x \in U^t \cup V^t$.

(3') $i^t \geq 3$, and if $\ell \neq 0$ then $D^t_{\ell} \geq 3$. 

(1') $G^t$ contains exactly one non-trivial connected component whose vertices are $U^t \cup V^t$.
We will say that $G^t$ satisfies (') if it satisfies (1'), (2'), and (3'). We construct the algorithm for B inductively.

**Lemma 4.4.** Assume that $G^{t-2}$ is in Phase $\ell$ with $0 \leq \ell < k$ and that $G^{t-2}$ satisfies ('). Then B can play so that $G^t$ satisfies (1') and (2').

**Proof.** To be concrete we'll describe the algorithm for $\ell$ even, where we'll have $D^t_\ell \subseteq V^t$ and $D^t_{\ell+1} \subseteq U^t$. The algorithm and analysis for $\ell$ odd is exactly the same except the roles of $U^t$ and $V^t$ are reversed throughout. To deal with the case $\ell = 0$, we define $D^0_0 = \{v\}$. It will always be clear that (1') is maintained, so we omit this from our analysis.

We classify $e^{t-1}$ as being of type $I$, $O$, $AU$, or $AV$ essentially as we did before, except now $I$ moves are defined to be an edge between two vertices of $U^t \cup V^t$ which aren't necessarily from distinct parts. It follows that $e^{t-1}$ must be one of these four types of moves. Since $G^{t-2}$ satisfies (3'), we have $i^{t-2} \geq 3$ and $|D^t_{i-2}| \geq 3$ when $\ell \neq 0$. Thus $i^{t-1} \geq 1$ (the number of isolated vertices always decreases by at most 2) and $|D^t_{i-1}| \geq 1$ by Lemma 4.1 (and we always have $|D^t_{i-1}| = 1$). Thus there exists $x' \notin U^{t-1} \cup V^{t-1}$ and $v' \in D^t_{i-1}$.

If $e^{t-1}$ is an $I$ move, B plays $xv'$. Then $d^t_a(x, v) = \ell + 1$ and no other distances increase, so (2') is maintained.

If $e^{t-1} = yz$ is an $O$ move (noting that $yz$ is an isolated edge in $G^{t-1}$ since $G^{t-2}$ satisfies (1')), then B plays $yz$. We have $d^t_a(y, v) = \ell + 1$ and $d^t_a(z, v) = \ell + 2$, so (2') is maintained.

If $e^{t-1} = yu'$ is an $AU$ move with $u' \in U^{t-1}$ and $y \notin U^{t-2} \cup V^{t-2}$, then B plays $xv'$. Then $d^t_a(y, v) \leq \ell + 2$ since $d^t_a(u', v) = \ell$ and hence at most $\ell + 1$, and $d^t_a(x, v) = \ell + 1$, so (2') is maintained.

Say $e^{t-1} = yw'$ is an $AU$ move with $w' \in V^{t-2}$ and $y \notin U^{t-2} \cup V^{t-2}$. If $d^t_a(w', v) \leq \ell$, then B plays $xv$, essentially skipping his turn and maintaining (2') (in the analogous situation with $\ell$ odd, B plays $xy$ to skip his turn). If instead $d^t_a(w', v) = \ell + 2$, let $w'w''w''' \cdots$ be a shortest alternating path from $w'$ to $v$. Then B adds the edge $yw''$, and if this is a legal move we have $d^t_a(y, v) = \ell + 1$, maintaining (2'). The following lemma shows that this move is indeed allowed.

**Lemma 4.5.** Let $w'$ be such that $d^{t-2}_a(w', v) \geq 2$, say with $w'ywy'' \cdots$ a shortest alternating path from $w'$ to $v$, and let $y$ be an isolated vertex in $G^{t-2}$. If $e^{t-1} = yw'$, then $e^t = yw''$ is a legal move.

**Proof.** Let $G^t$ denote $G^{t-1}$ with the edge $yw'$, and assume this created a forbidden odd cycle $C$. $yw'$ and $yw''$ must be two edges of $C$. If $y'$ is not a vertex of $C$, then let $C'$ be $C$ after replacing the edges $yw'$ and $yw''$ with $ywy'$ and $ywy''$. Then $C'$ is a forbidden odd cycle in $G^{t-1}$, a contradiction. Thus $y'$ must be a vertex of $C$, so $C$ contains in $G^{t-1}$ a path from $w'$ to $y'$ and a path from $y'$ to $w'$, with exactly one of these paths being of even length. Any path of even length from $w''$ to $y'$ in $G^{t-1}$ has length at least $2k$, since otherwise this path together with the edge $y'w''$ would create a forbidden odd cycle in $G^{t-1}$. A similar observation holds for paths of even length from $y'$ to $w'$. We conclude in this case that $C$ has length at least $2k + 3$, which is allowed.

During Phase $\ell$ define $g^t = |V^t| - |U^t|$ if $\ell$ is even and $g^t = |U^t| - |V^t|$ if $\ell$ is odd. It’s not too difficult to observe the following by using Lemma 4.1. We omit the details.

