Steinberg modules and Donkin pairs.

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1 Summary

We prove that in characteristic $p > 0$ a module with good filtration for a group of type $E_6$ restricts to a module with good filtration for a group of type $F_4$. Thus we confirm a conjecture of Brundan for one more case. Our method relies on the canonical Frobenius splittings of Mathieu. Next we settle the remaining cases, in characteristic not 2, with a computer-aided variation on the old method of Donkin.

2 Preliminaries

Our base field $k$ is algebraically closed of characteristic $p$. Let $G$ be a connected semisimple group and $H$ a connected semisimple subgroup. (Good filtrations with more general groups are treated in [3].) We refer to [5] and [16] for unexplained terminology and notation.

Now choose a Borel subgroup $B$ in $G$ and a maximal torus $T$ in $B$ so that, if $B^-$ is the opposite Borel subgroup, then $B \cap H$ and $B^- \cap H$ are a Borel subgroups in $H$ and $T \cap H$ is a maximal torus in $H$.

We follow the convention that the roots of $B$ are positive. If $\lambda \in X(T)$ is dominant, then $\text{ind}^G_H(-\lambda)$ is the dual Weyl module $\nabla_G(\lambda^*)$ with highest weight $\lambda^* = -w_0\lambda$ and lowest weight $-\lambda$. Its dual is the Weyl module $\Delta_G(\lambda)$. In a good filtration of a $G$-module the layers are of the form $\nabla_G(\mu)$.

Definition 2.1 We say that $(G, H)$ is a Donkin pair if for any $G$-module $M$ with good filtration, the $H$-module $\text{res}_H^G M$ has good filtration.
Let $U(U)$ denote the hyperalgebra of the unipotent radical $U$ of $B$. We recall the presentation of Weyl modules.

**Lemma 2.2** Let $\lambda$ be dominant and let $v_{-\lambda^*}$ be a nonzero weight vector of lowest weight $-\lambda^*$ in $\Delta_G(\lambda)$. Then $v_{-\lambda^*}$ generates $\Delta_G(\lambda)$ as a $U(U)$-module, and the annihilator of $v_{-\lambda^*}$ equals the left ideal of $U(U)$ generated by the $X_{\alpha}^{(n)}$ with $\alpha$ simple and $n > (\lambda^*, \alpha^\vee)$.

**Proof** Note that $U(U)$ is a graded algebra graded by height. Therefore the left ideal in the lemma is the intersection of all ideals $I$ of finite codimension that contain it and that lie inside the annihilator. But by the proof of [12, Proposition Fondamentale] such ideals $I$ are equal to the annihilator. \[ \square \]

Let $X$ be a smooth projective $B$-variety with canonical bundle $\omega$. (Generalizations to other varieties will be left to the reader.) There is by [10, §2] a natural map $\epsilon : H^0(X, \omega^{1-p}) \to k$ so that $\phi \in H^0(X, \omega^{1-p})$ determines a Frobenius splitting if and only if $\epsilon(\phi) = 1$. Let $St_G$ be the Steinberg module of the simply connected cover $\tilde{G}$ of $G$. For simplicity of notation we further assume that $St_G$ is actually a $G$-module. Its $B$-socle is the highest weight space $k((p-1)\rho)$.

Recall that a Frobenius splitting of $X$ is called canonical if the corresponding $\phi$ is $T$-invariant and lies in the image of a $B$-module map $St_G \otimes k((p-1)\rho) \to H^0(X, \omega^{1-p})$. (Compare lemma 2.2 and [10, Definition 4.3.5].) If the group $G$ needs to be emphasized, we will speak of a $G$-canonical splitting. Now suppose $X$ is actually a $G$-variety.

**Lemma 2.3** $X$ has a canonical splitting if and only if there is a $G$-module map $\psi : St_G \otimes St_G \to H^0(X, \omega^{1-p})$ so that $\epsilon \psi \neq 0$.

**Proof** There is, up to scalar multiple, only one possibility for a map $St_G \otimes St_G \to k$. If $\epsilon \psi \neq 0$ then the subspace of $T$-invariants in $St_G \otimes k((p-1)\rho)$ maps isomorphically to $k$. Conversely, a map from $St_G \otimes k((p-1)\rho)$ to a $G$-module $M$ can be extended to $St_G \otimes St_G$ because the $G$-module generated by the image of $k((p-1)\rho)$ in $M \otimes St_G^*$ is $St_G$. \[ \square \]

We have the following fundamental result of Mathieu [8].
Theorem 2.4 [3, 6.2] Assume $X$ has a canonical splitting and $\mathcal{L}$ is a $G$-linearized line bundle on $X$. Then $H^0(X, \mathcal{L})$ has a good filtration.

