Core Shrinkage Covariance Estimation for Matrix-variate Data

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Abstract

A separable covariance model for a random matrix provides a parsimonious description of the covariances among the rows and among the columns of the matrix, and permits likelihood-based inference with a very small sample size. However, in many applications the assumption of exact separability is unlikely to be met, and data analysis with a separable model may overlook or misrepresent important dependence patterns in the data. In this article, we propose a compromise between separable and unstructured covariance estimation. We show how the set of covariance matrices may be uniquely parametrized in terms of the set of separable covariance matrices and a complementary set of “core” covariance matrices, where the core of a separable covariance matrix is the identity matrix. This parametrization defines a Kronecker-core decomposition of a covariance matrix. By shrinking the core of the sample covariance matrix with an empirical Bayes procedure, we obtain an estimator that can adapt to the degree of separability of the population covariance matrix.

Keywords: decorrelation, equivariance, Kronecker product, matrix decomposition, tensor, quadratic discriminant analysis, matrix square root, whitening.
1 Introduction

Many modern datasets include matrix-variate data, that is, a sample of \( n \) matrices \( Y_1, \ldots, Y_n \) having a common dimension \( p_1 \times p_2 \). Examples of such datasets include collections of images, networks, gene by tissue expression arrays, and multivariate time series, among others. One approach to the analysis of such data is to first vectorize each data matrix and then proceed with a method that is appropriate for generic multivariate data. For example, if \( Y_1, \ldots, Y_n \) is a random sample from a population of mean-zero matrices, the population covariance could be estimated by the sample covariance \( S = \frac{1}{n} \sum_{i=1}^{n} y_i y_i^\top / n \), where for \( i = 1, \ldots, n \), \( y_i \) is the vector of length \( p = p_1 \times p_2 \) obtained by vectorizing \( Y_i \).

However, in many applications the sample size \( n \) is insufficient for such unstructured estimates to be statistically stable. For example, even though \( p_1 \) and \( p_2 \) might be of moderate magnitude individually, a sample size of \( n \geq p_1 p_2 \) is necessary for \( S \) to be non-singular, and for the likelihood corresponding to a normal model to be bounded. Furthermore, even if the sample size is sufficient for estimation, an unstructured estimate such as \( S \) may be difficult to interpret, as it is not expressed in terms of conceptually simple row factors or column factors.

For these reasons, covariance models that are based on the matrix structure of the data have been developed. Most popular are the separable or Kronecker-structured covariance models that assume the \( p \times p \) population covariance matrix is the Kronecker product of two smaller covariance matrices of dimension \( p_1 \times p_1 \) and \( p_2 \times p_2 \), representing across-row and across-column covariance respectively. In particular, the separable covariance model for normally-distributed data (Dawid, 1981) has been used for a wide variety of applications including environmental monitoring (Mardia and Goodall, 1993), signal processing (Werner et al., 2008), image analysis (Zhang and Schneider, 2010), gene expression data (Yin and Li, 2012), radar detection (Greenewald et al., 2016) and many others.

In addition to its interpretability, a separable covariance model is appealing because of its statistical stability, which is a result of its parsimony as compared to an unstructured covariance model. Remarkably, the MLE in the separable normal
model exists uniquely for any sample size $n$ larger than $p_1/p_2 + p_2/p_1$ (Roš et al., 2016; Soloveychik and Trushin, 2016; Drton et al., 2021; Derksen and Makam, 2021). This is in contrast to a sample size requirement of $n \geq p_1p_2$ in a normal model with an unstructured covariance. However, the appropriateness of a separable covariance estimator depends on the extent to which the population covariance is truly separable. If the population covariance is not separable, a separable estimate might give an incomplete or misleading summary of the statistical dependencies in the data, or could lead to poor performance of statistical procedures, such as generalized least-squares or quadratic discriminant analysis, that rely on an accurate estimate of the population covariance. These and other concerns about the appropriateness of the separability assumption have been raised by Stein (2005) and Rougier (2017), specifically in the context of random spatio-temporal processes. To address these concerns, Masak et al. (2022) and Masak and Panaretos (2022) have proposed generalizations of the class of separable covariance operators for functional data analysis with two-dimensional domains (e.g., space and time). The first of these is based on an approximation of an arbitrary positive definite covariance operator by a sum of separable matrices. The second of these assumes the covariance operator is the sum of two positive definite operators, one of which is separable and the other is banded, where the banding is determined by the metrics of each of the two domains.

In this article we consider covariance estimation for random matrices with rows and columns that represent arbitrary factors, and so in particular do not necessarily correspond to points in a spatio-temporal domain. We develop a covariance estimation strategy that makes use of the parsimony and interpretability of a separable covariance model, yet can consistently describe covariance matrices that are non-separable. This is accomplished with a new matrix decomposition for positive definite matrices, which we call the “Kronecker-core decomposition”, or KCD. This decomposition expresses an arbitrary covariance matrix in terms of a low-dimensional separable covariance matrix and a complementary high-dimensional “core” covariance matrix. By adaptively shrinking the core of the sample covariance matrix, an estimator is obtained that can have a risk that is comparable to that of the separable estimator when the population covariance is truly separable, and otherwise has lower
risk than both the separable and unstructured estimators.

In the next section we define the Kronecker covariance and core covariance of an arbitrary $p_1p_2 \times p_1p_2$ covariance matrix. We show that the space of all $p_1p_2 \times p_1p_2$ covariance matrices can be identifiably parametrized by the product space of Kronecker and core covariance matrices using the Kronecker-core decomposition. In Section 3, we propose a class of core shrinkage estimators that are obtained by shrinking the core of the sample covariance matrix towards the identity matrix, or equivalently, shrinking the sample covariance matrix towards the space of separable covariance matrices. Such shrinkage estimators can be viewed as empirical Bayes estimators, where the amount of shrinkage is estimated from the data. We show that our proposed core shrinkage estimator is consistent, and in a simulation study in Section 4.1, we show that the loss of the core shrinkage estimator can be very close to that of an oracle Bayes estimator, and lower than that of both the separable and unstructured MLEs across a variety of conditions. In Section 4.2, we use core shrinkage estimators as inputs into a quadratic discriminant analysis for a speech recognition task. We observe that classifications using core shrinkage estimators have lower out-of-sample misclassification rates than those using separable or unstructured MLEs. A discussion of directions for further research follows in Section 5. Proofs of mathematical results are provided in an appendix. Replication code for the numerical results in this article are available at the first author’s website and from the R-package covKCD.

# Kronecker and core covariances

## 2.1 The Kronecker covariance of a random matrix

Let $Y$ be a mean-zero random matrix taking values in $\mathbb{R}^{p_1 \times p_2}$ with a non-singular covariance matrix $\Sigma \in \mathcal{S}_p^+$ where $p = p_1p_2$, meaning that $E[yy^\top] = \Sigma$ where $y = \text{vec}(Y)$. In what follows, we will use both $\text{Var}[Y]$ and $\text{Var}[y]$ to denote the $p \times p$ covariance matrix of the vectorization $y$ of $Y$. Recall that $\Sigma$ is Kronecker separable, or simply separable, if it can be expressed as $\Sigma = \Sigma_2 \otimes \Sigma_1$ for some matrices $\Sigma_1 \in \mathcal{S}_{p_1}^+$. 


\(\Sigma_2 \in S^+_p\), where “\(\otimes\)" is the Kronecker product. In this case, the matrices \(\Sigma_1, \Sigma_2\) (or matrices \(c\Sigma_1, \Sigma_2/c\) for any \(c > 0\)) are often referred to as the row covariance and column covariance of \(Y\) respectively. For example, the covariance of the \(p_2\) random variables in a common row of \(Y\) is proportional to \(\Sigma_2\), and so \(\Sigma_2\) represents the covariances of the elements of \(Y\) across its columns.

Let \(S^+_{p_1, p_2} = \{\Sigma_2 \otimes \Sigma_1 : \Sigma_1 \in S^+_{p_1}, \Sigma_2 \in S^+_{p_2}\} \subset S^+_p\) be the set of separable covariance matrices for given values of \(p_1\) and \(p_2\). A separable covariance model is a collection of probability distributions for \(Y\) for which it is assumed that \(\text{Var}[Y] \in S^+_{p_1, p_2}\). The most widely used separable model is the separable normal model, or “matrix normal" model (Dawid [1981]), which specifies that \(Y \sim N_{p_1 \times p_2}(0, \Sigma_2 \otimes \Sigma_1)\) for unknown \(\Sigma_2 \otimes \Sigma_1 \in S^+_{p_1, p_2}\). A separable covariance model can be thought of as a bilinear transformation model: Let \(Z\) be a \(p_1 \times p_2\) mean-zero random matrix with \(\text{Var}[Z] = I_{p_1}\), and let \(Y = A_1ZA_2^\top\) for non-singular matrices \(A_1 \in \mathbb{R}^{p_1 \times p_1}, A_1 \in \mathbb{R}^{p_2 \times p_2}\). Then \(\text{Var}[Y] = A_2A_2^\top \otimes A_1A_1^\top\), and the range of \(\text{Var}[Y]\) over all such matrices \(A_1, A_2\) is exactly equal to \(S^+_{p_1, p_2}\). More generally, separability is preserved under row and column transformations of \(Y\): If \(\text{Var}[Y] = \Sigma_2 \otimes \Sigma_1\), then

\[
\text{Var}[A_1YA_2^\top] \equiv \text{Var}[(A_2 \otimes A_1)y] = (A_2 \otimes A_1)\text{Var}[y](A_2 \otimes A_1)^\top = (A_2 \otimes A_1)(\Sigma_2 \otimes \Sigma_1)(A_2 \otimes A_1)^\top = (A_2\Sigma_2A_2^\top) \otimes (A_1\Sigma_1A_1^\top).
\]

In the language of group theory, let \(GL_{p_1, p_2} = \{A_2 \otimes A_1 : A_1 \in GL_{p_1}, A_2 \in GL_{p_2}\}\) be the separable subgroup of the general linear group \(GL_p\) of nonsingular \(p \times p\) matrices. The transformation in \([1]\) from \(\text{Var}[Y]\) to \(\text{Var}[A_1YA_2^\top]\) defines a transitive group action of \(GL_{p_1, p_2}\) on \(S^+_{p_1, p_2}\). The group structure of the separable normal model and related tensor normal models has been exploited to develop methods for statistical estimation (Gerard and Hoff [2015]) and testing (Gerard and Hoff [2016]; Hoff [2016]).

