Stacks of \textit{Ann}-Categories and their morphisms

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Abstract

We show that \textit{ann}-categories admit a presentation by crossed bimodules, and prove that morphisms between them can be expressed by special kinds spans between the presentations. More precisely, we prove the groupoid of morphisms between two \textit{ann}-categories is equivalent to that of bimodule butterflies between the presentations. A bimodule butterfly is a specialization of a butterfly, i.e. a special kind of span or fraction, between the underlying complexes.

Introduction

A categorical ring is a category carrying a bimonoidal structure which resembles that of a ring, up to natural isomorphisms and coherence conditions. Unlike the notion of categorical group and its higher categorical version, there are different notions of categorical rings, according to the strength of the commutativity axiom imposed on the underlying additive categorical group. Usually, the underlying categorical group is assumed to be a symmetric one [JP07]. This is a sort of “fixed point:” in a companion paper [Ald15] we have (among other things) explored the possibility of relaxing the commutativity of the additive structure, by assuming just a braiding. However, in the unital case, a categorical ring satisfying these more relaxed axioms turns out to be equivalent to one of the usual sort.

Here we take a different approach, and we explore the case where the commutativity law on the additive structure is actually stricter, namely we study categorical rings whose underlying categorical groups are actually Picard groupoids. These where introduced, under the name \textit{Ann}-categories, in a series of works [Qua03; Qua08; QHT08]; in parallel with the classical analysis of \textit{gr}-categories (i.e. categorical groups) carried out in [Sin75], \textit{Ann}-categories were found to be classified by the third Shukla cohomology of rings. (By contrast, categorical rings of the more general breed discussed above are classified by the third Mac Lane cohomology: degree three is the level at which the two theories begin to diverge, although precise comparisons exist [Mac58; BP06].)

It is well known the third Shukla cohomology occurs in the classification of two-terms extensions of the form

$$0 \longrightarrow A \longrightarrow M \xrightarrow{\partial} R \longrightarrow B \longrightarrow 0,$$

where $B$, $R$ are rings (or $k$-algebras, fixing a commutative ring $k$), $M$ an $R$-bimodule and $A$ a $B$-bimodule. These can be taken to be objects in a topos $T$, which we assume to be of the form $\text{Sh}(S)$, for a site $S$, whenever convenient. $A$, $B$, and $\partial : M \rightarrow R$ satisfy certain axioms, discussed below, which in particular define the structure of crossed bimodule for $\partial : M \rightarrow R$. The link with categorical groups—in fact, with \textit{Ann}-categories—is that the Picard groupoid associated to the complex $\partial : M \rightarrow R$ carries such a structure. We show below that this remains true for the stack $\mathfrak{R}$ associated to that groupoid.

We start by requiring that our categorical rings (see below for the precise terminology adopted here) be in particular Picard groupoids. Thus, we start from the monoidal structure on the 2-category $\text{Pic}$ of Picard stacks described by Deligne in the seminal [Del73], and define a categorical ring as a monoid object in this 2-category. We call it a ring, or ring-like stack, and in effect it is an object fibered in \textit{Ann}-categories.

Our main interest is the structure of the 2-category of monoids in $\text{Pic}$, rather than the classification issue. In general terms, our main results are that every ring-like stack is locally equivalent to the Picard groupoid associated to a crossed bimodule, and that the 2-category they form is equivalent to the bicategory $\text{XBiMod}$ of crossed bimodules of $T$. More precisely, we have, first:

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Theorem (Theorem 3.1.1). Let $\mathcal{R}$ be a ring-like stack. Then $\mathcal{R}$ is equivalent to the stack associated to a crossed bimodule $\partial : M \to R$.

Given this, there arises the question of calculating the groupoid $\text{Hom}(\mathcal{S}, \mathcal{R})$ of morphisms $F : \mathcal{S} \to \mathcal{R}$ of ring-like stacks in terms of the presentations that are guaranteed to exist by the theorem. Here the situation is similar to the one dealt with in the case of group-like stacks in [AN09], namely that $F$ above does not translate into a naïve morphism of crossed bimodules. The correct translation is that $F$ corresponds to a diagram of the form

\[
\begin{array}{ccc}
N & \xrightarrow{x} & M \\
\downarrow{\partial} & & \downarrow{\partial} \\
S & \xrightarrow{f} & E \\
\downarrow{\pi} & & \downarrow{\pi} \\
S & \xrightarrow{j} & R
\end{array}
\]

defined in sect. 4.1 below. This is what we call a butterfly of crossed bimodules. A salient feature is that the anti-diagonal is a ring (or $k$-algebra) extension, in general non-singular. Fixing the “wings,” one shows that these diagrams form a groupoid $\mathcal{B}(S_*, R_*)$, and we have

Theorem (Theorem 5.1). There are equivalences

\[\text{Hom}(\mathcal{S}, \mathcal{R}) \simeq \mathcal{B}(S_*, R_*) \quad \text{and} \quad \mathcal{H}om(\mathcal{S}, \mathcal{R}) \simeq \mathcal{B}(S_*, R_*).\]

The objects on the right are the stack versions, obtained by restricting the theorem to a variable object $S$ of $T$.

Define $\mathcal{XBiMod}$ as the bicategory whose objects are the crossed bimodules of $T$ and whose Hom-categories are the groupoids $\mathcal{B}(S_*, R_*)$. Analogously to [AN09], there is a composition $\mathcal{B}(S_*, R_*) \times \mathcal{B}(T_*, S_*) \to \mathcal{B}(T_*, R_*)$ defined in sect. 6.1. Then we have

Proposition (Propositions 6.2.1 and 6.2.2). $\mathcal{XBiMod}$ is equivalent to the 2-category of ring-like stacks, i.e. the 2-category of monoids in $\mathcal{Pic}$.

Furthermore the fibered bicategory defined by $U \to \mathcal{XBiMod}(S/U)$, $U \in \text{Ob}(S)$, is a (weak) 2-stack $\mathcal{XBiMod}$ which is equivalent to the 2-stack of Picard stacks.

Notation and terminology

For the hierarchy of commutativity conditions on monoidal (or actually group-like) categories and stacks we use the terms: braided, symmetric, and Picard as opposed to braided, Picard, and strictly Picard in force in, e.g. [Del73; Bre94; Bre99].

All complexes are cohomological, that is, the differential raises the degree. In order to simplify our notation, we use lower indices for negative degrees. In particular, for crossed (bi)modules we denote $\partial : M \to R$, or rather the corresponding complex, by $M$, with $R_0 = R$ and $R_1 = M$.

We fix a site $S$ and the topos $T$ of sheaves over $S$. A set-theoretic notation is employed. If $F$ is an object of $T$, then $x \in F$ means $x \in F(U)$ for an appropriate (but not relevant) $U \in \text{Ob}(S)$, or equivalently $x : U \to F$, identifying $U$ with the (pre)sheaf it represents. The same holds for the notation $x \in \mathcal{R}$ when $\mathcal{R}$ is a (pre)stack.

If $M$ is a monoidal category, $\Omega^{-1}M$—or $M^*$, especially if the monoidal structure of $M$ comes from a ring-like one—denotes the “suspension,” i.e. the corresponding bicategory with one object [Bén67].

For simplicial manipulations we use Duskin’s “opposite index convention” or “missing index” convention [Dus02, pages 207–210], with the variant that we reverse the indexing for the 1-simplices.

Finally, the important issue of the terminology. An Ann-category is a categorical ring whose underlying group-like groupoid is Picard. The term was coined and used in a series of works [QHT08; Qua08; Qua03], as an evident parallel to the better known “gr-category,” used to denote a 2-group, or categorical group, i.e. a group-like groupoid. Therefore using “ring-like” seems a justified alternative. However, very often “categorical ring” means a bimonoidal structure where the underlying categorical group is only required to be symmetric [see e.g. JP07], or even just braided, as done in a companion paper [Ald15]. Here we do not consider these more general alternatives and restrict ourselves to the Picard case. Hence we use “ring,” or “ring-like”-category, or “categorical ring,” as a strict synonym of ann-category, a term which may be awkward at times, at least to the author.

\[\text{Footnote:} \text{Here we are not complicating the issue with strict vs. lax questions.}\]
1 Ring-like stacks

A ring-stack, or stack with a ring-like structure, will be a stack \( R \) in groupoids over a site \( S \) equipped with a structure making it into a so-called categorical ring. There are different non-equivalent definitions of such a notion, according to whether the underlying category or stack is Picard, or merely symmetric. Our current stance is to assume \( R \) to be Picard, thereby the resulting ring-like fiber category will be akin to the Ann-categories of ref. [QHT08], as opposed to those of ref. [JP07].

