BEREZIN-TOEPLITZ QUANTIZATION ON K"{A}HLER MANIFOLDS

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Dedicated to Professor Jochen Br"{u}ning on the occasion of his 65th birthday

ABSTRACT. We study Berezin-Toeplitz quantization on K"{a}hler manifolds. We explain first how to compute various associated asymptotic expansions, then we compute explicitly the first terms of the expansion of the kernel of the Berezin-Toeplitz operators, and of the composition of two Berezin-Toeplitz operators. As an application, we estimate the norm of Donaldson’s $Q$-operator.

0. INTRODUCTION

Berezin-Toeplitz operators are important in geometric quantization and the properties of their kernels turn out to be deeply related to various problems in K"{a}hler geometry (see e.g. [16, 17]). In this paper, we will study the precise asymptotic expansion of these kernels. We refer the reader to the book [24] for a comprehensive study of the Bergman kernel, Berezin-Toeplitz quantization and its applications. See also the survey [23].

The setting of Berezin-Toeplitz quantization on K"{a}hler manifolds is the following. Let $(X, \omega, J)$ be a compact K"{a}hler manifold of dim$_C X = n$ with K"{a}hler form $\omega$ and complex structure $J$. Let $(L, h^L)$ be a holomorphic Hermitian line bundle on $X$, and let $(E, h^E)$ be a holomorphic Hermitian vector bundle on $X$. Let $\nabla^L, \nabla^E$ be the holomorphic Hermitian connections on $(L, h^L), (E, h^E)$ with curvatures $R^L = (\nabla^L)^2, R^E = (\nabla^E)^2$, respectively. We assume that $(L, h^L, \nabla^L)$ is a prequantum line bundle, i.e.,

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L.$$ (0.1)

Let $g^T^X(\cdot, \cdot) := \omega(\cdot, J \cdot)$ be the Riemannian metric on $TX$ induced by $\omega$ and $J$. The Riemannian volume form $dv_X$ of $(X, g^T^X)$ has the form $dv_X = \omega^n / n!$. The $L^2$–Hermitian product on the space $\mathcal{C}^\infty(X, L^p \otimes E)$ of smooth sections of $L^p \otimes E$ on $X$, with $L^p := L^{0,p}$, is given by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1, s_2 \rangle(x) \, dv_X(x).$$ (0.2)

We denote the corresponding norm by $\| \cdot \|_{L^2}$ and by $L^2(X, L^p \otimes E)$ the completion of $\mathcal{C}^\infty(X, L^p \otimes E)$ with respect to this norm.

Given a continuous smoothing linear operator $K : L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E)$, the Schwartz kernel theorem [24, Th. B.2.7] guarantees the existence of an integral kernel with respect to $dv_X$, denoted by $K(x, x') \in (L^p \otimes E)_x \otimes (L^p \otimes E)_{x'}$, for $x, x' \in X$, i.e.,

$$K S(x) = \int_X K(x, x') S(x') \, dv_X(x'), \quad S \in L^2(X, L^p \otimes E).$$ (0.3)

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Consider now the space $H^0(X, L^p \otimes E)$ of holomorphic sections of $L^p \otimes E$ on $X$ and let $P_p : L^2(X, L^p \otimes E) \rightarrow H^0(X, L^p \otimes E)$ be the orthogonal (Bergman) projection. Its kernel $P_p(x, x')$ with respect to $d\nu_X(x')$ is smooth; it is called the Bergman kernel. The Berezin-Toeplitz quantization of a section $f \in \mathcal{C}^\infty(X, \text{End}(E))$ is the Berezin-Toeplitz operator $\{T_{f, p}\}_{p \in \mathbb{N}}$ which is a sequence of linear operators $T_{f, p}$ defined by

$$T_{f, p} : L^2(X, L^p \otimes E) \rightarrow L^2(X, L^p \otimes E), \quad T_{f, p} = P_p f P_p.$$  

The kernel $T_{f, p}(x, x')$ of $T_{f, p}$ with respect to $d\nu_X(x')$ is also smooth. Since $\text{End}(L) = \mathbb{C}$, we have $T_{f, p}(x, x') \in \text{End}(E)_x$ for $x \in X$.

We introduce now the relevant geometric objects used in Theorems 0.1, 0.2 and 0.3. Let $T^{(1,0)}X$ be the holomorphic tangent bundle on $X$, and $T^{*,(1,0)}X$ its dual bundle. Let $\nabla^{TX}$ be the Levi-Civita connection on $(X, g^{TX})$. We denote by $R^{TX} = (\nabla^{TX})^2$ the curvature, by $\text{Ric}$ the Ricci curvature and by $r$ the scalar curvature of $\nabla^{TX}$ (cf. (3.3)).

We still denote by $\nabla^E$ the connection on $\text{End}(E)$ induced by $\nabla^E$. Consider the (positive) Laplacian $\Delta$ acting on the functions on $(X, g^{TX})$ and the Bochner Laplacian $\Delta^E$ on $\mathcal{C}^\infty(X, E)$ and on $\mathcal{C}^\infty(X, \text{End}(E))$. Let $\{e_k\}$ be a (local) orthonormal frame of $(TX, g^{TX})$. Then

$$\Delta^E = -\sum_k (\nabla^E_{e_k} \nabla^E_{e_k} - \nabla^E_{\nabla^{TX} e_k}).$$

Let $\Omega^{q, r}(X, \text{End}(E))$ be the space of $(q, r)$-forms on $X$ with values in $\text{End}(E)$, and let

$$\nabla^{1, 0} : \Omega^{q, r}(X, \text{End}(E)) \rightarrow \Omega^{q+1, r}(X, \text{End}(E))$$

be the $(1, 0)$-component of the connection $\nabla^E$. Let $(\nabla^E^*)_*, \nabla^{1, 0}*, \overline{\nabla}^E*$ be the adjoints of $\nabla^E$, $\nabla^{1, 0}$, $\overline{\nabla}^E$, respectively. Let $D^{1, 0}, D^{0, 1}$ be the $(1, 0)$ and $(0, 1)$ components of the connection $\nabla^{T^*X} : \mathcal{C}^\infty(X, T^*X) \rightarrow \mathcal{C}^\infty(X, T^*X \otimes T^*X)$ induced by $\nabla^{TX}$.

In the following, we denote by

$$\langle \cdot, \cdot \rangle_\omega : \Omega^{n, *}(X, \text{End}(E)) \times \Omega^{n, *}(X, \text{End}(E)) \rightarrow \mathcal{C}^\infty(X, \text{End}(E))$$

the $\mathbb{C}$-bilinear pairing $\langle \alpha \otimes f, \beta \otimes g \rangle_\omega = \langle \alpha, \beta \rangle f \cdot g$, for forms $\alpha, \beta \in \Omega^{n, *}(X)$ and sections $f, g \in \mathcal{C}^\infty(X, \text{End}(E))$ (cf. (0.14), (0.16), (0.17)). Put

$$R^E_A = \langle R^E, \omega \rangle_\omega.$$ 

Let $\text{Ric}_\omega = \text{Ric}(J, \cdot, \cdot)$ be the $(1, 1)$-form associated to $\text{Ric}$. Set

$$|\text{Ric}_\omega|^2 = \sum_{i < j} |\text{Ric}_\omega(e_i, e_j)|^2, \quad |R^{TX}|^2 = \sum_{i < j} \sum_{k < l} |R^{TX}(e_i, e_j)e_k, e_l|^2,$$

and let

$$b_{2C} = -\frac{\Delta r}{48} + \frac{1}{96} |R^{TX}|^2 - \frac{1}{24} |\text{Ric}_\omega|^2 + \frac{1}{128} r^2,$$

$$b_{2E} = \frac{\sqrt{-1}}{32} \left( 2r R^E_A - 4 \langle \text{Ric}_\omega, R^E \rangle_\omega + \Delta^E R^E_A \right),$$

$$b_1 = \frac{r}{8\pi} + \frac{\sqrt{-1}}{2\pi} R^E_A, \quad b_2 = \frac{1}{\pi^2} (b_{2C} + b_{2E}).$$
We use now the notation from (3.6). By our convention (cf. (3.5)), we have at \( x_0 \in X \),
\[
\langle \alpha m \, dz_l \wedge dz_m, \beta_{k\bar{q}} \, dz_k \wedge dz_{\bar{q}} \rangle = -4 \alpha m \beta m, \quad \langle \alpha m \bar{q} \, dz_m \otimes dz_{\bar{q}}, \beta_{k\bar{q}} \, dz_k \otimes dz_{\bar{q}} \rangle = 4 \alpha m \bar{q} \beta m \bar{q}
\]
(note that \(|dz_{\bar{q}}|^2 = 2\)). Then by Lemma 3.1, (3.3) and (5.4), we have at \( x_0 \in X \),
\[
\text{Ric}_{\bar{k}k} = 2 R_{\bar{k}q\bar{q}k} = 2 R_{\bar{k}q\bar{q}} \quad r = 8 R_{\bar{q}q}, \quad \text{Ric}_\omega = \sqrt{-1} \text{Ric}_{\bar{k}k} \, dz_k \wedge d\bar{z}_k,
\]
\[
\sqrt{-1} R_{\Lambda}^E = 2 R_{kk}^E, \quad b_1 = \frac{1}{\pi} \left( R_{kkmm}^\Lambda + R_{mm}^\Lambda \right),
\]
\[
(0.9) \quad b_{2c} = -\frac{\Delta r}{48} + \frac{1}{6} R_{k\bar{m}q} R_{\ell kq} - \frac{2}{3} R_{kq\bar{m}} R_{\ell kq} + \frac{1}{2} R_{k\bar{q}q} R_{\ell km},
\]
\[
(0.10) \quad b_{2E} = R_{q\bar{q}}^E R_{k\ell km} - R_{m\bar{q}}^E R_{k\ell qm} + \frac{1}{2} \left( R_{q\bar{q}}^E R_{mk}^\Lambda - R_{m\bar{q}}^E R_{qk}^\Lambda \right)
\]
\[
+ \frac{1}{4} \left( - R_{kk;m\bar{m}} + 3 R_{m;k\bar{m}}^E \right).
\]

We say that a sequence \( \Theta_p \in \mathcal{C}^\infty(X, \text{End}(E)) \) has an asymptotic expansion of the form
\[
(0.11) \quad \left| \Theta_p(x) - \sum_{r=0}^{k} A_r(x) p^{n-r} + O(p^{-\infty}) \right|_{\mathcal{C}^k(X)} \leq C_{k,l} p^{n-k-1},
\]
where \( | \cdot |_{\mathcal{C}^k(X)} \) is the \( \mathcal{C}^k \)-norm on \( X \).

**Theorem 0.1.** For any \( f \in \mathcal{C}^\infty(X, \text{End}(E)) \), we have
\[
(0.12) \quad T_{f,p}(x,x) = \sum_{r=0}^{\infty} b_{n,r}(x) p^{n-r} + O(p^{-\infty}), \quad b_{n,r} \in \mathcal{C}^\infty(X, \text{End}(E)).
\]

Moreover,
\[
(0.13) \quad b_{0,0} = f, \quad b_{1,1} = \frac{r}{8 \pi} f + \frac{\sqrt{-1}}{4 \pi} \left( R^E_{\Lambda} f + f R^E_{\Lambda} \right) - \frac{1}{4 \pi} \Delta^E f.
\]

If \( f \in \mathcal{C}^\infty(X) \), then
\[
(0.14) \quad \pi^2 b_{2,2} = \pi^2 b_{2} f + \frac{1}{32} \Delta^2 f - \frac{1}{32} r \Delta f - \frac{\sqrt{-1}}{8} \langle \text{Ric}_\omega, \partial \bar{\partial} f \rangle + \frac{\sqrt{-1}}{24} \langle df, \nabla^E R^E_{\Lambda} \rangle \omega
\]
\[
+ \frac{1}{24} \langle \partial f, \nabla^{1,0} R^E \rangle \omega - \frac{1}{24} \langle \bar{\partial} f, \nabla^{0,1} R^E \rangle \omega - \frac{\sqrt{-1}}{8} (\Delta f) R^E_{\Lambda} + \frac{1}{4} \langle \partial \bar{\partial} f, R^E \rangle \omega.
\]

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1 The Bianchi identity reads \( [\nabla^E, R^E] = 0 \). Take the derivative of \( [\nabla^E, R^E] (\frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_n}) = 0 \), and use (3.10), (3.20), (3.31) to obtain \( \frac{\partial}{\partial z_n} \frac{\partial}{\partial \bar{z}_n} = -\frac{1}{2} \partial^X \frac{\partial}{\partial z_n} \partial \bar{\partial} R + O(|z|^2) \), \( \frac{\partial}{\partial z_n} \frac{\partial}{\partial \bar{z}_n} = O(|z|^2) \). We conclude that
\[
R^E_{mk;m+n} = R^E_{kk;m+n}.
\]

The term \( \frac{1}{4} \left( - R^E_{kk;m+n} + 3 R^E_{mk;m+n} \right) \) in (0.9) can be thus replaced by \( \frac{1}{4} R^E_{kk;m+n} \). Equivalently, by using (5.8), one can replace \( + \frac{\sqrt{-1}}{32} \Delta^E R^E_{\Lambda} + \frac{3}{16} \bar{\partial} \partial^E \nabla^{1,0} R^E \) in (0.8) by \( - \frac{\sqrt{-1}}{16} \Delta^E R^E_{\Lambda} \).
Theorem 0.2. For any \( f, g \in \mathcal{C}^\infty(X, \text{End}(E)) \), the kernel of the composition \( T_{f,p} \circ T_{g,p} \) has an asymptotic expansion on the diagonal

\[
(T_{f,p} \circ T_{g,p})(x, x) = \sum_{r=0}^{\infty} b_{n, f, g}(x)p^{n-r} + O(p^{-\infty}), \quad b_{n, f, g} \in \mathcal{C}^\infty(X, \text{End}(E)),
\]

in the sense of (0.11). Moreover, \( b_{0, f, g} = fg \) and

\[
b_{1, f, g} = \frac{1}{8\pi} \int \left[ f g + \frac{1}{4\pi} \left( \langle \overline{\partial} f, \partial g \rangle - \frac{1}{8} \langle \overline{\partial} \Delta f, \partial g \rangle + \frac{1}{16} \Delta f : \Delta g + \frac{1}{8} \langle D^{1,0} \overline{\partial} f, D^{1,0} g \rangle \right) \right].
\]

If \( f, g \in \mathcal{C}^\infty(X) \), then

\[
b_{2, f, g} = f b_{2, g} + g b_{2, f} - f g b_2 + \frac{1}{4\pi} \left\{ - \frac{1}{8} \langle \overline{\partial} f, \partial \Delta g \rangle - \frac{1}{8} \langle \overline{\partial} \Delta f, \partial g \rangle + \frac{1}{16} \Delta f : \Delta g + \frac{1}{8} \langle D^{0,0} \overline{\partial} f, D^{1,0} g \rangle \right\}.
\]

The existence of the expansions (0.12) and (0.15) and the formulas for the leading terms hold in fact in the symplectic setting and are consequences of [27, Lemma 4.6 and (4.79)] or [24, Lemma 7.2.4 and (7.4.6)] (cf. Lemma 2.2). The novel point of Theorems 0.1, 0.2 is the calculation of the coefficients \( b_{1, f}, b_{2, f}, b_{1, f, g}, b_{2, f, g} \). Note that the precise formula \( b_{1, f} \) for a function \( f \in \mathcal{C}^\infty(X) \) was already given in [24, Problem 7.2]. In Theorem 5.1 we find a general formula of \( b_{2, f} \) for any \( f \in \mathcal{C}^\infty(X, \text{End}(E)) \).

If \( f = 1 \), then \( T_{f, p} = P_p \), and the existence of the expansion (0.12) and the form of the leading term was proved by [30], [6], [33]. The terms \( b_1, b_2 \) were computed by Lu [22] (for \( E = \mathbb{C} \), the trivial line bundle with trivial metric), X. Wang [32], L. Wang [31], in various degree of generality. The method of these authors is to construct appropriate peak sections as in [30], using Hörmander’s \( L^2 \overline{\partial} \)-method. In [20, §5.1], Dai-Liu-Ma computed \( b_1 \) by using the heat kernel, and in [25, §2], [26, §2] (cf. also [24, §4.1.8, §8.3.4]), we computed \( b_1 \) in the symplectic case.

The expansion of the Bergman kernel \( P_p(x, x) \) on the diagonal, for \( E = \mathbb{C} \), was derived by Douglas and Klevtsov [11] by using path integral and perturbation theory. They give physics interpretations in terms of supersymmetric quantum mechanics, the quantum Hall effect and black holes (cf. also [12]).

An interesting consequence of Theorem 0.2 is the following precise computation of the expansion of the composition of two Berezin-Toeplitz operators.

Theorem 0.3. Let \( f, g \in \mathcal{C}^\infty(X, \text{End}(E)) \). The product of the Toeplitz operators \( T_{f, p} \) and \( T_{g, p} \) is a Toeplitz operator, more precisely, it admits the asymptotic expansion

\[
T_{f, p} \circ T_{g, p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f, g), p} + O(p^{-\infty}),
\]

where \( C_r \) are bidifferential operators, in the sense that for any \( k \geq 0 \), there exists \( c_k > 0 \) with

\[
\left\| T_{f, p} \circ T_{g, p} - \sum_{i=0}^{k} p^{-i} T_{C_r(f, g), p} \right\| \leq c_k p^{-k-1},
\]
where \( \| \cdot \| \) denotes the operator norm on the space of bounded operators. We have

\[
\begin{align*}
C_0(f, g) &= fg, \\
C_1(f, g) &= -\frac{1}{2\pi} \langle \nabla^{1,0} f, \overline{\partial} g \rangle_\omega \in \mathcal{C}^\infty(X, \text{End}(E)), \\
C_2(f, g) &= b_{2, f, g} - b_{2, f, g} - b_{1, C_1(f, g)}.
\end{align*}
\]

If \( f, g \in \mathcal{C}^\infty(X) \), then

\[
C_2(f, g) = \frac{1}{8\pi^2} \langle D^{1,0} \partial f, D^{0,1} \overline{\partial} g \rangle + \frac{\sqrt{-1}}{4\pi^2} \langle \text{Ric}_\omega, \partial f \wedge \overline{\partial} g \rangle - \frac{1}{4\pi^2} \langle \partial f \wedge \overline{\partial} g, R^E \rangle_\omega.
\]

The existence of the expansion (0.18) is a special case of [27, Th. 1.1] (cf. also [24, Th. 7.4.1, 8.1.10]), where we found a symplectic version in which the Toeplitz operators (0.4) are constructed by using the projection to the kernel of the Dirac operator. Note that the precise values of \( b_2 \) are not used to derive (0.21) (cf. Section 5.3).

The existence of the expansion (0.18) for \( E = \mathbb{C} \) was first established by Bordemann, Meinrenken and Schlichenmaier [3], Schlichenmaier [29] (cf. also [19]) using the theory of Toeplitz structures by Boutet de Monvel and Guillemin [4]. Charles [7] calculated \( C_1(f, g) \) for \( E = \mathbb{C} \). The asymptotic expansion (0.18) with a twisting bundle \( E \) was derived by Hawkins [18, Lemma 4.1] up to order one (i.e., (0.19) for \( k = 0 \)).

Also, there is related work of Engliš [13, 14] dealing with expressing asymptotic expansions of Bergman kernel and coefficients of the Berezin-Toeplitz expansion (0.18) in terms of the metric. Engliš [14, Cor. 15] computed \( C_1(f, g) \) and \( C_2(f, g) \) for a smoothly bounded pseudoconvex domain \( X = \{ z \in \mathbb{C}^n : \varphi(z) > 0 \} \), where \( \varphi \) is a defining function such that \( -\log \varphi \) is strictly plurisubharmonic, and for the trivial line bundle \( L = \mathbb{C} \) over \( X \), equipped with the nontrivial metric \( h^L = \varphi \) of positive curvature.

Note that we work throughout the paper with a non-trivial twisting bundle \( E \). Moreover, we have shown in [27, §5-6] (cf. also [24, §7.5]) that Berezin-Toeplitz quantization holds for complete Kähler manifolds and orbifolds endowed with a prequantum line bundle. The calculations of the coefficients in the present paper being local in nature, they hold also for the above cases.

For some applications of the results of this paper to Kähler geometry see the paper [17] by Fine.

We close the introduction with some remarks about the Berezin-Toeplitz star-product. Following the ground-breaking work of Berezin [11], one can define a star-product by using Toeplitz operators. Note that formal star-products are known to exist on symplectic manifolds by [9, 15]. The Berezin-Toeplitz star-product gives a very concrete and geometric realization of such product. For general symplectic manifolds this was realized in [24, 27] by using Toeplitz operators obtained by projecting on the kernel of the Dirac operator.

Consider now a compact Kähler manifold \( (X, \omega) \) and a prequantum line bundle \( L \). For every \( f, g \in \mathcal{C}^\infty(X) \) one defines the Berezin-Toeplitz star-product (cf. [19, 29] and [24, 27] for the symplectic case) by

\[
f * g := \sum_{k=0}^{\infty} C_k(f, g) h^k \in \mathcal{C}^\infty(X)[[h]].
\]
This star-product is associative. Moreover, for \( f, g \in \mathcal{C}^\infty(X) \) we have (cf. [24, (7.4.3)], [27, (4.89)])

\[
C_0(f, g) = fg = C_0(g, f), \quad C_1(f, g) - C_1(g, f) = \sqrt{-1}\{f, g\},
\]
where \( \{f, g\} \) is the Poisson bracket associated to \( 2\pi\omega \). Therefore

\[
[T_{f,p}, T_{g,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + \mathcal{O}(p^{-2}), \quad p \to \infty.
\]

Consider a twisting holomorphic Hermitian vector bundle \( E \) and \( f, g \in \mathcal{C}^\infty(X, \text{End}(E)) \) as in Theorem 0.3. This corresponds to matrix-valued Berezin-Toeplitz quantization, which models a quantum system with \( r = \text{rank} E \) degrees of freedom. By (0.18), this Berezin-Toeplitz quantization has the expected semi-classical behaviour. Moreover, by [24, Th. 7.4.2], [27, Th. 4.19] we have

\[
\lim_{p \to \infty} \|T_{f,p}\| = \|f\|_\infty := \sup_{0 \neq u \in \mathcal{E}, x \in X} |f(x)(u)|_{h^E}/|u|_{h^E}.
\]

**Corollary 0.4.** Let \( f, g \in \mathcal{C}^\infty(X, \text{End}(E)) \). Set

\[
f \ast g := \sum_{k=0}^{\infty} C_k(f, g) h^k \in \mathcal{C}^\infty(X, \text{End}(E))[h^i].
\]

where \( C_r(f, g) \) are determined by (0.18). Then (0.26) defines an associative star-product on \( \mathcal{C}^\infty(X, \text{End}(E)) \). Set moreover

\[
\{f, g\} := \frac{1}{2\pi\sqrt{-1}} (\langle \nabla_{1,0} g, \overline{f}^E \rangle_\omega - \langle \nabla_{1,0} f, \overline{g}^E \rangle_\omega).
\]

If \( fg = gf \) on \( X \) we have

\[
[T_{f,p}, T_{g,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + \mathcal{O}(p^{-2}), \quad p \to \infty.
\]

The associativity of the star-product (0.26) follows immediately from the associativity rule for the composition of Toeplitz operators, \( (T_{f,p} \circ T_{g,p}) \circ T_{k,p} = T_{f,p} \circ (T_{g,p} \circ T_{k,p}) \) for any \( f, g, k \in \mathcal{C}^\infty(X, \text{End}(E)) \), and from the asymptotic expansion (0.18) applied to both sides of the latter equality.

Due to the fact that \( \{f, g\} = \{g, f\} \) if \( E \) is trivial and comparing (0.24) to (0.28), one can regard \( \{f, g\} \) defined in (0.27) as a non-commutative Poisson bracket.

**Remark 0.5.** Throughout the paper we suppose that \( g^{TX}(u, v) = \omega(u, Jv) \). The results presented so far still hold for a general non-Kähler \( J \)-invariant Riemannian metric \( g^{TX} \). To explain this point we follow [24, §4.1.9].

Let us denote the metric associated to \( \omega \) by \( g^{TX}_{\omega} := \omega(\cdot, J\cdot) \). We identify the 2-form \( R^X \) with the Hermitian matrix \( \hat{R}^L \in \text{End}(\text{T}(1,0)X) \) via \( g^{TX} \). Then the Riemannian volume form of \( g^{TX}_{\omega} \) is given by \( dv_X,\omega = (2\pi)^{-n}\det(\hat{R}^L)dv_X \) (where \( dv_X \) is the Riemannian volume form of \( g^{TX} \)). Moreover, \( h^E_{\omega} := \det(\frac{\hat{R}^L}{2\pi})^{-1}h^E \) defines a metric on \( E \). We add a subscript \( \omega \) to indicate the objects associated to \( g^{TX}_{\omega} \), \( h^L \) and \( h^E_{\omega} \). Hence \( \langle \cdot, \cdot \rangle_{\omega} \) denotes the \( L^2 \) Hermitian product on \( \mathcal{C}^\infty(X, L^p \otimes E) \) induced by \( g^{TX}_{\omega} \), \( h^L \), \( h^E_{\omega} \). This product is equivalent to the product \( \langle \cdot, \cdot \rangle \) induced by \( g^{TX}, h^L, h^E \).

