SKEWNESS OF COSMIC MICROWAVE BACKGROUND
TEMPERATURE FLUCTUATIONS DUE TO
NON-LINEAR GRAVITATIONAL INSTABILITY

Dipak Munshi, Tarun Souradeep
Inter-University Center for Astronomy and Astrophysics,
Post Bag 4, Ganeshkhind, Pune 411007, India
munshi@iucaa.ernet.in
tarun@iucaa.ernet.in

and

Alexei A. Starobinsky
Yukawa Institute for Theoretical Physics, Kyoto University, Uji 611, Japan
and
Landau Institute for Theoretical Physics, Kosygina St. 2, Moscow 117334, Russia
alstar@landau.ac.ru

ABSTRACT

Skewness of temperature fluctuations of the cosmic microwave background (CMB) produced by initially Gaussian adiabatic perturbations with the flat (Harrison-Zeldovich) spectrum, which arises due to non-linear corrections to a gravitational potential at the matter-dominated stage, is calculated quantitatively. For the standard CDM model, the effect appears to be smaller than expected previously and lies below the cosmic variance limit even for small angles. The sign of the skewness is opposite to that of the skewness of density perturbations.

Subject headings: cosmology: theory — cosmic microwave background — large-scale structure of the Universe
1. Introduction

The reliable detection of fluctuations of the CMB temperature $\Delta T(\theta, \varphi)/T$ at large angular scales by the COBE group (with the data being in a very good agreement with a prediction made on the basis of the inflationary scenario of the early Universe 10 years before the detection) stimulates further investigation of subtler effects. Among the most important of them are possible deviations of the statistics of these fluctuations from the Gaussian one. The basic result which follows from all sufficiently simple variants of the inflationary scenario (contrary to that of rival theories based on topological defects such as cosmic strings, etc.) is that the statistics of the $\Delta T/T$ fluctuations is Gaussian because they are linearly connected to quantum vacuum fluctuations of a very weakly interacting scalar field (the inflaton). Thus, in the leading (linear) approximation, the mean CMB skewness $C_3(0) = \langle (\Delta T/T)^3 \rangle = 0$ where $\langle \rangle$ denotes averaging with respect to different realizations of stochastic space-time metric perturbations of the Friedmann-Robertson-Walker cosmological model which produce $\Delta T/T$. Note that the corresponding observable quantity is $\tilde{C}_3(0) = (4\pi)^{-1} \int (\Delta T(\theta, \varphi)/T)^3 \ d\Omega$, i.e., the average over the sky of one particular realization. Therefore, $\tilde{C}_3(0)$ generally differs from $C_3(0)$ by a so-called cosmic variance which is of the order of $(\langle (\Delta T/T)^2 \rangle)^{3/2}$ (see, e.g., Srednicki 1993). In practice, even $\tilde{C}_3(0)$ is unachievable because of a finite beam width of antennas, incomplete sky coverage, etc.

There exist different physical effects which may result in the appearance of a small, but non-zero mean skewness. Some of them are connected with non-linear corrections to the initial spectrum of scalar (adiabatic) metric perturbations which were generated during inflation (Falk et al. 1993, Gangui et al. 1994), the corresponding part of the CMB mean skewness may be called primordial. Other effects which take place after recombination (Luo & Schramm 1993) produce a secondary skewness. In this paper, we will concentrate on a detailed calculation of the secondary skewness produced by non-linear corrections to the primordial gravitational potential $\Phi$ which arises due to the same gravitational instability at the matter-dominated stage that leads to formation of galaxies and the large-scale structure of the Universe. The corresponding contribution to $\Delta T/T$ is contained in the non-local term of the Sachs-Wolfe expression for $\Delta T/T$ (Sachs & Wolfe 1967), it is also called the Rees-Sciama effect (Rees & Sciama 1968, a different view on this effect is presented in Zeldovich & Sazhin 1987).

The reason for our primary interest in this effect is that its contribution to $\Delta T/T$, though formally being of second order in powers of a small initial gravitational
potential $\Phi(0) = \phi_0(r)$, is not much less than the main linear effect. Its value may be estimated as $\Delta T/T \sim \Phi \delta \rho/\rho$ where $\delta \rho/\rho$ is the present rms density perturbation inside some linear characteristic size $L$. For $L \sim R_{eq} \approx 30h^{-1}\text{Mpc}$ corresponding to the angular scale $\vartheta \sim LH_0/c \sim 30'$ (if viewed from the redshift $z \sim 1$), it is $\sim \phi_0^2 z_{eq}$ where $z_{eq} \approx 4 \cdot 10^4 h^2 \kappa^{-1}$ is the redshift of matter-radiation energy equality; therefore, it may reach $10^{-6}$ (Martinez-Gonzales, Sanz & Silk 1992). Here $h$ is the value of the Hubble constant $H_0$ in units of 100 km/s/Mpc and $\kappa = \epsilon_{\text{rad}}/\epsilon_{\gamma} = 1.68$ for 3 types of light ($m \ll 1$ eV) neutrinos with standard concentrations.

On the basis of the above argument, it was conjectured (Luo & Schramm 1993) that the secondary CMB skewness produced by the non-local part of the Sachs-Wolfe effect will dominate the primordial skewness generated in the inflationary scenario. However, our calculations presented in the next section show that this is not the case for the standard CDM model. The secondary skewness appears to be small and of the order of primordial skewness. A possibility of getting a larger skewness without introducing late-time phase transitions is discussed in the Sec. 3. The section also contains our conclusions, as well as comparison with the cosmic variance of the skewness in the case of purely Gaussian perturbations. Details of our calculations are displayed in Appendices A and B.