Even if \(\text{Var}[Y]\) is not separable, it still may be of interest to define some notion of row covariance and column covariance for \(Y\). To this end, we identify a separable covariance matrix \(K \in S^+_{p_1, p_2}\) that summarizes the row and column covariance of \(Y\) when \(\text{Var}[Y]\) is an arbitrary covariance matrix \(\Sigma \in S^+_p\):
**Definition 1.** Let $E[Y] = 0$ and $\text{Var}[Y] = \Sigma \in S^+_p$. The Kronecker covariance of $\Sigma$ is $k(\Sigma) = \Sigma_2 \otimes \Sigma_1$, where $(\Sigma_1, \Sigma_2)$ are any matrices in $S^+_{p_1} \times S^+_{p_2}$ that satisfy

\begin{align}
\Sigma_1 &= E[Y \Sigma_2^{-1} Y^\top]/p_2 \quad (2) \\
\Sigma_2 &= E[Y^\top \Sigma_1^{-1} Y]/p_1.
\end{align}

Matrices $\Sigma_1$ and $\Sigma_2$ that solve (2) are weighted averages of across-row and across-column covariance matrices of whitened versions of $Y$. For example, $\Sigma_1$ is obtained from $Y$ by first whitening across its columns by $\Sigma_2$.

Solutions to (2) exist for all $\Sigma \in S^+_p$, and all solutions have the same Kronecker product, and so the Kronecker covariance function $k : S^+_p \to S^+_{p_1,p_2}$ is well defined. The existence of solutions and uniqueness of their Kronecker product follow from existing results for the separable normal model, and the following alternative definition of $k(\Sigma)$ as the element of $S^+_{p_1,p_2}$ that is closest to $\Sigma$ in terms of a standard divergence function:

**Proposition 1.** $(\Sigma_1, \Sigma_2)$ is a solution to (2) if and only if $\Sigma_2 \otimes \Sigma_1$ minimizes $d(K : \Sigma) = \ln |K| + \text{trace}(K^{-1} \Sigma)$ over $K \in S^+_{p_1,p_2}$.

The divergence function $d(K : \Sigma)$ is related to Stein’s loss for covariance estimation and to the Kullback-Leibler divergence between two normal distributions. Specifically, $k(\Sigma)$ is the covariance matrix of the separable normal distribution that minimizes the Kullback-Leibler divergence to the $N_{p_1 \times p_2}(0, \Sigma)$ distribution. This means, for example, that if $Y_1, \ldots, Y_n \sim \text{i.i.d. } N_{p_1 \times p_2}(0, \Sigma)$ then the maximum likelihood estimator (MLE) of $\Sigma_2 \otimes \Sigma_1$ under the potentially misspecified model $Y_1, \ldots, Y_n \sim \text{i.i.d. } N_{p_1 \times p_2}(0, \Sigma_2 \otimes \Sigma_1)$ converges in probability to $k(\Sigma)$ as $n \to \infty$ (Huber, 1967). In the language of misspecified models, $k(\Sigma)$ is the “pseudo-true” parameter under the separable normal model in the case that $\Sigma$ is not necessarily separable.

That the minimizer of the divergence function is unique follows from uniqueness results for the MLE in the separable normal model. The MLE for this model is obtained by minimizing over $\Sigma_2 \otimes \Sigma_1 \in S^+_{p_1,p_2}$ the scaled log-likelihood

\[ (-2/n) \times \ln p(Y_1, \ldots, Y_n|\Sigma_2 \otimes \Sigma_1) = \ln |\Sigma_2 \otimes \Sigma_1| + \text{trace}((\Sigma_2 \otimes \Sigma_1)^{-1} S) + p \ln 2\pi \]
where \( S = \sum_{i=1}^{n} y_i y_i^\top / n \) is the sample covariance matrix. Clearly, the conditions on \( S \) for there to exist a unique MLE of \( \Sigma_2 \otimes \Sigma_1 \) are the same as those on \( \Sigma \) for there to exist a unique minimizer of \( d(K : \Sigma) \) over \( K \in S_{p \times p}^+ \). In particular, \( k(S) \) is the MLE of \( \Sigma_2 \otimes \Sigma_1 \) under the separable normal model when \( S \) is the sample covariance matrix. Srivastava et al. (2008) show that this MLE exists uniquely if \( S \) is strictly positive definite, which implies that \( k(\Sigma) \) exists uniquely for any \( \Sigma \in S_{p}^+ \). We note that solutions may also exist uniquely when \( S \), or analogously \( \Sigma \), is singular (Soloveychik and Trushin, 2016; Drton et al., 2021; Derksen and Makam, 2021).

Numerical methods for finding the separable normal MLE may be used to compute the Kronecker covariance function. As shown in Dutilleul (1999), \( \hat{\Sigma}_2 \otimes \hat{\Sigma}_1 \) is an MLE of \( \Sigma_2 \otimes \Sigma_1 \) if (\( \hat{\Sigma}_1 \), \( \hat{\Sigma}_2 \)) satisfy

\[
\begin{align*}
\left( \sum_{i=1}^{n} Y_i \hat{\Sigma}_2^{-1} Y_i^\top / n \right) / p_2 &= \hat{\Sigma}_1 \\
\left( \sum_{i=1}^{n} Y_i^\top \hat{\Sigma}_1^{-1} Y_i / n \right) / p_1 &= \hat{\Sigma}_2.
\end{align*}
\]

Dutilleul also provided a block coordinate descent algorithm that converges to the MLE when it exists uniquely. Because this system of equations is analogous to the system (2) that define \( k(\Sigma) \), Dutilleul’s algorithm may be implemented to numerically compute the Kronecker covariance \( k(\Sigma) \) of any \( \Sigma \in S_{p}^+ \). In this context, given a starting value \( \Sigma_2 \in S_{p_2}^+ \), the algorithm is to iterate the following steps until a convergence criteria is met:

1. Set \( \Sigma_1 = E[Y \Sigma_2^{-1} Y^\top] / p_2 \);
2. Set \( \Sigma_2 = E[Y^\top \Sigma_1^{-1} Y] / p_1 \).

An algorithm to compute \( k(\Sigma) \) is provided in the replication material for this article.

An important property of the Kronecker covariance function is how it is affected by transformations of \( \Sigma \), or equivalently, of \( Y \). Recall that if \( Y \) has a separable covariance \( \Sigma_2 \otimes \Sigma_1 \), then \( A_1 Y A_2^\top \) has separable covariance \((A_2 \Sigma_2 A_2^\top) \otimes (A_1 \Sigma_1 A_1^\top)\), and so in this sense a linear transformation across the rows of \( Y \) changes the row covariance and not the column covariance, and analogously for a column transformation.
The following result shows that the Kronecker covariance function transforms in the same way, even if the covariance matrix of $Y$ is not separable:

**Proposition 2.** For $A_2 \otimes A_1 \in \text{GL}_{p_1,p_2}$ and $\Sigma \in S_p^+$ with $k(\Sigma) = \Sigma_2 \otimes \Sigma_1$,

$$
k((A_2 \otimes A_1)\Sigma(A_2 \otimes A_1)\top) = (A_2 \otimes A_1)k(\Sigma)(A_2 \otimes A_1)\top.
= (A_2\Sigma_2A_2\top) \otimes (A_1\Sigma_1A_1\top).
$$

From the perspective of group theory, the group action of $\text{GL}_{p_1,p_2}$ on $\mathbb{R}^{p_1 \times p_2}$ defined by $Y \mapsto A_1YA_2\top$ induces a group action of $\text{GL}_{p_1,p_2}$ on $S_p^+$ given by $\Sigma \mapsto (A_2 \otimes A_1)\Sigma(A_2 \otimes A_1)\top$. The result is that the Kronecker covariance function $k$ is equivariant with respect to this group action - the Kronecker covariance of the separably-transformed $\Sigma$ is the separably-transformed Kronecker covariance of $\Sigma$. This property will be used throughout the remainder of this article. Additional properties of the Kronecker covariance function include the following:

**Corollary 1.**

1. $k(I_p) = I_p$.
2. If $\Sigma \in S_{p_1,p_2}^+$ then $k(\Sigma) = \Sigma$.
3. For $a > 0$, $k(a\Sigma) = ak(\Sigma)$.
4. If $\Sigma$ is diagonal then $k(\Sigma)$ is diagonal.

The third item indicates that $k$ is a scale-equivariant function. As a result, the shrinkage estimator we propose in the Section 3 will be scale-equivariant.

### 2.2 The Kronecker-core parametrization and decomposition

The Kronecker covariance function $k$ defined above is a surjection from $S_p^+$ to $S_{p_1 \times p_2}^+$ that describes the row covariance and column covariance of an arbitrary element of $S_p^+$. We now use this function to define, for each $\Sigma \in S_p^+$, a “core” covariance matrix
that is complementary to $k(\Sigma)$ in that the core lacks across-row and across-column covariance in some sense. We then show that, taken together, the product space of separable and core covariance matrices identifiably parametrizes $S_p^+$.  

