Yamabe solitons on three-dimensional normal almost paracontact metric manifolds

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Abstract
The purpose of the paper is to study Yamabe solitons on three-dimensional para-Sasakian, paracosymplectic and para-Kenmotsu manifolds. Mainly, we prove that the following:

• If the semi-Riemannian metric of a three-dimensional para-Sasakian manifold is a Yamabe soliton, then it is of constant scalar curvature, and the flow vector field \( V \) is Killing.
  
  In the next step, we prove that either the manifold has constant curvature \(-1\) or \( V \) is an infinitesimal automorphism of the paracontact metric structure on the manifold.

• If the semi-Riemannian metric of a three-dimensional paracosymplectic manifold is a Yamabe soliton, then it has constant scalar curvature. Furthermore, either the manifold is \( \eta \)-Einstein, or Ricci flat.

• If the semi-Riemannian metric on a three-dimensional para-Kenmotsu manifold is a Yamabe soliton, then the manifold is of constant sectional curvature \(-1\). Furthermore, Yamabe soliton is expanding with \( \lambda = -6 \).

Finally, we construct examples to illustrate the results obtained in previous sections.

Keywords Para-Sasakian manifold · Paracosymplectic manifold · Para-Kenmotsu manifold · Yamabe soliton · Ricci soliton · Infinitesimal automorphism · Constant scalar curvature

Mathematics Subject Classification 53C25 · 53C21 · 53C44 · 53D15

1 Introduction
Several years ago, the notion of the Yamabe flow was introduced by Richard Hamilton at the same time as the Ricci flow (see [9,10]), as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on \((M^n, g)\) \((n \geq 3)\). On a smooth semi-Riemannian manifold, the Yamabe flow can be defined as the evolution of the semi-Riemannian metric \(g_0\) in time \(t\) to \(g = g(t)\) by the equation

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\[ \frac{\partial}{\partial t} g = -rg, \quad g(0) = g_0. \]

where \( r \) denotes the scalar curvature which corresponds to \( g \).

The significance of the Yamabe flow lies in the fact that it is a natural geometric deformation to metrics of constant scalar curvature. One notes that Yamabe flow corresponds to the fast diffusion case of the porous medium equation (the plasma equation) in mathematical physics. In dimension \( n = 2 \) the Yamabe flow is equivalent to the Ricci flow (defined by \( \frac{\partial}{\partial t} g(t) = -2S(t) \), where \( S \) stands for the Ricci tensor). However in dimension \( n > 2 \) the Yamabe and Ricci flows do not agree, since the first one preserves the conformal class of the metric but the Ricci flow does not in general. Just as a Ricci soliton is a special soliton of the Ricci flow, a Yamabe soliton is a special solution of the Yamabe flow that moves by one parameter family of diffeomorphisms \( \phi_t \) generated by a fixed (time-independent) vector field \( V \) on \( M \), and homotheties, i.e. \( g(.,t) = \sigma(t)\phi_t^*g_0 \).

A semi-Riemannian manifold \((M^n, g)\) is a Yamabe soliton [5] if it admits a vector field \( V \) such that

\[ \mathcal{L}_V g = (\lambda - r)g, \quad (1.1) \]

where \( \mathcal{L}_V \) denotes the Lie derivative in the direction of the vector field \( V \) and \( \lambda \) is a real number. Moreover, a vector field \( V \) as in the definition is called a soliton vector field (for \((M^n, g)\)). In the particular case of \( V \) is a gradient, i.e., \( V = \nabla f \) for some potential function \( f \), \((M^n, g)\) is said to be a gradient Yamabe soliton and \( V \) is called a gradient soliton vector field (for \((M^n, g)\)). A Yamabe soliton is said to be shrinking, steady or expanding if it admits a soliton vector field for which \( \lambda > 0, \lambda = 0, \lambda < 0 \), respectively.

A Ricci soliton (see [9]) is a natural generalization of Einstein metric (that is, the Ricci tensor is a constant multiple of the semi-Riemannian metric \( g \)). A Ricci soliton \((g, V, \mu)\) is defined on a semi-Riemannian manifold \((M^n, g)\) by

\[ \mathcal{L}_V g + 2S + 2\mu g = 0, \quad (1.2) \]

where \( \mathcal{L}_V g \) denotes the Lie derivative of semi-Riemannian metric \( g \) along a vector field \( V \) and \( \mu \) is a constant. It is clear that a Ricci soliton with \( V = 0 \) or a Killing vector field reduces to an Einstein metric. A Ricci soliton is said to be shrinking, steady and expanding according as \( \mu \) is negative, zero and positive, respectively. Also, Ricci solitons have been studied extensively in the context of semi-Riemannian geometry; we refer to [1–3,13] and references therein.

Yamabe solitons coincide with Ricci solitons (defined by (1.2)) in dimension \( n = 2 \). In higher dimensions Ricci solitons and Yamabe solitons have different behaviours.

In this study, we make the first contribution to investigate Yamabe solitons on paraccontact geometry. Yamabe solitons on three dimensional Sasakian manifolds and Kenmotsu manifolds were studied respectively by Sharma [14] and Wang [15].

