FROM MOMENT GRAPHS TO INTERSECTION COHOMOLOGY

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Abstract. We describe a method of computing equivariant and ordinary intersection cohomology of certain varieties with actions of algebraic tori, in terms of structure of the zero- and one-dimensional orbits. The class of varieties to which our formula applies includes Schubert varieties in flag varieties and affine flag varieties. We also prove a monotonicity result on local intersection cohomology stalks.

1. Statement of the Main Results

To a variety $X$ with an appropriate torus action ($\S 1.1$), we will associate a moment graph ($\S 1.2$), a combinatorial object which reflects the structure of the 0 and 1-dimensional orbits. There is a canonical sheaf ($\S 1.3$) on the moment graph, combinatorially constructed from it ($\S 1.4$), which we denote by $\mathcal{M}$. The main result ($\S 1.5$) uses the sheaf $\mathcal{M}$ to compute the local and global equivariant and ordinary intersection cohomology of $X$ functorially.

1.1. Assumptions on the Variety $X$. We assume that $X$ is an irreducible complex algebraic variety endowed with two structures:

1. An action of an algebraic torus $T \cong (\mathbb{C}^*)^d$. We assume that
(a) for every fixed point $x \in X^T$ there is a one-dimensional subtorus which is contracting near $x$, i.e. there is a homomorphism $i : \mathbb{C}^* \to T$ and a Zariski open neighborhood $U$ of $x$ so that $\lim_{\alpha \to 0} i(\alpha)y = x$ for all $y \in U$ (this implies $X^T$ is finite), and
(b) $X$ has finitely many one-dimensional orbits
2. A $T$-invariant Whitney stratification by affine spaces.

It follows that each stratum contains exactly one fixed point, since a contracting $\mathbb{C}^*$ action on an affine space must act linearly with respect to some coordinate system (see [2], Theorem 2.5). Let $C_x$ denote the stratum containing the fixed point $x$, so $X = \bigcup_{x \in X^T} C_x$. Every one dimensional orbit $L$ has exactly two distinct limit points: the $T$ fixed

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point $x$ in the stratum $C_x$ containing $L$ and another fixed point lying in some stratum in the closure of $C_x$.

The main case we are interested in is when $X$ is a Schubert variety in a flag variety or affine flag variety. More generally, if $M$ is a smooth projective variety with a $T$ action satisfying (a) and (b) above, one can take a homomorphism $\mathbb{C}^* \to T$ for which $M^{\mathbb{C}^*} = M^T$ and consider the corresponding Bialnicki-Birula decomposition of $M$ into cells. If it is a stratification, then the closure of any cell satisfies our hypotheses.

1.2. Moment graphs. Let $\mathfrak{t}$ be a complex vector space. A $\mathfrak{t}$ moment graph $\Gamma$ is a finite graph with two additional structures:

1. for each edge $L$, a one dimensional subspace $V_L$ of the dual vector space $\mathfrak{t}^*$ called the direction of $L$, and
2. a partial order $\leq$ on the set of vertices with the property that if an edge $L$ connects vertices $x$ and $y$, then either $x \leq y$ or $y \leq x$ (but $y \neq x$).

We denote the set of vertices of $\Gamma$ by $V$, and the set of edges by $E$. For a vertex $x \in V$, we denote by $U_x$ (for “up”) the set of edges connecting $x$ to a vertex $y$ where $x \leq y$, and by $D_x$ (for “down”) the set of edges connecting $x$ to a vertex $y$ where $y \leq x$.

Constructing a moment graph from $X$. Given a variety $X$ as in §1.1, we construct a moment graph $\Gamma$ as follows. The vertices of $\Gamma$ are the $T$ fixed points in $X$, and the edges of $\Gamma$ are the one dimensional orbits of $X$. The vector space $\mathfrak{t}$ is the Lie algebra of $T$. For an edge $L \in E$, every point on the one dimensional orbit has the same stabilizer in $T$; its Lie algebra is a hyperplane in $\mathfrak{t}$. The direction $V_L$ is the annihilator of that hyperplane in $\mathfrak{t}^*$. The partial order is defined by saying that for $x$ and $y$ in $V$, $x \leq y$ if and only if the stratum $C_y$ is in the closure of $C_x$. Note that $D_x \subset E$ is the set of one dimensional orbits contained in $C_x$.

Remarks. Similar structures (for smooth varieties) are considered by Guillemin and Zara in [13], [14], [15].

The term moment graph is motivated by the following. If $X$ is projective, there is a moment map $\mu : X \to \mathfrak{t}_K^*$ to the dual of the Lie algebra of the maximal compact torus $T_K \subset T$. For $L \in E$, the image $\mu(L)$ is a line segment joining $\mu(x)$ and $\mu(y)$, where $\{x, y\} = L \cap V$. The vector $\mu(x) - \mu(y)$ spans the space $V_L$, using the identification $\mathfrak{t}^* \cong \mathfrak{t}_K^* \otimes_{\mathbb{R}} \mathbb{C}$. 

1.3. Sheaves on the moment graph. Let $A = \text{Sym}(\mathfrak{t}^*)$ be the ring of polynomial functions on $\mathfrak{t}$. Given $L \in E$, denote the quotient ring
Let $\Gamma$ be a moment graph. A $\Gamma$-sheaf $\mathcal{M}$ is a triple $\mathcal{M} = (\{M_x\}, \{M_L\}, \{\rho_{x,L}\})$ where $M_x$ is an $A$-module defined for each vertex $x \in \mathcal{V}$, $M_L$ is an $A_L$-module (also an $A$-module by the homomorphism $A \to A_L$) defined for each $L \in \mathcal{E}$, and $\rho_{x,L} : M_x \to M_L$ is a homomorphism of $A$-modules defined whenever the vertex $x$ lies on the edge $L$.

Let $S(\Gamma)$ be the finite set $S(\Gamma) = \mathcal{V} \cup \mathcal{E}$ of vertices and edges of $\Gamma$. Given a subset $Z \subset S(\Gamma)$, we define a module $\mathcal{M}(Z)$ of “sections” on $Z$ by

$$\mathcal{M}(Z) = \{(\{s_x\}, \{s_L\}) \in \bigoplus_{a \in Z} M_a \mid \rho_{x,L}(s_x) = s_L \text{ if } x \text{ lies on } L \}. $$

In other words, an element of $\mathcal{M}(Z)$ is a choice of an element of $M_x$ for each $x \in Z \cap \mathcal{V}$, together with a choice of an element of $M_L$ for each $L \in Z \cap \mathcal{E}$, such that these elements are compatible under the homomorphisms $\rho_{x,L}$.

In a similar way, we have a “sheaf of rings” $\mathcal{A} = (\{A_x\}, \{A_L\}, \{q_{x,L}\})$ on $\Gamma$, given by letting $A_x = A$ for all $x \in \mathcal{V}$, and letting the $q_{x,L} : A_x \to A_L = A/V_L A$ be the quotient homomorphisms. Then we can define a ring of sections $\mathcal{A}(Z)$ of $\mathcal{A}$ in the same way as above, and $\mathcal{M}(Z)$ becomes a module over $\mathcal{A}(Z)$.

