Analysis of lateral behaviour of monopiles considering principal stress rotation under coupled loading in sand

L L Mu¹,², T Zhou¹,², W Li¹,²

¹ Department of Geotechnical Engineering, Tongji University, Shanghai 200092, China
² Key Laboratory of Geotechnical and Underground Engineering of Ministry of Education, Tongji University, Shanghai 200092, China

Abstract. Monopiles for offshore wind turbine foundations are often subjected to vertical-horizontal combined loading. For the analysis of piles subjected to horizontal loads in sand, the p-γ curve method based on the Winkler model is usually employed, where p is the soil resistance of the pile per unit length and γ is the local horizontal displacement of the pile or the soil compression at the point of study. As the load bearing properties of horizontally loaded piles depend on the ultimate resistance Pₑ of the soil around the pile in the shallow soil layer, it is important to determine Pₑ of the shallow soil accurately. In this paper, the wedge model is employed to calculate Pₑ. The effect of vertical shear stress along the pile on the principal stress direction which decides the bottom angle of the wedge is taken into account. Then, the effects of the vertical loads on the p-γ curves, P-Δ effect, M₀ effect is considered in the horizontal equilibrium equation of the pile. The shear-displacement method is employed to calculated the vertical responses of the pile. The shear stress and horizontal soil resistance were redistributed when the coupled effect of the vertical load and the horizontal load is considered. Then, the horizontal responses of the monopile under vertical load, horizontal load and moment simultaneously can be calculated. Finally, the proposed method is used to calculate a centrifuge model test where a piles is subjected to coupled loads in sandy soils. The results showed that the pile moment curve after considering the bottom angle of the wedge β modification is closer to the experimental results than without the β modification, so it makes the modified β more reasonable in analyzing the behavior of laterally loaded single piles in sand. At the same time, the β modification is not required for pure horizontal loading.

1. Introduction
Pile foundations are widely used in geotechnical engineering, which not only can use to support large vertical loads, but also the horizontal loads and the combined loads of them. In the past simplified analysis approach on the characteristics of large diameter pile under simultaneous axial and lateral loading does not take the coupling effect of the vertical-horizontal loads into consideration, based on the assumption that the impact of axial and lateral loads is independent of each other [1]. However, offshore wind turbines are now widely developed, which often adopt large diameter pile as foundations to bear high magnitude horizontal loads and vertical loads, thus making the behavior of the pile under combined loads is substantially different compared to the superposition of pile responses under pure independent loads due to the interaction between these loads [2,3]. Scholar investigated the behavior of the monopiles under combined loads finding that the impact of a vertical
load on the lateral response of monopiles can be divided into three aspects [4-7]: the ultimate horizontal soil resistance $p_u$ around the pile of $p-y$ curve caused by the vertical loads, $P-\Delta$ effect and the resisting bending moment $M_r$ generated by the uneven distribution of vertical skin friction.

As shown in Figure 1, the $p-y$ curve method is essentially based on the Winkler model of an elastic foundation beam, which can be used for analyzing pile’s behavior under lateral loads, where $p$ is the soil resistance per unit length of pile and $y$ is the lateral displacement of the pile at the research point or the soil compression at this point. The $p-y$ curve method firstly was proposed by Mcclelland [8]. Reese [9] raised the $p-y$ curves expression firstly in sand by conducting the lateral load experiment on piles in 1974. The $p-y$ curve method is semi-empirical, which can better reflect the properties of soil spring. Since using $p-y$ curve method to analyze the behavior of pile under lateral load is simple and accurate, it has been compiled into the commercial software LPILE V2019 and is widely used in geotechnical engineering (such as API [10], FHWA etc).

![Figure 1. p-y curve of pile under lateral loads.](image1)

Kim [11] summarized the $p-y$ curve model and found that the shape of $p-y$ curve is mainly related to the initial reaction modulus $k_{im}$ and the $p_u$. Figure 2 shows that the soil around a pile may occur three failure modes, namely the wedge failure of shallow soil in front of the pile, the circumference failure of deep soil and the tensile failure of shallow soil behind the pile, when the pile subjected to large magnitude lateral loads. It is crucial to accurately determine the $p_u$ of the shallow soil, which the bearing capacity of horizontally loaded piles depend largely on.

