Unification of Gravity and Electromagnetism Revisited

Partha Ghose*
Centre for Astroparticle Physics and Space Science (CAPSS), Bose Institute, Block EN, Sector V, Salt Lake, Kolkata 700 091, India

Abstract
Unification of gravity and electromagnetism based on a theory with an affine non-symmetric connection $\Gamma^\lambda_{\mu\nu} \neq \Gamma^\lambda_{\nu\mu}$ and $\Gamma^\mu = \Gamma^\lambda_{[\mu\lambda]} \neq 0$, proposed by S. N. Bose in 1953, is revisited in the context of modern developments in high energy physics. It is shown that electrogravity unification can be achieved at a premetric stage, and that this unification based on projective invariance is broken by matter fields, opening up the possibility of a unified theory of all forces in which gravity emerges as a classical field. The theory predicts $\alpha_{\text{sym}} = \frac{1}{9}$ in the projective invariant limit, where $\alpha$ is the fine structure constant.

1 Introduction

The unification of gravity and electromagnetism has remained an unfulfilled goal in spite of many efforts by stalwarts like Einstein, Weyl, Kaluza and Schrödinger. This has been largely due to the unrealistic dream of a unitary classical field theory to achieve the unification of gravity and electromagnetism and at the same time that of waves and particles by having particles as spherically symmetric singularity-free solutions of the field equations. No singularity-free solutions of these field equations, however, have been found. Moreover, after the advent of quantum theory, the unification of particles

*partha.ghose@gmail.com
and fields has acquired a completely different significance. The quantum theoretic unification of the electromagnetic and weak nuclear forces has also opened up a new perspective on the unification of forces. Nevertheless, all attempts to unify gravity with other forces have remained unsuccessful so far because of a fundamental incompatibility between quantum mechanics and general relativistic gravity, namely, quantum theories can be constructed only on a fixed non-dynamical space-time background whereas General Relativity requires diffeomorphism invariance. This has led in recent years to the view that perhaps gravity is an emergent rather than a fundamental field, and hence does not need to be quantized [1]. In this situation it would be worthwhile once again to revisit the earlier attempts at unification of classical electromagnetism and gravity without invoking extra dimensions.

In this context an almost completely ignored paper of S. N. Bose [2] is of particular interest. In this paper Bose generalized the equations of Einstein’s unitary field theory and derived them from a variational principle. This resulted in interesting mathematical solutions. However, Bose included terms that broke an important symmetry of the Einstein action, namely, \(\Lambda\)-transformation invariance [3] or, in modern parlance, projective invariance [4] that is necessary for a true geometric unification. In what follows a slightly different approach will be taken that is in conformity with modern developments in unified theories.

## 2 A Projective Invariant Unified Theory

Let the starting point be a 4-dimensional manifold \(\mathcal{E}\) with signature \((+, +, +, -)\), a non-symmetric tensor \(g^{\mu \nu}\) and a non-symmetric affine connection \(\Gamma\) with the property

\[
\Gamma_\mu = \frac{1}{2} (\Gamma^\lambda_{\mu \lambda} - \Gamma^\lambda_{\lambda \mu}) \neq 0.
\]

Let

\[
\Gamma^\lambda_{\mu \nu} = \Gamma^\lambda_{(\mu \nu)} + \Gamma^\lambda_{[\mu \nu]}
\]

where

\[
\Gamma^\lambda_{(\mu \nu)} = \frac{1}{2} (\Gamma^\lambda_{\mu \nu} + \Gamma^\lambda_{\nu \mu}) ,
\]

\[
\Gamma^\lambda_{[\mu \nu]} = \frac{1}{2} (\Gamma^\lambda_{\mu \nu} - \Gamma^\lambda_{\nu \mu}) ,
\]

2
\Gamma_\mu = \Gamma^{\lambda}_{[\mu\lambda]} \neq 0. \quad (5)

This condition turns out to be of crucial importance as \Gamma_\mu acts, as we will see, as the common source term for the electromagnetic and gravitational fields. \Gamma^{\lambda}_{[\mu\nu]} is known as the Cartan torsion tensor which will be related to the electromagnetic field.

Consider a non-symmetric tensor of the form

\[ E_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \frac{1}{2} (\Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\lambda}_{\nu\lambda,\mu}) + 2\Gamma_{\mu\nu}^\xi \Gamma^\lambda_{\xi\nu} - 2\Gamma_{\mu\lambda}^\xi \Gamma_{\xi\nu}^\lambda. \quad (6) \]

This tensor is both transposition invariant and \Lambda-transformation invariant. These are two symmetries that restrict the number of possible covariant terms in a nonsymmetric theory \[3\].

