ON NON-SMOOTH SLOW–FAST SYSTEMS

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Abstract. We deal with non-smooth differential systems \( \dot{z} = X(z), z \in \mathbb{R}^n \), with discontinuity occurring in a codimension one smooth surface \( \Sigma \). A regularization of \( X \) is a 1-parameter family of smooth vector fields \( X^\delta, \delta > 0 \), satisfying that \( X^\delta \) converges pointwise to \( X \) in \( \mathbb{R}^n \setminus \Sigma \), when \( \delta \to 0 \). We work with two known regularizations: the classical one proposed by Sotomayor and Teixeira and its generalization, using non-monotonic transition functions. Using the techniques of geometric singular perturbation theory we study minimal sets of regularized systems. Moreover, non-smooth slow–fast systems are studied and the persistence of the sliding region by singular perturbations is analyzed.

1. Introduction

One finds in real life and in various branches of science distinguished phenomena whose mathematical models are expressed by piecewise smooth systems and deserve a systematic analysis, see for instance [10, 12, 18]. However sometimes the treatment of such objects is far from the usual techniques or methodologies found in the smooth universe. In fact, for such systems, everything we know from the qualitative theory of dynamical systems has its own versions, starting with the concept of solution.

Consider two smooth vector fields \( X^+, X^- \) defined in \( \mathbb{R}^n \). A piecewise-smooth system is \( \dot{x} = X(x) \) with

\[
X = \frac{1}{2} \left[ (1 + \text{sgn}(h))X^+ + (1 - \text{sgn}(h))X^- \right] ,
\]

\( h : \mathbb{R}^n \to \mathbb{R} \) smooth and 0 a regular value of \( h \). The set \( \Sigma = \{ x \in \mathbb{R}^n : h(x) = 0 \} \) is called switching manifold.

In order to define what a solution is, it is necessary, first of all, to agree on what happens in \( \Sigma \). The points in \( \Sigma \) are classified as regular (if \( X^+ \) and \( X^- \) are transversal to \( \Sigma \)) or singular (if \( X^+ \) or \( X^- \) is tangent to \( \Sigma \)). Moreover the regular points are classified according to Filippov’s terminology [7] (1)

(i) \( \Sigma^w = \{ x \in \Sigma : (X^+ h.X^- h)(x) > 0 \} \) is the sewing region;
(ii) \( \Sigma^s = \{ x \in \Sigma : (X^+ h.X^- h)(x) < 0 \} \) is the sliding region.

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1As usual, we denote \( Xf = \nabla f.X \).
To be more precise we subdivide $\Sigma_s$ in *attracting sliding* $\Sigma_a^s$ (if $X^+h < 0$ and $X^-h > 0$) and *repelling sliding or escape* $\Sigma_r^s$ (if $X^+h > 0$ and $X^-h < 0$).

The orbits of $X$ by $\Sigma^w$ are naturally concatenated. On $\Sigma_s$ is defined the *sliding vector field* $X^{\Sigma}$ as a linear convex combination of $X^+$ and $X^-$ which is tangent to $\Sigma$. The orbits by $\Sigma_s$ follow the flow of $X^{\Sigma}$, a linear convex combination of $X^+$ and $X^-$ tangent to $\Sigma$, that is

$$X^{\Sigma} = \frac{(X^+.h)X^- - (X^-h)X^+}{(X^+ - X^-).h}.$$ 

The vector field $X^{\Sigma}$ is called *sliding vector field*.

While Filippov said how the flow of a piecewise smooth vector field behaves when finding the set of discontinuity, Sotomayor and Teixeira (see [24]) addressed the problem by seeking smooth approximations which were called regularization. A *regularization* of $X$ is a family of smooth vector fields $X^\delta$ depending on a parameter $\delta > 0$ such that $X^\delta$ converges simply to $X$ in $\mathbb{R}^n \setminus \Sigma$ when $\delta$ goes to zero.

The Sotomayor-Teixeira regularization (*ST-regularization*) is the one parameter family $X^\delta$ given by

$$X^\delta = \left(\frac{1 + \varphi(h/\delta)}{2}\right)X^+ + \left(\frac{1 - \varphi(h/\delta)}{2}\right)X^-$$

where $\varphi : \mathbb{R} \to [-1, 1]$ is a smooth function satisfying that $\varphi(t) = 1$ for $t \geq 1$, $\varphi(t) = -1$ for $t \leq -1$ and $\varphi'(t) > 0$ for $t \in (-1, 1)$. The regularization is smooth for $\delta > 0$ and satisfies that $X^\delta = X^+$ on $\{h \geq \delta\}$ and $X^\delta = X^-$ on $\{h \leq -\delta\}$. The flow of the regularized vector field proposed by Sotomayor–Teixeira, after the limit process, is exactly the flow idealized by Filippov.

In 2005 Silva, Teixeira and Buzzi, strongly inspired by Freddy Dumortier, wrote the article [3]. They proved that the regularization proposed by Sotomayor and Teixeira generates a singular perturbation problem. This provides a very important application of GSP–theory (geometric singular perturbation theory). Joint with Llibre they published [13, 15, 16, 17]. They studied regularization problems in $\mathbb{R}^n$, the double regularization in the case in which the discontinuity has codimension 1 (intersection of two planes) and the regularization in more degenerate surfaces. Bonet-Revés, Larrosa, M-Seara, Kristiansen, Uldall and Hogan also studied the singular perturbation problem arising from regularization. More precisely, they analyzed the regularization of fold-fold singularities where bifurcation and canard boundary cycles can occur. In [1, 2] the authors use of asymptotic analysis and the extension of the critical manifold to non–normally hyperbolic points. In [11] the authors strongly use the blow–up techniques developed by Dumortier–Roussarie [5] to deal with the same problem.

Our contribution to the general theory is to investigate the following problems.
• Applying techniques of the geometric singular perturbation theory we study the limit periodic sets (equilibrium points and periodic orbits contained in the switching manifold) obtained as limit of orbits of $X^\delta$ when $\delta \downarrow 0$ and we give alternative proofs of some results of [23]. See Section 2.

• The effect of breaking the monotonicity condition of the transition function $\varphi$ used in the regularization process (3). We review the concepts of sliding and sewing and their dependence on the regularization process considered. What is the relation between such sets and those idealized by Filippov? See Section 3.

• Non–smooth slow–fast systems with sliding points in the critical manifold. We generalize the results of [4] considering the new concept of sliding and sewing. See Sections 4 and 5.

In Section 2 we show how the GSP-theory can be used to get information about the regularized vector field. From our previous papers we know that the trajectories of a piecewise smooth vector field are obtained solving a slow–fast system with critical manifold being the graphic of a smooth function defined in $\Sigma^s$. Moreover, the projection of the reduced flow, in $\Sigma$, is the sliding flow. We prove that hyperbolic equilibrium points $p$ (respectively periodic orbits $\gamma$) of the sliding vector field (2) are limit of sequences of hyperbolic equilibrium points $p_\delta$ (respectively periodic orbits $\gamma_\delta$) of the regularized vector field (3). Besides, the dimensions of stable and unstable manifolds of $p_\delta$ (respectively $\gamma_\delta$) are determined. See Theorem 1.

In Section 3 we propose a more general definition of sewing and sliding points (called $r$-sewing and $r$-sliding points, respectively). Sewing and sliding points are defined depending on the choice of the regularization. Roughly speaking, a point $p$ is a sewing point for a regularization $r$ if around $p$ the flow of $r$ is transversal to $\Sigma$. A point $p$ is a sliding point for $r$ if there exists a sequence of invariant manifolds of $r$ tending to a neighborhood of $p$ in $\Sigma$. We prove that the sewing region $\Sigma^w$ contains the $r$-sewing region $\Sigma^w_r$ and the $r$-sliding region $\Sigma^s_r$ contains the sliding region $\Sigma^s$. Besides the sliding vector field on $\Sigma^s$ can be smoothly extended on $\Sigma^s_r$. See Figure 1 and Theorem 2.