The outline of the article goes as follows. In Sect. 2, we recall basic facts and some results related with almost paraccontact manifolds which we will need throughout the paper. Section 3 is devoted to Yamabe solitons on three dimensional para-Sasakian manifolds. Our first main result about para-Sasakian manifolds is that if the semi-Riemannian metric of a three-dimensional para-Sasakian manifold is a Yamabe soliton, then it has constant scalar curvature, and the vector field \( V \) is Killing. Furthermore, either the manifold has constant curvature \(-1\) or \( V \) is an infinitesimal automorphism of the paraccontact metric structure on the manifold. Our second main result about para-Sasakian manifolds is that if the manifold admits a Yamabe soliton and \( V \) is a pointwise collinear vector field with the structure vector field \( \xi \), then \( V \) is a constant multiple of \( \xi \). Also, we show that if the
semi-Riemannian metric of a three dimensional para-Sasakian manifold with \( r = -6 \) is a Yamabe soliton, then it is also a Ricci soliton. Section 4 is devoted to Yamabe solitons on three dimensional para-symplectic manifolds. Our main result about paracompact symplectic manifolds is that if the semi-Riemannian metric of a three-dimensional paracompact symplectic manifold is a Yamabe soliton, then it has constant scalar curvature. Furthermore, the manifold is either \( \eta \)-Einstein, or Ricci flat. Section 5 is devoted to Yamabe solitons on three dimensional para-Kenmotsu manifolds. Our first main result about para-Kenmotsu manifolds is that if the semi-Riemannian metric of a three-dimensional para-Kenmotsu manifold is a Yamabe soliton, then the manifold is of constant sectional curvature \(-1\). Furthermore, Yamabe soliton is expanding with \( \lambda = -6 \). Our second main result about para-Kenmotsu manifolds is that if the semi-Riemannian metric of a three dimensional para-Kenmotsu manifold is a Yamabe soliton, then it is also a Ricci soliton. At last, we show that if three-dimensional para-Kenmotsu manifold admits a Yamabe soliton for a vector field \( V \) and a constant \( \lambda \), then \( V \) can not be pointwise collinear with \( \xi \). Finally, we construct examples to illustrate the results obtained in previous sections.

2 Preliminaries

In this section we collect the formulas and results which we need on paracontact metric manifolds. All manifolds are assumed to be connected and smooth. We may refer to [11,18] and references therein for more information about paracontact metric geometry.

Paracontact metric structures were introduced in [11] as a natural odd-dimensional counterpart to paraHermitian structures, like contact metric structures correspond to the Hermitian ones. Paracontact metric manifolds \((M^{2n+1}, \varphi, \xi, \eta, g)\) have been studied by many authors in recent years, particularly since the appearance of [18]. The curvature identities for different classes of almost paracontact metric manifolds were obtained e.g. in [6,16,18]. The importance of paracontact geometry, and in particular of para-Sasakian geometry, has been pointed out especially in the last years by several papers highlighting the interplays with the theory of para-Kähler manifolds and its role in semi-Riemannian geometry and mathematical physics (cf. e.g., [4,7]).

An \((2n + 1)\)-dimensional smooth manifold \( M \) is said to have an almost paracontact structure if it admits a \((1,1)\)-tensor field \( \varphi \), a vector field \( \xi \) and a 1-form \( \eta \) satisfying the following conditions:

(i) \( \eta(\xi) = 1, \varphi^2 = I - \eta \otimes \xi \),

(ii) the tensor field \( \varphi \) induces an almost paracomplex structure on each fibre of \( D = \ker(\eta) \), i.e. the \( \pm 1 \)-eigendistributions, \( D^\pm = D_\varphi(\pm 1) \) of \( \varphi \) have equal dimension \( n \).

The manifold \( M \) is said to be an almost paracontact manifold if it is endowed with an almost paracontact structure. From the definition it follows that \( \varphi \xi = 0, \eta \circ \varphi = 0 \) and the endomorphism \( \varphi \) has rank \( 2n \). We denote by \([\varphi, \varphi]\) the Nijenhuis torsion

\[
[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].
\]

When the tensor field \( N_\varphi = [\varphi, \varphi] - 2d\eta \otimes \xi \) vanishes identically the almost paracontact manifold is said to be normal. If an almost paracontact manifold admits a semi-Riemannian metric \( g \) such that

\[
g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),
\]

for all \( X, Y \in \chi(M) \), then we say that \((M, \varphi, \xi, \eta, g)\) is an almost paracontact metric manifold. Notice that any such a semi-Riemannian metric is necessarily of signature \((n +
1, n). For an almost paracontact metric manifold, there always exists an orthogonal basis \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi, \eta\}, such that \(g(X_i, X_j) = \delta_{ij}\), \(g(Y_i, Y_j) = -\delta_{ij}\), \(g(X_i, Y_j) = 0\), \(g(\xi, X_i) = g(\xi, Y_j) = 0\), and \(Y_i = \varphi X_i\), for any \(i, j \in \{1, \ldots, n\}\). Such basis is called a \(\varphi\)-basis.

