On the Solvability of a Class of Degenerate or Singular Strongly Coupled Parabolic Systems.

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Abstract

The existence of strong solutions to general class of strongly coupled parabolic systems will be discussed. These systems can be degenerate or singular as boundedness of theirs solutions are unavailable and not assumed. The results greatly improve those in a recent papers [11, 12, 13] as the systems can have quadratic growth in gradients. A unified proof for both cases is presented. Most importantly, the VMO assumption in [12, 13] will be replaced by a much versatile one thanks to a new local weighted Gagliardo-Nirenberg involving BMO norms. Degenerate and singular generalized SKT models in biology will be presented as a nontrivial application of the main theorem.

1 Introduction

In this paper, for any $T_0 > 0$ and bounded domain $\Omega$ with smooth boundary in $\mathbb{R}^n$, $n \geq 2$, we consider the following parabolic system of $m$ equations ($m \geq 2$) for the unknown $u : Q \rightarrow \mathbb{R}^m$, where $Q = \Omega \times (0, T_0)$

$$
\begin{align*}
&\begin{cases}
 &u_t - \text{div}(A(x, u)Du) = \hat{f}(x, u, Du), \quad (x, t) \in Q, \\
 &u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega, \ t \in (0, T_0), \\
 &u(x, 0) = U_0(x) \quad x \in \Omega.
\end{cases}
\end{align*}
$$

(1.1)

Here, $A(x, u)$ is a $m \times m$ matrix in $x \in \Omega$ and $u \in \mathbb{R}^m$, $\hat{f} : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \rightarrow \mathbb{R}^m$ is a vector valued function. The initial data $U_0$ is given in $W^{1,p_0}(\Omega, \mathbb{R}^m)$ for some $p_0 > n$, the dimension of $\Omega$. As usual, $W^{1,p}(\Omega, \mathbb{R}^m)$, $p \geq 1$, will denote the standard Sobolev spaces whose elements are vector valued functions $u : \Omega \rightarrow \mathbb{R}^m$ with finite norm

$$
\|u\|_{W^{1,p}(\Omega, \mathbb{R}^m)} = \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}.
$$

The strongly coupled system (1.1) appears in many physical applications, for instance, Maxwell-Stephan systems describing the diffusive transport of multicomponent mixtures, models in reaction and diffusion in electrolysis, flows in porous media, diffusion of polymers, or population dynamics, among others.

We will discuss the existence of strong solutions to (1.1). We say that $u$ is a strong solution if $u$ solves (1.1) a.e. on $Q$ with $Du \in L^p_{\text{loc}}(Q)$ and $D^2 u \in L^p_{\text{loc}}(Q)$.

It is always assumed that the matrix $A(x, u)$ is elliptic in the sense that there exist two scalar positive continuous functions $\lambda_1(x, u), \lambda_2(x, u)$ such that

$$
\lambda_1(x, u)|\zeta|^2 \leq \langle A(x, u)\zeta, \zeta \rangle \leq \lambda_2(x, u)|\zeta|^2 \quad \text{for all } x \in \Omega, \ u \in \mathbb{R}^m, \zeta \in \mathbb{R}^{mn}.
$$

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If there exist positive constants $c_1, c_2$ such that $c_1 \leq \lambda_1(x, u)$ and $\lambda_2(x, u) \leq c_2$ then we say that $A(x, u)$ is \textit{regular elliptic}. If $c_1 \leq \lambda_1(x, u)$ and $\lambda_2(x, u)/\lambda_1(x, u) \leq c_2$, we say that $A(x, u)$ is \textit{uniform elliptic}. On the other hand, if we allow $c_1 = 0$ and $\lambda_1(x, u)$ tend to zero (respectively, $\infty$) when $|u| \to \infty$ then we say that $A(x, u)$ is \textit{singular} (respectively, \textit{degenerate}).

We consider the following structural conditions on the data of (1.1).

