Fast two-beam collisions in a linear optical medium with weak cubic loss in spatial dimension higher than 1

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Abstract

We study the dynamics of fast two-beam collisions in linear optical media with weak cubic loss in spatial dimension higher than 1. For this purpose, we extend the perturbation theory that was developed for analyzing two-pulse collisions in spatial dimension 1 to spatial dimension 2. We use the extended two-dimensional version of the perturbation theory to show that the collision leads to a change in the beam shapes in the direction transverse to the relative velocity vector. Furthermore, we show that in the important case of a separable initial condition for both beams, the longitudinal part in the expression for the amplitude shift is universal, while the transverse part is not universal. Additionally, we demonstrate that the same behavior holds for collisions between pulsed optical beams in spatial dimension 3. We check these predictions of the perturbation theory along with other predictions concerning the effects on the collision of partial beam overlap and anisotropy in the initial condition by extensive numerical simulations with the weakly perturbed linear propagation model in spatial dimensions 2 and 3. The agreement between the perturbation theory and the simulations is very good. Therefore, our study significantly extends and generalizes the results of previous works, which were limited to spatial dimension 1.

Keywords: Optical beam propagation, cubic loss, beam collisions, perturbation theory, amplitude dynamics.
I. INTRODUCTION

Linear evolution models have an important role in many areas of science. Examples include the linear diffusion equation [1], the linear wave equation [2], the linear propagation equation [3–5], and the linear Schrödinger equation [6]. In many cases, the physical systems that are described by these linear evolution models include weak nonlinear dissipation [1, 7]. As a result, the latter physical systems are more accurately described by perturbed linear evolution models with weak nonlinear dissipation. The presence of the nonlinear dissipation induces new physical effects, which do not exist in the unperturbed linear physical systems. A major example is the change in the pulse mass or energy during fast collisions between pulses of the linear propagation model [8–10]. Since the pulses of the linear physical systems (and also of their weakly perturbed counterparts) are not shape preserving, one does not expect to observe simple dynamics in these collisions. As a result, one also does not expect to be able to make simple general statements about the collision-induced effects.

In two previous works [8, 10], we showed that the opposite is in fact true for fast two-pulse collisions. The latter are collisions, in which the collision length, i.e., the length of the interval where the two pulses overlap, is much smaller than all the other length scales in the problem [11]. In Refs. [8, 10], we showed that the amplitude shifts in fast two-pulse collisions in linear physical systems, weakly perturbed by nonlinear dissipation, exhibit simple soliton-like behavior. The behavior was demonstrated for the following two central cases: (1) systems described by the linear propagation equation with weak cubic loss; (2) systems described by the linear diffusion-advection equation with weak quadratic loss. We first developed a perturbation method for analyzing the fast two-pulse collision dynamics in these weakly perturbed linear systems. We then used the perturbation method to show that in both cases, the expressions for the collision-induced amplitude shifts in the presence of weak nonlinear loss have the same simple form as the expression for the amplitude shift in a fast collision between two solitons of the nonlinear Schrödinger equation in the presence of weak cubic loss. Furthermore, in Ref. [10], we showed that the expressions for the amplitude shifts are universal in the sense that they are independent of the details of the initial pulse shapes. In addition, we found that within the leading order of the perturbation theory, the pulse shapes are not changed by the collision. The perturbation theory predictions for the collision-induced amplitude shifts were verified by extensive numerical simulations.
with the two perturbed linear evolution models for a variety of initial pulse shapes \[8,10\]. Additionally, in Ref. \[9\], we showed that the amplitude shift in a fast two-pulse collision in systems described by the linear propagation model with high-order nonlinear loss can be calculated by the same perturbation method that was developed in Ref. \[8\].

The three studies in Refs. \[8–10\] were limited to spatial dimension 1 \[12\]. As a result, these studies did not consider important collisional effects, which exist only is spatial dimension higher than 1, such as anisotropy and partial pulse overlap. Additionally, it is not clear if the simple (universal) dependence of the expressions for the collision-induced amplitude shifts on the physical parameters that was found in Refs. \[8\] and \[10\] remains valid in the high-dimensional problem. It is also unclear if the pulse shapes remain unchanged in fast collisions in the presence of cubic (or quadratic) loss in the high-dimensional problem. Thus, all the key aspects of the fast two-pulse collision problem, which are associated with the collision dynamics in spatial dimension higher than 1, were not addressed in previous studies.

In the current paper, we address the important aspects of the high-dimensional fast two-pulse collision problem that were mentioned in the preceding paragraph. For this purpose, we first develop a perturbation method, which generalizes the perturbation method that was introduced in Refs. \[8,10\] for the one-dimensional problem in three major ways. (1) It extends the perturbative calculation from spatial dimension 1 to spatial dimension 2, and enables further extension of the calculation to a general spatial dimension in a straightforward manner. (2) It provides a perturbative calculation of the collision-induced dynamics of the beam shape both inside and outside of the collision interval. In contrast, the perturbative calculation of Refs. \[8,10\] was limited to the collision interval only. (3) It enables the discovery and analysis of several collision-induced effects, which exist only in the high-dimensional problem.

We use the generalized version of the perturbation method to derive formulas for the collision-induced changes in the beam shapes and amplitudes in spatial dimension 2. We find that for a general initial condition, the collision leads to a change in the beam shape in the direction transverse to the relative velocity vector between the beam centers. Additionally, we find that in the important case of an initial condition that is separable for both beams, the beam shape in the longitudinal direction is not changed by the collision within the leading order of the perturbation theory. Furthermore, we show that for a separable initial
condition, the longitudinal part in the expression for the amplitude shift is universal, while
the transverse part is not universal and is proportional to the integral of the product of the
beam intensities with respect to the transverse coordinate. We also show that the same
behavior of the longitudinal and transverse parts in the expression for the collision-induced
amplitude shift exists in collisions between pulsed-beams in spatial dimension 3.

We check these predictions of the perturbation theory together with other predictions
concerning the effects of partial beam overlap and anisotropy in the initial condition by
extensive numerical simulations with the perturbed linear propagation model in spatial
dimensions 2 and 3. The simulations in spatial dimension 2 are carried out for four different
two-beam collision setups. These setups demonstrate the following four major effects and
properties of the collision that either exist only in spatial dimension higher than 1, or are
qualitatively different from their one-dimensional counterparts. (1) The universality of the
longitudinal part in the expression for the collision-induced amplitude shift. (2) The effect
of partial beam overlap. (3) The effect of anisotropy in the initial condition. (4) The
collision-induced change in the beam shape in the transverse direction. The prediction for
universal behavior of the longitudinal part in the expression for the amplitude shift is also
checked in spatial dimension 3 by numerical simulations of collisions between pulsed optical
beams. In all the simulation setups we obtain very good agreement between the perturbation
theory and the numerical simulations. Therefore, the simulations validate the theoretical
predictions for the four high-dimensional effects and properties of the collision and show
that the extended perturbation approach can indeed be used for analyzing the effects of fast
two-beam collisions in spatial dimension higher than 1.

In a related work, we studied the dynamics of fast two-pulse collisions in systems described
by linear diffusion-advection models with weak quadratic loss in spatial dimension higher
than 1 [13]. We first developed a perturbation method for analyzing the collision dynamics,
which is similar to the one introduced in the current paper. Using this perturbation method
and numerical simulations, we showed that the collision-induced changes in pulse shapes and
amplitudes in these systems exhibit similar behavior to the one found in the current paper
[13]. Thus, the current paper and the related study of Ref. [13] significantly extend and
generalize the results of the previous works in Refs. [8–10], which were limited to spatial
dimension 1. We also comment that detailed analytic results on collisions between pulse
solutions of linear or nonlinear evolution models in the presence of nonlinear dissipation in
spatial dimension higher than 1 are quite scarce. Therefore, the current work and the work in Ref. [13] also significantly extend the understanding of the more general high-dimensional problem of two-pulse collisions in the presence of nonlinear dissipation.

We choose to study two-beam collisions in the presence of cubic loss, since cubic loss is important in many optical systems, and is therefore a major example for nonlinear dissipative perturbations. The optical medium’s cubic loss typically arises due to two-photon absorption (2PA) [7, 14–16]. Propagation of optical pulses and optical beams in the presence of cubic loss has been studied in many earlier works, both in weakly perturbed linear media [8, 10, 17–20], and in nonlinear media [21–32]. The subject attracted renewed attention in recent years due to the importance of 2PA in silicon nanowaveguides, which are expected to play a major role in many applications in optoelectronic devices [7, 14, 15, 33, 34]. In the current paper, we assume that the optical medium is weakly nonlinear and neglect the effects of cubic (Kerr) nonlinearity. We emphasize that this assumption was successfully used in previous experimental and theoretical works, see, e.g., Refs. [17–20]. For the same reason, we also neglect the effects of high-order nonlinear loss on the collision. We remark that the latter effects can be described by the same perturbation method that is introduced in the current paper (see also Ref. [9], where the calculation was carried out for spatial dimension 1).

The rest of the paper is organized as follows. In Section II we present the extended perturbation method for calculating the amplitude and beam shape dynamics in fast collisions between beams of the linear propagation model in spatial dimension 2. In Section III we present the perturbation theory predictions and the results of numerical simulations with the weakly perturbed linear propagation model for four major collision setups. These setups demonstrate four major effects and properties of the collision that exist only in spatial dimension higher than 1. In Section IV we present the main predictions of the perturbation theory for collisions between pulsed-beams of the linear propagation equation in spatial dimension 3. We also present a comparison between the perturbation theory prediction for the collision-induced amplitude shift and the results of numerical simulations with the weakly perturbed linear propagation model. In Section V we summarize our conclusions. The five Appendixes contain calculations that support the material in the main body of the paper.
II. THE PERTURBATION THEORY FOR FAST TWO-BEAM COLLISIONS IN SPATIAL DIMENSION 2

A. Introduction

We consider fast collisions between two optical beams in a three-dimensional linear optical medium with weak cubic loss. We assume that the beams propagate along the $z$ axis with beam-steering in the $xy$ plane, and that the propagation is accurately described by the paraxial approximation \[3–5\]. For each given value of $z$, the distribution of the electric field is a function of $x$ and $y$. Therefore, we can think about the $z$ coordinate as a dynamical coordinate, and about the $x$ and $y$ coordinates as the actual spatial coordinates, which help describe the distribution of the electric field for each value of $z$. We refer to the dimension of the space, in which the distribution of the electric field is described (for a given $z$) as the spatial dimension. Thus, in the current problem, the spatial dimension is 2 and the total dimension is 3. The propagation is described by a $(2 + 1)$-dimensional propagation model, where the 2 in this terminology corresponds to the spatial dimension, and the 1 is the dimension of the dynamical axis (the $z$ axis).

We take into account the effects of isotropic diffraction and weak cubic loss, as well as the velocity difference between the beam centers, which is controlled by beam-steering \[35–42\]. For simplicity and without loss of generality, we assume that the relative velocity vector between the beam centers lies along the $x$ axis. This choice enables one to obtain closed formulas for the collision-induced changes in beam shapes and amplitudes, and in this manner, enables a significantly deeper insight into the collision dynamics. Furthermore, in Appendix E, we show that the choice of the relative velocity vector along the $x$ axis does not change the value of the collision-induced amplitude shift obtained by our perturbation approach. That is, the latter value is invariant under rotations of the $x$ and $y$ axes. Thus, the dynamics of the two-beam collision is described by the following weakly perturbed linear propagation model:

$$i \partial_z \psi_1 + \partial_x^2 \psi_1 + \partial_y^2 \psi_1 = -i \varepsilon_3 |\psi_1|^2 \psi_1 - 2i \varepsilon_3 |\psi_2|^2 \psi_1,$$

$$i \partial_z \psi_2 + i \partial_1 \partial_x \psi_2 + \partial_x^2 \psi_2 + \partial_y^2 \psi_2 = -i \varepsilon_3 |\psi_2|^2 \psi_2 - 2i \varepsilon_3 |\psi_1|^2 \psi_2.$$  \hspace{1cm} (1)

In Eq. (1), $\psi_j$ with $j = 1, 2$ are proportional to the electric fields of the beams, and $x$, $y$, and $z$ are the spatial coordinates \[43\]. In addition, $d_{11}$ is the coefficient related to the
velocity difference between the beam centers (the beam-steering coefficient), and \( \epsilon_3 \) is the cubic loss coefficient, which satisfies \( 0 < \epsilon_3 \ll 1 \). The terms \( \partial_x^2 \psi_j \) and \( \partial_y^2 \psi_j \) on the left hand side of Eq. (1) describe the effects of isotropic diffraction, while \( id_{11} \partial_x \psi_2 \) is related to the velocity difference between the beam centers. The first and second terms on the right hand side of Eq. (1) describe intra-beam and inter-beam effects due to cubic loss. In the current paper, we do not take into account the effects of linear loss, since these effects do not change the form of the expressions for the collision-induced changes in beam amplitudes and shapes. Furthermore, the simple effects of linear loss on amplitude dynamics can be incorporated into the analysis in exactly the same manner as was done in Refs. [8, 10] for spatial dimension 1 (see also Appendix B). We remark that the same perturbed linear propagation model (with some changes in the physical variables) also describes the dynamics of a fast collision between two pulsed-beams in a two-dimensional linear optical medium (e.g., a planar waveguide) with weak cubic loss. In this case, the coordinate \( x \) is replaced by the time variable \( t \), the term \( id_{11} \partial_t \psi_2 \) describes the effects of the group velocity difference, and the terms \( \partial_t^2 \psi_j \) describe the effects of second-order dispersion. The more general case of fast collisions between pulsed-beams in a three-dimensional medium (i.e., in spatial dimension 3 and total dimension 4) is studied in section IV.

We consider fast collisions between beams with generic initial shapes and with tails that decay sufficiently fast, such that the values of the integrals \( \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_j(x, y, 0)|^2 \) are finite. We assume that the initial beams can be characterized by the following parameters. (1) The initial amplitudes \( A_j(0) \). (2) The initial beam widths, i.e., the widths of the maxima of \( |\psi_j(x, y, 0)| \), which can be expressed in terms of the widths along the \( x \) and \( y \) axes, \( W_{j0}^{(x)} \) and \( W_{j0}^{(y)} \), respectively. (3) The initial positions of the beam centers, i.e., the locations of the maxima of \( |\psi_j(x, y, 0)| \), which are denoted by \( (x_{j0}, y_{j0}) \). (4) The initial phases \( \alpha_{j0} \). Therefore, the initial electric fields of the optical beams can be expressed as:

\[
\psi_j(x, y, 0) = A_j(0) h_j(x, y) \exp(i\alpha_{j0}),
\]

where \( h_j(x, y) \) is real-valued. Note that for brevity of notation, we did not write the dependence of the function \( h_j(x, y) \) on the beam parameters explicitly. We are also interested in the important case, where the initial electric fields of both beams are separable, i.e., where each of the functions \( \psi_j(x, y, 0) \) can be expressed as a product of a function of \( x \) and a
function of \( y \). In this case, the initial electric fields are given by:

\[
\psi_j(x, y, 0) = A_j(0) h_j^{(x)} \left[ \frac{(x - x_{j0})}{W_{j0}^{(x)}} \right] h_j^{(y)} \left[ \frac{(y - y_{j0})}{W_{j0}^{(y)}} \right] \exp(i\alpha_{j0}).
\]

(3)

In what follows, we will also consider cases where the initial electric field is separable for one beam and nonseparable for the other beam.

In the current paper, we study the collision-induced dynamics of complete fast collisions. The complete collision assumption means that the beams are well-separated before and after the collision. More specifically, in these collisions, the values of the \( x \) coordinate of the beam centers at \( z = 0 \) and at the final propagation distance \( z_f \), \( x_{j0} \) and \( x_j(z_f) \), satisfy

\[ |x_{20} - x_{10}| \gg W_{10}^{(x)} + W_{20}^{(x)} \] and \[ |x_2(z_f) - x_1(z_f)| \gg W_1^{(x)}(z_f) + W_2^{(x)}(z_f), \]

where \( W_j^{(x)}(z_f) \) are the beam widths in the \( x \) direction at \( z = z_f \). To obtain the condition for a fast collision, we first define the collision length \( \Delta z_c \), as the distance along which the beam widths in the \( x \) direction overlap. From this definition it follows that \( \Delta z_c = 2\left( W_{10}^{(x)} + W_{20}^{(x)} \right) / |d_{11}|. \) For a fast collision, we require that \( \Delta z_c \) would be much smaller than the smallest diffraction length in the problem. We note that the diffraction lengths of the \( j \)th beam in the \( x \) and \( y \) directions are \( z_{Dj}^{(x)} = W_{j0}^{(x)2}/2 \) and \( z_{Dj}^{(y)} = W_{j0}^{(y)2}/2 \), respectively. Thus, the smallest diffraction length \( z_{D}^{(min)} \) is \( z_{D}^{(min)} = \min \left\{ z_{D1}^{(x)}, z_{D2}^{(x)}, z_{D1}^{(y)}, z_{D2}^{(y)} \right\} \). Requiring that \( \Delta z_c \ll z_{D}^{(min)} \) and using the definition of \( \Delta z_c \), we obtain that the condition for a fast collision can be expressed as

\[ 2(W_{10}^{(x)} + W_{20}^{(x)}) \ll |d_{11}|z_{D}^{(min)}. \]

B. Calculation of the collision-induced changes in the beam shape and amplitude for a general initial condition

1. Introduction

The perturbation method that we present here generalizes the perturbation method presented in Refs. \([8, 10]\) in three major ways. First, it extends the calculation from spatial dimension 1 to spatial dimension 2 \([45]\). Second, it provides a perturbative calculation and analytic expressions for the collision-induced change in the beam shape both in the collision interval and away from the collision interval, whereas the calculation of the change in the beam shape in Refs. \([8, 10]\) was limited to the collision interval only. Third, it helps uncover
several collision-induced effects, which exist only in spatial dimension higher than 1. In the first step in the perturbative calculation, we look for a solution of Eq. (1) in the form:

\[ \psi_j(x, y, z) = \psi_{j0}(x, y, z) + \phi_j(x, y, z), \]

where \( j = 1, 2 \), \( \psi_{j0} \) are the solutions of the weakly perturbed linear propagation equations without the inter-beam interaction terms, and \( \phi_j \) describe corrections to the \( \psi_{j0} \) due to the effects of inter-beam interaction on the collision. By their definition, the \( \psi_{j0} \) satisfy the following two equations:

\[ i \partial_z \psi_{10} + \partial_x^2 \psi_{10} + \partial_y^2 \psi_{10} = -i \epsilon_3 |\psi_{10}|^2 \psi_{10}, \]

and

\[ i \partial_z \psi_{20} + i d_{11} \partial_x \psi_{20} + \partial_x^2 \psi_{20} + \partial_y^2 \psi_{20} = -i \epsilon_3 |\psi_{20}|^2 \psi_{20}. \]

Substituting the ansatz (4) into Eq. (1) and using Eqs. (5) and (6), we obtain equations for the \( \phi_j \). We concentrate on the calculation of \( \phi_1 \), since the calculation of \( \phi_2 \) is similar. The equation for \( \phi_1 \) in the leading order of the perturbative calculation is

\[ i \partial_z \phi_1 + \partial_x^2 \phi_1 + \partial_y^2 \phi_1 = -2i \epsilon_3 |\psi_{20}|^2 \psi_{10}. \]

Note that in writing Eq. (7), we neglected the high-order terms containing \( \phi_j \) on the right hand side of the equation.

In solving the equation for \( \phi_1 \), we distinguish between two intervals along the \( z \) axis, the collision interval and the post-collision interval. These intervals are defined in terms of the collision distance \( z_c \), which is the distance at which the \( x \) coordinates of the beam centers coincide, i.e., \( x_1(z_c) = x_2(z_c) \). The collision interval is the small interval \( z_c - \Delta z_c/2 \leq z \leq z_c + \Delta z_c/2 \) centered about \( z_c \), in which the two beams are overlapping. The post-collision interval is the interval \( z > z_c + \Delta z_c/2 \), in which the beams are no longer overlapping.

