Projection Theorems, Estimating Equations and Power-Law Distributions

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Abstract

Projection theorems of divergence functionals reduce certain estimation problems under specific families of probability distributions to linear problems. In this paper, we study projection theorems concerning Kullback-Leibler, Rényi, density power, and logarithmic-density power divergences which are popular in robust inference. We first extend these projection theorems to the continuous case by directly solving the associated estimating equations. We then apply these ideas to solve certain estimation problems concerning Student and Cauchy distributions. Finally, we explore the projection theorems by a generalized notion of principle of sufficiency. In particular, we show that the statistics of the data that influence the projection theorems are also a minimal sufficient statistics with respect to this generalized notion.

I. INTRODUCTION

Minimum divergence (or minimum distance) method is popular in statistical inference because of its many desirable properties including robustness and efficiency [4], [44]. Divergence is a non-negative extended real valued function $D$ defined for any pair of probability distributions $(p, q)$ satisfying $D(p, q) = 0$ if and only if $p = q$. Minimization of information divergence ($I$-divergence) or relative entropy is closely related to the maximum likelihood estimation (MLE) [24 Lem. 3.1]. MLE is not a preferred method when the data is contaminated by outliers. However, $I$-divergence can be extended, replacing the logarithmic function by some power function, to produce divergences that are robust with respect to outliers [3], [34]. In this paper we consider three such families of divergences that are well-known in the context of robust statistics. They are defined as follows.
Let $X$ and $Y$ be $d$-dimensional random vectors (jointly discrete or jointly continuous) that follow the distributions $p$ and $q$ respectively and have a common support $S \subseteq \mathbb{R}^d$. Let $\alpha > 0, \alpha \neq 1$.

1) The $D_\alpha$-divergence (also known as Rényi divergence [48] or power divergence [45] and, upto a monotone function, same as Cressie-Read power divergence [15]):

$$D_\alpha(p, q) := \frac{1}{\alpha - 1} \log \int p(x)^\alpha q(x)^{1-\alpha} \, dx.$$  

(1)

2) The $B_\alpha$-divergence (also known as power pseudo-distance [9], [8], density power divergence [3]):

$$B_\alpha(p, q) := \frac{\alpha}{1 - \alpha} \int p(x) q(x)^{\alpha - 1} \, dx - \frac{1}{1 - \alpha} \int p(x)^\alpha \, dx + \int q(x)^\alpha \, dx.$$  

(2)

3) The $I_\alpha$-divergence [51], [42], [52], [29] (also known as relative $\alpha$-entropy [37], [38], Rényi pseudo-distance [8], [9], logarithmic density power divergence [43], projective power divergence [26], $\gamma$-divergence [29], [14]):

$$I_\alpha(p, q) := \frac{\alpha}{1 - \alpha} \log \int p(x) q(x)^{\alpha - 1} \, dx - \frac{1}{1 - \alpha} \log \int p(x)^\alpha \, dx + \log \int q(x)^\alpha \, dx.$$  

(3)

Throughout the paper we assume that $\log$ stands for the natural logarithm and all the integrals are well defined over $S$. The integrals are with respect to the Lebesgue measure on $\mathbb{R}^d$ in the continuous case and with respect to the counting measure in the discrete case. Many well-known divergences fall in the above classes of divergences. The chi-square divergence, the Bhattacharyya distance [7] and the Hellinger distance [5] fall in the $D_\alpha$-divergence class; Cauchy-Schwarz divergence [47, Eq. (2.90)] falls in the $I_\alpha$-divergence class; squared Euclidean distance falls in the $B_\alpha$-divergence class [3]. All the three classes of divergences coincide with the $I_\alpha$-divergence as $\alpha \to 1$ [14], where

$$I(p, q) := \int p(x) \log \frac{p(x)}{q(x)} \, dx.$$  

(4)

In this sense each of these three classes of divergences can be regarded as a generalization of the $I_\alpha$-divergence.

$D_\alpha$-divergences also arise as generalized cut-off rates in information theory [20]. The $B_\alpha$-divergences belong to the Bregman class which is characterized by transitive projection rules (see [19, Eq. (3.2) and Th. 3]). The $I_\alpha$-divergences (for $\alpha < 1$) arise in information theory as a redundancy measure in the mismatched cases of guessing [51], source coding [38] and encoding...
of tasks \[\mathcal{I}\]. The three classes of divergences are associated with robust estimation, for \(\alpha > 1\) in case of \(B_\alpha\) and \(I_\alpha\), and \(\alpha < 1\) in case of \(D_\alpha\), as we shall see now.

Let \(X_1, \ldots, X_n\) be an independent and identically distributed (i.i.d.) sample drawn from an unknown distribution \(p\) which is supposed to be a member of a parametric family of probability distributions \(\Pi = \{p_\theta : \theta \in \Theta\}\), where \(\Theta\) is an open subset of \(\mathbb{R}^k\). MLE picks the distribution \(p_\theta^* \in \Pi\) that would have most likely caused the sample. MLE solves the so-called score equation or estimating equation for \(\theta\), given by

\[
\frac{1}{n} \sum_{j=1}^{n} s(X_j; \theta) = 0, \tag{5}
\]

where \(s(x; \theta) := \nabla \log p_\theta(x)\), called the score function and \(\nabla\) stands for gradient with respect to \(\theta\). In the discrete case, the above equation can be re-written as

\[
\sum_{x \in S} p_n(x) s(x; \theta) = 0, \tag{6}
\]

where \(p_n\) is the empirical measure of the sample \(X_1, \ldots, X_n\).

Let us now suppose that the sample \(X_1, \ldots, X_n\) is from a mixture distribution of the form

\[
p_\epsilon = (1 - \epsilon)p + \epsilon \delta, \quad \epsilon \in [0, 1),
\]

where \(p\) is supposed to be a member of \(\Pi = \{p_\theta : \theta \in \Theta\}\); \(p\) is regarded as the distribution of “true” samples and \(\delta\), that of outliers. In robust estimation, the objective is to find a distribution from the family \(\Pi = \{p_\theta : \theta \in \Theta\}\) that would have most likely caused the true samples. Throughout the paper, the above will be the setup in all the estimation problems, unless otherwise stated. Thus, in this case, one needs to modify the estimating equation so that the effect of outliers is down-weighted. The following modified estimating equation, referred as generalized Hellinger estimating equation, was proposed where the score function was weighted by \(p_n(x)^\alpha p_\theta(x)^{1-\alpha}\) instead of \(p_n(x)\) in (6):

\[
\sum_{x \in S} p_n(x)^\alpha p_\theta(x)^{1-\alpha} s(x; \theta) = 0, \tag{7}
\]

where \(\alpha \in (0, 1)\). The above estimating equation was proposed based on the following intuition. If \(x\) is an outlier, then \(p_n(x)^\alpha p_\theta(x)^{1-\alpha}\) will be smaller than \(p_n(x)\) for sufficiently smaller values of \(\alpha\). Hence the terms corresponding to outliers in (7) are down-weighted. (c.f. [4] Sec. 4.3] and the references therein.)

Notice that (7) does not extend to the continuous case due to the appearance of \(p_n^\alpha\). However in the literature, to avoid this technical difficulty, some smoothing techniques such as kernel density estimation [5 Sec. 3], [4 Sec. 3.1, 3.2.1], Basu-Lindsay approach [4 Sec. 3.5], Cao
et. al. modified approach \cite{12} and so on are used for a continuous estimate of \( p_n \). The resulting estimating equation is of the form

\[
\int \tilde{p}_n(x)^\alpha p_\theta(x)^{1-\alpha} s(x; \theta) \, dx = 0, \tag{8}
\]

where \( \tilde{p}_n \) is some continuous estimate of \( p_n \). (To avoid this smoothing, Broniatowski et. al. use a duality technique. See \cite{8} – \cite{10} for details.)

The following estimating equation, where the score function is weighted by power of model density and equated to its hypothetical one, was proposed by Basu et. al. \cite{3}:

\[
\frac{1}{n} \sum_{j=1}^{n} p_\theta(X_j)^{\alpha-1} s(X_j, \theta) = \int p_\theta(x)^{\alpha} s(x, \theta) \, dx, \tag{9}
\]

where \( \alpha > 1 \). Motivated by the works of Field and Smith \cite{27} and Windham \cite{58}, an alternative estimating equation, where the weights in (9) are further normalized, was proposed by Jones et. al. \cite{34}:

\[
\frac{1}{n} \sum_{j=1}^{n} p_\theta(X_j)^{\alpha-1} s(X_j; \theta) \frac{1}{n} \sum_{j=1}^{n} p_\theta(X_j)^{\alpha-1} = \frac{\int p_\theta(x)^{\alpha} s(x, \theta) \, dx}{\int p_\theta(x)^{\alpha} \, dx}, \tag{10}
\]

where \( \alpha > 1 \). Notice that (9) and (10) do not require the use of empirical distribution. Hence no smoothing is required in these cases. The estimators of (8), (9) and (10) are consistent and asymptotically normal \cite[Th. 2]{3}, \cite[Sec. 3]{34}, \cite[Th. 3]{5}. They also satisfy two invariance properties, one when the underlying model is re-parameterized by a one-one function of the parameter \cite[Sec. 3.4]{3}, and the other when the samples are replaced by some of their linear transformation \cite[Th. 3.1]{53}, \cite[Sec. 3.4]{3}. They coincide with the usual score equation (5) when \( \alpha = 1 \), under the condition that \( \int p_\theta(x) s(x; \theta) \, dx = 0 \). The estimating equations (5), (8), (9) and (10) are, respectively, associated with the divergences in (4), (1), (2), and (3) in a sense that will be made clear in the following.

Observe that the estimating equations (5), (8), (9), and (10) are implications of the first order optimality condition of maximizing, respectively, the usual log-likelihood function

\[
L(\theta) := \frac{1}{n} \sum_{j=1}^{n} \log p_\theta(X_j), \tag{11}
\]

and the following generalized likelihood functions

\[
L_1^{(\alpha)}(\theta) := \frac{1}{1-\alpha} \log \left[ \int \tilde{p}_n(x)^{\alpha} p_\theta(x)^{1-\alpha} \, dx \right]. \tag{12}
\]
\[ L_2^{(\alpha)}(\theta) := \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{\alpha \theta(X_j)^{\alpha-1}}{\alpha - 1} - \int p_\theta(x)^\alpha dx \right], \quad (13) \]

and
\[ L_3^{(\alpha)}(\theta) := \frac{\alpha}{\alpha - 1} \log \left[ \frac{1}{n} \sum_{j=1}^{n} p_\theta(X_j)^{\alpha-1} \right] - \log \left[ \int p_\theta(x)^\alpha dx \right]. \quad (14) \]

The above likelihood functions (12), (13) and (14) are not defined for \( \alpha = 1 \). However it can be shown that they all coincide with \( L(\theta) \) as \( \alpha \to 1 \).

It is easy to see that the probability distribution \( p_\theta \) that maximizes the likelihood function in (12), (11), (13) or (14) is same as, respectively, the one that minimizes \( D_\alpha(\tilde{p}_n, p_\theta) \) or the empirical estimates of \( I(p, p_\theta), B_\alpha(p_\epsilon, p_\theta) \) or \( J_\alpha(p_\epsilon, p_\theta) \). Thus for MLE or “robustified MLE”, one needs to solve
\[ \inf_{p_\theta \in \Pi} D(\bar{p}_n, p_\theta), \quad (15) \]

where \( D \) is either \( I, D_\alpha, B_\alpha \) or \( J_\alpha \), \( \bar{p}_n = p_n \) when \( D \) is \( I \), \( B_\alpha \) or \( J_\alpha \) and \( \tilde{p}_n = \bar{p}_n \) when \( D \) is \( D_\alpha \). Notice that (8) for \( \alpha > 1 \), (9) and (10) for \( \alpha < 1 \), do not make sense in terms of robustness. However, they still serve as first order optimality condition for the minimization problem in (15).

A probability distribution that attains the infimum is known as a reverse \( D \)-projection of \( \bar{p}_n \) on \( \Pi \).

A “dual” minimization problem is the so-called forward projection problem, where the minimization is over the first argument of the divergence function. Given a set \( C \) of probability distributions with support \( S \) and a probability distribution \( q \) with same support, any probability distribution \( p^* \in C \) that attains
\[ \inf_{p \in C} D(p, q) \quad (16) \]
is called a forward \( D \)-projection of \( q \) on \( C \). Forward projection is usually on a convex set or on an \( \alpha \)-convex set of probability distributions. Forward projection on a convex set is motivated by the well-known maximum entropy principle of statistical physics [32]. Motivation for forward projection on \( \alpha \)-convex set comes from the so-called non-extensive statistical physics [37], [54], [55]. Forward \( I \)-projection on convex set was extensively studied by Csiszár [16], [17], [21], Csiszár and Matúš [23], [22], Csiszár and Shields [24], and Csiszár and Tusnády [18].

The forward projections of either of the divergences in (1) - (4) on convex (or \( \alpha \)-convex) sets of probability distributions yield a parametric family of probability distributions. A reverse
projection on this parametric family turns into a forward projection on a convex (or an $\alpha$-convex) set, which further reduces to solving a system of linear equations. We call such a result a **projection theorem** of the divergence. These projection theorems were mainly due to an “orthogonal” relationship between the convex (or the $\alpha$-convex) family and the associated parametric family. The **Pythagorean theorem** of the associated divergence plays a key role in this context.

The projection theorem of the $I$-divergence is due to Csiszár and Shields where the convex family is a linear family (see Definition 22) and the associated parametric family is an exponential family [24, Th. 3.3]. Projection theorem for $\mathcal{I}_\alpha$-divergence was established by Kumar and Sundaresan, where the so-called $\alpha$-power-law family plays the role of the exponential family [38, Th. 18 and Th. 21]. Projection theorem for $D_\alpha$-divergence was established by Kumar and Sason, where a variant of the $\alpha$-power-law family, called $\alpha$-exponential family, plays the role of the exponential family and the so-called $\alpha$-linear family plays the role of the linear family [39, Th. 6]. Projection theorem for more general class of Bregman divergences, in which $B_\alpha$ is a subclass, was established by Csiszár and Matúš [23] using techniques from convex analysis.

One of our goals in this paper is to establish the projection theorem of the $B_\alpha$-divergence using elementary tools. We identify the parametric family associated with the projection theorem of $B_\alpha$-divergence which turns out to be a sub-family of $\alpha$-power-law family. The associated convex family here is a linear family as in the case of $I$ and $\mathcal{I}_\alpha$. Thus projection theorems enable us to find the estimator (whether MLE or robustified MLE) as a forward projection if the estimation is done under a specific parametric family. While for MLE the required family is exponential, for robustified MLE it is one of the power-law families.

Our main contributions in this paper are the following.

(i) Extension of the power-law families to a more general setup including the continuous case.
(ii) Extension of projection theorems to the general power-law families.
(iii) Exploration of projection theorems by a generalized notion of principle of sufficiency.
(iv) Applications to estimation on Student and Cauchy distributions.
(v) Projection theorem of the $B_\alpha$-divergence.

Rest of the paper is organized as follows. In Section II, we first generalize the power-law families to the continuous case and show that the Student and Cauchy distributions belong to this class. In Section III we establish the projection theorems for the general power-law families.
In Section [IV] we apply the projection theorems to the Student and Cauchy distributions to find generalized estimators for their parameters. In Section [V] we study the projection theorems by a generalized notion of the principle of sufficiency. We end the paper with a summary and concluding remarks in Section [VI] In Appendix [A] we establish the projection theorem of the $B_\alpha$-divergence using elementary tools. In Appendix [B] we provide the proofs of the stated results. In Appendix [C] we provide counterexamples for the cases where our main results do not extend to.

II. THE POWER-LAW FAMILIES: DEFINITION AND EXAMPLES

The power-law families are defined in the canonical discrete case in [38, Def. 8], [39, Eq. (62)] and Definition 25 of this paper. In this section we first extend these to a more general setting including the continuous case. We then show that the Student and Cauchy distributions form a power-law family. We shall begin by defining the exponential family $E$ in a more general setting.

Definition 1: [31, Eq. (7.7.5)] Consider a family of probability distributions $\{p_\theta : \theta \in \Theta\}$ on $\mathbb{R}^d$, where $\Theta$ is an open subset of $\mathbb{R}^k$. Let $S$ be the support of $p_\theta$ (which may depend on $\theta$). Let $w = [w_1, \ldots, w_s]^T$ and $f = [f_1, \ldots, f_s]^T$, where $w_i : \Theta \to \mathbb{R}$ is differentiable for $i = 1, \ldots, s$, $f_i : \mathbb{R}^d \to \mathbb{R}$ for $i = 1, \ldots, s$ and $h : \mathbb{R}^d \to \mathbb{R}$. The family is said to form a $k$-parameter exponential family or an $E$-family characterized by $h, w, f, \Theta$ and $S$ if

$$p_\theta(x) = \begin{cases} 
\exp[h(x) + Z(\theta) + w(\theta)^T f(x)] & \text{for } x \in S \\
0 & \text{otherwise}, 
\end{cases} \quad (17)$$

for some $Z : \Theta \to \mathbb{R}$.

The family is said to be regular if, in addition, the following conditions are satisfied [31, Def. 7.7.2].

(i) The support $S$ does not depend on the parameter $\theta$,
(ii) number of $\theta_i$’s equals the number of $w_i$’s, that is, $s = k$,
(iii) the functions 1, $w_1, \ldots, w_s$ are linearly independent on $\Theta$,
(iv) the functions 1, $f_1, \ldots, f_s$ are linearly independent on $S$.

Further, the family is said to be in canonical form if $w_i(\theta) = \theta_i$ for $i = 1, \ldots, k$.

If the $\Theta$ in Definition 1 is characterized by all those $\theta \in \mathbb{R}^k$ for which $\int_S \exp[h(x) + w(\theta)^T f(x)]dx < \infty$, then $\Theta$ is called the natural parameter space of the exponential family.
Analogously, the power-law families can be generalized in the following way. In this section we assume, in general, $\alpha \in \mathbb{R}, \alpha \neq 1$.

A. The $\mathbb{B}^{(\alpha)}$-family

Definition 2: Let $h, w, f, \Theta$ and $\mathbb{S}$ be as in Definition 1. The family of probability distributions $\{p_\theta : \theta \in \Theta\}$ is said to form a $k$-parameter $\mathbb{B}^{(\alpha)}$-family characterized by $h, w, f, \Theta$ and $\mathbb{S}$ if

$$p_\theta(x) = \begin{cases} 
[ h(x) + F(\theta) + w(\theta)^T f(x) ]^{\frac{\alpha}{\alpha - 1}} & \text{if } x \in \mathbb{S} \\
0 & \text{otherwise,}
\end{cases}$$

for some differentiable function $F : \Theta \rightarrow \mathbb{R}$.

The family is said to be regular if the conditions (i)-(iv) of Definition 1 hold and it is said to be in canonical form if $w_i(\theta) = \theta_i$ for $i = 1, \ldots, k$. The natural parameter space in this case is given by the set of all $\theta \in \mathbb{R}^k$ such that $[ h(x) + F(\theta) + w(\theta)^T f(x) ]^{1/(\alpha - 1)} > 0$ on $\mathbb{S}$ and $
\int_{\mathbb{S}} [ h(x) + F(\theta) + w(\theta)^T f(x) ]^{1/(\alpha - 1)} dx = 1.

We shall now see some examples of $\mathbb{B}^{(\alpha)}$-family.

