EXACT LAGRANGIAN SUBMANIFOLDS IN $T^*S^n$
AND THE GRADED KRONECKER QUIVER

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1. Introduction

The topology of Lagrangian submanifolds gives rise to some of the most basic, and also most difficult, questions in symplectic geometry. We have many tools that can be brought to bear on these questions, but each one is effective only in a special class of situations, and their interrelation is by no means clear. The present paper is a piece of shamelessly biased propaganda for a relatively obscure approach, using Fukaya categories. We will test-ride this machinery in a particularly simple case, where the computations are very explicit, and where the outcome can be nicely compared to known results. For clarity, the level of technical sophistication has been damped down a little, and therefore the resulting theorem is not quite the best one can get. Also, in view of the expository nature of the paper, we do not take the most direct path to the conclusion, but instead choose a more scenic route bringing the reader past some classical questions from linear algebra.

Take $M = T^*S^n$, $n \geq 2$, with its canonical symplectic structure. We will consider compact connected Lagrangian submanifolds $L \subset M$ which satisfy $H^1(L) = 0$ and whose second Stiefel-Whitney class $w_2(L)$ vanishes. Note that such an $L$ is automatically orientable (because $w_1(L)$ is the reduction of the integer-valued Maslov class, which must vanish).

**Theorem 1.** For such $L$,

1. $[L] = \pm [S^n] \in H_n(M);
2. $H^*(L; \mathbb{C}) \cong H^*(S^n; \mathbb{C});$
3. $\pi_1(L)$ has no nontrivial finite-dimensional complex representations;
4. if $L' \subset M$ is another Lagrangian submanifold satisfying the same conditions, then $L \cap L' \neq \emptyset$.

As mentioned above, the problem of exact Lagrangian submanifolds in $M$ and more generally in cotangent bundles has been addressed previously by several authors, and their results overlap substantially with Theorem 1. Part
with somewhat weakened assumptions, was proved by Lalonde-Sikorav [12] in one of the first papers on the subject, which is still well worth reading today. For odd \( n \), Buhovsky [3] proves a statement essentially corresponding to (2) (he does not assume \( w_2(L) = 0 \), and works with cohomology with \( \mathbb{Z}/2 \) coefficients; in fact, his result can be used to remove that assumption from our theorem for odd \( n \)). Viterbo’s work [19, 18] has some relevant implications, for instance he proved that there is no exact Lagrangian embedding of a \( K(\pi, 1) \)-manifold in \( T^*S^n \). Finally, in the lowest nontrivial dimension \( n = 2 \) much sharper results are known. Viterbo’s theorem implies that any exact \( L \subset T^*S^2 \) must be a two-sphere; Eliashberg-Polterovich [2] showed that such a sphere is necessarily differentiably isotopic to the zero-section, and this was improved to Lagrangian isotopy by Hind [9]. Therefore, the new parts of Theorem 1 would seem to be (1), (2) for even \( n \geq 4 \), and (3) in so far as it goes beyond Viterbo’s results. These overlaps are quite intriguing, as the arguments used in proving them are widely different (even though all of the higher-dimensional methods involve Floer homology in some way).

We close with some forward-looking observations. Both Buhovsky’s and Viterbo’s approaches should still admit some technical refinements, as does ours, and one can eventually hope to prove that any exact \( L \subset T^*S^n \) is homeomorphic to \( S^n \). In contrast, the question of whether \( L \) needs to be diffeomorphic to the standard sphere is hard to attack, and issues about its Lagrangian isotopy class seem to be entirely beyond current capabilities (for \( n > 2 \)). In the course of proving Theorem 1 we will show that any submanifold \( L \) satisfying its assumptions is “Floer-theoretically equivalent” (in rigorous language, isomorphic in the Donaldson-Fukaya category) to the zero-section. This is a much weaker equivalence relation than Lagrangian isotopy, but still sufficiently strong to imply properties (1) and (4) above.

