CHASING MAXIMAL PRO-\(p\) GALOIS GROUPS
VIA 1-CYCLOTOMICITY

CLAUDIO QUADRELLI

Abstract. Let \(p\) be a prime. We prove that certain amalgamated free pro-\(p\) products of Demushkin groups with pro-\(p\)-cyclic amalgam cannot give rise to a 1-cyclotomic oriented pro-\(p\) group, and thus do not occur as maximal pro-\(p\) Galois groups of fields containing a root of 1 of order \(p\). We show that other cohomological obstructions which are used to detect pro-\(p\) groups that are not maximal pro-\(p\) Galois groups — the quadraticity of \(\mathbb{Z}/p\mathbb{Z}\)-cohomology and the vanishing of Massey products — fail with the above pro-\(p\) groups. Finally, we prove that the Minač-Tân pro-\(p\) group cannot give rise to a 1-cyclotomic oriented pro-\(p\) group, and we conjecture that every 1-cyclotomic oriented pro-\(p\) group satisfy the strong \(n\)-Massey vanishing property for \(n > 2\).

1. Introduction

Let \(p\) be a prime number, and let \(1 + p\mathbb{Z}_p\) denote the pro-\(p\) group of principal units of the ring of \(p\)-adic integers \(\mathbb{Z}_p\) — namely, \(1 + p\mathbb{Z}_p = \{1 + p\lambda \mid \lambda \in \mathbb{Z}_p\}\). An oriented pro-\(p\) group is a pair \((G, \theta)\) consisting of a pro-\(p\) group \(G\) and a morphism of pro-\(p\) groups \(\theta: G \to 1 + p\mathbb{Z}_p\), called an orientation of \(G\) (see [30]; oriented pro-\(p\) groups were introduced by I. Efrat in [7], with the name “cyclotomic pro-\(p\) pairs”). An oriented pro-\(p\) group \((G, \theta)\) gives rise to the continuous \(G\)-module \(\mathbb{Z}_p(\theta)\), which is equal to \(\mathbb{Z}_p\) as an abelian pro-\(p\) group, and which is endowed with the continuous \(G\)-action defined by

\[
g \cdot \lambda = \theta(g) \cdot \lambda \quad \text{for all } g \in G \text{ and } \lambda \in \mathbb{Z}_p(\theta).
\]

An oriented pro-\(p\) group \((G, \theta)\) is said to be Kummerian if the following cohomological condition is satisfied: for every \(n \geq 1\) the natural morphism

\[
H^1(G, \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)) \to H^1(G, \mathbb{Z}/p\mathbb{Z}),
\]

induced by the epimorphism of continuous \(G\)-modules \(\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta) \to \mathbb{Z}/p\mathbb{Z}\) is surjective (see [11]) — here we consider \(\mathbb{Z}/p\mathbb{Z}\) as a trivial \(G\)-module. Moreover, the oriented pro-\(p\) group \((G, \theta)\) is said to be 1-cyclotomic if the above cohomological condition is satisfied also for every closed subgroup of \(G\) — namely, the natural morphism \((1.1)\) is surjective also with \(H\) instead of \(G\), and the restriction \(\theta|_H: H \to 1 + p\mathbb{Z}_p\) instead of \(\theta\) for all closed subgroups \(H\) of \(G\) (in [26,27] a 1-cyclotomic oriented pro-\(p\) group is called a “1-smooth” oriented pro-\(p\) group). This cohomological condition was considered first by J. Labute, who showed ane litteram that for every Demushkin group \(G\) there exists

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precisely one orientation which completes $G$ into a Kummerian oriented pro-$p$ group, namely, the orientation induced by the dualizing module of $G$ (see [14]).

In case of trivial orientations, 1-cyclotomicity translates into a purely group-theoretical statement. Namely, an oriented pro-$p$ group $(G, 1)$ — where $1 : G \to 1 + p\mathbb{Z}_p$ denotes the orientation which is constantly equal to $1$ — is 1-cyclotomic if, and only if, the abelianization of every closed subgroup of $G$ is a free abelian pro-$p$ group. Pro-$p$ groups satisfying this group-theoretic condition are called absolutely torsion-free pro-$p$ groups, and they were introduced by T. Würfel in [37].

The main goal of this work is to produce new examples of pro-$p$ groups which no orientations can turn into a 1-cyclotomic oriented pro-$p$ group.

**Theorem 1.1.** Let $G$ be a pro-$p$ group with pro-$p$ presentation

\[(1.2) \quad G = \langle x, y_1, \ldots, y_{d_1}, z_1, \ldots, z_{d_2} \mid r_1 = r_2 = 1 \rangle,
\]

where $d_1, d_2$ are two positive odd integers, and either:

- $(1.1.a)$ $d_1 + d_2 \geq 4$ and
  \[
  r_1 = [x, y_1][y_2, y_3] \cdots [y_{d_1-1}, y_{d_1}],
  \]
  \[
  r_2 = [x, z_1][z_2, z_3] \cdots [z_{d_2-1}, z_{d_2}];
  \]
- $(1.1.b)$ $p$ is odd and
  \[
  r_1 = y_1^p[y_1, x][y_2, y_3] \cdots [y_{d_1-1}, y_{d_1}],
  \]
  \[
  r_2 = z_1^p[z_1, x][z_2, z_3] \cdots [z_{d_2-1}, z_{d_2}].
  \]

Then there are no orientations $\theta : G \to 1 + p\mathbb{Z}_p$ such that the oriented pro-$p$ group $(G, \theta)$ is 1-cyclotomic.

It is worth underlining that the pro-$p$ groups described in Theorem 1.1 are amalgamated free pro-$p$ products of two Demushkin groups — the subgroup generated by $x, y_1, \ldots, y_{d_1}$ and the subgroup generated by $x, z_1, \ldots, z_{d_2}$ —, with pro-$p$-cyclic amalgam, generated by $x$. Despite Demushkin groups and their free pro-$p$ products are some of the (extremely few) examples of pro-$p$ groups which are known to give rise to 1-cyclotomic oriented pro-$p$ groups, the presence of a pro-$p$-cyclic amalgam is enough to lose 1-cyclotomicity.

Oriented pro-$p$ groups satisfying 1-cyclotomicity have great prominence in Galois theory. Given a field $K$, let $\overline{K}$ and $K(p)$ denote respectively the separable closure of $K$, and the compositum of all finite Galois $p$-extensions of $K$. The maximal pro-$p$ Galois group of $K$, denoted by $G_K(p)$, is the maximal pro-$p$ quotient of the absolute Galois group $\text{Gal}(\overline{K}/K)$ of $K$, and it coincides with the Galois group of the Galois extension $K(p)/K$. Detecting maximal pro-$p$ Galois groups among pro-$p$ groups, are crucial problems in Galois theory. Already the pursuit of concrete examples of pro-$p$ groups which do not occur as maximal pro-$p$ Galois groups of fields is already considered a very remarkable challenge (see [12] § 25.16), and, e.g., [13, 34, 20, 34].

The maximal pro-$p$ Galois group $G_K(p)$ of a field $K$ containing a root of 1 of order $p$ gives rise to the oriented pro-$p$ group $(G_K(p), \theta_K)$, where

$\theta_K : G_K(p) \longrightarrow 1 + p\mathbb{Z}_p$
denotes the pro-$p$ cyclotomic character (see Example 2.3 below). By Kummer theory, the oriented pro-$p$ group $(G_K(p), \theta_K)$ is 1-cyclotomic (see [14, p. 131] and [11, § 4]) — in case $p = 2$ we need to assume further that $\sqrt{-1} \in K$. Therefore, a pro-$p$ group which cannot complete into a 1-cyclotomic oriented pro-$p$ group does not occur as the maximal pro-$p$ group of a field containing a root of 1 of order $p$ — and hence neither as the absolute Galois group of any field (see, e.g., [25, Rem. 3.3]). Hence, the following corollary may be deduced directly from Theorem 1.1.

**Corollary 1.2.** A pro-$p$ group $G$ as in Theorem 1.1 does not occur as the maximal pro-$p$ Galois group of any field containing a root of 1 of order $p$ (and also $\sqrt{-1}$ if $p = 2$). Hence, $G$ does not occur as the absolute Galois group of any field.

In the recent past, other cohomological properties have been used to study maximal pro-$p$ Galois groups — and to find examples of pro-$p$ groups which do not occur as maximal pro-$p$ Galois groups. By the Norm Residue Theorem — proved by M. Rost and V. Voevodsky, with the contribution by Ch. Weibel, see [13, 35] — one knows that if $K$ is a field containing a root of 1 of order $p$, the $\mathbb{Z}/p\mathbb{Z}$-cohomology algebra $H^\bullet(G_K(p), \mathbb{Z}/p\mathbb{Z})$, endowed with the cup-product

$$\cup : H^m(G_K(p), \mathbb{Z}/p\mathbb{Z}) \times H^n(G_K(p), \mathbb{Z}/p\mathbb{Z}) \to H^{m+n}(G_K(p), \mathbb{Z}/p\mathbb{Z}),$$

is quadratic, i.e., its ring structure is determined by the 1st and the 2nd cohomology groups (see, e.g., [23, § 2]). Moreover, it was shown by E. Matzri that if $K$ is a field containing a root of 1 of order $p$, then $G_K(p)$ satisfies the triple Massey vanishing property (see [3] and references therein) — for an overview on Massey products in Galois cohomology see [20]. These two cohomological properties were used to find examples of pro-$p$ groups which do not occur as maximal pro-$p$ Galois groups of fields containing a root of 1 of order $p$, for example in [3, § 8] and in [20, § 7].

We prove that the pro-$p$ groups described in Theorems 1.1 cannot be ruled out as maximal pro-$p$ Galois groups employing the above two cohomological obstructions.

**Proposition 1.3.** Let $G$ be a pro-$p$ group as in Theorem 1.1.

(i) The $\mathbb{Z}/p\mathbb{Z}$-cohomology algebra $H^\bullet(G, \mathbb{Z}/p\mathbb{Z})$ is quadratic.

(ii) The pro-$p$ group $G$ satisfies the cyclic $p$-Massey vanishing property — namely, the $p$-fold Massey product

$$\langle \alpha, \ldots, \alpha \rangle$$

contains 0 for every $\alpha \in H^1(G, \mathbb{Z}/p\mathbb{Z})$.

(iii.a) If $G$ is as in (1.1.a), then $G$ satisfies the 3- and the strong 4-Massey vanishing property.

(iii.b) If $G$ is as in (1.1.b) and $p > 3$ then $G$ satisfies the 3- and the strong 4-Massey vanishing property.

(We recall the basic notions on Massey products in Galois cohomology in § 6.1 below.) Hence, Corollary 1.2 provides brand new examples of pro-$p$ groups which do not occur as maximal pro-$p$ Galois groups of fields containing a root of 1 of order $p$, and as absolute Galois groups. Moreover, we remark that the relations which define the pro-$p$ groups described in Theorem 1.1 are rather “elementary” — just elementary commutators of
generator times, possibly, the $p$-power of a generator —, unlike the examples provided in [14][20][25], where the relations involve higher commutators.

Finally, we focus on the Minač-Tân pro-$p$ group, i.e., the pro-$p$ group $G$ with pro-$p$ presentation

$$G = \langle x_1, \ldots, x_5 \mid [[[x_1, x_2], x_3], x_4, x_5] = 1 \rangle.$$  

In [20, § 7], J. Minač and N.D. Tân showed that $G$ does not satisfy the 3-Massey vanishing property, and thus it does not occur as the maximal pro-$p$ Galois group of any field containing a root of 1 of order $p$. We prove that $G$ cannot complete into a 1-cyclotomic oriented pro-$p$ group.

**Theorem 1.4.** Let $p$ be an odd prime. Then there are no orientations turning the Minač-Tân pro-$p$ group into a 1-cyclotomic oriented pro-$p$ group.

Theorem 1.4 has been proved independently by I. Snopce and P. Zalesskiĭ (unpublished). Theorem 1.4 provides a negative answer to the question posed in [30, Rem. 3.7] — namely, the Minač-Tân pro-$p$ group may be ruled out as a maximal pro-$p$ Galois group of a field containing a root of 1 of order $p$ (and thus as an absolute Galois group) in a “Massey-free” way.

Altogether, 1-cyclotomicity of oriented pro-$p$ groups provides a rather powerful tool studying maximal pro-$p$ Galois groups, and it succeeds in detecting pro-$p$ groups which are not maximal pro-$p$ Galois groups when other methods fail, as underlined above. We believe that further investigations in this direction will lead to new obstructions for the realization of pro-$p$ groups as maximal pro-$p$ Galois group.

Actually, Theorem 1.4 and the main result in [34] (see in particular [34, p. 1907]), may lead to the suspect that 1-cyclotomicity is a more restrictive condition in comparison with the vanishing of Massey products. Thus, we formulate the following conjecture.

**Conjecture 1.5.** Let $(G, \theta)$ be an oriented pro-$p$ group, such that $\text{Im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$ if $p = 2$. If $(G, \theta)$ is 1-cyclotomic, then the pro-$p$ group $G$ satisfies the 3-Massey vanishing property; if moreover $G$ is finitely generated, then $G$ satisfies the strong $n$-Massey vanishing property for every $n \geq 3$.

After the publication on the arXiv of an earlier version of this paper, A. Merkurjev and F. Scavia proved the first statement of Conjecture 1.5 — see [17 Thm. 1.3] —; while, on the other hand, there are 1-cyclotomic oriented pro-$2$ groups $(G, \theta)$ such that $\text{Im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$, where $G$ is not finitely generated and does not satisfy the strong 4-Massey vanishing property — see [10 Thm. 1.6]. In particular, [17 Thm. 1.3] implies Theorem 1.4 (see also [17 Rem. 6.3]).

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2. Oriented pro-

2.1. Notation and preliminaries. Throughout the paper, every subgroup of a pro-

2.2. Oriented pro-

Fact 2.1. Let \( G \) be a finitely generated pro-

2.3. Kummerianity and 1-cyclotomicity. Let \((G, \theta)\) be an oriented pro-

An oriented pro-

Throughout the paper, we will make use of the following straightforward fact.