1.1 Tensor products of Picard stacks

In ref. [Del73] Deligne observes that the 2-category \( \mathcal{P}(\mathcal{S}) \) of Picard stacks over the site \( \mathcal{S} \) is equipped with a monoidal structure \( \otimes : \mathcal{P}(\mathcal{S}) \times \mathcal{P}(\mathcal{S}) \rightarrow \mathcal{P}(\mathcal{S}) \). Recall that each Picard stack \( \mathcal{A} \) admits a presentation

\[
[A_1 \xrightarrow{\varepsilon} A_0] \sim \mathcal{A}
\]

where \( A_1 \rightarrow A_0 \) is a complex of abelian sheaves on \( \mathcal{S} \) supported in degrees \([-1, 0]\) and the left hand side above denotes forming the associated stack. (This is the same as taking the stacky quotient \([A_0/A_1]\) by the action of \( A_1 \) on \( A_0 \) via the differential of the complex.) The tensor structure in \( \mathcal{P}(\mathcal{S}) \) is defined as follows. If \( \mathcal{A} \) and \( \mathcal{B} \) are Picard stacks with given presentations as above, define [Del73]:

\[
\mathcal{A} \otimes \mathcal{B} = \left[ \tau_{\leq -1}(A_1 \otimes B_1) \right] \sim,
\]

where \( \tau \) denotes the soft truncation. The construction of \( \mathcal{A} \otimes \mathcal{B} \) does indeed have the expected universal property with respect to biadditive functors from \( \mathcal{A} \times \mathcal{B} \). In slightly more details, for any Picard stacks \( \mathcal{P} \) and \( \mathcal{Q} \) let \( \mathcal{H}om(\mathcal{Q}, \mathcal{P}) \) denote the Picard stack of additive functors. Moreover, let \( \mathcal{H}om(\mathcal{A}, \mathcal{B}; \mathcal{P}) \) denote the Picard stack of biadditive functors. Then there is an equivalence (of Picard stacks)

\[
\mathcal{H}om(\mathcal{A}, \mathcal{B}; \mathcal{P}) \simeq \mathcal{H}om(\mathcal{A} \otimes \mathcal{B}, \mathcal{P}),
\]

and \( \otimes : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B} \) is a “universal biadditive functor.”

1.2 Definition of ring-like stacks

With these premises, define a ring-like stack by mimicking the well-known fact that a ring is a monoid in the monoidal category of abelian groups:

1.2.1 Definition. A ring-like stack over \( \mathcal{S} \) is a Picard stack \( \mathcal{R} \) over \( \mathcal{S} \) equipped with a morphism

\[
m : \mathcal{R} \otimes \mathcal{R} \rightarrow \mathcal{R}
\]

of Picard stacks and a (unit) object \( I \) which together combine into the structure of a (lax) monoid in \( (\mathcal{P}(\mathcal{S}), \otimes) \).

In the sequel we will usually suppress \( m \) from the notation, and write \( XY \) in place of \( m(X \otimes Y) \), etc.

1.2.2 Remark. If \( \mathcal{R} \) is a ring-like stack as above and \([\varepsilon : M \rightarrow R]\) is a presentation of the underlying Picard stack, then the objects of \( \mathcal{R} \) can be interpreted as \( M \)-torsors with a trivialization of their extension as \( R \)-torsors (cf. refs. [Bre90; AN09]). It is important to keep in mind that this interpretation pertains to the additive structure of \( \mathcal{R} \). Thus, if \( E_1 \) and \( E_2 \) are two objects of \( \mathcal{R} \), the object \( E_1 \wedge E_2 \) corresponds in this interpretation to the standard torsor contraction \( E_1 \wedge M E_2 \). On the other hand, the object \( E_1 E_2 \) is an altogether different one (see below for an explicit construction).

1.2.3 Example. Let \( \mathcal{A} \) be a Picard stack, and let \( \mathcal{R} = \mathcal{E}nd(\mathcal{A}) \), the Picard stack of endomorphisms with respect to the sum of additive functors induced by that of \( \mathcal{A} \). Then \( \mathcal{R} \) has a ring-like structure with multiplication given by composition.

A morphism of ring-like stacks \( F : \mathcal{F} \rightarrow \mathcal{R} \) is defined in the obvious way, that is, as a morphism of the underlying Picard stacks compatible with the \( \otimes \)-monoidal structures, see [JP07]— modulo the difference between the symmetric and the Picard conditions for the underlying categorical groups.
2 Crossed bimodules and their quotients

2.1 Crossed bimodules

A way to produce ring-like stacks in the above sense is to consider complexes equipped with some additional structure, and then take the associated stack in the usual way. The appropriate structure is that of a crossed bimodule, or crossed module in algebras over $S$. Let us use the notation $\text{Ch}^{-1,0}(S)$ for the category of complexes of abelian sheaves on $T$ supported in degrees $[-1,0]$. Let us recall the definition (cf. refs. [Lod98; BM02; BP06]).

2.1.1 Definition. A crossed bimodule, or algebra crossed module, of $\text{Ch}^{-1,0}(S)$ is a complex

$$\partial : M \longrightarrow R$$

where $R$ is a ring, $M$ is an $R$-bimodule, and $\partial$ is a morphism of $R$-bimodules such that

$$\begin{equation}
(\partial m_1)m_2 = m_1(\partial m_2).
\end{equation}$$

for all $m_1, m_2 \in M$.

It is clear that the definition works for $k$-algebras, where $k$ is a fixed commutative ring $T$.

2.1.3 Remark. In more intrinsic term, the last condition in the definition—the Pfeiffer identity in algebra form—amounts to the commutativity of

$$
\begin{array}{ccc}
M \times M & \xrightarrow{(id_M, \partial)} & M \times R \\
(\partial, id_M) & \downarrow & \downarrow \\
R \times M & \longrightarrow & M
\end{array}
$$

In fact the resulting morphism is $R$-bilinear, hence it induces a product map

$$M \otimes_R M \longrightarrow M$$

$$m \otimes m' \longmapsto m \partial m'$$

making $M$ into a non-unital ring (or $k$-algebra), with $\partial$ becoming a homomorphism of non-unital rings. We will denote by $\langle \cdot, \cdot \rangle$ this map.

The primary example of a crossed bimodule is that of a (bilateral) ideal $I$ in a ring $R$. Secondly, for any ring $R$ of $T$ and any $R$-bimodule $M$,

$$M^0 \longrightarrow R$$

is evidently a crossed bimodule.

A good supply of crossed bimodules is provided by differential rings or differential graded $k$-algebras or simplicial rings (or $k$-algebras), depending on the framework we choose, as follows.

2.1.4 Example (See [BFM]). Let $R^*$ be a $k$-DGA supported in negative degrees, that is, $R^i = 0$ for $i > 0$. Then the soft truncation

$$\tau_{\geq -1} R^* : (R^{-1} / \text{Im} \partial) \longrightarrow R^0$$

is a crossed bimodule.

2.1.5 Example. Let $R_0$ be a simplicial ring. Let $MR^*$ be its Moore complex (denoted cohomologically): in each degree $MR^{-n} = \bigcap_{i=1}^{n-1} \ker d_i$, with $d = d_n$ restricted to $MR^{-n}$. It is easily verified that the soft truncation

$$\tau_{\geq -1} MR^* : MR^{-1} / \text{Im} d \longrightarrow MR^0 = R_0$$

is a crossed bimodule. For this, let $R_0$ act on $MR^{-1} = \ker d_0$ by

$$r_0 \cdot m \cdot r_1 \overset{def}{=} s_0(r_0) m s_0(r_1),$$
where \( r_0, r_1 \in R_0 \) and \( m \in \ker d_0 \). In addition, if \( m, m' \in R_1 \), then the simplicial identities imply that the combination 
\[
d_1(m) \cdot m' - m \cdot d_1(m') = s_0 d_1(m) m' - m s_0 d_1(m')
\]
belongs to \( \text{Im} \, d_2 : R_2 \to R_1 \), since 
\[
s_0 d_1(m) m' - m s_0 d_1(m') = d_2(s_0(m) s_1(m') - s_1(m) s_0(m')).
\]
Furthermore, if \( m, m' \in \ker d_0 \), the combination within the parentheses on the right hand side above belongs to \( M R^{-2} \). Thus in the soft truncation the algebraic Pfeiffer identity \((2.1.2)\) is satisfied.

