Asymptotics for the best Sobolev constants and their extremal functions

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Let \( \Lambda_1^p(\Omega) = \inf \left\{ \| \nabla u \|_p^p : u \in W_0^{1,p}(\Omega) \text{ and } \| u \|_{L^\infty(\Omega)} = 1 \right\} \), where \( \Omega \) is a bounded domain of \( \mathbb{R}^N \), \( N \geq 2 \), and \( p > N \). We prove that \( \lim_{p \to \infty} \Lambda_1^p(\Omega) = |\rho|^{-1}_\infty \), where \( \rho \) denotes the distance function to the boundary.

Then, we show that, up to subsequences, the extremal functions of \( \Lambda_1^p(\Omega) \) converge (as \( p \to \infty \)) to the viscosity solutions of a specific Dirichlet problem involving the infinity Laplacian in the punctured domain \( \Omega \setminus \{x_0\} \), for some \( x_0 \in \Omega \).

1 Introduction

Let \( p > 1 \) and let \( \Omega \) be a bounded and smooth domain of \( \mathbb{R}^N, N \geq 2 \). It is well known that the Sobolev immersion \( W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \) is compact if

\[
1 \leq q < p^* := \begin{cases} Np \over N-p & \text{if } 1 < p < N, \\ \infty & \text{if } p \geq N. \end{cases}
\]

As consequence of this fact, for each \( q \in [1, p^*) \) there exists \( w_q \in L^q(\Omega) \) such that \( \| w_q \|_q = 1 \) and

\[
\lambda_q(\Omega) := \inf \left\{ \| \nabla u \|_p^p : u \in W_0^{1,p}(\Omega) \text{ and } \| u \|_q = 1 \right\} = \| \nabla w_q \|_p^p.
\] (1.1)

(Throughout this paper \( \| \cdot \|_s \) denotes the standard norm of \( L^s(\Omega), 1 \leq s \leq \infty. \) )

The value \( \lambda_q(\Omega) \) is, therefore, the best constant \( c \) in the Sobolev inequality

\[
c \| u \|_q^p \leq \| \nabla u \|_p^p, \quad u \in W_0^{1,p}(\Omega),
\]

and \( w_q \) is a corresponding extremal (or minimizer) function.

The Euler–Lagrange formulation associated with the minimizing problem (1.1) is

\[
\begin{cases}
-\Delta_p u = \lambda_q(\Omega) |u|^{p-2} u \quad & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\] (1.2)

where \( \Delta_p u := \text{div} (|\nabla u|^{p-2} \nabla u) \) is the \( p \)-Laplacian operator. It turns out that \( |w_q| \) is a nonnegative and nontrivial solution of (1.2), since \( |w_q| \) also minimizes \( \lambda_q(\Omega) \). Thus, the maximum principle (see [21]) assures that \( w_q \) does not change sign in \( \Omega \).
From now on, we denote by $w_q$ any positive extremal function of $\lambda_q(\Omega)$. Therefore, such a function enjoys the following properties

$$
\|w_q\|_q = 1, \quad \|\nabla w_q\|_p = \lambda_q(\Omega) \quad \text{and} \quad \begin{cases}
-\Delta_p w_q = \lambda_q(\Omega) w_q^{q-1} & \text{in } \Omega, \\
w_q = 0 & \text{on } \partial \Omega, \\
w_q > 0 & \text{in } \Omega.
\end{cases}
$$

It can be checked (see [9, Lemma 4.2]), as a simple application of the Hölder inequality, that the function

$$
q \in [1, p^*) \mapsto \lambda_q(\Omega) |\Omega|^\frac{q}{2}
$$

is decreasing for any fixed $p > 1$, where here and from now on $|D|$ denotes the Lebesgue volume of the set $D$, i.e., $|D| = \int_D dx$.

The monotonicity of the function in (1.3) guarantees that

$$
\Lambda_p(\Omega) := \lim_{q \rightarrow p^*} \lambda_q(\Omega)
$$

is well defined and also that

$$
0 \leq \Lambda_p(\Omega) = \inf_{q \geq 1} \left( \lambda_q(\Omega) |\Omega|^\frac{q}{2} \right) \left( \lim_{q \rightarrow p^*} |\Omega|^{-\frac{q}{2}} \right) \leq \lambda_1(\Omega) |\Omega|^p \left( \lim_{q \rightarrow p^*} |\Omega|^{-\frac{q}{2}} \right).
$$

It is known that

$$
\Lambda_p(\Omega) = \begin{cases}
S_p & \text{if } 1 < p < N, \\
0 & \text{if } p = N,
\end{cases}
$$

(1.4)

where $S_p$ is the Sobolev constant: the best constant $S$ in the Sobolev inequality

$$
S \|u\|_{L^p(\mathbb{R}^N)}^p \leq \|\nabla u\|_{L^p(\mathbb{R}^N)}^p, \quad u \in W_0^{1,p}(\mathbb{R}^N).
$$

It is explicitly given by (see [2], [20])

$$
S_p := \pi^{\frac{N}{2}} N \left( \frac{N - p}{p - 1} \right)^{p-1} \left( \frac{\Gamma(N/p) \Gamma(1 + N - N/p)}{\Gamma(1 + N/2) \Gamma(N)} \right)^\frac{q}{2}
$$

(1.5)

where $\Gamma(t) = \int_0^\infty s^{t-1} e^{-s} ds$ is the Gamma Function.

The case $1 < p < N$ in (1.4) can be seen in [8], whereas the case $p = N$ is a consequence of the following result proved in [18]

$$
\lim_{q \rightarrow \infty} q^{N-1} \lambda_q(\Omega) = \frac{N^{2N-1} \omega_N}{(N-1)^{N-1}} e^{N-1},
$$

where

$$
\omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2} + 1)}
$$

(1.6)

is the volume of the unit ball $B_1$. (From now on $B_p$ denotes the ball centered at the origin with radius $\rho$).

As we can see from (1.4) the value $\Lambda_p(\Omega)$ does not depend on $\Omega$, when $1 < p \leq N$. This property does not hold if $p > N$. Indeed, by using a simple scaling argument one can show that

$$
\Lambda_p(B_R) = \Lambda_p(B_1) R^{N-p}.
$$

In the first part of this paper, developed in Section 2, we consider a general bounded domain $\Omega$ and $p > N$ and show that

$$
\Lambda_p(\Omega) = \inf \left\{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega) \text{ and } \|u\|_\infty = 1 \right\}.
$$

(1.7)

Thus, $\Lambda_p(\Omega)$ is the best constant associated with the (compact) Sobolev immersion

$$
W_0^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}), \quad p > N,
$$

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We also show that there exists $q_s \to \infty$ such that $w_{q_s}$ converges strongly, in both Banach spaces $C(\overline{\Omega})$ and $W_0^{1,p}(\Omega)$, to a positive function $u_p$ satisfying $\|u_p\|_\infty = 1$. Moreover, we prove that this function attains the infimum at (1.7):

$$\|\nabla u_p\|_p^p = \Lambda_p(\Omega) = \min \left\{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega) \text{ and } \|u\|_\infty = 1 \right\}. \quad (1.8)$$

However, our main result in Section 2 is the complete characterization of the minimizers in (1.8), which we call extremal functions of $\Lambda_p(\Omega)$ and denote by $u_p$. More precisely, we prove that if $u_p \in W_0^{1,p}(\Omega)$ is such that

$$\|u_p\|_\infty = 1 \text{ and } \|\nabla u_p\|_p^p = \Lambda_p(\Omega)$$

then $u_p$ does not change sign in $\Omega$, attains its sup norm at a unique point $x_p$, and satisfies the equation

$$-\Delta_p u_p = u_p(x_p) \Lambda_p(\Omega) \delta_{x_p},$$

where $\delta_{x_p}$ is the Dirac delta distribution concentrated at $x_p$.

In the particular case where $\Omega = B_R$, a ball of radius $R$, we show that

$$\Lambda_p(B_R) = \frac{N \omega_N}{R^{p-N}} \left( \frac{p-N}{p-1} \right)^{p-1}$$

and that

$$\lim_{q \to \infty} w_q(|x|) = u_p(|x|) := 1 - \left( \frac{|x|}{R} \right)^{\frac{p-N}{p-1}} ; \quad 0 \leq |x| \leq R,$$

where $w_q(|\cdot|)$ is the positive extremal function of $\lambda_q(B_R)$. Moreover, we prove that the function $u_p$ defined in (1.10) is the unique minimizer of $\Lambda_p(B_R)$. Since $x_p = 0$, our main result in Section 2 implies that

$$-\Delta_p u_p = \Lambda_p(B_R) \delta_0.$$

It is convenient to recall the following consequence of [19, Theorem 2.E], due to Talenti:

$$N(\omega_N) \frac{1}{p} \left( \frac{p-N}{p-1} \right)^{p-1} |\Omega|^{\frac{1-p}{p}} \|u\|_\infty^p \leq \|\nabla u\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega). \quad (1.11)$$

We emphasize that, in view of (1.7), this inequality allows one to conclude that

$$N(\omega_N) \frac{1}{p} \left( \frac{p-N}{p-1} \right)^{p-1} |\Omega|^{\frac{1-p}{p}} \leq \Lambda_p(\Omega). \quad (1.12)$$

Note that when $\Omega = B_R$ the left-hand side of (1.12) coincides with the right-hand side of (1.9). Thus, equality in (1.11) holds when $\Omega$ is a ball and $u$ is a scalar multiple of the function defined in (1.10), as pointed out in [19]. In this paper we show that if $\Omega$ is not a ball, then the inequality in (1.11) has to be strict. This fact was not observed in [19].

