Bisimulations for intuitionistic temporal logics

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Abstract

We introduce bisimulations for the logic ITLe with ◯ (‘next’), U (‘until’) and R (‘release’), an intuitionistic temporal logic based on structures (W, ≼, S), where ≼ is used to interpret intuitionistic implication and S is a ≼-monotone function used to interpret the temporal modalities. Our main results are that ◯ (‘eventually’), which is definable in terms of U, cannot be defined in terms of ◯ and ◻, and similarly that ◻ (‘henceforth’), definable in terms of R, cannot be defined in terms of ◯ and U, even over the smaller class of here-and-there models.

1 Introduction

The definition and study of full combinations of modal [5] and intuitionistic [6, 23] logics can be quite challenging [30], and temporal logics, such as LTL [28], are no exception. Some intuitionistic analogues of temporal logics have been proposed, including logics with ‘past’ and ‘future’ tenses [9] or with ‘next’ [7, 19], and ‘henceforth’ [17]. We proposed an alternative formulation in [4], where we defined the logics ITLe and ITLp using semantics similar to those of expanding and persistent products of modal logics, respectively [13], and the tenses ◯ (‘next’), ◯ (‘eventually’), and ◻ (‘henceforth’). ITLe in particular differs from previous proposals (e.g. [9, 27]) in that we consider minimal frame

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conditions that allow for all formulas to be upward-closed under the intuitionistic preorder, which we denote \( \leq \). We then showed that \( \text{ITL}^e \) with \( \Diamond \) (‘next’), \( \Diamond \) (‘eventually’), and \( \Box \) (‘henceforth’) is decidable, thus obtaining the first intuitionistic analogue of LTL which contains the three tenses, is conservative over propositional intuitionistic logic, is interpreted over unbounded time, and is known to be decidable.

Note that both \( \Diamond \) and \( \Box \) are taken as primitives, in contrast with the classical case, where \( \Diamond \varphi \) may be defined by \( \Diamond \varphi \equiv \neg \Box \neg \varphi \), whereas the latter equivalence is not intuitionistically valid. The same situation holds in the more expressive language with \( \mathcal{U} \) (‘until’): while the language with \( \Diamond \) and \( \mathcal{U} \) is equally expressive to classical monadic first-order logic with \( \leq \) over \( \mathbb{N} \) [12], \( \mathcal{U} \) admits a first-order definable intuitionistic dual, \( \mathcal{R} \) (‘release’), which cannot be defined in terms of \( \mathcal{U} \) using the classical definition. However, this is not enough to conclude that \( \mathcal{R} \) cannot be defined in a different way. Thus, while in [4] we explored the question of decidability, here we will focus on definability: which of the modal operators can be defined in terms of the others? As is well-known, \( \Diamond \varphi \equiv \top \mathcal{U} \varphi \) and \( \Box \varphi \equiv \bot \mathcal{R} \varphi \); these equivalences remain valid in the intuitionistic setting. Nevertheless, we will show that \( \Box \) cannot be defined in terms of \( \mathcal{U} \), and \( \Diamond \) cannot be defined in terms of \( \mathcal{R} \); in order to prove this, we will develop a theory of bisimulations on \( \text{ITL}^e \) models.

Following Simpson [30] and other authors, we interpret the language of \( \text{ITL}^e \) using bi-relational structures, with a partial order \( \leq \) to interpret intuitionistic implication, and a function or relation, which we denote \( S \), representing the passage of time. Alternatively, one may consider topological interpretations [8], but we will not discuss those here. Various intuitionistic temporal logics have been considered, using variants of these semantics and different formal languages. The main contributions include:

- Davies’ intuitionistic temporal logic with \( \Diamond \) [7] was provided Kripke semantics and a complete deductive system by Kojima and Igarashi [19].
- Logics with \( \Diamond, \Box \) were axiomatized by Kamide and Wansing [17], where \( \Box \) was interpreted over bounded time.
- Nishimura [24] provided a sound and complete axiomatization for an intuitionistic variant of the propositional dynamic logic PDL.
- Balbiani and Diéguez axiomatized the here-and-there variant of LTL with \( \Diamond, \Box, \mathcal{U} \) [2], here denoted \( \text{ITL}^N \).
- Fernández-Duque [10] proved the decidability of a logic based on topological semantics with \( \Diamond, \Box \) and a universal modality.
- The authors [4] proved that the logic \( \text{ITL}^e \) with \( \Diamond, \Box, \mathcal{U} \) has the strong finite model property and hence is decidable, yet the logic \( \text{ITLP} \), based on a more restrictive class of frames, does not enjoy the fmp.

In this paper, we extend \( \text{ITL}^e \) to include \( \mathcal{U} \) (‘until’) and \( \mathcal{R} \) (‘release’). We will introduce different notions of bisimulation which preserve formulas with \( \Diamond \) and each of \( \Diamond, \Box, \mathcal{U} \) and \( \mathcal{R} \). With this, we will show that \( \mathcal{R} \) (or even \( \Box \)) may
not be defined in terms of $U$ over the class of here-and-there models, while $\Diamond$ can be defined in terms of $\Box$ and $U$ can be defined in terms of $R$ over this class. However, we show that over the wider class of expanding models, $\Diamond$ cannot be defined in terms of $\Box$.

2 Syntax and semantics

We will work in sublanguages of the language $L$ given by the following grammar:

$$
\varphi, \psi ::= p \mid \bot \mid \varphi \land \psi \mid \varphi \lor \psi \mid \varphi \rightarrow \psi \mid \Box \varphi \mid \Diamond \varphi \mid \varphi U \psi \mid \varphi R \psi
$$

where $p$ is an element of a countable set of propositional variables $P$. All sublanguages we will consider include all Boolean operators and $\Diamond$, hence we denote them by displaying the additional connectives as a subscript; for example, $L_{\Diamond \Box}$ denotes the $U$-free, $R$-free fragment. As an exception to this general convention, $L_{\Diamond}$ denotes the fragment without $\Diamond$, $\Box$, $U$ or $R$.

Given any formula $\varphi$, we define the length of $\varphi$ (in symbols, $|\varphi|$) recursively as follows:

- $|p| = |\bot| = 0$;
- $|\varphi \otimes \psi| = 1 + |\varphi| + |\psi|$, with $\otimes \in \{\lor, \land, \rightarrow, R, U\}$;
- $|\varphi| = 1 + |\psi|$, with $\otimes \in \{\neg, \Diamond, \Box, \Diamond\}$.

Broadly speaking, the length of a formula $\varphi$ corresponds to the number of connectives appearing in $\varphi$.