**Transposition symmetry**

Let \[ \tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \] and \[ \tilde{g}_{\mu\nu} = g_{\nu\mu} \]. Then terms that are invariant under the simultaneous replacements of \[ \Gamma^\lambda_{\mu\nu} \] and \[ g_{\mu\nu} \] by \[ \tilde{\Gamma}^\lambda_{\mu\nu} \] and \[ \tilde{g}_{\mu\nu} \], followed by the interchange of the two lower indices, are called transposition invariant. For example, the tensor

\[ E'_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\xi}_{\mu\nu} \Gamma^{\lambda}_{\xi\lambda} - \Gamma^{\xi}_{\mu\lambda} \Gamma^{\lambda}_{\xi\nu} \quad (7) \]

is not transposition invariant because it is transposed to

\[ \tilde{E}'_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu,\lambda} - \Gamma^{\lambda}_{\nu\lambda,\mu} + \Gamma^{\xi}_{\nu\mu} \Gamma^{\lambda}_{\xi\lambda} - \Gamma^{\xi}_{\nu\lambda} \Gamma^{\lambda}_{\xi\nu} \neq E'_{\mu\nu}. \quad (8) \]

But \[ (6) \] is transposition invariant.

**\Lambda-transformation or projective symmetry**

Define the transformations

\[ \Gamma^*_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu \Lambda_{\nu}, \]
\[ g^{*\mu\nu} = g^{\mu\nu}, \quad (9) \]

where \( \Lambda \) is an arbitrary function of the coordinates, \( \delta^\lambda_\mu \) is the Kronecker tensor, and the comma denotes the partial derivative. It is easy to check that \( E_{\mu\nu} \) given by \( (6) \) and hence \( g^{\mu\nu} E_{\mu\nu} \) are invariant under these transformations. What this means is that a theory characterized by \( E_{\mu\nu} \) cannot determine the
\( \Gamma \)-field completely but only up to an arbitrary function \( \Lambda \). Hence, in such a theory, \( \Gamma \) and \( \Gamma^* \) represent the same field. Further, this \( \Lambda \)-transformation produces a non-symmetric \( \Gamma^* \) from a \( \Gamma \) that is symmetric or anti-symmetric in the lower indices. Hence, the symmetry condition for \( \Gamma \) loses objective significance. This sets the ground for a genuine unification of the gravitational and electromagnetic fields, the former determined by the symmetric part of the tensor \( E_{\mu\nu} \) and the latter by its antisymmetric part.

Separating the symmetric and antisymmetric parts of \( E_{\mu\nu} \), and using the definitions

\[
R'_{\mu\nu} = \Gamma^\lambda_{(\mu\nu),\lambda} - \frac{1}{2} (\Gamma^\lambda_{(\mu\lambda),\nu} + \Gamma^\lambda_{(\nu\lambda),\mu}) + \Gamma^\xi_{(\mu\nu)} \Gamma^\lambda_{(\xi\lambda)} - \Gamma^\xi_{(\mu\lambda)} \Gamma^\lambda_{(\xi\nu)},
\]

(10)

\[
G^\lambda_{\mu\nu} = \Gamma^\lambda_{[\mu\nu]} + \frac{1}{3} \delta^\lambda_\mu \Gamma_\nu - \frac{1}{3} \delta^\lambda_\nu \Gamma_\mu,
\]

(11)

\[
G^\lambda_{\mu\nu,\lambda} = G^\lambda_{\mu\nu,\lambda} - G^\lambda_{\mu\xi} \Gamma^\xi_{(\lambda\nu)} - G^\lambda_{\nu\xi} \Gamma^\xi_{(\mu\lambda)} + G^\xi_{\mu\nu} \Gamma^\lambda_{(\xi\lambda)},
\]

(12)

one can show that

\[
E_{\mu\nu} = \frac{1}{2} \left[ R'_{\mu\nu} - G^\lambda_{\mu\xi} G^\xi_{\lambda\nu} + \frac{1}{3} \Gamma_\mu \Gamma_\nu - \frac{1}{2} (\Gamma_{\mu,\nu} + \Gamma_{\nu,\mu}) + \Gamma^\lambda_{(\mu\nu)} \Gamma^\lambda_{(\lambda)} \right]
\]

\[
+ \frac{1}{2} \left[ G^\lambda_{\mu\nu;\lambda} - \frac{1}{3} (\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}) \right].
\]

(13)

