Chiral Phase Transition at Strong Coupling in Lattice QCD

Dong Chen
Department of Physics
Columbia University

Abstract
We study the chiral phase transition with staggered fermions on
the lattice at finite temperature in the strong coupling limit. The ther-
modynamic potential is derived in the large $d$ approximation where
$d+1$ is the dimension of space time. Our calculation is simpler than
the conventional method and leads to a simple physical interpretation
for the approximation scheme.

1 Introduction

The chiral phase transition has been extensively studied in the context of
the large $d$ expansion at strong coupling. The mean field result at finite
temperature shows a second order phase transition [1–4]. The flavor de-
pendence and the critical exponents of the transition have also been studied
[5, 6]. However, in the conventional treatment of the large $d$ expansion, the
links in the temporal direction are treated quite differently from those in the
spatial directions, leading to a complicated determinant calculation. This
more accurate treatment of the temporal links raises a question about the
overall consistency of this application of strong coupling method to the QCD
phase transition. In this paper, a better motivated and simpler approach is
presented. The final result of this new approach agrees with that of previous
works to leading order in $d$.

We start from the partition function of lattice QCD with staggered fermions.
In section 2, we derive some basic formulae relating the chiral condensate to
the thermodynamic potential. In section 3, we take the strong coupling limit $g^2 \to \infty$. After integrating out the links in the spatial directions, we make a large $d$ expansion and derive the thermodynamic potential in a saddle point approximation to the integration over an intermediate variable. In section 4, we integrate over the time links and the fermion fields to explicitly evaluate the thermodynamic potential. The phase transition is studied afterwards. Our method is much simpler than the conventional treatment, yet it gives the same result to leading order in the $1/d$ expansion. Finally, we give a physical interpretation of the calculation which is quite obvious in this new approach.

## 2 Basic Formulation

We consider lattice QCD with staggered fermions on a $d + 1$ dimensional asymmetric lattice at finite temperature. The lattice size is $N_s^d \times N_t$ with $N_t$ kept fixed and $N_s \to \infty$. The lattice spacing is $a_s$ in the spatial directions and $a_t$ in the temporal direction. The partition function with an external source $\sigma(x)$ coupled to $\chi(x)$ is defined as

$$Z[\sigma] = \int [dU d\chi d\bar{\chi}] \exp\{- (S_G + S_F) + \sum_x \sigma(x) \bar{\chi}(x) \chi(x)\}, \quad (1)$$

where the gauge action $S_G$ and the fermion action $S_F$ are

$$S_G = \beta_s \sum_{U_s} \{ 1 - \frac{1}{2N} Tr(U_s + U_s^\dagger) \} + \beta_t \sum_{U_t} \{ 1 - \frac{1}{2N} Tr(U_t + U_t^\dagger) \}, \quad (2)$$

$$S_F = \frac{1}{2} \xi \sum_x \{ \bar{\chi}(x) U_t(x) \chi(x + i) - \bar{\chi}(x + i) U_t(x)^\dagger \chi(x) \}$$

$$+ \frac{1}{2} \sum_x \sum_{k=1}^d \eta_k(x) \{ \bar{\chi}(x) U_k(x) \chi(x + \hat{k}) - \bar{\chi}(x + \hat{k}) U_k(x)^\dagger \chi(x) \}. \quad (3)$$

In the above equations,

$$\beta_s = \frac{2N}{g_s^2} \frac{1}{\xi}, \quad \beta_t = \frac{2N}{g_t^2} \xi, \quad \xi = \frac{a_s}{a_t}, \quad (4)$$

CU–TP–700 [hep-lat/9506004]
and \( \eta_k(x) = (-1)^{x_1 + \cdots + x_k} \) is the staggered phase factor. Define

\[
Z[\sigma] = e^{W[\sigma]}.
\]  

(5)

The Legendre transform of the free energy \( W[\sigma] \) is the thermodynamic potential

\[
\Gamma[\Phi] = \sum_x \sigma(x) \Phi(x) - W[\sigma],
\]

(6)

where

\[
\Phi(x) = \frac{\delta W[\sigma]}{\delta \sigma(x)}, \quad \sigma(x) = \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)}.
\]

(7)

The normal chiral condensate is related to \( \Phi(x) \) by

\[
\langle \chi(x) \chi(x) \rangle = \frac{\delta W[\sigma]}{\delta \sigma(x)} \bigg|_{\sigma(x)=0} = \Phi(x)|_{\sigma(x)=0} = \Phi(x)|_{\Phi(x)} = 0.
\]

(8)

Thus the expectation value \( \langle \chi \chi \rangle \) equals the value of \( \Phi \) at the stationary points of the thermodynamic potential \( \Gamma[\Phi] \).

