Riemann curvature of a boosted spacetime geometry

Emmanuele Battista,1,2 ∗ Giampiero Esposito,2†
Paolo Scudellaro,1,2‡ and Francesco Tramontano1,2§

1Dipartimento di Fisica, Complesso Universitario di Monte S. Angelo,
Via Cintia Edificio 6, 80126 Napoli, Italy
2Istituto Nazionale di Fisica Nucleare, Sezione di Napoli,
Complesso Universitario di Monte S. Angelo,
Via Cintia Edificio 6, 80126 Napoli, Italy
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Abstract
The ultrarelativistic boosting procedure had been applied in the literature to map the metric of Schwarzschild-de Sitter spacetime into a metric describing de Sitter spacetime plus a shock-wave singularity located on a null hypersurface. This paper evaluates the Riemann curvature tensor of the boosted Schwarzschild-de Sitter metric by means of numerical calculations, which make it possible to reach the ultrarelativistic regime gradually by letting the boost velocity approach the speed of light. Thus, for the first time in the literature, the singular limit of curvature, through Dirac’s δ distribution and its derivatives, is numerically evaluated for this class of spacetimes. Eventually, the analysis of the Kretschmann invariant and the geodesic equation show that the spacetime possesses a “scalar curvature singularity” within a 3-sphere and it is possible to define what we here call “boosted horizon”, a sort of elastic wall where all particles are surprisingly pushed away, as numerical analysis demonstrates. This seems to suggest that such “boosted geometries” are ruled by a sort of “antigravity effect” since all geodesics seem to refuse to enter the “boosted horizon” and are “reflected” by it, even though their initial conditions are aimed at driving the particles towards the “boosted horizon” itself.

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∗E-mail: ebattista@na.infn.it
†E-mail: gesposit@na.infn.it
‡E-mail: scud@na.infn.it
§E-mail: tramonta@na.infn.it
I. INTRODUCTION

The subject of gravitational fields generated by sources which move at the speed of light has been extensively studied in the literature because of its close connection to the topic of gravitational waves, whose direct detection remains extremely difficult, since one normally deals with a very weak signal. The first who dealt with this aspect of general relativity was Tolman in 1934 \[1\], who studied the gravitational field of light beams and pulses in the linearized theory. But it was only in 1971 that Aichelburg and Sexl \[2\] developed a method to describe the gravitational field associated to a massless point particle moving at the speed of light (i.e. the gravitational field from a single photon). In fact in Ref. \[2\] the authors first derive this field by solving the linearized Einstein field equations for a particle with rest mass \(m\) moving uniformly with velocity \(v\). Then they take the limit \(v \to 1\) while the mass of the particle tends to zero in such a way that its energy remains finite. After that, they start with the full Einstein theory and the Schwarzschild metric (the exact metric describing a particle at rest), which written in isotropic coordinates reads as

\[
ds^2 = \frac{(1 - A)^2}{(1 + A)^2} dt^2 - (1 + A)^2 (dx^2 + dy^2 + dz^2), \tag{1.1}\]

with \(A = m/2r\) and \(r^2 = x^2 + y^2 + z^2\). Afterwards they apply to this metric a Lorentz transformation

\[
\bar{t} = (1 - v^2)^{-1/2} (t + vx), \tag{1.2}
\]
\[
\bar{x} = (1 - v^2)^{-1/2} (x + vt), \tag{1.3}
\]
\[
\bar{y} = y, \tag{1.4}
\]
\[
\bar{z} = z, \tag{1.5}
\]

to obtain the gravitational field as seen by an observer moving uniformly with velocity \(v\) relative to the mass. Once the limits \(v \to 1\) and \(m \to 0\) are taken, Aichelburg and Sexl obtain the remarkable result that both the linearized solution and the exact solution agree completely.

The method first developed by Aichelburg and Sexl is called in the literature “the boost of a metric”. With this procedure it is possible to show that the gravitational field of a null source is nonvanishing on a plane containing the particle and orthogonal to the direction of motion, i.e. plane-fronted gravitational waves. The Riemann curvature tensor is zero.
everywhere except on this plane, where it assumes a distributional nature. The intriguing fact is that the boosted metric in the ultrarelativistic regime \((v \to 1)\) has a new type of singularity, i.e. a distributional (Dirac-delta-like) singularity. The boosted ultrarelativistic metric obtained in Ref. \[2\] reads indeed as

\[
ds^2 = d\bar{t}^2 - d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2 - 4p\{(|\bar{t} - \bar{x}|)^{-1} - 2\delta(\bar{t}^2 - \bar{x}^2)\log \sqrt{y^2 + z} \}(d\bar{t} - d\bar{x})^2, \tag{1.6}
\]

with \(p \equiv m/\sqrt{1 - v^2}\).

Years after the work by Aichelburg and Sexl, more general impulsive waves were obtained by boosting other black hole spacetimes with rotation, charge and a cosmological constant \[3–6, 6–10\]. The technique of boosting a spacetime metric in fact has a lot of applications in theoretical physics. The work in Ref. \[11\], for instance, shows that the black hole formation caused by the collision of two particles with large relative velocity \((v \to 1)\), and considered in the rest frame of one of the particles, involves the concept of boosted metric: the gravitational field of the other particle is described by the ultrarelativistic boosted Schwarzschild-de Sitter metric. Moreover, the collisions of shock-waves and heavy ions as well as the entropy that is consequently produced \[12\] appeal to the boost procedure, also in the context of higher dimensions \[13\] and branes \[14–17\]. Furthermore, it is possible to study the formation of marginally trapped surfaces in the head-on collision both of two shock-waves \[18\] and of two ultrarelativistic charged particles \[19\] in de Sitter space by using the procedure of boosting a metric, since for example in the latter case the metric of the two charges is obtained by boosting Reissner-Nordström-de Sitter spacetime to the speed of light, while with similar arguments it is shown in Ref. \[20\] that the collision of two Reissner-Nordström gravitational shock-waves in anti-de Sitter space prevents the formation of marginally trapped surfaces of Penrose type. Finally, the concept of a boosted metric can be used as a tool to describe (de Sitter) spacetime from a quantum point of view \[21\].

