The Calabi functional on a ruled surface

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Abstract

We study the Calabi functional on a ruled surface over a genus two curve. For polarisations which do not admit an extremal metric we describe the behaviour of a minimising sequence splitting the manifold into pieces. We also show that the Calabi flow starting from a metric with suitable symmetry gives such a minimising sequence.

1 Introduction

In [3] Calabi introduced the problem of minimising the $L^2$-norm of the scalar curvature (this is called the Calabi functional) over metrics in a fixed Kähler class on a compact Kähler manifold. A critical point of the Calabi functional is called an extremal metric. The Euler-Lagrange equation is that the gradient of the scalar curvature is a holomorphic vector field. It is known that extremal metrics in fact minimise the Calabi functional (see [13], [1], [9]). Recently much progress has been made in understanding when extremal metrics exist, at least on a conjectural level. Kähler-Einstein metrics are a special case and when the first Chern class of the manifold is positive (the manifold is called Fano in this case), Yau conjectured that the existence of Kähler-Einstein metrics is related to the stability of the manifold in the sense of geometric invariant theory. In the case of negative or zero first Chern class Yau [20] and Aubin [2] have shown that Kähler-Einstein metrics always exist, answering a conjecture of Calabi. Tian [18] made significant progress towards understanding the Fano case, solving it completely in the case of surfaces in [17]. Donaldson [6] showed that the scalar curvature can be interpreted as a moment map (this was also observed by Fujiki [10]) and this enabled extending the conjectures about the existence of Kähler-Einstein metrics to more general constant scalar curvature and extremal metrics (see [7], [14], [15]).

In this paper we look at what we can say about minimising the Calabi functional in a Kähler class which admits no extremal metric, concentrating on
a concrete example. Let $\Sigma$ be a genus 2 curve and $\mathcal{M}$ a degree -1 line bundle on it. We consider the ruled surface $X = \mathbb{P}(\mathcal{M} \oplus \mathcal{O})$ with a family of polarisations $L_m = C + mS_\infty$, where $C$ is the class of a fibre, $S_\infty$ is the infinity section (with self-intersection 1), and $m > 0$. Technically we should take $m$ to be rational, especially when discussing test-configurations, but by an approximation and continuity argument we can take $m$ to be real. The aim is to study the problem of minimising the Calabi functional in these Kähler classes. Our main result is the following.

**Theorem 1.** There exist constants $k_1 \simeq 18.9, k_2 \simeq 5.03$, such that

1. If $0 < m < k_1$ then $X$ admits an extremal metric (this is due to Tønnesen-Friedman [19]).

2. If $k_1 \leq m \leq k_2(k_2+2)$ then there exists a minimising sequence of metrics which breaks $X$ into two pieces and converges to complete extremal metrics on both.

3. If $m > k_2(k_2+2)$ then there exists a minimising sequence of metrics which breaks $X$ into three pieces. It converges to complete extremal metrics on two of these and the third degenerates into a fibration of infinitely long and infinitely thin cylinders.

To construct metrics on our ruled surface, we use the momentum construction given in Hwang-Singer [12]. This construction has been used repeatedly in the past to find special metrics on ruled manifolds, in particular extremal metrics. See [1] for a unified treatment of these constructions or [12] for a historical overview and more references. The momentum construction allows us to construct circle invariant metrics from functions on an interval and it gives a convenient expression for the scalar curvature. More precisely, let $\phi : [0, m] \to \mathbb{R}$ be a smooth function, positive on the interior $(0, m)$, vanishing at the endpoints, and such that $\phi'(0) = 2, \phi'(m) = -2$. The momentum construction gives a metric $\omega_\phi$ in the Kähler class $L_m$, with scalar curvature

$$S(\omega_\phi) = \frac{-2}{1+\tau} - \frac{1}{2(1+\tau)}[(1+\tau)\phi]'' \cdot [1]$$

Here $\tau$ is the moment map for the $S^1$-action on the fibres and working with this coordinate is the central idea of the momentum construction. We will recall this construction in Section [2]. Of particular importance to us is the fact that we can consider momentum profiles which vanish on a subset of $(0, m)$. These
correspond to degenerate metrics and they arise as the limits of the minimising sequences in Theorem 1.

In Section 3 we consider the problem of directly minimising the Calabi functional on the set of metrics obtained by the momentum construction. Since the $L^2$-norm of the scalar curvature is equivalent to the $H^2$-norm of the momentum profiles, this is straightforward. We find that the Euler-Lagrange equation for a minimiser $\phi$ is $\phi S(\phi)'' = 0$ and $S(\phi)'$ must be a negative distribution, ie. $S(\phi)$ is concave. We show that a unique minimiser exists in each Kähler class and its momentum profile is in $C^2$. Note that $S(\phi)'' = 0$ is the equation for $\phi$ to define an extremal metric.

In Section 4 we explicitly construct the minimisers, which can be degenerate in the sense that the momentum profiles can vanish on a subset of $(0, m)$. Here we will see the three different kinds of behaviour stated in Theorem 1. In Section 5 we construct test-configurations for $X$ and calculate their Futaki invariants. This will clarify the role of the concavity of $S(\phi)$ for minimisers of the Calabi functional. In fact, rational, piecewise-linear convex functions on $[0, m]$ give test-configurations essentially by the construction in [7] as generalised to bundles of toric varieties in [10]. We can approximate $-S(\phi)$ by such functions, and Donaldson’s theorem on lower bounds for the Calabi functional in [9] shows that $-\omega_\phi$ actually achieves the infimum of the Calabi functional on the whole Kähler class, not just the metrics arising from the momentum construction. This will complete the proof of Theorem 1.

