Game positions of
Multiple Hook Removing Game

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Abstract

Multiple Hook Removing Game (MHRG for short) introduced in [1] is an impartial game played in terms of Young diagrams. In this paper, we give a characterization of the set of all game positions in MHRG. As an application, we prove that for \( t \in \mathbb{Z}_{\geq 0} \) and \( m, n \in \mathbb{N} \) such that \( t \leq m \leq n \), and a Young diagram \( Y \) contained in the rectangular Young diagram \( Y_{t,n} \) of size \( t \times n \), \( Y \) is a game position in MHRG with \( Y_{m,n} \) the starting position if and only if \( Y \) is a game position in MHRG with \( Y_{t,n} - m + t \) the starting position, and also that the Grundy value of \( Y \) in the former MHRG is equal to that in the latter MHRG.

1 Introduction.

The Sato-Welter game is an impartial game studied by Welter [8] and Sato [5], independently. This game is played in terms of Young diagrams. The rule is given as follows:

(i) The starting position is a Young diagram \( Y \).

(ii) Assume that a Young diagram \( Y' \) appears as a game position. A player chooses a box \( (i, j) \in Y' \), and moves game position from \( Y' \) to \( Y'(i, j) \), where \( Y'(i, j) \) is the Young diagram which is obtained by removing the hook at \( (i, j) \) from \( Y' \) and filling the gap between two diagrams (see Figure 1 below).

(iii) The (unique) ending position is the empty Young diagram \( \emptyset \). The winner is the player who makes \( \emptyset \) after his/her operation [1].

Kawanaka [2] introduced the notion of a plain game, as a generalization of the Sato-Welter game. A plain game is played in terms of \( d \)-complete posets which was introduced and classified by Proctor [3, 4], and can be thought of as a generalization of Young diagrams. It is known that \( d \)-complete posets are

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closely related to not only the combinatorial game theory, but also the representation theory and the algebraic geometry associated with simply-laced finite-dimensional simple Lie algebras. For example, the weight system of a minuscule representation (which is identical to the Weyl group orbit of a minuscule fundamental weight) for a simply-laced finite-dimensional simple Lie algebra can be described in terms of a $d$-complete poset. Applying the “folding” technique to this fact for the simply-laced case, Tada [7] described the Weyl group orbits of some fundamental weights for multiply-laced finite-dimensional simple Lie algebras in terms of $d$-complete posets with “coloring”.

Based on [7], Abuku and Tada [1] introduced a new impartial game, named Multiple Hook Removing Game (MHRG for short). MHRG is played in terms of Young diagrams with the unimodal numbering; for the definition of unimodal numbering, see Section 3 and Example 3.1. Let us explain the rule of MHRG.

(1) All game positions are some Young diagrams contained in $\mathcal{F}(Y_{m,n})$ with the unimodal numbering. The starting position is the rectangular Young diagram $Y_{m,n}$.

(2) Assume that $Y \in \mathcal{F}(Y_{m,n})$ appears as a game position. If $Y \neq \emptyset$ (the empty Young diagram), then a player chooses a box $(i, j) \in Y$, and remove the hook at $(i, j)$ in $Y$. We denote by $Y(i, j)$ the resulting Young diagram. Then we know from [11] Lemma 3.15 (see also Lemma 4.4 below) that $f := \| \{(i', j') \in Y(i, j) \mid \mathcal{H}_Y(i, j) \neq \mathcal{H}_Y(i', j') \}\|$ where $\mathcal{H}_Y(i, j)$ (resp., $\mathcal{H}_Y(i', j')$) is the numbering multiset for the hook at $(i, j) \in Y$ (resp., $(i', j') \in Y(i, j)$); see Section 3. If $f = 0$, then a player moves $Y$ to $Y(i, j) \in Y_{m,n}$.
\( \mathcal{O}(Y) \). If \( f = 1 \), then a player moves \( Y \) to \( (Y(i,j))(i',j') \in \mathcal{O}(Y) \), where \((i',j') \in Y(i,j) \) is the unique element such that \( H_{Y(i,j)}(i',j') = H_Y(i,j) \).

(3) The (unique) ending position is the empty Young diagram \( \emptyset \). The winner is the player who makes \( \emptyset \) after his/her operation (2).

In general, not all Young diagrams in \( \mathcal{F}(Y_{m,n}) \) appear as game positions of MHRG (see Example 4.3). The goal of this paper is to give a characterization of the set of all game positions in MHRG. Let us explain our results more precisely.

Let \( \left[ 1, m + n \right] \) denote the set of all subsets of \([1, m + n] := \{ x \in \mathbb{N} \mid 1 \leq x \leq m + n \} \) having \( m \) elements. Then there exists a bijection \( I \) from \( \mathcal{F}(Y_{m,n}) \) onto \( \left[ 1, m + n \right]^m \) (see Subsection 2.1 below). Let \( Y^D \) denote the dual Young diagram of \( Y \) in \( Y_{m,n} \) (see Subsection 2.1). We set \( c := (m + n - 1 + \chi) / 2 \), where \( \chi = 0 \) (resp., \( \chi = 1 \)) if \( m + n \) is odd (resp., even). For \( Y \in \mathcal{F}(Y_{m,n}) \), we set \( I_R(Y) := I(Y) \cap [c + 1 - \chi, m + n] \). We denote by \( S(Y_{m,n}) \) the set of all those Young diagrams in \( \mathcal{F}(Y_{m,n}) \) which appear as game positions of MHRG (with \( Y_{m,n} \) the starting position).

**Theorem 1.1** (= Theorem 5.1). Let \( Y \in \mathcal{F}(Y_{m,n}) \), and \( \lambda = (\lambda_1, \ldots, \lambda_m) \) the partition corresponding to \( Y \). The following (I), (II), (III), and (IV) are equivalent.

(I) \( Y \in S(Y_{m,n}) \). (II) \( Y^D \in S(Y_{m,n}) \). (III) \( I_R(Y) \cap I_R(Y^D) = \emptyset \).

(IV) \( \lambda_i + \lambda_j \neq n - m + i + j - 1 \) for all \( 1 \leq i \leq j \leq m \).

**Theorem 1.2** (= Theorem 6.1). Let \( t \in \mathbb{Z}_{\geq 0} \) and \( m, n \in \mathbb{N} \) such that \( t \leq m \leq n \). For a Young diagram \( Y \) having at most \( t \) rows, \( Y \in S(Y_{t,n}) \) if and only if \( Y \in S(Y_{t,n-m+t}) \). Moreover, the Grundy value of \( Y \) as an element of \( S(Y_{m,n}) \) is equal to the Grundy value of \( Y \) as an element of \( S(Y_{t,n-m+t}) \).

This paper is organized as follows. In Section 2, we fix our notation for Young diagrams, and recall some basic facts on the combinatorial game theory. In Section 3, we recall the definition of the unimodal numbering and the diagonal expression for Young diagrams. In Section 4, we recall the rule of MHRG, and a basic property (Lemma 4.4). In Sections 5 and 6, we prove Theorems 1.1 and 1.2 above, respectively.

