Equations of general relativistic radiation hydrodynamics in Kerr space–time

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ABSTRACT

Equations of fully general relativistic radiation hydrodynamics in Kerr space–time are derived. While the interactions between matter and radiation are introduced in the comoving frame, the derivates used when describing the global evolutions of both the matter and the radiation are given in the Boyer–Lindquist frame (BLF) which is a frame fixed to the coordinate describing the central black hole. Around a rotating black hole, both the matter and the radiation are influenced by the frame-dragging effects due to the black hole’s rotation. As a fixed frame, we use the locally non-rotating reference frame (LNRF) which is one of the orthonormal frame. While the special relativistic effects such as beaming effects are introduced by the Lorentz transformation connecting the comoving frame and the LNRF, the general relativistic effects such as frame dragging and gravitational redshift are introduced by the tetrads connecting the LNRF and the BLF.

Key words: accretion: accretion discs – black hole physics – hydrodynamics – radiative transfer – relativity.

1 INTRODUCTION

Accretion discs around black holes play one of the essential roles in the active universe. When the mass accretion rate is near or over the Eddington mass accretion rate, the radiation in the disc interacts with matter. For the supercritical accretion flows considered in, for example, Seyfert galaxies or the black hole binaries, the photons interact with matter. On the other hand, for the hypercritical accretion flows considered as one of the leading candidate for the central engine of gamma-ray bursts, the neutrinos interact with matter around the central black hole. In such situations, the dynamics and the energy balance in matter and radiation are influenced by each other. The radiative transfer in the curved space is investigated by many authors (e.g. Lindquist 1966; Anderson & Spiegel 1972; Schmid-Burgk 1978; Thorne 1981; Schinder 1988; Turolla & Nobili 1988; Anile & Romano 1992; Cardall & Mezzacappa 2003; Park 2006). Lindquist (1966) gives a general treatment of the radiation transfer equation and the radiation hydrodynamic equations derived from the zeroth and the first moments of the radiation transfer equations by using a comoving Lagrangian frame of reference. For the higher moments, Anderson & Spiegel (1972) and Thorne (1981) give a detailed treatments. Especially, Thorne (1981) derives general relativistic moment equations up to an arbitrary order by introducing projected symmetric trace-free (PSTF) tensors. This formalism has been used in spherically symmetric problems (e.g. Flammang 1982, 1984; Turolla & Nobili 1988). Other forms of the moment equations are presented by Schinder (1988) and Schinder & Bludman (1989) in a static, spherical space–time by using a Lagrangian comoving coordinate system. In these formalisms based on the comoving frames, the physical quantities in terms of matter and radiation and the directional derivatives are described in the comoving frame.

On the other hand, Mihalas (1980) introduces the alternative approach by using the radiation hydrodynamic equations in the Eulerian framework where the derivatives become much simpler form in this framework. In this approach, the physical quantities in terms of matter and radiation and the derivatives were introduced in the frame fixed to the coordinate of the central object, for example, a black hole, while the interactions between matter and radiation were calculated in the comoving frame, that is, the local processes like the interaction between the matter and the radiation are evaluated in the comoving frame, while the derivatives which are used for the calculations of the global dynamics of both the matter and the radiation are derived in the frame fixed to the coordinate describing the central object. In principle, this formalism can be extended to arbitrary space–time. Actually, by using the energy-momentum tensor in a covariant form, Park (1993) derived the radiation hydrodynamic equations for spherically symmetric systems, and recently, Park (2006) gives the explicit expressions for the basic equations.
of the general relativistic radiation hydrodynamics in Schwarzschild space–time. So far, the formalism in the space–time around a rotating black hole which is usually considered for the black holes in the real world has not been presented. One of the natural next step is to derive the explicit expressions for the general relativistic radiation hydrodynamics in space–time around a rotating black hole in this formalism. In this paper, we assume $c = 1$ in most equations except in a few cases where $c$ is explicitly used for clarity. After giving the general forms of the basic equations for the general relativistic radiation hydrodynamics in Section 2, in Section 3 by using the orthonormal tetrads fixed to the coordinate (Section 3.1), the radiation moments (Section 3.2), the radiation four-force (Section 3.3) and the radiation hydrodynamic equations (Section 3.4) in Kerr space–time are derived. Concluding remarks are given in the last section.

2 BASIC EQUATIONS FOR GENERAL RELATIVISTIC RADIATION HYDRODYNAMIC IN COVARIANT FORM

Here, we summarize the basic equations for the general relativistic radiation hydrodynamic in covariant form (e.g. Mihalas & Mihalas 1984).

The energy-momentum tensor for matter of an ideal gas is described as

$$ T^{\alpha\beta} = \rho_0 h^0_\alpha u^\beta + P_\beta g^{\alpha\beta}, $$

(1)

where $u^\alpha$, $\rho_0$, $h_\alpha^0$, and $P_\beta$ are the four-velocity, the rest-mass density, the relativistic specific enthalpy and the pressure of the gas, respectively. The specific enthalpy of the gas, $h_\alpha$, is calculated as $h_\alpha = (\epsilon_\alpha + P_\alpha)/\rho_0$ where $\epsilon_\alpha$ is the energy density of the gas. Here, the fluid quantities $h_\alpha$, $\epsilon_\alpha$, $P_\beta$ and $\rho_0$ are all being measured in the comoving frame of the fluid. The radiation stress-energy tensor is given as

$$ R^{\alpha\beta} = \int I(n, \nu) n^\alpha n^\beta d\nu d\Omega, $$

(2)

where $I(n^\alpha, n) = n^\nu$ is the specific intensity of photons moving in direction $n$ with the frequency $\nu$. The photon four-momentum $p^\nu = (h\nu)(1, n)$ and $n^\nu \equiv p^\nu/h\nu$.

