Noncommutative solitons of gravity

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Abstract
We investigate a three-dimensional gravitational theory on a noncommutative space which has a cosmological constant term only. We found various kinds of nontrivial solutions by applying a similar technique which was used to seek noncommutative solitons in noncommutative scalar field theories. Some of those solutions correspond to bubbles of spacetimes or represent dimensional reduction. The solution which interpolates $G_{\mu\nu} = 0$ and the Minkowski metric is also found. All solutions we obtained are non-perturbative in the noncommutative parameter $\theta$, therefore they are different from solutions found in other contexts of noncommutative theory of gravity and would have a close relation to quantum gravity.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
The construction of a consistent theory of spacetime at the Planck scale is one of the main issues in fundamental physics. There is an expectation that a noncommutativity among spacetime coordinates,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu},$$

(1.1)
emerges in such a scale. In fact, there are so many attempts at making this idea manifest, that is, to construct a consistent theory of gravity with a noncommutativity to be taken into account. For example, a noncommutative extension of the gauge theory of gravitation has been investigated [1, 2]. This formalism is based on gauging the noncommutative $SO(1, 4)$ de Sitter group [3] and using the Seiberg–Witten map [4] with subsequent contraction to the Poincaré group $ISO(1, 3)$. In that theory, corrections to cosmological and black hole solutions...
due to the noncommutativity have been found [5–7]. Applications of the Seiberg–Witten map to Chern–Simons theories have been carried out in [8, 9]. Utilizing the correspondence between three-dimensional Einstein gravity and three-dimensional Chern–Simons theory, the noncommutative gauge theory of gravitation is considered in [10–13]. Another approach to noncommutative spacetimes is considering noncommutative effects on gravitational sources [14–24]. The authors have found some solutions which solve the Einstein equation with gravitational sources of Gaussian type whose widths are related to the noncommutativity. This approach is directly connected to an expectation of smearing curvature singularities that appear in Einstein gravity. Also, the authors of [25, 26] proposed a theory of gravity on noncommutative spaces from the viewpoint of twisting the diffeomorphism. This theory has been extended to a theory which includes fermionic terms, i.e., a supergravity on noncommutative spaces [27, 28]. There are some trials to give classical solutions for those theories and actually a few solutions have been found [29–32]. Other approaches to noncommutative gravity can also be found in [33–35].

Although these approaches are different in the basic hypothesis, they are aiming to construct a consistent noncommutative gravitational theory by deforming the Einstein–Hilbert action by the noncommutative parameter $\theta$, that comes back to the ordinary Einstein gravity in the commutative limit $\theta \to 0$. Moreover, the solutions already found are also deformations of solutions for the ordinary Einstein gravity, namely, we have not had any nontrivial solutions particular to the gravitational theories on noncommutative spaces so far.

In this paper, we take a rather different approach to investigate the effect of a noncommutativity, by finding classical solutions that cannot be obtained by deformations of solutions of commutative theories but are non-perturbative in the noncommutativity $\theta$. To this end, we would like to work with a three-dimensional noncommutative gravitational theory that consists of a cosmological constant term only, that is to say, a noncommutative gravitational theory without the Ricci scalar. We adopt the first-order (vielbein) formalism, and the action of our theory reduces to just the three-dimensional determinant of the vielbein. One reason to work with this situation is that the cosmological constant term is made by the $\star$-multiplication only and that would be common for many approaches to noncommutative gravity, i.e., it is model independent.

This set-up is also motivated by the idea of [36], where some noncommutative solitons have been derived in noncommutative scalar field theories. The authors of [36] take a limit that the space noncommutativity is very large, which makes the kinetic term negligible compared with the potential term of that scalar field theory. Since all derivatives disappear, we naively expect that we cannot find nontrivial solutions, but this is not the case due to the noncommutativity. Actually they found some classically stable solutions called noncommutative solitons. Soon after [36], their theory was extended to that which includes the kinetic term, and by a solution-generating technique, the solutions which solve the equation of motion including the kinetic term have been explicitly constructed [37]. Our case, as the determinant is made of the $\star$-multiplication of vielbein, is analogous to the noncommutative $\phi^3$-theory investigated in [36], and we can apply a similar technique to find nontrivial solutions, namely, by switching to the operator formulation and using projection operators or their generalization. One of the purposes of this paper is to construct such noncommutative solitons of gravity.

By comparing to the noncommutative scalar solitons, our theory is regarded naturally as the situation where the scalar curvature can be negligible in comparison with the cosmological constant, but we will argue that there is another possibility in which the theory would be interpreted in a more radical way as the emergence of spacetime only from the cosmological constant term without the scalar curvature. In any case, the solutions we found suggest a
close connection to quantum gravity, where degenerate metrics play important roles. Such a
degenerate metric that satisfies det $E_\mu^\nu = 0$ or det $G_{\mu\nu} = 0$ represents a non-classical phase of
the theory and contributes to the path integral. In particular, the diffeomorphism invariant phase
$E_\mu^\nu = 0$ is considered an unbroken vacuum, while the metricity condition does not restrict the
spin connection $\omega_{\mu}^b$ to be the Christoffel symbol, which becomes a completely independent
variable. This implies that the first-order formalism using vielbein is not equivalent to the
ordinary second-order formalism using metrics. Another characteristic feature of quantum
gravity is that topology and signature-changing solutions are allowed [38]. The solutions
obtained in this paper share the same features as above.

The organization of this paper is as follows. In the following section, we give our action
and derive the equation of motion. In section 3, we will construct solutions for the equation of
motion by using projection operators. We first give examples to show typical structures of the
solutions (bubbles of spacetime, dimensional reduction). Then the most general solutions of
this class are presented. In section 4, we construct another class of solutions by using Gamma
matrices, which are more close to conventional spacetimes. The final section is devoted to
discussion and future directions.

2. The noncommutative gravity of cosmological constant

2.1. Action and equation of motion

Let us start with a three-dimensional noncommutative plane $\mathbb{R}^3$ with coordinates $x^\mu (\mu =
0, 1, 2)$ or $(t, x, y)$. The star product is defined for any functions on $\mathbb{R}^3$ as
\[
(f \ast g)(x) = \exp \left( \frac{i}{2} \theta_{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right) f(x) g(y) \bigg|_{y \to x},
\]
where $\theta_{\mu\nu}$ is a constant, anti-symmetric matrix which represents a noncommutativity. In this
paper, for simplicity, we introduce the noncommutativity purely in the spatial coordinates\(^3\)
\[
[x, y]_\ast = x \ast y - y \ast x = i \theta,
\]
by choosing $\theta^0 = 0$ ($i = 1, 2$) and $\theta^{12} \equiv \theta$.