Example 1: Student distributions: Let $\mu := [\mu_1, \ldots, \mu_d]^T \in \mathbb{R}^d, \Sigma := (\sigma_{ij})$ be a symmetric, positive-definite matrix of order $d$ and $\nu \in \mathbb{R} \setminus \{0\}$. The $d$-dimensional Student distribution with location parameter $\mu$, scale parameter $\Sigma$ and degrees of freedom parameter $\nu$, with $\nu \notin [2 - d, 0]$ when $d \geq 3$, is given by

$$p_{\mu, \Sigma}(x) = N_{\Sigma, \nu} \left[ 1 + \frac{1}{\nu} (x - \mu)^T \Sigma^{-1} (x - \mu) \right]^{-\frac{\nu + d}{2}},$$

(19)

where $[r]_+ = \max\{r, 0\}$. The support of this distribution is given by

$$\mathbb{S} = \begin{cases} 
\{ x : (x - \mu)^T \Sigma^{-1} (x - \mu) < -\nu \} & \text{if } \nu \in (-\infty, \min\{0, 2 - d\}) \\
\mathbb{R}^d & \text{if } \nu \in (0, \infty),
\end{cases}$$

and the normalizing factor $N_{\Sigma, \nu}$ is given by

$$N_{\Sigma, \nu} := \begin{cases} 
\frac{\Gamma(1-\nu/2)}{\Gamma(1-\nu/2)(-\nu\pi^{d/2})^{d/2}} & \text{if } \nu \in (-\infty, \min\{0, 2 - d\}) \\
\frac{\Gamma((\nu+d)/2)}{\Gamma((\nu)/2)(\nu\pi^{d/2})^{d/2}} & \text{if } \nu \in (0, \infty).
\end{cases}$$

It should be noted that Student distributions are not defined for $\nu \in [2 - d, 0]$ when $d \geq 3$ as (19) is not integrable in this case. While these distributions do not have finite mean for $\nu \in [0, 1]$, they do not have finite variance for $\nu \in [0, 2]$. For all other values of $\nu$, the mean and covariance
matrix of these distributions are given by $\mu$ and $[\nu/(\nu-2)] \cdot \Sigma$ respectively. Further, (19) coincides with a normal distribution when $\nu \to \pm \infty$.

Let $\alpha = 1 - \frac{2}{\nu+d}$ and $\theta = [\mu_i, \sigma_{ij}]_{i,j=1,\ldots,d,i\leq j}^T$. Then (19) can be re-written as

$$p_{\theta}(x) = N_{\theta,\alpha}[1 + b_{\alpha}(x - \mu)^T \Sigma^{-1}(x - \mu)]^{\frac{1}{\alpha-1}}, \quad (20)$$

where $\alpha \in (-\infty, \min\{0, (d-2)/d\}) \cup ((d-2)/d, 1) \cup (1, \infty)$, $b_{\alpha} = 1/\nu = (1-\alpha)/(2-d(1-\alpha))$ and $N_{\theta,\alpha} = N_{\Sigma,\nu}$, the normalizing factor. Notice that, the Student distribution with $\nu = -d$ is not considered in (20) as $\nu = -d$ corresponds to an infinite value of $\alpha$.

For a matrix $A = (a_{ij})_{d \times d}$, we use the following notations.

$$\text{Tr}(A) := \sum_{i=1}^{d} a_{ii}, \quad \text{vec}(A) := [a_{11}, \ldots, a_{1d}, a_{21}, \ldots, a_{2d}, \ldots, a_{d1}, \ldots, a_{dd}]^T,$$

that is, vec($A$) is a column vector of dimension $d^2$ and its $[(i-1)d + j]$-th element is $a_{ij}$ for $i, j = 1, \ldots, d$. With these notations (20) can be re-written, for $x \in \mathbb{S}$, as

$$p_{\theta}(x) = N_{\theta,\alpha}[1 + b_{\alpha}(x^T \Sigma^{-1}x - 2\mu^T \Sigma^{-1}x + \mu^T \Sigma^{-1}\mu)]^{\frac{1}{\alpha-1}},$$

where equality (a) follows because $x^T \Sigma^{-1}x$ is a scalar, (b) follows because $\text{Tr}(AB) = \text{Tr}(BA)$, and (c) follows because $\text{Tr}(AB) = \text{vec}(A)^T \text{vec}(B^T)$. Comparing (21) with (18), we conclude that the Student distributions form a $d(d+3)/2$-parameter $\mathbb{B}^{(\alpha)}$-family with

$$\theta = [\mu_i, \sigma_{ij}]_{i,j=1,\ldots,d,i\leq j}^T, \quad F(\theta) = N_{\theta,\alpha}^{\alpha-1} + b_{\alpha} N_{\theta,\alpha}^{\alpha-1} \mu^T \Sigma^{-1} \mu - 1,$$

$$h(x) \equiv 1, \quad w(\theta) = [w^{(1)}(\theta), w^{(2)}(\theta)]^T, \quad f(x) = [f^{(1)}(x), f^{(2)}(x)]^T,$$

where

$$w^{(1)}(\theta) = -2 b_{\alpha} N_{\theta,\alpha}^{\alpha-1} \cdot \Sigma^{-1} \mu, \quad f^{(1)}(x) = x,$$

$$w^{(2)}(\theta) = b_{\alpha} N_{\theta,\alpha}^{\alpha-1} \cdot \text{vec}(\Sigma^{-1}), \quad f^{(2)}(x) = \text{vec}(xx^T). \quad (22)$$

The distributions in (20) were studied by Johnson and Vignat as the maximizer of Rényi entropy of order $\alpha$ under covariance constraint for $\alpha > d/(d+2)$, where they classified them as Student-t
for $d/(d + 2) < \alpha < 1$ and \textit{Student-r} for $\alpha > 1$ \cite[Def. 1]{33}. For simplicity we just call them \textit{Student distributions}. Observe that \cite[(19)]{19} for $\nu > 0$ is the usual $d$-dimensional $t$-distribution.

We now show that the Student distributions for $\alpha \in ((d - 2)/d, 1)$ (that is, $\nu \in (0, \infty)$) form a \textit{regular} $B(\alpha)$-family. Let $\Sigma^{-1} := (\sigma^{ij})_{d \times d}$ be the inverse of $\Sigma$. The characterizing functions $w^{(i)}$’s and $f^{(i)}$’s in (22) are given by

$$w^{(1)}(\theta) = [w_1(\theta), \ldots, w_d(\theta)]^T, \quad f^{(1)}(x) = [f_1(x), \ldots, f_d(x)]^T,$$

where

$$w_i(\theta) = -2b_\alpha N_{\theta, \alpha}^{-1} \sum_{j=1}^d \sigma^{ij} \mu_j, \quad f_i(x) = x_i, \quad \text{for } i = 1, \ldots, d,$$

and

$$w^{(2)}(\theta) = [w_{ij}(\theta)]^T_{i,j=1,\ldots,d, \ i \leq j}, \quad f^{(2)}(x) = [f_{ij}(x)]^T_{i,j=1,\ldots,d, \ i \leq j},$$

where

$$w_{ij}(\theta) = b_\alpha N_{\theta, \alpha}^{-1} \sigma^{ij} \quad \text{for } i, j = 1, \ldots, d, \ i \leq j,$$

$$f_{ii}(x) = x_i^2, \quad f_{ij}(x) = 2x_i x_j \quad \text{for } i, j = 1, \ldots, d, \ i < j.$$

Note that the number of $w_i$’s and $w_{ij}$’s $= d + d + (d - 1) + (d - 2) + \cdots + 1 = d(d + 3)/2$, which is same as the number of unknown parameters $\theta_i$’s. Also $1$, $f_i$’s and $f_{ij}$’s are linearly independent on $S$. Hence it remains to show only that $1$, $w_i$’s and $w_{ij}$’s are linearly independent on $\Theta$. Suppose that

$$c_1 + \sum_{i=1}^d c_i w_i(\theta) + \sum_{i=1}^d \sum_{j=i}^d c_{ij} w_{ij}(\theta) = 0 \quad \text{for some } c, c_i, c_{ij} \in \mathbb{R}.$$

Dividing both sides by $b_\alpha N_{\theta, \alpha}^{\alpha - 1}$, we get

$$cb_\alpha N_{\theta, \alpha}^{\alpha - 1} - 2 \sum_{i=1}^d c_i \left[ \sum_{j=1}^d \sigma^{ij} \mu_j \right] + \sum_{i=1}^d \sum_{j=i}^d c_{ij} \sigma^{ij} = 0. \quad (23)$$

Taking partial derivative with respect to $\mu$ in (23), we get

$$[c_1, \ldots, c_d] \cdot \Sigma^{-1} = 0^T, \quad (24)$$

where $0$ is the zero vector in $\mathbb{R}^d$. Since $|\Sigma^{-1}| \neq 0$, from (24) we must have $c_1 = \cdots = c_d = 0$. Thus (23) becomes

$$cb_\alpha N_{\theta, \alpha}^{\alpha - 1} + \sum_{i=1}^d \sum_{j=i}^d c_{ij} \sigma^{ij} = 0. \quad (25)$$
For $i, j = 1, \ldots, d$, $i \leq j$, we have

$$
\partial_{\sigma^{ij}} (c_{\alpha}^{-1} N_{\theta, \alpha}^{1-\alpha}) = c_{\alpha}^{-1} (1 - \alpha) N_{\theta, \alpha}^{-\alpha} \partial_{\sigma^{ij}} (N_{\theta, \alpha})
$$

$$
= \left( [c_{\alpha}^{-1} (1 - \alpha) N_{\theta, \alpha}^{1-\alpha}] / 2 |\Sigma^{-1}| \right) \partial_{\sigma^{ij}} (|\Sigma^{-1}|)
$$

$$
= -k_{\theta} \partial_{\sigma^{ij}} (|\Sigma^{-1}|),
$$

where $k_{\theta} := [c_{\alpha}^{-1} (\alpha - 1) N_{\theta, \alpha}^{1-\alpha}] / [2 |\Sigma^{-1}|]$. Thus differentiating (25) with respect to $\sigma^{ij}$, for $i, j = 1, \ldots, d$, $i \leq j$, we get

$$
c_{ij} = k_{\theta} \partial_{\sigma^{ij}} (|\Sigma^{-1}|).
$$

Using these values in (25), we get

$$
c_{ij} = k_{\theta} \partial_{\sigma^{ij}} (|\Sigma^{-1}|).
$$

Using these values in (25), we get

$$
c_{ij} = k_{\theta} \partial_{\sigma^{ij}} (|\Sigma^{-1}|).
$$

(26)

Since $\Sigma^{-1}$ is symmetric, we have

$$
\sum_{i=1}^{d} \sum_{j=i}^{d} \sigma^{ij} \partial_{\sigma^{ij}} (|\Sigma^{-1}|) = \sum_{i=1}^{d} \sum_{j=i}^{d} \sigma^{ij} \cdot (\text{cofactor of } \sigma^{ij} \text{ in } \Sigma^{-1}) = d \cdot |\Sigma^{-1}|.
$$

Using this in (26), we get

$$
c_{ij} = k_{\theta} \partial_{\sigma^{ij}} (|\Sigma^{-1}|).
$$

(27)

Since $\alpha > (d - 2)/d$, then $c = 0$. This implies $k_{\theta} = 0$ and thus $c_{ij} = 0$ for all $i, j = 1, \ldots, d$, $i \leq j$. Thus 1, $w_i$’s and $w_{ij}$’s are linearly independent. Hence Student distributions form a $d(d + 3)/2$-parameter regular $\mathcal{B}^{(\alpha)}$-family for $\alpha \in ((d - 2)/d, 1)$.

Note that Student distributions for $\alpha \notin ((d - 2)/d, 1)$ is not regular as the support in this case depends on the unknown parameters.

**Example 2:** Wigner semi-circle distributions [57] form a $\mathcal{B}^{(\alpha)}$-family.

**B. The $\mathcal{M}^{(\alpha)}$-family**

**Definition 3:** Let $h, w, f, \Theta$ and $\mathcal{S}$ be as in Definition 1. The family of probability distributions $\{p_{\theta} : \theta \in \Theta\}$ is said to form a $k$-parameter $\alpha$-power-law family or an $\mathcal{M}^{(\alpha)}$-family characterized by $h, w, f, \Theta$ and $\mathcal{S}$ if

$$
p_{\theta}(x) = \begin{cases} 
Z(\theta) [h(x) + w(\theta)^T f(x)]^{\frac{1}{1-\alpha}} & \text{for } x \in \mathcal{S} \\
0 & \text{otherwise,}
\end{cases}
$$

(28)
for some differentiable function $Z : \Theta \rightarrow \mathbb{R}$.

The family is said to be \textit{regular} if, along with \textit{(i)-(iii)} of Definition \[1\] also the functions $f_1, \ldots, f_s$ are linearly independent on $\mathbb{S}$. Further, it is said to be \textit{canonical} if $w_i(\theta) = \theta_i$ for $i = 1, \ldots, k$. The \textit{natural parameter space} of this family is given by the set of all $\theta \in \mathbb{R}^k$ such that $[h(x) + w(\theta)^T f(x)]^{1/(\alpha - 1)} > 0$ on $\mathbb{S}$ and $\int_{\mathbb{S}}[h(x) + w(\theta)^T f(x)]^{1/(\alpha - 1)} \, dx < \infty$.

\textbf{Example 3:} The Student distributions in (20) can be re-written as

$$p_\theta(x) = N_{\theta, \alpha \theta}[1 + b_\alpha \{x^T \Sigma^{-1} x - 2 \mu^T \Sigma^{-1} x + \mu^T \Sigma^{-1} \mu\}]_{+}^{1/\alpha - 1}$$

$$= N_{\theta, \alpha \theta}[1 + b_\alpha \{\text{vec}(\Sigma^{-1})^T \text{vec}(xx^T) - 2(\Sigma^{-1})^T x + \mu^T \Sigma^{-1} \mu\}]_{+}^{1/\alpha - 1}. \quad (29)$$

Let $S(\theta) := 1 + b_\alpha \mu^T \Sigma^{-1} \mu$. Note that $S(\theta) > 0$ if $\alpha \in ((d - 2)/d, 1)$. However, when $\alpha \notin ((d - 2)/d, 1)$, we consider the restricted parameter space such that $S(\theta) > 0$. Thus (29) can be re-written, for $x \in \mathbb{S}$, as

$$p_\theta(x) = S(\theta)^{1/\alpha - 1} N_{\theta, \alpha \theta}[1 + b_\alpha S(\theta)^{-1} \{\text{vec}(\Sigma^{-1})^T \text{vec}(xx^T) - 2(\Sigma^{-1})^T x + \mu^T \Sigma^{-1} \mu\}]_{+}^{1/\alpha - 1}. \quad (30)$$

Comparing (30) and (28), we see that Student distributions form a $d(d + 3)/2$-parameter $\mathbb{M}^{(\alpha)}$-family with

$$\theta = [\mu_i, \sigma_{ij}]_{i,j=1,\ldots,d,i\neq j}^T, \quad Z(\theta) = S(\theta)^{1/\alpha - 1} N_{\theta, \alpha}, \quad h(x) \equiv 1,$$

$$w(\theta) = [w^{(1)}(\theta), w^{(2)}(\theta)]^T, \quad f(x) = [f^{(1)}(x), f^{(2)}(\theta)]^T,$$

where

$$w^{(1)}(\theta) = -2b_\alpha S(\theta)^{-1} \cdot \Sigma^{-1} \mu, \quad f^{(1)}(x) = x,$$

$$w^{(2)}(\theta) = b_\alpha S(\theta)^{-1} \cdot \text{vec}(\Sigma^{-1}), \quad f^{(2)}(x) = \text{vec}(xx^T).$$

\textbf{Example 4:} Wigner semi-circle distributions also form an $\mathbb{M}^{(\alpha)}$-family.

Observe that any $p_\theta$ in (28) can be re-written, for $x \in \mathbb{S}$, as

$$p_\theta(x) = [1 + F(\theta) + \tilde{w}(\theta)^T \tilde{f}(x)]_{+}^{1/\alpha - 1} \quad (31)$$

with $F(\theta) \equiv -1$, $\tilde{w}(\theta) = [Z(\theta)^{\alpha - 1}, Z(\theta)^{\alpha - 1} w_1(\theta), \ldots, Z(\theta)^{\alpha - 1} w_s(\theta)]^T$ and $\tilde{f}(x) = [h(x), f_1(x), \ldots, f_s(x)]^T$. This implies that these $p_\theta$ also form a $k$-parameter $\mathbb{B}^{(\alpha)}$-family but characterized by $1, \tilde{f}$ and $\tilde{w}$. On the other hand, any $p_\theta \in \mathbb{B}^{(\alpha)}$ as in (18) can be re-written, for $x \in \mathbb{S}$, as

$$p_\theta(x) = Z(\theta)[h(x) + \tilde{w}(\theta)^T \tilde{f}(x)]_{+}^{1/\alpha - 1} \quad (32)$$
with \( Z(\theta) \equiv 1 \), \( \bar{w}(\theta) = [F(\theta), w_1(\theta), \ldots, w_s(\theta)]^T \) and \( \tilde{f}(\mathbf{x}) = [1, f_1(\mathbf{x}), \ldots, f_s(\mathbf{x})]^T \), or

\[
p_\theta(\mathbf{x}) = Z(\theta)[1 + \bar{w}(\theta)^T \tilde{f}(\mathbf{x})]^{-\frac{1}{\alpha-1}}
\]

with \( Z(\theta) = F(\theta)^{\frac{\alpha}{\alpha-1}} \), \( \bar{w}(\theta) = [1/F(\theta), w_1(\theta)/F(\theta), \ldots, w_s(\theta)/F(\theta)]^T \) and \( \tilde{f}(\mathbf{x}) = [h(\mathbf{x}), f_1(\mathbf{x}), \ldots, f_s(\mathbf{x})]^T \), provided \( F(\theta) > 0 \). This implies that \( p_\theta \) forms a \( k \)-parameter \( \mathbb{M}(\alpha) \)-family as in (32) or in (33). However, as before, the characterizing entities of \( p_\theta \) when we view it as a member of \( \mathbb{M}(\alpha) \) are not the same as we view it as \( \mathbb{B}(\alpha) \). Notice also that the number of \( w_i \)'s (and \( f_i \)'s) is increased in either case. Thus in general (32) or (33) need not define a regular \( \mathbb{M}(\alpha) \)-family even if (18) defines a regular \( \mathbb{B}(\alpha) \)-family. This can be seen in the following example.

Consider the 1-dimensional Student distributions with unit variance and \( 1/3 < \alpha < 1 \):

\[
p_\mu(x) = N_\alpha[1 + b_\alpha(x - \mu)^2]^{\frac{1}{\alpha-1}},
\]

where \( N_\alpha \) is the normalizing factor which is independent of the unknown parameter \( \mu \). This can be viewed as a regular \( \mathbb{B}(\alpha) \)-family as

\[
p_\mu(x) = [(N_\alpha^{-1} + N_\alpha^{-1}b_\alpha x^2) + N_\alpha^{-1}b_\alpha \mu^2 + (-2N_\alpha^{-1}b_\alpha \mu)x]^{\frac{1}{\alpha-1}}
\]

with \( h(x) = N_\alpha^{-1} + N_\alpha^{-1}b_\alpha x^2, F(\mu) = N_\alpha^{-1}b_\alpha \mu^2, w_1(\mu) = -2N_\alpha^{-1}b_\alpha \mu \) and \( f_1(x) = x \).

Observe that (34) can be re-written as an \( \mathbb{M}(\alpha) \)-family as

\[
p_\mu(x) = N_\alpha[(1 + b_\alpha x^2) + b_\alpha \mu^2 + (-2b_\alpha \mu)x]^{\frac{1}{\alpha-1}}
\]

with \( h(x) = 1 + b_\alpha x^2, Z(\mu) = N_\alpha, w_1(\mu) = b_\alpha \mu^2, f_1(x) = 1, w_2(\mu) = -2b_\alpha \mu \) and \( f_2(x) = x \). However, this does not define a regular \( \mathbb{M}(\alpha) \) as number of unknown parameters (which is one) is not equal to the number of \( w_i \)'s (which is two).

Thus a \( \mathbb{B}(\alpha) \)-family is unique on its own. Moreover, unlike the normalizing factor \( F(\theta) \) in \( \mathbb{B}(\alpha) \), the \( Z(\theta) \) in an \( \mathbb{M}(\alpha) \) is positive for all \( \theta \) (see, for example, Example 10 in Appendix A and Example 3 for a comparison). Nonetheless the two families are more closely related when \( h \) is identically a constant.

**Proposition 4:** A regular \( \mathbb{B}(\alpha) \)-family as in Definition 2 with \( h \) being a non-zero constant also forms a regular \( \mathbb{M}(\alpha) \)-family characterized by the same functions \( h \) and \( f_i \)'s, if \( 1 + [F(\theta)/h] > 0 \) for \( \theta \in \Theta \) and one of the following conditions holds.