The continuity equation, that is, particle number conservation equation, in the absence of particle creation and annihilation is given as

$$ (n u^\alpha)_\alpha = 0, $$

(3)

where $n$ is the particle number density measured in the comoving frame. Here, $n$ is related to $\rho_0$ as $n = \rho_0/m_\gamma$ where $m_\gamma$ is the mass of the gas particle. On the other hand, the conservation equation for the total energy momentum of gas plus radiation is given as

$$ (T^{\alpha\beta} + R^{\alpha\beta})_\beta = 0. $$

(4)

The radiation four-force density acting on the matter is given as

$$ G^\alpha = \frac{1}{c} \int \left[ \chi \nu I(n, \nu) - \eta \right] n^\alpha d\nu d\Omega, $$

(5)

where $\chi$ and $\eta$ are the opacity and the emissivity, respectively. The invariant emissivity and the invariant opacity are $\eta_\nu/\nu^2$ and $\nu \chi_\nu$, respectively. The dynamical equations for the matter and the radiation field are described as

$$ T^{\alpha\beta} = G^\alpha, $$

(6)

and

$$ R^{\alpha\beta} = -G^\alpha, $$

(7)

respectively.

3 RADIATION HYDRODYNAMICS IN KERR SPACE–TIME

In order to see the correspondence with the work by Park (2006) deriving the equations for the radiation hydrodynamics in the Schwarzschild space–time, in this study we use the Boyer–Lindquist coordinate for the description of the rotating black hole’s space–time. The background geometry around the rotating black hole written by the Boyer–Lindquist coordinate is described as

$$ ds^2 = g_{\phi\phi} dx^\alpha dx^\beta, $$

$$ = -\alpha^2 dr^2 + \chi r (dx^i + \beta^i dr)(dx^j + \beta^j dr), $$

(8)

where $i, j = r, \theta, \phi$ and the non-zero components of the lapse function $\alpha$, the shift vector $\beta^i$ and the spatial matrix $\chi_{ij}$ are given in the geometric unit as

$$ \alpha = \sqrt{\frac{\Sigma \Delta}{A}}, \beta^j = -\alpha \chi_{ij}, \chi_{ij} = \Sigma, \gamma_{\theta\theta} = \Sigma, \gamma_{\phi\phi} = \frac{A \sin^2 \theta}{\Sigma}. $$

(9)

Here, we use the geometric mass $m = GM/c^2$, $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2$ and $A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta = \Sigma \Delta + 2m r (r^2 + a^2)$, where $M$ is the black hole mass, $G$ is the gravitational constant and $c$ is the speed of light. The position of the outer and the inner horizon, $r_+$, is calculated from $\Delta = 0$ as $r_+ = m \pm (m^2 - a^2)^{1/2}$. The angular velocity of the frame dragging due to the black hole’s rotation is calculated as $\Omega = -\phi_0^2/\phi_{\phi\phi} = 2m/\Delta$. When $a/m = 0$, we obtain $\alpha = \Gamma$, $\beta^\phi = 0$, $\gamma_{\phi\phi} = 1/\Gamma^2$, $\gamma_{\theta\theta} = r^2$ and $\gamma_{\phi\phi} = r^2 \sin^2 \theta$ where $\Gamma = (1 - 2m r^2)^{1/2}$ (Park 2006).
3.1 Orthonormal tetrads

In this study, we use three frames: (i) the Boyer–Lindquist frame (BLF) which is the frame based on the Boyer–Lindquist coordinate describing the metric and has been frequently used for the description of the global dynamics (e.g. Gammie & Popham 1998); (ii) the locally non-rotating reference frame (LNRF) which is the orthonormal frame fixed to the coordinate and is called as a fixed frame in Park (2006) and (iii) the comoving frame where the interactions between the matter and the radiation are introduced. As a fixed tetrad which is the orthonormal tetrad fixed with respect to the coordinates, we use the tetrad for the LNRF (Bardeen, Press & Teukolsky 1972; Frolov & Novikov 1998). This frame is also called as a zero angular momentum observer (ZAMO) frame. This frame corresponds to the generalization of the fixed frame used in Park (2006) for the space–time around a rotating black hole. When using the Boyer–Lindquist coordinate, the LNRF is the frame where the fiducial observer in this frame is rotating around the black hole with the angular velocity \( \omega \) of the frame dragging due to the black hole’s rotation with zero radial velocity. The base \([\varepsilon_\mu = \partial / \partial x^\mu]\) can be expressed by the coordinate base \( \partial / \partial x^\mu \) as

\[
\frac{\partial}{\partial t} = \frac{1}{\alpha} \left( \frac{\partial}{\partial t} - \beta^\phi \frac{\partial}{\partial \phi} \right), \quad \frac{\partial}{\partial \bar{r}} = \frac{1}{\sqrt{\gamma_{rr}}} \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial \theta} = \frac{1}{\sqrt{\gamma_{\theta\theta}}} \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial \phi} = \frac{1}{\sqrt{\gamma_{\phi\phi}}} \frac{\partial}{\partial \phi}.
\]  

(10)

Here, \((\theta^0, \theta^1, \theta^2, \theta^3) = (i, \bar{r}, \hat{\theta}, \hat{\phi})\).

In the LNRF, a fiducial observer who is fixed with respect to the coordinates sees the fluid with the three-velocity whose components are calculated as

\[
\vec{v} = \frac{u^i}{u^0}, \quad (i = r, \theta, \phi),
\]

(11)

where \(u^0\) is the four-velocity in the LNRF. The hat denotes the physical quantities measured in the LNRF. The components of the three-velocity are explicitly calculated as

\[
\vec{v}^r = \frac{\sqrt{\gamma_{rr}}}{\alpha u^0} u^r, \quad \vec{v}^\theta = \frac{\sqrt{\gamma_{\theta\theta}}}{\alpha u^0} u^\theta, \quad \vec{v}^\phi = \frac{\sqrt{\gamma_{\phi\phi}}}{\alpha} (\Omega + \beta^\phi),
\]

(12)

where \(\Omega \equiv u^\theta / u^r\) is the angular velocity and we have used \(u^i = -u^i\). The Lorentz factor \(\gamma\) for this three-velocity is calculated as

\[
\gamma \equiv (1 - \vec{v}^2)^{-1/2} = au^i,
\]

(13)

where \(\vec{v}^2 = v \cdot v = \vec{v}^r + \vec{v}^\theta + \vec{v}^\phi\).