We exploit the first-order formulation of a three-dimensional theory of gravity on a
noncommutative $\mathbb{R}^3$ which has a cosmological constant term only,
\[
S = -\frac{\Lambda}{\kappa^2} \int dt d^2 x \ E^\ast,
\]
where $\Lambda$ is a cosmological constant. Here $E^\ast$ is the $\ast$-determinant defined by
\[
E^\ast = \det E^\ast = \frac{1}{3!} \epsilon^{\mu\nu\rho} e_{abc} E_\mu^a \ast E_\nu^b \ast E_\rho^c,
\]
where $E_\mu^a(x)$ is a vielbein. We denote spacetime indices by $\mu, \nu, \rho$ and tangent space indices
by $a, b, c$. All indices run from 0 to 2. The metric is also defined through the star product in
a similar way [11, 25],
\[
G_{\mu\nu} = \frac{1}{2} \left( E_\mu^a \ast E_\nu^b + E_\nu^a \ast E_\mu^b \right) \eta_{ab},
\]
where $\eta_{ab}$ is an $SO(1, 2)$ invariant metric of the local Lorentz frame. We do not assume that
$E_\mu^a$ or $G_{\mu\nu}$ are invertible as 3 $\times$ 3 matrices, that is, we allow degenerate metrics. Through

\(^3\) In general, we can choose one of the coordinates which remains commutative by changing $\theta_{\mu\nu}$ to the Jordan form.
Time direction is usually chosen in order to avoid infinite time derivatives. However, for static classical solutions,
(2.1) reduces automatically to (2.2) only.
this paper, $G_{\mu\nu}$ is assumed to be real for simplicity. The solutions we discuss later will not contradict with this assumption, but complex metrics can also be treated in a similar manner.

Here, we would like to point out that there are two possibilities (a) and (b) to see this simple setting in a full gravitational theory on the noncommutative space: (a) the action given in (2.3) is a part of a full theory, that is, we need to add a noncommutative generalization of the Einstein–Hilbert term to (2.3). This is of course the common belief. In this case, our theory (2.3) is considered to be valid when the scalar curvature term is negligible compared with the cosmological constant. However, as opposed to the noncommutative scalar field theory, this is not achieved by taking the large noncommutativity limit $\theta \to \infty$ of a certain full theory\footnote{We recall the argument in [36]: by rescaling the coordinates $x \to x/\sqrt{\theta}$, $y \to y/\sqrt{\theta}$, all $\theta$ in the star product disappear, while any other derivatives (and also gauge fields) acquire $1/\sqrt{\theta}$. Then all the derivative terms become negligible in the large $\theta$ limit. However, as opposed to the scalar theory, the vielbein and other quantities in (2.6) are also transformed under the rescaling keeping the action invariant. Thus, the simple rescaling argument cannot be directly applied to our case. Note also that we should take a static or a slowly time-varying approximation as well, in order to drop time derivatives.}

\[ S = \frac{1}{2\kappa^2} \int dt \, d^3x \, E^\mu_0 \{ E^\nu_1, E^\rho_2 \} , \]  

where $R_\nu$ is a suitably defined scalar curvature, which may be model dependent.

On the other hand, we propose another possibility in this paper: (b) the action given in (2.3) is already a full theory. In this case, the metric and other quantities like a scalar curvature are considered to be composite quantities made from the vielbein. For a quantity in ordinary Einstein theory, we can define several different quantities in the noncommutative case. For example, another metric rather than (2.5) can be defined by $g_{\mu\nu} = E^a_\mu \cdot E^b_\nu \eta_{ab}$ using ordinary product\footnote{This is possible because a product $f \cdot g$ of two functions is also written by the $\star$-product [25]. To this end, first apply the bi-differential exponential operator inverse to that appears in (2.1) to $f$ and $g$, then take the $\star$-product.}.

We call the latter a 'commutative' metric in this paper, but do not confuse! It is just a quantity in the noncommutative theory. Both 'commutative' and 'noncommutative' quantities are used for capturing the spacetime structures given by a classical solution of the vielbein. In this paper, we will use two kinds of determinants $\det G$ and $\det G^\star$ of the metric (2.5) and 'commutative' scalar curvatures. In this way, we switch effectively from the first-order (vielbein) formalism to the second-order (metric) formalism without introducing a spin connection. We emphasize that the noncommutativity makes it possible. Such kind of prescription would have never appeared in the literature to our knowledge. This is motivated by the disagreement between the first and the second-order formalism in phases with degenerate metrics in quantum gravity. Of course this possibility itself should be justified, but we will see in this paper that the solutions in this interpretation possess interesting properties, suggesting a connection to the very notion of quantum gravity.

Now let us derive the equation of motion of our theory (2.3). We use the fact that the cyclic permutation of the star product is allowed in the integral,

\[
\int f \star g \star h = \int f (g \star h) = \int (g \star h) f = \int g \star h \star f ,
\]  

which comes from a property of the star product

\[
\int f \star g = \int g \star f = \int fg .
\]  

Taking this into account, the action (2.3) can be rewritten as

\[
S = -\lambda \int dt \, d^3x \, \epsilon_{abc} E^a_0 \{ E^b_1, E^c_2 \} ,
\]  

(2.9)
where $\lambda = \frac{\sqrt{2}}{\Lambda_{1}}$. Here, we used the star-anti-commutator defined by $\{ f, g \}_\star = f \star g + g \star f$. Varying the action (2.9) with respect to $E^\nu_a$ and using the cyclic symmetry of the star product, we have nine equations of motion for $\forall \mu$ and $\forall a$,

$$\epsilon^{\mu\nu\rho}\epsilon_{abc}\{ E^\nu_b, E^\rho_c \}_\star = 0.$$  \hspace{1cm} (2.10)

Clearly the action (2.9) will be zero if the vielbein solves (2.10), that is, all classical solutions give degenerate vielbein that satisfies $\text{det} E = 0$. Nevertheless, as we will explicitly show, there are in fact nontrivial solutions other than $\text{det} G = 0$. This is in contrast to the theory only with the cosmological constant term defined on a commutative space, where only $\text{det} G = 0$ is allowed due to the absence of the kinetic term. This is because the star product has an infinite number of derivatives in it, which act as an effective kinetic term.

### 2.2. Star product and operator formulation

In the following sections, we explicitly give solutions of equation (2.10). For simplicity, we will consider static or stationary solutions there. In order to find solutions, we exploit the recipe used in [36], i.e., the usage of the connection between the star product and the operator formulation, an analog of the Weyl–Wigner correspondence in quantum mechanics (see also [39] for a review). The vielbein $E^\mu_a(x, y)$ is a function on $\mathbb{R}^2$ if it is static. Recall that, given a (suitably defined) function $f(x, y)$ on $\mathbb{R}^2$, there is a map which uniquely assigns to it an operator $O_f(\hat{x}, \hat{y})$ that acts on the corresponding one-dimensional quantum mechanical Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ with $[\hat{x}, \hat{y}] = i\theta$. By choosing the Weyl ordering prescription, the Weyl map is given by

$$O_f(\hat{x}, \hat{y}) = \frac{1}{(2\pi)^2} \int d^2k \tilde{f}(k) e^{ik(x\hat{x} + y\hat{y})},$$  \hspace{1cm} (2.11)

where

$$\tilde{f}(k) = \int d^2x e^{-i(kx + ky)} f(x, y)$$  \hspace{1cm} (2.12)

is the Fourier transformation. Then the algebra of functions with the $\star$-multiplication is isomorphic to the operator algebra with relations

$$O_f \cdot O_g = O_{f \star g},$$  \hspace{1cm} (2.13)

$$\text{Tr} O_f = \int \frac{d^2x}{2\pi \theta} f.$$  \hspace{1cm} (2.14)

The creation and the annihilation operator are defined by

$$\hat{a} = \frac{\hat{x} + i\hat{y}}{\sqrt{2\theta}}, \quad \hat{a}^\dagger = \frac{\hat{x} - i\hat{y}}{\sqrt{2\theta}}.$$  \hspace{1cm} (2.15)