Our main attention here will be devoted to the work in Refs. \[5, 10\] where it has been shown in detail how to map, through a boosting procedure, the Schwarzschild-de Sitter metric

\[
ds^2 = -\left(1 - \frac{2m}{r} - \frac{r^2}{a^2}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r} - \frac{r^2}{a^2}\right)} + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2), \tag{1.7}
\]
into the highly singular form (with $v \to 1$)

\[
\begin{align*}
\text{d}s^2 &= -\text{d}Y_0^2 + \text{d}Y_1^2 + \text{d}Y_2^2 + \text{d}Y_3^2 + \text{d}Y_4^2 \\
&\quad + 4p \left[ -2 + \frac{Y_1}{a} \log \left( \frac{a + Y_4}{a - Y_4} \right) \right] \delta(Y_0 + Y_1)(\text{d}Y_0 + \text{d}Y_1)^2,
\end{align*}
\]

(1.8)

where the first line describes de Sitter space viewed as a four-dimensional hyperboloid of radius $a$ having equation

\[
Y_0^2 = -a^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2,
\]

(1.9)

embedded into flat five-dimensional space, while the second line of (1.8) describes a shock-wave singularity located on the null hypersurface having equations

\[
Y_0 + Y_1 = 0,
\]

(1.10)

\[
Y_2^2 + Y_3^2 + Y_4^2 - a^2 = 0,
\]

(1.11)

equation (1.11) being obtained by the joint effect of the hyperboloid constraint (1.9) and the Dirac-delta condition (1.10). Since the metric is turned into a mathematical object having distributional nature, the usual spacetime picture is no longer valid, but it would be very interesting to evaluate the effect of these shock-wave singularities on curvature. The great revolution introduced by Einstein’s theory consists in fact in viewing the gravitational field as the curvature of spacetime. Such a curvature is directly coupled to the energy and momentum of whatever matter and radiation are present, as specified by the Einstein field equations whose content states that “the matter and the energy say to the spacetime how to curve, and the curvature of spacetime says to the matter how to move” [22]. Thus, one of the most important objects of the theory of the gravitational field is the Riemann tensor, since it is an intrinsic object that catches in an elegant and covariant way the features of spacetime curvature. Therefore it could be of great physical importance to evaluate the Riemann tensor for this type of geometries, i.e. “the boosted geometries”. (To fully appreciate the importance of this tensor see Appendix A).

Since “gravitation is a manifestation of spacetime curvature, and curvature shows up in the deviation of one geodesic from a nearby geodesic” [22], the concept of spacetime curvature is directly related to the geodesic completeness of spacetime, as we say that a spacetime manifold is geodesically complete if any geodesic can be extended to arbitrary
values of the affine parameter. Thus, knowledge of the Riemann curvature tensor is an 
essential step towards the description of topological features of spacetime and this motivates 
the effort we made in calculating the Riemann tensor for the boosted Schwarzschild-de Sitter 
metric.

We stress that the definitions (A1)-(A5) are given in terms of objects that, unlike the 
ones we will handle, have no distributional singularities (cfr (1.8)). Thus, in this article 
we are interested in a sort of generalization of the usual concept of Riemann tensor, which 
enlarges the notion of curvature, i.e. what we call “the boosted Riemann tensor”, with a 
particular interest in the ultrarelativistic regime, where distributional singularities show up. 
By virtue of the high difficulty of dealing with the metric (1.8), we decided to start from its 
low-velocity limit and then to reach the ultrarelativistic regime via numerical calculations. 
For this purpose, Sec. II evaluates the procedure to obtain the boosted Schwarzschild-de 
Sitter metric in manifestly four-dimensional form. Then, it is shown that the basis defined by 
the boost procedure is a coordinate basis, a property that greatly simplifies the calculations 
performed. In Sec. III the analysis of both the Kretschmann invariant and the geodesic 
equation allows us to characterize the features of curvature. Concluding remarks and open 
problems are presented in Sec. IV.

II. THE “BOOSTED” RIEMANN CURVATURE TENSOR

Following Refs. [5], [10] we can express a de Sitter spacetime in four dimensions as a 
four-dimensional hyperboloid of radius \(a\) embedded in five-dimensional Minkowski spacetime 
having metric

\[
d\sigma_M^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 + dZ_4^2,
\]

with coordinates satisfying the hyperboloid constraint

\[
a^2 = -Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2.
\]
By exploiting the relations between the $Z_i$ ($i = 0, 1, 2, 3, 4$) coordinates and the spherical static coordinates $(t, r, \theta, \phi)$

\[
Z_0 \equiv \sqrt{a^2 - r^2} \sinh(t/a), \quad (2.3)
\]
\[
Z_1 \equiv r \cos \theta, \quad (2.4)
\]
\[
Z_2 \equiv r \sin \theta \cos \phi, \quad (2.5)
\]
\[
Z_3 \equiv r \sin \theta \sin \phi \quad (2.6)
\]
\[
Z_4 \equiv \pm \sqrt{a^2 - r^2} \cosh(t/a), \quad (2.7)
\]

and on defining

\[
f^2 \equiv a^2 - r^2 = Z_4^2 - Z_0^2, \quad (2.8)
\]
\[
F_m \equiv 1 - \frac{2a^2m}{f^2r} - \frac{a^2/r^2}{1 - \frac{2a^2m}{f^2r}}, \quad (2.9)
\]
\[
Q \equiv 1 + \frac{2Z_0^2}{f^2}, \quad (2.10)
\]

we can express the Schwarschild-de Sitter metric (1.7) in the form

\[
ds^2 = h_{00} dZ_0^2 + h_{44} dZ_4^2 + 2h_{04} dZ_0 dZ_4 + dZ_1^2 + dZ_2^2 + dZ_3^2, \quad (2.11)
\]

where

\[
h_{00} \equiv -\frac{1}{2} (Q - 1) F_m - \left(1 - \frac{2a^2m}{f^2r}\right) \frac{Z_0^2}{r^2}, \quad (2.12)
\]
\[
h_{44} \equiv -\frac{1}{2} (Q + 1) F_m + \left(1 - \frac{2a^2m}{f^2r}\right) \frac{Z_4^2}{r^2}, \quad (2.13)
\]
\[
h_{04} \equiv \frac{Z_0 Z_4}{f^2} F_m + \frac{Z_0 Z_4}{r^2}. \quad (2.14)
\]

At this stage, we introduce a boost in the $Z_1$-direction by defining a new set of coordinates independent of $v$, i.e. the $Y_i$ coordinates, such that (hereafter $\gamma \equiv 1/\sqrt{1 - v^2}$)

\[
Z_0 = \gamma (Y_0 + vY_1), \quad (2.15)
\]
\[
Z_1 = \gamma (vY_0 + Y_1), \quad (2.16)
\]
\[
Z_2 = Y_2, \quad Z_3 = Y_3, \quad Z_4 = Y_4. \quad (2.17)
\]
Thus, starting from (2.11) jointly with (2.15)–(2.17) we eventually obtain the boosted Schwarzschild-de Sitter metric

\[ ds^2 = \gamma^2 (h_{00} + v^2) dY_0^2 + \gamma^2 (1 + v^2h_{00}) dY_1^2 + dY_2^2 + dY_3^2 + h_{44} dY_4^2 + 2v\gamma (1 + h_{00}) dY_0 dY_1 + 2\gamma h_{04} dY_0 dY_4 + 2v\gamma h_{04} dY_1 dY_4, \]

(2.18)

whose singular ultrarelativistic limit is expressed by (1.8). Thus, we can interpret (2.18) as the low-velocity limit of (1.8).

