Critical Behavior of Hierarchical Ising Models

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Abstract: We consider the critical behavior of two-dimensional layered Ising models where the exchange couplings between neighboring layers follow hierarchical sequences. The perturbation caused by the non-periodicity could be irrelevant, relevant or marginal. For marginal sequences we have performed a detailed study, which involved analytical and numerical calculations of different surface and bulk critical quantities in the two-dimensional classical as well as in the one-dimensional quantum version of the model. The critical exponents are found to vary continuously with the strength of the modulation, while close to the critical point the system is essentially anisotropic: the correlation length is diverging with different exponents along and perpendicular to the layers.

PACS-numbers: 05.50.+q, 64.60.Cn, 64.60.Fr
I. Introduction

Since the discovery of quasicrystals[1] there is a growing interest to understand their structure and physical properties. Theoretically it is a challenging problem to understand the properties of phase transitions in these quasiperiodic or more generally non-periodic structures. Since a non-periodic system shares some aspects with a system with quenched disorder, from these studies one hopes to get a better understanding about the critical behavior of random systems, too.

Early studies on this field were numerical investigations on specific problems (Ising model[2-4], percolation[5,6], random walks[7] etc.) on two- and three-dimensional quasiperiodic lattices, whereas analytical results were obtained on layered two-dimensional Ising models with one-dimensional aperiodicity[8,9] or on the corresponding Ising quantum chain[10-14]. The nature of phase transitions is better understood, since Luck[15] generalized the Harris criterion[16] for non-periodic systems. In this relevance-irrelevance criterion the strength of fluctuations in the couplings in a scale of the correlation length is of primary importance. The perturbation caused by the non-periodicity is irrelevant (relevant) if the local energy fluctuations are smaller (greater) than the corresponding thermal energy. Theoretically most interesting is the borderline case, when the thermal and fluctuating energy contributions are in the same order of magnitude, i.e. the perturbation is marginal. In this case - according to exact results on two-dimensional layered Ising models[17-20] - the critical behavior is non-universal, the critical exponents are continuous functions of the strength of the modulation. Furthermore, close to the critical point the system becomes essentially anisotropic[20], i.e. the correlation length is diverging with different exponents along and perpendicular to the layers.

The Luck criterion is obtained in the frame of a linear stability analysis therefore its validity is restricted to such non-periodic systems where the perturbation in the local couplings is small. Thus the criterion is valid for quasiperiodic lattices and for such one-dimensional aperiodic sequences which are generated by substitutional rules. On the other hand the criterion is no longer valid in its original form, if the modulation of the couplings follows some hierarchical sequence, such that certain couplings could become arbitrarily large or small. This type of hierarchical sequence[21] was introduced first by Huberman and Kerszberg[22] and generalized by others[23,24] to study anomalous diffusion and the properties of the spectrum of the Hamiltonian[25] in one-dimension.
As far as the critical properties of Ising quantum chains with a hierarchical structure in the couplings are concerned two conflicting results exist. From a study of the low-lying excitations of the system Ceccatto[26] has drawn the conclusion, that the perturbation is irrelevant, if the hierarchical parameter $r$ is smaller than some critical value $r_c$. Whereas for $r > r_c$ the perturbation is relevant, the Ising-type critical point of the homogeneous system is washed out by the perturbation and the critical behavior of the system is similar to that of the McCoy-Wu model[8], i.e. to a layered Ising model with quenched randomness. In contrary to this results Lin and Goda[27] obtained continuously varying surface magnetisation exponent of the hierarchical quantum Ising model, thus according to this study the perturbation caused by this non-priodicity is marginal in the whole range of the parameter $r$. Similar conclusion is drawn from renormalization group and Monte Carlo simulation studies on the two-dimensional Ising model with layered hierarchical couplings[28].

Our aim in the present paper is to clarify this controversial and to present a comprehensive picture about the critical behavior of hierarchical Ising models. For this purpose we consider general hierarchical sequences and investigate the condition for relevance (irrelevance) of the perturbation. For marginal hierarchical modulation we perform a detailed study, which includes analytical and numerical calculations of different bulk and surface critical quantities both in the two-dimensional classical and in the one-dimensional quantum version of the model. Finally, the results are discussed in the frame of a general scaling theory, which is then compared to the analogous one of aperiodic systems.