Broms [12] obtained an empirical formula to calculate the $p_u$ based on horizontal loading pile field tests. Reese et al [9] and Matlock [13] proposed a semi-empirical and semi-theoretical formula to calculate the $p_u$ based on the formula derivation of theoretical failure model. The semi-empirical and semi-theoretical formula for $p_u$ was also proposed by Barton [14] based on the results of centrifuge experiments. As shown in Figure 3, the shallow passive soil in front of the pile could be simplified as a 3D wedge to calculate the $p_u$. Norris [15] first proposed the strain wedge model (SW model) in 1986 for predicting the response of flexible long piles subjected to lateral loads in homogeneous sand. The SW model for solving horizontal pile foundations had continuously been improved by Ashour et al [16], who extended the SW model to layered soils.

![Figure 2. Failure pattern of soil.](image2)

![Figure 3. Definition of failure zone](image3)
As shown in Figure 4, a 3D wedge to calculate the $p_u$ is described by the bottom angle $\beta$, the sector angle $\alpha$ and the depth $x$. The wedge bottom angle $\beta$ is a very important factor in determining $p_u$. Under pure lateral loading, the $\beta$ is generally taken $45^\circ + \phi/2$, where $\phi$ is the internal friction angle of sand; While under the vertical-horizontal loading, the principal stress direction will be rotated and no longer along the horizontal and vertical directions, which lead to the $\beta$ will not be $45^\circ + \phi/2$. Because the principal stress direction determines the $\beta$, it’s necessary to consider the influence of vertical shear stress along the pile on the principal stress rotation to correct the $\beta$. Li [17] corrected the $\beta$ in undrained saturated clay. This paper is carried out to shed further light on the wedge base angle $\beta$ in sand.

In this paper, the effect of vertical shear stress along the pile on the principal stress direction is taken into account. Then, the effects of the vertical loads on the $p-y$ curves, $P-\Delta$ effect and $M_r$ effect are considered in the horizontal equilibrium equation of the pile. The shear-displacement method is employed to calculated the vertical responses of the pile. Then, the horizontal responses of the monopile under vertical load, horizontal load and moment simultaneously can be calculated. Finally, the proposed method is used to calculate a centrifuge model test where a pile is subjected to coupled loads in sand.

2. Theoretical analysis

2.1. $p-y$ curve modification in sand

2.1.1. Calculation of $p_u$ around the pile

In this paper, based on the utilization of three-dimensional wedges method to calculate $p_u$, the friction effects between pile and sand is further considered and the formula to calculate $p_u$ is improved. The mechanical analysis of 3D soil wedge is shown in Figure 4 and Table 1.

![Figure 4. Schematic diagram of wedge failure.](image)

| Force                        | Expressions                                                                 |
|------------------------------|----------------------------------------------------------------------------|
| Self-weight of wedge $W$     | $W = \gamma \left( \frac{1}{3} x^3 \tan^2 \beta \tan \alpha + \frac{1}{2} x^2 D \tan \beta \right)$ |
| Lateral normal force of wedge $F_n$ | $F_n = \frac{\gamma x^3 K_0 \tan \beta}{6 \cos \alpha}$                  |
Lateral friction of wedge $F_s$
$$F_s = \frac{\gamma x^3 K_o \tan \beta \tan \varphi}{6 \cos \alpha}$$

Bottom force of wedge $F_b$
$$F_b = \frac{W + 2F_s \cos \beta + F_t}{\sin(\beta - \varphi)}$$

Pile-soil contact friction $F_t$
$$F_t = \frac{\pi D \tau x}{2}$$

Passive earth pressure $F_p$
$$F_p = 2F_s \cos \alpha \sin \beta + F_s \cos(\beta - \varphi) - 2F_a \sin \alpha$$

Active earth pressure $F_a$
$$F_a = K_a \frac{\gamma x^2 D}{2}$$

From the equilibrium of horizontal forces, the total resistance $F_{hr}$ is given by:
$$F_{hr} = F_p - F_a$$  \hspace{1cm} (1)

