Abstract: Four-qubit cluster states of two photons entangled in polarization and linear momentum have been used to realize a complete set of single qubit rotations and the C-NOT gate for equatorial qubits with high values of fidelity. By the computational equivalence of the two degrees of freedom our result demonstrate the suitability of two photon cluster states for rapid and efficient one-way quantum computing.

Measurement pattern for single qubit rotations (a) and C-NOT gate (b) realized in the one-way model of quantum computation

One-way quantum computation via manipulation of polarization and momentum qubits in two-photon cluster states

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1. Introduction

The relevance of cluster states in quantum information and quantum computation (QC) has been emphasized in several papers in recent years [1–4]. By these states novel significant tests of quantum nonlocality, which are more resistant to noise and show significantly larger deviations from classical bounds can be realized [5–7,2]. Besides that, cluster states represent today the basic resource for the realization of a quantum computer operating in the one-way model [1]. In the standard QC approach any quantum algorithm can be realized by a sequence of single qubit rotations and two qubit gates, such as C-NOT and C-Phase [8]. A deterministic one-way QC is based on the initial preparation of entangled qubits in a cluster state, followed by a temporally ordered pattern of single qubit measurements and feed-forward operations. Indeed these operations, depending on the outcome of the measured qubits [1], correspond either to intermediate feed-forward measurements or to Pauli matrix feed-forward corrections on the final output state. Two qubit gates can be realized by exploiting the existing entanglement between qubits. In this way the difficulties of standard QC, related to the implementation of two qubit gates, are transferred to the preparation of the state.

Concerning the experimental realizations, complete one-way QC has been realized by 4-photon cluster states [3,4] and, more recently, the first proof of principle has been given by employing 2-photons entangled in 4-qubit cluster states [9].

The detection rate in such experiments, approximately 1 Hz, is limited by the fact that four photon events in a standard spontaneous parametric down conversion (SPDC) process are rare. Moreover, four-photon clus-
ter states are characterized by limited values of fidelity, while efficient computation requires highly faithful prepared states.

Cluster states at high level of brightness and fidelity were realized by entangling two photons in more degrees of freedom [7]. Precisely, this was demonstrated by entangling the polarization (π) and linear momentum (k) degrees of freedom of one of the two photons belonging to a hyperentangled state [10,11]. The high fidelity and detection rate of the prepared two-photon four-qubit cluster states make them suitable for the realization of high speed one-way QC.

In this letter we present the realization of arbitrary single qubit rotations and of the C-NOT gate for equatorial qubits in the one-way model of quantum computation, obtained by using the four qubits two photon cluster states. We have also verified the equivalence existing between the four degrees of freedom for qubit rotations, by using either k or π as output qubit, demonstrating that all four qubits can be adopted for computational applications.

2. Experimental setup

Entanglement of two photons generated by Spontaneous Parametric Down Conversion (SPDC) [12–14] in more than one degree of freedom allows the encoding of more qubits in the same particle. By the source given in Fig. 1, we can generate the polarization-momentum hyperentangled states:

\[ |\Xi^{\pm\pm}\rangle = |\Phi^{\pm\pm}\rangle_\pi \otimes |\psi^{\pm\pm}\rangle_k, \]

where we have defined the polarization and momentum Bell states:

\[ |\Phi^{\pm\pm}\rangle_\pi = \frac{1}{\sqrt{2}} (|H\rangle_A|H\rangle_B \pm |V\rangle_A|V\rangle_B), \]

\[ |\psi^{\pm\pm}\rangle_k = \frac{1}{\sqrt{2}} (|V\rangle_A|r\rangle_B \pm |r\rangle_A|V\rangle_B). \]

In the above expressions |H⟩, |V⟩ respectively correspond to the horizontal and vertical polarization states, while |r⟩, |ℓ⟩ refer to the left or right path states of the photon A (Alice) or B (Bob) (cfr. Fig. 1) [10,15]. The state |Ξ⟩ is created in a thin Type I β-barium-borate BBO crystal operating under double (back and forth) excitation of a cw Argon laser (λp=364 nm) and selecting two pairs of correlated k modes at the degenerate wavelength \( \lambda = 2\lambda_p \) within the conical emission of the crystal. For a detailed description of the source we refer to the caption of Fig. 1 and to previous published papers [10,15].