2.1.6. The crossed bimodule \( \bar{\partial} : M \to R \) determines a groupoid 
\[
\mathcal{R}_0 : R \oplus M \to R,
\]
objectwise over \( S \), which is a presheaf of strict categorical rings: the additive structure is standard, and the multiplicative one is given, at the level of objects, by the ring structure of \( R \), and at the level of morphisms by
\[
(r_0, m_0)(r_1, m_1) = (r_0 r_1, r_0 m_1 + m_0 r_1 + m_0 \bar{\partial}(m_1)),
\]
for all \( r_0, r_1 \in R \) and \( m_0, m_1 \in M \). The verification of the axioms is straightforward. The nerve of \( \mathcal{R}_0 \) (again, objectwise), is a simplicial presheaf \( N_* \mathcal{R}_0 \) where, for each \( n \geq 0 \),
\[
N_n \mathcal{R}_0 = R \oplus M \oplus \cdots \oplus M, \quad \text{\( n \)-times}.
\]
It is easy to see that \( N_* \mathcal{R} \) is a simplicial ring. For this, analogously to ref. [Bre90], inductively define \( u_n : N_n \mathcal{R}_0 \to N_0 \mathcal{R}_0 = R \) by
\[
u_0 = \text{id}_R, \quad u_n(y, m) = u_{n-1}(y) + \bar{\partial} m,
\]
where we write an object of \( N_n \mathcal{R}_0 \cong N_{n-1} \mathcal{R}_0 \oplus M \) as \((y, m)\), with \( y \in N_{n-1} \mathcal{R}_0 \) and \( m \in M \). Then the ring structure is obtained by inductively generalizing \((2.1.7)\), namely with the same conventions:
\[
(y_0, m_0)(y_1, m_1) = (y_0 y_1, u_{n-1}(y_0)m_1 + m_0 u_{n-1}(y_1) + m_0 \bar{\partial}(m_1)).
\]
In particular, crossed bimodules are seen in this way to be equivalent to simplicial rings whose Moore complexes are supported in degrees \([-1,0]\).

2.1.8. If \( \bar{\partial} : M \to R \) is a crossed bimodule, one considers \( A = \pi_1(R_*) = \ker \bar{\partial} \) and \( B = \pi_0(R_*) = \text{coker} \bar{\partial} \). It is well known and easy to see that \( B \) is a ring (or \( k \)-algebra) and \( A \) a \( B \)-bimodule. One refers to the complete exact sequence
\[
0 \to A \to M \xrightarrow{\bar{\partial}} R \to B \to 0
\]
as a crossed extension of \( B \) by \( A \). Of course \( A \) and \( B \) are the homotopy objects of the simplicial ring determined by the crossed bimodule, in other words the homology objects of the associated Moore complex.

2.2 Strict morphisms

The notion of morphism between crossed bimodules has a straightforward definition.

2.2.1 Definition. Let \( \bar{\partial} : M \to R \) and \( \bar{\partial} : N \to S \) be two crossed bimodules. A morphism of crossed bimodules between them is a morphism of complexes, i.e. a pair \((\alpha, \beta)\) such that in the commutative diagram
\[
\begin{array}{ccc}
N & \xrightarrow{\beta} & M \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
S & \xrightarrow{\alpha} & R
\end{array}
\]
\( \alpha \) is a ring homomorphism and \( \beta \) is \( \alpha \)-equivariant, that is, we have \( \beta(s_0 n s_1) = \alpha(s_0) \beta(n) \alpha(s_1) \) for all \( s_0, s_1 \in S, n \in N \).
By a standard procedure, a morphism of crossed bimodules will induce a functor between the corresponding groupoids. It is straightforward to verify that it is a morphism of ring-like structures. For, let $\mathcal{H}_0$ and $\mathcal{R}_0$ be the groupoids corresponding to the complexes $N \to S$ and $M \to R$, respectively. It is standard that $\alpha$ and $(\alpha, \beta): S \oplus N \to R \oplus M$ combine to give an additive functor $\mathcal{H}_0 \to \mathcal{R}_0$. In addition, we have, for all $(s, n)$ and $(s', n') \in S \oplus N$,

$$(a(s), \beta(n)) \otimes (a(s'), \beta(n')) = (a(s)\alpha(s'), a(s)\beta(n) + \beta(n)\alpha(s') + \beta(n)\partial(n'))$$

and the latter is just the image of $(s, n) \otimes (s', n')$. Thus $(\alpha, \beta)$ gives a morphism of strict categorical rings.

We recall the notion of homotopy. Let $(\alpha, \beta)$ and $(\alpha', \beta')$ be two morphisms between $\partial: N \to S$ and $\partial: M \to R$, as in Definition 2.2.1 above.

2.2.2 Definition. A homotopy $h: (\alpha', \beta') \Rightarrow (\alpha, \beta)$ is a $k$-linear map $h: S \to M$ such that:

(2.2.3a) $\alpha' - \alpha = -\partial \circ h$,

(2.2.3b) $\beta' - \beta = -h \circ \partial$,

and, for all $s, s' \in S$,

(2.2.3c) $h(s) = (\alpha, h(s')) + (h(s)\alpha(s') - \partial h(s)h(s'))$.

2.2.4 Remark. The first two conditions amount to the standard definition of chain homotopy (for complexes supported in degrees $[-1, 0]$). The third can be given the following interpretation. Consider the complex of Hochschild cochains for $S$ with values in the $S$-bimodule $M$ (cf. [Lod98]), the bimodule structure is via the homomorphism $\alpha$. Let us denote the Hochschild coboundary by $\delta$. Then (2.2.3c) can be recast as

$$\delta h = (h, h),$$

where the right-hand side is the product introduced in Remark 2.1.3.

A homotopy $h$ determines a morphism of functors $(\alpha', \beta') \Rightarrow (\alpha, \beta)$ between the categorical rings $\mathcal{H}_0$ and $\mathcal{R}_0$. Also, it is easily verified that $h = -h$ is a homotopy from $(\alpha, \beta)$ to $(\alpha', \beta')$, thus morphisms of crossed bimodules and homotopies between them form a groupoid, denoted $\text{Hom}(S, R_*)$.

2.3 Associated ring-like stacks

Let $\partial: M \to R$ be a crossed bimodule, and let $\mathcal{R}_0$ the corresponding groupoid

$$\mathcal{R}_0: R \oplus M \longrightarrow R,$$

as above. We have observed that it is a presheaf of categorical rings on $S$, with corresponding strict additive bifunctor

$m_0: \mathcal{R}_0 \times \mathcal{R}_0 \to \mathcal{R}_0$. Let $\mathcal{R} = [M \to R]^\sim$ be the associated Picard stack, and $j: \mathcal{R}_0 \to \mathcal{R}$ the corresponding local equivalence. By the usual universal property argument, we have the equivalence

$$\mathcal{H}om(\mathcal{R}_0 \times \mathcal{R}_0, \mathcal{R}) \cong \mathcal{H}om(\mathcal{R} \times \mathcal{R}, \mathcal{R}),$$

cf. [Del73, §1.4.10], hence the composite morphism $j \circ m_0: \mathcal{R}_0 \times \mathcal{R}_0 \longrightarrow \mathcal{R}$ yields

$$m: \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}.$$

Thus $\mathcal{R}$ is a ring-like stack.

2.3.1 Remark. The (truncated) tensor product of $\partial: M \to R$ with itself

$$M \otimes M \longrightarrow M \otimes R$$

$R \otimes M \longrightarrow (\bullet)$

defines by push-out a complex (the portion in red in the diagram) representing $R \otimes R$. 

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As we have observed, a morphism of crossed bimodules \( (\alpha, \beta): S_* \to R_* \) induces one between groupoids \( F_0: \mathcal{S}_0 \to \mathcal{R}_0 \) (cf. sect. 2.2). If we compose the latter with \( j: \mathcal{R}_0 \to \mathcal{R} \), again by standard arguments, there results a morphism \( F: \mathcal{S} \to \mathcal{R} \)

between the associated Picard stacks, which is easily seen to be a morphism of ring-like stacks.

We regard these morphisms as strict in the following sense.

**2.3.2 Definition.** Let \( \mathcal{S} \cong [N \to S]^\sim \) and \( \mathcal{R} \cong [M \to R]^\sim \) be the associated stacks. A morphism \( F: \mathcal{S} \to \mathcal{R} \) of ring-like stacks is strict if it arises from a crossed bimodule morphism between the presentations.

