Quadrature and polarization squeezing in a dispersive optical bistability model

Ferran V. García–Ferrer\textsuperscript{1}, Isabel Pérez–Arjona\textsuperscript{2}, Germán J. de Valcárcel\textsuperscript{1}, and Eugenio Roldán\textsuperscript{1}

\textsuperscript{1}Departament d’Òptica, Universitat de València, Dr. Moliner 50, 46100–Burjassot, Spain and
\textsuperscript{2}Departament de Física Aplicada, Escola Politècnica Superior de Gandia, Universitat Politècnica de València, Ctra. Nazaret–Oliva S/N, 46730–Grau de Gandia, Spain

Abstract

We theoretically study quadrature and polarization squeezing in dispersive optical bistability through a vectorial Kerr cavity model describing a nonlinear cavity filled with an isotropic $\chi^{(3)}$ medium in which self-phase and cross-phase modulation, as well as four-wave mixing, occur. We derive expressions for the quantum fluctuations of the output field quadratures as a function of which we express the spectrum of fluctuations of the output field Stokes parameters. We pay particular attention to study how the bifurcations affecting the non-null linearly polarized output mode squeezes the orthogonally polarized vacuum mode, and show how this produces polarization squeezing.

PACS numbers: 42.50.-p, 42.50.Lc, 42.65.Sf

I. INTRODUCTION

Squeezing has received continued attention since the early eighties \textsuperscript{2, 3, 8} because it is a beautiful quantum phenomenon with amazing potential applications \textsuperscript{3}, the most recent of which are connected with continuous variable quantum information \textsuperscript{4}. One particular aspect of squeezing that is receiving more attention at present is polarization squeezing, probably because its detection does not need homodyning, as it is the case with quadrature squeezing, i.e., photodetectors suffice for detecting polarization squeezing \textsuperscript{5, 6, 7}. Here we are concerned with the study of both quadrature and polarization squeezing in the field exiting a particular nonlinear optical cavity.

One system that attracted early attention for quadrature squeezing is dispersive optical bistability \textsuperscript{8, 9, 10}, which consists in an optical cavity filled with a nonlinear $\chi^{(3)}$ medium which is fed with an external (pumping) field. For appropriate values of the system parameters, the output field exhibits bistability and at the turning points of the hysteresis cycle, perfect squeezing is obtained. Here we revisit this system because we are interested not only in quadrature but also in polarization squeezing.

The generalization of this simple dispersive optical bistability model we consider in this article consists in taking into account the existence, inside the optical cavity, of a mode with orthogonal polarization with respect to that of the pumping field, which is also close to cavity resonance. For low pump, this extra mode remains empty, i.e., the output cavity field has the same polarization as the pumping field, but for large enough pump this orthogonally polarized mode can switch on (what is known as polarization instability) and then the field exiting the cavity changes from linearly to elliptically polarized. This system is known as vectorial Kerr cavity \textsuperscript{11, 12} and its mathematical modeling consists in taking into account self– and cross–phase modulation, as well as four–wave mixing, for the two orthogonally polarized intracavity modes, the weight of each of the three nonlinear processes depending on the particular nonlinear medium under consideration \textsuperscript{13}.

The vectorial Kerr cavity has been studied in the past from the viewpoints of stability theory \textsuperscript{16} and pattern formation \textsuperscript{13, 17, 18, 19, 20}. Also quantum fluctuations have been studied in this system but, as far as we know, only from the perspective of pattern formation \textsuperscript{21}, i.e., by considering a continuum of transverse modes. Here we consider the single transverse mode model and study quadrature and polarization squeezing in the field exiting the nonlinear cavity. As we show below, the vectorial Kerr cavity exhibits polarization squeezing. In particular, and most interestingly, this polarization squeezing is optimum at the bistability bifurcation points, when the orthogonally polarized field is off. Thus the vectorial Kerr cavity model, which can be taken as a more accurate model of dispersive optical bistability, reveals that this well known process can act as a polarization squezeer.

The rest of the article is structured as follows. In Section II we derive the Langevin equations for quantum fluctuations from the quantum Hamiltonian of a Kerr cavity pumped by a linearly polarized coherent field, which are derived from the Fokker–Planck equation verified by the generalized-P distribution. In Sec. III, in order to keep in mind the scenario of the different solutions exhibited by the system, their classical stationary solutions are given and their stability properties are briefly reviewed. The linearized Langevin equations around the semiclassical stationary solutions and the spectrum of quantum fluctuations are introduced in Sec. IV. Sec. V is devoted to the analysis of quadrature squeezing. Then, in Section VI quantum Stokes operators and the definition of polarization squeezing are introduced. In Sec. VII we state specifically the relation between the Stokes parameters variances and the quadrature squeezing spectra and analyze polarization squeezing in this system, paying special attention to the squeezing occurring at the bifurcations affecting the linearly polarized solution, but
considering also polarization squeezing in the elliptically polarized solution in a particular case. Finally, in Sec. VIII we show the most relevant conclusions of our work.

II. MODEL

Consider an optical cavity filled with an isotropic Kerr medium and assume that cavity losses occur only at one of the cavity mirrors being the same for all possible field polarizations (i.e., assume that the cavity is also isotropic). An external laser field, whose frequency is far enough from any resonance in the nonlinear medium (what makes possible that the interaction be described by a real cubic nonlinear susceptibility) is injected into the cavity. We shall assume that the injected field is linearly polarized, say along the $x$-direction. As cavity losses do not favour the injected field polarization, under certain circumstances the intracavity field can experience a polarization instability leading to the spontaneous generation of a field orthogonally polarized with respect to the injected field ($y$-polarized in our case), hence the name vectorial Kerr cavity for such a nonlinear system.

The quantum Hamiltonian which describes this system is given in the interaction picture, by

$$\hat{H} = \hat{H}_{\text{free}} + \hat{H}_{\text{ext}} + \hat{H}_{\text{int}},$$

with

$$\hat{H}_{\text{free}} = \hbar (\omega_c - \omega_0) \left( \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \right),$$

$$\hat{H}_{\text{ext}} = i\hbar E_0 \left( \hat{a}_1 - \hat{a}_1^\dagger \right),$$

$$\hat{H}_{\text{int}} = -\hbar \eta g \left[ \frac{1}{2} \left( \hat{a}_1^2 \hat{a}_2^2 + \hat{a}_2^2 \hat{a}_1^2 \right) + \frac{B}{4} \left( \hat{a}_2^\dagger \hat{a}_2^2 + \hat{a}_1^\dagger \hat{a}_1^2 \right) \right],$$

