Angular Gelfand–Tzetlin Coordinates for the Supergroup $UOSp(k_1/2k_2)$

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(Dated: March 28, 2022)

We construct Gelfand–Tzetlin coordinates for the unitary orthosymplectic supergroup $UOSp(k_1/2k_2)$. This extends a previous construction for the unitary supergroup $U(k_1/k_2)$. We focus on the angular Gelfand–Tzetlin coordinates, i.e. our coordinates stay in the space of the supergroup. We also present a generalized Gelfand pattern for the supergroup $UOSp(k_1/2k_2)$ and discuss various implications for representation theory.

I. INTRODUCTION

If the symmetries of a physical problem are simple enough, proper coordinates are easy to find. However, already the Schrödinger equation for a particle in a potential with spherical symmetry leads to non–trivial group theory, such as parametrization of the Lie group $SO(3)$ with Euler angles, spherical harmonics, Wigner representation functions and, in the case of the Hydrogen atom, additional symmetries and the Lie group $SO(4)$. The coordinates mostly used distinguish, for a good physics reason, certain directions and thus do not treat all coordinates on an equal footing. Group theoretically, such parametrizations are called non–canonical. The Euler angles, for example, describe three subsequent rotations, first, about the footing. Group theoretically, such parametrizations are called non–canonical. The Euler angles, for example, describe three subsequent rotations, first, about the $z$–axis, second, about the new $y$–axis, and, third, about the new $z$–axis.

Nevertheless, there are many problems, particularly in statistical mechanics, in many–body physics and in matrix models, where one does not want to distinguish certain directions. Rather, all variables parametrizing the group should be treated on an equal footing. Gelfand–Tzetlin coordinates are such a coordinate system. Their construction is based on a group chain or coset decomposition. Thus Gelfand–Tzetlin coordinates have a clear recursive structure. From a physics point of view, it is important that matrix elements, measures and other quantities reflect this clear recursive structure and can be given very explicitly. The generality of this chain construction makes Gelfand–Tzetlin coordinates powerful tools in applications, see for example Refs. [4,5,6], but also for conceptual studies, see for example Refs. [7,8]. A particularly intriguing aspect is the intimate and direct connection between Gelfand–Tzetlin coordinates on the group manifold and representations of this group. Their rich features and their relevance for different types of studies ranging from physics applications to pure mathematics render Gelfand–Tzetlin coordinates important objects in their own right.

In Ref. [9], Gelfand–Tzetlin coordinates were constructed for the unitary supergroup $U(k_1/k_2)$. In the present contribution, we further extend this and construct Gelfand–Tzetlin coordinates for the unitary orthosymplectic supergroup $UOSp(k_1/2k_2)$. As this group contains the symplectic group $USp(2k_2)$ and the orthogonal group $SO(k_1)$ as subgroups, our construction also includes coordinate systems for these two groups in ordinary space. The construction for the orthogonal group was implicitly also done in Ref. [10].

For the sake of clarity, an important remark is in order: We distinguish between angular and radial Gelfand–Tzetlin coordinates. In the present work, we construct angular ones. By that we mean, that they never leave the space of the group and its algebra. In previous contributions, we constructed radial Gelfand–Tzetlin coordinates to study certain types of group integrals. These radial Gelfand–Tzetlin coordinates are capable of mapping the integral over a group onto integrals over the radial part of a different symmetric space. Hence, in this sense, these coordinates leave the space of the group and its algebra. Here, we always stay with the angular Gelfand–Tzetlin coordinates.

The appreciated explicit formulæ resulting from the Gelfand–Tzetlin construction imply the unavoidable disadvantage that a reader, not familiar with the subject, can quickly lose his orientation. Therefore, we decided to skip several detailed calculations if, in our opinion, it would not be too cumbersome for the reader to recover the missing steps by properly adjusting the corresponding ones in Ref. [9]. In any case, we recommend that an interested but unexperienced reader studies first Refs. [4,5,6] and then Ref. [9] before reading the present contribution.

The paper is organized as follows: In Sec. II, we construct the angular Gelfand–Tzetlin coordinates. We state the generalized Gelfand pattern in Sec. III and discuss some issues related to representation theory. Summary and conclusions are given in Sec. IV.

II. CONSTRUCTION OF THE COORDINATE SYSTEM

In Sec. II A, we collect some properties of the supergroup $UOSp(k_1/2k_2)$ needed in the sequel. We set up the proper Gelfand–Tzetlin equations and their recursion to all levels in Secs. II B and II C, respectively. We solve these equations...
in Sec. II.D. We summarize the construction of the Gelfand–Tzetlin coordinates for the ordinary unitary symplectic group in Sec. II.E. The invariant measure of the supergroup is worked out in Sec. II.F. The matrix elements of the supergroup are obtained in Sec. II.G.

A. The supergroup UOSp(\(k_1/2k_2\))

The classification of superalgebras and supergroups can be found in Refs. 13, 14, 15. Here, we restrict ourselves to summarizing features of the supergroups OSp(\(k_1/2k_2\)) and UOSp(\(k_1/2k_2\)). We will refer to \(k_1\) and \(2k_2\) as to the bosonic and fermionic dimensions, respectively. We introduce the notation \((k_1/2k_2)\) for the resulting superdimension. The elements of OSp(\(k_1/2k_2\)) are those elements \(u\) of the general linear supergroup GL(\(k_1/2k_2\)) which satisfy \(u^\dagger Lu = L\). The metric \(L\) is given by

\[
L = \text{diag}(1_{k_1}, 1_{k_2} \otimes \tau^{(1)}) ,
\]

where \(1_{k_1}\) and \(1_{k_2}\) are the \(k_1 \times k_1\) and the \(k_2 \times k_2\) unit matrices and where \(\tau^{(1)}\) is one of the Pauli matrices,

\[
\tau^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tau^{(1)} = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix},
\]

\[
\tau^{(2)} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad \tau^{(3)} = \begin{bmatrix} +i & 0 \\ 0 & -i \end{bmatrix} .
\]

The supergroup UOSp(\(k_1/2k_2\)) is the compact subgroup of OSp(\(k_1/2k_2\)). By construction, the direct product SO(\(k_1\)) \(\otimes\) USp(\(k_2\)) of the ordinary orthogonal and the ordinary unitary symplectic group is a subgroup of UOSp(\(k_1/2k_2\)). As is well known, the ordinary orthogonal group SO(\(k_1\)) has slightly different features for even and odd dimension \(k_1\). Thus, these differences are also present in the supergroup UOSp(\(k_1/2k_2\)).

The group elements act on a graded space, which we denote by \(\mathcal{L} = 0\mathcal{L} \oplus 1\mathcal{L}\). It decomposes into a sum of an even \(0\mathcal{L}\) and an odd \(1\mathcal{L}\) subspace according to its transformation properties under the parity automorphism \(\hat{\sigma}\). We define a basis \(e_j = e_{j1}, j = 1, \ldots, k_1\) for \(0\mathcal{L}\), and \(e_{k_1+j} = e_{j2}, j = 1, \ldots, 2k_2\) for \(1\mathcal{L}\) respectively.

The supergroup UOSp(\(k_1/2k_2\)) can be obtained by the exponential mapping of the superalgebra uosp(\(k_1/2k_2\)), such that \(\sigma \in \text{uosp}(k_1/2k_2)\) leads to \(u = \exp(\sigma) \in \text{UOSp}(k_1/2k_2)\). The construction of the angular Gelfand–Tzetlin coordinates uses as the starting point the Cartan subalgebra uosp(\(0\))(\(k_1/2k_2\)) of uosp(\(k_1/2k_2\)). For even bosonic dimension \(2k_1\), the elements of uosp(\(0\))(\(2k_1/2k_2\)) are the matrices

\[
s = \text{diag}(is_{11}\tau^{(1)}, \ldots, is_{k_11}\tau^{(1)}, s_{12}\tau^{(3)}, \ldots, s_{k_21}\tau^{(3)}) ,
\]

while for odd bosonic dimension \(2k_1 + 1\), the uosp(\(0\))(\(2k_1 + 1/2k_2\)) consists of the matrices

\[
s = \text{diag}(is_{11}\tau^{(1)}, \ldots, is_{k_11}\tau^{(1)}, 0, s_{12}\tau^{(3)}, \ldots, s_{k_21}\tau^{(3)}) .
\]

Naturally, uosp(\(0\))(\(k_1/2k_2\)) is the direct sum of the Cartan subalgebras of so(\(k_1\)) and usp(\(2k_2\)).

B. Derivation of the angular Gelfand–Tzetlin equations

Gelfand–Tzetlin coordinates are based on a group chain or, equivalently, on a coset decomposition. The coset decomposition needed for the supergroup UOSp(\(k_1/2k_2\)) is

\[
\text{UOSp}(k_1/2k_2) = \frac{\text{UOSp}(k_1/2k_2)}{\text{USp}(k_1-1/2k_2)} \otimes \frac{\text{USp}(k_1-1/2k_2)}{\text{USp}(k_1-2/2k_2)} \otimes \cdots \otimes \frac{\text{USp}(1/2k_2)}{\text{SU}(2)} \otimes \text{SU}(2) .
\]

(2.5)

Every coset space describes a unit sphere. The first coset UOSP(\(k_1/2k_2\))/USp(\(k_1-1/2k_2\)) is a sphere in a superspace with dimension \((k_1-1)/2k_2\). The dimension of the space in which the sphere lives is lowered by one in every step. The sphere UOSP(\(1/2k_2\))/USp(\(2k_2\)) is the last one living in a superspace, the following spheres in the second line of Eq. (2.5) are spheres in ordinary spaces. Coordinate systems will be constructed on all these spheres under the non–trivial requirement that the orthogonality, more precisely the equation \(u^\dagger Lu = L\), is always respected. Thus, once the coordinate system on one sphere has been obtained, the orthogonal complement to every fixed vector on this
sphere has to be constructed, the next sphere lives in this smaller space. Hence, loosely speaking, the spheres in the coset decomposition are orthogonal to each other. The construction to follow is an extension of the one in Ref. [5] for the unitary supergroup. For simplicity, we consider the case of even \(k_1\) first. The differences occurring for odd \(k_1\) will be dealt with in Sec. [11].

