QUANTUM CORRECTIONS TO THE ENTROPY
FOR HIGHER SPIN FIELDS IN HYPERBOLIC SPACE

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Abstract

We calculate the one-loop corrections to the free energy and to the entropy for fields with arbitrary spins in the space $S^1 \otimes H^N$. For conformally invariant fields by means of a conformal transformation of the metric the results are valid in Rindler space with $D = N + 1$ dimensions. We use the zeta regularization technique which yields an ultraviolet finite result for the entropy per unit area. The problem of the infinite area factor in the entropy which arises equally in Rindler space and in the black hole background is addressed in the light of a factor space $H^N/\Gamma$. 

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1 Introduction

Quantum fields in black hole backgrounds have been actively investigated during the last years [1], but only recently an interest to thermodynamic properties and in particular to the entropy have been renewed [2]. Several attempts have been undertaken to calculate quantum corrections to the thermodynamic entropy in spaces with horizons. A large number of works have been carried out in the limit of infinite black hole mass when the Schwarzschild spherical horizon surface becomes planar and the metric becomes the Rindler metric.

It is known that the black hole entropy stored in quanta near the horizon is ultraviolet divergent. In Rindler space several authors have found such a divergence [3, 4, 5, 6]. This is in contradiction with the finiteness of the Bekenstein-Hawking entropy [7].

Recently the statistical mechanics mode counting was carried out for the entropy spectrum of scalars in Rindler space with the help of WKB approximation [4]. The free energy for a massless field in D=4 dimensional spacetime, in accordance with previous results, comes out proportional to $\beta^{-4}$ ($\beta$ is the inverse temperature) and to the area of the horizon. It has been shown that the entropy per unit area is quadratically divergent near the horizon. Since the level density diverges due to the infinite shift of frequencies, a cutoff parameter should be introduced [4, 5, 8]. Such quantum corrections to the density of entropy, as has been pointed out in [4], are equivalent to the quantum corrections to the gravitational coupling. Therefore, the entropy divergencies obtained by state counting are closely related to the conventional ultraviolet divergencies of canonical quantum gravity.

An independent calculation of the finite temperature stress energy tensor gives the same results [5]. An interesting method to compute the entropy mostly for $D = 2$ was developed in [6]. In two dimensions the divergence is logarithmic and the coefficients are cutoff independent. For $D > 2$ the heat kernel of the Laplace-Beltrami operator in Rindler space have been used. As a result, the free energy lower integration limit needs a short distance regularization. The heat kernel techniques in computing quantum corrections in Rindler space have been used in [8], where the computation was done in geometry with different topology (without the conical singularity).

In this paper we suggest a powerful method for the computation of the first quantum correction to the free energy associated with fields of arbitrary spins on the manifold $M = S^1 \otimes H^N$. Such a manifold can be obtained in the result of a conformal transformation of the Euclidean Rindler space. In Sect. 2 we discuss the general technique based on the zeta regularization approach for the calculation of the one-loop free energy. We show in Sect. 3 that the ultraviolet divergencies of the free energy are removed by the zeta-regularization approach. The free energy per unit area is represented by a series in inverse powers of $\beta$ (i.e., as a high temperature expansion) whose coefficients are cutoff independent. This series has been analytically continued to all $\beta$. In the conclusions a comparison of the results is done with the corresponding corrections obtained for the conformal invariant and minimally coupled fields. We discuss the problem of regularizing the divergencies in the entropy which result from the infinite horizon area.
2 The Zeta Functional Regularization of the Free Energy

Let us start with the functional integration of the partition function associated with Rindler space. The geometry of the $D$-dimensional Euclidean Rindler space can be written as follows

$$ds^2 = \xi^2 d\tau^2 + d\xi^2 + \sum_{i=1}^{N-1} dy_i^2.$$  \hspace{1cm} (1)

Here $y_i$ are the $N-1$ transverse flat coordinates ($D = N + 1$), $\tau$ is the Euclidean time (periodically identified with period $\beta$) and the lines $\xi = \text{const}$ correspond to uniformly accelerated observers. It is convenient to use the optical metric \cite{9} $g_{\pm} = g_{00}^{-1} = g\xi^{-2}$ in order to define the appropriate functional integration \cite{8, 10}

$$Z[g, \beta] = \int D[\Phi] e^{-S[\Phi]},$$ \hspace{1cm} (2)

where $S[\Phi]$ is the Euclidean action related to the quantum field $\Phi$ in the $D$-dimensional manifold of the form $M = S^1 \otimes H^N$ and $H^N$ is the simply connected real hyperbolic space.