**Lemma 4.6.** Assume that $G^{t-2}$ satisfies (') and is in Phase $\ell$ for $0 \leq \ell < k$. If B uses the algorithm described in Lemma 4.4, then the following holds for $\ell$ even (where whenever we write $\Delta(|D^t_\ell|)$ we assume that $\ell \neq 0$).

- If $e^{t-1}$ is of type $I$: $\Delta(|D^t_{\ell+1}|) \geq -1$, $\Delta(|D^t_\ell|) \geq -3$, $\Delta(g^t) = -1$, $\Delta(i^t) = -1$. 



Analogous results hold for \( t \) odd by switching \( AU \) with \( AV \).

Let \( t_\ell \) denote the smallest value such that \( G^{t_\ell} \) is in Phase \( \ell \), with \( t_\ell = \infty \) if no such value exists. Let \( s_\ell = \min \{t_\ell, t_{\ell-1}\} \).

**Lemma 4.7.** If \( n \) is sufficiently large and \( B \) uses the algorithm described in Lemma 4.4, then \( G^t \) satisfies \((3')\) for all even \( 0 \leq t < s_1 \).

**Proof.** Since \( \Delta(i') \geq -2 \) and \( i'' = 0 \) for some sufficiently large \( T \), there exists a (smallest) even \( t_0' \) such that \( t_0' = \lfloor c_1n \rfloor \) or \( \lfloor c_1n \rfloor + 1 \), and moreover \( t_0' \geq \frac{1}{2}((1-c_1)n - 1) \). We claim that \( t_0' \geq s_1 \), which will give the result when \( \lfloor c_1n \rfloor \geq 3 \).

Let \( r_1 \) denote the number of even \( t \) with \( 0 \leq t < t_0' \) such that \( e^{t+1} \) is of type \( I \) or \( AU \), and similarly define \( r_2 \) for \( O \) and \( AV \) moves. Note that \( r_1 + r_2 = \frac{1}{2}t_0' \geq \frac{1}{4}((1-c_1)n - 1) \). By Lemma 4.6 we have \( g^{t_0'} \leq -r_1 \) and

\[
|D^{t_0'}_1| \geq r_2 - r_1 \geq \frac{1}{4}((1-c_1)n - 1) - 2r_1.
\]

If \( r_1 > c_{-1}n \), then \( G^{t_0'} \) is in Phase \(-1\), and if \( r_1 \leq c_{-1}n \) then \( |D^{t_0'}_1| \geq c_2n + 9c_{-1}n \) by (2). We conclude that \( G^{t_0'} \) is either in Phase \(-1\) or Phase 1, i.e. \( s_1 \leq t_0' \). \( \square \)

**Lemma 4.8.** Assume that \( t_\ell \) is finite with \( 1 \leq \ell < k \) and that \( n \) is sufficiently large. If \( G^{t_\ell-2} \) satisfies \((')\) and if \( B \) uses the algorithm described in Lemma 4.4, then \( G^t \) satisfies \((3')\) for all even \( t \leq t < s_{\ell+1} \).

**Proof.** Assume first that \( \ell \) is even. Let \( r_1 \) denote the number of \( t_\ell \leq t < s_{\ell+1} \) such that \( G^t \) satisfies \((3')\) and such that \( e^{t+1} \) is an \( I \) or \( AU \) move, and similarly define \( r_2 \) for moves of type \( O \) and \( AV \). Let \( t'_\ell = t_\ell + 2(r_1 + r_2) \). We claim that \( t'_\ell = s_{\ell+1} \), which would imply the desired result. We trivially have \( t'_\ell \leq s_{\ell+1} \), so assume for contradiction that \( t'_\ell < s_{\ell+1} \).

Since \( G^t \) satisfies \((3')\) for \( t_\ell \leq t < t'_\ell \), \( B \) can indeed use the algorithm up to and including the point where the game reaches \( G^{t'_\ell} \), and further, \( t'_\ell \) is the smallest value such that \( G^{t'_\ell} \) is in Phase \( \ell \) and doesn’t satisfy \((3')\). By Lemma 4.6 we have \( g^{t'_\ell} - g^t \leq r_1 \). Since \( g^{t'_\ell} < c_{-1}n \) and \( g^t > -c_{-1}n \), \( g^{t'_\ell} - g^t \leq c_{-1}n \) by (1). We can assume that \( r_1 \leq 2c_{-1}n \).