2.1 Dynamic posets

Formulas of $L$ are interpreted over dynamic posets. A dynamic poset is a tuple $D = (W, \preceq, S)$, where $W$ is a non-empty set of states, $\preceq$ is a partial order, and $S$ is a function from $W$ to $W$ satisfying the forward confluence condition that for all $w, v \in W$, if $w \preceq v$ then $S(w) \preceq S(v)$. An intuitionistic dynamic model, or simply model, is a tuple $M = (W, \preceq, S, V)$ consisting of a dynamic poset equipped with a valuation function $V$ from $W$ to sets of propositional variables that is $\preceq$-monotone, in the sense that for all $w, v \in W$, if $w \preceq v$ then $V(w) \subseteq V(v)$. In the standard way, we define $S^0(w) = w$ and, for all $k > 0$, $S^k(w) = S(S^{k-1}(w))$.

Then we define the satisfaction relation $\models$ inductively by:

1. $M, w \models p$ iff $p \in V(w)$;
2. $M, w \not\models \bot$;
3. $M, w \models \varphi \land \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$;
4. $M, w \models \varphi \lor \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$;  
5. $M, w \models \Diamond \varphi$ iff $M, S(w) \models \varphi$;  
6. $M, w \models \varphi \rightarrow \psi$ iff $\forall v \preceq w, \ M, v \models \varphi$, then $M, v \models \psi$;
7. $M, w \models \Diamond \varphi$ iff there exists $k$ s.t. $M, S^k(w) \models \varphi$;  
8. $M, w \models \Box \varphi$ iff for all $k$, $M, S_k(w) \models \varphi$;  
9. $M, w \models \varphi \cup \psi$ iff there exists $k \geq 0$ s.t. $M, S^k(w) \models \psi$ and $\forall i \in [0, k), M, S^i(w) \models \varphi$;  
10. $M, w \models \varphi R \psi$ iff for all $k \geq 0$, either $M, S^k(w) \models \psi$, or $\exists i \in [0, k)$ s.t. $M, S^i(w) \models \varphi$.

As usual, a formula $\varphi$ is satisfiable over a class of models $\Omega$ if there is a model $M \in \Omega$ and a world $w$ of $M$ so that $M, w \models \varphi$, and valid over $\Omega$ if, for every world $w$ of every model $M \in \Omega$, $M, w \models \varphi$. Satisfiability (validity) over the class of models based on an arbitrary dynamic poset will be called satisfiability (validity) for $\text{ITL}^e$, or expanding domain linear temporal logic.

The relation between dynamic posets and expanding products of modal logics is detailed in [4], where the following is also shown. Below, we use the notation $\llbracket \varphi \rrbracket = \{ w \in W \mid M, w \models \varphi \}$.

**Lemma 1.** Let $D = (W, \preceq, S)$, where $(W, \preceq)$ is a poset and $S : W \to W$ is any function. Then, $D$ is a dynamic poset if and only if, for every valuation $V$ on $W$ and every formula $\varphi$, $\llbracket \varphi \rrbracket$ is $\preceq$-monotone, i.e., if $w \in \llbracket \varphi \rrbracket$ and $v \succcurlyeq w$, then $v \in \llbracket \varphi \rrbracket$.

The proof that all valuations on a dynamic poset are $\preceq$-monotone proceeds by a standard structural induction on formulas, and the cases for $U, R$ are similar to those for $\Diamond, \Box$ in [4]. This suggests that dynamic posets provide suitable semantics for intuitionistic $\text{LTL}$. Moreover, dynamic posets are convenient from a technical point of view:

**Theorem 1 ([4]).** There exists a computable function $B$ such that any formula $\varphi \in \text{L}_{\Diamond}^{\Box}$ satisfiable (resp. falsifiable) on an arbitrary model is satisfiable (resp. falsifiable) on a model whose size is bounded by $B(|\varphi|)$.

It follows that the $\text{L}_{\Diamond}^{\Box}$-fragment of $\text{ITL}^e$ is decidable. Moreover, as we will see below, many of the familiar axioms of classical $\text{LTL}$ are valid over the class of dynamic posets, making them a natural choice of semantics for intuitionistic $\text{LTL}$.

### 2.2 Persistent posets

Despite the appeal of dynamic posets, in the literature one typically considers a more restrictive class of frames, similar to persistent frames, as we define them below.

**Definition 1.** Let $(W, \preceq)$ be a poset. If $S : W \to W$ is such that, whenever $v \succeq S(w)$, there is $u \succeq w$ such that $v = S(u)$, we say that $S$ is backward confluent. If $S$ is both forward and backward confluent, we say that it is persistent. A tuple $(W, \preceq, S)$ where $S$ is persistent is a persistent intuitionistic temporal frame, and the set of valid formulas over the class of persistent intuitionistic temporal frames is denoted $\text{ITL}^p$, or persistent domain $\text{LTL}$.

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1Note that in [4] we used $\text{ITL}^e$ to denote the fragment of this logic without $U, R$. 

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As we will see, persistent frames do have some technical advantages over arbitrary dynamic posets. Nevertheless, they have a crucial disadvantage:

**Theorem 2** ([4]). The logic ITL does not have the finite model property, even for formulas in \( \mathcal{L}_{\Diamond \Box} \).

### 2.3 Temporal here-and-there models

An even smaller class of models which, nevertheless, has many applications is that of temporal here-and-there models [2]. Some of the results we will present here apply to this class, so it will be instructive to review it. Recall that the logic of here-and-there is the maximal logic strictly between classical and intuitionistic propositional logic, given by a frame \( \{0,1\} \) with \( 0 \preceq 1 \). The logic of here-and-there is obtained by adding to intuitionistic propositional logic the axiom \( p \lor (p \rightarrow q) \lor \neg q \).

A temporal here-and-there frame is a persistent frame that is ‘locally’ based on this frame. To be precise:

**Definition 2.** A temporal here-and-there frame is a persistent frame \((W,\leq,S)\) such that \( W = T \times \{0,1\} \) for some set \( T \), and there is a function \( f : T \rightarrow T \) such that for all \( t,s \in T \) and \( i,j \in \{0,1\} \), \((t,i) \preceq (s,j)\) if and only if \( t = s \) and \( i \leq j \) and \( S(t,i) = (f(t),i) \).

The prototypical example is the frame \((W,\equiv,S)\), where \( W = \mathbb{N} \times \{0,1\} \), \((i,j) \preceq (i',j')\) if \( i = i'\) and \( j \leq j'\), and \( S(i,j) = (i + 1,j) \). Note, however, that our definition allows for other examples (see Figure 1). In [2], this logic is axiomatized, and it is shown that \( \Diamond \) cannot be defined in terms of \( \Box \), a result we will strengthen here to show that \( \Diamond \) cannot be defined even in terms of \( U \). It is also claimed in [2] that \( \Diamond \) is not definable in terms of \( \Box \) over the class of here-and-there models, but as we will see in Proposition 5, this claim is incorrect.