An alternative approach to minimising the Calabi functional is using the Calabi flow introduced in [3]. This is the flow which deforms the Kähler potential in the direction of the scalar curvature. It is expected (see [7], [8]) that the Calabi flow should minimise the Calabi functional and if there is no extremal metric in a given Kähler class, then it should break up the manifold into pieces which admit complete extremal metrics or collapse in some way. In Sections 6 and 7 we will verify this, showing

**Theorem 2.** *If the initial metric is given by the momentum construction then the Calabi flow exists for all time and converges to the minimiser of the Calabi functional.*

The Calabi flow on ruled manifolds has been previously studied in [11], where the long time existence and convergence is proved for the Kähler classes which admit an extremal metric. We use similar techniques, the main difference being that we introduce some variants of the Mabuchi functional when no extremal metric exists. In particular in the unstable case where $k_1 \leq m \leq k_2(k_2 + 2)$
we define a functional which decreases along the Calabi flow, is bounded below, and whose derivative is given by the difference between the Calabi functional and its infimum. This leads to the convergence result. The case $m > k_2(k_2 + 2)$ is more delicate since the analogous Mabuchi-type functional is not bounded from below. Nevertheless it has at worst logarithmic decay along the Calabi flow and this is enough to show that the flow minimises the Calabi functional. This is discussed in Section 6.

Note that throughout the paper we have ignored factors of $2\pi$, for example in the definition of the Calabi functional. Also we normalise the Futaki invariant slightly differently from usual in Section 5. Hopefully this will lead to no confusion.

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2 Metrics on the ruled surface

In this section we describe the momentum construction for metrics on the ruled surface (see Hwang-Singer [12]). Let $X$ be the ruled surface as above, so that $X = P(M \oplus \mathcal{O}) \to \Sigma$, where $\Sigma$ is a genus 2 curve, and $M$ is a degree -1 line bundle over $\Sigma$. Let $\omega_\Sigma$ be a metric on $\Sigma$ with area $2\pi$ and constant scalar curvature $-2$. Also, let $h$ be a Hermitian metric on $M$ with curvature form $i\omega_\Sigma$. We consider metrics on the total space of $M$ of the form

$$\omega = p^*\omega_\Sigma + 2i\partial\bar{\partial}f(s),$$

where $p : M \to \Sigma$ is the projection map, $s = \frac{1}{2} \log |z|^2_h$ is the logarithm of the fibrewise norm and $f(s)$ is a suitable strictly convex function that makes $\omega$ positive definite. The point of the momentum construction is the change of coordinate from $s$ to $\tau = f'(s)$. The metric $\omega$ is invariant under the $U(1)$-action on $M$, and $\tau$ is just the moment map for this action. Let $I \subset \mathbb{R}$ be the image of $\tau$, and let $F : I \to \mathbb{R}$ be the Legendre transform of $f$. By definition this means that

$$f(s) + F(\tau) = s\tau,$$
and $F$ is a strictly convex function. The *momentum profile* is defined to be the function

$$
\phi(\tau) = \frac{1}{F''(\tau)}.
$$

We have the following relations:

$$
s = F'(\tau), \quad \frac{ds}{d\tau} = F''(\tau), \quad \phi(\tau) = f''(s).
$$

**The metric in local coordinates**

Let us now see what the metric $\omega$ looks like in local coordinates. Choose a local coordinate $z$ on $\Sigma$ and a fibre coordinate $w$ for $M$. The fibrewise norm is given by $|zw|^2 h(z)$ for some positive function $h$, so that

$$
s = \frac{1}{2} \log |w|^2 + \frac{1}{2} \log h(z).
$$

We can choose the local trivialisation of $M$ in such a way that at a point $(z_0, w_0)$ we have $d \log h(z) = 0$. We can then compute at the point $(z_0, w_0)$

$$
2i\partial\bar{\partial} f(s) = if'(s)d\partial \log h(z) + f''(s)\frac{i dw \wedge d\bar{w}}{2|w|^2} = \tau p^* \omega_\Sigma + \phi(\tau) \frac{i dw \wedge d\bar{w}}{2|w|^2}.
$$

The metric at the point $(z_0, w_0)$ is therefore given by

$$
\omega = (1 + \tau)p^* \omega_\Sigma + \phi(\tau) \frac{i dw \wedge d\bar{w}}{2|w|^2}. \quad (1)
$$

In order to compute the scalar curvature of $\omega$, note that the determinant of the metric $g$ corresponding to $\omega$ is

$$
det(g) = \frac{1}{|w|^2} (1 + \tau)\phi(\tau) \det(g_\Sigma),
$$

which is valid for all points, not just $(z_0, w_0)$. The Ricci form at $(z_0, w_0)$ is given by

$$
\rho = -i\partial\bar{\partial} \log \det g
$$

$$
= p^* \rho_\Sigma - \frac{[(1 + \tau)\phi]'}{2(1 + \tau)} p^* \omega_\Sigma - \frac{\phi}{2} \frac{(1 + \tau) (1 + \tau) \phi'' - [(1 + \tau)\phi]'^2}{(1 + \tau)^2} \frac{i dw \wedge d\bar{w}}{|w|^2},
$$

where the derivatives are all with respect to $\tau$ (note that $d/ds = \phi(\tau) d/d\tau$) and $\rho_\Sigma$ is the Ricci form of the metric $\omega_\Sigma$. Taking the trace of this, we find that the scalar curvature $S(\omega)$ is given by

$$
S(\omega) = \frac{-2}{1 + \tau} - \frac{1}{2(1 + \tau)} [(1 + \tau)\phi]''. \quad (2)
$$
In [12] the extendability of the metrics to the projective completion of \( \mathcal{M} \) is studied. The proposition we need is the following.

**Proposition 3** (see [12]). For some \( m > 0 \) let \( \phi : [0, m] \rightarrow \mathbb{R} \) be a smooth function such that \( \phi \) is positive on \((0, m)\), and

\[
\phi(0) = \phi(m) = 0, \quad \phi'(0) = 2, \quad \phi'(m) = -2. \tag{3}
\]

Then the momentum construction defines a smooth metric \( \omega_\phi \) on \( X \) in the Kähler class \( C + mS_\infty \), with scalar curvature \( S(\omega)(\tau) \) given by Equation 2. Here \( C \) is the class of a fibre, and \( S_\infty \) the infinity section.