The spacetime metric (2.18) is apparently expressed by a 5 \times 5 matrix while the original metric (1.7) is expressed through 4 local coordinates \( t, r, \theta, \phi \). Hence also the metric (2.18) should be eventually expressed through 4 coordinates only, if one wants to arrive at a formula for the curvature, since our reference spacetime remains four-dimensional. To restore the usual four-dimensional form of the metric, we have to exploit the constraint (2.2) expressed in terms of \( Y_i \) coordinates, i.e. Eq. (1.9). By virtue of this condition we can write

\[ Y_0 = \sqrt{-a^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2} \equiv \sqrt{\sigma(Y_\mu)}, \]

(2.19)

\[ dY_0 = \frac{\prod_{\mu=1}^4 Y_\mu dY_\mu}{\sqrt{\sigma(Y_\mu)}}, \]

(2.20)

and eventually, using (2.19) and (2.20), we obtain the manifestly four-dimensional form of
the boosted metric \((2.18)\), which can be expressed by the relations

\[
\begin{align*}
g_{11} &= \frac{\gamma^2(h_{00} + v^2)}{\sigma} Y_1^2 + \gamma^2(1 + v^2 h_{00}) + \frac{2\nu \gamma^2 \sqrt{1 + h_{00}}}{\sqrt{\sigma}} Y_1, \\
g_{22} &= \frac{\gamma^2(h_{00} + v^2)}{\sigma} Y_2^2 + 1, \\
g_{33} &= \frac{\gamma^2(h_{00} + v^2)}{\sigma} Y_3^2 + 1, \\
g_{44} &= \frac{\gamma^2(h_{00} + v^2)}{\sigma} Y_4^2 + h_{44} + \frac{2\gamma h_{04}}{\sqrt{\sigma}} Y_4, \\
g_{12} &= \frac{\gamma^2(h_{00} + v^2)}{\sigma} Y_1 Y_2 + \frac{\nu \gamma^2 \sqrt{1 + h_{00}}}{\sqrt{\sigma}} Y_2, \\
g_{13} &= \frac{\gamma^2(h_{00} + v^2)}{\sigma} Y_1 Y_3 + \frac{\nu \gamma^2 \sqrt{1 + h_{00}}}{\sqrt{\sigma}} Y_3, \\
g_{14} &= \frac{\gamma^2(h_{00} + v^2)}{\sigma} Y_1 Y_4 + \frac{\nu \gamma^2 \sqrt{1 + h_{00}}}{\sqrt{\sigma}} Y_4 + \frac{\gamma h_{04}}{\sqrt{\sigma}} + v \gamma h_{04}, \\
g_{23} &= \frac{\gamma^2(h_{00} + v^2)}{\sigma} Y_2 Y_3, \\
g_{24} &= \frac{\gamma^2(h_{00} + v^2)}{\sigma} Y_2 Y_4 + \frac{\gamma h_{04}}{\sqrt{\sigma}} Y_2, \\
g_{34} &= \frac{\gamma^2(h_{00} + v^2)}{\sigma} Y_3 Y_4 + \frac{\gamma h_{04}}{\sqrt{\sigma}} Y_3.
\end{align*}
\]

Having obtained the formulas \((2.21)-(2.30)\), we can evaluate the Riemann-Christoffel symbols and consequently the Riemann curvature tensor of the boosted Schwarzschild-de Sitter metric by using the familiar relations of classical general relativity. The most general form of Riemann-Christoffel symbols reads as follows \((22)\) (\(a, b, c\) being abstract indices):

\[
\Gamma_{abc} = \frac{1}{2} \left( g_{ab,c} + g_{ac,b} - g_{bc,a} + c_{abc} + c_{acb} - c_{bca} \right),
\]

where the “commutation coefficients” \(c_{abc}\) are defined by

\[
[e_b, e_c] \equiv c_{bc}^a e_a,
\]

with \(\{e_a\}\) being any noncoordinate basis. Last, the components of the Riemann tensor are given by

\[
R_{bcd}^a = \Gamma_{bd,c}^a - \Gamma_{bc,d}^a + \Gamma_{bd}^c \Gamma_{ec}^a - \Gamma_{be}^c \Gamma_{cd}^a - \Gamma_{bd}^e \Gamma_{ce}^a - \Gamma_{be}^c \Gamma_{cd}^e.
\]

We can somewhat simplify the relations \((2.31), (2.33)\) in the case in which \(\frac{\partial}{\partial Y_\mu}\) (\(\mu\) being a coordinate index such that \(\mu = 1, 2, 3, 4\)) is a coordinate basis. As we know, the static
spherical basis \((t, r, \theta, \phi)\) is indeed a coordinate basis. Bearing in mind definitions \((2.3)\)–\((2.7)\), the Jacobian of the transformation between the spherical coordinates and the \(\{\partial / \partial Z^\mu\}\) is expressed by

\[
J_{\mu}^\lambda = \begin{pmatrix}
0 & \cos \theta & -r \sin \theta & 0 \\
0 & \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
0 & \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
\sqrt{a^2 - r^2} \frac{\sinh(t/a)}{a} & \frac{-r}{\sqrt{a^2 - r^2}} \cosh(t/a) & 0 & 0
\end{pmatrix}, \tag{2.34}
\]

while the inverse Jacobian reads as

\[
(J^{-1})^\mu_\lambda = \begin{pmatrix}
a r \cos \theta \coth(t/a) & a r \cos \phi \sin \theta \coth(t/a) & a r \sin \phi \cos \phi / r & a (\sinh(t/a))^{-1} \\
(a^2 - r^2) \cos \theta & (a^2 - r^2) \cos \phi \sin \theta & \sin \theta \sin \phi & 0 \\
- \sin \theta / r & \cos \theta \cos \phi / r & \cos \theta \sin \phi / r & 0 \\
0 & - \sin \phi / r \sin \theta & \cos \phi / r \sin \theta & 0
\end{pmatrix}. \tag{2.35}
\]

