Local smoothing estimates of fractional Schrödinger equations in $\alpha$-modulation spaces with some applications

YUFENG LU

Abstract. We show some new local smoothing estimates of the fractional Schrödinger equations with initial data in $\alpha$-modulation spaces via decoupling inequalities. Furthermore, our necessary conditions show that the local smoothing estimates are sharp in some cases. As applications, the local smoothing estimates could show some new local well-posedness on modulation spaces of the fourth-order nonlinear Schrödinger equations on the line.

1. Introduction

Let $\beta > 0$, denote $S_\beta(t) := \mathcal{F}^{-1}e^{it|\xi|^\beta} \mathcal{F}$, the fractional Schrödinger semigroup, $I = [0, 1].$ Here $(\mathcal{F}^{-1}) \mathcal{F}$ is the (inverse) Fourier transform. The local smoothing estimate, based on $L^p$-Sobolev spaces in harmonic analysis, is as follows.

$$\|S_\beta(t)u\|_{L^p(I \times \mathbb{R}^d)} \lesssim \|u\|_{W^s,p}, \quad \forall u \in W^s,p,$$  \hspace{1cm} (1)

where $W^s,p = (I - \Delta)^{-s}L^p$ is the $L^p$-Sobolev space when $1 < p < \infty.$ In contrast to the fixed time estimate:

$$\|S_\beta(t_0)u\|_{L^p(\mathbb{R}^d)} \lesssim \|u\|_{W^{s_0},p}, \quad \forall u \in W^{s_0},p,$$

we know that when we take the integration of $t$ over $I$, the optimal regularity index $s$ could be less than $s_0$. So, we call it the local smoothing estimate.

One of the significant open problems in harmonic analysis is the local smoothing conjecture of the wave equation ($\beta = 1$). Until now, we only know this conjecture is true when $d = 2.$ One can refer to [18, 28, 38]. For $\beta \neq 1$, the $L^p$-smoothing estimate conjecture is that (1) holds for any

$$p > 2 + 2/d, \quad s/\beta > d(1/2 - 1/p) - 1/p.$$

Rogers in [32] obtained that (1) holds for $p > 2 + 4/(d+1), \quad s/\beta > d(1/2 - 1/p) - 1/p$ in case of $\beta = 2.$ Later, Rogers and Seeger in [33] further extended this result to the

Mathematics Subject Classification: 35R11, 35Q55, 42B37

Keywords: Local smoothing estimates, $\alpha$-modulation spaces, Fractional Schrödinger equations, Decoupling.
case where $\beta > 1, s/\beta \geq d(1/2 - 1/p) - 1/p$. The local smoothing conjecture and the restriction conjecture are closely related. Some techniques of the restriction conjecture have driven the development of the local smoothing conjecture. One can refer to [11,12,15].

In this paper, we consider the local smoothing estimate of fractional Schrödinger semigroup on $\alpha$-modulation spaces as follows.

$$\|S_\beta(t)u\|_{L^p(I \times \mathbb{R}^d)} \lesssim \|u\|_{\mathcal{M}^{s,\alpha}_{p,q}}, \quad \forall u \in \mathcal{M}^{s,\alpha}_{p,q}. \quad (2)$$

The modulation spaces $\mathcal{M}^{s,\alpha}_{p,q}$ introduced by Feichtinger in [7], are used to measure the decay and the regularity of the function by the short time Fourier transform (STFT). In contrast to the dyadic decomposition in the definition of Besov spaces, the modulation spaces can be characterized by the bounded admission partition of unity (BAPU). The $\alpha$-modulation spaces $\mathcal{M}^{p,\alpha}_{p,q}$, introduced by Feichtinger’s student Gröbner in [13], are proposed as the intermediate function spaces to connect modulation and Besov spaces with respect to a parameter $\alpha \in [0, 1]$. The $\alpha$-modulation spaces $\mathcal{M}^{s,\alpha}_{p,q}$ could be regarded as the generalized modulation spaces. Similarly, they could be defined by $\alpha$-BAPU. One can refer to [8,9,35]. The precise definitions of these spaces will be given in Sect. 2. One can also refer to the textbooks by Gröchenig [14] and Wang et al. [37].

Some recent work has been devoted to studying $\alpha$-modulation spaces. Borup and Nielsen in [2], also Fornasier in [10] studied the Banach frames for $\alpha$-modulation spaces. Kobayashi et al. in [27] discussed the boundedness for a class of pseudo-differential operators with symbols in $\alpha$-modulation spaces. Wang and Han in [19] described some basic properties of these spaces including the dual spaces, embeddings, scaling and algebraic structure. The relations between $\alpha$-modulation spaces and some classical function spaces such as Sobolev spaces were given by Kato in [22]. One can also refer [17,42].

One of the most significant differences between $\alpha$-modulation spaces and Sobolev spaces is the boundedness of the unimodular Fourier multiplier such as $S_\beta(t) = \mathcal{F}^{-1} e^{i t |\xi|^\beta} \mathcal{F}$. For example, consider the case of $\alpha = 0, \beta = 2$. Miyachi showed in [30] that $S_2(t)$ is bounded on $L^p$ if and only if $p = 2$. However, as shown by Bényi et al. in [1], $S_2(t)$ is bounded on all $\mathcal{M}^{s}_{p,q}$ with $1 \leq p, q \leq \infty, s \in \mathbb{R}$. For general cases, we refer to [31,40,41]. In this sense, the $\alpha$-modulation spaces are better spaces for the initial data of the Cauchy problems for some nonlinear dispersive equations in contrast with the Sobolev spaces. One can refer to [16,20,21,23,24,36,39]. Recently, Schippa in [34] gave some sufficient and necessary conditions for the local smoothing estimates of the Schrödinger semigroup in modulation spaces when $p \geq 2$ using the decoupling inequality. The aim of our paper is to extend these results to the general cases with $\beta > 0, \alpha < 1, 1 \leq p \leq \infty$.

Denote $M_\beta = \{ (\xi, |\xi|^\beta) : \xi \in \mathbb{R}^d, 1/2 \leq |\xi| \leq 1 \}$, the compact surface in $\mathbb{R}^{d+1}$. One can easily know that the Gauss curvature of $M_\beta$ is nonzero when $\beta \neq 1$. The second fundamental form is positive definite when $\beta > 1$. 

For the case of $0 < \beta < 1$, our main result is

**Theorem 1.** Let $0 < \beta < 1$, $1 \leq p, q \leq \infty, s \in \mathbb{R}$. Denote $\alpha = 1 - \beta/2 \in (0, 1), p_0 = 2 + 4/d$. Then (2) holds if one of the following conditions is satisfied:

(A) $p_0 \leq p, 1/q \leq -(d + 4)/dp + 1, s > \beta d(1 - 1/p - 1/q)/2 - \beta/p$.

(B) $2 \leq p \leq p_0, q \geq p, s > \beta d(1/2 - 1/q)/2$.

(C) $1/q \geq -(d + 4)/dp + 1, p' \leq q \leq p, s > \beta d(1 - 1/p - 1/q)/4$.

(D) $q \leq p', q \leq p, s \geq 0$.

(E) $p < 2, q \geq p, s > \beta d(1/p - 1/q)/2$.

where $p'$ is the dual index of $p$ with $1/p' + 1/p = 1$.

**Remark 1.** For the relation of $(p, q, s)$ in Theorem 1, see Fig. 1, where $p_0 = 2 + 4/d$.

For the case of $1 < \beta \leq 2$, our main result is

**Theorem 2.** Let $1 < \beta \leq 2, 1 \leq p, q \leq \infty, s \in \mathbb{R}$. Denote $\alpha = 1 - \beta/2 \in [0, 1), p_0 = 2 + 4/d$. Then (2) holds if one of the following conditions is satisfied:

(A) $p_0 \leq p, 1/q \leq -(d + 2)/dp + 1, s > \beta d(1 - 1/p - 1/q)/2 - \beta/p$.

(B) $2 \leq p \leq p_0, q \geq 2, s > \beta d(1/2 - 1/q)/2$.

(C) $1/q \geq -(d + 2)/dp + 1, p' \leq q \leq 2, s > 0$.

(D) $q \leq p, q \leq p', s \geq 0$.

(E) $p \leq 2, q > p, s > \beta d(1/p - 1/q)/2$.

**Remark 2.** When $\beta = 2, \alpha = 0$, we know that $M^\varepsilon_{p, q} = M^\delta_{p, q}$, the theorem above is just the Theorem 1.1 in [34]. For the relation of $(p, q, s)$ in Theorem 2, see Fig. 2, where $p_0 = 2 + 4/d$. 

![Figure 1. Relation of $(p, q, s)$ in Theorem 1](image-url)
Remark 3. The proof of Theorem 2 is similar to the proof of Theorem 1, where we use the $\ell^2$-decoupling instead of the $\ell^p$-decoupling. We omit it here for simplicity. One can also refer the proof of Theorem 1.1 in [34].

Also, we can prove the following necessary conditions when $0 < \beta \leq 2$.

**Theorem 3.** Let $0 < \beta \leq 2, 1 \leq p, q \leq \infty, s \in \mathbb{R}$, denote $\alpha = 1 - \beta/2 \in (0, 1)$. If (2) holds, then we have

\[
s \geq 0 \vee \frac{\beta d}{2} \left( \frac{1}{p} - \frac{1}{q} \right) \vee \left( \frac{\beta d}{2} \left( 1 - \frac{1}{p} - \frac{1}{q} \right) - \frac{\beta}{p} \right),
\]

where $a \vee b = \max \{a, b\}$.

**Remark 4.** By the necessary condition above, we can see that conditions (A), (D), (E) in Theorem 1 and conditions (A),(C),(D),(E) in Theorem 2 are sharp ignoring the endpoint cases.

**Remark 5.** We will explain why we refer to the estimates in the theorems above as “local smoothing estimates”. Take estimates (2) for example. Fix $t \in I$, we consider the sharp condition of the estimate

\[
\| S_\beta(t)u \|_p \lesssim \| u \|_{M^{s,\alpha}_{p,q}}, \forall u \in M^{s,\alpha}_{p,q}.
\]

By Theorem 4.2 in [40], we know that $S_\beta(t) : M^{s,\alpha}_{p,q} \to L^p$ is bounded for any $1 \leq p, q \leq \infty$ when $0 < \beta \leq 2, \alpha = 1 - \beta/2$. Therefore, (3) is equivalent to $M^{s,\alpha}_{p,q} \hookrightarrow L^p$. By the main result in [42] (one can also see Lemma 3 in the next section), we know that the sharp condition of this embedding is $s \geq -\beta \sigma (p, q)/2$ ignoring the endpoint case, where $\sigma (p, q)$ will be denoted in the Preliminary Section below.
Under this condition, we can easily see that (2) holds by integrating the both sides with respect to \( t \) over \( I \). If we compare this condition with the results in Theorem 1, we see that there are some smoothing effects on conditions (A), (B), (C).

For \( \beta > 2 \), we consider the estimates as below:

\[
\| S_\beta(t)u \|_{L^p(I \times \mathbb{R}^d)} \lesssim \| u \|_{M^s_{p,q}}, \quad \forall u \in M^s_{p,q}.
\]

Our main result is

**Theorem 4.** Let \( \beta > 2, 1 \leq p, q \leq \infty, s \in \mathbb{R} \), denote \( p_0 = 2 + 4/d \). Then (4) holds if one of the following conditions is satisfied:

(A) \( p_0 \leq p, 1/q \leq -(d+2)/dp + 1, s > d(\beta - 2)(1/2 - 1/p) - d(1/p + 1/q - 1) - \beta/p \).

(B) \( 2 \leq p \leq p_0, q \geq 2, s > d(\beta - 2)(1/2 - 1/p) - d(1/p + 1/q - 1) - \beta d (1/2 - 1/p) /2 \).

(C) \( p_0 \leq p, -(d+2)/dp + 1 \leq 1/q, s > (\beta - 2)(d/2 - (d+1)/p) \).

(D) \( 2 \leq p \leq p_0, q \leq 2, s > d(\beta - 2)(1/2 - 1/p)/2 \).

(E) \( p \leq 2, q \leq p, s \geq d(\beta - 2)(1/p - 1/2) \).

(F) \( p \leq 2, q > p, s > d(\beta - 2)(1/p - 1/2) - d(1/q - 1/p) \).

**Remark 6.** For the relation of \( (p, q, s) \) in Theorem 4, see Fig. 3, where \( p_0 = 2 + 4/d, a_1 = (\beta - 2)/p_0, a_2 = d(\beta - 2)/2, a_3 = a_1 + d/2, a_4 = a_2 + d \).

**Remark 7.** As a contrast, for a fixed \( t \in I \), we consider the sharp conditions for the estimates.

\[
\| S_\beta(t)u \|_p \lesssim \| u \|_{M^s_{p,q}}, \quad \forall u \in M^s_{p,q}.
\]
Kobayashi and Sugimoto [26] gave the almost sharp condition of this estimate. By Theorems 5.3 and 5.6 in their paper we know that (5) holds if and only if $s \geq d(\beta - 2) |1/2 - 1/p| - \sigma(p, q)$ if we ignore the endpoint cases. Therefore, in Theorem 4, we have some smoothing effects on conditions (A),(B),(C),(D), when we take the integration of $t$ over $I$.

