DYNAMICAL SYSTEMS WITH A PRESCRIBED GLOBALLY BP-ATTRACTING SET AND APPLICATIONS TO CONSERVATIVE DYNAMICS

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Abstract. Given an arbitrary fixed closed subset $C \subset \mathbb{R}^n$, we provide an explicit method to construct a dynamical system which admits the regular part of $C$ as globally bp-attracting set, i.e., a closed and invariant set which attracts every bounded positive orbit of the dynamical system. As application, we provide an explicit method of leafwise asymptotic bp-stabilization of the regular part of an a-priori given invariant set of a conservative system. The theoretical results are illustrated for the completely integrable case of the Rössler dynamical system.

1. Introduction. This paper is concerned with an inverse problem of smooth dynamical systems, regarding a class of attracting sets, namely the so-called globally bp-attracting sets. The theory of attractors of finite or infinite dimensional dynamical systems is an intensively studied domain (see e.g. [9], [2], and the references therein), with direct applications in all sciences using mathematical models described by dynamical systems. One of the problems in the study of attractors is related to their shape (see e.g. [3], [9], and the references therein). Together with the analysis of possible shapes of an attractor, it comes natural the associated inverse problem, studied e.g. in [4].

The aim of this article is to analyze the inverse problem of finite dimensional smooth dynamical systems regarding the globally bp-attracting sets. More precisely, given a nonempty closed subset of $\mathbb{R}^n$ defined as $\Sigma_{d_1, \ldots, d_p} := D^{-1}(\{(d_1, \ldots, d_p)\}) \subset U \subseteq \mathbb{R}^n$ (where $p$ is some natural number, $1 \leq p \leq n$, $U \subseteq \mathbb{R}^n$ is an open subset, $D := (D_1, \ldots, D_p) : U \to \mathbb{R}^p$ is a smooth function, and $(d_1, \ldots, d_p) \in \mathbb{R}^p$ is some point in the image of $D$) we construct a smooth vector field $X$, defined on the open set $\text{Mrk}(D) \subseteq U$ (consisting of the maximal rank points of $D$), such that $\Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)$, if not empty, is a globally bp-attracting set of $X$ relative to $\text{Mrk}(D)$. Moreover, in the case when $(d_1, \ldots, d_p)$ is a regular value of $D$, then the same construction provides a smooth vector field $X$ such that $\Sigma_{d_1, \ldots, d_p}$ is a globally bp-attracting set of $X$ relative to $\text{Mrk}(D)$. Note that the above results are valid for

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an arbitrary closed proper subset of $\mathbb{R}^n$, as any closed proper subset of $\mathbb{R}^n$ might be written as a level set of some smooth function (see e.g. [7]).

As application, we shall provide an explicit method of leafwise asymptotic bp-stabilization of the regular part of an a-priori given invariant set of a conservative system. More precisely, given a conservative $n$-dimensional dynamical system (i.e. a dynamical system which admits a $(k+p)$-dimensional vector type first integral, where $k+p < n$) and an invariant set $\mathcal{S}$ (given as a level set of a $p$-dimensional first integral defined by some $p$-dimensional projection of the initial $(k+p)$-dimensional first integral), we construct a curve of dynamical systems starting from the original system, such that each system on this curve is still conservative (admitting the $k$-dimensional first integral which together with the $p$-dimensional first integral, forms the initial $(k+p)$-dimensional first integral), keeps invariant the set $\mathcal{S} \cap \text{Mrk}$ (where Mrk is the open set consisting of the points where the rank of the $(k+p)$-dimensional first integral is maximal), and moreover, the intersection of $\mathcal{S} \cap \text{Mrk}$ with each level set (corresponding to regular values) of the $k$-dimensional first integral, is a (leafwise) globally bp-attracting set of each system on the curve (except for the original system) restricted to the corresponding level set. In the non-conservative case, a similar approach was used in [12], in order to globally asymptotically bp-stabilize a given closed invariant set of a finite dimensional dynamical system generated by a smooth vector field. The theoretical results obtained in this section are illustrated for the completely integrable case of the Rössler dynamical system.

The structure of the article is as follows: in the second section we recall a result from [11] which provides an explicit method to construct the class of smooth vector fields (defined on a smooth Riemannian manifold) which admit an a-priori given set of first integrals and, in the same time, dissipate a given set of scalar quantities, with a-priori defined dissipation rates. The third section represents the main part of this work, and gives an explicit method to construct a dynamical system which admits an a-priori defined globally bp-attracting set. The only requirement needed in order to construct the vector field associated to a given closed proper subset of $\mathbb{R}^n$, is to know a representation of this set as a level set of some smooth function. The fourth section presents an application of the results given in the previous section, in order to construct perturbations of conservative dynamical systems, which admit an a-priori prescribed leafwise globally bp-attracting set. More precisely, let $\mathfrak{S}$ be a given dynamical system (defined on an open subset $U \subseteq \mathbb{R}^n$) which admits $k+p$ smooth first integrals, $I_1, \ldots, I_k, D_1, \ldots, D_p$ (or equivalently, it admits two vector type first integrals, $I := (I_1, \ldots, I_k)$ and $D := (D_1, \ldots, D_p)$). Let $\Sigma_{D_1, \ldots, D_p}$ be a dynamically invariant set of $\mathfrak{S}$, given by the level set of the vector type first integral $D$ corresponding to some (regular or singular) value $d := (d_1, \ldots, d_p) \in \text{Im}(D)$. Starting with these data, we construct a family of smooth dynamical systems $\{\mathfrak{S}_\lambda\}_{\lambda > 0}$ (defined on the open subset $\text{Mrk}((D,I)) \subseteq U$ consisting of the points of maximum rank of the smooth function $(D,I)$), such that $\mathfrak{S}_0 = \mathfrak{S}|_{\text{Mrk}((D,I))}$, and for all $\lambda > 0$, the associated dynamical system, $\mathfrak{S}_\lambda$, admits also the vector type first integral $I|_{\text{Mrk}((D,I))}$, keeps dynamically invariant the set $\Sigma_{D_1, \ldots, D_p} \cap \text{Mrk}((D,I))$ and moreover, the invariant set $\Sigma_{D_1, \ldots, D_p} \cap (I|_{\text{Mrk}((D,I))})^{-1}(\{\mu\})$ (if not empty) is a globally bp-attracting set of $\mathfrak{S}_\lambda|_{(I|_{\text{Mrk}((D,I))})^{-1}(\{\mu\})}$, for every regular value $\mu \in \text{Im}(I|_{\text{Mrk}((D,I))})$. In particular, if $\mu$ is a regular value of $I|_{\text{Mrk}((D,I))}$ such that
the intersection of some connected component of \((I|\text{Mrk}(\mathcal{D}, I))^{-1}(\{\mu\})\) with the invariant set \(\Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(\mathcal{D}, I)\), contains a single orbit/cycle of the dynamical system \(\mathcal{S}\), e.g. equilibrium point, periodic orbit, homoclinic or heteroclinic cycle (if any such \(\mu\) exists), then this orbit/cycle preserves its nature as an orbit/cycle of the dynamical system \(\mathcal{S}_\lambda\) (for each \(\lambda > 0\)), and moreover it attracts every bounded positive orbit of the dynamical system \(\mathcal{S}_\lambda|\{I|\text{Mrk}(\mathcal{D}, I))^{-1}(\{\mu\})\) sharing the same connected component. In the case of an equilibrium state, this becomes asymptotically stable, as an equilibrium of the dynamical system \(\mathcal{S}_\lambda|\{I|\text{Mrk}(\mathcal{D}, I))^{-1}(\{\mu\})\). In the last section we illustrate the theoretical results obtained in the fourth section, for the completely integrable case of the Rössler dynamical system.

2. Dynamical systems with prescribed conserved and dissipated scalar quantities. In this section we recall a result from [11] which provides a constructive method to obtain the class of smooth vector fields defined on a smooth Riemannian manifold, which admits an a-priori given set of first integrals and, in the same time, dissipate a given set of scalar quantities, with a-priori defined dissipation rates.

More precisely, we have the following result, which is the key ingredient to obtain the main results of this article.

**Theorem 2.1** ([11]). Let \((M, g)\) be an \(n\)-dimensional smooth Riemannian manifold, and fix \(k, p \in \mathbb{N}\) two natural numbers such that \(p > 0\) and \(0 < k + p \leq n\). Let \(h_1, \ldots, h_p \in C^\infty(U, \mathbb{R})\) be a given set of smooth functions defined on an open subset \(U \subseteq M\), and respectively let \(I_1, \ldots, I_k, D_1, \ldots, D_p \in C^\infty(U, \mathbb{R})\) be given, such that

\[
\{\nabla_g I_1, \ldots, \nabla_g I_k, \nabla_g D_1, \ldots, \nabla_g D_p\} \subset \mathcal{X}(U)
\]

forms a set of pointwise linearly independent vector fields on \(U\).

