Mixed Finite Element Method for Geometrically Nonlinear Buckling Analysis of Truss with Member Length Imperfection

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Abstract. This paper focuses on the numerical method for the geometrically nonlinear buckling analysis of truss with initial member length imperfection. The solution of nonlinear buckling problem of truss with imperfection using a displacement-based finite element is dependent on the imperfection implemented. Generally, the operation of incorporating the initial imperfection to the master stiffness equation develops by master-slave elimination method, penalty augmentation method or Lagrange multiplier adjunction methods. Obviously, the initial imperfection considerably increases the difficulty in finite element formulation nonlinear buckling problem. This research proposes a novel approach to formulate the nonlinear buckling problem of truss with imperfection using mixed finite element method. The mixed balanced equation of truss is formulated using the principle of stationary potential energy. The paper presents novel mixed finite truss element, including initial member length imperfection, considering large displacement. Using the arc length technique, the research develops a new incremental-iterative algorithm for solving the nonlinear buckling problem of truss with initial imperfection in different cases of model formulation, including displacement-based finite element and mixed finite element formulation. The numerical test is presented to investigate the equilibrium path for plan truss with initial member length imperfection. The calculation results in solving problem formulated in both displacement and mixed finite model are converged showing the efficiency and reliability of the proposed method.

1. Introduction

Truss structures have been widely used in various public buildings due to their outstanding advantages in material saving and maximum utilization of structure load capacity. Many truss members have initial geometric imperfection as results of manufacturing, transporting, and handling processes. The member initial imperfection probably will affect the buckling behaviour of structures. In geometrical nonlinear finite element analysis, the nodal equivalent force is impossible to use for replacing the imperfection. Using a displacement-based finite element, the solution of buckling analysis of truss with imperfection is dependent on the implementation of imperfection. For incorporating member imperfection to the master stiffness matrix, the master-slave elimination method, penalty augmentation method or Lagrange multiplier adjunction methods are usually used [1]. Obviously, the operation of imposing initial member imperfection considerably increases the difficulty in finite element formulation nonlinear buckling. This paper proposes a novel approach to formulate the nonlinear
buckling problem of truss with imperfection to escape difficulties of the mathematical treatment of imperfection. The contribution deals with mixed variational formulation with initial imperfection condition. Based on the principle of stationary potential energy, the mixed form balanced equation of truss is constructed. For solving the proposed problem, the arc length method is employed, which is a very efficient method to predict the proper response and follow the nonlinear equilibrium path through the limit. Using length technique, the word established the incremental-iterative algorithm for solving nonlinear buckling problem of truss with initial imperfection. Based on the proposed algorithm, the calculation procedure and programs for determining the nonlinear equilibrium path are established. For investigating the nonlinear equilibrium path presented a numerical test for plane truss with member initial geometrical imperfection.

2. Mixed finite element approach for formulation truss problem with member length imperfection considering large displacement

2.1. Balanced equation of truss element
Consider the two-node mixed truss element shown in Figure 1. Designate the following

- \( L_e \) and \( \Delta_e \) - initial length of truss element and imperfection;
- \( L_0 \) and \( L \) - distance between \( i \)-th and \( j \)-th node before and after deformation;
- \( u_1, u_2, u_3, u_4 \) and - \( P_1, P_2, P_3, P_4 \) nodal displacements and forces in a global coordinate system;
- \( P_e \) - resultant external force at the \( i \)-th cross section after deformation;
- \( N \) - axial load of truss element;
- \( A \) - cross sectional area of truss element, \( E \) - elastic modulus of the material.

The length of the truss element after deformation can be given by

\[
L = \sqrt{(L_0 \sin \alpha_0 + u_4 - u_2)^2 + (L_0 \cos \alpha_0 + u_3 - u_1)^2}
\]  

The axial deformation of truss element is obtained

\[
\Delta L = L - L_e = L - L_0 + \Delta_e
\]  

Work of internal axial force can be computed by
\[
\delta V = - \int_0^L \sigma_s \varepsilon_s \, dV = - \int_0^L \sigma_s \, dA \int_0^L \varepsilon_s \, dx = - N \int_0^L \frac{\delta (\Delta dx)}{\delta} \, dx = - N \int_0^\Delta \delta (\Delta dx) =
\]

\[= - N \delta \int_0^\Delta \delta dx = - N \delta \Delta L = - N \left\{ \sum_{i=1}^4 \frac{\partial \Delta L}{\partial u_i} \delta u_i + \frac{\partial \Delta L}{\partial \Delta e} \delta \Delta e \right\} \]

