GROWTH OF REGULATORS IN FINITE ABELIAN COVERINGS

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ABSTRACT. We show that the regulator, which is the difference between the homology torsion and the combinatorial Ray-Singer torsion, of finite abelian coverings of a fixed complex has sub-exponential growth rate.

1. INTRODUCTION

1.1. Based free complex over group ring and its quotients. Suppose \( \pi \) is a finitely presented group and \( \mathbb{Z}[\pi] \) is the group ring of \( \pi \) over the ring \( \mathbb{Z} \) of integers.

Let \( \mathcal{C} \) be a finitely-generated based free \( \mathbb{Z}[\pi] \)-complex

\[
0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} C_{m-2} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0.
\]

Here “based free” means each \( C_k \) is a free \( \mathbb{Z}[\pi] \)-module equipped with a preferred base.

For a normal subgroup \( \Gamma \triangleleft \pi \) let \( \mathcal{C}_\Gamma := \mathbb{Z}[\pi/\Gamma] \otimes_{\mathbb{Z}[\pi]} \mathcal{C} \). Assume that the index \( [\pi : \Gamma] \) is finite.

Then \( \mathcal{C}_\Gamma \) is a finitely-generated based free \( \mathbb{Z} \)-complex, where the preferred base of \( \mathbb{Z}[\pi/\Gamma] \otimes_{\mathbb{Z}[\pi]} C_k \) is defined using the one of \( C_k \) in a natural way.

A prototypical case is the following. Suppose \( \tilde{X} \to X \) is a regular covering with \( \pi \) the group of deck transformations and \( X \) a finite CW-complex. Choose a lift in \( \tilde{X} \) of each cell of \( X \). Then the CW-complex \( \mathcal{C} \) of \( \tilde{X} \) induced from that of \( X \) is a finitely-generated based free \( \mathbb{Z}[\pi] \)-complex. For a normal subgroup \( \Gamma \triangleleft \pi \), \( \mathcal{C}_\Gamma \) is the CW-complex of the covering \( X_\Gamma \), corresponding to the group \( \Gamma \), and \( H_k(C_\Gamma) \) is the \( k \)-th homology of the covering \( X_\Gamma \). Usually interesting invariants do not depend on the choice of the lifts of cells of \( X \).

1.2. Two torsions. We can define two torsions of the quotient complex \( \mathcal{C}_\Gamma \), the homology torsion \( \tau^H(C_\Gamma) \) and the combinatorial Ray-Singer torsion \( \tau^{RS}(C_\Gamma) \), as follows. The homology torsion is

\[
\tau^H(C_\Gamma) := \left( \prod_{k}^* |\text{tor}_{\mathbb{Z}}(H_k(C_\Gamma))| \right)^{-1} \in \mathbb{R}_+,
\]

where \( \text{tor}_{\mathbb{Z}}(M) \) is the \( \mathbb{Z} \)-torsion part of the finitely-generated abelian group \( M \), and \( \prod_k^* a_k \) is the alternating product

\[
\prod_{k}^* a_k = \prod_{k}^* a_k^{(-1)^k}.
\]

The Ray-Singer torsion of \( \mathcal{C}_\Gamma \) is

\[
\tau^{RS}(C_\Gamma) = \prod_{k}^* \det'(\partial_k) \in \mathbb{R}_+,
\]

where \( \det' \) is the geometric determinant of linear maps between based Hermitian spaces. We recall the definition of \( \det' \) in Section 2.

1.3. Comparison: general question. We want to compare the asymptotics of the two torsions as \( \Gamma \) becomes “thinner and thinner in \( \pi \)”, so that \( \pi/\Gamma \) approximates \( \pi \) in the following sense. A finite set \( S \) of generators of \( \pi \) defines a word length function \( l_S \) (and hence a metric) on \( \pi \). Define

\[
(\Gamma) := \min \{ l_S(x) \mid x \in \Gamma \setminus \{ e \} \}.
\]

Here \( e \) is the unit of \( \pi \). In all what follow, statements do not depend on the choice of the generator set \( S \), since the metrics of two different generator sets are quasi-isometric.
We are interested in the following question: Suppose $\mathcal{C}$ is $L^2$-acyclic (see e.g. [Li2]). Under what conditions does it hold that

$$
\lim_{(\Gamma) \to \infty, |\pi: \Gamma| < \infty} \frac{\ln(\tau^H(\mathcal{C}_\Gamma)) - \ln(\tau^{RS}(\mathcal{C}_\Gamma))}{|\pi : \Gamma|} = 0.
$$

The motivation of this question comes from the question [Ln2]: can one approximate $L^2$-torsions by finite-dimensional analogs? In some favorable conditions, one expects that the growth rate of each of $\tau^H$ and $\tau^{RS}$ is the $L^2$-torsion, hence they must be the same.

**Remark 1.1.** (a) If $\{\Gamma_n, n = 1, 2, \ldots\}$ is a sequence of exhausted nested normal subgroups of $\pi$, i.e. $\Gamma_{n+1} \subset \Gamma_n$ and $\cap_n \Gamma_n = \{e\}$, then $\lim_{n \to \infty} (\Gamma_n) = \infty$. The limit in $[1]$ is more general (stronger) than the limit of an exhausted nested sequence, as we don’t have “nested” property.

(b) There exists a sequence $\Gamma_n < \pi$ such that $\lim_{n \to \infty} (\Gamma_n) = \infty$ if and only if $\pi$ is residually finite. Hence, the left hand side of $[1]$ makes sense only when $\pi$ is residually finite.

(c) Define $\tau(x) = \delta_{x,e}$ for $x \in \pi$. This functional trace is the base for the definition of many combinatorial $L^2$-invariants. For a fixed $x \in \pi$, we have

$$
\lim_{(\Gamma) \to \infty} \tau_{\Gamma}(x) = \tau_{\pi}(x).
$$

This is the reason why one expects that as $(\Gamma) \to \infty$, many $L^2$-invariants (under some technical conditions) can be approximated by the corresponding invariants of $\pi/\Gamma$.

### 1.4. Main results.

The main result of the paper treats the case $\pi = \mathbb{Z}^n$.

**Theorem 1.** Suppose $\mathcal{C}$ is an $L^2$-acyclic finitely generated based free $\mathbb{C}[\mathbb{Z}^n]$-complex. Then $[1]$, with $\pi = \mathbb{Z}^n$, holds true.

We will not give the definition of $L^2$-acyclicity. Instead, for $\pi = \mathbb{Z}^n$, we will use an equivalent definition [El, Lü2]: the $L^2$-homology $H_k^{(2)}(\mathcal{C})$ vanishes if and only if $H_k(C \otimes_{\mathbb{Z}[\mathbb{Z}^n]} F) = 0$. Here $F$ is the fractional field of the commutative domain $\mathbb{Z}[\mathbb{Z}^n]$.

It is expected that $[1]$ holds for a large class of residually non-abelian finite groups (like hyperbolic groups), but a proof is probably very difficult.

