Intertwining operators for Sklyanin algebra and elliptic hypergeometric series

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Abstract

Intertwining operators for infinite-dimensional representations of the Sklyanin algebra with spins $\ell$ and $-\ell - 1$ are constructed using the technique of intertwining vectors for elliptic $L$-operator. They are expressed in terms of elliptic hypergeometric series with operator argument. The intertwining operators obtained (W-operators) serve as building blocks for the elliptic $R$-matrix which intertwines tensor product of two $L$-operators taken in infinite-dimensional representations of the Sklyanin algebra with arbitrary spin. The Yang-Baxter equation for this $R$-matrix follows from simpler equations of the star-triangle type for the $W$-operators. A natural graphic representation of the objects and equations involved in the construction is used.

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1 Introduction

Central to the theory of quantum integrable systems is quantum $R$-matrix satisfying the celebrated Yang-Baxter equation. General $R$-matrices with additive spectral parameter are parametrized via elliptic functions. The simplest elliptic $R$-matrix is

$$R(\lambda) = \sum_{a=0}^{3} \frac{\theta_{a+1}(2\lambda + \eta|\tau)}{\theta_{a+1}(\eta|\tau)} \sigma_a \otimes \sigma_a,$$

(1.1)

where $\theta_a(z|\tau)$ are Jacobi $\theta$-functions, $\sigma_a$ are Pauli matrices, and $\sigma_0$ is the unit matrix. This $R$-matrix is associated with the celebrated 8-vertex model solved by Baxter [1], being the matrix of local Boltzmann weights at the vertex. The transfer matrix of this model is the generating function of conserved quantities for the integrable anisotropic (of $XYZ$ type) spin-$1/2$ chain. Integrable spin chains of $XYZ$-type and their higher spin generalizations can be solved by the generalized algebraic Bethe ansatz [2, 3].

In lattice integrable models with elliptic $R$-matrix (1.1), the algebra of local observables is the Sklyanin algebra [4, 5] which is a special 2-parametric deformation of the universal enveloping algebra $U(gl(2))$. A concrete model is defined by fixing a particular representation of this algebra. Such representations can be realized by difference operators. Similar to the $sl(2)$-case (models of $XXX$-type), the representations are labeled by a continuous parameter which is called spin, and for positive half-integer values of this parameter the operators representing the Sklyanin algebra generators are known to have a finite-dimensional invariant space. However, we allow the spin to take any complex value, so we are going to work in a general infinite-dimensional representation of the algebra of observables.

Integrable spin chains of $XXX$-type with infinite-dimensional representations of symmetry algebra at the sites were first studied in the seminal papers [6, 7] in the context of high energy QCD, see also [8]. Later, lattice models with trigonometric $R$-matrix (of $XXZ$-type) with non-compact quantum group symmetry were considered [9]. A representation-theoretical approach to models with elliptic $R$-matrix and “non-compact” Sklyanin algebra symmetry is presently not available but there is no doubt that it should exist.

In this paper we present a direct construction of the elliptic $R$-matrix intertwining the tensor product of two arbitrary infinite-dimensional representations of the Sklyanin algebra. It can be realized as a difference operator in two variables, in general of infinite order, so we often call this object $R$-operator rather than $R$-matrix. Another important object is a face type $R$-matrix related to the $R$-operator via a functional version of the vertex-face correspondence. The latter $R$-matrix provides an elliptic analog of $6j$-symbols.

Our method closely follows the similar construction in the chiral Potts model [10, 11, 12] and the broken $\mathbb{Z}_N$-symmetric model [13, 14]. It is based on the observation that the elementary $L$-operator is in fact a composite object built from simpler entities called “intertwining vectors” [14, 15]. Then the proof of the Yang-Baxter equation and other properties of the $L$-operator can be reduced to simple manipulations with the intertwining vectors using basic relations between them. Remarkably, all elements of this procedure have a nice graphic interpretation which makes them rather clear and greatly simplifies
the arguments. It provides simultaneously a very good illustration and an important heuristic tool. This graphical technique resembles both the one developed for the Chiral Potts and broken $\mathbb{Z}_N$-symmetric models and the one known in the representation theory of $q$-deformed algebras, in particular in connection with $q$-deformation of $6j$-symbols [16].

However, practical realization of these ideas in the infinite-dimensional setting is by no means obvious. Technically, it is rather different from what is customary in the $8$-vertex model and its relatives. Our construction goes along the line of our earlier work [17] devoted to the $Q$-operator for spin chains with infinite-dimensional representations of the Sklyanin algebra at each site and extensively uses such really special functions as elliptic gamma-function and elliptic generalization of hypergeometric series. The theory of elliptic hypergeometric functions originated by Frenkel and Turaev in [18] is now an actively developing new branch of mathematics (see, e.g., [19, 20, 21] and references therein).

The elliptic $R$-operator appears to be a composite object whose building blocks are operators which intertwine representations of the Sklyanin algebra with spins $\ell$ and $-\ell - 1$ (the $W$-operator). They can be expressed through the elliptic hypergeometric series $\omega_3$ with an operator argument. The kernels of the $W$-operators are expressed through ratios of the elliptic gamma-function. These intertwining operators were found in our earlier paper [17] as a by-product of the general elliptic $Q$-operator construction. Here we re-derive this result with the help of the intertwining vectors using much more direct arguments. We also give a construction of vacuum vectors for the elliptic $L$-operator using the graphic technique and show how they are related to the kernel of the $W$-operator.

It should be remarked that similar results, in one or another form, can be found in the existing literature. In particular, the elliptic $R$-operator has been found [22] in terms of operators which implement elementary permutations of parameters entering the $RLL = LLR$ relation. A solution to the star-triangle equation built from ratios of the elliptic gamma-functions was recently suggested in [23]. Some closely related matters are discussed in the recent paper [24]. It seems to us that our approach may be of independent interest since it emphasizes the connection with the Sklyanin algebra and allows one to obtain more detailed results in a uniform way.

The paper is organized as follows. Section 2 contains the necessary things related to the Sklyanin algebra, its realization by difference operators and representations. In section 3 we describe a space of discontinuous functions of special form, where the Sklyanin algebra acts, and which are identified with kernels of difference operators. Here we follow [17]. The technique of intertwining vectors developed in Section 4 is used in Section 5 to construct operators which intertwine representations of the Sklyanin algebra with spins $\ell$ and $-\ell - 1$. They appear to be the most important constituents of the elliptic $R$-operator. In section 6 we show how the vacuum vectors for the $L$-operator constructed in [17] emerge within the approach of the present paper. The construction of the elliptic $R$-operator and related objects for arbitrary spin is presented in Section 7, where the Yang-Baxter and star-triangle relations are also discussed. Some concluding remarks are given in Section 8. Appendix A contains necessary information on the special functions involved in the main part of the paper. In Appendix B some details of the calculations with elliptic hypergeometric series are presented.
2 Representations of the Sklyanin algebra

The aim of this section is to give the necessary preliminaries on representations of the Sklyanin algebra. We begin with a few formulas related to the quantum L-operator with elliptic dependence on the spectral parameter.

The elliptic quantum L-operator is the matrix

$$L(\lambda) = \frac{1}{2} \begin{pmatrix}
\theta_1(2\lambda)s_0 + \theta_4(2\lambda)s_3 & \theta_2(2\lambda)s_1 + \theta_3(2\lambda)s_2 \\
\theta_2(2\lambda)s_1 - \theta_3(2\lambda)s_2 & \theta_1(2\lambda)s_0 - \theta_4(2\lambda)s_3
\end{pmatrix}$$

(2.1)

with non-commutative matrix elements. Specifically, $s_a$ are difference operators in a complex variable $z$:

$$s_a = \frac{\theta_{a+1}(2z - 2\ell \eta)}{\theta_1(2z)} e^{\eta \partial_z} - \frac{\theta_{a+1}(-2z - 2\ell \eta)}{\theta_1(2z)} e^{-\eta \partial_z}$$

(2.2)

introduced by Sklyanin [5]. Here $\theta_a(z) \equiv \theta_a(z|\tau)$ are Jacobi $\theta$-functions with the elliptic module $\tau$, $\Im \tau > 0$, $\ell$ is a complex number (the spin), and $\eta \in \mathbb{C}$ is a parameter which is assumed to belong to the fundamental parallelogram with vertices 0, 1, $\tau$, $1+\tau$, and to be incommensurate with 1, $\tau$. Definitions and transformation properties of the $\theta$-functions are listed in Appendix A.

The four operators $s_a$ obey the commutation relations of the Sklyanin algebra [4]:

$$(-1)^{a+1}I_{a0}s_as_\gamma = I_{\beta\gamma}s_\beta s_\gamma - I_{\gamma\beta}s_\gamma s_\beta,$$

(2.3)

$$(-1)^{a+1}I_{a0}s_\alpha s_a = I_{\gamma\beta}s_\beta s_\gamma - I_{\beta\gamma}s_\gamma s_\beta$$

with the structure constants $I_{ab} = \theta_{a+1}(0)\theta_{b+1}(2\eta)$. Here $a, b = 0, \ldots, 3$ and $\{\alpha, \beta, \gamma\}$ stands for any cyclic permutation of $\{1, 2, 3\}$. The relations of the Sklyanin algebra are equivalent to the condition that the L-operator satisfies the "RLL = LLR" relation with the elliptic $R$-matrix (1.1).

The parameter $\ell$ in (2.2) is called the spin of the representation. If necessary, we write $s_a = s_a^{(\ell)}$ or $L^{(\ell)}(\lambda)$ to indicate the dependence on $\ell$. When $\ell \in \frac{1}{2}\mathbb{Z}_+$, these operators have a finite-dimensional invariant subspace, namely, the space $\tilde{\Theta}^+_{2\ell}$ of even $\theta$-functions of order $4\ell$ (see Appendix A). This is the representation space of the $(2\ell + 1)$-dimensional irreducible representation (of series a) of the Sklyanin algebra. For example, at $\ell = \frac{1}{2}$ the functions $\tilde{\theta}_1(z), \tilde{\theta}_3(z)$ (hereafter we use the notation $\tilde{\theta}_a(z) \equiv \theta_a(z|\frac{\pi i}{2})$) form a basis in $\Theta^+_{2\ell}$, and the generators $s_a$, with respect to this basis, are represented by $2 \times 2$ matrices $(-i)^{\delta_{a2}}(\theta_{a+1}(\eta))^{-1}\sigma_a$. In this case, $L(\lambda) = R(\lambda - \frac{1}{2}\eta)$, where $R$ is the 8-vertex model $R$-matrix (1.1). In general, the representation space of the Sklyanin algebra where the operators $s_a$ act is called quantum space while the two-dimensional space in which the $L$-operator is the $2 \times 2$ matrix is called auxiliary space.

As is proved in [25], the space $\tilde{\Theta}^+_{2\ell}$ for $\ell \in \frac{1}{2}\mathbb{Z}_+$ is annihilated by the operator

$$W_\ell = e^{2\ell+1} \sum_{k=0}^{2\ell+1} (-1)^{k} \begin{vmatrix}
2\ell + 1 \\
k
\end{vmatrix} \frac{\theta_1(2z + 2(2\ell - 2k + 1)\eta)}{\prod_{j=0}^{2\ell+1} \theta_1(2z + 2(j - k)\eta)} e^{(2\ell - 2k + 1)\eta \partial_z}.$$  

(2.4)

The standard generators of the Sklyanin algebra [4], $s_a$, are related to ours as follows: $s_a = (i)^{\delta_{a2}}\theta_{a+1}(\eta)s_a$. 

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where $c$ is a normalization constant to be fixed below. Hereafter, we use the “elliptic factorial” and “elliptic binomial” notation:

$$[j] \equiv \theta_1(2j\eta), \quad [n]! = \prod_{j=1}^{n}[j], \quad \left[ \begin{array}{c} n \\ m \end{array} \right] \equiv \frac{[n]!}{[m]![n-m]!}.$$ \hspace{1cm} (2.5)

The defining property of the operator $W_\ell$ established in [25] is that $W_\ell$ intertwines representations of spin $\ell$ and of spin $-(\ell + 1)$:

$$W_\ell s^{(\ell)}_a = s^{(-\ell-1)}_a W_\ell, \quad a = 0, \ldots, 3.$$ \hspace{1cm} (2.6)

The same intertwining relation can be written for the quantum $L$-operator (2.1):

$$W_\ell L^{(\ell)}(\lambda) = L^{(-\ell-1)}(\lambda) W_\ell.$$ \hspace{1cm} (2.7)

Note that the operator $W_\ell$ serves as an elliptic analog of $(d/dz)^{2\ell+1}$ in the following sense. In the case of the algebra $sl(2)$, the intertwining operator between representations of spins $\ell$ and $-\ell - 1$ (realized by differential operators in $z$) is just $(d/dz)^{2\ell+1}$. It annihilates the linear space of polynomials of degree $\leq 2\ell$ (which results in the rational degeneration of the elliptic space $\Theta^+_{\ell+\ell}$).

