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Immersion of transitive tournaments in digraphs with large minimum outdegree*

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Abstract

We prove the existence of a function $h(k)$ such that every simple digraph with minimum outdegree greater than $h(k)$ contains an immersion of the transitive tournament on $k$ vertices. This solves a conjecture of Devos, McDonald, Mohar and Scheide.

In this note, all digraphs are without loops. Let $D$ be a digraph. We denote by $V(D)$ its vertex set and $A(D)$ its arc set. A digraph $D$ is \textit{simple} if there is at most one arc from $x$ to $y$ for any $x, y \in V(D)$. Note that arcs in opposite directions are allowed. The \textit{multiplicity} of a digraph $D$ is the maximum number of parallel arcs in the same direction in $D$. For an arc $a = (u, v)$ of a digraph $D$, we say that $u$ is the \textit{tail} of $a$ and $v$ its \textit{head}. The \textit{outdegree} (resp. \textit{indegree}) of a vertex $v$, denoted by $d^+(v)$ (resp. $d^-(v)$), is equal to the number of arcs $a$ of $D$ such that $v$ is the tail (resp. head) of $a$. The \textit{outneighbourhood} (resp. \textit{inneighbourhood}) of a vertex $v$, denoted by $N^+(v)$ (resp. $N^-(v)$), is the set of vertices $y$ such that $(v, y)$ (resp. $(y, v)$) is an arc of $D$. A \textit{directed path} $P$ in a digraph $D$ is a set of vertices $x_1, \ldots, x_k$ such that $(x_i, x_{i+1}) \in A(D)$ for all $1 \leq i \leq k - 1$. A \textit{directed cycle} $C$ in a digraph $D$ is a set of vertices $x_1, \ldots, x_k$ such that $(x_i, x_{i+1}) \in A(D)$ for all $1 \leq i \leq k - 1$ and $(x_k, x_1) \in A(D)$. A digraph $D$ is a \textit{tournament} if exactly one of $(u, v)$ and $(v, u)$ is an arc of $D$ for all distinct $u, v \in V(D)$. The \textit{transitive tournament} on $k$ vertices, denoted by $TT_k$, is the unique tournament on $k$ vertices without any directed cycle. The \textit{complete digraph} on $k$ vertices is the simple digraph on $k$ vertices with every possible arc.

We say that a digraph $D$ contains an \textit{immersion} of a digraph $H$ if there exists a mapping such that vertices of $H$ are mapped to distinct vertices of $D$, and the arcs of $H$ are mapped to directed paths joining the corresponding pairs of vertices of $D$, in such a way that these paths are pairwise arc-disjoint. If the directed paths are pairwise internally vertex-disjoint, we say that $D$ contains a \textit{subdivision} of $H$.

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Understanding the necessary conditions for undirected graphs to contain a subdivision of a clique is a very natural and well-studied question. One of the most important examples is the following result by Mader [6]:

**Theorem 1** ([6]). For every \( k \geq 1 \), there exists an integer \( f(k) \) such that every undirected graph with minimum degree greater than \( f(k) \) contains a subdivision of \( K_k \).

Bollobás and Thomason [1] as well as Komlós and Szemerédi [4] proved that \( f(k) = O(k^2) \). In the case of digraphs, there exist examples of digraphs with large out- and indegree without a subdivision of the complete digraph on three vertices, as shown by Thomassen [7] (see also [3] for a simpler construction). However Mader [5] conjectured that an analogue should hold for transitive tournaments \( TT_k \) in digraphs with large minimum outdegree.

**Conjecture 2** ([5]). For every \( k \geq 1 \), there exists an integer \( g(k) \) such that every simple digraph with minimum outdegree greater than \( g(k) \) contains a subdivision of \( TT_k \).

The question turned out to be much more difficult than the undirected case, as the existence of \( g(5) \) remains unknown. Weakening the statement, Devos, McDonald, Mohar and Scheide [3] made the following conjecture replacing subdivision with immersion and proved it for the case of Eulerian digraphs.

