Syntactic completeness of proper display calculi

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Abstract

A recent strand of research in structural proof theory aims at exploring the notion of analytic calculi (i.e. those calculi that support general and modular proof-strategies for cut elimination), and at identifying classes of logics that can be captured in terms of these calculi. In this context, Wansing introduced the notion of proper display calculi as one possible design framework for proof calculi in which the analyticity desiderata are realized in a particularly transparent way. Recently, the theory of properly displayable logics (i.e. those logics that can be equivalently presented with some proper display calculus) has been developed in connection with generalized Sahlqvist theory (aka unified correspondence). Specifically, properly displayable logics have been syntactically characterized as those axiomatized by analytic inductive axioms, which can be equivalently and algorithmically transformed into analytic structural rules so that the resulting proper display calculi enjoy a set of basic properties: soundness, completeness, conservativity, cut elimination and subformula property. In this context, the proof that the given calculus is complete w.r.t. the original logic is usually carried out syntactically, i.e. by showing that a (cut free) derivation exists of each given axiom of the logic in the basic system to which the analytic structural rules algorithmically generated from the given axiom have been added. However, so far this proof strategy for syntactic completeness has been implemented on a case-by-case base, and not in general. In this paper, we address this gap by proving syntactic completeness for properly displayable logics in any normal (distributive) lattice expansion signature. Specifically, we show that for every analytic inductive axiom a cut free derivation can be effectively generated which has a specific shape, referred to as pre-normal form.

Keywords: Proper display calculi, properly displayable logics, unified correspondence, analytic inductive inequalities, lattice expansions.

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1 Introduction

In recent years, research in structural proof theory has focused on analytic calculi [37, 4, 29, 1, 43, 44], understood as those calculi supporting a robust form of cut elimination, i.e. one which is preserved by adding rules of a specific shape (the analytic rules). Important results on analytic calculi have been obtained in the context of various proof-theoretic formalisms: classes of axioms have been identified for which equivalent correspondences with analytic rules have been established algorithmically or semi-algorithmically. Without claiming to be exhaustive, we briefly review this strand of research as it has been developed in the context of sequent and labelled calculi [38, 36, 37], sequent and hypersequent calculus [4, 33, 34], and (proper) display calculi [32, 7, 29].

In [38], a methodology is established, sometimes referred to as axioms-as-rules, for transforming universal axioms in the language of first order classical (or intuitionistic) logics into analytic sequent rules. As remarked in the same paper, this methodology has a precursor in [35] for the intuitionistic theories of apartness and order. The rules so generated are then used to expand the sequent calculus G3c for first order classical logic. In [36], the axioms-as-rules methodology is generalized so as to capture the so-called geometric implications in the language of first order classical logic, i.e. formulas of the form $\forall x (A \to B)$ where $A$ and $B$ are geometric formulas (i.e. first-order formulas not containing $\rightarrow$ or $\forall$). In [37], the axioms-as-rules methodology is applied to capture various normal modal logic axioms via equivalent analytic labelled-calculus rules over the basic labelled calculus G3K for the modal logic K; moreover, following the standard methods as for the G3-style sequent calculi, the admissibility of cut, substitution and contraction is established. Although these calculi do not satisfy the full subformula property, decidability is established thanks to their enjoying the so-called subterm property (requiring all the terms in minimal derivations to occur in the endsequent) and height-preserving admissibility of contraction.

In [4], a hierarchy (sometimes referred to as substructural hierarchy) is defined of classes of substructural formulas, and it is shown how to translate substructural axioms up to level $N_2$ of the hierarchy into equivalent rules of a Gentzen-style sequent calculus, and axioms up to a subclass of level $P_3$ into equivalent rules of a hypersequent calculus; the rules so generated are then transformed into equivalent analytic rules whenever they satisfy an additional condition or the base calculus admits weakening; cut elimination is proved via a semantic argument extending the semantic proof of [39] to hypersequent calculi (and in [5], this approach is generalized to multi-conclusions hypersequents, and a heuristic is proposed to go beyond $P_3$ axioms). In [33], $n$-simple formulas, a particularly well-behaved proper subset of geometric formulas [37], are identified, and a method is introduced which transforms $n$-simple formulas into equivalent hypersequent rules for a variety of normal modal logics extending the modal logics $K$, $K_4$, or $KB$; cut admissibility is proved for $n$-simple extensions of $K$ and $K_4$, and decidability (via standard sub-formula property) is established for $n$-simple extensions of $KB$. In [34], the format of hypersequent rules with context restrictions is introduced, and transformations are studied between rules and modal axioms on a classical or intuitionistic base; decidability and complexity results are proved for a variety of modal logics, as well as uniform cut elimination extending the proof in [4]. In [6], hypersequent calculi are studied capturing analytic extensions of the full Lambek calculus FLe, and a procedure is introduced for translating structural rules into equivalent formulas in disjunction form. This approach is also applied to some normal modal logics on a classical base. The main goal of [6] is to show that cut-free derivations in hypersequent calculi can be transformed into derivations in sequent calculi satisfying various weaker versions of the subformula property which still guarantee decidability (although not necessarily cut elimination). Specifically, [6, Theorem 12(i)] shows how to construct a derivation in hypersequent calculi of formulas in disjunction form which are equivalent to structural rules.

In [32], the syntactic shape of primitive axioms in the language of tense modal logic on a classical base is characterized as the one which can be equivalently captured as analytic structural rules extending the minimal display calculus for tense logic. In [7], an analogous characterization is provided in a more general setting for a given but not fixed display calculus, by introducing a procedure for transforming axioms into analytic structural rules and showing the converse direction whenever the calculus satisfies additional conditions.

In [29], which is the contribution in the line of research described above to which the results of the present paper most directly connect, a characterization, analogous to the one of [7], of the property of being properly displayable [6] is obtained for arbitrary normal (D)LE-logic [4] via a systematic connection between proper display calculi and generalized Sahlqvist correspondence theory (aka unified correspondence [9, 10, 11, 17]). Thanks to this connection, general meta-theoretic results are established for properly displayable (D)LE-logics. In particular, in [29], the properly displayable (D)LE-logics

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1 For a comparison between the characterizations in [1] and in [29], see [29] Section 9.
2 The adjective ‘proper’ singles out a subclass of Belnap's display calculus [1] identified by Wansing in [33] Section 4.1. A display calculus is proper if every structural rule is closed under uniform substitution. This requirement strengthens Belnap's conditions $C_4$ and $C_7$. In [19], this requirement is extended to multi-type display calculi. A logic is (properly) displayable if it can be captured by some (proper) display calculus (see [29] Section 2.2).
3 Normal (D)LE-logics are those logics algebraically captured by varieties of normal (distributive) lattice expansions, i.e. (distributive) lattices endowed with additional operations that are finitely join-preserving or meet-reversing in each coordinate, or are finitely meet-preserving or join-reversing in each coordinate.
are syntactically characterized as the logics axiomatised by *analytic inductive axioms* (cf. Definition 2.10); moreover, the same algorithm ALBA which computes the first-order correspondent of (analytic) inductive (D)LE-axioms can be used to effectively compute their corresponding analytic structural rule(s).

The *semantic* equivalence between each analytic inductive axiom $\varphi \vdash \psi$ and its corresponding analytic structural rule(s) $R_1, \ldots, R_n$ is an immediate consequence of the soundness of the rules of ALBA on perfect normal (distributive) lattice expansions (cf. Footnote 8). However, on the *syntactic* side, an effective procedure was still missing for building *cut-free* derivations of $\varphi \vdash \psi$ in the proper display calculus obtained by adding $R_1, \ldots, R_n$ to the basic proper display calculus D.LE (resp. D.DLE) of the basic normal (D)LE-logic. Such an effective procedure would establish, via syntactic means, that for any properly displayable (D)LE-logic $L$, the proper display calculus for $L$—i.e., the calculus obtained by adding the analytic structural rules corresponding to the axioms of $L$ to the basic calculus D.LE (resp. D.DLE)—derives all the theorems (or derivable sequents) of $L$. This is what we refer to as the *syntactic completeness* of the proper display calculus for $L$ with respect to any analytic (D)LE-logic $L$. This syntactic completeness result for all properly displayable logics in arbitrary (D)LE-signatures is the main contribution of the present paper. It is perhaps worth to emphasize that we do not just show that any analytic inductive axiom is derivable in its corresponding proper display calculus, but we also provide an algorithm to generate a *cut-free* derivation of a particular shape that we refer to as being in *pre-normal form* (see Section 2.6).

### 2 Preliminaries

The present section adapts material from [12, Section 2], [29, Section 2], [23, Section 2], and [15, Section 2].

#### 2.1 Basic normal LE-logics and their algebras

Our base language is an unspecified but fixed language $L_{LE}$, to be interpreted over lattice expansions of compatible similarity type. This setting up accounts for many well known logical systems, such as full Lambek calculus and its axiomatic extensions, full Lambek-Grishin calculus, and other lattice-based logics.

In our treatment, we make use of the following auxiliary definition: an *order-type over $n \in \mathbb{N}$* is an $n$-tuple $\varepsilon \in \{1, \partial\}^n$. For every order type $\varepsilon$, we denote its *opposite order type* by $\varepsilon^d$, that is, $\varepsilon^d_i = \varepsilon^d(i) = 1$ iff $\varepsilon_i = \varepsilon(i) = \partial$ for every $1 \leq i \leq n$, and $\varepsilon^d = \varepsilon^d(i) = \partial$ iff $\varepsilon_i = \varepsilon(i) = 1$ for every $1 \leq i \leq n$. For any lattice $A$, we let $A^\varepsilon := A$ and $A^{\varepsilon^d}$ be the dual lattice, that is, the lattice associated with the converse partial order of $A$. For any order type $\varepsilon$, we let $A^{\varepsilon^d} := \Pi_{i=1}^n A^{\varepsilon_i}$. The language $L_{LE}(\mathcal{F}, \mathcal{G})$ (from now on abbreviated as $L_{LE}$) takes as parameters: a denumerable set of proposition letters $AtProp$, elements of which are denoted $p$, $q$, $r$, possibly with indexes, and disjoint sets of connectives $\mathcal{F}$ and $\mathcal{G}$. Each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ has arity $n_f \in \mathbb{N}$ (resp. $n_g \in \mathbb{N}$) and is associated with some order-type $\varepsilon_f$ over $n_f$ (resp. $\varepsilon_g$ over $n_g$). Unary $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) are sometimes denoted $\Diamond$ (resp. $\Box$) if their order-type is 1, and $\varphi \vdash \psi$ if their order-type is $\varepsilon$.

The terms (formulas) of $L_{LE}$ are defined recursively as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi(f(\varphi_1, \ldots, \varphi_n)) \mid g(\varphi_1, \ldots, \varphi_n)$$

where $p \in AtProp$. Terms in $L_{LE}$ are denoted either by $s$, $t$, or by lowercase Greek letters such as $\varphi$, $\psi$, $\gamma$.

**Definition 2.1.** For any tuple $(\mathcal{F}, \mathcal{G})$ of disjoint sets of function symbols as above, a *lattice expansion* (abbreviated as LE) is a tuple $A = (L, \mathcal{F}^A, \mathcal{G}^A)$ such that $L$ is a bounded lattice, $\mathcal{F}^A = \{ f^A \mid f \in \mathcal{F} \}$ and $\mathcal{G}^A = \{ g^A \mid g \in \mathcal{G} \}$, such that every $f^A \in \mathcal{F}^A$ (resp. $g^A \in \mathcal{G}^A$) is an $n_f$-ary (resp. $n_g$-ary) operation on $A$. An LE $A$ is *normal* if every $f^A \in \mathcal{F}^A$ (resp. $g^A \in \mathcal{G}^A$) 4

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4 The connectives in $\mathcal{F}$ (resp. $\mathcal{G}$) correspond to those referred to as positive (resp. negative) connectives in 4. This terminology is not adopted in the present paper to avoid confusion with positive and negative nodes in signed generation trees, defined later in this section. Our assumption that the sets $\mathcal{F}$ and $\mathcal{G}$ are disjoint is motivated by the desideratum of generality and modularity. Indeed, for instance, the order theoretic properties of Boolean negation $\neg$ guarantee that this connective belongs both to $\mathcal{F}$ and to $\mathcal{G}$. In such cases we prefer to define two copies $\neg \varphi \in \mathcal{F}$ and $\neg \varphi \in \mathcal{G}$, and introduce structural rules which encode the fact that these two copies coincide. Another possibility is to admit a non empty intersection of the sets $\mathcal{F}$ and $\mathcal{G}$. Notice that only unary connectives can be both left and right adjoints. Whenever a connective belongs both to $\mathcal{F}$ and to $\mathcal{G}$ a completely standard solution in the display calculi literature is also available (c.f. Remark 2.1 and 2.2).

5 The adjoints of the unary connectives $\ominus$, $\ominus$, $\lt$ and $\gt$ are denoted $\top$, $\bot$, $\lt$ and $\gt$, respectively.
preserves finite – hence also empty – joins (resp. meets) in each coordinate with $e_i(i) = 1$ (resp. $e_i(i) = 0$) and reverses finite – hence also empty – meets (resp. joins) in each coordinate with $e_i(i) = 0$ (resp. $e_i(i) = 1$). Let $\mathbb{LE}$ be the class of LEs. Sometimes we will refer to certain LEs as $L_{\text{LE}}$-algebras when we wish to emphasize that these algebras have a compatible signature with the logical language we have fixed.

In the remainder of the paper, we will often simplify notation and write e.g. $f$ for $f^\mathbb{L}$, $n$ for $n_f$ and $e_i$ for $e_i$. We also extend the $[1,\partial]$-notation to the symbols $\lor, \land, \bot, \leq, \vdash$ by stipulating that the superscript $^\partial$ denotes the identity map, defining

$\lor^\partial = \land, \quad \land^\partial = \lor, \quad \bot^\partial = \top, \quad \leq^\partial = \geq$,

and stipulating that $\varphi \vdash^\partial \psi$ stands for $\psi \vdash \varphi$.

Henceforth, the adjective ‘normal’ will typically be dropped. The class of all LEs is equational, and can be axiomatized by the usual lattice identities and the following equations for any $f \in \mathcal{F}$, $g \in \mathcal{G}$ and $1 \leq i \leq n$:

\[
\begin{align*}
  f(p_1, \ldots, q \lor^\mathbb{L} p_n) &= f(p_1, \ldots, q \lor p_n) \lor f(p_1, \ldots, p_n), \\
  f(p_1, \ldots, \bot^\mathbb{L} p_n) &= f(p_1, \ldots, \bot p_n) \lor f(p_1, \ldots, p_n), \\
  g(p_1, \ldots, q \land^\mathbb{L} p_n) &= g(p_1, \ldots, q \land p_n) \land g(p_1, \ldots, p_n), \\
  g(p_1, \ldots, \bot^\mathbb{L} p_n) &= g(p_1, \ldots, \bot p_n),
\end{align*}
\]

Each language $L_{\text{LE}}$ is interpreted in the appropriate class of LEs. In particular, for every LE $\mathbb{L}$, each operation $f^\mathbb{L} \in \mathcal{F}^\mathbb{L}$ (resp. $g^\mathbb{L} \in \mathcal{G}^\mathbb{L}$) is finitely join-preserving (resp. meet-preserving) in each coordinate when regarded as a map $f^\mathbb{L} : \mathbb{L}^\mathbb{L} \to \mathbb{L}$ (resp. $g^\mathbb{L} : \mathbb{L}^\mathbb{L} \to \mathbb{L}$).

The generic LE-logic is not equivalent to a sentential logic. Hence the consequence relation of these logics cannot be uniformly captured in terms of theorems, but rather in terms of sequents, which motivates the following definition:

**Definition 2.2.** For any language $L_{\text{LE}} = L_{\text{LE}}(\mathcal{F}, \mathcal{G})$, the basic, or minimal $L_{\text{LE}}$-logic is a set of sequents $\varphi \vdash \psi$, with $\varphi, \psi \in L_{\text{LE}}$, which contains as axioms the following sequents for lattice operations and additional connectives:

\[
\begin{align*}
  p \vdash p, & \quad \bot \vdash p, & \quad p \vdash \top, & \quad p \vdash q, & \quad p \lor q \vdash p, & \quad p \land q \vdash p, & \quad p \lor q \vdash q, \\
  f(p_1, \ldots, q \lor^\mathbb{L} p_n) & \vdash f(p_1, \ldots, q \lor p_n) \lor f(p_1, \ldots, p_n), & \quad f(p_1, \ldots, \bot^\mathbb{L} p_n) & \vdash f(p_1, \ldots, \bot p_n) \lor f(p_1, \ldots, p_n), \\
  g(p_1, \ldots, q \land^\mathbb{L} p_n) & \vdash g(p_1, \ldots, q \land p_n) \land g(p_1, \ldots, p_n), & \quad g(p_1, \ldots, \bot^\mathbb{L} p_n) & \vdash g(p_1, \ldots, \bot p_n),
\end{align*}
\]

and is closed under the following inference rules (note that $\varphi \vdash^\partial \psi$ means $\psi \vdash \varphi$):

\[
\begin{align*}
  & \varphi \vdash \psi & \chi \vdash \psi & \varphi \lor \psi \vdash \psi \\
  \frac{\varphi \vdash \chi \vdash \psi}{\varphi \vdash \chi / \psi \vdash \psi} & \frac{\varphi \lor \psi \vdash \chi}{\chi \lor \varphi \vdash \psi} & \frac{\chi \lor \varphi \vdash \psi}{\varphi \lor \psi \vdash \chi} \\
  & \varphi \lor^\mathbb{L} \psi \vdash \psi & g(p_1, \ldots, q \land^\mathbb{L} p_n) & \vdash g(p_1, \ldots, q \land p_n) \land g(p_1, \ldots, p_n), \\
  \frac{\varphi \lor^\mathbb{L} \psi}{f(p_1, \ldots, q \lor^\mathbb{L} p_n) + f(p_1, \ldots, q \lor p_n) \lor f(p_1, \ldots, p_n)} & \frac{g(p_1, \ldots, q \land^\mathbb{L} p_n) \lor g(p_1, \ldots, q \land p_n)}{g(p_1, \ldots, q \land p_n) \land g(p_1, \ldots, p_n)}
\end{align*}
\]

We let $L_{\text{LE}}$ denote the minimal $L_{\text{LE}}$-logic. We typically drop reference to the parameters when they are clear from the context. By an LE-logic we understand any axiomatic extension of $L_{\text{LE}}$ in the language $L_{\text{LE}}$. If all the axioms in the extension are analytic inductive (cf. Definition 2.10) we say that the given LE-logic is analytic.

