Dispersion and Continuum Models
of powder Elasticity †

Hirosuke Okamoto
Engineering Reserch Institute
Faculty of Eng., Univ.of Tokyo *

Abstract

A granular medium is idealized here by a model composed of stress/strain field deviations in a random continuum. Total stress-strain relationships for this model are derived by integrating relationships. Further more, the theory is developed to explain the dispersion of internal microstress, initial elastic strain, and local yield stress on the macroscopic behavior of a granular medium. Predictions of the theory are reported and agree with the results of the experimental work.

1. Introduction

As is known, granular samples, though produced by a same material, are remarkably showing different deformation behavior and strength characteristics when the preparation procedures are different. Starting from such a condition, which suggests that the initial anisotropy and the physical quantity are showing material property of granule which varies with place, in general. This means that granule must be microscopically treated as a non-homogeneous system consisting of many different structures. One of the characteristics featuring the construction of irregular system formed by this spatial variation is the statistical structure. Here, information on the geometrical configuration of particles around a particle and information on their interface are among the important parameters too. In addition, granule is given the stochastic character derived from an uncertainty coming from the discrete state where a granule is composed of particles. To deal with the mechanical property of granule, therefore, statistical and stochastic techniques are necessary and effective as well.

Regarding such a granule composed of irregular and uncertain elements, the physical quantity can be locally defined if assuming a continuum, where the conventional continuum mechanics can be applied. Moreover, if variations of various quantities are much less than the typical quantities expressing the features of an entire system, or if the optional portions of a system indicate the situation of statistically same variations (distributions), the statistical mechanics will be applicable.

Theories dealing with granule as continuum so far furnished less direct discussions by considering deformation as a principal, and treating it according to the stress indication. In understanding the deformation in the light of subsequent observations; the connection and correlation of forces measured externally as even continuum and the microscopic quantities according to the deformation itself were treated in discussions in the multidimensional space generalized, and the relevant equations were examined statistically.

2. Expression As A Random Continuum

In an actual behavior to give force to granule, it is assumed that an even strain is not spread entirely at a moment, but the relative displacement takes place at contacting points

* Yayoi Bunkyo-ku, Tokyo 113 Japan
TEL 03-3812-2111
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of finite number, which then appear and where slipping is caused among particles in unstable places; the particles move there in separating or contacting. Portions pulled by the movement and rotate around the particle bulks; the equilibrium and compatibility of the forces are formed entirely.

The writer showed, in the previous report\textsuperscript{1}, that the noise caused by a flow of particles reflects the contacting state among internal particles, and proved, in an observation then made, that the time interval of a generating sound (collisional sound) caused in a certain time zone took the shape of Poisson distribution. This means that contacts among particles appeared nearly at random.

A static test points out that it is to be regarded rational that when the equilibrium state under a combination of some principal stresses is transferred to the equilibrium state by a combination of other succeeding principal stresses through slipping on a contacting point of particle, the distribution of contacting points among particles in an aggregate, spread in all directions at random prior to the initial condition before causing slipping; and this was determined by observing the distribution of contacting point angles.\textsuperscript{11}

Regarding many granules, it is observed that the relation between the ratio of principal stresses and ratio of principal strain increments is linear.\textsuperscript{11}

Linear relation means that the present condition can be considered as standard point irrespective of the former history; in combination with the test results mentioned before, the random characteristics in the strain direction immediately before slipping occurred on the contacting point of particle is also reproduced in deformation process, namely, the same random characteristics on the optional point as an original one will be assumed.

From the above, a step to consider the mechanical process of granule as a continuum giving a mean value of certain random variable that considered as observed quantity, could also be taken into account. Equation of motion of a granule to be stochastically non-homogeneous continuous medium becomes a non-linear stochastic partial differential equation containing elastic coefficient such as random stochastic variable, where it can be said in general, that it is a difficult problem to obtain its strict analytic solution.