Let \((G, \theta)\) be an oriented pro-

(a) If \( N \) is a normal subgroup of \( G \) contained in \( \text{Ker}(\theta) \), one has the oriented pro-

(b) If \( A \) is an abelian pro-

Observe that the \( p \)-power \( g^p \), \( g \in G \) (cf., e.g., [5, Prop. 1.9]). A minimal generating set of \( G \) gives rise to a basis of the \( \mathbb{Z}/p\mathbb{Z}\)-vector space \( G/\Phi(G) \), and conversely (cf., e.g., [5 Prop. 1.9]).

Fact 2.1. Let \( G \) be a finitely generated pro-

An orientation \( \theta : G \to 1 + p\mathbb{Z}_p \) is said to be torsion-free if \( p \) is odd, or if \( p = 2 \) and \( \text{Im}(\theta) \subseteq 1 + 4\mathbb{Z}_2 \). Observe that one may have an oriented pro-

A morphism of oriented pro-

Within the family of oriented pro-

Let \((G, \theta)\) be an oriented pro-

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An oriented pro-

G, \theta Groth, with \( \Phi(G) \) being the canonical projection.

Observe that the \( \theta \)-action on the \( G \)-module \( \mathbb{Z}_p(\theta)/p\mathbb{Z}_p(\theta) \) is trivial, as \( \theta(g) \equiv 1 \mod p \) for all \( g \in G \). Thus, \( \mathbb{Z}_p(\theta)/p\mathbb{Z}_p(\theta) \) is isomorphic to \( \mathbb{Z}/p \) as a trivial \( G \)-module.

An oriented pro-

G, \theta Groth, with \( \Phi(G) \) being the canonical projection.
The subgroup $K_θ(G)$ is normal in $G$, and it is contained in both $\text{Ker}(θ)$ and $\Phi(G)$. On the other hand, $K_θ(G) \supseteq \text{Ker}(θ)'$, so that $\text{Ker}(θ)/K_θ(G)$ is an abelian pro-$p$ group. Moreover, if $θ$ is a torsion-free orientation, $G/\text{Ker}(θ) \cong \text{Im}(θ)$ is torsion-free, and thus either trivial or isomorphic to $\mathbb{Z}_p$. Hence, the epimorphism $G \to G/\text{Ker}(θ)$ splits, and since $ghg^{-1} ≡ h^{θ(g)} \pmod{K_θ(G)}$ for every $g \in G$ and $h \in \text{Ker}(θ)$, one concludes that
\[
(G/K_θ(G), θ/Κ_θ(G)) ≃ \frac{\text{Ker}(θ)}{K_θ(G)} \times (G/\text{Ker}(θ), θ/\text{Ker}(θ))
\]
(c.f., e.g., [31 § 2.2, eq. (2.1)]).

One has the following result relating the subgroup $K_θ(G)$ and the surjectivity of the maps (1.1) (cf. [11, Thm. 7.1], see also [31, Prop. 2.6]).

**Proposition 2.2.** Let $(G, θ)$ be an oriented pro-$p$ group with $θ$ a torsion-free orientation. The following are equivalent.

1. The natural map
   \[
   H^1(G, \mathbb{Z}_p(θ)/p^n\mathbb{Z}_p(θ)) \twoheadrightarrow H^1(G, \mathbb{Z}/p\mathbb{Z}),
   \]
   is surjective for every positive integer $n$.
2. The quotient $\text{Ker}(θ)/K_θ(G)$ is a free abelian pro-$p$ group.

If an oriented pro-$p$ group $(G, θ)$ with torsion-free orientation satisfies the above two equivalent properties, then it is said to be Kummerian. Moreover, $(G, θ)$ is said to be 1-cyclotomic if $(H, θ|_H)$ is Kummerian for every subgroup $H \subseteq G$.

**Remark 2.3.** The original definition of 1-cyclotomic oriented pro-$p$ group requires only that for every open subgroup $U$ of $G$, the oriented pro-$p$ group $(U, θ|_U)$ is Kummerian (cf. [11 § 1]). By a continuity argument, this is enough to imply that the oriented pro-$p$ group $(H, θ|_H)$ is Kummerian also for every subgroup $H \subseteq G$ (cf. [31 Cor. 3.2]).

If $(G, 1)$ is an oriented pro-$p$ group with $1: G \to 1 + p\mathbb{Z}_p$ the orientation constantly equal to 1, then $K_1(G) = G'$, and by Proposition 2.2 $(G, θ)$ is Kummerian if, and only if, $G/G' = \text{Ker}(1)/K_1(G)$ is a free abelian pro-$p$ group (cf. [11 Ex. 3.5–(1)])). Hence, $(G, 1)$ is 1-cyclotomic if, and only if, $H/H'$ is a free abelian pro-$p$ group for every subgroup $H \subseteq G$, i.e., $G$ is absolutely torsion-free (cf. [26 Rem. 2.3]).

**2.4. Examples.**

**Example 2.4.** Let $K$ be a field containing a root of 1 of order $p$, and also $\sqrt{-1}$ if $p = 2$. Then the pro-$p$ cyclotomic character $θ_K$ of $G_K(p)$ — induced by the action of $G_K(p)$ on the roots of 1 of $p$-power order contained in $K(p)$ — has image contained in $1 + p\mathbb{Z}_p$. Observe that $\text{Im}(θ_K) = 1 + p^f\mathbb{Z}_p$, where $f \in \mathbb{N} \cup \{∞\}$ is maximal such that $K$ contains a root of 1 of order $p^f$ (if $f = ∞$, we set $p^∞ = 0$). In particular, $θ_K$ is a torsion-free orientation. The module $\mathbb{Z}_p(θ_K)$ is called the 1st Tate twist of $\mathbb{Z}_p$ (cf., e.g., [21 Def. 7.3.6]).

For the convenience of the reader, here we recall J. Labute’s argument to show that the oriented pro-$p$ group $(G_K(p), θ_K)$ is Kummerian — and thus also 1-cyclotomic, as every subgroup $H \subseteq G_K(p)$ is the maximal pro-$p$ Galois group of an extension of $K$, with pro-$p$ cyclotomic character $θ_K|_H$ —, as it is presented in [131 p. 131] (where the module
\[ \mathbb{Z}_p(\theta_K) \text{ is denoted by } I = I(\chi'). \] For every \( n \geq 1 \) one has an isomorphism of continuous \( G_K(p) \)-modules
\[ \mathbb{Z}_p(\theta_K)/p^n\mathbb{Z}_p(\theta_K) \cong \mu_{p^n} = \left\{ \zeta \in \mathbb{K}(p) \mid \zeta^{p^n} = 1 \right\}. \]
Let \( K^\times \) and \( \mathbb{K}(p)^\times \) denote the multiplicative groups of units of \( K \) and \( \mathbb{K}(p) \) respectively. By Hilbert 90, the short exact sequence of continuous \( G_K(p) \)-modules
\[ (2.1) \quad \{1\} \longrightarrow \mu_{p^n} \longrightarrow \mathbb{K}(p)^\times \longrightarrow \mathbb{K}(p)^\times \longrightarrow \{1\} \]
induces a commutative diagram
\[ \begin{array}{ccc} \mathbb{K}^\times/(\mathbb{K}^\times)^{p^n} & \longrightarrow & H^1(G_K(p), \mu_{p^n}) \longrightarrow \longrightarrow H^1(G_K(p), \mathbb{Z}_p(\theta_K)/p^n\mathbb{Z}_p(\theta_K)) \\ \pi_n \downarrow & & \downarrow \pi_n \\ \mathbb{K}^\times/(\mathbb{K}^\times)^{p} & \longrightarrow & H^1(G_K(p), \mu_p) \longrightarrow \longrightarrow H^1(G_K(p), \mathbb{Z}/p\mathbb{Z}) \end{array} \]
where the left-side vertical arrow \( \pi_n \) and the central vertical arrow are induced by the \( p^{n-1} \)-th power map \( p^n : \mathbb{K}(p)^\times \rightarrow \mathbb{K}(p)^\times \), and the right-side vertical arrow is induced by the epimorphism of continuous \( G_K(p) \)-modules \( \mathbb{Z}_p(\theta_K)/p^n\mathbb{Z}_p(\theta_K) \rightarrow \mathbb{Z}/p\mathbb{Z} \). Since the map \( \pi_n \) is surjective, also the other vertical arrows are surjective.

**Example 2.5.** Let \( G \) be a free pro-\( p \) group. Then the oriented pro-\( p \) group \( (G, \theta) \) is 1-cyclotomic for any orientation \( \theta : G \rightarrow 1+p\mathbb{Z}_p \) (cf. [30] § 2.2)).

**Example 2.6.** Let \( G \) be an infinite Demushkin group (cf., e.g., [21] Def. 3.9.9)). By [14] Thm. 4, \( G \) comes endowed with a canonical orientation \( \chi : G \rightarrow 1+p\mathbb{Z}_p \) which is the only one completing \( G \) into a 1-cyclotomic oriented pro-\( p \) group. In particular, if \( d = \dim(H^1(G, \mathbb{Z}/p\mathbb{Z})) \) is even (which is always the case if \( p \neq 2 \)), then \( G \) has a presentation
\[ G = \left\langle x_1, \ldots, x_d \mid x_1^{p^f}[x_1, x_2] \cdots [x_{d-1}, x_d] = 1 \right\rangle, \]
with \( f \geq 1 \) (\( f \geq 2 \) if \( p = 2 \)). In this case \( \chi(x_2) = (1-p^f)^{-1} \) and \( \chi(x_i) = 1 \) for \( i \neq 2 \).

**Example 2.7.** Let \( (G, \theta) \) be an oriented pro-\( p \) group, with \( \theta \) a torsion-free orientation. The oriented pro-\( p \) group \( (G, \theta) \) is said to be \( \theta \)-abelian if the subgroup \( K_\theta(G) \) is trivial and if \( \text{Ker}(\theta) \) is a free abelian pro-\( p \) group — in this case \( G \) is a free abelian-by-cyclic pro-\( p \) group, i.e.,
\[ G \cong \text{Ker}(\theta) \times \frac{G}{\text{Ker}(\theta)} \]
(cf. [31] Rem. 2.2)). In other words, \( G \) has a presentation
\[ G = \left\langle x_0, x_i \mid i \in I, \ x_0^{\theta(x_0)} = x_i^{\theta(x_i)}, [x_i, x_j] = 1 \forall i, j \in I \right\rangle, \]
for some set of indices \( I \), and \( \theta(x_i) = 1 \) for all \( i \in I \) (cf. [28] Prop. 3.4]). A \( \theta \)-abelian oriented pro-\( p \) group \( (G, \theta) \) is Kummerian by Proposition [22] as by definition \( K_\theta(G) \) is trivial and \( \text{Ker}(\theta) \) is a free abelian pro-\( p \) group. Moreover, if \( H \) is a subgroup of \( G \), then one has
\[ H \cong (H \cap \text{Ker}(\theta)) \times \frac{H}{\text{Ker}(\theta)|_H} \]
(cf. [31] Rem. 2.4)), so that the oriented pro-\( p \) group \( (H, \theta|_H) \) is \( \theta|_H \)-abelian, and thus Kummerian, and consequently \( (G, \theta) \) is 1-cyclotomic.
One has the following result to check whether an oriented pro-$p$ group is Kummerian (cf. [31] Prop. 2.6, Prop. 3.6).

**Proposition 2.8.** Let $(G, \theta)$ be an oriented pro-$p$ group, with $\theta$ a torsion-free orientation. Then $(G, \theta)$ is Kummerian if, and only if, there exists a normal subgroup $N$ of $G$ such that $N \subseteq \text{Ker}(\theta) \cap \Phi(G)$, and the quotient $(G/N, \theta|_N)$, is a $\theta|_N$-abelian oriented pro-$p$ group. If such a normal subgroup $N$ exists, then $N = K_\theta(G)$.

2.5. **Kummerianity and 1-cocyles.** Let $(G, \theta)$ be an oriented pro-$p$ group. Recall that for $n \in \mathbb{N} \cup \{\infty\}$, a 1-cocycle $c: G \to \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ is a continuous map satisfying

$$(2.2) \quad c(gh) = c(g) + \overline{\theta(g)}c(h),$$

for every $g, h \in G,$ where $\overline{\theta(g)}$ denotes the image of $\theta(g)$ modulo $p^n$. From (2.2) one deduces

$$(2.3) \quad c([g, h]) = \overline{\theta(gh)}^{-1} \left( c(g)(1 - \theta(h)) - c(h)(1 - \theta(g)) \right).$$

For $n \in \mathbb{N} \cup \{\infty\}$, every element of $H^1(G, \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta))$ is represented by a 1-cocycle $c: G \to \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$. The following result is due to J. Labute (cf. [14] Prop. 6).

**Lemma 2.9.** Let $(G, \theta)$ be a finitely generated oriented pro-$p$ group with torsion-free orientation, and let $X = \{x_1, \ldots, x_d\}$ be a minimal generating set of $G$. The following are equivalent.

(i) $(G, \theta)$ is Kummerian.

(ii) For all $n \in \mathbb{N} \cup \{\infty\}$ and for any sequence $\lambda_1, \ldots, \lambda_d$ of elements of $\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ there exists a continuous 1-cocycle $G \to \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ satisfying $c(x_i) = \lambda_i$ for all $i = 1, \ldots, d$.

**Proposition 2.10.** Let $G$ be a finitely generated pro-$p$ group, and let $(G, \theta)$ be a Kummerian oriented pro-$p$ group with torsion-free orientation. If $N$ is a normal subgroup of $G$ such that $N \subseteq \text{Ker}(\theta)$ and the restriction map

$$\text{res}_{G, N}^1: H^1(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(N, \mathbb{Z}/p\mathbb{Z})^G$$

is surjective, then also $(G/N, \theta|_N)$ is Kummerian.

In order to prove Proposition 2.10 we need the following fact, whose proof — rather straightforward — is left to the reader.

