Transitive points in CR-dynamical systems

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Abstract
We study different types of transitive points in CR-dynamical systems \((X, G)\) with closed relations \(G\) on compact metric spaces \(X\). We also introduce transitive and dense orbit transitive CR-dynamical systems and discuss their properties and the relations between them. This generalizes the notion of transitive topological dynamical systems \((X, f)\).

Keywords: Closed relations; CR-dynamical systems; Transitive points; Intransitive points, Transitive and dense orbit transitive CR-dynamical systems
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1 Introduction
In topological dynamical systems theory, the study of the chaotic behaviour of a dynamical system is often based on some properties of topological spaces or some properties of continuous functions. One of the commonly studied properties in the theory of topological dynamical systems is the transitivity of a dynamical system \((X, f)\) or the transitivity of the function \(f\). This often reduces to studying transitive or intransitive points of dynamical systems. Regarding this, the study of forward orbits and backward orbits of points is also required. Backward orbits of points are actually forward orbits of points in the dynamical system \((X, f^{-1})\), if \(f^{-1}\) is well-defined. But usually, \(f^{-1}\) is not a well-defined function, therefore, a more general tool is needed to study these properties. Note that for a continuous function \(f : X \rightarrow X\), the set

\[
\Gamma(f)^{-1} = \{(y, x) \in X \times X \mid y = f(x)\}
\]

is a closed relation on \(X\) that describes best the dynamics of \((X, f)\) in the backward direction when \(f^{-1}\) is not well-defined. So, generalizing topological dynamical
systems \((X,f)\) to topological dynamical systems \((X,G)\) with closed relations \(G\) on \(X\) by making the identification \((X,f) = (X,\Gamma(f))\) is only natural.

In this paper, we generalize the notion of transitivity from topological dynamical systems to topological dynamical systems with closed relations. We introduce notions of different types of transitive points, different types of dense orbit transitivity as well as the transitivity of dynamical systems with closed relations. In the past, many similar generalizations of topological or dynamical objects have already been introduced. One of them was presented in 2004 by Ingram and Mahavier [IM, M] introducing inverse limits of inverse sequences of compact metric spaces \(X\) with upper semi-continuous set-valued bonding functions \(f\). Their graphs \(\Gamma(f)\) are examples of closed relations on \(X\). These inverse limits provide a valuable extension to the role of inverse limits in the study of dynamical systems and continuum theory. For example, Kennedy and Nall have developed a simple method for constructing families of \(\lambda\)-dendroids [KN]. Their method involves inverse limits of inverse sequences with upper semi-continuous set-valued functions on closed intervals with simple bonding functions. Such generalizations have proven to be useful (also in applied areas); frequently, when constructing a model for empirical data, continuous (single-valued) functions fall short, and the data are better modelled by upper semi-continuous set-valued functions, or sometimes, even closed relations that are not set-valued functions are required. The Christiano-Harrison model from macroeconomics is one such example [CH]. The study of inverse limits of inverse sequences with upper semi-continuous set-valued functions is rapidly gaining momentum - the recent books by Ingram [I], and by Ingram and Mahavier [IM], give a comprehensive exposition of this research prior to 2012.

Also, several papers on the topic of dynamical systems with (upper semi-continuous) set-valued functions have appeared recently, see [BEGK, CP, LP, LYY, LWZ, KN, KW, MRT, R, SS, SS2], where more references may be found. However, there is not much known of such dynamical systems and therefore, there are many properties of such set-valued dynamical systems that are yet to be studied. We proceed as follows. In sections that follow Section 2, where basic definitions are given, we discuss the following topics:

- Transitive and intransitive points in CR-dynamical systems (Section 3).
- 3-transitive points that are not 2-transitive (Section 4).
- Transitivity trees (Section 5).
- Dense orbit transitive CR-dynamical systems (Section 6).
- Transitive CR-dynamical systems (Section 7).
In each section when introducing a new object in CR-dynamical systems, we first revisit transitive dynamical systems \((X, f)\) and then, we generalize this property from dynamical systems \((X, f)\) to dynamical systems with closed relations \((X, G)\) by making the identification \((X, f) = (X, \Gamma(f))\). Results about dynamical systems \((X, f)\), presented in this paper are well-known. Therefore, we omit them. The reader can track their proofs by a little help from S. Kolyada’s and L. Snoha’s paper “Topological Transitivity” [KS], where a wonderful overview of transitive dynamical systems is given, or by E. Akin’s book “General Topology of Dynamical Systems” [A], where dynamical systems using closed relations are presented.

At the end of the paper, two examples are given proving that the statement of [SS, Theorem 9, page 3], saying that any dynamical system with an upper semi-continuous set-valued function is transitive if and only if there is a point with a dense orbit, is incorrect.

2 Definitions and notation

In this section, basic definitions and well-known results that are needed later in the paper are presented. All spaces in this paper are non-empty compact metric spaces.

**Definition 2.1.** Let \(X\) and \(Y\) be metric spaces, and let \(f : X \to Y\) be a function. We use
\[
\Gamma(f) = \{(x, y) \in X \times Y \mid y = f(x)\}
\]
to denote the graph of the function \(f\).

**Definition 2.2.** If \(X\) is a compact metric space, then \(2^X\) denotes the set of all non-empty closed subsets of \(X\).

**Definition 2.3.** Let \(X\) be a compact metric space and let \(G \subseteq X \times X\) be a relation on \(X\). If \(G \in 2^{X \times X}\), then we say that \(G\) is a closed relation on \(X\).

**Definition 2.4.** Let \(X\) be a set and let \(G\) be a relation on \(X\). Then we define
\[
G^{-1} = \{(y, x) \in X \times X \mid (x, y) \in G\}
\]
to be the inverse relation of the relation \(G\) on \(X\).

**Definition 2.5.** Let \(X\) be a compact metric space and let \(G\) be a closed relation on \(X\). Then we call
\[
\star_{i=1}^m G = \left\{(x_1, x_2, x_3, \ldots, x_{m+1}) \in \prod_{i=1}^{m+1} X \mid \text{for each } i \in \{1, 2, 3, \ldots, m\}, (x_i, x_{i+1}) \in G\right\}
\]
for each positive integer \( m \), the \( m \)-th Mahavier product of \( G \), and

\[
\star_{i=1}^\infty G = \{(x_1, x_2, x_3, \ldots) \in \prod_{i=1}^\infty X \mid \text{for each positive integer } i, (x_i, x_{i+1}) \in G\}
\]

the infinite Mahavier product of \( G \).

**Observation 2.6.** Let \( X \) be a compact metric space, let \( f : X \to X \) be a continuous function. Then

\[
\star_{n=1}^\infty \Gamma(f)^{-1} = \lim_{\leftarrow}(X, f).
\]

**Definition 2.7.** Let \( X \) be a compact metric space. For each positive integer \( k \), we use \( \pi_k : \prod_{i=1}^\infty X \to X \) to denote the \( k \)-th standard projection from \( \prod_{i=1}^\infty X \) to \( X \). We also use \( \pi_k : \prod_{i=1}^n X \to X \) to denote the \( k \)-th standard projection from \( \prod_{i=1}^n X \) to \( X \), where \( n \) is a positive integer and \( k \in \{1, 2, 3, \ldots, n\} \).

**Definition 2.8.** Let \( G \) be a relation on \( X \), let \( A \subseteq X \) and let \( x \in X \). Then we define

\[
G(x) = \{y \in X \mid (x, y) \in G\}
\]

and

\[
G(A) = \bigcup_{x \in A} G(x).
\]

For each positive integer \( n \), we also define

\[
G^n(x) = \{y \in X \mid \text{there is } x \in \star_{i=1}^n G \text{ such that } \pi_1(x) = x \text{ and } \pi_{n+1}(x) = y\}
\]

and

\[
G^n(A) = \bigcup_{x \in A} G^n(x).
\]

For each positive integer \( n \), we also define

\[
G^{-n}(x) = (G^{-1})^n(x)
\]

and

\[
G^{-n}(A) = \bigcup_{x \in A} G^{-n}(x).
\]

Finally, we define

\[
G^{\omega}(X) = \bigcap_{n=1}^\infty G^n(X)
\]

and

\[
G^{-\omega}(X) = \bigcap_{n=1}^\infty G^{-n}(X).
\]
We conclude this section by stating the following observations.

**Observation 2.9.** Let \((X,G)\) be a CR-dynamical system, let \(x \in X\) and let \(n\) be a positive integer. Then \(G^{n+1}(x) = G(G^n(x))\).

**Observation 2.10.** Let \((X,G)\) be a CR-dynamical system, let \(x,y \in X\), and let \(n\) be a non-negative integer. The following statements are equivalent.

1. \(x \in G^n(y)\).
2. There is \(x = (x_1, x_2, x_3, \ldots, x_n, x_{n+1}) \in \bigstar_{i=1}^n G\) such that \(x_1 = y\) and \(x_{n+1} = x\).

**Observation 2.11.** Let \((X,G)\) be a CR-dynamical system, let \(x,y \in X\), and let \(n\) be a non-negative integer. The following statements are equivalent.

1. \(x \in G^{-n}(y)\).
2. There is \(x = (x_1, x_2, x_3, \ldots, x_n, x_{n+1}) \in \bigstar_{i=1}^n G^{-1}\) such that \(x_1 = y\) and \(x_{n+1} = x\).
3. There is \(x = (x_1, x_2, x_3, \ldots, x_n, x_{n+1}) \in \bigstar_{i=1}^n G\) such that \(x_1 = x\) and \(x_{n+1} = y\).

3 Transitive and intransitive points in CR-dynamical systems

First, we revisit dynamical systems, and then, transitive and intransitive points in dynamical systems.

**Definition 3.1.** Let \(X\) be a non-empty compact metric space and \(f : X \to X\) a function. If \(f\) is continuous, then we say that \((X,f)\) is a dynamical system.

**Definition 3.2.** Let \((X,f)\) be a dynamical system and let \(x \in X\). The sequence
\[
x = (x, f(x), f^2(x), f^3(x), \ldots) \in \bigstar_{i=1}^\infty \Gamma(f)
\]
is called the trajectory of \(x\) in \((X,f)\). The set
\[
\mathcal{O}_f^0(x) = \{x, f(x), f^2(x), f^3(x), \ldots\}
\]
is called the forward orbit of \(x\) in \((X,f)\).
Definition 3.3. Let \((X,f)\) be a dynamical system and let \(x \in X\). If \(\text{Cl}(O^0(x)) = X\), then \(x\) is called a transitive point in \((X,f)\). Otherwise it is an intransitive point in \((X,f)\). We use \(\text{tr}(f)\) to denote the set
\[
\text{tr}(f) = \{x \in X \mid x \text{ is a transitive point in } (X,f)\}
\]
and \(\text{intr}(f)\) to denote the set
\[
\text{intr}(f) = \{x \in X \mid x \text{ is an intransitive point in } (X,f)\}.
\]

Next, we generalize these to dynamical systems with closed relations.

Definition 3.4. Let \(X\) be a non-empty compact metric space and let \(G\) be a closed relation on \(X\). We say that \((X,G)\) is a dynamical system with a closed relation or, briefly, a CR-dynamical system.

Observation 3.5. Let \(X\) be a non-empty compact metric space and \(f : X \to X\) a function. Then \((X,f)\) is a dynamical system if and only if \((X,\Gamma(f))\) is a CR-dynamical system.

Definition 3.6. Let \((X,G)\) be a CR-dynamical system and let \(x_0 \in X\). We use \(T^+_G(x_0)\) to denote the set
\[
T^+_G(x_0) = \{x \in \bigstar_{i=1}^\infty G \mid \pi_1(x) = x_0\} \subseteq \bigstar_{i=1}^\infty G.
\]

Observation 3.7. Let \((X,f)\) be a dynamical system and let \(x_0 \in X\). Then
\[
T^+_{\Gamma(f)}(x_0) = \{x\},
\]
where \(x\) is the trajectory of \(x_0\) in \((X,f)\).

Definition 3.8. Let \((X,G)\) be a CR-dynamical system and let \(x \in X\). We say that

1. \(x\) is an illegal point in \((X,G)\), if \(T^+_G(x) = \emptyset\). We use \(\text{illegal}(G)\) to denote the set of illegal points in \((X,G)\).

2. \(x\) is a legal point in \((X,G)\), if \(T^+_G(x) \neq \emptyset\). We use \(\text{legal}(G)\) to denote the set of legal points in \((X,G)\).

Observation 3.9. Let \((X,f)\) be a dynamical system. Then \(\text{illegal}(\Gamma(f)) = \emptyset\) and \(\text{legal}(\Gamma(f)) = X\).

Example 3.10. Let \(X = \{1,2\}\) and let \(G = \{(1,1)\}\). Then \(\text{illegal}(G) = \{2\}\) and \(\text{legal}(G) = \{1\}\).
Theorem 3.11. Let \((X, G)\) be a CR-dynamical system and let \(x \in X\). The following statements are equivalent.

1. \(x \in \text{illegal}(G)\).

