An efficient implementation of the Shamir secret sharing scheme

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Abstract

The Shamir secret sharing scheme requires a Maximum Distance Separable (MDS) code, and in its most common implementation, a Reed-Solomon (RS) code is used. In this paper, we observe that the encoding procedure can be made simpler and faster by dropping the MDS condition and specifying the possible symbols that can be shared. In particular, the process can be made even faster by using array codes based on XOR operations instead of RS codes.

Index Terms

Shamir secret sharing scheme, erasure correcting codes, MDS codes, Reed-Solomon codes, array codes.

I. INTRODUCTION

Storage systems are under continuous cyber attacks like ransomware, which have become endemic. It is extremely important to protect these systems using the most advanced tools in key distribution. The Shamir secret sharing scheme is adapted to the cyber challenges of the present, is one of such advanced tools. The secret may consist of a large file and the pieces of information distributed to the participants may continuously change in order to enhance security, hence the whole process requires very fast encoding and decoding algorithms. The purpose of this paper is to present some methods achieving this goal.

The Shamir secret sharing scheme consists of a secret symbol \( D \) that can be reconstructed by sharing \( n − 1 \) symbols among \( n \) different participants. Given \( k < n \), the secret symbol can be reconstructed from any \( k \) of the \( n − 1 \) shared symbols by an interpolation process. However, knowledge of any \( k − 1 \) symbols gives no information about the secret \( D \). It was observed that the Shamir scheme is equivalent to implementing an \([n, k]\) MDS code \([10]\) such that the secret is one of the data symbols (for example, the first symbol, a convenient assumption for implementation, as we will see in the next section), the remaining \( k − 1 \) data symbols are random symbols, and the \( n − k \) parity symbols are obtained by encoding the \( k \) data symbols into the given MDS code. Then, the \( n − 1 \) symbols excluding the secret symbol are distributed among \( n − 1 \) participants. Any \( k \) participants can then reconstruct the secret symbol by performing erasure correction, but less than \( k \) participants are unable to do so.

We will present a modification of the Shamir scheme in which the parity symbols are not assigned to participants, but are known by everybody. Only the \( k − 1 \) data symbols excluding the secret symbol are assigned. The method will make the encoding faster, since the parities will be independent from each other, no linear system needs to be solved and they can even be computed in parallel. The decoding will be as fast as the one of the traditional Shamir scheme. The encoding and decoding can be made even faster by using array codes based on XOR operations, a feature that has been used in RAID-type architectures.

The paper is structured as follows: in Section II we describe the modified Shamir scheme and we discuss its advantages during encoding. In particular, we illustrate this modified scheme with RS codes. In Section III we consider the advantages of using the modified Shamir scheme of Section II with array codes as opposed to RS codes. In particular, we illustrate the ideas with Generalized EVENODD codes \([3]\). In Section IV we address other possibilities, like adapting the modified Shamir scheme to Generalized Row-Diagonal Parity (GRDP) codes \([11]\), \([12]\) and identifying cases in which some participants report incorrect symbols. We divide those cases into two categories: one in which some participants are traitors and deliberately present the wrong symbol, and another in which a few errors are involuntary. In the second case, we propose mitigation by using array codes with local properties.

II. MODIFIED SHAMIR SECRET SHARING SCHEME

Assume that \( D_0 \) is a secret symbol, there are \( k − 1 \) participants and we want this secret symbol to be recoverable as long as \( k − r \) participants are present, where \( r < k \), but a gathering of at most \( k − r − 1 \) participants provides no information about \( D_0 \). In this section, we will assume that \( D_0 \) is a symbol in a finite field \( GF(q) \) \([10]\) (for simplicity, we assume that \( q \) is a power of 2 throughout the paper, although this assumption is not necessary). Assume that the \( k − 1 \) participants are each assigned a random symbol, say \( D_i ∈ GF(q) \), \( 1 ≤ i ≤ k − 1 \). Let \( C \) be a \([k, k − r]\) MDS code over \( GF(q) \) (for example, a RS code),

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and \( H \) an \( r \times k \) parity-check matrix of \( C \). Construct a new \([k + r, k]\) code \( C'\) whose parity-check matrix is the \( r \times (k + r) \) (systematic) matrix

\[
H' = (H | I_r),
\]

where \( I_r \) is the \( r \times r \) identity matrix. It is well known that if \( r \leq 3 \), then code \( C' \) is MDS \([10]\), but this is not the case for \( r > 4 \). However, it does not matter if \( C' \) is not MDS, the result will be valid for any \( r < k \).

