Infrared Divergence Separated for Stochastic Force
- Langevin Evolution in the Inflationary Era -

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Inflation in the early Universe is a grand phase transition which have produced the seeds of all the structures we now observe. We focus on the non-equilibrium aspect of this phase transition especially the inevitable infrared (IR) divergence associated to the quantum and classical fields during the inflation. There is a long history of research for removing this IR divergence for healthy perturbation calculations. On the other hand, the same IR divergence is quite relevant and have developed the primordial density fluctuations in the early Universe. We develop a unified formalism in which the IR divergence is clearly separated from the microscopic quantum field theory but only appear in the statistical classical structure. We derive the classical Langevin equation for the order parameter within the quantum field theory through the instability of the de Sitter vacuum during the inflation. This separation process is relevant in general to develop macroscopic structures and to derive the basic properties of statistical mechanics in the quantum field theory.

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I. INTRODUCTION

Primordial quantum fluctuations have produced the present macroscopic structures of the Universe[1]. Quantum fields are squeezed in the exponentially expanding de Sitter Universe (i.e. inflation[2, 3]) and develop the seeds of density perturbations, which eventually grow into the galaxies and the clusters we now observe. These seed fluctuations have almost the Zeldovich spectrum which diverges in the infrared realm. This infrared (IR) divergence is induced by the massless minimally coupled scalar condensation degrees of freedom in the de Sitter space. The IR divergence is thus necessary and unavoidable process for producing macroscopic realm.

On the other hand, the same IR divergence of the same quantum fields in de Sitter space destroys the perturbation evaluation of higher order corrections. There are many references trying to avoid this catastrophic of the theory[4, 5]. Thus the IR divergence is highly unfavorable in the microscopic realm.

The above dilemma of IR divergence opposing in micro-macro realms with each other may be deeply related with the problem of the transition from quantum fluctuations into classical density perturbations in the inflationary era, which itself is a grand phase transition. In this phase transition, the density perturbation is the order parameter, which violates the translational invariance in 3D real space[6]. This transition problem has been often discussed in various aspects such as the classicalization when passing through the horizon, the frozen fluctuations, squeezing of the vacuum, decoherence, or dis-entanglement,... However any comprehensive description has not yet given so far[7].

The above problems would become clear if we adopt that a quantum field has two phases each represents micro- and macro- degrees of freedom. The macroscopic classical degrees of freedom is the condensation of the quantum field $\varphi$ and the microscopic quantum degrees of freedom is the quantum excitations $\phi$ on the classical filed $\bar{\varphi}$.

The separation of them becomes clear if we use the generalized effective action method [8]. This method is briefly introduced in section 2. According to this formalism, the two kinds of degrees of freedom $\varphi$ and $\phi$ interact with each other. Furthermore the IR divergence turns out to appear only in the statistical part of the Langevin dynamics for $\varphi$. This separation therefore makes the ordinary perturbation calculations possible in the quantum field theory for $\phi$.

The popular stochastic method[10] fits well with this formalism (section 3) although the artificial separation of the field is necessary. This problem is resolved if we introduce genuine interaction term (section 4), which clearly defines the classical order parameter. We further examine the IR property of the general massive non-minimally coupled scalar field in the de Sitter spacetime (section 5). Lastly we summarize our work and comment on the general generation of statistical mechanics (section 6).

II. LANGEVIN EQUATION FROM QUANTUM FIELD THEORY

We will derive Langevin equation in de Sitter space later. In this section, we start from the classical Langevin equation back to the field theory.

Langevin equation is a typical description of a particle motion exerted by both the potential force $-V'$ and the random force $\xi$ with friction $\gamma$:

$$\ddot{x}(t) = -\gamma \dot{x}(t) - V'(x(t)) + \xi(t)$$

(1)

where the random field obeys the statistical property determined by the weight functional $P[\xi]$,

$$\langle...\rangle_\xi = \int D[\xi]...P[\xi]$$

(2)

where we temporally assume the Gaussian form $P[\xi] = e^{-\int \xi(t)^2/(2\sigma^2)}$. Then the $x-$correlation can be generated by

$$Z[J] = \left\langle e^{-\int dt J(t)x(t)} \right\rangle_\xi$$

(3)