II. The model

We consider the Ising model on the square lattice with different layered structures: the system is translationally invariant either along the columns (Fig 1a) or along the diagonals (Fig 1b). In the first case the interaction along the layers $K_1$ is constant, whereas it is modulated in the other direction and given as $K_2(k)$ in units of $1/k_BT$ between neighboring layers at $k$ and $k + 1$. In the extreme anisotropic limit $K_1 \to \infty$, $K_2(k) \to 0$ the transfer matrix of the problem involves the Hamiltonian of a quantum Ising chain:

$$H = -\frac{1}{2} \sum_k [\sigma^z_k + \lambda_k \sigma^x_k \sigma^x_k]$$

where $\sigma^x_k$, $\sigma^z_k$ are Pauli matrices at site $k$ and

$$\lambda_k = -2K_2(k)/\ln(\tanh K_1).$$
In the second situation the $K_d(k)$ diagonal couplings are the same within one column and the quantities

$$Y_k = \sinh[2K_d(k)]$$

are assumed to vary in a hierarchical way.

The hierarchical sequences we use in this paper were first introduced in economical problems[21] and later applied to study the so called hyperdiffusion process. For an integer number $m$ the sequence $a_1, a_2, \ldots$ is defined as:

$$a_k = ar^{f_k}$$

where $r$ is the ratio of the sequence, $a$ is some reference value and the $f_k$-s are natural numbers satisfying the relation:

$$k = m^{f_k}(ml + \mu) \ , \ l = 0, 1, \ldots , \mu = 1, 2, \ldots , m - 1$$

As pointed out in Ref[29] it is possible to generalize the sequence by modifying eq(3) as $a_k = ar^{g(f_k)}$, where $g(x)$ is some analytical function. Here we shall consider power functions: $g(x) = x^\omega$, $\omega > 0$, thus the original sequence in eq(3) corresponds to $\omega = 1$.

For the layered Ising models we assume the hierarchical variation in the couplings $\lambda_k$ and in the parameters $Y_k$, respectively.

**III. Surface magnetisation**

For the two-dimensional Ising model the surface magnetisation $m_s$ is the simplest order parameter, which can be most easily determined in the extreme anisotropic limit eq(1) through the formula:

$$m_s = \left(1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} \lambda_k^{-2}\right)^{-1/2}$$

For the general hierarchical sequence containing the exponent $\omega$ the surface magnetisation is rewritten in the form:

$$m_s = [S(\lambda, r)]^{-1/2} \ , \ S(\lambda, r) = \sum_{j=0}^{\infty} \lambda^{-2j}r^{-2n_j} \ , \ n_j = \sum_{k=1}^{j} (f_k)\omega \ , \ n_0 = 0$$

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The critical coupling $\lambda_c$ is such that

$$\lim_{j \to \infty} \frac{1}{j} \sum_{k=1}^{j} \ln \lambda_k = 0$$

(7)

and related to the hierarchical parameter $r$ as

$$\lambda_c = r^{-\delta(\omega,m)}$$

(8)

where $\delta(\omega,m) = \lim_{j \to \infty} n_j / m^j$. As shown in the Appendix:

$$\delta(\omega,m) = \left(1 - \frac{1}{m}\right) \sum_{j=1}^{\infty} \frac{j^\omega}{m^j}$$

(9)

which can be expressed in closed form for integer $\omega$s. For $\omega = 1, 2$ and 3 it is given as:

$$\delta(1,m) = \frac{1}{m - 1}, \quad \delta(2,m) = \frac{m + 1}{(m - 1)^2}, \quad \delta(3,m) = \frac{m^2 + 4m + 1}{(m - 1)^3}$$

(10)

To calculate the surface magnetisation we do separately for $\omega = 1$ and $\omega \neq 1$. In the first case, for $\omega = 1$ one can verify that $f_{mp} = f_p + 1$ and $f_{mp+\mu} = 0$, $\mu = 1, 2, \ldots, m - 1$, from which the relations $n_{mp} = n_p + p$, $n_{mp+\mu} = n_{mp}$ follow. Then dividing the sum for $S(\lambda, r)$ in eq(6) into $m$ parts:

$$S(\lambda, r) = \sum_{p=0}^{\infty} \lambda^{-2mp}p^{-2n_{mp}} + \sum_{\mu=1}^{m-1} \sum_{p=0}^{\infty} \lambda^{-2(mp+\mu)p-2n_{mp+\mu}}$$