In effect, assume that \( k - r \) participants are present, say, those holding

\[
D_{j_0}, D_{j_1}, \ldots, D_{j_{k-r-1}}, \quad \text{where} \quad 1 \leq j_0 < j_1 < \cdots < j_{k-r-1} \leq k - 1,
\]

while the symbols \( D_0, D_1, D_2, \ldots, D_{i_r} \) are missing, where

\[1 \leq i_1 < i_2 < \cdots < i_{r-1} \leq k - 1 \quad \text{and} \quad \{j_0, j_1, \ldots, j_{k-r-1}\} \cup \{0, i_1, \ldots, i_{r-1}\} = \{0, 1, \ldots, k - 1\}.
\]

The missing \( r \) symbols can be recovered if the corresponding \( r \times r \) submatrix of the parity-check matrix \( H' \) is invertible. Specifically, assume that \( H' = (\xi_0, \xi_1, \ldots, \xi_{k+r-1}) \), where \( \xi_i \) is column \( i \) of \( H' \), hence, we have to show that the \( r \times r \) submatrix \( H_r \) of \( H' \) given by \( H_r = (\xi_0, \xi_1, \ldots, \xi_{r-1}) \) is invertible. Since \( i_{r-1} \leq k - 1 \), by \([10]\), this submatrix is also a submatrix of \( H \), and since \( H \) is the parity-check matrix of an MDS code, then \( H_r \) is invertible \([10]\). In particular, symbol \( D_0 \), which corresponds to the secret, can be recovered. This is impossible if less than \( k - r \) participants are present.

Obtaining the parity symbols using the parity-check matrix \( H' \) according to \([11]\) is very simple, since matrix \( H' \) is in systematic form. Specifically,

\[
(D_k, D_{k+1}, \ldots, D_{k+r-1}) = H(D_0, D_1, \ldots, D_{k-1})^T,
\]

a process that is very fast for codes such as RS codes (it is equivalent to computing \( r \) syndromes in a RS code). Encoding in a regular RS code is a special case of the decoding, thus, it involves solving a linear system of \( r \) equations with \( r \) unknowns. This is not the case for the systematic encoding of code \( C' \), since the parities are computed independently and no linear system needs to be solved, they may be even computed in parallel. The resulting code is not MDS when \( r \geq 4 \) and \( C \) is a RS code, but in our case it does not matter, since the erasures are in the data and, as we have seen, the system is always solvable.

**Example 1.** Consider the finite field \( GF(8) \) with primitive polynomial \([10]\) \(1 + x + x^3\). Let \( k = 7 \) and \( r = 4 \), so, according to the description above, let \( C \) be a \([7, 3]\) RS code over \( GF(8) \) with parity-check matrix

\[
H = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1
1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6
1 & \alpha^2 & \alpha^4 & \alpha^6 & \alpha^8 & \alpha^{10} & \alpha^{12}
1 & \alpha^3 & \alpha^6 & \alpha^9 & \alpha^{12} & \alpha^{15} & \alpha^{18}
\end{pmatrix}.
\]

Also, according to \([11, 7]\), \( C' \) is the \([11, 7]\) code whose parity-check matrix \( H' \) is

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 & 0 & 1 & 0 & 0
1 & \alpha^2 & \alpha^4 & \alpha^6 & \alpha^8 & \alpha^{10} & \alpha^{12} & 0 & 0 & 1 & 0
1 & \alpha^3 & \alpha^6 & \alpha^9 & \alpha^{12} & \alpha^{15} & \alpha^{18} & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Next, assume that the secret is the symbol \( D_0 = \alpha^2 \) and the 6 participants are assigned the symbols \( D_1 = \alpha^3, D_2 = \alpha, D_3 = 1, D_4 = 0, D_5 = \alpha^5 \) and \( D_6 = \alpha^3 \). The first step is computing the parity symbols as

\[
(D_7, D_8, D_9, D_{10}) = H \begin{pmatrix}
\alpha^2 \\
\alpha^3 \\
\alpha \\
1 \\
0 \\
\alpha^6 \\
\alpha^3
\end{pmatrix} = (\alpha, 0, \alpha^5, \alpha^2),
\]

i.e., \( D_7 = \alpha, D_8 = 0, D_9 = \alpha^5 \) and \( D_{10} = \alpha^2 \). The parity symbols \( D_7, D_8, D_9 \) and \( D_{10} \) are known by all the participants.