$$= \int D[\xi]D[x]P[\xi]\delta[\dot{x}(t) + \gamma \dot{x}(t) + V'(x(t)) - \xi(t)]e^{-\int dt J(t)x(t)}$$

$$= \int D[\xi]D[x]D[x']P[\xi]e^{\int dt \{\dot{x}(t) + \gamma \dot{x}(t) + V'(x(t)) - \xi(t)\}} \epsilon^{-\int dJ x(t)}$$

$$\equiv \int D[\xi]D[x]D[x']P[\xi]e^{i\tilde{S}[x,x'] - i\int dJ x}$$

where the integral form of the delta functional is utilized, and

$$\tilde{S}[x,x'] = \int dt \{-\dot{x'}(t)\dot{x}(t) + \gamma x'(t)\dot{x}(t) + x'(t)V'(x(t)) - x'(t)\xi(t)}$$

(4)
where the boundary term is dropped. This is the ‘action’ because the application of the least action principle for the variable \(x'(t)\) yields the original Langevin equation Eq. (1). There is another expression for \(Z[J]\) given by integrating out \(\xi\),

\[
Z[J] = \int D[x]D[x']e^{i\tilde{\Gamma}[x,x']-i\int Jx}
\]  

(5)

where the ‘complex action’ is,

\[
\tilde{\Gamma}[x,x'] \equiv \int dt\{-\dot{x}'(t)\dot{x}(t) + \gamma x'(t)\dot{x}(t) + x'(t)V'(x(t)) - i\sigma^2 x'(t)^2/2\}.
\]  

(6)

It is apparent that the imaginary part of \(\tilde{\Gamma}[x,x']\) represents statistical fluctuations.

It is possible to reverse the logic. If we have a complex action including an extra degrees of freedom like \(x'\) above, we can derive a Langevin equation. This is the formalism of the generalized effective action method utilizing the closed time-contour [9, 10]. This formalism is a slight generalization of the ordinary quantum field theory but particularly suitable for the dissipative dynamics for the condensed classical variables [11, 12].

Let us consider the quantum field theory generalizing the above considerations. The generating functional of the many point functions is defined as

\[
\tilde{Z}[\tilde{J}] \equiv \text{Tr}\left[\tilde{T}\left[\exp[i\int \tilde{J}\tilde{\phi}]\rho\right]\right] = \exp[i\tilde{W}[\tilde{J}]],
\]  

(7)

where the tildes mean that the associated quantities are defined on the closed time-contour: from \(-\infty \) to \(+\infty\) and than back to \(-\infty\) again. \(T\) means the time ordering operation on this contour, \(J\) is an external source, and \(\rho\) is the initial density matrix for the field \(\phi\). The trace operation is over the functions on the closed time-contour. In the two by two matrix representation, \(\tilde{\phi}(x) = (\phi_+(x), \phi_-(x))\), \(\tilde{J}[x] = (J_+(x), J_-(x))\), and \(\int \tilde{J}\tilde{\phi} = \int dxJ_+(x)\phi_+(x) - \int dxJ_-(x)\phi_-(x)\). Note the extra minus sign in the above comes from the reversed time contour part that has negative measure. A pair of variables \(\tilde{\phi}_\Delta \equiv \phi_+(x) - \phi_-(x)\) and \(\tilde{\phi}_C \equiv (\phi_+(x) + \phi_-(x))/2\) are also often used. In the interaction picture: \(\mathcal{L}[\phi] = \mathcal{L}_0[\phi] - V[\phi]\), we have,

\[
\tilde{Z}[\tilde{J}] = \exp\left[-i\int V\left[\frac{\delta}{i\delta \tilde{J}}\right]\right]\exp[-i\int \tilde{J}(x)\tilde{G}_0(x,y)\tilde{J}(y)]\text{Tr}:(\exp(i\int \tilde{J}\tilde{\phi}) : \rho),
\]  

(8)

where \(\tilde{\phi}\) is in the interaction picture and \(\tilde{G}_0\) is a free propagator. We can develop perturbative calculations based on the last expression. The C-number order parameter \(\tilde{\varphi}\) is defined by

\[
\tilde{\varphi}(x) = \frac{\delta \tilde{W}}{\delta \tilde{J}(x)}
\]  

(9)

