Non-colocated Time-Reversal MUSIC: High-SNR Distribution of Null Spectrum

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Abstract—We derive the asymptotic distribution of the null spectrum of the well-known Multiple Signal Classification (MUSIC) in its computational Time-Reversal (TR) form. The result pertains to a single-frequency non-colocated multistatic scenario and several TR-MUSIC variants are here investigated. The analysis builds upon the 1st-order perturbation of the singular value decomposition and allows a simple characterization of null-spectrum moments (up to the 2nd order). This enables a comparison in terms of spectrums stability. Finally, a numerical analysis is provided to confirm the theoretical findings.

Index Terms—Time-Reversal (TR), Radar imaging, Null-spectrum, Resolution, TR-MUSIC.

I. INTRODUCTION

TIME-REVERSAL (TR) refers to all those methods which exploit the invariance of the wave equation (in lossless and stationary media) by re-transmitting a time-reversed version of the scattered (or radiated) field measured by an array to focus on a scattering object (or radiating source), by physical or synthetic means. In the latter case (computational TR), it consists in numerically back-propagating the field data by using a known Green’s function, representative of the propagation medium. Since the employed Green function depends on the object (or source) position, an image is formed by varying the probed location (this procedure is referred to as “imaging”). Computational TR has been successfully applied in different contexts such as subsurface prospecting, through-the-wall and microwave imaging.

The key entity in TR-imaging is the Multistatic Data Matrix (MDM), whose entries are the scattered field due to each Transmit-Receive (Tx-Rx) pair. Two popular methods for TR-imaging are the decomposition of TR operator (DORT) and the TR Multiple Signal Classification (TR-MUSIC). DORT imaging exploits the MDM spectrum by back-propagating each eigenvector of the so-called signal subspace, thus allowing to selectively focus on each (well-resolved) scatterer. On the other hand, TR-MUSIC imaging is based on a complementary point of view and relies on the noise subspace (viz. orthogonal-subspace), leading to satisfactory performance as long as the data space dimension exceeds the signal subspace dimension and sufficiently high Signal-to-Noise-Ratio (SNR) is present. TR-MUSIC was first introduced for a Born Approximated (BA) linear scattering model and, later, successfully applied to the Foldy-Lax (FL) non-linear model. Also, it became popular mainly due to: (a) algorithmic efficiency; (b) no need for approximate scattering models; and (c) finer resolution than the diffraction limits (especially in scenarios with few scatterers). Recently, TR-MUSIC has been expanded to extended scatterers in.

Though a vast literature on performance analysis of MUSIC for Direction-Of-Arrival (DOA) estimation exists (see references for resolution studies and for asymptotic Mean Squared Error (MSE) derivation, with more advanced studies presented in), such results cannot be directly applied to TR-MUSIC. Indeed, in TR framework scatterers/sources are generally assumed deterministic and more importantly a single snapshot is used, whereas MUSIC results for DOA refer to a different asymptotic condition (i.e. a large number of snapshots). Also, to our knowledge, no corresponding theoretical results have been proposed in the literature for TR-MUSIC, except for providing the asymptotic (high-SNR) localization MSE for point-like scatterers. Yet, a few works have tackled achievable theoretical performance both for BA and FL models via the Cramér-Rao lower-bound.

In this letter we provide a null-spectrum analysis of TR-MUSIC for point-like scatterers, via a 1st-order perturbation of Singular Value Decomposition (SVD), thus having asymptotic validity (i.e. meaning a high SNR regime). The present results are based on a homogeneous background assumption and neglecting mutual coupling, as well as polarization or antenna pattern effects. Here we build upon (tackling the simpler colocated case) and consider a general non-colocated multistatic setup with BA/FL models where several TR-MUSIC variants, proposed in the literature, are here investigated. The obtained results complement those found in DOA literature and allow to obtain both the mean and the variance of each null-spectrum, as well as to draw-out its pdf. Also, they highlight performance dependence of null-spectrum on the scatterers/arrays configurations and compare TR-MUSIC variants in terms of spectrum stability.