A tetrad base for the comoving frame \(\partial / \partial x^\mu\) is calculated by the Lorentz transformation as

\[
\frac{\partial}{\partial x^\mu} = \Lambda^\mu_\nu (\vec{v}) \frac{\partial}{\partial x^\nu},
\]

(14)

where the bar denotes the physical quantities measured in the comoving frame. The components of the Lorentz transformation \(\Lambda^\mu_\nu (\vec{v})\) are given as

\[
\Lambda^\mu_\nu = \hat{\gamma} \Lambda^\mu_\nu, \quad \Lambda^\mu_0 = \hat{\gamma} \hat{\nu}, \quad \Lambda^0_\nu = \hat{\gamma} \hat{\nu}.
\]

Here, \((x^0, x^1, x^2, x^3) = (\bar{t}, \bar{r}, \hat{\theta}, \hat{\phi})\). The components of the base of the comoving tetrad \(\partial / \partial x^\mu\) can be expressed by the coordinate base \(\partial / \partial x^\mu\) as

\[
\frac{\partial}{\partial t} = \gamma \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \frac{\partial}{\partial \bar{r}} + \vec{v} \cdot \nabla \frac{\partial}{\partial \theta} + \vec{v} \cdot \nabla \frac{\partial}{\partial \phi} + \gamma \left( \frac{v^\phi}{\sqrt{\gamma_{\phi\phi}}} - \frac{\beta^\phi}{\alpha} \right) \frac{\partial}{\partial \phi},
\]

\[
\frac{\partial}{\partial \bar{r}} = \frac{\gamma v^\phi}{\alpha} \frac{\partial}{\partial \bar{r}} + \frac{1}{\sqrt{\gamma_{rr}}} \left[ 1 + v^\theta \left( \frac{\gamma^2}{\gamma + 1} \right) \right] \frac{\partial}{\partial \theta} + \frac{v^\theta}{\sqrt{\gamma_{rr}}} \left( \frac{\gamma^2}{\gamma + 1} \right) \frac{\partial}{\partial \theta} + \frac{v^\phi}{\sqrt{\gamma_{\phi\phi}}} \left( \frac{\gamma^2}{\gamma + 1} \right) \frac{\partial}{\partial \phi}.
\]

\[
\frac{\partial}{\partial \theta} = \frac{\gamma v^\phi}{\alpha} \frac{\partial}{\partial \theta} + \frac{v^\phi}{\sqrt{\gamma_{rr}}} \left( \frac{\gamma^2}{\gamma + 1} \right) \frac{\partial}{\partial \theta} + \frac{1}{\sqrt{\gamma_{\theta\theta}}} \left[ 1 + v^\phi \left( \frac{\gamma^2}{\gamma + 1} \right) \right] \frac{\partial}{\partial \theta} + \frac{v^\phi}{\sqrt{\gamma_{\phi\phi}}} \left( \frac{\gamma^2}{\gamma + 1} \right) \frac{\partial}{\partial \phi}.
\]

\[
\frac{\partial}{\partial \phi} = \frac{\gamma v^\phi}{\alpha} \frac{\partial}{\partial \phi} + \frac{v^\phi}{\sqrt{\gamma_{rr}}} \left( \frac{\gamma^2}{\gamma + 1} \right) \frac{\partial}{\partial \phi} + \frac{v^\phi}{\sqrt{\gamma_{\theta\theta}}} \left( \frac{\gamma^2}{\gamma + 1} \right) \frac{\partial}{\partial \theta} + \frac{1}{\sqrt{\gamma_{\phi\phi}}} \left[ 1 + v^\phi \left( \frac{\gamma^2}{\gamma + 1} \right) \right] \frac{\partial}{\partial \phi}.
\]

(15)

In the same way, the inverse transformation from the base of the comoving tetrad to the coordinate base is calculated by using the inverse Lorentz transformation \(\Lambda_{\mu\nu} (-\vec{v})\), and the coordinate base \(\partial / \partial x^\mu\) is calculated from the base of the comoving tetrad \(\partial / \partial x^\mu\).
as
\[
\frac{1}{\alpha} \frac{\partial}{\partial t} = \left( \hat{\gamma} - \frac{\beta^\theta \sqrt{\gamma_{00}}}{\alpha} \hat{\gamma} \hat{v}_\phi \right) \frac{\partial}{\partial t} + \left[ -\hat{\gamma} \hat{v}_r + \frac{\beta^\theta \sqrt{\gamma_{00}}}{\alpha} \hat{v}_r \hat{v}_\phi \left( \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] \frac{\partial}{\partial r} + \left[ -\hat{\gamma} \hat{v}_\theta + \frac{\beta^\theta \sqrt{\gamma_{00}}}{\alpha} \hat{v}_\theta \hat{v}_\phi \left( \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] \frac{\partial}{\partial \theta} + \left[ 1 + \hat{\beta}_v \right] \frac{\partial}{\partial \phi}. \]

\[
= \frac{1}{\sqrt{\gamma_{rr}}} \frac{\partial}{\partial r} \left[ -\hat{\gamma} \hat{v}_r + \frac{\beta^\theta \sqrt{\gamma_{00}}}{\alpha} \hat{v}_r \hat{v}_\phi \left( \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] \frac{\partial}{\partial r} + \left[ -\hat{\gamma} \hat{v}_\theta + \frac{\beta^\theta \sqrt{\gamma_{00}}}{\alpha} \hat{v}_\theta \hat{v}_\phi \left( \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] \frac{\partial}{\partial \theta} + \left[ 1 + \hat{\beta}_v \right] \frac{\partial}{\partial \phi}. \]

\[
= \frac{1}{\sqrt{\gamma_{\theta\theta}}} \frac{\partial}{\partial \theta} \left( -\hat{\gamma} \hat{v}_\theta + \frac{\beta^\theta \sqrt{\gamma_{00}}}{\alpha} \hat{v}_\theta \hat{v}_\phi \left( \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right) \frac{\partial}{\partial \theta} + \left[ 1 + \hat{\beta}_v \right] \frac{\partial}{\partial \phi}. \]

\[
(16) \]

