PARALLEL TRANSPORT FOR HIGGS BUNDLES OVER $p$-ADIC CURVES

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Abstract. Faltings conjectured that under the $p$-adic Simpson correspondence, finite dimensional $p$-adic representations of the geometric étale fundamental group of a smooth proper $p$-adic curve $X$ are equivalent to semi-stable Higgs bundles of degree zero over $X$. In this article, we establish, over a $p$-adic curve of genus $g \geq 2$, an equivalence between these representations and Higgs bundles, whose underlying bundles potentially admit a strongly semi-stable reduction of degree zero. We show that these Higgs bundles are semi-stable of degree zero and investigate some evidence for the aforementioned conjecture.

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1. Introduction

1.1.1. In his seminal work [49], Simpson established a correspondence between finite dimensional complex linear representations of the topological fundamental group of a connected projective complex manifold and semi-stable Higgs bundles with vanishing Chern classes. This result generalizes the Narasimhan–Seshadri correspondence [41] between unitary irreducible representations of the fundamental group of a compact Riemann surface of genus $\geq 2$ and stable vector bundles of degree zero.

Inspired by these works, Faltings developed an analogue theory between (generalized) $p$-adic local systems and Higgs bundles over $p$-adic smooth proper varieties, called the $p$-adic Simpson correspondence [18]. Faltings’ short paper has been systematically studied by Abbes–Gros and Tsuji [4], each introducing their own new approach and laying the foundations of this theory.

Simultaneously and independently, Deninger and Werner developed a $p$-adic analogue of the Narasimhan-Seshadri correspondence: they associated functorially to every vector bundle over a $p$-adic smooth projective curve whose reduction is strongly semi-stable of degree zero a $p$-adic representation of the geometric fundamental group of the curve [13]. The author showed that the construction of Deninger–Werner is compatible with Faltings’ theory in the case where the Higgs field is trivial [55]. These results have been generalized to the case of higher dimensional $p$-adic varieties by Deninger–Werner [16] and Würthen [54].

Based on Scholze’s work [48], Liu and Zhu [38] developed a $p$-adic Riemann-Hilbert correspondence for $\mathbb{Q}_p$-local systems on a smooth rigid analytic variety over a finite extension of $\mathbb{Q}_p$, which allows them to associate nilpotent Higgs bundles to these arithmetic local systems. This work is recently extended to the
logarithmic case by Diao–Lan–Liu–Zhu [17]. Y. Wang compared Faltings’ approach in the good reduction case with the work of Liu–Zhu in his thesis [53].

Faltings imbedded the category of $p$-adic representations of the geometric fundamental group into the much larger category of so called “generalized representations” that he expected to be equivalent to the category of all Higgs bundles. The fundamental problem is then to understand which Higgs bundles arise from true $p$-adic representations. He formulated, for curves, the hope that true $p$-adic representations correspond to semi-stable Higgs bundles of slope zero. Faltings sketched a strategy of two steps to establish the equivalence between generalized representations and Higgs bundles. First, one establishes an equivalence between the subcategories of objects that are $p$-adically close from the trivial objects, that he qualifies as being “small”. Second, one extends the equivalence to all objects by descent, as any generalized representation étale locally becomes small. The first step has been developed by Abbes–Gros and Tsuji [4] in any dimension. The second step is only sketched for smooth proper curves by Faltings.

Recently, Heuer [30] established a new approach to the $p$-adic Simpson correspondence for smooth proper varieties. In this approach, generalized representations are considered as bundles for $v$-topology. Heuer and the author [31] geometrized the $p$-adic Simpson correspondence as a (twisted) isomorphism between moduli stacks of Higgs $G$-bundles and of $v$-$G$-bundles for a reductive group $G$ in the case curve. These works are based on the Hitchin fibration, the geometric Sen operator and are different to Faltings’ original approach.

1.1.2. In this article, we provide for curves a characterization of the Higgs bundles associated to true $p$-adic representations and we compare it with Faltings’ expected description. Our approach is inspired by Tsuji’s approach to the $p$-adic Simpson correspondence and Deninger–Werner’s theory.

Let us first describe these Higgs bundles. Let $K$ be a finite extension of $\mathbb{Q}_p$, $\overline{K}$ an algebraic closure of $K$, $\mathbb{C}$ (resp. $\mathbb{O}$) the $p$-adic completion of $\overline{K}$ (resp. $\mathbb{O}_\mathbb{K}$), $X$ a geometrically connected, smooth proper curve over $K$ and $\mathfrak{p}$ a geometric generic point of $X$. Roughly speaking, we establish an equivalence between the category $\text{Rep}(\pi_1(X_{\overline{K}}, \mathfrak{p}), \mathbb{C})$ of continuous finite dimensional $\mathbb{C}$-representations of $\pi_1(X_{\overline{K}}, \mathfrak{p})$ and a category of Higgs bundles over $X\mathbb{C}$, whose underlying bundles admit a strongly semi-stable reduction of degree zero as in Deninger–Werner’s theory, after taking a twisted inverse image over a finite étale cover of $X_{\overline{K}}$. This equivalence is compatible with the $p$-adic Simpson correspondence when restricted to small objects. We prove that these Higgs bundles are semi-stable of degree zero. It is conjectured that the converse holds. We discuss these results in more details in the following.

1.1.3. We first briefly explain the $p$-adic Simpson correspondence following Faltings’ original approach.

We set $S = \text{Spec}(\mathcal{O}_K)$. Let $X$ be a projective semi-stable $S$-scheme with geometrically connected generic fiber $X_{\mathbb{K}}$ and $\mathfrak{p}$ a geometric generic point of $X_{\mathbb{K}}$. The $p$-adic Simpson correspondence applies to certain objects more general than those of $\text{Rep}(\pi_1(X_{\overline{K}}, \mathfrak{p}), \mathbb{C})$, called generalized representations. They are certain modules of the Faltings ringed topos and locally can be expressed as semi-linear representations of $\pi_1(X_{\overline{K}}, \mathfrak{p})$ over finite projective modules over a certain $p$-adic ring equipped with a continuous action of $\pi_1(X_{\overline{K}}, \mathfrak{p})$. A generalized representation is called small, if locally, the associated representation admits a basis, whose $\pi_1(X_{\overline{K}}, \mathfrak{p})$-action is trivial modulo $p^\beta$ for some $\beta > \frac{1}{p-1}$ (definition 4.1.8). The category $\text{GRep}(X)$ of generalized representations over $X$ contains $\text{Rep}(\pi_1(X_{\overline{K}}, \mathfrak{p}), \mathbb{C})$ as a full subcategory and we denote by $\text{GRep}_{\text{small}}(X)$ its full subcategory of small objects.

On the other hand, a Higgs bundle over $X_{\mathbb{C}}$ is a pair $(M, \theta)$ consisting of a vector bundle $M$ over $X_{\mathbb{C}}$ together with a Higgs field $\theta: M \to M \otimes_{\mathcal{O}_{X_{\mathbb{C}}}} \Omega^1_{X_{\mathbb{C}}}(−1)$ such that $\theta \wedge \theta = 0$, where $−1$ denotes the inverse of the Tate twist. We say a Higgs bundle $(M, \theta)$ over $X_{\mathbb{C}}$ is small if it admits an integral model $(M^\circ, \theta)$ over $X_\mathbb{S}$, $X = X \otimes_{\mathcal{O}_K} \mathfrak{p}$ such that $\theta(M^\circ) \subset p^\alpha M^\circ \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ for some $\alpha > \frac{1}{p^\beta}$. Here, $\Omega^1_{X/S}$ denotes the $\mathcal{O}_X$-module of logarithmic differentials for the log structures $\mathcal{M}_X$ and $\mathcal{M}_S$. We denote by $\text{HB}(X_{\mathbb{C}})$ the category of Higgs bundles over $X_{\mathbb{C}}$ and by $\text{HB}_{\text{small}}(X_{\mathbb{C}})$ its full subcategory of small Higgs bundles.

Set $\mathcal{O}_{\mathfrak{p}} = \varprojlim_{x \to x} \mathcal{O}_{\mathfrak{p}}/p^\alpha \mathcal{O}_{\mathfrak{p}}$ and $A_{\text{inf}} = W(\mathcal{O}_{\mathfrak{p}})$ and let $\theta: A_{\text{inf}} \to \mathfrak{p}$ be Fontaine’s theta map. Let $\tilde{X}$ be the $p$-adic completion of $X_\mathbb{S}$. We take a family of log-smooth liftings $\mathcal{X} = (\mathcal{X}_n)_{n \geq 1}$ of $\tilde{X}$ over $(A_{\text{inf}}/\text{Ker}(\theta^n))_{n \geq 1}$.

\footnote{We restrict ourself to this case for the simplicity of presentation.}
with respect to log structures extending by $\mathcal{M}_X$ and $\mathcal{M}_S$ (see § 2.1.9 for more details). Based on the works of Faltings [18], Abbes–Gros [4], Tsuji [4, 52], the lifting $\mathcal{X}$ induces an equivalence of categories:

(1.1.3.1) \[ H_{\mathcal{X}} : G\text{Rep}_{\text{small}}(X) \simeq HB_{\text{small}}(X_C), \]

called the $p$-adic Simpson correspondence.

1.1.4. In the following, we assume moreover that $X$ is a stable curve over $S$ and that the genus of $X_R$ is $\geq 2$. In this article, we construct a functor (5.2.2.1) via $H_{\mathcal{X}}$:

(1.1.4.1) \[ H_{\mathcal{X}, \text{Exp}} : \text{Rep}(\pi_1(X_R, \overline{\tau}), C) \to HB(X_C). \]

The construction relies on a section $\text{Exp} : C \to 1 + m$ (A.0.0.2) of the $p$-adic logarithmic homomorphism $\log : 1 + m \to C$, $x \mapsto \sum_{n=1}^{\infty} (-1)^{n+1} (x-1)^n$. For small $C$-representations, the above functor is explicitly defined by gluing a local version of the $p$-adic Simpson correspondence ([4] § II.13, II.14).

To extend the construction for all $C$-representations, an important tool is the twisted inverse image functor for Higgs bundles, sketched in Faltings’ paper [18]. We present this functor in a more general setting in the appendix, joint with T. He. Let $Y$ be a projective stable $S$-curve and $f : Y \to X$ a proper $S$-morphism, which is finite over $X_K$, and $\mathcal{Y}$ a family of log-smooth liftings of $\mathcal{Y}$ over $(A_{fil} / \text{Ker}(\theta))^n_{n \geq 1}$. There exists a twisted inverse image functor associated to these data (A.4.3.1):

(1.1.4.2) \[ f_{\mathcal{Y}, \text{Exp}}^Y : HB(X_C) \to HB(Y_C), \]

which is different from the usual inverse image functor by twisting a line bundle on the spectral curve. Its restriction to small Higgs bundles is compatible with the natural inverse image functor of generalized representations via $p$-adic Simpson correspondences $H_{\mathcal{X}}$ and $H_{\mathcal{Y}}$. Its restriction to vector bundles (Higgs bundles with zero Higgs field) is compatible with the usual inverse image functor for vector bundles.

The twisted inverse image functor and the étale descent for Higgs bundles allow us to define the functor $H_{\mathcal{X}, \text{Exp}}$. The same argument also extends $H_{\mathcal{X}}$ to a functor $H_{\mathcal{X}, \text{Exp}}$ from generalized representations over $X$ to Higgs bundles over $X_C$.

In this article, we describe the essential image of $H_{\mathcal{X}, \text{Exp}}$, in terms of Deninger–Werner’s theory [13], and then construct a quasi-inverse of $H_{\mathcal{X}, \text{Exp}}$ (1.1.13).

1.1.5. In summary, we have the following commutative diagram

(1.1.5.1)

\[
\begin{array}{ccc}
\text{GRep}_{\text{small}}(X) & \xrightarrow{H_{\mathcal{X}}} & HB_{\text{small}}(X_C) \\
\downarrow \quad \text{GRep}(X) & & \downarrow \\
\text{Rep}(\pi_1(X_R, \overline{\tau}), C) & \xrightarrow{H_{\mathcal{X}, \text{Exp}}} & HB(X_C) \\
& & \downarrow \\
& & VB_DW^C(X_C)
\end{array}
\]

Let us briefly explain how Deninger–Werner’s functor $V_{DW}$ [13] fits into this picture. Recall that a vector bundle over a smooth proper curve $C$ of characteristic $p > 0$ is strongly semi-stable if its inverse images by all the power of the absolute Frobenius $F_C$ of $C$ are semi-stable. A vector bundle $M$ over $X_s$ is called Deninger–Werner if its restriction to the normalisation of each irreducible component of $X_R$ is strongly semi-stable of degree zero. A vector bundle over $X_C$ is Deninger–Werner if it admits a Deninger–Werner vector bundle over $X_s$ as an integral model. For every $n \in \mathbb{N}$, the reduction modulo $p^n$ of a Deninger–Werner vector bundle over $X_s$ can be trivialized by a proper morphism of $S$-curves $Y \to X$, which is finite étale over $X_K$ (after taking a finite extension of $K$). By parallel transport, Deninger and Werner constructed the functor $V_{DW}$ in the above diagram. The vertical functor $VB_DW^C(X_C) \to HB(X_C)$ is given by equipping each vector bundle with the trivial Higgs field, and factors through $HB_{\text{small}}(X_C)$. The compatibility between $V_{DW}$ and $H_{\mathcal{X}}$ is proved in the author’s thesis [55].
1.1.6. In the following, we characterize the essential image of $\mathbb{H}_{X,\text{Exp}}$ (1.1.4.1) in terms of certain Higgs bundles related to Deninger–Werner vector bundles.

We denote by $\text{HB}^{\text{DW}}(X_o)$ the category of pairs $(M, \theta)$ consisting of a Deninger–Werner vector bundle $M$ on $X_o$ together with a small Higgs field $\theta$ on $M$, and by $\text{HB}^{\text{DW}}(X_C)$ the full subcategory of $\text{HB}(X_C)$ consisting of Higgs bundles which admits an integral model in $\text{HB}^{\text{DW}}(X_o)$. A Higgs bundle over $X_C$ is called potentially Deninger–Werner, if it belongs to $\text{HB}^{\text{DW}}(Y_C)$ after taking a twisted inverse image functor (1.1.4.2). We denote by $\text{HB}^{\text{DW}}_{X,\text{Exp}}(X_C)$ the category these Higgs bundles over $X_C$. Here is our main result.

**Theorem 1.1.7** (5.2.6, 6.3.1). (i) The functor $\mathbb{H}_{X,\text{Exp}}$ factors through the category $\text{HB}^{\text{DW}}_{X,\text{Exp}}(X_C)$:

$$\mathbb{H}_{X,\text{Exp}} : \text{Rep}(\pi_1(X_{\overline{F}}, \overline{\tau}), C) \to \text{HB}^{\text{DW}}_{X,\text{Exp}}(X_C).$$

(ii) There exists a quasi-inverse $\mathbb{V}_{X,\text{Exp}}$ of $\mathbb{H}_{X,\text{Exp}}$:

$$\mathbb{V}_{X,\text{Exp}} : \text{HB}^{\text{DW}}_{X,\text{Exp}}(X_C) \rightarrow \text{Rep}(\pi_1(X_{\overline{F}}, \overline{\tau}), C).$$

**Remark 1.1.8.**

(i) A Higgs bundle $(M, 0)$ with zero Higgs field belongs to $\text{HB}^{\text{DW}}_{X,\text{Exp}}(X_C)$ if and only if $M$ potentially belongs to $\text{VB}^{\text{DW}}(X_C)$ (3.1.2, theorem 6.4.1). The functor $\mathbb{V}_{X,\text{Exp}}$ is compatible with $\mathbb{V}^{\text{DW}}$. We abusively call that these vector bundles are potentially of Deninger–Werner.

(ii) For an object $(E, \theta)$ of $\text{HB}^{\text{DW}}_{X,\text{Exp}}(X_C)$, there exists an $S$-morphism $f : Y \to X$ and a lifting $\mathcal{Y}$ as in (1.1.4) such that $f_K$ is finite étale and that $f^*_{\overline{\mathcal{Y}}, X,\text{Exp}}(E, \theta)$ belongs to $\text{HB}^{\text{DW}}(Y_C)$ (corollary 6.3.8).

**Theorem 1.1.9** (§ 5.3, 6.4.6). (i) [Faltings] Every object of $\text{HB}^{\text{DW}}_{X,\text{Exp}}(X_C)$ is semi-stable of degree zero.

(ii) Every Higgs line bundle of degree zero over $X_C$ belongs to $\text{HB}^{\text{DW}}_{X,\text{Exp}}(X_C)$.

(iii) The category $\text{HB}^{\text{DW}}_{X,\text{Exp}}(X_C)$ is abelian and is closed under extension.

In ([18] § 5), Faltings claimed that Higgs bundles over $X_C$, associated to $\text{Rep}(\pi_1(X_{\overline{F}}, \overline{\tau}), C)$, are semi-stable of degree zero and gave a brief sketch of the proof. Our proofs of theorems 1.1.7(i), 1.1.9(i) are inspired by Faltings’ sketched argument. Moreover, he expressed the hope that all semi-stable Higgs bundles of degree zero can be obtained in this way:

**Question 1.1.10** (Faltings). Does every semi-stable Higgs bundle of degree zero over $X_C$ belong to $\text{HB}^{\text{DW}}_{X,\text{Exp}}(X_C)$?

**Remark 1.1.11.** (i) For vector bundles (as Higgs bundles with zero Higgs field), the above question asks whether every semi-stable vector bundle of degree zero is potentially of Deninger–Werner (remark 1.1.8(i)). Andreatta [5] recently gave a negative answer to the above question for vector bundles over a smooth projective curve with good reduction. The above question is still widely open for Higgs bundles with non-zero Higgs fields.

(ii) Our article restricts to curves with genus $\geq 2$. Heuer–Mann–Werner [27] studied the $p$-adic Simpson correspondence and described the essential image of $C$-representations of the fundamental group over elliptic curves (and more generally over abeoids).

(iii) Heuer [26, 28] studied the $p$-adic Simpson correspondence for line bundles over smooth proper varieties and a geometrisation of this correspondence. A similar statement of theorem 1.1.9(ii) was proved in [28].

Moreover, we can compare the cohomologies of two sides in this correspondence by considering extensions:

**Proposition 1.1.12.** Let $(M, \theta)$ be a Higgs bundle of $\text{HB}^{\text{DW}}_{X,\text{Exp}}(X_C)$ and $V = \mathbb{V}_{X,\text{Exp}}(M, \theta)$ the associated $C$-representation. There exists a canonical $C$-linear isomorphism:

$$H^1(X_{\overline{F}, \text{ét}}, V) \simeq H^1(X_C, M \rightarrow M \otimes_{\mathcal{O}_{X_C}} \mathcal{O}_{X_C}(-1)).$$

1.1.13. We outline the proof of above theorems.

Assertion (i) of theorem 1.1.7 follows from the explicit construction of $\mathbb{H}_{X,\text{Exp}}$ (proposition 5.1.1).
The construction of the functor $\mathbb{V}_{\mathcal{X}, \exp}$ is inspired by $\mathbb{V}^{\text{DW}}$. Recall (1.1.5) that, for every $n \geq 1$, the reduction modulo $p^n$ of a Deninger–Werner vector bundle over $X_{\mathcal{S}}$ can be trivialized by a proper morphism of $S$-curves $f : Y \to X$ such that $f_K$ is finite étale. We extend this trivializing property to $\text{HB}^{\text{DW}}(X_{\mathcal{S}})$ (see theorem 3.2.2 for a precise statement in terms of Higgs crystals). Via Tsuji’s approach to the $p$-adic Simpson correspondence ([4] Chapter IV), we obtain a similar trivializing property for generalized representations over $X$ associated to $\text{HB}^{\text{DW}}(X_{\mathcal{C}})$. This allows us to apply a parallel transport functor ([55] § 8) from these generalized representations over $X$ to $\text{Rep}(\pi_1(X_{\mathcal{F}}), \mathcal{C})$ (theorem 6.2.1(i)). Combined with these constructions, we obtain a functor

$$\mathbb{V}_X : \text{HB}^{\text{DW}}(X_{\mathcal{C}}) \to \text{Rep}(\pi_1(X_{\mathcal{F}}), \mathcal{C}).$$

We extend $\mathbb{V}_X$ to the functor $\mathbb{V}_{\mathcal{X}, \exp}$ by descent.

When $\mathbb{V}_{\mathcal{X}, \exp}$ (resp. $\mathbb{H}_{\mathcal{X}, \exp}$) restricts to small Higgs bundles (resp. small $\mathcal{C}$-representations), we show that it is compatible with the $p$-adic Simpson correspondence $\mathbf{H}_X$ (and is independent of the choice of $\exp$). This assertion for $\mathbb{V}_{\mathcal{X}, \exp}$ (theorem 6.2.1(iii)) relies on a recent work of T. He on the cohomological descent in Faltings topos [23] (another approach to this result is also obtained by Abbes–Gros [2] proposition 4.6.30). Then, we deduce that $\mathbb{V}_{\mathcal{X}, \exp}$ is a quasi-inverse of $\mathbb{H}_{\mathcal{X}, \exp}$ via the $p$-adic Simpson correspondence. Finally, we prove theorem 1.1.9 using similar properties for Deninger–Werner vector bundles [13].

1.1.14. We briefly go over the structure of this article. Section 2 contains a brief review of Tsuji’s interpretation of small Higgs bundles as Higgs crystals over a site and some complements. Section 3 is devoted to the trivializing property of Deninger–Werner Higgs bundles (1.1.13). In section 4, we briefly review Tsuji’s approach to $p$-adic Simpson correspondence and a local description as well. In section 5, we present the construction of $\mathbb{H}_{\mathcal{X}, \exp}$ and prove theorem 1.1.7(i) and theorem 1.1.9. Section 6 is devoted to the construction of $\mathbb{V}_{\mathcal{X}, \exp}$ and the proof of theorem 1.1.7(ii). In the appendix, jointly with T. He, we define the twisted inverse image functor (1.1.4.2) in a more general setting.

1.2. Notations and conventions. In this article, $\mathcal{N}$ denotes the set of positive integers.

1.2.1. Let $K$ be a complete discrete valuation field of characteristic zero, with an algebraically closed residue field $k$ of characteristic $p > 0$, $\mathcal{O}_K$ the valuation ring of $K$, $\overline{K}$ an algebraic closure of $K$, $\mathcal{O}_\mathcal{F}$ the integral closure of $\mathcal{O}_K$ in $\overline{K}$, $\mathfrak{o}$ the $p$-adic completion of $\mathcal{O}_\mathcal{F}$ and $\mathcal{C}$ the fraction field of $\mathfrak{o}$. In sections 3, 5 and 6, we assume moreover that $k$ is an algebraic closure of $\mathbb{F}_p$.

We choose a compatible system $(p^{\varepsilon_n})_{n \geq 1}$ of $n$-th roots of $p$ in $\mathcal{O}_\mathcal{F}$. For any rational number $\varepsilon > 0$, we set $p^{\varepsilon} = (p^{\frac{\varepsilon}{n}})^{n\varepsilon}$, where $n$ is a positive integer such that $n\varepsilon$ is an integer, and $\mathfrak{o}_e = \mathfrak{o}/p^{\varepsilon}\mathfrak{o}$.

We set $S = \text{Spec}(\mathcal{O}_K)$, $\overline{S} = \text{Spec}(\mathcal{O}_\mathcal{F})$. We denote by $s$, $\eta$ and $\overline{s}$ the special point, generic point and geometrically generic point of $S$ respectively. We equip $S$ with the log structure $\mathcal{M}_S$ defined by the closed point (i.e. $\mathcal{M}_S = j_* (\mathcal{O}_S^*) \cap \mathcal{O}_S$). Let $A_{\text{inf}}$ be $\text{W}(\mathcal{O}_\mathcal{F}, p) = (p, p^{\frac{1}{\varepsilon}}, \cdots) \in (\mathcal{O}_\mathcal{F})^\mathfrak{o}$, $\xi = (p, p^{\frac{1}{\varepsilon}}, \cdots)$ a generator of $\text{Ker}(\theta : A_{\text{inf}} \to \mathfrak{o})$ and $B^+_{\text{dr}, 2}$ the completion of $A_{\text{inf}}[\frac{1}{p}]$ with respect to $\text{Ker}(\theta[\frac{1}{p}])$. We set $B^+_{\text{dr}, 2} = B^+_{\text{dr}}/\xi^2$.

Let $X$ be a scheme (resp. formal scheme, resp. a rigid analytic space over $\mathcal{C}$) such that $\mathcal{O}_X$ is coherent. Then, we say a coherent $\mathcal{O}_X$-module is a vector bundle if it is locally free of finite type. We denote by $\text{Coh}(X)$ (resp. $\text{VB}(X)$) the category of coherent modules (resp. vector bundles) over $X$.

1.2.2. Given an $S$-scheme $X$, we set $\overline{X} = X \times_S \overline{S}$. We say an $S$-scheme of finite type $X$ is semi-stable (or $X$ is a semi-stable $S$-scheme) if, étale locally on $X$, $X$ is isomorphic to

$$\text{Spec}(\mathcal{O}_K[t_0, \cdots, t_b, t_{b+1}, \cdots, t_c] / (t_1 \cdots t_b - \pi^{e_0})),
$$

for some integers $0 \leq b \leq c$, $e \geq 1$ and an uniformizer $\pi$ of $\mathcal{O}_K$.

We denote by $\mathcal{M}_X$ the log structure on $X$ associated with the open immersion $\eta : S \to X$ ([43] III.1.6.1). The structure morphism $f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ admits a semi-stable chart as follow ([25] example 5.6), is a smooth, saturated morphism between fine, saturated log schemes and moreover adequate in the sense of ([4] III.4.7) (c.f. [25] lemma 12.3).
Let $P$ be the submonoid of $\mathbb{Z}_{\geq 0}^{1+b} \oplus \mathbb{Z}^{c-b}$ generated by $(c\mathbb{Z}_{\geq 0})^{1+b} \oplus \mathbb{Z}^{c-b}$ and $(1,1,\ldots,1,0,\ldots,0) \in \{1\}^{1+b} \times \{0\}^{c-b}$. In other words, 

$$P = \{(a_0,\ldots,a_c) \in \mathbb{Z}_{\geq 0}^{1+b} \oplus \mathbb{Z}^{c-b} | a_0 \equiv a_1 \equiv \cdots \equiv a_b \mod e\}.$$ 

Then, $P$ is a fine, saturated monoid. Let $\alpha : \mathbb{Z}_{\geq 0} \rightarrow \mathcal{O}_K$ be the homomorphism of monoids sending 1 to $\pi$, and $\gamma : \mathbb{Z}_{\geq 0} \rightarrow P$ the homomorphism of monoids sending 1 to $(1,1,\ldots,1,0,\ldots,0) \in \{1\}^{1+b} \times \{0\}^{c-b}$. Then, there exists an isomorphism of $\mathcal{O}_K$-algebra:

$$\mathcal{O}_K[t_0,\ldots,t_b,t_{b+1},\ldots,t_c]/(t_1\cdots t_b - \pi^e) \overset{\sim}{\rightarrow} \mathcal{O}_K \otimes_{\mathbb{Z}[\mathbb{Z}_{\geq 0}]} \mathbb{Z}[P],$$

sending $t_i$ to $1 \otimes (0,\ldots,e,\ldots,0)$ where $e$ appears on the $i$-th position for $0 \leq i \leq c$. 

1.2.3. A variety over a field $F$ means a geometrically connected, separated scheme of finite type over $F$. A curve over $F$ is a variety of dimension one.

Let $T = \text{Spec}(R)$ be the spectrum of a valuation ring $R$ of height one with fraction field $F$, and $\mathbb{F}$ an algebraic closure of $F$. We denote by $\tau$ (resp. $\pi$) the generic point (resp. special) point of $T$. Given a proper $F$-scheme $X$, an $T$-model of $X$ is a flat, proper $T$-scheme of finite presentation $Y$ with generic fiber $Y_\tau \simeq X$.

A $T$-curve is a flat, proper $T$-scheme of finite presentation $X$ of relative dimension one whose generic fiber is a smooth curve over $F$.

We say an $S$-curve $X$ is semi-stable (or $X$ is a semi-stable $S$-curve) if it is semi-stable in the sense of 1.2.2. In particular, its special fiber $X_s$ is reduced, and its singular points are ordinary double points.

We say a semi-stable $S$-curve $X$ is stable if moreover, the following conditions are verified: (i) $X_s$ has genus $g \geq 2$; (ii) Any irreducible component of $X_s$, which is isomorphic to $\mathbb{P}^1$, intersects the other components at least three points.

By an abuse of language, we say an $\mathbb{F}$-curve $X$ is semi-stable (resp. stable) if there exists a finite extension $K'$ of $K$, a semi-stable (resp. stable) $S' = \text{Spec}(\mathcal{O}_{K'})$-curve $Y$ such that $X \simeq Y \times_{S'} \mathbb{F}$. Given a smooth proper $\mathbb{F}$-curve $C$ of genus $\geq 2$, the stable $\mathbb{F}$-model of $C$ exists and is unique (up to isomorphisms). Given a finite morphism $f : C' \rightarrow C$ of smooth proper $\mathbb{F}$-curves of genus $\geq 2$ and $Y,X$ the stable $\mathbb{F}$-models of $C',C$ respectively, the morphism $f$ extends uniquely to an $\mathbb{F}$-morphism $g : Y \rightarrow X$ ([37] proposition 4.4).

A morphism $\varphi : Y \rightarrow X$ of $T$-curves is called $\tau$-cover (resp. generic $\tau$-cover), if $\varphi$ is proper of finite presentation and $\varphi_\tau : Y_\tau \rightarrow X_\tau$ is finite étale (resp. finite).

Let $\varphi : Y \rightarrow X$ be a generic $\tau$-cover. We say $\varphi$ is a Galois generic $\tau$-cover, if there exists a finite group $G$ and an action $\mu$ of $G$ on $Y$ over $X$ such that over the unramified locus of $\varphi_\tau$, $\varphi^{-1}_\tau(U) \rightarrow U$ is a $G$-torsor with action $\mu$. When $\varphi$ is a $\tau$-cover (i.e. $U = X_\tau$), we simply say $\varphi$ is a Galois $\tau$-cover.

1.2.4. Almost mathematics. In this article, we consider the category of almost $\mathfrak{a}$-modules with respect to the maximal ideal $\mathfrak{m}$ of $\mathfrak{a}$. We denote by $(\_\_\_\_\_\_)_{\mathfrak{a}}$ the functor

$$\text{Mod}(\mathfrak{a}) \rightarrow \text{Mod}(\mathfrak{a}), \quad M \mapsto M_{\mathfrak{a}} = \text{Hom}_{\mathfrak{a}}(\mathfrak{m}, M).$$

Recall that it sends almost isomorphisms to isomorphisms.

Let $\mathcal{A}$ be an abelian category, $\text{End}(\text{id}_{\mathcal{A}})$ the ring of endomorphisms of the identity functor and $\varphi : \mathfrak{a} \rightarrow \text{End}(\text{id}_{\mathcal{A}})$ a homomorphism. We say an object $M$ of $\mathcal{A}$ is almost-zero if it is annihilated by every element of $\mathfrak{m}$. We denote by $\alpha_{\mathcal{A}}$ the quotient of $\mathcal{A}$ by the thick full subcategory of almost zero objects. We denote by $\alpha$ the canonical functor

$$\alpha : \mathcal{A} \rightarrow \alpha_{\mathcal{A}}, \quad M \mapsto M^\alpha.$$

We say a morphism $f$ of $\mathcal{A}$ is an almost isomorphism if $\alpha(f)$ is an isomorphism and we use $\overset{\sim}{\rightarrow}$ to denote an almost isomorphism.

Given an additive category $\mathcal{C}$, we denote by $\mathcal{C}_Q$ the category of objects of $\mathcal{C}$ up to isogeny, i.e. the localized category of $\mathcal{C}$ with respect to isogenies.

1.2.5. Let $T$ be a topos. The projective systems of objects of $T$ indexed by the ordered set of positive integers $\mathbb{N}$ forms a topos, that we denote by $T^{\mathbb{N}}$. 
Given a ring $\tilde{A} = (A_i)_{i \geq 1}$ of $\mathbb{T}^n$, we say an $\tilde{A}$-module $M = (M_i)_{i \geq 1}$ is adic if for every integers $1 \leq i < j$, the morphism $M_j \otimes_{A_i} A_i \rightarrow M_i$, induced by $M_j \rightarrow M_i$, is an isomorphism.

For any scheme $X$, we denote by $\tilde{\mathbf{E}}_X$ (resp. $X_{et}$) the étale site (resp. topos) of $X$, and by $\mathbf{F\acute{e}t}_X$ (resp. $X_{\acute{e}t}$) the finite étale site (resp. topos) of $X$.

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2. Higgs crystals and Higgs bundles

In this section, we briefly review the notion of Higgs crystal introduced by Tsuji ([4] IV.2), which provides a site-theoretic description of small Higgs bundles. We work with the base $(A_{inf}, (\xi))$ (c.f. [4] IV.1 Notation).

2.1. Higgs envelope. We first review the notion of Higgs envelope of algebras.

Definition 2.1.1 ([4] IV.2.1.1). (1) We denote by $\mathscr{A}$ the category of $A_{inf}$-algebras $A$ with a decreasing filtration $\{F^nA\}_{n \geq 0}$ of ideals such that $F^0A = A$ and $\xi^nA \subset F^nA$ for $n \geq 0$. $A$ morphism in $\mathscr{A}$ is a homomorphism of $A_{inf}$-algebras compatible with the filtrations.

(2) For $r \in \mathbb{N}$ (resp. $r = \infty$), we denote by $\mathscr{A}_{alg}^r$ the full subcategory of $\mathscr{A}$ consisting of $(A, F^nA)$ satisfying the following conditions:

(i) The rings $A$ and $A/F^nA (n \in \mathbb{N})$ are $p$-torsion free.

(ii) The ring $A$ is $\xi$-torsion free.

(iii) For every $s \in \mathbb{Z} \cap [0, r]$ (resp. $s \in \mathbb{Z}_{\geq 0}$), we have $pF^sA \subset \xi^sA$ (resp. $F^sA = \xi^sA$).

(iv) The inverse image of $F^nA$ under $\xi : A \rightarrow A$ is $F^{n-1}A$ for every $n \in \mathbb{N}$.

(3) We denote by $\mathscr{A}_{alg}^r_p$ the full subcategory of $\mathscr{A}_{alg}^r$ consisting of $(A, F^nA) \in \text{Ob} \mathscr{A}_{alg}^r$ satisfying the following condition: The rings $A$ and $A/F^nA (n \in \mathbb{N})$ are $p$-adically complete and separated.

(4) For $r \in \mathbb{N} \cup \{\infty\}$, we define $\mathscr{A}^r$ to be the full subcategory of $\mathscr{A}_{alg}^r$ consisting of $(A, F^nA) \in \text{Ob} \mathscr{A}_{alg}^r$ satisfying the following condition: The natural homomorphism $A \rightarrow \lim_{\rightarrow n} A/F^nA$ is an isomorphism.

Definition 2.1.2 ([4] IV.2.1.5). (1) We denote by $\mathscr{A}_{p, *}^r$ the category of inverse systemes of $A_{inf}$-algebras $(A_N)_{N \geq 1}$ such that $A_N$ are $p$-adically complete and separated, $\xi^N A_N = 0$ and the transition maps $\pi : A_{N+1} \rightarrow A_N$ are surjective.

(2) For $r \in \mathbb{N}$ (resp. $r = \infty$), we denote by $\mathscr{A}_{p, *}$ the full subcategory of $\mathscr{A}_{p, *}$ consisting of $(A_N)$ satisfying the following conditions ([4] IV.2.1.5):

(i) The rings $A_N (N \in \mathbb{N})$ are $p$-torsion free.

(ii) For every $s \in \mathbb{Z} \cap [0, r]$ (resp. $s \in \mathbb{N}$) and $N \in \mathbb{N}$, we have $pF^sA_N \subset \xi^sA_N$ (resp. $F^sA_N = \xi^sA_N$), where $F^nA_N (n \in \mathbb{N})$ is the decreasing filtration of $A_N$ by ideals defined by $F^0A_N = A_N$, $F^nA_N = \text{Ker} (\pi : A_N \rightarrow A_n) (1 \leq n \leq N)$ and $F^nA_N = 0 (n > N)$.

(iii) The kernel of the homomorphism $A_N \rightarrow A_N; x \mapsto \xi x$ is $F^{N-1}A_N$ for every $N \in \mathbb{N}$.

For $N \geq n$, we set $F^nA_N = \text{Ker} (A_N \rightarrow A_n)$. Then, the functor $\mathscr{A}_{p, *} \rightarrow \mathscr{A}$, defined by $(A_N) \mapsto (\lim_{\rightarrow N} A_N, \lim_{\rightarrow N} F^nA_N)$ is fully faithful ([4] IV.2.1.7). The projective limit functor induces an equivalence of categories $\mathscr{A}_{p, *} \xrightarrow{\sim} \mathscr{A}^r$ ([4] IV.2.1.7). In summary, we have the following diagram of fully faithful functors:
The functors in the second row have left adjoint functors ([4] IV.2.1.2, IV.2.1.8). Then, so is the inclusion functor in the first row. We denote the left adjoint functor by \( D_{\text{Hig}} : \mathcal{A} \rightarrow \mathcal{A}' \) (resp. \( D_{\text{Hig}} : \mathcal{A}_{\mathfrak{p}, \bullet} \rightarrow \mathcal{A}_{\mathfrak{p}}' \)) and call it Higgs envelope (of level \( r \)).