**Remark 1.2.** (a) Our result does not imply that

$$
\lim_{(\Gamma) \to \infty, |\pi: \Gamma| < \infty} \frac{\ln(\tau^H(\mathcal{C}_\Gamma))}{|\pi : \Gamma|} = \lim_{(\Gamma) \to \infty, |\pi: \Gamma| < \infty} \frac{\ln(\tau^{RS}(\mathcal{C}_\Gamma))}{|\pi : \Gamma|},
$$

as we cannot prove the existence of each of the limits. As mentioned above, for a large class of (abelian and non-abelian groups) and maybe under some restrictions on $\Gamma$, one expects that both limits exist, are the same, and equal to the $L^2$-torsion of $\mathcal{C}$. This holds true for $\pi = \mathbb{Z}$, with no restrictions on $\Gamma$, see [GS, KL, Li2]. Even for the case $\pi = \mathbb{Z}^2$ and $\mathcal{C}$ is a 2-term complex $0 \to C_1 \to C_0 \to 0$, so that only $H_0(\mathcal{C})$ is non-trivial, there is still no proof of the conjecture that the $L^2$-torsion is equal to either of the above limits. For results and discussions of this conjecture, see [Li2, Li1, BV, Le1, Le2, FJ, SW].

(b) It should be noted that the exact calculation of the torsion part of the homology of finite coverings, even in the abelian case, is very difficult, see [HS, MM, Po] for some partial results.

### 1.5. Refinement.

Suppose $\pi$ is residually finite and the $L^2$-homology $H_k^{(2)}(\mathcal{C}) = 0$ for some $k$. For any normal subgroup $\Gamma < \pi$ of finite index, the homology group $H_k(\mathcal{C}_\Gamma)$ is a finitely-generated abelian group. Because $H_k^{(2)}(\mathcal{C}) = 0$ one should expect that $H_k(\mathcal{C}_\Gamma)$ is negligible. In fact, a theorem of Lück [Li1] (and Kazhdan for this case) says that

$$
\lim_{(\Gamma) \to \infty, |\pi: \Gamma| < \infty} \frac{\text{rk}_\mathbb{Z} H_k(\mathcal{C}_\Gamma)}{|\pi : \Gamma|} = 0.
$$

This means the free part $H_k(\mathcal{C}_\Gamma)_{\text{free}}$ of $H_k(\mathcal{C}_\Gamma)$ is small compared to the index. There is another measure of the free part $H_k(\mathcal{C}_\Gamma)_{\text{free}}$, denoted by $R_k(\mathcal{C}_\Gamma)$ and called the regulator, or volume, see [BV] and Section 3. Another expression of the fact that $H_k(\mathcal{C}_\Gamma)_{\text{free}}$ is small compared to the index is expressed in the following statement, which complements the result of Kazhdan-Lück.
**Theorem 2.** Suppose $C$ is a finitely generated based free $\mathbb{Z}[\pi]$-complex with $\pi = \mathbb{Z}^n$ and $H_k^{(2)}(C) = 0$ for some index $k$. Then

$$\lim_{(\Gamma) \to \infty, ||\pi \cdot \Gamma|| < \infty} \frac{\ln \text{vol}(H_k(\mathcal{C})_{\text{free}})}{||\pi \cdot \Gamma||} = 0.$$  

One conjectures that (2) holds for a large class of groups including hyperbolic groups.

**Remark 1.3.** In [Ra], it is proved that there is a sequence $\Gamma_n$, with $(\Gamma_n) \to \infty$, such that the limit on the left hand side of (2) along $\Gamma_n$ is 0. Theorem 2 implies that the limit is 0 for any such sequence.

1.6. **On the proofs.** For the proofs we use tools in commutative algebras and algebraic geometry. In particular, we make essential use of the theory of torsion points in $\mathbb{Q}$-algebraic set (a simple version of the Manin-Mumford principle). We hope that the methods and results can be adapted to the case of elementary amenable groups.

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1.8. **Organization of the paper.** In Section 2 we recall the notions of geometric determinant and volume. We discuss the relation between homology and Ray-Singer torsions in Section 3. An overview of the theory of torsion points in algebraic set is given in Section 4. Section 5 contains a version of the Manin-Mumford principle. We hope that the methods and results can be adapted to the case of elementary amenable groups.

2. **Geometric determinant, lattices and volume in based Hermitian spaces**

In this section we recall the definition of geometric determinant and basic facts about volumes of lattices in based Hermitian spaces.

2.1. **Geometric determinant.** For a linear map $f : V_1 \to V_2$, where each $V_i$ is a finite-dimensional Hermitian space the geometric determinant $\det'(f)$ is the product of all non-zero singular values of $f$. Recall that $x \in \mathbb{R}$ is singular value of $f$ if $x \geq 0$ and $x^2$ is an eigenvalue of $f^*f$. By convention $\det'(f) = 1$ if $f$ is the 0 map. Thus we always have $\det'(f) > 0$.

Since the maximal singular value of $f$ is the norm $||f||$, we have

$$\det'(f) \leq ||f||^\dim V_2 \quad \text{if } f \text{ is non-zero.}$$

**Remark 2.1.** The geometric meaning of $\det'f$ is the following. The map $f$ restricts to a linear isomorphism $f'$ from $\text{Im}(f^*)$ to $\text{Im}(f)$, each is a Hermitian space. Then $\det'f = |\det(f')|$, where the ordinary determinant $\det(f')$ is calculated using orthonormal bases of the Hermitian spaces.

2.2. **Based Hermitian space and volume.** Suppose $W$ is a finite-dimensional based Hermitian space, i.e. a $\mathbb{C}$-vector space equipped with an Hermitian product $(.,.)$ and a preferred orthonormal basis. The $\mathbb{Z}$-submodule $\Omega \subset W$ spanned by the basis is called the fundamental lattice.

For a $\mathbb{Z}$-submodule (also called a lattice) $\Lambda \subset W$ with $\mathbb{Z}$-basis $v_1, \ldots, v_l$ define

$$\text{vol}(\Lambda) = |\det((v_i, v_j)_{i,j=1}^l)|^{1/2}.$$  

By convention, the volume of the 0 space is 1. If $\Lambda \subset \Omega$, we say that $\Lambda$ is an integral lattice. It is clear that $\text{vol}(\Lambda) \geq 1$ if $\Lambda$ is an integral lattice.

For a $\mathbb{C}$-subspace $V \subset W$ the lattice $V^{(\mathbb{Z})} := V \cap \Omega$ is called the $\mathbb{Z}$-support of $V$. We define

$$\text{vol}(V) := \text{vol}(V^{(\mathbb{Z})}).$$

A lattice $\Lambda \subset \Omega$ is primitive if is cut out from $\Omega$ by some subspace, i.e. $\Lambda = V^{(\mathbb{Z})}$ for some subspace $V \subset W$. By definition, any primitive lattice is integral.

As usual, we say that a subspace $V \subset W$ is is defined over $\mathbb{Q}$ if it is defined by some linear equations with rational coefficients (using the coordinates in the preferred base). It is easy to see that $V$ is defined over $\mathbb{Q}$ if and only if it is spanned by its $\mathbb{Z}$-support.