We claim that \( r_2 > r := |4ckn + 9(k - \ell - 1)c_{-1}n + 2c_{-1}n| \). Indeed assume otherwise. Note that we have \( |D^{t'_\ell}_1| \geq 4ckn + 9(k - \ell)c_{-1}n \) for each \( 2 \leq \ell < k \), with the \( \ell = 2 \) case coming from (3). By this and Lemma 4.6, for \( n \) sufficiently large,

\[
|D^{t'_\ell}_1| \geq |D^{t'_\ell}_1| - 3r_1 - r_2 \\
\geq 4ckn + 9(k - \ell)c_{-1}n - 6c_{-1}n - 4ckn - 9(k - \ell - 1)c_{-1}n - 2c_{-1}n = c_{-1}n \geq 3.
\]
Similarly, \(i^t \geq 8(k - \ell)c_k n + \sum_{j=\ell}^{k} 27(k - j)c_{-1}n\) for each \(2 \leq \ell < k\) with the \(\ell = 2\) case coming from (4). Thus

\[
i^{t'} \geq 8(k - \ell)c_k n + \sum_{j=\ell}^{k} 27(k - j)c_{-1}n - 4c_{-1}n - 8c_k n - 18(k - \ell)c_{-1}n - 4c_{-1}n \geq
\]

\[
8(k - \ell - 1)c_k n + \sum_{j=\ell+1}^{k} 27(k - j)c_{-1}n + c_{-1}n \geq
\]

\[
8(k - \ell - 1)c_k n + \sum_{j=\ell+1}^{k} 27(k - j)c_{-1}n + 3,
\]

It follows that (3') holds for \(G_t^t\), a contradiction to how \(t'_t\) is defined so we conclude that \(r_2 > r\).

Let \(t''_t = t_t + 2(r_1 + r + 1)\), noting that this is at most \(t'_t\). Repeating our above computations gives

\[
i^{t''} \geq 8(k - \ell - 1)c_k n + \sum_{j=\ell+1}^{k} 27(k - j)c_{-1}n,\]

and we also have by Lemma 4.6

\[
|D_{t'_{\ell+1}}^t| \geq r + 1 - r_1 \geq 4c_k n + 9(k - \ell - 1)c_{-1}n.
\]

We conclude that \(G_t^t\) is either in Phase \(-1\) or Phase \(\ell + 1\), and hence \(s_{t+1} \leq t''_t \leq t'_t\), a contradiction to \(s_{t+1} > t'_t\). We conclude that \(s_{t+1} = t'_t\).

The analysis for \(\ell \geq 3\) odd is essentially the same as above after switching the roles of \(AU\) and \(AV\). The analysis is almost the same for \(\ell = 1\), except we redefine \(r = [c_2 n + 2c_{-1} n]\) and use that \(|D_{t_1}^t| \geq c_2 n + 9c_{-1}n\) and \(i^t \geq c_1 n\). \(\square\)

**Proof of Theorem 4.3.** \(G^0\) satisfies ('). If \(G^{t-2}\) satisfies (') and B uses the algorithm described in Lemma 4.4, then Lemma 4.4 shows that \(G^t\) satisfies (1') and (2'), and Lemma 4.7 and Lemma 4.8 shows that it satisfies (3') as well. \(\square\)

**Corollary 4.9.** For \(n\) sufficiently large there exists a strategy for B in the \(C_{2k-1}\)-saturation game for \(k \geq 5\) such that \(G^t\) is in Phase \(k\) or Phase \(-1\) for some sufficiently large \(t\).

**Proof.** For some large \(t'\), \(i^t < 3\). By Theorem 4.3 we know that B can play so that \(i^t \geq 3\) whenever \(G^t\) is in Phase \(\ell\) for \(0 \leq \ell < k\). Thus if B uses this strategy, \(G^t\) must be in Phase \(k\) or Phase \(-1\). \(\square\)

### 4.4. Endgame

**Theorem 4.10.** There exists a strategy for B in the \(C_{2k-1}\)-saturation game for \(k \geq 5\) such that if the game eventually reaches Phase \(k\) or Phase \(-1\), then \(G^\infty\) contains at most \((\frac{1}{2} - c_{-1}^2)n^2 + o(n^2)\) edges.

We first describe B’s strategy for Phase \(k\), recalling that once the game enters this Phase it never leaves it (and similarly for Phase \(-1\)). In Phase \(k\), B connects two vertices of \(D_{k-1}^t\) as long as \(|D_{k-1}^t| \geq 2\), which is a legal move by Lemma 4.2. If \(|D_{k-1}^t| \leq 1\, B\) plays arbitrarily. Note that \(\Delta(|D_{k}^t|) \geq -4\) by Lemma 4.1. Since \(|D_{k}^t| \geq 4c_k n\), B is able to create at least \(c_k n - 1\) \(C_{2k+1}\)‘s which all share a common vertex \(v\) and with no other vertices shared between the cycles. We will call such a structure a \((c_k n - 1)\)-bouquet.

**Lemma 4.11.** Let \(G\) be a graph that contains no odd cycles smaller than \(C_{2k+1}\) with \(k \geq 2\). If \(C\) is a \(C_{2k+1}\) in \(G\), then no vertex of \(G\) has more than two neighbors in \(C\).
Proof. Let \( v \) be a vertex with neighbors \( v_1, v_2, v_3 \in C \). Let \( d_C(x, y) \) denote the length of the shortest path from \( x \) to \( y \) using only edges of \( C \). First assume \( v \in C \), and without loss of generality that \( d_C(v, v_1) = d_C(v, v_2) = 1 \). Thus \( d_C(v, v_3) = \ell \) with \( 2 \leq \ell \leq k \). In this case \( G \) contains cycles of length \( \ell + 1 \) and \( 2k + 1 - \ell + 1 \), one of which must be an odd number that is at most \( 2k - 1 \) since \( k \geq 1 \), a contradiction.