For the necessary conditions of (4), our results are as follows.

**Theorem 5.** Let $\beta > 2$, $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$. Then if (4) holds, we have

$$s \geq \left( d \left( 1 - \frac{1}{p} - \frac{1}{q} \right) - \frac{\beta}{p} \right) \vee \left( d(\beta - 2) \left( \frac{1}{p} - \frac{1}{2} \right) + d \left( \frac{1}{p} - \frac{1}{q} \right) \right).$$

Moreover, when $p \leq 2$, $p \geq q$, we have $s \geq d(\beta - 2)(1/p - 1/2)$.

**Remark 8.** We can also consider the estimate (4) when $0 < \beta \leq 2$, which follows from the sharp embedding between $M^s_{p,q}$ and $M^{s,\alpha}_{p,q}$. One can refer to [17,19] for more details. The results are also sharp in some cases, as shown in Theorems 1 and 2. In fact, one can see this idea from the proof of Theorem 4, but we omit it here for simplicity.

The paper is organized as follows. In Sect. 2, we will recall some basic notations and definitions, and also present some useful lemmas, which will be used frequently in our proof. The proofs of our main theorems are given in Sects. 3–6. Finally, we apply Theorem 4 to solve nonlinear PDEs in Sect. 7. We obtain the local well-posedness of the fourth-order cubic nonlinear Schrödinger equations on $M^s_{p,2}$ for some $0 < s < 1/2$. Furthermore, we extend the local results to global when the regularity of the initial data is higher.

2. Preliminary

2.1. Basic notations

The following notations will be used throughout this article. For $0 < p, q \leq \infty$, we denote

$$\sigma(p, q) := d \left( 0 \wedge \left( \frac{1}{q} - \frac{1}{p} \right) \wedge \left( \frac{1}{q} + \frac{1}{p} - 1 \right) \right);$$

$$\tau(p, q) := d \left( 0 \vee \left( \frac{1}{q} - \frac{1}{p} \right) \vee \left( \frac{1}{q} + \frac{1}{p} - 1 \right) \right),$$

where $a \wedge b = \min \{a, b\}$, $a \vee b = \max \{a, b\}$.

We write $\mathcal{S}(\mathbb{R}^d)$ to denote the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$, and $\mathcal{S}'(\mathbb{R}^d)$ to denote the dual space of $\mathcal{S}(\mathbb{R}^d)$, called the space of tempered distributions. For simplification, we omit $\mathbb{R}^d$ without causing ambiguity. The (inverse) Fourier transform $(\mathcal{F}^{-1}) \mathcal{F}$ can be defined as follows:

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} d\xi;$$
\( \mathcal{F}^{-1} f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} f(\xi) e^{ix\xi} d\xi. \)

For \( x \in \mathbb{R}^d \), denote \( \langle x \rangle = (1 + |x|^2)^{1/2} \). For \( 1 \leq p \leq \infty \), recall the \( L^p \) norm:

\[
\|f\|_p = \begin{cases} 
(f_{\mathbb{R}^d} |f(x)|^p \, dx)^{1/p}, & 1 \leq p < \infty; \\
\text{ess sup}_{x \in \mathbb{R}^d} |f(x)|, & p = \infty.
\end{cases}
\]

For \( 1 < p < \infty \), we define the \( L^p \)-Sobolev norm:

\[
\|f\|_{W^{s,p}} = \left\| (I - \Delta)^{s/2} f \right\|_p,
\]

where \( \Delta \) is the Laplace operator, \( (I - \Delta)^{s/2} = \mathcal{F}^{-1} (\xi)^s \mathcal{F} \). Recall that the Sobolev space \( W^{s,p} \) is defined by \( W^{s,p} = \{ f \in \mathcal{S} : \|f\|_{W^{s,p}} < \infty \} \). When \( p = 2 \), we denote \( W^{s,2} \) by \( H^s \).

We use the notation \( I \lesssim J \) if there is an independent constant \( C \) such that \( I \leq CJ \) and the notation \( I \lesssim_a J \) if the constant \( C \) depends on \( a \). We also denote \( I \approx J \) if \( I \lesssim J \) and \( J \lesssim I \). For \( 1 \leq p \leq \infty \), we denote the dual index \( p' \) such that \( 1/p + 1/p' = 1 \).

**Definition 1.** (Dyadic decomposition, [37], Chapter 1) Choose a smooth radial bump function \( \psi : \mathbb{R}^d \to [0, 1] \) adapted to the ball \( B(0, 2) \), such that \( \psi(\xi) = 1 \) as \( |\xi| \leq 1 \) and \( \psi(\xi) = 0 \) as \( |\xi| \geq 2 \). Let \( \varphi(\xi) = \psi(\xi) - \psi(2\xi) \), and \( 1 \leq j \in \mathbb{Z} \), and set \( \varphi_0(\xi) = 1 - \sum_{j \geq 1} \varphi_j(\xi) \). We denote \( \Delta_j = \mathcal{F}^{-1} \varphi_j \mathcal{F} \) and refer to \( \{\Delta_j\}_{j \geq 0} \) as the dyadic decomposition operators.

2.2. \( \alpha \)-modulation spaces

**Definition 2.** (\( \alpha \)-covering, [2], Section 2) Let \( \alpha < 1 \). A countable set \( \{Q_i\}_i \), where \( Q_i \subseteq \mathbb{R}^d \), is called a \( \alpha \)-covering of \( \mathbb{R}^d \) if:

(i) \( \mathbb{R}^d = \bigcup_i Q_i \),

(ii) \( \# \{Q' \in \{Q_i\}_i : Q' \cap Q \neq \emptyset\} \leq c(d) \), uniformly for \( Q \in Q_i \),

(iii) \( \langle x \rangle^{\alpha d} \approx |Q_i| \) uniformly for \( x \in Q_i \).

**Definition 3.** (\( \alpha \)-modulation spaces, [19], Section 1) Let \( \alpha < 1 \), denote \( \beta = \alpha/(1 - \alpha) \), suppose that \( C > c > 0 \) are two appropriate constants such that \( \{B_k\}_{k \in \mathbb{Z}^d} \) is a \( \alpha \)-covering of \( \mathbb{R}^d \), where \( B_k = B(\langle k \rangle^\beta, C \langle k \rangle^\beta) \). We can choose a Schwartz function sequence \( \{\eta_k^\alpha\}_{k \in \mathbb{Z}^d} \) satisfying

\[
\begin{align*}
|\eta_k^\alpha(\xi)| &\geq 1, \quad \text{if} \quad |\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k| \leq c \langle k \rangle^{\frac{\alpha}{1-\alpha}}; \\
\text{supp} \eta_k^\alpha &\subseteq \left\{\xi : \left|\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k\right| < C \langle k \rangle^{\frac{\alpha}{1-\alpha}}\right\}; \\
\sum_{k \in \mathbb{Z}^d} \eta_k^\alpha(\xi) &\equiv 1, \quad \forall \xi \in \mathbb{R}^d; \\
|\partial^\gamma \eta_k^\alpha(\xi)| &\leq C_\alpha \langle k \rangle^{-\frac{\alpha|\gamma|}{1-\alpha}}, \quad \forall \xi \in \mathbb{R}^d, \gamma \in \mathbb{N}^d,
\end{align*}
\]
where $C_\alpha$ is a positive constant depending only on $d$ and $\alpha$. We usually call these \( \{ \eta_k^\alpha \}_{k \in \mathbb{Z}^d} \) the bounded admission partition of unity ($\alpha$-BAPU) corresponding to the $\alpha$-covering \( \{ B_k \}_{k \in \mathbb{Z}^d} \). The operators of this frequency decomposition can be defined as follows:

\[
\square_k^\alpha := \mathcal{F}^{-1} \eta_k^\alpha \mathcal{F}.
\]

Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $\alpha \in [0, 1)$, the $\alpha$-modulation space $M^{s,\alpha}_{p,q}$ can be defined by

\[
M^{s,\alpha}_{p,q} = \left\{ f \in \mathcal{S}' : \| f \|_{M^{s,\alpha}_{p,q}} = \left( \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{s q / (1-\alpha)} \| \square_k^\alpha f \|_p \right)^{1/q} < \infty \right\},
\]

with the usual modification when $q = \infty$.

When $\alpha = 0$, we usually denote $M^{s,\alpha}_{p,q}$ by $M^{s}_{p,q}$, when $s = 0$, we denote $M^{s}_{p,q}$ by $M_{p,q}$.

**Remark 9.** In previous literature, researchers usually consider $\alpha$-modulation spaces in the case of $\alpha \in [0, 1)$, but here we extend the definition to include $\alpha < 0$. These generalized $\alpha$-modulation spaces and their embedding properties will be used in the proof of Theorem 4. For the convenience of readers, we provide the specific definition in “Appendix A”.

### 2.3. Some lemmas

In this subsection, we gather some useful results.

**Lemma 1.** ([19], Proposition 2.4, embedding) Let $\alpha \in [0, 1)$, $1 \leq p, q_1, q_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$.

1. If $q_1 > q_2$, $s_1 + d(1-\alpha)/q_1 > s_2 + d(1-\alpha)/q_2$, then $M^{s_1,\alpha}_{p,q_1} \hookrightarrow M^{s_2,\alpha}_{p,q_2}$.
2. If $q_1 \leq q_2$, $s_1 \geq s_2$, then $M^{s_1,\alpha}_{p,q_1} \hookrightarrow M^{s_2,\alpha}_{p,q_2}$.

**Lemma 2.** ([19], Theorem 2.2, interpolation) Let $\alpha \in [0, 1)$, $1 \leq p, p_0, p_1, q, q_0, q_1 \leq \infty$, $s, s_0, s_1 \in \mathbb{R}$. Suppose $0 < \theta < 1$ and

\[
s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},
\]

then we have

\[
\left( M^{s_0,\alpha}_{p_0,q_0}, M^{s_1,\alpha}_{p_1,q_1} \right)_\theta = M^{s,\alpha}_{p,q}.
\]

**Lemma 3.** ([42], Theorem 1.2, 1.4, 1.6) Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $\alpha \in [0, 1)$. Then $M^{s,\alpha}_{p,q} \hookrightarrow L^p$ if and only if one of the following conditions is satisfied:

1. $q \leq p < \infty$, $s \geq -(1-\alpha)\sigma(p, q)$;
Moreover, if we denote $J. Evol. Equ. Local smoothing in \alpha$-modulation spaces
Proof. The estimates of $F_{\beta, t}$ follows from the Young’s convolution inequality and the interpolation of the $(2, 2)$ boundedness.

For $F_{\beta, t}(\xi)$, if we denote $\rho_{\beta, t}(\xi) = e^{i|\xi|^2/2} \sigma((\xi - k) - t) \sigma((k - \alpha)/(1 - \alpha) \xi - k)$, then $F_{\beta, t} \rho_{d} \ast u = F_{\beta, t} \rho_{d} \ast u$. Therefore, we only need to estimate $\|F_{\beta, t} \rho_{d} \ast u\|_1$. By the scaling invariance of the $F L^1$ norm, we have
\[
\|F_{\beta, t} \rho_{d} \ast u\|_1 = \|F_{\beta, t} \rho_{d} \ast u\|_1 \approx \\langle k \beta(\alpha) \xi \rangle \approx (k - \alpha)/(1 - \alpha) \xi \rangle \approx 2t \langle k \beta(\alpha) \xi \rangle \approx 2 \langle k \beta(\alpha) \xi \rangle \approx 2 \|u\|_p.
\]
where $\beta\alpha/(1 - \alpha) + \beta - 2 = 0$ when $\alpha = 1 - \beta/2$.

Taking the $\ell^q_k$ norm on both sides of the above estimates, we have

Lemma 5. ([31], Theorem 1.1) Let $\beta > 2, 1 \leq p, q \leq \infty, s \in \mathbb{R}, s \geq d(\beta - 2)/2 - 1/p$. Then for any $u \in M^{s, \alpha}_p$, we have
\[
\|S_{\beta}(t) u\|_{M^{s, \alpha}_p} \approx \|u\|_{M^{s, \alpha}_p}.
\]

Lemma 6. ([40], Theorem 4.10) Let $0 < \beta \leq 2, \beta \neq 1, 1 \leq p, q \leq \infty, s \in \mathbb{R}$, denote $\alpha = 1 - \beta/2 \in [0, 1)$. Then for any $u \in M^{s, \alpha}_{p, q}$, we have
\[
\|S_{\beta}(t) u\|_{M^{s, \alpha}_{p, q}} \approx \|u\|_{M^{s, \alpha}_{p, q}}.
\]

Lemma 7. ([5], Theorem 1, Littlewood–Paley theory of $M^{s, \alpha}_{p, q}$) Let $\alpha < 1, 1 \leq p, q \leq \infty, s \in \mathbb{R}$. Then for any $u \in M^{s, \alpha}_{p, q}$, we have
\[
\|u\|_{M^{s, \alpha}_{p, q}} \approx \|2^j \Delta_j u\|_{M^{0, \alpha}_{p, q}}\|_{\ell^q_j},
\]
where $\Delta_j$ is the dyadic decomposition operator.
Remark 10. In [5], the authors presented the above argument for the modulation space, which is the special case of $M_{p,q}^{\alpha}$ with $\alpha = 0$. The main observation there is that for any $k \in \mathbb{Z}^d$, the number of $j \in \mathbb{N}$, where $\square_k \triangle_j \neq 0$, is bounded by a number depends only on the dimension $d$. For $\square_k^p$, this property is also true. One can refer to Subsection 4.2 in [19]. Therefore, with the same discussion in [5], we can obtain the lemma above. For simplicity, we omit the proof.