2. Method of calculation

For large angular scales $\vartheta \geq 2^\circ$, CMB temperature fluctuations produced by adiabatic perturbations are given by the Sachs-Wolfe formula:

$$\frac{\Delta T}{T}(\theta, \varphi) = \left(\frac{\Delta T}{T}\right)_{loc} + \left(\frac{\Delta T}{T}\right)_{non-loc} = \frac{1}{3} \Phi(r_0, \theta, \varphi) + 2 \int_{\eta_{rec}}^{\eta_0} \left(\frac{\partial \Phi(\eta, r)}{\partial \eta}\right)_{r=\eta_0-\eta} d\eta,$$

$$r = (r, \theta, \varphi), \quad r_0 = \eta_0 - \eta_{rec}, \quad \eta = \int \frac{dt}{a(t)} $$

(1)

where $a(t)$ is the scale factor of the FRW model, $\eta_0$ is the present conformal time and $\eta_{rec}$ is the conformal recombination time, $\eta_0/\eta_{rec} \approx z_{rec}^{1/2} \sim 30$. More exact calculation for the CDM model with 3 light neutrino species and $z_{rec} = 1100$ gives $\eta_0/\eta_{rec} = 49.6$. At the matter-dominated stage in the absence of spatial curvature and the cosmological constant, $a(t) \propto t^{2/3}$, $\eta = 3t/a(t)$. For smaller angles $5' < \vartheta < 2^\circ$ where standard recombination may be still considered as instantaneous, additional local terms in Eq. (1) appear which describe the Doppler and the Silk
effects (see, e.g., Starobinsky 1988), but the non-local term remains the same. In the linear approximation, this term is exactly zero for any power-law \( a(t) \), though it may contribute significantly if \( a(t) \) deviates from a power-law behaviour after recombination, e.g., due to the cosmological constant (Kofman & Starobinsky 1985), decaying relativistic particles (Kofman, Pogosyan & Starobinsky 1986) or spatial curvature (Wilson 1983, Abbott & Schaeffer 1986). However, for the purely matter-dominated stage in the flat Universe that we consider, the non-local term produces a non-zero contribution to \( \Delta T/T \) if non-linear corrections to the gravitational potential \( \Phi \), arising due to gravitational instability in the Universe (see, e.g., Peebles 1980), are taken into account. These corrections are also responsible for the appearance of a non-zero skewness of CMB in the case of a Gaussian initial (linear) potential.

So, we expand the (peculiar) gravitational potential into a series in powers of a density enhancement: \( \Phi(r, t) = \Phi^{(1)} + \Phi^{(2)} + \ldots \). The linear term \( \Phi^{(1)} = \phi_0(r) \) is assumed to be Gaussian. Further, we consider the case of a flat (Harrison-Zeldovich) initial spectrum. Then the two-point correlation function of the linear potential is given by

\[
\xi_\phi(r) = \langle \phi_0(0)\phi_0(r) \rangle = \int_0^\infty P_\phi(k) \frac{\sin kr}{kr} k^2 dk = B \int c^2(k) \frac{\sin kr}{kr} \frac{dk}{k},
\]

where \( c(k) \) is the standard transfer function of the CDM model. For physical scales \( R \equiv ra(t_0) \gg R_{eq} \), \( \xi_\phi(r) \approx \xi_\phi(0) - B \ln(r/r_{eq}) \) (strictly speaking, the constant \( \xi_\phi(0) \) is infinite but it is unobservable, so no difficulties arise). The constant \( B \) is related to the amplitude normalization \( A \) introduced in Starobinsky 1983 by the relation \( B = 9A^2/200\pi^2 \). The \( rms \) CMB quadrupole value is expressed through it by \( Q_{rms-PS}^2/T^2 = 5B/108 \), so that \( B = 1.16 \times 10^{-9} \) for the presently preferred values \( T = 2.726 \, K \) and \( Q_{rms-PS} = 20 \, \mu K \) (see, e.g., Gorski et al. 1994).

The second-order term is given by (Peebles 1980, notations of the paper by Munshi & Starobinsky 1994 are used below):

\[
\Phi^{(2)} = \frac{\eta^2}{42} (5\Delta^{-1}P + Q);
\]

\[
P(\vec{r}) = (\Delta \phi_0)^2 + \nabla \phi_0 \nabla (\Delta \phi_0) = \nabla (\nabla \phi_0 \Delta \phi_0), \quad Q(\vec{r}) = (\nabla \phi_0)^2.
\]

To get a finite expression for \( C_3(0) \), it is sufficient to use the “renormalized” quantity:

\[
\Phi^{(2)}_{ren}(\vec{r}, \eta) = \Phi^{(2)}(\vec{r}, \eta) - \langle \Phi^{(2)}(\vec{r}, \eta) \rangle,
\]

(the last term in \( [3] \) depends on \( \eta \) only) that corresponds to the subtraction of an unobservable constant (monopole) term from \( \Delta T/T \). More rigorous justification of
this prescription follows from the fact that the difference $\Phi(r, \eta) - \Phi(r_0, \eta)$, where $r_0$ denotes an observer (e.g., our) location, is an observable and finite quantity both in linear and non-linear regimes. Further, we omit the subscript “ren”. Note that the subtraction (4) does not remove the whole contribution to $C_3(0)$ from monopole terms, in each of $\Delta T/T$ some finite part remains.

A non-zero contribution to the CMB mean skewness in the lowest (fourth) order in $\Phi$ is given by

$$C_3^{(4)}(0) = 3\langle (\frac{\Delta T}{T})^2_{loc} (\frac{\Delta T}{T})_{non-loc} \rangle .$$

(5)

Let us now remove a monopole component from each of $\Delta T/T$ completely:

$$\left(\frac{\Delta T}{T}\right)_S = \left(\frac{\Delta T}{T}\right) - \frac{1}{4\pi} \int \frac{\Delta T}{T} d\Omega .$$

(6)

Then the term (3) can be represented as

$$C_3^{(4)}(0) = I_{UUU} - I_{UUM} - 2I_{UMU} + 2I_{UMM} + I_{MMU} - I_{MMM}$$

(7)

where the subscript $U$ means substitution of an unsubtracted $\Delta T/T$ into (5) and $M$ - substitution of a monopole term there (the gravitational potential in both terms is assumed to be renormalized according to (4)). Strictly speaking, a dipole component should be subtracted from $\left(\frac{\Delta T}{T}\right)_S$ further, but this correction appears to be smaller than the correction due to the monopole subtraction (3) (e.g., $I_{UUD} \approx 0.74 I_{UUM}$, see Appendix A) and practically does not change the final result for $C_3(0)$.