We remark that (1.9) provides the following upper bound to $\Lambda_p(\Omega)$:

$$\Lambda_p(\Omega) \leq N(\omega_N) \frac{1}{p} \left( \frac{p-N}{p-1} \right)^{p-1} |B_{R_{\Omega}}|^{\frac{1-p}{p}},$$

where $R_{\Omega}$ denotes the inradius of $\Omega$, that is, the radius of the largest ball inscribed in $\Omega$. We use the bounds (1.12) and (1.9) and the explicit expression of $S_p$ in (1.5) to conclude that the function $p \mapsto \Lambda_p(\Omega)$ is continuous at $p = N$.

In the second part, developed in Section 3, we study the asymptotic behavior, as $p \to \infty$, of the pair $(\Lambda_p(\Omega), u_p)$, where $u_p \in W_0^{1,p}(\Omega)$ will denote a positive extremal function of $\Lambda_p(\Omega)$. First we prove that

$$\lim_{p \to \infty} \Lambda_p(\Omega) \frac{1}{\|p\|_{\infty}}.$$
where
\[ \rho(x) := \inf_{y \in \Omega} |y - x|, \quad x \in \Omega, \]
is the distance function to the boundary. We recall the well-known fact:
\[ \frac{1}{\|\rho\|_{\infty}} = \min \left\{ \frac{\|\nabla \phi\|_{\infty}}{\|\phi\|_{\infty}} : \phi \in W^{1,\infty}_0(\Omega) \setminus \{0\} \right\}. \] (1.13)

Then, we prove that there exist a sequence \( p_n \to \infty \), a point \( x_* \in \Omega \) and a function \( u_\infty \in W^{1,\infty}_0(\Omega) \cap C(\overline{\Omega}) \) such that: \( x_{p_n} \to x_* \), \( \|\rho\|_{\infty} = \rho(x_*) \), \( u_\infty \leq \frac{\rho}{\|\rho\|_{\infty}} \) and \( u_{p_n} \to u_\infty \), uniformly in \( \overline{\Omega} \) and strongly in \( W^{1,p}_0(\Omega) \) for all \( r > N \). Moreover, \( x_* \) is the unique maximum point of \( u_\infty \), this function is also a minimizer of (1.13) and satisfies
\[
\begin{cases}
\Delta_\infty u_\infty = 0 & \text{in } \Omega \setminus \{x_*\}, \\
u_\infty = \frac{\rho}{\|\rho\|_{\infty}} & \text{on } \partial (\Omega \setminus \{x_*\}) = \{x_*\} \cup \partial \Omega,
\end{cases}
\]
in the viscosity sense, where \( \Delta_\infty \) denotes the well-known \( \infty \)-Laplacian operator (see [4], [6], [15]), defined formally by
\[ \Delta_\infty u := \sum_{i,j=1}^N \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \frac{\partial^2 u}{\partial x_i \partial x_j}. \]

Still in Section 3 we characterize the domains \( \Omega \) for which \( u_\infty = \frac{\rho}{\|\rho\|_{\infty}} \) in \( \overline{\Omega} \) and show that each maximum point of the distance function \( \rho \) gives rise to a minimizer of (1.13). We then use this latter fact to conclude that if \( \Omega \) is an annulus, then there exist infinitely many positive and nonradial minimizers of (1.13).

2 \( \Lambda_p(\Omega) \) and its extremal functions

In this section, \( p > N \geq 2 \) and \( \Omega \) denotes a bounded and smooth domain of \( \mathbb{R}^N \). We recall the well-known Morrey’s inequality
\[ \|u\|_{C^{\gamma}(\overline{\Omega})} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad \text{for all } u \in W^{1,p}(\Omega), \]
where \( \gamma := 1 - \frac{N}{p} \) and \( C \) depends only on \( \Omega \), \( p \) and \( N \). This inequality implies immediately that the immersion \( W^{1,p}_0(\Omega) \hookrightarrow C(\overline{\Omega}) \) is compact.

Let us also recall that
\[ \Lambda_p(\Omega) := \lim_{q \to \infty} \lambda_q(\Omega) \]
where \( \lambda_q(\Omega) \) is defined in (1.1).

**Theorem 2.1** There holds
\[ \Lambda_p(\Omega) = \inf \left\{ \|\nabla u\|^p_p : u \in W^{1,p}_0(\Omega) \text{ and } \|u\|_{\infty} = 1 \right\}. \] (2.1)

**Proof.** Let
\[ \mu := \inf \left\{ \|\nabla u\|^p_p : u \in W^{1,p}_0(\Omega) \text{ and } \|u\|_{\infty} = 1 \right\}. \]

Let us take \( u \in W^{1,p}_0(\Omega) \) such that \( \|u\|_{\infty} = 1 \). Since \( \lim_{q \to \infty} \|u\|_q = \|u\|_{\infty} = 1 \) we have
\[ \Lambda_p(\Omega) = \lim_{q \to \infty} \lambda_q(\Omega) \leq \lim_{q \to \infty} \frac{\|\nabla u\|^p_p}{\|u\|^p_q} = \|\nabla u\|^p_p, \]
implying that \( \Lambda_p(\Omega) \leq \mu \).
Now, for each $q \geq 1$ let $w_q$ be a positive extremal function of $\lambda_q(\Omega)$. Since

$$\mu \leq \|\nabla (w_q/\|w_q\|_\infty)\|_p = \frac{\lambda_q(\Omega)}{\|w_q\|_\infty^p},$$

in order to verify that $\mu \leq \Lambda_p(\Omega)$ we need only check that

$$\lim_{q \to \infty} \|w_q\|_\infty = 1.$$  \tag{2.2}

Since $1 = \|w_q\|_q \leq |\Omega|^{\frac{1}{q'}} \|w_q\|_\infty$, $\|\nabla w_q\|_p = \lambda_q(\Omega)$ and $\Lambda_p(\Omega) \leq \mu$ we have

$$|\Omega|^{-\frac{1}{q'}} \leq \|w_q\|_\infty \leq \frac{\|\nabla w_q\|_p^p}{\mu} \leq \frac{\lambda_q(\Omega)}{\Lambda_p(\Omega)},$$

which leads to (2.2), after making $q \to \infty$. \hfill \Box

Taking into account (2.1), we make the following definition:

**Definition 2.2** We say that $v \in W^{1,p}_0(\Omega)$ is an extremal function of $\Lambda_p(\Omega)$ if and only if

$$\|\nabla v\|_p = \Lambda_p(\Omega) \quad \text{and} \quad \|v\|_\infty = 1.$$

In the sequel we show that an extremal function of $\Lambda_p(\Omega)$ can be obtained as the limit of $w_{q_n}$ for some $q_n \to \infty$, where $w_{q_n}$ denotes the extremal function of $\lambda_{q_n}(\Omega)$.

**Theorem 2.3** There exists $q_n \to \infty$ and a nonnegative function $w \in W^{1,p}_0(\Omega) \cap C(\overline{\Omega})$ such that $w_{q_n} \to w$ strongly in $C(\overline{\Omega})$ and also in $W^{1,p}_0(\Omega)$. Moreover, $w$ is an extremal function of $\Lambda_p(\Omega)$.

**Proof.** Since $w_{q_n}$ is uniformly bounded in $W^{1,p}_0(\Omega)$ and also in $C^{0,1-N/2}(\overline{\Omega})$ there exist $q_n \to \infty$ and a nonnegative function $w \in W^{1,p}_0(\Omega) \cap C(\Omega)$ such that $w_{q_n} \to w$ weakly in $W^{1,p}_0(\Omega)$ and strongly in $C(\Omega)$. Thus, $\|w\|_\infty = \lim \|w_{q_n}\|_\infty = 1$ (because of (2.2)) and hence

$$\Lambda_p(\Omega) \leq \|\nabla w\|_p \leq \liminf \|\nabla w_{q_n}\|_p = \lim \lambda_{q_n}(\Omega) = \Lambda_p(\Omega).$$

This implies that $\Lambda_p(\Omega) = \lim \|\nabla w_{q_n}\|_p = \|\nabla w\|_p$, so that $w_{q_n} \to w$ strongly in $W^{1,p}_0(\Omega)$ and also that $w$ is an extremal function of $\Lambda_p(\Omega)$. \hfill \Box

**Remark 2.4** As we will see in the sequel, any nonnegative extremal function of $\Lambda_p(\Omega)$ must be strictly positive in $\Omega$.