Then the effective action \(\tilde{\Gamma}\) is defined as the Legendre transformation of \(\tilde{W}\):

\[
\tilde{\Gamma}[\tilde{\varphi}] \equiv \tilde{W}[\tilde{J}] - \int \tilde{J}\tilde{\varphi}.
\]  

(10)

The propagator part in the above \(\tilde{J}(x)\tilde{G}_0(x,y,\tilde{J}(y)\) becomes

\[
J_\Delta(x)G_R(x,y)J_C(y) + J_C(x)G_A(x,y)J_\Delta(y) - iJ_\Delta(x)G_C(x,y)J_\Delta(y)
\]  

(11)

where

\[
G_R(x,y) = i\theta (x^0 - y^0) \langle \phi(x), \phi(y) \rangle,
\]  

(12)

\[
G_A(x,y) = -i\theta (y^0 - x^0) \langle \phi(x), \phi(y) \rangle,
\]  

(13)

\[
G_C(x,y) = \langle \phi(x), \phi(y) \rangle.
\]  

(14)

The last term in Eq. (11) is special and imaginary. It comes from the symmetric part of the propagator, while the rest comes from the anti-symmetric part of the propagator. Thus they differ by factor \(i\). If used in the original equation, it yields the pure Gaussian factor. Functionally Fourier transforming this term, we obtain

\[
\exp[i\tilde{\Gamma}[\varphi_\Delta, \varphi_C]] = \int D\xi P[\xi] \exp[i\tilde{S}_{eff}[\varphi_\Delta, \varphi_C, \xi]],
\]  

(15)
This extreme exponential expansion is modeled to be caused by the scalar field.

The full effective action in Eq. (15) is a bundle of effective actions though only the massless minimally coupled case (\(\xi = 0\)) on this space-time, Eq. (16). This function has a weight \(P\) which yields the equation of motion for \(\varphi\). Let us consider the inflationary era in the early universe when the cosmic expansion is exponential \(a(t) = e^{Ht} = -(H\eta)^{-1}\), i.e. the de Sitter space-time

\[
ds^2 = dt^2 - e^{2Ht}dx^2 = (H\eta)^{-1}(dy^2 - dx^2).
\]

This extreme exponential expansion is modeled to be caused by the scalar field

\[
S[\phi] = \int d^4x (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 - \xi R\phi)
\]

though only the massless minimally coupled case \((m = \xi = 0)\) is relevant. The field is expanded in the normal mode on this space-time,

\[
\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\pi H\eta^{3/2}}{2}} \left( \hat{a}_k H^{(1)}(k|\eta|)e^{i \vec{k} \cdot \vec{x}} + \hat{a}_k H^{(2)}(k|\eta|)e^{-i \vec{k} \cdot \vec{x}} \right)
\]

where

\[
P[\xi] = \exp\left[ -\frac{1}{2} \int dx \int dy \xi(x)GC(x, y)^{-1}\xi(y) \right],
\]

and

\[
\hat{S}_{\text{eff}}[\varphi, \varphi, \xi] = -\int dx dy \varphi(x)G_R(x, y)^{-1} \varphi(y) - \int dx dy \varphi(x)G_A(x, y)^{-1} \varphi(y) - \int dx \xi(x)\varphi(x).
\]

The full effective action in Eq. (15) is a bundle of effective actions \(\hat{S}_{\text{eff}}[\varphi, \xi]\) that depends on the field \(\xi(x)\). This field can be interpreted as the classical random field since the correlations of them is generated by the Gaussian functional Eq. (10). This function has a weight \(P[\xi]\) in the average \(\int D[\xi]\). The real part \(\hat{S}_{\text{eff}}[\varphi, \xi]\) represents the time evolution

\[
\frac{\delta \hat{S}_{\text{eff}}[\varphi, \xi]}{\varphi(x)}|_{\varphi(x) = 0} = -jC,
\]

which yields the equation of motion for \(\varphi\):

\[
\int dy 2G_R(x, y)^{-1} \varphi(y) + \xi = jC.
\]

In this equation, the first term in the left hand side often yields the friction term \(\gamma \dot{\varphi}(x) + \gamma \ddot{\varphi}(x) + \ldots\) originated from the time asymmetric part in the propagator. This time reversal asymmetry comes from the choice of our initial condition to choose in state, i.e. the closed time-contour from \(-\infty\) to \(+\infty\) and than back to \(-\infty\) again.