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2 Preliminaries.

2.1 Young diagrams.

Let \( \mathbb{N} \) denote the set of positive integers. For \( a, b \in \mathbb{Z} \), we set \([a, b] := \{ x \in \mathbb{Z} \mid a \leq x \leq b \} \). Throughout this paper, we fix \( m, n \in \mathbb{N} \) such that \( m \leq n \). For
a positive integer \( x \in \mathbb{N} \), we set \( \mathfrak{m} := m + n + 1 - x \). Let \( \mathcal{Y}_m(m+n) \) be the set of partitions \( \lambda = (\lambda_1, \ldots, \lambda_m) \) of length at most \( m \) such that \( n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0 \). We can identify \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{Y}_m(m+n) \) with the Young diagram \( Y_{\lambda} := \{(i, j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq m, 1 \leq j \leq \lambda_i \} \) of shape \( \lambda \); if \( \lambda = (0, 0, \ldots, 0) \in \mathcal{Y}_m(m+n) \), then we denote \( Y_{\lambda} \) by \( \emptyset \), and call it the empty Young diagram. We identify \((i, j) \in Y_{\lambda} \) with the square in \( \mathbb{R}^2 \) whose vertices are \((i-1, j-1), (i-1, j), (i, j-1), \) and \((i, j)\); elements in \( Y_{\lambda} \) are called boxes in \( Y_{\lambda} \). Let \( \mathcal{Y}_{m,n} := \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq m, 1 \leq j \leq n \} \) be the rectangular Young diagram of size \( m \times n \), which corresponds to \((n, n, \ldots, n) \in \mathcal{Y}_m(m+n) \). Set \( \mathcal{F}(\mathcal{Y}_{m,n}) := \{ Y_{\lambda} \mid \lambda \in \mathcal{Y}_m(m+n) \} \); notice that \( \mathcal{F}(\mathcal{Y}_{m,n}) \) is identical to the set of all Young diagrams contained in the rectangular Young diagram \( Y_{m,n} \). We set \( \lambda^D := (n-\lambda_m, n-\lambda_{m-1}, \ldots, n-\lambda_1) \in \mathcal{Y}_m(m+n) \). The Young diagram \( Y_{\lambda}^D := Y_{\lambda^D} \) is called the dual Young diagram of \( Y_{\lambda} \) (in \( Y_{m,n} \)).

Let \( \{(1, m+n)\}_{m+1} \) denote the set of all subsets of \([1, m+n]\) having \( m \) elements. For \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{Y}_m(m+n) \), we set \( i'_t := \lambda_m-t+1 + t \) for \( 1 \leq t \leq m \); observe that \( I_\lambda := \{i'_1 < \cdots < i'_m \} \in \{(1, m+n)\}_{m+1} \). It is well-known that the map \( \lambda \mapsto I_{\lambda} \) is a bijection from \( \mathcal{Y}_m(m+n) \) onto \( \{(1, m+n)\}_{m+1} \). By the composition of this bijection and the inverse of the bijection \( \mathcal{Y}_m(m+n) \rightarrow \mathcal{F}(\mathcal{Y}_{m,n}) \), \( \lambda \mapsto Y_{\lambda} \), we obtain a bijection \( I \) from \( \mathcal{F}(\mathcal{Y}_{m,n}) \) onto \( \{(1, m+n)\}_{m+1} \). Let \( Y \in \mathcal{F}(\mathcal{Y}_{m,n}) \). For \((i, j) \in \mathcal{F}(\mathcal{Y}_{m,n}) \), we set \( H_Y(i,j) := \{(i,j)\} \cup \{(i',j') \in Y \mid j < j'\} \cup \{(i',j) \in Y \mid i < i'\} \), and call it the hook at \((i,j)\) in \( Y \). Also, for \((i, j) \in \mathcal{F}(\mathcal{Y}_{m,n}) \), we set

\[ Y(i,j) := \{(i',j') \mid (i',j') \in Y, \text{ and } i' < i \text{ or } j' < j \} \]
\[ \cup \{(i'-1,j'-1) \mid (i',j') \in Y, \text{ and } i' > i \text{ and } j' > j \} . \]

The procedure which obtains \( Y(i,j) \) from \( Y \) is called removing the hook at \((i,j)\) from \( Y \) (see Figure[1] below).

### 2.2 Combinatorial game theory.

For the general theory of combinatorial games, we refer the reader to [6, Chapters 1 and 2]. In this subsection, we fix an impartial game in normal play whose game positions are all short (in the sense of [6, pages 4 and 9]).
Definition 2.1. A game position of an impartial game is called an \(N\)-position (resp., a \(P\)-position) if the next player (resp., the previous player) has a winning strategy.

Definition 2.2. For a (proper) subset \(X\) of \(\mathbb{Z}_{\geq 0}\), we set \(\text{mex}\ X := \min (\mathbb{Z}_{\geq 0} \setminus X)\).

For a game position \(G\) of an impartial game, we denote by \(\mathcal{O}(G)\) the set of all options of \(G\).

Definition 2.3. Let \(G\) be a game position. The Grundy value \(G(G)\) of \(G\) is defined by

\[
G(G) := \begin{cases} 
0 & \text{if } G \text{ is an ending position}, \\
\text{mex} \{G(P) \mid P \in \mathcal{O}(G)\} & \text{if } G \text{ is not an ending position}.
\end{cases}
\]

Recall from [6, page 6] that each game position of an impartial game is either an \(N\)-position or a \(P\)-position. The following result is well-known in the combinatorial game theory.

Theorem 2.4 ([6, Theorem 2.1]). A game position \(G\) is a \(P\)-position if and only if \(G(G) = 0\).
3 Unimodal numbering on Young diagrams.

Let $Y \in \mathcal{F}(Y_{m,n})$. For each box $(i, j) \in Y$, we write $c(i, j) := \min (j - i + m, i - j + n)$ on it; we call this numbering on $Y$ the unimodal numbering on $Y$.