We can now define the \textit{fundamental 2-form} of the almost paracontact metric manifold by \(\Phi(X, Y) = g(X, \varphi Y)\). If \(d\eta(X, Y) = g(X, \varphi Y)\), then \((M, \varphi, \xi, \eta, g)\) is said to be paracontact metric manifold. In a paracontact metric manifold one defines a symmetric, trace-free operator \(h = \frac{1}{2}L_\xi \varphi\), where \(L_\xi\), denotes the Lie derivative. It is known [18] that \(h\) anti-commutes with \(\varphi\) and satisfies \(h^2 \xi = 0\), \(tr h = tr h \varphi = 0\) and

$$\nabla \xi = -\varphi + \varphi \cdot h, \tag{2.2}\$$

where \(\nabla\) is the Levi-Civita connection of the semi-Riemannian manifold \((M, g)\).

Moreover \(h = 0\) if and only if \(\xi\) is Killing vector field. In this case \((M, \varphi, \xi, \eta, g)\) is said to be a \(K\)-paracontact manifold. A normal paracontact metric manifold is called a para-Sasakian manifold. Also in this context the para-Sasakian condition implies the \(K\)-paracontact condition and the converse holds only in dimension 3. We also recall that any para-Sasakian manifold satisfies

\[
R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y), \tag{2.3}
\]

\[
(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.4}
\]

\[
\nabla_X \xi = -\varphi X, \tag{2.5}
\]

\[
R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.6}
\]

\[
S(X, \xi) = -(n - 1)\eta(X), \tag{2.7}
\]

where \(Q\) is the Ricci operator, \(R\) is the Riemannian curvature tensor and \(S\) is Ricci tensor defined by \(S(X, Y) = g(QX, Y)\).

From [12] we know that if \(M\) has constant curvature \(c\), then

\[
R(X, Y)Z = c(g(Z, X)Y - g(Z, Y)X). \tag{2.8}
\]

On an almost paracontact metric manifold \(M\), if the Ricci operator satisfies

\[
Q = \alpha \text{id} + \beta \nabla \otimes \xi, \tag{2.9}
\]

where both \(\alpha\) and \(\beta\) are smooth functions, then the manifold is said to be an \(\eta\)-Einstein manifold. An \(\eta\)-Einstein manifold with \(\beta\) a constant is obviously an Einstein manifold. An \(\eta\)-Einstein manifold is said to be proper \(\eta\)-Einstein if \(\beta \neq 0\).

We recall that the curvature tensor of a 3-dimensional semi-Riemannian manifold satisfies

\[
R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y - \frac{r}{2}(g(Y, Z)X - g(X, Z)Y) \tag{2.10}
\]

where \(Q\) is the Ricci operator of \(M\).

An infinitesimal automorphism is a vector field such that Lie derivatives along it of all objects of some tensor structure vanish. For an almost paracontact, metric structure, the condition that a vector \(V\) is an infinitesimal automorphism is as follows:

\[
\mathcal{L}_V \eta = \mathcal{L}_V \xi = \mathcal{L}_V \Phi = \mathcal{L}_V g = 0. \tag{2.11}
\]
3 Yamabe solitons on three dimensional para-Sasakian manifolds

In this section, before presenting our main results about Yamabe solitons on three dimensional para-Sasakian manifolds, we will give some lemmas which will be used later.

A vector field $V$ on an $n$-dimensional semi-Riemannian manifold $(M, g)$ is said to be conformal vector field if,

$$\mathcal{L}_V g = 2\rho g,$$  \hspace{1cm} (3.1)

where $\rho$ is called the conformal coefficient (from (1.1) we get $\rho = \frac{\lambda - r}{2}$). If conformal coefficient is zero, we will say that conformal vector field is Killing vector field.

**Lemma 3.1** \cite{17} On an $n$-dimensional semi-Riemann manifold $(M^n, g)$ endowed with a conformal vector field $V$, we have

$$(\mathcal{L}_V S)(X, Y) = -(n - 2)g(\nabla_X D\rho, Y) + (\Delta \rho)g(X, Y),$$  \hspace{1cm} (3.2)

$$\mathcal{L}_V r = -2\rho r + 2(n - 1)\Delta \rho$$  \hspace{1cm} (3.3)

for any vector fields $X$ and $Y$, where $D$ denotes the gradient operator and $\Delta := -\text{div} D$ denotes the Laplacian operator of $g$.

**Lemma 3.2** For a para-Sasakian manifold, the following relations are valid:

$$\eta(\mathcal{L}_V \xi) = r - \lambda,$$  \hspace{1cm} (3.4)

$$\mathcal{L}_V \eta)(\xi) = \lambda - r$$  \hspace{1cm} (3.5)

**Proof** Since the Reeb vector field $\xi$ is a unit vector field we have $g(\xi, \xi) = 1$. Taking the Lie-derivative of this relation along the vector field $V$ and using (1.1) we get (3.4). Using $\eta(\xi) = 1$ and (3.4), we have (3.5). \hfill $\Box$

**Lemma 3.3** For any three-dimensional para-Sasakian manifold $(M^3, \varphi, \xi, \eta, g)$, we have

$$\xi(r) = 0.$$  \hspace{1cm} (3.6)

**Proof** If we replace $Y = Z$ by $\xi$ in (2.10) and use (2.3), (2.7) we get

$$QX = \left(\frac{r}{2} + 1\right)X - \left(\frac{r}{2} + 3\right)\eta(X)\xi$$  \hspace{1cm} (3.7)

for any vector field $X \in \chi(M)$. If we use (3.7), (2.5) and (2.7) in the following well known formula for semi-Riemannian manifolds

$$\text{trace} \{Y \to (\nabla Y Q)X\} = \frac{1}{2}\nabla_X r$$

we obtain

$$\xi(r) = 0.$$ \hfill $\Box$

From (3.7), we get

**Lemma 3.4** For a three-dimensional para-Sasakian manifold $(M^3, \varphi, \xi, \eta, g)$, the Ricci tensor $S$ is given by

$$S(X, Y) = \left(\frac{r}{2} + 1\right)g(X, Y) - \left(\frac{r}{2} + 3\right)\eta(X)\eta(Y)$$  \hspace{1cm} (3.8)

for any vector fields $X, Y \in \chi(M)$. 