By taking the derivative of $F_{hr}$, we can obtain the ultimate resistance of the shallow soil $P_{ue}$:
$$P_{ue} = K_0 \gamma x^3 \tan \beta \left[ \sin \beta \tan \varphi + \frac{\cos \beta \tan \varphi}{\cos \alpha \tan (\beta - \varphi)} - \tan \alpha \right] + \frac{\gamma x \tan \beta (D + x \tan \beta \tan \alpha)}{\tan (\beta - \varphi)} + \frac{\pi D \tau}{2 \tan (\beta - \varphi)} - \gamma K_a D x$$  \hspace{1cm} (2)

Where $K_0$ is coefficient of earth pressure at rest, take $K_0 = 1 - \sin \psi_1$; $K_a$ is coefficient of active earth pressure; $\varphi$ is the internal friction angle of sand; $\gamma$ is unit weight of sand; $x$ is the depth of wedge; $\beta$ is the bottom angle of wedge; $\alpha$ is the sector angle of wedge, take $\alpha = \varphi / 5$; $D$ is the pile diameter; $\tau$ is the vertical shear stress along the pile.

Therefore, the ultimate horizontal soil resistance $p_u$ per unit pile length in sand will be of the form:
$$\begin{bmatrix} p_u = (C_1 x + C_2 D) \gamma x + C_3 \\ p_{ue} = C_4 D y x \end{bmatrix}$$  \hspace{1cm} (3)

$$p_u = \min \left( p_{ue}, p_{uu} \right)$$  \hspace{1cm} (4)

where constant $C_1, C_2, C_3$ and $C_4$ can be expressed as:
$$C_1 = K_0 \tan \beta \left[ \sin \beta \tan \varphi + \frac{\cos \beta \tan \varphi}{\cos \alpha \tan (\beta - \varphi)} - \tan \alpha \right] + \tan^2 \beta \tan \alpha \tan (\beta - \varphi)$$
$$C_2 = \frac{\tan \beta}{\tan (\beta - \varphi)} - K_a$$
$$C_3 = \frac{\pi D \tau}{2 \tan (\beta - \varphi)}$$
$$C_4 = K_0 \tan \varphi \tan^4 \beta + K_a \left( \tan^8 \beta - 1 \right)$$

2.1.2. The wedge bottom angle $\beta$ modification

The wedge bottom angle $\beta$, which is the complementary angle between the failure plane and the horizontal plane, is a very important factor in determining the ultimate horizontal soil resistance $p_u$. Previous studies did not consider the rotation of the principal stress caused by vertical shear stress along the pile when taking the wedge bottom angle $\beta$.

Taking a horizontal differential soil layer for analysis, as shown in Figure 5. When the pile is under pure horizontal load $H$, there is no vertical shear stress along the pile and the direction of major and minor principal stresses in the soil is vertical and horizontal respectively. As the vertical load $V$ is
applied there is the possibility that the shear stress along the pile make the direction of principal stress rotate and thus change the $\beta$ value.

According to the Mohr-Coulomb failure criterion, the stress state of soil differential element can be represented in Figure 6. Noting that $|\sigma_1 b|=|\sigma_1 a|\sin\phi'$, we obtain:

$$
(\sigma_1' + \sigma_3') \sin \phi' + 2c' \cos \phi' = \sigma_1' - \sigma_3' 
$$

(5)

Furthermore, the main stress can be determined using plane stress principle as:

$$
\sigma_1' = \frac{\sigma_h + \sigma_v}{2} + \sqrt{\left(\frac{\sigma_h - \sigma_v}{2}\right)^2 + \tau_s^2} 
$$

(6)

$$
\sigma_3' = \frac{\sigma_h + \sigma_v}{2} - \sqrt{\left(\frac{\sigma_h - \sigma_v}{2}\right)^2 + \tau_s^2} 
$$

(7)

Substituting equations (6) and (7) into (5) and simplifying we obtain :

$$
(\sigma_h + \sigma_v) \sin \phi' + 2c' \cos \phi' = 2\sqrt{\left(\frac{\sigma_h - \sigma_v}{2}\right)^2 + \tau_s^2} 
$$