### 3 Large \( d \) expansion at strong coupling

We work in the strong coupling limit, \( g_s \to \infty, g_t \to \infty \). To leading order in \( \beta_s \) and \( \beta_t \), we will neglect \( S_G \) in the functional integral. We will start by integrating out the gauge links in the spatial directions and applying a self-consistent large \( d \) expansion to the fermion fields. The leading quadratic terms in the quark bilinear \( \chi \chi \) will be linearized by introducing a Gaussian integral over an intermediate scalar field \( \lambda(x) \). We will then make a saddle point approximation to the \( \lambda(x) \) integral and derive the thermodynamic potential in a simple form. The integrations over the time links and fermion fields are left to the next section.

After integrating over all the \( U_k \)'s in the spatial directions, we get

\[
Z[\sigma] = \int [dU_t d\chi d\bar{\chi}] \exp \left\{ \sum_x \sigma(x) \bar{\chi}(x) \chi(x) \right. \\
\left. + \frac{\xi}{2} \sum_x (\bar{\chi}(x) U_t(x) \chi(x + \hat{i}) - \bar{\chi}(x + \hat{i}) U^\dagger_t(x) \chi(x)) \\
+ \sum_{x, k=1}^d F(x, x + \hat{k}) \right\},
\]

(9)
where
\[
F(x, y) = \frac{1}{4N} \chi(x) \chi(y) + \frac{1}{32N^2(N - 1)} (\chi(x) \chi(y))^2
+ \cdots .
\] (10)

\(F(x, y)\) is a polynomial of the composite fields \(\chi(x) \chi(y)\) and \(\epsilon_{ij1\ldots iN} \epsilon_{j1j2\ldots jN} \chi^{i1}(x) \chi^{i2}(x) \cdots \chi^{iN}(x) \chi^{j1}(y) \chi^{j2}(y) \cdots \chi^{jN}(y)\) which is symmetrical between the two spatially neighboring sites \(x\) and \(y\). \(N\) is the number of colors. This is a result from the \(U_k\) integral and from the fact that we are neglecting the gauge action at strong coupling. The summation over \(d\) neighbors for \(F(x, y)\) in equation (9) leads to a large \(d\) expansion [7]. It turns out that \(\langle \chi \chi \rangle\) is on the order of \(\sqrt{1/d}\) and the higher order terms in \(F(x, y)\) are suppressed by powers of \(1/d\). A naive justification for this \(1/d\) expansion would be the mean field approximation. In that approximation, there would be a common factor \(d\) in front of all terms in \(F\) and the \(1/d\) expansion would amount to a standard loopwise expansion. The actual derivation is similar but more involved. For our current calculation, it is sufficient to keep only the leading term in \(F(x, y)\). Follow the standard treatment in [4], we define
\[
V(x, x') = \frac{1}{2d} \sum_k (\delta_{x+k,x'} + \delta_{x-k,x'}),
\] (11)
which satisfies the following equations
\[
V(x, x') = V(x', x), \quad \sum_x V(x, x') = 1, \quad \sum_x V(x, x')^{-1} = 1.
\] (12)

We rewrite the leading term of \(\sum F(x, y)\) in the exponent as
\[
\exp \left\{ \frac{1}{4N} \sum_x \sum_{k=1}^d \chi(x) \chi(x + \hat{k}) \chi(x + \hat{k}) \right\}
= \exp \left\{ \frac{d}{2N} \sum_{x,x'} \chi(x) \chi(x') V(x, x') \chi(x') \right\}
= \int [d\lambda] \exp \left\{ -\frac{N}{d} \sum_{x,x'} \lambda(x) V(x, x')^{-1} \lambda(x') + \sum_x \lambda(x) \chi(x) \chi(x) \right\} .
\] (13)