Suppose $V_1, V_2$ are subspaces of $W$ defined over $\mathbb{Q}$, and $f : V_1 \to V_2$ is a $\mathbb{C}$-linear map. We say that $f$ is integral if $f(V_1^{(\mathbb{Z})}) \subset V_2^{(\mathbb{Z})}$.

We summarize some well-known properties of volumes of lattices (see e.g. [Ber]).
Proposition 2.1. Suppose $V_1, V_2$ are subspaces of $W$ defined over $\mathbb{Q}$ of a based Hermitian space $W$ and $f : V_1 \to V_2$ is an integral, non-zero $\mathbb{C}$-linear map. Then

\[ \text{vol}(V_1 + V_2) \leq \text{vol}(V_1) \text{vol}(V_2) \]
\[ \text{vol}(\ker f) \text{vol}[f(V_1^{(Z)})] = \det'(f) \text{vol}(V_1). \]

For a detailed discussion of (5) and its generalizations to lattices in $\mathbb{Z}[\mathbb{Z}^n]$, see [Ra].

3. Regulator, homology torsion, and Ray-Singer torsion

In this section we explain the relation between the homology torsion and the combinatorial Ray-Singer torsion. Proposition 3.1 of this section will be used in the proof of main theorems.

Throughout this section we fix a finitely-generated based free $\mathbb{Z}$-complex $\mathcal{E}$

\[ 0 \to E_m \xrightarrow{d_{m-1}} E_{m-1} \xrightarrow{d_{m-2}} \cdots \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \to 0. \]

Define a Hermitian product on $E \otimes \mathbb{C}$ such that the preferred base is an orthonormal base. Now $E_k \otimes \mathbb{C}$ becomes a based Hermitian space.

We use the notations

\[ Z_k = \ker d_k, \quad B_k = \text{Im} d_{k+1}, \quad \overline{B}_k = (B_k \otimes \mathbb{Z}) \cap E_k. \]

Let $d_k^* : E_{k-1} \to E_k$ be adjoint of $d_k$ and $D_k : E_k \to E_k$ be defined by

\[ D_k = d_k^* d_k + d_{k+1} d_{k+1}^*. \]

3.1. Ray-Singer torsion and homology torsion. Define the Ray-Singer torsion and the homology torsion of $\mathcal{E}$ by

\[ \tau^{RS}(\mathcal{E}) = \prod_k \det'(d_k) \in \mathbb{R}+, \]
\[ \tau^H(\mathcal{E}) = \left( \prod_k \text{vol}(H_k(\mathcal{E})) \right)^{-1}. \]

Remark 3.1. The Ray-Singer torsion and the homology torsion can be defined through the classical Reidemeister torsion as follows.

Let $\mathfrak{h}_k$ be an orthonormal basis of $\ker(Dk) \otimes \mathbb{Z} \mathbb{C} = H_k(\mathcal{E} \otimes \mathbb{Z} \mathbb{C})$. With the bases $\{ \mathfrak{h}_k \}$ of the homology of $\mathcal{E} \otimes \mathbb{Z} \mathbb{C}$, one can define the Reidemeister torsion $\tau^R(\mathcal{E} \otimes \mathbb{Z} \mathbb{C}, \{ \mathfrak{h}_k \})$, defined up to signs (see e.g. [Tu]). It is not difficult to show that

\[ \tau^{RS}(\mathcal{E}) = \left|\tau^R(\mathcal{E} \otimes \mathbb{Z} \mathbb{C}, \{ \mathfrak{h}_k \})\right|. \]

Both $\overline{B}_k$ and $Z_k$ are primitive lattices in $E_k$, and $\overline{B}_k \subset Z_k$. There is a collection $\mathfrak{h}_k$ of elements of $Z_k \subset E_k$ which descend to a basis of the group $Z_k/\overline{B}_k$, the free part of $H_k(\mathcal{E})$. Since $\mathfrak{h}_k$ is a basis of $H_k(\mathcal{E} \otimes \mathbb{Z} \mathbb{C})$, there is defined the Reidemeister torsion $\tau^R(\mathcal{E} \otimes \mathbb{Z} \mathbb{C}, \{ \mathfrak{h}_k \})$. It is not difficult to prove the following generalization of the Milnor-Turaev formula [Mi] [Tu].

\[ \tau^H(\mathcal{E}) = \left|\tau^R(\mathcal{E} \otimes \mathbb{Z} \mathbb{C}, \{ \mathfrak{h}_k \})\right|. \]

3.2. Regulators. By definition, $H_k(\mathcal{E}) = Z_k/B_k$. The $\mathbb{Z}$-torsion of $H_k(\mathcal{E})$ is $\overline{B}_k/B_k$, and the free part $H_k(\mathcal{E})_{\text{free}}$ is isomorphic to $Z_k/\overline{B}_k$. For this reason, we define the volume $\text{vol}(H_k(\mathcal{E})_{\text{free}})$ to be

\[ R_k(\mathcal{E}) := \frac{\text{vol}(Z_k)}{\text{vol}(\overline{B}_k)}. \]

Here we follow the notation of [BV], where $R_k$ is called the regulator. Using Identity (5), one can prove (see [BV] Formula 2.2.4)

\[ \tau^{RS}(\mathcal{E}) = \tau^H(\mathcal{E}) \left( \prod_k R_k(\mathcal{E}) \right). \]

We will use the following estimate of the regulator.
**Proposition 3.1.** Let $\tilde{R}_k := \text{vol}(\ker D_k)$. For every $k$ one has

$$\tilde{R}_k \geq R_k \geq \frac{1}{\tilde{R}_k}.$$ 

**Proof.** Let $W$ be the orthogonal complement of $B_k \otimes \mathbb{Z} \subset Z_k \otimes \mathbb{Z} \subset C$, and $p : Z_k \otimes \mathbb{Z} \rightarrow W$ be the orthogonal projection. Then

$$R_k = \text{vol}(p(Z_k)).$$

By Hodge theory (for finitely-generated $\mathbb{Z}$-complex),

$$\ker(D_k) = E_k \cap W = W^{(Z)}.$$ 

It follows that $\ker(D_k) \subset p(Z_k)$, and hence $\text{vol}(p(Z_k)) \leq \text{vol}(\ker(D_k))$, or

(7) $$R_k \leq \tilde{R}_k.\quad \Box$$ 

By [Ber, Proposition 1(ii)],

(8) $$R_k = \frac{\text{vol}(Z_k)}{\text{vol}(B_k)} = \left[\frac{[W \cap Z_k^*] : D_k]{R_k}}{R_k}\right],$$

where $Z_k^*$ is the $\mathbb{Z}$-dual of $Z_k$ in $Z_k \otimes \mathbb{Z} \subset C$ under the inner product. Note that $Z_k^*$ is also the orthogonal projection of $E_k$ on to $Z_k \otimes \mathbb{Z} \subset C$.

Since the numerator of (8) is $\geq 1$, we have $R_k \geq 1/\tilde{R}_k$, which, together with (7), proves the proposition.

