On the Approximation Relationship between Optimizing Ratio of Submodular (RS) and Difference of Submodular (DS) Functions

Pierre Perrault
Adobe Research, San Jose, CA — ENS Paris-Saclay — Inria Lille

Jennifer Healey
Adobe Research, San Jose, CA

Zheng Wen
DeepMind

Michal Valko
DeepMind — Inria Lille

Abstract
We demonstrate that from an algorithm guaranteeing an approximation factor for the ratio of submodular (RS) optimization problem, we can build another algorithm having a different kind of approximation guarantee — weaker than the classical one — for the difference of submodular (DS) optimization problem, and vice versa. We also illustrate the link between these two problems by analyzing a GREEDY algorithm which approximately maximizes objective functions of the form \( \Psi(f, g) \), where \( f, g \) are two non-negative, monotone, submodular functions and \( \Psi \) is a quasiconvex 2-variables function, which is non decreasing with respect to the first variable. For the choice \( \Psi(f, g) \doteq f/g \), we recover RS, and for the choice \( \Psi(f, g) \doteq f - g \), we recover DS. To the best of our knowledge, this greedy approach is new for DS optimization. For RS optimization, it reduces to the standard GREEDY algorithm that has already been analyzed in Bai et al. (2016). However, our analysis is novel for this case.

Keywords: Difference of Submodular, Ratio of Submodular, Approximation, Greedy

1. Introduction
Combinatorial optimization is important in many areas of engineering, economics, operations research, computer vision and machine learning. An increasing number of problems coming from this last two areas have been identified as an optimization one involving submodular functions (see e.g., Kempe et al. (2003); Krause and Guestrin (2005); Lin and Bilmes (2011); Bach (2011)). Submodular functions can indeed be used in a wide range of applications, such as feature selection, data summarization, or active learning. Learning and maximizing these functions from data is therefore of particular interest (Tschiatschek et al., 2018; Joseph et al., 2019). We can define submodular functions in the following way. If \( [n] \doteq \{1, \ldots, n\} \) refers a ground set of \( n \) elements, and \( \mathcal{P}([n]) \) is the set of all the subsets of \( [n] \), a set function (i.e., a function defined on sets) \( f : \mathcal{P}([n]) \to \mathbb{R} \) is said to be submodular if for all \( A \in \mathcal{P}([n]) \) and \( B \in \mathcal{P}([n]) \), \( f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \) (Fujishige, 2005). Submodular functions can be characterized by an intuitive diminishing return property, where the marginal gain \( f(i|A) \doteq f(A \cup \{i\}) - f(A) \) of an element \( i \in [n] \) in the context of a set \( A \in \mathcal{P}([n] \setminus \{i\}) \) is non increasing with respect to \( A \). This property is naturally and initially found in economics, where \( f(A) \) can measures the output of a production process involving a set \( A \) of different factors.

© P. Perrault, J. Healey, Z. Wen & M. Valko.
The class of submodular functions is special on many aspects. In particular, the problem of minimizing a submodular function is known to be solvable in time polynomial in \( n \) (Iwata et al., 2001), hence submodularity is also often called the discrete analog of convexity (Lovász, 1983). In contrast, general submodular maximization is NP-Hard (Schrijver, 2008), with many existing algorithms for constant factor approximation (see Buchbinder and Feldman (2017) for a survey).

In addition to problems involving a single submodular function, there exist many problems that are aimed at maximizing (or minimizing) a combination of two submodular functions. The two functions in this kind of objective generally model two desired behaviours that are more or less complementary to each other. For example, one of the two functions may represent a regularization term on the other (the main objective). Another common scenario is when one term encodes the gain associated to a particular feasible set \( S \), whereas the other represents a cost associated to \( S \).

There are two well known examples of such problems that we are interested in here, involving objectives that is a difference of submodular (DS) functions, or a ratio of submodular (RS) functions.

**RS and DS optimization problems** Given two monotone\(^1\), submodular, but not necessarily normalized\(^2\) functions \( f, g : \mathcal{P}([n]) \to \mathbb{R}_+ \) such that \( g \) is positive, we investigate the two following problems involving \( f \) and \( g \).

**Problem 1 (DS Optimization)** (Kawahara and Washio, 2011; Narasimhan and Bilmes, 2012; Iyer and Bilmes, 2012)

\[
\max_{S \in \mathcal{P}([n])} f(S) - g(S).
\]

**Problem 2 (RS Optimization)** (Bai et al., 2016; Qian et al., 2017; Wang et al., 2019)

\[
\max_{S \in \mathcal{P}([n])} \frac{f(S)}{g(S)}.
\]

For DS optimization (but not RS), we can assume that both \( f \) and \( g \) are normalized without loss of generality. We review in the following some recent examples of such optimization problems. We can already see that both problems model the same willingness to maximize the function \( f \) while minimizing the function \( g \).

**Applications** RS and DS has been extensively studied in the literature, and can be applied to some real world optimization problems such as sensor placement and feature selection (Iyer and Bilmes, 2012; Bai et al., 2016). Bai et al. (2016) also highlighted that RS can be used for maximizing the F-measure in information retrieval or optimizing normalized cuts and ratio cuts (Shi and Malik, 2000). On the other hand, Iyer and Bilmes (2012) considered modular\(^3\) upper and lower bounds in order to tackle DS optimization. They also presented some applications such as discriminatively structured graphical models and neural computation or probabilistic inference. Another application of RS and DS is influence maximization (Kempe et al., 2003; Chen et al., 2010), where the submodular function \( f \) (called spread function) is defined on the sets \( S \) of vertices of a social network. \( f(S) \) represents a gain corresponding to the expected number of users influenced by the seed set \( S \) of

---

1. A set function \( f \) is monotone if \( A \subset B \Rightarrow f(A) \leq f(B) \).
2. A set function \( f \) is said to be normalized if \( f(\emptyset) = 0 \).
3. A function \( f \) is modular if both \( f \) and \( -f \) are submodular. This is equivalent to \( f(S) = f(\emptyset) + \sum_{i \in S} f(\{i\}) \) for all \( S \in \mathcal{P}([n]) \).
initial influencers. Traditionally, the influence maximization problem aims to maximize \( f \) under a cardinality constraint on the seed set. Another formulation is to include a modular function \( g \) in the objective of the influence maximization problem, that represents the cost of triggering seed influencers. Notice, both subtracting or dividing \( f \) by \( g \) have a sense to build the final objective function, that thus falls either in Problem 1 or Problem 2.

**Already known connection between RS and DS** DS and RS are already known to be connected (Bai et al., 2016; Chen et al., 2019), since if \( \lambda^* = \max f/g \), then the following are equivalent:

- \( S \in \mathcal{P}([n]) \) is a maximizer for DS with the objective function \( f - \lambda^* g \),
- \( S \in \mathcal{P}([n]) \) is a maximizer for RS with the objective \( f/g \).