Lemma 8. (Embedding between $\alpha$-modulation spaces) Let $s \in \mathbb{R}$, $\alpha < 1$, $0 < p, q \leq \infty$.

1. If $\alpha > 0$, then $M_{p,q}^s \hookrightarrow M_{p,q}^{0,\alpha}$ if and only if $s \geq -\alpha \sigma(p,q)$.

2. If $\alpha < 0$, then $M_{p,q}^s \hookrightarrow M_{p,q}^{0,\alpha}$ if and only if $s \geq -\alpha \tau(p,q)$.

Proof. The proof of (1) can be found in Theorem 4.1 of [19], which is the special case of the embeddings between $\alpha$-modulation spaces. For characterization of these embedding relationships, one can refer to Theorem 1.2 in [17]. As for (2), the proof is similar to the proof of sharp embedding $M_{p,q}^{s_1,\alpha_1} \hookrightarrow M_{p,q}^{s_2,\alpha_2}$ when $\alpha_1 > \alpha_2$. One can see Appendix A or Section 4 in [19] for details. □

Lemma 9. ([3,4], Theorem 1.1, decoupling)

1. Let $S = \{(\xi, \psi(\xi)) : \xi \in \mathbb{R}^d\}$ be a compact $C^2$ hypersurface in $\mathbb{R}^{d+1}$ with positive definite second fundamental form. Denote $E_S f(t,x) = \int_{\mathbb{R}^d} e^{it\psi(\xi) + i\xi \cdot x} f(\xi) d\xi$, the Fourier extension operator of $S$. Then when $p = p_0 = 2 + 4/d$, for any $R \geq 1$, the following estimate holds for any $\varepsilon > 0$:

$$\|E_S f(t,x)\|_{L^p(B_{R}^{d+1})} \lesssim R^\varepsilon \left( \sum_{\square : R^{-1/2}-cube} \|E_S f\|_{L^p(\omega_{R}^{d+1})}^2 \right)^{1/2}.$$  

Here $B_{R}^{d+1}$ denotes any ball in $\mathbb{R}^{d+1}$ with radius $R$. $\omega_{R}^{d+1}$ denotes a smooth version of the indicator function on $B_{R}^{d+1}$.

2. Let $S = \{(\xi, \psi(\xi)) : \xi \in \mathbb{R}^d\}$ be a compact $C^2$ hypersurface in $\mathbb{R}^{d+1}$ with nonzero Gaussian curvature. The Fourier extension operator is also defined above. Then when $p = p_0 = 2 + 4/d$, for any $R \geq 1$, the following estimate holds for any $\varepsilon > 0$:

$$\|E_S f(t,x)\|_{L^p(B_{R}^{d+1})} \lesssim R^{d - \frac{d+1}{p} + \varepsilon} \left( \sum_{\square : R^{-1/2}-cube} \|E_S f\|_{L^p(\omega_{R}^{d+1})}^p \right)^{1/p}.$$  

3. Proof of Theorem 1

We first prove the proposition below, which is the key part of the proof of Theorem 1.
**Proposition 1.** Let $0 < \beta < 1$, denote $\alpha = 1 - \beta/2 \in (0, 1)$, $p_0 = 2 + 4/d$. If $p \geq p_0$, $s > \beta d(1 - 2/p)/2 - \beta/p$, then we have

$$
\| S_\beta'(t) u \|_{L^p(I \times \mathbb{R}^d)} \lesssim \| u \|_{M^{\alpha, \alpha}_{p, p}}, \forall u \in M^{\alpha, \alpha}_{p, p}.
$$

**Proof.** We only need to prove that when $s > \beta d(1 - 2/p)/2 - \beta/p$, $p \geq p_0$, the estimate

$$
\| S_\beta'(t) u \|_{L^p(I \times \mathbb{R}^d)} \lesssim \lambda^s \| u \|_{M^{\alpha, \alpha}_{p, p}},{(6)}
$$

is true for any $\lambda \geq 1$ and $u \in M^{\alpha, \alpha}_{p, p}$ with supp $\hat{u} \subseteq \{ \xi \in \mathbb{R}^d : \lambda/2 \leq |\xi| \leq \lambda \}$.

In fact, if we have the estimate above, then by the dyadic decomposition of $u$, we have $u = \sum_{j \geq 0} \Delta_j u$. By the triangle inequality, we have

$$
\| S_\beta'(t) u \|_{L^p(I \times \mathbb{R}^d)} \leq \sum_{j \geq 0} \| S_\beta'(t) \Delta_j u \|_{L^p(I \times \mathbb{R}^d)}.
$$

We can use (6) to estimate $\| S_\beta'(t) \Delta_j u \|_{L^p(I \times \mathbb{R}^d)}$. So we have

$$
\| S_\beta'(t) \Delta_j u \|_{L^p(I \times \mathbb{R}^d)} \lesssim 2^{js} \| \Delta_j u \|_{M^{\alpha, \alpha}_{p, p}}.
$$

Combining the two estimates above, we have

$$
\| S_\beta'(t) u \|_{L^p(I \times \mathbb{R}^d)} \lesssim \sum_{j \geq 0} 2^{js} \| \Delta_j u \|_{M^{\alpha, \alpha}_{p, p}} \lesssim \| \| \Delta_j u \|_{M^{\alpha, \alpha}_{p, p}} 2^{j(s+\varepsilon)} \|_{\ell^p_j},
$$

for any $\varepsilon > 0$. Here we use the Hölder’s inequality of $\ell^p_j$. Then by Lemma 7, we know that

$$
\| \Delta_j u \|_{M^{\alpha, \alpha}_{p, p}} 2^{j(s+\varepsilon)} \|_{\ell^p_j} \approx \| u \|_{M^{s+\varepsilon, \alpha}_{p, p}},
$$

which is the result as desired.

Next, we give the proof of (6). Denote $u(x) = v_\lambda(x) = v(\lambda x)$, then $\hat{u}(\xi) = \lambda^{-d} \hat{v}(\lambda^{-1} \xi)$, which means that supp $\hat{v} \subseteq \{ \xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 1 \}$.

By the homogeneity of $|\xi|^\beta$, it is easy to verify that $S_\beta'(t) v_\lambda(x) = S_\beta'(\lambda^\beta t) v(\lambda x)$. So, we have

$$
\| S_\beta'(t) u \|_{L^p(I \times \mathbb{R}^d)} = \| S_\beta'(\lambda^\beta t) v(\lambda x) \|_{L^p(I \times \mathbb{R}^d)} = \lambda^{-(\beta+d)/p} \| S_\beta'(t) v \|_{L^p(\lambda^\beta I \times \mathbb{R}^d)}.$$

In order to estimate $\| S_\beta'(t) v \|_{L^p(\lambda^\beta I \times \mathbb{R}^d)}$ by decoupling, we need to localize the physical space $\mathbb{R}^d$. We decompose $\mathbb{R}^d$ into balls with radius $\lambda^\beta$, which are bounded overlapped.

$$
\mathbb{R}^d = \bigcup B_{\lambda^\beta}.
$$
So, we have
\[ \| S_\beta(t) v \|_{L^p(\lambda^\beta I \times \mathbb{R}^d)} \leq \| S_\beta(t) v \|_{L^p(\lambda^\beta I \times B_{\beta})} \| f \|_{L^p_{\beta} B_{\beta}}. \] (8)

Recall that \( S_\beta(t) v(x) = E_\beta \hat{v}(t, x) \). By \( \ell^p \)-decoupling of \( M_\beta \) (Lemma 9), for any \( \varepsilon > 0 \), we have
\[ \| S_\beta(t) v \|_{L^p(\lambda^\beta I \times B_{\beta})} \leq \varepsilon \lambda^{-\beta \left( \frac{d+1}{p} - \frac{d}{2} \right) + \varepsilon} \| S_\beta(t) \Box^{\beta/2} k v \|_{L^p(\alpha \beta^I \times \omega_{B_{\beta}})} \| f \|_{L^p_{\beta} B_{\beta}}, \]
where \( \Box^{\beta/2} k u = \mathcal{F}^{-1} \sigma(\lambda^{\beta/2} \xi - k) \hat{u}(\xi) \), is the decomposition of \( \lambda^{\beta/2} \)-cubes. Substituting this into (8), we have
\[ (8) \leq \varepsilon \lambda^{-\beta \left( \frac{d+1}{p} - \frac{d}{2} \right) + \varepsilon} \| S_\beta(t) \Box^{\beta/2} k v \|_{L^p(\alpha \beta^I \times \omega_{B_{\beta}})} \| f \|_{L^p_{\beta} B_{\beta}} \]
\[ (9) \leq \varepsilon \lambda^{-\beta \left( \frac{d+1}{p} - \frac{d}{2} \right) + \varepsilon} \left( \lambda^d \int e^{i\lambda^{\beta} \xi} \sigma(\lambda^{\beta/2} \xi - k) \lambda^d \hat{u}(\lambda \xi) d\xi \right) \]
\[ = \int e^{i\lambda^{\beta - \beta} \xi} \sigma(\lambda^{\beta/2} \xi - k) \lambda^d \hat{u}(\lambda \xi) d\xi. \] (10)

Notice that when \( \text{supp} \sigma(\lambda^{\beta/2 - 1} \cdot -k) \cap \text{supp} \hat{u} \neq \emptyset \), we have \( \langle k \rangle \approx \lambda^{\beta/2} \). So, if we denote \( \alpha = 1 - \beta/2 \), we have \( \sigma(\lambda^{\beta/2 - 1} \xi - k) = \sigma((k)^{\beta/(1-\alpha)} \xi - k) \). Substituting these into (10) and (9), by scaling, we have
\[ (9) \leq \varepsilon \lambda^{-\beta \left( \frac{d+1}{p} - \frac{d}{2} \right) + \varepsilon} \left( \lambda^d \int e^{i\lambda^{\beta - \beta} \xi} \sigma(\lambda^{\beta/2} \xi - k) \lambda^d \hat{u}(\lambda \xi) d\xi \right) \]
\[ (10) \leq \varepsilon \lambda^{-\beta \left( \frac{d+1}{p} - \frac{d}{2} \right) + \varepsilon} \left( \lambda^d \int e^{i\lambda^{\beta - \beta} \xi} \sigma(\lambda^{\beta/2} \xi - k) \lambda^d \hat{u}(\lambda \xi) d\xi \right) \]
\[ (11) \leq \varepsilon \lambda^{-\beta \left( \frac{d+1}{p} - \frac{d}{2} \right) + \varepsilon} \lambda^d \| u \|_{M^{\alpha, \beta}_{p, p}}. \]

By Lemma 6, we have
\[ \| S_\beta(t) \Box^{\alpha} k u \| \leq \langle t \rangle^{d/2} \| \Box^{\alpha} k u \|_p. \]

Substituting this into (11), we have
\[ (11) \leq \varepsilon \lambda^{-\beta \left( \frac{d+1}{p} - \frac{d}{2} \right) + \varepsilon} \lambda^d \| u \|_{M^{\alpha, \beta}_{p, p}}. \]
Combining the estimate above and estimates (7) through (11), we have
\[ \|S_\beta(t)u\|_{L^p(I \times \mathbb{R}^d)} \lesssim \lambda^{-\beta\left(\frac{d+1}{p} - \frac{d}{2}\right) + \epsilon} \|u\|_{M^{p,\alpha}_{p,p}}, \]
which is (6) as desired. \( \square \)

Next, we are ready to prove Theorem 1.

**Proof of Theorem 1.** (A) By the embedding of \( M_{p,q}^{s,\alpha} \) (Lemma 1) and Proposition 1, when \( p = p_0, \alpha = 1 - \beta/2 \), we have
\[ \|S_\beta(t)u\|_{L^p(I \times \mathbb{R}^d)} \lesssim \|u\|_{\frac{\beta d}{p} \lambda^\left(\frac{1}{2} - \frac{d}{p}\right) - \frac{\beta}{p} + \alpha} \lesssim \|u\|_{\frac{\beta d}{q} \lambda^\left(\frac{1}{2} - \frac{d}{q}\right) - \frac{\beta}{q} + \alpha}. \]
When \( p = \infty, s > \frac{\beta d}{2} (1 - 1/q) \), by Lemma 3, we have \( M_{\infty,q}^{s,\alpha} \hookrightarrow L^\infty \). Then, by Lemma 6, we have
\[ \|S_\beta(t)u\|_{L^\infty(I \times \mathbb{R}^d)} \leq \|S_\beta(t)u\|_{L^\infty(M_{\infty,q}^{s,\alpha})} \lesssim \|t\|^{d/2} \|u\|_{L^\infty_{\infty,q}} \lesssim \|u\|_{M_{\infty,q}^{s,\alpha}}. \]
Then, the condition (A) follows by taking the interpolation (Lemma 2) between \((p, q) = (\infty, \infty), (\infty, 1), (p_0, \infty), (p_0, p_0)\).