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2. THE DONALDSON-FUKAYA CATEGORY

In the first place, Fukaya categories provide a convenient way of packaging the information obtained from Lagrangian Floer cohomology groups. We will find it convenient to include non-compact Lagrangian submanifolds with a fixed behaviour at infinity, so we choose a point on \( S^n \) and consider the corresponding cotangent fibre \( F_0 \subset M \). The objects of the Donaldson-Fukaya category \( \mathcal{F}(M) \) are constructed from connected Lagrangian submanifolds \( L \subset M \) subject to the following restrictions:

- \( L \) is either compact, or else agrees with \( F_0 \) outside a compact subset.
• For any smooth disc with boundary on \( L \), \( u : (D, \partial D) \to (M, L) \), the symplectic area \( \int u^* \omega \) vanishes;

• Similarly, the Maslov number of any disc \( u \), which is a relative Chern class \( \int u^*(2c_1(M, L)) \in \mathbb{Z} \), must vanish. As remarked above, this implies orientability of \( L \);

• \( w_2(L) = 0 \).

To be more precise, the objects are Lagrangian branes \( L^b \), which means Lagrangian submanifolds \( L \) as described above coming with certain choices of additional data:

• A grading \( \tilde{\alpha}_L \), which is a lift of the Lagrangian phase function \( \alpha_L : L \to S^1 \) to a real-valued function. This exists because of the zero Maslov condition, but clearly there are infinitely many different choices, differing by integer constants. We write \( L^b[k] \) for the brane obtained by shifting the grading down by \( k \).

• A spin structure on \( L \), and a flat complex vector bundle \( \xi_L \). These are actually coupled, in the sense that changing the spin structure by an element of \( H^1(L; \mathbb{Z}/2) \), and simultaneously tensoring \( \xi_L \) with the corresponding complex line bundle (with monodromy \( \pm 1 \)), does not change our Lagrangian brane.

This list is no doubt baffling to the non-specialist reader, but its purpose is just to make the Lagrangian Floer cohomology groups well-defined and as nicely behaved as possibly. We need these to provide the rest of the category structure, namely, the morphisms between two objects are the Floer cohomology groups in degree zero, with twisted coefficients in our flat vector bundles:

\[
\text{Hom}_{HF^0(M)}(L^b_0, L^b_1) = HF^0(L^b_0, L^b_1),
\]

and composition of morphisms is provided by the pair-of-pants (or more appropriately pseudo-holomorphic triangle) product

\[
(1) \quad HF^0(L^b_1, L^b_2) \otimes HF^0(L^b_0, L^b_1) \to HF^0(L^b_0, L^b_2).
\]

Some basic reminders about Floer cohomology are in order. First of all, one can recover the full Floer groups by considering morphisms into shifted objects, \( Hom(L^b_0, L^b_1[k]) = HF^k(L^b_0, L^b_1) \). The endomorphism group of any brane equipped with the trivial line bundle is its usual cohomology

\[
(2) \quad HF^*(L^b, L^b) \cong H^*(L; \mathbb{C}) \quad \text{for} \quad \xi_L \cong \mathbb{C} \times L.
\]

More generally, if we take \( L \) with a fixed grading and spin structure, and equip it with two different flat vector bundles \( \xi_L, \xi'_L \), the resulting two branes satisfy

\[
(3) \quad HF^*(L^b, L^{b'}) \cong H^*(L; \xi_L \otimes \xi'_L).
\]
In our arguments, three simple Lagrangian submanifolds in $M$ will be prominent. One is the zero-section $Z$, which we make into a brane by choosing some grading and equipping it with the trivial line bundle. The second is the fibre $F_0$, treated in the same way, and the third is the image $F_1 = \tau Z(F_0)$ under the Dehn twist $\tau_Z$, which inherits a brane structure from $F_0$.

**Lemma 2.**

1. The groups $HF^*(Z, F^k_0)$, $HF^*(F^k_0, Z)$ for $k = 0, 1$ are all one-dimensional;
2. $HF^*(F^0_0, F^1_1) \cong H^*(S^{n-1}; \mathbb{C})$;
3. $HF^*(F^1_1, F^0_0) = 0$.

Part (1) is obvious, because the Lagrangian submanifolds intersect in a single point. For the rest, one needs to remember that Floer cohomology of a pair of non-compact Lagrangian submanifolds is defined by moving the first one slightly by the normalized geodesic flow $\phi_t$, which is the Hamiltonian flow of the function $H$ with $H(\xi) = |\xi|$ outside a compact subset. As a consequence, the intersections 

$$\phi_t(L_0) \cap L_1, \quad \text{for } t > 0 \text{ small}$$

are relevant for computing $HF^*(L^0, L^1)$. In the case of $(F_0, F_1)$, the effect of this perturbation is make the submanifolds intersect cleanly along an $S^{n-1}$, and standard Bott-Morse methods yield (2). For $(F_1, F_0)$, in contrast, the perturbation will make them disjoint, so (3) follows.