2.1.3. We present an example of Higgs envelope following ([4] IV.2.3). Let \( A = (A_N) \) be an object of \( \mathcal{A}' \). If \( r \in \mathbb{N} \), for integers \( N \geq 1, m, n \geq 0 \), we set

\[
A^{(m), n}_N = p^{-n_1} \xi^{n-n_2} F^m A_N \subset A_N, \quad \text{and} \quad A^{(m)}_N = \sum_{n \geq 0} A^{(m), n}_N,
\]

where \( n_1 = \min\{\lfloor n - m \rfloor / r, 0\} \) and \( n_2 = n - m - n_1 r \). If \( r = \infty \), we set \( A^{(m)}_N = A_N \).

Let \( d \) be an integer \( \geq 1 \), and define an object \( B = (B_N) \) of \( \mathcal{A}_{\mathfrak{p}, \bullet} \) over \( A \) by

\[
B_1 = A_1, \quad B_N = A_N\{T_1, \ldots, T_d\} = \lim_{m} A_N/p^m A_N[T_1, \ldots, T_d], \quad (N \geq 2),
\]

\[
B_2 \rightarrow B_1, T_i \mapsto 0, \quad B_{N+1} \rightarrow B_N, T_i \mapsto T_i.
\]

We define the \((\text{alg}/\xi^N)\)-submodule \( A_N|W_1, \ldots, W_d|_r \) of \( A_N|W_1, \ldots, W_d|_r \)) to be

\[
\bigoplus_{I \in \mathbb{Z}^d_{\geq 0}} A^{(l)}_N W^I,
\]

which is an \( A_N \)-subalgebra ([4] IV.2.3.13(1)). We define \( A_N|W_1, \ldots, W_d|_r \) to be the \( p \)-adic completion of \( A_N|W_1, \ldots, W_d|_r \) and we obtain an object \((A_N|W_1, \ldots, W_d|)\) of \( \mathcal{A}_{\mathfrak{p}, \bullet} \), denoted by \( A|W_1, \ldots, W_d|_r \).

By ([4] IV.2.3.15), \( A|W_1, \ldots, W_d|_r \) is actually an object of \( \mathcal{A}' \) and \( D_{\text{Hig}}(B) \) with the adjunction morphism \( B \rightarrow D_{\text{Hig}}(B) \) is isomorphic to the following homomorphism in \( \mathcal{A}_{\mathfrak{p}, \bullet} \) over \( A \):

\[
B_N = A_N\{T_1, \ldots, T_d\} \rightarrow A_N|W_1, \ldots, W_d|_r, \quad T_i \mapsto \xi W_i, \quad N \geq 2.
\]

In the following, we mainly use the description (2.1.3.1) in the case \( r = \infty \), where \( A_N|W_1, \ldots, W_d|_\infty \) is the \( p \)-adic completion of the polynomial algebra \( A_N|W_1, \ldots, W_d|_\infty \).

For the convenience of readers, we present a simple proof of (2.1.3.1) for \( r = \infty \) in lemma 2.1.5.

2.1.4. Let \((C, F^n C)\) be an object of \( \mathcal{A} \) such that \( F^1 C \) is finitely generated by \((\xi, a_i)_{i=1}^m\), \( F^n C = \xi^n C \) for \( n \geq 2 \) and that \((\xi, a_i, p)_{i=1}^m\) is a regular sequence. We denote by \( C_0 = C[\xi^\infty] \) the dilatation of the ideal \( F^1 C \) with respect to \( \xi \) ([56] 3.3). We set \( \widetilde{C}_0 = \lim_{n \geq 1} C_0/p^n C_0 \) and \( D = \lim_{n \geq 1} \widetilde{C}_0/\xi^n \widetilde{C}_0 \).

Then, the description (2.1.3.1) in the case \( r = \infty \) follows from the following lemma.

**Lemma 2.1.5.** With the above assumption, \((D, \xi^n D)\) belongs to \( \mathcal{A}' \) and is isomorphic to Higgs envelope \( D_{\text{Hig}}(C, F^n C) \).

**Proof.** In view of the proof of ([4] IV.2.1.2), it suffices to show that \((C_0, \xi^n C_0)\) belongs to \( \mathcal{A}_{\mathfrak{p}, \bullet}^\infty \) and is isomorphic to the image of \((C, F^n C)\) via the left adjoint of \( \mathcal{A}_{\mathfrak{p}, \bullet} \rightarrow \mathcal{A} \). We have an isomorphism of \( C \)-algebras:

\[
C_0 \simeq C[x_1, \ldots, x_m]/(a_i - \xi x_i)_{i=1}^m.
\]

As \( C \) is \( \xi \)-torsion free, then \( C_0 \) is a sub-ring of \( C[\xi] \) and is also \( \xi \)-torsion free. Since \((\xi, a_i, p)_{i=1}^m\) is a regular sequence of \( C \), then the quotient \( C_0/\xi C_0 \simeq (C/F^1 C)[x_1, \ldots, x_n]\) is \( p \)-torsion free, i.e. \((\xi, p)\) is a regular sequence of \( C_0 \). Then, we deduce that for every \( N \geq 1 \), \( C_0/\xi^N C_0 \) is \( p \)-torsion free. Therefore, \((C_0, \xi^n C_0)\) is an object of \( \mathcal{A}_{\mathfrak{p}, \bullet}^\infty \).

Let \((E, F^n E)\) be an object of \( \mathcal{A}_{\mathfrak{p}, \bullet} \) and \( f : (C, F^n C) \rightarrow (E, F^n E) \) a homomorphism compatible with filtrations. Since \( E \) is \( \xi \)-torsion free and \( f(F^1 C) \subset \xi E \), \( f \) induces a homomorphism of \( C \)-algebras \( g : C_0 \rightarrow E \) compatible with filtrations. This finishes the proof.
2.1.6. Let $X \to \text{Sp}(\mathbb{Z}_p)$ be a $p$-adic formal scheme (i.e., $p\mathcal{O}_X$ is an ideal of definition) over $\mathbb{Z}_p$. For $m \geq 1$, we denote by $X_m$ the reduction modulo $p^m$ of $X$. A fine log structure $\mathcal{M}$ on $X$ is a family of fine log structures $\mathcal{M}_m$ on $X_m$ and exact closed immersions $(X_m, \mathcal{M}_m) \to (X_{m+1}, \mathcal{M}_{m+1})$ ($m \geq 1$) extending the closed immersion $X_m \to X_{m+1}$ (IV.2.2). We define a $p$-adic fine log formal scheme to be a $p$-adic formal scheme endowed with a fine log structure. A morphism of $p$-adic fine log formal schemes $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is a family of morphisms of fine log formal schemes $f_m : (X_m, \mathcal{M}_m) \to (Y_m, \mathcal{N}_m)$ ($m \in \mathbb{N}$) compatible with the exact closed immersions $(X_m, \mathcal{M}_m) \to (X_{m+1}, \mathcal{M}_{m+1})$ and $(Y_m, \mathcal{N}_m) \to (Y_{m+1}, \mathcal{N}_{m+1})$. We say $f$ is smooth (resp. étale, resp. an exact closed immersion, resp. a closed immersion, resp. strict, resp. integral ([4] II.5.18)) if for every $m \geq 1$, $f_m$ is smooth (resp. étale, . . .).

Given a $p$-adic fine log formal scheme $X$, the étale site $\mathcal{E}_X$ is defined by the category of $p$-adic fine log formal schemes strict étale over $X$, equipped with the étale topology. We define the sheaf of rings $\mathcal{O}_X$ (resp. sheaf of monoids $\mathcal{M}_X$) on $X_\mathbb{Z}$ by $\Gamma(U, \mathcal{O}_X) = \varprojlim_m \Gamma(U_m, \mathcal{O}_{U_m})$ (resp. $\Gamma(U, \mathcal{M}_X) = \varprojlim_m \Gamma(U_m, \mathcal{M}_{U_m})$). We refer to loc.cit for more details on $p$-adic fine log formal schemes.

2.1.7. Next, we review the notion of Higgs envelope of $p$-adic fine log formal schemes following loc.cit.

We set $\Sigma_N = \text{Spf}(\text{Ann}_f/\xi_N)$, equipped with the trivial log structure. We refer to ([4] IV.2.2.1) for the definition of category $C$ (over $(\Sigma_N, N_2 \geq 1)$) and its full subcategory $C^r$ for $r \in \mathbb{N} \cup \{\infty\}$. Recall that an object of $C$ is a sequence of morphisms $\mathcal{Y} = (\mathcal{Y}_1 \to \mathcal{Y}_2 \to \cdots \to \mathcal{Y}_N \to \cdots)_{N \geq 1}$ of $p$-adic fine log formal schemes $\mathcal{Y}_N$ over $\Sigma_N$, compatible with $\Sigma_N \to \Sigma_{N+1}$ such that:

- the morphism $\mathcal{Y}_1 \to \mathcal{Y}_2$ is an immersion, and for $N \geq 2$, the morphism $\mathcal{Y}_N \to \mathcal{Y}_{N+1}$ is an exact closed immersion;
- for $N \geq 2$, the morphism of the modulo $p$ reduction of $\mathcal{Y}_N \to \mathcal{Y}_{N+1}$ is a nilpotent immersion.

An object $\mathcal{Y} = (\mathcal{Y}_N)$ of $C$ belongs to $C^\infty$, if it satisfies moreover the following conditions:

- $\mathcal{Y}_1 \to \mathcal{Y}_2$ is an exact closed immersion.
- The morphism $p : \mathcal{O}_\mathcal{Y}_N \to \mathcal{O}_\mathcal{Y}_N$ is injective for all $N \geq 1$.
- For every $N \geq 1$, $\mathcal{O}_\mathcal{Y}_N$ is killed by $\xi_N$ and Ker$(\mathcal{O}_\mathcal{Y}_N \to \mathcal{O}_\mathcal{Y}_N)$ = $\xi_N^{n} \mathcal{O}_\mathcal{Y}_N$ for $1 \leq n \leq N$.
- Ker$(\xi : \mathcal{O}_\mathcal{Y}_N \to \mathcal{O}_\mathcal{Y}_N)$ = $\xi_{N-1} \mathcal{O}_\mathcal{Y}_N$.

Given an object $\mathcal{Y} = (\mathcal{Y}_N)_{N \geq 1}$ of $C$ and an immersion $X \to \mathcal{Y}_1$ over $\Sigma_1$, we also denote by $(X \to \mathcal{Y})$ the object $(X \to \mathcal{Y}_2 \to \mathcal{Y}_3 \to \cdots)$ of $C$. If a caligraphic letter $\mathcal{Y} = (\mathcal{Y}_N)_{N \geq 1}$ denotes an object of $C$, we use the correspondence capital letter $Y$ to denote $\mathcal{Y}$.

For $N \geq 1$, we denote by $\Sigma_{N, S}$ the $p$-adic formal scheme $\Sigma_N$, equipped with the fine log structure extended by $\mathcal{M}_S$ on $S$ (c.f. [4] II.9.6), and by $\Sigma_S$ the object $(\Sigma_{N,S})_{N \geq 1}$ of $C$. Note that, $\Sigma_S$ belongs to $C^\infty$, for every $r \in \mathbb{N} \cup \{\infty\}.

Let $f = (f_N)_{N \geq 2} : Y = (Y_N) \to Y' = (Y'_N)$ be a morphism of $C$. We say $f$ is smooth (resp. étale, resp. integral) if $f_N$ is smooth (resp. étale, resp. integral) for every $N \geq 1$. We say $f$ is Cartesian if the morphism $Y'_N \to Y'_{N+1} \times_{Y_{N+1}} Y_N$ induced by $f$ is an isomorphism for every $N \geq 1$.

2.1.8. For $r \in \mathbb{N} \cup \{\infty\}$, the natural inclusion $C^r \to C$ admits a right adjoint functor $D^r_{Hig} : C \to C^r$, called Higgs envelope of level $r$ ([4] IV.2.2.2, IV.2.2.9). When $r = \infty$, we simply call it Higgs envelope.

If $Y$ is an object of $C$ such that $\mathcal{Y}_N$ are affine and $Y_1 \to Y_2$ is an exact closed immersion, the Higgs envelope of level $r$ of $Y$ is constructed by the functor $D^r_{Hig} : \mathcal{D}(p, \mathcal{M}) \to \mathcal{D}(p, \mathcal{M})$ applying to $(\mathcal{Y}_N, \mathcal{O}_{\mathcal{Y}_N})_{N \geq 1}$, and the log structure induced by $(\mathcal{M}_N, \mathcal{M}_S)_{N \geq 1}$ ([4] IV.2.2.14). We refer to ([4] IV.2.2) for more details.

2.1.9. Let $X \to \Sigma_{1, S}$ be a smooth integral morphism of $p$-adic fine log formal schemes and $\mathcal{X} = (\mathcal{X}_N)_{N \geq 1}$ $\to \Sigma_S$ a smooth Cartesian lifting of $X$ (i.e., $\mathcal{X}_1 = X$). Then, $\mathcal{X}$ belongs to $C^r$ for every $r \in \mathbb{N} \cup \{\infty\}$.

Let $s$ be an integer $\geq 1$. We denote by $\mathcal{X}^{s+1}$ the product of $(s + 1)$-copies of $\mathcal{X}$ over $\Sigma_S$ in $C$ and $p(s) : \mathcal{D}(s) = (\mathcal{D}(s)_N)_{N \geq 1} \to (X \to \mathcal{X}^{s+1})$ the Higgs envelope of level $r$ of the diagonal embedding $(X \to \mathcal{X}^{s+1})$. We set $\mathcal{D}(s) = \mathcal{D}(s)_1$. We denote by $q_i : \mathcal{D}(1) \to X$ (resp. $p_{ij} : \mathcal{D}(2) \to \mathcal{D}(1)$) be the projection in the $i$-th component for $i = 1, 2$ (resp. $i,j$-th component for $1 \leq i < j \leq 3$) and by $\Delta_s : X \to \mathcal{D}(s)$ the morphism induced by the diagonal immersion $X \to \mathcal{X}^{s+1}$.
Suppose that $\mathcal{X}$ is affine and that there exist $x_i = (x_i, N) \in \varprojlim \Gamma(\mathcal{X}_N, \mathcal{M}_{\mathcal{X}_N})$ ($1 \leq i \leq d$) ([4] IV.2.2) such that $\{ d \log(x_i, N) \}_{1 \leq i \leq d}$ is a basis of $\Omega^1_{\mathcal{X}_{\text{aff}}} / \Sigma_{\text{aff}}$ for $N \geq 1$. By ([4] IV.2.3.17), for $i \in \{1, \cdots, d\}$ and $N \geq 2$, there exists a unique element $u_{i,N} \in 1 + F^I \mathcal{O}_{\mathcal{D}(1)}$, such that

$$p(1)_N(x_i, N) = p(1)_N(x_{i,N} \otimes 1) u_{i,N} \in \mathcal{O}_{\mathcal{D}(1)}$$

and that there exists canonical isomorphisms of $(\mathcal{O}_{\mathcal{X}_N})_{N \geq 1}$-algebras in $\mathcal{O}_X^{\infty}$

$$(2.1.9.1) \quad \mathcal{O}_{\mathcal{X}_N} \{ W_1, \cdots, W_d \}_r \overset{\sim}{\rightarrow} q_{j_*}(\mathcal{O}_{\mathcal{D}(1)}), \quad W_i \mapsto \frac{u_{i,N}^{-1}}{\xi}, \quad 1 \leq i \leq d, \; j = 1, 2.$$

2.2. Higgs crystals.

2.2.1. Let $Y$ be a $p$-adic fine log formal scheme over $\Sigma_{1,S}$. For $r \in \mathbb{N} \cup \{\infty\}$, we denote by $(Y/\Sigma_S)_\text{HIG}$ (resp. $(Y/\Sigma_S)_\text{HIG}^{\infty}$) the Higgs site (resp. Higgs topos) defined in ([4] IV.3.1.4). Recall ([4] IV.3.1.1) that an object of $(Y/\Sigma_S)_\text{HIG}$ is a pair $(T, z)$ consisting of an object $T = (T_N)_{N \geq 1}$ of $\mathcal{O}_{\Sigma_S}$ and a $\Sigma_{1,S}$-morphism $z : T(= T_1) \rightarrow Y$. A morphism $(T', z') \rightarrow (T, z)$ is a morphism $u : T' \rightarrow T$ in $\mathcal{O}_{\Sigma_S}$ such that $z \circ u_1 = z'$. For an object $(T, z)$ of $(Y/\Sigma_S)_\text{HIG}$, the set of coverings Cov$((T, z))$ is defined to be

$$\left\{ (u_\alpha : (T_\alpha, z_\alpha) \rightarrow (T, z))_{\alpha \in A} \mid \begin{array}{l}
(i) \text{ u}_\alpha \text{ is strict étale and Cartesian for all } \alpha \in A \\
(ii) \bigcup_{\alpha \in A} u_{\alpha,1}(T_\alpha) = T
\end{array} \right\}$$

The functor $(T, z) \mapsto \Gamma(T, \mathcal{O}_T)$ defines a sheaf of rings, denoted by $\mathcal{O}_{Y/\Sigma_S}$ on $(Y/\Sigma_S)_\text{HIG}$. For each rational number $\alpha \in \mathbb{Q}_{>0}$, we set $\mathcal{O}_{Y/\Sigma_S, \alpha} = \mathcal{O}_{Y/\Sigma_S}/p^\alpha \mathcal{O}_{Y/\Sigma_S}$ and $\mathcal{O}_{Y/\Sigma_S, \alpha} = \mathcal{O}_{Y/\Sigma_S[p]}/p^\alpha$. An $\mathcal{O}_{Y/\Sigma_S}$-module $\mathcal{M}$ is equivalent to a collection of data $\{ \mathcal{M}(T, z), \gamma_u \}$, where for each object $(T, z)$ of $(Y/\Sigma_S)_\text{HIG}$, $\mathcal{M}(T, z)$ is an $\mathcal{O}_T$-module of $T_\text{ét}$ and for each morphism $u : (T', z') \rightarrow (T, z)$ of $(Y/\Sigma_S)_\text{HIG}$, $\gamma_u$ is an $\mathcal{O}_T$-linear morphism

$$\gamma_u : u_1^*(\mathcal{M}(T, z)) \rightarrow \mathcal{M}(T', z'),$$

satisfying following conditions:

(i) for any morphism $u : (T', z') \rightarrow (T, z)$ such that the underlying morphism $u : T' \rightarrow T$ in $\mathcal{O}$ is strict étale and Cartesian, the morphism $\gamma_u$ is an isomorphism;

(ii) for any morphisms $(T'', z'') \rightarrow (T', z') \rightarrow (T, z)$, we have $\gamma_u \circ v_{z'}^*(\gamma_u) = \gamma_{uv}$.

A Higgs isocrystal of level $r$ is an $\mathcal{O}_{Y/\Sigma_S, \alpha}^{\infty}$-module $\mathcal{M}$ on $(Y/\Sigma_S)_\text{HIG}$ such that for every morphism $u$ of $(Y/\Sigma_S)_\text{HIG}$, $\gamma_u$ is an isomorphism. A Higgs crystal is an $\mathcal{O}_{Y/\Sigma_S}$-module $\mathcal{M}$ on $(Y/\Sigma_S)_\text{HIG}$ such that for every morphism $u$ of $(Y/\Sigma_S)_\text{HIG}$, $\gamma_u$ is an isomorphism.

2.2.2. To introduce a finiteness condition on Higgs (iso-)crystals in a simple way, we assume $X$ is a semi-stable $S$-scheme ($\S 1.2.2$) in the following of this section. Let $\tilde{X}$ be the $p$-adic completion of $X \otimes_{\mathcal{O}_K} \mathcal{O}$, equipped with the fine log structure induced by $\mathcal{M}_{\mathcal{X}}$. Then, $\tilde{X}$ is smooth over $\Sigma_{1,S}$.

We denote by $\mathcal{Q}/X$ (or simply $\mathcal{Q}$ if there is no confusion) ([4] III.10.5) the full subcategory of $\tilde{\text{Et}}/X$ étale schemes over $X$ consisting of affine schemes $U$ such that there exists a fine and saturated chart $\Gamma(U, \mathcal{M}_{\mathcal{X}}) \rightarrow \Gamma(U, \mathcal{O}_U)$. Then, $\tilde{X}$ is smooth over $\Sigma_{1,S}$.

A Higgs isocrystal of level $r$ (resp. a Higgs crystal) $\mathcal{M}$ is called finite, if for any object $U$ of $\mathcal{Q}$ and any smooth Cartesian lifting $\mathcal{U}$ of $U$ over $\Sigma_S$ (regarded as an object of $(\tilde{X}/\Sigma_S)_\text{HIG}$), the $\mathcal{O}_{\tilde{U}}^{[\frac{1}{p}]}$-module $\mathcal{M}_{\tilde{U}}$ is finitely generated and projective (resp. coherent and $\mathcal{M}_{\tilde{U}}^{[\frac{1}{p}]}$ is projective). In particular, when $\tilde{X}$ admits a smooth Cartesian lifting $\mathcal{X}$ over $\Sigma_S$, this condition is equivalent to the fact that $\mathcal{M}_{\mathcal{X}}$ is finitely generated and locally projective (resp. coherent and $\mathcal{M}_{\mathcal{X}}^{[\frac{1}{p}]}$ is locally projective).

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2This sheaf of rings is denoted by $\mathcal{O}_{Y/\Sigma_S, 1}$ in ([4] IV.3.1.4).

3Our definition of (iso-)crystals is slightly different to that of ([4] IV.3.3), where certain finiteness condition is imposed.
We denote by $\HC^0_{\mathbb{Q}}(\tilde{X}/\Sigma_S)$ (resp. $\HC_{\mathbb{Z}}(\tilde{X}/\Sigma_S)$) the category of Higgs isocrystals on $(\tilde{X}/\Sigma_S)_{\text{HIG}}$ (resp. Higgs crystals on $(\tilde{X}/\Sigma_S)_{\text{HIG}}$) and by $\HC^0_{\mathbb{Q},\text{fin}}(\tilde{X}/\Sigma_S)$ (resp. $\HC_{\mathbb{Z},\text{fin}}(\tilde{X}/\Sigma_S)$) the full subcategory consisting of objects which are finite.

The canonical functor $\HC^0_{\mathbb{Q},\text{fin}}(\tilde{X}/\Sigma_S)_\mathbb{Q} \to \HC^0_{\mathbb{Q},\text{fin}}(\tilde{X}/\Sigma_S)$ is fully faithful ([4] IV.3.5.1).

**Remark 2.2.3.** Under the assumption of §2.2.2, the above finiteness condition is stronger than the condition in ([4] IV.3.3.1(ii) and IV.3.3.2(iii)). Hence we can apply results of ([4] IV) to our setting.

### 2.2.4.
Let $\mathcal{X} \to \Sigma_S$ be a smooth Cartesian lifting of $\tilde{X}$ in $\mathcal{C}$. We keep the notation in §2.1.9.

We denote by $\HC^0_{\mathbb{Q}}(\tilde{X},\mathcal{X}/\Sigma_S)$ (resp. $\HC^0_{\mathbb{Z}}(\tilde{X},\mathcal{X}/\Sigma_S)$) the category of pairs $(M,\varepsilon)$ consisting of an $\mathcal{O}_\mathcal{X}$-module (resp. $\mathcal{O}_\tilde{X}$-module) $M$ together with an $\mathcal{O}_\mathcal{D}(1)$-linear isomorphism $\varepsilon : q_2^*\mathbb{M}(M) \sim \mathbb{M}(\tilde{M})$, called Higgs stratification, such that: (a) $\Delta_1^*(\varepsilon) = \id_M$; (b) $p_{12}^*(\varepsilon) \circ p_{23}^*(\varepsilon) = p_{13}^*(\varepsilon)$.

By a standard argument ([4] IV.3.4.4), the lifting $\mathcal{X}$ induces equivalences of categories between

$(\mathcal{X}/\Sigma_S) \to \HC^0_{\mathbb{Q}}(\tilde{X},\mathcal{X}/\Sigma_S)$ (resp. $\HC^0_{\mathbb{Z}}(\tilde{X},\mathcal{X}/\Sigma_S)$) and $\HC^0_{\mathbb{Q},\text{fin}}(\tilde{X},\mathcal{X}/\Sigma_S)$, $\HC_{\mathbb{Z},\text{fin}}(\tilde{X},\mathcal{X}/\Sigma_S)$, which $\varepsilon$ is constructed by the composition $q_2^*(M) \sim \mathbb{M}(\tilde{M})$.

We denote by $\HC^0_{\mathbb{Q},\text{fin}}(\tilde{X},\mathcal{X}/\Sigma_S)$ (resp. $\HC^0_{\mathbb{Q},\text{fin}}(\tilde{X},\mathcal{X}/\Sigma_S)$) the full subcategory of $\HC^0_{\mathbb{Q}}(\tilde{X},\mathcal{X}/\Sigma_S)$ (resp. $\HC^0_{\mathbb{Z}}(\tilde{X},\mathcal{X}/\Sigma_S)$) corresponding to finite Higgs (iso-)crystals (of level $r$).

### 2.2.5.
We denote by $\Omega^1_{\mathcal{X}/\mathcal{S}}$ the module of log differentials $\Omega^1_{(\mathcal{X}\times\mathcal{M})(/\mathcal{S})}$ and by $\Omega^1_{\mathcal{X}/\Sigma_S}$ the $p$-adic completion of $\Omega^1_{\mathcal{X}/\mathcal{S}}$ over $\mathcal{X}$. Given an $\mathcal{O}_\mathcal{X}$-module $M$ and an $\mathcal{O}_\mathcal{X}$-linear morphism $\theta : M \to \mathcal{O} = \mathcal{O}_\mathcal{X} \otimes_{\mathbb{Q}} \xi^{-1}\Omega^1_{\mathcal{X}/\Sigma_S}$, we can define an $\mathcal{O}_\mathcal{X}$-linear morphism, for $q \geq 1$:

$$\theta^q : M \otimes_{\mathcal{O}_\mathcal{X}} \xi^{-q}\Omega^1_{\mathcal{X}/\Sigma_S} \to M \otimes_{\mathcal{O}_\mathcal{X}} \xi^{-q-1}\Omega^1_{\mathcal{X}/\Sigma_S}, \quad x \otimes \xi^{-q}\omega \mapsto \theta(x) \wedge \xi^{-q}\omega.$$ 

If $\theta^1 \circ \theta = 0$, we say $\theta$ is a Higgs field and $(M,\theta)$ is a Higgs module.

Since $\mathcal{X}$ is semi-stable over $\mathcal{S}$, there exists a strict étale morphism $U \to \mathcal{X}$ and $(t_i)_{i=1}^d \in \Gamma(U,\mathcal{M})$ such that $\{d\log(t_i)\}_{i=1}^d$ is a basis of $\Omega^1_{\mathcal{U}/\Sigma_S}$. Let $\theta_i$ (1 $\leq i \leq d$) be the endomorphism of $M$ defined by $\theta(x) = \sum_{1 \leq i \leq d} \xi^{-1}\theta_i(x) \otimes d\log(t_i)$. The condition $\theta^1 \circ \theta = 0$ is equivalent to the fact that $\theta_i$ and $\theta_j$ commutes to each other.

Suppose $M$ is a coherent $\mathcal{O}_U(1)[1/p]$-module (resp. a $p$-torsion free coherent $\mathcal{O}_U$-module). We endow $M$ with the $p$-adic topology and for $x \in \Gamma(U,M)$ and $m \in \mathbb{M}_d$, we set $\theta_m(x) = \prod_{1 \leq i \leq d} \theta_i^{m_i}(x)$. Consider the following convergent condition (Conv), (resp. integral condition (Int)) on $M$ ([4] IV.3.4.12):

(Conv): $p^{-\frac{2r}{p-1}} \frac{\partial}{\partial m} \theta_m(x)$ converges to 0 as $|m| \to \infty$, where $|m|^p r^{-1} = 0$ if $r = \infty$.

(Int): we have $\frac{\partial}{\partial m} \theta_m(x) \in M$.

**Definition 2.2.6.** (i) We denote by $HM_{\mathbb{Q}}^{\text{conv}}(\tilde{X}/\Sigma_S)$ (resp. $HM_{\mathbb{Z}}^{\text{conv}}(\tilde{X}/\Sigma_S)$) the full subcategory of $\HC_{\mathbb{Q}}^{\text{conv}}(\tilde{X}/\Sigma_S)$ consisting of an $\mathcal{O}_\mathcal{X}$-module (resp. $\mathcal{O}_\tilde{X}$-module) together with a Higgs field $\theta$, and by $HM_{\mathbb{Q}}^{\text{fin}}(\tilde{X}/\Sigma_S)$ the full subcategory of $HM_{\mathbb{Q}}^{\text{conv}}(\tilde{X}/\Sigma_S)$ whose underlying $\mathcal{O}_\mathcal{X}$-module is locally projective of finite type.

(ii) ([4] IV.3.4.14) We denote by $HM_{\mathbb{Q},\text{conv}}^{\text{fin}}(\tilde{X}/\Sigma_S)$ (resp. $HM_{\mathbb{Z},\text{conv}}^{\text{fin}}(\tilde{X}/\Sigma_S)$) the full subcategory of $HM_{\mathbb{Q}}^{\text{conv}}(\tilde{X}/\Sigma_S)$ (resp. $HM_{\mathbb{Z}}^{\text{conv}}(\tilde{X}/\Sigma_S)$) consisting of $(M,\theta)$, where $M$ is a finitely generated locally projective $\mathcal{O}_\mathcal{X}$-module (resp. a $p$-torsion free coherent $\mathcal{O}_\mathcal{X}$-module such that $M(1/p)$ is locally projective) and $\theta$ satisfies:

There exist a strict étale covering $(U_\alpha \to \tilde{X})_{\alpha \in A}$ and $(t_{i,\alpha})_{i=1}^d \in \Gamma(U_\alpha,\mathcal{M}_{U_\alpha})$ such that:

- $d\log t_{i,\alpha}$ is a basis of $\Omega^1_{U_\alpha/\Sigma_S}$ for every $N \geq 1$,
- $U_\alpha$ is affine
- the pair $(\Gamma(U_\alpha,M),\Gamma(U_\alpha,\theta))$ satisfies (Conv), (resp. (Int)) with respect to $t_{i,\alpha}$.

(iii) We denote by $HM_{\mathbb{Q},\text{small}}^{\text{fin}}(\tilde{X}/\Sigma_S)$ the full subcategory of $HM_{\mathbb{Q}}^{\text{conv}}(\tilde{X}/\Sigma_S)$ consisting of objects belonging to $HM_{\mathbb{Q},\text{conv}}^{\text{fin}}(\tilde{X}/\Sigma_S)$ for some integer $r \in \mathbb{N}$. We set $HM_{\mathbb{Q},\text{conv}}^{\text{fin}}(\tilde{X}/\Sigma_S) = HM_{\mathbb{Q},\text{conv}}^{\text{fin}}(\tilde{X}/\Sigma_S)$.
Proposition 2.2.7 ([4] IV.3.4.16). Let $\mathcal{X}$ be a smooth Cartesian lifting of $\tilde{X}$ over $\Sigma_S$, and $r \in \mathbb{N} \cup \{\infty\}$.

(i) The lifting $\mathcal{X}$ induces a canonical functor:

$$\text{HS}^\ast_{\mathbb{Z}_p}(\tilde{X}, \mathcal{X}/\Sigma_S) \to \text{HM}^\ast_{\mathbb{Z}_p}(\tilde{X}/\Sigma_{1,S}), \quad (M, \varepsilon) \mapsto (M, \theta).$$

(ii) The functor (2.2.7.1) induces an equivalence of categories between $\text{HC}^\ast_{\mathbb{Z}_p}(\tilde{X}/\Sigma_S)$ and $\text{HC}^\ast_{\mathbb{Z}_p,\text{fin}}(\tilde{X}/\Sigma_S)$ (resp. full subcategory of $\text{HC}^\ast_{\mathbb{Z}_p,\text{fin}}(\tilde{X}/\Sigma_S)$ such that $\mathcal{M}_\mathcal{X}$ is $p$-torsion free).

We briefly review the construction of functor (2.2.7.1). Let $\mathcal{D}(1)_N^\mathcal{X}$ be the first infinitesimal neighborhood of $\Delta_{1,N} : \mathcal{X}_N \to \mathcal{D}(1)_N$. Then, there exists a canonical isomorphism (see [4] IV.2.4.7)

$$\text{Ker}(\mathcal{O}_{\mathcal{D}(1)_N^\mathcal{X}} \to \mathcal{O}_{\mathcal{X}_N}) \cong \xi^{-1}\Omega^1_{\mathcal{X}_N/\Sigma_{N,S}}.$$

Given $(M, \varepsilon)$ as above, $\varepsilon$ induces an isomorphism $\varepsilon^1 : \mathcal{O}_{\mathcal{D}(1)_N^\mathcal{X}} \otimes M \to M \otimes \mathcal{O}_{\mathcal{D}(1)_N^\mathcal{X}}$, and $\theta$ is defined by

$$\theta : M \to \xi^{-1}M \otimes \varepsilon^1 \Omega^1_{\mathcal{X}_N/\Sigma_{1,S}}, \quad x \mapsto \varepsilon^1(1 \otimes x) - x \otimes 1.$$

2.2.8. Assume moreover that $X$ is proper $\mathcal{O}_K$. We have a canonical isomorphism of rigid spaces $X^\text{an}_C \simeq \tilde{X}^\text{rig}$. Let $\tilde{X}^\text{ad}$ be the rigid space $\tilde{X}^\text{rig}$, equipped with the admissible topology and

$$\rho_X : (\tilde{X}^\text{rig}, \mathcal{O}_{\tilde{X}^\text{rig}}) \to (\tilde{X}, \mathcal{O}_{\tilde{X}})$$

the morphism of ringed topoi ([1] 4.7.5.2). We have the following commutative diagram:

$$\begin{array}{ccc}
\text{Coh}(\mathcal{O}_{X^\text{an}}) & \xrightarrow{\sim} & \text{Coh}(\mathcal{O}_X) \\
\downarrow{\rho_X^\ast} & & \downarrow{\sigma_X^\ast} \\
\text{Coh}(\mathcal{O}_{X^\text{an}}) & \xrightarrow{\sim} & \text{Coh}(\mathcal{O}_{X^\text{an}})
\end{array}$$

where horizontal functors are the $p$-adic completion functor and analytification functor and induce equivalences of categories, the vertical functor is essentially surjective and is fully faithful up to isogeny.

Proposition 2.2.9. Assume moreover that $X$ is proper over $\mathcal{O}_K$.

(i) The analytification functor induces an equivalence of categories $\text{VB}(X^\text{an}_C) \xrightarrow{\sim} \text{VB}(X^\text{an}_C)$.

(ii) The functor $\rho_X^\ast$ induces an equivalence between the category $\text{LP}(\mathcal{O}_X^\text{rig} |_{\mathcal{X}})$ of locally projective $\mathcal{O}_X^\text{rig} |_{\mathcal{X}}$-modules of finite type and $\text{VB}(X^\text{an}_C)$.

Proof. Assertion (i) is due to Köpf [34] (see [55] proposition 4.7).

(ii) By ([55] 5.15), $\rho_X^\ast$ sends $\text{LP}(\mathcal{O}_X^\text{rig} |_{\mathcal{X}})$ to $\text{VB}(X^\text{an}_C)$. It remains to show it is essentially surjective.

Let $E$ be a vector bundle over $X^\text{rig}$ and $\mathcal{E}$ a torsion-free coherent $\mathcal{O}_X^\text{rig}$-module with generic fiber $E$. For any affine open subscheme $U = \text{Spec}(R)$ of $X_\mathfrak{p}$ such that $U \cap X_\mathfrak{p}$ is not-empty, the $\hat{R}$-module $\mathcal{E}(U) |_{\mathcal{X}}$ is finite projective. Then, the $\tilde{R}$-module $\mathcal{E}(U) |_{\mathcal{X}} \otimes \mathcal{E}(U) |_{\mathfrak{p}} \otimes_R \tilde{R}$ is also finite projective and $\mathcal{E}(U) |_{\mathcal{X}}$ is therefore locally projective of finite type. Then, the essential surjectivity follows.

□

Definition 2.2.10 ([18]). (i) Let $M$ be a coherent $\mathcal{O}_X$-module and $\theta$ a Higgs field on $M$.

- For $\alpha \in \mathbb{Q}_{>0}$, we say $\theta$ is $\alpha$-small if $\theta(M) \subset p^\alpha M \otimes_{\mathcal{O}_X} \xi^{-1}\Omega^1_{\mathcal{X}/\Sigma_{1,S}}$.

- We say $\theta$ is small if it is $\alpha$-small for some $\alpha > \frac{1}{p^\ell}$. (ii) Let $\mathcal{N}$ be a locally projective $\mathcal{O}_X^\text{rig} |_{\mathcal{X}}$-module and $\theta$ a Higgs field on $\mathcal{N}$. We say $\theta$ is small if there exists a coherent sub-$\mathcal{O}_X$-module $\mathcal{N}^\circ$ of $\mathcal{N}$ such that it generates $\mathcal{N}$ over $\mathcal{O}_X^\text{rig} |_{\mathcal{X}}$ and that $\theta$ preserves $\mathcal{N}^\circ$ and the restriction of $\theta$ on $\mathcal{N}^\circ$ is a small in the sense of (i).
Given a $p$-torsion free coherent $\mathcal{O}_X$-module $M$, a small Higgs fields on $M$ satisfies condition (Int).