Now assume \( v \notin C \). Then \( G \) contains cycles of length \( 2 + d_C(v_i, v_j) \) and \( 2k + 1 - d_C(v_i, v_j) + 2 \) for each \( i \neq j \). The only way these values can both be either even or at least \( 2k + 1 \) is if \( d_C(v_i, v_j) = 2 \) for all \( i \neq j \). This is impossible since \( C \) is not a \( c_k \).

Lemma 4.12. Let \( G \) be an \( n \)-vertex graph that contains no odd cycles smaller than \( C_{2k+1} \) with \( k > 1 \). If \( G \) contains a \( (c_k n - 1) \)-bouquet, then \( G \) contains at most \( (\frac{1}{4} + (2 - k)c_k^2 + k^2 c_k^2) n^2 + o(n^2) \) edges.

Note that in the statement and proof of this lemma we implicitly assume that \( 2k(c_k n - 1) + 1 \leq n \), which certainly holds for our choice of \( c_k \).

Proof. For simplicity assume that \( G \) contains a \( (c_k n, C_{2k+1}) \)-bouquet and that \( c_k n \) is an integer (this won’t affect the asymptotics). The number of edges involving some vertex of a given cycle of the bouquet is at most \( 2n \) by Lemma 4.11, so the number of edges involving some vertex of the bouquet is at most \( 2c_k n^2 \). The number of edges involving vertices that are not part of the bouquet is at most

\[
\frac{1}{4}(n - 2kc_k n - 1)^2 = \left(\frac{1}{4} - k^2 c_k^2\right) n^2 + o(n^2).
\]

Adding these two values gives the desired result. \( \square \)

We now describe B’s strategy when the game reaches Phase \( \ell - 1 \) from Phase \( \ell < k \). B will play arbitrarily if \( G^{\ell-1} \) is connected. Otherwise B plays the same strategy he did for Phase \( \ell \), except instead of having \( v' \in D^{\ell-1}_t \) we take \( v' = v \) if \( t \) is even and \( v' = u \) if \( t \) is odd. If one goes back through the algorithm one can verify that with this strategy, for all even \( t \geq t-1 \), \( G^t \) satisfies (1’), (2’) for \( \ell \), and that \( ||U^t| - |V^t|| \geq c-1 - n - 1 \).

Let \( D^t_\ell \) denote the set of vertices that were in \( D^t_\ell \) for some \( t \). Let \( U' = \{ u' \in U : \exists u'' \in E(G^\infty) \} \) and \( V' = \{ v' \in V : \exists v'' \in E(G^\infty) \} \).

Lemma 4.13. No vertex \( w \in U' \cup V' \) has \( d^\infty_a(w, v) < k - 1 \). No vertex of \( U' \) is adjacent to any vertex of \( D^t_2 \), and no vertex of \( V' \) is adjacent to any vertex of \( D^t_1 \).

Proof. Let \( u', u'' \in U' \) be such that \( u'u'' \) is an edge in \( G^\infty \). If \( d^\infty_a(u', v) < k - 1 \), then \( G \) contains an odd cycle of length \( d^\infty_a(u'', v) + d^\infty_a(v, u') + d(u', u'') < 2k + 1 \) since \( d^\infty_a(u'', v) \leq k + 1 \) by (2’), a contradiction.

Now assume that \( u'v' \) is an edge in \( G^\infty \) for some \( v' \in D^t_2 \). Then \( G^\infty \) contains an odd cycle of length \( d^\infty_a(u'', v) + d(v, v') + d(v', v'') + d(u'', u'') \leq k + 5 \) by (2’). Since \( k + 5 < 2k + 1 \) by assumption of \( k \geq 5 \), this is a forbidden odd cycle, a contradiction. The proof for \( V' \) is analogous. \( \square \)

Proof of Theorem 4.10. If B uses the above algorithm and the game is in Phase \( k \), then \( G^\infty \) will contain a \( (c_k - 1) \)-bouquet, and hence it will contain at most \( (\frac{1}{4} c_k - c_k^2) n^2 + o(n^2) \) edges by Lemma 4.12 and (5).