The cluster state

\[ |C_4\rangle = \frac{1}{2} (|H\ell\rangle_A|Hr\rangle_B - |Hr\rangle_A|H\ell\rangle_B + |V\ell\rangle_A|Vr\rangle_B + |Vr\rangle_A|V\ell\rangle_B) \]

is created by starting from the state

\[ |\Xi^{+-}\rangle = |\Phi^{+-}\rangle_\pi \otimes |\psi^{+-}\rangle_k \]

and introducing a π phase shift in one of the four output modes of the SPDC source \(^1\). Precisely, a zero-order half

\(^1\) The state (3) is equivalent to that generated in [7] up to single qubit transformations.
wave (HW) plate with the optical axis oriented along the vertical direction is inserted on the $r_A$ mode. The operation represented by the insertion of the HW plate in the Alice’s side corresponds to a Controlled Phase (CP) between the control ($k_A$) and the target ($\pi_A$) qubits, and creates a genuine four-partite entanglement, without any kind of post-selection. Cluster states were observed at 1 kHz detection rate with fidelity $F = 0.880 \pm 0.013$, obtained from the measurement of the whole set of observables associated to the 16 stabilizer operators of $|C_4\rangle$, i.e. operators whose eigenstate is $|C_4\rangle$. They can be evaluated by local measurements [2].

By using the correspondence $|H\rangle \leftrightarrow |0\rangle, |V\rangle \leftrightarrow |1\rangle, |\ell\rangle \leftrightarrow |0\rangle, |r\rangle \leftrightarrow |1\rangle$ \footnote{States $|0\rangle$ and $|1\rangle$ represent the usual computational (orthogonal) qubit basis.}, the generated state $|C_4\rangle$ is equivalent to the cluster state \footnote{The state $|\Phi_A^{lin}\rangle$ is obtained by preparing a chain of qubits all prepared in the state $|+\rangle$ and then applying the gate $CP = |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes \sigma_\pi$ for each link.}

$$|\Phi_A^{lin}\rangle = \frac{1}{2} \left((+)_1|0\rangle_2|0\rangle_3|+\rangle_4 + (+)_1|0\rangle_2|1\rangle_3|-\rangle_4 +
-|+\rangle_1|1\rangle_2|0\rangle_3|+\rangle_4 - |−\rangle_1|1\rangle_2|1\rangle_3|-\rangle_4\right)$$

with

$$|\pm\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle \pm |1\rangle\right)$$

up to single qubit unitaries:

$$|C_4\rangle = U_1 \otimes U_2 \otimes U_3 \otimes U_4 |\Phi_A^{lin}\rangle \equiv U|\Phi_A^{lin}\rangle.$$ (4)

Here $|\Phi_A^{lin}\rangle$ and $|C_4\rangle$ are respectively expressed in the so-called “computational” and “laboratory” basis, while the $U_j$'s ($j = 1, \ldots, 4$) are products of Hadamard gates $H = (\sigma_x + \sigma_z)/\sqrt{2}$ and Pauli matrices $\sigma_i$. Their explicit expressions depend on the ordering of the four physical qubits, namely $k_A, k_B, \pi_A, \pi_B$. In this work we use three different ordering (we also report the corresponding operators $U_1, \ldots, U_4$):

$$U = U_1 \otimes U_2 \otimes U_3 \otimes U_4,$$

\text{a) (1, 2, 3, 4) = (k_B, k_A, \pi_A, \pi_B),}

$$U = \sigma_\pi H \otimes \sigma_x \otimes I \otimes H,$$

\text{b) (1, 2, 3, 4) = (\pi_B, \pi_A, k_A, k_B),}

$$U = H \otimes \sigma_\pi \otimes \sigma_x \otimes \sigma_z H,$$

\text{c) (1, 2, 3, 4) = (k_A, k_B, \pi_B, \pi_A),}

$$U = \sigma_\pi H \otimes \sigma_x \otimes I \otimes H.$$

In the following we refer to these expressions depending on the logical operation we will consider.