The Hilbert space $\mathcal{H}$ is now spanned by orthonormal basis $| n \rangle$ ($n = 0, 1, 2, \ldots$), which is the energy eigenstate of the one-dimensional harmonic oscillator given in (2.15). Thus, a general operator $O$ acting on $\mathcal{H}$ can be written as the linear combination of the matrix elements of the form

$$O = \sum_{i, j = 0}^{\infty} O^j_i | i \rangle \langle j |.$$  \hspace{1cm} (2.16)
In particular, the projection operator $|i⟩⟨i|$ will be important to construct solutions in the following sections. The function (symbol) $φ_i$ corresponding to the projection operator (that is $O_{φ_i} = |i⟩⟨i|$) can be expressed as [36, 39]

$$φ_i(x, y) = (-1)^i e^{-r^2/θ} L_i \left( \frac{2r^2}{θ} \right),$$  \hspace{1cm} (2.17)

where $L_i(x)$ is the $i$th Laguerre polynomial and $r^2 = x^2 + y^2$. By construction, $φ_i$ is the orthogonal projection

$$φ_i ⋆ φ_j = δ_{ij} φ_i$$  \hspace{1cm} (2.18)

and satisfies the completeness relation\(^6\)

$$\sum_{i=0}^{∞} φ_i = 1.$$  \hspace{1cm} (2.21)

In the following, we sometimes use a loose notation not to distinguish $O_f$ and $f$.

3. Noncommutative solitons by projection operators

In this section, we will give various static solutions of the equation of motion (2.10) using projection operators. We begin by simple solutions of two types and then move to more general solutions.

3.1. Diagonal solution

As a warm-up, let us first consider the case with a diagonal vielbein, namely, we take an ansatz

$$E^a_μ = \begin{pmatrix} E^0_0 & 0 & 0 \\ 0 & E^1_1 & 0 \\ 0 & 0 & E^2_2 \end{pmatrix}$$  \hspace{1cm} (3.1)

as a $3 \times 3$ matrix. In this case, the equation of motion (2.10) reduces to three equations

$$E^0_0 : 0 = \{ E^1_1, E^2_2 \} \ast,$$

$$E^1_1 : 0 = \{ E^2_2, E^0_0 \} \ast,$$

$$E^2_2 : 0 = \{ E^0_0, E^1_1 \} \ast.$$  \hspace{1cm} (3.2)

Therefore, if each component of the vielbein is given by a projection operator and they are orthogonal among them, then they solve (3.2). The simplest choice is

$$E^a_μ = \begin{pmatrix} α_0 φ_0 & 0 & 0 \\ 0 & α_1 φ_1 & 0 \\ 0 & 0 & α_2 φ_2 \end{pmatrix},$$  \hspace{1cm} (3.3)

where $α_0, α_1$ and $α_2$ are arbitrary complex numbers. Of course, any other choice of three different projection operators (say $φ_3, φ_16$ and $φ_51$, etc) is also a solution. More generally, arbitrary mutually orthogonal three groups of projection operators are allowed.

\(^6\) By using the generating function for the Laguerre polynomials

$$\sum_{i=0}^{∞} L_i(α) t^i = \frac{1}{(1 - t e^t)^{α}} \exp \left( -\frac{xt}{1 - t} \right),$$  \hspace{1cm} (2.19)

it is shown explicitly:

$$\sum_{i=0}^{∞} φ_i = 2 e^{-r^2/θ} \sum_{i=0}^{∞} (-1)^i L_i \left( \frac{2r^2}{θ} \right) = 1.$$  \hspace{1cm} (2.20)
Figure 1. The value of the Ricci scalar $R$ of spacetime (3.4). The left and right graphs are the $y = 0$ and the $x = 0$ section of $R$, respectively. Here, we set $\theta = 1$ and $\alpha_0 = \alpha_1 = 1/\sqrt{2}$ and $\alpha_2 = i/\sqrt{2}$ as an example.

This simple example already possesses some interesting features, as we will see below. In order to give insight into the solution (3.3), we apply the prescription announced in the previous section to this example. First, we can see that all solutions of this type give non-zero metric $G_{\mu\nu}$. In fact, (3.3) gives the following line element:

$$
\frac{dr^2}{G_{\mu\nu} dx^\mu dx^\nu} = \frac{1}{2} \left( E^a_\mu \ast E^b_\nu + E^b_\nu \ast E^a_\mu \right) \eta_{ab} dx^\mu dx^\nu
$$

$$
= -\alpha_0^2 \phi_0^2 dr^2 + \alpha_1^2 \phi_1^2 dx^2 + \alpha_2^2 \phi_2^2 dy^2
$$

$$
= -\alpha_0^2 \phi_0^2 dr^2 + \alpha_1^2 \phi_1^2 dx^2 + \alpha_2^2 \phi_2^2 dy^2
$$

$$
= 2 e^{-r^2/\theta} \left( -\alpha_0^2 L_0(2r^2/\theta) dr^2 - \alpha_1^2 L_1(2r^2/\theta) dx^2 + \alpha_2^2 L_2(2r^2/\theta) dy^2 \right)
$$

$$
= 2 e^{-r^2/\theta} \left( -\alpha_0^2 dr^2 - \alpha_1^2 \left( 1 - \frac{2r^2}{\theta} \right) dx^2 + \alpha_2^2 \left( 1 - \frac{4r^2}{\theta} + \frac{2r^4}{\theta^2} \right) dy^2 \right),
$$

(3.4)

where we used the property of the projection operators (2.18). We clearly see that the metric becomes singular when we take the commutative limit $\theta \to 0$. This means that this solution cannot exist if we start from the commutative theory. Furthermore, for finite $\theta$, other ‘commutative’ quantities defined from this metric (3.4) are now computable, because it is non-degenerate in the sense that it gives $\det G \neq 0$ except for some points. Of course, as mentioned in the previous section, this treatment itself should be justified.

By adopting the remark above, we can evaluate the Ricci scalar $R$ and the Kretschmann invariant $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ from (3.4) by the standard analysis. The results are shown in figures 1 and 2. The explicit forms of them are given in the appendix. All of them diverge at $r = \infty$ coming from the overall factor $e^{-r^2/\theta}$ appeared in (3.4) and also diverge at several values of $r$ which comes from the zero points of the Laguerre polynomials. As seen from these figures, the spacetime is divided into several radial regions by the walls of curvature singularities. The divergent points agree with those that satisfy $\det G = 0$. In each region, the Ricci scalar evaluated by the ordinary GR method is meaningful because $\det G \neq 0$. The result is not exactly but very close to 0, and moreover, is almost constant for finite $\theta$. Because $\theta$ is a free parameter, we can take a commutative limit $\theta \to 0$. Then we see that all of the walls shrink to $r = 0$, and the space measured by the metric concentrates on one point with a curvature singularity. Conversely, the metric at the finite $\theta$ can be viewed as a resolution of such a ‘one-point space’. This solution suggests that the bubbles of several spacetimes with small
cosmological constants would emerge as a fine structure of a single point. This fact might give a new direction for the cosmological constant problem.\(^7\)

### 3.2. Nondiagonal solutions and dimensional reduction

Next, let us slightly generalize the above and take a non-diagonal ansatz for the vielbein of the form

\[
\begin{pmatrix}
E_0^0 & 0 & 0 \\
0 & E_1^1 & E_1^2 \\
0 & E_2^1 & E_2^2
\end{pmatrix}.
\]  

(3.5)

The equation of motion (2.10) reduces to five equations

\[
0 = \{E_1^1, E_2^2\} - \{E_1^2, E_2^1\},
\]

(3.6)

\[
0 = \{E_0^0, E_\mu^\mu\} (\alpha, \mu = 1, 2).
\]

(3.7)

We will give solutions that represent effectively two-dimensional spacetime.