(11)

and using the relations between the $n_j$s one gets the following equation:

$$S(\lambda, r) = S(\lambda^m, r) \frac{1 - \lambda^{-2m}}{1 - \lambda^{-2}}$$

(12)

From this expression the surface magnetisation exponent $\beta_s$ defined by $m_s(t) \sim (t)^{\beta_s}$ as $t = 1 - (\lambda_c/\lambda)^2 \to 0^+$ can be evaluated as in Ref[18]. According to eq(6) close to the critical point $S(t) \sim (t)^{-2\beta_s}$, and the critical exponent corresponding to eq(12) is given by:

$$\beta_s = \ln \left[ \frac{1-\lambda_c^{-2m}}{1-\lambda_c^{-2}} \right] \frac{1}{2 \ln m}$$

(13)

In the special case $m = 2$ one recovers the result in Ref[27]. According to eq(13) for $\omega = 1$ the critical behavior of the hierarchical Ising model is non-universal, since $\beta_s$ is
a continuous function of the ratio $r$. This functional dependence is shown on Fig 2. for several values of the parameter $m$.

Next we turn to discuss the situation for $\omega \neq 1$, in which case the surface magnetisation is studied on large finite sequences. Let us consider the system of length $j = m^N - 1 \equiv L - 1$, i.e. after $N$ generations, and define in analogy with eq(6):

$$S_N(\lambda, r) = 1 + \sum_{j=1}^{m^N-1} \prod_{k=1}^{j} \lambda_k^{-2} = \sum_{j=0}^{m^N-1} \lambda^{-2j} r^{-2n_j}$$

This quantity satisfies the relation:

$$S_{N+1}(\lambda, r) = \frac{1 - X^{2m}}{1 - X^2} S_N(\lambda, r), \quad X = \lambda^{-L} r^{-n_L}$$

As shown in the Appendix in the large $L$ limit the finite size corrections to the criticality condition in eq(8) are logarithmic:

$$n_L = \delta(\omega, m)L - \frac{\omega}{m-1} N^{\omega-1} + O\left(N^{\omega-2}\right)$$

thus the parameter $X$ in eq(15) in leading order is given as $X = (\lambda_c/\lambda)^L r^{[\omega/(m-1)]} N^{\omega-1}$ and at the critical point one gets the relation:

$$S_{N+1}(\lambda_c) = \frac{1 - Z^{2m}}{1 - Z^2} S_N(\lambda_c), \quad Z = r^{[\omega/(m-1)]} N^{\omega-1}$$

where $r$ is related to $\lambda_c$ through eq(8).

For large $N$ the parameter $Z$ in eq(17) scales differently for $\omega = 1$, $\omega < 1$ and $\omega > 1$, respectively, and the corresponding functional form of $S_N(\lambda_c)$ is also different in the three cases. In the borderline case $\omega = 1$, which is studied already before, $S_N(\lambda_c)$ has a power law dependence on the size of the system:

$$S_N(\lambda_c) \sim (L)^{2x_s}, \quad \omega = 1$$

where according to finite size scaling[31] $x_s$ is the scaling dimension of surface spins. Using the value of $\delta(1, m)$ in eq(10) one gets from eqs(17) and (18) that $x_s$ corresponds to $\beta_s$ in eq(13), thus the scaling relation $\beta_s = \nu x_s$ is satisfied, since the correlation length critical exponent for the two-dimensional Ising model is $\nu = 1$. 

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By finite size scaling one may investigate the surface critical behavior of the system at the right end of the chain taking $L = m^N$ spins. Denoting the inverse square of the finite size surface magnetisation as $\bar{S}_N(\lambda, r)$ one can write the simple relation:

$$\bar{S}_N(\lambda, r) = 1 + \lambda^{-2} r^{-2N} S_N(\lambda, r)$$ \hspace{1cm} (19)

from which $\bar{x}_s$ the scaling dimension of surface spins on the right end of the chain is given by:

$$\bar{x}_s = x_s - \frac{\ln r}{\ln m} = \ln \left[ \frac{1 - \lambda^2}{1 - \lambda^2 \omega} \right] \frac{2}{2 \ln m}$$ \hspace{1cm} (20)

thus $\bar{x}_s(\lambda_c) = x_s(\lambda_c^{-1})$.