The main term in the right-hand side of Eq. (3) is $I_{UUU}$. After substitution of (5) into Eq. (3), it takes the form

$$I_{UUU} = \frac{4}{63} \int_{\eta_{rec}}^{\eta_0} \eta \left(\xi_\phi^2 + 5\Delta^{-1}((\Delta \xi_\phi)^2 + \xi_\phi^2(\Delta \xi_\phi)')\right)_{r=\eta-\eta_{rec}} d\eta$$

(8)

where $\xi_\phi$ is given in Eq. (2) and the prime means derivative with respect to $r$. Note that for $r >> r_{eq}$, $\Delta^{-1}((\Delta \xi_\phi)^2 + \xi_\phi^2(\Delta \xi_\phi)') = -\frac{B^2}{2\eta}$. Thus, if we are speaking about the skewness of temperature fluctuations smoothed over a scale $R_a > R_{rec} = a(\eta_0)\eta_{rec}$ (e.g., due to a finite antenna beam width), then the integral in (8) diverges at both large and small $\eta$ logarithmically: $I_{UUU} = -\frac{2}{21} B^2 \ln \frac{R_a}{R_{eq}}$, where $R_a = a(\eta_0)\eta_0$ is a present cosmological horizon of the FRW model. This means that the growth of $\left(\frac{\Delta T}{T}\right)_{non-loc}$ in (3) at late times due to $\Phi(2)$ is exactly cancelled by decay of correlations between $\left(\frac{\Delta T}{T}\right)_{loc}$ and $\left(\frac{\Delta T}{T}\right)_{non-loc}$ at large spatial separations. Another conclusion following from the behaviour of the integrand in Eq. (8) is that the contribution of the small redshift region $z \sim 1$, $\eta \sim \eta_0$ to the skewness is
subdominant, it is smaller than the total effect in a few \((\sim 1/\ln(\eta_0/\eta_{\text{rec}}))\) times. That is why higher-order non-linear corrections do not noticeably change the fourth order result for the skewness (we check it explicitly for the leading sixth order term, see Appendix B). Clearly, the same statement is true for the case of a blue-tilted initial spectrum \(n > 1\). However, in the opposite case of a significantly red-tilted spectrum \((n \approx 0.9)\) higher-order non-linear corrections may become crucial since the main effect originates at recent times.

In terms of the multipole decomposition, this behaviour corresponds to equal contributions to \(C_3(0)\) from each logarithmic interval of \(l\) for \(l \gg 1\). In other words, if \(C_3(0) = \sum_l C_l\), then \(C_l = -\frac{2B^2}{21(l+0.5)}\) for \(l \gg 1\). Using the monopole value \(C_0\) obtained below (see Appendix A and Eq. (1)), this fit may be made better by changing \((l + 0.5)\) to \((l + 1/3)\). So, if we take the smoothing angle \(\theta_{\text{FWHP}} = 10^\circ\) as in the smoothed COBE maps \((\theta_g = 0.425\theta_{\text{FWHP}} = 1/13.5)\), then

\[
C_3(10^\circ) = -0.16B^2, \quad \langle (\Delta T(10^\circ))^3 \rangle = 4.4 (\mu K)^3
\]

(9)

(with monopole and dipole terms subtracted). Since \(C_2(10^\circ) = 0.18B\), the smoothed large-angle skewness parameter \(S_3(10^\circ) \equiv \frac{C_3(10^\circ)}{C_2(10^\circ)} \approx -5\).

For the unsmoothed distribution of temperature fluctuations, the main contribution to \(I_{UUU}\) comes from the vicinity of the recombination surface \(\eta - \eta_{\text{rec}} \approx \eta_{\text{eq}}\) as a result of properties of the transfer function \(c(k)\). However, this distance is still significantly larger than the thickness of the recombination surface. Hence, the latter may be neglected for our problem. The corresponding angular range is \(10^\prime - 30^\prime\). Here lies the first acoustic, or Doppler peak, so the first term alone in the right-hand side of Eq. (1) should be corrected by accounting for the Doppler and the Silk effects (but not the second one). As a result, we obtain (details of the calculation are presented in the Appendix A):

\[
I_{UUU} \approx -2.2 B^2.
\]

(10)

Other terms in the right-hand side of (1) may be calculated in the approximation \(\eta_{\text{rec}} = 0, c(k) = 1\). From symmetry considerations, \(I_{UMM} = I_{MMU} = I_{MMM}\). It is shown in the Appendix A that \(I_{UUM} = -\frac{2}{21}(\frac{2}{8} - \ln 2)B^2 \approx -0.05B^2\). Other auxiliary integrals entering into Eq. (7) are also given there. The dipole contribution can be estimated to be \(\sim -0.07B^2\) using the abovementioned fit \(C_l = -\frac{2B^2}{21(l+1/3)}\). As a result,

\[
C_3^{(4)}(0) = I_{UUU} - I_{UUM} - 2(I_{UMU} - I_{UMM}) - C_1
\]
\[\langle (\Delta T)^2 \rangle = -2.2B^2 + 0.05B^2 + 0.24B^2 + 0.07B^2 \approx -1.8B^2,\]
\[\langle (\Delta T)^3 \rangle \approx 50 (\mu K)^3. \] (11)

With a good accuracy, \[\sigma_T^2 \equiv C_2(0) = \langle \left(\frac{\Delta T}{T}\right)^2 \rangle \approx 10^{-9} \approx B \] where \[C_2(\vartheta) \equiv \langle \frac{\Delta T}{T}(0) \frac{\Delta T}{T}(\vartheta) \rangle \] is the 2-point angular temperature correlation function. Therefore, the skewness parameter \[S_3(0) \equiv \frac{C_3(0)}{C_2(0)} \sim -2 \] is of the same order as the primordial skewness considered in Gangui et al. 1994. The unsmoothed value of the skewness \[C_3(0) \sim 11\] times more than \[C_3(10^\circ)\] because the main effect comes from the angles \[10' \sim 30'. \] However, the unsmoothed skewness parameter \[S_3(0)\] is smaller than \[S_3(10^\circ)\] because fluctuations themselves (i.e. \[C_2\]) are significantly larger at small angles.

Note the negative sign of \[S_3. \] Its physical explanation is that it reflects the existence of large regions with \(\Phi\) positive and growing with time (corresponding to voids) which produce positive \(\left(\frac{\Delta T}{T}\right)_{\text{non-loc}}\) and relatively small regions of negative and decreasing \(\Phi\) which produce smaller cold spots but with larger absolute values of \(\frac{\Delta T}{T}\). It is these cold spots that make the main contribution to the skewness and determine its sign. The sign is opposite to the sign of a skewness of density perturbations \(\frac{\delta \rho}{\rho}\). In the latter case, the main contribution to the skewness is produced by regions with \(\frac{\delta \rho}{\rho} > 0\) which are smaller in volume but larger in amplitude of \(\frac{\Delta T}{T}\).