Now, assume that we have \( k - r = 3 \) participants, say, \( D_2, D_3 \) and \( D_5 \), who want to compute \( D_0 \). The parity-check matrix \( H' \) gives the following system of 4 equations with 4 unknowns:
\[ D_0 + D_1 + D_4 + D_6 = S_0 \]
\[ D_0 + \alpha D_1 + \alpha^4 D_4 + \alpha^6 D_6 = S_1 \]
\[ D_0 + \alpha^2 D_1 + \alpha^8 D_4 + \alpha^{12} D_6 = S_2 \]
\[ D_0 + \alpha^3 D_1 + \alpha^{12} D_4 + \alpha^{18} D_6 = S_3, \]

where \( S_0 = D_2 \oplus D_3 \oplus D_5 \oplus D_7 = \alpha^2, \quad S_1 = \alpha^2 D_2 \oplus \alpha^3 D_3 \oplus \alpha^5 D_5 \oplus D_8 = \alpha^4, \quad S_2 = \alpha^4 D_2 \oplus \alpha^6 D_3 \oplus \alpha^{10} D_5 \oplus D_9 = 1 \) and \( S_3 = \alpha^6 D_2 \oplus \alpha^9 D_3 \oplus \alpha^{15} D_5 \oplus D_{10} = 0. \)

We need to solve the system above only for the secret symbol \( D_0 \). For example, using Cramer’s rule, we have

\[
D_0 = \frac{\det \begin{pmatrix}
\alpha^2 & 1 & 1 & 1 & 1 \\
\alpha^4 & \alpha & \alpha^4 & \alpha^6 & \\
1 & \alpha^2 & \alpha^8 & \alpha^{12} & \\
0 & \alpha^3 & \alpha^{12} & \alpha^{18} & 
\end{pmatrix}}{\det \begin{pmatrix}
1 & 1 & 1 & 1 & \\
1 & \alpha & \alpha^2 & \alpha^6 & \\
1 & \alpha^2 & \alpha^8 & \alpha^{12} & \\
1 & \alpha^3 & \alpha^{12} & \alpha^{18} & 
\end{pmatrix}} = \alpha^2,
\]

since the determinant of the numerator is \( \alpha \), while the determinant of the denominator is a Vandermonde determinant, which is equal to

\[(1 \oplus \alpha)(1 \oplus \alpha^4)(1 \oplus \alpha^6)(\alpha \oplus \alpha^4)(\alpha \oplus \alpha^6)(\alpha^4 \oplus \alpha^6) = \alpha^6.\]

We will see next a more efficient method for computing the erased symbol \( D_0 \). \( \qed \)

Example 1 illustrates the simplicity of the encoding method: each parity symbol is the syndrome of the \( k \) data symbols with respect to the parity-check matrix \( H \). For RS codes, there is ample literature on how to efficiently compute the syndromes. Regarding the decoding, mainly the computation of the secret symbol \( D_0 \), we will next describe a method that is similar to the one presented in [2].

In effect, assume the conditions described above with the codes \( C \) and \( C' \), where the \( r \) erased symbols are \( D_0, D_{14}, D_{15}, \ldots, D_{k-r-1} \), the symbols corresponding to the \( k-r \) participants that are present are \( D_{j_0}, D_{j_1}, \ldots, D_{j_{k-r-1}} \), while the parity symbols are \( D_k, D_{k+1}, \ldots, D_{k+r-1} \). Moreover, we assume that \( C \) is a (shortened) RS code with parity-check matrix

\[
H = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & \alpha & \alpha^2 & \cdots & \alpha^{k-1} & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
1 & \alpha^{r-1} & \alpha^{2(r-1)} & \cdots & \alpha^{(k-1)(r-1)} & 
\end{pmatrix}. \tag{2}
\]

We are interested in computing only \( D_0 \). The syndrome \( S_u \), \( 0 \leq u \leq r-1 \), is given by

\[
S_u = \bigoplus_{v=0}^{k-r-1} \alpha^{u+j_v} D_{j_v} \oplus D_{k+u} = \bigoplus_{s=0}^{r-1} \alpha^{u+j_s} D_{j_s}. \tag{3}
\]