(8)

where $\sigma_h$ and $\sigma_v$ are the horizontal stress and vertical stress respectively; $\tau_s$ is the vertical shear stress along the pile; $c'$ and $\phi'$ are the effective cohesion and effective internal friction angle of soil respectively. For sand, taking $c'=0$ and $\phi' = \phi$, the above leads to the simple solution:

$$
\sigma_h = \sigma_v (1 + \sin^2 \phi) + 2\sqrt{\sigma_v^2 \sin^2 \phi - \tau_s^2 \cos^2 \phi} 
$$

(9)

From the Figure 6, we obtain the principal plane orientation where the principal stress $\sigma_1$ is located as:

$$
\alpha_0 = \frac{1}{2} \arctan \left( \frac{2\tau_s}{\sigma_h - \sigma_v} \right) 
$$

(10)
As shown in Figure 5, the maximum principal stress is rotated clockwise by $\alpha_0$ from the starting horizontal direction. Therefore, for sand, the bottom angle of wedge $\beta$ is modified as:

$$\beta = \frac{\pi}{4} + \frac{\phi}{2} + \alpha_0$$  \hspace{1cm} (11)

Kondner [18] proposed that $p-y$ curve could be fitted by the hyperbolic formulation (12) based on the results of soil stress-strain relationship under triaxial compression test. Therefore, in this paper, the hyperbolic linearity is still used in the $p-y$ curve in sand as follows:

$$p = \frac{y}{k_{ini} + \frac{y}{p_u}}$$  \hspace{1cm} (12)

Where $p_u$ is the ultimate horizontal soil resistance; $y$ is the lateral displacement of monopile; $k_{ini}$ is the initial modulus of subgrade reaction. For monopile in sand, Terzaghi [19] and Zhang [20] proposed that the initial modulus of subgrade reaction increases linearly with the depth of the soil, then:

$$k_{ini} = n_h \cdot z$$  \hspace{1cm} (13)

Where $n_h$ is the coefficient of subgrade reaction, as shown in Figure 7, which can be determined by factors such as the relative density $D_r$.

2.2. Analysis method of monopile under vertical loads

As shown in Figure 8, when the monopile in sand is subjected to vertical loading, the differential equation of pile can be obtained as:

$$E_p A_p \frac{d^2 w(z)}{dz^2} - U_p \tau(z) = 0$$  \hspace{1cm} (14)

Where $E_p$ is the elastic modulus of pile; $A_p$ is the cross-sectional area of pile; $U_p$ is the perimeter of pile; $w(z)$ is the vertical deformation of pile at depth $z$; $\tau(z)$ is the vertical shear stress at depth $z$.

According to the shear displacement method proposed by Cooke [21], the soil deformation around the pile can be ideally considered as a concentric cylinder. In the elastic analysis stage, the pile-soil contact is always tied without relative slip, so it can be obtained that the vertical displacement of pile...
is equal to the vertical soil displacement at the same depth. The differential equation of the monopile will be of the form:

\[ E_p A_p \frac{d^2 w(z)}{dz^2} - k_z w(z) = 0 \]  

(15)

Where \( k_z \) is the soil spring stiffness obtained by the shear displacement method.

The finite difference method is used to solve the differential equation (15) and the vertical displacement of pile can be obtained as:

\[ \{w\} = [K_z]^{-1} \{F_z\} \]  

(16)

where \( \{w\} \) is the vertical displacement vector of the pile nodes; \([K_z]\) is the vertical stiffness matrix of the pile; \(\{F_z\}\) is the vertical loading vector.

2.3. Analysis method of monopile under coupled loads
Taking a micro element of pile for analysis, as shown in Figure 9, we can obtain the fourth order differential equilibrium equation of single pile under the coupled loading considering the vertical shear stress along pile and soil resistance as follows:

\[ EI \frac{d^4 y}{dz^4} - V_y \frac{dy}{dz} + F_y \frac{d^2 y}{dz^2} + p(z) + \frac{dM_y}{dz} = 0 \]  

(17)

Figure 9. Analysis on micro element of pile

Figure 10. Schematic diagram of pile differential process

From the Figure 10, the pile is discretized in units along the pile based on the one-dimensional elastic foundation beam. Then, the differential equilibrium equation (17) is solved using the finite difference method to obtain the monopile lateral displacement under coupled loads as:

\[ \{y\} = [K]^{-1} \{F\} \]  

(18)

where \( \{y\} \) is the horizontal displacement vector of the pile nodes; \([K]\) is the horizontal stiffness matrix of the pile; \(\{F\}\) is the load column vector.