**References.** All that we have discussed belongs to the basics of Floer homology theory. Gradings and twisted coefficients were both introduced in [10]; for the use of spin structures see [6]; for the product [11] see [5] (these are by no means the only possible references).

## 3. Triangulated Categories

A triangulated category lurks wherever long exact sequences of (any kind of) cohomology groups appear. The axioms of exact triangles formalize some nontrivial properties of such sequences, thus allowing one to manipulate them more efficiently. We will give an informal description of the axioms, which is neither exhaustive nor totally rigorous, but which suffices for our purpose. Let $\mathcal{C}$ be a category in which the morphism spaces $Hom(X_0, X_1)$ between any two objects are complex vector spaces. Having a triangulated structure on $\mathcal{C}$ gives one certain ways of constructing new objects out of old ones. First of all, we assume that $\mathcal{C}$ is additive, which means that one can form the direct sum $X_0 \oplus X_1$ of two objects, with the expected properties. Secondly, one can shift or translate an object by an integer amount, $X \mapsto X[k]$, which allows one to define higher degree morphism spaces $Hom^k(X_0, X_1) = Hom(X_0, X_1[k])$. We will denote the direct sum of all these spaces by $Hom^*(X_0, X_1)$. As a side-remark which will be useful
later, note that by using direct sums and shifts, one can define the tensor product \( V \otimes X \) of any object \( X \) with a finite-dimensional graded complex vector space \( V \): choose a homogeneous basis \( v_j \) for \( V \), and set

\[
V \otimes X = \bigoplus_j V[-\deg(v_j)],
\]

which is easily seen to be independent of the choice of basis up to isomorphism. The final and most important construction procedure for objects is the following: to any morphism \( a : X_0 \to X_1 \) one can associated a new object, the mapping cone \( \text{Cone}(a) \), which is unique up to isomorphism. This can be interpreted as measuring the failure of \( a \) to be an isomorphism: at one extreme, if \( a = 0 \) then \( \text{Cone}(a) = X_0[1] \oplus X_1 \), and on the other hand, \( a \) is an isomorphism iff \( \text{Cone}(a) \) is the zero object.

Mapping cones come with canonical maps \( i : X_1 \to \text{Cone}(a), \pi : \text{Cone}(a) \to X_0[1] \), such that the composition of any two arrows in the diagram

\[
X_0 \xrightarrow{a} X_1 \xrightarrow{i} \text{Cone}(a) \xrightarrow{\pi} X_0[1] \xrightarrow{a[1]} X_1[1]
\]

is zero. By applying \( \text{Hom}(Y, -) \) or \( \text{Hom}(-, Y) \) for some object \( Y \) one gets long exact sequences of vector spaces,

\[
\cdots \text{Hom}^k(Y, X_0) \to \text{Hom}^k(Y, X_1) \to \text{Hom}^k(Y, \text{Cone}(a)) \to \cdots
\]

\[
\cdots \text{Hom}^k(X_0, Y) \leftarrow \text{Hom}^k(X_1, Y) \leftarrow \text{Hom}^k(\text{Cone}(a, Y)) \leftarrow \cdots
\]

which is what we were talking about at the beginning of the section. Diagrams of the form (5) (and isomorphic ones) are called exact triangles, and drawn rolled up like this:

\[
\begin{array}{c}
X_1 \\
\downarrow a \\
X_0
\end{array} \quad \quad \quad \quad \quad \quad \begin{array}{c}
\text{Cone}(a) \\
\uparrow [1]
\end{array}
\]

where the [1] reminds us that this arrow is really a morphism to \( X_0[1] \). One remarkable thing is that triangles can be rotated:

\[
\begin{array}{c}
\text{Cone}(a) \\
\uparrow i \\
X_1
\end{array} \quad \quad \quad \quad \quad \quad \begin{array}{c}
X_0[1] \\
\downarrow [1]
\end{array}
\]

is again an exact triangle, which means that \( X_0[1] \cong \text{Cone}(i) \), and similarly \( X_1[1] \cong \text{Cone}(\pi) \). We know that the cone of the zero morphism is a direct sum, and therefore in \( \mathbb{Z} \)

\[
a = 0 \implies \text{Cone}(a) \cong X_0[1] \oplus X_1,
\]

\[
i = 0 \implies X_0 \cong X_1 \oplus \text{Cone}(a)[-1],
\]

\[
\pi = 0 \implies X_1 \cong \text{Cone}(a) \oplus X_0.
\]
Loosely speaking, the formalism of triangulated categories puts the boundary operator in long exact sequences on the same footing as the other two maps. To round off our picture, we look at the situation where one has two morphisms

\[
X_0 \xrightarrow{a} X_1 \xrightarrow{b} X_2.
\]