In this paper, for the sake of practical treatment, a large sample is employed and an approximation is used for the interaction effect of stress field of particles, where the most neighboring particles are considered. Variate here is nearly random, where one portion representing sample on an optional point, namely, a part in which sufficiently many particles to allow thinking the average number of contact points distributed at random in a contacting condition, can be considered. Now, in order to find how shearing stress $\tau$ occurred through the contacting point on such part would affect a macroscopic stress, probability density function $p(\tau)$ has been introduced. Macroscopic strain and stress are expressed as in the following equations:\textsuperscript{11}

\begin{align}
\langle \varepsilon_{ij} \rangle &= \int_{0}^{\infty} \varepsilon_{ij}(\tau)p(\tau) \, d\tau \\
\langle \tau_{ij} \rangle &= \int_{0}^{\infty} \tau_{ij}p(\tau) \, d\tau
\end{align}

### 3. Balance of Internal Stress and Strain

A mechanism to cause internal stress is explained. By nature, a substance does not generate stress, if deformed, without external restriction. Strain caused in such a deformation is called 'plastic strain'. Strain not causing internal stress, even if the change of shape (generation of strain) is generated, is a non-elastic strain.

Such a strain, which does not generate stress by itself, can generate internal stress if the strain is produced uneven. On the other hand, however, there is a strain causing an even deformation to material but not generating stress. A few strains, such as those by thermal expansion or change of shape according to the phase transition, are known. The occurred strains, when observed only by means of thermal expansion and crystal lattice structure, are of same character as those defined above.
Obviously, the internal stresses generated therein are derived from a given unevenness. In this respect, in considering the change of force giving the relative displacement of particles, a definite difference becomes apparent, where components of internal strain can meet a cause.

To investigate the condition of internal strain generated by such unevenness, the strain deviator defined by subtracting the mean strain from the actual strain, which expresses the adequate stress-strain relation, is taken up. This shows distortion of the system, on which the shearing deformation is completely dependent.

Assume that stress deviator $\sigma^D$ is the same as to the shearing stress causing a relative displacement between two particles, namely, $\tau = \sigma^D$, $\sigma^D = \sigma_{ij} - \frac{1}{3} \sigma_{kk}$  

As illustrated in Fig. 1, within an interface between two particles it is observed that a slipping band on periphery and an internal elastic contacting point do exist together.

Until an external force becomes and the following condition is satisfied, the slipping annulus on the interface increases gradually, becomes finally a macroscopic slipping and is separated into two units.

When a granule is pressed, on the interface of two contacting particles to be components or in the deformation of infinitesimal granules, an elastic deformation phase and a plastic deformation phase co-exist together, and the deformation process is advanced in parallel. That is, before reaching the yield point observed macroscopically, the deformation is progressing containing local non-elastic deformation. Apparently, the deformation is running with elastic behavior in relation to the entire balance and adaptation, and it reaches the so-called yield condition as one influential force of non-elastic element increases.

Naturally, the stress-strain relation is different on the elastic and plastic phases, being regarded as a gradually changing expression. In this paper, the relation dealt with is up to immediately before the macroscopic yield point. In the following, $\tau$ in Eq.(4)will be treated by dividing into an elastic resisting part and a non-elastic resisting part.

Express an elastic component of shearing force $\tau$ acting on i-plane in j-direction as $\tau_{ij}^e$ and a non-elastic strain component by $\varepsilon_{ij}^p$. That is, $\tau_{ij} = \sigma^D - a \varepsilon_{ij}^p$  

where, $a$ is a quantity expressing the contribution rate to the entire elastic modulus affected by the non-elastic deformation part, which is fixed according to the balance of internal stress and non-elastic deformation, namely, a constant $0 < a \leq 1$.

For complete plasticity, $a = 1$, which is consistent with the expression of conventional mathematical theory of plasticity.

Non-elastic strain here shall be defined as a difference of deflective strain provided that an actual deflective strain and the Hooke's law are realized. Shearing stress-strain relation is obtained by letting a shearing elastic coef-
\[ \tau_{ij} = 2G \varepsilon_{ij} \]

\[ = 2G \left( \varepsilon_{ij}^p - \beta \varepsilon_{ij}^p \right) \tag{6} \]

where, \( \beta \) is a proportional constant.

When an initial elastic microscopic strain \( \varepsilon^o \) is contained, the following equations are utilized by introducing initial strain field \( \varepsilon^* \)

\[ \tau_{ij} = \sigma_{ij}^p - d \varepsilon_{ij}^* \tag{7} \]

\[ \left( \varepsilon_{ij}^* = \varepsilon_{ij}^p + \frac{2G}{d} \varepsilon_{ij}^o \right) \tag{8} \]

Condition until a shearing drift of two particles in contact is caused by

\[ \varepsilon^o \leq \tau_o / 2G \tag{9} \]

where, \( \tau_o \) means a yield shearing strain.