The virtual external work can be defined as

\[
\delta \tilde{V} = P_1 \delta u_1 + P_2 \delta u_2 + P_3 \delta u_3 + P_4 \delta u_4 + P_e \delta \Delta e = \sum_{i=1}^4 P_i \delta u_i + P_e \delta \Delta e
\]

Adding Eq. (3) to Eq. (4), total work done by the applied forces and the inertial forces of a mechanical system can be given

\[
\delta V + \delta \tilde{V} = - N \left\{ \sum_{i=1}^4 \frac{\partial \Delta L}{\partial u_i} \delta u_i + \frac{\partial \Delta L}{\partial \Delta e} \delta \Delta e \right\} + \left\{ \sum_{i=1}^4 P_i \delta u_i + P_e \delta \Delta e \right\} =
\]

\[= \sum_{i=1}^4 \left\{ - N \frac{\partial \Delta L}{\partial u_i} + P_i \right\} \delta u_i + \left\{ - N \frac{\partial \Delta L}{\partial \Delta e} + P_e \right\} \delta \Delta e = 0 \]

Based on the principle of virtual work, in equilibrium the virtual work of the forces applied to a system is zero, getting

\[
\begin{cases}
- N \frac{\partial \Delta L}{\partial u_i} + P_i = 0 & (i = 1, 2, 3, 4) \\
N \frac{\partial \Delta L}{\partial \Delta e} - P_e = 0
\end{cases}
\]

Adding deformation from Eq. (2) to Eq. (6), expressing axial force through deformation, getting

\[
\frac{E A}{L_e} (L - L_0 + \Delta_e) \frac{\partial (L - L_0 + \Delta_e)}{\partial u_i} = P_i & (i = 1, 2, 3, 4)
\]

\[
\frac{E A}{L_e} (L - L_0 + \Delta_e) \frac{\partial (L - L_0 + \Delta_e)}{\partial \Delta e} - P_e = 0
\]

From Eq. (7) and Eq. (1), having

\[
\begin{cases}
\frac{E A}{L_e} (L - L_0 + \Delta_e) \frac{\partial L}{\partial u_i} = P_i & (i = 1, 2, 3, 4) \\
\frac{E A}{L_e} (L - L_0 + \Delta_e) - P_e = 0
\end{cases}
\]

Designate

\[
\left\{ q^{(e)}_1 = \frac{E A}{L_e} (L - L_0 + \Delta_e) \frac{\partial L}{\partial u_i} & (i = 1, 2, 3, 4) \right\}
\]

Designate

\[
\left\{ q^{(e)}_5 = \frac{E A}{L_e} (L - L_0 + \Delta_e) - P_e \right\}
\]

The Eq. (8) can be compactly written as

\[
q^{(e)}_k (u) = P^{(e)}_k \quad k = 1, 2, \ldots, 5
\]
Where

\[ u \equiv u = \{u_1, u_2, u_3, u_4, u_5\}^T \] is the vector consists of nodal variables (primary unknowns) in the global coordinate system;

\[ u_5 \equiv P_e \] is external force unknown at the \( i \)th cross section after deformation, obviously \( u_3 \equiv P_e = N \) from equilibrium condition.

Based on mixed finite element formulation, the balanced equation truss element can be written as

\[ q^{(e)}(u) = P^{(e)} \] (10)

Designating: \( q^{(e)}(u) \equiv \{q_1^{(e)}(u), q_2^{(e)}(u), q_3^{(e)}(u), q_4^{(e)}(u), q_5^{(e)}(u)\}^T \); \( P^{(e)} \equiv \{P_1^{(e)}, P_2^{(e)}, P_3^{(e)}, P_4^{(e)}, P_5^{(e)}\}^T \)

Input incremental loading into the Eq. (10), getting

\[ q^{(e)}(u) + \Delta q^{(e)}(u) = P^{(e)} + \Delta P^{(e)} \] (11)

\( \Delta q^{(e)}(u) \) is second order infinitesimal and it can be negligible, the Eq. (11) is becoming

\[ q^{(e)}(u) + \frac{\partial q^{(e)}(u)}{\partial u} \delta u = P^{(e)} + \Delta P^{(e)} \] (12)

Setting

\[ k^{(e)}(u) = \frac{\partial q^{(e)}(u)}{\partial u} \]

The Eq. (12) can be compactly written as

\[ k^{(e)}(u) \delta u = (P^{(e)} + \Delta P^{(e)}) - q^{(e)}(u) \] (13)

In the Eq. (13), \( k^{(e)}(u) \) is a mixed matrix of truss element considering the initial length imperfection, is given by

\[ k^{(e)}(u) = \begin{bmatrix}
    k_{11} & k_{12} & \ldots & k_{15} \\
    k_{21} & k_{22} & \ldots & k_{25} \\
    \vdots & \vdots & \ddots & \vdots \\
    k_{51} & k_{52} & \ldots & k_{55}
\end{bmatrix}, k_{ij} = \frac{\partial q^{(e)}(u)}{\partial u_j}, \quad (i, j = 1, 2, \ldots, 5) \] (14)