Then the set of solutions \(X \in \mathcal{X}(U)\) of the system

\[
\begin{align*}
\mathcal{L}_X I_1 &= \cdots = \mathcal{L}_X I_k = 0, \\
\mathcal{L}_X D_1 &= h_1, \ldots, \mathcal{L}_X D_p = h_p,
\end{align*}
\]

forms the affine distribution

\[
\mathcal{A}[X_0; \nabla_g I_1, \ldots, \nabla_g I_k, \nabla_g D_1, \ldots, \nabla_g D_p]
\]

\[
= X_0 + \mathcal{X}[\nabla_g I_1, \ldots, \nabla_g I_k, \nabla_g D_1, \ldots, \nabla_g D_p],
\]

locally generated by the set of vector fields

\[
\{X_0\} \bigcup \left\{ \left. \left( \sum_{i=1, i \neq a}^{n-(k+p)} Z_i \bigwedge \bigwedge_{j=1}^p \nabla_g D_j \wedge \bigwedge_{l=1}^k \nabla_g I_l \right) : a \in \{1, \ldots, n-(k+p)\} \right\}
\]

where

\[
X_0 = \left| \bigwedge_{i=1}^p \nabla_g D_i \wedge \bigwedge_{j=1}^k \nabla_g I_j \right|^{-2} \cdot \sum_{i=1}^p (-1)^{n-i} h_i \Theta_i,
\]

\[
\Theta_i = \left| \bigwedge_{j=1, j \neq i}^p \nabla_g D_j \wedge \bigwedge_{l=1}^k \nabla_g I_l \wedge \left( \bigwedge_{j=1}^p \nabla_g D_j \wedge \bigwedge_{l=1}^k \nabla_g I_l \right) \right|,
\]

and respectively the set of locally defined vector fields

\[
\{\nabla_g I_1, \ldots, \nabla_g I_k, \nabla_g D_1, \ldots, \nabla_g D_p, Z_1, \ldots, Z_{n-(k+p)}\}
\]
forms a moving frame. The notation * stands for the Hodge star operator on multivector fields, and $\mathcal{L}_X F := g(\nabla g F, X)$ stands for the Lie derivative of the smooth function $F \in C^\infty(U, \mathbb{R})$ along the vector field $X$.

Note that in contrast with the vector fields $\nabla g I_1, \ldots, \nabla g I_k, \nabla g D_1, \ldots, \nabla g D_p$, which are globally defined on $U$, the vector fields $Z_1, \ldots, Z_{n-(k+p)}$ exist in general only locally around each point $x \in U$, in some open neighborhood $U_x \subseteq U$. Nevertheless, the equations (1) have a globally defined particular solution in $U$, given by the vector field $X_0$. Moreover, if $X$ is a vector field which conserves $I_1, \ldots, I_k, D_1, \ldots, D_p$ (i.e. $X$ is a solution of the homogeneous system associated to (1)), then $X + X_0$ is a global solution of (1). More precisely, we have the following result from [11].

**Theorem 2.2 ([11]).** Let $\dot{x} = X(x)$ be the dynamical system generated by a vector field $X \in \mathfrak{X}(U)$ which conserves the smooth functions $I_1, \ldots, I_k, D_1, \ldots, D_p \in C^\infty(U, \mathbb{R})$. Assume that $\nabla g I_1, \ldots, \nabla g I_k, \nabla g D_1, \ldots, \nabla g D_p$ are pointwise linearly independent on some open subset $V \subseteq U$.

Then the perturbed dynamical system

$$\dot{x} = X(x) + X_0(x), \quad x \in V,$$

with $X_0 \in \mathfrak{X}(V)$ given in Theorem 2.1, is a dissipative dynamical system, generated by the dissipative vector field $X + X_0 \in \mathfrak{X}(V)$ which conserves $I_1, \ldots, I_k$ (i.e. $\mathcal{L}_{X+X_0} I_1 = \cdots = \mathcal{L}_{X+X_0} I_k = 0$) and dissipates $D_1, \ldots, D_p$ with (corresponding) dissipation rates $h_1, \ldots, h_p$ (i.e. $\mathcal{L}_{X+X_0} D_1 = h_1, \ldots, \mathcal{L}_{X+X_0} D_p = h_p$).

3. **Dynamical systems with a prescribed globally bp-attracting set.** This section is the main part of this paper and gives an explicit method to construct a dynamical system which admits an a-priori defined globally bp-attracting set. The only requirement needed in order to construct the vector field associated to a given closed proper subset of $\mathbb{R}^n$, is to know a representation of this set as the level set of some smooth function. As any closed proper subset of $\mathbb{R}^n$ might be expressed as a level set of some smooth function (see e.g. [7]), this method makes possible the explicit construction of a vector field which have as a globally bp-attracting set, a general a-priori prescribed closed proper subset of $\mathbb{R}^n$.

Let us start by fixing a nonempty closed subset of $\mathbb{R}^n$ given by

$$\Sigma_{d_1, \ldots, d_p} := D^{-1}(\{(d_1, \ldots, d_p)\}) \subset U \subseteq \mathbb{R}^n,$$

where $p$ is a natural number, $1 \leq p \leq n$, $U \subseteq \mathbb{R}^n$ is an open subset, $D := (D_1, \ldots, D_p) : U \to \mathbb{R}^p$ is a smooth function, and $(d_1, \ldots, d_p) \in \mathbb{R}^p$ is some point in the image of $D$. Note that if $(d_1, \ldots, d_p) \in \mathbb{R}^p$ is a regular value of $D$, then $\Sigma_{d_1, \ldots, d_p}$ is a smooth $(n-p)$-dimensional submanifold of $\mathbb{R}^n$, and hence for every $x \in \Sigma_{d_1, \ldots, d_p}$, the vectors $\nabla D_1(x), \ldots, \nabla D_p(x)$ are linearly independent, where $\nabla$ stands for the gradient operator associated with respect to the canonical inner product on $\mathbb{R}^n$. By Sard’s theorem we know that almost all points in the image of $D$ are regular values, i.e. the set of singular values of $D$ is a set of Lebesgue measure zero in $\mathbb{R}^p$. Let us denote by $\text{Mrk}(D) \subseteq U$ the set of maximal rank points of $D$, i.e. the points $x \in U$ such that the vectors $\nabla D_1(x), \ldots, \nabla D_p(x)$ are linearly independent. Recall that $\text{Mrk}(D)$ is an open subset of $U$ contained in the set of regular points of $D$. Recall that a point $x_0 \in U$ is a regular point of $D$ if there exists an open neighborhood $U_{x_0} \subseteq U$ such that $\text{rank}(dD(x)) = \text{rank}(dD(x_0))$, for
all \( x \in U_{x_0} \). Recall also that the set of regular points of \( D \) is an open dense subset of \( U \) in contrast with \( \text{Mrk}(D) \) which is open but not necessarily dense. The rank of \( dD(\cdot) \) is constant on each connected component of the set of regular points of \( D \). Concerning the set \( \Sigma_{d_1,\ldots,d_p} \), if \( (d_1,\ldots,d_p) \) is a regular value of \( D \), then \( \Sigma_{d_1,\ldots,d_p} \subseteq \text{Mrk}(D) \).

Before stating the main theorem of this section, let us recall briefly some terminology and also some classical results we shall need in the sequel. For details see e.g. [6], [5]. In order to do that, let us consider a smooth vector field \( X \in \mathfrak{X}(U) \) defined on an open set \( U \subseteq \mathbb{R}^n \). Then, for each \( \mathfrak{E} \in U \) we shall denote by \( t \in I_{\mathfrak{E}} \subseteq \mathbb{R} \mapsto x(t;\mathfrak{E}) \in U \) the integral curve of \( X \) starting from \( \mathfrak{E} \) at \( t = 0 \), i.e. the solution of the Cauchy problem \( dx/dt = X(x(t)), x(0) = \mathfrak{E} \), defined on the maximal domain \( I_{\mathfrak{E}} \subseteq \mathbb{R} \), where \( I_{\mathfrak{E}} \) is an open interval of \( \mathbb{R} \) containing the origin.

For each \( \mathfrak{E} \in U \) we associate the set \( \mathcal{O}_{\mathfrak{E}}^+ := \{ y \in U : y = x(t;\mathfrak{E}), \ t \geq 0 \} \), called the positive orbit of \( \mathfrak{E} \). Consequently, a subset \( \mathcal{C} \subseteq U \) is called positively invariant if for every \( \mathfrak{E} \in \mathcal{C} \) we have that \( \mathcal{O}_{\mathfrak{E}}^+ \subseteq \mathcal{C} \). If a set \( \mathcal{C} \) is positively invariant, then so are the sets \( \mathcal{C} \) and \( \mathcal{C} \). Let us recall that if \( \mathcal{O}_{\mathfrak{E}}^+ \) is contained in some compact subset of \( U \), then the solution \( x(t;\mathfrak{E}) \) is defined for all \( t \in [0,\infty) \). If one denotes by \( \omega(\mathfrak{E}) := \{ y \in U : \exists (t_n)_{n \in \mathbb{N}} \subset [0,\infty), \ t_n < t_{n+1}, \ t_n \to \infty \text{ s.t.} \ \lim_{n \to \infty} x(t_n;\mathfrak{E}) = y \} \) the \( \omega \)-limit set of \( \mathfrak{E} \), then we have that \( \mathcal{O}_{\mathfrak{E}}^+ = \mathcal{O}_{\mathfrak{E}}^- \cup \omega(\mathfrak{E}) \), and \( \omega(\mathfrak{E}) = \omega(x(t;\mathfrak{E})) \), for all \( t \geq 0 \). Note that for all points \( y \in \omega(\mathfrak{E}) \), we have that \( \mathcal{O}_{y}^- \subseteq \omega(\mathfrak{E}) \), and hence the \( \omega \)-limit set of \( \mathfrak{E} \) can be expressed as \( \omega(\mathfrak{E}) = \bigcap \{ \mathcal{O}_{y}^+ : y \in \mathcal{O}_{\mathfrak{E}}^+ \} \). Moreover, for every \( \mathfrak{E} \in U \) such that the set \( \{ x(t;\mathfrak{E}) : t \geq 0 \} \) is bounded, the associated \( \omega \)-limit set, \( \omega(\mathfrak{E}) \), is a not empty, invariant, compact and connected subset of \( U \), and \( x(t;\mathfrak{E}) \) approaches \( \omega(\mathfrak{E}) \) as \( t \to \infty \), i.e. \( x(t;\mathfrak{E}) \to \omega(\mathfrak{E}) \) as \( t \to \infty \). Recall that given a closed and invariant set \( \mathcal{C} \subseteq U \), we say that the solution starting from a point \( \mathfrak{E} \in U \) approaches the set \( \mathcal{C} \) (and we denote \( x(t;\mathfrak{E}) \to \mathcal{C} \)), if for every \( \varepsilon > 0 \) there exists \( T > 0 \) such that \( \text{dist}(x(t;\mathfrak{E}), \mathcal{C}) < \varepsilon \), for all \( t > T \), where for every point \( p \in U \), \( \text{dist}(p, \mathcal{C}) := \inf_{x \in \mathcal{C}} \text{dist}(p,x) \). In what follows, in order to show that a solution starting from a point \( \mathfrak{E} \in U \) approaches some closed and invariant set \( \mathcal{C} \) as \( t \to \infty \), we shall prove that \( \omega(\mathfrak{E}) \subseteq \mathcal{C} \), and then using the attracting property of an \( \omega \)-limit set, i.e. \( x(t;\mathfrak{E}) \to \omega(\mathfrak{E}) \) as \( t \to \infty \), we get that \( x(t;\mathfrak{E}) \to \mathcal{C} \) as \( t \to \infty \).

