We study the Čerenkov effect in the context of the Maxwell-Chern-Simons (MCS) limit of the Standard Model Extension. We present a method to determine the exact radiation rate for a point charge.

1. The Čerenkov effect in the MCS model

In recent years the so-called Standard-Model Extension (SME)\(^1\) has provided a convenient framework for studying minute Lorentz and CPT violations that may be low-energy signatures for Planck-scale physics.\(^2\) In this work we will study a subsector of the SME describing pure electrodynamics, where Maxwell theory has been modified with Chern–Simons-like term in the Lagrangian parametrized by dimensionful parameter \((k AF)\mu\):

\[
L_{M CS} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (k AF)_{\mu} A_{\nu} \tilde{F}^{\mu\nu} - A_{\mu} j^\mu.
\]  

The Chern–Simons term explicitly violates Lorentz invariance, as well as PT and CPT invariance. (For an explicit mechanism generating it see Ref. 3.) We have explicitly included a coupling to an external current \(j^\mu\), which we take to satisfy \(\partial_\mu j^\mu = 0\).

As will become clear below, the inclusion of the \((k AF)_\mu\) term results in a modification of the photon dispersion relation, with the possibility of phase speeds smaller than the conventional speed of light in vacuum \(c\). If realized in Nature, this opens up the possibility that ordinary charged matter could move with a velocity exceeding the phase velocity of radiation, and thus should emit Čerenkov radiation \(in \text{ vacuum}\). This effect is well established experimentally and theoretically in conventional macroscopic media.\(^4\) Recently, some unexpected features have been encountered in observations involving lead ions\(^5\) and in exotic condensed-matter systems.\(^5\)
Some of these issues have been studied theoretically. In this talk, we will present recent work by the present authors in which vacuum Čerenkov radiation was investigated in detail. Our approach provides a new conceptual perspective on Čerenkov radiation, exploiting the fact that we have a fully relativistic Lagrangian, that allows arbitrary observer Lorentz transformations. In particular, going to the charge’s rest frame turns out to simplify the analysis.

The dispersion relation that follows from (1) is given by:

$$D(p^\mu) = p^4 + 4p^2k^2 - 4(p \cdot k)^2 = 0.$$  

(2)

where $p^\mu = (\omega, \vec{p})$ corresponds to the photon 4-momentum and $k^\mu \equiv (k_{AF})^\mu$. Generally, this dispersion relation includes time-like as well as spacelike solutions for $p^\mu$. In figure 1 the case of space-like $k^\mu$ is depicted. It can be shown that the spacelike and timelike branches of the dispersion relation correspond to deformed elliptical polarizations. At high momenta, they become left- and right circular polarizations.

In order to determine the rate of emission of Čerenkov radiation, it will be necessary to determine the solution of the equations of motion in the

![Figure 1](image-url)
presence of a charge, that is, with nonzero four-current. The solution of the equation of motion that follows from lagrangian (1) is:

\[ A^{\mu}(x) = A^{\mu}_{0}(x) + \int_{C_\omega} \frac{d^4p}{(2\pi)^4} \hat{G}^{\mu\nu} \hat{j}_\nu \exp(-ip \cdot x), \]

(3)

where \( A^{\mu}_{0}(x) \) is any solution to the free equations of motion (with \( j^{\mu} = 0 \)), \( \hat{j}_\nu \) is the Fourier transform of the current, while the momentum space Green’s function equals

\[ \hat{G}^{\mu\nu} = -\frac{p^2 \eta^{\mu\nu} + 2i\varepsilon^{\mu\nu\rho\sigma} k_\rho p_\sigma + 4k^\mu k^\nu}{p^2 - k^2} + 4\hat{G}^{\mu\nu}_{0}, \]

(4)

where

\[ \hat{G}^{\mu\nu}_{0} = \frac{(p \cdot k)(p^\mu k^\nu + k^\mu p^\nu) - k^2 p^\mu p^\nu}{[D(p^\mu)]p^2}. \]

(5)

can be ignored as it yields a total derivative upon contraction with a conserved current, thus giving rise to a gauge artifact. The integration contour \( C_\omega \) has to be chosen judiciously to insure retarded boundary conditions.