2. Collision-induced effects in the collision interval

We substitute \( \psi_{j0}(x, y, z) = \Psi_{j0}(x, y, z) \exp[i \chi_{j0}(x, y, z)] \) and \( \phi_1(x, y, z) = \Phi_1(x, y, z) \exp[i \chi_{10}(x, y, z)] \), where \( \Psi_{j0} \) and \( \chi_{j0} \) are real-valued, into Eq. (7), and obtain
the following equation for $\Phi_1$:

$$
\begin{align*}
&i \partial_z \Phi_1 - (\partial_z \chi_{10}) \Phi_1 + [\partial_x^2 \Phi_1 + 2i (\partial_x \chi_{10}) \partial_x \Phi_1 \\
&+ i (\partial_x^2 \chi_{10}) \Phi_1 - (\partial_x \chi_{10})^2 \Phi_1] + [\partial_y^2 \Phi_1 + 2i (\partial_y \chi_{10}) \partial_y \Phi_1 \\
&+ i (\partial_y^2 \chi_{10}) \Phi_1 - (\partial_y \chi_{10})^2 \Phi_1] = -2i \epsilon_3 \Psi_{20}^2 \Psi_{10}. \\
\end{align*}
$$

(8)

Since the collision length $\Delta z_c$ is of order $1/|d_{11}|$, the term $i \partial_z \Phi_1$ is of order $|d_{11}| \times O(\Phi_1)$. Additionally, the term $-2i \epsilon_3 \Psi_{20}^2 \Psi_{10}$ is of order $\epsilon_3$. Equating the orders of $i \partial_z \Phi_1$ and $-2i \epsilon_3 \Psi_{20}^2 \Psi_{10}$, we find that $\Phi_1$ is of order $\epsilon_3/|d_{11}|$. In addition, $\Phi_1$ does not contain any fast dependence on $x$ and $y$, and $\chi_{10}$ does not contain any fast dependence on $x$, $y$, and $z$. As a result, all the other terms in Eq. (8) are of order $\epsilon_3/|d_{11}|$ or higher, and can therefore be neglected. It follows that in the leading order of the perturbative calculation, the equation for $\Phi_1$ is

$$
\partial_z \Phi_1 = -2 \epsilon_3 \Psi_{20}^2 \Psi_{10}.
$$

(9)

Equation (9) has the same form as the equation obtained for a fast collision between two pulses of the linear propagation equation in the presence of weak cubic loss in spatial dimension 1 [8, 10]. It also has the same form as the equation obtained for a fast collision between two solitons of the NLS equation in the presence of weak cubic loss in spatial dimension 1 [29].

We now introduce the following approximations to the solutions $\psi_{j0}(x, y, z)$ of Eqs. (5) and (6):

$$
\psi_{j0}(x, y, z) \simeq A_j(z) \tilde{\psi}_{j0}(x, y, z),
$$

(10)

where $A_j(z)$ are the $z$ dependent beam amplitudes, and

$$
\tilde{\psi}_{j0}(x, y, z) = \tilde{\Psi}_{j0}(x, y, z) \exp[i \chi_{j0}(x, y, z)],
$$

(11)

are the solutions to the unperturbed linear propagation equation with the initial condition (2) with unit amplitude. From Eqs. (10) and (11), it follows that

$$
\Psi_{j0}(x, y, z) \simeq A_j(z) \tilde{\Psi}_{j0}(x, y, z).
$$

(12)

Using Eqs. (11) and (11), we find that the total electric fields of the beams can be approximated by

$$
\psi_j(x, y, z) \simeq A_j(z) \tilde{\psi}_{j0}(x, y, z) + \phi_j(x, y, z).
$$

(13)
Note that the approximate expressions (10), (12), and (13) are used both inside and outside of the collision interval. In addition, the dynamics of the $A_j(z)$ that is associated with single-beam propagation is described in Appendix B.

The collision-induced amplitude shift of beam 1 is calculated from the collision-induced change in $\Phi_1$ in the collision interval, $\Delta \Phi_1(x, y, z_c) = \Phi_1(x, y, z_c + \Delta z_c/2) - \Phi_1(x, y, z_c - \Delta z_c/2)$. To calculate $\Delta \Phi_1(x, y, z_c)$, we substitute the approximations (12) for the $\Psi_j$ into Eq. (9), and integrate with respect to $z$ over the collision interval. This calculation yields

$$\Delta \Phi_1(x, y, z_c) = -2\epsilon_3 \int_{z_c - \Delta z_c/2}^{z_c + \Delta z_c/2} dz' A_1(z') A_2^2(z') \bar{\Psi}_{10}(x, y, z_c - d_{11} z').$$

(14)

Note that $\bar{\Psi}_{20}$ is the only function in the integrand on the right hand side of Eq. (14) that contains fast variations with respect to $z$, which are of order 1. Therefore, we can approximate the other functions $A_1(z)$, $A_2(z)$, and $\bar{\Psi}_{10}(x, y, z)$ by $A_1(z_c^-)$, $A_2(z_c^-)$, and $\bar{\Psi}_{10}(x, y, z_c)$, where $A_j(z_c^-)$ is the limit from the left of $A_j$ at $z_c$. Furthermore, in calculating the integral, we can take into account in an exact manner only the fast dependence of $\bar{\Psi}_{20}$ on $z$, i.e., the dependence on $z$ that is contained in the factors $\bar{x} = x - x_{20} - d_{11} z$, and replace $z$ by $z_c$ everywhere else in the expression for $\bar{\Psi}_{20}$. We denote this approximation of $\bar{\Psi}_{20}(x, y, z)$ by $\bar{\Psi}_{20}(\bar{x}, y, z_c)$. Carrying out all the aforementioned approximations in Eq. (14), we obtain:

$$\Delta \Phi_1(x, y, z_c) = -2\epsilon_3 A_1(z_c^-) A_2^2(z_c^-) \bar{\Psi}_{10}(x, y, z_c) \int_{z_c - \Delta z_c/2}^{z_c + \Delta z_c/2} dz' \bar{\Psi}_{20}(x - x_{20} - d_{11} z', y, z_c).$$

(15)

We assume that the integrand on the right hand side of Eq. (15) is sharply peaked at a small interval around the collision distance $z_c$. Under this assumption, we can extend the integral’s limits to $-\infty$ and $\infty$. We also change the integration variable from $z'$ to $\bar{x} = x - x_{20} - d_{11} z'$ and obtain:

$$\Delta \Phi_1(x, y, z_c) = -\frac{2\epsilon_3 A_1(z_c^-) A_2^2(z_c^-)}{|d_{11}|} \bar{\Psi}_{10}(x, y, z_c) \int_{-\infty}^{\infty} d\bar{x} \bar{\Psi}_{20}(\bar{x}, y, z_c).$$

(16)

We see that the $y$ dependence of beam 2 at $z = z_c$ affects the $y$ dependence of $\Delta \Phi_1(x, y, z_c)$, while the $x$ dependence of beam 2 does not affect the $x$ dependence of $\Delta \Phi_1(x, y, z_c)$. Thus,
inside the collision interval, the beam shape in the longitudinal direction is preserved, while
the beam shape in the transverse direction is changed by the collision. We also point out
that the collision-induced change in the beam shape is an effect that exists only in spatial
dimension higher than 1. Indeed, it was shown in Refs. [8, 10] that in the one-dimensional
case, the beam shape is preserved by the collision within the leading order of the perturbative
calculation.

In Appendix A, we show that the collision-induced amplitude shift of beam 1 $\Delta A_1^{(c)}$ is
related to $\Delta \Phi_1(x, y, z_c)$ by:

$$\Delta A_1^{(c)} = C_{p1}^{-1} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \tilde{\Psi}_{10}(x, y, z_c) \Delta \Phi_1(x, y, z_c),$$

(17)

where

$$C_{p1} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \tilde{\Psi}_{10}^2(x, y, 0).$$

(18)

Substituting Eq. (16) into Eq. (17), we obtain the following expression for the collision-
induced amplitude shift of beam 1 for the general initial condition (2):

$$\Delta A_1^{(c)} = -\frac{2\epsilon_3 A_1(z_c^-)A_2(z_c^-)}{C_{p1}|d_{11}|} \times \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \tilde{\Psi}_{10}^2(x, y, z_c) \int_{-\infty}^{\infty} dx \tilde{\Psi}_{20}^2(x, y, z_c).$$

(19)

3. Dynamics of $\phi_1(x, y, z)$ in the post-collision interval

In the post collision interval, i.e., for $z > z_c + \Delta z_c/2$, the two beams are no longer
overlapping. As a result, the inter-beam interaction terms $-2i\epsilon_3|\psi_2|^2\psi_1$ and $-2i\epsilon_3|\psi_1|^2\psi_2$
are negligible in this interval. Therefore, in the leading order of the perturbative calculation,
the equation describing the dynamics of $\phi_1(x, y, z)$ in the post-collision interval is the
unperturbed linear propagation equation

$$i\partial_z \phi_1 + \partial_x^2 \phi_1 + \partial_y^2 \phi_1 = 0.$$ 

(20)

To find the initial condition for Eq. (20), we first note that for $|d_{11}| \gg 1$, $\Delta \Phi_1(x, y, z_c)$ can
be written as

$$\Delta \Phi_1(x, y, z_c) \simeq \Phi_1(x, y, z_1^+) - \Phi_1(x, y, z_1^-) \simeq \Phi_1(x, y, z_1^+),$$

(21)
where $\Phi_1(x, y, z^+)$ is the limit from the right of $\Phi_1(x, y, z)$ at $z = z_c$. Thus, using the relation $\phi_1(x, y, z) = \Phi_1(x, y, z) \exp[i\chi_{10}(x, y, z)]$, the initial condition for Eq. (20) is:

$$\phi_1(x, y, z^+) = \Phi_1(x, y, z^+) \exp[i\chi_{10}(x, y, z_c)], \quad (22)$$

where $\Phi_1(x, y, z^+)$ is given by Eq. (16). The solution of Eq. (20) with the IC (22) can be written as

$$\phi_1(x, y, z) = \mathcal{F}^{-1}\left(\hat{\phi}_1(k_1, k_2, z^+_c) \exp[-i(k_1^2 + k_2^2)(z - z_c)]\right), \quad (23)$$

where $\hat{\phi}_1(k_1, k_2, z^+_c) = \mathcal{F}(\phi_1(x, y, z^+_c))$, and $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the inverse Fourier transform with respect to $x$ and $y$. In addition, the total electric field of beam 1 in the post-collision interval is given by Eq. (13), where $A_1(z)$ is given by Eq. (13) in Appendix A.

C. Calculation of the collision-induced changes in the beam shape and amplitude for a separable initial condition

1. Introduction

We now describe the collision-induced dynamics in the important case, where the initial condition for both beams is separable, i.e., it is given by Eq. (3). This case is of special importance for two main reasons. First, this initial condition describes the output electric field from many types of lasers [4, 5]. Second, in this case, it is possible to further simplify the expressions for the collision-induced changes of the beam shape and amplitude, and in this manner, obtain deeper insight into the collision dynamics.

It is straightforward to show that the solutions of the unperturbed linear propagation equation with the separable initial condition (3) and with unit amplitude can be written as

$$\tilde{\psi}_{j0}(x, y, z) = g_j^{(x)}(x, z)g_j^{(y)}(y, z) \exp(i\alpha_{j0}) =$$

$$G_j^{(x)}(x, z)G_j^{(y)}(y, z) \exp \left\{ i \left[ \chi_{j0}^{(x)}(x, z) + \chi_{j0}^{(y)}(y, z) + \alpha_{j0} \right] \right\}, \quad (24)$$

where

$$g_1^{(x)}(x, z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dk_1 f_1^{(x)}(k_1) \exp[-ik_1^2z + ik_1x],$$

$$g_2^{(x)}(x, z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dk_1 f_2^{(x)}(k_1) \exp[-id_{11}k_1z - ik_1^2z + ik_1x], \quad (25)$$

13
\[ g_j^{(y)}(y, z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dk_2 \hat{f}_j^{(y)}(k_2) \exp[-ik_2^2 z + ik_2 y], \]  \hspace{1cm} (26)

and \( G_j^{(x)}, G_j^{(y)}, \chi_{j0}^{(x)}, \) and \( \chi_{j0}^{(y)} \) are real-valued. The functions \( \hat{f}_j^{(x)} \) and \( \hat{f}_j^{(y)} \) in Eqs. (25) and (26) are defined by:

\[
\hat{f}_j^{(x)}(k_1) = \mathcal{F} \left( h_j^{(x)}[(x - x_{j0})/W_{j0}^{(x)}] \right), \hspace{1cm} (27)
\]

and

\[
\hat{f}_j^{(y)}(k_2) = \mathcal{F} \left( h_j^{(y)}[(y - y_{j0})/W_{j0}^{(y)}] \right). \hspace{1cm} (28)
\]

Using Eqs. (11) and (24), we obtain

\[
\tilde{\Psi}_{j0}(x, y, z) = G_j^{(x)}(x, z)G_j^{(y)}(y, z), \hspace{1cm} (29)
\]

and

\[
\chi_{j0}(x, y, z) = \chi_{j0}^{(x)}(x, z) + \chi_{j0}^{(y)}(y, z) + \alpha_{j0}. \hspace{1cm} (30)
\]

In addition, using the conservation of the total energy for the unperturbed linear propagation equation, the definitions of \( G_j^{(x)} \) and \( G_j^{(y)} \), and the initial condition (3), we obtain

\[
\int_{-\infty}^{\infty} dx G_j^{(x)}(x, z) = \int_{-\infty}^{\infty} dx G_j^{(x)}(x, 0) = W_{j0}^{(x)} \int_{-\infty}^{\infty} ds h_j^{(x)}(s) = W_{j0}^{(x)} c_{pj}^{(x)}, \hspace{1cm} (31)
\]

and

\[
\int_{-\infty}^{\infty} dy G_j^{(y)}(y, z) = \int_{-\infty}^{\infty} dy G_j^{(y)}(y, 0) = W_{j0}^{(y)} \int_{-\infty}^{\infty} ds h_j^{(y)}(s) = W_{j0}^{(y)} c_{pj}^{(y)}, \hspace{1cm} (32)
\]

where \( c_{pj}^{(x)} \) and \( c_{pj}^{(y)} \) are constants.

2. Collision-induced effects in the collision interval

We first obtain the general expression for \( \Delta \Phi_1(x, y, z_c) \) for an initial condition that is separable for both beams. For this purpose, we note that from the definition of \( \tilde{\Psi}_{20}(\tilde{x}, y, z) \)
it follows that at \( z = z_c \), \( \bar{\Psi}_{20}(\tilde{x}, y, z_c) = \bar{\Psi}_{20}(x, y, z_c) \). Using this relation along with Eq. (29), we obtain \( \bar{\Psi}_{20}(\tilde{x}, y, z_c) = G_2^{(x)}(x, z_c)G_2^{(y)}(y, z_c) \). It follows that:

\[
\int_{-\infty}^{\infty} d\tilde{x} \, \bar{\Psi}_{20}^{(x)}(\tilde{x}, y, z_c) = G_2^{(x)}(y, z_c) \int_{-\infty}^{\infty} dx \, G_2^{(x)}(x, z_c).
\]  

(33)

Employing the conservation law (31) in Eq. (33), we obtain:

\[
\int_{-\infty}^{\infty} d\tilde{x} \, \bar{\Psi}_{20}^{(x)}(\tilde{x}, y, z_c) = c_p^2 W_{20}^{(x)} G_2^{(y)}(y, z_c) \int_{-\infty}^{\infty} dx \, G_2^{(x)}(x, z_c).
\]  

(34)

Substitution of Eq. (34) into Eq. (16) yields the following expression for \( \Delta \Phi_1(x, y, z_c) \), which is valid for an initial condition that is separable for beam 2:

\[
\Delta \Phi_1(x, y, z_c) = -\frac{2\epsilon_3 A_1(z_c^-) A_2(z_c^-)}{|d_{11}|} c_p^2 W_{20}^{(x)} G_2^{(y)}(y, z_c) \bar{\Psi}_{10}(x, y, z_c).
\]  

(35)

Equation (35) is valid for a general initial condition for beam 1. When the initial condition for beam 1 is also separable, Eq. (35) takes the form

\[
\Delta \Phi_1(x, y, z_c) = -\frac{2\epsilon_3 A_1(z_c^-) A_2(z_c^-)}{|d_{11}|} c_p^2 W_{20}^{(x)} G_1^{(x)}(x, z_c) G_2^{(y)}(y, z_c) G_1^{(y)}(y, z_c) G_2^{(y)}(y, z_c).
\]  

(36)

We see that as in the case of the general initial condition (2), the shape of the beam in the longitudinal direction does not change inside of the collision interval. Moreover, it follows from Eq. (36) that for a separable initial condition, the shape of the beam in the longitudinal direction is not changed by the collision at all, i.e., for any \( z > z_c \) (within the leading order of the perturbative calculation). Indeed, for \( |d_{11}| \gg 1 \), Eq. (36) is also the initial condition for the dynamics of \( \phi_1(x, y, z) \) in the post-collision interval. We observe that this initial condition is separable. In addition, at \( z = z_c \) the \( x \) dependences of \( \phi_1 \) and \( \bar{\Psi}_{10} \) are identical.

Since in the post-collision region \( \phi_1 \) and \( \bar{\Psi}_{10} \) satisfy the same linear propagation equation with separable initial conditions, which have the same dependence on \( x \), the \( x \) dependences of \( \phi_1 \) and \( \bar{\Psi}_{10} \) remain identical for any \( z > z_c \). Thus, for a separable initial condition, the shape of the beam in the longitudinal direction is not changed by the collision at all.

We now turn to obtain the expression for \( \Delta A_1^{(c)} \) for an initial condition that is separable for both beams. Using the conservation of the total energy and Eqs. (31) and (32), we
obtain \( C_{p1} = c_{p1}^{(x)} c_{p1}^{(y)} W_{10}^{(x)} W_{10}^{(y)} \). In addition, using Eqs. (29) and (34), we find

\[
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \tilde{\Psi}_{10}^2(x, y, z_c) \int_{-\infty}^{\infty} d\tilde{x} \, \tilde{\Psi}_{20}^2(\tilde{x}, y, z_c) =
\]

\[
c_{p1}^{(x)} c_{p2}^{(x)} W_{10}^{(x)} W_{20}^{(x)} \int_{-\infty}^{\infty} dy \, G_1^{(y)2}(y, z_c) G_2^{(y)2}(y, z_c). \tag{37}
\]

Substituting Eq. (37) and the expression for \( C_{p1} \) into Eq. (19), we obtain the following expression for the collision-induced amplitude shift for a separable initial condition:

\[
\Delta A_1^{(c)} = -2\epsilon_3 A_1(z_c^-) A_2^2(z_c^-) \frac{c_{p1}^{(x)} W_{20}^{(x)}}{|d_{11}|} \int_{-\infty}^{\infty} dy \, G_1^{(y)2}(y, z_c) G_2^{(y)2}(y, z_c). \tag{38}
\]

Note that the expression for \( \Delta A_1^{(c)} \) has the form

\[
\Delta A_1^{(c)} = -(\text{overall factor}) \times (\text{longitudinal factor}) \times (\text{transverse factor}), \tag{39}
\]

where the overall factor is equal to \( 2\epsilon_3 A_1(z_c^-) A_2^2(z_c^-)/|d_{11}| \), and the longitudinal factor is \( c_{p2}^{(x)} W_{20}^{(x)} \). This form of the expression for \( \Delta A_1^{(c)} \) is expected to be valid for a general spatial dimension, when the initial condition is separable in the longitudinal direction for both beams. It is interesting to note that Eq. (38) is a generalization of the equation obtained for a fast two-pulse collision in spatial dimension 1. Indeed, using the notation of the current paper, the latter equation, which is Eq. (19) in Ref. [10], can be written as:

\[
\Delta A_1^{(c)(1D)} = -2\epsilon_3 A_1(z_c^-) A_2^2(z_c^-) c_{p2}^{(x)} W_{20}^{(x)}. \tag{40}
\]

We observe that the overall and longitudinal factors in the equation for the amplitude shift in spatial dimension 1 have the same form as the overall and longitudinal factors in spatial dimension 2, while the transverse factor in the one-dimensional case is equal to 1. We also observe that the longitudinal factor \( c_{p2}^{(x)} W_{20}^{(x)} \) is universal in the sense that it does not depend on the exact details of the initial pulse shapes and on the collision distance \( z_c \). In contrast, the transverse factor is not universal since it does depend on the details of the initial pulse shapes and on the collision distance. Therefore, the universality of the expression for \( \Delta A_1^{(c)} \) in the one-dimensional case, which was first demonstrated in Ref. [10], is extended to spatial dimension 2 (and to spatial dimension \( n \)), but in a somewhat restricted manner. That is, in
the two-dimensional (and the n-dimensional) case, only the overall and longitudinal parts of
the expression for $\Delta A_1^{(c)}$ are universal, and this is true when the initial condition is separable
in the longitudinal direction for both beams.