A sequent $\varphi \vdash \psi$ is valid in an LE $\mathbb{L}$ if $h(\varphi) \leq h(\psi)$ for every homomorphism $h$ from the $L_{\text{LE}}$-algebra of formulas over $\text{AtProp}$ to $\mathbb{L}$. The notation $\mathbb{L} \models \varphi \vdash \psi$ indicates that $\varphi \vdash \psi$ is valid in every LE of the appropriate signature. Then, by means of a routine Lindenbaum-Tarski construction, it can be shown that the minimal LE-logic $L_{\text{LE}}$ is sound and complete with respect to its corresponding class of algebras $\mathbb{LE}$, i.e. that any sequent $\varphi \vdash \psi$ is provable in $L_{\text{LE}}$ iff $\mathbb{LE} \models \varphi \vdash \psi$.

\footnote{Normal LEs are sometimes referred to as *lattices with operators* (LOs). This terminology comes from the setting of Boolean algebras with operators, in which operators are operations which preserve finite joins in each coordinate. However, this terminology is somewhat ambiguous in the lattice setting, in which primitive operations are typically maps which are operators if seen as $\mathbb{A}^\mathbb{LE} \to \mathbb{A}$ for some order-type $e$ on $n$ and some order-type $\eta \in [1,\partial]$.}
2.2 The fully residuated language $L^*_{LE}$

Any given language $L_{LE} = \mathcal{L}_{LE}(\mathcal{F}, \mathcal{G})$ can be associated with the language $L^*_{LE} = \mathcal{L}_{LE}(\mathcal{F}^*, \mathcal{G}^*)$, where $\mathcal{F}^* \supset \mathcal{F}$ and $\mathcal{G}^* \supset \mathcal{G}$ are obtained by expanding $L_{LE}$ with the following connectives:

1. the $n_f$-ary connective $f_i^\delta$ for $0 \leq i \leq n_f$, the intended interpretation of which is the right residual of $f \in \mathcal{F}$ in its $i$th coordinate if $e_f(i) = 1$ (resp. its Galois-adjoint if $e_f(i) = \partial$);
2. the $n_g$-ary connective $g_i^\epsilon$ for $0 \leq i \leq n_g$, the intended interpretation of which is the left residual of $g \in \mathcal{G}$ in its $i$th coordinate if $e_g(i) = 1$ (resp. its Galois-adjoint if $e_g(i) = \partial$).

We stipulate that $f_i^\delta \in \mathcal{G}^*$ if $e_f(i) = 1$, and $f_i^\delta \in \mathcal{F}^*$ if $e_f(i) = \partial$. Dually, $g_i^\epsilon \in \mathcal{F}^*$ if $e_g(i) = 1$, and $g_i^\epsilon \in \mathcal{G}^*$ if $e_g(i) = \partial$. The order-type assigned to the additional connectives is predicated on the order-type of their intended interpretations. That is, for any $f \in \mathcal{F}$ and $g \in \mathcal{G}$,

1. if $e_f(i) = 1$, then $e_{f_i^\delta}(i) = 1$ and $e_{f_i^\delta}(j) = e_f(i)$ for any $j \neq i$.
2. if $e_f(i) = \partial$, then $e_{f_i^\delta}(i) = \partial$ and $e_{f_i^\delta}(j) = e_f(j)$ for any $j \neq i$.
3. if $e_g(i) = 1$, then $e_{g_i^\epsilon}(i) = 1$ and $e_{g_i^\epsilon}(j) = e_g(i)$ for any $j \neq i$.
4. if $e_g(i) = \partial$, then $e_{g_i^\epsilon}(i) = \partial$ and $e_{g_i^\epsilon}(j) = e_g(j)$ for any $j \neq i$.

For instance, if $f$ and $g$ are binary connectives such that $e_f = (1, \partial)$ and $e_g = (\partial, 1)$, then $e_{f_1^\delta} = (1, 1)$, $e_{f_2^\delta} = (1, \partial)$, $e_{g_1^\epsilon} = (\partial, 1)$ and $e_{g_2^\epsilon} = (1, 1)$.

**Definition 2.3.** For any language $L_{LE}(\mathcal{F}, \mathcal{G})$, its associated basic $L^*_{LE}$-logic is defined by specializing Definition 2.2 to the language $L^*_{LE} = \mathcal{L}_{LE}(\mathcal{F}^*, \mathcal{G}^*)$ and closing under the following additional residuation rules for $f \in \mathcal{F}$ and $g \in \mathcal{G}$:

\[
\begin{align*}
\varphi \vdash f(\varphi_1, \ldots, \varphi_n) & \quad \varphi_1^{e_f(i)} \vdash f_i^\delta(\varphi_1, \ldots, \varphi_n) \\
\varphi \vdash g(\varphi_1, \ldots, \varphi_n) & \quad \varphi_1^{e_g(i)} \vdash g_i^\epsilon(\varphi_1, \ldots, \varphi_n)
\end{align*}
\]

The double line in each rule above indicates that the rule is invertible (i.e., bidirectional). Let $L^*_{LE}$ be the basic $L^*_{LE}$-logic.

The algebraic semantics of $L^*_{LE}$ is given by the class of fully residuated $L_{LE}$-algebras, defined as tuples $A = (\mathbb{L}, \mathcal{F}^*, \mathcal{G}^*)$ such that $\mathbb{L}$ is a lattice and moreover,

1. for every $f \in \mathcal{F}$ with $n_f \geq 1$, all $a_1, \ldots, a_{n_f}, b \in \mathbb{L}$ and $1 \leq i \leq n_f$, $f(a_1, \ldots, a_i, \ldots, a_{n_f}) \leq b \iff a_i \leq f_i^\delta(a_1, \ldots, b, \ldots, a_{n_f})$.
2. for every $g \in \mathcal{G}$ with $n_g \geq 1$, all $a_1, \ldots, a_{n_g}, b \in \mathbb{L}$ and $1 \leq i \leq n_g$, $b \leq g(a_1, \ldots, a_i, \ldots, a_{n_g}) \iff g_i^\epsilon(a_1, \ldots, b, \ldots, a_{n_g}) \leq a_i$.

It is also routine to prove using the Lindenbaum-Tarski construction that $L^*_{LE}$ (as well as any of its canonical axiomatic extensions) is sound and complete with respect to the class of fully residuated $L_{LE}$-algebras (or a suitably defined equational subclass, respectively).

**Theorem 2.4.** The logic $L^*_{LE}$ is a conservative extension of $L_{LE}$, i.e. every $L_{LE}$-sequent $\varphi \vdash \psi$ is derivable in $L_{LE}$ if and only if $\varphi \vdash \psi$ is derivable in $L^*_{LE}$.

\[\text{Note that this notation depends on the connective which is taken as primitive, and needs to be carefully adapted to well known cases. For instance, consider the ‘fusion’ connective $\odot$, which, when denoted as $f$, is such that $e_f = (1, 1)$. Its residuals $f_i^\delta$ and $f_i^\epsilon$ are commonly denoted $\backslash$ and $\backslash$ respectively. However, if $\backslash$ is taken as the primitive connective $g$, then $g_i^\epsilon = f$, and $g_i^\epsilon(x_1, x_2) = x_2/x_1 = f_i^\delta(x_2, x_1)$. This example shows that, when identifying $g_i^\epsilon$ and $f_i^\delta$, the conventional order of the coordinates is not preserved, and depends on which connective is taken as primitive.}\]
Proof. We only outline the proof. Clearly, every \( L_{\text{LE}} \)-sequent which is \( L_{\text{LE}} \)-derivable is also \( L^*_\text{LE} \)-derivable. Conversely, if an \( L_{\text{LE}} \)-sequent \( \varphi \vdash \psi \) is not \( L_{\text{LE}} \)-derivable, then by the completeness of \( L_{\text{LE}} \) with respect to the class of \( L_{\text{LE}} \)-algebras, there exists an \( L_{\text{LE}} \)-algebra \( A \) and a variable assignment \( \nu \) under which \( \varphi \not\leq \psi \). Consider the canonical extension \( A^\ast \). Since \( A \) is a subalgebra of \( A^\ast \), the sequent \( \varphi \vdash \psi \) is not satisfied in \( A^\ast \) under the variable assignment \( \nu \circ (\iota \text{ denoting the canonical embedding } A \hookrightarrow A^\ast) \). Moreover, since \( A^\ast \) is a perfect \( L_{\text{LE}} \)-algebra, it is naturally endowed with a structure of \( L^*_\text{LE} \)-algebra. Thus, by the completeness of \( L^*_\text{LE} \) with respect to the class of \( L^*_\text{LE} \)-algebras, the sequent \( \varphi \vdash \psi \) is not derivable in \( L^*_\text{LE} \), as required.

Notice that the algebraic completeness of the logics \( L_{\text{LE}} \) and \( L^*_\text{LE} \) and the canonical embedding of LEs into their canonical extensions immediately give completeness of \( L_{\text{LE}} \) and \( L^*_\text{LE} \) with respect to the appropriate class of perfect LEs.

2.3 Analytic inductive LE-inequalities

In this section we recall the definitions of inductive LE-inequalities introduced in [12] and their corresponding ‘analytic’ restrictions introduced in [29] in the distributive setting and then generalized to the setting of LEs of arbitrary signatures in [23]. Each inequality in any of these classes is canonical and elementary (cf. [12 Theorems 7.1 and 6.1]).

Definition 2.5 (Signed Generation Tree). The positive (resp. negative) generation tree of any \( L_{\text{LE}} \)-term \( s \) is defined by labelling the root node of the generation tree of \( s \) with the sign \( + \) (resp. \( - \)), and then propagating the labelling on each remaining node as follows:

- For any node labelled with \( \lor \) or \( \land \), assign the same sign to its children nodes.
- For any node labelled with \( \epsilon \in F \cup G \) of arity \( n \), assign the same (resp. the opposite) sign to its \( i \)th child node if \( e_\epsilon(i) = 1 \) (resp. \( e_\epsilon(i) = 0 \)).

Nodes in signed generation trees are positive (resp. negative) if they are signed \( + \) (resp. \( - \)).

Signed generation trees will be mostly used in the context of term inequalities \( s \leq t \). In this context we will typically consider the positive generation tree \( +s \) for the left-hand side and the negative one \( -t \) for the right-hand side. We will also say that a term-inequality \( s \leq t \) is uniform in a given variable \( p \) if all occurrences of \( p \) in both \( +s \) and \( -t \) have the same sign, and that \( s \leq t \) is \( \epsilon \)-uniform in a (sub)array \( A \) of its variables if \( s \leq t \) is uniform in \( r \), occurring with the sign indicated by \( \epsilon \), for every \( r \) in \( A \).

For any term \( s(p_1, \ldots, p_n) \), any order type \( \epsilon \) over \( n \), and any \( 1 \leq i \leq n \), an \( \epsilon \)-critical node in a signed generation tree of \( s \) is a leaf node \( +p_i \) with \( e_\epsilon(i) = 1 \) or \( -p_i \) with \( e_\epsilon(i) = 0 \). An \( \epsilon \)-critical branch in the tree is a branch from an \( \epsilon \)-critical node. Variable occurrences corresponding to \( \epsilon \)-critical nodes are those used in the runs of the various versions of the algorithm ALBA (cf. [10] [9] [12] [13]) to compute the minimal valuations. For every term \( s(p_1, \ldots, p_n) \) and any order type \( \epsilon \), we say that \( +s \) (resp. \( -s \)) agrees with \( \epsilon \), and write \( \epsilon(s +) \) (resp. \( \epsilon(s -) \)), if every leaf in the signed generation tree of \( +s \) (resp. \( -s \)) is \( \epsilon \)-critical. We will also write \( +s' \prec \prec +s \) (resp. \( -s' \prec \prec -s \)) to indicate that the subterm \( s' \) inherits the positive (resp. negative) sign from the signed generation tree \( +s \). Finally, we will write \( \epsilon(\gamma) \prec \prec \epsilon(s) \) (resp. \( \epsilon(\delta) \prec \prec \epsilon(s) \)) to indicate that the signed subtree \( \gamma, \delta \) with the sign inherited from \( s \) agrees with \( \epsilon \) (resp. with \( \epsilon^\prime \)).

Notation 2.6. In what follows, we will often need to use placeholder variables to e.g. specify the occurrence of a subformula within a given formula. In these cases, we will write e.g. \( \varphi(\gamma) \) (resp. \( \varphi(\delta) \)) to indicate that the variable \( \gamma \) (resp. each variable \( z \) in vector \( \gamma \)) occurs exactly once in \( \varphi \). Accordingly, we will write \( \varphi(\gamma/\epsilon\gamma) \) (resp. \( \varphi(\delta/\epsilon\delta) \)) to indicate the formula obtained from \( \varphi \) by substituting \( \gamma \) (resp. each formula \( \gamma \) in \( \delta \)) for the unique occurrence of (its corresponding variable) \( z \) in \( \varphi \). Also, in what follows, we will find sometimes useful to group placeholder variables together according to certain assumptions we make about them. So, for instance, we will sometimes write e.g. \( \varphi(\gamma/\delta) \) to indicate that \( \epsilon(\gamma) < \epsilon(\delta) \) for

8 The canonical extension of a bounded lattice \( L \) is a complete lattice \( L^\ast \) with \( L \) as a sublattice, satisfying denseness: every element of \( L^\ast \) can be expressed both as a join of meets and as a meet of joins of elements from \( L \), and compactness: for all \( S, T \subseteq L \), if \( S \subseteq T \) and \( T \subseteq L^\ast \), then \( S \subseteq L^\ast \). It is well known that the canonical extension of \( L \) is unique up to isomorphism fixing \( L \) (cf. e.g. [22] Section 2.2), and that the canonical extension is a perfect bounded lattice, i.e. a complete lattice which is completely join-generated by its completely join-irreducible elements (cf. e.g. [22] Definition 2.14). The canonical extension of an \( L_{\text{LE}} \)-algebra \( A = (L, F^A, G^A) \) is the perfect \( L_{\text{LE}} \)-algebra \( A^\ast = (L^\ast, F^\ast_A, G^\ast_A) \) such that \( F^\ast_A \) and \( G^\ast_A \) are defined as the \( \epsilon \)-extension of \( F^A \) and as the \( \epsilon \)-extension of \( F^A \) respectively, for all \( f \in F \) and \( g \in G \) (cf. [20][31]).

9 In the context of sequents \( s \vdash t \), signed generation trees \( +s \) and \( -t \) can also be used to specify when subformulas of \( s \) (resp. \( t \)) occur in precedent or succedent position. Specifically, a given occurrence of formula \( \gamma \) is in precedent (resp. succedent) position in \( s \vdash t \) if \( \epsilon(\gamma) < \epsilon(s) \) or \( \epsilon(\gamma) > \epsilon(t) \) (resp. \( \epsilon(\gamma) < \epsilon(s) \) or \( \epsilon(\gamma) > \epsilon(t) \)).

10 If a term inequality \( s \leq t \) is \( \epsilon \)-uniform in all variables in \( \gamma \) (cf. discussion after Definition 2.6), then the validity of \( s \leq t \) is equivalent to the validity of \( s \leq t \), where \( \gamma_{\epsilon(i)} = \gamma \) if \( e(\epsilon(i)) = 1 \) and \( \gamma_{\epsilon(i)} = \gamma^{\ominus} \), if \( e(\epsilon(i)) = 0 \).
all variables x in \( \overline{x} \) and \( s^\partial(y) \prec \ast \varphi \) for all variables y in \( \overline{y} \), or we will write e.g. \( f(\overline{x}, \overline{y}) \) to indicate that \( f \) is monotone (resp. antitone) in the coordinates corresponding to every variable x in \( \overline{x} \) (resp. y in \( \overline{y} \)). We will provide further explanations as to the intended meaning of these groupings whenever required. Finally, we will also extend these conventions to inequalities or sequents, and thus write e.g. \( f ! x, ! y \) to indicate that \( f \) is monotone (resp. antitone) in the coordinates corresponding to every variable x in \( x \) (resp. y in \( y \)).

**Definition 2.7.** Nodes in signed generation trees will be called \( \Delta \)-adjoints, syntactically left residuals (SLR), syntactically right residuals (SRR), and syntactically right adjoints (SRA), according to the specification given in Table 1. A branch in a signed generation tree \( *s \), with \( * \in \{ - , + \} \), is called a good branch if it is the concatenation of two paths \( P_1 \) and \( P_2 \), one of which may possibly be of length 0, such that \( P_1 \) is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes, and \( P_2 \) consists (apart from variable nodes) only of Skeleton-nodes. A good branch is Skeleton if the length of \( P_1 \) is 0, and is SLR, or definite, if \( P_2 \) only contains SLR nodes.

| Skeleton | PIA |
|----------|-----|
| \( \Delta \)-adjoints | Syntactically Right Adjoint (SRA) |
| + \( \lor \) | + \( \land g \) with \( n_\varphi = 1 \) |
| - \( \land \) | - \( \lor f \) with \( n_\varphi = 1 \) |
| Syntactically Left Residual (SLR) | Syntactically Right Residual (SRR) |
| + \( f \) with \( n_\varphi \geq 1 \) | + \( g \) with \( n_\varphi \geq 2 \) |
| - \( g \) with \( n_\varphi \geq 1 \) | - \( f \) with \( n_\varphi \geq 2 \) |

Table 1: Skeleton and PIA nodes for LE-languages.

We refer to [12, Remark 3.3] and [42] for a discussion of the notational conventions and terminology. We refer to [12, Section 3.2] and [9, Section 1.7.2] for a comparison with [10] and [22] where the nodes of the signed generation tree were classified according to the choice and universal terminology.

**Definition 2.8** (Inductive inequalities). For any order type \( \varepsilon \) and any irreflexive and transitive relation (i.e. strict partial order) \( \Omega \) on \( p_1, \ldots, p_n \), the signed generation tree \( *s \) (\( * \in \{ - , + \} \)) of a term \( s(p_1, \ldots, p_n) \) is \((\Omega, \varepsilon)\)-inductive if

1. for all \( 1 \leq i \leq n \), every \( \varepsilon \)-critical branch with leaf \( p_i \) is good (cf. Definition 2.7);
2. every \( m \)-ary SRR-node occurring in the critical branch is of the form \( \ast(\gamma_1, \ldots, \gamma_{j-1}, \beta, \gamma_{j+1}, \ldots, \gamma_m) \), where for any \( h \in \{ 1, \ldots, m \} \setminus j \):
   a. \( s^\partial(\gamma_h) \prec \ast s \) (cf. discussion before Definition 2.7), and
   b. \( p_h \prec_\Omega p_i \) for every \( p_h \) occurring in \( \gamma_h \) and for every \( 1 \leq k \leq n \).