In Section 4 we study non–smooth slow–fast systems. Let $F,G : \mathbb{R}^{n+1} \times [0, +\infty) \rightarrow \mathbb{R}^n$, $H : \mathbb{R}^{n+1} \times [0, +\infty) \rightarrow \mathbb{R}$ and $h : \mathbb{R}^{n+1} \times [0, +\infty) \rightarrow \mathbb{R}$ be smooth. We consider systems of the kind

$$\dot{x} = \begin{cases} F(x, y, \varepsilon), & \text{if } h(x, y, \varepsilon) \geq 0 \\ G(x, y, \varepsilon), & \text{if } h(x, y, \varepsilon) \leq 0 \end{cases}, \quad \varepsilon \dot{y} = H(x, y, \varepsilon).$$

We prove that the $r$-sliding region in the critical manifold $H(x, y, 0) = 0$ persists for small $\varepsilon > 0$. See Theorem 3.

In Section 5 we give one more definition of sewing and sliding points using any continuous combination (not necessary convex) of $X^+$ and $X^-$. We state and prove that sliding regions obtained via continuous combination
Figure 1. Figure (A) exhibits the flow of a piecewise smooth vector fields. Figure (B) is obtained using the ST–regularization. The sliding region is given in red. Figure (C) is obtained using a $r$–regularization. Note that $\Sigma^s \subseteq \Sigma^r$.

of $X^+$ and $X^-$ also are persistent by singular perturbation. See Theorem 4.

2. Fenichel’s Theory and Sliding Vector Fields.

In order to simplify our explanation we take local local coordinates such that

$$\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}, \quad \Sigma = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y = 0\}.$$  

Let $X^+, X^- : \mathbb{R}^n \to \mathbb{R}^n$ be smooth vector fields. Denote $X^+ = (f_1, g_1)$ and $X^- = (f_2, g_2)$. The piecewise smooth vector field which we consider in this section is

$$X = \frac{1}{2} \left[ (1 + \text{sgn}(y))X^+ + (1 - \text{sgn}(y))X^- \right].$$

The sliding vector field (2) becomes

$$X^\Sigma = \frac{(X^+, y)X^- - (X^-, y)X^+}{(X^+ - X^-).y} = \left( \frac{f_2g_1 - f_1g_2}{g_1 - g_2}, 0 \right).$$

The trajectories of the ST-regularized vector field $X^\delta$ given by (3) are the solutions of the differential system

$$\mathcal{P}_\delta : \begin{cases} \dot{x} &= (f_1 + f_2)/2 + \varphi(y/\delta)(f_1 - f_2)/2 \\ \dot{y} &= (g_1 + g_2)/2 + \varphi(y/\delta)(g_1 - g_2)/2. \end{cases}$$

Consider the polar blow up $(x, y, \delta) = \Phi(x, \theta, r) = (x, r \cos \theta, r \sin \theta)$. We get the system

$$\mathcal{P}_r : \begin{cases} \dot{x} &= (f_1 + f_2)/2 + \varphi(\cot \theta)(f_1 - f_2)/2, \\ \dot{\theta} &= -\sin \theta [(g_1 + g_2)/2 + \varphi(\cot \theta)(g_1 - g_2)/2]. \end{cases}$$
For $r = 0$, $\overline{P}_r$ has two limit problems, the reduced (7) and the layer, as we will see below.

(7) $\overline{P}_0\text{\,\texttt{\bf reduced}} : \begin{cases} \dot{x} &= (f_1 + f_2)/2 + \varphi(\cot\theta)(f_1 - f_2)/2, \\ 0 &= -\sin\theta[(g_1 + g_2)/2 + \varphi(\cot\theta)(g_1 - g_2)/2]. \end{cases}$

$\overline{P}_0\text{\,\texttt{\bf layer}} : \begin{cases} x' &= 0, \\ \theta' &= -\sin\theta[(g_1 + g_2)/2 + \varphi(\cot\theta)(g_1 - g_2)/2]. \end{cases}$

The polar blow-up is not the most suitable for calculations. For this reason, we perform the directional blow-up. The directional blow-up consists in the following change of coordinates $(x, y, \delta) = \Gamma(x, \bar{y}, \delta) = (x, \delta\bar{y}, \delta)$. We observe that the directional blow-up and the polar blow-up are essentially the same. In fact, if we consider the map $G(x, \theta, r) = (x, \cot\theta, r\sin\theta)$ then $\Gamma \circ G = \Phi$. The directional blow-up applied in system (6) gives

(8) $\overline{P}_\delta : \dot{x} = \alpha(x, \bar{y}, \delta), \quad \delta\dot{\bar{y}} = \beta(x, \bar{y}, \delta)$

with

$\alpha = (f_1 + f_2)/2 + \varphi(\bar{y})(f_1 - f_2)/2, \quad \beta = (g_1 + g_2)/2 + \varphi(\bar{y})(g_1 - g_2)/2$

and the $f_1, f_2, g_1, g_2$ evaluated at $(x, \delta\bar{y})$.

2.1. GSP-theory. Systems as (8) are known in the literature as slow-fast systems. General slow-fast systems are systems of the kind

(9) $\dot{x} = \alpha(x, y, \delta), \quad \delta\dot{y} = \beta(x, y, \delta)$,
where \( x \in \mathbb{R}^n, y \in \mathbb{R}^k, \delta \geq 0 \) and \( \alpha \) and \( \beta \) are smooth functions. Taking \( \delta = 0 \) in (9) we obtain the reduced system

\[
\dot{x} = \alpha(x, y, 0), \quad \beta(x, y, 0) = 0.
\]

The set \( S = \{\beta(x, y, 0) = 0\} \) is called critical (or slow) manifold. The time scale \( \tau = t/\delta \) transforms system (9) in the fast system

\[
(10) \quad \dot{x} = \delta \alpha(x, y, \delta), \quad \dot{y} = \beta(x, y, \delta).
\]

Taking \( \delta = 0 \) in (10) we get the layer system. We say that a point \((x_0, y_0) \in S \) is normally hyperbolic if the real parts of the eigenvalues of \( D_y \beta(x_0, y_0, 0) \) are nonzero.

Let \( N \subset S \) be a compact normally hyperbolic set. Consider \((x, y) \in N \) and suppose that \( D_y \beta(x, y, 0) \) has \( k^s \) eigenvalues with negative real parts and \( k^u \) eigenvalues with positive real parts. The following result ensures the persistence of normally hyperbolic sets in \( S \) as invariant manifolds of system (9), for small values of \( \delta > 0 \).

**Proposition 1** (Fenichel, [6]). Let \( N \subset S \) be a \( j \)-dimensional compact normally hyperbolic manifold with a \((j + j^s)\)-dimensional local stable manifold \( W^s \) and a \((j + j^u)\)-dimensional local unstable manifold \( W^u \). Then there exists a family \( \mathcal{N}_\delta \) such that the following statements hold.

(a) \( N_0 = N \).

(b) \( \mathcal{N}_\delta \) is an invariant manifold of (9) with a \((j + j^s + k^s)\)-dimensional local stable manifold \( N^s_\delta \) and a \((j + j^u + k^u)\)-dimensional local unstable manifold \( N^u_\delta \).