The correlation function for the random field \(\xi\) is given by

\[
<...>_{\xi} = \int D[\xi]...P[\xi],
\]

and

\[
<\xi(x)\xi(y)>_{\xi} = GC(x, y).
\]

The above separation of the full dynamics into the two parts, deterministic \(\hat{S}_{\text{eff}}[\varphi, \varphi, \xi]\) and stochastic \(P[\xi]\), is general. The arguments are formal so far and actually nothing special in equilibrium system. However in the non-equilibrium settings, such as in the evolving background spacetime, the system actually yields fluctuations and dissipation.

Furthermore in our context, the IR divergence is only in the stochastic part and the deterministic part is safe from the IR divergence. Therefore the ordinary perturbation calculation is possible using \(\hat{S}_{\text{eff}}[\varphi, \varphi, \xi]\) and the stochastic part \(P[\xi]\) agitates the system intermittently. In the following sections we will see the detail of this structure.

III. BI-LINEAR INTERACTION

Let us consider the inflationary era in the early universe when the cosmic expansion is exponential \(a(t) = e^{Ht} = -(H\eta)^{-1}\), i.e. the de Sitter space-time

\[
S[\phi] = \int d^4x (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 - \xi R\phi)
\]

though only the massless minimally coupled case \((m = \xi = 0)\) is relevant. The field is expanded in the normal mode on this space-time,
where \( \nu = (2 - m^2 \omega^2 + \xi R \phi)^{1/2} \) and \( H^{(s)}_k \) are the Hankel functions. This normal mode is selected by the requirement that the mode function reduces to the Minkowski form locally \( k \to \infty \). We now restrict our considerations to the most relevant massless minimally coupled case \( \nu = 3/2 \),

\[
H^{(1)}_{3/2}(k|\eta|) = -\frac{\sqrt{\frac{2}{\pi}}e^{ik|\eta|}(k|\eta| + i)}{(k|\eta|)^{3/2}}. \tag{25}
\]

The standard method is to introduce the separation of the field \( \phi = \phi_+ + \phi_- \) at around the scale of the horizon: \( \phi_+ \equiv \int dk \theta(k-H) \phi \). And consider the interaction of them \( \phi_+(x)\phi_-(x) \). Then the effective dynamics for the large scale mode \( \phi_- \) is given by integrating \( \phi_+ \) first.

\[
\tilde{Z}[\tilde{J}] = \int D\tilde{\phi}_- D\tilde{\phi}_+ \exp[i\tilde{S}[\tilde{\phi}]] + i \int d^4x \tilde{J}(x)\tilde{\phi}(x), \tag{26}
\]

\[
= \int D\tilde{\phi}_- \exp[i \int d^4x \tilde{\phi}_-(x)\tilde{G}_0(x-y)\tilde{\phi}_-(y) + i \int d^4x \tilde{J}(x)\tilde{\phi}(x)],
\]

\[
= \int D\tilde{\phi}_- \exp[-\frac{1}{4} \int d^4k \phi_-(\vec{k})G_C(\vec{k})\phi_<(\vec{k}) + \frac{i}{2} \int d^4k \phi_<(\vec{k})\theta(\Delta|\eta|)G_R(\vec{k})\phi_<(\vec{k}) + i \int d^4k \tilde{J}(\vec{k})\phi(\vec{k})],
\]

\[
G_0 = -\int \frac{d^3k}{(2\pi)^3} \frac{\eta H^2}{2k^3} e^{-ik(\eta-\eta')} + i \vec{k} \cdot (\vec{z} - \vec{x}) (i - k\eta)(i + k\eta'),
\]

\[
G_C(\vec{k}) = \frac{H^2}{k^3} (1 + k^2\eta)(\cos(k\Delta|\eta|) + k\Delta|\eta| \sin(k\Delta|\eta|)) \propto \frac{H^2}{k^3} \tag{28}
\]

\[
G_R(\vec{k}) = -i \frac{H^2}{k^3} (\cos(k\Delta|\eta|) + (1 + k^2\eta) \sin(k\Delta|\eta|)) \propto \frac{iH^2\Delta|\eta|}{k^3} \tag{29}
\]