We can however give one key difference between the two: Although the problem of finding an optimal solution is NP-hard in both cases (Bai et al., 2016; Iyer and Bilmes, 2012), RS optimization can be solved with bounded approximation factors (indeed, Bai et al. (2016) prove an approximation ratio of \( O(\sqrt{n \log(n)}) \) for the general RS maximization problem, which is tight up to the log factor), whereas DS optimization is, in the worst case, inapproximable (Iyer and Bilmes, 2012) (actually, they show that any set function maximization problem can be written as a DS optimization problem). The goal of the present paper is to highlight that both problems are actually equivalent in some weaker sense of approximability.

**Contributions** In Section 2, we first prove that the two more general problems of maximizing a ratio \( f/g \) and a difference \( f - g \) over a general set \( \mathcal{X} \) are actually equivalent in some sense of approximability, inducing an approximability equivalence result for RS and DS optimization. More precisely, we show that the problem of finding \( x \in \mathcal{X} \) having an \( \alpha \leq 1 \) approximation ratio for \( \max f/g \) is equivalent to the problem of finding \( x \in \mathcal{X} \) such that \( f(x) - g(x) \geq \alpha f(y^*) - g(y^*) \), where \( y^* \in \arg \max_{y \in \mathcal{X}} f(y) - g(y) \). It should be emphasizes here that this last approximation guarantee is weaker than the classical \( f(x) - g(x) \geq \alpha(f(y^*) - g(y^*)) \). Such weaker notion of approximation have been considered for instance in Filmus and Ward (2012); Sviridenko et al. (2013); Harshaw et al. (2019); Perrault et al. (2019); Feldman (2020); Ene et al. (2020). In the case \( g \) has a low curvature, we further demonstrate in Section 3 the strong relationship between DS and RS by giving approximations guarantees for a simple GREEDY algorithm that tackles both problems in a very similar way. These approximations are new, to the best of our knowledge. This GREEDY algorithm was indeed already studied by Bai et al. (2016) for the RS problem, where they claim the same approximation as ours, however, there seems to be an error in their demonstration of Theorem 3.4, since their inequality (16) does not hold for non normalized set functions. In addition, our analysis allows us to extend the result to objectives of the form \( \Psi(f, g) \), where \( \Psi \) is a quasiconvex 2-variables function, which is non decreasing with respect to the first variable.

### 2. Approximability of general difference and ratio optimization problems

The idea behind the connection between RS and DS is actually orthogonal to the fact that the functions \( f \) and \( g \) are submodular. We thus consider a more general framework here. Let \( \mathcal{X} \) be a given set, and let \( \mathcal{F} \) be a set of non-negative functions on \( \mathcal{X} \), and \( \mathcal{G} \) be a cone4 containing positive functions \( g \) on \( \mathcal{X} \), closed under addition by a scalar \( c \in \mathbb{R} \), as long as \( g + c \) remains positive. Assume

---

4. A cone is a subset of a vector space that is closed under multiplication by a positive scalar.
that for \( f \in \mathcal{F} \) and \( g \in \mathcal{G} \), we can efficiently compute \( x_f = \arg \max_{X} f \) and \( x_g = \arg \min_{X} g \). Notice, \( \mathcal{F} \) can be the set of non-negative monotone submodular functions and \( \mathcal{G} \) can be the set of positive monotone submodular functions. We consider the two following problems given \((f, g) \in \mathcal{F} \times \mathcal{G}\):

\[
\begin{align*}
\text{(Difference)} & \quad \text{Find } x \in \arg \max_{X} f(x') - g(x') \\
\text{(Ratio)} & \quad \text{Find } x \in \arg \max_{X} f(x')/g(x').
\end{align*}
\]

Assume an algorithm \( A \) that solves exactly Difference is available. Then it is already known that one can use a binary search method in order to solve Ratio with an additive \( \varepsilon \) error, using a number of calls to \( A \) that is \( \mathcal{O}(\log(1/\varepsilon)) \) (Gondran and Minoux, 1984). Indeed, to this aim, we start (using \( x_f \) and \( x_g \)) from an initial interval \([\lambda_-, \lambda_+]\) that contains the optimal ratio \( \lambda^* = f(x^*)/g(x^*) \). Then, we take \( \lambda = (\lambda_- + \lambda_+)/2 \) and call \( A \) for Difference with the objective \( f - \lambda g \), to get a solution \( x_\lambda \). If \( f(x_\lambda)/g(x_\lambda) \leq \lambda \), then by definition of \( x_\lambda \), we necessarily have \( f(x^*) - \lambda g(x^*) \leq f(x_\lambda) - \lambda g(x_\lambda) \leq 0 \) and \( \lambda^* \leq \lambda \). Then, \( \lambda_- \) is updated as \( \lambda_- \leftarrow \lambda \). Else, by definition of \( \lambda^* \), \( f(x_\lambda) - \lambda g(x_\lambda) \geq 0 \geq f(x_\lambda) - \lambda^* g(x_\lambda) \), so \( \lambda \leq \lambda^* \) and \( \lambda_+ \) is updated as \( \lambda_+ \leftarrow \lambda \). Then we go back to the first step with the new shorter interval \([\lambda_-, \lambda_+]\) and iterate this process until the interval is of length less than \( \varepsilon \). The last update of \( \lambda \) corresponds to a solution \( x_{\lambda} \) which satisfies \( f(x_\lambda)/g(x_\lambda) \geq \lambda_- \geq \lambda_+ - \varepsilon \geq \lambda^* - \varepsilon \). The number of iterations needed is clearly of order \( \mathcal{O}(\log(1/\varepsilon)) \), since at each iteration, the size of the interval \([\lambda_-, \lambda_+]\) is divided by 2. In summary, we have a fully polynomial-time approximation scheme (FPTAS) algorithm for reducing Difference to Ratio.

**Remark 1** In the previous binary search method, ratios \( \lambda \) considered are not necessarily of the form \( f(x)/g(x) \) for some \( x \in X \). There exist another iterative method (Isbell and Marlow, 1956) which works as follows. Starting from \( x_0 \in X \) and \( \lambda = f(x_0)/g(x_0) \), we call the algorithm \( A \) to get a maximizer \( x_\lambda \) of \( f - \lambda g \). If the optimum is lower than \( \varepsilon g(x_0) \), then \( f(x^*) - (\lambda + \varepsilon)g(x^*) \leq 0 \) and \( \lambda \geq \lambda^* - \varepsilon \). Otherwise, we set \( \lambda \leftarrow f(x_\lambda)/g(x_\lambda) \) and repeat. We obtain an increasing sequence of ratios \( \lambda_k \) such that \( \lambda_k \rightarrow \lambda^* \). We are not aware of a work that takes the reverse path, i.e., that starts from an algorithm \( A \) that solves exactly Ratio and that uses it for solving Difference. The reason is probably because it is usually easier to deal with a difference than a ratio in optimization. However, in the case of RS and DS optimization, it is rather the contrary that happens, since any set function can be written as a difference of two monotone submodular functions (Narasimhan and Bilmes, 2012), whereas a ratio of monotone submodular functions is a more specific type of set function. In the following, we will provide such reverse path process. We will also state our method for more general approximation versions of the problems Ratio and Difference, allowing us to make a connection between RS and DS approximability.