**Fact 2.11.** Let $G$ be a finitely generated pro-$p$ group, and let $(G, \theta)$ be an oriented pro-$p$ group with torsion-free orientation.

(i) If $c: G \to \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ is a continuous 1-cocycle, with $n \in \mathbb{N} \cup \{\infty\}$, then $c^{-1}(0) \cap \text{Ker}(\theta)$ is a normal subgroup of $G$.

(ii) Let $N \subseteq G$ be a normal subgroup satisfying $N \subseteq \text{Ker}(\theta)$, with canonical projection $\pi: G \to G/N$. For $n \in \mathbb{N} \cup \{\infty\}$ one has the following:

(a) a continuous 1-cocycle $c: G \to \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ satisfying $c|_N \equiv 0$ induces a continuous 1-cocycle $\tilde{c}: G/N \to \mathbb{Z}_p(\theta|_N)/p^n\mathbb{Z}_p(\theta|_N)$ such that $c = \tilde{c} \circ \pi$;

(b) a continuous 1-cocycle $c: G/N \to \mathbb{Z}_p(\theta|_N)/p^n\mathbb{Z}_p(\theta|_N)$ induces a continuous 1-cocycle $c: G \to \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ satisfying $c|_N \equiv 0$ and $c = \tilde{c} \circ \pi$. 

Remark 2.12. Proposition 2.10 may be proved also in a purely group-theoretic way, more in general to describe the \( \mathbb{Z} \)-cohomology of \( G \).

Set \( \bar{G} = G/N \) and \( \bar{\theta} = \theta/N \). For every \( n \geq 1 \), the canonical projection \( \pi: G \rightarrow \bar{G} \) induces the inflation maps

\[
\begin{align*}
    f_n &: H^1(G, \mathbb{Z}_p(\bar{\theta})/p^n\mathbb{Z}_p(\bar{\theta})) \rightarrow H^1(G, \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)), \\
    f &: H^1(G, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^1(G, \mathbb{Z}/p\mathbb{Z}),
\end{align*}
\]

which are injective by \cite{21} Prop. 1.6.7. Also, the epimorphisms (respectively of continuous \( \bar{G} \)-modules and continuous \( G \)-modules) \( \mathbb{Z}_p(\bar{\theta})/p^n\mathbb{Z}_p(\bar{\theta}) \rightarrow \mathbb{Z}/p\mathbb{Z} \) and \( \mathbb{Z}_p(\theta)/p^n \rightarrow \mathbb{Z}/p\mathbb{Z} \) induce, respectively, the morphisms

\[
\begin{align*}
    \tau^N_n &: H^1(G, \mathbb{Z}_p(\theta)/p^n) \rightarrow H^1(G, \mathbb{Z}/p), \\
    \tau_n &: H^1(G, \mathbb{Z}_p(\theta)/p^n) \rightarrow H^1(G, \mathbb{Z}/p).
\end{align*}
\]

Altogether, by \cite{21} Prop. 1.5.2 one has the commutative diagram

\[
\begin{array}{ccc}
    H^1(G, \mathbb{Z}_p(\bar{\theta})/p^n\mathbb{Z}_p(\bar{\theta})) & \xrightarrow{\tau^N_n} & H^1(G, \mathbb{Z}/p) \\
    f_n \downarrow & & \downarrow f \\
    H^1(G, \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)) & \xrightarrow{\tau_n} & H^1(G, \mathbb{Z}/p)
\end{array}
\]

Since \( (G, \theta) \) is Kummerian, \( \tau_n \) is surjective for every \( n \geq 1 \). Given \( \beta \in H^1(G, \mathbb{Z}/p\mathbb{Z}) \), \( \beta \neq 0 \), our goal is to find \( \alpha \in H^1(G, \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)) \) such that \( \bar{\beta} = \tau^N_n(\alpha) \).

Set \( \beta = \beta \circ \pi = f(\bar{\beta}) \). Then \( \beta: G \rightarrow \mathbb{Z}/p \) is a non-trivial continuous homomorphism such that \( \text{Ker}(\beta) \supseteq N \). By hypothesis, the morphism \( N^\Phi[G, N] \rightarrow G/\Phi(G) \) induced by the inclusion \( N \hookrightarrow G \), and dual to \( \text{res}_{G,N} \), is injective. Thus, one may find a minimal generating set \( \mathcal{X} \) of \( G \) such that \( \mathcal{Y} = \mathcal{X} \cap N \) generates \( N \) as a normal subgroup of \( G \).

By Lemma 2.9 there exists a continuous 1-cocycle \( c: G \rightarrow \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta) \) satisfying

\[
c(x) \equiv \beta(x) \mod p\mathbb{Z}_p(\theta)
\]

— i.e., \( \tau_n([c]) = \beta \), where \( [c] \in H^1(G, \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)) \) denotes the cohomology class of \( c \) —, and moreover \( c(x) = 0 \) for every \( x \in \mathcal{Y} \). Therefore, by Fact 2.11 (i), the restriction

\[
c\mid_N: N \rightarrow \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)
\]

is the map constantly equal to 0. By Fact 2.11 (ii), \( c \) induces a continuous 1-cocycle

\[
\bar{c}: \bar{G} \rightarrow \mathbb{Z}_p(\bar{\theta})/p^n\mathbb{Z}_p(\bar{\theta})
\]

such that \( \bar{c} \circ \pi = c \), and \( [c] = f_n([\bar{c}]) \), where \( [\bar{c}] \in H^1(\bar{G}, \mathbb{Z}_p(\bar{\theta})/p^n\mathbb{Z}_p(\bar{\theta})) \) denotes the cohomology class of \( \bar{c} \). Altogether, one has

\[
f(\bar{\beta}) = \beta = \tau_n([c]) = \tau_n \circ f_n([\bar{c}]) = f \circ \tau^N_n([\bar{c}]).
\]

Since \( f \) is injective, one obtains \( \bar{\beta} = \tau^N_n([\bar{c}]) \). \( \Box \)

**Remark 2.12.** Proposition 2.10 may be proved also in a purely group-theoretic way, see \cite{3} Rem. 3.9.

### 3. The \( \mathbb{Z}/p\mathbb{Z} \)-cohomology of \( G \)

The purpose of this section is to prove the first statement of Proposition 1.3, and more in general to describe the \( \mathbb{Z}/p\mathbb{Z} \)-cohomology algebra \( H^*(G, \mathbb{Z}/p\mathbb{Z}) \) with \( G \) as in Theorem 1.1.
3.1. **Degree 1 and 2.** Let \( G \) be a pro-\( p \) group. We set the subgroup \( G_{(3)} \) of \( G \) as follows:

\[
G_{(3)} = \begin{cases} 
G^p[G, G'] & \text{if } p \neq 2, \\
G^4(G')^2[G, G'] & \text{if } p = 2,
\end{cases}
\]

i.e., \( G_{(3)} \) is the third term of the \( p \)-Zassenhaus filtration of \( G \) (cf., e.g., [24, § 3.1]). In particular, \( G_{(3)} \) is a normal subgroup of the Frattini subgroup \( \Phi(G) \), and the quotient \( \Phi(G)/G_{(3)} \) is a \( p \)-elementary abelian pro-\( p \) group — and thus also a \( \mathbb{Z}/p \)-vector space.

Recall that the cohomology group \( H^1(G, \mathbb{Z}/p\mathbb{Z}) \) is equal to the group of pro-\( p \) group homomorphisms from \( G \) to \( \mathbb{Z}/p \), namely, one has

\[
(\Phi(G)/G_{(3)})^* \cong H^1(G, \mathbb{Z}/p\mathbb{Z}),
\]

where \( \cdot^* \) denotes the \( \mathbb{Z}/p \)-dual (cf., e.g., [33, Ch. I, § 4.2]). Thus, the dimension of \( H^1(G, \mathbb{Z}/p\mathbb{Z}) \) is equal to the cardinality \( d(G) \) of any minimal generating set of \( G \). On the other hand, the dimension of \( H^2(G, \mathbb{Z}/p\mathbb{Z}) \) is equal to the number \( r(G) \) of defining relations of \( G \) (cf. [33, Ch. I, § 4.3]). Moreover, if both \( H^1(G, \mathbb{Z}/p\mathbb{Z}) \) and \( H^2(G, \mathbb{Z}/p\mathbb{Z}) \) are finite, and if the cup-product yields an epimorphism \( H^1(G, \mathbb{Z}/p\mathbb{Z}) \cong H^2(G, \mathbb{Z}/p\mathbb{Z}) \), one has an isomorphism of elementary abelian \( p \)-groups

\[
(\Phi(G)/G_{(3)})^* \xrightarrow{\text{trg}} H^2(G, \mathbb{Z}/p\mathbb{Z})
\]

(cf. [18, Thm. 7.3]). For further properties of the cohomology of pro-\( p \) groups we refer to [33, Ch. I, § 4] and to [21, Ch. III, § 9].

3.2. **Amalgams.** Henceforth \( G \) will denote a pro-\( p \) group as in Theorem [11]. Set

\[
G_1 = \langle x, y_1, \ldots, y_{d_1} \mid x^{p^j}[x, y_1] \cdots [y_{d_1-1}, y_{d_1}] = 1 \rangle,
\]

\[
G_2 = \langle x, z_1, \ldots, z_{d_2} \mid x^{p^j}[x, z_1] \cdots [z_{d_2-1}, z_{d_2}] = 1 \rangle,
\]

with \( \epsilon = 0, 1 \) depending on whether we are considering case (1.1.a) or (1.1.b). Then \( G_1, G_2 \) are Demushkin groups, and \( G \) is the amalgamated free pro-\( p \) product

\[
G = G_1 \amalg G_2,
\]

with amalgam the subgroup \( X \subseteq G_1, G_2 \) generated by \( x \). Observe that \( X \cong \mathbb{Z}/p \), as \( X \) has infinite index in both \( G_1, G_2 \), and subgroups of infinite index of Demushkin groups are free pro-\( p \) groups (cf. [33, Ch. I, § 4.5, Ex. 5–(b)]). Therefore, the amalgamated free pro-\( p \) product is proper, i.e., \( G_1, G_2 \subseteq G \) (cf. [32]).

3.3. **Quadratic cohomology.** Let

\[
\mathcal{B} = \{ \chi, \varphi_1, \ldots, \varphi_{d_1}, \psi_1, \ldots, \psi_{d_2} \}
\]

be the basis of \( H^1(G, \mathbb{Z}/p\mathbb{Z}) = \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \) dual to \( X = \{ x, y_1, \ldots, z_{d_2} \} \) — i.e.,

\[
\chi(w) = \begin{cases} 
1 & \text{if } w = x \\
0 & \text{if } w = y_i, z_j
\end{cases}
\]

and

\[
\varphi_i(w) = \begin{cases} 
\delta_{i,i} & \text{if } w = y_i \\
0 & \text{if } w = x, z_j
\end{cases}
\]

\[
\psi_j(w) = \begin{cases} 
\delta_{j,j'} & \text{if } w = z_j' \\
0 & \text{if } w = x, y_i,
\end{cases}
\]
for every $1 \leq i, i' \leq d_1$ and $1 \leq j, j' \leq d_2$ (cf. (3.1)). With an abuse of notation, we may consider the subsets $B_1 = \{\chi, \varphi_1, \ldots, \varphi_{d_1}\}$, $B_2 = \{\chi, \psi_1, \ldots, \psi_{d_2}\}$, and $B_X = \{\chi\}$, as bases of $H^1(G_1, \mathbb{Z}/p\mathbb{Z})$, $H^1(G_2, \mathbb{Z}/p\mathbb{Z})$, and $H^1(X, \mathbb{Z}/p\mathbb{Z})$ respectively.

**Proposition 3.1.** The algebra $H^\bullet(G, \mathbb{Z}/p\mathbb{Z})$ is quadratic.

**Proof.** As stated in § 3.2, $G = G_1 \Pi^p X G_2$ is a proper amalgamated free pro-$p$ product. Since $B_X \subseteq B_1, B_2$, the restriction maps

$$\text{res}^1_{G_i, X} : H^1(G_i, \mathbb{Z}/p\mathbb{Z}) \to H^1(X, \mathbb{Z}/p\mathbb{Z}), \quad \text{with } i = 1, 2,$$

are surjective. Moreover, $H^2(X, \mathbb{Z}/p\mathbb{Z}) = 0$, as $X \simeq \mathbb{Z}_p$, and thus $\text{Ker}(\text{res}^2_{G_i, X}) = H^2(G_i, \mathbb{Z}/p\mathbb{Z})$ for both $i = 1, 2$. On the other hand, $H^1(G_1, \mathbb{Z}/p\mathbb{Z})$ and $H^1(G_2, \mathbb{Z}/p\mathbb{Z})$ are generated by $\chi \varphi_1$ and $\chi \psi_1$ respectively, as $G_1, G_2$ are Demushkin groups (cf., e.g., [21] Prop. 3.9.16), and thus

$$\text{Ker}(\text{res}^2_{G_i, X}) = H^2(G_i, \mathbb{Z}/p\mathbb{Z}) = \text{Ker}(\text{res}^1_{G_i, X}) \sim H^1(G_i, \mathbb{Z}/p\mathbb{Z}), \quad \text{with } i = 1, 2,$$

as $\text{res}^1_{G_i, X}(\varphi_1) = 0$ and $\text{res}^1_{G_i, X}(\psi_1) = 0$. Finally, Demushkin groups are well-known to yield a quadratic $\mathbb{Z}/p\mathbb{Z}$-cohomology algebra, while $H^\bullet(X, \mathbb{Z}/p\mathbb{Z})$ is obviously quadratic, as $X \simeq \mathbb{Z}_p$. Therefore, we may apply [29] Thm. B, so that also $H^\bullet(G, \mathbb{Z}/p\mathbb{Z})$ is quadratic. 