Notice that by construction \( G^\lambda_{\mu\lambda} = 0 \). One can now write an invariant Lagrangian density

\[
\mathcal{L} = \frac{1}{\kappa} \sqrt{|g|} g^{\mu\nu} E_{\mu\nu} = \frac{1}{\kappa} (s^{\mu\nu} + a^{\mu\nu}) E_{\mu\nu}
\]

\[
= \frac{1}{\kappa} s^{\mu\nu} \left[ R_{\mu\nu} - G^\lambda_{\mu\xi} G^\xi_{\lambda\nu} + \frac{1}{3} \Gamma_\mu \Gamma_\nu + \Gamma^\lambda_{(\mu\nu)} \Gamma^\lambda_{(\lambda)} - \Gamma_{\nu,\nu} \right]
\]

\[
+ \frac{1}{\kappa} a^{\mu\nu} \left[ G^\lambda_{\mu\nu;\lambda} - \frac{1}{3} (\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}) \right],
\]

(14)

where

\[
R_{\mu\nu} = \Gamma^\lambda_{(\mu\nu),\lambda} - \Gamma^\lambda_{(\mu\lambda),\nu} + \Gamma^\xi_{(\mu\nu)} \Gamma^\lambda_{(\xi\lambda)} - \Gamma^\xi_{(\mu\lambda)} \Gamma^\lambda_{(\xi\nu)},
\]

(15)

\[
s^{\mu\nu} = \frac{1}{2} \sqrt{|g|} (g^{\mu\nu} + g^{\nu\mu}) \equiv \sqrt{|g|} g^{(\mu\nu)},
\]

(16)

\[
a^{\mu\nu} = \frac{1}{2} \sqrt{|g|} (g^{\mu\nu} - g^{\nu\mu}) \equiv \sqrt{|g|} g^{[\mu\nu]},
\]

(17)

\[
|g| = |\bar{g}|.
\]

(18)
and $\kappa$ is an arbitrary constant of the dimension of inverse force. Let us therefore consider the variation
\[
\delta I = \delta \int L \, d^4x = 0. \tag{19}
\]

Arbitrary variations of $s^{\mu \nu}$ and $a^{\mu \nu}$ while keeping the connections fixed (generalized Palatini variations) give rise to the field equations
\[
R_{\mu \nu} - G^\lambda_{\mu \xi} G^\xi_{\lambda \nu} + \frac{1}{3} \Gamma_{\mu \nu} \Gamma_{\nu} + \Gamma^\lambda_{(\mu \nu)} \Gamma^\lambda_{\nu} = 0, \tag{20}
\]
\[
G^\lambda_{\mu \nu ; \lambda} - \frac{1}{3} (\Gamma_{\mu \nu} - \Gamma_{\nu \mu}) = 0. \tag{21}
\]

The coefficients of $s^{\mu \nu}$ and $a^{\mu \nu}$ in the Lagrangian (14) are respectively the symmetric and anti-symmetric curvature tensors in the theory. These variational equations show that these tensors vanish. They also show that $\Gamma_{\mu}$ acts as the common source of $G^\lambda_{\mu \nu ; \lambda}$ and $R_{\mu \nu}$. In a theory in which $\Gamma_{\mu} = 0$, $G^\lambda_{\mu \nu ; \lambda}$ and $R_{\mu \nu}$ would have no common source, and the two cannot be said to be genuinely unified.