To project onto a smaller subspace, we write \(u \in \text{UOSp}(k_1/2k_2)\) as \(u = [u_1 u_2 \cdots u_{k_1+2k_2}]\) where the columns \(u_i\) are normalized supervectors. We denote by \(u_{ji}\) their entries in the basis \(e_{j_1}, j_1 = 1, \ldots k_1\) and \(e_{j_2}, j_2 = 1, \ldots 2k_2\). The orthogonality condition requires the vectors \(u_i, i \leq k_1\) to be real

\[
u_{ji} = u_{ji}^*, \quad \text{for } 1 \leq j \leq k_1 \quad \text{and} \quad u_{(k_1+2j)i} = u_{(k_1+2j-1)i}^*, \quad \text{for } 1 \leq j \leq k_2. \quad (2.6)
\]

We consider the first vector, it is parametrized by \(k_1\) real commuting variables \(u_{j1}\) and \(2k_2\) complex anticommuting variables, for the latter we write

\[
u_{(k_1+2j)i} = \alpha_i^*, \quad 1 \leq j \leq k_2. \quad (2.7)
\]

We also define \(|\alpha_j|^2 = \alpha_j^* \alpha_j\). The supervector \(u_1\) describes the coset space \(\text{UOSp}(k_1/2k_2)/\text{UOSp}(k_1-1/2k_2)\) which is similar to ordinary spaces – isomorphic to the surface of the \((k_1/2k_2)\) dimensional sphere \(S^{(k_1-1)/2k_2}\). We go from Cartesian coordinates to a new set of coordinates for \(u_1\) by projecting a fixed element \(s\) of the Cartan subalgebra on a space of superdimension \((k_1-1)/2k_2)\) to \(u_1\),

\[
v^{(1)} = (1_{k_1+2k_2} - u_1 u_1^t)s(1_{k_1+2k_2} - u_1 u_1^t). \quad (2.8)
\]

The eigenvalues and eigenvectors of this projected matrix are obtained by solving the supersymmetric Gelfand–Tzetlin equation

\[
v_p^{(1)} = (1_{k_1+2k_2} - u_1 u_1^t)s(1_{k_1+2k_2} - u_1 u_1^t)v_p^{(1)} = (1_{k_1+2k_2} - u_1 u_1^t)s e_p^{(1)}, \quad (2.9)
\]

which extends the equation in Ref. [5] for the unitary supergroup to \(\text{UOSp}(k_1/2k_2)\). It is convenient to rotate the basis in such a way that \(s\) becomes diagonal before solving Eq. (2.9), we introduce the primed basis

\[
e^{(1)}_i = \frac{1}{\sqrt{2}} (e^{(2i-1)}_1 + i e^{(2i)}_1),
\]

\[
e^{(2i)}_1 = \frac{1}{\sqrt{2}} (i e^{(2i-1)}_1 + e^{(2i)}_1), \quad i = 1 \ldots k_1/2,
\]

\[
e^{i}_{p2} = e^{i}_{p2}, \quad i = k_1 + 1, \ldots, k_1 + 2k_2. \quad (2.10)
\]

The rotation only affects the bosonic degrees of freedom, not the fermionic ones. Due to this rotation, the bosonic entries of \(u_1'\) are now complex variables which we write in the form

\[
u^{(i)}_{p2j1} = i u^{(i)}_{p2j-1} = \frac{i}{\sqrt{2}} |v^{(1)}_j| \exp \left(-i \tilde{\theta}^{(1)}_j\right), \quad j = 1, \ldots, k_1/2. \quad (2.11)
\]

The fermionic entries are, also in the primed basis, given by Eq. (2.7). To calculate the eigenvalues, we need the characteristic function of the eigenvalue equation (2.9),

\[
z \left(s^{(1)}_p\right) = \det g \left(1_{k_1+2k_2} - u_1 u_1^t\right) s - s^{(1)}_p = -s^{(1)}_p \det g \left(s - s^{(1)}_p\right) u_1 t^*_{1_{k_1+2k_2} - s^{(1)}_p} u_1. \quad (2.12)
\]

Importantly, the function \(z(s^{(1)}_p)\) behaves differently for the \(k_1\) bosonic eigenvalues, i.e. for those in the boson–boson block \(s^{(1)}_p = s^{(1)}_{p1}, \quad p = 1, \ldots, k_1\), and for the \(2k_2\) fermionic eigenvalues, i.e. for those in the fermion–fermion block \(s^{(1)}_{k_1+p} = i s^{(1)}_{p2}, \quad p = 1, \ldots, 2k_2\). The equation above has therefore to be discussed in the limits

\[
z \left(s^{(1)}_p\right) \longrightarrow \begin{cases} 0 & \text{for } p = 1, \ldots, k_1 \\ \infty & \text{for } p = k_1 + 1, \ldots, k_1 + 2k_2. \end{cases} \quad (2.13)
\]

Together with the normalization condition \(u_1^t u_1 = 1\) we find the following set of equations,

\[
1 = \sum_{p=1}^{k_1/2} |v^{(1)}_p|^2 + \sum_{p=1}^{k_2} |e^{(1)}_p|^2, \quad (2.14)
\]
Thus, we can write down the rotated \( u \) on the subspace orthogonal to \( \lambda \) and belongs to the Cartan subalgebra of \( \text{UOSP}((k_1-1)/2k_2) \). This is crucial for the recursion. The system (2.14) to (2.16) is overdetermined, out of the \( k_1 + 2k_2 + 1 \) equations in (2.14) to (2.16), only \( k_1/2 + k_2 \) are independent. The system yields the moduli squared of the entries of the vector \( u' \) expressed in terms of the eigenvalues \( s^{(1)} \). We call the latter bosonic eigenvalues, if they satisfy Eq. (2.17) and fermionic eigenvalues if they satisfy Eq. (2.16). With the substitutions \( \hat{s}^{(j)}_{q_1} \rightarrow (s^{(j)}_{q_1})^2 \) and \( \hat{s}^{(j)}_{q_2} \rightarrow (s^{(j)}_{q_2})^2 \), \( j = 1, 2 \), the set of independent equations is equivalent to the corresponding set of equations for the unitary supergroup. Thus, we can directly read off the solutions from Ref. [5]. They will be stated in Sec. III.

C. Recursion to all levels in superspace

The construction just outlined for the first coset space has to be continued recursively to cover the entire group manifold. For the ordinary groups, this recursion can be found in Ref. [1,2,3]. In the present case, we extend the recursion for the unitary supergroup in Ref. [3]. As the Cartan subalgebra \( \text{UOSP}((k_1/2k_2)) \) is slightly different for even and odd bosonic dimension \( k_1 \) according to Eqs. (2.3) and (2.4), we have to distinguish these two cases for the recursion. For brevity, we refer to a level as even, if \( (k_1 - n + 1) \) is even, and as odd otherwise.

In the \( n \)-th step the vector \( u'_n \) is expanded in a set of \( k_1 - n + 1 + 2k_2 \) basis vectors \( e'_j^{(n-1)} \), which span the subspace of \( \mathcal{L} \) orthogonal to \( u = [u_1 u_2 \cdots u_{n-1}] \). This set splits into two disjoint subsets. The first subset contains \( k_1 - n + 1 \) vectors \( e'_j^{(n-1)} \) spanning some subspace of \( \mathcal{L} \). The second one contains \( 2k_2 \) basis vectors \( e'_j^{(n-1)} \) spanning \( \mathcal{L} \). The entries of \( u'_n \) in this basis are complex variables

\[
e'_j^{(2p-1)}(n-1) u_n = i \left( e'_j^{(2p-1)}(n-1) u_n \right)^* = \frac{i}{\sqrt{2}} \left| v_{(n)}^p \right| \exp \left( -i \theta_{(n)}^p \right), \quad p \leq \frac{(k_1 - n + 1)/2}{(k_1 - n)/2} \quad \text{for} \quad k_1 - n + 1 \quad \text{even,}
\]

\[
e'_j^{(2p)}(n-1) u_n = \left( e'_j^{(2p-1)}(n-1) u_n \right)^* = \frac{1}{\sqrt{2}} \left| v_{(n)}^p \right| \quad \text{for} \quad k_1 - n + 1 \quad \text{odd,}
\]

(2.17)

For \( k_1 - n + 1 \) odd, the remaining entry is parametrized by a real variable and an integer \( r \in \{0, 1\} \) as

\[
e'_j^{(k_1-n+1)}(n-1) u_n = (-1)^r v_{(n)}^{(k_1-n+1)} u_n
\]

(2.18)

Thus, we can write down the rotated \( n \)-th eigenvector on the \((n-1)\)-th level

\[
u_n^{(n-1)} = \sum_{j=1}^{k_1+2k_2-n+1} e_j^{(n-1)} u_n e_j^{(n-1)}.
\]

(2.19)

The projection of \( s \) onto this subspace after the \( n \)-th step is given by

\[
s^{(n-1)} = \left( \sum_{i=n}^{k_1+2k_2} u_i u_i^\dagger \right) s \left( \sum_{i=n}^{k_1+2k_2} u_i u_i^\dagger \right) = \left( 1_{k_1+2k_2} - \sum_{i=1}^{n-1} u_i u_i^\dagger \right) s \left( 1_{k_1+2k_2} - \sum_{i=1}^{n-1} u_i u_i^\dagger \right)
\]

(2.20)

and belongs to the Cartan subalgebra of \( \text{UOSP}((k_1 - n)/2k_2) \). The new coordinates are obtained by projecting \( s^{(n-1)} \) on the subspace orthogonal to \( u_n \) by

\[
s_p^{(n)} e_p^{(n)} = (1_{k_1+2k_2} - u_n u_n^\dagger) s^{(n-1)} (1_{k_1+2k_2} - u_n u_n^\dagger) e_p^{(n)} = (1_{k_1+2k_2} - u_n u_n^\dagger) s^{(n-1)} e_p^{(n)}.
\]