where $\hat{H}_{\text{free}}$ corresponds to the free Hamiltonian of the two orthogonally polarized intracavity modes, $\hat{H}_{\text{ext}}$ describes the injected coherent field, and $\hat{H}_{\text{int}}$ is the interaction Hamiltonian. In the above expressions $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ are, respectively, the creation and annihilation operators corresponding to the $x$-polarized ($i = 1$) and the $y$-polarized ($i = 2$) fields; $\omega_0$ and $\omega_c$ are the injected field frequency and the frequency of the cavity mode closest to $\omega_0$ (which is assumed to be the same for the two orthogonally polarized modes as the cavity is isotropic), respectively; $E_0$ is proportional to the amplitude of the injected coherent field (which we take as a real quantity without loss of generality); $A$ and $B$ are the Maker–Terhune coefficients (which, for isotropic media, satisfy $A + B/2 = 1$) governing the relative strength of cross-phase modulation and four-wave mixing, respectively; $g = 2\hbar \omega_0 \chi^{(3)}_{\text{xxx}}/\varepsilon^2 V$ is the radiation–matter coupling constant ($V$ is the cavity volume, $\varepsilon$ is the medium dielectric constant and $\chi = \chi^{(3)}_{\text{xxx}}$, $i = x,y,z$, is the nonlinear susceptibility); and, finally, $\eta = \pm 1$ takes account of the self-focusing ($\eta = +1$) or self-defocusing ($\eta = -1$) cases.

The intracavity field exits the cavity through the single output mirror. Treating, as it is usual, the external vacuum modes as a reservoir, the master equation governing the evolution of density matrix $\rho$ of the intracavity modes is

$$\frac{\partial}{\partial t} \hat{\rho} = \frac{1}{i\hbar} [\hat{H}, \hat{\rho}] + \hat{\Lambda} \rho,$$

$$\hat{\Lambda} \rho = \gamma \sum_{i=1,2} \left( [\hat{a}_i, \hat{\rho} \hat{a}_i^\dagger] + [\hat{a}_i^\dagger, \hat{\rho}] \right),$$

the Liouvillian term $\hat{\Lambda} \rho$ modeling the coupling between the system and the external reservoir through the output mirror ($\gamma$ denotes cavity losses; $2\gamma$ is the photon number loss rate).

A. Fokker–Planck and Langevin equations

In order to obtain Langevin equations describing c-numbers evolution from the master equation, one needs an appropriate quantum quasiprobability distribution. We choose the generalized-P representation, which although also gives normally ordered products of operators which provides the calculation of fluctuations spectra outside the resonator itself, on the other hand, avoids the problems associated with other quasiprobability distributions, such as the Glauber–P representation, which although also gives normally ordered correlation products, can exhibit non positive–semi definite diffusion matrix in its dynamical equation.

The generalized-P representation sets a correspondence between quantum operators $\hat{a}_i(t)$ and $\hat{a}_i^\dagger(t)$ and independent c-numbers $a_i(t)$ and $a_i^\dagger(t)$, respectively, satisfying in their mean value that $(a_i(t))^\dagger = (a_i^\dagger(t))$. We have derived the equation of evolution for the generalized-P distribution $P(a,t)$ by using standard techniques, and have obtained the following well–behaved Fokker–Planck equation

$$\frac{\partial P}{\partial t} = \left[ -\sum_{i=1}^{4} \frac{\partial}{\partial a_i} A_i(a) + \frac{1}{2} \sum_{i,j=1}^{4} \frac{\partial^2}{\partial a_i \partial a_j} D_{i,j}(a) \right] P,$$

with $a = \text{col}(a_1, a_1^\dagger, a_2, a_2^\dagger)$ and

$$A_1(a) = E_0 - \gamma (1 + i\eta \Delta) a_1 + i\eta g \times \left[ a_1^2 a_2^\dagger + A_1 a_2 a_2^\dagger + \frac{B}{2} \alpha_1^2 a_2^\dagger \right],$$

$$A_2(a) = E_0 - \gamma (1 - i\eta \Delta) a_1^\dagger - i\eta g \times \left[ a_1^\dagger a_2^\dagger + A_1 a_1 a_2^\dagger + \frac{B}{2} \alpha_1^2 a_2 \right]^2,$$

$$A_3(a) = -\gamma (1 + i\eta \Delta) a_2 + i\eta g \times \left[ a_2^\dagger a_2^\dagger + A_1 a_1 a_2^\dagger + \frac{B}{2} \alpha_1^2 a_2 \right]^2,$$

where $\Delta = \omega_c - \omega_0$
\[ A_4(a) = -\gamma (1 - i\eta \Delta) \alpha_1^\dagger - i\eta g \times \]
\[ \times \left[ \alpha_2 \alpha_2^\dagger + A\alpha_1 \alpha_2^\dagger + \frac{B}{2} (\alpha_1^\dagger)^2 \alpha_2 \right], \]

where the new detuning parameter \( \Delta \) is

\[ \Delta = \frac{1}{\eta \gamma} \left[ \bar{\omega}_c - \omega_0 - \eta g \left( 1 + \frac{A}{2} \right) \right] = \frac{1}{\eta \gamma} \left( \bar{\omega}_c - \omega_0 \right). \]

The Fokker-Planck equation (4) can be transformed into an equivalent set of classical–looking stochastic differential Langevin equation via the Ito rules [25]. These Langevin equations read

\[ \frac{d\alpha_1}{dt} = E_0 - \gamma (1 + i\eta \Delta) \alpha_1 + i\eta g \left[ |\alpha_1|^2 \alpha_1 + A |\alpha_2|^2 \alpha_1 + \frac{B}{2} \alpha_1^\dagger \alpha_2 \right], \]

\[ \frac{d\alpha_2}{dt} = -\gamma (1 + i\eta \Delta) \alpha_2 + i\eta g \left[ |\alpha_2|^2 \alpha_2 + A |\alpha_1|^2 \alpha_2 + \frac{1}{2} B \alpha_1^\dagger \alpha_2 \right]. \]

The above equations exhibit the symmetry \( \{ \alpha_1, \alpha_2, \Delta, \eta \} \leftrightarrow \{ \alpha_1^\dagger, \alpha_2^\dagger, -\Delta, -\eta \} \) (remember that \( E_0 \) has been taken to be real without loss of generality). Then passing from positive \( \eta \) (self–focusing) to negative \( \eta \) (self–defocusing) is equivalent to changing \( \Delta \leftrightarrow -\Delta \) for what concerns the steady states and their stability properties.

By introducing the following changes

\[ \tau = \gamma t, \quad E = \frac{1}{\gamma} \sqrt{\frac{g}{\gamma}} E_0, \quad a_i = \sqrt{\frac{g}{\gamma}} \alpha_i, \]

Eqs. (11) transform into an equivalent set of equations coinciding with those given in [16]. Then we can use the results previously derived in [16] by taking into account the above changes. (We notice that along the following sections it will be more convenient to give some of the expressions in terms of \( a_i, E \), and \( \tau \) than in terms of the original quantities, so we shall refer later to the above transformations).