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Lemma 3.5 Suppose that the semi-Riemannian metric of a three-dimensional para-Sasakian manifold \((M^3, \varphi, \xi, \eta, g)\) is a Yamabe soliton. If the scalar curvature of \(M^3\) is harmonic, that is \(\Delta r = 0\), then \(\lambda = r\).

Proof Since \(V\) is a conformal vector field with \(\rho = \frac{\lambda - r}{2}\), for \(n = 3\) from Eqs. (3.2) and (3.3), we obtain

\[
(\mathcal{L}_V S)(X, Y) = \frac{1}{2}g(\nabla_X D_r, Y) - \frac{1}{2}(\Delta r)g(X, Y), \tag{3.9}
\]

\[
\mathcal{L}_V r = r(r - \lambda) - 2\Delta r \quad \tag{3.10}
\]

for any vector fields \(X, Y \in \chi(M)\). If we take the Lie-derivative of (3.8) in the direction of \(V\) and use (1.1), (3.10), we get

\[
(\mathcal{L}_V S)(X, Y) = (-\Delta r + \lambda - r)g(X, Y) + \left(\Delta r + \frac{r}{2}(\lambda - r)\right)\eta(X)\eta(Y)
- \left(\frac{r}{2} + 3\right)\{((\mathcal{L}_V \eta)(X))\eta(Y) + (\mathcal{L}_V \eta)(Y)\eta(X)\} \tag{3.11}
\]

for any vector fields \(X, Y \in \chi(M)\). In view of (3.9) and (3.11), we obtain

\[
g(\nabla_X D_r, Y) = (-\Delta r + 2(\lambda - r))g(X, Y) + (2\Delta r + r(\lambda - r))\eta(X)\eta(Y)
- (r + 6)\{((\mathcal{L}_V \eta)(X))\eta(Y) + (\mathcal{L}_V \eta)(Y)\eta(X)\} \tag{3.12}
\]

for any vector fields \(X, Y \in \chi(M)\). Setting \(X = Y = \xi\) in (3.12) and using (3.5) and (3.6), we get

\[
0 = \Delta r + 4(r - \lambda). \tag{3.13}
\]

The last equation completes the proof.

Theorem 3.6 If the semi-Riemannian metric of a three-dimensional para-Sasakian manifold \((M^3, \varphi, \xi, \eta, g)\) is a Yamabe soliton, then it has constant scalar curvature, and the vector field \(V\) is Killing. Furthermore, either \((M^3, \varphi, \xi, \eta, g)\) has constant curvature \(-1\) or \(V\) is an infinitesimal automorphism of the paracontact metric structure on \((M^3, \varphi, \xi, \eta, g)\).

Proof Differentiating covariantly (3.6) along the direction of an arbitrary vector field \(X\) and using (2.5) we have

\[
\varphi X(r) = g(\nabla_X D_r, \xi). \tag{3.14}
\]

Putting \(\xi\) for \(Y\) in (3.12), using (3.5), (3.13), (3.14), we obtain

\[
(\lambda - r)\left(\frac{r}{2} + 3\right)\eta(X) - \varphi X(r) = (r + 6)(\mathcal{L}_V \eta)(X) \tag{3.15}
\]

for any vector field \(X \in \chi(M)\). Using the last equation and (3.13) in (3.12), we deduce

\[
\nabla_X D_r = -2(\lambda - r)(X - \eta(X)\xi) + \varphi X(r)\xi - \eta(X)\varphi D_r \tag{3.16}
\]

for any vector field \(X \in \chi(M)\).

Differentiating covariantly along the direction of \(Y\), we get

\[
\nabla_Y \nabla_X D_r = 2Y(r)\{X - \eta(X)\xi\}
- 2(\lambda - r)\{\nabla_Y X + g(\varphi Y, X)\xi - g(\nabla_Y X, \xi)\xi + \eta(X)\varphi Y\}
+ Y(\varphi X(r))\xi - (\varphi X(r))\varphi Y
- \eta(X)\{-g(Y, D_r)\xi + \varphi \nabla_Y D_r\}
- \{-g(\varphi Y, X) + g(\nabla_Y X, \xi)\}\varphi D_r. \tag{3.17}
\]
Replacing $X$ and $Y$ in (3.17), we obtain
\[
\nabla_X \nabla_Y D_r = 2X(r) [Y - \eta(Y) \xi] \\
- 2(\lambda - r) [\nabla_X Y + g(\varphi X, Y) \xi - g(\nabla_X Y, \xi) \xi + \eta(Y) \varphi X] \\
+ X(\varphi Y(r)) \xi - (\varphi Y(r)) \varphi X \\
- \eta(Y) [\nabla_r \eta] [\nabla_r \eta] + \varphi \nabla_X D_r] \\
- [-g(\varphi X, Y) + g(\nabla_X Y, \xi)] \varphi D_r.
\]
(3.18)