**A)** $A(x, u)$ is $C^1$ in $x \in \Omega$, $u \in \mathbb{R}^m$ and there exist a constant $C_\ast > 0$ and scalar $C^1$ positive functions $\lambda(u), \omega(x)$ such that for all $u \in \mathbb{R}^m$, $\zeta \in \mathbb{R}^{mn}$ and $x \in \Omega$

$$\lambda(u)\omega(x)|\zeta|^2 \leq \langle A(x, u)\zeta, \zeta \rangle \text{ and } |A(x, u)| \leq C_\ast \lambda(u)\omega(x). \quad (1.3)$$

In addition, there is a constant $C$ such that $|\lambda_u(u)||u| \leq C\lambda(u)$ and

$$|A_u(x, u)| \leq C|\lambda_u(u)||\omega(x)| \text{, } |A_x(x, u)| \leq C|\lambda(u)||D\omega|. \quad (1.4)$$

Here and throughout this paper, if $B$ is a $C^1$ (vector valued) function in $u \in \mathbb{R}^m$ then we abbreviate its derivative $\frac{\partial B}{\partial u}$ by $B_u$. Also, with a slight abuse of notations, $A(x, u)\zeta$, $\langle A(x, u)\zeta, \zeta \rangle$ in (1.2), (1.3) should be understood in the following way: For $A(x, u) = \{a_{ij}(x, u)\}$, $\zeta \in \mathbb{R}^{mn}$ we write $\zeta = [\zeta_i]_{i=1}^m$ with $\zeta_i = (\zeta_{i,1}, \ldots, \zeta_{i,n})$ and

$$A(x, u)\zeta = \{\sum_{j=1}^m a_{ij}\zeta_j\}_{i=1}^m, \langle A(x, u)\zeta, \zeta \rangle = \sum_{i,j=1}^m a_{ij}\zeta_i\zeta_j.$$  

We also assume that $A(x, u)$ is regular elliptic for \textit{bounded} $u$.

**AR)** $\omega \in C^1(\Omega)$ and there are positive numbers $\mu_\ast, \mu_{\ast\ast}$ such that

$$\mu_\ast \leq \omega(x) \leq \mu_{\ast\ast}, \quad |D\omega(x)| \leq \mu_{\ast\ast} \quad \forall x \in \Omega. \quad (1.5)$$

For any bounded set $K \subset \mathbb{R}^m$ there is a constant $\lambda_\ast(K) > 0$ such that

$$\lambda_\ast(K) \leq \lambda(u) \quad \forall u \in K. \quad (1.6)$$

Concerning the reaction term $\tilde{f}(x, u, Du)$, which may have linear or \textit{quadratic} growth in $Du$, we assume the following condition.

**F)** There exist a constant $C$ and a nonegative differentiable function $f: \mathbb{R}^m \to \mathbb{R}$ such that $\tilde{f}$ satisfies:

$$f(u) \leq C|f_u(u)|(1 + |u|). \quad (1.7)$$

For any differentiable vector valued functions $u : \mathbb{R}^n \to \mathbb{R}^m$ and $p : \mathbb{R}^n \to \mathbb{R}^{mn}$ we assume either that

**f.1)** $\tilde{f}$ has a linear growth in $p$

$$|\tilde{f}(x, u, p)| \leq C\lambda(u)||p||\omega(x) + f(u)\omega(x), \quad (1.8)$$

$$|D\tilde{f}(x, u, p)| \leq C(\lambda(u)||p||Dp| + |\lambda_u(u)||p||^2)\omega + C\lambda(u)||p||D\omega| + C|D(f(u)\omega(x))|;$$
or

\[ f.2) \quad \lambda_{uu}(u) \text{ exists and } \hat{f} \text{ has a quadratic growth in } p \]

\[ |\hat{f}(x, u, p)| \leq C|\lambda_u(u)||p|^2\omega(x) + f(u)\omega(x), \quad (1.9) \]

\[ |D\hat{f}(x, u, p)| \leq C(|\lambda_u(u)||p||Dp| + |\lambda_{uu}(u)||p|^3)\omega + C|\lambda_u(u)||p|^2|D\omega| + C|D(f(u)\omega(x))|. \]