### 2.1. FPTAS connection between Difference and Ratio approximation

In this subsection, we present a method for designing approximation schemes that make the link between problems Difference and Ratio. Instead of assuming that an \textit{exact algorithm} is available for one problem, we rather assume that some sort of approximation algorithm is available, which is more general and makes more sense in the context of RS and DS optimization, where existing
algorithms are not exact. Specifically, we will provide a FPTAS algorithm for moving from one problem to the other. For $\alpha \in [0, 1]$, we consider the following approximation problems: Given $(f, g) \in \mathcal{F} \times \mathcal{G}$, find $x \in \mathcal{X}$ such that $\forall x' \in \mathcal{X}$:

(\alpha\text{-approximation of Difference}) \quad f(x) - g(x) \geq \alpha f(x') - g(x').

(\alpha\text{-approximation of Ratio}) \quad f(x)/g(x) \geq \alpha f(x')/g(x').

We state the following Theorem 1 which is the main result of this section, and gives a reduction from $\alpha$-approximation of Difference to $\alpha$-approximation of Ratio and vice versa.

**Theorem 1** Problems $\alpha$-approximation of Difference and $\alpha$-approximation of Ratio are equivalent, in the sense that the first can be reduced with a FPTAS to the second, and conversely.

The proof of Theorem 1 is postponed to Appendix A. We use the same dichotomy based FPTAS as we saw previously, for both reductions. For the reduction from $\alpha$-approximation of Ratio to $\alpha$-approximation of Difference, we use an additive adjustable constant $c$ to $g$ in a similar way as we used the multiplicative constant $\lambda$ previously for the $\alpha$-approximation of Difference to $\alpha$-approximation of Ratio reduction. The parameter $c$ aims to approach the value $c^* \triangleq \max_{y \in \mathcal{X}} f(y) - g(y^*)$. Indeed, if $x$ is a maximizer of $f/(g + c^*)$, then $f(x)/(g(x) + c^*) \geq f(y^*)/(g(y^*) + c^*) = 1$, so $f(x) - g(x) \geq c^*$ and $x$ is a maximizer of $f - g$ as well.

Since we consider approximation algorithms, we will not maintain an interval containing the optimal parameter $\lambda^* = \max_{x \in \mathcal{X}} f(x)/g(x) = f(x^*)/g(x^*)$ (resp. $c^* = \max_{y \in \mathcal{X}} f(y) - g(y) = f(y^*) - g(y^*)$), but rather a lower bound on the optimal parameter, and an upper bound on the approximated parameter $\alpha f(x^*)/g(x^*)$ (resp. $\alpha f(y^*) - g(y^*)$).

From Theorem 1, we now clearly have an equivalence between RS and DS in terms of approximability, taking the choices for $\mathcal{F}$ and $\mathcal{G}$ as indicated at the beginning of this section. In the following, we refer to the above new concept of approximability for DS optimization as weak DS approximability. We can give a DS example where $\alpha$-approximation of Difference can benefit from an approximation factor from the problem $\alpha$-approximation of Ratio. Indeed, Bai et al. (2016) has provided an algorithm for $\alpha$-approximation of Ratio with an approximation ratio of $\alpha = \mathcal{O}(n^{-1/2} \log(n)^{-1})$, using the $\mathcal{O}(n^{-1/2} \log(n)^{-1})$-approximation of any monotone submodular function in polynomial time by a surrogate function that is the square root of a modular function (Goemans et al., 2009). We thus have the following result as a corollary of Theorem 1, giving a solution to the problem $\alpha$-approximation of Difference.

**Corollary 1** Given two monotone, submodular, functions $f, g : \mathcal{P}([n]) \to \mathbb{R}_+$, we have an algorithm outputting $S$ with the following weak DS approximability guarantee: For any $S^* \in \mathcal{P}([n])$, we have

$$f(S) - g(S) \geq \mathcal{O}\left(n^{-1/2} \log^{-1}(n)\right) f(S^*) - g(S^*).$$

In the following section, we further investigate approximation for RS and DS in the case where $g$ as a low curvature, i.e., when $g$ is “close” to be a modular function.
Algorithm 1 \(\Psi\)-GREEDY

**Input:** set functions \(f, g, \Psi\).

\(S_0 \leftarrow \emptyset\).

**for** \(k \in [n]\) **do**

\(i_k \leftarrow \arg \max_{i \in [n] \setminus S_{k-1}} f(i|S_{k-1})/g(i|S_{k-1}).\)

\(S_k \leftarrow S_{k-1} \cup \{i_k\}.\)

**end for**

\(S \leftarrow \arg \max_{k \in \{0, \ldots, n\}} \Psi(f(S_k), g(S_k)).\)

**Output:** \(S\).

3. RS and DS problems through GREEDY algorithms

We further illustrate the link between RS and DS problems by proving that a simple GREEDY strategy can tackle these two problems in a same way under low curvature assumption on \(g\). We recall the notion of *curvature* (Conforti and Cornuëjols, 1984; Vondrak, 2010; Iyer et al., 2013; Sviridenko et al., 2013) of a monotone submodular function \(f\):

\[
c_f \triangleq 1 - \min_{i \in [n]} \frac{f(i|\{i\}\setminus[n])}{f(i|\emptyset)} \in [0, 1].
\]

Intuitively, the curvature measures how close to modular a submodular set function is. In particular, if \(f\) is monotone submodular, then \(c_f = 0\) if and only if \(f\) is modular. We will not state any assumption on the curvature but make the approximation ratio we consider depends on this quantity.

We provide our method in Algorithm 1. Depending on the function \(\Psi\) taken in input, we can deal with a more general class of optimization problems with submodular functions, where RS corresponds to \(\Psi(f, g) = f/g\) and DS to \(\Psi(f, g) = f - g\). Remark that before the last step, Algorithm 1 is exactly the standard GREEDY (Bai et al., 2016). Perhaps surprisingly, we will see that this algorithm can provide guarantees for many kind of “last step” without changing the loop part.

We begin by a very structured case where both RS and DS problems are “easy”. We show in the following Proposition 1 that Algorithm 1 is *exact* when functions \(f\) and \(g\) are modular (but not necessarily monotone), and \(\Psi(f, g) = f - g\) or \(\Psi(f, g) = f/g\).