For example, we can easily find a solution which consists of the two projections \(\phi_0\) and \(\phi_1\) as

\[
E_v^b = \begin{pmatrix}
\alpha_0 \phi_0 & 0 & 0 \\
0 & \alpha_1 \phi_1 & \alpha_1 \phi_1 \\
0 & \alpha_1 \phi_1 & \alpha_1 \phi_1
\end{pmatrix},
\]  

(3.8)

where \(\alpha_0\) and \(\alpha_1\) are arbitrary constants as before. This implies the metric\(^8\)

\[
dx^2 = -\alpha_0^2 \phi_0^2 \, dt^2 + 2 \alpha_1^2 \phi_1 (dx^2 + 2 \, dx \, dy + dy^2)
\]

\[
= 2 \, e^{-r^2/\theta} \left( -\alpha_0^2 \, dr^2 - 2 \alpha_1^2 \left( 1 - \frac{2r^2}{\theta} \right) (dx + dy)^2 \right).
\]  

(3.9)

(3.10)

---

\(^7\) Note also the signature of the metric. Due to the nature of the Laguerre polynomials, the sign of each component of the metric oscillates as \(r\) increases. This is not surprising because such a sign-changing solution is also typical in the black hole spacetime, where in the interior of the event horizon \(dt^2\) becomes spacelike while \(dr^2\) becomes timelike. The signs of the coefficients of \(dr^2\) and \(dx^2\) change independently in our solution.

\(^8\) We assume \(dx \, dy = dy \, dx\).
As seen in the second term of the metric, the line element effectively consists of $dt$ and $dx + dy$. In other words, the metrical dimension of this metric is 2. The disagreement between the naive (manifold) dimension and the metrical dimension would be a sign of quantum gravity again [38]. In particular, it would be interesting to compare it with the results obtained in the analysis by causal dynamical triangulation [40–43] or spontaneous dimensional reduction in short-distance quantum gravity [44].

A similar solution only with a single projection operator $\phi_0$ is obtained from the above solution by replacing $\phi_0 \rightarrow 1 - \phi_0$ and $\phi_1 \rightarrow \phi_0$. Its metric is given by

$$ds^2 = -(1 - \phi_0) dt^2 + 2\phi_0(dx^2 + 2 dx dy + dy^2)$$

(3.11)

$$= -(1 - 2 e^{-r^2/\theta}) dt^2 + 4 e^{-r^2/\theta} (dx + dy)^2.$$  

(3.12)

This is again an effectively two-dimensional metric.

On the other hand, effectively one-dimensional solutions are obtained in the most general ansatz and the corresponding equation of motion (2.10). For example, by using a single projection operator $\phi_0$, the vielbein

$$E^b_v = \begin{pmatrix} \phi_0 & \phi_0 & \phi_0 \\ \phi_0 & \phi_0 & \phi_0 \\ \phi_0 & \phi_0 & \phi_0 \end{pmatrix}$$

(3.13)

solves equation (2.10). The line element of this solution,

$$ds^2 = \phi_0(dt + dx + dy)^2$$

(3.14)

$$= 2 e^{-r^2/\theta} (dt + dx + dy)^2,$$

(3.15)

shows that the metric effectively reduces to a one-dimensional metric. The disagreement between the naive dimension and the metrical dimension appears again. Clearly this happens because of the degeneracy of the vielbein. In other words, the rank or the invertibility of the vielbein determines whether such a dimensional reduction occurs or not. We discuss this point again in the following section.

### 3.3. General solutions by projection operators

The structure of the dimensional reduction in the above examples suggests a systematic construction of solutions. In general, each component of vielbein is a function on noncommutative $\mathbb{R}^2$ (we refer to time-independent metrics only) and is written as an operator acting on the Hilbert space of a harmonic oscillator. Therefore, the most general expression of the vielbein is written as

$$E^a_\mu = \sum_{i,j=0}^{\infty} (C^a_\mu)^i_j |i\rangle\langle j|,$$

(3.16)

where $(C^a_\mu)_j^i$ is a (complex) number. Now a (star) product of two components is written by using $\langle j|k\rangle = \delta^j_k$ as a matrix multiplication for $i$, $j$:

$$E^a_\mu \star E^b_\nu = \sum_{i,j=0}^{\infty} (C^a_\mu C^b_\nu)^i_j |i\rangle\langle j|.$$  

(3.17)

Thus, the metric is given by using the anti-commutator as

$$G_{\mu\nu} = \frac{1}{2} \eta_{ab} \sum_{i,j=0}^{\infty} [C^a_\mu, C^b_\nu]^i_j |i\rangle\langle j|.$$  

(3.18)
Similarly, the determinant (for $\mu$ and $a$)
\[
\det (E^a_\mu) = \sum_{i,j=0}^{\infty} [\det (C^a_\mu)]^i_j |i\rangle \langle j|
\]
(3.19)
reduces to the determinant of the matrix $C^a_\mu$. Correspondingly, the equation of motion (2.10) reduces to the following constraint for $(C^a_\mu)^i_j$:
\[
\epsilon^{\mu\nu\rho} \epsilon_{\alpha\beta\gamma} (C^b_\nu C^c_\rho)_{ij} = 0.
\]
(3.20)

As a particular situation, let us assume the diagonality in $i, j$, that is, each vielbein is written in the linear combination of the projection operators as
\[
E^a_\nu = \sum_{j=0}^{\infty} C(j)^a_\nu \phi_j, \quad C(j)^a_\nu \equiv (C^a_\mu)^i_j.
\]
(3.21)

Then, (3.20) becomes
\[
\epsilon^{\mu\nu\rho} \epsilon_{\alpha\beta\gamma} C(j)^b_\nu C(j)^c_\rho = 0,
\]
(3.22)
for an arbitrary $j$ (no summation). For a fixed $j$, this is an ordinary (commutative) matrix equation and $C(j)^a_\nu$ is seen as a $3 \times 3$ matrix for $\nu$ and $b$,\(^9\)
\[
C(j) = \begin{pmatrix}
C(j)^0_0 & C(j)^0_1 & C(j)^0_2 \\
C(j)^1_0 & C(j)^1_1 & C(j)^1_2 \\
C(j)^2_0 & C(j)^2_1 & C(j)^2_2 
\end{pmatrix}.
\]
(3.23)