**Definition 3.1.** Let \( U \subseteq \mathbb{R}^n \) be an open subset of \( \mathbb{R}^n \) and let \( X \in \mathfrak{X}(U) \) be a smooth vector field. A closed and invariant subset \( \mathcal{A} \subset U \) will be called **globally bp-attracting set** of the dynamical system generated by \( X \) if for every point \( \mathfrak{E} \in U \) such that the set \( \{ x(t;\mathfrak{E}) : t \geq 0 \} \) is bounded, the integral curve of \( X \) starting from \( \mathfrak{E} \) approaches \( \mathcal{A} \) as \( t \to \infty \), i.e. \( x(t;\mathfrak{E}) \to \mathcal{A} \) as \( t \to \infty \).

Note that in contrast to global attractors, a globally bp-attracting set is not required to be compact, and also needs not be connected, its connectivity being related to the connectivity of the open set \( U \). Regarding unbounded attractors, we recall the study of unbounded sets of attraction in the context of iteration of maps, carried out in [1]. As mentioned in [1], unbounded chaotic trajectories are observed in the iteration of maps which are not globally defined (due to the presence of a denominator which vanishes on a zero-measure set).

Next result points out an important property of globally bp-attracting sets.

**Remark 1.** Assume that \( \mathcal{A} \subset U \) is a globally bp-attracting set of the dynamical system generated by a smooth vector field \( X \in \mathfrak{X}(U) \). Then for every positively
invariant compact set \( K \subset U \) (if any), and every point \( \bar{x} \in K \), the integral curve of \( X \) starting from \( \bar{x} \) approaches \( A \) as \( t \to \infty \).

At this point we have all ingredients required in order to start the approach of the main result of this section. In order to do that, pick some nonempty closed subset of \( \mathbb{R}^n \) given as

\[
\Sigma_{d_1, \ldots, d_p} := D^{-1}(\{(d_1, \ldots, d_p)\}) \subset U \subset \mathbb{R}^n,
\]

where \( p \) is some natural number, \( 1 \leq p \leq \mathbb{N} \), \( U \) is an open subset of \( \mathbb{R} \), \( D := (D_1, \ldots, D_p) : U \to \mathbb{R}^p \) is a smooth function, and \( (d_1, \ldots, d_p) \in \mathbb{R}^p \) is some point in the image of \( D \).

Starting with these data, we shall construct a smooth vector field \( X \), defined on the open set \( \text{Merk}(D) \subset U \), such that \( \Sigma_{d_1, \ldots, d_p} \cap \text{Merk}(D) \), if not empty, is a globally bp-attracting set of \( X \), i.e., for every \( \bar{x} \in \text{Merk}(D) \), such that the set \( \{x(t; \bar{x}) : t \geq 0\} \) is bounded, we have that \( x(t; \bar{x}) \to \Sigma_{d_1, \ldots, d_p} \cap \text{Merk}(D) \) as \( t \to \infty \). Note that if \( (d_1, \ldots, d_p) \) is a regular value of \( D = (D_1, \ldots, D_p) \), then \( \Sigma_{d_1, \ldots, d_p} \subset \text{Merk}(D) \), and hence in this case \( \Sigma_{d_1, \ldots, d_p} \) is a globally bp-attracting set of the vector field \( X \).

In order to do that, let us fix a strictly positive real number \( \lambda > 0 \). Then using the Theorem 2.1, we construct a smooth vector field \( X \in \mathfrak{X}(\text{Merk}(D)) \) such that

\[
\mathcal{L}_X D_1 = (-\lambda)(D_1 - d_1), \ldots, \mathcal{L}_X D_p = (-\lambda)(D_p - d_p).
\]

(2)

Note that by construction, \( \Sigma_{d_1, \ldots, d_p} \cap \text{Merk}(D) \) is a dynamically invariant set of \( X \). By Theorem 2.1, a particular solution of the equation (2) with maximal domain of definition, is given by the vector field \( X^\lambda_0 \in \mathfrak{X}(\text{Merk}(D)) \) defined as follows:

\[
X^\lambda_0 = \left\| \bigwedge_{i=1}^p \nabla D_i \right\|_p^{-2} \cdot \sum_{i=1}^p (-1)^{n-i}(-\lambda)(D_i - d_i) \Theta_i,
\]

where

\[
\Theta_i = \bigstar \left[ \bigwedge_{j=1,j\neq i}^p \nabla D_j \bigwedge \bigstar \left( \bigwedge_{j=1}^p \nabla D_j \right) \right].
\]

Let us state now the main result of this section.

**Theorem 3.2.** Let \( C \subset \mathbb{R}^n \) be a nonempty closed subset of \( \mathbb{R}^n \), given as the preimage \( \Sigma_{d_1, \ldots, d_p} := D^{-1}(\{(d_1, \ldots, d_p)\}) \subset U \subset \mathbb{R}^n \) corresponding to some value \( (d_1, \ldots, d_p) \) of a smooth map \( D := (D_1, \ldots, D_p) : U \to \mathbb{R}^p \), \( 1 \leq p \leq \mathbb{N} \), defined on an open subset \( U \subset \mathbb{R}^n \). Assume that \( \Sigma_{d_1, \ldots, d_p} \cap \text{Merk}(D) \neq \emptyset \).

Then for each real number \( \lambda > 0 \), we associate a smooth vector field \( X^\lambda_0 \in \mathfrak{X}(\text{Merk}(D)) \) given by

\[
X^\lambda_0 = \left\| \bigwedge_{i=1}^p \nabla D_i \right\|_p^{-2} \cdot \sum_{i=1}^p (-1)^{n-i}(-\lambda)(D_i - d_i) \Theta_i,
\]

where

\[
\Theta_i = \bigstar \left[ \bigwedge_{j=1,j\neq i}^p \nabla D_j \bigwedge \bigstar \left( \bigwedge_{j=1}^p \nabla D_j \right) \right], \quad i \in \{1, \ldots, p\},
\]
(a) The set of the equilibrium states of the vector field $X_0^λ \in \mathcal{X}(\text{Mrk}(D))$, i.e.
$\mathcal{E}(X_0^λ) := \{x \in \text{Mrk}(D) : X_0^λ(x) = 0\}$, is given by $\mathcal{E}(X_0^λ) = \Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}(D)$.

(b) The vector field $X_0^λ \in \mathcal{X}(\text{Mrk}(D))$ admits the globally bp-attracting set $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}(D)$. More precisely, for every $\overline{x} \in \text{Mrk}(D)$, such that the set $\{x(t;\overline{x}) : t \geq 0\}$ is bounded, $x(t;\overline{x}) \rightarrow \Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}(D)$ as $t \rightarrow \infty$.

Proof. Let us define the smooth function $F : \text{Mrk}(D) \rightarrow [0, \infty)$ given by

$$F(x) := (D_1(x) - d_1)^2 + \cdots + (D_p(x) - d_p)^2, \; \forall x \in \text{Mrk}(D).$$

Let $\overline{x} \in \text{Mrk}(D)$ be given, and let $t \in I_\overline{x} \subseteq \mathbb{R} \mapsto x(t;\overline{x}) \in \text{Mrk}(D)$ be the integral curve of the vector field $X_0^λ \in \mathcal{X}(\text{Mrk}(D))$ such that $x(0;\overline{x}) = \overline{x}$, where $I_\overline{x} \subseteq \mathbb{R}$ stands for the maximal domain of definition of the solution $x(\cdot;\overline{x})$.