Since the mapping cone measures the failure of a map to be an isomorphism, it seems intuitive that these defects should somehow add up under composition, and indeed there is an exact triangle

\[
\begin{array}{c}
\text{Cone}(a) \\
\downarrow \\
\text{Cone}(b)[-1].
\end{array}
\]

This is part of the “octahedral axiom” of triangulated structures (the name comes from a more complicated diagram, which describes various compatibility conditions between the maps in \((8)\) and those in the triangles defining \(\text{Cone}(a), \text{Cone}(b)\)).

The following simple construction arose first in algebraic geometry as part of the theory of mutations. Suppose that \(\mathcal{C}\) is a triangulated category in which the graded vector spaces \(V = \text{Hom}^*(X,Y)\) are finite-dimensional. One can define the tensor products \(V \otimes X\) and \(V^\vee \otimes Y\) as in \((4)\), and these come with canonical evaluation morphisms \(ev : \text{Hom}^*(X,Y) \otimes X \to Y\), \(ev' : X \to \text{Hom}^*(X,Y)^\vee \otimes Y\). We define \(T_X(Y) = \text{Cone}(ev), T'_X(X) = \text{Cone}(ev')[-1]\). By definition, this means that \(T_X(Y)\) sits in an exact triangle

\[
\begin{array}{c}
Y \\
\downarrow \\
T_X(Y)
\end{array}
\]

\[
\begin{array}{c}
 \text{Hom}^*(X,Y) \otimes X \\
\uparrow \\
T'_X(X)
\end{array}
\]

and similarly for \(T'_X\). Unfortunately, the axioms of triangulated categories are not quite strong enough to make \(T_X, T'_X\) into actual functors. Still, these operations can be shown to have nice behaviour, such as taking direct sums to direct sums, and cones to cones (up to isomorphism).

We will now come to the geometric interpretation of \(T, T'\). Contrary to what the reader may have hoped, the Donaldson-Fukaya category \(H^0\mathcal{F}(M)\) is not triangulated. It does have a shift operation with the correct properties, namely the change of grading for a Lagrangian brane, but neither direct sums nor cones exist in general. It is an open question whether one can define a triangulated version of this category by geometric means, such as including Lagrangian submanifolds with self-intersections. Meanwhile, there is a purely algebraic construction in terms of twisted complexes, which yields a triangulated category \(D^b\mathcal{F}(M)\) containing \(H^0\mathcal{F}(M)\) as a full subcategory.
The gist of this is simply to add on new objects in a formal way, so that the requirements of a triangulated category are satisfied. The details are slightly less straightforward than this description may suggest, and involve the use of Fukaya’s higher order \((A_\infty)\) product structures on Floer cohomology, but for the purposes of this paper, all we need is the knowledge that \(D^b\mathcal{F}(M)\) exists, and the following fact:

**Theorem 3.** For any object \(L^\flat\) of \(H^0\mathcal{F}(M)\), the “algebraic twist” \(T_{Z^\flat}(L^\flat)\) and the “geometric (Dehn) twist” \(\tau_{Z}(L^\flat)\) are isomorphic objects of \(D^b\mathcal{F}(M)\). In the same vein, one has \(T'_{Z^\flat}(L^\flat) \cong \tau^{-1}_{Z}(L^\flat)\). □

**References.** [7] is an accessible presentation of the abstract theory of triangulated categories. For mutations see the papers in [14]. The construction of \(D^b\mathcal{F}(M)\) is outlined in [10]. The long exact sequence in Floer cohomology which is a consequence of Theorem 3 was introduced in [17]. The argument given there can easily be adapted to prove the result as stated, see [16] for an explanation.