(2.21)
For $k_1 - n + 1$ even, this leads to a system of equations as in (2.14) to (2.16) reduced by $(n-1)/2$ unknown variables. For $(k_1 - n + 1)$ odd, the equations have a slightly different form,

$$1 = \sum_{p=1}^{k_1-n} |v_p^{(n)}|^2 + |v_{k_1-n+1}^{(n)}|^2 + \sum_{p=1}^{k_2} |\gamma_p^{(n)}|^2, \quad (2.22)$$

$$0 = \sum_{q=1}^{k_1-n} \left( \frac{(s_{q1}^{(n)})^2 |\epsilon_q^{(n)}|^2}{\left( s_{q1}^{(n-1)} \right)^2 - (s_{q2}^{(n)})^2} + |v_{k_1-n+1}^{(n)}|^2 + \sum_{q=1}^{k_2} \frac{(s_{q1}^{(n)})^2 |\gamma_q^{(n)}|^2}{(is_{q2}^{(n-1)})^2 - (s_{q1}^{(n)})^2} \right), \quad p = 1, \ldots, (k_1-n), \quad (2.23)$$

$$z_p = -i s_{p2}^{(n)} \prod_{q=1}^{k_1-n} \frac{1}{\left( s_{q1}^{(n-1)} \right)^2 - (is_{q2}^{(n)})^2} \left( \sum_{q=1}^{k_1-n} \frac{(s_{q1}^{(n)})^2 |\epsilon_q^{(n)}|^2}{(is_{q2}^{(n-1)})^2 - (is_{q2}^{(n)})^2} + |v_{k_1-n+1}^{(n)}|^2 + \sum_{q=1}^{k_2} \frac{(s_{q1}^{(n)})^2 |\gamma_q^{(n)}|^2}{(is_{q2}^{(n-1)})^2 - (is_{q2}^{(n)})^2} \right), \quad z_p \to \infty, \quad p = 1, \ldots, 2k_2. \quad (2.24)$$

The difference between Eqs. (2.22) to (2.24) and the corresponding equations (2.14) to (2.16) for the even levels is due to the isolated entry (2.18), which has to be treated separately. This reflects the difference between the even orthogonal group and the odd orthogonal group in ordinary space.

The new basis vectors $\hat{e}_j^{(n)}$ are related to the basis vectors of the foregoing level by a $(k_1-n+2k_2) \times (k_1-n+1+2k_2)$ rectangular supermatrix $\hat{b}^{(n)}$. The moduli squared of its entries $\hat{b}_{p,m}^{(n)}$ are determined by rewriting Eq. (2.21) and multiplying it from the left hand side with $e_m^{(n-1)^\dagger}$

$$e_m^{(n-1)^\dagger} s^{(n-1)} e_p^{(n)} = s_p^{(n)} e_m^{(n-1)^\dagger} e_p^{(n)} + e_m^{(n-1)^\dagger} u_n b_p^{(n)}, \quad (2.25)$$

where we defined $b_p^{(n)} = u_n s^{(n-1)} e_p^{(n)}$. On the other hand we have

$$e_m^{(n-1)^\dagger} s^{(n-1)} e_p^{(n)} = s_m^{(n-1)} e_m^{(n-1)^\dagger} e_p^{(n)}, \quad (2.26)$$

which yields for the matrix elements of $\hat{b}^{(n)}$ the expression

$$\hat{b}_{p,m}^{(n)} = \frac{1}{s_m^{(n-1)} - s_p^{(n)}} \sum_{m'}^{(k_1-n-1)/2} \alpha_m^{(n)} \beta_m^{(n)} \quad . \quad (2.27)$$

The modulus squared of $b_p^{(n)}$ is determined by the normalization of the rotated basis vectors $e_m^{(n-1)^\dagger} e_p^{(n)} = \delta_{mp}$, i.e. by the condition that the matrix $\hat{b}^{(n)^\dagger} \hat{b}^{(n)}$ is unity in the $k_1-n+2k_2$ dimensional subspace orthogonal to $u = [u_1, u_2, \cdots, u_n]$. Due to the block structure of the supermatrix $\hat{b}^{(n)}$, the vector $\hat{b}^{(n)}$ has commuting and anticommuting elements. For $k_1 - n + 1$ even we define $|b_p^{(n)}|^2 = |b_{p2}^{(n)}|^2 = |b_{p1}^{(n-1)}|^2, p = 1, \ldots, (k_1-n+1)/2$ for the commuting and $|b_p^{(n)}|^2 = |b_{k_1-n+2p}^{(n)}|^2 = |b_{k_1-n+2p-1}^{(n)}|^2, p = 1, \ldots, k_2$ for the anticommuting elements. For $k_1 - n + 1$ odd we define $|b_p^{(n)}|^2$ and $|b_p^{(n)}|^2$ correspondingly. Again, there is a difference in the determining equations of $|w_p^{(n)}|^2$ and $|\beta_p^{(n)}|^2$ between the even and the odd levels of the recursion. For $k_1 - n + 1$ even we have

$$\frac{1}{|w_p^{(n)}|^2} = \sum_{m=1}^{(k_1-n+1)/2} \frac{(s_m^{(n-1)})^2 + (s_{m'}^{(n)})^2}{\left( s_m^{(n-1)} \right)^2 - (s_{m'}^{(n)})^2} |\epsilon_m^{(n)}|^2 + \sum_{m'=1}^{(k_1-n-1)/2} \frac{(s_{m'}^{(n-1)})^2 + (s_{m'}^{(n)})^2}{(is_{m'}^{(n-1)})^2 - (s_{m'}^{(n)})^2} |\alpha_{m'}^{(n)}|^2, \quad p = 1, \ldots, \frac{k_1-n-1}{2}. \quad (2.28)$$

For the remaining modulus squared we obtain

$$\frac{1}{|w_p^{(n)}|_{k_1-n+1}^2} = \sum_{m=1}^{(k_1-n+1)/2} \frac{1}{(s_m^{(n-1)})^2} |\epsilon_m^{(n)}|^2 + \sum_{m'=1}^{(k_1-n-1)/2} \frac{1}{(is_{m'}^{(n-1)})^2} |\alpha_{m'}^{(n)}|^2. \quad (2.29)$$
The moduli squared of the anticommuting coordinates of \( b^{(n)} \) fulfil a formally similar equation. However, it is mathematically more precise to write it in the inverted form to avoid the appearance of purely nilpotent variables in the denominator,

\[
1 = |\beta_p^{(n)}|^2 \left( \sum_{m=1}^{(k_1-n+1)/2} \left( \frac{(s_{m1}^{(n-1)})^2 + (i s_{m2}^{(n)})^2}{(s_{m1}^{(n-1)})^2 - (i s_{m2}^{(n)})^2} \right)^2 |v_m^{(n)}|^2 + \sum_{m'=1}^{k_2} \left( \frac{(i s_{m2}^{(n-1)})^2 + (s_{m1}^{(n)})^2}{(i s_{m2}^{(n-1)})^2 - (s_{m1}^{(n)})^2} \right)^2 |\alpha_{m'}^{(n)}|^2 \right) ,
\]

\[\text{for } p = 1, \ldots, k_2 . \tag{2.30}\]

The corresponding equations for the odd levels are obtained from Eqs. (2.28) and (2.29) by making the following formal replacements. In Eq. (2.28), the sum over \( m \) runs only to \((k_1 - n)/2\) and, in addition, the term \(|v_m^{(n)}|^{2}/|s_{m1}^{(n)}|^2\) is subtracted. In Eq. (2.30), the first sum runs only to \((k_1 - n)/2\) and the term \(|v_m^{(n)}|^{2}/|s_{m1}^{(n)}|^2\) is subtracted. Moreover, Eq. (2.29) does not exist for the odd levels.

### D. Solution of the angular Gelfand–Tzetlin equations

Up to the \(k_1\)-th level both sets of equations (2.14) to (2.16) and (2.28) to (2.30) have to be solved for even and odd levels separately. For the even levels, there is, as already mentioned above, a direct correspondence to the case of the unitary supergroup. Thus, we find the results of Ref. 3

\[
|v_p^{(n)}|^2 = \prod_{q=1}^{k_1-n-1} \left( \frac{(s_{p1}^{(n-1)})^2 - (s_{q1}^{(n)})^2}{(s_{p1}^{(n-1)})^2 - (i s_{q2}^{(n)})^2} \right) \prod_{q=1}^{k_2} \left( \frac{(i s_{p2}^{(n-1)})^2 - (s_{q1}^{(n)})^2}{(i s_{p2}^{(n-1)})^2 - (i s_{q2}^{(n)})^2} \right) ,
\]

\[p = 1, \ldots, (k_1 - n + 1)/2 , \ \ \ k_1 - n + 1 \text{ even} , \tag{2.31}\]

\[
|\alpha_p^{(n)}|^2 = \left( (i s_{p2}^{(n-1)})^2 - (i s_{p2}^{(n-2)})^2 \right) \prod_{q=1}^{k_1-n-1} \left( \frac{(i s_{p2}^{(n-1)})^2 - (s_{q1}^{(n)})^2}{(i s_{p2}^{(n-1)})^2 - (i s_{q1}^{(n)})^2} \right) \prod_{q=1}^{k_2} \left( \frac{(i s_{p2}^{(n-1)})^2 - (i s_{q2}^{(n)})^2}{(i s_{p2}^{(n-1)})^2 - (i s_{q2}^{(n)})^2} \right) ,
\]

\[p = 1, \ldots, k_2 . \]