3 Pairings

Now apply this to $X = G/B$. Of course the $\nabla_G(\mu)$ are of the form $H^0(X, \mathcal{L})$, see [3, I 5.12]. It follows that $(G, H)$ is a Donkin pair if $X$ has an $H$-canonical splitting. We also have a surjection $\text{St}_G \otimes \text{St}_G \to H^0(X, \omega^{1-p})$, by [3, II 14.20]. The composite with $H^0(X, \omega^{1-p}) \to k$ may be identified as in [3, I] with the natural pairing on the self dual representation $\text{St}_G$. Thus we get

Theorem 3.1 (Pairing criterion) Assume there is an $H$-module map

$$\text{St}_H^* \otimes \text{St}_H \to \text{St}_G^* \otimes \text{St}_G$$

whose composite with the evaluation map $\text{St}_G^* \otimes \text{St}_G \to k$ is nonzero. Then $(G, H)$ is a Donkin pair.

Remark 3.2 Despite the notation, the $\tilde{H}$-module $\text{St}_H$ need not be an $H$-module. Even if $\text{St}_H$ is not an $H$-module, $\text{St}_H^* \otimes \text{St}_H$ is one. It may be better to replace $H \subset G$ with the homomorphism $\tilde{H} \to \tilde{G}$. Thus an operation like $\text{res}_H^G$ would really mean restriction along $\tilde{H} \to \tilde{G}$.

Remark 3.3 The pairing criterion is satisfied if and only if $G/B$ has an $H$-canonical splitting. Indeed suppose we are given a map $\text{St}_H^* \otimes \text{St}_H = \text{St}_H \otimes \text{St}_H \to H^0(G/B, \omega^{1-p})$ as in Lemma 2.3. We have to factor it through the surjection $\pi : \text{St}_G^* \otimes \text{St}_G \to H^0(G/B, \omega^{1-p})$. But the kernel $K$ of $\pi$ has good filtration by [3] (or by the proof in [3, II 4.16]), so $\text{Ext}^1_H(\text{St}_H^* \otimes \text{St}_H, \text{res}_H^G K)$ vanishes by theorem 2.4 and the main properties of good filtrations ([3, Theorem 1], [3, II 4.13]).

Now we illustrate the criterion with some old examples of Donkin pairs.

Example 3.4 Let $G$ still be semisimple and connected. It is easy to see from the formulas in the proof of [3, 3.2] that the pairing criterion applies to the diagonal $G$ inside a product $G \times \cdots \times G$. 

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Example 3.5 Let $H$ be the commutator subgroup of a Levi subgroup of a parabolic in the semisimple connected group $G$. Then, after passing to simply connected covers if necessary, $\text{St}_H$ is a direct summand of $\text{res}^G_H \text{St}_G$, so again the pairing criterion applies.

Lemma 3.6 Let $(G, H)$ satisfy the pairing criterion and let $X$ be a smooth projective $G$-variety. If $X$ has a $G$-canonical splitting, then it has an $H$-canonical one.

Proof Use lemma 2.3.

The following lemma was pointed out to me by Jesper Funch Thomsen.

Lemma 3.7 Let $X, Y$ be smooth projective $G$-varieties with canonical splitting. Then $X \times Y$ has a $G$-canonical splitting.

Proof Use example 3.4.

Remark 3.8 For the users of our book, let us now point out how to get theorem 2.4. We have $G \times^B X = G/B \times X$ by remark [16, 1.2.2], so [16, lemma 4.4.2] applies with $Y = X$ in the notations of that lemma.

Remark 3.9 In lemma 3.7 one cannot replace $G$ with $B$. Here is an example. Take $G = SL_3$ in characteristic 2 and let $Z$ be the Demazure resolution of a Schubert divisor. Then $H^0(Z, \omega_Z^{-1})$ is a nine dimensional $B$-module. There is a fundamental representation $V$ so that $H^0(Z, \omega_Z^{-1})$ is isomorphic to a codimension one submodule of the degree three part of the ring of regular functions on $V$. Using this, one checks with computer assisted computations that $Z$, $Z \times Z$, $Z \times Z \times Z$ have $B$-canonical splittings, while $Z \times Z \times Z \times Z$ does not have one.

Our next aim is to treat the following example.

Example 3.10 For $G$ we take the simply connected group of type $E_6$. From the symmetry of its Dynkin diagram we have a graph automorphism which is an involution. For $H$ we take the group of fixed points of the involution. It is connected ([13, 8.2]) of type $F_4$. It has been conjectured by Brundan [3, 4.4] that $(G, H)$ is a Donkin pair.
More generally, with our usual notations we have.

**Theorem 3.11** Assume there are dominant weights $\sigma_1, \sigma_2, \sigma_3$, so that

1. The highest weight $(p-1)\rho_G$ of $\text{St}_G$ equals $\sigma_1 + \sigma_2 + \sigma_3$.
2. $\sigma_1 + \sigma_2$ and $\sigma_2 + \sigma_3$ both restrict to the highest weight $(p-1)\rho_H$ of $\text{St}_H$.
3. The natural map $\nabla_G(\sigma_1) \rightarrow \nabla_H(\text{res}_{B\cap H}^G \sigma_1)$ is surjective.

Then $(G, H)$ is a Donkin pair. In fact it satisfies the pairing criterion.