Example 3.1. Assume that $m = 3$ and $n = 5$. The Young diagram $Y = Y_{(4,4,2)} \in \mathcal{F}(Y_{3,5})$ with the unimodal numbering is as follows:

![Young diagram](image)

It can be easily checked that $c := (m + n - 1 + \chi) / 2$ is the maximum number appearing in the unimodal numbering, where

$$\chi := \begin{cases} 
1 & \text{if } m + n \in 2\mathbb{N}, \\
0 & \text{if } m + n \in 2\mathbb{N} + 1.
\end{cases}$$

We define $\mathbb{D}_{m,n} \subset \mathbb{Z}_{\geq 0}^{m+n+1}$ by

$$\mathbb{D}_{m,n} := \{(a_1, a_2, a_3, \ldots, a_{m+n-1}, a_m+n, a_{m+n+1}) \in \mathbb{Z}_{\geq 0}^{m+n+1} | \\
\quad a_1 = a_{m+n+1} = 0, 0 \leq a_k - a_{k-1} \leq 1 \text{ for } 2 \leq k \leq m + 1, \\
\quad 0 \leq a_k - a_{k+1} \leq 1 \text{ for } m + 1 \leq k \leq m + n\}.$$

For $Y \in \mathcal{F}(Y_{m,n})$, we set $d_k = d_k(Y) := \#\{(i, j) \in Y | j - i = -m - 1 + k\}$ for each $1 \leq k \leq m + n + 1$; note that $d_1 = d_{m+n+1} = 0$. We know from [1, Proposition 3.6] that

$$D_{m,n}(Y) := (d_1, d_2, d_3, \ldots, d_{m+n-1}, d_{m+n}, d_{m+n+1})$$

is an element of $\mathbb{D}_{m,n}$. Thus we obtain the map $D_{m,n} : \mathcal{F}(Y_{m,n}) \to \mathbb{D}_{m,n}$, $Y \mapsto D_{m,n}(Y)$. An element $D_{m,n}(Y) \in \mathbb{D}_{m,n}$ is called the diagonal expression of $Y$. For simplicity of notation, we denote $D_{m,n}$ by $D$.

Example 3.2. Assume that $m = 3$ and $n = 5$. Let $\lambda = (4, 3, 1) \in \mathcal{Y}_3(8)$. Then we have $D_{3,5}(Y_{\lambda}) = (0, 1, 1, 2, 2, 1, 1, 0, 0) \in \mathbb{D}_{3,5}$.

Proposition 3.3 ([1 Proposition 3.6]). The map $D_{m,n} : \mathcal{F}(Y_{m,n}) \to \mathbb{D}_{m,n}$ is bijective.

Here we recall from [1 Subsection 3.3] the relation between “removing a hook” (see Figure [1] and the diagonal expression (see Example 3.5 below). For a subset $S$ of $Y \in \mathcal{F}(Y_{m,n})$, we define $\mathcal{H}_Y(S)$ to be the multiset consisting of $c(i, j)$ for $(i, j) \in S$. The multiset $\mathcal{H}_Y(S)$ is called the numbering multiset for
Definition 3.4. Let $a \in Y$ and $d \in H$ be such that $(i, j) \in S$. In particular, if $S = H_Y(i, j)$ for some $(i, j) \in Y$, then we denote $H_Y(S)$ by $H_Y(i, j)$. We deduce that $H_Y(Y) = H_Y(Y(i, j)) \cup H_Y(i, j)$ (the union of multisets). Now, let $Y \in \mathcal{F}(Y_{m,n})$, and fix $(i, j) \in Y$. Let $i'$ (resp., $j'$) be such that $(i', j) \in Y$ and $(i' + 1, j) \notin Y$ (resp., $(i, j') \in Y$ and $(i, j' + 1) \notin Y$).

Then we see that

$$
\#\{(x, y) \in Y \mid y - x = -m + k\} - \#\{(x, y) \in Y(i, j) \mid y - x = -m + k\} = \begin{cases} 1 & \text{if } m + j - i' \leq k \leq m + j' - i, \\ 0 & \text{otherwise}. \end{cases}
$$

Therefore, if

$$D(Y) = (d_1, \ldots, d_{m+j-i'}, d_{m+j-i'+1}, d_{m+j-i'+2}, \ldots, d_{m+j-i'+i}, d_{m+j-i'+i+1}, d_{m+j-i'+2}, \ldots, d_{m+n+1}),$$

then

$$D(Y(i, j)) = (d_1, \ldots, d_{m+j-i'}, d_{m+j-i'+1} - 1, d_{m+j-i'+2} - 1, \ldots, d_{m+j-i'-i} - 1, d_{m+j-i'+1} - 1, d_{m+j-i'+2} - 1, \ldots, d_{m+n+1}).$$

Thus, if we remove a hook from $Y \in \mathcal{F}(Y_{m,n})$, then 1 is subtracted from some consecutive entries in $D(Y)$; in the case above, the consecutive entries are $d_l, d_{l+1}, \ldots, d_r$, with $l = m + j - i' + 1$ and $r = m + j' - i + 1$.

**Definition 3.4.** Let $\mathbf{a} = (a_1, a_2, \ldots, a_{m+n}, a_{m+n+1}) \in \mathbb{D}_{m,n}$; recall that $a_1 = a_{m+n+1} = 0$. For $2 \leq l \leq r \leq m + n$, we write $\mathbf{a} \xrightarrow{l,r} \mathbf{a}'$ if $a_k \geq 1$ for all $l \leq k \leq r$, and $\mathbf{a}' = (a_1, a_2, \ldots, a_{l-1}, a_l - 1, a_{l+1}, a_{l+2}, \ldots, a_{r-1}, a_r - 1, a_{r+1}, \ldots, a_{m+n}, a_{m+n+1}) \in \mathbb{Z}_{\geq 0}^{m+n+1}$.

Recall that the map $D = D_{m,n} : \mathcal{F}(Y_{m,n}) \to \mathbb{D}_{m,n}$ is bijective. Let $\mathbf{a}, \mathbf{a}' \in \mathbb{D}_{m,n}$, and set $Y := D^{-1}(\mathbf{a})$, $Y' := D^{-1}(\mathbf{a}')$. If $\mathbf{a} \xrightarrow{l,r} \mathbf{a}'$ for some $2 \leq l \leq r \leq m + n$, then we write $Y \xrightarrow{l,r} Y'$.

**Example 3.5.** Keep the notation and setting in Example 3.2. It follows that

$$Y_\lambda(2, 1) = \{(1, 1), (1, 2), (1, 3), (1, 4)\},$$

and hence $D(Y_\lambda(2, 1)) = (0, 0, 0, 1, 1, 1, 1, 0, 0)$. Thus we have $D(Y_\lambda) \xrightarrow{2,5} D(Y_\lambda(2, 1))$ (and hence $Y_\lambda \xrightarrow{2,5} Y_\lambda(2, 1)$).
4 Multiple Hook Removing Game.

Abuku and Tada [1] introduced an impartial game, named Multiple Hook Removing Game (MHRG for short), whose rule is given as follows; recall that $m$ and $n$ are fixed positive integers such that $m \leq n$:

(1) All game positions are some Young diagrams contained in $\mathcal{F}(Y_{m,n})$ with the unimodal numbering. The starting position is the rectangular Young diagram $Y_{m,n}$.

(2) Assume that $Y \in \mathcal{F}(Y_{m,n})$ appears as a game position. If $Y \neq \emptyset$ (the empty Young diagram), then a player chooses a box $(i, j) \in Y$, and remove the hook at $(i, j)$ in $Y$; recall from Subsection 2.1 that the resulting Young diagram is $Y\langle i, j \rangle$. Then we know from [1, Lemma 3.15] (see also Lemma 4.4 below) that $f := \#\{(i', j') \in Y\langle i, j \rangle \mid H_{Y\langle i, j \rangle}(i', j') = H_Y(i, j) \text{ (as multisets)}\} \leq 1$. If $f = 0$, then a player moves $Y$ to $Y\langle i, j \rangle \in \mathcal{O}(Y)$; we call this case and this operation (MHR 1). If $f = 1$, then a player moves $Y$ to $(Y\langle i, j \rangle\langle i', j' \rangle \in \mathcal{O}(Y))$, where $(i', j') \in Y\langle i, j \rangle$ is the unique element such that $H_{Y\langle i, j \rangle}(i', j') = H_Y(i, j)$; we call this case and this operation (MHR 2).