A $\Gamma$-sheaf $\mathcal{M}$ can be thought of as a sheaf in the usual sense. Put a topology on $S(\Gamma)$ by declaring $O \subseteq S(\Gamma)$ to be open if whenever $x \in O \cap \mathcal{V}$ is a vertex, all edges $L \in \mathcal{E}$ adjacent to $x$ are in $O$ as well. Given a $\Gamma$-sheaf $\mathcal{M}$, sending an open set $O$ to $\mathcal{M}(O)$ defines a sheaf on $S(\Gamma)$; restriction homomorphisms are defined in the obvious way. In the same way $\mathcal{A}$ defines a sheaf of rings on $S(\Gamma)$, and the sheaf $\mathcal{M}$ is a sheaf of modules over $\mathcal{A}$.

**Proposition 1.1.** This association gives a bijection between $\Gamma$-sheaves and sheaves of modules over $\mathcal{A}$ on the topological space $S(\Gamma)$.

Because of this, we will also refer to $\Gamma$-sheaves as $\mathcal{A}$-modules.

**Proof.** If $\Sigma \subset S(\Gamma)$, we define $\Sigma^0$ to be the minimal open set with the same vertices as $\Sigma$. If $\Sigma$ is a complete subgraph of $\Gamma$, then restriction gives an isomorphism $\mathcal{M}(\Sigma^0) = \mathcal{M}(\Sigma)$.

The proposition now follows immediately, since the $\Gamma$-sheaf can be recovered from the sheaf on $S(\Gamma)$ as follows:

$$M_x = \mathcal{M}(x^0), \ M_L = \mathcal{M}(L),$$

and $\rho_{x,L}$ is given by restriction $\mathcal{M}(x^0) \to \mathcal{M}(L)$. $\square$
1.4. Construction of the canonical \( \Gamma \)-sheaf \( \mathcal{M} \). For an \( A \)-module \( M \), we denote by \( \overline{M} \) the graded vector space \( \overline{M} = M/(t^*M) \).

Recall that a projective cover \( P \) of an \( A \)-module \( M \) is a free \( A \)-module on the smallest number of generators with a surjection \( P \to M \). This is equivalent to saying that the induced homomorphism \( P \to \overline{M} \) is an isomorphism.

A projective cover \( P \) may be constructed by setting \( P = M \otimes A \), and defining the map to \( M \) by choosing any splitting of the quotient \( M \to \overline{M} \). In particular, projective covers of \( M \) are isomorphic up to a non-unique isomorphism.

Given a \( t \) moment graph \( \Gamma \) arising from a variety \( X \), there is a canonical \( \Gamma \)-sheaf \( \mathcal{M} \) constructed by the following inductive procedure. Begin at the “top” of \( \Gamma \): since \( X \) is irreducible by assumption, there is a unique vertex \( x_0 \) which is maximal in the partial order; put \( M_{x_0} = A \).

Now suppose \( \mathcal{M} \) is known on the full subgraph \( \Gamma_{>x} \) consisting of all vertices \( y > x \), together with all edges joining them. We extend it to \( \Gamma_{\geq x} \), the full subgraph of all vertices \( y \geq x \). First extend it to the set \( \Gamma_{\geq x} = \Gamma_{>x} \cup U_x = \Gamma_{>x} \setminus \{x\} \) as follows. If \( L \in U_x \) and \( y \in \Gamma_{>x} \) is the upper vertex of \( L \), put \( M_L = M_y/V_L M_y \) and let \( \rho_{y,L} \) be the quotient homomorphism.

Next, define a module \( M_{\partial x} \) to be the image of the restriction homomorphism
\[
\phi: \mathcal{M}(\tilde{\Gamma}_{>x}) \to \mathcal{M}(U_x).
\]
Then let \( M_x \) be the projective cover of \( M_{\partial x} \). The composition
\[
M_x \to M_{\partial x} \subset \mathcal{M}(U_x) = \bigoplus_{L \in U_x} M_L
\]
defines the homomorphisms \( \rho_{x,L} \).

Since projective covers are always isomorphic, this defines a sheaf uniquely up to isomorphism. To get a strong functorial result, we need to show our sheaves are “rigid”. This follows from the following local result.

**Proposition 1.2.** If \( M_x \to M_{\partial x} \) and \( N_x \to M_{\partial x} \) are two projective covers, then there is a unique isomorphism \( M_x \to N_x \) commuting with the projective cover homomorphisms.

The proof, which we postpone, uses the algebraic geometry of \( X \). Denote by \( \text{Aut}_A(\mathcal{M}) \) the group of automorphisms of \( \mathcal{M} \) as a graded \( A \)-module.

**Corollary 1.3.** The restriction \( \text{Aut}_A(\mathcal{M}) \to \text{Aut}_A(\mathcal{M}(x_0)) \) is a bijection, where the second group is the group of automorphisms of \( \mathcal{M}(x_0) \).
as a graded $A$-module. As a result, the group of automorphisms of $\mathcal{M}$ is just multiplication by scalars in $\mathbb{C}^*$.

**Another definition of $\mathcal{M}$.** There is another way to describe the sheaf $\mathcal{M}$. Call an $A$-module $\mathcal{N}$ pure if for all $x \in \mathcal{V}$

1. $\mathcal{N}(x)$ is a free $A$-module,
2. $\mathcal{N}(L) = \mathcal{N}(x)/V_L \mathcal{N}(x)$ whenever $L \in D_x$, and
3. the restrictions of $\mathcal{N}(x^o) \to \mathcal{N}(U_x)$ and $\mathcal{N}(\bar{\Gamma}_{>x}) \to \mathcal{N}(U_x)$ have the same image.

**Theorem 1.4.** $\mathcal{M}$ is the unique indecomposable pure sheaf with $\mathcal{M}(x_0) = A$. Any pure $A$-module is a direct sum of sheaves obtained by applying the same construction to the subgraphs $\Gamma_{\leq x}$ consisting of all vertices $y \leq x$ and all edges joining them, or by applying shifts to these sheaves.

**Proof.** The first statement is clear. The second follows by an easy induction from the following statement: if $S$ is a pure $A$-module on $\Gamma$, and $S|_{\Gamma_{>x}}$ splits as a direct sum of two pure sheaves, then this splitting can be extended to $\Gamma_{\geq x}$ (see [7], Theorem 2.3 for the analogous result for toric varieties).

1.5. **The main results.** Suppose that a torus $T$ acts on a variety $X$ as in §1.1, that the $t$ graph $\Gamma$ is constructed from $X$ as in §1.2, and the $\Gamma$-sheaf $\mathcal{M}$ is constructed from $\Gamma$ as in §1.4.