2. Conditions for the emission of Čerenkov radiation

We will now determine the rate of emission of Čerenkov radiation by a pointlike charge. As it turns out, the calculation is simplest in the rest frame of the charge. As the current is time-independent in that frame, we have for its Fourier transform

\[ \hat{j}^{\mu}(\vec{p}) = 2\pi \delta(\omega) \tilde{j}^{\mu}(\vec{p}) \]

(6)

where \( \tilde{j}^{\mu}(\vec{p}) \) is the Fourier transform in 3-space. It follows

\[ A^{\mu} = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{N^{\mu\nu} \hat{j}_\nu(\vec{p}) \exp(i\vec{p} \cdot \vec{r})}{D(0, \vec{p})} \]

(7)

with

\[ N^{\mu\nu}(\vec{p}) = p^2 \eta^{\mu\nu} - 2i\varepsilon^{\mu\nu\rho\sigma} k_\rho p_\sigma - 4k^\mu k^\nu. \]

(8)

As the source is independent of time, the resulting electromagnetic fields are expected to be stationary as well. Only spatial oscillations of the fields can occur. This time independence suggests that the radiated energy should be zero in the rest frame of the charge.

Evaluating (7), it is advantageous, as usual, to extend the \( |\vec{p}| \) integral to the complex plane, and use residue calculus. It follows then directly that this integral yields a factor

\[ \exp(i\vec{p}_0 \cdot \vec{r}), \]

(9)
where $\vec{p}_0$ satisfies the dispersion relation:

$$D(0, \vec{p}_0) = 0.$$  \hspace{1cm} (10)

We conclude that a nonzero imaginary part of $p_0$ implies exponential decay of the fields with increasing $r$, while a nonzero real part corresponds to an oscillatory behavior. As transport of energy-momentum to infinity can only occur in the presence of long-range fields, it follows that we can expect vacuum Čerenkov radiation only if there are real four-momenta $p^\mu = (0, \vec{p})$ satisfying the plane wave dispersion relation in the charge’s rest frame.

In a general frame, where charge’s velocity is $\vec{β}'$, the four-momentum $p^\mu = (0, \vec{p})$ is transformed into $(\vec{β}' \cdot \vec{p}', \vec{p}')$, where $\vec{p}' = \vec{p} + (\gamma - 1)(\vec{p} \cdot \vec{β}') \vec{β}' / |\vec{β}'|^2$. It follows that the phase velocity equals

$$c_{\text{ph}}' = |\vec{β}' \cdot \vec{p}'| / |\vec{p}'| \leq |\vec{β}'|$$ \hspace{1cm} (11)

so that the velocity of the particle must exceed the phase velocity of the waves. This corresponds exactly to the conventional condition for emission of Čerenkov radiation.

It is useful to consider the analogue of a boat in still water. If the boat is in motion relative to the water, a v-shaped wavefront appears. For an observer on the boat, the wave pattern is stationary, while for a general observer on the shore it oscillates with decaying frequency (after the boat has passed).

Figure 2 depicts a quantity related to the potential as a function of position, which clearly shows the nontrivial directional dependence of the emitted waves. Note that the MCS lagrangian implies a nontrivial dispersion relation (10). Consequently, the direction of the Čerenkov waves is frequency dependent, resulting in the absence of a sharp shock-wave.

3. Calculation of the emission rate

The usual way to determining Čerenkov rate involves integration of the $r^{-2}$ piece of Poynting vector over the boundary surface of space at infinity. However, this procedure is intractable in the present case, because determination of the asymptotic fields turns out to be difficult.