Moreover, \( H^0(X, L^p \otimes E) \) does not depend on the Riemannian metric on \( X \) or on the Hermitian metrics on \( L, E \). Therefore, the orthogonal projections from \( (\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle) \) and \( (\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle) \) onto \( H^0(X, L^p \otimes E) \) are the same. Hence \( P_p = P_{p,\omega} \) and therefore \( T_{f,p} = T_{f,p,\omega} \) as operators. However, their kernels are different. If
$T_{f,p,\omega}(x, x')$, $(x, x' \in X)$, denotes the smooth kernels of $T_{f,p,\omega}$ with respect to $du_{X,\omega}(x')$, we have

$$(0.29) \quad T_{f,p}(x, x') = (2\pi)^{-n}\det(\hat{R}^L)(x')T_{f,p,\omega}(x, x').$$

For the kernel $T_{f,p,\omega}(x, x')$, we can apply Theorem \[0.1\] since $g^T_{\omega}(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is a Kähler metric on $TX$. We obtain in this way the expansion of $T_{f,p}(x, x)$ for a non-Kähler metric $g^T_{\omega}$ on $X$. By (0.29), the coefficients of these expansions (0.12), (0.18) satisfy

$$(0.30) \quad b_{n,f} = (2\pi)^{-n}\det(\hat{R}^L)b_{n,\omega}, \quad C_{r}(f, g) = C_{r,\omega}(f, g).$$

This paper is organized as follows. In Section 1 we recall the formal calculus on $\mathbb{C}^n$ for the model operator $L$, which is the main ingredient of our approach. In Section 2 we review the asymptotic expansion of the kernel of Berezin-Toeplitz operators and explain the strategy of our computation. In Section 3 we obtain explicitly the first terms of the Taylor expansion of our rescaled operator $L_\epsilon$. In Section 4 we study in detail the contribution of $O_2, O_4$ to the term $F_4$ from (2.20). In Section 5 by applying the formal calculi on $\mathbb{C}^n$ and the results from Section 4 we establish Theorems 0.1, 0.2 and 0.3. We also verify that our calculations are compatible with Riemann-Roch-Hirzebruch Theorem. In Section 6 we estimate the $C^m$-norm of Donaldson’s $Q$-operator, thus continuing [20, 21].

We shall use the following notations. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^m$, $Z \in \mathbb{C}^m$, we set $|\alpha| := \sum_{j=1}^{m} \alpha_j$ and $Z^\alpha := Z_1^{\alpha_1} \cdots Z_m^{\alpha_m}$. Moreover, when an index variable appears twice in a single term, it means that we are summing over all its possible values.

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1. Kernel calculus on $\mathbb{C}^n$

In this section we recall the formal calculus on $\mathbb{C}^n$ for the model operator $L$ introduced in [27, §2], [24, §7.1] (with $a_j = 2\pi$ therein), and we derive the properties of the calculus of the kernels $(F, P)(Z, Z')$, where $F \in \mathbb{C}[Z, Z']$ and $P(Z, Z')$ is the kernel of the projection on the null space of the model operator $L$. This calculus is the main ingredient of our approach.

Let us consider the canonical coordinates $(Z_1, \ldots, Z_{2n})$ on the real vector space $\mathbb{R}^{2n}$. On the complex vector space $\mathbb{C}^n$ we consider the complex coordinates $(z_1, \ldots, z_n)$. The two sets of coordinates are linked by the relation $z_j = Z_{2j-1} + \sqrt{-1}Z_{2j}$, $j = 1, \ldots, n$.

We consider the $L^2$-norm $\| \cdot \|_{L^2} = \left( \int_{\mathbb{R}^{2n}} | \cdot |^2 dZ \right)^{1/2}$ on $\mathbb{R}^{2n}$, where $dZ = dZ_1 \cdots dZ_{2n}$ is the standard Euclidean volume form. We define the differential operators:

$$(1.1) \quad b_i = -2\frac{\partial}{\partial z_i} + \pi z_i, \quad b_i^\dagger = 2\frac{\partial}{\partial \bar{z}_i} + \pi \bar{z}_i, \quad b = (b_1, \ldots, b_n), \quad L = \sum_i b_i b_i^\dagger,$$

which extend to closed densely defined operators on $(L^2(\mathbb{R}^{2n}), \| \cdot \|_{L^2})$. As such, $b_i^\dagger$ is the adjoint of $b_i$ and $L$ defines as a densely defined self-adjoint operator on $(L^2(\mathbb{R}^{2n}), \| \cdot \|_{L^2})$. 

The following result was established in [25, Th. 1.15] (cf. also [24, Th. 4.1.20]).

**Theorem 1.1.** The spectrum of $\mathcal{L}$ on $L^2(\mathbb{R}^{2n})$ is given by

$$\text{Spec}(\mathcal{L}) = \left\{ 4\pi |\alpha| : \alpha \in \mathbb{N}^n \right\}.$$ (1.2)

Each $\lambda \in \text{Spec}(\mathcal{L})$ is an eigenvalue of infinite multiplicity and an orthogonal basis of the corresponding eigenspace is given by

$$B_\lambda = \left\{ b_\alpha (z^\beta e^{-\pi \sum_i |z_i|^2/2}) : \alpha \in \mathbb{N}^n \text{ with } 4\pi |\alpha| = \lambda, \beta \in \mathbb{N}^n \right\}.$$ (1.3)

and $\bigcup \{ B_\lambda : \lambda \in \text{Spec}(\mathcal{L}) \}$ forms a complete orthogonal basis of $L^2(\mathbb{R}^{2n})$. In particular, an orthonormal basis of $\text{Ker}(\mathcal{L})$ is

$$\left\{ \varphi_\beta(z) = \left( \frac{2|\beta|!}{\pi} \right)^{1/2} z^\beta e^{-\pi \sum_i |z_i|^2/2} : \beta \in \mathbb{N}^n \right\}. \quad (1.4)$$

Let $\mathcal{P}(Z, Z')$ denote the kernel of the orthogonal projection $\mathcal{P} : L^2(\mathbb{R}^{2n}) \rightarrow \text{Ker}(\mathcal{L})$ with respect to $dZ'$. Let $\mathcal{P}^\perp = \text{Id} - \mathcal{P}$. We call $\mathcal{P}(\cdot, \cdot)$ the Bergman kernel of $\mathcal{L}$.

Obviously $\mathcal{P}(Z, Z') = \sum_\beta \varphi_\beta(z) \overline{\varphi_\beta(z')} \neq 0$ we infer from (1.4) that

$$\mathcal{P}(Z, Z') = \exp \left( -\frac{\pi}{2} \sum_{i=1}^n \left( |z_i|^2 + |z'_i|^2 - 2z_i z'_i \right) \right). \quad (1.5)$$

In the calculations involving the kernel $\mathcal{P}(\cdot, \cdot)$, we prefer however to use the orthogonal projection, rather than integrating against the expression (1.5) of $\mathcal{P}(\cdot, \cdot)$. This point of view helps simplify a lot the computations and understand better the operations. As an example, if $\varphi(Z) = b_\alpha (z^\beta e^{-\pi \sum_i |z_i|^2/2})$ with $\alpha, \beta \in \mathbb{N}^n$, then Theorem 1.1 implies immediately that

$$\mathcal{P}(\varphi)(Z) = \left\{ \begin{array}{ll}
 z^\beta e^{-\pi \sum_i |z_i|^2/2}, & \text{if } |\alpha| = 0, \\
 0, & \text{if } |\alpha| > 0.
\end{array} \right. \quad (1.6)$$

The following commutation relations are very useful in the computations. Namely, for any polynomial $g(z, \bar{z})$ in $z$ and $\bar{z}$, we have

$$\begin{align*}
[b_i, b_j^+] &= b_i b_j^+ - b_j^+ b_i = -4\pi \delta_{ij}, \\
[b_i, b_j] &= [b_i^+, b_j^+] = 0, \\
[g(z, \bar{z}), b_j] &= 2\frac{\partial}{\partial z_j} g(z, \bar{z}), \\
[g(z, \bar{z}), b_j^+] &= -2\frac{\partial}{\partial z_j} g(z, \bar{z}).
\end{align*} \quad (1.7)$$

For a polynomial $F$ in $Z, Z'$, we denote by $F \mathcal{P}$ the operator on $L^2(\mathbb{R}^{2n})$ defined by the kernel $F(Z, Z') \mathcal{P}(Z, Z')$ and the volume form $dZ$ according to (0.3).

The following very useful Lemma [24, Lemma 7.1.1] describes the calculus of the kernels $(F \mathcal{P})(Z, Z') := F(Z, Z') \mathcal{P}(Z, Z')$.

**Lemma 1.2.** For any $F, G \in \mathbb{C}[Z, Z']$ there exists a polynomial $\mathcal{K}[F, G] \in \mathbb{C}[Z, Z']$ with degree $\deg \mathcal{K}[F, G]$ of the same parity as $\deg F + \deg G$, such that

$$((F \mathcal{P}) \circ (G \mathcal{P}))(Z, Z') = \mathcal{K}[F, G](Z, Z') \mathcal{P}(Z, Z'). \quad (1.8)$$
2. Expansion of the Kernel of Berezin-Toeplitz Operators

In this section, we review some results from [25], [27] (cf. also [24, § 7.2]). We explain then how to compute the coefficients of various expansions considered in this paper. We keep the notations and assumptions from the Introduction.

Kodaira-Laplace operator. Let $\delta^{Lp \otimes E,*}$ be the adjoint of the Dolbeault operator $\delta^{Lp \otimes E}$. Let $\square_p = \delta^{Lp \otimes E,*} \delta^{Lp \otimes E}$ be the restriction of the Kodaira Laplacian to $\mathcal{C}^\infty(X, L^p \otimes E)$. Let $\Delta^{Lp \otimes E}$ be the Bochner Laplacian on $\mathcal{C}^\infty(X, L^p \otimes E)$ associated to $\nabla^L$, $\nabla^E$, $g^TX$, defined as in (0.5). Then we have (cf. [24, Remark 1.4.8])

\begin{equation}
2 \square_p = \Delta^{Lp \otimes E} - \frac{\pi^2}{a^2} R^E(e_0, Je_i) - 2np. \tag{2.1}
\end{equation}

Moreover, by Hodge theory (cf. [24, Th. 1.4.1]) we have

\begin{equation}
\text{Ker}(\square_p) = H^0(X, L^p \otimes E). \tag{2.2}
\end{equation}

This identification is important since the computations are performed by rescaling $\square_p$ and expanding the rescaled operator.

Normal coordinates. Let $a^X$ be the injectivity radius of $(X, g^TX)$. We denote by $B^X(x, \varepsilon)$ and $B^{TX}(0, \varepsilon)$ the open balls in $X$ and $TX$ with center $x$ and radius $\varepsilon$, respectively. Then the exponential map $T_xX \ni Z \mapsto \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{TX}(0, \varepsilon)$ onto $B^X(x, \varepsilon)$ for $\varepsilon \leq a^X$. From now on, we identify $B^{TX}(0, \varepsilon)$ with $B^X(x, \varepsilon)$ via the exponential map for $\varepsilon \leq a^X$. Throughout what follows, $\varepsilon$ runs in the fixed interval $]0, a^X/4[.$

Basic trivialization. We fix $x_0 \in X$. For $Z \in B^{T^*X}(0, \varepsilon)$ we identify $(L_Z, h_Z^2)$, $(E_Z, h_Z^E)$ and $(L^p \otimes E)_Z$ to $(L_{x_0}, h_{x_0}^2)$, $(E_{x_0}, h_{x_0}^E)$ and $(L^p \otimes E)_{x_0}$ by parallel transport with respect to the connections $\nabla^L$, $\nabla^E$ and $\nabla^{Lp \otimes E}$ along the curve $\gamma_Z : [0, 1] \ni u \mapsto \exp_{x_0}^X(uZ)$. This is the basic trivialization we use in this paper.

Using this trivialization we identify $f \in \mathcal{C}^\infty(X, \text{End}(E))$ to a family $\{f_{x_0}\}_{x_0 \in X}$ where $f_{x_0}$ is the function $f$ in normal coordinates near $x_0$, i.e., $f_{x_0} : B^{T^*X}(0, \varepsilon) \ni Z \mapsto \exp_{x_0}^X(Z) \circ f \in \text{End}(E_{x_0})$, $f_{x_0}(Z) = f \circ \exp_{x_0}^{-1}(Z)$. In general, for functions in the normal coordinates, we will add a subscript $x_0$ to indicate the base point $x_0 \in X$. Similarly, $P_{x_0}(x, x')$ induces in terms of the basic trivialization a smooth section $(Z, Z') \mapsto P_{x_0}(Z, Z') = \pi^* \text{End}(E)$ over $\{(Z, Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon\}$, which depends smoothly on $x_0$. Here we identify a section $\mathcal{S} \in \mathcal{C}^\infty(TX \times_X TX, \pi^* \text{End}(E))$ with the family $(S_x)_{x \in X}$, where $S_x = S|_{x=\gamma_x(Z)}$.

Coordinates on $T_{x_0}X$. Let us choose an orthonormal basis $\{w_i\}_{i=1}^n$ of $T^{(1,0)}_{x_0}X$. Then $e_{2i-1} = \frac{1}{\sqrt{2}}(w_i + \overline{w}_j)$ and $e_{2j} = \frac{1}{\sqrt{2}}(w_i - \overline{w}_j), j = 1, \ldots, n$ form an orthonormal basis of $T_{x_0}X$. We use coordinates on $T_{x_0}X \simeq \mathbb{R}^{2n}$ given by the identification

\begin{equation}
\mathbb{R}^{2n} \ni (Z_1, \ldots, Z_{2n}) \mapsto \sum_i Z_i e_i \in T_{x_0}X. \tag{2.3}
\end{equation}

In what follows we also use complex coordinates $z = (z_1, \ldots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

Volume form on $T_{x_0}X$. If $d\nu_X$ is the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$, there exists a smooth positive function $\kappa_{x_0} : T_{x_0}X \rightarrow \mathbb{R}, Z \mapsto \kappa_{x_0}(Z)$ defined by

\begin{equation}
\nu_X(Z) = \kappa_{x_0}(Z) d\nu_X(Z), \quad \kappa_{x_0}(0) = 1, \tag{2.4}
\end{equation}

where the subscript $x_0$ of $\kappa_{x_0}(Z)$ indicates the base point $x_0 \in X$.

Sequences of operators. Let $\Theta_p : L^2(X, L^p \otimes E) \rightarrow L^2(X, L^p \otimes E)$ be a sequence of continuous linear operators with smooth kernel $\Theta_p(\cdot, \cdot)$ with respect to $d\nu_X$ (e.g. $\Theta_p = \ldots$)
Lemma 2.2. Let \( T : TX \times_X TX \to X \) be the natural projection from the fiberwise product of \( TX \) on \( X \). In terms of our basic trivialization, \( \Theta_p(x, y) \) induces a family of smooth sections \( Z, Z' \mapsto \Theta_p(x_0, Z, Z') \) of \( \pi^* \text{End}(E) \) over \( \{(Z, Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon \} \), which depends smoothly on \( x_0 \).

We denote by \( \| \Theta_p(x_0, Z, Z') \|_{C^l} \) the \( \mathcal{C}^l \) norm with respect to the parameter \( x_0 \in X \). We say that \( \Theta_p(x_0, Z, Z') = O(p^{-\infty}) \) if for any \( l, m \in \mathbb{N} \), there exists \( C_{l,m} > 0 \) such that \( \| \Theta_p(x_0, Z, Z') \|_{C^m(x)} \leq C_{l,m} p^{-l} \).

Notation 2.1. Recall that \( \mathcal{P}_{x_0} = \mathcal{P} \) was defined in (1.5). Fix \( k \in \mathbb{N} \) and \( \varepsilon' \in (0, a^X \mathbb{N}] \). Let \( \{Q_{r,x_0} \}_{0 \leq r \leq k, x_0 \in X} \) be a family of polynomials \( Q_{r,x_0} \in \text{End}(E)_{x_0}[Z, Z'] \) in \( Z, Z' \), which is smooth with respect to the parameter \( x_0 \in X \). We say that

\[
(2.5) \quad p^{-n} \Theta_p(x_0, Z, Z') \equiv \sum_{r=0}^{k} (Q_{r,x_0} \mathcal{P}_{x_0})(\sqrt{pZ}, \sqrt{pZ'})p^{-r/2} + O(p^{-(k+1)/2}),
\]

on \( \{(Z, Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon' \} \) if there exist \( C_0 > 0 \) and a decomposition

\[
(2.6) \quad p^{-n} \Theta_p(x_0, Z, Z') \kappa_{x_0}^{1/2}(Z) \kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k} (Q_{r,x_0} \mathcal{P}_{x_0})(\sqrt{pZ}, \sqrt{pZ'})p^{-r/2} = \Psi_{p,k,x_0}(Z, Z') + O(p^{-\infty}),
\]

where \( \Psi_{p,k,x_0} \) satisfies the following estimate on \( \{(Z, Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon' \} \) for every \( l \in \mathbb{N} \) there exist \( C_{k,l} > 0 \), \( M > 0 \) such that for all \( p \in \mathbb{N}^* \)

\[
(2.7) \quad \| \Psi_{p,k,x_0}(Z, Z') \|_{C^l} \leq C_{k,l} p^{-(k+1)/2} (1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^M \varepsilon^{-C_0 \sqrt{p}|Z-Z'|}.
\]

The sequence \( P_p \). By [3], Proposition 4.1 we know that the Bergman kernel decays very fast outside the diagonal of \( X \times X \). Namely, for any \( l, m \in \mathbb{N} \), \( \varepsilon > 0 \), there exists \( C_{l,m,\varepsilon} > 0 \) such that for all \( p \geq 1 \) we have

\[
(2.8) \quad \| P_p(x, x') \|_{C^m} \leq C_{l,m,\varepsilon} p^{-l} \quad \text{on} \quad \{(x, x') \in X \times X : d(x, x') \geq \varepsilon \}.
\]

Here the \( \mathcal{C}^m \)-norm is induced by \( \nabla^L, \nabla^E, \nabla^{TX} \) and \( h^L, h^E, g^{TX} \).

By [3], Th. 4.18', there exist polynomials \( J_{r,x_0}(Z, Z') \in \text{End}(E)_{x_0} \) in \( Z, Z' \) with the same parity as \( r \), such that for any \( k \in \mathbb{N}, \varepsilon \in [0, a^X/4] \), we have

\[
(2.9) \quad p^{-n} J_{r,x_0}(Z, Z') \equiv \sum_{r=0}^{k} (J_{r,x_0} \mathcal{P}_{x_0})(\sqrt{pZ}, \sqrt{pZ'})p^{-r/2} + O(p^{-(k+1)/2}),
\]

on the set \( \{(Z, Z') \in TX \times_X TX : |Z|, |Z'| < 2\varepsilon \} \), in the sense of Notation 2.1.

The sequence \( T_{f,p} \). From (2.9), we get the following result (cf. [27], Lemma 4.6, [24], Lemma 7.2.4).

Lemma 2.2. Let \( f \in \mathcal{C}^\infty(X, \text{End}(E)) \). There exists a family \( \{Q_{r,x_0}(f)\}_{r \in \mathbb{N}, x_0 \in X} \), depending smoothly on the parameter \( x_0 \in X \), where \( Q_{r,x_0}(f) \in \text{End}(E)_{x_0}[Z, Z'] \) are polynomials with the same parity as \( r \) and such that for every \( l \in \mathbb{N}, \varepsilon \in [0, a^X/4] \),

\[
(2.10) \quad p^{-n} T_{f,p}(Z, Z') \equiv \sum_{r=0}^{k} (Q_{r,x_0}(f) \mathcal{P}_{x_0})(\sqrt{pZ}, \sqrt{pZ'})p^{-r/2} + O(p^{-(k+1)/2}),
\]
on the set \( \{(Z, Z') \in TX \times TX : |Z|, |Z'| < 2\varepsilon\} \), in the sense of Notation 2.1. Moreover, \( Q_{r, z_0}(f) \) are expressed by

\[
Q_{r, z_0}(f) = \sum_{r_1 + r_2 + |\alpha| - r} \mathcal{K} \left[ J_{r_1, z_0} \frac{\partial^\alpha f(z_0)}{\partial Z^\alpha} \frac{Z^\alpha}{\alpha!} J_{r_2, z_0} \right].
\]

Especially,

\[
Q_{0, z_0}(f) = f(z_0).
\]

Our goal is of course to compute the coefficients \( Q_{r, z_0}(f) \). For this we need \( J_{r, z_0} \), which are obtained by computing the operators \( \mathcal{F}_{r, z_0} \) defined by the smooth kernels

\[
\mathcal{F}_{r, z_0}(Z, Z') = J_{r, z_0}(Z, Z') \mathcal{P}(Z, Z')
\]

with respect to \( dZ' \). Our strategy (already used in [24, 25]) is to rescale the Kodaira-Laplace operator, take the Taylor expansion of the rescaled operator and apply resolvent analysis. In the remaining of this section we outline the main steps and continue the Rescaling \( \Box_p \) and Taylor expansion. For \( s \in \mathcal{C}^\infty(\mathbb{R}^{2n}, E_{z_0}) \), \( Z \in \mathbb{R}^{2n}, |Z| \leq 2\varepsilon \), and for \( t = \frac{1}{\sqrt{r}} \), set

\[
(S_t s)(Z) := s(Z/t), \quad V_t := S_t^{-1} t V^{L^p E} S_t, \quad \mathcal{L}_t := S_t^{-1} t^{1/2} \nabla^{L^p E} t^{1/2} \mathcal{L}_t = \kappa^{1/2}(tZ) \nabla V_t \kappa^{-1/2}(tZ), \quad \mathcal{O}_1 = 0.
\]

Resolvent analysis. We define by recurrence \( f_r(\lambda) \in \text{End}(L^2(\mathbb{R}^{2n}, E_{z_0})) \) by

\[
f_0(\lambda) = (\lambda - \mathcal{L}_0)^{-1}, \quad f_r(\lambda) = (\lambda - \mathcal{L}_0)^{-1} \sum_{j=1}^{r} \mathcal{O}_j f_{r-j}(\lambda).
\]

Let \( \delta \) be the counterclockwise oriented circle in \( \mathbb{C} \) of center 0 and radius \( \pi/2 \). Then by [25, (1.110)] (cf. also Theorem 3.2), we obtain

\[
\mathcal{L}_0 = \sum_j b_j b_j^+ = \mathcal{L}, \quad \mathcal{O}_1 = 0.
\]

Resolvent calculation of \( \mathcal{L} \) in terms of differential operators. From Theorem 1.11 (2.16), if \( \mathcal{L}_0 = \sum_j b_j b_j^+ = \mathcal{L}, \quad \mathcal{O}_1 = 0 \), the resolvent is

\[
\mathcal{F}_{r, z_0} = \frac{1}{2\pi r^{1/2}} \int_{\delta} f_r(\lambda) d\lambda.
\]

Recall that \( \mathcal{P}^\perp = \text{Id} - \mathcal{P} \). From Theorem 1.11 (2.16) and (2.18), we get

\[
\mathcal{F}_{0, z_0} = \mathcal{P}, \quad \mathcal{F}_{1, z_0} = 0,
\]

\[
\mathcal{F}_{2, z_0} = -\mathcal{L}^\perp \mathcal{P} \mathcal{O}_2 \mathcal{P} - \mathcal{P} \mathcal{O}_2 \mathcal{L}^\perp \mathcal{P}^\perp,
\]

\[
\mathcal{F}_{3, z_0} = -\mathcal{L}^\perp \mathcal{P} \mathcal{O}_3 \mathcal{P} - \mathcal{P} \mathcal{O}_3 \mathcal{L}^\perp \mathcal{P}^\perp.
\]
and
\[
\mathcal{F}_{4, z_0} = \frac{1}{2\pi \sqrt{-1}} \int_\delta \left[ (\lambda - L)^{-1} P_1 (O_2 f_2 + O_4 f_0) (\lambda) + \frac{1}{\lambda} P (O_2 f_2 + O_4 f_0) (\lambda) \right] d\lambda \\
= L^{-1} P_1 O_2 L^{-1} P_1 O_2 P - L^{-1} P_1 O_4 P \\
+ P O_2 L^{-1} P_1 O_2 L^{-1} P_2 P - P O_4 L^{-1} P_1 P \\
+ L^{-1} P_1 O_2 O_2 L^{-1} P_1 - P O_2 L^{-2} P_1 O_2 P \\
- P O_2 P O_2 L^{-2} P_1 - P L^{-2} O_2 P O_2 P.
\]

(2.20)

In particular, the first two identities of (2.19) imply
\[
J_0, z_0 = 1, \quad J_1, z_0 = 0.
\]

Remark 2.3. \(L_t\) is a formally self-adjoint elliptic operator on \(C^\infty(\mathbb{R}^{2n}, E_{z_0})\) with respect to the norm \(\| \cdot \|_{L^2}\) induced by \(h_{E_{z_0}}, dZ\). Thus \(L_0\) and \(O_r\) are also formally self-adjoint with respect to \(\| \cdot \|_{L^2}\). Therefore the third and fourth terms in (2.20) are the adjoints of the first and second terms, respectively. In Lemma 4.1, we will show that \(P O_2 P = 0\), hence the last two terms in (2.20) vanish. Set
\[
(2.22) \quad \mathcal{F}_{41} = L^{-1} P_1 O_2 L^{-1} P_1 O_2 P - L^{-1} P_1 O_4 P.
\]

3. TAYLOR EXPANSION OF THE RESCALED OPERATOR \(L_t\)

In this section we compute the operators \(L_0\) and \(O_i\) (for \(1 \leq i \leq 4\)) from (2.15) (see Theorem 3.2), which will be used in Sections 4, 5 for the evaluation of the coefficients of the expansion of the kernels of the Berezin-Toeplitz operators.

We denote by \(\langle \cdot, \cdot \rangle\) the \(\mathbb{C}\)-bilinear form on \(TX \otimes_{\mathbb{R}} \mathbb{C}\) induced by \(g^{TX}\). Let \(R^{TX}\) be the curvature of the Levi-Civita connection \(\nabla^{TX}\). Let \(\text{Ric}\) and \(r\) be the Ricci and scalar curvature of \(\nabla^{TX}\). Then we have the following well know facts: for \(U, V, W, Y\) vector fields on \(X\),
\[
R^{TX}(U, V)W + R^{TX}(V, W)U + R^{TX}(W, U)V = 0,
\]
\[
\langle R^{TX}(U, V)W, Y \rangle = \langle R^{TX}(W, Y)U, V \rangle = -\langle R^{TX}(V, U)W, Y \rangle.
\]

Now we work on \(T_{z_0}X \simeq \mathbb{R}^{2n}\) as in (2.3). Recall that we have trivialized \(L_t E\). Let \(\nabla_U\) denote the ordinary differentiation operator on \(T_{z_0}X\) in the direction \(U\).