We describe now more in detail the structure of $H^\bullet(X, \mathbb{Z}/p\mathbb{Z})$. By duality — cf. [18] Thm. 3.2 and (3.2) —, the set $\{\chi \varphi_1, \chi \psi_1\}$ is a basis of $H^2(G, \mathbb{Z}/p\mathbb{Z})$, and in $H^2(G, \mathbb{Z}/p\mathbb{Z})$ one has the relations

$$\chi \varphi_{i'} = \chi \psi_{j'} = \varphi_i \psi_j = 0$$

for all $1 \leq i, i' \leq d_1$ and $1 \leq j, j' \leq d_2$, with $i', j' \neq 1$, and

$$\psi_j \psi_{j'} = \begin{cases} (-1)^{j} \chi \psi_1 & \text{if } 2 \mid j \neq j' - 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_i \varphi_{i'} = \begin{cases} (-1)^{i} \chi \varphi_1 & \text{if } 2 \mid i \neq i' - 1, \\ 0 & \text{otherwise} \end{cases}$$

(see also [24] § 3.2).

Finally, one has an exact sequence

$$\xymatrix{ \cdots \ar[r] & H^2(X, \mathbb{Z}/p\mathbb{Z}) \ar[r] & H^3(G, \mathbb{Z}/p\mathbb{Z}) \ar[r] & H^3(G_1, \mathbb{Z}/p\mathbb{Z}) \oplus H^3(G_2, \mathbb{Z}/p\mathbb{Z}) \ar[r] & \cdots }$$

(cf. [29] p. 653). Since $H^2(X, \mathbb{Z}/p\mathbb{Z}) = H^3(G_i, \mathbb{Z}/p\mathbb{Z}) = 0$ for both $i = 1, 2$, one has $H^3(G, \mathbb{Z}/p\mathbb{Z}) = 0$, and thus by quadraticity also $H^n(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $n \geq 3$.

**Remark 3.2.** It is well-known that if a pro-$p$ group has non-trivial torsion, then its $n$-th $\mathbb{Z}/p$-cohomology group is non trivial for every $n > 0$; hence, $G$ is torsion-free.
4. Proof of Theorem 1.1 Case (1.1.a)

Let $G$ be a pro-$p$ group as defined in Theorem 1.1 with defining relations as in (1.1.a) — namely,

$$G = \langle x, y_1, \ldots, y_{d_1}, z_1, \ldots, z_{d_2} \mid r_1 = r_2 = 1 \rangle,$$

with $d_1 + d_2 \geq 4$ and

$$r_1 = [x, y_1] \cdots [y_{d_1-1}, y_{d_1}],$$
$$r_2 = [x, z_1] \cdots [z_{d_2-1}, z_{d_2}].$$

Without loss of generality, we may assume that $d_1 \geq 3$.

4.1. Kummerianity. Let $G_1, G_2$ be the two Demushkin groups as in §3.2 with $\epsilon = 0$. By Example 2.6 if

$$\theta_1: G_1 \to 1 + p\mathbb{Z}_p \quad \text{and} \quad \theta_2: G_2 \to 1 + p\mathbb{Z}_p$$

are two torsion-free orientations completing respectively $G_1$ and $G_2$ into Kummerian oriented pro-$p$ groups, then necessarily $\theta_1(x) = \theta_1(y_1) = \ldots = \theta_1(y_{d_1}) = 1$, and analogously $\theta_2(x) = \theta_2(z_1) = \ldots = \theta_2(z_{d_2}) = 1$.

Proposition 4.1. Let $\theta: G \to 1 + p\mathbb{Z}_p$ be a torsion-free orientation. Then the oriented pro-$p$ group $(G, \theta)$ is Kummerian if, and only if, $\theta$ is constantly equal to 1.

Proof. If $\theta \equiv 1$, then $(G, 1)$ is Kummerian if, and only if, the abelianization $G^{ab}$ is a free abelian pro-$p$ group. But this is easily verified, as clearly $G^{ab} \simeq \mathbb{Z}_p^{d_1 + d_2 - 1}$.

Conversely, suppose that $(G, \theta)$ is Kummerian. Let $N_1$ and $N_2$ denote the normal subgroups of $G$ generated as normal subgroups by $z_1, \ldots, z_{d_2}$, and $y_1, \ldots, y_{d_1}$, respectively. Then $G/N_1 \simeq G_1$ and $G/N_2 \simeq G_2$. Moreover, Proposition 2.10 implies that $(G/N_i, \theta/N_i)$ is Kummerian for both $i = 1, 2$. Since $G/N_i \simeq G_i$ for both $i$, Example 2.6 and the argument before the statement of the proposition imply that the torsion-free orientations $\theta/N_1$ and $\theta/N_2$ are constantly equal to 1. Hence, also $\theta$ is constantly equal to 1, as $\theta(w) = \theta(N_1 \cdot u \cdot N_1)$ for every $w \in G_1$, and analogously $\theta(w) = \theta(N_2 \cdot w \cdot N_2)$ for every $w \in G_2$. \hfill \square

Therefore, if $G$ may complete into a 1-cyclotomic oriented pro-$p$ group, then necessarily $G$ is absolutely torsion-free. In order to prove Theorem 1.1 in case (1.1.a), we aim at exhibiting an open subgroup $H$ of $G$, of index $p^2$, whose abelianization $H^{ab}$ has non-trivial torsion.

4.2. The subgroup $U$. Set $u = y_3^p$, $t_0 = z_1^{-1}y_3$, and $t_h = t_0y_3^s \cdots y_3^h$ for all $h = 0, \ldots, p - 1$. A straightforward computation shows that

$$z_h^h = y_3^h \cdot (t_0^{-1})y_3^{h-1} \cdots (t_0^{-1})y_3 \cdot t_0^{-1} = y_3^h t_0^{-1}$$
for all $h = 0, \ldots, p - 1$.

Let $\phi_G: G \to \mathbb{Z}/p$ be the homomorphism of pro-$p$ groups defined by $\phi_G(z_j) = 1$ and $\phi_G(x) = \phi_G(y_i) = \phi_G(z_j) = 0$ for all $i = 1, 2, 4, \ldots, d_1$ and $j = 2, \ldots, d_2$, and set $U = \text{Ker}(\phi)$. Then $U$ is an open subgroup of $G$ of index $p$, generated as a normal subgroup by the subset

$$X = \{ u, x, t_0, y_i, z_j \mid i = 1, 2, 4, \ldots, d_1, j = 2, \ldots, d_2 \},$$
and $G/U = \{ U, y_3U, \ldots, y_3^{p-1}U \}$. 
Lemma 4.2. The subset

\[ \mathcal{Y}_U = \left\{ u, x, y_2, t_h, y_h^h, z_j^h \mid i = 1, 4, \ldots, d_1, j = 2, \ldots, d_2, h = 0, \ldots, p - 1 \right\} \]

of \( U \) is a minimal generating set of \( U \) as a pro-\( p \) group.

Proof. Since \( U \) is normally generated by \( \mathcal{X} \) and \( G/U = \left\{ U, \ldots, y_3^{p-1}U \right\} \), \( U \) is generated as a pro-\( p \) group by the set \( \left\{ w^h \mid w \in \mathcal{X}, h = 0, \ldots, p - 1 \right\} \). Also, \( U \) is subject to the relations

\[
\begin{align*}
\gamma_1^h &= \left[ y_5^h, y_1^h \right] \cdots \left[ y_{d_1-1}^h, y_{d_1}^h \right] = 1, \\
\gamma_2^h &= \left[ z_3^h, z_1^h \right] \cdots \left[ z_{d_2-1}^h, z_{d_2}^h \right] = 1,
\end{align*}
\]

with \( h = 0, \ldots, p - 1 \).

Consider the abelianization \( U^{ab} \). Since the only factor in \( \mathcal{Y}_U \) which does not lie in \( U' \) is \( y_3^h, y_5^h \), the relation \( \mathcal{Y}_U \) implies that \( y_3^h, y_5^h \in U' \) as well, and therefore

\[ y_3^h \equiv y_2 \mod U' \quad \text{for all } h = 0, \ldots, p - 1. \]

Analogously, the only factor in \( \mathcal{Y}_U \) which does not lie in \( U' \) is \( x^{y_3^h}, z_1^h \), so that the relation \( \mathcal{Y}_U \) implies that \( x^{y_3^h}, z_1^h \in U' \) as well. Hence, one has

\[
\begin{align*}
[x, z_1] \equiv 1 \mod U' &\Rightarrow x^{y_3^h t_0^{-1}} \equiv x \mod U' \\
&\Rightarrow x^{y_3^h} \equiv x^{t_0} \mod U', \\
[x^{y_3^h}, z_1^{y_3^h}] \equiv 1 \mod U' &\Rightarrow (x^{y_3^h})^{(z_1^{y_3^h})} = x^{y_3^h (t_0^{-1}) y_3^h} \equiv x^{y_3^h} \mod U' \\
&\Rightarrow x^{y_3^h} \equiv x^{t_1} \mod U',
\end{align*}
\]

and so on. Thus

\[ x^{y_3^h} \equiv x^{t_{h-1}} \mod U' \quad \text{for all } h = 1, \ldots, p - 1. \]

Altogether, \( U^{ab} \) is the free abelian pro-\( p \) group generated by the cosets \( \left\{ wU' \mid w \in \mathcal{Y}_U \right\} \), so that Fact 2.1 yields the claim. \( \square \)

Now set \( U_1 = G_1 \cap U \) and \( U_2 = G_2 \cap U \). Then \( U_1, U_2 \) are open subgroups of \( G_1, G_2 \) respectively of index \( p \), and thus they are again Demushkin groups, on \( 2 + p(d_1 - 1) \) and \( 2 + p(d_2 - 1) \) generators respectively (cf. [9]). In particular, the defining relation of \( U_1 \) is

\[
s_1 = \prod_{h=p-1}^{0} \left( [y_4^h, y_5^h] \cdots [y_{d_1-1}^h, y_{d_1}^h] [x^{y_3^h}, y_1^h] \right) [y_2, u] = 1,
\]

while the defining relation of \( U_2 \) is

\[
s_2 = \prod_{h=p-1}^{0} \left( [z_2^h, z_3^h] \cdots [z_{d_2-1}^h, z_{d_2}^h] [x, z_1^h] \right) \prod_{h=p-1}^{0} \left( [y_2^{t_{h-1}}, z_3^h] \cdots [z_{d_2-1}^h, z_{d_2}^h] \right) [x, u t_{p-1}^{-1}] = 1.
\]
Also, from the relations (4.4)–(4.5) and from (4.1), one computes
\[ x^y = x^{z_1 t_0} = x^b ([z_{d_2}, z_{d_2 - 1}] \cdots [z_3, z_2])^{t_0}, \]
and so on. In fact, the two relations (4.4)–(4.5) — with the relations (4.6) and subject to the 2 \[ p \]
and so on. In fact, the two relations (4.4)–(4.5) — with the \( x^y \)'s replaced using (4.6) — are all the defining relations we need to get \( U \), as shown in the following.

**Lemma 4.3.** The pro-\( p \) group \( U \) has \( r(U) = 2 \) defining relations.

**Proof.** Since \( H^n(G, \mathbb{Z}/p\mathbb{Z}) = 0 \) for every \( n \geq 3 \) (cf. § 3.3) and \( |G : U| = p \), one has \( H^n(U, \mathbb{Z}/p\mathbb{Z}) = 0 \) for every \( n \geq 3 \) as well (cf. [21, Prop. 3.3.5]). Moreover, one has
\[ r(U) - d(U) = 1 = p (r(G) - d(G) + 1) \]
(cf. [21] Prop. 3.3.13). By definition, \( r(G) = 2 \) and \( d(G) = 1 + d_1 + d_2 \), while \( d(U) = 3 + p(d_1 + d_2 - 2) \) by Lemma 4.2. Therefore, from (4.7) one computes \( r(U) = 2 \). \( \square \)

4.3. The subgroup \( H \). Let \( \phi_U : U \to \mathbb{Z}/p \) be the homomorphism of pro-\( p \) groups defined by \( \phi_U(y_1) = 1 \), and \( \phi_U(w) = 0 \) for any other element \( w \) of \( \mathcal{Y}_U \), and put \( H = \text{Ker}(\phi_U) \). Then \( H \) is an open subgroup of \( U \) of index \( p \). Set \( v = y_1 \). Since \( U/H = \{ H, vH, \ldots, v^p H \} \), \( H \) is the pro-\( p \) group (non-minimally) generated by
\[ X_H = \{ v^p, (vy_1^{y_1})^v, w^v \mid w \in \mathcal{Y}_U, w \neq v, y_1^{y_1}, h = 0, \ldots, p - 1 \}, \]
and subject to the 2p relations \( s_1^{y_1} = 1 \) and \( s_2^{y_1} = 1 \), with \( h = 0, \ldots, p - 1 \). We claim that the abelianization \( H^{ab} \) yields non-trivial torsion.

**Proposition 4.4.** The abelian pro-\( p \) group \( H^{ab} \) is not torsion-free.

**Proof.** Since all the elements of \( \mathcal{Y}_U \) showing up in the last terms of the equalities (4.6) belong to \( H \), one deduces that \( x^{y_1} \equiv x \mod H' \) for all \( h = 0, \ldots, p - 1 \).