From (3.16), we have
\[
\nabla_{[Y,X]} D_r = -2(\lambda - r) [\nabla_Y X - \nabla_X Y - g(\nabla_Y X, \xi) \xi + g(\nabla_X Y, \xi) \xi] \\
+ g(\varphi \nabla_Y X, D_r) \xi - g(\varphi \nabla_X Y, D_r) \xi \\
- [g(\nabla_Y X, \xi) - g(\nabla_X Y, \xi)] \varphi D_r.
\]
(3.19)

Putting (3.17), (3.18) and (3.19) in the Riemannian curvature tensor $R$ equation
\[
R(X, Y) D_r = \nabla_X \nabla_Y D_r - \nabla_Y \nabla_X D_r - \nabla_{[X,Y]} D_r
\]
and contracting over $Y$ (we assume $(\epsilon_i) (i = 1, 2, 3)$ to be a local orthonormal frame on $M$), we obtain
\[
S(X, D_r) = \sum_{i=1}^{3} \epsilon_i g(R(\epsilon_i, X) D_r, \epsilon_i)
\]
\[
= -\eta(X) \sum_{i=1}^{3} g(\varphi \nabla_{\epsilon_i} D_r, \epsilon_i)
\]
(3.20)

where $i$ is summer over 1, 2, 3. Using (3.16) in the last equation, we have $S(X, D_r) = 0$. Taking account (3.8) and the fact that $\eta(D_r) = g(D_r, \xi) = \xi(r) = 0$, we have
\[
S(X, D_r) = (r + 1) g(X, D_r) - \left(\frac{r}{2} + 3\right) \eta(X) \eta(D_r),
\]
\[
0 = \left(\frac{r}{2} + 1\right) X_r
\]
where $g(X, D_r) = X_r$. Hence $r$ is constant. By virtue of (3.13), we get $\lambda = r$. The Eq. (1.1) leads to $\mathcal{L}_V g = 0$, namely $V$ is Killing.

If $r = -6$, from (3.8), we have $S = -2g(X, Y)$, hence $M$ is an Einstein manifold. Taking account of (2.8) and (2.10), $M$ has constant curvature $-1$.

If $r \neq -6$, (3.15) gives $\mathcal{L}_V \eta = 0$. By virtue of this and being $V$ is Killing, we have $\mathcal{L}_V \xi = 0$. If we take the Lie-derivative of the well known equation $\Phi(X, Y) = g(X, \varphi Y)$ in the direction of $V$, we get $\mathcal{L}_V \Phi = 0$. From (2.11), we can conclude that $V$ is an infinitesimal automorphism of the paracontact metric structure on $(M^3, \varphi, \xi, \eta, g)$. So, the proof is completed. \hfill $\Box$

**Proposition 3.7** Let $(M^3, \varphi, \xi, \eta, g)$ be a three-dimensional para-Sasakian manifold. If $M^3$ admits a Yamabe soliton and $V$ is a pointwise collinear vector field with the structure vector field $\xi$, then $V$ is a constant multiple of $\xi$.

**Proof** Let $V$ be a pointwise collinear vector field with the structure vector field $\xi$, that is $V = b\xi$, where $b$ is a smooth function on $M^3$. From (1.1), we obtain
\[
g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0
\]
(3.21)
for any vector fields $X, Y \in \chi(M)$. \hfill $\Box$
Taking $V = b\xi$ in (3.21) and using (2.5), we have

$$X(b)\eta(Y) + Y(b)\eta(X) = 0$$

(3.22)

for any vector fields $X, Y \in \chi(M)$. Putting $Y = \xi$ in (3.22), we get

$$X(b) + \xi(b)\eta(X) = 0.$$ (3.23)

Replacing $X$ by $\xi$ in the last equation, we obtain

$$\xi(b) = 0.$$ (3.24)

If we use (3.24) in (3.23), we have

$$X(b) = 0$$

which yields $db = 0$, that is, $b = \text{constant}$. This completes the proof. \hfill \Box

**Theorem 3.8** If the semi-Riemannian metric of a three dimensional para-Sasakian manifold with $r = -6$ is a Yamabe soliton, then it is also a Ricci soliton.

**Proof** Let $(M^3, \varphi, \xi, \eta, g)$ be a three-dimensional para-Sasakian manifold. From Theorem 3.6 one can say that if $g$ is a Yamabe soliton with $r = -6$ then the Ricci operator of $(M^3, \varphi, \xi, \eta, g)$ is $Q = -2\text{id}$. Therefore (1.2) returns to

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 4g(X, Y) = 0.$$ (4.1)

Hence, $g$ is an expanding Ricci soliton with $\mu = 2$. \hfill \Box

### 4 Yamabe solitons on three dimensional paracosymplectic manifolds

An almost paracontact metric manifold $M^{2n+1}$, with a structure $(\varphi, \xi, \eta, g)$ is said to be an almost $\alpha$-paracosymplectic manifold, if

$$d\eta = 0, \quad d\varphi = 2\alpha\eta \wedge \varphi,$$ (4.2)

where $\alpha$ may be a constant or function on $M$.