**Theorem 1.5.** There is a canonical identification

$$IH^*_T(X) = \mathcal{M}(\Gamma)$$

of the $T$-equivariant intersection cohomology of $X$ with the space of the global sections of $\mathcal{M}$. They are free $A$-modules. The intersection cohomology of $X$ is given by

$$IH^*(X) = \mathcal{M}(\Gamma) = \mathcal{M}(\Gamma) \otimes_A \mathbb{C}.$$ 

The local intersection homology groups of $X$ at $x \in X$ are invariants of the singularity type of $X$ at $x$. Since these are constant along a stratum $C_x \subset X$, to know them all it is enough to compute them at the fixed point $x \in C_x$.

**Theorem 1.6.** The local equivariant intersection cohomology at $x \in X$ is canonically isomorphic to the stalk $M_x$:

$$IH^*_T(X)_x = \mathcal{M}({x}) = M_x$$

The local intersection cohomology of $X$ is given by

$$IH^*(X)_x = \mathcal{M}({x}) = \mathbb{M}_x.$$
It follows from results in [12] that similar calculations hold in ordinary cohomology if the sheaf \( \mathcal{M} \) is replaced by the sheaf \( \mathcal{A} \). We have \( H^*_T(X) = A(\Delta) \); \( H^*_X = A(\Delta) \); and (trivially) \( H^*_T(X)_x = A(\{x\}) = A \), and \( H^*_X = A(\{x\}) = C \).

**Theorem 1.7.** The module structures over the cohomology ring of the intersection cohomology groups mentioned above are given by the module structure over \( \mathcal{A} \) of \( \mathcal{M} \). For example, the module structure of \( IH^*_X \) over \( H^*_X \) is the module structure of \( \mathcal{M}(\Delta) \) over \( A(\Delta) \).

Finally, we also prove an unrelated result, Theorem 3.6, that says the intersection cohomology stalks of \( X \) can only grow larger at smaller strata. In the case of Schubert varieties, this gives another proof of an inequality on Kazhdan-Lusztig polynomials originally proved by Irving [17].

### 1.6. Remarks on the proof.

There is an equivariant intersection homology \( \Gamma \)-sheaf \( \mathcal{M} \) defined by

\[
M_x = IH^*_T(X)_x, \quad M_L = IH^*_T(X)_L;
\]

these are free modules over \( A \), \( A_L \) respectively. The homomorphism \( \rho_{x,L} : M_x \to M_L \) is the composition

\[
IH^*_T(X)_x \xrightarrow{\sim} IH^*_T(X)_{x \cup L} \to IH^*_T(X)_L.
\]

We will prove the following slight improvement of Theorem 1.6:

**Theorem 1.8.** The equivariant intersection homology \( \Gamma \)-sheaf is canonically isomorphic to the \( \Gamma \)-sheaf constructed in §1.4.

Using results of [12], this result implies all of the others in §1.5 (the action of \( T \) on \( X \) is equivariantly formal, [12], for weight reasons). Note also that because of Corollary 1.3, the identifications in section §1.5 are all canonical. Because of this, we can use these sheaves to study how the intersection homology sheaves extend each other to form more complicated perverse sheaves – this will be explored in §3.

For the equivariant intersection homology \( \Gamma \)-sheaf, we have \( M_{x_0} = A \) for the maximal vertex \( x_0 \) because \( x_0 \) is a smooth point of \( X \). If \( L \in D_y \), we have \( M_L = M_y/V_{LM_y} \) because \( L \) and \( y \) lie in the same stratum \( C_y \). So everything comes down to the calculation of \( M_x \) in terms of the sheaf \( \mathcal{M}|_{\Delta_x} \).

Let \( N \subset X \) be a \( T \)-invariant normal slice to \( C_x \) through \( x \). It can be embedded as a \( T \)-invariant closed subvariety of an affine space \( \mathbb{C}^r \) with a linear action of \( T \). If \( T \) has a subtorus contracting \( \mathbb{C}^r \) to \( \{0\} \), then \( IH^*_T(N) \) is the projective cover of \( IH^*_T(N_0) \) where \( N_0 = N \setminus \{0\} \).
This was originally proved by Bernstein and Lunts in [8]; we will give a somewhat simpler proof.

Thus we see that our theorem amounts to showing that $M_{\partial x} = IH^*_T(N_0)$. The localization theorem of [12] says that for nice enough (e.g. projective) $T$-varieties $Y$ with isolated fixed points, restriction gives an injection $IH^*_T(Y) \subset IH^*_T(Y^T)$, and the submodule is cut out by relations determined by the one-dimensional orbits. We can apply this to the projective variety $N_0/C^*$ for a contracting subtorus $C^* \subset T$; the result is that the restriction

$$\text{(2) } IH^*_T(N_0) \to \bigoplus_{L \in U_x} IH^*_T(N_0)_L$$

is an injection. We then use a surjectivity result coming from the weight filtration of mixed Hodge theory to argue that $M_{\partial x}$ is the image of (2).

This calculation is similar to the calculation of equivariant IH for toric varieties described in [6], [8], [9]. In both cases there is an induction from larger strata to smaller ones, at each step calculating the equivariant IH of a punctured neighborhood of the singularity at a new stratum and taking the projective cover.

There are two main differences between our situation and the toric case. First, in our case we only need data from the zero and one-dimensional orbits — since the strata are contractible, knowledge of the stalk at $x$ is as good as knowledge of the stalks on all of $C_x$. Second, in the toric case strata have affine neighborhoods which are themselves unions of strata. So the definition of sheaves on fans, which are parallel to our $\Gamma$-sheaves, uses only one module for each stratum, and the computation of the module analogous to $M_{\partial x}$ is simpler.

We remark that the definition of $\mathcal{M}$ in §1.4 makes sense for general moment graphs, whether or not they arise from a variety $X$. However, we do not know how to show that it satisfies the degree restrictions of intersection cohomology, or that Corollary 1.3 holds, without using the variety. Such a result might be useful, for instance, in interpreting the the coefficients of Kazhdan-Lusztig polynomials for non-crystallographic Coxeter groups such as $H_3$ and $H_4$. In this case there is a clear definition of a moment graph (see §2.1), but no underlying variety.

1.7. **Computational simplifications.** The main difficulty in computing the sheaf $\mathcal{M}$ is in taking the image of the homomorphism $\phi$ from (1). Fortunately, there is a major simplification, which we give as Theorem 1.3. Essentially it says that to check whether an element of $\mathcal{M}(U_x)$ is in the image of $\phi$ it is enough to check that it can be extended to give sections of $\mathcal{M}$ on planar subgraphs of $\Gamma_{\geq x}$. 
Some of the relations cutting out the image of $\phi$ are easy to describe. Suppose $x < y$, and take a subspace $V \subset t^*$. If we have an increasing path $x = x_1 < x_2 < \cdots < x_n = y$ with $x_i$ joined to $x_{i+1}$ by an edge $L_i$, we call it a $V$-path if $V_{L_i} \subset V$ for all $i$.