(a) \( F(\theta) \) is identically a constant, or

(b) \( 1, F(\theta), w_1(\theta), \ldots, w_k(\theta) \) are linearly independent.
As a consequence of the above proposition, we see that Student distributions also form a regular $\mathcal{M}^{(\alpha)}$-family for $\alpha \in ((d - 2)/d, 1)$. Recall that, for $\alpha \in ((d - 2)/d, 1)$, Student distributions form a regular $\mathcal{B}^{(\alpha)}$-family with $h(x) \equiv 1$ (see Example I). Hence, in view of Proposition 4, these also form a regular $\mathcal{M}^{(\alpha)}$-family if $1$, $F(\theta)$, $w_i(\theta)$’s and $w_{ij}(\theta)$’s as described in Example I are linearly independent. Let

$$c.1 + c_0F(\theta) + \sum_{i=1}^{d}c_iw_i(\theta) + \sum_{i=1}^{d}\sum_{j=i}^{d}c_{ij}w_{ij}(\theta) = 0 \quad (35)$$

for some $c, c_i$ and $c_{ij} \in \mathbb{R}$, where $F(\theta) = N^{\alpha - 1}_\theta + b_\alpha N^{\alpha - 1}_{\theta,c} \mu^T \Sigma^{-1} \mu - 1$ and $w_i$’s are as defined in Example I. Note that $\partial_\mu[m^T \Sigma^{-1} \mu] = 2(\Sigma^{-1} \mu)$ and $\partial_\mu[(\Sigma^{-1} \mu)^T] = \Sigma^{-1}$. Hence taking partial derivative with respect to $\mu$ in (35), we get

$$2b_\alpha N^{\alpha - 1}_{\theta,c}[c_0 \Sigma^{-1} \mu - \Sigma^{-1} \bar{c}] = 0, \quad (36)$$

where $\bar{c} = [c_1, \ldots, c_d]^T$. Since $|\Sigma^{-1}| \neq 0$, from (36), we have $c_0 \mu - \bar{c} = 0$, which, further upon taking partial derivative with respect to $\mu$, implies $c_0 = c_1 = \cdots = c_d = 0$. Thus (35) reduces to (25) of Example I. Hence proceeding as in Example I we get $c_{ij} = 0$ for $i, j = 1, \ldots, d, i \leq j$. This proves that Student distributions also form a regular $\mathcal{M}^{(\alpha)}$-family for $\alpha \in ((d - 2)/d, 1)$. However, these do not form a regular $\mathcal{M}^{(\alpha)}$ for $\alpha \notin ((d - 2)/d, 1)$ as their support depends on the unknown parameters in this case.

C. The $\mathcal{E}^{(\alpha)}$-family

**Definition 5:** Let $h, w, f, \Theta$ and $\mathcal{S}$ be as in Definition I. The family of probability distributions\{p_\theta : \theta \in \Theta\} is said to form a $k$-parameter $\alpha$-exponential family or an $\mathcal{E}^{(\alpha)}$-family characterized by $h, w, f, \Theta$ and $\mathcal{S}$ if

$$p_\theta(x) = \begin{cases} Z(\theta)[h(x) + w(\theta)^T f(x)]^{-1/\alpha} & \text{for } x \in \mathcal{S} \\ 0 & \text{otherwise} \end{cases} \quad (37)$$

for some differentiable function $Z : \Theta \rightarrow \mathbb{R}$.

The family is said to be regular if, along with (i)-(iii) of Definition I, also the functions $f_1, \ldots, f_s$ are linearly independent on $\mathcal{S}$. Further, it is said to be canonical if $w_i(\theta) = \theta_i$ for $i = 1, \ldots, k$. The natural parameter space in this case is given by the set of all $\theta \in \mathbb{R}^k$ such that $[h(x) + w(\theta)^T f(x)]^{1/(1-\alpha)} > 0$ on $\mathcal{S}$ and $\int_{\mathcal{S}}[h(x) + w(\theta)^T f(x)]^{1/(1-\alpha)} dx < \infty$. 

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DRAFT
In the following lemma we establish a connection between $\mathcal{M}^{(\alpha)}$-family and $\mathcal{E}^{(\alpha)}$-family. This is due to Karthik and Sundaresan [35, Th. 2], where it was proved for the discrete, canonical case.

Lemma 6: Let $\alpha \neq 0$. The map $p \mapsto p^{(\alpha)}$ establishes a one-to-one correspondence between an $\mathcal{E}^{(\alpha)}$-family characterized by $h, f, w, \Theta$ and $S$, and the $\mathcal{M}^{(1/\alpha)}$-family characterized by the same entities, where $p^{(\alpha)}(x) = p(x)^{\alpha} \int p(y)^{\alpha}dy$, the $\alpha$-scaled measure associated with $p$, which is referred as escort measures among physicists.

Remark 1: As a consequence of Lemma 6, the $\alpha$-scaled Student distributions form an $\mathcal{E}^{(1/\alpha)}$-family.

Remark 2: Observe that (37) can be re-written, for $x \in S$, as

$$p_\theta(x) = Z(\theta)[h(x) + w(\theta)^T f(x)]^{\frac{1}{(2-\alpha)}}.$$  

(38)

Let $\alpha' = 2 - \alpha$. Then (38) defines an $\mathcal{M}^{(\alpha')}$-family characterized by the same entities that characterize the $\mathcal{E}^{(\alpha)}$-family in (37). The converse is also true in the sense that an $\mathcal{M}^{(\alpha)}$-family, as in (28), is also an $\mathcal{E}^{(\alpha')}$-family.

Example 5: Cauchy distributions: Let us consider the $d$-dimensional Student distributions $p_\theta$ as in (20). The $\alpha$-scaled measure of $p_\theta$ is given by

$$q_\eta(x) := p_\eta^{(\alpha)}(x) = \tilde{N}_{\eta,\alpha}[1 + b_\alpha(x - \mu)^T \Sigma^{-1}(x - \mu)]^\frac{\alpha}{2-\alpha},$$

(39)

where

$$\tilde{N}_{\eta,\alpha} := \frac{1}{\int [1 + b_\alpha(x - \mu)^T \Sigma^{-1}(x - \mu)]^{\frac{\alpha}{2-\alpha}} dx}$$

is the normalizing factor and $\eta = [\mu_i, \sigma_{ij}]^{T}_{i,j=1,...,d, i \leq j}$. Observe that $q_\eta$ is a valid density function for $\alpha \in (-\infty, \min\{0, (d - 2)/d\}) \cup (d/(d + 2), 1) \cup (1, \infty)$ and it has full support for $\alpha \in (d/(d + 2), 1)$. Notice that the $q_\eta$ in (39) can be re-written, for $x \in S$, as

$$q_\eta(x) = \tilde{N}_{\eta,\alpha}[1 + b_\alpha \{\text{vec}(\Sigma^{-1})^T \text{vec}(xx^T) - 2(\Sigma^{-1})^T x + \mu^T \Sigma^{-1} \mu\}]^\frac{\alpha}{2-\alpha}$$

$$= S(\eta)^{\frac{\alpha}{2-\alpha}} \tilde{N}_{\eta,\alpha}[1 + b_\alpha S(\eta)^{-1} \{\text{vec}(\Sigma^{-1})^T \text{vec}(xx^T) - 2(\Sigma^{-1})^T x\}]^{\frac{1}{1-\alpha}},$$

where $S(\eta) := 1 + b_\alpha \mu ^T \Sigma^{-1} \mu$. Using the notations $\beta = 1/\alpha$, $c_\beta = b_{1/\beta}$ and $M_{\eta,\beta} = \tilde{N}_{\eta,1/\beta}$, for $x \in S$, we have

$$q_\eta(x) = S(\eta)^{\frac{1}{\beta}} M_{\eta,\beta}[1 + c_\beta S(\eta)^{-1} \{\text{vec}(\Sigma^{-1})^T \text{vec}(xx^T) - 2(\Sigma^{-1})^T x\}]^{\frac{1}{1-\alpha}},$$

(40)
where $\beta \in (d/(d-2), 0) \cup (0, 1) \cup (1, (d+2)/d)$ for $d \leq 2$ and $\beta \in (-\infty, 0) \cup (0, 1) \cup (1, (d+2)/d)$ for $d \geq 3$. Comparing (40) with (37), we see that $q_\eta$’s form a $d(d+3)/2$-parameter $\mathcal{E}(\beta)$-family with

$$\eta = [\mu_i, \sigma_{ij}]_{i,j=1,\ldots,d,i\leq j}, \quad Z(\eta) = S(\eta)^{-1} S M_{\eta, \beta}, \quad h(x) \equiv 1,$$

$$w(\eta) = \begin{bmatrix} w^{(1)}(\eta), w^{(2)}(\eta) \end{bmatrix}^T, \quad f(x) = \begin{bmatrix} f^{(1)}(x), f^{(2)}(x) \end{bmatrix}^T,$$

where

$$w^{(1)}(\eta) = -2c_\beta S(\eta)^{-1} \cdot \Sigma^{-1} \mu, \quad f^{(1)}(x) = x,$$

$$w^{(2)}(\eta) = c_\beta S(\eta)^{-1} \cdot \text{vec}(\Sigma^{-1}), \quad f^{(2)}(x) = \text{vec}(xx^T).$$

Some special cases of (40) include the following:

(a) The usual $d$-dimensional Cauchy distributions correspond to $\beta = (d + 3)/(d + 1)$.

(b) The generalized Cauchy distributions studied by Rider [49] correspond to $\beta = (1 + \omega)/\omega$ and $\beta \in (1, 3)$.

(c) The multivariate truncated generalized Cauchy distributions studied by Ateya and Madhagi [1, Eq. (2.3)] correspond to $\beta = 1 + 2/(2\kappa + d)$ where $\kappa$ equals to the $\alpha$ in their paper and $\beta \in (1, (d + 2)/d)$.

While studying the diffusion problem under Lévy distributions, Prato and Tsallis found (40) as the maximizer of Rényi (or Tsallis) entropy subject to linear constraints on the $\alpha$-scaled measure of the distribution [46, Eq. (10)-(11)]. Vignat and Plastino [56], Ghoshdastidar et al. [30] studied these distributions as $q$-Gaussian distributions. However, we shall call them simply Cauchy distributions. Here $\mu$ and $\Sigma$, respectively, are the location and scale parameters.

Observe that the functions $w$ and $f$ in the Cauchy distribution as in (40) are the same as the ones in the Student distribution in (30). Thus by a similar argument as described in Example 3 we can show that the conditions for a regular $\mathcal{E}(\alpha)$-family hold for the family of Cauchy distributions. Hence Cauchy distributions form a $d(d + 3)/2$-parameter regular $\mathcal{E}(\beta)$-family for $\beta \in (1, (d + 2)/d)$. Note that for $\beta \notin (1, (d + 2)/d)$, they do not define a regular family as, in this case, the support depends on the unknown parameters.

**Example 6:** Consider the Student distributions as in (30). In view of Remark 2, these form an $\mathcal{E}(\alpha')$-family, where $\alpha' = 2 - \alpha$ (that is, $\alpha' \in (-\infty, 1) \cup (1, (d + 2)/d) \cup ((d + 2)/d, \infty)$ when $d \leq 2$, and $\alpha' \in (-\infty, 1) \cup (1, (d + 2)/d) \cup (2, \infty)$ otherwise), characterized by the same functions as in (30). Note that it has full support for $\alpha' \in (1, (d + 2)/d)$. In this case they indeed form a regular $\mathcal{E}(\alpha')$ as this corresponds to $\alpha \in ((d - 2)/d, 1)$ in their $M(\alpha)$ form.
Remark 3: Observe that the exponential family (17) can be re-written in any of the following equivalent forms:

\[ p_\theta(x) = Z'(\theta) \exp \left[ h(x) + w(\theta)^T f(x) \right], \]  

(41)

or

\[ p_\theta(x)^{-1} = Z'(\theta)^{-1} \exp \left[ -h(x) - w(\theta)^T f(x) \right], \]  

(42)

and

\[ p_\theta(x)^{-1} = Z'(\theta)^{-1} \exp \left[ -h(x) - Z(\theta) - w(\theta)^T f(x) \right], \]  

(43)

where \( Z'(\theta) = \exp Z(\theta) \). Analogous to (41), (42) and (43), respectively, the probability distributions in \( \mathcal{E}^{(\alpha)} \), \( \mathcal{M}^{(\alpha)} \) and \( \mathcal{B}^{(\alpha)} \)-families can be expressed, for \( x \in S \), as

\[ p_\theta(x) = Z'(\theta)e_\alpha \left[ h(x) + w(\theta)^T f(x) \right], \]

\[ p_\theta(x)^{-1} = Z'(\theta)^{-1} e_\alpha \left[ -h(x) - w(\theta)^T f(x) \right], \]

and

\[ p_\theta(x)^{-1} = e_\alpha \left[ -h(x) - Z(\theta) - w(\theta)^T f(x) \right], \]

where the \( \alpha \)-exponential function \( e_\alpha : [-\infty, \infty] \to (-\infty, \infty] \) is defined as

\[ e_\alpha(r) = \begin{cases} 
\max\{1 + (1 - \alpha)r, 0\}^{1/(1-\alpha)} & \alpha \neq 1 \\
\exp(r) & \alpha = 1.
\end{cases} \]

Observe that the \( \alpha \)-exponential function coincides with the usual exponential function as \( \alpha \to 1 \). Hence the families \( \mathcal{E}^{(\alpha)} \), \( \mathcal{M}^{(\alpha)} \) and \( \mathcal{B}^{(\alpha)} \) coincide with the usual exponential family as \( \alpha \to 1 \). Thus these three power-law families can be seen as generalizations of the exponential family.

III. PROJECTION THEOREMS FOR GENERAL POWER-LAW FAMILIES

Projection theorems of \( I \), \( B_\alpha \), \( \mathcal{A}_\alpha \) and \( D_\alpha \) divergences enable us to transform the reverse projections of these divergences, respectively on \( \mathcal{E} \), \( \mathcal{B}^{(\alpha)} \), \( \mathcal{M}^{(\alpha)} \) and \( \mathcal{E}^{(\alpha)} \) families, into solving a system of linear equations, called projection equations. However, these projection theorems in the literature are known only for the discrete, canonical, regular families and in their natural parameter space.

In the following, we assume that the families \( \mathcal{E} \), \( \mathcal{B}^{(\alpha)} \), \( \mathcal{M}^{(\alpha)} \) and \( \mathcal{E}^{(\alpha)} \) are canonical and regular with support \( S \) being finite and the parameter space \( \Theta \) being the natural parameter space. Let \( p_n \) denote the empirical distribution of the sample \( X_1, \ldots, X_n \).

1) Projection theorem of \( I \)-divergence:
Let $E$ be an exponential family characterized by $f$ and a probability distribution $h$ with support $S$. The MLE under $E$ satisfies

$$\mathbb{E}_\theta[f(X)] = \bar{f},$$

where $\bar{f} := [\bar{f}_1, \ldots, \bar{f}_k]^T$, $\bar{f}_i := \frac{1}{n} \sum_{j=1}^n f_i(X_j)$ for $i = 1, \ldots, k$ and $\mathbb{E}_\theta[\cdot \cdot \cdot]$ denotes expected value with respect to $p_\theta$. This is due to [24, Th. 3.3].

2) *Projection theorem for $B_\alpha$-divergence*:

Consider a $B^{(\alpha)}$-family characterized by $f$ and $h = q^{\alpha-1}$, where $q$ is a probability distribution with support $S$. The reverse $B_\alpha$-projection of $p_n$ on $\mathbb{B}^{(\alpha)}$ satisfies (44), by Theorem 28 and Remark 11 of this paper.

3) *Projection theorem of $I_\alpha$-divergence*:

Consider an $I^{(\alpha)}$-family characterized by $f$ and $h = q^{\alpha-1}$, where $q$ is a probability distribution with support $S$. The reverse $I_\alpha$-projection of $p_n$ on $M^{(\alpha)}$ satisfies

$$\mathbb{E}_\theta[f(X)] \mathbb{E}_\theta[h(X)] = \bar{f} h,$$

where $\bar{h} := \frac{1}{n} \sum_{j=1}^n h(X_j)$. This result is due to [38, Th. 18 and Th. 21].

4) *Projection theorem of $D_\alpha$-divergence*:

Consider an $D^{(\alpha)}$-family characterized by $f$ and $h = q^{1-\alpha}$, where $q$ is a probability distribution with support $S$. The reverse $D_\alpha$-projection of $p_n$ on $E^{(\alpha)}$ satisfies

$$\mathbb{E}_{\theta^{(\alpha)}}[f(X)] \mathbb{E}_{\theta^{(\alpha)}}[h(X)] = \bar{f}^{(\alpha)} h^{(\alpha)},$$

where $\mathbb{E}_{\theta^{(\alpha)}}[\cdot \cdot \cdot]$ denotes expectation with respect to $p_{\theta}^{(\alpha)}$; $\bar{h}^{(\alpha)}$ and $\bar{f}_i^{(\alpha)}$ are respectively averages of $h$ and $f_i$ with respect to $p_n^{(\alpha)}$. This result is due to [39, Th. 6].

In this section we extend the above projection theorems to the general power-law families by solving the respective estimating equations. We also find conditions under which the new projection equations reduce to the ones as in the canonical case.

First we have the following lemma that establishes a connection between the generalized Hellinger estimating equation (8) and the Jones et. al. estimating equation (10).

**Lemma 7**: The estimating equations (7) and (10) are the same up to the transformation $p \mapsto p^{(\alpha)}$ when $S$ is discrete. In the continuous case the same is true between (8) and (10) if $\int \nabla[p_\theta(x)]dx = 0$ and in (10) the empirical measure $p_n$ is replaced by a continuous estimate $\tilde{p}_n$.

This, together with Lemma [6] establishes the following.
Corollary 8: Suppose that $E^{(a)}$ is an $\alpha$-exponential family characterized by $h, w, f, \Theta$ where all the distributions have a common support $S$. Then solving the generalized Hellinger estimation problem under $E^{(a)}$-family is equivalent to solving the Jones et al. estimation problem under the $M^{(1/\alpha)}$-family characterized by the same entities.

We now prove the main result of this section.

Theorem 9: Let $X_1, \ldots, X_n$ be $n$ i.i.d. samples. Let $\Pi$ be one of the families $E$, $B^{(a)}$, $M^{(a)}$, or $E^{(a)}$ and assume that support of $\Pi$ does not depend on the parameter space $\Theta$. In (a) and (d), also assume that $\int \partial_r[p_\theta(x)]dx = 0$ for $r = 1, \ldots, k$. Then the following hold.

(a) MLE under $E$ must satisfy
\[
\partial_r[w(\theta)]^T \mathbb{E}_\theta[f(X)] = \partial_r[w(\theta)]^T \bar{f} \quad \text{for } r = 1, \ldots, k. \tag{47}
\]

(b) Basu et al. estimator under $B^{(a)}$ must satisfy (47).

(c) Jones et al. estimator under $M^{(a)}$ must satisfy
\[
\frac{\partial_r[w(\theta)]^T \mathbb{E}_\theta[f(X)]}{\mathbb{E}_\theta[h(X) + w(\theta)^T f(X)]} = \frac{\partial_r[w(\theta)]^T \bar{f}}{h + w(\theta)^T \bar{f}} \quad \text{for } r = 1, \ldots, k. \tag{48}
\]

(d) Generalized Hellinger estimator under $E^{(a)}$ must satisfy
\[
\frac{\partial_r[w(\theta)]^T \mathbb{E}_{\theta^{(a)}}[f(X)]}{\mathbb{E}_{\theta^{(a)}}[h(X) + w(\theta)^T f(X)]} = \frac{\partial_r[w(\theta)]^T \bar{f}^{(a)}}{\bar{\theta}^{(a)} + w(\theta)^T \bar{f}^{(a)}} \quad \text{for } r = 1, \ldots, k. \tag{49}
\]

Here $\partial_r[w(\theta)] := \left[\frac{\partial}{\partial \theta_1}[w_1(\theta)], \ldots, \frac{\partial}{\partial \theta_k}[w_k(\theta)]\right]^T$ for $r = 1, \ldots, k$. In (d), $\bar{\theta}^{(a)} := \mathbb{E}_{\bar{p}_n^{(a)}}[h(X)]$ and $\bar{f}^{(a)} := \mathbb{E}_{\bar{p}_n^{(a)}}[f(X)]$, where $\bar{p}_n$ is the empirical distribution $p_n$ in the discrete case; a suitable continuous estimate of $p_n$ in the continuous case.