Then in the last equation, the statistical and deterministic parts are separated as

\[
\tilde{Z}[\tilde{J}] = \int D\xi P(\xi) \int D\phi_+ \exp[i \int d^4k \phi_<(\vec{k})\theta(\Delta|\eta|)G_R(\vec{k})\phi_<(\vec{k}) + \frac{i}{2} \int d^4k \xi(\vec{k})\phi_<(\vec{k}) + i \int d^4k \tilde{J}(\vec{k})\phi(\vec{k})],
\]

where the statistical weight becomes the Gaussian form,

\[
P(\xi) = \exp[-\frac{1}{4} \int d^4k \xi(\vec{k})G_C(\vec{k})^{-1}\xi(\vec{k})]. \tag{31}
\]

The Langevin equation is derived by the variation by \( \phi_<(\vec{k}) \) to yield,

\[
3H \frac{d\phi_<(\vec{k})}{d\eta} = \xi \tag{32}
\]

and the correlation function of \( \phi_<(\vec{k}) \) becomes

\[
\langle \phi_<(\vec{k})\phi_<(\vec{k'}) \rangle \approx \frac{H^2}{k^3} \tag{33}
\]

at the Horizon crossing \( \eta = -k^{-1} \), the standard evaluation point. This is the stochastic method [10, 13].

However, artificial separation of free field \( \phi = \phi_- + \phi_+ \) at the Horizon does not resolve the quantum-classical transition problem. The field \( \phi_- \) is still quantum. Something equivalent to a detector degrees of freedom is needed to discuss the statistical and classical nature of the fluctuations in this formalism [7]. We will further consider this point introducing the self interaction of the scalar field in the effective action formalism in the next section.
IV. SELF-COUPLED INTERACTION

In the above, we have no idea why the field $\phi_<$ behaves classically. We would like to solve this problem together with the IR problem in de Sitter space. We introduce the non-linearity of the scalar field and the condensation of this quantum field. The action is given by

$$S[\phi] = \int d^4x (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 - \xi R \phi - \lambda \phi^4 / 4!) .$$

The partition function becomes

$$\tilde{Z}[\tilde{J}] = \int D\tilde{\phi} \exp[i S[\phi_+] - i S[\phi_-] + i \int d^4x \tilde{J}(x) \tilde{\phi}(x)] \equiv \exp i \tilde{W} ,$$

and its Legendre transform, i.e. the effective action becomes

$$\exp [i \tilde{\Gamma}[\tilde{\varphi}]] = \exp [i\tilde{\Gamma}[\tilde{\varphi}]] = \int D\tilde{\phi} \exp [i S[\tilde{\phi}]] = \int D\tilde{\phi} \exp [i S[\tilde{\phi}]] + \int d^4x \tilde{J}(x) (\tilde{\phi}(x) - \tilde{\varphi}(x)) ,$$

where the integration field is shifted by $\varphi$. Then expanding the action around $\varphi$, we further have

$$\exp [i \tilde{\Gamma}[\tilde{\varphi}]] = \exp [i W[\tilde{\varphi}]] - \int d^4x \tilde{J}(x) \tilde{\varphi}(x)$$

$$= \int D\tilde{\phi} \exp [i S_{int}[\tilde{\phi}]] + \frac{1}{2} \int d^4x \tilde{\phi}(x) \tilde{G}_0^{-1}(x-y) \tilde{\phi}(y) - \int d^4x \tilde{J}(x) \tilde{\varphi}(x) ,$$

where $S_{int}[\phi; \varphi]$ is the Taylor expansion of $\phi$ around $\varphi$. The first order term does not vanish because we do not assume the $\varphi$ solves the free equation of motion from $S_0$ as in the ordinary stationary approach. The second order term is absorbed into the propagator $G_0(x-y)$. The third order or higher terms are genuine interactions which yields the one particle irreducible graphs as usual. The first term yields the factor in the effective action

$$\exp [i \tilde{\Gamma}[\tilde{\varphi}]] \int D\tilde{\phi} \exp [i \lambda \varphi^3 \tilde{\phi}] + \frac{1}{2} \int d^4x \phi(x) \tilde{G}_0^{-1}(x-y) \phi(y)]$$