**Conjecture 3** ([3]). For every \( k \geq 1 \), there exists an integer \( h(k) \) such that every simple digraph with minimum outdegree greater than \( h(k) \) contains an immersion of \( TT_k \).

As with subdivisions, Devos et al. showed in [3] the existence of digraphs with large out- and indegree without an immersion of the complete digraph on three vertices. Finding the right value for \( h(k) \) in the case of undirected graphs is an interesting question on its own (see [2] for more details).

The goal of this note is to present a proof of this conjecture. Let \( F(k, l) \) be the digraph consisting of \( k \) vertices \( x_1, \ldots, x_k \) and \( l \) arcs from \( x_i \) to \( x_{i+1} \) for every \( 1 \leq i \leq k - 1 \). It is clear that \( F(k, \binom{k}{2}) \) contains an immersion of \( TT_k \), so the following theorem implies Conjecture 3.

**Theorem 4.** For every \( k \geq 1 \) and \( l \geq 2 \), there exists a function \( f(k, l) \) such that every digraph with minimum outdegree greater than \( f(k, l) \) and multiplicity at most \( kl \) contains an immersion of \( F(k, l) \).

**Proof.** We prove the result for \( f(k, l) = 2k^3l^2 \) and \( l \geq 2 \). We proceed by induction on \( k \). For \( k = 1 \) this is trivial because \( F(1, l) \) is one vertex. Suppose now that the result holds for \( k \) and assume for a contradiction that it does not hold for \( k + 1 \). Let \( D \) be the digraph with the smallest number of arcs and vertices such that \( D \) has multiplicity at most \( (k + 1)l \), all but at most \( c_1 = k + (k + 1)l \) vertices have outdegree at least \( f(k + 1, l) \) and without an immersion of \( F(k + 1, l) \). By minimality of \( D \), every vertex has outdegree exactly \( f(k + 1, l) \), expect \( c_1 \) of them with outdegree 0. Call \( T \) the set of vertices of outdegree 0. Suppose we want to remove arcs from \( D \) such that the multiplicity of the remaining digraph is at most \( kl \), while keeping the minimum outdegree as large as possible. For a vertex \( v \), the worst case is when, for every vertex \( y \in N^+(v) \), the multiplicity of \( (v, y) \) is equal to \( (k + 1)l \). In this case we have to remove at most \( l \) arcs for each of the \( \frac{f(k + 1, l)}{(k + 1)l} \) vertices of \( N^+(v) \). Therefore, removing \( T \) and some of the parallel arcs, we obtain a digraph of outdegree greater than \( d' = f(k + 1, l) - c_1(k + 1)l - \frac{f(k + 1, l)}{(k + 1)l} \) with multiplicity \( kl \). Because \( f(k + 1, l) - f(k, l) = 2(3k^3 + 3k + 1)l^2 \) and \( c_1(k + 1)l + \frac{f(k + 1, l)}{(k + 1)l} = k(k + 1)l + 3(k + 1)^2l^2 \), we get that \( d' \geq f(k, l) \) and by induction there
exists an immersion of $F(k, l)$ in $D - T$. Call $X = \{x_1, \ldots, x_k\}$ the set of vertices of the immersion and, numbering the paths arbitrarily, $P_{i,j}$ the $j$th directed path of this immersion from $x_i$ to $x_{i+1}$. We can assume this immersion is of minimum size, so that every vertex in $P_{i,j}$ has exactly one outgoing arc in $P_{i,j}$. Let $D'$ be the digraph obtained from $D$ by removing all the arcs of the $P_{i,j}$ and the vertices $x_1, \ldots, x_{k-1}$. By the previous remark, the outdegree of each vertex in $D'$ is either 0 if this vertex belongs to $T$ or at least $f(k+1, l) - (k-1)l - (k-1)(k+1)l$.