For a particular choices of the function $\alpha = 0$, we have almost paracosymplectic manifolds. If additionally normality condition is fulfilled, then manifolds are called paracosymplectic. We may refer to [6,8] and references therein for more information about paracosymplectic manifolds. We also recall that any paracosymplectic manifold satisfies

$$R(X, Y)\xi = 0,$$ (4.3)

$$\nabla_X \varphi Y = 0,$$ (4.4)

$$\nabla_X \xi = 0,$$ (4.5)

where $Q$ is the Ricci operator, $R$ is the Riemannian curvature tensor and $S$ is Ricci tensor defined by $S(X, Y) = g(QX, Y)$.

Lemma 3.2 also valid for paracosymplectic manifolds. For a three-dimensional paracosymplectic manifold, using (4.2), (4.5) in (2.10), we have

$$QX = \frac{r}{2}(X - \eta(X)\xi),$$ (4.6)

$$S(X, Y) = \frac{r}{2}(g(X, Y) - \eta(X)\eta(Y))$$ (4.7)
for any vector field $X \in \chi(M)$. If we use (4.6) and use the same procedure in the proof of Lemma 3.3, we obtain $\xi(r) = 0$.

**Lemma 4.1** For a three-dimensional paracosymplectic manifold which admits a Yamabe soliton has always harmonic scalar curvature, that is $\Delta r = 0$.

**Proof** If we take the Lie-derivative of (4.7) in the direction of $V$ and using (1.1), (3.10), we get

$$
(L_V S)(X, Y) = (-\Delta r)g(X, Y) + \left(\Delta r + \frac{r}{2}(\lambda - r)\right)\eta(X)\eta(Y)
- \frac{r}{2}\left[(L_V \eta)(X)\eta(Y) + (L_V \eta)(Y)\eta(X)\right]
$$

(4.8)

for any vector fields $X, Y \in \chi(M)$. In view of (3.9) and (4.8), we obtain

$$
g(\nabla_X D_r, Y) = (-\Delta r)g(X, Y) + (2\Delta r + r(\lambda - r))\eta(X)\eta(Y)
- r\left[(L_V \eta)(X)\eta(Y) + (L_V \eta)(Y)\eta(X)\right]
$$

(4.9)

for any vector fields $X, Y \in \chi(M)$. Setting $X = Y = \xi$ in (4.9) and using (3.5), $\xi(r) = 0$, we get

$$
\Delta r = 0.
$$

(4.10)

Unlike the case of contact, we can not deduce that $r$ is constant. In other words, on compact semi-Riemannian manifold, there may be non-constant $r$ such that $\Delta r = 0$. So we prove following:

**Theorem 4.2** If the semi-Riemannian metric of a three-dimensional paracosymplectic manifold $(M^3, \varphi, \xi, \eta, g)$ is a Yamabe soliton, then it has constant scalar curvature. So, if $r \neq 0$, then the manifold is $\eta$-Einstein, if $r = 0$, then the manifold is Ricci flat.

**Proof** If we differentiate covariantly $\xi(r) = 0$ along the direction of an arbitrary vector field $X$ and use (4.4), we have

$$
0 = g(\nabla_X D_r, \xi).
$$

(4.11)

Putting $\xi$ for $Y$ in (4.9), using (3.5), (4.10), (4.11), we obtain

$$
\left((\lambda - r)\frac{r}{2}\right)\eta(X) = r(L_V \eta)(X)
$$

(4.12)

for any vector field $X \in \chi(M)$. Making use of last equation and (4.10) in (4.9), we deduce

$$
\nabla_X D_r = 0
$$

(4.13)

for any vector field $X \in \chi(M)$. If we use (4.13) in the Riemannian curvature tensor $R$ equation

$$
R(X, Y)D_r = \nabla_X \nabla_Y D_r - \nabla_Y \nabla_X D_r - \nabla_{[X,Y]} D_r,
$$

we obtain $R(X, Y)D_r = 0$. So, it is clear that $S(X, D_r) = 0$. Taking account of (4.7), we have $\frac{r}{2}X(r) = 0$ which shows that $r$ is constant. By (4.7), one can easily deduce that if $r \neq 0$, then manifold is $\eta$-Einstein, if $r = 0$, then manifold is Ricci flat. 

\(\square\)
5 Yamabe solitons on three-dimensional para-Kenmotsu manifolds

For a particular choices of the function $\alpha = 1$ in (4.1) we have almost para-Kenmotsu manifolds. If in addition the normality condition is fulfilled, then manifolds are called para-Kenmotsu. We may refer to [8] and references therein for more information about para-Kenmotsu manifolds. We also recall that any para-Kenmotsu manifold satisfies

\[
R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (5.1)
\]

\[
(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad (5.2)
\]

\[
\nabla_X \xi = X - \eta(X)\xi, \quad (5.3)
\]

\[
S(X, \xi) = -(n - 1)\eta(X), \quad (5.4)
\]

where $Q$ is the Ricci operator, $R$ is the Riemannian curvature tensor and $S$ is a Ricci tensor defined by $S(X, Y) = g(QX, Y)$.

Lemma 3.2 also valid for para-Kenmotsu manifolds.