The measurement apparatus is sketched in Fig. 2. The k modes corresponding to photons A or B, are respectively matched on the up and down side of a common symmetric beam splitter (BS) (see inset), which can be also finely moved in the vertical direction such that one or both photons don’t pass through it.

Polarization analyses are performed by standard π tomographic apparatus ($\lambda/4$, $\lambda/2$ and polarizing beam splitter PBS). Two thin glass plates before the BS allow to set the basis of momentum measurement for each photon.

Radiation belonging to each output mode, filtered by a $\Delta \lambda = 6$ nm interference filter, is detected by a single photon counting module (SPCM) AQR14 (Pelkin Elmer).

3. Realization of one-way algorithms

In this section we will describe two algorithms realized in the framework of one-way quantum computation, namely the single qubit rotations and the C-NOT gate for equatorial target qubit. These two algorithms require the four qubit cluster state described in the previous section.

3.1. Single qubit rotations

In the one-way model a three-qubit linear cluster state is sufficient to realize arbitrary single qubit rotations [16, 17]. Starting from the four qubit linear cluster, this operation can be implemented through three different steps and single qubit measurements. According to the measurement basis for a generic qubit $j$,

$$|\varphi_{\pm}\rangle_j = \frac{1}{\sqrt{2}} \left(|0\rangle_j \pm \exp\{ -i \varphi \} |1\rangle_j\right),$$

we define $s_j = 0$ ($s_j = 1$), when the $|\varphi_{+}\rangle_j$ ($|\varphi_{-}\rangle_j$) outcome is obtained. The parameter $s_j$ here introduced identifies the two possible outcomes that can be obtained in each
single qubit measurement. In each step, the choice of the measurement basis is fixed by the values of the previously obtained $s_j$’s. Moreover the whole set of $s_j$’s determines the necessary Pauli error corrections to be implemented at the end of the process for deterministic one-way QC (see the following for details).

With the four-qubit cluster expressed in the computational basis the following procedure must be performed (see Fig. 3a):

I: A three-qubit linear cluster is generated by measuring the first qubit in the basis $\{0\}_1, \{1\}_1$. The input logical qubit $|\chi_{in}\rangle$ is then encoded in qubit 2. If the outcome of the first measurement is $\{0\}_1$ then $|\chi_{in}\rangle = |+\rangle$, otherwise $|\chi_{in}\rangle = |\rangle$.

II: Measuring qubit 2 in the basis $|\alpha\beta\rangle_2$, with $\alpha$ corresponding to a particular value of $\varphi$, the computational qubit (now encoded in qubit 3) is transformed into $|\chi'_\varphi\rangle = (\sigma_3)^{s_2}HR_z(\alpha)|\chi_{in}\rangle$, with $HR_z(\alpha) = \exp\{-i(\pi/2)\alpha \sigma_z\}$.

III: Measurement of qubit 3 in the basis $|\beta\rangle_3$ (if $s_2 = 0$) or $|\pm\rangle_3$ (if $s_2 = 1$) (also in this case $\beta$ corresponds to a particular value of $\varphi$) leaves the last qubit in the state $|\chi_{out}\rangle = (\sigma_3)^{s_3}HR_z(-1)^{s_2}\beta)|\chi'_\varphi\rangle = \sigma_3^{s_3}\sigma_z^{s_2}\beta R_z(\beta)|\chi_{in}\rangle$,

with $R_z(\beta) = \exp\{-i(\pi/2)\beta \sigma_z\}$.

In this way, by suitable choosing the values of $\alpha$ and $\beta$, we can perform any arbitrary single qubit rotation $|\chi_{in}\rangle \rightarrow R_z(\beta)R_z(\alpha)|\chi_{in}\rangle$ up to Pauli errors $(\sigma_3^{s_3}\sigma_z^{s_2})$, that can be corrected by proper feed-forward operations [4]. In our case we applied this procedure by considering as output qubit either the polarization or momentum of photon $B$, demonstrating the QC equivalence of the two degrees of freedom. This corresponds to respectively choose the order a) or b) for the physical qubits.