By virtue of \((2.34)\) and \((2.35)\), if we adopt the concise notation \(x_\lambda \equiv (t, r, \theta, \phi)\) we can write

\[
\frac{\partial}{\partial Z^\mu} = (J^{-1})^\mu_\lambda \frac{\partial}{\partial x_\lambda}, \tag{2.36}
\]

and, by exploiting the fact that \(\{\partial / \partial x_\lambda\}\) is a coordinate basis, after a lengthy calculation we arrive at the conclusion that also the basis \(\{\partial / \partial Z^\mu\}\) is a coordinate basis, or in other words we have that

\[
\left[ \frac{\partial}{\partial Z^\mu}, \frac{\partial}{\partial Z^\lambda} \right] = 0. \tag{2.37}
\]

The relations \((2.15)\)–\((2.17)\) for the boost show that the transformations between \(Z^\mu\) and \(Y^\mu\) are linear, therefore we can easily conclude that

\[
\left[ \frac{\partial}{\partial Y^\mu}, \frac{\partial}{\partial Y^\lambda} \right] = 0, \tag{2.38}
\]

hence the basis \(\{\partial / \partial Y^\mu\}\) is a coordinate basis as well.

This means that we can evaluate the Riemann-Christoffel symbols and the Riemann curvature tensor for the boosted spacetime metric \((2.21)\)–\((2.30)\) by setting \(c_{abc} = 0\) in the relations \((2.31)\), \((2.33)\). Nevertheless, these relations are still too complicated to be computed
analytically, and therefore a numerical calculation has been necessary. Formulas (2.21)–(2.30) show indeed that we are dealing with a spacetime metric represented by a $4 \times 4$ matrix whose elements are given by some complicated nonvanishing functions of the $Y_\mu$ coordinates. That is why we first tried to compute the Riemann curvature tensor analytically in terms of tetrads (see Appendix B) before realizing that even this solution was far too complicated. Thus, the only way we had to compute the Riemann-Christoffel symbols and the Riemann tensor was represented by numerical calculations. In this way we can evaluate the behavior of spacetime curvature also in the ultra-relativistic regime, which is the one we are mainly interested in, by letting the velocity defined by the boost relations (2.15)–(2.17) approach gradually the speed of light.

In what follows we discuss the results of our computation meanly by studying curvature invariants and the behavior of geodesics in our reference spacetime. We in fact think that these features represent the best tools to describe physically the concept of spacetime curvature.

III. THE KRETSCCHMANN INVARIANT

Intuitively, a spacetime singularity is a "place" where the curvature "blows up" or, by analogy with electrodynamics, a point where the metric tensor is either not defined or not suitably differentiable. Regrettably, both these statements are not rigorous definitions that can characterize the concept of singularity. First of all, since in general relativity we do not know the manifold and the metric structure in advance (they are solutions of Einstein field equations), we are not able to give a physical sense to the notion of an event until we solve Einstein equations, and hence the idea of a singularity as a "place" has not a satisfactory meaning. Moreover, also the notion of curvature becoming larger and larger as a general criterion for singularities has pathological problems. In fact, the bad behavior of components or derivatives of the Riemann tensor could be ascribed to the coordinate or tetrad basis used. To avoid this problem, one might examine scalar curvature invariants constructed from the Riemann tensor or its covariant derivatives, which in some cases can completely characterize the spacetime (see Refs. for further details). However, even if the value of some scalar invariants is unbounded, curvature might blow up only "as one goes to infinity", a case that we would interpret as a singularity-free spacetime.
Furthermore, spacetimes may be singular without any bad behavior of the curvature tensor (the so-called “conical singularities” [23]). Lastly, the bad behavior of the metric tensor at some spacetime points cannot be a way to define singularities, as one could always cut out such points and hence the remaining manifold, representing the whole spacetime, would turn out to be nonsingular.

A more satisfactory idea to define singularities is to use the notion of incompleteness of timelike geodesics, i.e. geodesics which are inextendible in at least one direction and hence have only a finite range of affine parameter. This has the immediate physical interpretation that there exist freely moving observers or particles whose histories did not exist after (or before) a finite interval of proper time. Although the physical meaning of affine parameter on null geodesics is different from the case of timelike geodesics, we could also regard null geodesic incompleteness as a good criterion to define spacetime singularities. Thus, timelike and null geodesic completeness are minimum conditions for spacetime to be considered singularity-free [24]. However, as there are examples of geodesically complete spacetimes which contain an inextendible timelike curve of bounded acceleration and finite length [27], we should generalize the concept of affine parameter to all $C^1$ curves, no matter whether they are geodesics or not. This fact is linked to the concept of b-completeness (short for bundle completeness), which we shortly describe following Refs. [24, 28] in Appendix C.

Therefore, we can classify a singularity represented by the presence of at least one incomplete geodesic according to whether [23]

1. a curvature invariant blows up along a geodesic (“scalar curvature singularity”),

2. a component of the Riemann tensor or its covariant derivatives in a parallelly propagated tetrad blows up along a geodesic (“parallelly propagated curvature singularity”),

3. no such invariant or component blows up (“noncurvature singularity”).

We can therefore understand the importance of scalar curvature invariants in the analysis of spacetime singularities. Being coordinate independent, curvature invariants can describe the size of curvature and its growth along timelike curves, and can also characterize curvature singularities [29], while providing important information about the nature of singularities. For example, in the case of Schwarzschild metric, which can be obtained from (1.7) if we put $a = \infty$ (for an unambiguous definition of the notion of limit applied to spacetimes
see Ref. [30]), the Kretschmann invariant (i.e. the Riemann tensor squared) is such that
\[ R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 48m^2/r^6, \]
in agreement with the fact that in all coordinate systems the real singularity is located only at \( r = 0 \) and not also at \( r = 2M \) (i.e. the event horizon).

In order to study the features of the Riemann curvature of spacetime described by the metric (1.8), we therefore decided to plot the Kretschmann invariant at different values of boost velocity \( v \) and study the geodesic equation
\[ \ddot{Y}^\mu(s) + \Gamma^\mu_{\nu\lambda} \dot{Y}^\nu(s) \dot{Y}^\lambda(s) = 0, \]
(3.1)
s being the affine parameter of the geodesic having parametric equation \( Y^\mu = Y^\mu(s) \).

