Stability of charged black holes in string theory under charged massive scalar perturbations

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Abstract

Similar to the superradiant effect in Reissner-Nordström black hole, a charged scalar field can be amplified when impinging on the charged black hole in string theory. According to the black-hole bomb mechanism, the mass term of the incident field can works as the reflecting mirror, which may trigger the instability of black hole. We study the possible instability triggered by superradiant effect and demonstrate that the charged black hole in string theory is stable against the massive charged scalar perturbation. The reason is that there is no trapping potential well in the black hole exterior which is separated from the horizon by a potential barrier and there is no bound states in the superradiant regime.

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The stability tests of black holes become an important topics since the initial work of Regge and Wheeler [1]. The Schwarzschild black hole is well known to be stable [1, 2]. The studying of stability for the charged or the rotating black holes becomes involved because of the significant effect of classical black holes, i.e. superradiance. If an incident bosonic wave whose frequency lies in the superradiant regime, the wave scattered by the event horizon get amplified [3–6]. This classical effect allows extracting rotational energy efficiently from the rotating black holes or extracting the Coulomb energy from charged black holes [7]. The stability analysis of these types of black holes should be adequately addressed.

Press and Teukolsky [8] proposed to use the superradiance to built the black-hole bomb by adding an reflecting mirror outside the black hole (see also [9–15] for a recent studies on this topic). In this mechanism, the amplitudes of superradiant modes trapped in the potential well between the mirror and the event horizon will grow exponential. Later, it is found that the mass term of the scalar field or the boundary of anti de-sitter (AdS) spacetime can play the roll of the reflecting mirror [16–37]. In these cases, the superradiance of the scalar field perturbation will lead to the instability of the black hole.

Recently, it is proved by Hod in [38, 39] that the existence of a trapping potential well outside the black hole and superradiant amplification of the trapped modes cannot be satisfied simultaneously. This means that the Reissner-Nordström (RN) black holes are stable under the perturbations of massive charged scalar fields. However, whether all of the charged black holes are stable is still an open question worth studying.

In this work, we will study the stability of the charged black hole in string theory. Similar to the superradiant effect in Reissner-Nordström black hole, a charged scalar field can be amplified when impinging on the charged black hole in string theory. According to the black-hole bomb mechanism, the mass term of the incident field can effectively works as the reflecting mirror, which may trigger the instability of black hole. However, we will demonstrate that the charged black hole in string theory is stable against the massive charged scalar perturbation. By carefully studying the behavior of the effective potential outside the horizon, we will show that there is no trapping potential well in the black hole exterior which is separated from the horizon by a potential barrier. So there is no bound states in the superradiant regime which can leat to the instability of charged stringy black hole.

We consider the static charged black holes in low energy effective theory of heterotic string theory in four dimensions. Besides the Einstein-Hilbert term, the action also includes
the contributions from the Dilaton field and the Maxwell’s field. The charged black hole in
string theory which is firstly found by Gibbons and Maeda in [40] and independently found
by Garfinkle, Horowitz, and Strominger in [41] a few years later is described by the metric
\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r\left(1 - \frac{Q^2}{M}\right)(d\theta^2 + \sin^2\theta d\phi^2) , \]
and the electric field and the dilaton field
\[ A_t = -\frac{Q}{r} , \quad e^{2\Phi} = 1 - \frac{Q^2}{Mr} . \]

The parameters \( M \) and \( Q \) are the mass and electric charge of the black hole respectively. The event horizon of black hole is located at \( r = 2M \). The area of the sphere of the charged stringy black hole approaches to zero when \( r = Q^2/M \). Therefore, the sphere surface of the radius \( r = Q^2/M \) is singular. When \( Q^2 \leq 2M^2 \), this singular surface is surrounded by the event horizon. We will consider the black hole with the parameters satisfying the condition \( Q^2 \leq 2M^2 \) in this paper. When \( Q^2 = 2M^2 \), the singular surface coincides with the event horizon. This is the case of extremal black hole.