To derive the equations of connection, one can use a variational principle with an undetermined Lagrange multiplier $k^\mu$, namely
\[
\delta \int (\kappa L - 2k^\mu G^\lambda_{\mu \lambda}) \, d^4x = 0, \tag{22}
\]
in which all the 24 components of $G^\lambda_{\mu \nu}$ are treated to be independent although, as we have seen, they are not because $G^\lambda_{\mu \nu} = 0$. One then obtains the equation (see Appendix for details)
\[
g^\mu_{\lambda \alpha} \Gamma^\nu_{\lambda \alpha} + g^\alpha_{\mu \nu} \Gamma^\tau_{\alpha \lambda} = 3g^{\mu \nu} \Phi_{\lambda} \tag{23}
\]
with the affine connections $\Gamma'$ given by Eqs. (61) and (62) in the Appendix, and
\[
\Phi_{\lambda} = \frac{g[\lambda \beta]k^\beta}{\sqrt{|g|}}, \tag{24}
\]
\[
k^\beta = \frac{1}{3} \left( s^\beta_{\mu \nu} \Gamma^\nu_{\mu} + \frac{3}{2} s^\lambda_{\mu \nu} \Gamma^\beta_{(\lambda \nu)} \right), \tag{25}
\]
and
\[
s^\mu_{\alpha ; \lambda} + s^\alpha_{\beta ; (\alpha \beta)} + a^\alpha_{\beta ; \lambda} G^\mu_{\alpha \lambda} = 0, \tag{26}
\]
\[
a^\mu_{\lambda ; \nu} = 3k^\mu. \tag{27}
\]
The last equation implies
\[ k_{\mu} = 0. \] (28)
Eqn. (25) determines this 4-vector \( k^\mu \) and constrains the number of independent components of \( G_{\mu\nu}^\lambda \) to be 20 in accordance with the property \( G_{\mu\lambda}^\lambda = 0 \). If the determinant
\[ \|g_{[\lambda\beta]}\| = (g_{12}g_{34} + g_{23}g_{14} + g_{31}g_{24})^2 = 0, \] (29)
one can have \( \Phi_\lambda = 0 \) but \( k^\mu \neq 0 \). It is possible in this case to relate the electromagnetic field intensity with \( a^{\mu\nu} \) through the relation
\[ F_{\mu\nu} = ecR a^{\mu\nu} \] (30)
with a non-zero curvature
\[ R = g^{[\mu\nu]}E_{[\mu\nu]}. \] (31)
Thus, \( (ecR\sqrt{|g|})(g_{23}, g_{31}, g_{12}) \) are components of the magnetic field \( \vec{B} \) and \( (ecR\sqrt{|g|})(g_{41}, g_{42}, g_{43}) \) those of the electric field \( i\vec{E} \) which satisfy the condition \( \vec{E} \cdot \vec{B} = 0 \), \( e \) being the electric charge and \( c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \). In the absence of electrically charged matter, \( e = 0 \), and hence \( i\vec{E} \) and \( \vec{B} \) vanish, but \( a^{\mu\nu} \neq 0 \), and the geometric structure of the electromagnetic field remains. It is only with the introduction of charged matter, as we will see, that this geometric structure acquires physical dimensions.

The equations of connection are then of the form
\[ g_{,\lambda}^{\mu\nu} + g^{\mu\alpha}\Gamma_{\lambda\alpha}^{\nu} + g^{\alpha\nu}\Gamma_{\alpha\lambda}^{\mu} = 0, \] (32)
and one also has
\[ \mathfrak{F}_{i\nu}^{\mu\nu} = ecR a_{i\nu}^{\mu\nu} + ecR a_{,\nu}^{\mu\nu} = e\mathfrak{F}_{i\nu}^{\mu}. \] (33)
When \( \mathfrak{F}_{i\nu}^{\mu} = 0 \),
\[ a_{i\nu}^{\mu\nu} = -R^{-1}R_{,\nu}a_{i\nu}^{\mu\nu} = 3k^\mu. \] (34)
This is the case in the projective invariant limit with no particles present.

However, as we have seen, \( R_{\mu\nu} \) and \( G_{\mu\nu;\lambda}^\lambda \) cannot be objectively separated and identified with the physical gravitational and electromagnetic fields respectively because of projective invariance. In the observable universe at present, however, this symmetry is badly broken in the sense that the electromagnetic and gravitational fields can be objectively separated and
identified, and the electric charge $e$ and the gravitational charge $\kappa = 8\pi G/c^4$ are widely different. Furthermore, there are no charged particles in the theory which can be shown to be singularity-free solutions of the classical field equations and which can act as the source $J_{\mu}$ of the electromagnetic field. This makes the projective invariant classical theory with $\Gamma_\mu \neq 0$ incomplete. Furthermore, there are strong and weak nuclear interactions and quantum mechanical effects that need to be taken into account. These are the issues that will be addressed in the following sections.

3 Matter and Projective Symmetry Breaking

Let us first consider projective symmetry breaking. In the perspective of modern developments in unified theories, it would be natural to think of a symmetry breaking transition at some appropriate stage of the evolution of the universe that separates the gravitational and electromagnetic fields objectively and physically. Such a scenario would be possible provided there is some natural mechanism to break the $\Lambda$-transformation or projective symmetry of the action so that the symmetric and anti-symmetric parts of the connection can be objectively separated. The symmetry condition for the connection $\Gamma$ characteristic of Riemann manifolds and Einstein’s gravitational theory based on them would then acquire objective significance. From such a symmetry breaking would then emerge the observed space-time world endowed with a symmetric dynamical metric field $g_{(\mu\nu)}$ encoding gravity as well as the anti-symmetric field $g_{[\mu\nu]}$ (resulting from torsion) encoding electromagnetism. The most natural stage for such a symmetry breaking to occur would be the emergence of matter fields at the end of a non-symmetric affine field dominated phase of a premetric universe. This is because a matter Lagrangian $\mathcal{L}_m$ obtained by minimally coupling the matter to the connection is not generally projective invariant.