**Proposition 1** If \(f\) and \(g\) are modular, then Algorithm 1 is exact for \(\Psi(f, g) = f - g\) or \(\Psi(f, g) = f/g\).

Proposition 1 is proved in Appendix B. We now provide in Theorem 2 an approximation guarantee for Algorithm 1 in the general case where \(f\) and \(g\) are non-negative, monotone and submodular, with a general choice of \(\Psi\) that covers both RS and DS. The proof of Theorem 2 is delayed to Appendix C.

**Theorem 2** Assume \(f\) and \(g\) are two non-negative, monotone, submodular functions, and assume \(\Psi\) is a quasiconvex 2-variables function that is non-decreasing with respect to the first variable. Let \(S^*\) be any subset of \([n]\). Algorithm 1 is guaranteed to obtain a solution \(S\) such that:

\[
\Psi\left((1 - e^{c_g - 1})f(S^*), g(S^*)\right) \leq \Psi(f(S), g(S)).
\]
Assume Theorem 3 notice that we also slightly changed the assumption on initialization (Khuller et al., 1999). We provide in Algorithm 2 the modified algorithm. In addition, the approach is slightly different in this case, since it relies on a partial enumeration step for the initialization. We provide here an approximation guarantee similar to that of Theorem 2, but under a knapsack constraint \( g(S) \leq B \), for some budget \( B \geq 0 \). The approach is slightly different in this case, since it relies on a partial enumeration step for the initialization (Khuller et al., 1999). We provide in Algorithm 2 the modified algorithm. In addition, notice that we also slightly changed the assumption on \( \Psi \). We get the following theorem, proved in Appendix D.

**Theorem 3** Assume \( f \) and \( g \) are two non-negative, monotone, submodular functions, and assume that \( \Psi \) is non-decreasing with respect to the first variable, and non-increasing with respect to the second variable. Then, the output \( S \) of Algorithm 2 satisfies

\[
\forall S^* \in \mathcal{P}([n]) \text{ s.t. } g(S^*) \leq B, \quad \Psi \left( (1 - e^{c_g-1}) f(S^*), g(S^*) \right) \leq \Psi(f(S), g(S) - \varepsilon).
\]

\[\text{Algorithm 2 Knapsack } \Psi\text{-Greedy}
\]

**Input:** set functions \( f, g, \Psi, B, \varepsilon \).

\[S_1 \leftarrow \arg \max_{|S| \leq 2, g(S) \leq B} \Psi(f(S), g(S)), \quad S_2 \leftarrow \emptyset\]

**for** \( b \in \{1, \ldots, \lfloor B/\varepsilon \rfloor, B/\varepsilon \} \) **do**

**for** \( |S| = 3, g(S) \leq B \) **do**

\[N' \leftarrow N \setminus S\]

\[S_G \leftarrow \Psi\text{-Greedy on } N' \text{ and initialization with } S, \text{ with the condition that } g(S_G) \leq b\varepsilon.\]

**if** \( \Psi(f(S_G), g(S_G)) \geq \Psi(f(S_2), g(S_2)) \) **then**

\[S_2 \leftarrow S_G\]

**end if**

**end for**

**end for**

\[S \leftarrow \arg \max_{k \in \{1,2\}} \Psi(f(S_k), g(S_k)).\]

**Output:** \( S \).
4. Experiments

In this section, we experimentally evaluate our Algorithm 1 for DS maximization, i.e., with \( \Psi(f, g) = f - g \). Indeed, an empirical evaluation for \( \Psi(f, g) = f/g \) is already given in Bai et al. (2016) (the algorithm is called GREEDRATIO), and its superiority over other methods such as the Majorization-Minimization and Ellipsoidal Approximation algorithms (Bai et al., 2016) has been noticed.

We compare Algorithm 1 to the state-of-the-art\(^5\) Modular-Modular (ModMod) procedure of Iyer and Bilmes (2012) on the problem of influence maximization with costly influencers (IMC). In IMC, we consider a social network where some influence interactions between users occur. We record these interactions with a sequence of directed graphs \( G_t = (V, E_t), t \in [T] \), where \( e = (u, v) \in E_t \) means that user \( u \) influences user \( v \) on the instance \( t \), in other words that some behavior of \( u \) (e.g., retweeting or buying a certain product) induced the same behavior for \( v \) on the \( t \) instance. For a subset of nodes \( S \subset V \), we let \( f(S) \triangleq \frac{1}{T} \sum_{t=1}^{T} F(S, G_t) \), where \( F(S, G_t) \) is the number of nodes \( u \in V \) such that there is a forward path in \( G_t \) starting from some node of \( S \) and ending to \( u \) (i.e., \( F(S, G_t) \) is the number of nodes that are influenced by \( S \)). It is well known that \( S \mapsto F(S, G_t) \) is submodular (Kempe et al., 2015), so is \( f \). The function \( f \) evaluates how good it is to initially influence the set of node \( S \) in order to let the influence spread throughout the network. We also define the modular function \( g(S) = \lambda \sum_{u \in S} c_u \), where values \( c_u \) can be interpreted as a cost to pay to initially influence \( u \). The goal is to maximize the resulting revenue \( f - g \).

We consider a subgraph of Facebook network (Leskovec and Krevl, 2014), with \( n = |V| = 333 \) and \( |E| = 5038 \), as in Wen et al. (2017), and generate \( G_t \) using an independent cascade model (Kempe et al., 2003) with weights \( w_e \) taken uniformly in \([0, 0.1]\), i.e., \( (w_e)_{e \in E} \sim U(0, 0.1)^{\otimes E} \). We use \( T = 10 \) in our experiments, and \((c_u)_{u \in V} \sim U(0, 1)^{\otimes V} \). For each value of \( \lambda \) considered, we run 50 independent simulations with costs and weights randomly chosen as just described.

Before presenting the result of the experiment, we give more details about the two algorithms we are comparing. We implement a lazy evaluations (Minoux, 1978) version of our greedy algorithm, where instead of maximizing the marginal gain \( (f(S \cup \{u\}) - f(S))/c_u \), we keep an upper bound \( \rho(u) \) (initially \( \infty \)) on it, sorted in decreasing order. In each iteration \( i \), we evaluate the element on top of the list, say \( u \), and updates its upper bound with the marginal gain at \( S_{i-1} \). If after the update we have \( \rho(u) \geq \rho(v) \) for all \( u \neq v \), submodularity guarantees that \( u \) is the element with the largest marginal gain. The ModMod Algorithm is described as follows. We start with \( S_0 = \emptyset \), and at each iteration \( i \geq 1 \), \( S_i \) maximizes the modular function \( A \mapsto m_{S_{i-1}}(A) - g(A) \), with \( A \mapsto m_X(A) \) the modular lower bound on \( f \) defined by

\[
m_X(\sigma(i)) \triangleq \begin{cases} f(\{\sigma(1)\}) & \text{if } i = 1 \\ f(\sigma(i)|\{\sigma(1), \ldots, \sigma(i-1)\}) & \text{otherwise,} \end{cases}
\]

where \( \sigma \) is a random permutation of \([n]\) such that \( \{\sigma(1), \ldots, \sigma(|X|)\} = X \). We stop the procedure when \( S_{i-1} = S_i \).