### 3 Some valid and non-valid ITL\(_e\)-formulas

In this section we explore which axioms of classical LTL are still valid in our setting. We start by showing that the intuitionistic version of the interaction and induction axioms used in [2] remain valid in our setting. However, not all Fisher-Servi axioms [11], which are valid in the here-and-there LTL of [2], are valid in ITL\(_e\).

**Proposition 1.** The following formulas:

1. \( \Diamond \bot \leftrightarrow \bot \)
2. \( \Diamond (\varphi \land \psi) \leftrightarrow (\Diamond \varphi \land \Diamond \psi) \)
3. \( \Diamond (\varphi \lor \psi) \leftrightarrow (\Diamond \varphi \lor \Diamond \psi) \)
4. \( \Diamond (\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi) \)
5. \( \Diamond (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \)
6. \( \Diamond (\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi) \)
7. \( \Diamond (\varphi \lor \psi) \rightarrow (\Diamond \varphi \lor \Diamond \psi) \)
8. \( \Diamond \varphi \leftrightarrow \varphi \land \Diamond \varphi \)
9. \( \varphi \lor \diamond\top \equiv \diamond\varphi \);
11. \((\diamond\varphi \rightarrow \varphi) \rightarrow (\top \varphi \rightarrow \varphi)\)

are \(\text{ITL}^e\)-valid.

Proof. Let us consider (10) and (11). For (10), let \(M = (W, \leq, S)\) be any \(\text{ITL}^e\) model and \(w \in W\) be such that \(M, w \models (\varphi \rightarrow \diamond\varphi)\). Let \(v \succ w\) be arbitrary and assume that \(M, v \models \varphi\). Then, by induction on \(i\) we obtain that \(S^i(w) \equiv S^i(v)\) for all \(i\); since \(M, S^i(w) \models \varphi \rightarrow \diamond\varphi\) for all \(i\), it follows that \(M, S^i(v) \models \varphi \rightarrow \diamond\varphi\) for all \(i\) as well. Hence an easy induction shows that \(M, S^i(v) \models \varphi\) for all \(i\), which means that \(M, v \models \varphi\). Since \(w\) was arbitrary, we conclude that the formula (10) is valid.

For (11), let \(M\) be as above and \(w \in W\) be such that \(M, w \models (\diamond\varphi \rightarrow \varphi)\). Let \(v \succ w\) be such that \(M, v \models \diamond\varphi\). It follows that \(M, v \models \diamond\varphi\), so \(M, v \models \top\varphi\). Since \(w, v\) were arbitrary, the formula (11) is valid as well.

The proofs for the rest of formulas are left to the reader. 

Some of the well-known Fisher Servi axioms (11) are only valid on the class of persistent frames.

**Proposition 2.** The formulas

1. \((\diamond\varphi \rightarrow \diamond\psi) \rightarrow (\diamond\varphi \rightarrow \psi)\),
2. \((\diamond\varphi \rightarrow \top\psi) \rightarrow (\diamond\varphi \rightarrow \psi)\)

are not \(\text{ITL}^e\)-valid. However they are \(\text{ITL}^p\)-valid.

Proof. Let \(\{p, q\}\) be a set of propositional variables and let us consider the \(\text{ITL}^e\) model \(M = (W, \leq, S, V)\) defined as: 1) \(W = \{w, v, u\}\); 2) \(S(w) = v, S(v) = v\) and \(S(u) = u\); 3) \(v \leq u\); 4) \(V(p) = \{u\}\). Clearly, \(M, u \not\models p \rightarrow q\), so \(M, v \not\models p \rightarrow q\). By definition, \(M, w \not\models \diamond(p \rightarrow q)\) and \(M, w \not\models \top(p \rightarrow q)\); however, it can easily be checked that \(M, w \models \top(p \rightarrow q)\) and \(M, w \models (\diamond(p \rightarrow q) \rightarrow (p \rightarrow q)\).

Let us check their validity over the class of persistent frames. For (11), let \(M = (W, \leq, S, V)\) be an \(\text{ITL}^p\) model and \(w\) a world of \(M\) such that \(M, w \models \diamond\varphi \rightarrow \top\psi\). Suppose that \(v \succ S(w)\) satisfies \(M, v \models \varphi\). By backward confluence, there exists \(u \succ w\) such that \(v = S(u)\), so that \(M, u \models \diamond\varphi\) and thus \(M, u \models \top\psi\). But this means that \(M, v \models \psi\), and since \(v \succ S(w)\) was arbitrary, \(M, S(w) \models \varphi \rightarrow \psi\), i.e. \(M, w \models (\varphi \rightarrow \psi)\).

Similarly, for (2) let us assume that \(M = (W, \leq, S, V)\) is an \(\text{ITL}^p\) model and \(w\) a world of \(M\) such that \(M, w \models \diamond\varphi \rightarrow \top\psi\). Consider arbitrary \(k \in \mathbb{N}\), and suppose that \(v \succeq S^k(w)\) is such that \(M, v \models \varphi\). Then, it is readily checked that the composition of backward confluent functions is backward confluent, so that in particular \(S^k\) is backward confluent. This means that there is \(u \succeq w\) such that \(S^k(u) = v\). But then, \(M, u \models \diamond\varphi\), hence \(M, u \models \top\psi\), and \(M, v \models \psi\). It follows that \(M, S^k(w) \models \varphi \rightarrow \psi\), and since \(k\) was arbitrary, \(M, w \models (\varphi \rightarrow \psi)\).
We make a special mention of the schema $\Box (\Box \varphi \to \psi) \lor \Box (\Box \psi \to \varphi)$, which characterises the class of weakly connected frames in classical modal logic. We say that a frame $(W, R, V)$ is weakly connected if it satisfies the following first-order property: for every $x,y,z \in W$, if $x R y$ and $x R z$, then either $y R z$, $y = z$, or $z R y$.

**Proposition 3.** The axiom schema $\Box (\Box \varphi \to \psi) \lor \Box (\Box \psi \to \varphi)$ is not $\mathsf{ITL}^\mathsf{ht}$-valid.

**Proof.** Let us consider the set of propositional variables $\{p,q\}$ and the $\mathsf{ITL}^\mathsf{ht}$ model $\mathcal{M} = (W,\varepsilon,S,V)$ defined as: 1) $W = \{w,t,u,v\}$; 2) $S(w) = v$, $S(v) = v$, $S(t) = u$ and $S(u) = u$; 3) $v \leq u$ and $w \leq t$; 4) $V(p) = \{v,u\}$ and $V(q) = \{t,u\}$.