$$= \exp [i S_0[\tilde{\varphi}]] \int D\tilde{\phi} \exp [i (\lambda \varphi(x))^3 \Delta G_R(x-y)(\lambda \varphi(y))^3 \Delta C + (\lambda \varphi(x))^3 \Delta G_A(x-y)(\lambda \varphi(y))^3 \Delta G_R(x-y)(\lambda \varphi(y))^3 \Delta$$

where $G_R(x-y)$, $G_C(x-y)$ have the form Eq. (29). In the interaction, the remaining kinetic terms becomes irrelevant in the IR limit and the mass term does not exist. The IR divergent term $G_C(x-y)$ becomes pure Gaussian and therefore can be separated as in the previous way to yield statistical fluctuations. The finite terms $G_R(x-y)$, $G_A(x-y)$ yield friction term. However the macroscopic friction term $-3H \tilde{\phi}$ directly associated with the cosmic expansion in the equation of motion dominates this friction. Thus the full effective action turns out to be

$$\exp [i \tilde{\Gamma}[\tilde{\varphi}]] = \int D\xi P(\xi) \exp [i \Gamma[\varphi; \xi ]] ,$$

where

$$\exp [i \Gamma[\varphi; \xi ]] = \exp [i S_0[\tilde{\varphi}]] \exp [i S_{int}[\frac{\delta}{i \delta \tilde{J} \tilde{\phi}}, \varphi]] \exp [i \frac{1}{2} \int d^4x \tilde{J}(x) G_0'(x-y) \tilde{J}(y) + \int d^4x \xi(x)(\lambda \varphi(x))^3 \Delta] .$$

where $S_{int}'$ is the interaction term with the linear term removed and $G_0'(x-y)$ is the propagator with the IR divergence removed.
This allows the ordinary perturbation calculations, infrared safe, for higher order quantum corrections; the infrared diverging term is fully separated in the fluctuation kernel $P(\xi)$. The statistical fluctuations represented by $\xi$ acts on the local quantum dynamics intermittently. However the effect is mostly limited in the long range IR region.

We can obtain the equation of motion for the order parameter $\tilde{\phi}$:

$$\frac{\delta \tilde{T}}{\delta \tilde{\phi}(x)} = -\tilde{J}(x).$$

This becomes in the lowest order of $\xi_k$ in the strong damping regime,

$$3H\tilde{\varphi}_k + (\lambda/2)\tilde{\varphi}_0^2\tilde{\varphi}_k = (\lambda/2)\tilde{\varphi}_0^2\xi_k.$$  \hfill (42)

This equation yields the same power spectrum for $\varphi_k$ but a slightly different amplitude:

$$\langle \varphi_k \varphi_k \rangle_\xi \approx \lambda^2 \varphi_0^2 \frac{H^2}{k^3}.$$ \hfill (43)

The spectrum does not change because the statistical fluctuations showing IR divergence dominate in the full quantum propagators. The amplitude does change because the statistical fluctuations are extracted through the non-linear interactions.

There are variety of applications of the obtained Langevin equation to the actual inflationary dynamics. This can be used to select the correct model among fair amount of inflationary models presently proposed. Interestingly, the Langevin analysis on the original standard model of inflation yields the same result as the bi-linear case. The detail will be reported elsewhere.

V. NON-MINIMAL MASSIVE CASE

The massless minimal coupling scalar field is most useful for inflation. Therefore the IR divergence has been considered first in this case. However the IR anomalous enhancement is not restricted to this case. We will see the general scalar field in de Sitter spacetime now. The same approach is enough for this purpose.

The propagator becomes \[14]\]

$$\langle \phi(x)\phi(x') \rangle = \int \frac{d^3k}{(2\pi)^3} H^{(1)}(k\eta)H^{(2)}(k\eta'),$$ \hfill (44)

where the real and imaginary part of the Hankel functions are manifest,

$$H^{(1)}(z) = J_\nu(z) + i\nu Y_\nu(z), \quad H^{(2)}(z) = J_\nu(z) - i\nu Y_\nu(z),$$ \hfill (45)

and behaves, in small argument, as

$$J_\nu(z) = \frac{2}{\Gamma(1+\nu)} z^\nu + O(z^{2+\nu}), \quad Y_\nu(z) = -\frac{2\nu\Gamma(\nu)}{\pi} z^{-\nu} + O(z^{2-\nu}).$$ \hfill (46)