(B) By taking the interpolation with \( p = p_0 \), we only need to consider the case of \( p = 2 \). Notice that
\[ \|S_\beta(t)u\|_{L^2(I \times \mathbb{R}^d)} = \|u\|_2. \]
So, (2) is equivalent to \( M_{2,q}^{s,\alpha} \hookrightarrow L^2 = M_{2,2}^{0,\alpha} \), which is true when \( s > \beta d (1/2 - 1/p)/2 \) according to Lemma 1.

(C) The cases of \( 1/q = -(d + 4)/dp + 1 \) and \( 2 \leq p \leq q \leq p_0 \) are already considered in (A)(B). By taking the interpolation, we only need to consider the case of \( 1/p + 1/q = 1, p \geq 2 \). When \( s > 0 \), by Lemma 3, we have \( M_{p,q}^{s,\alpha} \hookrightarrow L^p \). Then, we have
\[ \|S_\beta(t)u\|_{L^p(I \times \mathbb{R}^d)} \lesssim \|S_\beta(t)u\|_{L^p(M_{p,q}^{s,\alpha})} \lesssim \|u\|_{M_{p,q}^{s,\alpha}}. \]

(D, E) When \( q \leq p, \beta \leq p', s > 0 \) or \( p < 2, q < p, s > \frac{\beta d}{2} (1/p - 1/q) \), by Lemma 3, we have \( M_{p,q}^{s,\alpha} \hookrightarrow L^p \). Then, by an argument similar to the one used in (C), we have the result as desired. \( \square \)

**4. Proof of Theorem 3**

**Proof.** Choose \( \varphi \in \mathcal{S} \) with supp \( \hat{\varphi} \subseteq \{ \xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 1 \} \). Then we have \( \|\varphi\|_{M^{s,\alpha}_{p,q}} = \|\varphi\|_p \approx 1 \). Recall that
\[ S_\beta(t)\varphi(x) = \int_{\mathbb{R}^d} e^{-i|\xi|^{\beta} + i\xi \cdot \xi} \hat{\varphi}(\xi) d\xi. \]
Then we know that \( |S_\beta(t)\varphi(x)| \gtrsim 1 \) when \( |x| \ll 1, |t| \ll 1 \).
(a) For any $k \geq 10$, $k \in \mathbb{Z}^d$, denote $c_k = (k)^{1/(1-\alpha)}$. Take $u(x) = M_{c_k} \varphi(x) = e^{i c_k x} \varphi(x)$. Then we have $\hat{u}(\xi) = \hat{\varphi}(\xi - c_k)$ with $\text{supp} \hat{u} \subseteq c_k + [-1, 1]^d$. So, we know that $\|u\|_{M_{p,q}^{c_k}} = (k)^{\frac{s}{1-\alpha}}$.

Next, we estimate $S_\beta(t)u$:

$$S_\beta(t)u(x) = \int_{\mathbb{R}^d} e^{it|\xi|^\beta + i x \xi} \varphi(\xi - c_k) d\xi = e^{ixc_k} \int_{\mathbb{R}^d} e^{it|\xi + c_k|^\beta + i x \xi} \hat{\varphi}(\xi) d\xi = e^{it|c_k|^\beta + i x c_k} \int_{\mathbb{R}^d} e^{it(|\xi + c_k|^\beta - |c_k|^\beta |c_k|^\beta)} + i(x + \beta |c_k|^\beta c_k t) \hat{\varphi}(\xi) d\xi.$$  

Denote $h(\xi) = |\xi|^\beta$, then by Taylor’s expansion, we have

$$|\xi + c_k|^\beta - |c_k|^\beta |c_k|^\beta = h(\xi + c_k) - h(c_k) - \nabla h(c_k) \xi = \sum_{|\gamma|=2} \int_0^1 (1-t) \partial^\gamma h(c_k + t\xi) \xi^\gamma dt.$$  

Notice that when $0 < \beta \leq 2$, for any $|\gamma| = 2$, $|\xi| \geq 1$, we have $|\partial^\gamma h(\xi)| \leq 1$.

Therefore, we have $|S_\beta(t)u| \geq 1$ when $|t| \ll 1, |x + \beta |c_k|^\beta c_k t| / 1$, which means that $\|S_\beta(t)u\|_{L^p(I \times \mathbb{R}^d)} \gg 1$. Combining the estimates of $u$, we have $s \geq 0$ if (2) holds.

(b) For any $\lambda \gg 1$, take $u(x) = \varphi_\lambda(x) = \varphi(\lambda x)$. Then we have $\hat{u}(\xi) = \lambda^{-d} \hat{\varphi}(\lambda^{-1} \xi)$, which means that $\text{supp} \hat{u} \subseteq \{\xi \in \mathbb{R}^d : \lambda/2 \leq |\xi| \leq \lambda\}$. Denote

$$\landar{\lambda} = \left\{ k \in \mathbb{Z}^d : \Box^\alpha_k u \neq 0 \right\}.$$  

Then by orthogonality we know that for any $k \in \landar{\lambda}$, we have $(k)^{\frac{1}{1-\alpha}} \approx \lambda$, $\# \landar{\lambda} \approx \lambda^{d(1-\alpha)}$. So, we have

$$\|u\|_{M_{p,q}^{c_k}} = \|\Box^\alpha_k u\|_{L^p} (k)^{s/(1-\alpha)}$$  

$$\lesssim \lambda^s \|\varphi\|_1 \|\mathcal{F}^{-1} \sigma ((k)^{-\alpha} - k)\|_{L^q_{c_k \in \landar{\lambda}}}$$  

$$\lesssim \lambda^s \lambda^{-d} (k)^{(1-\alpha)p} \lambda^{(1-\alpha)d/q} \approx \lambda^{s-d + d\alpha/p + d(1-\alpha)/q}.$$  

By scaling, we have

$$\|S_\beta(t)u\|_{L^p(I \times \mathbb{R}^d)} = \|S_\beta(\lambda^\beta t)\varphi(\lambda x)\|_{L^p(I \times \mathbb{R}^d)}$$  

$$= \lambda^{-\left(\beta + d\right)/p} \|S_\beta(t)\varphi\|_{L^p(\lambda^\beta I \times \mathbb{R}^d)}$$  

$$\gtrsim \lambda^{-\left(\beta + d\right)/p},$$  

where we use the estimate of $|S_\beta(t)\varphi|$ at the last inequality. Taking the estimates of $u$ into (2), we have $s \geq d - \frac{1}{p} - \frac{\alpha}{p'} - d(1-\alpha)/q - (\beta + d)/p = \frac{\beta d}{2} (1 - \frac{1}{p} - \frac{1}{q} - \frac{\beta}{p}).$
(c) Fix $N \in \mathbb{N}$, which will be chosen later. For any $\lambda \gg 1$, denote
\[
\wedge_{\lambda} = \left\{ k \in \mathbb{Z}^d : \langle k \rangle^{1/(1-\alpha)} \approx \lambda \right\}.
\]
Take $u_N = \sum_{k \in \wedge_{\lambda}} T_{Nk}(M_{ck}\varphi)$, where $M_{ck}f(x) = f(x)e^{ixck}$, $T_{Nk}f(x) = f(x-Nk)$. Then by the same discussion as in (a), we have the following estimate:
\[
\|u_N\|_{M_{p,q}^{s,\alpha}} = \left\| \langle k \rangle^{s/(1-\alpha)} \|\Box^q_k u_N\|_p \right\|_q \lesssim \lambda^s \left\| T_{Nk}(M_{ck}\varphi) \right\|_p \|\Box^q_k u_N\|_p \lesssim \lambda^s \lambda^{(1-\alpha)d/p}.
\]
On the other hand, $S_{\beta}(t)u_N = \sum_{k \in \wedge_{\lambda}} S_{\beta}(t)T_{Nk}(M_{ck}\varphi)$. By the same calculation as in (a), we know that
\[
|S_{\beta}(t)T_{Nk}(M_{ck}\varphi)| \gtrsim 1 \text{ in } D_k,
\]
where $D_k = \{(x, t) : |t| \ll 1, |x-Nk+\beta|ck|^{\beta-2}ck| \ll 1\}$. We can choose $N \gg 1$ so that $\{D_k\}_{k \in \wedge_{\lambda}}$ are disjoint. Then we have
\[
\left\| S_{\beta}(t)u_N \right\|_{L^p(I \times \mathbb{R}^d)} \gtrsim \left\| S_{\beta}(t)T_{Nk}(M_{ck}\varphi) \right\|_p \|\Box^q_k u_N\|_p \gtrsim \lambda^{(1-\alpha)d/p}.
\]
Taking the estimates of $u_N$ into (2), we have
\[
s \geq d(1-\alpha) \left( \frac{1}{p} - \frac{1}{q} \right) = \frac{\beta d}{2} \left( \frac{1}{p} - \frac{1}{q} \right).
\]
\[\square\]

5. Proof of Theorem 4

We first prove the proposition below.

**Proposition 2.** Let $\beta > 2$, denote $p = p_0 = 2 + 4/d$. If $s > d(\beta - 1)(1/2 - 1/p) - \beta/p$, then we have
\[
\left\| S_{\beta}(t)u \right\|_{L^p(I \times \mathbb{R}^d)} \lesssim \|u\|_{M_{p,2}^s}
\]
holds for any $u \in M_{p,2}^s$.

**Proof.** Denote $\alpha = 1 - \beta/2 < 0$, by the same method as in the proof of Theorem 1, for any $\lambda \geq 1$ and any $u \in \mathscr{S}$ with supp $\hat{u} \subseteq \{\xi : \lambda/2 \leq |\xi| \leq \lambda\}$, denote $v(x) = u_{\lambda^{-1}}(x) := u(\lambda^{-1}x)$. Then for any $\varepsilon > 0$, we have
\[
\left\| S_{\beta}(t)u \right\|_{L^p(I \times \mathbb{R}^d)} = \lambda^{-\frac{\beta+d}{p}} \left\| S_{\beta}(t)v \right\|_{L^p(\lambda^\beta I \times \mathbb{R}^d)}
\]
\[
\lambda^{-\frac{\beta+d}{p}+\varepsilon} \left\| S_\beta(t) \Box_k^{\beta/2} k u \right\|_{L^p(Q_t)} \lesssim \| u \|_{M_{p,2}^{0,\alpha}}.
\]

By Lemma 4, for any \( t \in I \), we have
\[
\left\| S_\beta(t) \Box_k^\alpha u \right\|_{L^p(I \times \mathbb{R}^d)} \lesssim t^{d(1/2-1/p)} \| \Box_k^\alpha u \|_{L^p}.
\]
Substituting this into the estimate of \( S_\beta(t)u \) above, we have
\[
\left\| S_\beta(t)u \right\|_{L^p(I \times \mathbb{R}^d)} \lesssim \lambda^\varepsilon \left\| \Box_k^\alpha u \right\|_{L^p} = \lambda^\varepsilon \| u \|_{M_{p,2}^{0,\alpha}}.
\]

So, by the dyadic decomposition, for any \( u \in \mathcal{S} \), we have
\[
\left\| S_\beta(t)u \right\|_{L^p(I \times \mathbb{R}^d)} \lesssim \| u \|_{M_{p,2}^{r,\alpha}}.
\]

Then by Lemma 8, we know that \( M_{p,2}^r \hookrightarrow M_{p,2}^{0,\beta} \) if \( s \geq -\beta \tau(p, 2) = (\beta - 1)d(1/2 - 1/p) - \beta/p \).

Then we can get the result as desired in the proposition. \( \square \)

Next, we could give the proof of Theorem 4.

**Proof of Theorem 4.** (a) By Remark 7, when \( p \leq 2 \) or \( p = \infty \), we do not have any smooth effects. So, in these cases, the argument in Theorem 4 follows by the results in [26].

(b) Proposition 2 above is just the case of \( (p, q) = (p_0, 2) \). Then, by Lemma 1, we know that for any \( \varepsilon > 0 \), we have \( M_{p_0,1}^s \hookrightarrow M_{p_0,2}^s \) and \( M_{p_0,\infty}^{s+d/2+\varepsilon} \hookrightarrow M_{p_0,2}^s \), which means that (4) is true under the conditions of Theorem 4 when \( (p, q) = (p_0, 1), (p_0, \infty) \).