Experiments are presented in Figure 1. We first observe that our Greedy algorithm (Algorithm 1) is always better than ModMod. More precisely, we see that the larger the value of \( \lambda \), the greater the ratio between the two algorithms. On the other hand, the computation time of the two algorithms seems to be similar, with a slight tendency for the ModMod to be faster when \( \lambda \) is bigger (but the performance is worse).

---

5. The authors noticed that ModMod is fast at each iteration and experimentally does about as well as the Supermodular-Submodular (SupSub) and the Submodular-Supermodular (SubSup) procedures (Iyer and Bilmes, 2012).
Approximation relationship between RS and DS optimization

Figure 1: Performance comparisons between Algorithm 1 and ModMod. the ”+” corresponds to the mean point. The computation time difference is given in seconds.

5. Conclusion

In this paper, we studied the problem of maximizing the ratio $f/g$ (RS) and the difference $f - g$ (DS), where $f$ and $g$ are two monotonous submodular set functions, as well as the relationship between these two problems. We showed that for a weaker approximation guarantee, these two problems are equivalent. We also provided a FPTAS to switch from one to the other and generalized the GreedRatio algorithm of Bai et al. (2016) for a larger family of problems including RS and DS. An interesting research direction would be to further generalize the results to combinations of more than two submodular functions. Another interesting question is whether assumptions on $\Psi$ in Theorem 2 can be further reduced.

References

Francis Bach. Learning with Submodular Functions: A Convex Optimization Perspective. 2011. URL http://arxiv.org/abs/1111.6453.

Wenruo Bai, Rishabh Iyer, Kai Wei, and Jeff Bilmes. Algorithms for optimizing the ratio of submodular functions. In International Conference on Machine Learning, pages 2751–2759, 2016.

Niv Buchbinder and Moran Feldman. Submodular functions maximization problems. Handbook of Approximation Algorithms and Metaheuristics, 1:753–788, 2017.

Huiping Chen, Grigorios Loukides, Jiashi Fan, and Hau Chan. Limiting the influence to vulnerable users in social networks: A ratio perspective. In International Conference on Advanced Information Networking and Applications, pages 1106–1122. Springer, 2019.

Wei Chen, Chi Wang, and Yajun Wang. Scalable influence maximization for prevalent viral marketing in large-scale social networks. In Knowledge Discovery and Data Mining, 2010.
Michele Conforti and Gérard Cornuéjols. Submodular set functions, matroids and the greedy algorithm: Tight worst-case bounds and some generalizations of the Rado-Edmonds theorem. *Discrete Applied Mathematics*, 7(3):251–274, 1984. ISSN 0166218X. doi: 10.1016/0166-218X(84)90003-9.

Werner Dinkelbach. On nonlinear fractional programming. *Management science*, 13(7):492–498, 1967.

Alina Ene, Sofia Maria Nikolakaki, and Evimaria Terzi. Team formation: Striking a balance between coverage and cost, 2020.

Moran Feldman. Guess free maximization of submodular and linear sums. *Algorithmica*, pages 1–26, 2020.

Yuval Filmus and Justin Ward. A tight combinatorial algorithm for submodular maximization subject to a matroid constraint. In *Proceedings - Annual IEEE Symposium on Foundations of Computer Science, FOCS*, pages 659–668, 2012. ISBN 978-0-7695-4874-6. doi: 10.1109/FOCS.2012.55.

Satoru Fujishige. *Submodular functions and optimization*, volume 58. Elsevier, 2005.

Michel X Goemans, Nicholas JA Harvey, Satoru Iwata, and Vahab Mirrokni. Approximating submodular functions everywhere. In *Proceedings of the twentieth annual ACM-SIAM symposium on Discrete algorithms*, pages 535–544. Society for Industrial and Applied Mathematics, 2009.

Michel Gondran and Michel Minoux. *Graphs and algorithms*. Wiley, 1984.

Christopher Harshaw, Moran Feldman, Justin Ward, and Amin Karbasi. Submodular maximization beyond non-negativity: Guarantees, fast algorithms, and applications. *arXiv preprint arXiv:1904.09354*, 2019.

JR Isbell and WH Marlow. Attrition games. *Naval Research Logistics Quarterly*, 3(1-2):71–94, 1956.

S Iwata, L Fleischer, and Satoru Fujishige. A combinatorial strongly polynomial algorithm for minimizing submodular functions. *Journal of the ACM*, 48(4):761–777, 2001.

Rishabh Iyer and Jeff Bilmes. Algorithms for approximate minimization of the difference between submodular functions, with applications. *arXiv preprint arXiv:1207.0560*, 2012.

Rishabh K Iyer, Stefanie Jegelka, and Jeff A Bilmes. Curvature and optimal algorithms for learning and minimizing submodular functions. In *Advances in Neural Information Processing Systems*, pages 2742–2750, 2013.

K J Joseph, Vinam Teja R, Krishnakant Singh, and Vineeth N Balasubramanian. Submodular batch selection for training deep neural networks. In *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19*, pages 2677–2683. International Joint Conferences on Artificial Intelligence Organization, 7 2019. doi: 10.24963/ijcai.2019/372. URL https://doi.org/10.24963/ijcai.2019/372.
Yoshinobu Kawahara and Takashi Washio. Prismatic algorithm for discrete dc programming problem. In Advances in Neural Information Processing Systems, pages 2106–2114, 2011.

David Kempe, Jon Kleinberg, and Éva Tardos. Maximizing the spread of influence through a social network. Knowledge Discovery and Data Mining, page 137, 2003.

David Kempe, Jon Kleinberg, and Éva Tardos. Maximizing the spread of influence through a social network. Theory of Computing, 11(4):105–147, 2015.

Samir Khuller, Anna Moss, and Joseph Seffi Naor. The budgeted maximum coverage problem. Information processing letters, 70(1):39–45, 1999.

A Krause and C. Guestrin. Near-optimal nonmyopic value of information in graphical models. In Proc. UAI, 2005.

Jure Leskovec and Andrej Krevl. {SNAP Datasets}: {Stanford} Large Network Dataset Collection. \url{http://snap.stanford.edu/data}, jun 2014.

H Lin and J Bilmes. A Class of Submodular Functions for Document Summarization. In North American chapter of the Association for Computational Linguistics/Human Language Technology Conference (NAACL/HLT-2011), Portland, OR, 2011.

László Lovász. Submodular functions and convexity. Mathematical programming the state of the art, pages 235–257, 1983. URL http://www.cs.elte.hu/\%7B~\%7Dlovasz/scans/submodular.pdf.