**Proposition 2.** Let \( X \) be the piecewise smooth vector field (4) and \( X^\delta \) its ST-regularization (3). If \( \Sigma = \Sigma^s \) then all points on the critical manifold \( S \) of system (8) are normally hyperbolic. In particular \( S \) is a graphic of a smooth function \( \bar{y} = h(x) \) with \((x, 0) \in \Sigma^s \). Moreover, the projection of the reduced flow, on \( \Sigma \), is the sliding flow of (5).

**Proof.** For completeness of the proof we rewrite the proof of Fenichel’s original result in [14]. The critical manifold \( S \) is defined by the equation 

\[
(g_1 + g_2) + \varphi(\bar{y})(g_1 - g_2) = 0,
\]

with \( g_1, g_2 \) evaluated at \((x, 0)\). If \((x, 0) \in \Sigma^s \) then \( g_1 g_2 < 0 \) and thus

\[
(x, \bar{y}) \in S \iff \varphi(\bar{y}) = -\frac{g_1 + g_2}{g_1 - g_2}.
\]

Since \( \left| \frac{g_1 + g_2}{g_1 - g_2} \right| \leq 1 \) and \( \left| \frac{g_1 + g_2}{g_1 - g_2} \right| = 1 \) if and only if \( g_1 g_2 = 0 \) it follows that 

\[
D_y \beta(x, \bar{y}, 0) = \varphi'(\bar{y})(g_1 - g_2)(x, 0) \neq 0.
\]

In fact, for \((x, 0) \in \Sigma^s \) we have 

\[
\left| \frac{g_1 + g_2}{g_1 - g_2} \right| < 1, \quad \bar{y} \in (-1, 1) \text{ and thus } \varphi'(\bar{y}) > 0.
\]

It concludes the proof of the assertion about the critical manifold.

To see the relation between the reduced flow and the sliding vector vector field observe that taking \( \varphi(\bar{y}) = -\frac{g_1 + g_2}{g_1 - g_2} \) the system (8) becomes

\[
\dot{x} = \frac{f_2 g_1 - f_1 g_2}{g_1 - g_2}.
\]
which is exactly the same equation of the trajectories of $X^\Sigma$.

**Theorem 1.** Let $X$ be the piecewise smooth vector field (4). If $Q$ is a $\ell-$ dimensional compact invariant manifold of $X^\Sigma$ given by (5) with a $\ell+\ell^s-$ dimensional local stable manifold, a $\ell+\ell^u-$ dimensional local unstable manifold and $\ell^s+\ell^u=n-1$. If $Q \in \Sigma^s_\delta$ then there are a neighborhood $V$ of $Q$ in $\mathbb{R}^n$ and $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, $X^\delta$ has a invariant manifold $Q_\delta \in V$ with $(\ell + \ell^s + 1)-$dimensional stable manifold and $(\ell + \ell^u)-$dimensional unstable manifold.

**Proof.** Assume $\ell = 0$. Thus $Q = (x_0, 0)$ is a hyperbolic equilibrium point of $X^\Sigma$. The trajectories of the ST-regularization $X^\delta$ are the solutions of system (6). Performing the directional blow up we get system (8). Moreover, $Q$ is an equilibrium of $\dot{x} = \frac{f_2 g_1 - f_1 g_2}{g_1 - g_2}$, $\dot{y} = h(x)$ with $S : \dot{y} = h(x)$ defined implicitly by $(g_1 + g_2)/2 + \varphi(y)(g_1 - g_2)/2 = 0$. The hypothesis $g_1(Q) < 0$ and $g_2(Q) > 0$ implies that $S$ satisfies the attractiveness condition:

- all points in $S$ are normally hyperbolic;
- $k^s = 1$ and $k^u = 0$.

Since the reduced system and the sliding system have the same equations we conclude that $(x_0, y_0)$, with $y_0 = h(x_0)$, is an equilibrium of the reduced system with $j^s = \ell^s$ and $j^u = \ell^u$. Then, applying Proposition 1 we conclude that there exists $Q_\delta$, an equilibrium point of system (6), with $(\ell^s + 1)$-dimensional stable manifold and $(\ell^u)$-dimensional unstable manifolds.

**Example 1.** Let $X : \mathbb{R}^2 \to \mathbb{R}^2$ be the piecewise smooth vector field (4) with $X^+ = (0, -1)$ and $X^- = (x, -y + 1)$. The sliding vector field (5) is

$$X^\Sigma = \left( \frac{x}{2}, 0 \right).$$

$Q = (0, 0) \in \Sigma^s_\delta$ is an repelling equilibrium point of $X^\Sigma$, that is, $\ell^s = 0$ and $\ell^u = 1$. Thus, Theorem 1 says that for small $\delta$, $X^\delta$ has an equilibrium $Q_\delta$ of the kind saddle.

Note that this can be verified directly with simple calculations. In fact,

$$X^\delta = \left( \frac{x}{2} \left( 1 - \varphi \left( \frac{y}{\delta} \right) \right), -\frac{y}{2} + \varphi \left( \frac{y}{\delta} \right) \left( \frac{y}{2} - 2 \right) \right)$$

and

$$Q_\delta = (0, y_0), \quad \text{with} \quad \varphi \left( \frac{y_0}{\delta} \right) = \frac{y_0}{y_0 - 2}.$$ 

The existence of $Q_\delta$ is guaranteed by the fact that the graphics of $\varphi(y/\delta)$ and $\frac{y}{y - 2}$ intersect at some value of $y \in (-1, 1)$. Moreover the eigenvalues of the linearized system at $(0, y_0)$ are

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{y_0 - 2} + \varphi' \left( \frac{y_0}{\delta} \right) \cdot \frac{1}{\delta} \frac{y_0 - 2}{2}.$$ 

Since $y_0 \in (-1, 1)$ we have $\lambda_2 < 0$. Soon $Q_\delta$ is a saddle.
Example 2. Let \( X : \mathbb{R}^3 \to \mathbb{R}^3 \) be the piecewise smooth vector field (4) with
\[
X^+ = \left( 0, 2x_1 + 2x_2(\sqrt{x_1^2 + x_2^2} - 1), -1 \right)
\]
and
\[
X^- = \left( -2x_2 + 2x_1(\sqrt{x_1^2 + x_2^2} - 1), 0, 1 \right).
\]
The sliding region is \( \Sigma = (x_1, x_2) \)-plane and the sliding vector field is
\[
X^\Sigma = \left( -x_2 + x_1(\sqrt{x_1^2 + x_2^2} - 1), x_1 + x_2(\sqrt{x_1^2 + x_2^2} - 1), 0 \right).
\]
The equilibrium point \( p_0 = (0, 0, 0) \) and the limit cycle \( \gamma_0 : x_1^2 + x_2^2 = 1 \), of \( X^\Sigma \) satisfy the hypotheses of Theorem 1 with \( \ell = 0 \) and \( \ell = 1 \) respectively.

3. Non–smooth Systems and Regularization

Consider piecewise smooth system (1) defined in an open set \( \mathcal{U} \subset \mathbb{R}^n \) with \( X^+, X^- : \mathcal{U} \to \mathbb{R}^n \), \( h : \mathcal{U} \to \mathbb{R} \) smooth and assume that 0 is a regular value of \( h \).

The following regularization will be refered as \( r \)-regularization:

\[
X^r_\delta = \frac{1 + \psi(p, h/\delta)}{2} X^+ + \frac{1 - \psi(p, h/\delta)}{2} X^-,
\]
where \( \psi : \Sigma \times \mathbb{R} \to [-1, 1] \) is a more general smooth transition function satisfying that \( \psi(p, t) = -1 \) for \( t \leq -1 \) and \( \psi(p, t) = 1 \) for \( t \geq 1 \).