We have included the first level by setting \( s = s^{(0)} \). To find the solution of Eqs. (2.28) to (2.30), one cannot directly make use of the results in the unitary case. An explicit calculation is necessary which is given in App. A. It yields

\[
|w_p^{(n)}|^2 = -\prod_{m=1}^{k_1-n-1} \left( \frac{(s_{m1}^{(n-1)})^2 - (s_{p1}^{(n)})^2}{(s_{m1}^{(n-1)})^2 - (i s_{p2}^{(n)})^2} \right) \prod_{q=1}^{k_2} \left( \frac{(i s_{p2}^{(n-1)})^2 - (s_{q1}^{(n)})^2}{(i s_{p2}^{(n-1)})^2 - (i s_{q1}^{(n)})^2} \right) , \ \ p = 1, \ldots, (k_1 - n - 1)/2 ;
\]

\[
|w_p^{(n)}|_{k_1-n-1}^2 = \frac{\prod_{m=1}^{k_1-n-1} (s_{m1}^{(n-1)})^2 \prod_{q=1}^{k_2} (i s_{q2}^{(n)})^2}{\prod_{m=1}^{k_1-n-1} (s_{m1}^{(n-1)})^2 \prod_{q=1}^{k_2} (i s_{q2}^{(n)})^2} ,
\]

\[
|\beta_p^{(n)}|^2 = \left( (i s_{p2}^{(n-1)})^2 - (i s_{p2}^{(n-2)})^2 \right) \prod_{q=1}^{k_1-n-1} \left( \frac{(i s_{p2}^{(n-1)})^2 - (s_{q1}^{(n)})^2}{(i s_{p2}^{(n-1)})^2 - (i s_{q1}^{(n)})^2} \right) \prod_{q=1}^{k_2} \left( \frac{(i s_{p2}^{(n-1)})^2 - (i s_{q2}^{(n)})^2}{(i s_{p2}^{(n-1)})^2 - (i s_{q2}^{(n)})^2} \right) ,
\]

\[p = 1 \ldots k_2 . \tag{2.32}\]

We observe that the squares of the fermionic eigenvalues of the different levels \((i s_{p2}^{(n)})^2\) differ only by a nilpotent variable. Hence, we introduce complex anticommuting variables \( \xi_p^{(n)} \) such that

\[
|\xi_p^{(n)}|^2 = (i s_{p2}^{(n)})^2 - (i s_{p2}^{(n-1)})^2 .
\]

We emphasize that this feature is highly non–trivial: the difference of the squared fermionic eigenvalues for two neighbouring levels can be expressed as the modulus squared of one anticommuting variable.
The solutions of Eqs. (2.22) to (2.24) for the odd levels, i.e., for $k_1 - n + 1$ odd, cannot directly be obtained by adjusting the results of Ref. [2]. However, as the necessary modifications are intuitively clear, we do not derive the solutions for the odd levels in detail. We simply state the results,

$$|v_p^{(n)}|^2 = \frac{\prod_{q=1}^{k_1-n} ((s_{p1}^{(n-1)})^2 - (s_{q1}^{(n)})^2) \prod_{q=1}^{k_2} ((s_{p1}^{(n-1)})^2 - (is_{q2}^{(n-1)})^2)}{(s_{p1}^{(n)})^2 \prod_{q=1}^{k_1-n} ((s_{p1}^{(n-1)})^2 - (s_{q1}^{(n)})^2) \prod_{q=1}^{k_2} ((s_{p1}^{(n-1)})^2 - (is_{q2}^{(n)})^2)} \ , \quad p = 1, \ldots, (k_1-n)/2 \ ,$$

$$|\alpha_p^{(n)}|^2 = \frac{\prod_{q=1}^{k_1-n} ((is_{p2}^{(n-1)})^2 - (s_{q1}^{(n)})^2) \prod_{q=1}^{k_2} ((s_{p2}^{(n-1)})^2 - (is_{q2}^{(n-1)})^2)}{(s_{p2}^{(n-1)})^2 \prod_{q=1}^{k_1-n} ((is_{p2}^{(n-1)})^2 - (s_{q1}^{(n)})^2) \prod_{q=1}^{k_2} ((s_{p2}^{(n-1)})^2 - (is_{q2}^{(n)})^2)} \ , \quad p = 1, \ldots, k_2 \ ,$$

The solutions of Eqs. (2.28) to (2.31) for the odd levels read

$$|w_p^{(n)}|^2 = \frac{1}{2} \frac{\prod_{q=1}^{k_1-n} ((s_{p1}^{(n-1)})^2 - (s_{q1}^{(n)})^2) \prod_{q=1}^{k_2} ((s_{p1}^{(n-1)})^2 - (is_{q2}^{(n-1)})^2)}{(s_{p1}^{(n)})^2 \prod_{q=1}^{k_1-n} ((s_{p1}^{(n-1)})^2 - (s_{q1}^{(n)})^2) \prod_{q=1}^{k_2} ((s_{p1}^{(n-1)})^2 - (is_{q2}^{(n)})^2)} \ , \quad p = 1, \ldots, (k_1-n)/2 \ ,$$

$$|\beta_p^{(n)}|^2 = \frac{1}{2} \frac{\prod_{q=1}^{k_1-n} ((is_{p2}^{(n-1)})^2 - (is_{q2}^{(n-1)})^2) \prod_{q=1}^{k_2} ((is_{p2}^{(n-1)})^2 - (s_{q1}^{(n)})^2)}{(is_{p2}^{(n-1)})^2 \prod_{q=1}^{k_1-n} ((is_{p2}^{(n-1)})^2 - (is_{q2}^{(n-1)})^2) \prod_{q=1}^{k_2} ((is_{p2}^{(n-1)})^2 - (s_{q2}^{(n)})^2)} \ , \quad p = 1, \ldots, k_2 \ .$$

From the solutions stated in Eqs. (2.31) to (2.34) one derives the corresponding formulae for the group $SO(k_1)$ in ordinary space by setting all anticommuting variables to zero.

A comparison with the results for the unitary supergroup $U(k_1/2k_2)$ in Ref. [3] reveals an interesting formal connection. The Cartan subalgebras $u^{(0)}(k_1 - n + 1/2k_2)$ and $u^{(0)}(k_1 - n - 1/2k_2)$ of $U(k_1 - n + 1/2k_2)$ and $U(k_1 - n - 1/2k_2)$, respectively, are all diagonal matrices

$$s^{(n-1)} = \text{diag} \left(s_{11}^{(n-1)}, \ldots, s_{k_1-n+1}^{(n-1)}, is_{12}^{(n-1)}, \ldots, is_{2k_2}^{(n-1)}\right) \ ,$$

$$s^{(n)} = \text{diag} \left(s_{11}^{(n)}, \ldots, s_{k_1-n+1}^{(n)}, is_{12}^{(n)}, \ldots, is_{2k_2}^{(n)}\right) \ .$$

(2.36)

If one now formally replaces, in the results for the unitary supergroup, these matrices $s^{(n-1)}$ and $s^{(n)}$ with elements of the Cartan subalgebra of $u^{(0)}(k_1 - n + 1/2k_2)$ and $u^{(0)}(k_1 - n - 1/2k_2)$ according to

$$s^{(n-1)} \leftrightarrow \text{diag} \left(s_{11}^{(n-1)}, \ldots, s_{k_1-n+1}^{(n-1)}, is_{12}^{(n-1)}, \ldots, is_{k_2}^{(n-1)}\right) \otimes \tau(3) \ ,$$

$$s^{(n)} \leftrightarrow \text{diag} \left(s_{11}^{(n)}, \ldots, s_{k_1-n+1}^{(n)}, is_{12}^{(n)}, \ldots, is_{k_2}^{(n)}\right) \otimes \tau(3) \ , \quad k_1 - n + 1 \text{ even} \ .$$

(2.37)

the results in Eqs. (2.34) and (2.35) and in Eqs. (2.34) and (2.35) are recovered. This formal connection between the unitary supergroup and the unitary orthosymplectic one is natural and plausible. Unfortunately, we could not make a sound mathematical reasoning out of the replacement (2.37) which would go beyond the a posteriori observation. However, the formal connection stated above illustrates the deep relationship between the groups which will become even more apparent in the generalized Gelfand pattern given in Sec. [V].

E. Ordinary unitary symplectic group

The $k_1$-th step of the recursion is the last one in a superspace. We now approach the second line of Eq. (2.5). The following steps do not involve anticommuting variables anymore. We are left with the ordinary unitary symplectic group $USp(2k_2)$ and its coset decomposition. We make use of the isomorphism $USp(2k_2) \cong U(k_2; 4)$ where $U(k_2; 4)$ is
the unitary group in $k_2$ dimensions parametrized over the quaternions. Since $U(k_2; 4)$ can be parametrized analogously to $U(k_2; 2) \cong U(k_2)$, i.e. to the unitary group over the complex numbers, we only have to adjust the results of Refs. 1,4 where Gelfand–Tzetlin coordinates for the ordinary unitary group were constructed. We write $U \in U(k_2; 4)$ as $U = [U_1 U_2 \cdots U_{k_2}]$. The normalized vectors $U_i$ have quaternionic entries. The Cartan subalgebra is of the form $s^{(k_1)}_2 = \text{diag}(s^{(k_1)}_{12}, \ldots, s^{(k_1)}_{k_22}) \otimes \tau^{(3)}$. The Gelfand–Tzetlin eigenvalue equation reads for the first level of the USp($2k_2$) recursion, i.e. for the level $k_1 + 1$ of the USp($k_1/2k_2$) recursion

$$
(1_{k_2} - U_1 U_1^\dagger)_{s^{(k_1)}_2}(1_{k_1} - U_1 U_1^\dagger)E_n^{(1)} = is^{(k_1+1)}E_n^{(1)}.
$$

We introduced capital letters $U_1 = u_{k_1+1}$ and $E_n^{(1)} = e_n^{(k_1+1)}$ in order to highlight that the vectors and matrices used here live in ordinary space. Since the operator on the left hand side is not Hermitian selfdual, Eq. (2.38) has not a unique solution, see for example Ref. 17. This can be cured by multiplying Eq. (2.38) on both sides with $1_{k_2}$ from the right. A well defined eigenvalue equation for a selfdual matrix obtains. It is known to have $k_2$ scalar eigenvalues $is^{(k_1+1)}_21_2$ which, to keep the notation simple, we also denote by $is^{(k_1+1)}_2$. After this adjustment, we can proceed along the same lines which led to Eq. (2.12). The equation reduces to the well known Gelfand–Tzetlin equations of the unitary group $U(k_2; 2) \cong U(k_2)$,

$$
1 = \sum_{n=1}^{k_2} |U_{n1}|^2,
$$

$$
0 = \sum_{m=1}^{k_2} \frac{|U_{m1}|^2}{is^{(k_1)}_2 - is^{(k_1+1)}_2}, \quad p = 1, \ldots, k_2 - 1.
$$

(2.39)

This establishes a one–to–one correspondence between the $(k_2 - 1)$ eigenvalues $is^{(k_1+1)}_2$ and the moduli squared of the quaternionic entries

$$
|U_{n1}|^2 = \text{Tr} U^\dagger_{m1}U_{m1}.
$$

(2.40)

All formulae derived in Refs. 1,4 for the unitary group can now be adopted to the unitary symplectic one.