Since the vector field $X_0^λ \in \mathcal{X}(\text{Mrk}(D))$ satisfies by construction the relations (2), we have that

$$\begin{align*}
\mathcal{L}_{X_0^λ}F = & \sum_{i=1}^{p} \mathcal{L}_{X_0^λ}(D_i - d_i)^2 = \sum_{i=1}^{p} 2(D_i - d_i)\mathcal{L}_{X_0^λ}(D_i - d_i) \\
= & \sum_{i=1}^{p} 2(D_i - d_i)(-λ)(D_i - d_i) = (-2λ)\sum_{i=1}^{p} (D_i - d_i)^2 \\
= & (-2λ)F.
\end{align*}$$

Using the relation (4), we obtain that

$$\frac{d}{dt} F(x(t;\overline{x})) = (-2λ)F(x(t;\overline{x})), \; \forall t \in I_\overline{x},$$

and hence

$$F(x(t;\overline{x})) = \exp(-2λt) \cdot F(\overline{x}), \; \forall t \in I_\overline{x}. \tag{5}$$

Moreover, since the set of zeros of $F$ coincides with $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}(D)$, the following set equality holds true:

$$\{x \in \text{Mrk}(D) : (\mathcal{L}_{X_0^λ}F)(x) = 0\} = \Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}(D). \tag{6}$$

(a) In order to prove that $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}(D) = \mathcal{E}(X_0^λ)$, note that from the definition of $X_0^λ$ one obtains directly that $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}(D) \subseteq \mathcal{E}(X_0^λ)$. For proving the converse inclusion, let $\overline{x} \in \mathcal{E}(X_0^λ)$ be an equilibrium point of $X_0^λ$. Hence, the associated integral curve satisfies the relation $x(t;\overline{x}) = \overline{x}$, for all $t \in \mathbb{R}$, and so by relation (5) we get that $F(\overline{x}) = 0$. Since $F(\overline{x}) = \sum_{i=1}^{p} (D_i(\overline{x}) - d_i)^2 = 0$, and $\overline{x} \in \text{Mrk}(D)$ by definition, we get that $\overline{x} \in \Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}(D)$, and so as $\overline{x}$ was arbitrary chosen from $\mathcal{E}(X_0^λ)$, we obtain the converse inclusion, i.e. $\mathcal{E}(X_0^λ) \subseteq \Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}(D)$.

(b) In order to prove the second item, let $\overline{x} \in \text{Mrk}(D)$ be an arbitrary point of the open set $\text{Mrk}(D)$ such that the set $\{x(t;\overline{x}) : t \geq 0\}$ is bounded. We shall show now that the $\omega$–limit set $\omega(\overline{x})$ is a subset of $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}(D)$. Indeed, let $y \in \omega(\overline{x})$ be arbitrary chosen. Then, there exists an increasing sequence $(t_n)_{n \in \mathbb{N}} \subset [0, \infty)$, $\lim_{n \rightarrow \infty} t_n = \infty$, such that $\lim_{n \rightarrow \infty} x(t_n;\overline{x}) = y$. Since
the set \{x(t; \bar{x}) : t \geq 0\} is bounded, we get that \([0, \infty) \subset I_{\bar{x}}\), and hence the relation (5) implies that

\[
F(x(t; \bar{x})) = \exp(-2\lambda t) \cdot F(\bar{x}), \ \forall t \in [0, \infty).
\] (7)

Consequently, for \(t = t_n \geq 0, n \in \mathbb{N}\), the equality (7) becomes

\[
F(x(t_n; \bar{x})) = \exp(-2\lambda t_n) \cdot F(\bar{x}), \ \forall n \in \mathbb{N}.
\]

Since \(\lim_{n \to \infty} t_n = \infty, \lambda > 0, \lim_{n \to \infty} x(t_n; \bar{x}) = y, \) and \(F\) is continuous, we obtain that \(F(y) = 0\), and hence taking into account that the set of zeros of \(F\) is \(\Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\), it follows that \(y \in \Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\). As \(y \in \omega(\bar{x})\) was arbitrary chosen, we obtain that \(\omega(\bar{x}) \subseteq \Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\).

Since \(x(t; \bar{x}) \to \omega(\bar{x}) \subseteq \Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\) as \(t \to \infty\), it follows that \(x(t; \bar{x}) \to \Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\) as \(t \to \infty\).

\(\square\)

Using the properties of \(\omega\)-limit sets and the Theorem 3.2 we get the following result.

**Proposition 1.** In the hypothesis of Theorem 3.2, the following assertions hold true:

(a) For every positively invariant compact set \(K \subset \text{Mrk}(D)\) (if any), and for every \(\bar{x} \in K\) we have that \(\omega(\bar{x}) \subseteq \Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\), and consequently \(x(t; \bar{x}) \to \Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\) as \(t \to \infty\).

(b) If \(\Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\) contains isolated points, then each such a point \(\bar{x} \in \Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\) is an asymptotically stable equilibrium point of \(X^\lambda\).

**Proof.** The item (a) follows directly from Theorem 3.2 and the Remark 1. In order to prove the statement (b) it is enough to construct a strict Lyapunov function associated to each isolated equilibrium state \(x_e \in \Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\) (recall from Theorem 3.2 that the set of equilibrium points of \(X^\lambda\) equals to \(\Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\)). In order to do that, pick an isolated point \(x_e \in \Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\). Hence, there exists \(U_{x_e} \subseteq \text{Mrk}(D)\), an open neighborhood of \(x_e\), such that \(U_{x_e} \cap (\Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)) = \{x_e\}\), since \(x_e\) is an isolated point of \(\Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)\).

Let us define now the smooth function \(F : U_{x_e} \to [0, \infty)\) given by

\[
F(x) := (D_1(x) - d_1)^2 + \cdots + (D_p(x) - d_p)^2, \ \forall x \in U_{x_e}.
\]

Since by hypothesis we have that \(U_{x_e} \cap (\Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D)) = \{x_e\}\), and the set of zeros of \(F\) in \(U_{x_e}\) is the set \(U_{x_e} \cap (\Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(D))\), it follows that \(x_e\) is the unique solution of the equation \(F(x) = 0\) in \(U_{x_e}\). Recall from (4) that

\[
(\mathcal{L}X^\lambda_0)F(x) = (-2\lambda)F(x), \ \forall x \in U_{x_e}.
\] (8)

Hence, we get that \(F(x_e) = 0, F(x) > 0, (\mathcal{L}X^\lambda_0)F(x) < 0\), for every \(x \in U_{x_e} \setminus \{x_e\}\), and consequently \(F\) is a strict Lyapunov function associated to the equilibrium point \(x_e\). \(\square\)
Remark 2. Regarding Theorem 3.2 and Proposition 1, in the case when $\Sigma_{d_1,\ldots,d_p}$ is a closed subset of $\mathbb{R}^n$ given by $\Sigma_{d_1,\ldots,d_p} := D^{-1}((d_1,\ldots,d_p)) \subset U \subset \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is an open set, $D := (D_1,\ldots,D_p) : U \to \mathbb{R}^p$ is a smooth function, and $(d_1,\ldots,d_p) \in \mathbb{R}^p$ is a regular value of $D$, it follows that $\Sigma_{d_1,\ldots,d_p} \subset \text{Mrk}(D)$, and hence $\Sigma_{d_1,\ldots,d_p} \cap \text{Mrk}(D) = \Sigma_{d_1,\ldots,d_p}$. Moreover, in this case, $\Sigma_{d_1,\ldots,d_p}$ is a $(n-p)$-dimensional smooth submanifold of $\mathbb{R}^n$.

4. Application to conservative dynamics. The aim of this section is to apply the main results of the above section in order to provide an answer to the following asymptotic stabilization problem: given a conservative $n$-dimensional dynamical system (i.e. a dynamical system which admits a $(k+p)$-dimensional vector type first integral, where $k+p < n$; for a brief introduction see, e.g. [8]) and an invariant set $S$ (given as a level set of a $p$-dimensional first integral defined by some $p$-dimensional projection of the initial $(k+p)$-dimensional first integral), construct a curve of dynamical systems starting from the original system, such that each system on this curve is still conservative (admitting the $k$-dimensional first integral which together with the $p$-dimensional first integral, forms the initial $(k+p)$-dimensional first integral), keeps invariant the set $S \cap \text{Mrk}$ (where Mrk is the open set consisting of the points where the rank of the $(k+p)$-dimensional first integral is maximal), and moreover, the intersection of $S \cap \text{Mrk}$ with each level set (corresponding to regular values) of the $k$-dimensional first integral, is a (leafwise) globally bp-attracting set of each system on the curve (excepting the original system).

More precisely, let $\mathcal{G}$ be a dynamical system generated by a smooth vector field $X \in \mathfrak{X}(U)$ defined on an open subset $U \subset \mathbb{R}^n$, which admits $k+p$ smooth first integrals $(0 < k + p < n, k \geq 0)$, $I_1,\ldots,I_k, D_1,\ldots,D_p \in C^\infty(U,\mathbb{R})$. In order to simplify the notations, we shall denote by $I, D$, and $(D,I)$, the vector type first integrals, $(I_1,\ldots,I_k)$, $(D_1,\ldots,D_p)$, and respectively $(D_1,\ldots,D_p, I_1,\ldots,I_k)$. Let $S$ be a closed invariant set of $\mathcal{G}$, given by the level set of the vector type first integral $D$ corresponding to some (regular or singular) value $d := (d_1,\ldots,d_p) \in \text{Im}(D)$, i.e. $S = \Sigma_{d_1,\ldots,d_p} := D^{-1}((d_1,\ldots,d_p))$.

In this settings, following mimetically the approach given in the previous section, we construct a family of smooth vector fields, $X_0^\lambda, \lambda > 0$ (defined on the open set Mrk($(D, I)) \subset U$ consisting of those points of $U$ such that the rank of the differential of $(D, I) : U \to \mathbb{R}^{p+k}$ evaluated at them is maximal), with properties similar to those of the vector field analyzed in Theorem 3.2.