Remark 4:

(a) Recall that a $B^{(a)}$-family can be expressed as an $M^{(a)}$-family as in (32) or (33). This implies that the Jones et al. estimator under $B^{(a)}$-family satisfies (48) with $w, f$ replaced by $\tilde{w}, \tilde{f}$ as defined in (32) or (33). Further, in view of Remark 2 (32) or (33) is also an $E^{(\alpha')}$-family where $\alpha' = 2 - \alpha$. Thus the $\alpha'$-generalized Hellinger estimator under $B^{(a)}$-family must satisfy (49) with $w, f$ and $\alpha$ replaced, respectively, by $\tilde{w}, \tilde{f}$ and $\alpha'$.

(b) An $M^{(a)}$-family can be expressed as a $B^{(a)}$-family as in (31). Thus the Basu et al. estimator under an $M^{(a)}$-family must satisfy (47) with $w$ and $f$ replaced by $\tilde{w}, \tilde{f}$ as defined in (31).
Further, in view of Remark 2 the \( \alpha' \)-generalized Hellinger estimator under \( \mathcal{M}(\alpha) \)-family must satisfy (49) with \( \alpha \) replaced by \( \alpha' \).

(c) An \( \mathcal{E}(\alpha) \)-family can be expressed as an \( \mathcal{M}(\alpha') \)-family as in (38), and hence can be expressed as a \( \mathbb{B}(\alpha') \)-family as in (31) with \( \alpha \) replaced by \( \alpha' \). Thus the \( \alpha' \)-Jones et. al. estimator under \( \mathcal{E}(\alpha) \)-family satisfies (48). Similarly the \( \alpha' \)-Basu et. al. estimator under \( \mathcal{E}(\alpha) \)-family satisfies (47) with \( w \) and \( f \) replaced by \( \tilde{w}, \tilde{f} \) as in (31).

When the families are regular, the projection equations reduce to the one as in the canonical case.

**Corollary 10:** Under the assumptions of Theorem 9 the following results hold. Let \( \bar{f}, \bar{h}, \bar{f}(\alpha) \) and \( \bar{h}(\alpha) \) be as defined in Theorem 9.

(a) The estimating equations for MLE under a regular exponential family \( \mathcal{E} \) and that for Basu et. al. under a regular \( \mathbb{B}(\alpha) \)-family reduce to (44).

(b) The Jones et. al. estimating equation under a regular \( \mathcal{M}(\alpha) \)-family reduces to (45).

(c) The generalized Hellinger estimating equation under a regular \( \mathcal{E}(\alpha) \)-family reduces to (46).

Theorem 9 fails if the support of the underlying family depends on the parameters. We show these by some examples in C1 and C2.

Basu et. al. estimating equation (9) differs from the Jones et. al. estimating equation (10) in which the weights are normalized. Much research has been done to compare these two estimations (for example, see [34]). We saw in Section II that a regular \( \mathbb{B}(\alpha) \)-family can be viewed as a regular \( \mathcal{M}(\alpha) \)-family under some conditions. In the following we show that the two estimations coincide on a regular \( \mathbb{B}(\alpha) \)-family with \( h \) being a non-zero constant (or on a regular \( \mathcal{M}(\alpha) \)-family with \( h \) being a non-zero constant).

**Theorem 11:** For a regular \( \mathbb{B}(\alpha) \)-family with \( h \) being identically a non-zero constant, the Basu et. al. and the Jones et. al. estimating equations are the same.

**Corollary 12:** For a regular \( \mathcal{M}(\alpha) \)-family of the form

\[
p_\theta(x) = Z(\theta)[h + w(\theta)^T f(x)]^{1-\alpha}
\]

for \( x \in \mathbb{S} \), where \( h \) is a non-zero constant, the Basu et. al. and the Jones et. al. estimating equations are the same if \( Z^{1-\alpha} \) is linearly independent with \( w_i \)'s.
IV. APPLICATIONS: GENERALIZED ESTIMATION UNDER STUDENT AND CAUCHY DISTRIBUTIONS

In this section we use Corollary 10 of Section III to find generalized estimators for the parameters of Student and Cauchy distributions.

A. Basu et. al. and Jones et. al. estimation under Student distributions

In Example 1 we saw that for \( \alpha \in ((d-2)/d, 1) \) (that is, for \( \nu \in (0, \infty) \)) Student distributions form a \( d(d+3)/2 \)-parameter regular \( \mathbb{B}^{(\alpha)} \)-family with \( f^{(1)}(x) = x \) and \( f^{(2)}(x) = \text{vec}(xx^T) \). Hence to find the Basu et. al. estimators of the parameters of a Student distribution, its mean and variance should be finite. However, as we saw in Example 1, (21) does not have finite mean and variance for \( \alpha \in ((d-2)/d, d/(d+2)) \]. Hence we restrict ourselves to Student distributions for \( \alpha \in (d/(d+2), 1) \). The mean and the covariance of a Student distribution for \( \alpha \in (d/(d+2), 1) \) are given by \( \mu \) and \( K := (k_{ij})_{d \times d} = [\nu/(\nu-2)] \cdot \Sigma \) respectively. Let \( X_1, \ldots, X_n \) be an i.i.d. sample where each \( X_i = [X_{1i}, \ldots, X_{di}]^T \) for \( i = 1, \ldots, n \). Suppose also that the true distribution \( p \) is a Student distribution as in (21). Using Corollary 10(a), the Basu et. al. estimators of \( \mu \) and \( K \) are given by

\[
\hat{\mu} = \bar{X}, \quad \hat{k}_{ij} = \frac{1}{n} \sum_{l=1}^{n} X_{il}X_{jl} - \hat{\mu}_i\hat{\mu}_j,
\]

(50)

for \( i, j = 1, \ldots, d \) and \( i \leq j \), where \( \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \).

Next consider the Student distributions as in (30) with \( \alpha \in (d/(d+2), 1) \). We saw that it forms a \( d(d+3)/2 \)-parameter regular \( \mathbb{M}^{(\alpha)} \)-family with \( h \equiv 1 \). Hence, from Theorem 11, the Jones et. al. estimators for \( \mu \) and \( K \) are the same as the Basu et. al. estimators as in (50). We summarize these in the following theorem.

**Theorem 13:** For \( \alpha \in (d/(d+2), 1) \), the Basu et. al. and the Jones et. al. estimators of the mean and covariance parameters of a \( d \)-dimensional Student distribution as in (20) are the same and are given by (50).

**Remark 5:** For \( \alpha \notin ((d-2)/d, 1) \), the support of Student distributions depend on the parameters. Thus Theorem 9 can not be used to find the estimators. However, in this case, one can find the estimators by maximizing the respective likelihood function as described in C1.

**Remark 6:** It can be shown that, as \( \alpha \to 1 \), the Student distributions coincide with a normal distribution with mean \( \mu \) and covariance matrix \( K \). Similarly when \( \alpha = 1 \), the Basu et. al. estimating equation or the Jones et. al. estimating equation becomes the estimating equation (5).
of MLE. Thus there is a continuity on the $\alpha$-estimations of the mean and covariance parameters for $\alpha \in (0, 1]$.

**B. Generalized Hellinger estimation under Cauchy distributions**

Let $X_i = [X_{1i}, \ldots, X_{di}]^T$, $i = 1, \ldots, n$ be an i.i.d. sample. Let the true distribution $p$ be a Cauchy distribution as in (40). Let $\hat{p}_n$ be a suitable continuous density estimator of the empirical measure $p_n$ of the sample $X_i$'s. When $\beta \in (1, (d+2)/d)$, we saw that Cauchy distributions form a regular $\mathcal{E}^{(\beta)}$-family. Thus in this case we will use Corollary 10(c) to estimate its parameters. But for $\beta \notin (1, (d+2)/d)$, the support of this distribution depends on the unknown parameters. In this case one can estimate the parameters by maximizing the associated likelihood function (12) as described in [22].

Let $\beta \in (1, (d+2)/d)$. In Example 5 we saw that Cauchy distributions form a $d(d+3)/2$-parameter regular $\mathcal{E}^{(\beta)}$-family with $h(x) \equiv 1$, $f^{(1)}(x) = x$ and $f^{(2)}(x) = \text{vec}(xx^T)$. Using (46), therefore we have the following estimating equations for the Cauchy distribution.

$$
\mathbb{E}_{\eta^{(\beta)}}[X] = \mathbb{E}_{\eta^{(\beta)}}[\text{vec}(XX^T)] = \mathbb{E}_{\eta^{(\beta)}}[\text{vec}(XX^T)].
$$

(51)

Let us find $\mathbb{E}_{\eta^{(\beta)}}[X]$ and $\mathbb{E}_{\eta^{(\beta)}}[\text{vec}(XX^T)]$. In Example 5 we saw that $p^{(\alpha)}_{\theta} = q_{\eta}$, where $\alpha = 1/\beta$. Hence $q^{(\beta)}_{\eta} = p_{\theta}$. Thus

$$
\mathbb{E}_{\eta^{(\beta)}}[X] = \mathbb{E}_{\theta}[X] = \mu \quad \text{and} \quad \mathbb{E}_{\eta^{(\beta)}}[X_iX_j] = \mathbb{E}_{\theta}[X_iX_j] = k_{ij} + \mu_i\mu_j,
$$

(52)

where $X = [X_1, \ldots, X_d]^T$ and $k_{ij} = [\nu/(\nu - 2)] \cdot \sigma_{ij}$. Using this in (51), we get

$$
\mu = \mathbb{E}_{\eta^{(\beta)}}[X], \quad k_{ij} = \mathbb{E}_{\eta^{(\beta)}}[X_iX_j] - \hat{\mu}_i\hat{\mu}_j,
$$

(53)

for $i, j = 1, \ldots, d$ and $i \leq j$. Thus the generalized Hellinger estimators for the the location and scale parameters are

$$
\hat{\mu} = \mathbb{E}_{\eta^{(\beta)}}[X], \quad \hat{\sigma}_{ij} = \left( \mathbb{E}_{\eta^{(\beta)}}[X_iX_j] - \hat{\mu}_i\hat{\mu}_j \right) / \left( \nu / [\nu - 2] \right)
$$

(54)

for $i, j = 1, \ldots, d$ and $i \leq j$. We summarize these in the following theorem.

**Theorem 14:** For $\beta \in (1, (d+2)/d)$, the generalized Hellinger estimators for the location and scale parameters of a $d$-dimensional Cauchy distribution as in (40) are given by (54).

Notice that the estimators involve a continuous estimate $\tilde{p}_n$ of $p_n$. In the following we present an example where we use a ‘kernel density estimation’ to find such $\tilde{p}_n$ and use that to find the
estimators. In the literature the commonly used kernel to estimate the $d$-dimensional empirical measure is of the following form (see [53]):

$$\tilde{p}_n(x) = \frac{1}{nh_n^d} \sum_{i=1}^{n} \xi \left( \frac{x - X_i}{h_n} \right),$$  \hspace{1cm} (55)$$

where $\xi$ is a symmetric distribution on $\mathbb{R}^d$ and $\{h_n\}$ is a sequence of real numbers with suitable properties, called bandwidth. These properties of the kernel $\xi$ and bandwidth $h_n$ influence the performance of the estimators greatly. There is no general theory in the literature to choose the right continuous estimate for a given problem. However authors like Beran [5], Tamura and Boos [53] and Simpson [50] imposed conditions on $k$ and $h_n$ so that the estimators perform better in their specific setting. We shall use the following kernel, called uniform kernel, to find the estimators. The $d$-dimensional uniform kernel is defined as follows.

$$\xi(x) = \begin{cases} \frac{1}{2^d} & \text{if } x \in [-1, 1]^d \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\xi \left( \frac{x - X_i}{h_n} \right) = \begin{cases} \frac{1}{2^d} & \text{if } (x - X_i) \in [-h_n, h_n]^d \\ 0 & \text{otherwise}, \end{cases}$$

for $i = 1, \ldots, n$. Let us choose the bandwidth $h_n = n^{-1/2d}$ for $n \geq 1$. Then $\xi$ and $h_n$ satisfy the following conditions which guarantee the $L_1$ convergence of $\tilde{p}_n$ to the true density (see [4, Sec. 3.3] and the references therein):

(i) $\xi$ is symmetric about 0 and has compact support.

(ii) $\lim_{n \to \infty} h_n = 0$ and $\lim_{n \to \infty} [h_n + (nh_n^d)^{-1}] = 0$.

Let us denote $[X_{1i} - n^{-1/2d}, X_{1i} + n^{-1/2d}] \times \cdots \times [X_{di} - n^{-1/2d}, X_{di} + n^{-1/2d}]$ by $[\mathbf{X}_i - n^{-1/2d}, \mathbf{X}_i + n^{-1/2d}]$ for $i = 1, \ldots, n$ and call them rectangles. Assume that all these rectangles are disjoint (these are actually disjoint for large enough $n$). Then from (55) we have

$$\tilde{p}_n(x) = n^{-1/2} \sum_{i=1}^{n} \xi \left( n^{1/2d} [x - \mathbf{X}_i] \right).$$

That is,

$$\tilde{p}_n(x) = \begin{cases} n^{-1/2} 2^{-d} & \text{if } x \in [\mathbf{X}_i - n^{-1/2d}, \mathbf{X}_i + n^{-1/2d}] \text{ for } i = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$
Thus \( \tilde{p}_n \) is the uniform distribution on \( \bigcup_{i=1}^{n} [X_i - n^{-1/2d}, X_i + n^{-1/2d}] \). This implies that the \( \beta \)-scaled distribution \( \tilde{p}_n^{(\beta)} \) is the same as \( \tilde{p}_n \). Therefore, we have

\[
E_{\tilde{p}_n^{(\beta)}}[X] = \frac{1}{n} \sum_{i=1}^{n} X_i,
\]

\[
E_{\tilde{p}_n^{(\beta)}}[X^2] = \frac{1}{n^{1/2d}} \sum_{l=1}^{n} \frac{(2n^{-1/2d})^{d-1}[(X_{jl} + n^{-1/2d})^3 - (X_{jl} - n^{-1/2d})^3]}{3}
= \frac{1}{n^{1/2d}} \sum_{l=1}^{n} \frac{(2n^{-1/2d})^{d-1}[2n^{-1/2d}(3X_{jl}^2 + n^{-2/2d})]}{3}
= \frac{1}{n^{1/2d}} \sum_{l=1}^{n} \frac{(2n^{-1/2d})^d[(3X_{jl}^2 + n^{-2/2d})]}{3}
= \frac{1}{n} \sum_{l=1}^{n} X_{jl}^2 + \frac{1}{3n^{1/d}}, \text{ for } j = 1, \ldots, d,
\]

and

\[
E_{\tilde{p}_n^{(\beta)}}[X_i X_j] = \frac{1}{n^{1/2d}} \sum_{l=1}^{n} \frac{(2n^{-1/2d})^{d-2}[4n^{-1/2d}X_{il}][4n^{-1/2d}X_{jl}]}{4}
= \frac{1}{n} \sum_{l=1}^{n} X_{il} X_{jl} \text{ for } i, j = 1, \ldots, d \text{ and } i < j.
\]

Hence the estimators for the parameters are given by

\[
\hat{\mu} = \overline{X}, \quad \hat{\kappa}_{ij} = \left[\nu/(\nu - 2)\right] \cdot \hat{\sigma}_{ij} = \frac{1}{n} \sum_{l=1}^{n} X_{il} X_{jl} - \hat{\mu}_i \hat{\mu}_j + \epsilon_n,
\]

for \( i, j = 1, \ldots, d \) and \( i \leq j \), where \( \epsilon_n = \frac{1}{3n^{1/d}} \) if \( i = j \) and equals to zero otherwise.

**Remark 7:** In the view of Example 6 we know that Student distributions form a regular \( \mathcal{E}^{(\alpha')} \)-family for \( \alpha' \in (1, (d + 2)/d) \). Thus one can do the \( \alpha' \)-generalized Hellinger estimation on Student distributions as well.

## V. Projection Theorems and Principle of Sufficiency

In the previous section we noticed that the projection (or estimating) equations for the power-law families depended only on some specific statistics of the sample. In this section we explore this phenomenon further from the principle of sufficiency. In a parameter estimation problem, if there is no prior information about the unknown parameter \( \theta \), an experimenter uses only the given sample to estimate \( \theta \). However, if there is a statistic \( T \) such that the estimator depends on the sample only through \( T \), then one who knows the value of \( T \) can find the estimator...
without having the knowledge of the entire sample. In other words, the statistic \( T \) contains all the relevant information about the unknown parameter \( \theta \) that the entire sample can supply. Fisher named such statistic sufficient for estimating the parameter in question. He gave the following definition for sufficiency \[28\]. A statistic \( T \) is sufficient for \( \theta \) if the conditional distribution of the sample given the value of \( T \) does not depend on \( \theta \). However, it is not clear if Fisher had MLE in mind for estimation while proposing this definition. Later, Koopman gave another mathematical formulation for sufficient statistic based on Fisher’s idea \[36, pp. 400\]. It is the following. Let \( X^n_1 := (X_1, \ldots, X_n) \) and \( Y^n_1 := (Y_1, \ldots, Y_n) \) be two i.i.d. samples from some \( p_\theta \in \Pi \). A system of statistics \( T \) is said to be a sufficient statistic for \( \theta \) if
\[
\frac{p_\theta(X^n_1)}{p_\theta(Y^n_1)}
\]
is independent of \( \theta \) whenever \( T(X^n_1) = T(Y^n_1) \). However, still it is not very clear whether the estimation in question is MLE. The following result, known as factorization theorem \[13, Th. 6.2.6\], which is an equivalent criterion of Koopman’s (and Fisher’s) definition, helps us answer this question.

A system of statistics \( T \) is a sufficient statistic for \( \theta \) if and only if there exists functions \( g : \Theta \times \mathcal{T} \to \mathbb{R} \) and \( h : \mathbb{R}^n \to \mathbb{R} \), where \( \mathcal{T} := \{ t : T(X^n_1) = t \text{ for some } X^n_1 \} \), such that the log-likelihood function can be written as
\[
L(X^n_1; \theta) = g(\theta, T(X^n_1)) + h(X^n_1),
\]
for all samples \( X^n_1 \) and for all \( \theta \in \Theta \). This implies that MLE of \( \theta \) is given by
\[
\arg \max_{\theta} L(X^n_1; \theta) = \arg \max_{\theta} [g(\theta, T(X^n_1)) + h(X^n_1)] = \arg \max_{\theta} g(\theta, T(X^n_1)).
\]
That is, MLE depends on the sample only through \( T \). Thus it is now clear that the estimation in question while formulating the classical notion of sufficiency was MLE.

However, when the data set is contaminated, as we saw in Section \[II\] one should use some generalized likelihood function \( L_G \) for inference instead of the usual log-likelihood function. Hence, if we can analogously write \( L_G(\theta) \) as in \( (57) \) for some functions \( g \) and \( h \), then the estimator would depend on the sample only through the statistic \( T \). Thus, if the estimation is by maximizing a likelihood function \( L_G \), it is reasonable to call such \( T \) a sufficient statistic for \( \theta \) with respect to the likelihood function \( L_G \). In the following we first extend the notion of sufficient statistics with respect to a generalized likelihood function, along the lines of Koopman. We then derive a factorization theorem that is appropriate for this generalized notion. We also prove that, as in the exponential family \[36, Th. I\], the power-law families admit a fixed number of sufficient statistics irrespective of the sample size with respect to this generalized notion.
**Definition 15:** Let \( X_1^n \) and \( Y_1^n \) be two i.i.d. samples and \( L_G \) be any generalized likelihood functions as in (12) - (14). Then \( T \) is a sufficient statistic for \( \theta \) with respect to \( L_G \) if \( T(X_1^n) = T(Y_1^n) \) implies \( [L_G(X_1^n; \theta) - L_G(Y_1^n; \theta)] \) is independent of \( \theta \) for every \( \theta \in \Theta \).