Where

\[ k_{11} = \frac{EA}{L_e} \begin{pmatrix}
    (L_0 \cos \alpha_0 - u_i + u_1)^2 \\
    (L_0 \sin \alpha_0 + u_4 - u_2)^2 + (L_0 \cos \alpha_0 + u_1 - u_i)^2 + \frac{(L_0 \sin \alpha_0 + u_4 - u_2)^2 + (L_0 \cos \alpha_0 + u_1 - u_i)^2}{\frac{1}{2} (L_0 \sin \alpha_0 + u_4 - u_2)^2 + (L_0 \cos \alpha_0 + u_1 - u_i)^2}\end{pmatrix} + \frac{\Delta x}{L_e} \begin{pmatrix}
    (L_0 \sin \alpha_0 + u_4 - u_2)^2 + (L_0 \cos \alpha_0 + u_1 - u_i)^2 \\
    (L_0 \sin \alpha_0 + u_4 - u_2)^2 + (L_0 \cos \alpha_0 + u_1 - u_i)^2\end{pmatrix} \]
2.2. Balanced equation of the global system

Based on the Eq. (10), the equilibrium condition of the system can be expressed as

\[ \mathbf{q}(\mathbf{u}) = \mathbf{P} \]  

(15)

Where

“m” is a number of truss elements and “n” is number of unknowns;

\[ \mathbf{u} = [u_1, u_2, \ldots, u_n]^T; \mathbf{q}(\mathbf{u}) = [q_1(u), q_2(u), \ldots, q_n(u)]^T; \mathbf{P} = \{P_1, P_2, \ldots, P_n\}^T \]

The Eq. (15) is a nonlinear system, it can be expressed in incremental form as

\[ \mathbf{q}(\mathbf{u}) + \frac{\partial \mathbf{q}(\mathbf{u})}{\partial \mathbf{u}} \delta \mathbf{u} = \mathbf{P} + \Delta \mathbf{P} \]  

(16)

Setting

\[ \mathbf{K}(\mathbf{u}) = \frac{\partial \mathbf{q}(\mathbf{u})}{\partial \mathbf{u}} \]  

(17)

The Eq. (16) & (17) can be compactly written as
\[
\mathbf{K}(u)\delta\mathbf{u} = (\mathbf{P} + \Delta\mathbf{P}) - \mathbf{q}(u); \quad \mathbf{K}_{ij}(u) = \sum_{e=1}^{m} \mathbf{k}_{ij}^{(e)}(u), \quad (i, j = 1, 2, ..., n) \quad (18) \& (19)
\]

Where: \( \mathbf{K}(u) \) is a global mixed matrix, developed by assembling mixed matrix of truss elements.

3. Displacement-based finite element approach for formulation of geometrical nonlinear truss problem with member length imperfection

This section is focused on the displacement-based finite element approach for formulating truss problem with member length imperfection. For incorporating member length imperfection to the master stiffness matrix is employed Lagrange multiplier method [1-3, 4].

Consider the truss system having “n” degree of freedom, \( \mathbf{u} = [u_1, u_2, ..., u_n]^T, \mathbf{u} \in \mathbb{R}^n \) is nodal displacement vector, \( \mathbf{P} = [P_1, P_2, ..., P_n]^T \) is a nodal force vector, \( \Pi(u) \) is the total potential energy of the system. The imperfections are expressed as constraints equation \( g_k(u) = 0, k = 1..m \)

The governing system of equations can be developed by minimization of the augmented potential energy function by incorporating the constraints as

\[
\min \left\{ \Pi(u) : g(u) = [g_1(u), g_2(u), ..., g_m(u)]^T = 0, \mathbf{u} \in \mathbb{R}^n \right\} \quad (20)
\]

The potential energy of the unconstrained finite element model is

\[
\Pi(u) = U(u) - \mathbf{u}^T \mathbf{P} \quad (21)
\]

Converting a constrained problem into unconstrained problem a form using the Lagrange multipliers method [4]. To impose the constraint, adjoin additional unknowns - Lagrange multipliers \( \lambda = [\lambda_1, \lambda_2, ..., \lambda_m]^T, \lambda \in \mathbb{R}^m \). Form the Lagrangian \( L(u, \lambda) \), then extremize \( L \) with respect to \( u \) and \( \lambda \) yields the multiplier-augmented form as below

\[
L(u, \lambda) = U(u) - \mathbf{u}^T \mathbf{P} + \sum_{k=1}^{m} \lambda_k g_k(u); \quad \min \left\{ L(u, \lambda) : \mathbf{u} \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \right\} \quad (22) \quad (23)
\]

