Form-invariant solution to quantum state on the sphere

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Abstract
This paper investigates the quantum states that emerge from the transformation design of conformal mapping on the two-dimensional sphere. Three results are reported. First, the construction of form-invariant spherical harmonics labelled by the fractional quantum number through a scalar potential interaction is given. Second, the form-invariant equation of the charge-monopole system is studied. Rather than the half-integer classification of the monopole harmonics, the quantization of the monopole harmonics here can be any fractional number specified by the conformal index. The gauge equivalent condition of the vector potentials which result in the invariant equation shows that the monopole field and the quantization condition of the pole strength due to Dirac can be generalized to more general vector fields and values in the conformal space. Finally, we explore the quadratic conformal image of the charged particle coupling to the constant monopole field on the sphere. It is shown that the lowest order approximation of the image is the magnetic Hooke-Newton transmutation.

1. Introduction
An elegant way to get to interesting solutions to a physical equation is to search for its form-invariant (FI) equation under a transformation. Nevertheless, from the viewpoint of the real world, not all transformations are meaningful since a transformation may alter the geometry and topology of the original space, and the solutions in the transformation space do not always occur in nature. Recently, the development of the transformation optics has shown the possibility of creating the solution to the FI equation by a device with novel material parameters specified by the reinterpretation of the geometric structure of the transformation (e.g. [1, 2]). Since different wave equations associate with the different parameters, the meanings of the reinterpretations in the different wave phenomena also differ. For the matter wave, Chang et al., in the construction of a radial FI solution for a quantum cloak, illustrated that the solution is realizable by endowing the transformation structure with the meaning of the particle with anisotropic mass parameters while experiencing an action of a scalar potential [3].

The wave function of a particle on the sphere ($S^2$) is a striking result in quantum mechanics. For the free particle, it is the well-known spherical harmonics labelled by the pair of the integer quantum number ($l, m$) with $l = 0, 1, 2, \cdots$, and $m = 0, \pm 1, \pm 2, \cdots$. Attracted by the form-invariant feature of the equation in the transformation space, and the concept of the transformation design, the paper explores the FI states on $S^2$ and their realization. The content is arranged as follows: In section 2 we study the FI solution of the particle on the sphere. It is explained how one can construct the fractional spherical harmonics using a local action of a scalar field. Section 3 is used to search for the FI solution for the monopole harmonics by introducing a generalized monopole field. Lately, it was reported that the quadratic conformal image of the action of the uniform magnetic field on a charged particle in the two dimensional plane is the Coulomb force, i.e., the magnetic Hooke-Newton transmutation [4]. The transmuted system exhibits a novel quantum spectrum and behavior depending on the signature of the angular momentum. Section 4 of the paper focuses on discussing the transmutation of a charged
particle on the sphere interacting with the uniform magnetic field generated by a monopole placed at the center of the sphere. Finally, several remarks are made in the final section.

2. The form-invariant equation for the fractional spherical harmonics

Consider the orthogonal curvilinear coordinate system \((q^1, q^2, q^3)\) whose relation to the rectangular coordinates \(x, y, z\) is denoted by \(x = x(q^1, q^2, q^3), y = y(q^1, q^2, q^3), z = z(q^1, q^2, q^3)\). The square of the line element between two neighboring points of the curvilinear system is

\[
d s^2 = \sum_i g_{ii}(d q^i)^2,
\]

where the metric coefficients are defined by

\[
g_{ii}(q) = \frac{\partial x}{\partial q^i} \frac{\partial x}{\partial q^i} + \frac{\partial y}{\partial q^i} \frac{\partial y}{\partial q^i} + \frac{\partial z}{\partial q^i} \frac{\partial z}{\partial q^i}.
\]

A state of a free particle moving in the space satisfies the time independent Schrödinger equation

\[
-\frac{\hbar^2}{2M} \sum_i \frac{\partial^2}{\partial q^i} \left( \sqrt{g} g^{ii} \frac{\partial}{\partial q^i} \right) \Psi = E \Psi,
\]

where \(g^{ii}\) is the inverse of the component \(g_{ii}\) and \(g = \det|g_{ii}|\) is the determinant of the coefficient \(g_{ii}\). We are interested in the construction of the form-invariant states satisfying

\[
-\frac{\hbar^2}{2M} \sum_i \frac{\partial}{\partial q^i} \left( \sqrt{\tilde{g}} \tilde{g}^{ii} \frac{\partial}{\partial q^i} \right) \Psi = E \Psi,
\]

in which the metric coefficients are defined by \(\tilde{g}_{ii} = g_{ii}(q)\) with the new variable \(\tilde{q}^i\) being any function of \(q^i\), i.e., \(\tilde{q}^i = \tilde{q}^i(q^i)\), and \(\tilde{g} = \det|\tilde{g}_{ii}|\). Seemingly, the wave function \(\tilde{\Psi}\) does indeed satisfy the equation. Nevertheless, the equation is illegal since the defined metric coefficients \(\tilde{g}_{ii}\) are in general not the genuine coefficients

\[
G_{ii} = \frac{\partial x}{\partial q^i} \frac{\partial x}{\partial q^i} + \frac{\partial y}{\partial q^i} \frac{\partial y}{\partial q^i} + \frac{\partial z}{\partial q^i} \frac{\partial z}{\partial q^i}
\]

for the space parametrized by \((q^1, q^2, q^3)\). Thus, the operator on the left hand side (L.H.S.) of (4) is not the kinematic operator of the quantum particle. To endow the form-invariant state with a physical meaning, one can insert identities \(\sqrt{G}/\sqrt{\tilde{G}} = 1\) with \(G = \det|G_{ii}|\) in L.H.S. of the first derivative \(\partial/\partial \tilde{q}^i\), and \(\sqrt{G} G^{ii}/\sqrt{\tilde{G}} \tilde{G}^{ii} = 1\) in L.H.S. of the second derivative \(\partial^2/\partial \tilde{q}^i \partial \tilde{q}^i\) in (4) such that equation (4) is expressed in terms of

\[
-\frac{\hbar^2}{2} \sum_i \frac{\partial}{\partial q^i} \left( \sqrt{G} G^{ii} \frac{1}{M_{ii}} \frac{\partial}{\partial \tilde{q}^i} \right) \Psi = (E - U) \tilde{\Psi}
\]

with the defined mass parameters

\[
M_{ii} = \frac{\sqrt{G}}{\sqrt{\tilde{G}}} \frac{\tilde{g}_{ii}}{G_{ii}} M,
\]

and the effective potential

\[
U = \left(1 - \frac{\sqrt{G}}{\sqrt{\tilde{G}}}\right) E.
\]