A core covariance of $\Sigma \in S_p^+$ is obtained by applying a transformation to $\Sigma$ that whitens its Kronecker covariance. Specifically, let $H = H_2 \otimes H_1$ be any matrix in $GL_{p_1,p_2}$ such that $HH^\top = k(\Sigma)$. By the equivariance of $k$, we have

\[ k(H^{-1} \Sigma H^{-\top}) = H^{-1} k(\Sigma) H^{-\top} = H^{-1} HH^\top H^{-\top} = I_p. \]

We define Kronecker-whitened versions of $\Sigma$ as follows:

**Definition 2.** Let $H = H_2 \otimes H_1 \in GL_{p_1,p_2}$ satisfy $HH^\top = k(\Sigma)$. Then the matrix $C$ given by $C = H^{-1} \Sigma H^{-\top}$ is a core of $\Sigma$.

We call the matrix $C = H^{-1} \Sigma H^{-\top}$ a core of $\Sigma$ because the four-way tensor $\tilde{\Sigma} \in \mathbb{R}^{p_1 \times p_2 \times p_1 \times p_2}$ with entries corresponding to $\Sigma$ may be expressed in terms of $C$, $H_1$ and $H_2$ via the multilinear operation

\[ \tilde{\Sigma} = \tilde{C} \times \{H_1, H_2, H_1, H_2\}, \]

where “×” is the multilinear product and $\tilde{C}$ is the four-way tensor corresponding to $C$. Equivalently, we have $\text{vec}(\Sigma) = (H_2 \otimes H_1 \otimes H_2 \otimes H_1)\text{vec}(C)$. In the context of Tucker products, the tensor that gets multiplied along each mode by a matrix is called the “core”.

There are multiple cores for a given $\Sigma$, as there are multiple separable matrices $H$ for which $HH^\top = k(\Sigma)$. Conversely, a core of $\Sigma$ is also a core of $H \Sigma H^\top$ for any $H \in GL_{p_1,p_2}$. More generally, we say that a covariance matrix $C \in S_p^+$ is a core covariance matrix if it is the core of some $\Sigma \in S_p^+$, and so any core covariance matrix satisfies $k(C) = I_p$. Furthermore, suppose $C \in S_p^+$ satisfies $k(C) = I_p$. Then $C$ is a core of any covariance matrix $H CH^\top$ for $H \in S_{p_1,p_2}$. As such, for a given $p_1$ and $p_2$, we define the set of core matrices as follows:

**Definition 3.** For a given $p_1$ and $p_2$ with $p_1 \times p_2 = p$, the set of core covariance matrices is $C_{p_1,p_2}^+ = \{C \in S_p^+ : k(C) = I_p\}$. 

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The condition \( k(C) = I_p \) defining \( C^+_{p_1,p_2} \) can alternatively be expressed as follows:

**Proposition 3.** Let \( Y \) have covariance matrix \( C \in S^+_p \), and let \( \tilde{C} \) be the \( p_1 \times p_2 \times p_1 \times p_2 \) tensor where \( \tilde{C}_{i,j,i',j'} = \text{Cov}[Y_{i,j}, Y_{i',j'}] \). Then \( k(C) = I_p \) if and only if

\[
E[YY^\top]/p_2 \equiv \sum_{j=1}^{p_2} \tilde{C}_{j,j}/p_2 = I_{p_1} \\
E[Y^\top Y]/p_1 \equiv \sum_{i=1}^{p_1} \tilde{C}_{i,i}/p_1 = I_{p_2}.
\]

So for a core covariance matrix, the across-column average of the across-row covariance is the identity matrix, and analogously for the across-column covariance. Intuitively, a core covariance has no across-row or across-column correlation or heteroscedasticity, on average.

From the proposition we see that \( C^+_{p_1,p_2} \) is defined by a system of linear constraints, and that \( \text{trace}(C) = p \) for any core covariance matrix \( C \), so \( C^+_{p_1,p_2} \) is a compact convex subset of \( S^+_p \). Additionally, the core covariances \( C^+_{p_1,p_2} \) and the separable covariances \( S^+_{p_1,p_2} \) are nearly non-overlapping: If \( C \) is core then \( k(C) = I_p \), and if \( C \) is separable, then \( k(C) = C \) by Corollary [1]. Therefore, if \( C \) is core and separable, then \( C = I_p \). Thus \( S^+_{p_1,p_2} \cap C^+_{p_1,p_2} = I_p \).

For every \( \Sigma \in S^+_p \) there is a core matrix \( C \in C^+_{p_1,p_2} \) and separable matrix \( K \in S^+_{p_1,p_2} \) such that \( \Sigma = HCH^\top \) for some separable \( H \in GL_{p_1,p_2} \) such that \( K = HH^\top \). Conversely, to every \( C \in C^+_{p_1,p_2} \) and \( K \in S^+_{p_1,p_2} \) we can define an element of \( S^+_p \) as \( HCH^\top \) where \( H \in GL_{p_1,p_2} \) and \( K = HH^\top \). This suggests that there is a bijection between \( S^+_p \) and \( S^+_{p_1,p_2} \times C^+_{p_1,p_2} \). In fact, there are many such bijections, including one for each way to define a separable matrix square root \( H \) of \( K \), or equivalently, one for each way to define a row and column whitening matrix from \( K \). To specify a particular bijection, we need to specify a separable square root function.

**Definition 4.** Let \( \mathcal{H} \) be a subset of \( GL_{p_1,p_2} \) such that the function \( s : \mathcal{H} \to S^+_{p_1,p_2} \) defined by \( s(H) = HH^\top \) is a bijection. Then \( h = s^{-1} \) is a separable matrix square root function.

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Essentially, a separable square root function is defined by a set of separable matrices $\mathcal{H}$ with unique crossproducts, the set of which equals the set of separable covariance matrices. The defining feature of such a function is that $h(\mathbf{H} \mathbf{H}^\top) = \mathbf{H}$ for $\mathbf{H} \in \mathcal{H}$. Examples include the following:

- **Symmetric square root:** $h(\Sigma_2 \otimes \Sigma_1) = \Sigma_2^{1/2} \otimes \Sigma_1^{1/2}$, where $\Sigma_j^{1/2}$ is the symmetric square root of $\Sigma_j$, $j \in \{1, 2\}$.

- **Cholesky square root:** $h(\Sigma_2 \otimes \Sigma_1) = L_2 \otimes L_1$ where $L_j L_j^\top$ is the lower triangular Cholesky factorization of $\Sigma_j$, $j \in \{1, 2\}$.

- **PCA square root:** $h(\Sigma_2 \otimes \Sigma_1) = E_2 \Lambda_2^{1/2} \otimes E_1 \Lambda_1^{1/2}$ where $E_j \Lambda_j E_j^\top$ is the eigendecomposition of $\Sigma_j$, $j \in \{1, 2\}$. Note that conventions on the signs and column orderings of $E$ need to be specified in order for $h$ to be a bijection.

A non-example would be $GL_{p_1,p_2}$: While the set of crossproducts of this set is equal to $\mathcal{S}_{p_1,p_2}^+$, elements of the set do not have unique crossproducts.

For a given separable square root function $h$ we define the core covariance function $c : \mathcal{S}_p^+ \rightarrow \mathcal{C}_p^+$ as $c(\Sigma) = H^{-1} \Sigma H^{-\top}$ where $H = h(k(\Sigma))$. Since the core represents “non-separable” covariance, we would hope the core function to be invariant to bilinear transformations of the form $\Sigma \mapsto (A_2 \otimes A_1) \Sigma (A_2 \otimes A_1)^\top$ that induce separable covariance. This property partly holds:

**Proposition 4.** Let $\Sigma \in \mathcal{S}_p^+$ and $A \in GL_{p_1,p_2}$. Then

1. $c(A \Sigma A^\top) = (R_2 \otimes R_1)c(\Sigma)(R_2 \otimes R_1)^\top$ for some $R_1 \in \mathcal{O}_{p_1}$, $R_2 \in \mathcal{O}_{p_2}$.

2. $c(A \Sigma A^\top) = c(\Sigma)$ if $A \in \mathcal{H}$ and $\mathcal{H}$ is a group.

Item 2 of the proposition says that if $\mathcal{H}$ is a group then $c$ is a maximal invariant function of $\Sigma \in \mathcal{S}_p^+$ under the group action $\Sigma \mapsto H \Sigma H^\top$ for $H \in \mathcal{H}$, while the Kronecker covariance function $k$ is an equivariant function by Proposition 2. One such group $\mathcal{H}$ is the set of Kronecker products of lower-triangular matrices with positive diagonal entries, with $h$ being the Cholesky square root. However, the results on covariance estimation in the remainder of the article are unaffected by
the choice of $h$ as long as it is continuous, as is the case for the symmetric and Cholesky square root functions mentioned above. We assume use of one of these two continuous square root function for the remainder of the article.

We now arrive at the main result of this section - an identifiable parametrization of the set of covariance matrices in terms of Kronecker and core covariance matrices:

**Proposition 5.** The function $f : S^+_p \rightarrow S^+_{p_1,p_2} \times C^+_{p_1,p_2}$ defined by $f(\Sigma) = (k(\Sigma), c(\Sigma))$ is a homeomorphism with inverse $g : S^+_{p_1,p_2} \times C^+_{p_1,p_2} \rightarrow S^+_p$ given by $g(K, C) = h(K)Ch(K)^\top$.