The definitions of \( r \)-sewing and \( r \)-sliding points were introduced in [21] and depend of the regularization \( r \) considered. More precisely, \( p \in \Sigma \) is a \( r \)-sewing point if there exist an open neighborhood \( \mathcal{U} \subset \mathbb{R}^{n-1} \times \mathbb{R} \) of \( p \) and local coordinates \( (x, y) \) defined in \( \mathcal{U} \) such that:
(a) \( \Sigma = \{ y = 0 \} \);
(b) for each sufficiently small \( \delta > 0 \), the vector field \( v(x, y) = (0, 1) \) is a generator of (11) in \( \mathcal{U} \).

\( p \in \Sigma \) is a \( r \)-sliding point if there exist an open neighborhood \( \mathcal{U} \subset \mathbb{R}^n \) of \( p \) and a family of smooth manifolds \( S_\delta \subset \mathcal{U} \) satisfying:
(a) \( S_\delta \) is invariant for \( r \);
(b) for each compact \( K \subset \mathcal{U} \), the sequence \( S_\delta \cap K \) converges to \( \Sigma \cap K \), as \( \delta \) goes to zero according to Hausdorff distance.

We denote \( \Sigma^w_r \) and \( \Sigma^s_r \) the \( r \)-sewing and \( r \)-sliding regions, respectively and we assume local coordinates \( (x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \) such that \( h(x, y) = y \), \( X^+ = (f_1, g_1) \), \( X^- = (f_2, g_2) \) with
\[
(f_i(x, y), g_i(x, y)) \in \mathbb{R}^{n-1} \times \mathbb{R}, \quad i = 1, 2.
\]
Theorem 2. Consider a non-smooth system (1) and a $r$-regularization (11). Suppose that $\frac{\partial \psi}{\partial \bar{y}}(x,t) \neq 0$, for all $x \in \Sigma^s$ and $|t| < 1$. Then the following statements hold.

(a) $\Sigma^w_w \subseteq \Sigma^w$ and $\Sigma^s \subseteq \Sigma^s_r$.

(b) If $g_1 \neq g_2$ in $\Sigma^s_r \setminus \Sigma^s$, then the sliding vector field $X^S$ can be smoothly extended from $\Sigma^s$ to $\Sigma^s_r$.

Proof. Consider $x_0 \in \Sigma^w_w$. Since $v(x,y) = (0,1)$ is a generator of (11), $X^-(x_0,0)$ and $X^+(x_0,0)$ point to the same hand side. So

$$(X^+.y)(X^-y)(x_0,0) > 0$$

and we conclude that $x_0 \in \Sigma^w_w$. Therefore $\Sigma^w_w \subseteq \Sigma^w$.

Now, consider $x_0 \in \Sigma^s$. Taking the directional blow-up $y = \delta \bar{y}$, the $r$-regularization (11) becomes the slow–fast system

\begin{equation}
\dot{x} = \alpha(x, \bar{y}, \delta), \quad \delta \dot{\bar{y}} = \beta(x, \bar{y}, \delta)
\end{equation}

where

$$\alpha(x, \bar{y}, \delta) = 1/2((1 + \psi(x, \bar{y}))f_1 + (1 - \psi(x, \bar{y})))f_2$$

and

$$\beta(x, \bar{y}, \delta) = 1/2((1 + \psi(x, \bar{y}))g_1 + (1 - \psi(x, \bar{y})))g_2,$$ 

with the functions $f_i$ and $g_i$, $i = 1, 2$, evaluated at $(x, \delta \bar{y})$. The reduced system

\begin{equation}
\dot{x} = \alpha(x, \bar{y}, 0),
\end{equation}

is defined for $(x, \bar{y})$ in the critical manifold $S = \{\beta(x, \bar{y}, 0) = 0\}$. Consider $|y_0| < 1$ such that $(x_0, y_0) \in S$. Note that $(g_1 - g_2)(x_0,0) \neq 0$ because $g_1g_2(x_0,0) < 0$. So,

$$\frac{\partial \beta}{\partial \bar{y}}(x_0, y_0, 0) = \frac{\partial \psi}{\partial \bar{y}}(x_0, y_0). (g_1 - g_2)(x_0,0) \neq 0$$
and thus \((x_0, y_0)\) is a normally hyperbolic point. The Fenichel’s result ensures the existence of an invariant manifold \(S_\delta\) of (12) such that \(S_\delta \to S_0\) as \(\delta \to 0\), according to Hausdorff’s distance. Thus \(x_0 \in \Sigma^s_r\) and it concludes the proof of item (a).

Since \((g_1 - g_2)(x,0) \neq 0\) for \(x \in \Sigma^s_r\), the equation \(\beta(x, \bar{y}, 0) = 0\) provides

\[
\psi(x, \bar{y}) = -\frac{g_1 + g_2}{g_1 - g_2}.
\]

The dynamics at \((x, \bar{y})\) obtained from (13) and (14) is given by

\[
\dot{x} = \frac{f_2g_1 - f_1g_2}{g_1 - g_2}.
\]

Hence \(X^\Sigma\) can be smoothly extended on \(\Sigma^s_r\). It concludes the proof of item (b).

The following example illustrates Theorem 2.

**Example 3.** Consider the non–smooth system

\[
\dot{x} = (\dot{x}_1, \dot{x}_2) = \begin{cases} 
(x_2 - 1, -1), & \text{if } x_1 \geq 0, \\
(x_2, 1), & \text{if } x_1 \leq 0.
\end{cases}
\]

The sliding region is \(\Sigma^s = ]0, 1[\) and the sliding vector field (2) is

\[
X^\Sigma = (0, -2x_2 + 1).
\]

The \(r\)-regularization of (15) is the 1–parameter family

\[
x_1' = 1/2(2x_2 - 1 - \psi(x_1/\delta, x_2)), \quad x_2' = -\psi(x_1/\delta, x_2).
\]

Suppose that partial derivative \(\frac{\partial \psi}{\partial \bar{y}}(t, x_2)\) vanishes only at \((t, x_2) = (a_0, b_0)\) with \(-1 < a_0 < 1\) and \(b_0 > 1\). Applying the directional blow–up \(x_1 = \delta \bar{x}_1\) we obtain the slow–fast system

\[
\dot{\delta} = \frac{1}{2}(2x_2 - 1 - \psi(\bar{x}_1, x_2)), \quad \dot{x}_2 = -\psi(\bar{x}_1, x_2).
\]

The critical manifold \(S\) is given implicitly by

\[
x_2 = \frac{1}{2} \psi(\bar{x}_1, x_2) + 1
\]

and it is a curve connecting \((\bar{x}_1, x_2) = (-1, 0)\) to \((\bar{x}_1, x_2) = (1, 1)\). All points in \(S_0 = S \setminus (a_0, b_0)\) are normally hyperbolic. So Fenichel’s result ensures the existence of an invariant manifold \(S_\delta\) of (16) converging to \(S_0\) according to Hausdorff’s distance. Thus \(\Sigma^s_r = ]0, b_0[\). The reduced system in \(S_0\) is

\[
\dot{x}_2 = -2x_2 + 1.
\]

Thus \(X^\Sigma\) can be smoothly extended on \(\Sigma^s_r = ]0, b_0[\). Note that the \(r\)-sliding region \(\Sigma^s_r\) obtained contains the sliding region \(\Sigma^s = ]0, 1[\) defined by Filippov. See Figure 4.
4. NON–SMOOTH SLOW–FAST SYSTEMS

A non–smooth slow–fast system is

\[ \dot{x} = \begin{cases} F(x, y, \varepsilon), & \text{if } h(x, y, \varepsilon) \geq 0, \\ G(x, y, \varepsilon), & \text{if } h(x, y, \varepsilon) \leq 0, \end{cases}, \quad \varepsilon \dot{y} = H(x, y, \varepsilon) \]

where \( x \in \mathbb{R}^n, y \in \mathbb{R} \) and \( \varepsilon > 0 \) is a small parameter. For each \( \varepsilon \geq 0 \) let \( \Sigma_\varepsilon \) be the switching manifold of (17), i.e. \( \Sigma_\varepsilon = \{ h(x, y, \varepsilon) = 0 \} \). We assume that \( \Sigma_0 \) and \( S = \{ H(x, y, 0) = 0 \} \) are transversal and \( H_y(x, y, 0) \neq 0 \) on \( S \).