For $0 < \omega < 1$ $Z$ in eq(17) goes to zero, and the relation in eq(17) in leading order reads as

$$S_{N+1}(\lambda_c) = m(1 + \omega N^{\omega-1} \ln r) S_N(\lambda_c), \hspace{0.5cm} 0 < \omega < 1$$ \hspace{1cm} (21)

which is solved by the function $S_N(\lambda_c) \sim m^N r^{N \omega}$. Now using from finite size scaling theory[31] that $m^N \approx L \sim |t|^{-\nu}$ one obtains with $\nu = 1$ for the temperature dependence of the surface magnetisation:

$$m_s(t) \sim t^{1/2} r^{-\left(\frac{1}{2} + \omega \frac{1}{2}\text{ln}m\right)/\omega}, \hspace{0.5cm} 0 < \omega < 1$$ \hspace{1cm} (22)

Thus the surface magnetisation in this case vanishes with the same exponent $\beta_s = 1/2$ as the homogeneous model with $r = 1$, however there is a logarithmic correction. Consequently the hierarchical perturbation in the couplings for $0 < \omega < 1$ is marginally irrelevant.

The behavior of the system is completely different for $\omega > 1$. In this case one has to study separately $r > 1$, when $Z$ in eq(17) goes to infinity and $r < 1$, when $Z$ goes to zero. In the first case the asymptotic relation

$$S_{N+1}(\lambda_c) = r^{2\omega N^{\omega-1}} S_N(\lambda_c)$$ \hspace{1cm} (23)

is satisfied by the function

$$S_N(\lambda_c) \sim r^{2N \omega}$$ \hspace{1cm} (24)

† If the chain consists of $m^N - 1$ spins as before, then it is symmetric to its center and the surface magnetisation is the same at both ends.
Therefore the coupling dependence of the surface magnetisation is anomalous:

\[ m_s(t) \sim r^{-\left(\log t/\log m\right)^\omega}, \quad \omega > 1, \quad r > 1 \quad (25) \]

it vanishes faster than any power as \( \lambda \to \lambda_c \).

On the other hand for \( r < 1 \) (and \( \omega > 1 \)) \( Z \) goes to one, and the asymptotic relation

\[ S_{N+1}(\lambda_c) = \left( 1 + r \frac{2^\omega}{m^\omega-1} \right) S_N(\lambda_c) \quad (26) \]

implies \( \lim_{N \to \infty} S_N(\lambda_c) < \infty \), since the product \( \prod_{N=N_0}^{\infty} \left( 1 + r \frac{2^\omega}{m^\omega-1} \right) \) is convergent. Consequently the surface magnetisation stays finite at the critical point and the phase transition at the surface is of first order. According to eq(26) the surface magnetisation approaches its limiting value \( m_s(0) \) in an anomalous way:

\[ m_s(t) - m_s(0) \sim r^{\left|\log t/\log m\right|^\omega-1} 2^\omega/(m-1) \quad (27) \]

where again the correspondence \( L \sim t^{-1} \) was used. The surface magnetisation for \( r = .92 \) and different values of \( \omega \geq 1 \) are shown on Fig 3.

Next we turn to present results about the critical behavior of the model at the \((1,1)\) surface, when the non-periodicity in the couplings is given in the diagonal direction (Fig 1b). The criticality condition now reads as\[32\]:

\[ \lim_{j \to \infty} \frac{1}{j} \sum_{k=1}^{j} \ln Y_k = 0 \quad (28) \]

where the parameter \( Y_k = \sinh[2K_d(k)] \) plays a similar role as \( \lambda_k \) for the quantum Ising model in eq(7). To make use further this analogy we assign the hierarchical modulation in eq(3) to the generalized \( Y_k \) parameters. As a consequence the critical value \( Y_c \) and the ratio of the sequence \( r \) are related as in eq(8) for the quantum model.