It should be noted that the measure of the path integral (2) is formally regularized with respect to some inner product of the fields. Generally speaking, the inner product is ill-defined at the horizon (see, for example, \cite{10}). But actually such a choice of the measure is suited to obtain the same thermodynamics as the one associated with Rindler mode counting \cite{1, 8, 11, 12}.

The partition function can be given by

$$\log Z^{\pm}[g, \beta] = \pm\frac{1}{2} \log \det A^\pm(\beta),$$ \hspace{1cm} (3)

where the sign $(\pm)$ (resp. $(-)$) refer to Dirac spinor (resp. integer spin-s fields), $A^\pm(\beta) = \partial_\tau^2 + L^\pm_N$, where $L^\pm_N$ is the self-adjoint Laplace-Beltrami operator acting in the space $H^N$. The eigenvalues of the operator $A^\pm(\beta)$ have the form

$$\omega_n^{(s)} = \left[\frac{2\pi(n + s)}{\beta}\right]^2 + \lambda^2, \hspace{0.2cm} n \in \mathbb{Z}. \hspace{1cm} (4)$$

Here $\lambda^2$ are the eigenvalues of the operator $L_N$.

The generalized (Riemann) zeta function related to the operator $A(\beta)$ can be represented using the Mellin transform of the heat kernel $K(t\mid A(\beta)) = \text{Tr} \exp(-t(A(\beta))):

$$\zeta(x \mid A^\pm(\beta)) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{x-1} K(t \mid A^\pm(\beta)).$$ \hspace{1cm} (5)

Note that the “global” function $\zeta(x\mid A(\beta))$ depends on $x \in M$ only, since the manifold $M$ is a homogeneous space.

Using the relations

$$\sum_{n=-\infty}^{\infty} e^{-n^2 \beta^2/4t} = 2\sqrt{\frac{\pi t}{\beta}} \sum_{n=-\infty}^{\infty} e^{-4\pi n^2 t / \beta^2}$$

3
which are identities for the Jacobi’s elliptic theta functions $\Theta_3(v,q) = \sum_{n=-\infty}^{\infty} q^n e^{2i\pi n v}$ and $\Theta_4(v,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^n e^{2i\pi n v}$, one can rewrite $\zeta(x | A(\beta))$ in the form (see [3] for more details)

$$
\zeta(x | A^\pm(\beta)) = \sum_{n=-\infty}^{\infty} \zeta(x | L_N^\pm + [2\pi(n + s)/\beta]^2) = \frac{\beta \Gamma(x - 1/2)}{2\sqrt{\pi} \Gamma(x)} \zeta(x - 1/2 | L_N^\pm) + \frac{\beta}{2\sqrt{\pi} \Gamma(x)} \int_{0}^{\infty} dt \ t^{x-3/2} K(t | L_N^\pm) \left( \Theta^\pm(\beta, t) - 1 \right),
$$

where

$$
\Theta^\pm(\beta, t) = \begin{cases} 
\Theta_3(0, e^{-\beta^2/4t}), & s = 0, 1, 2, \\
\Theta_4(0, e^{-\beta^2/4t}), & s = 1/2 
\end{cases}.
$$

A complex integral representation for the zeta function can be obtained with the help of Mellin-Parseval identity

$$
\int_{0}^{\infty} dt \ f(t) g(t) = \frac{1}{2\pi i} \int_{R_\sigma = \infty} f(z) g(1 - z) \ dz,
$$

where $\hat{f}(z)$ ($\hat{g}(z)$) is the Mellin transform of $f(t)$ ($g(t)$), $t^{z-1} f(t) \in L(0, \infty)$ and $\sigma$ being a real number in the strip in which $\hat{f}(z)$ and $\hat{g}(1 - z)$ are analytic. Let us suppose that $f(t) = t^{z-3/2} K(t | L_N^\pm)$, $g(t) = (-1)^{2s} \exp(-n^2 \beta^2/4t)$. Then, using Eq. (6) and the Mellin-Parceval identity (8) one can obtain

$$
\zeta(x | A^\pm(\beta)) = \frac{\beta \Gamma(x - 1/2)}{2\sqrt{\pi} \Gamma(x)} \zeta(x - 1/2 | L_N^\pm) + \frac{1}{\sqrt{\pi} \Gamma(x) 2\pi i} \int_{R_\sigma = c} \ dz \ \zeta^\pm(z) \Gamma(z - 1) \Gamma(z) \zeta_{R}(z) \left( \frac{\beta}{2} \right)^{-(z-1)},
$$

where $\zeta^\pm(z) = \zeta_{R}(z)$ is the Riemann zeta function, $\zeta^+(z) = (1 - 2^{1-z}) \zeta_{R}(z)$ and $c > N + 1$.