**Remark 3.12** If $(G, H)$ is a Donkin pair and $\lambda$ is dominant, then one knows that $\nabla_G(\lambda) \rightarrow \nabla_H(\text{res}_{B\cap H}^G \lambda) = \text{ind}_{H\cap B^-}^H(\text{res}_{H\cap B^-}^B \lambda)$, induced by the projection of $\nabla_G(\lambda)$ onto its highest weight space, is surjective. (Exercise. Use a good filtration as in the proof of [5, II 4.16].)

**Remark 3.13** Our theorem 3.11 also applies to the Levi subgroup case of example 3.5 (take $\sigma_1 = 0$). One hopes to find a more general method to attack at least all graph automorphisms. Theorem 3.11 applies if the graph automorphism is an involution and different simple roots in an orbit are perpendicular to each other. But for the graph automorphism of a group of type $A_{2n}$ in characteristic $p > 2$ there are no $\sigma_1, \sigma_2, \sigma_3$ as in the theorem. The coefficient of $\text{res}_{B\cap H}^G \rho_G$ with respect to the fundamental weight that corresponds to the short root is four, which is too high.

**Proof of Theorem 3.11.**

We will often write the restriction of a weight to $T \cap H$ with the same symbol as the weight. We will repeatedly use basic properties of Weyl modules and their duals. See [3, II 14.20] for surjectivity of cup product between dual Weyl modules and [3, II 2.13] for Weyl modules as universal highest weight modules. We first need a number of nonzero maps of $H$-modules. They are natural up to nonzero scalars that do not interest us.

The first map is the map

$$\epsilon_H : \nabla_H(2(p-1)\rho_H) \rightarrow k$$
which detects Frobenius splittings on $H/(H \cap B)$. Together with the surjection
\[
\nabla_H(\sigma_2) \otimes \nabla_H((p-1)\rho_G) \to \nabla_H(2(p-1)\rho_H)
\]
it gives a nonzero map $\nabla_H(\sigma_2) \otimes \nabla_H((p-1)\rho_G) \to k$ and hence a nonzero
\[
\eta_1 : \nabla_H(\sigma_2) \to \nabla_H((p-1)\rho_G)^*.
\]

The map $\nabla_G(\sigma_2 + \sigma_3) \to \text{St}_H$ is nonzero, hence surjective. The map $\nabla_G(\sigma_1) \to \nabla_H(\sigma_1)$ is surjective by assumption. In the commutative diagram
\[
\begin{array}{ccc}
\nabla_G(\sigma_2 + \sigma_3) \otimes \nabla_G(\sigma_1) & \to & \text{St}_G \\
\downarrow & & \downarrow \\
\text{St}_H \otimes \nabla_H(\sigma_1) & \to & \nabla_H((p-1)\rho_G)
\end{array}
\]
the horizontal maps are also surjective. So the map
\[
\eta_2 : \nabla_H((p-1)\rho_G)^* \to \text{St}_G^*
\]
is injective. We obtain a nonzero
\[
\eta_2 \eta_1 : \nabla_H(\sigma_2) \to \text{St}_G^*.
\]
The nonzero $\text{St}_H \to \nabla_G(\sigma_2 + \sigma_3)$ combines with the map
\[
\nabla_G(\sigma_1) \otimes \nabla_G(\sigma_2 + \sigma_3) \to \text{St}_G
\]
to yield
\[
\nabla_G(\sigma_1) \otimes \text{St}_H \to \text{St}_G
\]
and combining this with $\eta_2 \eta_1$ we get
\[
\eta_3 : \nabla_H(\sigma_2) \otimes \nabla_G(\sigma_1) \otimes \text{St}_H \to \text{St}_G^* \otimes \text{St}_G.
\]
We claim that its image is detected by the evaluation map
\[
\eta_4 : \text{St}_G^* \otimes \text{St}_G \to k.
\]
This is because $\eta_3$ factors through $\nabla_H((p-1)\rho_G)^* \otimes \text{St}_G$, on which the restriction of $\eta_4$ factors through $\nabla_H((p-1)\rho_G)^* \otimes \nabla_H((p-1)\rho_G)$, the map $\eta_1$ is nonzero, the image of $\nabla_G(\sigma_1) \otimes \text{St}_H \to \text{St}_G$ maps onto $\nabla_H((p-1)\rho_G)$. 


From the nontrivial \( \eta_4 \eta_3 \) we get a nontrivial
\[
\eta_5 : \nabla_H(\sigma_2) \otimes \nabla_G(\sigma_1) \to \text{St}_H^*.
\]
Then \( \eta_5 \) must be split surjective. Choose a left inverse
\[
\eta_6 : \text{St}_H^* \to \nabla_H(\sigma_2) \otimes \nabla_G(\sigma_1)
\]
of \( \eta_5 \). It leads to
\[
\eta_7 : \text{St}_H^* \otimes \text{St}_H \to \nabla_H(\sigma_2) \otimes \nabla_G(\sigma_1) \otimes \text{St}_H
\]
and the map we use in the pairing criterion is \( \eta_3 \eta_7 \). Indeed the map \( \text{St}_H^* \to \text{St}_H^* \) defined by \( \eta_4 \eta_3 \eta_7 \) equals \( \eta_5 \eta_6 \), hence is nonzero. \( \square \)

4 The \( \text{E}_6-\text{F}_4 \) pair.

We turn to the \( \text{E}_6-\text{F}_4 \) pair of example 3.10. First observe that for \( p > 13 \) one could simply follow the method of [2] to prove that the pair is a Donkin pair. Indeed the restriction to \( \text{F}_4 \) of a fundamental representation then has its dominant weights in the bottom alcove. Looking a little closer and applying the linkage principle one can treat \( p \geq 11 \) in the same manner.