Very little is known about infinite-dimensional representations of the Sklyanin algebra. The difference operators (2.2) do provide such a representation but any characterization of the space of functions where they are going to act is not available at the moment, at least for continuous functions. On the other hand, the difference character of the operators (2.2) suggests to consider their action on a space of discontinuous functions of a special form. The latter are naturally identified with kernels of difference operators. This formalism was used in our earlier paper [17]. It is reviewed in the next section.

3 Kernels of difference operators

Let $\delta(z)$ be the function equal to zero everywhere but at $z = 0$, where it equals 1: $\delta(z) = 0$, $z \neq 0$, $\delta(0) = 1$. (We hope that the same notation as for the conventional delta-function will cause no confusion because the latter will not appear in what follows.) Clearly, $z\delta(z) = 0$ and $\delta^2(z) = \delta(z)$.

Consider the space $\mathcal{C}$ of functions of the form

$$f(z) = \sum_{k \in \mathbb{Z}} f_k \delta(z - \nu + 2k\eta), \quad f_k \in \mathbb{C},$$ \hspace{1cm} (3.1)

where $\nu \in \mathbb{C}$. This space is isomorphic to the direct product of $\mathbb{C}$ and the linear space of sequences $\{f_k\}_{k \in \mathbb{Z}}$. We call functions of the form (3.1) combs. Clearly, the Sklyanin algebra realized as in (2.2) acts in this space (shifting $\nu \rightarrow \nu \pm \eta$).
A comb is said to be finite from the right (respectively, from the left) if there exists \( M \in \mathbb{Z} \) such that \( f_k = 0 \) as \( k > M \) (respectively, \( k < M \)). Let \( \mathcal{C}^r \) (respectively, \( \mathcal{C}^l \)) be the space of combs finite from the left (respectively, from the right).

We define the pairing
\[
(F(z), \delta(z - a)) = F(a) \tag{3.2}
\]
for any function \( F(z) \), not necessarily of the form (3.1). In particular,
\[
(\delta(z - a), \delta(z - b)) = \delta(a - b). \tag{3.3}
\]
Formally, this pairing can be written as an integral:
\[
(F(z), \delta(z - a)) = \int dz F(z) \delta(z - a) \tag{3.4}
\]
(perhaps a \( q \)-integral symbol would be more appropriate). We stress that the integral here means nothing more than another notation for the pairing, especially convenient in case of many variables. By linearity, the pairing can be extended to the whole space of combs. We note that the pairing between the spaces \( \mathcal{C}^r \) and \( \mathcal{C}^l \) is well defined since the sum is always finite.

Combs are to be thought of as kernels of difference operators. By a difference operator in one variable we mean any expression of the form
\[
D = \sum_{k \in \mathbb{Z}} c_k(z) e^{(\mu + 2k\eta) \partial_z}, \quad \mu \in \mathbb{C}. \tag{3.5}
\]
The comb
\[
D(z, \zeta) = \sum_{k \in \mathbb{Z}} c_k(z) \delta(z - \zeta + \mu + 2k\eta), \tag{3.6}
\]
regarded as a function of any one of the variables \( z, \zeta \), is the kernel of this difference operator in the following sense. Using the pairing introduced above, we can write:
\[
(Df)(z) = \int D(z, \zeta) f(\zeta) d\zeta = \sum_{k \in \mathbb{Z}} c_k(z) f(z + \mu + 2k\eta). \tag{3.7}
\]
The kernel \( D(z, \zeta) \) can be viewed as an infinite matrix with continuously numbered rows \( (z) \) and columns \( (\zeta) \). Then the convolution with respect to the second argument of the kernel, as in (3.7), defines action of the operator from the left. The convolution with respect to the first argument defines the action from the right,
\[
(fD)(z) = \int f(\zeta) D(\zeta, z) d\zeta, \tag{3.8}
\]
equivalent to the action of the transposed difference operator from the left:
\[
D^t = \sum_{k \in \mathbb{Z}} e^{-\mu + 2k\eta) \partial_z} c_k(z) = \sum_{k \in \mathbb{Z}} c_k(z - \mu - 2k\eta) e^{-(\mu + 2k\eta) \partial_z}. \tag{3.9}
\]
The transposition \( t \) is the anti-automorphism of the algebra of difference operators such that \((c(z)e^{\alpha \partial_z})^t = e^{-\alpha \partial_z} c(z)\). In terms of the above pairing we can write \((f, Dg) = (D^t f, g)\).
The following simple remarks will be useful in what follows. Let \( F(z), G(z) \) be any functions, then \( F(z)D(z, \zeta)G(\zeta) \), with \( D(z, \zeta) \) as above, is the kernel of the difference operator

\[
FDG = \sum_{k \in \mathbb{Z}} c_k(z) F(z) G(z + \mu + 2k\eta) e^{(\mu+2k\eta)\partial_z}
\]

which is the composition of the multiplication by \( G \), action of the operator \( D \) and subsequent multiplication by \( F \). Let \( D^{(1)}(z, \zeta), D^{(2)}(z, \zeta) \) be kernels of difference operators \( D^{(1)}, D^{(2)} \) respectively, then the convolution

\[
\int d\xi D^{(2)}(z, \xi) D^{(1)}(\xi, \zeta)
\]

is the kernel of the difference operator \( D^{(2)}D^{(1)} \). If the kernels \( D^{(1)}(z, \zeta), D^{(2)}(z, \zeta) \) are combs finite from the left (right) as functions of \( z \), then the convolution is always well defined and the resulting kernel belongs to the same space of combs.

The kernels of Sklyanin’s operators \([2.2] \) are:

\[
s_a(z, z') = \frac{\theta_{a+1}(2z - 2\ell\eta)}{\theta_1(2z)} \delta(z - z' + \eta) - \frac{\theta_{a+1}(-2z - 2\ell\eta)}{\theta_1(2z)} \delta(z - z' - \eta).
\] (3.10)

Note that \( s_a(-z, -z') = s_a(z, z') \). Let us find the kernel of the \( L \)-operator \([2.1] \). Using identities for theta-functions, it is easy to see that

\[
L_\xi^z(\lambda) = \theta_1(2\lambda + 2\ell\eta)V^{-1}(\lambda + \ell\eta, z) \begin{pmatrix} \delta(z-\zeta+\eta) & 0 \\ 0 & \delta(z-\zeta-\eta) \end{pmatrix} V(\lambda - \ell\eta, z),
\] (3.11)

where \( V(\lambda, z) \) is the matrix

\[
V(\lambda, z) = \begin{pmatrix} \bar{\theta}_4(z + \lambda) & \bar{\theta}_3(z + \lambda) \\ \bar{\theta}_4(z - \lambda) & \bar{\theta}_3(z - \lambda) \end{pmatrix}
\]

and \( V^{-1}(\lambda, z) \) is its inverse:

\[
V^{-1}(\lambda, z) = \frac{1}{2\theta_1(2z)} \begin{pmatrix} \bar{\theta}_3(z - \lambda) & -\bar{\theta}_3(z + \lambda) \\ -\bar{\theta}_4(z - \lambda) & \bar{\theta}_4(z + \lambda) \end{pmatrix}
\]

(recall that \( \bar{\theta}_a(z) \equiv \theta_a(z|\frac{\pi}{2}) \)). A crucial point is that the diagonal matrix with delta-functions factorizes into the product of column and row vectors:

\[
\begin{pmatrix} \delta(z-\zeta+\eta) & 0 \\ 0 & \delta(z-\zeta-\eta) \end{pmatrix} = \begin{pmatrix} \delta(z-\zeta+\eta) \\ \delta(z-\zeta-\eta) \end{pmatrix} \begin{pmatrix} \delta(z-\zeta+\eta) & \delta(z-\zeta-\eta) \end{pmatrix}
\]

and thus so does \( L_\xi^z(\lambda) \). The vectors which represent the factorized kernel of the \( L \)-operator are “intertwining vectors” introduced in the next section.

### 4 Intertwining vectors

We introduce the 2-component (co)vector

\[
|\zeta\rangle = \begin{pmatrix} \bar{\theta}_4(\zeta) \\ \bar{\theta}_3(\zeta) \end{pmatrix}, \quad \langle \zeta| = (\bar{\theta}_4(\zeta), \bar{\theta}_3(\zeta)).
\] (4.1)
The vector orthogonal to $\langle \zeta | \rangle$ is $|\zeta\rangle^\perp = \begin{pmatrix} \bar{\theta}_3(\zeta) \\ -\bar{\theta}_4(\zeta) \end{pmatrix}$, the covector orthogonal to $|\zeta\rangle$ is $\langle \zeta | \rangle^\perp = (\bar{\theta}_3(\zeta), -\bar{\theta}_4(\zeta))$, so $\langle \zeta | \rangle^\perp \cdot \langle \zeta | \rangle = 0$. More generally, we have:

$$\langle \xi | \zeta \rangle^\perp = 2\theta_1(\xi + \zeta)\theta_1(\xi - \zeta) = -\langle \xi | \zeta \rangle.$$ (4.2)

Note also that

$$|\zeta + \frac{1}{2}(1 + \tau)| = e^{-\pi i z^2 - 2\pi i \zeta} |\zeta\rangle^\perp.$$ (4.3)

Introduce now the intertwining vectors

$$|\phi^\zeta_z(\lambda)\rangle = \frac{1}{\sqrt{2\theta_1(2z)}}((z + \lambda)\delta(z - z' + \eta) + |z - \lambda\rangle \delta(z - z' - \eta)), \quad (4.4)$$

$$|\bar{\phi}^\zeta_z(\lambda)\rangle = -\frac{1}{\sqrt{2\theta_1(2z)}}(\langle \zeta - \lambda | \rangle^\perp \delta(z - z' + \eta) - |z + \lambda\rangle^\perp \delta(z - z' - \eta)) \quad (4.5)$$

and the corresponding covectors

$$\langle \phi^\zeta_z(\lambda)\rangle = \frac{1}{\sqrt{2\theta_1(2z)}}(|z + \lambda\rangle \delta(z - z' + \eta) + \langle z - \lambda | \delta(z - z' - \eta)), \quad (4.6)$$

$$\langle \bar{\phi}^\zeta_z(\lambda)\rangle = \frac{1}{\sqrt{2\theta_1(2z)}}(|z - \lambda\rangle^\perp \delta(z - z' + \eta) - \langle z + \lambda |^\perp \delta(z - z' - \eta)) \quad (4.7)$$

It is easy to check that

$$|\phi^\zeta_z(-\lambda)\rangle = \sqrt{\frac{\theta_1(2z')}{\theta_1(2z)}} |\phi^\zeta_z(\lambda + \eta)\rangle,$$

$$|\bar{\phi}^\zeta_z(-\lambda)\rangle = -\sqrt{\frac{\theta_1(2z')}{\theta_1(2z)}} |\bar{\phi}^\zeta_z(\lambda - \eta)\rangle.$$
Figure 2: The graphic representation of the relation $W^{z,z'}(\lambda - \mu)\langle \phi^z_\lambda(\lambda + \frac{\eta}{2}) \big| \phi^{\lambda'}_\epsilon(\mu - \frac{\eta}{2}) \rangle = W^{z,z'}(\lambda - \mu)\langle \phi^z_\lambda(\mu + \frac{\eta}{2}) \big| \phi^{\lambda'}_\epsilon(\lambda - \frac{\eta}{2}) \rangle$. The horizontal bold line segment common for the covector to the left and the vector to the right means taking scalar product of the two-dimensional (co)vectors. The intersection point of the spectral parameter lines corresponds to the "vertex" $W^{z,z'}(\lambda - \mu)$.