For an $A$-module $M$, we put
$$M_V = M \otimes_A (A/V A) = M/V M.$$ If we have a $V$-path from $x$ to $y$ as above, then $(M_{x_{i+1}})_V \rightarrow (M_{L_i})_V$ are isomorphisms, so we can compose their inverses with $(M_{x_i})_V \rightarrow (M_{L_i})_V$ to get a homomorphism $(M_x)_V \rightarrow (M_y)_V$.

**Proposition 1.9.** This homomorphism depends only on $x$, $y$, and $V$, and not on the path.

In particular, taking $V = t^*$, we get a homomorphism $m_{y,x} : \overline{M}_x \rightarrow \overline{M}_y$.

Similarly, by composing all but the first homomorphism we can get a homomorphism $m_{y,L_1}^V : (M_{L_1})_V \rightarrow (M_y)_V$. It is independent of the path chosen.

**Corollary 1.10.** If $\{\alpha_L\}_{L \in U_x}$ is in $M_{\partial x}$, then $m_{y,L}^V(\alpha_L)$ is independent of $L$ and $V$ for any $y$.

If there are only finitely many two dimensional orbits in the punctured neighborhood $N_0$ of a fixed point $x$, the image of the map ([3]) is exactly the set of $\{\alpha_L\}$ satisfying these relations. This will hold if and only if for every three distinct edges $L_1$, $L_2$, $L_3$ in $U_x$, the total span of the $V_{L_i}$ is three-dimensional.

This happens, for instance, when $X$ is a Grassmannian, i.e. $X = G/P$ where $P$ is a maximal parabolic in a semisimple complex algebraic group $G$. In general, however, $N_0$ may have infinitely many two-dimensional orbits, and there are additional relations beyond those imposed by the corollary above. We will see an example of these extra relations in §2.3.

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2. **Schubert varieties**

Our main motivation for this work was the case of Schubert varieties. A flag variety $M$ is stratified by Schubert cells $C_x$, whose closures $\overline{C}_x$ are the Schubert varieties. Our results give a functorial calculation of $IH^*(\overline{C}_x)_y$ for a $T$-fixed point $y \in \overline{C}_x$. The Poincaré polynomials
of these groups are the Kazhdan-Lusztig polynomials $P_{x,y}$, which are important in representation theory.

Our calculation uses only data (the moment graph) from the interval $[y, x]$ in the Bruhat order. Brenti [4] has given a formula for the Kazhdan-Lusztig polynomials using only data from this graph (whereas the original Kazhdan-Lusztig algorithm used the entire interval $[0, x]$). We have not been able to understand Brenti’s formulas in terms of our construction.

2.1. Schubert varieties for a complex algebraic group. Let $G$ be a semisimple complex algebraic group, $B$ a Borel subgroup, $P \supseteq B$ a parabolic subgroup, and $T \subseteq B$ a maximal torus. Then $M = G/P$ is a flag variety. The Schubert cells $C_x$ of $M$ are the orbits of $B$ on $M$. Let $X = \overline{C_x} \subseteq M$ be a Schubert variety. Then the action of $T$ on $X$ satisfies the assumptions of §1.1, taking as strata the Schubert cells in $X$.

To calculate the local or global intersection homology of $X$ as in §1.3, we need to determine the moment graph for $X$, as defined in §1.2. Let $W$ be the Weyl group of $G$, and $W_P$ the parabolic subgroup of $W$ corresponding to $P$ ($W_P$ is the Weyl group of the Levi of $P$). Then $W$ acts on $\mathfrak{t}^*$, the dual of the Lie algebra of $T$. Let $v \in \mathfrak{t}^*$ be a vector whose stabilizer is $W_P$. Then the following sets are canonically equivalent, and we abuse notation by identifying them: the orbit $O$ of $v$ under $W$, the quotient set $W/W_P$, the set of Schubert cells of $M$, and the set of fixed points $M^T$ of $M$. There is a Bruhat partial order on this set (given by the usual Bruhat order on the maximal elements of the cosets of $W/W_P$), which corresponds to the closure relation on the Schubert cells. The moment graph $\Gamma$ of $X$ is determined as follows:

- The vertices of $\Gamma$ are those $y \in O$ such that $y \leq x$.
- Edges $L$ connect pairs of vertices $y$ and $z$ such that $y = Rz$ where $R$ is a reflection (not necessarily simple) in $W$.
- The direction $V_L \subset \mathfrak{t}^*$ is spanned by $y - z$.
- The partial order is the Bruhat order.

So the embedding of $O$ in $\mathfrak{t}^*$ gives a linear map of the moment graph to $\mathfrak{t}^*$ in which the direction of $L$ is the angle of the image of $L$. Such a graph is drawn below in §2.3.

2.2. Affine Schubert varieties and the loop group. Let $G$ be a semisimple complex algebraic group, $G(\mathbb{C}((t)))$ the corresponding loop group, $I$ an Iwahori subgroup, $P \supseteq I$ a parahoric subgroup. Then $M = G/P$ is an affine flag variety. The Schubert cells $C_x$ of $M$ are the orbits of $I$ on $M$. Let $X = \overline{C_x} \subseteq M$ be an affine Schubert variety. It
is a finite dimensional projective algebraic variety, even though $M$ is infinite dimensional. Let $A \subseteq G(\mathbb{C})$ be a maximal torus whose inclusion in $G(\mathbb{C}((t)))$ lies in $I$. Let $T$ be the torus $A \times \mathbb{C}^*$ which acts on $M$ as follows: $A$ acts through $G(\mathbb{C})$ and $\mathbb{C}^*$ acts by “rotating the loop”, i.e. $\lambda \in \mathbb{C}^*$ sends the variable $t$ to $\lambda t$. Then $T$ preserves $X$, and the action of $T$ on $X$ satisfies the assumptions of §1.1, taking as strata the Schubert cells in $X$.

As before, to calculate the local or global intersection homology of $X$ we need to specify the moment graph for $X$. Let $W$ be the affine Weyl group $W$ of $G(\mathbb{C}((t)))$, and $W_P$ the parabolic subgroup of $W$ corresponding to $P$ (note that $W_P$ is a finite group). Then $W$ acts on $t^*$, the dual of the Lie algebra of $T$ in a somewhat nonstandard way satisfying the following properties:

1. The projection of $t^*$ to $a^*$ is $W$ equivariant, where the action of $W$ on $a^*$, the dual to the Lie algebra of $A$, is the standard one.
2. Reflections in $W$ act by pseudoreflections on $t^*$, i.e. order two affine maps that fix a hyperplane.

Up to affine equivalence, there are only two actions satisfying these properties, and the action in question is the one that is not the product action.