Taking \( \varepsilon = 0 \) in (17) we have the reduced system

\[ \dot{x} = \begin{cases} \bar{F}(x) & \text{if } \bar{h}(x) \geq 0, \\ \bar{G}(x) & \text{if } \bar{h}(x) \leq 0, \end{cases}, \quad H(x) = 0, \]

where \( \bar{F}(x) = F(x, y(x), 0), \bar{G}(x) = G(x, y(x), 0), \bar{h}(x) = h(x, y(x), 0) \) and \( H(x) = H(x, y(x), 0) \). System (18) is defined in the critical manifold \( S = \{ H(x) = 0 \} \).

We denote \( \Sigma_{r,0}^w, \Sigma_{r,0}^s \) and \( \Sigma_{r,\varepsilon}^w, \Sigma_{r,\varepsilon}^s \) the \( r \)-sewing and \( r \)-sliding regions of systems (17) and (18) respectively. The \( r \)-regularizations of (17) and (18) are

\[ \dot{x} = 1/2 \left( (1 + \psi(h/\delta, p)) F + (1 - \psi(h/\delta, p)) G \right), \quad \varepsilon \dot{y} = H \]

and

\[ \dot{x} = 1/2 \left( (1 + \psi(\bar{h}/\delta, p)) \bar{F} + (1 - \psi(\bar{h}/\delta, p)) \bar{G} \right), \quad \bar{H} = 0, \]

respectively, where \( \psi : \mathbb{R} \times \Sigma \rightarrow [-1, 1] \) is a more general smooth transition function satisfying that \( \psi(t, p) = -1 \) for \( t \leq -1 \) and \( \psi(t, 1) = 1 \) for \( t \geq 1 \).
The next theorem states that $r$-sliding points $p_0 \in \Sigma_{r,0}$ persist under effect of singular perturbations with the additional assumptions:

\begin{equation}
\frac{\partial H}{\partial y}(p_0, 0) \neq 0, \quad \frac{\partial h}{\partial x}(p_0, 0) \neq 0, \quad \frac{\partial h}{\partial y} \equiv 0,
\end{equation}

\begin{equation}
\tilde{F}.h(p_0, 0) \neq \tilde{G}.h(p_0, 0), \quad \frac{\partial \psi}{\partial t}(t, p_0) \neq 0 \quad \text{for } -1 < t < 1.
\end{equation}

The assumptions are necessary for applying the change of coordinates described in the proof and for ensuring the normal hyperbolicity of $p_0$. In [22] Sieber and Kowalczyk show that stable periodic motion with sliding is not robust under effect of singular perturbations. Fridman ([8, 9]) also studies periodic motion considering the last assumption of (19). In [4] the authors provide examples showing that sliding regions are not persistent with respect singular perturbations if this assumption is not considered.

**Theorem 3.** Consider a non–smooth slow–fast system (17) and $p_0 \in \Sigma_{r,0}$ satisfying the assumptions (19) and (20). Then the following statements hold.

(a) There exist sufficiently small $\varepsilon_0 > 0$ and a family of $r$-sliding points \( \{p_\varepsilon : \varepsilon \in (0, \varepsilon_0)\} \) of system (17) such that $p_\varepsilon \to p_0$ as $\varepsilon \to 0$, according to Hausdorff distance.

(b) If $p_0$ is an equilibrium point (or periodic orbit) of the sliding vector field associated to reduced system (18) then there exist sufficiently small $\varepsilon_1 > 0$ and a family of equilibrium points (or periodic orbits) \( \{p_\varepsilon : \varepsilon \in (0, \varepsilon_1)\} \) of the sliding vector field associated to system (17) such that $p_\varepsilon \to p_0$ as $\varepsilon \to 0$, according to Hausdorff distance.

**Example 4.** Consider the non–smooth slow–fast system

\begin{equation}
(\dot{x}_1, \dot{x}_2) = \begin{cases}
(x_2 - 1, -1 + \varepsilon), & \text{if } x_1 \geq 0, \\
(x_1 + x_2 + \varepsilon, x_2 + 1 - \varepsilon), & \text{if } x_1 \leq 0,
\end{cases} \quad \varepsilon \dot{y} = y.
\end{equation}

The corresponding reduced system

\begin{equation}
(\dot{x}_1, \dot{x}_2) = \begin{cases}
(x_2 - 1, -1), & \text{if } x_1 \geq 0, \\
(x_1 + x_2, 1), & \text{if } x_1 \leq 0,
\end{cases}
\end{equation}

is defined on the plane \( \{y = 0\} \). Note that $(0, 0, 0)$ and $(0, 1, 0)$ are fold points and $\Sigma^s = [0, 1]$. See Figure 5–(A).

The $r$-regularization of (21) is

\begin{align*}
\dot{x}_1 &= 1/2(-1 + x_1 + 2x_2 + \varepsilon - (x_1 + \varepsilon + 1)\psi(x_1/\delta, x_2, y)), \\
\dot{x}_2 &= 1/2(x_2 - (2 + 2x_2 - 2\varepsilon)\psi(x_1/\delta, x_2, y)), \\
\varepsilon \dot{y} &= y.
\end{align*}

Assume that the partial derivative $\frac{\partial \psi}{\partial t}(t, x_2, 0)$ vanishes only at $(t, x_2, 0) = (a_0, b_0, 0)$ with $a_0 = 0$ and $b_0 = -2$. Applying the directional blow-up
Figure 5. In (A) we have the folds points (0, 0, 0) and (0, 1, 0). The sliding region is $\Sigma_s^s = [0, 1]$. In (B) we obtain after the blow-up a smooth curve (critical manifold) connecting the fold points. The point (0, −2, 0) is a non normal hyperbolic point.