(3) The (unique) ending position is the empty Young diagram $\emptyset$. The winner is the player who makes $\emptyset$ after his/her operation (2).

**Definition 4.1.** We denote by $S(Y_{m,n})$ the set of all those Young diagrams in $\mathcal{F}(Y_{m,n})$ which appear as game positions of MHRG (with $Y_{m,n}$ the starting position); in general, $S(Y_{m,n}) \subset \mathcal{F}(Y_{m,n})$ as Example 4.3 below shows.

**Definition 4.2.** Let $Y \in S(Y_{m,n})$, and $Y' \in \mathcal{O}(Y)$. If a player moves $Y$ to $Y'$ by operation (MHR 1) (resp., (MHR 2)), then we write $Y \xrightarrow{\text{MHR 1}} Y'$ (resp., $Y \xrightarrow{\text{MHR 2}} Y'$).

**Example 4.3.** Assume that $m = 2$ and $n = 3$. The elements of $S(Y_{2,3})$ are
The following elements of \( \mathcal{F}(Y_{2,3}) \) are not contained in \( \mathcal{S}(Y_{2,3}) \):

\[
\begin{array}{cccccc}
2 & 2 & 1 & , & 2 & 2 \\
1 & 2 & , & 1 & 2 & ,
\end{array}
\begin{array}{cccc}
2 & 2 & 1 & ,
1 & 2 & , & 2
\end{array}
\]

**Lemma 4.4** ([1, Lemma 3.15]). Let \( Y \in \mathcal{F}(Y_{m,n}) \), and \( (i,j) \in Y \). Assume that there exists a box \((i',j') \in Y(i,j)\) such that \( \mathcal{H}_{Y(i,j)}(i',j') = \mathcal{H}_{Y}(i,j) \) (as multisets). If \( Y \xrightarrow{lr} Y(i,j) \), then \( Y(i,j) \xrightarrow{\text{lr}} (Y(i,j))(i',j') \). In particular, \( \#\{(i',j') \in Y(i,j) \mid \mathcal{H}_{Y(i,j)}(i',j') = \mathcal{H}_{Y}(i,j) \text{ (as multisets)}\} \leq 1 \).

**Remark 4.5.** In fact, the following holds (see [1, Lemma 3.15]), although we do not use these facts in this paper.

1. Keep the notation and setting in Lemma 4.4. There does not exist \((i'',j'') \in (Y(i,j))(i',j')\) such that \( \mathcal{H}_{Y(i,j)}(i'',j'') = \mathcal{H}_{Y}(i,j) \).

2. Let \((i,j),(k,l) \in Y\). Assume that \( \mathcal{H}_{Y}(i,j) = \mathcal{H}_{Y}(k,l) \). If there exists a box \((i',j') \in Y(i,j)\) such that \( \mathcal{H}_{Y(i,j)}(i',j') = \mathcal{H}_{Y}(i,j) \), then there exists a (unique) box \((k',l') \in Y(k,l)\) such that \( \mathcal{H}_{Y(k,l)}(k',l') = \mathcal{H}_{Y}(i,j) \). Moreover, in this case, we have \((Y(i,j))(i',j') = (Y(k,l))(k',l')\).

### 5 Description of \( \mathcal{S}(Y_{m,n}) \).

Recall that \( m, n \in \mathbb{N} \) are such that \( m \leq n \), and that \( c = \max \{c(i,j) \mid (i,j) \in Y_{m,n}\} \) is equal to \( (m+n-1+\chi)/2 \), where \( \chi = 0 \) (resp., \( \chi = 1 \)) if \( m+n \) is odd (resp., even). Also, we have a canonical bijection \( I : \mathcal{F}(Y_{m,n}) \to \binom{[1,m+n]}{m} \) (see Subsection 2.1).

Let \( Y \in \mathcal{F}(Y_{m,n}) \). We set \( I_{R}(Y) := I(Y) \cap [c+1-\chi,m+n] \); note that \( c+1-\chi = m+n+1-(c+1-\chi) = c+1 \geq c+1-\chi \).

**Theorem 5.1.** Let \( Y \in \mathcal{F}(Y_{m,n}) \), and \( \lambda = (\lambda_{1}, \ldots, \lambda_{m}) \) the partition corresponding to \( Y \), that is, \( Y = Y_{\lambda} \). The following (I), (II), (III), and (IV) are equivalent.

(I) \( Y \in \mathcal{S}(Y_{m,n}) \). \hspace{1cm} (II) \( Y^{D} \in \mathcal{S}(Y_{m,n}) \). \hspace{1cm} (III) \( I_{R}(Y) \cap I_{R}(Y^{D}) = \emptyset \). \hspace{1cm} (IV) \( \lambda_{i} + \lambda_{j} \neq n-m+i+j-1 \) for all \( 1 \leq i, j \leq m \).

The rest of this section is devoted to a proof of Theorem 5.1. We can easily show the following lemma.

**Lemma 5.2.** (A) It holds that \( I(Y^{D}) = \{i = m+n+1-i \mid i \in I(Y)\} = \overline{I(Y)} \) for \( Y \in \mathcal{F}(Y_{m,n}) \).

(B) Let \( Y \in \mathcal{F}(Y_{m,n}) \), and let \( l, r \in [2, m+n] \) such that \( l \leq r \). Then, \( l-1 \notin I(Y) \) and \( r \in I(Y) \) if and only if there exists a (unique) box \((i,j) \in Y \) such that \( Y \xrightarrow{lr} Y(i,j) \); in this case, \( I(Y(i,j)) = (I(Y) \setminus \{r\}) \cup \{l-1\} \) and \( I(Y(i,j)^{D}) = (I(Y^{D}) \setminus \{r\}) \cup \{l-1\} \).
Remark 5.3. Let \( Y \in \mathcal{F}(Y_{m,n}) \), and \((i, j) \in Y\). Let \( 2 \leq l \leq r \leq m + n \) be such that \( Y \xrightarrow{\ell} Y_{i,j} \). By Lemmas 5.2 and 5.3, it follows that \( \mathcal{R} \notin \mathcal{I}(Y_{i,j}) \) and \( l - 1 \in \mathcal{I}(Y_{i,j}) \) if and only if there exists a (unique) box \((i', j') \in Y_{i,j}\) such that \( Y_{i,j} \xrightarrow{l-1} (Y_{i,j})(i', j') \); in particular, in this case, it holds that \( H_{Y_{i,j}}(i', j') = H_{Y_{i,j}}(i, j) \) (as multisets).