An equivalent way of stating the notion of strict morphism would be to say that \( F: \mathcal{S} \to \mathcal{R} \) is strict whenever it arises from a morphism of the underlying prestacks \( \mathcal{S}_0 \) and \( \mathcal{R}_0 \). Due to Theorem 3.1.1 below, the notion of strict morphism makes sense for all ring-like stacks.

## 3 Ring-like stacks and their presentations

Every Picard stack of \( T \) has a presentation by a complex of abelian sheaves supported in degrees \([-1, 0]\). If \( \mathcal{R} \) is a ring-like stack, we prove the presentation is a crossed bimodule. We use them to discuss the forms of the descent data (i.e. the cocycles) and the monoidal structures. Later, in section 6, we discuss the significance from the point of view of the 2-category of Picard stacks.

### 3.1 Presentations of ring-like stacks

**3.1.1 Theorem.** Let \( \mathcal{R} \) be a ring-like stack. Then \( \mathcal{R} \) admits a presentation by a crossed bimodule \( \partial: M \to R \).

**Proof (Sketch).** Take a presentation \( \mathcal{R} \cong [B \to A]^\sim \) of the underlying Picard stack of \( \mathcal{R} \). The complex \( B \to A \cong A_1 \to A_0 \) is just a complex of abelian groups of \( T \).

Consider the tensor algebra \( T(A) \) over \( A \), where \( T(\cdot) \) is taken over \( Z \). We claim the projection \( \pi: A \to \mathcal{R} \) factors through \( T(A) \). To see this, define \( \sigma: T(A) \to \mathcal{R} \) by

\[
\sigma(a_1 \otimes \cdots \otimes a_n) = (\pi(a_1)\pi(a_2)\ldots\pi(a_n)),
\]

using the left bracketing for the expression on the right. We want this to be unital, namely for \( n = 0 \) we send 1 to the \( I_\mathcal{R} \), the multiplicative unit object of \( \mathcal{R} \). One can view the \( a_i, i = 1, \ldots, n \) as parametrizing a collection of objects of \( \mathcal{R} \) via \( \pi \). Thus, by [Lap72a; Lap72b], \( \sigma \) is well defined. It is also essentially surjective, since \( \pi \) is. Now, define \( M \) as the homotopy kernel of \( \sigma: T(A) \to \mathcal{R} \). An element of \( M \) is a pair \((b_1 \otimes \cdots \otimes b_n, \lambda)\) where \( \lambda: \sigma(b_1 \otimes \cdots \otimes b_n) \sim 0_\mathcal{R} \). Forgetting \( \lambda \) gives the differential \( \partial: M \to T(A) \). As it is easily seen, \( M \) is a \( T(A) \)-bimodule, and computations similar to those in [AN09, p. 5.3.8] show that the Pfeiffer identity holds. Thus the complex \( \partial: M \to R \), with \( R = T(A) \), is a crossed bimodule. We also have a commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\partial} & A \\
\eta_1 & & \pi \\
M & \xrightarrow{\partial} & T(A) & \xrightarrow{\pi} & \mathcal{R}
\end{array}
\]

where \( \eta_0, \eta_1 \) are monomorphisms. It is easily seen that \( \partial: B \to A \) and \( \partial: M \to T(A) \) have the same kernel and cokernel, which then coincide with \( \pi_1(\mathcal{R}) \) and \( \pi_0(\mathcal{R}) \).

In the following we will always use a presentation \( \partial: M \to R \) by a crossed bimodule.
3.2 Objects and products in a ring-like stack

The standard geometric interpretation of $\mathcal{R} = [M \to R]$ is obtained by observing that it is equivalent (as a Picard stack) to $\text{Tors}(M, R)$, the stack of $M$-torsors $E$ equipped with a trivialization $s : E \wedge M \to R$. If $E$ and $E'$ are two $M$-torsors, it is standard that their sum $E + E'$ is given by the $M$-torsor $E \wedge M E'$ equipped with the trivialization $s + s'$. The projection morphism $\pi : R \to \text{Tors}(M, R)$ assigns to $r \in R$ the trivial torsor $M$ equipped with the $M$-equivariant map that sends $0$ to $r$. (Thus $m \in M$ is sent to $r + \partial m$.) In particular, $(M, 0) = \pi(0)$ will be identified with the zero object $0$ (the unit of the sum operation).

Less standard is the product $EE' = m(E, E')$ induced by the second monoidal structure of the ring-like stack $\mathcal{R}$. This structure can be described as follows. First, a local description. To local data (i.e. sections) $e \in E$ and $e' \in E'$ we assign the trivial $M$-torsor, which we can think as being generated by the symbol $\{e, e'\}$. Recall that $E$ and $E'$ have trivializations of their push-outs as $R$-torsors via $\partial$, by way of $M$-equivariant maps $s$ and $s'$, respectively. The trivial $M$-torsor associated to $(e, e')$ is equipped with the map denoted $ss'$ sending the generator $\{e, e'\}$ to $s(e)s'(e') \in R$. Replacing the pair $(e, e')$ with $(e + m, e' + m')$, results in the isomorphism of trivial torsors such that:

$$\{e, e'\} \mapsto \{e + m, e' + m'\} - (s(e)m' + m s'(e') + m \partial m').$$

The sought-after $M$-torsor $EE'$ will be obtained by gluing the above trivial torsors by way of this isomorphism. (It is clear that it satisfies the appropriate cocycle condition.)

An alternative, more global description simply is obtained by observing that the above construction presents $EE'$ as the quotient of $E \times E' \times M$ by the action of $M \times M$ given by:

$$(e, e', m_0) \mapsto (e + m, e' + m', m_0 + s(e)m' + m s'(e') + m \partial m'), \quad m, m' \in M.$$

The correspondence between the two pictures is that $\{e, e'\}$ is the class of $(e, e', 0)$, and that in the resulting $M$-torsor we have:

$$\{e + m, e' + m'\} = \{e, e'\} + (s(e)m' + m s'(e') + m \partial m').$$

Note that the map $ss$ defined above is compatible with this relation, hence it is well defined as an $M$-equivariant map $ss : EE' \to R$.

The unit object $I_\mathcal{R}$ for the just defined multiplicative structure can be identified with $(M, 1) = \pi(1)$. Indeed, if $E$ is any $(M, R)$-torsor, we have the standard structure isomorphisms:

$$\lambda_\mathcal{R} : I_\mathcal{R} E \to E \quad \rho_\mathcal{R} : E I_\mathcal{R} \to E$$

$$[0, e] \mapsto e, \quad \{e, 0\} \mapsto e.$$

It is easily checked that they are well defined and functorial.

3.3 Cocycles

Objects of $\mathcal{R}$ can be described in terms of descent data. Given a presentation, descent data become just cocycle representations for such torsors as described above. Using these data, the ring structure on $\mathcal{R}$ is very concretely described by localized versions of the formulas for $\mathcal{R}_0$, as follows.

Let $V \to U$ be a hypercover of $T$. An object $E$ over $U$ will be represented by a triple $(V, r, m)$, where $r \in R(V_0)$, and $m \in M(V_1)$ such that:

$$d^*_0 r - d^*_1 r = \partial m$$

$$d^*_2 m + d^*_3 m = d^*_1 m.$$

If now $E, E'$ are two objects of $\mathcal{R}_U$, and $(V, r, m)$ and $(V, r', m')$ the corresponding descent data, where the hypercover $V \to U$ is assumed for simplicity to be same for both objects, the object $E + E'$ is represented by $(V, r_0 + r_1, m_0 + m_1)$, whereas the multiplication $EE' = m(E, E')$ is represented by the triple:

$$(V, rr', (d^*_1 r)m' + m(d^*_0 r') + m \partial m') = (V, rr', (d^*_1 r)m' + m(d^*_0 r')).$$
These formulas are most transparent in the Čech formalism. Assuming $T = \text{Sh}(S)$, and $S$ to have (finite) limits, if $(U_i)_{i \in I}$ is a cover of $U \in S$, we write the data for $E$ as a collection $(r_i, m_{ij})$, where $r_i \in R(U_i)$ and $m_{ij} \in M(U_i \times_U U_j)$, such that

$$r_j - r_i = \partial m_{ij},$$
$$m_{ij} + m_{jk} = m_{ik},$$

and similarly for $E'$. Therefore $E + E'$ is represented by

$$(r_i + r'_i, m_{ij} + m'_{ij}),$$

whereas $EE'$ by

$$(r_i r'_i, r_i m'_{ij} + m_{ij} r'_i).$$

The cocycle, that is, the triple $(V, r, m)$ corresponds to a simplicial map

$$\xi: V \rightarrow N \otimes_0 R,$$

see [Jar09] and [AN09, §3.3.1-3.4.4]. The simplicial ring structure of $N \otimes_0 R$ gives pointwise sum and product operations for cocycles. Hence $E + E'$ and $EE'$ give rise and are determined by the simplicial maps $\xi + \xi'$ and $\xi \xi'$, defined by

$$(\xi + \xi')_n \eqdef \xi_n + \xi'_n \quad \text{and} \quad (\xi \xi')_n \eqdef \xi_n \xi'_n.$$

By explicitly writing down the simplicial maps (see loc. cit. or, e.g., [May92]) we arrive at the formulas for the addition and multiplication of cocycles given above.