Then this constraint simply shows that all minors (the determinants of cofactor matrices) of each matrix element $C(j)^a_\nu$ should be zero. The most general form of such a matrix is given by
\[
C(j) = \begin{pmatrix}
\alpha_j \\
\beta_j \\
\gamma_j
\end{pmatrix}
\begin{pmatrix}
s_j & t_j & u_j
\end{pmatrix}
= \begin{pmatrix}
\alpha_j s_j & \alpha_j t_j & \alpha_j u_j \\
\beta_j s_j & \beta_j t_j & \beta_j u_j \\
\gamma_j s_j & \gamma_j t_j & \gamma_j u_j
\end{pmatrix},
\]
(3.24)
where $\alpha_j, \beta_j, \gamma_j, s_j, t_j$ and $u_j$ are arbitrary constants. This means that $C(j)$ is a matrix whose rank is 1, parametrized by $C^6$. Here, the remarkable fact is that any linear combination (3.21), with each $C(j)$ given by (3.24), is also a solution due to the orthogonality of $\phi_j$’s. Therefore, we can in fact generate an infinite number of classical solutions easily by assigning a set of degenerate matrices $\{C(j)\}_{j \in \mathbb{Z}}$. We conclude that the most general solution of the vielbein and the corresponding metric written by the projection operators are as follows:
\[
E^a_\mu = \begin{pmatrix}
E^0_0 & E^1_0 & E^2_0 \\
E^0_1 & E^1_1 & E^2_1 \\
E^0_2 & E^1_2 & E^2_2
\end{pmatrix}
= \sum_{j=0}^{\infty} \left(\begin{array}{ccc}
\alpha_j & \alpha_j t_j & \alpha_j u_j \\
\beta_j & \beta_j t_j & \beta_j u_j \\
\gamma_j & \gamma_j t_j & \gamma_j u_j
\end{array}\right) \phi_j,
\]
(3.25)
\[
G_{\mu\nu} = \begin{pmatrix}
G_{00} & G_{01} & G_{02} \\
G_{10} & G_{11} & G_{12} \\
G_{20} & G_{21} & G_{22}
\end{pmatrix}
= \sum_{j=0}^{\infty} \left(-\alpha_j^2 + \beta_j^2 + \gamma_j^2\right) \begin{pmatrix}
s_j^2 & s_j t_j & s_j u_j \\
t_j^2 & t_j^2 & t_j u_j \\
u_j^2 & u_j^2 & u_j^2
\end{pmatrix} \phi_j.
\]
(3.26)

It is immediately shown that any metric of the form (3.26) satisfies $\det G = 0$.\(^9\) It is also equivalent to the equation of motion for the vielbein $e_\nu^a$ in the commutative theory that has only one matrix $C$.\(^{10}\)
Of course, all the solutions obtained so far are characterized in this way. In fact, the first example giving (3.4) is characterized by the following three degenerate matrices:

\[
C(0) = \begin{pmatrix} \alpha_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}.
\] (3.27)

As we mentioned before, the fact that some solutions have the discrepancy between the dimension of the manifolds and that of the metrics can be explained by the degeneracy of these matrices. The examples (3.8) and (3.13) given in the previous section are characterized by

\[
C(0) = \begin{pmatrix} \alpha_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_1 & \alpha_1 \\ 0 & \alpha_1 & \alpha_1 \end{pmatrix},
\] (3.28)

and

\[
C(0) = \begin{pmatrix} \alpha_0 & \alpha_0 & \alpha_0 \\ \alpha_0 & \alpha_0 & \alpha_0 \\ \alpha_0 & \alpha_0 & \alpha_0 \end{pmatrix}, \quad C(1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_1 & \alpha_1 \\ 0 & \alpha_1 & \alpha_1 \end{pmatrix},
\] (3.29)

respectively. Since each matrix \(C(j)\) carries rank 1, the sum of two such terms in the former gives effective two dimension, while the latter gives one dimension. In other words, we need at least three non-zero matrices \(C(j)\) in order to construct a three-dimensional solution as (3.27). As noted above, the commutative theory corresponds to a single matrix \(C\). Along the argument here, it is clear that the metric in the commutative theory is at most one dimensional.

In summary, even for restricting the diagonal (projection) operators in \(i, j\), we have found infinitely many solutions characterized by the infinite set of degenerate matrices \(C(j)\). Dividing by the symmetry, we would obtain the vacuum moduli space of the theory in this diagonal sector.

We close this section by a remark. Although we consider the three-dimensional theory in this paper, the extension to the \((2n + 1)\)-dimensional theory is straightforward. Then the construction of the solutions in this section is also applied to the higher dimensional case. To be more precise, the vielbein is represented as operators on the \(n\)-dimensional harmonic oscillator basis \(|j_1, j_2, \ldots, j_n\rangle\). The eom \(\epsilon^{\mu_1 \cdots \mu_n} \epsilon_{\mu_1 \cdots \mu_n} E_{j_1}^{a_1} \cdots \cdots E_{j_n}^{a_n} = 0\) is solved in the same way as (3.25) but now the sum is over any projection operators \(\phi_{j_1, \ldots, j_n} = |j_1, j_2, \ldots, j_n\rangle \langle j_1, j_2, \ldots, j_n|\), because each matrix \(C(j_1, j_2, \ldots, j_n)\) is independently solved similarly to (3.24). The structure of the dimensional reduction is also same, which means that the invertibility of the vielbein might be a key to the mechanism of compactification of higher dimensional theories.

4. Noncommutative solitons by clifford algebras

In this section, we will give another class of solutions represented by various dimensional Clifford algebras. Here all solutions are proportional to the Minkowski metric and satisfy \(\det G \neq 0\), as opposed to the solutions in section 3.

4.1. First solution

Let us come back to the ansatz (3.1) for the vielbein. In the previous section, we found solutions using the projection operators, which correspond to the diagonal matrix elements \(|i\rangle \langle i|\) in the harmonic oscillator basis. However, because the equation of motion (3.2) shows
that the vielbein should be mutually anti-commuting, the vielbein obeying the Clifford algebra relation solves (3.2). Such a solution is generally represented by a non-diagonal matrix element $|i⟩⟨j|$ in that basis.

To be more precise, let us for example focus on the indices $i = 0, 1$ and define the $SO(3)$ gamma matrices (Pauli matrices) as

$$
\begin{align*}
\gamma^0 &= \sigma^3 = |0⟩⟨0| - |1⟩⟨1|, \\
\gamma^1 &= \sigma^1 = |1⟩⟨0| + |0⟩⟨1|, \\
\gamma^2 &= \sigma^2 = i|1⟩⟨0| - i|0⟩⟨1|.
\end{align*}
$$

They satisfy the Clifford algebra relation $[\gamma^\mu, \gamma^\nu] = 2\delta^{\mu\nu} I_2$. Here, $I_2 = |0⟩⟨0| + |1⟩⟨1|$ is a unit matrix in the two-dimensional subspace spanned by $|0⟩$ and $|1⟩$, which is equivalent to the projection operator $\phi_0 + \phi_1$ in the full Hilbert space. Then the vielbein of the form

$$
E^a_\mu = \begin{pmatrix}
\gamma^0 & 0 & 0 \\
0 & \gamma^1 & 0 \\
0 & 0 & \gamma^2
\end{pmatrix}
$$

is evidently a solution for (3.2). The metric for this vielbein is

$$
G_{\mu\nu} = \eta_{\mu\nu} (|0⟩⟨0| + |1⟩⟨1|)
= \eta_{\mu\nu} (\phi_0 + \phi_1)
= \frac{4r^2}{\theta} e^{-r^2/\theta} \eta_{\mu\nu}.
$$

In the last line, we rewrote $\phi_0$ and $\phi_1$ in terms of the Laguerre polynomials.