2. There is a positive integer \(n\) such that \(G^n(x) = \emptyset\).

3. \(x \in X \setminus G^{-\omega}(X)\).

Proof. To show that 1 implies 2, let \(x \in \text{illegal}(G)\) and suppose that for each positive integer \(n\), \(G^n(x) \neq \emptyset\). For each positive integer \(n\), let \(x_n \in \bigstar_{i=1}^{\infty} G\) be such that \(\pi_1(x_n) = x\) and \(\pi_{n+1}(x_n) = x_n\) (by Observation 2.10, such a point \(x_n\) does exist). Next, for each positive integer \(n\), let \(z_n \in \bigprod_{i=1}^{\infty} X\) be such a point that for each \(k \in \{1, 2, 3, \ldots, n+1\}\), \(\pi_k(z_n) = \pi_k(x_n)\). Let \(z = (z_1, z_2, z_3, \ldots) \in \bigprod_{i=1}^{\infty} X\) and let \((z_{i_0})\) be a convergent subsequence of the sequence \((z_n)\) such that \(\lim_{n \to \infty} z_{i_0} = z\).

Then \(z \in \bigstar_{i=1}^{\infty} G\) and \(\pi_1(z) = x\). It follows that \(x \in \text{illegal}(G)\) – a contradiction.

To prove the implication from 2 to 3, let \(n\) be a positive integer such that \(G^n(x) = \emptyset\) and suppose that \(x \in G^{-\omega}(X)\). Then \(x \in G^{-n}(X)\) for each positive integer \(n\). For each positive integer \(n\), let \(x_n \in \bigstar_{i=1}^{\infty} G\) be such a point that \(\pi_{n+1}(x_n) = x\) (by Observation 2.11, such a point \(x_n\) exists). Also, for each positive integer \(n\), let \(y_n = (\pi_{n+1}(x_n), \pi_n(x_n), \pi_{n-1}(x_n), \ldots, \pi_1(x_n))\).

Then \(y_n \in \bigstar_{i=1}^{\infty} G\) for each positive integer \(n\). It follows that for each positive integer \(n\), \(G^n(x) \neq \emptyset\) – a contradiction.

To show the implication from 3 to 1, let \(x \in X \setminus G^{-\omega}(X)\) and show that \(x \in \text{illegal}(G)\). Suppose that \(x \in \text{legal}(G)\). Let \(x = (x_1, x_2, x_3, \ldots) \in \bigstar_{i=1}^{\infty} G\) be such that \(x_1 = x\). For each positive integer \(n\), let \(x_n = (x_{n+1}, x_n, x_{n-1}, \ldots, x_3, x_2, x_1)\).

Then \(x_n \in \bigstar_{i=1}^{\infty} G^{-1}\) for each positive integer \(n\). Therefore, for each positive integer \(n\), \(x \in G^{-n}(X)\). It follows that \(x \in G^{-\omega}(X)\) – a contradiction. □
Lemma 3.12. Let \((X,G)\) be a dynamical system and let \(A\) be a closed subset of \(X\). Then for each positive integer \(n\), \(G^n(A)\) and \(G^{-n}(A)\) are closed in \(X\).

Proof. To prove that for each positive integer \(n\), \(G^n(A)\) is closed in \(X\), we use induction on \(n\). First, we prove that \(G(A)\) is closed in \(X\). Let \(p_2 : X \times X \to X\) be the second standard projection, \(p_2(x,y) = y\) for any \((x,y) \in X \times X\). Note that

\[
G(A) = p_2(G \cap (A \times X)).
\]

Since \(G \cap (A \times X)\) is compact and \(p_2\) is continuous, it follows that \(p_2(G \cap (A \times X))\) is compact. Therefore, \(G(A)\) is closed in \(X\).

Next, let \(n\) be a positive integer and suppose that \(G^n(A)\) is closed in \(X\). By the previous step of the proof, it follows that \(G(G^n(A))\) is closed in \(X\). To finish the proof, note that it follows from Observation 2.9 that \(G(G^n(A)) = G^{n+1}(A)\). We have just proved that for each positive integer \(n\), \(G^n(A)\) is closed in \(X\).

Next, let \(n\) be a positive integer. Since \(G^{-n}(A) = (G^{-1})^n(A)\) and since also \(G^{-1}\) is a closed relation on \(X\), it follows that \(G^{-n}(A)\) is closed in \(X\).

\(\square\)

Corollary 3.13. Let \((X,G)\) be a CR-dynamical system. Then for each positive integer \(n\),

\[
G^{n+1}(X) \subseteq G^n(X),
\]

so \(G^\omega(X)\) is a non-empty compactum.

Proof. The corollary follows directly from Lemma 3.12. \(\square\)

Theorem 3.14. Let \((X,G)\) be a CR-dynamical system. The set \(\text{illegal}(G)\) is open in \(X\).

Proof. By Theorem 3.11

\[
\text{illegal}(G) = X \setminus G^{-\omega}(X).
\]

It follows from

\[
X \setminus G^{-\omega}(X) = X \setminus \left( \bigcap_{n=1}^{\infty} G^{-n}(X) \right) = \bigcup_{n=1}^{\infty} \left( X \setminus G^{-n}(X) \right)
\]

that

\[
\text{illegal}(G) = \bigcup_{n=1}^{\infty} \left( X \setminus G^{-n}(X) \right).
\]

It follows from Lemma 3.12 that for each positive integer \(n\), \(G^{-n}(X)\) is closed in \(X\), and \(\text{illegal}(G)\) is open in \(X\). \(\square\)

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Corollary 3.15. Let $(X, G)$ be a CR-dynamical system. The set $\text{legal}(G)$ is closed in $X$.

Proof. Since $\text{legal}(G) = X \setminus \text{illegal}(G)$, the corollary follows directly from Theorem 3.14. □

Definition 3.16. Let $(X, G)$ be a CR-dynamical system, let $x \in \bigstar_{i=1}^{\infty} G$, and let $x_0 \in X$. We

1. say that $x$ is a trajectory of $x_0$ in $(X, G)$, if $\pi_1(x) = x_0$.
2. use $O^G_{G}(x)$ to denote the set

   $$O^G_{G}(x) = \{\pi_k(x) \mid k \text{ is a positive integer}\}.$$ 

   We call the set $O^G_{G}(x)$ a forward orbit of $x_0$ in $(X, G)$
3. use $U^G_{G}(x_0)$ to denote the set

   $$U^G_{G}(x_0) = \bigcup_{y \in T^+_G(x_0)} O^G_{G}(y).$$

Definition 3.17. Let $(X, G)$ be a CR-dynamical system and let $x \in \text{legal}(G)$. We say that

1. $x$ is a type 1 transitive point or 1-transitive point in $(X, G)$, if for each $x \in T^+_G(x)$, $O^G_{G}(x)$ is dense in $X$. We use $\text{trans}_1(G)$ to denote the set of 1-transitive points in $(X, G)$.
2. $x$ is a type 2 transitive point or 2-transitive point in $(X, G)$, if there is $x \in T^+_G(x)$ such that $O^G_{G}(x)$ is dense in $X$. We use $\text{trans}_2(G)$ to denote the set of 2-transitive points in $(X, G)$.
3. $x$ is a type 3 transitive point or 3-transitive point in $(X, G)$, if $U^G_{G}(x)$ is dense in $X$. We use $\text{trans}_3(G)$ to denote the set of 3-transitive points in $(X, G)$.
4. $x$ is an intransitive point in $(X, G)$, if $x$ is not 3-transitive. We use $\text{intrans}(G)$ to denote the set of intransitive points in $(X, G)$.

Observation 3.18. Let $(X, G)$ be a CR-dynamical system. Note that

$$\text{trans}_1(G) \subseteq \text{trans}_2(G) \subseteq \text{trans}_3(G).$$

Observation 3.19. Let $(X, G)$ be a CR-dynamical system. Note that

$$\text{legal}(G) = \text{trans}_3(G) \cup \text{intrans}(G).$$

Therefore, by Corollary 3.15, the set $\text{trans}_3(G) \cup \text{intrans}(G)$ is closed in $X$. 9
**Observation 3.20.** Let \((X, f)\) be a dynamical system. Then
\[
\text{tr}(f) = \text{trans}_1(\Gamma(f)) = \text{trans}_2(\Gamma(f)) = \text{trans}_3(\Gamma(f)).
\]

In the following example, we show that there are CR-dynamical systems \((X, G)\) that have type 2 transitive points which are not type 1 transitive.

**Example 3.21.** Let \(X = [0, 1]\) and let
\[
G = ([0, 1] \times \{\frac{1}{2}\}) \cup (\{\frac{1}{2}\} \times [0, 1])
\]
see Figure 1 Then \((X, G)\) is a CR-dynamical system. Let \(x = \frac{1}{2}\). Then \(x = \)

![Figure 1: The relation \(G\) from Example 3.21](image)

\((x, x, x, \ldots) \in T^+_G(x)\) is such a point that \(O^G_G(x) = \{x\}\) is not dense in \(X\). Let
\[
[0, 1] \cap \mathbb{Q} = \{q_1, q_2, q_3, \ldots\}
\]
be the set of rationals in \([0, 1]\), let \(x_1 = x\), for each positive integer \(n\), let \(x_{2n} = \frac{1}{2}\) and \(x_{2n+1} = q_n\), and let \(x = (x_1, x_2, x_3, \ldots)\). Then \(x \in T^+_G(x)\) is such a point that \(\text{Cl}(O^G_G(x)) = X\). Therefore,
\[
x \in \text{trans}_2(G) \setminus \text{trans}_1(G).
\]
In the following example, we show that there are CR-dynamical systems \((X, G)\) that have type 3 transitive points which are not type 2 transitive.

**Example 3.22.** Let \(X = [0, 1]\) and let
\[
G = ([0, 1] \times \{1\}) \cup (\{0\} \times [0, 1]),
\]
see Figure 2.

![Figure 2: The relation \(G\) from Example 3.22](image)

Then \((X, G)\) is a CR-dynamical system. Let \(x = 0\). Note that for each \(x \in T^+_G(x)\), the set \(O^+_G(x)\) is not dense in \(X\). Also note, that for each \(y \in [0, 1]\), there is \(y \in T^+_G(x)\) such that \(\pi_2(y) = y\). Therefore, \(U^+_G(x)\) is dense in \(X\). Therefore,
\[
x \in \text{trans}_3(G) \setminus \text{trans}_2(G).
\]

Note that the relation \(G\) in Example 3.22 contains a vertical line that is essential in the example for the existence of type 3 transitive points in \(([0, 1], G)\) that are not type 2 transitive. The following example shows that there are closed relations \(G\) on \([0, 1]\) such that \(G\) contains no vertical lines and there are type 3 transitive points in \(([0, 1], G)\) that are not type 2 transitive.

**Example 3.23.** Let \(X = [0, 1]\), let \(C\) be the standard ternary Cantor set in \(X\), let \(f : X \to X\) be the Cantor function or the devil’s staircase function; see [DMRV].
for the exact definition and the basic properties of the function $f$. Note that $f(C) = [0,1]$. Let

$$G = \left( \left\{ \frac{1}{2} \right\} \times C \right) \cup \Gamma(f),$$

see Figure 3 where a sketch of an approximation of $G$ is presented. Then $(X,G)$

![Figure 3: The relation $G$ from Example 3.23 is a CR-dynamical system. Let $x = \frac{1}{2}$. For each $t \in [0,1]$ let $c_t \in C$ be such a point that $f(c_t) = t$ and let $x_t = \left( \frac{1}{2}, c_t, t, f(t), f^2(t), f^3(t), \ldots \right)$

Then

$$[0,1] = \bigcup_{t \in [0,1]} \mathcal{O}_G^\text{b}(x_t) \subseteq \mathcal{U}_G^\text{b}(x).$$

It follows that $x \in \text{trans}_3(G)$. Note that for any $x \in T_+^+(x)$, $\mathcal{O}_G^\text{b}(x)$ is not dense in $[0,1]$. Therefore, $x \in \text{trans}_3(G) \setminus \text{trans}_2(G)$.

**Definition 3.24.** Let $X$ be a compact metric space. We use $\text{isolated}(X)$ to denote the set of isolated points of $X$.

**Lemma 3.25.** Let $X$ be a compact metric space, let $A \subseteq X$ be such that $A$ is not dense in $X$, and let $x \in X$. If $A \cup \{x\}$ is dense in $X$, then $x \in \text{isolated}(X)$. 

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Proof. Suppose that \( A \cup \{x\} \) is dense in \( X \). Then \( \text{Cl}(A) \neq X \) while \( \text{Cl}(A \cup \{x\}) = X \). Since
\[
X = \text{Cl}(A \cup \{x\}) = \text{Cl}(A) \cup \text{Cl}(\{x\}) = \text{Cl}(A) \cup \{x\},
\]
it follows that \( \{x\} = X \setminus \text{Cl}(A) \). Therefore, \( x \) is an isolated point in \( X \). \( \square \)

In the following two examples, \( \text{isolated}(X) \neq \emptyset \) and \( \text{trans}_2(G) \neq \emptyset \). In both examples \( \text{isolated}(X) \cap \text{trans}_2(G) \neq \emptyset \). In Example 3.26 \( \text{isolated}(X) \subseteq \text{trans}_2(G) \) while in Example 3.27 \( \text{isolated}(X) \not\subseteq \text{trans}_2(G) \).