Given a locally projective $\mathcal{O}_X[1/p]$-module $\mathcal{N}$ of finite type, $\theta$ is small if and only if $(\mathcal{N}, \theta)$ belongs to $\text{HB}_{\mathbb{Q}_p, \text{conv}}(\hat{X}/\Sigma_1, S)$ for some $r \in \mathbb{N}$, i.e. $(\mathcal{N}, \theta)$ belongs to $\text{HB}_{\mathbb{Q}_p, \text{small}}(\hat{X}/\Sigma_1, S)$ \cite[IV.3.6.6]{[4]}.

We denote by $\text{HB}(\mathcal{X})$ (resp. $\text{HB}(\mathcal{X}_{\mathbb{C}})$) the category of pairs $(\mathcal{M}, \theta)$ of a vector bundle $\mathcal{M}$ and a Higgs field $\theta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{O}_{X_c} \xi^{-1}\Omega^1_{X_c/C}$ (resp. $\theta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{O}_{X_{c}} \xi^{-1}\Omega^1_{X_{c}/C}$). By proposition 2.2.9, we have equivalences of categories

\begin{equation}
\text{HB}_{\mathbb{Q}_p}(\hat{X}/\Sigma_1, S) \xrightarrow{\sim} \text{HB}(\mathcal{X}_{\mathbb{C}}), \quad \text{HB}(\mathcal{X}) \xrightarrow{\sim} \text{HB}(\mathcal{X}_{\mathbb{C}}).
\end{equation}

We say a Higgs bundle $(\mathcal{M}, \theta)$ of $\text{HB}(\mathcal{X}_{\mathbb{C}})$ (resp. $\text{HB}(\mathcal{X})$) is small (with respect to $X$) if there exists a small Higgs bundle of $\text{HB}_{\mathbb{Q}_p}(\hat{X}/\Sigma_1, S)$ associated to $(\mathcal{M}, \theta)$ via the above equivalences. We denote by $\text{HB}_{\text{small}}(\mathcal{X})$ the full subcategory of small Higgs bundles.

**Corollary 2.2.11.** Let $\mathcal{M}$ be an object of $\text{HC}_{\mathbb{Q}_p, \text{un}}(\hat{X}/\Sigma_1, S)$ such that $M = \mathcal{M}_X$ is $p$-torsion free and $\varepsilon$ (resp. $\theta$) the associated Higgs stratification (resp. field) on $M$. If $\theta$ is $\alpha$-small and set $\beta = \alpha - \frac{1}{p^r}$, then for any local sections $a \in \mathcal{O}_{D(1)}, x \in M$, we have

\begin{equation}
\varepsilon(a \otimes x) \equiv x \otimes a \mod p^\beta.
\end{equation}

In particular, if $M/p^\beta M$ is moreover free, then there exists an isomorphism of Higgs crystals modulo $p^\beta$:

\begin{equation}
\mathcal{M}/p^\beta \mathcal{M} \simeq (\mathcal{O}_{\hat{X}/\Sigma_1, S})^\Sigma_{\mathbb{C}}.
\end{equation}

**Proof.** Let $U \to \hat{X}$ be a strict étale morphism as in 2.2.5. With the notations of 2.1.9 and 2.2.5, the Higgs stratification $\varepsilon$ can be written, for any local section $m \in M$, as:

\[ \varepsilon(1 \otimes x) = \sum_{\mathfrak{m} \in \mathfrak{A}_{\Sigma_0}} \frac{\theta_{\mathfrak{m}}(x)}{m!} \otimes \prod_{1 \leq i \leq d} \left( \frac{u_{i,2} - 1}{\xi} \right)^{m_i}. \]

Since $\theta$ is $\alpha$-small, we deduce that $\frac{\theta_{\mathfrak{m}}(x)}{m!} \equiv 0 \mod p^\beta$ except $m = (0, \cdots, 0)$. Then, the congruence (2.2.11.1) follows. The second assertion follows from the equivalence (2.2.4.1). \qed

### 2.3. Inverse image of Higgs crystals

In this subsection, we discuss the inverse image functor for Higgs crystals and its realization in terms of modules with Higgs stratification (2.2.4) and of Higgs modules.

**2.3.1.** Let $K'$ be a finite extension of $K$ in $\bar{K}$ and $S' = \text{Spec}(\mathcal{O}_{K'})$. With the notations of § 2.1.7, there exists a canonical morphism $g : \Sigma_{S'} \to \Sigma_{S} \in \mathcal{C}$.

Let $X'$ be a semi-stable $S'$-scheme, $\hat{X}'$ the $p$-adic completion of $X' \otimes_{\mathcal{O}_{K'}} \mathfrak{o}$, equipped with the fine log structure induced $\mathcal{M}_{X'}$. Let $f : X' \to X_{S'} = X \times_S S'$ be an $S'$-morphism. We denote by $\hat{f} : \hat{X}' \to \hat{X}$ the morphism of $p$-adic fine log formal schemes over $\Sigma_{1,S}$ induced by $f$. For $r', r \in \mathbb{N} \cup \{\infty\}$ such that $r' \geq r$, the morphisms $\hat{f}$ and $g$ induces a cocontinuous functor:

\begin{equation}
(\hat{X}'/\Sigma_{S'})_{\text{HIG}} \to (\hat{X}/\Sigma_{S})_{\text{HIG}}, \quad (T, z) \mapsto (T \to \Sigma_{S'} \to \Sigma_{S}, g \circ z),
\end{equation}

and then a morphism of topoi \cite[IV.3.1.5]{[4]}

\[ \hat{f}_{\text{HIG}} : (\hat{X}'/\Sigma_{S'})_{\text{HIG}} \to (\hat{X}/\Sigma_{S})_{\text{HIG}}, \]

such that $\hat{f}_{\text{HIG}}(\mathcal{O}_{\hat{X}/\Sigma_{S}}) = \mathcal{O}_{\hat{X}'/\Sigma_{S'}}$. Its inverse image functors preserve Higgs (iso-)crystals:

\[ \hat{f}_{\text{HIG}} : \text{HC}_{\mathbb{Q}_p}(\hat{X}/\Sigma_{S}) \to \text{HC}_{\mathbb{Q}_p}(\hat{X}'/\Sigma_{S'}), \quad \text{HC}_{\mathbb{Q}_p}(\hat{X}/\Sigma_{S}) \to \text{HC}_{\mathbb{Q}_p}(\hat{X}'/\Sigma_{S'}).
\]

If a Higgs crystal $\mathcal{M}$ is finite, then so is its inverse image $\hat{f}_{\text{HIG}}(\mathcal{M})$. When $S' = S$ and $X' = X$, the above functor, defined for Higgs isocrystals of different levels, is fully faithful \cite[IV.3.5.3]{[4]}.

In the following, we focus on the inverse image for Higgs (iso-)crystals (i.e. $r' = r = \infty$). We ignore the superscript $\infty$ from the notations $\text{HS}^{\infty}_{\mathbb{Q}_p}$.
Let \( \mathcal{X} \) (resp. \( \mathcal{X}' \)) be a smooth Cartesian lifting of \( \tilde{X} \) over \( \Sigma_S \) (resp. \( \tilde{X}' \) over \( \Sigma_S' \)) and \( D(s) \) (resp. \( D'(s) \)) the Higgs envelope of \( X \to \mathcal{X}'^{s+1} \) (resp. \( \mathcal{X}' \to \mathcal{X}'^{s+1} \)) for \( s \geq 1 \). By (2.2.4.1) and 2.2.7(ii), for \( \bullet \in \{\mathbb{Z}_p, \mathbb{Q}_p\} \), we obtain twisted inverse image functors for modules with stratifications and Higgs bundles (definition 2.2.6)

\[
(2.3.1.2) \quad \tilde{f}_{\mathcal{H}(\mathcal{X},\mathcal{X}')}^{*}: \mathcal{H}_{\mathcal{S},\text{fin}}(\tilde{X},\mathcal{X}/\Sigma_S) \to \mathcal{H}_{\mathcal{S},\text{fin}}(\tilde{X}',\mathcal{X}'/\Sigma_{S'}), \quad \mathcal{H}_{\mathcal{S},\text{conv}}(\tilde{X}/\Sigma,\mathcal{X}) \to \mathcal{H}_{\mathcal{S},\text{conv}}(\tilde{X}'/\Sigma,\mathcal{X}'.)
\]

The above functors for Higgs bundles are different to the usual inverse image functors for Higgs bundles unless the Higgs field is zero (see remark 2.3.5).

**Lemma 2.3.2.** Suppose \( \tilde{f} : \mathcal{X}' \to \mathcal{X} \) is a lifting of \( f : \mathcal{X}' \to \mathcal{X} \) over \( \Sigma_S \) (which locally exists). Then, for \( s \geq 1 \), it induces canonical morphisms \( \tilde{f}(s) : D'(s) \to D(s) \) compatible with natural projections and the diagonal embedding \( \mathcal{X} \to D(s) \) (c.f. § 2.1.9).

**Proof.** We have a commutative diagram

\[
\begin{array}{ccc}
D'(s) & \xrightarrow{\tilde{f}} & \tilde{X}' \\
\downarrow \Delta & & \downarrow \Delta \\
D'(s) & \xrightarrow{\tilde{f}^{s+1}} & \tilde{X}^{s+1}
\end{array}
\]

Then, by the universal property of Higgs envelope, we deduce a canonical morphism \( D'(s) \to D(s) \) above \( \tilde{f}^{s+1} \). The compatibility can be verified in a similar way. \( \square \)

**2.3.3.** Keep the assumption of lemma 2.3.2. Then, for \( \bullet \in \{\mathbb{Z}_p, \mathbb{Q}_p\} \), the morphism \( \tilde{f} \) induces functors:

\[
(2.3.3.1) \quad \tilde{f}^{*} : \mathcal{H}_{\mathcal{S},\text{fin}}(\tilde{X},\mathcal{X}/\Sigma_S) \to \mathcal{H}_{\mathcal{S},\text{fin}}(\tilde{X}',\mathcal{X}'/\Sigma_{S'}), \quad (M, \varepsilon) \mapsto (\tilde{f}^{*}M, \varepsilon \otimes \varepsilon_{\mathcal{D}(1)}, \tilde{f}(1) \varepsilon_{\mathcal{D}(1)}),
\]

\[
(2.3.3.2) \quad \tilde{f}^{*} : \mathcal{H}_{\mathcal{S},\text{conv}}(\tilde{X}/\Sigma,\mathcal{X}) \to \mathcal{H}_{\mathcal{S},\text{conv}}(\tilde{X}'/\Sigma,\mathcal{X}'.), \quad (M, \theta) \mapsto \tilde{f}^{*}(M, \theta),
\]

which are compatible with the twisted inverse image functor (2.3.1.2) via functors (2.2.4.1) and (2.2.7.1). The restriction of \( \tilde{f}^{*} \) to underlying \( \mathcal{O}_{\tilde{X}} \)-modules coincides with the usual inverse image functor \( f^{*} \).

**Proposition 2.3.4.** (i) Suppose \( \tilde{f}' \) is another lifting of \( f \). There exists a natural isomorphism of functors

\[
\alpha_{\tilde{f}', \tilde{f}} \sim \tilde{f}'^{*} \sim \tilde{f}^{*} \quad (2.3.3.1), \quad (2.3.3.2).
\]

(ii) The isomorphisms \( \alpha_{\tilde{f}', \tilde{f}} \) satisfy a cocycle condition for three liftings of \( \tilde{f} \).

(iii) Let \( \{U_i\}_{i \in I} \) be an open covering of \( \mathcal{X}' \) and \( \{\tilde{f}_i : U_i \to \mathcal{X}\}_{i \in I} \) a collection of liftings of \( \tilde{f}|_{U_i} \). By gluing local constructions, we obtain inverse image functors:

\[
(2.3.4.1) \quad \tilde{f}_{\mathcal{X}', \mathcal{X}, (\tilde{f}_i), \in I}^{*} : \mathcal{H}_{\mathcal{S},\text{fin}}(\tilde{X},\mathcal{X}/\Sigma_S) \to \mathcal{H}_{\mathcal{S},\text{fin}}(\tilde{X}',\mathcal{X}'/\Sigma_{S'}), \quad (M, \varepsilon) \mapsto (\tilde{f}_i^{*}(M, \varepsilon), \alpha_{\tilde{f}', \tilde{f}}),
\]

\[
(2.3.4.2) \quad \tilde{f}_{\mathcal{X}', \mathcal{X}, (\tilde{f}_i), \in I}^{*} : \mathcal{H}_{\mathcal{S},\text{conv}}(\tilde{X}/\Sigma,\mathcal{X}) \to \mathcal{H}_{\mathcal{S},\text{conv}}(\tilde{X}'/\Sigma,\mathcal{X}'.), \quad (M, \theta) \mapsto (\tilde{f}_i^{*}(M, \theta), \alpha_{\tilde{f}', \tilde{f}}),
\]

which are independent of the choice of \( (\tilde{f}_i)_{i \in I} \) up to canonical isomorphisms. We denote the resulting functors by \( \tilde{f}_{\mathcal{X}', \mathcal{X}} \).

(iv) The functors \( \tilde{f}_{\mathcal{X}', \mathcal{X}} \) are canonically isomorphic to functors (2.3.1.2).

**Proof.** By proposition 2.2.7, it suffices to prove the proposition for the categories \( \mathcal{H}_{\mathcal{S}} \).

(i) By the universal property of Higgs envelope, the canonical morphism \( (f', f) : \mathcal{X}' \to \mathcal{X}'^{2} \) induces a canonical morphism over \( \Sigma_S \):

\[
(2.3.4.3) \quad \alpha : \mathcal{X}' \to \mathcal{D}(1).
\]

Given an object \((M, \varepsilon)\) of \( \mathcal{H}_{\mathcal{S},\text{fin}}(\tilde{X},\mathcal{X}/\Sigma_S)\), then we obtain an isomorphism

\[
\alpha^{*}(\varepsilon) : \tilde{f}_{\mathcal{S}}^{*}(M) \sim \tilde{f}^{*}(M).
\]
It remains to show that this isomorphism is compatible with $\mathcal{O}_{D(1)}$-stratifications defined in (2.3.3.1). Indeed, we need to show the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{O}_{D(1)} \otimes \epsilon, \tilde{f}^*(M) & \xrightarrow{\tilde{f}^*(\epsilon)} & \tilde{f}^*(M) \otimes \epsilon, \mathcal{O}_{D(1)} \\
\mathcal{O}_{D(1)} \otimes \alpha & \xrightarrow{\alpha \otimes \mathcal{O}_{D(1)}} & \tilde{f}^*(M) \\
\end{array}
$$

The morphism $\tilde{f}^*(\epsilon)$ is a canonical isomorphism $\tilde{f}^*(M) \cong \mathcal{O}_{D(1)}$. Then, assertion (iii) follows.

We present a local description of the isomorphism following two compositions:

$$
(\tilde{f}^*, \tilde{f}) : \mathcal{X} \rightarrow \mathcal{X}'
$$

induces a canonical isomorphism, which coincides with assertion (ii), we deduce a canonical isomorphism

$$
\Phi_{\tilde{f}^*, \tilde{f}} : \tilde{f}^*(M) \cong \mathcal{O}_{D(1)}
$$

for functors (2.3.3.1) and finish the proof of assertion (i).

Assertion (ii) follows from the cocycle condition for stratifications.

(iii) By assertion (ii) and gluing local constructions defined by $\tilde{f}$, we obtain functors (2.3.4.2).

If we have another collection of lifting $\{\tilde{g}_j : V_j \rightarrow \mathcal{X} \}_{j \in J}$, we may assume that $\{V_j\}_{j \in J}$ refines $\{U_i\}_{i \in I}$. By assertions (i-ii), we deduce a canonical isomorphism $\Phi_{\tilde{f}^*, \tilde{f}} : \tilde{f}^*(M) \cong \tilde{f}^*(M)$, satisfying a cocycle condition. Then, assertion (iii) follows.

(iv) Via the equivalence between Higgs crystals and modules with Higgs stratification, we obtain an isomorphism between $\tilde{f}^*_{\mathcal{X}, \mathcal{X}'}(M)$ and $\tilde{f}^*_{\mathcal{H}, \mathcal{X}, \mathcal{X}'}$, which is compatible with $\Phi_{\tilde{f}^*, \tilde{f}}$ in (iii). Then, assertion (iv) follows.

**Remark 2.3.5.** We present a local description of the isomorphism $\alpha^*(\epsilon)$. Suppose $\mathcal{X}$ is affine and there exist $t_i = (t_i, N) \in \lim_{\to N} \Gamma(X_N, \mathcal{M}_{X_N})$ ($1 \leq i \leq d$) such that $\{d \log(t_i, N)\}_{1 \leq i \leq d}$ is a basis of $\Omega^1_{X_N/S_N}$ for $N \geq 1$. We keep the notation of 2.1.9. Then, the homomorphism $\mathcal{O}_{D(1)} \rightarrow \mathcal{O}_{X}$ (2.3.4.3) sends

$$
u_{i, 2} - 1 \rightarrow \tilde{f}^*(t_{i, 2}) \tilde{f}^*(t_{i, 2})^{-1} - 1$$

where $\tilde{f}^*(t_{i, 2}) \tilde{f}^*(t_{i, 2})^{-1}$ denotes an element $v_{i, 2} = 1 + \xi \mathcal{O}_{\mathcal{X}'}$ such that $\tilde{f}^*(t_{i, 2}) = v_{i, 2} \tilde{f}^*(t_{i, 2})$. Let $(M, \xi)$ be an object of $\mathcal{H}_{\mathcal{X}, \mathcal{M}}(X, \mathcal{M}_{X_N})$ and $\theta$ the associated Higgs field. The $\mathcal{O}_{D(1)}$-stratification $\xi$ on $M$ is locally defined, for $x \in \tilde{M}$, by

$$
\varepsilon(1 \otimes x) = \sum_{\mathcal{O}_{\mathcal{X}'}(x)} \frac{\xi}{\theta_{m}(x)} \otimes d \left( \frac{u_{i, 2} - 1}{\xi} \right)^{m_i}.
$$

Note that this formula $p$-adically converges as $\theta$ satisfies (Conv)$_r$ (resp. (Int)) (2.2.5).

Then, the isomorphism $\alpha^*(\epsilon) : \tilde{f}^*(M) \cong \tilde{f}^*(M)$ is locally given by:

$$
\alpha^*(\epsilon)(\tilde{f}^*(x)) = \sum_{\mathcal{O}_{\mathcal{X}'}(x)} \tilde{f}^* \left( \frac{\theta_{m}(x)}{m!} \right) \otimes \prod_{i=1}^{d} \left( \frac{\tilde{f}^*(t_{i, 2}) \tilde{f}^*(t_{i, 2})^{-1} - 1}{\xi} \right)^{m_i}.
$$

**Corollary 2.3.6.** Let $(M, \theta)$ be a Higgs bundle of $\mathcal{H}_{\mathcal{X}, \mathcal{M}}(X_{1/S}, \mathcal{M}_{X_{1/S}})$ and $\alpha \in \mathbb{Q}_{>0}$.

(i) If the image of the morphism $df : f^* \Omega^1_{X/S} \rightarrow \Omega_{\mathcal{X}, \mathcal{X}'}^1$ of log differentials (2.2.5) is contained in $p^n \Omega^1_{X_{1/S}'}$, then the Higgs field of $f^*_{\mathcal{X}, \mathcal{X}'}(M, \theta)$ is trivial modulo $p^n$. 


(ii) If the Higgs field \( \theta \) is \( \alpha \)-small for \( \alpha > \frac{1}{p-1} \), for \( \beta = \alpha - \frac{1}{p-1} \), then the reduction modulo \( p^\beta \) of the underlying bundle of \( \tilde{f}_{\mathcal{X}, \mathcal{X}}(M, \theta) \) is canonically isomorphic to that of \( \tilde{f}^*(M) \).

(iii) If \( \theta = 0 \), then \( \tilde{f}_{\mathcal{X}, \mathcal{X}}(M, 0) \simeq (\tilde{f}^*(M), 0) \).

Proof. (i) The assertion being local property, we may assume that there exists a lifting \( \tilde{f} : \mathcal{X}' \to \mathcal{X} \) of \( \tilde{f} \). Then, the assertion follows from the formula (2.3.3.2).

(ii) If a local lifting \( \tilde{f} \) of \( \tilde{f} \) exists, the underlying bundle of \( \tilde{f}_{\mathcal{X}, \mathcal{X}}(M, \theta) \) is isomorphic to \( \tilde{f}^*(M) \). These local constructions are glued by the formula (2.3.5.1), which is congruent to the identity modulo \( p^\beta \). Then, the assertion follows.

(iii) If \( \theta = 0 \), then the associated stratification \( \varepsilon \) sends \( 1 \otimes m \) to \( m \otimes 1 \), and so is \( \alpha^*(\varepsilon) \) (2.3.5.1). Then, the assertion follows. \( \square \)

Remark 2.3.7. The above construction also applies to the inverse image functoriality of Cartier transform of Ogus–Vologodsky in characterizer \( p > 0 \) and its lifting modulo \( p^n \) via Oyama topos [44, 45].

3. Small Higgs bundles with strongly semi-stable reduction of degree zero

In this section, we assume moreover that \( k \) is an algebraic closure of \( \mathbb{F}_p \). We will generalize some results in [13] to certain Higgs bundles.

3.1. Review on Deninger–Werner’s construction [13].

Definition 3.1.1. (i) Let \( C \) be a smooth proper curve over \( k \) and \( F_C : C \to C \) the absolute Frobenius of \( C \). We say a vector bundle \( E \) over \( C \) is strongly semi-stable if \( (F_C^n)^*(E) \) is semi-stable for all \( n \geq 0 \).

(ii) Let \( C \) be an irreducible proper curve over \( k \) and \( \pi : \tilde{C} \to C \) the normalization of the reduced subscheme of \( C \). We say a vector bundle \( E \) over \( C \) is strongly semi-stable of degree zero if \( \pi^*(E) \) is strongly semi-stable of degree zero on each irreducible connected component of \( \tilde{C} \).

Definition 3.1.2. (i) Let \( X \) be an \( \overline{\mathcal{S}} \)-model of a smooth proper curve \( X_\mathfrak{p} \) over \( \overline{K} \) (1.2.3) and \( \tilde{X} \) the \( p \)-adic completion of \( X \). We denote by \( \text{VB}^{\text{DW}}(\tilde{X}) \) the full subcategory of vector bundles \( E \) over \( \tilde{X} \) whose special fiber \( E_s \) is strongly semi-stable of degree zero on each irreducible component of \( X_s \).

(ii) Let \( Y \) be a smooth proper \( \overline{K} \)-curve. We denote by \( \text{VB}^{\text{DW}}(Y^\text{an}_\mathfrak{p}) \) the full subcategory of vector bundles \( E \) over \( Y^\text{an}_\overline{k} \) such that there exists an \( \overline{\mathcal{S}} \)-model \( X \) of \( Y \) and an object \( \mathcal{E} \) of \( \text{VB}^{\text{DW}}(\tilde{X}) \) with rigid generic fiber \( E \). We also denote by \( \text{VB}^{\text{DW}}(Y^\text{an}_\mathfrak{p}) \) the essential image of \( \text{VB}^{\text{DW}}(Y^\text{an}_\mathfrak{p}) \) via equivalence \( \text{VB}(Y^\text{an}_\mathfrak{p}) \simeq \text{VB}(Y^\text{an}_\mathfrak{p}) \).

We denote by \( \text{VB}^{\text{DDW}}(Y^\text{an}_\mathfrak{p}) \) the full subcategory of vector bundles \( E \) over \( Y^\text{an}_\mathfrak{p} \) consisting of vector bundles \( E \), which admits a finite map \( g : Z \to Y \) of smooth proper \( \overline{\mathcal{S}} \)-curves such that \( g^\mathfrak{p}_\mathfrak{p}(E) \) belongs to \( \text{VB}^{\text{DDW}}(Z^\mathfrak{p}) \).

Theorem 3.1.3 ([13] theorems 16, 17, [51] théorème 2.4). Let \( X \) be an \( \overline{\mathcal{S}} \)-model of a smooth proper \( \overline{\mathcal{K}} \)-curve. A vector bundle \( E \) over \( \tilde{X} \) belongs to \( \text{VB}^{\text{DW}}(\tilde{X}) \) if and only if, for each \( n \geq 1 \), there exists an \( \overline{\mathcal{S}} \)-cover (1.2.3) \( \varphi : X' \to X \) such that \( \varphi_\mathfrak{p}^\mathfrak{p}(E) \) is free of finite type, where \( (-)_n \) denotes the reduction modulo \( p^n \).

Let \( \overline{\mathfrak{p}} \) be a geometric generic point of \( X_\mathfrak{p} \). Deninger and Werner [13, 15] constructed functors:

\[
\forall^\mathfrak{p}_X^{\text{DDW}} : \text{VB}^{\text{DW}}(\tilde{X}) \to \text{Rep}(\mathcal{M}_1(\overline{\mathfrak{p}}, \overline{\mathfrak{p}}), \mathfrak{p}), \quad \forall^\mathfrak{p}_X^{\text{DDW}} : \text{VB}^{\text{DDW}}(X^\mathfrak{p}) \to \text{Rep}(\pi_1(X_\mathfrak{p}, \overline{\mathfrak{p}}), \mathfrak{p}).
\]

3.2. Covers trivializing small Higgs bundles with strongly semi-stable degree zero reduction.

3.2.1. Let \( X \) be a semi-stable \( S \)-curve whose generic fiber \( X_\mathfrak{p} \) has genus \( g \geq 1 \). We denote by \( \tilde{X} \) the \( p \)-adic completion of \( X \times_S \overline{\mathcal{S}} \), equipped with the fine log structure induced by \( \mathcal{M}_X \). We fix a smooth Cartesian lifting \( \tilde{X} \) of \( X \) over \( \Sigma_S \), which always exists ([55] § 11.2). We denote by \( \text{HB}^{\text{DW}}_{\text{small}}(\tilde{X}/\Sigma_1, S) \) the full subcategory of \( \text{HB}_{\text{small}}(\tilde{X}/\Sigma_1, S) \) (2.2.6) consisting of Higgs modules \( (M, \theta) \) such that \( M \) belongs to \( \text{VB}^{\text{DW}}(\tilde{X}) \) and that \( \theta \) is small (2.2.10).

Our main result in this section is the following, generalizing the description of \( \text{VB}^{\text{DW}}(\tilde{X}) \) in theorem 3.1.3:
Theorem 3.2.2. Let $M$ be a vector bundle of rank $r$ over $\tilde{X}$, $\theta$ a small Higgs field on $M$, and $\mathcal{M}$ the associated Higgs crystal on $(\tilde{X}/\Sigma S)_{\text{HG}}^\infty$. The following conditions are equivalence:

(i) $(M, \theta)$ belongs to $\text{HB}_{z_p, \text{small}}^\infty(\tilde{X}/\Sigma S)$.

(ii) For every $n \in \mathbb{N}$, there exists a finite extension $K'$ of $K$, a semi-stable $S'(=\text{Spec}(\mathcal{O}_{K'}))$-curve $Y$ and an $\eta'$-cover $f : Y \to X_{S'}$ satisfying the following property: if $\tilde{f} : \tilde{Y} \to \tilde{X}$ denotes the $p$-adic completion of $f \times_{S'} \mathcal{S}$ and the associated morphism of $p$-adic fine log formal schemes, then there exists an isomorphism of reduction modulo $p^n$ of Higgs crystals

\[
\tilde{f}_{\text{HG}}(M)_n \simeq (\mathcal{O}_{Y/\Sigma S',n})^{\oplus r}.
\]

We will use the following argument of Faltings on covers of $X$ whose associated inverse image of log differentials is divisible by a power of $p$.

Proposition 3.2.3 (Faltings, [18] Proof of theorem 6). For every $n \in \mathbb{N}$, there exists a finite extension $K'$ of $K$, a semi-stable $S'(=\text{Spec}(\mathcal{O}_{K'}))$-curve $Y$ and an $\eta'$-cover $f : Y \to X_{S'}$ such that the image of the morphism $df : f^*(\Omega^1_{X/\mathcal{S}}) \to \Omega^1_{Y/\mathcal{S}'}$ of log differentials is contained in $p^n\Omega^1_{Y/\mathcal{S}'}$.

Proof. Let $J_K$ be the Jacobian of $X_K$ and $J$ its Néron model over $S$, which is isomorphic to the neutral component of the relative Picard scheme $\text{Pic}^0_{X/\mathcal{S}}$ ([11] 9.5 theorem 1). After taking an extension of $K$ and choosing an $S$-point of $X$, we obtain a closed immersion $X_K \to J_K$ defined by this point. The multiplication by $p^n$ on $J_K$ induces an étale cover $f_K : X_{n,K} \to X_K$ by base change. After replacing $K$ by a finite extension, there exist a semi-stable $S$-model $X_n$ of $X_{n,K}$ and a model $f : X_n \to X$ of $f_K$.

If $J_{n,K}$ denotes the Jacobian of $X_{n,K}$ and $J_n$ its Néron model over $S$, $f_K$ induces a morphism $g_K : J_{n,K} \to J_K$ by the Albanese property and $g_K$ factors through $p^n : J_K \to J_K$. Moreover, by the universal property of Néron models, $g_K$ induces an $S$-morphism $g : J_n \to J$, which factors through $p^n$. Now we consider the following diagram of global sections of differentials:

Note that $\Omega^1_{X/\mathcal{S}}$ is isomorphic to the dualizing sheaf $\omega_{X/\mathcal{S}}$. The outer isomorphisms in the above diagram are induced by $H^1(X, \mathcal{O}_X) \simeq \text{Lie}J$ and the coherent duality.

Since $g$ factors through $p^n$, the image of $g^*$ is divisible by $p^n$ and so is $f^*$. Since $\Omega^1_{X_{K}/K}$ is generated by global sections, the sub-$\mathcal{O}_X$-module of $\Omega^1_{X/\mathcal{S}}$, generated by global sections $\Gamma(X, \Omega^1_{X/\mathcal{S}})$, contains $p^n\Omega^1_{X/\mathcal{S}}$ for some integer $n_0$, which is independent of $n$. Hence the image of $df : f^*(\Omega^1_{X/\mathcal{S}}) \to \Omega^1_{X_{n}/\mathcal{S}}$ is divisible by $p^{n-n_0}$. This finishes the proof. \hfill \square

Proof of theorem 3.2.2. (ii) \Rightarrow (i): Take $n = 1$ and let $\mathcal{Y}$ be a smooth Cartesian lifting of $\tilde{Y}$ over $\Sigma S'$. By corollary 2.3.6(ii), $f^*(M)_s$ is isomorphic to the special fiber of the underlying bundle of $f_{\mathcal{Y},X}(M, \theta)$, which is trivial by (3.2.2.1). Then, we conclude that $M_s$ is strongly semi-stable of degree zero by ([13] theorem 18).
(i) ⇒ (ii): We first take an $\eta'$-cover $f : Y \rightarrow X_{S'}$ as in proposition 3.2.3 such that the image of $df$ is divisible by $p^{n+1}$ and a smooth Cartesian lifting $Y$ of $Y$ over $\Sigma_{S'}$. By corollary 2.3.6(i), the Higgs field of $\tilde{f}_{Y,X}^*(M, \theta)$ is $(n + 1)$-small. By corollary 2.3.6(ii), the special fiber $(\mathcal{M}_Y)_s$ of $\mathcal{M}_Y$ is isomorphic to $\tilde{f}^*_s(M_s)$. Since $M_s$ is strongly semi-stable of degree zero, then so is $\tilde{f}^*_s(M_s)$ by ([13] theorem 18).

By theorem 3.1.3 and the semi-stable reduction theorem, there exists a finite extension $K''$ of $K'$ a projective $S''(= \text{Spec}(\mathcal{O}_{K''}))$-curve $Z$ with semi-stable reduction and an $\eta'$-cover $g : Z \rightarrow Y_{S''}$ satisfying following property: If $\tilde{g}$ denotes the $p$-adic completion of $g \times_{S''} \mathcal{S}$, then $\tilde{g}^*((\mathcal{M}_Y)_{n+1})$ is trivial.

In the following, we show that the functor $(\tilde{f} \circ \tilde{g})^*_{Z,X}(M, \theta)$ is étale and $\tilde{g}$, we obtain a canonical isomorphism of Higgs modules modulo $p^n$ in $\text{HM}(\tilde{Z}, Z/\Sigma_{S''})$ (2.3.4.2):

$$((\tilde{f} \circ \tilde{g})^*_{Z,X}(M, \theta))_{n}.$$

Then, we conclude the assertion from corollary 2.2.11. □

4. Review on the $p$-adic Simpson correspondence

4.1. Faltings topos.

4.1.1. Let $Y \rightarrow X$ be a morphism of coherent schemes ("coherent" means "quasi-compact and quasi-separated"). We denote by $E_{Y \rightarrow X}$ the category of morphisms of schemes $(V \rightarrow U)$ above $(Y \rightarrow X)$, i.e., commutative diagrams

$$
\begin{array}{ccc}
V & \longrightarrow & U \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
$$

such that $U \rightarrow X$ is étale and $V \rightarrow Y \times_X U$ is finite étale. We endow $E_{Y \rightarrow X}$ with the topology generated by the following types of morphisms

(v) $\{(V_m \rightarrow U) \rightarrow (V \rightarrow U)\}_{m \in M}$, where $\{V_m \rightarrow V\}_{m \in M}$ is a finite étale covering,

(c) $\{(V \times_U U_n \rightarrow U_n) \rightarrow (V \rightarrow U)\}_{n \in N}$, where $\{U_n \rightarrow U\}_{n \in N}$ is an étale covering.

We denote by $\overline{E}_{Y \rightarrow X}$ the topos of sheaves of sets on $E_{Y \rightarrow X}$ and call it Faltings topos associated to $Y$.

The presheaf $\overline{\mathcal{F}}_{Y \rightarrow X}$ on $E_{Y \rightarrow X}$ defined by

$$\overline{\mathcal{F}}_{Y \rightarrow X}(V \rightarrow U) = \Gamma(U^V, \mathcal{O}^V),$$

where $U^V$ is the integral closure of $U$ in $V$ ([57] OBAK), is a sheaf of rings ([23] proposition 7.6). For $n \geq 1$, we set $\overline{\mathcal{F}}_{Y \rightarrow X, n} = \overline{\mathcal{F}}_{Y \rightarrow X}/p^n\overline{\mathcal{F}}_{Y \rightarrow X}$.

We denote by $\overline{\mathcal{M}}_{Y \rightarrow X}$ the sheaf of rings of $\overline{E}_{Y \rightarrow X}^{\mathcal{M}}$ (§ 1.2.5), defined by the projective system $(\overline{\mathcal{F}}_{Y \rightarrow X, n})_{n \geq 1}$.

Proposition 4.1.2 ([4] VI.5.10). A sheaf $F$ on $E_{Y \rightarrow X}$ is equivalent to the datum of a sheaf $F_U$ of $U_{Y, \text{ét}}$, with $U_Y = U \times_X Y$ for every object $U$ of $\overset{\sim}{\textbf{Ét}}_X$ and a morphism $u_f : F_U \rightarrow (f_Y)_{\text{ét},*}(F_{U'})$ for every morphism $f : U' \rightarrow U$ of $\overset{\sim}{\textbf{Ét}}_X$ satisfying the following condition:

(i) a cocycle condition for morphisms $u_f$;

(ii) for every covering $(f_i : U_i \rightarrow U)_{i \in I}$ of $\overset{\sim}{\textbf{Ét}}_X$, if we set $U_{ij} = U_i \times_U U_j$ and $f_{ij} : U_{ij} \rightarrow U$ the canonical morphism, then the following sequence of morphisms of sheaves of $U_{Y, \text{ét}}$ is exact:

$$F_U \rightarrow \prod_{i \in I} (f_i)_{\text{ét},*}(F_{U_i}) \Rightarrow \prod_{i,j \in I} (f_{ij})_{\text{ét},*}(F_{U_{ij}}).$$
4.1.3. We briefly review the cohomological descent in Faltings topos. We assume that $X$ is a coherent $\overline{S}$-scheme, $Y$ is a coherent $\overline{\eta}$-scheme and moreover that in the following diagram:

$\begin{array}{ccc}
Y & \rightarrow & X^Y \\
\downarrow & & \downarrow \\
\overline{\eta} & \rightarrow & \overline{S}
\end{array}$

where $X^Y$ denotes the integral closure of $X$ in $Y$ ([57] 0BAK), the square is Cartesian.

Let $X_\bullet \rightarrow X$ be a proper hypercovering of $\overline{S}$-schemes, and $\varepsilon : \overline{E}_{Y \rightarrow X_\bullet} \rightarrow \overline{E}_{Y \rightarrow X}$ the augmentation of simplicial topos with $Y_\bullet = X_\bullet \times_X Y$. For any $\overline{\mathcal{F}}_{Y \rightarrow X}$-module $\mathcal{N}$, we have a canonical morphism:

(4.1.3.1) $\mathcal{N} \rightarrow R\varepsilon_*(\varepsilon^* \mathcal{N})$.

We say $\mathcal{N}$ satisfies the cohomological descent (in Faltings topos), if for any proper hypercovering $X_\bullet \rightarrow X$ of $\overline{S}$-schemes, the above morphism is an almost isomorphism.