Eqs. (11) have two different steady state solutions (see [16] and references therein): The singlemode (or linearly polarized) solution, which corresponds to the pure Kerr solution [10]

\[ E_0^2 = \gamma^2 \left[ 1 + \left( \frac{\Delta - \frac{g}{\gamma} I_{1s}}{I_{1s}} \right)^2 \right] I_{1s}, \]

\[ \phi_{1s} = \arccos \left( \sqrt{\frac{I_{1s}}{E_0}} \right) \]

\[ I_{2s} = 0, \]

where \( \alpha_j = \sqrt{I_{1s}} e^{i\phi_{js}} \) at steady state (\( j = 1, 2 \); and the bimode (or elliptically polarized) solution

\[ E_0^2 I_{1s} = \gamma^2 (I_{1s} + I_{2s})^2 \]

\[ + g^2 (I_{1s} - I_{2s})^2 (I_{1s} + I_{2s} - \frac{\gamma \Delta}{g})^2, \]

\[ I_{2s} = \frac{\gamma \Delta}{g} - A I_{1s} \pm \sqrt{\left( \frac{B}{2} \right)^2 I_{1s}^2 - \left( \frac{\gamma \Delta}{g} \right)^2}, \]

\[ \phi_{1s} = \arccos \left[ \sqrt{\frac{I_{1s}}{E_0}} \left( 1 + \frac{I_{2s}}{I_{1s}} \right) \right], \]

\[ \phi_{2s} = \phi_{1s} + \frac{1}{2} \arccos \left( \frac{\gamma \Delta - g I_{1s} - A g I_{2s}}{\frac{g}{2} g I_{1s}} \right). \]

III. CLASSICAL MODEL

Let us write the model equations [3] in the classical limit (i.e., when \( \alpha_1^\dagger \) is interpreted as \( \alpha_1^* \) and the noise terms are ignored). They read
Notice that the bimode solution with the same intensity and phase \( \phi_2 = \phi_{2s} + \pi \) also exists, i.e., Eqs. (11) exhibit phase bistability for the cross-polarized mode.

The stability of these two steady state solutions has been extensively studied in [16] and turns out to be quite involved. Fortunately for our present purposes it will suffice to consider the stability of the singlemode solution and consider parameter values where the bimode solution does not experience secondary bifurcations, an information we can extract from [16]. From now on we shall restrict ourselves to the case of liquids, for which \( \mathcal{A} = 1/4 \) and \( \mathcal{B} = 3/2 \), and we refer the interested reader to [16] for full details and general expressions (including possible anisotropies in the cavity losses). It is interesting to say here that there is not a threshold value for the Maker-Feschbach solution (and consider parameter values where the bimode solution exists, or, equivalently, for intracavity intensity values between \( E_{01}^2 \), \( E_{02}^2 \)), see Eqs. (12).

We thus write

\[
\frac{d}{dt} \delta \mathbf{a} = \mathbf{\dot{A}} \cdot \delta \mathbf{a} + \mathbf{\dot{B}} \cdot \mathbf{\xi}(t),
\]

where matrix \( \mathbf{\dot{A}} \) corresponds to

\[
\dot{A}_{ij} = \left( \frac{\partial A_{ij}}{\partial a_i} \right)_{a=a_s},
\]

and \( \mathbf{\dot{B}} = \mathbf{B} (\mathbf{a} = \mathbf{a}_s) \).

Quantum fluctuations are well characterized by their spectrum [26, 27] (i.e., by the Fourier transform of the two-time correlation functions), which are given by the spectral matrix \( \mathbf{M}(\omega) \) defined as

\[
\mathbf{M}(\omega) = \int_{-\infty}^{+\infty} dt \ e^{i\omega t} \langle \delta \mathbf{a}_i(0) \delta \mathbf{a}_j(t) \rangle.
\]

Chaturvedi et al. [26] showed that \( \mathbf{M}(\omega) \) can be directly obtained from Eq. (20)

\[
\mathbf{M}(\omega) = (\mathbf{\hat{A}} + i\omega \mathbf{I})^{-1} \mathbf{\hat{D}} (\mathbf{A}^T - i\omega \mathbf{I})^{-1}
\]

with \( \mathbf{\hat{D}} = \mathbf{\hat{B}} \mathbf{B}^T \) and \( \mathbf{I} \) the identity matrix. We do not give the explicit expression of the matrix elements of \( \mathbf{M}(\omega) \) because they are very lengthy.

As stated, the stability properties of the elliptically polarized solution are much more involved than those of the singlemode solution: It can undergo secondary tangent bifurcations as well as Hopf bifurcations and no analytical expressions exist for the parameter values at these secondary bifurcations. However these scenarios are out of the scope of the present work as we will consider parameter values for which these secondary bifurcations do not occur.

IV. LINEARIZED QUADRATURE SQUEEZING SPECTRA

We return now to the quantum description of the system. In order to describe in a simple way the dynamics of quantum fluctuations, the Langevin equations (8) are linearized around the stationary classical mean values of the intracavity fields, given by Eqs. (13) for the single-mode solution (only the field parallel to the injected one is present in the cavity) and Eqs. (17) for the bimode solution (the orthogonal polarized fields switches on and the output field is elliptically polarized). We thus write

\[
\alpha_j = \sqrt{I_{js}} e^{i\phi_{js}} + \delta \alpha_j,
\]

\[
\alpha_j^+ = \sqrt{I_{js}} e^{-i\phi_{js}} + \delta \alpha_j^+,
\]

with \( j = 1, 2 \) and where the \( I_{js} \) and \( \phi_{js} \) are, respectively, the intensity and the phase values corresponding to the analyzed stationary solution. Then we obtain the linearized Langevin equations describing the evolution of quantum fluctuations, \( \delta \mathbf{a} = \text{col}(\delta \alpha_1, \delta \alpha_1^+, \delta \alpha_2, \delta \alpha_2^+) \), which read

\[
\frac{d}{dt} \delta \mathbf{a} = \mathbf{\dot{A}} \cdot \delta \mathbf{a} + \mathbf{\dot{B}} \cdot \mathbf{\xi}(t),
\]
Now, as we are interested in the quantum fluctuations of the fields outside the cavity, we must use the input–output theory for calculating the spectrum of quantum fluctuations of the output fields. As we have used the generalized-P representation and the cavity is pumped by a coherent field, the input–output theory [27] shows that
\[ M^{\text{out}}(\omega) = \int_{-\infty}^{+\infty} dt \, e^{i\omega t} \langle \delta a_i^{\text{out}}(0) \delta a_j^{\text{out}}(t) \rangle = 2\gamma M(\omega). \]

We write down now expressions for the quadratures’ squeezing spectra. We define the quadratures corresponding to the quantum fluctuations as
\[ \delta X^{\text{out}}_{j,\beta}(t) = \delta a_j^{\text{out}}(t) e^{i\beta} + \delta a_j^{\text{out}}(t) e^{-i\beta}, \]
where we remind that \( j = 1, 2 \) denotes the parallel (\( j = 1 \)) or orthogonal (\( j = 2 \)) field and \( \beta \) is the (arbitrary) quadrature angle.