**Example 3.26.** Let \( X = \{1, 2, 3\} \) and let \( G = \{(1, 2), (2, 3), (3, 1)\} \). Then \( \text{isolated}(X) = \text{trans}_2(G) = \{1, 2, 3\} \). Therefore, \( \text{isolated}(X) \neq \emptyset \), \( \text{trans}_2(G) \neq \emptyset \), and \( \text{isolated}(X) \subseteq \text{trans}_2(G) \).

**Example 3.27.** Let \( X = [0, 1] \cup \{2, 3\} \) and let \( f : [0, 1] \to [0, 1] \) be the tent function defined by \( f(t) = 2t \) for \( t \in [0, \frac{1}{2}] \) and \( f(t) = 2 - 2t \) for \( t \in [\frac{1}{2}, 1] \). Also, let \( x \in \text{tr}(f) \), and let
\[
G = \Gamma(f) \cup \{(2, 3), (3, x)\},
\]
see Figure 4.

![Figure 4: The relation G from Example 3.27](image)

Then \( \text{trans}_2(G) = \{2\} \) and \( \text{isolated}(X) = \{2, 3\} \). It follows that \( \text{isolated}(X) \neq \emptyset \), \( \text{trans}_2(G) \neq \emptyset \) and
\[
\text{isolated}(X) \not\subseteq \text{trans}_2(G).
\]

Note that \( \text{trans}_2(G) \) contain exactly one isolated point.
In Example 3.28, we construct an example of a CR-dynamical system \((X, G)\) such that \(X\) is infinite, isolated\((X) \cap \text{trans}_2(G) \neq \emptyset\), and trans\(_2(G)\) contains more than just one isolated point.

**Example 3.28.** Let \(X = [0, 1] \cup \{2, 3\}\) and let \(f : [0, 1] \to [0, 1]\) be the tent function defined by \(f(t) = 2t\) for \(t \in [0, \frac{1}{2}]\) and \(f(t) = 2 - 2t\) for \(t \in [\frac{1}{2}, 1]\). Also, let \(x \in \text{tr}(f)\), and let
\[
G = \Gamma(f) \cup \{(2, 3), (3, 2), (3, x)\},
\]
see Figure 5.

![Figure 5](image_url)

Figure 5: The relation \(G\) from Example 3.28

Then \(\text{trans}_1(G) = \emptyset\), \(\text{trans}_2(G) = \{2, 3\}\) and \(\text{isolated}(G) = \{2, 3\}\). It follows that \(\text{isolated}(X) \cap \text{trans}_2(G) \neq \emptyset\), and \(\text{trans}_2(G)\) contains more than just one isolated point.

In Theorem 3.29, we show that if \(\text{isolated}(X) \neq \emptyset\), \(\text{trans}_1(G) \neq \emptyset\) and \(\text{trans}_2(G) \neq \emptyset\), then \(\text{isolated}(X) \cap \text{trans}_1(G) \neq \emptyset\) and \(\text{isolated}(X) \cap \text{trans}_2(G) \neq \emptyset\) is always the case. Also, if \(X\) is infinite, then \(\text{trans}_1(G)\) contains exactly one isolated point.

**Theorem 3.29.** Let \((X, G)\) be a CR-dynamical system such that \(\text{isolated}(X) \neq \emptyset\). Then the following hold.

1. If \(\text{trans}_1(G) \neq \emptyset\) then \(\text{trans}_1(G) \cap \text{isolated}(X) \neq \emptyset\). In addition, if \(X\) is not finite, then \(\text{trans}_1(G)\) contains exactly one isolated point.
2. If \( \text{trans}_2(G) \neq \emptyset \) then \( \text{trans}_2(G) \cap \text{isolated}(X) \neq \emptyset \) contains at least one isolated point.

**Proof.** Let \( k \in \{1, 2\} \) and suppose that \( \text{trans}_k(G) \neq \emptyset \). Let \( y \in \text{trans}_k(G) \) and let \( x \in T^+_G(y) \) such that \( O^\beta_G(y) \) is dense in \( X \). Since \( O^\beta_G(y) \) is dense in \( X \) all isolated points of \( X \) are elements of \( O^\beta_G(y) \). Choose the smallest \( m \in \mathbb{N} \) such that \( \pi_m(y) \in \text{isolated}(X) \). We show that \( \pi_m(y) \in \text{trans}_1(G) \) by showing that the set \( \{\pi_m(y), \pi_{m+1}(y), \pi_{m+2}(y), \ldots\} \) is dense in \( X \). If \( m = 1 \), then \( \{\pi_m(y), \pi_{m+1}(y), \pi_{m+2}(y), \ldots\} \) is dense in \( X \). Suppose that \( m > 1 \). To show that \( \{\pi_m(y), \pi_{m+1}(y), \pi_{m+2}(y), \ldots\} \) is dense in \( X \), let \( U \) be any non-empty open set in \( X \). We show that

\[
\{\pi_m(y), \pi_{m+1}(y), \pi_{m+2}(y), \ldots\} \cap U \neq \emptyset.
\]

Since for each \( k \in \{1, 2, 3, \ldots, m-1\}, \pi_k(y) \notin \text{isolated}(X) \), it follows that

\[
U \setminus \{\pi_1(y), \pi_2(y), \pi_3(y), \ldots, \pi_{m-1}(y)\}
\]

is also a non-empty open set in \( X \). Since \( O^\beta_G(y) \) is dense in \( X \), it follows that

\[
O^\beta_G(y) \cap (U \setminus \{\pi_1(y), \pi_2(y), \pi_3(y), \ldots, \pi_{m-1}(y)\}) \neq \emptyset.
\]

It follows from \( O^\beta_G(y) = \{\pi_1(y), \pi_2(y), \pi_3(y), \ldots\} \) that there is a positive integer \( k \) such that

\[
\pi_k(y) \in U \setminus \{\pi_1(y), \pi_2(y), \pi_3(y), \ldots, \pi_{m-1}(y)\}.
\]

Choose such a positive integer \( k \). Obviously, \( k \geq m \). It follows that

\[
\{\pi_m(y), \pi_{m+1}(y), \pi_{m+2}(y), \ldots\} \cap U \neq \emptyset.
\]

Therefore, \( \{\pi_m(y), \pi_{m+1}(y), \pi_{m+2}(y), \ldots\} \) is dense in \( X \).

For the rest of the proof suppose that \( X \) is not finite. We prove that \( \text{trans}_1(G) \) contains exactly one of the isolated points. Assume that \( x_1, x_2 \in \text{trans}_1(G) \cap \text{isolated}(X) \) such that \( x_1 \neq x_2 \). Pick any trajectories \( x_1 \in T^+_G(x_1) \) and \( x_2 \in T^+_G(x_2) \). Since \( O^\beta_G(x_1) \) and \( O^\beta_G(x_2) \) are dense in \( X \) there are positive integers \( m \) and \( n \) such that \( \pi_m(x_1) = x_2 \) and \( \pi_n(x_2) = x_1 \). Note that \( m, n > 1 \) since \( x_1 \neq x_2 \). Also, note that \( \pi_m(x_1) = \pi_1(x_2) \) and \( \pi_n(x_2) = \pi_1(x_1) \). Let \( z \in \prod_{i=1}^{\infty} X \) be defined by

\[
z = (\pi_1(x_1), \pi_2(x_1), \ldots, \pi_m(x_1), \pi_2(x_2), \pi_3(x_2), \ldots, \pi_n(x_2), \pi_2(x_1), \pi_3(x_1), \ldots),
\]

Then \( z \in T^+_G(x_1) \) and

\[
O^\beta_G(z) = \{\pi_1(x_1), \pi_2(x_1), \ldots, \pi_m(x_1), \pi_2(x_2), \pi_3(x_2), \ldots, \pi_n(x_2), \pi_2(x_1), \pi_3(x_1), \ldots\}.
\]

It follows that \( O^\beta_G(z) \) is finite and since \( X \) is infinite, \( O^\beta_G(z) \) is not dense in \( X \), which is a contradiction since \( x_1 \in \text{trans}_1(G) \) and \( z \in T^+_G(x_1) \). \(\square\)
In the following example we show that it may happen that $\text{trans}_3(G) \neq \emptyset$ and $\text{trans}_3(G)$ contains no isolated points (even in the case when isolated($X$) $\neq \emptyset$).

**Example 3.30.** Let $X = [0, 1] \cup \{2\}$, let $y \in (0, 1]$ and let

$$G = ((0, 1] \times \{1\}) \cup ((0] \times [0, 1]) \cup \{(0, 2), (2, y)\},$$

see Figure 6. Note that isolated($X$) $= \{2\}$. Also, note that

![Figure 6: The relation $G$ from Example 3.30](image)

$$\mathcal{U}_G^0(2) = O_G^0(2) = \{2, y, 1\}$$

and, therefore, $\mathcal{U}_G^0(2)$ is not dense in $X$. So, $2 \notin \text{trans}_3(G)$. Next, let $x \in (0, 1]$. Then

$$\mathcal{U}_G^0(x) = O_G^0(x) = \{x, 1\}$$

which is not a dense subset of $X$. Therefore, $x \notin \text{trans}_3(G)$. It follows that $\text{trans}_3(G) = \{0\}$. Therefore, $\text{trans}_3(G) \cap \text{isolated}(X) = \emptyset$.

Note that there are examples of CR-dynamical systems $(X, G)$ such that $X$ is infinite, with an isolated point $x$ such that for some $y \in X$, $(x, y) \in G$ and $y \in \text{trans}_2(G)$. The following example demonstrates this. In Theorem 3.32, we show that for 1-transitive points, this is not the case.
Example 3.31. Let $X = [0,1] \cup \{2\}$ and let $f : [0,1] \to [0,1]$ be the tent-map defined by $f(t) = 2t$ for $t \leq \frac{1}{2}$ and $f(t) = 2 - 2t$ for $t \geq \frac{1}{2}$, let $t_0 \in \text{tr}(f)$ and let

$$G = \Gamma(f) \cup \{(2,t_0),(t_0,2)\},$$

see Figure 7. Note that for $x = 2$ and $y = t_0$.

![Figure 7: The relation $G$ from Example 3.31](image)

Figure 7: The relation $G$ from Example 3.31

$x \in \text{isolated}(X), (x,y) \in G$ and $y \in \text{trans}_2(G)$.

Theorem 3.32. Let $(X, G)$ be a CR-dynamical system such that $X$ is infinite, let $x \in \text{isolated}(X)$ and let $y \in X$ be any point.

1. If $y \in \text{trans}_2(G)$, then either $x = y$ or there is a positive integer $n$ and a point $x \in \star_{i=1}^n G$ such that $\pi_1(x) = y$ and $\pi_{n+1}(x) = x$.

2. If $(x,y) \in G$, then $y \notin \text{trans}_1(G)$.

Proof. First, we prove the first statement. If $x = y$, there is nothing to prove. Suppose that $x \neq y$. Let $y \in \text{trans}_2(G)$ and let $y \in \star_{i=1}^\infty G$ be a trajectory of $y$ such that the forward orbit $O^G_{\infty}(y)$ is dense in $X$. Since $x \in \text{isolated}(X)$, $\{x\}$ is open in $X$ and, therefore, $O^G_{\infty}(y) \cap \{x\} \neq \emptyset$. It follows that there is a positive integer $m$ such that $\pi_m(y) = x$. Let $m$ be such a positive integer and let $n = m - 1$ (note that $n > 0$). Then let

$$x = (\pi_1(y),\pi_2(y),\pi_3(y),\ldots,\pi_{n+1}(y)).$$
Obviously, \( \pi_1(x) = y \) and \( \pi_{n+1}(x) = x \).

To prove the second claim, suppose that \( y \in \text{trans}_1(G) \). It follows that \( y \in \text{trans}_2(G) \) and we have just proved that either \( x = y \) or there is a positive integer \( n \) and a point \( x \in \star_{i=1}^n G \) such that \( \pi_1(x) = y \) and \( \pi_{n+1}(x) = x \). Next, we observe each of these two possibilities.

1. If \( x = y \), then \( y = (y, y, y, \ldots) \in T^+_{G_1}(y) \) such that \( O^G_{\pi_1}(y) = \{y\} \) is not dense in \( X \) since \( X \) is not degenerate – a contradiction.

2. Let \( n \) be a positive integer and let \( x \in \star_{i=1}^n G \) be such a point that \( \pi_1(x) = y \) and \( \pi_{n+1}(x) = x \). Let

\[
y = (\pi_1(x), \pi_2(x), \pi_3(x), \ldots, \pi_{n+1}(x), \pi_1(x), \pi_2(x), \pi_3(x), \ldots, \pi_{n+1}(x), \pi_1(x), \ldots).
\]

Then \( O^G_{\pi_1}(y) = \{\pi_1(x), \pi_2(x), \pi_3(x), \ldots, \pi_{n+1}(x)\} \) and since \( X \) is infinite, the orbit \( O^G_{\pi_1}(y) \) of \( y \) is not dense in \( X \) – a contradiction.