The function $g$ can be viewed as a parametrization of $S^+_p$ in terms of $S^+_{p_1,p_2} \times C^+_{p_1,p_2}$. Conversely, the function $f$ provides a matrix decomposition for elements of $S^+_p$: Every $\Sigma \in S^+_p$ has a unique representation as $\Sigma = K^{1/2}CK^{1/2}$ for some $K \in S^+_{p_1,p_2}$ and $C \in C^+_{p_1,p_2}$. We refer to this representation as the Kronecker-core decomposition, or KCD.

### 3 Core shrinkage via empirical Bayes

#### 3.1 Core shrinkage estimators

Let $Y_1, \ldots, Y_n$ be an i.i.d. random sample from a mean-zero normal population of $p_1 \times p_2$ matrices, that is

$$Y_1, \ldots, Y_n \sim \text{i.i.d. } N_{p_1 \times p_2}(0, \Sigma)$$

for some unknown $\Sigma \in S^+_p$ where $p = p_1 \times p_2$. In this case where no assumptions are made on the structure of $\Sigma$, the standard estimator is the sample covariance matrix $S = \frac{1}{n} \sum_{i=1}^{n} y_i y_i^\top$ where $y_i = \text{vec}(Y_i)$. This estimator is unbiased, and if $n \geq p$ it is the MLE. However, as is well known, the risk of $S$ can be suboptimal, and substantially so if $n$ is not somewhat larger than $p$.

As an alternative to $S$, we propose an estimator obtained by shrinking $S \in S^+_p$ towards the lower-dimensional subset $S^+_{p_1,p_2}$ of separable covariance matrices, using the parametrization of $S^+_p$ described in the previous section. Let $K = k(\Sigma)$ and
$C = c(\Sigma)$ be the unknown Kronecker and core covariance matrices of $\Sigma$, so that $\Sigma = K^{1/2}CK^{1/2}$ where $K^{1/2}$ is the symmetric square root of $K$. Because MLEs are parametrization invariant, the MLE $(\hat{K}, \hat{C})$ of $(K, C)$ is

$$
\hat{K} = k(S) \\
\hat{C} = c(S) = \hat{K}^{-1/2}S\hat{K}^{-1/2}.
$$

Note that $\hat{K}$ is also the MLE of $\Sigma$ under the separable normal model that assumes $\Sigma \in S^{+}_{p_1,p_2}$. The number of parameters needed to define $K$ and to define $C$ are on the order of $p_1^2 + p_2^2$ and $p_1^2p_2^2$ respectively, and so heuristically we expect that $\hat{K}$ is a better estimate of $K$ than $\hat{C}$ is of $C$. For this reason, we consider shrinkage estimators of the form

$$
\hat{\Sigma} = \hat{K}^{1/2}\hat{C}_w\hat{K}^{1/2}, \\
\hat{C}_w = (1 - w)\hat{C} + wI_p
$$

for some choice of $w \in [0, 1]$. Because the space of core matrices is convex and includes $I_p$, the value $\hat{C}_w$ is itself a core matrix and is a linear shrinkage estimator of $C$, shrinking the core $\hat{C}$ of the sample covariance matrix towards $I_p$, or equivalently, shrinking the sample covariance matrix $\hat{\Sigma}$ towards the MLE $\hat{K}$ of the separable submodel:

$$
\hat{\Sigma} = \hat{K}^{1/2}[(1 - w)\hat{C} + wI_p]\hat{K}^{1/2} \\
= (1 - w)\hat{K}^{1/2}\hat{C}\hat{K}^{1/2} + w\hat{K} \\
= (1 - w)S + w\hat{K}.
$$

In particular, $w = 1$ gives the MLE under the assumption that $\Sigma$ is separable, whereas $w = 0$ gives the sample covariance, or equivalently, the unrestricted MLE in the case that $n \geq p$. Furthermore, if $w > 0$ then the estimator is positive definite even if $n$ is much smaller than $p$, as long as $n$ is large enough for $\hat{K}$ to be the MLE for the separable submodel.
3.2 Empirical Bayes estimation

An estimator having the form (5) can be viewed as an empirical Bayes estimator. Consider an inverse-Wishart prior distribution for the unknown covariance \( \Sigma \),

\[
\Sigma^{-1} \sim \text{Wishart}_p\left(\left((\nu - p - 1)\Sigma_2 \otimes \Sigma_1\right)^{-1}, \nu\right),
\]

which is parametrized so that \( \mathbb{E}[\Sigma] = \Sigma_2 \otimes \Sigma_1 \). The hyperparameter \( \nu \) partly controls how concentrated the prior distribution of \( \Sigma \) is around the separable covariance matrix \( \Sigma_2 \otimes \Sigma_1 \). Under this prior distribution, the posterior distribution of \( \Sigma \) is

\[
\Sigma^{-1}|S \sim \text{Wishart}_p\left([nS + (\nu - p - 1)\Sigma_2 \otimes \Sigma_1]^{-1}, n + \nu\right),
\]

and the Bayes estimator under squared-error loss is the posterior mean,

\[
\mathbb{E}[\Sigma|S] = (1 - w)S + w\Sigma_2 \otimes \Sigma_1
\]

where \( w = (\nu - p - 1)/(n + \nu - p - 1) \). An empirical Bayes estimator that replaces the hyperparameter \( \Sigma_2 \otimes \Sigma_1 \) with \( \hat{K} = k(S) \) gives the estimator

\[
\mathbb{E}[\Sigma|S] = (1 - w)S + w\hat{K},
\]

which is the same as in (5). While \( \hat{K} \) is not the marginal MLE of \( \Sigma_2 \otimes \Sigma_1 \) under the prior distribution (6), \( \hat{K} \) can be seen as a marginal moment estimator in the following sense: Because \( \mathbb{E}[\Sigma] = \Sigma_2 \otimes \Sigma_1 \) the marginal variance of a generic \( Y \) is \( \Sigma_2 \otimes \Sigma_1 \) as well. By Corollary 1 we then have

\[
\Sigma_1 = \mathbb{E}[Y\Sigma_2^{-1}Y^\top]/p_2
\]
\[
\Sigma_2 = \mathbb{E}[Y^\top\Sigma_1^{-1}Y]/p_1,
\]

where the expectation is with respect to the marginal distribution of \( Y \). The value \( \hat{K} = k(S) \) is a solution to these equations with the marginal expectation replaced by expectation with respect to the empirical distribution of \( Y_1, \ldots, Y_n \), and so \( \hat{K} \) is a generalized method of moments estimator of \( \Sigma_2 \otimes \Sigma_1 \).

The amount of shrinkage \( w \) is determined by the hyperparameter \( \nu \). Our proposed empirical Bayes estimator of \( \nu \) is the maximizer in \( \nu \) of the marginal density
\( p(S|\nu, \Sigma_2 \otimes \Sigma_1) \) with \( \hat{K} \) plugged-in for \( \Sigma_2 \otimes \Sigma_1 \). This density has an essentially closed-form expression due to the conjugacy of the inverse-Wishart prior distribution. Using standard calculations, we obtain the marginal density of \( S \) as

\[
p(S|\nu, \Sigma_2 \otimes \Sigma_1) = r \times \frac{k(\nu)((\nu - p - 1)\Sigma_2 \otimes \Sigma_1)^{\nu/2}}{k(\nu + n)|nS + (\nu - p - 1)\Sigma_2 \otimes \Sigma_1|^{(\nu+n)/2}},
\]

where \( r \) does not depend on \( \nu \) and \( k(\nu)^{-1} = 2^{\nu p/2}\Gamma_p(\nu/2) \), with \( \Gamma_p \) being the multivariate gamma function. Now we plug-in \( \hat{K} \) for \( \Sigma_2 \otimes \Sigma_1 \), and utilize the fact that \( S = \hat{K}^{1/2}\hat{\Sigma}\hat{K}^{1/2} \) to obtain

\[
p(S|\nu, \hat{K})/b = |\hat{K}|^{-n/2} \times \frac{k(\nu)}{k(\nu + n)}(\nu - p - 1)^{-np/2}|I_p + \frac{n}{\nu - p - 1}\hat{\Sigma}|^{-(\nu+n)/2} = \left(|\hat{K}|^{-n/2}2^{np/2}\right) \times \frac{\Gamma_p((\nu + n)/2)}{\Gamma_p(\nu/2)}(\nu - p - 1)^{-np/2}|I_p + \frac{n}{\nu - p - 1}\hat{\Sigma}|^{-(\nu+n)/2}.
\]

After some additional manipulation, we have that \( p(S|\nu, \hat{K}) \propto \nu, L(\nu) \) where

\[
L(\nu) = \frac{\Gamma_p((n + \nu)/2)}{\Gamma_p(\nu/2)} \times w^{np/2}(1-w)^{np/2} \times |(1-w)\hat{\Sigma} + wI_p|^{-(\nu+n)/2}, \tag{8}
\]

with \( w = (\nu - p - 1)/(n + \nu - p - 1) \) as before. Computation of \( L(\nu) \) is facilitated by noting that the determinant term can be expressed as \( |(1-w)\hat{\Sigma} + wI_p| = \prod_{j=1}^p(w + (1-w)c_j) \), where \( c_1, \ldots, c_p \) are the eigenvalues of \( \hat{\Sigma} \). Our proposed empirical Bayes estimator of \( \nu \) is the maximizer \( \hat{\nu} \) of \( L \), which gives \( \hat{\nu} = (\hat{\nu} - p - 1)/(n + \hat{\nu} - p - 1) \) as the amount of shrinkage. The resulting empirical Bayes core shrinkage estimator is given by

\[
\hat{\Sigma} = \hat{K}^{1/2}[(1-\hat{\nu})\hat{\Sigma} + \hat{\nu}I_p]\hat{K}^{1/2} \tag{9}
\]

To understand how the data influence the value of \( \hat{\Sigma} \) through \( \hat{\nu} \), write \( L(\nu) = a(\nu) \times b(\nu) \) where \( b(\nu) = |(1-w)\hat{\Sigma} + wI_p|^{-(\nu+n)/2} \) is the part of \( L \) that depends on the data, and \( a(\nu) = L(\nu)/b(\nu) \). The function \( a(\nu) \) is generally increasing, and so this part of \( L \) “favors” large values of \( \hat{\nu} \) (and \( \hat{\nu} \)). If the sample covariance \( S \) is very close to being separable, then \( \hat{\Sigma} \) is very close to the identity matrix and so \( b(\nu) \) is
roughly constant in $\nu$. In this case, $a(\nu)$ dominates $L(\nu)$, resulting in a large $\hat{w}$ and strong shrinkage of $S$ towards the sample Kronecker covariance $\hat{K}$. However, if $\hat{C}$ is far from the identity then $b$ can be strongly decreasing in $\nu$, which results in $\hat{\Sigma}$ being close to $S$. In summary, the degree of shrinkage towards the space of separable covariance matrices depends on how close $S$ is to being separable, as measured by how close $\hat{C}$ is to the identity matrix.