Accordingly, the quantum state \(\tilde{\Psi}\) can be physically constructed by reinterpreting the particle with the coordinate-dependent mass parameters \(M_{ii}\) while interacting with the local potential field \(U\). The representation of the Schrödinger equation for the design of a quantum state with transformation means was presented by Chen et al in [5]. Now introduce the transformation \((\theta, \varphi)\) on \(S^2\) defined by (see appendix A for details)

\[
\sin \tilde{\theta} = \frac{2(\sin \theta)^a}{(1 + \cos \theta)^a + (1 - \cos \theta)^a}, \text{ and } \tilde{\varphi} = a \varphi,
\]

where \(a\) is an arbitrary real number. It is easy to show that the line element in the transformation space of \(S^2\) is

\[
d s^2 = d \tilde{\theta}^2 + \sin^2 \tilde{\theta} d \varphi^2.
\]

In the usual coordinates \((\theta, \varphi)\) of \(S^2\), one sees that it is a conformal mapping of \(S^2\). Explicitly,

\[
d s^2 = d \tilde{\theta}^2 + \sin^2 \tilde{\theta} d \varphi^2 = \left(\frac{d \tilde{\theta}}{d \theta}\right)^2 d \theta^2 + \left(\frac{\sin \tilde{\theta} d \varphi}{\sin \theta d \varphi}\right)^2 \sin^2 \theta d \varphi^2 = \left(\frac{d \tilde{\theta}}{d \theta}\right)^2 (d \theta^2 + \sin^2 \theta d \varphi^2) \equiv (\tilde{\theta})^2 ds^2
\]
with

$$\frac{\theta'}{d\theta} = \frac{2a(\sin \theta)^{a-1}}{(1 + \cos \theta)^a + (1 - \cos \theta)^a}. \quad (12)$$

The transformation can also be defined by

$$\cos \tilde{\theta} = \frac{(1 + \cos \theta)^a - (1 - \cos \theta)^a}{(1 + \cos \theta)^a + (1 - \cos \theta)^a}, \quad \text{and} \quad \tilde{\varphi} = a\varphi. \quad (13)$$

It is easy to show that the trigonometric functions with the conformal coordinates also satisfy the familiar equality

$$\cos^2 \tilde{\theta} + \sin^2 \tilde{\theta} = 1, \quad (14)$$

and the derivatives

$$\frac{d}{d\tilde{\theta}} \cos \tilde{\theta} = \frac{d\theta}{d\tilde{\theta}} \cos \theta = -\sin \tilde{\theta}, \quad (15)$$

and

$$\frac{d}{d\tilde{\theta}} \sin \tilde{\theta} = \cos \tilde{\theta}. \quad (16)$$

Thus, we have the Euler’s identities \( \cos \tilde{\theta} = (e^{i\theta} + e^{-i\theta})/2, \) and \( \sin \tilde{\theta} = (e^{i\theta} - e^{-i\theta})/2i, \) and the relationships among the trigonometric functions of plane geometry are applicable to the conformal coordinates \( (\tilde{\theta}, \tilde{\varphi}). \) Figures 1 and 2 show the graphs of \( \cos \tilde{\theta} \) and \( \sin \tilde{\theta} \) for some values of positive conformal index \( a. \) They are periodic functions with range \([-1, 1]\) and \([0, 1]\) except the trivial case \( a = 1 \) for the latter. A striking result is that the locations of the extreme values of these functions coincide with those given by \( \cos \theta \) and \( \sin \theta. \) An alternative interesting result is that \( \cos \tilde{\theta} \) (\( \sin \tilde{\theta} \)) is an anti-symmetric (symmetric) function with respect to the negative value of \( a. \) Explicitly,

$$\cos \tilde{\theta}|_{-a} = \frac{(1 + \cos \theta)^a - (1 - \cos \theta)^a}{(1 + \cos \theta)^a + (1 - \cos \theta)^a} \times \frac{(1 + \cos \theta)^a (1 - \cos \theta)^a}{(1 + \cos \theta)^a (1 - \cos \theta)^a} = -\cos \theta|_{-a}, \quad (17)$$

and, analogously, one has the symmetric relation \( \sin \tilde{\theta}|_{-a} = \sin \theta|_{-a}. \) It can be proved that the transformation stretches the line element, but preserves the form of the Gaussian curvature of \( S^2. \) We are in the position to calculate the mass parameters for the realization of the conformally geometric effect. Without loss of generality, the radius of the sphere is taken as \( R = 1. \) The metric for \( S^2 \) is \( (g_{\varphi}) = \text{diag}(1, \sin^2 \theta). \) So we have

$$\sqrt{g} = \sqrt{|g_{\varphi}|} = \sin \tilde{\theta}. \quad (18)$$

The metric coefficients of the conformal space are calculated by

$$G_{\tilde{\theta}\tilde{\theta}} = \left( \frac{\partial x}{\partial \tilde{\theta}} \right)^2 + \left( \frac{\partial y}{\partial \tilde{\theta}} \right)^2 + \left( \frac{\partial z}{\partial \tilde{\theta}} \right)^2 = \frac{1}{(\theta')^2} \left( \left( \frac{\partial x}{\partial \theta} \right)^2 + \left( \frac{\partial y}{\partial \theta} \right)^2 + \left( \frac{\partial z}{\partial \theta} \right)^2 \right) = \frac{1}{(\theta')^2}, \quad (19)$$

Figure 1. The graphs of \( \cos \tilde{\theta} \) for several different values of conformal index \( a. \) The function is antisymmetric with respect to \(-a. \) For any value of \( a, \) the positions of the maxima of the function coincide with the regular cosine function \( \cos \theta, \) corresponding to \( a = 1. \)
We have

\[ q = \sin \theta \]

The nonzero components of the mass parameters are

\[ M_{11} = \sqrt{\frac{G}{g}} \frac{G_{11}}{G} = \frac{\theta' \sin \theta}{\varphi' \sin \theta} M = M, \]

and

\[ M_{22} = \sqrt{\frac{G}{g}} \frac{G_{22}}{G} = \frac{\varphi' \sin \theta}{\theta' \sin \theta} M = M. \]

They are independent of coordinates and just the proper mass of the particle. Thus, the realization of the geometric effect can be achieved only by offering the effective potential \( U \) on the sphere that could reach through a corresponding electromagnetic force. By substituting the related quantities into (6), we obtain the Schrödinger equation of the conformal space

\[ \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] \Psi = -k^2 \Psi, \]

where \( k^2 = 2ME/b^2 \). It is form-invariant with respect to that of \( S^2 \). Let \( \Psi(\theta, \varphi) = \Theta(\theta) \Phi(\varphi) \) with \( \Phi(\varphi) = \exp \{i\mu \varphi\} \). It is shown that the function \( \Theta(\theta) \) satisfies the equation

\[ \left\{ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) + \left[ \nu(\nu + 1) - \frac{\mu^2}{\sin^2 \theta} \right] \right\} \Theta = 0. \]

Here, we have defined \( k^2 = \nu(\nu + 1) \) with \( \nu \) being the quantum number to be determined later. The parameter \( k \) is dimensionless since the radius is supposed to be \( R = 1 \). The unit is recovered by making the replacement \( M \to MR^2 \). Now, let \( z = \cos \theta \). The equation becomes

\[ \left\{ (1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} + \left[ \nu(\nu + 1) - \frac{\mu^2}{1 - z^2} \right] \right\} \Theta = 0. \]