**Proposition 16:** Let \( L_G \) be a generalized likelihood function. Then a system of statistics \( T \) is a sufficient statistic for \( \theta \) with respect to \( L_G \) if and only if there exists functions \( g \) and \( h \) such that

\[
L_G(X_1^n; \theta) = g(\theta, T(X_1^n)) + h(X_1^n),
\]
for all sample points \( X_1^n \) and for all \( \theta \in \Theta \).

We now find sufficient statistics for the parameters of each of the power-law families with respect to the likelihood function associated with them.

**Theorem 17:** Let \( \Pi \) be any of the families \( \mathcal{E}, \mathbb{B}^{(\alpha)}, \mathcal{M}^{(\alpha)} \) or \( \mathcal{S}^{(\alpha)} \) with support \( S \) as defined in Section II and let \( L(\theta), L_2^{(\alpha)}(\theta), L_3^{(\alpha)}(\theta) \) and \( L_1^{(\alpha)}(\theta) \) be the associated likelihood functions. Suppose that \( S \) does not depend on the parameter \( \theta \). Then \( T \) is sufficient statistics for \( \theta \) with respect to the associated likelihood function if there exists real valued functions \( \psi_1, \ldots, \psi_s \) defined on \( \mathcal{T} \), such that for any i.i.d. sample \( X_1^n \) and for \( i = 1, \ldots, s \),

(a) \( \psi_i(T(X_1^n)) = f_i(X_1^n) \), when \( \Pi = \mathcal{E} \) or \( \mathbb{B}^{(\alpha)} \).

(b) \( \psi_i(T(X_1^n)) = f_i(X_1^n)/h(X_1^n) \), when \( \Pi = \mathcal{M}^{(\alpha)} \).

(c) \( \psi_i(T(X_1^n)) = f_i(X_1^n) / h(X_1^n) \), when \( \Pi = \mathcal{S}^{(\alpha)} \).

Here \( f_i(X_1^n) = \sum_{j=1}^n f_i(X_j) / n, h(X_1^n) = \sum_{j=1}^n h(X_j) / n \), \( f_i(X_1^n) = \mathbb{E}_{p_i^{(\alpha)}}[f_i(X)] \) and \( h(X_1^n) = \mathbb{E}_{p_1^{(\alpha)}}[h(X)] \), where \( p_n \) is the empirical measure of \( X_1^n \). In (c), \( \bar{p}_n \) is same as the empirical measure \( p_n \) if \( S \) is finite; otherwise it is a suitable continuous estimate of \( p_n \).

**Remark 8:** Theorem 17 tells us that \( \bar{f}, \bar{f} / \bar{h} \) and \( \bar{f}^{(\alpha)} / \bar{h}^{(\alpha)} \) are respectively the sufficient statistics for \( \mathbb{B}^{(\alpha)}, \mathcal{M}^{(\alpha)} \) and \( \mathcal{S}^{(\alpha)} \)-family with respect to the associated likelihood functions. Interestingly, these are precisely the statistics that influence the respective estimating equations (or the projection equations). (See Theorem 9 and Corollary 10)

**Example 7:**

(i) **Sufficient statistics for Student distributions under Basu et. al. or Jones et. al. estimation:**

Consider the student distributions as in (20) for \( \alpha \in (d/(d+2), 1) \). We saw that they form a \( \mathbb{B}^{(\alpha)} \) and an \( \mathcal{M}^{(\alpha)} \)-family with \( h(x) \equiv 1, f(x) = [x, \text{vec} (xx^T)]^T \), and the support does not depend on the parameters. Let \( \bar{X} := \frac{1}{n} \sum_{i=1}^n X_i \) and \( XX^T \) be the \( d \times d \) matrix whose
(i, j)-th entry is $\frac{1}{n} \sum_{l=1}^{n} X_{il} X_{jl}$. Then the statistics $[\bar{X}, \text{vec}(\bar{X}X^T)]^T$ is a sufficient statistics for $\theta$ with respect to $L_2^{(\alpha)}$ and $L_3^{(\alpha)}$, by Theorem 17 (a) and (b).

(ii) **Sufficient statistics for Cauchy distributions under generalized Hellinger estimation:** Consider the Cauchy distribution as in (40) for $\beta \in (1, (d + 2)/d)$. We saw that they form an $\mathcal{E}^{(\beta)}$-family with the same $h$ and $f$ as in (i), and the support does not depend on the parameters. Let $\tilde{p}_n$ be a continuous estimator for the empirical measure $p_n$. Then $[\mathbb{E}_{\tilde{p}_n(a)}[X], \mathbb{E}_{\tilde{p}_n(a)}[\text{vec}(XX^T)]]^T$ is a sufficient statistics for $\eta$ with respect to $L_1^{(\alpha)}$, by Theorem 17 (c). In particular, if $\tilde{p}_n$ is the uniform kernel density as in Section [V-B] then $[\bar{X}, \text{vec}(\bar{XX}^T)]^T$ is a sufficient statistics for $\eta$, where $\bar{X}$ and $\bar{XX}^T$ are as defined in (i).

Theorem 17 fails when the support of $\Pi$ depends on the parameters. An example showing this is given in [C3].

While sufficient statistics are used for data reduction, minimal sufficient statistics is one that achieves maximum such data reduction. We next show that the statistics in Theorem 17 are actually a minimal sufficient statistic with respect to the associated likelihood function. Let us first recall the definition of a minimal sufficient statistic.

**Definition 18:** [13, Def. 6.2.11] A system of statistics $T$ is said to be a minimal sufficient statistic if $T$ is a function of any other sufficient statistic. In other words, $T$ is minimal if the following holds: For any sufficient statistic $\tilde{T}$ and for any two i.i.d. samples $X^n_i$ and $Y^n_i$, $\tilde{T}(X^n_i) = \tilde{T}(Y^n_i)$ implies $T(X^n_i) = T(Y^n_i)$.

There is an easy criterion to find a minimal sufficient statistic from the log-likelihood function due to Lehmann and Scheffé [13 Th. 6.2.13]. Here we first generalize this criterion for the generalized likelihood functions and then use it to find a minimal sufficient statistic for the power-law families.

**Proposition 19:** Let $L_G$ be a generalized likelihood function. Suppose that the following condition holds for $T$. For any two i.i.d. samples $X^n_i$ and $Y^n_i$, $T(X^n_i) = T(Y^n_i)$ if and only if $[L_G(X^n_i; \theta) - L_G(Y^n_i; \theta)]$ is independent of $\theta$ for $\theta \in \Theta$. Then $T$ is a minimal sufficient statistics with respect to $L_G$.

It is well known that the system of statistics $f(X^n_i)$ is a minimal sufficient statistic for a regular exponential family [40 Cor. 6.16]. Here we have an analogous result for the regular power-law families.

**Theorem 20:** The statistics $f(X^n_i)$, $f(X^n_i)/h(X^n_i)$, $f(X^n_i)^{(\alpha)}/h(X^n_i)^{(\alpha)}$ as defined in Theorem 17 are minimal sufficient statistics, respectively, for the regular $\mathcal{E}^{(\alpha)}$, $\mathcal{M}^{(\alpha)}$, $\mathcal{E}^{(\alpha)}$ families with
respect to the respective likelihood functions.

Example 8:

(i) Minimal sufficient statistics for Student distribution under Basu et. al. or Jones et. al. estimation: Consider the Student distributions as in \( (20) \) for \( \alpha \in \left( \frac{d}{d+2}, 1 \right) \). Then \( \begin{bmatrix} X, \text{vec}(XX^T) \end{bmatrix}^T \) is a minimal sufficient statistics for \( \theta \) with respect to \( L_2^{(\alpha)} \) and \( L_3^{(\alpha)} \), by Theorem 20 and Example 7(i).

(ii) Minimal sufficient statistics for Cauchy distribution under generalized Hellinger estimation: Consider the Cauchy distribution as in \( (40) \) for \( \beta \in \left( 1, \frac{(d+2)}{d} \right) \). Then \( \begin{bmatrix} \hat{E}_{\tilde{p}_n}^{(\beta)}[X], \tilde{E}_{\tilde{p}_n}^{(\beta)}[\text{vec}(XX^T)] \end{bmatrix}^T \) is a minimal sufficient statistics for \( \eta \) with respect to \( L_1^{(\alpha)} \), by Theorem 20 and Example 7(ii). Further, if we use the uniform kernel density estimator for \( \tilde{p}_n \), then \( \begin{bmatrix} X, \text{vec}(XX^T) \end{bmatrix}^T \) is a minimal sufficient statistics for \( \eta \).

VI. Summary and Concluding Remarks

Projection theorems concerning divergences \( I, B_\alpha, J_\alpha \) and \( D_\alpha \) tell us that the reverse projection of \( I, B_\alpha, J_\alpha \) or \( D_\alpha \), respectively, on \( E, B^{(\alpha)}, J^{(\alpha)} \) or \( E^{(\alpha)} \)-families turns out to be a forward projection of the respective divergence on a “simpler” family (linear or \( \alpha \)-linear) which, in turn, reduces to a linear problem on the underlying probability distribution. The applicability of projection theorems known in the literature were limited as they dealt only models in the discrete, canonical set-up. Here we first generalized the power-law families to a more general set-up including the continuous case. We then extended the projection theorems to these families by solving the respective estimating equations. We observed that, for regular families, the new estimating equations coincide with the respective projection equations, similar to the ones in the canonical case. We further tried to understand the projection theorems via the principle of sufficiency. We arrived at a generalized notion of principle of sufficiency based on the likelihood functions arising from the generalized estimating equations. We showed that the sufficient statistics for each of the power-law families are precisely the statistics that influence the respective estimating (or projection) equations. We showed that, if the family is further regular, there always exists a sufficient statistics of fixed length (irrespective of the sample size) that is equal to the number of unknown parameters. The converse of this result is also true as in the case of exponential family. This will be the focus in one of our forthcoming works. Similar to the result of Lehmann and Scheffé for the exponential family \( [40, \text{Cor. 6.16}] \), we also showed that these sufficient statistics for the respective regular power-law families are minimal too.
We finally applied the above ideas to find generalized estimators for the Student and Cauchy distributions. Interestingly, both the Basu et. al. and Jones et. al. estimators for the mean and covariance parameters of a Student distribution for \( \nu > 2 \) are the same as the MLE of the respective parameters of a normal distribution. However it is not the case for \( \nu < -d \). We showed by an example that, when \( \nu < -d \), the generalized estimators for the mean parameter could be different from the sample mean. In a similar fashion we found the generalized Hellinger estimators for the Cauchy distributions. Here too we found estimators of its parameters by using a kernel density estimator for the empirical measure. It is well-known that the MLE for the Student or for the Cauchy distributions do not have a closed form solution. To overcome this, standard iterative methods such as Newton-Raphson, Gauss-Newton, EM are used in the literature [2], [25]. However, the sequence of estimators in these iterative methods may converge to a local maximum and the rate of convergence is also slow [2], [41]. Later some generalized iterative methods such as ECM, ECME were proposed, for example, in [41], where the rate of convergence was made faster than the previous methods. But again, they converge only to a local maximum. The generalized estimators that we studied in this paper might help us overcome some of these issues as we have closed form formula for the estimators (see Eqs. (50), (53)).

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**APPENDIX**

A. *Projection Theorem for Density Power Divergence*

The Projection theorem and the Pythagorean property of the more general class of Bregman divergences were established by Csiszár and Matúš [23] using tools from convex analysis. The density power divergences \( B_\alpha \) is a subclass of the Bregman divergences. However, it is not easy to extract the results for the \( B_\alpha \)-divergence from [23]. In this section we derive the projection results for the \( B_\alpha \)-divergence in the discrete case using some elementary tools. We must point out that the geometry of \( B_\alpha \)-divergence is quite a natural extension of that of \( I \)-divergence. Let
Let $S$ be a finite alphabet set and $\mathcal{P} := \mathcal{P}(S)$ be the space of all probability distributions on $S$. Then for any $p, q \in \mathcal{P}$, from (2), the $B_\alpha$-divergence in the discrete case can be written as

$$B_\alpha(p, q) = \frac{\alpha}{1 - \alpha} \sum_{x \in S} p(x)q(x)^{\alpha-1} - \frac{1}{1 - \alpha} \sum_{x \in S} p(x)^\alpha + \sum_{x \in S} q(x)^\alpha.$$  

(59)

Let us also recall the definitions of reverse and forward projections given in (15) and (16). For $p \in \mathcal{P}$, we shall denote the support of $p$ as $\text{Supp}(p)$. For $\mathcal{C} \subset \mathcal{P}$, $\text{Supp}(\mathcal{C})$ is defined as the union of support of members of $\mathcal{C}$.

We now show the Pythagorean inequality of $B_\alpha$-divergence in connection with the forward projection on a closed, convex set. In the sequel we assume that $\text{Supp}(q) = S$.

**Theorem 21:** Let $p^*$ be the forward $B_\alpha$-projection of $q$ on a closed and convex set $\mathcal{C}$. Then

$$B_\alpha(p, q) \geq B_\alpha(p, p^*) + B_\alpha(p^*, q) \quad \forall p \in \mathcal{C}.$$  

(60)

Further if $\alpha < 1$, $\text{Supp}(\mathcal{C}) = \text{Supp}(p^*)$.

**Proof 1:** Let $p \in \mathcal{C}$ and define, for $t \in [0, 1]$ and $x \in S$,

$$p_t(x) = (1 - t)p^*(x) + tp(x).$$

Since $\mathcal{C}$ is convex, $p_t \in \mathcal{C}$. By mean-value theorem, for each $t \in (0, 1)$,

$$0 \leq \frac{1}{t} [B_\alpha(p_t, q) - B_\alpha(p^*, q)] = \frac{1}{t} [B_\alpha(p_t, q) - B_\alpha(p_0, q)] = \frac{d}{dt} B_\alpha(p_t, q) \bigg|_{t=\bar{t}} \quad \text{for some } \bar{t} \in (0, t).$$  

(61)

Using (59) we have

$$\frac{d}{dt} B_\alpha(p_t, q) = \frac{\alpha}{\alpha - 1} \sum_{x \in S} \left[ p(x) - p^*(x) \right] \left[ p_t(x)^{\alpha-1} - q(x)^{\alpha-1} \right].$$

Therefore (61) implies

$$\frac{\alpha}{\alpha - 1} \sum_{x \in S} \left[ p(x) - p^*(x) \right] \left[ p_t(x)^{\alpha-1} - q(x)^{\alpha-1} \right] \geq 0.$$  

(62)

Hence, as $t \downarrow 0$, we have

$$\frac{\alpha}{\alpha - 1} \sum_{x \in S} \left[ p(x) - p^*(x) \right] \left[ p^*(x)^{\alpha-1} - q(x)^{\alpha-1} \right] \geq 0,$$

(63)

which is equivalent to (60).

If $\text{Supp}(p^*) \neq \text{Supp}(\mathcal{C})$, then there exists $p \in \mathcal{C}$ and $x \in S$ such that $p(x) > 0$ but $p^*(x) = 0$. Hence if $\alpha < 1$, then the left-hand side of (62) goes to $-\infty$ as $t \downarrow 0$, which contradicts (62). This proves the claim.
Remark 9: If $\alpha > 1$, in general, $\text{Supp}(p^*) \neq \text{Supp}(C)$. [38 Example 2] serves as a counterexample here as well. It follows from the following fact. Since $\mathbb{S}$ is finite, the $B_\alpha$-divergence can be written as

$$B_\alpha(p, u) = \frac{1}{1 - \alpha} |\mathbb{S}|^{1 - \alpha} - \frac{1}{1 - \alpha} \sum_{x \in \mathbb{S}} p(x)\alpha,$$

where $u$ is the uniform distribution on $\mathbb{S}$ and $|\mathbb{S}|$ denotes the cardinality of $\mathbb{S}$. This implies

$$\arg \min_{p \in C} B_\alpha(p, u) = \arg \max_{p \in C} H_\alpha(p),$$

where $H_\alpha(p) := \frac{1}{1 - \alpha} \log \sum_{x \in \mathbb{S}} p(x)\alpha$, the Rényi entropy of $p$ of order $\alpha$. That is, forward $B_\alpha$-projection of the uniform distribution on $C$ is same as the maximizer of Rényi entropy on $C$. The same is true when $B_\alpha$ is replaced by $J_\alpha$ or $D_\alpha$.

We will now show that equality holds in (60) when $\alpha < 1$ and $C$ is a linear family.

Definition 22: The linear family, determined by $k$ real valued functions $f_i, i = 1, \ldots, k$ on $\mathbb{S}$ and $k$ real numbers $a_i, i = 1, \ldots, k$, is defined as

$$\mathbb{L} := \{ p \in \mathcal{P} : \sum_{x \in \mathbb{S}} p(x)f_i(x) = a_i, \ i = 1, \ldots, k \}.$$

Theorem 23: Let $p^*$ be the forward $B_\alpha$-projection of $q$ on $\mathbb{L}$. The following hold.

(a) If $\alpha < 1$ then the Pythagorean equality holds, that is,

$$B_\alpha(p, q) = B_\alpha(p, p^*) + B_\alpha(p^*, q) \quad \forall p \in \mathbb{L}. \quad (65)$$

(b) If $\alpha > 1$ and $\text{Supp}(p^*) = \text{Supp}(\mathbb{L})$ then the Pythagorean equality (65) holds.

Proof 2: (a) Let $p_t$ be as in Theorem 21. Since $\text{Supp}(p^*) = \text{Supp}(\mathbb{L})$, there exists $t' < 0$ such that $p_t = (1 - t)p^* + tp \in \mathbb{L}$ for $t \in (t', 0)$. Hence, proceeding as in Theorem 21 for every $t \in (t', 0)$, there exists $\tilde{t} \in (t, 0)$ such that

$$\frac{\alpha}{\alpha - 1} \sum_{x \in \mathbb{S}} [p(x) - p^*(x)] [p_{\tilde{t}}(x)^{\alpha - 1} - q(x)^{\alpha - 1}] \leq 0.$$ 

Hence we get (63) with a reversed inequality. Thus we have equality in (63). Hence (65) holds.

(b) Similar to (a).

When $\alpha > 1$, equality in (65) does not hold in general. We have the following counterexample.

Example 9: Consider the same setting as in [38 Example 2], that is, $\alpha = 2, \mathbb{S} = \{1, 2, 3, 4\}$ and

$$\mathbb{L} := \{ p \in \mathcal{P} : p(1) - 3p(2) - 5p(3) - 6p(4) = 0 \}.$$
In view of Remark 9 and Example 2, we see that \( p^* = [3/4, 1/4, 0, 0]^T \) is the forward \( B_\alpha \)-projection of the uniform distribution \( u \) on \( \mathbb{L} \). It is easy to see that there exists \( p \in \mathbb{L} \) (for example \( p = [471/600, 97/600, 12/600, 20/600]^T \)) that satisfies only the strict inequality in (60).

The issue here is that \( \text{Supp}(p^*) \subsetneq \text{Supp}(\mathbb{L}) \).

We will now find an explicit expression of the forward \( B_\alpha \)-projection in both the cases \( \alpha < 1 \) and \( \alpha > 1 \) separately.

**Theorem 24:** Let \( q \in \mathcal{P} \) and let \( \mathbb{L} \) be a linear family of probability distributions as in (64).

(a) If \( \alpha < 1 \), the forward \( B_\alpha \)-projection \( p^* \) of \( q \) on \( \mathbb{L} \) satisfies

\[
p^*(x) = \left[ q(x)^{\alpha - 1} + F + \theta^T f(x) \right]^{\frac{1}{\alpha - 1}} \quad \forall x \in \text{Supp}(\mathbb{L}),
\]

with \( \theta := [\theta_1, \ldots, \theta_k]^T \), \( f := [f_1, \ldots, f_k]^T \) where \( \theta_1, \ldots, \theta_k \) are some scalars and \( F \) is a constant.

(b) If \( \alpha > 1 \), the forward \( B_\alpha \)-projection \( p^* \) of \( q \) on \( \mathbb{L} \) satisfies

\[
p^*(x) = \left[ q(x)^{\alpha - 1} + F + \theta^T f(x) \right]^{\frac{1}{\alpha - 1}} \quad \forall x \in \mathbb{S},
\]

where \( \theta, f \) and \( F \) are as in (a).