Let us point out the contribution $Z[\bar{g}, \beta]$ of conformally invariant fields to the partition function $Z[g, \beta]$ in the Rindler space. The renormalized free energy $\mathcal{F}[g, \beta]$ for two conformally related static spaces with metrics $\bar{g}_{\mu\nu} = e^{-2\omega} g_{\mu\nu}$ can be rewritten in the form

$$
\mathcal{F}^\pm[g, \beta] = \pm \beta^{-1} \log Z^\pm[g, \beta] = \mathcal{F}^\pm[\bar{g}, \beta] + \Delta \mathcal{F}^\pm[\omega, g].
$$

We may apply further Eq. (10) to the particular case when $\omega = \frac{1}{2} \log \xi^2$ and $\bar{g}_{\mu\nu} = e^{-2\omega} g_{\mu\nu}$ is an ultrastatic metric. It should be noted that the term $\Delta \mathcal{F}^\pm[\omega, g]$ for two conformally related theories is proportional to $\beta$ and, hence, does not contribute to the entropy. So, we will suppress this term below.

From Eq. (11) one can obtain the representation for the free energy which is related to the free energy by means of equation

$$
\mathcal{F}^\pm[\bar{g}, \beta] = \mp \frac{1}{2} \beta^{-1} \zeta'(0 | A^\pm(\beta)).
$$
As a result, we have
\[
F^\pm [g, \beta] = \pm \frac{1}{2} \zeta^{(r)}(-1/2 | L_N^\pm) \pm \frac{1}{2\pi i} \int_{\Re z = c} dz \, \zeta^\pm(z) \Gamma(z - 1) \zeta\left(\frac{z - 1}{2} | L_N^\pm\right) \beta^{-z} \\
\equiv F^\pm_0 + F^\pm(\beta).
\] (12)

Here \(F^\pm_0\) is the vacuum energy, \(F^\pm(\beta)\) is the temperature dependent part of \(F^\pm [g, \beta]\), and we have introduced the notation
\[
\zeta^{(r)}(-1/2 | L_N^\pm) = \text{FP} \zeta(-1/2 | L_N^\pm) + (2 - 2 \log 2) \text{Res}_{z = -1/2} \zeta(z | L_N^\pm). \tag{13}
\]

In Eq. (13) the symbols FP and Res denote the finite part and residue of the function at the specified point, respectively (for more details see [13]). As it has been pointed out in [14, 13], the contribution to the free energy of the fermionic field \(F^+(\beta)\) can be obtained from the bosonic part of \(F^-(\beta)\). This relation can be easily reproduced from Eq. (12) and the result is
\[
F^+(\beta) = 2F^-(2\beta) - F^-(\beta). \tag{14}
\]

3 Quantum Corrections Associated with Arbitrary Spin Fields in \(S^1 \otimes H^N\)

For a noncompact rank one symmetric space \(H^N\) (the rank \(M\) is the dimension of the commutative algebra of invariant differential operators, Laplace operators for instance) the related zeta function can be constructed by the help of the spectral function \(\mu(\lambda)\) known as the Plancherel measure [15]. In the case of a Riemann noncompact symmetric space with negative curvature the explicit form of \(\mu(\lambda)\) is given by
\[
\mu(\lambda) = [C(\lambda)C(-\lambda)]^{-1},
\]
where \(C(\lambda)\) is the Harish-Chandra function [16, 13]. It can be given in terms of a product over the positive roots of the symmetric space. The spectral function is essential in the construction of the zeta function in a noncompact space. It takes the form
\[
\zeta(z | L_N^\pm) = \int_0^\infty \frac{d\lambda \, \mu(\lambda)}{(\lambda^2 + C_s^2)^z} \tag{14}
\]
and \(C_s\) are known constants depending on the mass of the fields and on the curvature \(R = -N(N - 1)a^{-2}\). We choose the radius \(a = 1\) and in the final results the dependence on \(a\) can be restored.

Here, we add a remark concerning the role of the coefficients \(C_s\). As it is known, for a conformal invariant massless scalar field it holds \(C_0 = 0\) and the corresponding Laplace-Beltrami operator has no gap in its spectrum. In that case we keep the coefficient \(C_0 \neq 0\) until the end of the calculations and use it as a regularization parameter. In this way we get a well defined zeta function which is suited for analytical continuation.
The Plancharel measure for spin-\(s\) fields in \(H^N\) has been calculated recently in Refs. [16, 17]. It reads

\[
\mu^\pm(\lambda, s) = \frac{\pi \lambda}{[2^{N-2}\Gamma(N/2)]^2} \tanh[\pi(\lambda + is)] \sigma^\pm(\lambda, s) \tag{15}
\]