Finally, we note that $\hat{\Sigma}$ does not depend on the choice of separable square root function: This is because if $\hat{C}$ and $\hat{C}'$ are core matrices of $S$ obtained from different square root functions, they still must satisfy $\hat{C}' = R\hat{C}R^\top$ for some orthogonal matrix $R$. This difference does not affect the empirical Bayes estimator of $\nu$, since

$$|(1 - w)\hat{C}' + wI_p| = |(1 - w)R\hat{C}R^\top + wRR^\top|$$

$$= |RR^\top||(1 - w)\hat{C} + wI_p| = |(1 - w)\hat{C} + wI_p|.$$ 

### 3.3 Consistency

We now provide some consistency results for the components of the KCD and the core shrinkage estimator. First, we have the very general result that a consistent estimator of $\Sigma$ can be used to obtain consistent estimators of $k(\Sigma)$ and $c(\Sigma)$, and vice versa:

**Corollary 2.** $k(S) \overset{p}{\to} k(\Sigma)$ and $c(S) \overset{p}{\to} c(\Sigma)$ if and only if $S \overset{p}{\to} \Sigma$.

This follows directly from the continuity result in Proposition 5 and the continuous mapping theorem. This result can be used to show the consistency of the core shrinkage estimator $\hat{\Sigma}$: Recall that our core shrinkage estimator can be written as $\hat{\Sigma} = (1 - \hat{w})S + \hat{w}\hat{K}$, where $S$ is the sample covariance matrix and $\hat{K}$ is the Kronecker covariance of $S$. Consistency of $\hat{\Sigma}$ will follow if $S$ is consistent and the weight $\hat{w}$ on the separable matrix $\hat{K}$ converges to zero if $\Sigma$ is not exactly separable. This is because if $\Sigma \notin \mathcal{S}^+_p$ but $\hat{w} \overset{p}{\to} 0$ then $\hat{\Sigma} \overset{p}{\to} \Sigma$ because $S$ is consistent for $\Sigma$, regardless of the separability of $\Sigma$. Conversely, if $\Sigma$ is separable then $k(\Sigma) = \Sigma$ and so $\hat{K}$ is consistent for $\Sigma$ by the above proposition. To summarize, we have the following:
Proposition 6. If \( S \stackrel{p}{\to} \Sigma \) then \( \hat{\Sigma} \stackrel{p}{\to} \Sigma \) for any \( \Sigma \in S^+_p \). If \( S \stackrel{p}{\to} \Sigma \) and \( \Sigma \notin S^+_{p_1,p_2} \) then \( \hat{w} \stackrel{p}{\to} 0 \).

Note that the results given in Corollary 2 and Proposition 6 only assume that \( S \) converges in probability to \( \Sigma \) as some index \( n \), used in the definition of \( \hat{w} \), goes to infinity. The estimator \( S \) need not be Wishart-distributed or even a sample covariance matrix, and \( n \) need not be a sample size.

4 Numerical examples

4.1 Monte Carlo study

Because of its adaptive nature, we expect that the core shrinkage estimator \( \hat{\Sigma} \) outperforms the unrestricted MLE \( S \) in general, and performs nearly as well as the separable MLE \( \hat{K} \) when the true covariance is exactly separable. We examine this in a finite sample setting with a small simulation study. We considered two dimensions for the sample space, \((p_1, p_2) = (5, 7)\) and \((p_1, p_2) = (13, 17)\) which correspond to values of \( p = p_1 \times p_2 \) being 35 and 221, respectively. For each dimension, eight sample sizes \( n \) were considered, ranging from \( p_2 \) to \( 3p_1p_2/2 \). For each dimension and each sample size, population covariance matrices were generated under four scenarios, three of which were simulated from the inverse-Wishart prior distribution (6) with three values of the degrees of freedom parameter \( \nu \) ranging from \( p + 2 \) to \( 3p + 1 \). In the fourth scenario, which we refer to as \( \nu = \infty \), \( \Sigma \) was set to a separable matrix (the identity matrix). To summarize, our simulation scenarios include \( 8 \times 4 = 32 \) combinations of \( n \) and \( \nu \) for each of 2 different values of \( (p_1, p_2) \).

For each of these 64 scenarios, 200 matrices \( \Sigma \) were simulated from (6), and from each a sample of \( n \) random matrices from the corresponding multivariate normal distribution (4) were generated. From each sample, we computed four estimators: the sample covariance or MLE \( S \), the separable MLE \( \hat{K} \), the core shrinkage estimator \( \hat{\Sigma} \), and the oracle Bayes estimator (7) which uses perfect knowledge of the hyperparameters \( \nu \) and \( \Sigma_2 \otimes \Sigma_1 \) of the prior distribution (6). For each sample and estimator, the squared error loss in estimating \( \Sigma \) was computed.
Before comparing the estimators in terms of loss, we first examine the performance of the empirical Bayes estimator of \( \hat{w} \) of \( w \), which determines the amount of shrinkage towards \( \hat{K} \). Results for all simulation scenarios are shown in Figure 1, where sample means of the 200 values of \( \hat{w} \) are plotted as a function of the sample size. On average, \( \hat{w} \) overestimates \( w \) with the bias decreasing with increasing sample size and dimension \( p \), and also being smaller for the smaller values of \( w \). Our intuition regarding the overestimation is that the ideal estimate of \( w \) would be obtained by evaluating how close \( S \) is to \( \Sigma_2 \otimes \Sigma_1 \). In contrast, \( \hat{w} \) is obtained by evaluating how close \( S \) is to \( \hat{K} \). Since \( \hat{K} \) is the closest element of \( S_{p_1,p_2}^+ \) to \( S \) by construction, \( \hat{w} \) overestimates how close \( S \) is to \( \Sigma_2 \otimes \Sigma_1 \).

Loss comparisons for the four estimators are displayed in Figures 2 and 3 for the \((p_1,p_2) = (5,7)\) and \((p_1,p_2) = (13,17)\) scenarios respectively. The performance comparisons among the four estimators are similar in each of these two cases. The oracle Bayes estimator has the best performance for each value of \( \nu \). For the smallest values of \( \nu \), for which \( \Sigma \) is not close to being separable, the performance of the
Figure 2: Loss comparisons for \((p_1, p_2) = (5, 7)\). Plotting symbols are given for 200 simulated datasets for each scenario. Lines are averages of log-loss. The estimators include the sample covariance matrix (MLE), the separable MLE (KMLE), the core shrinkage estimator (CSE) and the oracle Bayes estimator (OBayes).
Figure 3: Loss comparisons for \((p_1, p_2) = (13, 17)\). Plotting symbols are given for 200 simulated datasets for each scenario. Lines are averages of log-loss. The estimators include the sample covariance matrix (MLE), the separable MLE (KMLE), the core shrinkage estimator (CSE) and the oracle Bayes estimator (OBayes).
unrestricted MLE is nearly identical to that of the oracle Bayes estimator. This is because the value of the oracle shrinkage weight is $1/n$ and so these two estimators are nearly the same. The core shrinkage estimator (CSE) has a loss performance nearly identical to these two estimators, since for small values of $\nu$, the estimate $\hat{\nu}$ is quite good. In contrast, the Kronecker separable MLE (KMLE) has worse performance on average than the other estimators, and its loss does not improve with increasing sample size. The explanation for this is that the Kronecker covariance $k(\Sigma)$ does not require a large sample size to be well-estimated, and so $\hat{K}$ is close to $k(\Sigma)$ for all sample sizes, but this is far from $\Sigma$ since $\Sigma$ is not close to being separable.

The pattern changes somewhat for the larger values of $\nu$. In general, the loss of the KMLE is good for small sample sizes, but does not improve much with increasing sample size since it converges to $k(\Sigma)$, which is not equal to $\Sigma$. In contrast, the unrestricted MLE is poor for small sample sizes but, since it is a consistent estimator, has a loss that steadily decreases with increasing sample size. The core shrinkage estimator is generally as good or better than either of these estimators across the different sample sizes: For small $n$ it is about as good as the KMLE, and for large $n$, where both $k(\Sigma)$ and $\nu$ can be well-estimated, it performs nearly as well as the oracle Bayes estimator.