3.2 Radiation moments

The radiation energy density $E$, the radiation flux $F^i$ and the radiation pressure tensor $P^{ij}$ are defined as the zeroth, the first and the second moments of the specific intensity $I_s(x^a, \eta)$, respectively. We denote the radiation moments as

\[
E = \int \int I_s \, d\Omega, \quad F^i = \int \int I_s \hat{n}^i \, d\Omega, \quad P^{ij} = \int \int I_s \hat{n}^i \hat{n}^j \, d\Omega, \quad (17)
\]

when measured in the LNRF, and

\[
E = \int \int I_s \, d\Omega, \quad F^i = \int \int I_s \hat{n}^i \, d\Omega, \quad P^{ij} = \int \int I_s \hat{n}^i \hat{n}^j \, d\Omega, \quad (18)
\]

when measured in the comoving frame. Correspondingly, the radiation stress tensors for the LNRF and the comoving frame are given as

\[
\tilde{R}^{\hat{a}\hat{b}} = \begin{pmatrix}
\tilde{E} & \tilde{F}^r & \tilde{F}^\theta & \tilde{F}^\phi \\
\tilde{F}^r & \tilde{p}^{rr} & \tilde{p}^{r\theta} & \tilde{p}^{r\phi} \\
\tilde{F}^\theta & \tilde{p}^{\theta r} & \tilde{p}^{\theta\theta} & \tilde{p}^{\theta\phi} \\
\tilde{F}^\phi & \tilde{p}^{\phi r} & \tilde{p}^{\phi\theta} & \tilde{p}^{\phi\phi}
\end{pmatrix}, \quad (19)
\]

and

\[
\tilde{R}^\hat{a}\hat{b} = \begin{pmatrix}
\tilde{E} & \tilde{F}^r & \tilde{F}^\theta & \tilde{F}^\phi \\
\tilde{F}^r & \tilde{p}^{rr} & \tilde{p}^{r\theta} & \tilde{p}^{r\phi} \\
\tilde{F}^\theta & \tilde{p}^{\theta r} & \tilde{p}^{\theta\theta} & \tilde{p}^{\theta\phi} \\
\tilde{F}^\phi & \tilde{p}^{\phi r} & \tilde{p}^{\phi\theta} & \tilde{p}^{\phi\phi}
\end{pmatrix}, \quad (20)
\]

respectively. The contravariant components of the radiation stress tensor $R^{\hat{a}\hat{b}}$ are calculated from $\tilde{R}^{\hat{a}\hat{b}}$ by the transformation as

\[
R^{\hat{a}\hat{b}} = \frac{\partial x^a}{\partial \hat{x}^\alpha} \frac{\partial x^b}{\partial \hat{x}^\beta} \tilde{R}^{\hat{a}\hat{b}}, \quad (21)
\]

and explicitly given as

\[
R^{\hat{a}\hat{b}} = \begin{pmatrix}
\frac{\tilde{E}}{\alpha^2} & \frac{\tilde{F}^r}{\alpha \sqrt{\gamma_{rr}}} & \frac{\tilde{F}^\theta}{\alpha \sqrt{\gamma_{\theta\theta}}} & \frac{1}{\alpha} \left( \frac{\tilde{F}^\phi}{\sqrt{\gamma_{00}}} - \beta^\phi \tilde{E} \right) \\
\frac{\tilde{F}^r}{\alpha \sqrt{\gamma_{rr}}} & \frac{\tilde{p}^{rr}}{\sqrt{\gamma_{rr}}} & \frac{\tilde{p}^{r\theta}}{\sqrt{\gamma_{rr\theta}}} & \frac{1}{\sqrt{\gamma_{rr}}} \left( \frac{\tilde{p}^{r\phi}}{\sqrt{\gamma_{00}}} - \beta^\phi \tilde{F}^r \right) \\
\frac{\tilde{F}^\theta}{\alpha \sqrt{\gamma_{\theta\theta}}} & \frac{\tilde{p}^{\theta r}}{\sqrt{\gamma_{rr\theta}}} & \frac{\tilde{p}^{\theta\theta}}{\sqrt{\gamma_{\theta\theta}}} & \frac{1}{\sqrt{\gamma_{\theta\theta}}} \left( \frac{\tilde{p}^{\theta\phi}}{\sqrt{\gamma_{00}}} - \beta^\phi \tilde{F}^\theta \right) \\
\frac{1}{\alpha} \left( \frac{\tilde{F}^\phi}{\sqrt{\gamma_{00}}} - \beta^\phi \tilde{E} \right) & \frac{1}{\sqrt{\gamma_{rr}}} \left( \frac{\tilde{p}^{r\phi}}{\sqrt{\gamma_{00}}} - \beta^\phi \tilde{F}^r \right) & \frac{1}{\alpha \sqrt{\gamma_{00}}} \left( \frac{\tilde{p}^{\theta\phi}}{\sqrt{\gamma_{00}}} - \beta^\phi \tilde{F}^\theta \right) & \frac{2 \beta^\phi}{\alpha} \tilde{E} + \left( \frac{\tilde{p}^{\phi\phi}}{\alpha} \right) \tilde{E}
\end{pmatrix}. \quad (22)
\]

By using the Lorentz transformations, the radiation moments measured in the comoving frame are calculated from those measured in the LNRF as

\[
R^{\hat{a}\hat{b}} = \Lambda_{\hat{a}}^a (\nu) \Lambda_{\hat{b}}^b (\nu) \tilde{R}^{\hat{a}\hat{b}}, \quad (23)
\]
and are explicitly given as (Mihalas & Mihalas 1984; Park 2006)

$$\hat{E} = \hat{\gamma}^2 \left( \hat{E} - 2\hat{v}_i \hat{v}^i + \hat{v}_i \hat{v}_j \hat{P}^{ij} \right),$$

(24)

$$\bar{F}^i = -\hat{\gamma}^2 \hat{v}^i \hat{E} + \hat{\gamma} \left[ \hat{\gamma}^2 + \left( \hat{\gamma}^2 + 2 \hat{v}^2 \right) \hat{v}^i \hat{v}_j \right] \hat{F}^j - \hat{\gamma} \hat{v}_j \left( \delta^i_j + \frac{\hat{\gamma}^2}{\hat{\gamma}^2 + 1} \hat{v}^i \hat{v}_k \right) \hat{P}^{jk},$$

(25)

$$\bar{P}^{ij} = \hat{\gamma}^2 \hat{v}^i \hat{v}^j \hat{E} - \hat{\gamma} \left( \hat{v}^i \hat{v}^j + \hat{v}^j \hat{v}^i + 2 \hat{v}^j \hat{v}^k \hat{v}^i \hat{v}_k \right) \hat{F}^k + \left( \delta^i_j + \frac{\hat{\gamma}^2}{\hat{\gamma}^2 + 1} \hat{v}^i \hat{v}_k \right) \left( \delta^j_i + \frac{\hat{\gamma}^2}{\hat{\gamma}^2 + 1} \hat{v}^j \hat{v}_k \right) \hat{P}^{ki}.$$  

(26)