In such a theory one would have

$$ J_{\mu}^{em} = \left[ \sum_i \bar{\psi}_i \gamma^\mu \psi_i + \sum_j \bar{\phi}_j \beta^\mu \phi_j + \cdots \right] $$

(35)

where $\psi_i$ are Dirac wavefunctions describing spin-$\frac{1}{2}$ particles, $\phi_j$ are Kemmer-Duffin wavefunctions describing spin-0 and spin-1 particles, the $\beta$s being the Kemmer-Duffin-Petiau matrices, and the dots represent higher spin wavefunctions if any.
The broken symmetric Lagrangian density including the matter wave-functions can then take the general form

\[ L' = \frac{1}{\kappa} s^\mu{}^\nu \left[ R_{\mu\nu} - k' \left( G^\lambda_{\mu\xi} G^\xi_{\lambda\nu} - \frac{1}{3} \Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} \right) \right] + \frac{1}{\kappa} a^\mu{}^\nu \left[ G^\lambda_{\mu\nu,\lambda} - \frac{k}{3} (\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}) \right] + \mathcal{L}_m(\psi,g,\Gamma) \]  

(36)

where now \( \kappa = 8\pi G/c^4 \) and \( k \) and \( k' \) are arbitrary dimensionless constants to be determined by experiments. Hence, varying \( s^\mu{}^\nu \) and \( a^\mu{}^\nu \) together with the connections, one obtains

\[ R_{\mu\nu} - \frac{1}{2} g_{(\mu\nu)} R = -\kappa \left[ T^m_{(\mu\nu)} + T^{em}_{(\mu\nu)} \right] , \]  

(37)

\[ G^\lambda_{\mu\nu,\lambda} = \frac{k}{3} (\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}) , \]  

(38)

where

\[ T^m_{(\mu\nu)} = -\frac{2}{\sqrt{|g|}} \frac{\delta \left( \sqrt{|g|} \mathcal{L}_m \right)}{\delta g^{(\mu\nu)}}, \]  

(39)

\[ T^{em}_{(\mu\nu)} = \frac{k'}{\kappa} \left[ \frac{1}{3} \Gamma_{\mu,\nu} + \frac{\Gamma_{\lambda,\mu}}{g^{\lambda\nu}} G^\xi_{\lambda\nu} - \frac{1}{3} G^\lambda_{\mu\xi} G^\xi_{\lambda\nu} \right] \]  

(40)

and

\[ \Gamma_\mu = \frac{1}{c R} \delta^\mu - \frac{1}{c R^2} R_{,\nu} \delta^\nu - \frac{3}{2} s^\lambda{}^\nu \Gamma_{\mu,(\lambda\nu)} ; \]  

(41)

\[ \Gamma^\lambda_{(\mu\nu)} = \frac{1}{2} g^{(\lambda\rho)} \left( g_{(\rho,\mu,\nu)} + g_{(\rho,\nu,\mu)} - g_{(\mu,\nu,\rho)} \right) ; \]  

(42)

\[ \Gamma_{\mu,(\lambda\nu)} = \frac{1}{2} \left( g_{(\mu,\lambda,\nu)} + g_{(\mu,\nu,\lambda)} - g_{(\lambda,\nu,\mu)} \right) . \]  

(43)

The first of these equations follows from Eqn. (33) and Eqns. (25), (27) and (30), and the other two are consequences of Eqn. (32) for the symmetric part. The constant \( k' \) must be chosen to fit experimental data on electromagnetic contributions to the total stress-energy tensor. A comparison of Eqns. (37) and (38) suggests that we identify \( k = 3\sqrt{\alpha} \) where \( \alpha = 1/137 \) is the dimensionless fine structure constant so that the two fundamental constants \( \kappa \) and
\( \alpha \) determine the strengths of the couplings of the sources to the symmetric and antisymmetric curvature tensors in the theory.

In the projective invariant limit, \( k' = k = 1 \) and \( \kappa \to 0 \) so that the matter Lagrangian density can be ignored in comparison with the other terms. Hence, the theory predicts that in the invariant limit \( \alpha_{sym} = \frac{1}{9} \).

Notice also that \( T^{\text{em}}_{(\mu\nu)} \) differs from the standard general relativistic form of the electromagnetic stress-energy tensor

\[
\frac{1}{\mu_0} \left( F^{\mu\nu} F_{\nu}^{\alpha} - \frac{1}{4} g^{(\mu\nu)} F_{\alpha\beta} F^{\alpha\beta} \right) .
\]

Hence, the predictions of the theory regarding the effects of the electromagnetic stress tensor on gravity differ from those of standard General Relativity, and hence can be tested in principle. These effects are being investigated and will be reported elsewhere.