The intertwining vectors satisfy the following orthogonality relations:

$$\langle \phi^z_\lambda(\lambda) \big| \phi^{\lambda'}_\epsilon(\lambda) \rangle = \theta_1(2\lambda) \delta(z' - z') \left( \delta(z - z' + \eta) + \delta(z - z' - \eta) \right),$$

(4.8)

$$\langle \phi^z_\lambda(\lambda + \eta) \big| \phi^{\lambda'}_\epsilon(\lambda - \eta) \rangle = \theta_1(2\lambda) \frac{\theta_1(2z)}{\theta_1(2z')} \delta(z' - z'') \left( \delta(z - z' + \eta) + \delta(z - z' - \eta) \right),$$

(4.9)

$$\int d\zeta \left| \delta^z_\lambda(\lambda) \right\rangle \left\langle \phi^z_\lambda(\lambda) \right| = \theta_1(2\lambda) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(4.10)

$$\int d\zeta \frac{\theta_1(2\zeta)}{\theta_1(2z')} \left| \delta^z_\lambda(\lambda - \eta) \right\rangle \left\langle \phi^z_\lambda(\lambda + \eta) \right| = \theta_1(2\lambda) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. $$

(4.11)

The general scalar product of two intertwining vectors is

$$\left\langle \phi^z_\lambda(\lambda) \big| \phi^{\lambda'}_\epsilon(\mu) \right\rangle = \frac{1}{\sqrt{4\theta_1(2z)\theta_1(2\zeta)}} \times \left\{ \theta_1(z + \zeta + \lambda - \mu)\theta_1(z - \zeta + \lambda + \mu)\delta(z - z' + \eta)\delta(\zeta - \zeta' + \eta) \\
- \theta_1(z + \zeta + \lambda + \mu)\theta_1(z - \zeta + \lambda - \mu)\delta(z - z' + \eta)\delta(\zeta - \zeta' - \eta) \\
+ \theta_1(z + \zeta - \lambda - \mu)\theta_1(z - \zeta - \lambda + \mu)\delta(z - z' - \eta)\delta(\zeta - \zeta' + \eta) \\
- \theta_1(z + \zeta - \lambda + \mu)\theta_1(z - \zeta - \lambda - \mu)\delta(z - z' - \eta)\delta(\zeta - \zeta' - \eta) \right\}.$$  

(4.12)

It is a matter of direct verification to see that such scalar products satisfy the "intertwining relation":

$$W^{z,z'}(\lambda - \mu)\langle \phi^z_\lambda(\lambda + \frac{\eta}{2}) \big| \phi^{\lambda'}_\epsilon(\mu - \frac{\eta}{2}) \rangle = W^{z,z'}(\lambda - \mu)\langle \phi^z_\lambda(\mu + \frac{\eta}{2}) \big| \phi^{\lambda'}_\epsilon(\lambda - \frac{\eta}{2}) \rangle,$$  

(4.12)
Figure 3: The kernel of the $L$-operator $L^z(\lambda, \mu) = \left| \phi^z(\lambda - \frac{\eta}{2}) \langle \phi^z(\mu + \frac{\eta}{2}) \right|$. 

where the quantities $W^z(\lambda)$ solve the following difference equations in $z, \zeta$:

$$W^{z+\eta,\zeta+\eta}(\lambda) = \frac{\theta_1(z + \zeta + \lambda + \eta)}{\theta_1(z + \zeta + \lambda + \eta)} W^{z,\zeta}(\lambda),$$

$$(4.13)$$

These equations can be solved in terms of the elliptic gamma-function $\Gamma(z|\tau, 2\eta) := \Gamma(z)$ [26, 27] (see Appendix A):

$$W^z(\lambda) = e^{-2\pi i z / \eta} \frac{\Gamma(z + \zeta + \lambda + \eta)\Gamma(z - \zeta + \lambda + \eta)\Gamma(z + \zeta - \lambda + \eta)\Gamma(z - \zeta - \lambda + \eta)}{\Gamma(z + \zeta + \lambda + \eta)\Gamma(z - \zeta + \lambda + \eta)\Gamma(z + \zeta - \lambda + \eta)\Gamma(z - \zeta - \lambda + \eta)}.$$  \hspace{1cm} (4.14)

There is a freedom to multiply the solution by an arbitrary $2\eta$-periodic function of $z + \zeta$ and $z - \zeta$. We put this function equal to 1. (However, this does not mean that this is the best normalization; other possibilities will be discussed elsewhere.) In our normalization

$$W^z(\lambda)W^z(-\lambda) = 1$$  \hspace{1cm} (4.15)

but $W^z(\lambda)$ is not symmetric under permutation of $z$ and $\zeta$.

The intertwining vectors can be represented graphically as shown in Fig. 1. The vertical line carries the spectral parameter and serves as a line of demarcation between the “real” (transparent) world and the “shadow” world. Then the relation (4.12) means that the horizontal line in Fig. 2 can be moved through the intersection point of the two spectral parameter lines. This intersection point is a new graphic element which corresponds to $W^{z,\zeta}(\lambda - \mu)$.

The kernel of the $L$-operator for the representation of spin $\ell$ can be written in the factorized form as the product of intertwining vectors:

$$L^{(\ell)}^z(\lambda) = \left| \phi^z(\lambda_+ - \frac{\eta}{2}) \langle \phi^z(\lambda_- + \frac{\eta}{2}) \right|, \quad \lambda_\pm = \lambda \pm (\ell + \frac{1}{2})\eta.$$  \hspace{1cm} (4.16)

It clear that the spectral parameter $\lambda$ and the representation parameter $\ell\eta$ enter here on equal footing, so the notation $L^{(\ell)}^z(\lambda) = L^z(\lambda_+, \lambda_-)$ is sometimes also convenient. Graphically, the kernel of the $L$-operator is shown in Fig. 3.
5 Intertwining operators for arbitrary spin

There is a relation which is “dual” to (4.12) (see also Fig. 2) meaning that it can be read from the same configuration of lines in the figure by exchanging the real and shadow world pieces of the plane (see Fig. 5). Two new elements appear: first, the vertex \( W_\zeta(\lambda - \mu) \) is different from the one in Fig. 2 and, second, one should take convolution \( \int d\zeta \) with respect to the “intermediate” variable \( \zeta \) associated to the finite triangle in the shadow world. The two vertices, \( W^{\zeta,\xi}(\lambda - \mu) \) and \( W_\zeta(\lambda - \mu) \), are shown separately in Fig. 4.

According to Fig. 5, the dual relation has the form

\[
\int d\zeta W_\zeta(\lambda - \mu) \bar{\phi}_\zeta'(\lambda - \eta) \phi_\zeta'(\mu + \eta) = \int d\zeta W_\zeta(\lambda - \mu) \bar{\phi}_\zeta'(\mu - \eta) \phi_\zeta'(\lambda + \eta).
\] (5.1)

Changing the notation \( \lambda \rightarrow \lambda_+ \), \( \mu \rightarrow \lambda_- \), one can write it as

\[
\int d\zeta W_\zeta(\lambda_+ - \lambda_-) L_\zeta^\xi(\lambda_+, \lambda_-) = \int d\zeta W_\zeta(\lambda_+ - \lambda_-) L_\zeta^\xi(\lambda_-, \lambda_+)
\]

which is just the intertwining relation for the \( L \)-operator \( L^{(\ell)}(\xi) = L(\lambda_+, \lambda_-) \) (2.7), with \( W_\zeta(\lambda_+ - \lambda_-) \) being the kernel of the difference operator \( W_\zeta \). Taking this into account, we are going to find solutions for the \( W_\zeta^\xi \) in the space of combs finite either from the right or from the left.

Let us take the scalar product of both sides of equation (5.1) with the covector \( \langle \phi_\zeta''(\lambda + \frac{3\eta}{2}) \rangle \) from the left and the vector \( \langle \bar{\phi}_\zeta'(\lambda + \frac{\eta}{2}) \rangle \) from the right. Using the orthogonality relations (4.8), (4.9), we obtain:

\[
\frac{W_\zeta^\xi(\lambda - \mu)}{\theta_1(2\zeta''\eta)} \langle \phi_\zeta''(\mu + \frac{\eta}{2}) \bar{\phi}_\zeta'(\lambda + \frac{\eta}{2}) \rangle = \frac{W_\zeta^\xi(\lambda - \mu)}{\theta_1(2\zeta''\eta)} \langle \phi_\zeta''(\lambda + \frac{3\eta}{2}) \bar{\phi}_\zeta'(\mu - \frac{\eta}{2}) \rangle.
\] (5.2)

This functional relation for \( W_\zeta^\xi \) can be solved in terms of \( W^{\zeta,\xi} \) with the help of (4.12): \( W_\zeta(\lambda)W^{\zeta,\xi}(\lambda + \eta) = \theta_1(2\zeta) \). However, this solution is not exactly what we need because it is not a comb-like function. Proceeding in a slightly different way, one can rewrite (5.2) as a system of difference equations for \( W_\zeta^\xi(\lambda) \):

\[
\begin{align*}
W^\zeta_{\xi+\eta}(\lambda) &= \frac{\theta_1(2\zeta + 2\eta)}{\theta_1(2\zeta)} \frac{\theta_1(z + \zeta - \lambda)}{\theta_1(z + \zeta + \lambda + 2\eta)} W_\zeta^\xi(\lambda), \\
W^\zeta_{\xi-\eta}(\lambda) &= \frac{\theta_1(2\zeta + 2\eta)}{\theta_1(2\zeta)} \frac{\theta_1(z - \zeta - \lambda)}{\theta_1(z - \zeta + \lambda + 2\eta)} W_\zeta^\xi(\lambda).
\end{align*}
\] (5.3)
Comparing with (4.13), one immediately finds a solution in the space of combs:

\[ W_z^\zeta(\lambda) = \frac{c(\lambda)\theta_1(2\zeta)}{W_\zeta^z(\lambda + \eta)} \sum_{k \in \mathbb{Z}} \delta(z - \zeta - \nu + 2k\eta) \]

with \( W_\zeta^z \) given by (4.14) and arbitrary \( \nu \). (The factor in front of the sum is also a solution but in the space of meromorphic functions.) The function \( c(\lambda) \) introduced here for the proper normalization is not determined from the difference equations. It will be fixed below. One may truncate the comb from the left choosing \( \nu = \lambda \); then the coefficients in front of \( \delta(z - \zeta - \lambda + 2k\eta) \) with \( k < 0 \) vanish because the function \( W_z^\zeta(\lambda + \eta) \) has poles at \( \zeta = z - \lambda + 2k\eta, \ k \leq -1 \). Another possibility is to truncate the comb from the right choosing \( \nu = -\lambda \); then the arguments of the delta-functions at \( k \leq 0 \) exactly coincide with the half-infinite lattice of zeros of the function \( W_\zeta^z(\lambda + \eta) \), and so one can make the truncated comb by taking residues. Below we use the first possibility and consider the solution

\[
W_z^\zeta(\lambda) = c(\lambda) \sum_{k \geq 0} \theta_1(2z - 2\lambda + 4k\eta) \frac{\Gamma(2z - 2\lambda + 4k\eta)}{\Gamma(2z + 2\eta + 2k\eta)} \delta(z - \zeta - 2\lambda + 2k\eta) e^{(-\lambda + 2k\eta)\partial_z}.
\]
Rewriting the coefficients in terms of the elliptic Pochhammer symbols with the help of (A15), (A17) and extracting a common multiplier, we obtain

$$W(\lambda) = \tilde{c}(\lambda) e^{2\pi i \lambda (z-\lambda)/\eta} \frac{\Gamma(-2\lambda) \Gamma(2z-2\lambda+2\eta)}{\Gamma(2\eta) \Gamma(2\lambda+2\eta)} \sum_{k \geq 0} \frac{[\frac{-\lambda}{\eta}]^k [\frac{-2\lambda}{\eta}]_k \frac{z}{\eta} + 1}{[\frac{-\lambda}{\eta}]^k} e^{(-\lambda+2k\eta) \partial_z},$$

where \( \tilde{c}(\lambda) = ie^{\frac{2\pi i}{\eta} \eta_D(\tau)} c(\lambda) \). The infinite sum can be written in terms of the elliptic hypergeometric series \(_4\omega_3\) (see Appendix A for the definition) with operator argument:

$$W(\lambda) = \tilde{c}(\lambda) e^{2\pi i \lambda (z-\lambda)/\eta} \frac{\Gamma(-2\lambda) \Gamma(2z-2\lambda+2\eta)}{\Gamma(2\eta) \Gamma(2\lambda+2\eta)} \cdot _4\omega_3 \left( \frac{z - \lambda}{\eta}; -\frac{\lambda}{\eta}; e^{2\eta \partial_z}; e^{-\lambda \partial_z} \right). \quad (5.5)$$