Furthermore, we assume that

\[ |\lambda_{uu}(u)\lambda(u) \leq C|\lambda_u(u)|^2. \quad (1.10) \]

By a formal differentiation of (1.8) and (1.9), one can see that the growth conditions for \( \hat{f} \) naturally implies those of \( D\hat{f} \) in the above assumption. The condition (1.10) is verified easily if \( \lambda(u) \) has a polynomial growth in \( |u| \).

The first fundamental problem in the study of (1.1) is the local and global existence of its solutions. One can decide to work with either weak or strong solutions. In the first case, the existence of a weak solution can be achieved via Galerkin, time discretization or variational methods but its regularity (e.g., boundedness, Hölder continuity of the solution and its higher derivatives) is still an open issue. Several works have been done along this line to improve the early work [8] of Giaquinta and Struwe and establish partial regularity of bounded weak solutions to (1.1).

Otherwise, if strong solutions are considered then their existence can be established via semigroup theories as in the works of Amann [1, 2]. Combining with interpolation theories of Sobolev’s spaces, Amann established local and global existence of a strong solution \( u \) of (1.1) under the assumption that one can control \( \|u\|_{W^{1,p}(\Omega, \mathbb{R}^m)} \) for some \( p > n \). His theory did not apply to the case where \( \hat{f} \) has quadratic growth in \( Du \) as in f.2).

In both forementioned approaches, the assumption on the boundedness of \( u \) must be the starting point and the techniques in both cases rely heavily on the fact that \( A(x, u) \) is regular elliptic. For strongly coupled systems like (1.1), as invariant/maximum principles for cross diffusion systems are generally unavailable, the boundedness of the solutions is already a hard problem. One usually needs to use ad hoc techniques on the case by case basis to show that \( u \) is bounded (see [7, 19]). Even for bounded weak solutions, we know that they are only Hölder continuous almost everywhere (see [8]). In addition, there are counter examples for systems \( (m > 1) \) which exhibit solutions that start smoothly and remain bounded but develop singularities in higher norms in finite times (see [6]).

In our recent work [13, 12], we choose a different approach making use of fixed point theory and discussing the existence of strong solutions of (1.1) under the weakest assumption that they are a-priori VMO, not necessarily bounded, and general structural conditions on the data of (1.1) which are independent of \( x \), we assumed only that \( A(u) \) is uniformly elliptic. Applications were presented in [12] when \( \lambda(u) \) has a positive polynomial growth in \( |u| \) and, without the boundedness assumption on the solutions, so (1.1) can be degenerate as \( |u| \to \infty \). The singular case, \( \lambda(u) \to 0 \) as \( |u| \to \infty \), was not discussed there.

In this paper, we will establish much stronger results than those in [13] under much more general assumptions on the structure of (1.1) as described in A) and F). Beside the minor fact that the data can depend on \( x \), we allow further that:
• $A(x,u)$ can be either degenerate or singular as $|u|$ tends to infinity;
• $\hat{f}(x,u,Du)$ can have a quadratic growth in $Du$ as in f.2);
• no a-priori boundedness of solutions is assumed but a a very weak integrability of strong solutions of (1.1) is considered.

Most remarkably, the key assumption in [12, 13] that the BMO norm of $u$ is small in small balls will be replaced by a more versatile one in this paper: $K(u)$ is has small BMO norm in small balls for some suitable map $K : \mathbb{R}^m \rightarrow \mathbb{R}^m$. This allows us to consider the singular case where one may not be able to estimate the BMO norm of $u$ but that of $K(u)$. Examples of this case in applications will be provided in Section 2 where $|K(u)| \sim \log(|u|)$.