$x_1 = \delta x_1$ the previous system becomes

$$
\begin{align*}
\delta \dot{x}_1 &= 1/2 (-1 + \delta x_1 + 2 x_2 + \varepsilon - (\delta x_1 + \varepsilon + 1)\psi(x_1, x_2, y)), \\
\dot{x}_2 &= 1/2 (x_2 - (2 + x_2 - 2\varepsilon)\psi(x_1, x_2, y)), \\
\varepsilon \dot{y} &= y.
\end{align*}
$$

The reduced system ($\varepsilon = \delta = 0$) associated to (23) is

$$
\begin{align*}
0 &= -1 + 2 x_2 - \psi(x_1, x_2, 0), \\
\dot{x}_2 &= 1/2 (x_2 - (2 + x_2)\psi(x_1, x_2, 0)), \\
0 &= y,
\end{align*}
$$

and it is a $r$–regularization of (22). The slow manifold $\mathcal{S} = \{(x_1, (1 + \psi(x_1, x_2, 0))/2, 0)\}$ is a curve connecting the points (−1, 0, 0) and (1, 1, 0). All points in $\mathcal{S}$ are normally hyperbolic (parameter \(\delta\)), except the point (0, −2, 0). According to Fenichel’s result $\mathcal{S}_0 = \mathcal{S} - \{(0, -2, 0)\}$ persists for the system

$$
\begin{align*}
\delta \dot{x}_1 &= 1/2 (-1 + \delta x_1 + 2 x_2 - (\delta x_1 + 1)\psi(x_1, x_2, y)), \\
\dot{x}_2 &= 1/2 (x_2 - (2 + x_2)\psi(x_1, x_2, 0)), \\
0 &= y,
\end{align*}
$$

i.e, there exists an invariant manifold $\mathcal{S}_{0,\delta}$ of the previous system converging to $\mathcal{S}_0$, as $\delta \to 0$. So, $\Sigma_{r,0}^s = [-2, 1]$. For each small $\delta > 0$, $\mathcal{S}_{0,\delta}$ is normally hyperbolic (parameter $\varepsilon$) for the previous system. Thus, there exists an invariant manifold $\mathcal{S}_{\varepsilon,\delta}$ of (23) such that

$$
\mathcal{S}_{\varepsilon,\delta} \xrightarrow{\varepsilon \to 0} \mathcal{S}_{0,\delta} \xrightarrow{\delta \to 0} \mathcal{S}_0,
$$

that is, there exists a $r$-sliding region $\Sigma_{r,\varepsilon}^s$ of (21) converging to $\Sigma_{r,0}^s$.

The sliding vector field of (22) is

$$
(24) \quad x_1 = 0, \quad \dot{x}_2 = 1 - x_2 - x_2^2, \quad y = 0.
$$
It has the equilibrium points

\[ p_0^\pm = \left(0, (-1 \pm \sqrt{5})/2, 0 \right). \]

The sliding vector field of (21) is the slow–fast system

\[
\begin{align*}
    x_1 &= 0, \\
    \dot{x}_2 &= -\frac{x_2^2 + 2\varepsilon x_2 - x_2 + \varepsilon^2 - 2\varepsilon + 1}{\varepsilon + 1}, \quad \varepsilon \dot{y} = y.
\end{align*}
\]

Note that its reduced system coincides with (24). It has the equilibrium points

\[ p_\varepsilon^\pm = \left(0, \left(-1 + 2\varepsilon \pm \sqrt{5 - 12\varepsilon + 8\varepsilon^2}\right)/2, 0 \right), \]

converging respectively to \( p_0^+ \) and \( p_0^- \) as \( \varepsilon \) goes to zero.

**Proposition 3.** Consider the non–smooth slow–fast system (17) and \( p_0 \in \Sigma^{s}_{r,0} \) satisfying the last two assumptions of (19). Then there exist local co-ordinates around \( (p, \varepsilon) = (p_0, 0) \) such that \( h(x, y, \varepsilon) = x_1 \).

**Proof.** Without lost of generality we can suppose that \( \partial h/\partial x_1(p_0, 0) \neq 0 \). Applying the change of coordinates \( x_1 = h(x, y, \varepsilon) \), \( x_i = x_i \) and \( y = y \), for \( i = 2, ..., n \), we obtain

\[
\begin{align*}
    \dot{x}_1 &= \frac{\partial h}{\partial x_1} \dot{x}_1 + \ldots + \frac{\partial h}{\partial x_n} \dot{x}_n + \frac{\partial h}{\partial y} \dot{y} + \frac{\partial h}{\partial \varepsilon} \dot{\varepsilon} \\
    &= \frac{\partial h}{\partial x_1} \dot{x}_1 + \ldots + \frac{\partial h}{\partial x_n} \dot{x}_n + \frac{\partial h}{\partial y} \dot{y} + \frac{\partial h}{\partial \varepsilon} \dot{\varepsilon}.
\end{align*}
\]

Since \( \partial h/\partial y \equiv 0 \) and \( \dot{\varepsilon} = 0 \), the previous expression becomes

\[
\dot{x}_1 = \frac{\partial h}{\partial x_1} \dot{x}_1 + \ldots + \frac{\partial h}{\partial x_n} \dot{x}_n.
\]

Since \( \partial h/\partial x_1 \neq 0 \) the determinant of the change of coordinates matrix is nonzero and system (17) becomes

\[
\dot{x} = \begin{cases} F(\overline{x}, \overline{y}, \varepsilon), & \text{if } \overline{x}_1 \geq 0, \\
G(\overline{x}, \overline{y}, \varepsilon), & \text{if } \overline{x}_1 \leq 0, \end{cases}, \quad \varepsilon \dot{y} = H(\overline{x}, \overline{y}, \varepsilon).
\]

where \( \overline{x} = (x_1, ..., x_n) \).

**Proof of Theorem 3.** Suppose that \( p_0 \in \Sigma^{s}_{r,0} \). According to Proposition 3 we can assume that \( h(x, y, \varepsilon) = x_1 \) in (17) around \( (p, \varepsilon) = (p_0, 0) \). Denote \( p = (x_2, ..., x_n, y) \). The switching manifold becomes \( \Sigma_\varepsilon = \{(0, p)\} \) for each \( \varepsilon \geq 0 \). The \( r \)-regularization of (17) is the 2–parameters (\( \varepsilon \) and \( \delta \)) family

\[
\dot{x} = 1/2((1 + \psi(x_1/\delta, p))F + (1 - \psi(x_1/\delta, p))G), \quad \varepsilon \dot{y} = H.
\]

Note that it is a slow–fast system (parameter \( \varepsilon \)) and its reduced system is a \( r \)-regularization of reduced system (18). Applying the directional blow-up
\( x_1 = \delta \bar{x}_1 \) we obtain the \textit{three time scale singular perturbation problem}

\[
\begin{align*}
\delta \dot{\bar{x}}_1 &= \alpha_1(\bar{x}_1, p, \varepsilon, \delta), \\
\dot{x}_i &= \alpha_i(\bar{x}_1, p, \varepsilon, \delta), \\
\varepsilon \dot{y} &= \overline{H}(\bar{x}_1, p, \varepsilon),
\end{align*}
\]

for \( i = 2, ..., n \), where

\[
\alpha_i(\bar{x}_1, p, \varepsilon, \delta) = 1/2((1 + \psi(\bar{x}_1, p))F_1 + (1 - \psi(\bar{x}_1, p))G_i),
\]

for \( i = 1, ..., n \), with the function \( F_i \) and \( G_i \) evaluated at \((\delta \bar{x}_1, p, \varepsilon)\) and \( \overline{H}(\bar{x}_1, p, \varepsilon, \delta) = H(\delta \bar{x}_1, p, \varepsilon) \). The reduced system associated to (25) \((\varepsilon = \delta = 0)\) is the following

\[
\begin{align*}
0 &= \alpha_1(\bar{x}_1, p, 0, 0), \\
\dot{x}_i &= \alpha_i(\bar{x}_1, p, 0, 0), \\
0 &= \overline{H}(\bar{x}_1, p, 0, 0),
\end{align*}
\]

for \( i = 2, ..., n \). Let \( S \) be the critical manifold given by equations (26) and (27) and \((t_0, p_0) \in S\). Since

\[
\frac{\partial \overline{H}}{\partial y}(t_0, p_0, 0, 0) = \frac{\partial H}{\partial y}(0, p_0, 0) \neq 0,
\]