In order to do that, let us fix a strictly positive real number $\lambda > 0$. Then using the Theorem 2.1, a particular solution of the system of equations

$$\mathcal{L}_X I_1 = \cdots = \mathcal{L}_X I_k = 0, \mathcal{L}_X D_1 = (-\lambda)(D_1 - d_1), \ldots, \mathcal{L}_X D_p = (-\lambda)(D_p - d_p), \quad (9)$$

is given by the vector field $X = X_0^\lambda \in \mathfrak{X}(\text{Mrk}((D, I)))$ defined by

$$X_0^\lambda = \left\| \bigwedge_{i=1}^p \nabla D_i \wedge \bigwedge_{j=1}^k \nabla I_j \right\|^{-2} \sum_{i=p}^{k} (-1)^{n-i} (-\lambda)(D_i - d_i) \Theta_i, \quad (10)$$

where

$$\Theta_i = \star \left( \bigwedge_{j=1, j \neq i}^p \nabla D_j \wedge \bigwedge_{l=1}^k \nabla I_l \wedge \star \left( \bigwedge_{j=1}^p \nabla D_j \wedge \bigwedge_{l=1}^k \nabla I_l \right) \right).$$
Theorem 4.1. Let $X_0^\lambda \in \mathfrak{X}(\text{Mrk}(\{(D, I)\}))$, $\lambda > 0$, be the vector field defined by the relation (10). Assuming that $\Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(\{(D, I)\}) \neq \emptyset$, the following statements hold true.

(a) The set of the equilibrium states of the vector field $X_0^\lambda$, i.e., $E(X_0^\lambda) := \{x \in \text{Mrk}(\{(D, I)\)) : X_0^\lambda(x) = 0\}$, is given by $E(X_0^\lambda) = \Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(\{(D, I)\})$.

(b) For each regular value $\mu$ of $I|_\text{Mrk}(\{(D, I)\})$, the vector field $X_0^\lambda|_{I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})}$ admits the globally bp-attracting set $\Sigma_{d_1, \ldots, d_p} \cap (I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\}))$, if not empty. More precisely, for every $\mathfrak{X} \in (I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})$, such that the set $\{x(t; \mathfrak{X}) : t \geq 0\}$ is bounded, $x(t; \mathfrak{X}) \to \Sigma_{d_1, \ldots, d_p} \cap (I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})$ as $t \to \infty$.

(c) Let $\mu$ be an arbitrary fixed regular value of $I|_\text{Mrk}(\{(D, I)\})$. Then for every positively invariant compact set $K \subset (I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})$ (if any), and for every $\mathfrak{X} \in K$ we have that $\omega(\mathfrak{X}) \subseteq \Sigma_{d_1, \ldots, d_p} \cap (I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})$, and consequently $x(t; \mathfrak{X}) \to \Sigma_{d_1, \ldots, d_p} \cap (I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})$ as $t \to \infty$. Particularly, if $I|_\text{Mrk}(\{(D, I)\})$ is a proper map, then the set $(I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})$, is compact in $\text{Mrk}(\{(D, I)\})$, dynamically invariant, and consequently for every $\mathfrak{X} \in (I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})$, we have that $x(t; \mathfrak{X}) \to \Sigma_{d_1, \ldots, d_p} \cap (I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})$ as $t \to \infty$.

(d) Assume that $\mu$ is a regular value of the vector type first integral $I|_\text{Mrk}(\{(D, I)\})$ such that $\Sigma_{d_1, \ldots, d_p} \cap (I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})$ contains isolated points. Then each such a point $\mathfrak{X} \in \Sigma_{d_1, \ldots, d_p} \cap (I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})$ is an asymptotically stable equilibrium point of $X_0^\lambda|_{I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})}$.

Proof. The proof follows mimetically those of Theorem 3.2 and Proposition 1. □

At this point we have all necessary ingredients to construct a family of smooth dynamical systems $\{\mathfrak{S}_\lambda\}_{\lambda \geq 0}$ (defined on the open subset $\text{Mrk}(\{(D, I)\}) \subseteq U$), such that $\mathfrak{S}_0 = \mathfrak{S}|_\text{Mrk}(\{(D, I)\})$, and for all $\lambda > 0$, the associated dynamical system, $\mathfrak{S}_\lambda$, admits the vector type first integral $I|_\text{Mrk}(\{(D, I)\})$, keeps dynamically invariant the set $\Sigma_{d_1, \ldots, d_p} \cap \text{Mrk}(\{(D, I)\})$ and moreover, the invariant set $\Sigma_{d_1, \ldots, d_p} \cap (I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})$, if not empty, is a globally bp-attracting set of the dynamical system $\mathfrak{S}_\lambda|_{I|_\text{Mrk}(\{(D, I)\})^{-1}(\{\mu\})}$, for every regular value $\mu \in \text{Im}(I|_\text{Mrk}(\{(D, I)\}))$.

The above construction is explicitly given in the following theorem, which is the main result of this section.

Theorem 4.2. Let $X \in \mathfrak{X}(U)$ be a smooth vector field defined on an open subset $U \subseteq \mathbb{R}^n$, which admits $k + p$ smooth first integrals ($0 < k + p < n$, $k \geq 0$), $I_1, \ldots, I_k, D_1, \ldots, D_p \in C^\infty(U, \mathbb{R})$. Let $S \subseteq U$ be a given nonempty closed and invariant set, defined as the level set of the vector type first integral $D :=$
(D_1, \ldots, D_p) \in C^\infty(U, \mathbb{R}^p) corresponding to some value d := (d_1, \ldots, d_p) \in \operatorname{Im}(D), i.e. \mathcal{S} = \sum_{d_1, \ldots, d_p} \forall D^{-1} \{ (d_1, \ldots, d_p) \}.

Then, to each \lambda \geq 0, we associate a smooth vector field, \( X_\lambda := X + X_\lambda \), defined on the open set \( \operatorname{Mrk}(D, I) \subseteq U \), where the smooth vector field \( X_\lambda \in \mathfrak{X}(\operatorname{Mrk}(D, I)) \) is given by

\[
X_\lambda = \left[ \prod_{i=1}^p \nabla D_i \wedge \bigwedge_{j=1}^k \nabla I_j \right]^{-2} \cdot \sum_{i=1}^{p+k} (-1)^{n-i} (-\lambda)(D_i - d_i) \Theta_i,
\]

where

\[
\Theta_i = \left[ \prod_{j=1, j \neq i}^p \nabla D_j \wedge \bigwedge_{l=1}^k \nabla I_l \wedge \left( \prod_{j=1}^p \nabla D_j \wedge \bigwedge_{l=1}^k \nabla I_l \right) \right].
\]

In the above settings, the following assertions hold true.

(a) \( \mathcal{E}(X_\lambda) = \mathcal{E}(X) \cap \mathcal{E}(X_\lambda) = \mathcal{E}(X) \cap \sum_{D_1, \ldots, D_p} \mathcal{Mrk}(D, I), (\mathcal{E}) \) stands for the set of equilibrium points of the vector field \( Z \).

(b) The set \( \sum_{D_1, \ldots, D_p} \mathcal{Mrk}(D, I), \) if not empty, is a closed dynamically invariant set of the vector field \( X_\lambda \), for every \( \lambda \geq 0 \). Moreover, for each regular value \( \mu \) of the vector type first integral

\[
I_{\operatorname{Mrk}(D, I)} := (I_1|_{\operatorname{Mrk}(D, I)}), \ldots, I_K|_{\operatorname{Mrk}(D, I)}),
\]

the set \( \sum_{D_1, \ldots, D_p} \mathcal{Mrk}(D, I) \cap (I_{\operatorname{Mrk}(D, I)})^{-1}(\{ \mu \}) \), if not empty, is a closed and dynamically invariant set of the vector field \( X_\lambda \), for every \( \lambda \geq 0 \).

(c) For each regular value \( \mu \) of the vector type first integral \( I_{\operatorname{Mrk}(D, I)}, \) the vector field \( X_\lambda(1|_{\operatorname{Mrk}(D, I)}), \lambda > 0 \), admits the globally \( \mu \)-attracting set

\[
\sum_{D_1, \ldots, D_p} \mathcal{Mrk}(D, I) \cap (I_{\operatorname{Mrk}(D, I)})^{-1}(\{ \mu \}),
\]

if not empty. More precisely, for every \( \overline{x} \in (I_{\operatorname{Mrk}(D, I)}^{-1}(\{ \mu \}), \) such that the set \( \{ x(t; \overline{x}) : t \geq 0 \} \) is bounded, \( x(t; \overline{x}) \to \sum_{D_1, \ldots, D_p} \mathcal{Mrk}(D, I) \cap (I_{\operatorname{Mrk}(D, I)})^{-1}(\{ \mu \}) \) as \( t \to \infty \).

(d) Suppose there exists \( \mu \), a regular value of \( I_{\operatorname{Mrk}(D, I)}, \) such that the intersection of some connected component of \( (I_{\operatorname{Mrk}(D, I)})^{-1}(\{ \mu \}) \) with the invariant set \( \sum_{D_1, \ldots, D_p} \mathcal{Mrk}(D, I), \) contains a single orbit \( \gamma \) of \( X \). Then \( \gamma \) preserves its nature as an orbit of \( X_\lambda(1|_{\operatorname{Mrk}(D, I)}), (\lambda > 0), \) and moreover, \( \gamma \) attracts every bounded positive orbit of \( X_\lambda(1|_{\operatorname{Mrk}(D, I)}), \) sharing the same connected component.

(e) Let \( \mu \) be an arbitrary fixed regular value of \( I_{\operatorname{Mrk}(D, I)}, \) then for every positively invariant compact set \( K \subset (I_{\operatorname{Mrk}(D, I)}^{-1}(\{ \mu \}), \) (if any), and for every \( \overline{x} \in K \) we have that \( u(\overline{x}) \subseteq \sum_{D_1, \ldots, D_p} \mathcal{Mrk}(D, I) \cap (I_{\operatorname{Mrk}(D, I)})^{-1}(\{ \mu \}), \) and consequently \( x(t; \overline{x}) \to \sum_{D_1, \ldots, D_p} \mathcal{Mrk}(D, I) \cap (I_{\operatorname{Mrk}(D, I)})^{-1}(\{ \mu \}) \) as \( t \to \infty \). Particularly, if \( I_{\operatorname{Mrk}(D, I)} \) is a proper map, then the set \( (I_{\operatorname{Mrk}(D, I)}^{-1}(\{ \mu \}), \) is compact in \( \mathcal{Mrk}(D, I), \) dynamically invariant, and consequently for every \( \overline{x} \in (I_{\operatorname{Mrk}(D, I)}^{-1}(\{ \mu \}), \) we have that \( x(t; \overline{x}) \to \sum_{D_1, \ldots, D_p} \mathcal{Mrk}(D, I) \cap (I_{\operatorname{Mrk}(D, I)})^{-1}(\{ \mu \}) \) as \( t \to \infty \).