1Such term underlines that it is orthogonal to the signal subspace.
2We underline that the MUSIC imaging function is commonly referred to as “pseudo-spectrum” in DOA literature. Though less used, in this paper we will instead adopt to the term “null-spectrum” employed in, as the latter work represents the closest counterpart in DOA estimation to the present study.
We recall that stability property is important for TR-MUSIC, and has been investigated by numerical means [26], [27] or using compressed-sensing based approaches [28]. Finally, there are a few numerical examples, for a 2-D geometry with scalar scattering, are presented to confirm our findings.

The letter is organized as follows: Sec. II describes the system model and reviews classic results on SVD perturbation analysis. Sec. III presents the theoretical characterization of TR-MUSIC null-spectrum, whereas its validation is shown in Sec. IV via simulations. Finally, conclusions are in Sec. V.

II. SYSTEM MODEL

We consider localization of $M$ point-like scatterers at unknown positions $\{x_k\}_{k=1}^M$ in $\mathbb{R}^p$ with unknown scattering potentials $\{r_k\}_{k=1}^M \in \mathbb{C}$. The Tx (resp. Rx) array consists of $N_T$ (resp. $N_R$) isotropic point elements (resp. receivers) located at $\{\mathbf{r}_j\}_{j=1}^{N_T}$ in $\mathbb{R}^p$ (resp. $\{\mathbf{r}_j\}_{j=1}^{N_R}$ in $\mathbb{R}^p$). The illuminators first send signals to the probed scenario (in a known homogeneous background with wavenumber $\kappa$) and the transducer array records the received signals. The (single-frequency) measurement model is then [20]:

$$K_n = K(x_1:M, \tau) + W$$

(1)

$$G_i(\mathbf{x}_1:M) M(\mathbf{x}_1:M, \tau) G_i(\mathbf{x}_1:M)^T + W$$

(2)

where $K_n \in \mathbb{C}^{N_R \times N_T}$ (resp. $K(\mathbf{x}_1:M, \tau)$) denotes the measured (resp. noise-free) MDM. Differently $W \in \mathbb{C}^{N_R \times N_T}$ is a noise matrix s.t. $\text{vec}(W) \sim N_C(0, \sigma_w^2 I_N)$, where $N \triangleq N_T N_R$. We have, however: (i) the vector of scattering coefficients as $\tau = [\tau_1 \cdots \tau_M]^T \in \mathbb{C}^{M \times 1}$; (ii) (b) the Tx (resp. Rx) array matrix as $G_i(\mathbf{x}_1:M) \in \mathbb{C}^{N_T \times M}$ (resp. $G_i(\mathbf{x}_1:M) \in \mathbb{C}^{N_R \times M}$), whose $(i,j)$th entry equals $\mathcal{G}(\mathbf{r}_i, \mathbf{x}_j)$ (resp. $\mathcal{G}(\mathbf{r}_i, \mathbf{x}_j)$), where $\mathcal{G}$(, ) denotes the (scalar) background Green function [7]. Also, $\mathbf{g}_i(\mathbf{x}_i)$ (resp. $\mathbf{g}_i(\mathbf{x}_i)$) of $G_i(\mathbf{x}_1:M)$ (resp. $G_i(\mathbf{x}_1:M)$) denotes the Tx (resp. Rx) Green’s function vector evaluated at $\mathbf{x}_i$. In Eq. 2, the scattering matrices $M(\mathbf{x}_1:M, \tau) \in \mathbb{C}^{M \times M}$ equals $M(\mathbf{x}_1:M, \tau) \triangleq \text{diag}(\tau)$ for BA model [7], while $M(\mathbf{x}_1:M, \tau) \triangleq \left[\text{diag}^{-1}(\tau) - S(\mathbf{x}_1:M)\right]^{-1}$ in the case of FL model [22], where the $(m,n)$th entry of $S(\mathbf{x}_1:M)$ equals $\mathcal{G}(\mathbf{x}_m, \mathbf{x}_n)$ when $m \neq n$ and zero otherwise. We recall that our null-spectrum characterization of TR-MUSIC is general and can be applied to both scattering models.

