Graphs, lattices and deconstruction hierarchies

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Abstract

The mathematics underlying the connection between deconstruction lattices and locality diagrams of conformal models is developed from scratch, with special emphasis on classification issues. In particular, the notions of equilocality classes, deflation map, essential vertices and stem graphs are introduced in order to characterize those graphs that may arise as locality diagrams.

1 Introduction

An important construction procedure in 2D Conformal Field Theory [10, 18], leading to new consistent models from known ones, is orbifolding [15, 14], which involves identifying states related by some group of symmetries (the ‘twist group’), supplemented by the introduction of new states (grouped into so-called ‘twisted sectors’ labeled by the conjugacy classes of the twist group) to ensure modular invariance. Orbifold models have found different applications over the years, ranging from string model building [16, 17] (based on the observation that
the CFT describing the world-sheet dynamics of strings moving on a quotient of Minkowski space by some group action can be obtained in such a way) to the description of second-quantized strings (via so-called symmetric products orbifolds) [13, 12, 5], not to mention their role in the celebrated FLM construction of the Moonshine module [19], and the (physicist) proof of the congruence subgroup property of rational conformal models [4].

Orbifold deconstruction [6], the inverse procedure of orbifolding, aims at recognizing whether a given conformal model could be realized as an orbifold of another one, and if so, to determine this original model and the relevant twist group. That orbifold deconstruction does not only make sense, but that it can be performed effectively has been explained in [6, 8]: starting from some readily available data (conformal weights, fusion rules and conformal characters) characterizing a conformal model, one can identify all possible realizations of the model as an orbifold, possibly up to some finite (usually rather small) ambiguity that can, as a last resort, be resolved through a case-by-case analysis. Of course, the computational need grows steadily with the complexity of the model, but judicious algorithms allow to deal with models having as much as several hundreds primary fields.

An interesting aspect of the theory is that there is actually a whole hierarchy of orbifold deconstructions [7]. This is natural to some extent: indeed, if a model may be obtained from another one by orbifolding with respect to some twist group, it may also be reached in steps, by first orbifolding with respect to a normal subgroup of the twist group, and then orbifolding the result by the corresponding factor group. This means that for any deconstruction with a given twist group one has a full hierarchy of partial deconstructions corresponding to the different factor groups of the twist group, and these form a lattice isomorphic to the lattice of normal subgroups of the latter. Actually, the situation is more subtle, because in general one and the same model can be
obtained in genuinely different ways as an orbifold\textsuperscript{1}, hence the different deconstructions of a given model do not form a lattice, but only some more general kind of ordered structure \[7\].

A major result of \[7\] is that to each conformal model is naturally associated an algebraic lattice (with nice properties, like being modular, which is crucial in view of the group theoretic interpretation) into which one can embed the whole deconstruction hierarchy, with the actual deconstructions corresponding to special lattice elements, the so-called 'twisters' of the model \[6\]. Most importantly for us, this 'deconstruction lattice' is self-dual, i.e. it comes equipped with an order-reversing involutory self-map. From a vantage point, it is fair to say that the deconstruction lattice is an important combinatorial characteristic of the conformal model.

Unfortunately, determining the deconstruction lattice from scratch can be pretty involved, as one should consider all subsets of primaries, and filter out those that are closed under the fusion product, a procedure whose computational cost grows exponentially with the number of primaries, prohibiting actual computations for models with more than a couple of dozens of them, while interesting examples involve hundreds, if not thousands. Fortunately, as explained in \[9\], there is a way out, exploiting the relationship between self-dual lattices and undirected graphs (with possible loops): to each graph corresponds a well-defined self-dual lattice, its 'associated lattice', and there is an undirected graph naturally related to any given conformal model, its so-called 'locality graph' (whose vertices correspond to the primary fields, with two of them adjacent if the corresponding primaries are mutually local), whose associated lattice can be shown to coincide with the

\textsuperscript{1}That this behavior is the rule rather than the exception is clearly exemplified by the FLM construction \[19\] of the Moonshine module from the Leech lattice model, in which one first constructs a \(\mathbb{Z}_2\)-orbifold of the latter, and then deconstructs this holomorphic orbifold: as it turns out, there are two different non-trivial deconstructions, one giving back (as expected) the Leech lattice model, while the other results in the Moonshine module.
deconstruction lattice \( \mathcal{L} \). This observation makes it much more easy to determine the deconstruction lattice, by first considering the locality graph, and then computing its associated lattice. What is more, by using the natural isomorphism between the associated lattice of a graph and that of its deflation, the computations become even simpler.

The present paper aims at giving a mathematically sound presentation of the above ideas. We start by describing the basic facts on the relation between undirected graphs and their associated lattices, and discuss such notions as equilocality classes and the deflation map. We then show, by introducing 'duality graphs', that every self-dual lattice is indeed the associated lattice of some undirected graph. Next, we turn to the question of describing all (irreducible) graphs with isomorphic associated lattices: this involves such notions as essential vertices and the radical of a graph. Finally, we present the physics motivation behind all the foregoing, by showing that the associated lattice of the locality graph does indeed coincide with the deconstruction lattice of the conformal model, and expound on some of the most interesting consequences of this result.

\[ \text{2 Graphs and their lattices} \]

Let \( \nabla \) denote an undirected graph [2, 11] with (finite) vertex set \( \mathcal{V} \). To each vertex \( x \in \mathcal{V} \) is associated its neighborhood \( \nabla(x) \), the set of all vertices adjacent to it. More generally, for any subset \( X \subseteq \mathcal{V} \) its dual

\[
\nabla(X) = \{ x \in \mathcal{V} \mid X \subseteq \nabla(x) \} = \bigcap_{x \in X} \nabla(x) \tag{2.1}
\]

is the set of all vertices that are adjacent to each element of \( X \).

**Lemma 1.** For any \( X, Y \subseteq \mathcal{V} \) one has \( X \subseteq \nabla(Y) \) iff \( Y \subseteq \nabla(X) \).

**Proof.** \( X \subseteq \nabla(Y) \) means that all vertices in \( X \) are adjacent to all vertices in \( Y \), i.e. \( Y \subseteq \nabla(X) \) by the symmetry of the adjacency relation. \( \square \)
Lemma 2. \( \nabla(X \cup Y) = \nabla(X) \cap \nabla(Y) \) for \( X, Y \subseteq V \).

Proof. \( x \in \nabla(X \cup Y) \iff X \cup Y \subseteq \nabla(x) \), i.e. \( X \subseteq \nabla(x) \) and \( Y \subseteq \nabla(x) \), and this is equivalent to \( x \in \nabla(X) \) and \( x \in \nabla(Y) \), proving the claim. \( \square \)

Lemma 3. The assignment \( X \mapsto \nabla(X) \) is order-reversing, i.e. \( X \subseteq Y \) implies \( \nabla(Y) \subseteq \nabla(X) \).

Proof. One has \( X \subseteq Y \iff X \cup Y = Y \), and by Lemma 2 this implies \( \nabla(Y) = \nabla(X \cup Y) = \nabla(X) \cap \nabla(Y) \), that is \( \nabla(Y) \subseteq \nabla(X) \). \( \square \)

Lemma 4. For a subset \( X \subseteq V \) let
\[
\Delta(X) = \nabla(\nabla(X)) = \{ x \in V \mid \nabla(X) \subseteq \nabla(x) \} \tag{2.2}
\]
Then
\[
X \subseteq \Delta(X) \tag{2.3}
\]
\[
\Delta(\nabla(X)) = \nabla(\Delta(X)) = \nabla(X) \tag{2.4}
\]
\[
\Delta(X) \subseteq \Delta(Y) \tag{2.5}
\]
for \( X \subseteq Y \subseteq V \).