With this set-up, the construction of $\Gamma$ is identical to the construction for semisimple algebraic groups above. Let $v \in t^*$ be a vector whose stabilizer is $W_P$. We identify the following sets, which are canonically equivalent: the orbit $O$ of $v$ under $W$, the quotient set $W/W_P$, the set of Schubert cells of $M$, and the set of fixed points $M^T$ of $T$. There is a Bruhat partial order on this set, defined as above, which corresponds to the closure relations of the Schubert cells. The moment graph $\Gamma$ of $X$ is determined by the same procedure: The vertices of $\Gamma$ are those $y \in O$ such that $y \leq x$; edges $L$ connect pairs of vertices $y$ and $z$ such that $y = Rz$ where $R$ is a reflection in $W$; the direction $V_L \subset t^*$ is spanned by $y - z$; and the partial order is the Bruhat order.

As before, the embedding of $O \cap X$ in $t^*$ gives the structure. The points of $O$ lie on a paraboloid in $t^*$. The case of the loop Grassmannian (an affine flag manifold for a particular parahoric $P$), is worked out in [1], which also has some pictures of $O$.

2.3. Example. Take $G = SL_3(\mathbb{C})$, and take $X = G/B$. The moment graph is pictured in Figure 1. Since $X$ is smooth, we must have $M_w = A$ for all $w$. Still, it is instructive to see what Theorem 1.8 says in this case.
The induction begins with $M_{w_0} = A$ for the longest word $w_0 = sts$. If $w = st$ or $ts$, there is only one edge $L$ in $U_w$, giving $M_{\partial w} = A/V_L A$. Since this module is generated in degree 0, we have $M_w \cong A$.

If $w = s$ or $t$, there are two edges, say $L$ and $L'$, in $U_w$. $M_{\partial w}$ consists of pairs of polynomials in $M_L$ and $M_{L'}$ whose constant terms agree — this is exactly the relation implied by Corollary 1.10. As a module this is just $A/V_L V_{L'} A$, which again has a single generator in degree zero.

We see a new phenomenon when we look at $w = 1$. The relations from Corollary 1.10 only affect the degree zero part; without further relations we would have $\dim(M_{\partial w})_2 = 3$, which would imply that $M_w$ has a generator in degree 2.

The fact that we get the right relation from Theorem 1.8 follows from the projective dual of Pappus’ theorem in projective plane geometry. Degree two sections of $\mathcal{M} = \mathcal{A}$ can be seen as infinitesimal motions of the vertices which preserve the parallelism classes of the edges. If you remove the vertex labeled 1 from Figure 1 and allow the remaining vertices to move, the constraints imposed by the edges ensure that the three lines through 1 will still meet in a point.

2.4. For a Schubert variety $X \subset G/B$, there is another description of $M_X = IH^*_T(X)$ as a module over $R = H^*_T(G/B)$, coming from results due to Soergel ([20], see [19] for a non-equivariant version). In essence, he shows how to compute the equivariant cohomology of a resolution of $X$; by the decomposition theorem it is a direct sum of $M_X$ and shifted copies of $M_{X'}$ for smaller Schubert varieties $X'$. He proves that the $M_X$ are indecomposable $R$-modules, so in principle it is possible to compute the desired submodule.
3. Equivariant intersection cohomology

3.1. Definitions and conventions. All our sheaves and cohomology groups will be taken with complex coefficients. For \( X \) a complex algebraic variety, let \( \text{IC}'(X) \) be the intersection cohomology object in the derived category \( D^b(X) \), shifted so that it restricts to the constant local system in degree 0 on the smooth locus of \( X \). Its hypercohomology \( H^d(\text{IC}'(X)) = IH^d(X) \) is the intersection cohomology of \( X \).

If \( i: Y \to X \) is the inclusion of a subvariety, we put
\[
\text{IH}^d(X)_Y = H^d(i^* \text{IC}'(X)).
\]
The adjunction \( \text{IC}'(X) \to i_* i^* \text{IC}'(X) \) gives rise to a homomorphism \( \text{IH}^*(X) \to \text{IH}^*(X)_Y \). If \( i \) is a normally nonsingular inclusion, then there is a canonical isomorphism \( i^* \text{IC}'(X) \cong \text{IC}'(Y) \), giving a homomorphism
\[
\text{IH}^*(X) \to \text{IH}^*(Y).
\]

3.2. Equivariant IH. Now suppose an algebraic torus \( T \) acts on \( X \). More sophisticated treatments of equivariant intersection cohomology can be found in \([3], [8], [16]\), but the following is enough for our purposes. Fix an isomorphism \( T \cong (\mathbb{C}^*)^d \), and let \( E_k = (\mathbb{C}^k \setminus \{0\})^d \) carry the \( T \)-action given by termwise multiplication. Let \( E_k \) sit inside \( E_{k+1} \) as the set of points whose \((k+1)\)st coordinates are all zero. The quotient \( B_k = E_k/T \cong (\mathbb{CP}^{k-1})^d \) is a finite-dimensional approximation to the classifying space \( BT = \bigsqcup B_k \). The cohomology ring \( H^*(BT) \) is canonically isomorphic to the symmetric algebra \( A = S(t^*) \).

Let \( X_k = (X \times E_k)/T \). The inclusion \( X_k \subset X_{k+1} \) is normally nonsingular. This gives a natural homomorphism \( IH^n(X_{k+1}) \to IH^n(X_k) \); it is an isomorphism when \( 2(k-1) \geq n \). We define the equivariant intersection cohomology by
\[
IH^*_T(X) = \lim_{\leftarrow} IH^n(X_k).
\]
Similarly, if \( Y \subset X \) is a \( T \)-invariant subvariety, we put
\[
IH^*_T(X)_Y = \lim_{\leftarrow} IH^n(X_k)_Y.
\]
The natural map \( X_k \to B_k \) makes \( IH^*(X_k) \) into a module over \( H^*(B_k) \). Taking limits, \( IH^*_T(X) \) becomes an \( A \)-module. Since \( X_1 = X \), we have a homomorphism \( IH^*_T(X) \to IH^*(X) \); it factors to give a homomorphism
\[
\overline{IH}^*_T(X) \to IH^*(X).
\]
The following lemma gives the isomorphism used in §1.6 to define the homomorphisms \( \rho_{x,L} \) in the sheaf \( \mathcal{M} \).
Lemma 3.1. Suppose $X$ has an algebraic $\mathbb{C}^*$ action which commutes with the action of $T$ and contracts a locally closed subvariety $Y \subset X$ onto another subvariety $Y'$. Then $IH^*_T(X)_Y \to IH^*_T(X)_{Y'}$ is an isomorphism.

3.3. Localization. We recall the result from [12] that we will need.

Theorem 3.2. If either (a) $X$ is projective, or (b) $IH^i(X)$ vanishes for $i$ odd, then $IH^*_T(X)$ is a free $A$-module, and the localization homomorphism

$$\lambda: IH^*_T(X) \to IH^*_T(X)_{X^T}$$

is an injection.

If $X$ has finitely many one-dimensional orbits, then

$$(s_x) \in \bigoplus_{x \in X^T} IH^*_T(X)_x = IH^*_T(X)_{X^T}$$

is in the image of $\lambda$ if and only if $s_x$ and $s_y$ map to the same element of $IH^*_T(X)_L$ whenever $L$ is a one-dimensional orbit with $\{x, y\} \subset L$.