To study the magnetisation on the \((1,1)\) surface we use a numerical procedure by Hilhorst and van Leeuwen (HvL), which is originally developed for the triangular lattice and based on a repeated use of the star-triangle transformation\[33\]. From the square lattice on Fig 1b one can obtain the triangular lattice by connecting vertically next-nearest neighbors with non-vanishing couplings. The numerical procedure also works for this triangular lattice, however, we shall only study the problem on the square lattice. Details of the method together with results on the surface critical behavior of different type of non-periodic triangular lattice Ising models will be presented elsewhere\[34\].
The HvL method enables one to obtain very accurate numerical estimates on the magnetisation at the (1,1) surface as well as to study the decay of surface correlations. The obtained results on the surface critical behavior for marginally irrelevant \((\omega < 1)\) and marginally relevant \((\omega > 1)\) perturbations are qualitatively the same as we found for the corresponding quantum chain. In the truly marginal case - \(\omega = 1\) - the results even quantitatively agree with those obtained on the (1,0) surface. We are going to discuss these results in more detail.

First investigating the surface magnetisation one can see that in the limit \(r \to 0\) \(m_s(T) = 1\) for \(T \leq T_c\) and \(m_s(T) = 0\) for \(T > T_c\), i.e. it stays finite at the critical point. This anomalous behavior is due to the topology of the lattice in the diagonal direction. As \(r \to 0\) in the first layer (and also in any even layers) \(Y_1 = Y_c = 1/r\), which together with \(K_d(1)\) goes to infinity, resulting an ordered surface layer. For \(r > 0\) according to numerical results the surface magnetisation vanishes at the critical point. For large number of iterations \(I\) the critical surface magnetisation goes to zero as a power: \(m_s(t = 0) \sim I^{-\gamma}\), where the exponent \(\gamma > 0\) is related to the decay exponent \(\eta_\parallel\) in eq(29) as \(\gamma = \eta_\parallel/4\).

The surface magnetisation exponent \(\beta_s\) determined from the relation \(m_s(t, r) \sim t^{\beta_s}\) is found to be the same function of \(r\) as that at the (1,0) surface in eq(13). Similar conclusion holds for the magnetisation exponent on the right end of the chain. We obtained \(\bar{x}_s(Y_c) = x_s(Y_c^{-1})\) in close analogy with the results on the (1,0) surface. Numerical estimates on the surface magnetisation exponents are drawn on Fig 2. We note that the accuracy of the calculation is limited by the fact that the error in the magnetisation decreases very slowly with the number of iterations \(I\) as \(I^{-1/2}\).

We have also investigated the decay of critical surface spin correlations. The decay is given as a power:
\[G_s(r_\parallel, t = 0) \sim r_\parallel^{-\eta_\parallel}\]
and according to the numerical estimates the critical exponent \(\eta_\parallel\) can be described with high accuracy by the formula:
\[\eta_\parallel = \frac{2x_s}{x_s + \bar{x}_s}, \quad \bar{\eta}_\parallel = \frac{2\bar{x}_s}{x_s + \bar{x}_s}\]
both for \(r < 1\) and \(r > 1\). In eq(30) \(\bar{\eta}_\parallel\) is the decay exponent on the right surface of the system.

The relations in eq(30) are in conflict with ordinary scaling, which would imply \(\eta_\parallel = 2x_s\). They could be explained, however, within the frame of anisotropic scaling theory[35].
Then the critical spin-spin correlation function at the left side of the system is expected to behave as:

$$G_s(r_\parallel, t) = b^{-2x_s}G_s(r_\parallel / b^z, b^{1/\nu}t)$$  \hspace{1cm} (31)

when lengths perpendicular to the layers transform as $r_\perp \rightarrow r_\perp / b$, whereas scaling along the layers involves the dynamical exponent $z$: $r_\parallel \rightarrow r_\parallel / b^z$. As a consequence the correlation length close to the critical point diverges with different exponents in the two directions: $\xi_\perp \sim t^{-\nu}$ and $\xi_\parallel \sim t^{-\nu z}$. According to eq(31) the decay exponent is given by $\nu_\parallel = 2x_s/z$, thus the numerical results in eq(30) imply that the dynamical exponent is expressed by the sum of the two surface magnetisation exponents:

$$z = x_s + \bar{x}_s = \frac{\ln \left[ \frac{\lambda_{c}^{1-\lambda_{c}}}{\lambda_{c}^{1-\lambda_{c}} - \lambda_{c}} \right]}{\ln m}$$  \hspace{1cm} (32)

In the following we check the validity of the anisotropic scaling hypothesis by calculating other critical quantities.