4. The graded Kronecker quiver

We will now switch gears slightly. The representation theory of quivers is a subject with deceptively humble appearance. Superficially, it is no more than a convenient way of reformulating certain questions in linear algebra, but in many cases these reveal themselves to be equivalent to much less elementary problems in other areas, such as algebraic geometry. We will only need a very simple instance of the theory, namely the following graded Kronecker quiver, with nonzero \(d \in \mathbb{Z}\):

\[
\begin{array}{c}
\bullet \\
0 \\
\downarrow \\
d \\
\bullet \\
\end{array}
\]

By definition, a representation of (9) consists of two finite-dimensional graded \(\mathbb{C}\)-vector spaces \(V, W\), together with linear maps \(\alpha : V \to W\) of degree zero and \(\beta : V \to W[d]\) of degree \(d\), respectively. Two representations \((V, W, \alpha, \beta)\) and \((V', W', \alpha', \beta')\) are isomorphic if there are graded linear isomorphisms \(\phi : V \to V', \psi : W \to W'\) such that \(\alpha' \phi = \psi \alpha, \beta' \phi = \psi \beta\). Of course, any representation splits into a direct sum of indecomposable ones. The Kronecker quiver is nice in that the latter can be classified explicitly:

**Proposition 4.** Any indecomposable representation is isomorphic, up to a common shift in the gradings of both vector spaces, to one of the following:
• (\(\mathcal{O}_X(k), k < 0\)): \(\dim(V) = -k, \dim(W) = -k - 1\),
\[
V = \mathbb{C} \oplus \mathbb{C}[-d] \oplus \cdots \oplus \mathbb{C}[(k - 1)d]
\]
\[
\alpha = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \ddots \\ 1 & \cdots & 0 \end{pmatrix}
\]
\[
W = \mathbb{C}[-d] \oplus \mathbb{C}[-2d] \oplus \cdots \oplus \mathbb{C}[(k - 1)d]
\]

• (\(\mathcal{O}_X(k), k \geq 0\)): \(\dim(V) = \dim(W) = k + 1\),
\[
V = \mathbb{C}[d] \oplus \mathbb{C}[2d] \oplus \cdots \oplus \mathbb{C}[kd]
\]
\[
\alpha = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \ddots & \ddots \iddots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}
\]
\[
W = \mathbb{C} \oplus \mathbb{C}[d] \oplus \cdots \oplus \mathbb{C}[kd];
\]

• (\(\mathcal{O}_X/\mathcal{O}_X^k, k \geq 1\)): \(\dim(V) = \dim(W) = k\),
\[
V = \mathbb{C}[d] \oplus \mathbb{C}[2d] \oplus \cdots \oplus \mathbb{C}[kd]
\]
\[
\alpha = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \ddots & \ddots \iddots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}
\]
\[
W = \mathbb{C} \oplus \mathbb{C}[d] \oplus \cdots \oplus \mathbb{C}[(k - 1)d]
\]

• (\(\mathcal{O}_X/\mathcal{O}_X^\infty, k \geq 1\)): \(\dim(V) = \dim(W) = k\),
\[
V = \mathbb{C} \oplus \mathbb{C}[d] \oplus \cdots \oplus \mathbb{C}[(k - 1)d]
\]
\[
\alpha = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \ddots & \ddots \iddots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots \iddots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}
\]
\[
W = \mathbb{C} \oplus \mathbb{C}[d] \oplus \cdots \oplus \mathbb{C}[(k - 1)d].
\]

This is actually simple enough to be tackled using only linear algebra, but the meaning of the classification becomes much clearer in algebro-geometric terms. To any representation \((V, W, \alpha, \beta)\) of (9) one can associate a map of algebraic vector bundles on \(X = \mathbb{C}P^1\),
\[
\mathcal{O}_X(-1) \otimes V \xrightarrow{\alpha x + \beta y} \mathcal{O}_X \otimes W.
\]
Take the \(\mathbb{C}^*\)-action on \(\mathbb{C}^2\) given by \(\zeta \cdot (x, y) = (x, \zeta^{-d}y)\), and the induced action on \(X\). Then the sheaves \(\mathcal{O}_X, \mathcal{O}_X(-1)\) are naturally equivariant, and if one equips \(V, W\) with the \(\mathbb{C}^*\)-actions whose weights are given by the
grading, \( \alpha x + \beta y \) becomes an equivariant map. At this point, the usual procedure would be to look at the kernel and cokernel of (10), and to use Grothendieck’s splitting theorem for holomorphic vector bundles on \( \mathbb{CP}^1 \) to derive Proposition [11] but we prefer a sleeker and more high-tech approach using triangulated categories. The category \( \text{Coh}_{\mathbb{C}^*}(X) \) of equivariant coherent sheaves admits a full embedding into a triangulated category, its derived category \( D^b \text{Coh}_{\mathbb{C}^*}(X) \). One can look at the cone of (10) as an object \( \mathcal{E} \) in that category. A computation using the long exact sequences (6) shows that the morphisms between two such cones are precisely the homomorphisms of quiver representations, which means that the cone construction embeds the category of representations of our quiver as a full subcategory into \( D^b \text{Coh}_{\mathbb{C}^*}(X) \). In particular, indecomposable representations must give rise to indecomposable objects \( \mathcal{E} \).