Inversely, the radiation moments measured in the LNRF are calculated from those measured in the comoving frame as $R^{ij} = \Lambda^i_\nu (\nu) \Lambda^j_\nu (\nu) R^{ij}$, and explicitly given as

$$\hat{E} = \hat{\gamma}^2 \left( \hat{E} + 2 \hat{v}_i \hat{F}^i + \hat{v}_i \hat{v}_j \hat{P}^{ij} \right),$$

(28)

$$\hat{F}^i = \hat{\gamma}^2 \hat{v}^i \hat{E} + \hat{\gamma} \left[ \hat{\gamma}^2 + \left( \hat{\gamma}^2 + 2 \hat{v}^2 \right) \hat{v}^i \hat{v}_j \right] \hat{F}^j + \hat{\gamma} \hat{v}_j \left( \delta^i_j + \frac{\hat{\gamma}^2}{\hat{\gamma}^2 + 1} \hat{v}^i \hat{v}_k \right) \hat{P}^{jk},$$

(29)

$$\hat{P}^{ij} = \hat{\gamma}^2 \hat{v}^i \hat{v}^j \hat{E} + \hat{\gamma} \left( \hat{v}^i \hat{v}^j + \hat{v}^j \hat{v}^i + 2 \hat{v}^j \hat{v}^k \hat{v}^i \hat{v}_k \right) \hat{F}^k + \left( \delta^i_j + \frac{\hat{\gamma}^2}{\hat{\gamma}^2 + 1} \hat{v}^i \hat{v}_k \right) \left( \delta^j_i + \frac{\hat{\gamma}^2}{\hat{\gamma}^2 + 1} \hat{v}^j \hat{v}_k \right) \hat{P}^{ki}.$$  

(30)

### 3.3 Radiation four-force density

The radiation four-force density measured in the comoving frame is given as (Mihalas & Mihalas 1984)

$$G^a = \frac{1}{c} \int \int \left( \bar{\mathcal{I}} - \bar{\eta} \right) d^3 v d\Omega,$$

(31)

where $\bar{\mathcal{I}}$ and $\bar{\eta}$ are the mean opacity and the emissivity measured in the comoving frame, respectively. The time component $G^t$ has the dimension $c^{-1}$ times the net rate of the radiation energy per unit volume, and the spatial component $G^i$ has the dimension of the net rate of the momentum exchange between the matter and the radiation. The time component of the radiation four-force density measured in the comoving frame can be calculated as (Mihalas & Mihalas 1984; Park 2006)

$$G^i = \tilde{\Gamma} - \tilde{\Lambda},$$

(32)

where the heating function $\tilde{\Gamma}$ and the cooling function $\tilde{\Lambda}$ are defined as

$$\tilde{\Gamma} = \frac{1}{c} \int \int \bar{\mathcal{I}} d^3 v d\Omega, \quad \tilde{\Lambda} = \frac{1}{c} \int \int \bar{\eta} d^3 v d\Omega.$$  

(33)

The components of the radiation force $G^a$ are calculated from those in the comoving frame $G^{\tilde{a}}$ by the transformation

$$G^a = \frac{\partial \tilde{\Lambda}^a}{\partial \tilde{\chi}^\tilde{a}} G^{\tilde{a}},$$

(34)

and are explicitly given as

$$G^t = \frac{\hat{\gamma}}{\alpha} \left( G^t + \hat{v}_i G^i \right),$$

$$G^i = \frac{1}{\sqrt{\gamma}} \left( G^i + \hat{\gamma} \hat{v}_i G^i + \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \hat{v}_i \hat{v}_j G^{jk} \right),$$

$$G^j = \frac{1}{\sqrt{\gamma}} \left( G^j + \hat{\gamma} \hat{v}_j G^j + \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \hat{v}_i \hat{v}^i G^j \right),$$

$$G^\theta = \frac{\hat{\gamma}}{\sqrt{\gamma}} \left( \hat{v}_\theta - \frac{\beta^\theta}{\alpha} \right) G^r + \hat{\gamma} \left[ \frac{\hat{\gamma} \hat{v}_\phi}{\sqrt{\gamma}} - \frac{\beta^\phi}{\alpha} \right] \hat{v}_\phi G^\theta.$$  

(35)

### 3.4 Radiation hydrodynamic equations

#### 3.4.1 Continuity equation

As a continuity equation, now we consider the particle number conservation, $(n u^a)_{,a} = 0$. This equation can be calculated as

$$\frac{\partial}{\partial t} \left( n u^a \sqrt{\gamma} \right) + \frac{\partial}{\partial x^i} \left( n u^a \sqrt{\gamma} \right) = 0,$$

(36)

where $\gamma = \det g_{ij}$. In the case of the Kerr metric written by the Boyer–Lindquist coordinate $\gamma = (\Sigma \Delta) \sin^2 \theta$. 

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3.4.2 Hydrodynamic equations

The relativistic Euler equations are obtained by the projection of the equation of the energy-momentum conservation $T_{\mu}^{\nu} = G^\nu$ on the specific directions by using the projection tensor $P_{\mu}^{\nu} = g^{\mu\nu} + u^{\mu} u^{\nu}$ as $P_{\mu}^{\gamma} T_{\gamma}^{\beta} = P_{\mu}^{\beta} G^\beta$. From this, we can obtain $\rho_0 \ h_x \ u_x^2 \ u^\theta + g^{\mu\nu} P_{\mu,\beta} + u^{\mu} u^{\nu} P_{\beta,\mu} = G^\nu + u^\alpha u^\theta \ n^\mu G^\mu$.