The reader can check that $\mathcal{M}, v \not\vDash p \to q$ and $\mathcal{M}, t \not\vDash q \to p$. Consequently $\mathcal{M}, w \not\vDash (p \to q) \lor (q \to p)$. \hfill \Box

Finally, we show that $\Diamond \varphi$ (resp. $\Box \varphi$) can be defined in terms of $\mathcal{U}$ (resp. $\mathcal{R}$) and the LTL axioms involving $\mathcal{U}$ and $\mathcal{R}$ are also valid in our setting:

**Proposition 4.** The following formulas are $\mathsf{ITL}^\mathsf{ht}$-valid:

1. $\varphi \mathcal{U} \psi \leftrightarrow \psi \lor (\varphi \land (\varphi \mathcal{U} \psi))$
2. $\varphi \mathcal{R} \psi \leftrightarrow \psi \land (\varphi \lor (\varphi \mathcal{R} \psi))$
3. $\varphi \mathcal{U} \psi \to \Diamond \psi$
4. $\Box \psi \to \varphi \mathcal{R} \psi$
5. $\Diamond \varphi \leftrightarrow \mathcal{U} \varphi$
6. $\Box \varphi \leftrightarrow \mathcal{R} \varphi$
7. $\Diamond(\varphi \mathcal{U} \psi) \leftrightarrow \Diamond \varphi \mathcal{U} \Diamond \psi$
8. $\Box(\varphi \mathcal{R} \psi) \leftrightarrow \Box \varphi \mathcal{R} \Box \psi$

**Proof.** We consider some cases below. For [1], from left to right, let us assume that $\mathcal{M}, w \vDash \varphi \mathcal{U} \psi$. Therefore there exists $k \geq 0$ s.t. $\mathcal{M}, S^k(w) \vDash \psi$ and for all $j$ satisfying $0 \leq j < k$, $\mathcal{M}, S^j(w) \vDash \varphi$. If $k = 0$ then $\mathcal{M}, w \vDash \psi$ while, if $k > 0$ it follows that $\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, S(w) \vDash \varphi \mathcal{U} \psi$. Therefore $\mathcal{M}, w \vDash \psi \lor (\varphi \land \Diamond \varphi \mathcal{U} \psi)$. From right to left, if $\mathcal{M}, w \vDash \psi$ then $\mathcal{M}, w \vDash \varphi \mathcal{U} \psi$ by definition. If $\mathcal{M}, w \vDash \varphi \land \Diamond \varphi \mathcal{U} \psi$ then $\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, S(w) \vDash \varphi \mathcal{U} \psi$ so, due to the semantics, we conclude that $\mathcal{M}, w \vDash \varphi \mathcal{U} \psi$. In any case, $\mathcal{M}, w \vDash \varphi \mathcal{U} \psi$.

For [2], we work by contraposition. From right to left, let us assume that $\mathcal{M}, w \not\vDash \varphi \mathcal{R} \psi$. Therefore there exists $k \geq 0$ s.t. $\mathcal{M}, S^k(w) \not\vDash \psi$ and for all $j$ satisfying $0 \leq j < k$, $\mathcal{M}, S^j(w) \not\vDash \varphi$. If $k = 0$ then $\mathcal{M}, w \not\vDash \psi$ while, if $k > 0$ it follows that $\mathcal{M}, w \not\vDash \varphi$ and $\mathcal{M}, S(w) \not\vDash \varphi \mathcal{R} \psi$. In any case, $\mathcal{M}, w \not\vDash \psi \lor (\varphi \lor \Diamond \varphi \mathcal{R} \psi)$. From left to right, if $\mathcal{M}, w \not\vDash \psi$ then $\mathcal{M}, w \not\vDash \varphi \mathcal{R} \psi$ by definition. If $\mathcal{M}, w \not\vDash \varphi \lor \Diamond \varphi \mathcal{R} \psi$ then $\mathcal{M}, w \not\vDash \varphi$ and $\mathcal{M}, S(w) \not\vDash \varphi \mathcal{U} \psi$ so, due to the semantics of $\mathcal{R}$, we conclude that $\mathcal{M}, w \not\vDash \varphi \mathcal{R} \psi$. In any case, $\mathcal{M}, w \not\vDash \varphi \mathcal{R} \psi$.

The remaining items are left to the reader. \hfill \Box

As in the classical case, over the class of persistent models we can ‘push down’ all occurrences of $\Box$ to the propositional level. Say that a formula $\varphi$ is in $\Box$-normal form if all occurrences of $\Box$ are of the form $\Diamond p$, with $p$ a propositional variable.

**Theorem 3.** Given $\varphi \in \mathcal{L}$, there exists $\bar{\varphi}$ in $\Box$-normal form such that $\varphi \leftrightarrow \bar{\varphi}$ is valid over the class of persistent models.
Proof. The claim can be proven by structural induction using the validities in Propositions 1, 2, and 4.

We remark that the only reason that this argument does not apply to arbitrary ITL models is the fact that $(\Box \varphi \rightarrow \varphi) \rightarrow \Box (\varphi \rightarrow \varphi)$ is not valid in general (Proposition 2).

4 Bounded bisimulations for $\Diamond$ and $\square$

In this section we adapt the classical definition of bounded bisimulations for modal logic to our case. To do so we combine the ordinary definition of bounded bisimulations with the work of [26] on bisimulations for propositional intuitionistic logic. Such work introduces extra conditions involving the partial order $\leq$. In our setting, we combine both approaches in order to define bisimulation for a language involving $\Diamond$, $\Box$, and $\circ$ as modal operators plus an intuitionistic $\rightarrow$. Since all languages we consider contain Booleans and $\circ$, it is convenient to begin with a ‘basic’ notion of bisimulation for this language.

Definition 3. Given $n > 0$ and two ITL models $M_1$ and $M_2$, a sequence of binary relations $Z_n \subseteq \cdots \subseteq Z_0 \subseteq W_1 \times W_2$ is said to be a bounded $\Diamond$-bisimulation if for all $(w_1, w_2) \in W_1 \times W_2$ and for all $0 \leq i < n$, the following conditions are satisfied:

Atoms. If $w_1 \ Z_i \ w_2$ then for all propositional variables $p$, $M_1, w_1 \models p$ iff $M_2, w_2 \models p$.

Forth $\rightarrow$. If $w_1 \ Z_{i+1} \ w_2$ then for all $v_1 \in W_1$, if $v_1 \geq w_1$, there exists $v_2 \in W_2$ such that $v_2 \geq w_2$ and $v_1 \ Z_i \ v_2$.