We start with analysing the scalar field perturbation in the background of the charged stringy black hole. The dynamics of the charged massive scalar field perturbation is governed by the Klein-Gordon equation
\[ \left[ (\nabla_\nu - iqA_\nu)(\nabla^\nu - iqA^\nu) - \mu^2 \right] \Psi = 0 , \]
where \( q \) and \( \mu \) denote the charge and the mass of the scalar field. By taking the ansatz of the scalar field \( \Psi = e^{-i\omega t}R(r)Y_{lm}(\theta,\phi) \), where \( \omega \) is the conserved energy of the mode, \( l \) is the spherical harmonic index, and \( m \) is the azimuthal harmonic index with \( -l \leq k \leq l \), one can deduce the radial wave equation in the form of
\[ \Delta \frac{d}{dr} \left( \Delta \frac{dR}{dr} \right) + UR = 0 , \]
where we have introduced a new function \( \Delta = (r - Q^2/M)(r - 2M) \) and the potential function is given by
\[ U = \left(r - \frac{Q^2}{M}\right)^2(\omega r - qQ)^2 - \Delta \left[ \mu^2 r \left(r - \frac{Q^2}{M}\right) + l(l + 1) \right] . \]

Firstly, we want to demonstrate that the classical superradiance is present for scalar field perturbation in charged stringy black hole. In particular, we shall study the asymptotic
solutions of the radial wave equation near the horizon and at spatial infinity with the appropriate boundary conditions and obtain the superradiant condition of the charged scalar field.

To this end, it is convenient to define the tortoise coordinate \( r^* \) by the equation \( \frac{dr^*}{dr} = \frac{r^2}{\Delta} \), and introduce the new radial function as \( \tilde{R} = r^* R(r) \). The domain of the tortoise coordinate \( r^* \) is \((-\infty, +\infty)\) corresponding to the domain \((2M, +\infty)\) of coordinate \( r \). In terms of the tortoise coordinate and the new radial function, the radial wave equation can be rewritten as

\[
\frac{d^2 \tilde{R}}{dr^*_2} + \tilde{U} \tilde{R} = 0 ,
\]

with

\[
\tilde{U} = \frac{U}{r^3} - \frac{\Delta}{r^3} \frac{d}{dr} \left( \frac{\Delta}{r^2} \right).
\]

It can be easily checked that when \( Q = 0 \) the above equation reduced to the Regge-Wheeler equation \[1\] as expected.

It is easy to obtain the asymptotic behavior of the potential \( \tilde{U} \) as

\[
\lim_{r \to 2M} \tilde{U} = \left(1 - \frac{Q^2}{2M^2}\right)^2 \left(\omega - \frac{gQ}{2M}\right)^2 ,
\]

\[
\lim_{r \to \infty} \tilde{U} = \omega^2 - \mu^2 .
\]

So, the radial wave equation has the following asymptotic behavior when \( \omega^2 > \mu^2 \)

\[
\tilde{R} = \begin{cases} 
  e^{i\sigma r^*} + A e^{-i\sigma r^*} , & r \to 2M , \\
  B e^{i\sqrt{\omega^2 - \mu^2} r^*} , & r \to \infty ,
\end{cases}
\]

with the parameter \( \sigma = \left(1 - \frac{Q^2}{2M^2}\right) \left(\omega - \frac{gQ}{2M}\right) \). By calculating the radial flux of the scalar field modes respectively, one can see that the modes of the form \( e^{i\sigma r^*} \) and \( e^{-i\sigma r^*} \) are outcoming from the event horizon and ingoing to the event horizon respectively, while the modes of the form \( e^{i\sqrt{\omega^2 - \mu^2} r^*} \) correspond to the outgoing flux at the spatial infinity. This boundary conditions then correspond to the flux coming from the event horizon of the black hole which is partially reflected back to the black hole and partially sent to the spatial infinity.

Given any two linearly independent solutions \( \varphi_1(r^*) \) and \( \varphi_2(r^*) \) of the radial equation (10), their Wronskian is constant, i.e.

\[
W(\varphi_1, \varphi_2) \equiv \varphi_1 \frac{d\varphi_2}{dr^*} - \varphi_2 \frac{d\varphi_1}{dr^*} = \text{constant}
\]
Because the potential $\tilde{U}$ is real, the asymptotic solution (14) and its complex conjugate are linearly independent to each other. Evaluating the Wronskian for the asymptotic solution (14) and its complex conjugate at the horizon and spatial infinity respectively, we can get

$$\sigma(1 - |A|^2) = \sqrt{\omega^2 - \mu^2}|B|^2.$$ \hspace{1cm} (12)

Now, we can see that if $\sigma < 0$, we have $|A|^2 > 1$. This means that an incident wave from the horizon is reflected back to the horizon with an increased amplitude. This phenomenon is known as superradiance. From $\sigma < 0$, we can get the condition to occur the superradiance

$$\omega < q\Phi_H,$$ \hspace{1cm} (13)

with $\Phi_H = \frac{Q^2}{2M}$ being the electric potential at the horizon. In fact, this condition has already been obtained in [42], where the superradiant effects for the dilaton black holes are studied.