Grothendieck’s theorem extends easily to the derived category and to the equivariant case: each indecomposable object of \( D^b \text{Coh}_{\mathbb{C}^*}(X) \) is, up to a shift, either a line bundle \( \mathcal{O}_X(k) \), or else a torsion sheaf of the form \( \mathcal{O}_X/\mathcal{J}^k \), where \( p \in \{0, \infty\} \) is one of the two fixed points \( 0 = [0 : 1] \) or \( \infty = [1 : 0] \) of the \( \mathbb{C}^* \)-action. Going back to the objects \( \mathcal{E} \) constructed above, one finds that only two essentially different possibilities can occur. One is that \( \mathcal{E}[-1] \cong \mathcal{O}_X(k) \) with \( k < 0 \), in which case the map (10) is surjective, with kernel \( \mathcal{E} \). The usual long exact sequence in sheaf cohomology shows that one can recover the representation from \( \mathcal{E} \) as follows:

\[
V \cong H^1(\mathcal{E}[-1] \otimes \mathcal{O}_X(-1)), \quad W \cong H^1(\mathcal{E}[-1]),
\]

and the maps \( \alpha, \beta \) are the (Yoneda) products with the standard generators of \( H^0(\mathcal{O}_X(1)) \). The other case is where \( \mathcal{E} \) is either \( \mathcal{O}_X(k) \) with \( k \geq 0 \), or a torsion sheaf; then (10) is surjective, with cokernel \( \mathcal{E} \), and this time one finds that

\[
V \cong H^0(\mathcal{E} \otimes \mathcal{O}_X(-1)), \quad W \cong H^0(\mathcal{E}).
\]

A straightforward computation of cohomology groups identifies the various \( \mathcal{E} \) with the corresponding cases in Proposition [11] thereby concluding our proof of that result.

**References.** The analogue of Proposition [11] for the ungraded quiver is due to Kronecker, and is explained in textbooks on the representation theory of finite-dimensional algebras [1, 2]. The connection with coherent sheaves is well-known. The fact that an indecomposable object of the derived category is actually a single sheaf is a general property of abelian categories of homological dimension one, and is used extensively in papers about mirror symmetry on elliptic curves [13, 14]. Grothendieck’s paper is [8].
5. Proof of Theorem 1

Consider the three basic Lagrangian branes $Z^\flat$, $F_0^\flat$ and $F_1^\flat$ as objects in $D^b\mathcal{F}(M)$. Because $F_1$ is the image of $F_0$ under $\tau_Z$, Theorem 3 can be applied, and we use this to prove:

Lemma 5. $T_{F_0^\flat}T_{F_1^\flat}(Z^\flat)$ is the zero object.

Proof. From Lemma 2(1) we know that $HF^*(F_1^\flat, Z^\flat)$ is one-dimensional. For simplicity, assume that the gradings have been chosen in such a way that the nontrivial Floer group lies in degree zero, and denote a nonzero element of it by $c$. Using the definitions of $T, T'$, and Theorem 3, one gets

$$T_{F_1^\flat}(Z^\flat) = \text{Cone}(F_1^\flat \xrightarrow{c} Z^\flat) = T_{Z^\flat}(F_1^\flat)[1] \cong \tau_Z^{-1}(F_1^\flat)[1] \cong F_0^\flat[1].$$

By (2) $HF^*(F_0^\flat, F_0^\flat) \cong H^*(F_0; \mathbb{C}) \cong \mathbb{C}$, and therefore

$$T_{F_0^\flat}(F_0^\flat) = \text{Cone}(F_0^\flat \xrightarrow{id} F_0^\flat) = 0. \quad \square$$