3. Dynamics of $\phi_1(x, y, z)$ in the post-collision interval

We now turn to analyze the dynamics of $\phi_1(x, y, z)$ in the post-collision interval. This
analysis is especially interesting for two main reasons. First, we showed in subsection II.B
that the collision induces a change of the beam shape in the transverse direction. Even
though this effect exists for a general initial condition, its simplest and clearest demonstration
is realized in the case of an initial condition that is separable for both beams. Furthermore,
since in both experiments and simulations the change in the beam shape is measured in the
post-collision interval, we must analyze the evolution of the beam shape in this interval.
Second, we claimed in section 2.3.2 that for a separable initial condition, the shape of the
beam in the longitudinal direction does not change at all due to the collision. This claim
can be directly proved by analyzing the dynamics of $\phi_1(x, y, z)$ in the post-collision interval.

In the post-collision interval, $\phi_1$ satisfies the unperturbed linear propagation equation
(20). Using Eqs. (22), (29), and (36), we find that the initial condition for Eq. (20) is

$$
\phi_1(x, y, z_c^+) = -\hat{a}_1(z_c^-)g_1^{(x)}(x, z_c^-)g_1^{(y)}(y, z_c^+)G_2^{(y)2}(y, z_c^-) \exp(i\alpha_{10}).
$$

(41)

This initial condition can be written as:

$$
\phi_1(x, y, z_c^+) = -\tilde{a}_1(z_c^-)g_1^{(x)}(x, z_c^-)g_1^{(y)}(y, z_c^+) \exp(i\alpha_{10}),
$$

(42)

where

$$
\tilde{a}_1(z_c^-) = 2\epsilon_3 A_1(z_c^-)A_2^2(z_c^-)c_{y2}^{(x)} W_2^{(x)} / |d_{11}|,
$$

(43)

and

$$
g_{12}^{(y)}(y, z_c^-) = g_1^{(y)}(y, z_c^-)C_2^{(y)2}(y, z_c^-).
$$

(44)

The Fourier transform of the initial condition (42) is

$$
\hat{\phi}_1(k_1, k_2, z_c^+) = -\hat{a}_1(z_c^-)g_1^{(x)}(k_1, z_c^-)g_{12}^{(y)}(k_2, z_c^-) \exp(i\alpha_{10}),
$$

(45)
where $\hat{g}_1^{(x)}$ and $\hat{g}_{12}^{(y)}$ are the Fourier transforms of $g_1^{(x)}$ and $g_{12}^{(y)}$ with respect to $x$ and $y$, respectively. Substituting Eq. (45) into Eq. (23), we obtain:

$$\phi_1(x, y, z) = -\tilde{a}_1(z_c) \mathcal{F}^{-1} \left( \hat{g}_1^{(y)}(k_1, z_c) \exp[-ik_1^2(z - z_c)] \right) \times \mathcal{F}^{-1} \left( \hat{g}_{12}^{(y)}(k_2, z_c) \exp[-ik_2^2(z - z_c)] \right) \exp(i\alpha_{10}).$$

(46)

Note that when the initial condition for beam 1 is separable, $\hat{g}_1^{(x)}(k_1, z_c) \exp[-ik_1^2(z - z_c)]$ is equal to $\hat{g}_1^{(x)}(k_1, z)$:

$$\hat{g}_1^{(x)}(k_1, z_c) \exp[-ik_1^2(z - z_c)] = \hat{g}_1^{(x)}(k_1, 0) \exp(-ik_1^2z_c) \exp[-ik_1^2(z - z_c)] = \hat{g}_1^{(x)}(k_1, 0) \exp(-ik_1^2z) = \hat{g}_1^{(x)}(k_1, z).$$

(47)

Substituting this relation into Eq. (46), we obtain the expression for $\phi_1(x, y, z)$ in the post-collision interval for a separable initial condition:

$$\phi_1(x, y, z) = -\tilde{a}_1(z_c) \hat{g}_1^{(x)}(x, z) \times \mathcal{F}^{-1} \left( \hat{g}_{12}^{(y)}(k_2, z_c) \exp[-ik_2^2(z - z_c)] \right) \exp(i\alpha_{10}).$$

(48)

We see that when the initial condition is separable for both beams, the $x$ dependences of $\phi_1(x, y, z)$ and $\tilde{\psi}_{10}(x, y, z)$ are identical for $z > z_c$. Therefore, as argued in subsection 2.3.2, the shape of the beam in the longitudinal direction does not change at all by the collision. Furthermore, the calculation of the modified beam shape in the transverse direction amounts to the calculation of the inverse Fourier transform of $\hat{g}_{12}^{(y)}(k_2, z_c) \exp[-ik_2^2(z - z_c)]$.

III. PERTURBATIVE CALCULATION AND NUMERICAL SIMULATIONS FOR NEW COLLISIONAL EFFECTS IN SPATIAL DIMENSION 2

A. Introduction

We now use the perturbation method of subsections II B and II C along with numerical simulations with Eq. (4) to demonstrate four important effects and properties of the collision, which either exist only in spatial dimension higher than 1, or are qualitatively different from their one-dimensional counterparts. These four effects and properties are: (1) universality of the longitudinal part in the expression for the collision-induced amplitude shift, (2) the effect of partial beam overlap, (3) the effect of anisotropy in the initial condition,
the collision-induced change in the beam shape in the transverse direction. For each effect or property, we first use the perturbation theory to obtain explicit formulas, which demonstrate the collisional effect or property. Since these formulas are only approximate expressions, which are based on a number of simplifying assumptions of the perturbative calculation, it is important to check their validity by numerical simulations with the perturbed linear propagation equation (1). Therefore, in the current subsection, we also take on this important numerical investigation, by carrying out extensive numerical simulations with Eq. (1), and by comparing the simulations results with the approximate predictions of the perturbation theory for each of the four collisional effects and properties. We solve Eq. (1) numerically by the split-step method with periodic boundary conditions [46, 47].

B. Universality of the longitudinal part in the expression for the amplitude shift

In subsection II C, we showed that for a separable initial condition, the longitudinal factor in the expression for the collision-induced amplitude shift is universal in the sense that it does not depend on the exact details of the initial beam shapes. In contrast, the transverse factor is not universal, since it does depend on the details of the beam shapes and on the collision distance. Thus, according to the perturbative calculation, the universality of the expression for \( \Delta A_1^{(c)} \) is extended from spatial dimension 1 to higher spatial dimensions, but in a somewhat restricted manner.

In the current subsection, we demonstrate the universality of the longitudinal part in the expression for the collision-induced amplitude shift. For this purpose, we first obtain explicit expressions for \( \Delta A_1^{(c)} \) for two initial beam shapes that have widely different dependences on the \( x \) coordinate. Moreover, we verify the validity of the expressions for \( \Delta A_1^{(c)} \) by extensive numerical simulations with Eq. (1). This numerical investigation is especially important, since it shows that the approximations used in the perturbative calculation are indeed valid for widely different beam shapes. In this manner, the extensive numerical simulations with Eq. (1) help validate the universal nature of the longitudinal part in the expression for \( \Delta A_1^{(c)} \). The initial \( x \) dependence for the first beam type that we consider is Gaussian, i.e., it is rapidly decreasing with increasing distance from the beam center. In contrast, the initial \( x \) dependence for the second beam type that we consider is given by a Cauchy-Lorentz distribution, i.e., it decreases slowly (as a power-law) with increasing distance from the
beam center. The initial beam profile in the transverse direction is taken as Gaussian, as this choice enables the explicit calculation of the integral with respect to $y$ on the right hand side of Eq. (38). Therefore, the two initial conditions for the two-beam collision problem are

$$\psi_1(x, y, 0) = A_1(0) \exp \left[ -\frac{x^2}{2W_{10}^2} - \frac{y^2}{2W_{10}^2} + i\alpha_{10} \right],$$

$$\psi_2(x, y, 0) = A_2(0) \exp \left[ -\frac{(x - x_{20})^2}{2W_{20}^2} - \frac{y^2}{2W_{20}^2} + i\alpha_{20} \right],$$

(49)

for Gaussian beams, and

$$\psi_1(x, y, 0) = A_1(0) \left[ 1 + \frac{2x^4}{W_{10}^4} \right]^{-1} \exp \left[ -\frac{y^2}{2W_{10}^2} + i\alpha_{10} \right],$$

$$\psi_2(x, y, 0) = A_2(0) \left[ 1 + \frac{2(x - x_{20})^4}{W_{20}^4} \right]^{-1} \exp \left[ -\frac{y^2}{2W_{20}^2} + i\alpha_{20} \right],$$

(50)

for Cauchy-Lorentz-Gaussian beams.

Let us obtain the expression for the collision-induced amplitude shift for the initial conditions (49) and (50). From Eq. (D11) in Appendix D it follows that for both initial conditions

$$G_j(y, z_c) = \frac{W_j'(y)}{(W_j')^4 + 4z_c^2} \exp \left[ -\frac{W_j'(y)^2 y^2}{2(W_j')^4 + 4z_c^2} \right],$$

(51)

where $j = 1, 2$. In addition, $c_{j1}^{(y)} = \pi^{1/2}$ for both initial conditions, $c_{j2}^{(x)} = \pi^{1/2}$ for Gaussian beams, and $c_{j2}^{(x)} = 3\pi/2^{11/4}$ for Cauchy-Lorentz-Gaussian beams. We now substitute Eq. (51) along with the values of $c_{j1}^{(y)}$ and $c_{j2}^{(x)}$ into Eq. (38), and perform the integration with respect to $y$. This calculation yields the following expression for $\Delta A_1^{(c)}$:

$$\Delta A_1^{(c)} = -\frac{2b_{13}A_1(z_c^-)A_2^2(z_c^-)}{|d_{11}|} \times \frac{W_{10}'W_{20}'(W_{10}')^2}{(W_{10}'^2 + W_{20}'^2)^{1/2}(W_{10}'^2W_{20}'^2 + 4z_c^2)^{1/2}},$$

(52)

where $b = \pi^{1/2}$ for Gaussian beams, and $b = 3\pi/2^{11/4}$ for Cauchy-Lorentz-Gaussian beams. The longitudinal part in the expression for $\Delta A_1^{(c)}$, $c_{j2}^{(x)}W_{20}' = bW_{20}'$, is clearly universal. In contrast, the transverse part in the expression, which is given by:

$$\text{transverse factor} = \frac{W_{10}'W_{20}'}{(W_{10}'^2 + W_{20}'^2)^{1/2}(W_{10}'^2W_{20}'^2 + 4z_c^2)^{1/2}}.$$

(53)
depends on \( z_c \), and does not seem to have a simple universal form. One aspect of the nonuniversal nature of the expression for \( \Delta A_1^{(c)} \) in spatial dimension 2 is the deviation of the dependence on \( |d_{11}| \) from the \( 1/|d_{11}| \) scaling, which is observed in the one-dimensional case \[8, 10\], and also in fast collisions between NLS solitons in the presence of nonlinear dissipation in spatial dimension 1 \[29, 48\]. Note that the collision distance \( z_c \) satisfies \( z_c = (x_{10} - x_{20})/d_{11} \). Therefore, the deviation of the \( |d_{11}| \) dependence of \( \Delta A_1^{(c)} \) from the \( 1/|d_{11}| \) scaling is due to the term \( 4(x_{10} - x_{20})^2/d_{11}^2 \) inside the factor \( (W_{10}^{(y)2}W_{20}^{(y)2} + 4(x_{10} - x_{20})^2/d_{11}^2)^{1/2} \) on the right hand side of Eq. \[52\]. It is useful to define the quantity \( \Delta A_1^{(c)(s)} \), which is the approximate expression for the amplitude shift that is obtained from the full expression by neglecting the \( (x_{10} - x_{20})^2/d_{11}^2 \) term. Carrying out the latter approximation, we find:

\[
\Delta A_1^{(c)(s)} = - \frac{2 \epsilon_3 A_1(z_-)A_2^2(z_-)}{|d_{11}|} \frac{W_{20}^{(x)}W_{20}^{(y)}}{(W_{10}^{(y)2} + W_{20}^{(y)2})^{1/2}}.
\]

(54)

Therefore, the difference \( |\Delta A_1^{(c)} - \Delta A_1^{(c)(s)}| \) is a measure for the deviation of the \( d_{11} \) dependence of \( \Delta A_1^{(c)} \) from the \( 1/|d_{11}| \) scaling observed in the one-dimensional case. Since in a complete collision \( |x_{20} - x_{10}| \gg 1 \), the term \( 4(x_{10} - x_{20})^2/d_{11}^2 \) is not necessarily small for intermediate values of \( |d_{11}| \). As a result, the deviation from the \( 1/|d_{11}| \) scaling might be significant even for intermediate \( |d_{11}| \) values.

To check the perturbation theory predictions for universality of the longitudinal part in the expression for \( \Delta A_1^{(c)} \), we carry out numerical simulations with Eq. \[11\] with the two initial conditions \[49\] and \[50\], which possess widely different initial beam profiles in the longitudinal direction. The extensive simulations with these initial conditions provide a careful test for the validity of the perturbation theory approximations for widely different beam shapes, and in this manner, help confirm the universality of the longitudinal part in the expression for \( \Delta A_1^{(c)} \). We carry out the simulations for \( d_{11} \) values in the intervals \( 4 \leq |d_{11}| \leq 60 \). For concreteness, we present the results of the simulations with \( \epsilon_3 = 0.01 \). The parameter values of the initial conditions \[49\] and \[50\] are \( A_j(0) = 1, \alpha_{j0} = 0, x_{20} = \pm 20, W_{10}^{(x)} = 3, W_{10}^{(y)} = 5, W_{20}^{(x)} = 4, \) and \( W_{20}^{(y)} = 6 \). The final propagation distance is \( z_f = 2z_c = -2x_{20}/d_{11} \). The values of \( x_{20} \) and \( z_f \) ensure that the centers of the two beams are well separated at \( z = 0 \) and at \( z = z_f \). We point out that results similar to the ones described in the current subsection are obtained in simulations with other parameter values. For each initial condition, we compare the dependence of \( \Delta A_1^{(c)} \) on \( d_{11} \) obtained in the simulations with the perturbation theory prediction of Eq. \[52\], and with the more crude approximation
$\Delta A_1^{(c)(s)}$ of Eq. (54). We also discuss the behavior of the relative errors in the approximation of $\Delta A_1^{(c)}$ (in percentage), which are defined by $E_r^{(1)} = |\Delta A_1^{(c)(num)} - \Delta A_1^{(c)(th)}| \times 100/|\Delta A_1^{(c)(th)}|$ and $E_r^{(2)} = |\Delta A_1^{(c)(num)} - \Delta A_1^{(c)(s)}| \times 100/|\Delta A_1^{(c)(s)}|$, respectively.

We start by discussing the results of the simulations for Gaussian beams, which represent beams with rapidly decaying tails. The initial beam shapes $|\psi_j(x,y,0)|$, and the beam shapes $|\psi_j(x,y,z)|$ obtained in the simulation with $d_{11} = 10$ at the intermediate distance $z_i = 2.4 > z_c [49]$, and at the final distance $z_f = 4$ are shown in Fig. 1. We observe that the beams undergo broadening due to diffraction without developing significant tails. In addition, the maximum values of $|\psi_j(x,y,z)|$ decrease with increasing $z$, mainly due to diffraction. The dependence of $\Delta A_1^{(c)}$ on $d_{11}$ that is obtained in the simulations is shown in Fig. 2 along with the analytic prediction of Eq. (52) and the more crude approximation $\Delta A_1^{(c)(s)}$ of Eq. (54). We observe that despite the diffraction-induced broadening of the beams, the agreement between the simulations result and the analytic prediction of Eq. (52) is very good. In particular, the relative error $E_r^{(1)}$ is less than 3.5% for $10 \leq |d_{11}| \leq 60$ and less than 5.1% for $4 \leq |d_{11}| < 10$. We also observe that the two theoretical curves for $\Delta A_1^{(c)}$ and $\Delta A_1^{(c)(s)}$ are close to each other. As a result, the relative error $E_r^{(2)}$ is only somewhat larger than $E_r^{(1)}$. More specifically, the value of $E_r^{(2)}$ is smaller than 4.4% for $10 \leq |d_{11}| \leq 60$ and smaller than 10.0% for $4 \leq |d_{11}| < 10$. This means that the deviation of the $d_{11}$ dependence of $\Delta A_1^{(c)}$ from the $1/|d_{11}|$ scaling is not significant for $|d_{11}| \geq 10$. However, it should be noted that the latter result is due to the choice of the values of $W_{10}^{(g)}$ and $W_{20}^{(g)}$, i.e., for smaller values of these parameters we can observe significantly larger deviations from the $1/|d_{11}|$ scaling.

We now turn to discuss the simulations results for beams, whose tails exhibit slow decay in the longitudinal direction. For such beams it is unclear if the sharp-peak approximation, which is used in the derivation of Eq. (16) from Eq. (14), is valid. Therefore, in this case, the numerical simulations of the two-beam collision serve as an important check of the perturbation theory predictions. We use the Cauchy-Lorentz-Gaussian beams of Eq. (50) as prototypical examples for beams, whose tails exhibit slow (power-law) decay with increasing distance from the beam center. Figure 3 shows the initial beam shapes $|\psi_j(x,y,0)|$, and the beam shapes $|\psi_j(x,y,z)|$ obtained in the simulation with $d_{11} = 10$ at $z = z_i = 2.4$, and at $z = z_f = 4$. We observe that the beams experience broadening and develop extended tails due to diffraction. In addition, the maximum values of $|\psi_j(x,y,z)|$ decrease with increasing...
FIG. 1: (Color online) Contour plots of the beam shapes $|\psi_j(x, y, z)|$ at $z = 0$ (a), $z = z_i = 2.4$ (b), and $z = z_f = 4$ (c) in a fast collision between two Gaussian beams with parameter values $\epsilon_3 = 0.01$ and $d_{11} = 10$. The plots represent the beam shapes obtained by numerical solution of Eq. (1) with the initial condition (49).