Proof. Eq.(2.3) follows from Lemma 1 with \( Y = \nabla(X) \). As to Eq.(2.4), substituting \( X \) by \( \nabla(X) \) in Eq.(2.3) gives \( \nabla(X) \subseteq \Delta(\nabla(X)) \), while applying \( \nabla \) to both sides of it and using Lemma 3 leads to \( \nabla(\Delta(X)) \subseteq \nabla(X) \). Since the composition of mappings is associative \( \nabla(\Delta(X)) = \Delta(\nabla(X)) \), hence \( \nabla(X) \subseteq \Delta(\nabla(X)) = \nabla(\Delta(X)) \subseteq \nabla(X) \), proving Eq.(2.4). Finally, Eq.(2.5) is a consequence of Lemma 3 applied twice. \( \square \)

Corollary 1. \( \Delta(\Delta(X)) = \Delta(X) \) for any \( X \subseteq V \), hence the assignment \( X \mapsto \Delta(X) \) is a closure operator on subsets of \( V \).

Proof. Substituting \( X \) by \( \nabla(X) \) in Eq.(2.4) gives \( \Delta(\Delta(X)) = \Delta(X) \), and combined with Eqs.(2.3) and (2.5) this proves that \( X \mapsto \Delta(X) \) is indeed a closure operator. \( \square \)
Recall [20] that a lattice $L$ is a partially ordered set in which every collection $X \subseteq L$ of lattice elements has a least upper bound (their join $\vee X$) and a greatest lower bound (their meet $\wedge X$). The lattice is bounded if it has a maximal and a minimal element, and it is self-dual if there exists a map $a \mapsto a^\perp$ of $L$ onto itself (the 'duality map'), which is involutive, i.e. $(a^\perp)^\perp = a$, and order-reversing, i.e. $a \leq b$ implies $b^\perp \leq a^\perp$ for all $a, b \in L$. Note that in a self-dual lattice join and meet are related by the duality map via de Morgan’s law $(a \lor b)^\perp = a^\perp \land b^\perp$.

Two self-dual lattices $L_1$ and $L_2$ are isomorphic iff there exists a lattice isomorphism (an order-preserving bijective map whose inverse is also order-preserving) $\phi : L_1 \to L_2$ compatible with the respective duality maps, i.e. such that $\phi(a^\perp) = \phi(a)^\perp$ for all $a \in L_1$. In particular, the automorphism group $\text{Aut}(L)$ of the self-dual lattice $L$ consists of those lattice automorphisms of $L$ that commute with the duality map.

**Theorem 1.** The collection $\mathbb{L} (\nabla) = \{X \subseteq V \mid \Delta(X) = X\}$ ordered by inclusion is a self-dual lattice, with duality map given by $X \mapsto \nabla(X)$.

**Proof.** That $\mathbb{L} (\nabla)$ is a lattice is a general feature of closure operators, so what we have to prove is that the assignment $X \mapsto \nabla(X)$ is a duality map. That $\nabla(X) \in \mathbb{L} (\nabla)$ if $X \in \mathbb{L} (\nabla)$ follows from Eq.(2.4), hence $X \mapsto \nabla(X)$ does indeed map $\mathbb{L} (\nabla)$ into itself; it is order-reversing according to Lemma 3, and involutive by Eq.(2.2).

The lattice $\mathbb{L} (\nabla)$ has as maximal element the set of all vertices, while its minimal element is the set of those (so-called ‘universal’) vertices that are adjacent to every vertex. The meet of $X, Y \in \mathbb{L} (\nabla)$ is their intersection $X \cap Y$, while their join can be expressed using de Morgan’s law as

$$X \lor Y = \nabla(\nabla(X) \cap \nabla(Y)) \quad (2.6)$$

**Lemma 5.** $\Delta(X) = X$ iff $X = \nabla(Y)$ for some $Y \subseteq V$, hence every closed set $X \in \mathbb{L} (\nabla)$ is an intersection of vertex-neighborhoods.
Proof. If $\Delta(X) = X$, then $X = \nabla(Y)$ holds with $Y = \nabla(X)$ according to Eq.(2.2). Conversely, $X = \nabla(Y)$ implies $\Delta(X) = \Delta(\nabla(Y)) = \nabla(Y) = X$ by Eq.(2.4), proving the claim. \hfill $\square$

Lemma 6. The least element of $\mathbb{L}(\nabla)$ that contains $x \in V$ is
$$\Delta(x) = \{y \in V \mid \nabla(x) \subseteq \nabla(y)\} \quad \text{(2.7)}$$
Consequently, for all $X \subseteq V$
$$\Delta(X) = \bigvee_{x \in X} \Delta(x) \quad \text{(2.8)}$$

Proof. It is clear that $x \in \Delta(x)$, and because $\Delta(x)$ equals $\nabla(\nabla(x))$ according to Eq.(2.1), it belongs to $\mathbb{L}(\nabla)$ by Lemma 5. On the other hand, if $X \in \mathbb{L}(\nabla)$ contains $x \in V$, then $\nabla(X) \subseteq \nabla(x)$ by Eq.(2.1), hence $\Delta(x) \subseteq \Delta(X) = X$ by Lemma 3, proving that $\Delta(x)$ is indeed the least element of $\mathbb{L}(\nabla)$ that contains $x \in V$. As to Eq.(2.8), one has
$$\Delta(X) = \nabla(\nabla(X)) = \nabla \left( \bigcap_{x \in X} \nabla(x) \right) = \bigvee_{x \in X} \nabla(\nabla(x)) = \bigvee_{x \in X} \Delta(x)$$
taking into account Eq.(2.6). \hfill $\square$

Lemma 7. An isomorphism between two graphs induces an isomorphism between their associated lattices.

Proof. The graphs $\nabla_1$ and $\nabla_2$ are isomorphic in case there exists a one-to-one correspondence $\phi$ between their vertices such that two vertices of $\nabla_1$ are adjacent iff their images are adjacent vertices of $\nabla_2$, hence
$$\phi(\nabla_1(X)) = \{\phi(y) \mid y \in \nabla_1(x) \text{ for all } x \in X\}$$
$$= \{y \mid y \in \nabla_2(\phi(x)) \text{ for all } x \in X\} = \nabla_2(\phi(X)) \quad \text{(2.9)}$$
for any set $X$ of vertices of $\nabla_1$, implying that $\phi(X)$ belongs to $\mathbb{L}(\nabla_2)$ precisely when $X \in \mathbb{L}(\nabla_1)$ by Lemma 5. As a result, $X \mapsto \phi(X)$ is a one-to-one order-preserving map from $\mathbb{L}(\nabla_1)$ to $\mathbb{L}(\nabla_2)$, and it is compatible with the respective duality maps thanks to Eq.(2.9). \hfill $\square$
As the above result shows, the isomorphism class of the self-dual lattice $\mathbb{L}(\nabla)$ is completely determined by the isomorphism class of the graph $\nabla$, but the converse is far from being true: there are infinitely many non-isomorphic graphs with isomorphic associated lattices. To understand what happens, we need the following notion.