Define the polynomials of degree at most \( r-1 \)

\[
S(x) = S_0 \oplus S_1 x \oplus \cdots \oplus S_{r-1} x^{r-1} \quad \text{and} \quad G(x) = (x \oplus \alpha^{i_1})(x \oplus \alpha^{i_2}) \cdots (x \oplus \alpha^{i_{r-1}}) = g_{r-1} \oplus g_{r-2} x \oplus \cdots \oplus g_0 x^{r-1}. \tag{4}
\]

Notice that \( G(1) = (1 \oplus \alpha^{i_1})(1 \oplus \alpha^{i_2}) \cdots (1 \oplus \alpha^{i_{r-1}}) \) while \( G(\alpha^{i_s}) = 0 \) for \( 1 \leq s \leq r-1 \). Then, assuming \( i_0 = 0 \), by (3), (4) and (5), we have
\[
\sum_{u=0}^{r-1} S_u g_{r-u-1} = \sum_{u=0}^{r-1} \left( \sum_{s=0}^{r-1} \alpha^{u+s} D_{is} \right) g_{r-u-1}
= \sum_{s=0}^{r-1} D_{is} \left( \sum_{u=0}^{r-1} \alpha^{u+s} g_{r-u-1} \right)
= \sum_{s=0}^{r-1} D_{is} G(\alpha^{is})
= D_0 \prod_{s=1}^{r-1} (1 + \alpha^{is}).
\]

Thus, by (5) and (6)

\[
D_0 = \frac{\sum_{u=0}^{r-1} S_u g_{r-u-1}}{\prod_{s=1}^{r-1} (1 + \alpha^{is})} = \frac{\sum_{u=0}^{r-1} S_u g_{r-u-1}}{\sum_{u=0}^{r-1} g_u},
\]

since by (5), \(G(1) = \prod_{s=1}^{r-1} (1 + \alpha^{is}) = \sum_{u=0}^{r-1} g_u\). Both the numerator and the denominator in (7) can be easily computed.

Example 2. Let us revisit Example 1 and find \(D_0\) using (7). Using the syndromes obtained in Example 1 and (4), we obtain

\[
S(x) = \alpha^2 + \alpha^4 x + x^2.
\]

From Example 1 we have \(i_1 = 1\), \(i_2 = 4\) and \(i_3 = 6\), so, by (5), we obtain

\[
G(x) = (x + \alpha)(x + \alpha^4)(x + \alpha^6) = \alpha^4 + \alpha^6 x + x^2 + x^3,
\]

i.e., \(g_3 = \alpha^4\), \(g_2 = \alpha^6\), \(g_1 = 1\) and \(g_0 = 1\) in (5). Hence, the numerator in (7) is given by

\[
g_3 S_0 \oplus g_2 S_1 \oplus g_1 S_2 \oplus g_0 S_3 = \alpha^4 \oplus \alpha^6 \oplus 1 = \alpha^5,
\]

while the denominator equals

\[
G(1) = \alpha^4 + \alpha^6 + 1 + 1 = \alpha^3,
\]

so \(D_0 = \alpha^2\), which coincides with the value obtained in Example 1. \(\square\)

III. USE OF ARRAY CODES IN THE MODIFIED SHAMIR SECRET SHARING SCHEME

The purpose of using array codes in RAID-type architectures [2], [5] was to replace finite field operations, which usually require a look-up table, by XOR operations. In an application like the Shamir scheme described in Section II, if the size of the secret is pretty large, implementation of a RS code has to be done multiple times. Array codes like the ones described in [1], [3], [5], [7] can have symbols (which correspond to columns in the array) of size \(p\), where \(p\) is a prime number. Certainly, \(p\) can be as large as needed, while large symbols in a RS code require a large look-up table in the corresponding finite field and may not be practical.

An example of an MDS array code is given by Blaum-Roth (BR) codes [5]. We are not the first to point out the usefulness of array codes in the context of the Shamir secret sharing scheme. For example, in [13], the use of BR codes is proposed.