## 4 Bimodule butterflies

Butterflies ([Noo07; AN09]) are certain kind of diagrams computing morphisms between length 2-complexes in the homotopy category. We specialize the concept to the present situation.

### 4.1 Bimodule butterflies

Let $S_\bullet: N \xrightarrow{\bar{e}} S$ and $R_\bullet: M \xrightarrow{\bar{e}} R$ be two crossed bimodules of $T$.

#### 4.1.1 Definition. A crossed bimodule butterfly, or simply a butterfly, for short, from $S_\bullet$ to $R_\bullet$ is a diagram

![Diagram](image-url)

where:

1. $E$ is a ring (or $k$-algebra);
2. The NE-SW diagonal $M \rightarrow E \rightarrow S$ is an extension, namely it is an exact sequence of the underlying modules, and $\pi: E \rightarrow S$ is a ring (or $k$-algebra) homomorphism;
3. The NW-SE diagonal $N \rightarrow E \rightarrow R$ is a complex of abelian groups (or $k$-modules), namely $j \circ \kappa = 0$; $j: E \rightarrow E$ is a ring ($k$-algebra) homomorphism;
4. For all $m \in M$, $n \in N$, and $e \in E$, the following compatibility conditions hold:
   
   | Condition | Expression |
   |-----------|------------|
   | (4.1.3a)  | $i(m j(e)) = i(m) e$ |
   | (4.1.3b)  | $i(j(e) m) = e i(m)$ |
   | (4.1.3c)  | $\kappa(n \pi(e)) = \kappa(n) e$ |
   | (4.1.3d)  | $\kappa(\pi(e) n) = e \kappa(n)$. |
There are some elementary consequences of the definition.

4.1.4 Lemma. In the butterfly defined above:

1. $M$ is a bilateral ideal in $E$.

2. The images of $N$ and $M$ in $E$ multiply to zero: $\kappa(N)\iota(M) = 0$ in $E$.

Proof. The first is obvious (it is just a restatement of the second condition in the definition). The second easily follows from (4.1.3).

4.1.5 Remark. 1. The NW-SW diagonal is not necessarily an abelian extension, namely $M^2 \neq 0$ in general, as an ideal in $E$. Indeed, for all $m, m' \in M$ we have

$$\iota(m)\iota(m') = \iota(m \cdot m') = \iota(m \partial m'),$$

and $m \partial m'$ is in general nonzero.

2. The multiplication on $M$ induced by $E$ is the same as that induced by the crossed bimodule structure (cf. Remark 2.1.3).

A shorthand notation for a butterfly from $S$ to $R$, with centerpiece $E$ will be $[S, E, R]$.

4.1.6 Definition. A morphism $\alpha: [S, E, R_*] \to [S', E', R_*]$ is a ring (or $k$-algebra) isomorphism $\alpha: E \overset{\sim}{\to} E'$ compatible with the structural maps of both butterflies in the sense that the following diagram commutes

$$\begin{array}{ccc}
N & \xleftarrow{\kappa'} & E' \\
\downarrow{\varrho} & & \downarrow{\varrho'} \\
S & \xleftarrow{\pi} & E \\
\downarrow{\iota} & & \downarrow{\iota'} \\
M & \xleftarrow{\kappa} & E \\
\downarrow{\varrho} & & \downarrow{\varrho'} \\
R & \xleftarrow{\pi} & E
\end{array}$$

With the notion of morphism just introduced, butterflies from $S_\ast$ to $R_\ast$ clearly form a groupoid, denoted $\mathcal{B}(S_\ast, R_\ast)$. Analogously to [AN09, §5 and §8], we can consider a local version with respect to $S$, namely form the fibered category $\mathcal{B}(S, R_\ast)$ from $U \to \mathcal{B}(S_\ast|U, R_\ast|U)$, where $U \in \text{Ob}(S)$. These groupoids are subgroupoids of the corresponding ones constructed by forgetting the multiplicative structures and considering just the underlying abelian sheaves (or $k$-modules). Denote them by $\mathcal{B}_k(S_\ast, R_\ast)$ and $\mathcal{B}_k(S_\ast|U, R_\ast|U)$, respectively. By loc. cit., the latter are the fibers of a stack $\mathcal{B}_k(S_\ast, R_\ast)$. This implies that $\mathcal{B}(S_\ast, R_\ast)$ forms a stack as well. Note, however, that it will not be closed with respect to the symmetric structure given by “addition” of butterflies (cf. [AN09, §8]).

4.2 Fractions

The diagram (4.1.2) can be completed to

$$\begin{array}{ccc}
N \oplus_S E & \xleftarrow{\kappa} & M \\
\downarrow{\varrho} & & \downarrow{\iota} \\
S & \xleftarrow{\pi} & E \\
\downarrow{\iota} & & \downarrow{\iota'} \\
R & \xleftarrow{\pi} & E
\end{array}$$

where the left wing is a pull-back. With set-theoretic notation, $N \oplus_S E = \{(n, e) \in N \oplus E \mid \partial n = \pi e\}$. As in the abelian case, $\kappa: N \to E$ gives a splitting of $\pi$, so that we have an isomorphism

$$(\text{id}_N, \text{id}_E - \kappa): N \oplus_S E \overset{\sim}{\to} N \oplus M,$$

with inverse $(\text{id}_N, \iota + \kappa)$. In addition, the complex $E_*: N \oplus_S E \to E$ is a crossed bimodule: first, $N \oplus_S E$ is an $E$-bimodule with the operations (written set-theoretically as):

$$e_0 \cdot (n, e) = (\pi(e_0)n, e_0e) \quad \text{and} \quad (n, e) \cdot e_1 = (n \pi(e_1), ee_1),$$
for all \( e, e_0, e_1 \in E \) and \( n \in N \). An elementary verification shows that the Pfeiffer identity (cf. Remark 2.1.3)

\[
\partial(n_0, e_0)(n_1, e_1) = (n_0, e_0)\partial(n_1, e_1)
\]

holds.

**4.2.1 Lemma.** Each wing of the above diagram determines a morphism of crossed bimodules, the left one being a quasi-isomorphism.

**Proof.** The first statement is an elementary verification and it is left to the reader. The second follows from considering the pullback of extensions

\[
\begin{array}{c}
0 \to M \overset{i}{\to} N \oplus E \overset{\pi}{\to} N \to 0 \\
0 \overset{i}{\to} M \to E \overset{\pi}{\to} S \to 0
\end{array}
\]

along \( \partial_3 \). An elementary application of the snake lemma yields \( \pi_i(S) \cong \pi_i(E) \), for \( i = 0, 1 \). \qed

### 4.3 Split butterflies

A morphism \((\alpha, \beta)\) of crossed bimodules determines a butterfly in which the NE-SW diagonal is a trivial extension, namely

\[
E = S \oplus M,
\]

where \( M \) is considered as an \( S \)-bimodule via \( \alpha: S \to R \). The ring structure on \( E \) is given by

\[
(s, m)(s', m') = (ss', \alpha(s)m' + m\alpha(s') + m\partial(m')),
\]

and the four maps in the butterfly diagram are given by:

- \( \iota: M \to S \oplus M \):
  \[
iota: \begin{cases} 
  \iota: M \to S \oplus M \\
  m \mapsto (0, m)
\end{cases}
\]

- \( \pi: S \oplus M \to S \):
  \[
  \pi: \begin{cases} 
  \pi: S \oplus M \to S \\
  (s, m) \mapsto s
\end{cases}
\]

- \( \kappa: N \to S \oplus M \):
  \[
  \kappa: \begin{cases} 
  \kappa: N \to S \oplus M \\
  n \mapsto (\partial n, -\beta(n))
\end{cases}
\]

- \( \jmath: S \oplus M \to R \):
  \[
  \jmath: \begin{cases} 
  \jmath: S \oplus M \to R \\
  (s, m) \mapsto \alpha(s) + \partial m.
\end{cases}
\]

The map \( \sigma = (\id_S, \id_M): S \to S \oplus M \) is evidently a splitting of the exact diagonal. More generally we have:

**4.3.1 Definition.** A butterfly (4.1.2) is strongly split if its NE-SW diagonal is equipped with an algebra extension splitting homomorphism \( \sigma: S \to E \). Equivalently, it is isomorphic in the sense of Definition 4.1.6 to one arising from a morphism of crossed bimodules.