Two linearly independent solutions to the equation are the Legendre functions, i.e.,

\[ \Theta(z) = C_1 P^\nu_\mu(z) + C_2 Q^\nu_\mu(z) \]

with \( C_1 \) and \( C_2 \) being constants. It is well-known that for the trivial case \( \alpha = 1 \) that the uniquely bounded solution is given by \( \Theta(z) = C_1 P^\nu_\mu(z) \). Here we are interested in the case \( \alpha \approx 1 \). The value of \( \mu \) can be determined by the single valuedness condition of the wave function as follows:
\[ e^{i\mu \varphi} = e^{i\mu \varphi - \mu \pi} = e^{i\mu (\varphi - 2\pi)}, \]

that requires \( \exp(i \mu 2\pi) = 1 \), or equivalently,

\[ \mu = \frac{n_0}{a}, \quad n_1 = 0, \pm 1, \pm 2, \ldots. \]

To determine the value of \( \nu \), it is beneficial to use the hypergeometric function representation of the Legendre functions defined in the domain \(-1 < z < 1\) for any values of \( \mu \) and \( \nu \) (p.166, [6]):

\[ P^\mu_\nu(z) = \frac{1}{\Gamma(1 - \mu)} \left(\begin{array}{c} 1 + z \\ 1 - \mu \\ \nu + 1; 1 - \mu; 1 - 2z \end{array}\right), \]

and

\[ Q^\mu_\nu(z) = -\frac{\pi}{2 \sin \mu \pi} \left[ \cos \mu \pi P^\mu_\nu(z) - \frac{\Gamma(\mu + \mu + 1)}{\Gamma(\nu - \mu + 1)} \Gamma(\nu + 1) \right]. \]

Since the wave function needs to be bounded in the whole domain \(-1 \leq z \leq 1\), the condition requires that the allowed quantum number \( \mu \) and \( \nu \) can only be classified into two types (see appendix B for details). (i) The solution is given by

\[ \Theta(z) = C_1 P^\mu_\nu(z) \]

with the quantum number \( \mu = (n_1/a) < 0 \) and \( \nu = n + |\mu|, \quad n = 0, 1, 2, \ldots \).

(ii) The solution is

\[ \Theta(z) = C_1 P^\mu_\nu(z) + C_2 Q^\mu_\nu(z) \]

with the allowed quantum number \( \mu = (n_1/a) < 0 \) and \( \nu = -n - |\mu| - 1, \quad n = 0, 1, 2, \ldots \).

Obviously, the solution of (32) reduces to the familiar Legendre polynomials \( P^\mu_\nu \) when \( a = 1 \) (\( P^\mu_\nu \) implies that \( P^{\mu - n}_{\nu + n} \) is also a solution since both are linearly dependent). A physically acceptable solution should be equation (32) as we consider that the solution needs to include the trivial case of \( a = 1 \). The corresponding energy levels are

\[ E = \frac{\hbar^2}{2MR^2} \nu(\nu + 1). \]

The radius \( R \) is here retrieved to give the correct physical units. According to equation (6), the solution can be realized via offering the interacting potential

\[ U = \left(1 - \sqrt{\frac{\mathcal{G}}{\mathcal{G}}}\right) E = \left(1 - \frac{4a^2 (\sin \theta)^{2a-2}}{[(1 + \cos \theta)^a + (1 - \cos \theta)^a]^2} \right) E \]

on \( S^2 \). It provides a force along \( \hat{e}_3 \) that alters the quantization values of the angular momentum.

Before finishing the discussion of this section, we would like to point out that the potential can also be found out by directly performing the inverse transformation of the FI equation (24). Explicitly, we have

\[ \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \]

and

\[ \frac{\partial^2}{\partial \phi^2} = \frac{1}{a^2} \frac{\partial^2}{\partial \phi^2}, \]

where

\[ \hat{\theta}' = \frac{a \sin \hat{\theta}}{\sin \theta} \]

from (12), and

\[ \hat{\theta}'' = a \left( -\frac{\cos \theta}{\sin^2 \theta} \sin \hat{\theta} + \frac{a}{\sin^2 \theta} \sin \theta \cos \hat{\theta} \right) \]

So we have the expression of the factor

\[ \frac{\hat{\theta}''}{(\hat{\theta}')^2} = \frac{1}{a^2} \left( -\frac{\sin \theta \cos \theta}{\sin^2 \hat{\theta}} + a \frac{\sin \theta \cos \hat{\theta}}{\sin^2 \hat{\theta}} \right). \]
A substitution of (36)–(41) into (24) gives
\[
\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] \Psi = -\frac{2M}{\hbar^2} (E - U) \Psi
\]
with \( U \) given by (35). The result shows that the FI solution of equation (24) is caused by the action of the effective potential in the original space \( S^2 \).

3. The form-invariant equation for the fractional monopole harmonics

In the last section, we demonstrated how the form-invariant equation of free particles on \( S^2 \) and its solution are constructed from the conformal mapping (9). In this section, we investigate the construction of the form-invariant equation of the charge-monopole system and its solution through the interaction of a charged particle and a vector potential on the sphere. Consider the vector potential
\[
A_I = \frac{\mathbf{g}}{r} \left( \frac{1 - \cos \bar{\theta}}{\sin \bar{\theta}} \right) \mathbf{\hat{e}}_r,
\]
where \( g \) is the coupling constant. Since the locations of the extrema of \( \cos \bar{\theta} \) are the same as \( \cos \theta \), the function has a singular string along the negative \( z \)-axis. It is easy to show that the magnetic field caused by the potential is radial, and is given by
\[
B = \nabla \times A_I = \frac{\mathbf{g \bar{\theta}}}{r^2} \left( \frac{\sin \bar{\theta}}{\sin \bar{\theta}} \right)^2 \mathbf{\hat{e}}_r.
\]
Only the trivial case of \( a = 1 \) makes it become the monopole field in the traditional meaning. The singular point of (44) only appears at \( r = 0 \), and is regular at any point on the sphere. At the points \( \theta = 0, \pi/2, \) and \( \pi \), (44) is isotropic. But it is deformed anywhere else by the conformal mapping \( \bar{\theta} \) when \( \theta \neq 0, \pi/2, \) and \( \pi \). It can be shown that the magnetic field can be understood as generated by a generalized monopole in the conformal space (see remarks (c) and (d) for details). The field (44) can also be generated by the vector potentials
\[
A_{III} = -\frac{\mathbf{g \bar{\theta}}}{r} \left( \frac{1 + \cos \bar{\theta}}{\sin \bar{\theta}} \right) \mathbf{\hat{e}}_r,
\]
and
\[
A_{III} = -\frac{\mathbf{g \bar{\theta}}}{r} \left( \frac{\cos \bar{\theta}}{\sin \bar{\theta}} \right) \mathbf{\hat{e}}_\varphi.
\]
The former has the singular string along the positive \( z \)-axis, and the latter has a pair of symmetric singular strings at \( \theta = 0, \pi \). According to the gauge theory, they are equivalent, up to a gauge transformation due to the \( \mathbf{B} \) field being gauge independent. The physically coupling condition of a charge with the magnetic field can be determined by the nonintegrable phase factor [7]
\[
\exp \left\{\frac{ie}{\hbar c} \oint (A_I - A_{III}) \cdot dr \right\}
\]
along the path of \( \theta = \pi/2 \) on \( S^2 \). It is easy to evaluate the phase integral, and the phase factor is given by
\[
\exp \{ \cdots \} = \exp \left\{ \frac{ie}{\hbar c} 4\pi g \right\}.
\]
Since \( A_I \) and \( A_{III} \) have the same physical effect, it requires the phase to satisfy the condition
\[
\frac{2eg}{\hbar c} = n_2, \quad n_2 = 0, \pm 1, \pm 2, \cdots.
\]
The rule was first pointed out by Dirac in discussing the quantization of pole strength in [8]. The angular part of the Schrödinger equation for the charge-monopole system can be solved by the monopole harmonics (e.g. [9]). In this section we construct its form-invariant equation by the conformal mapping on \( S^2 \). To manifest the influence of the transformation on the dynamics of a charge interacting with a vector potential, one expresses equation (24) as
\[
\frac{1}{\sin \theta} \left( \frac{\partial}{\partial \theta} \right) \left( \frac{\sin \bar{\theta} \frac{\partial}{\partial \theta}}{\sin \bar{\theta} \frac{\partial}{\partial \varphi}} \right) \Psi = -k^2 \Psi.
\]
It is known that the electromagnetic interaction is through the minimal coupling \( \frac{\partial}{\partial \tau} - (ie/\hbar c) A_\tau \). So the influence of the mapping on the charge in the magnetic field on \( S^2 \) can be expressed as...
To solve the equation, let \( \Psi = \Theta(\hat{\theta}) \exp \{i\mu \hat{\varphi} \} \). It results in \( \partial^2 \Psi / \partial \varphi \partial \varphi = i \mu \varphi \partial \varphi \Psi \), and we have \( \partial (1/\varphi') / \partial \varphi = - \varphi'' / (\varphi')^2 \). The equation has the reduction