Taking the derivative of a function \( L(u, \lambda) \) with respect to \( u \) and \( \lambda \) yields, setting equal to zero, getting system of \((n + m)\) equations having \((n + m)\) unknowns \( u_i, \lambda_k \) as

\[
\begin{align*}
\frac{\partial U(u)}{\partial u_i} - \mathbf{P} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k(u)}{\partial u_i} &= 0, \quad i = 1..n \quad \text{(24.1)} \\
g(u) = [g_1(u), g_2(u), ..., g_m(u)]^T &= 0 \quad \text{(24.2)}
\end{align*}
\]

The system (24) is nonlinear, it can be expressed in incremental form as

\[
\begin{bmatrix}
\frac{\partial U(u)}{\partial u_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k(u)}{\partial u_i} \\
\frac{\partial g(u)}{\partial u}
\end{bmatrix} =
\begin{bmatrix}
P + \Delta P \\
0
\end{bmatrix} - \begin{bmatrix}
\frac{\partial U(u)}{\partial u_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k(u)}{\partial u_i} \\
\frac{\partial g(u)}{\partial u}
\end{bmatrix}, \quad (i, j = 1..n)
\]

System (25) is written in compact form as
\[ R(u) \delta u = P + \Delta P - q(u) \]  

(26)

4. Algorithm for solving the nonlinear buckling problem based on the arc length method

The arc length method [5-9], both load control and displacement control, is a very efficient method in solving non-linear systems of equations when the problem under consideration exhibits one or more critical points.

Based on spherical arc length method, the block diagram of the algorithm for solving the nonlinear buckling problems is established (shown in figure 2).

**Figure 2.** Incremental-iterative procedure for solving nonlinear buckling problem based on arc length technique
5. Test example

5.1. Example formulation
The system is composed of bars made of the same material and had the same geometrical properties (the system is shown in figure 3), having length imperfection $\Delta_1$. The geometric parameters, material parameters and loading parameters are given

$$E = 2 \times 10^4 \text{kN} / \text{cm}^2, \ A = 4 \text{cm}^2, \ L = 400 \text{cm}, \ L_0 = 300 \text{cm}, \Delta_1 = 100 \text{cm}$$

The unknowns of truss system are designated as shown in Figure 3 for solving the nonlinear equation based on mixed formulation, including $(u_1, u_2)$ - nodal displacement unknowns and $(u_3, u_4, u_5)$ - axial force unknowns.

![Figure 3](image)

**Figure 3.** Examined system, designating unknowns of the system for the mixed formulation

The unknowns of truss system are designated as shown in Figure 4 for solving the nonlinear equation based on displacement-based formulation.

The imperfections are expressed as constraints equation

$$g_1(u) = u_3 + \Delta_1 \cos(15^\circ) = 0; g_2(u) = u_4 + \Delta_1 \sin(15^\circ) = 0$$

![Figure 4](image)

**Figure 4.** Examined system, designating unknowns of the system for displacement-based formulation

5.2. Numerical results
The calculating results are load-displacement and load-internal force equilibrium path in two cases of solving, shown in Figure 5 and 6.
Designating as below:

\( P_{u1}, P_{u2}, P_{N1}, P_{N2}, P_{N3}, u_1, u_2, N_1, N_2, N_3 \) – Results of mixed formulation problem

\( P_{u1cv}, P_{u2cv}, P_{N1cv}, P_{N2cv}, P_{N3cv}, u_{1cv}, u_{2cv}, N_{1cv}, N_{2cv}, N_{3cv} \) – Results of displacement-based formulation problem

\[ u_1^{cv}, u_2^{cv} \]

\[ P_{N1}, P_{N1cv}, P_{N2}, P_{N2cv}, P_{N3}, P_{N3cv} \]

\[ N_1, N_1^{cv}, N_2, N_2^{cv}, N_3, N_3^{cv} \]

\[ \text{Figure 5. Load-displacement equilibrium path} \]

\[ \text{Figure 6. Load-internal force equilibrium path} \]

5.3. Comments

The calculation results of solving problem formulated in both displacement and mixed finite model have a negligible difference showing the efficiency and reliability of the proposed method.

6. Conclusions

In solving the geometrical nonlinear problem of truss with the member imperfection, the mixed formulation model is simpler and more effective than the displacement-based formulation model.

The mixed model of the finite element formulation has a remarkable advantage in the analysis of problems with nonlinear displacement constraint. Using presented mixed formulation helps to escape difficulties of the mathematical treatment of imperfection in nonlinear finite element model.

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