If instead \( \phi \) satisfies the boundary conditions

\[
\phi(0) = \phi(m) = 0, \quad \phi'(0) = 0, \quad \phi'(m) = -2,
\]

and \( \phi(\tau) \leq O(\tau^2) \) for small \( \tau \), then the momentum construction gives a complete metric with finite volume on the complement of the zero section in \( X \). Similarly if \( \phi'(0) = 2 \) and \( \phi'(m) = 0 \) then we obtain a complete metric on the complement of the infinity section.

The metrics are extremal, i.e., their scalar curvature has holomorphic gradient, when \( S(\phi)'' = 0 \).

Let us also note the definition

**Definition 4.** A momentum profile is a \( C^2 \) function \( \phi : [0, m] \rightarrow \mathbb{R} \) which is positive on \((0, m)\) and satisfies the boundary conditions (3). A singular momentum profile is the same except we only require it to be non-negative instead of positive, i.e. it can vanish on a subset of \((0, m)\).

Let us write \( \Phi \) for the unique solution of \( S(\Phi)'' = 0 \) satisfying the same boundary conditions as a momentum profile. Then \( \Phi \) is positive on \((0, m)\) precisely when the polarisation admits an extremal metric. We define the Calabi functional to be

\[
\text{Cal}(\phi) = \int_0^m (S(\phi) - S(\Phi))^2 (1 + \tau) \, d\tau = \int_0^m \frac{1}{4(1 + \tau)} [(1 + \tau)(\Phi - \phi)]''^2 \, d\tau.
\]

This differs from the \( L^2 \)-norm of \( S(\phi) \) by a constant, since

\[
\int_0^m (S(\phi) - S(\Phi)) S(\Phi) (1 + \tau) \, d\tau = \int_0^m \frac{1}{2} [(1 + \tau)(\Phi - \phi)]'' S(\Phi) \, d\tau = 0,
\]
integrating by parts, so

\[ \text{Cal}(\phi) = \int_0^m S(\phi)^2 (1 + \tau) d\tau - \int_0^m S(\Phi)^2 (1 + \tau) d\tau. \]

Throughout the paper when we integrate a function over \( X \) which only depends on \( \tau \) we will often use the volume form \((1+\tau)d\tau\). From the formula (1) we see that this is a constant multiple of the integral with respect to the volume form \( \omega^2 \). Because of the boundary conditions on \( \phi \) the Poincaré inequality shows that the Calabi functional is equivalent to the \( H^2 \)-norm of \( \phi \). This makes it easy to minimise the Calabi functional directly as we do in the next section.

3 Minimising the Calabi functional

It is fairly simple to directly minimise the Calabi functional on the set of metrics which are given by momentum profiles. We introduce the set of functions

\[ A = \left\{ \phi : [0, m] \to \mathbb{R} \mid \phi \in H^2, \ \phi \geq 0 \text{ and } \phi \text{ satisfies the boundary conditions in Proposition 3} \right\}, \]

and we want to minimise the Calabi functional on this space. Let us choose a minimising sequence \( \phi_k \in A \). We have a bound \( \|\phi_k\|_{H^2} \leq C \cdot \text{Cal}(\phi_k) \), so we can choose a subsequence converging weakly to some \( \phi \in H^2 \). Weak convergence in \( H^2 \) implies convergence in \( C^1 \) so the boundary conditions and non-negativity hold in the limit, ie. \( \phi \in A \). Moreover \( \text{Cal} \) is lower-semicontinuous because the \( H^2 \)-norm is, so \( \phi \) is the required minimiser.

**Proposition 5.** The minimiser \( \phi \) in \( A \) satisfies \( \phi S(\phi)'' = 0 \) and \( S(\phi)'' \) is a negative distribution. In particular \( S(\phi) \) is continuous, so \( \phi \in C^2 \). Conversely if \( \psi S(\psi)' = 0 \) and \( S(\psi) \) is concave, then \( \psi = \phi \).

**Proof.** The variation of \( \text{Cal} \) at \( \phi \) is given by

\[ D\text{Cal}_\phi(\tilde{\phi}) = -\int_0^m (S(\phi) - S(\Phi)) \left[(1 + \tau)\tilde{\phi}''\right]'' d\tau. \]

We are considering variations inside \( A \), so \( \tilde{\phi} \) and its first derivative vanishes at the endpoints. We can therefore integrate by parts, and find that

\[ -\int_0^m S(\phi)''\tilde{\phi}(1 + \tau) d\tau \geq 0 \]

7
for all \( \tilde{\phi} \) such that \( \phi + \epsilon \tilde{\phi} \in A \) for small enough \( \epsilon \). We can choose \( \tilde{\phi} \) to be an arbitrary non-negative smooth function which vanishes along with its first derivative at the endpoints. This shows that \( S(\phi)'' \) is a negative distribution. On the open set where \( \phi \) is positive we can choose \( \tilde{\phi} \) to be negative or positive, so it follows that \( S(\phi)'' = 0 \) at these points. Therefore \( \phi S(\phi)'' = 0 \) on \( (0, m) \).

The continuity of \( S(\phi) \) follows from it being concave, and this implies that \( \phi \in C^2 \).

The converse follows from the following computation.

\[
\text{Cal}(\psi) \leq \text{Cal}(\psi) + \int_0^m (S(\phi) - S(\psi))^2 (1 + \tau) \, d\tau
\]

\[
= \text{Cal}(\phi) + 2 \int_0^m (S(\psi) - S(\phi)) S(\psi) (1 + \tau) \, d\tau
\]

\[
= \text{Cal}(\phi) + \int_0^m [(1 + \tau)\phi - (1 + \tau)\psi]'' S(\psi) \, d\tau
\]

\[
= \text{Cal}(\phi) + \int_0^m \phi S(\psi)'' (1 + \tau) \, d\tau
\]

\[
\leq \text{Cal}(\phi).
\]

Since \( \text{Cal}(\phi) \) is minimal we must have equality, i.e.

\[
\int_0^m (S(\phi) - S(\psi))^2 (1 + \tau) \, d\tau = 0.
\]

This implies that \( S(\phi) = S(\psi) \), from which it follows that \( \phi = \psi \). \( \square \)

4 Explicit minimisers

In this section we compute explicitly the minimisers of the Calabi functional for all polarisations. For each \( m \) we are looking for a singular momentum profile (Definition 4) such that \( S(\phi)'' = 0 \) whenever \( \phi \) does not vanish, and in addition \( S(\phi) \) is concave.