\[
\frac{1}{\sin \hat{\theta}} \frac{\partial}{\partial \hat{\theta}} \left( \sin \hat{\theta} \frac{\partial}{\partial \hat{\theta}} \Psi \right) - \frac{1}{\sin^2 \hat{\theta}} \times \left\{ \frac{\varphi''}{(\varphi')^2} \left[ i(q(1 - \cos \hat{\theta})) + \frac{2\mu q}{\varphi'} (1 - \cos \hat{\theta}) + \frac{1}{(\varphi')^2} \right] \right\} \Psi = -k^2 \Psi,
\]

(52)

where \( q = (-eg / \hbar) \). Since \( \varphi = a \varphi \), so \( \varphi' = a \), and \( \varphi'' = 0 \). Together with the single valuedness condition \( \mu = n_1/a \), the equation can be expressed as

\[
\frac{1}{\sin \hat{\theta}} \frac{d}{d\hat{\theta}} \left( \sin \hat{\theta} \frac{d}{d\hat{\theta}} \Theta \right) - \frac{1}{\sin^2 \hat{\theta}} \left[ \left( \frac{n_1}{a} \right) + \frac{q}{a} \right] \left( 1 - \cos \hat{\theta} \right)^2 \Theta = -k^2 \Theta.
\]

(53)

A further simplification is achieved by letting \( \bar{n} = (n_1 + q)/a \), and \( \bar{q} = q/a \). We finally have the representation

\[
\frac{1}{\sin \hat{\theta}} \frac{d}{d\hat{\theta}} \left( \sin \hat{\theta} \frac{d}{d\hat{\theta}} \Theta \right) - \frac{1}{\sin^2 \hat{\theta}} \left( \bar{n}^2 - 2\bar{n} \bar{q} \cos \hat{\theta} + \bar{q}^2 \right) \Theta = -k^2 \Theta,
\]

(54)

where \( k^2 = k^2 + \bar{q}^2 \). This is the form-invariant equation for the function \( \Theta \) of monopole harmonics (e.g., see equation (22) in [9], and equation (B9) in [10]). The normalized solution is given by (see appendix C)

\[
\Theta_{q_{J,0}a}(\hat{\theta}) = \sqrt{\frac{(2^J + 1) \Gamma(J - \bar{q} + 1) \Gamma(J + \bar{q} + 1)}{2^J \Gamma(J - \bar{n} + 1) \Gamma(J + \bar{n} + 1)}} \times \left( \frac{1 - \cos \hat{\theta}}{2} \right)^{(q - \bar{n})/2} \left( \frac{1 + \cos \hat{\theta}}{2} \right)^{(q + \bar{n})/2} P_{\bar{n}}^{(q - \bar{n}, q + \bar{n})}(\cos \hat{\theta}),
\]

(55)

where \( P_{\bar{n}}^{(q - \bar{n}, q + \bar{n})}(\cos \hat{\theta}) \) is the Jacobi polynomials with \( n = 0, 1, 2, \ldots \). The allowed quantum number is subjected to the rules

\[
J - \bar{q} = n, \quad n = 0, 1, 2, \ldots,
\]

(56)

\[
J = |\bar{q}|, \quad |\bar{q}| + 1, \quad |\bar{q}| + 2, \ldots,
\]

(57)

and for any values of \( \bar{q} \), the range of \( \bar{n} \) satisfies the inequality

\[
-(J + 1) < \bar{n} < (J + 1).
\]

(58)

Specifically, when \( a = 1, J = 0, 1/2, 1, \ldots \), and for each value of \( J, -J \leq \bar{n} \leq J \) in integral steps of increment, which is for the charge-monopole system (e.g., see [9]). The energy level corresponding to the state (55) is given by

\[
E = \frac{\hbar^2}{2MR^2} (J(J + 1) - \bar{q}^2).
\]

(59)

The entire wave function reads

\[
\Psi(\hat{\theta}, \varphi) = \Theta_{q_{J,0}a}(\hat{\theta}) \Phi_{\bar{q}}(\varphi)
\]

(60)

with

\[
\Phi_{\bar{q}}(\varphi) = \frac{1}{\sqrt{2\pi\bar{a}}} e^{i\mu \varphi}.
\]

(61)

Physically, the form-invariant solution can be created by the interaction of the charge with the scalar and vector potentials (35) and (43) on \( S^2 \).