As noted above, the smallness of \(|S_3|\) is due to the fact that the region where \(|\Phi^{(2)}|\) along a photon trajectory becomes comparable to \(|\phi_0| (z \sim 1, \eta \sim \eta_0)\) is widely spatially separated from the recombination surface \(\eta = \eta_{\text{rec}}\) where \(\left(\frac{\Delta T}{T}\right)_{\text{loc}}\) is located. Thus, to check that there is no other significant contribution to \(|S_3|\), we consider sixth-order terms that need not possess this property. The most important of them is the following term:

\[C_3^{(6)}(0) = \langle \left(\frac{\Delta T}{T}\right)_{\text{non-loc}}^3 \rangle, \] (12)

where \(\left(\frac{\Delta T}{T}\right)_{\text{non-loc}}\) is calculated with the use of \(\Phi^{(2)}\). The main contribution to \(\langle \rangle\) is mainly produced at recent times \(\eta \sim \eta_0\) and it is not attenuated by a small value of the spatial correlation function. However, it contains one more power of \(B\) as compared to \(\langle \rangle\). The term \(\langle \rangle\) can be represented as

\[C_3^{(6)}(0) = -\frac{16n_0^4}{423} \frac{8}{(2\pi)^7} \int \int \int d^3k_1d^3k_2d^3k_3 \phi^2(k_1)\phi^2(k_2)\phi^2(k_3)\delta(k_{1z} - k_{2z})\delta(k_{2z} - k_{3z})\delta(k_{1z} - k_{3z})\times M(k_1, -k_2)M(k_2, -k_3)M(k_3, -k_1)(13)\]
where the kernel $M(k_1,k_2)$ is defined in Eq. (A8) below (see Appendix B for a detailed derivation). Numerical evaluation of this integral gives

$$C_3^{(6)}(0) = -2.15 \times 10^6 B^3, \quad S_3^{(6)} \approx -10^{-3}. \quad (14)$$

Thus, the sixth order contribution is much smaller than the fourth order one.

3. Conclusions

We have calculated the mean CMB skewness generated due to leading non-linear corrections to an initially Gaussian adiabatic perturbations with the flat spectrum. The unsmoothed value $S_3(0) \approx -2$ that we found lies much below the cosmic variance of this quantity $\delta S_3 \approx (\sigma_T l_c)^{-1} \approx 130$ (here we model $C_2(\theta)$ by the Gaussian $C_2 = \sigma_T^2 \exp \left(-\frac{(\theta l_c)^2}{2}\right)$ with $\sigma_T = 3 \cdot 10^{-5}$ and $l_c = 250$ corresponding to the first Doppler peak). The situation for smoothed large-angle maps is even worse (e.g., $S_3 \approx -5$ but $\delta S_3 \approx 3600$ for $\theta_{FWHM} = 10^\circ$). Therefore, as regarding observations, the prediction is that there should be no noticeable mean skewness above the noise level due to cosmic variance in the case of the standard CDM model with the flat (Harrison-Zeldovich) initial spectrum of adiabatic perturbations. This conclusion seems to be in a good agreement with existing data (Hinshaw et al. 1994, Kogut et al. 1994). It may be considered as one more confirmation of predictions of the inflationary scenario, though, of course, this fact does not close the way for other theories leading to the same prediction.

It is clear from our derivation that $|S_3|$ can be substantially larger if there exists a first order contribution to $\left(\frac{\Delta T}{T}\right)_{non-loc}$ at late times ($z \sim 1$), because then correlations between first and second order terms in $C_3(0)$ are not small. It may happen, as mentioned above, for the flat CDM+Λ cosmological model and other ones. We shall consider this case elsewhere.

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A. Fourth order contribution

We outline some of the details of our estimation of the skewness. The lowest order term leading to skewness (4th. order in $\phi_0$) is given by

$$C_3^{(4)}(0) = \frac{2}{63} \int_{\eta_{rec}}^{\eta_0} d\eta \frac{\partial}{\partial \eta} \langle \phi_0^2(r_1)\Phi^{(2)}(r_2) \rangle, \quad |r_1| = r_0, \quad r_2 = r_1 \frac{\eta_0 - \eta}{r_0},$$

(A1)

where $\phi_0$ and $\Phi^{(2)}$ represent values of potential fluctuations at the linear and second order, respectively, and $\Phi^{(2)}$ is regularized according to Eq. (4); $r_0$ is defined in Eq. (1). In the paper, we follow the notation $\phi = \phi_U - \phi_M - \phi_D$ and $\Phi^{(2)} = \Phi_U - \Phi_M - \Phi_D$, where the subscripts $U$, $M$ and $D$ represent unsubtracted, monopole and dipole fields, correspondingly.

The CMB mean skewness at the lowest order can be then expressed (see Eq. (7)) as

$$C_3^{(4)}(0) = I_{UUU} - I_{UUM} - I_{UUD} - 2I_{UMU} + 2I_{UMM} - I_{MMU} + 2I_{UMD} - I_{MMD}$$

(A2)

where we have ignored terms involving $\phi_D$ since they are expected to be small. In addition, we also expect that $I_{UMD}$ and $I_{MMD}$ are smaller in magnitude.

All the terms contain an average of the type $\langle \phi_0(r_1)\phi_0(r_1')\Phi^{(2)}(r_2) \rangle$ which can be expressed in terms of the potential-potential correlation function $\xi_\phi$ and its first derivative $\xi'_\phi$ as

$$\langle \phi_0(r_1)\phi_0(r_1')\Phi^{(2)}(r_2) \rangle = 2\nabla \xi_\phi(r_1 - r_2) \nabla \xi_\phi(r_1' - r_2) + 5\Delta^{-1} \left[ 2\xi'_\phi(r_1 - r_3)\xi'_\phi(r_1' - r_3) 
\right. \\
\left. + \nabla \xi_\phi(r_1 - r_3) \nabla \xi'_\phi(r_1' - r_3) + \nabla \xi'_\phi(r_1 - r_3) \nabla \xi_\phi(r_1' - r_3) \right].$$