(f) Suppose there exists \( \mu \), a regular value of \( I_{\operatorname{Mrk}(D, I)}, \) such that the intersection of some connected component of \( (I_{\operatorname{Mrk}(D, I)})^{-1}(\{ \mu \}) \) with the invariant set
\[ \sum_{d_1, \ldots, d_p}^D \cap \text{Mrk}((D, I)), \text{contains isolated points. Then each such a point is an asymptotically stable equilibrium point of } X_\lambda|_{\text{Mrk}((D, I))^{-1}(\{\mu\})}, \text{for every } \lambda > 0. \]

**Proof.** Let us define the smooth function \( F : \text{Mrk}((D, I)) \to [0, \infty) \) given by
\[
F(x) := (D_1(x) - d_1)^2 + \cdots + (D_p(x) - d_p)^2, \quad \forall x \in \text{Mrk}((D, I)).
\]
Let \( \varpi \in \text{Mrk}((D, I)) \) be given, and let \( t \in I_{\varpi} \subseteq \mathbb{R} \to x(t; \varpi) \in \text{Mrk}((D, I)) \) be the integral curve of the vector field \( X_\lambda \in \mathfrak{X}(\text{Mrk}((D, I))) \) such that \( x(0; \varpi) = \varpi \), where \( I_{\varpi} \subseteq \mathbb{R} \) stands for the maximal domain of definition of the solution \( x(\cdot; \varpi) \).

Since \( D = (D_1, \ldots, D_p) \) and \( I = (I_1, \ldots, I_k) \) are first integrals of \( X \), the vector field \( X_0^\lambda \in \mathfrak{X}(\text{Mrk}((D, I))) \), \( \lambda > 0 \) satisfies by construction the relations (9), then using the Theorem 2.2, it follows that the vector field \( X_\lambda = X + X_0^\lambda \in \mathfrak{X}(\text{Mrk}((D, I))) \), \( \lambda \geq 0 \), satisfies the relations (9) too.

Hence, for each \( \lambda \geq 0 \), the vector field \( X_\lambda \in \mathfrak{X}(\text{Mrk}((D, I))) \) admits the vector type first integral \( I_{\text{Mrk}((D, I))} \), and keeps dynamically invariant the set \( \mathcal{S} \cap \text{Mrk}((D, I)) = \Sigma_{d_1, \ldots, d_p}^D \cap \text{Mrk}((D, I)) = D^{-1}(\{d_1, \ldots, d_p\}) \cap \text{Mrk}((D, I)) \). Moreover, the following equalities hold true:
\[
\begin{align*}
\mathcal{L}_{X_\lambda} F &= \mathcal{L}_{X + X_0^\lambda} F = \sum_{i=1}^p \mathcal{L}_{X + X_0^\lambda} \left[(D_i - d_i)^2\right] = \sum_{i=1}^p 2(D_i - d_i)\mathcal{L}_{X + X_0^\lambda}(D_i - d_i) \\
&= \sum_{i=1}^p 2(D_i - d_i)\mathcal{L}_X(D_i - d_i) + \sum_{i=1}^p 2(D_i - d_i)\mathcal{L}_{X_0^\lambda}(D_i - d_i) \\
&= \sum_{i=1}^p 2(D_i - d_i)\mathcal{L}_X(D_i) + \sum_{i=1}^p 2(D_i - d_i)\mathcal{L}_{X_0^\lambda}(D_i) \\
&= \sum_{i=1}^p 2(D_i - d_i) \cdot 0 + \sum_{i=1}^p 2(D_i - d_i)\mathcal{L}_{X_0^\lambda}(D_i) \\
&= \sum_{i=1}^p 2(D_i - d_i)(-\lambda)(D_i - d_i) = (-2\lambda)\sum_{i=1}^p (D_i - d_i)^2 \\
&= (-2\lambda)F. \tag{11}
\end{align*}
\]

Using the relation (11), we obtain that
\[
\frac{d}{dt} F(x(t; \varpi)) = (-2\lambda)F(x(t; \varpi)), \quad \forall t \in I_{\varpi},
\]
and hence
\[
F(x(t; \varpi)) = \exp(-2\lambda t) \cdot F(\varpi), \quad \forall t \in I_{\varpi}. \tag{12}
\]
Moreover, since the set of zeros of \( F \) coincides with \( \Sigma_{d_1, \ldots, d_p}^D \cap \text{Mrk}((D, I)) \), the following sets equality holds true:
\[
\{x \in \text{Mrk}((D, I)) : (\mathcal{L}_{X_\lambda} F)(x) = 0\} = \Sigma_{d_1, \ldots, d_p}^D \cap \text{Mrk}((D, I)). \tag{13}
\]

(a) Since by Theorem 4.1 we have that \( \mathcal{E}(X_0^\lambda) = \Sigma_{d_1, \ldots, d_p}^D \cap \text{Mrk}((D, I)) \), in order to complete the proof of the first statement, it is enough to show that \( \mathcal{E}(X_\lambda) = \mathcal{E}(X) \setminus \mathcal{E}(X_0^\lambda) \). We shall prove this equality by double inclusion. The inclusion \( \mathcal{E}(X) \setminus \mathcal{E}(X_0^\lambda) \subseteq \mathcal{E}(X_\lambda) \) is trivial since \( X_\lambda = X + X_0^\lambda \). In order to show the converse inclusion, \( \mathcal{E}(X_\lambda) \subseteq \mathcal{E}(X) \setminus \mathcal{E}(X_0^\lambda) \), let us pick some element
Then, since $X_\lambda = X + X_0^\lambda$, $X_\lambda \in \text{Mrk}((D,I))$, it follows that $x_e \in \text{Mrk}((D,I))$ and $X(x_e) + X_0^\lambda(x_e) = 0$. On the other hand, since $x_e \in \mathcal{E}(X_\lambda)$, it follows that the integral curve of $X_\lambda$ starting from $x_e$, is constant, i.e. $x(t;x_e) = x_e$, for all $t \in \mathbb{R}$. Consequently, the relation (12) implies that $F(x_e) = \exp(-2\lambda t) \cdot F(x_e)$, for all $t \in \mathbb{R}$, and hence $F(x_e) = 0$, which is in turn equivalent to $x_e \in \Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}((D,I))$. As $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}((D,I)) = \mathcal{E}(X_0^\lambda)$, it follows that $x_e \in \mathcal{E}(X_0^\lambda)$, and consequently, since $X(x_e) + X_0^\lambda(x_e) = 0$ we get that $X(x_e) = 0$, and hence $x_e \in \mathcal{E}(X) \cap \mathcal{E}(X_0^\lambda)$. Since $x_e \in \mathcal{E}(X_\lambda)$ was arbitrary chosen, it follows that $\mathcal{E}(X_\lambda) \subseteq \mathcal{E}(X) \cap \mathcal{E}(X_0^\lambda)$.

(b) In order to prove that $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}((D,I))$ is a dynamically invariant set of each vector field $X_\lambda$, for every $\lambda \geq 0$, note that for $\lambda = 0$ we obtain $X_\lambda = X$, and hence $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}((D,I))$ is an invariant set of $X$, since $(D,I)$ is by definition a vector type first integral of $X$ and consequently both sets, $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p}$ and $\text{Mrk}((D,I))$, are invariant. Hence, in order to complete the proof of this statement, it remains to show that $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}((D,I))$ is a dynamically invariant set of each vector field $X_\lambda$, for every $\lambda > 0$. In order to do that, let us fix some $\lambda > 0$ and $\overline{x} \in \Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}((D,I))$. We shall show that the integral curve of $X_\lambda$ starting from $\overline{x}$ at $t = 0$, verifies that $x(t;\overline{x}) \in \Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}((D,I))$, for all $t \in I_{\overline{x}}$, where $I_{\overline{x}} \subseteq \mathbb{R}$ stands for the maximal domain of definition of the solution $x(\cdot;\overline{x})$. Using the relation (12), we get that $F(x(t;\overline{x})) = \exp(-2\lambda t) \cdot F(\overline{x})$, $\forall t \in I_{\overline{x}}$. Since $\overline{x} \in \Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}((D,I))$ it follows that $F(\overline{x}) = 0$, and consequently we obtain that $F(x(t;\overline{x})) = 0$, $\forall t \in I_{\overline{x}}$, and so $x(t;\overline{x}) \in \Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}((D,I))$, for all $t \in I_{\overline{x}}$. Since $\overline{x}$ was arbitrary chosen in $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}((D,I))$, it follows that $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap \text{Mrk}((D,I))$ is a dynamically invariant set for $X_\lambda$.

Moreover, since $I_{\text{Mrk}((D,I))}$ is a vector type first integral of $X_\lambda$ it follows that for each regular value $\mu$ of $I_{\text{Mrk}((D,I))}$, the set $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap (I_{\text{Mrk}((D,I))})^{-1}(\{\mu\})$ is a closed and dynamically invariant set of the vector field $X_\lambda$, since both sets, $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p}$ and $(I_{\text{Mrk}((D,I))})^{-1}(\{\mu\})$, are closed and invariant.