4. The magnetic Hooke-Newton transmutation of the electron on the sphere

In the recent investigations [4, 11, 12], it was pointed out that the Hooke-Newton (HN) transmutation generated by a quadratic conformal mapping in mechanics (Cor. III, Prop. VII, [13], see also [14, 15], and section 6 in [16]) can also appear in the Landau level’s system. If the charge is on the sphere while moving in the field of a magnetic monopole placed at the center of the sphere, it would experience a uniform magnetic field that may cause the HN transmutation under a mapping. In the coming content, we shall prove that the HN transmutation does
indeed appear on the sphere in the lowest order approximation under the quadratic conformal mapping of (9). Consider the minimal coupling interaction $\partial_{\alpha} - (ie/c)A_{\alpha}$ of the charge with the vector potential

$$A = \frac{q}{r} \left( \frac{1 - \cos \theta}{\sin \theta} \right) \hat{\varphi}$$

(62)
due to the magnetic monopole. Resembling the treatment in (50) and (51), the influence of conformal mapping on the action of the vector potential on the charged particle is described by the equation

$$\frac{1}{\sin \theta} \left( \partial_{\varphi} - ie \frac{\Theta}{\varphi} \right) \frac{\partial}{\partial \varphi} \sin \theta \left( \frac{1}{\sin \theta} \partial_{\varphi} \Psi - ie \frac{\Theta}{\varphi} \left( \frac{1 - \cos \theta}{\sin \theta} \right) \right)$$

$$\times \left( \frac{\sin \vartheta}{\partial \varphi} \frac{\partial}{\partial \varphi} \sin \theta \left( \frac{1}{\sin \theta} \partial_{\varphi} \Psi - ie \frac{\Theta}{\varphi} \left( \frac{1 - \cos \theta}{\sin \theta} \right) \right) \right) \Psi = -k^2 \Psi,$$

(63)

and, like (51), with the ansatz $\Psi = \Theta(\vartheta) \exp \{i\mu \varphi\}$, the equation can be reduced to the expression

$$\frac{1}{\sin \theta} \left( \partial_{\varphi} - ie \frac{\Theta}{\varphi} \right) \frac{\partial}{\partial \varphi} \sin \theta \left( \frac{1}{\sin \theta} \partial_{\varphi} \Psi - ie \frac{\Theta}{\varphi} \left( \frac{1 - \cos \theta}{\sin \theta} \right) \right)$$

$$\times \left[ \frac{\varphi^n}{(\varphi^2)^3} \left( i q (1 - \cos \theta) \right) + \frac{2 \mu q}{\varphi} (1 - \cos \theta) + \frac{1}{(\varphi^2)^2} \left( i q (1 - \cos \theta) \right)^2 + \mu^2 \right] \Psi = -k^2 \Psi.$$  

(64)

A further simplification is given by the single valuedness condition $n = n/a$ of $\exp \{i\mu \varphi\}$ which makes the conformal image $\Theta(\vartheta)$ of the charge-monopole system satisfy

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \Theta \right) - \frac{1}{\sin^2 \theta} \left( \frac{n}{a} + \frac{q}{a} \right) (1 - \cos \theta) \right)^2 \Theta = -k^2 \Theta.$$  

(65)

For the quadratic mapping $a = 2$, one can show from (9) that

$$\cos \theta = \left( \frac{1 + \sin \vartheta}{\cos \vartheta} \right).$$  

(66)

Since the ranges $\cos \theta \in [-1, 1]$, $\cos \vartheta \in [-1, 1]$, and $\sin \vartheta \in [0, 1]$, only the negative sign above is acceptable for the inverse transformation. The conformal mapping of the equation then becomes

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \Theta \right) - \frac{1}{\sin^2 \theta} \left( \frac{n}{2} + \frac{q}{2} \right) (1 - \sin \vartheta) \right)^2 \Theta = -k^2 \Theta.$$  

(67)

So far, there is no closed form solution for it to obtain. Here we only consider the solution around $\vartheta \approx 0$. Using (14), one has $\cos \theta = \sqrt{(1 - \sin \vartheta) / (1 + \sin \vartheta)}$. Near the considered region, there is an approximation

$$\cos \theta \approx 1 - \sin \vartheta + \frac{1}{2} \sin^2 \vartheta + \cdots.$$  

(68)

Substituting the approximation into (65), it yields

$$\frac{d^2}{d\vartheta^2} \Theta + \cot \vartheta \frac{d}{d\vartheta} \Theta \left( \frac{n}{2} \right)^2 \frac{q}{\sin \vartheta} + \sin \vartheta \left( \frac{q}{2} \right)^2 \sin \vartheta + \cdots \right) \Theta = -k^2 \Theta.$$  

(69)

For $\vartheta \approx 0$, it is known that

$$\left\{ \begin{array}{l} \cot \vartheta \approx (1/\vartheta - \vartheta/3 + \cdots), \\
\sin \vartheta \approx \vartheta(1 - \vartheta^2/3 + \cdots), \\
\sin^2 \vartheta \approx \vartheta^2(1 - \vartheta^2/3 + \cdots). \end{array} \right.$$  

(70)

With the approximations, one finds from (69) that the first order approximation is

$$\left[ \frac{d^2}{d\vartheta^2} + \frac{1}{\vartheta} \frac{d}{d\vartheta} - \left( \frac{n}{2} \right)^2 - \frac{q}{\vartheta} \left( \frac{n}{2} \right) \right] \Theta = -E \Theta,$$  

(71)
where \( E = k^2 + \lambda^2 \) with

\[
\lambda^2 = \left[ \frac{1}{3} \left( \frac{n_1}{2} \right)^2 - \frac{q}{2} \left( \frac{q}{2} - \frac{n_1}{2} \right) \right].
\]

Equation (71) is the Schrödinger equation for the Coulomb interaction in the variable \( \tilde{\theta} \). Instead of the charge-charge interaction of the regular Coulomb system, here the interaction strength is dictated by both the quantized angular momentum \( n_1/2 \) and monopole strength \( q = n_2/2 \). This feature is also different from the case of the HN transmutation in the 2D plane \([4, 11]\), where the interaction strength is determined by the constant Larmor frequency coupling to quantized angular momentum. The condition for the existence of bound states is the interaction \( q \cdot \left( n_1/2 \right) < 0 \). So it must have (i) \( q < 0 \) and \( n_1/2 > 0 \). Let us define the positive quantum number \( n_1/2 \) of angular momentum corresponding to the motion of the charge counterclockwise. So the signature \( q = -e\gamma/hc = |e|\gamma/hc < 0 \) for an electron, i.e., the existence of bound states for an electron moving in the uniform field of a monopole with negative magnetic charge is only when the electron’s motion is counterclockwise. If the electron moves clockwise, it would suffer a scattering. The signature of the angular momentum here plays a significant role. (ii) \( q > 0 \) and \( n_1/2 < 0 \). It implies that bound states appear only when the electron is moving clockwise in the uniform field of a positive magnetic monopole. An electron with counterclockwise motion in the field would experience a repulsive force. Conditions (i) and (ii) tell that, given the electric and magnetic charges of the system, the signature of the angular momentum determines the type of interaction. To obtain the quantization levels and the wave function of (71), let us make the conjecture that the bound states have the form

\[
\Theta(\tilde{\theta}) = \tilde{\theta}^{n_1/2} e^{-\tilde{\theta}b} u(\tilde{\theta}),
\]

where \( \beta = \sqrt{-E} \). The substitution into (71) yields

\[
\tilde{\theta} \frac{d^2 u}{d\tilde{\theta}^2} + \left( |n_1| + 1 - 2\tilde{\theta} \right) \frac{du}{d\tilde{\theta}} + \left[ \alpha - (|n_1| + 1)\beta \right] u = 0,
\]

where the parameter \( \alpha \equiv |q| \cdot n_1/2 \) was introduced. Change the variable by \( y = 2\tilde{\theta} \). It turns into

\[
y \frac{d^2 u}{dy^2} + \left( |n_1| + 1 - y \right) \frac{du}{dy} - \left[ \left( \frac{n_1}{2} \right) + \frac{1}{2} \right] u = 0.
\]