Back $\rightarrow$. If $w_1 \ Z_{i+1} \ w_2$ then for all $v_2 \in W_2$ if $v_2 \geq w_2$ then there exists $v_1 \in W_1$ such that $v_1 \geq w_1$ and $v_1 \ Z_i \ v_2$.

Forth $\circ$. If $w_1 \ Z_{i+1} \ w_2$ then $S(w_1) \ Z_i \ S(w_2)$.

Note that there is not ‘back’ clause for $\circ$; this is simply because $S$ is a function, so its ‘forth’ and ‘back’ clauses are identical. Bounded $\circ$-bisimulations are useful because the preserve the truth of relatively small $\mathcal{L}_\circ$-formulas.

Lemma 2. Given two ITL models $M_1$ and $M_2$ and a bounded $\circ$-bisimulation $Z_0 \subseteq \cdots \subseteq Z_n$ between them, for all $i \leq n$ and $(w_1, w_2) \in W_1 \times W_2$, if $w_1 \ Z_i \ w_2$ then for all $\varphi \in \mathcal{L}_\circ$ satisfying $|\varphi| \leq i$, $M_1, w_1 \models \varphi$ iff $M_2, w_2 \models \varphi$.

Proof. We proceed by induction on $i$. Let $0 \leq i \leq n$ be such that for all $j < i$ the lemma holds. Let $w_1 \in W_1$ and $w_2 \in W_2$ be such that $w_1 \ Z_n \ w_2$ and let us consider $\varphi \in \mathcal{L}_\circ$ such that $|\varphi| \leq i$. The cases where $\varphi$ is an atom or of the forms $\theta \land \psi$, $\theta \lor \psi$ are as in the classical case and we omit them. Thus we focus on the following:

\footnote{Although not optimal, we use the length of the formula in this lemma to simplify its proof. More precise measures like counting the number of modalities and implications could be equally used.}
CASE $\varphi = \theta \to \psi$. We proceed by contrapositive to prove the left-to-right implication. Note that in this case we must have $i > 0$.

Assume that $M_2, w_2 \not\models \theta \to \psi$. Therefore there exists $v_2 \in W_2$ such that $v_2 \geq w_2$, $M_2, v_2 \models \theta$, and $M_2, v_2 \not\models \psi$. By the BACK $\to$ condition, it follows that there exists $v_1 \in W_1$ such that $v_1 \geq w_1$ and $v_1 S_{i-1} v_2$. Since $|\theta| \leq i - 1$ and $|\psi| < n$, by the induction hypothesis, it follows that $M_1, v_1 \models \theta$ and $M_1, v_1 \not\models \psi$. Consequently, $M_1, w_1 \not\models \theta \to \psi$. The converse direction is proved in a similar way but using the FORTH $\to$.

CASE $\varphi = \Diamond \psi$. Once again we have that $i > 0$. Assume that $M_1, w_1 \models \Diamond \psi$, so that $M_1, S(w_1) \models \psi$. By FORTH $\Diamond$, $S_1(w_1) Z_{i-1} S_2(w_2)$. Moreover, $|\psi| \leq i - 1$, so that by the induction hypothesis, $M_2, S(w_2) \models \psi$, and $M_2, w_2 \models \Diamond \psi$. The right-to-left direction is analogous.

Next, we will extend the notion of a bounded $\Diamond$-bisimulation to include other tenses. Let us begin with $\Box$.

**Definition 4.** Given $n > 0$ and two ITL$^\Box$ models $M_1$ and $M_2$, a bounded $\Diamond$-bisimulation $Z_0 \subseteq \cdots \subseteq Z_n \subseteq W_1 \times W_2$ is said to be a bounded $\Diamond$-bisimulation if for all $(w_1, w_2) \in W_1 \times W_2$ and for all $0 \leq i < n$, if $w_1 Z_i w_2$, then the following conditions are satisfied:

FORTH $\Box$. For all $k_1 \geq 0$ there exist $k_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that $S^{k_2}(w_2) \geq v_2$, $v_1 \geq S^{k_1}(w_1)$ and $v_1 Z_i v_2$.

BACK $\Box$. For all $k_2 \geq 0$ there exist $k_1 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that $S^{k_1}(w_1) \geq v_1$, $v_2 \geq S^{k_2}(w_2)$ and $v_1 Z_i v_2$.

As was the case of Lemma 3 if two worlds are related by a bounded $\Diamond$-bisimulation, then they satisfy the same $L_\Box$-formulas of small length.

**Lemma 3.** Given two ITL$^\Box$ models $M_1$ and $M_2$ and a bounded $\Diamond$-bisimulation $Z_0 \subseteq \cdots \subseteq Z_n$ between them, for all $i \leq n$ and $(w_1, w_2) \in W_1 \times W_2$, if $w_1 Z_n w_2$ then for all $\varphi \in L_\Box$ satisfying $|\varphi| \leq n$, $M_1, w_1 \models \varphi$ iff $M_2, w_2 \models \varphi$.

**Proof.** We proceed by induction on $n$. Let $0 \leq i \leq n$ be such that for all $j < i$ the lemma holds. Let $w_1 \in W_1$ and $w_2 \in W_2$ be such that $w_1 Z_i w_2$ and let us consider $\varphi \in L_\Box$ such that $|\varphi| \leq i$. We only consider the case where $\varphi = \Diamond \psi$, as other cases are covered by Lemma 2.

From left to right, if $M_1, w_1 \models \Diamond \psi$ then there exists $k_1 \geq 0$ such that $M_1, S^{k_1}(w_1) \models \psi$. By FORTH $\Diamond$, there exists $k_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that $S^{k_2}(w_2) \geq v_2$, $v_1 \geq S^{k_1}(w_1)$ and $v_1 Z_{i-1} v_2$. By $\Leftarrow$-monotonicity, $M_1, v_1 \models \psi$. Then, by the induction hypothesis and the fact that $|\psi| \leq i - 1$, it follows that $M_2, v_2 \models \psi$, thus by $\Leftarrow$-monotonicity once again, $M_2, S^{k_2}(w_2) \models \psi$, so that $M_2, w_2 \models \Diamond \psi$. The converse direction is proved similarly by using BACK $\Diamond$.

We can define bounded $\Box$-bisimulations in a similar way.