4. **Abelian groups, algebraic subgroups of $(\mathbb{C}^*)^n$, and torsion points**

We review some facts about representation theory of finite abelian groups in subsection 4.1 and the theory of torsion points on rational algebraic sets (a simple version of Manin-Mumford principle) in subsections 4.2 and 4.3.

4.1. **Decomposition of the group ring of a finite abelian group.** Suppose $A$ is a finite abelian group. The group ring $\mathbb{C}[A]$ is an $A$-module (the regular representation) and is a $\mathbb{C}$-vector space of dimension $|A|$. Equip $\mathbb{C}[A]$ with a Hermitian product so that $A$ is an orthonormal basis. This makes $\mathbb{C}[A]$ a based Hermitian space, with $\mathbb{Z}[A]$ the fundamental lattice.

Let $A = \text{Hom}(A, \mathbb{C}^*)$, known as the Pontryagin dual of $A$, be the group of all characters of $A$. Here $\mathbb{C}^*$ is the multiplicative group of non-zero complex numbers. We have $|\hat{A}| = |A|$.

The theory of representations of $A$ over $\mathbb{C}$ is easy: $\mathbb{C}[A]$ decomposes as a direct sum of mutually orthogonal one-dimensional $A$-modules:

(9) $$\mathbb{C}[A] = \bigoplus_{\chi \in \hat{A}} \mathbb{C}e_{\chi},$$

where $e_{\chi}$ is the idempotent

(10) $$e_{\chi} = \frac{1}{|A|} \sum_{a \in A} \chi(a^{-1})a.$$ 

The vector subspaces $\mathbb{C}e_{\chi}$’s are not only orthogonal with respect to the Hermitian structure, but also orthogonal with respect to the ring structure in the sense that $e_{\chi} e_{\chi'} = 0$ if $\chi \neq \chi'$. Each $\mathbb{C}e_{\chi}$ is an ideal of the ring $\mathbb{C}[A]$.

From the trace identity (see e.g. [Se, Section 2.4]) we have, for every $a \in A$,

(11) $$\sum_{\chi \in \hat{A}} \chi(a) = \begin{cases} 0 & \text{if } a \neq e \\ |A| & \text{if } a = e. \end{cases}$$

Here $e \in A$ is the trivial element.

4.2. **Algebraic subgroups of $(\mathbb{C}^*)^n$ and lattices in $\mathbb{Z}^n$.**
4.2.1. **Algebraic subgroups of** \((\mathbb{C}^*)^n\). An algebraic subgroup of \((\mathbb{C}^*)^n\) is a subgroup which is closed in the Zariski topology.

For a lattice \(\Lambda\), i.e. a subgroup \(\Lambda\) of \(\mathbb{Z}^n\), not necessarily of maximal rank, let \(G(\Lambda)\) be the set of all \(z \in \mathbb{C}^n\) such that \(z^k = 1\) for every \(k \in \Lambda\). Here for \(k = (k_1, \ldots, k_n) \in \mathbb{Z}^n\) and \(z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n\) we set \(z^k = \prod_{j=1}^nz_{j}^{k_j}\).

It is easy to see that \(G(\Lambda)\) is an algebraic subgroup. The converse holds true: Every algebraic subgroup is equal to \(G(\Lambda)\) for some lattice \(\Lambda\), see [Sch]. If \(\Lambda\) is primitive, then \(G(\Lambda)\) is connected, and in this case it is called a torus.

4.2.2. **Automorphisms of** \((\mathbb{C}^*)^n\). An example of a torus of dimension \(l\) is the standard \(l\)-torus \(T = (\mathbb{C}^*)^l \times \mathbb{C}^{n-l} \subset (\mathbb{C}^*)^n\), which is \(G(\Xi_{n-l})\), where

\[
\Xi_{n-l} = \{(k_1, \ldots, k_n) \in \mathbb{Z}^n | k_1 = \cdots = k_l = 0\}.
\]

The following trick shows that each torus is isomorphic to the standard torus. For details see [Sch].

For matrix \(K \in GL(n, \mathbb{Z})\) with entries \((K_{ij})_{i,j=1}^n\), one can define an automorphism \(\varphi_K\) of \((\mathbb{C}^*)^n\) by

\[
\varphi_K(z_1, z_2, \ldots, z_n) = \left( \prod_{j=1}^n z_j^{K_{1j}}, \prod_{j=1}^n z_j^{K_{2j}}, \ldots, \prod_{j=1}^n z_j^{K_{nj}} \right).
\]

For any lattice \(\Lambda \subset \mathbb{Z}^n\), \(\varphi_K(G(\Lambda)) = G(\Lambda)\). When \(\Lambda\) is a primitive lattice of rank \(n-l\), there is \(K \in GL_n(\mathbb{Z})\) such that \(K(\Lambda) = \Xi_{n-l}\). Then \(\varphi_K(G(\Lambda))\) is the standard \(l\)-torus.

4.2.3. **Algebraic subgroups and character groups.** Fix generators \(t_1, \ldots, t_n\) of \(\mathbb{Z}^n\). We will write \(\mathbb{Z}^n\) multiplicatively and use the identification \(\mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]\).

Suppose \(\Gamma \subset \mathbb{Z}^n\) is a lattice. Every element \(z \in G(\Gamma)\) defines a character \(\chi_z\) of the quotient group \(A_\Gamma := \mathbb{Z}^n/\Gamma\) via

\[
\chi_z(t_1^{k_1} \cdots t_n^{k_n}) = z^k, \quad \text{where} \quad k = (k_1, \ldots, k_n).
\]

Conversely, every character of \(A_\Gamma\) arises in this way. Thus one can identify \(G(\Gamma)\) with the Pontryagin dual \(\hat{A}_\Gamma\) via \(z \mapsto \chi_z\).

We will write \(e_z\) for the idempotent \(e_{\chi_z}\), and the decomposition \([A_\Gamma] = \bigoplus_{\chi \in G(\Gamma)} \mathbb{C} e_\chi\).

4.3. **Torsion points in \(\mathbb{Q}\)-algebraic sets.**

4.3.1. **Torsion points.** With respect to the usual multiplication \(\mathbb{C}^* := \mathbb{C} \setminus \{0\}\) is an abelian group, and so is the direct product \((\mathbb{C}^*)^n\). The subgroup of torsion elements of \(\mathbb{C}^*,\) denoted by \(U\), is the group of roots of unity, and \(U^n\) is the torsion subgroup of \((\mathbb{C}^*)^n\).

If \(\Gamma \subset \mathbb{Z}^n\) is a lattice of maximal rank, then \(G(\Gamma)\) is finite, and \(G(\Gamma) \subset U^n\).

4.3.2. **Torsion points and torsion coset.** A torsion coset is any set of the form \(z G(\Lambda)\), where \(z \in U^n\) and \(\Lambda \subset \mathbb{Z}^n\) is primitive, i.e. \(G(\Lambda)\) is a torus.

Suppose \(Z \subset \mathbb{C}^n\). A torsion coset \(X \subset Z\) is called a maximal torsion coset in \(Z\) if it is not a proper subset of any torsion coset in \(Z\).