Now we define the output quadrature squeezing spectra as
\[ q^{\text{out}}_{\beta}(\beta, \omega) = \int_{-\infty}^{+\infty} \delta X^{\text{out}}_{j,\beta}(t) \delta X^{\text{out}}_{j,\beta}(t + \tau) \, e^{i\omega \tau} \, d\tau, \]
where \( : \) denotes normal and time ordering. By making use of the input–output formalism and using the spectral matrix [27] the output quadrature squeezing spectra can be written as
\[ q^{\text{out}}_{1}(\beta, \omega) = 2\gamma (M_{11} e^{-i\beta} + M_{22} e^{i\beta}) + 2\gamma (M_{12} + M_{21}) (27a), \]
\[ q^{\text{out}}_{2}(\beta, \omega) = 2\gamma (M_{33} e^{-i\beta} + M_{44} e^{i\beta}) + 2\gamma (M_{34} + M_{43}) (27b), \]
It will be useful for later purposes to introduce cross-squeezing spectra, through the generalization of (26)
\[ q^{\text{out}}_{jk}(\beta_j, \beta_k, \omega) = \int_{-\infty}^{+\infty} \delta X^{\text{out}}_{j,\beta_j}(t) \delta X^{\text{out}}_{k,\beta_k}(t + \tau) \, e^{i\omega \tau} \, d\tau, \]
which are given by
\[ q^{\text{out}}_{12}(\beta_1, \beta_2, \omega) = 2\gamma (M_{13} e^{-i\sigma_12} + M_{24} e^{i\sigma_12}) + 2\gamma (M_{14} e^{i\sigma_12} + M_{23} e^{-i\sigma_12}), \]
\[ q^{\text{out}}_{21}(\beta_2, \beta_1, \omega) = 2\gamma (M_{31} e^{-i\sigma_21} + M_{42} e^{i\sigma_21}) + 2\gamma (M_{32} e^{i\sigma_21} + M_{41} e^{-i\sigma_21}), \]
where
\[ \sigma_{jk} = \beta_j + \beta_k, \quad \delta_{jk} = \beta_j - \beta_k. \]
Given the normal ordering, complete absence of fluctuations in a field quadrature means: \( q^{\text{out}}_j(\beta, \omega) = -1 \). Moreover, by taking into account the shot noise level outside the cavity it turns out that the symmetrically ordered squeezing spectra are
\[ q^{\text{out}}_j(\beta, \omega) = 1 + q^{\text{out}}_j(\beta, \omega), \]
\[ q^{\text{out}}_{21}(\beta_2, \beta_1, \omega) = : q^{\text{out}}_{21}(\beta_2, \beta_1, \omega) :. \]

V. ANALYSIS OF QUADRATURE SQUEEZING AT THE SINGLEMODE SOLUTION BIFURCATIONS

We analyze here quadrature squeezing at the bifurcations affecting the linearly polarized solution. Following the steps we have outlined, after some algebra one finds that the quantities of interest at the bifurcations (where \( I_{2s} = 0 \)) can be written as
\[ q^{\text{out}}_{1}(\beta, \omega) = \frac{4 |a_{1s}|^2 Q_{1n}}{(Q_{1d} - \omega^2)^2 + 4\omega^2}, \]
\[ q^{\text{out}}_{2}(\beta, \omega) = \frac{6 |a_{1s}|^2 Q_{2} ((Q_{2d}+2\omega^2)^2+16\omega^2), \]
\[ q^{\text{out}}_{21}(\beta_1, \beta_2, \omega) = q^{\text{out}}_{21}(\beta_1, \beta_2, \omega) = 0, \]
where
\[ Q_{1n} = - \left( 3 |a_{1s}|^4 - 4 |a_{1s}|^2 \Delta + \Delta^2 - 1 - \omega^2 \right) \sin \psi + 2 \left( \Delta - 2 |a_{1s}|^2 \right) \cos \psi + 2 |a_{1s}|^2, \]
\[ Q_{1d} = 3 |a_{1s}|^4 - 4 |a_{1s}|^2 \Delta + \Delta^2 + 1, \]
\[ Q_{2} = \left[ |a_{1s}|^4 + \Delta |a_{1s}|^2 - 2 \left( \Delta^2 - 1 - \omega^2 \right) \right] \sin \psi + 4 \Delta - |a_{1s}|^2 \cos \psi + 3 |a_{1s}|^2, \]
\[ Q_{2d} = |a_{1s}|^4 + \Delta |a_{1s}|^2 - 2 \Delta^2 - 2, \]
and
\[ \psi = 2 (\beta - \phi_{1s}), \]
is twice the phase difference between the analyzed quadrature and the steady state phase, and \( |a_{1s}|^2 \) is the normalized intensity given in Eqs. (12) and (13). In the above equations the frequency \( \omega \) has been normalized to \( \gamma \), but we do not change the symbol for not complicating unnecessarily the notation.

Of course a well known general result is that at the bifurcations there is a field quadrature (of the field directly affected by the bifurcation) that is perfectly squeezed. A different question is how does the bifurcation affecting one mode influence the quadrature squeezing of the other (orthogonal) mode. Let us consider the different bifurcations separately.

A. Polarization bifurcation

At the polarization bifurcation \( I_{1s} = I_{1s,\text{pol}}, \) Eq. (13), and there is a quadrature of field \( \alpha_2 \) (the field that switches on at this bifurcation) that is perfectly squeezed at frequency \( \omega = 0 \) as expected (see the inset in Fig. 2). It can be shown that the squeezed quadrature is \( \delta X^{\text{out}}_{2,\beta}(t) \) with \( \beta = (\psi_{\text{pol, opt}}/2 + \phi_{1s}) \) and
\[ \psi_{\text{pol, opt}} = -\frac{1}{2} \arccos \left[ \frac{1 - \Delta \sqrt{8 + 9\Delta^2}}{3(1 + \Delta^2)} \right], \]
which tends to zero for large negative $\Delta$ and to $-\pi/2$ for large positive $\Delta$ (Fig. 2). It is interesting to note that the range of $\psi$ values for which squeezing occurs is very narrow.

As for the field $\alpha_1$ (the mode that is on) it undergoes large levels of quadrature noise reduction at this bifurcation, even if the analyzed bifurcation does not affect the field $\alpha_1$ but the field $\alpha_2$. These levels tend to perfect squeezing for increasing positive $\Delta$. In Fig. 3 the squeezing spectrum is shown for two values of the cavity detuning. The optimum squeezing is reached when $\psi = 0$, i.e. for $\beta = \phi_{1s}$, i.e., it is an amplitude squeezing, and occurs at the frequency

$$\omega_{\text{opt}} = \sqrt{5 - \frac{7}{2} \Delta \left( -3\Delta + \sqrt{8 + 9\Delta^2} \right)}.$$  \hspace{1cm} (36)

In Fig. 4 both the optimum squeezing level and the frequency at which this squeezing occurs are represented as a function of cavity detuning. At this stage it is convenient to remember that as $\Delta$ increases, the polarization and one of the bistability bifurcations approach, see Fig. 1, what undoubtedly is connected with these large squeezing levels.