**Lemma 5.1** For any three-dimensional para-Kenmotsu manifold $(M^3, \varphi, \xi, \eta, g)$, we have

\[
\xi(r) = -2(r + 6). \quad (5.5)
\]

**Proof** If we replace $Y = Z$ by $\xi$ in (2.10) and use (5.1), (5.4) we get

\[
QX = \left(\frac{r}{2} + 1\right)X - \left(\frac{r}{2} + 3\right)\eta(X)\xi \quad (5.6)
\]

for any vector field $X \in \chi(M)$. If we use (5.6), (5.3) and (5.4) in the following well known formula for semi-Riemannian manifolds

\[
\text{trace} \{ Y \rightarrow (\nabla_X Q)X \} = \frac{1}{2} \nabla_X r
\]

we obtain the requested equation. \qed

The following Lemma and Theorem can be obtained following the same argument used in Lemma 3.3 and Theorem 1.1 in [15] and, for this reason, we omit their proofs.

**Lemma 5.2** If the semi-Riemannian metric on a three-dimensional para-Kenmotsu manifold $(M^3, \varphi, \xi, \eta, g)$ is a Yamabe soliton, then the Yamabe soliton is expanding with $\lambda = -6$ and the scalar curvature of $M^3$ is harmonic, that is $\Delta r = 0$.

**Theorem 5.3** If the semi-Riemannian metric on a three-dimensional para-Kenmotsu manifold $(M^3, \varphi, \xi, \eta, g)$ is a Yamabe soliton, then the manifold is of constant sectional curvature $-1$. Furthermore, Yamabe soliton is expanding with $\lambda = -6$.

The proof of following theorem is similar to Theorem 3.8.

**Theorem 5.4** If the semi-Riemannian metric of a three dimensional para-Kenmotsu manifold is a Yamabe soliton, then it is also a Ricci soliton.

**Proposition 5.5** Let $(M^3, \varphi, \xi, \eta, g)$ be a three-dimensional para-Kenmotsu manifold. If $M^3$ admits a Yamabe soliton for a vector field $V$ and a constant $\lambda$, then $V$ can not be pointwise collinear with $\xi$. 

\[\text{Springer}\]
Proof Let $V$ be a pointwise collinear vector field with the structure vector field $\xi$, that is $V = b\xi$, where $b$ is a smooth function on $M^3$. From (1.1), we obtain

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0 \tag{5.7}$$

for any vector fields $X, Y \in \chi(M)$.

Taking $V = b\xi$ in (5.7) and using (5.3), we have

$$X(b)\eta(Y) + Y(b)\eta(X) + 2bg(X, Y) - 2b\eta(X)\eta(Y) = 0 \tag{5.8}$$

for any vector fields $X$ and $Y$. We have $b = 0$, for $X$ and $Y$ belongs to the contact distribution, in the last equation. $\square$

6 Examples

Now, we will give examples which support Theorems 3.6 and 5.3.

Example 6.1 We consider the 3-dimensional manifold

$$M^3 = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$$

and the vector fields

$$\varphi e_2 = e_1 = 2y \frac{\partial}{\partial x} + z \frac{\partial}{\partial z}, \quad \varphi e_1 = e_2 = \frac{\partial}{\partial y}, \quad \xi = e_3 = \frac{\partial}{\partial x}.$$

The 1-form $\eta = -\frac{2y}{z}dz$ defines an almost paracontact structure on $M$ with characteristic vector field $\xi = \frac{\partial}{\partial x}$. Let $g, \varphi$ be the semi-Riemannian metric and the $(1,1)$-tensor field respectively given by

$$g = \begin{pmatrix} 1 & 0 & -\frac{y}{z} \\ 0 & -1 & 0 \\ -\frac{y}{z} & 0 & \frac{1 + 4y^2}{z^2} \end{pmatrix}, \quad \varphi = \begin{pmatrix} 0 & 2y & 0 \\ 0 & 0 & \frac{1}{z} \\ 0 & z & 0 \end{pmatrix},$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

Using (2.5) we have

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_1 = \xi, \quad \nabla_\xi e_1 = -e_2,$$

$$\nabla_{e_1} e_2 = -\xi, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_\xi e_2 = -e_1,$$

$$\nabla_{e_1} \xi = -e_2, \quad \nabla_{e_2} \xi = -e_1, \quad \nabla_\xi \xi = 0.$$

Hence the manifold is a 3-dimensional para-Sasakian manifold. One can easily compute,

$$R(e_1, e_2)\xi = 0, \quad R(e_2, \xi)\xi = -e_2, \quad R(e_1, \xi)\xi = -e_1,$$

$$R(e_1, e_2)\xi = -3e_1, \quad R(e_2, \xi)\xi = -\xi, \quad R(e_1, \xi)\xi = 0, \tag{6.1}$$

$$R(e_1, e_2)\xi = -3e_2, \quad R(e_2, \xi)\xi = 0, \quad R(e_1, \xi)\xi = \xi.$$