Therefore, \( y \notin \text{trans}_1(G) \). \( \square \)

**Theorem 3.33.** Let \((X, G)\) be a CR-dynamical system and let \( x \in \text{legal}(G) \). Then the following hold.

1. If \( x \in \text{intrans}(G) \), then for each \( y \in X \),

\[
(x, y) \in G \implies y \in \text{intrans}(G).
\]

2. If \( x \in \text{intrans}(G) \), then for each \( x \in T^+_{G_1}(x) \) and for each positive integer \( n \),

\[
\pi_n(x) \in \text{intrans}(G).
\]

3. If \( x \) is not an isolated point and if \( x \in \text{trans}_1(G) \), then for each \( y \in \text{legal}(G) \),

\[
(x, y) \in G \implies y \in \text{trans}_1(G).
\]

4. If \( X \) has no isolated points and if \( x \in \text{trans}_1(G) \), then for each \( x \in T^+_{G_1}(x) \) and for each positive integer \( n \),

\[
\pi_n(x) \in \text{trans}_1(G).
\]

5. If \( x \) is not an isolated point and if \( x \in \text{trans}_2(G) \), then there is \( y \in \text{legal}(G) \) such that

\[
(x, y) \in G \text{ and } y \in \text{trans}_2(G).
\]
6. If $X$ has no isolated points and if $x \in \text{trans}_2(G)$, then there is $x \in T^+_G(x)$ such that $O^\oplus_G(x)$ is dense in $X$ and for each positive integer $n$,

$$\pi_n(x) \in \text{trans}_2(G).$$

**Proof.** To prove 1, let $x \in \text{intrans}(G)$ and let $y \in X$ such that $(x,y) \in G$. If $y \notin \text{intrans}(G)$, then $y \in \text{trans}_3(G)$. It follows that $U^\oplus_G(y)$ is dense in $X$. Since $U^\oplus_G(y) \subseteq U^\oplus_G(x)$, it follows that $U^\oplus_G(x)$ is dense in $X$, meaning that $x \in \text{trans}_3(G)$ – a contradiction.

To prove 2, let $x \in \text{intrans}(G)$ and let $x \in T^+_G(x)$. Suppose that $n$ is a positive integer such that $\pi_n(x) \notin \text{intrans}(G)$. It follows that $U^\oplus_G(\pi_n(x))$ is dense in $X$. Since $U^\oplus_G(\pi_n(x)) \subseteq U^\oplus_G(x)$, it follows that $U^\oplus_G(x)$ is dense in $X$. Therefore, $x \notin \text{intrans}(G)$ – a contradiction. It follows that for each positive integer $n$, $\pi_n(x) \in \text{intrans}(G)$.

To prove 3, suppose that $x$ is not an isolated point in $X$ and that $x \in \text{trans}_1(G)$ and let $y \in \text{legal}(G)$ be such that $(x,y) \in G$. Suppose that $y \notin \text{trans}_1(G)$. Then there is $y \in T^+_G(y)$ such that $O^\oplus_G(y)$ is not dense in $X$. Let

$$x = (x,y,\pi_2(y),\pi_3(y),\pi_4(y),\ldots).$$

Then $x \in T^+_G(x)$ is such a point that $\text{Cl}(O^\oplus_G(x)) = \text{Cl}(O^\oplus_G(y)) \cup \{x\}$. Therefore, by Lemma 3.25, $O^\oplus_G(x)$ is not dense in $X$ – a contradiction.

To prove 4, suppose that $X$ has no isolated points and that $x \in \text{trans}_1(G)$ and let $x \in T^+_G(x)$. Suppose that there is a positive integer $n$ such that $\pi_n(x) \notin \text{trans}_1(G)$. Let $n$ be the smallest such integer. Note that $n > 1$. Let $t = \pi_{n-1}(x)$ and $y = \pi_n(x)$. Then $t \in \text{trans}_1(G)$, $y \in \text{legal}(G)$, $(t,y) \in G$, and $y \notin \text{trans}_1(G)$. This contradicts 3.

To prove 5, suppose that $x$ is not an isolated point in $X$ and that $x \in \text{trans}_2(G)$. We show that there is $y \in \text{legal}(G)$ such that

$$(x,y) \in G \text{ and } y \in \text{trans}_2(G).$$

Since $x \in \text{trans}_2(G)$, there is $x \in T^+_G(x)$ such that $O^\oplus_G(x)$ is dense in $X$. Let $x$ be such a point and let $y = \pi_2(x)$. Then $(x,y) \in G$. Suppose that $y \notin \text{trans}_2(G)$ and let

$$y = (\pi_2(x),\pi_3(x),\pi_4(x),\ldots).$$

Then $O^\oplus_G(x) = O^\oplus_G(y) \cup \{x\}$. Since $O^\oplus_G(x)$ is dense in $X$ while $O^\oplus_G(y)$ is not, it follows from Lemma 3.25 that $x$ is an isolated point in $X$ – a contradiction.

Finally, we prove 6. Suppose that $X$ has no isolated points and suppose that $x \in \text{trans}_2(G)$. Since $x \in \text{trans}_2(G)$, there is $x \in T^+_G(x)$ such that $O^\oplus_G(x)$ is dense in $X$. Let $x$ be such a point and let for each positive integer $n$,

$$y_n = (\pi_n(x),\pi_{n+1}(x),\pi_{n+2}(x),\ldots).$$
We show that for each positive integer \( n \), \( \pi_n(x) \in \text{trans}_2(G) \) by showing that for each positive integer \( n \), \( O_G^{\beta}(y_n) \) is dense in \( X \). Suppose that there is a positive integer \( n \) such that \( O_G^{\beta}(y_n) \) is not dense in \( X \). Let \( n \) be the smallest such integer. Note that \( n > 1 \). Then \( O_G^{\beta}(y_{n-1}) \) is dense in \( X \) and \( O_G^{\beta}(y_n) \) is not dense in \( X \). Since \( O_G^{\beta}(y_{n-1}) = O_G^{\beta}(y_n) \cup \{ \pi_{n-1}(x) \} \), it follows from Lemma 3.25 that \( \pi_{n-1}(x) \) is an isolated point in \( X \) – a contradiction. This completes the proof. \( \square \)

In the following example, we show that from Theorem 3.33 cannot be generalized to the case when \( x \in \text{isolated}(X) \).

**Example 3.34.** Let \( X = \{0, 1\} \) and let \( G = \{(0, 1), (1, 1)\} \). Let \( x = 0 \) and \( y = 1 \). Note that \( (x, y) \in G \) and that \( x \) is an isolated point of \( X \) and that \( x \in \text{trans}_1(G) \) while \( y \notin \text{trans}_1(G) \).

In Example 3.35, we demonstrate why Theorem 3.33 does not deal with the set \( \text{trans}_3(G) \).

**Example 3.35.** Let \( X = [0, 1] \) and let \( G = ([0, 1] \times \{1\}) \cup (\{0\} \times [0, 1]) \), see Figure 8. We proved in Example 3.22 that \( 0 \in \text{trans}_3(G) \). Note that for each \( y \in (0, 1) \),

\[
\begin{array}{c}
0 \\
\hline
1 \\
0 \\
1
\end{array}
\]

Figure 8: The relation \( G \) from Example 3.35

\((0, y) \in G \) while \( y \notin \text{trans}_3(G) \).

**Theorem 3.36.** Let \((X, G)\) be a CR-dynamical system and let \( k \in \{1, 2\} \). If \( \text{isolated}(X) = \emptyset \) and \( \text{trans}_k(G) \neq \emptyset \), then \( \text{trans}_k(G) \) is dense in \( X \).
Proof. Let $k = 1$ and suppose that $X$ does not have isolated points and that $\text{trans}_1(G) \neq \emptyset$. Let $x \in \text{trans}_1(G)$ and let $\pi_1(x) \in T^+_G(x)$. Then $\text{Cl}(O^\beta_G(x)) = X$. By 4 from Theorem 3.33 for each positive integer $n,$

$$\pi_n(x) \in \text{trans}_1(G).$$

Therefore, $O^\beta_G(x) \subseteq \text{trans}_1(G)$ and $\text{Cl(\text{trans}_1(G))} = X$ follows.

Let $k = 2$ and suppose that $X$ does not have isolated points and that $\text{trans}_2(G) \neq \emptyset$. Let $x \in \text{trans}_2(G)$ and let $\pi_1(x) \in T^+_G(x) \subseteq \text{trans}_1(G)$ be such that $\text{Cl}(O^\beta_G(x)) = X$ and such that for each positive integer $n$,

$$\pi_n(x) \in \text{trans}_2(G).$$

Such a point does exist by Theorem 3.33. Therefore, $O^\beta_G(x) \subseteq \text{trans}_2(G)$ and $\text{Cl(\text{trans}_2(G))} = X$ follows. □

Observation 3.37. In Example 3.34 a CR-dynamical system is given such that $\text{trans}_1(G) \neq \emptyset$ and $\text{trans}_2(G) \neq \emptyset$ while neither $\text{trans}_1(G)$ or $\text{trans}_2(G)$ is dense in $X$. Note that in this example isolated$(X) \neq \emptyset$.

The following corollary easily follows from Theorem 3.36.

Corollary 3.38. Let $(X,G)$ be a CR-dynamical system and let $k \in \{1,2\}$. If isolated$(X) = \emptyset$ and trans$_k(G) \neq \emptyset$, then illegal$(G) = \emptyset$.

Proof. Assume that illegal$(G) \neq \emptyset$. By Theorem 3.14 the set illegal$(G)$ is an open set in $X$. Choose any point $x \in \text{trans}_k(G)$ and let $\pi_1(x) \in T^+_G(x)$ be such that $O^\beta_G(x)$ is dense in $X$. Since illegal$(G)$ is open in $X$ and $O^\beta_G(x)$ is dense in $X$, it follows that $\text{illegal}(G) \cap O^\beta_G(x) \neq \emptyset$.

Since $O^\beta_G(x) = \{\pi_1(x), \pi_2(x), \pi_3(x), \ldots\}$, there exists $m \in \mathbb{N}$ such that

$$\pi_m(x) \in \text{illegal}(G).$$

But notice that $T^+_G(\pi_m(x)) \neq \emptyset$. Hence, $\pi_m(x) \in \text{legal}(G)$ and we get a contradiction. □

Note that there are CR-dynamical systems $(X,G)$ such that $\text{trans}_3(G)$ is non-empty and $\text{trans}_3(G)$ is not dense in $X$; Example 3.35 is such an example. Next, we give more examples of CR-dynamical systems such that $\text{trans}_3(G) \neq \emptyset$ and $\text{trans}_3(G)$ is not dense in $X$. Example 3.39 does not have isolated points, Example 3.40 does have isolated points. In addition to that, we show in both examples that for each $x \in \text{trans}_3(G)$, $x \notin \text{trans}_2(G)$. This serves as a motivation for Definition 4.3.
Example 3.39. Let $X = [0, 1]$, and let

$$f_1 : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$$

be defined by $f_1(t) = 2t$ if $t \in [0, \frac{1}{4}]$ and $f_1(t) = 1 - 2t$ if $t \in [\frac{1}{4}, \frac{1}{2}]$ and let

$$f_2 : [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]$$

be defined by $f_2(t) = 2t - \frac{1}{2}$ if $t \in [\frac{1}{2}, \frac{3}{4}]$ and $f_2(t) = \frac{5}{2} - 2t$ if $t \in [\frac{3}{4}, 1]$. Let $x_1 \in \text{tr}(f_1)$ and $x_2 \in \text{tr}(f_2)$, and let $G$ be defined as follows:

$$G = \Gamma(f_1) \cup \Gamma(f_2) \cup \{(0, x_1), (0, x_2)\},$$

see Figure 9.

Let $x = 0$. We show first that $x$ is a type 3 transitive point in $([0, 1], G)$ and is not a type 2 transitive point in $([0, 1], G)$. To see that $x \notin \text{trans}_2(G)$, let $x \in T^+_G(x)$ be any point. Then there are the following possible cases.

1. $x = (0, 0, 0, \ldots)$. In this case $O^\oplus_G(x) = \{0\}$ and, therefore, $O^\oplus_G(x)$ is not dense in $X$. 

Figure 9: The relation $G$ from Example 3.39.
2. There is a positive integer \( n \) such that \( \pi_n(x) = x_1 \). In this case, \( O^2_G(x) \subseteq [0, \frac{1}{2}] \) and, therefore, \( O^2_G(x) \) is not dense in \( X \).

3. There is a positive integer \( n \) such that \( \pi_n(x) = x_2 \). In this case, \( O^2_G(x) \subseteq [\frac{1}{2}, 1] \) and, therefore, \( O^2_G(x) \) is not dense in \( X \).