There are three cases to consider depending on the polarisation.

Case 1. There exists an extremal metric, \( m < k_1 \approx 18.889 \)

In this case we want to solve the equation \( S(\phi)'' = 0 \). By the Formula (2) for the scalar curvature, this is the ODE

\[
\frac{1}{2(1 + \tau)}(-4 - [(1 + \tau)\phi]'') = A\tau + B,
\]
for some constants $A, B$. Rearranging this and integrating twice we obtain

$$(1 + \tau)\phi = \frac{-A\tau^4}{6} - \frac{(A + B)\tau^3}{3} - B\tau^2 - 2\tau + C\tau + D,$$  \hspace{1cm} (4)$$

where $C$ and $D$ are also constants. The boundary conditions on $\phi$ on the interval $[0, m]$ give a system of linear equations on $A, B, C, D$ which we can solve to obtain

$$\phi(\tau) = \frac{2\tau(m - \tau)}{m(m^2 + 6m + 6)(1 + \tau)} \left[ \tau^2(2m + 2) + \tau(-m^2 + 4m + 6) + m^2 + 6m + 6 \right].$$

This will give a metric when it is positive on the interval $(0, m)$ which happens if and only if the quadratic expression in square brackets is positive on this interval. This is the case for $m < k_1$ where $k_1$ is the only positive real roof of the quartic $m^4 - 16m^3 - 52m^2 - 48m - 12$. Approximately $k_1 \simeq 18.889$, which is the result obtained by Tønnessen-Friedman [19]. See Figure 1 for a graph of $\phi(\tau)$ for $m = 17$.

![Figure 1: Momentum profile of an extremal metric on $X$ when $m = 17$.](image)

**Case 2. $X$ breaks up into two pieces, $k_1 \leq m \leq k_2(k_2 + 2) \simeq 35.33$**

When $m \geq k_1$ we can no longer find a positive solution of $S(\phi)'' = 0$ on the whole interval $[0, m]$ so we split the interval into two pieces $[0, c]$ and $[c, m]$. 

We would like to find $\phi$ which vanishes at $c$, but on the intervals $(0,c)$ and $(c,m)$ we have $S(\phi)'' = 0$, and $S(\phi)$ is concave on $[0,m]$. We first let $\phi_1$ be the solution of the equation

$$S(\phi_1)'' = 0 \text{ on the interval } (0,c)$$

$$\phi_1(0) = \phi_1(c) = 0, \quad \phi_1'(0) = 2, \quad \phi_1'(c) = 0.$$ 

We obtain

$$\phi_1(\tau) = \frac{2\tau(c - \tau)^2}{c^2(c^2 + 6c + 6)(1 + \tau)} \left[\tau(-c^2 + 2c + 3) + c^2 + 6c + 6\right].$$

This is positive on $(0,c)$ if the linear expression in square brackets is positive on this interval. This happens for $c \leq k_2$ where $k_2$ is the only positive real root of the cubic $c^3 - 3c^2 - 9c - 6$. Approximately $k_2 \approx 5.0275$. The scalar curvature is given by

$$S(\phi_1) = \frac{12(c^2 - 2c - 3)}{c^2(c^2 + 6c + 6)} \tau - \frac{6(2c^3 - c - 4)}{c(c^2 + 6c + 6)}.$$

To deal with the interval $[c,m]$ we first solve the equation

$$S(\psi)'' = 0 \text{ on the interval } (0,d)$$

$$\psi(0) = \psi(d) = 0, \quad \psi'(0) = 0, \quad \psi'(d) = -2.$$ 

for some constant $d$, and then shift the solution to $[c,m]$. The solution on $[0,d]$ is given by

$$\psi(\tau) = \frac{2\tau^2(d - \tau)}{d^2(d^2 + 6d + 6)(1 + \tau)} \left[\tau(2d^2 + 4d + 3) - d^3 + 3d^2 + 9d + 6\right].$$

As before, this is positive on $(0,d)$ if the linear term in square brackets is positive on this interval. This is the case for $d \leq k_2$, for the same $k_2$ as above. The scalar curvature is given by

$$S(\psi) = \frac{12(2d^2 + 4d + 3)}{d^2(d^2 + 6d + 6)} \tau - \frac{6(3d^2 + 5d + 2)}{d(d^2 + 6d + 6)}.$$

Now note that if we define $\phi_2$ by

$$\phi_2(\tau) = (c + 1) \psi \left(\frac{\tau - c}{c + 1}\right),$$

then $\phi_2$ solves the equation

$$S(\phi_2)'' = 0 \text{ on the interval } (c,(c + 1)d + c)$$

$$\phi_2(c) = \phi_2((c + 1)d + c) = 0, \quad \phi_2'(c) = 0, \quad \phi_2'((c + 1)d + c) = -2.$$
The scalar curvature is given by

\[ S(\phi_2)(\tau) = \frac{1}{c + 1} S(\psi) \left( \frac{\tau - c}{c + 1} \right). \]

We now define \( \phi \) by

\[ \phi(\tau) = \begin{cases} 
\phi_1(\tau) & \tau \in [0, c], \\
\phi_2(\tau) & \tau \in [c, (c + 1)d + c]. 
\end{cases} \]

We can check that \( S(\phi) \) will be continuous at \( \tau = c \) precisely when \( c = d \). We also want \( (c + 1)d + c = m \), which implies that \( c = \sqrt{m + 1} - 1 \). With these choices a simple computation shows that \( S(\phi) \) is concave for \( m \geq k_1 \) (note that it is linear for \( m = k_1 \), and convex for \( m < k_1 \)). Finally recall that the condition that \( \phi \) is non-negative means that \( c \leq k_2 \), which in turn implies \( m \leq k_2(k_2 + 2) \). See Figure 2 for a graph of \( \phi \) for \( m = 24 \).