Now assume that B uses the above algorithm and the game is in Phase \( -1 \). Further assume that \( G^\infty \) is not bipartite, or equivalently that \( U' \cup V' \) is non-empty. Since \( d^\infty_a(w, v) \geq k - 1 \geq 4 \) for any \( w \in U' \cup V' \) by Lemma 4.13, \( U' \cup V' \) will be empty unless the game enter Phase \( -1 \) from Phase \( \ell \geq 2 \).
since B maintains that \( d^\infty(x,y) \leq \ell + 2 \). In particular, at some point the game reaches Phase 2 with \(|D^2| \geq c_2n\). By Lemma 4.13, (1'), and (2') for \( \ell = 1 \), every vertex of \( U' \cup V' \) must have been isolated during all of Phase 1 in order to have \( d^\infty(w,v) \geq 4 \). Hence \( s := |U' \cup V'| \leq c_1n + 1 - c_2n \) (Phase 1 starts with at most \( c_1n + 1 \) isolated vertices, and at least \( c_2n \) vertices are in \( D^2_1 \)).

Let \( G' \) be the complete bipartite graph with bipartition \( U^\infty \cup V^\infty \) (which will be \( G^\infty \) if \( s = 0 \)). The only edges of \( G^\infty \) that aren’t in \( G' \) are those contained in \( U' \cup V' \), and there are at most \( \frac{1}{4}s^2 + 1 \) such edges by Mantel’s theorem. However, \( G' \) contains all of the edges from \( D^2_2 \) to \( U' \) and \( D^1_1 \) to \( V' \), and none of these edges are in \( G^\infty \) by Lemma 4.13. There are at least \( |D^2_2||U'| + |D^1_1||V'| \geq c_2ns \) edges of this type, so in total \( G' \) contains at least \( c_2ns - \frac{1}{4}s^2 - 1 \) more edges than \( G^\infty \) does. This number is non-negative if \( s \neq 0 \) by (6). It’s thus enough to give an upper bound for the number of edges of \( G' \), which is exactly \( |U^\infty||V^\infty| \). Since \( |U^\infty| - |V^\infty| \geq c_1n - 1 \), we have

\[
|U^\infty||V^\infty| \leq ([n/2] - c_{-1}n + 1)([n/2] - 1 + c_{-1}n - 1) = \left( 1 - c_{-1}^2 \right) n^2 + o(n^2).
\]

\( \square \)

**Proof of the upper bound of Theorem 1.1.** Corollary 4.9 and Theorem 4.10 show that for \( k \geq 4 \), B has a strategy for the \( C_{2k+1} \)-saturation game guaranteeing \( G^\infty \) contains at most \( \left( \frac{1}{4} - c^{-2}_{-1} \right)n^2 + o(n^2) \) edges, where \( c_{-1} = (1000(k + 1)^{-2} \) (recalling that Theorem 4.3, Theorem 4.10, and \( c_{-1} \) were defined in this section in terms of the \( C_{2k-1} \)-saturation game). \( \square \)

**5. Proof of Theorem 1.3**

**Lemma 5.1.** Let \( k \geq 2 \) and \( \ell = \max(3, \lfloor \sqrt{2k} \rfloor) \). There exists a constant \( t_0 \) such that, for \( n \) sufficiently large, B can play in the \( \{C_{2k+1}\} \)-saturation game such that \( G^{t_0} \) contains a clique on the vertex set \( U = \{u_1, \ldots, u_\ell\} \), and such that there exists \( \ell \) vertex disjoint paths of length \( k - 2 \), each with a distinct \( u_i \) as its endpoint.

**Proof.** Implicitly we assume throughout the proof that \( G^{t-1} \) contains an isolated vertex, which will certainly be true if \( t \) is bounded by a constant and \( n \) is sufficiently large. First assume \( \ell = \lfloor \sqrt{2k} \rfloor \geq 3 \). We claim that any choice of \( e^t \) for \( t \leq 2 \lfloor \ell/2 \rfloor \) is a legal move. Indeed, for any \( t \leq 2 \lfloor \ell/2 \rfloor - 1 \), \( G^t \) will contain at most \( 2 \lfloor \ell/2 \rfloor - 1 \leq \ell^2 - 1 \leq 2k - 1 \) edges, and hence any choice of \( e^{t+1} \) won’t create a \( C_{2k+1} \) in \( G^{t+1} \). With this in mind, B spends his first \( \lfloor \ell/2 \rfloor \) moves creating a \( K_{\ell} \) on \( U = \{u_1, \ldots, u_\ell\} \).

For his next move after forming this \( K_{\ell} \), B adds the edge \( e^t = u_1x_1 \) where \( x_1 \) is an isolated vertex of \( G^{t-1} \). For his \( i \)th move after adding the edge \( u_1x_1 \), B adds the edge \( e^t = x_ix_{i+1} \) with \( x_{i+1} \) an isolated vertex of \( G^{t-1} \). B continues this up to \( i = k - 3 \), creating a path of length \( k - 2 \) with \( u_1 \) as an endpoint. The next edge B adds is \( e^t = u_2y_1 \) with \( y_1 \) an isolated vertex of \( G^{k-1} \). As before B will extend this path until its length reaches \( k - 2 \). B repeats this process for each vertex of \( U \), giving the desired subgraph by time \( t_0 = 2 \lfloor \ell/2 \rfloor + 2(\ell(k - 2)) \).