The condition decides the correlation among two random parameters and \( \varepsilon^o \), which also shows that an initial microscopic stress having the strength exceeding the local yield point is not considered as the cause. Actual slipping is caused on the part unstable to infinitesimal variation of stress on contacting points in the critical condition as mentioned in the beginning. To perform actual calculation, therefore, application of equal signs to Eq.(9) regards the effect of microscopic force to be higher from a physical point of view.

4. Distribution of Initial Elastic Microscopic Strain

Express initial elastic microscopic strain \( \varepsilon_{ij}^o \) within a range not to destroy the homogeneity and isotropy in an initial macroscopic condition and also express strain field distribution with such strain as in the following variate

\[ \varepsilon^o = \sqrt{\varepsilon_{ij}^o \cdot \varepsilon_{ij}^o} \tag{10} \]

For a shear giving the relative displacement of particle, a stochastic variate of five strain components is to be considered by deleting the term of volume variation

\[ \varepsilon_1^o = \sqrt{\frac{3}{2}} \varepsilon_{11}^o, \quad \varepsilon_2^o = \sqrt{2} \left( \varepsilon_{22}^o + \frac{1}{2} \varepsilon_{11}^o \right) \]

\[ \varepsilon_3^o = \sqrt{2} \varepsilon_{12}^o, \quad \varepsilon_4^o = \sqrt{2} \varepsilon_{33}^o, \quad \varepsilon_5^o = \sqrt{2} \varepsilon_{13}^o \tag{11} \]

From the above, vector length \( \varepsilon_0 \) in a five-dimensional strain space is defined as

\[ \varepsilon_0 = \sqrt{\varepsilon_{ij}^o \cdot \varepsilon_{ij}^o} = \sqrt{\varepsilon_1^o + \varepsilon_2^o + \cdots + \varepsilon_5^o} \tag{12} \]

Consequently, a so-called yield curved surface is expressed as a five-dimensional hypersurface of the five-dimensional strain vector space. As this \( \varepsilon_{ij}^o \) is a random quantity, the mathematical expected value(mean value) of this five-dimensional vector is zero.

Assume, now, the distribution of each \( \varepsilon_{ij}^o \) follows the normal distribution. It is found by means of the probability theory that a variable fixed according to the square sum of independent variable following an identical normal distribution of a mean value at zero has a chi-square distribution density function of five degrees of freedom in such a case.

Accordingly, probability density of \( \varepsilon_0 \) is obtained

\[ p(\varepsilon_0) = \frac{2}{3 \sqrt{2\pi} a^5} \cdot \varepsilon_0^3 \cdot \exp \left( -\frac{\varepsilon_0^2}{2a^2} \right) \tag{13} \]

where, \( a^5 \) shows a dispersion of \( \varepsilon_0 \).

5. Microscopic Strain Distribution and Macroscopic Deformation

By utilizing the results obtained, an entire strain in five-dimensional strain space employed in the previous section, namely,

\[ \varepsilon = \varepsilon_{ij}^p + \varepsilon_{ij}^* \tag{14} \]

is to be treated, which is

\[ \langle \varepsilon_{ij} \rangle = \int \int \int \int \int \varepsilon_{ij} p(\varepsilon_{ij}) d\varepsilon_1 d\varepsilon_2 \cdots d\varepsilon_5 \tag{15} \]

where,

\[ p(\varepsilon^o) = \frac{1}{(2\pi)^{5/2}} \cdot \frac{1}{a^5} \exp \left( -\frac{\varepsilon_0^2}{2a^2} \right) \tag{16} \]

Change the above equation to a spherical coordinates system. Jacobian \( J \) in this five-dimensional coordinates system is

\[ J = r^4 \sin^4 \theta_1 \sin^4 \theta_2 \sin \theta_3 \tag{17} \]
and, Eq. (15) becomes
\[
\langle \varepsilon_j \rangle = \frac{1}{(2\pi)^{3/2}} \cdot \frac{1}{(2G\alpha)^3} \int_0^\infty \int_0^\infty \int_0^\infty \varepsilon_j \times \exp \left(-\frac{\varepsilon_j^3}{2\alpha^3}\right) Jd\tau d\theta d\phi d\theta_i \tag{18}
\]

This relation is not simple.