F. Invariant measure

According to the coset decomposition (2.5), the invariant measure $du(u)$ of $u \in \text{OSp}(k_1/2k_2)$ is the product of all measures on the cosets, i.e. on the spheres described by them. Of course, these measures are conditioned, because the orthogonality of the vectors $u_n$ in $u = [u_1 u_2 \cdots u_{k_1+2k_2}]$ has to be respected. As we will see, the Gelfand–Tzetlin coordinates take care of this condition in a most convenient way. We evaluate the squared invariant length element $du_\dagger du_n$. For $(k_1 - n + 1)$ even, it reads,

$$
du_\dagger du_n = du_\dagger du_n = \sum_{m=1}^{k_1-n+1} \frac{1}{4|v^{(n)}_m|^2} (d|v^{(n)}_m|^2)^2 + \sum_{m=1}^{k_1-n+1} (d|v^{(n)}_m|^2)^2 + \sum_{m'=1}^{k_2} d(\alpha^{(n)}_{m'})^* d\alpha^{(n)}_{m'}.
$$

(2.41)

where we use the parametrization (2.11). The first equality is due to the basis independence of the invariant length element. It is a highly welcome feature of the Gelfand–Tzetlin coordinates for the unitary group in ordinary space and in superspace that the metric remains diagonal. This holds also in the present problem. Extending the corresponding calculation of Ref. 3, we find for the even levels, i.e. for $k_1 - n + 1$ even,

$$
d\mu(u_n) = 2^{k_2} \prod_{p=1}^{k_2} B_{k_1-n-k_2}^{p} \frac{B_{k_1-n-k_2}^{p}}{B_{k_1-n-k_2}^{p}} \frac{(s^{(n)})^2}{(s^{(n-1)})^2} |d[s^{(n)}_1]| |d[q^{(n)}]| |d[x^{(n)}]|, n \leq k_1, \quad k_1 - n + 1 \text{ even}.
$$

(2.42)

and for the odd levels, i.e. for $k_1 - n + 1$ odd, we have

$$
d\mu(u_n) = 2^{k_2} \prod_{p=1}^{k_2} (s^{(n)})^2 \frac{B_{k_1-n-k_2}^{p}}{B_{k_1-n-k_2}^{p}} \frac{(s^{(n)})^2}{(s^{(n-1)})^2} |d[s^{(n)}_1]| |d[q^{(n)}]| |d[x^{(n)}]|, n \leq k_1, \quad k_1 - n + 1 \text{ odd}.
$$

(2.43)
Here, we introduced the function

$$B_{nm}(s) = \frac{\prod_{p>q}(s_{pq} - s_{qp}) \prod_{p>q}^{m}(is_{pq} - is_{qp})}{\prod_{p<q}(s_{pq} - is_{qp})} = \frac{\Delta_n(s_1)\Delta_m(is_2)}{\prod_{p<q}^{m}(s_{pq} - is_{qp})}.$$  (2.44)

It contains the ordinary Vandermonde determinants $\Delta_n(s_1)$ and $\Delta_m(is_2)$ and can be viewed as the supersymmetric generalization of the Vandermonde determinant. Furthermore, we defined

$$d[s_1^{(n)}] = \prod_{p=1}^{k_1-n+1} ds_{p1}^{(n)}, \quad d[\hat{\vartheta}^{(n)}] = \prod_{p=1}^{k_1-n+1} d\hat{\vartheta}_p^{(n)} \quad \text{and} \quad d[\xi^{(n)}] = \prod_{p=1}^{k_2} d\xi_p^{(n)} d\xi_p^{(n)}.$$  (2.45)

Remarkably, Eqs. (2.42) and (2.43) imply that the measures on all cosets factorize. Collecting all these measures up to the $k_1$–th step, we obtain the invariant measure of $u \in \mathrm{USp}(k_1/2k_2)$ in the form

$$d\mu(u) = 2^{k_1 k_2} \frac{\Delta_{k_1}(is_1^{(k_1)})^2 \prod_{i=1}^{k_1} i_{p}^2 d[s_1^{(i)}] d[\hat{\vartheta}^{(i)}] d[\xi^{(i)}] d\mu(U)}{B_{k_2/2k_2}(s^2)},$$  (2.46)

where $d\mu(U)$ is the invariant measure on $U \in \mathrm{USp}(2k_2)$. We mention in passing that the measure of the orthogonal group in ordinary space can be obtained by setting all anticommuting variables to zero in the invariant length (2.41), and skipping all couplings between the bosonic and fermionic eigenvalues in Eq. (2.46).

The measure (2.46) on $\mathrm{USp}(k_1/2k_2)$ has an important feature: Most conveniently, it is, apart from $d\mu(U)$, flat. This follows directly from the factorization of the measures (2.42) and (2.43) on the coset spaces. This is also true for the Gelfand–Tzetlin coordinates of the unitary group in ordinary space as well as for the ones of the orthogonal group in ordinary space. However, this important feature does not continue beyond the $k_1$–th level. We will see that now in working out the measure $d\mu(U)$ for $U \in \mathrm{USp}(2k_2)$. We write $u = [U \ U_2 \cdots U_{k_2}]$ and decompose the entry $U_{nm}$ as $U_{nm} = |U_{nm}| \tilde{U}_{nm}$, with $\tilde{U}_{nm}$ a unimodular quaternion. We introduce a parametrization of the unimodular quaternion,

$$\tilde{U}_{nm} = \begin{bmatrix} \cos \psi^{(m)}_n \exp(-i\tilde{\gamma}^{(m)}_n) \\ \sin \psi^{(m)}_n \exp(i\tilde{\gamma}^{(m)}_n) \end{bmatrix},$$  (2.47)

which allows us to write the invariant length element squared as

$$\text{Tr} \ dU^*_m dU_m = \sum_{n=1}^{k_2} \left[ \frac{1}{4|U_{nm}|^2} (|U_{nm}|^2)^2 + \sum_{i=1}^{2} |U_{nm}|^2 (d\gamma^{(m)}_n)^2 + |U_{nm}|^2 (d\cos \psi^{(m)}_n)^2 \right], \quad m = 1 \ldots k_2.$$  (2.48)

Employing and properly adjusting the results of Refs. 1,4 one finds the measure of the coset in the $m$–th level

$$d\mu(U_m) = \frac{\Delta_{k_2-m}(is_2^{(k_1+m)}) \prod_{p,q}^{m}(is_2^{(k_1+m-1)} - is_2^{(k_1+m)}) d[s_2^{(k_1+m)}] d[\cos \psi^{(m)}] d[\gamma^{(m)}]}{2^{k_2-m} \Delta_{k_2-m+1}(is_2^{(k_1+m-1)})},$$  (2.49)

with the definitions

$$d[s_2^{(k_1+m)}] = \prod_{p=1}^{k_2-m+1} ds_{p2}^{(k_1+m)}, \quad d[\cos \psi^{(m)}] = \prod_{p=1}^{k_2-m+1} d\cos \psi^{(m)}_p \quad \text{and} \quad d[\gamma^{(m)}] = \prod_{p=1}^{k_2-m+1} d\gamma^{(m)}_p.$$  (2.50)

One clearly sees that the factorization property does not hold for the unitary symplectic group. This is a peculiarity of the Gelfand–Tzetlin parametrization for the unitary symplectic group. Collecting all levels we arrive at the invariant measure on $U \in \mathrm{USp}(2k_2)$,

$$d\mu(U) = \frac{1}{2^{k_2(k_2-1)/2} \Delta_{k_2}^3(is_2^{(k_1)})} \prod_{m=1}^{k_2} \prod_{n=1}^{k_2-m+1} \prod_{n'=1}^{k_2-m} \frac{is_2^{(k_1+m-1)} - is_2^{(k_1+m)}}{is_2^{(k_1+m)} - is_2^{(k_1+m)}} d[s_2^{(k_1+m)}] d[\cos \psi^{(m)}] d[\gamma^{(m)}].$$  (2.51)

which combines with Eq. (2.46) to the full measure on $\mathrm{UOSp}(k_1/2k_2)$. 
G. Matrix elements

With the results of the previous sections, we can express an arbitrary column \( u_p \) of a matrix \( u = [u_1 \, u_2 \, \cdots \, u_{k_1+2k_2}] \) in the unitary orthosymplectic supergroup \( \text{UOSp}(k_1/2k_2) \) in terms of our angular Gelfand–Tzetlin coordinates. In the rotated primed basis \((2.10)\), we have

\[
u_p' = \tilde{b}^{(1)}T \tilde{b}^{(2)}T \cdots \tilde{b}^{(n-1)}T u_p^{(n-1)},
\]