We recall a well-known fact: $(-\Delta_p)^{-1} : W^{-1,p'}(\Omega) \mapsto W^{1,p}_0(\Omega)$ is bijective. Thus, if $p > N$ the equation

$$-\Delta_p u = c \delta_y$$  \tag{2.3}

has a unique solution $u \in W^{1,p}_0(\Omega)$ for each fixed $y \in \Omega$ and $c \in \mathbb{R}$. Note that if $p > N$ then $\delta_y \in W^{-1,p'}_0(\Omega)$, since

$$|\delta_y(\phi)| = |\phi(y)| \leq \|\phi\|_\infty \leq \Lambda_p(\Omega)^{-\frac{1}{p'}} \|\nabla \phi\|_p, \quad \text{for all} \quad \phi \in W^{1,p}_0(\Omega).$$

The equation (2.3) is to be interpreted in sense of the distributions:

$$\int_{\Omega} |\nabla \phi|^{p-2} \nabla u \cdot \nabla \phi \, dx = c \phi(y), \quad \text{for all} \quad \phi \in W^{1,p}_0(\Omega).$$

**Theorem 2.5** Let $u_p \in W^{1,p}_0(\Omega)$ be an extremal function of $\Lambda_p(\Omega)$ and let $x_p \in \Omega$ be such that

$$|u_p(x_p)| = \|u_p\|_\infty = 1.$$  

We claim that

(i) $-\Delta_p u_p = u_p(x_p) \Lambda_p(\Omega) \delta_{x_p}$ in $\Omega$,
(ii) $x_p$ is the unique global maximum point of $|u_p|$,
(iii) $u_p$ does not change sign in $\Omega$, and
(iv) for each $0 < t < 1$, there exists $\alpha_t \in (0,1)$ such that $u_p \in C^{1,\alpha_t}(\overline{E_t})$, where $E_t = \{x \in \Omega : 0 < |u_p(x)| < t\}$. 


Proof. For the sake of simplicity, we will assume throughout this proof that \( u_p(x_p) = 1 \) (otherwise, if \( u_p(x_p) = -1 \), we replace \( u_p \) by \(-u_p\).)

Let \( v \in W^{1,p}_0(\Omega) \) be such that

\[
-\Delta_p v = \Lambda_p(\Omega)\delta_{x_p} \quad \text{in} \quad \Omega.
\]

Since \( u_p(x_p) = 1 \),

\[
\Lambda_p(\Omega) = \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla u_p \, dx \leq \int_{\Omega} |\nabla v|^{p-1} |\nabla u_p| \, dx. \tag{2.4}
\]

Hence, since \( \Lambda_p(\Omega) = \|\nabla u_p\|_p^p \) and

\[
\|\nabla v\|_p^p = \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla u_p \, dx = \Lambda_p(\Omega)v(x_p) \tag{2.5}
\]

we apply the Hölder inequality to (2.4) in order to get

\[
\int_{\Omega} |\nabla v|^{p-1} |\nabla u_p| \, dx \leq \|\nabla v\|^{p-1} \|\nabla u_p\|_p
\]

\[
= (\Lambda_p(\Omega)v(x_p))^{\frac{p-1}{p}} \Lambda_p(\Omega)^{\frac{1}{p}} = \Lambda_p(\Omega) (v(x_p))^{\frac{p-1}{p}}. \tag{2.6}
\]

It follows from (2.4) and (2.6) that \( 1 \leq v(x_p) \leq \|v\|_{\infty} \).

On the other hand, (2.1) and (2.5) yield

\[
\Lambda_p(\Omega) \leq \frac{\|\nabla v\|_p^p}{\|\nabla v\|_{\infty}^p} = \frac{\Lambda_p(\Omega)v(x_p)}{\|\nabla v\|_{\infty}^p} \leq \frac{\Lambda_p(\Omega)}{\|\nabla v\|_{\infty}^{p-1}}. \tag{2.7}
\]

Hence, \( v(x_p) \leq \|v\|_{\infty} \leq 1 \) and then we conclude that

\[
1 = v(x_p) = \|v\|_{\infty}. \tag{2.8}
\]

Combining (2.8) with (2.7) we obtain

\[
\Lambda_p(\Omega) = \|\nabla v\|_p^p,
\]

showing that \( v \) is an extremal function of \( \Lambda_p(\Omega) \).

In order to prove that \( u_p = v \) we combine (2.8) with (2.6) and (2.4) to get

\[
\Lambda_p(\Omega) = \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla u_p \, dx = \int_{\Omega} |\nabla v|^{p-1} |\nabla u_p| \, dx = \|\nabla v\|_p^{p-1} \|\nabla u_p\|_p. \tag{2.9}
\]

The third equality in (2.9) is exactly the case of an equality in the Hölder inequality. It means that

\[
|\nabla v| = |\nabla u_p| \quad \text{a.e. in} \quad \Omega. \tag{2.10}
\]

(Note that \( \|\nabla v\|_p = \|\nabla u_p\|_p \).)

We still obtain from (2.9) that

\[
0 = \int_{\Omega} |\nabla v|^{p-2} (|\nabla v||\nabla u_p| - \nabla v \cdot \nabla u_p) \, dx.
\]

Since \( |\nabla v||\nabla u_p| \geq \nabla v \cdot \nabla u_p \) this yields

\[
\nabla v \cdot \nabla u_p = |\nabla v||\nabla u_p| \quad \text{a.e. in} \quad \Omega. \tag{2.11}
\]

Note that this equality occurs even at the points where \( |\nabla v|^{p-2} = 0 \).

It follows from (2.11) and (2.10) that

\[
\nabla v = \nabla u_p \quad \text{a.e. in} \quad \Omega,
\]

implying that \( \|\nabla (v - u_p)\|_p = 0 \). Since both \( v \) and \( u_p \) belong to \( W^{1,p}_0(\Omega) \) we conclude that

\[
v = u_p \quad \text{a.e. in} \quad \Omega
\]

so that \( -\Delta_p u_p = \Lambda_p(\Omega)\delta_{x_p} \). Thus, the proof of (i) is complete.
The claim (ii) follows directly from (i). In fact, another global maximum point, say \(x_1\), would lead to the following absurd: \(\Lambda_p(\Omega)\delta_{x_1} = -\Delta_p u_p = \Lambda_p(\Omega)\delta_{x_1}\).

Let us prove (iii). First we observe that \(u_p \geq 0\) in \(\Omega\). This is a consequence of the weak comparison principle since

\[
\int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \phi \, dx = \Lambda_p(\Omega)\phi(x_p) \geq 0, \quad \text{for all } \phi \in W^{1,p}_0(\Omega), \ \phi \geq 0.
\]

Now, we argue that \(u_p\) is \(p\)-harmonic in \(\Omega \setminus \{x_p\}\). Indeed, for each ball \(B \subset \Omega \setminus \{x_p\}\) and each \(\phi \in W^{1,p}_0(B) \subset W^{1,p}_0(\Omega)\) (here we are considering \(\phi = 0\) in \(\Omega \setminus B\)) we have

\[
\int_B |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \phi \, dx = \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \phi \, dx = \Lambda_p(\Omega)\phi(x_p) = 0,
\]

implying that \(u_p\) is \(p\)-harmonic in \(B\).

Let us consider the following subset \(Z := \{x \in \Omega : u_p(x) = 0\}\). Of course, \(Z\) is closed in \(\Omega\). Moreover, \(Z\) is also open in \(\Omega\). In fact, if \(z \in Z\) then \(z \in B\) for some ball \(B \subset \Omega \setminus \{x_p\}\). Since \(u_p\) is nonnegative in \(\Omega\) we can conclude that \(u_p\) restricted to \(B\) assumes its minimum value 0 at \(z \in B\). Since \(u_p\) is \(p\)-harmonic in \(B\) it must assume its minimum value only on the boundary \(\partial B\), unless it is constant on \(B\) (see [16]). So, we conclude that \(u_p\) is null in \(B\), proving that \(B \subset Z\). Since \(\Omega\) is connected (because it is a domain) the only possibility to \(Z\) to be empty. This fact implies that \(u_p > 0\) in \(\Omega\).

In order to prove (iv) let us take \(0 < t < 1\) and consider the set \(E_t = \{x \in \Omega : 0 < u_p(x) < t\}\), which is open, since \(u_p\) is continuous. We remark that \(u_p\) is constant on \(\partial E_t\). Moreover, by following the reasoning made in the proof of the third claim, \(u_p\) is \(p\)-harmonic in \(E_t\), because this set is away from \(\{x_p\}\) (recall that \(t < u_p(x_p)\)). Thus, \(u_p\) is constant on \(\partial E_t\) and satisfies \(-\Delta_p u_p = 0\) in \(E_t\). This fact allows us to apply the regularity result of Lieberman (see [14, Theorem 1]) to each connected component of \(E_t\) to conclude that there exists \(\alpha_t \in (0, 1)\) such that \(u_p \in C^{1,\alpha_t}(E_t)\).