Next, let \( x_1, x_2 \in T_G^+(x) \) be such that \( \pi_2(x_1) = x_1 \) and \( \pi_2(x_2) = x_2 \). Then \( O^2_G(x_1) \cup O^2_G(x_2) \) is dense in \( X \). Therefore, \( x \in \text{trans}_3(G) \). Note that \( \text{trans}_3(G) = \{0\} \) and, therefore, \( \text{trans}_3(G) \) is not dense in \( X \).

**Example 3.40.** For each positive integer \( n \), let

\[
a^n_1, a^n_2, a^n_3, \ldots, a^n_n \in \left( \frac{2^{n+1} - 1}{2^n + 1}, \frac{2^{n+2} - 1}{2^n + 2} \right)
\]

be \( n \) distinct points and let

\[
A_n = \{a^n_1, a^n_2, a^n_3, \ldots, a^n_n\}
\]

and

\[
G_n = \left\{ \left( \frac{2^{n+1} - 1}{2^n + 1} \right) \times A_n \right\}
\]

Let

\[
X = \{1\} \cup \left\{ \frac{2^{n-1} - 1}{2^{n-1}} \mid n \text{ is a positive integer} \right\} \cup \bigcup_{n=1}^{\infty} A_n
\]

and let

\[
G = \{(x, x) \mid x \in X\} \cup \left\{ \left( 0, \frac{2^{n-1} - 1}{2^{n-1}} \right) \mid n \text{ is a positive integer} \right\} \cup \bigcup_{n=1}^{\infty} G_n,
\]

see Figure 70.

Then \( 0 \in \text{trans}_3(G) \) but \( 0 \notin \text{trans}_2(G) \). To see this, let \( x \in T_G^+(0) \). If \( \pi_k(x) = 0 \) for each positive integer \( k \), then \( x = (0, 0, 0, \ldots) \) and \( O_G^2(x) = \{0\} \), which is not dense in \( X \). Suppose that there is a positive integer \( k \) such that \( \pi_k(x) \neq 0 \) and let

\[
m = \min\{k \mid k \text{ is a positive integer such that } \pi_k(x) \neq 0\}.
\]

Note that \( m \geq 2 \). Then there is a non-negative integer \( n \) such that

\[
\pi_m(x) = \frac{2^{n+1} - 1}{2^n + 1} \text{ or } \pi_m(x) = 1.
\]

If \( \pi_m(x) = 1 \), then \( x = (0, 0, 0, \ldots, 0, 1, 1, 1, \ldots) \) and \( O_G^2(x) = \{0, 1\} \), which is not dense in \( X \). If \( \pi_m(x) = \frac{2^{n+1} - 1}{2^n + 1} \) for some non-negative integer \( n \), then we distinguish the following possible cases.
1. \( n = 0 \). Then \( x = (0, 0, 0, \ldots, 0, 1, \frac{1}{2}, \frac{1}{2}, \ldots) \) and \( O_G^0(x) = \{0, \frac{1}{2}\} \), which is not dense in \( X \).

2. \( n \geq 1 \). Then there is \( k \in \{1, 2, 3, \ldots, n\} \) such that
\[
x = (0, 0, 0, \ldots, 0, \frac{2^n - 1}{2^n}, a^n_k, a^n_k, a^n_k, \ldots)
\]
and \( O_G^0(x) = \{0, \frac{2^n - 1}{2^n}, a^n_k\} \), which is not dense in \( X \).

We have just proved that 0 \( \not\in \text{trans}_2(G) \). Note that for any \( x \in X \), there is a point \( x \in T^+_G(0) \) such that \( x \in O_G^0(x) \). Therefore, \( U_G^0(0) \) is dense in \( X \). It follows that 0 \( \in \text{trans}_3(G) \). Note that \( \text{trans}_3(G) = \{0\} \) and, therefore, \( \text{trans}_3(G) \) is not dense in \( X \). Also, note that \( \text{illegal}(G) \neq \emptyset \).

4 3-transitive points that are not 2-transitive

In this section, we study true 3-transitive points; i.e., the 3-transitive points that are not 2-transitive. We begin with the following definition.
**Definition 4.1.** Let \((X, G)\) be a CR-dynamical system and let \(x \in X\). For each positive integer \(n\), we define

\[ R_n(x) = \{x\} \cup G(x) \cup G^2(x) \cup G^3(x) \cup \ldots \cup G^n(x) \]

to be the \(n\)-reach of the point \(x\). We also define

\[ R_\omega(x) = \{x\} \cup G(x) \cup G^2(x) \cup G^3(x) \cup \ldots \]

to be the \(\omega\)-reach of the point \(x\).

**Observation 4.2.** Let \((X, G)\) be a CR-dynamical system and let \(x \in \text{trans}_3(G)\). Then

\[ R_\omega(x) = U \oplus G(x) \]

Also, note that \(x \in \text{trans}_3(G)\) if and only if \(R_\omega(x)\) is dense in \(X\).

**Definition 4.3.** Let \((X, G)\) be a CR-dynamical system and let \(x \in \text{trans}_3(G) \setminus \text{trans}_2(G)\). We say that \(x\) is

1. \((3, n)\)-transitive in \((X, G)\), if there is a positive integer \(m\) such that \(R_m(x)\) is dense in \(X\) and

   \[ n = \min\{m \mid m \text{ is a positive integer such that } R_m(x) \text{ is dense in } X\} \]

   We use \(\text{trans}_{(3, n)}(G)\) to denote the set of \((3, n)\)-transitive points in \((X, G)\).

2. \((3, \omega)\)-transitive in \((X, G)\), if

   \[ x \notin \bigcup_{n=1}^{\infty} \text{trans}_{(3, n)}(G) \]

   We use \(\text{trans}_{(3, \omega)}(G)\) to denote the set of \((3, \omega)\)-transitive points in \((X, G)\).

**Observation 4.4.** Let \((X, G)\) be a CR-dynamical system and let \(x \in X\). For all positive integers \(m, n\),

\[ x \in \text{trans}_{(3, n)}(G) \text{ and } m \geq n \implies R_m(x) \text{ is dense in } X. \]

Also, if \(x\) is illegal, then there is a positive integer \(n\) such that for all integers \(m \geq n\), \(R_m(x) = R_n(x)\).

**Example 4.5.** Let \(X = [0, 1]\) and let \(G = ([0, 1] \times \{1\}) \cup (\{0\} \times [0, 1])\). We have seen in Example 3.35 that \(0 \in \text{trans}_3(G)\) and that for each \(y \in (0, 1)\), \((0, y) \in G\). It follows that

\[ 0 \in \text{trans}_{(3, 1)}(G). \]
Note that there are CR-dynamical systems \((X, G)\) such that \(\text{trans}^{(3, n)}(G) \neq \emptyset\) and \(\text{trans}^{(3, n)}(G)\) is not dense in \(X\); Example 4.5 is such an example. We continue by examining even more such examples.

**Example 4.6.** Let \(X = [0, 1]\), and let \(G\) be defined as in Example 3.39. We have seen in Example 3.39 that \(0 \in \text{trans}^3(G) \setminus \text{trans}^2(G)\). Note that for each positive integer \(n\),
\[
0 \not\in \text{trans}^{(3, n)}(G).
\]
Therefore,
\[
0 \in \text{trans}^{(3, \omega)}(G).
\]
Note that the union of two orbit sets, \(O^p_G(x_1)\) and \(O^p_G(x_2)\) (see Example 3.39 for more details) guaranties the density of the set \(U^p_G(0)\).

**Example 4.7.** Let \((X, G)\) be the CR-dynamical system from Example 3.40. We have seen in that example that \(0 \in \text{trans}^3(G) \setminus \text{trans}^2(G)\). Note that for each positive integer \(n\),
\[
0 \not\in \text{trans}^{(3, n)}(G).
\]
Therefore,
\[
0 \in \text{trans}^{(3, \omega)}(G).
\]
Note that the union of (countably) infinitely many orbit sets guaranties the density if the set \(U^p_G(0)\) while it is not possible to achieve this with any union of just finitely many orbit sets, see Example 3.40 for more details.

Examples 4.6 and 4.7 also serve as a motivation for the following definition.

**Definition 4.8.** Let \((X, G)\) be a CR-dynamical system, let \(x \in \text{trans}^{(3, \omega)}(G)\) and let \(n\) be a positive integer. We say that \(x\) is

1. \((3, \omega, n)\)-transitive in \((X, G)\), if \(x \in \text{trans}^{(3, \omega)}(G)\) and there are 
\[
x_1, x_2, x_3, \ldots, x_n \in T^+_G(x)
\]
such that
   (a) for all \(i, j \in \{1, 2, 3, \ldots, n\}\),
   
   \[
x_i = x_j \iff i = j,
   \]
   (b) \(O^p_G(x_1) \cup O^p_G(x_2) \cup O^p_G(x_3) \cup \ldots O^p_G(x_n)\) is dense in \(X\),
   
   and for each positive integer \(m\), if \(x_1, x_2, x_3, \ldots, x_m \in T^+_G(x)\) are such that
(a) for all \( i, j \in \{1, 2, 3, \ldots, m\} \),
\[
x_i = x_j \iff i = j,
\]

(b) \( O_G^b(x_1) \cup O_G^b(x_2) \cup O_G^b(x_3) \cup \ldots \cup O_G^b(x_m) \) is dense in \( X \),

then \( m \geq n \). We use \( \text{trans}_{(3, \omega, n)}(G) \) to denote the set of \( (3, \omega, n) \)-transitive points in \( (X, G) \).

2. \( (3, \omega, \omega) \)-transitive in \( (X, G) \), if \( x \in \text{trans}_{(3, \omega)}(G) \) and for each positive integer \( n \), \( x \notin \text{trans}_{(3, \omega, n)}(G) \). We use \( \text{trans}_{(3, \omega, \omega)}(G) \) to denote the set of \( (3, \omega, \omega) \)-transitive points in \( (X, G) \).

We conclude this section with the following two examples.

**Example 4.9.** Let \( (X, G) \) be the CR-dynamical system from Example 3.39. Then \( 0 \in \text{trans}_{(3, \omega, 2)}(G) \).

**Example 4.10.** Let \( (X, G) \) be the CR-dynamical system from Example 3.40. Then \( 0 \in \text{trans}_{(3, \omega, \omega)}(G) \).

## 5 Transitivity trees

In the previous section, we saw that 3-transitivity leads to many different possible and quite complicated orbit structures. These structures naturally form trees of countable height.

In this section, we present the notion of transitive points by a set-theoretical model, a tree, as a tool to help provide an intuitive (visual) guide to transitivity in this setting, to potentially aid future research.

We restrict our definition of trees to connected trees of height \( \omega \). We employ \( \mathbb{N} \) in place of the ordinal \( \omega \).

**Definition 5.1.** Let \( (T, \leq) \) be a partially ordered set. We say that \( (T, \leq) \) is a **tree**, if there is a unique point \( x \in T \), the root of \( T \), such that

1. for each \( y \in T \), \( x \leq y \) and
2. for each \( y \in T \), \( (\{z \in T \mid z \leq y\}, \leq) \) is well ordered.

Maximal well-ordered sets in a tree are called **branches**. If \( n \) is the cardinality of a branch \( B \), then we say that the length of the branch \( B \) is \( n - 1 \) and write \( \ell_T(B) = n - 1 \). If \( n \) is finite, we say that the length of \( B \) is finite and write \( \ell_T(B) < \infty \). If \( n \) is not finite, we say that the length of \( B \) is infinite and write \( \ell_T(B) = \infty \). We
denote by $\mathcal{B}(T)$ the set of branches of the tree $(T, \leq)$ and by $\mathcal{B}_\infty(T)$ the set of branches of the tree $(T, \leq)$ with infinite length. We also define the height of $(T, \leq)$ as

$$\text{height}(T) = \sup\{\ell_T(B) \mid B \text{ is a branch of } (T, \leq)\}.$$

**Definition 5.2.** Let $(T, \leq)$ be a tree with root $x \in T$. We define $\text{level}_0(T)$ by

$$\text{level}_0(T) = \{x\}.$$

Let $n$ be a positive integer. Then we say that $\text{level}_n(T)$ is the set of all points $y \in T$ such that there are points $z_0, z_1, z_2, z_3, \ldots, z_n \in T$ satisfying

1. $z_0 = x$ and $z_n = y$,

2. for each $i \in \{0, 1, 2, 3, \ldots, n-1\}$,

$$z_i \leq z_{i+1} \text{ and } z_i \neq z_{i+1},$$

3. for each $i \in \{0, 1, 2, 3, \ldots, n-1\}$ and for each $w \in T$,

$$z_i \leq w \leq z_{i+1} \implies w \in \{z_i, z_{i+1}\}.$$

In Figure 11, there is an example of a tree $(T, \leq)$ with root $x$. A branch of the tree is coloured blue. The points in $\text{level}_1(T)$ are coloured pink, the points in $\text{level}_2(T)$ are coloured green, the points in $\text{level}_3(T)$ are coloured red, and the points in $\text{level}_4(T)$ are coloured blue.