Here the double dots mean normal ordering such that the shift operator \( e^{2\eta \partial_z} \) is moved to the right. By construction, this operator satisfies the intertwining relation

$$W(\lambda - \mu) L(\lambda, \mu) = L(\mu, \lambda) W(\lambda - \mu). \quad (5.6)$$

The intertwining property (5.6) suggests that \( W(\lambda) W(-\lambda) = id \) or, equivalently,

$$\int d\zeta W^\xi(\lambda) W^\zeta(-\lambda) = \delta(z - z') \quad (5.7)$$

which is a shadow world analog of (4.15). This is indeed true provided that the function \( c(\lambda) \) is fixed to be

$$c(\lambda) = \frac{\rho_0 e^{\pi i \lambda^2 /\eta}}{\Gamma(-2\lambda)}, \quad (5.8)$$

where the constant \( \rho_0 \) is

$$\rho_0 = \frac{\Gamma(2\eta)}{ie^{\frac{2\pi i}{\eta} \eta_D(\tau)} \cdot \frac{e^{\frac{\pi i}{\eta} (2\eta-3\tau)}}{i\eta_D(2\eta)}} \quad (5.9)$$

(clearly, there is still a freedom to multiply \( c(\lambda) \) by a function \( \varphi(\lambda) \) such that \( \varphi(\lambda) \varphi(-\lambda) = 1 \)). It should be noted that the very fact that the product \( W(\lambda) W(-\lambda) \) is proportional to the identity operator is by no means obvious from the infinite series representation (5.5). This fact was explicitly proved in [22] with the help of the Frenkel-Turaev summation formula. For completeness, we present some details of this calculation in Appendix B. It is this calculation that allows one to find \( c(\lambda) \) explicitly.

We thus conclude that the properly normalized intertwining operator \( W(\lambda) \) reads

$$W(\lambda) = e^{-\pi i \lambda^2 /\eta} \frac{\Gamma(2z-2\lambda+2\eta)}{\Gamma(2\lambda+2\eta)} \cdot _4\omega_3 \left( \frac{z - \lambda}{\eta}; -\frac{\lambda}{\eta}; e^{2\eta \partial_z}; e^{-\lambda \partial_z} \right), \quad (5.10)$$

or, in terms of the parameter \( d \equiv 2\ell + 1 \in \mathbb{C} \) related to the spin \( \ell \) of the representation,

$$W_\ell \equiv W(d\eta) = e^{-\pi id^2 \eta + 2\pi idz} \frac{\Gamma(2z-2(d-1)\eta)}{\Gamma(2\lambda+2\eta)} \cdot _4\omega_3 \left( \frac{z - d}{\eta}; -d; e^{2\eta \partial_z}; e^{-d \eta \partial_z} \right). \quad (5.11)$$

It is not difficult to see that the change of sign \( z \rightarrow -z \) transforms \( W(\lambda) \) to another intertwining operator for the Sklyanin algebra, which is an infinite series in shifts in the opposite direction. (It is this latter operator which was constructed in the paper [17].)
It can be obtained within the same approach if one uses the other possibility to truncate the comb which has been discussed above. If \( \ell \in \frac{1}{2} \mathbb{Z}_+ \) (i.e., \( d \in \mathbb{Z}_+ \)), then the elliptic hypergeometric series is terminating and both operators are represented by finite sums (containing \( d+1 \) terms). Moreover, they coincide with each other and are explicitly given by the formula

\[
W_\ell = \left( i e^{\pi i (-\eta + \frac{\pi}{4})} \right) \frac{d}{k} \sum_{k=0}^{d} (-1)^k \left[ \begin{array}{c} d \\ k \end{array} \right] \frac{\theta_1(2z - 2(2k)\eta)}{\prod_{j=0}^{d} \theta_1(2z + 2(k - j)\eta)} e^{(-d+2k)\eta \partial_z} \tag{5.12}
\]

which coincides with equation (2.4).

Let us conclude this section by summarizing the graphic elements of the diagrams and rules of their composing. The plane is divided into “transparent” and “shadow” pieces by a number of straight dashed lines in such a way that each segment of any line is a border between pieces of the different kind. Each dashed line carries a spectral parameter denoted by \( \lambda, \mu, \) etc. There may be also bold straight lines which become dotted when they go through shadow pieces of the plane. Each shadow piece (bounded by dashed or dotted lines or by infinity) carries a complex variable denoted by \( z, \zeta, \) etc. Those which sit on infinite pieces are fixed while those which sit on finite pieces bounded by lines of any type should be “integrated” in the sense of the pairing (3.4). The intersection points of the dashed lines are of two types depending on the way how the transparent and shadow parts are adjacent to it. Correspondingly, there are two types of vertex functions shown in Fig. 4. The intersection of a dashed line with a bold one corresponds to an intertwining (co)vector as shown in Fig. 1. Finite bold segments mean taking scalar products of (co)vectors associated with their endpoints.

6 Vacuum vectors

In order to make a closer contact with our earlier work [17], it is useful to demonstrate how the vacuum vectors for the \( L \)-operator can be constructed within the approach developed in the previous sections. Let us recall the general definition of the vacuum vectors. Consider an arbitrary \( L \)-operator \( L \) with two-dimensional auxiliary space \( \mathbb{C}^2 \), i.e., an arbitrary \( \times 2 \times 2 \) operator-valued matrix

\[
L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}.
\]

The operators \( L_{ij} \) act in a linear space \( \mathcal{H} \) which is called the quantum space of the \( L \)-operator. For the moment, let \( \phi, \psi, \) etc denote vectors from \( \mathbb{C}^2 \) and \( X, X_1, \) etc vectors from \( \mathcal{H} \), then acting by the quantum \( L \)-operator on the tensor product \( X \otimes \phi \), we, generally speaking, obtain a mixed state in the quantum space: \( LX \otimes \phi = X_1 \otimes \phi_1 + X_2 \otimes \phi_2 \). The special case of a pure state,

\[
LX \otimes \phi = X' \otimes \psi,
\tag{6.1}
\]

is of prime importance. The relation (6.1) (in the particular case \( \mathcal{H} \cong \mathbb{C}^2 \)) was the key point for Baxter in his solution of the 8-vertex model [1]. (This is what he called
the “pair-propagation through a vertex” property.) Taking the scalar product with the vector \( \psi^\perp \) orthogonal to \( \psi \), we get:

\[
(\psi^\perp L \phi)X = 0 ,
\]

i.e., the operator \( K = (\psi^\perp L \phi) \) (acting in the quantum space only) has a zero mode \( X \in \mathcal{H} \). Suppose \( (6.1) \) (or \( (6.2) \)) holds with some vectors \( \phi, \psi \); then the vector \( X \) is called a vacuum vector of the \( L \)-operator. An algebro-geometric approach to the equation \( (6.1) \) for finite-dimensional matrices \( L_{ik} \) was suggested by Krichever \[28\] and further developed in \[29, 30\]. In our paper \[17\] the Baxter’s method of vacuum vectors was adopted to the infinite-dimensional representations of the Sklyanin algebra.

For \( L \)-operators with elliptic spectral parameter it is convenient to pass to the elliptic parametrization of the components of the vectors \( \phi, \psi \) as is given by \( (4.1) \). Writing \( L(\lambda)|\zeta \rangle \) (respectively, \( \langle \zeta|L(\lambda) \rangle \)) we mean that the 2×2 matrix \( L \) acts on the 2-component vector from the left (respectively, on the 2-component covector from the right). Similarly, we introduce right and left vacuum vectors \( X_R, X_L \) according to the relations

\[
\langle \zeta|L(\lambda)X_R = \langle \xi|X'_R, \quad X_L\langle \zeta|L(\lambda) = X'_L\langle \xi| .
\]

(6.3)

In the latter formula the matrix elements of \( L \) act on \( X_L \) from the right. Introducing the operator

\[
K = K(\zeta, \xi) = \langle \zeta|L(\lambda)|\xi \rangle^\perp ,
\]

(6.4)

we can rewrite \( (6.3) \) as \( K X_R = X_L K = 0 \). The explicit form of the operator \( K \) can be found from \( (2.1), (2.2) \):

\[
K = K(\zeta, \xi) = \rho(z)e^{\eta \partial z} + \rho(-z)e^{-\eta \partial z} ,
\]

(6.5)

where

\[
\rho(z) = \frac{1}{\theta_1(2z)} \prod_{\epsilon = \pm} \theta_1 \left( z + \epsilon \zeta - \lambda_+ + \frac{\eta}{2} \right) \theta_1 \left( z + \epsilon \xi + \lambda_- + \frac{\eta}{2} \right) .
\]

These difference operators appeared in \[30, 17\] and later were independently introduced in \[31, 32\]. So, the equations for the right and left vacuum vectors read

\[
\rho(z)X_R(z + \eta) = -\rho(-z)X_R(z - \eta) ,
\]

(6.6)

\[
\rho(-z - \eta)X_L(z + \eta) = -\rho(z - \eta)X_L(z - \eta) .
\]

(6.7)

Instead of solving these equations explicitly, below we show how the vacuum vectors emerge within the approach of the present paper. The key relation is (see Fig. 6)

\[
\int d\zeta \langle \phi^\zeta_\xi (\mu + \frac{\eta}{2})|\xi|\phi^\zeta_\xi (\lambda_+ - \frac{\eta}{2}) \rangle \langle \phi^\zeta_\xi (\lambda_+ - \frac{\eta}{2})|W^{\xi,\zeta}(\lambda_+ - \mu)W^{\zeta,z}(\lambda_- - \mu) \rangle = \int d\zeta \langle \phi^\zeta_\xi (\lambda_+ + \frac{\eta}{2})|\xi|\phi^\zeta_\xi (\lambda_- - \frac{\eta}{2}) \rangle \langle \phi^\zeta_\xi (\mu + \frac{\eta}{2})|W^{\xi,\zeta}(\lambda_+ - \mu)W^{\zeta,z}(\lambda_- - \mu) \rangle .
\]

(6.8)

The left-hand side represents the action of the \( L \)-operator \( L(\zeta_\xi (\lambda_+ + \frac{\eta}{2}) \) to the covector \( \langle \phi^\zeta_\xi (\lambda_- - \frac{\eta}{2}) \rangle \) in the auxiliary space from the right and to the vector \( W^{\xi,\zeta}(\lambda_+ - \mu)W^{\zeta,z}(\lambda_- - \mu) \) in the quantum space from the left. It is convenient to denote

\[
X_R^{\xi,\zeta}(z|\lambda_+, \lambda_-) = W^{\xi,\zeta}(\lambda_+ - \frac{\eta}{2})W^{\zeta,z}(\lambda_- - \frac{\eta}{2}) ,
\]

(6.9)
then relation (6.8) can be rewritten (after setting \( z' = \xi \pm \eta \) and some transformations) as the system of equations

\[
\begin{align*}
\langle \xi - \mu \lvert \mathcal{L}(\lambda_+ + \mu, \lambda_- + \mu) X_{R}^{\xi,\xi' \pm \eta} \rangle &= a \langle \xi' - \mu \lvert X_{R}^{\xi+\eta,\xi'+\eta} + b \langle \xi' + \mu \lvert X_{R}^{\xi+\eta,\xi'-\eta}, \\
\langle \xi + \mu \lvert \mathcal{L}(\lambda_+ + \mu, \lambda_- + \mu) X_{R}^{\xi,\xi'} &= c \langle \xi' - \mu \lvert X_{R}^{\xi-\eta,\xi'+\eta} + d \langle \xi' + \mu \lvert X_{R}^{\xi-\eta,\xi'-\eta},
\end{align*}
\]