We adopt the convention that all tensors will be evaluated at the base point \(z_0 \in X\) and most of time, we will omit the subscript \(z_0\).

For \(W \in T_{z_0}X, Z \in \mathbb{R}^{2n}\), let \(\tilde{W}(Z)\) be the parallel transport of \(W\) with respect to \(\nabla^{TX}\) along the curve \([0, 1] \ni u \to uZ\). Because the complex structure \(J\) is parallel with respect to \(\nabla^{TX}\), we know that
\[
(3.2) \quad J_2 \tilde{W}(Z) = \tilde{J}_{z_0} W(Z).
\]

Recall that \(\{e_i\}\) is a fixed orthonormal basis of \((T_{z_0}X, g^{TX})\). Then for \(U, V \in T_{z_0}X,\)
\[
(3.3) \quad \text{Ric}_{z_0}(U, V) = -\langle R^{TX}(U, e_j)V, e_j \rangle_{z_0}, \quad r_{z_0} = -\langle R^{TX}(e_i, e_j)e_i, e_j \rangle_{z_0}.
\]
We define
\[ R_{i}^{TX} \in (T^*X \otimes \Lambda^2(T^*X) \otimes \text{End}(T^*X))_{x_0}, \]
\[ R_{(i,j)}^{TX} \in ((T^*X)^{\otimes 2} \otimes \Lambda^2(T^*X) \otimes \text{End}(T^*X))_{x_0}, \]
\[ \text{Ric}_{(i,j)} \in (T^*X \otimes (T^*X)^{\otimes 2})_{x_0}, \]
\[ R_{i}^{E} \in (T^*X \otimes \Lambda^2(T^*X) \otimes \text{End}(E))_{x_0}, \]
\[ R_{i,(i,j)}^{E} \in ((T^*X)^{\otimes 2} \otimes \Lambda^2(T^*X) \otimes \text{End}(E))_{x_0}, \]
by
\[
\left\langle R_{i}^{TX}(e_m, e_j)e_q, e_i \right\rangle = \left\langle \nabla_{e_q} \left( R_{i}^{TX}(\bar{e}_m, \bar{e}_j)\bar{e}_q, \bar{e}_i \right) \right\rangle_{x_0},
\]
\[
\left\langle R_{(i,j)}^{TX}(e_m, e_j)e_q, e_i \right\rangle = \left\langle \nabla_{e_q} \nabla_{e_q} \left( R_{(i,j)}^{TX}(\bar{e}_m, \bar{e}_j)\bar{e}_q, \bar{e}_i \right) \right\rangle_{x_0},
\]
\[
(3.4)
\]
\[
\text{Ric}_{i}(e_j) = (\nabla_{e_q} \text{Ric}(\bar{e}_i, \bar{e}_j))_{x_0},
\]
\[
R_{i}^{E}(e_j) = (\nabla_{e_q} R_{i}^{E}(\bar{e}_i, \bar{e}_j))_{x_0},
\]
\[
R_{i,(i,j)}^{E}(e_j) = (\nabla_{e_q} \nabla_{e_q} R_{i}^{E}(\bar{e}_i, \bar{e}_j))_{x_0}.
\]
We will also use the complex coordinates \( z = (z_1, \ldots, z_n) \). Note that
\[
(3.5) \quad e_{2j-1} = \frac{\partial}{\partial z_{2j-1}} = \frac{\partial}{\partial z_j} + \frac{\partial}{\partial \overline{z}_j}, \quad e_{2j} = \frac{\partial}{\partial z_j} = \sqrt{-1} \left( \frac{\partial}{\partial z_j} - \frac{\partial}{\partial \overline{z}_j} \right), \quad \left| \frac{\partial}{\partial \overline{z}_j} \right|^2 = \frac{1}{2}.
\]
Set
\[
R_{k\overline{m}\overline{n}} = \left\langle R_{i}^{TX} \left( \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \overline{z}_m} \right) \frac{\partial}{\partial z_l}, \frac{\partial}{\partial \overline{z}_q} \right\rangle_{x_0}, \quad R_{k\overline{l}}^{E} = R_{x_0}^{E} \left( \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \overline{z}_l} \right),
\]
\[
(3.6)
\]
and in the same way, we define \( R_{k\overline{m}\overline{n};\overline{r}} \), \( R_{k\overline{m}\overline{n};\overline{s}} \), \( \text{Ric}_{k\overline{l}} \), \( R_{k\overline{l}}^{E} \), \( R_{k\overline{k};\overline{i}} \), \( R_{k\overline{k};\overline{i}}^{E} \).

Since \( R_{i}^{TX} \) is a \( (1, 1) \)-form and \( \nabla^{E} \) is the Chern connection on \((E, h^{E})\), we deduce from (3.1)–(3.3) the following.

**Lemma 3.1.**

1. \( R_{k\overline{m}\overline{n}} = R_{m\overline{k}\overline{n}} = R_{k\overline{n}\overline{m}} = R_{m\overline{q}\overline{m}}, \quad r = 8 \), \( R_{m\overline{q}\overline{m}} = R_{q\overline{m}\overline{m}} \),
2. \( R_{k\overline{m}\overline{n};\overline{s}} = R_{m\overline{k}\overline{n};\overline{s}} = R_{k\overline{q}\overline{m};\overline{s}} = R_{q\overline{k}\overline{m};\overline{s}} \),
3. \( \text{Ric} \) is a symmetric \( (1, 1) \)-tensor and \( \text{Ric}_{k\overline{q}} = 2 R_{m\overline{q}k}, \quad \text{Ric}_{m\overline{q};\overline{k}} = 2 R_{m\overline{q}k;\overline{i}} \),
4. \( R_{i}^{TX}(\bar{e}_m, \bar{e}_j), R_{(i,j)}^{TX}(\bar{e}_m, \bar{e}_j) \) are \( (1, 1) \)-forms with values in \( \text{End}(T_{x_0}X) \) which commute with \( J_{x_0} \),
5. \( R_{i}^{E}(\bar{e}_m), R_{i,(i,j)}^{E}(\bar{e}_m, \bar{e}_j) \in \text{End}(E_{x_0}) \).

Let \( \text{div}(\text{Ric}) \) be the divergence of \( \text{Ric} \). By [28, §2.3.4, Prop. 6],
\[
(3.7) \quad dr = 2 \text{div}(\text{Ric}) = 2 \left( \nabla^{L_{\overline{m}}X}_{\overline{e}_m} \text{Ric}(\bar{e}_m, \cdot) \right).
\]
Lemma 3.1 and (3.7) entail
\[
R_{\bar{\xi} \bar{\eta} \bar{m} \bar{n}} = R_{\bar{\xi} \bar{m} \bar{n}} \bar{\eta}, \quad R_{\bar{\xi} \bar{m} \bar{n} \bar{k}} = R_{\bar{\xi} \bar{n} \bar{k} \bar{m}},
\]
\[
-(\Delta r)_{x_0} = 2\epsilon_\eta \epsilon_m (\text{Ric}(\bar{\eta}, \bar{e}_m))_{x_0} = 32R_{\bar{m} \bar{n} \bar{k} \bar{l}} = 32R_{\bar{n} \bar{k} \bar{l} \bar{m}} = 32R_{\bar{m} \bar{n} \bar{k} \bar{l}}.
\]

Set
\[
(3.9) \quad \mathcal{R} := \sum_i Z_i e_i = Z, \quad \nabla_{\mathcal{R}} := \nabla + \frac{1}{2} R_{x_0}(\mathcal{R}, \mathcal{R}).
\]

Thus \(\mathcal{R}\) is the radial vector field on \(\mathbb{R}^{2n}\). We also introduce the vector fields \(z = \sum_i z_i \frac{\partial}{\partial z_i}\) and \(\bar{z} = \sum_i \bar{z}_i \frac{\partial}{\partial \bar{z}_i}\). By [24, Prop. 1.2.2], (3.2), we have
\[
(3.10) \quad \mathcal{R} = \sum_i Z_i \bar{e}_i, \quad z = \sum_i z_i \frac{\partial}{\partial z_i}, \quad \bar{z} = \sum_i \bar{z}_i \frac{\partial}{\partial \bar{z}_i}.
\]

By (3.1) and (1.1), we get
\[
(3.11) \quad b_i = -2\nabla_{0, \frac{\partial}{\partial z_i}}, \quad b_i^+ = 2\nabla_{0, \frac{\partial}{\partial \bar{z}_i}}, \quad R_{x_0} = -2\pi \sqrt{-1} (J \mathcal{R}, \mathcal{R})_{x_0}.
\]

Let \(A_{1z}, A_{2z} \in (T^*X)^{\otimes 2}\) be polynomials in \(z\) with values symmetric tensors, defined by
\[
(3.12) \quad A_{1z} (e_i, e_j) = \left( R_{x_0}^{TX} (\mathcal{R}, e_i) \mathcal{R}, e_j \right)_{x_0}, \quad A_{2z} (e_i, e_j) = \left( R_{x_0}^{TX} (\mathcal{R}, e_i) \mathcal{R}, R_{x_0}^{TX} (\mathcal{R}, e_j) \mathcal{R} \right)_{x_0}.
\]

Recall that the operator \(\mathcal{L}\) was defined in (1.1). Set
\[
(3.13) \quad \mathcal{O}_2 = \frac{1}{3} \left( R_{x_0}^{TX} (\mathcal{R}, e_i) \mathcal{R}, e_j \right) \nabla_{0, e_i} \nabla_{0, e_j} - 2R_{x_0}^{\mathcal{R}} + \frac{\pi}{3} R_{x_0}^{TX} (z, \mathcal{R}, e_j) + \frac{2}{3} \text{Ric}_{x_0} (\mathcal{R}, e_j) - R_{x_0}^{\mathcal{R}} (\mathcal{R}, e_j) \nabla_{0, e_j},
\]
and
\[
\mathcal{O}_4 = \frac{1}{20} \left( A_{1z} - \frac{4}{3} A_{2z} \right) (e_i, e_j) \nabla_{0, e_i} \nabla_{0, e_j},
\]
\[
\mathcal{O}_{42} = \mathcal{L} - \left( \frac{1}{80} A_{1z} - \frac{1}{360} A_{2z} \right) (e_j, e_j) - \frac{1}{288} \text{Ric} (\mathcal{R}, \mathcal{R})^2 + \frac{\mathcal{L}}{144} \text{Ric} (\mathcal{R}, \mathcal{R})^2,
\]
\[
(3.14) \quad \mathcal{O}_{43} = -\frac{1}{144} \text{Ric} (\mathcal{R}, \mathcal{R}) \mathcal{L} \text{Ric} (\mathcal{R}, \mathcal{R}),
\]
\[
\mathcal{O}_{44} = \left\{ \frac{\pi}{30} A_{1z} (\mathcal{R}, e_i) - \frac{\pi}{10} A_{2z} (\mathcal{R}, e_i) + \frac{1}{20} A_{1z} + \frac{2}{45} A_{2z} \right\} (e_i, e_j) \nabla_{0, e_i} + \frac{8}{25} \left( \frac{1}{40} A_{1z} + \frac{1}{45} A_{2z} \right) (e_i, e_j) \nabla_{0, e_i}.
\]
The following result extends [24, Th. 4.1.25] where $L_0, O_1, O_2$ were computed.

**Theorem 3.2.** The following identities hold for the operators $O_*$ introduced in (2.15):

$$
L_0 = \sum_j b_j b_j^* = L' = -\sum_i \nabla_{v, e_i} \nabla_{v, e_i} - 2\pi n, \quad O_1 = 0,
$$

$$
O_2 = O_2' - \frac{1}{3} \text{Ric}_{x_0}(R, e_j) \nabla_{v, e_j} - \frac{r_{x_0}}{6},
$$

and

$$
O_3 = \frac{1}{6} \left\langle R_{x_0}^{TX}(R, e_j)\mathcal{R}, e_j \right\rangle_{x_0} \nabla_{v, e_i} \nabla_{v, e_j} + \frac{2\pi}{15} \left\langle R_{x_0}^{TX}(z, \bar{z})\mathcal{R}, e_i \right\rangle_{x_0} + \frac{1}{6} \text{Ric}_{x_0}(R, e_i)
+
\frac{1}{6} \left\langle R_{x_0}^{TX}(R, e_j)\mathcal{R}, e_i \right\rangle_{x_0} - \frac{2}{3} R_{x_0}^{E}(R, e_i) \nabla_{v, e_i} + \frac{\pi}{15} \left\langle R_{x_0}^{TX}(z, \bar{z})\mathcal{R}, e_j \right\rangle_{x_0} - \frac{1}{6} \text{Ric}_{x_0}(R, e_i)
-
\frac{1}{12} \text{Ric}_{x_0}(R, e_i) - \frac{1}{3} R_{x_0}^{E}(R, e_i) - \frac{\sqrt{-1}}{2} R_{x_0}^{E}(e_i, J e_i),
$$

$$
O_4 = O_{41} + O_{42} + O_{43} + O_{44} + O_{45} + O_{46}.
$$

Proof. Recall that $\tilde{e}_i(Z)$ is the parallel transport of $e_i$ with respect to $\nabla_{v}^T X$ along the curve $[0, 1] \ni u \to uZ$. Let $\tilde{\theta}(Z) = (\theta_j(Z))_{j=1}^{2n}$ be the $2n \times 2n$-matrix such that

$$
e_i = \sum_j \theta_j^*(Z) \tilde{e}_j(Z), \quad \tilde{e}_j(Z) = (\tilde{\theta}(Z)^{-1})^j_k e_k.
$$

Taking into account the Taylor expansion of $\theta_j^i$ at 0 we have (cf. [24, (1.2.27)])

$$
\sum_{|\alpha| \leq 1} (|\alpha|^2 + |\alpha|)(\partial^\alpha \theta_j^i)(0) \frac{Z^\alpha}{\alpha!} = \left\langle R_{x_0}^{TX}(R, e_j)\mathcal{R}, e_i \right\rangle_{x_0}.
$$

From this equation, we obtain first that

$$
e_j(Z) = \tilde{e}_j(Z) + \frac{1}{6} \left\langle R_{x_0}^{TX}(R, e_j)\mathcal{R}, e_k \right\rangle_{x_0} \tilde{e}_k(Z) + O(|Z|^3).
$$
From (3.4), (3.10), (3.12), (3.19) and (3.20), we get further

\[ \theta^i_j = \delta_{ij} + \frac{1}{6} \left\langle R_{\sigma_0}^T (\mathcal{R}, e_i) \mathcal{R}, e_j \right\rangle_{\sigma_0} + \frac{1}{12} \left\langle R_{\tau_0}^T (\mathcal{R}, e_i) \mathcal{R}, e_j \right\rangle_{\tau_0} \\
+ \frac{1}{20} \frac{1}{2} A_{1Z} (e_i, e_j) + \frac{1}{6} A_{2Z} (e_i, e_j) \right) + O(|Z|^5). \]

(3.21)

Set \( g_{ij}(Z) = g^{TX}(e_i, e_j)(Z) = \left\langle e_i, e_j \right\rangle_{\mathcal{Z}} \) and let \((g^{ij}(Z))\) be the inverse of the matrix \((g_{ij}(Z))\). Then by (3.12), (3.13) and (3.21), we have

\[ g^{ij}(Z) = \theta^k_i(Z) \theta^l_j(Z) = \delta_{ij} + \frac{1}{3} \left\langle R_{\tau_0}^T (\mathcal{R}, e_i) \mathcal{R}, e_j \right\rangle_{\tau_0} - \frac{1}{6} \left\langle R_{\tau_0}^T (\mathcal{R}, e_i) \mathcal{R}, e_j \right\rangle_{\tau_0} \\\n- \frac{1}{20} A_{1Z} (e_i, e_j) + \frac{1}{15} A_{2Z} (e_i, e_j) + O(|Z|^5). \]

(3.22)

In view of the expansion \((1 + a)^{-1} = 1 - a + a^2 + \ldots\), we obtain

\[ g^{ij}(Z) = \delta_{ij} + \frac{1}{3} \left\langle R_{\tau_0}^T (\mathcal{R}, e_i) \mathcal{R}, e_j \right\rangle_{\tau_0} - \frac{1}{6} \left\langle R_{\tau_0}^T (\mathcal{R}, e_i) \mathcal{R}, e_j \right\rangle_{\tau_0} \\\n- \frac{1}{20} A_{1Z} (e_i, e_j) + \frac{1}{15} A_{2Z} (e_i, e_j) + O(|Z|^5). \]

(3.23)

If \( \Gamma^i_{j}(Z) \) are the Christoffel symbols of \( \nabla^TX \) with respect to the frame \( \{e_i\} \), then \((\nabla^TX e_j)(Z) = \Gamma^i_{j}(Z)e_i \). By the explicit formula for \( \nabla^TX \), we get (cf. [24] (4.1.102)) with \( \partial_j := \frac{\partial}{\partial z_j} \)

\[ \Gamma^i_{j}(Z) = \frac{1}{2} g^{lj}(\partial_j g_{jk} + \partial_k g_{lj} - \partial_k g_{ij})(Z) \]

\[ = \frac{1}{3} \left[ \left\langle R_{\tau_0}^T (\mathcal{R}, e_j) e_i, e_l \right\rangle_{\tau_0} + \left\langle R_{\tau_0}^T (\mathcal{R}, e_i) e_j, e_l \right\rangle_{\tau_0} \right] + O(|Z|^5). \]

(3.24)

For \( j \) fixed, \( \Gamma^i_{jj}(Z) = \frac{1}{2} g^{lj}(2 \partial_j g_{lk} - \partial_k g_{lj})(Z) \), thus by (3.22) and (3.23),

\[ \Gamma^i_{jj}(Z) = \frac{2}{3} \left\langle R_{\tau_0}^T (\mathcal{R}, e_j) e_i, e_l \right\rangle_{\tau_0} \\\n+ \frac{1}{12} \left[ 4 \left\langle R_{\tau_0}^T (\mathcal{R}, e_j) e_i, e_l \right\rangle_{\tau_0} + 2 \left\langle R_{\tau_0}^T (\mathcal{R}, e_j) \mathcal{R}, e_l \right\rangle_{\tau_0} + \left\langle R_{\tau_0}^T (\mathcal{R}, e_j) e_i, \mathcal{R} \right\rangle_{\tau_0} \right] \\\n- \frac{2}{9} \left\langle R_{\tau_0}^T (\mathcal{R}, e_l) \mathcal{R}, R_{\tau_0}^T (\mathcal{R}, e_j) e_j \right\rangle_{\tau_0} + \frac{\partial}{\partial Z_j} \left( \frac{1}{20} A_{1Z} + \frac{2}{45} A_{2Z} \right) (e_j, e_l) \\\n- \frac{1}{2} \frac{\partial}{\partial Z_j} \left( \frac{1}{20} A_{1Z} + \frac{2}{45} A_{2Z} \right) (e_j, e_j) + O(|Z|^4). \]

(3.25)

Note that

\[ \det(\delta_{ij} + a_{ii}) = 1 + \sum_i a_{ii} + \sum_{i<j} (a_{ii} a_{jj} - a_{ij} a_{ji}) + \ldots \]

and

\[ (1 + a)^{1/4} = 1 + \frac{1}{4} a - \frac{3}{32} a^2 + \ldots \]
Thus by (3.28), (3.29) and since (3.30)

\[
\begin{align*}
\kappa(Z)^{1/2} &= |\mathrm{det}(g_{ij}(Z))|^{1/4} = 1 + \frac{1}{12} \langle R^{TX}_{x_0}(\mathcal{R}, e_j)\mathcal{R}, e_j \rangle_{x_0} \\
- \frac{1}{24} \mathrm{Ric}_{z}(\mathcal{R}, \mathcal{R}) + \frac{1}{80} A_{1Z}(e_j, e_j) + \frac{1}{90} A_{2Z}(e_j, e_j) \\
+ \frac{1}{36} \sum_{i<j} \left( \langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_i \rangle_{x_0} \langle R^{TX}_{x_0}(\mathcal{R}, e_j)\mathcal{R}, e_j \rangle_{x_0} - \langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j \rangle_{x_0}^2 \right) \\
- \frac{1}{96} \left( \sum_{j} \langle R^{TX}_{x_0}(\mathcal{R}, e_j)\mathcal{R}, e_j \rangle_{x_0} \right)^2 + \mathcal{O}(|Z|^5) \\
= 1 - \frac{1}{12} \mathrm{Ric}(\mathcal{R}, \mathcal{R}) - \frac{1}{24} \mathrm{Ric}_{z}(\mathcal{R}, \mathcal{R}) + \left( \frac{1}{80} A_{1Z} - \frac{1}{360} A_{2Z} \right)(e_j, e_j) \\
+ \frac{1}{288} \mathrm{Ric}(\mathcal{R}, \mathcal{R})^2 + \mathcal{O}(|Z|^5).
\end{align*}
\]

(3.26)

Thus

\[
\begin{align*}
\kappa(Z)^{-1/2} &= 1 + \frac{1}{12} \mathrm{Ric}(\mathcal{R}, \mathcal{R}) + \frac{1}{24} \mathrm{Ric}_{z}(\mathcal{R}, \mathcal{R}) - \left( \frac{1}{80} A_{1Z} - \frac{1}{360} A_{2Z} \right)(e_j, e_j) \\
+ \left( \frac{1}{144} - \frac{1}{288} \right) \mathrm{Ric}(\mathcal{R}, \mathcal{R})^2 + \mathcal{O}(|Z|^5).
\end{align*}
\]

(3.27)

Observe that \( J \) is parallel with respect to \( \nabla^{TX} \), thus \( \langle J\bar{e}_i, e_j \rangle_Z = \langle Je_i, e_j \rangle_{x_0} \). From (3.9), (3.10), (3.18) and (3.21), we get

\[
\begin{align*}
\frac{\sqrt{-1}}{2\pi} R^L_Z(\mathcal{R}, e_i) &= \theta_i^*(Z) \langle J\bar{e}_i, \bar{e}_j \rangle_Z Z_i = \theta_i^*(Z) \langle J\mathcal{R}, e_j \rangle_{x_0} \\
&= \langle J\mathcal{R}, e_i \rangle_{x_0} + \frac{1}{6} \langle R^{TX}_{x_0}(\mathcal{R}, J\mathcal{R})\mathcal{R}, e_i \rangle_{x_0} \\
+ \frac{1}{12} \langle R^{TX}_{x_0}(\mathcal{R}, J\mathcal{R})\mathcal{R}, e_i \rangle_{x_0} + \frac{1}{40} \langle R^{TX}_{x_0}(\mathcal{R}, J\mathcal{R})\mathcal{R}, e_i \rangle_{x_0} \\
+ \frac{1}{120} \langle R^{TX}_{x_0}(\mathcal{R}, J\mathcal{R})\mathcal{R}, R^{TX}_{x_0}(\mathcal{R}, e_i) \rangle_{x_0} + \mathcal{O}(|Z|^5).
\end{align*}
\]

(3.28)

Let \( \Gamma^* = \Gamma^L, \Gamma^E \) and \( R^* = R^{E}, R^{L} \), respectively. By [24, Lemma 1.2.4], the Taylor coefficients of \( \Gamma^*(e_i)(Z) \) at \( x_0 \) up to order \( r \) are only determined by those of \( R^* \) up to order \( r - 1 \), and

\[
\sum_{|\alpha| = r} \left( \frac{\partial^{\alpha} \Gamma^*}{\alpha!} \right)(e_i) \frac{Z^\alpha}{\alpha!} = \frac{1}{r + 1} \sum_{|\alpha| = r - 1} \left( \frac{\partial^{\alpha} R^*}{\alpha!} \right)(e_i) \frac{Z^\alpha}{\alpha!}.
\]

(3.29)

Thus by (3.28), (3.29) and since \( R^{TX} \) is a (1,1)-form, we obtain

\[
R^{TX}(\mathcal{R}, J\mathcal{R}) = -2\sqrt{-1} R^{TX}(z, \bar{Z}), \quad \langle R^{TX}_{x_0}(\mathcal{R}, J\mathcal{R})\mathcal{R}, e_i \rangle_{x_0} = -2\sqrt{-1} A_{1Z}(\bar{Z}, e_i)
\]

and

\[
\begin{align*}
t^{-1} \Gamma^L(\bar{e}_i)(tZ) &= -\pi \sqrt{-1} \langle J\mathcal{R}, e_i \rangle_{x_0} - t^2 \pi \frac{15}{6} \langle R^{TX}_{x_0}(z, \bar{Z})\mathcal{R}, e_i \rangle_{x_0} \\
- t^3 \frac{\pi}{15} \langle R^{TX}_{x_0}(z, \bar{Z})\mathcal{R}, e_i \rangle_{x_0} + \pi \frac{4}{60} A_{1Z}(\bar{Z}, e_i) - t^4 \frac{\pi}{180} A_{2Z}(\bar{Z}, e_i) + \mathcal{O}(t^5).
\end{align*}
\]

(3.30)
By (2.14), (3.4), (3.20), (3.29) and (3.30), for $t = \frac{1}{\sqrt{p}}$, we get

$$\Gamma^E(e_i)(Z) = \frac{1}{2} R^E_{z_0}(\mathcal{R}, e_i) + \frac{1}{3} R^E_{z; z}(\mathcal{R}, e_i) + \frac{1}{8} \left( R^E_{z, z}(\mathcal{R}, e_i) + \frac{1}{3} \left( R^{TX}(\mathcal{R}, e_i) \mathcal{R}, e_k \right)_{x_0} R^E_{x_0}(\mathcal{R}, e_k) \right) + O(|Z|^4),$$

(3.31)

$$\nabla_{t, e_i} = \nabla_{e_i} + \frac{1}{t} \Gamma^E(e_i)(tZ) + t \Gamma^E(e_i)(tZ),$$

$$= \nabla_{0, e_i} - \frac{t^2}{6} \left( \pi R^{TX}(z, \overline{z}) \mathcal{R}, e_i \right)_{x_0} + \frac{t^2}{2} R^E_{z_0}(\mathcal{R}, e_i) + O(t^3).$$