**Definition 1.** Two vertices of a graph are equilocal if their vertex-neighborhoods coincide.

That is, the vertices $x, y \in V$ of the graph $\nabla$ are equilocal when $\nabla(x) = \nabla(y)$. Equilocality is clearly an equivalence relation, whose equivalence classes (the ‘equilocality classes’) partition the set of all vertices. Equilocality is compatible with adjacency in the sense that if two vertices are adjacent, then every pair of vertices from their respective equilocality classes are also adjacent; put another way, the equilocality classes provide a modular partition [2] of the graph.

**Lemma 8.** The neighborhood of a vertex is a union of equilocality classes.

*Proof.* We have to show that if $y \in \nabla(x)$ and $\nabla(y) = \nabla(z)$, then $z \in \nabla(x)$. But this is immediate, since (by the symmetric nature of the adjacency relation) $y \in \nabla(x)$ iff $x \in \nabla(y)$ iff $x \in \nabla(z)$ iff $z \in \nabla(x)$. \hfill \Box

**Corollary 2.** Any $X \in \mathbb{L}(\nabla)$ is a union of equilocality classes.

*Proof.* This follows at once from Lemmas 5 and 8. \hfill \Box

**Definition 2.** The deflation $\nabla^b$ of an undirected graph $\nabla$ is the graph whose vertices are the equilocality classes of $\nabla$, with two of them adjacent if their elements are adjacent in $\nabla$.

**Lemma 9.** The deflation $\nabla^b$ is an irreducible graph, i.e. each of its equilocality classes contains precisely one vertex.
Proof. The neighborhood in $\nabla^b$ of an equilocality class $\mathcal{E}$ consists of those equilocality classes all of whose elements are adjacent to every element of $\mathcal{E}$, hence the union of these classes gives the neighborhood of any element of $\mathcal{E}$. This means that, should two equilocality classes have the same neighborhood in $\nabla^b$, their respective elements would also have equal neighborhoods in $\nabla$, hence the two classes would coincide.

Clearly, the graph determines both its deflation and the collection of its equilocality classes, and conversely, the knowledge of the deflation and of the equilocality classes determines completely the graph. In this sense, the study of arbitrary undirected graphs may be reduced to that of irreducible ones that are isomorphic to their deflation.

For $X \in \mathbb{L}(\nabla)$, let's consider the set $\partial(X)$ of those equilocality classes that are contained in $X$: this is well-defined according to Corollary 2, and satisfies trivially $\bigcup \partial(X) = X$.

**Lemma 10.** For any $X \in \mathbb{L}(\nabla)$ one has
\[
\nabla^b(\partial(X)) = \partial(\nabla(X))
\] (2.10)

**Proof.** A class belongs to $\nabla^b(\partial(X))$ precisely if it is adjacent with every equilocality class contained in $\partial(X)$, that is, if all of its elements are adjacent to every element of $X$, i.e. if it is contained in $\nabla(X)$. \hfill \Box

**Corollary 3.** $\partial(X) \in \mathbb{L}(\nabla^b)$ for every $X \in \mathbb{L}(\nabla)$.

**Proof.** This follows from Lemma 5, since $\partial(X) = \nabla^b(\partial(\nabla(X)))$ for any $X \in \mathbb{L}(\nabla)$ according to Lemma 10. \hfill \Box

**Theorem 2.** The map $X \mapsto \partial(X)$ that assigns to each $X \in \mathbb{L}(\nabla)$ the collection of all equilocality classes contained in it is an isomorphism between the self-dual lattices $\mathbb{L}(\nabla)$ and $\mathbb{L}(\nabla^b)$. 

Proof. The map $X \mapsto \partial(X)$ between $\mathbb{L}(\nabla)$ and $\mathbb{L}(\nabla^b)$ is obviously order-preserving and injective. It is also surjective (hence an isomorphism) because $X = \partial(\cup X)$ for every $X \in \mathbb{L}(\nabla^b)$, and it is compatible with the respective duality maps according to Eq. (2.10).

The importance of Theorem 2 is that it allows to reduce the study of the lattice $\mathbb{L}(\nabla)$ to that of $\mathbb{L}(\nabla^b)$ and the deflation isomorphism $X \mapsto \partial(X)$. In particular, the structure of $\mathbb{L}(\nabla)$ as a self-dual lattice is completely captured by that of $\mathbb{L}(\nabla^b)$. For this reason, we shall usually restrict our attention to irreducible graphs (unless indicated otherwise). It is also of great practical importance for explicit computations, since the cost of determining the equilocality classes and the deflation map grows polynomially with size (= number of vertices), while in case of the associated lattice the growth is exponential.

Let’s go back to the question of classifying all non-isomorphic graphs with isomorphic associated lattices. Theorem 2 goes a long way in answering it, since it shows that graphs with isomorphic deflations have isomorphic associated lattices. Since the deflation of a graph is always irreducible according to Lemma 9, the problem can be reduced to that of classifying all non-isomorphic irreducible graphs with isomorphic associated lattices. To attack this problem, we need to show first that every self-dual lattice is the associated lattice of some irreducible graph.

3 The duality graph

As we have seen in the previous section, one can associate to any undirected graph a self-dual lattice. It is now time to look at the inverse procedure that associates to a self-dual lattice an undirected graph whose associated lattice is isomorphic with the one that we started with.
**Definition 3.** Let \( L \) be a self-dual lattice with duality map \( a \mapsto a^\perp \). The duality graph \( \nabla_L \) is the undirected graph whose vertices are the elements of \( L \), with \( a, b \in L \) adjacent if \( a \leq b^\perp \).

**Remark 1.** One could consider a variant \( \nabla'_L \) of the duality graph that is obtained by reversing the order relation in the definition of \( \nabla_L \), i.e. by declaring the elements \( a, b \in L \) adjacent if \( a \geq b^\perp \), but this won’t lead to anything new, as the duality map \( a \mapsto a^\perp \) provides a natural graph isomorphism between \( \nabla_L \) and \( \nabla'_L \).

**Lemma 11.** The image of \( a \in L \) under the duality map equals the join of its vertex-neighborhood \( \nabla_L(a) \), i.e. \( a^\perp = \bigvee \nabla_L(a) \).

**Proof.** This follows at once from \( \nabla_L(a) = \{ b \in L \mid b \leq a^\perp \} \).

**Corollary 4.** The duality graph \( \nabla_L \) is irreducible.

**Proof.** By Lemma 11, \( \nabla_L(a) = \nabla_L(b) \) implies \( a^\perp = b^\perp \), i.e. \( a = b \).

**Lemma 12.** \( \nabla_L(X) = \nabla_L(\bigvee X) \) for \( X \subseteq L \).

**Proof.** Indeed, 
\[
\nabla_L(\bigvee X) = \{ b \in L \mid \bigvee X \leq b^\perp \} = \{ b \in L \mid a \leq b^\perp \text{ for all } a \in X \} = \nabla_L(X)
\]
by the very definition of the join \( \bigvee X \).

**Corollary 5.** Every element \( X \subseteq \mathbb{L}(\nabla_L) \) is a vertex-neighborhood.

**Proof.** By Lemma 5, any \( X \subseteq \mathbb{L}(\nabla_L) \) can be written as \( X = \nabla_L(Y) \) with a suitable \( Y \subseteq L \), hence \( X = \nabla_L(a) \) by Lemma 12 (with \( a = \bigvee Y \)).