(c) The rest of the cases follow by taking the interpolation between (a) and (b) above. For example, Condition (A) in Theorem 4 can be obtained by taking the interpolation between \( (p, q) = (\infty, \infty), (\infty, 1), (p_0, 2), (p_0, \infty) \). The other conditions are similar. \( \square \)

**6. Proof of Theorem 5**

**Proof.** Choose \( \varphi \in \mathcal{S} \) with supp \( \hat{\varphi} \subseteq \{ \xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 1 \} \). Then we have \( \| \varphi \|_{M_{p,q}^s} = \| \varphi \|_p \approx 1 \). Recall that
\[
S_\beta(t)\varphi(x) = \int_{\mathbb{R}^d} e^{it|\xi|^\beta + ix\xi} \hat{\varphi}(\xi) d\xi,
\]
so, we know that \( |S_\beta(t)\varphi(x)| \gtrsim 1 \) when \( |x| \ll 1, |t| \ll 1 \).
(a) For any $\lambda \gg 1$, take $u(x) = \varphi_\lambda(x) = \varphi(\lambda x)$, then we have $\hat{u}(\xi) = \lambda^{-d} \hat{\varphi}(\lambda^{-1} \xi)$, which means that $\text{supp} \hat{u} \subseteq \{ \xi \in \mathbb{R}^d : \lambda/2 \leq |\xi| \leq \lambda \}$. Denote

$$\wedge_\lambda = \{ k \in \mathbb{Z}^d : \Box_k u \neq 0 \}.$$  

Then by the orthogonality, we know that for any $k \in \wedge_\lambda$, we have $\langle k \rangle \approx \lambda$. Also $\# \wedge_\lambda \approx \lambda^d$. So, by Young’s convolution inequality, we have

$$\|S_\beta(t)u\|_{L^p(I \times \mathbb{R}^d)} \lesssim \lambda^s \|\Box_k u\|_{L^q(I \times \mathbb{R}^d)} \lesssim \lambda^s \|u\|_1 \|\mathcal{F}^{-1} \sigma(\cdot - k)\|_{L^q} \lesssim \lambda^{s-d+d/q}.$$  

By scaling, we have

$$\|S_\beta(t)u\|_{L^p(I \times \mathbb{R}^d)} = \lambda^{-(\beta-\gamma)/p} \|S_\beta(\lambda^{\beta-\gamma} t)\varphi(\lambda x)\|_{L^p(I \times \mathbb{R}^d)}$$

Taking these two estimates into (4), we have $s \geq d(1 - 1/p - 1/q) - \beta/p$.

(b) Take $u = \varphi_\lambda$ as above. By scaling, we have

$$\|S_\beta(t)u\|_{L^p(I \times \mathbb{R}^d)} = \lambda^{d(\beta-1)/p} \|S_\beta(\lambda^{\beta-1} t)\varphi(\lambda x)\|_{L^p(I \times \mathbb{R}^d)}$$

where

$$I(t, x) = S_\beta(\lambda^{\beta-1} t)\varphi(\lambda x) = \int_{\mathbb{R}^d} e^{i\lambda^{\beta-1} t \xi\cdot x + \xi d} \hat{\varphi}(\xi) d\xi.$$  

By the method of stationary phase, for any $t \in [1/2, 1]$, we have

$$|I(t, x)| \lesssim \begin{cases} \lambda^{-\beta d/2}, & |x| \leq 10; \\ \lambda^\beta |x|^{100}, & |x| \geq 10. \end{cases}$$

So, we have $\|I(t, x)\|_{L^p([1/2, 1] \times \mathbb{R}^d)} \lesssim \lambda^{-\beta d/2}$ when $p = 1, \infty$. By the $L^2$ isometric of $S_\beta(t)$, we have $\|I(t, x)\|_{L^2([1/2, 1] \times \mathbb{R}^d)} = \lambda^{-\beta d/2}$. Then by the convex inequality of the $L^p$ norm:

$$\|I(t, x)\|_{L^p([1/2, 1] \times \mathbb{R}^d)} \leq \|I(t, x)\|_{L^\infty([1/2, 1] \times \mathbb{R}^d)}^{1-\theta} \|I(t, x)\|_{L^\infty([1/2, 1] \times \mathbb{R}^d)}^\theta,$$  

for $p < 2, \theta = p/2$.  


\[ \|I(t, x)\|_{L^2([1/2, 1] \times \mathbb{R}^d)} \leq \|I(t, x)\|_{L^p([1/2, 1] \times \mathbb{R}^d)}^\theta \|I(t, x)\|_{L^1([1/2, 1] \times \mathbb{R}^d)}^{1-\theta}, \] for \( p > 2, \theta = p'/2. \]

Then we have \( \|I(t, x)\|_{L^p([1/2, 1] \times \mathbb{R}^d)} \geq \lambda^{-\beta d/2}. \) Substituting this into (12), we have

\[ \|S_\beta(t)u\|_{L^p(I \times \mathbb{R}^d)} \gtrsim \lambda^{d(\beta-1)/p-\beta d/2}. \]

Substituting this estimate into (4), we have \( s - d + d/q \geq d(\beta - 1)/p - \beta d/2, \) which means that \( s \geq d(\beta - 2)/(1/p - 1/2) + d(1/p - 1/q). \)

(c) When \( p \leq 2, \) for any \( |k| \geq 10, k \in \mathbb{Z}^d, \) take \( u = M_k \varphi, \) then \( \hat{u}(\xi) = \hat{\varphi}(\xi - k). \)

So, we know \( \|u\|_{M^p_{\beta,q}} \approx \langle k \rangle^s. \) By the change of variables, we have

\[ S_\beta(t)u(x) = \int_{\mathbb{R}^d} e^{it|\xi|^\beta + ix\xi} \hat{\varphi}(\xi - k) d\xi \]

\[ = e^{it|k|^\beta + i|x|} \int_{\mathbb{R}^d} e^{it(h(\xi + k) - h(k) - \nabla h(k)\xi)} e^{ix(\xi + t\nabla h(k))} \varphi(\xi) d\xi, \]

where \( h(\xi) = |\xi|^\beta. \) Denote \( h_k(\xi) = h(\xi + k) - h(k) - \nabla h(k)\xi, \) one can easily get that

\[ |\partial^\gamma h_k(\xi)| \lesssim |k|^{\beta-2}. \] \( (13) \)

Taking the \( L^p \) norm on both sides above, we have

\[ \|S_\beta(t)u\|_{L^p(I \times \mathbb{R}^d)} = \|H(t, x)\|_{L^p(I \times \mathbb{R}^d)}, \] \( (14) \)

where \( H(t, x) = \int_{\mathbb{R}^d} e^{i th_k(\xi) + ix\xi} \hat{\varphi}(\xi) d\xi. \) We can choose \( \eta \in \mathcal{S} \) such that \( \eta(\xi) = 1 \) when \( |\xi| \leq 1, \) and \( \eta(\xi) = 0, \) when \( |\xi| > 2. \) So, we have \( \eta \hat{\varphi} = \hat{\varphi}. \)

Denote \( \eta_t(\xi) = e^{-it h_k(\xi)} \eta(\xi). \) So, we have \( \eta_t(\xi) \cdot \mathcal{F} H(t, \cdot)(\xi) = \hat{\varphi}(\xi), \) then

\[ \varphi(x) = \mathcal{F}^{-1} \eta_t \ast H(t, \cdot)(x). \] \( (15) \)

By the method of stationary phase and the estimates (13), we know that for any \( t \in [1/2, 1], \) we have

\[ \|\mathcal{F}^{-1} \eta_t\|_\infty = \left\| \int_{\mathbb{R}^d} e^{i \langle k \rangle^{\beta-2} (th_k(\xi)/\langle k \rangle^{\beta-2} + x\xi)} \eta(\xi) d\xi \right\|_\infty \lesssim \langle k \rangle^{-d(\beta-2)/2}. \]

By Young’s convolution inequality, for any \( t \in [1/2, 1], \) we have

\[ \|\mathcal{F}^{-1} \eta_t \ast f\|_\infty \lesssim \langle k \rangle^{-d(\beta-2)/2} \|f\|_1. \]

Obviously, \( \|\mathcal{F}^{-1} \eta_t \ast f\|_2 = \|f\|_2. \) Then, by taking the interpolation, for any \( p \leq 2, \) we have

\[ \|\mathcal{F}^{-1} \eta_t \ast f\|_{p'} \lesssim \langle k \rangle^{-d(\beta-2)(1/p-1/2)} \|f\|_p. \]
Substituting this into (15), we have

$$1 \approx \| \varphi \|_{p'} \lesssim \langle k \rangle^{-d(\beta - 2)(1/p - 1/2)} \| H(t, \cdot) \|_p.$$ 

Taking integration of $t$ over $[1/2, 1]$, we have

$$1 \lesssim \langle k \rangle^{-d(\beta - 2)(1/p - 1/2)} \| H(t, x) \|_{L^p([1/2, 1] \times \mathbb{R}^d)}.$$ 

Then by (14), we have

$$\| S_\beta(t)u \|_{L^p(I \times \mathbb{R}^d)} \gtrsim \langle k \rangle^{d(\beta - 2)(1/p - 1/2)}.$$ 

Then substituting the estimates of $u$ into (4), we have

$$s \geq d(\beta - 2)(1/p - 1/2),$$ 

which is the result as desired.

\[\square\]

7. Applications

As applications of the local smoothing estimates in Theorem 4, we could obtain the local well-posedness of the following one dimensional fourth order nonlinear Schrödinger equation (4NLS) on modulation spaces with lower regularity request of the initial data. Then, by the energy argument, we could obtain the global well-posedness of (4NLS).

$$\begin{cases}
  iu_t + u_{xxxx} = -|u|^2 u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
  u(0, x) = u_0, & x \in \mathbb{R}.
\end{cases}$$

(4NLS)

Our main results are:

**Theorem 6.** For any $D \in \{M^s_{p,2} + H^{-s(p)} : 10/3 \leq p \leq 6, s > s(p) := 1/2 - 1/p\}$, (4NLS) is local well-posedness in $D$. In detail, for any $u_0 \in D$, there exists $T(\|u_0\|_D) > 0$, and a unique solution $u$ of (4NLS) with $u \in L^a(p)(I_T \times \mathbb{R}^d)$, where $a(p) = 8p/(3p - 2)$, $I_T = [0, T]$. Also, the data-to-solution map above is Lipschitz continuous.

**Proof.** For any $u_0 \in D$, denote $I_T = [0, T]$, $X = L^a(p)(I_T \times \mathbb{R}^d)$, where $0 < T \leq 1$, depends on $\|u_0\|_D$, will be chosen later. Denote $B_X(R) = \{u \in X : \|u\|_X \leq R\}$, the closed ball in $X$, centering at 0 with radius of $R$. Notice that when $10/3 \leq p \leq 6$, we have $a(p) \leq p$.

For any decomposition of $u_0 = u_0^{(1)} + u_0^{(2)}$, with $u_0^{(1)} \in M^s_{p,2}, u_0^{(2)} \in H^{-s(p)}$. By Hölder’s inequality and the local smoothing estimates in Theorem 4, we have

$$\left\| S_4(t)u_0^{(1)} \right\|_X \leq |T|^{1/a(p) - 1/p} \left\| S_4(t)u_0^{(1)} \right\|_{L^p(I \times \mathbb{R}^d)} \lesssim |T|^{(3p - 10)/8p} \left\| u_0^{(1)} \right\|_{M^s_{p,2}}.$$ 

(16)
Denote $A f(x, t) = \int_0^t S_4(t - \tau) f(\tau) d\tau$. By Strichartz estimates of $S_4(t)$ with smoothing effects (Proposition 2 in [29]), we have

$$\| S_4(t)u_0^{(2)} \|_{L_t^{\gamma_1(p)} L_x^p} \lesssim \| u_0^{(2)} \|_{H^{-s(p)}},$$

where $(\gamma_1(p), p)$ is the classical Strichartz pair, satisfying:

$$\frac{1}{\gamma_1(p)} = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{p} \right).$$

Note that when we consider the estimate of the local time, the estimate above also holds for inhomogeneous spaces:

$$\| S_4(t)u_0^{(2)} \|_{L_t^{\gamma_1(p)} L_x^p} \lesssim \| u_0^{(2)} \|_{H^{-s(p)}}, \quad (17)$$

By Strichartz estimates of $S_4(t)$ without smoothing effects (Proposition 1 in [29]), we have

$$\| A f \|_{L_t^{\gamma(p)} L_x^p} \lesssim \| f \|_{L_t^{a(p)/3} L_x^{p/3}}, \quad (18)$$

where $(\gamma(p), p)$ is the biharmonic Strichartz pair, satisfying:

$$\frac{1}{\gamma(p)} = \frac{1}{4} \left( \frac{1}{2} - \frac{1}{p} \right).$$

Notice that when $10/3 \leq p \leq 6$, we have $a(p) \leq \gamma_1(p)$. So, by Hölder’s inequality and (17), we have

$$\left\| S_4(t)u_0^{(2)} \right\|_X \leq |T|^{1/a(p) - 1/\gamma_1(p)} \left\| S_4(t)u_0^{(2)} \right\|_{L_t^{\gamma_1(p)} L_x^p} \lesssim |T|^{1/8 + 1/4p} \left\| u_0^{(2)} \right\|_{H^{-s(p)}}, \quad (19)$$

Combining (16) and (19), we know that there exists $M > 0$, depends only on $p, s, d$, such that

$$\| S_4(t)u_0 \|_X \leq M \| u_0 \|_D. \quad (20)$$

Substituting $f = |u|^2 u$ into the Strichartz estimate (18), also by Hölder’s inequality, we have

$$\| Af \|_X \leq |T|^{1/a(p) - 1/\gamma(p)} \| Af \|_{L_t^{\gamma(p)} L_x^p} \lesssim |T|^{1/4} \| f \|_{L_t^{a(p)/3} L_x^{p/3}} \leq C |T|^{1/4} \| u \|_X^3. \quad (21)$$
If we choose $T \ll 1$ such that $C |T|^{1/4} (2M \|u_0\|_D)^2 \leq 1/10$. Then for any $u_0 \in B_X (2M \|u_0\|_D)$, by (20) and (21), we have

$$\|S_4(t)u_0\|_X + \left\| A \left( |u|^2 u \right) \right\|_X \leq M \|u_0\|_D + C |T|^{1/4} \|u\|_X^3 \leq M \|u_0\|_D + 10 \|u\|_X \leq 2M \|u_0\|_D.$$ 