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3 We remind the reader that, as per our convention, $L_\Box$ is the $\Diamond, \forall, \exists$-free fragment. A similar comment applies to other sublanguages of $L$ mentioned below.
Definition 5. A bounded $\circ$-bisimulation $Z_n \subseteq \cdots \subseteq Z_0 \subseteq W_1 \times W_2$ is said to be a bounded $\Box$-bisimulation if for all $(w_1, w_2) \in W_1 \times W_2$ and for all $0 \leq i < n$, if $w_1 Z_i w_2$, then:

FORTH $\Box$. For all $k_2 \geq 0$ there exist $k_1 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ s.t. $S^{k_2}(w_2) \geq v_2, v_1 \geq S^{k_1}(w_1)$ and $v_1 Z_i v_2$.

BACK $\Box$. For all $k_1 \geq 0$ there exist $k_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ s.t. $S^{k_1}(w_1) \geq v_1, v_2 \geq S^{k_2}(w_2)$ and $v_1 Z_i v_2$.

Lemma 4. Given two ITL$^e$ models $\mathcal{M}_1$ and $\mathcal{M}_2$ and a bounded $\Box$-bisimulation $Z_n \subseteq \cdots \subseteq Z_0$ between them, for all $(w_1, w_2) \in W_1 \times W_2$ and $i \leq n$, if $w_1 Z_i w_2$ then for all $\varphi \in \mathcal{L}_\Box$ such that $|\varphi| \leq i$, then $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$.

Proof. We proceed by induction on $i$. Let $i \geq 0$ be such that for all $j < i$ the lemma holds. Let $w_1 \in W_1$ and $w_2 \in W_2$ be such that $w_1 Z_i w_2$ and let us consider $\varphi \in \mathcal{L}_\Box$ such that $|\varphi| \leq i$. Note that the cases for atoms as well as propositional and $\circ$ connectives are proved as in Lemma 2 so we only consider $\varphi = \Box \psi$.

For the left-to-right implication, we work by contrapositive, and assume that $\mathcal{M}_2, w_2 \not\models \Box \psi$. Then, there exists $k_2 \geq 0$ such that $\mathcal{M}_2, S^{k_2}(w_2) \not\models \psi$. By FORTH $\Box$, there exist $k_1 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ s.t. $S^{k_2}(w_2) \geq v_2, v_1 \geq S^{k_1}(w_1)$ and $v_1 Z_{i-1} v_2$. As in the proof of Lemma 3 by $\leftarrow$-monotonicity, the induction hypothesis and the fact that $|\psi| \leq i-1$, it follows that $\mathcal{M}_1, v_1 \not\models \psi$; thus $\mathcal{M}_1, S^{k_1}(w_1) \not\models \psi$, and again by $\leftarrow$-monotonicity $\mathcal{M}_1, w_1 \not\models \Box \psi$. The converse direction follows a similar reasoning but using BACK $\Box$.

5 Bounded bisimulations for $\mathcal{U}$ and $\mathcal{R}$

In this section we adapt the bisimulations defined for a language with until and since [13] presented by Kurtonina and de Rijke [20] to our case. Let us begin with bounded bisimulations for $\mathcal{U}$.

Definition 6. Given $n \in \mathbb{N}$ and two ITL$^e$ models $\mathcal{M}_1$ and $\mathcal{M}_2$, a bounded $\circ$-bisimulation $Z_n \subseteq \cdots \subseteq Z_0 \subseteq W_1 \times W_2$ is said to be a bounded $\mathcal{U}$-bisimulation iff for all $(w_1, w_2) \in W_1 \times W_2$, $w_1 Z_n w_2$ and for all $0 \leq i < n$:

FORTH $\mathcal{U}$. For all $k_1 \geq 0$ there exist $k_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that

1. $S^{k_2}(w_2) \geq v_2, v_1 \geq S^{k_1}(w_1)$ and $v_1 Z_i v_2$,
2. for all $j_2 \in [0, k_2)$ there exist $j_1 \in [0, k_1)$ and $(u_1, u_2) \in W_1 \times W_2$ such that $u_1 \geq S^{j_1}(w_1), S^{j_2}(w_2) \geq u_2$ and $u_1 Z_i u_2$.

BACK $\mathcal{U}$. For all $k_2 \geq 0$ there exist $k_1 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that

1. $S^{k_1}(w_1) \geq v_1, v_2 \geq S^{k_2}(w_2)$ and $v_1 Z_i v_2$,
2. for all $j_1 \in [0, k_1)$ there exist $j_2 \in [0, k_2)$ and $(u_1, u_2) \in W_1 \times W_2$ such that $u_2 \geq S^{j_2}(w_2), S^{j_1}(w_1) \geq u_1$ and $u_1 Z_i u_2$. 

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As was the case before, the following lemma states that two bounded $\mathcal{U}$-bisimilar models agree on small $L_{\mathcal{U}}$ formulas.

**Lemma 5.** Given two ITL$^\ast$ models $\mathcal{M}_1$ and $\mathcal{M}_2$ and a bounded $\mathcal{U}$-bisimulation $Z_0 \subseteq \cdots \subseteq Z_n$ between them, for all $m \leq n$ and $(w_1, w_2) \in W_1 \times W_2$, if $w_1 Z_m w_2$ then for all $\varphi \in L_{\mathcal{U}}$ such that $|\varphi| \leq m$, $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$.

**Proof.** Once again, proceed by induction on $n$. Let $m \leq n$ be such that for all $k < m$ the lemma holds. Let $w_1 \in W_1$ and $w_2 \in W_2$ be such that $w_1 Z_m w_2$ and let us consider $\varphi \in L_{\mathcal{U}}$ such that $|\varphi| \leq m$. As before, we only consider the ‘new’ case, where $\varphi = \theta \mathcal{U} \psi$. From left to right, assume that $\mathcal{M}_1, w_1 \models \varphi \mathcal{U} \psi$. Then, there exists $i_1 \geq 0$ such that $\mathcal{M}_1, S^{i_1}(w_1) \models \psi$ and for all $j_1$ satisfying $0 \leq j_1 < i_1$, $\mathcal{M}_1, S^{j_1}(w_1) \models \varphi$. By **Forth** $\mathcal{U}$, there exist $i_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that $1. S^{i_2}(v_2) \models v_2$, $v_1 \models S^{i_1}(w_1)$ and $v_1 Z_{m-1} v_2$; 2. for all $j_2$ satisfying $0 \leq j_2 < i_2$ there exist $j_1 \in [0, i_1]$ and $(u_1, u_2) \in W_1 \times W_2$ s. t. $u_1 \models S^{j_1}(w_1)$, $S^{j_2}(u_2) \models u_2$ and $u_1 Z_{m-1} u_2$.