$z$, mainly due to diffraction. The latter decrease is especially noticeable for beam 1. This can be explained by noting that $W_{10}^{(x)} < W_{20}^{(x)}$, and as a result, diffraction-induced beam broadening and generation of extended tails are stronger for beam 1 compared with beam 2. The $d_{11}$ dependence of $\Delta A_1^{(c)}$ obtained in the simulations is shown in Fig. 4 together with the analytic predictions $\Delta A_1^{(c)}$ and $\Delta A_1^{(c)(s)}$ of Eqs. (52) and (54). The agreement between the simulations result for $\Delta A_1^{(c)}$ and the perturbation theory prediction of Eq. (52) is very
FIG. 2: (Color online) Dependence of the collision-induced amplitude shift of beam 1 $\Delta A_1^{(c)}$ on the beam-steering coefficient $d_{11}$ in a fast collision between two Gaussian beams for $\varepsilon_3 = 0.01$. The red circles represent the result obtained by numerical simulations with Eq. (1) with the initial condition (49). The solid blue and dashed green curves represent the theoretical predictions of Eqs. (52) and (54), respectively.

good despite the beam broadening and the generation of extended beam tails. In particular, the value of the relative error $E_r^{(1)}$ is smaller than 2.8% for $10 \leq |d_{11}| \leq 60$ and smaller than 5.1% for $4 \leq |d_{11}| < 10$. Note that these values are comparable to the values of $E_r^{(1)}$ for collisions between Gaussian beams. Thus, based on the results of our simulations, we conclude that the longitudinal part in the expression for $\Delta A_1^{(c)}$, $c_p^{(p)} W_{0}^{(p)}$, is indeed universal in the sense that it is not sensitive to the exact form of the initial beam shapes. We also note that the value of $E_r^{(2)}$ is smaller than 4.3% for $10 \leq |d_{11}| \leq 60$ and smaller than 9.2% for $4 \leq |d_{11}| < 10$. Thus, the deviation of the $d_{11}$ dependence of $\Delta A_1^{(c)}$ from the $1/|d_{11}|$ scaling is not significant for $|d_{11}| \geq 10$ for the parameter values used in our simulations.

C. Fast collisions between partially overlapping beams

Another important property of a complete two-beam collision is related to the relative location of the beam centers at the collision distance $z_c$. When the $y$ coordinates of the beams are equal at $z_c$, $y_1(z_c) = y_2(z_c)$, we say that the beams are fully overlapping at $z_c$. In contrast, when the $y$ coordinates of the beams are not equal at $z_c$, $y_1(z_c) \neq y_2(z_c)$, we say that the beams are only partially overlapping at $z_c$. It is clear that complete collisions between two partially overlapping beams exist only in spatial dimension higher than 1, since
FIG. 3: (Color online) Contour plots of the beam shapes $|\psi_j(x, y, z)|$ at $z = 0$ (a), $z = z_i = 2.4$ (b), and $z = z_f = 4$ (c) in a fast collision between two Cauchy-Lorentz-Gaussian beams with parameter values $\epsilon_3 = 0.01$ and $d_{11} = 10$. The plots represent the beam shapes obtained by numerical solution of Eq. (1) with the initial condition (50).

in spatial dimension 1, the two beams are always fully overlapping at $z_c$ in a complete fast collision. It is therefore interesting to employ the perturbation theory for studying the effect of the partial overlap between the colliding beams in 2D on the collision-induced amplitude shift. This problem is of further interest, since we can use it for checking the perturbation theory prediction for the transverse part in the expression for the amplitude shift in a nontrivial setup. Thus, in the current subsection, we investigate the dynamics of
FIG. 4: (Color online) Dependence of the collision-induced amplitude shift of beam 1 $\Delta A_1^{(c)}$ on the beam-steering coefficient $d_{11}$ in a fast collision between two Cauchy-Lorentz-Gaussian beams for $\epsilon_3 = 0.01$. The red circles represent the result obtained by numerical simulations with Eq. (1) with the initial condition (50). The solid blue and dashed green curves represent the perturbation theory predictions of Eqs. (52) and (54), respectively.

To demonstrate the effects of partial overlap on the collision-induced amplitude shift, we consider an initial condition in the form of two Gaussian beams with different initial values of the $y$ coordinates of the beam centers, $y_{10} \neq y_{20}$. For simplicity and without loss of generality, we assume that the initial beam widths satisfy $W_{10}^{(x)} = W_{20}^{(x)} \equiv W_0^{(x)}$ and $W_{10}^{(y)} = W_{20}^{(y)} \equiv W_0^{(y)}$. Therefore, the initial condition for the two-beam collision problem is given by:

$$\psi_j(x, y, 0) = A_j(0) \exp \left[ -\frac{(x-x_{j0})^2}{2W_0^{(x)2}} - \frac{(y-y_{j0})^2}{2W_0^{(y)2}} + i\alpha_{j0} \right], \quad (55)$$

for $j = 1, 2$. Since the initial condition is separable for both beams, we can use Eq. (38) for calculating the collision-induced amplitude shift. For this initial condition, $c_{p1}^{(y)} = c_{p2}^{(x)} = \pi^{1/2}$, and the functions $G_j^{(y)}$ are given by Eq. (51) with $W_{10}^{(y)} = W_{20}^{(y)} = W_0^{(y)}$. Substituting these expressions into Eq. (38) and integrating with respect to $y$, we obtain:

$$\Delta A_1^{(c)} = -\left(2\pi\right)^{1/2}\epsilon_3 A_1(z_c^-)A_2(z_c^-)$$

$$\times \left|d_{11}\right| \frac{W_0^{(x)}W_0^{(y)2}}{(W_0^{(y)4} + 4z_c^2)^{1/2}} \exp \left[ -\frac{W_0^{(y)2}(y_{20} - y_{10})^2}{2(W_0^{(y)4} + 4z_c^2)} \right]. \quad (56)$$
Thus, the effect of partial beam overlap on the amplitude shift is contained in the transverse part of the expression for $\Delta A_1^{(c)}$:

$$\text{transverse factor} = \frac{W_0(y)^2}{2^{1/2}(W_0(y)^4 + 4z_c^2)^{1/2}} \exp \left[ -\frac{W_0(y)^2(y_{20} - y_{10})^2}{2(W_0(y)^4 + 4z_c^2)} \right].$$

(57)

We see that $\Delta A_1^{(c)}$ is a Gaussian function of the separation between the beam centers at $z_c$, $y_{20} - y_{10}$. The width of the Gaussian function is equal to $2^{1/2}W(y)(z_c)$, where $W(y)(z_c) = (W_0(y)^2 + 4z_c^2/W_0(y)^2)^{1/2}$ is the width of both beams in the transverse direction at $z_c$.

It is unclear if the approximations used by the perturbation theory hold when the separation between the beam centers at $z_c$ is relatively large. For this reason, it is important to check the predictions of Eqs. (56) and (57) by numerical solution of Eq. (1). We take on this important numerical investigation by carrying out simulations with Eq. (1) and by measuring the dependence of $\Delta A_1^{(c)}$ on $y_{20} - y_{10}$. For brevity, we describe the results of these simulations briefly without showing the corresponding figures. The physical parameter values are $\epsilon_3 = 0.01$ and $d_{11} = 20$. The initial values of the beam parameters in Eq. (55) are $A_j(0) = 1$, $\alpha_j = 0$, $x_{10} = 0$, $x_{20} = -20$, $y_{10} = 0$, $W_0(x) = 4$, $W_0(y) = 5$, and the value of $y_{20}$ is varied in the interval $-10 \leq y_{20} \leq 10$. The final propagation distance is $z_f = 2$, and the beam centers are well separated at $z = 0$ and at $z = z_f$. In addition to $\Delta A_1^{(c)}$, we measure the relative error (in percentage) $|\Delta A_1^{(c)}(\text{num}) - \Delta A_1^{(c)(th)}| \times 100/|\Delta A_1^{(c)(th)}|$, where $\Delta A_1^{(c)(th)}$ is given by Eq. (56). The agreement between the numerical simulations result and the perturbation theory prediction is very good. More specifically, the relative error is smaller than 7.9% in the entire interval $-10 \leq y_{20} \leq 10$. Thus, our numerical simulations confirm the prediction of Eqs. (56) and (57) for Gaussian dependence of $\Delta A_1^{(c)}$ and its transverse factor on $y_{20} - y_{10}$. Furthermore, these simulations demonstrate that the perturbation method of subsections II B and II C is applicable for fast collisions between partially overlapping beams, even when the beam centers are relatively far from each other at the collision distance $z_c$.

D. Dependence of the amplitude shift on the orientation angle between the beams

Another phenomenon that exists only in spatial dimension higher than 1 is associated with direction dependent collision-induced effects, i.e., with effects that exist due to some anisotropy in the system. In particular, we are interested in studying the effects of anisotropy
in the initial condition. In a simple setup, this anisotropy can be characterized by a single angle, e.g., the angle $\theta_0$ between a “preferred” direction in the initial condition and the $x$ axis. To illustrate this situation, consider the case where the initial shape of beam $1$ $|\psi_1(x, y, 0)|$ is wider along one direction that we denote by $x'$, and narrower along the perpendicular direction that we denote by $y'$. We can then define the angle $\theta_0$, as the angle that the $x'$ axis forms with the $x$ axis of our coordinate system. Thus, $\theta_0$ is the angle between the relative velocity vector (between the beam centers) and the $x'$ axis of beam 1. In addition, if beam 2 is circularly symmetric, or is elongated along the $x$ or the $y$ axes, then $\theta_0$ can also be regarded as the orientation angle between the two beams. An important question about the collision dynamics in this anisotropic setup concerns the dependence of the collision-induced amplitude shift $\Delta A_1^{(\text{c})}$ on the orientation angle $\theta_0$. In the current subsection, we address this important question by both analytic calculations and numerical simulations with Eq. (1).

We consider the following anisotropic collision setup, which consists of two initially well separated Gaussian beams. In this setup, beam 1 is elongated along its $x'$ axis, which forms an angle $\theta_0$ with the $x$ axis, while beam 2 is circular in the $xy$ plane. Figure 5 shows the contour plot of $|\psi_2(x, y, 0)|$ for this initial condition in the case where $\theta_0 = \pi/4$. The initial condition can be written as:

$$
\psi'_1(x', y', 0) = A_1(0) \exp \left[ -\frac{x'^2}{2W_{10}^{(x)^2}} - \frac{y'^2}{2W_{10}^{(y)^2}} + i\alpha_{10} \right],
$$

and

$$
\psi_2(x, y, 0) = A_2(0) \exp \left[ -\frac{(x - x_{20})^2}{2W_{20}^{(x)^2}} - \frac{y^2}{2W_{20}^{(y)^2}} + i\alpha_{20} \right],
$$

where $\psi'_1(x', y', z)$ denotes the electric field of beam 1 in the $(x', y', z)$ coordinate system, $W_{10}^{(x)}>W_{10}^{(y)}$, and

$$
x' = x \cos \theta_0 + y \sin \theta_0,
$$

$$
y' = y \cos \theta_0 - x \sin \theta_0.
$$

Substituting relation (60) into Eq. (58), we obtain:

$$
\psi_1(x, y, 0) = A_1(0) \exp \left[ -B_1 x^2 - B_2 y^2 - B_3 xy + i\alpha_{10} \right],
$$

where

$$
B_1 = \frac{\cos^2 \theta_0}{2W_{10}^{(x)^2}} + \frac{\sin^2 \theta_0}{2W_{10}^{(y)^2}}.
$$
FIG. 5: (Color online) A contour plot of the initial beam shapes $|\psi_j(x, y, 0)|$ for the anisotropic initial condition of Eqs. (59) and (61). In this example, the initial beam widths are $W_{10}^{(x)} = 8$, $W_{10}^{(y)} = 2$, and $W_{20} = 2$, and the orientation angle is $\theta_0 = \pi/4$.

$$B_2 = \frac{\sin^2 \theta_0}{2W_{10}^{(x)^2}} + \frac{\cos^2 \theta_0}{2W_{10}^{(y)^2}},$$

and

$$B_3 = \left(\frac{1}{W_{10}^{(x)^2}} - \frac{1}{W_{10}^{(y)^2}}\right) \sin \theta_0 \cos \theta_0.$$

Notice that the initial condition for beam 1 is not separable in the $(x, y, z)$ coordinate system. Therefore, the investigation described in the current subsection also provides an example for collision-induced dynamics in a collision with a nonseparable initial condition.

The initial condition for beam 1 is nonseparable, and therefore we need to calculate $\Delta A_{1}^{(c)}$ by using the general expression, which is given by Eq. (19). It is straightforward to show that for the current setup, $C_{p1} = \pi W_{10}^{(x)} W_{10}^{(y)}$. Furthermore, since the initial condition for beam 2 is separable, we can use Eq. (34), where $c_{p2} = \pi^{1/2}$. Substitution of these relations into Eq. (19) yields

$$\Delta A_{1}^{(c)} = -\frac{2\epsilon_3 A_1(z_c^-) A_2^2(z_c^-)}{\pi^{1/2}|d_{11}|} \frac{W_{20}}{W_{10}^{(x)} W_{10}^{(y)}} \times \int_{-\infty}^{\infty} dy G_{2}^{(y)^2}(y, z_c) \int_{-\infty}^{\infty} dx \tilde{\Psi}_{10}^{2}(x, y, z_c).$$

Since diffraction is isotropic, the unperturbed linear propagation equation for beam 1 in the $(x', y', z)$ coordinate system has the same form as in the $(x, y, z)$ coordinate system. Thus, the unperturbed propagation equation for $\tilde{\psi}_{10}'$ is:

$$i\partial_{x'} \tilde{\psi}'_{10} + \partial_{x'}^2 \tilde{\psi}'_{10} + \partial_{y'}^2 \tilde{\psi}'_{10} = 0.$$
Therefore, we can calculate $\tilde{\Psi}_{10}(x, y, z)$ by solving Eq. (63) with the initial condition (58) in the $(x', y', z)$ coordinate system, and by using Eq. (60) to express the solution in the $(x, y, z)$ coordinate system. The solution of Eq. (63) with the Gaussian initial condition (58) is described in Appendix D. Using Eqs. (D8)-(D12) in this Appendix, we obtain:

$$\tilde{\Psi}'_{10}(x', y', z) = \frac{W^{(x)}_{10} W^{(y)}_{10}}{(W^{(x)}_{10} + 4z^2)^{1/4}(W^{(y)}_{10} + 4z^2)^{1/4}} \times \exp \left[ - \frac{W^{(x)}_{10} x^2 + W^{(y)}_{10} y^2}{2(W^{(x)}_{10} + 4z^2)} - \frac{W^{(y)}_{10} y^2}{2(W^{(y)}_{10} + 4z^2)} \right].$$

(64)

Using the transformation relations (60) in Eq. (64), and using Eq. (51) with $W^{(y)}_{20} = W_{20}$ for $G^{(y)}_{2}(y, z_c)$, we obtain:

$$G^{(y)}_{2}(y, z_c)\tilde{\Psi}_{10}(x, y, z_c) = \frac{W^{(x)}_{10} W^{(y)}_{10} W^{(y)}_{20}}{(W^{(x)}_{10} + 4z^2)^{1/2}(W^{(y)}_{10} + 4z^2)^{1/2}(W^{(y)}_{20} + 4z^2)^{1/2}} \times \exp \left[ -b_1^2 x^2 - 2b_2 x y - b_3^2 y^2 \right],$$

(65)

where

$$b_1 = \left( \frac{W^{(x)}_{10} \cos^2 \theta_0 + W^{(y)}_{10} \sin^2 \theta_0}{W^{(x)}_{10} + 4z_c^2} \right)^{1/2},$$

(66)

$$b_2 = \left( \frac{W^{(x)}_{10} \cos^2 \theta_0}{W^{(x)}_{10} + 4z^2} - \frac{W^{(y)}_{10}}{W^{(y)}_{10} + 4z_c^2} \right) \sin \theta_0 \cos \theta_0,$$

(67)

and

$$b_3 = \left( \frac{W^{(y)}_{20}}{W^{(x)}_{20} + 4z_c^2} + \frac{W^{(y)}_{10} \sin^2 \theta_0}{W^{(y)}_{10} + 4z_c^2} + \frac{W^{(y)}_{10} \cos^2 \theta_0}{W^{(y)}_{10} + 4z_c^2} \right)^{1/2}.$$

(68)

Substituting Eq. (65) into Eq. (62) and carrying out the double integration, we obtain the following expression for the collision-induced amplitude shift:

$$\Delta A_1^{(c)} = -\frac{2\pi^{1/2} \epsilon_3 A_1^0(z_c^-) A_2^0(z_c^-)}{|d_{11}|} \times \frac{W^{(x)}_{10} W^{(y)}_{10} W^{3}_{20}}{(W^{(x)}_{10} + 4z_c^2)^{1/2}(W^{(y)}_{10} + 4z_c^2)^{1/2}(W^{(y)}_{20} + 4z_c^2)^{1/2}(b_1^2 b_2^2 - b_3^2)^{1/2}},$$

(69)

where

$$b_1^2 b_2^2 - b_3^2 = \frac{W^{(y)}_{20}}{W^{(y)}_{10} + 4z_c^2} \left( \frac{W^{(x)}_{10} \cos^2 \theta_0}{W^{(y)}_{10} + 4z_c^2} + \frac{W^{(y)}_{10} \sin^2 \theta_0}{W^{(y)}_{10} + 4z_c^2} \right) + \frac{W^{(y)}_{10} W^{(y)}_{20}}{(W^{(y)}_{10} + 4z_c^2)(W^{(y)}_{10} + 4z_c^2)}.$$

(70)
We see that even the relatively simple anisotropic setup of the two-beam collision considered in the current subsection leads to a nontrivial dependence of $\Delta A_1^{(c)}$ on the orientation angle $\theta_0$. This nontrivial dependence on $\theta_0$ can also be associated with the nonseparable nature of the initial condition for beam 1.

We check the predictions of Eq. (69) for the dependence of the collision-induced amplitude shift on $\theta_0$ by carrying out numerical simulations with Eq. (1) with the initial condition of Eqs. (59) and (61). The simulations are carried out for $\theta_0$ values in the interval $0 \leq \theta_0 \leq \pi/2$. The physical parameter values are $\epsilon_3 = 0.01$ and $d_{11} = 20$. The initial values of the beam parameters in Eqs. (59) and (61) are $A_j(0) = 1$, $\alpha_j(0) = 0$, $x_{20} = -20$, $W_1^{(x)} = 8$, $W_1^{(y)} = 2$, and $W_20 = 2$. The final propagation distance is $z_f = 2$, and therefore, the beam centers are well separated at $z_f$. Figure 6 shows the initial beam shapes $|\psi_j(x,y,0)|$, and the beam shapes $|\psi_j(x,y,z)|$ obtained in the numerical simulation with $\theta_0 = \pi/4$ at the intermediate distance $z_i = 1.2$, and at $z_f = 2$. We observe that both beams experience broadening due to diffraction but do not develop extended tails. The dependence of $\Delta A_1^{(c)}$ on the orientation angle $\theta_0$ is shown in Fig. 7. We observe very good agreement between the perturbation theory prediction and the numerical simulations result. In particular, the numerical simulations confirm the expectation that for the chosen parameter values, the value of $|\Delta A_1^{(c)}|$ would be larger for smaller orientation angles, since in this case, beam 2 traverses through the wider part of beam 1. Furthermore, the relative error (in percentage) $|\Delta A_1^{(c)(num)} - \Delta A_1^{(c)(th)}| \times 100/|\Delta A_1^{(c)(th)}|$ is smaller than 6.3% in the entire interval $0 \leq \theta_0 \leq \pi/2$. Therefore, the numerical simulations with Eq. (1) confirm the perturbation theory prediction for a nontrivial dependence of $\Delta A_1^{(c)}$ on $\theta_0$ due to the anisotropic (and nonseparable) nature of the initial condition.

E. Collision-induced change in the beam shape

It was shown in subsection II B that the two-beam collision in the presence of weak cubic loss leads to a change of the beam shape in the transverse direction. In contrast, it was shown in Refs. [8, 10] that within the leading order of the perturbative calculation for the one-dimensional case, the beam shape is preserved during a fast two-beam collision in the presence of cubic loss. Thus, the collision-induced change in the beam shape in the transverse direction is clearly a collisional effect that exists only in spatial dimension higher
FIG. 6: (Color online) Contour plots of the beam shapes $|\psi_j(x, y, z)|$ at $z = 0$ (a), $z = z_i = 1.2$ (b), and $z = z_f = 2$ (c) in a fast collision between two Gaussian beams with the anisotropic initial condition of Eqs. (59) and (61). The plots represent the beam shapes obtained by numerical solution of Eq. (1) with parameter values $\epsilon_3 = 0.01$ and $d_{11} = 20$. The orientation angle is $\theta_0 = \pi/4$.

than 1. In the current subsection, we investigate this effect in detail for a concrete two-beam setup by both analytic calculations and numerical simulations.