CU–TP–700 hep-lat/9506004
Substituting the above equation into eq (9) and making the change of variables $\lambda(x) \to \lambda(x) - \sigma(x)$, we have

\[
Z[\sigma] = \int [d\lambda dU_t d\chi d\overline{\chi}] \exp \left\{ \sum_x \lambda(x) \overline{\chi}(x) \chi(x) \right. \\
+ \frac{\xi}{2} \sum_x (\overline{\chi}(x) U_t(x) \chi(x + \hat{t}) - \overline{\chi}(x + \hat{t}) U_t^\dagger(x) \chi(x)) \\
- \frac{N}{d} \sum_{x,x'} (\lambda(x) - \sigma(x)) V(x, x')^{-1} (\lambda(x') - \sigma(x')) \right\} \\
= \int [d\lambda] \exp \{- \frac{N}{d} \sum_{x,x'} (\lambda(x) - \sigma(x)) V(x, x')^{-1} (\lambda(x') - \sigma(x')) \\
+ \sum_x A(\lambda(x)) \} ,
\]

(14)

where

\[
\exp \{ \sum_x A(\lambda(x)) \} \\
= \int [dU_t d\chi d\overline{\chi}] \exp \left\{ \sum_x \lambda(x) \overline{\chi}(x) \chi(x) \right. \\
+ \frac{\xi}{2} \sum_x (\overline{\chi}(x) U_t(x) \chi(x + \hat{t}) - \overline{\chi}(x + \hat{t}) U_t^\dagger(x) \chi(x)) \right\} .
\]

(16)

We now derive the thermodynamic potential. Make saddle point approximation to the $\lambda(x)$ integral,

\[
Z[\sigma] = e^{W[\sigma]} \\
= \exp \{- \frac{N}{d} \sum_{x,x'} (\lambda(x) - \sigma(x)) V(x, x')^{-1} (\lambda(x') - \sigma(x')) \\
+ \sum_x A(\lambda(x)) \} .
\]

(17)

The saddle point condition is

\[
- \frac{2N}{d} \sum_{x,x'} V(x, x')^{-1} (\lambda(x') - \sigma(x')) + A'(\lambda(x)) = 0 .
\]

(18)

We find

\[
\Phi(x) = \frac{\delta W[\sigma]}{\delta \sigma} = A'(\lambda(x)) ,
\]

(19)
\[
\Gamma[\Phi] = \sum_x \sigma(x) \Phi(x) - W[\sigma]
= -\frac{d}{4N} \sum_{x,x'} \Phi(x) V(x,x') \Phi(y) + \sum_x B(\Phi(x)) .
\] (20)

And \(B(\Phi(x))\) is the Legendre transform of \(A(\lambda(x))\)

\[
B(\Phi(x)) = \lambda(x) \Phi(x) - A(\lambda(x)) .
\] (21)

4 Mean Field Results

In this section, we integrate out the time links and the fermion fields to obtain \(A(\lambda)\) explicitly. We then study the thermodynamic function \(\Gamma[\Phi]\). We choose a different integration order from the conventional treatment and give a simpler derivation.

To calculate \(A(\lambda)\), we will first integrate out the \(U_t's\), keeping only the leading term in the \(\bar{\chi}\chi\). Then we will introduce a scalar field \(\lambda_t(x)\) in order to rewrite the \(\bar{\chi}\chi\chi\) term into a bilinear form. Finally we will integrate over the Grassmann fields \(\chi\bar{\chi}\).

\[
\exp\{\sum_x A(\lambda(x))\}
= \int [d\chi d\bar{\chi}] \exp \{\sum_x \lambda(x) \bar{\chi}(x) \chi(x) + \frac{\xi^2}{4N} \sum_x \bar{\chi}(x) \chi(x + \hat{t}) \chi(x + \hat{t})\}
= \int [d\chi d\bar{\chi}] \exp \{\sum_x \lambda(x) \bar{\chi}(x) \chi(x) + \frac{\xi^2}{4N} \sum_{x,x'} \bar{\chi}(x) \chi(x) V_t(x,x') \bar{\chi}(x') \chi(x')\}
= \int [d\lambda_t d\chi d\bar{\chi}] \exp \{-\frac{N}{\xi^2} \sum_{x,x'} \lambda_t(x) \bar{V}_t^{-1}(x,x') \lambda_t(x')
+ \sum_x (\lambda(x) + \lambda_t(x)) \bar{\chi}(x) \chi(x)\}
= \int [d\lambda_t] \exp\{-\frac{N}{\xi^2} \sum_{x,x'} \lambda_t(x) \bar{V}_t^{-1}(x,x') \lambda_t(x')\} \times \prod_x (\lambda(x) + \lambda_t(x))^N ,
\] (22)

where

\[
V_t(x,x') = 1/2(\delta_{x,x'+t} + \delta_{x,x'-t}) .
\] (23)
Making the assumption that both $\lambda(x)$ and $\lambda_t(x)$ are independent of $t$, we get

$$A(\lambda(\vec{x})) = \frac{1}{N_t} \log \left( \int d\lambda_t(\vec{x}) \exp \left\{ -\frac{N N_t}{\xi^2} \lambda_t(\vec{x})^2 \right\} \times (\lambda_t(\vec{x}) + \lambda(\vec{x}))(N N_t) \right).$$  \hfill (24) \vspace{0.25in}