From the analysis of the Kretschmann invariant we found that it is not defined unless the inequality (hereafter, numerical values of \( Y \) coordinates have downstairs indices, to be consistent with the notation in Sec. II)
\[ Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 > a^2, \]
(3.2)
is satisfied. Hence, we see that the hyperboloid constraint, condition (1.9), allows us to define a 3-sphere of radius \( a \) where the Kretschmann invariant is not defined. This peculiar feature of our “boosted spacetime geometry” is indeed obvious if we look at formulas (2.21)–(2.30), as here the quantities \( \sigma \) and \( \sqrt{\sigma} \) always appear at the denominator of the expressions of the metric tensor \( g_{\mu\nu} \), which means that the metric is defined only if the inequality (3.2) holds. Moreover, if we interpret \( Y_0 \) as the time coordinate (see (2.15)), we can view (3.2) as the condition on time.

In the \( Y_1 - Y_2 \) plane this 3-sphere becomes the circle with center at \( Y_1 = Y_2 = 0 \) and radius \( a \) of Fig. 1 which represents a contour plot of the Kretschmann invariant, i.e. a plot where each different color corresponds to different values of the Kretschmann invariant and these values increase as we approach this circle.

Another interesting feature of “boosted geometries” that we have found consists in the presence of a sort of barrier surrounding the 3-sphere, which we may call “boosted horizon”, in the sense that all geodesics, despite maintaining their completeness condition, are surprisingly pushed away from it.\(^1\) We have also discovered that the extension of the “boosted

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\(^1\) More precisely, one defines an “event horizon” as the boundary of the causal past of future null infinity \[ \text{[24]} \]. In the ultrarelativistic regime we cannot say if this concept is still valid and hence we talk about “boosted horizon” as the surface of spacetime where the Kretschmann invariant is not defined and where all geodesics, despite being complete, are pushed away.
FIG. 1: Contour plot of the Kretschmann invariant numerically obtained with the following values of parameters: $a = 1$, $m = 1$, $Y_3 = Y_4 = 0$ and $v = 0.9$. The dark purple zone represents the circle of radius $a$ where the Kretschmann invariant is not defined.

The horizon” depends on the boost velocity $v$, as we will shortly see. Since all geodesics are complete, according to standard definitions of general relativity the “boosted horizon” is not a singularity but, as we will show, it seems to be a sort of elastic wall which is hit by all particles before they get away. We have observed this effect numerically, by varying initial conditions of (3.1) and the boost velocity $v$, so as to reproduce different physical situations.

Figures 2 and 3 indeed represent one among the many situations analyzed which witness this “antigravity” effect. Figures 2 and 3 show in fact a particle initially lying on the $Y_1 = 0$ line of Fig. 1 and having an initial velocity directed toward the region where the Kretschmann invariant is not defined. Strikingly, the solution “refuses” to be attracted by the 3-sphere but, regardless of its initial velocity, the particle always arrives at a certain point and then it goes away from it, as if an elastic wall were present. We propose to call this elastic wall “boosted horizon”. The position of such a “boosted horizon” is independent of the initial velocity of the particle, but depends only on the boost velocity $v$. In fact, bearing in mind Fig. 1 both for particles coming from “above” (i.e. particles initially lying on the positive half-line $Y_2 > 0$, $Y_1 = 0$ and with $Y_2'(0) < 0$) and for those coming from “below” (i.e. particles initially lying on the negative half-line $Y_2 < 0$, $Y_1 = 0$ and with $Y_2'(0) > 0$), the position of the “boosted horizon” does not change, as Tab. 1 shows.

We have numerically checked, for each line of Tab. 1 that the minimum distance of the particle from the boundary of the 3-sphere is always bigger than its radius $a$, independently of the particle initial velocity. This means that the boosted horizon is always outside the
FIG. 2: Numerical solution of Eq. (3.1) for the function $Y_2(s)$ obtained in the $Y_1 - Y_2$ plane and with initial conditions $Y_1(0) = Y_3(0) = Y_4(0) = 0$, $Y_2(0) = 5$, $Y_1''(0) = Y_3''(0) = Y_4''(0) = 0$ and $Y_2''(0) = -0.7$. The values of parameters are $a = 1$, $m = 1$ and $v = 0.9$. It is possible to see an “antigravity effect”, since the function $Y_2(s)$ is pushed away from the “boosted horizon”, which is represented by the horizontal line located at $Y_2 = 2.12$.

FIG. 3: Numerical solution of Eq. (3.1) for the function $Y_2(s)$ obtained in the $Y_1 - Y_2$ plane and with initial conditions $Y_1(0) = Y_3(0) = Y_4(0) = 0$, $Y_2(0) = -5$, $Y_1''(0) = Y_3''(0) = Y_4''(0) = 0$ and $Y_2''(0) = 0.9$. The values of parameters are $a = 1$, $m = 1$ and $v = 0.9$. The function $Y_2(s)$ initially moves toward the “boosted horizon”, i.e. the horizontal line at $Y_2 = -2.12$, but then it is pushed away.

3-sphere. For example, we find that, when the boost velocity $v = 0.5$, the minimum distance $d_m = 3.1$ when $a = 1$, and it decreases monotonically as $v$ increases or decreases, reaching a minimum value of order $1.05 \div 1.10$.

The situation becomes somewhat intriguing when the particle lies initially on the $Y_2 = 0$ line (see Fig. 1). In fact, in the cases in which the particle lies initially on the positive half-line $Y_1 > 0$, $Y_2 = 0$, it always manages to hit the 3-sphere where the Kretschmann invariant is not defined, even if its initial velocity is extremely low, as we can see from Fig. 4. After the particle reaches the 3-sphere, its geodesic is not defined anymore and hence, according to definitions given above and those of Appendix C we can conclude that the
TABLE I: Position of the “boosted horizon” as a function of the boost velocity $v$. The positive sign refers to particles coming from “above” and the negative to those coming from “below”. The values of parameters are $a = 1$ and $m = 1$.

3-sphere of equation $Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = a^2$ defines a “scalar curvature singularity” for our “boosted geometry”.

When the particle lies initially on the negative half-line $Y_1 < 0$, $Y_2 = 0$, its geodesic is not defined even before it reaches the 3-sphere (see Fig. 5). This means that another “scalar curvature singularity” exists. Its position depends only on the boost velocity $v$ and not on the particle initial velocity.