The radial part of the Euler equation, that is, momentum conservation for $r$-direction, is calculated as

$$
\rho_0 \ h_x \ u^r \ \frac{\partial u^r}{\partial t} + \rho_0 h_x \ u^r \ \frac{\partial u^r}{\partial x^i} + \rho_0 h_x \ \frac{u^r}{2} \left[ (\ln \gamma_{xx})_x \ (u^r)^2 + 2(\ln \gamma_{xx})_x \ u^r u^\theta - (\ln \gamma_{xx})_x \ \frac{\gamma_{ox}}{\gamma_{xx}} (u^r)^2 \right] - \frac{\rho_0 h_x}{\gamma_{xx}} \left( \frac{(ln \gamma_{xx})_x (\Omega + \beta^\theta)^2 + 2\gamma_{ox} \beta_{,x}^\theta (\Omega + \beta^\theta) - 2\alpha^2 (\ln \alpha)_x}{\gamma_{xx}} + \frac{1}{\gamma_{xx}} \ \frac{\partial P_x}{\partial \theta} + u^\theta \left( u^r \ \frac{\partial P_x}{\partial \theta} + u^r \ \frac{\partial P_x}{\partial x^i} \right) \right] = -\alpha^2 u^r u^\theta G^1 + \gamma_{ox} u^r u^\theta G^1 + \left[ 1 + \gamma_{ox} (u^r)^2 \right] G^\theta + \gamma_{ox} (\Omega + \beta^\theta) (G^\theta + \beta^\theta G^1) u^r u^\theta. $$

The angular velocity $\Omega_\pm$ is the generalization of the Keplerian angular velocity $\Omega_\pm^\circ = \pm m^{1/2}/(r^{1/2} + a m^{1/2})$, and in the limit of $\theta = \pi/2$, we obtain $\Omega_\pm^\circ = \Omega_\pm^\theta$. In the right-hand side of equation (37), the terms not including the derivatives represent the gravitational acceleration and the centrifugal acceleration, respectively.

The Euler equation in $\theta$-direction, that is, momentum conservation in $\theta$-direction is calculated as

$$
\rho_0 \ h_x \ u^\theta \ \frac{\partial u^\theta}{\partial t} + \rho_0 h_x \ u^\theta \ \frac{\partial u^\theta}{\partial x^i} + \rho_0 h_x \ \frac{u^\theta}{2} \left[ (\ln \gamma_{xx})_x \ (u^\theta)^2 + 2(\ln \gamma_{xx})_x \ u^r u^\theta - (\ln \gamma_{xx})_x \ \frac{\gamma_{ox}}{\gamma_{xx}} (u^r)^2 \right] - \frac{\rho_0 h_x}{\gamma_{xx}} \left( \frac{(ln \gamma_{xx})_x (\Omega + \beta^\theta)^2 + 2\gamma_{ox} \beta_{,x}^\theta (\Omega + \beta^\theta) - 2\alpha^2 (\ln \alpha)_x}{\gamma_{xx}} + \frac{1}{\gamma_{xx}} \ \frac{\partial P_x}{\partial \theta} + u^\theta \left( u^r \ \frac{\partial P_x}{\partial \theta} + u^r \ \frac{\partial P_x}{\partial x^i} \right) \right] = -\alpha^2 u^r u^\theta G^1 + \gamma_{ox} u^r u^\theta G^1 + \left[ 1 + \gamma_{ox} (u^r)^2 \right] G^\theta + \gamma_{ox} (\Omega + \beta^\theta) (G^\theta + \beta^\theta G^1) u^r u^\theta. $$

For the local energy, conservation is obtained from $u_\theta \ T_{\theta}^{\nu} = u_\theta G^\nu$ calculated as

$$
-\nu_\theta \ \frac{\partial}{\partial t} \left( \frac{\rho h_x}{n} \right) - nu^r \ \frac{\partial}{\partial x^i} \left( \frac{\rho h_x}{n} \right) + u^\theta \ \frac{\partial P_x}{\partial t} + u^\theta \ \frac{\partial P_x}{\partial x^i} = -\alpha^2 u^r u^\theta G^1 + \gamma_{ox} u^r u^\theta G^1 + \gamma_{ox} u^r u^\theta G^1 + \gamma_{ox} (\Omega + \beta^\theta) (G^\theta + \beta^\theta G^1) u^r u^\theta. $$

The right-hand side of equation (41) is also calculated by the heating and the cooling function defined in the comoving frame as $u_\theta G^\nu = -G^\nu = \Lambda - \Gamma$. 

3.4.3 Radiation moment equations

The radiation moment equation $R_{\theta}^{\mu} = -G^\nu$ gives the equation for the energy density, the radiation flux, and the radiation pressure tensor. The radiation energy equation obtained from $t$-component of this equation $R_{\theta}^{\theta} = -G^\theta$ is calculated as