But for \( p = 5 \) one has \( \varpi_4 \uparrow \varpi_1 + \varpi_4 \) and for \( p = 7 \) one has \( \varpi_1 \uparrow \varpi_1 + \varpi_4 \). This makes that one has more trouble to see that the restriction of \( \nabla_G(\varpi_4) \) has a good filtration with respective layers \( \nabla_H(\varpi_1), \nabla_H(\varpi_3), \nabla_H(\varpi_3), \nabla_H(\varpi_2) \). For \( p = 2 \) or \( p = 3 \) it is even worse.

So let us apply theorem 3.11 instead. We take \( \sigma_1 = (p-1)(\varpi_1 + \varpi_3), \sigma_2 = (p-1)(\varpi_2 + \varpi_4), \sigma_3 = (p-1)(\varpi_5 + \varpi_6) \) in the notations of Bourbaki for \( \text{E}_6 \) [1, Planches]. Then \( \text{res}_{B \cap H}^B \varpi_i \) equals \( \varpi_4, \varpi_1, \varpi_3, \varpi_2, \varpi_3, \varpi_4 \) for \( i = 1, \ldots, 6 \) respectively.

First let let \( p = 2 \). Then \( \nabla_H(\text{res}_{B \cap H}^B \sigma_1) = \nabla_H(\varpi_3 + \varpi_4) \) is irreducible. Indeed its dominant weights come in two parts. The weights \( 0, \varpi_4, \varpi_1, \varpi_3, 2\varpi_4, \varpi_1 + \varpi_4, \varpi_2 \) lie in one orbit, and the highest weight lies in a different orbit under the affine Weyl group. To be more specific, \( \varpi_1 - \rho_H \uparrow \varpi_3 + \varpi_4, \) but \( \varpi_4 - \rho_H \uparrow 0 \uparrow \varpi_4 \uparrow \varpi_1 \uparrow \varpi_3 \uparrow 2\varpi_4 \uparrow \varpi_1 + \varpi_4 \uparrow \varpi_2 \). So \( \nabla_G(\sigma_1) \to \nabla_H(\text{res}_{B \cap H}^B \sigma_1) \) is surjective.

Remains the case \( p > 2 \). To see that \( \nabla_G(\lambda) \to \nabla_H(\text{res}_{B \cap H}^B \lambda) \) is surjective for \( \lambda = \sigma_1 \), it suffices to do this for \( \lambda = \varpi_1 \) and \( \lambda = \varpi_3 \). For \( p > 3 \) one could now use that \( \nabla_H(\text{res}_{B \cap H}^B \lambda) \) is irreducible for both \( \lambda = \varpi_1 \) and \( \lambda = \varpi_3 \),
because each of the dominant weights of $\nabla_H(\text{res}_{B/H}^H \lambda)$ is in a different orbit under the affine Weyl group.

But we need an argument that works for $p \geq 3$. Now $\nabla_G(\varpi_1)$ is a miniscule representation of dimension 27, and $\nabla_H(\varpi_4) = \nabla_H(\text{res}_{B/H}^B \varpi_4)$ has dimension 26. There are 24 short roots and they have multiplicity one in $\nabla_H(\varpi_4)$. So the map from $M := \nabla_G(\varpi_1)$ to $\nabla_H(\varpi_4)$ hits at least 24 dimensions and its kernel consists of $H$-invariants. Indeed there are three weights of $\nabla_G(\varpi_1)$ that restrict to zero. In Bourbaki notation they are

$$
\zeta_1 = \frac{1}{6}(\epsilon_8 - \epsilon_7 - \epsilon_6) + \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4 - \epsilon_5),
\zeta_2 = \frac{1}{6}(\epsilon_8 - \epsilon_7 - \epsilon_6) + \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4 - \epsilon_5),
\zeta_3 = -\frac{1}{3}(\epsilon_8 - \epsilon_7 - \epsilon_6) + \epsilon_5.
$$

Put

$$
\zeta_4 = \frac{1}{6}(\epsilon_8 - \epsilon_7 - \epsilon_6) + \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 + \epsilon_5),
\zeta_5 = \frac{1}{6}(\epsilon_8 - \epsilon_7 - \epsilon_6) + \frac{1}{2}(-\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4 - \epsilon_5).
$$

Then $X_{\alpha_1}$ induces an isomorphism $M_{\zeta_3} \to M_{\zeta_4}$ and it annihilates $M_{\zeta_1} + M_{\zeta_2}$. Similarly $X_{\alpha_6}$ induces an isomorphism $M_{\zeta_2} \to M_{\zeta_4}$ and annihilates $M_{\zeta_1} + M_{\zeta_3}$. The same space is annihilated by $X_{\alpha_3}$, which induces an isomorphism $M_{\zeta_2} \to M_{\zeta_5}$. Finally $X_{\alpha_5}$ induces an isomorphism $M_{\zeta_1} \to M_{\zeta_5}$ and annihilates $M_{\zeta_2} + M_{\zeta_3}$.