4 Quantization

Let us now see how the above scheme can be incorporated into the current understanding of gravitation and the quantum theory of matter and radiation. We first note that the electromagnetic gauge potential \( A_\mu \) has played no role so far in our considerations. That is because \( A_\mu \) is required for minimal coupling with charged matter which is absent in a projective invariant theory. Charged matter is represented by complex quantum mechanical wavefunctions whose imaginary parts are arbitrary local phases, a typical quantum mechanical feature that is wholly absent in classical theories of matter. \( A_\mu \) has the geometric significance of a connection associated with a horizontal subspace of a principal bundle \( P = (E, \Pi, M, G) \) (associated with the phase) whose projection is \( \Pi : E \to M \). The 1-form \( A = A_\mu dx^\mu \) transforms as \( hAh^{-1} + hdh \) under a group transformation \( g' = hg, g \in G \).

There is a curvature associated with this 1-form given by \( F' = F'_\mu\nu dx^\mu \wedge dx^\nu \) with \( F'_\mu\nu = \partial_\mu A_\nu - \partial_\nu A_\mu \). It transforms as \( hF'h^{-1} \). \( F'_\mu\nu \) is identified with the electromagnetic field. In this case \( G = U(1) \), and \( F' \) is invariant. The bundle is trivial since the base space \( M \) is flat and \( E = M \times G \) everywhere.

The first change that is needed is the replacement of the principal bundle \( P \) by \( P' = (E, \Pi, E, G) \) whose base space is the manifold \( E \) of the theory and whose local structure group \( G \) is ideally a simple Lie group which can be broken down to \( SU(3) \times SU(2) \times U(1) \), the symmetry group of the Standard
Model. Such a bundle is only locally $M \times G$. In keeping with current theory, we also require that matter wavefunctions be introduced as global sections of vector bundles associated with specific representations of $G$. Since the electromagnetic field is also defined on the manifold $\mathcal{E}$ by the relation (30), the compatibility requirement

$$\Pi \tilde{F}^{\mu\nu} = \tilde{F}^{\mu\nu} = e c R a^{\mu\nu}$$

must be satisfied, where $\tilde{F}^{\mu\nu} = \sqrt{|g|} g^{(\mu\alpha)} g^{(\nu\beta)} F_{\alpha\beta}$. This immediately implies that quantization of $F_{\mu\nu}$ leads to quantization of $a^{\mu\nu}$. Other gauge fields in the theory belonging to non-Abelian groups are not subject to this compatibility requirement.

Finally, let us consider the quantization of the gravitational field. It is important to emphasize that whereas the metric tensor $g_{(\mu\nu)}$ plays a fundamental role in a Riemannian manifold and the connections (Christoffel symbols) are derived from it, in an affine manifold like $\mathcal{E}$ the tensor $g_{\mu\nu}$ and the non-symmetric connections $\Gamma$ play equally fundamental roles, and the two are later related through the equations of connection. Now, quantization requires the existence of a symplectic manifold with a nondegenerate 2-form $\omega$ on which Poisson brackets can be defined. Following Dirac’s prescription, these Poisson brackets can then be replaced by commutators to quantize a classical theory. This is the canonical quantization procedure. Therefore, to quantize the fields in our theory we need to construct the 2-form $\omega = E_{\mu\nu} dx^\mu \wedge dx^\nu$. Because of the antisymmetry of the wedge product, $\omega = E_{[\mu\nu]} dx^\mu \wedge dx^\nu$ which shows that only the antisymmetric part of $E_{\mu\nu}$ contributes to $\omega$. However, as we have seen, the splitting of $E_{\mu\nu}$ into a symmetric and an antisymmetric part has no objective significance due to projective invariance. Hence, canonical quantization cannot be carried out with any objective significance on such a manifold. But, it can be done after the invariance is broken, and then it is at once clear why the symmetric part $E_{(\mu\nu)}$ associated with gravity cannot be quantized canonically while the antisymmetric part associated with electromagnetism can be. Thus, in the theory developed here, though both gravity and electromagnetism are emergent fields from a premetric, prequantum manifold $\mathcal{E}$, gravity remains classical while electromagnetism can be quantized.
5 Concluding Remarks

We have seen that the unification of electromagnetism and gravity into a single geometric entity is beautifully accomplished in a theory with non-symmetric connection and $\Gamma_\mu \neq 0$, the unifying symmetry being projective symmetry. Matter wavefunctions appear in the theory as global sections of vector bundles associated with specific representations of an appropriate simple Lie group $G$ that can be broken down to $SU(3) \times SU(2) \times U(1)$, the symmetry group of the Standard Model. The matter Lagrangian breaks projective invariance, generating classical relativistic gravity and quantum electromagnetism. This is possible because the original non-symmetric manifold $\mathcal{E}$ is assumed to be smooth. Hence, for the theory to be valid, the symmetry breaking transition must occur at a larger scale than the Planck length. The theory predicts $\alpha_{sym} = \frac{1}{9}$ below this scale.