Recently, T. He [23] showed that the cohomological descent holds for $\mathcal{N} = \mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathcal{F}}_{Y \rightarrow X}$, with a finite locally constant abelian sheaf $L$ of $\overline{E}_{Y \rightarrow X}$. A key ingredient of his proof is the arc-descent for perfectoid algebras due to Bhatt–Scholze [9]. In particular, we have the following corollary:

**Corollary 4.1.4.** Assume $Y = X_{\overline{\eta}}$. For every $n \in \mathbb{N}$, a locally free $\overline{\mathcal{F}}_{Y \rightarrow X, n}$-module $M$ of finite type satisfies the cohomological descent.

**Proof.** Let $\{(V_i \rightarrow U_i) \rightarrow (X_{\overline{\eta}} \rightarrow X)\}_{i \in I}$ be a covering such that $M|_{\{(V_i \rightarrow U_i)\}}$ is free of finite type. By further localization, we may assume each object $(V \rightarrow U) \rightarrow (X_{\overline{\eta}} \rightarrow X)$ in this covering that $V, U$ are coherent. By ([57] 035K), the generic fiber of $U^Y$ is isomorphic to $V$. The cohomological descent ([23] corollaries 8.14, 8.18) holds for the restriction of $M$ to the localization $E_{Y \rightarrow X}/(V \rightarrow U) \simeq E_{V \rightarrow U}$. As the assertion is local on $E_{Y \rightarrow X}$, the corollary follows.

4.1.5. In the following, let $X$ be a flat, normal, separated $S$-scheme (resp. $\overline{S}$-scheme) of finite presentation such that $\overline{X} = X \times_S \overline{S}$ (resp. $X$) is normal and $Y = X_{\overline{\eta}}$. We will use both of two settings in this article. Then, $\overline{X}$ (resp. $X$) is locally irreducible (cf. [4] III.3.1). Indeed, as $\overline{X}$ (resp. $X$) is flat over $\overline{S}$, its generic points are the generic points of $X_{\overline{\eta}}$, which are finite (cf. [4] III.3.2(ii)).

For simplicity, we denote the Faltings site $E_{X_{\overline{\eta}} \rightarrow X}$ (resp. the structure sheaf $\overline{\mathcal{F}}_{X_{\overline{\eta}} \rightarrow X}$) by $E_X$ or $E$ (resp. $\overline{\mathcal{F}}_X$ or $\overline{\mathcal{F}}$), if there is no confusion. The Faltings topos $\overline{E}_X$ admits a closed sub-topos $E_{X,s}$ ([6] IV.9.3.5, [4] III.9.3). For each $n \geq 1$, the ring $\overline{\mathcal{F}}_{X,n}$ belongs to $E_{X,s}$.

The functor $\mathbf{F}\mathbf{et}_{X_{\overline{\eta}}} \rightarrow E$, defined by $V \mapsto (V \rightarrow X)$, is continuous and left exact and induces a morphism of topoi $\beta : \overline{E}_X \rightarrow X_{\overline{\mathbf{f}\mathbf{et}}_{\overline{\eta}}}$. Moreover, it induces morphisms of ringed topos ([55] 7.13)

(4.1.5.1) $\beta_{X,n} : (\overline{E}_{X,s}, \overline{\mathcal{F}}_n) \rightarrow (X_{\overline{\mathbf{f}\mathbf{et}}_{\overline{\eta}}}, \mathcal{O}_n), \quad \forall \ n \geq 1, \quad \beta_X : (\overline{E}_X, \overline{\mathcal{F}}) \rightarrow (X_{\overline{\mathbf{f}\mathbf{et}}_{\overline{\eta}}}, \mathcal{O})$,

where $\mathcal{O}_n$ is the constant sheaf of $X_{\overline{\mathbf{f}\mathbf{et}}_{\overline{\eta}}}$ with value $\mathcal{O}_n$, and $\mathcal{O} = (\mathcal{O}_n)^{n \geq 1}$.

The functor $\mathbf{F}\mathbf{t}_{X} \rightarrow E$, defined by $U \mapsto (U_{\overline{\eta}} \rightarrow U)$, is continuous and left exact and induces a morphism of topoi $\sigma : \overline{E}_X \rightarrow X_{\mathbf{f}\mathbf{t}}$. Moreover, it induces morphisms of ringed topos ([4] III.9.9, III.9.10)

(4.1.5.2) $\sigma_{X,n} : (\overline{E}_{X,s}, \overline{\mathcal{F}}_n) \rightarrow (X_{\mathbf{f}\mathbf{t}}, \mathcal{O}_n^X), \quad \forall \ n \geq 1, \quad \sigma_X : (\overline{E}_X, \overline{\mathcal{F}}) \rightarrow (X_{\mathbf{f}\mathbf{t}}, \mathcal{O}_X)$,

where $\mathcal{O}_X$ is the étale sheaf of $X_{\mathbf{f}\mathbf{t}}$ associated to the coherent sheaf $\mathcal{O}_{\overline{X}}$ of $X_{zar}$ and $\mathcal{O}_X = (\mathcal{O}_X)^{n \geq 1}$.

When $X$ is an $S$-scheme, the canonical functor $E_X \rightarrow E_{X,s}$, defined by $(V \rightarrow U) \mapsto (V \rightarrow U)$, induces a morphism of topoi

(4.1.5.3) $h : \overline{E}_X \rightarrow \overline{E}_X$.

We have $h_*(\overline{\mathcal{F}}_{\overline{X}}) = \overline{\mathcal{F}}_X$ and the above morphism is ringed by $\overline{\mathcal{F}}_{\overline{X}}$ and $\overline{\mathcal{F}}_X$. 

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Lemma 4.1.6. (i) For any $\mathfrak{o}_n$-module $L$ of $X_{\mathfrak{f},\text{ét}}$, the following canonical morphism is an isomorphism:

$$\beta^*_{X,n}(L) \overset{\sim}{\rightarrow} h_*(\beta^*_{X,n}(L)).$$

(ii) For any $\Theta_{\mathfrak{o}_n}$-module $M$ of $X_{\mathfrak{f},\text{ét}}$, the following canonical morphism is an isomorphism:

$$\sigma^*_{X,n}(M) \overset{\sim}{\rightarrow} h_*(\sigma^*_{X,n}(M)).$$

Proof. (i) By proposition 4.1.2 and ([4] VI.8.9 and VI.10.9), both two sheaves are isomorphic to the sheaf associated to the presheaf:

$$(U \mapsto f^*_{\mathfrak{f},\text{ét}}(L) \otimes_{\mathfrak{o}_n} (\mathcal{O}_{\mathfrak{f},\text{ét}})_{|U}, \quad f : U \rightarrow X \in \text{Ét}_X),$$

where $\mathfrak{o}_n$ is the constant sheaf of $U_{\mathfrak{f},\text{ét}}$. Then, the assertion follows.

Assertion (ii) can be proved in a similar way using ([4] VI.5.34(ii)).

4.1.7. In the following, we assume $X$ is a semi-stable $S$-scheme, equipped with the log structure $\mathcal{M}_X$ (1.2.2). Then, $\overline{X}$ is normal. Let $U = \text{Spec}(R) \rightarrow X$ be an object of $\mathcal{Q}(2.2.2)$, $\overline{U} \rightarrow U_{\mathfrak{f}}$ a geometric generic point and $y$ the image of $\overline{U}$ in $U_{\mathfrak{f}}$, which is a generic point, such that $\overline{U} := \Gamma(U_{\mathfrak{f}}, \mathcal{O}_{U_{\mathfrak{f}}})$ is an algebraic closure of the residue field $K$ of $U_y$ at $y$. We define $K_{\text{ur}}$ to be the union of all finite extensions of $L$ of $K$ contained in $\overline{K}$ such that the integral closures of $R$ in $\overline{K}$ are étale over $\prod_{\mathfrak{p}}$. Let $\overline{R}$ be the integral closure of $R$ in $K_{\text{ur}}$ and $\overline{\mathcal{M}}$ the $p$-adic completion of $\overline{R}$.

Since $U_{\overline{U}}$ is smooth over $\overline{U}$, generic points of $U_{\overline{U}}$ are classified by the set $\text{Con}(U_{\overline{U}})$ of connected components of $U_{\overline{U}}$. Let $U_{\overline{U}}$ be the connected component of $U_{\overline{U}}$ containing $\overline{U}$. The morphism $\overline{U} \rightarrow U_{\overline{U}}$ induces an extension $\overline{K} \rightarrow \overline{K}$ above $K \rightarrow \overline{K}$. We denote by $G(U_{\overline{U}})$ the Galois group $\text{Gal}(K_{\text{ur}}/K)$, and by $\Delta(U_{\overline{U}})$ the kernel of $G(U_{\overline{U}}) \rightarrow \text{Gal}(\overline{K}/K)$, which is canonically isomorphic to $\pi_1(U_{\overline{U}}, \overline{U})$. Let $\mathcal{B}_{\Delta(U_{\overline{U}})}$ be the classifying topos of the profinite group $\Delta(U_{\overline{U}})$. Under the equivalence: $\nu_{\overline{U}} : (U_{\overline{U}})_{\text{ét}} \overset{\sim}{\rightarrow} \mathcal{B}_{\Delta(U_{\overline{U}})}$ defined by the fiber functor $F \rightarrow F_{\overline{U}}$, we have a canonical isomorphism ([4] III.8.15)

$$\nu_{\overline{U}}(\mathcal{O}_{\overline{U}}(U_{\overline{U}})_{\text{ét}}) \overset{\sim}{\rightarrow} \overline{R}.$$ 

Moreover, the action $\Delta(U_{\overline{U}})$ on $\overline{R}$ is continuous with respect to the $p$-adic topology.

For each pair $(U, \overline{U})$ as above, we associate to adic $\overline{\mathcal{M}}$-modules of finite type (1.2.5) continuous $\overline{R}$-representations of $\Delta(U_{\overline{U}})$ via $\nu_{\overline{U}}$. In the following, we collect some notions about representations:

Definition 4.1.8 ([4] II.13.1, II.13.2). Let $G$ be a topological group, $A$ an $\alpha$-algebra that is complete and separated for the $p$-adic topology, equipped with a continuous $G$-action (via homomorphisms of $\alpha$-algebras).

(i) A continuous $A$-representation of $G$ is an $A$-module $M$ of finite type, equipped with the $p$-adic topology and a continuous $A$-semi-linear action of $G$.

(ii) Let $M$ be a continuous $A$-representation of $G$ and $\alpha$ a rational number $> 0$.

We say $M$ is $\alpha$-small if $M$ is a free $A$-module of finite rank having a basis over $A$ consisting of elements which are $G$-invariant modulo $\alpha^p$.

We say $M$ is small if $M$ is $\alpha$-small for a rational number $\alpha > \frac{2}{p-1}$.

(iii) A continuous $A[\frac{1}{p}]$-representation of $G$ is an $A[\frac{1}{p}]$-module of finite type, equipped with the $p$-adic topology and a continuous $A[\frac{1}{p}]$-semi-linear action of $G$.

(iv) We say a continuous $A[\frac{1}{p}]$-representation $N$ of $G$ is small, if $N$ is moreover projective, and there exists a rational number $\alpha > \frac{2}{p-1}$, a sub-$A$-module of finite type $N^\circ$ which is $G$-invariant and generates $N$ over $A[\frac{1}{p}]$, and a finite number of generators of $N^\circ$ that are $G$-invariant modulo $\alpha^p$.

We denote by $\text{Rep}(G, A)$ (resp. $\text{Rep}(G, A[\frac{1}{p}])$) the category of continuous $A$-representations (resp. $A[\frac{1}{p}]$-representations) of $G$ and by $\text{Rep}_{\text{small}}(G, A)$ (resp. $\text{Rep}_{\text{small}}(G, A[\frac{1}{p}])$) the full subcategory of $\text{Rep}(G, A)$ (resp. $\text{Rep}(G, A[\frac{1}{p}])$) consisting of small representations.
4.1.9. We denote by $\text{LocSys}(X_{\pi, \text{fét}}, \mathfrak{o}_n)$ (resp. $\text{LocSys}(X_{\pi, \text{fét}}^n, \delta)$) the category of locally free of finite type $\mathfrak{o}_n$-modules of $X_{\pi, \text{fét}}$ (resp. adic $\delta$-modules $M = (M_n)_{n \geq 1}$ of finite type such that each $M_n$ belongs to $\text{LocSys}(X_{\pi, \text{fét}}, \mathfrak{o}_n)$). Suppose $X_{\pi}$ is connected. Let $\mathfrak{T}$ be a geometric generic point of $X_{\pi}$. The fiber functor $\nu_{\mathfrak{T}}$ at $\mathfrak{T}$ induces fully faithful functors:

$(4.1.9.1) \quad \text{LocSys}(X_{\pi, \text{fét}}, \mathfrak{o}_n) \to \text{Rep}(\pi_1(X_{\pi}, \mathfrak{T}), \mathfrak{o}_n), \quad \text{LocSys}(X_{\pi, \text{fét}}^n, \delta) \to \text{Rep}(\pi_1(X_{\pi}, \mathfrak{T}), \delta),$

and an equivalence of categories (c.f. [55] 3.29):

$(4.1.9.2) \quad \text{LocSys}(X_{\pi, \text{fét}}^n, \delta) \cong \text{Rep}(\pi_1(X_{\pi}, \mathfrak{T}), C).$

The essential images of (4.1.9.1) are representations whose underlying $\mathfrak{o}_n$-modules, $\mathfrak{o}$-modules are free.

Let $\rho : X_{\pi, \text{fét}} \to X_{\pi, \text{fét}}$ be the canonical morphism of topoi. The adjunction morphism $\text{id} \to \rho_* \rho^*$ is an isomorphism. The inverse image of $\rho$ induces a fully faithfull functor:

$(4.1.9.3) \quad \rho^* : \text{Mod}(X_{\pi, \text{fét}}, \mathfrak{o}_n) \to \text{Mod}(X_{\pi, \text{fét}}, \mathfrak{o}_n), \quad \forall \ n \geq 1.$

4.2. $p$-adic Simpson correspondence via Higgs crystals.

4.2.1. We keep the assumption and notation of 4.1.7. We denote by $\mathcal{R}$ the perfection $\lim_{\leftarrow \xi \to \xi'} \mathcal{R}/p\mathcal{R}$ of $\mathcal{R}/p\mathcal{R}$ and by $\theta : W(\mathcal{R}) \to \hat{\mathcal{R}}$ the canonical homomorphism defined by Fontaine, which is surjective ([4] II.9.5). The element $\xi$ ([1.2.1]) generates the ideal Ker($\theta$).

For $N \geq 1$, we set $A_N(\mathcal{R}) = W(\mathcal{R})/\xi^N W(\mathcal{R})$, which is also $p$-adically complete and separated. There exists a canonical and continuous action of $G_{(U,\overline{\mathcal{T}})}$ on $A_N(\mathcal{R})$ ([4] IV.5.1.2). We denote by $\overline{U}$ the $p$-adic formal scheme $\overline{U} = \text{Spf}(\mathcal{R})$ and for $N \geq 1$, $D_N(\overline{U})$ the $p$-adic formal scheme $\text{Spf}(A_N(\mathcal{R}))$. They are equipped with fine and saturated log structures induced by $\mathcal{M}_U$ and there is a canonical isomorphism $\overline{U} \cong D_1(\overline{U})$ (cf. [4] IV.5.1.4).

The inductive system $D(\overline{U}) = (D_N(\overline{U}))_{N \geq 1}$ together with a canonical morphism $\tau_{\overline{U}} : \overline{U} \to \hat{U}$ form an object of $(\hat{X}/\Sigma_S)^{rHIG}$ for $r \in \mathbb{N} \cup \{\infty\}$ (cf. [4] IV.5.2). Then, $\Delta_{(U,\overline{\mathcal{T}})}$ acts on $(D(\overline{U}), \tau_{\overline{U}})$.

4.2.2. We denote by $\text{LPM}(\mathcal{R})$ the category of finitely generated $\mathcal{R}$-modules $M$ such that $M[\frac{1}{p}]$ is projective over $\mathcal{R}[\frac{1}{p}]$, and by $\text{Rep}_{\text{LPM}}^{\text{PM}}(\Delta_{(U,\overline{\mathcal{T}})}, \mathcal{R})$ (resp. $\text{Rep}_{\text{PM}}^{\text{PM}}(\Delta_{(U,\overline{\mathcal{T}})}, \mathcal{R}[\frac{1}{p}])$) the full subcategory of $\text{Rep}(\Delta_{(U,\overline{\mathcal{T}})}, \mathcal{R})$ (resp. $\text{Rep}(\Delta_{(U,\overline{\mathcal{T}})}, \mathcal{R}[\frac{1}{p}])$) whose underlying module belongs to $\text{LPM}(\mathcal{R})$ (resp. is finitely generated and projective over $\mathcal{R}[\frac{1}{p}]$).

For an object $U$ of $\mathcal{Q}$, a geometric generic point $\overline{\mathcal{T}}$ of $U_{\overline{\mathcal{T}}}$ as in 4.1.7 and an object $\mathcal{M}$ of $\text{HC}_{\mathcal{Q}, \text{fin}}(\hat{X}/\Sigma_S)$ (resp. $\text{HC}_{\mathcal{Q}, \text{fin}}^C(\hat{X}/\Sigma_S)$), we associate an object of $\text{LPM}(\mathcal{R})$ (resp. a finite projective $\mathcal{R}[\frac{1}{p}]$-module):

$(4.2.2.1) \quad T_{(U, \overline{\mathcal{T}})}(\mathcal{M}) = \Gamma((D(\overline{U}), \tau_{\overline{U}}), \mathcal{M}), \quad \text{(resp. } V'_{(U, \overline{\mathcal{T}})}(\mathcal{M}) = \Gamma((D(\overline{U}), \tau_{\overline{U}}), \mathcal{M})),$

The action of $\Delta_{(U, \overline{\mathcal{T}})}$ on $(D(\overline{U}), \tau_{\overline{U}})$ provides a semi-linear action of $\Delta_{(U, \overline{\mathcal{T}})}$ on $T_{(U, \overline{\mathcal{T}})}(\mathcal{M})$. Then, we obtain the following functors ([4] IV.5.2):

$(4.2.2.2) \quad T_{(U, \overline{\mathcal{T}})} : \text{HC}_{\mathcal{Q}, \text{fin}}(\hat{X}/\Sigma_S) \to \text{Rep}_{\text{LPM}}^{\text{PM}}(\Delta_{(U, \overline{\mathcal{T}})}, \mathcal{R}), \quad \mathcal{M} \mapsto \Gamma((D(\overline{U}), \tau_{\overline{U}}), \mathcal{M}),$

Let $f : U' \to U$ be a morphism of $\mathcal{Q}$ and $\overline{\mathcal{T}}'$ a geometric generic point of $U'_{\overline{\mathcal{T}}}$ above $\overline{\mathcal{T}}$. It induces a homomorphism $\Delta_{(U', \overline{\mathcal{T}}')} \to \Delta_{(U, \overline{\mathcal{T}})}$ and a morphism $D(U') \to D(U)$ of $(\hat{X}/\Sigma_S)^{rHIG}$ compatible with $\Delta_{(U', \overline{\mathcal{T}}')}$-actions. Moreover, we deduce a $\mathcal{R}$-linear and $\Delta_{(U', \overline{\mathcal{T}}')}$-equivariant morphism:

$(4.2.2.2') \quad T_{(U, \overline{\mathcal{T}})}(\mathcal{M}) \to T_{(U', \overline{\mathcal{T}}')}(\mathcal{M}) \quad \text{(resp. } V'_{(U, \overline{\mathcal{T}})}(\mathcal{M}) \to V'_{(U', \overline{\mathcal{T}}')}(\mathcal{M})).$
Since $\mathcal{M}$ is a crystal, we deduce the following $R'$-linear $\Delta_{(U',\overline{\eta})}$-equivariant isomorphism:

\begin{equation}
T_{(U,\overline{\eta})}(\mathcal{M}) \otimes_{\overline{\eta}} R' \sim T_{(U',\overline{\eta})}(\mathcal{M}) \quad \text{(resp. } V_{(U,\overline{\eta})}(\mathcal{M}) \otimes_{\overline{\eta}} R' \sim V_{(U',\overline{\eta})}(\mathcal{M})).
\end{equation}

The above construction globalizes to a functor ([4] IV.6.4.5):

\begin{equation}
T : \text{HC}_{Z_p,\text{fin}}(\tilde{X}/\Sigma S) \to \text{Mod}(\overline{\mathcal{B}}), \quad \mathcal{M} \mapsto (T_{(U,\overline{\eta})}(\mathcal{M}))_{n \geq 1, U \in \mathcal{Q}, \overline{\eta} \in \text{Con}(U)}.
\end{equation}

where $\text{Mod}(\overline{\mathcal{B}})$ denotes the category of $\overline{\mathcal{B}}$-modules, geometric generic points $\overline{\eta}$ are taken over the set of connected components $\text{Con}(U)$ of $U$. For $r \in \mathbb{N}$, the category $(\mathcal{M} \to \text{Mod}(\overline{\mathcal{B}}))_\mathcal{Q}$ forms a $\overline{\mathcal{B}}_{\mathcal{Q},n}|_U$-module of $U$. The above functor is fully faithful up to isogeny ([4] theorem IV.6.4.9).

For $r \in \mathbb{N}$, we denote by $\text{HC}_{Z_p,\text{fin}}(\tilde{X}/\Sigma S)$ the category whose object is a triple consisting of an object $\mathcal{M}$ of $\text{HC}_{Z_p,\text{fin}}(\tilde{X}/\Sigma S)$, an object $\mathcal{M}^\circ$ of $\text{HC}_{Z_p,\text{fin}}(\tilde{X}/\Sigma S)$ and an isomorphism $\iota_{\mathcal{M}} : \mathcal{M}^\circ \simeq \mathcal{M}|_{\tilde{X}/\Sigma S}^\text{HIG}$. A morphism is defined in a natural way (c.f. [4] definition IV.3.5.4).

The functor $T$ induces a fully faithful functor:

\begin{equation}
V' : \text{HC}_{Z_p,\text{fin}}(\tilde{X}/\Sigma S)_\mathcal{Q} \to \text{Mod}(\overline{\mathcal{B}})_\mathcal{Q}, \quad (\mathcal{M}, \mathcal{M}^\circ) \mapsto T(\mathcal{M}^\circ)_\mathcal{Q},
\end{equation}

whose restriction to $(U,\overline{\eta})$ is compatible with the functor $V'_{(U,\overline{\eta})}$.

4.2.3. Next, we discuss the pullback functoriality of functors $T, V'$. We keep the assumption of § 2.3.1. Let $(\tilde{E}_X^\mathbb{N}, \overline{\mathcal{B}}_X)$ be the Faltings’ ringed topos associated to the $S'$-scheme $X'$. Then, the morphism $f : X' \to X'_S$ induces a morphism of ringed topos:

$$\Phi : (\tilde{E}_X^\mathbb{N}, \overline{\mathcal{B}}_X) \to (\tilde{E}_{X'}^\mathbb{N}, \overline{\mathcal{B}}_{X'})$$

Let $T' : \text{HC}_{Z_p,\text{fin}}(\tilde{X}'/\Sigma S') \to \text{Mod}(\overline{\mathcal{B}}_{X'})$ be the functor (4.2.2.4) defined by $X'$.

**Proposition 4.2.4.** (i) The following diagram is commutative up to canonical isomorphisms $\gamma_f$:

\[
\begin{array}{ccc}
\text{HC}_{Z_p,\text{fin}}(\tilde{X}/\Sigma S) & \xrightarrow{T} & \text{Mod}(\overline{\mathcal{B}}_X) \\
\downarrow f_{\text{HIG}} & & \downarrow \Phi_* \\
\text{HC}_{Z_p,\text{fin}}(\tilde{X}'/\Sigma S') & \xrightarrow{T'} & \text{Mod}(\overline{\mathcal{B}}_{X'})
\end{array}
\]

(ii) Let $K''$ be a finite extension of $K'$ in $K$, $X''$ a semi-stable $S''$ = Spec$(\mathcal{O}_{K''})$-scheme, and $g : X'' \to X'_{S''}$ an $S''$-morphism. Then, we have an identity of natural transforms:

\begin{equation}
\gamma_{gf} = \gamma_g(f_{\text{HIG}}(-)) \circ \Psi'(\gamma_f(-)),
\end{equation}

where $\Psi : (\tilde{E}_{X''}^\mathbb{N}, \overline{\mathcal{B}}_{X''}) \to (\tilde{E}_X^\mathbb{N}, \overline{\mathcal{B}}_X)$ is the canonical morphism of Faltings’ ringed topos associated to $g$.

**Proof.** (i) Let $\mathcal{M}$ be an object of $\text{HC}_{Z_p,\text{fin}}(\tilde{X}/\Sigma S)$. Let $U'$ be an object of $\mathcal{Q}_{X'}$, equipped with an $S'$-morphism $f : U' \to U_{S'} = U \times_{S'} S'$, and $\overline{\eta}$ a geometric generic point of $U'_{\mathcal{Q}}$ above $\overline{\eta}$. With the notation of 4.2.2, $D(U')$ defines an object of $\text{HC}_{Z_p,\text{fin}}(\overline{\mathcal{B}}_{\overline{\eta}})$ via functor (2.3.1.1). As in 4.2.2, the morphism $f$ induces a homomorphism $\Delta_{(U',\overline{\eta})} \to \Delta_{(U,\overline{\eta})}$ and a morphism $D(U') \to D(U)$ of $\text{HC}_{Z_p,\text{fin}}(\overline{\mathcal{B}}_{\overline{\eta}})$ compatible with $\Delta_{(U,\overline{\eta})}$-actions. Since $\mathcal{M}$ is a crystal, we deduce a $\overline{\eta}$-linear $\Delta_{(U,\overline{\eta})}$-equivariant isomorphism:

\begin{equation}
T_{(U,\overline{\eta})}(\mathcal{M}) \otimes_{\overline{\eta}} R' \sim T_{(U',\overline{\eta})}(\mathcal{M})).
\end{equation}
Then, we deduce a canonical isomorphism:

\[
\nu_{\mathcal{T}}(\gamma_f(M))_{U_{\mathcal{T}}} = \nu_{\mathcal{T}}(\Phi^*(T(M)))_{U_{\mathcal{T}}} \cong T'_{U_{\mathcal{T}}}(\tilde{f}_{\text{HIG}}(M)).
\]

The above isomorphism is natural on \((U', \mathcal{F})\) and \((U, \mathcal{F})\). Then, we obtain a canonical isomorphism \(\gamma_f(M) : \Phi^*(T(M)) \cong T'_{\tilde{f}_{\text{HIG}}(M)}\), which is functorial. Then the assertion follows.

(ii) Recall that the isomorphism \(\gamma_f(M)\) is constructed by the transition isomorphism of the evaluation of Higgs crystals on certain objects of Higgs site (4.2.4.2). Then, the assertion follows from the cocycle condition of transition isomorphisms.

\[\square\]

4.2.5. The functor \(T_{U_{\mathcal{F}}}\) (resp. \(V_{U_{\mathcal{F}}}\)) can be reinterpreted as an admissible isomorphism for a “period ring”.

Let \(X\) be a smooth Cartesian lifting of \(\tilde{X}\) over \(\Sigma_S\) Following ([4] IV.5.2), we denote by \(\mathcal{D}_{X,\mathcal{F}}(\mathcal{U})\) the object of \(\mathcal{C}^r\) defined by the Higgs envelope (§ 2.1.8)

\[\mathcal{D}_{X,\mathcal{F}}(\mathcal{U}) = D_{\text{HIG}}(\mathcal{U}) \to X \times_{\Sigma_S} D(\mathcal{U}),\]

which is affine. We define an object \(\mathcal{A}_{X,\mathcal{F}}(\mathcal{U}) = (\mathcal{A}_{X,N}(\mathcal{U}))\) of \(\mathcal{C}^r\) (§ 2.1.1) by

\[\mathcal{A}_{X,\mathcal{F}}(\mathcal{U}) = \Gamma(\mathcal{D}_{X,\mathcal{F}}(\mathcal{U}), \mathcal{D}_{X,\mathcal{F}}(\mathcal{U})).\]

We ignore the notation \(\tilde{X}, \mathcal{F}\) from \(\mathcal{A}_{X,\mathcal{F}}\) if there is no confusion and we focus on \(\mathcal{A}_{\mathcal{F}}\).

Let \(\mathcal{U} = \text{Spec}(\mathcal{O}_1)\) be the connected component (or equivalently, irreducible component) of \(\mathcal{U} = U \times S \mathcal{F}\) containing \(Y\) and \(\mathcal{U} = \text{Spec}(\mathcal{O}_1)\) its p-adic completion. The projections \(\mathcal{D}^r(\mathcal{U}) \to D(\mathcal{U})\) and \(\mathcal{D}^r \to X\) induce compatible homomorphisms \(\mathcal{R} \to \mathcal{A}^r(\mathcal{R})\) and \(\mathcal{R} \to \mathcal{A}^r(\mathcal{R})\). We refer to ([4] IV.5.2.5) for a description of \(\mathcal{R}\)-algebra structure of \(\mathcal{A}^r(\mathcal{R})\). The ring \(\mathcal{A}^r(\mathcal{R})\) is equipped with a continuous action of \(\Delta_{U_{\mathcal{F}}}\) with respect to the \(p\)-adic topology ([4] IV.5.3.6) and a Higgs field

\[\theta_{\mathcal{F}} : \mathcal{A}^r(\mathcal{R}) \to \xi^{-1} \mathcal{A}^r(\mathcal{R}) \otimes_{\mathcal{R}} \Omega^1_{\mathcal{R}/\mathcal{O}_K}.\]

Let \(\mathcal{M}\) be a Higgs isocrystal of \(\text{HC}_{C_2,\mathcal{F}}(\tilde{X}/\Sigma_S)\) (resp. a Higgs crystal of \(\text{HC}_{C_2,\mathcal{F}}(\tilde{X}/\Sigma_S)\)). We set \(M = \Gamma(\mathcal{U}, \mathcal{M}, \mathcal{N})\) and \(\theta : M \to \xi^{-1} M \otimes_{\mathcal{R}} \Omega^1_{\mathcal{R}/\mathcal{O}_K}\) the associated Higgs field on \(M\). We set \(V(\mathcal{M})\) to be \(V_{U_{\mathcal{F}}}(\mathcal{M})\) (resp. \(T_{U_{\mathcal{F}}}(\mathcal{M})\)). Then, there exists a canonical \(\Delta_{U_{\mathcal{F}}}\)-equivariant \(\mathcal{A}^r(\mathcal{R})\)-linear isomorphism compatible with Higgs fields ([4] IV.5.2.12):

\[\mathcal{A}^r(\mathcal{R}) \otimes_{\mathcal{R}} V(\mathcal{M}) \cong \mathcal{A}^r(\mathcal{R}) \otimes \mathcal{R} M,\]

where the Higgs field on \(V(\mathcal{M})\) is trivial and \(\Delta_{U_{\mathcal{F}}}\)-action on \(M\) is also trivial.

Moreover, Tsuchi compared the algebra \(\mathcal{A}^r(\mathcal{R})\) with the Higgs-Tate algebra introduced by Abbes–Gros ([4] II.10). More precisely, let \(\mathcal{E}_{X,\mathcal{F}}(\mathcal{U})\) be the Higgs-Tate \(\mathcal{R}\)-algebra defined by \(X_2\) over \(\Sigma_{2,S}\) associated to \((U, \mathcal{F})\) ([4] II.10.5) and let \(\mathcal{E}_{X,\mathcal{F}}(\mathcal{U})\) be its p-adic completion. There exists a canonical \(\Delta_{U_{\mathcal{F}}}\)-equivariant isomorphism compatible with Higgs fields ([4] IV.5.4.3)):

\[\mathcal{E}_{X,\mathcal{F}}(\mathcal{U}) \cong \mathcal{A}^r_{X,N}(\mathcal{R}).\]

For \(r \geq 1\), there exists an injection of \(\mathcal{R}\)-homomorphism from \(\mathcal{A}^r(\mathcal{R})\) to \(\mathcal{A}^r(\mathcal{R})\) p-adic completion of the Higgs-Tate algebra of thickness \(1/r\) ([4] II.12.1) whose cokernel is annihilated by \(p\) ([4] IV.5.4.4)).

4.2.6. The Higgs-Tate algebra admits a globalization as a \(\mathcal{R}\)-algebra \(\mathcal{E}_{X,\mathcal{F}}(\mathcal{U})\) for \(s \in \mathbb{Q}_{>0}\) ([4] III.10.31). For a pair of an adic \(\mathcal{R}\)-module of finite type \(\mathcal{N}\) and a Higgs bundle \((M, \theta)\) of \(\text{HB}_{\mathcal{R}}(\tilde{X}/\Sigma_{1,S})\) (2.2.6), Abbes and Gros introduced the notion of \(\mathcal{N}\) and \((M, \theta)\) are associated by an admissible condition defined by \(\mathcal{E}_{X,\mathcal{F}}(\mathcal{U})\) for some \(s \in \mathbb{Q}_{>0}\) ([4] III.12.11). An adic \(\mathcal{R}\)-module of finite type satisfying such an admissible
condition is called Dolbeault. This admissible condition establishes an equivalence between small Higgs bundles and Dolbeault $\mathcal{F}_{X,Q}$-modules:

\[(4.2.6.1) \quad T_X : \text{HB}_{\mathbb{Q}, \text{small}}(\hat{X}/\Sigma_1,S) \simeq \text{Mod}^{\text{Dolb}}(\mathcal{F}_{X,Q}) : \text{H}_X.\]

Their approach to the $p$-adic Simpson correspondence is compatible Tsuji's approach via $\iota_X$ (proposition 2.2.7(ii)). More precisely, in view of §4.2.5 and proposition 4.2.4, we have:

**Proposition 4.2.7.** (i) Given an object $(\mathcal{M}, \mathcal{M}^\circ)$ of $\text{HC}^r_{Z_p, \text{fin}}(\hat{X}/\Sigma_S)$, there exists a functorial isomorphism:

\[T(\mathcal{M}^\circ)_{\mathbb{Q}}(= V^r(\mathcal{M}, \mathcal{M}^\circ)) \simeq T_X(\iota_X^{-1}(\mathcal{M})).\]

(ii) The equivalence $T_X$ (resp. $H_{X_2}$) satisfies the inverse image functoriality with respect to the twisted inverse image functor for small Higgs bundles (2.3.1.2) and inverse image functor of Faltings topoi as in proposition 4.2.4.

**4.3. A local version of the $p$-adic Simpson correspondence.** In this subsection, we assume that $\text{id}_X$ belongs $\mathcal{Q}$ and that $X_\mathbb{F}$ is connected. We fix a geometric generic point $\mathbb{F}$ of $X_{\mathbb{F}}$ as in 4.1.7. We review a local description of the $p$-adic Simpson correspondence on $X$ following ([18], [4] II.13, II.14).

4.3.1. Note that $X \times_S \mathbb{F}$ is affine, and is denoted by $\text{Spec}(R_1)$ (§4.2.5). We denote by $\text{HB}_{Z_p, \text{small}}^\text{free}(\hat{R}_1)$ the category of pairs $(M, \theta)$ consisting of a free $\hat{R}_1$-module $M$ of finite rank together with a small Higgs field $\theta : M \to \xi^{-1}M \otimes R_1 \Omega^1_{1/R_1\mathbb{C}_K}$ (2.2.10). We note $\Delta_X$ simply by $\Delta$ and refer to ([4] II.6.10) for the definition of $\Delta_\infty$. There exist equivalences of categories ([18] lemma 1, theorem 3; [4] II.13.11, II.14.4):

\[(4.3.1.1) \quad \text{HB}_{Z_p, \text{small}}(\hat{R}_1) \xrightarrow{\sim} \text{Rep}_{\text{small}}(\Delta_\infty, \hat{R}_1) \xrightarrow{\sim} \text{Rep}_{\text{small}}(\Delta, \hat{R}),\]

where the first equivalence preserves the underlying $\hat{R}_1$-modules and the second one is defined by $M \mapsto M \otimes_{R_1} \hat{R}$. A Higgs bundle $(M, \theta)$ of $\text{HB}_{Z_p, \text{small}}^\text{free}(\hat{R}_1)$ and the corresponding small $\hat{R}$-representation of $\Delta$ are associated in the sense of 4.2.6 ([4] II.13.16).

If $\text{HB}_{Z_p, \text{small}}(\hat{X}/\Sigma_1,S)$ denotes the full subcategory of $\text{HB}_{Z_p, \text{conv}}(\hat{X}/\Sigma_1,S) (2.2.10)$ consisting of objects whose underlying $\theta\hat{X}$-module is free and $\theta$ is small, then there exists a canonical functor $\text{HB}_{Z_p, \text{small}}(\hat{X}/\Sigma_1,S) \to \text{HB}_{Z_p, \text{small}}(\hat{R}_1)$ defined by restriction to $\hat{U}_{\mathbb{F}}$. We denote the composition of above functors by

\[(4.3.1.2) \quad V : \text{HB}_{Z_p, \text{small}}(\hat{X}/\Sigma_1,S) \to \text{Rep}_{\text{small}}(\Delta, \hat{R}).\]

**Proposition 4.3.2.** Let $X$ be a smooth Cartesian lifting of $\hat{X}$ over $\Sigma_S$. Then, the following diagram is commutative up to isomorphisms

\[
\begin{array}{ccc}
\text{HB}_{Z_p, \text{small}}(\hat{X}/\Sigma_1,S) & \xrightarrow{V} & \text{Rep}_{\text{LPM}}(\Delta, \hat{R}) \\
\iota_X & \downarrow & \text{HC}_{Z_p, \text{fin}}(\hat{X}/\Sigma_S) \\
\end{array}
\]

where $\iota_X$ is defined in proposition 2.2.7(ii).