In any case, numerical analysis shows that this kind of singularities exists only if the
FIG. 4: Numerical solution of Eq. (3.1) for the function $Y_1(s)$ obtained in the $Y_1-Y_2$ plane and with initial conditions $Y_1(0) = 5$, $Y_2(0) = Y_3(0) = Y_4(0) = 0$, $Y_1'(0) = -0.01$, $Y_2'(0) = Y_3'(0) = Y_4'(0) = 0$. The values of parameters are $a = 1$, $m = 1$ and $v = 0.9$. The particle manages to hit the 3-sphere, which is represented by the horizontal line $Y_1 = 1$.

FIG. 5: Numerical solution of Eq. (3.1) for the function $Y_1(s)$ obtained in the $Y_1-Y_2$ plane and with initial conditions $Y_1(0) = -5$, $Y_2(0) = Y_3(0) = Y_4(0) = 0$, $Y_1'(0) = 0.7$, $Y_2'(0) = Y_3'(0) = Y_4'(0) = 0$. The values of parameters are $a = 1$, $m = 1$ and $v = 0.8$. The particle does not manage to hit the 3-sphere but disappears in correspondence of the $Y_1 = -2.5$ line.

We have repeated the same analysis also by putting $Y_1 = Y_2 = 0$ in the relations defining the curvature, i.e. in the $Y_3-Y_4$ plane, and we have found the same “antigravity effect” of the previous cases, as shown in Figs. 6 and 7, which represent some examples among the many situations numerically analyzed. Interestingly, in this case we have found no “scalar curvature singularities”.

As one might expect, the things change somewhat as we approach the ultrarelativistic regime with $v = 0.9999$. As a consequence of the occurrence of Dirac’s $\delta$ in the metric (1.8), the “antigravity effects” of previous Figures disappear, as we can see from Fig. 8 which represents just one of the many situations we have analyzed in the ultrarelativistic regime.
FIG. 6: Numerical solution of Eq. (3.1) for the function $Y_3(s)$ obtained in the $Y_3 - Y_4$ plane and with initial conditions $Y_1(0) = Y_2(0) = 0$, $Y_3(0) = Y_4(0) = -5$, $Y'_1(0) = Y'_2(0) = 0$, $Y'_3(0) = Y'_4(0) = 0.566$. The values of parameters are $a = 1$, $m = 1$ and $v = 0.9$. The “antigravity effect” is once again evident.

FIG. 7: Numerical solution of Eq. (3.1) for the function $Y_4(s)$ obtained in the $Y_3 - Y_4$ plane and with initial conditions $Y_1(0) = Y_2(0) = 0$, $Y_3(0) = Y_4(0) = -5$, $Y'_1(0) = Y'_2(0) = 0$, $Y'_3(0) = Y'_4(0) = 0.566$. The values of parameters are $a = 1$, $m = 1$ and $v = 0.9$. The “antigravity effect” is once again evident.

FIG. 8: Numerical solution of Eq. (3.1) for the function $Y_2(s)$ obtained with initial conditions $Y_1(0) = Y_3(0) = Y_4(0) = 0$, $Y_2(0) = -5$, $Y'_1(0) = Y'_3(0) = Y'_4(0) = 0$ and $Y'_2(0) = 0.9$. The values of parameters are $a = 1$, $m = 1$ and $v = 0.9999$. The effect of Dirac’s delta produces a straight line solution with angular coefficient given by the value of $Y'_i(s)$ at $s = 0$, for any $i = 1, \ldots, 4$.  

17
IV. CONCLUDING REMARKS AND OPEN PROBLEMS

We have numerically evaluated, for the first time in the literature, the Riemann curvature of a boosted spacetime in the ultrarelativistic limit $v \to 1$, starting from Schwarzschild-de Sitter spacetime metric (1.7). We have exploited the fact that a de Sitter spacetime can be seen as a four-dimensional hyperboloid embedded in a five-dimensional spacetime and satisfying the constraint (2.2). After that, we have introduced the boosting procedure through the relations (2.15)–(2.17) which make it possible to obtain the boosted Schwarzschild-de Sitter metric (2.18), whose ultrarelativistic limit is represented by (1.8). By exploiting the hyperboloid constraint (2.2) we have then expressed (2.18) in the manifestly four-dimensional form (2.21)–(2.30). By virtue of (2.2), the metric components (2.21)–(2.30) are defined only if $\sigma > 0$, $\sigma$ being defined by relation (2.19). This fact is strictly related to inequality (3.2). In fact, $\{\partial/\partial Y^\mu\}$ being a coordinate basis, we have numerically computed the Riemann curvature tensor by using the usual relations of general relativity, and to better understand the features of curvature we have studied both the Kretschmann invariant and the geodesic equation (3.1). We have indeed found that the Kretschmann invariant is not defined unless (3.2) holds and thus we have just concluded that there exists a 3-sphere of radius $a$ where the spacetime possesses a “scalar curvature singularity”. In fact, from the numerical analysis of the geodesic equation, we have found that if the particle lies initially on the positive half-line $Y_1 > 0, Y_2 = 0$ of Fig. 1 it always reaches the 3-sphere (Fig. 4). After that, its geodesic is not defined anymore and hence we can conclude that the 3-sphere of equation $Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = a^2$ defines a “scalar curvature singularity” for the “boosted geometry” under investigation. When the particle lies initially on the negative half-line $Y_1 < 0, Y_2 = 0$, its geodesic is not defined even before it manages to reach the 3-sphere (see Fig. 5): a “scalar curvature singularity” whose position depends on the boost velocity $v$ there exists.

We have also discovered that “boosted geometries” are characterized by the presence of a sort of elastic wall surrounding the 3-sphere whose coordinates depend only on the boost velocity (see Tab. 1). All geodesics indeed, despite being complete, are always pushed away from there, as Figs. 2 and 3 show. We propose to call this barrier “boosted horizon” because, as in the case of Schwarzschild geometry, it is not a singularity of spacetime, but it is related to a sort of “antigravity effect” that should rule “boosted geometries”.

As we know, boosted geometries are ruled by the fact that both the spacetime metric and
the Riemann curvature tensor assume a distributional nature in the ultrarelativistic regime. In fact in this regime the “antigravity effect” disappears (see Fig. 8), as a consequence of the presence of Dirac’s delta in the metric (1.8).