Remarkably, this metric is proportional to the three-dimensional Minkowski metric, so that it is natural to regard this solution as a soliton that interpolates two vacua $G_{\mu\nu} = 0$ and $G_{\mu\nu} = \eta_{\mu\nu}$. The overall factor of the projection operators means that the (noncommutative) Minkowski space exists only in the region where $\phi_0 + \phi_1$ has non-zero support in analogy with the interpretation of the noncommutative scalar solitons: on the noncommutative plane, each projection $\phi_i$ shares a region with a minimal area $2\pi \theta$, which is determined by the uncertainty relation. It is indeed seen by noting $\det G = (\det \eta) (\phi_0 + \phi_1)^2 = - (\phi_0 + \phi_1)$ and $\text{Tr} (\phi_0 + \phi_1) = 2$. This implies that an effective cosmological constant term defined by $\det G$ (it is a composite quantity different from our action) is given by

$$
S_{\text{eff}} = -\lambda \int dr \, d^2 x \sqrt{-\det G} = -2\pi \theta \lambda \int dr \text{Tr} (\phi_0 + \phi_1) = -4\pi \theta \lambda \int dr,
$$

which means the finite volume $4\pi \theta$ in the spatial direction. Because now $\det G \neq 0$, it is in principle possible to compute the noncommutative scalar curvature $R_s$, to capture the structure further. But it needs a proper definition of $R_s$, of course, and we will not perform it in this paper.

Nevertheless, the same qualitative feature can be observed in analyzing ‘commutative’ quantities. In this treatment, the support of $\phi_0 + \phi_1$ (non-degenerate region) as a function is $0 < r < \infty$. However, note that $r$ is the radial coordinate in the isotropic coordinates

$$
d\ell^2 = \frac{4r^2}{\theta} e^{-r^2/\theta} (-dr^2 + dr^2 + r^2 d\phi^2),
$$

which differs from the physical radial coordinate usually defined by $R = (2r^2/\sqrt{\theta}) e^{-r^2/2\theta}$. Both $r = 0$ and $r = \infty$ correspond to $R = 0$. This implies that the physical distance between $r = 0$ and $r = \infty$ is finite. It is consistent with the finite spatial integral in (4.5). Keeping this
in mind, we now check invariant scalars $R$ and $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ for this metric, and the results are given as (see also figure 3)

\begin{align}
R &= -\frac{e^{r^2/\theta}}{2r^4\theta}(\theta^2 - 6r^2\theta + r^4), \\
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} &= \frac{e^{2r^2}}{4r^8\theta^2}(5\theta^4 - 10r^2\theta^3 + 18r^4\theta^2 - 6r^6\theta + r^8).
\end{align}

All of them diverge both at $r = 0$ and $r = \infty$, where $\det G(r) = 0$. From figure 3, we see that the ‘interior’ region is actually $1.0 < r/\sqrt{\theta} < 1.5$, and in this region the spacetime seems to be a ‘warped’ Minkowski space in the sense that the scalar curvature behaves as if it were almost constant and the metric has an overall scaling factor $(4r^2/\theta) e^{-r^2/\theta}$. These analyses indicate that this spacetime is seen as a small bubble of ordinary space surrounded by the empty space (nothing state). The size of this bubble is approximately $\sqrt{\theta}$.

### 4.2. Generalizations

The above solution can be immediately generalized in two ways.

First, we can change the choice of the two indices from $i = 0, 1$ to any other pair, because it is not important to construct a solution. In particular, this generalization is related to the solution-generating technique [37]. Note that if $E_\mu^u$ is a solution then $S E_\mu^u S^\dagger$ is also a solution\(^{10}\). Here $S$ is a shift operator defined by

\begin{equation}
S = \sum_{i=0}^{\infty} |i+1\rangle \langle i|
\end{equation}

and satisfies $S^\dagger S = 1$ but $SS^\dagger = 1 - \phi_0$. Thus, the metric

\begin{align}
G_{\mu\nu} &= \eta_{\mu\nu} S (|0\rangle\langle 0| + |1\rangle\langle 1|) S^\dagger \\
&= \eta_{\mu\nu} S (|1\rangle\langle 1| + |2\rangle\langle 2|) S^\dagger \\
&= \eta_{\mu\nu} (\phi_1 + \phi_2)
\end{align}

is also a solution. The choice of the two indices does not affect the size of the ‘bubble’ on the noncommutative space because $\text{Tr} (\phi_1 + \phi_2) = 2$ is the same as above.

\(^{10}\) Note that the unitary transformation $E_\mu^u \rightarrow U E_\mu^u U^{-1}$ is a symmetry of the cosmological action.
Next, generalization is to enlarge the size of gamma matrices. To do this, choose the index $i = 0, 1, \ldots, q$, where $q = 2^{[d/2]} - 1$ and define $SO(d)$ gamma matrices $\gamma^0, \ldots, \gamma^d$ in the harmonic oscillator space as above. Then by selecting three of them, say, $E_0^0 = \gamma^0, E_1^1 = \gamma^1, E_2^2 = \gamma^2,$ (4.11) they also solve the equation of motion. Because now the size of gamma matrices is $2^{[d/2]}$, the corresponding metric is proportional to the rank $2^{[d/2]}$ projection operators as $G_{\mu\nu} = \eta_{\mu\nu} (|0\rangle\langle 0| + \cdots + |q\rangle\langle q|)$ (4.12)

The volume of the support which will contribute to the effective cosmological constant term is $\text{Tr} (\phi_0 + \cdots + \phi_q) = q + 1$ times larger than previous solutions, as expected. In particular, by taking a large matrix-size limit $q \to \infty$, we find that (4.12) actually reduces to the Minkowski metric because of the completeness relation $\sum_{i=0}^{\infty} \phi_i = 1$ (2.21). The derived second-order cosmological constant term in this limit

$$S = -2\lambda \int dt \, d^2x$$

is in fact divergent. It is surprising that the Minkowski spacetime can emerge only from the cosmological constant term. And it is rather confusing that the Minkowski metric carries the divergent cosmological constant, because that spacetime is a classical vacuum for the vanishing cosmological constant in the ordinary sense. The point is that we see the spacetime from the nothing $G_{\mu\nu} = 0$ as the ground state, where the Minkowski space has infinite volume, while in the ordinary Einstein equation it is implicitly assumed that the Minkowski space is a ground state. Therefore it is not a contradiction. In summary, we found a sequence of solutions that interpolates $G_{\mu\nu} = 0$ ($q = 0$) and the Minkowski space ($q \to \infty$).

Another interesting application is to choose the index now starting from 1, i.e., $i = 1, \ldots, q$ with $q = 2^{[d/2]}$ and to take the $q \to \infty$ limit. Then $E_0^0 = \gamma^0, E_1^1 = \gamma^1, E_2^2 = \gamma^2$ in this basis define a solution as above but now the metric becomes

$$G_{\mu\nu} = \eta_{\mu\nu} (|1\rangle\langle 1| + \cdots + |q\rangle\langle q|)$$

$$= \eta_{\mu\nu} (\phi_1 + \cdots + \phi_q)$$

$$\xrightarrow{q \to \infty} \eta_{\mu\nu} (1 - \phi_0).$$

(4.14)

Thus, the metric approaches $G_{\mu\nu} = (1 - 2e^{-r^2/\theta})\eta_{\mu\nu}$. More generally, by choosing the index $i = k, \ldots, \infty$ for some $k$, we have a class of solutions

$$G_{\mu\nu} \xrightarrow{q \to \infty} \eta_{\mu\nu} (1 - \phi_0 - \cdots - \phi_k).$$

(4.15)

As opposed to the solutions above, the metric (4.14) has a support for all over the spacetime except for that of $\phi_0$. Then as seen from the ‘noncommutative’ determinant, this spacetime is seen as a ‘hole’ of minimal size $\sim \sqrt{\theta}$ in the Minkowski space. Similarly, the metric (4.15) has a hole of radius $k\sqrt{\theta}$ in the Minkowski spacetime.