Finally, we define the SNR $\triangleq \|K(\mathbf{x}_1:M, \tau)\|^2_T / (\sigma_w^2 N_T N_R)$ and, for notational convenience, $N_{\text{ref}} \triangleq (N_R - M)$ and $N_{\text{Tdof}} \triangleq (N_T - M)$ as the dimensions of the left and right orthogonal subspaces, whereas $N_{\text{dof}} \triangleq (N_{\text{Tdof}} + N_{\text{ref}})$.

A. TR-MUSIC Spatial Spectrum

Several TR-MUSIC variants have been proposed in the literature for the non co-located setup [8]. As a first approach consists in using the so-called $Rx$ mode TR-MUSIC, which evaluates the null (or spatial) spectrum (assuming $M \leq N_R$):

$$\mathcal{P}_t(\mathbf{x}; \bar{U}_n) \triangleq \mathbf{g}_r(\mathbf{x})^\dagger \bar{P}_r \mathbf{g}_r(\mathbf{x}) = \|\bar{U}^\dagger \mathbf{g}_r(\mathbf{x})\|^2$$

(3)

The number of scatterers $M$ is assumed to be known, as usually done in array-processing literature [29].
N. Intuitively, a small perturbation is observed at high-SNR. The expressions for \( \Delta U_n \) and \( \Delta V_n \), at 1st-order, are\(^3\) \(^3\)

\[
\Delta U_n = -(A^\top)^T N^\top U_n; \quad \Delta V_n = -(A^\top)^T N V_n;
\]

where we have expressed \( A = V_s \Sigma_s^{-1} U_s^\top \) \(^4\)

### III. Null-spectrum analysis

First, we observe that the null spectrum at scatterer positions \( \mathcal{P}_r(x_k; \bar{U}_n) \), \( \mathcal{P}_r(x_k; \bar{V}_n) \) and \( \mathcal{P}_r(x_k; \bar{U}_n, \bar{V}_n) \), \( k \in \{1, \ldots, M \} \), in Eqs. (3), (4) and (5) can be simplified, using \( U_n = U_n + \Delta U_n \) and \( V_n = V_n + \Delta V_n \) and exploiting the properties\(^2\) \(^2\)

\[
\mathcal{P}_r(x_k; \bar{U}_n, \bar{V}_n) = (||\xi_{\bar{r},k}||^2)^2, \quad \mathcal{P}_r(x_k; \bar{V}_n) = (||\xi_{\bar{r},k}||^2)^2, \quad \mathcal{P}_r(x_k; \bar{U}_n, \bar{V}_n) = (||\xi_{\bar{r},k}||^2)^2,
\]

where \( \xi_{\bar{r},k} \triangleq \Delta U_k \sum_{i}^N \bar{g}_i(x_k) \in \mathbb{C}^{N_{\text{rdof}} \times 1} \) and \( \xi_{\bar{r},k} \triangleq \Delta V_k \sum_{i}^N \bar{g}_i(x_k) \in \mathbb{C}^{N_{\text{rdof}} \times 1} \), respectively. Similarly, \( \mathcal{P}_r(x_k; \bar{U}_n, \bar{V}_n) \), \( \mathcal{P}_r(x_k; \bar{V}_n) \) and \( \mathcal{P}_r(x_k; \bar{U}_n, \bar{V}_n) \) can be further simplified using Eqs. (8) and (9). The latter result is based on circularity of the entries of \( W \), along with their mutual independence. Interestingly these DOFs coincide with those available for TR-MUSIC localization through Rx and Tx modes, respectively.