The next theorem is contained in [19, Theorem 2.E].

**Theorem 2.6** Let \(R > 0\). Consider the function

\[
\lambda_p(|x|) := 1 - \left(\frac{|x|}{R}\right)^{\frac{pN}{p-N}} ; \quad 0 \leq |x| \leq R.
\]

One has,

\[
\|\nabla u_p\|_p^p = \frac{N \omega_N}{R^{p-N}} \left(\frac{p-N}{p-1}\right)^{p-1} = \Lambda_p(B_R).
\]

**Proof.** We have

\[
\|\nabla u_p\|_p^p = \int_{B_R} |\nabla u_p(|x|)|^p \, dx
\]

\[
= N \omega_N \int_0^R r^{N-1} |u'_p(r)|^p \, dr
\]

\[
= N \omega_N \left(\frac{p-N}{p-1}\right)^p R^{\frac{pN}{p-N}} \int_0^R r^{N-1+\frac{pN}{p-N} - 1} \, dr
\]

\[
= N \omega_N \left(\frac{p-N}{p-1}\right)^p R^{\frac{pN}{p-N}} \frac{p-1}{p-N} R^{\frac{pN}{p-N}} = \frac{N \omega_N}{R^{p-N}} \left(\frac{p-N}{p-1}\right)^{p-1},
\]

which gives the first equality in (2.13).

Of course, \(u_p(|\cdot|) \in W^{1,p}_0(B_R)\). Since \(\|u_p\|_\infty = 1\), it follows from Theorem 2.1 that

\[
\Lambda_p(B_R) \leq \|\nabla u_p\|_p^p = \frac{N \omega_N}{R^{p-N}} \left(\frac{p-N}{p-1}\right)^{p-1}.
\]
On the other hand, it follows from (1.11) that if \( v \in W_0^{1,p}(B_R) \) and \( \|v\|_\infty = 1 \) then

\[
\frac{N \omega_N}{R^{p-N}} \left(\frac{p-N}{p-1}\right)^{p-1} = N(\omega_N)^{\frac{2}{p}} \left(\frac{p-N}{p-1}\right)^{p-1} |B_R|^{1-\frac{2}{p}} \leq \|\nabla v\|_p.
\]

Taking into account Theorem 2.1, this means that

\[
\frac{N \omega_N}{R^{p-N}} \left(\frac{p-N}{p-1}\right)^{p-1} \leq \Lambda_p(B_R)
\]

and the proof is complete.

\[\square\]

**Corollary 2.7** The following estimates for \( \Lambda_p(\Omega) \) hold

\[
N(\omega_N)^{\frac{2}{p}} \left(\frac{p-N}{p-1}\right)^{p-1} |\Omega|^{1-\frac{2}{p}} \leq \Lambda_p(\Omega) \leq N(\omega_N)^{\frac{2}{p}} \left(\frac{p-N}{p-1}\right)^{p-1} |B_{R_\Omega}|^{1-\frac{2}{p}},
\]

where \( R_\Omega \) is the inradius of \( \Omega \) (i.e. the radius of the largest ball inscribed in \( \Omega \)).

**Proof.** The lower bound in (2.15) follows from (1.11). Let \( B_{R_\Omega}(x_0) \subset \Omega \) be a ball centered at a point \( x_0 \in \Omega \) with radius \( R_\Omega \). Since (it is easy to see)

\[
\Lambda_p(\Omega) \leq \Lambda_p(B_{R_\Omega}(x_p)) = \Lambda_p(B_{R_\Omega})
\]

we obtain the upper bound in (2.15) from (2.13) with \( R = R_\Omega \).

\[\square\]

**Remark 2.8** It follows from (2.15) that \( \limsup_{p \to \infty} \Lambda_p(\Omega)^{\frac{p}{2}} \leq R_\Omega^{-1} \). As we will see in Section 3, \( \Lambda_p(\Omega)^{\frac{p}{2}} \)
increases as \( p \) increases and really converges to \( R_\Omega^{-1} \) as \( p \to \infty \). This shows that the upper bound in (2.15) gets
asymptotically better as \( p \) increases.

**Corollary 2.9** The equality in (1.11) occurs for some \( 0 \neq u \in W_0^{1,p}(\Omega) \) if, and only if, \( \Omega \) is a ball.

**Proof.** When \( \Omega = B_R \) the equality holds true in (1.11) for the function \( u_p \) defined in (2.12), as (2.13) shows. On the other hand, if the equality in (1.11) is verified for some \( 0 \neq v \in W_0^{1,p}(\Omega) \), we can assume that

\[
\|v\|_\infty = 1. Thus,
\]

\[
N(\omega_N)^{\frac{2}{p}} \left(\frac{p-N}{p-1}\right)^{p-1} |\Omega|^{1-\frac{2}{p}} = \|\nabla v\|_p.
\]

But,

\[
N(\omega_N)^{\frac{2}{p}} \left(\frac{p-N}{p-1}\right)^{p-1} |\Omega|^{1-\frac{2}{p}} = \frac{N \omega_N}{(R^*)^{p-N}} \left(\frac{p-N}{p-1}\right)^{p-1} = \Lambda_p(B_{R^*})
\]

where, as before, \( R^* = (|\Omega|/\omega_N)^{\frac{1}{p}} \) is such that \( |B_{R^*}| = |\Omega| \). It follows that \( \Lambda_p(B_{R^*}) = \|\nabla v\|_p^p \).

Let \( v^* \in W_0^{1,p}(B_R) \) denote the Schwarz symmetrization of \( v \). We have \( \|v^*\|_\infty = \|v\|_\infty = 1 \) and

\[
\Lambda_p(B_{R^*}) \leq \|\nabla v^*\|_p \leq \|\nabla v\|_p^p = \Lambda_p(B_{R^*}),
\]

from which we conclude that \( \|\nabla v^*\|_p = \|\nabla v\|_p \). This fact implies that \( \Omega \) is a ball, according to [5, Lemma 3.2].

\[\square\]

**Corollary 2.10** One has

\[
\lim_{p \to N^{-}} \frac{\Lambda_p(\Omega)}{|p-N|^{p-1}} = \frac{N \omega_N}{(N-1)^{N-1}} = \lim_{p \to N^+} \frac{\Lambda_p(\Omega)}{|p-N|^{p-1}}.
\]

In particular, the function \( p \in (1, \infty) \mapsto \Lambda_p(\Omega) \) is continuous at \( p = N \).
\textbf{Proof.} It follows from (1.4), (1.5) and (1.6) that
\[
\lim_{p \to N^+} \frac{\Lambda_p(\Omega)}{|p - N|^{p-1}} = \lim_{p \to N^+} \frac{\pi^\frac{p}{2}}{(p-1)^{p-1}} \left( \frac{\Gamma(N/p)\Gamma(1+N-N/p)}{\Gamma(1+N/2)\Gamma(N)} \right)^\frac{1}{p} \leq \frac{\pi^\frac{p}{2}}{\Gamma(N)} \frac{1}{(N-1)^{N-1}} = \frac{N \omega_N}{(N-1)^{N-1}}.
\]
Now, by using (2.15) we obtain
\[
\frac{N \omega_N}{(N-1)^{N-1}} = \lim_{p \to N^+} \frac{N(\omega_N)^\frac{p}{2}}{(p-1)^{p-1}} |\Omega|^{1 - \frac{p}{2}} \leq \lim_{p \to N^+} \frac{\Lambda_p(\Omega)}{|p - N|^{p-1}}
\]
and
\[
\lim_{p \to N^+} \frac{\Lambda_p(\Omega)}{|p - N|^{p-1}} \leq \lim_{p \to N^+} \frac{N(\omega_N)^\frac{p}{2}}{(p-1)^{p-1}} |B_\delta|^1 \leq \frac{N \omega_N}{(N-1)^{N-1}}.
\]
The continuity follows, since
\[
\lim_{p \to N} \Lambda_p(\Omega) = \lim_{p \to N^+} \frac{\Lambda_p(\Omega)}{|p - N|^{p-1}} \lim_{p \to N^+} \frac{\Lambda_p(\Omega)}{|p - N|^{p-1}} = 0 = \Lambda_N(\Omega).
\]
\(\square\)

Theorem 2.6 says that the function \(u_p(|x|)\) defined in (2.12) is a positive extremal function of \(\Lambda_p(B_R)\). Let us prove that it is the unique.

\textbf{Theorem 2.11} Let \(R > 0\). The function \(u_p(|x|)\) defined in (2.12) is the unique positive extremal function of \(\Lambda_p(B_R)\).