with

\[
\tanh[\pi(\lambda + is)] = \begin{cases} 
\tanh(\pi\lambda), & s = 0, 1, \ldots \\
\coth(\pi\lambda), & s = 1/2 
\end{cases}
\]

and

\[
\sigma^-(\lambda, s) = \begin{cases} 
\left[\lambda^2 + (s + \frac{N-3}{2})^2\right] \prod_{j=0}^{q-2} (\lambda^2 + j^2) \equiv \sum_{k=1}^{q} a_{k,N}^{(s)} \lambda^{2k}, & N = 2q + 1, \\
\left[\lambda^2 + (s + \frac{N-3}{2})^2\right] \prod_{j=1/2}^{q-5/2} (\lambda^2 + j^2) \equiv \sum_{k=0}^{q-1} b_{k,N}^{(s)} \lambda^{2k}, & N = 2q, 
\end{cases} \tag{16}
\]

\[
\sigma^+(\lambda, 1/2) = \begin{cases} 
[\lambda \coth(\pi\lambda)]^{-1} \prod_{j=1/2}^{q-1/2} (\lambda^2 + j^2) \equiv [\lambda \coth(\pi\lambda)]^{-1} \sum_{k=0}^{q} a_{k,N}^+ \lambda^{2k}, & N = 2q + 1, \\
\prod_{j=1}^{q-1} (\lambda^2 + j^2) \equiv \sum_{k=0}^{q-1} b_{k,N}^+ \lambda^{2k}, & N = 2q, 
\end{cases}
\]

where \(q \in \mathbb{Z}_+\). The coefficients \(a_{k,N}^{(s)}, b_{k,N}^{(s)}, a_{k,N}^+, b_{k,N}^+\) are defined by expanding the products into polynomials in \(\lambda^2\) in Eqs. (16) and (17). In Eq. (13), for \(N = 3\), the product is to be omitted and \(d_{0,3}^{(s)} = \frac{s}{2}, d_{1,3}^{(s)} = 1\). For \(N = 4\), the product is also omitted and we have 
\(b_{0,4}^{(s)} = (s + 1/2)^2, b_{1,4}^{(s)} = 1\) (the spectral functions on \(H^2\) for spin 0 and 1 are both given by \(\mu^-(\lambda, s) = \pi \lambda \tanh(\pi\lambda), b_{0,2}^+ = 1\)).

Finally, the zeta function can be written as [17, 18]

\[
\zeta(z \mid L_N^-) = \begin{cases} 
\frac{1}{2} g(s) A(N) \sum_{k=1}^{q} a_{k,N}^{(s)} C_{s}^{2k-2s+1} B(k + 1/2, z - k - 1/2), & N = 2q + 1 \\
\frac{1}{2} g(s) A(N) \sum_{k=0}^{q-1} b_{k,N}^{(s)} \left[C_{s}^{2k-2s+2} B(k + 1, z - k - 1) - 4I_k^-(C_{s}, z)\right], & N = 2q 
\end{cases} \tag{18}
\]

\[
\zeta(z \mid L_N^+) = \begin{cases} 
2^{[N/2]-1} A(N) \sum_{k=0}^{q} a_{k,N}^+ C_{1/2}^{2k-2s+1} B(k + 1/2, z - k - 1/2), & N = 2q + 1 \\
2^{N/2-1} A(N) \sum_{k=0}^{q-1} b_{k,N}^+ \left[C_{1/2}^{2k-2s+2} B(k + 1, z - k - 1) + 4I_k^+(C_{1/2}, z)\right], & N = 2q 
\end{cases} \tag{19}
\]

where \(B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)\) is Euler’s beta function,

\[
I_k^\pm(C, z) = \int_0^\infty \frac{d\lambda}{e^{2\pi \lambda \mp 1}} \frac{\lambda^{2k+1}(\lambda^2 + C^2)^{-z}}{e^{2\pi \lambda \pm 1}}, \tag{20}
\]
\[ g(s) = \frac{(2s + N - 3)(s + N - 4)}{(N - 3)!s!}, \quad A(N) = \frac{A}{2^{N-1}\pi^{N/2}\Gamma(N/2)}, \quad (21) \]

and \( A \) is the area of the manifold \( M \). For \( N = 3 \) we should take \( g(0) = 1 \) and \( g(s) = 2 \) for \( s \geq 1 \).

For integer spins and odd \( N \), the zeta function (18) is meromorphic in the complex \( z \)-plane with simple poles at \( z = N/2, N/2 - 1, ... \) and exhibits trivial zeros at \( z = 0, -1, -2, ... \). For even \( N \), the integral term in (18) is analytic in \( z \), but the first term carries a finite number of first order poles at \( z = N/2, N/2 - 1, ... , 1 \).