Concerning the measurement apparatus, two HW oriented at 22.5° (Hd and Hb in Fig. 2) are inserted in photon A modes. They will be used together with the $\lambda/4$’s in order to transform the $\{|\varphi\rangle_\pi A, |\varphi^-\rangle_\pi A\}$ states into linearly polarized states.

Let’s consider ordering a). The output state, encoded in the polarization of photon $B$, can be written in the laboratory basis as

\[ |\chi_{out}\rangle_{\pi B} = (\sigma_3)^{s_3}(\sigma_x)^{s_2}HR_z(\beta)R_z(\alpha)|\chi_{in}\rangle, \]

where the $H$ gate derives from the change between the computational and laboratory basis. This also implies that the actual measurement bases are $|\pm\rangle_{k_B}$ for the momentum of photon $B$ and $|\alpha z\rangle_{k_A}$ for the momentum of photon A. The measurement basis on the third qubit ($\pi_{k_A}$) depends, according to the one-way model, on the results of the measurement on the second qubit ($k_B$). Note that in our scheme this simply corresponds to measure it in the bases $|\beta\rangle_{\pi A}$ or $|\pm\rangle_{\pi A}$ depending on the BS output mode (i.e. $s_2 = 0$ or $s_2 = 1$). This is a direct consequence of the possibility to encode two qubits ($k_A$ and $\pi_A$) in the same photon. As a consequence, differently from the case of four-photon cluster states, in this case active feed-forward measurements (e.g. adopting Pockels cells) are
not required, while Pauli errors corrections are in any case necessary for deterministic QC.

In Fig. 4a the results obtained in the case $s_2 = s_3 = 0$ (i.e. when the computation proceeds without errors) with $|\chi_{in}\rangle = |+\rangle$ are given. We report on the Bloch sphere the experimental qubits and their projections on the theoretical state $HR_z(\beta)R_x(\alpha)|+\rangle$ for different values of $\alpha$ and $\beta$. The corresponding fidelities are given in Table 1. We also performed the tomographic analysis on the output qubit $\pi_B$ for all the possible combinations of $s_2$ and $s_3$ and for input qubit $|\chi_{in}\rangle = |+\rangle$. In all the cases we obtained an average value of fidelity $F \approx 0.9$. As an example, we show in Fig. 4b the case $s_2 = 1$, $s_3 = 0$ (i.e. $|\chi_{in}\rangle = |+\rangle$). The high values of the fidelity obtained in these measurements represent the necessary condition to implement efficient active feed-forward corrections.

By considering ordering b) the same computation can be performed with output state $|\chi_{out}\rangle_k{B}$ encoded in the linear momentum of photon B, whose explicit expression in the laboratory basis is

$$|\chi_{out}\rangle_k{B} = (\sigma_2)^{s_2}(\sigma_3)^{s_3}HR_z(\beta)R_x(\alpha)|\chi_{in}\rangle.$$  

By the apparatus of Fig. 2 we measured $|\chi_{out}\rangle_k{B}$ by choosing different values of $\alpha$ (corresponding in the laboratory to the polarization basis $|\alpha{+}\rangle_{\pi_A}$) and $\beta = 0$ (corresponding in the laboratory to the momentum basis $|\pm\rangle_k{B}$). While the first qubit ($\pi_B$) was always measured in the basis $|\pm\rangle_{\pi_B}$. In this case only $H_A$ and $H_{B}$ Hadamard gates are inserted. The complete single qubit tomography on $k{B}$ requires the measurement of the $\sigma_x, \sigma_y$ operators, by proper setting of phase $\phi_B$, and $\sigma_z$, performed by removing the BS on photon B.