To enable a more accurate comparison between the perturbation theory predictions and the numerical simulations, we assume that the effects of the optical medium’s cubic loss on single-beam propagation are negligible compared with cubic loss effects on inter-beam
interaction. This situation can be realized, for example, in certain semiconductors, in which two-photon absorption (2PA) associated with the simultaneous absorption of two photons with the same wavelength (degenerate 2PA) is much weaker than 2PA associated with the simultaneous absorption of two photons with different wavelengths (nondegenerate 2PA) [50–52]. Under this assumption, the dynamics of the two-beam collision is described by the following perturbed linear propagation model, in which the perturbation terms are only due to the effects of weak cubic loss on two-beam interaction:

\[
\begin{align*}
    i\partial_z \psi_1 + \partial_x^2 \psi_1 + \partial_y^2 \psi_1 &= -2i\epsilon_3 |\psi_2|^2 \psi_1, \\
    i\partial_z \psi_2 + id_{11} \partial_x \psi_2 + \partial_x^2 \psi_2 + \partial_y^2 \psi_2 &= -2i\epsilon_3 |\psi_1|^2 \psi_2. \tag{71}
\end{align*}
\]

Similar to Eq. (1), we assume that the cubic loss coefficient satisfies \(0 < \epsilon_3 \ll 1\). We consider the change in the beam shape in a collision between two Gaussian beams as a concrete example. Therefore, the initial condition for the collision problem is given by Eq. (49). This choice of the initial condition allows us to obtain an explicit analytic expression for the collision-induced change in the shape of beam 1 \(\phi_1\) in the post-collision interval.

Since the initial condition (49) is separable for both beams, we can calculate \(\phi_1\) in the post-collision interval by employing Eq. (48). In addition, since the effects of cubic loss on single-beam propagation are negligible, we can replace \(A_j(z^-)\) by \(A_j(0)\) everywhere in the
calculation. Therefore, the coefficient $\tilde{a}_1$, which is defined in Eq. (43), takes the form:

$$\tilde{a}_1 = 2\pi^{1/2}\epsilon_3 A_1(0) A_2^2(0) W_{20}^{(x)} / |d_{11}|,$$

(72)

where $c_{p2}^{(x)} = \pi^{1/2}$ is used. The function $g_1^{(x)}(x, z)$ in Eq. (48) is obtained by the solution of the unperturbed linear propagation equation with the initial condition (49). This function is given by Eqs. (D4) and (D6) in Appendix D. Additionally, in Appendix C we show that the inverse Fourier transform of $\hat{g}_1^{(y)}(y, z_c) \exp[-ik_2^2(z - z_c)]$ is given by:

$$\mathcal{F}^{-1}\left(\hat{g}_1^{(y)}(y, z_c) \exp[-ik_2^2(z - z_c)]\right) = \frac{W_1^{(y)} W_2^{(y)} W_0^{(y)}}{(W_1^{(y)} W_2^{(y)} W_0^{(y)} + 4z_0^2)^{1/4}(W_1^{(y)} W_2^{(y)} W_0^{(y)} + 4z_0^2)^{1/2} R_1(z, z_c)} \times \exp\left[-\frac{q_1(z_c) y^2}{R_1(z, z_c)} + i\chi_1^{(tot)}(x, y, z)\right],$$

(73)

where $g_1(z_c), R_1(z, z_c)$, and $\chi_1^{(y)}(y, z)$ are given by Eqs. (C3), (C8), and (C9), respectively. Substituting Eqs. (72), (73), (D4), and (D6) into Eq. (48), we obtain the following expression for $\phi_1(x, y, z)$ in the post-collision interval:

$$\phi_1(x, y, z) = \frac{\tilde{a}_1 W_1^{(x)} W_1^{(y)} W_2^{(y)}}{(W_1^{(x)} W_1^{(y)} W_2^{(y)} + 4z^2)^{1/4}(W_1^{(x)} W_1^{(y)} W_2^{(y)} + 4z^2)^{1/2} R_1(z, z_c)} \times \exp\left[-\frac{q_1(z_c) y^2}{R_1(z, z_c)} + i\chi_1^{(tot)}(x, y, z)\right],$$

(74)

where the total phase factor $\chi_1^{(tot)}$ is given by:

$$\chi_1^{(tot)}(x, y, z) = \chi_1^{(x)}(x, z) + \chi_1^{(y)}(y, z) + \alpha_{10} + \pi.$$

(75)

Comparing Eqs. (75) and (D12), we observe that the beam’s phase factor in the post-collision interval is different from the phase factor of the unperturbed beam. We therefore define the collision-induced change in the beam’s phase factor by:

$$\Delta\chi_1^{(tot)}(x, y, z) = \chi_1^{(tot)}(x, y, z) - \chi_1^{(tot)}(x, y, z).$$

(76)

Using Eqs. (75) and (D12), we obtain

$$\Delta\chi_1^{(tot)}(y, z) = \chi_1^{(y)}(y, z) - \chi_1^{(y)}(y, z) - \pi,$$

(77)

where $\chi_1^{(y)}$ and $\chi_1^{(y)}$ are given by Eqs. (D7) and (C9). Thus, only the dependence of the phase factor on the transverse coordinate is changed by the collision, while the dependence
on the longitudinal coordinate remains unchanged. This change in the y dependence of the phase factor is a result of the change in the beam’s shape inside the collision interval, which leads to a change in the y-dependence of \( \phi_1(x, y, z) \) in the post-collision interval [see detailed discussions following Eqs. (36) and (48)].

The collision-induced change in the beam shape can be characterized by the fractional intensity reduction factor \( \Delta I_1^{(r)} \), which is defined by:

\[
\Delta I_1^{(r)}(x, y, z) = \frac{\tilde{I}_1(x, y, z) - I_1(x, y, z)}{\tilde{I}_1(x, y, z)} = 1 - \frac{I_1(x, y, z)}{A_1^2(0)\tilde{\Psi}_{10}^2(x, y, z)},
\]

(78)

where \( I_1(x, y, z) = |\psi_1(x, y, z)|^2 \) is the intensity of beam 1 at \((x, y, z)\) in the presence of cubic loss, and \( \tilde{I}_1(x, y, z) = A_1^2(0)\tilde{\Psi}_{10}^2(x, y, z) \) is the intensity of beam 1 in the absence of cubic loss. Thus, \( \Delta I_1^{(r)} \) measures the ratio between the intensity decrease of beam 1, which is induced solely by the effects of cubic loss on the collision, and the intensity of beam 1 in the unperturbed single-beam propagation problem. We obtain the perturbation theory prediction for the fractional intensity reduction factor by expressing \( \Delta I_1^{(r)} \) in terms of \( \phi_1 \).

Using Eq. (13) and keeping terms up to order \( \epsilon^3/d_{11} \), we arrive at

\[
I_1 \simeq A_1^2(0)|\tilde{\psi}_{10}|^2 + A_1(0)\left(\tilde{\psi}_{10}\phi_1^* + \tilde{\psi}_{10}\phi_1\right).
\]

(79)

Substitution of Eq. (79) together with the relation \( |\tilde{\psi}_{10}| = \tilde{\Psi}_{10} \) into Eq. (78) yields:

\[
\Delta I_1^{(r)} = -\frac{\tilde{\psi}_{10}\phi_1^* + \tilde{\psi}_{10}\phi_1}{A_1(0)\tilde{\Psi}_{10}^2}.
\]

(80)

Using Eq. (11) and the relation \( \phi_1 = |\phi_1|\exp[i\chi_1^{(tot)}] \), we obtain:

\[
\tilde{\psi}_{10}\phi_1^* + \tilde{\psi}_{10}\phi_1 = 2\tilde{\Psi}_{10}|\phi_1|\cos\left[\Delta \chi_1^{(tot)}\right].
\]

(81)

Therefore, we can also express \( \Delta I_1^{(r)} \) as:

\[
\Delta I_1^{(r)} = -\frac{2|\phi_1|\cos\left[\Delta \chi_1^{(tot)}\right]}{A_1(0)|\tilde{\Psi}_{10}|}.
\]

(82)

For a separable initial condition, \( \Delta \chi_1^{(tot)} \) is given by Eq. (77). In addition, the \( x \) dependences of \( \tilde{\Psi}_{10} \) and of the leading order of \( |\phi_1| \) are identical. As a result, in this case, the dependence on \( x \) cancels out on the right hand side of Eq. (82), and \( \Delta I_1^{(r)} \) becomes independent of \( x \). Therefore, for a separable initial condition, the expression for \( \Delta I_1^{(r)} \) takes the form:

\[
\Delta I_1^{(r)}(y, z) = -\frac{2|\phi_1(x, y, z)|\cos\left[\chi_1^{(y)}(y, z) - \chi_1(y, z) - \pi\right]}{A_1(0)|\tilde{\Psi}_{10}(x, y, z)|}.
\]

(83)
The fractional intensity reduction factor for the collision setup considered in the current subsection is given by Eq. (83), where \( \phi_1, \tilde{\Psi}_{10}, \chi_{10}^{(y)}, \) and \( \chi_1^{(y)} \) are given by Eqs. (74), (D9), (D7), and (C9), respectively. We note that the effects of the collision-induced change in the beam’s phase factor are included in Eq. (83) via the dependence of the expression on the right hand side of this equation on \[ \cos \left[ \chi_{10}^{(y)}(y, z) - \chi_1^{(y)}(y, z) - \pi \right]. \] We will see in the next paragraphs that these effects can lead to negative values of \( \Delta I_1^{(r)}(z) \) in certain intervals of \( y \), that is, to a localized increase in the intensity of beam 1 relative to the intensity in the unperturbed single-beam propagation problem.

We check the perturbation theory predictions for the collision-induced change in the beam shape by extensive numerical simulations with Eq. (71). The simulations are carried out with \( \epsilon_3 = 0.01 \) and with \( d_{11} \) values that are varied in the intervals \( 4 \leq |d_{11}| \leq 60 \). The parameter values of the initial condition (49) are \( A_j(0) = 1, \alpha_{j0} = 0, x_{20} = \pm 20, W_{10}^{(x)} = 3, W_{10}^{(y)} = 2, W_{20}^{(x)} = 2, \) and \( W_{20}^{(y)} = 1 \). The final propagation distance is \( z_f = 2z_c = -2x_{20}/d_{11}, \) and therefore, the beam centers are well separated at \( z_f \). Figure 8 shows the initial beam shapes \( |\psi_j(x, y, 0)| \), and the beam shapes \( |\psi_j(x, y, z)| \) obtained in the simulation with \( d_{11} = 25 \) at \( z_i = 0.96 > z_c \), and at \( z_f = 1.6 \). We observe that both beams experience significant broadening due to diffraction. Figure 9 shows the collision-induced change in the shape of beam 1 obtained in the simulation with \( d_{11} = 25 \) at \( z = z_f |\phi_1^{(num)}(x, y, z_f)| \). The perturbation theory prediction \( |\phi_1^{(th)}(x, y, z_f)| \), which is obtained by Eq. (74) is also shown. The agreement between the simulation result and the perturbation theory prediction is very good both near the beam’s maximum and at the tails. We quantify the difference between \( |\phi_1^{(num)}(x, y, z)| \) and \( |\phi_1^{(th)}(x, y, z)| \) by defining the relative error (in percentage) \( E_r^{(|\phi_1|)}(z) \) in the following manner:

\[
E_r^{(|\phi_1|)}(z) = 100 \times \left[ \int dx \int dy |\phi_1^{(th)}(x, y, z)|^2 \right]^{-1/2} \times \left\{ \int dx \int dy \left[ \left| \phi_1^{(th)}(x, y, z) - |\phi_1^{(num)}(x, y, z)| \right| \right]^2 \right\}^{1/2}, \tag{84}
\]

where the integration is carried out over the entire domain in the \( xy \) plane, which is used in the numerical simulation. We find that the value of \( E_r^{(|\phi_1|)}(z_f) \) for \( d_{11} = 25 \) is 7.0\%, in accordance with the good agreement between simulation and theory that is observed in Fig. 9.

We now turn to analyze the behavior of the fractional intensity reduction factor as a
FIG. 8: (Color online) Contour plots of the beam shapes $|\psi_j(x,y,z)|$ at $z = 0$ (a), $z = z_i = 0.96$ (b), and $z = z_f = 1.6$ (c) in a fast collision between two Gaussian beams with parameter values $\epsilon_3 = 0.01$ and $d_{11} = 25$. The plots represent the beam shapes obtained by numerical solution of Eq. (71) with the initial condition (49).

The perturbation theory prediction of Eqs. (83) and (74) is also shown. The agreement between the simulations result and the theoretical prediction is very good. Based on this result and on similar results that are obtained with other choices of the physical parameter values we conclude that the perturbation theory correctly captures the spatial distribution of the intensity reduction in fast two-beam colli-
FIG. 9: (Color online) The collision-induced change in the shape of beam 1 $|\phi_1(x, y, z_f)|$ at $z_f = 1.6$ in a fast two-beam collision with parameter values $\epsilon_3 = 0.01$ and $d_{11} = 25$. Top: the perturbation theory prediction of Eq. (74). Bottom: the result obtained by numerical solution of Eq. (71).

In particular, the numerically obtained $\Delta I_1^{(r)}(y, z_f)$ attains negative values in the intervals $3.85 \leq |y| \leq 6.25$ (for $d_{11} = 25$). This prediction is confirmed by the numerical simulation. In particular, the numerically obtained $\Delta I_1^{(r)}(y, z_f)$ attains negative values in the intervals $3.8 \leq |y| \leq 6.1$, in very good agreement with the result of Eqs. (83) and (74). The negative values of $\Delta I_1^{(r)}(y, z_f)$ correspond to a localized increase in the intensity of beam 1 relative to the intensity in the unperturbed single-beam propagation problem. According to the perturbation theory, these values are a result of the collision-induced change in the phase factor of beam 1, which affects the value of $\Delta I_1^{(r)}(y, z_f)$ via its dependence on $\cos[\Delta \chi_1^{(tot)}(y, z_f)]$ [see Eqs. (83) and (74)]. To check if this is indeed the case, we compare the numerical simulation result for the $y$ dependence of $\cos[\Delta \chi_1^{(tot)}(y, z_f)]$ with the perturbation theory prediction, which is obtained by using Eqs. (77), (D7), and (C9). This comparison is shown in Fig. 14. We observe good agreement between the results of the perturbative calculation and the numerical simulation for this quantity. In particular, the perturbation theory result for $\cos[\Delta \chi_1^{(tot)}(y, z_f)]$ attains positive values in the intervals $3.85 \leq |y| \leq 6.25$, while the simulation result attains positive values in the intervals $3.8 \leq |y| \leq 6.1$. These intervals are the same as the ones, in which the values of the theoretically and numerically obtained $\Delta I_1^{(r)}(y, z_f)$ are negative. We therefore conclude that the relative localized intensity increase for beam 1 at intermediate $y$ values is indeed a result of the collision-induced change in the phase factor of this beam.
FIG. 10: (Color online) The $y$ dependence of the fractional intensity reduction factor for beam 1 at $z = z_f$ $\Delta I_1^{(r)}(y, z_f)$ in a two-beam collision with parameter values $\epsilon_3 = 0.01$ and $d_{11} = 25$. The solid blue curve corresponds to the perturbation theory prediction of Eqs. (83) and (74). The other two curves represent the results obtained by numerical solution of Eq. (71). The dashed red curve is obtained by averaging $\Delta I_1^{(r)}(x, y, z_f)$ over the $x$-interval $[-2, 2]$. The dashed-dotted green curve is obtained by using the numerically computed value of $\Delta I_1^{(r)}(0, y, z_f)$.

FIG. 11: (Color online) The $y$ dependence of the cosine of the collision-induced change in the phase factor of beam 1 at $z = z_f$ $\cos[\Delta \chi_1^{(tot)}(y, z_f)]$ in a two-beam collision with parameter values $\epsilon_3 = 0.01$ and $d_{11} = 25$. The solid blue curve corresponds to the perturbation theory prediction, which is obtained by using Eqs. (77), (D7), and (C9). The other two curves represent the results obtained by numerical solution of Eq. (71). The dashed red curve is obtained by averaging $\cos[\Delta \chi_1^{(tot)}(x, y, z_f)]$ over the $x$-interval $[-2, 2]$. The dashed-dotted green curve is obtained by using the numerically computed value of $\cos[\Delta \chi_1^{(tot)}(0, y, z_f)]$. 

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FIG. 12: (Color online) The $d_{11}$ dependence of the fractional intensity reduction factor for beam 1 at $y = 0$ and $z = z_f$, $\Delta I_1^{(r)}(0, z_f)$, in a fast two-beam collision with $\epsilon_3 = 0.01$. The solid blue curve corresponds to the perturbation theory prediction of Eqs. (83) and (74). The other two curves are obtained from the numerical solution of Eq. (71). The red squares represent the result obtained by averaging $\Delta I_1^{(r)}(x, 0, z_f)$ over the $x$-interval $[-2, 2]$. The green circles represent the result obtained by using the numerically computed value of $\Delta I_1^{(r)}(0, 0, z_f)$.

Finally, we study the dependence of the fractional intensity reduction factor on the value of the beam-steering coefficient by measuring $\Delta I_1^{(r)}(0, z_f)$ as a function of $d_{11}$. Figure 12 shows the dependence of $\Delta I_1^{(r)}(0, z_f)$ on $d_{11}$ obtained in the numerical simulations together with the theoretical prediction of Eqs. (83) and (74). The agreement between the simulations result and the perturbation theory prediction is excellent over the entire interval of $d_{11}$ values. More specifically, the relative error in the approximation of $\Delta I_1^{(r)}(0, z_f)$ (in percentage), which is defined by $|\Delta I_1^{(r)(num)}(0, z_f) - \Delta I_1^{(r)(th)}(0, z_f)| \times 100/|\Delta I_1^{(r)(th)}(0, z_f)|$, is smaller than 0.6% for $10 \leq |d_{11}| \leq 60$, and smaller than 1.2% for $4 \leq |d_{11}| < 10$. We also checked the dependence of $\Delta A_1^{(c)}$ on $d_{11}$, and obtained very good agreement between the perturbation theory prediction and the numerical simulations result (similar to what is shown in Figs. 2 and 4). Based on these results and on the results shown in Figs. 9 - 11 we conclude that the perturbation theory of subsections II B and II C enables accurate calculation of both the change in the beam shapes and the dynamics of the beam amplitudes in fast two-beam collisions in the presence of weak cubic loss.
IV. FAST COLLISIONS BETWEEN PULSED-BEAMS IN SPATIAL DIMENSION

A. Introduction

We consider the dynamics of collisions between two pulsed-beams in a three-dimensional linear optical medium with weak cubic loss. Similar to sections II and III, we assume that the pulsed-beams propagate along the z axis, and that the propagation is accurately described by the paraxial approximation. For each value of z, the distribution of the optical field depends on the spatial coordinates x and y, and on time t. Therefore, using the terminology that was introduced in section II A, the spatial dimension is 3, and the propagation is described by a (3 + 1)-dimensional propagation model.