(2.52)

where \( b^{(n)} \) and the scalar products are defined in Eq. \((2.18)\) and \((2.17)\) and in Eqs. \((2.25)\) and \((2.27)\). So far, we have constructed a unitary representation of \( \text{UOSp}(k_1/2k_2) \). We also wish to obtain an orthosymplectic representation. To this end, we have to assure that the vectors \( u_j \), \( j \leq k_1 \) become real, when the matrix \( u' = [u_1' \, u_2' \, \cdots \, u_{k_1+k_2}] \) is rotated back into the unprimed basis. We only discuss the case \( k_1 - n + 1 \) even, the odd case is treated analogously. We recall that the vector \( b^{(n)} \) entering in the projection matrix in Eq. \((2.27)\) has been determined only up to a phase. There is an ambiguity in choosing the phase of \( b^{(n)} \). The Gelfand–Tzetlin coordinates parametrize the vector \( u_n \) only up to some phases associated with the action of the Cartan subgroup of \( \text{UOSp}((k_1 - n + 1)/2k_2) \). Thus, the projection matrix \( \tilde{b}^{(n)} \) is as well invariant under the action of this Cartan subgroup. We may multiply \( \tilde{b}^{(n)} \) with an arbitrary element of the Cartan subgroup without changing its projection properties. We set \( b^{(n)}_{2p'} = i b^{(n)}_{2p-1} \), \( p \leq (k_1 - n + 1)/2 \) and \( b^{(n)}_{k_1-n} = i[w_{1,\cdots,1}] \) in the commuting sector and \( b^{(n)}_{k_1-n+2p} = -\tilde{b}^{(n)*}_{k_1-n+2p}, \ p = 1, \ldots, k_2 \) in the anticommuting one. The remaining phases may be set to zero. With this choice of phases and after undoing the basis rotation, the columns as well as the rows of \( \tilde{b}^{(n)T} \) fulfill the reality condition \((2.6)\). The vectors \( u^{(n-1)} \) become real, too. An explicit form of the real matrices \( \tilde{b}^{(n)} \) is given in App. \( B \).

III. GENERALIZED GELFAND PATTERN

The unitary Lie group \( \text{U}(k; \beta) \) over the real \( (\beta = 1) \) and complex \( (\beta = 2) \) numbers and over the quaternions \( (\beta = 4) \) is isomorphic to the orthogonal, unitary and unitary symplectic group, \( \text{SO}(k) \cong \text{U}(k; 1), \ \text{U}(k) \cong \text{U}(1; 2) \) and \( \text{USp}(2k) \cong \text{U}(k; 4) \). The Gelfand–Tzetlin representation scheme obtains from the following procedure. An irreducible representation is defined by an ordered set of integers or half integers called highest weights. This irreducible representation can be decomposed in irreducible representations of \( \text{U}(k-1; \beta) \). In the decomposition each irreducible representation of \( \text{U}(k-1; \beta) \) occurs either exactly once or never. Only those irreducible representations appear whose highest weights satisfy certain betweenness conditions depending on the group under consideration. Going through all steps of the group chain or, equivalently, the coset decomposition,

\[
\text{U}(k; \beta) = \frac{\text{U}(k; \beta)}{\text{U}(k-1; \beta)} \otimes \frac{\text{U}(k-1; \beta)}{\text{U}(k-2; \beta)} \otimes \cdots \otimes \frac{\text{U}(2; \beta)}{\text{U}(1; \beta)} \otimes \text{U}(1; \beta),
\]

(3.1)

one has labelled all states in the irreducible representation of \( \text{U}(k; \beta) \) by a set of integers or half integers, arranged in a Gelfand pattern.

The analogue for the coordinates is as follows. We consider the adjoint group action \( \mathcal{O}_k = U^1 xU \) on an element \( x \) of the Cartan subalgebra \( \mathfrak{u}^{(0)}(k; \beta) \) with \( U \in \text{U}(k; \beta) \). Here, in this one instance, we use the symbol \( x \) for an element in the algebra, because we want to emphasize that the present discussion so far applies to ordinary groups and because we want to avoid confusion with the discussion to follow on the supergroups. This subset \( \mathcal{O}_k = U^1 xU \) of the complete algebra is called orbit. We can map the \( \text{U}(k; \beta) \) orbit labelled by an ordered set of eigenvalues \( x_i > x_{i+1} \) onto many different \( \text{U}(k-1; \beta) \) orbits by projecting \( \mathcal{O}_k \) onto a \( k-1 \) dimensional subspace. But only those \( \text{U}(k-1; \beta) \) orbits \( \mathcal{O}_{k-1} \) can be reached, whose eigenvalues interlace two neighboring eigenvalues of \( \mathcal{O}_k \). This is the so called minimax principle for selfadjoint operators. The Gelfand–Tzetlin method uses the eigenvalues of the projected matrix as coordinates of the coset \( \text{U}(k; \beta)/\text{U}(k-1; \beta) \). However, \( x \) is a fixed point of the action of the Cartan subgroup \( \exp(x_0) \), \( x_0 \in \mathfrak{u}^{(0)}(k; \beta) \). Hence, the coset \( \text{U}(k; \beta)/\text{U}(k-1; \beta) \) is parametrized by the eigenvalues of \( \mathcal{O}_{k-1} \) only up to equivalence classes with respect to the action of the Cartan subgroup of \( \text{U}(k; \beta) \), parametrized by \( x_0 \). In this way the set of variables describing the coset is split into two parts: One part consists of the eigenvalues of \( \mathcal{O}_{k-1} \), the other one of the independent elements of \( x_0 \). Guillemin and Sternberg introduced the concept of complete integrability by interpreting the entries of \( x \) as action and the elements of \( x_0 \) as angle coordinates of a generalized mechanical system. We emphasize that this usage of the term angles is different from the one introduced previously. We distinguish between radial and angular Gelfand–Tzetlin coordinates. However, both Gelfand–Tzetlin coordinates, the radial and the angular ones, allow for a further distinction between action and angle degrees of freedom, although the interpretation is slightly different in the two cases. The Guillemin–Sternberg theory applies only to the groups
U(k; β) for β = 1, 2 but not to the unitary symplectic group. This can be considered as the reason for the relatively complicated expression of the measure for USp(2k) ≅ U(k; 4).

The generalized Gelfand pattern for UOSp(k1/2k2) can be extracted from the positive definiteness of the moduli squared of the bosonic matrix elements |v|^{(n)}|^2. If one restricts oneself to the subgroup, which consists of the direct product SO(k1) ⊗ USp(2k2) the pattern of the SO(k1) and USp(2k2) are rederived which are well known from representation theory. We state them here in a different form which emphasizes the relation to the pattern of the unitary group U(k) which is the famous triangle [2]

\[ \begin{array}{cccccc}
  x^{(0)}_1 & x^{(0)}_2 & \cdots & x^{(0)}_k & x^{(0)}_{k+1} \\
  x^{(1)}_1 & x^{(1)}_2 & \cdots & x^{(1)}_{k-2} & x^{(1)}_k \\
  & x^{(1)}_{k-2} & \cdots & x^{(1)}_k & x^{(1)}_{k+1} \\
  & & \vdots & & \\
  x^{(k-1)}_1 & x^{(k-1)}_2 & \cdots & x^{(k-1)}_k & x^{(k-1)}_{k+1} \\
  & x^{(k-1)}_1 & \cdots & x^{(k-1)}_k & x^{(k-1)}_{k+1} \\
  & & & \vdots & \\
  x^{(k)}_1 & x^{(k)}_2 & \cdots & x^{(k)}_k & x^{(k)}_{k+1} \\
 \end{array} \]  

(3.2)

with the betweeness conditions

\[ x^{(j-1)}_{i+1} \leq x^{(j)}_i \leq x^{(j-1)}_i. \]  

(3.3)

The first row in the pattern (3.2) labels the orbit which was used as the starting point for the construction of the parametrization. We underline them to distinguish them from the coordinates of the group. From this pattern the pattern of the orthogonal group can be derived by the substitution rule (2.37), i.e. by assigning to the Cartan parametrization. We underline them to distinguish them from the coordinates of the group. From this pattern the pattern (3.2) acquires the form

\[ \begin{array}{cccccc}
  +x^{(0)}_1 & +x^{(0)}_2 & \cdots & +x^{(0)}_k & -x^{(0)}_{k+1} & -x^{(0)}_1 \\
  x^{(1)}_1 & x^{(1)}_2 & \cdots & x^{(1)}_k & -x^{(1)}_{k+1} & -x^{(1)}_1 \\
  & x^{(1)}_k & \cdots & x^{(1)}_2 & -x^{(1)}_2 & -x^{(1)}_1 \\
  & & \vdots & & & \\
  & x^{(k-1)}_k & \cdots & x^{(k-1)}_2 & -x^{(k-1)}_2 & -x^{(k-1)}_1 \\
  x^{(k)}_1 & x^{(k)}_2 & \cdots & x^{(k)}_k & -x^{(k)}_{k+1} & -x^{(k)}_1 \\
 \end{array} \]  

(3.4)

with the betweeness conditions

\[ x^{(j-1)}_{i+1} \leq x^{(j)}_i \leq x^{(j-1)}_i. \]  

(3.3)

We notice the symmetry along the middle axis. The variable space of the SO(2k + 2) is already covered by the left half of the triangle. The other half is shown to indicate its relation to the unitary case (3.2). Physically, this symmetry is due to time reversal invariance: A system which is not invariant under time reversal is modelled by hermitean operators. One can go to a time reversal invariant system by replacing these operators with real symmetric ones. Restricting the pattern (3.2) to the left half of the triangle, the patterns appear in their traditional form [2]. By construction, the pattern of the unitary symplectic group USp(2k) coincides with the one of the unitary group U(k), this is due to the fact that only the action variables are used in the pattern. The unitary group has only one, but the unitary symplectic group has three angle variables coming with every action.