**Proof 3:**

(a) The proof is similar to that of Csiszár and Shields for \( I \)-divergence [24, Th. 3.2]. The linear family in (64) can be re-written as

\[
\mathbb{L} := \left\{ p \in \mathcal{P} : \sum_{x \in \text{Supp}(\mathbb{L})} p(x)f_i(x) = a_i, \quad i = 1, \ldots, k \right\}.
\]

Let \( \mathbb{H} \) be the subspace of \( \mathbb{R}^{|\text{Supp}(\mathbb{L})|} \) spanned by the \( k \) vectors \( f_i(\cdot) - a_1, \ldots, f_k(\cdot) - a_k \). Then every \( p \in \mathbb{L} \) can be thought of a \( |\text{Supp}(\mathbb{L})| \)-dimensional vector in \( \mathbb{H} \). Hence \( \mathbb{H} \) is a subspace of \( \mathbb{R}^{|	ext{Supp}(\mathbb{L})|} \) that contains a vector whose components are strictly positive since \( p^* \in \mathbb{L} \) and \( \text{Supp}(p^*) = \text{Supp}(\mathbb{L}) \). It follows that \( \mathbb{H} \) is spanned by its probability vectors.

From (63) we see that (65) is equivalent to

\[
\sum_{x \in \mathbb{S}} [p(x) - p^*(x)] [p^*(x)^{\alpha - 1} - q(x)^{\alpha - 1}] = 0 \quad \forall p \in \mathbb{L}.
\]

This implies that the vector

\[ p^*(\cdot)^{\alpha - 1} - q(\cdot)^{\alpha - 1} - \sum_x p^*(x) [p^*(x)^{\alpha - 1} - q(x)^{\alpha - 1}] \in (\mathbb{H})^\perp = \mathbb{H}. \]
Hence
\[ p^*(x)^{\alpha-1} - q(x)^{\alpha-1} - \sum_x p^*(x) [p^*(x)^{\alpha-1} - q(x)^{\alpha-1}] \]
\[ = \sum_{i=1}^k c_i [f_i(x) - a_i] \quad \forall x \in \text{Supp}(\mathcal{M}) \]
for some scalars \( c_1, \ldots, c_k \). This implies (66) for appropriate choices of \( F \) and \( \theta_1, \ldots, \theta_k \).

(b) The proof of this case is similar to that of \( J_{\alpha} \)-divergence \[38, \text{Th. 14(b)}\]. The optimization problem concerning the forward \( B_{\alpha} \)-projection is
\[
\min_p B_{\alpha}(p, q) \tag{70}
\]
subject to
\[
\sum_x p(x) f_i(x) = a_i, \quad i = 1, \ldots, k, \tag{71}
\]
\[
\sum_x p(x) = 1, \tag{72}
\]
\[
p(x) \geq 0 \quad \forall x \in \mathcal{S}. \tag{73}
\]
Hence by \[6, \text{Prop. 3.3.7}\], there exists Lagrange multipliers \( \lambda_1, \ldots, \lambda_k, \nu \) and \((\mu(x), x \in \mathcal{S})\), respectively, associated with the above constraints such that, for \( x \in \mathcal{S} \),
\[
\frac{\partial}{\partial p(x)} B_{\alpha}(p, q) \bigg|_{p=p^*} = -\sum_{i=1}^k \lambda_i [f_i(x) - a_i] + \mu(x) - \nu, \tag{74}
\]
\[
\mu(x) \geq 0, \tag{75}
\]
\[
\mu(x) p^*(x) = 0. \tag{76}
\]
Since
\[
\frac{\partial}{\partial p(x)} B_{\alpha}(p, q) = \frac{\alpha}{\alpha - 1} [p(x)^{\alpha-1} - q(x)^{\alpha-1}], \tag{77}
\]
(74) can be re-written as
\[
\frac{\alpha}{\alpha - 1} [p^*(x)^{\alpha-1} - q(x)^{\alpha-1}] = -\sum_{i=1}^k \lambda_i [f_i(x) - a_i] + \mu(x) - \nu \quad \text{for} \quad x \in \mathcal{S}. \tag{78}
\]
Multiplying both sides by \( p^*(x) \) and summing over all \( x \in \mathcal{S} \), we get
\[
\nu = \frac{\alpha}{\alpha - 1} \sum_{x \in \mathcal{S}} p^*(x) [q(x)^{\alpha-1} - p^*(x)^{\alpha-1}].
\]
For \( x \in \text{Supp}(p^*) \), from (76), we must have \( \mu(x) = 0 \). Then, from (78), we have
\[
p^*(x)^{\alpha - 1} = q(x)^{\alpha - 1} - \frac{\alpha - 1}{\alpha} \sum_{i=1}^{k} \lambda_i [f_i(x) - a_i] - \frac{\alpha - 1}{\alpha} \nu. \tag{79}
\]
If \( p^*(x) = 0 \), from (78) we get
\[
q(x)^{\alpha - 1} - \frac{\alpha - 1}{\alpha} \sum_{i=1}^{k} \lambda_i [f_i(x) - a_i] - \frac{\alpha - 1}{\alpha} \nu = -\frac{\alpha - 1}{\alpha} \mu(x) \leq 0. \tag{80}
\]
Combining (79) and (80) we get (67).

Theorem 24 suggests us to define a parametric family of probability distributions that is a generalization of the usual exponential family. We call it a \( \mathbb{B}^{(\alpha)} \)-family. First we formally define this family and then show an orthogonality relationship between this family and the linear family. As a consequence we will also show that the reverse \( B_{\alpha} \)-projection on a \( \mathbb{B}^{(\alpha)} \)-family is same as the forward projection on a linear family.

**Definition 25:** Let \( q \in P \) where \( \text{Supp}(q) = S \) for \( \alpha > 1 \) and \( f = [f_1, \ldots, f_k]^T \) where \( f_i \) for \( i = 1, \ldots, k \) be real valued function on \( S \). The \( k \)-parameter canonical \( \mathbb{B}^{(\alpha)} := \mathbb{B}^{(\alpha)}(q, f) \) family of probability distributions characterized by \( q \) and \( f \) is defined by \( \mathbb{B}^{(\alpha)} = \{ p_\theta : \theta \in \Theta \} \subset P \) where
\[
p_\theta(x) = \left[ q(x)^{\alpha - 1} + F(\theta) + \theta^T f(x) \right]^{-\frac{1}{\alpha - 1}} > 0 \quad \text{for } x \in S, \tag{81}
\]
for some \( F : \Theta \to \mathbb{R} \) and \( \Theta \) is the subset of \( \mathbb{R}^k \) for which \( p_\theta \in P \).

**Remark 10:**
(a) Observe that \( \mathbb{B}^{(\alpha)} \)-family is a special case of the family \( \mathcal{F}_{[\beta h]} \) in [23, Eq. (28)] with \( h = q \) and \( \beta(\cdot, t) = \frac{1}{\alpha - 1} [t^\alpha - \alpha t + \alpha - t] \).
(b) The family depends on the reference measure \( q \) only in a loose manner in the sense that any other member of the family can play the role of \( q \). The change of reference measure only corresponds to a translation of the parameter space. (This fact is true for the \( \mathbb{M}^{(\alpha)} \)-family [38, Prop. 22].)

The following theorem and its corollary together establish an “orthogonality” relationship between the \( \mathbb{B}^{(\alpha)} \)-family and the associated linear family.

**Theorem 26:** Let \( \alpha \in (0, 1) \). Consider a \( \mathbb{B}^{(\alpha)} \)-family as in Definition 25 and let \( \mathbb{L} \) be the corresponding linear family determined by the same functions \( f_i, i = 1, \ldots, k \) and some constants \( a_i, i = 1, \ldots, k \) as in (64). If \( p^* \) is the forward \( B_{\alpha} \)-projection of \( q \) on \( \mathbb{L} \) then we have the following:
(a) \( \mathbb{L} \cap \text{cl}(\mathbb{B}^{(\alpha)}) = \{p^*\} \) and
\[
B_\alpha(p, q) = B_\alpha(p, p^*) + B_\alpha(p^*, q) \quad \forall p \in \mathbb{L}.
\] (82)

(b) Further, if \( \text{Supp}(\mathbb{L}) = \mathbb{S} \), then \( \mathbb{L} \cap \mathbb{B}^{(\alpha)} = \{p^*\} \).

Proof 4: By Theorem 24, the forward \( B_\alpha \)-projection \( p^* \) of \( q \) on \( \mathbb{L} \) is in \( \mathbb{B}^{(\alpha)} \). This implies that \( p^* \in \mathbb{L} \cap \mathbb{B}^{(\alpha)} \). Hence it suffices to prove the following:

(i) Every \( \tilde{p} \in \mathbb{L} \cap \text{cl}(\mathbb{B}^{(\alpha)}) \) satisfies (82) with \( \tilde{p} \) in place of \( p^* \).

(ii) \( \mathbb{L} \cap \text{cl}(\mathbb{B}^{(\alpha)}) \) is non-empty.

We now proceed to prove both (i) and (ii).

(i) Let \( \tilde{p} \in \mathbb{L} \cap \text{cl}(\mathbb{B}^{(\alpha)}) \). As \( \tilde{p} \in \text{cl}(\mathbb{B}^{(\alpha)}) \), this implies that there exists a sequence \( \{p_n\} \subset \mathbb{B}^{(\alpha)} \) such that \( p_n \to \tilde{p} \) as \( n \to \infty \). Since \( p_n \in \mathbb{B}^{(\alpha)} \), we can write
\[
p_n(x)^{\alpha-1} = q(x)^{\alpha-1} + F_n + \theta_n^T f(x) \quad \forall x \in \mathbb{S}
\] (83)
for some constants \( \theta_n := [\theta_n^{(1)}, \ldots, \theta_n^{(k)}]^T \in \mathbb{R}^k \) and \( F_n \). Now for any \( p \in \mathbb{L} \) we have, from the definition of linear family, \( \sum_{x \in \mathbb{S}} p(x) f_i(x) = a_i, i = 1, \ldots, k \). Since \( \tilde{p} \in \mathbb{L} \), we also have \( \sum_{x \in \mathbb{S}} \tilde{p}(x) f_i(x) = a_i, i = 1, \ldots, k \). Multiplying both sides of (83) by \( p \) and \( \tilde{p} \) separately, we get
\[
\sum_{x \in \mathbb{S}} p(x)p_n(x)^{\alpha-1} = \sum_{x \in \mathbb{S}} p(x)q(x)^{\alpha-1} + F_n + \sum_{i=1}^k \theta_n^{(i)} a_i
\]
and
\[
\sum_{x \in \mathbb{S}} \tilde{p}(x)p_n(x)^{\alpha-1} = \sum_{x \in \mathbb{S}} \tilde{p}(x)q(x)^{\alpha-1} + F_n + \sum_{i=1}^k \theta_n^{(i)} a_i.
\]
Combining the above two equations we get
\[
\sum_{x \in \mathbb{S}} [p(x) - \tilde{p}(x)] \left[p(x)^{\alpha-1} - q(x)^{\alpha-1}\right] = 0.
\]
As \( n \to \infty \), the above becomes
\[
\sum_{x \in \mathbb{S}} [p(x) - \tilde{p}(x)] \left[\tilde{p}(x)^{\alpha-1} - q(x)^{\alpha-1}\right] = 0,
\]
which is equivalent to (65).

(ii) Let \( p_n^* \) be the forward \( B_\alpha \)-projection of \( q \) on the linear family
\[
\mathbb{L}_n := \left\{ p : \sum_{x \in \mathbb{S}} p(x) f_i(x) = \left(1 - \frac{1}{n}\right) a_i + \frac{1}{n} \sum_{x \in \mathbb{S}} q(x) f_i(x), \quad i = 1, \ldots, k \right\}
\]
(see Figure 1). By construction \( \left(1 - \frac{1}{n}\right) p + \frac{1}{n} q \in \mathbb{L}_n \) for any \( p \in \mathbb{L} \). Hence, since \( \text{Supp}(q) = \mathbb{S} \),
Fig. 1: Orthogonality between \( B^{(\alpha)} \)-family and the linear family \( L_n \).

we have \( \text{Supp}(L_n) = S \). Since \( L_n \) is also characterized by the same functions \( f_i, i = 1, \ldots, k \), we have \( p_n^* \in B^{(\alpha)} \) for every \( n \in \mathbb{N} \). Hence limit of any convergent sub-sequence of \( \{p_n^*\} \) belongs to \( \text{cl}(B^{(\alpha)}) \cap L \). Thus \( \text{cl}(B^{(\alpha)}) \cap L \) is non-empty. This completes the proof.

**Corollary 27:** Let \( \alpha \in (0, 1) \). Let \( L \) and \( B^{(\alpha)} \) be characterized by the same functions \( f_i, i = 1, \ldots, k \). Then \( L \cap \text{cl}(B^{(\alpha)}) = \{p^*\} \) and

\[
B_\alpha(p, q) = B_\alpha(p, p^*) + B_\alpha(p^*, q) \quad \forall p \in L, \quad \forall q \in \text{cl}(B^{(\alpha)}). \tag{84}
\]

**Proof 5:** By Theorem 26, we have \( L \cap \text{cl}(B^{(\alpha)}) = \{p^*\} \). In view of Remark 10(b), notice that every member of \( B^{(\alpha)} \) has the same projection on \( L \), namely \( p^* \). Hence (84) holds for every \( q \in B^{(\alpha)} \). Thus we only need to prove (84) for every \( q \in \text{cl}(B^{(\alpha)}) \setminus B^{(\alpha)} \). Let \( q \in \text{cl}(B^{(\alpha)}) \setminus B^{(\alpha)} \). There exists \( \{q_n\} \subset B^{(\alpha)} \) such that \( q_n \to q \). Hence for any \( p \in L \), we have

\[
B_\alpha(p, q_n) = B_\alpha(p, p^*) + B_\alpha(p^*, q_n) \quad \forall n \in \mathbb{N}. \tag{85}
\]

Since for a fixed \( p, q \mapsto B_\alpha(p, q) \) is continuous as a function from \( \mathcal{P} \) to \([0, \infty]\), taking limit as \( n \to \infty \) on both sides of (85), we have (84). This completes the proof.

Theorem 26 does not hold, in general, for \( \alpha > 1 \). In the following example we show that \( \text{cl}(B^{(\alpha)}) \) may not intersect the associated linear family \( L \).

**Example 10:** Consider \( \alpha, S, L \) and \( u \) as in Example 9. Then the associated \( B^{(\alpha)} \)-family is given by

\[
B^{(\alpha)} = \{p_\theta : p_\theta(x) = u(x) + F(\theta) + \theta f(x), \text{for all } x \in S\},
\]
where \( f = [1, -3, -5, -6]^T \) and \( \theta = \frac{13\theta}{4} \) and \( \theta \in (-\frac{1}{17}, \frac{1}{17}) \). Then we have

\[
\mathbb{B}^{(\alpha)} = \{ p_\theta : \theta \in (-\frac{1}{17}, \frac{1}{17}) \}
\]

\[
\text{cl}(\mathbb{B}^{(\alpha)}) = \{ p_\theta : \theta \in [-\frac{1}{17}, \frac{1}{17}] \}
\]

where \( p_\theta = \left[ (\frac{1}{4} + \frac{17\theta}{4}), (\frac{1}{4} + \theta), (\frac{1}{4} - \frac{7\theta}{4}), (\frac{1}{4} - \frac{11\theta}{4}) \right]^T \). If \( p_\theta \in \text{cl}(\mathbb{B}^{(\alpha)}) \cap \mathbb{L} \) then \( \sum_{x \in \mathbb{S}} p_\theta(x) f(x) = 0 \). This implies \( \theta = \frac{13}{115} \), which is outside the range of \( \theta \). Hence \( \text{cl}(\mathbb{B}^{(\alpha)}) \cap \mathbb{L} = \emptyset \).

The following theorem tells us that a reverse \( B_\alpha \)-projection on a \( \mathbb{B}^{(\alpha)} \)-family can be turned into a forward \( B_\alpha \)-projection on the associated linear family. We shall refer this as the projection theorem for the \( B_\alpha \)-divergence. This theorem is analogous to the one for \( I \)-divergence [24, Th. 3.3], \( \mathcal{I}_\alpha \)-divergence [38, Th. 18] and \( D_\alpha \)-divergence [39, Th. 6].

**Theorem 28:** Let \( \alpha \in (0, 1) \). Let \( X_1^n := (X_1, \ldots, X_n) \in \mathbb{S}^n \). Let \( p_n \) be the empirical probability measure of \( X_1^n \) and let

\[
\hat{\mathbb{L}}_n := \left\{ p \in \mathcal{P} : \sum_{x \in \mathbb{S}} p(x) f_i(x) = \hat{f}_i, \quad i = 1, \ldots, k \right\}
\]

where \( \hat{f}_i = \frac{1}{n} \sum_{j=1}^n f_i(X_j), i = 1, \ldots, k \). Let \( p^* \) be the forward \( B_\alpha \)-projection of \( q \) on \( \hat{\mathbb{L}}_n \). Then the following hold.

(i) If \( p^* \in \mathbb{B}^{(\alpha)} \), then \( p^* \) is the reverse \( B_\alpha \)-projection of \( p_n \) on \( \mathbb{B}^{(\alpha)} \).

(ii) If \( p^* \notin \mathbb{B}^{(\alpha)} \), then \( p_n \) does not have a reverse \( B_\alpha \)-projection on \( \mathbb{B}^{(\alpha)} \). However, \( p^* \) is the reverse \( B_\alpha \)-projection of \( p_n \) on \( \text{cl}(\mathbb{B}^{(\alpha)}) \).

**Proof 6:** Let us first observe that \( \hat{\mathbb{L}}_n \) is constructed so that \( p_n \in \hat{\mathbb{L}}_n \). Since the families \( \hat{\mathbb{L}}_n \) and \( \mathbb{B}^{(\alpha)} \) are defined by the same functions \( f_i, i = 1, \ldots, k \), by Corollary 27 we have \( \hat{\mathbb{L}} \cap \text{cl}(\mathbb{B}^{(\alpha)}) = \{ p^* \} \) and

\[
B_\alpha(p_n, q) = B_\alpha(p_n, p^*) + B_\alpha(p^*, q) \quad \forall q \in \text{cl}(\mathbb{B}^{(\alpha)}).
\]

Hence it is clear that the minimizer of \( B_\alpha(p_n, q) \) over \( q \in \text{cl}(\mathbb{B}^{(\alpha)}) \) is same as the minimizer of \( B_\alpha(p^*, q) \) over \( q \in \text{cl}(\mathbb{B}^{(\alpha)}) \) (Notice that this statement is also true with \( \text{cl}(\mathbb{B}^{(\alpha)}) \) replaced by \( \mathbb{B}^{(\alpha)} \)). But \( B_\alpha(p^*, q) \) over \( q \in \text{cl}(\mathbb{B}^{(\alpha)}) \) is uniquely minimized by \( q = p^* \). Hence if \( p^* \notin \mathbb{B}^{(\alpha)} \), since minimum value of \( B_\alpha(p_n, q) \) over \( q \in \text{cl}(\mathbb{B}^{(\alpha)}) \) is same as that of \( B_\alpha(p_n, q) \) over \( q \in \mathbb{B}^{(\alpha)} \), the later is not attained on \( \mathbb{B}^{(\alpha)} \).

**Remark 11:** Theorems 26, 28 and Corollary 27 continue to hold for \( \alpha > 1 \) as well if attention is restricted to probability measures with strictly positive components and the existence of \( p^* \) is guaranteed.
B. Proofs

1) Proof of Proposition 4. Consider a regular $\mathbb{E}^{(\alpha)}$-family with $h$ being identically a constant. Then from (18), for $x \in S$,

$$p_\theta(x) = [h + F(\theta) + w(\theta)^T f(x)]^{\alpha/n},$$

(87) can be re-written as

$$p_\theta(x) = S(\theta)^{\alpha/n} [h + [w(\theta)/S(\theta)]^T f(x)]^{\alpha/n},$$

(88)

where $S(\theta) := 1 + [F(\theta)/h]$. Comparing (88) with (28) we see that $p_\theta$’s form an $\mathbb{M}^{(\alpha)}$-family characterized by $h, f_1, \ldots, f_k$. This family is regular if $1, w_1(\theta)/S(\theta), \ldots, w_k(\theta)/S(\theta)$ are linearly independent. Let

$$c_0 + c_1[w_1(\theta)/S(\theta)] + \cdots + c_k[w_k(\theta)/S(\theta)] = 0,$$

for some scalars $c_i$, $i = 0, 1, \ldots, k$. Using the value of $S(\theta)$, we get

$$c_0[h + F(\theta)] + hc_1w_1(\theta) + \cdots + hc_kw_k(\theta) = 0.$$

If $F(\theta)$ is identically a constant then $c_0 = c_1 = \cdots = c_k = 0$, since $1, w_1, \ldots, w_k$ are linearly independent. Otherwise also $c_0 = c_1 = \cdots = c_k = 0$, if $1, F(\theta), w_1(\theta), \ldots, w_k(\theta)$ are linearly independent. This completes the proof.