**Definition 5.3.** Let $(T, \leq)$ be a tree with root $x \in T$ and let $n$ be a non-negative integer. We define the set $\mathcal{L}_n(T)$ as follows:

$$\mathcal{L}_n(T) = \text{level}_0(T) \cup \text{level}_1(T) \cup \text{level}_2(T) \cup \ldots \cup \text{level}_n(T).$$

Next, we implement the trees into the theory of CR-dynamical systems.

**Definition 5.4.** Let $(X, G)$ be a CR-dynamical system and let $x \in X$. Then we define $(T(x), \leq)$ as follows. Let

$$T(x) = \{x\} \cup \bigcup_{n=1}^{\infty} G^n(x)$$

and for all $y, z \in X$, we define

$$y \leq z \iff z = y \text{ or there is a positive integer } n \text{ such that } z \in G^n(y).$$

**Observation 5.5.** Let $(X, G)$ be a CR-dynamical system and let $x \in X$. Then $(T(x), \leq)$ is a tree.
Definition 5.6. Let \((X,G)\) be a CR-dynamical system and let \(x \in X\). We call the tree \((T(x), \leq)\) the transitivity tree of \((X,G)\) with respect to \(x\).

Observation 5.7. Let \((X,G)\) be a CR-dynamical system, let \(x \in X\), and let \((T(x), \leq)\) be the transitivity tree of \((X,G)\) with respect to \(x\). The leaves of the tree \((T(x), \leq)\) are the maximal members of branches and they correspond to illegal points.

Also, the point \(x\) is illegal if and only if \((T(x), \leq)\) is a tree of finite height.

Observation 5.8. Let \((X,G)\) be a CR-dynamical system, let \(x \in X\), let \(n\) be a positive integer, and let \((T(x), \leq)\) be the transitivity tree of \((X,G)\) with respect to \(x\). Then the following holds.

1. \(x \in \text{legal}(G)\) if and only if \(B_\infty(T(x)) \neq \emptyset\).

2. \(x \in \text{trans}_1(G)\) if and only if \(B_\infty(T(x)) \neq \emptyset\) and for each \(B \in B_\infty(T(x))\), \(\text{Cl}(B) = X\).

3. \(x \in \text{trans}_2(G)\) if and only if there is \(B \in B_\infty(T(x))\) such that \(\text{Cl}(B) = X\).

4. \(x \in \text{trans}_3(G)\) if and only if \(\text{Cl}\left(\bigcup_{B \in B_\infty(T(x))} B\right) = X\).

5. \(x \in \text{intrans}(G)\) if and only if \(B_\infty(T(x)) \neq \emptyset\) and \(\text{Cl}\left(\bigcup_{B \in B_\infty(T(x))} B\right) \neq X\).
6. $x \in \text{trans}_{(3,0)}(G)$ if and only if $\text{Cl}(\mathcal{L}_n(T(x))) = X$ and $\text{Cl}(\mathcal{L}_{n-1}(T(x))) \neq X$.

7. $x \in \text{trans}_{(3,\omega)}(G)$ if and only if $\text{Cl}(T(x)) = X$, for each $B \in \mathcal{B}_\infty(T(x))$, $\text{Cl}(B) \neq X$, and for each positive integer $n$, $\text{Cl}(\mathcal{L}_n(T(x))) \neq X$.

8. $x \in \text{trans}_{(3,\omega,\omega)}(G)$ if and only if $x \in \text{trans}_{(3,\omega)}(G)$ and there are $T_1, T_2, T_3, \ldots, T_n \in \mathcal{B}_\infty(T(x))$ such that
   
   (a) for all $i, j \in \{1, 2, 3, \ldots, n\}$,
   
   $T_i = T_j \iff i = j$,

   (b) $T_1 \cup T_2 \cup T_3 \cup \ldots \cup T_n$ is dense in $X$,
   
   and for each positive integer $m$, if $T_1, T_2, T_3, \ldots, T_m \in \mathcal{B}_\infty(T(x))$ such that

   (a) for all $i, j \in \{1, 2, 3, \ldots, m\}$,

   $T_i = T_j \iff i = j$,

   (b) $T_1 \cup T_2 \cup T_3 \cup \ldots \cup T_m$ is dense in $X$,

   then $m \geq n$.

9. $x \in \text{trans}_{(3,\omega,\omega)}(G)$ if and only if statement 7 is satisfied and statement 8 is not.

**Observation 5.9.** Let $(X,G)$ be a CR-dynamical system. Then the following hold.

1. $G$ is a graph of a single valued function from a subspace of $X$ to $X$ if and only if for each $x \in X$, $|\mathcal{B}(T(x))| = 1$.

2. $G$ is a graph of a single valued function from $X$ to $X$ if and only if for each $x \in X$, $|\mathcal{B}_\infty(T(x))| = 1$.

### 6 Dense orbit transitive CR-dynamical systems

We begin the final section by recalling the definition of a dense orbit transitive dynamical system $(X,f)$.

**Definition 6.1.** Let $(X,f)$ be a dynamical system. We say that $(X,f)$ is **dense orbit transitive** or **DO-transitive**, if $\text{tr}(f) \neq \emptyset$. 

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Next, we generalize this definition to CR-dynamical systems. Since in CR-dynamical systems, there are many different types of transitive points, it is natural to introduce different types of dense orbit transitivity in this context, see Definition 6.2.

**Definition 6.2.** Let \((X, G)\) be a CR-dynamical system. We say that

1. \((X, G)\) is **type 1 dense orbit transitive** or **1-DO-transitive**, if \(\text{trans}_1(G) \neq \emptyset\).
2. \((X, G)\) is **type 2 dense orbit transitive** or **2-DO-transitive**, if \(\text{trans}_2(G) \neq \emptyset\).
3. \((X, G)\) is **type 3 dense orbit transitive** or **3-DO-transitive**, if \(\text{trans}_3(G) \neq \emptyset\).
4. For each positive integer \(n\), we say that \((X, G)\) is **type \((3, n)\) dense orbit transitive** or **\((3, n)\)-DO-transitive**, if \(\text{trans}_{(3, n)}(G) \neq \emptyset\).
5. We say that \((X, G)\) is **type \((3, \omega)\) dense orbit transitive** or **\((3, \omega)\)-DO-transitive**, if \(\text{trans}_{(3, \omega)}(G) \neq \emptyset\).
6. For each positive integer \(n\), we say that \((X, G)\) is **type \((3, \omega, n)\) dense orbit transitive** or **\((3, \omega, n)\)-DO-transitive**, if \(\text{trans}_{(3, \omega, n)}(G) \neq \emptyset\).
7. We say that \((X, G)\) is **type \((3, \omega, \omega)\) dense orbit transitive** or **\((3, \omega, \omega)\)-DO-transitive**, if \(\text{trans}_{(3, \omega, \omega)}(G) \neq \emptyset\).

**Observation 6.3.** Let \((X, G)\) be a CR-dynamical system. If \((X, G)\) is 1-DO-transitive, then \((X, G)\) is 2-DO-transitive. If \((X, G)\) is 2-DO-transitive, then \((X, G)\) is 3-DO-transitive.

**Observation 6.4.** Let \((X, f)\) be a dynamical system. The following statements are equivalent.

1. \((X, f)\) is DO-transitive.
2. \((X, \Gamma(f))\) is 1-DO-transitive.
3. \((X, \Gamma(f))\) is 2-DO-transitive.
4. \((X, \Gamma(f))\) is 3-DO-transitive.

It is a well-known fact that if \((X, f)\) is a DO-transitive dynamical system and \(\text{isolated}(X) = \emptyset\), then \(f\) is surjective. Theorem 6.5 generalizes this to CR-dynamical systems. Also, note that there are examples of DO-transitive dynamical system (with \(\text{isolated}(X) \neq \emptyset\)) such that \(f\) is not surjective; for an example see Example 6.7 letting \(f : X \to X\) be such a mapping that \(G = \Gamma(f)\).
Theorem 6.5. Let \((X,G)\) be a CR-dynamical system, such that \(X\) has no isolated points or \(X\) is degenerate. Then for each \(k \in \{1, 2, 3\}\),

\[(X,G)\text{ is } k\text{-DO-transitive} \implies p_1(G) = p_2(G) = X.\]

Proof. If \(X\) is degenerate, then there is nothing to prove. For the rest of the proof, assume that \(X\) is a non-degenerate space. Let \((X,G)\) be a 3-DO-transitive CR-dynamical system and let \(x \in \text{trans}_3(G)\). Then \(U^0_{\text{G}}(x)\) is dense in \(X\). To see that \(p_1(G) = p_2(G) = X\), let \(t \in X\) be any point. We show that \(t \in p_1(G) \cap p_2(G)\). First, we prove the following claim.

Claim. There is a sequence of points \((t_n)\) in \(U^0_{\text{G}}(x)\) such that

1. \(\lim_{n \to \infty} t_n = t\) and
2. for each positive integer \(n\), there is a positive integer \(j_n \geq n\) such that \(t_{j_n} \neq t\).

Proof of Claim. Suppose that for each sequence \((t_n)\) of points in \(U^0_{\text{G}}(x)\) such that \(\lim_{n \to \infty} t_n = t\), there is a positive integer \(n_0\) such that for each \(n \geq n_0\), \(t_n = t\). First, we show that there is an open set \(U\) in \(X\) such that

\[U \cap U^0_{\text{G}}(x) = \{t\}.\]

On the contrary, suppose that for each open set \(U\) in \(X\) such that \(t \in U\), there is \(z \in U \cap U^0_{\text{G}}(x)\) such that \(z \neq t\). Using that, let for each positive integer \(n\),

\[t_n \in \left(B(t, \frac{1}{n}) \cap U^0_{\text{G}}(x)\right) \setminus \{t\}.\]

Then \((t_n)\) is a sequence in \(U^0_{\text{G}}(x)\) such that \(\lim_{n \to \infty} t_n = t\) for each positive integer \(n\), \(t_n \neq t\) – a contradiction. Therefore, there is an open set \(U\) in \(X\) such that

\[U \cap U^0_{\text{G}}(x) = \{t\}.\]

Let \(U\) be such an open set in \(X\). Since \(X\) has no isolated points, \(U \neq \{t\}\). Then \(U \setminus \{t\}\) is a non-empty open set in \(X\). Since \(U^0_{\text{G}}(x)\) is dense in \(X\), it follows that \((U \setminus \{t\}) \cap U^0_{\text{G}}(x) \neq \emptyset\) – a contradiction. This completes the proof of the claim.

It follows from the above claim that there is a sequence of points \((t_n)\) in \(U^0_{\text{G}}(x)\) such that

1. \(\lim_{n \to \infty} t_n = t\) and
2. for each positive integer \(n\), \(t_n \neq t\).
Let \((t_n)\) be such a sequence. For each positive integer \(n\), let \(x_n \in T_{G}^{+}(x)\) and let \(i_n\) be a positive integer such that \(\pi_{i_n}(x_n) = t_n\). We consider the following possible cases.

(i) \(x \neq t\). Then there is a positive integer \(n_0\) such that for each positive integer \(n \geq n_0\), \(t_n \neq x\). It follows that for each positive integer \(n \geq n_0\), \(i_n > 1\). It follows that for each positive integer \(n \geq n_0\),

\[
(\pi_{i_n-1}(x_n), \pi_{i_n}(x_n), \pi_{i_n+1}(x_n)) \in \star_{i=1}^{2} G.
\]

Note that for each positive integer \(n\), \(\pi_{i_n}(x_n) = t_n\) and that \(\star_{i=1}^{2} G\) is compact. Let \((x_0, t_0, y_0) \in \star_{i=1}^{2} G\) and let \((\pi_{i_{n_k}-1}(x_{n_k}), t_{n_k}, \pi_{i_{n_k}+1}(x_{n_k}))\) be a convergent subsequence of the sequence \((\pi_{i_n-1}(x_n), t_n, \pi_{i_n+1}(x_n))\) such that

\[
\lim_{n \to \infty} (\pi_{i_{n_k}-1}(x_{n_k}), t_{n_k}, \pi_{i_{n_k}+1}(x_{n_k})) = (x_0, t_0, y_0).
\]

Then \(t_0 = t\) and it follows that \(t \in p_1(G) \cap p_2(G)\).

(ii) \(x = t\). Since for each positive integer \(n\), \(\pi_{i_n}(x_n) = t_n\), it follows that for each positive integer \(n\), \(i_n > 1\). Therefore, for each positive integer \(n\),

\[
(\pi_{i_n-1}(x_n), \pi_{i_n}(x_n), \pi_{i_n+1}(x_n)) \in \star_{i=1}^{2} G.
\]

Following the proof of (i) for \(n_0 = 1\), we also get here that \(t \in p_1(G) \cap p_2(G)\).

Next, let \((X, G)\) be a 1-DO-transitive or 2-DO-transitive CR-dynamical system. By Observation 6.3 \((X, G)\) is a 3-DO-transitive CR-dynamical system. It follows that \(p_1(G) = p_2(G) = X\).