Proposition 1 says that the point \((t_0, p_0)\) persists for the system

\[
\begin{align*}
0 &= \alpha_1(\bar{x}_1, p, \varepsilon, 0), \\
\dot{x}_i &= \alpha_i(\bar{x}_1, p, \varepsilon, 0), \\
\varepsilon \dot{y} &= \overline{H}(\bar{x}_1, p, \varepsilon, 0),
\end{align*}
\]

for \( i = 2, ..., n \). Indeed there exists a compact set \( S_0 \subset S \) containing \((t_0, p_0)\) and a family of invariant manifolds \( S_\varepsilon \) of (28), such that \( S_\varepsilon \to S_0 \) as \( \varepsilon \to 0 \), according to Hausdorff distance. See Figure 6. Since

\[
\frac{\partial \alpha_1}{\partial \bar{x}_1}(\bar{x}_1, p, 0, 0) = \frac{\partial \psi}{\partial \bar{x}_1}(\bar{x}_1, p). (F_1 - G_1)(0, p, 0) \neq 0,
\]

for \((\bar{x}_1, p) \in S_0\), by continuity we have that

\[
\frac{\partial \alpha_1}{\partial \bar{x}_1}(\bar{x}_1, p, 0, \varepsilon) \neq 0,
\]

for \((\bar{x}_1, p) \in S_\varepsilon \) and sufficiently small \( \varepsilon > 0 \). Thus Proposition 1 ensures the persistence of \( S_\varepsilon \) for system (25). More specifically, there exists a family of invariant manifolds \( S_{\varepsilon, \delta} \) of (25) such that \( S_{\varepsilon, \delta} \to S_\varepsilon \) as \( \delta \to 0 \).

Therefore there exists a family of \( r \)-sliding points \( p_\varepsilon \) of (17) satisfying that \( p_\varepsilon \to p_0 \) as \( \varepsilon \to 0 \), according to Hausdorff’s distance. It concludes the proof of statement (a).

Note that the sliding vector field associated to (17) is the slow–fast system

\[
\dot{x}_i = \frac{F_1 G_i - G_i F_1}{F_1 - G_1}(0, p, \varepsilon), \quad \varepsilon \dot{y} = H(0, p, \varepsilon),
\]
for $i = 2, ..., n$, and its reduced system
\[
\dot{x}_i = \frac{F_1G_i - G_1F_i}{F_1 - G_1}(0, p, 0), \quad H(0, p, 0) = 0,
\]
has the dynamics of the sliding vector field associated to (18). So the proof of statement (b) follows directly from Proposition 1.

5. Continuous Combinations of Non–Smooth Systems

Now we consider another way to define sliding points. Instead of considering a convex combination of vectors $X^+(p)$ and $X^-(p)$ we consider a continuous combination. This convention is given in [19] and [20].

Consider a non–smooth system (1). A continuous combination of $X^+$ and $X^-$ is a 1–parameter family of smooth vector fields $\tilde{X}(\lambda, p)$ with $(\lambda, p) \in [-1, 1] \times \Sigma$ satisfying that $\tilde{X}(1, p) = X^+(p)$ and $\tilde{X}(-1, p) = X^-(p)$. We denote
\[
[X^+, X^-]^c = \{ \tilde{X}(\lambda, p), \lambda \in [-1, 1] \}.
\]

Consider and a regular point $p \in \Sigma$.

(i) We say that $p$ is a c-sewing point and denote $p \in \Sigma^w_c$ if $\tilde{X}.h(\lambda, p) \neq 0$ for all $\lambda \in (-1, 1)$.

(ii) We say that $p$ is a c-sliding point and denote $p \in \Sigma^s_c$ if there exists $\lambda \in (-1, 1)$, such that $\tilde{X}.h(\lambda, p) = 0$.

We say that $\tilde{X}(\lambda(p), p)$ is a c-sliding vector field if for each $p \in \Sigma^s_c$ there exists $\lambda(p) \in (-1, 1)$ such that $\tilde{X}.h(\lambda(p), p) = 0$. 

Figure 6. Geometric situation in the blow–up process described in the proof of Theorem 3.
There may be more than one possible sliding on $p$, see Figure 7. In [20] the authors prove that $\Sigma^w_c \subseteq \Sigma^w$ and $\Sigma^s_c \subseteq \Sigma^s_c$.

Here we study the persistence of $c$-sliding points via slow–fast systems considering the continuous combination

$$\tilde{Y}(\lambda, x, y, \varepsilon) = (\tilde{X}(\lambda, x, y, \varepsilon), H(x, y, \varepsilon)/\varepsilon),$$

where $\tilde{X}(\lambda, x, y, \varepsilon)$ is a continuous combination of $X^+(x, y, \varepsilon)$ and $X^-(x, y, \varepsilon)$.

We denote $\Sigma^w_c, 0$, $\Sigma^s_c, 0$ and $\Sigma^w_c, \varepsilon$, $\Sigma^s_c, \varepsilon$ the $c$-sewing and $c$-sliding regions of systems (17) and (18) respectively.

The next theorem provides results like the ones given in Theorem 3 however for $c$-sliding points. We consider the assumption (19) and the following one

(29) $$\frac{\partial Q}{\partial \lambda}(\lambda^*, p_0, 0) \neq 0,$$

where $Q(\lambda, p, \varepsilon) = \tilde{Y}.\nabla h(\lambda, p, \varepsilon)$ and $\lambda^*$ satisfies the equation $Q(\lambda^*, p_0, 0) = 0$ for $p_0 \in \Sigma^s_c, 0$.

**Theorem 4.** Consider a non–smooth slow–fast system (17) and $p_0 \in \Sigma^s_r, 0$ satisfying the assumptions (19) and (29). Then the following statements hold.

(a) There exist sufficiently small $\varepsilon_0 > 0$ and a family of $c$-sliding points $\{p_\varepsilon : \varepsilon \in (0, \varepsilon_0)\}$ of system (17) such that $p_\varepsilon \to p_0$ as $\varepsilon \to 0$, according to Hausdorff distance.

(b) If $p_0$ is an equilibrium point (or periodic orbit) of the $c$-sliding vector field associated to reduced system (18) then there exist sufficiently small $\varepsilon_1 > 0$ and a family of equilibrium points (or periodic orbits) $\{p_\varepsilon : \varepsilon \in (0, \varepsilon_1)\}$ of the $c$-sliding vector field associated to system (17) such that $p_\varepsilon \to p_0$ as $\varepsilon \to 0$, according to Hausdorff distance.

**Example 5.** Consider the non–smooth slow–fast system

(30) $$\dot{x} = \begin{cases} (x_2 - 1 + \varepsilon, -1 + \varepsilon) & \text{if } x_1 \geq 0 \\ (x_2 + \varepsilon, 1 + \varepsilon) & \text{if } x_1 \leq 0 \end{cases}, \quad \varepsilon y = y$$

and the continuous combination

$$\tilde{Y}(\lambda, x_1, x_2, y, \varepsilon) = (\lambda^2(x_2 + \varepsilon) - (\lambda + 1)/2, \lambda^2 - \lambda - 1 + \varepsilon, y/\varepsilon).$$

The equation $\lambda^2(x_2 + \varepsilon) - (\lambda + 1)/2 = 0$ provides

$$\lambda_1^\varepsilon = \frac{1 + \sqrt{1 + 8x_2 + 8\varepsilon}}{4(x_2 + \varepsilon)}, \quad \lambda_2^\varepsilon = \frac{1 + \sqrt{1 + 8x_2 + 8\varepsilon}}{4(x_2 + \varepsilon)}.$$