The terms $I_{UUM}$ and $I_{UUD}$ involve a simpler expression with $r_1 \equiv r_1'$ and can be easily evaluated in the coordinate space. We find that the terms $I_{UUM}$, $I_{UUD}$, $I_{UMU}$ and $I_{UMM}$ are all insensitive to the form of the transfer function $c(k)$ and may be evaluated in the approximation $c(k) \equiv 1$, $\eta_{rec} = 0$. 
The term $I_{UUM}$ and $I_{UUD}$ involve the monopole and dipole term of $\Phi(2)$, respectively and can be expressed as

\[ I_{UUM} = \frac{2}{63} \frac{1}{4\pi} \int d\Omega_2 \int_{\eta_{rec}}^{\eta_0} d\eta \frac{\partial}{\partial \eta} \langle \Phi^2(r_1) \Phi^2(r_2) \rangle |_{r_1 = r_0, \ r_2 = \eta_0 - \eta}, \]
\[ I_{UUD} = \frac{2}{63} \frac{3}{4\pi} \int d\Omega_2 \cos \theta_2 \int_{\eta_{rec}}^{\eta_0} d\eta \frac{\partial}{\partial \eta} \langle \Phi^2(r_1) \Phi^2(r_2) \rangle |_{r_1 = r_0, \ r_2 = \eta_0 - \eta}. \quad (A3) \]

Assuming a scale invariant spectrum and $c(k) \equiv 1$, $\eta_{rec} = 0$, we set $\xi_{\phi}(r) \approx \xi_{\phi}(0) - B \ln(r/r_{eq})$ to reduce the expressions to

\[ I_{UUM} = -\frac{2B^2}{21} \frac{1}{4\pi} \int_0^{\eta_0} d\eta \int \frac{d\Omega}{r^2}, \]
\[ I_{UUD} = -\frac{2B^2}{21} \frac{3}{4\pi} \int_0^{\eta_0} d\eta \int \frac{\cos \theta}{r^2} d\Omega \quad (A4) \]

where $r^2 = (\eta_0 - \eta)^2 + \eta_0^2 - 2\eta_0(\eta_0 - \eta) \cos \theta$.

Performing the integration over angles, we get

\[ I_{UUM} = -\frac{B^2}{21} \int_0^1 \frac{x \ dx}{1-x} \ln \left( \frac{2-x}{x} \right) = -\frac{B^2}{21} \left( \frac{\pi^2}{8} - \ln 2 \right) \approx -0.0515B^2, \]
\[ I_{UUD} = -\frac{B^2}{7} \int_0^1 \frac{x \ dx}{(1-x)^2} \left[ \frac{(1-x)^2 + 1}{2} \ln \left( \frac{2-x}{x} \right) - 1 + x \right] = -\frac{B^2}{7} \left( \frac{3}{2} - \frac{\pi^2}{8} \right) \approx -0.0380B^2(A5) \]

Now we consider the terms $I_{UMU}$ and $I_{UMM}$ which involve the monopole term of $\phi_0$ and can be written as

\[ I_{UMU} = \frac{2}{63} \frac{1}{4\pi} \int d\Omega_1 \int_{\eta_{rec}}^{\eta_0} d\eta \frac{\partial}{\partial \eta} \langle \phi_0(r_1) \phi_0(r_1') \Phi^2(r_2) \rangle, \quad |r_1| = |r_1'| = r_0, \ r_2 = r_1 \frac{\eta_0 - \eta}{r_0}; \]
\[ I_{UMM} = \frac{2}{63} \frac{1}{16\pi^2} \int d\Omega_1 \int d\Omega_2 \int_{\eta_{rec}}^{\eta_0} d\eta \frac{\partial}{\partial \eta} \langle \phi_0(r_1) \phi_0(r_1') \Phi^2(r_2) \rangle, \quad |r_1| = |r_1'| = r_0, \ |r_2| = \eta_0 - \eta. \quad (A6) \]

It is more convenient to evaluate these terms in the momentum space. In the $k$-space, the unsubtracted, monopole and dipole terms for the temperature fluctuation along the line of sight $n$ can be expressed as

\[ \left( \frac{\Delta T}{T} \right)^{(1)}_U = \frac{1}{3} \int \frac{d^3k}{(2\pi)^3} \phi_0(k) e^{ikn \eta_0}, \]
where we have ignored the self-coupling term $\phi$ does not change in a dust-dominated, flat FRW universe. The function $j_T/T_n$ is just cancelled after the renormalization (4). Note that within the scope of this paper the first order temperature fluctuation $(\Delta T/T)^{(1)} \equiv (\Delta T/T)_{loc}$ and the second order temperature fluctuation $(\Delta T/T)^{(2)} \equiv (\Delta T/T)_{non-loc}$, owing to the fact that the linear gravitational potential does not change in a dust-dominated, flat FRW universe.

In evaluating expectation values in the $k$-space we invoke the Gaussian nature of the initial potential perturbations $\phi_0$ and implement the following relation

$$
\langle \phi_0(k_1)\phi_0(k_2)\Phi^{(2)}(k) \rangle \rightarrow \langle \phi_0(k_1)\phi_0(k_2)\phi_0(k_3)\phi_0(k_4) \rangle = \phi^2(k_1)\phi^2(k_2)
\times (\delta(k_1 + k_3) \delta(k_2 + k_4) + \delta(k_1 + k_4) \delta(k_2 + k_3)) ,
$$

$$
\phi^2(k) = \frac{2\pi^2 B_c k}{k^3} ,
$$

where we have ignored the self-coupling term $\phi^2(k_1) \delta(k_1 + k_2)\langle \Phi^{(2)}(k) \rangle$ because it is just cancelled after the renormalization (4).

The expressions for the two terms now read

$$
I_{UMU} = -\frac{B^2}{63} \int_{\eta_{rec}}^{\eta_0} d\eta \eta \int \frac{d^3k_1}{2\pi} \int \frac{d^3k_2}{2\pi} \frac{c^2(k_1)}{k_1^3} \frac{c^2(k_2)}{k_2^3} M(k_1, k_2) e^{-ik_1n(\eta-\eta_{rec})} e^{ik_2n(\eta_0-\eta)} j_0(k_2r_0) ,
$$
\[ I_{UMM} = -\frac{B^2}{63} \int_{\eta_{rec}}^{\eta_0} d\eta \int_{0}^{\infty} dk_1 c^2(k_1) j_1(k_1(\eta-\eta_{rec})) \int_{0}^{\infty} dk_2 c^2(k_2) j_0(k_2r_0) j_1(k_2(\eta_0-\eta)) \]