One of the key ingredients in the proof in [13, 12] is the local weighted Gagliardo-Nirenberg inequality involving BMO norm [13, Lemma 2.4]. In this paper, we make use of a new version of this inequality reported in our work [14] replacing the BMO norm of $u$ by that of $K(u)$ for some suitable map $K : \mathbb{R}^m \rightarrow \mathbb{R}^m$.

We organize our paper as follows. In Section 2 we state the main result, Theorem 2.1 of this paper and its application to the generalized SKT systems on planar domain. In Section 3 we recall the new version of the local weighted Gagliardo-Nirenberg inequality in [14] to prepare for the proof the main Theorem 2.1 in Section 4. The proof of solvability of the generalized SKT systems in Section 2 is provided in Section 5.

2 Preliminaries and Main Results

We state the main results of this paper in this section. The key assumption of these results is some uniform a priori estimate for the BMO norm of $K(u)$ where $K$ is some suitable map on $\mathbb{R}^m$ and $u$ is any strong solution to (1.1). To begin, we recall some basic definition in Harmonic Analysis.

Let $\omega \in L^1(\Omega)$ be a nonnegative function and define the measure $d\mu = \omega(x)dx$. For any $\mu$-measurable subset $A$ of $\Omega$ and any locally $\mu$-integrable function $U : \Omega \rightarrow \mathbb{R}^m$ we denote by $\mu(A)$ the measure of $A$ and $U_A$ the average of $U$ over $A$. That is,

$$U_A = \frac{1}{\mu(A)} \int_A U(x) \, d\mu.$$

We define the measure $d\mu = \omega(x)dx$ and recall that a vector valued function $f \in L^1(\Omega, \mu)$ is said to be in $\text{BMO}(\Omega, \mu)$ if

$$[f]_{\text{BMO}(\Omega, \mu)} = \sup_{B_R \subset \Omega} \frac{1}{\mu(B_R)} \int_{B_R} |f - f_{B_R}| \, d\mu < \infty, \quad f_{B_R} := \frac{1}{\mu(B_R)} \int_{B_R} f \, d\mu. \quad (2.1)$$

We then define

$$\|f\|_{\text{BMO}(\Omega, \mu)} := [f]_{\text{BMO}(\Omega, \mu)} + \|f\|_{L^1(\Omega, \mu)}.$$

For $\gamma \in (1, \infty)$ we say that a nonnegative locally integrable function $w$ belongs to the class $A_\gamma$ or $w$ is an $A_\gamma$ weight on $\Omega$ if the quantity

$$[w]_{A_\gamma} := \sup_{B \subset \Omega} \left( \frac{1}{\mu(B)} \int_B w \, d\mu \right) \left( \frac{1}{\mu(B)} \int_B w^{1-\gamma'} \, d\mu \right)^{\gamma-1}$$

is finite. \quad (2.2)
Here, $\gamma = \gamma/(\gamma - 1)$. For more details on these classes we refer the reader to [18, 21]. If the domain $\Omega$ is specified we simply denote $[w]_{\gamma,\Omega}$ by $[w]_{\gamma}$.

Throughout this paper, in our statements and proofs, we use $C, C_1, \ldots$ to denote various constants which can change from line to line but depend only on the parameters of the hypotheses in an obvious way. We will write $C(a, b, \ldots)$ when the dependence of a constant $C$ on its parameters is needed to emphasize that $C$ is bounded in terms of its parameters. We also write $a \lesssim b$ if there is a universal constant $C$ such that $a \leq Cb$. In the same way, $a \sim b$ means $a \lesssim b$ and $b \lesssim a$.

To begin, as in [13] with $A$ is independent of $x$, we assume that the eigenvalues of the matrix $A(x, u)$ are not too far apart. Namely, for $C_*$ defined in (1.3) of A) we assume

**SG)** $(n - 2)/n < C_*^{-1}$.

Here $C_*$ is, in certain sense, the ratio of the largest and smallest eigenvalues of $A(x, u)$. This condition seems to be necessary as we deal with systems, cf. [16].