An alternative approach has been developed in Ref. 8. We start with the following expression for the energy-momentum tensor

$$\Theta^{\mu\nu} = -F^{\mu\alpha} F^{\nu\alpha} + \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - k^{\nu} \tilde{F}^{\mu\alpha} A_\alpha$$ \hspace{1cm} (12)

which obeys the conservation condition

$$\partial_\mu \Theta^{\mu\nu} = j_\mu F^{\mu\nu}.$$ \hspace{1cm} (13)
Figure 2. General field pattern of a point charge resting at the origin. The function $|\vec{r}| I_{osc}(\vec{r})$ is shown for $\vec{r}$ in the $xz$ plane with $\vec{k}$ along the $z$ direction. This function was evaluated by an analytical $|\vec{p}|$-type integration followed by numerical angular integrations. Uninteresting nonoscillatory pieces $I_{non}$ have been subtracted for clarity, so that only the oscillatory part $I_{osc} \equiv I - I_{non}$ contributes to this plot. The wave pattern is reminiscent to that caused by a boat moving in water.

Integrating this equation over 3-volume yields

$$\int_{\sigma} d\sigma^{t} \Theta_{t\nu} = \int_{V} d^{3}\vec{r} \, j^{\mu} F_{\mu\nu} - \frac{\partial}{\partial t} \int_{V} d^{3}\vec{r} \, \Theta_{0\nu}.$$ (14)

We now take static point charge source $J^{\mu}(\vec{r}) = (q\delta(\vec{r}), 0)$. It follows from Eq. 14 that

$$\int_{\sigma} d\sigma^{t} \cdot \vec{S} = 0$$ (15)

for the Poynting vector $\Theta_{t0} = S_{t} = -S^{t}$, so the net radiated energy is always zero in the charge’s rest frame, as anticipated. There is, however, a nonzero rate of radiation of 3-momentum:

$$\dot{P}_{s} \equiv \int_{\sigma} d\sigma^{t} \Theta_{ts}$$ (16)
which becomes
\[ \dot{\vec{P}} = \int_V d^3r J^\mu \vec{\nabla} A^\mu. \]  

(17)

Using the explicit (retarded) solution (7) obtained for \( A^\mu \) one can calculate \( \dot{\vec{P}} \) by regularizing the delta-function defining the source, and performing the Fourier integral. It follows that
\[ \dot{\vec{P}} = -\text{sgn}(k_0) \frac{q^2 k_0^2}{4\pi k^2} \vec{e}_k. \]

(18)

Note that, as a consequence, \( \dot{\vec{P}} = 0 \) if \( k_0 = 0 \), that is, there is no radiation in the rest frame unless \( k_0 \) is nonzero.

Transforming to general frame in which the charge has an arbitrary velocity generally yields non-zero components for all components of \( \dot{\vec{P}}^\mu \) that depend on both \( \vec{\beta} \) and \( \vec{k} \).

Figure 3 indicates the polarization of the radiation as a function of the direction of the wave vector \( \vec{p} \) in relation to \( \vec{\beta} \) and \( \vec{k} \).

Figure 3. Dependence of the polarization on direction. For vectors \( \vec{p} \) pointing in the clear (shaded) direction, the associated waves are right (left) polarized. The radiation exhibits linear polarization only when \( \vec{p} \) lies on one of the dashed lines. Vacuum Čerenkov radiation may not be emitted into all directions. The wave 4-vector \( p^\mu = (\vec{\beta}, \vec{p}, p) \) is further constrained by the dispersion relation.

A natural question that presents itself is whether vacuum Čerenkov
radiation might be observed. As it turns out, in the laboratory frame the components of $k^\mu$ are observationally constrained by $O(k^\mu) \lesssim 10^{-42}$ GeV.\footnote{The smallness of this bound implies that deviations of the photon phase speed from $c$ are expected to be extremely small. Taking this bound to be saturated, it can be shown\footnote{One can show that:} that a proton at the end of the observed cosmic-ray spectrum ($10^{20}$ eV) will only emit radiation of wavelengths larger than $1.2 \times 10^5$ m. Conceivably, such radiation might be observable in high-energy astrophysical jets emitted in the direction of sight.