**Theorem 3.** Any self-dual lattice is isomorphic to the associated lattice of its duality graph.
Proof. Let $L$ denote a self-dual lattice. We claim that the map

\[
\Phi : L \rightarrow \mathbb{L}(\nabla_L)
\]

\[
a \mapsto \nabla_L(a^\perp)
\]

that assigns to each lattice element the vertex-neighborhood of its dual provides an isomorphism between $L$ and the associated lattice $\mathbb{L}(\nabla_L)$ of its duality graph $\nabla_L$. Clearly, $\Phi$ is an order-preserving map, being the composite of two (order-reversing) duality maps, and it is surjective by Corollary 5. Because the map

\[
\Psi : \mathbb{L}(\nabla_L) \rightarrow L
\]

\[
X \mapsto \bigvee X
\]

is inverse to $\Phi$ according to Lemma 11, $\Phi$ is actually bijective, hence a lattice isomorphism. Finally, $\Phi$ is compatible with the respective duality maps since clearly $\Phi(a^\perp) = \nabla_L(a)$ for $a \in L$.

It follows from the above that every self-dual lattice is the associated lattice of some irreducible graph (e.g. of its own duality graph). But it should be emphasized that this correspondence is far from being one-to-one, for there could (and usually does) exist several non-isomorphic irreducible graphs with isomorphic associated lattices. To get control over them, we shall need the notions of essential vertices and the radical of a graph, to be introduced in the next section.

4 Essential vertices and the radical

As we have seen previously, one can associate a self-dual lattice to any undirected graph, and it is natural to ask to what extent does this associated lattice characterize the graph itself. To some extent this has been answered by Theorem 2, showing that graphs with isomorphic deflations have isomorphic associated lattices, allowing to reduce the question to the case of irreducible graphs. But there are usually sev-
eral different irreducible graphs with isomorphic associated lattices: in particular, Theorem 3 asserts that this is the case for any irreducible graph that is not isomorphic with the duality graph of its associated lattice. To settle this issue, we have to look at the associated lattices of induced subgraphs.

Recall that the induced subgraph $\nabla^W$ corresponding to a collection $W$ of vertices of the (undirected) graph $\nabla$ has as vertices the elements of $W$, with two of them adjacent precisely when they are adjacent as vertices of $\nabla$: this means that $\nabla^W(x) = \nabla(x) \cap W$ for $x \in W$, and in general $\nabla^W(X) = \nabla(X) \cap W$ for any $X \subseteq W$. Note that an induced subgraph of an irreducible graph is not necessarily irreducible. The importance of induced subgraphs stems from the following result.

**Lemma 13.** Every irreducible graph is isomorphic with an induced subgraph of the duality graph of its associated lattice.

**Proof.** Let $L = \mathbb{L}(\nabla)$ denote the associated lattice of the irreducible graph $\nabla$, and consider the collection $W = \{\Delta(x) \mid x \in V\} \subseteq L$ of lattice elements. The vertices of $\nabla$ are in one-to-one correspondence with the elements of $W$ since $\Delta(x) = \Delta(y)$ implies $x = y$ by irreducibility, and the vertices $\Delta(x)$ and $\Delta(y)$ of $\nabla^W_L$ are adjacent iff $\Delta(x) \subseteq \nabla(\Delta(y)) = \nabla(y)$, which is equivalent to $x \in \nabla(y)$ by Lemma 6, i.e. the adjacency of the vertices $x$ and $y$ of $\nabla$, proving that $\nabla^W_L$ is indeed isomorphic to $\nabla$. \qed

The following result will be essential in the proof of Theorem 4.

**Lemma 14.** If $W$ denotes a collection of vertices of the graph $\nabla$, then $\mathbb{L}_W(\nabla) = \{\nabla(X) \mid X \subseteq W\}$ is a subset of the associated lattice $\mathbb{L}(\nabla)$, and $\mathbb{L}(\nabla^W) = \{X \cap W \mid X \in \mathbb{L}_W(\nabla)\}$. Moreover, $\Delta(X \cap W) = X$ and $\nabla^W(X \cap W) = \nabla^W(X)$ in case $X \in \mathbb{L}(\nabla)$ satisfies $\nabla(X) \in \mathbb{L}_W(\nabla)$.

**Proof.** $X \in \mathbb{L}_W(\nabla)$ means that there exists $X^* \subseteq W$ such that $X = \nabla(X^*)$, hence $X \in \mathbb{L}(\nabla)$ and $X \cap W = \nabla^W(X^*) \in \mathbb{L}(\nabla^W)$ by Lemma 5. As to
the second statement, $\nabla(X) \in \mathbb{L}_W(\nabla)$ if there exists $X^* \subseteq W$ such that $\nabla(X) = \nabla(X^*)$, which implies $X^* \subseteq \Delta(X^*) = X$ according to Eq.(2.3), i.e. $X^* \subseteq X \cap W \subseteq X$. As a consequence, $X = \Delta(X^*) \subseteq \Delta(X \cap W) \subseteq X$ by Eq.(2.4), proving that indeed $\Delta(X \cap W) = X$, and applying $\nabla$ to both sides leads to $\nabla(X \cap W) = \nabla(X)$, hence $\nabla^W(X \cap W) = \nabla^W(X)$. \hfill $\square$

Let’s recall that an element of a lattice is called (completely) meet-irreducible (resp. join-irreducible), if any collection of lattice elements whose meet (resp. join) is the given element necessarily contains that element; note that in a self-dual lattice the duality map interchanges meet-irreducible elements with join-irreducible ones. A meet-(resp. join-)irreducible decomposition of a lattice element $a$ is a collection $A$ of meet-(resp. join-)irreducible elements whose meet (resp. join) equals $a$, and the decomposition is irredundant if no proper subset of $A$ has this last property. In a finite lattice all lattice elements have (not necessarily unique) irredundant irreducible decompositions.

**Lemma 15.** Any meet-irreducible element of the lattice $\mathbb{L}(\nabla)$ is the neighborhood $\nabla(x)$ of some vertex $x \in V$.

**Proof.** According to Eq.(2.1), there should exist some $x \in \nabla(X)$ such that $X = \nabla(x)$ in case $X \in \mathbb{L}(\nabla)$ is meet-irreducible. \hfill $\square$

The above result justifies the following notion.

**Definition 4.** A vertex $x$ of an undirected graph $\nabla$ is essential if its neighborhood $\nabla(x)$ is a meet-irreducible (or, what is the same, $\Delta(x)$ is a join-irreducible) element of the associated lattice $\mathbb{L}(\nabla)$. A stem graph is an irreducible graph all of whose vertices are essential.

We shall denote by $\mathcal{E}(\nabla)$ the collection of all essential vertices of the graph $\nabla$. Notice that either none or all vertices in an equilocality class are essential, hence it makes sense to speak of essential classes. What is more, the deflation map provides a one-to-one correspondence
between the essential classes and the essential vertices of the deflation, i.e. \( \mathcal{E}(\nabla^b) = \partial \mathcal{E}(\nabla) \).

For duality graphs of self-dual lattices one has the following lattice-theoretic characterization of essential vertices.

**Lemma 16.** A vertex of the duality graph of a self-dual lattice is essential iff it is a join-irreducible lattice element.

**Proof.** According to Definition 4, \( a \in L \) belongs to \( \mathcal{E}(\nabla_L) \) iff \( \nabla_L(a) \) is a meet-irreducible element of \( L(\nabla_L) \), i.e. \( \nabla_L(X) = \nabla_L(a) \) with \( X \subseteq L \) would imply \( a \in X \); since \( \nabla_L(X) = \nabla_L(\sqcup X) \) by Lemma 12, and because the duality graph \( \nabla_L \) is irreducible by Corollary 4, this is equivalent to the requirement that \( a = \sqcup X \) should imply \( a \in X \), i.e. that \( a \in L \) is join-irreducible, as claimed.  