Thus, a strongly split butterfly in effect corresponds to a morphism of crossed bimodules. Note that such an object is in fact a pair \((E, \sigma)\), where \( E \) is an object of \( \mathcal{B}(S, R) \) and \( \sigma \) is an algebra splitting. It is easily seen that a homotopy \( h: (\alpha', \beta') \Rightarrow (\alpha, \beta) \) of morphisms of crossed bimodules determines a morphism \( \psi: (E, \sigma) \to (E', \sigma') \) of split butterflies. Explicitly, if both \( E \) and \( E' \) are identified with \( S \oplus M \), then the required homomorphism has the form

\[
\psi = (\id_S, \id_M + h): S \oplus M \to S \oplus M.
\]

Conversely, an isomorphism \( \psi: S \oplus M \to S \oplus M \) which fits into a morphism of (split) butterflies, necessarily has the above form, with \( h: S \to M \) satisfying (2.2.3).

Let us denote by \( \mathcal{B}_{str}(S, R) \) the resulting groupoid. By the foregoing, it is equivalent to the previously introduced groupoid \( \Hom(S, R) \). There is an obvious functor \( \mathcal{B}_{str}(S, R) \to \mathcal{B}(S, R) \), and hence \( \Hom(S, R) \to \mathcal{B}(S, R) \). A better characterization will be given below.
4.4 Butterflies and extensions

Let us denote by $\text{ExtAlg}(S, M)$ the category of algebra extensions of $S$ by $M$, whose objects are algebra extensions as above, and whose morphisms are commutative diagrams

$$
\begin{array}{c}
0 \\[-1ex] \downarrow \quad \downarrow \pi \quad \downarrow \alpha \\
M \\[-1ex] \downarrow \quad \downarrow i' \\
0
\end{array}
\begin{array}{c}
E \\[-1ex] \downarrow \quad \downarrow \pi' \\
S
\end{array}
\begin{array}{c}
0
\end{array}
$$

The extensions we consider are not assumed to be abelian, nor are they assumed to be $k$-split. Analogously to [AN09, §8], there is an obvious forgetful functor

$$
p : B(S_*, R_*) \longrightarrow \text{ExtAlg}(S, M)
$$

which is a fibration (cf. [Bre92]). For, if $[S_*, E', R_*]$ is such that its NE-SW diagonal is isomorphic to the extension $0 \to M \to E \to S \to 0$ with $\alpha : E \to E'$, then $[S_*, E, R_*]$ is an isomorphic butterfly with structure maps $j = j' \circ \alpha$ and $\kappa = \kappa' \circ \alpha^{-1}$. Evidently $\alpha$ gives the corresponding morphism of butterflies. Essential surjectivity also holds, since, rather trivially, in the extension $0 \to M \to E \to S \to 0$ the morphism $M \to E$ is a crossed bimodule, and so is $0 \to S$, therefore we can choose

Note that $[M \to E]^\sim \cong S$, where $S$ is considered as a discrete stack, since the groupoid determined by $M \to E$ is an equivalence relation. The groupoid $B_{\text{ext}}(S_*, R_*)$ is the homotopy kernel of the morphism $p$ above. In fact it is easy to see the whole sequence

$$
B_{\text{ext}}(S_*, R_*) \longrightarrow B(S_*, R_*) \xrightarrow{p} \text{ExtAlg}(S, M)
$$

is exact. Since the homotopy kernel is determined up to equivalence, we can rewrite this sequence as

$$
\text{Hom}(S_*, R_*) \longrightarrow B(S_*, R_*) \xrightarrow{p} \text{ExtAlg}(S, M).
$$

By forgetting the algebra structure we get (with corresponding meanings of the symbols)

$$
\text{Hom}_k(S_*, R_*) \longrightarrow B_k(S_*, R_*) \xrightarrow{p} \text{Ext}_k(S, M)
$$

which, as observed in [AN09], is also an extension. The first object on the left is identified with the groupoid of split butterflies, i.e. strict morphisms of complex of abelian sheaves.

We can also consider the (homotopy) kernel of the forgetful functor

$$
\text{ExtAlg}(S, M) \longrightarrow \text{Ext}_k(S, M),
$$

which we denote by $\text{ExtAlg}_0(S, M)$. It consists of those algebra extensions which possess a $k$-linear splitting. The pullback groupoid

$$
B_0(S_*, R_*) \cong B(S_*, R_*) \times_{\text{ExtAlg}(S_*, R_*)} \text{ExtAlg}_0(S_*, R_*)
$$

then consists of those butterflies whose NE-SW diagonal admits a $k$-linear splitting.

The above constructions can be sheafified (or actually stackified) over $S$. Putting all together, we can form the diagram of stacks over $S$:

$$
\begin{array}{c}
\mathcal{B}_0(S_*, R_*) \xrightarrow{p} \mathcal{E}_{\text{alg}}(S, M)
\end{array}
\begin{array}{c}
\downarrow \\
\mathcal{C}(S_*, R_*) \xrightarrow{p} \mathcal{E}_{\text{alg}}(S, M)
\end{array}
\begin{array}{c}
\downarrow \\
\mathcal{C}_k(S_*, R_*) \xrightarrow{p} \mathcal{E}_k(S, M)
\end{array}
$$
The objects on the leftmost column, as well as $\mathcal{R}_0(S_\ast, R_\ast)$, consist of locally split butterflies from $S_\ast$ to $R_\ast$ (cf. [AN09]).

5 Butterflies as morphisms of ring-like stacks

In this section we prove our main result, that analogously to the case of group-like stacks, bimodule butterflies compute morphisms between stacks equipped with a ring-like structure.

Let $\mathcal{F}$ and $\mathcal{R}$ be two ring-like stacks. We denote by $\operatorname{Hom}(\mathcal{F}, \mathcal{R})$ the groupoid of (homo)morphisms from $\mathcal{F}$ to $\mathcal{R}$, and by $\operatorname{Hom}_t(\mathcal{F}, \mathcal{R})$ the groupoid of morphisms of underlying Picard stacks. Similarly, we denote by $\mathcal{H}\operatorname{om}(\mathcal{F}, \mathcal{R})$ and $\mathcal{H}\operatorname{om}_t(\mathcal{F}, \mathcal{R})$ their respective stack analogs. Assume $\mathcal{R}$ and $\mathcal{F}$ have presentations by crossed bimodules $R_\ast : M \xrightarrow{\varepsilon} R$ and $S_\ast : N \xrightarrow{\varepsilon} S$, respectively.

5.1 Theorem. There are equivalences

$$\operatorname{Hom}(\mathcal{F}, \mathcal{R}) \simeq \mathcal{B}(S_\ast, R_\ast) \quad \text{and} \quad \mathcal{H}\operatorname{om}(\mathcal{F}, \mathcal{R}) \simeq \mathcal{B}(S_\ast, R_\ast).$$

This is the specialization of the context of ring-like stacks of the corresponding statements for Picard (or even just group-like) stacks proved in [AN09]. Indeed, forgetting the ring-like structures we get equivalences

$$\operatorname{Hom}_t(\mathcal{F}, \mathcal{R}) \simeq \mathcal{B}_t(S_\ast, R_\ast) \quad \text{and} \quad \mathcal{H}\operatorname{om}_t(\mathcal{F}, \mathcal{R}) \simeq \mathcal{B}_t(S_\ast, R_\ast).$$

The necessary ingredient we will need is the construction of two mutually quasi-inverse functors $\Phi : \mathcal{B}(S_\ast, R_\ast) \to \operatorname{Hom}_t(\mathcal{F}, \mathcal{R})$ and $\Psi : \operatorname{Hom}_t(\mathcal{F}, \mathcal{R}) \to \mathcal{B}(S_\ast, R_\ast)$. We will recall some of the details of their definition from loc. cit., then prove that they restrict to equivalences between $\operatorname{Hom}(\mathcal{F}, \mathcal{R})$ and $\mathcal{B}(S_\ast, R_\ast)$. Many of the “moves” in the new part of the proof would be a repeat of those already carried out in the original one, therefore we only sketch the main lines.