In order to study the thermodynamic function $\Gamma[\Phi]$, we shall find the Legendre transform of $A(\lambda)$. Let us first discuss two regions of values for $\lambda$ where we can make simple approximations:

1) If $\lambda$ is large, on the order of $\sqrt{d}$, the leading term in $A(\lambda)$ is $\lambda N N_t$. So we have

$$A(\lambda) = N \log \lambda + \text{const}, \quad \Phi = A'(\lambda) = \frac{N}{\lambda}. \hfill (25, 26)$$

It is natural to set $\Phi$ to a constant at this stage when we evaluate $\Gamma[\Phi]$, so that

$$\Gamma[\Phi] = -\frac{d}{4N} \sum_{x,x'} \Phi(x)V(x,x')\Phi(y) + \sum_x N \log(\Phi(x))$$

$$= \left( -\frac{d}{4N} \Phi^2 + N \log \Phi \right) N^d N_t, \quad \hfill (27)$$

$$\langle \chi\chi \rangle = \Phi(x) \bigg|_{\Phi(x)=0} = N \sqrt{\frac{2}{d}}. \hfill (28)$$

This is the known result giving chiral symmetry breaking at zero temperature. We see that both $N_t$ and $\xi$ have disappeared from the final answer for $\langle \chi\chi \rangle$, which means we are essentially working with a $d + 1$ dimensional system, identical to the original system at $T = 0$. The chiral condensate is proportional to $\sqrt{1/d}$, which justifies the large $d$ expansion. Chiral symmetry is always broken.

2) If $\lambda$ is small, we can expand $A(\lambda)$ in a Taylor series.

$$A(\lambda(\vec{x})) = \frac{1}{N_t} \log \{C(1 + a_2 \lambda^2 + a_4 \lambda^4 + \ldots)\}$$

$$= \frac{1}{N_t} \{ \log C + a_2 \lambda^2 + (a_4 - \frac{a_2^2}{2}) \lambda^4 + \ldots \}, \quad \hfill (29)$$

CU–TP–700 \(\text{hep-lat/9506004}\)
where

\[ C = \int d\lambda_t \exp\{-\frac{NN_t}{\xi^2}\lambda_t^2\} \lambda_t^{NN_t}, \]

\[ a_2 = \frac{1}{C} \left( \frac{NN_t}{2} \right) \int d\lambda_t \exp[-\frac{NN_t}{\xi^2}\lambda_t^2] \lambda_t^{NN_t} = \frac{(NN_t)^2}{\xi^2}, \]

\[ a_4 = \frac{1}{C} \left( \frac{NN_t}{4} \right) \int d\lambda_t \exp[-\frac{NN_t}{\xi^2}\lambda_t^2] \lambda_t^{NN_t-4} = \frac{(NN_t)^3(NN_t-2)}{6\xi^4}. \] (30)

Neglecting the constant term, and taking \( NN_t \gg 1 \) just for simplicity, we get

\[ A(\lambda(\vec{x})) = \frac{N^2N_t}{\xi^2} \lambda^2 - \frac{N^4N^3_t}{3\xi^4} \lambda^4, \] (31)

\[ B(\Phi(\vec{x})) = \frac{\xi^2}{4N^2N_t} \Phi^2 + \frac{\xi^4}{48N^4N_t} \Phi^4. \] (32)

Again we take \( \Phi(\vec{x}) \) to be a constant field in space,

\[ \Gamma[\Phi] = [(-\frac{d}{4N} + \frac{\xi^2}{4N^2N_t})\Phi^2 + \frac{\xi^4}{48N^4N_t} \Phi^4]N_s^dN_t. \] (33)

Normally, when \( d \) is large, the coefficient of the \( \Phi^2 \) term in the above equation has a negative sign and we have broken chiral symmetry. When the temperature is raised, \( \xi \) becomes large. If \( \xi \) is on the order of \( \sqrt{d} \), the coefficient can change sign and a second order phase transition occurs. Above the transition, \( \langle \overline{\chi}\chi \rangle = 0 \) and chiral symmetry is restored. The critical \( \xi \) for the chiral phase transition is, to leading order in \( 1/d \),

\[ \xi_c = \frac{a_s}{a_t} = \sqrt{NN_t d}. \] (34)