(c) In order to prove this item, pick an arbitrary element $\overline{x} \in (I_{\text{Mrk}((D,I))})^{-1}(\{\mu\})$, such that the set $\{x(t;\overline{x}) : t \geq 0\}$ is bounded, where $t \mapsto x(t;\overline{x})$ stands for the integral curve of the vector field $X_\lambda|_{(I_{\text{Mrk}((D,I))})^{-1}(\{\mu\})}$, $\lambda > 0$, starting from $\overline{x}$ at $t = 0$. We shall show now that the $\omega$-limit set $\omega(\overline{x})$ is a subset of $\Sigma_{d_1,\ldots,d_p}^{D_1,\ldots,D_p} \cap (I_{\text{Mrk}((D,I))})^{-1}(\{\mu\})$. Indeed, let $y \in \omega(\overline{x})$ be arbitrary chosen. Then, there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset [0,\infty)$, $\lim_{n \to \infty} t_n = \infty$, such that $\lim_{n \to \infty} x(t_n;\overline{x}) = y$. Since the set $\{x(t;\overline{x}) : t \geq 0\}$ is bounded, we get that $[0,\infty) \subset I_{\overline{x}}$, and hence the relation (12) implies that

$$F(x(t;\overline{x})) = \exp(-2\lambda t) \cdot F(\overline{x}), \quad \forall t \in [0,\infty). \quad (14)$$

Consequently, for $t = t_n \geq 0$, $n \in \mathbb{N}$, the equality (14) becomes

$$F(x(t_n;\overline{x})) = \exp(-2\lambda t_n) \cdot F(\overline{x}), \quad \forall n \in \mathbb{N}.$$
Since $\mathcal{I} \in (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$, and $(I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\}) \subset \operatorname{Mrk}(D,I)$ is closed and dynamically invariant, it follows that $y = \lim_{n \to \infty} x(t_n; \mathcal{I}) \in (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$, and consequently $y \in \Sigma_{d_1, \ldots, d_p} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$. As $y \in \omega(\mathcal{I})$ is arbitrary chosen, we obtain that

$$\omega(\mathcal{I}) \subseteq \Sigma_{d_1, \ldots, d_p} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\}).$$

Since $x(t; \mathcal{I}) \to \omega(\mathcal{I}) \subseteq \Sigma_{d_1, \ldots, d_p} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$ as $t \to \infty$, it follows that $x(t; \mathcal{I}) \to \Sigma_{d_1, \ldots, d_p} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$ as $t \to \infty$.

(d) Let $\gamma$ be an orbit of $X$ such that $\gamma \subseteq \Sigma_{d_1, \ldots, d_p} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$. Since $\Sigma_{d_1, \ldots, d_p} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\}) \subseteq \Sigma_{d_1, \ldots, d_p} \cap \operatorname{Mrk}(D,I) = \mathcal{E}(X^\lambda)$ it follows that $X^\lambda(\gamma) = \{0\}$. As $\Sigma_{d_1, \ldots, d_p} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$ is a closed and dynamically invariant set of the vector field $X^\lambda = X + X^\lambda$, for every $\lambda \geq 0$, it follows that $\gamma$ is also an orbit of the same nature of the vector field $X^\lambda$, for every $\lambda \geq 0$. The rest of the proof is a direct consequence of (c).

(e) The proof follows directly from (c) since $K \subset (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$ being a compact and positively invariant set, implies that for every $\mathcal{I} \in K$, the integral curve of $X^\lambda|_{(I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})}$ starting from $\mathcal{I}$ at $t = 0$, remains in $K$ for all $t \geq 0$, and hence the set $\{x(t; \mathcal{I}) : t \geq 0\}$ is bounded.

(f) Let us denote by $C^\mu$ a connected component of $(I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$ whose intersection with the invariant set $\Sigma_{d_1, \ldots, d_p} \cap \operatorname{Mrk}(D,I)$ contains isolated points. In order to complete the proof, it is enough to construct a strict Lyapunov function associated to each isolated equilibrium state $x_e \in \Sigma_{d_1, \ldots, d_p} \cap C^\mu$. Note that the set $\Sigma_{d_1, \ldots, d_p} \cap C^\mu$ is dynamically invariant, and since it contains isolated points, each isolated point must be an equilibrium point of the vector field $X^\lambda|_{(I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})}$, for every $\lambda > 0$. Let $x_e \in \Sigma_{d_1, \ldots, d_p} \cap C^\mu$ be such an equilibrium point of $X^\lambda|_{(I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})}$. Let us denote $C^\mu_{x_e} := \Sigma_{d_1, \ldots, d_p} \cap C^\mu$. Since $x_e \in C^\mu_{x_e}$ is an isolated point of $C^\mu_{x_e}$, there exists $U_{x_e} \subseteq \operatorname{Mrk}(D,I)$, an open neighborhood of $x_e$, such that $U_{x_e} \cap C^\mu_{x_e} = \{x_e\}$. By shrinking $U_{x_e}$ if necessary, we suppose that $U_{x_e} \cap \Sigma_{d_1, \ldots, d_p} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\}) = \{x_e\}$.

Let us consider now the restriction to $U_{x_e} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$ of the smooth function $F : \operatorname{Mrk}(D,I) \to [0, \infty)$ given by

$$F(x) := (D_1(x) - d_1)^2 + \cdots + (D_p(x) - d_p)^2, \quad \forall x \in \operatorname{Mrk}(D,I).$$

Since by hypothesis we have that

$$U_{x_e} \cap \Sigma_{d_1, \ldots, d_p} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\}) = \{x_e\},$$

and the set of zeros of $F$ located in $U_{x_e} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$ is given by

$$U_{x_e} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\}) \cap \Sigma_{d_1, \ldots, d_p},$$

it follows that $x_e$ is the unique solution of the equation $F(x) = 0$ in $U_{x_e} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$. Using the relation (11) and the identity

$$\mathcal{L}_{X^\lambda} |_{U_{x_e} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})} F|_{U_{x_e} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})} = \left(\mathcal{L}_{X^\lambda} F\right)|_{U_{x_e} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})},$$

where $\mathcal{L}_{X^\lambda}$ denotes the Lie derivative of $F$ with respect to $X^\lambda$, we obtain that

$$\mathcal{L}_{X^\lambda} F|_{U_{x_e} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})} = 0,$$

which implies that $F$ is constant on $U_{x_e} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$. Therefore, $F$ is a Lyapunov function for the vector field $X^\lambda$ in $U_{x_e} \cap (I|_{\operatorname{Mrk}(D,I)})^{-1}(\{\mu\})$.
it follows that
\[ \mathcal{L}_{X_\lambda}|_{U_{x_e}\cap(I_{\text{Mrk}}((D, I)))^{-1}(\{\mu\})} F|_{U_{x_e}\cap(I_{\text{Mrk}}((D, I)))^{-1}(\{\mu\})} = (-2\lambda) F|_{U_{x_e}\cap(I_{\text{Mrk}}((D, I)))^{-1}(\{\mu\})}. \]

Hence, we get that \( F|_{U_{x_e}\cap(I_{\text{Mrk}}((D, I)))^{-1}(\{\mu\})}(x_e) = 0 \) and
\[
\begin{cases}
F|_{U_{x_e}\cap(I_{\text{Mrk}}((D, I)))^{-1}(\{\mu\})}(x) > 0 \\
(\mathcal{L}_{X_\lambda}|_{I_{\text{Mrk}}((D, I)))^{-1}(\{\mu\})}) F|_{U_{x_e}\cap(I_{\text{Mrk}}((D, I)))^{-1}(\{\mu\})}(x) < 0,
\end{cases}
\]
for every \( x \in (U_{x_e}\cap(I_{\text{Mrk}}((D, I)))^{-1}(\{\mu\})) \setminus \{x_e\} \). Consequently, the restriction to \( U_{x_e}\cap(I_{\text{Mrk}}((D, I)))^{-1}(\{\mu\}) \) of the smooth function \( F \) is a strict Lyapunov function associated to the equilibrium point \( x_e \) of the vector field \( X_{\lambda}|(I_{\text{Mrk}}((D, I)))^{-1}(\{\mu\}) \).

5. **Example.** The aim of this section is to illustrate the theoretical results obtained in Theorem 4.2, for the completely integrable case of the Rössler dynamical system [10]. Let us recall from [10] that this particular case of the Rössler system is generated by the smooth vector field
\[
X(x, y, z) := (-y - z)\partial_x + x\partial_y + xz\partial_z \in \mathfrak{X}(\mathbb{R}^3).
\]
Following [10], the vector field \( X \) is completely integrable as it admits two independent first integrals, \( F_1, F_2 \in C^\infty(\mathbb{R}^3, \mathbb{R}) \), given by
\[
F_1(x, y, z) := \frac{1}{2}(x^2 + y^2) + z, \quad F_2(x, y, z) := ze^{-y}, \quad \forall (x, y, z) \in \mathbb{R}^3.
\]

In order to apply the Theorem 4.2 we have two possibilities to choose the functions \( I \) and \( D \), namely, \( I := F_1, D := F_2 \), and respectively, \( I := F_2, D := F_1 \).

Obviously, in both cases, the open set \( \text{Mrk}((D, I)) \subseteq \mathbb{R}^3 \) is given by \( \text{Mrk}((D, I)) = \mathbb{R}^3 \setminus \mathcal{E}(X) \), where \( \mathcal{E}(X) = \text{span}(\{0, 1, -1\}) \) is the set consisting of the equilibrium states of the vector field (15).