While our definition of essential vertices relies on lattice-theoretic notions, there is a purely graph-theoretic characterization of them.

**Proposition 1.** A vertex is essential iff its neighborhood is properly contained in the intersection of all vertex-neighborhoods that properly contain it.

**Proof.** Since \( y \in \Delta(x) \) iff \( \nabla(x) \subseteq \nabla(y) \) for any vertex \( x \in V \), the set \( \Delta^x_x = \{ y \in \Delta(x) | \nabla(y) \neq \nabla(x) \} \) consists of all those elements \( y \in \Delta(x) \) that do not belong to the equilocality class of \( x \), hence it is a proper subset of \( \Delta(x) \), and its dual \( \nabla(\Delta^x_x) \) equals the intersection of all the vertex neighborhoods that properly contain the neighborhood \( \nabla(x) \) of \( x \), hence \( \nabla(x) \subseteq \nabla(\Delta^x_x) \), and this containment is proper iff the containment \( \Delta(\Delta^x_x) \subseteq \Delta(x) \) is proper. Since the difference \( \Delta(x) \setminus \Delta^x_x = \{ y | \nabla(y) = \nabla(x) \} \) is nothing but the equilocality class of \( x \), and because \( \Delta(\Delta^x_x) \) is a union of equilocality classes by Lemma 8, either \( \Delta(\Delta^x_x) = \Delta^x_x \) or \( \Delta(\Delta^x_x) = \Delta(x) \), hence the containment \( \Delta(\Delta^x_x) \subseteq \Delta(x) \) is proper iff \( \Delta^x_x \in \mathbb{L}(\nabla) \), and we claim that \( \Delta^x_x \in \mathbb{L}(\nabla) \) precisely when
Δ(x) is join-irreducible, i.e. the vertex x is essential. Indeed, Δ(x) is join-irreducible iff any subset $X^* \subseteq Δ(x)$ such that $Δ(x) = Δ(X^*)$ contains a vertex from the equilocality class of x, i.e. has at least one element not contained in $Δ_x$, and this happens iff $Δ_x \in L(∇)$. □

The importance of essential vertices stems from the following result.

**Theorem 4.** The associated lattice $L(∇^W)$ of the induced subgraph $∇^W$ corresponding to a collection $W$ of vertices is isomorphic with the associated lattice of $∇$ iff $W$ contains all essential vertices.

**Proof.** To prove the only if part, notice that the isomorphism of two lattices implies that they have the same number of elements. But one has $L(∇^W) = \{X \cap W \mid X \in L_W(∇)\}$ and $L_W(∇) \subseteq L(∇)$ according to Lemma 14, consequently $|L(∇^W)| \leq |L_W(∇)| \leq |L(∇)|$, which implies that $L(∇^W)$ and $L(∇)$ cannot be isomorphic unless $L_W(∇) = L(∇)$, hence for every $X \in L(∇)$ there exists $X^* \subseteq W$ such that $X = ∇(X^*)$. This should be true in particular for the neighborhood $∇(x)$ of any essential vertex $x \in E(∇)$, and because $∇(x)$ is meet-irreducible in that case, $∇(x) = ∇(X^*)$ implies that $x \in X^* \subseteq W$, proving that indeed $E(∇) \subseteq W$ if $L(∇^W)$ and $L(∇)$ are isomorphic.

As to the if part, since any lattice element can be written as the meet of a suitable collection of meet-irreducible elements, and the latter are, according to Eq. (2.3), all of the form $∇(x)$ for some $x \in E(∇)$, the condition $E(∇) \subseteq W$ means that for any $X \in L(∇)$ there is a subset $X^* \subseteq W$ such that $X = ∇(X^*)$, i.e. $L_W(∇) = L(∇)$. This implies $Δ(X \cap W) = X$ for $X \in L(∇)$, according to Lemma 14, since $∇(X) \in L(∇) = L_W(∇)$. Consequently, we have a pair of surjective order-preserving maps

$$\text{res}_W : L(∇) \to L(∇^W) \quad \text{and} \quad Δ : L(∇^W) \to L(∇)$$

$$X \mapsto X \cap W \quad \text{and} \quad X \mapsto Δ(X)$$

that are mutually inverse to each other (i.e. lattice isomorphisms), and
which are compatible with the respective duality maps since
\[(\nabla^W \circ \text{res}_W)(X) = \nabla^W(X \cap W) = \nabla^W(X) = \nabla(X) \cap W = (\text{res}_W \circ \nabla)(X)\]
as follows once again from Lemma 14.

Theorem 4 highlights the importance of essential vertices and the corresponding induced subgraph, leading to the following notion.

**Definition 5.** The radical $\sqrt{\nabla}$ of the irreducible graph $\nabla$ is the induced subgraph corresponding to the collection of its essential vertices.

**Lemma 17.** The radical of an irreducible graph is itself irreducible.

*Proof.* According to Definition 4, a vertex is essential iff the smallest element of the associated lattice containing it is join-irreducible. \(\square\)

Note that the radical need not be connected, and that a stem graph (all of whose vertices are essential) coincides with its radical.

**Lemma 18.** An isomorphism between irreducible graphs induces an isomorphism between their radicals.

*Proof.* It follows from Proposition 1 that essential vertices are mapped to essential ones by a graph isomorphism, hence the corresponding induced subgraphs (the radicals) are isomorphic too. \(\square\)

**Theorem 5.** Two irreducible graphs have isomorphic associated lattices iff their radicals are isomorphic.

*Proof.* That irreducible graphs having isomorphic radicals have isomorphic associated lattices follows from Theorem 4 and Lemma 7. To prove the converse, we shall show that the radical of an irreducible graph is isomorphic with that of the duality graph of its associated lattice: this implies the assertion, because an isomorphism between self-dual lattices induces obviously an isomorphism between their duality graphs, hence between their radicals by Lemma 18.
Let $L = \mathbb{L}(\nabla)$ denote the associated lattice of the irreducible graph $\nabla$. The radical $\sqrt{\nabla_L}$ of the duality graph $\nabla_L$ is the unique induced subgraph of $\nabla_L$ with the least number of vertices that has an associated lattice isomorphic to $L$. On the other hand, the associated lattice of $\sqrt{\nabla}$ is also isomorphic to $L$ according to Theorem 4, hence the radical $\sqrt{\nabla}$ is isomorphic to an induced subgraph of $\nabla_L$ by Theorem 3 having $|\mathcal{E}(\nabla)|$ vertices. Since this equals the number $|\mathcal{E}(\nabla_L)|$ of vertices of $\sqrt{\nabla}_L$, the graphs $\sqrt{\nabla}$ and $\sqrt{\nabla}_L$ are isomorphic.

The above results show that there is a one-to-one correspondence between (isomorphism classes of) stem graphs and self-dual lattices: to each stem graph corresponds its associated lattice, and to each self-dual lattice corresponds a stem graph, the radical of its duality graph. This means that in order to classify self-dual lattices one could instead classify stem graphs, which could prove easier through the use of suitable graph-theoretic techniques. To this end one should note that, while a stem graph need not be connected, it is the disjoint union of its connected components. The connected stem graphs with less than 4 vertices are displayed on Figure 1 on page 18.