We find it convenient to split each of the terms into two pieces involving the local part \( Q \) and the non-local part \( P \) part of \( \Phi^{(2)} \), respectively. After carrying out some of the integrations analytically, the terms can be expressed as

\[ I_{UMUQ} = -\frac{4B^2}{63} \int_{\eta_{rec}}^{\eta_0} d\eta \int_{0}^{\infty} dk_1 c^2(k_1) j_1(k_1(\eta-\eta_{rec})) \int_{0}^{\infty} dk_2 c^2(k_2) j_0(k_2r_0) j_1(k_2(\eta_0-\eta)) \]

\[ I_{UMUP} = -\frac{5B^2}{63} \int_{\eta_{rec}}^{\eta_0} d\eta \int_{0}^{\infty} dk_1 c^2(k_1) \int_{0}^{\infty} dk_2 c^2(k_2) j_0(k_2r_0) \]

\[ \times \int_{-1}^{1} du \left[ \frac{(k_1^2 + k_2^2)u + 2k_1k_2}{k_1^2 + k_2^2 + 2k_1k_2u} \right] j_0 \left( \sqrt{k_1^2(\eta-\eta_{rec})^2 + k_2^2(\eta_0-\eta)^2 - 2k_1k_2(\eta-\eta_{rec})(\eta_0-\eta)u} \right) \]

\[ I_{UMMQ} = \frac{4B^2}{63} \int_{\eta_{rec}}^{\eta_0} d\eta \int_{0}^{\infty} dk c^2(k) j_0(kr_0) j_1(k(\eta_0-\eta)) \]

\[ I_{UMMP} = -\frac{5B^2}{63} \int_{\eta_{rec}}^{\eta_0} d\eta \int_{0}^{\infty} dk_1 c^2(k_1) j_0(k_1r_0) \int_{0}^{\infty} dk_2 c^2(k_2) j_0(k_2r_0) \]

\[ \times \int_{-1}^{1} du \left[ \frac{2k_1k_2 + (k_1^2 + k_2^2)u}{k_1^2 + k_2^2 + 2k_1k_2u} \right] j_0 \left( (\eta_0-\eta)\sqrt{k_1^2 + k_2^2 + 2k_1k_2u} \right). \tag{A11} \]

The expressions are further simplified if we take \( c(k) \equiv 1 \) and \( \eta_{rec} = 0 \) to obtain the final results

\[ I_{UMUQ} = -\frac{2B^2}{63} \int_{0}^{1} dx \left[ 1 + \frac{x(x-2)}{x-1} \tanh^{-1}(1-x) \right] = -\frac{2B^2}{63} (2 \ln 2 - 1) \approx -0.0123B^2, \]

\[ I_{UMUP} = -0.23B^2, \quad I_{UMU} = I_{UMUQ} + I_{UMUP} = -0.24B^2 \tag{A12} \]

and

\[ I_{UMMQ} = \frac{B^2}{63} \int_{0}^{1} dx \frac{dx}{(1-x)^2} \left[ 1 + \frac{x(x-2)}{x-1} \tanh^{-1}(1-x) \right]^2 \approx 7.30 \times 10^{-4}B^2, \]

\[ I_{UMMP} \approx -0.12B^2, \quad I_{UMM} = I_{UMMQ} + I_{UMMP} = -0.12B^2. \tag{A13} \]
We have numerically verified the above terms including the CDM transfer function and value of $\eta_{\text{rec}}$.

Now we deal with the most significant term $I_{UUU}$ which depends sensitively on small-scale power in the radiation perturbation spectrum and transfer functions for radiation (see below). As noted before, for the angular range $5' < \vartheta < 2^\circ$ that roughly corresponds to the scale range $10 \text{ Mpc} < a(\eta_0)(kh)^{-1} < 200 \text{ Mpc}$, all three effects - the Sachs-Wolfe, the Doppler and the Silk ones - should be taken into account in the local term $(\Delta T/T)_{\text{loc}}$. Moreover, only the sum of the three effects appears to be a gauge-invariant quantity. On the other hand, the recombination may be still considered as instantaneous (the width of the recombination surface in terms of the conformal time $\Delta \eta_{\text{rec}} \ll k^{-1}$). More exactly, the effect of the non-zero width $\Delta \eta_{\text{rec}}$ may be empirically accounted by the increase of the scale of Silk damping. Therefore, we use (partly unpublished) results of Starobinsky & Sahni 1984 and Sahni 1984 (see also Starobinsky 1987, 1988) obtained in the two-fluid approximation of the CDM model with radiation. Namely, the limiting case $\Omega_\gamma = 0$ was first considered and the matter before recombination was assumed to consist of two ideal fluids interacting through gravity only: dust with pressure $p = 0$ representing cold dark matter and radiation with $p = \epsilon/3$ representing photons (tightly coupled with baryons) and other massless particles. After the recombination, photons are described as free massless particles. This approach is equivalent to that used in Seljak 1994. Also, we assume that $\eta_{\text{eq}} \ll \eta_{\text{rec}}$. However, a small, but non-zero value of $\Omega_b$ should be finally taken into account because it is crucial for the determination of the Silk damping scale. Also, we accounted for it in the value of the sound velocity in the radiation component before recombination to get the right location of the acoustic peaks.

We use the following values in actual caculations: $H_0 = 50 \text{ km/s/Mpc}$, $T = 2.726 \text{ K}$, $\kappa = 1.681$, $\Omega_{\text{tot}} = 1$, $\Omega_b = 0.06$, $z_{\text{rec}} = 1100$. The scale factor of the two-fluid (dust and radiation) CDM model has the form $a(\eta) = a_1(\eta_1 + \eta)$ where $\eta_1 = 2(\sqrt{2} + 1)\eta_{\text{eq}} = 4.828\eta_{\text{eq}}$. Now we can find values of $\eta_{\text{eq}}$, $\eta_1$ and $\eta_{\text{rec}}$ using the present-day value of $\Omega_\gamma = 0.9895 \times 10^{-4}$:

$$\frac{\eta_0}{\eta_1} \approx (2\sqrt{\kappa\Omega_\gamma})^{-1}(1 - \sqrt{\kappa\Omega_\gamma}) = 38.3 \ , \ \frac{\eta_0}{\eta_{\text{eq}}} = 185 \ , \ \eta_{\text{eq}} = (\kappa\Omega_\gamma)^{-1} = 6012 \ ,$$

$$\frac{\eta_{\text{rec}}}{\eta_1} = \frac{-1 + \sqrt{1 + (\kappa\Omega_\gamma z_{\text{rec}})^{-1}}}{2} = 0.771 \ , \ \frac{\eta_0}{\eta_{\text{rec}}} = 49.6 \ . \ (A14)$$

It is clear that the approximation $\eta_{\text{eq}} \ll \eta_{\text{rec}}$ has $\approx 15\%$ accuracy.