5.1 Recollections from [AN09, §4.3 and §4.4]

Throughout the proof we will use the equivalences $\mathcal{F} \simeq \mathcal{T}ors(N, S)$ and $\mathcal{R} \simeq \mathcal{T}ors(M, R)$.

Let $(Y, y)$ be an object of $\mathcal{F}$. Thus, $Y$ is an $N$-torsor equipped with an $N$-equivariant map $y : Y \to S$. Let $E$ (by abuse of language) be a $k$-butterfly from $S_\ast$ to $R_\ast$. First, define the $M$-torsor of local $N$-equivariant liftings of $Y$ to $E$:

$$X_{\text{def}} = \operatorname{Hom}_N(Y, E)_y.$$

The $M$-action on $X$ takes the following form: if $y_0$ and $y_1$ are two different liftings defined over $U \in \operatorname{Ob} S$, we have $y_0 = y_1 + t(m)$, where, a priori, $m : Y \to M$. The $N$-equivariance of the lifts implies that $m$ is in fact $N$-invariant, thus it only depends on the two lifts and not on the specific points of $Y$. The torsor $X$ is equipped with the $M$-equivariant map $x : X \to R$ defined by sending a (local) lift $\tilde{y} : Y|U \to E$ to $f \circ \tilde{y}$. This is well defined: again, if $v, v + n$ are two points of $Y|U$, with $n \in N|U$, then

$$\tilde{y}(v + n) = \tilde{y}(v) + \kappa(n),$$

(see loc. cit.) so the post-composition with $f$ does not depend on the specific point of $Y|U$, but only on the lift itself. Then, by definition, $\Phi(E) : \mathcal{F} \to \mathcal{R}$ assigns to $(Y, y)$ the pair $(X, x)$ just defined. It is clear that if $f : (Y', y') \to (Y, y)$ is a morphism of $(N, S)$-torsors, then we get a corresponding morphism $(X, x) \to (X', x')$. Also, a morphism $\alpha : E \to E'$ of butterflies induces

$$\alpha_* : \operatorname{Hom}_N(Y, E)_y \to \operatorname{Hom}_N(Y, E')_y,$$

and hence a morphism of functors

$$\alpha_* : \Phi(E) \Longrightarrow \Phi(E') : \mathcal{F} \to \mathcal{R}.$$

We refer to loc. cit. for details.

In the opposite direction, if $F : \mathcal{F} \to \mathcal{R}$, then $E = \Psi(F)$ is the butterfly where:

$$E \defeq S \times_{\mathcal{R}} R,$$

where the (stack) fiber product is computed with respect to the maps $\pi_{\mathcal{R}} : R \to \mathcal{R}$ and $F \circ \pi_{\mathcal{F}} : S \to \mathcal{R}$. Thus $E$ consists of triples $(s, \varphi, r)$, where $s \in S, r \in R$, and $\varphi : F(\pi_{\mathcal{F}}(s)) \to \pi_{\mathcal{R}}(r)$. The maps $\pi : E \to S$ and $f : E \to R$ are just the
canonical projections to $S$ and $R$, respectively. The sequence $M \xrightarrow{\delta} R \xrightarrow{\pi} \mathscr{R}$ is homotopy exact, so its pullback along $F \circ \pi_{\mathscr{R}}$ gives rise to the exact sequence $M \xrightarrow{\iota} E \xrightarrow{\pi} S$, the NE-SW diagonal of the butterfly. The explicit form of the map $\iota : M \to E$ can be computed from the sequence: if $m \in M$, then we have:

$$\iota(m) = (0, \varphi_m, \partial m),$$

where $\varphi_m : F(\pi_{\mathscr{R}}(0)) = F(0_{\mathscr{R}}) \to \pi_{\mathscr{R}}(\partial m)$ is the composite of the structural morphism $F(0_{\mathscr{R}}) \to 0_{\mathscr{R}}$ with the (unique) isomorphism of torsors $\pi_{\mathscr{R}}(\partial m) = (M, \partial m) \to (M, 0) = \pi_{\mathscr{R}}(0) = 0_{\mathscr{R}}$. $\kappa$ can be defined along similar lines, bearing in mind that $N \xrightarrow{\delta} S \xrightarrow{E \circ \pi_{\mathscr{R}}} \mathscr{R}$ is only a complex, and so it will be its pullback along $\pi_{\mathscr{R}}$, giving rise to the NW-SE diagonal of the butterfly. We refer to loc. cit. for further details on $\iota : M \to E$ and $\kappa : N \to E$ as well as the various functoriality properties.

### 5.2 Proof of Theorem 5.1

We show that, given a butterfly $E \in \mathcal{B}(\mathscr{S}, R_*)$, the resulting morphism $\Phi(E) : \mathscr{S} \to \mathscr{R}$ of Picard stacks is in fact ring-like by constructing isomorphisms

$$\Phi(E)(Y, y)\Phi(E)(Y', y') \xrightarrow{\sim} \Phi(E)(YY', yy')$$

(5.2.1)

satisfying the standard properties. If $(Y, y)$ and $(Y', y')$ are objects of $\mathscr{S}$, we define the required isomorphism by sending two lifts $e : Y \to E$ and $e' : Y' \to E$ to the product $ee'$. This is well defined and compatible with the actions of $N$ on $Y$ and $Y'$, of $M$ on their images $X$ and $X'$, and with the definition of the product of torsors in sect. 3.2. Indeed, for $v \in Y$, $v' \in Y'$, and $n, n' \in N$, we have:

$$e(v + n)e'(v' + n') = e(v)e'(v') + e(v)\kappa(n') + \kappa(n)e'(v') + \kappa(n)e(n')$$

$$= e(v)e'(v') + \kappa(\pi_e(v)n' + n \pi_e(v') + n \pi(n'))$$

$$= e(v)e'(v') + \kappa(y(v)n' + n y'(v') + n \partial n'),$$

and we see the last line is just the equivariance of the lift $ee'$. Similarly, for $m, m' \in M$, we have:

$$(e + \iota(m))(e' + \iota(m')) = ee' + e \iota(m') + \iota(m)e' + \iota(m)\iota(m')$$

$$= ee' + i(j \circ e \cdot m' + m \cdot j \circ e' + m \cdot j \circ \iota(m'))$$

$$= ee' + i(x(e)m' + m x'(e') + m \partial m').$$

The verification that (5.2.1) is functorial and compatible with the associativity constraint follows the same steps as the proof in the group-like case of loc. cit., and it is left to the reader.

Conversely, if $F : \mathscr{S} \to \mathscr{R}$ is a morphism of ring-like stacks, then the resulting butterfly $E = \Psi(F)$ in $\mathcal{B}(\mathscr{S}, R_*)$ actually satisfies the conditions in Definition 4.1.1, with $E$ being equipped with a ring (or $k$-algebra) structure.

This is actually automatic, since $E = \Psi(F) = S \times_{\mathscr{R}} R$, so the pullback sequence

$$M \xrightarrow{\iota} S \times_{\mathscr{R}} R \xrightarrow{\pi} S$$

comes naturally equipped with the structure of an algebra extension. Explicitly, the product in $E$ reads:

$$(s, \varphi, r)(s', \varphi', r') \overset{\text{def}}{=} (ss', \varphi \varphi', rr'),$$

where $\varphi \varphi'$ stands for the composition:

$$F(\pi_{\mathscr{R}}(s)\pi_{\mathscr{R}}(s')) \simeq F(\pi_{\mathscr{R}}(s)F(\pi_{\mathscr{R}}(s'))) \overset{\varphi \varphi'}{\to} \pi_{\mathscr{R}}(r)\pi_{\mathscr{R}}(r') \simeq \pi_{\mathscr{R}}(rr').$$

Associativity holds for the same reason it does for the sum operation in $E$. Distributivity of the product with respect to the sum holds thanks to the fact that it (obviously) does in $S$ and $R$, and (weakly) in $\mathscr{S}$, $\mathscr{R}$ and preserved by $F$. For instance, for elements $e_i = (s_i, \varphi_i, r_i)$, for $i = 0, 1$ and $e = (s, \varphi, r)$ of $E$, the equality $(e_0 + e_1)e = e_0e + e_1e$ rests upon that of morphisms in $\mathscr{R}$

$$(\varphi_0 + \varphi_1)\varphi = \varphi_0\varphi + \varphi_1\varphi.$$
whereas
κ
where
E
morphisms of butterfly, it is easily verified that there results a morphism
(again, with shortened notation), which follows from the commutativity of structure diagrams as in [JP07, Definition 2.2].