It follows that in $M_{\zeta_1} + M_{\zeta_2} + M_{\zeta_3}$ there is just a one dimensional subspace of vectors annihilated by both $X_{\alpha_1} + X_{\alpha_6}$ and $X_{\alpha_3} + X_{\alpha_5}$. (These two operators come from the Lie algebra of $H$.) We conclude that $\text{res}_H^G M$ has a good filtration and that $M \to \nabla_H(\varpi_4)$ is surjective. As $p > 2$, we then also have that $M \wedge M$ and $\text{res}_H^G(M \wedge M)$ have a good filtration. It then follows from the character that $M \wedge M = \nabla_G(\varpi_3)$. (We use the program LiE [17].) So $\text{res}_H^G \nabla_G(\varpi_3)$ has a good filtration and therefore maps onto $\nabla_G(\text{res}_{B/H}^B \varpi_3)$.

Summing up, we have shown

**Theorem 4.1** The $E_6$-$F_4$ pair is a Donkin pair. In fact it satisfies the pairing criterion.

**Remark 4.2** When Steve Donkin received this proof, he proceeded to show that one could also prove $E_6$-$F_4$ to be a Donkin pair with the ‘ancient methods’ of his book [3]. Of course he had to treat more representations than we do. We will use his method in the last section to treat the remaining cases of Brundan’s conjecture in characteristic $p > 2$, where we have no alternative yet.
5 Induction and canonical splitting

We finish the discussion of canonical splittings with an analogue of proposition [8, 5.5]. It makes a principle from [8] more explicit. The result was explained to us by O. Mathieu at a reception of the mayor of Aarhus in August 1998. It shows once more that canonical splittings combine well with Demazure desingularisation of Schubert varieties.

Proposition 5.1 Let $X$ be a projective $B$-variety with canonical splitting. Let $P$ be a minimal parabolic. Then $P \times^B X$ has a canonical splitting.

Corollary 5.2 The same conclusion holds for any parabolic subgroup.

Proof If $P$ is not minimal, take a Demazure resolution $Z = P_1 \times^B P_2 \times^B \ldots \times^B P_r/B$ of $P/B$ and apply the proposition to get a canonical splitting on $P_1 \times^B P_2 \times^B \ldots \times^B P_r \times^B X$. Then push the splitting forward ([10, Prop. 4]) to $P \times^B X$.

Proof of Proposition We use notations as in [10, Ch. 4, A.4]. Let $\zeta$ be the highest weight of $St$ and $s$ the simple reflection corresponding with $P$. One checks as in [10, A.4.6] that

$$\mathcal{E}nd_F(P \times^B X) = (P \times^B \mathcal{E}nd_F(X)) \otimes \pi^* L(s\zeta - \zeta),$$

where $\pi : P \times^B X \to P/B$. We are given a map $\phi : k_\zeta \otimes St \to \mathcal{E}nd_F(X)$. The required map $\psi : k_\zeta \otimes St \to \mathcal{E}nd_F(P \times^B X)$ may be constructed by composing maps

$$k_\zeta \otimes St \cong k_{-s\zeta} \otimes \text{ind}^P_B(k_{\zeta + s\zeta} \otimes St) \to$$

$$k_{-s\zeta} \otimes \text{ind}^P_B(k_{s\zeta} \otimes \mathcal{E}nd_F(X)) \cong$$

$$k_{-s\zeta} \otimes H^0(P \times^B X, P \times^B (\mathcal{E}nd_F(X)[s\zeta])) \cong$$

$$\mathcal{E}nd_F(P \times^B X, B \times^B X) \to$$

$$\mathcal{E}nd_F(P \times^B X).$$

Here $k_{-s\zeta}$ is identified with the weight space of weight $-s\zeta$ of

$$H^0(P \times^B X, \pi^* L(-\zeta)).$$
An element of that weight space has divisor \((p - 1)B \times B\) = \((p - 1)X\).