We take into account the effects of first and second-order dispersion, isotropic diffraction, and weak cubic loss, and neglect beam-steering. Since beam-steering is neglected, the relative velocity vector between the beam centers lies along the t axis in the txy space. Therefore, the dynamics of the collision is described by the following weakly perturbed linear propagation model:

\begin{align*}
    i\partial_z \psi_1 + \partial_t^2 \psi_1 + d_2 \partial_x^2 \psi_1 + d_2 \partial_y^2 \psi_1 &= -i\epsilon_3 |\psi_1|^2 \psi_1 - 2i\epsilon_3 |\psi_2|^2 \psi_1, \\
    i\partial_z \psi_2 + id_{11} \partial_t \psi_2 + \partial_t^2 \psi_2 + d_2 \partial_x^2 \psi_2 + d_2 \partial_y^2 \psi_2 &= -i\epsilon_3 |\psi_2|^2 \psi_2 - 2i\epsilon_3 |\psi_1|^2 \psi_2,
\end{align*}

(85)

where \( \psi_j \) are proportional to the electric fields of the beams, x, y, and z are the spatial coordinates, and t is time \[ t \in \mathbb{R} \]. In Eq. (1), \( d_{11} \) is the first-order dispersion coefficient, \( d_2 \) is the diffraction coefficient, and \( \epsilon_3 \) is the cubic loss coefficient. The term \( id_{11} \partial_t \psi_2 \) describes the effects of first-order dispersion, the terms \( \partial_t^2 \psi_j \) describe the effects of second-order dispersion, and \( d_2 \partial_x^2 \psi_j \) and \( d_2 \partial_y^2 \psi_j \) describe the effects of isotropic diffraction. Additionally, the terms \( -i\epsilon_3 |\psi_j|^2 \psi_j \) and \( -2i\epsilon_3 |\psi_k|^2 \psi_j \) describe intra-beam and inter-beam effects due to weak cubic loss.

We consider collisions between pulsed-beams with general initial shapes and with tails that decrease sufficiently fast, such that the values of the integrals \( \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_j(t, x, y, 0)|^2 \) are finite. We assume that the initial pulsed-beams can be characterized by the following parameters. (1) The initial amplitudes \( A_j(0) \). (2) The
initial widths of the pulsed-beams along the $t$, $x$, and $y$ axes, $W_{j0}^{(t)}$, $W_{j0}^{(x)}$, and $W_{j0}^{(y)}$. \(3\)

The initial positions of the beam centers ($t_{j0}$, $x_{j0}$, $y_{j0}$). \(4\) The initial phases $\alpha_{j0}$. Thus, the initial electric fields can be written as:

$$\psi_j(t, x, y, 0) = A_j(0) h_j(t, x, y) \exp(i\alpha_{j0}),$$

where $h_j(t, x, y)$ is real-valued. We are equally interested in the important case, where the initial electric fields of the two pulsed-beams are separable, i.e., where each of the functions $\psi_j(t, x, y, 0)$ is a product of three functions of $t$, $x$, and $y$. \[44\]. In this case, the initial electric fields can be expressed as:

$$\psi_j(t, x, y, 0) = A_j(0) h_j^{(t)}[(t - t_{j0})/W_{j0}^{(t)}] h_j^{(x)}[(x - x_{j0})/W_{j0}^{(x)}] h_j^{(y)}[(y - y_{j0})/W_{j0}^{(y)}] \exp(i\alpha_{j0}).$$

We are interested in the collision-induced dynamics of complete fast collisions. We therefore obtain conditions on the physical parameter values for these collisions. As stated earlier, the relative velocity vector of the pulsed-beams centers lies along the $t$ axis in the $txy$ space. As a result, the conditions for a complete fast collision involve the initial widths of the pulsed-beams along the $t$ axis $W_{j0}^{(t)}$, as well as the initial and final values of the $t$ coordinate of the pulsed-beam centers $t_{j0}$ and $t_j(z_f)$, respectively. The conditions for a complete collision are obtained by requiring that the pulsed-beams are well-separated in time before and after the collision. This requirement yields the following inequalities: $|t_{20} - t_{10}| \gg W_{10}^{(t)} + W_{20}^{(t)}$ and $|t_2(z_f) - t_1(z_f)| \gg W_1^{(t)}(z_f) + W_2^{(t)}(z_f)$, where $W_j^{(t)}(z_f)$ are the pulsed-beam widths along the $t$ axis at $z = z_f$. The collision length $\Delta z_c$ is given by $\Delta z_c = 2(W_{10}^{(t)} + W_{20}^{(t)})/|d_{11}|$. The assumption of a fast collision means that $\Delta z_c$ is much smaller than the length scale $z_D^{(min)}$, which is the smallest dispersion length or diffraction length in the problem. By definition, $z_D^{(min)} = \min \{z_{d1}^{(t)}, z_{d2}^{(t)}, z_{D1}^{(x)}, z_{D2}^{(x)}, z_{D1}^{(y)}, z_{D2}^{(y)}\}$, where $z_{d_j}^{(t)}$ are the dispersion lengths of the pulsed-beams, and $z_{D_j}^{(x)}$ and $z_{D_j}^{(y)}$ are the diffraction lengths along the $x$ and $y$ axes. Requiring that $\Delta z_c \ll z_D^{(min)}$, we obtain $2(W_{10}^{(t)} + W_{20}^{(t)}) \ll |d_{11}|z_D^{(min)}$, as the condition for a fast collision.
B. The perturbation theory predictions for the collision-induced changes in the pulsed-beam shape and amplitude

The perturbative calculation of the collision-induced changes in the shapes and amplitudes of the pulsed-beams in spatial dimension 3 is very similar to the calculation that was presented in section II for the two-dimensional case. For this reason and for brevity, we do not present the entire derivation of the equations for the collision-induced dynamics. Instead, we present only the two main results of the perturbative calculation, namely, the expressions for the collision-induced changes in the shape and amplitude of pulsed-beam 1, $\Delta \Phi_1(t, x, y, z_c)$ and $\Delta A_1^{(c)}$. The notations for the physical quantities are the same as the ones used in section II, apart from the fact that $\psi_j$, $\psi_{j0}$, $\tilde{\psi}_j$, $\tilde{\Psi}_{j0}$, and $\chi_{j0}$ are now functions of $t$, $x$, $y$, and $z$.

We start by describing the results of the perturbative calculation for the general initial condition (86). In this case, the collision-induced change in the shape of pulsed-beam 1 inside the collision interval is given by:

$$\Delta \Phi_1(t, x, y, z_c) = -\frac{2\epsilon_3 A_1(z_c^-) A_2(z_c^-)}{|d_{11}|} \tilde{\psi}_{10}(t, x, y, z_c) \int_{-\infty}^{\infty} dt \tilde{\Psi}_{20}^2(\tilde{t}, x, y, z_c),$$

(88)

where $\tilde{t} = t - t_{20} - d_{11} z$. Additionally, the collision-induced change in the amplitude of pulsed-beam 1 is

$$\Delta A_1^{(c)} = -\frac{2\epsilon_3 A_1(z_c^-) A_2(z_c^-)}{C_{p1}|d_{11}|} \times\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \tilde{\Psi}_{10}^2(t, x, y, z_c) \int_{-\infty}^{\infty} \tilde{t} \tilde{\Psi}_{20}^2(\tilde{t}, x, y, z_c),$$

(89)

where $C_{p1}$ is given by:

$$C_{p1} = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \tilde{\Psi}_{10}^2(t, x, y, 0).$$

(90)

Further insight into the collision dynamics is obtained when the initial condition is of the form (87), which is completely separable for both beams. This case is also of special interest for practical reasons [44]. In this case, the collisional change in the shape of pulsed-beam 1 in the collision interval is given by:

$$\Delta \Phi_1(t, x, y, z_c) = -\frac{2\epsilon_3 A_1(z_c^-) A_2(z_c^-)}{|d_{11}|} C_{p2} W_{20}^{(t)} G_1^{(x)}(t, z_c) G_1^{(y)}(x, z_c) G_2^{(x)}(x, z_c) G_2^{(y)}(y, z_c).$$

(91)
The real-valued functions $G_j(t, z)$, $G_j(x, z)$, and $G_j(y, z)$ in Eq. (91) are defined by:

\[
\tilde{\psi}_{j0}(t, x, y, z) = G_j(t, z)G_j(x, z)G_j(y, z) \times \exp \left\{ i \left[ \chi_j(t, z) + \chi_j(x, z) + \chi_j(y, z) + \alpha_0 \right] \right\},
\]

where $\tilde{\psi}_{j0}$ are the solutions to the unperturbed linear propagation equations with the separable initial condition (87), and $\chi_j(t, z)$, $\chi_j(x, z)$, and $\chi_j(y, z)$ are the (real-valued) phase factors. In addition, the coefficient $c_{p2}$ is given by the equation

\[
\int_{-\infty}^{\infty} dt G_j(t, z)^2 = \int_{-\infty}^{\infty} dt G_j(t, 0) = W_j^{(t)} \int_{-\infty}^{\infty} ds h_j(s)^2 = W_j^{(t)} c_{p2}.
\]

Similar to the situation in the two-dimensional case, it follows from Eq. (91) that the $t$ dependence of the pulsed-beam is not changed by the collision at all (within the leading order of the perturbative calculation).

The collision-induced change in the amplitude of pulsed-beam 1 in the case of a completely separable initial condition is given by:

\[
\Delta A_1^{(c)} = -2 \epsilon_3 A_1(z_c^-) A_2(z^-) \frac{c_{p2} W_2^{(t)}}{c_{p1} W_1^{(x)} c_{p1} W_1^{(y)}} \times \int_{-\infty}^{\infty} dx G_1^{(x)2}(x, z_c) G_2^{(x)2}(x, z_c) \int_{-\infty}^{\infty} dy G_1^{(y)2}(y, z_c) G_2^{(y)2}(y, z_c).
\]

We observe that Eq. (94) has the same form as Eq. (39), where the longitudinal factor is now $c_{p2} W_2^{(t)}$ and the overall factor is $2 \epsilon_3 A_1(z_c^-) A_2(z^-) / |d_{11}|$. This finding is in accordance with the expectation that the form (39) is valid for a general spatial dimension when the initial condition is separable in the longitudinal direction for both beams. In addition, similar to the situation in spatial dimensions 1 and 2, the longitudinal factor $c_{p2} W_2^{(t)}$ is universal in the sense that it does not depend on the exact details of the initial pulsed-beam shapes and on the collision distance $z_c$.

C. Numerical simulations for pulsed-beam collisions

The predictions of the perturbative calculation in subsection IVB are based on several simplifying assumptions. In particular, it is assumed that the pulsed-beams are sharply peaked in the $txy$ space throughout the propagation, and that as a result, the integration
with respect to $z$ can be extended to $\pm\infty$. In addition, the explicit conditions for the validity of the approximations employed by the perturbative calculation are not known. For these reasons, it is important to check the predictions of the perturbative calculation by numerical simulations with the perturbed linear propagation model (85).

As explained in the beginning of subsection IV.B, the perturbative calculation of the collision-induced dynamics in spatial dimension 3 is very similar to the calculation for spatial dimension 2. For this reason and for brevity, we do not present the results of the numerical simulations for all the collisional setups that were considered in section III in the two-dimensional case. Instead, we present only the simulations results for the collisional setup, which is used for checking the theoretical predictions for universality of the longitudinal part in the expression for $\Delta A_1^{(c)}$ (the setup considered in subsection III.B for spatial dimension 2).

Similar to subsection III.B, we choose two initial conditions with widely different pulsed-beam profiles in the longitudinal (temporal) direction. More specifically, the $t$ dependence of the pulsed-beams in the first initial condition is Gaussian, i.e., it has rapidly decaying tails. In contrast, the $t$ dependence of the pulsed-beams in the second initial condition is given by a generalized Cauchy-Lorentz distribution, i.e., it has slowly decaying tails, whose decay is characterized by a power-law. The initial profiles of the pulsed-beams in the transverse direction (that is, the initial dependence on $x$ and $y$) is taken as Gaussian, since this choice enables the explicit calculation of the integrals with respect to $x$ and $y$ on the right hand side of Eq. (94). The numerical simulations with the two types of initial pulsed-beams, which possess widely different temporal (longitudinal) profiles, provide a careful test to the perturbation theory prediction for universal behavior of the longitudinal part in the expression for the collision-induced amplitude shift.

The initial condition for a collision between two Gaussian pulsed-beams is given by:

$$\psi_1(t, x, y, 0) = A_1(0) \exp \left[ -\frac{t^2}{2W_{10}^{(t)^2}} - \frac{x^2}{2W_{10}^{(x)^2}} - \frac{y^2}{2W_{10}^{(y)^2}} + i\alpha_{10} \right],$$

$$\psi_2(t, x, y, 0) = A_2(0) \exp \left[ -\frac{(t - t_{20})^2}{2W_{20}^{(t)^2}} - \frac{x^2}{2W_{20}^{(x)^2}} - \frac{y^2}{2W_{20}^{(y)^2}} + i\alpha_{20} \right].$$

Additionally, the initial condition for a collision between two Cauchy-Lorentz-Gaussian
pulsed-beams is given by:

\[
\psi_1(t, x, y, 0) = A_1(0) \left[ 1 + \frac{2t^4}{W_{10}^{(4)}} \right]^{-1} \exp \left[ -\frac{x^2}{2W_{10}^{(x)}} - \frac{y^2}{2W_{10}^{(y)}} + i\alpha_{10} \right],
\]

\[
\psi_2(t, x, y, 0) = A_2(0) \left[ 1 + \frac{2(t - t_{20})^4}{W_{20}^{(4)}} \right]^{-1} \exp \left[ -\frac{x^2}{2W_{20}^{(x)}} - \frac{y^2}{2W_{20}^{(y)}} + i\alpha_{20} \right].
\]

The calculation of the collision-induced amplitude shift by using Eq. (94) is similar to the one that was presented in subsection III B for the two-dimensional problem. This calculation yields the following expression for \( \Delta A_1^{(c)} \):

\[
\Delta A_1^{(c)} = -\frac{2b\varepsilon_3 A_1(z_c^-)A_2^2(z_c^-)}{|d_{11}|} \frac{W_{10}^{(x)}W_{10}^{(y)}W_{20}^{(t)}W_{20}^{(x)2}W_{20}^{(y)2}}{(W_{10}^{(x)})^2 + W_{20}^{(x)2})^{1/2}(W_{10}^{(y)})^2 + W_{20}^{(y)2})^{1/2}}
\]

\[\times \frac{1}{(W_{10}^{(x)}W_{20}^{(x)2} + 4d_{20}^2z_c^2)^{1/2}(W_{10}^{(y)}W_{20}^{(y)2} + 4d_{20}^2z_c^2)^{1/2}},\]

where \( b = \pi^{1/2} \) for Gaussian pulsed-beams, and \( b = 3\pi/2^{11/4} \) for Cauchy-Lorentz-Gaussian pulsed-beams. The longitudinal part of the expression for \( \Delta A_1^{(c)} \), \( c_{p2}^{(t)}W_{20}^{(t)} = bW_{20}^{(t)} \), is universal. On the other hand, the transverse part of the expression, which is given by

\[
\text{transverse factor} = \frac{W_{10}^{(x)}W_{10}^{(y)}W_{20}^{(x)2}W_{20}^{(y)2}}{(W_{10}^{(x)})^2 + W_{20}^{(x)2})^{1/2}(W_{10}^{(y)})^2 + W_{20}^{(y)2})^{1/2}}
\]

\[\times \frac{1}{(W_{10}^{(x)}W_{20}^{(x)2} + 4d_{20}^2z_c^2)^{1/2}(W_{10}^{(y)}W_{20}^{(y)2} + 4d_{20}^2z_c^2)^{1/2}},\]

is clearly not universal. We define the quantity \( \Delta A_1^{(c)(s)} \), which is used for measuring the deviation of the \( d_{11} \) dependence of \( \Delta A_1^{(c)} \) from the \( 1/|d_{11}| \) scaling, by a simple generalization of the definition in the two-dimensional case. For this purpose, we note that the collision distance \( z_c \) satisfies \( z_c = (t_{10} - t_{20})/d_{11} \). \( \Delta A_1^{(c)(s)} \) is defined as the approximate expression for the collision-induced amplitude shift that is obtained by neglecting the terms \( 4d_{20}^2(t_{10} - t_{20})^2/d_{11}^2 \) in the denominator of Eq. (97). Therefore, \( \Delta A_1^{(c)(s)} \) is given by:

\[
\Delta A_1^{(c)(s)} = -\frac{2b\varepsilon_3 A_1(z_c^-)A_2^2(z_c^-)}{|d_{11}|} \frac{W_{20}^{(t)}W_{20}^{(x)2}W_{20}^{(y)}}{(W_{10}^{(x)})^2 + W_{20}^{(x)2})^{1/2}(W_{10}^{(y)})^2 + W_{20}^{(y)2})^{1/2}}.
\]

It is clear from the definition of \( \Delta A_1^{(c)(s)} \) that the difference \( |\Delta A_1^{(c)} - \Delta A_1^{(c)(s)}| \) measures the deviation of the \( d_{11} \) dependence of \( \Delta A_1^{(c)} \) from the \( 1/|d_{11}| \) scaling observed in spatial dimension 1.
We check the perturbation theory predictions for $\Delta A_1^{(c)}$ and for the universality of the longitudinal part in the expression for $\Delta A_1^{(c)}$ by numerical simulations with Eq. (85) with the initial conditions (95) and (96). These initial conditions possess very different temporal (longitudinal) pulsed-beam profiles. In particular, the tails of the Gaussian pulsed-beams decay rapidly with increasing values of $|t|$ and $|t - t_{20}|$. For such pulsed-beams, the sharp-peak approximation that is used in the derivation of Eqs. (89) and (94) is expected to hold. In contrast, the tails of the Cauchy-Lorentz-Gaussian pulsed-beams are slowly decaying with increasing values of $|t|$ and $|t - t_{20}|$, and the decay is characterized by a power-law. It is unclear if the sharp-peak approximation is valid for pulsed-beams with such slowly decaying tails. Therefore, the numerical simulations with the initial conditions (95) and (96) provide a careful check for the validity of the perturbation theory approximations for pulsed-beams with widely different temporal (longitudinal) distributions. In this way, the simulations help to establish the universality of the longitudinal part in the expression for $\Delta A_1^{(c)}$.

Equation (85) is numerically solved by the split-step method with periodic boundary conditions [46, 47]. Since we are interested in fast collisions, we perform the simulations for $d_{11}$ values in the intervals $4 \leq |d_{11}| \leq 60$. The other physical parameters values in Eq. (85) are chosen as $\epsilon_3 = 0.01$ and $d_2 = 1.5$, as an example. The parameter values of the initial pulsed-beams are $A_j(0) = 1$, $\alpha_{j0} = 0$, $t_{20} = \pm 15$, $W_{10}^{(t)} = 2$, $W_{10}^{(y)} = 3$, $W_{10}^{(z)} = 4$, $W_{20}^{(t)} = 3$, $W_{20}^{(x)} = 4$, and $W_{20}^{(y)} = 5$. The final propagation distance is $z_f = 2z_c = -2t_{20}/d_{11}$. We emphasize that results similar to the ones presented in the current subsection are obtained in simulations with other values of the physical parameters. For each initial condition, we compare the numerically obtained dependence of $\Delta A_1^{(c)}$ on $d_{11}$ with the theoretical predictions of Eqs. (97) and (99). We also describe the behavior of the relative errors in the approximation of $\Delta A_1^{(c)}$ (in percentage), $E_r^{(1)}$ and $E_r^{(2)}$, which were defined in subsection III B.