**IV. Other critical quantities**

A direct evidence for anisotropic scaling can be obtained from the behavior of the correlation length. At the critical point the perpendicular correlation length on a strip of width $L$ is restricted to $\xi_\perp \sim L$, thus $\xi_\parallel \sim \xi_\perp^z \sim L^z$. In the extreme anisotropic limit $\xi_\parallel$ is given by the inverse gap of the critical Hamiltonian in eq(1), thus one expects for low laying states:

$$E_i - E_0 \sim L^{-z}$$  \hspace{1cm} (33)

To check this relation we have numerically studied the excitation spectrum of the Hamiltonian in eq(1) at the critical point. Using standard methods[36] a quadratic fermion Hamiltonian is obtained via the Jordan-Wigner transformation, which is then diagonalized by a Bogoliubov transformation. Using free boundary conditions we calculated the energy of the first fermion modes on finite systems of size $L = mN - 1$ ($m = 2, N = 2, \ldots, 16; m = 3, N = 2, \ldots, 10$). Their size dependence is found to be very accurately described by the relation in eq(33). The $z$-exponents obtained through sequence extrapolation methods[37] coincide with the values in eq(32) up to $4 - 5$ digits.

Even more accurate estimates can be obtained if the energy of fermion modes are calculated in a second order perturbation expansion[15]. At the critical point their size
dependence follows from the relation:

\[ \Lambda(L) \sim \left[ \sum_{j=1}^{L} \sum_{k=1}^{L} \lambda_{j+1}^2 \lambda_{j+2}^2 \ldots \lambda_{j+k}^2 \right]^{-1/2} \]  

(34)

Evaluating eq(34) up to \( L = 2^{24} \) and \( L = 3^{15} \) the accuracy of the estimates on the \( z \)-exponent has been increased up to 7-8 digits.

Next we turn to study the behavior of the specific heat on finite systems. According to anisotropic scaling the singular part of the bulk free energy density in a finite system of size \( L \) behaves as[35]:

\[ f(t, L) = b^{-(1+z)} f\left(b^{1/\nu} t, L/b\right) \]  

(35)

This expression is in accord with the finite size scaling behavior of the singular part of the critical bulk energy density \( \epsilon(L) \sim L^{-x_e} \), since according to our numerical estimates:

\[ x_e = z \]  

(36)

From eq(35) the finite size dependence of the specific heat at the critical point is given by \( C(t = 0, L) \sim L^{-\alpha/\nu} \), where the specific heat exponent

\[ \alpha = 1 - z \]  

(37)

is negative for \( r \neq 1 \). As shown in Table 1 this relation is also satisfied.

Finally, we report on our results on the scaling behavior of the surface energy density at the critical point: \( \epsilon_s(L) \sim L^{-x_{es}} \). On the left surface we obtain:

\[ x_{es} = z + 2x_s \]  

(38)

and a similar expression is valid on the right surface.
V. Discussion

In this paper the effect of a layered hierarchical structure on the critical properties of the two-dimensional Ising model is studied by analytical and accurate numerical methods. The perturbation is found to be marginal for $\omega = 1$, which corresponds to the original Huberman-Kerszberg series. In this case the critical exponents are continuous functions of the hierarchical parameter $r$, for the whole range of $0 < r < \infty$. These results are in accordance with the findings of Lin and Goda\[27\] on the surface magnetisation in the $m = 2$ model, however they are in conflict with the results of Ceccatto\[26\]. The failure of Ceccatto’s calculation is due to the fact that his perturbational method is only valid for irrelevant inhomogeneities and can not be used for the hierarchical sequence, which is a marginal perturbation. Our conclusions are also consistent with the RG and MC simulation results of Stella et al\[28\] on a two dimensional hierarchically layered Ising model.