It is also seen from the analysis of ‘commutative’ quantities. Indeed, a ‘hole’ for the metric (4.14) is roughly seen by its ‘step function’ profile (see figure 4) jumping at $r = \sqrt{(\ln 2)\theta} \sim 0.833 \sqrt{\theta}$, which is the zero point of $\det G$. Checking now the invariant scalars of this spacetime (4.14), we find the concrete forms of them as

$$R = -\frac{8 \, e^{r^2/\theta}}{(e^{r^2/\theta} - 2)^3 \theta^2} \{ (1 - 2 \, e^{-r^2/\theta}) r^2 + 2 (-2 + e^{r^2/\theta}) \theta \}$$

(4.16)
Figure 4. The profile of $1 - 2e^{-r^2/\theta}$. Here we set $\theta = 1$.

![Ricci scalar and Kretschmann invariant plots](image)

Figure 5. The Ricci scalar (left) and the Kretschmann invariant (right) of spacetime \((4.14)\) for $\theta = 1$.

\[
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{32 e^{2r^2/\theta}}{(e^{r^2/\theta} - 2)r^4} \times \left[ (2 + 4e^{2r^2/\theta})r^4 - 2(2 - 7e^{r^2/\theta} + 3e^{2r^2/\theta})r^2\theta + 3(-2 + e^{r^2/\theta})\theta^2 \right] \tag{4.17}
\]

Both of them are finite at the origin and diverge at $r \sim 0.833\sqrt{\theta}$. As shown in figure 5, the spacetime are divided into two regions by a wall where the curvature diverges. Therefore, two bubbles of the universes of different curvatures seem to be glued at the curvature wall. The outer region has an almost zero scalar curvature so that it is expected to be the Minkowski spacetime outside the ‘hole’, in order to be consistent with the ‘noncommutative’ quantities. On the other hand, the interior region has a negative scalar curvature. This naively indicates the AdS spacetime, which is not expected from the noncommutative viewpoint. However, we would not like to be serious about the precise value of the scalar curvature in this ‘commutative’ evaluation.

A remarkable feature of this class of solutions \((4.12), (4.14)\) and \((4.15)\) is that they do not shrink to a point in the commutative limit. In the limit, all the projection operators $\phi_i$ reduce to sharp, delta-function-type distributions, and all the degenerate points with $\det G = 0$ and $\det G = 0$ will concentrate on the origin. Then the metrics \((4.14)\) and \((4.15)\) approach the Minkowski spacetime except for the origin, which is the point-like curvature singularity\(^{11}\).

\(^{11}\) On the other hand, the metric \((4.12)\) approaches spacetime with a single point around the nothing.
Indeed, it is easily shown that $R \to 0$ for $r \neq 0$ and $R \sim 16/\theta \to \infty$ for $r = 0$. It is again interesting to see this as the resolution of singularities. When the spacetime is highly curved at a point in the second-order formulation of gravity, one would expect strong effects of quantum gravity to appear at this point. In our case, it is simply the degenerate point of the metric, where the second-order formulation becomes meaningless but they are still well defined in the first-order noncommutative formulation. This scenario is analogous to the stringy resolution of singularity in the instanton moduli space, where the small instanton singularity is not an end of the moduli space actually and is connected to other branches of the vacua. Moreover, each singularity carries a kind of index ($k$ in (4.15)), which has a definite meaning in the noncommutative space as the size of the singularity. This reminds us of the black hole microstates.

5. Conclusions and discussions

In this paper, we investigated the three-dimensional gravitational theory on the noncommutative space. We considered the setting where the action has the cosmological constant term only. Although the action has no kinetic term, we found infinitely many nontrivial classical solutions owing to the noncommutativity. In order to construct the solutions, we applied the recipe developed in [36], i.e., the usage of the connection between the star product and the operator formulation.

To understand the solutions, we proposed a new point of view for the cosmological constant term, that is, the action we gave here is already a full theory without introducing the scalar curvature term. When we adopt this idea, the metric, the Ricci scalar and other physical quantities can be constructed after the vielbein is obtained by solving the equation of motion (2.10). In other words, we switch the second-order formalism effectively. In this case, the vielbein which solves (2.10) can be regarded as a ‘meta’ spacetime or a seed vielbein that can work as a source for the commutative Ricci scalar $R$ or the noncommutative one $R^\star$. We would like to emphasize that this point of view has never appeared. One of the reasons for that is that on commutative spaces, a cosmological constant term itself cannot give a nontrivial solution, but it needs a kinetic term.

Let us now summarize the solutions that are classified into two classes. The solutions of the first class are constructed by using the projection operators. We constructed general solutions of this class. All of them satisfy $\det G = 0$ but $\det G \neq 0$, so we calculated commutative scalar curvatures produced by the metrics based on the solutions of the vielbein. We found that the spacetimes are divided into several regions by the walls of the curvature singularities where $\det G$ becomes zero. In that sense, they have structures of the bubbles of spacetimes with various cosmological constants. Another feature of this class is that they indicate dimensional reduction, that is, there are some solutions which are effectively one or two dimensional because of the degeneracy $\det G = 0$. In the context of quantum gravity, the possibility of dimensional reduction has been intensively discussed [38, 41], or in the other gravitational theory, a similar issue has been reported [45]. It would be interesting to investigate the relation of our theory to them.

The solutions of the second class are constructed by applying the Clifford algebra and the gamma matrices. They are noncommutative solitons interpolating $G_{\mu \nu} = 0$ and $G_{\mu \nu} = \eta_{\mu \nu}$. They satisfy $\det G \neq 0$ and $\det G \neq 0$, so both noncommutative and commutative quantities can be derived from the vielbein. This analysis indicates that the solutions are regarded as either a bubble of ordinary spacetime around the nothing $G_{\mu \nu} = 0$ or a hole (bubble of nothing) in the Minkowski space, where their regions with different scalar curvatures are partitioned by the wall of the curvature singularity. Interestingly, the Minkowski metric is included in
this class of solutions, in which the curvature singularities are absent in the large size limit of the gamma matrices. We also argued the possible mechanism for the resolution of point-like curvature singularities in the commutative limit.

Thus, we found a lot of nontrivial solutions which can be expected to have effects of quantum gravity, but there are many open questions to be investigated. We would like to note again that they depend on the two possible interpretations of the model discussed in section 2.

The first possibility is to regard the action we have used in this paper as a part of a full theory. In other words, we need to add a (noncommutative) spin connection to our theory. Along this interpretation, the solutions in this paper would not exact solutions in the full theory. However, they should be valid in a certain limit, where the spin connection term is negligible compared with the cosmological constant term. It is interesting if the existence of our solutions would restrict possible noncommutative extensions of the first-order formulation of gravity. Note that for noncommutative scalar field theories, the solutions obtained in the large noncommutativity limit can be extended to the so-called exact noncommutative solutions in the full theory with kinetic term by adding noncommutative gauge fields. The spin connection would play a similar role as gauge fields. It would also be useful to focus on the symmetry of our solutions for that purpose. We refer that the $E_\mu = 0$ solution preserves the full (twisted) diffeomorphism, while the Minkowski metric preserves the twisted Poincaré symmetry. What is the corresponding twisted symmetry in our case? Because of the static, rotational symmetric ansatz, a naive guess is the twisted version of $\mathbb{R} \times U(1) \times SO(2)$.