Then it can be shown (see, e.g. Starobinsky 1988) that, in the approximation
outlined above, \((\Delta T/T)_\text{loc}\) has the following structure:

\[
\frac{\Delta T}{T}_\text{loc}(k) = \frac{1}{3} \tilde{\phi}_0(k) \left( F(k) + 3i c_s \frac{kn}{k} G(k) \right) \exp \left( -\frac{k^2 R_S^2}{2} \right),
\]

\[
F(k) = f(k) \cos(kc_s \eta_{\text{rec}}) - g(k) \sin(kc_s \eta_{\text{rec}}),
\]

\[
G(k) = f(k) \sin(kc_s \eta_{\text{rec}}) + g(k) \cos(kc_s \eta_{\text{rec}}),
\]

\[
c_s = \frac{1}{\sqrt{3(1 + R)}} = 0.486, \quad R \equiv \left( \frac{3 \Omega_b}{4 \Omega_\gamma} \right)_{\text{rec}} = 0.413
\]  \( (A15) \)

where \(\tilde{\phi}_0(k)\) is the initial perturbation spectrum, i.e. \(\tilde{\phi}_0(k) = \phi_0(k)/c(k)\), \(c_s\) is the sound velocity in the radiation component coupled with baryons (in terms of the light velocity) and \(R_S\) is the Silk damping scale. Following Bond 1988, we assume \(R_S = 12.4\) Mpc (for the values of \(H_0\) and \(\Omega_b\) given above). This value of \(R_S\) partly accounts for an increase of the Silk damping scale during recombination, and the final result for temperature fluctuations appears to be in a good agreement with numerical calculations using the exact kinetic approach (Bond & Efstathiou 1987, Scott & White 1994, Sugiyama 1994). The dependence of the transfer functions \(f(k)\) and \(g(k)\) on \(\Omega_b\) is weak and we neglect it. Their form in the limit \(\Omega_b \to 0\) calculated in Sahni 1984 may be well approximated by the following analytical fits in terms of the variable \(x = k \eta_1 = ka^{-1}(\eta_0) \times 309\) Mpc:

\[
f(x) = \begin{cases} 
1 + 0.042 x^2 & x \leq 1.55 \\
0.8991 + 0.1302 x & 1.55 < x \leq 9.379 \\
\frac{5x}{x + 12.74} & x > 9.379
\end{cases}
\]

\[
g(x) = \frac{3 \ln \left(1 + \frac{x}{10}\right)}{1 + \frac{x}{10} + \frac{3x^2}{400}}.
\]  \( (A16) \)

The oscillating terms in \((A15)\) are the well-known acoustic oscillations in the baryon-photon plasma (see, e.g., Zeldovich & Novikov 1983 and references therein).

Substituting \((A15)\) into Eq. (5), we get after some algebra:

\[
I_{UUU} = -\frac{4}{63} \int_{\eta_{\text{rec}}}^{\eta_0} d\eta \frac{1}{(2\pi)^6} \int \frac{d^3 k_1}{k_1^3} \int \frac{d^3 k_2}{k_2^3} M(k_1, k_2) e^{-i(k_1 + k_2) n(\eta - \eta_{\text{rec}})}
\]

\[
\times (2\pi^2 B)^2 c(k_1) c(k_2) e^{-\frac{1}{2}(k_1^2 + k_2^2) R_S^2} \left( F(k_1) - 3i c_s \frac{k_1 n}{k_1} G(k_1) \right) \left( F(k_2) - 3i c_s \frac{k_2 n}{k_2} G(k_2) \right)
\]

\[
= -\frac{B^2}{63} \int_{\eta_{\text{rec}}}^{\eta_0} d\eta \int_0^{\infty} dk_1 \int_0^{\infty} dk_2 \int_{-1}^{1} du \left( 2u + \frac{5(2k_1 k_2 + (k_1^2 + k_2^2) u)}{k_1^2 + k_2^2 + 2k_1 k_2 u} \right) c(k_1) c(k_2) e^{-\frac{1}{2}(k_1^2 + k_2^2) R_S^2}
\]
\[ \times [F(k_1)F(k_2)j_0(w) - 3c_4 (F(k_1)G(k_2)(k_1u + k_2) + G(k_1)F(k_2)(k_1 + k_2u)) \frac{j_1(w)(\eta - \eta_{\text{rec}})}{w} \\
+ 9c_4^2G(k_1)G(k_2) \left( \frac{j_2(w)(\eta - \eta_{\text{rec}})^2(k_1 + k_2u)(k_1u + k_2)}{w^2} - \frac{j_1(w)u}{w} \right) ] , \]

\[ w = \sqrt{k_1^2 + k_2^2 + 2k_1k_2u (\eta - \eta_{\text{rec}})} . \quad (A17) \]

Calculating this integral numerically, we obtain

\[ I_{UUU} \approx -2.2 \, B^2. \quad (A18) \]

Note for comparison that if we did not take into account the increase of \( (\frac{\Delta T}{T})_{\text{loc}} \)
due to the Silk and Doppler effects, i.e. if we calculated the integral \( (A17) \) in the limit \( f(k) \equiv 1, \ g(k) \equiv 0 \) (but with the exact \( c(k) \)) we would get the answer

\[ I_{UUU} = -0.9 \, B^2. \] So, the account of all effects increase the answer by a factor of 2.4.

\[ \text{B. Sixth order contribution} \]

Beyond the fourth order term, the next contribution to the skewness arises at the sixth order in \( \phi_0 \). In this section, we outline the calculation for the most significant of the sixth order terms, \( \langle (\Delta T/T)^{(2)}_U (\Delta T/T)^{(2)}_U (\Delta T/T)^{(2)}_U \rangle \). Substituting the expression for \( (\Delta T/T)^{(2)}_U \) from \( (A8) \) and \( (A9) \) we obtain

\[ C_3^{(6)}(0) = -\frac{64}{(42)^3} \int_{\eta_{\text{rec}}}^{\eta_0} d\eta_1 d\eta_1 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k'_1}{(2\pi)^3} [5P(k_1, k'_1) + Q(k_1, k'_1)]e^{i(k_1 \cos \theta_1 + k'_1 \cos \theta'_1)(\eta_0 - \eta_1)} \]

\[ \times \int_{\eta_{\text{rec}}}^{\eta_0} d\eta_2 d\eta_2 \int \frac{d^3k_2}{(2\pi)^3} \int \frac{d^3k'_2}{(2\pi)^3} [5P(k_2, k'_2) + Q(k_2, k'_2)]e^{i(k_2 \cos \theta_2 + k'_2 \cos \theta'_2)(\eta_0 - \eta_2)} \]

\[ \times \int_{\eta_{\text{rec}}}^{\eta_0} d\eta_3 d\eta_3 \int \frac{d^3k_3}{(2\pi)^3} \int \frac{d^3k'_3}{(2\pi)^3} [5P(k_3, k'_3) + Q(k_3, k'_3)]e^{i(k_3 \cos \theta_3 + k'_3 \cos \theta'_3)(\eta_0 - \eta_3)} \]

\[ \times \langle \phi(k_1) \phi(k'_1) \phi(k_2) \phi(k'_2) \phi(k_3) \phi(k'_3) \rangle , \quad (B1) \]