By (2.1) and (2.14), we get,

$$L = -\frac{\kappa(tZ)^{1/2}}{2} g^{ij}(tZ) \left[ \nabla_{t, e_i} \nabla_{t, e_j} - t \Gamma^l_{ij}(t*) \nabla_{t, e_l} \right](Z) \kappa(tZ)^{-1/2}$$

$$- t^2 \sqrt{-\Omega} R^E(\xi, J\xi)(tZ) - 2\pi n.$$

We will derive now (3.16a) and (3.16b) (they were already obtained in [24, Th. 4.1.25]). By using the Taylor expansion of the expressions from (3.32) (see (3.23), (3.24), (3.26), (3.27), (3.31)) we obtain immediately the formulas for $L_0$ and $O_1$ given in (3.16a). In order to compute $O_2$, observe first that by (3.1) and the fact that $R^{TX}$ is a (1,1)-form with values in $\text{End}(TX)$, we get

$$\nabla_{e_j} \left( R^{TX}_{x_0}(z, \overline{z}) \mathcal{R}, e_j \right) = 2 \left( \frac{\partial}{\partial z_j} \left( R^{TX}_{x_0}(z, \overline{z}) \mathcal{R}, e_j \right) + \frac{1}{3} \left( R^{TX}_{x_0}(z, \overline{z}) \mathcal{R}, e_j \right) \right) = 0.$$

Thus from (3.13), (3.23), (3.24), (3.26), (3.31)–(3.33), we have

$$O_2 = O_2' + \left[ L_0, \frac{1}{12} \text{Ric}(\mathcal{R}, \mathcal{R}) \right].$$

By the formula of $L_0$ (see (3.16a)) and since

$$\left[ L_0, \text{Ric}(\mathcal{R}, \mathcal{R}) \right] = -4 \text{Ric}(\mathcal{R}, e_j) \nabla_{0, e_j} - 2 \text{Ric}(e_j, e_j),$$

we get from (3.34) the formula for $O_2$ given in (3.16b). From (3.32), we have also

$$O_3 = \frac{1}{6} \left( R^{TX}_{x_0}(\mathcal{R}, e_i) \mathcal{R}, e_j \right)_{x_0} \nabla_{0, e_i} \nabla_{0, e_j}$$

$$- \left[ - \frac{2\pi}{15} \left( R^{TX}_{x_0}(z, \overline{z}) \mathcal{R}, e_i \right)_{x_0} + \frac{2}{3} R^E_{z; z}(\mathcal{R}, e_i) \right] \nabla_{0, e_i}$$

$$- \frac{\partial}{\partial \overline{z}_i} \left[ - \frac{\pi}{15} \left( R^{TX}_{x_0}(z, \overline{z}) \mathcal{R}, e_i \right)_{x_0} + \frac{1}{3} R^E_{z; z}(\mathcal{R}, e_i) \right]$$

$$+ \frac{1}{12} \left[ 4 \text{Ric}_{; z}(\mathcal{R}, e_i) + 2 \left( R^{TX}_{x_0}(\mathcal{R}, e_j) \mathcal{R}, e_i \right)_{x_0} + \text{Ric}_{; z}(\mathcal{R}, \mathcal{R}) \right] \nabla_{0, e_i}$$

$$+ \left[ L_0, \frac{1}{12} \text{Ric}_{; z}(\mathcal{R}, \mathcal{R}) \right] - \frac{\sqrt{-1}}{2} R^E_{z; z}(e_i, Je_i).$$

In (3.35), the first (resp. second and third, resp. fourth, resp. fifth) term is the contribution of the coefficient of $t^5$ in $g^{ij}(tZ)$ (resp. $\nabla_{t, e_i}$, resp. $t \Gamma^l_{ij}(t*)$, resp. $\kappa^{1/2}(tZ)$). By the
same argument in (3.33) and the formula of \( \mathcal{L}_0 \) given in (3.16a), we get
\[
\frac{\partial}{\partial Z_i} \left\langle R_{x_0}^T(z, \mathcal{Z}) \mathcal{R}, e_i \right\rangle_{x_0} = \left\langle R_{x_0}^T(z, \mathcal{Z}) \mathcal{R}, e_j \right\rangle_{x_0},
\]
(3.36)
\[
[\mathcal{L}_0, \text{Ric}_{z}^{\mathcal{Z}}(\mathcal{R}, \mathcal{R})] = -2 \text{Ric}_{z}(\mathcal{R}, \mathcal{R}) + 2 \text{Ric}_{z}(\mathcal{R}, e_i) \nabla_{0, e_i} - 4 \text{Ric}_{z}(\mathcal{R}, e_i) - 2 \text{Ric}_{z}(e_i, e_i).
\]
From (3.35) and (3.36) we get the formula for \( \mathcal{O}_3 \) asserted in (3.17a). Moreover,
\[
\mathcal{O}_4 = \mathcal{O}_{42} + \left[ \mathcal{O}_{43} + \frac{1}{12} \text{Ric}(\mathcal{R}, \mathcal{R}) \right] + \mathcal{O}_{43} + \mathcal{O}_{44}
\]
\[
+ \frac{1}{3} \left\langle R_{x_0}^T(\mathcal{R}, e_i) \mathcal{R}, e_j \right\rangle_{x_0} \left\{ \left( -\frac{1}{3} \left\langle \pi R_{x_0}^T(z, \mathcal{Z}) \mathcal{R}, e_i \right\rangle_{x_0} + R_{x_0}^E(\mathcal{R}, e_i) \right) + \frac{1}{3} \frac{\partial}{\partial Z_i} \left\langle \pi R_{x_0}^T(z, \mathcal{Z}) \mathcal{R}, e_j \right\rangle_{x_0} \right\}
\]
\[
- \frac{1}{6} \frac{\partial}{\partial Z_i} \left( -\frac{1}{3} \left\langle \pi R_{x_0}^T(z, \mathcal{Z}) \mathcal{R}, e_j \right\rangle_{x_0} - \frac{2}{3} \left\langle R_{x_0}^T(\mathcal{R}, e_i) e_j, e_i \right\rangle_{x_0} \right)
\]
\[
- \left( -\frac{2}{9} \left\langle R_{x_0}^T(\mathcal{R}, e_i) \mathcal{R}, e_k \right\rangle_{x_0} \frac{R_{x_0}^E(\mathcal{R}, e_i)}{\nabla_{0, e_i}} \right)
\]
\[
- \frac{\partial}{\partial Z_i} \left( -\frac{1}{6} \left\langle \pi R_{x_0}^T(z, \mathcal{Z}) \mathcal{R}, e_i \right\rangle_{x_0} + \frac{1}{3} R_{x_0}^E(\mathcal{R}, e_i) \right)
\]
(3.37)
\[
- \left\langle \left( -\frac{1}{9} A_{x_0}^2(\mathcal{Z}, e_i) + \frac{1}{12} \left\langle R_{x_0}^T(\mathcal{R}, e_i) \mathcal{R}, e_m \right\rangle_{x_0} \frac{R_{x_0}^E(\mathcal{R}, e_m)}{\nabla_{0, e_i}} \right)
\]
\[
+ \frac{2}{9} \left\langle R_{x_0}^T(\mathcal{R}, e_i) \mathcal{R}, e_k \right\rangle_{x_0} \frac{R_{x_0}^E(\mathcal{R}, e_i)}{\nabla_{0, e_i}} \right) \mathcal{R}(\mathcal{R}, e_k) + \frac{1}{4} R_{x_0}^E(\mathcal{R}, e_i) \right)
\]
\[
- \frac{\sqrt{-1}}{4} R_{x_0}^E(\mathcal{R}, e_i, J e_i).
\]

Here
- \( \mathcal{O}_{42} \) is the contribution of the coefficients of \( t^4 \) in \( \kappa^{1/2}(t \mathcal{Z}) \) and \( \kappa^{-1/2}(t \mathcal{Z}) \),
- the second term is the contribution of the coefficients of \( t^3 \) in \( \kappa^{1/2}(t \mathcal{Z}) \), \( \kappa^{-1/2}(t \mathcal{Z}) \) and in \( -g^{ij}(t \mathcal{Z})(\nabla'_{t, e_i} \nabla'_{t, e_j} - t \Gamma^l_{ij}(t \mathcal{Z}) \nabla'_{t, e_l}) \),
- \( \mathcal{O}_{43} \) is the contribution of the coefficients of \( t^2 \) in \( \kappa^{1/2}(t \mathcal{Z}) \) and \( \kappa^{-1/2}(t \mathcal{Z}) \),
- \( \mathcal{O}_{44} \) is the contribution of the coefficients of \( t^4 \) in \( g^{ij}(t \mathcal{Z}) \),
- the fifth term is the contribution of the coefficients of \( t^2 \) in \( -g^{ij}(t \mathcal{Z}) \) and in \( (\nabla'_{t, e_i} \nabla'_{t, e_j} - t \Gamma^l_{ij}(t \mathcal{Z}) \nabla'_{t, e_l}) \),
- the sixth, seventh and eight terms are the contributions of the coefficients of \( t^4 \) in \( -(\nabla'_{t, e_i} \nabla'_{t, e_j} - t \Gamma^l_{ij}(t \mathcal{Z}) \nabla'_{t, e_l}) \): the sixth term is the contribution of the coefficients of \( t^2 \) in \( \nabla'_{t, e_i} \), \( -t \Gamma^l_{ij}(t \mathcal{Z}) \) and \( t^2 \) in \( \nabla'_{t, e_i} \); the seventh and eighth terms are the contributions of the coefficients of \( t^4 \) in \( \nabla'_{t, e_i} \) and \( -t \Gamma^l_{ij}(t \mathcal{Z}) \).

Now by (3.13),
\[
\frac{1}{12} \left\langle \mathcal{O}_{42}^2, \text{Ric}(\mathcal{R}, \mathcal{R}) \right\rangle = \frac{1}{36} \left\langle R_{x_0}^T(\mathcal{R}, e_i) \mathcal{R}, e_j \right\rangle_{x_0} (4 \text{Ric}(\mathcal{R}, e_j) \nabla_{0, e_i} + 2 \text{Ric}(e_i, e_j))
\]
(3.38)
\[
+ \frac{1}{6} \left( \left\langle \frac{\pi}{3} R_{x_0}^T(z, \mathcal{Z}) \mathcal{R}, e_j \right\rangle_{x_0} + \frac{2}{3} \text{Ric}(\mathcal{R}, e_j) - R_{x_0}^E(\mathcal{R}, e_j) \right) \text{Ric}(\mathcal{R}, e_j).
\]
and the same argument used to obtain (3.33) shows that
\[
\langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_j \rangle_{x_0} \frac{\partial}{\partial Z_i} \langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_i \rangle_{x_0} = \langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, R_{x_0}^{TX}(z, \overline{z})e_i \rangle_{x_0},
\]
(3.39)
\[
\frac{\partial}{\partial Z_i}(A_{1z}(\mathcal{R}, e_i)) = \frac{\partial}{\partial Z_i} \langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_i \rangle_{x_0} = 2 \langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, (z, \overline{z})e_i \rangle_{x_0},
\]
\[
\frac{\partial}{\partial Z_i}(A_{2z}(\mathcal{R}, e_i)) = \frac{\partial}{\partial Z_i} \langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R} \rangle_{x_0}
\]
\[
= \langle R_{x_0}^{TX}(z, \overline{z})e_i, R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R} \rangle_{x_0} + \langle R_{x_0}^{TX}(z, \overline{z})\mathcal{R}, e_j \rangle_{x_0} \text{Ric} (\mathcal{R}, e_j).
\]
Finally, by (3.31) and since $R^{TX}$ is a (1, 1)-form, we obtain
\[
\langle R_{x_0}^{TX}(z, \overline{z})e_i, R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R} \rangle = \langle R_{x_0}^{TX}(z, \overline{z})e_i, R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_m \rangle = 0,
\]
(3.40)
\[
\langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_m \rangle_{x_0} \text{Ric}(e_i, e_m) = -\delta_{i\overline{m}} \text{Ric}_m(z, \overline{z}).
\]
Thus
\[
O_4 = \sum_{a=1}^{5} O_{4a} - \frac{\pi^2}{36} \langle R_{x_0}^{TX}(z, \overline{z})\mathcal{R}, R_{x_0}^{TX}(z, \overline{z})\mathcal{R} \rangle - \frac{1}{4} R_{x_0}^{BG}(\mathcal{R}, e_j)^2
\]
\[
+ \frac{1}{6} \langle \pi R_{x_0}^{TX}(z, \overline{z})\mathcal{R}, e_j \rangle R_{x_0}^{BG}(\mathcal{R}, e_j) + \frac{\pi}{30} \langle R_{x_0}^{TX}(z, \overline{z})\mathcal{R}, e_i \rangle_{x_0}
\]
\[
- \frac{\pi}{20} \langle R_{x_0}^{TX}(z, \overline{z})\mathcal{R}, e_j \rangle_{x_0} \text{Ric}(\mathcal{R}, e_j) + \frac{1}{9} \text{Ric}(\mathcal{R}, e_j) \text{Ric}(\mathcal{R}, e_j)
\]
\[
+ \frac{1}{18} \langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_j \rangle_{x_0} \text{Ric}(e_i, e_j) + \frac{1}{8} \text{Ric}(\mathcal{R}, e_j) R_{x_0}^{BG}(\mathcal{R}, e_j)
\]
\[
- \frac{1}{4} R_{x_0}^{BG}(\mathcal{R}, e_i) - R_{x_0}^{BG}(z, \overline{z}) \left( \frac{\partial}{\partial Z_i}, \frac{\partial}{\partial Z_i} \right).
\]
(3.41)

Putting together Lemma 3.1, (3.12), (3.14), (3.40), (3.41) and the fact that $R^{TX}$ is a (1, 1)-form, we infer (3.17b). The proof of Theorem 3.2 is completed.

4. Evaluation of $\mathcal{T}_4$ from (2.20)

We calculate in this section an explicit formula for the operator $\mathcal{T}_4$, defined by (2.13) and appearing in the Bergman kernel expansion (2.9). This is necessary in Section 5 in order to evaluate the expansion of the kernel of the Berezin-Toeplitz operators. We use formula (2.20) to achieve our aim. Recall that explicit formulas for the operators $\mathcal{L}$, $O_2$, $O_4$ appearing in (2.20) were given in Theorem 3.2.

This section is organized as follows. In Section 4.1 we determine the terms in (2.20) which involve $O_2$. In Section 4.2 we calculate the terms in (2.20) which involve $O_4$. In Section 4.3 we obtain the formula for $\mathcal{T}_4(0, 0)$ (cf. Theorem 4.5).

We adopt the convention that all tensors will be evaluated at the base point $x_0 \in X$ and most of time, we will omit the subscript $x_0$.

4.1. Contribution of $O_2$ to $\mathcal{T}_4(0, 0)$.
Lemma 4.1. The following identities hold:

\begin{align}
(4.1a) \quad \mathcal{P} \mathcal{O}_2 \mathcal{P} &= 0, \\
(4.1b) \quad (\mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1})(0, 0) &= \frac{1}{4\pi^2} \left( \sum_{\lambda} R_{m\bar{m}k\bar{k}} + \sum_{k} R_{k\bar{k}}^E \right)^2, \\
(4.1c) \quad (\mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P})(0, 0) &= \frac{1}{36\pi^2} R_{m\bar{k}\bar{q}} R_{k\bar{m}q} + \frac{1}{4\pi^2} \left( \frac{4}{3} R_{q\bar{m}m\bar{q}} + R_{q\bar{q}}^E \right) \left( \frac{4}{3} R_{\bar{k}\bar{k}q\bar{q}} + R_{\bar{k}\bar{k}q}^E \right).
\end{align}

Proof. Note that by (1.1) and (1.5),

\begin{align}
(4.2) \quad (b_i^+ \mathcal{P})(Z, Z') = 0, \quad (b_i \mathcal{P})(Z, Z') = 2\pi(\bar{Z}_i - \bar{Z}'_i) \mathcal{P}(Z, Z').
\end{align}

For \( \phi \in T^* X \), by (3.5), (3.11) and (4.2), we have

\begin{align}
\phi(e_i) e_i &= 2\phi(\overline{e_j}) \frac{\partial}{\partial e_j} + 2\phi(\overline{e_j}) \frac{\partial}{\partial e_j}, \quad \phi(e_i) \nabla_{0_e_i} = \phi(\overline{e_j}) b_j^+ - \phi(\overline{e_j}) b_j, \\
\phi(e_i) \nabla_{0_e_i} \mathcal{P}(Z, 0) &= -2\pi \phi(\overline{e}_i) \mathcal{P}(Z, 0).
\end{align}

By Lemma 3.1, (3.1), (3.6), (3.16b), (4.3) and the fact that \( R^{TX} \) is a \((1, 1)\)-form, we get

\begin{align}
\mathcal{O}_2 &= \frac{1}{3} R_{k\bar{m}q\bar{q}} z_k z_{\bar{m}} b_q + \frac{1}{3} R_{k\bar{q}m\bar{q}} z_k \bar{z}_m b_q + \frac{1}{3} R_{k\bar{m}q\bar{q}} \bar{z}_m b_q^+ + \frac{1}{3} R_{k\bar{q}m\bar{q}} z_k \bar{z}_m b_q^+ - \frac{4}{3} R_{m\bar{m}q\bar{q}} \\
&\quad + \left( \frac{2}{3} R_{k\bar{m}q\bar{q}} \bar{z}_m - \frac{\pi}{3} R_{k\bar{m}q\bar{q}} z_k \bar{z}_m \bar{z}_q \right) b_q^+ - \left( \frac{2}{3} R_{k\bar{q}m\bar{q}} z_k + \frac{\pi}{3} R_{k\bar{q}m\bar{q}} z_k \bar{z}_m \bar{z}_q \right) b_q \\
&\quad - 2R_{q\bar{q}}^E - R^{E}(z, \frac{\partial}{\partial \bar{e}_q}) b_q^+ + R^{E}(z, \frac{\partial}{\partial \bar{e}_q}) b_q.
\end{align}

By Lemma 3.1, (1.7) and (4.4), we get \([R_{k\bar{m}q\bar{q}} z_k z_{\bar{m}} b_q] = 8b_q R_{k\bar{m}q\bar{q}} z_k + 8 R_{m\bar{m}q\bar{q}}, \) and

\begin{align}
\mathcal{O}_2 &= \frac{1}{3} b_q b_q R_{k\bar{m}q\bar{q}} z_k z_q + b_q \left( -\frac{\pi}{3} R_{k\bar{m}q\bar{q}} z_k z_q \bar{z}_m \bar{z}_q + 2R_{k\bar{q}m\bar{q}} z_k + R_{q\bar{q}}^E z_k \right) \\
&\quad + \left( \frac{2b_q}{3} R_{k\bar{m}q\bar{q}} \bar{z}_m - \frac{\pi}{3} R_{k\bar{m}q\bar{q}} z_k \bar{z}_m \bar{z}_q + 2R_{k\bar{q}m\bar{q}} \bar{z}_m \bar{z}_q + R_{q\bar{q}}^E \bar{z}_m \bar{z}_q \right) b_q^+ \\
&\quad + \frac{1}{3} R_{k\bar{m}q\bar{q}} \bar{z}_m b_q^+ b_q^+.
\end{align}

Thus Lemma 3.1, (1.7), (4.2) and (4.5) yield

\begin{align}
\mathcal{O}_2 \mathcal{P} &= \left\{ \frac{1}{3} b_q b_q R_{k\bar{m}q\bar{q}} z_k z_q + b_q \left( -\frac{\pi}{3} R_{k\bar{m}q\bar{q}} z_k z_q \bar{z}_m \bar{z}_q + 2R_{k\bar{q}m\bar{q}} z_k + R_{q\bar{q}}^E z_k \right) \right\} \mathcal{P} \\
&= \left\{ \frac{1}{6} b_q b_q R_{k\bar{m}q\bar{q}} z_k z_q + \frac{4}{3} b_q R_{k\bar{m}q\bar{q}} z_k + b_q R_{q\bar{q}}^E z_q \bar{z}_m + \frac{1}{4\pi} R_{q\bar{q}}^E z_q \right\} \mathcal{P}.
\end{align}

Now, (4.1a) follows from Theorem 1.1 (1.6) and (4.6). These imply also

\begin{align}
(4.7) \quad \mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P} &= \left\{ \frac{b_m b_q}{48\pi} R_{k\bar{m}q\bar{q}} z_k z_l + \frac{b_q}{3\pi} R_{k\bar{q}m\bar{q}} z_k z_l - \frac{b_q}{12} R_{k\bar{m}q\bar{q}} z_k z_l \bar{z}_m \bar{z}_q + \frac{b_q}{4\pi} R_{q\bar{q}}^E z_q \right\} \mathcal{P}.
\end{align}

Due to (1.7) and (4.7) we have

\begin{align}
(4.8) \quad (\mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P})(Z, 0) &= \left\{ \frac{b_m b_q}{48\pi} R_{k\bar{m}q\bar{q}} z_k z_l + \frac{b_q}{4\pi} \left( \frac{4}{3} R_{k\bar{q}m\bar{q}} + R_{q\bar{q}}^E \right) z_q \right\} \mathcal{P}(Z, 0), \\
(\mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P})(0, Z) &= -\frac{1}{2\pi} (R_{m\bar{m}q\bar{q}} + R_{q\bar{q}}^E) \mathcal{P}(0, Z).
\end{align}
Since $\mathcal{O}_2$, $\mathcal{L}$ are symmetric (as explained in Remark 2.3) and $(R^E_{\mathcal{Q}})^* = R^E_{\mathcal{Q}}$, we get by (4.11), (3.6) and (4.8),

$$
(\mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1})(0, Z) = ((\mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P})(Z, 0))^*
$$

(4.9)

$$
(\mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1})(Z, 0) = ((\mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P})(0, Z))^* = \frac{1}{2\pi}(R_{k^qkq} + R^E_{q^q})\mathcal{P}(Z, 0).
$$

Note that $\mathcal{P}(0, 0) = 1$ by (1.5). From (1.6), (1.7), (4.2), (4.8) and (4.9), we get (4.1b), and

$$
(\mathcal{P} \mathcal{O}_2 \mathcal{L}^{-2} \mathcal{O}_2 \mathcal{P})(0, 0) = \mathcal{P}\left\{ \frac{32\pi^2}{(48\pi^2) R_{m^qkq} R_{k^qm^q} z_k z_l} + \frac{1}{4\pi^2} (R_{q^q} + R^E_{q^q}) z_l \left( \frac{4}{3} R_{k^qk^q} + R^E_{q^q} \right) z_k \right\} \mathcal{P}(0, 0).
$$

(4.10)

Let $\phi \in \mathbb{C}[b, z]$ be a polynomial in $b, z$. By (1.7), (4.2), we have

$$
(\mathcal{L}_k \phi(b, z) \mathcal{P})(Z, 0) = \phi(b, z) \frac{b_k}{2\pi} \mathcal{P}(Z, 0) = \left( \frac{b_k}{2\pi} + \frac{1}{\partial \phi} \frac{\partial \phi}{\partial z_k} \right) \mathcal{P}(Z, 0),
$$

(4.11)

Let $F(Z)$ be a homogeneous degree 2 polynomial in $Z$. By (4.2),

$$
(F(Z) \mathcal{P})(Z, 0) = \left( \frac{1}{2 \partial z_j} \frac{\partial F}{\partial z_j} z_j z_j + \frac{\partial^2 F}{\partial z_j \partial z_j} z_j \frac{b_j}{2\pi} + \frac{1}{2 \partial z_j, \partial z_j} \frac{\partial^2 F}{\partial z_j \partial z_j} \right) \mathcal{P}(Z, 0).
$$

(4.12)

Thus from (1.6), (4.11) and (4.12), we have

$$
(F \mathcal{P})(Z, 0) = \left( \sum_{|\alpha| = 2} \frac{\partial^2 F}{\partial z^\alpha} \frac{z^\alpha}{\alpha!} + \frac{1}{\pi} \frac{\partial F}{\partial z_j, \partial z_j} \right) \mathcal{P}(Z, 0),
$$

(4.13)

$$
(F \mathcal{P} \mathcal{L}_j \mathcal{P})(0, 0) = \frac{1}{\pi^2} \frac{\partial^2 F}{\partial z_j \partial z_j} \mathcal{P}(Z, 0).
$$

By (4.10) and (4.13), we get (4.1c). The proof of Lemma 4.1 is completed. □

**Lemma 4.2.** The following identity holds

$$
\pi^2 (\mathcal{L}^{-1} \mathcal{P} \mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P})(0, 0) = \frac{25}{23 \cdot 33} R_{m^qkq} R_{k^qm^q} - \frac{47}{54} R_{k^qk^q} R_{m^qm^q} + \frac{1}{8} R_{k^qk^q} R_{m^qm^q} + \frac{1}{4} R^E_{k^qk^q} R_{m^qm^q} - \frac{7}{6} R^E_{q^q} R_{m^qm^q} + \frac{1}{8} (R^E_{k^qk^q} R_{q^q} - 3 R^E_{q^q} R^E_{q^q}).
$$

(4.14)
By Theorem 1.1, (1.6) and (1.7), we also have

\[
I_1 = \left\{ \frac{1}{3} b_i b_j R_{z_i z_j} z_i z_j + b_t \left[ -\frac{\pi}{3} R_{z_i z_j} z_i \frac{\partial}{\partial z_j} + \left( 2 R_{z_i z_j} + R_{z_i z_j}^\mu \right) z_i \right] \right\} \\
\times \left[ \frac{b_m b_n}{48\pi} R_{\mu z_j} z_j + \frac{b_1}{4\pi} \left( \frac{4}{3} R_{\mu z_j} + R_{\mu z_j}^\mu \right) z_j \right],
\]