In the projective invariant premetric phase the fourth dimension of the manifold $\mathcal{E}$ with negative signature cannot be identified with physical time. Also, this manifold is affine and has no origin. These features of the theory have important implications for the origin (and possibly also the dissolution) of the observed universe which need to be further explored but are beyond the scope of this paper.

6 Acknowledgement

I dedicate this paper to the memory of my research guide and teacher, the Late Professor Satyendranath Bose whose 1953 paper is its inspiration. I thank the National Academy of Sciences, India for the grant of a Senior Scientist Platinum Jubilee Fellowship which enabled this work to be undertaken.

7 Appendix

In order to derive the equations of connection, let us first write

$$\kappa \mathcal{L} = H + \frac{dX^\lambda}{dx^\lambda}$$

(46)
with

\[
X^\lambda = s^{\mu\nu}\Gamma^\lambda_{(\mu\nu)} - s^{\mu\lambda}\Gamma^\nu_{(\mu\nu)} + a^{\mu\nu}G^\lambda_{\mu\nu} + \frac{2}{3}a^\mu_\lambda\Gamma^\nu_\mu + \Gamma^\lambda
\]

\[
H = -s^{\mu\lambda}\Gamma^\lambda_{(\mu\nu)} + s^{\mu\lambda}\Gamma^\nu_{(\mu\nu)} + s^{\mu\nu}\left(\Gamma^\xi_{(\mu\nu)}\Gamma^\lambda_{(\xi\nu)} - \Gamma^\xi_{(\mu\lambda)}\Gamma^\lambda_{(\xi\nu)}\right) + s^{\mu\nu}\Gamma^\lambda_{(\mu\nu)}\Gamma^\lambda_{(\mu\nu)} - a^\mu_\lambda G^\lambda_{\mu\nu}
\]

\[
+ a^{\mu\nu}\left(-G^\lambda_{\mu\xi}\Gamma^\xi_{(\nu\lambda)} - G^\lambda_{\xi\nu}\Gamma^\xi_{(\mu\lambda)} + G^\xi_{\mu\nu}\Gamma^\lambda_{(\xi\lambda)}\right) - \frac{2}{3}a^\mu_\lambda\Gamma^\nu_\mu.
\]

(47)

Thus, \(H\) is free of the partial derivatives of \(\Gamma^\lambda_{(\mu\nu)}\), \(G^\lambda_{\mu\nu}\) and \(\Gamma^\nu_\mu\), and the four-divergence term in the integral \(I\) is equal to a surface integral at infinity on which all arbitrary variations are taken to vanish.

Now, it follows from the definition of \(G^\lambda_{\mu\nu}\) that \(G^\lambda_{\mu\lambda} = 0\), and hence all the 24 components of \(G^\lambda_{\mu\nu}\) are not independent. Remembering that these four relations must always hold good in the variations of the elements \(\Gamma^\lambda_{(\mu\nu)}\), \(G^\lambda_{\mu\nu}\), \(\Gamma^\nu_\mu\), one can use the method of undetermined Lagrange multipliers \(k^\mu\) to derive the equations of connection by varying the function

\[
H - 2k^\mu G^\lambda_{\mu\lambda}.
\]

(48)

The resulting equations are

\[
s^{\mu\nu} + s^{\nu\alpha}\Gamma^\alpha_{(\lambda\alpha)} + s^{\lambda\alpha}\Gamma^\nu_{(\alpha\lambda)} - s^{\mu\nu}\Gamma^\alpha_{(\lambda\alpha)} = -\left[a^{\mu\alpha}G^\nu_{\alpha\lambda} + a^{\alpha\nu}G^\mu_{\alpha\lambda}\right],
\]

(49)

\[
a^\mu_\lambda + a^{\mu\alpha}\Gamma^\nu_{(\lambda\alpha)} + a^{\alpha\nu}\Gamma^\mu_{(\alpha\lambda)} - a^\mu_\lambda\Gamma^\nu_{(\lambda\alpha)} - k^\mu\delta^\nu_\lambda + k^\nu\delta^\mu_\lambda = -\left[s^{\mu\alpha}G^\nu_{\alpha\lambda} + s^{\alpha\nu}G^\mu_{\alpha\lambda}\right],
\]

(50)

and

\[
a^{\mu\nu} - s^{\mu\nu}\Gamma^\nu_{(\lambda\nu)} - \frac{3}{2}s^{\lambda\nu}\Gamma^\mu_{(\lambda\nu)} = 0.
\]