\textbf{P r o o f.} It follows from Theorem 2.5 that
\[-\Delta_p u_p = \Lambda_p(B_R) \delta_0.\]

Now, let us suppose that \(v \in W_0^{1,p}(B_R)\) is an arbitrary, positive extremal function of \(\Lambda_p(B_R)\). Let \(v^* \in W_0^{1,p}(B_R)\) denote the Schwarz symmetrization of \(v\) (see [13]). It follows that \(v^*\) is radial and radially nonincreasing and, moreover, it satisfies \(\|v^*\|_\infty = \|v\|_\infty\) and \(\|\nabla v^*\|_p \leq \|\nabla v\|_p\). Therefore, \(v^*(0) = \|v^*\|_\infty = \|v\|_\infty = 1\) and
\[\Lambda_p(B_R) \leq \|\nabla v^*\|_p \leq \|\nabla v\|_p = \Lambda_p(B_R).\]
Thus, \(v^*\) is also a nonnegative extremal function of \(\Lambda_p(B_R)\). Theorem 2.5 yields \(-\Delta_p v^* = \Lambda_p(B_R) \delta_0 = -\Delta_p u_p\), which implies that \(v^* = u_p\). Since
\[|\nabla v^*(x)| = |\nabla u_p(|x|)| = \frac{p - N}{p - 1} R^{-\frac{N-1}{p-1}} |x|^{-\frac{N}{p-1}} > 0, \quad 0 < |x| \leq R,
\]
the set \(\{x \in B_R : \nabla v^* = 0\}\) has Lebesgue measure zero. Hence, we can apply a well-known result (see [5, Theorem 1.1]) to conclude that \(v = v^* = u_p\). \(\square\)

\textbf{Corollary 2.12} Let \(w_q\) denote the extremal function of \(\lambda_q(B_R)\). We have
\[\lim_{q \to \infty} w_q(|x|) = 1 - \frac{|x|}{|x| / R}^\frac{q}{p-1}, \quad (2.16)\]
strongly in \(C(B_R)\) and also in \(W_0^{1,p}(B_R)\). Moreover, (2.16) holds in \(C^1(B_{\epsilon R})\) for each \(\epsilon \in (0, R)\), where \(B_{\epsilon R} = \{\epsilon < |x| < R\}\).

\textbf{P r o o f.} It follows from Theorem 2.11 that \(1 - (|x| / R)^\frac{q}{p-1}\) is the only limit function of the family \(\{w_q(|\cdot| / q)|, q \to \infty\}\). Therefore, the convergence given by Theorem 2.3 is valid for any sequence \(q_n \to \infty\) and this guarantees that (2.16) happens strongly in \(C(B_R)\) and also in \(W_0^{1,p}(B_R)\).
The convergence in $C^1(B_{r,R})$ is consequence of the following fact
\[
\lim_{q \to \infty} \lambda_q w_q(|x|)^{q-1} = 0, \text{ uniformly in } B_{r,R},
\]
which occurs because of the uniform convergence of $w_q(|x|)$ to $1 - (|x|/R)^{\frac{q}{q-1}}$. (Note that $0 \leq w_q(|x|) \leq k < 1$ for some $k$, and for all $x \in B_{r,R}$ and all $q$ large enough.) Therefore, we can apply a result of Lieberman (see [14, Theorem 1]) to guarantee that, for all $q$ large enough, $w_q$ is uniformly bounded in the Hölder space $C^{1,q}(B_{r,R})$, for some $\alpha \in (0, 1)$ that does not depend on $q$. Then, we obtain the convergence (2.16) from the compactness of the immersion $C^{1,q}(B_{r,R}) \hookrightarrow C^1(B_{r,R})$ by taking into account that the limit function is always $1 - (|x|/R)^{\frac{q}{q-1}}$.

\section{Asymptotics as $p \to \infty$}

In this section, $u_p \in W_0^{1,p}(\Omega) \cap C^{0,1-\frac{2}{p}}(\overline{\Omega})$ denotes a positive extremal function of $\Lambda_p(\Omega)$ and $\rho \in W_0^{1,\infty}(\Omega)$ denotes the distance function to the boundary $\partial \Omega$. Thus, $0 < u_p(x) \leq \|u_p\| = 1$ for all $x \in \Omega$, and
\[
\Lambda_p(\Omega) = \min \left\{ \|\nabla u\|_p : u \in W_0^{1,p}(\Omega) \text{ and } \|u\|_\infty = 1 \right\} = \|\nabla u_p\|_p
\]
and
\[
\rho(x) = \inf_{y \in \Omega} |y - x|, \quad x \in \overline{\Omega}.
\]
As shown in Section 2, $u_p$ has a unique maximum point, denoted by $x_p$, and
\[
\begin{align*}
-\Delta_p u_p &= \Lambda_p(\Omega) \delta_{x_p} \quad \text{in } \Omega, \\
u_p &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
It is convenient to recall some properties of the distance function:

(P1) $\rho \in W_0^{1,r}(\Omega)$ for all $1 \leq r \leq \infty$.
(P2) $|\nabla \rho| = 1$ almost everywhere in $\Omega$.
(P3) $\|\rho\|_\infty = R_{1,1}^\rho$ is the radius of the largest ball contained in $\Omega$.
(P4) $1 - \|\rho\|_\infty \leq \frac{\|\nabla \phi\|_\infty}{\|\phi\|_\infty}$ for all $0 \neq \phi \in W_0^{1,\infty}(\Omega)$.

Let us, for a moment, consider $\Omega = B_R$. For this domain
\[
\rho(x) = R - |x|; \quad 0 \leq |x| \leq R,
\]
and, accordingly to (2.13) and (2.12): $x_p = 0$ for all $p > N$,
\[
\lim_{p \to \infty} \Lambda_p(B_R)^\frac{1}{p} = \lim_{p \to \infty} \left( \frac{N \omega_N}{R} \right)^\frac{1}{p} \left( \frac{p - N}{p - 1} \right)^\frac{p - 1}{p} = \frac{1}{R} = \frac{1}{\|\rho\|_\infty} \tag{3.2}
\]
and
\[
\lim_{p \to \infty} u_p(x) = \lim_{p \to \infty} 1 - \left( \frac{|x|}{R} \right)^{\frac{P}{p}} = 1 - \frac{|x|}{R} \left( \frac{\rho(x)}{\|\rho\|_\infty} \right); \quad 0 \leq |x| \leq R. \tag{3.3}
\]
As we will see in the sequel, (3.2) holds for any bounded domain, whereas (3.3) holds only for some special domains.

\textbf{Lemma 3.1} \textit{The function $p \in (N, \infty) \mapsto \Lambda_p(\Omega)^\frac{1}{p} |\Omega|^{-\frac{1}{p}}$ is increasing.}

\textbf{Proof.} Let $N < p_1 < p_2$ and, for each $i \in \{1, 2\}$ let $u_{p_i} \in W_0^{1,p_i}(\Omega)$ denote a positive extremal function of $\Lambda_{p_i}(\Omega)$. The Hölder inequality implies that
\[
\Lambda_{p_i}(\Omega) \leq \int_{\Omega} |\nabla u_{p_i}|^{p_i} \, dx \leq \left( \int_{\Omega} |\nabla u_{p_i}|^{p_i} \, dx \right)^\frac{p}{p_i} |\Omega|^{1-\frac{p}{p_i}} = \Lambda_{p_i}(\Omega)^\frac{p}{p_i} |\Omega|^{1-\frac{p}{p_i}},
\]

so that
\[ \Lambda_p(\Omega)^{\frac{1}{p}} |\Omega|^{\frac{1}{p}} \leq \Lambda_p(\Omega)^{\frac{1}{p}} |\Omega|^{\frac{1}{p}}. \]

An immediate consequence of this lemma is that the function \( p \in (N, \infty) \mapsto \Lambda_p(\Omega) \) is increasing. \( \square \)

**Theorem 3.2** One has
\[ \lim_{p \to \infty} \Lambda_p(\Omega)^{\frac{1}{p}} |\Omega|^{-\frac{1}{p}} = \frac{1}{\|\rho\|_\infty}. \]

*Proof.* It is enough to prove that
\[ \lim_{p \to \infty} \Lambda_p(\Omega)^{\frac{1}{p}} |\Omega|^{-\frac{1}{p}} = \frac{1}{\|\rho\|_\infty}. \]

It follows from (3.1) that
\[ \Lambda_p(\Omega)^{\frac{1}{p}} |\Omega|^{-\frac{1}{p}} \leq \frac{\|\nabla \rho\|_p}{\|\rho\|_\infty} |\Omega|^{-\frac{1}{p}} = \frac{1}{\|\rho\|_\infty}, \quad p > N. \]