(6.10)

where \( X_{R}^{\xi,\xi'} = X_{R}^{\xi,\xi'}(z|\lambda_+, \lambda_-) \) and

\[
\begin{align*}
a &= -\frac{\theta_1(\xi+\xi'-\lambda_+ + \lambda_- + \eta) \theta_1(\xi-\xi'-\lambda_+ - \lambda_- - 2\mu)}{\theta_1(2\xi' + 2\eta)}, \\
b &= \frac{\theta_1(\xi-\xi'-\lambda_+ + \lambda_- + \eta) \theta_1(\xi+\xi'-\lambda_+ - \lambda_- - 2\mu)}{\theta_1(2\xi' - 2\eta)}, \\
c &= -\frac{\theta_1(\xi-\xi'+\lambda_- - \lambda_- - \eta) \theta_1(\xi+\xi'+\lambda_+ + \lambda_- + 2\mu)}{\theta_1(2\xi' + 2\eta)}, \\
d &= \frac{\theta_1(\xi-\xi'+\lambda_- - \lambda_- - \eta) \theta_1(\xi-\xi'+\lambda_+ + \lambda_- + 2\mu)}{\theta_1(2\xi' - 2\eta)}.
\end{align*}
\]

We note that setting \( \mu = 0 \) one obtains from (6.10)

\[
\langle \xi \lvert \mathcal{L}(\lambda_+, \lambda_-) X_{R}^{\xi,\xi'} = \langle \xi' \lvert \left( a_0 X_{R}^{\xi+\eta,\xi'+\eta} + b_0 X_{R}^{\xi+\eta,\xi'-\eta} \right) = \langle \xi' \lvert \left( c_0 X_{R}^{\xi-\eta,\xi'+\eta} + d_0 X_{R}^{\xi-\eta,\xi'-\eta} \right),
\]

where \( a_0 = a(\mu = 0) \), etc. This means that \( X_{R}^{\xi,\xi'}(z|\lambda_+ - \frac{\eta}{2}, \lambda_- - \frac{\eta}{2}) \) is the right vacuum vector for the \( \mathcal{L} \)-operator (see the first equation in (6.3)). Moreover, we conclude that

\[
a_0 X_{R}^{\xi+\eta,\xi'+\eta} + b_0 X_{R}^{\xi+\eta,\xi'-\eta} = c_0 X_{R}^{\xi-\eta,\xi'+\eta} + d_0 X_{R}^{\xi-\eta,\xi'-\eta}. \quad (6.11)
\]
One can that the vacuum vector is in fact a composite object. It is a product of two \( W \)-functions. Equations (4.14), (5.4) together with the 3-term identity for the Jacobi theta-function imply the relation

\[
\langle \xi | L(\lambda_+, \lambda_-) X^{\xi, \xi'}_R(z | \lambda_+, \lambda_-) = \theta_4(2\lambda_- + \eta) \langle \xi' | X^{\xi, \xi'}_R(z | \lambda_+ + \eta, \lambda_- + \eta) \tag{6.12}
\]

which is equation (4.22) from our paper [17] written in the slightly different notation. The left vacuum vectors can be considered in a similar way.

### 7 The \( R \)-operator and related objects

The \( R \)-operator \( \check{R} = \check{R}(\lambda_+, \lambda_- | \mu_+, \mu_-) \) intertwines the product of two \( L \)-operators:

\[
\check{R}(\lambda_+, \lambda_- | \mu_+, \mu_-) L(\lambda_+, \lambda_-) \otimes L(\mu_+, \mu_-) = L(\mu_+, \mu_-) \otimes L(\lambda_+, \lambda_-) \check{R}(\lambda_+, \lambda_- | \mu_+, \mu_-). \tag{7.1}
\]
Passing to the different notation, $\lambda_\pm = \lambda \pm (\ell + \frac{1}{2})\eta$, $\mu_\pm = \mu \pm (\ell' + \frac{1}{2})\eta$, we can rewrite (7.1) in a more conventional form:

$$\hat{R}^{(\ell\ell')}(\lambda, \mu) L^{(\ell)}(\lambda) \otimes L^{(\ell')}(\mu) = L^{(\ell)}(\mu) \otimes L^{(\ell')}(\lambda) \hat{R}^{(\ell\ell')}(\lambda, \mu).$$  \hspace{1cm} (7.2)

Here $\hat{R}^{(\ell\ell')}(\lambda, \mu) = \hat{R}(\lambda_+, \lambda_-|\mu_+ , \mu_-)$ is a difference operator in two variables acting in the tensor product of the quantum spaces for the two $L$-operators. In terms of the kernels equation (7.1) reads:

$$\int d\zeta \int d\zeta' R^{zz'}_{\zeta\zeta'}(\lambda, \lambda_-|\mu_+ , \mu_-) L^{\zeta}(\lambda, \lambda_-) L^{\zeta'}(\mu_+, \mu_-)$$

$$= \int d\zeta \int d\zeta' L^{\zeta}(\mu_+, \mu_-) L^{\zeta'}(\lambda, \lambda_-) R^{\zeta\zeta'}_{\zeta\zeta'}(\lambda_+, \lambda_-|\mu_+ , \mu_-).$$  \hspace{1cm} (7.3)

Graphically it is shown in Fig. 7. The figure clarifies the structure of the kernel of the $R$-operator which is shown in more detail in Fig. 8. It is clear that the kernel is the product of four $W$-vertices: two of them are of the $W^{z,\zeta}$-type (meromorphic functions) and the other two are of the $W^{\zeta,\zeta'}$-type (comb-like functions). Specifically, we can write:

$$R^{zz'}_{\zeta\zeta'}(\lambda_+, \lambda_-|\mu_+ , \mu_-)$$

$$= W^{z,z'}(\lambda_+ - \mu_-) W^{\zeta}(\lambda_- - \mu_-) W^{\zeta}(\lambda_+ - \mu_+) W^{\zeta,\zeta'}(\lambda_- - \mu_+)$$

$$= W^{z,z'}(\lambda_+ - \mu_-) \left[ \frac{c(\lambda_- - \mu_-)}{W^{\zeta}(\lambda_- - \mu_- + \eta)} \sum_{k' \geq 0} \delta(z' - \zeta' - \lambda_- + \mu_ + 2k'\eta) \right]$$

$$\times \left[ \frac{c(\lambda_+ - \mu_+)}{W^{\zeta}(\lambda_+ - \mu_+ + \eta)} \sum_{k \geq 0} \delta(z - \zeta - \lambda_+ + \mu_ + 2k\eta) \right] W^{\zeta,\zeta'}(\lambda_- - \mu_+)$$

which is the kernel of the difference operator

$$\hat{R}(\lambda_+, \lambda_-|\mu_+ , \mu_-) = W^{z,z'}(\lambda_+ - \mu_-) W^{(z')}(\lambda_- - \mu_-) W^{(z)}(\lambda_+ - \mu_+) W^{z,z'}(\lambda_- - \mu_+)$$  \hspace{1cm} (7.4)

(here the notation $W^{(z)}$ means that the operator $W$ acts to the variable $z$. In full, the $R$-operator reads

$$\hat{R}(\lambda_+, \lambda_-|\mu_+ , \mu_-) = e^{-\frac{\delta}{\eta}(\lambda_+ - \mu_+)^2 - \frac{\delta}{\eta}(\lambda_- - \mu_-)^2 + \frac{2\delta}{\eta}(\lambda_+ - \mu_+)z + \frac{2\delta}{\eta}(\lambda_- - \mu_-)z'}$$

$$\times e^{-\frac{2\delta}{\eta}(\lambda_- - \mu_-)z} \frac{\Gamma(z + z' + \lambda_+ - \mu_ + \eta)}{\Gamma(z + z' + \lambda_- + \mu_- + \eta)} \frac{\Gamma(z + z' + \lambda_+ - \mu_ + + \gamma)}{\Gamma(z + z' + \lambda_- + \mu_- + \gamma)}$$

$$\times e^{\frac{\delta}{\eta}(\lambda_+ - \mu_+ + 2\eta)} \frac{\Gamma(2z + 2\eta)}{\Gamma(2z' + 2\eta)} \psi(2\omega_3 \left( \frac{z + \mu_+ - \lambda_+}{\eta} - \frac{\mu_- - \lambda_-}{\eta}; e^{2\eta p_+} \right); e^{-(\lambda_- - \mu_+)} \partial z}$$

$$\times e^{\frac{\delta}{\eta}(\lambda_- - \mu_- + 2\eta)} \frac{\Gamma(2z - 2\lambda_- + \mu_- + 2\eta)}{\Gamma(2z + 2\eta)} \psi(2\omega_3 \left( \frac{z + \mu_+ - \lambda_+}{\eta} - \frac{\mu_- - \lambda_-}{\eta}; e^{2\eta p_+} \right); e^{-(\lambda_+ - \mu_-)} \partial z}$$

$$\times e^{\frac{\delta}{\eta}(\lambda_- - \mu_- + 2\eta)} \frac{\Gamma(z + z' + \lambda_- - \mu_+ + \eta)}{\Gamma(z + z' + \lambda_- + \mu_ + + \eta)} \frac{\Gamma(z + z' + \lambda_- - \mu_ + + \eta)}{\Gamma(z + z' + \lambda_- + \mu_ + + \eta)}.$$

(7.5)
Figure 9: The Yang-Baxter equation for the \( R \)-operator.

The difference operators in the third and the fourth lines of the r.h.s. commute because they act in different variables but both of them do not commute with the operator of multiplication by the function \( W^z,z'(\lambda - \mu, \lambda - \nu) \). Note that the \( R \)-operator can be also written in terms of the \( 6\omega \) series due to the identity

\[
\omega_3 \left( \frac{z - \lambda}{\eta}; -\frac{\lambda}{\eta} , \zeta + \mu - \lambda + \frac{\eta}{2}; e^{2\eta \partial_z} \right) \cdot e^{-\lambda \partial_z} W^{\zeta,z}(\mu) = W^{\zeta-z-\lambda(\mu)} \cdot \omega_3 \left( \frac{z - \lambda}{\eta}; -\frac{\lambda}{\eta} , \zeta + \mu - \lambda + \frac{\eta}{2}; e^{2\eta \partial_z} \right) \cdot e^{-\lambda \partial_z}. \tag{7.6}
\]

The Yang-Baxter equation for the \( R \)-operator is schematically shown in the self-explanatory Fig. 9. One can see that as soon as the \( R \)-operator is a composite object, the Yang-Baxter equation can be reduced to simpler equations for its elementary constituents. The latter are the \( W \)-vertices of the two types. For them one can prove a sort of the star-triangle relations

\[
W^z,z'(\mu - \nu)W^z',z''(\lambda - \mu)W^z''(\lambda - \nu) = \int d\zeta W^\zeta,z(\lambda - \mu)W^z',z(\lambda - \nu)W^z''(\lambda - \nu) \tag{7.7}
\]

\[
W^z,z'(\mu - \nu)W^z'',z'(\mu - \nu)W^z''(\lambda - \mu) = \int d\zeta W^\zeta,z(\mu - \nu)W^z',z(\lambda - \nu)W^z''(\lambda - \mu) \tag{7.8}
\]

schematically shown in Fig. 10. The proof is given in Appendix B. As is seen from Fig. 9, the proof of the Yang-Baxter equation is reduced to sequential transferring of vertical lines from the left to the right through intersection points of the other lines with the use of the star-triangle relations (7.7) and (7.8) at each step. Let us note that the both sides of the star-triangle relations (7.7) and (7.8) represent the kernels of the difference operators explicitly written in Appendix B (\( B3 \) and \( B6 \) respectively).