Given an odd prime number \(p\), the codewords of a \([p, k]\) BR code consist of \((p-1) \times p\) arrays such that, when appending a zero row to such an array in the code, making it a \(p \times p\) array, the lines of slope \(i\) (with a toroidal topology), \(0 \leq i \leq p-k-1\), have even parity. For example, for the first four lines of the \(5 \times 5\) array below are in a \([5, 2]\) BR code: the horizontal lines (slope 0), the lines of slope 1 and the lines of slope 2 have all even parity. In the left array, we illustrate in bold the second line of slope 1, while in the right array, in bold is the third line of slope 2 (we assume that the individual symbols in the arrays are bits, although they can have any size. It is not necessary either that the number of columns is a prime number, since some columns may be assumed to be zero).
An equivalent algebraic definition of BR codes (and a very convenient one for decoding) is that they are RS codes over the ring of polynomials modulo $M_p(x) = 1 + x + x^2 + \cdots + x^{p-1}$ \cite{sh}. The parity-check matrix $H$ of such a code (shortened to $k$ columns) is given by \cite{sh}. Let us point out that the polynomial $M_p(x)$ may not be irreducible (for example, $M_5(x)$ is irreducible but $M_7(x) = (1 + x + x^3)(1 + x^2 + x^3)$). However, the code is always MDS \cite{sh}.

In order to apply our particular version of the Shamir scheme as described in Section II, we need to consider the parity-check matrix $H'$ as given by \cite{sh}, while $H$ is given by \cite{sh}. Such a resulting code is a generalization of the EVENODD code \cite{1979}, and has different names in literature: generalized EVENODD code \cite{1987}, independent parity (IP) code \cite{1994} or Blaum-Bruck-Vardy code \cite{2001}. The MDS condition of these codes has been extensively studied for $r \geq 4$ \cite{2001}, \cite{2011}, but for our purpose the modified Shamir scheme will always work for $r < k$, as in the case of RS codes we studied in Section III. Notice that for these generalized EVENODD codes, the horizontal lines always have even parity, while the lines of slope $i$, $1 \leq i \leq r-1$, may have either even or odd parity: the special line of slope $i$ starting in the last bit of the first column (which is 0 and not written) determines the parity of all the other lines of slope $i$. So, the encoding is very fast and convenient.

**Example 3.** The following array corresponds to a generalized EVENODD code with $p = 5$ and 3 parities:

\[
\begin{array}{cccc|c|c}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

In the array on the left we illustrate in bold the entries of the special line of slope 1 starting at the bottom of the first column. It has an even number of ones, so all the diagonals must have even parity, which is determined by the second parity column (the first parity column corresponds to horizontal parity, so it has always even parity). Similarly, in the array on the right, we illustrate in bold the entries corresponding to the special line of slope 2 starting at the bottom of the first column. In this case, the number of 1s of this special line is odd, so all the lines of slope 2 must have odd parity, and this is reflected in the last parity column. Notice that the parities are independent of each other, so, for that reason, these codes are also called Independent Parity (IP) codes \cite{2001}.

Denote the 8 columns in the array as $(\xi_0, \xi_1, \ldots, \xi_7)$ and assume that the secret is $\xi_0$, while the parities are $\xi_1, \xi_2, \xi_3, \xi_4$ and $\xi_5$. The four data columns $\xi_1, \xi_2, \xi_3, \xi_4$ and $\xi_5$ are assigned to participants, while the three parity columns are known by everybody. Assume that $k - r = 2$ participants get together, say, $\xi_1$ and $\xi_2$. Then, symbols $\xi_0$ (the secret), $\xi_1$ and $\xi_2$ are erased, and we have to use $\xi_3, \xi_4, \xi_5, \xi_6, \xi_7$ to retrieve them. We proceed similarly to the method described in Section III for RS codes. The first step is computing the syndromes using the parity-check matrix $H'$:

\[
S_0 = \xi_0 + \xi_4 + \xi_5 \\
S_1 = \alpha^2 \xi_0 + \alpha^4 \xi_4 + \xi_5 \\
S_2 = \alpha^4 \xi_0 + \alpha^8 \xi_4 + \xi_5.
\]