B. Bistability bifurcations

The bistability bifurcation has been studied several times in the past (see, e.g., [12]) and we do not find it necessary to repeat here these well known results, specially because we shall not need them when analyzing polarization squeezing in the following section. It will suffice to remember that as $\Delta$ increases, the polarizations and one of the bistability bifurcations approach, see Fig. 1, what undoubtedly is connected with these large squeezing levels.

1. Upper branch

After substituting $I_{1s} = I_{1s,+}$ in Eq. (32a) one can show that optimum squeezing occurs for $\psi = \pi$ at a frequency that is very close to zero for $\Delta$ close to $\sqrt{3}$ (remember that the bistability bifurcation requires $\Delta \geq \sqrt{3}$) and becomes zero for $\Delta > 1.89$, see Fig. 5. The squeezing level increases with detuning, as can be seen in Fig. 6, passing from $q_2^{\text{opt}}(\pi, \omega_{\text{opt}}) = -0.75$ at $\Delta = \sqrt{3}$ to $q_2^{\text{opt}}(\pi, \omega_{\text{opt}}) = -0.98$ for $\Delta$ tending to infinity. It is important to stress for later purposes that optimum squeezing occurs at $\psi = \pi$, i.e., for $\beta = \phi_{1s} + \pi/2$.

2. Lower branch

After substituting $I_{1s} = I_{1s,-}$ in Eq. (32b) one can show that optimum squeezing occurs for $\psi = \pi$ at $\omega_{\text{opt}} = \sqrt{\frac{1}{7} \left[ 7\Delta \left( \Delta + \sqrt{\Delta^2 - 3} \right) - 15 \right]}$, a frequency that increases linearly with $\Delta$ from its minimum value $\omega = 1/\sqrt{3}$ at $\Delta = \sqrt{3}$. Fig. 7 shows the squeezing spectrum for two values of the cavity detuning, and Fig. 8 shows the optimum squeezing as well as the frequency at which it occurs. The squeezing level is not perfect (the optimum is $q_2^{\text{opt}}(\pi, \omega_{\text{opt}} = 1/\sqrt{3}) = -0.75$ for $\Delta = \sqrt{3}$), and degrades with increasing detuning. As in the previous case, we stress that optimum squeezing occurs at $\psi = \pi$, i.e., for $\beta = \phi_{1s} + \pi/2$.

VI. QUANTUM STOKES PARAMETERS

The polarization state of light is classically described by using the Stokes parameters. The Hermitian Stokes operators are defined directly from the analogy with the classical parameters \([5]\)

\[
\hat{S}_0 = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2, \quad \hat{S}_1 = \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2, \\
\hat{S}_2 = \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1, \quad \hat{S}_3 = i \left( \hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2 \right). \hspace{1cm} (38a)
\]

Operator $\hat{S}_0$ refers to the beam intensity, whilst $\hat{S}_1, \hat{S}_2$ and $\hat{S}_3$ describe the polarization state. The parameter $\hat{S}_0$ commutes with all the others

\[
[\hat{S}_0, \hat{S}_j] = 0, \quad j = 1, 2, 3 \hspace{1cm} (39)
\]

whereas the remaining parameters satisfy the commutation relation of the SU(2) Lie algebra

\[
[\hat{S}_k, \hat{S}_l] = 2i \varepsilon_{klm} \hat{S}_m. \hspace{1cm} (40)
\]

Therefore, the simultaneous measurements of the Stokes parameters are impossible in general. The non–zero commutator in (40) implies a restriction to the variances of the Stokes operators, in the form of the uncertainty relations

\[
V_k V_l \geq \left| \left< \hat{S}_m \right> \right|^2, \quad k \neq m \neq k, \quad l \neq k, \hspace{1cm} (41)
\]
where
\[ V_k = \left\langle \hat{S}_k^2 \right\rangle - \left\langle \hat{S}_k \right\rangle^2, \]

is the variance of the quantum Stokes parameter \( \hat{S}_k \). We find it convenient to normalize the variances to the mean intensity and thus we shall use
\[ \tilde{V}_k = \frac{\left\langle \hat{S}_k^2 \right\rangle - \left\langle \hat{S}_k \right\rangle^2}{\left\langle \hat{S}_0 \right\rangle}, \]

and then the inequality (41) reads
\[ \tilde{V}_k \tilde{V}_l \geq \left\langle \hat{S}_m \right\rangle^2 \left\langle \hat{S}_0 \right\rangle, \quad l \neq m \neq k. \] (44)

We stress that the uncertainty relations (41) depend on the expected value of the operators, which makes non-trivial the definition of polarization squeezed states [6, 7].

In terms of field quadratures, a squeezed state is that which (a) is a minimum uncertainty state (MUS) and (b) has fluctuations in a particular field quadrature below those corresponding to the vacuum state (or a coherent state as it is synonymous). Obviously the reduction of fluctuations in one field quadrature occurs at the expense of increasing the orthogonal field quadrature fluctuations. In the case of polarization squeezing this is no more the case because of the already commented appearance of the expected value in inequality (41). Let us see this is some more detail. A coherent state verifies [6] \( \tilde{V}_k = 1 \) for all the Stokes parameters but a coherent state is not, in general, a MUS in terms of polarization (this depends on its polarization state): In effect, consider a coherent state such that for some \( m \) it verifies \( \left\langle \hat{S}_m \right\rangle < \left\langle \hat{S}_0 \right\rangle \), then obviously \( \tilde{V}_k \tilde{V}_l > \left\langle \hat{S}_m \right\rangle / \left\langle \hat{S}_0 \right\rangle \) for this coherent state as \( \tilde{V}_k \tilde{V}_l = 1 \). It is also obvious that there can be states with some \( \tilde{V}_k < 1 \) which are not polarization squeezed states: Think of a state with \( \left\langle \hat{S}_m \right\rangle = 0 \) for some \( m \) and \( \tilde{V}_k < 1 \). It is clear that is not experiencing any transfer of fluctuations from one Stokes parameter to another one, any rearrangement of quantum fluctuations. Then the quantum vacuum is not a reference for defining the squeezed polarization state.

These are the reasons why polarization squeezing is defined in a different way: A field state is said to be squeezed if [6]
\[ \tilde{V}_l < 1 \quad \Rightarrow \quad \tilde{V}_l \tilde{V}_k < \left\langle \hat{S}_m \right\rangle / \left\langle \hat{S}_0 \right\rangle, \quad l \neq m \neq k \neq 0. \] (45)

for some \( l \), i.e., a polarization squeezed state is a state for which the variance of one of the Stokes parameters lies not only below the coherent limit, but also below the corresponding minimum uncertainty state limit. In a state verifying such a condition there is a rearrangement of quantum fluctuations between the different Stokes parameters and is thus a proper squeezed state.