Now we shall analysis whether this superradiant effect will lead to the instability of the charged stringy black hole. The purpose can be achieved by studying whether there exists a trapping potential well outside the horizon that can trap the superradiant modes. If there exists a trapping potential well, superradiant amplification of the bound state in the potential well will trigger the instability of black hole. Otherwise, the black hole is stable although the superradiance is present.

As a matter of convenience, we want to introduce a new radial function as $\psi = \Delta^{1/2} R$. The radial wave equation (5) can be rewritten as

$$\frac{d^2 \psi}{dr^2} + (\omega^2 - V)\psi = 0,$$ \hspace{1cm} (14)

with the effective potential

$$V = \omega^2 - \frac{1}{\Delta^2} \left[ U + \frac{1}{4} \left( \frac{Q^2}{M} - 2M \right)^2 \right].$$ \hspace{1cm} (15)

Now, we analyze the behavior of the effective potential outside of the horizon. From Eq.(20), after some algebra, we can get the derivative of the effective potential as

$$V' = \frac{1}{\Delta^3} \left( r - \frac{Q^2}{M} \right) \left[ 2 \left( r - \frac{Q^2}{M} \right)^2 (\omega r - qQ)(2M\omega - qQ) - 2M\mu^2 \left( r - \frac{Q^2}{M} \right)^2 (r - 2M) - l(l + 1) \left( r - \frac{Q^2}{M} \right) (r - 2M) - l(l + 1)(r - 2M)^2 \right].$$ \hspace{1cm} (16)
It is convenient to define a new variable $z = r - \frac{Q^2}{M}$, in terms of which the derivative of the effective potential can be rewritten as

$$V' = \frac{z}{\Delta^3}(az^3 + bz^2 + cz + d),$$

with

$$a = 2\omega(2M\omega - qQ) - 2M\mu^2,$$

$$b = 2(2M\omega - qQ) \left( \frac{\omega Q^2}{M} - qQ \right) - 2M\mu^2 \left( \frac{Q^2}{M} - 2M \right) - 2l(l + 1),$$

$$c = -3l(l + 1) \left( \frac{Q^2}{M} - 2M \right),$$

$$d = -l(l + 1) \left( \frac{Q^2}{M} - 2M \right)^2.$$

(17) (18) (19) (20) (21)

The behavior of the effective potential can be roughly described by the properties of the roots of $V' = 0$. We now analysis the case of the nonextremal black hole. Obviously, $z = 0$ is a root of $V'(z) = 0$. The other three roots are denoted as $\{z_1, z_2, z_3\}$. It is well-known that the relations between the roots and the coefficients of the cubic equation $az^3 + bz^2 + cz + d = 0$ are given by

$$z_1 + z_2 + z_3 = -\frac{b}{a},$$

$$z_1z_2 + z_1z_3 + z_2z_3 = \frac{c}{a},$$

$$z_1z_2z_3 = -\frac{d}{a}.$$

(22) (23) (24)

For the scalar field satisfying the superradiant condition and the bound state condition, the frequency $\omega$ of the mode satisfies

$$0 \leq \omega < \min\{qQ/2M, \mu\}.$$

(25)

Then, it is obvious that

$$a < 0.$$

(26)

This implies that

$$V'(r \to \infty) \to 0^-.$$

(27)
FIG. 1: Qualitative shape of the effective potential $V$ for different $\omega$. The parameters are given by $M = 1, Q = 1, q = 1, \mu = 1$ and $l = 1$. From top to down, the three curves correspond to $\omega = 1/3, 1/4$ and $1/5$ respectively.

Note also that

$$V(r \to 2M) \to -\infty,$$

and

$$V(r \to Q^2/M) \to -\infty.$$  \hfill (28)

This implies that there exists at least one maximum point in the region $r > 2M$ and at least one maximum point in the region $Q^2/M < r < 2M$. We denote this two maximum points as $z_1$ and $z_2$ respectively. Then we have

$$z_1 > z_2 > 0.$$  \hfill (30)

For the nonextremal black hole, $Q^2 < 2M^2$. Then, we have

$$c > 0, \quad d < 0.$$  \hfill (31)

This implies

$$z_3 < 0.$$  \hfill (32)

This means that in the superradiant regime the effective potential $V(r)$ has only one maximum outside the event horizon. This implies that there is no trapping potential well outside of the event horizon which is separated from the horizon by a potential barrier.