The following fact, known as the Manin-Mumford theory for torsion points, is well-known, see [Lau, Sch].

**Proposition 4.1.** [Lau] Suppose \(Z \subset \mathbb{C}^n\) is an algebraic closed set defined over \(\mathbb{Q}\). There are in total only a finite number of maximal torsion cosets \(X_j \subset Z, \ j = 1, \ldots, q\). A torsion point \(z \in U^n\) belongs to \(Z\) if and only if \(z \in X_j\) for some \(j\), i.e.

\[
Z \cap U^n = \bigcup_{j=1}^q (X_j \cap U^n).
\]
4.3.3. \textit{Q-closure}. Let \( \text{Gal}(\mathbb{C}/\mathbb{Q}) \) be the set of all field automorphisms of \( \mathbb{C} \) fixing every point in \( \mathbb{Q} \). For \( \mathbf{z} = (z_1, \ldots, z_n) \), a Galois conjugate of \( \mathbf{z} \) is any point of the form \( \sigma(\mathbf{z}) = (\sigma(z_1), \ldots, \sigma(z_n)) \), where \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \).

Suppose \( X \subset \mathbb{C}^n \). Define the \textit{Q-closure} of \( X \) by

\[
\text{cl}_\mathbb{Q}(X) := \bigcup_{\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})} \sigma(X).
\]

If \( X \subset Z \), where \( Z \subset \mathbb{C}^n \) is an algebraic set defined over \( \mathbb{Q} \), then \( \text{cl}_\mathbb{Q}(X) \subset Z \).

If \( \mathbf{z} \in \mathbb{U}^n \) is a torsion point, then the \( \mathbb{Q} \)-closure of \( \{ \mathbf{z} \} \), also denoted by \( \text{cl}_\mathbb{Q}(\mathbf{z}) \), consists of a finite number of torsion points. One can prove that if torsion order of \( \mathbf{z} \) is \( k \), then \( |\text{cl}_\mathbb{Q}(\mathbf{z})| = \phi(k) \), where \( \phi \) is the Euler totient function. Though we don’t need this result.

**Lemma 4.2.** Let \( X \subset (\mathbb{C}^*)^n \) be a torsion coset. Then there exists a torsion point \( \mathbf{u} \in \mathbb{U}^n \) and a primitive lattice \( \Lambda \subset \mathbb{Z}^n \) such that

\[
\text{cl}_\mathbb{Q}(X) = \bigsqcup_{\mathbf{z} \in \text{cl}_\mathbb{Q}(\mathbf{u})} \mathbf{z}(\Lambda).
\]

**Proof.** By definition, there is a torus \( T \) of dimension \( l \) and a torsion point \( \mathbf{u}' \) such that \( X = \mathbf{u}'T \). Using an automorphism as described in Subsection \([4.2.2])\), we can assume that \( T \) is the standard \( l \)-torus, \( T := (\mathbb{C}^*)^l \times 1^{n-l} \). Let \( \mathbf{u} \in \mathbb{U}^n \) be the point whose first \( l \)-coordinates are 1, and the \( j \)-th coordinate is the same as that of \( \mathbf{u}' \) for \( j > l \). Alternatively, \( \mathbf{u} \) is the only intersection point of \( X \) and \( 1^l \otimes (\mathbb{C}^*)^{n-l} \).

Then \( \mathbf{u}T = \mathbf{u}'T = X \). Any Galois conjugate of \( \mathbf{u} \) is in \( 1^l \otimes (\mathbb{C}^*)^{n-l} \). If \( \mathbf{z}, \mathbf{z}' \) are two distinct Galois conjugates of \( \mathbf{u} \), then \( \mathbf{z}^{-1}\mathbf{z}' \in 1^l \otimes (\mathbb{C}^*)^{n-l} \), and hence \( \mathbf{z}^{-1}\mathbf{z}' \not\in \Lambda(\mathbf{u}) \). It follows easily that

\[
\text{cl}_\mathbb{Q}(X) = \text{cl}_\mathbb{Q}(\mathbf{u}T) = \bigsqcup_{\mathbf{z} \in \text{cl}_\mathbb{Q}(\mathbf{u})} \mathbf{z}T. \quad \square
\]

5. A GROWTH RATE ESTIMATE

In this section we prove Proposition \([5.4])\), establishing a crucial growth estimate of volumes of subspaces depending on a torsion coset \( X \) and a lattice \( \Gamma \subset \mathbb{Z}^n \) of maximal rank.

5.1. Settings and notations. Throughout this section fix a torsion coset \( X \subset (\mathbb{C}^*)^n, X \neq (\mathbb{C}^*)^n \), and a \( k \times l \) matrix \( D \) with entries in \( \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \).

By right multiplication, we consider \( D \) as a \( \mathbb{C}[\mathbb{Z}^n] \)-morphism

\[
D : \mathbb{C}[\mathbb{Z}^n]^k \to \mathbb{C}[\mathbb{Z}^n]^l.
\]

Suppose \( \Gamma \leq \mathbb{Z}^n \) is a lattice of maximal rank. In what follows we fix \( X, D \) but vary \( \Gamma \).

Let \( A = A_{\Gamma} := \mathbb{Z}^n/\Gamma \), a finite abelian group. Equip \( \mathbb{C}[A] \) with the structure of a based Hermitian space as in Section \([4.1])\). The fundamental lattice of \( \mathbb{C}[A] \) is \( \mathbb{Z}[A] \).

The map \( D \) in \([4.1])\) descends to an integral \( \mathbb{C}[A] \)-morphism

\[
D_{\Gamma} : \mathbb{C}[A]^k \to \mathbb{C}[A]^l.
\]

By Lemma \([4.2])\) there is a primitive lattice \( \Lambda \leq \mathbb{Z}^n \) and a torsion point \( \mathbf{u} \in \mathbb{U}^n \) such that

\[
\text{cl}_\mathbb{Q}(X) = \bigsqcup_{j=1}^{r} \mathbf{u}_j \Gamma(A),
\]

where \( \{ \mathbf{u}_j, j = 1, \ldots, r \} \) is the set of all Galois conjugates of \( \mathbf{u} \). Since \( X \neq (\mathbb{C}^*)^n \), \( \Lambda \) is not the trivial group, \( \Lambda \neq \{0\} \).

Let \( B = B_{\Gamma, \Lambda} := (\Gamma + \Lambda)/\Gamma \). Then \( B \) is a subgroup of \( A = \mathbb{Z}^n/\Gamma \).