Now, each factor of \( s_2 \) — cf. (4.3) — is a commutator of elements of \( H \), and thus the relations \( x^{y_1} \equiv 1 \) yield trivial relations in \( H^{ab} \). On the other hand, every factor of \( s_1 \) — cf. (4.4) —, but \( [x, y_1] \) and \( [x^{y_1}, y_1^{y_1}] \), is a commutator of elements of \( H \). From (4.3) one obtains
\[ [x^{y_1}, y_1^{y_1}] [x, y_1] \equiv [x, v^{-1} (vy_1^{y_1})] [x, v] \equiv [x, v^{-1}] [x, v] \equiv 1 \mod H', \]
as \( vy_1^{y_1} \in H \). Altogether, \( H^{ab} \) is the abelian pro-\( p \) group (non-minimally) generated by the set \( X_{H^{ab}} = \{ w H' \mid w \in X_H \} \), and subject to the \( p \) relations
\[ [x^{v^h} H', v^{-1} H'] [x^{v^h} H', v H'] = H', \quad \text{with } h = 0, \ldots, p - 1, \]
as $U/H = \{H, vH, \ldots, v^{p-1}H\}$. From these relations one deduces the equivalences:

\[
x^{v^2} \equiv (x^v)^2 \cdot x^{-1} \mod H' \quad \text{with } h = 1, \\
x^{v^3} \equiv (x^v)^2 \cdot (x^v)^{-1} \equiv (x^v)^3 \cdot x^{-2} \mod H' \quad \text{with } h = 2, \\
\vdots \\
x^{v^{p-1}} \equiv (x^{v^{p-2}})^2 \cdot (x^{v^{p-3}})^{-1} \equiv (x^v)^{p-1} \cdot x^{2-p} \mod H' \quad \text{with } h = p - 2, \\
x^{v^p} \equiv (x^{v^{p-1}})^2 \cdot (x^{v^{p-2}})^{-1} \equiv (x^v)^p \cdot x^{1-p} \mod H' \quad \text{with } h = p - 1.
\]

But $x^{v^p} \equiv x \mod H'$, as $v^p \in H$, and thus from the last of the above equivalences one obtains

\[
x \equiv (x^v)^p x^{1-p} \mod H' \implies (x^v)^p x^{-p} \equiv (x^v x^{-1})^p \equiv 1 \mod H'.
\]

Altogether, $H_{ab}$ is the abelian pro-$p$ group minimally generated by

\[
\mathcal{Y}_{H_{ab}} = \left\{ v^h H', xH', x^v H', (vy_1^{p_1})^{v^h} H', w^{v^h} H' \mid h = 0, \ldots, p-1 \right\},
\]

where $w \in \mathcal{Y}_U \setminus \{v, y_1^{p_1}, x\}$, and subject to the relation $((xH')^{-1} \cdot x^v H')^p = H'$ — in particular, $H_{ab}$ is isomorphic to $\mathbb{Z}_p^{2+2p+p(d_1+d_2-2)} \times \mathbb{Z}/p\mathbb{Z}$.

\[
5. \text{ Proof of Theorem 1.1 case (1.1.b)}
\]

Let $p$ be an odd prime, and let $G$ be a pro-$p$ group as defined in Theorem 1.1, with defining relations as in (1.1.b) — namely,

\[
G = \langle x, y_1, \ldots, y_{d_1}, z_1, \ldots, z_{d_2} \mid r_1 = r_2 = 1 \rangle,
\]

with

\[
\begin{align*}
    r_1 &= y_1^{p_1} [y_1, x] \cdots [y_{d_1-1}, y_{d_1}], \\
    r_2 &= z_1^{p_2} [z_1, x] \cdots [z_{d_2-1}, z_{d_2}].
\end{align*}
\]

5.1. Kummerianity. Let $G_1, G_2$ be the two Demushkin groups as in §3.2 with $\epsilon = 1$. By Example 2.6 if

\[
\theta_1: G_1 \rightarrow 1 + p\mathbb{Z}_p \quad \text{and} \quad \theta_2: G_2 \rightarrow 1 + p\mathbb{Z}_p
\]

are two torsion-free orientations completing respectively $G_1$ and $G_2$ into Kummerian oriented pro-$p$ groups, then necessarily $\theta_1(y_1) = \ldots = \theta_1(y_{d_1}) = 1$, and analogously $\theta_2(z_1) = \ldots = \theta_2(z_{d_2}) = 1$, while $\theta_1(x) = \theta_2(x) = (1 - p)^{-1}$.

**Proposition 5.1.** An orientation $\theta: G \rightarrow 1 + p\mathbb{Z}_p$ completes $G$ into a Kummerian oriented pro-$p$ group $(G, \theta)$ if, and only if,

\[
\theta(x) = (1 - p)^{-1} \quad \text{and} \quad \theta(y_i) = \theta(z_j) = 1
\]

for all $i = 1, \ldots, d_1$ and $j = 1, \ldots, d_2$. 

Proof. Suppose that $\theta : G \to 1 + p\mathbb{Z}_p$ is the orientation defined as above, and pick arbitrary $p$-adic integers $\lambda, \lambda_i, \lambda_j' \in \mathbb{Z}_p$ for $1 \leq i \leq d_1$ and $1 \leq j \leq d_2$. The assignment $x \mapsto \lambda, y_i \mapsto \lambda_i$ and $z_j \mapsto \lambda_j'$ for every $i, j$ yields a well-defined continuous 1-cocycle $c : G \to \mathbb{Z}_p(\theta)$, as (2.3) implies that
\[
c(r_1) = c(y_1^p) + c([y_1, x]) + c([y_2, y_3]) + \ldots + c([y_{d_1-1}, y_{d_1}])
= p \cdot \lambda_1 + \theta(x)^{-1}(\lambda_1(1 - \theta(x)) - 0) + 0 + \ldots + 0
= 0
\]
and
\[
c(r_2) = c(z_1^p) + c([z_1, x]) + c([z_2, z_3]) + \ldots + c([z_{d_2-1}, z_{d_2}])
= p \cdot \lambda_1' + \theta(x)^{-1}(\lambda_1'(1 - \theta(x)) - 0) + 0 + \ldots + 0
= 0
\]
Therefore, $(G, \theta)$ is Kummerian by Lemma 2.9.

Conversely, suppose that $(G, \theta)$ is Kummerian. Let $N_1$ and $N_2$ denote the normal subgroups of $G$ generated as normal subgroups by $z_1, \ldots, z_{d_2}$ and $y_1, \ldots, y_{d_1}$ respectively. Then $G/N_1 \simeq G_1$ and $G/N_2 \simeq G_2$. Moreover, Proposition 2.10 implies that $(G/N_1, \theta/N_1)$ is Kummerian for both $i = 1, 2$.

Since $G/N_1 \simeq G_1$ for both $i$, Example 2.6 and the argument before the statement of the proposition imply that $\theta/\theta_{N_1}(y_1N_1) = \ldots = \theta/\theta_{N_1}(y_{d_1}N_1) = 1$, and analogously $\theta/N_2(z_1N_2) = \ldots = \theta/N_2(z_{d_2}N_2) = 1$, while $\theta/N_1(xN_1) = \theta/N_2(xN_2) = (1 - p)^{-1}$. Hence, $\theta$ is as defined above, as $\theta(w) = \theta/\theta_{N_1}(wN_1)$ for every $w \in G_1$, and analogously $\theta(w) = \theta/N_2(wN_2)$ for every $w \in G_2$.

Henceforth, $\theta : G \to 1 + p\mathbb{Z}_p$ will denote the orientation as in Proposition 5.1.

5.2. The subgroup $H$. Let $\phi_1 : G_1 \to \mathbb{Z}/p \oplus \mathbb{Z}/p$ and $\phi_2 : G_2 \to \mathbb{Z}/p \oplus \mathbb{Z}/p$ be the homomorphisms of pro-$p$ groups defined by
\[
\phi_1(x) = \phi_2(x) = (1, 0),
\phi_1(y_1) = \phi_2(z_1) = (0, 1),
\phi_1(y_i) = \phi_2(z_j) = (0, 0) \text{ for } i, j \geq 2.
\]
(5.1)

Put $U_1 = \text{Ker}(\phi_1)$ and $U_2 = \text{Ker}(\phi_2)$, and also
\[
t = z_1^{-1}y_1, \quad u = x^p, \quad v = y_1^p, \quad w = z_1^p.
\]
Then $U_1$ is an open normal subgroup of $G_1$ of index $p^2$, and likewise for $U_2$ and $G_2$ — note that by (b) both $U_1$ and $U_2$ are Demushkin groups.

Finally, put $N_1 = \text{Ker}(\theta|_{U_1})$, $N_2 = \text{Ker}(\theta|_{U_2})$, and let $T$ be the subgroup of $G$ generated by $t$. Observe that $N_1$ and $N_2$ are free pro-$p$ groups, as they are subgroups of infinite index of Demushkin groups (cf. [33, Ch. I, § 4.5, Ex. 5–(b)]), while $T \simeq \mathbb{Z}_p$ as $G$ is torsion-free (cf. Remark 6.2).

Let $H$ be the subgroup of $G$ generated by $U_1, U_2$ and $T$, and let $M$ be the subgroup of $H$ generated by $N_1, N_2$ and $T$. Observe that $M \subseteq \text{Ker}(\theta)$. Our aim is to show that the oriented pro-$p$ group $(H, \theta|_H)$ is not Kummerian. For this purpose, we need the following.

Lemma 5.2. (i) $M = N_1 \Pi N_2 \Pi T$. 


(ii) $M$ is a normal subgroup of $H$, and $H \simeq M \rtimes X_p$

(iii) One has an isomorphism of $p$-elementary abelian groups

$$\frac{G}{\Phi(G)} \simeq \frac{X_p}{X_p^T} \times \frac{N_1}{N_1\langle U_1 \rangle} \times \frac{N_2}{N_2\langle U_2 \rangle} \times \frac{T}{T_p}.$$ 

Proof. Consider the pro-$p$ tree $T$ associated to the amalgamated free pro-$p$ product $T$ [3.3]. Namely, $T$ consists of a set vertices $V$ and a set of edges $E$, where

$$V = \{ hG_1, hG_2 \mid h \in G \} = G/G_1 \cup G/G_2,$$

$$E = \{ hX \mid h \in G \} = G/X,$$

and it comes endowed with a natural $G$-action, i.e.,

$$g.(hG_1) = (gh)G_1 \quad \text{for every } g \in G, hG_1 \in G/G_1 \subseteq V$$

$$g.(hG_1) = (gh)G_2 \quad \text{for every } g \in G, hG_2 \in G/G_2 \subseteq V,$$

$$g.(hX) = (gh)X \quad \text{for every } g \in G, hX \in G/X = E.$$ 

Pick $g \in M$ and $hX \in E$. Then $g.hX = hX$ if, and only if, $g \in hXh^{-1}$, i.e., $g = hx^\lambda h^{-1}$ for some $\lambda \in \mathbb{Z}_p$. Since $M \subseteq \operatorname{Ker}(\theta)$, it follows that

$$1 = \theta(g) = \theta(hx^\lambda h^{-1}) = \theta(x)^\lambda = (1 - p)^\lambda,$$

and therefore $\lambda = 0$, as $1 + p\mathbb{Z}_p$ is torsion-free. Hence, the subgroup $M$ intersects trivially the stabilizer $\operatorname{Stab}_G(hX)$ of every edge $hX \in E$. By [15] Thm. 5.6], $M$ decomposes as free pro-$p$ product as follows:

$$M = \prod_{\omega \in \mathcal{V}'} \operatorname{Stab}_M(\omega) \amalg F,$$

where $F$ is a free pro-$p$ group, and $\mathcal{V}' \subseteq \mathcal{V}$ is a continuous set of representatives of the space of orbits $M \setminus \mathcal{V}$. Clearly, the vertices $G_1$ and $G_2$ belong to different orbits, thus in the decomposition [5.3] one finds the two factors

$$\operatorname{Stab}_M(G_1) = \{ g \in M \mid gG_1 = G_1 \} = M \cap G_1,$$

$$\operatorname{Stab}_M(G_2) = \{ g \in M \mid gG_2 = G_2 \} = M \cap G_2.$$ 

Since $N_1 \subseteq M \cap G_1 \subseteq \operatorname{Ker}(\theta) \cap G_1 = N_1$, one has $\operatorname{Stab}_M(G_1) = N_1$, and analogously $\operatorname{Stab}_M(G_2) = N_2$. Therefore, from [5.5] one obtains

$$M = N_1 \amalg N_2 \amalg \left( \prod_{\omega \in \mathcal{V}'} \operatorname{Stab}_M(\omega) \amalg F \right).$$

It is straightforward to see that $t \notin N_1 \amalg N_2$. Since $M$ is generated as pro-$p$ group by $N_1$, $N_2$ and $t$, the right-side factor in [5.6] is necessarily $T$, and this proves (i).

In order to prove (ii), we need only to show that $uM u^{-1} = M$, as $H = \langle u, M \rangle$. Since $N_1$ is normal in $U_1$, and $u \in U_1$, then $uN_1 u^{-1} = N_1$ — analogously, $uN_2 u^{-1} = N_2$. Now, observe that the integer

$$(1 - p)^p - 1 = \left( 1 - \binom{p}{1} p + \binom{p}{2} p^2 - \ldots - p^p \right) - 1$$
is divisible by \( p^2 \) but not by \( p^3 \), so we put \((1 - p)^p = 1 + p^2 \lambda \), with \( \lambda \in 1 + p \mathbb{Z}_p \). From the relation \( r_1 = 1 \) one deduces
\[
y_1^p = y_1^{-1 - p} \cdot (\{y_2, y_3, \ldots, [y_{d_1 - 1}, y_{d_1}]\})^{-1},
\]
and by iterating \( (5.7) \) \( p \) times, one obtains \( y_1^n = y_1^{(1 - p)p} n_1 \) for some \( n_1 \in N_1' \) — for this purpose, observe that for every \( \nu \geq 0 \) and \( i \geq 1 \), the triple commutator
\[
[y_1^n, [y_i, y_{i+1}]] = \left[y_i^{y_1^n}, y_{i+1}^{y_1^n}\right]^{-1} \cdot [y_i, y_{i+1}]
\]
belongs to \( N_1' \), as \( y_1^{y_1^n} \in N_1 \). Analogously, \( z_1^n = z_1^{(1 - p)p} n_2 \) for some \( n_2 \in N_2' \). Altogether,
\[
t^u = (z_1^{-1} y_1)^u = y_1^{z_1 y_1^n} = n_2^{-1} \cdot w^{-p\lambda} \cdot t \cdot v^{p\lambda} \cdot n_1,
\]
which belongs to \( M \) — here we replaced \( z_1^{-1} (1 - p)^p = w^{-p\lambda} \cdot z_1^{-1} \) and \( y_1^{(1 - p)p} = y_1 \cdot v^{p\lambda} \).
Hence, \( M \leq H \). Finally, by definition \( H = M \times X^p \), and moreover
\[
M \cap X^p \subseteq \text{Ker}(\theta) \cap X^p = \{1\},
\]
so that \( H = M \times X^p \). This completes the proof of (ii).