$$
\frac{\partial R^\theta}{\partial t} + \frac{1}{\alpha \sqrt{\gamma}} \left\{ \frac{\partial}{\partial \varphi} \left( \alpha \sqrt{\gamma} \ R^\theta \right) + \frac{\partial}{\partial \varphi} \left( \alpha \sqrt{\gamma} \ R^{\theta} \right) \right\} + \frac{\partial R^\phi}{\partial \phi} + \left( g^{\mu \beta_{,x}} + g^{\phi \beta_{,x}} \right) R^\theta + \left( g^{\mu \beta_{,x}} + g^{\phi \beta_{,x}} \right) R^\phi = -G^\theta. $$
The radiation momentum equation in $r$-direction $R^r_r = -G^r$ is calculated as
\[
\frac{\partial R^r}{\partial t} + \frac{1}{\alpha \sqrt{\gamma}} \left[ \frac{\partial}{\partial r} (\alpha \sqrt{\gamma} R^r_r) + \frac{\partial}{\partial \theta} (\alpha \sqrt{\gamma} R^r_{\theta}) \right] + \frac{\partial R^\phi}{\partial \phi} + \frac{1}{2 \sqrt{\gamma}} \left( g_{r,rr} R^r_r + 2 g_{r,\theta} R^\phi_r + g_{\phi,\phi} R^\phi_r \right) = -G^r. \tag{44}
\]
In the similar manner, the radiation momentum equation in $\theta$-direction $R^\theta_r = -G^\theta$ is calculated as
\[
\frac{\partial R^\theta}{\partial t} + \frac{1}{\alpha \sqrt{\gamma}} \left[ \frac{\partial}{\partial r} (\alpha \sqrt{\gamma} R^\theta_r) + \frac{\partial}{\partial \theta} (\alpha \sqrt{\gamma} R^\theta_{\theta}) \right] + \frac{\partial R^r}{\partial \phi} + \frac{1}{2 \sqrt{\gamma}} \left( g_{\theta,rr} R^r_r + 2 g_{\theta,\theta} R^\phi_r + g_{\phi,\phi} R^\phi_r \right) = -G^\theta. \tag{45}
\]
The radiation momentum equation in $\phi$-direction $R^\phi_r = -G^\phi$ is calculated as
\[
\frac{\partial R^\phi}{\partial t} + \frac{1}{\alpha \sqrt{\gamma}} \left[ \frac{\partial}{\partial r} (\alpha \sqrt{\gamma} R^\phi_r) + \frac{\partial}{\partial \theta} (\alpha \sqrt{\gamma} R^\phi_{\theta}) \right] + \frac{\partial R^r}{\partial \phi} + \frac{1}{2 \sqrt{\gamma}} \left( g_{\phi,rr} R^r_r + 2 g_{\phi,\theta} R^\phi_r + g_{\phi,\phi} R^\phi_r \right) = -G^\phi. \tag{46}
\]
Finally, by inserting the components of the radiation stress tensor given by equation (22) into equations (43), (44), (45) and (46), we obtain the radiation moment equations corresponding to equations (49), (52), (53) and (54) in Park (2006) as
\[
\frac{\partial}{\partial t} \left( \frac{\dot{E}}{\alpha \gamma_r} \right) + \frac{1}{\alpha \sqrt{\gamma}} \frac{\partial}{\partial r} \left( \sqrt{\gamma} \dot{E} \right) + \frac{1}{\alpha \sqrt{\gamma}} \frac{\partial}{\partial \theta} \left( \sqrt{\gamma} \dot{E} \right) + \frac{\partial}{\partial \phi} \left[ \frac{\dot{\rho}}{\alpha \gamma_r} \right] = -G^r. \tag{47}
\]
\[
\frac{\partial}{\partial t} \left( \frac{\dot{F}^r}{\alpha \gamma_r} \right) + \frac{1}{\alpha \sqrt{\gamma}} \frac{\partial}{\partial r} \left( \alpha \sqrt{\gamma} \dot{F}^r \right) + \frac{1}{\alpha \sqrt{\gamma}} \frac{\partial}{\partial \theta} \left( \alpha \sqrt{\gamma} \dot{F}^r \right) + \frac{\partial}{\partial \phi} \left[ \frac{\dot{\rho}}{\alpha \gamma_r} \right] = -G^\theta. \tag{48}
\]
\[
\frac{\partial}{\partial t} \left( \frac{\dot{F}^\phi}{\alpha \gamma_r} \right) + \frac{1}{\alpha \sqrt{\gamma}} \frac{\partial}{\partial r} \left( \alpha \sqrt{\gamma} \dot{F}^\phi \right) + \frac{1}{\alpha \sqrt{\gamma}} \frac{\partial}{\partial \theta} \left( \alpha \sqrt{\gamma} \dot{F}^\phi \right) + \frac{\partial}{\partial \phi} \left[ \frac{\dot{\rho}}{\alpha \gamma_r} \right] = -G^\phi. \tag{49}
\]
where the derivatives for $\ln \alpha$, $\beta^\phi$, $\ln \gamma_{rr}$, $\ln \gamma_{\theta \theta}$ and $\ln \gamma_{\phi \phi}$ with respect to $r$ and $\theta$ are given in Appendix A.

4 CONCLUDING REMARKS

As is well known, the moment equations which are truncated at the finite order do not constitute a complete system of equations, that is, the number of equations is smaller than the number of variables to be solved. Thus, we need the additional equations to close the system of the equations (Chandrasekhar 1960; Mihalas 1970; Pomraning 1973; Rybicki & Lightman 1979; Mihalas & Mihalas 1984; Shu 1991) such as the Eddington factors. In the diffusion limit where the photon mean-free path is much smaller than the characteristic length of the system, that is, in the optically thick region, the so-called Eddington approximation $P^\phi = (\delta^\phi / 3)E$ is valid. On the other hand, in the optically thin region, the radiation becomes anisotropic in general. So, in such case, the full angle-dependent radiative transfer equations should be solved.
and sometimes the variable Eddington factors are introduced. Some authors have expressed the variable Eddington factors as a function of the optical depth. Additionally, it is pointed out that the influence of the flow velocity should also be taken into account when constructing the closure relations. These problems and their calculation methods are extensively studied in the past studies (Auer & Mihalas 1970; Hummer & Rybicki 1971; Tamazawa et al. 1975; Mihalas 1980; Mihalas & Mihalas 1984; Schinder & Bludman 1989; Nobili, Turolla & Zampieri 1993; Yin & Miller 1995; Kato, Fukue & Mineshige 1998; Fukue 2006). The similar analysis for the closure relations will be studied in the case of the Kerr space–time in future. In this study, we use the Boyer–Lindquist coordinate which is a frequently used coordinate in the astrophysics around a rotating black hole. However, there is a coordinate singularity at the horizon in this coordinate. Future studies also resolve this singularity by using another coordinate with no singularity at the horizon (see e.g. Takahashi 2007).

In this study, we have derived the radiation hydrodynamic equation in the Kerr space–time. While the interactions between the matter and the radiation are defined and calculated in the comoving frame, the derivatives which are used to describe the global evolutions of both the matter and the radiation are calculated in the BLF. Both the matter and the radiation are influenced by the frame dragging due to the black hole’s rotation. In this approach, as a fixed orthonormal frame, we use the so-called locally LNRF which is first calculated in the Kerr space–time by Bardeen et al. (1972) by using the Boyer–Lindquist coordinate. The special relativistic effects such as beaming effects are introduced by the Lorentz transformation between the LNRF and the comoving frame. On the other hand, the general relativistic effects such as frame-dragging effects due the black hole’s rotation and the gravitational redshift are introduced by the tetrads describing the BLF.