Notice that as a function of $\alpha$, from the array above, $\xi_2 = \alpha$, $\xi_4 = \alpha^3$, $\xi_5 = 1$, $\xi_6 = \alpha$ and $\xi_7 = \alpha^2 + \alpha^3$, where $M_5(\alpha) = 0$. Hence, $\alpha^4 = 1 + \alpha + \alpha^2 + \alpha^3$ and $\alpha^5 = 1$, and the syndromes can be easily calculated as $S_0 = 1 + \alpha + \alpha^3$, $S_1 = \alpha + \alpha^2 + \alpha^3$ and $S_2 = 1 + \alpha + \alpha^2 + \alpha^3 = \alpha^4$. Thus, by \cite{2001},

\[
S(x) = (1 + \alpha + \alpha^3) + (\alpha + \alpha^2 + \alpha^3)x + \alpha^4x^2.
\]

Next, using \cite{2001}, since $i_1 = 1$ and $i_2 = 3$,

\[
G(x) = (x + \alpha)(x + \alpha^3) = \alpha^4 + (\alpha + \alpha^3)x + x^2,
\]

i.e., in \cite{2001}, $g_2 = \alpha^4$, $g_1 = \alpha + \alpha^3$ and $g_0 = 1$. Next, we have to compute the right hand side in \cite{2001}, which gives
\[ S_0 g_2 \oplus S_1 g_1 \oplus S_2 g_0 = \alpha \oplus \alpha^3. \]

Using (6), we have to solve

\[ (1 \oplus \alpha)(1 \oplus \alpha^3)D_0 = \alpha \oplus \alpha^3, \]

Let \((1 \oplus \alpha^3)D_0 = X\), then we have to solve first

\[ (1 \oplus \alpha)X = \alpha \oplus \alpha^3, \quad (8) \]

which can be done using the following lemma [5]:

**Lemma 1.** Assume that we want to solve \((1 \oplus \alpha^3)X(\alpha) = Y(\alpha)\) over the ring of polynomials modulo \(M_p(x)\), where \(p\) is prime, \(1 \leq j \leq p - 1\), \(Y(\alpha) = \bigoplus_{i=0}^{p-2} y_i \alpha^i\) is given and \(X(\alpha) = \bigoplus_{i=0}^{p-2} x_i \alpha^i\). Then, for \(1 \leq u \leq p - 1\),

\[ x_{(-u-j)-1} = x_{-(u-1)(j-1)} \oplus \hat{y}_{-(u-1)(j-1)}, \quad (9) \]

where given any integer \(m\), \(\langle m \rangle\) denotes the unique integer \(v\), \(0 \leq v \leq p - 1\), such that \(v \equiv m \pmod p\) (for example, for \(p = 5\), \(\langle -2 \rangle = 3\)), \(y_{p-1} = 0\) and \(\hat{y}_j = y_j \oplus \bigoplus_{i=0}^{p-1} y_i\).

Applying recursion (9) in Lemma 1 to (8), we obtain \(X = \alpha \oplus \alpha^2\). Next we have solve \((1 \oplus \alpha^3)D_0 = \alpha \oplus \alpha^2\). Again, using recursion (9), we obtain \(D_0 = 1 \oplus \alpha^2 \oplus \alpha^3\), which coincides with column \(c_0\) of the array above, corresponding to the secret. □

**IV. Other Possibilities and Conclusions**

The Shamir scheme can have other implementations as well. Another array code that can be used is the GRDP code [11, 6, 2]. The GRDP code has a minimal number of encoding operations, so it is very convenient for the modified Shamir scheme described in sections [11] and [11]. A GRDP\((p, r)\) code consists of the arrays \(a_{i,j}\) \(0 \leq j \leq r - 1\) \(0 \leq i \leq p - 1\), such that

\[ a_{i,p-1} = \bigoplus_{j=0}^{p-2} a_{i,j} \quad \text{(10)} \]

\[ a_{i,k+u} = \bigoplus_{j=0}^{p-1} a_{(i-u),j} \quad \text{for} \quad 1 \leq u \leq r - 1, \quad \text{(11)} \]

where \(\langle m \rangle\) was defined in Lemma 1. For example, according to (10) and (11), the following is an array in GRDP\((5, 3)\):

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Column \(p + u\), \(1 \leq u \leq r - 1\), contains the parity of the lines of slope \(u\) in the \(p \times p\) array, computed using the horizontal parity (hence, the parities are not independent as in the extended EVENODD code described in Section [11]), and excluding the line starting at location \(p - 1\) of the first column. The encoding is simpler than the encoding of the extended EVENODD codes, since the lines of slope \(u\), \(1 \leq u \leq r - 1\), all have even parity and the parity of the lines starting at location \(p - 1\) of the first column do not need to be computed.