The number of connected stem graphs grows steadily with the number of vertices, as illustrated in Table 1 on page 19: according to the above, this is equal to the number of isomorphism classes of self-dual lattices with a given number of join-irreducible element. The table also gives the number of non-isomorphic irreducible graphs with a modular associated lattice; as we shall see later, these are those that can (in
Table 1: Number of non-isomorphic connected graphs of different types as a function of the number of vertices (the last column giving the number of irreducible graphs with a modular associated lattice).

| # vertices | # irreducible | # stem | # modular |
|------------|---------------|-------|----------|
| 1          | 2             | 2     | 2        |
| 2          | 2             | 1     | 2        |
| 3          | 6             | 3     | 6        |
| 4          | 31            | 18    | 24       |
| 5          | 230           | 140   | 95       |
| 6          | 2683          | 1716  | 439      |
| 7          | 50922         | 33448 | 2362     |

principle) arise as locality diagrams of conformal models.

The following result provides a partial converse to Lemma 13.

**Lemma 19.** If the collection $\mathcal{W} \subseteq L$ contains all join-irreducible elements of the self-dual lattice $L$, then the induced subgraph $\nabla_L^W$ of the duality graph is irreducible.

**Proof.** Since $\mathcal{W} \subseteq L$ contains all join-irreducible elements of $L$, for every $a \in L$ there exists a subset $A \subseteq \mathcal{W}$ such that $\bigvee A = a^\perp$, and because $\nabla_L^W(a) = \{ b \in \mathcal{W} \mid b \leq a^\perp \}$, one has $A \subseteq \nabla_L^W(a)$. But this implies that $a^\perp = \bigvee A \leq \bigvee \nabla_L^W(a) \leq a^\perp$, i.e. $\bigvee \nabla_L^W(a) = a^\perp$, and this proves that indeed $\nabla_L^W(a) = \nabla_L^W(b)$ iff $a = b$. \qed

While the induced subgraphs of the duality graph corresponding to different subsets $\mathcal{E}(\nabla_L) \subseteq \mathcal{W} \subseteq L$ comprise a full list of those irreducible graphs whose associated lattice is isomorphic with $L$, there might exist non-trivial isomorphisms between such graphs, corresponding to suitable automorphisms of the self-dual lattice $L$. This issue is settled by the following result.
Proposition 2. Given two subsets \( W_1, W_2 \subseteq L \) of a self-dual lattice \( L \) that contain all join-irreducible elements, the induced subgraphs of the duality graph \( \nabla_L \) corresponding to \( W_1 \) and \( W_2 \) are isomorphic iff there exists an automorphism \( \sigma \in \text{Aut}(L) \) such that \( W_2 = \sigma(W_1) \).

Proof. If \( \sigma \in \text{Aut}(L) \) satisfies \( W_2 = \sigma(W_1) \), then for \( a \in W_1 \) one has
\[
\sigma \left( \nabla_L^{W_1}(a) \right) = \{ \sigma(b) \mid b \in W_1, b \leq a^\perp \} = \{ b \in W_2 \mid b \leq \sigma(a^\perp) \} = \nabla_L^{W_2}(\sigma(a))
\]
hence the restriction of \( \sigma \) to \( W_1 \) provides a graph isomorphism between the induced subgraphs \( \nabla_L^{W_1} \) and \( \nabla_L^{W_2} \).

The other way round, if \( \nabla_L^{W_1} \) and \( \nabla_L^{W_2} \) are isomorphic, i.e. there exists a bijective map \( \phi : W_1 \to W_2 \) such that \( \phi \left( \nabla_L^{W_1}(a) \right) = \nabla_L^{W_2}(\phi(a)) \) for all elements \( a \in W_1 \), then this map can be extended to an isomorphism \( \hat{\phi} : \mathbb{L}(\nabla_L^{W_1}) \to \mathbb{L}(\nabla_L^{W_2}) \) of lattices compatible with the duality maps according to Lemma 7. On the other hand, by Theorem 4 there exists lattice isomorphisms \( \Omega_1 : L \to \mathbb{L}(\nabla_L^{W_1}) \) and \( \Omega_2 : L \to \mathbb{L}(\nabla_L^{W_2}) \). It follows that the composite map \( \sigma = \Omega_2^{-1} \circ \hat{\phi} \circ \Omega_1 \) is an automorphism of \( L \) that commutes with the duality map, \( \sigma(a^\perp) = \sigma(a)^\perp \) for \( a \in L \), and satisfies \( W_2 = \sigma(W_1) \) because \( \sigma(a) = \phi(a) \) for \( a \in W_1 \).

Remark 2. Based on the above, one can show that the number of non-isomorphic irreducible graphs whose associated lattice is isomorphic with a given the self-dual lattice \( L \) equals
\[
\frac{1}{|\text{Aut}(L)|} \sum_{\sigma \in \text{Aut}(L)} 2^{\lambda(\sigma) - \epsilon(\sigma)}
\]
where \( \lambda(\sigma) \), resp. \( \epsilon(\sigma) \) denotes the number of orbits of \( \sigma \in \text{Aut}(L) \) on the set of all (resp. join-irreducible) lattice elements, and the following 'mass formula' holds
\[
\sum_{\nabla} \frac{1}{|\text{Aut}(\nabla)|} = \frac{1}{|\text{Aut}(L)|} \left( \frac{|L| - |\mathcal{J}|}{n - |\mathcal{J}|} \right)
\]
where \( \mathcal{J} \) denotes the set of join-irreducible elements of \( L \), and the sum runs over all irreducible graphs with \( |\nabla| = n \) vertices and associated lattice isomorphic to \( L \).

Finally, let’s note the following interesting result, which proves important in physics applications.

**Lemma 20.** The length of a maximal chain ending at \( X \in \mathbb{L}(\nabla) \) cannot exceed the number of essential classes contained in it.

**Proof.** Let \( \delta(X) \) denote the number of equilocality classes contained in \( X \in \mathbb{L}(\nabla) \), which makes sense thanks to Lemma 15. We claim that the length of a maximal chain ending at \( X \) cannot exceed \( \delta(X) \). Indeed, \( \delta(Y) < \delta(X) \) for any \( Y \in \mathbb{L}(\nabla) \) properly contained in \( X \), hence for a maximal chain \( Y_0 \subset Y_1 \subset \cdots \subset Y_n = X \) ending at \( X \) one has \( \delta(Y_0) < \delta(Y_1) < \cdots < \delta(X) \), and by taking into account that the values \( \delta(X) \) are non-negative integers this proves \( n \leq \delta(X) \). Since the associated lattices of \( \nabla \) and \( \sqrt{\nabla} \) are isomorphic by Theorem 4, and all equilocality classes of the latter are essential, applying this result to the radical \( \sqrt{\nabla} \) instead of \( \nabla \) proves the claim. \( \square \)

## 5 Locality graphs and FC sets

Consider a (unitary) rational CFT [10, 18]. We shall denote by \( d_\rho \) and \( h_\rho \) the quantum dimension and conformal weight of a primary \( \rho \), and by \( N(\rho) \) the associated fusion matrix, whose matrix elements are given by the fusion rules

\[
[N(\rho)]^r_q = N^{r_{pq}}_{pq}
\]