We shall adopt the above criterion, but we must mention that there has been some debate concerning the most appropriate criterion for polarization squeezing and we refer the reader to [6, 7] and references therein.

VII. STOKES PARAMETERS’ FLUCTUATION SPECTRA

As stated, the polarization squeezing properties of light are provided by the knowledge of the variances of the quantum Stokes operators. As we commented in the previous section, polarization and quadrature squeezing are not equivalent, and a quadrature squeezed state does not imply necessarily a polarization squeezed state. Nevertheless, we find it much convenient to write down the spectra of the variances of the quantum Stokes parameters in terms of the quadrature squeezing spectra (26) and (28), showing up the relation between them. We see below that the cross-mode spectra (28) differentiates the polarization squeezing from a simple combination of quadrature spectra.

After lengthy but straightforward algebra one obtains that the variances of the quantum Stokes operators can be written as
\[ V_0(\omega) = I_{1s}q_1^{\text{out}}(\phi_{1s}, \omega) + I_{2s}q_2^{\text{out}}(\phi_{2s}, \omega) \]
\[ + \sqrt{I_{1s}I_{2s}q_1^{\text{out}}}(\phi_{1s}, \phi_{2s}, \omega) \]
\[ + \sqrt{I_{1s}I_{2s}q_2^{\text{out}}}(\phi_{2s}, \phi_{1s}, \omega), \] (46a)
\[ V_1(\omega) = I_{1s}q_1^{\text{out}}(\phi_{1s}, \omega) + I_{2s}q_2^{\text{out}}(\phi_{2s}, \omega) \]
\[ - \sqrt{I_{1s}I_{2s}q_1^{\text{out}}}(\phi_{1s}, \phi_{2s}, \omega) \]
\[ - \sqrt{I_{1s}I_{2s}q_2^{\text{out}}}(\phi_{2s}, \phi_{1s}, \omega), \] (46b)
\[ V_2(\omega) = I_{2s}q_1^{\text{out}}(\phi_{2s}, \omega) + I_{1s}q_2^{\text{out}}(\phi_{1s}, \omega) \]
\[ + \sqrt{I_{1s}I_{2s}q_1^{\text{out}}}(\phi_{2s}, \phi_{1s}, \omega) \]
\[ + \sqrt{I_{1s}I_{2s}q_2^{\text{out}}}(\phi_{1s}, \phi_{2s}, \omega), \] (46c)
\[ V_3(\omega) = I_{2s}q_1^{\text{out}}(\phi_{2s} + \frac{\pi}{2}, \omega) + I_{1s}q_2^{\text{out}}(\phi_{1s} + \frac{\pi}{2}, \omega) \]
\[ - \sqrt{I_{1s}I_{2s}q_1^{\text{out}}}(\phi_{2s} + \frac{\pi}{2}, \phi_{1s} + \frac{\pi}{2}, \omega) \]
\[ - \sqrt{I_{1s}I_{2s}q_2^{\text{out}}}(\phi_{1s} + \frac{\pi}{2}, \phi_{2s} + \frac{\pi}{2}, \omega), \] (46d)

where \( I_{1s} \) and \( \phi_{1s} \) correspond to the steady state values given by Eqs. (13) and Eqs. (14) for the singlemode and bimode solutions, respectively. As for the quadrature squeezing spectra elements appearing in the above equations, they have been written accordingly to the definitions in Eqs. (26) and (28).
Next we analyze the behavior of \( \tilde{V}_k(\omega) = V_k(\omega)/\langle \tilde{S}_0 \rangle \), with \( \langle \tilde{S}_0 \rangle = I_{1s} + I_{2s} \). We do this first at the bifurcations we have analyzed in Section 3, and then we analyze \( \tilde{V}_k(\omega) \) as a function of pump for a particular value of the detuning.

### A. Polarization squeezing at the bifurcations

Eqs. (47) simplify a lot at the bifurcation points [45] and (47) as at these particular pump values the steady state solution is the singlemode one, Eq. (13). Then, setting \( I_{2s} = 0 \) one immediately obtains

\[
\begin{align*}
\tilde{V}_0(\omega) &= \tilde{V}_1(\omega) = 1 + q_1(\phi_{1s}, \omega), \quad (47a) \\
\tilde{V}_2(\omega) &= 1 + |q_2(\phi_{1s}, \omega)|, \quad (47b) \\
\tilde{V}_3(\omega) &= 1 + |q_2(\phi_{1s} + \pi/2, \omega)|, \quad (47c)
\end{align*}
\]

where we have taken into account Eq. (41a).

The above expressions show that in the singlemode solution the variances of the two first Stokes parameters correspond to the quadrature fluctuations of the emitting (parallel) mode \( \alpha_1 \) at the particular quadrature angle \( \beta = \phi_{1s} \) (i.e., the intensity fluctuations as it must be) which, in general, is not the quadrature that exhibits less fluctuations. Contrarily, the last two Stokes parameters correspond to the quadrature fluctuations of the non-emitting (orthogonal) mode \( \alpha_2 \) at the particular quadrature angles \( \beta = \phi_{1s} \) and \( \beta = \phi_{1s} + \pi/2 \).

In this case, i.e., at the bifurcations, we can give the explicit expressions of the \( \tilde{V}_i(\omega) \) that read

\[
\begin{align*}
\tilde{V}_0(\omega) &= \tilde{V}_1(\omega) = 1 + \frac{4 |a_{1s}|^2 (\Delta - |a_{1s}|^2)}{C_1}, \quad (48a) \\
\tilde{V}_2(\omega) &= 1 + \frac{6 |a_{1s}|^2 (|a_{1s}|^2 + 2\Delta)}{C_2}, \quad (48b) \\
\tilde{V}_3(\omega) &= 1 + \frac{2 (|a_{1s}|^2 - \Delta)}{|a_{1s}|^2 + 2\Delta} \tilde{V}_2(\omega), \quad (48c)
\end{align*}
\]

where

\[
\begin{align*}
C_1 &= \left( \omega^2 + 1 - 3 |a_{1s}|^4 + 4 |a_{1s}|^2 \Delta - \Delta^2 \right)^2 \quad (49a) \\
&\quad + 4 \left( 3 |a_{1s}|^4 - 4 |a_{1s}|^2 \Delta + \Delta^2 \right), \\
C_2 &= \left( 2\omega^2 + 2 + |a_{1s}|^4 + |a_{1s}|^2 \Delta - 2\Delta^2 \right)^2 \quad (49b) \\
&\quad - 8 \left( |a_{1s}|^2 - \Delta \right) \left( |a_{1s}|^2 + 2\Delta \right),
\end{align*}
\]

with \( \Delta = \theta/\gamma \) and \( |a_{1s}|^2 = g I_{1s}/\gamma \), as introduced in Eq. (12a), and where \( \omega \) is normalized to \( \gamma \). Now the above expressions must be particularized for the different possible bifurcations affecting the singlemode solution.