Replacing $\lambda_1^\varepsilon$ and $\lambda_2^\varepsilon$ in $\tilde{Y}$ we obtain two $c$-sliding vector fields

$$X^\lambda_\varepsilon = \left(0, \frac{(\sqrt{8x_2 + 8\varepsilon + 1} + 1)^2}{16(x_2 + \varepsilon)^2} - \frac{\sqrt{8x_2 + 8\varepsilon + 1} + 1}{4(x_2 + \varepsilon)} + \varepsilon - 1, \frac{y}{\varepsilon}\right).$$
According to Proposition 3 we can assume that

\[ h(x, y, \varepsilon) = x_1 \] around 

\( (p, \varepsilon) = (p_0, 0) \). Denote 

\[ p = (x_2, \ldots, x_n, y) \] and 

\[ \tilde{X}(\lambda, x_1, p) = (s_1(\lambda, x_1, p), \ldots, s_n(\lambda, x_1, p)) \] 

a continuous combinations of \( \tilde{F} \) and \( \tilde{G} \).
The c-sliding regions of (17) and (18) become
\[
\Sigma_{c,\varepsilon}^s = \{(0, x_2, ..., x_n, y) : \exists \lambda \in (-1, 1), s_1(\lambda, x, y, \varepsilon) = 0\},
\]
\[
\Sigma_{c,0}^s = \{(0, x_2, ..., x_n, y) : \exists \lambda^* \in (-1, 1), s_1(\lambda^*, x, y, 0) = 0\}.
\]

Consider \(\lambda^*\) such that \(s_1(\lambda^*, p_0, 0) = 0\). Assumption (29) ensures the existence of a neighborhood \(V\) of \((p_0, 0)\) such that \(\lambda = \lambda(q, \varepsilon), \lambda(p_0, 0) = \lambda^*\) and \(s_1(\lambda(q, \varepsilon), q, \varepsilon) = 0\) for all \((q, \varepsilon) \in V\), in particular for \(q \in V \cap \Sigma_{c,\varepsilon}^s\). Therefore there exists \(p_\varepsilon \in \Sigma_{c,\varepsilon}^s\) such that \(p_\varepsilon \rightarrow p_0\) as \(\varepsilon \rightarrow 0\) and statement (a) is proved.

The dynamics of the c-sliding vector field \(\tilde{Y}\) is given by
\[
\dot{x}_i = s_i(\lambda, x, y, \varepsilon), \quad \varepsilon \dot{y} = H(x, y, \varepsilon), \quad i = 2, ..., n,
\]
with \(\lambda\) satisfying the equation \(s_1(\lambda, x, y, \varepsilon) = 0\). Note that system (33) is a singular perturbation problem. The dynamic of the c-sliding vector field associated to system (18) is given by
\[
\dot{x}_i = s_i(\lambda^*, x, y(x), 0), \quad i = 2, ..., n,
\]
with \(\lambda^*\) satisfying \(s_1(\lambda^*, x, y(x), 0) = 0\).

Since \(\lambda = \lambda(q, \varepsilon)\) the reduced system associated to system (33) is given by
\[
\dot{x}_i = s_i(\lambda(q, 0), x, y(x), 0), \quad i = 2, ..., n,
\]
Therefore system (34) coincides with system (35). Now we apply the Fenichel’s result for concluding the proof of item (b).

**Remark.** In [19] and [20] is defined a regularization of (1) called *nonlinear regularization* as

\[
\dot{x} = \tilde{X}(\varphi(h/\delta), p),
\]

where \(\varphi\) is a monotonic transition function. Next lemma says that \(\Sigma^c\) is the sliding region linked to nonlinear regularization (36). Thus Theorem 4 can be proved such as Theorem 3 but considering the nonlinear regularization of systems (17) and (18)

\[
\begin{align*}
\dot{x} &= \tilde{Y}(\varphi(h/\varepsilon), x, y, \varepsilon_2), \\
\dot{x} &= \tilde{X}(\varphi(h/\varepsilon), x, y(x)), \quad \tilde{H}(x) = 0,
\end{align*}
\]

respectively.

**Lemma 5.** Consider a non-smooth system (1) with \(K(\lambda, p) = \tilde{X}.h(\lambda, p)\) and \(\tilde{X}\) a continuous combination of \(X^+\) and \(X^-\). Suppose that there exists \(\lambda^* \in (-1, 1)\) such that

\[
\frac{\partial K}{\partial \lambda}(\lambda^*, p_0) \neq 0,
\]

for \(p_0 \in \Sigma\). Thus \(p_0\) is a c-sliding point if and only if \(p_0\) is a sliding point for nonlinear regularization (36).

**Proof.** Consider a system like (1) and a continuous combination \(\tilde{X}\) of \(X^+\) and \(X^-\). Take local coordinates \((x_1, p) = (x_1, ..., x_n)\) such that \(h(x) = x_1\) and \(\tilde{X}(\lambda, x) = (s_1(\lambda, x_1, p), ..., s_n(\lambda, x_1, p))\). The switching manifold and the c-sliding region become

\[
\begin{align*}
\Sigma &= \{(0, p)\}, \\
\Sigma^c &= \{(0, p) : \exists \lambda \in (-1, 1), s_1(\lambda, 0, p) = 0\},
\end{align*}
\]

respectively. Consider \(p_0 \in \Sigma^c\). Thus there exists \(\lambda^*\) satisfying \(s_1(\lambda^*, 0, p) = 0\). The nonlinear regularization (36) becomes

\[
\begin{align*}
\dot{x}_1 &= s_1(\varphi(x_1/\delta), x_1, p), \\
\dot{x}_i &= s_n(\varphi(h/\delta), x_1, p),
\end{align*}
\]

for \(i = 2, ..., n\). Taking the blow-up \(x_1 = \delta \overline{x}_1\) the previous system becomes the slow–fast system

\[
\begin{align*}
\delta \dot{\overline{x}}_1 &= s_1(\varphi(\overline{x}_1), \delta \overline{x}_1, p), \\
\dot{\overline{x}}_i &= s_n(\varphi(\overline{x}_1), \delta \overline{x}_1, p),
\end{align*}
\]

for \(i = 2, ..., n\). Define \(\lambda = \varphi(\overline{x}_1)\). Since \(\varphi'(t) > 0\) in \((-1, 1)\) there exists \(\overline{x}_1^*\) such that \(\lambda^* = \varphi(\overline{x}_1^*)\) with \((\overline{x}_1^*, p_0) \in \mathcal{M}\), where \(\mathcal{M} = \{s_1(\lambda, 0, p) = 0\}\) is the slow manifold. We claim that \((\overline{x}_1^*, p_0)\) is a normally hyperbolic point. In fact,

\[
\frac{\partial s_1}{\partial \overline{x}_1}(\lambda^*, 0, p_0) = \varphi'(\overline{x}_1^*) \frac{\partial s_1}{\partial \lambda}(\lambda^*, 0, p_0) \neq 0.
\]
Thus the Fenichel’s result ensures the existence of an invariant manifold $\mathcal{M}_\lambda$ of (38) converging to a compact manifold $\mathcal{M}_0 \subset \mathcal{M}$ containing $p_0$. Therefore $p_0$ is a sliding point for the nonlinear regularization (36).

Conversely, if $p_0$ is a sliding point for the nonlinear regularization (36) satisfying assumption (37) then $p_0$ is a $c$-sliding point. In fact, note that the slow manifold $\mathcal{M}$ and $\Sigma^c_s$ are defined by same equation and $\varphi$ is increasing in $(-1, 1)$. So, there exist neighborhoods $U \subset \mathcal{M}$ and $V \subset \Sigma^c_s$ of $(\lambda^*, p_0)$ and $p_0$, respectively, and a diffeomorphism $\xi : U \to V$ satisfying $\xi(\lambda^*, p_0) = p_0$.

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