If instead \( \ell = 3 \), then it’s not too difficult to see that B can create a \( K_3 \) in his first 3 moves for any \( k \geq 2 \). After this he does essentially the same strategy as before and creates the desired subgraph by time \( t_0 = 6 + 6(k - 2) \). \( \square \)

**Proof of Theorem 1.3.** B first uses the strategy in Lemma 5.1, making sure that \( G^{t_0} \) contains a clique on \( U = \{u_1, \ldots, u_\ell\} \) and vertex disjoint paths \( \{p_1, \ldots, p_\ell\} \), each of length \( k - 2 \) with \( p_i \) starting at \( u_i \) and ending at, say, \( v_i \). Let \( V = \{v_1, \ldots, v_\ell\} \), and let \( v^i \) denote a \( v_i \) with minimal degree in \( G^t \). Let \( i^t \) denote the number of isolated vertices of \( G^t \).
For all even $t > t_0$, B uses the following strategy. If $i^t < 2$, B plays arbitrarily. Otherwise if R plays $xy$ with $x, y$ isolated vertices of $G^{t-1}$, B plays $xv$ (a legal move since this doesn’t create a cycle). Otherwise B plays $xv$ with $x$ an isolated vertex of $G^{t-1}$.

We wish to bound the number of edges of $G^\infty$ when B uses this strategy. To this end, let $P$ denote the vertices that belong to some $p_i$ (including $u_i$ and $v_i$), let $V_i = N^\infty(v_i)$, and let $W = V(G^\infty) \setminus (P \cup V_i)$. Let $p'_i$ denote $p_i$ but treated as a path from $v_i$ to $u_i$. Lastly, for $X, Y \subseteq V(G^\infty)$, let $e(X, Y)$ denote the number of edges in $G^\infty$ where one vertex lies in $X$ and the other in $Y$.

Claim 5.2. $e(P, V(G^\infty)) \leq \ell(k - 1)n = o(n^2)$.

Proof. This is immediately from the fact that $|P| = \ell(k - 1)$. □

Claim 5.3. $e(V_i, V_i) \leq \frac{2k-1}{2}n = o(n^2)$.

Proof. If this were not the case, then by the Erdős-Gallai Theorem there would exist a path of length $2k$ in $V_i$. Since $v_i$ is adjacent to the two endpoints of this path, this would imply that $G^\infty$ contains a $C_{2k+1}$, a contradiction. □

Claim 5.4. $e(V_i, V_j) = 0$ for $i \neq j$.

Proof. Assume $G^\infty$ contained the edge $w_i w_j$ with $w_i \in V_i$, $w_j \in V_j$. Then for any $r \neq i, j$ (and such an $r$ exists since $\ell \geq 3$), $G^\infty$ would contain the cycle $w_i p'_i u_r p_j w_j$, but this is a $C_{2k+1}$, a contradiction. □

Claim 5.5. For any $w \in W$, $e(\{w\}, V_i) \neq 0$ for at most one $i$.

Proof. Assume $G^\infty$ contained the edges $ww_i$ and $ww_j$ with $w_i \in V_i$, $w_j \in V_j$, $i \neq j$. Then $G^\infty$ would contain the cycle $ww_i p'_i p_j w_j$, a $C_{2k+1}$, a contradiction. □

Claim 5.6. $e(W, \bigcup V_i) \leq |W| \max(|V_i|)$.

Proof. This is an immediate consequence of the previous claim. □

Claim 5.7. For all $i$, $|V_i| \geq \frac{n}{\ell} + O(1)$.

Proof. Note that $i^t \geq n - 2t_0$ and $\Delta(i^t) \geq -2$ for all even $t \geq t_0 + 2$ by the way the algorithm was constructed. It follows that there are at least $n/2 + O(1)$ values of $t$ with $i^t < 2$, and hence B adds an edge of the form $xv^t$ for at least this many values of $t$. Thus B ensures that each of the $\ell$ vertices $v_i$ have at least $\frac{n}{\ell} + O(1)$ neighbors in $G^\infty$. □

Claim 5.8. For all $i$, $|V_i| \leq (1 - \frac{\ell - 1}{2\ell})n - |W| + O(1)$.

Proof. This is an immediate consequence of the fact that $|V_i| = n - \sum_{j \neq i} |V_j| - |W| - |P|$ and the previous claim. □

Claim 5.9. $\max(|V_i|) \leq \frac{1}{2}(1 + \frac{1}{\ell})n - |W| + O(1)$.

Proof. This is an immediate consequence of the previous claim since $(1 - \frac{\ell - 1}{2\ell}) = \frac{1}{2}(1 + \frac{1}{\ell})$. □

We conclude that the number of edges in $G^\infty$ will be

$$e(W, W) + e(W, \bigcup V_i) + o(n^2) \leq \frac{1}{4}|W|^2 + |W| \left( \frac{1}{2}(1 + \frac{1}{\ell})n - |W| \right) + o(n^2).$$
This value is maximized when \( |W| = \frac{1}{8}(1 + \frac{1}{2})n \), giving an upper bound of

\[
\frac{1}{12} \left( 1 + \frac{1}{2} \right)^2 n^2 + o(n^2)
\]
as desired. □

6. Proof of Theorem 1.5

We will say that a vertex \( v \) is good if all but at most one edge incident to \( v \) is contained in a triangle. We will say that a graph \( G \) is \( k \)-good if there exists a set of edges \( B(G) \) with \( |B(G)| \leq k \) such that every vertex of \( G \setminus B(G) \) is good. Observe that if \( G \) is \( k \)-good and \( G' \) is \( G \) plus an edge, then \( G' \) is \((k + 1)\)-good.