Looking at our model from the observational point of view is very interesting as well. Concerning it, we note that there is an argument that $G_{\mu
u} = 0$ is an origin of the dark matter [46]. Here, the $E_\mu = 0$ does not constrain the spin connection and thus in the equation of motion for the fluctuation there is an extra integration constant, which behaves as the dark matter. If such a possibility would be applicable to our solutions as well, we might be able to see noncommutative effect by cosmological observations. In that sense, we need more ‘realistic’ solutions, e.g., a four-dimensional and time-dependent solution. The application of our model to black holes on noncommutative spaces is also an interesting direction. In the commutative limit $\theta \to 0$ of (4.14), there appears a sharp, delta-function-like singularity at the origin which behaves as a point-like source. There are black hole solutions on the noncommutative space with that kind of source term (smeared by the noncommutativity) [14, 17–20]. It is interesting to investigate the relation to our solutions. In the weakly noncommutative case, the $\theta$-expansion works so that we can approximately use the ordinary Einstein–Hilbert action. Note that for any finite $k$ the metric (4.15) also represents a point-like source but now with $k$ internal degrees of freedom. There might be a relation to black hole microstates.

On the other hand, when we regard our theory as a full theory, the most important issue is to show the validity of this approach, in other words, to show the relation to the second-order formalism without spin connection. Concerning this, we remind that there is already a similar situation in string field theory and in the context of quantum gravity [38]: there exists the solution which satisfies $\Phi_0 \ast \Phi_0 = 0$ of the pre-geometrical action $S \sim \int \Phi \ast \Phi \ast \Phi$ defines a BRST charge as $\Phi_0$ and the fluctuation theory becomes Witten’s SFT. This seems to be a very interesting scenario, if there is an analogous mechanism for our solutions to emerge gravity starting from the cosmological constant only.

This is the first paper that suggests the emergence of ‘meta’ spacetimes only from a cosmological constant and noncommutativity. Besides the ordinary expectation that the noncommutativity becomes important at the Planck scale, our model may suggest a more radical scenario that the noncommutativity would also be crucial for spacetimes even at a large scale. In this respect, this scenario gives also a new direction about the cosmological constant.
problem, that is, the cosmological constant is necessary for spacetimes to emerge. Both fundamental and phenomenological questions on this model have to be investigated further.

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Appendix. The explicit forms of the Ricci scalar and the Kretschmann invariant for the metric (3.4)

We give the explicit forms of the Ricci scalar and the Kretschmann invariant for the metric (3.4). They are given by

$$
R = 2 e^{r^2} (8 x^{12} + x^{10} (40 y^2 - 680 \theta + 8 x^8 (10 y^4 - 37 y^2 \theta + 24 \theta^2) \\
+ 4 x^6 (20 y^6 - 126 y^4 \theta + 172 y^2 \theta^2 - 65 \theta^4) \\
+ 2 x^4 (20 y^8 - 208 y^6 \theta + 456 y^4 \theta^2 - 359 y^2 \theta^3 + 89 \theta^5) \\
+ x^2 (8 y^{10} - 164 y^8 \theta + 528 y^6 \theta^2 - 656 y^4 \theta^3 + 330 y^2 \theta^4 - 65 \theta^5) \\
+ \theta (-24 y^{10} + 112 y^8 \theta - 198 y^6 \theta^2 + 152 y^4 \theta^3 - 61 y^2 \theta^4 + 10 \theta^5)) / \\
\left[\theta (-2 r^2 + \theta)^2 (2 r^4 - 4 r^2 \theta + \theta^2)^2\right]
$$

(A.1)

and

$$
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = 4 e^{r^2} (64 x^{24} + 64 x^{22} (10 y^2 - 13 \theta) + 16 x^{20} (180 y^4 - 476 y^2 \theta^2 + 305 \theta^4) \\
+ 64 x^{18} (120 y^6 - 486 y^4 \theta + 639 y^2 \theta^2 - 260 \theta^4) \\
+ 32 x^{16} (420 y^8 - 2328 y^6 \theta + 4744 y^4 \theta^2 - 3989 y^2 \theta^3 + 1148 \theta^5) \\
+ 16 x^{14} (1008 y^{10} - 7224 y^8 \theta + 20488 y^6 \theta^2 - 26914 y^4 \theta^3 \\
+ 16049 y^2 \theta^4 - 3487 \theta^5) + 8 x^{12} (1680 y^{12} - 15 120 y^{10} \theta + 56 812 y^8 \theta^2 \\
- 104 748 y^6 \theta^3 + 97 688 y^4 \theta^4 - 43 900 y^2 \theta^5 + 75 870 \theta^6) \\
+ 8 x^{10} (960 y^{14} - 10 752 y^{12} \theta + 52 640 y^{10} \theta^2 - 129 556 y^8 \theta^3 + 169 222 y^6 \theta^4 \\
- 118 388 y^4 \theta^5 + 42 087 y^2 \theta^6 - 60 710 \theta^7) \\
+ 4 x^8 (720 y^{16} - 10 176 y^{14} \theta + 65 744 y^{12} \theta^2 - 211 400 y^{10} \theta^3 \\
+ 365 560 y^8 \theta^4 - 355 060 y^6 \theta^5 + 194 909 y^4 \theta^6 - 57 396 y^2 \theta^7 + 7245 \theta^8) \\
+ 4 x^6 (160 y^{18} - 3024 y^{16} \theta + 27 232 y^{14} \theta^2 - 114 072 y^{12} \theta^3 + 252 764 y^{10} \theta^4 \\
- 320 300 y^8 \theta^5 + 241 376 y^6 \theta^6 - 108 758 y^4 \theta^7 + 27 825 y^2 \theta^8 - 3176 \theta^9) \\
+ x^4 (64 y^{20} - 1984 y^{18} \theta + 28 688 y^{16} \theta^2 - 157 984 y^{14} \theta^3 + 438 784 y^{12} \theta^4 \\
- 696 832 y^{10} \theta^5 + 675 216 y^8 \theta^6 - 413 272 y^6 \theta^7 + 160 576 y^4 \theta^8 \\
- 36 972 y^2 \theta^9 + 3897 \theta^{10}) - 2 x^2 \theta (64 y^{20} - 2208 y^{18} \theta + 16 112 y^{16} \theta^2 \\
- 54 952 y^{14} \theta^3 + 106 080 y^{12} \theta^4 - 126 580 y^{10} \theta^5 + 98 472 y^8 \theta^6 - 51 586 y^6 \theta^7 \\
+ 17 960 y^4 \theta^8 - 3815 y^2 \theta^9 + 3736 \theta^{10} + \theta^2 (320 y^{20} - 3008 y^{18} \theta + 12 256 y^{16} \theta^2 \\
- 27 984 y^{14} \theta^3 + 39 812 y^{12} \theta^4 - 37 688 y^{10} \theta^5 + 24 916 y^8 \theta^6 - 11 652 y^6 \theta^7 \\
+ 374 y^4 \theta^8 - 738 y^2 \theta^9 + 660 \theta^{10}) / \theta^2 (-2 r^2 + \theta)^4 (2 r^4 - 4 r^2 \theta + \theta^2)^4),
$$

(A.2)

respectively.
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