From the first item, $\prec$-monotonicity, the fact that $|\psi| \leq m-1$, and the induction hypothesis, it follows that $\mathcal{M}_2, S^{i_2}(w_2) \models \psi$. Take any $j_2$ satisfying $0 \leq j_2 < i_2$. By the second item, the fact that $|\theta| \leq m-1$, and the induction hypothesis, we conclude that $\mathcal{M}_2, S^{j_2}(u_2) \models \varphi$ so $\mathcal{M}_2, u_2 \models \varphi \mathcal{U} \psi$. The right-to-left direction is symmetric (but using **Back** $\mathcal{U}$).

Finally, we define bounded bisimulations for $\mathcal{R}$.

**Definition 7.** A bounded $\circ$-bisimulation $Z_0 \subseteq \cdots \subseteq Z_n \subseteq W_1 \times W_2$ is said to be a bounded $\mathcal{R}$-bisimulation if for all $(w_1, w_2) \in W_1 \times W_2$ such that $w_1 Z_{i+1} w_2$ and for all $0 \leq i < n$:

**Forth** $\mathcal{R}$. For all $k_2 \geq 0$ there exist $k_1 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that

1. $S^{k_2}(w_2) \models v_2$, $v_1 \models S^{k_1}(w_1)$ and $v_1 Z_i v_2$, and

2. for all $j_1$ satisfying $0 \leq j_1 < k_1$ there exist $j_2$ such that $0 \leq j_2 < k_2$ and $(u_1, u_2) \in W_1 \times W_2$ s. t. $u_1 \models S^{j_1}(w_1)$, $S^{j_2}(u_2) \models u_2$ and $u_1 Z_i u_2$.

**Back** $\mathcal{R}$. For all $k_1 \geq 0$ there exist $k_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that

1. $S^{k_1}(w_1) \models v_1$, $v_2 \models S^{k_2}(w_2)$ and $v_1 Z_i v_2$, and

2. for all $j_2$ satisfying $0 \leq j_2 < k_2$ there exist $j_1$ such that $0 \leq j_1 < k_1$ and $(u_1, u_2) \in W_1 \times W_2$ s. t. $u_2 \models S^{j_2}(w_2)$, $S^{j_1}(u_1) \models u_1$ and $u_1 Z_i u_2$.

Once again, we obtain a corresponding bisimulation lemma for $L_{\mathcal{R}}$.

**Lemma 6.** Given two ITL$^\ast$ models $\mathcal{M}_1$ and $\mathcal{M}_2$ and a bounded $\mathcal{R}$-bisimulation $Z_0 \subseteq \cdots \subseteq Z_n$ between them, for all $m \leq n$ and $(w_1, w_2) \in W_1 \times W_2$, if $w_1 Z_m w_2$ then for all $\varphi \in L_{\mathcal{U}}$ such that $|\varphi| \leq m$, $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$.

**Proof.** As before, we proceed by induction on $n$; the critical case where $\varphi = \theta \mathcal{R} \psi$ follows by a combination of the reasoning for Lemmas 4 and Lemma 5. Details are left to the reader. \[\square\]
6 Definability and undefinability of modal operators

In this section, we explore the question of when it is that the basic connectives can or cannot be defined in terms of each other. It is known that, classically, ♦ and □ are interdefinable, as are ℤ and ℬ; we will see that this is not the case intuitionistically. On the other hand, ℤ and hence ℬ is not definable in terms of ♦, □ in the classical setting [18], and this result immediately carries over to the intuitionistic setting, as the class of classical LTL models can be seen as the subclass of that of dynamic posets where the partial order is the identity.

Interdefinability of modal operators can vary within intermediate logics. For example, ∧, ∨ and → are basic connectives in propositional intuitionistic logic, but in the intermediate logic of here-and-there [15], ∧ [12] and → [11] are basic operators while ∨ is definable in terms of → and ∧ [22]. In first-order here-and-there [21], the quantifier ∃ is definable in terms of ∀ and → [24] while ∀ is not definable in terms of the other operators. In the modal case, Simpson [30] shows that modal operators are not interdefinable in the logic IK and Balbiani and Diéguez [2] proved the same result for the linear time temporal extension of here-and-there. This last proof is adapted to show that modal operators are not definable in ITL^e. Note, however, that here we correct the claim of [2] stating that ♦ is not here-and-there definable in terms of □.

Let us begin by studying the definability of □ in terms of ○ and ℤ. Below, if \( L' \subseteq L \), \( \varphi \in L \) and \( \Omega \) is a class of models, we say that \( \varphi \) is \( L' \)-definable over \( \Omega \) if there is \( \varphi' \in L' \) such that \( \Omega \vdash \varphi \leftrightarrow \varphi' \).

**Theorem 4.** The connective □ is not \( L_\mathcal{U} \)-definable, even over the class of finite here-and-there models.

**Proof.** For \( n > 0 \) consider a model \( \mathcal{H}_n = (W, \leq, S, V) \) with \( W = \{0, \ldots, n+1\} \times \{0, 1\} \), \( (i, j) \leq (i', j') \) if \( i = i' \) and \( j \leq j' \), \( S(i, j) = (i', j') \) if and only if \( i' = i \) and \( j' \equiv j + 1 \) (mod \( n+2 \)), and \( V(p) = W \setminus \{(n + 1, 0)\} \). Clearly \( \mathcal{H}_n \) is a here-and-there model. For \( m \leq n \), let \( \sim_m \) be the least equivalence relation such that
Consider the following cases.

Case \( (i, j) \sim_m (i', j') \) whenever

\[
\min\{i(j - 1), i'(j' - 1)\} \leq n - m
\]

(see Figure [1]). Then, it can easily be checked that \( (H_n, (0, 0)) \neq \diamond p \), \( (M, (0, 1)) \vDash \diamond p \), and \((0, 0) \sim_m (0, 1)\).

It remains to check that \( (\sim_m)_{m \leq n} \) is a bounded \( \mathcal{U} \)-bisimulation. The atoms, \( \rightarrow \) and \( \circ \) clauses are easily verified, so we focus on those for \( \mathcal{U} \). Since \( \sim_m \) is symmetric, we only check the forth \( \mathcal{U} \). Suppose that \((i_1, j_1) \sim_m (i_2, j_2)\), and fix \( k_1 \geq 0 \). Let \( S(i_1, j_1) = (i', j') \). Then, we can see that \( k_2 = 0 \), \( v_1 = (i', 1) \) and \( v_2 = (i_2, j_2) \) that the clause holds, where the intermediate condition for \( j_2 \in [0, k_2) \) holds vacuously since \( [0, k_2) = \emptyset \).

By letting \( n = |\varphi| \), we see using Lemma 5 that no \( \mathcal{L}_{\mathcal{U}} \)-formula \( \varphi \) can be equivalent to \( \Box p \).