3.4. Hodge Intersection Cohomology. The proofs of our results rely on the weight filtration on intersection cohomology of a complex variety. This filtration was constructed by Saito as part of his theory of mixed Hodge modules; see [18] for a good introduction. In this section we extract some simple results from the theory which suffice for our needs.

Given a complex variety $X$ and an open subvariety $U$, the weight filtration is an increasing filtration $W_i IH^*(X)$ and $W_i IH^*(X, U)$ on the intersection cohomology groups $IH^*(X)$ and $IH^*(X, U)$. It is strongly compatible with the homomorphisms in the long exact sequence for the pair $(X, U)$: taking the associated graded $Gr^W_k$ of all terms in the sequence gives another exact sequence.

The homomorphisms $IH^*(X) \to IH^*(Y)$ induced by a normally nonsingular inclusion $X \to Y$ are also strictly compatible with the weight filtration. Thus we get an induced weight filtration on the equivariant intersection cohomology $IH^*_T(X)$ of a variety with an algebraic $T$-action.

Lemma 3.3. We have $W_k IH^d(X, U) = 0$ if $k < d$.

Proof. We have $IH^*(X, U) = \mathbb{H}(i^! \mathcal{IC}^*(X))$, where $i: X \setminus U \to X$ is the inclusion. According to [18], the functors $i^!$ and $\mathbb{H}$ can only increase weights. \qed
Definition. We define the Hodge intersection cohomology of \((X, U)\) by

\[ HIH^d(X, U) = Gr^W_d IH^d(X, U) = W_d IH^d(X, U). \]

If \(X\) carries a \(T\)-action, we let \(HIH^d_T(X) = W_d IH^d_T(X)\).

If \(HIH^*(X) = IH^*(X)\), we say that \(X\) is pure. Projective varieties and quasiconical affine varieties are always pure. The following proposition generalizes these two cases.

Proposition 3.4. If \(X\) has an action of \(T = \mathbb{C}^*\) which contracts \(X\) onto \(X^T\), and \(X^T\) is proper, then \(X\) is pure.

Proof. We have \(IH^*(X) = H^*(IC^*(X)) = H^*(i^*IC^*(X))\), where \(i : X^T \to X\) is the inclusion. The middle expression vanishes for weights less than the degree, while the right one vanishes for weights greater than the degree (the functor \(i^*\) can only decrease weights, and hypercohomology of mixed Hodge modules on proper varieties preserves weights).

Theorem 3.5. If \(U \subset X\) is an open subvariety, then the restriction \(HIH^*(X) \to HIH^*(U)\) is a surjection. If \(X\) carries an action of \(T\) and \(U\) is \(T\)-invariant, then \(HIH^*_T(X) \to HIH^*_T(U)\) is a surjection.

For example, take \(X = \mathbb{CP}^1\), and \(U = X \setminus \{p, q\}\) for \(p \neq q\). Then \(IH^1(X) \to IH^1(U)\) is not surjective, but \(HIH^1(U) = 0\).

Proof. Lemma 3.3 implies that the connecting homomorphism \(Gr^W_d IH^d(U) \to Gr^W_d IH^{d+1}(X, U)\) vanishes. The equivariant case then follows from the nonequivariant case, since for \(k\) large enough we have \(HIH^n_T(X) = HIH^n(X_k)\) and \(HIH^n_T(U) = HIH^n(U_k)\).

3.5. Monotonicity for local stalks. Theorem 3.5 has the following consequence, which is independent from the rest of the paper. Let \(X\) be a \(T\)-variety satisfying the conditions of \(\S 1.1\). Take \(x, y \in X^T\) and assume that \(x \leq y\), so \(x \in C_y\).

Theorem 3.6. There is a surjection \(IH^*(X)_x \to IH^*(X)_y\).

Proof. For any \(x \in X^T\), let \(U\) be a \(T\)-invariant affine neighborhood of \(x\), and let \(\rho_x\) be the composition of restriction and localization homomorphisms

\[ IH^*(X) \to IH^*(U) \sim IH^*(U_x) = IH^*(X)_x. \]

It is a surjection, using Theorem 3.5 and Lemma 3.1. So it is enough to find a homomorphism \(m : IH^*(X)_x \to IH^*(X)_y\) with \(m\rho_x = \rho_y\). Such an \(m\) is given by the composition

\[ IH^*(X)_x \sim IH^*(U) \to IH^*(X)_y \sim IH^*(X)_C_y \sim IH^*(X)_y, \]
where \( y' \) is any point in \( C_y \cap U \). The last two isomorphisms result from the equisingularity of \( X \) along \( C_y \).

The homomorphism \( m \) does not depend on the choice of point \( y' \), and in fact it can be described in our moment graph language; it is the homomorphism \( m_{y,x} : M_x \rightarrow M_y \) given by Proposition 1.9.

If \( X \) is a Schubert variety in a flag variety or affine flag variety, this gives an inequality on Kazhdan-Lusztig polynomials: let \( P^i_{x,y} \) be the \( i \)th coefficient of \( P_{x,y} \).

**Corollary 3.7.** \( P^i_{x,z} \geq P^i_{y,z} \) if \( x \leq y \).

This was proved algebraically in the case of ordinary flag varieties by Irving ([17], Corollary 4), using the Koszul dual interpretation of Kazhdan-Lusztig polynomials as multiplicities of simple objects in the socle filtration of a Verma module. To our knowledge the corresponding statement for affine flag varieties was not previously known.

### 3.6. The local calculation

The following theorem describes the local \( IH^*_T \) groups of quasihomogeneous singularities. It was proved by Bernstein and Lunts in [8]; we will give a proof we feel is slightly simpler.

Suppose that a torus \( T \) acts linearly on \( C^r \), and a subtorus \( C^* \subset T \) contracts \( C^r \) to \( \{0\} \). Let \( X \subset C^r \) be a \( T \)-invariant variety, and put \( X_0 = X \setminus \{0\} \). By Lemma 3.1, we have \( IH^*_T(X)_x \cong IH^*_T(X) \).

Recall that for any \( A \)-module \( M \) we put \( \overline{M} = M \otimes_A \mathbb{C} \).

**Theorem 3.8.** The restriction homomorphism makes \( IH^*_T(X) \) into a projective cover of \( IH^*_T(X_0) \). Its kernel is isomorphic to the local equivariant intersection homology with compact supports

\[
IH^*_{T,c}(X) = IH^*_T(X, X_0);
\]

it is a free \( A \)-module, and \( IH^*_{T,c}(X) = IH^*_c(X) \).

**Proof.** The claim that \( IH^*_T(X) \) is a free module follows from the collapsing of the spectral sequence \( H^p(\bigcup B_k; IH^q(X)) \Longrightarrow IH_T^{p+q}(X) \). This in turn happens because the intersection cohomology of the varieties \( X_k = (X \times E_k)/T \) is pure, by Proposition 3.4. A similar argument shows that \( IH^*_{T,c}(X) \) is free.