2) Proof of Lemma 6. For any $p_\theta \in \mathcal{E}^{(\alpha)}$ characterized by the functions $h, f$ and $w$, we have from (5), for $x \in S$,

$$p_\theta^{(\alpha)}(x) = \frac{Z(\theta)^\alpha}{\|p_\theta\|^\alpha} [h(x) + w(\theta)^T f(x)]^{\frac{\alpha}{1-\alpha}},$$

(89)

where $\|p_\theta\|^\alpha = \int p(x)^\alpha dx$ and $Z'(\theta) = Z(\theta)^\alpha/\|p_\theta\|^\alpha$. Hence $p_\theta^{(\alpha)} \in \mathbb{M}^{(1/\alpha)}$ characterized by the same functions $h, f$ and $w$. So, the mapping is well-defined. The map is clearly one-one, since it is easy to see that if $p_\theta^{(\alpha)} = p_\eta^{(\alpha)}$ for some $\theta, \eta \in \Theta$ then $p_\theta = p_\eta$. To verify it is onto, let $p \in \mathbb{M}^{(1/\alpha)}$ be arbitrary. Then, for $x \in S$,

$$p(x) = Z(\theta) [h(x) + w(\theta)^T f(x)]^{\frac{1}{1-\alpha}},$$

which implies

$$p(x)^{1/\alpha} = Z(\theta)^{1/\alpha} [h(x) + w(\theta)^T f(x)]^{\frac{1}{1-\alpha}}$$
and hence
\[ p^{(1/\alpha)}(x) = \frac{Z(\theta)^{1/\alpha}}{\int p(y)^{1/\alpha} dy} [h(x) + w(\theta)^T f(x)]^{1-\alpha}. \]

Thus \( p^{(1/\alpha)} \in \mathcal{E}^{(\alpha)} \) and so \( p^{(1/\alpha)} = p_{\theta} \) for some \( \theta \in \Theta \). It is now easy to show that \( p_{\theta}^{(\alpha)} = p \).

Thus for any \( p \in \mathcal{M}^{(1/\alpha)} \) characterized by \( h, f \) and \( w \), there exists \( p_{\theta} \in \mathcal{E}^{(\alpha)} \) characterized by the same functions such that \( p_{\theta}^{(\alpha)} = p \). Hence the mapping is onto.

3) Proof of Lemma[7]: We present the proof for the discrete case. The proof for the continuous case follows by a similar argument if we replace \( p_n \) by \( \tilde{p}_n \) throughout in the proof.

The estimating equation in (8) can be re-written as
\[ \sum_{x \in S} p_n(x)^{\alpha} p_{\theta}(x)^{1-\alpha} s(x; \theta) = \sum_{x \in S} p_{\theta}(x) s(x; \theta), \tag{90} \]

since \( \sum_{x \in S} p_{\theta}(x) s(x; \theta) = 0 \). This can further be re-written as
\[ \frac{\sum_{x \in S} p_n^{(\alpha)}(x) [p_{\theta}^{(\alpha)}(x)]^{1/\alpha - 1} s(x; \theta)}{\sum_{x \in S} p_n^{(\alpha)}(x) [p_{\theta}^{(\alpha)}(x)]^{1/\alpha - 1}} = \frac{\sum_{x \in S} [p_{\theta}^{(\alpha)}(x)]^{1/\alpha} s(x; \theta)}{\sum_{x \in S} [p_{\theta}^{(\alpha)}(x)]^{1/\alpha}}. \tag{91} \]

Observe that
\[ s^{(\alpha)}(x; \theta) := \nabla \log p_{\theta}^{(\alpha)}(x) = \nabla \log \frac{p_{\theta}(x)^{\alpha}}{\|p_{\theta}\|^{\alpha}} = \nabla \left[ \log p_{\theta}(x)^{\alpha} - \log \|p_{\theta}\|^{\alpha} \right] = \alpha \left[ s(x; \theta) - \nabla \log \|p_{\theta}\| \right]. \]

Hence
\[ s(x; \theta) = \frac{1}{\alpha} s^{(\alpha)}(x; \theta) + A(\theta), \tag{92} \]

where \( A(\theta) = \nabla \log \|p_{\theta}\| \). Plugging (92) in (91), we get
\[ \frac{\sum_{x \in S} p_n^{(\alpha)}(x) [p_{\theta}^{(\alpha)}(x)]^{1/\alpha - 1} s^{(\alpha)}(x; \theta)}{\sum_{x \in S} p_n^{(\alpha)}(x) [p_{\theta}^{(\alpha)}(x)]^{1/\alpha - 1}} = \frac{\sum_{x \in S} [p_{\theta}^{(\alpha)}(x)]^{1/\alpha} s^{(\alpha)}(x; \theta)}{\sum_{x \in S} [p_{\theta}^{(\alpha)}(x)]^{1/\alpha}}. \tag{93} \]

This is same as [10] with \( p_n, p_{\theta} \) and \( \alpha \), respectively, replaced by \( p_n^{(\alpha)}, p_{\theta}^{(\alpha)} \) and \( 1/\alpha \).
4) Proof of Theorem 9. We will prove only (a), (b) and (c). The proof of (d) follows from Lemmas 6 and 7.

(a) Let $p_\theta \in \mathcal{E}$. Then from (17), for $x \in \mathcal{S}$, we have

$$\log p_\theta(x) = h(x) + Z(\theta) + w(\theta)^T f(x).$$

Taking derivative with respect to $\theta_r$ for $r = 1, \ldots, k$, we get

$$\partial_r \log p_\theta(x) = \partial_r [Z(\theta)] + \partial_r [w(\theta)]^T f(x).$$

Hence (5) implies that

$$-\partial_r [Z(\theta)] = \partial_r [w(\theta)]^T \left[ \frac{1}{n} \sum_{j=1}^n f(X_j) \right].$$

(94)

Since $\mathbb{E}_\theta [\partial_r \log p_\theta(X)] = 0$ by the regularity condition, we have

$$-\partial_r [Z(\theta)] = \partial_r [w(\theta)]^T \mathbb{E}_\theta [f(X)].$$

(95)

From (94) and (95), we get (47).

(b) If $p_\theta \in \mathbb{B}^{(\alpha)}$ then from (18), for $x \in \mathcal{S}$, we have

$$p_\theta(x)^{\alpha-1} = h(x) + F(\theta) + w(\theta)^T f(x).$$

Taking derivative with respect to $\theta_r$ for $r = 1, \ldots, k$, we get

$$(\alpha - 1)p_\theta(x)^{\alpha-2} \partial_r [p_\theta(x)] = \partial_r [F(\theta)] + \partial_r [w(\theta)]^T f(x).$$

(96)

The estimating equation (9) can be re-written as

$$\frac{1}{n} \sum_{j=1}^n p_\theta(X_j)^{\alpha-2} \partial_r [p_\theta(X_j)] = \int p_\theta(x)^{\alpha-1} \partial_r [p_\theta(x)] dx.$$  (97)

Substituting (96) in (97), we get (47).

(c) If $p_\theta \in \mathbb{M}^{(\alpha)}$, then from (28), for $x \in \mathcal{S}$, we have

$$p_\theta(x)^{\alpha-1} = Z(\theta)^{\alpha-1}[h(x) + w(\theta)^T f(x)].$$

(98)

Taking derivative with respect to $\theta_r$ for $r = 1, \ldots, k$, we get

$$(\alpha - 1)p_\theta(x)^{\alpha-2} \partial_r [p_\theta(x)] = \partial_r [Z(\theta)^{\alpha-1}] [h(x) + w(\theta)^T f(x)] + Z(\theta)^{\alpha-1} \partial_r [w(\theta)]^T f(x).$$

Substituting this in (10), we get (48).
5) Proof of Corollary [41]: Let us first observe that for a regular family the matrix \( [\partial_i(w_j(\theta))]_{k \times k} \) is non-singular for \( \theta \in \Theta \). To see this, let

\[
c_1 \partial_r[w_1(\theta)] + \cdots + c_k \partial_r[w_k(\theta)] = 0
\]

for some scalars \( c_1, \ldots, c_k \) and for each \( r = 1, \ldots, k \). Then

\[
c_1 w_1(\theta) + \cdots + c_k w_k(\theta) = c,
\]

for some constant \( c \). Now linear independence of \( 1, w_1, \ldots, w_k \) implies that \( c = c_1 = \cdots = c_k = 0 \).

(a) For a regular \( \mathcal{E} \)-family, or for a regular \( \mathbb{B}^{(\alpha)} \)-family, from (47) we have

\[
\partial_r[w(\theta)]^T(\mathbb{E}_\theta[f(X)] - \bar{f}) = 0 \quad \text{for} \quad r = 1, \ldots, k.
\]

Hence \( \mathbb{E}_\theta[f(X)] = \bar{f} \).

(b) (48) can be re-written as

\[
\partial_r[w(\theta)]^T\left[\frac{\mathbb{E}_\theta[f(X)]}{\mathbb{E}_\theta[h(X) + w(\theta)^T f(X)]} - \frac{\bar{f}}{\bar{h} + w(\theta)^T \bar{f}}\right] = 0.
\]

Since the family is regular, we have

\[
\frac{\mathbb{E}_\theta[f(X)]}{\mathbb{E}_\theta[h(X) + w(\theta)^T f(X)]} - \frac{\bar{f}}{\bar{h} + w(\theta)^T \bar{f}} = 0.
\]

This implies

\[
\bar{h} + w(\theta)^T \bar{f} = \bar{h} + \frac{\bar{h} + w(\theta)^T \bar{f}}{\mathbb{E}_\theta[h(X) + w(\theta)^T f(X)]} w(\theta)^T \mathbb{E}_\theta[f(X)].
\]

That is,

\[
\{\bar{h} + w(\theta)^T \bar{f}\} \{\mathbb{E}_\theta[h(X) + w(\theta)^T f(X)]\}
\]

\[
= \bar{h} \mathbb{E}_\theta[h(X) + w(\theta)^T f(X)] + [\bar{h} + w(\theta)^T \bar{f}] w(\theta)^T \mathbb{E}_\theta[f(X)].
\]

Hence

\[
\{\bar{h} + w(\theta)^T \bar{f}\} \mathbb{E}_\theta[h(X)] = \bar{h} \mathbb{E}_\theta[h(X) + w(\theta)^T f(X)].
\]

Substituting this back in (100), we get

\[
\frac{\mathbb{E}_\theta[f(X)]}{\mathbb{E}_\theta[h(X)]} = \frac{\bar{f}}{\bar{h}}.
\]

(c) This is analogous to (b).
6) **Proof of Theorem 11:** Consider the $\mathbb{B}^{(\alpha)}$-family as in (87). If it is regular, from Corollary 10(a), the Basu et. al. estimating equation is given by
\[
\mathbb{E}_h[f(X)] = \bar{f}.
\] (101)
We now show that the Jones et. al. estimating equation is also the same. Recall that (87) can be written as an $\mathbb{M}^{(\alpha)}$-family as in (88). We divide the proof into two parts.

(i) Suppose that $F'(\theta)$ is linearly independent with $1, w_i(\theta)$'s. Then (88) forms a regular $\mathbb{M}^{(\alpha)}$-family by Proposition 4. Therefore using Corollary 10(b) we see that the Jones et. al. estimating equation for (88) is same as (101) since $h$ is identically a constant.

(ii) Next let us suppose that $F'(\theta)$ is linearly dependent with $1, w_i(\theta)$'s. Then there exists scalars $c_0, c_1, \ldots, c_k$ (not all zero) such that
\[
F(\theta) = c_0 + c_1 w_1(\theta) + \cdots + c_k w_k(\theta).
\] (102)
Then
\[
\partial_r[F(\theta)] = c_1 \partial_r[w_1(\theta)] + \cdots c_k \partial_r[w_k(\theta)].
\] (103)
Using Theorem 9(b), the Jones et. al. estimating equation for (88) is given by
\[
\frac{\partial_r[w(\theta)]/S(\theta)^T \mathbb{E}_h[f(X)]}{\mathbb{E}_h[h + [w(\theta)/S(\theta)]^T f(X)]} = \frac{\partial_r[w(\theta)]/S(\theta)^T \bar{f}}{h + [w(\theta)/S(\theta)]^T \bar{f}} \quad \text{for } r = 1, \ldots, k.
\] (104)
Substituting the value of $S(\theta)$, an easy calculation yields
\[
\frac{\partial_r[F(\theta)] + \partial_r[w(\theta)]^T \mathbb{E}_h[f(X)]}{\mathbb{E}_h[h + F(\theta) + w(\theta)^T f(X)]} = \frac{\partial_r[F(\theta)] + \partial_r[w(\theta)]^T \bar{f}}{h + F(\theta) + w(\theta)^T \bar{f}} \quad \text{for } r = 1, \ldots, k.
\] (104)
Using (102) and (103) in (104) we get
\[
\partial_r[w(\theta)]^T \left[ \frac{\mathbb{E}_h[f(X) + c]}{h + c_0 + w(\theta)^T \mathbb{E}_h[f(X) + c]} - \frac{\bar{f} + c}{h + c_0 + w(\theta)^T [\bar{f} + c]} \right] = 0,
\] (105)
where $c = [c_1, \ldots, c_k]^T$. Since $1, w_1, \ldots, w_k$ are linearly independent, the matrix $[\partial_r(w_j(\theta))]_{k \times k}$ is non-singular. Using this, (105) becomes
\[
\frac{\mathbb{E}_h[f(X) + c]}{h + c_0 + w(\theta)^T \mathbb{E}_h[f(X) + c]} = \frac{\bar{f} + c}{h + c_0 + w(\theta)^T [\bar{f} + c]}.
\] (106)
Proceeding as in Corollary 10(b), we get
\[
(h + c_0 + w(\theta)^T \mathbb{E}_h[f(X) + c])(h + c_0) = (h + c_0 + w(\theta)^T [\bar{f} + c])(h + c_0).
\]
That is,
\[
\frac{h + c_0 + w(\theta)^T \mathbb{E}_h[f(X) + c]}{h + c_0 + w(\theta)^T [\bar{f} + c]} = 1.
\]
Hence
\[ \mathbb{E}_\theta[f(X) + c] = \bar{f} + c, \]
and thus
\[ \mathbb{E}_\theta[f(X)] = \bar{f}. \]

This completes the proof.

7) Proof of Proposition 16: Let $T$ be a sufficient statistics for $\theta$ with respect to $L_G$. Let $X^n_1$ and $Y^n_1$ be two samples such that $T(X^n_1) = T(Y^n_1)$. Then we have $[L_G(X^n_1; \theta) - L_G(Y^n_1; \theta)]$ is independent of $\theta$. Let us define a relation ‘∼’ on the set of all sample points of length $n$ by $X^n_1 \sim Y^n_1$ if and only if $T(X^n_1) = T(Y^n_1)$. Then ‘∼’ is an equivalence relation. Let us denote the equivalence classes by $S^i_t$, $t \in \mathcal{T}$. For each equivalence class $S^i_t$, designate an element $X^n_{1,t} \in S^i_t$. Let $X^n_1$ be any sample point such that $T(X^n_1) = t^*$. Then $X^n_1 \in S^{i^*}$ and
\[ T(X^n_{1,t}) = T(X^n_{1,t^*}). \]

This implies that $[L_G(X^n_1; \theta) - L_G(X^n_{1,t^*}; \theta)]$ is independent of $\theta$ by hypothesis. Let $h(X^n_1) := [L_G(X^n_1; \theta) - L_G(X^n_{1,t^*}; \theta)]$. Then
\[ L_G(X^n_1; \theta) = L_G(X^n_{1,t}; \theta) + h(X^n_1) = g(\theta, t^*) + h(X^n_1), \]
where $g(\theta, t) := L_G(X^n_{1,t}; \theta)$.
Conversely, let us assume that there exists two functions $g$ and $h$ such that
\[ L_G(X^n_1; \theta) = g(\theta, T(X^n_1)) + h(X^n_1) \]
for all sample points $X^n_1$ and for all $\theta \in \Theta$. Thus for any two sample points $X^n_1$ and $Y^n_1$ with $T(X^n_1) = T(Y^n_1)$, we have
\[ L_G(X^n_1; \theta) = g(\theta, T(X^n_1)) + h(X^n_1) = g(\theta, T(Y^n_1)) + h(Y^n_1) \]
\[ = h(X^n_1) - h(Y^n_1). \]
This implies that $[L_G(X^n_1; \theta) - L_G(Y^n_1; \theta)]$ is independent of $\theta$. By definition, $T$ is a sufficient statistics for $\theta$. This completes the proof.
8) Proof of Theorem [17]: Let $X^n_1$ and $Y^n_1$ be two i.i.d. samples and

$$T(X^n_1) = T(Y^n_1).$$

(a) For $\Pi = \mathcal{E}$, this was established by Koopman [36, Th. II].

Let $\Pi = \mathbb{B}(\alpha)$, $f_1(X^n_1) = \psi_1(T(X^n_1))$, and $f_1(Y^n_1) = \psi_1(T(Y^n_1))$ for $i = 1, \ldots, s$. Then

$$L_2^{(\alpha)}(X^n_1, \theta) - L_2^{(\alpha)}(Y^n_1, \theta)$$

$$= \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{\alpha \{ h(X_j) + F(\theta) + w(\theta)^T f(X_j) \} - 1}{\alpha - 1} - \frac{\alpha \{ h(Y_j) + F(\theta) + w(\theta)^T f(Y_j) \} - 1}{\alpha - 1} \right]$$

$$= \frac{\alpha}{n(\alpha - 1)} \sum_{j=1}^{n} \left[ h(X_j) - h(Y_j) \right] - \frac{\alpha}{\alpha - 1} \sum_{i=1}^{s} w_i(\theta) \left[ \psi_i(T(X^n_1)) - \psi_i(T(Y^n_1)) \right]$$

$$= \frac{\alpha}{n(\alpha - 1)} \sum_{j=1}^{n} \left[ h(X_j) - h(Y_j) \right],$$

which is independent of $\theta$. Hence by Proposition [16], $T$ is a sufficient statistic for $\theta$.

Taking $L_3^{(\alpha)}(\theta)$ and $L_1^{(\alpha)}(\theta)$ respectively, the assertions in (b) and (c) can be established in a similar fashion. This completes the proof.

9) Proof of Proposition [19]: The condition that if $T(X^n_1) = T(Y^n_1)$ then $[L_G(X^n_1; \theta) - L_G(Y^n_1; \theta)]$ is independent of $\theta$, for all $\theta \in \Theta$ implies that $T$ is a sufficient statistic for $\theta$ with respect to $L_G$, by Proposition [16]. Thus the only thing we need to prove is that $T$ is further minimal. Let $\tilde{T}$ be a sufficient statistics such that $\tilde{T}(X^n_1) = \tilde{T}(Y^n_1)$. Then by Proposition [16] $[L_G(X^n_1; \theta) - L_G(Y^n_1; \theta)]$ is independent of $\theta$. Hence by hypothesis, we have $T(X^n_1) = T(Y^n_1)$. Therefore $T$ is minimal, by definition.