As a consequence:

**Corollary 1.** The connective \( \mathcal{R} \) is not definable in terms of \( \circ \) and \( \mathcal{U} \), even over the class of persistent models.

**Proof.** If we could define \( q \mathcal{R} p \), then we could also define \( \Box p \equiv \perp \mathcal{R} p \).

---

**Proposition 5.** Over the class of here-and-there models, \( \diamond \) is \( \mathcal{L}_\mathcal{O} \)-definable. To be precise, \( \diamond p \) is equivalent to

\[
\varphi = (\Box(p \rightarrow \Box(p \lor \neg p)) \land (\Box\Box(p \lor \neg p) \rightarrow p \lor \neg p \lor \Box\Box\neg p)) \rightarrow (\Box(p \lor \neg p) \land \Box\Box\neg p).
\]

**Proof.** Let \( M = (T \times \{0, 1\}, \leq, S, V) \) be a here-and-there model with \( S(t, i) = (f(t), i) \) (see Section 2.3). First assume that \( x = (x_1, x_2) \) is such that \( (M, x) \vDash \diamond p \). To check that \( (M, x') \vDash \varphi \), let \( x' \succeq x \), so that \( x' = (x_1, x'_2) \) with \( x'_2 \geq x_1 \), and consider the following cases.

**CASE** \((M, x') \vDash \Box(p \lor \neg p)\). In this case, it is easy to see that we also have \((M, x) \vDash \Box\Box\neg p \) given that \((M, x) \vDash \diamond p \).

**CASE** \((M, x') \neq \Box(p \lor \neg p)\). Using the assumption that \((M, x) \vDash \diamond p \), choose \( k \) such that \((M, (f^k(x_1), x_2)) \vDash p \) and consider two sub-cases.

1. Suppose there is \( k' > k \) such that \((M, (f^{k'}(x_1), x'_2)) \neq p \lor \neg p \). Then, it follows that \((M, (f^k(x_1), x'_2)) \neq p \rightarrow \Box p \lor \neg p \) and hence \((M, x') \neq \Box(p \lor \neg p)\).

2. If there is not such \( k' \), then there must be a maximal \( k' < k \) such that \((M, (f^{k'}(x_1), x'_2)) \neq p \lor \neg p \) (otherwise, we would be in CASE \((M, x') \vDash \Box(p \lor \neg p)\)). It is easily verified that

\[
(M, (f^{k'}(x_1), x'_2)) \neq \Box\Box(p \lor \neg p) \rightarrow p \lor \neg p \lor \Box\Box\neg p,
\]

and hence

\[
(M, x') \neq \Box(\Box\Box(p \lor \neg p) \rightarrow p \lor \neg p \lor \Box\Box\neg p).
\]

---

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Theorem 5. The operator $\Diamond$ cannot be defined in terms of $\Box$ over the class of finite expanding models.

Proof. Given $n > 0$, consider a model $\mathcal{E}_n = (W, \leq, S, V)$ with $W = \{0, \ldots, n + 1\} \times \{0, 1\}$, $(i, j) \leq (i', j')$ if $i = i'$ and $j \leq j'$, $S(i, j) = (i + 1, j)$ if $i \leq n$, $S(n + 1, j) = (0, 0)$, and $V(p) = \{(n + 1, 1)\}$. For $m \leq n$, let $\sim_m$ be the least equivalence relation such that $(i, j) \sim_m (i', j')$ whenever $\max\{i, i'\} \leq n - m$. Then, it can easily be checked that $(\mathcal{M}, (0, 0)) \nmid p$, $(\mathcal{M}, (0, 1)) \equiv \Diamond p$, and $(0, 0) \sim_m (0, 1)$.
It remains to check that \((\sim_m)_{m \leq n}\) is a bounded \(\square\)-bisimulation. As before, we focus on the \(\square\) clauses, specifically \(\text{BACK} \, \square\). Suppose that \((i_1, j_1) \sim_m (i_2, j_2)\) and fix \(k_1 \geq 0\). Let \((i'_1, j'_1) = S^{k_1}(i_1, j_1)\). Choose \(k_2 > n + 1\) such that \(i_2 + k_2 \equiv i'_1 \pmod{n+1}\), and let \((i'_2, j'_2) = S^{k_2}(i_2, j_2)\). It is not hard to check that \(i'_1 = i'_2\) and \(j'_2 = 0\), from which we obtain \((i'_2, j'_2) \preceq (i'_1, j'_1)\). Hence, setting \(v_1 = v_2 = (i'_2, j'_2)\) gives us the desired witnesses.

By letting \(n\) vary, we see that no \(L_\square\)-formula can be equivalent to \(\Diamond p\).

7 Conclusions

In this paper we have investigated on \(\text{ITL}^e\), an intuitionistic analogue of \(\text{LTL}\) based on expanding domain models from modal logic. We have shown that, as happens in other modal intuitionistic logics or modal intermediate logics, modal operators are not interdefinable.

This work and [4] are the first attempts to study \(\text{ITL}^e\). Needless to say, many open questions remain. We know that \(\text{ITL}^e\) is decidable, but the proposed decision procedure is non-elementary. However, there seems to be little reason to assume that this is optimal, raising the following question:

**Question 1.** Are the satisfiability and validity problems for \(\text{ITL}^e\) without \(\mathcal{U}, \mathcal{R}\) elementary? Is the full logic still decidable?

Meanwhile, we saw in Theorems 1 and 2 that \(\text{ITL}^e\) has the strong finite model property, while \(\text{ITL}^p\) does not have the finite model property at all. However, it may yet be that \(\text{ITL}^p\) is decidable despite this.

**Question 2.** Is \(\text{ITL}^p\) decidable?

Regarding expressive completeness, it is known that \(\text{LTL}\) is expressively complete [18, 29, 12, 10]; there exists a one-to-one correspondence (over \(\mathbb{N}\)) between the temporal language and the monadic first-order logic equipped with a linear order and ‘next’ relation [12]. It is not known whether the same property holds between \(\text{ITL}^e\) and first-order intuitionistic logic.

**Question 3.** Is \(L\) equally expressive to monadic first-order logic over the class of dynamic or persistent models?

Finally, a sound and complete axiomatization for \(\text{ITL}^e\) remains to be found. The results we have presented here could be a first step in this direction, and we conclude with the following:

**Question 4.** Are the \(\text{ITL}^e\)-valid formulas listed in this work, together with the intuitionistic tautologies and standard inference rules, complete for the class of dynamic posets? Is the logic augmented with \((\Diamond p \rightarrow \Diamond q) \rightarrow \Diamond(p \rightarrow q)\) complete for the class of persistent models?
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