Finally, by (i) and (ii) one has the isomorphism of \( p \)-elementary abelian groups
\[
\frac{M/\Phi(M)}{\Phi(M)} \cong N_1/\Phi(N_1) \times N_2/\Phi(N_2) \times T/T^p,
\]
which embeds in \( H/\Phi(H) \). In particular, \( \{uN, tN\} \) is a minimal generating set of \( \tilde{H} \).
Thus, by Proposition 2.10 if the oriented pro-\( p \) group \((\tilde{H}, \tilde{\theta})\) is not Kummerian — where \( \tilde{\theta} = (\theta|_{H})/_{N} : \tilde{H} \to 1 + p \mathbb{Z}_p \) is the orientation induced by \( \theta|_{H} \) —, then also \((H, \theta|_{H})\) is not Kummerian.

By (5.8), in \( H \) one has that \( [t, u^{-1}] \equiv 1 \mod N \), and thus \( \tilde{H} \) is abelian. Moreover,
\[
\tilde{\theta}(uN) = \theta(u) = (1 - p)^p \quad \text{and} \quad \tilde{\theta}(tN) = \theta(t) = 1,
\]
so that \( \text{Ker}(\tilde{\theta}) = \langle tN \rangle \). Therefore, the subgroup \( K_{\tilde{\theta}}(\tilde{H}) \) is generated by
\[
\left(t^{-\tilde{\theta}(u)} ut u^{-1}\right) N = t^2 \lambda N.
\]
Thus, the quotient \( \text{Ker}(\tilde{\theta})/K_{\tilde{\theta}}(\tilde{H}) = \langle tN \rangle / \langle tN \rangle^p \) is not torsion-free, and by Proposition 2.2 \((\tilde{H}, \tilde{\theta})\) is not Kummerian.

This completes the proof of Theorem 1.1 case (1.1.b).
Remark 5.4. If $d_1 = d_2 = 1$, case (1.1.b) of Theorem [13] is a particular case of [3] Prop. 6.5).

6. Massey products

6.1. Massey products in Galois cohomology. Here we recall briefly what we need in order to prove Proposition [12]. For a detailed account on Massey products for pro-$p$ groups, we direct the reader to [8, 20, 36].

Let $G$ be a pro-$p$ group. For $n \geq 2$, the $n$-fold Massey product on $H^1(G, \mathbb{Z}/p\mathbb{Z})$ is a multi-valued map

$$H^1(G, \mathbb{Z}/p\mathbb{Z}) \times \ldots \times H^1(G, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2(G, \mathbb{Z}/p\mathbb{Z}).$$

For $n \geq 2$, given a sequence $\alpha_1, \ldots, \alpha_n$ of elements of $H^1(G, \mathbb{Z}/p\mathbb{Z})$ (with possibly $\alpha_i = \alpha_j$ for some $1 \leq i < j \leq n$), the (possibly empty) subset of $H^2(G, \mathbb{Z}/p\mathbb{Z})$ which is the value of the $n$-fold Massey product associated to the sequence $\alpha_1, \ldots, \alpha_n$ is denoted by $(\alpha_1, \ldots, \alpha_n)$. If $n = 2$, then the 2-fold Massey product coincides with the cup-product, i.e., for $\alpha_1, \alpha_2 \in H^1(G, \mathbb{Z}/p\mathbb{Z})$ one has

$$\langle \alpha_1, \alpha_2 \rangle = \{ \alpha_1 \cup \alpha_2 \} \subseteq H^2(G, \mathbb{Z}/p\mathbb{Z}).$$

A pro-$p$ group $G$ is said to satisfy:

(a) the $n$-Massey vanishing property (with respect to $\mathbb{Z}/p\mathbb{Z}$) if for every sequence $\alpha_1, \ldots, \alpha_n$ of elements of $H^1(G, \mathbb{Z}/p\mathbb{Z})$, $\langle \alpha_1, \ldots, \alpha_n \rangle \neq \emptyset$ implies $0 \in \langle \alpha_1, \ldots, \alpha_n \rangle$;

(b) the strong $n$-Massey vanishing property (with respect to $\mathbb{Z}/p\mathbb{Z}$) if for every sequence $\alpha_1, \ldots, \alpha_n$ of elements of $H^1(G, \mathbb{Z}/p\mathbb{Z})$, the condition on the cup-products

$$\alpha_1 \cup \alpha_2 = \alpha_2 \cup \alpha_3 = \ldots = \alpha_{n-1} \cup \alpha_n = 0$$

implies $0 \in \langle \alpha_1, \ldots, \alpha_n \rangle$ (cf. [22] Def. 1.2) — we remind that the triviality condition (6.2) is satisfied whenever $\langle \alpha_1, \ldots, \alpha_n \rangle \neq \emptyset$, cf., e.g., [20] § 2;

(c) the cyclic $p$-Massey vanishing property if for every element $\alpha \in H^1(G, \mathbb{Z}/p\mathbb{Z})$, the $p$-fold Massey product $\langle \alpha, \ldots, \alpha \rangle$ contains 0.

Remark 6.1. Given a sequence $\alpha_1, \ldots, \alpha_n$ of elements of $H^1(G, \mathbb{Z}/p\mathbb{Z})$, if an element $\omega$ of $H^2(G, \mathbb{Z}/p\mathbb{Z})$ is a value of the $n$-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$, then

$$\omega + \alpha_1 \cup \beta \in \langle \alpha_1, \ldots, \alpha_n \rangle \quad \text{and} \quad \omega + \alpha_n \cup \beta \in \langle \alpha_1, \ldots, \alpha_n \rangle$$

for any $\beta \in H^1(G, \mathbb{Z}/p\mathbb{Z})$ (cf. [20] Rem. 2.2).

In [13] Thm. 8.1, J. Minac and N.D. Tˆ an proved that the maximal pro-$p$ Galois group of a field $\mathbb{K}$ containing a root of 1 of order $p$ (and also $\sqrt{-1}$ if $p = 2$) satisfies the cyclic $p$-Massey vanishing property. The proof of the last property for a pro-$p$ group $G$ as in Theorem [13] is rather immediate.

Proof of Proposition [13] (ii). By Proposition [4.1] and Proposition [5.1] $G$ may complete into a Kummerian oriented pro-$p$ group with torsion-free orientation. Hence, $G$ satisfies the cyclic $p$-Massey vanishing property by [23] Thm. 3.10].
6.2. Massey products and unipotent upper-triangular matrices. Massey products for a pro-$p$ group $G$ may be translated in terms of unipotent upper-triangular representations of $G$ as follows. For $n \geq 2$ let

$$U_{n+1} = \left\{ \begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,n+1} \\ 1 & a_{2,3} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & a_{n,n+1} & \cdots & 1 \end{pmatrix} \mid a_{i,j} \in \mathbb{Z}/p \right\} \subseteq \text{GL}_{n+1}(\mathbb{Z}/p\mathbb{Z})$$

be the group of unipotent upper-triangular $(n+1) \times (n+1)$-matrices over $\mathbb{Z}/p$. Then $U_{n+1}$ is a finite $p$-group. Moreover, for $1 \leq h, l \leq n+1$ let $E_{h,l}$ denote the $(n+1) \times (n+1)$ matrix with the $(h,l)$-entry equal to 1, and all the other entries equal to 0.

Now let $\rho: G \to U_{n+1}$ be a homomorphism of pro-$p$ groups. Observe that for every $h = 1, \ldots, n$, the projection $\rho_{h,h+1}: G \to \mathbb{Z}/p$ of $\rho$ onto the $(h,h+1)$-entry is a homomorphism, and thus it may be considered as an element of $H^1(G, \mathbb{Z}/p\mathbb{Z})$. One has the following “pro-$p$ translation” of a result of W. Dwyer which interprets Massey product in terms of unipotent upper-triangular representations (cf., e.g., [11, Lemma 9.3]).

**Proposition 6.2.** Let $G$ be a pro-$p$ group, and let $\alpha_1, \ldots, \alpha_n$ be a sequence of elements of $H^1(G, \mathbb{Z}/p\mathbb{Z})$, with $n \geq 2$. Then the $n$-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$:

(i) is not empty if, and only if, there exists a morphism of pro-$p$ groups $\tilde{\rho}: G \to U_{n+1}/\mathbb{Z}(U_{n+1})$ such that $\tilde{\rho}_{h,h+1} = \alpha_h$ for every $h = 1, \ldots, n$;

(ii) vanishes if, and only if, there exists a morphism of pro-$p$ groups $\rho: G \to U_{n+1}$ such that $\rho_{h,h+1} = \alpha_h$ for every $h = 1, \ldots, n$.

We recall that $Z(U_{n+1}) = \{ I_{n+1} + aE_{1,n+1} \mid a \in \mathbb{Z}/p\mathbb{Z} \} \simeq \mathbb{Z}/p\mathbb{Z}$.

We use this fact to prove statements (iii.a)–(iii.b) of Proposition 13. First of all, let $G$ be as in Theorem 13 and let $\alpha_1, \ldots, \alpha_n$ be a sequence of elements of $H^1(G, \mathbb{Z}/p\mathbb{Z})$. Keeping the same notation as in §5.3, for $h = 1, \ldots, n$ one has

$$\alpha_h = \alpha_h(x) \cdot \chi + \sum_{i=1}^{d_1} \alpha_h(y_i) \cdot \varphi_i + \sum_{j=1}^{d_2} \alpha_h(z_j) \cdot \psi_j.$$

Therefore, for $h = 1, \ldots, n-1$ one obtains

$$\alpha_h \circ \alpha_h = S_h \cdot (\chi \circ \varphi_1) + S'_h \cdot (\chi \circ \psi_1),$$

where

$$S_h = (\alpha_h(x)\alpha_h(y_1) - \alpha_h(y_1)\alpha_h(x)) + (-1)^{\ell} \sum_{2 \mid i} (\alpha_h(y_i)\alpha_h(y_{i+1}) - \alpha_h(y_{i+1})\alpha_h(y_{i+1})),
$$

$$S'_h = (\alpha_h(x)\alpha_h(z_1) - \alpha_h(z_1)\alpha_h(x)) + (-1)^{\ell} \sum_{2 \mid j} (\alpha_h(z_j)\alpha_h(z_{j+1}) - \alpha_h(z_{j+1})\alpha_h(z_j)).$$
with $\epsilon = 0$ if $G$ is as in (1.1.a), and $\epsilon = 1$ if $G$ is as in (1.1.b). If the sequence $\alpha_1, \ldots, \alpha_n$ satisfies condition \((6.2)\), then one has $S_h = S'_h = 0$ for $h = 1, \ldots, n - 1$, as $\{\chi \sim \varphi_1, \chi \sim \psi_1\}$ is a basis of $H^2(G, \mathbb{Z}/p)$. 

From now on, we will assume that $p > 3$ while considering a pro-$p$ group $G$ as in (1.1.b), unless stated otherwise.

6.3. 3-fold Massey products. We are ready to prove the following.

**Proposition 6.3.** A pro-$p$ group $G$ satisfies the 3-Massey vanishing property in the following cases:

(a) if $G$ is as in (1.1.a);

(b) if $G$ is as in (1.1.b) and $p > 3$.

**Proof.** Let $\alpha_1, \alpha_2, \alpha_3$ be a sequence of elements of $H^1(G, \mathbb{Z}/p\mathbb{Z})$ satisfying \((6.2)\). Then $S_1 = S'_1 = S_2 = S'_2 = 0$ (cf. § 6.2). Our goal is to construct a morphism $\rho: G \to \mathbb{U}_4$ such that $\rho_{1,2} = \alpha_1, \rho_{2,3} = \alpha_2, \rho_{3,4} = \alpha_3$.

For every $w \in \mathcal{X}$ set

$$A(w) = I + \alpha_1(w)E_{1,2} + \alpha_2(w)E_{2,3} + \alpha_3(w)E_{3,4} \in \mathbb{U}_4,$$

where $I$ denotes the $4 \times 4$ identity matrix. If $G$ is as in (1.1.a), then one computes

$$C = [A(x), A(y_1)] \cdots [A(y_{d_1} \cdot 1), A(y_{d_1})]$$

$$= I + E_{1,4} \left( \alpha_1(y_1)\alpha_2(x)\alpha_3(y_1) + \sum_{2 | i} \alpha_1(y_i)\alpha_2(y_{i+1})\alpha_3(y_i) \right)$$

$$C' = [A(x), A(z_1)] \cdots [A(z_{d_2} \cdot 1), A(z_{d_2})]$$

$$= I + E_{1,4} \left( \alpha_1(z_1)\alpha_2(x)\alpha_3(z_1) + \sum_{2 | j} \alpha_1(z_j)\alpha_2(z_{j+1})\alpha_3(z_j) \right);$$

while if $G$ is as in (1.1.b), then one computes

$$C = A(y_1)^p[A(y_1), A(x)] \cdots [A(y_{d_1} \cdot 1), A(y_{d_1})]$$

$$= I + E_{1,4} \left( \alpha_1(x)\alpha_2(y_1)\alpha_3(x) + \sum_{2 | i} \alpha_1(y_i)\alpha_2(y_{i+1})\alpha_3(y_i) \right)$$

$$C' = A(z_1)^p[A(z_1), A(x)] \cdots [A(z_{d_2} \cdot 1), A(z_{d_2})]$$

$$= I + E_{1,4} \left( \alpha_1(x)\alpha_2(z_1)\alpha_3(x) + \sum_{2 | j} \alpha_1(z_j)\alpha_2(z_{j+1})\alpha_3(z_j) \right).$$

— observe that the exponent of $\mathbb{U}_4$ is $p$, as $p > 4$, and thus $A(y_1)^p = A(z_1)^p = I$.