Figure 2: Momentum profile of the minimiser on \( X \) when \( m = 24 \). The manifold breaks into two pieces both of which are equipped with a complete extremal metric.

**Case 3. \( X \) breaks up into three pieces, \( m > k_2(k_2 + 2) \)**

The previous construction no longer works for \( m > k_2(k_2 + 2) \) so we need to split the interval \([0, m]\) into three pieces. From the previous case we have a
solution $\phi_1$ to the equation
\[
S(\phi_1)'' = 0 \text{ on the interval } (0,k_1)
\]
\[
\phi_1(0) = \phi_1(k_1) = 0, \quad \phi_1'(0) = 2, \quad \phi_1'(k_1) = 0,
\]
and also a solution $\phi_2$ to
\[
S(\phi_2)'' = 0 \text{ on the interval } (c,m)
\]
\[
\phi_2(c) = \phi_2(m) = 0, \quad \phi_2'(c) = 0, \quad \phi_2'(m) = -2,
\]
where the constant $c$ is defined by
\[
c = \frac{m+1}{k_2+1} - 1. \tag{5}
\]
We define
\[
\phi(\tau) = \begin{cases} 
\phi_1(\tau) & \tau \in [0,k_2] \\
0 & \tau \in [k_2,c] \\
\phi_2(\tau) & \tau \in [c,m].
\end{cases}
\]
We can check that $c > k_2$ precisely when $m > k_2(k_2 + 2)$, and this choice of $\phi$ satisfies that $\phi S(\phi)'' = 0$ and $S(\phi)$ is concave. See Figure 3 for a graph of $\phi$ for $m \simeq 41.2$.

Conclusion

For any $m$ one of the previous 3 cases will hold, so we can construct a $\phi$ which satisfies the equation $\phi S(\phi)'' = 0$ and $S(\phi)$ is concave. According to Proposition 5 this $\phi$ will give the minimum of the Calabi functional on the space of singular momentum profiles. In the next section we will show that they give the infimum of the Calabi functional over all metrics in their Kähler class.

This will complete the proof of Theorem 1.

5 Test-configurations

In the previous section we have found a (possibly degenerate) metric in each Kähler class, which minimises the Calabi functional on the set of metrics which come from the momentum construction. In this section we want to show that these metrics minimise the Calabi functional on their entire Kähler class. For this we use the theorem of Donaldson [9] which gives a lower bound on the Calabi functional, given a destabilising test-configuration. We will not give a detailed explanation of the test-configurations that we use, and the computation of their Futaki invariants. For more details see [15] and [7].
Figure 3: Momentum profile of the minimiser on $X$ when $m \simeq 53.2$. The manifold breaks into three pieces, two of which, $A$ and $C$, admit complete extremal metrics, and in the third, $B$, the $S^1$-orbits collapse.

**Proposition 6** (Donaldson [9]). Suppose there exists a test-configuration $\chi$ for a polarised variety $(X, L)$ such that the Futaki invariant $F(\chi)$ is negative. Then for any metric $\omega$ in the class $c_1(L)$ we have the inequality

$$\|S(\omega) - \hat{S}\|_{L^2} \geq \frac{-F(\chi_i)}{\|\chi_i\|}.$$

The idea is to produce a sequence of test-configurations $\chi_i$ for which

$$\lim_{i \to \infty} \frac{-F(\chi_i)}{\|\chi_i\|} = \|S(\omega) - \hat{S}\|_{L^2},$$

where $\omega$ is the degenerate metric corresponding to the singular momentum profile in each Kähler class that we have found in the previous section. This will imply that this is the infimum of the Calabi functional and $\omega$ minimises the Calabi functional on its Kähler class.

To obtain test-configurations we use the construction in [16] Section 4.1 (Theorem 4.1.2), which is an extension of the construction of test-configurations for toric varieties by Donaldson [7] to bundles of toric varieties. For the case of our ruled surface we obtain
Proposition 7. Given a rational, piecewise-linear, convex function \( h : [0, m] \to \mathbb{R} \), there exists a test-configuration for \((X, L_m)\) with Futaki invariant given by
\[
F(h) = h(0) + (1 + m)h(m) - 2 \int_0^m h(\tau) \, d\tau - \hat{S} \int_0^m h(\tau)(1 + \tau) \, d\tau, \tag{6}
\]
and norm
\[
\|h\|^2 = \int_0^m (h(\tau) - \hat{h})^2(1 + \tau) \, d\tau,
\]
where \( \hat{h} \) is the average of \( h \) with respect to the measure \((1 + \tau) \, d\tau \).

To work with test-configurations we should restrict to polarisation \( L_m \) with \( m \) rational but an approximation argument gives us the conclusion of Proposition 6 for any real \( m \) as well. Given a continuous convex function \( h \) on \([0, m]\) which is not necessarily rational or piecewise-linear, we still define the “Futaki invariant” \( F(h) \) of \( h \) by Equation 6.

Lemma 8. Let \( \phi \) be a singular momentum profile, and \( h : [0, m] \to \mathbb{R} \) a piecewise-smooth convex function. Suppose that \( h \) is linear on any interval on which \( \phi \) does not vanish identically. Then
\[
F(h) = \int_0^m h(\tau)(S(\phi) - \hat{S})(1 + \tau) \, d\tau.
\]

This result is analogous to the fact that the Futaki invariant of a holomorphic vector field can be computed algebro-geometrically or differential geometrically (see [7]). Here if \( h \) is rational and piecewise-linear then it does not define a holomorphic vector field but the result says that we can still compute the Futaki invariant of the test-configuration it induces with a differential geometric formula as long as we use a metric which degenerates in a suitable way at points where \( h \) is not linear.