We first show (I) \( \Rightarrow \) (III). Since \( Y \in \mathcal{S}(Y_{m,n}) \) by (I), there exists a sequence of game positions of the form

\[
Y_{m,n} = Y_0 \xrightarrow{t_1} Y_1 \xrightarrow{t_2} Y_2 \xrightarrow{t_3} \cdots \xrightarrow{t_p} Y_p = Y,
\]

where \( t_i \) is either (MHR 1) or (MHR 2) for each \( 1 \leq i \leq p \). For \( 1 \leq i \leq p \) such that \( t_i \) is (MHR 2), we see from Lemmas 4.4 and 5.2 that \( Y_{i-1} \xrightarrow{t_i,r_i} Y_i \xrightarrow{r_i-1,l_i-1} Y_{i-1} \) for some \( 2 \leq l_i \leq r_i \leq m + n \) with \( l_i - 1 \notin \mathcal{I}(Y_{i-1}) \), \( r_i \in \mathcal{I}(Y_{i-1}) \), and \( Y'_i \in \mathcal{F}(Y_{m,n}) \). Similarly, for \( 1 \leq i \leq p \) such that \( t_i \) is (MHR 1), there exists \( 2 \leq l_i \leq r_i \leq m + n \) with \( l_i - 1 \notin \mathcal{I}(Y_{i-1}) \) and \( r_i \in \mathcal{I}(Y_{i-1}) \) such that \( Y_{i-1} \xrightarrow{t_i,r_i} Y_i \); we set \( Y'_i := Y_i \) by convention. We show by induction on \( p \) that \( I_{R}(Y_{p}) \cap I_{R}(Y_{p}) = \emptyset \). If \( p = 0 \), then it is obvious that \( I_{R}(Y_{m,n}) \cap I_{R}(Y_{m,n}) = \emptyset \), since \( I_{R}(Y_{m,n}) = \{ n + 1, n + 2, \ldots, m + n \} \).

Assume that \( p > 0 \); by the induction hypothesis,

\[
I_{R}(Y_{p-1}) \cap I_{R}(Y_{p-1}) = \emptyset. \tag{5.1}
\]

By Lemma 5.2, we have

\[
I_{R}(Y_{p}) \setminus \{l_p - 1\} = I_{R}(Y_{p-1}) \setminus \{r_p\}, \tag{5.2}
\]

\[
I_{R}(Y_{p}) \setminus \{l_p - 1\} = I_{R}(Y_{p-1}) \setminus \{r_p\}. \tag{5.3}
\]

Lemma 5.4. It holds that \( I_{R}(Y_{p}) \cap I_{R}(Y_{p}) \neq \emptyset \) if and only if \( l_p - 1 \notin \mathcal{I}(Y_{p-1}) \) \( \{r_p\} \text{ or } l_p - 1 = l_p - 1 \); notice that \( l_p - 1 = l_p - 1 \) if and only if \( \chi = 0 \) and \( l_p - 1 = c + 1 \).

Proof. Assume first that \( l_p - 1 < c + 1 - \chi \); recall that \( l_p - 1 > c + 1 - \chi = c + 1 \geq c + 1 - \chi \). It follows from 5.2 and 5.3 that

\[
I_{R}(Y_{p}) = I_{R}(Y_{p-1}) \setminus \{r_p\}, \quad I_{R}(Y_{p}) = (I_{R}(Y_{p-1}) \setminus \{r_p\}) \cup \{l_p - 1\}.
\]

Because \( I_{R}(Y_{p-1}) \cap I_{R}(Y_{p-1}) = \emptyset \) by the induction hypothesis, we see that \( I_{R}(Y_{p}) \cap I_{R}(Y_{p}) \neq \emptyset \) if and only if \( l_p - 1 \notin \mathcal{I}(Y_{p-1}) \) \( \{r_p\} \). Assume next that \( l_p - 1 \geq c + 1 - \chi \). It follows from 5.2 and 5.3 that

\[
I_{R}(Y_{p}) = (I_{R}(Y_{p-1}) \setminus \{r_p\}) \cup \{l_p - 1\},
\]
\[ I_R(Y_p^{D'}) = \begin{cases} I_R(Y_{p-1}^{D'}) \setminus \{r_p\} & \text{if } \overline{t_p - 1} < c + 1 - \chi, \\ (I_R(Y_{p-1}^{D'}) \setminus \{\overline{t_p - 1}\}) \cup \{t_p - 1\} & \text{if } \overline{t_p - 1} \geq c + 1 - \chi. \end{cases} \]

Here we note that \( \overline{t_p - 1} \in I(Y_{p-1}) \setminus \{r_p\} \) if and only if \( t_p - 1 \in I(Y_{p-1}^{D'}) \setminus \{r_p\} \) by Lemma 5.2 (A). If \( \overline{t_p - 1} < c + 1 - \chi \) (resp., \( \overline{t_p - 1} \geq c + 1 - \chi \)), then it holds that \( I_R(Y_p^{D'}) \cap I_R(Y_{p-1}^{D'}) \neq \emptyset \) if and only if \( \overline{t_p - 1} \in I(Y_{p-1}) \setminus \{r_p\} \) (resp., \( t_p - 1 \in I(R(Y_{p-1}) \setminus \{r_p\}) \) or \( t_p - 1 = \overline{t_p - 1} \)). Thus we have proved the lemma. \( \square \)

**Proposition 5.5.** (1) The operation \( t_p \) is (MHR 1) if and only if either of the following (a) or (b) holds.

(a) \( \overline{t_p - 1} \notin I(Y_{p-1}) \) and \( t_p - 1 \neq \overline{t_p - 1} \).

(b) \( t_p - 1 = \overline{t_p - 1} \) (notice that \( t_p - 1 \neq \overline{t_p - 1} \) also in this case since \( t_p - 1 \neq r_p = \overline{t_p - 1} \)).

(2) The operation \( t_p \) is (MHR 2) if and only if \( \overline{t_p - 1} \in I(Y_{p-1}) \setminus \{r_p\} \) or \( t_p - 1 = \overline{t_p - 1} \).