There is an object “dual” to the \( R \)-operator \( \tilde{R} \) in the sense that its kernel is graphically represented by the same pattern, with shadow parts of the plane being complimentary to those in Fig. 8. This duality provides a transformation which is an infinite-dimensional
version of the vertex-face correspondence. It sends the $R$-operator to a difference operator in one variable rather than two. We call it the $S$-operator. It acts in the variable $z$ and depends on $z'$ and $z''$ as parameters. Its kernel, $S_z(\xi, z'|\lambda_+; \lambda_-; \mu_+; \mu_-)$, or simply $S_z(\xi, z'|\lambda_+; \lambda_-; \mu_+; \mu_-)$, is shown in Fig. 11. This kernel is to be regarded as an $R$-matrix for a face-type model with complex variables associated to shadow parts of the plane. It generalizes the fused Boltzmann weights of the SOS-type 8-vertex model [33]. According to Fig. 11 it reads

$$S_z(\xi, z'|\lambda_+; \lambda_-; \mu_+; \mu_-) = \int d\eta W_{\xi}(z - \lambda_+; \lambda_-; \mu_+; \mu_- + 2n\eta) W_{z'}(z' - \lambda_+; \lambda_-; \mu_+; \mu_- + 2n\eta).$$

The convolution is taken with respect to the variable sitting in the finite parallelogram at the center of Fig. 11. Since each of the two $W_{\xi}$-vertices is represented by a half-infinite sum of the type (5.4), the whole expression (7.9) is a double sum. Performing the convolution and re-arranging the double sum, we can write

$$S_z(\xi, z'|\lambda_+; \lambda_-; \mu_+; \mu_-) = c(\lambda_+ - \mu_-) c(\lambda_- - \mu_+) \theta_1(2\xi) \sum_{n \geq 0} A_n(z, z', z'') \delta(z - \xi - \lambda_+ - \lambda_- + \mu_+ + \mu_- + 2n\eta),$$

where

$$A_n(z, z', z'') = \sum_{k=0}^{n} \frac{\theta_1(2z - 2\lambda_+ + 2\mu_- + 4k\eta) W_{\xi}(z - \lambda_+; \mu_+; \mu_- + 2k\eta) W_{z'}(z' - \lambda_+; \mu_+; \mu_- + 2k\eta) W_{z''}(z'' - \lambda_+; \mu_+; \mu_- + 2k\eta)}{W_{z - \lambda_+; \mu_+; \mu_- + 2k\eta}(\lambda_+ - \mu_- + 2k\eta) W_{z' - \lambda_+; \mu_+; \mu_- + 2k\eta}(\lambda_+ - \mu_- + 2k\eta) W_{z'' - \lambda_+; \mu_+; \mu_- + 2k\eta}(\lambda_+ - \mu_- + 2k\eta)}.$$
The kernel $S_\xi(z',z'')$ "dual" to the kernel of the $R$-operator (cf. Fig. 8).

\[ S_\xi(z',z'') = C_\theta(2\xi) e^{2\pi i \xi + \frac{2\pi i}{\eta}((\lambda_+ - \lambda_-)z + (\mu_+ - \mu_-)\xi + (\mu_+ - \lambda_+)z')} \]

\[ \times \frac{\tilde{\theta}_1(z - z' + \mu_- - \mu_+ + \eta) \tilde{\theta}_1(z - \xi + \mu_+ + \mu_- - \lambda_+ - \lambda_-)}{\tilde{\theta}_1(z - z' + \mu_- + \mu_+ - 2\lambda_+ + \eta) \tilde{\theta}_1(z - \xi - \mu_+ + \mu_- - \lambda_+ + \lambda_-)} \]

\[ \times \frac{\Gamma(2z + 2\mu_- - 2\lambda_+ + 2\eta)}{\Gamma(2z + 2\eta)} \prod_{j=5}^{10} \frac{\Gamma(2\alpha_j \eta)}{\Gamma(2(\alpha_1 - \alpha_j + 1) \eta)} \]

\[ \times 10\omega_9(\alpha_1; \alpha_4, \ldots, \alpha_{10}) \sum_{n \geq 0} \delta(z - \xi - \lambda_+ - \lambda_- + \mu_+ + \mu_- + 2n\eta). \]

Here $C$ is a constant which depends on the spectral parameters, $\tilde{\theta}_1(x) \equiv \theta_1(x|2\eta)$ and the values of the $\alpha_j$'s are

\[ \alpha_1 = \frac{z + \mu_- - \lambda_+}{\eta}, \quad \alpha_4 = \frac{\mu_- - \lambda_+}{\eta}, \quad \alpha_{5,6} = \frac{z \pm z + \mu_+ + \mu_- - \lambda_+ - \lambda_-}{2\eta}, \]

\[ \alpha_{7,8} = \frac{z \pm z' + \mu_- - \mu_+ + \eta}{2\eta}, \quad \alpha_{9,10} = \frac{z \pm z'' + \lambda_- - \lambda_+ + \eta}{2\eta}. \]

One can see that the series $10\omega_9$ is balanced (the balancing condition \[A21\] is satisfied) and terminating ($\alpha_6 = -n$ because of the $\delta$-function). Equation \[7.10\] is a version of the Frenkel-Turaev result \[18\] adapted to continuous values of parameters and obtained by a different method. The $S$-operator satisfies a sort of the Yang-Baxter equation which can be graphically represented like in Fig. 9 with transparent pieces of the plane being changed to the shadow ones and vice versa.

Another object closely related to the $R$-operator is the "transfer matrix on 1 site"

\[ T(\lambda_+, \lambda_-|\mu_+, \mu_-) = \text{tr}_\mu \left( \hat{R}(\lambda_+, \lambda_-|\mu_+, \mu_-) \mathcal{P} \right), \quad (7.11) \]
where $P$ is the permutation operator of the two quantum spaces and the trace is taken in
the space associated with the spectral parameters $\mu_\pm$. The kernel of this transfer matrix
is
\[
T_\xi^z(\lambda_+, \lambda_-|\mu_+, \mu_-) = \int d\xi R_\xi^z(\lambda_+, \lambda_-|\mu_+, \mu_-). \quad (7.12)
\]
It is not difficult to see that this kernel is expressed through the kernel $S_\xi^z(z'|z'')$ given
by (7.10) as follows:
\[
T_\xi^z(\lambda_+, \lambda_-|\mu_+, \mu_-) = S_\xi^z(\xi, z)(\lambda_-, \lambda_+|\mu_+, \mu_-). \quad (7.13)
\]
(Note the exchange of the spectral parameters $\lambda_+ \leftrightarrow \lambda_-$ in the right-hand side.) The
easiest way to see this is to draw the corresponding pictures.

8 Concluding remarks

In this paper we have presented a unified approach to intertwining operators for quantum
integrable models with elliptic $R$-matrix associated with the Sklyanin algebra. We work
in the most general setting of infinite-dimensional representations (with a complex spin
parameter $\ell$) realized by difference operators in the space of functions of a complex
variable $z$. The elementary building blocks are so-called intertwining vectors and
$W$-functions which are defined in terms of their scalar products. These elements have a
nice graphic representation as diagrams in the transparent/shadow plane which allows
one to easily construct more complicated objects like $L$-operators, their vacuum vectors
and different kinds of $R$-matrices and to prove relations between them. An important
constituent of the construction is the intertwining operator for representations with spins
$\ell$ and $-\ell-1$. For general values of $\ell$, it is given by the elliptic hypergeometric series $4_3\omega_3$ with operator argument.

In fact the material presented here is only the very beginning of the theory of inte-
grable “spin chains” with elliptic $R$-matrices and infinite-dimensional space of states at
each site. Indeed, our discussion has been focused on a single $L$ or $R$ operator which is
relevant to a spin chain of just one site. The next step is to construct the transfer matrix,
i.e., to consider a chain of the $R$-operators and to take trace in the auxiliary space. We
plan to address this problem elsewhere. It would be also very desirable to find a direct
connection of our approach with elliptic beta integrals [24, 34]. Presumably, the pairing
(3.2) or (3.4) should be replaced by a sum of residues.

Among other things, the results presented in this paper indicate convincingly that there
should exist a meaningful theory of infinite-dimensional representations of the
Sklyanin algebra. Such a theory is still to be developed and this paper may provide
some background in reaching this ambitious goal.

At last, one should keep in mind that the Sklyanin algebra is just a very particular
representative of a wide family of elliptic algebras [36] and, moreover, integrable sys-
tems associated to algebras from this class can be constructed [37]. It would be very
interesting to investigate to what extent the methods developed in the present paper
can be extended to other elliptic algebras and corresponding integrable models. Such an
extension will probably require a further generalization of elliptic hypergeometric series.
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Appendix A

Theta-functions

We use the following definition of the Jacobi $\theta$-functions:

\begin{align*}
\theta_1(z|\tau) &= -\sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau (k + \frac{1}{2})^2 + 2\pi i (z + \frac{1}{2})(k + \frac{1}{2}) \right), \\
\theta_2(z|\tau) &= \sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau (k + \frac{1}{2})^2 + 2\pi i z(k + \frac{1}{2}) \right), \\
\theta_3(z|\tau) &= \sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau k^2 + 2\pi i z k \right), \\
\theta_4(z|\tau) &= \sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau k^2 + 2\pi i (z + \frac{1}{2}) k \right).
\end{align*}

(A1)

They also can be represented as infinite products. The infinite product representation for the $\theta_1(z|\tau)$ reads:

\begin{align*}
\theta_1(z|\tau) &= i \exp \left( \frac{i\pi \tau}{4} - i\pi z \right) \prod_{k=1}^{\infty} \left( 1 - e^{2\pi i k \tau} \right) \left( 1 - e^{2\pi i (k-1)\tau + z} \right) \left( 1 - e^{2\pi i (k\tau - z)} \right). \quad (A2)
\end{align*}

Throughout the paper we write $\theta_a(x|\tau) = \theta_a(x)$, $\theta_a(z|\frac{\tau}{2}) = \bar{\theta}_a(z)$. The transformation properties for shifts by the periods are:

\begin{align*}
\theta_a(x \pm 1) &= (-1)^{\delta_{a,1} + \delta_{a,2}} \theta_a(x), \quad \theta_a(x \pm \tau) &= (-1)^{\delta_{a,1} + \delta_{a,4}} e^{-\pi i \tau + 2\pi i x} \theta_a(x). \quad (A3)
\end{align*}

Under the modular transformation $\tau \rightarrow -1/\tau$ the $\theta$-functions behave as follows:

\begin{align*}
\theta_1(z|\tau) &= i \sqrt{i/\tau} e^{-\pi i z^2/\tau} \theta_1(z/\tau| - 1/\tau), \\
\theta_2(z|\tau) &= \sqrt{i/\tau} e^{-\pi i z^2/\tau} \theta_4(z/\tau| - 1/\tau), \\
\theta_3(z|\tau) &= \sqrt{i/\tau} e^{-\pi i z^2/\tau} \theta_3(z/\tau| - 1/\tau), \\
\theta_4(z|\tau) &= \sqrt{i/\tau} e^{-\pi i z^2/\tau} \theta_2(z/\tau| - 1/\tau). \quad (A4)
\end{align*}
The identities often used in the computations are
\[
\begin{align*}
\bar{\theta}_4(x) \bar{\theta}_3(y) + \bar{\theta}_4(y) \bar{\theta}_3(x) &= 2\theta_4(x+y)\theta_4(x-y), \\
\bar{\theta}_3(x) \bar{\theta}_3(y) - \bar{\theta}_4(y) \bar{\theta}_3(x) &= 2\theta_1(x+y)\theta_1(x-y), \\
\bar{\theta}_3(x) \bar{\theta}_3(y) + \bar{\theta}_4(y) \bar{\theta}_4(x) &= 2\theta_3(x+y)\theta_3(x-y), \\
\bar{\theta}_3(x) \bar{\theta}_3(y) - \bar{\theta}_4(y) \bar{\theta}_4(x) &= 2\theta_2(x+y)\theta_2(x-y), \\
\theta_1(z-a-d)\theta_1(z-b-c)\theta_1(a-d)\theta_1(c-b) + \theta_1(z-b-d)\theta_1(z-a-c)\theta_1(b-d)\theta_1(a-c) &= 0.
\end{align*}
\]
(A5)

By \( \Theta_n \) we denote the space of \( \theta \)-functions of order \( n \), i.e., entire functions \( F(x), x \in \mathbb{C} \), such that
\[
F(x+1) = F(x), \quad F(x+\tau) = (-1)^n e^{-\pi i n \tau - 2\pi i n x} F(x).
\]
(A7)

It is easy to see that \( \dim \Theta_n = n \). Let \( F(x) \in \Theta_n \), then \( F(x) \) has a multiplicative representation of the form \( F(x) = c \prod_{i=1}^{n} \theta_1(x-x_i) \), \( \sum_{i=1}^{n} x_i = 0 \), where \( c \) is a constant. Imposing, in addition to (A7), the condition \( F(-x) = F(x) \), we define the space \( \Theta_n^+ \subset \Theta_n \) of even \( \theta \)-functions of order \( n \), which plays the important role in representations of the Sklyanin algebra. If \( n \) is an even number, then \( \dim \Theta_n^+ = \frac{1}{2} n + 1 \).