5.2. Decomposition of \( D_{\Gamma} \) and norm of \( D_{\Gamma} \). Recall that \( \hat{A} = G(\Gamma) \) and we have the decomposition \([12])\) of \( A \)-module

\[
\mathbb{C}[A] = \bigoplus_{\mathbf{z} \in \hat{A}} \mathbb{C}\mathbf{e}_\mathbf{z}.
\]

For each \( \mathbf{z} \in \hat{A} = G(\Gamma) \), \( D \) induces a \( \mathbb{C}[A] \)-map

\[
D(\mathbf{z}) : (\mathbb{C}\mathbf{e}_\mathbf{z})^k \to (\mathbb{C}\mathbf{e}_\mathbf{z})^l.
\]
Here $D(z)$ is simply the $k \times l$ matrix with entries in $\mathbb{C}$, obtained by evaluating $D$ at $z$. (Recall that each entry of $D$ is a Laurent polynomial in $n$ variables, and one can evaluate such a Laurent polynomial at any point $z \in (\mathbb{C}^*)^n$.) We have

$$D_\Gamma = \bigoplus_{z \in \hat{A}} D(z).$$

It follows that

$$||D_\Gamma|| = \max_{z \in \hat{A}} ||D(z)||.$$

It is easy to see that the norm of any $k \times l$ matrix is less than or equal to $kl$ times the sum of the absolute values of all the entries.

For a Laurent polynomial $f \in \mathbb{Z}[\mathbb{Z}^n]$ let the $\ell^1$-norm of $f$ be the sum of the absolute values of all the coefficients of $f$. Let $||D||_1$ be the sum of the $\ell^1$-norms of all of its entries. Then we have $||D(z)|| \leq kl||D||_1$ because each component of $z$ has absolute value 1. Thus we have the following uniform upper bound for $D_\Gamma$:

$$(16) \quad ||D_\Gamma|| \leq kl||D||_1.$$

5.3. Integral decomposition of $D_\Gamma$ along $X$. For each $z \in G(\Gamma)$, the 1-dimensional vector space $\mathbb{C}e_z$ is in general not defined over $\mathbb{Q}$. However, its $\mathbb{Q}$-closure is defined over $\mathbb{Q}$.

Consider the following $\mathbb{C}[A]$-submodule $\alpha(\Gamma, X)$ of $\mathbb{C}[A]$:

$$\alpha(\Gamma, X) := \bigoplus_{z \in \hat{A} \cap \text{cl}_\mathbb{Q}(X)} \mathbb{C}e_z \subset \mathbb{C}[A] = \bigoplus_{z \in \hat{A}} \mathbb{C}e_z.$$

Since the set $\hat{A} \cap \text{cl}_\mathbb{Q}(X)$ is closed under Galois conjugations, $\alpha(\Gamma, X)$ is defined over $\mathbb{Q}$. The orthogonal complement of $\alpha(\Gamma, X)$ is

$$\alpha^\perp(\Gamma, X) := \bigoplus_{z \in \hat{A} \setminus \text{cl}_\mathbb{Q}(X)} \mathbb{C}e_z.$$

We have $\mathbb{C}[A] = \alpha(\Gamma, X) \oplus \alpha(\Gamma, X)$, and each of $\{\alpha(\Gamma, X), \alpha^\perp(\Gamma, X)\}$ is an ideal of $\mathbb{C}[A]$, or an $A$-subspace of $\mathbb{C}[A]$.

The $A$-morphism $D_\Gamma$ restricts to $A$-morphisms

$$D_{\Gamma, X} : \alpha(\Gamma, X)^k \to \alpha(\Gamma, X)^l \quad \text{and} \quad D_{\Gamma, X}^\perp : (\alpha^\perp(\Gamma, X))^k \to (\alpha^\perp(\Gamma, X))^l,$$

which are integral. We have $D_\Gamma = D_{\Gamma, X} \oplus D_{\Gamma, X}^\perp$.

5.4. Projection onto $\alpha_{\Gamma, X}$.

Lemma 5.1. If $G(\Gamma) \cap X \neq \emptyset$, then

$$(17) \quad \dim_\mathbb{C}(\alpha_{\Gamma, X}) = r|A|/|B|$$

and the orthogonal projection from $\mathbb{C}[A]$ onto $\alpha_{\Gamma, X}$ is given by the idempotent

$$(18) \quad N_X := \frac{1}{|B|} \sum_{j=1}^r \sum_{b \in B} u_j (b^{-1})b.$$

Proof. Since $\hat{A} = G(\Lambda)$ is defined over $\mathbb{Q}$, if it intersects $X$, then it intersects every component $u_j G(\Lambda)$ of the decomposition $[13]$ of $\text{cl}_\mathbb{Q}(X)$. Let $u'_j \in G(\Gamma) \cap u_j G(\Lambda)$. Since $G(\Lambda)$ is a subgroup of $(\mathbb{C}^*)^n$ and $u'_j \in u_j G(\Lambda)$, we have $u_j G(\Lambda) = u'_j G(\Lambda)$ and

$$(19) \quad u_j^{-1}u'_j \in G(\Lambda).$$

Since $G(\Gamma)$ is a subgroup and $u'_j \in G(\Gamma)$, we have $G(\Gamma) = u'_j G(\Gamma)$, and hence

$$G(\Gamma) \cap u_j G(\Lambda) = u'_j |G(\Gamma) \cap G(\Lambda)| = u'_j |G(\Gamma + \Lambda)|.$$

From the above identity and the decomposition $[13]$ we have

$$(20) \quad G(\Gamma) \cap \text{cl}_\mathbb{Q}(X) = \bigcup_{j=1}^r u'_j |G(\Gamma + \Lambda)|.$$
Let $s = |A|/|B|$, which is the cardinality of the quotient group $A/B = \mathbb{Z}^n/(\Gamma + \Lambda)$. Then $|G(\Gamma + \Lambda)| = s$, since $G(\Gamma + \Lambda)$ is the Pontryagin dual of $A/B = \mathbb{Z}^n/(\Gamma + \Lambda)$. From (20) we have $|A \cap cl_q(X)| = rs$. Hence $\dim_{\mathbb{C}}(\alpha_{\Gamma,X}) = rs = r|A|/|B|$. This proves (17).

For each element of the quotient group $A/B$ choose a lift in $A$, and denote by $C$ the set of all such lifts. We assume that the chosen lift of the trivial element is $e$. Then

$$A = \{bc \mid b \in B, c \in C\}.$$ 

From (11), for every $c \in C$,

$$\sum_{z \in \hat{A} \cap cl_q(X)} \sum_{a \in A} z(a^{-1})a = \frac{1}{|A|} \sum_{z \in \hat{A} \cap cl_q(X)} \sum_{b \in B, c \in C} z(b^{-1})bc = \frac{1}{|A||C|} \sum_{z \in \hat{A} \cap cl_q(X)} \sum_{b \in B} z(b^{-1})b \left[ \sum_{c \in C} z(c^{-1})c \right]$$

$$= \frac{1}{|B||C|} \sum_{j=1}^{r} \sum_{z \in \hat{A} \cap cl_q(X)} \left[ \sum_{b \in B} z(b^{-1})u_j(b^{-1})b \right] \left[ \sum_{c \in C} z(c^{-1})u_j(c^{-1})c \right]$$

by (20)

$$= \frac{1}{|B||C|} \sum_{j=1}^{r} \left[ \sum_{b \in B} u_j(b^{-1})b \right] \sum_{b \in B} \sum_{c \in C} z(c^{-1})u_j(c^{-1})c$$

by (21)

$$= \frac{1}{|B|} \sum_{j=1}^{r} \sum_{b \in B} u_j(b^{-1})b$$

by (19).