Proof. The proof is a simple integration by parts, using the formulas for \( F(h) \) and \( S(\phi) \). \( \square \)

We can now complete the proof of Theorem 1.

Proof of Theorem 1. What remains to be shown is that for each polarisation, the minimiser \( \phi \) that we have constructed in the previous section minimises the Calabi functional over the whole Kähler class, not just over the set of metrics obtained from the momentum construction. Let \( \phi \) be one of these minimisers. Since \(-S(\phi)\) is convex, we can approximate it in the \( C^0\)-norm by a sequence
of rational, piecewise-linear convex functions \( h_i \). These define a sequence of test-configurations \( \chi_i \) such that

\[
\lim_{i \to \infty} \frac{-F(\chi_i)}{\|\chi_i\|} = \frac{-F(-S(\phi))}{\|S(\phi) - \hat{S}\|_{L^2}}.
\]

If we let \( h = -S(\phi) \), then \( \phi \) and \( h \) satisfy the conditions of Lemma 8 so that

\[
F(-S(\phi)) = -\int_0^m S(\phi)(S(\phi)(\tau) - \hat{S})(1 + \tau) \, d\tau = -\|S(\phi) - \hat{S}\|^2_{L^2}.
\]

Therefore

\[
\lim_{i \to \infty} \frac{-F(\chi_i)}{\|\chi_i\|} = \|S(\phi) - \hat{S}\|_{L^2},
\]

so that Proposition 6 now implies that this limit is the infimum of the Calabi functional on the Kähler class.

\[\square\]

6 The Calabi flow

We have seen that in the case of a ruled surface it is fairly simple to minimise the Calabi functional directly over the set of metrics given by momentum profiles. It is also interesting to see whether the Calabi flow converges to these minimisers. In this section we will prove that this is the case. In [11] Guan has shown that on a ruled manifold when an extremal metric exists, then starting from a metric given by the momentum construction the Calabi flow exists for all time and converges to the extremal metric exponentially fast. Our techniques are similar to his, but we need to introduce some new functionals which are modifications of the Mabuchi functional more suited for studying the unstable polarisations.

We consider a family of metrics \( \omega_s \) given by the momentum construction (see Section 2), ie.

\[
\omega_t = p^* \omega_\Sigma + 2i\partial \bar{\partial} f_t(s),
\]

for some family of suitably convex functions \( f_t \). This path of metrics satisfies the Calabi flow if

\[
\frac{\partial f_t}{\partial t} = S(\omega_t).
\]

If we denote by \( F_t \) the Legendre transforms of the \( f_t \), then from the definition of the Legendre transformation we find

\[
\frac{\partial F_t}{\partial t} = -\frac{\partial f_t}{\partial t},
\]

so that the path of momentum profiles \( \phi_t = 1/F''_t \) satisfies

\[
\frac{\partial \phi_t}{\partial t} = \phi_t^2 S(\phi_t)''.
\]
where $S(\phi_t)$ is given by Equation 2.

It is known that the flow exists for a short time with any smooth initial metric (see Chen-He [4]). Also, the Calabi functional is decreased under the flow:

**Lemma 9.** If $\phi$ is a solution to the Calabi flow, then

$$\frac{d\text{Cal}(\phi)}{dt} = - \int_0^m \phi^2 (S(\phi)')^2 (1 + \tau) d\tau \leq 0.$$  

In particular the $H^2$ norm of $\phi_t$ is uniformly bounded along the flow.

**Proof.** The result follows from the following computation of the variation.

$$\frac{d\text{Cal}(\phi)}{dt} = 2 \int_0^m (S(\phi) - S(\Phi)) \left( - \frac{1}{2(1 + \tau)} \left[(1 + \tau)\phi^2 S(\phi)''\right]' \right) (1 + \tau) d\tau$$

$$= - \int_0^m \phi^2 (S(\phi)')^2 (1 + \tau) d\tau.$$  

We can perform the integration by parts because $\phi^2$ and $(\phi^2)'$ vanish at the endpoints. Also recall that $S(\Phi)'' = 0$.  

In Section 7 we will show that there is a solution to the Calabi flow for all time for any polarisation. In this section we concentrate on proving the following.

**Proposition 10.** If the flow exists for all time then the momentum profiles converge in $H^2$ to the minimiser that we found in Section 4.

**Proof.** Let us write $\Psi$ for the minimiser, so when $m < k_1$ then $\Psi$ is the momentum profile of an extremal metric, when $m \leq m \leq k_2(k_2 + 2)$ then $\Psi$ vanishes at an interior point of $(0, m)$ and when $m > k_2(k_2 + 2)$ then $\Psi$ vanishes on an interval inside $(0, m)$.

Introduce the functional

$$M(\phi) = \int_0^m \left( \frac{\Psi}{\phi} + \log \phi \right) (1 + \tau) d\tau,$$

defined on momentum profiles $\phi$. When $m < k_1$ then in fact $M$ is the modified Mabuchi functional (see [1] Section 2.3).

The key point is that $M$ is decreasing under the flow (this is well-known for the modified Mabuchi functional, since the Calabi flow is its gradient flow).
This follows from the computation
\[
\frac{dM(\phi_t)}{dt} = \int_0^m (-\Psi S(\phi_t)'' + \phi_t S(\phi_t)') (1 + \tau) d\tau
\]
\[
= \int_0^m (\phi_t - \Psi) (S(\phi_t) - S(\Psi))'' (1 + \tau) d\tau + \int_0^m \phi_t S(\Psi)'' (1 + \tau) d\tau
\]
\[
\leq -2 \int_0^m (S(\phi_t) - S(\Psi))^2 (1 + \tau) d\tau,
\]
where we have used that \(\Psi S(\Psi)'' = 0\) and \(S(\Psi)''\) is a negative distribution.