(4.15)

\[
I_2 = \left\{ \frac{2b_1}{3} R_{\mu z_j} z_j \frac{\partial}{\partial z_j} - \frac{\pi}{3} R_{\mu z_j} z_j \frac{\partial}{\partial z_j} + \left( 2 R_{\mu z_j} + R_{\mu z_j}^\mu \right) z_j \right\} \\
\times \left( \frac{b_m}{6} R_{\mu z_j} z_j + \frac{4}{3} R_{\mu z_j} z_j + R_{z_i z_j}^\mu \right).
\]

Then by Lemma 3.1 (1.7), (4.2), (4.5) and (4.8), we get as in (4.10) that

\[
\left( \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P} \right) (Z, 0) = \left\{ I_1 + I_2 + \frac{2}{9} R_{\mu z_j} z_j \frac{\partial}{\partial z_j} + R_{\mu z_j} z_j \frac{\partial}{\partial z_j} \right\} \mathcal{P} (Z, 0).
\]

(4.16)

Let \( h(z) = \sum_i h_i z_i, \ h'(z) = \sum_i h'_i z_i \) with \( h_i, h'_i \in \mathbb{C} \), and let \( F(Z) \) be a homogeneous degree 2 polynomial in \( Z \). By Theorem 1.1 (1.6), (1.7), (4.2), (4.11) and (4.12), we have

\[
(L^{-1} P^F \mathcal{P})(0, 0) = -\frac{1}{\pi} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_i},
\]

(4.17a)

\[
(L^{-1} P^h h \mathcal{P})(0, 0) = \left( L^{-1} b_i h \mathcal{P} \right)(0, 0) = -\frac{1}{2\pi} h_i,
\]

(4.17b)

\[
\left( L^{-1} b_j F b_i \mathcal{P} \right)(0, 0) = \left( L^{-1} b_i b_j F \mathcal{P} \right)(0, 0) = -\frac{1}{2\pi} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j},
\]

(4.17c)

\[
\left( L^{-1} b_j F b_i \mathcal{P} \right)(0, 0) = \left( L^{-1} b_i b_j F \mathcal{P} \right)(0, 0) = -\frac{3}{8\pi^2} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j},
\]

(4.17d)

\[
\left( L^{-1} b_i F b_j \mathcal{P} \right)(0, 0) = \left( L^{-1} b_j F b_i \mathcal{P} \right)(0, 0) = -\frac{1}{4\pi^2} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j}.
\]

(4.17e)

In (4.17c) and (4.17d), we have used \( F b_i = b_i F + 2\frac{\partial F}{\partial z_i} \). Observe that (1.5), (1.7) imply that for every homogeneous degree \( k \) polynomial \( G \) in \( Z \), and every \( \alpha \in \mathbb{N}^n \), we have

\[
(b^\alpha G \mathcal{P})(0, 0) = \begin{cases} 0, & \text{if } |\alpha| \neq k, \\ (-2)^k \frac{\partial^\alpha G}{\partial z^\alpha}, & \text{if } |\alpha| = k. \end{cases}
\]

(4.18)

By Theorem 1.1 (1.6) and (1.7), we also have

\[
(L^{-1} b_i h b_j h' \mathcal{P})(0, 0) = \left( \frac{b_i b_j}{8\pi} hh' + \frac{b_i}{2\pi} h_j h' \right) \mathcal{P}(0, 0) = \frac{1}{2\pi} (h_i h_j' - h_j h_i'),
\]

(4.19a)

\[
(L^{-1} P^h h \mathcal{P})(0, 0) = -\frac{1}{2\pi} h_j h_i' - \frac{3}{2\pi} h_i h_j'.
\]

(4.19b)

where we use in the last equation that \( h_b h_j b_j = b_i h h' b_j + 2h_i b_j b_j \) and further (4.17b), (4.17d).
Let $\phi(z) = \phi_{ij}z_iz_j$, $\psi = \psi_{ij}z_iz_j$ be degree 2 polynomials in $z$ with symmetric matrices $(\phi_{ij}), (\psi_{ij})$. Then by Theorem 1.1 (1.7), (4.11), (4.17d), (4.17e) and (4.19a), we obtain

\[
(4.20a) \quad \left( \mathcal{L}^{-1} b_kb_k \phi b_j \mathcal{P} \right)(0, 0) = \left( \frac{b_kb_k}{12\pi} \phi h + \frac{b_kb_k}{4\pi} \frac{\partial \phi}{\partial z_j} h \right) \mathcal{P}(0, 0) = -2 \frac{\partial^3(\phi h)}{3\pi \partial z_\ell \partial z_k \partial z_j} + \frac{2}{\pi} (\phi_{\ell j} h_k + \phi_{kj} h_\ell),
\]

\[
(4.20b) \quad \left( \mathcal{L}^{-1} b_k \phi b_j \mathcal{P} \right)(0, 0) = \mathcal{L}^{-1} \left( \frac{b_kb_k}{4\pi} \frac{\partial \phi}{\partial z_j} b_j \mathcal{P}(0, 0) = -2 \frac{\partial^3(\phi h)}{3\pi \partial z_\ell \partial z_k \partial z_j} + \frac{2}{\pi} (\phi_{\ell j} h_k + \phi_{kj} h_\ell),
\]

\[
(4.20c) \quad -\pi \left( \mathcal{L}^{-1} b_k \phi \mathcal{P} \right)(0, 0) = -\frac{1}{2} \mathcal{L}^{-1} b_k \phi b_j \left( b_j h + 2h_\ell \right) \mathcal{P}(0, 0) = \frac{1}{3\pi} \frac{\partial^3(\phi h)}{\partial z_\ell \partial z_k \partial z_j} - \frac{1}{\pi} \phi_{\ell k} h_j,
\]

and

\[
(4.21a) \quad \left( \mathcal{L}^{-1} b_k b_j \phi \psi \mathcal{P} \right)(0, 0) = \left( \frac{b_kb_k}{12\pi} \phi h + \frac{b_k}{4\pi} (b_j h + b_j h_i) \psi \right) \mathcal{P}(0, 0) = -2 \frac{\partial^3(\phi h)}{3\pi \partial z_\ell \partial z_k \partial z_j} + \frac{2}{\pi} (\psi_{kj} h_j + \psi_{kj} h_i),
\]

\[
(4.21b) \quad \left( \mathcal{L}^{-1} b_k \phi \mathcal{P} \right)(0, 0) = \mathcal{L}^{-1} b_k b_j \left( b_j \psi + \frac{1}{\pi} \frac{\partial \psi}{\partial z_j} \right) \mathcal{P}(0, 0) = -\frac{1}{2} \frac{\partial^3(\phi h)}{\partial z_\ell \partial z_k \partial z_j} + \frac{1}{\pi^2} (\psi_{kj} h_j + \psi_{kj} h_k),
\]

\[
(4.21c) \quad -\pi \left( \mathcal{L}^{-1} \phi \mathcal{P} \right)(0, 0) = \frac{1}{6\pi^2} \frac{\partial^3(\phi h)}{\partial z_\ell \partial z_k \partial z_j} + \frac{3}{2\pi^2} \psi_{\ell k} h_j,
\]

In fact, by (4.2), (4.11) implies

\[
\pi(h \overline{\phi} \overline{\psi} \mathcal{P})(Z, 0) = \frac{1}{2} \left( b_k b_j \psi \overline{\phi} \overline{\psi} \right) \mathcal{P}(Z, 0),
\]

so from (4.17d), (4.19b) and (4.21b) we get (4.21c).

Set $R_{\overline{m} \overline{n}}(Z) = R_{k\overline{m}\overline{n}} z_\ell z_k$. By (1.7) and Lemma 3.1

\[
R_{\overline{m} \overline{n}}(Z) b_m b_q R_{\overline{m} \overline{n}}(Z) = \left( b_m b_q R_{\overline{m} \overline{n}}(Z) + 8b_m R_{\overline{m} \overline{k} \overline{q}} z_\ell + 8R_{\overline{m} \overline{q} \overline{k}} \right) R_{\overline{m} \overline{n}}(Z),
\]
thus Theorem 1.1 (4.11), (4.20b) and (4.21a) show that

\[
\left( \mathcal{L}^{-1} b_k b_l R_{qk_l} b_m b_n R_{m_n q_l} \right)(0, 0) \\
= b_k b_l \left( \frac{b_m b_n}{16 \pi} R_{qk_l} + \frac{2b_m}{3 \pi} R_{qk_l} \frac{1}{\overline{z_l}} + \frac{1}{\pi} R_{m_n q_l} \right) R_{m_n q_l} (0, 0) \\
= \frac{1}{\pi} \frac{\partial^4 (R_{qk_l} R_{m_n q_l})}{\partial z_l \partial z_n \partial z_m} - \frac{16}{3 \pi} \frac{\partial^3 (R_{qk_l} R_{m_n q_l})}{\partial z_l \partial z_n \partial z_m} + \frac{8}{\pi} R_{m_n q_l} R_{m_n q_l},
\]

(4.22)

\[
- \left( \pi \mathcal{L}^{-1} b_k \overline{z_l} R_{qk_l} b_m b_n R_{m_n q_l} \right)(0, 0) \\
= - \mathcal{L}^{-1} b_k \left( \frac{1}{2} b_l R_{qk_l} b_m b_n R_{m_n q_l} + \frac{\partial R_{qk_l}}{\partial z_l} b_m b_n R_{m_n q_l} + R_{qk_l} b_m b_n \frac{\partial R_{m_n q_l}}{\partial z_l} \right) (0, 0) \\
= \frac{1}{6 \pi} \frac{\partial^4 (R_{qk_l} R_{m_n q_l})}{\partial z_l \partial z_n \partial z_m \partial z_q} + \frac{8}{3 \pi} \frac{\partial^3 (R_{qk_l} R_{m_n q_l})}{\partial z_l \partial z_n \partial z_m \partial z_q} - \frac{4}{\pi} R_{m_n q_l} R_{m_n q_l} - \frac{16}{\pi} R_{m_n q_l} R_{m_n q_l}.
\]

Due to (4.15), (4.17a)–(4.22), we get

\[
\pi^2 (\mathcal{L}^{-1} I_1 \mathcal{P})(0, 0) = \frac{1}{144} \left[ \frac{7}{6} \frac{\partial^4 (R_{qk_l} R_{m_n q_l})}{\partial z_l \partial z_n \partial z_m \partial z_q} - 3 \frac{\partial^3 (R_{qk_l} R_{m_n q_l})}{\partial z_l \partial z_n \partial z_m \partial z_q} + 4 R_{m_n q_l} R_{m_n q_l} \\
- 16 R_{m_n q_l} R_{m_n q_l} + 12 \frac{\partial^3 ((2 R_{qk_l} + R_{qk_l}) R_{m_n q_l})}{\partial z_l \partial z_n \partial z_m \partial z_q} + \frac{1}{12} \left[ - \frac{1}{3} \frac{\partial^3 (R_{qk_l} R_{m_n q_l})}{\partial z_l \partial z_n \partial z_m \partial z_q} + 4 R_{qk_l} \left( \frac{4}{3} R_{qk_l} + R_{qk_l} \right) - R_{qk_l} \left( \frac{4}{3} R_{m_n q_l} + R_{m_n q_l} \right) \\
+ \frac{1}{8} \left( 2 R_{qk_l} R_{m_n q_l} - R_{qk_l} \left( \frac{4}{3} R_{m_n q_l} + R_{m_n q_l} \right) \right) \right].
\]

But from Lemma 3.1 we have

\[
\frac{\partial^4 (R_{qk_l} R_{m_n q_l})}{\partial z_l \partial z_n \partial z_m \partial z_q} = 4 R_{qk_l} R_{m_n q_l} + 4 R_{qk_l} R_{m_n q_l} + 16 R_{m_n q_l} R_{m_n q_l},
\]

(4.23)

\[
\frac{\partial^3 (R_{qk_l} R_{m_n q_l})}{\partial z_l \partial z_n \partial z_m \partial z_q} = 2 R_{qk_l} R_{m_n q_l} + 4 R_{qk_l} R_{m_n q_l},
\]

(4.24)

\[
\frac{\partial^3 (R_{qk_l} R_{m_n q_l})}{\partial z_l \partial z_n \partial z_m \partial z_q} = 2 R_{qk_l} R_{m_n q_l} + 4 R_{qk_l} R_{m_n q_l}.
\]

Plugging (4.24) in (4.23) we see that the coefficient of \( R_{qk_l} R_{m_n q_l} \) in the term \( \frac{1}{144} \cdot \cdots \) of (4.23) is \( \frac{1}{144} \left( \frac{66}{3} - \frac{32}{3} - 16 - 16 + 24 \right) = 0 \) and

\[
\pi^2 (\mathcal{L}^{-1} I_1 \mathcal{P})(0, 0) = \frac{1}{144} \left[ \frac{10}{3} R_{qk_l} R_{m_n q_l} - \frac{1}{27} R_{qk_l} R_{m_n q_l} \\
+ \left[ \frac{1}{144} \cdot \frac{10}{3} + \frac{1}{12} \cdot \frac{-20}{9} + \frac{1}{3} \cdot \frac{-5}{3} \right] R_{qk_l} R_{m_n q_l} + \left[ \frac{-4}{144} + \frac{1}{12} + \frac{5}{12} \right] R_{qk_l} R_{m_n q_l} \\
+ \frac{4}{144} \cdot \frac{1}{12} \cdot \frac{8}{3} - \frac{10}{3} \cdot \frac{8}{3} \right] R_{qk_l} R_{m_n q_l} + \frac{1}{8} \left( R_{qk_l} R_{qk_l} - R_{qk_l} R_{qk_l} \right) \\
= \frac{5}{23} \cdot 3 \cdot 3 R_{m_n q_l} R_{m_n q_l} - \frac{1}{27} R_{qk_l} R_{m_n q_l} + \frac{1}{8} \left( R_{qk_l} R_{qk_l} - R_{qk_l} R_{qk_l} \right) \\
+ \frac{1}{4} R_{qk_l} R_{m_n q_l} - \frac{1}{6} R_{qk_l} R_{m_n q_l} + \frac{1}{8} \left( R_{qk_l} R_{qk_l} - R_{qk_l} R_{qk_l} \right) \\
+ \frac{1}{4} R_{qk_l} R_{m_n q_l} - \frac{1}{6} R_{qk_l} R_{m_n q_l} + \frac{1}{8} \left( R_{qk_l} R_{qk_l} - R_{qk_l} R_{qk_l} \right).
\]
From (4.15), (4.17c), (4.17f), (4.17g), (4.21b), (4.21c) and (4.24), we get

\[
\pi^2(\mathcal{L}^{-1} \mathcal{P})(0, 0) = \frac{1}{18} \left[ -\frac{1}{2} \beta^3 \left( \frac{R_{\varphi \theta \bar{\varphi} \theta} z \bar{z}}{\partial z \bar{z} \partial z} \right) + 4 R_{\varphi \theta \bar{\varphi}} R_{\varphi m \bar{\varphi}} \\
+ \frac{3}{2} R_{m \varphi \bar{\varphi}} R_{\bar{n} \bar{\varphi}} + 3(2 R_{w \psi \bar{w}} + R_{\bar{w} \bar{w}} (-\frac{2}{4}) R_{m \bar{m} \psi \bar{m}}) \\
+ \left[ -\frac{1}{12} R_{\varphi \psi \bar{\varphi}} - \frac{1}{4} (2 R_{w \bar{w} \bar{w}} + R_{\bar{w} \bar{w}}) \right] \frac{2}{3} R_{\varphi \bar{\varphi} \psi \bar{\varphi}} + R_{\bar{w} \bar{w}} \right] \\
= \frac{1}{36} R_{m \varphi \bar{\varphi}} R_{\varphi \bar{m} \psi \bar{\varphi}} - \frac{5}{6} R_{s \varphi \bar{\varphi} \psi \bar{\varphi}} R_{\varphi \bar{m} \psi \bar{\varphi}} - R_{\bar{w} \bar{w}} R_{\varphi \bar{\varphi} \psi \bar{\varphi}} - \frac{1}{4} R_{\bar{w} \bar{w}} R_{\varphi \bar{\varphi} \psi \bar{\varphi}}.
\]

By (4.17f), we get

\[
(4.27) \quad \left( \mathcal{L}^{-1} \frac{2\pi}{9} R_{m \varphi \bar{\varphi} \psi \bar{\varphi} \psi \bar{\varphi}} R_{\bar{k} \bar{m} \bar{n} \bar{q}} z_k z_{\bar{k}} \mathcal{P} \right)(0, 0) = -\frac{1}{6\pi^2} R_{m \varphi \bar{\varphi}} R_{\varphi \bar{m} \psi \bar{\varphi}}.
\]

Relations (4.16), (4.25), (4.26) and (4.27) imply the desired formula (4.14).

\[\Box\]

4.2. Contribution of \( \mathcal{O}_4 \) to \( \mathcal{P}_4(0, 0) \). We will use the following remark repeatedly in our computation.

Remark 4.3. Let \( \Phi \) be a polynomial in \( b^+, z, \bar{z} \). Due to (1.7) and (4.2), the value of the kernels of \( \mathcal{P} \Phi \mathcal{P}, \mathcal{P}^{1} \Phi \mathcal{P}, \mathcal{L}^{-1} \mathcal{P}^{1} \Phi \mathcal{P} \) at \( (0, 0) \) consists of the terms of \( \Phi \) whose total degree in \( b \) and \( \bar{z} \) is the same as the total degree in \( b^+ \) and \( z \).

Lemma 4.4. We have the following identity :

\[
-\pi^2(\mathcal{L}^{-1} \mathcal{O}_4 \mathcal{P})(0, 0) = -\frac{\Delta r}{96} + \frac{23}{108} R_{m \varphi \bar{\varphi} \psi \bar{\varphi}} R_{\varphi \bar{m} \psi \bar{\varphi}} + \frac{41}{54} R_{s \varphi \bar{\varphi} \psi \bar{\varphi}} R_{\varphi \bar{m} \psi \bar{\varphi}} \\
+ R_{m \varphi \bar{\varphi} \psi \bar{\varphi}} R_{\bar{k} \bar{q} \bar{\varphi} \bar{\varphi}} + \frac{1}{8} (-R_{m \varphi \bar{\varphi} \psi \bar{\varphi}} + 3 R_{\bar{k} \bar{q} \bar{\varphi} \bar{\varphi}} + \frac{1}{4} R_{\bar{k} \bar{q} \bar{\varphi} \bar{\varphi}}.
\]

Proof. By (1.7), (3.14), (4.2) and (4.3), as in (4.4), we have

\[
-\mathcal{O}_4 \mathcal{P}(z, 0) = -\left( \frac{1}{20} \left( A_{12} - \frac{4}{3} A_{22} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) b_i b_j \right) \mathcal{P}(z, 0) \\
- \frac{1}{20} \left( A_{12} - \frac{4}{3} A_{22} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) b_i b_j \mathcal{P}(z, 0) \\
= -\frac{\pi^2}{6} \left( A_{12} - \frac{4}{3} A_{22} \right) (z, \bar{z}) + \frac{\pi}{5} \left( A_{12} - \frac{4}{3} A_{22} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) \mathcal{P}(z, 0).
\]

and

\[
-\mathcal{L}^{-1} \mathcal{O}_4 \mathcal{P} = \mathcal{P}^1 \left[ 4 \left( \frac{1}{80} A_{12} - \frac{1}{360} A_{22} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) - \frac{1}{72} \text{Ric}(z, \bar{z})^2 \right] \mathcal{P}.
\]
From (3.12), (3.14), (4.2) and (4.3), and since $R^{TX}$ is $(1,1)$-form, we have

$$-(\mathcal{O}_{44} \mathcal{P})(Z,0) = 2\pi \left\{ \frac{\pi}{30} A_{1z}(\overline{z}, z) + \left( \frac{\partial}{\partial z_j} \left( \frac{1}{20} A_{1z} + \frac{2}{45} A_{2z} \right) \right) (\overline{z}, e_j) 
- \frac{\pi}{10} A_{2z}(\overline{z}, z) - z^i \frac{\partial}{\partial \overline{z}_i} \left( \frac{1}{10} A_{1z} + \frac{4}{45} A_{2z} \right) \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j} \right) \right\} \mathcal{P}(Z,0),$$

(4.31)

$$-(\mathcal{O}_{55} \mathcal{P})(Z,0) = 2\pi \left\{ \frac{2}{9} \left\langle R^{TX}(\mathcal{R}, e_k) z, R_{z_0}^{TX}(\mathcal{R}, e_k) \overline{z} \right\rangle 
- \frac{1}{9} \left\langle R^{TX}(z, \overline{z}) \mathcal{R}, e_k \right\rangle \text{Ric}(\mathcal{R}, e_k) 
+ \frac{1}{4} \left\langle R^{TX}(z, \overline{z}) \mathcal{R}, e_j \right\rangle \mathcal{R}_k^{\mathcal{R}}(\mathcal{R}, e_j) - \frac{1}{4} R_k^{\mathcal{R}}(z, \overline{z}) \right\} \mathcal{P}(Z,0).$$

Let $\psi_{ijk}$ be degree 3 polynomials in $z$ which are symmetric in $i, j$. By (1.7), (4.2), we get

$$(\psi_{ijk} z_j z_k \mathcal{P})(Z,0) = \frac{1}{8\pi^2} (\psi_{ijk} b_i b_j b_k \mathcal{P})(Z,0) = \frac{1}{8\pi^3} \left\{ b_i b_j b_k \psi_{ijk} 
+ 2b_i b_j \frac{\partial \psi_{ijk}}{\partial z_k} + 4b_i b_k \frac{\partial^2 \psi_{ijk}}{\partial z_j \partial z_k} + 8b_i \frac{\partial^2 \psi_{ijk}}{\partial z_j \partial z_j} + 8 \frac{\partial^3 \psi_{ijk}}{\partial z_i \partial z_j \partial z_k} \right\} \mathcal{P}(Z,0).$$

(4.32)

Thus by Theorem 1.1, (1.1) and (4.32), we obtain

$$\pi^2 \left( \mathcal{L}^{-1} \mathcal{P} \right) \left( \psi_{ijk} z_j z_k \mathcal{P} \right)(0,0) = \frac{1}{8\pi^2} \left\{ \frac{b_i b_j b_k}{12} \psi_{ijk} + \frac{1}{4} b_i b_j \frac{\partial \psi_{ijk}}{\partial z_k} 
+ \frac{1}{2} b_i b_k \frac{\partial^2 \psi_{ijk}}{\partial z_j \partial z_k} + 2b_i \frac{\partial^2 \psi_{ijk}}{\partial z_j \partial z_j} + b_k \frac{\partial^3 \psi_{ijk}}{\partial z_i \partial z_j \partial z_k} \right\} \mathcal{P}(0,0) = -\frac{11}{24\pi^2} \frac{\partial^3 \psi_{ijk}}{\partial z_i \partial z_j \partial z_k}. $$

(4.33)

Since $|\frac{\partial}{\partial z_j}|^2 = \frac{1}{2}$, Lemma 3.1, (3.6), (3.12) and the fact that $R^{TX}$ is a $(1,1)$-form entail

$$A_{1z}(\overline{z}, z) = \left\langle R^{TX}_{(1z)}(z, \overline{z}) \mathcal{P} \right\rangle, $n
$$A_{2z}(\overline{z}, z) = 2 \left\langle R^{TX}(z, \overline{z}) z, R^{TX}(z, \overline{z}) \overline{z} \right\rangle = -4 R_{u_1 u_2} R_{u_3 u_4} z_u \overline{z}_u z_v \overline{z}_v \mathcal{P},$$

(4.34)

By Remark 4.3 we can replace $A_{1z}(\overline{z}, z)$ by $2 R_{k_m q} z_k \overline{z}_k z_m \overline{z}_m$ in our computation. We deduce from (4.33) and (4.34) that

$$\pi^2 \left( \mathcal{L}^{-1} \mathcal{P} \right) A_{1z}(\overline{z}, z) \mathcal{P}(0,0) = -\frac{11}{6\pi^2} \left( R_{m_0 m_1} + 2 R_{m_0 m_0} \right),$$

(4.35)

$$\pi^2 \left( \mathcal{L}^{-1} \mathcal{P} \right) A_{2z}(\overline{z}, z) \mathcal{P}(0,0) = \frac{11}{3\pi^2} \left( R_{m_0 q} R_{m_0 q} + 2 R_{m_0 z} R_{m_0 z} \right).$$

Let $F_{ij}$ be homogeneous degree 2 polynomials in $z$. Then by (1.1), (4.11) and (4.17), we obtain

$$\left( \mathcal{L}^{-1} \mathcal{P} \right) F_{ij} z_j \mathcal{P}(0,0) = -\frac{1}{\pi^2} \frac{\partial^2 F_{ij}}{\partial z_i \partial z_j},$$

(4.36a)

$$\left( \mathcal{L}^{-1} \mathcal{P} \right) z_k \frac{\partial (F_{ij} z_j \mathcal{P})}{\partial \overline{z}_k}(0,0) = -\frac{3}{4\pi^3} \frac{\partial^3 F_{ij}}{\partial z_i \partial z_j \partial z_k}. $$

(4.36b)
By Remark 4.3 and (4.34), only the term \(-2R_{\overline{m}q\overline{m}z}\overline{z}_mz_k\overline{z}_l\) from \(A_{12}(\frac{\partial}{\partial z^q}, \frac{\partial}{\partial z^l})\) has a nontrivial contribution in our computation at \((0, 0)\), and from (4.37), (4.36a) and (4.36b), we get

\[
(\mathcal{P}^\perp A_{12}(\frac{\partial}{\partial z^q}, \frac{\partial}{\partial z^l})\mathcal{P})(0, 0) = \frac{2}{\pi^2}(R_{\overline{m}m\overline{r}, r} + R_{\overline{m}\overline{q}m\overline{r}, r}),
\]

\[
\frac{1}{2}(\mathcal{L}^{-1}\mathcal{P}^\perp \overline{z}_m \frac{\partial}{\partial z^q} A_{12}(\frac{\partial}{\partial z^q}, \frac{\partial}{\partial z^l})\mathcal{P})(0, 0) = \left(\mathcal{L}^{-1}\mathcal{P}^\perp A_{12}(\frac{\partial}{\partial z^q}, \frac{\partial}{\partial z^l})\mathcal{P}\right)(0, 0)
= \frac{3}{4\pi^3}(R_{\overline{m}m\overline{r}, r} + R_{\overline{m}\overline{q}m\overline{r}, r}),
\]