We will refer to \( <_\Omega \) as the dependency order on the variables. An inequality \( s \leq t \) is \((\Omega, \varepsilon)\)-inductive if the signed generation trees \( +s \) and \( -t \) are \((\Omega, \varepsilon)\)-inductive. An inequality \( s \leq t \) is inductive if it is \((\Omega, \varepsilon)\)-inductive for some \( \Omega \) and \( \varepsilon \).
In what follows, we refer to formulas $\varphi$ such that only PIA nodes occur in $+\varphi$ (resp. $-\varphi$) as positive (resp. negative) PIA-formulas, and to formulas $\xi$ such that only Skeleton nodes occur in $+\xi$ (resp. $-\xi$) as positive (resp. negative) Skeleton-formulas. PIA formulas $+\varphi$ in which no binary SRA-nodes (i.e. $+\wedge$ and $-\vee$) occur are referred to as definite. Skeleton formulas $+\xi$ in which no $\Delta$-adjoint nodes (i.e. $-\wedge$ and $+\vee$) occur are referred to as definite. Hence, $+\xi$ (resp. $-\varphi$) is definite Skeleton (resp. definite PIA) iff all nodes of $+\xi$ (resp. $-\varphi$) are SLR (resp. SRR or unary SRA).

**Lemma 2.9.** For every LE-language $\mathcal{L}$,

1. if $\gamma$ is a positive PIA (resp. negative Skeleton) $\mathcal{L}$-formula, then $\gamma$ is equivalent to $\bigwedge_{i\in I} \gamma_i$ for some finite set of definite positive PIA (resp. negative Skeleton) formulas $\gamma_i$;
2. if $\delta$ is a negative PIA (resp. positive Skeleton) $\mathcal{L}$-formula, then $\delta$ is equivalent to $\bigvee_{j\in J} \delta_j$ for some finite set of definite negative PIA (resp. positive Skeleton) formulas $\delta_j$.

**Proof.** By simultaneous induction on $\gamma$ and $\delta$. The base cases are immediately true. If $\delta := f(\overline{\delta'}, \overline{\gamma'})$, then by induction hypothesis on each $\delta'$ in $\overline{\delta'}$ and each $\gamma'$ in $\overline{\gamma'}$, the formula $\delta$ is equivalent to $f(\bigvee_{j\in J} \delta'_j, \bigwedge_{i\in I} \overline{\gamma'_i})$ for some finite sets of definite positive PIA (resp. negative Skeleton) formulas $\gamma'_i$ and of definite positive Skeleton (resp. negative PIA) formulas $\delta'_j$. By the coordinatewise distribution properties of every $f \in \mathcal{F}$, the term $f(\bigvee_{j\in J} \delta'_j, \bigwedge_{i\in I} \overline{\gamma'_i})$ is equivalent to $\bigvee_{j\in J} f(\overline{\delta'_j}, \overline{\gamma'_i})$ with each $f(\overline{\delta'_j}, \overline{\gamma'_i})$ being definite positive Skeleton (resp. negative PIA), as required. The remaining cases are omitted. \qed

**Definition 2.10** (Analytic inductive inequalities). For every order type $\epsilon$ and every irreflexive and transitive relation $\Omega$ on the variables $p_1, \ldots, p_n$, the signed generation tree $+s$ ($s \in \{+, -\}$) of a term $s(p_1, \ldots, p_n)$ is analytic $(\Omega, \epsilon)$-inductive if

1. $+s$ is $(\Omega, \epsilon)$-inductive (cf. Definition 2.8);
2. every branch of $+s$ is good (cf. Definition 2.7).

An inequality $s \leq t$ is analytic $(\Omega, \epsilon)$-inductive if $+s$ and $-t$ are both analytic $(\Omega, \epsilon)$-inductive. An inequality $s \leq t$ is analytic inductive if is analytic $(\Omega, \epsilon)$-inductive for some $\Omega$ and $\epsilon$. An analytic inductive inequality is definite if no $\Delta$-adjoint nodes (i.e. $-\wedge$ and $+\vee$) occur in its Skeleton.

![Diagram](attachment:image.png)

**Notation 2.11.** We will sometimes represent $(\Omega, \epsilon)$-analytic inductive inequalities/sequents as follows:

\[
(\varphi \leq \psi)[\overline{\alpha}/\overline{\epsilon}/\overline{\beta}/\overline{\gamma}/\overline{\delta}/\overline{\epsilon}], \\
(\varphi + \psi)[\overline{\alpha}/\overline{\epsilon}/\overline{\beta}/\overline{\gamma}/\overline{\delta}/\overline{\epsilon}],
\]

where $(\varphi \leq \psi)[\overline{\alpha}/\overline{\epsilon}/\overline{\beta}/\overline{\gamma}/\overline{\delta}/\overline{\epsilon}]$ is the skeleton of the given inequality, $\overline{\alpha}$ (resp. $\overline{\beta}$) denotes the vector of positive (resp. negative) maximal PIA-subformulas, i.e. each $\alpha$ in $\overline{\alpha}$ and $\beta$ in $\overline{\beta}$ contains at least one $\epsilon$-critical occurrence of some propositional variable, and moreover:

1. for each $\alpha$ in $\overline{\alpha}$, either $+\alpha < +\varphi$ or $+\alpha < -\psi$;
2. for each $\beta$ in $\overline{\beta}$, either $-\beta < +\varphi$ or $-\beta < -\psi$,

and $\overline{\gamma}$ (resp. $\overline{\delta}$) denotes the vector of positive (resp. negative) maximal $\epsilon^0$-subformulas, and moreover:

1. for each $\gamma$ in $\overline{\gamma}$, either $+\gamma < +\varphi$ or $+\gamma < -\psi$;

8
2. for each δ in V ar, either −δ < +φ or −δ < −ψ.

For the sake of a more compact notation, in what follows we sometimes write e.g. \((ϕ ≤ ψ)[Γ/β, β/Γ, γ, γ/Γ] \) in place of \((ϕ ≤ ψ)[Γ/β, β/Γ, γ, γ/Γ, δ, δ/Γ] \). The colours are intended to help in identifying which subformula occurrences are in precedent (blue) or successor (red) position (cf. Footnote [9]).

**Lemma 2.12.** For any LE-language \(L\) any analytic inductive \(L\)-sequent \((ϕ + ψ)[Γ/β, β/Γ, γ, γ/Γ, δ, δ/Γ] \) is equivalent to the conjunction of definite analytic inductive \(L\)-sequents \((ϕ + ψ)[Γ/β, β/Γ, γ, γ/Γ] \).

**Proof.** Since by assumption \(ϕ(Γ, γ, γ, δ, δ)\) is positive Skeleton and \(ψ(Γ, γ, γ, δ, δ)\) is negative Skeleton, by Lemma [2.9] the given sequent is equivalent to \((∨_{j≠l} ϕ_j + ∧_{j≠l} ψ_j)[Γ/β, β/Γ, γ, γ/Γ, δ, δ/Γ] \), where every \(ψ_j\) is definite negative Skeleton and every \(ϕ_j\) is definite positive Skeleton, from which the statement readily follows.

**Remark 2.13.** We adopt the convention that in graphical representations of signed generation trees the squared variable occurrences are the ε-critical ones, the doubly circled nodes are the Skeleton ones and the single-circle ones are PIA-nodes.

**Example 2.14.** Let \(L := L(ο, G)\), where \(F := \{ο\} \) and \(G := \{ο, ⊕, →\} \) with the usual arity and order-type. The \(L\)-inequality \(p ≤ ⊕οp\) is e-Sahlqvist for \(ε(p) = 1\), but is not analytic inductive for any order-type, because the negative generation tree of \(ο⊕p\), which has only one branch, is not good. The Church-Rosser inequality \(ο⊕p ≤ ⊕οp\) is analytic e-Sahlqvist for every order-type.

The inequality \(p → (q → r) ≤ ((p → q) → (οp → r)) \) is not Sahlqvist for any order-type: indeed, both the positive and the negative occurrence of \(q\) occur under the scope of an SRR-connective. However, it is an analytic \((Ω, ε)\)-inductive inequality, e.g. for \(p <o q <o r\) and \(ε(p, q, r) = (1, 1, 0)\).

Below, we represent the signed generation trees pertaining to the inequalities above (see Notation [2.13]):

\[\begin{align*}
+p ≤ & 0 \quad -p \\
γ & \quad -q + r \\
\end{align*}\]

The following auxiliary definition was introduced in [29, Definition 48] as a simplified version of [8, Definition 5.1], and serves to calculate effectively the residuals of definite positive and negative PIA formulas (cf. [29], discussion after Definition 2.8) w.r.t. a given variable occurrence \(x\). The intended meaning of symbols such as \(ϕ(Γ, γ)\) is that the variable \(x\) occurs exactly once in the formula \(ϕ\) (cf. Notation [2.6]). In the context of the following definition, the variable \(x\) is used (and referred to) as the pivotal variable, i.e. the variable that is displayed by effect of the recursive residuation procedure.

**Definition 2.15.** For every definite positive PIA \(L_{\text{PIA}}\)-formula \(ϕ = ϕ(Γ, γ)\), and any definite negative PIA \(L_{\text{PIA}}\)-formula \(ξ = ξ(Γ, γ)\) such that \(x\) occurs in them exactly once, the \(L_{\text{PIA}}\)-formulas \(lαι(ϕ)(u, γ)\) and \(rαι(ξ)(u, γ)\) (for \(u ∈ V ar − (x ∪ γ)\)) are defined by simultaneous recursion as follows:

\[\begin{align*}
/& \quad α_p \\
−p & \quad β_{β_1} \\
\end{align*}\]

The use of colours in this notational convention is inspired by, but different from, the one introduced in [31], where the blue (resp. red) colour identifies the logical connectives algebraically interpreted as right (resp. left) adjoints or residuals. However, when restricted to the analytic inductive LE- inequalities, these two conventions coincide, since the main connective of a (non-atomic) positive (resp. negative) maximal PIA-subformula is a right (resp. left) adjoint/residual. Interestingly, the so-called (strong) focalization property of the focalized sequent calculi introduced in [31] can be equivalently formulated in terms of maximal PIA-subtrees.
ψ \rightarrow \psi, y \} \in \mathcal{H}

Above, symbols such as \( \psi \) denote the vector obtained by removing the \( j \)th coordinate of the vector \( \tilde{\psi} \).

**Example 2.16.** Let \( \mathcal{L} := \mathcal{L}(\mathcal{F}, \mathcal{G}) \), where \( \mathcal{F} := \{ \rightarrow \} \) and \( \mathcal{G} := \{ \bowtie, \rightarrow \} \) with the usual arity and order-type. Let \( \mathcal{F}^* := \{ \rightarrow, \otimes, \otimes \} \) and \( \mathcal{G}^* := \{ \bowtie, \rightarrow, \otimes \} \), where \( \rightarrow \) (resp. \( \otimes \)) has \( \rightarrow' \) (resp. \( \otimes' \)) as residual in its first coordinate and \( \otimes \) (resp. \( \otimes \)) as residual in its second coordinate. Consider the definite positive PIA \( \mathcal{L}(\mathcal{F} \cup \mathcal{G}) \) where the structures \( \mathcal{F} \) and \( \mathcal{G} \) with an arity and an order type which coincide with those of its associated operational connective in \( \mathcal{F}^* \) and \( \mathcal{G}^* \).

**Remark 2.17.** If \( f \in \mathcal{F} \) and \( g \in \mathcal{G} \) form a dual pair then \( n_f = n_g \) and \( e_f = e_g \). Then \( f \) and \( g \) can be assigned one and the same structural operator \( H \), which is interpreted as \( f \) when occurring in precedent position and as \( g \) when occurring in succedent position (cf. Footnote 9):

Moreover, for any \( 1 \leq i \leq n_f = n_g \), the residuals \( f_i^g \) and \( g_i^f \) are dual to one another. Hence they can also be assigned one and the same structural connective as follows:

This observation has made it possible to associate one structural connective with two logical connectives, which has become common in the display calculi literature. In this paper, we prefer to maintain a strict one-to-one correspondence between operational and structural symbols.

If we admit that the sets \( \mathcal{F} \) and \( \mathcal{G} \) have a non empty intersection (cf. Footnote 4), then a unary connective \( h \in \mathcal{F} \cap \mathcal{G} \) can be assigned one and the same structural operator \( \tilde{h} \), which is interpreted as \( h \) when occurring in precedent position and in succedent position:

For notational convenience, we let \( \mathcal{F}^\partial := \mathcal{G} \) and \( \mathcal{G}^\partial := \mathcal{F} \). Moreover, given the sets \( \text{Str}_{\mathcal{F}}, \text{Str}_{\mathcal{G}} \) defined below and any order type \( \varepsilon \) on \( n \), we let \( \text{Str}_{\mathcal{F}} := \prod^n_{i=1} \text{Str}_{\mathcal{F}_i} \) and \( \text{Str}_{\mathcal{G}} := \prod^n_{i=1} \text{Str}_{\mathcal{G}_i} \).

The calculus \( \text{D.LE} \) manipulates sequents \( \Pi \vdash \Sigma \) where the structures \( \Pi \) (for precedent) and \( \Sigma \) (for succedent) are defined by the following simultaneous recursion:

\[
\text{Str}_{\mathcal{F}} \ni \Pi := \phi \uparrow \hat{\rightarrow} \hat{\rightarrow} (\pi^{(e)})
\]

\[
\text{Str}_{\mathcal{G}} \ni \Sigma := \phi \downarrow \hat{\rightarrow} \hat{\rightarrow} (\pi^{(e)})
\]

---

12 Examples of dual pairs are \((\land, \lor), (\land, \rightarrow), (\land, \leftarrow), \) and \((\land, \lor)\) where \( \land \) is defined as \( \neg \land \).
with \( \varphi \in \mathcal{L}_{LE} \), and \( \hat{f} \in S_{\mathcal{F}}, \hat{g} \in S_{\mathcal{G}}, \Pi^{(\varphi)} \in \text{Str}_{\mathcal{F}}^{\varphi} \) and \( \Sigma^{(\varphi)} \in \text{Str}_{\mathcal{G}}^{\varphi} \). Notice that for any connective \( h \) of arity \( n \geq 1 \) the notational convention \( \hat{h} \) conveys also the information that \( h \) is a left-adjoint/residual and the notational convention \( \hat{h} \) conveys the information that \( h \) is a right-adjoint/residual.

In what follows, we use \( \Upsilon_1, \ldots, \Upsilon_n \) as structure metavariables in \( \text{Str}_{\mathcal{F}} \cup \text{Str}_{\mathcal{G}} \). The introduction rules of the calculus below will guarantee that \( \Upsilon \in \text{Str}_{\mathcal{F}} \) whenever it occurs in succedent position, and \( \Upsilon \in \text{Str}_{\mathcal{G}} \) whenever it occurs in succedent position. The calculus \( \text{D.LE} = \text{D.LE}_c \) consists of the following rules\(^{13}\):

- **Identity and cut rules\(^{14}\):**

  \[
  \frac{\Pi \vdash \varphi}{\Pi \vdash \varphi \cup \Sigma} \text{ Cut}
  \]

  \[
  \frac{p \vdash p}{\Pi \vdash \Sigma} \text{ Id}
  \]

- **Display postulates for \( f \in \mathcal{F} \) and \( g \in \mathcal{G} \):** for any \( 1 \leq i, j \leq n_f \) and \( 1 \leq h, k \leq n_g \).

  If \( e_f(i) = 1 \) and \( e_g(h) = 1 \),

  \[
  \frac{\hat{f}(\Upsilon_1, \ldots, \Upsilon_i, \ldots, \Upsilon_n) \supset \Sigma}{\Pi \vdash \hat{g}(\Upsilon_1, \ldots, \Upsilon_i, \ldots, \Upsilon_n) \vdash \Sigma} \quad \Pi \vdash \hat{g}(\Upsilon_1, \ldots, \Upsilon_i, \ldots, \Upsilon_n) \vdash \Sigma \]

  \[
  \frac{\Pi \vdash \hat{g}(\Upsilon_1, \ldots, \Upsilon_i, \ldots, \Upsilon_n) \vdash \Sigma}{\Pi \vdash \hat{f}(\Upsilon_1, \ldots, \Upsilon_i, \ldots, \Upsilon_n) \supset \Sigma} \quad \Pi \vdash \hat{f}(\Upsilon_1, \ldots, \Upsilon_i, \ldots, \Upsilon_n) \vdash \Sigma}
  \]

  If \( e_f(j) = 0 \) and \( e_g(k) = 0 \),

  \[
  \frac{\hat{f}(\Upsilon_1, \ldots, \Upsilon_i, \ldots, \Upsilon_n) \supset \Sigma}{\Pi \vdash \hat{g}(\Upsilon_1, \ldots, \Upsilon_i, \ldots, \Upsilon_n) \vdash \Sigma} \quad \Pi \vdash \hat{g}(\Upsilon_1, \ldots, \Upsilon_i, \ldots, \Upsilon_n) \vdash \Sigma}
  \]

- **Structural rules for lattice connectives:**

  \[
  \begin{array}{c|c|c}
  \Upsilon & \top & \Sigma \\hline
  \Pi & \Pi & \Sigma \end{array}
  \]

- **Logical introduction rules for lattice connectives:**

  \[
  \begin{array}{c|c|c|c|c}
  \Upsilon & \top & \Sigma & \top & \top \\hline
  \varphi & \psi & \varphi \land \psi & \varphi & \psi \\hline
  \varphi & \varphi & \varphi \land \psi & \varphi & \varphi \\hline
  \varphi & \psi & \varphi \land \psi & \varphi & \varphi \\hline
  \varphi & \psi & \varphi \land \psi & \varphi & \varphi \\hline
  \end{array}
  \]

- **Logical introduction rules for \( f \in \mathcal{F} \) and \( g \in \mathcal{G} \):**

  \[
  \hat{f}(\Upsilon_1, \ldots, \Upsilon_n) \supset (\varphi_1 \vdash \psi_1, \ldots, \varphi_n \vdash \psi_n) \quad \frac{\Pi \vdash \hat{f}(\Upsilon_1, \ldots, \Upsilon_n)}{} \]

  \[
  \hat{g}(\Upsilon_1, \ldots, \Upsilon_n) \vdash (\varphi_1 \vdash \psi_1, \ldots, \varphi_n \vdash \psi_n) \quad \frac{\Pi \vdash \hat{g}(\Upsilon_1, \ldots, \Upsilon_n)}{} \]

  \[
  \hat{f}(\varphi_1, \ldots, \varphi_n) \vdash (\Upsilon_1, \ldots, \Upsilon_n) \supset (\varphi_1 \vdash \psi_1, \ldots, \varphi_n \vdash \psi_n) \quad \frac{\Pi \vdash \hat{f}(\varphi_1, \ldots, \varphi_n)}{} \]

  \[
  \hat{g}(\varphi_1, \ldots, \varphi_n) \vdash (\Upsilon_1, \ldots, \Upsilon_n) \vdash (\varphi_1 \vdash \psi_1, \ldots, \varphi_n \vdash \psi_n) \quad \frac{\Pi \vdash \hat{g}(\varphi_1, \ldots, \varphi_n)}{} \]

If \( f \) and \( g \) are 0-ary (i.e. they are constants), the rules \( f_R \) and \( g_L \) above reduce to the axioms (0-ary rule) \( \hat{f} \vdash f \) and \( g \vdash \hat{g} \).