For every vertex $y \in D' - x_k$, there do not exist $l$ arc-disjoint directed paths from $x_k$ to $y$ in $D'$, for otherwise there would be an immersion of $F(k+1, l)$ in $D$. Hence, by Menger’s Theorem there exists a set $E_y$ of less than $l$ arcs such that there is no directed path from $x_k$ to $y$ in $D' \setminus E_y$. Define $C_y$ for every vertex $y \in D' - x_k$ as the set of vertices which can reach $y$ in $D' \setminus E_y$. Now take $Y$ a minimal set such that $\cup_{y \in Y} C_y$ covers $D' - x_k$. We claim that $Y$ consists of at least $c_2 \geq \frac{f(k+1, l) - (k-1)l - (k-1)(k+1)l}{c_2 - c_1}$ elements, as $\cup_{y \in Y} E_y$ must contain all the arcs of $D'$ with $x_k$ as tail.

For each $y \in Y$, define $S_y$ as the set of vertices which belong to $C_y$ and no other $C_y'$ for $y' \in Y$. Since $Y$ is minimal, every $S_y$ is non-empty. Note that for $u \in S_y$, if there exists $y' \in Y \setminus y$ and $v \in C_y'$ such that $uv \in A(D)$, then $uv \in E_{y'}$. Note that $T \subseteq Y$ as vertices in $T$ have outdegree 0 and if $y \in Y \setminus T$ then $S_y$ consists only of vertices of outdegree $f(k+1, l)$ in $D$.

Let $R$ be the digraph with vertex set $Y$ and arcs from $y$ to $y'$ if there is an arc from $S_y$ to $C_{y'}$. As noted before, $d_R(y) \leq |E_y| \leq l$. The average outdegree of the vertices of $Y \setminus T$ in $R$ is then at most $\frac{c_1 l + (c_2 - c_1)l}{c_2 - c_1} \leq 2l$. Let $y$ be a vertex of $R \setminus T$ with outdegree at most this average. Let $H$ be the digraph induced on $D'$ by the vertices in $S_y$ to which we add $X$, all the arcs that existed in $D$ (with multiplicity) from vertices of $S_y$ to vertices of $X$ and the following arcs: For each $P_{i,j}$, let $z_1, z_2, \ldots, z_l = P_{i,j} \cap S_y$, where $z_i$ appears before $z_{i+1}$ on $P_{i,j}$ and add all the arcs $(z_i, z_{i+1})$ to $H$. Note that, if $(x, y)$ is an arc of $D'$, then by minimality of the immersion of $F(k, l)$, every time $x$ appears before $y$ on some $P_{i,j}$, then $P_{i,j}$ uses one of the arcs $(x, y)$. Thus for each pair of vertices $x$ and $y$ in $H$, either $(x, y) \in A(D)$ and the number of $(x, y)$ arcs in $H$ is equal to the one in $D$, or $(x, y) \not\in A(D)$ and the number of $(x, y)$ arcs in $H$ is bounded by $(k-1)l$. This implies that $H$ has multiplicity at most $(k+1)l$.

Claim 4.1. $H$ is a digraph with multiplicity at most $(k+1)l$, such that all but at most $c_1$ vertices have outdegree greater than $f(k+1, l)$ and $H$ does not contain an immersion of $F(k+1, l)$.

Proof of the claim. Suppose $H$ contains an immersion of $F(k+1, l)$, then by replacing the new arcs by the corresponding directed paths along the $P_{i,j}$ we get an immersion of $F(k+1, l)$ in $D$. Moreover, we claim that the number of vertices in $H$ with outdegree smaller than $f(k+1, l)$ is at most $k + 2l + (k-1)l = c_1$. Indeed, the vertices of $H$ that can have outdegree smaller in $H$ than in $D$ are the $x_i$, or the vertices with outgoing arcs in $E_{y'}$ for some $y' \in Y \setminus y$, or the vertices along the $P_{i,j}$. But with the additions of the new arcs, we know that there is at most one vertex per path $P_{i,j}$ that loses some outdegree in $H$.

However, since $H$ is strictly smaller than $D$, we reach a contradiction.

\[\Box\]

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