We first discuss the simulations results for Gaussian pulsed-beams. Figure 13 shows the values of $|\psi_j(t, x, y, z)|$ obtained in the simulation with $d_{11} = 20$ at three specific planes (cross-sections) at the distances $z = 0$, $z = z_i = 0.9$, and $z = z_f = 1.5$ [49]. At each distance, we choose the three cross-sections (planes) such that the main bodies of the pulsed-beams are shown clearly [57]. In particular, one cross-section, which is denoted by $CS_{2}^{(0)}$, is located at the plane $x = 0$. Additionally, the other two cross-sections are located at the planes $t = t_j(z)$, where $j = 1, 2$, $t_j(z)$ is the $t$ coordinate of the $j$th pulsed-beam’s center, and $z$
can take the values 0, z₁, or z₂. The latter cross-sections are denoted by \( C S₁^{(t_j(z))} \), where \( j = 1, 2 \). It is seen that the pulsed-beams experience broadening due to the effects of second-order dispersion and diffraction. Additionally, the maximum values of \( |ψ_j(t, x, y, z)| \) decrease with increasing distance, mainly due to the broadening. The dependence of \( ΔA_1^{(c)} \) on \( d_{11} \) obtained in the simulations is shown in Fig. 14 together with the two theoretical predictions of Eqs. (97) and (99). The agreement between the simulations result and the prediction of Eq. (97) is very good. More specifically, the relative error \( E_r^{(1)} \) is smaller than 1.9% for 10 \( ≤ |d_{11}| \leq 60 \) and smaller than 4.5% for 4 \( ≤ |d_{11}| < 10 \). In addition, we observe good agreement between the simulations result and the more crude approximation \( ΔA_1^{(c)(s)} \) for large \( |d_{11}| \) values, but there is a noticeable difference between the results for intermediate \( |d_{11}| \) values. In particular, the relative error \( E_r^{(2)} \) is smaller than 10.4% for 10 \( ≤ |d_{11}| \leq 60 \) and smaller than 39.3% for 4 \( ≤ |d_{11}| < 10 \). Thus, as expected from Eqs. (97) and (99), the deviation of the \( d_{11} \) dependence of \( ΔA_1^{(c)} \) from the \( 1/|d_{11}| \) scaling increases with decreasing value of \( |d_{11}| \).

We now describe the results of the simulations for collisions between Cauchy-Lorentz-Gaussian pulsed-beams, which serve as an example for pulsed-beams with tails that exhibit slow (power-law) temporal decay. In this case, it is not clear if the sharp-peak approximation used in the perturbative calculation is valid. For brevity, we discuss the simulations results without showing the corresponding figures. The numerical simulation with \( d_{11} = 20 \) shows that the pulsed-beams experience considerable broadening and develop extended tails due to second-order dispersion and diffraction. As a result, the maximum values of \( |ψ_j(t, x, y, z)| \) decrease with increasing \( z \). Furthermore, despite the broadening of the pulsed-beams, the agreement between the numerical simulations result for \( ΔA_1^{(c)} \) and the theoretical prediction of Eq. (97) is very good. In particular, the relative error \( E_r^{(1)} \) is less than 1.3% for 10 \( ≤ |d_{11}| \leq 60 \) and less than 10.2% for 4 \( ≤ |d_{11}| < 10 \). The values of \( E_r^{(1)} \) are comparable to the values obtained for collisions between Gaussian pulsed-beams. Based on these findings and on similar results obtained with other parameter values, we conclude that the longitudinal part of the expression for \( ΔA_1^{(c)} \) is indeed universal, since it is not sensitive to the details of the initial pulsed-beams shapes. We also note that the relative error \( E_r^{(2)} \) in the approximation of the amplitude shift by \( ΔA_1^{(c)(s)} \) is smaller than 9.8% for 10 \( ≤ |d_{11}| \leq 60 \) and smaller than 42.9% for 4 \( ≤ |d_{11}| < 10 \). These values are significantly larger than the corresponding values of \( E_r^{(1)} \). Thus, the agreement between the numerical result and the perturbation theory
FIG. 13: (Color online) Contour plots of the pulsed-beam shapes $|\psi_j(t,x,y,z)|$ on three planes in the $txy$ space at $z = 0$ (a), $z = z_i = 0.9$ (b), and $z = z_f = 1.5$ (c) in a fast collision between two Gaussian pulsed-beams. The physical parameter values are $\epsilon_3 = 0.01$, $d_2 = 1.5$, and $d_{11} = 20$. The plots represent the beam shapes obtained by numerical solution of Eq. (85) with the initial condition (95). The plane $x = 0$ is denoted by $CS_2^{(0)}$, and the planes $t = t_j(z)$ with $j = 1, 2$ are denoted by $CS_1^{(t_j(z))}$.

The prediction for $\Delta A_1^{(c)}$ is significantly better than the agreement between the numerical result and the cruder approximation $\Delta A_1^{(c)(s)}$. Additionally, as in the case of Gaussian pulsed-beams, the deviation of the $d_{11}$ dependence of $\Delta A_1^{(c)}$ from the $1/|d_{11}|$ scaling increases with decreasing value of $|d_{11}|$. 
FIG. 14: (Color online) The collision-induced amplitude shift of pulsed-beam 1 $\Delta A_1^{(c)}$ vs the first-order dispersion coefficient $d_{11}$ in a fast collision between two Gaussian pulsed-beams with parameter values $\epsilon_3 = 0.01$ and $d_2 = 1.5$. The red circles represent the result obtained by numerical simulations with Eq. (85) with the initial condition (95). The solid blue and dashed green curves represent the theoretical predictions of Eqs. (97) and (99), respectively.

V. CONCLUSIONS

We studied the dynamics of fast collisions between two optical beams in linear optical media with weak cubic loss in spatial dimension higher than 1. For this purpose, we introduced a perturbation method, which generalizes the perturbation method developed in Refs. [8, 10] for the one-dimensional case in three major ways. First, it extends the perturbative calculation from spatial dimension 1 to spatial dimension 2, and enables the extension of the calculation to a general spatial dimension in a straightforward manner. Second, it provides a perturbative calculation of the collision-induced dynamics of the beam shape both in the collision interval and outside of the collision interval. In contrast, the perturbative calculation in Refs. [8, 10] was limited to the collision interval only. Third, it enables the discovery and analysis of several collision-induced effects, which exist only in spatial dimension higher than 1.

We used the generalized two-dimensional version of the perturbation method to obtain formulas for the collision-induced changes in the beam shapes and amplitudes. This was done both for a general initial condition and for the important case of an initial condition, which is separable for both beams. We found that for a general initial condition, the collision leads to a change in the beam shape in the direction transverse to the relative velocity vector.
between the beam centers. Additionally, we found that for a separable initial condition, the beam shape in the longitudinal direction is not changed by the collision within the leading order of the perturbative calculation. Moreover, we showed that for a separable initial condition, the longitudinal part in the expression for the collision-induced amplitude shift is universal, while the transverse part is proportional to the integral of the product of the beam intensities with respect to the transverse coordinate. We also demonstrated that the same behavior of the longitudinal and transverse parts of the expression for the amplitude shift exists in collisions between pulsed optical beams in spatial dimension 3.

We checked these predictions of the generalized perturbation method along with other predictions concerning the effects on the collision of partial beam overlap and anisotropy in the initial condition by extensive numerical simulations with the perturbed linear propagation model in spatial dimensions 2 and 3. The simulations in spatial dimension 2 were carried out for four different two-beam collision setups. These setups demonstrate four major collisional effects and properties that either exist only in spatial dimension higher than 1, or are qualitatively different from their one-dimensional counterparts. (1) The universality of the longitudinal part in the expression for the collision-induced amplitude shift. (2) The effect of partial beam overlap. (3) The effect of anisotropy in the initial condition. (4) The collision-induced change in the beam shape in the transverse direction. The prediction for universal behavior of the longitudinal part in the expression for the amplitude shift was also checked in spatial dimension 3 by numerical simulations of collisions between pulsed optical beams.

In all the simulation setups we obtained very good agreement between the perturbation theory and the numerical simulations. In particular, in setup (1), the simulations showed that the longitudinal part in the expression for the collision-induced amplitude shift is universal in the sense that it is not sensitive to the details of the initial beam shapes, and that this is true in both spatial dimensions 2 and 3. Additionally, the simulations in setup (2) verified the validity of the transverse part in the expression for the amplitude shift and demonstrated that the generalized perturbation method can be employed for fast collisions between partially overlapping beams. The simulations in the anisotropic setup [setup (3)] verified the complex dependence of the expression for the amplitude shift on the orientation angle $\theta_0$, which was predicted by the perturbation theory. We attributed this complex dependence to the nonseparable nature of the initial condition in the anisotropic setup. Moreover,
the numerical simulations in setup (4) confirmed the perturbation theory predictions for the collision-induced change in the beam shape in the transverse direction. Based on the results of the latter simulations we concluded that the generalized perturbation method that we developed in the current paper enables accurate calculation of both the change in the beam shape and the dynamics of the beam amplitude in fast two-beam collisions in the presence of weak cubic loss.

In summary, our study extended the results of the previous works in Refs. [8–10] in two major ways. First, it generalized the perturbation method of Refs. [8–10] from spatial dimension 1 to a general spatial dimension. Second, it demonstrated a variety of collision-induced physical effects, which exist only for spatial dimension higher than 1. We point out that in another study, we developed a similar perturbation method for analyzing fast two-pulse collisions in systems described by linear diffusion-advection equations with weak quadratic loss in spatial dimension higher than 1 [13]. Using the latter perturbation method and numerical simulations, we found that the collision-induced changes in pulse shapes and amplitudes in the perturbed linear diffusion-advection systems exhibit similar behavior to the one reported in the current paper. Thus, the perturbation methods developed in the current paper and in Ref. [13] are very valuable tools for analyzing fast-collision dynamics in linear physical systems with weak nonlinear dissipation. Indeed, as described in the current paper and in Ref. [13], these methods enabled deep insight into many collision-induced effects in the perturbed linear systems in spatial dimensions 2 and 3. We also comment that detailed analytic results on collisions between pulse (or beam) solutions of linear or nonlinear evolution models in the presence of nonlinear dissipation in spatial dimension higher than 1 are quite scarce. Therefore, the current work and the work in Ref. [13] also significantly extended the understanding of the more general high-dimensional problem of two-pulse (or two-beam) collisions in the presence of nonlinear dissipation.

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Appendix A: The relation between $\Delta A_1^{(c)}$ and $\Delta \Phi_1(x, y, z_c)$

In this Appendix, we present the derivation of the relation (17) between the collision-induced amplitude shift $\Delta A_1^{(c)}$ and the collision-induced change in the beam shape $\Delta \Phi_1(x, y, z_c)$. This relation is used in subsection II B to obtain Eq. (19) from Eq. (16). The derivation is carried out for a collision in spatial dimension 2, but it can be generalized in a straightforward manner to spatial dimension $n$.

We first recall that the amplitude dynamics of a single beam propagating in the presence of diffraction and linear or nonlinear loss can be approximately determined by an energy balance equation of the form

$$\partial_z \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_1(x, y, z)|^2 = \ldots,$$

where the right hand side of the equation is determined by the type of the loss perturbation. [See, for example, Eqs. (B1) and (B4) in Appendix B]. A fast collision in the presence of nonlinear loss at the distance $z = z_c$ leads to a jump in the value of $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_1(x, y, z)|^2$ at $z = z_c$. Therefore, for a fast collision, the term $\partial_z \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_1(x, y, z)|^2$ in the equation that determines the amplitude dynamics can be replaced by:

$$\Delta P = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_1(x, y, z_c^+)|^2 - \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_1(x, y, z_c^-)|^2.$$  \hspace{1cm} (A1)

The derivation of the relation between $\Delta A_1^{(c)}$ and $\Delta \Phi_1(x, y, z_c)$ is based on finding two expressions for $\Delta P$, one involving $\Delta A_1^{(c)}$ and the other involving $\Delta \Phi_1(x, y, z_c)$, and on equating the two expressions.

We note that in the limit of a fast collision, $\phi_1(x, y, z_c^-) \simeq 0$, and therefore, $\psi_1(x, y, z_c^-) \simeq \psi_{10}(x, y, z_c^-)$. Using Eqs. (10) and (11), we find: $\psi_1(x, y, z_c^-) \simeq A_1(z_c^-)\tilde{\Psi}_{10}(x, y, z_c) \exp[i\chi_{10}(x, y, z_c)]$. We use the latter approximation for $\psi_1(x, y, z_c^-)$ to evaluate the second integral on the right hand side of Eq. (A1). This calculation yields:

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_1(x, y, z_c^-)|^2 = C_{p1} A_1^2(z_c^-),$$  \hspace{1cm} (A2)

where $C_{p1}$ is given by Eq. (18). Since $\phi_1(x, y, z_c^-) \simeq 0$, $\Delta \phi_1(x, y, z_c)$ can be written as:

$\Delta \phi_1(x, y, z_c) \simeq \phi_1(x, y, z_c^+) - \phi_1(x, y, z_c^-) \simeq \phi_1(x, y, z_c^+)$. Using this relation together with Eq. (11) and the definition of $\psi_{10}$, we obtain:

$$\psi_1(x, y, z_c^+) = \psi_{10}(x, y, z_c) + \Delta \phi_1(x, y, z_c).$$  \hspace{1cm} (A3)
We now use Eqs. (A3), (10), and (11) along with the definition of $\Delta \Phi_1$ to obtain
\[
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_1(x, y, z_c^+)|^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left[ A_1(z_c^-) \tilde{\Psi}_{10}(x, y, z_c) + \Delta \Phi_1(x, y, z_c) \right]^2.
\] (A4)

We expand the integrand on the right hand side of Eq. (A4), while keeping only the first two leading terms. We obtain
\[
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_1(x, y, z_c^+)|^2 \simeq C_{p1} A_1^2(z_c^-) + 2A_1(z_c^-) \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \tilde{\Psi}_{10}(x, y, z_c) \Delta \Phi_1(x, y, z_c).
\] (A5)

Substitution of Eqs. (A2) and (A5) into Eq. (A1) yields the first expression for $\Delta P$:
\[
\Delta P = 2A_1(z_c^-) \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \tilde{\Psi}_{10}(x, y, z_c) \Delta \Phi_1(x, y, z_c).
\] (A6)

The second expression for $\Delta P$ is obtained by writing $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_1(x, y, z_c^+)|^2$ in terms of $\Delta A_1^{(c)}$ in the following manner:
\[
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_1(x, y, z_c^+)|^2 = \left( A_1(z_c^-) + \Delta A_1^{(c)} \right)^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \tilde{\Psi}_{10}^2(x, y, z_c) \simeq C_{p1} A_1^2(z_c^-) + 2C_{p1} A_1(z_c^-) \Delta A_1^{(c)}.
\] (A7)

Substitution of Eqs. (A2) and (A7) into Eq. (A1) yields the second expression for $\Delta P$:
\[
\Delta P = 2C_{p1} A_1(z_c^-) \Delta A_1^{(c)}.
\] (A8)

Equating the right hand sides of Eqs. (A6) and (A8), we arrive at:
\[
\Delta A_1^{(c)} = C_{p1}^{-1} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \tilde{\Psi}_{10}(x, y, z_c) \Delta \Phi_1(x, y, z_c),
\] (A9)

which is Eq. (17). We point out that the relation (A9) can be generalized to spatial dimension $n$ by replacing all the two-dimensional integrals in the equations in the current Appendix by $n$-dimensional integrals with respect to the $n$ spatial coordinates.

Appendix B: Amplitude dynamics in the perturbed single-beam propagation problem

In this Appendix, we derive the equation for the dynamics of the beam amplitudes in the perturbed single-beam propagation problem, i.e., for a single beam propagating in the
presence of weak cubic loss. This equation is used for calculating the amplitude values in the approximate expressions for the $\psi_{j0}$. It is also used for calculating the values of $A_j(z^-)$ in Eqs. (16), (19), (36), and (38) for $\Delta \Phi_1(x, y, z_c)$ and $\Delta A_1^{(c)}$, and in other equations in section II. We also show that the effects of weak linear loss can be incorporated in a straightforward manner in the equation for amplitude dynamics for single-beam propagation. Moreover, we show that the effects of weak linear loss do not change the form of the expressions for the collision-induced amplitude shifts.

Consider the propagation of a single beam in the presence of diffraction, beam steering, and weak cubic loss. The propagation is described by Eq. (5) for beam 1 and by Eq. (6) for beam 2. Employing energy balance calculations for these two equations, we obtain

$$\partial_z \int \int \mid \psi_{j0} \mid^2 = -2\epsilon_3 \int \int \mid \psi_{j0} \mid^4. \quad (B1)$$

We now substitute the approximations to the $\psi_{j0}$, which are given by Eqs. (10)-(12), into Eq. (B1). This substitution yields the following equation for the $A_j$:

$$C_{pj} \frac{dA_j^2}{dz} = -2\epsilon_3 H_{4j}(z)A_j^4, \quad (B2)$$

where $H_{4j}(z) = \int \int \tilde{\psi}_{j0}^4(x, y, z)$, $C_{p1}$ is given by Eq. (18), and $C_{p2}$ is given by a similar equation, in which $\tilde{\psi}_{20}^2(x, y, 0)$ is replaced by $\tilde{\psi}_{20}^2(x, y, 0)$ on the right hand side. The solution of Eq. (B2) on the interval $[0, z]$ is

$$A_j(z) = \frac{A_j(0)}{\left[1 + 2\epsilon_3 \tilde{H}_{4j}(0, z)A_j^2(0)/C_{pj}\right]^{1/2}}, \quad (B3)$$

where $\tilde{H}_{4j}(0, z) = \int_0^z d\zeta \tilde{H}_{4j}(\zeta')$.

It is straightforward to incorporate the effects of weak linear loss into the equation for the dynamics of the pulse amplitudes. In this case, single-beam propagation of beams 1 and 2 is described by Eqs. (5) and (6) with the terms $-i\epsilon_1 \psi_1$ and $-i\epsilon_2 \psi_2$ added on the right hand sides, where $0 < \epsilon_1 \ll 1$ is the linear loss coefficient. Energy balance calculations for the two modified propagation equations yield the following equation

$$\partial_z \int \int \mid \psi_{j0} \mid^2 = -2\epsilon_1 \int \int \mid \psi_{j0} \mid^2 - 2\epsilon_3 \int \int \mid \psi_{j0} \mid^4. \quad (B4)$$

Using the approximate expressions (10)-(12) for the $\psi_{j0}$ in Eq. (B4), we obtain

$$C_{pj} \frac{dA_j^2}{dz} = -2\epsilon_1 C_{pj} A_j^2 - 2\epsilon_3 H_{4j}(z)A_j^4. \quad (B5)$$
Equation (B5) is a Bernoulli equation for $A_j^2(z)$. Its solution on the interval $[0, z]$ is given by:

$$A_j(z) = \frac{A_j(0)e^{-\epsilon_3 z}}{\left[1 + 2\epsilon_3 \tilde{H}_4j(0, z)A_j^2(0)/C_{pj}\right]^{1/2}},$$

(B6)

where $\tilde{H}_4j(0, z) = \int_0^z dz' H_4j(z')e^{-2\epsilon_3 z'}$.

We now show that the addition of the weak linear loss terms to Eq. (1) does not alter the form of the expressions for the collision-induced changes in the beam shape and amplitude. For this purpose, we first note that the equation for $\phi_1$ in the leading order of the perturbative calculation is still Eq. (7). As a result, the evolution of $\Phi_1$ in the collision interval is described by Eq. (9). It follows that $\Delta \Phi_1(x, y, z_c)$ and $\Delta A_1^{(c)}$ are still given by Eqs. (16) and (19). In addition, the evolution of $\phi_1$ in the post-collision interval is described by Eq. (20), and as a result, $\phi_1$ is still given by Eq. (23) in this interval. Thus, the addition of the weak linear loss terms to Eq. (1) does not alter the form of the expressions for the collision-induced changes in the beam shape and amplitude in the leading order of the perturbative calculation. We point out that the weak linear loss affects the values of $\Delta \Phi_1(x, y, z_c)$, $\Delta A_1^{(c)}$, and $\phi_1(x, y, z)$ only via the dependence of these quantities on the beam amplitudes. More specifically, in the absence of linear loss, these quantities are calculated with amplitude values that are given by Eq. (B3), while in the presence of weak linear loss, these quantities are calculated with amplitude values that are given by Eq. (B6).