Therefore, if we define $\mathcal{T} : u \rightarrow S_4(t)u_0 + iA \left( |u|^2 u \right)$, we know that

$$\mathcal{T} : B_X (2M \|u_0\|_D) \rightarrow B_X (2M \|u_0\|_D)$$

is a contraction. Then by the contraction mapping theorem, there exists $u \in B_X (2M \|u_0\|_D)$,

such that $u = \mathcal{T} u$, which means that $u \in X$ is the solution of (4NLS), satisfying $\|u\|_X \leq 2M \|u_0\|_D$, which means that the data-to-solution map is Lipschitz continuous. \hfill \Box

If the initial $u_0$ has better regularity, the local solution $u$ above is actually global. Precisely, we have

**Theorem 7.** For any $E \in \{ M^s_{p,2} : 10/3 \leq p \leq 6, s > 7/2 - 2/p \}$, if initial data $u_0 \in E$, then the solution $u$ in Theorem 6 can be extended to $t \in [0, \infty)$. \hfill \Box

**Proof.** Our proof, named the energy argument, is based on the method in [34], one can also refer to [6].

For any $u_0 \in E$, choose $D$ in Theorem 6 with the same $p$ as $E$. By Theorem 6, we have a unique solution $u \in X = L_{t \in I_T}^a(p) L_x^p$, such that $u = S_4(t)u_0 + iA \left( |u|^2 u \right)$. Notice that when $s > 7/2 - 2/p$, by Lemma 5, we have

$$\|S_4(t)u_0\|_D \leq \|S_4(t)u_0\|_{M^{1/2 - 1/p+}_{p,2}} \lesssim (T)^{1/2} \|u_0\|_{M^{1/2 - 1/p+}_{p,2}} \lesssim (T)^{1/2} \|u_0\|_{M'_{p,2}}.$$

(22)

As for the nonlinear term $A \left( |u|^2 u \right)$, by Strichartz estimates (18), we have

$$\left\| A \left( |u|^2 u \right) \right\|_{L_{t \in I_T}^\infty(D)} \leq \left\| A \left( |u|^2 u \right) \right\|_{L_{t \in I_T}^\infty L_x^2} \lesssim \|u\|_X^3.$$ 

So, we know that $\|u(T)\|_D < \infty$. By the semigroup theory, we can also solve the equation (4NLS) at $t = T$. In this way, we have a maximal time $T^* > 0$, where the solution $u$ exists in $[0, T]$ for any $T < T^*$. By the blow-up criterion, if $T^* < \infty$, it must be

$$\lim_{t \rightarrow T^*} \|u(t)\|_D = \infty.$$
Denote \( w(t) = S_4(t)u_0, v(t) = u(t) - w(t) \), by (22), we know that

\[
\lim_{t \to T^*} \|W(t)\|_D \leq C(T^*) < \infty.
\]

Therefore, it must be

\[
\lim_{t \to T^*} \|v(t)\|_D = \infty.
\]

Notice that \( L^2 \hookrightarrow D \), so we have

\[
\lim_{t \to T^*} \|v(t)\|_2 = \infty.
\]

Then we only need to prove that

\[
\sup_{0 < t < T^*} \|v(t)\|_2 \leq C(T^*) < \infty.
\]

Denote \( \langle v, u \rangle = \int_{\mathbb{R}} v \bar{u} dx \), the inner product in \( L^2(\mathbb{R}) \). The mass and energy of \( v \) are defined below:

\[
M_v(t) = \frac{1}{2} \langle v, v \rangle;
\]

\[
E_v(t) = \frac{1}{2} \langle v_{xx}, v_{xx} \rangle + \frac{1}{4} \left( v^2, v^2 \right);
\]

\[
\tilde{E}_v(t) = \frac{1}{2} \langle v_{xx}, v_{xx} \rangle + \frac{1}{4} \left( (v + w)^2, (v + w)^2 \right) - \frac{1}{4} \left( w^2, w^2 \right).
\]

Note that a priori, it is not clear that the mass and energy of \( v \) is finite for \( t \neq 0 \).

The following computations are carried out for initial data from a priori class, say \( u_0 \in \mathcal{S} \). This ensures all quantities to be finite and allows to justify integration by parts arguments. Since we prove bounds depending only on \( \|u_0\|_{M^p_q} \) for \( p < \infty \), the arguments are a posteriori justified by density of \( \mathcal{S} \subseteq D \) and well-posedness.

By definition of \( v = u - w \), we know that \( v \) satisfies the equation below:

\[
\begin{cases}
iv_t + v_{xxxx} = -|v + w|^2 (v + w), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R} \\
v(0, x) = 0, \quad x \in \mathbb{R}
\end{cases}
\] (4NLSw)

Then we have

\[
\frac{d}{dt} M_v(t) = \text{Re} \langle v_t, v \rangle = \text{Re} \left( iv_{xxxx} + i |v + w|^2 (v + w), v \right)
\]

\[
= \text{Re} \langle iv_{xx}, v_{xx} \rangle + \text{Re} \left( i |v + w|^2 v, v \right) + \text{Re} \left( i |v + w|^2 w, v \right)
\]

\[
= -\text{Im} \left( |v + w|^2 w, v \right)
\]

\[
= -\text{Im} \left( \langle v \bar{w}, v \rangle + \langle vw \bar{w}, v \rangle + \langle \bar{v}w, v \rangle + \langle w \bar{w}, v \rangle \right).
\]

So, we have

\[
\left| \frac{d}{dt} M_v(t) \right| \lesssim \|v^3w\|_1 + \|v^2w^2\|_1 + \|vw^3\|_1.
\] (24)
Notice that by Lemma 3, we know that for any $p \leq 6 \leq q \leq \infty$, we have $M_{p,2}^{1/2+} \hookrightarrow M_{q,2}^{1/2+} \hookrightarrow L^q$, so we have

$$
\|w(t)\|_q = \|S_4(t)u_0\|_q \lesssim \|S_4(t)u_0\|_{M_{p,2}^{1/2+}} \lesssim (T^*)^{1/2} \|u_0\|_{M_{p,2}^{1/2++2(1/2-1/p)+}} \lesssim T^* 1.
$$

(25)

So, by Hölder’s inequality, we have

$$
\begin{align*}
\|v^3w\|_1 &\leq \|w\|_\infty \|v\|_2 \|v\|^2_4 \lesssim \|v\|^2_2 + \|v\|^4_4; \\
\|v^2w^2\|_1 &\leq \|w\|_\infty \|v\|^2_2 \lesssim \|v\|^2_2; \\
\|vw^3\|_1 &\leq \|w\|^3_6 \|v\|_2 \lesssim \|v\|^2_2 + 1.
\end{align*}
$$

Substituting these into (24), we have

$$
\left| \frac{d}{dt} M_v(t) \right| \lesssim T^* E_v(t) + M_v(t) + 1.
$$

(26)

By the definition of $\tilde{E}_v(t)$, we have

$$
\tilde{E}_v(t) - E_v(t) = \frac{1}{4} \left( (v + w)^2, (v + w)^2 \right) - \left( w^2, w^2 \right) - \left( v^2, v^2 \right).
$$

$$
= \left( v^2, vw \right) + \frac{1}{2} \left( v^2, w^2 \right) + \left( vw, vw \right) + \left( vw, w^2 \right).
$$

So, by the same estimates of $w$ above, we have

$$
|\tilde{E}_v(t) - E_v(t)| \leq \|v^3w\|_1 + \|v^2w^2\|_1 + \|vw^3\|_1 \lesssim T^* \|v\|_2 \|v\|^2_4 + \|v\|^2_2 + \|v\|_2.
$$

Then by Cauchy Schwarz inequality $ab \leq \varepsilon a^2 + b^2/4\varepsilon$, we have

$$
|\tilde{E}_v(t) - E_v(t)| \leq \frac{1}{2} \|v\|^4_4 + C(T^*) \left( \|v\|^2_2 + 1 \right) \leq \frac{1}{2} E_v(t) + C(T^*) (M_v(t) + 1).
$$

If we denote $A_v(t) = M_v(t) + 1$, then we have

$$
-\frac{1}{2} E_v(t) - C(T^*) A_v(t) \leq \tilde{E}_v(t) - E_v(t) \leq \frac{1}{2} E_v(t) + C(T^*) A_v(t),
$$

which means that

$$
\frac{1}{2} \left( E_v(t) + 2C(T^*) A_v(t) \right) \leq \tilde{E}_v(t) + 2C(T^*) A_v(t) \leq \frac{3}{2} \left( E_v(t) + 2C(T^*) A_v(t) \right),
$$

which is equivalent to

$$
E_v(t) + 2C(T^*) A_v(t) \approx \tilde{E}_v(t) + 2C(T^*) A_v(t).
$$

(27)
Substituting this into (26), we have
\[ \left| \frac{d}{dt} M_v(t) \right| \lesssim_{T^*} \tilde{E}_v(t) + 2C(T^*)A_v(t). \] (28)

As for the estimate of \( \tilde{E}_v(t) \), by (4NLSw), we have
\[
\frac{d}{dt} \tilde{E}_v(t) = \text{Re} \langle v_{xxx}, v_{xx} \rangle + \text{Re} \left( (v_t + w_t)(v + w), (v + w)^2 \right) - \text{Re} \left( w_t w, w^2 \right)
= \text{Re} \langle v_t, v_{xxx} \rangle + \text{Re} \left( v_t + w_t, |v + w|^2 (v + w) \right) - \text{Re} \left( w_t, |w|^2 w \right)
= \text{Re} \langle v_t, v_{xxx} \rangle + |v + w|^2 (v + w) + \text{Re} \left( w_t, |v + w|^2 (v + w) - |w|^2 w \right)
= \text{Re} \left( i w_{xxx}, |v + w|^2 (v + w) - |w|^2 w \right). \] (29)

Compute that
\[ |v + w|^2 (v + w) - |w|^2 w = \left( v^2 \overline{v} \right) + \left( 2v \overline{v} w + v^2 w \right) + \left( 2v w \overline{w} + \overline{w}^2 \right) := E_{3,0} + E_{2,1} + E_{1,2}. \]

Substituting this into (29), and taking integration by parts, we have
\[
\frac{d}{dt} \tilde{E}_v(t) = -\text{Im} \langle w_{xxx}, E_{3,0} + E_{2,1} + E_{1,2} \rangle
= -\text{Im} \langle w_{xx}, (E_{3,0})_{xx} \rangle - \text{Im} \langle w_{xx}, (E_{2,1})_{xx} \rangle - \text{Im} \langle w_{xx}, (E_{1,2})_{xx} \rangle. \] (30)

Next, we estimate these three parts separately.

For the \( E_{3,0} \) part, we have
\[
\left| \langle w_{xx}, (E_{3,0})_{xx} \rangle \right| \leq \left\| w_{xx} v_{xx} v^2 \right\| + \left\| w_{xx} v_x^2 \right\| \leq \left\| w_{xx} \right\| \left\| v_{xx} \right\|_2 \left\| v \right\|_4 + \left\| w_{xx} \right\|_\infty \left\| v \right\|_4 \left\| v_x \right\|_{8/3}. \] (31)

By (25), we know that for any \( p \leq 6 \leq q \leq \infty \),
\[
\left\| w_{xx} \right\|_q \lesssim \left( T^* \right)^{1/2} \left\| u_0 \right\|_{M^{2+1/2+2(1/2-1/p)+}}^{1/2} \lesssim T^* \left\| u_0 \right\|_{M^{7/2-2/p+}} \lesssim T^* 1.
\]
By Gagliardo–Nirenberg inequality (See Appendix C.1 in [37]), we have
\[ \|v_x\|_{8/3} \lesssim \|v\|_{1/2}^{1/2} \|v_{xx}\|_{1/2}^{1/2}. \]

Substituting these two estimates into (31), we have
\[ \|w_{xx}, (E_{3,0})_{xx}\| \lesssim T^* \|v_{xx}\|_2 \|v\|_4^2 \lesssim T^* \|v_{xx}\|_2^2 + \|v\|_4^4 \lesssim T^* E_v(t). \]

For the $E_{2,1}$ part, by the same calculation, we have
\[ \|w_{xx}, (E_{2,1})_{xx}\| \lesssim \|w_{xx} v_{xx} w\|_1 + \|w_{xx} v_x^2 w\|_1 + \|w_{xx} v_x w_x\|_1 + \|w_{xx} v_{xx} v_{xx}\|_1 \]
\[ \lesssim T^* \|v_{xx}\|_2 \|v\|_2 \|v_x\|^2 + \|v\|_2 \|v_x\|_2 + \|v\|_2^2 \]
\[ \lesssim T^* \|v\|^2_2 + \|v_{xx}\|^2_2 + \|v_x\|^2_2. \]

Also, by Gagliardo–Nirenberg inequality, we have
\[ \|v_x\|_2 \lesssim \|v\|_{1/2}^{1/2} \|v_{xx}\|_{1/2}^{1/2}. \]