To see that the image of \(\psi\) is not in the kernel of 
\[\epsilon_{P \times B X} : \text{End}_F(P \times B X) \to k,\]
it suffices to show that the diagram
\[
\begin{array}{ccc}
k_\zeta \otimes \text{St} & \to & \text{End}_F(P \times B X, B \times B X) \\
\| & & \downarrow \\
k_\zeta \otimes \text{St} & \xrightarrow{\phi} & \text{End}_F(X) \to k
\end{array}
\]
commutes. Now
\[
k_{-s_\zeta} \otimes \text{ind}_B^P(k_{s_\zeta} \otimes \text{St}) \quad \longrightarrow \quad k_\zeta \otimes \text{St}
\]
commutes and by restricting to the trivial fibration \(BsB \times B X \to BsB / B\) one shows through the following lemma that the bottom map in this last diagram agrees with the map that factors through \(\text{End}_F(P \times B X, B \times B X)\).

\[\blacksquare\]

**Lemma 5.3** Let \(A\) be a commutative \(k\)-algebra. Then
\[
\text{End}_F(A[t]) = \text{End}_F(A) \otimes \text{End}_F(k[t]) = \text{End}_F(A) \otimes k[t]
\]
and the map \(\text{End}_F(A[t], (t)) \to \text{End}_F(A)\) is induced by the map
\[
t^{p-1}k[t] = t^{p-1} * \text{End}_F(k[t]) = \text{End}_F(k[t], (t)) \to \text{End}_F(k) = k
\]
which sends \(t^{p-1}f(t)\) to \(f(0)\).

**Proof** Straightforward, provided one keeps in mind how \(\text{End}_F(R)\) is an \(R\)-module ([16, 4.3.3]). Compare also [16, A.4.5]. \[\blacksquare\]

### 6 More Donkin pairs

In this section we do not use the pairing criterion. Instead we return to the methods of Donkin’s book [3], combined with computer calculations of characters, of linkage, and of the Jantzen sum formula.
Let $G$, $H$ be as before, with $G$ simply connected. In fact $H$ will be the commutator subgroup of the group of fixed points of an involution of $G$ which leaves invariant the maximal torus $T$ and the Borel subgroup $B$. We refer to [14] for the classification of the possibilities, assuming $p > 2$. (Of course involutions of the simply connected $G$ are lifted [15, 9.16] from the involutions of the corresponding adjoint group, which are treated in [14].)

**Remark 6.1** Let $H$ be the fixed point group of an involution that leaves $T$ and $B$ invariant in the simply connected semisimple $G$. Then $H$ is connected reductive by [15, 8.2]. Now an $H$-module has good filtration in the sense of [3] if and only if its restriction to the commutator subgroup of $H$ has good filtration. That is why we look only at semisimple subgroups $H$.

Let $\mathcal{M}$ denote the set of finite dimensional $G$-modules $M$ with good filtration for which $\text{res}^G_H M$ has good filtration. Let $\mathcal{S}$ denote the set of dominant weights $\lambda$ of $G$ so that $\nabla^G(\lambda) \in \mathcal{M}$. As always we try to show that all dominant weights of $G$ are in $\mathcal{S}$. For this purpose we recall some useful lemmas.

**Lemma 6.2**

1. If $M_1 \oplus M_2 \in \mathcal{M}$, then $M_1 \in \mathcal{M}$.

2. If $0 \to M' \to M \to M'' \to 0$ is exact, and $M' \in \mathcal{M}$, then $M \in \mathcal{M}$ if and only if $M'' \in \mathcal{M}$.

3. If $M_1, M_2 \in \mathcal{M}$, then $M_1 \otimes M_2 \in \mathcal{M}$.

If $M$ is a $G$-module with good filtration, write $\text{supp}_\nabla(M)$ for the set of dominant weights $\lambda$ so that $\nabla^G(\lambda) \in \mathcal{M}$. We order the dominant weights of $G$ by the partial order in which $\mu \leq \lambda$ if and only if $\lambda - \mu$ is in the closed cone spanned by the positive roots. In particular, if $\lambda$ is dominant, then $0 \leq \lambda$, and all dominant weights $\mu$ of $\nabla^G(\lambda)$ satisfy $\mu \leq \lambda$. We say that a filtration of $M$ is a good filtration adapted to the partial order if there are $\lambda_i$ so that the $i$-th layer is a direct sum of copies of $\nabla^G(\lambda_i)$, and $i \leq j$ if $\lambda_i \leq \lambda_j$. (So we still call it a good filtration, even though $\nabla^G(\lambda_i)$ may have multiplicity in the $i$-th layer.) If $M$ has a good filtration, then it also has one adapted to the partial order, by the proof of [3, II 4.16].

**Lemma 6.3** Let $M \in \mathcal{M}$ and $\lambda \in \text{supp}_\nabla(M)$. Assume for every weight $\mu$ in $\text{supp}_\nabla(M)$, distinct from $\lambda$, that one of the following holds
1. \( \mu < \lambda \) and \( \mu \in \mathcal{S} \).

2. \( \mu \) and \( \lambda \) are in different orbits under the affine Weyl group.

Then \( \lambda \in \mathcal{S} \).

Proof We may replace \( M \) by an indecomposable direct summand \( M_1 \) with \( \lambda \in \text{supp}_\nabla(G)(M_1) \). The linkage principle tells that we thus get rid of the second possibility in the lemma. Then in a good filtration adapted to the partial order, the module \( \nabla_G(\lambda) \) occurs only as a summand of the top layer, which is in \( \mathcal{M} \) by lemma 6.2. \( \square \)

Lemma 6.4 Let \( \lambda \) be a dominant weight of \( G \). If \( \lambda \) is in the bottom alcove, or if the Jantzen sum formula yields zero, then \( \nabla_G(\lambda) \) is irreducible.