M Minoux. Accelerated greedy algorithms for maximizing submodular set functions. Optimization Techniques, pages 234–243, 1978.

Mukund Narasimhan and Jeff A. Bilmes. A submodular-supermodular procedure with applications to discriminative structure learning. arXiv preprint arXiv:1207.1404, 2012.

Pierre Perrault, Vianney Perchet, and Michal Valko. Exploiting structure of uncertainty for efficient matroid semi-bandits. In Kamalika Chaudhuri and Ruslan Salakhutdinov, editors, Proceedings of the 36th International Conference on Machine Learning, volume 97 of Proceedings of Machine Learning Research, pages 5123–5132, Long Beach, California, USA, 09–15 Jun 2019. PMLR. URL http://proceedings.mlr.press/v97/perrault19a.html.

Chao Qian, Jing-Cheng Shi, Yang Yu, Ke Tang, and Zhi-Hua Zhou. Optimizing ratio of monotone set functions. In Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI-17, pages 2606–2612, 2017. doi: 10.24963/ijcai.2017/363. URL https://doi.org/10.24963/ijcai.2017/363.

Siegfried Schaible. Fractional programming. ii, on dinkelbach’s algorithm. Management science, 22(8):868–873, 1976.

Alexander Schrijver. Combinatorial Optimization: Polyhedra and Efficiency. 2008. ISBN 3540204563. doi: 10.1007/978-3-642-24508-4. URL http://scholar.google.com/scholar?hl=en(&)btnG=Search(&)q=intitle:Algorithms+and+Combinatorics+24(#9).
J Shi and J Malik. Normalized Cuts and Image Segmentation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 22:888–905, 2000.

Maxim Sviridenko, Jan Vondrák, and Justin Ward. Optimal approximation for submodular and supermodular optimization with bounded curvature. nov 2013. URL http://arxiv.org/abs/1311.4728.

Sebastian Tschiatschek, Aytunc Sahin, and Andreas Krause. Differentiable submodular maximization. In *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI-18*, pages 2731–2738. International Joint Conferences on Artificial Intelligence Organization, 7 2018. doi: 10.24963/ijcai.2018/379. URL https://doi.org/10.24963/ijcai.2018/379.

Jan Vondrak. Submodularity and curvature: the optimal algorithm. *RIMS Kokyuroku Bessatsu*, pages 253–266, 2010. ISSN 8755-2930. doi: 10.1193/1.1623497.

Yi-Jing Wang, Da-Chuan Xu, Yan-Jun Jiang, and Dong-Mei Zhang. Minimizing ratio of monotone non-submodular functions. *Journal of the Operations Research Society of China*, Mar 2019. ISSN 2194-6698. doi: 10.1007/s40305-019-00244-1. URL https://doi.org/10.1007/s40305-019-00244-1.

Zheng Wen, Branislav Kveton, Michal Valko, and Sharan Vaswani. Online influence maximization under independent cascade model with semi-bandit feedback. In *Neural Information Processing Systems*, 2017.
Appendix A. Proof of Theorem 1

**Proof** Assume first we have access to an algorithm $A$ for $\alpha$-approximation of Ratio. We consider the procedure given in Algorithm 3, that take as input $f \in \mathcal{F}$, $g \in \mathcal{G}$ and $\varepsilon > 0$.

**Algorithm 3** DIFFERENCE FROM RATIO

| Initialization: $c_- \leftarrow f(x_g) - g(x_g)$, $c_+ \leftarrow \alpha f(x_f) - g(x_g)$, $x \leftarrow x_g$. |
| while $c_+ - c_- > \varepsilon$ do |
| $c \leftarrow (c_+ + c_-)/2$. |
| $y \leftarrow A(f, c + g)$. |
| if $f(y) - g(y) \leq c$ then |
| $c_+ \leftarrow c$. |
| else |
| $c_- \leftarrow c$. |
| $x \leftarrow y$. |
| endif |
| endwhile |
| Output: $x$. |

**Algorithm 4** RATIO FROM DIFFERENCE

| Initialization: $\lambda_- \leftarrow f(x_g)/g(x_g)$, $\lambda_+ \leftarrow \alpha f(x_f)/g(x_g)$, $x \leftarrow x_g$. |
| while $\lambda_+ - \lambda_- > \varepsilon$ do |
| $\lambda \leftarrow (\lambda_+ + \lambda_-)/2$. |
| $y \leftarrow A(f, \lambda g)$. |
| if $f(y)/g(y) \leq \lambda$ then |
| $\lambda_+ \leftarrow \lambda$. |
| else |
| $\lambda_- \leftarrow \lambda$. |
| $x \leftarrow y$. |
| endif |
| endwhile |
| Output: $x$. |

In Algorithm 3, we always have $c_- \leq f(x) - g(x)$ and $c_+ \geq \alpha f(x') - g(x')$, $\forall x' \in \mathcal{X}$. Indeed, these inequalities are clear at the initialization, and the first remains trivially true through the loop from the update of $x$ and $c_-$. The second remains true since for all $x' \in \mathcal{X}$, when $c_+$ is updated, we have

$$1 \geq \frac{f(y)}{g(y) + c_+} \geq \frac{\alpha f(x')}{g(x') + c_+},$$

where (3) is from $f(y) - g(y) \leq c_+$, and (4) from approximation guarantee of $A$. Once Algorithm 3 leaves the while loop, $c_+ - c_- \leq \varepsilon$, giving

$$\forall x' \in \mathcal{X}, \quad \alpha f(x') - g(x') \leq c_+ \leq c_- + \varepsilon \leq f(x) - g(x) + \varepsilon.$$

Assume now that the algorithm $A$ solves $\alpha$-approximation of Difference, and consider Algorithm 4 that takes as input $f \in \mathcal{F}$, $g \in \mathcal{G}$ and $\varepsilon > 0$.

In the same way as previously, in Algorithm 4, we always have $\lambda_- \leq f(x)/g(x)$ and $\lambda_+ \geq \alpha f(x')/g(x')$, $\forall x' \in \mathcal{X}$, due to the following when $\lambda_+$ is updated:

$$0 \geq f(y) - \lambda_+ g(y) \geq \alpha f(x') - \lambda_+ g(x'),$$

where (5) is from $f(y)/g(y) \leq \lambda_+$, and (6) from approximation guarantee of $A$. Again, once Algorithm 4 leaves the while loop, $\lambda_+ - \lambda_- \leq \varepsilon$, giving

$$\forall x' \in \mathcal{X}, \quad \alpha f(x')/g(x') \leq \lambda_+ \leq \lambda_- + \varepsilon \leq f(x)/g(x) + \varepsilon.$$

The number of iterations needed for both Algorithm 3 and 4 to converge is of order $O(\log(1/\varepsilon))$ since the quantity $c_+ - c_-$ (resp. $\lambda_+ - \lambda_-$) is divided by two at each step. ■
Remark 2 If one want to have an approximation factor in front of \(g\) rather than \(f\), an idea would be to consider Algorithm 3 with a call of \(A(f-c,g)\). However, the function \(f-c\) would not be guaranteed to be non-negative, whereas we know in our case that \(c+g>0\). Indeed, since \(c_"<c_+\) in the while loop, necessarily \(c>c_\geq f(x_g)-g(x_g)\), so \(c+g>f(x_g)+g-g(x_g)\geq 0\).