It remains to prove that the butterfly satisfies the conditions (4.1.3). Let us pick just one of them, \(i(m \ j(e)) = i(m) \ e\). Let \(e = (s, \varphi, r)\), as above. We have
\[
i(m) \ e = (0, \varphi_m, \partial m)(s, \varphi, r) = (0, \varphi_m \varphi, (\partial m) r) = (0, \varphi_m \varphi, \partial (m r)).
\]

On the other hand, \(j(e) = r\), therefore
\[
i(m \ j(e)) = (0, \varphi_m, \partial (m r)).
\]

Let \(\eta\) be the structural isomorphism \(F(0, \varphi) \to 0_{\mathcal{G}}\). The commutativity of the diagram
\[
\begin{array}{ccc}
F(0, \varphi) & \sim & F(0, \varphi) \ (\pi_{\varphi}(s)) \\
\varphi_m & \downarrow \eta & \varphi_m \varphi \\
0_{\mathcal{G}} & \sim & 0_{\mathcal{G}} \ (\pi_{\varphi}(r)) \\
\pi_{\varphi}(\partial (m r)) & \sim & \pi_{\varphi}(\partial m) \pi_{\varphi}(r) \\
\pi_{\varphi}(\partial (m r)) & \sim & \pi_{\varphi}(\partial m) \pi_{\varphi}(r)
\end{array}
\]

shows that, modulo the slight abuse of notation implied by omitting from it the standard isomorphisms, \(\varphi_{\partial r} = \varphi_m \varphi\), thereby implying the desired equality. The remaining ones in (4.1.3) are treated similarly.

\[\square\]

6 Compositions of bimodule butterflies and the 2-stack of crossed bimodules

6.1 Composition of butterflies

Let \(T_*, S_*\) and \(R_*\) be crossed bimodules. We define a composition operation
\[
\mathcal{B}(T_*, S_*) \times \mathcal{B}(S_*, R_*) \to \mathcal{B}(T_*, R_*)
\]

by restriction of the one for abelian sheaves defined in ref. in [AN09]. Consider the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\kappa} & N \\
\varphi & \downarrow \iota & \varphi \\
P & \xrightarrow{\kappa} & N \\
S & \xrightarrow{\varphi} & E \\
T & \xrightarrow{\iota} & R
\end{array}
\]

where \(E\) is a butterfly from \(S_*\) to \(R_*\) and \(F\) one from \(T_*\) to \(S_*\). As an abelian sheaf, the object \(F \oplus S E\) is obtained as the cokernel of the monomorphism

\[
(6.1.2)
\]

It is proved in loc. cit. that the right hand side of (6.1.1) is in \(\mathcal{B}_k(T_*, R_*)\), with \(\pi''\) and \(j''\) being the obvious projections, whereas \(\kappa''\) and \(\iota''\) are induced by \((\kappa', 0)\) and \((0, \iota)\), respectively. In addition, \(F \oplus S E\) has an obvious algebra structure, and it is immediately seen that \(N\) is an ideal via (6.1.2). It is also easy to see the four morphisms \(\kappa''\), \(\iota''\), \(\pi''\) and \(j''\) satisfy (4.1.3), so the right hand side of (6.1.1) indeed is a bimodule butterfly. If \(\beta: F' \to F\) and \(\alpha: E' \to E\) are (iso)morphisms of butterfly, it is easily verified that there results a morphism

\[
(\beta, \alpha): F \oplus S E' \to F \oplus S E,
\]

\[\begin{array}{ccc}
M & \xrightarrow{(\kappa', \iota')} & N \\
\pi & \downarrow \varphi & \pi \\
M & \xrightarrow{(\kappa', \iota')} & N \\
S & \xrightarrow{\pi} & E \\
T & \xrightarrow{\varphi} & R
\end{array}
\]

\[
\begin{array}{ccc}
P & \xrightarrow{\kappa} & N \\
\varphi & \downarrow \iota & \varphi \\
P & \xrightarrow{\kappa} & N \\
S & \xrightarrow{\varphi} & E \\
T & \xrightarrow{\iota} & R
\end{array}
\]
as a butterfly from $T_\ast \to R_\ast$.

This construction, analogously to the abelian sheaf case, can be sheafified over $S$, so we obtain a composition law

$$\mathcal{B}(T_\ast, S_\ast) \times \mathcal{B}(S_\ast, R_\ast) \to \mathcal{B}(T_\ast, R_\ast).$$

In view of the equivalence of Theorem 5.1, we have

**6.1.3 Lemma.** Let $\mathcal{T}, \mathcal{S}$ and $\mathcal{R}$ be the ring-like stacks corresponding to the above crossed bimodules. The composition law (6.1.1) is induced by that on ring-like stacks:

$$\mathcal{H}om(\mathcal{T}, \mathcal{S}) \times \mathcal{H}om(\mathcal{S}, \mathcal{R}) \to \mathcal{H}om(\mathcal{T}, \mathcal{R})$$

via the equivalence of Theorem 5.1.

**Proof.** Let $B: \mathcal{T} \to \mathcal{S}$ and $A: \mathcal{S} \to \mathcal{R}$ be two morphisms. Let $F$ and $E$ be the corresponding butterflies. We prove that the butterfly determined by $A \circ B$ is isomorphic to $F \oplus^N E$.

From the proof of Theorem 5.1 we have $E = S \oplus R$ and $F = T \oplus S$. Then

$$F \oplus^E S = F \oplus^E (S \oplus R) \simeq F \oplus^E R.$$

As a consequence, the morphism (6.1.2) equals

$$N \xrightarrow{(\gamma', 0)} F \oplus_{\mathcal{R}} R$$

and its cokernel is therefore $T \oplus_{\mathcal{R}} R$, since

$$N \xrightarrow{\gamma'} F \xrightarrow{\pi'} T$$

is exact. But $T \oplus_{\mathcal{R}} R$ is the center element of the butterfly determined by $A \circ B$, as wanted. Tracing the various steps shows that $F \oplus^N E \simeq T \oplus_{\mathcal{R}} R$ is a ring isomorphism.

We leave to the reader the tedious but straightforward verification that the above constructions are compatible with morphisms, namely that 2-morphisms $\alpha: A' \Rightarrow A: \mathcal{T} \to \mathcal{R}$ and $\beta: B' \Rightarrow B: \mathcal{T} \to \mathcal{S}$ give rise to a morphism $F' \oplus^N E' \to F \oplus^N E$ of butterflies as above corresponding to $\alpha \ast \beta: A' \circ B' \Rightarrow A \circ B$. \qed

### 6.2 The 2-stack of crossed bimodules

Let $\text{XBiMod}(S)$ the bicategory whose objects are crossed bimodules over $S$. The category (in fact, groupoid) of morphisms from the crossed bimodule $S_\ast$ to $R_\ast$ is the groupoid of butterflies $\mathcal{B}(S_\ast, R_\ast)$: since the composition (6.1.1) is obtained from the fiber product construction of the butterfly applied to the composite $\mathcal{T} \to \mathcal{S}$, the composition of butterflies is only associative up to isomorphism.

As a consequence of Theorem 3.1.1 and Theorem 5.1 we have:

**6.2.1 Proposition.** $\text{XBiMod}(S)$ is equivalent to the 2-category of ring-like stacks, i.e. the 2-category of monoids in $\text{Pic}(S)$.

This is the specialization of a similar equivalence holding for the corresponding larger 2-categories of complexes and Picard stacks. More precisely, we have a (faithful) forgetful functor $\text{XBiMod}(S) \to \text{Ch}^{-1,0}(S)$, where the latter is the bicategory of length 1-complexes of abelian sheaves over $S$, equipped with butterflies $B_1(-, -)$ as morphism groupoids. Similarly, we can consider the whole 2-category of Picard stacks, $\text{Pic}(S)$. Then we have an equivalence of bicategories

$$\text{Ch}^{-1,0}(S) \simeq \text{Pic}(S).$$

This equivalence can actually be sheafified over $S$ to yield an equivalence of 2-stacks

$$\mathcal{C}_h^{-1,0}(S) \xrightarrow{\sim} \mathcal{Pic}(S),$$

see [AN09, Thm. 8.5.2 and Prop. 8.5.4]. Note that $\mathcal{C}_h^{-1,0}(S)$ is a 2-stack in a weaker sense, as it is fibered in bicategories.

**6.2.2 Proposition.** The fibered bicategory defined by $U \to \text{XBiMod}(S/U)$, $U \in \text{Ob}(S)$, is a 2-stack $\text{XBiMod}(S)$. Moreover, there is an equivalence with the 2-stack of monoids in $\mathcal{Pic}(S)$. \qed
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