**Proof.** Let $(M, \theta)$ be an object of $\text{HB}_{Z_p, \text{small}}(\hat{X}/\Sigma_1,S)$ and $\mathcal{M}$ the associated Higgs crystal on $(\hat{X}/\Sigma_S)^\infty$. We denote abusively by $(M, \theta)$ the restriction of $(M, \theta)$ to $\text{HB}_{Z_p, \text{small}}(\hat{R}_1)$. By proposition 4.2.7 and ([4] II.13.16), we have the following $\Delta$-equivariant $\hat{\mathcal{F}}_{X_2, (X, \mathbb{F})}$-linear isomorphism

\[\hat{\mathcal{F}}_{X_2, (X, \mathbb{F})} \otimes_R V(M, \theta) \sim \hat{\mathcal{F}}_{X_2, (X, \mathbb{F})} \otimes_{\hat{R}_1} (M, \theta),\]

compatible with Higgs fields. We compare it with the isomorphism (4.2.5.1) via (4.2.5.2). Then, the assertion follows from taking Higgs field invariants and the fact that $(\hat{\mathcal{F}}_{X_2, (X, \mathbb{F})})^{\theta=0} = \hat{R}$ ([4] IV.5.2.10).
5. FROM $\mathbf{C}$-REPRESENTATIONS OF THE GEOMETRIC FUNDAMENTAL GROUP TO HIGGS BUNDLES

In this section, we assume moreover that $k$ is an algebraic closure of $\mathbb{F}_p$. Let $X$ be a semi-stable $S$-scheme and geometrically connected generic fiber $X_\eta$. Let $\breve{X}$ be the $p$-adic completion of $X_S = X \otimes_{\mathbb{O}_k} \mathbb{O}$, equipped with the log structure induced by $\mathcal{M}_X$ on $X$, and $\mathcal{X}$ a smooth Cartesian lifting of $\breve{X}$ over $\Sigma_S$. Let $\mathfrak{X}$ be a geometric generic point of $X_\eta$ and $\pi_1(X_\eta, \mathfrak{X})$ the étale fundamental group.

The $p$-adic logarithmic homomorphism $\log : 1 + m \to \mathbb{C}$, $x \mapsto \sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ admits a section $\exp : y \mapsto \sum_{n=0}^{\infty} \frac{y^n}{n!}$ on the open ball of radius $p^{-\frac{1}{p-1}}$ of $\mathbb{C}$. We fix a section $\operatorname{Exp} : \mathbb{C} \to 1 + m$ of log, extending $\exp \ (\text{A.0.0.2})$.

In this section, we globalise the construction in §4.3 and then apply the descent for the $p$-adic Simpson correspondence over curves to construct a functor (depending on $\mathcal{X}$ and $\operatorname{Exp}$):

$$\mathbb{H}_X, \operatorname{Exp} : \operatorname{Rep}(\pi_1(X_\eta, \mathfrak{X}), \mathfrak{c}) \to \operatorname{HB}(X_\mathfrak{c}).$$

We will provide a description of the essential image of $\mathbb{H}_X, \operatorname{Exp}$ in §6.

5.1. Higgs bundles associated to small $\mathfrak{c}$-representations.

**Proposition 5.1.1.** (i) There exists a functor defined by $\mathcal{X}$ (2.2.6, 4.1.8)

$$\mathbb{H}_\mathfrak{X} : \operatorname{Rep}_{\pi_1}(\pi_1(X_\eta, \mathfrak{X}), \mathfrak{c}) \to \operatorname{HB}_{\mathbb{Z}_p, \text{conv}}(\breve{X}/\Sigma_1, S).$$

(ii) If an $\mathfrak{c}$-representation $V$ of $\pi_1(X_\eta, \mathfrak{X})$ is $\alpha$-small for some $\alpha > \frac{2}{p-1}$ (4.1.8) and $\beta = \alpha - \frac{1}{p-1}$, then $\mathbb{H}_\mathfrak{X}(V)$ is $\beta$-small and is isomorphic to a trivial Higgs bundle modulo $p^{\beta - \frac{1}{p-1}}$.

**Proof.** (i) Let $V$ be an $\alpha$-small $\mathfrak{c}$-representation of $\pi_1(X_\eta, \mathfrak{X})$ for $\alpha > \frac{2}{p-1}$. We take $s \in \mathbb{Q}_{>0}$ such that $\beta > s + \frac{1}{p-1}$. Let $U$ be an object of $\mathbb{Q}$ and $\overline{y}$ a geometric generic point of $U_{\mathfrak{c}}$ as in 4.1.7.

By applying equivalences (4.3.1.1) to the $\alpha$-small representation $V \otimes_{\mathfrak{c}} \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}$ of $\Delta(U_{\mathfrak{c}})$, there exists a Higgs bundle $M(U_{\mathfrak{c}}) = (V \otimes_{\mathfrak{c}} \mathbb{R}_1, \theta(U_{\mathfrak{c}}))$ of $\operatorname{HB}_{\mathbb{Z}_p, \text{small}}(\mathbb{R}_1)$ and a canonical $\Delta(U_{\mathfrak{c}})$-equivariant $\mathcal{E}_{\mathcal{X}_2(U_{\mathfrak{c}})}^{\mathfrak{c}}$-linear isomorphism compatible with Higgs fields ([4] II.13.16):

$$(5.1.1.1) \quad M(U_{\mathfrak{c}}) \otimes_{\mathbb{R}_1} \mathcal{E}_{\mathcal{X}_2(U_{\mathfrak{c}})}^{\mathfrak{c}} \to V \otimes_{\mathfrak{c}} \mathcal{E}_{\mathcal{X}_2(U_{\mathfrak{c}})}^{\mathfrak{c}},$$

where $M(U_{\mathfrak{c}})$ is equipped with the trivial $\Delta(U_{\mathfrak{c}})$-action, $V$ is equipped with the trivial Higgs field, and $\mathcal{E}_{\mathcal{X}_2(U_{\mathfrak{c}})}^{\mathfrak{c}}$ is equipped with the canonical action of $\Delta$ and the Higgs field $p^s d \phi_{\mathfrak{c}}$ ([4] II.12.1). Moreover, $\theta(U_{\mathfrak{c}})$ is $\beta$-small and the above isomorphism is congruent to the identity map modulo $p^{\beta - \frac{1}{p-1}}$ in view of the proof of ([4] II.13.16).

If $g : U' \to U$ is a morphism of $\mathbb{Q}$ and $\overline{y}$ a geometric generic point of $U'_{\mathfrak{c}}$ above $\overline{y}$, then $g$ induces a $\Delta(U'_{\mathfrak{c}})$-equivariant homomorphism $\theta_{\mathcal{X}_2(U_{\mathfrak{c}})}^{\mathfrak{c}} \to \theta_{\mathcal{X}_2(U_{\mathfrak{c}})}^{\mathfrak{c}}$ via $\Delta(U'_{\mathfrak{c}}) \to \Delta(U_{\mathfrak{c}})$. We deduce the following $\Delta(U'_{\mathfrak{c}})$-equivariant isomorphism from (5.1.1.1) compatible with Higgs fields:

$$(5.1.1.2) \quad M(U_{\mathfrak{c}}) \otimes_{\mathbb{R}_1} \mathcal{E}_{\mathcal{X}_2(U'_{\mathfrak{c}})}^{\mathfrak{c}} \cong M(U'_{\mathfrak{c}}) \otimes_{\mathbb{R}_1} \mathcal{E}_{\mathcal{X}_2(U'_{\mathfrak{c}})}^{\mathfrak{c}}.$$
Corollary 5.1.2. (i) The functor \( \mathcal{H}_X \) factors through the full subcategory \( \mathcal{HB}_{Z_p,\text{small}}^D(\tilde{X}/\Sigma_1,1) \) (3.2.1).

(ii) The composition \((2.2.7(ii), \ (4.2.4.4))\)

\[
T \circ \iota_X \circ \mathcal{H}_X : \text{Rep}_{\text{small}}(\pi_1(X,\mathcal{T},1),\sigma) \rightarrow \mathcal{HB}_{Z_p,\text{conv}}(\tilde{X}/\Sigma_1,1) \sim \mathcal{HC}_{Z_p,\text{fin}}(\tilde{X}/\Sigma) \rightarrow \text{Mod}(\mathcal{F}_X)
\]

is canonically isomorphic to the functor \( V \mapsto \tilde{\beta}_{X}^{\ast}(V) \) (4.1.5.1), (4.1.9.1).

Proof. Assertion (i) follows from proposition 5.1.1(ii).

(ii) It suffices to show that for each pair \((U,\mathcal{F})\), the composition \( T \mid_{U,\mathcal{F}} \circ \iota_X \circ \mathcal{H}_X \) is isomorphic to the functor \( V \mapsto V \otimes_\sigma \tilde{\mathcal{R}} \). We may assume that \( U = X \) belongs to \( \mathcal{Q} \). In this case, the composition \( V \circ \mathcal{H}_X \) (4.3.1.2) is isomorphic to the functor \( V \mapsto V \otimes_\sigma \tilde{\mathcal{R}} \). Then, the assertion follows from proposition 4.3.2.

5.1.3. We denote by \( \mathcal{HB}_{Z_p,\text{small}}^D(\tilde{X}/\Sigma_1,1) \) the essential image of \( \mathcal{HB}_{Z_p,\text{small}}^D(\tilde{X}/\Sigma_1,1) \) in \( \mathcal{HB}_{\mathcal{Q}}(\tilde{X}/\Sigma_1,1) \). By passing to categories up to isogeny, the functor \( \mathcal{H}_X \) extends to the following functor, still denoted by \( \mathcal{H}_X \)

\[(5.1.3.1) \quad \mathcal{H}_X : \text{Rep}_{\text{small}}(\pi_1(X,\mathcal{T},1),\mathcal{C}) \rightarrow \mathcal{HB}_{Z_p,\text{small}}^D(\tilde{X}/\Sigma_1,1) \]

We also consider \( \mathcal{HB}_{\mathcal{Q},\text{small}}^D(\tilde{X}/\Sigma_1,1) \) as a full subcategory of \( \mathcal{HB}(X_{\mathcal{C}}) \) via equivalences (2.2.10.1).

Proposition 5.1.4. (i) The functor \( \tilde{\beta}_{X,Q}^{\ast} \) (4.1.5.1) sends small \( \mathcal{C} \)-representations to Dolbeault modules (4.2.6).

(ii) The functor \( \mathcal{H}_X \) (5.1.3.1) is canonically isomorphic to the composition of functors:

\[
\text{Rep}_{\text{small}}(\pi_1(X,\mathcal{T},1),\mathcal{C}) \xrightarrow{\tilde{\beta}_{X,Q}^{\ast}} \text{Mod}^{\text{Dolb}}(\mathcal{F}_X) \xrightarrow{\mathcal{H}_X} \mathcal{HB}_{\mathcal{Q},\text{small}}(\tilde{X}/\Sigma_1,1).
\]

Proof. Let \( V \) be a small \( \mathcal{C} \)-representation of \( \pi_1(X,\mathcal{T},1) \) and \( V^\circ \) a small \( \sigma \)-representation lattice of \( V \). By corollary 5.1.2(ii), there exists an integer \( r \geq 1 \) and an object \((\mathcal{M},\mathcal{M}^\circ)\) of \( \mathcal{HC}_{Z_p,\text{fin}}^D(\tilde{X}/\Sigma) \) (4.2.2) such that \( \mathcal{M}^\circ \simeq \iota_X(\mathcal{H}_X(V^\circ)) \). Then, the proposition follows from the equivalence (4.2.6.1) and proposition 4.2.7.

Corollary 5.1.5. Let \( K' \) be a finite extension of \( K \), \( Y \) a semi-stable \( S' = \text{Spec}(\mathcal{O}_{K'}) \)-curve, \( f : Y \rightarrow X_{S'} \) a generic \( \eta' \)-cover, \( \mathcal{Y} \) a geometric generic point of \( Y_{\mathcal{T}} \) above \( \mathcal{T} \) and \( Y \) a smooth Cartesian lifting of \( Y \) over \( \Sigma_{S'} \).

(i) The twisted inverse image functor \( \tilde{f}_{Y,X}^{\ast} \) (2.3.4.2) sends \( \mathcal{HB}_{Z_p,\text{small}}^D(\tilde{X}/\Sigma_1,1) \) (resp. \( \mathcal{HB}_{\mathcal{Q},\text{small}}^D(\tilde{X}/\Sigma_1,1) \)) to \( \mathcal{HB}_{\mathcal{Q},\text{small}}^D(\tilde{Y}/\Sigma_{1,S'}) \) (resp. \( \mathcal{HB}_{\mathcal{Q},\text{small}}^D(\tilde{Y}/\Sigma_{1,S'}) \)).

(ii) Via \( \pi_1(Y_{\mathcal{T}},\mathcal{Y}) \approx \pi_1(X_{\mathcal{T}},\mathcal{T}) \), the following diagram is commutative up to canonical isomorphisms \( \gamma_f \):

\[
\begin{array}{ccc}
\text{Rep}_{\text{small}}(\pi_1(X,\mathcal{T},1),\mathcal{C}) & \xrightarrow{\mathcal{H}_X} & \mathcal{HB}_{\mathcal{Q},\text{small}}^D(\tilde{X}/\Sigma_1,1) \\
\downarrow \gamma_f & & \downarrow \tilde{f}_{Y,X}^{\ast} \\
\text{Rep}_{\text{small}}(\pi_1(Y,\mathcal{T},1),\mathcal{C}) & \xrightarrow{\mathcal{H}_Y} & \mathcal{HB}_{\mathcal{Q},\text{small}}^D(\tilde{Y}/\Sigma_{1,S'})
\end{array}
\]

Moreover, \( \gamma_f \) satisfy a cocycle condition as in (4.2.4.1).

Proof. (i) The assertion follows from theorem 3.2.2.

(ii) The functor \( \tilde{\beta}_{X,Q}^{\ast} : \text{Rep}_{\text{small}}(\pi_1(X,\mathcal{T},1),\mathcal{C}) \rightarrow \text{Mod}(\mathcal{F}_X) \) is compatible with the inverse image functoriality, defined by morphisms of topoi. Then, the assertion follows from proposition 5.1.4 and the inverse image functoriality of \( \mathcal{H}_X \) (propositions 4.2.4, 4.2.7).
5.2. **Construction of $\mathbb{H}_{X,Y,\text{Exp}}$ via descent.** In the following of this section, we assume moreover that $X$ is a *stable $S$-curve*. To extend the construction of $\mathbb{H}_X$ to all $\mathbb{C}$-representations, we need the twisted inverse image functor for Higgs bundles, that we summarize in the following.

**5.2.1.** Let $\pi : Y_\mathfrak{F} \to X_\mathfrak{F}$ be a finite morphism of smooth proper $\mathfrak{F}$-curves. There exists a finite extension $K'$ of $K$ such that $Y_\mathfrak{F}$ admits a stable $S'$-model and that $\pi$ extends to a (unique) $S'$-morphism $f : Y \to X_{S'}$. We fix a smooth Cartesian lifting $\mathcal{Y}$ of $\tilde{Y}$ over $\Sigma_{1,S'}$.

In appendix, we construct the twisted inverse image functor (A.4.3.1) after Faltings:

$$f_{\pi_{\mathfrak{F},X,\text{Exp}}}^* : \text{HB}(X_\mathfrak{C}) \to \text{HB}(Y_\mathfrak{C}), \quad (M, \theta) \mapsto f_\mathfrak{C}^*(M, \theta) \otimes_{\mathcal{A}_c} \mathcal{L}_{f,\mathfrak{b}}^\text{Exp},$$

where $b$ is the Hitchin image of $(M, \theta)$, $\mathcal{A}_c$ is the spectral algebra over $\mathcal{O}_{Y_\mathfrak{C}}$ defined by $c = f_\mathfrak{C}^*(b)$ and $\mathcal{L}_{f,\mathfrak{b}}^\text{Exp}$ is an invertible $\mathcal{A}_c$-module defined by the obstruction of lifting $\tilde{f} : \tilde{Y} \to \tilde{X}$ to $\Sigma_{2,S}$ and Exp (§ A.2.3).

While restricting to small Higgs bundles, $f_{\pi_{\mathfrak{F},X,\text{Exp}}}^*$ is canonically isomorphic to the functor $f_{\pi_{\mathfrak{F},X}}^*$ (2.3.4.2), and is independent of the choice of Exp (see propositions A.3.5, A.4.4). In particular, it is compatible with the inverse image functoriality of the $p$-adic Simpson correspondence (proposition 4.2.4).

5.2.2. In the following, we construct a functor

$$\mathbb{H}_{X,Y,\text{Exp}} : \text{Rep}(\pi_1(X_\mathfrak{F}, \mathfrak{F}), \mathbb{C}) \to \text{HB}(X_\mathfrak{C}).$$

For each object $V$ of $\text{Rep}(\pi_1(X_\mathfrak{F}, \mathfrak{F}), \mathbb{C})$, there exists a Galois étale cover $\pi : Y_\mathfrak{F} \to X_\mathfrak{F}$ with Galois group $G$ and a geometric generic point $\mathfrak{F}$ above $\mathfrak{F}$ such that the restriction of $V$ to $\pi_1(Y_\mathfrak{F}, \mathfrak{F})$ is small. Let $K'$ be a finite extension of $K$ and $f : Y \to X_{S'}$ the extension of $\pi$ to the stable $S' = \text{Spec}(\mathcal{O}_{K'})$-models as in 5.2.1. We take a smooth Cartesian lifting $\mathcal{Y}$ of $\tilde{Y}$ over $\Sigma_{1,S'}$ as above.

Since $V$ is a small representation of $\pi_1(Y_\mathfrak{F}, \mathfrak{F})$, we obtain a small Higgs bundle $\mathbb{H}_Y(V)$ over $Y_\mathfrak{C}$ (5.1.3.1). The automorphism group $\text{Aut}(\mathcal{Y}/X_\mathfrak{F})$, denoted by $G$, extends to an action on $Y$ above $X_{S'}$ ([37] proposition 4.6). By corollary 5.1.5, the $G$-action on $V|_{\pi_1(Y_\mathfrak{F}, \mathfrak{F})}$ induces a $G$-action on $\mathbb{H}_Y(V)$:

$$\varphi \cdot g = f_{\varphi_{\mathfrak{F},Y,\text{Exp}}}^*(\mathbb{H}_Y(V)) \simeq \mathbb{H}_Y(V) \mid g \in G,$$

such that $\varphi \circ g_{\varphi} = \varphi_{\varphi \circ g}$. Let $c = (c_i) \in \otimes_{i=1}^{\Gamma} (\mathcal{O}_{X_\mathfrak{C}} \backslash \mathcal{O}_{Y_\mathfrak{C}})$ be the Hitchin image of $\mathbb{H}_Y(V)$. We have $g^*(c) = c$ for all $g \in G$. By descent, we obtain a point $b = (b_i)_{i=1}^{\Gamma}$ of the Hitchin base $\otimes_{i=1}^{\Gamma} (\mathcal{O}_{X_\mathfrak{C}} \backslash \mathcal{O}_{Y_\mathfrak{C}})$ of $X_\mathfrak{C}$ such that $f_\mathfrak{C}^*(b) = c$.

Since $f \circ g = f$, we have a canonical isomorphism (A.2.4.1)

$$\mathcal{L}_{f,\mathfrak{b}}^\text{Exp} = \mathcal{L}_{f_{g,b}}^\text{Exp} \simeq \mathcal{L}_{g,c}^\text{Exp} \otimes g_c^\text{Exp}(\mathcal{L}_{f,\mathfrak{b}}^\text{Exp}^{-1}).$$

Then, the data (5.2.2.2) defines a usual descent data on the Higgs bundle $\mathbb{H}_Y(V) \otimes_{\mathcal{A}_c} (\mathcal{L}_{f,\mathfrak{b}}^\text{Exp})^{-1}$:

$$\phi : g_c^\text{Exp}(\mathbb{H}_Y(V) \otimes_{\mathcal{A}_c} (\mathcal{L}_{f,\mathfrak{b}}^\text{Exp})^{-1}) \simeq \mathbb{H}_Y(V) \otimes_{\mathcal{A}_c} (\mathcal{L}_{f,\mathfrak{b}}^\text{Exp})^{-1} \mid g \in G,$$

such that $\phi \circ g_c^\text{Exp}(\phi_g) = \phi_{g \circ g}$. By étale descent, we obtain a Higgs bundle $\mathbb{H}_{X,Y,\text{Exp}}(V)$ on $X_\mathfrak{C}$ together with a canonical isomorphism

$$f_{\pi_{\mathfrak{F},X,\text{Exp}}}^*(\mathbb{H}_{X,Y,\text{Exp}}(V)) \simeq \mathbb{H}_Y(V),$$

which gives rise to the descent data $\{\varphi_g\}_{g \in G}$.

**Proposition 5.2.3.** (i) The construction $V \mapsto \mathbb{H}_{X,Y,\text{Exp}}(V)$ is independent of the choice of the $\pi$, $K'$ and $\mathcal{Y}$ up to canonical isomorphisms.

(ii) The functor $\mathbb{H}_{X,Y,\text{Exp}}$ is well-defined and is exact.

**Proof.** (i) The independence of $\mathcal{Y}$ follows from proposition A.2.5(iv). Let $K''$ be a finite extension of $K$, $Z$ a stable $S''$-curve, $g : Z \to X_{S''}$ a Galois $\eta$-cover and $\mathfrak{F}$ a geometric generic point of $Z_\mathfrak{F}$ above $\mathfrak{F}$ such that $V$ is small as a $\mathbb{C}$-representation of $\pi_1(Z_\mathfrak{F}, \mathfrak{F})$. To prove the assertion, we may assume that $g$ dominates $f$. Then, the independence of $\pi$, $K'$ follows from corollary 5.1.5.
(ii) Let \( u : V \to V' \) be a morphism of \( \text{Rep}(\pi_1(X_\overline{\mathbb{F}}, \overline{\mathbb{F}}), \mathbb{C}) \). By (i), we may choose a Galois \( \eta' \)-cover \( f : Y \to X_{S'} \) as in §5.2.2 such that \( V \) and \( V' \) are both small as \( \mathbb{C} \)-representation of \( \pi_1(Y_{\overline{\mathbb{F}}}, \overline{\mathbb{F}}) \). Then, the functoriality of \( \mathbb{H}_{X, \text{Exp}} \) follows from that of \( \mathbb{H}_Y \) and of \( f^*_{Y,X, \text{Exp}} \).

The exactness follows from the fact that functors \( \mathbb{H}_X \) and \( f^*_{Y,X, \text{Exp}} \) are exact (proposition A.2.5). \( \square \)

To describe the essential image of \( \mathbb{H}_{X, \text{Exp}} \), we introduce the following category:

**Definition 5.2.4.** We denote by \( \text{HB}^\text{pDW}_{X, \text{Exp}}(X_C) \) the full subcategory of \( \text{HB}(X_C) \) consisting of Higgs bundles \((M, \theta)\) such that there exists a finite morphism \( \pi : Y_{\overline{\mathbb{F}}} \to X_{\overline{\mathbb{F}}} \) of smooth proper \( \overline{\mathbb{F}} \)-curves, a finite extension \( K' \) of \( K \) such that \( Y_{\overline{\mathbb{F}}} \) admits a stable \( S' = \text{Spec}(O_{K'}) \)-model \( Y \), \( \pi \) extends to the \( S' \)-morphism \( f : Y \to X_{S'} \) of stable \( S' \)-models and that for every smooth Cartesian lifting \( \mathcal{Y} \) of \( \pi \) over \( \Sigma_{S'} \), \( f^*_{\mathcal{Y}, X, \text{Exp}}(M, \theta) \) belongs to the essential image of \( \text{HB}^\text{pDW}_{Q_p, \text{small}}(\mathcal{Y}/\Sigma_{S'}) \) (5.1.3) in \( \text{HB}(Y_C) \).

**Remark 5.2.5.** Compared to the construction of 5.2.2, we take \( \pi \) to be a finite morphism in the above definition to allow more flexibility. In corollary 6.3.8, we will show that any object of \( \text{HB}^\text{pDW}_{X, \text{Exp}}(X_C) \) becomes a small Higgs bundle with strongly semi-stable reduction after (twisted) pullback along a finite étale morphism \( \pi \). That is, we can replace “finite morphism” by “finite étale morphism” in the above definition.

**Corollary 5.2.6.** (i) The functor \( \mathbb{H}_{X, \text{Exp}} \) factors through the full subcategory \( \text{HB}^\text{pDW}_{X, \text{Exp}}(X_C) \):

\[
\mathbb{H}_{X, \text{Exp}} : \text{Rep}(\pi_1(X_{\overline{\mathbb{F}}}, \overline{\mathbb{F}}), \mathbb{C}) \to \text{HB}^\text{pDW}_{X, \text{Exp}}(X_C).
\]

(ii) With the assumption of 5.1.5, the functor \( f^*_{\mathcal{Y}, X, \text{Exp}} \) sends \( \text{HB}^\text{pDW}_{X, \text{Exp}}(X_C) \) to \( \text{HB}^\text{pDW}_{\mathcal{Y}, \text{Exp}}(Y_C) \).

(iii) The following diagram is commutative up to isomorphisms \( \gamma_f \):

\[
\begin{array}{ccc}
\text{Rep}(\pi_1(X_{\overline{\mathbb{F}}}, \overline{\mathbb{F}}), \mathbb{C}) & \xrightarrow{\mathbb{H}_{X, \text{Exp}}} & \text{HB}^\text{pDW}_{X, \text{Exp}}(X_C) \\
\downarrow{\gamma_f} & & \downarrow{f^*_{\mathcal{Y}, X, \text{Exp}}} \\
\text{Rep}(\pi_1(Y_{\overline{\mathbb{F}}}, \overline{\mathbb{F}}), \mathbb{C}) & \xrightarrow{\mathbb{H}_{\mathcal{Y}, \text{Exp}}} & \text{HB}^\text{pDW}_{\mathcal{Y}, \text{Exp}}(Y_C)
\end{array}
\]

Moreover, \( \gamma_f \) satisfy a cocycle condition as in (4.2.4.1).

**Proof.** Assertion (i) follows from the construction of \( \mathbb{H}_{X, \text{Exp}} \) and corollary 5.1.2(i). Assertion (ii) follows from proposition A.2.5, [13] theorem 16, and [55] corollaire 5.8. Assertion (iii) follows from corollary 5.1.5. \( \square \)

5.3. Some properties of \( \text{HB}^\text{pDW}_{X, \text{Exp}}(X_C) \).

**Proposition 5.3.1.** (i) Every Higgs bundle of \( \text{HB}^\text{pDW}_{X, \text{Exp}}(X_C) \) is semi-stable of degree zero.

(ii) The category \( \text{HB}^\text{pDW}_{X, \text{Exp}}(X_C) \) is closed under extension.

**Proof.** (i) Since Deninger–Werner vector bundles are semi-stable of degree zero ([13] theorem 13), we deduce that every object of \( \text{HB}^\text{pDW}_{Q_p, \text{small}}(\mathcal{X}/\Sigma_1, S) \) is a semi-stable Higgs bundle of degree zero over \( X_C \). Given a Higgs bundle \((M, \theta)\) of \( \text{HB}^\text{pDW}_{X, \text{Exp}}(X_C) \), its twisted inverse image along a generic \( \eta \)-cover is semi-stable of degree zero. By proposition A.2.5(ii), we conclude that \((M, \theta)\) is also semi-stable of degree zero.

(ii) The same assertion holds for Deninger–Werner vector bundles ([13] proposition 9, theorem 11). We deduce the corresponding result for \( \text{HB}^\text{pDW}_{Q_p, \text{small}}(\mathcal{X}/\Sigma_1, S) \). Then, the assertion follows from the fact that \( f^* \) is exact (propositions A.2.5(ii) and A.4.4). \( \square \)

**Proposition 5.3.2.** Every Higgs line bundle \((L, \theta)\) of degree zero over \( X_C \) belongs to \( \text{HB}^\text{pDW}_{X, \text{Exp}}(X_C) \).
Proof. By propositions 3.2.3 and A.2.5, we may assume \( \theta \) is small and \( L \) has degree zero after applying a twisted inverse image functor. Recall that a line bundle of \( \text{Pic}^0_{X/S}(\mathfrak{o}) \) is Deninger–Werner over \( \breve{X} \) ([13] claim of theorem 12). Since the cokernel of the inclusion

\[
\text{Pic}^0_{X/S}(\mathfrak{o}) \to \text{Pic}^0_{X_n/K}(\mathcal{C})
\]

is torsion ([12] theorem 4.1), there exists an integer \( N \geq 1 \) such that \( L^{\otimes N} \in \text{Pic}^0_{X/S}(\mathfrak{o}) \). By applying arguments of proposition 3.2.3 and of ([13] theorem 12), there exists a finite extension \( K' \) of \( K \), a stable \( S' = \text{Spec}(\mathcal{O}_{K'}) \)-curve \( Y \) and an \( \mathfrak{n}' \)-cover \( f : Y \to X_{S'} \) such that the usual inverse image \( f_*^\mathfrak{n}'(L) \) belongs to \( \text{Pic}^0_{Y/S'}(\mathfrak{o}) \). We take a smooth Cartesian lifting \( Y \) of \( \breve{Y} \) over \( \Sigma_S \) and show that \( f^\mathfrak{n}_*,\breve{X}(L, \theta) \in \text{HB}^{\mathfrak{D}W}(\breve{Y}/\Sigma_{1,S}) \).

By proposition A.3.3, the line bundle \( \mathcal{L}_{f,\mathfrak{n}} \) admits an integral model with trivial special fiber and belongs to the image of \( \text{Pic}^0_{Y/S'}(\mathfrak{o}) \). Then, so is the underlying bundle \( f^\mathfrak{n}_*(L) \otimes \mathcal{O}_Y \) \( \mathcal{L}_{f,\mathfrak{n}} \) of \( \text{HB}^{\mathfrak{D}W}(\breve{Y}/\Sigma_{1,S}) \) (proposition A.2.7). The proposition follows.

6. Parallel transport for Higgs bundles

In this section, \( k \) denotes an algebraic closure of \( \mathbb{F}_p \). We will construct a quasi-inverse functor \( \mathbb{V}_{X,\text{Exp}} \) of \( \mathbb{H}_{X,\text{Exp}} \) (5.2.6.1) (see theorem 6.3.1).

6.1. Parallel transport via Faltings topos. In [55] § 8, we construct the parallel parallel transport functor for certain modules in Faltings topos inspired by Deninger–Werner’s construction [13]. In this subsection, we present another approach to this functor via h-descent for étale sheaves and we apply this functor to modules associated to Higgs bundles of \( \text{HB}^{\mathfrak{D}W}_{\text{et},\text{small}}(\breve{X}/\Sigma_1,S) \) (§ 3.2.1).

In this subsection, let \( X \) be an \( \mathcal{S} \)-model of a smooth proper \( \mathbb{K} \)-variety (§1.2.3), \( (\breve{E}_X, \mathcal{F}_X) \) the Faltings ringed topos associated to the pair \( (X_{\overline{T}} \to X) \). If \( Y \) is another \( \mathcal{S} \)-model and \( \varphi : Y \to X \) an \( \mathcal{S} \)-morphism, we denote the functorial morphism of Faltings ringed topos associated to \( \varphi \) by:

\[
(\varphi)_* : (\breve{E}_Y, \mathcal{F}_Y) \to (\breve{E}_X, \mathcal{F}_X).
\]

Definition 6.1.1 ([55] définition 8.5). (i) We say a \( \mathcal{F}_X,n \)-module \( M_n \) is potentially free of finite type if it is of finite type and there exists a proper \( \mathcal{S} \)-morphism \( \varphi : Y \to X \) such that \( \varphi_{\overline{T}} \) is finite étale and that the inverse image \( \varphi^*(M_n) \) is isomorphic to a free \( \mathcal{F}_{Y,n} \)-module of finite type.

(ii) We say a \( \mathcal{F}_X \)-module \( M = (M_n)_{n \geq 1} \) is potentially free of finite type if it is adic of finite type (1.2.5) and for each \( n \geq 1 \), the \( \mathcal{F}_{X,n} \)-module \( M_n \) is potentially free of finite type.

We denote by \( \text{Mod}^{\text{pltf}}(\mathcal{F}_{X,n}) \) (resp. \( \text{Mod}^{\text{pltf}}(\mathcal{F}_X) \)) the full subcategory of \( \text{Mod}(\mathcal{F}_{X,n}) \) (resp. \( \text{Mod}(\mathcal{F}_X) \)) consisting of potentially free of finite type objects, and by \( \text{Mod}_{\text{coh}}^{\text{pltf}}(\mathcal{F}_{X,n}) \) (resp. \( \text{Mod}_{\text{coh}}^{\text{pltf}}(\mathcal{F}_X) \)) the full subcategory of \( \text{Mod}^{\text{pltf}}(\mathcal{F}_{X,n}) \) (resp. \( \text{Mod}^{\text{pltf}}(\mathcal{F}_X) \)) consisting of \( \mathcal{F}_{X,n} \)-modules satisfying cohomological descent (§ 4.1.3) (resp. \( \mathcal{F}_X \)-modules \( M = (M_n)_{n \geq 1} \) such that each \( M_n \) belongs to \( \text{Mod}_{\text{coh}}^{\text{pltf}}(\mathcal{F}_{X,n}) \)).

The inverse image of morphisms of ringed topos \( \beta_n : (\breve{E}_X, \mathcal{F}_{X,n}) \to (X_{\overline{T},\text{fét}}, \mathfrak{o}_n) \), \( \beta : (\mathcal{F}_{X,n}^{\text{pltf}}, \mathcal{F}_X) \to (X_{\overline{T},\text{fét}}, \mathcal{F}_X) \) induce functors (4.1.5, 4.1.9):

\[
\beta^* : \text{LocSys}(X_{\overline{T},\text{fét}}, \mathfrak{o}_n) \to \text{Mod}^{\text{pltf}}(\mathcal{F}_{X,n}), \quad \beta^* : \text{LocSys}(X_{\overline{T},\text{fét}}, \mathfrak{o}_n) \to \text{Mod}_{\text{coh}}^{\text{pltf}}(\mathcal{F}_X).
\]

Indeed, if \( L \) is an object of \( \text{LocSys}(X_{\overline{T},\text{fét}}, \mathfrak{o}_n) \) and \( Z \) is a finite étale cover of \( X_{\overline{T}} \) trivializing \( \mathfrak{o}_n \), then the integral closure \( X^Z \to X \) of \( X \) in \( Z \) is an \( \mathfrak{T} \)-cover. Then, the morphism of Faltings ringed topos, induced by \( X^Z \to X \), trivializes \( \beta^*(L) \) (c.f. [55] proposition 8.7 for more details). Moreover, \( \beta^*(L) \) satisfies the cohomological descent by corollary 4.1.4.

6.1.2. In the following, we construct in another direction, the parallel transport functor:

\[
L_n : \text{Mod}^{\text{pltf}}(\mathcal{F}_{X,n}) \to \text{Mod}(X_{\overline{T},\text{fét}}, \mathfrak{o}_n).
\]
Let \( \mathcal{N} \) be an object of \( \text{Mod}^{\text{pltf}}(\mathcal{X}_{n\eta}) \) and \( \varphi : Y \to X \) an \( \eta \)-cover trivializing \( \mathcal{N} \) as in 6.1.1. We consider the hypercovering \( \varepsilon_* : Y_\bullet \to X \), where for \( m \geq 0 \), \( Y_m \to X \) is defined by the \( (m+1) \)-th self product of \( Y \) over \( X \). Let \( \varepsilon_* \eta : (Y_\bullet, \pi)_{\text{et}} \to (X_\bullet, \pi)_{\text{et}} \) be the augmentation of simplicial étale topoi. In the following, we apply h-descent to \( \varepsilon_* \) to construct \( \mathbb{L}_n(\mathcal{N}) \).

For each integer \( m \geq 1 \), the morphism of Faltings topoi associated to \( \varepsilon_m : Y_m \to X \) trivializes \( \mathcal{N} \). We have a canonical decomposition of the cohomology \( \varepsilon_m^* (\mathcal{N}) \) in terms of the connected components of \( Y_m, \eta \):

\[
\Gamma(E_{Y_m}, \varepsilon_m^* (\mathcal{N})) = \bigoplus_{Z \in \text{Con}(Y_m, \eta)} \Gamma(E_{Z \to Y_m}, \varepsilon_m^* (\mathcal{N})),
\]

where the functor \( \varepsilon_* : \text{Mod}(\mathfrak{o}) \to \text{Mod}(\mathfrak{o}) \) is defined by \( M \mapsto \text{Hom}(\mathfrak{o}, M) \) (1.2.4.1).

By Faltings’ comparison theorem ([24] theorem 5.13), each component of (6.1.2.2) is isomorphic to \( \Gamma(E_{\mathfrak{o}(n)}, \varepsilon_m^* (\mathcal{N})) = \mathfrak{o}(n) \), where \( r \) is the rank of \( \mathcal{N} \). We denote abusively by \( \Gamma(E_{\mathfrak{o}(n)}, \varepsilon_m^* (\mathcal{N})) \) the \( \mathfrak{o}(n) \)-module of \( \eta \) defined by the constant \( \mathfrak{o}(n) \)-module \( \Gamma(E_{Z \to Y_m}, \varepsilon_m^* (\mathcal{N})) \) in each connected component \( Z \) of \( Y_m, \eta \).