Take a Lagrangian submanifold $L \subset M$ satisfying the assumptions of Theorem 1. Because $H^1(L; \mathbb{C}) = 0$, $L$ is automatically exact and has zero Maslov class, so we can make it into an object $L^\flat$ of $H^0\mathcal{F}(M)$ by choosing a grading and spin structure, as well as the trivial line bundle. Near $Z$, $\tau_Z$ is equal to the antipodal involution, and so $\tau_Z^2$ is equal to the identity. By an isotopy along the Liouville (compressing) flow, one can move $L$ arbitrarily close to $Z$, so $\tau_Z^2(L)$ is Lagrangian isotopic to $L$. When one considers the gradings on both submanifolds, however, there is a difference:

$$(11) \quad \tau_Z^2(L^\flat) \sim L^\flat[2 - 2n].$$

From Theorem 3 and the definition of $T$ as a cone, we get exact triangles

$$\xymatrix{ \bar{L}^\flat \ar[r] & \tau_Z(L^\flat) \\ HF^*(Z^\flat, \bar{L}^\flat) \otimes Z^\flat \ar[u] & \\ }$$

and

$$\xymatrix{ \tau_Z(L^\flat) \ar[r] & \tau_Z^2(L^\flat) \cong L^\flat[2 - 2n] \\ HF^*(Z^\flat, \tau_Z(L^\flat)) \otimes Z^\flat \ar[u] & \\ }$$

The bottom term can be simplified by noticing that $HF^*(Z^\flat, \tau_Z(L^\flat)) \cong HF^*(\tau_Z^{-1}(Z^\flat), L^\flat) \cong HF^*(Z^\flat[n - 1], L^\flat) \cong HF^{*+1-n}(Z^\flat, L^\flat)$. Using the
octahedral axiom (8), the two exact triangles can be spliced together to a single one,

\[
\begin{array}{c}
L^\flat \xrightarrow{[1]} L^\flat [2-2n] \\
\text{Cone}(HF^\ast(Z^\flat, L^\flat)[-n] \otimes Z^\flat \to HF^\ast(Z^\flat, L^\flat) \otimes Z^\flat)
\end{array}
\]

The \(\to\) must be given by an element of \(HF^{2-2n}(L^\flat, L^\flat)\), but because of (2) the Floer cohomology groups vanish in negative degrees. Hence the morphism is necessarily zero, and as explained in (7),

(12) \(L^\flat \oplus L^\flat [1-2n] \cong \text{Cone}(HF^\ast(Z^\flat, L^\flat)[-n] \otimes Z^\flat \to HF^\ast(Z^\flat, L^\flat) \otimes Z^\flat)\).

Lemma 6. \(T_{F_0^1}T_{F_1^0}(L^\flat)\) is the zero object.

Proof. From Lemma 5 we know that \(T_{F_0^1}T_{F_1^0}\) annihilates \(Z^\flat\). Since it carries direct sums to direct sums and cones to cones, this operation also annihilates the right hand side of (12), hence \(L^\flat\). \(\square\)

Lemma 6 gives us a powerful hold on the a priori unknown object \(L^\flat\). Namely, by putting together the two exact triangles coming from the definition of \(T\) (we omit the details, since they are parallel to the computation carried out above) one finds that

(13) \(L^\flat \cong \text{Cone}(\text{Hom}^\ast(F^\flat_0, T_{F_1^0}^\flat(L^\flat)))[-1] \otimes F^\flat_0 \to \text{Hom}^\ast(F^\flat_1, L^\flat) \otimes F^\flat_1)\).

Hence, the isomorphism class of \(L^\flat\) as an object of the Fukaya category is determined by two finite-dimensional graded vector spaces

\[V = \text{Hom}^\ast(F^\flat_0, T_{F_1^0}^\flat(L^\flat))[-1], \quad W = \text{Hom}^\ast(F^\flat_0, L^\flat)\]

and the arrow in (13), which is an element \(x \in \text{Hom}^\ast_c(V, W) \otimes HF^\ast(F^\flat_0, F^\flat_1)\) of degree zero. We computed the relevant Floer cohomology group in Lemma 2(2). After choosing generators \(a\) of degree zero and \(b\) of degree \(n-1\), one writes \(x = \alpha \otimes a + \beta \otimes b\) where \(\alpha : V \to W\) is a linear map of degree zero, and \(\beta : V \to W\) one of degree \(1-n\). We see that \((V, W, \alpha, \beta)\) is a representation of the graded Kronecker quiver with

(14) \(d = 1 - n < 0\).