Fig. 4c shows the results obtained for different output qubits, for $s_2 = s_3 = 0$ and $|\chi_{in}\rangle = |+\rangle$. The corresponding fidelities are given Table 1. Also in this case we performed the $k{B}$ tomographic analysis for all the possible values of $s_2$ and $s_3$ and of the input qubit, obtaining in average $F > 0.9$. The case $s_2 = 0, s_3 = 1$ is shown in Fig. 4d with fidelities given in Table 1.

### 3.2. C-NOT gate for equatorial qubits

Nontrivial two-qubit operations, such as the C-NOT gate, can be realized by the four-qubit horeseshoe (180° rotated) cluster state (see Fig. 3b), whose explicit expression is equal to $|\phi^4_{in}\rangle$. By simultaneously measuring qubits 1 and 4, it’s possible to implement the logical circuit shown in Fig. 3b. In the computational basis the input state is $|\pm\rangle_c \otimes |+\rangle_t$ ($c =$control, $t =$target), while the output state, encoded in qubits 2 (control) and 3 (target), is $|\Psi_{out}\rangle = H_{C-NOT}(O)|\pm\rangle_c \otimes R_z(\alpha)|\pm\rangle_t$ (for $s_1 = s_4 = 0$). In the above expression we have $O = \mathbb{I}$ ($O = H_B$) when qubit 1 is measured in the basis $\{|0\rangle, |1\rangle\}$, while $O = \pi_B$. Qubit 4 is measured in the basis $\{|\alpha{+}\rangle_{\pi_B}\}$. It is worth noting that this circuit realizes the C-NOT gate (up to the Hadamard $H_{B}$) for arbitrary equatorial target qubit and control qubit $|0\rangle, |1\rangle$ or $|\pm\rangle$ depending on the measurement basis of qubit 1.

The experimental realization of this gate is performed by adopting ordering c). In this case the control output qubit is encoded in the momentum $k{B}$, while the target output is encoded in the polarization $\pi_B$. In the actual experiment we inserted $H_A$ and $H_{B}$ on photon B to compensate $H_A$, (while $H_A$ and $H_{B}$ were removed, see Fig. 2). The output state in the laboratory basis is then

$$|\Psi_{out}\rangle = (\Sigma)^{s_1}\sigma_z^{s_2}C-NOT(\sigma_z^{s_1}|+\rangle_c \otimes R_z(\alpha)|+\rangle_t),$$

where all the possible measurement outcomes of qubits 1 and 4 are considered. The Pauli errors are $\Sigma = \sigma_z^{s_2}\sigma_z^{s_1}$, while the matrix $\sigma_z^{s_2}$ is due to the changing between computational and laboratory basis. Table 2 shows the experimental fidelities of the target qubit corresponding to the measurement of the output control qubit in the basis $\{|0\rangle, |1\rangle\}$.

### 4. Conclusions

The results of our experiment indicate that a two-photon four-qubit cluster state, which realizes the full entangle-
ment of two photons through two degrees of freedom (in our case polarization and linear momentum), represents a basic resource for one-way QC. Indeed, any kind of single qubit rotations on the Bloch sphere have been realized by these states with high fidelity. These transformations were performed at average repetition rates of \( \sim 1 \) kHz by equivalently using polarization or momentum as output qubit. The two photon approach allows to implement this algorithm without the need of active devices for feed-forward measurements, as demonstrated in the present work in the case of polarization output qubit. One-way QC requires highly efficient active feed-forward corrections at the end of the process [4]. The high fidelity of the output states obtained in this work, even in presence of Pauli errors, is a necessary condition for deterministic QC. By using the same cluster states, we also realized a C-NOT gate for target qubits located in the equatorial plane of the Bloch sphere.

More complex algorithms could be realized by increasing the number of entangled qubits in the state. For instance, six qubits are necessary to implement a C-NOT gate operating over the entire Bloch sphere. In our scheme more qubits could be entangled by using different degrees of freedom, such as time-energy or exploiting the continuous k-mode emission within the SPDC cone of a type I crystal, and/or increasing the number of photons. For example eight-qubit four-photon cluster state cloud be generated by linking together two \( |C_4\rangle\) states by a proper CP gate. These different approaches are at the moment under investigation.

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