\( \rho \) will denote the vacuum primary for which \( h_0 = 0 \) and \( N(\rho) \) is the identity matrix (hence \( d_0 = 1 \)). Note that, since

\[
N(\rho) N(q) = \sum_r N^{r_{pq}}_{pq} N(r)
\]
the fusion matrices generate a commutative matrix algebra over \( \mathbb{C} \), whose irreducible representations, all of dimension 1, are in one-to-one correspondence with the primaries. According to Verlinde’s famous result [22], to each primary \( p \) corresponds an irreducible representation \( \rho_p \) that assigns to the fusion matrix \( N(q) \) the complex number

\[
\rho_p(q) = \sum_r N_{pq}^r \frac{d_r}{d_p} e^{2\pi i (h_p + h_q - h_r)}
\]

(5.3)

Note that \( d_p = \rho_0(p) \), and one has the inequality

\[
|\rho_p(q)| \leq d_q
\]

(5.4)

The theory of orbifold deconstruction [6, 8] points to the importance of so-called fusion closed sets, or 'FC sets' for short, i.e. sets of primaries containing the vacuum 0 and closed under the fusion product. The collection \( \mathcal{L} \) of all FC sets (ordered by inclusion) forms a modular lattice [7, 3] that is also self-dual, i.e. comes equipped with a duality map sending each FC set \( g \in \mathcal{L} \) to \( g^\perp = \{ p \mid \rho_q(p) = d_p \text{ for all } q \in g \} \), its so-called 'trivial class', which is itself an FC set. An important task is to describe the structure of this lattice \( \mathcal{L} \) characterizing the different possible deconstructions of the model under study.

The problem is that a brute force approach to determine all FC sets of a given model can be prohibitively difficult. Indeed, the cost of such a procedure is exponential in the number of primaries, and it breaks down already for a couple of dozens of primary fields, while truly interesting examples come with several hundreds, if not thousands of them. This is where the previous graph-theoretic ideas come to the rescue, as we shall now explain.

Recall that two primaries of a conformal model are mutually local if their operator product expansion coefficients are single-valued functions of separation [9]. In other words, \( p \) and \( q \) are local if \( h_p + h_q \) differs by an integer from \( h_r \) for each primary \( r \) such that \( N_{pq}^r > 0 \); using Verlinde’s formula, this is equivalent to the requirement \( \rho_q(p) = d_p \). Clearly, mutual locality is a symmetric relation, to which is associated
an undirected graph $\Lambda$, the so-called locality graph of the conformal model, whose vertices correspond to the primary fields, with two vertices adjacent if the corresponding primaries are mutually local\footnote{There is a closely related notion, that of the 'quasi-locality graph', whose vertices still correspond to the primary fields, with two of them adjacent if they saturate the bound Eq.\((5.4)\), i.e. $|\rho_p(q)|=d_q$; at the level of OPE, this means that the expansion coefficients are no more necessarily single-valued, but some power of them is. One can show that the associated lattice of this graph is a sublattice of $L(\Lambda)$, with elements corresponding to maximal Abelian extensions of FC sets (c.f. Section 5 of \cite{7}).}

In what follows, we shall make free use of results from \cite{7} on FC sets and their lattice $\mathcal{L}$, especially the properties of the duality map. The basic observation, motivating the present work, is the following one.

**Lemma 21.** The vertex-neighborhood

$$\Lambda(\alpha) = \{ p | \rho_\alpha(p) = d_p \}$$

(5.5)

of a primary $\alpha$ in the locality graph is an FC set.

**Proof.** For $p, q \in \Lambda(\alpha)$ one has

$$\sum_r N'^r_{pq}(d_r - \rho_\alpha(r)) = \sum_r N'^r_{pq}d_r - \sum_r N'^r_{pq}\rho_\alpha(r) = d_pd_q - \rho_\alpha(p)\rho_\alpha(q) = 0$$

hence, after taking real parts,

$$\sum_r N'^r_{pq}(d_r - \Re(\rho_\alpha(r))) = 0$$

All terms on the lhs. are non-negative since $\Re(\rho_\alpha(r)) \leq |\rho_\alpha(r)| \leq d_r$, and this implies that $N'^r_{pq} = 0$ unless $\rho_\alpha(r) = d_r$, i.e. $r \in \Lambda(\alpha)$. \qed

**Lemma 22.** $\Lambda(\alpha)^\perp$ is the smallest FC set containing the primary $\alpha$.

**Proof.** $p \in \Lambda(\alpha)^\perp$ precisely when $\rho_p(q) = d_q$ for all $q \in \Lambda(\alpha)$, and since this holds for $\alpha$ by definition, one has $\alpha \in \Lambda(\alpha)^\perp$. Moreover, if $g \in \mathcal{L}$ contains $\alpha$, then $p \in g^\perp$ implies $\rho_p(\alpha) = d_\alpha$, hence $\rho_\alpha(p) = d_p$, that is $p \in \Lambda(\alpha)$; consequently, $g^\perp \subseteq \Lambda(\alpha)$, which gives $\Lambda(\alpha)^\perp \subseteq g$ by duality, proving the claim. \qed
Lemma 23.

\[ \Lambda(\alpha)^\perp = \{ \beta \mid \Lambda(\alpha) \subseteq \Lambda(\beta) \} \]

Proof. \( \beta \in \Lambda(\alpha)^\perp \) implies \( \Lambda(\beta)^\perp \subseteq \Lambda(\alpha)^\perp \) by Lemma 22, hence \( \Lambda(\alpha) \subseteq \Lambda(\beta) \); conversely, \( \Lambda(\alpha) \subseteq \Lambda(\beta) \) implies \( \Lambda(\beta)^\perp \subseteq \Lambda(\alpha)^\perp \), from which follows \( \beta \in \Lambda(\alpha)^\perp \) because of \( \beta \in \Lambda(\beta)^\perp \). \( \square \)

Corollary 6. If \( g \in \mathcal{L} \) is an FC set, then

\[ g = \bigvee_{\alpha \in g} \Lambda(\alpha)^\perp \]  \hspace{1cm} (5.6)

and

\[ g^\perp = \bigcap_{\alpha \in g} \Lambda(\alpha) \]  \hspace{1cm} (5.7)

Proof. Eq.(5.6) is a direct consequence of Lemma 22, and Eq.(5.7) follows from it by duality. \( \square \)

Since, according to Lemma 5, any element of an associated lattice can be obtained as a meet of vertex neighborhoods, it follows that \( \mathbb{L}(\Lambda) \) is a sublattice of \( \mathcal{L} \). Actually, a much stronger statement is true.

Theorem 6. The self-dual lattices \( \mathcal{L} \) and \( \mathbb{L}(\Lambda) \) coincide.

Proof. By the above, all elements of \( \mathbb{L}(\Lambda) \) belong to \( \mathcal{L} \). Since both lattices are ordered by inclusion, what we have to show is that, conversely, every \( g \in \mathcal{L} \) belongs to \( \mathbb{L}(\Lambda) \), and that the respective duality maps coincide, i.e. \( g^\perp = \Lambda(g) \) for \( g \in \mathcal{L} \). But \( g^\perp = \bigcap_{\alpha \in g} \Lambda(\alpha) = \Lambda(g) \) by Eq.(5.7), and because \( g^\perp \in \mathcal{L} \), one concludes that \( g = (g^\perp)^\perp = \Lambda(g^\perp) \) belongs to \( \mathbb{L}(\Lambda) \), proving the claim. \( \square \)

Theorem 6 was the main motivation behind the theory presented before, since it makes available the results about associated lattices of undirected graphs in the study of the deconstruction lattice \( \mathcal{L} \). Not only does it underline the importance of the locality graph, but also
draws attention onto the so-called ‘locality diagram’ of the model (the deflation of its locality graph) and the equilocality classes of primaries. As an illustration, let’s note the following interesting result.