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REFERENCES

Anderson J. L., Spiegel E. A., 1972, ApJ, 171, 127
Anile A. M., Romano V., 1992, ApJ, 386, 325
Auer L. H., Mihalas D., 1970, MNRAS, 149, 65
Bardeen J. M., Press W. H., Teukolsky S. A., 1972, ApJ, 178, 347
Cardall C. Y., Mezzacappa A., 2003, Phys. Rev. D, 68, 023006
Chandrasekhar S., 1960, Radiative Transfer. Dover Publishing, Inc., New York
Flammang R. A., 1982, MNRAS, 199, 833
Flammang R. A., 1984, MNRAS, 206, 589
Frolov, V. P., Novikov, I. D., 1998, Black Hole Physics: Basic Concepts and New Developments. Kluwer Academic, Dordrecht
Fukue J., 2006, PASJ, 58, 461
Gammie C., Popham R., 1998, ApJ, 498, 313
Hummer D. G., Rybicki G. B., 1971, MNRAS, 152, 1
Kato S., Fukue J., Mineshige S., 1998, Black-Hole Accretion Disks. Kyoto Univ. Press, Kyoto
Lindquist R. W., 1966, Ann. Phys., 37, 487
Mihalas D., 1970, Stellar Atmosphere. San Francisco, W. H. Freeman and Co., San Francisco
Mihalas D., 1980, ApJ, 237, 574
Miharas D., Mihalas B. W., 1984, Foundations of Radiation Hydrodynamics. Oxford Univ. Press, Oxford
Nobili L., Turolla R., Zampieri L., 1993, ApJ, 404, 686
Park M.-G., 1993, A&A, 274, 642
Park M.-G., 2006, MNRAS, 367, 1739
Park M.-G., Miller G. S., 1991, ApJ, 371, 708
Pomrnaning G. C., 1973, The Equations of Radiation Hydrodynamics. Pergamon Press, Oxford, New York
Rybicki G. B., Lightman A. P., 1979, Radiative Processes in Astrophysics. Wiley, New York
Schinder P. J., 1988, Phys. Rev. D, 38, 1673
Schinder P. J., Bludman S. A., 1989, ApJ, 346, 350
Schmid-Burgk J., 1978, Ap&SS, 56, 191
Shu, F. H., 1991, The Physics of Astrophysics Vol. 1: Radiation. University Science Books, California
Takahashi R., 2007, MNRAS, in press (astro-ph/07050048)
Tamazawa S., Toyama K., Kaneko N., Ono Y., 1975, Ap&SS, 32, 403
Thorne K. S., 1981, MNRAS, 194, 439
Turolla R., Nobili L., 1988, MNRAS, 235, 1273
Yin W.-W., Miller G. S., 1995, ApJ, 449, 826

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APPENDIX A: METRIC COMPONENTS AND DIFFERENTIAL VALUES

Here, we present the explicit expressions for the metric components and their differential values with respect to $r$ and $\theta$ used in this paper. Non-zero components of the metric are given as

\begin{equation}
 g_{rr} = -2r^2 + \beta_r \beta^r = - \left( 1 - \frac{2mr}{\Sigma} \right), \quad g_{\phi \phi} = \beta_{\phi \phi} = - \frac{2mar^2 \sin^2 \theta}{\Sigma}, \quad g_{\phi \theta} = \gamma_{\phi \theta} = \frac{A \sin^2 \theta}{\Sigma},
\end{equation}

and

\begin{equation}
 g_{\gamma \gamma} = - \frac{1}{\alpha^2} = - \frac{A}{\Sigma \Delta}, \quad g_{\phi \phi} = \gamma_{\phi \phi} = \frac{2mar}{\Sigma \Delta}, \quad g_{\phi \phi} = \gamma_{\phi \phi} = \frac{1}{\alpha^2} \left[ \frac{1}{\sin^2 \theta} \right],
\end{equation}

The differential values of the metric used in this paper are given as

\begin{equation}
 (\ln \alpha)_r = \frac{1}{2} \left[ (\ln \Sigma)_r + (\ln \Delta)_r - (\ln A)_r \right], \quad (\ln \beta_\phi)_r = - \omega \frac{1}{r} - (\ln A)_r,
\end{equation}

\begin{equation}
 (\ln \gamma_{rr})_r = (\ln \Sigma)_r + (\ln \Delta)_r, \quad (\ln \gamma_{\phi \phi})_r = (\ln \Sigma)_r, \quad (\ln \gamma_{\phi \theta})_r = (\ln A)_r - (\ln \Sigma)_r,
\end{equation}

\begin{equation}
 (\ln \alpha)_\theta = \frac{1}{2} \left[ (\ln \Sigma)_\theta + (\ln \Delta)_\theta - (\ln A)_\theta \right], \quad (\ln \beta_\phi)_\theta = \omega (\ln A)_\theta,
\end{equation}

\begin{equation}
 (\ln \gamma_{rr})_\theta = (\ln \Sigma)_\theta + (\ln \Delta)_\theta, \quad (\ln \gamma_{\phi \phi})_\theta = (\ln \Sigma)_\theta, \quad (\ln \gamma_{\phi \theta})_\theta = (\ln A)_\theta + \frac{2}{\tan \theta} - (\ln \Sigma)_\theta,
\end{equation}

where

\begin{equation}
 (\ln \Sigma)_r = \frac{2r}{\Sigma}, \quad (\ln \Delta)_r = \frac{2(r - m)}{\Delta}, \quad (\ln A)_r = \frac{2}{A} \left[ (r - m) \Sigma + (r + m)(r^2 + a^2) \right],
\end{equation}

\begin{equation}
 (\ln \Sigma)_\theta = - a^2 \sin 2\theta, \quad (\ln \Delta)_\theta = 0, \quad (\ln A)_\theta = - \frac{a^2 \Delta}{A} \sin 2\theta.
\end{equation}

Here, we also give the differential values of the metric as

\begin{equation}
 g_{rr} = \frac{2m}{\Sigma} \left( 1 - \frac{2r^2}{\Sigma} \right), \quad g_{\phi \phi} = \frac{2ma \sin^2 \theta}{\Sigma} \left( \frac{2r^2}{\Sigma} - 1 \right), \quad g_{\phi \theta} = 2 \sin^2 \theta \left[ r - m + \frac{(3r^2 + a^2)m}{\Sigma} - \frac{2mr^2(r^2 + a^2)}{\Sigma^2} \right],
\end{equation}

\begin{equation}
 g_{\phi \theta} = \frac{2}{\Delta} \left[ r - (r - m) \Sigma \right], \quad g_{\phi \theta} = 2r.
\end{equation}

and

\begin{equation}
 g_{rr} = \frac{2ma r}{\Sigma^2} \sin 2\theta, \quad g_{\phi \phi} = - \frac{2mar(r^2 + a^2)}{\Sigma^2} \sin 2\theta, \quad g_{\phi \theta} = \frac{\Delta}{\Sigma^2} \left[ \Delta + \frac{2mr^2(r^2 + a^2)}{\Sigma^2} \sin 2\theta \right],
\end{equation}

\begin{equation}
 g_{\phi \theta} = - \frac{a^2}{\Delta} \sin 2\theta, \quad g_{\phi \theta} = - a^2 \sin 2\theta.
\end{equation}

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