(4.37)

\[
\frac{1}{2}(\mathcal{L}^{-1}\mathcal{P}^\perp \overline{z}_m \frac{\partial}{\partial z^q} A_{22}(\frac{\partial}{\partial z^q}, \frac{\partial}{\partial z^l})\mathcal{P})(0, 0) = -\frac{2}{\pi^2}(R_{\overline{m}m\overline{r}}R_{\overline{q}\overline{r}m\overline{r}} + 3R_{\overline{m}m\overline{r}}R_{\overline{q}\overline{r}m\overline{r}}),
\]

\[
\frac{1}{2}(\mathcal{L}^{-1}\mathcal{P}^\perp \overline{z}_m \frac{\partial}{\partial z^q} A_{22}(\frac{\partial}{\partial z^q}, \frac{\partial}{\partial z^l})\mathcal{P})(0, 0) = \left(\mathcal{L}^{-1}\mathcal{P}^\perp A_{22}(\frac{\partial}{\partial z^q}, \frac{\partial}{\partial z^l})\mathcal{P}\right)(0, 0)
= -\frac{3}{4\pi^3}(R_{\overline{m}m\overline{r}}R_{\overline{q}\overline{r}m\overline{r}} + 3R_{\overline{m}m\overline{r}}R_{\overline{q}\overline{r}m\overline{r}}).
\]

Remark 4.3 (4.3), (4.17c) and (4.17f) yield

\[
\left(\mathcal{L}^{-1}\mathcal{P}^\perp \left\langle R^F_{\overline{z}_m, e_m}(z, \overline{z}) R, e_m \right\rangle_{z_0} \mathcal{P}\right)(0, 0)
= 2 \left(\mathcal{L}^{-1}\mathcal{P}^\perp \left(R_{\overline{m}q\overline{m}, m\overline{r}} - R_{\overline{m}q\overline{m}, m\overline{r}}\right)\overline{z}_m \overline{z}_q \overline{z}_m \mathcal{P}\right)(0, 0) = 0,
\]

(4.38)

\[
\left(\mathcal{L}^{-1}\mathcal{P}^\perp \left\langle R^F_{\overline{z}_m, e_m}(z, \overline{z}) R, e_k \right\rangle_{z_0} \text{Ric}(\mathcal{R}, e_{k}) \mathcal{P}\right)(0, 0)
= 2 \left(\mathcal{L}^{-1}\mathcal{P}^\perp \left(- \text{Ric}_{\overline{q}\overline{r}} + \text{Ric}_{\overline{q}\overline{r}} \text{Ric}_{\overline{q}\overline{r}}\right) \overline{z}_q \overline{z}_q \overline{z}_m \mathcal{P}\right)(0, 0) = 0,
\]

\[
\left(\mathcal{L}^{-1}\mathcal{P}^\perp \text{Ric}(\mathcal{R}, e_{q}) \mathcal{P}\right)(0, 0)
= 2 \left(\mathcal{L}^{-1}\mathcal{P}^\perp \left(- \text{Ric}_{\overline{q}\overline{r}} - \text{Ric}_{\overline{q}\overline{r}} \text{Ric}_{\overline{q}\overline{r}}\right) \overline{z}_q \overline{z}_r \mathcal{P}\right)(0, 0) = 0,
\]

By (3.39), (3.40), (4.3) and (4.38), we know that for \(\alpha = 1, 2\),

\[
\left(\mathcal{L}^{-1}\mathcal{P}^\perp \frac{\partial}{\partial z_q} \left(A_{1\alpha}\overline{z}_q e_{q}\right) \mathcal{P}\right)(0, 0) = 0,
\]

(4.39)

\[
\left(\mathcal{L}^{-1}\mathcal{P}^\perp \frac{\partial}{\partial z_q} \left(A_{2\alpha}\overline{z}_q e_{q}\right) \mathcal{P}\right)(0, 0) = -2\left(\mathcal{L}^{-1}\mathcal{P}^\perp A_{2\alpha}(\frac{\partial}{\partial z^q}, \frac{\partial}{\partial z^q})\mathcal{P}\right)(0, 0).
\]
By (3.5) and (4.17f), we have

\[
\left( \mathcal{L}^{-1} \mathcal{P}^\perp \left( R_{T_0} (R_0, e_k) z, R_{T_0} (R_0, e_k) \mathcal{Z} \right) \mathcal{P} \right)(0, 0) = 4 \left( \mathcal{L}^{-1} \mathcal{P}^\perp (Q_{k+1} R_{m+1} + Q_{k+1} R_{m+1}) \mathcal{Z}_{z, z} \mathcal{Z}_{\mathcal{Z}, \mathcal{Z}} \mathcal{P} \right)(0, 0)
\]

(4.40)

\[
= -\frac{3}{2\pi^3} (Q_{k+1} R_{m+1} + 3 Q_{k+1} R_{m+1}),
\]

\[
\left( \mathcal{L}^{-1} \mathcal{P}^\perp \left( R_{T_0} (z, \mathcal{Z}) R_0 (R_0, e_k) \mathcal{P} \right) \right)(0, 0) = 2\pi \left( \mathcal{L}^{-1} \mathcal{P}^\perp (Q_{k+1} R_{m+1} - Q_{k+1} R_{m+1}) \mathcal{Z}_{z, z} \mathcal{Z}_{\mathcal{Z}, \mathcal{Z}} \mathcal{P} \right)(0, 0) = \frac{3}{\pi^2} (Q_{k+1} R_{m+1} + 3 Q_{k+1} R_{m+1}).
\]

Note that by Lemma 3.1, Ric(\mathcal{R}, \mathcal{R}) = 2 Ric_{k+1} z_k \mathcal{Z}_{\mathcal{Z}, \mathcal{Z}}, and by (1.2), (1.7) and (4.2),

\[
\mathcal{L} \text{Ric}(\mathcal{R}, \mathcal{R}) \mathcal{P} = 2 b_m b_k \text{Ric}_{k+1} z_k \mathcal{Z}_{\mathcal{Z}, \mathcal{Z}} = 4 b_m \text{Ric}_{k+1} z_k \mathcal{P}.
\]

Thus by (3.14), (4.19b) and (4.41), we obtain

\[
-\left( \mathcal{L}^{-1} \mathcal{P}^\perp \mathcal{O}_{43} \mathcal{P} \right)(0, 0) = \frac{1}{18} \left( \mathcal{L}^{-1} \mathcal{P}^\perp \text{Ric}_{k+1} z_k b_m \text{Ric}_{k+1} z_k \frac{b_k}{2\pi} \mathcal{P} \right)(0, 0)
\]

(4.42)

\[
= -\frac{1}{72\pi^2} \left( \text{Ric}_{k+1} \text{Ric}_{k+1} + 3 \text{Ric}_{k+1} \text{Ric}_{k+1} \right).
\]

From (4.29)–(4.39) and (4.42), we get

\[
-\pi^2 \left( \mathcal{L}^{-1} \left( \mathcal{O}_{41} + \mathcal{O}_{42} + \mathcal{O}_{43} + \mathcal{O}_{44} \right) \mathcal{P} \right)(0, 0) = -\frac{2}{15} \cdot \frac{11}{6} \left( R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}, \mathcal{Z}} + 2 R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}, \mathcal{Z}} \right)
\]

(4.43)

\[
+ \frac{1}{15} \cdot \frac{11}{3} \left( R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}} + 2 R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}} \right)
\]

Moreover, (4.31), (4.38) and (4.40) yield

\[
-\pi^2 \left( \mathcal{L}^{-1} \mathcal{O}_{45} \mathcal{P} \right)(0, 0) = -\frac{2}{3} \left( R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}} + 3 R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}} \right)
\]

(4.44)

\[
+ \frac{3}{2} R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}} + \frac{3}{8} \left( R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}} + R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}} \right)
\]

Further, (3.15), (4.17c), (4.35), (4.38) and (4.40) imply

\[
-\pi^2 \left( \mathcal{L}^{-1} \mathcal{O}_{46} \mathcal{P} \right)(0, 0) = \frac{11}{108} \left( R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}} + 2 R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}} \right) + \frac{1}{9} \text{Ric}_{k+1} \text{Ric}_{k+1}
\]

(4.45)

\[
-\frac{1}{2} R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}} - \frac{1}{4} R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}} - \frac{1}{2} R_{\mathcal{Z}, \mathcal{Z}, \mathcal{Z}}.
\]
By (3.8), (3.17b), (4.43)–(4.45), we get (4.28). The proof of Lemma 4.4 is completed.

4.3. Evaluation of $\mathcal{F}_4(0,0)$.

**Theorem 4.5.** The following identity holds:

$$\mathcal{F}_{4, z_0}(0,0) = b_2.$$ (4.46)

**Proof.** By Lemmas 4.2, 4.4 and (2.22), we have

$$\pi^2 \mathcal{F}_{41}(0,0) = -\frac{\Delta r}{96} + \frac{7}{72} R_{\xi y u} R_{\xi y u q} - \frac{1}{9} R_{\xi y u} R_{\xi y u q} + \frac{1}{8} R_{\xi y u} R_{\xi y u q} + \frac{1}{4} R_{\xi y u} R_{\xi y u q} - \frac{1}{6} R_{\xi y u} R_{\xi y u q} + \frac{1}{8} \left( R_{\xi y u} R_{\xi y u q} - R_{\xi y u} R_{\xi y u q} - R_{\xi y u} R_{\xi y u q} + 3 R_{\xi y u} R_{\xi y u q} \right).$$ (4.47)

Remark 2.3, Lemmas 3.1, 4.1, (0.9), (2.20), (4.47) and formula $\mathcal{P}(0,0) = 1$ entail

$$J_{4, z_0}(0,0) = \mathcal{F}_{41}(0,0) + \mathcal{F}_{41}(0,0)^* + \frac{1}{4\pi^2} \left[ \sum_{m,q} R_{\xi y u q} + \sum_q R_{\xi y u q} \right]^2$$

$$- \frac{1}{36\pi^2} R_{\xi y u} - \frac{1}{4\pi^2} \left( \frac{4}{3} R_{\xi y u} + R_{\xi y u} \right) \left( \frac{4}{3} R_{\xi y u} + R_{\xi y u} \right) = b_2.$$ (4.48)

The proof of Theorem 4.5 is completed.

5. The First Coefficients of the Asymptotic Expansion

The lay-out of this section is as follows. In Section 5.1, we explain the expansion of the kernel of Berezin-Toeplitz operators and verify its compatibility with Riemann-Roch-Hirzebruch Theorem. In Section 5.2, we establish Theorem 0.1. The results from Sections 4.1, 4.2 play an important role here. In Section 5.3, we prove Theorems 0.2, 0.3 i.e., the expansion of the composition of two Berezin-Toeplitz operators.

We use the notations and assumptions from Introduction and Section 3.

5.1. Expansion of the Kernel of Berezin-Toeplitz Operators. For $U \in TX$, we have (cf. (3.2))

$$\nabla_U^{\tau} \bar{X} \bar{d}z_j = 2 \langle \nabla_U^{\tau} \bar{X} \bar{\frac{\partial}{\partial z_j}}, \frac{\partial}{\partial z_m} \rangle \bar{d}z_m, \quad \nabla_U^{\tau} \bar{X} \bar{d}z_j = 2 \langle \nabla_U^{\tau} \bar{X} \bar{\frac{\partial}{\partial z_j}}, \frac{\partial}{\partial z_m} \rangle \bar{d}z_m.$$ (5.1)

For $\sigma = \sum_{k,q} \sigma_{kq} \bar{d}z_k \wedge \bar{d}z_q \in \Omega^{1,1}(X, \text{End}(E))$, by [24] Lemma 1.4.4, (0.6) and (5.1), we get

$$\nabla^{1,0}\sigma = - \left( 2 \langle \nabla_{U}^{\tau} \frac{\partial}{\partial z_m}, \sigma_{kq} \rangle \left( \nabla_U^{\tau} \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_m} \right) \right) \frac{\partial}{\partial z_k}$$

$$\delta E^* \sigma = \left( 2 \langle \nabla_{U}^{\tau} \frac{\partial}{\partial z_m}, \sigma_{kq} \rangle \left( \nabla_U^{\tau} \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_m} \right) \right) \frac{\partial}{\partial z_k}.$$ (5.2)

We evaluate now (5.2) at the point $x_0$ (identified to $0 \in \mathbb{R}^{2n}$). By using (3.29) applied for $r = 0, 1$ associated with the vector bundles $E, T^{(1,0)} X$, we get

$$\nabla^{1,0}\sigma_{x_0} = -2 \frac{\partial \sigma_{m k}}{\partial z_j}(0) \bar{d}z_k, \quad \delta E^* \sigma_{x_0} = -2 \frac{\partial \sigma_{m k}}{\partial z_j}(0) \bar{d}z_k,$$

$$\delta E^* \nabla^{1,0}\sigma_{x_0} = 4 \frac{\partial \sigma_{m k}}{\partial z_j}(0) + 2 \left[ R^{E}_{m k}, \sigma_{m k}(0) \right].$$ (5.3)
Note that by (0.9), (3.6) and (5.3), we have at $x_0$,

\begin{equation}
\omega = \frac{-1}{2} dz_q \wedge d \bar{z}_q, \quad \text{Tr}[R^{r(0)}] = \text{Ric}_{kq} dz_k \wedge d \bar{z}_q = -\sqrt{-1} \text{Ric}_\omega,
\end{equation}

\begin{equation}
R^E = R^{kq}_{kJ} dz_k \wedge d \bar{z}_q, \quad \nabla^{1.0*} R^E = -2 R^E_{m\nu \overline{m} q} d \bar{z}_q, \quad \partial^E R^E = 2 R^E_{kq} dz_k.
\end{equation}

For $f \in \mathcal{C}_\infty(X, \text{End}(E))$, recall that $T_{f,p}(x, x')$ is the smooth kernel of the Berezin-Toeplitz operator $T_{f,p}$ defined according to (0.4). Then by (3.29), at $x_0$,

\begin{equation}
\frac{\partial^2 f_{x_0}}{\partial z_q \partial \bar{z}_t}(0) = (\nabla^E \partial f)(x_0) - \frac{1}{2} [R^E_q, f(x_0)], \quad \Delta^E f = -\frac{\partial^2 f_{x_0}}{\partial z_q \partial \bar{z}_t}(0).
\end{equation}

In view of Lemma 3.1 (0.7) and (5.5), we introduce the following coefficients:

\begin{equation}
b_{c,f} := R_{m \overline{m} q \overline{p}} \frac{\partial f_{x_0}}{\partial z_k \partial z_k}(0) - R^{kq}_{kJ} \frac{\partial^2 f_{x_0}}{\partial z_q \partial \bar{z}_t}(0)
\end{equation}

\begin{equation}
= -\frac{r}{32} \Delta^E f - \frac{1}{8} \left\langle \text{Ric}_\omega, \nabla^{1.0*} f - \frac{1}{2} [R^E, f] \right\rangle_\omega,
\end{equation}

and

\begin{align}
b_{E,f_1} &:= \frac{\partial f_{x_0}}{\partial z_u}(0) \left( \frac{1}{6} R^E_{k k; \overline{u}} - \frac{5}{12} R^E_{q \overline{u}; q} \right) + \frac{1}{4} R^E_{m \overline{m} q} \frac{\partial f_{x_0}}{\partial z_u}(0) \\
&+ \left( \frac{1}{6} R^E_{k k; \overline{u}} + \frac{5}{12} R^E_{q \overline{u}; q} \right) \frac{\partial f_{x_0}}{\partial \overline{z}_u}(0) + \frac{1}{4} \frac{\partial f_{x_0}}{\partial \overline{z}_u}(0) R^E_{w \overline{m}; m} \\
&= \frac{1}{48} \left( \nabla^{1.0} f, 2 \sqrt{-1} \partial^E R^E + 5 \nabla^{1.0*} R^E \right)_\omega - \frac{1}{16} \left( \nabla^{1.0} R^E, \nabla^{1.0} f \right)_\omega \\
&+ \frac{1}{16} \left( \partial^E f, \partial^E R^E \right)_\omega + \frac{1}{48} \left( 2 \sqrt{-1} \nabla^{1.0} R^E_R^E - 5 \partial^E R^E, \partial^E f \right)_\omega,
\end{align}

\begin{align}
b_{E,f_2} &:= \frac{1}{2} \frac{\partial^2 f_{x_0}}{\partial z_k \partial z_k}(0) R^E_{q \overline{u}} - \frac{1}{2} \frac{\partial^2 f_{x_0}}{\partial z_q \partial \overline{z}_t}(0) R^E_{q \overline{u}} + \frac{1}{2} \frac{\partial^2 f_{x_0}}{\partial z_k \partial \overline{z}_t}(0) - \frac{1}{2} R^E_{q \overline{u}} \frac{\partial^2 f_{x_0}}{\partial z_k \partial \overline{z}_t}(0) \\
&= -\frac{\sqrt{-1}}{16} \left[ R^E_{q \overline{u}} \Delta^E f + \left( \Delta^E f \right) R^E_{q \overline{u}} \right] + \frac{1}{8} \left\langle \nabla^{1.0*} f - \frac{1}{2} [R^E, f], R^E \right\rangle_\omega \\
&+ \frac{1}{8} \left( R^E, \nabla^{1.0*} f - \frac{1}{2} [R^E, f] \right)_\omega.
\end{align}

The following result implies Theorem 0.1.

**Theorem 5.1.** Let $f \in \mathcal{C}_\infty(X, \text{End}(E))$. There exist smooth sections $b_{r,f}(x) \in \text{End}(E)_x$ such that (0.12) and (0.13) hold and

\begin{equation}
\pi^2 b_{r,f} = b_{c,f}(x_0) + \frac{1}{2} \left( b_{2E} + \frac{1}{16} (R^E_R^E)^2 \right) f(x_0) + \frac{1}{2} f(x_0) \left( b_{2E} + \frac{1}{16} (R^E_R^E)^2 \right)
\end{equation}

\begin{equation}
- \frac{1}{16} R^E_{kq} f(x_0) R^E + \frac{1}{32} (\Delta^E)^2 f + b_{c,f} + b_{E,f_1} + b_{E,f_2}.
\end{equation}

Before giving the proof, we verify that Theorem 0.1 is compatible with the Riemann-Roch-Hirzebruch Theorem. Note that by (0.11), the first Chern class $c_1(L)$ of $L$ is represented by $\omega$. By the Kodaira vanishing Theorem and the Riemann-Roch-Hirzebruch
Theorem, we have for \( p \) large enough:

\[
\dim H^0(X, L^p \otimes E) = \int_X \text{Td}(T^{(1,0)} X) \, \text{ch}(E) \, e^{p \omega}
\]

\[
= \frac{\omega_n}{n!} p^n + \int_X \left( c_1(E) + \frac{\text{rk}(E)}{2} c_1(X) \right) \frac{\omega^{n-1} p^{n-1}}{(n-1)!}
\]

\[
+ \int_X \left( \text{rk}(E) \{ \text{Td}(T^{(1,0)} X) \}^{(4)} + \frac{1}{2} c_1(X) c_1(E) + \{ \text{ch}(E) \}^{(4)} \right) \frac{\omega^{n-2} p^{n-2}}{(n-2)!}
\]

\[
+ O(p^{n-3}).
\]

As usual, \( \text{ch}(\cdot), c_1(\cdot), \text{Td}(\cdot) \) are the Chern character, the first Chern class and the Todd class of the corresponding complex vector bundles, \( \{ \cdot \}^{(4)} \) is the degree 4-part of the corresponding differential forms. Note that

\[
\frac{x}{1 - e^{-x}} = 1 + \frac{x}{2} + \frac{x^2}{12} + \ldots,
\]

thus \( \{ \text{Td}(T^{(1,0)} X) \}^{(4)} = \frac{1}{12} (c_1(X)^2 + c_2(X)) \). Let \( R^{T^{(1,0)} X} \) be the curvature of the Chern connection on \( T^{(1,0)} X \) which is the restriction of the Levi-Civita connection in our case. Then by (5.4), we have the following identities at the cohomology level:

\[
\{ \text{ch}(E) \}^{(4)} = -\frac{1}{8 \pi^2} \text{Tr}[R^E]^{2}, \quad c_1(X) = \frac{1}{2 \pi} \text{Ric}_\omega,
\]

\[
\{ \text{Td}(T^{(1,0)} X) \}^{(4)} = \frac{1}{32 \pi^2} (\text{Ric}_\omega)^2 + \frac{1}{96 \pi^2} \text{Tr} \left[ (R^{T^{(1,0)} X})^2 \right].
\]

By Lemma 3.1 and (5.4), we have

\[
\frac{1}{32} \left\langle (\text{Ric}_\omega)^2, \omega^2 / 2 \right\rangle = \frac{1}{2} (R_{\alpha \bar{\nu} \nu \bar{\eta}} R_{\alpha \bar{q} \eta \bar{q}} - R_{\nu \bar{\nu} \bar{\eta} \eta} R_{\alpha \bar{k} \bar{q} \bar{k}}),
\]

\[
\frac{1}{96} \left\langle \text{Tr} \left[ (R^{T^{(1,0)} X})^2 \right], \omega^2 / 2 \right\rangle = \frac{1}{6} (R_{\alpha \bar{\nu} \nu \bar{\eta}} R_{\alpha \bar{k} \bar{q} \bar{\eta}} - R_{\nu \bar{\nu} \bar{\eta} \eta} R_{\alpha \bar{k} \bar{q} \bar{k}}),
\]

\[
\left\langle \{ \text{ch}(E) \}^{(4)}, \omega^2 / 2 \right\rangle = \frac{1}{2 \pi^2} \text{Tr} \left[ R_{\alpha \beta}^E R_{\alpha \beta}^E - R_{\alpha \beta}^E R_{\alpha \beta}^E \right],
\]

\[
\left\langle \frac{1}{2} c_1(X) c_1(E), \omega^2 / 2 \right\rangle = \frac{1}{2 \pi^2} \text{Tr} \left[ R_{\alpha \beta}^E \text{Ric}_\omega - \text{Ric}_\omega R_{\alpha \beta}^E \right].
\]

We set now \( f = 1 \) in Theorem 0.1 take the pointwise trace of the expansion (0.12) relative to \( E \) and then integrate the result over \( X \) with respect to the volume form \( \omega^n / n! \).

Taking into account (0.9) and (5.10), (5.11), we recover the expansion up to \( O(p^{n-3}) \) given in (5.9) for the Hilbert polynomial. Thus the value of \( b_2 \) obtained in Theorem 0.1 is compatible with the Riemann-Roch-Hirzebruch Theorem.

5.2. Proof of Theorem 5.1. The first part of Theorem 5.1 follows from Lemma 2.2. Moreover, by (2.10), we have for any \( r \in \mathbb{N} \),

\[
b_{r,f}(x_0) = Q_{2r,x_0}(f)(0,0).
\]

Thus by (2.12), the formula \( b_{0,f} = f \), and by (2.11) and (2.21), we get

\[
Q_{2,x_0}(f) = \mathcal{K}[1, f(x_0)J_{2,x_0}] + \mathcal{K}[J_{2,x_0}, f(x_0)] + \sum_{|\alpha| = 2} \mathcal{K}[1, \frac{\partial^2 f(x_0)}{\partial Z^\alpha} (0) \frac{Z^\alpha}{\alpha!}].
\]
Further, (1.8), (2.13) and (2.19), entail
\[ \mathcal{K} [1, f(x_0) J_{2,x_0}] \mathcal{P} = -f(x_0) \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P} \downarrow, \]
(5.14)
\[ \mathcal{K} [J_{2,x_0}, f(x_0)] \mathcal{P} = -(\mathcal{L}^{-1} \mathcal{P} \downarrow \mathcal{O}_2 \mathcal{P}) f(x_0). \]
From (1.8) and (4.13), we deduce
\[ \sum_{|\alpha|=2} \mathcal{K} \left[ 1, \frac{\partial f_{\alpha}}{\partial Z^\alpha}(0) \frac{Z^\alpha}{\alpha!} \right] \mathcal{P}(Z,0) = \left( \sum_{|\alpha|=2} \frac{\partial f_{\alpha}}{\partial Z^\alpha}(0) \frac{Z^\alpha}{\alpha!} + \frac{1}{\pi} \frac{\partial^2 f_{\alpha}}{\partial Z_i \partial Z^\alpha}(0) \right) \mathcal{P}(Z,0). \]
Lemma 3.1 (4.8), (4.9), (5.5) and (5.12) (5.15) yield the formula for \( b_{1,f} \) from (0.12).