\section*{4. Back reaction on the charge}

Denoting the charge’s 4-momentum by $Q^\mu$, momentum conservation yields

$$\dot{Q}^\mu = -\dot{P}^\mu(\vec{\beta})$$ \hspace{1cm} (19)

It is possible to continue to consistently use the usual definition $Q^\mu = mu^\mu$ so that one obtains the differential equation

$$-\dot{P}^\mu(\vec{\beta}) = mu^\mu(\vec{\beta})$$ \hspace{1cm} (20)

where $\dot{P}^\mu(\vec{\beta})$ has been determined in the previous section (transforming formula (18) to the appropriate frame).

For the important case of space-like $k^\mu$, this equation can be integrated explicitly in the laboratory frame in which $k^0 = 0$, yielding the charge’s velocity as a function of time.\footnote{One might speculate whether the slow-down effect of high-energy charges might lead to an effective cut-off in the cosmic-ray spectrum for primary particles carrying an electric charge. This idea has been raised in the literature to place bounds on Lorentz breaking. In the present model, however, the energy-loss rate is suppressed by two powers of the (experimentally tightly bounded) Lorentz-violating coefficient $k^\mu$.}

- The component $\beta_\perp$ normal to $\vec{k}$ is always constant in time;
- The charge is always slowed down by Čerenkov radiation;
- The characteristic time scale governing the time dependence is given by $	au = 4\pi m/q^2 k^2 \sqrt{1 - \beta_\perp^2}$;
- The trajectory is generally curved, with a characteristic scale size $\tau \beta_\perp$.

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\section*{5. Phase space estimate}

While a full quantum field theory extension of the classical results obtained above is beyond the scope of the current work, we will here present a phase...
space estimate of the radiation rate and show it is consistent with Eq. 18.

We start with the decay rate of a particle $a$ into two particles $b$ and $c$ in quantum fields theory:

$$d\Gamma = \frac{|M_{a\rightarrow b,c}|^2}{2E_a}(2\pi)^4\delta^{(4)}(p_a^\mu - p_b^\mu - p_c^\mu)d\Pi_b d\Pi_c. \tag{21}$$

Here $d\Pi_i \ (i = a, b, c)$ denote the phase-space elements. We take $p_a^2 = p_b^2 = m^2$, corresponding to a mass $m$ particle with a conventional Lorentz invariant dispersion relation, while $c$ denotes photons with the MCS dispersion relation.

It is possible to show the for light-like $k^\mu$,

$$d\Pi_c = \frac{d^3\vec{p}_c}{(2\pi)^3|\vec{p}_c + \text{sgn}(k^0)\vec{k}|} \tag{22}$$

is observer-invariant for the space-like branches of the photon dispersion relation.

For the amplitude we can take

$$M_{a\rightarrow b,c} = qE_aM \tag{23}$$

as the generic form of the amplitude, with $M$ a dimensionless function of external momenta and the Lorentz-violating parameters.

An order-of-magnitude estimate for expression (21) can be worked out in the $m \rightarrow \infty$ limit, yielding for the decay rate:

$$\dot{\vec{P}} \simeq -\text{sgn}(k^0)\frac{q^2|\vec{M}|^2}{8\pi} \vec{k}^2\vec{e}_k. \tag{24}$$

Here $|\vec{M}|^2$ denotes a suitable angular average of $|M|^2$. This result is in correspondence with classical result (18).

6. Conclusions

We considered the possibility of Čerenkov radiation in the Maxwell–Chern–Simons model, a particular limit of the SME. We showed how the Lorentz-violating modification of the plane-wave dispersion relation leads to the emission of radiation by moving charges. Our novel approach exploited the fact that observer Lorentz invariance always allows one to transform to the rest frame of the charge, where the calculations are less complicated. We investigated various properties of this radiation, and obtained the exact (classical) rate of emission of radiation by a point charge. The possibility of detection of vacuum Čerenkov radiation in astrophysical context
was considered, with the conclusion that the tight observational bounds on the \((k_AF)\) parameter render any possible effect highly suppressed. We note that it would be interesting to consider the dimensionless \(k_F\) term in the SME: some of its components are currently only bounded at the \(10^{-9}\) level,\(^{10}\) and a dynamical study paralleling the present one could yield less suppressed rates. We expect our methodology to have applicability in more general cases including macroscopic media.

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