Let us now pick an arbitrary value of \( D \), say \( d \in \text{Im}(D) = \mathbb{R} \). As \( D \) is a first integral of \( X \), the set \( \Sigma^D_\lambda \) is a dynamically invariant set of the vector field (15). Starting with these data, the Theorem 4.2 provides a constructive method in order to globally leafwise asymptotically bp-stabilize the invariant set \( \Sigma^D_\lambda \cap \text{Mrk}((D, I)) \) of \( X|_{\text{Mrk}((D, I))} \), with respect to the foliation given by the level sets of the first integral \( I_{\text{Mrk}((D, I))} \) of \( X|_{\text{Mrk}((D, I))} \). More precisely, we construct a family of smooth vector fields, \( X_\lambda := X + X^D_\lambda \in \mathfrak{X}(\text{Mrk}((D, I))) \), \( \lambda \geq 0 \), such that

- \( X_0 = X|_{\text{Mrk}((D, I))} \),
- \( I_{\text{Mrk}((D, I))} \) is a first integral of \( X_\lambda \), \( \forall \lambda \geq 0 \), i.e. \( X_\lambda \) are conservative vector fields for all \( \lambda \geq 0 \),
- \( \Sigma^D_\lambda \cap \text{Mrk}((D, I)) \) is a closed dynamically invariant set of \( X_\lambda \), \( \forall \lambda \geq 0 \),
- for any \( \mu \in \mathbb{R} \), the closed invariant set \( \Sigma^D_\lambda \cap (I_{\text{Mrk}((D, I)))^{-1}(\{\mu\}) \), if not empty, is a globally bp-attracting set of \( X_\lambda|_{(I_{\text{Mrk}}((D, I)))^{-1}(\{\mu\})} \) for all \( \lambda \geq 0 \), i.e. for every \( \bar{x} = (x_0, y_0, z_0) \in (I_{\text{Mrk}((D, I)))^{-1}(\{\mu\}) \), such that the set \( \{ (x(t; \bar{x}), y(t; \bar{x}), z(t; \bar{x}) : t \geq 0 \} \) is bounded, \( (x(t; \bar{x}), y(t; \bar{x}), z(t; \bar{x}) \rightarrow \Sigma^D_\lambda \cap (I_{\text{Mrk}}((D, I)))^{-1}(\{\mu\}) \) as \( t \rightarrow \infty \).
(a) Let us now analyze the first case, where \( I = \frac{1}{2}(x^2 + y^2) + z \) and \( D = ze^{-y} \). Pick \( d \in \mathbb{R} \) an arbitrary value of \( D \). As \( D \) is a first integral of \( X \), the set

\[
\Sigma_d = \{(x, y, z) \in \mathbb{R}^3 : ze^{-y} = d\},
\]

is a dynamically invariant set of the vector field \((15)\).

According to Theorem 4.2, for each \( \lambda \geq 0 \) there exists a smooth vector field, \( X^\lambda_0 \in \mathcal{X}(\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}) \), given by

\[
X^\lambda_0 = \frac{-\lambda(ze^{-y} - d)}{e^{-y}[x^2 + x^2z^2 + (y + z)^2]} \cdot \{(x(yz - 1)\partial_x - (y + z + x^2z)\partial_y + [x^2 + y(y + z)]\partial_z\},
\]

such that the perturbed Rössler system

\[
\begin{align*}
\dot{x} &= -y - z - \frac{\lambda(ze^{-y} - d)}{e^{-y}[x^2 + x^2z^2 + (y + z)^2]} x(yz - 1), \\
\dot{y} &= x + \frac{\lambda(ze^{-y} - d)}{e^{-y}[x^2 + x^2z^2 + (y + z)^2]} (y + z + x^2z), \\
\dot{z} &= xz - \frac{\lambda(ze^{-y} - d)}{e^{-y}[x^2 + x^2z^2 + (y + z)^2]} [x^2 + y(y + z)],
\end{align*}
\]

generated by \( X_\lambda := X + X^\lambda_0 \in \mathcal{X}(\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}) \), verifies the following assertions:

- \( X_0 = X|_{\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}} \),
- \( I|_{\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}} \) is a first integral of \((16)\),
- \( \Sigma_d \cap (\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}) = \{(x, y, z) \in \mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\} : ze^{-y} = d\} \) is a closed dynamically invariant set of \((16)\),
- if \( \lambda > 0 \) then for any \( \mu \in \mathbb{R} \), the closed invariant set

\[
\Sigma_d \cap (I|_{\text{Merk}(\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}))}^{-1}(\{\mu\})
\]

\[
= \{(x, y, z) \in \mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\} : ze^{-y} = d, \ \frac{1}{2}(x^2 + y^2) + z = \mu\},
\]

if not empty, is a globally bp-attracting set of \((16)\) restricted to the dynamically invariant set

\[
(I|_{\text{Merk}(\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}))}^{-1}(\{\mu\}) = \{(x, y, z) \in \mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\} : \frac{1}{2}(x^2 + y^2) + z = \mu\}
\]

i.e. for every \( \mathfrak{P} = (x_0, y_0, z_0) \in (I|_{\text{Merk}(\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}))}^{-1}(\{\mu\}) \) such that its positive orbit \( \{(x(t; \mathfrak{P}), y(t; \mathfrak{P}), z(t; \mathfrak{P})) : t \geq 0\} \) is bounded, \( (x(t; \mathfrak{P}), y(t; \mathfrak{P}), z(t; \mathfrak{P})) \to \Sigma_d \cap (I|_{\text{Merk}(\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}))}^{-1}(\{\mu\}) \) as \( t \to \infty \).

(b) Let us now analyze the second case, where \( I = ze^{-y} \) and \( D = \frac{1}{2}(x^2 + y^2) + z \). Pick \( d \in \mathbb{R} \) an arbitrary value of \( D \). As \( D \) is a first integral of \( X \), the set

\[
\Sigma_d = \{(x, y, z) \in \mathbb{R}^3 : \frac{1}{2}(x^2 + y^2) + z = d\},
\]

is a dynamically invariant set of the vector field \((15)\).
According to Theorem 4.2, for each \( \lambda \geq 0 \) there exists a smooth vector field, \( X_0^\lambda \in \mathfrak{X}(\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}) \), given by

\[
X_0^\lambda = -\lambda \frac{1}{2} \left( \frac{1}{x^2 + x^2 z^2 + (y + z)^2} \right) \begin{bmatrix} x(z^2 + 1) \partial_x + (y + z) \partial_y + z(y + z) \partial_z \end{bmatrix},
\]

such that the perturbed Rössler system

\[
\begin{align*}
\dot{x} &= -y - z - \frac{\lambda}{x^2 + x^2 z^2 + (y + z)^2} x(z^2 + 1) \\
\dot{y} &= x - \frac{\lambda}{x^2 + x^2 z^2 + (y + z)^2} (y + z) \\
\dot{z} &= xz - \frac{\lambda}{x^2 + x^2 z^2 + (y + z)^2} z(y + z),
\end{align*}
\]

(17)

generated by \( X_\lambda := X + X_0^\lambda \in \mathfrak{X}(\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}) \), verifies the following assertions:

- \( X_0 = X|_{\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}} \),
- \( I|_{\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}} \) is a first integral of (17),
- \( \Sigma_d^I \cap (\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}) = \{(x, y, z) \in \mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\} : \frac{1}{2} (x^2 + y^2) + z = d \} \) is a closed dynamically invariant set of (17),
- if \( \lambda > 0 \) then for any \( \mu \in \mathbb{R} \), the closed invariant set

\[
\Sigma_d^I \cap (I|_{\text{Mrk}((D, I))})^{-1}(\mu)
\]

if not empty, is a globally bp-attracting set of (17) restricted to the dynamically invariant set

\[
(I|_{\text{Mrk}((D, I))})^{-1}(\mu) = \{(x, y, z) \in \mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\} : ze^{-y} = \mu \}
\]

i.e. for every \( \varpi = (x_0, y_0, z_0) \in (I|_{\text{Mrk}((D, I))})^{-1}(\mu) \) such that its positive orbit \( \{(x(t; \varpi), y(t; \varpi), z(t; \varpi)) : t \geq 0\} \) is bounded, \( (x(t; \varpi), y(t; \varpi), z(t; \varpi)) \to \Sigma_d^I \cap (I|_{\text{Mrk}((D, I))})^{-1}(\mu) \) as \( t \to \infty \).

Remark 3. • Note that in the cases (a) and (b), the vector field \( X_\lambda = X + X_0^\lambda \in \mathfrak{X}(\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}) \) does not have equilibrium states, for any \( \lambda > 0 \). This follows directly from Theorem 4.2

\[
\mathcal{E}(X_\lambda) = \mathcal{E}(X) \cap \mathcal{E}(X_0^\lambda) = \mathcal{E}(X) \cap \Sigma_d^I \cap \text{Mrk}((D, I)) = \text{span}_\mathbb{R}\{(0, 1, -1)\} \cap \Sigma_d^I \cap (\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\}) = \emptyset.
\]

• For a topological classification of the sets \( \Sigma_d^I \cap I^{-1}(\mu) \), \( (d, \mu) \in \text{Im}((D, I)) \), see [10]. In order to obtain a topological classification of the sets of type \( \Sigma_d^I \cap (I|_{\text{Mrk}((D, I))})^{-1}(\mu) \), restrict the former classification to \( \text{Mrk}((D, I)) = \mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0, 1, -1)\} \).
Recall that all topological aspects related to the vector fields $X_{\lambda} \in \mathfrak{X}(\mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0,1,-1)\})$, $\lambda > 0$, are considered with respect to the relative topology of $\text{Mrk}((D,I)) = \mathbb{R}^3 \setminus \text{span}_\mathbb{R}\{(0,1,-1)\}$.

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