Therefore, we have
\[ \|w_{xx}, (E_{2,1})_{xx}\| \lesssim T^* \|v\|_2^2 + \|v_{xx}\|_2^2 \lesssim T^* E_v(t) + M_v(t). \]

For the $E_{1,2}$ part, we have
\[ \|w_{xx}, (E_{1,2})_{xx}\| \lesssim \|w_{xx} v_{xx} w^2\|_1 + \|w_{xx} v_x w_x\|_1 + \|w_{xx} v_x v_{xx}\|_1 + \|w_{xx}^2 v w\|_1 \]
\[ \lesssim \|v_{xx}\|_2 \|w_{xx}\|_6 \|w\|_6^2 + \|v_x\|_2 \|w_x\|_6 \|w_{xx}\|_6 \]
\[ + \|v\|_2 \|w_{xx}\|_6 \|w_{xx}\|_6 \|w_{xx}\|_6 \]
\[ \lesssim T^* \|v_{xx}\|_2 + \|v_x\|_2 + \|v\|_2 \]
\[ \lesssim T^* \|v_{xx}\|_2 + \|v_x\|_2 + \|v\|_2 \]
\[ \lesssim T^* \|v_{xx}\|_2^2 + \|v\|_2^2 + 1 \]
\[ \lesssim T^* E_v(t) + M_v(t) + 1. \]

Substituting all these three parts into (30), we have
\[ \left| \frac{d}{dt} \tilde{E}_v(t) \right| \lesssim T^* E_v(t) + M_v(t) + 1. \]

Also, by (27), we have
\[ \left| \frac{d}{dt} \tilde{E}_v(t) \right| \lesssim T^* \tilde{E}_v(t) + 2C(T^*) A_v(t). \]

Combining this estimate with (28), we have
\[ \frac{d}{dt} \left( \tilde{E}_v(t) + 2C(T^*) A_v(t) \right) \leq C(T^*) \left( \tilde{E}_v(t) + 2C(T^*) A_v(t) \right), \]
\[ \tilde{E}_v(0) + 2C(T^*)A_v(0) = 2C(T^*). \]

Then, by Gronwall’s inequality, we know that
\[ \tilde{E}_v(t) + 2C(T^*)A_v(t) \lesssim T^* \]
which means that \( M_v(t) \lesssim T^* \). This is contradictory to (23). \( \square \)

Acknowledgements

Thanks to Robert Schippa for some helpful discussions. Thanks to advisor Baoxiang Wang for inspiring advises. The author sincerely thank the anonymous reviewers for their valuable comments that have led to the present improved version of the original manuscripts.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Appendix A \( M_{s, \alpha}^{q, \sigma} \) with \( \alpha < 0 \)

We first give the existence of the \( \alpha \)-covering of \( \mathbb{R}^d \) when \( \alpha < 0 \).

**Proposition 3.** For \( \alpha < 0 \), there exists a \( \alpha \)-covering \( \{ B_k \}_{k \in \mathbb{Z}^d} \) of \( \mathbb{R}^d \) with a corresponding \( \alpha \)-BAPU \( \{ \eta_k^\alpha \}_{k \in \mathbb{Z}^d} \).

**Proof.** Our proof is based on Subsection 2.1 in [2]. Denote \( \beta = \alpha/(1-\alpha) \in (-1, 0) \). We can define a homeomorphism \( \delta_\beta \) of \( \mathbb{R}^d \) by \( \delta_\beta(x) = |x|^\beta x \). It is easy to check that \( \delta_\beta^{-1} = \delta_{\beta'} \), where \( \beta' = -\beta/(1+\beta) \). Also, we know that \( \delta_\beta \in C^1(\mathbb{R}^d \setminus \{0\}) \).

For any \( k \in \mathbb{Z}^d \), \( |k| \geq 2\sqrt{d} \), for any \( x \in B(k, \sqrt{d}) \), we know that \( |x| \approx |k| \gtrsim 1 \). Then by the mean value inequality, we have
\[
|\delta_\beta(x) - \delta_\beta(k)| \leq C_\beta \sup_{0 \leq t \leq 1} |tx + (1-t)k|^\beta |x - k| \leq C_\beta \langle k \rangle^\beta.
\]
So, we know that \( \delta_{\beta} \left( B(k, \sqrt{d}) \right) \subseteq B(\delta_{\beta}(k), C_{\beta}(k)^{\beta}) \), \( \forall |k| \geq 2\sqrt{d} \).

When \( |k| \leq 2\sqrt{d} \), it is obvious that \( \delta_{\beta} \left( B(k, \sqrt{d}) \right) \) is bounded. So, we could choose \( C'_{\beta} > 0 \), such that \( \delta_{\beta} \left( B(k, \sqrt{d}) \right) \subseteq B(\delta_{\beta}(k), C_{\beta}(k)^{\beta}) \), \( \forall |k| \leq 2\sqrt{d} \).

Combining all the cases, we know that there exists \( r > 0 \), such that

\[
\delta_{\beta} \left( B(k, \sqrt{d}) \right) \subseteq B(\delta_{\beta}(k), r \langle k \rangle^{\beta}) \), \( \forall k \in \mathbb{Z}^d \).
\]

Denote \( B_k = B(\delta_{\beta}(k), r \langle k \rangle^{\beta}) \), with the condition above, we know that \( \mathbb{R}^d = \bigcup_k B_k \), because that \( \{B(k, \sqrt{d})\}_{k \in \mathbb{Z}^d} \) is a covering of \( \mathbb{R}^d \) and \( \delta_{\beta} \) is a homeomorphism.

Similarly, for the \( r > 0 \) above, we have \( R > 0 \), such that

\[
\delta_{\beta'} \left( B(\delta_{\beta}(k), r \langle k \rangle^{\beta}) \right) \subseteq B(k, R) \), \( \forall k \in \mathbb{Z}^d \).
\]

which is equivalent to

\[
B(\delta_{\beta}(k), r \langle k \rangle^{\beta}) \subseteq \delta_{\beta} \left( B(k, R) \right) \), \( \forall k \in \mathbb{Z}^d \).
\]

Then by the bounded overlap of \( \{B(k, R)\}_{k \in \mathbb{Z}^d} \), we know that \( \{B_k\}_{k \in \mathbb{Z}^d} \) is also bounded overlapped.

One can easily check that condition (iii) in Definition 2 holds for \( B_k \) given above. So, we constructed a \( \alpha \)-covering of \( \mathbb{R}^d \), named \( \{B_k\}_{k \in \mathbb{Z}^d} \). The corresponding \( \alpha \)-BAPU of this \( \alpha \)-covering can be given as follows.

For any \( |k| \neq 0 \), we can choose \( r_0 > 0 \), such that \( r \langle k \rangle^{\beta} \leq r_0 |k|^\beta \). Choose \( \rho \in \mathcal{S} \) with \( \rho(\xi) = 1 \), when \( |\xi| \leq r_0 \), \( \rho(\xi) = 1 \) when \( |\xi| \geq 2r_0 \). Denote \( \rho_0 = \rho \), \( \rho_k(\xi) = \rho(|k|^{-\beta} \xi - k) \) for any \( |k| \neq 0 \). So, we know that \( \rho_k(\xi) = 1 \) when \( \xi \in B_k \) and \( \text{supp} \rho_k \subseteq B(\delta_{\beta}(k), 2r_0 \langle k \rangle^{\beta}) := B'_k \). Notice that \( \{B'_k\}_{k \in \mathbb{Z}^d} \) is bounded overlapped and \( \{B_k\}_{k \in \mathbb{Z}^d} \) is a covering of \( \mathbb{R}^d \). We have

\[
1 \leq \sum_{k \in \mathbb{Z}^d} \rho_k(\xi) \leq C(d).
\]

Denote

\[
\psi_k = \frac{\rho_k}{\sum_{k \in \mathbb{Z}^d} \rho_k},
\]

one can check that \( \{\psi_k\}_{k \in \mathbb{Z}^d} \) is a \( \beta \)-BAPU. \( \square \)

**Proposition 4.** Let \( \alpha < 0 \), \( s \in \mathbb{R} \), \( 0 < p, q \leq \infty \). Then \( M^s_{p, q} \hookrightarrow M^{0, \alpha}_{p, q} \) if and only if \( s \geq -\alpha \tau(p, q) \).

**Proof.** The proof is based on Section 4 in [19].

**Sufficiency:**

(1) When \( (p, q) = (2, 2) \), \( \tau(p, q) = 0 \), we have \( L^2 = M_{2, 2} = M^{0, \alpha}_{2, 2} \).
(2) For any $k, \ell \in \mathbb{Z}^d$, denote
\[
\wedge_k = \{ \ell \in \mathbb{Z}^d : \Box_\ell \Box_k^{\alpha} \neq 0 \}, \\
\vee_\ell = \{ k \in \mathbb{Z}^d : \Box_\ell \Box_k^{\alpha} \neq 0 \}.
\]
By the orthogonality of $\{\Box_\ell\}_{\ell \in \mathbb{Z}^d}$ and $\{\Box_k^{\alpha}\}_{k \in \mathbb{Z}^d}$, we know that
\[
\# \wedge_k \lesssim d^{1/2}, \quad \# \vee_\ell \lesssim d^{(\ell - a)/2}.
\]
When $p = 1$, or $\infty$, we know $\tau(p, q) = d/q$. By Young’s convolution inequality and the orthogonality of these decompositions, we have
\[
\| u \|_{M_{p, q}^{0, \alpha}} = \| \| \Box_k^{\alpha} u \|_p \|_{\ell_q^k} \| = \| \| \sum_{k \in \wedge_k} \Box_k^{\alpha} u \|_p \|_{\ell_q^k} \| \\
\lesssim \| \sum_{|k - k'| \leq 1} \| \Box_k^{\alpha} u \|_p \|_{\ell_q^{k \vee \ell}} \| \\
\lesssim \| \langle \ell \rangle^{-a d/q} \Box_\ell u \|_p \|_{\ell_q^\ell} = \| u \|_{M_{p, q}^{-a d/q}}.
\]
When $0 < p < 1$, we know $\tau(p, q) = d(1/p + 1/q - 1)$. By using the convolution inequality ([25], Lemma 2.6) below instead of Young’s convolution inequality, we could get the embedding as well.
\[
\| \Box_k^{\alpha} u \|_p \lesssim \| \mathcal{F}^{-1} \eta_k^{\alpha} \|_p \| \Box_\ell u \|_p \lesssim \langle k \rangle^{a d(1-1/p)/(1-\alpha)} \| \Box_\ell u \|_p.
\]
(3) When $p = 2, q < 2$, we know $\tau(2, q) = d(1/q - 1/2)$. By the Plancherel formula and Hölder’s inequality, we have
\[
\| u \|_{M_{2, q}^{0, \alpha}} = \| \| \Box_k^{\alpha} u \|_2 \|_{\ell_q^k} \leq \| \| \Box_k^{\alpha} u \|_2 \|_{\ell_q^{k \vee \ell}} \| \\
\leq \| \| \Box_k^{\alpha} u \|_2 \|_{\ell_q^{k \vee \ell}} \langle \ell \rangle^{-a d(1/2 - 1/2/2)} \|_{\ell_q^\ell} \| \\
\lesssim \| \| \Box_\ell u \|_2 \langle \ell \rangle^{-a d(1/2 - 1/2)} \|_{\ell_q^\ell} = \| u \|_{M_{2, q}^{-a d(2, 2)}}.
\]
(4) Taking the interpolation of the estimates in (1,2,3), we obtain the results for the remaining cases.

Necessity: If we know $M_{p, q}^* \hookrightarrow M_{p, q}^{0, \alpha}$, then we have
\[
\| u \|_{M_{p, q}^{0, \alpha}} \lesssim \| u \|_{M_{p, q}^*}.
\]
(a) For any $k \in \mathbb{Z}^d$, take $u = \mathcal{F}^{-1} \sigma((k)^{-\alpha/(1-\alpha)} \xi - k)$ into (32). We know that

$$
\|u\|_{M^{0,\alpha}_{p,q}} \approx \|u\|_p = \langle k \rangle^{ad(1-1/p)/(1-\alpha)} ;
$$

$$
\|u\|_{M^{s,\alpha}_{p,q}} \approx \langle k \rangle^{s/(1-\alpha)} \|u\|_p .
$$

Then we have $s \geq 0$.

(b) For any $\ell \in \mathbb{Z}^d$, take $u = \mathcal{F}^{-1} \sigma(\xi - \ell)$ into the inequality above. We know that

$$
\|u\|_{M^{s,\alpha}_{p,q}} \approx \langle \ell \rangle^s ;
$$

$$
\|u\|_{M^{0,\alpha}_{p,q}} = \|\Box_k^a u\|_p \ell^q_{\mathbb{R}^q} \gtrsim \|\mathcal{F}^{-1} \eta_k^a\|_p \ell^q_{\mathbb{R}^q} \approx \langle \ell \rangle^{ad(1-1/p-1/q)} .
$$

So, we have $s \geq -ad(1/2 - 1/q - 1)$.