We must now take into account that as \( I_{2s} = 0 \) at the bifurcations, at these points the mean values of the Stokes parameters are \( \langle \tilde{S}_0 \rangle = \langle \tilde{S}_1 \rangle = I_{1s} \) and \( \langle \tilde{S}_2 \rangle = \langle \tilde{S}_3 \rangle = 0 \), and then Eq. (45) implies that the only Stokes parameters that can be squeezed are \( \tilde{V}_2 \) and \( \tilde{V}_3 \), and for that they must verify

\[
\tilde{V}_i < \frac{\langle \tilde{S}_i \rangle}{\langle \tilde{S}_0 \rangle} < \tilde{V}_j, \quad i, j = 2, 3, \quad i \neq j.
\]

1. Polarization bifurcation

In this case \( I_{1s} = I_{1s, pol} \), Eq. (48) and it is not difficult to show that neither \( \tilde{V}_2 \) nor \( \tilde{V}_3 \) are squeezed. We saw in Subsection 5.1 that optimum quadrature squeezing occurs at this bifurcation for \( \psi = (\psi_{pol, opt}/2 + \phi_{1s}) \) with \( \psi_{pol, opt} \) tending to 0 for \( \Delta \to -\infty \) and to \(-\pi/2\) for \( \Delta \to \infty \). It is immediately seen from this that only for \( \Delta \to -\infty \) it could happen that \( \tilde{V}_2 \) be squeezed, see Eqs. (47) and Eq. (49), but it turns out that this never occurs for finite values of the detuning because, as commented in Subsection 5.1, the squeezing level is extremely sensitive to the \( \psi \) value. Then there is not any polarization squeezing at this bifurcation.

2. Bistability bifurcation

In this case \( I_{1s} = I_{1s, b} \), Eq. (10). The fact the optimum quadrature squeezing for the field \( \alpha_2 \) occurs for \( \beta = \phi_{1s} + \pi/2 \), see Subsection 5.2, makes that \( \tilde{V}_3 \) be squeezed, see Eqs. (47). Then in this particular case quadrature squeezing implies polarization squeezing. In Fig. 9 we represent the optimum (minimum) value of \( \tilde{V}_3 \) at the two bistability bifurcations, which occurs at the frequencies given in Subsection 5.2.

### B. Polarization squeezing as a function of pump

In this subsection we study the dependence of fluctuations on pump intensity for a particular value of the detuning. We have chosen \( \Delta = 1 \) as for this value the only bifurcation suffered by the singlemode solution is the polarization bifurcation (remember that \( \Delta > \sqrt{3} \) for bistability exists) and thus there is not polarization squeezing at the bifurcation (Section VII.A.1). Furthermore, from [10] we know that for \( \Delta = 1 \) there is not any secondary bifurcation affecting the bimode solution.

We concentrate on the fluctuations of the Stokes parameters of the bimode solution. In Fig. 10 the steady state intensity as well as the mean values of the Stokes parameters are represented as a function of pump, and in Fig. 11 the minimum value of the variances of the Stokes
parameters are represented together with the mean values of the Stokes parameters. We do not find it necessary to indicate at which frequency the minimum value of the variances is reached. Notice that polarization squeezing occurs when the represented variances fall below the mean value of the appropriate Stokes parameter, which is the one that is shown in each figure, see Eq. (23).

We see that in spite of the fact that no polarization squeezing exists at the bifurcation, as commented in the previous subsection, as pump is increased parameters \( S_1 \) and \( S_2 \) are squeezed within some pump value ranges (we note that \( V_3 \) increases its value for pump values larger than those represented in the figure). The observed polarization squeezing must be attributed in this case to the effect of the cross-mode spectra in Eqs. (??). We would like to note that in spite of the fact that there is always some parameter for which \( \tilde{V}_i < 1 \), see Fig. 11. Although this is not polarization squeezing, it is a fact that the output field exhibits less fluctuations than a coherent state and this could also be useful as it is a form of intensity squeezing.

These results are shown just to illustrate that the vectorial Kerr cavity exhibits polarization squeezing not only at the bifurcations and we have checked that this happens for other parameter values. As the behavior of the bimode solution can be quite involved we do not find it necessary to go in our analysis beyond this point.

VIII. CONCLUSIONS

In this article we have studied quadrature and polarization squeezing in a vectorial Kerr cavity model. After deriving Langevin equations for the quantum fluctuations from a suitable Fokker–Planck equation, namely that of the generalized \( P \)-distribution, we have derived expressions for the quantum fluctuations of the output field quadratures as a function of which we have expressed the spectrum of fluctuations of the output field Stokes parameters. These last expressions are particularly useful for understanding the connection between quadrature and polarization squeezing and facilitate the study of the latter.

Through the analysis of the quadrature and polarization fluctuations spectra, we have analyzed the conditions under which squeezing occur. We have concentrated mainly on the squeezing occurring at the bifurcations affecting the singlemode (linearly polarized) solution, but have considered also polarization squeezing in the bimode (elliptically polarized) solution in a particular case. In particular we have demonstrated that polarization squeezing occurs in this system, being specially large in the upper tangent (bistability) bifurcation of the singlemode solution.

Although the results we have shown have been derived for particular values of the Maker–Terhune coefficients (namely corresponding to the case of liquids), we have analyzed also other cases and have convinced ourselves that our results are qualitatively general: Except for the case \( B = 0 \) (that describes isotropic media in which the Kerr nonlinearity is solely due to electrostriction) in which the field \( \alpha_2 \) is a coherent vacuum irrespective of the pump value (the polarization instability does not exists in this case), for any \( B \neq 0 \) there exists polarization squeezing at the tangent bifurcations (of course the amount of squeezing does depend on the particular \( B \) value).

From these results we conclude that the vectorial Kerr cavity model (including both cross-phase modulation and four–wave mixing of the two orthogonally polarized intracavity modes) constitutes a particularly interesting generalization of the standard dispersive bistability model from the point of view of quantum fluctuations, as it predicts the existence of polarization squeezing, a phenomenon that, obviously, cannot be described with the standard model, and that exists whenever four–wave mixing occurs (i.e., whenever \( B \neq 0 \)). It is particularly interesting that polarization squeezing occurs in the singlemode solution, being optimum at the tangent (bistability) bifurcations, i.e. without the need of reaching the polarization bifurcation.

This work has been supported by the Spanish Ministerio de Educación y Ciencia (M.E.C.) and the European Union FEDER through Projects FIS2005-07931-C03-01 and -02. I.P.–A. also acknowledges financial support from the M.E.C. Juan de la Cierva programme.