Finally, the bottom-right panel of each figure gives the performance of the CSE and unrestricted and separable MLEs in the case that $\Sigma$ is truly separable (the oracle Bayes estimator in this case is exactly $\Sigma$). The performance of the CSE and KMLE are nearly identical, and much better than that of the unrestricted MLE. This is not too surprising given the observation from Figure 1 that $\hat{\nu}$ tends to overestimate $\nu$ when $\nu$ is a large (finite) value. Although any finite estimate $\hat{\nu}$ of $\nu$ is in some sense too small for this case where $\Sigma$ is exactly separable, $\hat{\nu}$ is generally large enough to make the shrinkage weight on $w$ nearly equal to one, which gives an estimate that is nearly identical to the KMLE.
4.2 Speech recognition

Many data analysis tasks rely on accurate covariance estimates, including tasks that are not specifically about covariance estimation. For example, quadratic discriminant analysis (QDA) is a simple and popular method of classification that relies on estimates of the population means and covariances of each potential class to which new observations are to be assigned. Specifically, the score of a new observation with feature vector $y \in \mathbb{R}^p$ with respect to category $k \in \{1, \ldots, K\}$ is

$$s_k(y) = (y - \hat{\mu}_k)^\top \hat{\Sigma}_k^{-1}(y - \hat{\mu}_k) + \ln |\hat{\Sigma}_k|,$$

where $(\hat{\mu}_k, \hat{\Sigma}_k)$ are estimates of the population mean and covariance of the feature vectors of objects in class $k$. If the frequencies of the different classes are equal, the classification rule is to assign the object with feature vector $y$ to the class with the minimum score. The accuracy of such a classification procedure will depend on, among other things, the accuracy of the mean and covariance estimates for each group. In cases where the feature vector $y$ is the vectorization of a matrix of features, we may consider using the core shrinkage estimator given by (9) to make classifications, as an alternative to either the unstructured MLE, the separable MLE, or other types of estimators.

As a numerical illustration, we consider classification of spoken-word audio samples for 10 command words (“yes”, “no”, “up”, “down”, “left”, “right”, “on”, “off”, “stop”, “go”), using the dataset provided by Warden (2017) and described in Warden (2018). The data we consider include 20,600 1-second long audio WAV files, with a per-word sample size ranging from 1,987 to 2,103 across the 10 words, representing between 989 and 1079 unique speakers for each word. We retain 100 audio samples per word for testing, and train our classifier on the remaining 19,600 audio samples. We do not make use of the fact that some speakers are represented multiple times in the dataset.

A standard set of features for audio classification are mel-frequency cepstral coefficients (MFCCs), which describe an audio sample in terms of a matrix whose dimensions represent periodicities in the power spectrum of the signal across time.
For each audio sample in the dataset, we computed a $p_1 \times p_2 = 99 \times 13$ matrix of the first 13 mel cepstral coefficients across 99 time bins using the function `melfcc` in the R-package `tuneR` (Ligges et al., 2018). Sample means and correlations for two of the words appear in Figures 4 and 5 (correlations instead of covariances are easier to visualize because of the large across-coefficient heteroscedasticity). The sample covariance matrices for these words are $p \times p = 1287 \times 1287$ matrices where, for example, the 99 $\times$ 99 block in the upper left corner is the sample covariance matrix for the first cepstral coefficient across the 99 time points.

From the training data, we computed sample means and several different covariance estimates for each of the ten words. Our primary interest is in comparing prediction accuracy of the core shrinkage estimator to that of the unstructured and separable MLEs, but we also compute predictions using estimates that are partially pooled across groups. Quadratic discriminant analysis using partially pooled covariance estimates often have better performance than using class-specific sample covariance matrices, particularly when the sample size $n$ is not large compared to the dimension $p$. A variety of methods exist for choosing the pooling weights (Greene and ...)
Here we use the approach outlined in Greene and Rayens (1989), which is based on an inverse-Wishart hierarchical model for Σ₁,...,Σ₁₀. The resulting partially pooled covariance estimates (PPEs) are each roughly equal to a 32%-68% weighted average of the word-specific sample covariance and the pooled sample covariance matrices respectively.

Classifications for the 100 training observations were made using each of the covariance estimates. Confusion matrices are displayed in Figure 6, with the true word classes along the rows, and the predicted classes along the columns. For example, the word “go” is most frequently misclassified as “no”. From the figure, QDA with the core shrinkage estimator appears to be substantially more accurate than using either the unstructured or separable MLEs, and is similar to using the partially pooled estimates. Rates of correct classification across all words for all four QDA classifiers are given in Table 1. The CSE performs better than the KMLE for all words, and better than the unrestricted MLE for all words except “left” and “stop”. However, this apparent good performance on these two words is misleading, as it is a result of this classifier assigning most words to being either in one of these two categories, as can be seen from Figure 6. Additionally, the CSE is as good or better than the
Figure 6: Confusion matrices resulting from the four covariance estimates. Rows correspond to target words and columns correspond to predictions.
|      | KMLE | CSE | MLE | PPE |
|------|------|-----|-----|-----|
| yes  | 0.69 | 0.79| 0.37| 0.82|
| no   | 0.74 | 0.82| 0.09| 0.66|
| up   | 0.38 | 0.51| 0.04| 0.46|
| down | 0.30 | 0.53| 0.24| 0.57|
| left | 0.44 | 0.60| 0.77| 0.51|
| right| 0.40 | 0.60| 0.45| 0.67|
| on   | 0.41 | 0.59| 0.16| 0.58|
| off  | 0.41 | 0.65| 0.22| 0.57|
| stop | 0.61 | 0.70| 0.75| 0.66|
| go   | 0.30 | 0.50| 0.14| 0.48|

Table 1: Rates of correct classification on the test dataset for the four classifiers

PPE for seven of the ten words. We note that the PPE is, like the CSE, a type of shrinkage estimator, although one that does not make use of the matrix structure of the data.

5 Discussion

Many classic estimators of covariance matrices are obtained by first computing the eigendecomposition of the sample covariance matrix and then regularizing the resulting eigenvalues (Stein, 1975; Takemura, 1983). The core shrinkage estimator proposed in this article can be viewed analogously: the Kronecker-core decomposition of the sample covariance matrix is computed, and then the resulting core is regularized. However, while existing distributional results for the sample eigenvalues permit theoretical risk calculations for unstructured covariance estimators, we lack such detailed knowledge of the distribution of sample core matrices. Further research on the distribution of sample cores could permit theoretical comparisons of different core shrinkage estimators.

Empirical results from the speech recognition task in Section 4.1 indicate that
in this multi-group setting, two distinct types of shrinkage - towards separability and towards a common unstructured covariance matrix - both provided performance improvements. This suggests that further improvements might be obtained with an estimator that combines these two types of shrinkage. Such an estimator could be obtained by empirically assessing the degree to which the covariance matrices are separable, as well as the degree to which they are similar to each other, and then shrinking the sample covariance matrices appropriately.

The results in this article extend naturally to separable covariance models for tensor-valued data, that is, data arrays having three or more index sets. For example, an empirical Bayes covariance estimator that shrinks a sample covariance matrix towards a Kronecker product of several smaller covariance matrices, one for each index set, can be derived as in Section 3.2, using the same objective function (8) to determine the amount of shrinkage. A less straightforward extension would be an estimator that adaptively shrinks towards an appropriate separable submodel, that is, submodels that are separable after various index sets of the data array have been collapsed.

**Proofs**

*Proof of Proposition 1.* We first obtain an identity that relates the expectations in (2) to the trace term in the divergence function \( d(K : \Sigma) \). Letting \( y \) be the vectorization of \( Y \), for \((K_1, K_2) \in S^+_{p_1} \times S^+_{p_2}\) we have

\[
\text{trace}((K_2^{-1} \otimes K_1^{-1}) \Sigma) = E[\text{trace}((K_2^{-1} \otimes K_1^{-1}) yy^\top)]
= E[y^\top (K_2^{-1} \otimes K_1^{-1}) y]
= E[\text{trace}(Y^\top K_1^{-1}Y K_2^{-1})]
= \text{trace}(K_1^{-1}E[Y K_2^{-1}Y^\top]) = \text{trace}(K_2^{-1}E[Y^\top K_1^{-1}Y]).
\]

Therefore, for \( K = K_2 \otimes K_1 \) the divergence function may be written

\[
d(K : \Sigma) = p_2 \ln |K_1| + p_1 \ln |K_2| + \text{trace}(K_1^{-1}E[Y K_2^{-1}Y^\top])
= p_2 \ln |K_1| + p_1 \ln |K_2| + \text{trace}(K_2^{-1}E[Y^\top K_1^{-1}Y]).
\]
Now suppose that $\Sigma_2 \otimes \Sigma_1 \in S^+_{p_1,p_2}$ minimizes the divergence. Then $\Sigma_1$ must also be the minimizer of the divergence in $K_1$ when $K_2$ is fixed at $\Sigma_2$, that is, $\Sigma_1$ minimizes $p_2 \ln |K_1| + \text{trace}(K_1^{-1}E[Y\Sigma_2^{-1}Y^\top])$ over $K_1 \in S^+_{p_1}$. It is well known (Anderson 2003, Section 4.1) that this function of $K_1$ is uniquely minimized by $E[Y\Sigma_2^{-1}Y^\top]/p_2$, and so $\Sigma_1 = E[Y\Sigma_2^{-1}Y^\top]/p_2$. Similarly, $\Sigma_2$ must equal $E[Y^\top\Sigma_1^{-1}Y]/p_1$, and so $(\Sigma_1, \Sigma_2)$ is a solution to (2).