Using (6.1), we have constant scalar curvature

$$r = S(e_1, e_1) - S(e_2, e_2) + S(\xi, \xi) = 2.$$

After writing $V = ae_1 + be_2 + ce_3$; $a, b, c$ are real numbers and using the equation

$$(\mathcal{L}_V g)(X, Y) = \mathcal{L}_V g(X, Y) - g(\mathcal{L}_V X, Y) - g(X, \mathcal{L}_V Y)$$
we have
\[ (\mathcal{L}_{ae_1+be_2+ce_3}g)(X, Y) = a[g(\nabla_X e_1, Y) + g(X, \nabla_Y e_1)] \\
+ b[g(\nabla_X e_2, Y) + g(X, \nabla_Y e_2)] \\
+ c[g(\nabla_X e_3, Y) + g(X, \nabla_Y e_3)]. \]

Using the covariant derivatives, we obtain
\[
(\mathcal{L}_V g)(e_1, e_1) = 0,
(\mathcal{L}_V g)(e_1, e_2) = (\mathcal{L}_V g)(e_2, e_1) = 0,
(\mathcal{L}_V g)(e_1, e_3) = (\mathcal{L}_V g)(e_3, e_1) = -2b,
(\mathcal{L}_V g)(e_2, e_2) = 0,
(\mathcal{L}_V g)(e_2, e_3) = (\mathcal{L}_V g)(e_3, e_2) = 2a,
(\mathcal{L}_V g)(e_3, e_3) = 0.
\]

So, for being \( V \) is an infinitesimal automorphism, \( V \) should satisfy \( \mathcal{L}_V g = 0 \). Hence, from the above equations for being \( \mathcal{L}_V g = 0 \), we get \( a = b = 0 \). Choosing \( c = 1 \), we have \( V = \xi \). On the other hand, in our example, we have \( r = 2 \). So it satisfies the Eq. (3.13) and so \( \lambda = r \). So Yamabe solution Eq. (1.1) \( \mathcal{L}_V g = (\lambda - r)g \) is satisfied. Also using \( r = 2 \) and \( \lambda = r \) in Eq. (3.15), we have \( 0 = \mathcal{L}_\xi \eta \). By \( \mathcal{L}_\xi g = 0 \) and \( g(X, \varphi Y) = \Phi(X, Y) \), we obtain \( \mathcal{L}_\xi \Phi = 0 \). It is clear that \( \mathcal{L}_\xi \xi = 0 \). So, we can conclude that \( V = \xi \) is an infinitesimal automorphism of the paracontact metric structure on \( M^3 \).

**Example 6.2** We consider the 3-dimensional manifold 
\[
M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}
\]
and the vector fields
\[
e_1 = \frac{\partial}{\partial x}, \quad \varphi e_1 = e_2 = \frac{\partial}{\partial y}, \quad \xi = (x + 2y) \frac{\partial}{\partial x} + (2x + y) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.
\]
The 1-form \( \eta = dz \) defines an almost paracontact structure on \( M \) with characteristic vector field \( \xi = (x + 2y) \frac{\partial}{\partial x} + (2x + y) \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \). Let \( g, \varphi \) be the semi-Riemannian metric and the (1, 1)-tensor field given by
\[
g = \begin{pmatrix}
1 & 0 & -\frac{1}{2}(x + 2y) \\
0 & -1 & \frac{1}{2}(2x + y) \\
-\frac{1}{2}(x + 2y) & \frac{1}{2}(2x + y) & 1 - (2x + y)^2 + (x + 2y)^2
\end{pmatrix},
\]
\[
\varphi = \begin{pmatrix}
0 & 1 & -(2x + y) \\
1 & 0 & -(x + 2y) \\
0 & 0 & 0
\end{pmatrix},
\]
with respect to the basis \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \).

Using (5.3) we have
\[
\nabla_{e_1} e_1 = -\xi, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_\xi e_1 = -2e_2,
\]
\[
\nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = \xi, \quad \nabla_\xi e_2 = -2e_1,
\]
\[
\nabla_{e_1} \xi = e_1, \quad \nabla_{e_2} \xi = e_2, \quad \nabla_\xi \xi = 0.
\]
Hence the manifold is a para-Kenmotsu manifold. One can easily compute,
\[
R(e_1, e_2)\xi = 0, \quad R(e_2, \xi)e_2 = -e_2, \quad R(e_1, \xi)e_1 = -e_1,
\]
\[
R(e_1, e_2)e_2 = e_1, \quad R(e_2, \xi)e_2 = -\xi, \quad R(e_1, \xi)e_2 = 0,
\]
\[
R(e_1, e_2)e_1 = e_2, \quad R(e_2, \xi)e_1 = 0, \quad R(e_1, \xi)e_1 = \xi.
\]
Using (6.2), we have constant scalar curvature as follows,
\[
S(e_1, e_1) = -g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = -1 - 1 = -2 = -2g(e_1, e_1),
\]
\[
S(e_2, e_2) = g(R(e_2, e_1)e_1, e_2) + g(R(e_2, e_3)e_3, e_2) = 1 + 1 = 2 = -2g(e_2, e_2),
\]
\[
S(e_3, e_3) = g(R(e_3, e_1)e_1, e_3) - g(R(e_3, e_2)e_2, e_3) = -1 - 1 = -2 = -2g(e_3, e_3)
\]
and
\[
r = S(e_1, e_1) - S(e_2, e_2) + S(\xi, \xi) = -6.
\]
So, \(S(e_i, e_i) = -2e_ig(e_i, e_i)\). We can conclude that \(M\) is an Einstein manifold.

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