It remains to compute \( b_{2,f} \) for a self-adjoint section \( f \in C^\infty(X, \text{End}(E)) \) in order to complete the proof of Theorem 5.1. Set
\[ \mathcal{K}_{2f} = \sum_{|\alpha|=2} \mathcal{K} \left[ 1, \frac{\partial f_{\alpha}}{\partial Z^\alpha}(0) \frac{Z^\alpha}{\alpha!} J_{2,x_0} \right]. \]
By (2.11) and (2.21), we get
\[ Q_{4, x_0}(f) = \mathcal{K} [1, f(x_0) J_{4,x_0}] + \mathcal{K} [J_{2,x_0}, f(x_0) J_{2,x_0}] + \mathcal{K} [J_{4,x_0}, f(x_0)] \]
(5.17)
\[ + \sum_{|\alpha|=2} \mathcal{K} \left[ J_{2,x_0}, \frac{\partial f_{\alpha}}{\partial Z^\alpha}(0) \frac{Z^\alpha}{\alpha!} \mathcal{K}_{2f} \right] \]
\[ + \mathcal{K} \left[ 1, \frac{\partial f_{\alpha}}{\partial Z^\alpha}(0) J_{3,x_0} \right] \]
\[ + \mathcal{K} \left[ J_{3,x_0}, \frac{\partial f_{\alpha}}{\partial Z^\alpha}(0) \frac{Z^\alpha}{\alpha!} \right] \sum_{|\alpha|=4} \mathcal{K} \left[ 1, \frac{\partial f_{\alpha}}{\partial Z^\alpha}(0) \frac{Z^\alpha}{\alpha!} \mathcal{K}_{2f} \right]. \]

Since \( \mathcal{L}_0 \) and \( \mathcal{O}_2 \) are formally self-adjoint, (2.17) and (2.18) show that \( (\mathcal{F}_{n,x_0})^* = \mathcal{F}_{n,x_0} \). Hence, in the right hand side of (5.17), the first, fourth and sixth terms are adjoints of the third, fifth and seventh terms, respectively. When we take \( f = 1 \) in (5.17), we get
\[ J_{4,x_0} = \mathcal{K} [1, J_{4,x_0}] + \mathcal{K} [J_{2,x_0}, J_{2,x_0}] + \mathcal{K} [J_{4,x_0}, 1], \]
(5.18)
which is also a direct consequence of (2.19), (2.20) and (4.1a), as by (1.8),
\[ \mathcal{K} [1, J_{4,x_0}] \mathcal{P} = \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P} \downarrow \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P} \downarrow - \mathcal{P} \mathcal{O}_4 \mathcal{L}^{-1} \mathcal{P} \downarrow \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P} \downarrow - \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P} \downarrow \mathcal{O}_2 \mathcal{P}, \]
(5.19)
\[ \mathcal{K} [J_{4,x_0}, 1] \mathcal{P} = \mathcal{L}^{-1} \mathcal{P} \downarrow \mathcal{O}_2 \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P} \downarrow - \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P} \downarrow \mathcal{O}_2 \mathcal{P}, \]
\[ \mathcal{K} [J_{2,x_0}, J_{2,x_0}] \mathcal{P} = \mathcal{L}^{-1} \mathcal{P} \downarrow \mathcal{O}_2 \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P} \downarrow + \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P} \downarrow \mathcal{O}_2 \mathcal{P}. \]

Set
\[ K_{41} := -\Delta r \frac{5}{96} + \frac{5}{72} R_{\mu\nu} R_{\mu\nu} - \frac{5}{9} R_{\mu
u
\tau\nu} R_{\mu\nu\tau\nu} + \frac{1}{8} R_{\mu\nu\tau\nu} R_{\mu\nu\tau\nu}, \]
\[ K_{42} := \frac{1}{4} R_{\mu\nu\tau\tau}^E R_{\mu\nu\tau\tau}^E - \frac{5}{6} R_{\mu\nu\tau\nu} R_{\mu\nu\tau\nu} + \frac{1}{8} (R_{\mu\nu\tau\nu}^E R_{\mu\nu\tau\nu}^E - 3 R_{\mu\nu\tau\nu}^E R_{\mu\nu\tau\nu}^E + 3 R_{\mu\nu\tau\nu}^E R_{\mu\nu\tau\nu}^E), \]
(5.20)
\[ K_{2f} := \frac{1}{4} (R_{\mu\nu\tau\nu} R_{\mu\nu\tau\nu}^E f(x_0)(R_{\mu\nu\tau\nu} + R_{\mu\nu\tau\nu}^E) + \frac{1}{36} R_{\mu\nu\tau\nu} R_{\mu\nu\tau\nu} f(x_0) + \frac{1}{3} R_{\mu\nu\tau\nu} R_{\mu\nu\tau\nu} f(x_0) \]
\[ + \frac{1}{4} \left( \frac{4}{3} R_{\mu\nu\tau\nu} + R_{\mu\nu\tau\nu}^E \right) f(x_0) \left( \frac{4}{3} R_{\mu\nu\tau\nu} + R_{\mu\nu\tau\nu}^E \right). \]

By (2.22), (4.10), (4.47) and (5.19), we have
\[ \mathcal{K} [J_{4,x_0}, 1](0,0) = \frac{1}{\pi^2} (K_{41} + K_{42}). \]
By (1.8) and (2.19), we see as in (5.19) that
\[
\mathcal{K} \left[ J_{2, z_0}, f(x_0) J_{2, z_0} \right] \mathcal{P} = \mathcal{L}^{-1} \mathcal{P} \mathcal{O}_2 f(x_0) \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P} + \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} f(x_0) \mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P}.
\]
(5.22)

Thus by (4.8), (4.9) and (5.22), as in Lemma 4.1, we get
\[
\mathcal{K} \left[ J_{2, z_0}, f(x_0) J_{2, z_0} \right] (0, 0) = \frac{1}{\pi^2} K_{2f}.
\]
(5.23)

We next turn to the computation of the sixth term in (5.17). Set
\[
\mathcal{K}_2 = \mathcal{P} \sum_{|\alpha| = 2} \frac{\partial^2 f_{z_0}}{\partial Z^\alpha} (0) \frac{Z^\alpha}{\alpha!} \left( - \mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P} - \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P} \right).
\]
(5.24)

For a degree 2 polynomial $F(Z)$ we have by Remark 4.3, (1.6), (1.7), (4.2) and (4.8)
\[
-(\mathcal{P} F(L_2 \mathcal{P})(0, 0)
\]
(5.25)

By (4.9), (4.13), (5.24) and (5.25), we get
\[
\mathcal{K}_2(0, 0) = \frac{1}{2\pi^2} \frac{\partial^2 f_{z_0}}{\partial z_k \partial \bar{z}_l} (0) (R_{m\bar{m}q\bar{q}} + R_{q\bar{q}}^E) - \frac{1}{2\pi^2} \frac{\partial^2 f_{z_0}}{\partial z_k \partial \bar{z}_l} (0) \left( \frac{4}{3} R_{k\bar{k}q\bar{q}} + R_{q\bar{q}}^E \right).
\]
(5.26)

By Lemma 3.1, (5.6), (5.7) and (5.26), we get
\[
\mathcal{K}_2(0, 0) + \mathcal{K}_2(0, 0)^* = \frac{1}{\pi^2} (b_{c2} + b_{s2}) - \frac{1}{3\pi^2} R_{k\bar{k}q\bar{q}} \frac{\partial^2 f_{z_0}}{\partial z_k \partial \bar{z}_l} (0).
\]
(5.27)

We next turn to the computation of the sixth term in (5.17). Set
\[
K_{3f} := \left( \frac{1}{6} R_{k\bar{k}m\bar{m}} - \frac{1}{3} R_{k\bar{k}m\bar{m}} \right) \frac{\partial f_{z_0}}{\partial z_k} (0) + \left( \frac{1}{6} R_{k\bar{k}m\bar{m}} - \frac{1}{3} R_{k\bar{k}m\bar{m}} \right) \frac{\partial f_{z_0}}{\partial \bar{z}_k} (0)
\]
(5.29)

Lemma 5.2. The following identity holds with $K_{3f}$ defined in (5.29):
\[
\mathcal{K} \left[ J_{3, z_0}, \frac{\partial f_{z_0}}{\partial z_k} (0) Z_k \right] (0, 0) + \mathcal{K} \left[ 1, \frac{\partial f_{z_0}}{\partial z_k} (0) Z_k J_{3, z_0} \right] (0, 0) = \frac{1}{\pi^2} K_{3f}.
\]
(5.30)
Proof of Lemma 5.2 Set

\begin{align}
(5.31a) \quad B_1(b, Z) := \frac{1}{6} R_{k \bar{k} q \bar{q}, \bar{m}} z_k z_q b_q b_{\bar{m}} + \frac{2 \pi}{3} R_{q \bar{q}, \bar{m}} z_k
- \frac{2 \pi}{15} R_{k \bar{k} q, \bar{m}} z_q z_k z_{\bar{q}} + \frac{1}{3} R_{k \bar{k} \bar{q}, \bar{m}} - \frac{1}{3} R_{k \bar{k} q, \bar{m}} - \frac{2}{3} R_{k \bar{k} q, \bar{m}} z_k b_q
- \frac{2 \pi}{15} R_{k \bar{k} q, \bar{m}} z_k z_q - \frac{2}{3} R_{k \bar{k} q, \bar{m}} - \frac{2}{3} R_{k \bar{k} q, \bar{m}} + 2 R_{k \bar{k} q, \bar{m}} - 2 R_{q \bar{q}, \bar{m}} \right] z_m,
\end{align}

\begin{align}
(5.31b) \quad B_1(Z) := \frac{2 \pi^2}{5} R_{k \bar{k} q \bar{q}, \bar{m}} z_k z_q z_q z_k + \frac{8 \pi}{15} R_{k \bar{k} q, \bar{m}} z_k z_q z_m
+ \frac{4 \pi}{3} R_{k \bar{k} q, \bar{m}} z_k z_q z_m
- \frac{2}{3} \left[ R_{k \bar{k} q, \bar{m}} + R_{k \bar{k} q, \bar{m}} - R_{q \bar{q}, \bar{m}} + 3 R_{q \bar{q}, \bar{m}} \right] z_m \\
= B_1(Z) \mathcal{P}(Z, 0).
\end{align}

Then by Lemma 3.1 (4.2), we have

\begin{align}
B_1(b, Z) \mathcal{P}(Z, 0) = \left\{ \frac{4 \pi^2}{5} R_{k \bar{k} q \bar{q}, \bar{m}} z_k z_q z_q z_k + \frac{2 \pi}{3} R_{q \bar{q}, \bar{m}} z_k z_q z_k
- \frac{2 \pi}{15} R_{k \bar{k} q, \bar{m}} z_k z_q + \frac{1}{3} R_{k \bar{k} \bar{q}, \bar{m}} - \frac{1}{3} R_{k \bar{k} q, \bar{m}} - \frac{2}{3} R_{k \bar{k} q, \bar{m}} z_k b_q
- \frac{2 \pi}{15} R_{k \bar{k} q, \bar{m}} z_k z_q - \frac{2}{3} R_{k \bar{k} q, \bar{m}} - \frac{2}{3} R_{k \bar{k} q, \bar{m}} + 2 R_{k \bar{k} q, \bar{m}} - 2 R_{q \bar{q}, \bar{m}} \right\} \mathcal{P}(Z, 0)
= B_1(Z) \mathcal{P}(Z, 0).
\end{align}

Observe that the commutation relations (1.7) imply that

\begin{align*}
R_{k \bar{k} q \bar{q}, \bar{m}} z_k z_q b_q b_{\bar{m}} = b_q b_{\bar{m}} R_{k \bar{k} q \bar{q}, \bar{m}} z_k z_q + b_q b_{\bar{m}} \left( 8 R_{q \bar{q}, \bar{m}} + 4 R_{k \bar{k} q, \bar{m}} \right) z_k
+ b_{\bar{m}} \left( 8 R_{q \bar{q}, \bar{m}} + 16 R_{k \bar{k} q, \bar{m}} \right).
\end{align*}

By (1.7), (4.2) and (5.31b), we have

\begin{align}
B_1(Z) \mathcal{P}(Z, 0) = \frac{1}{\pi} \left\{ \left( \frac{1}{20} R_{k \bar{k} q \bar{q}, \bar{m}} z_k z_q b_q + \frac{2}{15} R_{k \bar{k} q, \bar{m}} z_k z_q + \frac{1}{3} R_{k \bar{k} q, \bar{m}} z_k \right) b_q b_{\bar{m}}
- \frac{1}{3} \left[ R_{k \bar{k} q, \bar{m}} + R_{k \bar{k} q, \bar{m}} - R_{q \bar{q}, \bar{m}} + 3 R_{q \bar{q}, \bar{m}} \right] b_m \right\} \mathcal{P}(Z, 0)
= \frac{1}{\pi} \left\{ \frac{b_q b_{\bar{m}}}{20} R_{k \bar{k} q \bar{q}, \bar{m}} z_k z_q + b_q b_{\bar{m}} \left( \frac{2}{5} R_{k \bar{k} q \bar{q}, \bar{m}} + \frac{1}{3} R_{k \bar{k} q, \bar{m}} + \frac{1}{3} R_{k \bar{k} q, \bar{m}} \right) z_k
+ b_{\bar{m}} \left( \frac{1}{15} R_{k \bar{k} q, \bar{m}} + R_{k \bar{k} q, \bar{m}} - \frac{1}{3} R_{q \bar{q}, \bar{m}} + R_{q \bar{q}, \bar{m}} \right) \right\} \mathcal{P}(Z, 0).
\end{align}

Note that by Theorem 1.1 and (4.2), we have \((\mathcal{P} \perp z_u \mathcal{P})(Z, 0) = (\mathcal{P} z_u \mathcal{P})(Z, 0) = 0\). Taking into account that \(\mathcal{P}(0, 0) = 1\) and relations (1.8), (2.13) and (2.19), we get

\begin{align}
\mathcal{H} \left[ J_{\alpha_3, \alpha_4} \frac{\partial f_{\alpha_0}}{\partial Z_u} (0) Z_u \right](0, 0) = - (\mathcal{L}^{-1} \mathcal{P} \perp \mathcal{P} \frac{\partial f_{\alpha_0}}{\partial z_u} (0) z_u \mathcal{P})(0, 0)
- (\mathcal{P} \mathcal{L} \mathcal{L}^{-1} \mathcal{P} \perp \mathcal{P} \frac{\partial f_{\alpha_0}}{\partial z_u} (0) Z_u \mathcal{P})(0, 0).
\end{align}
By Remark 2.3 \((\mathcal{D}O_3L^{-1}\overline{\mathcal{Z}}u\mathcal{P})(0, 0)\) is the adjoint of \((\mathcal{D}z_uL^{-1}\mathcal{P}^\perp O_3\mathcal{P})(0, 0)\), thus we will compute only the latter. By Lemma 3.1 (1.7), (3.17a) and (4.3), we get as in (4.4),
\[
\mathcal{O}_3 = \frac{1}{6} R_{k\ell\\q\\\bar{q},z} z_k z_\ell b_k b_\ell + \frac{2\pi}{3} R_{zq\\\bar{q},z} z_k + c_3(b, b^+, Z) b^+
\]
\[
- \left( \frac{2\pi}{15} R_{zq\\\bar{q},z} z_k + \frac{1}{3} R_{zq\\\bar{q},z} z_k - R_{q\\\bar{m},q} z_k - \frac{2}{3} R_{q\\\bar{m},z} z_k b_s \right)
\]
\[
\left( \frac{2\pi}{15} R_{\ell\\q\\\bar{q},z} z_\ell + \frac{1}{3} R_{\ell\\q\\\bar{q},z} z_\ell - R_{q\\\bar{m},q} z_\ell - \frac{2}{3} R_{q\\\bar{m},z} z_\ell b_s \right)
\]
\[
- \frac{2}{3} R_{\ell\\q\\\bar{q},z} - \frac{2}{3} R_{q\\\bar{m},z} - R_{q\\\bar{m},q} m - \frac{2}{3} R_{m\\\bar{m},z},
\]
and \(c_3(b, b^+, Z)\) are polynomials in \(b, b^+, Z\), whose precise formula will not be used, and \(R_{zq\\\bar{q},z}, R_{\ell\\q\\\bar{q},z}\) are defined by replacing \(\frac{\partial}{\partial z}\) by \(Z\) in (3.6).

From Lemma 3.1 (1.7), (5.31a) and (5.35) we deduce that the only term in \(\mathcal{O}_3\) not containing \(b^+\) and having total degree in \(b, Z\) bigger than its degree in \(z\), is \(B_1(b, Z)\). Now, Theorem 1.1, Remark 4.3 (1.6), (1.7), (5.32), (5.33) and (5.35) imply
\[
-(\mathcal{D}L^{-1}\mathcal{P}^\perp \mathcal{O}_3\mathcal{P})(0, 0) = -(\mathcal{D}L^{-1}\mathcal{P}^\perp B_1(b, Z)\mathcal{P})(0, 0)
\]
\[
= -\left( \mathcal{L}^{-1} \mathcal{P}^\perp z_u B_1(b, Z) z_u \mathcal{P} \right) (0, 0)
\]
\[
(5.37)
\]
By Remark 4.3 (1.6), (1.7) and (5.35), we also have
\[
-(\mathcal{L}^{-1}\mathcal{P}^\perp z_u B_1(b, Z)\mathcal{P})(0, 0) = -(\mathcal{L}^{-1}\mathcal{P}^\perp z_u B_1(b, Z)\mathcal{P})(0, 0)
\]
\[
(5.38)
\]
Lemma 3.1 (4.17c), (4.17f), (4.33), (5.31b) and (5.32) yield
\[
-(\mathcal{L}^{-1}\mathcal{P}^\perp z_u B_1(b, Z)\mathcal{P})(0, 0) = -(\mathcal{L}^{-1}\mathcal{P}^\perp z_u B_1(b, Z)\mathcal{P})(0, 0)
\]
\[
(5.39)
\]
Moreover, by \(\mathcal{L}^{-1}\mathcal{P}^\perp (\frac{\partial}{\partial z} B_1(b, Z)\mathcal{P})(0, 0)\),
\[
(5.40)
\]
Formulas (5.37)–(5.40) entail altogether

\( (5.41) \quad -\pi^2(\mathcal{L}^{-1}P_3^{1}\mathcal{P}_{\mathcal{L}}\mathcal{P}_{\mathcal{L}})(0, 0) = \frac{1}{6} R_{\text{symm}} + \frac{1}{5} R_{\text{symm}} + \frac{1}{3} R_{\text{symm}} \)

Combining (5.34), (5.36) with (5.41), we get

\( (5.42) \quad \pi^2 \mathcal{K} \left[ J_{3, z_0}, \frac{\partial f_{z_0}}{\partial Z_u}(0)Z_u \right](0, 0) = \left( \frac{1}{6} R_{\text{symm}} + \frac{1}{5} R_{\text{symm}} + \frac{1}{3} R_{\text{symm}} \right) \frac{\partial f_{z_0}}{\partial Z_u}(0) + \left( -\frac{1}{30} R_{s_0\bar{q}_0} + \frac{1}{2} R_{s_0\bar{q}_0} + \frac{1}{6} R_{s_0\bar{q}_0} - \frac{1}{2} R_{s_0\bar{q}_0} \right) \frac{\partial f_{z_0}}{\partial Z_u}(0). \)

Since \( \mathcal{K} \left[ 1, \frac{\partial f_{z_0}}{\partial Z_u}(0)Z_uJ_{3, z_0} \right] \) is the adjoint of \( \mathcal{K} \left[ J_{3, z_0}, \frac{\partial f_{z_0}}{\partial Z_u}(0)Z_u \right] \), Lemma 3.1 yield

\( (5.43) \quad \pi^2 \mathcal{K} \left[ 1, \frac{\partial f_{z_0}}{\partial Z_u}(0)Z_uJ_{3, z_0} \right](0, 0) = \frac{\partial f_{z_0}}{\partial Z_u}(0) \left( \frac{1}{6} R_{\text{symm}} + \frac{1}{5} R_{\text{symm}} + \frac{1}{3} R_{\text{symm}} \right) \frac{\partial f_{z_0}}{\partial Z_u}(0) + \left( \frac{1}{30} R_{s_0\bar{q}_0} + \frac{1}{2} R_{s_0\bar{q}_0} + \frac{1}{6} R_{s_0\bar{q}_0} - \frac{1}{2} R_{s_0\bar{q}_0} \right) \frac{\partial f_{z_0}}{\partial Z_u}(0). \)

Finally, (5.42) and (5.43) deliver (5.30). The proof of Lemma 5.2 is completed. \( \square \)

We continue with the proof of Theorem 5.1. We'll write now the formulas in terms of connections. For \( f \in \mathcal{C}^\infty(X, \text{End}(E)) \), we obtain by (2.1) (as in (3.32)) the following formula in normal coordinates:

\( (5.44) \quad (\Delta^E f)(Z) = -g^{ij}(\nabla^E_{e_i} \nabla^E_{e_j} - \Gamma^l_{ij} \nabla^E_{e_l}) f, \quad \nabla^E f = df + [\Gamma^E(\cdot), f]. \)

By (3.31),

\( (5.45) \quad \nabla^E_{e_i} \nabla^E_{e_i} = \frac{\partial^2}{\partial Z_i^2} + \frac{R^E(\mathcal{L}_i, e_i)}{\partial Z_i} + \frac{1}{4} R^E(\mathcal{L}_i, e_i) R^E(\mathcal{L}_i, e_i) + \frac{2}{3} R^E_{i\bar{e}_i}(\mathcal{L}_i, e_i) \frac{\partial}{\partial Z_i} + \frac{1}{6} R^E_{e_i, e_i} + \frac{1}{8} \left( 2 R^E_{i\bar{e}_i}(\mathcal{L}_i, e_i) + \frac{1}{2} \left( R^E(\mathcal{L}_i, e_i) e_i, e_i \right) R^E(\mathcal{L}_i, e_i) \right) + O(|Z|^3). \)

Note that by (3.4), \( \frac{\partial^2}{\partial Z_m^2} R^E_{i\bar{e}_i}(\mathcal{L}_i, e_i) = 2 R^E_{i\bar{e}_i}(e_i, e_i) = 0. \) Thus from (3.1), (4.3), (5.45), and taking into account that \( R^E(e_i, e_i) \) is anti-symmetric in \( m, i \), and \( \text{Ric}(e_i, e_i) \) is symmetric in \( m, i \), we infer

\( (5.46) \quad \left( \frac{\partial^2}{\partial Z_m^2} \nabla^E_{e_i} \nabla^E_{e_i} f \right)(0) = \frac{\partial^4 f_{z_0}}{\partial Z_m^2 \partial Z_i^2}(0) + \frac{1}{2} \left[ R^E(e_i, e_i), \frac{\partial^2 f_{z_0}}{\partial Z_i^2}(0) \right] + \frac{2}{3} \left[ R^E(e_i, e_i), \frac{\partial f_{z_0}}{\partial Z_i}(0) \right] + \frac{1}{6} \left[ R^E(e_i, e_i), f(x_0) \right] + \frac{8}{3} \left[ R^E_{m\bar{m}} \frac{\partial f_{z_0}}{\partial Z_i}(0) - \frac{8}{3} \left[ R^E_{\bar{m}m}, \frac{\partial f_{z_0}}{\partial Z_i}(0) \right. \right. \)
By (3.3), (3.25), (3.31) and (5.44),

\[
- \frac{\partial^2}{\partial \bar{Z}_m^2} (\Gamma^l_{\bar{m}} \nabla^E_{\bar{e}_l} f)(0)
\]

\[
= - \frac{4}{3} \text{Ric}(e_m, e_l) \left( \frac{\partial^2 f_{x_0}}{\partial \bar{Z}_l \partial Z_m}(0) + \frac{1}{2} \left[ R^E_{l \bar{k} l} \right] \right)
\]

\[
- \frac{1}{6} \left( 4 \text{Ric}_{e_m}(e_m, e_l) - 2 \text{Ric}_{e_m}(e_q, e_l) + \text{Ric}_{e_m}(e_m, e_m) \right) \frac{\partial f_{x_0}}{\partial Z_l}(0)
\]

(5.47)

\[
= - \frac{32}{3} \text{Ric} \frac{\partial^2 f_{x_0}}{\partial Z_l \partial \bar{Z}_m}(0)
\]

\[- \frac{4}{3} \left( \left( \text{Ric}_{m \bar{l} m} + \text{Ric}_{m \bar{m} l} \right) \frac{\partial f_{x_0}}{\partial Z_l}(0) + \left( \text{Ric}_{m \bar{m} m} + \text{Ric}_{m \bar{m} \bar{m}} \right) \frac{\partial f_{x_0}}{\partial \bar{Z}_l}(0) \right).
\]

Formulas (3.3), (3.23), (3.31), (5.5) and (5.44)–(5.47) yield

\[
((\Delta^E)^2 f)(x_0) = -\left( \frac{\partial^2}{\partial Z_m^2} \Delta^E f \right)(0)
\]

\[
= \frac{2}{3} \text{Ric}(e_m, e_l) (\nabla^E_{\bar{e}_l} \nabla^E_{\bar{e}_m} f)(0) + \left( \frac{\partial}{\partial \bar{Z}_l} \nabla^E_{\bar{e}_m} \nabla^E_{\bar{e}_l} f \right)(0) - \frac{\partial^2}{\partial \bar{Z}_m^2} (\Gamma^l_{\bar{m}} \nabla^E_{\bar{e}_l} f)(0)
\]

(5.48)

By Lemma 5.2, the discussion after (5.17), formulas (5.17), (5.21), (5.23), (5.27) and (5.28), we have

\[
\pi^2 Q_{A, x_0}(f)(0, 0) = 2K_{41}(x_0) + K_{42}(x_0) + f(x_0)K_{42} + K_{2f} + K_{3f}
\]

(5.49)

\[+ bc_f + b_{E/2} - \frac{1}{3} R^E_{l \bar{k} l \bar{k} \bar{l}} \frac{\partial^2 f_{x_0}}{\partial Z_l \partial Z_q}(0) + \frac{1}{2} \frac{\partial^4 f_{x_0}}{\partial Z_l \partial \bar{Z}_m \partial Z_q \partial \bar{Z}_l}(0).
\]

Note that \[ [R^E_{l \bar{k} l}, R^E_{l \bar{k} l}, f(x_0)] = R^E_{l \bar{k} l} R^E_{l \bar{k} l} f(x_0) - 2 R^E_{l \bar{k} l} f(x_0) R^E_{l \bar{k} l} + f(x_0) R^E_{l \bar{k} l} R^E_{l \bar{k} l}, \] so by (3.8), (5.12), (5.48) and (5.49), we get (5.8). The proof of Theorem 5.1 is completed.