Finally, the spinor zeta function (19) has first order poles at the same points (at \( z = N/2, N/2 - 1, ... \) for odd \( N \), and at \( z = N/2, N/2 - 1, ... , 1 \) for even \( N \)).

In order to obtain the Laurent series representation for the statistical sum it is convenient to use the Mellin-Barnes representation (12). Using the zeta functions related to the operators \( L_N^\pm \) given by eqs. (18) and (19) we obtain

\[ F^\pm(\beta) = \pm \frac{1}{2\pi i} \int_{Rz=c_0} \text{d}z \varphi^\pm(z, N), \quad (c_0 > N/2). \quad (22) \]

Here we introduced the notations

\[ \varphi^-(z, 2q+1) \]

\[ = -\frac{g(s)A(2q+1)}{4\sqrt{\pi}} \sum_{k=1}^{q} a_{k,N}^{(s)} C_s^{2k+2} \Gamma(k+1/2) \Gamma(z+1/2) \zeta_R(2z+1) \Gamma(z-k-1/2) \left( \frac{C_s \beta}{2} \right)^{-(2z+1)}, \quad (23) \]

\[ \varphi^-(z, 2q) \]

\[ = -\frac{g(s)A(2q)}{4\sqrt{\pi}} \sum_{k=0}^{q-1} b_{k,N}^{(s)} C_s^{2k+3} \Gamma(k+1) \Gamma(z+1/2) \zeta_R(2z+1) \Gamma(z-k-1) \left( \frac{C_s \beta}{2} \right)^{-(2z+1)} \]

\[ + 4g(s)A(2q) \sum_{k=0}^{q-1} b_{k,N}^{(s)} \zeta_R(2z+1) \Gamma(2z) I_k^-(C_s, z) \beta^{-(2z+1)}, \quad (24) \]

\[ \varphi^+(z, 2q+1) \]

\[ = \frac{2^{q-2}A(2q+1)}{\sqrt{\pi}} \sum_{k=0}^{q} a_{k,N}^+ C_{1/2}^{2k+2} \Gamma(k+1/2) \Gamma(z+1/2) \zeta^+(2z+1) \Gamma(z-k-1/2) \left( \frac{C_{1/2} \beta}{2} \right)^{-(2z+1)}, \quad (25) \]

\[ \varphi^+(z, 2q) \]

\[ = \frac{2^{q-2}A(2q)}{\sqrt{\pi}} \sum_{k=0}^{q-1} b_{k,N}^+ C_{1/2}^{2k+3} \Gamma(k+1) \Gamma(z+1/2) \zeta^+(2z+1) \Gamma(z-k-1) \left( \frac{C_{1/2} \beta}{2} \right)^{-(2z+1)} \]

\[ + 2^{q+2}A(2q) \sum_{k=0}^{q-1} b_{k,N}^+ \zeta^+(2z+1) \Gamma(2z) I_k^+(C_{1/2}, z) \beta^{-(2z+1)}. \quad (26) \]
For odd (even) dimension \( N = 2q + 1 \) (\( N = 2q \)), the function \( \varphi^-(z, N) \) is meromorphic in \( z \). It has first order poles at \( z = 0 \), \( z = N/2, N/2 - 1, N/2 - 2, \ldots \) \((z = -1/2, z = N/2, N/2 - 1, N/2 - 2, \ldots)\) and one second order pole at \( z = -1/2 \) \((z = 0)\). The function \( \varphi^+(z, N) \) has the same properties with the only difference that for even \( N \) the poles at \( z = 0, -1/2 \) are simple ones, while for odd \( N \) the pole at \( z = -1/2 \) is of second order and there is no pole at \( z = 0 \).

Now it is useful to move the integration contour in (22) to the left up to infinity. Thereby it will cross all the poles just mentioned which results in contributions from the corresponding residues. We obtain the following series representation:

\[
F^- (\beta)_{2q+1} = -\frac{g(s)A(2q + 1)}{4\sqrt{\pi}} \sum_{k=1}^{q} a_{k,N}^{(s)} C_s^{2k+2}\Gamma(k + 1/2)
\]

\[
\times \left\{ \sum_{j=0}^{k} (-1)^j \zeta_R(2k - 2j + 2) \frac{\Gamma(k - j + 1)}{\Gamma(j + 1)} \left( \frac{C_s\beta}{2} \right)^{(2k-2j+2)} \right. \\
\left. + \frac{(-1)^{k+1}\pi^{3/2}}{\Gamma(k + 3/2)} \left( \frac{C_s\beta}{2} \right)^{-1} + \frac{(-1)^{k+1}}{\Gamma(k + 2)} \left( \gamma + \log \left( \frac{C_s\beta}{4\pi} \right) \right) \right\} + \Xi_{k,2q+1} \left( \frac{C_s\beta}{2\pi} \right),
\]