If this representation was decomposable, it would give rise to a corresponding decomposition of the object \(L^\flat\) into direct summands in \(D^b_{\text{fd}}(M)\), but that is impossible since \(\text{Hom}(L^\flat, L^\flat) = HF^0(L^\flat, L^\flat) = H^0(L^\flat; \mathbb{C}) = \mathbb{C}\). More generally, the long exact sequences (13) applied to the cone (13) show that
$H^*(L; \mathbb{C}) \cong HF^*(L^\flat, L^\flat)$ is the cohomology of the two-step complex

$$C = \left\{ 0 \to Hom_C(V, V) \oplus Hom_C(W, W) \to \begin{array}{c}
\left( -\alpha \circ \cdots \circ \alpha \right) \\
\left( -\beta \circ \cdots \circ \beta \right)
\end{array} Hom_C(V, W) \oplus Hom_C(V, W)[d] \to 0 \right\}$$

This is actually a complex of graded vector spaces, so its cohomology $H^*(C)$ is bigraded, and one obtains $H^*(L; \mathbb{C})$ by summing up the two gradings.

One can compute $H^*(C)$ directly for each of the cases in Proposition 4, or alternatively, one can go back to our algebro-geometric proof of that result and note that $H^*(C)$ is equal to $Ext^*_{X}(\mathcal{E}, \mathcal{E})$ for the associated sheaf $\mathcal{E}$ (which is bigraded by the cohomological degree and $\mathbb{C}$-action). In either way, one sees that

- in the case where the representation is of type $\mathcal{O}_X(k)$, $H^*(C)$ is one-dimensional and concentrated in degree zero;
- for $\mathcal{O}_X/\mathcal{J} X, 0$, $H^*(C)$ is $2k$-dimensional, with generators in degrees $0, -d, \ldots, (1-k)d$ and $d+1, 2d+1, \ldots, kd+1$.
- for $\mathcal{O}_X/\mathcal{J} X, \infty$, $H^*(C)$ is $2k$-dimensional, with generators in degrees $0, d, \ldots, (k-1)d$ and $1-d, 1-2d, \ldots, 1-kd$.

Bearing in mind (14), one sees that $H^*(C)$ cannot be the cohomology of an $n$-dimensional oriented manifold except in one case, $\mathcal{O}_X/\mathcal{J} X, \infty$. We have proved:

**Lemma 7.** Up to a shift, $L^\flat$ is isomorphic to Cone$(a : F^0_\delta \to F^1_\delta)$ in the derived Fukaya category.

Part (2) of Theorem 1 follows immediately, since in the case of $\mathcal{O}_X/\mathcal{J} X, \infty$, $H^*(C) \cong H^*(S^n; \mathbb{C})$. Lemma 7 also implies that there is a long exact sequence

$$\ldots HF^*(F^0_1, F^0_1) \to HF^*(F^0_1, L^\flat) \to HF^{*+1}(F^0_1, F^0_0) \to \ldots$$

which in view of Lemma 2 shows that $HF^*(F^0_1, L^\flat)$ is one-dimensional. Because the Euler characteristic of Floer cohomology is the ordinary intersection number, it follows that $F_1 \cdot L = \pm 1$, which implies part (1) of Theorem 1. Next, if we had two Lagrangian submanifolds satisfying the conditions of that theorem, they would give rise to isomorphic objects in the Fukaya category by Lemma 7, hence their Floer cohomology would be equal to the ordinary cohomology of each. Since that is nonzero, the two Lagrangian submanifolds necessarily intersect, which is part (4).

The remaining statement (3) about the fundamental group is slightly more complicated. Take an indecomposable flat complex vector bundle $\xi_L$ on $L$, and use that to define a brane $L^{b, \prime}$. From (3) we see that $HF^0(L^{b, \prime}, L^{b, \prime}) = \ldots$
$H^0(\xi_L^* \otimes \xi_L)$ cannot contain any nontrivial idempotents; therefore the representation of $\Xi$ associated to $L^{,'}$ must still be indecomposable. Again, one finds that $O_X/\mathcal{J}_X,\infty$ is the only possibility, so $L^{,'}$ is isomorphic to the previously considered brane $L^\flat$. Now applying (3) to this pair of branes, one finds that $\xi_L$ must be the trivial line bundle.

References. Equation (11) is taken from [15].

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