Hence, the monotonicity proved in Lemma 3.1 guarantees that
\[ \Lambda_p(\Omega)^{\frac{1}{p}} |\Omega|^{-\frac{1}{p}} \leq L := \lim_{s \to \infty} \Lambda_s(\Omega)^{\frac{1}{s}} |\Omega|^{-\frac{1}{s}} = \lim_{s \to \infty} \Lambda_s(\Omega)^{\frac{1}{s}} \leq \frac{1}{\|\rho\|_\infty}, \quad \text{for all } p > N. \]

We are going to show that \( L = \frac{1}{|\rho|_\infty} \). For this, let us fix \( r > N \). Since
\[ \|\nabla u_p\|_r \leq \|\nabla u_p\|_r |\Omega|^{-\frac{1}{r}} = \Lambda_p(\Omega)^{\frac{1}{r}} |\Omega|^{-\frac{1}{r}} |\Omega|^{\frac{1}{r}} \leq L |\Omega|^{\frac{1}{r}}, \quad p > r, \]
the family \( \{u_p\}_{p > r} \) is uniformly bounded in \( W_0^{1,r}(\Omega) \). It follows that there exist \( p_n \to \infty \) and \( u \in W_0^{1,r}(\Omega) \) such that
\[ u_n \to u (\text{weakly}) \text{ in } W_0^{1,r}(\Omega). \]

Thus,
\[ \|\nabla u\|_r \leq \liminf_{n} \|\nabla u_n\|_r \leq L |\Omega|^{\frac{1}{r}}. \]

After passing to another subsequence, if necessary, the compactness of the immersion \( W_0^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega}) \) yields
\[ u_p \to u (\text{strongly}) \text{ in } C(\overline{\Omega}). \]

Note that \( \|u\|_{\infty} = 1 \) since \( \|u\|_{\infty} = 1 \) for all \( p > N \).

The uniform convergence \( u_p \to u \) implies that, if \( s > r \), then \( u \in W_0^{1,s}(\Omega) \) of a subsequence of \( \{u_n\} \). Therefore,
\[ u_\infty \in W_0^{1,s}(\Omega) \quad \text{and} \quad \|\nabla u_\infty\|_s \leq L |\Omega|^{\frac{1}{s}}, \quad \text{for all } s > r, \]

implying that \( u_\infty \in W_0^{1,\infty}(\Omega) \) and
\[ \|\nabla u_\infty\|_\infty \leq L \leq \frac{1}{\|\rho\|_\infty}. \]

Combining this fact with Property P4 (recall that \( \|u_\infty\|_\infty = 1 \)) we conclude that
\[ \|\nabla u_\infty\|_\infty \leq L \leq \frac{1}{\|\rho\|_\infty} \leq \|\nabla u_\infty\|_\infty, \]
from which we obtain
\[ L = \frac{1}{\|\rho\|_\infty} = \|\nabla u_\infty\|_\infty. \]

\( \square \)
It is interesting to notice that $\Lambda_p(\Omega)^{\frac{1}{p}}$ and $\lambda_p(\Omega)^{\frac{1}{p}}$ have the same asymptotic behavior as $p \to \infty$, since

$$\lim_{p \to \infty} \lambda_p(\Omega)^{\frac{1}{p}} = \frac{1}{\|\rho\|_\infty},$$

as proved in [10], [12], where the infinity-eigenvalue problem was studied as the limit problem of the standard eigenvalue problem for the $p$-Laplacian, as $p \to \infty$.

**Theorem 3.3** There exist $p_\infty \to \infty$, $x_\infty \in \Omega$ and $u_\infty \in W^{1,\infty}_0(\Omega)$ such that:

(i) $u_{p_n}$ converges to $u_\infty$ weakly in $W^{1,\infty}_0(\Omega)$, for any $r > N$, and uniformly in $\Omega$;

(ii) $\|u_\infty\|_\infty = \|\rho\|_\infty$;

(iii) $0 \leq u_\infty \leq \frac{\rho}{\|\rho\|_\infty}$ a.e. in $\Omega$;

(iv) $x_{p_n} \to x_\infty$;

(v) $u_\infty(x_{p_n}) = 1 = \|u_\infty\|_\infty$ and $\rho(x_{p_n}) = \|\rho\|_\infty$.

**Proof.** Items (i) and (ii) follow from the proof of the previous theorem. In particular, (ii) says that the Lipschitz constant of $\|\rho\|_\infty u_\infty$ is $\|\nabla(\|\rho\|_\infty u_\infty)\|_\infty = 1$. Thus,

$$0 < \|\rho\|_\infty u_\infty(x) \leq |x - y|,$$

for almost all $x \in \Omega$ and $y \in \partial \Omega$,

and hence we obtain $\|\rho\|_\infty u_\infty \leq \rho$ a.e. in $\Omega$, as affirmed in (iii). Of course, $\{p_n\}$ can be chosen such that $x_{p_n} \to x_\infty$ for some $x_\infty \in \Omega$, yielding (iv). Since $u_{p_n}(x_{p_n}) = 1$, the uniform convergence $u_{p_n} \to u_\infty$ implies that $u_\infty(x_{p_n}) = 1$. Therefore, (iii) implies that $\|\rho\|_\infty = \rho(x_{p_n})$, what concludes the proof of (v).

**Remark 3.4** We will prove in the sequel that $x_\infty$ is the only maximum point of $u_\infty$ and that $u_\infty$ is infinity harmonic in the punctured domain $\Omega \setminus \{x_\infty\}$.

**Remark 3.5** Item (ii) of Theorem 3.3 and property (P4) above imply that $u_\infty$ minimizes the Rayleigh quotient

$$\frac{\|\nabla u_\infty\|_\infty^2}{\|u_\infty\|_\infty^2}$$

among all nontrivial functions $u$ in $W^{1,\infty}_0(\Omega)$. This property is also shared with the distance function $\rho$ and the first eigenfunctions of the $\infty$-Laplacian (see [12]). In the sequel (see Theorem 3.14) we will prove that $u_\infty(\cdot) = \frac{\rho(\cdot)}{\|\rho\|_\infty}$ for some special domains. For such domains $u_\infty$ is also a first eigenfunction of the $\infty$-Laplacian, according to [22, Theorem 2.7].

In order to gain some insight on which equation $u_\infty$ satisfies, let us go back to the case $\Omega = B_R$. It follows from (3.3) that:

$$u_\infty = \frac{\rho}{\|\rho\|_\infty} = 1 - \frac{|x|}{R},$$

$x_\infty = 0$ and $u_\infty(0) = 1 = \frac{\rho(0)}{\|\rho\|_\infty}$. Moreover, it is easy to check that $u_\infty \in C(\overline{B_R}) \cap C^2(B_R \setminus \{0\})$, $\nabla u_\infty \neq 0$ in $B_R \setminus \{0\}$ and

$$\Delta_\infty u_\infty(x) = 0, \quad x \in B_R \setminus \{0\},$$

where $\Delta_\infty$ denotes the $\infty$-Laplacian (see [1], [4], [6], [7], [15]), defined by

$$\Delta_\infty \phi := \frac{1}{2} \left\{ \nabla \phi, \nabla |\nabla \phi|^2 \right\} = \sum_{i,j=1}^N \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j}.$$

After this motivation, let us to show that the function $u_\infty$ given by Theorem 3.3 is $\infty$-harmonic in $\Omega \setminus \{x_\infty\}$, i.e. that it satisfies $\Delta_\infty u = 0$ in $\Omega \setminus \{x_\infty\}$ in the viscosity sense. First, we need to recall some definitions regarding the viscosity approach for the equation $\Delta_p u = 0$, where $N < p \leq \infty$.

**Definition 3.6** Let $u \in C(\overline{\Omega})$, $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$. We say that $\phi$ touches $u$ at $x_0$ from below if

$$\phi(x) - u(x) < 0 = \phi(x_0) - u(x_0), \quad \text{for all } x \in \Omega \setminus \{x_0\}.$$
Analogously, we say that \( \phi \) touches \( u \) at \( x_0 \) from above if
\[
\phi(x) - u(x) > 0 = \phi(x_0) - u(x_0), \quad \text{for all} \quad x \in \Omega \setminus \{x_0\}.
\]

**Definition 3.7** Let \( N < p \leq \infty \) and \( u \in C(\Omega) \). We say that \( u \) is \( p \)-subharmonic in \( \Omega \) in the viscosity sense, if
\[
\Delta_p \phi(x_0) \geq 0
\]
whenever \( x_0 \in \Omega \) and \( \phi \in C^2(\Omega) \) are such that \( \phi \) touches \( u \) from above at \( x_0 \). Analogously, we say that \( u \) is \( p \)-superharmonic in \( \Omega \) in the viscosity sense, if
\[
\Delta_p \phi(x_0) \leq 0
\]
whenever \( x_0 \in \Omega \) and \( \phi \in C^2(\Omega) \) are such that \( \phi \) touches \( u \) from below at \( x_0 \).