Flowchart of the MATLAB script language is shown in Figure 11.
3. Validation of the modified method
The monopile centrifugal model experiments under vertical and horizontal loads from the literature [22] were selected as an arithmetic example to validate the modified method. The parameters of the prototype experiment corresponding to the centrifugal model experiment were substituted into the MATLAB scripts for calculation. Then, the theoretical calculation results were compared with the experimental results. The following analytical results in this paper are based on the prototype dimensions. The sand and pile parameters are shown in Tables 2 and 3, respectively. The monopile loading conditions in the literature [22] are shown in Table 4. The vertical loads $V$ is applied firstly, and then the horizontal loads $H$ is applied to 1MN in 10 steps. The model set-up for coupled loading is shown in Figure 12. 1# experiment only consider the effect of horizontal loads. The vertical dead loads and horizontal loads were applied with pile in 2# and 3# experiments in order to simulate the large diameter monopile suffering self-weight load and also subjected to wind or wave loads.

| Table 2. Sand properties. |
|---------------------------|
| $\gamma$ (kN/m$^3$) | $D_r$ (%) | $E_s$ (MPa) | $\phi$ (°) | $\nu$ |
|---------------------------|
| 18 | 0.3 | 50 | 32 | 0.3 |

| Table 3. Pile parameters. |
|---------------------------|
| Diameter $D$ (m) | Wall thickness $t$ (m) | Embedded depth $L$ (m) | Flexural stiffness $E_{p/l}$ (GN m$^2$) | Young’s modulus $E_p$ (Gpa) |
|---------------------------|
| 1 | 0.05 | 16.5 | 3.43 | 200 |
Table 4. Testing program.

| Test no. | V (MN) | H (MN) |
|----------|--------|--------|
| 1#       | 0      | 1      |
| 2#       | 1.25   | 1      |
| 3#       | 2.50   | 1      |

Figure 13 shows the distribution of bending moments of the pile under different coupling loads. From Figure 13, it can be seen that the bending moment of pile is triangularly distributed along the pile, with a peak value at 4 times the pile diameter from the ground surface, which is consistent with the maximum bending moment occurring at a depth of 3-10 times the pile diameter in general.

When the horizontal loads $H$ is small, as shown in Figure 13 (a), $H=0.25$MN, and the bending moment decreases as the vertical loads $V$ increases; While when the horizontal load $H$ is large, as shown in Figure 13 (b), $H=1$MN, and the bending moment increases with the increase of the vertical loads. Therefore, there is a coupling effect of vertical load and horizontal load on the pile bending moment. From Figure 13, we can clearly know that the pile bending moment obtained from the theoretical calculation is slightly larger in the shallow soil layer and smaller in the deep soil layer.

As is shown in Figure 13, the pile moment curve after considering the modification of the wedge bottom angle $\beta$ is closer to the experimental results of the centrifugal model than the pile moment curve without, which verifies that it’s reasonable to consider the modification of $\beta$ under the vertical load. And when vertical load $V=0$, the wedge bottom angle does not need to be corrected.

![Figure 13. Pile bending moment under coupled loading](image-url)

4. Conclusion

In this paper, the effect of vertical shear stress along the pile under combined loads in sand on the principal stress direction is considered. The bottom angle of wedge $\beta$ is corrected. The modified wedge method is used to calculate the ultimate soil resistance $p_u$. Then the effects of the vertical loads on the $p$-$y$ curves, $P$-$\Delta$ effect and $M_r$ effect are considered in the horizontal equilibrium equation of the pile. Then, the lateral response of monopile under the simultaneous action of vertical, horizontal and bending moment loads is solved. Finally, the reasonable of improved wedge method is verified by the monopile centrifugal model experiments.