5.3. Composition of Berezin-Toeplitz operators: proofs of Theorems 0.2, 0.3

**Proof of Theorem 0.2** By Lemma 2.2, we deduce as in the proof of Lemma 2.2 that for \( Z, Z' \in T_\omega X, \, |Z|, |Z'| < \varepsilon/4, \) we have (cf. [27] (4.79), [24] (7.4.6))

(5.50) \[ p^{-n}(T_{f, p} \circ T_{g, p})(x_0)(Z, Z') \simeq \sum_{r=0}^{k} (Q_{r, x_0}(f, g) \Phi_{x_0})(\sqrt{p} Z, \sqrt{p} Z') p^{-\frac{r}{2}} + O(p^{-\frac{r+1}{2}}), \]

where

(5.51) \[ Q_{r, x_0}(f, g) = \sum_{r_1 + r_2 = r} N^{x_0} [Q_{r_1, x_0}(f), Q_{r_2, x_0}(g)] \in \text{End}(E)_{x_0}[Z, Z'], \]

is a polynomial in \( Z, Z' \) with the same parity as \( r. \)
The existence of the expansion (5.52) and the expression of \( b_{0, f, g} \) follow from (2.12), (5.50) and (5.51); we get also

\[
(5.52) \quad b_{0, f, g}(x_0) = Q_{2r, x_0}(f, g)(0, 0).
\]

By (5.51),

\[
(5.53) \quad Q_{2, x_0}(f, g) = \mathcal{K}[f(x_0), Q_{2, x_0}(g)] + \mathcal{K}[Q_{1, x_0}(f), Q_{1, x_0}(g)] + \mathcal{K}[Q_{2, x_0}(f), g(x_0)].
\]

Formulas (1.8), (5.13)–(5.14) yield

\[
(5.54) \quad \mathcal{K}[Q_{2, x_0}(f), g(x_0)](\mathcal{P}) = (Q_{2, x_0}(f) \mathcal{P}) g(x_0) = \left( \mathcal{P} \sum_{|\alpha|=0}^{\infty} \frac{\partial^\alpha f_{x_0}}{\partial \bar{z}_i^\alpha} (0) \frac{\bar{z}_i^\alpha}{\alpha!} \mathcal{P} - \mathcal{L}^{-1} \mathcal{O}_{2, f} \mathcal{P} \right) g(x_0),
\]

Thus from (4.13), (5.55), we get as in \( \mathcal{K}[Q_{2, x_0}(f), Q_{1, x_0}(g)](0, 0) = \mathcal{K}[f(x_0), Q_{2, x_0}(g)](0, 0)
\]

\[
(5.56) \quad Q_{1, x_0}(f)(Z, Z') = \mathcal{K} \left[ 1, \frac{\partial f_z}{\partial \bar{Z}_q} (0) Z_q \right](Z, Z') = \frac{\partial f_{x_0}}{\partial \bar{z}_i} (0) Z_i + \frac{\partial f_{x_0}}{\partial \bar{z}_j} (0) Z_j.
\]

Now, (5.52), (5.53), (5.55) and (5.57) imply the formula for \( b_{1, f, g} \) given in (0.16).

We prove next (0.17). It suffices to consider \( f, g \in \mathcal{C}^\infty(X, \mathbb{R}) \), which we henceforth assume. By (2.9), we have \( Q_{r, x_0}(1) = J_{r, x_0} \). Hence, taking \( f = 1 \) in (5.51), we get

\[
(5.58) \quad Q_{4, x}(g) = \mathcal{K}[1, Q_{4, x}(g)] + \mathcal{K}[J_{4, x}, Q_{2, x}(g)] + \mathcal{K}[J_{3, x}, Q_{1, x}(g)] + g(x) \mathcal{K}[J_{4, x}, 1].
\]

Taking \( g = 1 \) in (5.51) yields

\[
(5.59) \quad Q_{4, x}(f) = f(x) \mathcal{K}[1, J_{4, x}] + \mathcal{K}[Q_{1, x}(f), J_{3, x}] + \mathcal{K}[Q_{2, x}(f), J_{2, x}] + \mathcal{K}[Q_{4, x}(f), 1].
\]

By (2.12) and (5.51), we get

\[
(5.60) \quad Q_{4, x}(f, g) = \mathcal{K}[f(x_0), Q_{4, x}(g)] + \mathcal{K}[Q_{1, x}(f), Q_{3, x}(g)] + \mathcal{K}[Q_{2, x}(f), Q_{2, x}(g)] + \mathcal{K}[Q_{3, x}(f), Q_{1, x}(g)] + \mathcal{K}[Q_{4, x}(f), g(x_0)].
\]
Set
\[
\tilde{Q}_{3, z_0}(g) = \mathcal{K}\left[1, \frac{\partial g_{z_0}}{\partial Z_q}(0)Z_q J_{2, z_0}\right] + \mathcal{K}\left[J_{2, z_0}, \frac{\partial g_{z_0}}{\partial Z_q}(0)Z_q\right]
\]
\[
+ \mathcal{K}\left[1, \sum_{|\alpha| = 3} \frac{\partial^3 g_{z_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}\right],
\]
(5.61)
\[
I_{4, f, g} = \mathcal{K}\left[\mathcal{K}\left[1, \sum_{|\alpha| = 2} \frac{\partial^2 g_{z_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}\right] + \mathcal{K}\left[1, \sum_{|\alpha| = 2} \frac{\partial^2 g_{z_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}\right]\right].
\]

Note that by (2.13) and (2.19), we have
(5.62)
\[
J_{3, z_0} = \mathcal{K}\left[1, J_{3, z_0}\right] + \mathcal{K}\left[J_{3, z_0}, 1\right].
\]
and (2.11), (2.21) together with (5.62) imply
(5.63)
\[
Q_{3, z_0}(g) = g(x_0)J_{3, z_0} + \tilde{Q}_{3, z_0}(g).
\]
Since \(g \in \mathcal{C}^\infty(X, \mathbb{R})\), (5.13) entails
(5.64)
\[
Q_{2, z_0}(g) = g(x_0)J_{2, z_0} + \mathcal{K}\left[1, \sum_{|\alpha| = 2} \frac{\partial^2 g_{z_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}\right].
\]
By Remark 4.3, (5.18), (5.58)–(5.64), we get
(5.65)
\[
Q_{4, z_0}(f, g) = f(x_0)Q_{4, z_0}(g) + g(x_0)Q_{4, z_0}(f) - f(x_0)g(x_0)J_{4, z_0}
\]
\[
+ \mathcal{K}\left[Q_{1, z_0}(f), \tilde{Q}_{3, z_0}(g)\right] + \mathcal{K}\left[\tilde{Q}_{3, z_0}(f), Q_{1, z_0}(g)\right] + I_{4, f, g}.
\]
By (4.13) and (5.15), we get
(5.66)
\[
\left(I_{4, f, g}(P)(0, 0)\right)
\]
\[
= \left\{P, \sum_{|\beta| = 2} \frac{\partial^2 f_{z_0}}{\partial Z^\beta}(0)\frac{Z^\beta}{\beta!}\left[\sum_{|\alpha| = 2} \frac{\partial^2 g_{z_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!} + \frac{1}{\pi} \frac{\partial^2 g_{z_0}}{\partial z_i \partial z_j}(0)\right]\right\}(0, 0)
\]
\[
= \frac{1}{\pi^2} \left(\frac{1}{2} \frac{\partial^2 f_{z_0}}{\partial z_i \partial z_j}(0)\frac{\partial^2 g_{z_0}}{\partial z_i \partial z_j}(0) + \frac{\partial^2 f_{z_0}}{\partial z_i \partial z_j}(0)\frac{\partial^2 g_{z_0}}{\partial z_i \partial z_j}(0)\right).
\]
By Remark 2.3, \(Q_{1, z_0}(f), Q_{3, z_0}(g)\) are self-adjoint for \(f, g \in \mathcal{C}^\infty(X, \mathbb{R})\), thus by (5.63),
(5.67)
\[
\mathcal{K}\left[\tilde{Q}_{3, z_0}(f), Q_{3, z_0}(g)\right] = \mathcal{K}\left[Q_{1, z_0}(g), \tilde{Q}_{3, z_0}(f)\right]^*.
\]
Thus we only need to compute the fourth term in (5.65).
An examination of (4.4) shows that in each term of the sum giving \(O_2\), the total degree in \(b^*, z\) equals the total degree in \(b, \bar{z}\). Hence Remark 4.3, (1.8), (2.19), (4.8), (4.9), (4.13), (5.25) and (5.56) yield
(5.68)
\[
\mathcal{K}\left[Q_{1, z_0}(f), \mathcal{K}\left[1, \frac{\partial g_{z_0}}{\partial Z_q}(0)Z_q J_{2, z_0}\right]\right](0, 0)
\]
\[
= \left(\mathcal{K}\frac{\partial f_{z_0}}{\partial z_u}(0)\frac{\partial g_{z_0}}{\partial z_v}(0)z_u J_{2, z_0}\right)(0, 0)
\]
\[
= \frac{1}{2\pi^2} \delta_{uv} \delta_{z_u} \delta_{z_v} R_{z_u z_y} + R_{z_y z_u} - \frac{4}{3} R_{u k k' l} - R_{u v}.
\]
Using (2.19), and the formula \((\mathcal{P} \mathcal{z}_i \mathcal{P})(Z, 0) = (\mathcal{P} z_i \mathcal{P})(Z, 0) = 0\) (cf. (1.6)), we get
\[
(\mathcal{K} [J_{2,x_0}, \frac{\partial g_{zz}}{\partial z_q}(0) Z_q], \mathcal{P})(Z, 0) = (\mathcal{P} [J_{2,x_0}, \frac{\partial g_{zz}}{\partial z_q}(0) Z_q], \mathcal{P})(Z, 0)
\]
(5.69)
\[
= (-\mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P} \frac{\partial g_{zz}}{\partial z_q}(0) Z_q \mathcal{P} - \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \frac{\partial g_{zz}}{\partial z_q}(0) Z_q \mathcal{P})(Z, 0).
\]
By Remark 4.3 (1.7), (1.8), (5.56) and (5.69), we obtain as in (5.68)
\[
\mathcal{K} \left[ Q_{1,x_0}(f), \mathcal{K} \left[ J_{2,x_0}, \frac{\partial g_{zz}}{\partial z_q}(0) Z_q \right] \right] \mathcal{P}(0, 0)
\]
(5.70)
\[
= \left( \mathcal{P} \frac{\partial f_{zz}}{\partial z_u}(0) \frac{\partial g_{zz}}{\partial z_q}(0) \mathcal{P} \right)(0, 0)
\]
\[
= -\left( \mathcal{P} \frac{\partial f_{zz}}{\partial z_u}(0) \mathcal{O}_2 \mathcal{P} \frac{\partial g_{zz}}{\partial z_q}(0) Z_q \mathcal{P} \right)(0, 0, 0) = 0,
\]
since by (1.6) and (4.11) we have \((\mathcal{P} z_u \mathcal{b}^u z^β \mathcal{P})(Z, 0) = 0\) for \(|α| \geq 1\).

Note that for homogeneous degree 3 polynomials \(H\) in \(Z\) the analogue of formula (4.12) holds for \((H \mathcal{P})(Z, 0)\). Using this analogue together with (1.6) and (4.11) we obtain
\[
(\mathcal{P} H \mathcal{P})(Z, 0) = \left( \sum_{|α| = 3} \frac{\partial H}{\partial Z^α} \frac{Z^α}{|α|} \right) \mathcal{P}(Z, 0).
\]
Finally, (1.8), (4.13), (5.56), (5.71) and the equality \(\mathcal{P}(0, 0) = 1\) imply
\[
\mathcal{K} \left[ Q_{1,x_0}(f), \mathcal{K} \left[ 1, \sum_{|α|=3} \frac{\partial^3 g_{zz}}{\partial Z^α}(0) \frac{Z^α}{|α|} \right] \right](0, 0) = \frac{1}{π^2} \frac{\partial f_{zz}}{\partial z_u}(0) \frac{\partial^3 g_{zz}}{\partial z_u \partial z_i \partial z_i}(0).
\]
By Lemma 3.1, (3.23), (3.25) and (5.44) for \(E = \mathbb{C}\), we get
\[
\frac{∂}{∂ z_q}(\Delta g)(0) = -4 \frac{\partial^3 g_{zz}}{\partial z_q \partial z_i \partial z_i}(0) + \frac{4}{3} \text{Ric}_q \frac{\partial g_{zz}}{\partial z_l}(0).
\]
Lemma 3.1, (5.61) and (5.68)–(5.73) entail
\[
π^2 \mathcal{K} [Q_{1,x_0}(f), Q_{3,x_0}(g)](0, 0)
\]
(5.74)
\[
= \frac{∂ f_{zz}}{∂ z_u}(0) \frac{∂^3 g_{zz}}{∂ z_q \partial z_i \partial z_i}(0) + \frac{1}{2} \frac{∂ f_{zz}}{∂ z_u}(0) \frac{∂ g_{zz}}{∂ z_u}(0) \left[ δ_{uu}(R_{qşq} + R_{qś}) - \frac{4}{3} R_{uqş} - R_{qş} \right]
\]
\[
= -\frac{1}{8} (\mathcal{O} f, ∆g) + \frac{1}{4} (\mathcal{O} f, g)(R_{qşq} + R_{qś}) - \frac{1}{8} (\mathcal{O} f, ∆g, R^E)ω.
\]
From (5.67) and (5.74), we have
\[
π^2 \mathcal{K} [Q_{3,x_0}(f), Q_{1,x_0}(g)](0, 0)
\]
(5.75)
\[
= -\frac{1}{8} (∆f, g) + \frac{1}{4} (∆f, g)(R_{qşq} + R_{qś}) - \frac{1}{8} (∆f, g, R^E)ω.
\]
By (0.9), (5.3), (5.65), (5.66), (5.74) and (5.75), we get (0.17). The proof of Theorem 0.2 is completed.

**Proof of Theorem 0.3** The existence of the expansion (0.18) and formula \(C_0(f, g) = f g\) were established in [27, Th. 1.1] (cf. also [24, Th. 7.4.1]) in general symplectic settings.
By (0.18), (2.10), (5.12), (5.50) and (5.52), we obtain (cf. also [24, (7.4.9)]),
\begin{align}
C_1(f, g) = (Q_{2, a}(f, g) - Q_{2, a}(fg))(0, 0) = b_{1, f, g} - b_{1, g}.
\end{align}

Hence (0.5), (0.13), (0.16) and (5.76) yield the formula for $C_1(f, g)$ given in (0.20). Moreover, (0.18), (5.12) and (5.52) imply the formula for $C_2(f, g)$ from (0.20).

We will prove (0.21) now. Let $\{e_i\}$ be an orthonormal frame of $(TX, g^{TX})$, and $\{w_i\}$ be an orthonormal frame of $T^{(1, 0)}X$. Let $\Box = \delta \overline{\delta} + \overline{\delta} \delta$ be the Kodaira Laplacian on $\Lambda(T^{*}X) \otimes \mathbb{R}$, and let $\Delta$ be the Bochner Laplacian on $\Lambda(T^{*}X) \otimes \mathbb{C}$ associated with the connection $\nabla^{A(T^{*}X)}$ on $\Lambda(T^{*}X)$ induced by $\nabla^{TX}$ (cf. (0.5)). Let $R^{A(T^{*}(1, 0)X)}$ be the curvature of the holomorphic Hermitian connection on $\Lambda(T^{*}(1, 0)X)$. By Lichnerowicz formula [24, Remark 1.4.8], and (5.4), we have
\begin{align}
R^{A(T^{*}(1, 0)X)} &= -\langle R^{TX} w_i, \overline{w}_k \rangle w^i \wedge i_{w_k}, \\
\Box &= \Delta - R^{A(T^{*}(1, 0)X)}(w_i, \overline{w}_j) + (2 \nabla^{A(T^{*}(1, 0)X)}(w_i, \overline{w}_j) + \text{Ric})(w_i, \overline{w}_j) w^k \wedge i_{\overline{w}_k}.
\end{align}

Since $(X, \omega, J)$ is Kähler, $\Box$ commutes with the operators $\partial, \overline{\partial}, d$ (cf. [24, Cor. 1.4.13]), and (5.77) shows that $\Box f = \Delta f$ for any $f \in \mathcal{C}^{\infty}(X)$. From Lemma 3.1, (5.4) and (5.77), we have for any $f \in \mathcal{C}^{\infty}(X)$:
\begin{align}
\Delta \partial f &= \partial \Delta f - \text{Ric}(\cdot, \overline{w}_k) w_k(f), \\
\Delta \overline{\partial} f &= \overline{\partial} \Delta f - \text{Ric}(\cdot, w_k) \overline{w}_k(f), \\
\Delta df &= d \Delta f - \text{Ric}(e_i, e_j) e_j(f).
\end{align}

Thus (0.5), (5.78) yield for any $f, g \in \mathcal{C}^{\infty}(X)$:
\begin{align}
\Delta(fg) &= g \Delta f + f \Delta g - 2\langle df, dg \rangle, \\
(5.79) \quad \Delta(\partial f, \overline{\partial} g) &= \langle \delta \Delta f, \overline{\delta} g \rangle + \langle \partial f, \overline{\delta} \Delta g \rangle - 2\langle \nabla^{TX} \partial f, \nabla^{TX} \overline{\delta} g \rangle \\
&= \langle \delta \Delta f, \overline{\delta} g \rangle + \langle \partial f, \overline{\delta} \Delta g \rangle - 2\langle \nabla^{TX} \partial f, \nabla^{TX} \overline{\delta} g \rangle - 2 \text{Ric}(w_m, \overline{w}_q) \overline{w}_m(g) w_q(f).
\end{align}

Using (0.5), (5.78) and (5.79), we infer
\begin{align}
\Delta^2(fg) &= f \Delta^2 g + g \Delta^2 f + 2(\Delta f) \Delta g - 4\langle df, dg \rangle - 4\langle df, d\Delta g \rangle \\
&+ 4\langle \nabla^{TX} df, \nabla^{TX} dg \rangle + 4 \text{Ric}(e_i, e_j) e_i(g) e_j(f).
\end{align}

We examine now closely the expression of $\pi^2 C_2(f, g)$ given by (0.20). Using (0.14), (0.17), (0.20), we see that the term of differential order 0 in $f, g$ from the expression of $\pi^2 C_2(f, g)$ is zero, and the term of total differential order 2 in $f, g$, disregarding the term involving $R^{E}$, in $\pi^2 C_2(f, g)$ is
\begin{align}
C_{22} = \frac{\sqrt{-1}}{8} \langle \text{Ric}_\omega, -f \overline{\delta} g - g \delta \overline{f} + \delta \overline{\delta} (fg) \rangle.
\end{align}

The term of total differential order 4 in $f, g$ in the expression of $\pi^2 C_2(f, g)$ is
\begin{align}
C_{24} &= \frac{1}{32} f \Delta^2 g + \frac{1}{32} g \Delta^2 f - \frac{1}{8} \langle \delta f, \delta \Delta g \rangle - \frac{1}{8} \langle \overline{\delta} f, \overline{\delta} \Delta g \rangle + \frac{1}{16} \langle \Delta f \Delta g \rangle \\
&+ \frac{1}{8} \langle D^{0.1} \delta f, D^{1.0} \overline{\delta} g \rangle - \frac{1}{32} \delta^2(fg) - \frac{1}{8} \langle \delta f, \overline{\delta} g \rangle.
\end{align}

By (5.79), (5.80), (5.82) and by the formula $\langle D^{1.0} \delta f, D^{1.0} \overline{\delta} g \rangle = \langle D^{0.1} \delta f, D^{1.0} \overline{\delta} g \rangle$, we get
\begin{align}
C_{24} &= \frac{1}{8} \langle D^{1.0} \delta f, D^{0.1} \overline{\delta} g \rangle + \frac{\sqrt{-1}}{8} \langle \text{Ric}_\omega, \delta f \wedge \overline{\delta} g - \partial g \wedge \overline{\delta} f \rangle.
\end{align}
Finally, by inspecting (6.14), (6.17), (6.20), we see that the term involving $R^\omega$ in the expression of $\pi^2 C_2(f, g)$ is

$$
\frac{\sqrt{-1}}{8} \left( -f \Delta g - g f - f \Delta(g f) \right) R^\omega_A + \frac{1}{4} \left( g \delta \overline{f} + f \delta g - \nabla(f g) \right) R^\omega_A \omega

+ \frac{\sqrt{-1}}{4} \left( \overline{f} \nabla f, \omega \right) R^A - \frac{1}{4} \left( \overline{f} \nabla g, \omega \right) R^A + \frac{\sqrt{-1}}{4} \left( \delta f, \overline{g} \right) R^A_B.

(5.84)

Combining (5.81), (5.83) and (5.84), we get (6.21). The proof of Theorem 0.3 is completed.

6. DONALDSON’S $Q$-OPERATOR

In this section we study the asymptotics of the sequence of operators introduced by Donaldson [10]. We suppose henceforth that $E = \mathbb{C}$. Set $\text{Vol}(X, dv_X) := \int_X dv_X$. Following [10] §4, set

$$(6.1) \quad K_p(x, x') := |P_p(x, x')|^2_{h_{\varphi^p} \otimes h_{\varphi^p}}, \quad R_p := (\text{dim} H^0(X, L^p))/\text{Vol}(X, dv_X).$$

Let $K_p, QK_p$ be the integral operators associated to $K_p$, defined for $f \in C^\infty(X)$ by

$$
(6.2) \quad (K_p(f))(x) := \int_X K_p(x, y)f(y)dv_X(y), \quad QK_p(f) = \frac{1}{R_p}K_pf.
$$

Recall that, just as the Bergman kernel appears when comparing a Kähler metric $\omega$ to its algebraic approximations $\omega_p$ (i.e. pull-backs of the Fubini-Study metrics by the Kodaira embeddings), the operators $QK_p$ appear when one relates infinitesimal deformations of the metric $\omega$ to the corresponding deformations of the approximations $\omega_p$. The asymptotics of the operator $K_p$ were obtained in [21] Th. 26 and used in [16]. The following result refines [21] Th. 26 and is applied in the recent paper [17].

**Theorem 6.1.** For every $m \in \mathbb{N}$, there exists $C > 0$ such that for any $f \in C^\infty(X), p \in \mathbb{N}^*$,

$$
(6.3) \quad \left| \frac{1}{p^n} K_pf - f + \frac{1}{8\pi p}(-rf + 2\Delta f) \right|_{C^m(X)} \leq C p^{-3/2} |f|_{C^{m+3}(X)} \quad \text{or} \quad C p^{-2} |f|_{C^{m+2}(X)}.
$$

**Proof.** By (2.9) with $Z = 0$, (2.13), (2.19) and (2.21), we get

$$
(6.4) \quad \left| \left( \frac{1}{p^{2n}} K_p, \nu \right)_{C^m(X)} - \left( 1 + \sum_{r=2}^{k} p^{-r/2} J^r(p, \sqrt{p}Z') \right) e^{-\pi p |Z'|^2} \right|_{C^m(X)} \leq C p^{-(k+1)/2} \left( 1 + |\sqrt{p}Z'|^N \right) \exp(-C_0 \sqrt{p} |Z'|) + \mathcal{O}(p^{-\infty}),
$$

with

$$
(6.5) \quad J^2(0, Z') = (J_2 + J_2)(0, Z').
$$

Note that in the present context [21] Th.26 should be modified as follows: $\lim_{p \to \infty} QK_p f = \frac{\text{Vol}(X, dv_X)}{\text{Vol}(X, dv_X)} \eta f$ in $C^m(X)$, or

$$
\left| QK_p f - \frac{\text{Vol}(X, \nu)}{\text{Vol}(X, dv_X)} \eta f \right|_{C^m(X)} \leq C p^{-1/2} |f|_{C^{m+1}(X)} \quad \text{or} \quad C p^{-1} |f|_{C^{m+2}(X)},
$$

since the right hand side of the second equation of [21] (33) and [21] (34)) should read as convergence in $C^m(X)$ without the speed, or $C p^{-1/2} |f|_{C^{m+1}(X)}$ or $C p^{-1} |f|_{C^{m+2}(X)}$. 
Now we have the analogue of [21 (32)],

\[ p^{-n} K_p f - p^n \left| \int_{|Z'| \leq \varepsilon} \left( 1 + \sum_{r=2}^{k} p^{-r/2} J_r(0, \sqrt{p} Z') \right) e^{-\pi p |Z'|^2} f_{x_0}(Z') d\nu_{T_{x_0}} x(Z') \right|_{\varepsilon^m(X)} \leq C p^{-(k+1)/2} |f|_{\varepsilon^m(X)}. \]

But as in the proof of [2, Th.2.29 (2)], we get

\[ p^n \left| \int_{|Z'| \leq \varepsilon} J'_r(0, \sqrt{p} Z') e^{-\pi p |Z'|^2} f_{x_0}(Z') d\nu_{T_{x_0}} x(Z') \right|_{\varepsilon^m(X)} \leq C |f|_{\varepsilon^m(X)}. \]

Moreover

\[ p^n \left| \int_{|Z'| \leq \varepsilon} e^{-\pi p |Z'|^2} f_{x_0}(Z') d\nu_{T_{x_0}} x(Z') - f(x_0) + \frac{1}{4\pi p} (\Delta f)(x_0) \right|_{\varepsilon^m(X)} \leq C p^{-3/2} |f|_{\varepsilon^{m+1}(X)} \text{ or } C p^{-2} |f|_{\varepsilon^{m+2}(X)}. \]

Finally, by (4.9) (cf. also [24 (4.1.110)]), we have

\[ \int_{Z' \in \mathbb{C}^n} J_2(0, Z') |\mathcal{P}|^2(0, Z') dZ' = \int_{Z' \in \mathbb{C}^n} \mathcal{P}(0, Z') J_2(Z', 0) \mathcal{P}(Z', 0) dZ' = (\mathcal{P} J_2 \mathcal{P})(0, 0) = -(\mathcal{P} O_2 Z^{-1} \mathcal{P}^{-1})(0, 0) = \frac{1}{16\pi} r. \]

Thus

\[ p^n \left| \int_{|Z'| \leq \varepsilon} J'_r(0, \sqrt{p} Z') e^{-\pi p |Z'|^2} f_{x_0}(Z') d\nu_{T_{x_0}} x(Z') - \left( \frac{r}{8\pi} f \right)(x_0) \right|_{\varepsilon^m(X)} \leq C p^{-1/2} |f|_{\varepsilon^{m+1}(X)} \text{ or } C p^{-1} |f|_{\varepsilon^{m+2}(X)}. \]

The proof of Theorem 6.1 is completed. \(\square\)

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