(51)

It follows from these equations that

\[
s^{\mu\alpha} + s^{\alpha\beta}\Gamma^\mu_{(\alpha\beta)} + a^{\alpha\beta}G^\mu_{\alpha\beta} = 0,
\]

(52)

\[
a^{\mu\nu} = 3k^\mu,
\]

(53)

which imply

\[
k^\mu_{,\mu} = 0,
\]

(54)

\[
k^\mu = \frac{1}{3}\left(s^{\mu\nu}\Gamma^\nu_{(\lambda\nu)} + \frac{3}{2}s^{\lambda\nu}\Gamma^\mu_{(\lambda\nu)}\right).
\]

(55)
Adding (49) and (50), we get

\[
g'_{\mu\nu} = g'_{\mu\alpha} \left( \Gamma'_{(\lambda\alpha)} + G'_{\lambda\alpha} \right) + g'_{\alpha\nu} \left( \Gamma'_{(\alpha\lambda)} + G'_{\alpha\lambda} \right) - g'_{\mu\nu} \Gamma'_{(\lambda\alpha)}
\]

where \( g'^{\mu\nu} = \sqrt{|g|} g^{\mu\nu} \). Multiplying (56) by \( g'_{\mu\nu} \) and using the results

\[
g'_{\mu\nu} g_{\mu\lambda} = \delta^\nu_\lambda, \quad g'_{\mu\nu} g_{\lambda\nu} = \delta^\mu_\lambda, \quad G_{\lambda\alpha} = 0, \quad (57)
\]

we first observe that

\[
\Gamma'_{(\lambda\alpha)} = \frac{|g|_{\lambda\alpha}}{2\sqrt{|g|}} + \frac{1}{2} \left( g'_{\lambda\beta} - g'_{\beta\lambda} \right) k^\beta
\]

\[
= \frac{|g|_{\lambda\alpha}}{2\sqrt{|g|}} + g'_{[\lambda\beta] k^\beta} \quad (58)
\]

Hence, dividing (56) by \( \sqrt{|g|} \), and also using (58) and the results

\[
g'^{\mu\alpha} g_{\beta\alpha} k^\beta = k^\mu \quad \text{and} \quad g'^{\alpha\nu} g_{\alpha\beta} k^\beta = k^\nu, \quad (59)
\]

we get

\[
g'^{\mu\nu} + g'^{\mu\alpha} \left( \Gamma'_{(\lambda\alpha)} + G'_{\lambda\alpha} \right) + g'^{\alpha\nu} \left( \Gamma'_{(\alpha\lambda)} + G'_{\alpha\lambda} \right) - g'^{\mu\nu} \Gamma'_{(\lambda\alpha)}
\]

\[
= 3g'^{\mu\nu} g_{[\lambda\beta] k^\beta} \frac{1}{\sqrt{|g|}} \quad (60)
\]

Now, define the new affine coefficients

\[
\Gamma'_{\lambda\alpha} = \left( \Gamma_{(\lambda\alpha)} + G_{\lambda\alpha} \right) + \frac{1}{\sqrt{|g|}} \left( g_{\lambda\beta} k^\beta \delta^\nu_\alpha - g_{\beta\alpha} k^\beta \delta^\nu_\lambda \right) \quad (61)
\]

\[
\Gamma'_{\alpha\lambda} = \left( \Gamma_{(\alpha\lambda)} + G_{\alpha\lambda} \right) + \frac{1}{\sqrt{|g|}} \left( g_{\alpha\beta} k^\beta \delta^\mu_\lambda - g_{\beta\lambda} k^\beta \delta^\mu_\alpha \right) \quad (62)
\]
and

\[ \Phi_\lambda = \frac{g^{[\lambda \beta]} k^\beta}{\sqrt{|g|}} \]  

(63)

Then, Eqn. (60) can be written in the form

\[ g^{\mu \nu}_{,\lambda} + g^{\mu \alpha} \Gamma^\nu_{\lambda \alpha} + g^{\alpha \nu} \Gamma^\mu_{\alpha \lambda} = 3g^{\mu \nu} \Phi_\lambda. \]  

(64)

References

[1] See, for example, T. Padmanabhan (2007), arXiv:0706.1654 [gr-qc] June.

[2] S. N. Bose (1953), Le Jour de Phys et le Radium (Paris) 14, 641-644.

[3] A. Einstein, The Meaning of Relativity, Methuen & Co., London, Sixth Edition, 1956, Appendix II.

[4] See, for example, F. H. Hehl, E. A. Lord and L. L. Smalley (1981), Gen. Rel. and Gravitation 13, 1037-1056 and references therein.