**Theorem 6.1.** There exists a strategy for \( B \) in the \((C_\infty \setminus C_3)\)-saturation game such that for all even \( t \), either \( G^{t-1} \) is \((C_\infty \setminus C_3)\)-saturated or \( G^t \) is \( 1 \)-good.

**Lemma 6.2.** Let \( G \) be a 2-good graph that contains no cycle \( C_{2k+1} \) for any \( k \geq 2 \). Then \( G \) contains no \( C_t \) for any \( t \geq 5 \).

**Proof.** Let \( C \) denote a \( C_{2k} \) in \( G \) with \( k \geq 3 \) on the vertex set \( \{v_1, \ldots, v_{2k}\} \), and let \( C' = C \setminus B(G) \). Since \( k \geq 3 \), there exists an \( i \) such that \( C' \) contains the edges \( v_{i-1}v_i \) and \( v_iv_{i+1} \). Since these edges are in \( G \setminus B(G) \), at least one of these edges is in a triangle, say \( v_iv_{i+1}w \) is a triangle in \( G \). If \( w \) is not in \( C \), then \( v_1v_2 \cdots v_iwv_{i+1} \cdots v_{2k} \) is a \( C_{2k+1} \) in \( G \), a contradiction. Thus \( w = v_j \) for some \( j \neq i, i+1 \).

Note that \( v_j \neq i+2, i+3 \). Indeed if, say, \( j = i+2 \), then \( v_1v_2 \cdots v_iv_{i+2} \cdots v_{2k} \) would be a \( C_{2k-1} \) in \( G \), a contradiction. A similar result holds if \( j = i+3 \). Observe that \( G \) contains the cycles \( v_iv_{i+1} \cdots v_j \) and \( v_{i+1}v_{i+2} \cdots v_j \). One of these cycles must have odd parity with length at least 5 since \( j \neq i+2, i+3 \), a contradiction. We conclude that \( G \) contains no \( C_{2k} \) with \( k \geq 3 \), proving the result. □

**Proof of Theorem 6.1.** \( G^0 \) is 1-good, so inductively assume that \( B \) has been able to play so that \( G^{t-2} \) is 1-good. If \( G^{t-1} \) is saturated then the game is over and \( B \) doesn’t play anything, so assume this is not the case. If \( G^{t-1} \) is 0-good, then \( B \) plays \( e^t \) arbitrarily and \( G^t \) will be 1-good.

Now assume that \( G^{t-1} \) isn’t 0-good. That is, there exists edges \( v_1x \) and \( v_2x \) with \( v_1 \neq v_2 \) such that neither of these edges are contained in triangles. We claim that adding \( e^t = v_1v_2 \) is a legal move. If it were not, then there must exist a path \( P \) of length \( 2k \) with \( k \geq 2 \) from \( v_1 \) to \( v_2 \) in \( G^{t-1} \). If \( x \) is not a vertex of \( P \), then \( G^{t-1} \) contains the cycle formed by taking \( P \) and adding the edges \( xv_1 \) and \( xv_2 \), which is a \( C_{2k+2} \). Since inductively \( G^{t-2} \) is 1-good, \( G^{t-1} \) is 2-good, and hence doesn’t contain such a \( C_{2k+2} \) by Lemma 6.2. Thus \( x \) must be a vertex of \( P \). Let \( P_t \) denote the path from \( v_1 \) to \( x \) in \( P \), and let \( k_i \) denote the length of \( P_i \).

\( G^{t-1} \) contains a \( C_{k_t+1} \), namely by taking \( P_t \) together with the edge \( xv_1 \). Thus \( k_t \leq 3 \) by Lemma 6.2. Also \( k_t \neq 2 \), since this would contradict \( xv_1 \) not being contained in a triangle. Since \( k_1 + k_2 = 2k \geq 4 \), we must have, say, \( k_1 = 3 \). Let \( C = v_1abx \) be the 4-cycle formed from \( P_1 \) and \( xv_1 \). If, say, \( ab \) were contained in a triangle \( abc \), then we must have \( c = v_1 \) or \( c = x \), as otherwise \( v_1acbx \) defines a \( C_5 \) in \( G^{t-1} \). But if \( c = v_1 \) or \( x \), then \( v_1x \) is contained in a triangle, a contradiction. A similar analysis shows that no edge of \( C \) is contained in a triangle. This is only possible if \( B(G^{t-1}) \) consists of two edges of \( C \) that aren’t both incident to \( x \), as otherwise one of \( ab \) and \( v_1a \) would be contained in a triangle. In particular, two of the edges \( \{xv_1, xv_2, xb\} \) are not in \( B(G^{t-1}) \), and we conclude that at least one of these edges must be contained in a triangle. But we’ve assumed that none of these edges are in triangles, a contradiction. We conclude that \( v_1v_2 \) is a legal move to play.
Note that at least one of the edges $xv_1$ and $xv_2$ must be in $B(G^{t-1})$, as otherwise $G^{t-1} \setminus B(G^{t-1})$ wouldn’t have all good vertices (namely, $x$ wouldn’t be a good vertex). Since $v_1x$, $v_2x$, and the new edge $v_1v_2$ are contained in a triangle of $G^t$, the set $B(G^t) := B(G^{t-1}) \setminus \{v_1x, v_2x\}$ shows that $G^t$ is 1-good as desired.