For \( m' > m \), every morphism \( \phi : Y_{m'} \to Y_m \) in the hypercovering \( Y_\bullet \to X \) induces a canonical morphism:

\[
\Phi^* : \Gamma(E_{Y_m}, \varepsilon_m^* (\mathcal{N})) = \bigoplus_{Z \in \text{Con}(Y_m, \eta)} \Gamma(E_{Z \to Y_m}, \varepsilon_m^* (\mathcal{N})).
\]

In summary, we obtain a descent datum of étale sheaves \( \Gamma(E_{Y_m}, \varepsilon_m^* (\mathcal{N})) \) in \( \mathcal{Y} \) with respect to the hypercovering \( Y_\bullet \to X_\bullet \). By h-descent ([57] 0G0F), this descent datum gives rise to an étale \( \mathfrak{o}(n) \)-module on \( X_\bullet, \eta \), denoted by \( \mathbb{L}_n(\mathcal{N}) \).

**Proposition 6.1.3.** (i) The construction \( \mathcal{N} \mapsto \mathbb{L}_n(\mathcal{N}) \) defines a functor

\[
\mathbb{L}_n : \text{Mod}^{\text{pltf}}(\mathcal{X}_{n\eta}) \to \text{Mod}(X_{\text{et}}, \mathfrak{o}(n))
\]

(ii) The functor \( \mathbb{L}_n \) factors through the full subcategory \( \text{Mod}(X_{\text{et}}^{\text{pltf}}, \mathfrak{o}_n) \) (4.1.9.3).

**Proof.** (i) We first show that \( \mathcal{N} \mapsto \mathbb{L}_n(\mathcal{N}) \) is independent of the choice of the \( \eta \)-cover \( f : Y \to X \). Let \( Z \to X \) be another \( \eta \)-cover trivializing \( \mathcal{N} \). After replacing \( Z \) by \( Z \times_X Y \), we may assume that \( Z \to X \) factors through \( X \to Y \). Let \( \varepsilon'_*: Z_\bullet \to X_\bullet \) be the hypercovering defined by self products of \( Z \) over \( X \).

The inverse image of the descent data \( \Gamma(E_{Z_m}, \varepsilon_m^* (\mathcal{N})) \) defined by \( Y_\bullet \) via \( \pi_* : Z_\bullet \to Y_\bullet \) is isomorphic to the descent data \( \Gamma(E_{Z_m}, \varepsilon_m^* (\mathcal{N})) \) defined by \( \varepsilon'_* \). Then, the claim follows.

Given a morphism \( u : \mathcal{N} \to \mathcal{N}' \) of \( \text{Mod}^{\text{pltf}}(\mathcal{X}_{n\eta}) \), we may take an \( \eta \)-cover \( Y \to Z \) trivializing both \( \mathcal{N} \) and \( \mathcal{N}' \). Then, the functoriality of \( \mathbb{L}_n \) follows.

Assertion (ii) follows from the fact that \( \mathbb{L}_n(\mathcal{N}) \) is trivialized by an finite étale cover \( Y_\eta \to X_\eta \).

By a similar argument of ([55] proposition 8.16), we conclude the following proposition:

**Proposition 6.1.4.** Let \( \mathbb{L}_n \) be an object of \( \text{LocSys}(X_{\eta, \text{et}}, \mathfrak{o}_n) \). Then, there exists a canonical and functorial almost isomorphism of \( \mathfrak{o}_n \)-modules of \( X_{\eta, \text{et}} \):

\[
\mathbb{L}_n(\beta_\eta^*(\mathbb{L}_n)) \cong \mathbb{L}_n.
\]

**Proposition 6.1.5.** The functors \( \beta_\eta^* \) and \( \mathbb{L}_n \) induce equivalences of categories quasi-inverse to each other up to almost isomorphisms (1.2.1):

\[
(\beta_\eta^*)^\alpha \circ \alpha - \text{LocSys}(X_{\eta, \text{et}}, \mathfrak{o}_n) \cong \alpha - \text{Mod}^{\text{pltf}}(\mathcal{X}_{n\eta}) : (\mathbb{L}_n)^\alpha.
\]
Remark 6.1.6. In ([39], Theorem 0.1(i)), Mann–Werner obtained a similar result between local systems with integral models and certain trivitilizable modules over a proper adic space of finite type over \( \mathbb{C} \) via \( v \)-descent.

The above proposition follows from proposition 6.1.4 and the following lemma.

Lemma 6.1.7. Let \( \mathcal{N} \) be an object of \( \text{Mod}_{\text{coh}}^{\text{pltf}}(\mathcal{F}_{X,n}) \). Then, there exists a canonical isomorphism of \( (\mathcal{F}_{X,n})^\alpha \)-modules:

\[
(\beta^\alpha_n(\mathcal{L}_n(\mathcal{N}))^\alpha \cong \mathcal{N}^\alpha.
\]

**Proof.** Set \( \mathcal{L} = \mathcal{L}_n(\mathcal{N}) \). By the construction in § 6.1.2, we have an isomorphism of \( (\mathcal{F}_{Y,n}) \)-modules:

\[
\varepsilon^*(\mathcal{L} \otimes_{\mathcal{O}_n} \mathcal{F}_{X,n}) \cong \Gamma(\mathcal{E}_{Y,n}, \varepsilon^*(\mathcal{N})) \otimes_{\mathcal{O}_n} \mathcal{F}_{Y,n}.
\]

Since \( \varepsilon^*(\mathcal{N}) \) is isomorphic to a free \( (\mathcal{F}_{Y,n}) \)-module of finite type, the following canonical morphisms are almost isomorphisms by Faltings’ comparison theorem ([24] theorem 1.13):

\[
\Gamma(\mathcal{E}_{Y,n}, \varepsilon^*(\mathcal{N})) \otimes_{\mathcal{O}_n} \mathcal{F}_{Y,n} \to \varepsilon^*(\mathcal{N}).
\]

Then, these almost isomorphisms fit into the following diagram:

\[
\begin{array}{ccc}
\mathcal{L} \otimes_{\mathcal{O}_n} \mathcal{F}_{X,n} & \xrightarrow{\approx} & \mathcal{N} \\
\downarrow \approx & & \downarrow \approx \\
R \varepsilon_*(\varepsilon^*(\mathcal{L} \otimes_{\mathcal{O}_n} \mathcal{F}_{X,n})) & \xrightarrow{\sim} & R \varepsilon_*(\varepsilon^*(\mathcal{N}) \otimes_{\mathcal{O}_n} \mathcal{F}_{Y,n}) \approx R \varepsilon_*(\varepsilon^*(\mathcal{N})).
\end{array}
\]

Two vertical arrows are almost isomorphisms due to the cohomological descent and corollary 4.1.4. Then, we conclude the isomorphism of \( (\mathcal{F}_{X,n})^\alpha \)-modules (6.1.7.1).

\( \square \)

6.1.8. For \( m \geq n \geq 1 \), the functors \( \mathbb{L}_m \) and \( \mathbb{L}_n \) are compatible up to almost isomorphisms via reduction ([55] lemma 8.13). By considering the projective system \( (\mathbb{L}_n)_{n \geq 1} \), we obtain following functor ([55] 8.14.1):

\[
\mathbb{L} : \text{Mod}_{\text{pltf}}(\mathcal{F}) \rightarrow \text{Mod}(X_{\eta_{\text{fét}},\delta}, M = (M_n)_{n \geq 1} \mapsto (m \otimes_{\mathbb{L}_n(M_n)})_{n \geq 1}.
\]

By passing to categories up to isogeny, the functor \( \mathbb{L} \) factors through the full subcategory \( \text{LocSys}(X_{\eta_{\text{fét}},\delta})_Q \), which is equivalent to \( \text{Rep}(\pi_1(X_{\eta_{\text{fét}},\delta}), C) \) (4.1.9.2) (c.f. [55] 8.14):

\[
\mathbb{L}_Q : \text{Mod}_{\text{pltf}}(\mathcal{F}_Q) \rightarrow \text{Rep}(\pi_1(X_{\eta_{\text{fét}},\delta}), C).
\]

6.2. Parallel transport for Deninger–Werner Higgs bundles. In this subsection, \( X \) denotes a semi-stable \( S \)-curve. We fix a smooth Cartesian lifting \( X' \) of \( X \) over \( \Sigma_S \) as in the beginning of §5.

**Theorem 6.2.1.** Let \( (M, \theta) \) be an object of \( HB^{DW}_{\Sigma,\text{smal}l}(\tilde{X}/\Sigma_1, S) \), \( M \) the associated Higgs crystal and \( \tilde{h} : (\tilde{E}_{\Sigma}^{W}, \mathcal{F}_{X}) \rightarrow (\tilde{E}_{\Sigma}^{W}, \mathcal{F}_{X}) \) the canonical morphism of ringed topoi (4.1.5.3).

(i) Then, \( \tilde{h}^*(T(M)) \) is potentially free of finite type. In particular, this allows us to define the parallel transport functor \( \mathcal{V}_X \) as the composition (4.2.2.4):

\[
(\mathcal{V}_X : HB^{DW}_{\Sigma,\text{small}}(\tilde{X}/\Sigma_1, S) \xrightarrow{\mathcal{L}} HC_{\Sigma,\text{fin}}(\tilde{X}/\Sigma) \xrightarrow{\tilde{h}^*} T(M) \xrightarrow{\mathcal{L}^{-1}} \text{Mod}_{\text{pltf}}(\mathcal{F}_{X}) \rightarrow \text{Mod}(X_{\eta_{\text{fét}},\delta}).
\]

(ii) The \( \mathcal{F}_X \)-module \( \tilde{h}^*(T(M)) \) satisfies the cohomological descent.

(iii) There exists a functorial isomorphism of \( (\mathcal{F}_X)^\alpha \)-modules on \( (M, \theta) \):

\[
(\beta^\alpha_X(\mathcal{V}_X(M, \theta)))^\alpha \cong (T(M))^\alpha.
\]
Proof. (i) Let $n$ be an integer $\geq 1$ and $\varphi : Y \to X_{\mathbb{P}}$ be an $\eta'$-cover of semi-stable curves trivializing $\mathcal{M}_n$, as in theorem 3.2.2. Since the functor $T$ is compatible with the inverse image functor of Higgs crystals and $\Phi^*$ (6.1.0.1) (proposition 4.2.4), then assertion (i) follows.

Assertion (ii) follows from corollary 4.1.4 and the following lemma.

**Lemma 6.2.2.** The $\mathcal{B}_{X,n}$-module $T(M)_n$ is locally free of finite type.

Proof. By ([4] II.5.17), there exists an étale covering $\{U_i \to X\}_{i \in I}$ of $Q$ such that $M|_{U_i}$ is a trivial vector bundle of rank $r$. Let $\overline{y}$ be a geometric generic point of $U_i \otimes_{\mathbb{P}} U_i$ the connected component of $U_i$ containing $y$ and we take again the notation of §4.3 for the pair $(U_i, \overline{y})$. In view of equivalences (4.3.1.1), there exists a trivialization of the underlying $R/p^nR$-module of $T(M)_n|_{U_i \otimes_{\mathbb{P}} U_i}$ (c.f. proposition 4.1.2):

$$\nu_1(T(M)_n|_{U_i \otimes_{\mathbb{P}} U_i}) \sim (R/p^nR)^\oplus r.$$ 

Moreover, there exists a finite étale Galois cover $V_i^1 \to U_i \otimes_{\mathbb{P}} U_i$ such that the action of $\Delta_{(U_i, \overline{y})} = \pi_1(U_i \otimes_{\mathbb{P}} U_i, \overline{y})$ on the above module factors through $\text{Gal}(V_i^1/U_i \otimes_{\mathbb{P}} U_i)$. In view of (4.2.2.3) and ([4] III.10.5), the above isomorphism induces a trivialization of $\mathcal{B}_{V_i^1 \to U_i, n}$-modules of $\tilde{E}_{V_i^1 \to U_i}$:

$$T(M)_n|_{(V_i^1 \to U_i)} \sim (\mathcal{B}_{V_i^1 \to U_i, n})^{\oplus r}.$$ 

Since $\{(V_i^1 \to U_i) \to (X_{\mathbb{P}} \to X)\}_{i \in I, \overline{y} \in \text{Con}(U_i, \overline{y})}$ forms a covering of Faltings site, the lemma follows. \qed

(ii) By assertion (ii) and lemma 6.1.7, we conclude the following $(\mathcal{B}_X)^\alpha$-linear isomorphism:

$$(\tilde{\beta}_X^\alpha(V_X(M, \theta)))^\alpha \sim (\tilde{h}^*(T(M)))^\alpha.$$ 

On the one hand, the following canonical morphism is an isomorphism in view of lemma 4.1.6(ii)

$$\tilde{\beta}_X^\alpha(V_X(M, \theta)) \sim \tilde{h}_*(\tilde{\beta}_X^\alpha(V_X(M, \theta))).$$

On the other hand, by lemma 6.2.2, the following canonical morphism is an isomorphism:

$$T(M) \sim \tilde{h}_*\tilde{h}^*(T(M)).$$

Then, the assertion follows from applying $\tilde{h}_*$ to the isomorphism (6.2.2.1). \qed

By passing to categories up to isogeny and (§ 6.1.8), we deduce from $V_X$ the following functor, that we abusively denote by $\mathbb{V}_X$:

$$V_X : \text{HD}_{Q_p, \text{small}}(\tilde{X}/\Sigma_{1,S}) \to \text{Rep}(\pi_1(X_{\mathbb{P}}, \overline{\pi}), C).$$

**Corollary 6.2.3.** The functor $T_{X_2} : \text{HD}_{Q_p, \text{small}}(\tilde{X}/\Sigma_{1,S}) \to \text{Mod}(\mathcal{B}_{X,q})$ (4.2.6.1) is canonically isomorphic to the composition:

$$\text{HD}_{Q_p, \text{small}}(\tilde{X}/\Sigma_{1,S}) \xrightarrow{\mathbb{V}_X} \text{Rep}(\pi_1(X_{\mathbb{P}}, \overline{\pi}), C) \xrightarrow{\tilde{\beta}_X^\alpha} \text{Mod}(\mathcal{B}_{X,q}).$$

Proof. It follows from proposition 4.2.7 and theorem 6.2.1(iii). \qed

**Corollary 6.2.4.** Let $K'$ be a finite extension of $K$, $Y$ a semi-stable $S' = \text{Spec}(O_{K'})$-curve, $f : Y \to X_{K'}$ a generic $\eta'$-cover, $\overline{y}$ a geometric generic point of $V_Y$ above $\overline{\pi}$ and $\overline{Y}$ a smooth Cartesian lifting of $Y$ over $\Sigma_{S'}$. Via $\pi_1(Y_{\mathbb{P}}, \overline{y}) \to \pi_1(X_{\mathbb{P}}, \overline{\pi})$, the following diagram is commutative up to canonical isomorphisms $\gamma_f$

$$\begin{array}{ccc}
\text{HD}_{Q_p, \text{small}}(\tilde{X}/\Sigma_{1,S}) & \xrightarrow{\mathbb{V}_X} & \text{Rep}(\pi_1(X_{\mathbb{P}}, \overline{\pi}), C) \\
\text{HD}_{Q_p, \text{small}}(\tilde{Y}/\Sigma_{1,S'}) \xrightarrow{f_{Y,X}} & \gamma_f & \xrightarrow{\gamma} \\
\end{array}$$

$$\begin{array}{ccc}
\text{HD}_{Q_p, \text{small}}(\tilde{Y}/\Sigma_{1,S'}) & \xrightarrow{\mathbb{V}_Y} & \text{Rep}(\pi_1(Y_{\mathbb{P}}, \overline{y}), C) \\
\end{array}$$
Moreover, $\gamma_f$ satisfy a cocycle condition as in (4.2.4.1).

Proof. By Faltings’ comparison theorem ([24] theorem 5.13), the following functor
\begin{equation}
\tilde{\beta}_{X,Q} : \text{Rep}(\pi_1(X_\eta, \mathcal{T}), \mathbb{C}) \simeq \text{LocSys}(X_{\eta, \text{ét}}, \delta)_{Q} \to \text{Mod}(\mathcal{F}_X_{\mathbb{Q}})
\end{equation}
is fully faithful (c.f. [55] proposition 8.8). Composing the above functor with the above diagram, the assertion follows from propositions 4.2.4, 4.2.7 and corollary 6.2.3.

6.3. Construction of $\mathbb{V}_{X, \text{Exp}}$ via descent, d’après [15]. In the following of this section, $X$ denotes a stable $S$-curve whose geometric generic fiber $X_\eta$ has genus $\geq 2$. We keep the notation and assumption as in the beginning of § 5. We extend the functor $\mathbb{V}_X$ to $\mathbb{HP}^{BDW}_{X, \text{Exp}}(X_C)$ (definition 5.2.4):

**Theorem 6.3.1.** There exists a quasi-inverse functor $\mathbb{V}_{X, \text{Exp}}$ to $\mathbb{H}_{X, \text{Exp}}$ (5.2.6.1):

\begin{equation}
\mathbb{V}_{X, \text{Exp}} : \mathbb{HP}^{BDW}_{X, \text{Exp}}(X_C) \to \text{Rep}(\pi_1(X_\eta, \mathcal{T}), \mathbb{C}).
\end{equation}

**6.3.2.** We revise ([15] § 3) on the fundamental groupoids. Let $Z$ be a variety over $\eta$. We denote by $\Pi_1(Z)$ the following topological groupoid. Its set of objects is $Z(\mathbb{C})$, and for $z, z' \in Z(\mathbb{C})$, $\text{Hom}_{\Pi_1(Z)}(z, z')$ is the set of isomorphisms of étale fibre functors $F_z \to F_{z'}$, where $F_z = \text{Mor}_Z(z, -)$ is the functor from the category $\text{Fét}_Z$ (1.2.5) to the category of finite sets. The pro-finite set $\text{Hom}_{\Pi_1(Z)}(z, z')$ is equipped with the pro-finite topology. If $z \in Z(\mathbb{C})$ is a base point, then $\text{Hom}_{\Pi_1(Z)}(z, z)$ is isomorphic to the étale fundamental group $\pi_1(Z, z)$. A morphism $f : Z \to Z'$ over $\eta$ induces a natural functor $f_* : \Pi_1(Z) \to \Pi_1(Z')$.

We denote by $\text{Rep}(\Pi_1(Z), \text{Vec}_\mathbb{C})$ the category of continuous functors from $\Pi_1(Z)$ to the category $\text{Vec}_\mathbb{C}$ of finite dimensional $\mathbb{C}$-vector spaces, equipped with the $p$-adic topology, whose morphisms are natural transforms. There exists a natural functor:

\begin{equation}
\text{LocSys}(Z_{\text{ét}, \delta})_{Q} \to \text{Rep}(\Pi_1(Z), \text{Vec}_\mathbb{C}),
\end{equation}

where $\rho_i$ sends any morphism $\gamma \in \text{Hom}_{\Pi_1(Z)}(z, z')$ to the composition, defined for every $i \geq 1$, $Y_i \in \text{Fét}_Z$ trivializing $L_i$ and a system of compatible morphisms $\gamma_i : F_i(Y_i)$ by

$\gamma \mapsto \gamma_L : L_z \xrightarrow{\lim_{i \to \infty} \gamma_i} \lim_{i \to \infty} L_y(Y_i) \xrightarrow{\lim_{i \to \infty} (\gamma_i(\nu_i))^{-1}} L_{z'}$.

The functor is an equivalence of categories. Indeed, for any geometric point $z \in Z$, we have functors:

\begin{equation}
\text{LocSys}(Z_{\text{ét}, \delta})_{Q} \to \text{Rep}(\Pi_1(Z), \text{Vec}_\mathbb{C}) \to \text{Rep}_\mathbb{C}(\pi_1(Z, z), \mathbb{C}),
\end{equation}

where the second functor is fully faithful ([14] Lemma 21) and the composition is the equivalence (4.1.9.2). In particular, the above functors are equivalences of categories.

**6.3.3.** Let $\alpha : V \to U$ be a finite étale Galois covering of varieties over $\eta$, with Galois group $G$. Let $B : \Pi_1(V) \to \text{Vec}_\mathbb{C}$ be a continuous functor, equipped with a system of isomorphism $\varphi_{\sigma} : B \circ \alpha_* \cong B$ for $\sigma \in G$ satisfying $\varphi_e = \text{id}$ and $\varphi_{\sigma\tau} = \varphi_{\tau} \circ (\varphi_{\sigma})$. By ([15] construction 7), we can define a continuous functor $A : \Pi_1(U) \to \text{Vec}_\mathbb{C}$ as follows: for $x \in \text{Ob}\Pi_1(U) = U(\mathbb{C})$, we set:

$A(x) = \{(f_y) \in \prod_{y \in \alpha^{-1}(x)} B(y) | \varphi_{\alpha_{\sigma}}(f_{\sigma y}) = f_y\}$

for all $\sigma \in G$ and $y \in \alpha^{-1}(x)$.

Let $\gamma$ be an étale path in $U$ from $x_1$ to $x_2$, and $y_1 \in V(\mathbb{C})$ a point above $x_1$. Then, there is a unique path $\delta$ from $y_1$ to a point $y_2 \in V(\mathbb{C})$ over $x_2$ such that $\alpha_{\delta}(\delta) = \gamma$. The product of $B(\sigma_{\delta}) : B(\sigma y_1) \to B(\sigma y_2)$ over $\sigma \in G$ induces an isomorphism:

$A(\gamma) : A(x_1) \subset \prod_{\sigma \in G} B(\sigma y_1) \to A(x_2) \subset \prod_{\sigma \in G} B(\sigma y_2)$.

Moreover, we have a canonical isomorphism of functors $\Phi : A \circ \alpha_* \cong B$, which gives rise to the above descent data $(\varphi_{\sigma})\sigma \in G$. The construction from $(B, \{\varphi_{\sigma}\}_{\sigma \in G}) \mapsto A$ is clearly functorial.
6.3.4. With the notation of § 6.2, the functor $V_X$ (6.2.2.2) factor through a functor:
\[(6.3.4.1) \quad V_X : \text{HB}^\text{DW}_{q_p, \text{small}}(X/\Sigma_1, S) \to \text{Rep}(\Pi_1(X_\mathbf{\Gamma}), \text{Vec}_C)\]

By corollary 6.2.4, we see that for every object $(M, \theta)$ of $\text{HB}^\text{DW}_{q_p, \text{small}}(X/\Sigma_1, S)$ and $x \in X_\mathbf{\Gamma}(C)$, we have a canonical isomorphism of $C$-vector spaces
\[V_X(M, \theta)(x) \cong M_x.\]

For each object $(M, \theta)$ of $\text{HB}^\text{DW}_{X, \text{Exp}}(X_C)$, there exists a finite morphism $\pi : Y_\mathbf{\Gamma} \to X_\mathbf{\Gamma}$ of smooth proper $\mathbf{\Gamma}$-curves, a finite extension $K'$ of $K$ such that $Y_\mathbf{\Gamma}$ admits a stable $S' = \text{Spec}(\mathcal{O}_{K'})$-model $Y$, $\pi$ extends to the $S'$-morphism $f : Y \to X_{S'}$ of stable $S'$-models and that $(N, \vartheta) := f^*_Y X_{\text{Exp}}(M, \theta)$ belongs to $\text{HB}^\text{DW}_{q_p, \text{small}}(Y/\Sigma_1, S')$ for a smooth Cartesiian lift $Y$ of $Y$ over $\Sigma_{S'}$. We may replace $\pi$ by its Galois closure and then assume that $f$ is a Galois generic $\eta'$-cover by the unique extension property of stable models (1.2.3).

We set $G = \text{Gal}((\mathcal{K}(Y_\mathbf{\Gamma}), \mathcal{K}(X_\mathbf{\Gamma}))$. We have an action of $G$ on $(N, \vartheta)$ by twisted inverse image functoriality:
\[(6.3.4.2) \quad \phi_\sigma : \sigma^*_\gamma Y, (N, \vartheta) \xrightarrow{\sim} (N, \vartheta), \quad \text{such that} \quad \phi_\sigma \circ \phi_\tau = \phi_{\sigma \tau}, \quad \forall \sigma, \tau \in G.\]

Then, we obtain a continuous functor $V_Y(N, \vartheta) : \Pi_1(Y_\mathbf{\Gamma}) \to \text{Vec}_C$, equipped with an action of $G$:
\[\phi_\sigma : V_Y(N, \vartheta) \circ \sigma_* \xrightarrow{\sim} V_Y(N, \vartheta), \quad \text{such that} \quad \phi_\sigma \circ \phi_\sigma = \phi_{\sigma \tau}, \quad \forall \sigma, \tau \in G.\]

Let $U$ be the open subset of $X_\mathbf{\Gamma}$ over which $\pi$ is unramified and $V = \pi^{-1}(U)$. By 6.3.3, we deduce from $V_Y(N, \vartheta)|_{\Pi_1(V)}$ a continuous functor:
\[V : \Pi_1(U) \to \text{Vec}_C, \quad x \mapsto M_x.\]

**Proposition 6.3.5.** The functor $V$ factors through $\Pi_1(U) \to \Pi_1(X_\mathbf{\Gamma})$ and induces a continuous functor \[V_{X, \text{Exp}}(M, \theta) : \Pi_1(X) \to \text{Vec}_C.\]

**Proof.** We follow ([15] theorem 9). Let $S = X_\mathbf{\Gamma} - U$ be the ramification loci and $T = \pi^{-1}(S)$. For every point $t \in T(C)$, we denote by $G_t$ the subgroup of $G$ consisting of elements fixing $t$. Since $f^*(M, \theta) = (N, \vartheta)$, for $\sigma \in G_t$, the fiber $\sigma_{\tau t} : N_t = (\sigma^t N_t) \xrightarrow{\sim} N_t$ of (6.3.4.2) at $t$ coincides with the identity map $\text{id}_{N_t}$.

We fix a base point $x_0 \in U(C)$ and a preimage $y_0 \in V(C)$ of $x_0$. Let $\gamma_0 \in \pi_1(U, x_0)$ be an element sending to an element $\sigma \in G_t$, and let $\gamma$ be an $\eta$-path in $\Pi_1(V)$ with starting point $y_0$ and endpoint $\sigma(y_0)$, lifting $\gamma_0$. Following the proof of ([15] theorem 9), for any $\eta$-path $\delta$ in $Y$ from $y_0$ to $t$ and $j : V \to Y_\mathbf{\Gamma}$ the open immersion, we show
\[\sigma_*((\delta)_{\eta, \lambda} j_\gamma(\delta)^{-1} = 1 \in \pi_1(Y, t)\]

In particular, we have $\forall(\sigma_\lambda j_\gamma(\delta)^{-1} = \text{id}_{X_t}$. Then, we deduce that $\forall(\gamma_0) = \text{id}_{V(x_0)}$ by ([15] Lemma 8). Hence, the representation $\forall_{x_0} : \pi_1(U, x_0) \to \text{Aut}(N_{x_0})$ factors through the quotient $\pi_1(X_\mathbf{\Gamma}, x_0)$. \qed

We also denote by $V_{X, \text{Exp}}(M, \theta) \in \text{Rep}(\pi_1(X_\mathbf{\Gamma}, \mathbf{\Gamma}), C)$ the $C$-representation associated to the functor in above proposition. Theorem 6.3.1 follows from the following proposition:

**Proposition 6.3.6.** (i) The construction $(M, \theta) \mapsto V_{X, \text{Exp}}(M, \theta)$ is independent of the choice of $\pi$, $K'$ and $\mathcal{Y}$ up to canonical isomorphisms.

(ii) The functor $V_{X, \text{Exp}} : \text{HB}^\text{DW}_{X, \text{Exp}}(X_C) \to \text{Rep}(\pi_1(X_\mathbf{\Gamma}, \mathbf{\Gamma}), C)$ is well-defined.

(iii) The functor $V_{X, \text{Exp}}$ is compatible with twisted inverse images of Higgs bundles and the restriction of representations as in corollary 6.2.4.

(iv) The functors $H_{X, \text{Exp}}$ and $V_{X, \text{Exp}}$ are quasi-inverse to each other.

**Proof.** (i) Let $K''$ be a finite extension of $K$, $Z$ a stable $S'' = \text{Spec}(\mathcal{O}_{K''})$-curve, $g : Z \to X_{S''}$ a generic $\eta''$-cover and $Z$ a smooth Cartesian lifting of $\tilde{Z}$ over $\Sigma_{S''}$ such that $f^*_{\tilde{Z}, \text{Exp}}(M, \theta)$ belongs to $\text{HB}^\text{DW}_{q_p, \text{small}}(\tilde{Z}/\Sigma_1, S'')$. To prove the assertion, we may assume that $g$ dominates $f$. Then, the independence of $\mathcal{Y}$ and of $\pi, K'$ follows from corollary 6.2.4 and proposition A.2.5(iv).
(ii) Let \( u : (M, \theta) \to (M', \theta') \) be a morphism of \( \text{HB}_{\mathcal{X}, \text{Exp}}^{\text{pDW}}(X_C) \). By (i), we may choose a generic \( \eta' \)-cover \( f : Y \to X_{S'} \) as in § 6.3.4 and a smooth Cartesian lift \( \tilde{Y} \) of \( Y \) over \( \Sigma_S \) such that both \( f_{\tilde{Y}}^*(M, \theta) \) and \( f_{\tilde{Y}}^*(M', \theta') \) belong to \( \text{HB}_{Q_{p, \text{small}}}^{\text{pDW}}(\tilde{X}/\Sigma_1, S) \). Then, the functoriality of \( \forall_{\mathcal{X}, \text{Exp}} \) follows from that of \( \forall_Y \) and of descent for \( C \)-representations (6.3.3).

(iii) The assertion follows from corollary 6.2.4 and the construction of \( \forall_{\mathcal{X}, \text{Exp}} \).

(iv) We ignore \( \mathcal{X}, \text{Exp} \) from the notations \( \forall_{\mathcal{X}, \text{Exp}}, \mathbb{H}_{\mathcal{X}, \text{Exp}} \) for simplicity. We first show that for a small object \( V \) of \( \text{Rep}(\pi_1(X_{\overline{\mathbb{F}}, \mathcal{F}}), C) \) (resp. \( (M, \theta) \) of \( \text{HB}_{Q_{p, \text{small}}}^{\text{pDW}}(X/\Sigma_1, S) \)), there exist functorial isomorphisms

\[
(6.3.6.1) \quad V \sim \forall(\mathbb{H}(V)), \quad (M, \theta) \sim \mathbb{H}(\forall(M, \theta)).
\]

On the other hand, let \( V \) be a small \( C \)-representation of \( \pi_1(X_{\overline{\mathbb{F}}, \mathcal{F}}) \). In view of corollary 5.1.2(ii), proposition 6.1.4 and theorem 6.2.1(i), we have a functorial isomorphism of \( \text{HB}_{\mathcal{X}, \text{p, \text{Q}}} \)-module:

\[
\tilde{\beta}_{X, Q}(V) \sim \tilde{\beta}_{X, Q}(\forall_{\mathcal{X}, \text{H}}(V)).
\]

In view of the full faithfulness of functor \( \tilde{\beta}_{X, Q} \) (6.2.4.1), the first isomorphism of (6.3.6.1) follows.

On the other hand, let \( (M, \theta) \) be a Higgs bundle of \( \text{HB}_{Q_{p, \text{small}}}^{\text{pDW}}(\tilde{X}/\Sigma_1, S) \) such that \( \mathbb{H}(M, \theta) \) is \( \alpha \)-small for some \( \alpha > \frac{\pi}{p-1} \). By proposition 5.1.4 and corollary 6.2.3, we obtain the following canonical isomorphisms:

\[
\mathbb{T}_{\mathcal{X}_2}(M, \theta) \sim \tilde{\beta}_{X, Q}(\forall(M, \theta)), \quad \mathbb{T}_{\mathcal{X}_2}(\mathbb{H}(\forall(M, \theta))) \sim \tilde{\beta}_{X, Q}^{*}(\forall(M, \theta)).
\]

Then, we deduce a functorial isomorphism \( (M, \theta) \sim \mathbb{H}(\forall(M, \theta)) \) via the equivalence \( \mathbb{T}_{\mathcal{X}_2} \) (4.2.6.1).

In general, let \( V \) be an object of \( \text{Rep}(\pi_1(X_{\overline{\mathbb{F}}, \mathcal{F}}), C) \). By (iii), corollary 5.2.6(ii) and finite étale descent for \( C \)-representations, we deduce from small \( C \)-representations case an isomorphism \( V \sim \forall(\mathbb{H}(V)) \).

Let \( (M, \theta) \) be an object of \( \text{HB}_{\mathcal{X}, \text{Exp}}^{\text{pDW}}(X_C) \). We take a Galois generic \( \eta' \)-cover \( f \) as in 6.3.4. By applying (6.3.6.1) for \( \text{HB}_{Q_{p, \text{small}}}^{\text{pDW}}(\tilde{Y}/\Sigma_1, S') \) and the following lemma to \( f_C \), we deduce a canonical isomorphism between \( M \) and the underlying vector bundle of \( \mathbb{H}(\forall(M, \theta)) \). Since \( f^*(M, \theta) \) comes from a Higgs bundle of \( \mathbb{H}(Y_C, \xi^{-1}f_C(\Omega_{X_C})) \) (proposition A.2.5), we conclude an isomorphism \( (M, \theta) \to \mathbb{H}(\forall(M, \theta)) \) by descent from the following lemma.

**Lemma 6.3.7.** Let \( g : C' \to C \) be a finite surjective map of smooth proper \( C \)-curves, \( G \) a finite group and \( \mu : G \times C' \to C' \) a \( G \)-action over \( C' \) such that over the unramified locus \( U \subset C \) of \( g \), \( g \) is a \( G \)-torsor with the action \( \mu \). The functor \( g^* \) induces a fully faithful functor from the category of vector bundles on \( C \) to the category of descent data \( (E, \{\varphi_\sigma\}_{\sigma \in G}) \) consisting of a vector bundle \( E \) over \( C' \) and isomorphisms \( \varphi_\sigma : \sigma^*(E) \sim E \) such that \( \varphi_{\sigma^{-1}} = \varphi_\sigma \circ (\varphi_\sigma)^* \) and \( \varphi_\epsilon = \text{id}_E \).

**Proof.** We may assume \( C = \text{Spec}(A) \) and \( C' = \text{Spec}(A') \) are affine and set \( B = \prod_{\sigma \in G} A' \). It suffices to show that for every projective \( A \)-module \( M \) of finite rank, the following sequence is exact:

\[
0 \to M \to M \otimes_A A' \stackrel{\iota_1-\iota_2}{\to} M \otimes_A B,
\]

where \( \iota_1 : A' \to B \) is the diagonal embedding and \( \iota_2 \) is induced by \( \mu \).

It suffices to show the exactness after taking \( - \otimes_A \hat{A}_x \), where \( \hat{A}_x \) is the completed local ring of \( A \) at a closed point \( x \in C \). When \( x \) lies in the unramified locus of \( g \), the exactness follows from fpfp descent.

Suppose \( g \) is ramified at \( x \) and we set \( I = g^{-1}(x) \). Since \( g \) is a \( G \)-torsor over \( U \), the \( G \)-action on \( I \) is transitive by continuity. For \( y \in I \), \( G_y = \{\sigma \in G|\sigma(y) = y\} \) is a cyclic group of order \( e > 1 \) (c.f. [15] proof of theorem 9) and we have \( (G/G_y) \cdot y = I \).

We replace \( A \) (resp. \( A', \text{ resp } B \)) by its tensor product \( - \otimes_A \hat{A}_x \). Then the ring \( A' \) decomposes as a product of completed local rings \( \prod_{y \in I} A'_y \). Each homomorphism \( A \to A'_y \) is isomorphic to the homomorphism \( A = \mathbb{C}[t] \to A'_y = \mathbb{C}[s] \), defined by \( t \mapsto s^e \), and the action of \( G_y \) is given by \( \sigma(s) = \zeta_{\sigma} s \) for an \( e \)-th root of
Proof. The assertion is true for $S = (i.e. M_A$ with the diagonal map $\eta)$. Let $\eta$ that there exists a canonical fully faithful functor from $\text{HB}_{\text{Q},\text{small}}(\hat{Y}/\Sigma_{1,S'})$. Then, we deduce the inverse image functor (proposition A.4.14). Then, we deduce

$$\prod_{y \in I} A'_y \simeq \prod_{y \in I} (\prod_{y \in I} A'_y),$$

with the diagonal map $A \to \prod_{y \in I} A'_y$. Then, the lemma follows. \qed

Corollary 6.3.8. Let $(M, \theta)$ be an object of $\text{HB}_{X,\text{Exp}}^{\text{PDW}}(X_C)$. There exists a finite extension $K'$ of $K$, a stable $S' = \text{Spec}(\partial_K)$-curve $Y$, an $\eta'$-cover $f : Y \to X_{S'}$, and a smooth Cartesian lift $\mathcal{Y}$ of $\hat{Y}$ over $\Sigma_{S'}$ such that $f_{\mathcal{Y},X,\text{Exp}}^0(M, \theta)$ belongs to $\text{HB}_{\mathcal{Y},\text{Exp}}^{\text{PDW}}(X_C)$. Moreover, we partially generalize the above result to the case of small Higgs fields.

6.4. Comparison with Deninger–Werner’s theory.

Theorem 6.4.1. (i) Let $M$ be a vector bundle over $X_C$. The following properties are equivalent:

(a) The Higgs bundle $(M, 0)$ belongs to $\text{HB}^{\text{PDW}}_{X,\text{Exp}}(X_C)$.

(b) The vector bundle $M$ belongs to $\text{VB}^{\text{PDW}}_{X,\text{Exp}}(X_C)$ (3.1.2).

(ii) Via (i), Deninger–Werner’s functor $\mathbb{V}^{\text{PDW}}_{X_C}$ (3.1.3.1) is compatible with functor $\mathbb{V}_{X,\text{Exp}}$.