\[
(27)
\]

\[
F^- (\beta)_{2q} = -\frac{g(s)A(2q)}{4\sqrt{\pi}} \sum_{k=0}^{q-1} b_{k,N}^{(s)} C_s^{2k+3}\Gamma(k + 1)
\]

\[
\times \left\{ \sum_{j=0}^{k} (-1)^j \zeta_R(2k - 2j + 3) \frac{\Gamma(k - j + \frac{3}{2})}{\Gamma(j + 1)} \left( \frac{C_s\beta}{2} \right)^{(2k-2j+3)} \right. \\
\left. + \frac{(-1)^k\pi^{1/2}}{\Gamma(k + 2)} \log (C_s\beta) \left( \frac{C_s\beta}{2} \right)^{-1} + \frac{(-1)^{k+1}\pi}{2\Gamma(k + 5/2)} + \Xi_{k,2q} \left( \frac{C_s\beta}{2\pi} \right) \right\} + g(s)A(2q) \sum_{k=0}^{q-1} b_{k,N}^{(s)} \left[ \frac{d}{dz} I^-(k, z) \right]_{z=0} + \frac{(-1)^k (1 - 2^{-2k-1})}{4(k + 1)} \left( \gamma - \log \beta^2 \right) B_{2k+2} \right\} \beta^{-1}
\]

\[
-2I^-(k, -\frac{1}{2}) + \Psi^- \left( \frac{\beta}{2\pi} \right),
\]

\[
(28)
\]

\[
F^+ (\beta)_{2q+1} = \frac{2^{q-2}A(2q + 1)}{\sqrt{\pi}} \sum_{k=0}^{q} a_{k,N}^{(s)} C_s^{2k+2}\Gamma(k + 1/2)
\]

\[
\times \left\{ \sum_{j=0}^{k} (-1)^j \xi^+(2k - 2j + 2) \frac{\Gamma(k - j + 1)}{\Gamma(j + 1)} \left( \frac{C_{1/2}\beta}{2} \right)^{(2k-2j+2)} \right. \\
\left. + \frac{(-1)^k}{\Gamma(k + 2)} \left( \gamma + \log \left( \frac{C_{1/2}\beta}{\pi} \right) \right) + \Xi_{k,2q+1} \left( \frac{C_{1/2}\beta}{2\pi} \right) \right\},
\]

\[
(29)
\]
\[ F^+(\beta)_{2q} = \frac{2^{q-2} A(2q)}{\sqrt{\pi}} \sum_{k=0}^{q-1} b^+_k C_{1/2}^{2k+3} \Gamma(k+1) \left\{ \sum_{j=0}^{k} (-1)^j \zeta^+(2k-2j+3) \frac{\Gamma(k-j+\frac{3}{2}) \Gamma(j+1)}{\Gamma(k+1)} \left(\frac{C_{1/2} \beta}{2}\right)^{-(2k-2j+3)} \right. \\
+ \left. (-1)^{k+1} \pi \frac{\Gamma(k+2)}{\Gamma(k+1)} \log 2 \left(\frac{C_{1/2} \beta}{2}\right)^{-1} + \frac{(-1)^k \pi}{2\Gamma(k+5/2)} \right\} \\
+ 2^q A(2q) \sum_{k=0}^{q-1} b^+_{k,N} \left\{ (-1)^k \frac{\log 2 B_{2k+2}}{2(k+1)} \beta^{-1} \right. \\
+ \left. 2I^+_k \left(\frac{C_{1/2}}{2}, -\frac{1}{2}\right) + \Psi^+ \left(\frac{\beta}{2\pi}\right) \right\}, \] 

with

\[ \Xi^\pm_{k,2q+1}(X) = \frac{1}{\sqrt{\pi}} \sum_{j=k+2}^{\infty} (-1)^j \zeta^\pm(2j-2k-1) \frac{\Gamma(j-k-\frac{1}{2}) \Gamma(j+1)}{\Gamma(j+1)} X^{2j-2k-2}, \] 

\[ \Xi^\pm_{k,2q}(X) = \frac{1}{\sqrt{\pi}} \sum_{j=2+k}^{\infty} (-1)^j \zeta^\pm(2j-2k-2) \frac{\Gamma(j-k-1) \Gamma(j+1)}{\Gamma(j+1)} X^{2j-2k-3}, \] 

\[ \Psi^-(X) = \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \zeta_R(2j) I_k^-(C_s, -j) (X)^{2j-1}, \] 

\[ \Psi^+(X) = \frac{2}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} (1 - 2^{2j}) \zeta_R(2j) I_k^+(C_{1/2}, -j) (X)^{2j-1} \] 

and \( \gamma \) is the Euler-Mascheroni constant, \( B_{2k} \) are the Bernoulli numbers. Eqs. (27) - (30) give a series representation for the free energy which is very convenient for high temperature expansion. Let us remark, that the regularization has been already removed in these expressions. They are finite. That means the pole at \( s = 0 \) is absent, as it happens usually in the zeta regularization techniques.