1 R. Loudon and P. L. Knight, J. Mod. Opt. 34, 709 (1987).
2 P. Meystre and D. F. Walls eds., Non classical effects in quantum optics, (American Institute of Physics, New York, 1991).
3 P. D. Drummond and Z. Ficek eds., Quantum Squeezing, (Springer Verlag, Berlin, 2004).
4 S.L. Braunstein and P. van Loock, Rev. Mod. Phys. 77, 513 (2005).
5 N. Korolkova, G. Leuchs, R. Loudon, T.C. Ralph and Ch. Silberhorn, Phys. Rev. A 65, 052306 (2002).
6 J. Heersink, T. Gaber, S. Lorenz, O. Glöckl, N. Korolkova, and G. Leuchs, Phys. Rev. A 68, 013815 (2003).
7 A. Luís and N. Korolkova, Phys. Rev. A 74, 043817 (2006).
8 P. D. Drummond and D. F. Walls, J. Phys. A 13, 725 (1980).
9 K. Vogel and H. Risken, Phys. Rev. A 38, 2409 (1988); \textit{ibid} 39, 4675 (1989).
10 L.A. Lugiato, in \textit{Progress in Optics XXI}, ed. by E. Wolf (North Holland, Amsterdam, 1984).
11 L.A. Orozco, M.G. Raizen, M. Xiao, R.J. Brecha, and H.J. Kimble, J. Opt. Soc. Am. B 4, 1490 (1987).
12 D.F. Walls and G.J. Milburn, \textit{Quantum Optics} (Springer, Berlin, 1994).
13 J.B. Geddes, J.V. Molo.)
ne, E.M. Wright, and W.J. Firth, Opt. Commun. 111, 623 (1994).
15 R.W. Boyd, *Nonlinear Optics* (Academic Press, San Diego, 2003).
16 V.J. Sánchez-Morcillo, G.J. de Valcárcel, and E. Roldán, Opt. Commun. 173, 381 (2000).
17 M. Hoyuelos, P. Colet, M. San Miguel, and D. Walgraef, Phys. Rev. E 58, 2992 (1998).
18 R. Gallego, M. San Miguel, and R. Toral, Phys. Rev. E 61, 2241(2000).
19 V.J. Sánchez-Morcillo, I. Pérez–Arjona, F. Silva, G.J. de Valcárcel, and E. Roldán, Opt. Lett. 25, 957 (2000).
20 I. Pérez-Arjona, V.J. Sánchez-Morcillo, G.J. de Valcárcel and E. Roldán, J. Opt. B: Quantum Semiclass. Opt. 3, S118–S123 (2001).
21 R. Zambrini, M. Hoyuelos, A. Gatti, P. Colet, L. A. Lugiato, and M. San Miguel, Phys. Rev. A 62, 063801 (2002).
22 H. Carmichael, *Statistical Methods in Quantum Optics I: Master Equations and Fokker-Planck Equations* (Springer; Berlin, 1999).
23 P D Drummond and C W Gardiner, J. Phys. A: Math. Gen. 13, 2353 (1980).
24 F.V. García–Ferrer, I. Pérez–Arjona, G.J. de Valcárcel, and E. Roldán, J. Mod. Opt. 52, 763 (2005).
25 C.W. Gardiner and F. Zoller, *Quantum Noise*, (Springer; Berlin, 2000).
26 S. Chaturvedi, C. W. Gardiner, I.S. Matheson and D.F. Walls, J. Stat. Phys. 17, 469 (1977).
27 M.J. Collet and C.W. Gardiner Phys. Rev. A 30, 1386 (1984); ibid 30, 2887 (1985).
28 M. Born and E. Wolf, *Principles of Optics*, (Cambridge University Press, Cambridge, 1999).

Figure Captions

Fig. 1. (a) Representation on the \( \langle E^2, \Delta \rangle \) plane of the two tangent bifurcations (full lines) and the polarization bifurcation (dashed line). The lower (down) and upper (up) tangent bifurcations receive these names because they occur, respectively, in the lower and upper branches of the hysteresis cycle. In (b) the output intensities are shown as a function of pump \( E^2 \) for \( \Delta = 2 \). The dashed lines in (b) indicate unstable solutions. Another example of stationary solutions is shown, later, in Fig. 10(a).

Fig. 2. The optimally squeezed quadrature of the field \( \alpha_2 \) (orthogonally polarized with respect to the pump) at the polarization bifurcation is \( \beta = (\psi_{\text{pol, opt}}/2 + \varphi_1) \). \( \psi_{\text{pol, opt}} \) is represented as a function of detuning. In the inset, the squeezing spectrum of this quadrature is represented for \( \Delta = 0 \).

Fig. 3. Squeezing spectrum of the field \( \alpha_1 \) (the mode with parallel polarization with respect to that of the pump) of the optimally squeezed quadrature at the polarization bifurcation for the two values of the cavity detuning indicated in the figure.

Fig. 4. Optimum squeezing level (full line, left vertical axis) and the frequency at which this squeezing occurs (dashed line, right vertical axis) for the field \( \alpha_1 \) at the polarization bifurcation.

Fig. 5. Squeezing spectrum of the field \( \alpha_2 \) (the mode with orthogonal polarization with respect to that of the pump) of the optimally squeezed quadrature at the upper tangent bifurcation suffered by the field \( \alpha_1 \), for the two values of the cavity detuning indicated in the figure.

Fig. 6. Optimum squeezing level for the field \( \alpha_2 \) at the upper tangent bifurcation.

Fig. 7. As Fig. 5 but for the lower tangent bifurcation suffered by the field \( \alpha_1 \), for the two values of the cavity detuning indicated in the figure.

Fig. 8. Optimum squeezing level (full line, left vertical axis) and the frequency at which this squeezing occurs (dashed line, right vertical axis) for the field \( \alpha_2 \) at the lower tangent bifurcation.

Fig. 9. Normalized variance of the \( S_3 \) Stokes parameter at the upper and lower tangent bifurcations as a function of cavity detuning. The optimum squeezed value is shown.

Fig. 10. (a) Intensity values of the two intracavity modes and (b) normalized mean value of the Stokes parameters for \( \Delta = 1 \). The polarization instability is marked with an arrow in the horizontal axis. The full and dashed lines in (b) correspond to the two possible values of the Stokes parameters due to the two phase bistability of Eqs. for the \( \alpha_2 \) field.

Fig. 11. Minimum value of the normalized Stokes parameters’ variances (full lines) and mean values of the Stokes parameters (dashed lines) as a function of pump intensity for \( \Delta = 1 \). Parameter \( S_2 \) is squeezed within a small pump parameter domain values around \( E^2 = 3 \), as can be seen in (a) and (c), while \( S_1 \) is squeezed for \( E^2 > 5 \), as can be seen (c). For larger pump intensity values, polarization squeezing dissapears.