The two patterns of the orthogonal and the unitary symplectic groups together represent the subgroup SO(k1) ⊗ USp(2k2) of UOSp(k1/2k2). What represents the coset UOSp(k1/2k2)/(SO(k1) ⊗ USp(2k2)) ? – We observe that the lengths squared |ξ_{p}^{(n)}|^2 of the anticommuting variables ξ_{p}^{(n)} introduced in Eq. (2.33) have a distinguished meaning. We may identify these lengths of the anticommuting variables as the analogues of the actions stemming from the commuting degrees of freedom. We can organize the lengths squared |ξ_{p}^{(n)}|^2 in a rectangular pattern. Thus, the
generalized Gelfand pattern for the unitary orthosymplectic supergroup UOSp($k_1/2k_2$) obtains,

\[
\begin{array}{cccccc}
+s_{11}^{(0)} & +s_{21}^{(0)} & \ldots & +s_{k1}^{(0)} & -s_{k1+1}^{(0)} & \ldots & -s_{21}^{(0)} & -s_{11}^{(0)} \\
^s_{11}^{(1)} & s_{21}^{(1)} & \ldots & s_{k1}^{(1)} & 0 & \ldots & -s_{21}^{(1)} & -s_{11}^{(1)} \\
-s_{11}^{(2k-2)} & s_{21}^{(2k-2)} & \ldots & -s_{k1}^{(2k-2)} & 0 & \ldots & -s_{11}^{(2k-2)} & -s_{11}^{(2k-2)} \\
|\xi_1^{(1)}|^2 & |\xi_2^{(1)}|^2 & \ldots & |\xi_{k2-1}^{(1)}|^2 & |\xi_{k2}^{(1)}|^2 & \ldots & |\xi_{k2-1}^{(2)}|^2 & |\xi_{k2}^{(2)}|^2 \\
|\xi_1^{(2)}|^2 & |\xi_2^{(2)}|^2 & \ldots & |\xi_{k2-1}^{(2)}|^2 & |\xi_{k2}^{(2)}|^2 & \ldots & |\xi_{k2-1}^{(1)}|^2 & |\xi_{k2}^{(1)}|^2 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
|\xi_1^{(k_1)}|^2 & |\xi_2^{(k_1)}|^2 & \ldots & |\xi_{k2-1}^{(k_1)}|^2 & |\xi_{k2}^{(k_1)}|^2 & \ldots & |\xi_{k2-1}^{(k_2)}|^2 & |\xi_{k2}^{(k_2)}|^2 \\
\end{array}
\]

(3.6)

with the betweeness conditions

\[
\begin{align*}
& s_{i1}^{(m-1)} \leq s_{i1}^{(m)} \leq s_{i1}^{(m-1)} \\
& s_{i1}^{(k_1+l)} \leq s_{i2}^{(k_1+l+1)} \leq s_{i2}^{(k_1+l)} \\
& -s_{j1}^{(k_1-2j-1)} \leq s_{j1}^{(k_1-2j)} \leq s_{j1}^{(k_1-2j-1)}
\end{align*}
\]

(3.7)

where $1 \leq j \leq k_1/2-1$, $1 \leq m \leq k_1-2$ and $0 \leq l \leq k_2-1$. It was shown in Ref. 22 that the unitary supergroup U(1/1) can be represented by supersymmetric generalizations of Wigner functions. This representation of the supergroup U(1/1) is labelled by the length of an anticommuting variable. Therefore, we want to interpret the generalized Gelfand pattern (3.6) as labelling another kind of representation which involves anticommuting variables as labels. The two triangles label the basis of an irreducible representation of the product SO($k_1 \otimes USp(2k_2)$), whereas the remaining coset UOSp($k_1/2k_2$)/SO($k_1 \otimes USp(2k_2)$) is represented by the rectangular block of the lengths squared of anticommuting variables. This extends the corresponding considerations for the unitary supergroup in Ref. 1. It is challenging to find a further interpretation of these new representations of supergroups, possibly by generalizing the Guillemin–Sternberg theory.

IV. SUMMARY AND CONCLUSIONS

We constructed Gelfand–Tzetlin coordinates for the unitary orthosymplectic supergroup UOSp($k_1/2k_2$). To this end, we further extended the construction for the unitary supergroup U($k_1/k_2$). We obtained **angular** Gelfand–Tzetlin coordinates, which always live in the space of the unitary orthosymplectic supergroup. They ought to be distinguished from **radial** Gelfand–Tzetlin which map group degrees of freedom onto those of another space. We also calculated the invariant Haar measure on UOSp($k_1/2k_2$) and obtained an expression that is fairly simple due to the recursive structure of the coordinates. As the orthogonal and the unitary symplectic groups are subgroups of the unitary orthosymplectic supergroup, our construction also includes **angular** Gelfand–Tzetlin coordinates on these ordinary groups.

The Gelfand–Tzetlin coordinates can be arranged in a generalized Gelfand pattern. A remarkable feature of this pattern is the appearance of moduli squared of anticommuting variables. We argued that an interpretation of these anticommuting variables as eigenvalues of a set of invariant operators is likely to exist. It is an interesting task to clarify the rôle of these anticommuting variables in the representation theory for supergroups.

So far, Gelfand–Tzetlin coordinates were only constructed for compact groups. But there is no apparent obstacle to construct them also for non–compact groups. It would be interesting to see if such a construction is indeed possible.
and how the non–compactness of some variables is reflected in the corresponding Gelfand pattern, in ordinary and in superspace.

Acknowledgments

TG and HK acknowledge financial support from the Swedish Research Council and from the RNT Network of the European Union with Grant No. HPRN–CT–2000-00144, respectively. HK also thanks the division of Mathematical Physics, LTH, for its hospitality during his visit to Lund.

APPENDIX A: SOLUTION OF Eqs. (2.28) TO (2.30)

We consider Eq. (2.28) and insert the solutions for $|\alpha_m(n)|^2$ and $|\alpha_m(n)|^2$ given in (2.31). The right hand side of Eq. (2.28) can then be expanded in a sum of monomials in the nilpotent Gelfand–Tzetlin variables $|\xi_q^{(n)}|^2$, $q = 1, \ldots, k_2$. Since each of the $|\xi_q^{(n)}|^2$ only appears linearly, the rank of the monomials cannot exceed $k_2$. Thus, we can rewrite Eq. (2.28) in the form

$$\frac{1}{|w_p^{(n)}|^2} = \sum_{r=0}^{k_2} M(r),$$  \hspace{1cm} (A1)

Where $M(r)$ is the nilpotent part of $1/|w_p^{(n)}|^2$, consisting of monomials in $|\xi_q^{(n)}|^2$ with rank $r$. Explicitly we have

$$M(r) = \sum_{j_1 \leq j_2 \leq \ldots \leq j_r} \sum_{m} \frac{\prod_{q=1, q \neq m}^{k_1-1} (s_{m1}^{-(n-1)})^2 - (s_{q1}^{(n)})^2}{\prod_{q=1}^{k_1-1} (s_{m1}^{-(n-1)})^2 - (s_{q1}^{(n)})^2} \frac{\prod_{i=1}^{r} |\xi_{j_i}|^2}{\prod_{i=1}^{r} (s_{m1}^{-(n-1)})^2 - (s_{j_i1}^{(n)})^2} \prod_{j_i \neq j_1} \prod_{i=1}^{r} (s_{m1}^{-(n-1)})^2 - (s_{j_i1}^{(n)})^2 \prod_{i=1}^{r} |\xi_{j_i}|^2, \hspace{1cm} (A2)$$

for $r = 1, \ldots, k_2$. The sum over $m$ is the Laplace expansion of a determinant. For its evaluation we use the formula

$$\frac{1}{\prod_{i=1}^{r} (s_{j_i1}^{(n-1)})^2 - (s_{j_i2}^{(n)})^2} = \sum_{r=1}^{k_2} \frac{1}{\prod_{i=1}^{r} (s_{j_i1}^{(n-1)})^2 - (s_{j_i2}^{(n)})^2} \prod_{i=1}^{r} \prod_{i=1}^{r} |\xi_{j_i}|^2, \hspace{1cm} (A3)$$

which is well known from complex analysis. After symmetrizing the second sum in the indices, $j_i, i = 1, \ldots, r$, we arrive at the following expression for $M(r)$.

$$M(r) = \sum_{j_1 \leq j_2 \leq \ldots \leq j_r} \sum_{i=1}^{r} \prod_{i \neq i}^{r} \frac{1}{(s_{j_i2}^{(n-1)})^2 - (s_{j_i1}^{(n)})^2} \frac{\prod_{q=1, q \neq m}^{k_1-1} (s_{m1}^{-(n-1)})^2 - (s_{q1}^{(n)})^2}{\prod_{q=1}^{k_1-1} (s_{m1}^{-(n-1)})^2 - (s_{q1}^{(n)})^2} \prod_{i=1}^{r} |\xi_{j_i}|^2$$

$$+ \sum_{j_1 \leq j_2 \leq \ldots \leq j_r} \sum_{i=1}^{r} \prod_{i \neq i}^{r} \frac{1}{(s_{j_i2}^{(n-1)})^2 - (s_{j_i1}^{(n)})^2} \frac{\prod_{q=1, q \neq m}^{k_1-1} (s_{m1}^{-(n-1)})^2 - (s_{q1}^{(n)})^2}{\prod_{q=1}^{k_1-1} (s_{m1}^{-(n-1)})^2 - (s_{q1}^{(n)})^2} \prod_{i=1}^{r} |\xi_{j_i}|^2. \hspace{1cm} (A4)$$
Now the determinant mentioned above can be evaluated by using the translational invariance of the differences. The second term in the squared bracket cancels completely, we are left with

\[
M(r) = \sum_{j_1 \leq j_2 \leq \ldots \leq j_r} \prod_{i=1}^{k_2} (is_{j_i,2}^{(n-1)})^2 - (is_{j_i,2}^{(n-1)})^2 \overline{2(s_{p_1}^{(n)})^2 \prod_{q=1, q \neq p}^{k_1-1-n} (s_{q_1}^{(n)})^2 - (s_{q_1}^{(n)})^2} \prod_{m=1}^{k_1-n-1} (s_{m_1}^{(n-1)})^2 - (s_{m_1}^{(n-1)})^2 \prod_{i=1}^{r} |\xi_{j_i}|^2 .
\] (A5)