---

\(^{13}\)For any \( \text{LE} \)-language \( \mathcal{L} \), we will sometimes let \( \text{D.LE} \) := \( \text{D.LE}_c \), i.e. we will let \( \text{D.LE} \)' denote the calculus obtained by instantiating the general definition of the basic calculus \( \text{D.LE}_c \) to \( \mathcal{L} := \mathcal{L}_c \).

\(^{14}\)In the display calculi literature, the identity rule is sometimes defined as \( \varphi \vdash \varphi \), where \( \varphi \) is an arbitrary, possibly complex, formula. The difference is inessential, given that, in any display calculus, \( p \vdash p \) is an instance of \( \varphi \vdash \varphi \), and \( \varphi \vdash \varphi \) is derivable for any formula \( \varphi \) whenever \( p \vdash p \) is the Identity rule.
Remark 2.18. If we admit that the sets \( \mathcal{F} \) and \( \mathcal{G} \) have a non empty intersection (c.f. Footnote 4), then the rules capturing a generic connective \( h \in (\mathcal{F} \cap \mathcal{G}) \) of arity \( n = 1 \) are as follows (notice that the notational convention \( \hat{h} \) conveys also the information that \( h \) is both a left-adjoint and a right-adjoint):

- Display postulates for \( h \in (\mathcal{F} \cap \mathcal{G}) \) occurring in precedent and in succedent position:

  If \( \varepsilon_h(1) = 1, \)
  \[
  \hat{h} \vdash h^+ \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma}
  \]
  If \( \varepsilon_h(1) = \partial, \)
  \[
  (h^+, h) \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma}
  \]

- Structural rules for \( h \in (\mathcal{F} \cap \mathcal{G}) \):

  If \( \varepsilon_h(1) = 1, \)
  \[
  \hat{h} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma}
  \]
  If \( \varepsilon_h(1) = \partial, \)
  \[
  \Pi \vdash \Sigma \quad \Pi \vdash \Sigma
  \]

- Logical introduction rules for \( h \in (\mathcal{F} \cap \mathcal{G}) \) occurring in precedent and in succedent position:

  \[
  h_L \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma}
  \]

Let \( \text{DLE} \) denote the calculus obtained by removing Cut in \( \text{DLE} \). In what follows, we indicate that the sequent \( \varphi \vdash \psi \) is derivable in \( \text{DLE} \) (resp. in \( \text{DLE} \)) by \( \vdash_{\text{DLE}} \varphi \vdash \psi \) (resp. \( \vdash_{\text{DLE}} \varphi \vdash \psi \)).

Proposition 2.19 (Soundness). The calculus \( \text{DLE} \) (hence also \( \text{DLE} \)) is sound w.r.t. the class of complete \( L \)-algebras.

Proof. The soundness of the basic lattice rules is clear. The soundness of the remaining rules is due to the monotonicity (resp. antitonicity) of the algebraic connectives interpreting each \( f \in \mathcal{F} \) and \( g \in \mathcal{G} \), and their adjunction/residuation properties, which hold since any complete \( L \)-algebra is an \( L \)-algebra.

\[ \square \]

Proposition 2.20. The calculus \( \text{DLE} \) is a proper display calculus (cf. [29, Theorem 26]), and hence cut elimination holds for it as a consequence of a Belnap-style cut elimination meta-theorem (cf. [29, Section 2.2 and Appendix A] and [27, Theorem 2]).

2.5 The setting of distributive LE-logics

In this section we discuss how the general setting presented above can account for the assumption that the given LE-logic is distributive, i.e. that the distributive laws \( (p \lor r) \land (p \lor q) \lor p \lor (r \land q) \) and \( p \land (r \lor q) \lor (p \lor r) \lor (p \lor q) \) are valid. Such logics will be referred to as DLE-logics, since they are algebraically captured by varieties of normal distributive lattice expansions (DLEs), i.e. LE-algebras as in Definition 2.1 such that \( \mathbb{L} \) is assumed to be a bounded distributive lattice.

For any (D)LE-language, the basic \( L_{\text{DLE}} \)-logic is defined as in Definition 2.2 augmented with the distributive laws above.

Since \( \land \) and \( \lor \) distribute over each other, besides being \( \Delta \)-adjoints, they can also be treated as elements of \( \mathcal{F} \) and \( \mathcal{G} \) respectively. In particular, the binary connectives \( \leftarrow \) and \( \rightarrow \) occur in the fully residuated language \( L_{\text{DLE}} \), the intended interpretations of which are the right residuals of \( \land \) in the first and second coordinate respectively, as well as the binary connectives \( \leftarrow \) and \( \rightarrow \), the intended interpretations of which are the left residuals of \( \lor \) in the first and second coordinate, respectively. Following the general convention discussed in Section 2.2 we stipulate that \( \rightarrow, \leftarrow \in \mathcal{F}^* \) and \( \rightarrow, \leftarrow \in \mathcal{G}^* \).
The basic fully residuated $L_{DLE}$-logic, which will sometimes be referred to as the basic bi-intuitionistic 'tense' DLE-logic, is given as per Definition 2.13. In particular, the residuation rules for the lattice connectives are specified as follows:

| $\phi \land \psi \land \chi$ | $\psi \land \phi \land \chi$ |
|--------------------------------|--------------------------------|
| $\psi \land \chi \land \phi \leftarrow \psi$ | $\psi \land \chi \land \phi \leftarrow \psi$ |
| $\psi \rightarrow \phi \Land \chi$ | $\psi \rightarrow \phi \Land \chi$ |
| $\phi \rightarrow \psi \Land \chi$ | $\phi \rightarrow \psi \Land \chi$ |
| $\phi \land \psi \land \chi$ | $\phi \land \psi \land \chi$ |

When interpreting LE-languages on perfect distributive lattice expansions (perfect DLEs, cf. Footnote 8), the logical disjunction is interpreted by means of the coordinatewise completely \&-preserving join operation of the lattice, and the logical conjunction with the coordinatewise completely \lor-preserving meet operation of the lattice. Hence we are justified in listing $\lor \land \land \lor$ among the SLRs, and $\lor \land \land \lor$ among the SRRs, as is done in Table 2. Consequently, the classes of (analytic) inductive $L_{DLE}$-inequalities are obtained by simply applying Definitions 2.7 and 2.8 with respect to Table 2 below.

| Skeleton | PIA |
|----------|-----|
| Δ-adjoints | SRA |
| $\lor$ with $n_e = 1$ | $\lor g$ with $n_e = 1$ |
| $\land$ with $n_e = 1$ | $\land f$ with $n_e = 1$ |
| SLR | SRR |
| $\lor$ with $n_e \geq 1$ | $\lor g$ with $n_e \geq 2$ |
| $\land$ with $n_e \geq 1$ | $\land f$ with $n_e \geq 2$ |

Table 2: Skeleton and PIA nodes for $L_{DLE}$.

Precisely because, as reported in Table 2, the nodes $\land$ and $\lor$ are now also SLR nodes, and $\lor$ and $\land$ are also SRR nodes (see also Remark 2.23), the classes of (analytic) inductive $L_{DLE}$-inequalities are strictly larger than the classes of (analytic) inductive $L_{LE}$-inequalities in the same signature, as shown in the next example.

**Example 2.21.** The inequality $\Box(\Box(p \lor q) \leq \Box \Box p \lor \Box \Box q)$ is not an inductive $L_{LE}$-inequality for any order-type, but it is an $\varepsilon$-Sahlqvist $L_{DLE}$-inequality e.g. for $\varepsilon(p, q) = (\overline{0}, \overline{1})$. The classification of nodes in the signed generation trees of $\Box(\Box(p \lor q) \leq \Box \Box p \lor \Box \Box q)$ is an $L_{DLE}$-inequality is on the left-hand side of the picture below, and the one as an $L_{LE}$-inequality is on the right (see Notation 2.13). In the classification on the right, no branch is good, therefore $\Box(\Box(p \lor q) \leq \Box \Box p \lor \Box \Box q)$ is not an inductive $L_{LE}$-inequality for any order-type.

\[
\gamma \begin{cases}
\lor \\
-\lor \bigg\{ \\
-\lor \bigg\}
\end{cases}
\]

The inequality $p \land (q \lor r) \leq q \lor (p \land r)$ is an $\varepsilon$-Sahlqvist $L_{DLE}$-inequality e.g. for $\varepsilon(p, q, r) = (1, 1, 1)$ but is not an inductive $L_{LE}$-inequality for any order-type. The classification of nodes in the signed generation trees of $p \land (q \lor r) \leq q \lor (p \land r)$ is an $L_{DLE}$-inequality is on the left-hand side of the picture below, and the one as an $L_{LE}$-inequality is on the right (see Notation 2.13). The squared variable occurrences are the $\varepsilon$-critical ones, the doubly circled nodes are Skeleton and the single-circle ones are PIA. In the classification on the right, no branch is good leading to occurrences of $r$, therefore $p \land (q \lor r) \leq q \lor (p \land r)$ is not an inductive $L_{LE}$-inequality for any order-type.

\[
\begin{array}{c}
p \land \lor \\
q \lor \land \\
r \lor \land \\
q \lor \land \\
r \lor \land
\end{array}
\]

\[
\begin{array}{c}
p \land \lor \\
q \lor \land \\
r \lor \land \\
q \lor \land \\
r \lor \land
\end{array}
\]

13Notice that $\phi \rightarrow \chi$ and $\chi \leftarrow \phi$ are interderivable for any $\phi$ and $\psi$, since $\land$ is commutative; similarly, $\psi \rightarrow \phi$ and $\psi \leftarrow \phi$ are interderivable, since $\lor$ is commutative. Hence in what follows we consider explicitly only $\rightarrow$ and $\leftarrow$.  

13
Also, definite Skeleton and definite PIA $L_{DLE}$-formulas are defined verbatim in the same way as in the setting of $L_{LE}$-formulas. Namely, $\ast \xi \ (\ast \varphi)$ is definite Skeleton (resp. definite PIA) iff all nodes of $\ast \xi \ (\ast \varphi)$ are SLR (resp. SRR). However, the classification of nodes we need to consider is now the one of Table 2, where $\ast \land$, $\ast \lor$, $\ast \top$, and $\ast \bot$ are also SRR-nodes. Definition 2.15 is specified for $\land$, $\lor$, $\top$, and $\bot$ as follows:

\[
\begin{align*}
\text{la}(\xi(x, z)) &= \text{la}(\psi(u \land \xi(z, z))); \\
\text{la}(\psi_1(x) \lor \psi_2(x, z)) &= \text{la}(\psi_2(a \land \psi_1(z, z))); \\
\text{la}(\xi(x, z) \rightarrow \psi(z)) &= \text{ra}(\xi(u \land \psi(z, z))); \\
\text{ra}(\xi(x, z) \rightarrow \psi(z)) &= \text{ra}(\xi(\psi(z) \lor u, z)); \\
\text{ra}(\xi_1(z) \land \xi_2(x, z)) &= \text{ra}(\xi_2(z) \land \xi_1(z) \rightarrow u, z); \\
\text{ra}(\xi(z) \rightarrow \psi(x, z)) &= \text{la}(\psi(\xi(z) \land u, z));
\end{align*}
\]

Finally, as to the display calculus $D_{DLE}$ for the basic $L_{DLE}$-logic, its language is obtained by augmenting the language of $DLE$ with the following structural symbols for the lattice operators and their residuals:

| Structural symbols | $\top$ | $\bot$ | $\land$ | $\lor$ | $\top$ | $\bot$ |
|-------------------|-------|-------|--------|-------|--------|-------|
| Operational symbols | $\top$ | $\bot$ | $\land$ | $\lor$ | $\top$ | $\bot$ |

Display postulates for lattice connectives and their residuals are specified as follows:

\[
\hat{\land} \vdash \Pi_1 \land \Pi_2 \vdash \Sigma \\
\Pi_2 \vdash \Pi_1 \vdash \Sigma
\]

Moreover, $D_{DLE}$ is augmented with the following structural rules encoding the characterizing properties of the lattice connectives:

\[
\begin{align*}
\tag{\text{T}_L}
\Pi \vdash \Sigma \\
\Pi \vdash \Pi \vdash \Sigma \\
\tag{\text{T}_R}
\Pi \vdash \Sigma \\
\Pi \vdash \Pi \vdash \Sigma
\end{align*}
\]

\[
\begin{align*}
\tag{\text{W}_L}
\Pi_1 \vdash \Pi_2 \vdash \Sigma \\
\Pi_1 \vdash \Pi_2 \vdash \Sigma
\end{align*}
\]

\[
\begin{align*}
\tag{\text{W}_R}
\Pi \vdash \Pi \vdash \Sigma \\
\Pi \vdash \Pi \vdash \Sigma
\end{align*}
\]

\[
\begin{align*}
\tag{\text{A}_L}
\Pi_1 \vdash \Pi_2 \vdash \Pi_3 \vdash \Sigma \\
(\Pi_1 \land \Pi_2) \vdash \Pi_3 \vdash \Sigma
\end{align*}
\]

\[
\begin{align*}
\tag{\text{A}_R}
\Pi \vdash (\Pi_1 \lor \Pi_2 \lor \Pi_3) \vdash \Sigma \\
\Pi \vdash (\Pi_1 \lor (\Pi_2 \lor \Pi_3)) \vdash \Sigma
\end{align*}
\]

and the introduction rules for the lattice connectives (and their residuals) follow the same pattern as the introduction rules of any $f \in F$ and $g \in G$:

\[
\begin{align*}
\tag{\text{L}_L}
\Pi \vdash \Pi \\
\Pi \vdash \Sigma \\
\tag{\text{L}_R}
\Pi \vdash \Pi \\
\Pi \vdash \Sigma
\end{align*}
\]

\[
\begin{align*}
\tag{\text{L}_L}
\varphi \vdash \psi \vdash \Sigma \\
\varphi \land \psi \vdash \Sigma \\
\tag{\text{L}_R}
\varphi \vdash \psi \vdash \Sigma \\
\varphi \land \psi \vdash \Sigma
\end{align*}
\]

\[
\begin{align*}
\Pi \vdash \varphi \vdash \psi \\
\Pi \vdash \varphi \lor \psi \lor \varphi \land \psi \\
\Pi \vdash \varphi
\end{align*}
\]

**Remark 2.22.** Rules $\land_L$, $\land_L$, $\land_L$, $\lor_R$, $\lor_R$ and $\lor_R$ in $DLE$ are derivable in $D_{DLE}$ as follows:

\[
\begin{align*}
\tag{\text{L}_L}
\Pi \vdash \varphi \lor \psi \lor \varphi \land \psi \\
\Pi \vdash \varphi \lor \psi \lor \varphi \land \psi \\
\Pi \vdash \varphi \lor \psi \lor \varphi \land \psi \\
\Pi \vdash \varphi \lor \psi \lor \varphi \land \psi
\end{align*}
\]

In the presence of the exchange rules $E_L$ and $E_R$, the structural connectives $\vdash$, $\vdash$ and the corresponding operational connectives $\vdash$, $\vdash$ are redundant. For simplicity, we consider languages and calculi where the operational connectives $\vdash$, $\vdash$ and their introduction rules are not included.
Remark 2.23. In what follows, we will work in the non-distributive setting with the calculus D.LE and its extensions. However, all the results we obtain about derivations in D.LE straightforwardly transfer to D.DLE using the following procedure: all applications of \( \land \_1, \land \_2, \lor \_1, \lor \_2 \) will be replaced by their derivations in D.LE (cf. Remark 2.22).

All occurrences of \( \land \) (resp. \( \lor \)) in an inductive L\(_{DLE}\)-inequality which are classified as SLR (resp. SRR) will be treated as connectives in \( T \) (resp. \( G \)).

2.6 Derivations in pre-normal form

In Section 4, we will show that any analytic inductive LE-axiom \( \varphi \vdash \psi \) can be effectively derived in the corresponding basic cut-free calculus D.LE enriched with the structural analytic rules \( R_1, \ldots, R_n \) corresponding to \( \varphi \vdash \psi \). In fact, the cut-free derivation we produce has a particular shape, referred to as pre-normal form, which we define in the present section. Informally, in a derivation in pre-normal form, a division of labour is effected on the applications of rules \( i \) some rules are applied only before the application of \( R_i \) and some rules are applied only after the application of \( R_i \).

Before moving on to the definitions, we highlight the following fact: when using ALBA to compute the analytic structural rule(s) corresponding to a given analytic inductive LE-axiom \( x \vdash y \), if \( + \land \) and \( - \lor \) occur as SRA nodes in a non-critical maximal PIA subtree of \( \varphi \vdash \psi \), then this subtree will generate two or more premises of one of the corresponding rules (depending on the number of occurrences of \( + \land \) and \( - \lor \)). If \( - \land \) and \( + \lor \) occur as \( \Delta \)-adjoints in the Skeleton of \( \varphi \vdash \psi \), then the axiom is non-definite, and by exhaustively permuting those occurrences upwards, i.e. towards the roots of the signed generation trees, and then applying the ALBA splitting rules, the given axiom can be equivalently transformed into a set of definite axioms, each of which will correspond to one analytic structural rule.

Definition 2.24. A derivation \( \pi \) in D.LE of the analytic inductive axiom \( \varphi \vdash \psi \) (also indicated as \( Ax \)) is in pre-normal form if the unique application of each rule in its corresponding set of analytic structural rules \( R_1(Ax), \ldots, R_m(Ax) \) computed by ALBA splits \( \pi \) into the following components:

\[
\begin{align*}
\text{PIA}(\pi) & \left\{ p_{11} \vdash p_{11} \cdots p_{1k} \vdash p_{1k} \cdots p_{n1} \vdash p_{n1} \cdots p_{nt} \vdash p_{nt} \right\} \\
\text{Skeleton}(\pi) & \left\{ \Pi_1^1 \vdash \Sigma_1 \cdots \Pi_1^t \vdash \Sigma_t \right\} R_1(Ax)
\end{align*}
\]

where:

(i) Skeleton(\( \pi \)) is the proof-subtree of \( \pi \) containing the root of \( \pi \) and applications of invertible rules for the introduction of all connectives occurring in the Skeleton of \( \varphi \vdash \psi \) (possibly modulo applications of display rules);

(ii) PIA(\( \pi \)) is a collection of proof-subtrees of \( \pi \) containing the initial axioms of \( \pi \) and all the applications of non-invertible rules for the introduction of connectives occurring in the maximal PIA-subtrees in the signed generation trees of \( \varphi \vdash \psi \) (possibly modulo applications of display rules) and such that

(iii) the root of each proof-subtree in PIA(\( \pi \)) coincides with a premise of the application of \( R(ax) \) in \( \pi \), where the atomic structural variables are suitably instantiated with operational maximal PIA-subtrees of \( \varphi \vdash \psi \).