The Laurent series in inverse powers of \( \beta \) we obtained is analogous to the one-loop contributions to the free energy in string theory [19], which is actively investigated in recent time.

The infinite series (31) and (32) are convergent for

\[ \beta < \beta_C^\pm, \quad \begin{cases} \beta_C^- = \frac{2\pi}{C_s} \\
\beta_C^+ = \frac{\pi}{C_{1/2}}. \end{cases} \] 

The series (31) and (32) can be analytically continued in \( \beta \) in the following way. Using the integral representation

\[ \zeta_R(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1} \]
for the Riemann zeta function and interchanging the orders of summation and integration, the sums over $k$ can be performed to yield Bessel functions $J_{k+1}(z)$,

$$
\Xi_{k,2q+1}(X) = (-1)^{k+1} \int_0^\infty \frac{dt}{e^t - 1} \left( \left( \frac{2}{Xt} \right)^{k+1} J_{k+1}(Xt) - \frac{1}{\Gamma(k+2)} \right),
$$

and Struve functions $H_{k+3/2}(z)$,

$$
\Xi_{k,2q}(X) = (-1)^k \int_0^\infty \frac{dt}{e^t - 1} \left( \left( \frac{2}{Xt} \right)^{k+\frac{1}{2}} H_{k+\frac{3}{2}}(Xt) \right).
$$

Taking into account their asymptotic behaviour for large argument, it is clear that the convergence of the integration over $t$ breaks down for $x \to \pm i$. For real $X$, the functions $\Xi_{k}(X)$ are smooth. By expanding them back into a series with respect to powers of $X$, the convergence radius is given by the nearest pole in the complex $X$-plane which lies in $X = \pm i$ in accordance with (35). For $\beta = \beta C (X = 1)$ and $\beta = \beta^+ (X = 1/2)$ the functions (36) and (37) remain finite. From the representations (36) and (37), the behaviour for $X \to \infty$ can be obtained:

$$
\Xi_{k}^{-}(X) \sim \frac{(-1)^k}{2\Gamma(k+2)} \log X, \quad (38)
$$

and

$$
\Xi_{k}^{+}(X) \sim \frac{(-1)^k}{\sqrt{\pi\Gamma(k+2)}} \log X. \quad (39)
$$

The functions $\Psi^{\pm}(X)$ (33) and (34) have similar properties. These series are convergent for $X \leq 1/(C\pi)$. Taking into account that their argument is $\beta/2\pi$ (in difference to $C\beta/2\pi$ in case of the functions $\Xi^{\pm}$), this is equivalent to (35). These functions can be analytically continued and can be represented in the form:

$$
\Psi^-(X) = -\frac{1}{\pi X} \int_0^\infty \frac{d\lambda}{e^{\pi\lambda} + 1} \log \left( \frac{\sinh(\pi X \sqrt{\lambda^2 + C^2})}{\pi X \sqrt{\lambda^2 + C^2}} \right)
$$

$$
\Psi^+(X) = -\frac{1}{\pi X} \int_0^\infty \frac{d\lambda}{e^{\pi\lambda} - 1} \log \left( \coth(\pi X \sqrt{\lambda^2 + C^2}) \right).
$$

They are of order $O\left(\frac{1}{X}\right)$ for $X \to \infty$.

### 4 Conclusions

Here we would like to make some final remarks concerning the obtained results. We have developed a formalism for the calculation of the one-loop free energy associated with fields of arbitrary spin in the manifold $M = S^1 \otimes H^N$.