**Definition 3.8** Let \( N < p \leq \infty \) and \( u \in C(\Omega) \). We say that \( u \) is \( p \)-harmonic in \( \Omega \), in the viscosity sense, if \( u \) is both: \( p \)-subharmonic and \( p \)-superharmonic in \( \Omega \), in the viscosity sense. We write \( \Delta_{\infty} u = 0 \) in \( \Omega \) to mean that \( u \) is \( \infty \)-harmonic in \( \Omega \), in the viscosity sense.

In Definitions 3.6 and 3.7, we mean
\[
\Delta_p \phi(x_0) := |\nabla \phi(x_0)|^p \left\{ |\nabla \phi(x_0)|^2 \Delta \phi(x_0) + (p - 2) \Delta_{\infty} \phi(x_0) \right\}, \quad N < p < \infty,
\]
and
\[
\Delta_{\infty} \phi(x_0) := \sum_{i,j=1}^N \frac{\partial \phi}{\partial x_i}(x_0) \frac{\partial \phi}{\partial x_j}(x_0) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x_0).
\]

The following two lemmas can be found in [15].

**Lemma 3.9** Suppose \( u \in C(\Omega) \cap W^{1,p}(\Omega) \) satisfies \( \Delta_p u \geq 0 \) (resp. \( \Delta_p u \leq 0 \)) in \( \Omega \), in the weak sense, then \( u \) is \( p \)-subharmonic (resp. \( p \)-superharmonic) in \( \Omega \), in the viscosity sense.

**Lemma 3.10** Suppose that \( f_n \to f \) uniformly in \( \Omega \), \( f_n, f \in C(\Omega) \). If \( \phi \in C^2(\Omega) \) touches \( f \) from below at \( y_0 \), then there exists \( y_{n_j} \to y_0 \) such that
\[
\phi(y_{n_j}) - \phi(y_0) = \min_{\Omega} \left\{ f_{n_j} - \phi \right\}.
\]

From now on, \( u_{\infty} \) and \( x_s \) are as in Theorem 3.3.

**Theorem 3.11** The function \( u_{\infty} \) satisfies
\[
\begin{align*}
\Delta_{\infty} v &= 0 \quad \text{in} \quad \Omega \setminus \{x_s\}, \\
v &= \frac{\rho}{|\rho|_{\infty}} \quad \text{on} \quad \{x_s\} \cup \partial \Omega,
\end{align*}
\]
in the viscosity sense.

**Proof.** Since \( u_{\infty} = \frac{\rho}{|\rho|_{\infty}} \) on \( \{x_s\} \cup \partial \Omega \), it remains to check that \( \Delta_{\infty} u_{\infty} = 0 \) in \( \Omega \setminus \{x_s\} \). Let \( \xi \in \Omega \setminus \{x_s\} \) and take \( \phi \in C^2(\Omega \setminus \{x_s\}) \) touching \( u_{\infty} \) from below at \( \xi \). Thus,
\[
\phi(x) - u_{\infty}(x) < 0 = \phi(\xi) - u_{\infty}(\xi), \quad \text{if} \quad x \neq \xi.
\]

If \( |\nabla \phi(\xi)| = 0 \) then we readily obtain
\[
\Delta_{\infty} \phi(\xi) = \sum_{i,j=1}^N \frac{\partial \phi}{\partial x_i}(\xi) \frac{\partial \phi}{\partial x_j}(\xi) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(\xi) = 0.
\]

Otherwise, if \( |\nabla \phi(\xi)| \neq 0 \) let us take a ball \( B_r(\xi) \subset \Omega \setminus \{x_s\} \) such that \( |\nabla \phi| > 0 \) in \( B_r(\xi) \). Let \( n_0 > N \) be such that \( x_{n_0} \not\in B_r(\xi) \) for all \( n > n_0 \). This is possible because \( x_{p_n} \to x_s \neq \xi \). It follows that \( u_{p_n} \) is \( p_n \)-harmonic in \( B_r(\xi) \) in the viscosity sense.

According Lemma 3.10, let \( \{\xi_{n_j}\} \subset B_r(\xi) \) such that \( \xi_{n_j} \to \xi \) and
\[
m_j := \min_{B_r(\xi)} \left\{ u_{p_{n_j}} - \phi \right\} = u_{p_{n_j}}(\xi_{n_j}) - \phi(\xi_{n_j}) \leq u_{p_{n_j}}(x) - \phi(x), \quad x \neq \xi_{n_j}.
\]
The function \( \psi(x) := \phi(x) + m_j - |x - \xi_n|^4 \) belongs to \( C^2(B_\epsilon(\xi)) \) and touches \( u_n \) from below at \( \xi_n \). Indeed,

\[
\psi(x) - u_{n_j}(x) = \phi(x) - u_{n_j}(x) + m_j - |x - \xi_n|^4
\]

\[
\leq - |x - \xi_n|^4 < 0 = \psi(\xi_n) - u_{n_j}(\xi_n), \quad x \neq \xi_n.
\]

Thus, \( \Delta_{n_j} \psi(\xi_n) \leq 0 \), since \( u_{n_j} \) is \( p_{n_j} \)-harmonic in \( B_\epsilon(\xi) \). Hence,

\[
0 \geq \Delta_{n_j} \psi(\xi_n) = \left| \nabla \psi(\xi_n) \right|^4 \left| \nabla \psi(\xi_n) \right|^2 \Delta \psi(\xi_n) + \left( p_{n_j} - 2 \right) \Delta \psi(\xi_n)
\]

from which we obtain

\[
\Delta_{\infty} \psi(\xi_n) = \Delta_{\infty} \psi(\xi_n) \leq - \frac{\left| \nabla \psi(\xi_n) \right|^2}{p_{n_j} - 2} \Delta \psi(\xi_n).
\]

So, by making \( j \to \infty \) we conclude that \( \Delta_{\infty} \phi(\xi) \leq 0 \).

We have proved that \( u_\infty \) is \( \infty \)-superharmonic in \( \Omega \setminus \{ x_\ast \} \), in the viscosity sense. Analogously, we can prove that \( u_\infty \) is also \( \infty \)-subharmonic in \( \Omega \setminus \{ x_\ast \} \), in the viscosity sense. \( \square \)

We recall that \( u_\infty \) is the only solution of the Dirichlet problem (3.4). This uniqueness result is a consequence of the following comparison principle (see [3], [11]):

**Theorem 3.12 (Comparison Principle)** Let \( D \) be a bounded domain and let \( u, v \in C(\overline{D}) \) satisfying \( \Delta u \geq 0 \) in \( D \) and \( \Delta v \leq 0 \) in \( D \). If \( u \leq v \) on \( \partial D \), then \( u \leq v \) in \( D \).

**Theorem 3.13** The function \( u_\infty \) is strictly positive in \( \Omega \) and attains its maximum value 1 only at \( x_\ast \).

**Proof.** Let \( D := \Omega \setminus \{ x_\ast \} \). Since \( u_\infty(x_\ast) > 0 \) and \( u_\infty \) is nonnegative and \( \infty \)-harmonic in \( D \), it follows from the Harnack inequality for the infinity harmonic functions (see [17]) that \( Z_\infty := \{ x \in \Omega : u_\infty(x) = 0 \} \) is open in \( \Omega \). Since \( Z_\infty \) is also closed and \( Z_\infty \neq \Omega \), we conclude that \( Z_\infty \) is not empty, so that \( u_\infty > 0 \) in \( \Omega \).

Let \( m := \max \{ |x - x_\ast| : x \in \partial \Omega \} \) and \( v(x) := 1 - \frac{1}{m} |x - x_\ast|, x \in \Omega \). It is easy to check that \( \Delta v = 0 \) in \( D \) and that \( v \geq u_\infty \) on \( \partial D = \{ x_\ast \} \cup \partial \Omega \). Therefore, by the comparison principle above, we have

\[
u_\infty(x) \leq v(x) = 1 - \frac{1}{m} |x - x_\ast| < 1 = \| u_\infty \|_{\infty}, \text{ for all } x \in \Omega \setminus \{ x_\ast \}.
\]

\( \square \)

Since \( x_\ast \) is also a maximum point of the distance function \( \rho \), an immediate consequence of the previous theorem is that if \( \Omega \) is such that \( \rho \) has a unique maximum point, then the family \( \{ u_p \}_{p > N} \) converges, as \( p \to \infty \), to the unique solution \( u_\infty \) of the Dirichlet problem (3.4). However, this property of \( \Omega \) alone does not assure that \( u_\infty = \frac{\rho}{\| \rho \|_\infty} \). For example, for the square \( S = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1 \} \) the origin is the unique maximum point of the distance function \( \rho \), but one can check from [12, Proposition 4.1] that \( \rho \) is not \( \infty \)-harmonic at the points of \( \Omega \) on the coordinate axes. As a matter of fact, for a general bounded domain \( \Omega \) the distance function fails to be \( \infty \)-harmonic exactly on the ridge of \( \Omega \), the set \( \mathcal{R}(\Omega) \) of all points in \( \Omega \) whose distance to the boundary is reached at least at two points in \( \partial \Omega \). This well-known fact can be proved by combining Corollaries 3.4 and 4.4 of [7], as pointed out in [22, Lemma 2.6]. Note that \( \mathcal{R}(S) \) is set of the points in \( S \) that are on the coordinate axes. As we will see in the sequel, the complementary condition to guarantee that \( u_\infty = \frac{\rho}{\| \rho \|_\infty} \) is \( \mathcal{R}(\Omega) = \{ x_0 \} \), where \( x_0 \) denotes the unique maximum point of \( \rho \).