Proof See [5, II Cor. 5.6 and 8.21] \( \square \)

6.5 The pairs \( E_8, D_8 \) and \( E_8, E_7A_1 \)

Say \( G \) is of type \( E_8 \) in characteristic \( p > 2 \) and \( H \) is the fixed point group of an involution. There are two cases, up to conjugacy. One may have \( H \) of type \( D_8 \) or one may have \( H \) of type \( E_7A_1 \).

In either case we wish to show that \( G, H \) is a Donkin pair. In other words, we want that all dominant weights are in \( \mathcal{S} \). We will argue by induction along the partial order. Thus when trying to prove that \( \lambda \in \mathcal{S} \), we shall always assume that \( \mu \in \mathcal{S} \) for \( \mu < \lambda \). Of course the zero weight is in \( \mathcal{S} \), so say \( \lambda \) is nonzero. If \( \lambda \) is not a fundamental weight, write \( \lambda = \lambda_1 + \lambda_2 \) where \( \lambda_i \) are nonzero dominant weights. As \( \lambda_i < \lambda \), we may apply lemma 6.3 with \( M = \nabla_G(\lambda_1) \otimes \nabla_G(\lambda_2) \) to conclude that \( \lambda \in \mathcal{S} \).

Remain the fundamental weights. Observe that \( \varpi_8 < \varpi_1 < \varpi_7 < \varpi_2 < \varpi_6 < \varpi_3 < \varpi_5 < \varpi_4 \). But we will not discuss them in this exact order.

To see that \( \varpi_8 \in \mathcal{S} \) we compute the character of \( \text{res}^G_H \nabla_G(\varpi_8) \) with the program LiE [17], decompose this character in terms of Weyl characters, and use lemma 6.3 to see that \( \text{res}^G_H \nabla_G(\varpi_8) \) has a composition series whose factors are irreducible (dual) Weyl modules. Here we use a Java applet of Lauritzen for the Jantzen sum formula.

To see that \( \varpi_1 \in \mathcal{S} \) we may argue the same way if \( H \) is of type \( D_8 \). If \( H \) is of type \( E_7A_1 \), let \( K \) be the subgroup of type \( E_7 \) in \( H \), and \( F \) the
subgroup of type $A_1$. Then $G, K$ is a Donkin pair (Levi subgroup case), so we may consider a good filtration adapted to the partial order (on weights for $K$) of $\text{res}^G_K \nabla_G(\varpi_1)$. It is a filtration by $H$-modules. It suffices to show that its layers have good filtration as $H$-modules. Let $N$ be such a layer. Its character is the character of some $\nabla_H(\lambda_1, \lambda_2)$ where $\lambda_1$ is a dominant weight for $K$ and $\lambda_2$ is one for $F$. (From now on we do not mention the computer calculations that are needed to support such statements.) Moreover, $\nabla_F(\lambda_2)$ is irreducible, so that the $K \cap B$-socle of $N$ is an irreducible $F$-module. It follows that the natural map $N \to \nabla_H(\lambda_1, \lambda_2)$ is an isomorphism, and thus $\varpi_1 \in S$.

To see that $\varpi_2 \in S$, we apply lemma 6.3 with $M = \nabla_G(\varpi_8) \otimes \nabla_G(\varpi_8)$. (Recall $\varpi_1, \varpi_8 < \varpi_2$, so that indeed $M \in \mathcal{M}$ by the inductive assumption.) One may find the necessary statement about linkage in Donkin’s book. (This is no accident, as we follow him in our choices.) We also checked the non-linkage with a straightforward Mathematica program.

To see that $\varpi_7 \in S$, we similarly use $M = \nabla_G(\varpi_3) \otimes \nabla_G(\varpi_3)$. To get $\varpi_4 \in S$, use $M = \nabla_G(\varpi_3) \otimes \nabla_G(\varpi_3)$ if $p = 3$, and $M = \wedge^3 \nabla_G(\varpi_8)$ if $p > 3$. To get $\varpi_6 \in S$, use $M = \nabla_G(\varpi_1) \otimes \nabla_G(\varpi_1)$ if $p = 3$, and $M = \wedge^4 \nabla_G(\varpi_8)$ if $p > 3$.

6.6 The pair $E_6, A_5 A_1$

Let $G$ be the simply connected group of type $E_6$ in characteristic $p > 2$ and let $H$ be the fixed point group of such an inner involution that $H$ is of type $A_5 A_1$ and the involution commutes with the graph automorphism. We wish to show again this is a Donkin pair. We argue as in the $E_8, E_7 A_1$ case.

If $\lambda = \varpi_1$ or $\varpi_2$, then we argue with socles as we did to show $\varpi_1 \in S$ for the $E_8, E_7 A_1$ pair.

To treat $\varpi_3$ we use $M = \nabla_G(\varpi_3) \wedge \nabla_G(\varpi_3)$. To get $\varpi_4 \in S$, use $M = \nabla_G(\varpi_2) \wedge \nabla_G(\varpi_2)$. The remaining two fundamental weights are in $S$ by symmetry.