We suppose that “antigravity effects” may result from the term $\Lambda = 3/a^2 > 0$ occurring in the Schwarzschild-de Sitter metric (1.7) (a positive cosmological constant $\Lambda$ represents a repulsive interaction), while “scalar curvature singularities” might be related to the presence of a more exotic object, i.e. a firewall [31], which can be a possible solution to an apparent inconsistency in black hole complementarity [32, 33].

Appendix A: The Riemann curvature tensor

Since our paper is addressed to a wide physics audience, we recall here some basic properties of pseudo-Riemannian geometry. The Riemann tensor can be defined in various alternative (and equivalent) ways [23], [34]. First, it can be defined as the map

$$R(X, Y, Z) \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$  \hfill (A1)

In the case in which $[X, Y] = 0$, the previous formula reduces to

$$R(X, Y, Z) \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z.$$  \hfill (A2)

Therefore, we can say that the Riemann tensor measures the failure of successive operations of differentiation to commute when applied to a dual vector field (which can be interpreted as the integrability obstruction for the existence of an isometry with Euclidean space), that is (in abstract index notation)

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = -R^{d}_{\;\;\;\;abc} \omega_d.$$  \hfill (A3)

Moreover, we can say that the failure of a vector to return to its original value when parallel transported around a small closed loop is directly connected to the Riemann tensor, which is in this way related to the path dependence of parallel transport. We can easily construct a small closed loop at $p \in M$ by choosing a two-dimensional surface $S$ through $p$ and choosing coordinates $t$ and $s$ in the surface. Next we construct the loop by moving of a quantity $\Delta t$
along the $s = 0$ curve, followed by moving $\Delta s$ along the $t = \Delta t$ curve and then we revert by $\Delta t$ and $\Delta s$. If we consider the vector $v^a$ at $p$ and parallel transport it around the closed loop we have just constructed, the change $\delta v^a$ to second order in the displacement $\Delta t$, $\Delta s$ that we register when we revert to the starting point involves once again the Riemann tensor, because we have

$$\delta v^a = \Delta t \Delta s \ v^d \ T^c \ S^b \ R^{a}_{debc}, \quad (A4)$$

where $T^c$, $S^b$ indicate the tangent to the curves of constant $s$ and $t$, respectively. Finally, the Riemann tensor appears also in the geodesic deviation equation, the equation which measures the tendency of geodesics to accelerate toward or away from each other. If $\gamma_s(t)$ denotes a smooth 1-parameter family of geodesics such that for each $s \in \mathbb{R}$ the curve $\gamma_s$ is a geodesic with affine parameter $t$, the geodesic deviation equation reads as

$$\alpha^c \equiv T^a \nabla_a (T^b \nabla_b X^c) = R^c_{def} T^d T^e X^f, \quad (A5)$$

where $\alpha^c$ is the relative acceleration of an infinitesimally nearby geodesic in the family, $X^a = \partial x^a(s,t)/\partial s$ is the deviation vector ($x^a(s,t)$ being the coordinates of one geodesic of the family $\gamma_s(t)$) and $T^b = \partial x^a(s,t)/\partial t$ represents the vector tangent to the geodesic. Therefore the equation $(A5)$ states that, if the curvature does not vanish, some initially parallel geodesics will fail to remain parallel: in the presence of a gravitational field the fifth postulate of Euclidean geometry is no longer valid.

**Appendix B: The tetrad formalism**

In most situations a curvature calculation that relies upon Christoffel symbols is extremely lengthy and not obviously feasible or readable. However, the tetrad formalism is known to simplify such a task, at least when the metric does not possess distributional singularities. Thus this appendix is devoted to some effort we made to express the highly singular ultrarelativistic boosted metric (1.8) in terms of tetrads.

As in the case of the boosted metric (2.18), starting from the ultrarelativistic metric (1.8) we can arrive at its manifestly four-dimensional form by exploiting (2.19) and (2.20) and hence we can eventually write the covariant metric components in the concise form

$$g_{kk} = 1 - \frac{Y_k^2}{\sigma(Y_j)} + \left( \frac{Y_k^2}{\sigma(Y_j)} + \delta_{1k} \right) f(Y_4) \delta \left( Y_1 + \sqrt{\sigma(Y_j)} \right), \ \forall k = 1, 2, 3, 4, \quad (B1)$$
\[ g_{1k} = -\frac{Y_1 Y_k}{\sigma(Y_j)} + \left( \frac{Y_1}{\sqrt{\sigma(Y_j)}} + 1 \right) \frac{Y_k}{\sqrt{\sigma(Y_j)}} f(Y_4) \delta \left( Y_1 + \sqrt{\sigma(Y_j)} \right), \quad \forall k = 2, 3, 4, \tag{B2} \]

\[ g_{2k} = -\frac{Y_2 Y_k}{\sigma(Y_j)} + \frac{Y_2 Y_k}{\sigma(Y_j)} f(Y_4) \delta \left( Y_1 + \sqrt{\sigma(Y_j)} \right), \quad \forall k = 3, 4, \tag{B3} \]

\[ g_{34} = -\frac{Y_3 Y_4}{\sigma(Y_j)} + \frac{Y_3 Y_4}{\sigma(Y_j)} f(Y_4) \delta \left( Y_1 + \sqrt{\sigma(Y_j)} \right), \tag{B4} \]

where

\[ f(Y_4) \equiv 4p \left[ -2 + \frac{Y_4}{a} \log \left( \frac{a + Y_4}{a - Y_4} \right) \right]. \tag{B5} \]

Since all components of this metric are nonvanishing, at this stage we still assume the existence of tetrad covectors \( e^a_{\mu} \) such that the covariant form of the metric reads as

\[ g_{\mu\nu} = e^a_{\mu} e^b_{\nu} \eta_{ab}, \tag{B6} \]