Based on these considerations, the means of the null-spectrum for Rx and Tx modes are \( \mathbb{E}(||\xi_{\bar{r},k}||^2)^2 = \sigma_w^2 ||t_{\bar{r},k}||^2 N_{\text{rdof}} \) and \( \mathbb{E}(||\xi_{\bar{r},k}||^2)^2 = \sigma_w^2 ||t_{\bar{r},k}||^2 N_{\text{rdof}} \), respectively, i.e. they are independent proper Gaussian vectors. Clearly, since \( \xi_{\bar{r},k} \) and \( \xi_{\bar{r},k} \) have zero mean and scaled-identity covariance, the corresponding \( \mathbb{E}(\xi_{\bar{r},k}^2)^2/\sigma_w^2 ||t_{\bar{r},k}||^2 N_{\text{rdof}} \) and \( \mathbb{E}(\xi_{\bar{r},k}^2)^2/\sigma_w^2 ||t_{\bar{r},k}||^2 N_{\text{rdof}} \) are \( \mathcal{N}(0, \sigma_w^2 ||t_{\bar{r},k}||^2 N_{\text{rdof}}) \), respectively, i.e. they are chi-square distributed. Interestingly these DOFs coalesce with those available for TR-MUSIC localization through Rx and Tx modes, respectively.

### IV. Conclusion

In summary, we can conclude that the null spectrum orthogonality property led to a significant simplification of the null-spectrum expressions. This simplification allows for a more tractable analysis of the DOA estimation error, which is a critical aspect of the MUSIC algorithm. The results presented here provide a solid foundation for further research into the performance of MUSIC in non-coherent settings, particularly in high-SNR environments.

\[^3\] We notice that in obtaining Eq. (3), "in-space" perturbations (e.g. the contribution to \( \Delta U_n \) depending on \( U_n \)) are not considered, though they have been shown to be linear with \( N \) (and thus not negligible at first-order)\(^1\). The reason is that these terms do not affect performance analysis of TR-MUSIC null-spectrum when evaluated at scatterers positions \( \{x_k\}_k \) due to the null spectrum orthogonality property.

\[^4\] Such conditions directly follow from orthogonality between left (resp. right) signal and orthogonal subspaces \( U_s \) and \( V_s \) (resp. \( V_r \) and \( U_r \)).

\[^1\] In the following of the letter we will implicitly mean that the results hold "approximately" in the high-SNR regime.

\[^2\] The reason is that these terms do not affect performance analysis of TR-MUSIC null-spectrum when evaluated at scatterers positions \( \{x_k\}_k \) due to the null spectrum orthogonality property.
Eq. (14) underlines (i) a clear dependence of generalized null-spectrum NSD on scatterers and measurement setup and (ii) independence from the noise level $\sigma_n^2$. Also, it is apparent that when $\|t_{x,k}\| \approx 0$ (resp. $\|t_{x,k}\| \approx 0$) the expression reduces to $\text{NSD}_k \approx 1/\sqrt{N_{\text{T dof}}}$. (resp. $\text{NSD}_k \approx 1/\sqrt{N_{\text{R dof}}}$), i.e. the NSD is dominated by Tx (resp. Rx) mode stability. Finally, the same equation is exploited to obtain the conditions ensuring that generalized spectrum is “more stable” than Tx and Rx modes ($\text{NSD}_k \leq \text{NSD}_{x,k}$ and $\text{NSD}_k \leq \text{NSD}_{r,k}$, respectively), expressed as the pair of inequalities

$$\frac{1}{2} \left[ 1 - \frac{N_{\text{T dof}}}{N_{\text{T dof}}} \right] \leq \left( \frac{\|t_{x,k}\|}{\|t_{x,k}\|} \right)^2 \quad \text{(Tx)}$$

$$\frac{1}{2} \left[ 1 - \frac{N_{\text{T dof}}}{N_{\text{R dof}}} \right] \leq \left( \frac{\|t_{x,k}\|}{\|t_{x,k}\|} \right)^2 \quad \text{(Rx)}$$

(15)

Clearly, when $N_R > N_T$ (resp. $N_T > N_R$) the inequality regarding the Tx (resp. Rx) mode is always verified as the left-hand side is always negative. Also, in the special case $N_T = N_R$ the left-hand side is always zero for both inequalities.