(c) For any $\ell \in \mathbb{Z}^d$, $N \in \mathbb{N}^+$, take

$$
u^N = \sum_{k \in \mathbb{V}_\ell} T_{Nk} \left( \mathcal{F}^{-1} \sigma((k)^{-\alpha/(1-\alpha)} \xi - k) \right),
$$

where $T_{Nk} f(x) = f(x - Nk)$ is the translation operator. Then we have

$$
\left\| \nu^N \right\|_{M^{0,\alpha}_{p,q}} = \left\| \mathcal{F}^{-1} \sigma((k)^{-\alpha/(1-\alpha)} \xi - k) \right\|_p \ell^q_{\mathbb{R}^q} \approx \langle \ell \rangle^{ad(1-1/p-1/q)} .
$$

On the other hand, we have $\|\nu^N\|_{M^{s,\alpha}_{p,q}} \approx \langle \ell \rangle^s \|f^N\|_p$. Let $N \to \infty$, by the almost orthogonality of $T_{Nk} \left( \mathcal{F}^{-1} \sigma((k)^{-\alpha/(1-\alpha)} \xi - k) \right)$, we have

$$
\lim_{N \to \infty} \left\| \nu^N \right\|_p = \left\| \mathcal{F}^{-1} \sigma((k)^{-\alpha/(1-\alpha)} \xi - k) \right\|_p \ell^q_{\mathbb{R}^q} \approx \langle \ell \rangle^{ad(1-1/p)} \langle \ell \rangle^{-ad/p} = \langle \ell \rangle^{ad(1/2 - 1/p)} .
$$

Substituting these two estimates into the inequality (32), we have $s \geq -ad(1/2 - 1/p)$.

Combining the three conditions above, we have $s \geq -\alpha \tau(p, q)$, as desired. \hfill \Box

As for $M^{s,\alpha}_{p,q} \hookrightarrow M_{p,q}$, we also have

**Proposition 5.** Let $\alpha < 0$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$. Then $M^{s,\alpha}_{p,q} \hookrightarrow M_{p,q}$ if and only if $s \geq \alpha \sigma(p, q)$.

**Proof.** The necessity is the same as the proof of Proposition 4. The sufficiency can be done by taking the interpolation of the following cases.

(a) When $p = 2, q > 2$, we know that $\sigma(2, q) = d(1/q - 1/2)$. By the Plancherel’s formula and the Hölder’s inequality of the sequences, we have

$$
\|u\|_{M_{2,q}} = \|\Box_k u\|_2 \ell^q_{\mathbb{R}^q} = \left\| \sum_{k \in \mathbb{V}_\ell} \Box_k^a \Box_{q,\ell} u \right\|_2 \ell^q_{\mathbb{R}^q} .
$$
\[
\left( \sum_{k \in \nu} \| \Box_k u \|_2^q \right)^{1/2} \| \ell^q \leq \left\| \sum_{k \in \nu} \| \Box_k u \|_2 \right\|_{\ell^q} \right)^{1/2} \leq \left\| (k)^{-\alpha d(1/2-1/q)/(1-\alpha)} \right\|_{\ell^q} = \left\| u \right\|_{M^{\alpha d(1/2-1/2),q}}.
\]

(b) When \( q \leq 1 \wedge p \), we have \( \sigma(p, q) = 0 \). By the (quasi-)triangle inequality and the embedding of \( \ell^r \) spaces, we have

\[
\| u \|_{M^{p,q}} = \left\| \Box u \right\|_{\ell^p} = \left\| \sum_{k \in \nu} \Box_k \Box u \right\|_{\ell^q} \leq \left\| \sum_{k \in \nu} \Box_k \Box u \right\|_{\ell^p} \leq \left\| \sum_{k \in \nu} \Box_k \Box u \right\|_{\ell^p} \leq \left\| \sum_{k \in \nu} \Box_k \Box u \right\|_{\ell^p} \leq \| u \|_{M^{0,q}}.
\]

(c) When \((p, q) = (\infty, \infty)\), we have \( \sigma(p, q) = -d \). By the triangle inequality, we have

\[
\| u \|_{M_{\infty,\infty}} = \left\| \Box u \right\|_{\ell^\infty} = \left\| \sum_{k \in \nu} \Box_k \Box u \right\|_{\ell^\infty} \leq \sup_{k \in \nu} \left\| \Box_k \Box u \right\|_{\ell^\infty} \leq \left\| \sum_{k \in \nu} \Box_k \Box u \right\|_{\ell^\infty} = \| u \|_{M_{\infty,\infty}}.
\]

(d) When \( p < 1 \), \( q = \infty \), we have \( \sigma(p, q) = -d/p \). By the quasi-triangle inequality, we have

\[
\| u \|_{M_{p,\infty}} = \left\| \Box u \right\|_{\ell^\infty} = \left\| \sum_{k \in \nu} \Box_k \Box u \right\|_{\ell^\infty} \leq \sup_{k \in \nu} \left\| \Box_k \Box u \right\|_{\ell^\infty} \leq \left\| \sum_{k \in \nu} \Box_k \Box u \right\|_{\ell^\infty} = \| u \|_{M_{p,\infty}}.
\]
REFERENCES

[1] Árpád Bényi, Karlheinz Gröchenig, Kasso A. Okoudjou, and Luke G. Rogers. Unimodular Fourier multipliers for modulation spaces. *J. Funct. Anal.*, 246(2):366–384, 2007.

[2] Lasse Borup and Morten Nielsen. Banach frames for multivariate α-modulation spaces. *J. Math. Anal. Appl.*, 321(2):880–895, 2006.

[3] Jean Bourgain and Ciprian Demeter. The proof of the $l^2$ decoupling conjecture. *Ann. of Math. (2)*, 182(1):351–389, 2015.

[4] Jean Bourgain and Ciprian Demeter. Decouplings for curves and hypersurfaces with nonzero Gaussian curvature. *J. Anal. Math.*, 133:279–311, 2017.

[5] Leonid Chaichenets, Dirk Hundertmark, Peer Christian Kunstmann, and Nikolaos Pattakos. Local well-posedness for the nonlinear Schrödinger equation in the intersection of modulation spaces $M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$. *arXiv e-prints*, arXiv:1610.08298, 2016.

[6] Benjamin Dodson, Avraham Soffer, and Thomas Spencer. Global well-posedness for the cubic nonlinear Schrödinger equation with initial data lying in $L^p$-based Sobolev spaces. *J. Math. Phys.*, 62(7):1–13, 2021.

[7] Hans Georg Feichtinger. Modulation spaces on locally compact abelian groups. Technical Report, University of Vienna, 1983.

[8] Hans Georg Feichtinger. Banach spaces of distributions defined by decomposition methods. II. *Math. Nachr.*, 132:207–237, 1987.

[9] Hans Georg Feichtinger and Peter Gröbner. Banach spaces of distributions defined by decomposition methods. I. *Math. Nachr.*, 123:97–120, 1985.

[10] Massimo Fornasier. Banach frames for α-modulation spaces. *Appl. Comput. Harmon. Anal.*, 22(2):157–175, 2007.

[11] Shengwen Gan, Changkeun Oh, and Shukun Wu. A note on local smoothing estimates for fractional Schrödinger equations. *J. Funct. Anal.*, 283(5):1–36, 2022.

[12] Chuanwei Gao, Changxing Miao, and Jiqiang Zheng. Improved local smoothing estimates for the fractional Schrödinger operator. *Bull. Lond. Math. Soc.*, 54(1):54–70, 2022.

[13] Peter Gröbner. *Banachräume glatter Funktionen und Zerlegungsmethoden*. PhD thesis, University of Vienna, 1992.

[14] Karlheinz Gröchenig. *Foundations of time-frequency analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2001.

[15] Shaoming Guo, Joris Roos, and Po-Lam Yung. Sharp variation-norm estimates for oscillatory integrals related to Carleson’s theorem. *Anal. PDE*, 13(5):1457–1500, 2020.

[16] Weichao Guo and Jiecheng Chen. Strichartz estimates on α-modulation spaces. *Electron. J. Differential Equations*, 2013(118):1–13, 2013.

[17] Weichao Guo, Dashan Fan, and Guoping Zhao. Full characterization of the embedding relations between α-modulation spaces. *Sci. China Math.*, 61(7):1243–1272, 2018.

[18] Larry Guth, Hong Wang, and Ruixiang Zhang. A sharp square function estimate for the cone in $\mathbb{R}^3$. *Ann. of Math. (2)*, 192(2):551–581, 2020.

[19] Jinsheng Han and Baoxiang Wang. α-modulation spaces (I) scaling, embedding and algebraic properties. *J. Math. Soc. Japan*, 66(4):1315–1373, 2014.

[20] Keiichi Kato, Masaharu Kobayashi, and Shingo Ito. Estimates on modulation spaces for Schrödinger evolution operators with quadratic and sub-quadratic potentials. *J. Funct. Anal.*, 266(2):733–753, 2014.

[21] Tomoya Kato. The global Cauchy problems for the nonlinear dispersive equations on modulation spaces. *J. Math. Anal. Appl.*, 413(2):821–840, 2014.

[22] Tomoya Kato. The inclusion relations between α-modulation spaces and $L^p$-Sobolev spaces or local Hardy spaces. *J. Funct. Anal.*, 272(4):1340–1405, 2017.

[23] Tomoya Kato. On applications of modulation spaces to dispersive equations. In *Harmonic analysis and nonlinear partial differential equations*, RIMS Kôkyûroku Bessatsu, B65, 63–78. Res. Inst. Math. Sci. (RIMS), Kyoto, 2017.

[24] Friedrich Klaus. Wellposedness of NLS in Modulation Spaces. *J. Fourier Anal. Appl.*, 29(1):1–37, 2023.
[25] Masaharu Kobayashi. Modulation spaces $M^{p,q}$ for $0 < p, q \leq \infty$. *J. Funct. Spaces Appl.*, 4(3):329–341, 2006.

[26] Masaharu Kobayashi and Mitsuru Sugimoto. The inclusion relation between Sobolev and modulation spaces. *J. Funct. Anal.*, 260(11):3189–3208, 2011.

[27] Masaharu Kobayashi, Mitsuru Sugimoto, and Naohito Tomita. On the $L^2$-boundedness of pseudo-differential operators and their commutators with symbols in $\alpha$-modulation spaces. *J. Math. Anal. Appl.*, 350(1):157–169, 2009.

[28] Izabella Łaba and Thomas Wolff. A local smoothing estimate in higher dimensions. *J. Anal. Math.*, 88:149–171, 2002. Dedicated to the memory of Tom Wolff.

[29] Changxing Miao and Bo Zhang. Global well-posedness of the Cauchy problem for nonlinear Schrödinger-type equations. *Discrete Contin. Dyn. Syst.*, 17(1):181–200, 2007.

[30] Akihiko Miyachi. On some singular Fourier multipliers. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 28(2):267–315, 1981.

[31] Akihiko Miyachi, Fabio Nicola, Silvia Rivetti, Anita Tabacco, and Naohito Tomita. Estimates for unimodular Fourier multipliers on modulation spaces. *Proc. Amer. Math. Soc.*, 137(11):3869–3883, 2009.

[32] Keith M. Rogers. A local smoothing estimate for the Schrödinger equation. *Adv. Math.*, 219(6):2105–2122, 2008.

[33] Keith M. Rogers and Andreas Seeger. Endpoint maximal and smoothing estimates for Schrödinger equations. *J. Reine Angew. Math.*, 640:47–66, 2010.

[34] Robert Schippa. On smoothing estimates in modulation spaces and the nonlinear Schrödinger equation with slowly decaying initial data. *J. Funct. Anal.*, 282(5):1–46, 2022.

[35] Felix Voigtlaender. *Embedding theorems for decomposition spaces with applications to wavelet coorbit spaces*. PhD thesis, RWTH Aachen University, 2015.

[36] Baoxiang Wang and Chunyan Huang. Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations. *J. Differential Equations*, 239(1):213–250, 2007.

[37] Baoxiang Wang, Zhaohui Huo, Chengchun Hao, and Zihua Guo. Harmonic analysis method for nonlinear evolution equations. *I. World Scientific Publishing Co. Pte. Ltd.*, Hackensack, NJ, 2011.

[38] Thomas Wolff. Local smoothing type estimates on $L^p$ for large $p$. *Geom. Funct. Anal.*, 10(5):1237–1288, 2000.

[39] Chunjie Zhang. Strichartz estimates in the frame of modulation spaces. *Nonlinear Anal.*, 78:156–167, 2013.

[40] Guoping Zhao, Jiecheng Chen, Dashan Fan, and Weichao Guo. Sharp estimates of unimodular multipliers on frequency decomposition spaces. *Nonlinear Anal.*, 142:26–47, 2016.

[41] Guoping Zhao, Jiecheng Chen, and Weichao Guo. Remarks on the unimodular Fourier multipliers on $\alpha$-modulation spaces. *J. Funct. Spaces*, 2014:1–8, 2014.

[42] Guoping Zhao, Guilian Gao, and Weichao Guo. Sharp embedding relations between local hardy and $\alpha$-modulation spaces. *Anal. Math.*, 47:451–481, 2021.

Yufeng Lu
School of Mathematics Sciences
Peking University
Beijing 100871
People’s Republic of China
E-mail: luyufeng@pku.edu.cn

Present Address
Yufeng Lu
School of Sciences
Jimei University
Xiamen 361021
People’s Republic of China

Accepted: 12 April 2023