From the above discussion, the modified Shamir scheme with GRDP codes is as follows: let the secret be a symbol of length \(p - 1\), where \(p\) is prime, then take \(p - 2\) random symbols of length \(p - 1\), and encode these \(p - 1\) symbols into a GRDP\((p, r)\) code. The \(p - 2\) random symbols together with the horizontal parity symbol are distributed to \(p - 1\) participants, while the \(r - 1\) parity symbols corresponding to lines of slope \(u\), \(1 \leq u \leq r - 1\), are known by all the participants. Then, if any \(p - r\) participants share their symbols, \(r\) erasures can be corrected by the code.
It has been shown [11, 8] that a GRDP code is MDS if and only if a corresponding generalized EVENODD code is also MDS. This property also helps with the decoding in the recovery of the secret: once the transformation is established, there are efficient methods to decode the generalized EVENODD code [11, 8] that can be used in our context.

As pointed out in [11], since an MDS code can correct errors together with erasures, the Shamir scheme can handle cases in which a number of participants, for a variety of reasons, incorrectly report their symbols. Specifically, an \( [n, k] \) MDS code can correct \( s \) errors together with \( t \) erasures as long as \( 2s + t \leq n - k \) [10]. In the Shamir scheme, this means that if \( n - t \) participants get together and \( s \) of them report the wrong symbol, then the secret can be recovered as long as \( 2s + t \leq n - k \).

This scheme works also for our modified Shamir scheme: in this case the \( r \) parity symbols are known by everybody, the secret is the first symbol and the remaining \( k - 1 \) symbols are distributed among participants. If \( k - t \) participants share their symbols but \( s \) of them provide the wrong symbol, then the secret can be recovered as long as \( 2s + t \leq r \).

The decoding of RS codes containing both errors and erasures is well known. However, there is no known efficient decoding algorithm correcting more than three errors for array codes such as BR codes. For example, an efficient algorithm correcting one error and any number of erasures was presented in [5]. Efficient algorithms correcting two and three errors with any number of erasures can be found in [3]. Beyond that, the problem is open, though correction of up to three errors may be enough for most applications of the Shamir scheme.

The inaccuracy of sharing symbols with other participants, as stated above, may be due to a few different reasons. One such cause involves a traitor among the participants, who may exploit the information from the other participants either to have sole access to the secret or to sabotage the entire enterprise. Provided that there is enough redundancy, the scheme for correcting errors and erasures prevents this scenario, allowing for the identification of up to \( s \) traitors. However, such a scheme is costly if the participant providing erroneous information did not have nefarious purposes. The information may have been corrupted by a few erroneous or erased bits through normal noise during transmission of the symbol.

Recently, an expansion of the BR, generalized EVENODD and GRDP codes was presented [4]. In these expansions, the arrays have column size \( p \) as opposed to \( p - 1 \). The expanded codes continue to be MDS, but each column is in a cyclic code with generator polynomial \( (1 \oplus x)g(x) \), where \( g(x) \) divides \( 1 \oplus x^p \). If the cyclic code has minimum distance \( d \), then \( s \) bits in error together with \( t \) erased bits can be corrected in every column as long as \( 2s + t \leq d - 1 \). Hence, a few errors and erasures can be corrected locally in each column of the array without invoking the other columns. The full power of the code is reserved for cases in which traitors deliberately misrepresent the column they had been assigned. A further generalization was obtained in [14], which describes a generalization of the expanded BR codes to powers of prime numbers.

We presented the decoding algorithm to obtain the secret as a result of repeated recursions. There are more efficient decoding algorithms reducing the number of recursions when obtaining all the erasures, mainly through the LU factorization of Vandermonde matrices [14]. For our purpose, however, we only need to obtain one erasure, the one corresponding to the secret.

The modified Shamir secret sharing scheme presented in this paper consists of assigning \( k - 1 \) random data symbols to participants (excluding the secret), while the parity symbols are independent from each other and known by everyone. This method simplifies the encoding since computing the parity does not require solving a system of linear equations and can be done in parallel, while the decoding remains the same. We studied this modified scheme with RS and with array codes. By using array codes with local properties, we showed that the cases in which participants report their symbols with involuntary errors can be mitigated.

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