Conversely, let $f(K_1, K_2 : \Sigma) = \ln |K_2 \otimes K_1| + \text{trace}(K_2 \otimes K_1)^{-1} \Sigma)$ be the divergence written as a real-valued function on $S^+_{p_1} \times S^+_{p_2}$. Differentiating $f$ with respect to $(K_1, K_2)$ shows that the stationary points of $f$ are the solutions to (2). Although $f$ is not convex, it is geodesically convex (Wiesel 2012), and so by Corollary 3.1 of Rapcsák (1991), every stationary point of $f$ is a global minimizer of $f$. Thus if $(K_1, K_2)$ is a solution to (2) then $K_2 \otimes K_1$ is a minimizer of $d$. \hfill \Box

\textit{Proof of Proposition 2}. Let $A = A_2 \otimes A_1$. For each $K$, we have

$$
d(K : A \Sigma A^\top) = \ln |K| + \text{trace}(K^{-1}A \Sigma A^\top)
= \ln |A^{-1}KA^{-\top}| + \text{trace}((A^{-1}KA^{-\top})^{-1} \Sigma) + \ln |AA^\top|
\equiv d(\tilde{K} : \Sigma) + \ln |AA^\top|
$$

where $\tilde{K} = A^{-1}KA^{-\top}$. Note that for $A \in GL_{p_1,p_2}$, $\{A^{-1}KA^{-\top} : K \in S^+_{p_1,p_2}\} = S^+_{p_1,p_2}$. By Proposition 1, $d(\tilde{K} : \Sigma)$ is minimized by $\tilde{K} = \Sigma_2 \otimes \Sigma_1$, and so $d(K : A \Sigma A^\top)$ is minimized by $K = AKA^\top = (A_2 \Sigma_2 A_2^\top) \otimes (A_1 \Sigma_1 A_1^\top)$. \hfill \Box

\textit{Proof of Corollary 4}. Items 1 and 2 can be shown by noting that the unconstrained minimizer of $\ln |K| + \text{trace}(K^{-1} \Sigma)$ over $K \in S^+_{p}$ is $\Sigma$, and so if $\Sigma \in S^+_{p_1,p_2}$ then the minimizer over $K \in S^+_{p_1,p_2}$ is $\Sigma$ as well. Alternatively, item 1 can be shown by noting that $I_p = I_{p_2} \otimes I_{p_1}$, and confirming that $(I_{p_1}, I_{p_2})$ provide a solution to (2) when $\text{Var}[Y] = I_p$. Item 2 can also be shown this way, or with Proposition 2. If $\Sigma = \Sigma_2 \otimes \Sigma_1$ then

$$
k(\Sigma) = k((\Sigma_2^{1/2} \otimes \Sigma_1^{1/2})I_p(\Sigma_2^{1/2} \otimes \Sigma_1^{1/2}))
= (\Sigma_2^{1/2} \otimes \Sigma_1^{1/2})k(I_p)(\Sigma_2^{1/2} \otimes \Sigma_1^{1/2})
= (\Sigma_2^{1/2} \otimes \Sigma_1^{1/2})(\Sigma_2^{1/2} \otimes \Sigma_1^{1/2}) = \Sigma.
$$

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Item 3 can also be obtained from Proposition 2 by choosing (for example) $A_1 = aI_{p_1}$ and $A_2 = I_{p_2}$. Finally, if $\text{Var}[Y] = \Sigma$ is diagonal then $\text{E}[y_i^\top A_1 y_{i'}] = 0$ for rows $y_i$ and $y_{i'}$ of $Y$ for any matrix $A_1 \in \mathbb{R}^{p_2 \times p_2}$ unless $i = i'$. As a result, $\text{E}[YA_1 Y^\top]$ is diagonal, as is $\text{E}[Y^\top A_2 Y^\top]$ for the same reason. This implies that if $(\Sigma_1, \Sigma_2)$ is a solution to (2) then both matrices are diagonal, as is their Kronecker product.

**Proof of Proposition 3.** If $\text{E}[YY^\top]/p_2 = I_{p_1}$ and $\text{E}[Y^\top Y]/p_1 = I_{p_2}$ then $(I_{p_1}, I_{p_2})$ is a solution to (2) and so $\text{E}[YY^\top] = I_{p_2} \otimes I_{p_1} = I_{p_2}$. Conversely, if $\text{E}[YY^\top]/p_2 = I_{p_1}$ and $\text{E}[Y^\top Y]/p_1 = I_{p_2}$ then any solution to (2) must be of the form $(cI_{p_1}, c^{-1}I_{p_2})$ for some $c > 0$, which then implies that $\text{E}[YY^\top]/p_2 = I_{p_1}$ and $\text{E}[Y^\top Y]/p_1 = I_{p_2}$. Finally, let $y_j$ be the $j$th column vector of $Y$. Then

$$\text{E}[YY^\top] = \sum_{j=1}^{p_2} \text{E}[y_j y_j^\top] = \sum_{j=1}^{p_2} \tilde{C}_{j,j},$$

**Proof of Proposition 4.** Let $c(\Sigma) = C$, $k(\Sigma) = K$ and $h(K) = H$, so $\Sigma = HCH^\top$. By Proposition 2, $k(A\Sigma A^\top) = AKA^\top = AHH^\top A^\top$. Let $\tilde{K} = AKA^\top$ and $\tilde{H} = h(\tilde{K})$. Then $c(A\Sigma A^\top) = \tilde{H}^{-1}(AH)C(AH)^\top \tilde{H}^{-\top}$. But by the definition of the square root function, we must have $\tilde{H}\tilde{H}^\top = \tilde{K} = AHH^\top A^\top$, and so $\tilde{H} = AHR^\top$ for some $R \in O_p$. Furthermore this $R$ must be separable because both $\tilde{H}$ and $AH$ are separable. Thus $\tilde{H}^{-1} = RH^{-1}A^{-1}$ and item 1 of the result follows. If $A \in \mathcal{H}$ and $\mathcal{H}$ is a group, then $AH \in \mathcal{H}$, and so $\tilde{H} \equiv h(AHH^\top A^\top) = AH$, giving item 2.

**Proof of Proposition 5.** First we show that $f$ is a bijection. For any $\Sigma \in \mathcal{S}_p^+$, let $H = h(k(\Sigma))$ and $C = c(\Sigma)$. Then

$$g(f(\Sigma)) = HCH^\top = H(H^{-1}\Sigma H^{-\top})H^\top = \Sigma.$$

Conversely, let $(C, K) \in \mathcal{C}_{p_1, p_2}^+ \times \mathcal{S}_{p_1, p_2}^+$. Then with $H = h(K)$, we have

$$f(g(C, K)) = f(HCH^\top) = (k(HCH^\top), c(HCH^\top)).$$

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Since $H \in S_{p_1,p_2}^+$, by Proposition 2 we have

$$
k(HCH^\top) = Hk(C)H^\top
= HIH^\top
= HH^\top = K.
$$

Finally,

$$
c(HCH^\top) = h(K)^{-1}(HCH^\top)h(K)^{-\top}
= H^{-1}(HCH^\top)H^{-\top} = C,
$$

and so $f(g(C,K)) = (C,K)$.

We now show that the Kronecker covariance function $k$ is continuous, from which the continuity results for $f$ and $g$ follow. The space $S_p^+$ is a complete Riemannian manifold with respect to the affine invariant metric $d_A : S_p^+ \times S_p^+ \to \mathbb{R}^+$ given by

$$
d_A(\Sigma, \tilde{\Sigma}) = \| \log(\Sigma^{-1/2} \tilde{\Sigma} \Sigma^{-1/2}) \|,
$$

where “log” is the matrix logarithm \cite{Bhatia2007, Higham2008}. Note that by the form of $d_A$ and the fact that $d_A(\Sigma, \tilde{\Sigma}) \leq d_A(\Sigma, I_p) + d_A(I_p, \tilde{\Sigma})$, a subset of $S_p^+$ is bounded under this metric if and only if the eigenvalues of its elements are bounded away from zero and infinity.

Let $\{S_n\}$ be a sequence in $S_p^+$ that converges to $\Sigma \in S_p^+$ in this metric. Convergence of the sequence implies it is bounded, and so there exists an interval $[a, b] \subset (0, \infty)$ that contains the eigenvalues of $S_n$ for all $n$. We now show that boundedness of $\{S_n\}$ implies that the sequence $\{K_n\} \equiv \{k(S_n)\}$ is bounded. Recall that $K_n$ is the minimizer of the divergence $d$ over $S_{p_1,p_2}^+$, and so $d(K_n : S_n) \leq d(I_p : S_n)$. Using this fact and the bounds on the eigenvalues of $\{S_n\}$, we have

$$
\sum_{j=1}^p (\log l_{n,j} + a/l_{n,j}) = d(K_n : aI_p) \leq d(K_n : S_n) \leq d(I_p : S_n) \leq pb,
$$

where $l_{n,j}$ is the $j$th largest eigenvalue of $S_n$. Noting that $\log x + a/x$ is a convex function with a minimum at $x = a$, we have for each $k \in \{1, \ldots, p\}$

$$
\log l_{n,k} + a/l_{n,k} \leq pb - \sum_{j \neq k} (\log l_{n,j} + a/l_{n,j}) \leq pb - (p-1) \times (\log a + 1).
$$
Since $\log x + a/x$ diverges as $x$ goes to zero or infinity, the above bound implies that there exists $[c, d] \subset (0, \infty)$ that contains $l_{n,k}$ for all $n$ and $k$, that is, $\{K_n\}$ is bounded.