Theorem 9.1 of [8] shows that \( IH^*_T(X_0) = IH^*_T/C^*(X_0/C^*) \). Since \( X_0/C^* \) is projective, we have \( HIH^*_T(X_0) = IH^*_T(X_0) \), and so Theorem 3.5 implies that \( IH^*_T(X) \rightarrow IH^*_T(X_0) \) is a surjection.
All that remains to prove the first statement is to show that $\overline{IH_T^*(X)} \to \overline{IH_T^*(X_0)}$ is an injection. But in the commutative square:

\[
\begin{array}{ccc}
\overline{IH_T^*(X)} & \longrightarrow & \overline{IH_T^*(X_0)} \\
\downarrow & & \downarrow \\
IH^*(X) & \longrightarrow & IH^*(X_0)
\end{array}
\]

the left homomorphism is an isomorphism, and the lower one is an injection — it is an isomorphism in degrees $< \dim_{\mathbb{C}}(X)$ and $IH^*(X)$ vanishes in higher degrees.

Finally, the second statement of the theorem follows from the long exact sequence for $IH_T^*$ of the pair $(X, X_0)$.

4. Proofs

4.1. The main theorem. We now have all the ingredients to prove Theorem 1.8. Let $\mathcal{M}$ be the $\Gamma$-sheaf defined by the inductive construction of §1.4, and let $\mathcal{M}'$ be the equivariant intersection cohomology $\Gamma$-sheaf of §1.6. We need to show that these sheaves are isomorphic.

Their restrictions to the top vertex $x_0$ clearly agree, since $M_{x_0} = A$ by definition and $X$ is smooth at $x_0$. Further, if $\mathcal{M}$ and $\mathcal{M}'$ agree at a vertex $y$, they agree at all edges $L \in D_y$, since $M_L = M_y/V_L M_y$ and $X$ is equisingular along $y \cup L \subset C_y$.

Now take a vertex $x \in V$ and assume inductively that $\mathcal{M}$ and $\mathcal{M}'$ have isomorphic restrictions to $\Gamma > x$. We need to show they agree on all of $\Gamma > x$. By the previous remark, they agree on $\widetilde{\Gamma} > x = \Gamma > x \cup U_x$.

By our assumptions, $x$ has a $T$-invariant affine neighborhood $U$.

**Lemma 4.1.** There is a $T$-invariant closed subvariety $N \subset U$ which is a normal slice to $C_x$ through $x$.

**Proof.** We can find a diagonal linear action of $T$ on some affine space $\mathbb{C}^r$, and an equivariant embedding $U \subset \mathbb{C}^r$. The tangent space $T_x C_x$ will be generated by a subset of the coordinate directions. Take the linear span of the remaining coordinates and intersect with $U$.

Applying Lemma 3.3 to $U$ and to $N$ and using the fact that $N \subset U$ is a normally nonsingular inclusion, we obtain isomorphisms $IH^*(U) \cong IH^*(U)_x \cong IH^*(X)_x$ and $IH^*(U) \to IH^*(N)$.

If we can show that $IH_T^*(N_0) \cong M_{\partial x}$, where $N_0 = N \setminus \{x\}$, then Theorem 3.8 implies that $M_x \cong IH_T^*(X)_x$, as required. Let $X_{> x} = \bigcup_{y > x} C_y$. 

Consider the following diagram of restriction homomorphisms:

\[
\begin{array}{ccc}
\text{IH}_T^*(X) & \longrightarrow & \text{IH}_T^*(N) \\
\downarrow & & \downarrow \\
\text{IH}_T^*(X_{>x}) & \longrightarrow & \text{IH}_T^*(N_0) \\
\gamma & & \beta \\
\phi \end{array}
\]

We will show that $\alpha$ is an isomorphism, $\beta$ is an injection, and $\gamma$ is a surjection; the result follows.

To see that $\alpha$ is an isomorphism, we apply Theorem 3.2 to $X_{>x}$; we claim that $\text{IH}_d^*(X_{>x})$ vanishes for $d$ odd. It follows from the previous steps of the induction that the local stalks $\text{IH}_*^*(X)_{y} \cong M_y$ vanish in odd degrees for $y > x$. Applying Verdier duality we see that $\mathbb{H}^*(i_y^! \text{IC}^*(X))$ vanishes in odd degrees if $i_y: C_y \to X$ is the inclusion and $y > x$. Since this can also be written $\text{IH}_*^*(X_{>y} \cup C_y, X_{>y})$, the claim follows by induction using the long exact sequence of a pair.

To see that $\beta$ is an injection, note that any contracting subtorus $C^* \subset T$ acts almost freely (only finite stabilizers) on $N_0$. By [8], Theorem 9.1, we have isomorphisms

\[
\text{IH}_T^*(N_0) \cong \text{IH}_T^*/C^*(N_0/C^*),
\]

\[
\mathcal{M}(U_x) \cong \bigoplus_{y \in (N_0/C^*)^T} \text{IH}_T^*/C^*(N_0/C^*)_y.
\]

Since $N_0/C^*$ is projective, we can apply Theorem [3.2].

Finally, $\gamma$ is a surjection because $\text{IH}_T^*(X) \to \text{IH}_T^*(N)$ and $\text{IH}_T^*(N) \to \text{IH}_T^*(N_0)$ are surjections. The first homomorphism factors as $\text{IH}_T^*(X) \to \text{IH}_T^*(U) \cong \text{IH}_T^*(N)$, so the surjectivity follows from Theorem [3.3] and Proposition [3.4]. The second surjection is part of Theorem [3.8].

Note that we have shown that $\text{IH}_*^*(X)$ vanishes in odd degrees, so Theorem [3.2] can be applied to deduce the theorems in §1.5 from Theorem [1.8].

4.2. Automorphisms. Proposition [1.2] now follows from Theorems [1.8] and [3.8], the degree vanishing conditions for local intersection cohomology and compactly supported intersection cohomology, and the following lemma.

**Lemma 4.2.** Let $M_i, M'_i, i = 1, 2$ be graded modules over a polynomial ring $A$, with $M_i$ free, and let $\phi_i: M_i \to M'_i$ be homomorphisms with $\bar{\phi}: \overline{M_i} \to \overline{M'_i}$ an isomorphism. Also suppose that for some $d \in \mathbb{Z}$, $M_1$ is generated in degrees $< d$ and $\ker \phi_2$ is generated in degrees $\geq d$.

Then if $f': M'_i \to M'_2$ is a homomorphism of graded modules, there is a unique $f: M_1 \to M_2$ so that $\phi_2 f = f' \phi_1$. 

4.3. **Planar relations.** For the results of this last section, we need to assume that the moment graph $\Gamma$ is constructed from a projective variety $X$.