In both cases, $C, C' \in \mathbb{Z}\langle \mathbb{U}_4 \rangle$, and therefore the assignment $w \mapsto A(w)$ for every $w \in \mathcal{X}$ yields a morphism $\rho: G \to \mathbb{U}_4/\mathbb{Z}\langle \mathbb{U}_4 \rangle$ satisfying $\rho_{h, h+1} = \alpha_h$ for $h = 1, 2, 3$. Thus, $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \neq \emptyset$ by Proposition 6.2.

Moreover, if $C = C'$ then the same assignment yields a morphism $\rho: G \to \mathbb{U}_4$ with the desired properties. In particular, by \((6.3)-(6.4)\) one has $C = I$ if $\alpha_1(w) = \alpha_3(w) = 0$ for every $w = y_1, \ldots, y_{d_1}$, or for every $w = y_2, \ldots, y_{d_1}$ and $w = x$; and analogously
Moreover, put
\[ C = (c_{hi}) = A(y_1)^{sp} \cdot [A(x), A(y_1)]^{(-1)^*} \cdots [A(y_{d_1-1}), A(y_{d_1})], \]
\[ C' = (c'_{hi}) = A(z_1)^{sp} \cdot [A(x), A(z_1)]^{(-1)^*} \cdots [A(z_{d_2-1}), A(z_{d_2})]. \]
We will consider the matrix \( C \) as a function of the matrices \( A(x), \ldots, A(y_{d_1}) \), and the matrix \( C' \) as a function of the matrices \( A(x), A(z_1), \ldots, A(z_{d_2}) \).
Then \( I \) implies that where we apply (6.7) to obtain the first equality, and in the second one the even positive integers between 2 and \( d \) straightforward to see that \( \alpha \) is 0. Moreover, for \( h = 1, 2, 3 \) one has \( c_{h,h+2} = S_h \) and \( c_{h,h+2} = S_h' \) — which are equal to 0 by (6.2).

We split the proof in the analysis of the following three cases. Our aim is to modify suitably the matrices \( A(w) \) — without modifying the \( (h,h+1) \)-entries with \( h = 1, \ldots, 4 \) — in order to obtain \( C = C' = I \).

**Case 1.** Suppose first that:

(1.a) \( \alpha_2(x) = \alpha_2(y_i) = 0 \) for all \( 2 \leq i \leq d_1 \); or

(1.b) \( \alpha_3(x) = \alpha_3(y_i) = 0 \) for all \( 2 \leq i \leq d_1 \).

Since \( S_1 = S_2 = S_3 = 0 \) by (6.2), one has

\[
\alpha_1(x)\alpha_2(y_1) = \alpha_2(y_1)\alpha_3(x) = 0,
\]

respectively in case (1.a) and in case (1.b). Applying (6.5) — (6.6), one computes

\[
[A(y_1), A(x)] = \begin{cases} I + (\alpha_3(y_1)\alpha_4(x) - \alpha_3(x)\alpha_4(y_1)) E_{3,5} & \text{in case (1.a),} \\ I + (\alpha_1(y_1)\alpha_2(x) - \alpha_2(x)\alpha_1(y_1)) E_{1,3} & \text{in case (1.b),} \end{cases}
\]

and

\[
[A(y_i), A(y_{i+1})] = \begin{cases} I + (\alpha_3(y_i)\alpha_4(y_{i+1}) - \alpha_3(y_{i+1})\alpha_4(y_i)) E_{3,5} & \text{in case (1.a),} \\ I + (\alpha_1(y_i)\alpha_2(y_{i+1}) - \alpha_2(y_{i+1})\alpha_1(y_i)) E_{1,3} & \text{in case (1.b),} \end{cases}
\]

for \( i = 2, 4, \ldots, d_1 - 1 \). Altogether, one has \( C = I + S_3 E_{3,5} \) in case (1.a) and \( C = I + S_1 E_{1,3} \) in case (1.b), so that in both cases \( C = I \) by (6.2).

Analogously, if \( \alpha_2(x) = \alpha_2(y_j) = 0 \) for all \( 2 \leq j \leq d_2 \), or if \( \alpha_3(x) = \alpha_3(y_j) = 0 \) for all \( 2 \leq j \leq d_2 \), then \( C' = I \). This completes the analysis of case 1.

**Case 2.** Now suppose that \( \alpha_1(x) = \alpha_4(x) = \alpha_1(y_i) = \alpha_4(y_i) = 0 \) for all \( 2 \leq i \leq d_1 \).

Since \( S_1 = S_2 = S_3 = 0 \) by (6.2), one has

\[
\alpha_1(y_1)\alpha_2(x) = \alpha_3(x)\alpha_4(y_1) = 0.
\]

Then one computes

\[
[A(y_1), A(x)] = I + (\alpha_2(y_1)\alpha_3(x) - \alpha_2(x)\alpha_3(y_1)) E_{2,4} + \alpha_2(x)\alpha_3(y_1)\alpha_4(y_1) E_{2,5},
\]

\[
[A(y_i), A(y_{i+1})] = I + (\alpha_2(y_i)\alpha_3(y_{i+1}) - \alpha_2(y_{i+1})\alpha_3(y_i)) E_{2,4},
\]

where we apply (6.7) to obtain the first equality, and in the second one \( r \) runs through the even positive integers between 2 and \( d_1 - 1 \). If \( \alpha_2(x)\alpha_3(y_1)\alpha_4(y_1) = 0 \) then it is straightforward to see that \( C = I + S_2 E_{2,4} = I \). Otherwise, \( \alpha_2(x) \neq 0 \), so that (6.7) implies that \( \alpha_1(y_1) = 0 \). In this case, set

\[
\hat{A} = I - \alpha_3(y_1)\alpha_4(y_1) E_{3,5}.
\]

Then

\[
[\hat{A}, A(x)] = I - \alpha_2(x)\alpha_3(y_1)\alpha_4(y_1) E_{2,5},
\]
Suppose we are in case (3.a). If needed — if $A$ odd, or one has $A$ and replace $A$ and replace $A$ with $w$, $w$ (3.b) there are $x$, $x$, $y$, $y$ with $z$, $z$ — possibly $w = w'$ — such that $\alpha_1(w) \neq 0$ and $\alpha_2(w') \neq 0$, or (3.a) there are $x$, $x$, $y$, $y$ with $z$, $z$ — possibly $w = w'$ — such that $\alpha_4(w) \neq 0$ and $\alpha_3(w') \neq 0$.

Suppose we are in case (3.a). If $w = x$ or $w = y_i$ with $i$ odd, set

$$\tilde{A} = \begin{cases} I + \frac{c_{1,4}}{\alpha_1(w)} E_{2,4} & \text{if } w \in \{ x, y_3, \ldots, y_{d_1} \} \\ I - \frac{c_{1,4}}{\alpha_1(w)} E_{2,4} & \text{if } w \in \{ y_i \ | \ i \text{ is even} \} \end{cases}$$

and replace $A(y_1)$ with $A(y_1) \tilde{A}$, if $w = x$, or $A(y_{i-1})$ with $A(y_{i-1}) \tilde{A}$ if $w = y_i$ with $i$ odd, or $A(y_{i+1})$ with $A(y_{i+1}) \tilde{A}$, if $w = y$ with $i$ even. After the replacement, one has $c_{ht} = 0$ for $h < l \leq h + 2$, and for $(h, l) = (1, 4)$. Then, set

$$\tilde{A}' = \begin{cases} I + \frac{c_{2,5}}{\alpha_1(w)} E_{3,5} & \text{if } w' \in \{ x, y_3, \ldots, y_{d_1} \} \\ I - \frac{c_{2,5}}{\alpha_1(w)} E_{3,5} & \text{if } w' \in \{ y_i \ | \ i \text{ is even} \} \end{cases}$$

and replace $A(y_1)$ with $A(y_1) \tilde{A}'$, if $w = x$, or $A(y_{i-1})$ with $A(y_{i-1}) \tilde{A}'$ if $w = y_i$ with $i$ odd, or $A(y_{i+1})$ with $A(y_{i+1}) \tilde{A}'$, if $w = y$ with $i$ even. After this further replacement, one has $c_{h\ell} = 0$ for $h < l \leq h + 3$. Finally, set

$$\tilde{A}'' = \begin{cases} I + \frac{c_{2,5}}{\alpha_1(w)} E_{2,5} & \text{if } w \in \{ x, y_3, \ldots, y_{d_1} \} \\ I - \frac{c_{2,5}}{\alpha_1(w)} E_{2,5} & \text{if } w \in \{ y_i \ | \ i \text{ is even} \} \end{cases}$$

and replace $A(y_1)$ with $A(y_1) \tilde{A}''$, if $w = x$, or $A(y_{i-1})$ with $A(y_{i-1}) \tilde{A}''$ if $w = y_i$ with $i$ odd, or $A(y_{i+1})$ with $A(y_{i+1}) \tilde{A}''$, if $w = y$ with $i$ even. After this last replacement, one has $C = I$.

Now suppose we are in case (3.b). If $w = x$ or $w = y_i$ with $i$ odd, set

$$\tilde{A} = \begin{cases} I - \frac{c_{2,5}}{\alpha_1(w)} E_{3,4} & \text{if } w \in \{ x, y_3, \ldots, y_{d_1} \} \\ I + \frac{c_{2,5}}{\alpha_1(w)} E_{3,4} & \text{if } w \in \{ y_i \ | \ i \text{ is even} \} \end{cases}$$

and replace $A(y_1)$ with $A(y_1) \tilde{A}$, if $w = x$, or $A(y_{i-1})$ with $A(y_{i-1}) \tilde{A}$ if $w = y_i$ with $i$ odd, or $A(y_{i+1})$ with $A(y_{i+1}) \tilde{A}$, if $w = y$ with $i$ even. After the replacement, one has
with the desired properties.

Question 6.6. (a) Let \( G \) be as in (1.1.a). Does \( G \) satisfy the strong \( n \)-Massey vanishing property for every \( n \geq 3 \)?

(b) Let \( G \) be as in (1.1.b). Does \( G \) satisfy the strong \( n \)-Massey vanishing property for every \( 3 \leq n < p \)?

7. The Minač-Tân pro-\( p \) group

We focus now on the Minač-Tân pro-\( p \) group

\[ G = \langle x_1, \ldots, x_5 \mid r = 1 \rangle \quad \text{with} \quad r = [[x_1, x_2], x_3] [x_4, x_5]. \]

Using Proposition 6.2, one may show that \( G \) does not satisfy the 3-Massey vanishing property (cf. [20, Ex. 7.2]). Our aim is to show that \( G \) cannot complete into a 1-cyclotomic oriented pro-\( p \) group with torsion-free orientation.

7.1. Kummerianity and 1-cyclotomicity.

Proposition 7.1. Let \( G \) be the Minač-Tân pro-\( p \) group, and let \( \theta : G \to 1 + p\mathbb{Z}_p \) be a torsion-free orientation. Then the oriented pro-\( p \) group \( (G, \theta) \) is Kummerian if, and only if, \( x_4, x_5 \in \ker(\theta) \), and:

(a) \( x_3 \in \ker(\theta) \); or
(b) \( x_1, x_2 \in \ker(\theta) \).
Proof. Let $c: G \to \mathbb{Z}_p(\theta)$ be an arbitrary continuous 1-cocycle, and set $c(x_i) = \lambda_i$ for $i = 1, \ldots, 5$. Applying (2.2)–(2.3) one computes $c(r) = c([x_i, x_j]) + c([x_4, x_5])$, and
\begin{equation}
\begin{aligned}
c([x_1, x_2], x_3)) &= \theta(x_1 x_2)^{-1} (\theta(x_3)^{-1} - 1) (\lambda_1 (1 - \theta(x_2)) - \lambda_2 (1 - \theta(x_1))) , \\
c([x_4, x_5]) &= \theta(x_4 x_5)^{-1} (\lambda_4 (1 - \theta(x_5)) - \lambda_5 (1 - \theta(x_4))) .
\end{aligned}
\end{equation}
(7.1)

On the other hand, $c(r) = 0$ as $r = 1$.

Suppose that $(G, \theta)$ is Kummerian. Then by Lemma 2.9 we may prescribe arbitrary values to $\lambda_1, \ldots, \lambda_5$. If $\lambda_4 = 1$ and $\lambda_i = 0$ for $i \neq 4$, from (7.1) and from the fact that $c(r) = 0$ one obtains $0 = 1 \cdot (1 - \theta(x_5))$, and thus $\theta(x_5) = 1$. Analogously, if $\lambda_5 = 1$ and $\lambda_i = 0$ for $i \neq 5$, one deduces $\theta(x_4) = 1$. Finally, if $\lambda_4 = \lambda_5 = 0$ from (7.1) one obtains
\begin{equation*}
0 = c(r) = \theta(x_1 x_2)^{-1} (\theta(x_4)^{-1} - 1) (\lambda_1 (1 - \theta(x_2)) - \lambda_2 (1 - \theta(x_1))) ,
\end{equation*}
and the arbitrariness of $\lambda_1, \lambda_2$ implies that $\theta(x_3) = 1$ or $\theta(x_1) = \theta(x_2) = 1$.

Conversely, suppose that $x_4, x_5 \in \text{Ker}(\theta)$, and at least one of the hypothesis (i)–(ii) holds true. Then for any choice for $\lambda_4, \lambda_5$, by (7.1) one has $c([x_4, x_5]) = 0$. On the other hand, one has
\begin{equation*}
c([x_1, x_2], x_3)) = \begin{cases} 0 \cdot (\lambda_1 (1 - \theta(x_2)) - \lambda_2 (1 - \theta(x_1))) & \text{if } x_3 \in \text{Ker}(\theta), \\
(\theta(x_3)^{-1} - 1) (\lambda_1 \cdot 0 - \lambda_2 \cdot 0) & \text{if } x_1, x_2 \in \text{Ker}(\theta).
\end{cases}
\end{equation*}

Altogether, any choice for $\lambda_1, \ldots, \lambda_5$ yields a well-defined continuous 1-cocycle $c: G \to \mathbb{Z}_p(\theta)$, and thus $(G, \theta)$ is Kummerian by Lemma 2.9.