For the minimally coupled scalar field we have $C_0^2 = \rho_N^2 + a^2 m^2$, where $\rho_N = (N-1)/2$ and $m$ is the mass of the field. For the massless field in $D = 4$ dimensions we have $N = 3$, $C_0 = 1$ and $\beta R = 2\pi$ in agreement with the Rindler temperature $T_R = 1/2\pi$, well known in the theory of conformal invariant fields. In addition, the leading term of the Laurent series has the form $-A\pi^2/90\beta^4$, which is also well known.
On the other hand the constant $C_{1/2}$ is the mass of the Dirac spinor field. For the vector (spin-1) field, the Hodge-de Rham operator $d\delta + \delta d$ acting on the exact one-forms is associated with the massless operator $[-\Delta^\mu \Delta^\nu + (N-1)a^{-2}]g_{\mu\nu}$. The eigenvalues of that operator are $\lambda^2 + (\rho_N - 1)^2$ and for the Proca-field of mass $m$ we find $C_1^2 = (\rho_N - 1)^2 + a^2 m^2$. In general, for the spin-$s$ field the wave operator has the form $L_N^+ + m^2 + Qa^{-2}$, where $Q$ is a given constant.

The renormalized free energy $F^-(\mathcal{g}, \beta)$ of the conformal invariant scalar field in the ultrastatic space with the metric $\mathcal{g}$ is given by Eqs. (27) and (28). For the conformal massless field we have $C_0 = 0$ and hence

$$F^-(\beta)_{2q+1} = - \frac{g(0)A(2q+1)}{4\pi} \sum_{k=1}^{q} a_{k,N}^{(0)} \zeta_R(2k+2)\Gamma(k+\frac{1}{2})\Gamma(k+1) \left( \frac{\beta}{2} \right)^{-2k-2},$$

$$F^-(\beta)_{2q} = - \frac{g(0)A(2q)}{4\pi} \sum_{k=0}^{q-1} b_{k,N}^{(0)} \zeta_R(2k+3)\Gamma(k+\frac{3}{2})\Gamma(k+1) \left( \frac{\beta}{2} \right)^{-2k-3} + g(0)A(2q) \sum_{k=0}^{q-1} b_{k,N}^{(0)} \left\{ \left[ \frac{d}{dz} I_k(0, z) \right]_{z=0} + \frac{(-1)^k(1-2^{-2k-1})}{4(k+1)} (\gamma - \log \beta^2) B_{2k+2} \right\} \beta^{-1}$$

$$- 2I_k(0, -\frac{1}{2}) + \sum_{j=1}^{\infty} \frac{(-1)^j}{\pi j} \zeta_R(2j) I_k(0, -j) \left( \frac{\beta}{2\pi} \right)^{2j-1}.$$

Using eq. (40) for $D = 4$ dimensions we have $g(0) = 1$, $A(3) = A/2\pi^2$ and $F^-(\beta) = -A\pi^2/90\beta^4$. Thus, there is no term proportional to $\beta^{-2}$, a standard result [14, 20]. The renormalized stress energency tensor can be obtained by the variation of $F^-[\mathcal{g}, \beta]$ with respect to the metric $g$. This tensor remains finite at the horizon since the divergence of the thermal contribution obtained by the variation of $\Delta F^-[\omega, g]$ is compensated by the vacuum polarization obtained by variation of $\Delta F^-[\omega, g]$. In general, for a massive conformal field the coefficient reads $C_0 = (\xi - \frac{1}{4})R + m^2$ [14].

The dominant contribution to the one-loop entropy density

$$S(\beta) = h^{-1}\beta^2 \frac{\partial F}{\partial \beta}$$

comes from the region near the horizon. In order to regulate the divergencies one can put in an infrared cutoff by defining a smooth compact manifold $M$ as $M = S^1 \otimes H^N/\Gamma$ and them impose suitable boundary conditions on the fields. Here, $H^N/\Gamma$ is the quotient of $H^N$ by a discontinuous group $\Gamma$ of isometries. Under this assumptions the volume of the compact manifold $H^N/\Gamma$ is $V(\mathcal{F}_N)A^N$, $V(\mathcal{F}_N)$ beeing the volume of the fundamental domain $\mathcal{F}_N$. By making use of the Selberg trace formula associated with $H^N/\Gamma$ [14], one obtaines the zeta function $\zeta(z \in H^N/\Gamma | L^\pm_N) = V(\mathcal{F}_N)\zeta(z | L^\pm_N) + \zeta'(z | L^\pm_N)$, where the topological term (the analytical part of the zeta function) $\zeta'(z | L^\pm_N)$ can be written as a sum over closed geodesics of $H^N/\Gamma$. The leading behaviour of the free energy [27]-[30] is independent on the topological contributions, of course. So we obtain the same leading behaviour as above with a finite volume factor.
Moreover, there exist discrete groups $\Gamma$ of special kind which are related to some noncompact hyperbolic manifolds with finite volume (in that case the group $\Gamma$ contains parabolic elements as well) \cite{21}. For the symmetric space of rank-1 some concrete examples of such groups $\Gamma$ are known. Therefore the fundamental domain of $\Gamma$ is noncompact and the finite invariant volume of the domain can be connected with the area of the horizon.

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