This is the standard form of the confluent hypergeometric equation. The regular solution (around \( y \to 0 \)) is the confluent hypergeometric function

\[
u = F\left( \left( \frac{n_1}{2} + \frac{1}{2} \right) - \frac{\alpha}{2\beta}, \; |n_1| + 1; \; y \right).
\]

A possible quantization solution meets the condition

\[
\left( \frac{n_1}{2} + \frac{1}{2} \right) - \frac{\alpha}{2\beta} = -n, \quad \text{with} \quad n = 0, 1, 2, \ldots,
\]

which reduces the solution \( u(y) \) to a polynomial of degree \( n \), and we get the energy levels

\[
E = -\frac{\hbar^2}{2MR^2} \left\{ \left( |n_1|/2 \right)^2 + \left( \frac{1}{3} \frac{m_1}{2} \right)^2 - \frac{q}{2} \left( \frac{q}{2} - \frac{m_1}{2} \right) \right\},
\]

where \( m_1 = 0, \pm 1, \pm 2, \ldots \), and \( q = -e\gamma/hc = -n_2/2, n_2 = 0, \pm 1, \pm 2, \ldots \). The corresponding wave functions have the closed representation

\[
\tilde{\Psi}(\tilde{\theta}, \varphi) = e^{in_1\gamma/2} [\tilde{\theta}^{n_1/2} e^{-\tilde{\theta}b} F(-n, \; |n_1| + 1; \; 2\tilde{\theta})]
\]

up to a normalization constant. Equation (78) shows a restriction to the value of \( q \). It cannot be too large. Otherwise, the value in the middle bracket may become negative enough such that it alters the signature of the curly bracket. In this situation, the bound state disappears. The situation reflects the restriction of the lowest order approximation.

5. Remarks

The paper investigates the transformation design of the form-invariant quantum states on a sphere via conformal mapping. The fractional spherical and monopole harmonics are constructed from the interaction of the charge with a scalar potential that is specified by the transformation structure. We also show that the lowest order approximation of the quadratic conformal image of the charge-monopole system is the magnetic Hooke-Newton transmutation on the sphere. Several remarks are made as follows:
(a) The equivalence between the geometric effect of the presented conformal mapping and the action of the scalar potential offers an alternative way to obtain the fractional quantization rule of the angular momentum in the 2D system. It is known that a state with fractional angular momentum can emerge from a particle interacting with a vector potential such as a charge coupling to the vector potential producing the Aharonov–Bohm (AB) effect. Section 2 exhibits the possibility of obtaining a fractional state by the coupling of the particle to the scalar potential determined by the conformal mapping. There is an intrinsic difference between these two kinds of fraction. The effect of the transformation method fractionalizes the integer quantum number by the quotient of each integer and the conformal index \( a \) such as the change of the radial wave, \( J_m(kr) \rightarrow J_{m/a}(kr) \), for a free particle, where \( J_m(kr) \) is the Bessel function. The fractionalizing effect of angular momentum by the vector potential is through adding the magnitude \( \Omega \) of the flux to each integer such as \( J_m(kr) \rightarrow J_{m+a}(kr) \) for the AB effect, where \( \alpha = \Omega / \Omega_0 \), with \( \Omega_0 = hc/e \) as the fundamental flux quantum.

(b) The quantization rule of the monopole strength due to Dirac is appropriate to the coupling of a more general singular anisotropic magnetic field. As shown in section 3, the coupling condition of the vector potential that generates the \( \theta \)-dependent magnetic field (44) satisfies Dirac’s monopole quantization rule. It is not difficult to see that a magnetic field defined by the more general vector potentials

\[
A_1 = \frac{g}{r} \left( \frac{1 - f(\theta)}{\sin \theta} \right) \hat{e}_r, \quad A_{II} = -\frac{g}{r} \left( \frac{1 + f(\theta)}{\sin \theta} \right) \hat{e}_r, \tag{80}
\]

where \( f(\theta) \) is any function of \( \theta \) with values \( f(\theta)|_{\theta = \pi} = -1 \), and \( f(\theta)|_{\theta = 0} = 1 \), has the same coupling condition. The statement is easy to prove by the single valuedness condition of the phase factor \( \exp(ie/hc \oint (A_1 - A_{II}) \cdot d\tilde{r}) \) along the path \( \theta = \pi/2 \) on \( S^2 \).

(c) The magnetic field (44) can be understood as being generated by a generalized monopole in the conformal space. We show this by the verification of Gauss’s law of magnetism as follows:

\[
B = \frac{ag}{r^2} \left( \sin \tilde{\theta} \right)^2 \hat{e}_r, \quad \tilde{\theta} = \frac{a \sin \tilde{\theta}}{\sin \theta} \text{ from } (9) \text{ and } (12).
\]

The integral form of Gauss’s law for the field with respect to the closed surface of a sphere is given by

\[
\int \int B \cdot da = \frac{ag}{a^2} \int \int \tilde{\theta}^2 \sin \tilde{\theta} d\tilde{\theta} d\tilde{\varphi} = \frac{ag}{a^2} \int \int \frac{\sin \tilde{\theta} d\tilde{\theta}}{d\tilde{\varphi}} \sin \tilde{\theta} d\tilde{\varphi} = \frac{\pi a}{2} \int \int \sin \tilde{\theta} d\tilde{\varphi} = 4\pi g.
\]  

This is the familiar statement of Gauss’s law for a pole, but in the conformal space.

(d) The fractional quantization rule for the form-invariant solution of the charge-monopole system can also be proved by the allowed gauge condition in the conformal space. It is known that the unit vector \( \hat{e}_q \) along an orthogonal coordinate curve can be calculated by the formula

\[
\hat{e}_q = \frac{1}{H_q} \frac{\partial r}{\partial q'},
\]

where \( H_q = \sqrt{g_{qq'}} \). For our consideration of the unit vector along the coordinate \( \varphi' \), \( H_{\varphi'} = \sqrt{g_{\varphi'\varphi'}} = \sin \theta' / \varphi' \) from (20). So we have

\[
\hat{e}_{\varphi'} = \frac{1}{H_{\varphi'}} \frac{\partial r}{\partial \varphi'} = \frac{\varphi' \partial r}{\sin \theta' \partial \varphi'} = \frac{1}{\sin \theta' \partial \varphi'} = \hat{e}_{\varphi'}.
\]  

This implies that (43) is also a vector potential in the conformal space along \( \hat{e}_{\varphi'} \)

\[
A_1 = \frac{g}{r} \left( \frac{1 - \cos \theta}{\sin \theta} \right) \hat{e}_{\varphi'}.
\]