In FIG. 1, we have plotted the shape of the effective potential $V$ given in Eq.(15) for different $\omega$. The analytical conclusion for the nonextremal black hole case is explicitly
shown in this figure. The mass of the scalar field is never able to generate a potential well outside of the horizon to trap the superradiant modes. Thus, there are no meta-stable bound states of the charged massive scalar field in the superradiant regime. In other words, the superradiance in the nonextremal charged stringy black hole can not trigger the instability.

For the case of the extremal black hole, \( Q^2 = 2M^2 \). The effective potential becomes

\[
V = \omega^2 - \frac{1}{(r-2M)^2} \left[ (\omega r - qQ)^2 - \mu^2 r(r - 2M) - l(l+1) \right].
\]  

(33)

The derivative of the effective potential is then given by

\[
V' = \frac{2}{z^3} (az + b),
\]

(34)

with

\[
a = 2\omega(2M\omega - qQ) - 2\mu^2,
\]

(35)

\[
b = 2(2M\omega - qQ)^2 - 2l(l + 1).
\]

(36)

The coefficient \( a \) is still unchanged, i.e. \( a < 0 \) in the superradiant region. The root of \( V'(z) = 0 \) is simply given by

\[
z_0 = -\frac{b}{a}.
\]

(37)

We shall analyze the following two situations in detail.

Case I: \( (2M\omega - qQ)^2 < l(l + 1) \) or \( b < 0 \)

The effective potential behaves

\[
V(r \to 2M) \to +\infty.
\]

(38)

But \( z_0 < 0 \), i.e. the root of \( V'(z) = 0 \) locates at the non-physical regime \( r < 2M \). This implies that there is neither an maximum point nor an minimum point outside the horizon. In the black-hole exterior, the effective potential will gradually bring down to a finite value.

In FIG. 2, we have plotted the shape of the effective potential given in Eq.(33) for different \( \omega \). The parameters are selected to satisfying the extremal condition and \( b < 0 \) simultaneously. One can see that, outside the horizon, there exists neither a potential barrier nor a potential well. In this case, the superradiance can not lead to the instability.

Case II: \( (2M\omega - qQ)^2 > l(l + 1) \) or \( b > 0 \)
FIG. 2: Qualitative shape of the effective potential $V$ for different $\omega$. The parameters are given by $M = 1, Q = \sqrt{2}, q = 1, \mu = 1$ and $l = 1$. From right to left, the three curves correspond to $\omega = 1/2, 1/3$ and $1/5$ respectively.

The effective potential behaves

$$V(r \to 2M) \to -\infty.$$ 

(39)

But $z_0 > 0$, i.e. there is a root of $V'(z) = 0$ in the region of $r > 2M$. This implies that the effective potential have only one maximum point outside the horizon, i.e. there is only a potential barrier outside of the horizon.

In FIG. 3, we have plotted the effective potential for different $\omega$ which satisfy the condition $b > 0$. The shape of the effective potential in this case is very similar to that of the nonextremal black hole. So for the same reason, in the present case, the superradiant modes can not be trapped as well and the black hole is also stable.

In summary, we have firstly shown that the classical superradiance phenomenon presents
in the charged stringy black holes for the charged scalar field perturbations. The superradiant condition is also obtained by analyzing the asymptotic solutions near the horizon and at the spatial infinity, which is similar to that of RN black hole. Then we investigate the possibility of instability triggered by the superradiance. It is shown by analyzing the behavior of the effective potential that for both the nonextremal black holes and the extremal black holes there is no potential well which is separated from the horizon by a potential barrier. Thus, the superradiant modes of charged massive scalar field can not be trapped and lead to the instabilities of the black holes. This indicates that the extremal and the nonextremal charged black holes in string theory are stable against the massive charged scalar field perturbations.

At last, we should note that although the mass of the scalar field can not provide an effective potential well outside the black hole, one can still make the black hole unstable by placing a reflecting mirror around the black hole [8–10, 14, 15]. It would be interesting to study the behavior of the scalar field in this black hole-mirror system in the future work.

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