Appendix B. Proof of Proposition 1

Proof From modularity, a maximizer of \(\Psi(f,g) = f-g\) is clearly

\[
S^* = \{i \in [n], \ f(i|\emptyset) - g(i|\emptyset) > 0\} = \left\{i \in [n], \ \frac{f(i|\emptyset)}{g(i|\emptyset)} > 1\right\}.
\]

Again by modularity, \(f(i|\emptyset)/g(i|\emptyset) = f(i|S_{k-1})/g(i|S_{k-1})\) for any \(k \in [n]\). So

\[
S^* = \left\{i_k, k \in [n], \ \frac{f(i_k|S_{k-1})}{g(i_k|S_{k-1})} > 1\right\},
\]

which is of the form \(\{i_1, \ldots, i_k\}\) for some \(k \in \{0, \ldots, n\}\), so is inspected at the end of Algorithm 1, which thus outputs a maximizer of \(\Psi(f,g) = f-g\).

Let’s see now the case \(\Psi(f,g) = f/g\). Let \(\lambda \triangleq \max_{S \in \mathcal{P}([n])} f(S)/g(S)\). By the same argument as above, a maximizer of \(f - \lambda g\) is

\[
S^* = \left\{i_k, k \in [n], \ \frac{f(i_k|S_{k-1})}{g(i_k|S_{k-1})} > \lambda\right\},
\]

which is of the form \(\{i_1, \ldots, i_k\}\) for some \(k \in \{0, \ldots, n\}\), so will be inspected at the end of Algorithm 1. Since the maximum of \(f - \lambda g\) is 0, \(S^*\) is also a maximizer of \(f/g\), so again Algorithm 1 outputs a maximizer of \(\Psi(f,g) = f/g\).

Appendix C. Proof of Theorem 2

Proof For all \(k \in [n]\),

\[
f(S^*) - f(S_{k-1}) \overset{(7)}{\leq} \sum_{i \in S^* \setminus S_{k-1}} f(i|S_{k-1})
\]

\[
\overset{(8)}{\leq} \frac{f(i_k|S_{k-1})}{g(i_k|S_{k-1})} \sum_{i \in S^* \setminus S_{k-1}} g(i|S_{k-1})
\]

\[
\overset{(9)}{\leq} \frac{f(i_k|S_{k-1})}{g(i_k|S_{k-1})} \sum_{i \in S^*} g(i|\emptyset)
\]

\[
\overset{(10)}{\leq} \frac{f(i_k|S_{k-1})}{g(i_k|S_{k-1})} \sum_{i \in S^*} \frac{g(i|S^* \setminus \{i\})}{1-c_g}
\]

\[
\overset{(11)}{\leq} \frac{f(i_k|S_{k-1})}{g(i_k|S_{k-1})} \cdot \frac{g(S^*) - g(\emptyset)}{1-c_g}
\]
Where (7) is from submodularity, monotonicity of \( f \), (8) is from Algorithm 1, (9) is from submodularity, monotonicity of \( g \), (10) from the curvature of \( g \), and (11) is from submodularity of \( g \). In other word, we have for all \( k \in [n] \) such that \( f(S^*) - f(S_{k-1}) \geq 0 \) that

\[
\frac{(1 - c_g)g(\ell)S_{k-1})}{g(S^*) - g(\emptyset)} \leq \frac{f(\ell)S_{k-1})}{f(S^*) - f(S_{k-1})}.
\]  

(12)

By monotonicity of \( g \), there must be an index \( \ell \in \{0, 1, \ldots, n-1\} \) such that \( g(S_{\ell}) \leq g(S^*) \leq g(S_{\ell+1}) \). Let \( \alpha \in [0, 1] \) be such that

\[
g(S^*) = \alpha g(S_{\ell+1}) + g(S_{\ell}).
\]  

(13)

Now, we claim that \((1 - e^{c_g - 1})f(S^*) \leq (1 - \alpha)f(S_{\ell}) + \alpha f(S_{\ell+1}) \) holds. To this end, we will look at two cases: If \( f(S^*) - (1 - \alpha)f(S_{\ell}) - \alpha f(S_{\ell+1}) \leq 0 \), then we trivially have

\[
(1 - e^{c_g - 1})f(S^*) \leq f(S^*) \leq (1 - \alpha)f(S_{\ell}) + \alpha f(S_{\ell+1}) .
\]

Otherwise,

\[
\begin{align*}
& f(S^*) - (1 - \alpha)f(S_{\ell}) - \alpha f(S_{\ell+1}) > 0 \\
\text{and} & f(S^*) - f(S_k) > 0 \text{ for all } k \in [\ell].
\end{align*}
\]  

(14)

Thus, we can upper bound \( \frac{f(S^*) - (1 - \alpha)f(S_{\ell}) - \alpha f(S_{\ell+1})}{f(S^*)} \) by the following quantity:

\[
\begin{align*}
& \frac{f(S^*) - (1 - \alpha)f(S_{\ell}) - \alpha f(S_{\ell+1})}{f(S^*)} \\
& = f(S^*) - f(S_{\ell}) - \alpha f(i_{\ell+1}S_{\ell}) \\
& f(S^*) - f(S_{\ell}) \sum_{k \in [\ell]} f(S^*) - f(S_k) \\
& = \left(1 - \frac{\alpha f(i_{\ell+1}S_{\ell})}{f(S^*) - f(S_{\ell})}\right) \left(1 - \frac{f(i_k|S_{k-1})}{f(S^*) - f(S_{k-1})}\right) \\
& \leq \left(1 - \frac{(1 - c_g)g(i_{\ell+1}S_{\ell})}{g(S^*) - g(\emptyset)}\right) \left(1 - \frac{(1 - c_g)g(i_k|S_{k-1})}{g(S^*) - g(\emptyset)}\right) \\
& \leq \exp\left(- (1 - c_g)\frac{\alpha g(i_{\ell+1}S_{\ell}) + \sum_{k \in [\ell]} g(i_k|S_{k-1})}{g(S^*) - g(\emptyset)}\right) \\
& = \exp\left(- (1 - c_g)\frac{\alpha g(i_{\ell+1}S_{\ell}) + g(S_{\ell}) - g(\emptyset)}{g(S^*) - g(\emptyset)}\right) \\
& \equiv e^{c_g - 1}.
\end{align*}
\]