Definition 2.25. A derivation \( \pi \) in D.DLE of the analytic inductive axiom \( \varphi \vdash \psi \) (also indicated as \( Ax \)) is in pre-normal form if the unique application of each rule in its corresponding set of analytic structural rules \( R_1(Ax), \ldots, R_m(Ax) \) computed by ALBA splits \( \pi \) into the following components: 17

17The name ‘pre-normal’ is intended to remind of a similar division of labour, among rules applied in derivations in normal form of the well known natural deduction systems for classical and intuitionistic logic.
where:

(i) \( \text{Skeleton}(\pi) \) is the proof-subtree of \( \pi \) containing, possibly modulo applications of display rules, the root of \( \pi \) and applications of

(a) invertible rules for the introduction of all connectives occurring as SLR nodes in the Skeleton of \( \varphi \vdash \psi \);
(b) non-invertible rules and Contraction for the introduction of all connectives occurring as \( \Delta \)-adjoint nodes in the Skeleton of \( \varphi \vdash \psi \);

(ii) \( \text{PIA}(\pi) \) is a collection of proof-subtrees of \( \pi \) containing, possibly modulo applications of display rules, the initial axioms of \( \pi \) and applications of

(a) non-invertible rules for the introduction of all connectives occurring as unary SRA nodes or as SRR nodes in the maximal PIA-subtrees in the signed generation trees of \( \varphi \vdash \psi \);
(b) invertible rules and Weakening for the introduction of all lattice connectives occurring as SRA nodes in the maximal PIA-subtrees in the signed generation trees of \( \varphi \vdash \psi \);

and such that

(iii) the root of each proof-subtree in \( \text{PIA}(\pi) \) coincides with a premise of the application of \( R(\alpha, \lambda) \) in \( \pi \), where the atomic structural variables are suitably instantiated with operational maximal PIA-subtrees of \( \varphi \vdash \psi \).

The key tools for obtaining the sub-derivations in \( \text{PIA}(\pi) \) introducing the connectives occurring as unary SRA nodes or as SRR nodes are given in Proposition 3.3 and Corollary 3.9. An inspection on the proofs of these results reveals that indeed only non-invertible logical rules and display rules are applied. The key tools involving the introduction of the lattice connectives occurring as SRA nodes in \( \text{PIA}(\pi) \) (resp. as \( \Delta \)-adjoint nodes in \( \text{Skeleton}(\pi) \)) are given in Proposition 3.6 (resp. Proposition 3.12). Again, inspecting the proofs of these results reveals that only introduction rules of one type are applied in each component.

**Remark 2.26.** The binary introduction rules of \( \text{DLE} \) for lattice connectives are invertible, while the corresponding rules of \( \text{DILE} \) are not, and Contraction is needed to derive these rules of \( \text{DLE} \) in \( \text{DILE} \). Likewise, the unary introduction rules of \( \text{DLE} \) for lattice connectives are not invertible, while the corresponding rules of \( \text{DILE} \) are, and so Weakening is needed to derive these rules of \( \text{DLE} \) in \( \text{DILE} \). This is why derivations in pre-normal form of analytic inductive axioms in the general lattice setting of Definition 2.24 can be described purely in terms of invertible and non-invertible introduction rules, while in the distributive lattice setting of Definition 2.25, the occurrences of lattice connectives in \( \Delta \)-adjoint/SRA-position in the signed generation trees of a given analytic inductive axiom need to be accounted for separately (cf. clauses (b) of Definition 2.25). However, if \( \varphi \vdash \psi \) is an analytic inductive axiom in the general lattice setting, applying the process described in Remark 2.23 to a derivation of \( \varphi \vdash \psi \) in \( \text{DLE} \) in pre-normal form according to Definition 2.24 results in a derivation of \( \varphi \vdash \psi \) in \( \text{DILE} \) which is in pre-normal form according to Definition 2.25.

**Remark 2.27.** If \( \varphi \vdash \psi \) is a definite analytic inductive axiom, then ALBA yields a single analytic structural rule corresponding to it. So, both in the general lattice and in the distributive settings, the Skeleton part of the derivation of \( \varphi \vdash \psi \) in pre-normal form will only have one branch, yielding the following simpler shape of \( \pi \):
All derivations in Examples 4.7 and 4.11 are derivations of definite analytic inductive axioms in pre-normal form.

## 3 Properties of the basic display calculi \( D \text{LE} \)

In this section, we will state and prove the key lemmas needed for the proof of the syntactic completeness. Throughout this section, we let \( \mathcal{L}_{\text{LE}} \) (resp. \( \mathcal{L}_{\text{DLE}} \)) be an arbitrary but fixed (D)LE-language, and D.LE (resp. D.DLE) denote the proper display calculi for the basic \( \mathcal{L}_{\text{LE}} \)-logic (resp. \( \mathcal{L}_{\text{DLE}} \)-logic).

### Notation 3.1
For any definite Skeleton (resp. definite PIA) formula \( \varphi \) (resp. \( \psi, \gamma, \delta, \xi, \ldots \)), we let its corresponding capital Greek letter \( \Phi \) (resp. \( \Psi, \Gamma, \Delta, \Xi, \ldots \)) denote its structural counterpart, defined by induction as follows (cf. Notation 2.6):

1. if \( \varphi := p \in \text{AtProp} \), then \( \Phi := p \);
2. if \( \varphi := f(\xi, \psi) \), then \( \Phi := f(\Xi, \Psi) \);
3. if \( \varphi := g(\psi, \xi) \), then \( \Phi := g(\Psi, \Xi) \).

**Proposition 3.2.** In what follows, we let \( \sigma, S \) and \( \sigma + S \) (resp. \( U, \tau \) and \( U + \tau \)) denote finite vectors of formulas, of structures in \( \text{Str}_G \) (resp. \( \text{Strr} \) and of \( D \text{(D)} \text{LE} \)-sequents).

**Proposition 3.3.** For every definite positive PIA (i.e. definite negative Skeleton) formula \( \gamma(\mathfrak{x}, \mathfrak{y}) \) and every definite negative PIA (i.e. definite positive Skeleton) formula \( \delta(\mathfrak{x}, \mathfrak{y}) \),

1. if \( \sigma + S \) and \( \tau + U \) are derivable in \( D \text{LE} \) (resp. \( D \text{DLE} \)), then so is \( \gamma(\sigma, \mathfrak{x}, \tau, \mathfrak{y}) \) and \( \Gamma(\sigma, \mathfrak{x}, \tau, \mathfrak{y}) \);
2. if \( \sigma + S \) and \( \tau + U \) are derivable in \( D \text{LE} \) (resp. \( D \text{DLE} \)), then so is \( \Delta(\sigma, \mathfrak{x}, \tau, \mathfrak{y}) \) and \( \delta(\sigma, \mathfrak{x}, \tau, \mathfrak{y}) \).

with derivations such that, if any rules are applied other than right-introduction rules for negative SRR-connectives and negative unary SRA-connectives (cf. Tables 1 and 2 and Definition 2.5), and left-introduction rules for positive SRR-connectives and positive unary SRA-connectives, then they are applied only in the derivations of \( \sigma + S \) and \( \tau + U \).

**Proof.** By simultaneous induction on \( \gamma \) and \( \delta \). If \( \gamma := x \), then \( \Gamma := x \). Hence, \( \gamma(\sigma, \mathfrak{x}, \tau, \mathfrak{y}) \) reduces to \( \sigma + S \), which is derivable by assumption. The case of \( \delta := x \) is shown similarly. As to the inductive steps, let \( \gamma(\mathfrak{x}, \mathfrak{y}) := g(\mathfrak{y}, \mathfrak{x}) \) with \( \psi \) definite positive PIA-formulas and \( \xi \) definite negative PIA-formulas. Then \( \gamma(\mathfrak{x}, \mathfrak{y}) = g(\mathfrak{y}, \mathfrak{x}) \) and \( \delta(\mathfrak{x}, \mathfrak{y}) \) and \( \Delta(\mathfrak{x}, \mathfrak{y}) \) and \( \delta(\mathfrak{x}, \mathfrak{y}) \).

By induction hypothesis, all sequents in the following vectors are derivable in \( D \text{LE} \) (resp. \( D \text{DLE} \)):

\[
\psi(\sigma, \mathfrak{x}, \tau, \mathfrak{y}) \quad \text{and} \quad \xi(\sigma, \mathfrak{x}, \tau, \mathfrak{y}).
\]

Then we can derive the required sequent \( \gamma(\sigma, \mathfrak{x}, \tau, \mathfrak{y}) \) by prolonging all these derivations with an application of \( g_L \) as follows:

\[
g_L(\sigma, \mathfrak{x}, \tau, \mathfrak{y}) = g(\mathfrak{y}, \mathfrak{x}).
\]

Let \( \delta(\mathfrak{x}, \mathfrak{y}) := f(\xi(\mathfrak{x}, \mathfrak{y}), \psi(\mathfrak{x}, \mathfrak{y})) \) with \( \xi \) definite negative PIA-formulas (i.e. positive Skeleton-formulas) and \( \psi \) definite positive PIA-formulas (i.e. negative Skeleton-formulas). Then \( \delta(\mathfrak{x}, \mathfrak{y}) \) and \( \delta(\mathfrak{x}, \mathfrak{y}) \) and \( \Delta(\mathfrak{x}, \mathfrak{y}) \) and \( \delta(\mathfrak{x}, \mathfrak{y}) \). By induction hypothesis, all sequents in the following vectors are derivable in \( D \text{LE} \) (resp. \( D \text{DLE} \)):

\[
\psi(\sigma, \mathfrak{x}, \tau, \mathfrak{y}) \quad \text{and} \quad \xi(\sigma, \mathfrak{x}, \tau, \mathfrak{y}).
\]
Then we can derive the required sequent $\Delta([U/\xi, S/\tau] \vdash \delta[\tau/\xi, \tau/\xi])$ by prolonging all these derivations with an application of $f_R$ as follows:

$$\xi([U/\xi, S/\tau] \vdash \xi[\tau/\xi, \tau/\tau]) \quad \psi([\tau/\tau, \tau/\tau] \vdash \psi[\tau/\tau, U/\tau])$$

The proof, specific to the setting of D.DLE, of the case in which $\gamma := \gamma_1 \lor \gamma_2$ (resp. $\delta := \delta_1 \land \delta_2$) goes like the case of arbitrary $g \in G$ (resp. $f \in F$) discussed above, using the D.DLE-rule $\lor_L$ (resp. $\land_R$).

By instantiating $\sigma \vdash S$ and $U \vdash \tau$ in the proposition above to identity axioms, we immediately get the following

**Corollary 3.4.** Any calculus D.L.E (resp. D.DLE) derives the following sequents (cf. Notation 3.1):

1. $\gamma \vdash \Gamma$ for every definite positive PIA (i.e. definite negative Skeleton) formula $\gamma$;
2. $\Delta \vdash \delta$ for every definite negative PIA (i.e. definite positive Skeleton) formula $\delta$,

with derivations which only consist of identity axioms, and applications of right-introduction rules for negative SRR-connectives and negative unary SRA-connectives (cf. Tables 7 and 2, and Definition 2.5), and left-introduction rules for positive SRR-connectives and unary SRA-connectives.

**Example 3.5.** The formula $\Diamond(p \otimes q) \rightarrow (q \oplus p)$ is definite positive PIA in any (D)LE-language such that $\Diamond, \otimes \in F$ and $\otimes, \rightarrow \in G$ with $n_1 = 1$ and $\varepsilon(1) = 1$, and $n_0 = n_\Diamond = n_{\rightarrow} = 2$ and $\varepsilon(i) = 1$ for every $\diamond \in \{\Diamond, \otimes, \rightarrow\}$ and every $1 \leq i \leq 2$ except $\varepsilon_{\rightarrow}(1) = \delta$. Then, instantiating the argument above, we can derive the sequent $\Diamond(p \otimes q) \rightarrow (q \oplus p) + \Diamond(p \otimes q) \rightarrow (q \oplus p)$ in D.L.E (resp. D.DLE) as follows:

$$\frac{p \vdash p \quad q \vdash q}{p \otimes q \vdash p \otimes q} \quad \frac{p \vdash p \quad q \vdash q}{p \oplus q \vdash p \oplus q}$$

The formula $\Diamond(p \otimes q)$ in the same language is definite negative PIA. Then, instantiating the argument above, we can derive the sequent $\Diamond(p \otimes q) \vdash \Diamond(p \otimes q)$ in D.L.E (resp. D.DLE) as follows:

$$\frac{p \vdash p}{\Diamond p \vdash \Diamond p} \quad \frac{q \vdash q}{\Diamond p \vdash \Diamond p}$$

**Proposition 3.6.** Let $\gamma = \gamma(\tau, \tau)$ and $\delta = \delta(\tau, \tau)$ be a positive and a negative PIA formula, respectively, and let $\land_{i \in I} \gamma_i$ and $\lor_{j \in J} \delta_j$ be equivalent rewritings as per Lemma 2.3, so that each $\gamma_i$ (resp. each $\delta_j$) is definite positive (resp. negative) PIA.

1. If $\sigma \vdash S$ and $U \vdash \tau$ are derivable in D.L.E (resp. D.DLE), then so is $\gamma(\sigma/\tau, \tau/\tau) \vdash \Gamma(\sigma/\tau, \tau/\tau)$ for each $i \in I$;
2. If $\sigma \vdash S$ and $U \vdash \tau$ are derivable in D.L.E (resp. D.DLE), then so is $\Delta(\sigma/\tau, \tau/\tau) \vdash \delta(\tau/\tau, \tau/\tau)$ for each $j \in J$,

with derivations such that, if any rules are applied other than right-introduction rules for negative PIA-connectives (cf. Tables 7 and 2, and Definition 2.5), and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of D.DLE), then they are applied only in the derivations of $\sigma \vdash S$ and $U \vdash \tau$.

**Proof.** Let $n_\gamma(\land)$ (resp. $n_\delta(\lor)$) be the number of occurrences of $\land$ in $+\gamma$ (resp. $-\delta$), and let $n_\gamma(\lor)$ (resp. $n_\delta(\land)$) be the number of occurrences of $\lor$ in $+\gamma$ (resp. $-\delta$). The proof is by simultaneous induction on $n_\gamma = n_\gamma(\land) + n_\gamma(\lor)$ and $n_\delta = n_\delta(\lor) + n_\delta(\land)$.

If $n_\gamma = n_\delta = 0$, then $\gamma$ (resp. $\delta$) is definite positive (resp. negative) PIA. Then the claims follow from Proposition 3.3.

If $n_\gamma \geq 1$, then let us consider one occurrence of $\land$ or $\lor$ in $+\gamma$, which we will refer to as 'the focal occurrence'. Let us assume that the focal occurrence of $\land$ or $\lor$ in $+\gamma$ is an occurrence of $\lor$ (the case in which it is an occurrence of $\land$ is argued similarly).

Let $\xi'$ and $\xi''$ be the two subtrees under the focal occurrence of $\lor$. Then $\xi' \lor \xi''$ is a subformula of $\gamma$ such that $\xi'$ and $\xi''$ are negative PIA formulas, and $n_{\xi'}$ and $n_{\xi''}$ are strictly smaller than $n_\gamma$. Let $u$ be a fresh variable which does not occur in $\gamma$, and let $\gamma'$ be the formula obtained by substituting the occurrence of $\xi' \lor \xi''$ in $\gamma$ with $u$. Then $\gamma'$ is a positive
PIA formula such that \( n_r \) is strictly smaller than \( n_r \), and \( \gamma = \gamma'((\xi' \vee \xi'')/u) \). Let \( \bigwedge_{i \in J} \gamma_i \cdot \bigwedge_{j \in J} \gamma_j' \cdot \bigwedge_{h \in H} \xi'_h \) and \( \bigwedge_{k \in K} \xi''_k \) be the equivalent rewritings of \( \gamma, \gamma', \xi' \) and \( \xi'' \), respectively, resulting from distributing exhaustively \(+ \wedge \) and \( -\vee \) over each connective in \( \gamma, \gamma', \xi' \) and \( \xi'' \). Then,

\[
\{ \gamma_i \mid i \in I \} = \{ \gamma'_j /u \} \mid j \in J \text{ and } h \in H \} \cup \{ \gamma''_k /u \} \mid j \in J \text{ and } k \in K \}.
\]

By induction hypothesis, the following sequents are derivable in DLE (resp. D.DLE) for every \( h \in H \) and \( k \in K \):

\[
\Xi_h [\Gamma / \psi, \delta / \xi / \chi] + \xi' [\psi / \psi, \delta / \xi / \chi] \quad \text{and} \quad \Xi_h [\Gamma / \psi, \delta / \xi / \chi] + \xi'' [\psi / \psi, \delta / \xi / \chi].
\]

Then, by prolonging the derivations of the two sequents above with suitable applications of \((\vee R_1)\) and \((\vee R_2)\), we obtain derivations in DLE (resp. D.DLE)\(^{10}\) of the following sequents for every \( h \in H \) and \( k \in K \):

\[
\Xi_h [\Gamma / \psi, \delta / \xi / \chi] + (\xi' \vee \xi'') [\psi / \psi, \delta / \xi / \chi] \quad \text{and} \quad \Xi_h [\Gamma / \psi, \delta / \xi / \chi] + (\xi' \vee \xi'') [\psi / \psi, \delta / \xi / \chi].
\] (3.1)

By induction hypothesis, \( \xi' \) are also derivable in DLE (resp. D.DLE) for every \( j \in J, h \in H \) and \( k \in K \):

\[
\gamma / \psi, \delta / \xi / \chi, (\xi' \vee \xi'') + \Gamma_j, \delta / \xi / \chi, \Xi_h [\Gamma / \psi, \delta / \xi / \chi] !u \quad \text{and} \quad \gamma / \psi, \delta / \xi / \chi, (\xi' \vee \xi'') + \Gamma_j, \delta / \xi / \chi, \Xi_h [\Gamma / \psi, \delta / \xi / \chi] !u.
\]

which is enough to prove the statement, since \( \gamma = \gamma'((\xi' \vee \xi'')/u) \), and for every \( i \in I \), either \( \gamma_i = \gamma'_j [\xi'_h /u] \) for some \( j \in J \) and \( h \in H \), or \( \gamma_i = \gamma''_k [\xi''_k /u] \) for some \( j \in J \) and \( k \in K \). The induction step for \( n_h \geq 1 \) is similar to the induction step above.