Appendix C: Derivation of Eq. (73)

In this Appendix, we derive Eq. (73) for the inverse Fourier transform of $\hat{g}_{12}^{(y)}(k_2, z_c) \exp[-ik_2^2(z - z_c)]$ in the case where the initial condition for the collision problem is given by Eq. (49). Equation (73) is used in the calculation of $\phi_1(x, y, z)$ in the post-collision interval in subsection III E.

We first employ Eq. (44) along with Eqs. (49), (55), (77), and (114) to obtain an expression for the function $g_{12}^{(y)}(y, z_c)$. We find:

$$g_{12}^{(y)}(y, z_c) = \frac{W_{10}^{(y)}W_{20}^{(y)^2}}{(W_{10}^{(y)^4} + 4z_c^2)^{1/4}(W_{20}^{(y)^4} + 4z_c^2)^{1/2}} \times \exp \left[ -\tilde{a}_2^2(z_c)y^2 - i \frac{2z_c}{2z_c} \arctan \left( \frac{2z_c}{W_{10}^{(y)^2}} \right) \right],$$

(C1)
where
\[ \tilde{a}_2^2(z_c) = q_1(z_c) + i q_2(z_c), \]  
\[ q_1(z_c) = \frac{W^{(y)2}_{10} W^{(y)2}_{20}(2W^{(y)2}_{10} + W^{(y)2}_{20}) + 4z_c^2(W^{(y)2}_{10} + 2W^{(y)2}_{20})}{2(W^{(y)4}_{10} + 4z_c^2)(W^{(y)4}_{20} + 4z_c^2)}, \]  
\[ q_2(z_c) = -\frac{z_c}{W^{(y)4}_{10} + 4z_c^2}. \]  
\[ (C2) \]
\[ (C3) \]
and
\[ \hat{g}_{12}(y, z_c) = \frac{W^{(y)2}_{10} W^{(y)2}_{20}(W^{(y)4}_{10} + 4z_c^2)^{1/4}}{\tilde{a}_3(z_c)} \times \exp \left[ -\frac{k_2^2}{4\tilde{a}_2^2(z_c)} - \frac{i}{2} \arctan \left( \frac{2z_c}{W^{(y)2}_{10}} \right) \right], \]  
\[ (C5) \]
where
\[ \tilde{a}_3(z_c) = \left[ 2(W^{(y)4}_{10} + 4z_c^2)(W^{(y)4}_{20} + 4z_c^2) \right]^{1/2} \tilde{a}_2(z_c). \]  
\[ (C6) \]
The Fourier transform of \( g_{12}^{(y)}(y, z_c) \) is:
\[ \hat{g}_{12}^{(y)}(k_2, z_c) = \frac{W^{(y)2}_{10} W^{(y)2}_{20}(W^{(y)4}_{10} + 4z_c^2)^{1/4}}{\tilde{a}_3(z_c)} \times \exp \left[ -\frac{k_2^2}{4\tilde{a}_2^2(z_c)} - \frac{i}{2} \arctan \left( \frac{2z_c}{W^{(y)2}_{10}} \right) \right], \]  
\[ (C5) \]
where
\[ \tilde{a}_3(z_c) = \left[ 2(W^{(y)4}_{10} + 4z_c^2)(W^{(y)4}_{20} + 4z_c^2) \right]^{1/2} \tilde{a}_2(z_c). \]  
\[ (C6) \]
Therefore, the inverse Fourier transform of \( \hat{g}_{12}^{(y)}(k_2, z_c) \exp[-ik_2^2(z - z_c)] \) is given by:
\[ \mathcal{F}^{-1} \left( \hat{g}_{12}^{(y)}(k_2, z_c) \exp[-ik_2^2(z - z_c)] \right) = \frac{W^{(y)2}_{10} W^{(y)2}_{20} \exp \left[ -q_1(z_c)y^2/R_1^4(z, z_c) + i\chi_1^{(y)}(y, z) \right]}{(W^{(y)4}_{10} + 4z_c^2)^{1/4}(W^{(y)4}_{20} + 4z_c^2)^{1/2}R_1(z, z_c)}, \]  
\[ (C7) \]
where
\[ R_1(z, z_c) = \left\{ 1 - 8q_2(z_c)(z - z_c) + 16[q_1^2(z_c) + q_2^2(z_c)](z - z_c)^2 \right\}^{1/4}, \]  
\[ (C8) \]
and
\[ \chi_1^{(y)}(y, z) = -\frac{1}{2} \arctan \left[ \frac{2z_c}{W^{(y)2}_{10}} \right] - \frac{1}{2} \arctan \left[ \frac{4q_1(z_c)(z - z_c)}{1 - 4q_2(z_c)(z - z_c)} \right] - \left\{ q_2(z_c) - 4[q_1^2(z_c) + q_2^2(z_c)](z - z_c) \right\} \frac{y^2}{R_1^4(z, z_c)}. \]  
\[ (C9) \]
Equation (C7) is Eq. (73) of subsection III E.
Appendix D: The solution of the unperturbed linear propagation equation with a Gaussian initial condition

In Section III, we extensively used the solution of the unperturbed linear propagation equation with a Gaussian initial condition as an example. We therefore present here a brief summary of the different forms of this solution.

We consider the unperturbed linear propagation equation

$$i\partial_z \psi + \partial_x^2 \psi + \partial_y^2 \psi = 0 \tag{D1}$$

with the separable Gaussian initial condition

$$\psi(x, y, 0) = A \exp \left[ \frac{-(x-x_0)^2}{2W_0^{(x)}^2} - \frac{(y-y_0)^2}{2W_0^{(y)}^2} + i\alpha_0 \right]. \tag{D2}$$

The solution of Eq. (D1) with the initial condition (D2) can be written as:

$$\psi(x, y, z) = Ag^{(x)}(\bar{x}, z)g^{(y)}(\bar{y}, z) \exp(i\alpha_0), \tag{D3}$$

where $\bar{x} = x - x_0$, $\bar{y} = y - y_0$,

$$g^{(x)}(\bar{x}, z) = \frac{W_0^{(x)}}{(W_0^{(x)} + 4z^2)^{1/4}} \exp \left[ -\frac{W_0^{(x)} \bar{x}^2}{2(W_0^{(x)} + 4z^2)} + i\chi^{(x)}_0(\bar{x}, z) \right], \tag{D4}$$

and

$$g^{(y)}(\bar{y}, z) = \frac{W_0^{(y)}}{(W_0^{(y)} + 4z^2)^{1/4}} \exp \left[ -\frac{W_0^{(y)} \bar{y}^2}{2(W_0^{(y)} + 4z^2)} + i\chi^{(y)}_0(\bar{y}, z) \right]. \tag{D5}$$

The phase factors $\chi^{(x)}_0(\bar{x}, z)$ and $\chi^{(y)}_0(\bar{y}, z)$ in Eqs. (D4) and (D5) are given by:

$$\chi^{(x)}_0(\bar{x}, z) = -\frac{1}{2} \arctan \left( \frac{2z}{W_0^{(x)} + 4\bar{x}^2} \right) + \frac{\bar{x}^2 z}{W_0^{(x)} + 4\bar{x}^2}. \tag{D6}$$

and

$$\chi^{(y)}_0(\bar{y}, z) = -\frac{1}{2} \arctan \left( \frac{2z}{W_0^{(y)} + 4\bar{y}^2} \right) + \frac{\bar{y}^2 z}{W_0^{(y)} + 4\bar{y}^2}. \tag{D7}$$

One can also write the solution (D3) in the form:

$$\psi(x, y, z) = A\Psi(x, y, z) \exp[i\chi_0(\bar{x}, \bar{y}, z)], \tag{D8}$$

58
where

\[
\Psi(x, y, z) = G^{(x)}(\bar{x}, z)G^{(y)}(\bar{y}, z),
\]  
(D9)

\[
G^{(x)}(\bar{x}, z) = \frac{W_0^{(x)}}{(W_0^{(x)}4 + 4z^2)^{1/4}} \exp \left[ -\frac{W_0^{(x)2} \bar{x}^2}{2(W_0^{(x)}4 + 4z^2)} \right],
\]  
(D10)

\[
G^{(y)}(\bar{y}, z) = \frac{W_0^{(y)}}{(W_0^{(y)}4 + 4z^2)^{1/4}} \exp \left[ -\frac{W_0^{(y)2} \bar{y}^2}{2(W_0^{(y)}4 + 4z^2)} \right],
\]  
(D11)

and

\[
\chi_0(\bar{x}, \bar{y}, z) = \chi_0^{(x)}(\bar{x}, z) + \chi_0^{(y)}(\bar{y}, z) + \alpha_0.
\]  
(D12)

We also note that the solution of Eq. (D1) with the term \(id_{11}\partial_x\psi\) on its left hand side and with the initial condition (D2) is given by Eqs. (D3)-(D7) [or by Eqs. (D8)-(D12)] with \(\bar{x} = x - x_0 - d_{11}z\), and \(\bar{y} = y - y_0\).

**Appendix E: Invariance of \(\Delta A_1^{(c)}\) under rotations**

In this Appendix, we show that the change in the coordinate system, in which we rotate the \(x'\) and \(y'\) axes by an angle \(\Delta \theta\), such that in the new coordinate system the relative velocity vector between the beam centers lies on the \(x\) axis, does not change the value of \(\Delta A_1^{(c)}\). That is, the value of the collision-induced amplitude shift is invariant under this rotation transformation. This calculation provides the justification for choosing the direction of the relative velocity vector between the beam centers along the direction of the \(x\) axis in sections III and III.

We consider the fast two-beam collision problem in the \((x', y', z)\) coordinate system, in which the relative velocity vector \(d'_1 = (d'_{11}, d'_{12})\) does not lie on the \(x'\) or \(y'\) axes. We assume that \(d'_1 = |d'_1| \gg 1\). The perturbed linear propagation model in the \((x', y', z)\) coordinate system is

\[
\begin{align*}
&i\partial_x \psi'_1 + \partial^2_x \psi'_1 + \partial^2_y \psi'_1 = -i\epsilon_3 |\psi'_1|^2 \psi'_1 - 2i\epsilon_3 |\psi'_2|^2 \psi'_1, \\
&i\partial_x \psi'_2 + i d'_{11} \partial_x \psi'_2 + i d'_{12} \partial_y \psi'_2 + \partial^2_x \psi'_2 + \partial^2_y \psi'_2 = -i\epsilon_3 |\psi'_2|^2 \psi'_2 - 2i\epsilon_3 |\psi'_1|^2 \psi'_2,
\end{align*}
\]  
(E1)
where $\psi_j'(x',y',z)$ is the electric field of the $j$th beam in this coordinate system. The initial condition is:

$$\psi_j'(x',y',0) = A_j(0)h_j'(x',y') \exp(i\alpha_{j0}),$$  \hspace{1cm} (E2)

where $h_j'(x',y')$ is real-valued.

We assume that the solution to the unperturbed propagation equation

$$i\partial_z\psi_1' + \partial_{x'}^2\psi_1' + \partial_{y'}^2\psi_1' = 0$$  \hspace{1cm} (E3)

does not contain any fast dependence on $z$. In addition, we assume that the only fast dependence on $z$ in the solution to the unperturbed propagation equation

$$i\partial_z\psi_2' + id_{11}'\partial_{x'}\psi_2' + id_{12}'\partial_{y'}\psi_2' + \partial_{x'}^2\psi_2' + \partial_{y'}^2\psi_2' = 0$$  \hspace{1cm} (E4)

is contained in factors of the form $x' - x'_{20} - d_{11}'z$ and $y' - y'_{20} - d_{12}'z$, where $(x'_{20}, y'_{20})$ is the initial location of beam 2 in the $x'y'$ plane. Under these assumptions, we can use the perturbation method of subsection II B to show that within the leading order of the method, the equation for $\Phi_1'$ in the collision interval is

$$\partial_z\Phi_1' = -2\epsilon_3\Psi_{20}'\Psi_{10}'.$$  \hspace{1cm} (E5)

In addition, in a similar manner to the calculation in subsection II B, we can show that $\Delta\Phi_1'(x', y', z_c)$ can be approximated by:

$$\Delta\Phi_1'(x', y', z_c) = -2\epsilon_3 A_1(z_c^-)A_2(z_c^-)\tilde{\Psi}_{10}'(x', y', z_c)$$
$$\times \int_{-\infty}^{\infty} dz' \tilde{\Psi}_{20}'(x' - x'_{20} - d_{11}'z', y' - y'_{20} - d_{12}'z', z_c).$$  \hspace{1cm} (E6)

It follows that the collision-induced amplitude shift in the $(x', y', z)$ coordinate system is

$$\Delta A_1^{(c)} = -\frac{2\epsilon_3 A_1(z_c^-)A_2(z_c^-)}{C_{p1}'}$$
$$\times \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \tilde{\Psi}_{10}'(x', y', z_c) \int_{-\infty}^{\infty} dz' \tilde{\Psi}_{20}'(x' - x'_{20} - d_{11}'z', y' - y'_{20} - d_{12}'z', z_c).$$  \hspace{1cm} (E7)

where

$$C_{p1}' = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \tilde{\Psi}_{10}'(x', y', 0).$$  \hspace{1cm} (E8)
We now make a change of variables by going to the \((x, y, z)\) coordinate system, in which the relative velocity vector \(d'_1\) lies on the \(x\) axis. The \((x, y, z)\) system is found by rotating the \(x'\) and \(y'\) axes by an angle \(\Delta \theta = \arctan(d'_{12}/d'_{11})\) about the \(z\) axis. The equations that define this change of variables are:

\[
\begin{align*}
x' &= x \cos \Delta \theta - y \sin \Delta \theta, \\
y' &= x \sin \Delta \theta + y \cos \Delta \theta,
\end{align*}
\]

(E9)

and

\[
\psi_j'(x', y', z) = \psi_j(x, y, z).
\]

(E10)

It is straightforward to show that the perturbed linear propagation model in the \((x, y, z)\) coordinate system is Eq. (11) and that \(d_{11} = d'_1\). The initial condition for the collision problem is given by Eq. (2), where \(h_j(x, y) = h'_j(x', y')\). We observe that the only large parameter in Eq. (11) is \(d_{11}\). Thus, the change of variables in Eqs. (E9) and (E10) does not change the properties of the fast dependence on \(z\) of the solutions to the unperturbed propagation equations (E3) and (E4). This means that the solution to the unperturbed equation

\[
i \partial_z \psi_1 + \partial_x^2 \psi_1 + \partial_y^2 \psi_1 = 0
\]

(E11)

does not contain any fast dependence on \(z\). In addition, the only fast dependence on \(z\) in the solution to the equation

\[
i \partial_z \psi_2 + i d_{11} \partial_x \psi_2 + \partial_x^2 \psi_2 + \partial_y^2 \psi_2 = 0
\]

(E12)

is contained in factors of the form \(x - x_{20} - d_{11} z\). If follows that we can employ the perturbation method of subsection 11.3 to calculate the collision-induced amplitude shift \(\Delta A^{(c)}_1\) in the \((x, y, z)\) system, and that \(\Delta A^{(c)}_1\) is given by Eq. (19), where \(C_{p1}\) is given by Eq. (18).

Let us show that the value of the amplitude shift \(\Delta A^{(c)}_1\) in Eq. (E7) is equal to the value \(\Delta A^{(c)}_1\) in Eq. (19). For this purpose we note that the determinant of the Jacobian matrix for the transformation (E9) is \(|J| = 1\). Using this together with Eqs. (18), (E8), and (E10), we obtain \(C'_{p1} = C_{p1}\). In addition, from Eq. (E10) it follows that \(\tilde{\Psi}_{j0}(x', y', z_c) = \tilde{\Psi}_{j0}(x, y, z_c)\). Furthermore, since the transformation in Eqs. (E9) and (E10) does not change the properties of the fast dependence on \(z\) of the solutions to the unperturbed propagation equations, and
since \( d_{12} = 0 \), we obtain

\[
\tilde{\Psi}_{20}'(x' - x_{20}' - d_{11}' z, y' - y_{20}' - d_{12}' z, z_c) = \tilde{\Psi}_{20}(x - x_{20} - d_{11} z, y, z_c).
\]  

(E13)

Using all the relations mentioned in the current paragraph in Eq. (E7), we arrive at:

\[
\Delta A_1^{(c)} = \frac{-2\epsilon_3 A_1(z^-_c) A_2^2(z^-_c)}{C_{pl}|d_{11}|} \times \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \tilde{\Psi}_{10}^2(x, y, z_c) \int_{-\infty}^{\infty} d\tilde{x} \tilde{\Psi}_{20}^2(\tilde{x}, y, z_c) = \Delta A_1^{(c)}.
\]  

(E14)

Thus, the value of \( \Delta A_1^{(c)} \) is indeed invariant under rotation transformations in the \( xy \) plane.

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In the current paper, we define the spatial dimension of the problem as the number of the coordinates on which the electric field depends for a given distance $z$. Therefore, in the three works in Refs. [8–10], for example, the spatial dimension was 1, since for each value of $z$, the electric field was a function of one coordinate only.

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[44] The separable initial condition is of special importance since it describes the electric fields that are produced by many lasers [4, 5].

[45] A method similar to the one presented in the current subsection can be used for calculating the collision-induced effects in any spatial dimension $n$, where $n \geq 2$. In section IV we carry out this calculation for spatial dimension 3.

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[53] Since in subsection III E the coefficient $\tilde{a}_1$ depends on $A_1(0)$ and $A_2(0)$ instead of on $A_1(z^-)$ and $A_2(z^-)$, we replace the notation $\tilde{a}_1(z^-)$ by $\tilde{a}_1$ in this subsection.

[54] Since the fractional intensity reduction factor obtained in the simulation $\Delta I_1^{(r)(num)}(x, y, z)$ shows weak dependence on $x$, we use two different methods to obtain $\Delta I_1^{(r)}(y, z)$ from the simulation result. In the first method, we calculate $\Delta I_1^{(r)}(y, z)$ by averaging $\Delta I_1^{(r)(num)}(x, y, z)$ over the $x$-interval $[-2, 2]$, and in the second method, we use the value of $\Delta I_1^{(r)(num)}(0, y, z)$.

[55] The dimensionless distance in Eq. (85) is $z = Z/(2L_D)$, where $Z$ is the dimensional distance, $L_D = \tau_0^2/|\tilde{\beta}_2|$ is the dispersion length, $\tau_0$ is the temporal width of a reference pulsed-beam, and $\tilde{\beta}_2$ is the dimensional second-order dispersion coefficient. The dimensionless coordinates $x$ and $y$ are $x = X/x_0$ and $y = Y/x_0$, where $X$ and $Y$ are the dimensional coordinates, and $x_0$ is the width of a reference pulsed-beam along the $x$ axis. $\psi_j = E_j/\sqrt{P_0}$, where $E_j$ is the electric field of the $j$th beam and $P_0$ is the peak power. The coefficients $d_{11}$ and $d_2$ are given by: $d_{11} = 2\tilde{\beta}_1\tau_0/|\tilde{\beta}_2|$ and $d_2 = 2\tilde{d}_2\tau_0^2/(|\tilde{\beta}_2|x_0^2)$, where $\tilde{\beta}_1$ is the dimensional first-order dispersion coefficient, and $\tilde{d}_2$ is the dimensional diffraction coefficient. In addition, the coefficient $\epsilon_3$ is given by: $\epsilon_3 = 2P_0\tau_0^2\tilde{\rho}_3/|\tilde{\beta}_2|$, where $\tilde{\rho}_3$ is the dimensional cubic loss coefficient.

[56] In writing Eq. (85) we assume (without loss of generality) $\text{sgn}(\tilde{\beta}_2) = -1$. The case $\text{sgn}(\tilde{\beta}_2) = 1$ can be treated in exactly the same manner.

[57] We choose to present the pulsed-beam shapes using cross-sections, since this enables a clear presentation of the dynamics of the main bodies of the pulsed-beams. Indeed, in a conventional contour plot of the pulsed-beams shapes in dimension 3, the main bodies are typically obscured by the outer parts of the pulsed-beams (i.e., by the pulsed-beams tails).