In consideration, now, of homogeneous, isotropic and infinitesimal displacement, a case of one dimension is taken up to briefly determine the tendency of the object.

To begin with, a case without initial strain is treated. In this case, \( \varepsilon_0^* \) becomes a function only of \( \tau \) as a scalar parameter. The relation with Eq. (18) is
\[
\langle \varepsilon_j \rangle = \int_0^\infty \varepsilon_j(\tau) p(\tau) d\tau \tag{19}
\]

In
\[
\tau_i > \tau \tag{20}
\]
it is expected, from the condition Eq. (9), that the probability density of \( \tau \) has a same shape of \( \varepsilon_0^* \); hence the following equations are formed.
\[
p(\tau) = \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{b^3} \cdot \frac{\tau_i^3}{b^3} \exp \left(-\frac{\tau^3}{2b^2}\right) \tag{21}
\]

\[
b^3 = (2G)^3 \cdot \alpha^3 \tag{22}
\]

From Eqs. (7) and (19)
\[
\langle \varepsilon_j \rangle = \frac{1}{d} \int_0^\infty (\sigma^0 - \tau) p(\tau) d\tau \tag{23}
\]

by which the changing rate of \( \varepsilon \) for the change of \( \sigma \)
\[
\frac{d\langle \varepsilon \rangle}{d\sigma} = \frac{1}{d} \int_0^\infty \langle \varepsilon_j \rangle d\tau \tag{24}
\]
is obtained.

From Eqs. (23) and (24), an equation of asymptote on the yielding point is formed
\[
\langle \sigma \rangle = d\langle \varepsilon \rangle + \langle T \rangle \tag{25}
\]

where, \( \langle T \rangle \) is a quantity
\[
\langle T \rangle = \int_0^\infty \tau p(\tau) d\tau
\]

\[
= \frac{8}{3\sqrt{2\pi}} \cdot b \cdot \int_0^\infty \exp(-\tau^3) \cdot \tau^3 d\tau (\tau = \sqrt{2} b t) \tag{26}
\]

which is a value of slice on the tangential stress axis of stress-strain curve. From the above, if the material is brittle, its yielding stress value can nearly be expressed by the value \( \langle T \rangle \) on the yielding point.

Eq. (26) indicates that \( \langle T \rangle \) is proportional to the standard deviation value of distribution of \( \tau \), and according to Eq. (22), the value is proportional to the standard deviation value of the initial strain distribution. That is, this relation suggests somehow an approach to increase the strength of material.

Now, in order to improve the tenacity of a brittle material (ceramics, for instance), a method known to be effective to disperse or deposit the heterogeneous phase as energy dissipative source into material has been proposed. It can be assumed that if there is an uneven part in structure, loading stress is concentrated around the boundary of the part, and a magnitude of the concentrated stress would appear in multiplication of shape, size, arrangement, distribution, difference of modulus of elasticity and thermal expansion to the surroundings.

Moreover, these elements act reversely depending on materials, which may reduce stress.

K_n showing the breaking renacity of material is expressed by the following equation.
\[
K_n = (2E \gamma)^{1/3} \tag{27}
\]

As Young’s modulus \( E \) is a given material constant, the material is made stronger by increasing the level of fracture energy \( \gamma \). This performance is available by controlling the structure of material. Result of Eq. (26) indicates a strain dispersion effect to the stress concentration. This indication furnishes that various elements mentioned before produce their own compound effects, work as the energy absorbing mechanisms, and increase the tenacity of material. Applying properly the parameters of normal distribution as-
Reduced strain, $a \cdot \frac{\sigma}{b}$

Fig. 2 Stress and strain curves picturing for three different value of the standard deviation $b$ (Y; effective yield point)

sumed, a sufficient approximation of the Weibull distribution showing the breaking strength characteristics of many materials is given, where actual application can be expected.

In Fig. 2 graphs and from the above analysis, a tendency of influence of strain distribution to the stress-strain relation becomes obvious.