\( \square \)

By instantiating \( \sigma = S \) and \( U = \tau \) in the proposition above to identity axioms, we immediately get the following

**Corollary 3.7.** For any positive (resp. negative) PIA formula \( \gamma \) (resp. \( \delta \)), let \( \bigwedge_{i \in J} \gamma_i \) (resp. \( \bigwedge_{j \in J} \delta_j \)) be its equivalent rewriting as per Lemma 2.9, so that each \( \gamma_i \) (resp. \( \delta_j \)) is definite positive (resp. negative) PIA. Then the following sequents are derivable in DLE (resp. D.DLE, cf. Notation 3.7):

1. \( \gamma \vdash \Gamma_i \) for every \( i \in I \);
2. \( \Delta_j \vdash \delta \) for every \( j \in J \);

with derivations which only consist of identity axioms, and applications of right-introduction rules for negative PIA-connectives (cf. Tables 7 and 2 and Definition 2.5), and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of D.DLE).

**Example 3.8.** The formula \( \Box (p \wedge q) \) is a positive PIA in any (D)LE-language such that \( \Box \in \mathcal{G} \), and is equivalent to \( \Box p \wedge \Box q \). Since \( p \vdash \Box p \) and \( q \vdash \Box q \) are derivable sequents

\[
\begin{align*}
p \vdash p &\quad \phi R \quad \Box p \vdash \Box p &\quad \Box R \quad q \vdash q &\quad \Box q \vdash \Box q \quad \Box q
\end{align*}
\]

instantiating the argument on Proposition 3.6, we can derive the sequents \( \Box (p \wedge q) \vdash \Box \Box p \) and \( \Box (p \wedge q) \vdash \Box \Box q \) in D.LE and in D.DLE as follows:

\[
\begin{array}{c}
\Xi_h [\Gamma / \psi, \delta / \xi / \chi] + \xi' [\psi / \psi, \delta / \xi / \chi] \quad \text{and} \quad \Xi_h [\Gamma / \psi, \delta / \xi / \chi] + \xi'' [\psi / \psi, \delta / \xi / \chi].
\end{array}
\]
In the remainder of the present section, if \( \varphi(\bar{x}, \bar{y}) \) (resp. \( \psi(\bar{y}, \bar{z}) \)) is a definite positive (resp. negative) PIA formula, we will need to fix one variable in \( \bar{x} \) or in \( \bar{y} \) and make it the pivotal variable for the computation of the corresponding \( \text{la}(\varphi)(u, \bar{x}) \) (resp. \( \text{ra}(\psi)(u, \bar{z}) \)), where the vector \( \bar{x} \) of parametric variables exactly includes all the placeholder variables in \( \bar{x} \) and in \( \bar{y} \) different from the pivotal one. So we write e.g. \( \varphi_x \) (resp. \( \varphi_y \)) to indicate that we are choosing the pivotal variable among the variables in \( \bar{x} \) (resp. \( \bar{y} \)).

In order to simplify the notation, we leave it to be understood that the set of parametric variables does not contain the pivotal one, although we do not make this fact explicit in the notation. In the remainder of the paper, we will let e.g. \( \text{LA}(\varphi)(u, \bar{x}) \) denote the structural counterpart of \( \text{la}(\varphi)(u, \bar{x}) \) (cf. Definition 2.5).

**Corollary 3.9.** Let \( \psi(\bar{x}, \bar{y}) \) and \( \xi(\bar{y}, \bar{z}) \) be a positive and a negative PIA formula respectively, and let \( \bigwedge j \psi_j \) and \( \bigvee j \xi_j \) be their equivalent rewritings as per Lemma 2.9 so that each \( \psi_j \) (resp. \( \xi_j \)) is a definite positive (resp. negative) PIA formula. Then:

1. If \( \sigma \vdash S \) and \( U \vdash \tau \) are derivable in D.LE (resp. D.DLE), then so is \( \text{LA}(\psi_j)[\psi_j(\bar{y}, \bar{z}, \bar{y})]/u, \bar{x}, \bar{y}, U/\bar{y}] \vdash S_x \), where \( \psi_j \) is the definite positive PIA formula in which the pivotal variable \( x \) occurs;

2. If \( \sigma \vdash S \) and \( U \vdash \tau \) are derivable in D.LE (resp. D.DLE), then so is \( U_j \vdash \text{LA}(\psi_j)[\psi_j(\bar{y}, \bar{y}, \bar{y})]/u, \bar{x}, \bar{y}, \bar{y}] \vdash S_x \), where \( \psi_j \) is the definite positive PIA formula in which the pivotal variable \( y \) occurs;

3. If \( \sigma \vdash S \) and \( U \vdash \tau \) are derivable in D.LE (resp. D.DLE), then so is \( \text{RA}(\xi_j)[\xi_j(\bar{y}, \bar{y}, \bar{z})]/u, \bar{y}, \bar{y}, \bar{z}] \vdash S_x \), where \( \xi_j \) is the definite negative PIA formula in which the pivotal variable \( x \) occurs;

4. If \( \sigma \vdash S \) and \( U \vdash \tau \) are derivable in D.LE (resp. D.DLE), then so is \( U_j \vdash \text{RA}(\xi_j)[\xi_j(\bar{y}, \bar{y}, \bar{z})]/u, \bar{y}, \bar{y}, \bar{z}] \vdash S_x \), where \( \xi_j \) is the definite negative PIA formula in which the pivotal variable \( y \) occurs,

with derivations such that, if any rules are applied other than display rules, right-introduction rules for negative PIA-connectives (cf. Tables 1 and 2 and Definition 2.5), and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of D.DLE), then they are applied only in the derivations of \( \sigma \vdash S \) and \( U \vdash \tau \).

**Proof.** 1. Let \( \Psi_j \) denote the structural counterpart of \( \psi_j \) (cf. Notation 3.1). The assumptions imply, by Proposition 3.6, that the sequent \( \psi_j[\bar{y}, \bar{y}, \bar{z}]/u, \bar{y}, \bar{y}] \vdash \Psi_j[S/\bar{x}, \bar{x}, \bar{y}] \) is derivable in D.LE (resp. D.DLE) with a derivation such that, if any rules are applied other than right-introduction rules for negative PIA-connectives and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of D.DLE), then they are applied only in the derivations of \( \sigma \vdash S \) and \( U \vdash \tau \). Then, we can prolong this derivation by applying display rules to each node of the branch of \( \Psi_j \), leading to the pivotal variable \( x \), so as to obtain a derivation of the required sequent \( \text{LA}(\psi_j)[\psi_j(\bar{y}, \bar{y}, \bar{y})]/u, \bar{x}, \bar{x}, \bar{y}] \vdash S_x \), the remaining items are proved similarly.

By instantiating \( \sigma \vdash S \) and \( U \vdash \tau \) in the corollary above to identity axioms, we immediately get the following

**Corollary 3.10.** The following sequents are derivable in D.LE (resp. D.DLE) for any positive PIA (i.e. negative Skeleton) formula \( \psi(\bar{x}, \bar{y}) \) and any negative PIA (i.e. positive Skeleton) formula \( \xi(\bar{y}, \bar{z}) \) such that \( \bigwedge j \psi_j \) and \( \bigvee j \xi_j \) are their equivalent rewritings as per Lemma 2.9 so that each \( \psi_j \) (resp. \( \xi_j \)) is a definite positive (resp. negative) PIA formula.
1. \[ \text{LA}(\phi_j)[\psi_i/u] \vdash x, \] where \( \phi_i \) is the definite positive PIA formula in which the pivotal variable \( x \) occurs;  
2. \( y \vdash \text{LA}(\phi_j)[\psi_i/u] \), where \( \phi_i \) is the definite positive PIA formula in which the pivotal variable \( y \) occurs;  
3. \[ \text{RA}(\xi_j)[\psi_i/u] \vdash x, \] where \( \xi_j \) is the definite negative PIA formula in which the pivotal variable \( x \) occurs;  
4. \( y \vdash \text{RA}(\xi_j)[\psi_i/u] \), where \( \xi_j \) is the definite negative PIA formula in which the pivotal variable \( y \) occurs,  

derivations which only consist of identity axioms, and applications of display rules, right-introduction rules for negative PIA-connectives (cf. Tables 1 and 2 and Definition 2.5), and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of D.DLE).

Example 3.11. The formula \( \Box\Box[p \land q] \lor r \) is a positive PIA in any (D)LE-language such that \( \Box, \lor \in G \), and is equivalent to \( \Box\Box(p \lor r) \land \Box\Box(q \lor r) \). Let \( x = p \), then \( \text{LA}(\Box\Box[p \lor r])\Box\Box((p \land q)) \lor r) \models \phi = \Box\Box((p \land q)) \lor r \quad \prec \quad r \), where \( \prec \) is the left residual of \( \lor \) on the first coordinate. As shown in Example 3.3, \( p \models \Box \lor p \) and \( r \vdash \Box \lor r \) are derivable sequents. Instantiating the argument in Corollary 3.9 we can derive the sequent \( \Box\Box((p \land q) \lor r) \vdash \Box\Box\lor p \) as follows:

D.DLE-derivation of \( \Box\Box((p \land q) \lor r) \vdash \Box\Box\lor p \):

\[
\begin{align*}
\frac{p \vdash p}{\Box p \lor p} & \quad \varnothing R \\
p \vdash \Box p & \quad \Box R \\
\Box(p \land q) \vdash \Box p & \quad \Box L \\
r \vdash r & \quad \Box R \\
\Box(p \land q) \lor r & \vdash \Box \Box p \lor \Box r & \quad \Box R \\
\Box \Box((p \land q) \lor r) & \vdash \Box \Box p \lor \Box r & \quad \Box R \\
\end{align*}
\]

Proposition 3.12. Let \( \varphi = \varphi(\Xi, \Psi) \) and \( \psi = \psi(\Xi, \Psi) \) be a positive and a negative Skeleton formula, respectively, and let \( \bigwedge_{j \in J} \varphi_j \) and \( \bigwedge_{i \in I} \psi_i \) be their equivalent rewritings as per Lemma 2.9 so that each \( \varphi_j \) (resp. each \( \psi_i \)) is definite positive (resp. negative) Skeleton. Then:

1. if \( \Phi_j(\Xi / \Psi, \Xi / \Psi) \vdash \Sigma \) for every \( j \in J \), then so is \( \varphi(\Xi / \Psi, \Xi / \Psi) \vdash \Sigma \);  
2. if \( \Pi \vdash \Psi_i(\Xi / \Psi, \Xi / \Psi) \) is derivable in D.DLE (resp. D.DLE) for every \( i \in I \), then so is \( \Pi \vdash \psi(\Xi / \Psi, \Xi / \Psi) \).

with derivations such that, if any rules are applied other than display rules, left-introduction rules for positive Skeleton-connectives (cf. Tables 1 and 2 and Definition 2.5), right-introduction rules for negative Skeleton-connectives, (and contraction in the case of D.DLE), then they are applied only in the derivations of \( \Phi_j(\Xi / \Psi, \Xi / \Psi) \vdash \Sigma \) and \( \Pi \vdash \Psi_i(\Xi / \Psi, \Xi / \Psi) \).
Proof. Let \( n_\varphi (+\lor) \) (resp. \( n_\varphi (+\lor, 0) \)) be the number of occurrences of \( +\lor \) in \( +\varphi \) (resp. \( -\varphi \)), and let \( n_\varphi (-\land) \) (resp. \( n_\varphi (-\land, 0) \)) be the number of occurrences of \( -\land \) in \( +\varphi \) (resp. \( -\varphi \)). The proof is by simultaneous induction on \( n_\varphi = n_\varphi (+\lor) + n_\varphi (-\land) \) and \( n_\varphi = n_\varphi (+\lor, 0) + n_\varphi (-\land) \).

If \( n_\varphi = n_\varphi = 0 \), then \( \varphi \) (resp. \( -\varphi \)) is definite positive (resp. negative) Skeleton. Then from a derivation of \( \Phi([\varphi]_*/[\tau]_*/[\tau]_*/[\tau]) \) we obtain a derivation of \( \varphi([\varphi]_*/[\tau]_*/[\tau]_*/[\tau]) \) by applications of left-introduction rules for positive SLR-connecitives, and right-introduction rules for negative SLR-connecitives, interleaved with applications of display rules.

If \( n_\varphi = 1 \), then let us consider one occurrence of \( -\land \) or \( +\lor \) in \( -\varphi \), which we will refer to as ‘the focal occurrence’. Let us assume that the focal occurrence of \( -\land \) or \( +\lor \) in \( -\varphi \) is an occurrence of \( +\lor \) (the case in which it is an occurrence of \( -\land \) is argued similarly). Let \( +\xi' \) and \( +\xi'' \) be the two subtrees under the focal occurrence of \( +\lor \). Then \( +\xi' \lor +\xi'' \) is a subformula of \( \varphi \) such that \( +\xi' \) and \( +\xi'' \) are positive Skeleton formulas, and \( n_\varphi + +\xi' \) and \( n_\varphi + +\xi'' \) are strictly smaller than \( n_\varphi \). Let \( u \) be a fresh variable which does not occur in \( \varphi \), and let \( +\xi' \) be the formula obtained by substituting the occurrence of \( +\xi' \lor +\xi'' \) in \( \varphi \) with \( u \). Then \( +\xi' \) is a negative Skeleton formula such that \( n_\varphi' \) is strictly smaller than \( n_\varphi \), and \( \varphi = +\psi'([\xi' \lor \xi'']/u]) \).

Let \( \land_{\land \varphi} \psi_1 \land_{\lor \varphi} \psi_2 \land_{\lor \varphi} \psi_3 \land_{\lor \varphi} \psi_4 \land_{\lor \varphi} \psi_5 \land_{\lor \varphi} \psi_6 \land_{\lor \varphi} \psi_7 \land_{\lor \varphi} \psi_8 \) be the equivalent rewritings of \( \psi_1, \psi_2, \xi', \xi'' \), respectively, resulting from applying Lemma \ref{lem:rewriting} to \( \psi, \psi' \), \( \xi', \xi'' \), respectively. Then,

\[
\{ \psi_i \mid i \in I \} = \{ [\psi_i]/[\xi_i'/u] \mid j \in J \} \cup \{ [\psi_i]/[\xi_i'/u] \mid j \in J \} \cup \{ [\psi_i]/[\xi_i'/u] \mid j \in J \} \cup \{ [\psi_i]/[\xi_i'/u] \mid j \in J \}.
\]

Hence, the assumptions can be equivalently reformulated as the following sequents being derivable in DLE (resp. DLE)

\[
\Pi \vdash \Psi_j ([\xi_j'/u]) \quad \Pi \vdash \Psi_j ([\xi_j'/u]).
\]

By prolonging those derivations with consecutive applications of display rules, we obtain derivations in DLE (resp. DLE) of the following sequents, for every \( j \in J \), \( h \in H \), and \( k \in K \):

\[
\Xi_{\xi'} + \Lambda(\psi_2)[[\Psi_j'/v]] \quad \Xi_{\xi''} + \Lambda(\psi_2)[[\Psi_j'/v]].
\]

Hence, by induction hypothesis on \( +\xi' \) and \( +\xi'' \), the following sequents are derivable in DLE (resp. DLE) for every \( j \in J \):

\[
+\xi' + \Lambda(\psi_2)[[\Psi_j'/v]] \quad +\xi'' + \Lambda(\psi_2)[[\Psi_j'/v]].
\]

Then, by prolonging the derivations of the two sequents above with suitable applications of \((\lor_2)\), we obtain derivations in DLE (resp. DLE) of the following sequents for every \( j \in J \):

\[
+\xi' + \Lambda(\psi_2)[[\Psi_j'/v]] + \Lambda \left( +\psi_1 \right)[[[\tau]/[\tau]/[\tau]/[\tau]]] \quad +\xi'' + \Lambda(\psi_2)[[\Psi_j'/v]] + \Lambda \left( +\psi_1 \right)[[[\tau]/[\tau]/[\tau]/[\tau]]],
\]

By prolonging the derivations above with consecutive applications of display rules, we obtain derivations in DLE (resp. DLE) of the following sequents for every \( j \in J \):

\[
\Pi \vdash \Psi_j ([\xi_j'/v], [\xi_j'/v], [\xi_j'/v], [\xi_j'/v], [\xi_j'/v]).
\]

By induction hypothesis on \( +\psi \), and recalling that \( +\psi'([\xi' \lor \xi'']/u]) \), we can conclude that \( +\psi([\tau'/[\tau]/[\tau]/[\tau]]) \) is derivable, as required.

\[\square\]

**Example 3.13.** The formula \( +p \lor +p \) is a negative PIA in any (D)LE-language such that \( + \in \mathcal{F} \), and is equivalent to \( +p \lor +p \). Assuming that \( +p \) and \( +p \) are derivable sequences, instantiating the argument in Proposition 3.12, we can derive the sequent \( +p \lor +p \) in DLE as follows:

\[
\begin{array}{c}
\vdash +p \lor +p \\
\vdash +p \lor +p \\
\vdash +p \lor +p \\
\vdash +p \lor +p \\
\end{array}
\]

In the calculus DLE, we obtain the derivations of the sequents in \( \Pi \) by replacing the applications of the derivable rules \((\lor_1)\) with the derivations of those applications as shown in Remark 3.2, thereby implementing the general strategy outlined in Remark 3.2 which in this specific case involves the application of structural contraction rule.
4 Syntactic completeness

In the present section, we fix an arbitrary LE-language $\mathcal{L}_{LE}$, for which we prove our main result (cf. Theorem 4.10), via an effective procedure which generates cut-free derivations in pre-normal form (cf. Definitions 2.24 and 2.25) of any analytic inductive $\mathcal{L}_{LE}$-sequent in $\mathcal{D}_{LE}$ (resp. $\mathcal{D}_{DE}$) augmented with the analytic structural rule(s) corresponding to the given sequent. In Section 4.1 we will first illustrate some of the main ideas of the proof in the context of a proper subclass of analytic inductive sequents, which we refer to as quasi-special inductive. Then in Section 4.2 we state and prove this result for arbitrary analytic inductive sequents.

Notation 4.1. In this section, we will often deal with vectors of formulas $\vec{\gamma}$ and $\vec{\delta}$ such that each $\gamma$ in $\vec{\gamma}$ (resp. $\delta$ in $\vec{\delta}$) is a positive (resp. negative) PIA formula, and hence, by Lemma 2.9, is equivalent to $\bigwedge_\lambda \gamma^\lambda$ (resp. $\bigvee_\mu \delta^\mu$). To avoid overloading notation, we will slightly abuse it and write $\gamma^\lambda$ (resp. $\delta^\mu$), understanding that, for each element of these vectors, each $\lambda$ and $\mu$ range over different sets.

4.1 Syntactic completeness for quasi-special inductive sequents

Definition 4.2. For every analytic ($\Omega, \epsilon$)-inductive inequality $s \leq t$, if every $\epsilon$-critical branch of the signed generation trees $+s$ and $-t$ consists solely of Skeleton nodes, then $s \leq t$ is a quasi-special inductive inequality. Such an inequality is definite if none of its Skeleton nodes is $+\lor$ or $-\land$.