Fix a vertex $x \in V$ of the moment graph. If $H \subset t^*$ is a sub-vector space, consider the graph with the same vertex set as $\Gamma$, but with only those edges $L$ of $\Gamma$ for which $V_L \subset H$. Denote by $\Gamma^H$ the connected component of this graph containing $x$, and let $\tilde{\Gamma}^H_{>x}$, $U^H_x$ be the intersections of $\tilde{\Gamma}^H_{>x}$, $U_x$ with $\Gamma^H$.

Let $\phi: M(\tilde{\Gamma}^H_{>x}) \to M(U^H_x)$ and $\phi^H: M(\tilde{\Gamma}^H_{>x}) \to M(U^H_x)$ be the restrictions. Given $\xi \in M(U^H_x)$, let $\xi^H$ be its restriction to $M(U^H_x)$.

**Theorem 4.3.** Take $\xi \in M(U^H_x)$. Then $\xi \in \text{Im}(\phi)$ if and only if $\xi^H \in \text{Im}(\phi^H)$ for all $H \in \mathcal{H}$.

The “only if” direction is trivial. Note that for two-planes $H \notin \mathcal{H}$, $\phi^H$ is automatically surjective.

Pick a subtorus $\mathbb{C}^* \subset T$ which is contracting near $x$. Its Lie algebra is a one-dimensional subspace $t_0 \subset t$; let $t_0^\perp \subset t^*$ be its annihilator. Since the action is contracting, we have $V_L \subset t_0^\perp$ for all $L \in U_x$. Let $A_0 = \text{Sym}(t_0^\perp)$; it is a subring of $A$. Note that the set of all possible $t_0$ forms an open subset of the set of points in the projective space $\mathbb{P}(t)$ which are rational with respect to the lattice of characters. Thus $t_0$ can be chosen to avoid any finite collection of vectors.

**Lemma 4.4.** $M_{\partial x}$ and $M(U^H_x)$ are free $A_0$-modules.

**Proof.** The result for $M(U^H_x)$ is clear from the construction of $\mathcal{M}$. Since in §4.1 we showed that

$$M_{\partial x} \cong IH_T^*(N_0) \cong IH_T^*/\mathbb{C}^*(N_0/\mathbb{C}^*)$$

we can apply the first part of Theorem 3.2.

Now take $\xi \in M(U^H_x)$. Define an ideal $I(\xi)$ in $A_0$ by

$$I(\xi) = \{ a \in A_0 \mid a\xi \in \text{Im}(\phi) \}.$$ 

The previous lemma plus the injectivity of $M_{\partial x} \to M(U^H_x)$ implies the following.

**Proposition 4.5.** (Chang and Skjelbred [10]) The ideal $I(\xi)$ is principal.

Take a vector space $H \in \mathcal{H}\cup\{\{0\}\}$. We say a vector $v \in t_0^\perp$ is $H$-good if $v \notin H$, and if, in the case $H = \{0\}$, $v$ is in some plane $J \in \mathcal{H}$. 
**Lemma 4.6.** If $\xi^H \in \text{Im}(\phi^H)$, then there is a nonzero $p \in I(\xi)$ which is a product of $H$-good linear factors.

Note that the condition $\xi^{(0)} \in \text{Im}(\phi^{(0)})$ is vacuous.

Theorem 4.3 immediately follows from the lemma: a generator of $I(\xi)$ must be a product of linear factors, but if $\xi \in \text{Im}(\phi^H)$ for all $H \in \mathcal{H}$, none of the possible factors can actually occur, and so $\xi \in \text{Im}(\phi)$.

Before proving the lemma, we need the following easy consequence of the projectivity of $X$. We say a moment graph $\Gamma$ is **flexible at $x$** if for any $H \subset t^*$ and any $y \in \Gamma^H$, $y \neq x$, there is a degree two section $\zeta \in \mathcal{A}(\Gamma^H)_2$ so that $\zeta_x = 0$, $\zeta_y \neq 0$, and $\zeta_z \in H$ for all vertices $z \in \Gamma^H$.

**Proposition 4.7.** The moment graph $\Gamma$ of a projective variety is flexible at all its vertices.

**Proof.** The moment map gives an embedding $\mu$ of the vertices of $\Gamma$ into $t^*$ so that if $z$ and $w$ are joined by an edge $L$, then $\mu(z) - \mu(w)$ is a nonzero vector in $V_L$. If we choose a linear projection $p: t^* \to H$ which does not kill $\mu(y) - \mu(x)$, then letting $\zeta_z = p(\mu(z) - \mu(x))$ provides the required section.

**Proof of Lemma 4.6.** Let $\widetilde{\Gamma}$ be the union of $U_x \cup \Gamma^H_{>x}$ with the set of upper vertices of edges in $U_x$, and let $\widetilde{\zeta}$ be any extension of $\xi$ to $\widetilde{\Gamma}$. We will construct an element $\tilde{p} \in \mathcal{A}(\tilde{\Gamma})$ so that

1. $\tilde{p}|_{U_x}$ comes from an element $p \in A_0$ which is a product of $H$-good linear factors, and
2. for any vertex $y \in \tilde{\Gamma}$ and any adjacent edge $L \notin \tilde{\Gamma}$, $\tilde{p}_y \in A \cdot V_L$.

If we can do this, $\tilde{p}\widetilde{\zeta}$ can be extended to $\Gamma_{>x}$ by placing a 0 on all vertices and edges outside of $\tilde{\Gamma}$, and so $p \in I(\xi)$, as claimed.

Assume that we have chosen $t_0$ so that the lines $H \cap t_0^H$ for $H \in \mathcal{H}$ are all distinct. Pick a vertex $y \in \tilde{\Gamma}$ and an adjacent edge $L \notin \tilde{\Gamma}$. We will construct a degree two section $a \in \mathcal{A}(\tilde{\Gamma})_2$ satisfying property (1) above and for which $a_y \in V_L$. The section $\tilde{p}$ we want is the product of these sections over all choices of $y$ and $L$.

If $y \notin \Gamma^H_{>x}$, then $y$ is the upper vertex of an edge $L' \in U_x$. Since $L' \not\subset t_0^H$, there are nonzero vectors $v \in V_L$, $v' \in t_0^H$ with $v - v' \in V_L$. The section which is $v$ on $y$ and $v'$ everywhere else does the trick. Note that $V_L + V_{L'} \in \mathcal{H}$, so $v'$ lies in a plane in $\mathcal{H}$.

Now suppose $y \in \Gamma^H_{>x}$, so $H \neq \{0\}$. Let $\zeta \in \mathcal{A}(\Gamma^H)_2$ be the section guaranteed by Proposition 4.7. It extends to a section $\zeta \in \mathcal{A}(\tilde{\Gamma})_2$ which is zero on all of $U_x$. We can assume that $t_0$ has been chosen so $\zeta_y \notin t_0^H$. Thus we can find $v \in V_L$ so that $v' = v - \zeta_y \in t_0^H$, and putting $a = v' + \zeta$ gives the required section. \( \square \)
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