10) Proof of Theorem [20]: Let us consider a regular $\mathbb{B}^{(\alpha)}$-family as in (18). We will prove that the statistic $\bar{f}(X^n_1)$ is minimal. Let us consider two sample points $X^n_1$ and $Y^n_1$ such that $[L_2^{(\alpha)}(X^n_1; \theta) - L_2^{(\alpha)}(Y^n_1; \theta)]$ is independent of $\theta$. Our aim is to prove that $\bar{f}(X^n_1) = \bar{f}(Y^n_1)$.

$$L_2^{(\alpha)}(X^n_1, \theta) - L_2^{(\alpha)}(Y^n_1, \theta)$$

$$= \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{\alpha \{ h(X_j) + F(\theta) + w(\theta)^T f(X_j) \} - 1}{\alpha - 1} - \frac{\alpha \{ h(Y_j) + F(\theta) + w(\theta)^T f(Y_j) \} - 1}{\alpha - 1} \right]$$

$$= \frac{\alpha}{n(\alpha - 1)} \sum_{j=1}^{n} \left[ h(X_j) - h(Y_j) \right] - \frac{\alpha}{\alpha - 1} w(\theta)^T \left[ \bar{f}(X^n_1) - \bar{f}(Y^n_1) \right].$$
Since the above is independent of $\theta$, we must have
\[ w(\theta)^T [f(X^n) - f(Y^n)] \]
is independent of $\theta$. Since the given family is regular, the functions $1, w_1, \ldots, w_k$ are linearly independent. Hence we have $f(X^n) - f(Y^n) = 0$. That is $f(X^n) = f(Y^n)$. Thus $f(X^n)$ is a minimal sufficient statistics for the regular $\mathbb{B}^{(\alpha)}$-family.

The results for $\mathbb{M}^{(\alpha)}$ and $\mathcal{E}^{(\alpha)}$ families can be derived by a similar argument if respectively the likelihood functions $L^{(\alpha)}_3(\theta)$ and $L^{(\alpha)}_1(\theta)$ are considered.

C. Counterexamples

1) Jones et. al. estimation under Student distributions for $\alpha > 1$: Suppose that $X_1, \ldots, X_n$ is an i.i.d. sample where $X_1 \leq \cdots \leq X_n$. Suppose also that the true distribution $p$ is a Student distribution with some known $\alpha > 1$ and variance, say $\sigma^2 = 1$:
\[ p_\mu(x) = N_\alpha [1 + b_\alpha(x - \mu)^2]^{\frac{1}{\alpha-1}}, \quad (107) \]
where $N_\alpha$ is the normalizing factor. The support of $p_\mu$ is given by
\[ S = \{ x : \mu - c_\alpha \leq x \leq \mu + c_\alpha \}, \]
where $c_\alpha := \sqrt{-1/b_\alpha}$. (Recall that $b_\alpha < 0$ for $\alpha > 1$). Observe that (107) defines a 1-parameter $\mathbb{M}^{(\alpha)}$-family whose support depends on the unknown parameter. We now show that the Jones et. al. estimator of $\mu$ could be different from $\bar{X}$. Since the support of $p_\mu$ depends on $\mu$, we cannot apply Theorem 9(b). Hence we resort to the maximization of the associated likelihood function:
\[ L^{(\alpha)}_3(\theta) = \frac{\alpha}{\alpha - 1} \log \left[ \frac{1}{n} \sum_{i=1}^{n} p_\theta(X_i)^{\alpha-1} \right] - \log \left[ \int p_\theta(x)^\alpha dx \right]. \]
The likelihood function, for (107), is
\[ L^{(\alpha)}_3(\mu | X_1, \ldots, X_n) \]
\[ = \frac{\alpha}{\alpha - 1} \log \left[ \frac{1}{n} \sum_{i=1}^{n} N_\alpha^{\alpha-1} \left[ 1 + b_\alpha(X_i - \mu)^2 \right] I(\mu - c_\alpha \leq X_i \leq \mu + c_\alpha) \right] \]
\[ - \log \left( E_\mu \left[ N_\alpha^{\alpha-1} \{ 1 + b_\alpha(X - \mu)^2 \} \right] \right) \]
\[ = \frac{\alpha}{\alpha - 1} \log \left[ \frac{N_\alpha^{\alpha-1}}{n} \sum_{i=1}^{n} \left[ 1 + b_\alpha(X_i - \mu)^2 \right] I(X_i - c_\alpha \leq \mu \leq X_i + c_\alpha) \right] - \log[N_\alpha^{\alpha-1}(1 + b_\alpha)] \]
\[ = \frac{\alpha}{\alpha - 1} \log \left[ \frac{N_\alpha^{\alpha-1}}{n} \sum_{i=1}^{n} \left[ 1 + b_\alpha(X_i - \mu)^2 \right] I(\mu \in I_i) \right] - \log[N_\alpha^{\alpha-1}(1 + b_\alpha)], \]
where $1(\cdot)$ denotes the indicator function and $I_i = [X_i - c_{\alpha}, X_i + c_{\alpha}]$ for $i = 1, \ldots, n$. Observe that the maximizer of $I^{(\alpha)}_3(\mu)$ is same as the maximizer of

$$
\ell^{(\alpha)}(\mu) := \sum_{i=1}^{n}[1 + b_{\alpha}(X_i - \mu)^2] \ 1(\mu \in I_i).
$$

It is clear from (108) that $\ell^{(\alpha)}(\mu)$ is positive if and only if $\mu$ lies at least in one $I_i$ for $i = 1, \ldots, n$. Thus, to find the maximizer $\hat{\mu}$ of $\ell^{(\alpha)}(\mu)$, we only need to consider the cases when $\mu$ lies in one of the $I_i$’s.

Consider $I_1$. If $I_1$ is disjoint from all other $I_j$ for $j \neq 1$, then $\ell^{(\alpha)}(\mu)$ equals to $[1 + b_{\alpha}(X_1 - \mu)^2]$ for $\mu \in I_1$. Similarly, if $I_1 \cap I_2 = \emptyset$ but all other $I_j$’s for $j \neq 1, 2$, are disjoint from $I_1$, then the value of $\ell^{(\alpha)}(\mu)$ in $I_1$ is given by:

$$
\ell^{(\alpha)}(\mu) = \begin{cases} 
[1 + b_{\alpha}(X_1 - \mu)^2] & \text{for } \mu \in I_1 \setminus I_2 \\
[2 + b_{\alpha}(X_1 - \mu)^2 + b_{\alpha}(X_2 - \mu)^2] & \text{for } \mu \in I_1 \cap I_2.
\end{cases}
$$

In general, if $I_2, I_3, \ldots, I_k$ for some $k \in \{2, \ldots, n\}$ satisfy $\cap_{i=1}^{k} I_i = \emptyset$ and all other $I_j$’s for $j \neq 1, \ldots, k$ are disjoint from $I_1$, then $I_1$ can be divided into $k$ disjoint sub-intervals and in each of these sub-intervals the value of $\ell^{(\alpha)}(\mu)$ is given by

$$
\ell^{(\alpha)}(\mu) = \sum_{i=1}^{j}[1 + b_{\alpha}(X_i - \mu)^2]
$$

for $\mu \in \left( \cap_{i=1}^{j} I_i \right) \setminus \left( \cup_{i=j+1}^{k+1} I_i \right)$, $j = 1, \ldots, k$, where $\cup_{k+1}^{n} I_i := \emptyset$.

Let us define $1(\mu \in \emptyset) = 0$. Then for $\mu \in I_1$, we can write

$$
\ell^{(\alpha)}(\mu) = [1 + b_{\alpha}(X_1 - \mu)^2] \ 1(\mu \in I_1 \setminus \cup_{i=2}^{n} I_i) \\
+ \left\{ \sum_{i=1}^{2}[1 + b_{\alpha}(X_i - \mu)^2] \right\} \ 1(\mu \in \left( \cap_{i=1}^{2} I_i \right) \setminus \left( \cup_{i=3}^{n} I_i \right)) \\
+ \cdots + \left\{ \sum_{i=1}^{k}[1 + b_{\alpha}(X_i - \mu)^2] \right\} \ 1(\mu \in \left( \cap_{i=1}^{k} I_i \right) \setminus \left( \cup_{i=k+1}^{n} I_i \right)) \\
+ \cdots + \left\{ \sum_{i=1}^{n}[1 + b_{\alpha}(X_i - \mu)^2] \right\} \ 1(\mu \in \left( \cap_{i=1}^{n} I_i \right)) \\
= \sum_{j=1}^{n} \left\{ \left[ \sum_{i=1}^{j}[1 + b_{\alpha}(X_i - \mu)^2] \right] \ 1(\mu \in \left( \cap_{i=1}^{j} I_i \right) \setminus \left( \cup_{i=j+1}^{n} I_i \right)) \right\}. \tag{110}
$$
Let $I'_2 := I_2 \setminus I_1$. Then proceeding as above, for $\mu \in I'_2$, we have

$$\ell^{(\alpha)}(\mu) = [1 + b_\alpha(X_2 - \mu)^2] \mathbf{1}(\mu \in I'_2 \setminus \bigcup_{i=3}^{n} I_i) + \left\{ \sum_{i=2}^{3} [1 + b_\alpha(X_i - \mu)^2] \right\} \mathbf{1}(\mu \in (I'_2 \cap I_3) \setminus \bigcup_{i=4}^{n} I_i) + \cdots + \left\{ \sum_{i=2}^{k} [1 + b_\alpha(X_i - \mu)^2] \right\} \mathbf{1}(\mu \in (I'_2 \cap (\bigcap_{i=3}^{k} I_i)) \setminus \bigcup_{i=k+1}^{n} I_i) + \cdots + \left\{ \sum_{i=2}^{n} [1 + b_\alpha(X_i - \mu)^2] \right\} \mathbf{1}(\mu \in (I'_2 \cap (\bigcap_{i=2}^{n} I_i)) \setminus \bigcup_{i=j+1}^{n} I_i).$$

(111)

In general, let $I'_k := I_k \setminus \bigcup_{i=1}^{k-1} I_i$ for $k = 1, \ldots, n$. Then for $\mu \in I'_k$, we have

$$\ell^{(\alpha)}(\mu) = \sum_{j=k}^{n} \left\{ \sum_{i=k}^{j} [1 + b_\alpha(X_i - \mu)^2] \right\} \mathbf{1}(\mu \in (I'_k \cap (\bigcap_{i=k}^{j} I_i)) \setminus \bigcup_{i=j+1}^{n} I_i).$$

(112)

Hence for $\mu \in I'_k$, $k \in \{1, \ldots, n\}$, we can divide $I'_k$ into at most $(n - k + 1)$ sub-intervals $I_{kj} := [(I'_k \cap (\bigcap_{i=k}^{j} I_i)) \setminus \bigcup_{i=j+1}^{n} I_i]$, $j \in \{k, \ldots, n\}$, such that the indicator functions in (112) will be positive for $\mu$ in either of these sub-intervals For example, in Figure 2 we considered a case where $I'_1$ can be divided into three disjoint sub-intervals, namely $I_{11}$, $I_{12}$ and $I_{13}$, and $I'_2$ can be divided into two disjoint sub-intervals $I_{22}$ and $I_{23}$, and so on. The maximizer of $\ell^{(\alpha)}(\mu)$ for $\mu$ in each of these sub-intervals $I_{kj}$ can be found in the following way.

Let $k$ and $j$ be such that $I_{kj} \neq \emptyset$. Then we have the following cases.

(i) $j = k$ and $I_{kj+1} \neq \emptyset$:

$$I_{kj} = \begin{cases} [X_{k-1} + c_\alpha, X_{k+1} - c_\alpha] & \text{if } I_k' \neq I_k, \\ [X_k - c_\alpha, X_{k+1} - c_\alpha] & \text{if } I_k' = I_k, \end{cases}$$
Fig. 3: Two different cases of maximizers in \([X_2 - \sqrt{5}, X_3 - \sqrt{5}]\).

(ii) \(j = k\) and \(I_k^{j+1} = \emptyset\):

\[
I_k^j = \begin{cases} 
[X_{k-1} + c_\alpha, X_k + c_\alpha] & \text{if } I_k' \neq I_k \\
[X_k - c_\alpha, X_k + c_\alpha] & \text{if } I_k' = I_k,
\end{cases}
\]

(iii) \(j > k\) and \(I_k^{j+1} \neq \emptyset\):

\[
I_k^j = \begin{cases} 
\max\{X_j - c_\alpha, X_{k-1} + c_\alpha\}, X_{j+1} - c_\alpha & \text{if } I_k' \neq I_k \\
[X_j - c_\alpha, X_{j+1} - c_\alpha] & \text{if } I_k' = I_k,
\end{cases}
\]

(iv) \(j > k\) and \(I_k^{j+1} = \emptyset\):

\[
I_k^j = \begin{cases} 
\max\{X_j - c_\alpha, X_{k-1} + c_\alpha\}, X_k + c_\alpha & \text{if } I_k' \neq I_k \\
[X_j - c_\alpha, X_k + c_\alpha] & \text{if } I_k' = I_k.
\end{cases}
\]

Also from (112), we have \(\ell(\alpha)(\mu) = \sum_{i=k}^j [1 + b_\alpha(X_i - \mu)^2]\) for \(\mu \in I_k^j\). Since

\[
\frac{\partial \ell(\alpha)(\mu)}{\partial \mu} = \sum_{i=k}^j \frac{\partial}{\partial \mu}[1 + b_\alpha(X_i - \mu)^2] = -2b_\alpha \sum_{i=k}^j (X_i - \mu),
\]

(113)
\(\ell(\alpha)(\mu)\) is monotone increasing for \(\mu \leq \frac{1}{j-k+1} \sum_{i=k}^j X_i\), and monotone decreasing otherwise. Thus the local maximizer of \(\ell(\alpha)(\mu)\) in any non-empty \(I_k^j\) for \(j \in \{k, k+1, \ldots, n\}\) is given by

the median of \(\{I_L, I_R, \frac{1}{j-k+1} \sum_{i=k}^j X_i\}\),

(114)

where \(I_L\) and \(I_R\) are respectively the left and right ends of the sub-interval \(I_k^j\) as in (i)-(iv). Figure 3 shows two different cases of maximizer of \(\ell(\alpha)(\mu)\) for \(\alpha = 2\) (\(c_\alpha = \sqrt{5}\) here) in the sub-interval \([X_2 - \sqrt{5}, X_3 - \sqrt{5}]\).
Observe that, in this process we divide the interval $\cup_{i=1}^{n} I_i$ into a finite number of non-empty disjoint sub-intervals such that $\ell^{(\alpha)}(\mu)$ is positive if and only if $\mu$ lies in one of these sub-intervals. In each of these sub-intervals, $\ell^{(\alpha)}(\mu)$ has a unique maximizer. Thus we have a finite number of local maximizers of $\ell^{(\alpha)}(\mu)$ in $\cup_{i=1}^{m} I_i$, and hence the global maximizer is one among these local maximizers. Notice that the global maximizer can be different from the sample mean as it is going to be some median value as in (114).

To demonstrate this, we generated the following random sample from the mixture $0.8p + 0.2N(10, 1)$, where $p$ is the Student distribution with $\alpha = 2$, $\mu = 0$ and $\sigma = 1$:

$$
\begin{align*}
-1.7287, & -1.1761, -1.0597, -0.3236, -0.2340, \\
0.4706, & 0.4712, 0.5435, 0.6309, 0.7533, \\
0.8020, & 0.9237, 1.1394, 1.4373, 1.5351, \\
1.6941, & 8.6501, 10.7254, 12.7694, 13.0349.
\end{align*}
$$

(115)

Consider $\mu \in I_1' = [-3.9648, 0.5074]$. Then $I_1^j \neq \emptyset$ for all $j \in \{1, \ldots, 16\}$. Using the formula (114), we have the following local maximizers of $\ell^{(\alpha)}(\mu)$ in each of these sub-intervals $I_1^j$:

$$
-3.4122, -3.2958, -2.5597, -2.4701, -1.7655, -1.7649, -1.6926, -1.6052, \\
-1.4828, -1.4341, -1.3124, -1.0967, -0.7988, -0.7010, -0.5420, 0.3674.
$$

Next consider $\mu \in I_2' = [0.5074, 1.06]$. Then the indicator functions in (111) are positive only when $j = 16$, that is, $I_2^j \neq \emptyset$ only for $j = 16$. Using (114), we get the maximizer of $\ell^{(2)}(\mu)$ in $I_2^{16} = I_2'$ is 0.5074. Similarly, we have only one maximizer in each $I_i'$ for $i = 3, \ldots, 16$ and they respectively are:

$$
1.0600, 1.1764, 1.9125, 2.0021, 2.7067, 2.7073, 2.7796, \\
2.8670, 2.9894, 3.0389, 3.1598, 3.3755, 3.6734, 3.7712.
$$

Next consider $\mu \in I_{17}' = [6.4140, 10.8862]$. Then $I_{17}^j \neq \emptyset$ for $j \in \{17, \ldots, 20\}$. We then have four local maximizers of $\ell^{(2)}(\mu)$ in four sub-intervals $I_{17}^j$ for $j \in \{17, \ldots, 20\}$ of $I_{17}'$ and they respectively are 8.4893, 9.6878, 10.7150, 10.8862. Similarly for $\mu \in I_i', i = 18, 19, 20$, $\ell^{(2)}(\mu)$ has only one local maximizer in each $I_i'$ and they respectively are 12.1766, 12.9615, 15.0055.

Comparing the values of $\ell^{(2)}(\mu)$ at each of the local maximizers, we get 0.3674 is the global maximizer of $\ell^{(2)}(\mu)$. Hence $\hat{\mu} = 0.3674$, which is different from $\overline{X} = 1.2940$. 

Observe that \( c_\alpha \to \infty \) when \( \alpha \to 1 \). Thus for \( \alpha \to 1 \), the length of the intervals \( I_i = [X_i - c_\alpha, X_i + c_\alpha] \) for \( i = 1, \ldots, n \) increases, and hence all the indicator functions in (108) become positive for any \( \mu \in \mathbb{R} \). This implies that the maximizer of \( \ell(\alpha)(\mu) \) is the usual sample mean \( \bar{X} \). Notice also that this complies with the case \( \alpha = 1 \) as the MLE of the mean parameter of a normal distribution is \( \bar{X} \). Recall also that the likelihood function \( L(\alpha) \) coincides with the usual log likelihood function and the Student distribution coincides with the normal distribution as \( \alpha \to 1 \).

2) Generalized Hellinger estimation under Cauchy distributions for \( 0 < \beta < 1 \): Let us consider the random sample in (115). Suppose that the true distribution is a Cauchy distribution with \( \beta = 1/2 \) and \( \sigma = 1 \) as in (40). We claim that the generalized Hellinger estimator of \( \mu \) could be different from \( \bar{X}_{p_n(1/2)}[X] \). Since the support of the Cauchy distribution depends on the parameters when \( 0 < \beta < 1 \), to find the estimator we maximize the likelihood function (12) as we did in [C1] for Student distributions. Thus, if \( q_\mu \) denotes the 1-dimensional Cauchy distributions with \( \beta = 1/2 \) and \( \sigma = 1 \), then the Hellinger estimator for \( \mu \) is the maximizer of

\[
\ell^{(1/2)}_1(\mu) = \int \tilde{p}_n(x)^{1/2} q_\mu(x)^{1/2} dx = \int \tilde{p}_n(x)^{1/2} q_\mu(x)^{1/2} \mathbf{1}(\mu - \sqrt{5} \leq x \leq \mu + \sqrt{5}) dx,
\]

where \( \mathbf{1}(\cdot) \) denotes the indicator function. In view of Lemmas [6] and [7], this problem is equivalent to finding the maximizer of \( L_3^{(2)} \) under the Student distributions with \( \alpha = 2 \) as discussed in [C1]. However one needs to use some continuous estimate \( \tilde{p}_n \) for the empirical measure \( p_n \) here.

3) Sufficient statistics for Student distribution when \( \alpha > 1 \): Consider the Student distributions for \( d = 1, \alpha = 2 \) and \( \sigma = 1 \) as in [C1]. Consider this as a \( \mathbb{B}(\alpha) \)-family. The support of this family is \( [\mu - \sqrt{5}, \mu + \sqrt{5}] \), which depends on \( \mu \). Let \( X_1^\alpha \) be an i.i.d. sample. Here \( \bar{f}(X_1^\alpha) = \bar{X} \), which cannot be a sufficient statistic for \( \mu \) with respect to \( L_2^{(2)} \) as knowledge of the entire sample is required to estimate \( \mu \), as we saw in [C1].

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