On the other hand we have that
\[
M(\phi) \geq \int_0^m \log \phi \cdot (1 + \tau) d\tau \geq -C_1 \int_0^m \log \frac{\Theta}{\phi} d\tau - C_2,
\]
where \(\Theta\) is a fixed momentum profile and \(C_1, C_2\) are constants. Since \(\log\) is concave we obtain
\[
M(\phi) \geq -C_3 \log \int_0^m \frac{\Theta}{\phi} d\tau - C_4,
\]
for some constants \(C_3, C_4\). The lemma that follows now implies that along the flow
\[
M(\phi_t) \geq -C \log(1 + t) - D.
\]
Since \(M(\phi_t)\) is decreasing, we necessarily have that along a subsequence its derivative tends to zero, i.e. \(S(\phi_t) \to S(\Psi)\) in \(L^2\) (integrating with respect to \((1 + \tau)d\tau\) as usual). Since \(\|S(\phi_t)\|_{L^2}\) is decreasing along the flow, it follows that
\[
\lim_{t \to \infty} \|S(\phi_t)\|_{L^2} = \|S(\Psi)\|_{L^2}. \tag{8}
\]

Let us now take any subsequence \(\phi_i\). Because of the uniform \(H^2\)-bound there is a subsequence also denoted by \(\phi_i\) which converges weakly in \(H^2\) to some limit. Now Equation \(\text{8}\) implies the convergence of the \(H^2\)-norms, which together with the weak convergence implies strong convergence in \(H^2\). The limit then has to be \(\Psi\) since the minimiser of the Calabi functional is unique (Proposition 5).

**Lemma 11.** Let \(\Theta : [0, m] \to \mathbb{R}\) be a momentum profile. For the solution \(\phi_t\) to the Calabi flow we have
\[
\int_0^m \frac{\Theta}{\phi_t} d\tau < C(1 + t), \tag{9}
\]
for some constant \(C\).
Proof. Let us define the functional
\[ F(\psi) = \int_0^m \Theta \psi - \log \Theta \psi \, d\tau \]
for any momentum profile \( \psi \). Along the Calabi flow we have
\[
\frac{d}{dt} F(\phi_t) = \int_0^m (\phi_t - \Theta) S(\phi_t)'' \, d\tau = \int_0^m (\phi_t - \Theta)'' S(\phi_t) \, d\tau \leq \left( \int_0^m (\phi_t'' - \Theta'')^2 \, d\tau \right)^{1/2} (\text{Cal}(\phi_t) + C)^{1/2}.
\]
The uniform \( H^2 \) bound on \( \phi_t \) now implies that \( F(\phi_t) \leq C(1 + t) \) for some \( C > 0 \). The result follows from the inequality \( x - \log x > x/2 \).

Remark. Note that when \( m \leq k_2(k_2 + 2) \) the functional \( M \) is bounded below on the set of momentum profiles. This is because we can write
\[
M(\phi) = \int_0^m \left( \frac{\Psi}{\phi} - \log \frac{\Psi}{\phi} \right) (1 + \tau) \, d\tau + \int_0^m \log \Psi \cdot (1 + \tau) \, d\tau.
\]
Since \( \Psi \) only vanishes at isolated points and to finite order, the integral of \( \log \Psi \) is finite, so the inequality \( \log x < x \) implies
\[
M(\phi) \geq \int_0^m \log \Psi \cdot (1 + \tau) \, d\tau.
\]
In the case \( m > k_2(k_2 + 2) \) however \( M \) is not bounded from below since now \( \Psi \) vanishes on an interval. In particular as \( \phi \to \Psi \), it is clear that \( M(\phi) \to -\infty \).

7 Long time existence

The existence of the Calabi flow for a short time has been proved by Chen-He [4] (also Guan [11] for ruled manifolds). In the case when an extremal metric exists, the long time existence has also been shown in [11] for ruled manifolds.

To show that the flow exists for all time we first need to show that \( \phi_t(x) \) does not become zero in finite time for \( x \in (0, m) \). Let \( \Theta \) be a fixed momentum profile, i.e., a non-negative function on \([0, m]\), strictly positive on the interior, and satisfying the usual boundary conditions. We want to show

**Proposition 12.** If \( \phi_t \) is the solution to the Calabi flow, then \( \sup_{\phi_t(x)} \frac{\Theta(x)}{\phi_t(x)} \) does not blow up in finite time.

**Proof.** This follows from Lemma [11] and the following lemma. \( \square \)
Lemma 13. Given a constant $C > 0$ there exists a constant $D > 0$ such that if for a momentum profile $\psi$ we have
\[
\int_0^m \frac{\Theta}{\psi} d\tau < C \quad \text{and} \quad \|\psi\|_{C^{1,1/2}} < C,
\]
then\[
\sup \Theta/\psi < D.
\]

Proof. Let us derive the estimate near the boundary first. Because of the $C^{1,1/2}$ bound on $\psi$, there exists a constant $C_1$ such that\[
|\psi'(x) - \psi'(0)| < C_1 \sqrt{x},
\]
ie.
\[
\psi'(x) > 2 - C_1 \sqrt{x}.
\]
This implies that\[
\psi(x) > x \left(2 - \frac{2}{3} C_1 \sqrt{x}\right),
\]
so that for $x < (3/2C_1)^2$ we have $\psi(x) > x$. We can apply the same argument around $x = m$ as well, so we obtain a small constant $\delta$ such that\[
\text{if } x < \delta \text{ or } x > m - \delta, \text{ then } \frac{\Theta(x)}{\psi(x)} < D.
\]
Now we concentrate on the set $(\delta, m - \delta)$. On this set we have a uniform lower bound $\Theta(x) > \epsilon > 0$ so we just need a lower bound on $\psi$. There is a constant $C_2$ such that $|\psi'(x)| < C_2$ for all $x$. Suppose that for some $x \in (\delta, m - \delta)$ we have $\psi(x) < \epsilon/k$ where $k$ is large. Assume for simplicity that $x < m/2$. Then for $y < m/2 - \delta$ we have\[
\psi(x + y) < \frac{\epsilon}{k} + C_2 y.
\]
Writing $a = m/2 - \delta$, this implies that\[
C > \int_0^m \frac{\Theta}{\psi} d\tau > \epsilon \int_0^a \frac{1}{\frac{k}{k} + C_2 y} dy > \frac{\epsilon}{C_2} \left[\log C_2 a - \log \frac{\epsilon}{k}\right].
\]
Since this tends to infinity as $k \to \infty$, we get the required lower bound on $\psi(x)$ for $x \in (\delta, m - \delta)$. Combining this with the boundary estimate we obtain the statement of the lemma.