Using identity (A3) once more and summing over \(r\) gives

\[
\frac{1}{|w_p^{(n)}|^2} = -2(s_{p_1}^{(n)})^2 \prod_{q=1, q \neq p}^{k_1-1-n} (s_{q_1}^{(n)})^2 - (s_{q_1}^{(n)})^2 \left( \sum_{r=0}^{k_2} \sum_{j_1 \leq j_2 \leq \ldots \leq j_r} \prod_{i=1}^{r} (s_{j_i}^{(n)})^2 - (is_{j_i,2}^{(n-1)})^2 \right) .
\] (A6)

The double sum in Eq. (A6) simply amounts to

\[
\prod_{q=1}^{k_2} \left( 1 + \frac{|\xi_{q_1}^{(n)}|^2}{(s_{q_1}^{(n)})^2 - (is_{q_2}^{(n-1)})^2} \right) .
\] (A7)

Employing the definition of \(|\xi_{q_1}^{(n)}|^2\), we arrive at the final result for \(|w_p^{(n)}|^2\) in Eq. (2.32). Equation (2.29) and the corresponding equation for the odd levels are evaluated similarly, yielding the results stated in Sec. 1D. Equation (2.30) has to be treated differently due to the Grassmann singularities, occurring on the left hand side. Inserting the expressions Eq. (2.31) into Eq. (2.30) we have

\[
1 = |\beta_p^{(n)}|^2 \left( \sum_{m} \frac{(s_{m_1}^{(n-1)})^2 + (is_{m_2}^{(n-1)})^2}{(s_{m_1}^{(n-1)})^2 - (is_{m_2}^{(n-1)})^2} |\alpha_{m_1}^{(n)}|^2 + \sum_{m_1=1}^{k_1-n-1} \frac{(s_{m_1}^{(n-1)}) - (is_{m_2}^{(n-1)})^2}{(s_{m_1}^{(n-1)})^2 - (is_{m_2}^{(n-1)})^2} |\alpha_{m_1}^{(n)}|^2 \right) \\
+ \prod_{q=1, q \neq p}^{k_1-1-n} \left( (is_{p_1}^{(n-1)})^2 - (is_{q_2}^{(n-1)})^2 \right) \prod_{q=1, q \neq p}^{k_1-1-n} \left( (is_{p_1}^{(n-1)})^2 - (is_{q_2}^{(n-1)})^2 \right) \left( (is_{p_1}^{(n-1)})^2 + (is_{p_2}^{(n-1)})^2 \right) \left( (is_{p_1}^{(n-1)})^2 + (is_{p_2}^{(n-1)})^2 \right) .
\] (A8)

To cancel the singularity, \(|\beta_p^{(n)}|^2\) has to be expanded in terms of \(c_p^{(n)}|\xi_p^{(n)}|^2\). The expansion coefficient \(c_p^{(n)}\) now contains a nonzero part and its inverse is therefore well defined. Dividing both sides by \(c_p^{(n)}\) and ordering the right hand side by powers of \(|\xi_p^{(n)}|^2\), one finds

\[
\frac{1}{c_p^{(n)}} = 2(is_{p_2}^{(n-1)})^2 \prod_{q=1, q \neq p}^{k_1-1-n} \left( (is_{p_1}^{(n-1)})^2 - (is_{q_2}^{(n-1)})^2 \right) \prod_{q=1, q \neq p}^{k_2} \left( (is_{p_1}^{(n-1)})^2 - (is_{q_2}^{(n-1)})^2 \right) \\
\left( \sum_{m} \frac{(s_{m_1}^{(n-1)})^2 + (is_{m_2}^{(n-1)})^2}{(s_{m_1}^{(n-1)})^2 - (is_{m_2}^{(n-1)})^2} |\alpha_{m_1}^{(n)}|^2 + \sum_{m_1=1}^{k_1-n-1} \frac{(s_{m_1}^{(n-1)}) - (is_{m_2}^{(n-1)})^2}{(s_{m_1}^{(n-1)})^2 - (is_{m_2}^{(n-1)})^2} |\alpha_{m_1}^{(n)}|^2 \right) \\
- \prod_{q=1, q \neq p}^{k_1-1-n} \left( (is_{p_1}^{(n-1)})^2 - (is_{q_2}^{(n-1)})^2 \right) \prod_{q=1, q \neq p}^{k_2} \left( (is_{p_1}^{(n-1)})^2 - (is_{q_2}^{(n-1)})^2 \right) \left( (is_{p_1}^{(n-1)})^2 - (is_{q_2}^{(n-1)})^2 \right) |\xi_p^{(n)}|^2 .
\] (A9)

Since \(c_p^{(n)}\) and thus \(1/c_p^{(n)}\) are of order zero in \(|\xi_p^{(n)}|^2\), the whole term in round brackets can be neglected. It can be shown by straightforward manipulations that this term leads just to a shift of \((is_{p_2}^{(n-1)})^2 \rightarrow (is_{p_2}^{(n)})^2\) in the resulting expression for \(c_p^{(n)}\). This does not affect \(|\beta_p^{(n)}|^2\). Hence, we immediately arrive at the result for \(|\beta_p^{(n)}|^2\) given in Eq. (2.32). The equations for the odd levels are treated accordingly.
APPENDIX B: REAL FORM OF THE PROJECTION MATRICES $\hat{b}^{(n)}$

We restrict ourselves to the case $n \leq k_1$, $(k_1 - n + 1)$ even. The odd case can be treated accordingly. The rectangular $(k_1 - n + 1 + k_2) \times (k_1 - n + k_2)$ matrix $\hat{b}^{(n)\mathsf{T}}$ can schematically be written as

$$
\hat{b}^{(n)} = \begin{bmatrix}
\hat{b}^{(n)}_{11} & \hat{b}^{(n)}_{12} \\
\hat{b}^{(n)}_{21} & \hat{b}^{(n)}_{22}
\end{bmatrix}.
$$

(B1)

Here, $\hat{b}^{(n)}_{11}$ is a $(k_1 - n + 1)/2 \times (k_1 - n - 1)/2$ matrix with entries

$$
(\hat{b}^{(n)}_{11})_{ij} = \sqrt{\frac{v_j^{(n)}}{(s_{11}^{(n-1)})^2 - (s_{12}^{(n)})^2}} \begin{bmatrix}
\beta_j^{(n)} \cos \theta_i^{(n)} + s_j^{(n-1)} \sin \theta_i^{(n)} \\
-\beta_j^{(n)} \cos \theta_i^{(n)} - s_j^{(n-1)} \sin \theta_i^{(n)}
\end{bmatrix}.
$$

(B2)

The matrix $\hat{b}^{(n)}_{12}$ has dimension $(k_1 - n + 1)/2 \times k_2$ and entries

$$
(\hat{b}^{(n)}_{12})_{ij} = \sqrt{\frac{|v_i^{(n)}|}{(s_{11}^{(n-1)})^2 - (s_{12}^{(n)})^2}} \begin{bmatrix}
\alpha_i^{(n)} s_j^{(n)} & i \alpha_i^{(n)} s_j^{(n)} \\
-i \alpha_i^{(n)} s_j^{(n)} & \alpha_i^{(n)*} s_j^{(n)}
\end{bmatrix}.
$$

(B3)

Moreover, $\hat{b}^{(n)}_{21}$ is a $k_2 \times (k_1 - n + 1)/2$ matrix with entries

$$
(\hat{b}^{(n)}_{21})_{ij} = \frac{|w_j^{(n)}|}{(s_{12}^{(n-1)})^2 - (s_{11}^{(n)})^2} \begin{bmatrix}
\alpha_i^{(n)*} s_j^{(n-1)} & i \alpha_i^{(n)*} s_j^{(n)} \\
-i \alpha_i^{(n)*} s_j^{(n-1)} & \alpha_i^{(n)} s_j^{(n-1)}
\end{bmatrix},
$$

(B4)

and $\hat{b}^{(n)}_{22}$ is a $k_2 \times (k_1 - n + 1)/2$ matrix with entries

$$
(\hat{b}^{(n)}_{22})_{ij} = \sqrt{2} \begin{bmatrix}
\alpha_i^{(n)} s_j^{(n)} / (is_{12}^{(n-1)} - is_j^{(n-1)}) & \alpha_i^{(n)*} s_j^{(n-1)} / (is_{12}^{(n-1)} + is_j^{(n-1)}) \\
-\alpha_i^{(n)*} s_j^{(n)} / (is_{12}^{(n-1)} + is_j^{(n-1)}) & -\alpha_i^{(n)} s_j^{(n-1)} / (is_{12}^{(n-1)} - is_j^{(n-1)})
\end{bmatrix}.
$$

(B5)

Finally, the entries of $\hat{b}^{(n)}_1$ and $\hat{b}^{(n)}_2$ are given by

$$
(\hat{b}^{(n)}_1)_i = \sqrt{2} \frac{|v_i^{(n)}| |w_i^{(n)-n+1}|}{s_{11}^{(n-1)}} \begin{bmatrix}
\sin \theta_i^{(n)} \\
\cos \theta_i^{(n)}
\end{bmatrix}, \quad i = 1, \ldots, k_1 - n + 1/2,
$$

$$
(\hat{b}^{(n)}_2)_i = \frac{1}{\sqrt{2}} \frac{|w_i^{(n)-n+1}|}{s_{11}^{(n-1)}} \begin{bmatrix}
i \alpha_i^{(n)} \\
-i \alpha_i^{(n)*}
\end{bmatrix}, \quad i = 1, \ldots, k_2.
$$

(B6)

We notice that all elements of $\hat{b}^{(n)}$ are real.

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