Then, a case having the initial elastic microscopic strain is examined. To simplify, distribution density of shearing strength $p(\tau)$ and distribution density of initial elastic microscopic strain $p(\varepsilon_0)$ are approximated respectively as in the following.

$$p(\tau) \approx \delta(\tau - \langle \tau \rangle)$$
$$p(\varepsilon_0) \approx \delta(\varepsilon_0 - \langle \varepsilon_0 \rangle)$$

Consequently, from Eq.(18)

$$\langle \varepsilon_n \rangle = \oiint \cdots \oiint \varepsilon_n \delta(\tau - \langle \tau \rangle) \delta(\varepsilon_0 - \langle \varepsilon_0 \rangle) \times \sin^2 \theta_1 \sin^2 \theta_2 \sin \theta_3 \sin \theta_4 d\varepsilon_0 d\theta_1 \cdots d\theta_4$$

A case of one dimension is also examined. From Eq.(29)

$$\langle \varepsilon_n \rangle = \frac{3}{4} \int_0^\infty \varepsilon_n \sin^2 \theta_1 d\theta_1$$

In this case, as $\theta_1$ of $\varepsilon_n$, namely, dependence of $\varepsilon_n$ can be regarded nearly linear, an equation of asymptote around the yielding point of its stress-strain curve is formed by adding a term regarding the initial strain to Eq.(25), that is

$$2G\langle \varepsilon_n \rangle$$

which is added linearly, where an increase of the initial strain increases the initial modulus of elasticity within the range of Eq.(9).

Fig. 3 illustrates the shearing test results of quartz sand (average grain diameter: 0.18 mm) by changing pressure $p$, giving thus an initial strain. In a considerable range of shear velocity, it is found that a shearing stress is increased as an initial pressure is increased.

It is also found that the grain size affects strength. Grain diameter is relative to the magnitude of surface curvature, and the magnitude of curvature gives an influence to the effect of stress concentration of contacting point.

As the dislocation density is increased, the curvature of particle becomes larger, and consequently it has widely been recognized

$$\tau_\theta \propto d^{-1/2}$$

that there is relation between yielding strength $\tau_\theta$ and crystal grain diameter $d$ (Hall-Petch relation) on polycrystalline material.

Meanwhile, the theory of comminution of granule indicates that energy used to comminute is inversely proportional to the square root of grain diameter. Stiffness of material is measured by the variation of elastic energy to displacement. Consequently,

$$G \propto d^{-1/3}$$

According to the relation of the above
Shear velocity (mm/mm).

Fig. 3 The relation between initial pressure $p_1$, frictional stress $\tau$ and shear velocity equation to Eqs. (22) and (26)

$$\tau = \langle T \rangle \propto 2Gd \propto d^{-1/2} \quad (34)$$

is obtained, which is approximate to Eq. (32).

Fig. 4 is prepared by re-arranging the test results on the relation of fracture energy of $\text{Al}_2\text{O}_3$ and grain size along $d^{-1/2}$-axis. As plotted, the relation is linear.

Fracture energy and fracture form of this case vary with the condition of microcrack inside. Density function $p(\cdot)\text{in Eq. (26)}$ is reasonable to be treated as a distribution density of microcrack inside grain. Such being the circumstances, the above relation and the conclusion of Eq. (34) indicate that an optimum grain size having the highest strength is existing, which can be estimated.

6. Conclusion

The contacting condition of the grain forming granule was examined; the system of non-linear elastic action as an expression of interacting mechanism was built in, normal distribution fulfilling an actual strength characteristics was used as distribution model to decide the combination law of acting elements, and a granule was re-constructed a stochastically non-homogeneous continuous medium, which was compared with an actual example. The main results were

1) influences affecting strength of internal strain distribution were found coincident qualitatively,

2) increase of initial strain coincided with increase of initial elastic modulus experimentally, and

3) relation of elastic limit and grain size were incident analytically. The availability of a technique to treat granule as stochastically inhomogeneous non-individual body non-linear elastic element has been confirmed.

Nomenclature

- $a^2$: variance of $\varepsilon^*$
- $b^2$: variance of $\tau$
- $d$: diameter of grain [m]
- $G$: elastic shear modulus [Pa]
- $J$: Jacobian [-]
- $\sigma_n$: normal component of traction [Pa]
- $p(\tau)$: density of the distribution of $\tau$
- $\alpha$, $\beta$: scalar coefficient [-]
- $\varepsilon$: strain [-]
- $\varepsilon^*$: initial elastic microstrain [-]
- $\sigma^p$: non-elastic microstrain [-]
- $\sigma$: stress [Pa]
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