Therefore the IR behavior is obvious:

$$G_C(\vec{k}) = J_\nu(k\eta)J_\nu(k'\eta') + Y_\nu(k\eta)Y_\nu(k'\eta') \propto H^{2\nu}k^{-2\nu},$$ \hfill (47)

$$G_R(\vec{k}) = i(J_\nu(k\eta)Y_\nu(k'\eta') - J_\nu(k'\eta)Y_\nu(k\eta)) \propto k^0.$$  

Since $\nu = (\frac{3}{4} - \frac{m^2\phi^2+\xi R}{H^2})^{1/2}$, and therefore $0 \leq \nu \leq 3/2$, it is apparent that the IR dangerous term exists only in $G_C(\vec{k})$. This term is isolated from the microscopic dynamics as a statistical fluctuations as before. In the present general case, the IR behavior is milder than the massless minimal case.

However higher loop contributions and/or higher point functions may yield severe IR behavior\[3\]. Even in those cases, it may happen that the IR divergence is associated with the imaginary part of the effective action. For example, the graph is associated with the real particle emission process. Then these IR divergent terms can be separated from the quantum dynamics as the statistical weight. In this case the statistical weight is no longer the Gaussian form. Therefore unusual statistical mechanics is expected. We don’t know how extent various IR divergence can be absorbed into the imaginary part of the effective action at present. We hope we can report this interesting problem soon in our future publications.
VI. CONCLUSIONS AND PROSPECTS

We studied the non-equilibrium aspect of the inflationary phase transition in the early Universe.

The massless minimally couple scalar field yields the exponential expansion of the Universe and the quantum fields on this spacetime becomes peculiar and yields IR divergence. This IR divergence is quite relevant to produce the seed fluctuations of all the structures in the Universe. On the other hand this IR divergence destroys the quantum field theory and the perturbation method.

In this paper, we clarified that this dilemma comes from the mixing up the finite quantum part and the diverging statistical part in the formalism. By separating these two contributions, we could derive the classical Langevin equation of motion for the order parameter of the inflationary phase transition. This IR divergence simply reflects that the statistical fluctuations have long-time correlation. This Langevin equation is the manifestly classical evolution and is adequate to describe the macroscopic dynamics such as the large scale structures in the Universe. On the other hand the remaining quantum evolution is free from IR divergence. This separation of statistical and quantum fluctuations has been the crucial point of the problem.

We further need to clarify the problem why and how extent the IR divergence is associated with the imaginary part or the statistical part of the effective action. For the scalar fields in de Sitter space, this association was general. Probably the IR divergence and the related peculiar non-equilibrium behavior in the curved space comes from the violent particle production process from the vacuum. Applying the Bogoliubov transformation formalism, we would like to generalize the present work to other evolving space-times.

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[1] D. Baumann, [http://arxiv.org/abs/0907.5424](http://arxiv.org/abs/0907.5424).
[2] A. H. Guth, Phys. Rev. D 23, 347 (1981).
[3] K. Sato, Monthly Notices of Royal Astronomical Society, 195, 467, (1981).
[4] T. Arai, Phys. Rev. D86 104064 (2012).
[5] A. M. Polyakov, Nucl. Phys. B, Proc. Suppl. B797, 199 (2008).
[6] M. Morikawa, Prog. Theor. Phys. 77, 1163 (1987).
[7] Y. Nambu and Y. Ohsumi, PRD80, 124031 (2009); PRD84, 044028 (2011).
[8] T. Fukuyama and M. Morikawa, Phys. Rev. D80, 063520 (2009).
[9] M. Morikawa, Phys. Rev. D33 (1986), 3607.
[10] Starobinsky, Alexei A. (1982). Phys. Lett. B117: 175–8.
[11] Keldysh, L. V., 1964, Zh. Eksp. Teor. Fiz., 47, 1515; [Sov. Phys. JETP, 1965, 20, 1018].
[12] Kadanoff, L. P., and Baym, G., 1962, Quantum Statistical Mechanics, (Benjamin, New York).
[13] A. Hosoya, M. Morikawa, and K. Nakayama, Int. J. Mod. Phys. A 04, 2613 (1989).
[14] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. Lond. A 360, 117 (1978).