**Lemma 24.** If the quantum dimension of a primary is a rational integer, then the same is true for all primaries in its equilocality class.

**Proof.** According to Corollary 11 of [7], the set \( \mathcal{I} = \{ p \mid d_p \in \mathbb{Z} \} \) of all primaries with integer quantum dimension is an FC set, hence it belongs to \( \mathbb{L}(\Lambda) \) according to Theorem 6. Consequently, \( \mathcal{I} \) is a union of equilocality classes by Corollary 2, hence \( p \in \mathcal{I} \) implies that all primaries in the equilocality class of \( p \) also belongs to \( \mathcal{I} \).

**Remark 3.** Let us mention that, using some elementary Galois theory together with the results of [7], one can prove the following generalization of Lemma 24: the algebraic number field generated by the quantum dimension of a primary is the same for all primaries in the same equilocality class.

As to the locality diagram, i.e. the deflation of the locality graph, it is clear that its knowledge, together with that of the primary content of the individual equilocality classes (i.e. the deflation map) does completely determine the locality graph itself. But the associated lattice of a graph and of its deflation are isomorphic according to Theorem 2, and this means that the locality diagram does determine the structure (as a self-dual lattice) of the deconstruction lattice \( \mathcal{L} \). In particular, fairly different conformal models could have isomorphic deconstruction lattices provided their locality diagrams are the same. From a practical point, since the locality diagram has usually much less vertices than the locality graph, these observations lead to a dramatic simplification in the computation of the deconstruction lattice.

But there is more to all this, as computational evidence suggests that many different conformal models of similar origin have the same
(up to isomorphism) locality diagram: for example, all unitary Virasoro minimal models, except for two, share the same 'generic Virasoro diagram', while for superconformal minimal models one has two such generic diagrams. The situation gets more complicated for other classes (like parafermionic or Gepner models), but there is still a clear pattern in the structure of the diagrams, and this suggests that locality diagrams can provide a (coarse) classification of conformal models.\footnote{Actually, since the notion of mutual locality does rely exclusively on the modular data of the conformal model, locality diagrams make sense in the larger context of Modular Tensor Categories \cite{21,1}, and can be used to classify the latter.}

Finally, let us note that locality diagrams of conformal models have a fairly restricted structure, since each

1. is irreducible, being the deflation of the locality graph;
2. has an associated lattice that has to be modular according to Theorem 2 of \cite{7};
3. has a 'universal' vertex adjacent to every vertex (including itself) that corresponds to the equilocality class containing the vacuum.

Of all the irreducible graphs of a given size, these criteria select out a small number of graphs: for example, the number of non-isomorphic irreducible graphs of a given size satisfying suitable combinations of these criteria can be read off Table 2 on page 27. What is really interesting is that computational evidence suggests that only a small fraction of all those graphs that satisfy these criteria does actually correspond to the locality diagram of some conformal model. Actually, an even stronger statement seems to be true: there is a one-to-one correspondence between locality diagrams of conformal models and their deconstruction lattices. This observation is indeed intriguing, as there are in principle several different irreducible graphs with a given associated lattice satisfying all the above criteria, but only one of these
TABLE 2: Statistics of connected graphs with a given number of vertices satisfying different combinations of the criteria for locality diagrams, namely criterion 1 (irreducibility), criterion 2 (modularity of the associated lattice) and criterion 3 (existence of universal vertex), the last column giving the number of known locality diagrams of the given size.

| # vertices | 1 | 1 & 2 | 1 & 2 & 3 | known locality diagrams |
|------------|---|-------|-----------|-------------------------|
| 3          | 6 | 6     | 3         | 1                       |
| 4          | 31| 24    | 8         | 4                       |
| 5          | 230| 95  | 27        | 2                       |
| 6          | 2683| 439 | 98        | 3                       |
| 7          | 50922| 2362| 443       | 2                       |

seems to be realized as the locality diagram of some suitable conformal model. Clarifying this issue could lead to a better understanding of conformal models (and Modular Tensor Categories).

Actually, there is a fourth, more subtle criterion for the realizability of an irreducible graph $\nabla$ as the locality diagram of some conformal model, that follows from Lemma 9 of [7]: there should exits a positive function $\mu$ on the set of vertices (in case of locality diagrams, $\mu$ equals the sum of dimensions squared of the primaries in the corresponding equilocality class) such that the product

$$\left( \sum_{x \in X} \mu(x) \right) \left( \sum_{y \in \nabla(X)} \mu(y) \right)$$

is the same for every $X \in \mathbb{L}(\nabla)$. This is equivalent to the existence of a strictly positive solution for a set of quadratic equations determined by the graph, and that this criterion is meaningful is exemplified by the graphs depicted on Figure 2 on page 28, which satisfy all the above criteria except for this last one. This allows to effectively reduce the number of allowed diagrams with 3 vertices from 3 to 1, and from 8 to
5 in case of 4 vertices. Unfortunately, this ’positivity’ criterion can be prohibitively difficult to check for graphs with 5 or more vertices.

An interesting numerical attribute of locality diagrams is their ’dimension’, the length of the longest chain between the minimal and the maximal elements of their associated lattices, whose existence is guaranteed by the modularity of the latter. It follows from Lemma 20 that the dimension cannot exceed the number of essential vertices, hence it can be quite small even for large irreducible graphs. In particular, there are infinitely many locality diagrams of dimension 2, corresponding to holomorphic orbifolds whose twist group is cyclic of prime order.

6 Summary and outlook

As we have seen, there is an intimate relationship between undirected graphs (with loops) and self-dual lattices: each graph determines a lattice, its associated lattice, and conversely, to each self-dual lattice corresponds a graph, its duality graph, whose associated lattice is the original one. While several (as a matter of fact, infinitely many) different graphs share the same associated lattice, one has a nice description of all these using Theorem 2, Lemma 13 and Theorem 4. The utility of this for physics stems from the observation that the deconstruction lattice of a conformal model, of prime interest for orb-
ifold deconstruction, is nothing but the associated lattice of the locality graph of the model, making available graph-theoretic tools in the study of the deconstruction lattice. Not only does one get access to graph theory as a powerful tool, but this connection brings to front such new concepts as locality diagrams and equilocality classes in the description of conformal models (and more generally, of Modular Tensor Categories).

There are many interesting questions related to this circle of ideas that merit further elaboration. An obvious one is to understand the pattern that governs the structure of the locality diagrams of models from some specified class, e.g. parafermionic or superconformal. Closely related to the previous question is to explain why there seems to be a unique locality diagram for all conformal models with the same deconstruction lattice. Another problem is related to the study of the possible degenerations of locality diagrams: for example, in case of unitary Virasoro minimal models, all diagrams but two are isomorphic with each other, but even the two exceptions may be understood as degenerations of the generic diagram, due to the fact that the corresponding models have too few primaries to effectively fill all the equilocality classes of the latter. Similar phenomena show up in all cases known to us, making it an interesting point to describe the different possible degenerations of a given diagram. There is also the question of what are common the attributes of primaries from the same equilocality class, and to which extent do locality diagrams offer a meaningful classification scheme for conformal models. We strongly believe that the study of these questions could lead to important new developments in the field.
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