This completes the proof of the lemma.

\[ \text{□} \]

5.5. Upper bound for the volume of $\alpha_{\Gamma,X}$.

Lemma 5.2. One has

$$\text{vol}(\alpha_{\Gamma,X}) \leq (r|B|)^{r|A|/|B|}. \tag{22}$$

Proof. We assume that $G(\Gamma) \cap X \neq \emptyset$, because otherwise $\alpha_{\Gamma,X} = 0$, and the statement is trivial.

From (18),

$$|B|N_X = \sum_{b \in B} \left( \sum_{j=1}^{r} u_j(b^{-1}) \right) b \in \mathbb{Z}[B].$$

If $a \in A$, then from (18) we have

$$|B|N_X a = \sum_{j=1}^{r} \sum_{b \in B} u_j(b^{-1}) ba.$$ 

Because $|u_j(b^{-1}) ba| = 1$, we see that $|B|N_X a$ has length $\leq r|B|$.

By Lemma 5.1, $\alpha(\Gamma, X)$ has dimension $rs$. Since $A$ spans $\mathbb{C}[A]$ and $N_X$ is the projection onto $\alpha_{\Gamma,X}$, the set $\{N_X a \mid a \in A\}$ spans $\alpha_{\Gamma,X}$. Hence, there are $a_1, \ldots, a_{rs} \in A$ such that $\{N_X a_j, j = 1, \ldots, rs\}$ is a basis of the vector space $\alpha_{\Gamma,X}$. Since $|B|N_X a_j \in \mathbb{Z}[A]$ is integral, $\text{vol}(\alpha_{\Gamma,X})$ is less than or equal to the volume of the parallelepiped spanned by $\{B|N_X a_j, j = 1, \ldots, rs\}$. The length of each $|B|N_X a_j$ is $\leq r|B|$. It follows that $\text{vol}(\alpha) \leq (r|B|)^{rs} = (r|B|)^{r|A|/|B|}$. \[ \text{□} \]
5.6. Growth of $|B|$. The following statement has been used in \[Le2\]. This is the place we use the assumption $\langle \Gamma \rangle \to \infty$ and $\Lambda \neq 0$.

**Lemma 5.3.** Suppose $\Lambda$ is a non-trivial lattice. Then

\[
\lim_{\langle \Gamma \rangle \to \infty} |B| = \infty.
\]

**Proof.** For the length of $x \in \pi = \mathbb{Z}^n$ (in the definition of $\langle \Gamma \rangle$) we will use the standard metric $|x|$ derived from the Hermitian structure.

Fix an element $x \in \Lambda$, $x \neq 0$, and look at the degree of $x$ in $B = (\Lambda + \Gamma)/\Gamma$. If $M|x| < \langle \Gamma \rangle$ for some positive integer $M$, then $M|x|$ does not belong to $\Gamma$ by the definition of $\langle \Gamma \rangle$, and hence $Mx$ is not 0 in $B = (\Lambda + \Gamma)/\Gamma$. This means the cyclic subgroup of $B$ generated by $x$ has order at least $\langle \Gamma \rangle/|x|$. It follows that $|B| \geq \langle \Gamma \rangle/|x|$. Hence $\lim_{\langle \Gamma \rangle \to \infty} |B| = \infty$. \hfill $\square$

5.7. Growth of $\ker(D_{\Gamma,X})$.

**Proposition 5.4.** For a fixed torsion coset $X \subset (\mathbb{C}^*)^n$ and a matrix $D$ with entries in $\mathbb{Z}[\mathbb{Z}^n]$, we have

\[
\lim_{\langle \Gamma \rangle \to \infty, |\mathbb{Z}^n : \Gamma| < \infty} \frac{\ln \left[ \text{vol}(\ker(D_{\Gamma,X})) \right]}{|\mathbb{Z}^n : \Gamma|} = 0.
\]

**Proof.** To simplify notations we will write $\alpha = \alpha_{\Gamma,X}$. By \[Le\],

\[
\text{vol}(\ker(D_{\Gamma,X})) \text{ vol}(\text{Im}(D_{\Gamma,X}^{(\mathbb{Z})}) = \det' D_{\Gamma,X} \text{ vol}(\alpha).
\]

Because $\text{vol}(\text{Im}(D_{\Gamma,X}^{(\mathbb{Z})})) \geq 1$, we have

\[
\text{vol}(\ker(D_{\Gamma,X})) \leq \text{det}' D_{\Gamma,X} \text{ vol}(\alpha)
\]

\[
\leq ||D_{\Gamma,X}||^{\dim(\alpha)} \text{ vol}(\alpha) \quad \text{ by (3)}
\]

\[
\leq ||D_{\Gamma,X}||^{|rA|/|B|} (r|B|)^{r|A|/|B|} \quad \text{ by (17) and (22)}
\]

Since $D_{\Gamma,X}$ is a restriction of $D_{\Gamma}$ on a subspace, we have

\[
||D_{\Gamma,X}|| \leq ||D_{\Gamma}|| \leq kl||D||_1,
\]

where the second inequality is \[Le\].

It follows that

\[
\text{vol}(\ker(D_{\Gamma,X})) \leq (klr ||D||_1 |B|)^{r|A|/|B|},
\]

and

\[
\frac{\ln \left[ \text{vol}(\ker(D_{\Gamma,X})) \right]}{|A|} \leq \frac{r \ln(klr ||D||_1 |B|)}{|B|}.
\]

Because $|B| \to \infty$ as $\langle \Gamma \rangle \to \infty$ by \[Le\], the right hand side of (24) goes to 0 as $\langle \Gamma \rangle \to \infty$. \hfill $\square$

6. Proofs of Theorems \[1\] and \[2\].

It is clear that Theorem \[1\] follows from Theorem \[2\] and Identity \[6\]. We will prove Theorem \[2\] in this section.

6.1. Preliminaries. Recall that $\mathcal{C}$ is a finitely-generated based free $\mathbb{Z}[\mathbb{Z}^n]$-complex

\[
0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} C_{m-2} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0.
\]

Using the bases of $C_j$’s we identify $C_j$ with $\mathbb{Z}[\mathbb{Z}^n]^{b_j}$ and $\partial_j$ with a $b_j \times b_{j-1}$ matrix with entries in $\mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[\mathbb{Z}^{t_j^1}, \ldots, \mathbb{Z}^{t_j^{n_j}}]$.

For $f = \sum a_j g_j \in \mathbb{Z}[\mathbb{Z}^n]$, where $a_j \in \mathbb{Z}$ and $g_j \in \mathbb{Z}^n$ let $f^* = \sum a_j g_j^{-1}$. If $f$ acts on $\ell^2(\mathbb{Z}^n)$, the Hilbert space with basis $\mathbb{Z}^n$, by multiplication, then $f^*$ is the adjoint operator of $f$. As usual, for a matrix $O = (O_{ij})$ with entries in $\mathbb{Z}[\mathbb{Z}^n]$ let the adjoint $O^*$ be defined by $(O^*)_{ij} = (O_{ji})^*$.