It is trivial to note that when dealing with a standard dynamical systems \((X, f)\) it is always the case that \(p_1(G) = X\), if \(G = \Gamma(f)\). However, it is not true for any CR-dynamical system \((X, G)\) that \(p_1(G) = X\). In Theorem 6.6 we show that if \((X, G)\) is a \(k\)-DO-transitive CR-dynamical system \(k \in \{1, 2, 3\}\), then \(p_1(G) = X\) no matter if isolated\((X) = \emptyset\) or isolated\((X) \neq \emptyset\).

**Theorem 6.6.** Let \((X, G)\) be a CR-dynamical system. Then for each \(k \in \{1, 2, 3\}\),

\((X, G)\) is \(k\)-DO-transitive \(\iff p_1(G) = X\).

**Proof.** First, suppose that \((X, G)\) is 3-DO-transitive and let \(x \in \text{trans}_3(G)\). Then \(U_{G}^{3}(x)\) is dense in \(X\), it means that

\[
\text{Cl}(U_{G}^{3}(x)) = X.
\]
To see that \( p_1(G) = X \), let \( t \in X \) be any point. We show that \( t \in p_1(G) \). Since 
\[ \text{Cl}(U \oplus G(x)) = X, \]
for each positive integer \( n \), there is 
\[ t_n \in B(x, \frac{1}{n}) \cap U \oplus G(x). \]
Since \( U \oplus G(x) = \bigcup_{y \in T_G(x)} \mathcal{O}^+_G(y) \), it follows that for each positive integer \( n \), there is \( y_n \in T_G(x) \) such that 
\[ t_n \in B(x, \frac{1}{n}) \cap \mathcal{O}^+_G(y_n). \]
Since \( p_1(G) \) is closed in \( X \) and since \( t_n \to t \), it follows that \( t \in p_1(G) \).

Next, suppose that \((X, G)\) is 1-DO-transitive or 2-DO-transitive. It follows from Observation 6.3 that \((X, G)\) is also 3-DO-transitive and we have just proved that in this case \( p_1(G) = X \) follows. This completes the proof.

We conclude this section with the following example.

**Example 6.7.** Let \( X = [0, 1] \cup \{ 2 \} \) and let \( f: [0, 1] \to [0, 1] \) be the tent-map defined by \( f(t) = 2t \) for \( t \leq \frac{1}{2} \) and \( f(t) = 2 - 2t \) for \( t \geq \frac{1}{2} \), let \( x \in \text{tr}(f) \) and let \( G = \Gamma(f) \cup \{(2, x)\} \), see Figure 12. Note that \( \text{trans}_1(G) = \text{trans}_2(G) = \text{trans}_3(G) = \{ 2 \} \) and that \( p_2(G) \neq X \).

### 7 Transitive CR-dynamical systems

First, we revisit the definition of a transitive dynamical system.

**Definition 7.1.** Let \((X, f)\) be a dynamical system. We say that \((X, f)\) is transitive, if for all non-empty open sets \( U \) and \( V \) in \( X \), there is a non-negative integer \( n \) such that 
\[ f^n(U) \cap V \neq \emptyset. \]

**Theorem 7.2.** Let \((X, f)\) be a dynamical system. The following statements are equivalent.
1. $(X, f)$ is transitive.

2. For all non-empty open sets $U$ and $V$ in $X$, there is a positive integer $n$ such that
   
   $$f^n(U) \cap V \neq \emptyset.$$ 

3. For each non-empty open set $U$ in $X$, $\bigcup_{k=0}^{\infty} f^k(U)$ is dense in $X$.

4. For each non-empty open set $U$ in $X$, $\bigcup_{k=1}^{\infty} f^k(U)$ is dense in $X$.

5. For all non-empty open sets $U$ and $V$ in $X$, there is a non-negative integer $n$ such that
   
   $$f^{-n}(U) \cap V \neq \emptyset.$$ 

6. For all non-empty open sets $U$ and $V$ in $X$, there is a positive integer $n$ such that
   
   $$f^{-n}(U) \cap V \neq \emptyset.$$ 

7. For each non-empty open set $U$ in $X$, $\bigcup_{k=0}^{\infty} f^{-k}(U)$ is dense in $X$.

8. For each non-empty open set $U$ in $X$, $\bigcup_{k=1}^{\infty} f^{-k}(U)$ is dense in $X$.

In the following definition, we generalize Definition 7.1 and in Theorem 7.5, we generalize Theorem 7.2 to CR-dynamical systems.
Definition 7.3. Let \((X,G)\) be a CR-dynamical system. We say that \((X,G)\) is transitive, if for all non-empty open sets \(U\) and \(V\) in \(X\), there is a non-negative integer \(n\) such that
\[
G^n(U) \cap V \neq \emptyset.
\]

We use the following lemma in the proof of Theorem 7.5.

Lemma 7.4. Let \((X,G)\) be a CR-dynamical system. If \(X\) has no isolated points, then [7] is equivalent to [2]

1. \((X,G)\) is transitive.

2. For any non-empty open set \(U\) in \(X\),
\[
\text{Cl} \left( \bigcup_{k=1}^{\infty} G^{-k}(U) \right) = X.
\]

Proof. Suppose that \((X,G)\) is transitive. Let \(U\) be a non-empty open subset of \(X\). To show that \(\text{Cl} \left( \bigcup_{k=1}^{\infty} G^{-k}(U) \right) = X\), let \(V\) be a non-empty open subset of \(X\). We show that
\[
V \cap \left( \bigcup_{k=1}^{\infty} G^{-k}(U) \right) \neq \emptyset.
\]

We treat the following possible cases.

1. \(V \setminus \text{Cl}(U) \neq \emptyset\). Let \(W = V \setminus \text{Cl}(U)\). Then \(W\) is a non-empty open subset of \(X\). Next, let \(m\) be a non-negative integer such that
\[
G^m(W) \cap U \neq \emptyset,
\]
and let \(x = (x_1, x_2, x_3, \ldots, x_m, x_{m+1}) \in \bigstar_{i=1}^m G\) such that \(x_1 \in W\) and \(x_{m+1} \in U\). Note that \(m \geq 1\). Then
\[
(x_{m+1}, x_m, x_{m-1}, \ldots, x_2, x_1) \in \bigstar_{i=1}^m G^{-1}
\]
and it follows that \(x_1 \in V \cap G^{-m}(U)\). Therefore, \(V \cap \left( \bigcup_{k=1}^{\infty} G^{-k}(U) \right) \neq \emptyset\).

2. \(V \setminus \text{Cl}(U) = \emptyset\). It follows that \(V \subseteq \text{Cl}(U)\). Since \(X\) has no isolated points, it follows that \(V\) has at least two points. Let \(x, y \in V\) such that \(x \neq y\). Let
\[
W_1 = B \left( x, \frac{d(x,y)}{2} \right) \cap V \text{ and } W_2 = B \left( y, \frac{d(x,y)}{2} \right) \cap V.
\]
Then \(W_1\) and \(W_2\) are non-empty open subsets of \(X\) such that
\[
W_1 \cap W_2 = \emptyset \text{ and } W_1 \cup W_2 \subseteq V.
\]
It follows from $W_1 \subseteq \text{Cl}(U)$ and $W_2 \subseteq \text{Cl}(U)$ that $W_1 \cap U \neq \emptyset$ and $W_2 \cap U \neq \emptyset$.
Let $m$ be a non-negative integer such that
$$G^m(W_1 \cap U) \cap (W_2 \cap U) \neq \emptyset.$$  

Such a non-negative integer $m$ does exist since $(X, G)$ is transitive. Let $x = (x_1, x_2, x_3, \ldots, x_m, x_{m+1}) \in \bigstar_{i=1}^m G$ such that $x_1 \in W_1 \cap U$ and $x_{m+1} \in W_2 \cap U$.

Note that $m \geq 1$ since $W_1 \cap W_2 = \emptyset$. It follows that $x_1 \in V \cap G^{-m}(x_{m+1}) \subseteq V \cap G^{-m}(U) \subseteq V \cap \left( \bigcup_{k=1}^{\infty} G^{-k}(U) \right)$

and therefore $V \cap \left( \bigcup_{k=1}^{\infty} G^{-k}(U) \right) \neq \emptyset$.

This proves the implication from 1 to 2. To prove the implication from 2 to 1, let $U$ and $V$ be non-empty open sets in $X$. Then $\left( \bigcup_{k=1}^{\infty} G^{-k}(V) \right) \cap U \neq \emptyset$. Let $x \in \left( \bigcup_{k=1}^{\infty} G^{-k}(V) \right) \cap U$ and let $n$ be a positive integer such that $x \in G^{-n}(V) \cap U$.

Next, let $x = (x_1, x_2, x_3, \ldots, x_n, x_{n+1}) \in \bigstar_{i=1}^n G$
such that $x_{n+1} \in V$ and $x_1 = x$. Since $x_1 \in U$, it follows that $x_{n+1} \in G^n(U)$. Therefore, $G^n(U) \cap V \neq \emptyset$ and this completes the proof.

\[ \square \]

**Theorem 7.5.** Let $(X, G)$ be a CR-dynamical system. Consider the following statements.

1. $(X, G)$ is transitive.

2. For all non-empty open sets $U$ and $V$ in $X$, there is a positive integer $n$ such that
$$G^n(U) \cap V \neq \emptyset.$$  

3. For each non-empty open set $U$ in $X$, $\bigcup_{k=0}^{\infty} G^k(U)$ is dense in $X$.

4. For each non-empty open set $U$ in $X$, $\bigcup_{k=1}^{\infty} G^k(U)$ is dense in $X$.

5. For all non-empty open sets $U$ and $V$ in $X$, there is a non-negative integer $n$ such that
$$G^{-n}(U) \cap V \neq \emptyset.$$  

6. For all non-empty open sets $U$ and $V$ in $X$, there is a positive integer $n$ such that
$$G^{-n}(U) \cap V \neq \emptyset.$$
7. For each non-empty open set \( U \) in \( X \), \( \bigcup_{k=0}^{\infty} G^{-k}(U) \) is dense in \( X \).

8. For each non-empty open set \( U \) in \( X \), \( \bigcup_{k=1}^{\infty} G^{-k}(U) \) is dense in \( X \).

Then the following holds.

- The statements 1, 3, 5 and 7 are equivalent.
- The statements 2, 4, 6 and 8 are equivalent.
- If \( X \) has no isolated points, then all statements are equivalent.

Proof. First, we prove the implication from 8 to 6. Let \( U \) and \( V \) be non-empty open subsets in \( X \). Since \( \bigcup_{k=1}^{\infty} G^{-k}(U) \) is dense in \( X \),

\[
\bigcup_{k=1}^{\infty} G^{-k}(U) \cap V \neq \emptyset.
\]

Then 6 follows. The prove of the implication from 7 to 5 is analogous.

Next, we prove the implication from 6 to 4. Let \( U \) be a non-empty open subset in \( X \). To show that \( \bigcup_{k=1}^{\infty} G^{k}(U) \) is dense in \( X \), let \( V \) be a non-empty open set in \( X \) and let \( n \) be a positive integer such that \( G^{-n}(V) \cap U \neq \emptyset \). Also, let \( x \in G^{-n}(V) \cap U \) and let

\[
x = (x_1, x_2, x_3, \ldots, x_n, x_{n+1}) \in \star_{i=1}^{n} G^{-1}
\]

such that \( x_{n+1} \in V \) and \( x_1 = x \). Since \( x_1 \in U \), it follows that \( x_{n+1} \in G^n(U) \). Therefore, \( \bigcup_{k=1}^{\infty} G^{k}(U) \cap V \neq \emptyset \) and 4 follows. The prove of the implication from 5 to 3 is analogous.

Next, we prove the implication from 4 to 2. Let \( U \) and \( V \) be non-empty open subsets in \( X \). Since \( \bigcup_{k=1}^{\infty} G^{k}(U) \) is dense in \( X \), it follows that there is a positive integer \( n \) such that \( G^n(U) \cap V \neq \emptyset \) and 2 follows. The prove of the implication from 3 to 1 is analogous.

Next, we prove the implication from 2 to 8. Let \( U \) be a non-empty open subset in \( X \). To show that \( \bigcup_{k=1}^{\infty} G^{-k}(U) \) is dense in \( X \), let \( V \) be a non-empty open set in \( X \) and let \( n \) be a positive integer such that \( G^n(V) \cap U \neq \emptyset \). Also, let \( x \in G^n(V) \cap U \) and let

\[
x = (x_1, x_2, x_3, \ldots, x_n, x_{n+1}) \in \star_{i=1}^{n} G^{-1}
\]

such that \( x_{n+1} \in V \) and \( x_1 = x \). Since \( x_1 \in U \), it follows that \( x_{n+1} \in G^{-n}(U) \). Therefore, \( \bigcup_{k=1}^{\infty} G^{-k}(U) \cap V \neq \emptyset \) and 8 follows. The prove of the implication from 1 to 7 is analogous.

We have just proved that the statements 1, 3, 5 and 7 are equivalent and that the statements 2, 4, 6 and 8 are equivalent.