**Theorem 3.14** One has \( u_\infty = \frac{\rho}{\| \rho \|_\infty} \) in \( \overline{\Omega} \) if, and only if:

(i) \( \rho \) has a unique maximum point, say \( x_0 \), and

(ii) for each \( x \in \Omega \setminus \{ x_0 \} \) there exists a unique \( y_x \in \partial \Omega \) such that \( |x - y_x| = \rho(x) \).

**Proof.** If \( u_\infty = \frac{\rho}{\| \rho \|_\infty} \) then \( x_\ast \) is the only maximum point of the distance function \( \rho \), according Theorems 3.3 and 3.13. It follows from Theorem 3.11 that \( \Delta_{\infty} \rho = 0 \) in \( \Omega \setminus \{ x_\ast \} \). Hence, \( \mathcal{R}(\Omega) = \{ x_0 \} \), which is equivalent to (i).
Conversely, item (i) and Theorem 3.3 imply that \( x_0 = x_* \), whereas item (ii) implies that \( R(\Omega) = \{x_0\} \). It follows that \( \frac{\rho}{\|\rho\|_{\infty}} \) satisfies (3.4). Hence, uniqueness of the viscosity solution of this Dirichlet problem guarantees that \( u_\infty = \frac{\rho}{\|\rho\|_{\infty}} \).

It follows that \( \rho \|\rho\|_{\infty} \) satisfies (3.4). Hence, uniqueness of the viscosity solution of this Dirichlet problem guarantees that \( u_\infty = \rho \|\rho\|_{\infty} \).

\( \square \)

Balls, ellipses and other convex domains satisfy conditions (i) and (ii).

### 3.1 Multiplicity of minimizers of the quotient \( \frac{\|\nabla \phi\|_{\infty}}{\|\phi\|_{\infty}} \) in \( W_0^{1,\infty}(\Omega) \setminus \{0\} \)

In this subsection we show that each maximum point \( x_0 \) of the distance function \( \rho \) gives rise to a positive function \( u \in W_0^{1,\infty}(\Omega) \setminus \{0\} \) satisfying

\[
\|u\|_{\infty} = 1 \quad \text{and} \quad \|\nabla u\|_{\infty} = \frac{1}{\|\rho\|_{\infty}} = \min \left\{ \frac{\|\nabla \phi\|_{\infty}}{\|\phi\|_{\infty}} : \phi \in W_0^{1,\infty}(\Omega) \setminus \{0\} \right\}. \tag{3.5}
\]

Moreover, such a function attains its maximum value only at \( x_0 \). In particular, we conclude that for an annulus, there exist infinitely many positive and nonradial functions satisfying (3.5).

**Proposition 3.15** Let \( x_0 \in \mathbb{R}^N \) and let \( u_\infty \in C(\overline{\Omega}) \) be the unique viscosity solution of the following Dirichlet problem

\[
\begin{cases}
\Delta_\infty u = 0 & \text{in } \Omega \setminus \{x_0\}, \\
u = 0 & \text{on } \partial \Omega, \\
u(x_0) = 1.
\end{cases} \tag{3.6}
\]

Then,

(i) \( 0 < u_\infty(x) < 1 \) for all \( x \in \Omega \setminus \{x_0\} \).

(ii) If \( x_0 \) is a maximum point of the distance function \( \rho \), then \( \|u_\infty\|_{\infty} = 1 \) and

\[
\|\nabla u_\infty\|_{\infty} = \frac{1}{\|\rho\|_{\infty}}. \tag{3.7}
\]

**Proof.** Following the proof of Theorem 3.13, we obtain item (i) by combining the Harnack inequality and the comparison principle in \( D := \Omega \setminus \{x_0\} \).

In order to prove (ii) we first show that

\[ u_\infty = \lim_{p \to \infty} u_p, \quad \text{uniformly in } \Omega \]

where

\[
\begin{cases}
-\Delta_p u_p = \Lambda_p(\Omega)\delta_{x_0} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

It is easy to check that \( -\Delta_p u_p \geq 0 \) in \( \Omega \), in the weak sense. Hence, according the weak comparison principle, \( u_p \geq 0 \) in \( \Omega \).

Since

\[
\Lambda_p(\Omega)\|u_p\|_{p}^{p} \leq \|\nabla u_p\|_{p}^{p} = \Lambda_p(\Omega)u_p(x_0) \leq \Lambda_p(\Omega)\|u_p\|_{\infty},
\]

we conclude that

\[
u_p(x_0) \leq \|u_p\|_{\infty} \leq 1 \quad \text{and} \quad \|\nabla u_p\|_{p} \leq \Lambda_p(\Omega)^{\frac{1}{p}}.
\]

Let \( r \in (N, p) \). Since

\[
\|\nabla u_p\|_{r} \leq \|\nabla u_p\|_{p}\|\Omega\|^{\frac{1}{p}} \leq \Lambda_p(\Omega)^{\frac{1}{p}}\|\Omega\|^{\frac{1}{p}} \leq \frac{\|\Omega\|^{\frac{1}{p}}}{\|\rho\|_{\infty}}, \quad p > r,
\]

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the family \( \{u_p\}_{p > p_0} \) is uniformly bounded in \( W^{1,r}_0(\Omega) \). It follows, as in the proof of Proposition 3.2, that there exist \( p_n \to \infty \) and \( U_\infty \in W^{1,\infty}_0(\Omega) \) such that \( u_{p_n} \to U_\infty \) (strongly) in \( C(\overline{\Omega}) \) with

\[
\| \nabla U_\infty \|_\infty \leq \frac{1}{\| \rho \|_\infty} \quad \text{and} \quad U_\infty \leq \frac{\rho}{\| \rho \|_\infty} \quad \text{a.e. in } \Omega.
\] (3.8)

Now we are going to show that \( U_\infty(x_0) = \| U_\infty \|_\infty = 1 \).

Since

\[
\Lambda_p(\Omega) \rho(x_0) = \int_{\Omega} |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \rho \, dx
\]

\[
\leq \| \nabla u_p \|_p^{p-1} \| \nabla \rho \|_p = (\Lambda_p(\Omega) u_p(x_0)) \frac{\rho}{\| \rho \|_\infty} |\Omega|^{\frac{1}{p}}.
\]

we have

\[
\rho(x_0) \leq \Lambda_p(\Omega)^{-\frac{1}{p}} (u_p(x_0)) \frac{\rho}{\| \rho \|_\infty} |\Omega|^{\frac{1}{p}}.
\]

Hence, after making \( p \to \infty \), we obtain

\[
\rho(x_0) \leq \| \rho \|_\infty U_\infty(x_0).
\]

The second inequality in (3.8) then implies that \( \rho(x_0) = \| \rho \|_\infty U_\infty(x_0) \). Thus, if \( \rho(x_0) = \| \rho \|_\infty \) we conclude that \( U_\infty(x_0) = 1 \). Therefore, (3.7) holds for \( U_\infty \), since

\[
\| \nabla U_\infty \|_\infty \leq \frac{1}{\| \rho \|_\infty} \leq \frac{\| \nabla U_\infty \|_\infty}{\| U_\infty \|_\infty} = \| \nabla U_\infty \|_\infty.
\]

Repeating the arguments in the proof of Theorem 3.11 we can check that \( U_\infty \) is a viscosity solution of (3.6), so that \( U_\infty = u_\infty \) and (3.7) holds true.

The following corollary is an immediate consequence of Theorem 3.15.

**Corollary 3.16** Suppose the distance function of \( \Omega \) has infinitely many maximum points. Then, there exist infinitely many positive functions \( u \in C(\overline{\Omega}) \cap W^{1,\infty}_0(\Omega) \) satisfying

\[
0 < u(x) \leq 1 = \| u \|_\infty \quad \text{and} \quad \| \nabla u \|_\infty = \min \left\{ \| \nabla \phi \|_\infty : \phi \in W^{1,\infty}_0(\Omega) \text{ and } \| \phi \|_\infty = 1 \right\}.
\] (3.9)

Moreover, each one of these functions assumes its maximum value 1 only at one point, which is also a maximum point of the distance function \( \rho \).

In particular, there exist infinitely many nonradial functions satisfying (3.9) for the annulus \( \Omega_{a,b} := \{ x \in \mathbb{R}^N : 0 < a < |x| < b \} \).

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