The nature of the hierarchical perturbation is found to be different for $\omega < 1$ and $\omega > 1$. In the first case the perturbation is irrelevant, however there are logarithmic scaling corrections. In the other regime, for $\omega > 1$, the perturbation is marginally relevant and the surface magnetisation behaves anomalously at the critical point. In the following we present a relevance-irrelevance criterion, which explains the above results.

This criterion is actually a modification of the Harris criterion\[16\] for random systems, which is then generalized by Luck\[15\] to aperiodic systems. In this criterion one compares the energy $E_f$ due to fluctuations in the couplings with the thermal excess energy $E_t$. If the couplings $J_k$ follow a one-dimensional aperiodicity and their average is $\bar{J}$ the fluctuating energy in a domain of size of the bulk correlation length $L \sim \xi$ behaves asymptotically as:

$$E_{fl}(L) = \sum_{k=1}^{L} (J_k - \bar{J}) \sim L^\Omega$$

(39)

where $\Omega < 1$ is the wandering exponent characteristic to the sequence\[38\]. The fluctuating energy per spin then scales as $\epsilon_{fl}(L) \sim L^{\Omega-1}$. The excess thermal energy per spin is proportional to the reduced temperature $\epsilon_t(L) \sim t \sim L^{-1/\nu}$. Comparing $\epsilon_{fl}(L)$ with $\epsilon_t(L)$ one assumes irrelevant perturbation for $\epsilon_{fl}(L) \ll \epsilon_t(L)$, i.e. for $\Omega < 1 - \nu$ and relevant modulation in the opposite limit $\Omega > 1 - \nu$. Finally, the perturbation is marginal for $\Omega = 1 - \nu$, which condition is satisfied for the Ising model with $\Omega = 0$.

For the hierarchical perturbation considered in this paper the above criterion can not be used directly. The perturbation is locally not small and for $r > 2$ even the average
value of the coupling $\bar{\lambda}$ is divergent. To remedy this problem we consider first the diagonal hierarchy in eq(2), where the parameters are given as $Y_k = Y_1 r^n$. For a large $n$ $Y_k \sim \exp[2K_d(k)] \sim r^n$, thus the coupling itself is given as $K_d(k) \sim \log Y_k \sim nK_d(1)$. Therefore the fluctuating energy in this case is expressed in terms of the logarithms of the parameters:

$$E_{fl}(L) \simeq \sum_{k=1}^{L} \log(Y_k/\bar{Y}) \sim L^\Omega$$

(40)

Similar conclusion is valid for the extreme anisotropic system, where $K_2(k) \to 0$, $K_1 \to \infty$ and $\lambda_k = K_2(k) \exp(2K_1) = \lambda_1 r^n$. To that quantum system one can assign another two-dimensional classical system, if $K_2$ is kept constant and the vertical couplings $K_1(k)$ vary with the column index $k$ such that their ratio remains $\lambda_k$. Then $K_1(k) \sim \log \lambda_k \sim nK_1(1)$, and the fluctuating energy is again given by:

$$E_{fl}(L) \simeq \sum_{k=1}^{L} \log(\lambda_k/\bar{\lambda}) \sim L^\Omega$$

(41)

Since eqs(40) and (41) remain valid for small aperiodic perturbations they can be considered as the general definition of the fluctuating energy both for bounded and unbounded sequences.

For generalized hierarchical sequences the fluctuating energy can be obtained by noticing that at the critical point $\prod_{k=1}^{L} \lambda_k = Z$, which is defined in eq(17). Since $\bar{\lambda} = \lim_{L \to \infty} [\prod_{k=1}^{L} \lambda_k]^{1/L} = 1$ we obtain

$$E_{fl}(L) \simeq \log r \left[ \frac{\omega}{m - 1} \left( \frac{\log L}{\log m} \right)^{\omega - 1} \right]$$

(42)

Thus the wandering exponent $\Omega = 0$, for any value of $\omega$ and a linear stability analysis can not decide about the relevance-irrelevance of the perturbation. In second order of the analysis the perturbation in the $\omega > 1$ region (where $E_{fl}(L)$ logarithmically divergent) is predicted to be relevant, whereas for $\omega < 1$ it is marginally irrelevant. Finally, for $\omega = 1$ $E_{fl}(L)$ is independent of the size of the system and the perturbation is truely marginal. These predictions of the modified Harris criterion are in complete agreement with our analytical results.