Now let $\{K_{n_s}\}$ be any convergent subsequence of $\{K_n\}$ and let $K_{n_s} \to K^*$, and so $d(K_{n_s} : S_{n_s}) \leq d(K : S_n)$. Since $d$ is jointly continuous in both of its arguments, taking the limit of the previous inequality gives $d(K^* : \Sigma) \leq d(K : \Sigma)$, which implies that $K^* = K$. This implies that $K_n \to K$ because the closure of the bounded set $\{K_n\}$ is itself bounded and therefore sequentially compact by the completeness of $S_p^+$. Thus $k$ is continuous. Furthermore, since the topology of $S_p^+$ under the affine invariant metric is the same as that under the Euclidean metric [Lee 2018, Theorem 2.55], $k$ is continuous for this metric space as well. Finally, the functions $f$ and $g$ are continuous because they are both compositions of the continuous function $k$ with other continuous functions.

Proof of Proposition 6. We first find a limiting form for an objective function from which $\hat{\nu}$ is obtained. To facilitate our analysis, we use the objective function $l_n(r, \hat{C}) = -2 \log L(nr)/n + p(\log \frac{n}{2} - 1)$ with $L$ defined in (8), so that the estimated value of $\nu$ is $\hat{\nu} = n \times \arg \min_{r \geq (p+1)/n} l_n(r, \hat{C}) = \arg \max_{\nu \geq p+1} L(\nu)$, where now we make explicit the dependence of the objective function on the sample core matrix $\hat{C}$. As a function of $(r,C) \in \mathbb{R}^+ \times C_{p_1,p_2}^+$, the objective function is then $l_n(r, C) = a_n + b_n(r) + c_n(r, C)$ where $a_n = p(\log \frac{n}{2} - 1)$ and

$$b_n = -\frac{2}{n} \log \left( \frac{\Gamma_p(n(1+r)/2)}{\Gamma_p(nr/2)} \right) + p(1+r) \log(1+r+\delta) - pr \log(r+\delta)$$

$$c_n = (1+r) \log |(1-w)C + wI_p|$$

where $w = (r+\delta)/(1+r+\delta)$ with $\delta = -(p+1)/n$. We will show that as $n \to \infty$, $a_n + b_n(r)$ converges uniformly to zero for $r \in [\epsilon, \infty)$ and $c_n(r, C)$ converges uniformly to $l(r, C)$, where

$$l(r, C) = (1+r) \log |C/(1+r) + rI_p/(1+r)|.$$
determinant of a matrix is a continuous function, and so is uniformly continuous on the compact set of convex combinations of core matrices and the identity. Next, we have that 

\[(1 - w)C + wI_p\] converges uniformly to \[C/(1 + r) + rI_p/(1 + r)\], because the norm of their difference is \[\|w/\left(\frac{r}{1 + r}\right)(I_p - C)\| < \sqrt{p(1 + r)}\|w - \frac{r}{1 + r}\|\]

and \[|w - \frac{r}{1 + r}| = \frac{\delta}{(1 + r)(1 + \delta)}\] converges to zero uniformly in \(r\) for \(r > 0\).

Next we use Stirling’s approximation \[
\log(\Gamma(z)) = z\log(z) - z + \frac{1}{2}\log(2\pi/z) + O(z^{-1})
\]
on the multivariate gamma terms of \(b_n(r)\). Letting \(\delta_j = (1 - j)/n\), we have

\[-\frac{2}{n}\left(\log\left(\Gamma_p(n(1 + r)/2)\right) - \log\left(\Gamma_p(nr/2)\right)\right)
\]

\[-\frac{2}{n}\sum_{j=1}^{p}\left(\log\left(\Gamma\left(\frac{n(1 + r) + 1 - j}{2}\right)\right) + \log\left(\Gamma\left(\frac{nr + 1 - j}{2}\right)\right)\right)
\]

\[\sum_{j=1}^{p}\left(1 + r + \delta_j\right)\log\left(\frac{n(1 + r + \delta_j)/2 + (r + \delta_j)\log\left(n(r + \delta_j)/2\right) + 1}{n(n(1 + r)/2)}\right)
\]

\[-\frac{1}{n}\sum_{j=1}^{p}\log\left(\frac{r + \delta_j}{1 + r + \delta_j}\right) + O\left(n^{-1}(1 + r)^{-1}\right) + O\left((nr)^{-1}\right).
\]

The last three terms in the above expression converge uniformly to 0 over \(r \in [\epsilon, \infty)\) for any \(\epsilon > 0\). Adding \(a_n\) and the remaining terms of \(b_n(r)\) gives \(a_n + b_n(r)\) being approximately equal to

\[-r\sum_{j=1}^{p}\log\left(\frac{(1 + r + \delta_j)(r + \delta)}{(1 + r + \delta)(r + \delta_j)}\right) - \sum_{j=1}^{p}\log\left(\frac{1 + r + \delta_j}{1 + r + \delta}\right) - \sum_{j=1}^{p}\delta_j\log\left(\frac{1 + r + \delta_j}{r + \delta_j}\right).
\]

The second and third sums above converge uniformly to zero over \(r \in [\epsilon, \infty)\). Regarding the first sum, consider the ratio

\[
\left(\frac{1 + r + \delta_j}{1 + r + \delta}\right)^r = \left(1 + \frac{\delta_j - \delta}{1 + r + \delta}\right)^{r + 1 + \delta} \left(1 + \frac{\delta_j - \delta}{1 + r + \delta}\right)^{-(1 + \delta)}.
\]

The log of the second factor on the right converges to zero uniformly in \(r\). For the first factor we have

\[1 \leq \left(1 + \frac{\delta_j - \delta}{1 + r + \delta}\right)^{1 + r + \delta} \leq e^{\left|\delta_j\right| r + |\delta|} \to 1
\]
as \( n \to \infty \), where the first inequality follows from \( \delta_j - \delta \geq 0 \). Similarly,

\[
1 \geq \left( \frac{r + \delta_j}{r + \delta_j} \right)^r = \left( 1 + \frac{\delta - \delta_j}{r + \delta_j} \right)^r \geq \left( 1 + \frac{\delta - \delta_j}{\epsilon + \delta_j} \right)^{\epsilon + \delta_j} \left( 1 + \frac{\delta - \delta_j}{r + \delta_j} \right)^{-\delta_j} \to 1
\]
as \( n \to \infty \). Thus \( a_n + b_n(r) \) converges uniformly to zero on \( r \in [\epsilon, \infty) \) for any \( \epsilon > 0 \).

The above calculation shows that our objective function \( l_n(r, C) \) converges uniformly to \( l(r, C) \) for \( (r, C) \in [\epsilon, \infty) \times C_{p1, p2}^+ \). We want to show that this limiting objective function is strictly increasing in \( r \) if \( C \neq I_p \), so in this scenario where \( \Sigma \) is not separable the estimated weight on the sample Kronecker covariance converges to zero. To see that this is the case, let \( c_1, \ldots, c_p \) be the eigenvalues of \( C \), so that

\[
l(r, C) = \sum_{j=1}^{p} (1 + r) \log \left( \frac{r + c_j}{1 + r} \right) = \sum_{j=1}^{p} (1 + r) \log \left( 1 + \frac{c_j - 1}{1 + r} \right).
\]
The derivative of the \( j \)th term of the sum with respect to \( r \) is \( \log(1 + x) \geq x/(1 + x) \) for \( x > -1 \) (with strict inequality for \( x \neq 0 \)) this derivative is positive for \( (c_j - 1)/(r + 1) > -1 \), or equivalently, for \( c_j > -r \), which holds for each \( j = 1, \ldots, p \) because \( C \) is positive definite. Additionally, because \( C \neq I_p \) there is at least one \( j \) for which \( c_j \neq 1 \), so at least one term in the sum has a strictly positive derivative, making our objective function a strictly increasing function of \( r \).

Finally, let \( \hat{C} = c(S) \) and \( C = c(\Sigma) \). We want to show that \( \hat{r} \), the minimizer of \( l_n(r, \hat{C}) \) over \( r \geq (p + 1)/n \), converges in probability to zero if \( C \neq I_p \), or equivalently \( \Pr(\hat{r} > \epsilon) \to 0 \) for any \( \epsilon > 0 \). By the result in the previous paragraph, \( l(\epsilon/2, C) < l(\epsilon, C) \) and by the continuity of \( l \) there is a ball \( B \) around \( C \) that does not contain \( I_p \) such that

\[
\inf_{\hat{C} \in B} l(\epsilon, \hat{C}) - \sup_{\hat{C} \in B} l(\epsilon/2, \hat{C}) = \delta > 0.
\]

By the uniform convergence of \( l_n \) to \( l \), there is an \( N \) such that \( |l_n(r, \hat{C}) - l(r, \hat{C})| < \delta/2 \)
for \( n > N \) and all \( \hat{C} \in B \) and \( r \geq \epsilon/2 \). If \( \hat{C} \in B \) then for any \( r \geq \epsilon \)

\[
l_n(\epsilon/2, \hat{C}) < l(\epsilon/2, \hat{C}) + \delta/2 \leq \sup_{\tilde{C} \in B} l(\epsilon/2, \tilde{C}) + \delta/2
= \inf_{\tilde{C} \in B} l(\epsilon, \tilde{C}) - \delta/2
\leq l(\epsilon, \hat{C}) - \delta/2
\leq l(r, \hat{C}) - \delta/2
< l_n(r, \hat{C})
\]

and so \( \hat{r} < \epsilon \) for \( n > N \) and \( \hat{C} \in B \). Thus \( \Pr(\hat{r} > \epsilon) \leq \Pr(\hat{C} \notin B) \to 0 \) as \( n \to \infty \), because \( B \) is a neighborhood of \( C \) and \( \hat{C} \) is consistent for \( C \) by Corollary 2. Thus \( \hat{r} \) and \( \hat{w} \) converge in probability to zero as \( n \to \infty \) if \( C \neq I \), that is, if \( \Sigma \) is not separable.

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