According to the assumption of Theorem \[2\]

\[
H_k(\mathcal{C} \otimes \mathbb{Z}[\mathbb{Z}^n]F) = 0,
\]

where $F$ is the fractional field of $\mathbb{Z}[\mathbb{Z}^n]$. Let

\[
D = \partial_k^* \partial_k + \partial_{k+1}^* \partial_{k+1} : C_k \to C_k.
\]
Then (28) is equivalent to the fact that \( D \) is an injective map, or that \( \det(D) \neq 0 \). Here \( \det(D) \) is the usual determinant of a square matrix with entries in a commutative ring. In our case \( \det(D) \) is a Laurent polynomial in \( t_1, \ldots, t_n \).

Let \( Z \) be the zero set of the Laurent polynomial \( \det(D) \) i.e. \( Z = \{ z \in \mathbb{C}^n \mid \det(D)(z) = 0 \} \). In other words, \( Z \) is the set of \( z \in \mathbb{C}^n \) such that the square matrix \( D(z) \) is singular. Since \( \det(D) \neq 0 \), \( Z \) is not the whole \( \mathbb{C}^n \).

For every subgroup \( \Gamma \leq \mathbb{Z}^n \), \( D \) induces a map \( D_\Gamma : \mathbb{C}[A]^{b_k} \to \mathbb{C}[A]^{b_k} \), where \( A = \mathbb{Z}^n / \Gamma \).

### 6.2. Growth of \( \text{vol}(\ker(D_\Gamma)) \)

**Proposition 6.1.** In the above setting, we have

\[
\lim_{(\Gamma) \to \infty, |\pi : \Gamma| < \infty} \frac{\ln[\text{vol}(\ker(D_\Gamma))]}{|\pi : \Gamma|} = 0.
\]

**Proof.** By Proposition 4.1, there are in total a finite number of maximal torsion cosets \( X_j \subset Z \), sets \( j = 1, \ldots, q \), and

\[
Z \cap U^n = \bigcup_{j=1}^q (X_j \cap U^n).
\]

Since \( Z \neq \mathbb{C}^n \), none of the lattices associated to the torsion cosets \( X_j \) is trivial.

Because \( Z \) is defined over \( \mathbb{Q} \), any Galois conjugate of \( X_j \) is also a maximal torsion coset in \( Z \), i.e. among \( \{ X_1, \ldots, X_q \} \).

Since \( G(\Gamma) \subset U^n \), one has

\[
Z \cap G(\Gamma) = (Z \cap U^n) \cap G(\Gamma) = \bigcup_{j=1}^q (X_j \cap U^n) \cap G(\Gamma) = \bigcup_{j=1}^q (X_j \cap G(\Gamma)).
\]

Because the Galois conjugate of \( X_j \’s \) are among the \( X_j \’s \), we also have

\[
Z \cap G(\Gamma) = \bigcup_{j=1}^q (\text{cl}_\mathbb{Q}(X_j) \cap G(\Gamma)).
\]

Let

\[
\alpha := \bigoplus_{z \in G(\Gamma) \cap Z} \mathbb{C} e_z, \quad \alpha^\perp := \bigoplus_{z \in G(\Gamma) \setminus Z} \mathbb{C} e_z.
\]

Then \( \mathbb{C}[A] = \alpha \oplus \alpha^\perp \), and each of \( \{ \alpha, \alpha^\perp \} \) is an \( A \)-subspace of \( \mathbb{C}[A] \). The linear operator \( D_\Gamma \) restricts to

\[
D_{\Gamma, \alpha} : \alpha^{b_k} \to \alpha^{b_k} \quad \text{and} \quad D_{\Gamma, \alpha}^\perp : (\alpha^\perp)^{b_k} \to (\alpha^\perp)^{b_k},
\]

and

\[
D_\Gamma = D_{\Gamma, \alpha} \oplus D_{\Gamma, \alpha}^\perp,
\]

where

\[
D_{\Gamma, \alpha} = \bigoplus_{z \in G(\Gamma) \cap Z} D(z), \quad D_{\Gamma, \alpha}^\perp = \bigoplus_{z \in G(\Gamma) \setminus Z} D(z).
\]

When \( z \not\in Z \), \( D(z) \) is non-singular. It follows that \( D_{\Gamma, \alpha}^\perp \) is non-singular. Hence,

\[
\ker(D_\Gamma) = \ker(D_{\Gamma, \alpha})
\]

\[
= \bigoplus_{z \in G(\Gamma) \cap Z} \ker(D(z)) \quad \text{by (28)}
\]

\[
= \sum_{j=1}^q \left( \bigoplus_{z \in \text{cl}_\mathbb{Q}(X_j) \cap G(\Gamma)} \ker(D(z)) \right) \quad \text{by (27)}
\]

From (27), we have

\[
\alpha = \sum_{j=1}^q \left[ \bigoplus_{z \in G(\Gamma) \cap \text{cl}_\mathbb{Q}(X_j)} \mathbb{C} e_z \right] = \sum_{j=1}^q \alpha(\Gamma, X_j),
\]
where $\alpha(\Gamma, X_j)$ is defined as in Section 5.3. One also has

$$\text{ker}(D_{\Gamma, X_j}) = \bigoplus_{z \in G(\Gamma) \cap \text{cl}_0(X_j)} \text{ker}(D(z)).$$

From (29) and (30), one has

$$\text{ker}(D_{\Gamma}) = \sum_{j=1}^{q} \text{ker}(D_{\Gamma, X_j}).$$

Hence, by (4),

$$\text{vol}(\text{ker}(D_{\Gamma})) \leq \prod_{j=1}^{q} \text{vol}(\text{ker}(D_{\Gamma, X_j})).$$

Now (26) follows from the above inequality and Proposition 5.4. □

6.3. **Proof of Theorem 2.** Applying Proposition 3.1 to the $Z$-complex $E = C_{\Gamma}$, we get

$$\text{vol}(\text{ker}(D_{\Gamma})) \geq R_k(C_{\Gamma}) \geq \frac{1}{\text{vol}(\text{ker}(D_{\Gamma}))}.$$ 

Thus we have

$$\frac{\ln(\text{vol}(\text{ker}(D_{\Gamma})))}{|Z^n : \Gamma|} \geq \frac{\ln(R_k(C_{\Gamma}))}{|Z^n : \Gamma|} \geq -\frac{\ln(\text{vol}(\text{ker}(D_{\Gamma}))}{|Z^n : \Gamma|}.$$ 

By Proposition 6.1, the limits of the two boundary terms, when $\langle \Gamma \rangle \to \infty$ and $|Z^n : \Gamma| < \infty$, are 0. Hence we also have

$$\lim_{\langle \Gamma \rangle \to \infty, |Z^n : \Gamma| < \infty} \frac{\ln(R_k(\Gamma))}{|Z^n : \Gamma|} = 0.$$ 

This completes the proof of Theorem 2.

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