To conclude the proof, assume that \( X \) has no isolated points. To show that the statements 1, 2, 3, 4, 5, 6, 7 and 8 are equivalent, it suffices to see that 1 is equivalent to 8. This follows from Lemma 7.4. \( \square \)
Observation 7.6. Let \((X, G)\) be a CR-dynamical system. It follows from Theorem 7.5 that \((X, G)\) is transitive if and only if \((X, G^{-1})\) is transitive.

In the following example, we show that the statements 1–8 from Theorem 7.5 may not be equivalent if \(X\) has an isolated point. Explicitly, we show that there is a CR-dynamical system \((X, G)\), which satisfies 3 but not 4. We need the following definition.

Definition 7.7. Let \((X, f)\) be a dynamical system. We say that \(f\) is locally eventually onto if for each non-empty open set \(U\) in \(X\) there is a positive integer \(n\) such that \(f^n(U) = X\).

Observation 7.8. Let \(X = [0, 1]\) and let \(f : [0, 1] \to [0, 1]\) be the tent-map defined by \(f(t) = 2t\) for \(t \leq \frac{1}{2}\) and \(f(t) = 2 - 2t\) for \(t \geq \frac{1}{2}\). Then \(f\) is locally eventually onto.

Example 7.9. Let \(X = [0, 1] \cup \{2\}\) and let \(f : [0, 1] \to [0, 1]\) be the tent-map defined by \(f(t) = 2t\) for \(t \leq \frac{1}{2}\) and \(f(t) = 2 - 2t\) for \(t \geq \frac{1}{2}\), let \(x \in \text{tr}(f)\) and let

\[G = \Gamma(f) \cup \{(2, x), (0, 2)\},\]

see Figure 13. To see that \((X, G)\) satisfies 3 of Theorem 7.5 let \(U\) be a non-empty open set in \(X\). To see that \(\bigcup_{k=0}^{\infty} G^k(U)\) is dense in \(X\), let \(V\) be a non-empty open set in \(X\). To show that \(\bigcup_{k=0}^{\infty} G^k(U) \cap V \neq \emptyset\), we consider the following cases.

![Figure 13: The relation G from Example 7.9](image-url)
1. \(2 \in U \cap V\). Since \(U \cap V \neq \emptyset\), it follows that \((\bigcup_{k=0}^{\infty} G^k(U)) \cap V \neq \emptyset\).

2. \(2 \notin U\) and \(2 \in V\). Since \(f\) is locally eventually onto, it follows that there is a positive integer \(n\) such that \(0 \in f^n(U)\). Then \(0 \in G^n(U)\) and it follows that \(2 \in G^{n+1}(U)\). Therefore, \((\bigcup_{k=0}^{\infty} G^k(U)) \cap V \neq \emptyset\).

3. \(2 \in U\) and \(2 \notin V\). Then \(x \in G(U)\). Since \(x \in \text{tr}(f)\), it follows that there is a positive integer \(n\) such that \(G^n(U) \cap V \neq \emptyset\). Therefore, \((\bigcup_{k=0}^{\infty} G^k(U)) \cap V \neq \emptyset\).

4. \(2 \notin U\) and \(2 \notin V\). Since \(([0,1],f)\) is transitive, it follows from [B, Theorem 4.2, page 10] that \((\bigcup_{k=0}^{\infty} f^k(U)) \cap V \neq \emptyset\). Therefore, \((\bigcup_{k=0}^{\infty} G^k(U)) \cap V \neq \emptyset\).

To see that \((X,G)\) does not satisfy 4 of Theorem 7.5, let \(U = \{2\}\). It follows that \(\text{Cl}\left(\bigcup_{k=1}^{\infty} G^k(U)\right) = \text{Cl}\left(\{x,f(x),f^2(x),\ldots\}\right) = [0,1] \neq X\).

**Definition 7.10.** Let \((X,G)\) be a CR-dynamical system. We say that \((X,G)\) is +transitive, if for all non-empty open sets \(U\) and \(V\) in \(X\), there is a positive integer \(n\) such that

\[G^n(U) \cap V \neq \emptyset.\]

**Observation 7.11.** Note that the following hold.

1. For any CR-dynamical system \((X,G)\) it holds that if \((X,G)\) is +transitive, then \((X,G)\) is transitive.

2. There are transitive CR-dynamical systems that are not +transitive (see Example 7.9).

3. Let \((X,G)\) be a CR-dynamical system. It follows from Theorem 7.5 that if \(\text{isolated}(X) = \emptyset\), then

\[(X,G)\) is transitive \iff (X,G) is +transitive.\]

4. Let \((X,G)\) be a CR-dynamical system. It follows from Theorem 7.5 that

\[(X,G)\) is +transitive \iff (X,G^{-1}) is +transitive.\]

**Theorem 7.12.** Let \((X,G)\) be a CR-dynamical system. If \((X,G)\) is transitive or +transitive, then

\[p_1(G) = p_2(G) = X.\]

**Proof.** First, suppose that \((X,G)\) is transitive. If \(X\) is degenerate, there is nothing to prove. So, suppose that \(X\) is non-degenerate. First, we prove that \(p_1(G) = X\). To show this, let \(U = X \setminus p_1(G)\). Suppose that \(U \neq \emptyset\). Then \(U\) is a non-empty
and what we have just proved, it follows that
\[ B(x, \varepsilon) \subseteq U \] and \( B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset. \)

Since \((X, G)\) is transitive, there is a non-negative integer \( n \) such that
\[ G^n(B(x, \varepsilon)) \cap B(y, \varepsilon) \neq \emptyset. \]

Note that since \( B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset, \) it follows that \( n > 0. \) Therefore, \( B(x, \varepsilon) \cap p_1(G) \neq \emptyset, \) which is a contradiction, since \( B(x, \varepsilon) \subseteq U. \) It follows that \( U = \emptyset \) and, therefore, \( p_1(G) = X. \) By Observation 7.6, \((X, G^{-1})\) is also transitive. Using this and what we have just proved, it follows that
\[ p_2(G) = p_1(G^{-1}) = X. \]

Next, suppose that \((X, G)\) is +transitive. By Observation 7.11, \((X, G)\) is also a transitive CR-dynamical system, therefore \( p_1(G) = p_2(G) = X \) follows.

Next, we give an example of a transitive CR-dynamical system \((X, G)\) such that \(X\) has no isolated points and \(\text{trans}_2(G) = \emptyset.\) This is an example that proves that the statement of [SS, Theorem 9, page 3] is incorrect.

**Example 7.13.** Let \(X = [0, 1],\) and let \(f_1 : [0, \frac{1}{2}] \to [0, \frac{1}{2}]\) be defined by \(f_1(t) = 2t \) if \( t \in [0, \frac{1}{4}] \) and \( f_1(t) = 1 - 2t \) if \( t \in [\frac{1}{4}, \frac{1}{2}] \) and let \(f_2 : [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]\) be defined by \(f_2(t) = 2t - \frac{1}{2} \) if \( t \in [\frac{1}{4}, \frac{3}{4}] \) and \( f_2(t) = \frac{3}{2} - 2t \) if \( t \in [\frac{3}{4}, 1].\) Let \(x_1 \in \text{tr}(f_1)\) and \(x_2 \in \text{tr}(f_2)\), and let \(G\) be defined by
\[ G = \Gamma(f_1) \cup \Gamma(f_2) \cup \{(0, x_2), (1, x_1)\}, \]

see Figure 14. Note that \(x_1 \in (0, \frac{1}{2})\) while \(x_2 \in (\frac{1}{2}, 1).\) We first show that \((X, G)\) is transitive. Let \(U\) and \(V\) be non-empty open sets in \(X.\) We consider the following possible cases.

1. \(U \cap [0, \frac{1}{2}] \neq \emptyset\) and \(V \cap [0, \frac{1}{2}] \neq \emptyset.\) Then there is a non-negative integer \(n\) such that \(f_1^n(U) \cap V \neq \emptyset,\) since \(([0, \frac{1}{2}], f_1)\) is transitive. Let \(n\) be such a non-negative integer. Since \(f_1^n(U) \subseteq G^n(U)\) it follows that \(G^n(U) \cap V \neq \emptyset.\)

2. \(U \cap [\frac{1}{2}, 1] \neq \emptyset\) and \(V \cap [\frac{1}{2}, 1] \neq \emptyset.\) Then there is a non-negative integer \(n\) such that \(f_2^n(U) \cap V \neq \emptyset,\) since \(([\frac{1}{2}, 1], f_2)\) is transitive. Let \(n\) be such a non-negative integer. Since \(f_2^n(U) \subseteq G^n(U)\) it follows that \(G^n(U) \cap V \neq \emptyset.\)
3. \( U \cap [0, \frac{1}{2}] \neq \emptyset \) and \( V \cap [\frac{1}{2}, 1] \neq \emptyset \). Since \( f_1 \) is locally eventually onto, there is a positive integer \( n \) such that \( 0 \in f_n^1(U) \). Let \( n \) be such an integer. Then \( x_2 \in G^{n+1}(U) \). Since \( x_2 \in \text{tr}(f_2) \), it follows that there is a non-negative integer \( k \) such that \( f_2^k(x_2) \in V \). Let \( k \) be such an integer. It follows that \( G^{n+k+1}(U) \cap V \neq \emptyset \).

4. \( U \cap [\frac{1}{2}, 1] \neq \emptyset \) and \( V \cap [0, \frac{1}{2}] \neq \emptyset \). Since \( f_2 \) is locally eventually onto, there is a positive integer \( n \) such that \( 1 \in f_n^2(U) \). Let \( n \) be such an integer. Then \( x_1 \in G^{n+1}(U) \). Since \( x_1 \in \text{tr}(f_1) \), it follows that there is a non-negative integer \( k \) such that \( f_1^k(x_1) \in V \). Let \( k \) be such an integer. It follows that \( G^{n+k+1}(U) \cap V \neq \emptyset \).

To show that \( \text{trans}_2(G) = \emptyset \), suppose that this is not the case and let \( x \in \text{trans}_2(G) \). Let \( y = (y_1, y_2, y_3, \ldots) \in \bigstar \bigcup_{i=1}^{\infty} G \) be such that \( y_1 = x \) and \( O^y_G \) is dense in \( X \). We distinguish the following possible cases.

1. \( y_1 \leq \frac{1}{2} \). Let \( n \) be the smallest positive integer such that \( y_n > \frac{1}{2} \). Then \( y_n = x_2 \). Since \( x_2 \in \text{tr}(f_2) \), it follows that for each non-negative integer \( k \), \( f_2^k(x_2) \neq 1 \). Therefore, for each positive integer \( k \geq n \), \( y_k \geq \frac{1}{2} \). It follows that

\[
\text{Cl}(O^y_G) = \{y_1, y_2, y_3, \ldots, y_{n-1}\} \cup \left[ \frac{1}{2}, 1 \right] = \{0\} \cup \left[ \frac{1}{2}, 1 \right] \neq [0, 1]
\]
and this is a contradiction since $O^G_G(y)$ is dense in $[0,1]$.

2. $y_1 \geq \frac{1}{2}$. Let $n$ be the smallest positive integer such that $y_n < \frac{1}{2}$. Then $y_n = x_1$. Since $x_1 \in \text{tr}(f_1)$, it follows that for each non-negative integer $k$, $f_1^k(x_1) \neq 0$. Therefore, for each positive integer $k \geq n$, $y_k \leq \frac{1}{2}$. It follows that

$$\text{Cl}(O^G_G(y)) = \{y_1, y_2, y_3, \ldots, y_{n-1}\} \cup \left[0, \frac{1}{2}\right) \neq [0,1]$$

and this is a contradiction since $O^G_G(y)$ is dense in $[0,1]$.

Next, we give an example of a non-transitive CR-dynamical system $(X,G)$ such that $\text{trans}_2(G) \neq \emptyset$. This is an example that proves that the statement of [SS, Proposition 8, page 2] is incorrect.

**Example 7.14.** Let $X = \{1,2\}$ and let $G = \{(1,2), (2,2)\}$. Then $\text{trans}_2(G) = \{1\}$. Let $U = \{2\}$ and $V = \{1\}$. Then $G^n(U) = U$ for each non-negative integer $n$. Therefore, for each non-negative integer $n$, $G^n(U) \cap V = \emptyset$. It follows that $(X,G)$ is not transitive.

We conclude the paper by stating and proving the following theorem.

**Theorem 7.15.** Let $(X,G)$ be a CR-dynamical system such that $\text{trans}_2(G) \neq \emptyset$. If $X$ has no isolated points, then $(X,G)$ is transitive.

**Proof.** To see that $(X,G)$ is transitive, let $U$ and $V$ be non-empty open sets in $X$. Also, let $x \in \text{trans}_2(G)$ and let $x = (x_1, x_2, x_3, \ldots) \in \star_{i=1}^\infty G$ such that $x_1 = x$. Then $O^G_G(x)$ is dense in $X$. Let $n$ be a positive integer such that $x_n \in U$. Let $W = V \setminus \{x_1, x_2, x_3, \ldots, x_n\}$. Then there is a positive integer $m > n$ such that $x_m \in W$. It follows that $G^{m-n}(U) \cap V \neq \emptyset$. □

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