**Proof.** It suffices to show only part (2). We first show the “only if” part of (2). Assume that \( t_p \) is (MHR 2); recall that \( Y_{p-1}^{Y_p', Y_{p-1}^Y} \xrightarrow{t_p - r_p} Y_p^{r_p} \xrightarrow{t_p - \overline{t_p - 1}} Y_p \). It follows from Lemma 5.2 (B) (applied to \( Y = Y_p' \) and \( i, j = Y_p \) that \( \overline{t_p - 1} \in I(Y_p') = (I(Y_{p-1}) \setminus \{r_p\}) \cup \{t_p - 1\} \). Thus we have \( \overline{t_p - 1} \in I(Y_{p-1}) \setminus \{r_p\} \) or \( t_p - 1 = \overline{t_p - 1} \). We next show the “if” part of (2); by Remark 5.3 and Lemmas 4.4 and 5.2 (B), it suffices to show that \( t_p \notin I(Y_p') \) and \( t_p - 1 \in I(Y_{p-1}) \). Because \( I(Y_p') = (I(Y_{p-1}) \setminus \{r_p\}) \cup \{t_p - 1\} \), it is obvious from the assumption that \( t_p - 1 \notin I(Y_p') \). Let us show that \( t_p \notin I(Y_p') \). Suppose, for a contradiction, that \( t_p \in I(Y_p') \). Since \( I(Y_p') = (I(Y_{p-1}) \setminus \{r_p\}) \cup \{t_p - 1\} \), and since \( t_p \neq \overline{t_p - 1} \), we have \( t_p \in I(Y_{p-1}) \setminus \{r_p\} \subset I(Y_{p-1}) \), and hence \( r_p \in I(Y_{p-1}^{D'}) \) by Lemma 5.2 (A). If \( c + 1 - \chi \leq r_p \), then \( r_p \in I(Y_{p-1}^{D'}) \). Since \( r_p \in I(Y_{p-1}^{D'}) \) by Lemma 5.2 (B), we get \( r_p \in I(Y_{p-1}^{D'}) \cap I(R(Y_{p-1}^{D'})) \), which contradicts the induction hypothesis (5.1). If \( c + 1 - \chi > r_p \), then \( c + 1 - \chi = c + 1 - \chi < t_p \), which implies that \( t_p \notin I(Y_{p-1}) \). Since \( r_p \in I(Y_{p-1}) \), we have \( t_p \in I(Y_{p-1}^{D'}) \) by Lemma 5.2 (A). Hence we get \( t_p \in I(Y_{p-1}^{D'}) \cap I(R(Y_{p-1}^{D'})) \), which contradicts the induction hypothesis (5.1). Therefore we obtain \( t_p \notin I(Y_p') \), as desired. Thus we have proved the proposition. \( \square \)

If \( t_p \) is (MHR 1) (recall that \( Y_p' = Y_p \) and \( Y_{p-1}^{D'} = Y_{p-1}^{D} \) in this case), then we see by Lemma 5.4 and Proposition 5.5 (1) that \( I_R(Y_p^{D'}) \cap I_R(Y_{p-1}^{D'}) = \emptyset \). Assume that \( t_p \) is (MHR 2), or equivalently, \( t_p - 1 \in I(Y_{p-1}^{D'}) \setminus \{r_p\} \) or \( t_p - 1 = \overline{t_p - 1} \) by Proposition 5.5 (2). Because \( Y_{p-1}^{Y_p', Y_{p-1}^Y} \xrightarrow{t_p - r_p} Y_p^{r_p} \xrightarrow{t_p - \overline{t_p - 1}} Y_p \) in this case, it follows from Lemma 5.2 (B) that

\[ I_R(Y_p) \setminus \{t_p, r_p - 1\} = I_R(Y_{p-1}) \setminus \{r_p, l_p - 1\}, \tag{5.4} \]

\[ I_R(Y_p^{D'}) \setminus \{r_p, t_p - 1\} = I_R(Y_{p-1}^{D'}) \setminus \{\overline{t_p - 1}, l_p - 1\}. \tag{5.5} \]
Hence, by (5.4) and (5.5), together with the induction hypothesis (5.1), we obtain \( I_R(Y_p) \cap I_R(Y'_p) = \emptyset \). Thus we have proved (I) \( \Rightarrow \) (III) in Theorem 5.4.

Conversely, we prove (III) \( \Rightarrow \) (I), that is, \( Y \in S(Y_{m,n}) \) if \( I_R(Y) \cap I_R(Y^D) = \emptyset \). We show by (descending) induction on \( |I(Y)| = \sum_{i \in I(Y)} i \). It is obvious that \( Y_{m,n} \in S(Y_{m,n}) \). Assume that \( \langle I(Y) \rangle < \langle I(Y_{m,n}) \rangle \). Since \( I(Y_{m,n}) = [n+1, m+n] \), and \( I(Y) \neq I(Y_{m,n}) \) with \( \#I(Y) = m \), there exists \( r \notin I(Y) \) such that \( n+1 \leq r \). Also, there exists \( l \leq r \) such that \( l-1 \in I(Y) \); note that \( l-1 < r \). Here we show that \( \overline{l-1} \notin I(Y) \). Suppose, for a contradiction, that \( \overline{l-1} \in I(Y) \). If \( c+1 - \chi \geq l-1 \), then \( c+1 - \chi \leq c+1 - \chi \leq \overline{l-1} \), and hence \( \overline{l-1} \in I_R(Y) \). By Lemma 5.2 (A) applied to \( l-1 \in I(Y) \), it follows that \( \overline{l-1} \in I_R(Y^D) \). Thus we obtain \( l-1 \in I_R(Y) \cap I_R(Y^D) = \emptyset \). If \( c+1 - \chi < l-1 \), then \( l-1 \in I_R(Y^D) \) because \( \overline{l-1} \notin I(Y) \). Since \( l-1 \in I_R(Y) \), we get \( l-1 \in I_R(Y) \cap I_R(Y^D) \), which contradicts the assumption that \( I_R(Y) \cap I_R(Y^D) = \emptyset \). Therefore we obtain \( \overline{l-1} \notin I(Y) \).

**Proposition 5.6.** Keep the setting above.

1. If \( \overline{r} \notin I(Y) \) or \( \overline{r} = l-1 \), then there exists a (unique) Young diagram \( Y' \) such that \( I(Y') = \langle I(Y) \rangle \setminus \{l-1\} \cup \{r\} \) and \( I(Y'^D) = \langle I(Y^D) \rangle \setminus \{l-1\} \cup \{\overline{r}\} \).

   Furthermore, \( Y' \in S(Y_{m,n}) \), and \( Y' \overset{\text{(MHR 1)}}{\rightarrow} Y \).

2. If \( \overline{r} \in I(Y) \) and \( \overline{r} \neq l-1 \), then there exists a (unique) Young diagram \( Y'' \) such that \( I(Y'') = \langle I(Y) \rangle \setminus \{\overline{r}, l-1\} \cup \{r, \overline{l-1}\} \) and \( I(Y''^D) = \langle I(Y^D) \rangle \setminus \{\overline{r}, l-1\} \cup \{r, \overline{l-1}\} \).

   Furthermore, \( Y'' \in S(Y_{m,n}) \), and \( Y'' \overset{\text{(MHR 2)}}{\rightarrow} Y \).