**Elliptic gamma-function**

Here we collect the main formulas on the elliptic gamma-function [26, 27]. We use the (slightly modified) notation of [27]. The elliptic gamma-function is defined by the double-infinite product
\[
\Gamma(z|\tau, \tau') = \prod_{k,k'=0}^{\infty} \frac{1 - e^{2\pi i(k+1)\tau+(k'+1)\tau'-z}}{1 - e^{2\pi i(k\tau+k'\tau'+z)}}.
\]
(A8)

A sufficient condition for the product to be convergent is \( \Im \tau > 0, \Im \tau' > 0 \). We need the following properties of the elliptic gamma-function:
\[
\begin{align*}
\Gamma(z+1|\tau, \tau') &= \Gamma(z|\tau, \tau'), \\
\Gamma(z+\tau|\tau, \tau') &= -ie^{-\frac{\pi iz}{\tau}}\eta_D^{-1}(\tau')e^{\pi iz}\theta_1(z|\tau')\Gamma(z|\tau, \tau'), \\
\Gamma(z+\tau'|\tau, \tau') &= -ie^{-\frac{\pi iz}{\tau}}\eta_D^{-1}(\tau)e^{\pi iz}\theta_1(z|\tau)\Gamma(z|\tau, \tau'), \quad \text{(A10, A11)}
\end{align*}
\]
where
\[
\eta_D(\tau) = e^{\frac{\pi iz}{\tau}} \prod_{k=1}^{\infty} \left(1 - e^{2\pi ik\tau}\right)
\]
is the Dedekind function. Another useful property is
\[
\Gamma(z|\tau, \tau') \Gamma(\tau' - z|\tau, \tau') = \frac{ie^{\pi iz'/6}\eta_D(\tau')}{e^{\pi iz}\theta_1(z|\tau')}.
\]
(A12)
Note also that \( \Gamma(z|\tau,\tau')\Gamma(\tau + \tau' - z|\tau,\tau') = 1. \)

Under the modular transformation \( \tau \to -1/\tau \) the elliptic gamma-function behaves as follows [27]:

\[
\Gamma(z|\tau,\tau') = e^{i\pi P(z)} \frac{\Gamma((z/\tau-1/\tau',\tau'/\tau')}{\Gamma((z/\tau'/\tau'-1/\tau)'}),
\]

where

\[
P(z) = -\frac{1}{3\tau\tau'} z^3 + \frac{\tau + \tau'-1}{2\tau\tau'} z^2 - \frac{\tau^2 + \tau'^2 + 3\tau\tau' - 3\tau - 3\tau' + 1}{6\tau\tau'} z - \frac{(\tau + \tau' - 1)(\tau + \tau' - \tau\tau')}{12\tau\tau'}.
\]

Let us list the most frequently used formulas for \( \Gamma(z) \equiv \Gamma(z|\tau,2\eta) \). Using (A11) several times, we obtain:

\[
\frac{\Gamma(x + 2k\eta)}{\Gamma(x)} = e^{\pi i\eta k^2} R^{-k} e^{\pi i kx} \prod_{j=0}^{k-1} \theta_1(x + 2j\eta),
\]

\[
\frac{\Gamma(x - 2k\eta)}{\Gamma(x)} = (-1)^k e^{\pi i\eta k^2} R^k e^{-\pi i kx} \prod_{j=0}^{k-1} \theta_1(-x + 2\eta + 2j\eta)^{-1},
\]

where \( R = i e^{\pi i (\eta + \tau/6)} \eta_D(\tau) \). In particular, ratios of such functions are expressed through the elliptic Pochhammer symbols as

\[
\frac{\Gamma(2\alpha\eta + 2k\eta)}{\Gamma(2\beta\eta + 2k\eta)} = e^{2\pi i (\alpha - \beta) k\eta} \frac{\Gamma(2\alpha\eta)}{\Gamma(2\beta\eta)} \frac{[\alpha]_k}{[\beta]_k},
\]

\[
\frac{\Gamma(2\alpha\eta - 2k\eta)}{\Gamma(2\beta\eta - 2k\eta)} = e^{-2\pi i (\alpha - \beta) k\eta} \frac{\Gamma(2\alpha\eta)}{\Gamma(2\beta\eta)} \frac{[1 - \beta]_k}{[1 - \alpha]_k}.
\]

As is seen from (A8), the function \( \Gamma(z|\tau,2\eta) \) has zeros at the points \( z = 2(k + 1)\eta + (m + 1)\tau + n \), and simple poles at the points \( z = -2k\eta - m\tau + n \), where \( k, m \) run over non-negative integers and \( n \) over all integers. The residues of the elliptic gamma-function at the poles at \( z = -2k\eta, k = 0, 1, 2, \ldots \) are:

\[
\text{res} \left|_{z=-2k\eta} \Gamma(z) = (-1)^k e^{\pi i k^2} R_0 \prod_{j=1}^{k} \left( \theta_1(2j\eta) \right)^{-1},
\]

where

\[
r_0 = \text{res} \left|_{z=0} \Gamma(z) = -\frac{e^{\pi i (\tau + 2\eta)/12}}{2\pi i\eta_D(\tau)\eta_D(2\eta)}.
\]

**Elliptic hypergeometric series**

Here we follow [18]. We define the elliptic Pochhammer symbol (the shifted elliptic factorial) by

\[
[x]_k \equiv [x][x+1] \ldots [x+k-1],
\]
where \([x] = \theta_1(2x\eta)\) (cf. [25]). By definition, the elliptic hypergeometric series is

\[
 r + 1 \omega_r(\alpha_1; \alpha_4, \alpha_5, \ldots, \alpha_{r+1}; z | 2\eta, \tau) = \sum_{k=0}^{\infty} z^k \frac{[\alpha_1 + 2k][\alpha_1]}{[\alpha_1][k]!} \prod_{m=1}^{r-2} \frac{[\alpha_{m+3}]_k}{[\alpha_1 - \alpha_{m+3} + 1]_k}. \tag{A20}
\]

This is an elliptic analog of the very-well-poised basic hypergeometric series [35]. The series is said to be balanced if \(z = 1\) and

\[
 r - 5 + (r - 3)\alpha_1 = 2 \sum_{m=1}^{r-2} \alpha_{m+3}. \tag{A21}
\]

For a series \(\sum_{k=0}^{\infty} c_k\) of the form (A20), the balancing condition (A21) means that the ratio \(c_{k+1}/c_k\) of the coefficients is an elliptic function of \(k\). For balanced series (A20), we drop the argument \(z = 1\) and the parameters \(\eta, \tau\) writing it simply as \(r + 1 \omega_r(\alpha_1; \alpha_4, \ldots, \alpha_{r+1})\).

For instance, \(\omega_7(\alpha_1; \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8) = \sum_{k=0}^{\infty} \frac{[\alpha_1 + 2k][\alpha_1]}{[\alpha_1][k]!} \prod_{m=1}^{5} \frac{[\alpha_{m+3}]_k}{[\alpha_1 - \alpha_{m+3} + 1]_k}. \tag{A22}
\]

The series is called terminating if at least one of the parameters \(\alpha_4, \ldots, \alpha_{r+1}\) is equal to a negative integer number. In this case the sum is finite and there is no problem of convergence. If, say \(\alpha_{r+1} = -n\), then the series terminates at \(k = n\). The terminating balanced series were shown [18] to possess nice modular properties. That is why they were called modular hypergeometric series.

The modular hypergeometric series obey a number of impressive identities. One of them is the elliptic analog of the Jackson summation formula:

\[
 s \omega_7(\alpha_1; \alpha_4, \ldots, \alpha_7, -n) = \frac{[\alpha_1 + 1]_n[\alpha_1 - \alpha_4 - \alpha_5 + 1]_n[\alpha_1 - \alpha_4 - \alpha_6 + 1]_n[\alpha_1 - \alpha_5 - \alpha_6 + 1]_n}{[\alpha_1 - \alpha_4 + 1]_n[\alpha_1 - \alpha_5 + 1]_n[\alpha_1 - \alpha_6 + 1]_n[\alpha_1 - \alpha_4 - \alpha_5 - \alpha_6 + 1]_n} \tag{A23}
\]

which is valid provided that the balancing condition \(2\alpha_1 + 1 = \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 - n\) is satisfied (the Frenkel-Turaev summation formula [18]).

A remark on the notation is in order. In the modern notation [19], what we call \(r + 1 \omega_r(\alpha_1; \alpha_4, \ldots, \alpha_{r+1}; \eta, \tau)\) (following [18]), would be \(r + 3 V_{r+1}(\alpha_1; a_6, \ldots, a_{r+3} | q^2, p)\) with \(q = e^{2\pi i\eta}, p = e^{2\pi i\tau}, a_j = e^{4\pi i\eta a_j - 2}\). In particular, our \(4 \omega_3\) would be \(6 V_5\). We understand that the modern notation is better justified by the meaning of the elliptic very-well-poisedness condition than the old one and is really convenient in many cases. However, we decided to use the old Frenkel-Turaev notation for the reason that the additive parameters \(\alpha_j\) are more convenient for us than their exponentiated counterparts. We think that it is simpler than to introduce a version of \(r + 1 V_r\) with additive parameters.

**Appendix B**

In this appendix we give some details of calculations which involve modular hypergeometric series.
The normalization of the $W^\zeta$-kernel

Let us consider convolution of the kernels $W^\zeta_\zeta(\lambda)$ and $W^\zeta_\zeta(-\lambda)$ given by equation $(5.4)$:

$$
\int d\zeta W^\zeta_\zeta(\lambda)W^\zeta_\zeta(-\lambda)
= \int d\zeta \frac{c(\lambda)c(-\lambda)\theta_1(2\zeta)\theta_1(2z')}{W^\zeta_\zeta(\lambda + \eta)W^\zeta_\zeta(-\lambda + \eta)} \sum_{k,k' \geq 0} \delta(z - \zeta - \lambda + 2k\eta)\delta(\zeta - z' + \lambda + 2k'\eta)
= \sum_{k,k' \geq 0} \frac{c(\lambda)c(-\lambda)\theta_1(2z - 2\lambda + 4k\eta)\theta_1(2z')}{W^{z - \lambda + 2k\eta,z}(\lambda + \eta)W^{z',z - \lambda + 2k\eta}(-\lambda + \eta)} \delta(z - z' + \lambda + 2(k + k')\eta)
= c(\lambda)c(-\lambda) \sum_{n \geq 0} \left( \sum_{k=0}^{n} \frac{\theta_1(2z - 2\lambda + 4k\eta)\theta_1(2z + 4n\eta)}{W^{z - \lambda + 2k\eta,z}(\lambda + \eta)W^{z + 2n\eta,z - \lambda + 2k\eta}(-\lambda + \eta)} \right) \delta(z - z' + 2n\eta).
$$

In order to calculate it explicitly, consider the sum

$$
S_n(z) = \sum_{k=0}^{n} \frac{\theta_1(2z - 2\lambda + 4k\eta)}{W^{z - \lambda + 2k\eta,z}(\lambda + \eta)W^{z + 2n\eta,z - \lambda + 2k\eta}(-\lambda + \eta)}
,$$

where the $W$-functions are given by $(1.14)$:

$$
W^{z - \lambda + 2k\eta,z}(\lambda + \eta) = e^{-\frac{2\pi i}{\eta}(\lambda + \eta)(z - \lambda + 2k\eta)} \frac{\Gamma(2z + 2\eta + 2k\eta)\Gamma(2\eta + 2k\eta)}{\Gamma(2z - 2\lambda + 2k\eta)\Gamma(-2\lambda + 2k\eta)},
$$

$$
W^{z + 2n\eta,z - \lambda + 2k\eta}(-\lambda + \eta) = e^{\frac{2\pi i}{\eta}(\lambda - \eta)(z + 2n\eta)} \frac{\Gamma(2z - 2\lambda + 2\eta + 2n\eta + 2k\eta)\Gamma(2\eta + 2n\eta - 2k\eta)}{\Gamma(2z + 2n\eta + 2k\eta)\Gamma(2\lambda + 2n\eta - 2k\eta)}.
$$