In terms of the convention introduced in Notation 2.11, quasi-special inductive sequents can be represented as $(\varphi \vdash \psi)[\alpha, \beta, \gamma, \delta]$, i.e. as those $(\varphi \vdash \psi)[\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}]$ such that each $\alpha$ in $\vec{\alpha}$ and $\beta$ in $\vec{\beta}$ is an atomic proposition.

Example 4.3. Let $\mathcal{L} := \mathcal{L}(\mathcal{F}, \mathcal{G})$, where $\mathcal{F} := \{\land, \otimes\}$ and $\mathcal{G} := \{\lor, \oplus, \Box\}$. The $\mathcal{L}_{LE}$-inequality $\Diamond p \leq \Box \Diamond p$, known in the modal logic literature as axiom 5, is a definite quasi-special inductive inequality, e.g. for $<_{\Omega} = \emptyset$ and $\epsilon(p) = 1$, as can be seen from the signed generation tree below (see Notation 2.13):

The $\mathcal{L}_{LE}$-inequality $p \otimes (p \otimes \Box q) \leq q \oplus (q \oplus \Diamond p)$ is a definite quasi-special inductive inequalities, e.g. for $p <_{\Omega} q$ and $\epsilon(p, q) = (1, \partial)$, as can be seen from the signed generation tree below (cf. Notation 2.13):
The $L_{DK}$-inequality $\Diamond (p \vee \Diamond p) \leq \Box \Diamond p$ is a *non-definite* quasi-special inductive inequality, e.g. for $<_{\Omega} = \emptyset$ and $e(p) = 1$, as can be seen from the signed generation tree below (see Notation 2.13):

![Diagram showing the signed generation tree for the $L_{DK}$-inequality $\Diamond (p \vee \Diamond p) \leq \Box \Diamond p$.]

Finally, in the distributive case, the $L_{DK}$-inequality $p \wedge q \leq q \vee p$, is a definite quasi-special inductive inequalities, e.g. for $p <_{\Omega} q$ and $e(p, q) = (1, \delta)$, as can be seen from the signed generation tree below (cf. Notation 2.13):

![Diagram showing the signed generation tree for the distributive $L_{DK}$-inequality $p \wedge q \leq q \vee p$.]

**Lemma 4.4.** If $(\varphi \vdash \psi)[\bar{p}, \bar{q}]$ is a quasi-special inductive inequality (cf. Notation 2.7), then each of its corresponding rules has the following shape:

\[
\begin{align*}
\frac{(Z + \Gamma^d)_{\bar{y}}[\bar{x} / \bar{p}, \bar{y} / \bar{q}]}{\Phi_i \vdash \Psi_j}[\bar{x}, \bar{y}, \bar{z}, \bar{w}] & \\
\end{align*}
\]

where for each $\gamma$ in $\bar{y}$ (resp. each $\delta$ in $\bar{z}$), each $\Gamma^d$ (resp. $\Delta^t$) is the structural counterpart of some conjunct (resp. disjunct) $\gamma^i$ (resp. $\delta^j$) of the equivalent rewriting of $\gamma$ (resp. $\delta$) as $\Lambda \gamma^i$ (resp. $\vee^j \delta^j$), as per Lemma 2.7 with each $\gamma^i$ (resp. $\delta^j$) being a definite positive (resp. negative) PIA formula, and each $\Phi_i$ (resp. $\Psi_j$) is the structural counterpart of some disjunct (resp. conjunct) $\varphi_i$ (resp. $\psi_j$) of the equivalent rewriting of $\varphi$ (resp. $\psi$) as $\vee_{j \in J} \varphi_j$ (resp. $\wedge_{i \in I} \psi_i$), as per Lemma 2.9 with each $\varphi_i$ (resp. $\psi_j$) being a definite positive (resp. negative) Skeleton formula.

**Proof.** Let us apply the algorithm ALBA to $(\varphi \vdash \psi)[\bar{p}, \bar{q}]$ to compute its corresponding analytic rules. Modulo pre-processing, we can assume w.l.o.g. that $(\varphi \vdash \psi)[\bar{p}, \bar{q}, \bar{y}, \bar{z}]$ is definite \[30\] and hence we can proceed with first approximation:

\[
\forall x y z w [(x \t p \wedge q \t y \wedge z \t y \wedge \delta \t w) \Rightarrow (\varphi \vdash \psi)[x, y, z, w]] \tag{4.1}
\]

Since $\gamma$ (resp. $\delta$) can be equivalently rewritten as $\Lambda \gamma^i$ (resp. $\vee^j \delta^j$), as per Lemma 2.9 the quasi-inequality above can be equivalently rewritten as follows:

\[
\forall x y z w [(x \t p \wedge q \t y \wedge (z \t \gamma^i)_{\bar{y}} \wedge (\delta \t w)_{\bar{z}}) \Rightarrow (\varphi \vdash \psi)[x, y, z, w]]. \tag{4.2}
\]

\[30\] If $(\varphi \vdash \psi)[\bar{p}, \bar{q}, \bar{y}, \bar{z}]$ is not definite, then by Lemma 2.12 it can equivalently be transformed into the conjunction of *definite* quasi-special inductive sequents which we can treat separately as shown in the proof.
If every $p$ in $\mathcal{P}$ and $q$ in $\mathcal{Q}$ has one critical occurrence, then we are in Ackermann-shape and hence we can eliminate the variables $\bar{p}$ and $\bar{q}$ as follows (since by assumption $\mathcal{P}$ and $\mathcal{Q}$ agree with $\delta^h$):

$$\forall x_1 y_1 \ldots x_n y_n \left[ (z \vdash \gamma^i) \cup (\delta^o \vdash w) \Rightarrow (\varphi \vdash \psi) \right]$$

which yields a rule of the desired shape. If there are multiple critical occurrences of some $p$ in $\mathcal{P}$ or $q$ in $\mathcal{Q}$, then the Ackermann-shape looks as in (4.2), but with $\forall_{k^2} x_k \vdash p_k$ and $q_1 \vdash \wedge_{h^1} y_h$, where $n_1$ (resp. $m_1$) is the number of critical occurrences of $p_1$ (resp. $q_1$). Hence, by applying the Ackermann rule we obtain a quasi-inequality similar to (4.3), except that the sequents in the antecedent have the following shape:

$$\left((z \vdash \gamma^i) \cup (\delta^o \vdash w) \Rightarrow (\varphi \vdash \psi)\right)$$

(4.4)

Since by assumption $\varepsilon(p) = 1$ for every $p$ in $\mathcal{P}$ and $\varepsilon(q) = \partial$ for every $q$ in $\mathcal{Q}$, recalling that $+\gamma^t$ and $-\delta^t$ agree with $\varepsilon^t$ for each $\gamma^t$ in $\mathcal{P}$ and $\delta^t$ in $\mathcal{Q}$, and moreover every $\gamma^t$ in $\mathcal{P}$ (resp. $\delta^t$ in $\mathcal{Q}$) is positive (resp. negative) PIA, the following semantic equivalences hold for each $\gamma^t$ in $\mathcal{P}$ and $\delta^t$ in $\mathcal{Q}$:

$$\forall x_1 y_1 \ldots x_n y_n \left[ (z \vdash \gamma^i) \cup (\delta^o \vdash w) \Rightarrow (\varphi \vdash \psi) \right]$$

(4.5)

Hence, for every $\gamma^t$ in $\mathcal{P}$ and $\delta^t$ in $\mathcal{Q}$, the corresponding sequents in (4.4) can be equivalently replaced by (at most) $\Sigma_{n_1 m_1} (n_1 m_1)$ sequents of the form

$$z \vdash \gamma^i \left[ x_k / p_k, y_h / q_h \right]$$

(4.6)

yielding again a rule of the desired shape.

As discussed, the Lemma above applies for both the non-distributive and distributive setting, following Remark 2.23.

**Example 4.5.** Let us illustrate the procedure described in the lemma above by applying it to the sequents discussed in Example 4.3.

**ALBA-run computing the structural rule for $\Diamond p \vdash \Box p$:**

$$\begin{array}{c}
\Diamond p \vdash \Box p \\
\text{iff} \\
\forall p \forall x w \left[ x \vdash p \land \Diamond p \vdash w \Rightarrow \Box x \vdash \Box w \right] \\
\text{Instance of (4.1)}
\end{array}$$

Hence, the analytic rule corresponding to $\Diamond p \vdash \Box p$ is

$$\begin{array}{c}
\Diamond X \vdash W \\
\Box X \vdash \Box W \\
R_1
\end{array}$$

**ALBA-run computing the structural rule for $p \land \Box q \vdash q \lor \Diamond p$:**

$$\begin{array}{c}
p \land \Box q \vdash q \lor \Diamond p \\
\text{iff} \\
\forall q \forall x y z w \left[ (z \vdash q \land \Box q \land \Diamond p \vdash w) \Rightarrow x \land z \vdash y \lor w \right] \\
\text{Instance of (4.1)}
\end{array}$$

Hence, the analytic rule corresponding to $p \land \Box q \vdash q \lor \Diamond p$ is

$$\begin{array}{c}
Z \vdash Y \\
\Diamond X \vdash W \\
X \land Z \vdash Y \lor W \\
R_2
\end{array}$$

**ALBA-run computing the structural rule for $p \land (p \land \Box q) \vdash q \lor (q \lor \Diamond p)$:**

$$\begin{array}{c}
p \land (p \land \Box q) \vdash q \lor (q \lor \Diamond p) \\
\text{iff} \\
\forall p q \forall x y z w \left[ (z \vdash p \land x \vdash \Box q \land y \vdash (q \lor \Diamond p) \lor (q \land \Diamond p) \lor w) \Rightarrow x \lor Z \vdash X \lor Y \lor W \right] \\
\text{Instance of (4.1)}
\end{array}$$

Hence, the analytic rule corresponding to $p \land (p \land \Box q) \vdash q \lor (q \lor \Diamond p)$ is

$$\begin{array}{c}
\Box X \vdash W \\
X \land Z \vdash Y \lor W \\
R_3
\end{array}$$

25
Hence, the analytic rule corresponding to \( p \otimes (p \otimes \Box q) + q \otimes (q \otimes p) \) is \[
\frac{Z + \Box Y_1}{\Box X_1 + W} \quad \frac{Z + \Box Y_2}{\Box X_2 + W} \quad R_3
\]
For the last inequality \( \Diamond (p \lor \Diamond p) \leq \Box \Diamond p \), we first need to preprocess the inequality and obtain two definite inequalities \( \Diamond p \leq \Box \Diamond p \) and \( \Diamond \Diamond p \leq \Box \Diamond p \). We now need to compute the rule for the second one (the first was already computed above):

**ALBA-run computing the structural rule for \( \Box \Diamond p \lor \Diamond \Diamond p \):**

\[
\begin{align*}
\Box \Diamond p & \lor \Diamond \Diamond p \\
\text{iff} \quad \forall \mathbf{x} p \land w [x + p & \land \Diamond p \lor w \Rightarrow \Diamond \Diamond x + \Box w] & \quad \text{Instance of (4.1)} \\
\text{iff} \quad \forall \mathbf{x} w [x + w \Rightarrow \Diamond \Diamond x + \Box w] & \quad \text{Instance of (4.3)}
\end{align*}
\]

Hence, the analytic rule corresponding to \( \Box \Diamond p \lor \Diamond \Diamond p \) is

\[
\frac{\Box X + W}{\Box \Box X + \Box W} \quad \text{R_4}
\]

**Theorem 4.6.** If \((\varphi \lor \psi)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\) is a quasi-special inductive inequality, then a cut-free derivation in pre-normal form exists of \((\varphi \lor \psi)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\) in \( \mathcal{D}_L^E + \mathcal{R} \) (resp. \( \mathcal{D}_L^E + \mathcal{R} \)), where \( \mathcal{R} \) denotes the finite set of analytic structural rules corresponding to \((\varphi \lor \psi)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\) as in Lemma **4.4**.

**Proof.** Recall that each \( \gamma \) in \( \mathbb{P} \) (resp. \( \delta \) in \( \mathbb{R} \)) is a positive (resp. negative) PIA formula. Hence, let \( \land \gamma \) (resp. \( \lor \delta \)) denote the equivalent rewriting of \( \gamma \) (resp. \( \delta \)) as conjunction (resp. disjunction) of definite positive (resp. negative) PIA formulas, as per Lemma **4.2**. Let us assume that \((\varphi \lor \psi)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\) is definite\(^{21}\) and hence \( \mathcal{R} \) has only one element \( \mathcal{R} \), which has the following shape (cf. Lemma **4.3**):

\[
(\Phi + \Psi)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})
\]

Then, modulo application of display rules, we can apply left-introduction (resp. right-introduction) rules to positive (resp. negative) SLR-connectives bottom-up, so as to transform all Skeleton connectives into structural connectives:

\[
(\Phi + \Psi)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})
\]

Notice that \((\Phi + \Psi)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\) is an instance of the conclusion of \( \mathcal{R} \) with \( \mathbb{P} / X, \mathbb{Q} / Y, \mathbb{R} / Z \) and \( \mathbb{S} / W \). Hence, we can apply \( \mathcal{R} \) bottom-up and obtain:

\[
(\gamma + \Gamma^3)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})
\]

By Corollary **3.7**, the sequents \((\gamma + \Gamma^3)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\) and \((\Lambda^\mu + \delta)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\) are cut-free derivable in \( \mathcal{D}_L^E \) (resp. \( \mathcal{D}_L^E \)), with derivations which only contain identity axioms, and applications of right-introduction rules for negative PIA-connectives, and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of \( \mathcal{D}_L^E \)). Moreover, as discussed above (cf. also Proposition **3.12**), the rules applied after applying the rules in \( \mathcal{R} \) are only display rules, left-introduction rules for positive Skeleton-connectives and right-introduction rules for negative Skeleton-connectives (plus contraction in the case of \( \mathcal{D}_L^E \)). This completes the proof that the cut-free derivation in \( \mathcal{D}_L^E + \mathcal{R} \) (resp. \( \mathcal{D}_L^E + \mathcal{R} \)) of \((\varphi \lor \psi)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\) is in pre-normal form. \( \square \)

**Example 4.7.** Let us illustrate the procedure described in the proposition above by deriving the sequents in Example **4.3**

\(^{21}\)If \((\varphi \lor \psi)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\) is not definite, then, by Lemma **4.12**, it can equivalently be transformed into the conjunction of definite quasi-special inductive sequents \((\varphi_i \lor \psi_i)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\), where \( \varphi \) is equivalent to \( \lor \varphi \), and \( \psi \) is equivalent to \( \land \psi \), which we can treat separately as shown in the proof. Then, a derivation of the original sequent can be obtained by applying the procedure indicated in the proof of Proposition **3.12** twice: by applying the procedure once, from derivations of \((\Phi_i + \Psi_i)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\) for every \( i \) and \( j \) we obtain derivations of \((\varphi_i \lor \psi_i)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\) for every \( j \). Then, by applying the procedure again on these sequents, we obtain the required derivation of \((\varphi \lor \psi)(\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S})\).
Modulo distribution and splitting (cf. Lemma 2.9), the quasi-inequality above can be equivalently rewritten as follows:

\[ \text{Lemma 4.8.} \quad \text{If } \varphi \rightarrow \psi \mid \overline{\beta, \gamma, \delta}, \text{ then each of its corresponding rules has the following shape:} \]

\[
\begin{align*}
\frac{\varphi \rightarrow \psi \mid \overline{\beta, \gamma, \delta}}{\varphi \rightarrow \psi \mid \overline{\beta, \gamma, \delta}} & \quad \text{Footnote [21]} \\
\end{align*}
\]

\[ \text{Footnote [21]} \]

\[ \text{4.2 Syntactic completeness for analytic inductive sequents} \]


\[ \text{Lemma 4.8.} \quad \text{If } \varphi \rightarrow \psi \mid \overline{\beta, \gamma, \delta}, \text{ then each of its corresponding rules has the following shape:} \]

\[
\begin{align*}
\frac{\varphi \rightarrow \psi \mid \overline{\beta, \gamma, \delta}}{\varphi \rightarrow \psi \mid \overline{\beta, \gamma, \delta}} & \quad \text{Footnote [21]} \\
\end{align*}
\]

\[ \text{Footnote [21]} \]

\[ \text{4.2 Syntactic completeness for analytic inductive sequents} \]

\[ \frac{\varphi \rightarrow \psi \mid \overline{\beta, \gamma, \delta}}{\varphi \rightarrow \psi \mid \overline{\beta, \gamma, \delta}} & \quad \text{Footnote [21]} \\
\end{align*}
\]

\[ \text{Footnote [21]} \]

\[ \text{4.2 Syntactic completeness for analytic inductive sequents} \]

\[ \frac{\varphi \rightarrow \psi \mid \overline{\beta, \gamma, \delta}}{\varphi \rightarrow \psi \mid \overline{\beta, \gamma, \delta}} & \quad \text{Footnote [21]} \\
\end{align*}
\]

\[ \text{Footnote [21]} \]

\[ \text{4.2 Syntactic completeness for analytic inductive sequents} \]

\[ \frac{\varphi \rightarrow \psi \mid \overline{\beta, \gamma, \delta}}{\varphi \rightarrow \psi \mid \overline{\beta, \gamma, \delta}} & \quad \text{Footnote [21]} \\
\end{align*}
\]

\[ \text{Footnote [21]} \]
where each $\alpha_p$ and $\alpha_q$ (resp. $\beta_p$ and $\beta_q$) is definite positive (resp. negative) PIA and contains a unique $\varepsilon$-critical propositional variable occurrence, which we indicate in its subscript. By applying adjunction and residuation ALBA-rules on all definite PIA-formulas $\alpha_p, \alpha_q, \beta_p$ and $\beta_q$ using each $\varepsilon$-critical propositional variable occurrence as the pivotal variable, the antecedent of the quasi-inequality above can be equivalently written as follows:

$$\frac{\{ \alpha_p[\chi/u, \overline{p}, \overline{q}] \vdash \rho \} \& \{ \beta_p[\chi/u, \overline{p}, \overline{q}] \vdash \rho \} \& q + \{ \alpha_q[\chi/u, \overline{p}, \overline{q}] \vdash \sigma \} \& \{ \beta_q[\chi/u, \overline{p}, \overline{q}] \vdash \sigma \} \& \varepsilon \vdash \gamma}{z \vdash \gamma \& \delta \vdash \omega}$$  \hspace{1cm} (4.12)

Since each $\gamma$ (resp. $\delta$) is a positive (resp. negative) PIA formula, by Lemma 4.10 it is equivalent to $\bigwedge_{\chi} \gamma^\chi$ (resp. $\bigvee_{\mu} \delta^\mu$), where each $\gamma^\chi$ (resp. $\delta^\mu$) is definite positive (resp. negative) PIA. Therefore (4.12) can be equivalently rewritten as follows:

$$\frac{\{ \alpha_p[\chi/u, \overline{p}, \overline{q}] \vdash \rho \} \& \{ \beta_p[\chi/u, \overline{p}, \overline{q}] \vdash \rho \} \& q + \{ \alpha_q[\chi/u, \overline{p}, \overline{q}] \vdash \sigma \} \& \{ \beta_q[\chi/u, \overline{p}, \overline{q}] \vdash \sigma \} \& \varepsilon \vdash \gamma^\chi}{z \vdash \gamma^\chi \& \delta \vdash \omega^\mu}$$  \hspace{1cm} (4.13)

Notice that the ‘parametric’ (i.e. non-critical) variables in $\overline{p}$ and $\overline{q}$ actually occurring in each formula $\{ \alpha_p[\chi/u, \overline{p}, \overline{q}] \}$, $\{ \beta_p[\chi/u, \overline{p}, \overline{q}] \}$, $\{ \alpha_q[\chi/u, \overline{p}, \overline{q}] \}$, and $\{ \beta_q[\chi/u, \overline{p}, \overline{q}] \}$ are those that are strictly $\varepsilon$-smaller than the (critical and pivotal) variable indicated in the subscript of the given PIA-formula. After applying adjunction and residuation as indicated above, the quasi-inequality (4.11) is in Ackermann shape relative to the $<_{\omega}$-minimal variables.