\[\square\]

**Proof of Theorem 1.5.** If B uses the strategy of Theorem 6.1, then $G^\infty$ will be 2-good. Lemma 6.2 shows, in particular, that $G^\infty$ can’t contain two even cycles of the form $C_{2r}$ and $C_{2r+2}$ for any $r$. The result then follows from Theorem 12 of [10] after taking $k = 2$.

\[\square\]

7. Concluding Remarks

- Our analysis was far from sharp in various places, and one could certainly improve the bounds of Theorem 1.1 with a more careful analysis.

- We do not have any strong upper bounds for $\text{sat}_g(C_5; n)$ or $\text{sat}_g(C_7; n)$, though we can prove that both are strictly less than $\frac{1}{4}n^2$. Specifically, we can prove the following bound for $k \geq 2$:

  \[
  \text{sat}_g(C_{2k+1}; n) \leq \frac{1}{4}n^2 - \left(k - \frac{1}{2}\right)n + O(1).
  \]

In order to get this bound, B uses the algorithm in Section 4 with $c_k$ and $c_{-1}$ scaled down by a factor of 25. All of the analysis we did before holds if $G^t$ is bipartite for all $t$, so we can assume that $G^\infty$ contains an odd cycle. In this case, the number of edges involving a vertex from the smallest odd cycle of $G$ will be at most $2n$ by Lemma 4.11, and there will be at most $\frac{1}{7}(n - 2k - 3)^2$ other edges in $G$. Adding these two values gives the desired bound.

We note that this bound is non-trivial. In particular, the algorithm of Theorem 2.1 shows that for $k \geq 2$, B has no strategy that guarantees the creation of any odd cycles in the $C_{2k+1}$-saturation game.

- Theorem 1.1 shows that for all $k \geq 4$, $\text{sat}_g(C_{2k+1}) \leq (\frac{1}{4} - c_k)n^2 + o(n^2)$ for some $c_k > 0$. We suspect that such a bound also holds for $\text{sat}_g(C_5; n)$ and $\text{sat}_g(C_7; n)$.

- As a consequence of the bounds of Theorem 1.1, we know that $\text{sat}_g(C_{2k+1}; n) \leq \text{sat}_g(C_{2k'+1}; n)$ when $k'$ is sufficiently larger than $k$ and $n$ is sufficiently large. We conjecture that this remains true when $k' = k + 1$.

**Conjecture 7.1.** For all $k \geq 2$,

\[
\text{sat}_g(C_{2k-1}; n) \leq \text{sat}_g(C_{2k+1}; n)
\]

for $n$ sufficiently large.

A positive answer to this conjecture would in particular provide a desirable upper bound for $\text{sat}_g(C_5; n)$ and $\text{sat}_g(C_7; n)$. Note that the bound $\text{sat}_g(C_5; n) \leq \frac{26}{121}n^2 + o(n^2)$ of [1] together with Theorem 1.2 shows that the conjecture is true for $k = 2$, and moreover that $\text{sat}_g(C_5; n) \leq \text{sat}_g(C_{2k+1}; n)$ for all $k \geq 2$ and $n$ sufficiently large.

- One can verify that the proof of Theorem 1.3 generalizes to bounding $\text{sat}_g(F; n)$ where $F$ is any set of odd cycles whose smallest cycle is a $C_{2k+1}$ with $k \geq 2$. Similarly the lower bounds of Theorem 1.1 and Theorem 1.2 generalize to bounding $\text{sat}_g(F; n)$ where $F$ is any set of odd cycles containing $C_{2k+1}$. This shows that $\text{sat}_g(C_\infty \setminus C_{2k+1}; n)$ is quadratic for any $k \geq 3$ since the set contains $C_{2k-1}$. Theorem 1.5 shows that this value is linear when $k = 1$, and it would be of interest to know whether $\text{sat}_g(C_\infty \setminus C_5; n)$ is quadratic or sub-quadratic.

- Further investigations of the $\{C_{2k+1}\}$-saturation games for $k \geq 2$ could be of interest. For example, is $\text{sat}_g(C_5; n) = o(n^2)$?
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