**Proof.** (1) Recall that \( l-1 \in I(Y) \) and \( r \notin I(Y) \), which implies that \( \langle I(Y) \rangle \setminus \{l-1\} \cup \{r\} \in \{1, m+n\} \). Since \( I \colon F(Y_{m,n}) \to \{1, m+n\} \) is a bijection, there exists unique \( Y' \in F(Y_{m,n}) \) such that \( I(Y') = \langle I(Y) \rangle \setminus \{l-1\} \cup \{r\} \); note that \( I(Y'^D) = \langle I(Y^D) \rangle \setminus \{l-1\} \cup \{\overline{r}\} \) by Lemma 5.2 (A). Then it follows from Lemma 5.2 (B) that \( Y' \overset{\text{t}}{\rightarrow} Y \). Because \( \overline{r} \notin I(Y) \) or \( \overline{r} = l-1 \) by the assumption of (1), and \( I_R(Y) \cap I_R(Y^D) = \emptyset \) by assumption, it can be easily verified that \( I_R(Y') \cap I_R(Y'^D) = \emptyset \). Since \( l-1 < r \), we have \( \langle I(Y') \rangle \geq \langle I(Y) \rangle \), and hence \( Y' \in S(Y_{m,n}) \) by the induction hypothesis. Because \( \overline{l-1} \notin I(Y) \), we see from Remark 5.3 that there does not exist a box \((i, j) \in Y\) such that \( Y' \overset{\text{t}}{\rightarrow} Y(i, j) \). Thus we obtain \( Y' \overset{\text{(MHR 1)}}{\rightarrow} Y \), as desired.

(2) Let \( Y' \) be as in the proof of part (1). Since \( \overline{r} \in I(Y) \) and \( \overline{r} \neq l-1 \) by the assumption of (2), and \( \overline{l-1} \notin I(Y) \) as seen above,

\[
\langle I(Y') \rangle \setminus \{\overline{r}\} \cup \{l-1\} = \langle I(Y) \rangle \setminus \{\overline{r}, l-1\} \cup \{r, \overline{l-1}\} \in \{1, m+n\}.
\]

Thus there exists \( Y'' \in F(Y_{m,n}) \) such that \( I(Y'') = \langle I(Y) \rangle \setminus \{\overline{r}, l-1\} \cup \{r, \overline{l-1}\} \); note that \( I(Y''^D) = \langle I(Y^D) \rangle \setminus \{r, \overline{l-1}\} \cup \{\overline{r}, l-1\} \) by Lemma 5.2 (A). It follows from Lemma 5.2 (B) that \( Y'' \overset{\text{t}}{\rightarrow} Y' \). Because \( \overline{r} \in I(Y) \) and \( \overline{r} \neq l-1 \)
by the assumption of (2), and $I_R(Y) ∩ I_R(Y^D) = ∅$ by assumption, it can be easily verified that $I_R(Y'') ∩ I_R(Y''D) = ∅$. Since $l - 1 < r$ and $l - 1 > r$, we have $⟨I(Y'')⟩ > ⟨I(Y)⟩$, and hence $Y'' ∈ S(Y_{m,n})$ by the induction hypothesis.

We see from Lemma 4.4 that $Y'' \xrightarrow{\text{MR.2}} Y$, as desired. 

By Proposition 5.4, we obtain $Y ∈ S(Y_{m,n})$. This completes the proof of (III) ⇒ (I), and hence (I) ⇔ (III). The equivalence (II) ⇔ (III) follows from the equivalence (I) ⇔ (III) since $I_R(Y^P) ∩ I_R(Y^D) = I_R(Y) ∩ I_R(Y^D)$.

Finally, let us show the equivalence (III) ⇔ (IV). Let $Y ∈ F(Y_{m,n})$, and $λ = (λ_1, ..., λ_m) ∈ Y_m(m + n)$ be such that $Y = Y_λ$. We first show (IV) ⇒ (III). Obviously, if $I_R(Y) ∩ I_R(Y^D) ≠ ∅$, then $I(Y) ∩ I(Y^D) ≠ ∅$. It follows from Subsection 2.1 that

$I(Y) = \{λ_p + m - p + 1 | 1 ≤ p ≤ m\}$,
$I(Y^D) = \{n - λ_q + q | 1 ≤ q ≤ m\}$.

Hence, $I(Y) ∩ I(Y^D) ≠ ∅$ if and only if $λ_i + m - i + 1 = n - λ_j + j$ (or equivalently, $λ_i + λ_j = n - m + i + j - 1$) for some $1 ≤ i, j ≤ m$. Thus we have shown (IV) ⇒ (III).

We next show (III) ⇒ (IV). Assume that $λ_i + λ_j = n - m + i + j - 1$ for some $1 ≤ i, j ≤ m$; we may assume that $i ≤ j$. As seen above, we have $λ_i + m - i + 1 ∈ I(Y) ∩ I(Y^D)$. Hence it suffices to show that if $λ_i + m - i + 1 ∈ [c + 1 - χ, m + n]$. Indeed, suppose, for a contradiction, that $λ_i + m - i + 1 \notin [c + 1 - χ, m + n]$. Then, $λ_i + m - i + 1 < c + 1 - χ$ or $m + n < λ_i + m - i + 1$. Because $λ_i + m - i + 1 ≤ n - m + i + j - 1 ≤ n + m$, we get $λ_i + m - i + 1 < c + 1 - χ$. Since $i ≤ j$ (and hence $λ_i ≥ λ_j$) and $λ_i < c - m - χ + i$, we have $λ_i + λ_j ≤ 2λ_i < (m + n + 1 - χ) - 2m - 2χ + 2i = n - m - χ - 2i - 1 ≤ n - m + i + j - 1 = 2λ_i + λ_j$, which is a contradiction. Therefore, we conclude that $λ_i + m - i + 1 ∈ [c + 1 - χ, m + n]$. Thus we have shown (III) ⇒ (IV), thereby completing the proof of (III) ⇔ (IV).

6 Application.

Let $t ∈ Z_{≥ 0}$ and $m, n ∈ N$ such that $t ≤ m ≤ n$. For $(λ_1, ..., λ_t) ∈ Y_t(t + n)$, we set

$[λ_1, ..., λ_t] : = (λ_1, ..., λ_t, λ_{t+1}, ..., λ_m) ∈ Y_m(m + n)$,

with $λ_k := 0$ for $t + 1 ≤ k ≤ m$.

Theorem 6.1. Under the notation and setting above, $Y_{[λ_1, ..., λ_t]} ∈ S(Y_{m,n})$ if and only if $Y_{(λ_1, ..., λ_t)} ∈ S(Y_{t,n-m+t})$. Moreover, the Grundy value of $Y_{[λ_1, ..., λ_t]} ∈ S(Y_{m,n})$ is equal to the Grundy value of $Y_{(λ_1, ..., λ_t)} ∈ S(Y_{t,n-m+t})$.

Proof. Since $λ_k = 0$ for $t + 1 ≤ k ≤ m$, it follows from Theorem 5.1 that $Y_{[λ_1, ..., λ_t]} ∈ S(Y_{m,n})$ if and only if $λ_i + λ_j ≠ n - m + i + j - 1$ for all $1 ≤ i ≤ j ≤ t$ and

$λ_s ≠ n - m + s - k - 1$ for all $1 ≤ s ≤ t$ and $t + 1 ≤ k ≤ m$; (6.1)
Note that \( 0 \neq n - m + k + l - 1 \) for all \( t + 1 \leq k, l \leq m \) since \( m \leq n \). Also, notice that (6.1) is equivalent to \( \lambda_1 \leq n - m + t \). Therefore, we deduce that \( Y_{[\lambda_1, \ldots, \lambda_t]} \in S(Y_{m,n}) \) if and only if \( Y_{(\lambda_1, \ldots, \lambda_t)} \in S(Y_{t,n-m+t}) \).