6.7 The pair $E_6, C_4$

Let $G$ be the simply connected group of type $E_6$ in characteristic $p > 2$ and let $H$ be the fixed point group of such an outer involution that $H$ is of type $C_4$ and the involution commutes with the graph automorphism.
The module $\nabla_G(\varpi_2)$ is the Lie algebra of the adjoint form of $G$. Its restriction $\text{res}_H^G \nabla_G(\varpi_2)$ has a six dimensional weight space for the weight zero, just like $\nabla_G(\varpi_2)$ itself. We claim that it contains no nonzero invariant. Indeed we may choose the involution so that $X_{a_3} + X_{a_5}, X_{a_1} + X_{a_6}, X_{a_3+a_4} + X_{a_5+a_4}, X_{a_2}$ are in the Lie algebra of $H$, where we have put $X_{a_3+a_4} = [X_{a_3}, X_{a_4}]$ and $X_{a_5+a_4} = [X_{a_5}, X_{a_4}]$. The only element in the weight zero weight space of $\nabla_G(\varpi_2)$ that is annihilated by all these elements is the zero vector. Now $\text{res}_H^G \nabla_G(\varpi_2)$ contains an irreducible $\nabla_H(2\varpi_1)$ and the quotient by that submodule is either irreducible, or $p = 3$ and there are two composition factors, one of which is one dimensional. (This also uses the Jantzen sum formula.) As there is no invariant in $\text{res}_H^G \nabla_G(\varpi_2)$ and there is no extension between $\nabla_H(2\varpi_1)$ and the other composition factors ([4, II 4.13, 4.14]), we get $\varpi_2 \in S$.

As $\text{res}_H^G \nabla_G(\varpi_1)$ is irreducible, we also have $\varpi_1 \in S$. The rest goes as for the pair $E_6, A_5A_1$.

### 6.8 The pairs $E_7, A_7$ and $E_7, D_6A_1$

Say $G$ is simply connected of type $E7$ in characteristic $p > 2$ and $H$ is the fixed point group of an involution. There are two cases, up to conjugacy. One may have $H$ of type $A_7$ or one may have $H$ of type $D_6A_1$.

We argue as before. If $H$ is of type $D_6A_1$ we show that $\varpi_1, \varpi_2, \varpi_7 \in S$ by the argument with socles used to show $\varpi_1 \in S$ for the $E_8, E_7A_1$ pair. If $H$ is of type $A_7$ we have $\varpi_1, \varpi_7 \in S$ for the same reason, involving the sum formula, as why $\varpi_8 \in S$ for the $E_8, D_8$ pair. If $p = 7$ we see in the same manner that $\varpi_2 \in S$. If $p \neq 7$ use $M = \nabla_G(\varpi_1) \otimes \nabla_G(\varpi_7)$ to get $\varpi_2 \in S$. To get $\varpi_3 \in S$ use $M = \nabla_G(\varpi_1) \wedge \nabla_G(\varpi_1)$. To get $\varpi_4 \in S$ use $M = \nabla_G(\varpi_2) \wedge \nabla_G(\varpi_2)$. To get $\varpi_5 \in S$ use $M = \nabla_G(\varpi_1) \wedge \nabla_G(\varpi_2)$ otherwise. To get $\varpi_6 \in S$ use $M = \nabla_G(\varpi_7) \wedge \nabla_G(\varpi_7)$.

### 6.9 The pairs $F_4, B_4$ and $F_4, C_3A_1$

Say $G$ is of type $F_4$ in characteristic $p > 2$ and $H$ is the fixed point group of an involution. There are two cases, up to conjugacy. One may have $H$ of type $B_4$ or one may have $H$ of type $C_3A_1$.

We argue as before. If $H$ is of type $C_3A_1$ we show that $\varpi_1, \varpi_4 \in S$ by the argument with socles used to show $\varpi_1 \in S$ for the $E_8, E_7A_1$ pair. If $H$ is of type $B_4$ we have $\varpi_1, \varpi_4 \in S$ for the same reason, involving the
sum formula, as why \( \varpi_8 \in S \) for the \( E_8, D_8 \) pair. To get \( \varpi_3 \in S \) use \( M = \nabla_G(\varpi_4) \land \nabla_G(\varpi_4) \). To get \( \varpi_2 \in S \) use \( M = \nabla_G(\varpi_1) \land \nabla_G(\varpi_1) \).

**Theorem 6.10 (Brundan’s Conjecture [2])** Let \( G \) be semisimple simply connected. If either

(i) \( H \) is the centralizer of a graph automorphism of \( G \); or

(ii) \( H \) is the centralizer of an involution of \( G \) and the characteristic is at least three,

then \( G, H \) is a Donkin pair.

**Proof**

We have either a Levi subgroup case, first settled in [3] (see also remarks 3.5, 6.1), or a case treated in [2], or a case treated above, up to conjugacy. \( \square \)

**Remark 6.11** Of course we would much prefer a case-free proof, based on the pairing criterion say.

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