Plugging this into $(B1)$ and representing ratios of elliptic gamma-functions through elliptic Pochhammer symbols with the help of $(A15), (A16)$, we obtain:

$$
S_n(z) = e^{4\pi iz - \frac{2\pi i}{\eta}\lambda(\lambda + n) - 4\pi i(\lambda - \eta)n} \times \theta_1(2z - 2\lambda) \frac{\Gamma(-2\lambda)\Gamma(2z - 2\lambda)\Gamma(2z + 2n\eta)\Gamma(2\lambda + 2n\eta)}{\Gamma(2\eta)\Gamma(2\eta + 2n\eta)\Gamma(2z + 2\eta)\Gamma(2z - 2\lambda + 2\eta + 2\eta)}
\times \sum_{k=0}^{n} \frac{[\frac{z - \lambda}{\eta} + 2k][\frac{z - \lambda}{\eta}]_k}{[\frac{z - \lambda}{\eta}][1]_k} \frac{[-\frac{\lambda}{\eta}]_k [\frac{z}{\eta} + n + 1]_k}{[\frac{z}{\eta} + 1]_k [\frac{z - \lambda}{\eta} + n + 1]_k [-\frac{\lambda}{\eta} - n + 1]_k}.
$$

The sum in the last line is the terminating balanced elliptic hypergeometric series

$$
\omega_7\left( \frac{z - \lambda}{\eta}; \frac{-\lambda}{\eta}, \frac{z}{\eta} + n, \frac{z - \lambda + \eta}{2\eta}, \frac{z - \lambda + \eta}{2\eta}, -n \right)
$$

which is equal to

$$
\frac{[\frac{z - \lambda}{\eta} + 1]_n [1 - n]_n [\frac{z + \lambda + \eta}{2\eta}]_n [-\frac{z + \lambda + \eta}{2\eta} - n + 1]_n}{[\frac{z}{\eta} + 1]_n [-\frac{\lambda}{\eta} - n + 1]_n [\frac{z - \lambda + \eta}{2\eta}]_n [-\frac{z - \lambda + \eta}{2\eta} - n + 1]_n}
$$

\[27\]
We thus have
\[ S_0(z) = e^{4\pi i z - \frac{2\pi i}{\eta} \lambda (\alpha + \eta)} \frac{\Gamma(2\lambda) \Gamma(-2\lambda) \Gamma(2z - 2\lambda) \theta_1(2z - 2\lambda)}{\Gamma(2\eta) \Gamma(2z + 2\eta)\Gamma(2z - 2\lambda + 2\eta)}. \]

We thus have
\[ \int d\zeta W_\zeta^\nu(\lambda) W_\zeta^\eta(-\lambda) = c(\lambda) c(-\lambda) \theta_1(2z) S_0(z) \delta(z - z'). \]

Using identities for the elliptic gamma-function the product \( \theta_1(2z) S_0(z) \) can be simplified to
\[ \theta_1(2z) S_0(z) = \rho_0^{-1} e^{-2\pi i \lambda^2/\eta} \Gamma(2\lambda) \Gamma(-2\lambda), \]
where
\[ \rho_0 = \frac{\Gamma(2\eta)}{ie^{\frac{\pi i}{\eta} \eta D} \Gamma(-2\lambda)}. \]

So, setting
\[ c(\lambda) = \frac{\rho_0 e^{\pi i \lambda^2/\eta}}{\Gamma(-2\lambda)} \]
we obtain the relation \( \text{[5.1]} \): \( \int d\zeta W_\zeta^\nu(\lambda) W_\zeta^\eta(-\lambda) = \delta(z - z'). \)

The star-triangle relations

Let us verify the star-triangle relation \( \text{[1.1]} \)
\[ W^{z',z}(\mu - \nu) W^{z',z''}(\lambda - \mu) W_z^\nu(\lambda - \nu) = \int d\zeta W_\zeta^\nu(\lambda - \mu) W_\zeta^\eta(\lambda - \nu) W_{z'}^\lambda(\mu - \nu) \]
(see Fig. \( \text{[10]} \)). We use formulas \( \text{[1.14]}, \text{[5.4]} \). The left hand side is
\[
c(\lambda - \nu) \theta_1(2z') \frac{W^{z',z}(\mu - \nu) W^{z',z''}(\lambda - \mu)}{W^{z',z}(\lambda - \nu + \eta)} \sum_{n \geq 0} \delta(z - z'' - \lambda + \nu + 2n\eta)
\]
\[ = c(\lambda - \nu) \sum_{n \geq 0} \theta_1(2z') \frac{W^{z',z}(\mu - \nu) W^{z',z''}(\lambda - \mu)}{W^{z,-\lambda + \nu + 2n\eta}(\lambda - \nu + \eta)} \delta(z - z'' - \lambda + \nu + 2n\eta)
\]
\[ = c(\lambda - \nu) \sum_{n \geq 0} C_n(z', z) \delta(z - z'' - \lambda + \nu + 2n\eta), \]
where \( c(\lambda) \) is given by \( \text{[B3]} \) and
\[
C_n(z', z) = e^{-\frac{2\pi i}{\eta} [(\lambda - \nu) z' - (\lambda - \nu + \eta) z + (\lambda - \nu)(\lambda - \nu + \eta)]} \theta_1(2z - 2\lambda + 2\nu + 4n\eta)
\]
\[ \times \frac{\Gamma(2\nu - 2\lambda) \Gamma(2z - 2\lambda + 2\nu) \Gamma(z + z' + \mu - \nu + \eta) \Gamma(z' - z + 2\lambda - \mu - \nu + \eta) \Gamma(z' - z - \mu + \nu + \eta)}{\Gamma(2\eta) \Gamma(2z + 2\eta) \Gamma(z + z' - 2\lambda + \mu + \nu + \eta) \Gamma(z' - z - \mu + \nu + \eta)}
\]
\[ \times \frac{[z - \frac{\lambda + \nu}{\eta}]_n [\frac{\nu}{\eta}]_n [z + z' + \mu + \eta]_n [\frac{\nu}{\eta}]_n [z - z' + \mu + \eta]_n [\frac{\nu}{\eta}]_n [z - z' + \mu + \eta]_n [\frac{\nu}{\eta}]_n [z - z' + \mu + \eta]_n [\frac{\nu}{\eta}]_n. \]
One can see from this expression that the left hand side of (B4) is the kernel of the difference operator
\[
e^{\frac{2\pi i}{\eta}((\lambda-\nu)(z-z') - \frac{2\pi i}{\eta}(\lambda-\nu)^2)} \frac{\Gamma(2z - 2\lambda + 2\nu + 2\eta) \Gamma(z + z' + \mu - \nu + \eta) \Gamma(z' - z + 2\lambda - \mu - \nu + \eta)}{\Gamma(2\eta) \Gamma(z + z' - 2\lambda + \mu + \nu + \eta) \Gamma(z' - z - \mu + \nu + \eta)} \\
\times \cdot \omega_5 \left( \frac{z - \lambda + \nu}{\eta}, \frac{\nu - \lambda}{\eta}, \frac{z + z' + \nu - \mu + \eta}{2\eta}, \frac{z - z' + \nu - \mu + \eta}{2\eta}; e^{2\eta \partial_z} \right); e^{(\nu-\lambda)\partial_z}. \tag{B5}
\]

Let us turn to the right hand side of (B4). It is
\[
c(\lambda - \mu)c(\mu - \nu) \int d\zeta \frac{W^{z',\zeta}(\lambda - \nu) \theta_1(2\zeta) \theta_1(2z'')}{W^{z,\zeta}(\lambda - \mu + \eta) W^{z'',\zeta}(\mu - \nu + \eta)} \\
\times \sum_{k,k' \geq 0} \delta(z - \zeta - \lambda + \mu + 2k\eta) \delta(\zeta - z' - \mu + \nu + 2k'\eta)
\]
\[
= c(\lambda - \mu)c(\mu - \nu) \sum_{n \geq 0} B_n(z', z) \delta(z - z'' - \lambda + \nu + 2n\eta),
\]
where
\[
B_n(z', z) = \sum_{k=0}^n \frac{\theta_1(2z - 2\lambda + 2\mu + 4k\eta) \theta_1(2z - 2\lambda + 2\nu + 4n\eta)}{W^{z-\lambda+\mu+2k\eta,z}(\lambda - \mu + \eta) W^{z-\lambda+\nu+2n\eta,z-\lambda+\mu+2k\eta}(\mu - \nu + \eta)} \theta_1(2z - 2\lambda + 2\mu)
\times \frac{\Gamma(2\mu - 2\lambda) \Gamma(2z - 2\lambda + 2\mu) \Gamma(z + z' + \mu - \nu + \eta) \Gamma(z' - z + 2\lambda - \mu - \nu + \eta)}{\Gamma(2\eta) \Gamma(z + 2\eta) \Gamma(z' - z + 2\lambda + \mu + \nu + \eta) \Gamma(z - z - \mu + \nu + \eta)} \\
\times \frac{\Gamma(2z - 2\lambda + 2\nu + 2n\eta) \Gamma(2\nu - 2\mu + 2n\eta)}{\Gamma(2\eta + 2n\eta) \Gamma(2z - 2\lambda + 2\mu + 2\eta + 2n\eta)} \theta_1(2z - 2\lambda + 2\mu)
\times s\omega_7(\alpha_1; \alpha_4, ..., \alpha_7, -n),
\]
where the parameters \( \alpha_i \) are:
\[
\alpha_1 = \frac{z - \lambda + \mu}{\eta}, \quad \alpha_4 = \frac{\mu - \lambda}{\eta}, \quad \alpha_5 = \frac{z - \lambda + \nu}{\eta} + n, \\
\alpha_6 = \frac{z + z' + \mu - \nu + \eta}{2\eta}, \quad \alpha_7 = \frac{z - z' + \mu - \nu + \eta}{2\eta}, \quad \alpha_8 = -n.
\]
The series with these parameters is balanced, so one can apply the Frenkel-Turaev summation formula \( \text{(A23)} \). The result is
\[
s\omega_7(\alpha_1; \alpha_4, ..., \alpha_7, -n) = \frac{[z - \lambda + \mu + 1]_n [\lambda - \nu + 1 - n]_n [z - z' - \mu + \nu + \eta]_n}{[z - \eta + 1]_n [\mu - \nu + 1 - n]_n [z - z' - 2\lambda + \mu + \nu + \eta]_n} \cdot \frac{[z - z' - \mu + \nu + \eta + 1 - n]_n}{[z - z' - 2\lambda + \mu + \nu + \eta + 1 - n]_n}.
\]
Now it is straightforward to calculate the ratio \( C_n(z', z)/B_n(z', z) \). One can see that all \( z, z' \) and \( n \) dependent factors cancel in the ratio and one is left with

\[
\frac{C_n(z', z)}{B_n(z', z)} = \frac{\Gamma(2\eta)}{ie^{\pi \eta} \eta \Gamma(2\mu - 2\lambda)} e^{\frac{2\pi i}{\eta}(\lambda - \mu)(\mu - \nu)} \frac{\Gamma(2\nu - 2\lambda)}{\Gamma(2\mu - 2\lambda)} = \frac{c(\lambda - \mu)c(\mu - \nu)}{c(\lambda - \nu)},
\]

where \( c(\lambda) \) is given by (B3). This means that the left and right hand sides of (B4) are indeed equal to each other.

The other star-triangle relation, (7.8), is proved in a similar way. We note that its both sides are kernels of the difference operator

\[
e^{\pi i \eta (\lambda - \nu)(2\mu - \lambda - \nu)} \frac{\Gamma(2z - 2\lambda + 2\nu + 2\eta)\Gamma(z + z' + \lambda - \mu + \eta)\Gamma(z - z' + \lambda - \mu + \eta)}{\Gamma(2z + 2\eta)\Gamma(z + z' + 2\nu - \lambda - \mu + \eta)\Gamma(z - z' + 2\nu - \lambda - \mu + \eta)} \times 6\omega_5(z - \lambda + \nu, \nu - \lambda, z + z' - \lambda + \mu + \eta, \frac{z - z' - \lambda + \mu + \eta}{2\eta}, \frac{z - z' + \lambda - \mu + \eta}{2\eta}; e^{2\pi \eta} e^{(\nu - \lambda)\partial_z}).
\]

(B6)

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