Proposition 6.4.2. Let $M$ be a vector bundle of $\text{VB}^{\text{PDW}}_{X,\text{Exp}}(X_C)$ and $\theta$ a small Higgs field on $M$ (2.2.10). Then $(M, \theta)$ belongs to $\text{HB}^{\text{PDW}}_{X,\text{Exp}}(X_C)$.

Lemma 6.4.3. Let $Y$ be a semi-stable $\bar{S}$-curve, $\mathcal{Y}$ a smooth Cartesian lifting of $\hat{Y}$ over $\Sigma_S$, $\mathcal{M}$ an object of $\text{VB}^{\text{PDW}}_{X}(\mathcal{Y})$ such that $\mathbb{V}^{\text{PDW}}_{\mathcal{Y}}(\mathcal{M})$ (3.1.3.1) is small. Then, we have a canonical isomorphism:

$$\mathbb{V}_{\mathcal{Y}}(\mathcal{M}|_{\frac{1}{p^j}}) \simeq \mathbb{V}^{\text{PDW}}_{\mathcal{Y}}(\mathcal{M})|_{\frac{1}{p^j}}.$$

Proof. Let $M = \mathcal{M}|_{\frac{1}{p^j}}$ and $W = \mathbb{V}_{\mathcal{Y}}(M, 0)$. By corollary 6.2.3, we have a $\mathbb{B}_{\mathcal{Y},\mathbb{Q}}$-linear isomorphism $T_{\mathcal{Y}}(M, 0) \simeq \beta^*(W)$. By ([55] proposition 11.7), we obtain a $\mathbb{B}_{\mathcal{Y},\mathbb{Q}}$-linear isomorphism

$$\beta^*(W) \simeq \sigma^*(M),$$

(i.e. $M$ is Weil-Tate in the sense of [55] definition 10.3). By ([55] corollaire 14.5, proposition 14.6), the functor $\mathbb{V}^{\text{PDW}}_{\mathcal{Y}}$ can be reconstructed from the above $\mathbb{B}_{\mathcal{Y},\mathbb{Q}}$-admissibility. We obtain $W \simeq \mathbb{V}^{\text{PDW}}_{\mathcal{Y}}(M)$. \qed

6.4.4. Proof of theorem 6.4.1. (i) We denote by $\text{HB}^{\text{PDW}, 0}_{X,\text{Exp}}(X_C)$ the full subcategory of $\text{HB}^{\text{PDW}}_{X,\text{Exp}}(X_C)$ consisting of Higgs bundles with zero Higgs field. When the twisted inverse image functor applies to Higgs bundles with zero Higgs field, it coincides with the usual inverse image functor (proposition A.4.14). Then, we deduce that there exists a canonical fully faithful functor from $\text{HB}^{\text{PDW}, 0}_{X,\text{Exp}}(X_C)$ to $\text{VB}^{\text{PDW}}_{X,\text{Exp}}(X_C)$ by forgetting the zero Higgs field.

It remains to show that for any object $M$ of $\text{VB}^{\text{PDW}}_{X,\text{Exp}}(X_C)$, $(M, 0)$ belongs to $\text{HB}^{\text{PDW}}_{X,\text{Exp}}(X_C)$. Let $V = \mathbb{V}^{\text{PDW}}_{X_C}(M)$. There exists a finite cover $\pi : Y_{\mathfrak{f}} \to X_{\mathfrak{f}}$ of smooth proper $\mathfrak{f}$-curves such that

- the restriction of $V$ to $\pi_1(Y_{\mathfrak{f}})$ has a small $\mathfrak{a}$-lattice $V^\circ$;
• $Y^*_\eta$ admits a semi-stable $S' = \text{Spec}(O_{K'})$-model $Y'$ for a finite extension $K'$ of $K$ such that $\pi^*(M)$ extends to a Deninger–Werner vector bundle $\mathcal{N}$ over $\hat{Y'}$.

Let $Y$ be the stable $S'$-model of $Y^*_\eta$. After replacing $Y'$ by the regular desingularization of $Y'$, we may assume that $Y'$ dominates $Y$ by a morphism $\varphi$. We take smooth Cartesian liftings $\mathcal{Y}', \mathcal{Y}$ of $\hat{Y}', \hat{Y}$ over $\Sigma_{S'}$ respectively. We obtain Higgs bundles $\mathbb{H}_{\mathcal{Y}'}(V^\circ), \mathbb{H}_{\mathcal{Y}}(V^\circ)$ on $\hat{Y}', \hat{Y}$ satisfying:

$$g^\alpha_{\mathcal{Y}'}(\mathbb{H}_{\mathcal{Y}'}(V^\circ)) \cong \mathbb{H}_{\mathcal{Y}}(V^\circ).$$

Applying theorem 6.3.1 and lemma 6.4.3 to $\mathcal{N}$ over $\hat{Y'}$, we have a canonical isomorphism on $Y_C(\cong Y'_C)$:

$$(\pi^*(M), 0) \cong \mathbb{H}_{\mathcal{Y}}(V).$$

Then, we deduce that $\mathbb{H}_{\mathcal{Y}'}(V^\circ)$ has no Higgs field and so is $\mathbb{H}_{\mathcal{Y}}(V^\circ)$. In particular, $\pi^*(M)$ admits a Deninger–Werner model $\mathbb{H}_{\mathcal{Y}}(V^\circ)$ over $\hat{Y}$ and assertion (i) follows.

(ii) By corollary 6.3.8 and descent for $C$-representations along finite étale covers, we may reduce to case where $M$ admits a model in $\text{VB}^{\text{DW}}(\hat{X})$. In this case, assertion (ii) follows from lemma 6.4.3.

6.4.5. **Proof of proposition 6.4.2.** By proposition 3.2.3, there exists a finite extension $K'$ of $K$, a stable $S'(=\text{Spec}(O_{K'}))$-curve $Y$ and a generic $\eta'$-cover $f : Y \to X_{S'}$ such that $f^*_C(M)$ admits an integral model $\mathcal{M}$ on $\hat{Y}$ with trivial special fiber and that $f^*_C(\theta)$ is $\alpha$-small on $\mathcal{M}$ for some $\alpha \in \mathbb{Q}_{>0}$ (definition 2.2.10). We fix a smooth lifting $\mathcal{Y}$ of $\hat{Y}$ over $\Sigma_{S'}$. By proposition A.3.3(ii), the line bundle $\mathcal{L}_{f, \eta}$ has the trivial special fiber on the relative spectral curve defined by $c = f^*_C(h(M, \theta)) \in \bigoplus_{i=1}^N \mathbb{P}V(\hat{Y}, \xi^{-i}\hat{f}^*(\Omega^1_X))$ (A.3.2). We deduce that $f^*_\mathcal{M}_{\mathcal{Y}}(M, \theta)$ has an integral model $(\mathcal{M} \otimes \mathcal{L}_{f, \eta}, \vartheta)$ with trivial special fiber $\mathcal{M}$, and an $\alpha$-small Higgs field $\vartheta$. Then $f^*_\mathcal{M}_{\mathcal{Y}}(M, \theta)$ belongs to $\text{HB}^{\text{DW}}_{\eta', \text{small}}(\hat{Y}/\Sigma_{1, S'})$ (5.1.3) and the proposition follows.

6.4.6. **Corollary.** The categories $\text{HB}^{\text{DW}}_{\mathcal{X}, \text{Exp}}(X_C)$ and $\text{VB}^{\text{DW}}(X_C)$ are abelian.

**Proof.** Since $\text{Rep}(\pi_1(X_{\bar{\eta}}, \bar{\mathcal{Y}}), C)$ is abelian and $\text{HB}^{\text{Exp}}_{\mathcal{X}, \text{Exp}}$ is an exact equivalence, the assertion for $\text{HB}^{\text{DW}}_{\mathcal{X}, \text{Exp}}(X_C)$ follows. The assertion for $\text{VB}^{\text{DW}}(X_C)$ follows from that of $\text{HB}^{\text{DW}}_{\mathcal{X}, \text{Exp}}(X_C)$ and theorem 6.4.1.

6.4.7. **Proof of proposition 1.1.12.** The equivalence $\text{HB}^{\text{Exp}}_{\mathcal{X}, \text{Exp}}$ induces an isomorphism of extension classes:

$$\text{Ext}^1_{\text{Rep}(\pi_1(X_{\bar{\mathcal{Y}}}, \mathcal{Y}), C)}(V, C) \cong \text{Ext}^1_{\text{HB}(X_C)}((M, \theta), (\theta_{X_C}, 0)).$$

The left hand side is isomorphic to $H^1_{\text{et}}(X_{\mathcal{Y}}, V)$. And the right hand side is calculated by the first Higgs cohomology. Then, we deduce an isomorphism (1.1.12.1):

$$H^1_{\text{et}}(X_{\mathcal{Y}}, V) \cong H^1(X_C, M \otimes_{X_C} \xi^{-1}\Omega^1_{X_C}).$$

6.4.8. When the Higgs bundle $(M, 0)$ has the trivial Higgs field, the right hand side of above isomorphism degenerates and gives a decomposition

$$H^1_{\text{et}}(X_{\mathcal{Y}}, V) \cong H^1(X_C, M) \oplus H^0(X_C, M \otimes_{X_C} \xi^{-1}\Omega^1_{X_C}).$$

The first component $H^1(X_C, M)$ represents the extension classes of vector bundles. The above isomorphism generalizes the Hodge–Tate decomposition.

**Appendix A. Twisted inverse image for Higgs bundles over curves, d’après Faltings**

Tongmu He, Daxin Xu

In this appendix, we define a twisted inverse image functor for Higgs bundles along a finite morphism between smooth proper $p$-adic curves. We present its construction for curves over $C$, which is different to the original definition of Faltings [18], defined for curves over a discrete valuation ring.
Let \( f : C' \to C \) be a finite morphism between smooth proper \( \mathbf{C} \)-curves of genus \( \geq 2 \). Let \( \mathcal{C}', \mathcal{C} \) be two flat liftings of \( C', C \) over \( \mathbb{B}^\dagger_{\text{DR,2}} \) respectively. The \( p \)-adic logarithmic map \( \log : 1 + \mathfrak{m} \to \mathcal{C} \), defined by 
\[
x \mapsto \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n},
\]
gives rise to a short exact sequence:
\[
(A.0.0.1) \quad 0 \to \mu_{p^n} \to 1 + \mathfrak{m} \xrightarrow{\log} \mathcal{C} \to 0.
\]
For any \( \varepsilon \in \mathbb{Q} \), we denote by \( B^\circ_x \) (resp. \( B_x \)) the open (resp. closed) ball \( \{ x \in \mathcal{C} \mid |x| < \varepsilon \} \) (resp. \( \{ x \in \mathcal{C} \mid |x| \leq \varepsilon \} \)) of radius \( \varepsilon \). We set \( r_p = p^{-\frac{1}{p-1}} \). The exponential map \( \exp : B^\circ_{r_p} \to 1 + \mathfrak{m} \), defined by \( x \mapsto \sum_{n \geq 0} \frac{x^n}{n!} \), is a section of \( \log \) on \( B^\circ_{r_p} \). Since \( (1 + \mathfrak{m}, \times) \) is divisible and is therefore an injective abelian group, we can choose a homomorphism, called \textit{exponential map}
\[
(A.0.0.2) \quad \exp : \mathcal{C} \to 1 + \mathfrak{m},
\]
satisfying: \( \log \circ \exp = \text{id} \) and \( \exp |_{B^\circ_{r_p}} = \exp \). In particular, \( \exp \) is continuous for the \( p \)-adic topologies.

With the above data, we will define the twisted inverse image functor between Higgs bundles (§2.2.10):
\[
(A.0.0.3) \quad f^*_{\mathcal{C}', \mathcal{C}, \exp} : \text{HB}(\mathcal{C}) \to \text{HB}(\mathcal{C}').
\]

\textbf{A.1. Splitting of the \( p \)-adic logarithm for commutative smooth algebraic groups.} We review the \( p \)-adic logarithm of rigid groups, following Fargues [19]. A more general approach via \( v \)-sheaves for the \( p \)-adic logarithm is developed by Heuer [28] §2.3. We define a functorial splitting of the \( p \)-adic logarithm, which is a key ingredient in the construction of \( f^*_{\mathcal{C}', \mathcal{C}, \exp} \).

\textbf{A.1.1.} Let \( H \) be a (classical) commutative rigid group over \( \mathbf{C} \), \( \text{Lie}\mathcal{H} \) its Lie algebra. If \( V \) is a finite dimensional \( \mathbf{C} \)-vector space, we denote by \( \mathbb{G}^a_n \otimes_{\mathbf{C}} V \) the associated rigid group, which represents the functor \( \mathcal{X} \mapsto (\text{Hom}_{\mathcal{X}}(\mathbb{G}^a_n, \mathcal{X})) \). After choosing a basis of \( V \), it is isomorphic to a sum of copies of \( \mathbb{G}^a_n \). For a rigid space \( X \) over \( \mathcal{C} \), we denote by \( |X| \) the set of classical points \( X(\mathbf{C}) \).

By [19] théorème 1.2, there exists an open rigid subgroup \( U_H \) of \( H \) such that the associated Berkovich space satisfies
\[
|U_H|_{\text{Ber}} = \{ g \in |H|_{\text{Ber}} \mid \lim_{n \to \infty} p^ng = 0 \}.
\]

There exists a unique étale morphism of rigid groups
\[
\log_H : U_H \to \mathbb{G}^a_n \otimes \text{Lie} H,
\]
inducing the identity map on Lie algebras and an isomorphism on a neighborhood of the identity element (cf. [19] proposition 9). Let \( H[p^\infty] \) be the \( p \)-divisible subgroup defined by \( p \)-power torsions of \( H \). We obtain a short exact sequence of rigid analytic groups (see the following proposition and [19] théorème 1.2):
\[
(A.1.1.1) \quad 0 \to H[p^\infty] \to U_H \xrightarrow{\log_H} \mathbb{G}^a_n \otimes \text{Lie} H.
\]
When \( H = \mathbb{G}^a_n \), \( |U_H| = 1 + \mathfrak{m} \) and the underlying points of the above sequence recovers (A.0.0.1). We refer to ([28] §2) for a generalization of above constructions in the context of \( v \)-sheaves.

A commutative rigid group \( H \) over \( \mathbf{C} \) is called \textit{\( p \)-divisible}, if \( [p] : H \to H \) is finite, faithfully flat and satisfies \( \forall g \in |H|_{\text{Ber}} \mid \lim_{n \to \infty} p^ng = 0 \) for the topology of the Berkovich space (c.f. [19] définition 2).

\textbf{Proposition A.1.2.} \textit{Let} \( G \) \textit{be a commutative algebraic group over} \( \mathbf{C} \), \( \mathbb{G}^a_n \) \textit{the associated rigid group.}

\textit{(i)} \textit{The rigid group} \( U_{G^{\text{an}}} \) \textit{is} \( p \)-\textit{divisible. We simply denote} \( U_{G^{\text{an}}} \) \textit{(resp.} \( \log_{G^{\text{an}}} \text{)} \textit{by} \( U_G \) \textit{(resp.} \( \log_G \)).

\textit{(ii)} \textit{The morphisms} \( \log_G \) \textit{and} \( \exp (A.0.0.2) \) \textit{induce a continuous homomorphism of topological groups:}
\[
(A.1.2.1) \quad \exp_G : \text{Lie} G \to |U_G|,
\]
\textit{which is a section of} \( |\log_G| \) \textit{(A.1.1.1) on underlying points. Moreover,} \( \exp_G \) \textit{is functorial in} \( G \).

\textit{Proof.} (i) By the definition of \( U_G \), it suffices to prove that \( [p] \) is finite and faithfully flat. By ([20] VII.8.4), \( [p] : G \to G \) is étale. By Chevalley’s structure theorem ([(20] VI.12.4), \( G \) is an extension of an abelian variety \( A \) by a commutative linear algebraic group \( D \). On \( A \) and \( D \), \( [p] \) is surjective and \( \text{Ker}[p] \) is finite. Hence, the
same holds for $G$ and $[p] : G \to G$ is finite and faithfully flat. Then, the same properties hold for $G^{an}$ and $U_G$.

(ii) Let $\Lambda = T_p G$ be the Tate module of $G$. By [19] théorèm 3.3, there exists a unique $\mathbb{C}$-linear morphism $f : \text{Lie} G \to \Lambda \otimes_{\mathbb{Z}_p} \mathbb{C}(-1)$ and a commutative diagram with exact rows ([19] proposition 16)

\[
\begin{array}{cccccc}
0 & \rightarrow & G[p^\infty] & \rightarrow & U_G & \rightarrow \mathbb{G}_a^{an} \otimes \text{Lie} G & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow f & & \downarrow f \\
0 & \rightarrow & \Lambda(-1) \otimes \mu_{p^\infty} & \rightarrow & \Lambda(-1) \otimes U_{G_m} & \rightarrow \mathbb{G}_a^{an} \otimes (\Lambda \otimes_{\mathbb{Z}_p} \mathbb{C}(-1)) & \rightarrow 0
\end{array}
\]

such that the top extension is isomorphic to the pullback of the lower one by $f$. If we consider the underlying points of the above sequence, we obtain

\[
\begin{array}{cccccc}
0 & \rightarrow & \lvert G[p^\infty] \rvert & \rightarrow & \lvert U_G \rvert & \rightarrow \text{Lie} G & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \boxed{f} & & \downarrow \boxed{f} \\
0 & \rightarrow & \Lambda(-1) \otimes_{\mathbb{Z}_p} \mu_{p^\infty} & \rightarrow & \Lambda(-1) \otimes_{\mathbb{Z}_p} (1 + m) \rightarrow \Lambda \otimes_{\mathbb{Z}_p} \mathbb{C}(-1) & \rightarrow 0
\end{array}
\]

where the right square is Cartesian. Then, the section $\text{Exp} : \mathbb{C} \rightarrow 1 + \mathfrak{m}$ of log induces an exponential map $\text{Exp}_G : \text{Lie} G \to \lvert U_G \rvert$. The continuity of $\text{Exp}_G$ follows from that of $\text{Exp}$ and of $f$. In view of the construction, the map $\text{Exp}_G$ is functorial on $G$. \hfill $\Box$

**Remark A.1.3.** The above result is generalized for $p$-divisible rigid groups in ([29] theorem 6.12).

**A.1.4.** Let $G$ be a commutative algebraic group over $\mathbb{C}$. Suppose that there exists a smooth group scheme $\mathcal{G}$ over $\mathfrak{o}$ with generic fiber $G$. Let $\mathcal{G} \rightarrow \mathcal{G}^{rig}$ be the $p$-adic completion of $\mathcal{G}$, $\mathcal{G}^{rig}$ the associated rigid group and $sp : \mathcal{G}^{rig} \rightarrow \mathcal{G}$ the specialization morphism. Then, we have ([19] Exemple 5.3(3)): $U_G = \bigcup_{n \geq 1} sp^{-1}(\mathcal{G}^{a}[p^n])$.

The Lie algebra $\text{Lie} \mathcal{G}$ of $\mathcal{G}$ is an $\mathfrak{o}$-lattice of $\text{Lie} G$. We abusively denote by $\text{Lie} \mathcal{G}$ the associated affine scheme over $\mathfrak{o}$. Let $\alpha$ be a positive rational number. Let $\mathcal{F}$ be the ideal sheaf of $\text{Lie} \mathcal{G}$ defined by the unit section over $\mathfrak{o}$ and $\text{Lie} \mathcal{G}^{\alpha} = \text{Spec}_o \mathcal{O}_{\text{Lie} \mathcal{G}}(\mathcal{F}_{\text{Lie} \mathcal{G}}[\frac{1}{p^\alpha}])$ the dilatation of $\text{Lie} \mathcal{G}$ of $\mathcal{F}$ with respect to $p^\alpha$. The rigid generic fiber $\text{Lie} \mathcal{G}^{\alpha, rig}$ of the $p$-adic completion of $\text{Lie} \mathcal{G}^{\alpha}$ is an open subspace of $\text{Lie} G \otimes \mathbb{G}_a^{an}$.

In this case, the $p$-adic logarithm $\log_{G^{\alpha}}$ is compatible by the logarithm of the formal group defined by the completion of $\mathcal{G}$ along the unit section ([50] §2.4, [19] exemple 5 (4)). When $\alpha > \frac{1}{p^\alpha}$, $\log_{G^{\alpha}}$ induces an isomorphism ([50] §2.4):

\[\log_{G^{\alpha}} : \text{Lie} \mathcal{G}^{\alpha, rig} \xrightarrow{\sim} \text{Lie} \mathcal{G}^{\alpha, rig}.\]

We denote the composition of the inverse of this map and the natural inclusion to $U_G$ by

\[ (A.1.4.1) \quad \exp_{G^{\alpha}} : \text{Lie} \mathcal{G}^{\alpha, rig} \rightarrow U_G.\]

The differential map of $\exp_{G^{\alpha}}$ induces the identity map on $\text{Lie} G$. 
A2. Construction of $f_{E,C,Exp}^\circ$. We keep the notation in the beginning of appendix A.

A2.1. We briefly review the notion of spectral cover over a curve following [8]. Let $L$ be a line bundle over $C$. We denote by $\text{HB}(C, L)$ the category of pairs $(M, \theta)$ consisting of a vector bundle $M$ and an $\mathcal{O}_C$-linear map $\theta : M \to M \otimes L$. Let $(M, \theta)$ be such a pair of rank $r$. For $1 \leq i \leq r$, the map $\theta$ induces a canonical morphism $\wedge^i M \to \wedge^i M \otimes L^{\otimes i}$ and we denote by $c_i(\theta)$ (or simply $c_i$, if there is no confusion) the image of $(-1)^i \text{id}_{\wedge^i M} \in \text{End}(\wedge^i M)$ in $\Gamma(C, L^{\otimes i})$ via the composition:

$$\theta^* + c_i \theta^{-1} + \cdots + c_r = 0 \in \text{Hom}(M, M \otimes L^{\otimes r}).$$

Then, $\theta$ satisfies the characteristic polynomial:

$$\theta^r + c_1 \theta^{r-1} + \cdots + c_r = 0 \in \text{Hom}(M, M \otimes L^{\otimes r}).$$

The above construction underlies to the Hitchin map $h$ sending each object $(M, \theta)$ of $\text{HB}(C, L)$ of rank $r$, to a point $(c_i(\theta))_{i=1}^r$ of the Hitchin base $B := (\oplus_{i=1}^r \Gamma(C, L^{\otimes i})) \otimes \mathbb{G}_m$. The spectral cover, reviewed below, implies that the Hitchin map is surjective. For a closed point $c = (c_i)_{i=1}^r$ of the Hitchin base $B$, we denote by $\mathcal{F}_c$ the ideal sheaf of $\text{Sym}_{\mathcal{O}_C}(L^{-1})$ locally generated by

$$(A2.1.1) \quad \{(a, c_1 \cdot a, \cdots, c_r \cdot a) | a \in L^{-\otimes r} \} \subset \oplus_{i=1}^r L^{-\otimes i} \oplus L^{-1} \oplus \mathcal{O}_C,$$

and we set

$$\mathcal{J}_c = \text{Sym}_{\mathcal{O}_C}(L^{-1})/\mathcal{F}_c.$$ 

The finite flat morphism $\pi_c : C_c := \text{Spec}(\mathcal{O}_C(\mathcal{J}_c)) \to C$ (resp. the algebra $\mathcal{A}_c$) is called spectral cover (resp. spectral algebra) of $c$. Since $\pi_c$ is finite flat, $\pi_c^*$, $\pi^{-1}_c$ induce an equivalence between invertible $\mathcal{A}_c$-modules over $C$ and invertible $\pi_c^{-1}(\mathcal{A}_c)(= \mathcal{O}_{C_c})$-modules over $C_c$. In particular, we have

$$(A2.1.2) \quad \text{H}^1(C, \mathcal{A}_c^\times) \simeq \text{H}^1(C_c, \mathcal{O}_{C_c}^\times).$$

An invertible $\mathcal{A}_c$-module defines an object $(M, \theta)$ of $\text{HB}(C, L)$ such that $h(M, \theta) = c$ ([8] proposition 3.6). On the other hand, for an object $(M, \theta)$ of $\text{HB}(C, L)$ with Hitchin image $c$, $M$ is equipped with an $\mathcal{A}_c$-module structure induced by $\theta$. The tensor product over $\mathcal{A}_c$ with an invertible $\mathcal{A}_c$-module defines a natural action on Higgs bundles with Hitchin image $c$. For such a Higgs bundle $(M, \theta)$, we also denote the associated ideal sheaf and the spectral algebra by $\mathcal{J}_\theta$ and $\mathcal{A}_\theta$ respectively.

Let $\Omega_C$ (resp. $T_C$) be the sheaf of differentials of $C$ over $\mathbb{C}$ (resp. tangent sheaf of $C$ over $\mathbb{C}$). We mainly work with $L = \xi^{-1} \Omega_C$, where an object $(M, \theta)$ of $\text{HB}(C, L)$ is the same as a Higgs bundle considered in § 2.2.

Lemma A2.2. Let $r \in \mathbb{N}$, $b = (b_i)_{i=1}^r \in \oplus_{i=1}^r \Gamma(C, \xi^{-1} \Omega_C^{\otimes i})$ a point of the Hitchin base, $C'_{f,b} \to C'$ the spectral cover of $c = f^*(b) \in \oplus_{i=1}^r \Gamma(C', \xi^{-1} f^*(\Omega_C^{\otimes i}))$ and $\mathcal{A}_c$ the spectral algebra of $c$ over $C'$.

(i) There exists a divisor $i : D \to C'$ such that $f^*(b)$ vanishes on the image of $D$.

(ii) The divisor $D$ of $C'$ can be uniquely lifted to a divisor $i : D \to C'_{f,b}$ such that, if we set $I = \text{Ker}(\mathcal{A}_c \to i_*(\mathcal{O}_D))$, the canonical morphism $\xi f^*(T_C) \to \mathcal{A}_c$ factors through a morphism of vector bundles over $C'$

$$\xi f^*(T_C) \to I.$$ 

Proof. (i) Since $f^*(\Omega_C^{\otimes r})$ has a positive degree (as $g(C) \geq 2$), the divisor of zeros of $f^*(b_i)$ is non-empty.

(ii) We may assume that $D = \text{Spec}(\mathbb{C})$ is a closed point. Then, $D \times C' C'_{f,b} \simeq \text{Spec}(\mathbb{C}[X]/P_D(X))$, where $P_D(X)$ is the restriction of $X^r + f^*(b_1)X^{r-1} + \cdots + f^*(b_r)$ to $D$ and $X$ is a local basis of $\xi f^*(T_C)$ around $D$. By (i), there is a unique surjection $\mathbb{C}[X]/P_D(X) \to \mathbb{C}$ sending $X$ to 0. The assertion follows. □

A2.3. We fix a divisor $i : D \to C'$ as in the above lemma and take its unique lift $i : D \to C'_{f,b}$. Since $g(C) \geq 2$ and $C$ is connected, we have $\Gamma(C', \xi f^*(T_C)) = 0$ and $\Gamma(C'_{f,b}, \mathcal{O}_{C'_{f,b}}) \simeq \mathbb{C}$. By ([46] corollaire 2.2.2), the divisor $i$ defines a rigidification of the Picard functor $\text{Pic}_{C'_{f,b}/\mathbb{C}}$ ([46] § 2.1). We consider the rigidified Picard functor $\mathcal{P}$, defined for any $\mathbb{C}$-scheme $T$ by

$$(A2.3.1) \quad \mathcal{P}(T) = \{(\mathcal{L}, \alpha) | \text{\mathcal{L} line bundle over } C'_{f,b,T}, \alpha : i_!\mathcal{L} \overset{\sim}{\rightarrow} i_!\mathcal{O}_{C'_{f,b,T}}\}.$$
where \(i_T : D_T \to C'_{f,T} \) denotes the base change of \(i \) by \(T \).

Then, \(\mathcal{P} \) is represented by a smooth algebraic group over \(\mathbb{C} \) ([46] proposition 2.4.3). Note that \(i^*(\mathcal{A}_c) \cong \mathcal{O}_D \) and \(I = \text{Ker}(\mathcal{A}_c \to i_*i^*(\mathcal{A}_c)) \). We have an exponential map (A.1.2.1):

\[
\text{Exp}_\mathcal{P} : \text{Lie}(\mathcal{P}(\cong H^1(C', I))) \to \mathcal{P}(\mathcal{C}).
\]

Flat liftings of \(f \) to a \(B^+_{\text{dR},2}\)-morphism from \(\mathcal{C}' \) to \(\mathcal{C} \) define a \(\xi f^*(T_{C'})\)-torsor \(\mathcal{L}_f \) over \(C' \). This torsor is unique up to a unique isomorphism as \(\Gamma(C', \xi f^*(T_{C'})) = 0 \). By pushout along \(\xi f^*(T_{C'}) \to I \) (lemma A.2.2) and taking \(\text{Exp}_\mathcal{P} \), we obtain an object

\[
(\mathcal{L}_{f,b}^{\text{Exp}}, \alpha) \in \mathcal{P}(\mathcal{C}).
\]

We abusively view \(\mathcal{L}_{f,b}^{\text{Exp}} \) as an invertible \(\mathcal{A}_c\)-module via direct image of \(\pi : C'_{f,b} \to C' \). If we set \(J = \text{Ker}(\mathcal{A}_c \to i_*i^*(\mathcal{A}_c)) \), we deduce from (A.2.1.2) that \(\mathcal{P}(\mathcal{C})(\cong H^1(C', J)) \).

**Proposition A.2.4.** (i) The line bundle \(\mathcal{L}_{f,b}^{\text{Exp}} \) on \(C'_{f,b} \) is independent of the choice of the rigidification \((i, D)\) in lemma A.2.2 up to a unique isomorphism.

(ii) Let \(\mathcal{C}' \to C' \) be a flat lifting of \(C'' \) to \(B^+_{\text{dR},2} \). Then, we have a canonical isomorphism of line bundles on \(C''_{f,g,b} \):

\[
\mathcal{E}_{f,g} : \mathcal{L}_{f,b}^{\text{Exp}} \cong \pi^*(\mathcal{L}_{g,f}^{\text{Exp}}(\mathcal{V})) \otimes (g \times \text{id}_{C''_{f,b}})^*(\mathcal{L}_{f,b}^{\text{Exp}}),
\]

where \(\pi^*(\mathcal{V}) \) is the pullback differential forms, \(\pi : C''_{f,g,b} \to C''_{f,b} \) is the canonical morphism induced by \(T_{C'} \to f^*(T_{C'}) \) and \(g \times \text{id}_{C''_{f,b}} : C''_{f,g,b} \to C''_{f,b} \) is the canonical morphism.

Proof. (i) Let \((i', D')\) be another rigidification satisfying conditions of A.2.2. \(\mathcal{P}' \) the associated rigidified Picard functor, \(I' = \text{Ker}(\mathcal{A}_c \to i^*i'_*(\mathcal{A}_c)) \), \(J' = \text{Ker}(\mathcal{A}_c \to i^*i'_*(\mathcal{A}_c)) \) and \((\mathcal{L}_{f,b}^{\text{Exp}}, \alpha') \) the associated rigidified line bundle. We may assume that \((i, D)\) is a sub-divisor of \((i', D')\). Then, the canonical morphism \(I' \to I, J' \to J \) induce compatible surjections \(\mathcal{P}' \to \mathcal{P} \) and \(\text{Lie} \mathcal{P}' \to \text{Lie} \mathcal{P} \). The canonical morphisms \(\xi^f(T_{C'}) \to I, \xi^f(T_{C'}) \to I' \) are compatible with \(I' \to I \). By the functoriality of \(\text{Exp}_\mathcal{P} \) (theorem A.1.2) and \(\Gamma(C', J) = 0 \), we obtain an identity in \(\mathcal{P}(\mathcal{C}) \):

\[
(\mathcal{L}_{f,b}^{\text{Exp}}, \alpha|_D) = (\mathcal{L}_{f,b}^{\text{Exp}}, \alpha).
\]

By forgetting the rigidifications, the assertion follows.

(ii) We lift \(i \) to a rigidification \(D \to C'_{f,b} \) of \(C'_{f,g,b} \) as in A.2.2. Let \(\mathcal{L}_{f,g} \) and \(\mathcal{L}_g \) be the torsor of liftings of \(f \circ g \) and of \(g \) to \(B^+_{\text{dR},2} \) respectively. We have a canonical isomorphism of \(\xi g^*f^*(T_{C'})\)-torsors over \(C''\):

\[
\mathcal{L}_{f,g} \cong (\mathcal{L}_g \otimes g^*(\mathcal{L}_f)),
\]

where \(\otimes \) denotes the pushout along \(\xi g^*f^*(T_{C'}) \to \xi g^*f^*(T_{C'}) \), and \(\otimes \) denotes the contracted product of \(\xi g^*f^*(T_{C'})\)-torsors ([10], § 2.3). By taking pushout of above isomorphism and exponential, we obtain an identity of line bundles equipped with rigidifications on \(C''_{f,b} \) as in (A.2.4.1). Then, we obtain the canonical isomorphism (A.2.4.1) by forgetting rigidifications.

Let \((M, \theta) \) be a Higgs bundle of rank \(r \) over \(C \). We also denote by \(C'_{f,\theta} \) (resp. \(\mathcal{L}_{f,\theta}^{\text{Exp}} \)) the associated rigidified Picard functor, \(C'_{f,b} \) (resp. line bundle \(\mathcal{L}_{f,b}^{\text{Exp}} \)) for \(b = h(M, \theta) \) (A.2.1). We set \(c = f^*(b) \in \oplus_{i=1}^r \Gamma(C', \xi^{-i}f^*(\Omega_C)^{\otimes i}) \).

**Proposition A.2.5.** (i) The association \((M, \theta) \to f^*(M, \theta) \otimes_{\mathcal{A}_c} \mathcal{L}_{f,\theta}^{\text{Exp}} \) defines a functor

\[
f_{\mathcal{E},\mathcal{L},\text{Exp}}^f : \text{HB}(C) \to \text{HB}(C', \xi^{-1}f^*(\Omega_C)),
\]

also denoted by \(f^\circ \) if there is no confusion. We define the twisted inverse image functor (A.0.0.3) by the composition of the above functor with \(\text{HB}(C', \xi^{-1}f^*(\Omega_C)) \to \text{HB}(C') \) induced by \(f^*(\Omega_C) \to \Omega_{C'} \).

(ii) The functor \(f^\circ \) (resp. functor (A.0.0.3)) is exact.

(iii) The underlying vector bundles of \(f^*(M, \theta) \) and \(f^\circ(M, \theta) \) have the same degree.
(iv) Under the assumption of A.2.4(ii), there exists a canonical isomorphism of functors (A.0.0.3):

\[(f \circ g)^! \simeq g^! \circ f^!\, .\]

Proof. (i-ii) Since \(\mathcal{L}_{f, \theta}^{\text{Exp}}\) is an invertible \(\mathcal{O}_\mathcal{X}\)-module, \(f^! (M, \theta) = f^*(M, \theta) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}_{f, \theta}^{\text{Exp}}\) is a Higgs bundle over \(\mathcal{X}'\). Let \(0 \to (M'', \theta'') \to (M, \theta) \to (M', \theta') \to 0\) be a short exact sequence of Higgs bundles over \(\mathcal{X}\). Since \(\theta\) and \(\theta'\) are compatible, the action of the spectral algebra \(\mathcal{A}_0\) on \(M\) induces an action on \(M'\), which is compatible with that of \(\text{Sym}_{\mathcal{O}_{\mathcal{X}}}(\xi T_{\mathcal{X}})\) via \(\theta'\). Then, we obtain a surjective homomorphism: \(\mathcal{A}_0 \to \mathcal{A}_0'\).

On the other hand, for the dual Higgs bundle \((M', \theta')\), we have \(c_1(\theta) = c_1(\theta')\) and \(\mathcal{A}_0 = \mathcal{A}_0'\) (A.2.1). By considering the dual of \((M'', \theta'') \to (M, \theta) \to (M', \theta') \to 0\) be the inverse image of the above sequence to \(\mathcal{X}'\) via \(f^\circ\). In view of the construction, we have canonical isomorphisms of \(\mathcal{A}_0\)-modules (resp. \(\mathcal{A}_0'\)-modules) on \(\mathcal{X}'\):

\[(A.2.5.2) \quad \mathcal{A}_0^\circ \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}_{f, \theta}^{\text{Exp}} \simeq \mathcal{L}_{f, \theta}^{\text{Exp}}, \quad \mathcal{A}_0'^\circ \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}_{f, \theta}^{\text{Exp}} \simeq \mathcal{L}_{f, \theta}^{\text{Exp}}.\]

By applying \(- \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}_{f, \theta}^{\text{Exp}}\), we deduce the following short exact sequence of Higgs bundles:

\[0 \to f^!(M'', \theta'') \to f^!(M, \theta) \to f^!(M', \theta') \to 0.\]

In this way, we obtain the construction of \(f^!(u)\) for a morphism \(u\) of \(\text{HB}(\mathcal{C})\), whose cokernel is still a Higgs bundle. We also show that \(f^\circ\) is exact, i.e., assertion (ii).

If \(u : (M_1, \theta_1) \to (M_2, \theta_2)\) is a monomorphism of Higgs bundles whose quotient is a torsion \(\mathcal{O}_\mathcal{C}\)-modules. Since \(u\) is an isomorphism on an open dense subset of \(\mathcal{C}\), we deduce that \(h(M_1, \theta_1) = h(M_2, \theta_2) = b\). Since \(f\) is flat, this allows us to define a monomorphism \(f^\circ(u) : f^!(M_1, \theta_1) \to f^!(M_2, \theta_2)\) by tensor with \(\mathcal{L}_{f, \theta}^{\text{Exp}}\).