Calculating the phase of the phase factor

\[
\exp \left\{ ie \frac{h}{c} \oint (A_1 - A_{II}) \cdot d\tilde{r} \right\}
\]

along the curve \( \tilde{\theta} = \pi/2 \) yields

\[
\frac{e}{hc} \oint (A_1 - A_{II}) \cdot d\tilde{r} = \frac{g}{h} \frac{a}{4\pi a}.
\]

Gauge equivalence requires that the phase can only be unity. It results in the quantization rule for the coupling of the singular field
\[ \frac{eg}{\hbar c} = \frac{n_2}{2a}, \quad n_2 = 0, \pm 1, \pm 2, \ldots. \]  

(88)

This is the fractional value \( q \) in (57) for the form-invariant monopole harmonics. Together with the remark (c), we see that the field \( A_\xi \) can be understood as being generated by the magnetic monopole in the conformal space with the strength of the fractional quantization values \( g \equiv (n_2/2a)(/\hbar c) \) of \( \Omega_6 \). Not long ago, the resembling fractional values were discovered in a superconducting film carried by the magnetic flux [17], and were expected in multicomponent superconductors [18]. The latter can have a magnetic field which is not localized in space.

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**Appendix A. The form-invariant conformal mapping of the sphere \( S^2 \)**

Here we explain how to find the conformal mapping \((\vartheta, \varphi)\). Consider the form-invariant line element of \( S^2 \)

\[ ds^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2. \]  

(A1)

Here we assume \( \vartheta = \bar{\vartheta}(\theta) \) and \( \varphi = \bar{\varphi}(\varphi) \). So the line element can be expressed as

\[ ds^2 = \left( \frac{d\vartheta}{d\theta} \right)^2 d\theta^2 + \frac{\sin \vartheta d\varphi^2}{\sin \theta d\varphi} \sin^2 \theta d\varphi^2. \]  

(A2)

The condition of the line element (A1) generated by a conformal mapping is

\[ \frac{d\vartheta}{d\theta} = \frac{\sin \bar{\vartheta} d\varphi}{\sin \theta d\varphi}. \]  

(A3)

This is equivalent to the statement

\[ \frac{d\vartheta \sin \theta}{d\theta \sin \bar{\vartheta}} = \frac{d\varphi}{d\varphi} = a, \]  

(A4)

where \( a \) is a constant. So we have

\[ \tan(\bar{\vartheta}/2) = \tan^a(\theta/2), \quad \text{and} \quad \bar{\varphi} = a\varphi. \]  

(A5)

With the formulas of the half angle, it is easy to find the relation

\[ \cos \bar{\vartheta} = \frac{(1 + \cos \theta)^a - (1 - \cos \theta)^a}{(1 + \cos \theta)^a + (1 - \cos \theta)^a}. \]  

(A6)

or, explicitly, since

\[ \sin(\bar{\vartheta}/2) = \pm \sqrt{\frac{1 - \cos \bar{\vartheta}}{2}}, \quad \text{and} \quad \cos(\bar{\vartheta}/2) = \pm \sqrt{\frac{1 + \cos \bar{\vartheta}}{2}}, \]  

(A7)

it results in the transformation

\[ \tan(\bar{\vartheta}/2) = \frac{\sin(\bar{\vartheta}/2)}{\cos(\bar{\vartheta}/2)} = \sqrt{\frac{1 - \cos \bar{\vartheta}}{1 + \cos \bar{\vartheta}}} = \tan^a(\theta/2). \]  

(A8)

The function \( \sin \bar{\vartheta} \) can be obtained by the resembling argument.

**Appendix B. The quantum number \( \nu \) for the physical solutions of \( P_{\nu}^n(z) \) and \( Q_{\nu}^n(z) \)**

The FI solutions on the sphere generated by the conformal mapping are the Legendre functions \( P_{\nu}^n(z) \) and \( Q_{\nu}^n(z) \). According to the single valuedness condition of the wave function, it has been explained in section 2 that the physically acceptable value of \( \mu = n_1/a \), with \( n_1 = 0, \pm 1, \pm 2, \ldots \). We shall only consider the case of \( a = 1 \) here. To determine the allowed values of \( \nu \), it is noted that, in the open interval \(-1 < z < +1\), the functions are associated with the hypergeometric function \( F(a, b; c; x) \) by (see, e.g., p.166, [6])

\[ P_{\nu}^n(z) = \frac{1}{\Gamma(1 - \mu)} \left( \frac{1 + z}{1 - z} \right)^{\mu/2} F\left(-\nu, \nu + 1, 1 - \mu; \frac{1 - z}{2} \right). \]  

(B1)
The details of the solution to the equation are presented here. Let us now turn our attention to the function \( Q_{\nu}^{(\mu)}(z) \). (a) When \( z \to 1^- \), we see from the expression of \( Q_{\nu}^{(\mu)} \) (p.260, [19]),

\[
Q_{\nu}^{(\mu)}(z) = 1 \frac{\Gamma(\mu)}{2 \Gamma(\nu + 1)} \cos \left( \frac{1 + z}{1 - z} \right)^{\nu/2} \Gamma\left( \nu, \nu + 1; 1 - \mu; \frac{1 - z}{2} \right) + 1 \frac{\Gamma(1 + \nu + \mu)}{\Gamma(\nu - 1 + \mu)} \Gamma\left( \frac{1 + z}{1 - z} \right)^{-\nu/2} \times \Gamma\left( \nu, \nu + 1; 1 + \mu; \frac{1 - z}{2} \right),
\]

that when \( \mu < 0 \), the function \( Q_{\nu}^{(\mu)} \to \infty \) unless \( \nu = -\mu + 1 = -n \). An alternative possibility is \( \mu = (2n + 1)/2 \), then the first term vanishes, and \( \lim_{z \to 1^-} Q_{\nu}^{(\mu)}(z) \sim (1 - z)^{\nu/2} \to 0 \). (b) For \( z \to 1^+ \), making use of the expression of \( Q_{\nu}^{(\mu)} \) (p.260, [19]),

\[
Q_{\nu}^{(\mu)}(z) = -1 \frac{\Gamma(\mu)}{2 \Gamma(\nu + 1)} \cos \left( \frac{1 + z}{1 - z} \right)^{-\nu/2} \Gamma\left( \nu, \nu + 1; 1 - \mu; \frac{1 - z}{2} \right) - 1 \frac{\Gamma(1 + \nu + \mu)}{\Gamma(\nu - 1 + \mu)} \cos(\nu + \mu) \pi \left( \frac{1 + z}{1 - z} \right)^{-\nu/2} \times \Gamma\left( \nu, \nu + 1; 1 + \mu; \frac{1 - z}{2} \right),
\]

we see that the condition of a finite solution for \( \mu < 0 \) is \( (\nu - \mu + 1) = -n \) or \( (\nu + \mu) = \pm(2n + 1)/2 \). Another possibility is \( \mu > 0 \), and \( \nu = \pm(2n + 1)/2 \). The situation makes the first term vanish, and \( \lim_{z \to 1^+} Q_{\nu}^{(\mu)}(z) \to 0 \). Accordingly, the finite conditions at both ends of \( z = \pm 1 \) for \( \mu < 0 \) are

\[
(\nu - \mu + 1) = -n \text{ with } n = 0, 1, 2, \cdots, \text{ and } \mu = n_1/a < 0, \ n_1 = 0, 1, 2, \cdots.
\]