Next we would like to estimate the derivatives of $\phi$ following the calculation in Guan [11]. Let us introduce the functional\[
L(\phi) = \int_0^m (\phi S(\phi)')^2 (1 + \tau) d\tau.
\]
We want to show
Lemma 14 (Guan [11]). For \( \phi_t \), a solution of the Calabi flow we have that \( L(\phi_t) \leq C(t) \) for some function \( C(t) \) defined for all \( t \).

Proof. All our constants will depend on \( t \) but will be finite for all \( t \). All the integral norms will be with respect to the measure \( d\tau \) and not \( (1 + \tau)d\tau \) as before.

In the proof we will repeatedly use the Hardy-type inequality

\[
\|f\|_{L^2(0,m)} \leq C \|\phi_t^{-k+1}(\phi_t^k f)'\|_{L^2(0,m)}
\]

for \( k \geq 1 \) and any \( f \in \mathcal{C}^1[0,m] \) with the constant \( C \) depending on \( t \). Using Proposition 12, this is easy to derive from the inequality

\[
\int_{-1}^{1} f(x)^2 \, dx \leq C \int_{-1}^{1} [(1 - x^2)^{-k+1}((1 - x^2)^k f(x))']^2 \, dx.
\]

This in turn follows from the inequality

\[
\int_{0}^{1} f(x)^2 \, dx \leq C \int_{0}^{1} (x^{-k+1}(x^k f)')^2 \, dx
\]

for \( f \) with \( f(1) = 0 \), applied to the intervals \([-1,0]\) and \([0,1]\) separately (see [11]).

Let us compute the derivative of \( L(\phi_t) \).

\[
\frac{d}{dt} L(\phi_t) = 2 \int_{0}^{m} (\phi_t S(\phi_t)^n)^3 (1 + \tau) d\tau - \int_{0}^{m} \left[ (1 + \tau)\phi_t^2 S(\phi_t)^n \right]^{\prime 2} \, d\tau \frac{1 + \tau}{1 + \tau} \leq C_1 \int_{0}^{m} (\phi_t S(\phi_t)^n)^3 \, d\tau - C_2 \left\| (\phi_t^2 S(\phi_t)^n)' \right\|_{L^2}^2.
\]

Let us estimate the cubed term. We have

\[
\int_{0}^{m} (\phi_t S(\phi_t)^n)^3 \, d\tau \leq C_5 \left\| \phi_t S(\phi_t)^n \right\|_{C^0 L(\phi_t)} \leq C_4 \left\| (\phi_t S(\phi_t)^n)' \right\|_{L^2 L(\phi_t)} \leq C(\epsilon) L(\phi_t)^2 + \epsilon \left\| (\phi_t S(\phi_t)^n)' \right\|_{L^2}^2,
\]

for any \( \epsilon > 0 \) using Young’s inequality. Using the uniform \( H^2 \)-bound on \( \phi_t \) and the Hardy-type inequality twice we obtain

\[
\left\| (\phi_t^{-1} \cdot \phi_t^2 S(\phi_t)^n)' \right\|_{L^2} \leq \left\| \phi_t^{-1} (\phi_t^2 S(\phi_t)^n)' \right\|_{L^2} + \left\| \phi_t' S(\phi_t)^n \right\|_{L^2} \leq C_5 \left\| \phi_t^{-1} (\phi_t^2 S(\phi_t)^n)' \right\|_{L^2} \leq C_6 \left\| \phi_t^2 S(\phi_t)^n \right\|_{L^2}.
\]

20
so if we choose $\epsilon$ small enough (depending on $t$), then we obtain the inequality

$$\frac{d}{dt} L(\phi_t) \leq C_1(t) L(\phi_t)^2.$$ 

This implies that

$$\frac{d}{dt} \log L(\phi_t) \leq C_1(t) L(\phi_t),$$

ie. for any $T > 0$ we have

$$\log L(\phi_T) \leq \log L(\phi_0) + \sup_{t \in [0,T]} C_1(t) \int_0^T L(\phi_t) \, dt.$$ 

Now Lemma 9 gives a bound on the integral of $L(\phi_t)$ since the Calabi functional is non-negative, so the proof is complete.

Now we need to use the inequality

$$\|f\|_{L^2}^2 \leq C(\|\phi f'\|_{L^2}^2 + f(m/2)^2)$$

for all $f \in C^1(0, m)$ which can be proved in the same way as the Hardy-type inequalities we used before. This implies that

$$\|S(\phi_t)\|_{L^\infty}^2 \leq C_1 \|S(\phi_t)'\|_{L^2}^2 \leq C_2 \left[ \|\phi_t S(\phi_t)''\|_{L^2}^2 + (S(\phi_t)'(m/2))^2 \right].$$

The bound on $\|\phi_t S(\phi_t)''\|_{L^2}$ gives a bound on $|S(\phi_t)'(x) - S(\phi_t)'(m/2)|$ for $x$ inside the interval $(\frac{m}{3}, \frac{2m}{3})$. The bound on $\|S(\phi_t)\|_{L^2}$ (the Calabi functional decreases along the flow) then gives an apriori bound on $S(\phi_t)'(m/2)$. Therefore as long as $L(\phi_t)$ remains bounded, we have a $C^2$ bound on $\phi_t$ (depending on $t$). To obtain estimates for the higher derivatives of $\phi_t$ we could either continue with similar integral estimates in the manner of [11] or we can note that a $C^2$ bound on the momentum profile implies a uniform bound on the Ricci curvature. According to Chen-He [11] the Calabi flow exists for all time as long as the Ricci curvature remains uniformly bounded.

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