\( a, b \) being Lorentz-frame indices, and \( \eta_{ab} \) being the familiar Minkowski metric \( \text{diag}(-1, 1, 1, 1) \). By comparison of the formulae (B1)–(B4) with (B6) we find that one can set

\[ e^0_k = \frac{Y_k}{\sqrt{\sigma(Y_j)}}, \quad \forall k = 1, 2, 3, 4, \tag{B7} \]

while the other components of the singular, distribution-valued limit of tetrad covectors solve the following nonlinear algebraic system:

\[ \left( e^1_k \right)^2 + \left( e^2_k \right)^2 + \left( e^3_k \right)^2 = 1 + \left( \frac{Y_k^2}{\sigma(Y_j)} + \delta_{1k} \right) f(Y_4) \delta \left( Y_1 + \sqrt{\sigma(Y_j)} \right), \quad \forall k = 1, 2, 3, 4, \tag{B8} \]

\[ \sum_{i=1}^{3} e^i_1 e^i_k = \left( \frac{Y_1}{\sqrt{\sigma(Y_j)}} + 1 \right) \frac{Y_k}{\sqrt{\sigma(Y_j)}} f(Y_4) \delta \left( Y_1 + \sqrt{\sigma(Y_j)} \right), \quad \forall k = 2, 3, 4, \tag{B9} \]

\[ \sum_{i=1}^{3} e^i_2 e^i_k = \frac{Y_2 Y_k}{\sigma(Y_j)} f(Y_4) \delta \left( Y_1 + \sqrt{\sigma(Y_j)} \right), \quad \forall k = 3, 4, \tag{B10} \]

\[ \sum_{i=1}^{3} e^i_3 e^i_4 = \frac{Y_3 Y_4}{\sigma(Y_j)} f(Y_4) \delta \left( Y_1 + \sqrt{\sigma(Y_j)} \right). \tag{B11} \]

Since the system (B8)–(B11) consists of 10 equations for the 12 unknown tetrad covectors, it is possible to find at least a particular solution. Now, once we get such a solution, the procedure should be as follows. As we know from general relativity, whenever the spacetime manifold is parallelizable, we can always introduce a set of Lorentz frames \([35]\), so that the
spin-connection 1-form $\omega^{ab}_\mu = \omega^{ab}_\mu dx^\mu$ obtained from requiring that the torsion 2-form should vanish has components \[36\]

$$
\omega^{ab}_\mu = \frac{1}{2} e^{au} (e^b_{\nu,\mu} - e^b_{\mu,\nu}) - \frac{1}{2} e^{bu} (e^a_{\nu,\mu} - e^a_{\mu,\nu}) + \frac{1}{2} e^{au} e^{b\sigma} (e^c_{\nu,\sigma} - e^c_{\sigma,\nu}) e_{c\mu},
$$

(B12)

where

$$
e^{au} = \eta^{ab} e^u_b, 
 e_{c\mu} = e^a_{c\mu} \eta_{ac},
$$

(B13)

the tetrad vectors $e^a_{\mu}$ being computable by comparison from the relation

$$
dx^\mu = e^a_{\mu} e^a,
$$

(B14)

which holds by virtue of the definition of tetrad 1-forms

$$
e^a \equiv e^a_{\mu} dx^\mu,
$$

(B15)

jointly with \[36\]

$$
e^a_{\mu} e^a_{\mu} = \delta^a_{\mu}.
$$

(B16)

At this stage, we should be able to perform the curvature calculation bearing in mind that the Riemann curvature is described by the 2-form

$$
R^{ab} = \frac{1}{2} R^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu,
$$

(B17)

where the components are given by

$$
R^{ab}_{\mu\nu} = (\omega^{ab}_{\nu,\mu} - \omega^{ab}_{\mu,\nu}) + \eta_{cd} (\omega^{bd}_{\mu} \omega^{ca}_{\nu} - \omega^{ad}_{\mu} \omega^{cb}_{\nu}).
$$

(B18)

By virtue of Secs. 2 and 3, the singular limit of the curvature 2-form is a nontrivial mathematical object, since it involves the Dirac delta distribution, its fractional powers and its derivatives. Finally, the Riemann curvature tensor $R^{\mu}_{\nu\rho\sigma}$ can be obtained from the identity

$$
R^{\mu}_{\nu\rho\sigma} e^a_{\mu} = R^a_{b\rho\sigma} e^b_{\nu}.
$$

(B19)

Appendix C: The b-completeness of spacetime

The b-boundary construction is a device to attach to any spacetime a set of boundary points. Such a boundary point can be considered as an equivalence class of inextendible curves in a spacetime, whose affine length is finite \[28\].
Let \( \lambda(t) \) be a \( C^1 \) curve through a point \( p \) of a manifold \( M \) and let \( \{E_\mu\} \) (as before \( \mu = 1, 2, 3, 4 \)) be a basis for the tangent vector space at \( p \) to the manifold, \( T_p M \). We can propagate \( \{E_\mu\} \) along \( \lambda(t) \) to obtain a basis for \( T_{\lambda(t)} \), \( \forall t \). Then any \( V = (\partial/\partial t)_{\lambda(t)} \in T_{\lambda(t)} M \) can be expressed as \( V = V^\mu(t) E_\mu \) and we can define a generalized affine parameter \( u \) on the curve \( \lambda(t) \) by

\[
u = \int_p \left( \sum_\mu V_\mu V^\mu \right)^{1/2} \, dt. \quad (C1)
\]

Let \( \{E'_\mu\} \) be another basis of \( T_p M \). Then there exists some nonsingular matrix \( A^\mu_\nu \) such that

\[
E_\nu = \sum_{\mu'} A^\mu_{\nu'} E_{\mu'}. \quad (C2)
\]

As \( \{E'_\mu\} \) and \( \{E_\mu\} \) are parallelly transported along \( \lambda(t) \), this relation is valid with constant \( A^\mu_\nu \) and hence we have

\[
V'^\mu(t) = \sum_\nu A^\mu_{\nu'} V^\nu(t). \quad (C3)
\]

Since \( A^\mu_\nu \) is nonsingular, there exists some constant \( C > 0 \) such that

\[
C \sum_\mu V_\mu V^\mu \leq \sum_{\mu'} V'_\mu V'^\mu \leq C^{-1} \sum_\mu V_\mu V^\mu. \quad (C4)
\]

Thus, the length of a curve \( \lambda \) is finite in the parameter \( u \) if and only if it is finite in the parameter \( u' \). If \( \lambda \) is a geodesic then \( u \) becomes its affine parameter, but the definition given above is still valid since it has been formulated in terms of a general parameter \( u \) defined on any \( C^1 \) curve. Therefore, we say that a spacetime \( (M, g) \) is b-complete if there exists an endpoint for every \( C^1 \) curve of finite length as measured by a generalized affine parameter.

We have that b-completeness implies g-completeness (short for geodesic completeness), but the converse is not true. Therefore, we can define a spacetime to be singularity-free if it is b-complete. This means that g-completeness represents the minimum condition for a spacetime to be considered singularity-free.

23
Acknowledgments

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