The critical behavior found at $\omega = 1$ is very similar to that obtained in the two dimensional Ising model with marginally aperiodic layered interactions. In both type of problems the system is essentially anisotropic at the critical point and the corresponding
$z$ exponent is related to the surface scaling dimensions, as in eq(38). The surface spin correlations follow the scaling law in eq(31), whereas the thermodynamic singularities are consistent with the anisotropic scaling form of the free-energy density in eq(35). Finally, we note that some exponents, including $z$, $x_e$ and $x_{es}$ can be obtained exactly by an analytical method, which can be used both for aperiodic and hierarchical marginal Ising chains. Details of the calculations will be presented elsewhere[39].

Acknowledgement: This work has been supported by the Hungarian National Research Fund under grant No OTKA TO12830. F.I. thanks for interesting discussions with L. Turban, B. Berche and A. Maritan.
Appendix

We calculate the $n_j$ parameter in eq(6) for sequences of lengths $j = m, m^2, \ldots, m^N, \ldots$:

\[ n_m = 1 \]
\[ n_{m^2} = n_m m + 2^\omega - 1^\omega \]
\[ \vdots \]
\[ n_{m^N} = n_{m^{N-1}} m + N^\omega - (N - 1)^\omega \]

which relations are satisfied with

\[ n_{m^N} = m^N \left( 1 - \frac{1}{m} \right) \sum_{j=1}^{N-1} \frac{j^\omega}{m^j} + N^\omega \]

From this expression the value of $\delta(\omega, m)$ as given in eq(9) follows. For a positive integer $\omega$ $\delta(\omega, m)$ can be expressed as:

\[ \delta(\omega, m) = \left\{ (1 - x) \left( x \frac{d}{dx} \right)^\omega \left[ \frac{1}{1 - x} \right] \right\} \bigg|_{x=1/m} \]

The finite size corrections to $n_{m^N}$ are given by:

\[ n_{m^N} - \delta(\omega, m)m^N = - \left( 1 - \frac{1}{m} \right) \sum_{j=N}^{\infty} \frac{j^\omega}{m^j} + N^\omega \]
\[ = N^\omega \left[ 1 - \left( 1 - \frac{1}{m} \right) \sum_{j=0}^{\infty} \frac{(1 + j/N)^\omega}{m^j} \right] \]

Expanding $(1 + j/N)^\omega$ in Taylor series and performing the summation the leading term is given in eq(16).
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Figure captions

Fig.1: Two-dimensional Ising model with layered hierarchical couplings: (a) along the columns and (b) along the diagonals.

Fig.2: Surface magnetisation exponent for marginally hierarchical Ising chains at the left surface ($\beta_s$) and at the right boundary ($\bar{\beta}_s$) as a function of the parameter $r$. The lines correspond to the analytical results in eqs(13) and (20) for the (1,0) surface and the squares are numerical estimates for the (1,1) surface.

Fig.3: The surface magnetisation for relevant perturbations ($r = .92$) at different values of $\omega$. 
Table caption

Table 1: Finite size estimates for the specific heat exponent of the $m = 2$ marginally hierarchical Ising model for different values of the critical coupling $\lambda_c$. The values in the table were calculated from finite size results on chains of length $N, N/4$ and $N/16$. In the last row the conjectured results from anisotropic scaling eq(37) are given.
| N     | $\lambda_c^2 = 2$ | $\lambda_c^2 = 3$ | $\lambda_c^2 = 4$ | $\lambda_c^2 = 5$ |
|-------|-------------------|-------------------|-------------------|-------------------|
| 32    | -.11269           | -.27620           | -.43191           | -.58306           |
| 64    | -.08671           | -.23422           | -.37389           | -.50684           |
| 128   | -.08008           | -.21749           | -.34636           | -.46609           |
| 256   | -.08007           | -.21118           | -.33350           | -.44517           |
| 512   | -.08154           | -.20886           | -.32748           | -.43454           |
| 1024  | -.08288           | -.20801           | -.32460           | -.42922           |
| 2048  | -.08378           | -.20771           | -.32319           | -.42679           |
| 4096  | -.08434           | -.20765           | -.32285           | -.42507           |
| $1 - z$ | -.08496           | -.20752           | -.32193           | -.42400           |

Table 1