Since \(\dim \mathcal{C} = 1\), a coherent \(\mathcal{O}_\mathcal{C}\)-module can be decomposed as a direct sum of a vector bundle over \(\mathcal{C}\) and a torsion \(\mathcal{O}_\mathcal{C}\)-module. From previous constructions, we establish the functoriality of \(f^\circ\).

(iii) Let \(\text{HB}(\mathcal{C}', \xi^{-1} f^*(\Omega_{\mathcal{C}}))\) be the algebraic stack of objects of \(\text{HB}(\mathcal{C}', \xi^{-1} f^*(\Omega_{\mathcal{C}}))\) of rank \(r\) ([35] § 1), \(\text{Pic}(\mathcal{A}_\mathcal{C})\) the stack of line bundles on \(\mathcal{C}'_{f, b}\). Since the connected components of the moduli stack of vector bundles are classified by their degrees, objects in the same connected component of \(\text{HB}(\mathcal{C}', \xi^{-1} f^*(\Omega_{\mathcal{C}}))\) have the same degrees. The tensor product on the spectral curve defines a natural action of \(\text{Pic}(\mathcal{A}_\mathcal{C})\) on the fiber \(\text{HB}(\mathcal{C}', \xi^{-1} f^*(\Omega_{\mathcal{C}}))_{c} = \text{HB}(\mathcal{C}', \xi^{-1} f^*(\Omega_{\mathcal{C}})) \times_{h, b} c\) of the Hitchin map \(h\) over \(c\) ([42] § 4).

By construction, there exits an integer \(\geq 1\) such that \(\mathcal{L}_{f, \theta}^{\text{Exp}}\) lies in the neutral component of \(\mathcal{D}\). By ([11] §9 corollary 14), \(\mathcal{L}_{f, \theta}^{\text{Exp}}\) lies in the neutral component of the relative Picard scheme \(\text{Pic}_{\mathcal{C}'_{f, b}/\mathcal{C}}\) and hence in that of \(\text{Pic}(\mathcal{A}_\mathcal{C})\). Then, \(f^!(M, \theta)\) and \(f^!(M, \theta)\) lie in the same connected component of \(\text{HB}(\mathcal{C}', \xi^{-1} f^*(\Omega_{\mathcal{C}}))\). Hence their underlying vector bundles have the same degree.

(iv) Proposition A.2.5(iv) follows from proposition A.2.4(ii). \(\square\)

Remark A.2.6. For a torsion coherent \(\mathcal{O}_\mathcal{C}\)-module \(M\) with a Higgs field \(\theta\), we can canonically define the twisted inverse image \(f^\circ(M, \theta)\) by its usual inverse image \(f^*(M, \theta)\). Together with the above construction, we can slightly extend \(f^\circ\) to the category of coherent \(\mathcal{O}_\mathcal{C}\)-modules with a Higgs field.

Proposition A.2.7. The functor \(f^\circ\) sends a Higgs line bundle \((L, \theta)\) to \((f^*(L) \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{L}_{f, \theta}^{\text{Exp}}, f^*(\theta))\).

Proof. Note that \(\theta\) is a section of \(\Gamma(\mathcal{C}, \xi^{-1}\Omega_{\mathcal{C}})\). Then, \(\mathcal{A}_{f^\circ(\theta)}\) is isomorphic to \(\mathcal{O}_\mathcal{C}\) and an invertible \(\mathcal{A}_{f^\circ(\theta)}\)-module is equivalent to a line bundle over \(\mathcal{C}'\) together with the Higgs field \(f^*(\theta) \in \Gamma(\mathcal{C}', \xi^{-1} f^*(\Omega_{\mathcal{C}}))\). The tensor product of invertible \(\mathcal{A}_{f^\circ(\theta)}\)-modules is the same as the tensor product of underlying line bundles together with the Higgs field \(f^*(\theta)\). Then, the proposition follows. \(\square\)

A.3. Twisted inverse image for small Higgs bundles via exponential. In the following, we discuss the twisted inverse image functor for small Higgs bundles. Let \(X\) be a semi-stable \(S\)-curve (1.2.3) such that \(X_{\mathcal{C}}\) has genus \(\geq 2\) and \(\mathcal{X}\) the \(p\)-adic completion of \(X_{\mathcal{C}} = X \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{O}\) equipped with the fine log structure induced by \(\mathcal{A}_X\). We simply write \(\Omega^1_{\mathcal{X}/\Sigma_{1,s}}\) as \(\Omega_{\mathcal{X}}\) and we denote its dual by \(T_{\mathcal{X}}\).

Here is a criterion for the smallness of a Higgs bundle in terms of its image in the Hitchin base.
Proposition A.3.1. A Higgs bundle $(M, \theta)$ of rank $r$ over $X_C$ is small (2.2.10) if and only if there exists a rational number $\alpha > \frac{1}{p-1}$ such that $c_i(\theta) \in p^{r+1} \Gamma(X, \xi^{-i} \Omega^*_X) \subset \Gamma(X_C, \xi^{-i} \Omega^*_X)$ for every $1 \leq i \leq r$.

Proof. Assume $(M, \theta)$ is small. There exists a coherent $\mathcal{O}_X$-module $\mathcal{M}$ with generic fiber $M$ and a rational number $\alpha > \frac{1}{p-1}$ such that $\theta(\mathcal{M}) \subset p^r \mathcal{M} \otimes \mathcal{O}_X \xi^{-1} \Omega^*_X$. For each $i \geq 1$, $\wedge^i \theta$ sends $\wedge^i \mathcal{M}$ to $p^{r+1} \wedge^i \mathcal{M} \otimes \mathcal{O}_X \xi^{-i} \Omega^*_X$.

Then, we deduce $c_i(\theta) \in p^{r+1} \Gamma(X, \xi^{-i} \Omega^*_X)$.

On the other hand, assume $c_i(\theta) \in p^{r+1} \Gamma(X, \xi^{-i} \Omega^*_X)$ for every $1 \leq i \leq r$. We take local sections $m$ of $M$, $d \log(x)$ of $\Omega^*_X$ and its dual $\partial$ of $T_X$. And we write

$$\theta(m) = \partial(m) \otimes \xi^{-1} d \log(x).$$

Locally, let $\mathcal{N}$ be the $\mathcal{O}_X$-submodule of $\mathcal{M}$ generated by $\{m, \partial(m), \ldots, (\partial(m))^{-1}(m)\}$. In view of the characteristic polynomial of $\theta$ and the assumption, for $n \geq r$, the section $(\partial(m))^n(m)$ belongs $p^{n-r} \mathcal{N}$. We take a positive integer $s$ such that $\alpha > \frac{1}{s} + \frac{1}{p-1}$. Then, we deduce that $p^{-s-1}(\partial(m))^n(m)/n!$ tends to zero, when $n$ tends to infinity, i.e. $(M, \theta)$ satisfies (Conv)\_$_x$ (2.2.5).

By ([4] IV.3.6.6), the Higgs field $\theta$ is small. $\square$

A.3.2. Let $K'$ be a finite extension of $K$, $Y$ a semi-stable $S'(= Spec(\mathcal{O}_{K'}))$-curve, $f : Y \to X_{S'}$ a generic $\eta'$-cover (1.2.3) and $\tilde{f} : \tilde{Y} \to X$ the associated morphism of $p$-adic log formal $\mathfrak{o}$-schemes. Let $\mathcal{Y}$ be a smooth lifting of $Y$ over $\Sigma_{S'}$. Let $\alpha \in \mathbb{Q}_{>0} / \mathbb{Z}$ be a rational number, $b = (b_i)_{i=1}^{r} \in \oplus_{i=1}^{r} p^{i} \Gamma(X, \xi^{-i} \Omega^*_X)$ and $c = f^\ast(b) = \oplus_{i=1}^{r} p^{i} \Gamma(\tilde{Y}, \xi^{-i} f^\ast(\Omega^*_X))$.

We denote by $\mathcal{J}_c$ the ideal sheaf of $\text{Sym}^{\bullet}_{\mathcal{O}_{\tilde{Y}}}(p^{-\alpha} \xi f^\ast\Omega^*_X)$ defined by a similar formula of (A.2.1.1) with coefficients $c$. We denote by $\mathcal{A}_c$ the finite, locally free $\mathcal{O}_{\tilde{Y}}$-algebra

$$\mathcal{A}_c = \text{Sym}^{\bullet}_{\mathcal{O}_{\tilde{Y}}}(p^{-\alpha} \xi f^\ast\Omega^*_X) / \mathcal{J}_c.$$

Flat liftings of $\tilde{f}$ to a $\Sigma_{2,S}$-morphism $\mathcal{Y} \to \mathcal{X}$ defines a $\xi f^\ast(\Omega^*_X)$-torsor $\mathcal{L}_f$ on $\tilde{Y}$. We set $\hat{Y}_{f,b} = \text{Spf}_{\mathcal{O}_{\tilde{Y}}}(\mathcal{A}_c)$ and consider the following composition:

$$\begin{align*}
\text{H}^1(\tilde{Y}, \xi f^\ast(\Omega^*_X)) & \to \text{H}^1(\tilde{Y}, p^r \mathcal{A}_c) \xrightarrow{\text{exp}} \text{H}^1(\tilde{Y}, (\mathcal{A}_c)^\times) \xrightarrow{(\varphi_{ij})} \text{H}^1(\hat{Y}_{f,b}, \mathcal{A}_c^\times), \\
\varphi = (\varphi_{ij}) & \mapsto \exp(\varphi) = (\exp(\varphi_{ij}))
\end{align*}$$

where the second map is well-defined due to $\alpha > \frac{1}{p-1}$, and the isomorphism can be verified in a similar way as in (A.2.1.2). By applying the above composition to a Čech cocycle of $[\mathcal{L}_f] \in \text{H}^1(\tilde{Y}, \xi f^\ast(\Omega^*_X))$, we obtain an invertible $\mathcal{A}_c$-module $\mathcal{L}_{f,b}$.

Proposition A.3.3. (i) The line bundle $\mathcal{L}_{f,b}$ is independent of the choice of the Čech cocycle of $[\mathcal{L}_f] \in \text{H}^1(\tilde{Y}, \xi f^\ast(\Omega^*_X))$ up to a unique isomorphism.

(ii) The special fiber of the line bundle $\mathcal{L}_{f,b}$ over $\tilde{Y}_{f,b}$ is trivial.

Proof. (i) Since $X_C$ has genus $\geq 2$, $\Gamma(Y, \xi f^\ast(\Omega^*_X))$ is a $p$-torsion-free submodule of $\Gamma(Y_C, f_C^\ast(\Omega^*_X_C))$ and therefore vanishes. Hence, two Čech cocycles of $\mathcal{L}_f$ is different by a unique Čech 1-coboundary $(g_i)$. Then, $\exp(g_i)$ defines a unique isomorphism between $\mathcal{L}_{f,b}$ defined by two Čech cocycles.

(ii) The line bundle $\mathcal{L}_{f,b}$ can be described as $\exp(\varphi_{ij})$ in (A.3.2.1). Since $p^n$ divides each $\varphi_{ij}$ and $\alpha > \frac{1}{p-1}$, the convergent series $\exp(\varphi_{ij}) \equiv \text{id}$ modulo the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$. Then, the assertion follows. $\square$

Proposition A.3.4. Let $(M, \theta)$ be an object of $\text{HB}_{\mathfrak{b}, \text{small}}(\tilde{X}/\Sigma_{1,s})$ (definition 2.2.6) with Hitchin image $b \in \oplus_{i=1}^{r} p^{i} \Gamma(\tilde{X}, \xi^{-i} \Omega^*_X)$. Via $\xi f^\ast(\Omega^*_X) \to \Omega^*_X$, the correspondence $(M, \theta) \mapsto f^\ast(M, \theta) \otimes \mathcal{A}_c \mathcal{L}_{f,b}$ defines the twisted inverse image functor for small Higgs bundles:

$$f^\ast_{\mathcal{X}, \mathcal{Y}} : \text{HB}_{\mathfrak{b}, \text{small}}(\tilde{X}/\Sigma_{1,s}) \to \text{HB}_{\mathfrak{b}, \text{small}}(\tilde{Y}/\Sigma_{1,s'}).$$
Proof. By proposition A.3.1, \( f_{\Y,\X}^*(M, \theta) \) is still small.

As in (A.2.5.2), a short exact sequence of small Higgs bundles \( 0 \to (M'', \theta'') \to (M, \theta) \to (M', \theta') \to 0 \) induces canonical isomorphisms of \( \mathcal{A}_c \)-modules, \( \mathcal{A}_{c'} \)-modules respectively:

\[
\mathcal{L}_{f,b} \otimes \mathcal{A}_c \simeq \mathcal{L}_{f,b'}, \quad \mathcal{L}_{f,b} \otimes \mathcal{A}_{c'} \simeq \mathcal{L}_{f,b'}.
\]

Then, we verify the functoriality of \( f_{\Y,\X}^* \) as in proposition A.2.5. \( \square \)

**Proposition A.3.5.** (i) By restricting \( f_{\Y,\X}^* \) (2.3.4.2) to small Higgs bundles, there exists a canonical isomorphism of functors \( \Psi_f : f_{\Y,\X}^* \simeq f_{\Y,\X}^* \).

(ii) Moreover, \( \Psi_f \) satisfies a cocycle condition as in (4.2.4.1).

**Proof.** We first make an explicit description of \( \mathcal{L}_{f,b} \). Let \( \{U_i\}_{i \in I} \) be an affine open coverings of \( \tilde{Y} \) and \( \mathcal{U}_i \to \tilde{Y} \) the associated open formal subschemes. We take a family of local liftings \( \tilde{f}_i : \mathcal{U}_i \to \X \) of \( f \). The difference between \( \tilde{f}_i \) and \( \tilde{f}_j \) (modulo \( \xi^2 \)) on the intersection \( \mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j \) defines a section

\[
\varphi_{ij} \in \text{Hom}(\tilde{f}_i^*(\Omega_{\X})|_{\mathcal{U}_{ij}}, \xi^2|_{\mathcal{U}_{ij}}).
\]

The cocyle \( \varphi = (\varphi_{ij}) \) defines the class \( [\mathcal{L}] \in H^1(\tilde{Y}, \xi f^*(T_\X)) \). The invertible \( \mathcal{A}_c \)-module \( \mathcal{L}_{f,b} \) is equivalent to \( \{\mathcal{A}_c|_{U_i}\}_{i \in I} \) equipped with the gluing data \( \{\Phi_{ij} = \text{id} \otimes \exp(\varphi_{ij})\}_{i,j \in I} \).

We may assume that each \( \mathcal{U}_i \) is affine and there exists \( x \in \mathcal{U}_{ij} \) such that \( \log(x_N) \) of \( \Omega_{\mathcal{U}_{ij,N}/\Sigma_{1,S}} \) for \( N \geq 1 \) as in 2.3.5. Then, we have

\[
\varphi_{ij}(d\log(x_1)) = \xi \left( \tilde{f}_j^*(x_2) \tilde{f}_j^*(x_2)^{-1} - 1 \right).
\]

As an isomorphism of \( \mathcal{O}_{\mathcal{U}_{ij}} \)-modules, the transition isomorphism \( \Phi_{ij} : \tilde{f}_i^*(M)|_{\mathcal{U}_{ij}} \simeq \tilde{f}_j^*(M)|_{\mathcal{U}_{ij}} \) is given by:

\[
\tilde{f}_i^*(m) \mapsto \sum_{n \geq 0} \tilde{f}_i^*(\theta^{\varphi_{ij}}(m)) \otimes \frac{1}{n!} \left( \tilde{f}_j^*(x_2) \tilde{f}_j^*(x_2)^{-1} - 1 \right)^n \xi.
\]

which coincides with the transition isomorphism of \( f_{\Y,\X,(\tilde{f}_j)_{j \in J}}^*(M, \theta) \) (2.3.5.1) and we obtain

\[
\Psi_{(\tilde{f}_j)_{j \in J}} : f_{\Y,\X,(\tilde{f}_j)_{j \in J}}^*(M, \theta) \simeq f_{\Y,\X,(\tilde{f}_j)_{j \in J}}^*(M, \theta).
\]

By (A.3.4.2), the transition isomorphisms \( \Phi_{ij} \) are functorial. Hence \( \Psi_{(\tilde{f}_j)_{j \in J}} \) defines an isomorphism of functors \( f_{\Y,\X}^*(M, \theta) \) and \( f_{\Y,\X,(\tilde{f}_j)_{j \in J}}^*(M, \theta) \).

(A.4.4.1) The cocycle condition follows from the cocycle condition of the Higgs stratification.

**Comparison of twisted inverse image functors.**

**A.4.1.** We set \( S^0 = \text{Spa}(\mathcal{C}, \mathfrak{c}) \) and \( S = \text{Spa}(B_{\text{Adic}}^+, \mathcal{A}_{\text{Adic}}) \), which are strongly Noetherian (lemma A.4.2).

Then, there exists a canonical morphism of locally ringed spaces induced by the identity map of \( B_{\text{Adic}}^+ \):

\[
\varphi : S \to \text{Spec}(B_{\text{Adic}}^+).
\]

Following [58] definition 6.1, a *relative analytification* of a locally finite type \( B_{\text{Adic}}^+ \)-scheme \( X \) is an adic \( S \)-space \( X^{\text{an}}/S \to S \) with a morphism of locally ringed \( \phi_X : X^{\text{an}}/S \to X \) above \( \varphi \) such that, for every adic \( S \)-space \( U, \phi_X \) induces a bijection:

\[
\text{Map}_{\text{Adic}}/S(U, X^{\text{an}}/S) \simeq \text{Map}_{\text{LRS}}/\text{Spec}(B_{\text{Adic}}^+)(U, X).
\]
Here Adic/S denotes the category of adic spaces over S and LRS / Spec(B_{dr,2}^+) denotes the category of locally ringed spaces over Spec(B_{dr,2}^+).

Such a relative analytification exists ([33] proposition 3.8) and is unique. When X is a locally of finite type C-scheme, the relative analytification is compatible with the anlytification functor to rigid spaces over C in the classical sense.

Let C be a smooth proper curve over C and C^{an} the associated rigid curve over S^o. Then, the relative analytification over B_{dr,2}^+ induces a natural map:

\{\text{flat liftings of } C \text{ over } B_{dr,2}^+\} \to \{\text{flat liftings of } C^{an} \text{ over } S\}.

Note that both side is a torsor under the cohomology group H^1(C; \xi T_C) (\simeq H^1(C^{an}; \xi T_{C^{an}})). Then, the above map is a bijection. In particular, any flat lifting of C^{an} over S comes from an algebraic lifting of C.

**Lemma A.4.2.** The ring B_{dr,2}^+ (equipped with the p-adic topology) is strongly Noetherian ([33] § 2).

**Proof.** We prove the lemma using the fact that C is strongly Noetherian. Let n \geq 1 be an integer and I an ideal of B_{dr,2}^+(T_1, \ldots, T_n). The image \overline{I} of the composition I \to B_{dr,2}^+(T_1, \ldots, T_n) \to C(T_1, \ldots, T_n) (as B_{dr,2}^+(T_1, \ldots, T_n)-modules) is an ideal of C(T_1, \ldots, T_n) and is therefore finitely generated over C(T_1, \ldots, T_n).

The kernel Ker(I \to \overline{I}) is a submodule of \xi B_{dr,2}^+(T_1, \ldots, T_n) and is finitely generated over B_{dr,2}^+(T_1, \ldots, T_n). Then, I is generated by liftings of generators of I and Ker(I \to \overline{I}). Hence the lemma follows. \qed

**A.4.3.** We keep the notation and assumption of § A.3.2. The adic spaces \Xi = X^{rig}_2, \mathcal{Y} = Y^{rig}_2 over Spa(B_{dr,2}^+, A_{inf,2}) associated to formal schemes \mathcal{X}_2, \mathcal{Y}_2 are flat liftings of \hat{X}^{rig}, \hat{Y}^{rig} respectively. By § A.4.1, \hat{X}^{rig}, \hat{Y}^{rig} and their liftings are algebrizable. Then, we have a twisted inverse image functor (A.2.5.1):

\begin{equation}
\text{Proposition A.4.4. Assume moreover that } X \text{ is a stable } S\text{-curve.}
\end{equation}

(i) Via the equivalence (2.2.10.1), there exists a canonical isomorphism \Psi_f between the restriction of functor f_{Y,X}^0 \exp on the category of small Higgs bundles and the functor f_{Y,X}^0 \exp (A.3.4.1).

(ii) For morphisms f, g between stable S, \Psi_f satisfies a cocycle condition as in (4.2.4.1).

**A.4.5.** We need to investigate the integral structure of the rigidified Picard functor \mathcal{P} (A.2.3.1). The \mathcal{O}_{\mathcal{Y}}-algebra \mathcal{A}_c comes from the p-adic completion of a finite locally free \mathcal{O}_{\mathcal{Y}}-algebra, denoted by \mathcal{A}_c. We set Y_{f,b} = Spec(\mathcal{O}_{\mathcal{Y}}(\mathcal{A}_c)) (whose p-adic formal completion is \hat{Y}_{f,b}) and take a divisor i : D = \square Spec(\mathcal{o}) \to \hat{Y}_{f,b} such that its generic fiber i_{\mathcal{C}} satisfies conditions in §A.2.2.

**Proposition A.4.6.** (i) The \mathcal{O}_{\mathcal{Y}_s}-module \mathcal{A}_c is cohomologically flat (of dimension 0) over \mathcal{o}.

(ii) The divisor (i, D) is a rigidification of the relative Picard functor Pic_{\mathcal{A}_c/\mathcal{o}} ([46] §2.1).

**Proof.** (i) Recall that \mathcal{A}_c \simeq \bigoplus_{i=0}^{r-1} p^{-i}\xi f_i^*(\hat{T}_{\mathcal{X}_s}) as \mathcal{O}_{\mathcal{Y}_s}-modules, where \hat{T}_{\mathcal{X}_s} is the dual of the dualizing sheaf \omega_{\mathcal{X}_s/S} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{o}. Since \mathcal{Y} is semi-stable over \mathcal{S}, \mathcal{O}_{\mathcal{Y}_s} is cohomologically flat along \pi : \mathcal{Y} \to \text{Spec}(\mathcal{o}).

Set \mathcal{L} = p^{-\alpha}\xi f_i^*(\hat{T}_{\mathcal{X}_s}). We will show that \pi_*(\mathcal{L}^{\otimes i}) = 0 for \alpha \geq 1 after any base change \mathcal{S} \to \text{Spec}(\mathcal{o}). By ([40] II.5 corollary 2), it suffices to show that for any point s \to \text{Spec}(\mathcal{o}), \Gamma(Y_s, L_s^{\otimes i}) = 0. And we may reduce to the case s is the closed point of the generic point of \text{Spec}(\mathcal{o}). In either case, the vanishing of \Gamma(Y_s, L_s^{\otimes i}) follows from the fact that \omega_{\mathcal{X}_S/s} is an ample line bundle on \mathcal{X}_s ([36] corollary 10.3.13).

Assertion (ii) follows from the vanishing results in (i) and ([46] corollaire 2.2.2). \qed

**A.4.7.** The rigidified Picard functor \mathcal{P} associated with \hat{Y}_{f,b} and D, is defined, for any \mathcal{o}-scheme \mathcal{T}, by

\begin{equation}
\text{The functor } \mathcal{P} \text{ is represented by a smooth algebraic space locally of finite presentation over } \mathcal{o} ([46] théorème 2.3.1 et corollaire 2.3.2). The generic fiber } \mathcal{P}_C \text{ of } \mathcal{P} \text{ is denoted by } \mathcal{P} \text{ in (A.2.3.1).}
Let $T$ be an affine $\mathfrak{o}$-scheme. We set $A_{c,T} = A_c \otimes \mathfrak{o} \mathcal{O}_T$, $I_T = \text{Ker}(A_{c,T} \to i_{T,*}i_T^*(A_{c,T}))$, $J_T = \text{Ker}(A^\times_{c,T} \to i_{T,*}i_T^*(A^\times_{c,T}))$. By considering the long exact sequences and ([46] corollaire 2.2.2), we deduce $H^0(Y_T, I_T) = 0$, $H^0(Y_T, J_T) = 0$, and short exact sequences:

\begin{equation}
A.4.7.2 \quad 0 \to H^0(D_T, \mathcal{O}_{D_T}) / H^0(Y_T, A_{c,T}) \to H^1(Y_T, I_T) \to H^1(Y_T, A_{c,T}) = 0,
\end{equation}

\begin{equation}
A.4.7.3 \quad 0 \to H^0(D_T, \mathcal{O}_{D_T}^\times) / H^0(Y_T, A^\times_{c,T}) \to H^1(Y_T, J_T) \to H^1(Y_T, A^\times_{c,T}) = H^1(D_T, \mathcal{O}_D^\times).
\end{equation}

By ([11] 8.4 theorem 1) and ([46] proposition 2.4.1), the Lie algebra of $\mathcal{P}$ is isomorphic to the functor:

\[ T \mapsto H^0(Y_T, I_T)(\simeq H^0(T, \pi_{T,*}(I_T))). \]

Moreover, by proposition A.4.6 and ([40] II.5 corollaries 1, 2), we deduce that $\text{Lie } \mathcal{P} \simeq R^1 \pi_{T,*}A_c$ is represented by a free $\mathfrak{o}$-module of finite rank.

By (A.4.7.3) and ([46] § 1.2, proposition 2.4.1), the functor $\text{Pic}_{A_c/\mathfrak{o}}$ (resp. $\mathcal{P}$) is isomorphic to the sheaf associated to the presheaf $T \mapsto H^1(Y_T, A^\times_{c,T})$ (resp. $H^1(Y_T, J_T)$) for the big Zariski topology. In particular, we have a natural map:

\[ H^1(Y_T, J_T) \to \mathcal{P}(T), \]

which is an isomorphism when $T = \text{Spec}(\mathfrak{o}_n)$ or $\text{Spec}(C)$ in viewed of ([46] proposition 2.1.2 c).

**Lemma A.4.8.** (i) We have $H^1(Y_T, A^\times_{c,T}) \simeq H^1(Y_{f,b,T}, \mathcal{O}_{Y_{f,b,T}}^\times)$.

(ii) Let $\hat{I} = \lim\limits_{\leftarrow n} I_{n\alpha}$, $\hat{J} = \lim\limits_{\leftarrow n} J_{n\alpha}$ be their $p$-adic completion on $\hat{Y}$. We have canonical isomorphisms:

\[ H^1(Y_\alpha, J_\alpha) \simeq H^1(\hat{Y}, \hat{J}) \simeq \lim\limits_{\leftarrow n} H^1(Y_{n\alpha}, J_{n\alpha}). \]

**Proof.** (i) Since $Y_{f,b} \to Y$ is finite flat, the assertion follows from the same proof of (A.2.1.2).

(ii) We first prove the assertion replacing $J$ by $A^\times_c$. By ([1] 2.8.9, 2.13.8), we have natural isomorphisms $H^1(Y_{f,b}, \mathcal{O}_{f,b,T}^\times) \simeq H^1(Y_{f,b,\alpha}, \mathcal{O}_{f,b,\alpha}^\times) \simeq \lim\limits_{\leftarrow n} H^1(Y_{f,b,\alpha_n}, \mathcal{O}_{f,b,\alpha_n}^\times)$. By (i), we obtain natural isomorphisms

\begin{equation}
A.4.8.1 \quad H^1(Y_\alpha, A^\times_c) \simeq H^1(\hat{Y}, A^\times_c) \simeq \lim\limits_{\leftarrow n} H^1(Y_{n\alpha}, A^\times_{c,n\alpha}).
\end{equation}

The transition map $J_{n\alpha} \to J_{\alpha}$ is surjective and hence $R^1 \lim\limits_{\leftarrow n} J_{n\alpha} = 0$. We have a short exact sequence:

\[ 0 \to \hat{J} \to A^\times_c \to \lim\limits_{\leftarrow n} J_{n\alpha} \to 0. \]

Note that $H^1(D_T, \mathcal{O}_{D_T}^\times) = 0$ for $T = \text{Spec}(\mathfrak{o})$ and $H^1(D, \mathcal{O}_D^\times) = 0$. By comparing the long exact sequence associated to the above sequence and (A.4.7.3), we deduce isomorphisms in (ii) from (A.4.8.1). \( \square \)

**A.4.9.** For $\alpha > \frac{1}{p^n}$ and $n \geq 1$, we have the exponential map:

\[ \exp_{\mathcal{P}, n} : H^1(Y_{n\alpha}, p^n I_{n\alpha}) \to H^1(Y_{n\alpha}, J_{n\alpha}), \quad \varphi = (\varphi_{i,j}) \mapsto \exp(\varphi) = (\exp(\varphi_{i,j})), \]

where $p^n I_{n\alpha} = \text{Im}(p^n : I_{n\alpha} \to I_{n\alpha})$ and the above exponential is a finite sum.

We have $H^1(\hat{Y}, p^n \hat{T}) \simeq \lim\limits_{\leftarrow n} H^1(Y_{n\alpha}, p^n I_{n\alpha})$ ([1] 2.12.3). By lemma A.4.8, we obtain the exponential map

\begin{equation}
A.4.9.1 \quad \exp_{\mathcal{P}} : H^1(\hat{Y}, p^n \hat{T}) \to H^1(\hat{Y}, \hat{J}), \quad \varphi = (\varphi_{i,j}) \mapsto \exp(\varphi) = (\exp(\varphi_{i,j})).
\end{equation}

As in lemma A.2.2, the natural map $\xi f^* (T_X) \to p^n A_c$ factors through $p^n \hat{T} \subset p^n A_c$. We apply the composition $H^1(\hat{Y}, \xi f^* (T_X)) \to H^1(\hat{Y}, p^n \hat{T})$ with $\exp_{\mathcal{P}}$ to a Šilov cocycle of $L_f$ (A.3.2) and obtain an object

\[ (L_{f,b}, \alpha) \in \lim\limits_{\leftarrow n} \mathcal{P}(\mathfrak{o}_n), \]

that is a line bundle $L_{f,b}$ on $\hat{Y}_{f,b}$ with a compatible system of rigidifications $\alpha = (\alpha_n)_{n\geq 1}$. 
Proposition A.4.10. (i) The line bundle $L_{f,b}$ is independent of the choice of (i, $D$) and of Čech cocycle of $\mathcal{L}_f$ up to a unique isomorphism.

(ii) There exists a unique isomorphism between line bundles $L_{f,b}$ and $\mathcal{L}_{f,b}$ (A.3.3) on $\hat{Y}_{f,b}$.

Proof. Assertion (i) can be verified in the same way as in propositions A.2.4 and A.3.3.

Assertion (ii) follows from the fact that $\text{exp}_{\mathcal{P}}$ is compatible with (A.3.2.1) via forgetting rigidification. □

Proposition A.4.11. The exponential map $\text{exp}_{\mathcal{P}}$ is compatible with the Exponential map of $\mathcal{P}_C$ (A.1.2.1)

\begin{equation}
\text{exp}_{\mathcal{P}_C} : \text{Lie} \mathcal{P}_C(\mathbb{C}) \simeq H^1(Y_C, I_C) \to |U_{\mathcal{P}_C}|(\subset \mathcal{P}_C(\mathbb{C}) \simeq H^1(Y_C, J_C)).
\end{equation}

Proof. Let $\mathcal{E}$ be the scheme-theoretic closure of the unit section of $\mathcal{P}_C$ in $\mathcal{P}$ ([46] §3.2 c)). Then, $\mathcal{E}$ is a sub-algebraic space of groups of $\mathcal{P}$. Moreover, the quotient $\mathcal{G} = \mathcal{P}/\mathcal{E}$ is represented by a smooth group scheme over $\mathfrak{o}$ ([46] proposition 3.3.5). By proposition A.4.6 and ([46] proposition 5.2), we have $\mathcal{E}^0 \simeq 0$ and hence $\text{Lie} \mathcal{E} = 0$. We deduce an isomorphism $\text{Lie} \mathcal{P} \xrightarrow{\sim} \text{Lie} \mathcal{G}$, which is isomorphic to an affine space over $\mathfrak{o}$.

Let $n$ be an integer $\geq 1$ and $A$ an $\mathfrak{o}_n$-algebra. Then, $\text{Lie} \mathcal{G}(A)$ is a $\mathfrak{o}$-module. The image of $\text{Lie} \mathcal{G}^\mathfrak{o}(A) \to \text{Lie} \mathcal{G}(A)$ (§A.1.4) is contained in $p^n \text{Lie} \mathcal{G}(A)$ (the image of multiplication by $p^n$). This allows us to define a homomorphism by the exponential

\begin{equation}
\exp^\mathfrak{o} : \text{Lie} \mathcal{G}^\mathfrak{o}(A) \to p^n \text{Lie} \mathcal{G}(A) \simeq p^n H^1(Y_A, I_A) \to H^1(Y_A, J_A) \to \mathcal{P}(A),
\end{equation}

\begin{equation}
\varphi = (\varphi_{i,j}) \mapsto \exp(\varphi) = (\exp(\varphi_{i,j})).
\end{equation}

Note that the above exponential is a finite sum due to $\alpha > \frac{1}{p-1}$ and is therefore well-defined.

Together with the quotient $\mathcal{P} \to \mathcal{G}$, the above morphism induces a morphism of group $\mathfrak{o}_n$-schemes:
\begin{equation}
\exp^\mathfrak{o}_n : (\text{Lie} \mathcal{G}^\mathfrak{o})_n \to \mathcal{G}_n.
\end{equation}

The above morphisms induce a morphism of formal group schemes over $\mathfrak{o}$ and of rigid groups over $\mathbb{C}$:

\begin{equation}
\widehat{\text{exp}}^\mathfrak{o}_n : \text{Lie} \mathcal{G}^\mathfrak{o}_n \to \mathcal{G}^\mathfrak{o}_n.
\end{equation}

In view of the construction (A.4.11.2), its differential is the identity $\text{id}_{\text{Lie} \mathcal{G}_C}$, and the underlying classical points of the above morphism is compatible with (A.4.9.1).

By the functoriality of log, we deduce that $\widehat{\text{exp}}^\mathfrak{o}_n$ is a section of $\log \mathcal{G}^\mathfrak{o}_n$ on $\text{Lie} \mathcal{G}^\mathfrak{o}_n$. By the uniqueness of the $p$-adic logarithm ([19] Théorème 1.2), we deduce that $\widehat{\text{exp}}^\mathfrak{o}_n$ is compatible with $\text{exp}_\mathcal{G}$ (A.1.4.1) and $\text{exp}_{\mathcal{P}_C}$ (A.4.11.1) as well. Then, the proposition follows. □

A.4.12. Proof of proposition A.4.4. By proposition A.4.11, we have the following commutative diagrams:

\begin{equation}
\begin{align*}
H^1(\hat{Y}, \xi f^*(T_X)) & \xrightarrow{\exp^\mathfrak{o}_n} H^1(\hat{Y}, p^n \hat{f}) \xrightarrow{\text{exp}_{\mathcal{P}_C}} H^1(\hat{Y}, \hat{f})(\simeq H^1(Y_\mathcal{F}, J_\mathcal{F})) \\
H^1(Y_C, \xi f^*(T_{X_C})) & \xrightarrow{\text{exp}_{\mathcal{P}_C}} H^1(Y_C, J_C)
\end{align*}
\end{equation}

By lemma A.4.8, $(L_{f,b}, \alpha) \in \varprojlim \mathcal{P}(\mathfrak{o}_n)$ comes from the $p$-adic completion of an object of $\mathcal{P}(\mathfrak{o})$, that we abusively denote by $(L_{f,b}, \alpha)$. The $\xi f^*(T_X)$-torsor $\mathcal{L}_f$ (§A.3.2) is sent to the $\xi f^*(T_{X_C})$-torsor $\mathcal{L}_f$ (§A.2.2) via the left vertical arrow. Hence there exists a unique isomorphism between the line bundle $L_{f,b}$ and $\mathcal{L}^\mathfrak{o}_n$ on $Y_{f,b,C}$ compatible with rigidifications via the right vertical arrow. Then, the proposition follows from proposition A.4.10. □

A.4.13. In the end, we consider the case $b = 0$. Since the point 0 is small (proposition A.3.1), we have a canonical isomorphism $\mathcal{L}_{f,0} \simeq \mathcal{L}_{f,0}^\mathfrak{o}_n$ (proposition A.4.10). In this case, we have a canonical $\Theta_\mathcal{F}$-homomorphism $\iota : \mathcal{A}_0 \to \Theta_\mathcal{F}$ with a nilpotent kernel.
Proposition A.4.14. (i) There exists a canonical isomorphism $\mathcal{L}_{f,0} \otimes_{A_0} \mathcal{O}_Y \simeq \mathcal{O}_Y$.

(ii) The restriction of $f^*_X \exp$ to vector bundles (viewed as Higgs bundles with zero Higgs field) is canonically isomorphic to the usual inverse image functor $f^*_C$ for vector bundles.

Proof. (i) Since $\text{Ker}(\iota)$ is nilpotent, we have $\iota(\exp(\phi_{ij})) = 1$ for a Čech cocycle $\exp(\phi_{ij})$ of $\mathcal{L}_{f,0}$. Then, the assertion follows.

(ii) Given a vector bundle $M$ on $X_C$, the action of $A_0$ on $(f^*_C(M), 0)$ factors through $\iota$. Then, the assertion follows from (i). \hfill \Box

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