Acknowledgments

The authors would like to thank financial support provided by the National Natural Science Foundation of China [grant numbers 51208378 and 41572260].
References

[1] Karthikeyan S, Ramakrishna V V G S T, Rajagopal K. Influence of vertical load on the lateral response of piles in sand[J]. Computers and geotechnics, 2006, 33(2):121-131.

[2] Madhusudan Reddy K, Ayothisraman R. Experimental studies on behavior of single pile under combined uplift and lateral loading[J]. Journal of geotechnical and geoenvironmental engineering, 2015, 141(7): 04015030-1-10.

[3] Kershaw, K, Luna R. Scale model investigation of the effect of vertical load on the lateral response of micropiles in sand[J]. The journal of the deep foundations institute. 2018, 12(1): 3-15.

[4] Liang F Y, Zhang H, Wang J L. Variational solution for the effect of vertical load on the lateral response of offshore piles[J]. Ocean engineering, 2015, 99: 23-33.

[5] Mu L L, Kang X Y, Feng K, et al. Influence of vertical loads on lateral behaviour of monopiles in sand[J]. European journal of environmental and civil engineering, 2018, 22(s1): 286-301.

[6] Mu L L, Kang X Y, Li W. Analytical method for single pile under V-H-M combined loads in sand[J]. Chinese Journal of Geotechnical Engineering, 2017, 39(s2): 153-156.

[7] Liang F Y, Chen H B, Chen S L. Influences of axial load on the lateral response of single pile with integral equation method[J]. International journal for numerical and analytical methods in geomechanics, 2012, 36(16): 1831-1845.

[8] Mcclelland B, Focht J A. Soil modulus for laterally loaded piles[J]. Transactions of the american society of civil engineers, 1958, 123(10):49-63.

[9] Reese L C, Cox W R, Koop F D. Analysis of laterally loaded pile in sand[C]. Proceedings of the sixth annual offshore technology conference, Houston, 1974.

[10] American Petroleum Institute. Recommended practice for planning, designing and constructing fixed offshore platforms[M]. American Petroleum Institute, 2000.

[11] Kim B T, Kim N K, Lee W J, et al. Experimental load-transfer curves of laterally loaded piles in Nak-Dong river sand[J]. Journal of geotechnical and geoenvironmental engineering, 2004, 130(4): 416-425.

[12] Broms B. Lateral resistance of piles in cohesionless soils[J]. Soil mechanics and foundation division journal, 1964, 90(3): 123-156.

[13] Matlock H, Reese L C. Generalized solutions for laterally loaded piles[J]. Geotechnical special publication, 1960, 127(118): 1220-1248.

[14] Barton, Y O. Laterally loaded model piles in sand: centrifuge tests and finite element analyses[D]. University of Cambridge, 1982.

[15] Norris G M. Theoretically based BEF laterally loaded pile analysis[C]. Proceedings of the 3rd International conference on numerical methods in offshore piling. Paris: Technip, 1986: 361-386.

[16] Ashour M, Norris G, Pilling P. Lateral loading of a pile in layered soil using the strain wedge model[J]. Journal of geotechnical and geoenvironmental engineering, 1998, 124(4): 303-315.

[17] Li W. Research on the analysis method of large diameter single pile under combined loads[D]. Tongji University, 2016.

[18] Kondner R L. Hyperbolic stress-strain response: Cohesive soils[J]. Journal of soil mechanical and foundation division, 1963, 89(1):115-143.

[19] Terzaghi K. Evaluation of coefficient of subgrade reaction[J]. Geotechnique, 1955, 5(4): 297-326.

[20] Zhang L Y. Nonlinear analysis of laterally loaded rigid piles in cohesionless soil[J]. Computers and geotechnics, 2009, 36(5): 718-724.

[21] Cooke R W. The settlement of friction pile foundations[C]. International conference on tall buildings, Kuala Lumpur, 1974.

[22] Lu W J, Zhang G. Influence mechanism of vertical-horizontal combined loads on the response of a single pile in sand[J]. Soils and foundations, 2018, 58: 1228-1239.