The conditions (B4) and (B7) give two acceptable solutions in the whole domain \(-1 \leq z \leq 1\): (i)

\[
\Theta(z) = C_1 P_{\nu}^{(\mu)}(z)
\]

with the quantum number \( \mu = (n_1/a) < 0 \) and \( \nu = n + |\mu|, n = 0, 1, 2, \cdots \), (ii)

\[
\Theta(z) = C_2 P_{\nu}^{(\mu)}(z) + C_3 Q_{\nu}^{(\mu)}(z)
\]

with the allowed quantum number \( \mu = (n_1/a) < 0 \) and \( \nu = -n - |\mu| - 1, n = 0, 1, 2, \cdots \). The case (i) reduces to the known Legendre polynomials \( P_{\nu}^{(\mu)}(z) \) when the mapping is the identical transformation, i.e., \( a = 1 \).

Appendix C. The solution to the form-invariant equation of the charge-monompoles system

The FL equation of the coordinate \( \Theta \) for the charge-monompoles system is given by equation (54)

\[
\frac{1}{\sin \bar{\theta}} \frac{d}{d \bar{\theta}} \left( \sin \bar{\theta} \frac{d}{d \bar{\theta}} \Theta \right) - \frac{1}{\sin^2 \bar{\theta}} [\bar{n}^2 - 2 \bar{n} \bar{q} \cos \bar{\theta} + \bar{q}^2] \Theta = -k^2 \Theta.
\]

The details of the solution to the equation are presented here. Let \( z = \cos \bar{\theta} \), it turns into

\[
\left[ (1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} - \frac{n^2 - 2nqz + q^2}{(1 - z^2)} \right] \Theta = -k^2 \Theta.
\]
Using the ansatz
\[ \Theta = (1 - z)^{(q - n)/2}(1 + z)^{(q + n)/2} W(z), \] (C2)
the function \( W(z) \) is shown to satisfy
\[
\left\{ \left( 1 - z^2 \right) \frac{d^2}{dz^2} - \left[ 2\tilde{n} - 2(\tilde{q} + 1)z \right] \frac{d}{dz} - \tilde{q}(\tilde{q} + 1) \right\} W = -\tilde{k}^2 W. \] (C3)
Change the variable by \( y = (1 - z)/2 \) yielding
\[
\left\{ y(1 - y) \frac{d^2}{dy^2} + \left[ (\tilde{q} - \tilde{n} + 1) - 2(\tilde{q} + 1)y \right] \frac{d}{dy} - \left[ -\tilde{k}^2 + \tilde{q}(\tilde{q} + 1) \right] \right\} W = 0. \] (C4)
This is the standard form of the hypergeometric equation. The solution to the equation is the hypergeometric function (p.37, [6])
\[ W = F(A, B; C; y), \] (C5)
with the parameters given by
\[
A = (\tilde{q} + 1/2) - \sqrt{\tilde{k}^2 + 1/4}, \quad B = (\tilde{q} + 1/2) + \sqrt{\tilde{k}^2 + 1/4}, \quad \text{and} \quad C = \tilde{q} - n + 1. \] (C6)
A physically quantized condition can be given by \( A = -n, n = 0, 1, 2, \cdots \). The solution then becomes a polynomial of degree \( n \). It is associated with the Jacobi polynomial by the equality (p.212, [6])
\[ P_n^{(\alpha,\beta)}(z) = C F\left( -n, n + \alpha + \beta + 1 ; \alpha + 1; \frac{(1 - z)}{2} \right). \] (C7)
where \( C' \) is a constant. Let
\[ \alpha = (\tilde{q} - \tilde{n}), \quad \beta = (\tilde{q} + \tilde{n}), \quad \text{and} \quad J = -\frac{1}{2} + \sqrt{\tilde{k}^2 + \frac{1}{4}}. \] (C8)
equation (C5) has the expression proportional to the Jocobi polynomial
\[ W = F(A, B; C; y) \sim P_n^{(\tilde{q} - \tilde{n}, \tilde{q} + \tilde{n})}(z), \] (C9)
where \( n = J - \tilde{q} \), and we get
\[ \Theta(z) \sim \left( \frac{1 - z}{2} \right)^{(q - n)/2} \left( \frac{1 + z}{2} \right)^{(q + n)/2} P_n^{(\tilde{q} - \tilde{n}, \tilde{q} + \tilde{n})}(z). \] (C10)
With the orthogonal relation (p.212, [6])
\[
\int_{-1}^{1} P_n^{(\alpha,\beta)}(z) P_{n'}^{(\alpha',\beta')}(z)(1 - z)^\alpha(1 + z)^\beta dz = \frac{\Gamma(\alpha + n + 1)\Gamma(\beta + n + 1)}{n!\Gamma(\alpha + \beta + n + 1)} \frac{2^{\alpha + \beta + 1}}{\Gamma(\alpha + \beta + 2n + 1)} \delta_{n,n'}, \] (C11)
the normalized solution is found to be
\[ \Theta_{\tilde{q},J,a}(z) = \sqrt{\frac{(2J + 1)\Gamma(J - \tilde{q} + 1)\Gamma(J + \tilde{q} + 1)}{\Gamma(J - \tilde{n} + 1)\Gamma(J + \tilde{n} + 1)}} \times \left( \frac{1 - z}{2} \right)^{(q - n)/2} \left( \frac{1 + z}{2} \right)^{(q + n)/2} P_n^{(\tilde{q} - \tilde{n}, \tilde{q} + \tilde{n})}(z). \] (C12)
Since \( J - \tilde{q} = n = 0, 1, 2, \cdots \), the quantization value of \( J \) is according to
\[ J = |\tilde{q}|, |\tilde{q}| + 1, |\tilde{q}| + 2, \cdots, \] (C13)
which can be any fractional value, depending on the conformal index \( a \). We can determine the range of \( \tilde{n} \) by the requirements \( \alpha > -1 \), and \( \beta > -1 \) of the Jacobi polynomial \( P_n^{(\alpha,\beta)} \) (p.210, [6]). It is easy to show that both cases \( \tilde{q} > 0 \) and \( \tilde{q} < 0 \) give the same inequality
\[ -(J + 1) < \tilde{n} < (J + 1). \] (C14)
Only for the condition of \( a = 1 \), do we have \( J = 0, 1/2, 1, \cdots \), and \( -J \leq \tilde{n} \leq J \) integral steps of increment, a striking result of the charge-monopole system. The energy levels of the form-invariant system can be found by the equation \( \tilde{k}^2 = J(J + 1) \) which shows
\[ E = \frac{\hbar^2}{2MR^2} \sqrt{J(J + 1) - \tilde{q}^2}. \] (C15)
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