Next, we show the assertion on the Grundy values. Assume that \( Y_{(\lambda_1, \ldots, \lambda_t)} \in S(Y_{t,n-m+t}) \), or equivalently, \( Y_{[\lambda_1, \ldots, \lambda_t]} \in S(Y_{m,n}) \). If \( t = 0 \) or \( \lambda_1 = 0 \), then \( Y_{[\lambda_1, \ldots, \lambda_t]} = Y_{(\lambda_1, \ldots, \lambda_t)} = \emptyset \) (the empty Young diagram). Thus, both the Grundy value of \( Y_{[\lambda_1, \ldots, \lambda_t]} = \emptyset \) in \( S(Y_{m,n}) \) and the Grundy value of \( Y_{(\lambda_1, \ldots, \lambda_t)} = \emptyset \) in \( S(Y_{t,n-m+t}) \) are equal to 0. Assume that \( 1 \leq t \) and \( 1 \leq \lambda_1 \). Since \( m \leq n \) and \( 1 \leq t \), we get \( m - t + 1 \leq n + t - 1 \). Hence, we have \( c(t, 1) = \min (1 - t + m, t - 1 + n) = m - t + 1 \). Since \( m - t + 1 \leq m + \lambda_1 - 1 \), and since \( \lambda_1 \leq n - m + t \) as seen above, we have \( c(1, \lambda_1) = \min (\lambda_1 - 1 + m, 1 - \lambda_1 + n) \geq m - t + 1 \). Thus, we obtain \( \min \{c(p, q) \mid (p, q) \in Y_{[\lambda_1, \ldots, \lambda_t]}\} \geq m - t + 1 \):

\[
\begin{array}{c}
| m < \ldots < t > \ldots > c(1, \lambda_t) | \\
| \vdots | \\
| \vdots | \\
| m-t+1 < \ldots |
\end{array}
\]

\[
\begin{array}{c}
| m < \ldots < c(1, h) | \\
| \vdots | \\
| \vdots | \\
| m-t+1 < \ldots |
\end{array}
\]

Figure 2. Numbering of \( Y_{[\lambda_1, \ldots, \lambda_t]} \) in \( S(Y_{m,n}) \).

We notice that

(i) in \( Y_{[\lambda_1, \ldots, \lambda_t]} \in S(Y_{m,n}) \) with the unimodal numbering \( c(p, q) \) for \( (p, q) \in Y_{[\lambda_1, \ldots, \lambda_t]} \), if we replace \( c(p, q) \) by \( c(p, q) - m + t \), then we get \( Y_{(\lambda_1, \ldots, \lambda_t)} \in S(Y_{t,n-m+t}) \) with the unimodal numbering:

(ii) in \( Y_{(\lambda_1, \ldots, \lambda_t)} \in S(Y_{t,n-m+t}) \) with the unimodal numbering \( c'(p, q) \) for \( (p, q) \in Y_{(\lambda_1, \ldots, \lambda_t)} \), if we replace \( c'(p, q) \) by \( c'(p, q) + m - t \), then we get \( Y_{[\lambda_1, \ldots, \lambda_t]} \in S(Y_{m,n}) \) with the unimodal numbering.

Here we give an example. Let \( m = 3, n = 5 \), and \( t = 2 \). Let \( \lambda = (3, 2, 0) \in \mathcal{Y}_3(8) \). In \( Y_{[3,2]} \in S(Y_{3,5}) \) (resp., \( Y_{[3,2]} \in S(Y_{2,4}) \) with the unimodal numbering \( c(p, q) \) for \( (p, q) \in Y_{[3,2]} \) (resp., \( c'(p, q) \) for \( (p, q) \in Y_{[3,2]} \)), if we replace \( c(p, q) \) by \( c(p, q) - 1 \) (resp., \( c'(p, q) \) by \( c'(p, q) + 1 \)), then we get \( Y_{(3,2)} \in S(Y_{2,4}) \) (resp., \( Y_{[3,2]} \in S(Y_{3,5}) \)) with the unimodal numbering:
where

\[ \lambda \in \begin{cases} C \cup \{(c_1(g), c_0(g)), (c_2(g), c_1(g)) \mid 0 \leq q \leq (p - 1)/2 \} & \text{if } n - 2 = 4p, \\ C \cup \{(c_2(g), c_1(g)), (c_3(g), c_2(g)) \mid 0 \leq q \leq (p - 1)/2 \} & \text{if } n - 2 = 4p + 1, \\ C \cup \{(c_0(g), c_{-1}(g)), (c_1(g), c_0(g)) \mid 0 \leq q \leq p/2 \} & \text{if } n - 2 = 4p + 2, \\ C \cup \{(2p + 4, 2p + 2), (2p + 5, 2p + 4)\} \\ \cup \{(c_1(g), c_0(g)), (c_2(g), c_1(g)) \mid 1 \leq q \leq p/2 \} & \text{if } n - 2 = 4p + 3, \\ \end{cases} \]

(6.2)

where \( p \in \mathbb{Z}_{>0} \), and \( C = C(p) := \{(2q, 2q) \mid 0 \leq q \leq p\} \).

The following is an immediate consequence of Theorem 6.1 and (6.2).

**Corollary 6.2.** We set \( d_i(q) := c - m + 2 + i + 4q \) for \( i \in \mathbb{Z} \) and \( q \geq 0 \). A Young diagram \( Y_\lambda \in S(Y_{m,n}) \) having at most two rows is a \( \mathcal{P} \)-position if and only if

\[ \lambda \in \begin{cases} D \cup \{[d_1(q), d_0(q)], [d_2(q), d_1(q)] \mid 0 \leq q \leq (p - 1)/2 \} & \text{if } n - m = 4p, \\ D \cup \{[d_2(q), d_1(q)], [d_3(q), d_2(q)] \mid 0 \leq q \leq (p - 1)/2 \} & \text{if } n - m = 4p + 1, \\ D \cup \{[d_0(q), d_{-1}(q)], [d_1(q), d_0(q)] \mid 0 \leq q \leq p/2 \} & \text{if } n - m = 4p + 2, \\ D \cup \{[2p + 4, 2p + 2], [2p + 5, 2p + 4] \} \\ \cup \{[d_1(q), d_0(q)], [d_2(q), d_1(q)] \mid 1 \leq q \leq p/2 \} & \text{if } n - m = 4p + 3, \end{cases} \]

where \( p \in \mathbb{Z}_{>0} \), and \( D = D(p) := \{[2q, 2q] \mid 0 \leq q \leq p\} \).

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