Assuming the initial potential fluctuations to be a gaussian random field (as predicted by most inflationary scenarios) we have

\[ \langle \phi(k_1) \phi(k'_1) \phi(k_2) \phi(k'_2) \phi(k_3) \phi(k'_3) \rangle = \phi^2(k_1) \phi^2(k_2) \phi^2(k_3) \]
\[ \times (\delta(k_1 + k'_2) \delta(k_2 + k'_3) \delta(k_3 + k'_1) + 7 \text{ permutations}) \]  

(B2)

where we ignore the self-coupling terms. We assume a scale-invariant initial spectrum with \( \phi^2(k) = 2\pi^2 B c^2(k)/k^3 \), where \( B \) is a normalisation constant, and introduce the notations \( u \equiv \cos \theta_1, v \equiv \cos \theta_2 \) and \( w \equiv \cos \theta_3 \) for brevity. The expressions can be simplified by making the approximation

\[ \int_{\eta_0}^{\eta_0} \eta_1 d\eta_1 \int_{\eta_0}^{\eta_0} \eta_2 d\eta_2 \int_{\eta_0}^{\eta_0} \eta_3 d\eta_3 e^{ik_1(\eta_2-\eta_1)u} e^{ik_2(\eta_3-\eta_2)v} e^{ik_3(\eta_1-\eta_3)w} \approx \int_{0}^{\eta_0^2} \eta_1^3 \times \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ e^{ik_1 ux} e^{ik_2(y-x)v} e^{-ik_3 wy} \]

\[ \left( \frac{\pi^2}{16} \right)^2 \delta(k_1 u - k_2 v) \delta(k_2 v - k_3 w) = \pi^2 \eta_0^4 \delta(k_{1z} - k_{2z}) \delta(k_{2z} - k_{3z}) \]  

(B3)

where \( x = \eta_2 - \eta_1, \ y = \eta_3 - \eta_1 \). Substituting (B3) and (B2) into expression (B1), \( C_3^{(6)}(0) \) can be written as (13).

As in the case of the fourth order calculations, it is convenient to split \( C_3^{(6)}(0) \) into pieces involving the local and the non-local parts of \( \Phi^{(2)} \). We express

\[ C_3^{(6)}(0) = \mathcal{I}_{QQQ} + 3\mathcal{I}_{QQP} + 3\mathcal{I}_{QP P} + \mathcal{I}_{PPP}. \]  

(B4)

Splitting the expression for \( C_3^{(6)}(0) \) as described by (B4), we obtain the following results for the constituent terms:

\[ \mathcal{I}_{QQQ} = 4 \left( \frac{B}{21} \right)^3 \int dk \ k^3 \ c^2(k) \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta \int_0^1 du \int_0^1 dv \int_0^1 dw \ c^2(k \frac{v}{u}) \ c^2(k \frac{w}{v}) \ \frac{v^2}{uw} \times \cos \theta_1 \ \cos \theta_2 \ \cos \theta_3 = 1.4 \times 10^6 B^3, \]  

(B5)

\[ \mathcal{I}_{QQP} = 10 \left( \frac{B}{21} \right)^3 \int dk \ k^3 \ c^2(k) \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta \int_0^1 du \int_0^1 dv \int_0^1 dw \ c^2(k \frac{v}{u}) \ c^2(k \frac{w}{v}) \ \frac{v^2}{uw} \times \left[ \frac{r^2 \cos \theta_1 - 2uv}{r^2 - 2uv \cos \theta_1} \right] \cos \theta_2 \ \cos \theta_3 = -8.1 \times 10^5 B^3, \]  

(B6)

\footnote{Our calculations can be trivially extended to non-scale-invariant spectra by incorporating the appropriate form for \( \phi^2(k) \).}
\[ I_{QPP} = 25 \left( \frac{B}{21} \right)^3 \int dk \ k^3 \ c^2(k) \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta \int_0^1 du \int_0^1 dv \int_0^1 dw \ c^2\left( \frac{kv}{u} \right) c^2\left( \frac{kv}{w} \right) \frac{v^2}{uw} \times \left[ \frac{r^2 \cos \theta_1 - 2uv}{r^2 - 2uv \cos \theta_1} \right] \left[ \frac{s^2 \cos \theta_2 - 2uv}{s^2 - 2uv \cos \theta_2} \right] \cos \theta_3 = 1.19 \times 10^7 B^3, \quad (B7) \]

\[ I_{PPP} = \frac{1}{2} \left( \frac{5B}{21} \right)^3 \int dk \ k^3 \ c^2(k) \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta \int_0^1 du \int_0^1 dv \int_0^1 dw \ c^2\left( \frac{kv}{u} \right) c^2\left( \frac{kv}{w} \right) \frac{v^2}{uw} \times \left[ \frac{r^2 \cos \theta_1 - 2uv}{r^2 - 2uv \cos \theta_1} \right] \left[ \frac{s^2 \cos \theta_2 - 2uv}{s^2 - 2uv \cos \theta_2} \right] \left[ \frac{t^2 \cos \theta_3 - 2uv}{t^2 - 2uv \cos \theta_3} \right] = -3.68 \times 10^7 B^3, \quad (B8) \]

where we have used the following notations

\[
\begin{align*}
\cos \theta_1 &= uv + \bar{u} \bar{v} \cos \alpha, \quad \cos \theta_2 = uv + \bar{u} \bar{v} \cos \beta, \quad \cos \theta_3 = wv + \bar{w} \bar{v} \cos \beta, \\
r^2 &= u^2 + v^2, \quad s^2 = v^2 + w^2, \quad t^2 = w^2 + u^2.
\end{align*}
\quad (B9)
\]

We have evaluated the expressions (B5), (B6), (B7) and (B8) numerically to estimate the contribution to the skewness at the 6th order in the potential fluctuation \( \phi_0 \).
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