And the transition temperature \( T_c \) is

\[ T_c = \frac{1}{N_t a_t} = \frac{\xi_c}{N_t a_s} = \frac{1}{a_s} \sqrt{\frac{Nd}{N_t}}, \] (35)

CU–TP–700 hep-lat/9506004
In general, large $d$ and high temperature are two competing factors which determine the phase of the system. This amounts to finding the true minimum of the effective potential in eq (17),

$$V_{\text{eff}}(\lambda) = \frac{N}{d} (\lambda(x) - \sigma(x))^2 - A(\lambda(x)),$$

in the saddle point approximation. For a fixed $d$ at low temperature, the system is in the chiral symmetry broken phase. A second order phase transition to the chirally symmetric phase occurs when the temperature is raised to the order of $\sqrt{d}$.

Our result above agrees with the conventional calculation although our method is different. In the conventional treatment, $A(\lambda(x))$ is calculated in the following order: One first integrates over the Grassmann variables $\chi\chi$, resulting in a determinant involving the links in the temporal direction. One then performs the integral over these $U_i$’s. Thus one performs an exact integration of the time links after one make the large $d$ expansion. However, this final integral is quite elaborate and seems unnecessary. In this paper, we treat the temporal and spatial links similarly in that they are all integrated out at the beginning. We then do the $\chi\chi$ integral and finally integrate over the intermediate field $\lambda_t$. We arrive at an $A(\lambda)$ that has the same structure as that given by the conventional treatment,

$$A(\lambda) = C \log(a_0 + a_2\lambda^2 + \cdots + \lambda^{N_tN}).$$

The difference lies only in the higher order $1/d$ corrections to the coefficients in the logarithm of $A(\lambda)$. Also, instead of using an effective potential which depends on the intermediate integration field, we derive the thermodynamic potential which is directly related to $\langle \chi\chi \rangle$ via eq (8).

5 Discussion

When $d$ is large, one normally expects zero temperature physics because the thermal fluctuations tend to average out in this limit. In terms of QCD, this leads to the chiral symmetry broken phase. However, even in the large $d$ limit, a chiral phase transition can occur if the temperature is high enough. From eq (33), there are two terms contributing to the coefficient of the $\Phi^2$ term in the thermodynamic potential $\Gamma[\Phi]$. The first term comes from the $d$
neighboring sites which tends to drive the system into the chiral symmetry broken phase, and the second term comes from the thermal fluctuations in the Euclidean time direction. When the temperature is high enough, on the order of $\sqrt{d}$, the thermal fluctuations from a single time direction will be comparable to the total effect from all $2d$ spatially neighboring sites and the chiral phase transition occurs. In the conventional strong coupling calculation, the role of the thermal fluctuation in the time direction is somewhat hidden which leads to an unnecessarily complicated derivation. In this paper, a simpler and more unified approach is presented. The only difference in the treatment of the coupling between fermion bilinears in spatial and temporal directions is that for the spatial direction, the intermediate integration variables are replaced by a constant determined from a saddle point condition; while in the temporal direction, the corresponding quantity must be explicitly integrated over to take the fluctuations into account. The chiral phase transition is second order in the large $d$ limit at strong coupling.

6 Acknowledgment

I wish to thank Professor Norman Christ for many helpful discussions.

References

[1] P. H. Damgaard, N. Kawamoto, and K. Shigemoto, Phys. Rev Lett. 53 (1984), 2211

[2] A. Gocksek and Michael Ogilvie, Phys. Rev. D 31 (1985), 877

[3] P. H. Damgaard, N. Kawamoto, and K. Shigemoto, Nucl. phys. B 264 (1986), 1-28

[4] Göran Fäl dt, and Bengt Petersson, Nucl. Phys. B 264 [FS15] (1986), 197-222

[5] Neven Bilić, Krešimir Demeterfi, and Bengt Petersson, Nucl. Phys. B 377 (1992), 651-665

[6] Neven Bilić, Frithjof Karsch, and Krzysztof Redlich, Phys. Rev. D 45 (1992), 3228

CU–TP–700 [hep-lat/9506004]
[7] Hannah Kluberg-Stern, André Morel, and Bengt Petersson, Nucl. Phys. B215 [FS7] (1983), 527-554

[8] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, 2nd Ed., Oxford Univ. Press, (1989, 1993)