(15)

Where (15) is from (12) and (14), (16) is from \( 1 - x \leq e^{-x} \) and (17) is from (13). Rearranging the inequality, we obtain our claim:

\[
(1 - e^{c_g - 1})f(S^*) \leq (1 - \alpha)f(S_{\ell}) + \alpha f(S_{\ell+1}) .
\]
We now use the fact that $\Psi$ is non-decreasing with respect to the first variable to get

$$
\Psi\left((1 - e^{c_0^{-1}})f(S^*) , \cdot \right) \leq \Psi((1 - \alpha)f(S_\ell) + \alpha f(S_{\ell + 1}), \cdot ).
$$

In the previous inequality, we evaluate the second variable to $g(S^*) = (1 - \alpha)g(S_\ell) + \alpha g(S_{\ell + 1})$, and use the quasiconvexity of $\Psi$ to get that $\Psi((1 - e^{c_0^{-1}})f(S^*), g(S^*))$ is upper bounded by

$$
\Psi((1 - \alpha)f(S_\ell) + \alpha f(S_{\ell + 1})), (1 - \alpha)g(S_\ell) + \alpha g(S_{\ell + 1})) \\
\leq \max_{k \in \{\ell, \ell + 1\}} \Psi(f(S_k), g(S_k)).
$$

To conclude, see that the output $S$ of Algorithm 1 maximizes $\Psi(f(S_k), g(S_k))$ over $k \in \{0, \ldots, n\}$, so we have $\max_{k \in \{\ell, \ell + 1\}} \Psi(f(S_k), g(S_k)) \leq \Psi(f(S), g(S))$. 

\[\square\]

**Appendix D. Proof of Theorem 3**

**Proof** Without loss of generality, we assume that $|S^*| \geq 3$, since otherwise the algorithm find the optimal solution. Let us write $S^*_i = \{e^*_1, \ldots, e^*_i\}$ where the elements are ordered such that:

$$
e^*_j = \arg\max_{e \in S^*_i \cup \{e\}} f(S^*_i \cup \{e\}).
$$

We consider the iteration where $b$ is the largest such that $(b - 1)\varepsilon \leq g(S^*) \leq b\varepsilon$ and where $S = S^*_3$. We thus have $g(S_G) - \varepsilon \leq (b - 1)\varepsilon \leq g(S^*) \leq b\varepsilon \leq g(S_G \cup \{x^*\})$ for $x^* = \arg\max_{i \in S^*_G \setminus S_G} f(i|S_G)/g(i|S_G)$. Replacing $i_{\ell + 1}$ by $x^*$ in the proof of Theorem 2 and letting $\alpha$ being such that $g(S^*) = (1 - \alpha)(g(S_G) - \varepsilon) + \alpha g(S_G \cup \{x^*\})$, we can replace equality (17) with an inequality and have that

$$
(1 - e^{c_0^{-1}})f(S^*) \leq (1 - \alpha)\tilde{f}(S_G) + \alpha \tilde{f}(S_G \cup \{x^*\}) \leq \tilde{f}(S_G \cup \{x^*\}),
$$

where $\tilde{f}(S) \triangleq f(S \cup S^*_G) - f(S^*_G) + f(\emptyset)$. Notice now that from the definition of $S^*_3$, and the submodularity of $f$, we have $f(S_G \cup \{x^*\}) - f(S_G) \leq \frac{f(S_G^*_{\emptyset}) - f(\emptyset)}{3}$, we thus have

$$
(1 - e^{c_0^{-1}})f(S^*) \leq (e^{c_0^{-1}} - 1/3)(f(S^*_G) - f(\emptyset)) + (1 - e^{c_0^{-1}})f(S^*) \leq f(S_G).
$$

We now use the fact that $\Psi$ is non-decreasing with respect to the first variable, and non-increasing with respect to the second variable

$$
\Psi\left((1 - e^{c_0^{-1}})f(S^*), g(S^*)\right) \leq \Psi(f(S_G), g(S_G) - \varepsilon).
$$

Notice that $S_G$ satisfies the knapsack constraint since $g(S_G) \leq b\varepsilon \leq B$. We thus have the desired result. 

\[\square\]
Appendix E. Optimality of the approximation when $g$ is modular

In this section, we show that the approximations results given in (1) and (2) are optimal. To this end, we shall prove that if there exists an algorithm with an improved $\alpha = 1 - 1/e + \varepsilon$ ratio, for some $\varepsilon > 0$, then we can use it to solve a randomized version of submodular maximization under cardinality constraint, with a factor $\alpha$. We consider the DS problem only, noticing that the same proof holds for RS. Let $f$ be a monotone normalized submodular function. Given some $\lambda > 0$, we can use a linear search to find the value $\lambda' = h(\lambda)$ that maximizes $f(S_{\lambda'}) - \lambda|S_{\lambda'}|$, where $S_{\lambda'}$ is the $\alpha$-approximate maximizer such that $f(S_{\lambda'}) - \lambda'|S_{\lambda'}| \geq \alpha f(S^*) - \lambda'|S^*|$. We notice that we have

$$f(S_{h(\lambda)}) - \lambda|S_{h(\lambda)}| \geq f(S_\lambda) - \lambda|S| \geq \alpha f(S^*) - \lambda|S^*|.$$ 

In addition, we also have that the function $\lambda \mapsto \max_{S_{\lambda'}} f(S_{\lambda'}) - \lambda|S_{\lambda'}|$ is continuous, and non-increasing. If $\lambda$ is some branching point between two maximizers $S_{h(\lambda^+)}$ and $S_{h(\lambda^-)}$, then looking at the derivative with respect to $\lambda$, we see that the cardinality of the maximizer dictates which one dominates if we increase/decrease $\lambda$. This means that $\lambda \mapsto |S_{h(\lambda)}|$ is non-increasing. Thus, given an integer $m \in [n]$, we have that there is a $\lambda$ (that we can find with a binary search) such that either $|S_{h(\lambda)}| = m$, or $|S_{h(\lambda^+)}| < m < |S_{h(\lambda^-)}|$ (in such case, we use a randomized maximizer $\tilde{S}_{h(\lambda)}$ so that $\mathbb{E}[|\tilde{S}_{h(\lambda)}|] = m$). We finally see that

$$\mathbb{E}[f(\tilde{S}_{h(\lambda)})] \geq \alpha f(S^*), \text{ with } \mathbb{E}[|\tilde{S}_{h(\lambda)}|] = m, \forall S^* \text{ s.t. } |S^*| = m.$$ 

Note that the linear and binary search gives a slight error in the $\alpha$ factor, but that this error can be made strictly smaller than $\varepsilon$, thus contradicting the optimality of the factor $1 - 1/e$ for the above problem.