For every $p \in \overline{p}$ and $q \in \overline{q}$, let us define the sets $Mv(p)$ and $Mv(q)$ by recursion on $<_{\omega}$ as follows:

- $Mv(p) := \{ \{ \alpha_p[\chi/u, \overline{p}, \overline{q}] \vdash \rho \} \& \{ \beta_p[\chi/u, \overline{p}, \overline{q}] \vdash \rho \} \& q \vdash \{ \alpha_q[\chi/u, \overline{p}, \overline{q}] \vdash \sigma \} \& \{ \beta_q[\chi/u, \overline{p}, \overline{q}] \vdash \sigma \} \& \varepsilon \vdash \gamma \}$ \hspace{1cm} (4.14)

Since by assumption $\varepsilon(p) = 1$ for every $p \in \overline{p}$ and $\varepsilon(q) = \delta$ for every $q \in \overline{q}$, recalling that $\gamma^\chi$ and $\delta^\mu$ agree with $\varepsilon^\chi$, and moreover every $\gamma^\chi$ in $\overline{\gamma}^\chi$ (resp. $\delta^\mu$ in $\overline{\delta}^\mu$) is positive (resp. negative) PIA, the following semantic equivalences hold for each $\gamma^\chi$ in $\overline{\gamma}^\chi$ and $\delta^\mu$ in $\overline{\delta}^\mu$:

$$\overline{\gamma}^\chi \cup \{ Mv(p)/\overline{p}, \overline{Mv(q)}/\overline{q} \} = \overline{\gamma}^\chi \cup \{ Mv(p)/\overline{p}, Mv(q)/\overline{q} \}$$

$$\overline{\delta}^\mu \cup \{ Mv(p)/\overline{p}, \overline{Mv(q)}/\overline{q} \} = \overline{\delta}^\mu \cup \{ Mv(p)/\overline{p}, Mv(q)/\overline{q} \}.$$  

Hence for every $\gamma^\chi$ in $\overline{\gamma}^\chi$ and $\delta^\mu$ in $\overline{\delta}^\mu$, the corresponding sequents in (4.14) can be equivalently replaced by (at most) $\Sigma_{n,m}(n,m)$ sequents of the form

$$z \vdash \gamma^\chi \cup \{ Mv(p)/\overline{p}, Mv(q)/\overline{q} \} \cup \delta^\mu \cup \{ Mv(p)/\overline{p}, Mv(q)/\overline{q} \} \vdash w,$$  \hspace{1cm} (4.15)

yielding a rule of the desired shape.

Example 4.9. Let us illustrate the procedure described in the lemma above by applying it to the sequents discussed in Example 2.14.
Then the analytic rule corresponding to $\Diamond\Box p + \Box p \Diamond q$ is:

$$
\begin{align*}
\Diamond & \Box X \vdash W & R_4 \\
\Diamond X & \vdash \Box W
\end{align*}
$$

ALBA-run computing the structural rule for $p \rightarrow (q \rightarrow r) \leq ((p \rightarrow q) \rightarrow (\Box p \rightarrow r)) \oplus \Diamond r$:

$$
\begin{align*}
\text{if} \quad & \quad \forall \forall \forall W \forall \forall \forall \forall \forall \forall [x_1 + \forall & \Rightarrow p \rightarrow q & & \oplus r \Rightarrow y_1 & & y_2 & & z + p \rightarrow (q \rightarrow r) & & \Rightarrow z + (x_1 \rightarrow y_2) \oplus y_1] \quad (4.10) \\
\text{if} \quad & \quad \forall \forall \forall W \forall \forall \forall \forall \forall \forall [\bullet x_1 + p & & & & & & & & & & \Rightarrow \bullet x_1 \rightarrow (q \rightarrow r) & & \Rightarrow z + (x_1 \rightarrow y_2) \oplus y_1] \quad (4.12) \\
\end{align*}
$$

Then the analytic rule corresponding to $p \rightarrow (q \rightarrow r) \leq ((p \rightarrow q) \rightarrow (\Box p \rightarrow r)) \oplus \Diamond r$ is:

$$
\begin{align*}
Z \vdash \bullet X_1 & \rightarrow (\bullet X_1 \hat{\land} X_2 \rightarrow \Box Y_1) & & Z \vdash \bullet X_1 & \rightarrow (\bullet X_1 \hat{\land} X_2 \rightarrow Y_2) & & R_5
\end{align*}
$$

ALBA-run computing the structural rule for $\Diamond\Box(p \land q) \land \Box p \lor \Box q$:

$$
\begin{align*}
\text{if} \quad & \quad \forall \forall \forall \forall \forall \forall W [x + \Box (p \land q) & & & & \lor \Box y & & \lor z & & \Rightarrow \Box x + \Box y \lor \Box z] & & \text{Instance of (4.10)} \\
\text{if} \quad & \quad \forall \forall \forall \forall \forall \forall W [x + \Box (p \land q) & & & & \lor \Box y & & \lor z & & \Rightarrow \Box x + \Box y \lor \Box z] & & \text{Instance of (4.12)} \\
\text{if} \quad & \quad \forall \forall \forall \forall \forall \forall W [x + \Box (p \land q) & & & & \lor \Box y & & \lor z & & \Rightarrow \Box x + \Box y \lor \Box z] & & \text{Instance of (4.14)} \\
\text{if} \quad & \quad \forall \forall \forall \forall \forall \forall W [x + \Box q & & & & \lor \Box z & & \lor \Box x & & \Rightarrow \Box x + \Box y \lor \Box z] & & \text{Instance of (4.15)}
\end{align*}
$$

Then the analytic rule corresponding to $\Diamond\Box(p \land q) \land \Box p \lor \Box q$ is:

$$
\begin{align*}
\Box X & \vdash \Box Y & & \Box X & \vdash \Box Z & & R_6
\end{align*}
$$

**Theorem 4.10.** If $(\varphi \lor \psi)[\varphi, \beta, \gamma, \delta]$ is an analytic inductive sequent, then a cut-free derivation in pre-normal form exists of $(\varphi \lor \psi)[\varphi, \beta, \gamma, \delta]$ in $\mathcal{D}_{\text{LE}} + \mathcal{R}$ (resp. $\mathcal{D}_{\text{LE}} + \mathcal{R}$), where $\mathcal{R}$ denotes the finite set of analytic structural rules corresponding to $(\varphi \lor \psi)[\varphi, \beta, \gamma, \delta]$ as in Lemma 4.8.

**Proof.** Recall that each $\gamma$ in $\beta$ (resp. $\delta$ in $\bar{\beta}$) is a positive (resp. negative) PLA formula. Let $\land_1 \gamma^i$ (resp. $\lor_1 \delta^i$) denote the equivalent rewriting of $\gamma$ (resp. $\delta$) as disjunction (resp. disjunction) of definite positive (resp. negative) PLA formulas as per Lemma 2.9. Let us assume that $(\varphi \lor \psi)[\varphi, \beta, \gamma, \delta]$ is definite$^{22}$ and hence $\mathcal{R}$ has only one element $\Gamma$, with the following shape (cf. Lemma 4.8):

$$
\begin{align*}
(Z + \Gamma^i)_{\Delta_1} & [\mathcal{M}(p)/\mathcal{P}, \mathcal{M}(q)/\mathcal{Q}] & & (Z + \Gamma^i)_{\Delta_1} & [\mathcal{M}(p)/\mathcal{P}, \mathcal{M}(q)/\mathcal{Q}] & & (\Phi + \Psi)[\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}]
\end{align*}
$$

Then, modulo application of display rules, we can apply left-introduction (resp. right-introduction) rules to positive (resp. negative) SLR-connectives bottom-up, so as to transform all skeleton connectives into structural connectives:

$$
(\Phi + \Psi)[\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}]
$$

(4.16)

Notice that $(\Phi + \Psi)[\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}]$ is an instance of the conclusion of $\mathcal{R}$. Hence we can apply $\mathcal{R}$ bottom-up and obtain:

$$
(\varphi \lor \psi)[\varphi, \beta, \gamma, \delta]
$$

(4.17)

$$
(\varphi \lor \psi)[\varphi, \beta, \gamma, \delta]
$$

Notice that $(\Phi + \Psi)[\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}]$ is an instance of the conclusion of $\mathcal{R}$. Hence we can apply $\mathcal{R}$ bottom-up and obtain:

---

$^{22}$If $(\varphi \lor \psi)[\varphi, \beta, \gamma, \delta]$ is not definite, then, by Lemma 2.12, it can equivalently be transformed into the conjunction of definite analytic inductive sequents $(\varphi \lor \psi)[\varphi, \beta, \gamma, \delta]$, where $(\varphi \lor \psi)$ is equivalent to $\lor_1 \delta^i$ and $(\varphi \lor \psi)$ is equivalent to $\land_1 \gamma^i$, which we can treat separately as shown in the proof. Then, a derivation of the original sequent can be obtained by applying the procedure indicated in the proof of Proposition 4.12 twice: by applying the procedure once, from derivations of $(\Phi + \Psi)[\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}]$ for every $i$ and $j$ we obtain derivations of $(\Phi + \Psi)[\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}]$ for every $j$. Then, by applying the procedure again on these sequents, we obtain the required derivation of $(\varphi \lor \psi)[\varphi, \beta, \gamma, \delta]$. 

29
To finish the proof that \((\varphi \rightarrow \psi)[\alpha, \beta, \gamma, \delta]\) is derivable in D\(_{\text{LE}}\) + R (resp. D\(_{\text{DLE}}\) + R), it is enough to show that the sequents

\[
(\forall \gamma \rightarrow \Gamma)_\gamma [\text{MV}(p)[\exists \alpha, \beta, \gamma]/\overline{\varphi}, \text{MV}(q)[\exists \alpha, \beta, \gamma]/\overline{\psi}, \text{MV}(q)[\exists \alpha, \beta, \gamma]/\overline{\psi}]
\]

are derivable in D\(_{\text{LE}}\) (resp. D\(_{\text{DLE}}\)). Recalling that each \(\gamma\) (resp. \(\delta\)) is a positive (resp. negative) PIA formula, by Proposition 3.6, it is enough to show that for every \(p\) and \(q\), the sequents MV\((p)[\exists \alpha, \beta, \gamma]/\overline{\varphi}\) \(\vdash p\) and \(q \vdash \text{MV}(q)[\exists \alpha, \beta, \gamma]/\overline{\psi}\) are derivable in D\(_{\text{LE}}\) (resp. D\(_{\text{DLE}}\)) for all formulas \(\text{MV}(p) \in \text{MV}(p)\) and \(\text{MV}(q) \in \text{MV}(q)\). Let us show this latter statement. Each sequent MV\((p)[\exists \alpha, \beta, \gamma]/\overline{\varphi}\) \(\vdash p\) is of either of the following forms:

\[
\text{LA}(\alpha_p)[(\alpha[\exists \beta, \gamma]/\overline{\varphi}, \text{MV}(p)[\exists \alpha, \beta, \gamma]/\overline{\psi}, \text{MV}(q)[\exists \alpha, \beta, \gamma]/\overline{\psi}]/\overline{\varphi} \vdash \alpha_p \text{MV}(p)/\overline{\psi}, \text{MV}(q)/\overline{\psi} \vdash p, \text{RA}(\beta_p)[\beta[\exists \alpha, \beta, \gamma]/\overline{\varphi}, \text{MV}(p)/\overline{\psi}, \text{MV}(q)/\overline{\psi}]/\overline{\varphi} \vdash \beta_p]
\]

where \(\alpha_p\) (resp. \(\beta_p\)) denotes the definite positive (resp. negative) PIA formula which occurs as a conjunct (resp. disjunct) of \(\alpha\) (resp. \(\beta\)) as per Lemma 2.9 which contains the \(e\)-critical occurrence of \(p\) as a subformula (cf. discussion around 4.11). By Corollary 3.9 it is enough to show that \(\text{MV}(p') \vdash p'\) and \(q' \vdash \text{MV}(q')\) are derivable in D\(_{\text{LE}}\) (resp. D\(_{\text{DLE}}\)) for each \(p', q' \leq \alpha\) \(p\) (which is true by the induction hypothesis, while the basis of the induction holds by Corollary 3.10), and \(p \vdash p\) is derivable in D\(_{\text{LE}}\) (resp. D\(_{\text{DLE}}\)). Since it is of course the case. Likewise, one shows that the sequents \(q \vdash \text{MV}(q)[\exists \alpha, \beta, \gamma]/\overline{\psi}\) are derivable in D\(_{\text{LE}}\) (resp. D\(_{\text{DLE}}\)), which completes the proof that \((\varphi \rightarrow \psi)[\alpha, \beta, \gamma, \delta]\) is derivable in D\(_{\text{LE}}\) + R (resp. D\(_{\text{DLE}}\) + R). Finally, the derivation so generated only consists of identity axioms, and applications of display rules, right-introduction rules for negative PIA-connectives, and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of D\(_{\text{DLE}}\)) before the application of a rule in \(R\) (cf. Proposition 3.10 and Corollaries 3.9 and 3.11). Moreover, after applying a rule in \(R\), the only rules applied are display rules, left-introduction rules for negative Skeleton-connectives and right-introduction rules for positive Skeleton-connectives (plus contraction in the case of D\(_{\text{DLE}}\)), cf. Proposition 3.12 and Footnote 23. This completes the proof that the cut-free derivation in D\(_{\text{LE}}\) + R (resp. D\(_{\text{DLE}}\) + R) of \((\varphi \rightarrow \psi)[\alpha, \beta, \gamma, \delta]\) is in pre-normal form.

\[\square\]

Example 4.11. Let us illustrate the procedure described in the proposition above by deriving the sequents in Example 2.14 using the rules computed in Example 4.9. In the last derivation below, the symbol \(\gamma\) denotes the left residual of \(\phi\) in its first coordinate.

\[\text{D\(_{\text{LE}}\)}\text{-derivation of } \\
\text{D\(_{\text{DLE}}\)}\text{-derivation of } \\
\text{D\(_{\text{LE}}\)}\text{-derivation of } \\
\text{D\(_{\text{DLE}}\)}\text{-derivation of } \\
\text{D\(_{\text{LE}}\)}\text{-derivation of } \\
\text{D\(_{\text{DLE}}\)}\text{-derivation of }
\]
5 Conclusions

Main contribution In this article we showed that, for any properly displayable (D)LE-logic L (i.e. a (D)LE-logic axiomatized by analytic inductive axioms, cf. Definition 2.10), the proper display calculus for L—i.e. the calculus obtained by adding the analytic structural rules corresponding to the axioms of L to the basic calculus DLE (resp. D.DLE)—derives all the theorems of L. This is what we refer to as the syntactic completeness of the proper display calculus for L with respect to L. We achieved this result by providing an effective procedure for generating a derivation—which is not only cut-free but also in pre-normal form (cf. Definitions 2.24 and 2.25)—of any analytic inductive axiom in any (D)LE-signature in the proper display calculus D.L (resp. D.DLE) augmented with the analytic structural rule(s) corresponding to the given axiom.

Scope Since (D)LE-logics encompass a wide family of well known logics (modal, intuitionistic, substructural), and since analytic inductive axioms provide a formulation of the notion of analyticity based on the syntactic shape of formulas/sequents, the results of the present paper directly apply to all logical settings for which analytic (proper display) calculi have been defined, such as those of [29] [1] [43] [44] [32] [7]. Moreover, in the present paper we have worked in a single-type environment, mainly for ease of exposition. However, all the results mentioned above straightforwardly apply also to properly displayable multi-type calculi, which have been recently introduced to extend the scope and benefits of proper display calculi also to a wide range of logics that for various reasons do not fall into the scope of the analytic inductive definition. These logics crop up in various areas of the literature and include well known logics such as linear logic [10], dynamic epistemic logic [20], semi De Morgan logic [26] [27], bilattice logic [28], inquisitive logic [21], non normal modal logics [3], the logics of classes of rough algebras [25] [24]. Interestingly, the multi-type framework can be
also usefully deployed to introduce logics specifically designed to describe and reason about the interaction of entities of different types, as done e.g. in [2, 18].

The syntax-semantics interface on analytic calculi  The main insight developed in the research line to which the present paper pertains is that there is a close connection between semantic results pertaining to correspondence theory and the syntactic theory of analytic calculi. This close connection, which has been observed and also exploited by several authors in various proof-theoretic settings (cf. e.g. [32, 37, 4]), gave rise in [29] to the notion of analytic inductive inequalities as the uniform and independent identification, across signatures, of the syntactic shape (semantically motivated by the order-theoretic properties of the algebraic interpretation of the logical connectives) which guarantees the desiderata of analyticity. In this context, the core of the “syntax-semantics interface” is the algorithm ALBA, which serves to compute both the first-order correspondent of analytic inductive axioms and their corresponding analytic structural rules. In this paper, we saw the analytic structural rules computed by ALBA at work as the key cogs of the machinery of proper display calculi to derive the axioms that had generated them. This result can be understood as the purely syntactic counterpart of the proof that ALBA preserves semantic equivalence on complete algebras (cf. [10, 16, 14]), which has been used in [29] to motivate the semantic equivalence of any given analytic inductive axiom with its corresponding ALBA-generated structural rules. This observation paves the way to various questions, among which, whether information about the derivation of a given analytic inductive axiom can be extracted directly from its successful ALBA-run, or conversely, whether information about (optimal) ALBA-runs of analytic inductive axioms can be extracted from its derivation in pre-normal form, or whether the recent independent topological characterization of analytic inductive inequalities established in [17] can be exploited for proof-theoretic purposes.

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