Now consider the subgroup $H$ of $G$ generated by $x_3, x_4, x_5$ and by $y = [x_1, x_2]$. Then $H$ is subject to the relation
\begin{equation*}
r = [y, x_3] [x_4, x_5] = 1 .
\end{equation*}

If $(G, \theta)$ is a 1-cyclotomic oriented pro-$p$ group, with $\theta$ a torsion-free orientation, then $(H, \theta|_H)$ is Kummerian. Therefore, if $c': H \to \mathbb{Z}_p(\theta|_H)$ is a continuous 1-cocycle, applying (2.2)–(2.3) one obtains
\begin{equation*}
\begin{aligned}
0 &= c'(r) = c'([y, x_3]) + c'([x_4, x_5]) \\
&= \theta(y x_3)^{-1} (c'(y) (1 - \theta(x_3)) - c'(x_3) (1 - \theta(y))) + 0 \\
&= \theta(y x_3)^{-1} c'(y) (1 - \theta(x_3)),
\end{aligned}
\end{equation*}
as $\theta(x_4) = \theta(x_3) = 1$ by Proposition 7.1 and $y \in G' \subseteq \text{Ker}(\theta)$. Since $c'(y)$ may be arbitrarily chosen by Lemma 2.9 one deduces $\theta(x_3) = 1$. This proves the following.

Lemma 7.2. Let $G$ be the Minaˇc-Tân pro-$p$ group, and let $\theta: G \to 1 + p\mathbb{Z}_p$ be a torsion-free orientation. If the oriented pro-$p$ group $(G, \theta)$ is 1-cyclotomic then $x_3, x_4, x_5 \in \text{Ker}(\theta)$.

Moreover, if $(G, \theta)$ is 1-cyclotomic we may suppose without loss of generality that $x_2 \in \text{Ker}(\theta)$, too. Indeed, let $v_p: \mathbb{Z}_p \to \mathbb{N}$ denote the $p$-adic valuation, and let $k \geq 1$ be such that $\text{Im}(\theta) = 1 + p^k \mathbb{Z}_p$.

Suppose first that $v_p(\theta(x_2) - 1) = k$ and $v_p(\theta(x_1) - 1) > k$, and set $z = x_2 x_1$. Then $\{z, x_2, x_3, x_4, x_5\}$ is a minimal generating set of $G$, $v_p(\theta(z) - 1) = k$, and $G$ is subject to the relation
\begin{equation*}
[[z, x_2], x_3] [x_4, x_5] = 1 ,
\end{equation*}
as $[x_2 x_1, x_2] = [x_1, x_2]$. Hence, we may assume $v_p(\theta(x_1) - 1) = k$. 


Consequently, there exists \( \lambda \in \mathbb{Z}_p \) such that \( \theta(x_2) = \theta(x_1)^{\lambda} \). Now set \( z = x_1^{-\lambda}x_2 \). Then \( \{x_1, z, x_3, x_4, x_5\} \) is a minimal generating set of \( G \), \( \theta(z) = \theta(x_2)\theta(x_1)^{-\lambda} = 1 \), and \( G \) is subject to the relation

\[
[[x_1, z], x_3] [x_4, x_5] = 1,
\]

as \( [x_1, x_1^{-\lambda}x_2] = [x_1, x_2] \).

Therefore, from now on \( \theta: G \to 1 + p\mathbb{Z}_p \) will denote a torsion-free orientation satisfying \( x_2, \ldots, x_5 \in \text{Ker}(\theta) \).

### 7.2. The subgroup \( U \)

Put \( u = x_1^p \) and \( t = x_1^{-1}x_3 \). Let \( \phi: G \to \mathbb{Z}/p \) be the homomorphism defined by \( \phi(x_1) = \phi(x_3) = 1 \) and \( \phi(x_i) = 0 \) for \( i = 2, 4, 5 \), and let \( U \) be the kernel of \( \phi \). Then \( U \) is a normal subgroup of \( G \) of index \( p \), and it is generated as a normal subgroup of \( G \) by \( \{u, t, x_2, x_4, x_5\} \). In fact, \( U \) is generated as a pro-\( p \) group by the set

\[
\mathcal{X}_U = \left\{ u, t, x_2, x_4^h, x_4^h, x_5^h \mid h = 0, \ldots, p - 1 \right\},
\]

as \( G/U = \{U, x_1U, \ldots, x_1^{p-1}U\} \). We need to find a subset of \( \mathcal{X}_U \) which minimally generates \( U \) as a pro-\( p \) group.

#### Proposition 7.3.

The set

\[
\mathcal{Y}_U = \left\{ t, x_2, x_2^x, t^{x_2^h}, x_4^h, x_5^h \mid h = 0, \ldots, p - 1 \right\},
\]

is a minimal generating set of \( U \) as a pro-\( p \) group. Moreover, the abelian pro-\( p \) group \( U^{ab} \) is not torsion-free.

**Proof.** The subgroup \( U \) is the pro-\( p \) group generated by \( \mathcal{X}_U \) and subject to the \( p \)-relations \( r^{x_i^h} = 1, h = 0, \ldots, p - 1 \). Since \( x_3 = x_1t \), one computes

\[
[[x_1, x_2], x_3] = [x_1, x_2]^{-1} \cdot [x_1, x_2]^{x_3} = [x_2, x_1] \cdot [x_1, x_2^{x_3}]^t = x_2^{-1} \cdot x_2 \cdot \left( x_2^{x_2^h} \right)^t.
\]

From (7.2), and from the relation \( r = 1 \), one deduces the equivalence

\[
\left( x_2^{x_2^h} \right)^{-1} \cdot (x_2^{x_1^h})^2 \cdot x_1^{-1} \equiv 1 \mod U',
\]

as \( [x_4, x_5] \in U' \) and \( t \in U \).

Hence, \( U^{ab} \) is the abelian pro-\( p \) group generated by \( \mathcal{X}_{U^{ab}} = \{wU' \mid w \in \mathcal{X}_U\} \) and subject to the \( p \) relations induced by the equivalences \( (x_2^{x_2^h} x_1^{-1})^{x_1^{-1}} \equiv 1 \mod U' \), namely

\[
x_2^{x_2^h} \equiv (x_2^{x_1^h})^2 x_1^{-1} \mod U', \quad \text{for } h = 0,
\]

\[
x_2^{x_2^h} \equiv (x_2^{x_1^h})^2 (x_1^{x_2^h})^{-1} \equiv (x_2^{x_1^h})^3 x_1^{-2} \mod U', \quad \text{for } h = 1,
\]

\[
\vdots
\]

\[
x_2^{x_2^h} \equiv (x_2^{x_1^h})^2 (x_1^{x_2^h})^{-1} \equiv (x_2^{x_1^h})^p x_1^{-1-p} \mod U', \quad \text{for } h = p - 2,
\]

\[
x_2^{x_2^h} \equiv (x_2^{x_1^h})^2 x_1^{-1} \equiv (x_2^{x_1^h})^p x_1^{-p} \mod U', \quad \text{for } h = p - 1.
\]
On the one hand, from (7.4) one deduces that the coset $x_2^{x_1} U'$ is generated by $x_2 U'$ and $x_2^{x_1} U'$ for every $h = 2, \ldots, p - 1$, so that $\mathcal{Y}_{U}^{ab} = \{w U' \mid w \in \mathcal{Y}_U\}$ generates $U^{ab}$ as an abelian pro-$p$ group. On the other hand, from the equivalences with $h = p - 2$ and $h = p - 1$ one deduces that
\[
(x_2^{x_1})^p x_1^{1-p} (x_2^{x_1})^{-1} \equiv (x_2^{x_1})^p x_1^{1-p-1} \equiv (x_2^{x_1} x_1^{-1})^p \equiv 1 \mod U',
\]
\[
(x_2^{x_1})^{p+1} x_1^{-p} (x_2^{x_1})^{-1} \equiv (x_2^{x_1})^{p+1-1} x_1^{-p} \equiv (x_2^{x_1} x_1^{-1})^p \equiv 1 \mod U',
\]
as $x_2^h \equiv x_2 \mod U'$; therefore they yield equivalent relations in $U^{ab}$. Altogether, $U^{ab}$ is the abelian pro-$p$ group minimally generated by $\mathcal{X}_{U}^{ab}$ and subject to the relation
\[
((x_2 U')^{-1} \cdot x_2^{x_1} U')^p = 1.
\]
Hence $U^{ab}$ is not torsion-free, and $\mathcal{Y}_U$ is a minimal generating set of $U$ by Fact 2.1. □

From Proposition 7.3 one deduces that $G$ is not absolutely torsion-free, and thus the oriented pro-$p$ group $(G, 1)$ is not 1-cyclotomic.

### 7.3. 1-cyclotomicity and the Minač-Tǎn pro-$p$ group

We are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** Suppose for contradiction that there exists a torsion free orientation $\theta : G \to 1 + \mathbb{Z}_p$ such that the oriented pro-$p$ group $(G, \theta)$ is 1-cyclotomic. Then by § 7.1 we may assume without loss of generality that $x_2, \ldots, x_5 \in \text{Ker}(\theta)$, while $\theta(x_1) \neq 1$ by § 7.2. Set $\lambda \in \mathbb{Z}_p \setminus \{0\}$ such that $\theta(x_1) = 1 + \lambda$.

Consider the oriented pro-$p$ group $(U, \theta|_U)$, and set $K = K_{\theta|_U}(U)$. Let $U = U/K$. Our goal is to show that the oriented pro-$p$ group $(\bar{U}, (\theta|_U)'|_K)$ is not $(\theta|_U)'|_K$-abelian, so that $(U, \theta|_U)$ is not Kummerian by Proposition 2.8 and thus $(G, \theta)$ is not 1-cyclotomic.

Since $K \subseteq \Phi(U)$, by Proposition 7.3 the set $\mathcal{Y}_\bar{U} = \{wK \mid w \in \mathcal{Y}_U\}$ is a minimal generating set of $\bar{U}$. Now, since $\theta(t) = \theta(x_1) = (1 + \lambda)^{-1}$, one has $w^t \equiv w^{1+\lambda} \mod K$ for every $w \in U$. Therefore, from (7.2), and from the fact that $[x_4, x_5] \in \text{Ker}(\theta|_U)' \subseteq K$, one obtains
\[
[x_1, x_2]^{-1} ([x_1, x_2]^{x_1})^t \equiv [x_1, x_2]^{-1} ([x_1, x_2]^{x_1})^{(1+\lambda)^{-1}} \equiv 1 \mod K,
\]
and consequently
\[
[x_1, x_2]^{x_1} \equiv [x_1, x_2]^{1+\lambda} \mod K,
\]
\[
[x_1, x_2]^{x_2} \equiv [x_1, x_2]^{(1+\lambda)^{2}} \mod K,
\]
\[
\vdots
\]
\[
[x_1, x_2]^{x_1^{p-1}} \equiv [x_1, x_2]^{(1+\lambda)^{p-1}}.
\]
Set
\[
\mu = (1 + \lambda)^0 + (1 + \lambda)^1 + \ldots + (1 + \lambda)^{p-1} = \frac{(1 + \lambda)^p - 1}{\lambda}.
\]
Then $\mu \neq 0$ (as $\lambda \neq 0$), and $p \mid \mu$. Since $[x_1, x_2] = (x_2^{x_1})^{-1} x_2$, replacing the coset $x_2^{x_1} K$ with the coset $[x_1, x_2] K$ in $\mathcal{Y}_\bar{U}$ yields another minimal generating set — let us call it $\mathcal{Y}_\bar{U}$.
— of $\bar{U}$. Now, from (7.3) one obtains
\[
[u, x_2] = [x_1, x_2]^{x_1^{-1}} \cdots [x_1, x_2] \cdot [x_1, x_2] \\
\equiv [x_1, x_2]^{1 + \lambda} \cdots [x_1, x_2]^{1 + \lambda} \cdot [x_1, x_2] \mod K \\
\equiv [x_1, x_2]^{\mu} \mod K
\]
observe that $[x_1, x_2]^{x_1^h} \in \text{Ker}(\theta|_U)$ for every $h$, and thus all such elements commute modulo $K$. Therefore, one has the relation
\[
([x_1, x_2]K)^{\mu} = [uK, x_2K]
\]
between elements of the minimal generating set $\bar{U}$, and by [11 Thm. 8.1] this relation prevents the oriented pro-$p$ group $(\bar{U}, (\theta|_U)/K)$ from being Kummerian — and thus also $(\theta|_U)/K$-abelian.

□

From Theorem 1.4 we obtain a new family of pro-$p$ groups which cannot complete into 1-cyclotomic oriented pro-$p$ groups.

**Corollary 7.4.** Let $G$ be the pro-$p$ group with presentation
\[
G = \langle x_1, \ldots, x_n, x_{n+1}, x_{n+2} \mid \ldots[[x_1, x_2], \ldots x_{n-1}], x_n][x_{n+1}, x_{n+2}] = 1 \rangle,
\]
with $n \geq 3$. Then $G$ cannot complete into a 1-cyclotomic oriented pro-$p$ group with torsion-free orientation.

**Proof.** Set $y = \ldots[x_1, x_2]\ldots x_{n-2}$, and let $H$ be the subgroup of $G$ generated by $\{y, x_{n-1}, \ldots, x_{n+2}\}$. Then
\[
H = \langle y, x_{n-1}, \ldots, x_{n+2} \mid [[y, x_{n-1}], x_n][x_{n+1}, x_{n+2}] \rangle
\]
is isomorphic to the Minač-Tân pro-$p$ group, and hence it cannot complete into a 1-cyclotomic oriented pro-$p$ group with torsion-free orientation by Theorem 1.4. □

The following question remains open (cf. [2 Ex. 3.2]).

**Question 7.5.** Is the Minač-Tân pro-$p$ group $G$ a Bloch-Kato pro-$p$ group? Namely, is the $\mathbb{Z}/p\mathbb{Z}$-cohomology algebra of every closed subgroup of $G$ a quadratic algebra?

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Department of Science and High-Tech, University of Insubria, Como, Italy EU
Email address: claudio.quadrelli@uninsubria.it