IV. NUMERICAL RESULTS

In this section we confirm our findings through simulations, focusing on 2-D localization, with Green function $G(x',x) = H_0^{(1)}(\kappa\|x' - x\|)$. Here $H_n^{(1)}(\cdot)$ and $\kappa = 2\pi/\lambda$ denote the $n$th order Hankel function of the 1st kind and the wavenumber ($\lambda$ is the wavelength), respectively. First, we consider a setup with $\lambda$-spaced Tx/Rx arrays ($N_T = 11$ and $N_R = 17$, respectively, see Fig. 1). Secondly, to quantify the level of multiple scattering (as in [8]) we define the index $\eta \triangleq \|K_f(x;1,M,\tau) - K_b(x;1,M,\tau)\|_F^2$, where $K_b(x;1,M,\tau)$ and $K_f(x;1,M,\tau)$ denote the MDMs generated via Ba and Fl models, respectively. Finally, for simplicity we discard the irrelevant constant term $j/4$.

We provided an asymptotic (high-SNR) analysis of TR-MUSIC null-spectrum in a non-colocated multistatic setup, by taking advantage of the 1st-order perturbation of the SVD of the MDM. Three different variants of TR-MUSIC were analyzed (i.e. Tx mode, Rx mode and generalized), based on the characterization of a certain complex-valued Gaussian vector. This allowed to obtain the asymptotic NSD (a measure of null-spectrum stability) for all the three imaging procedures. While similar results as the DOA setup were obtained for Tx and Rx modes, it was shown a clear dependence of generalized null-spectrum for brevity. To this end, Fig. 2 depicts the null-spectrum NSD vs. SNR for the two targets being considered, both for FL and BA models. It is apparent that, as the SNR increases, the theoretical results tightly approximate the MC-based ones, with approximations deemed accurate above $\text{SNR} \approx 10\,\text{dB}$. Differently, in Fig. 3 we plot the asymptotic NSD of the three TR-MUSIC variants vs. $d$, where $(x_1/\lambda) = [(-1 - d) -6]^T$ and $(x_2/\lambda) = [(1 - d) -6]^T$ (i.e. a rigid shift of the two scatterers), in order to investigate the potentially improved asymptotic stability (viz. NSD) of the generalized spectrum in comparison to Tx and Rx modes. It is apparent that the gain is significant when $d \in (-5,5)$, while outside this interval the NSD expression is either dominated by Tx or Rx mode, which for the present case $\text{NSD}_{x,k} = 1/\sqrt{N_T - 2} \approx 0.53$ and $\text{NSD}_{r,k} = 1/\sqrt{N_R - 2} \approx 0.26$, with the generalized NSD never above that of $\text{NSD}_{r,k}$ (as dictated from Eq. (15)).

V. CONCLUSIONS

We provided an asymptotic (high-SNR) analysis of TR-MUSIC null-spectrum in a non-colocated multistatic setup, by taking advantage of the 1st-order perturbation of the SVD of the MDM. Three different variants of TR-MUSIC were analyzed (i.e. Tx mode, Rx mode and generalized), based on the characterization of a certain complex-valued Gaussian vector. This allowed to obtain the asymptotic NSD (a measure of null-spectrum stability) for all the three imaging procedures. While similar results as the DOA setup were obtained for Tx and Rx modes, it was shown a clear dependence of generalized null-spectrum NSD on the scatterer and measurement setup. Finally, its potential stability advantage was investigated in propagation in inhomogeneous (random) media [35].
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