First of all, we will assume that the system (4.1) satisfies the structural conditions A) and F). Additional assumptions serving the purpose of this paper then follow so that the local weighted Gagliardo-Nirenberg inequality of [14] can applies here.

**H)** There is a $C^1$ map $K : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that $\mathbb{K}(u) = (K_u(u)^{-1})^T$ exists and $\mathbb{K}_u \in L^\infty(\mathbb{R}^m)$. Furthermore, for all $u \in \mathbb{R}^m$

$$\|\mathbb{K}(u)\| \lesssim \lambda(u)\|\lambda_u(u)|^{-1}. \quad (2.3)$$

We consider the following system

$$\begin{cases}
  u_t - \text{div}(A(x, u)Du) = f(x, u, Du), & x \in \Omega, \\
  u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{cases} \quad (2.4)$$

We imbed this system in the following family of systems

$$\begin{cases}
  u_t - \text{div}(A(x, \sigma u)Du) = \hat{f}(x, \sigma u, \sigma Du), & x \in \Omega, \sigma \in [0, 1], \\
  u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{cases} \quad (2.5)$$

For any strong solution $u$ of (2.5) we will consider the following assumptions.

**M.0** There exists a constant $C_0$ such that for some $r_0 > 1$ and $\beta_0 \in (0, 1)$

$$\sup_{T \in (0, T_0)} \|f_u(\sigma u)|\lambda^{-1}(\sigma u)\|_{L^{r_0}(\Omega, \mu)}, \quad \sup_{T \in (0, T_0)} \|u^{\beta_0}\|_{L^1(\Omega, \mu)} \leq C_0, \quad (2.6)$$

$$\int_Q \left( |f_u(\sigma u)| + \lambda(\sigma u)|Du|^2 + |u|^2 \right) d\mu d\tau \leq C_0. \quad (2.7)$$

**M.1** For any given $\mu_0 > 0$ there is positive $R_{\mu_0}$ sufficiently small in terms of the constants in A) and F) such that

$$\sup_{x_0 \in \Omega, r \in (0, T_0)} \|K(\sigma u)\|_{BMO(B_R(x_0) \cap \Omega, \mu)}^2 \leq \mu_0. \quad (2.8)$$

Furthermore, for $W_p(\sigma, x, \tau) := \lambda^{p+\frac{1}{2}}(\sigma u)|\lambda_u(\sigma u)|^{-p}$ and any $p \in [1, n/2]$ there exist some $\alpha > 2/(p + 2), \beta < p/(p + 2)$ such that $\sup_{T \in (0, T_0)} [W_p(\sigma, x, \tau)]^{\beta+1, B_{R_0}(x_0) \cap \Omega} \leq C_0$. 

5
The main theorem of this paper is the following.

**Theorem 2.1** Assume \( A), F), AR) and H). Moreover, if \( \hat{f} \) has a quadratic growth in \( Du \) as in f.2) then we assume also that \( n \leq 3 \). Suppose also that any strong solution \( u \) to (2.3) satisfies M.0), M.1) uniformly in \( \sigma \in [0,1] \).

Then the system (2.4) has a unique strong solution on \( \Omega \times (0,T_0) \).

The condition (2.8) on the smallness of the BMO norm of \( K(u) \) in small balls is the most crucial one in applications. In [13, 12], we consider the case \( \lambda(u) \sim (\lambda_0 + |u|^k \) with \( k > 0 \) and assume that \( K = Id \), the identity matrix. We assumed that \( K(u) = u \) has small BMO norm in small balls, which can be verified by establishing that \( \| Du \|_{L^p(\Omega)} \) is bounded. These results already improve those of Amann in [1] where boundedness of solutions was assumed and uniform estimates for \( \| Du \|_{L^p(\Omega)} \) for some \( p > n \) is needed. Both of such conditions seems to be very difficult to be verified in applications.

We should remark that all the assumptions on strong solutions of the family (2.5) can be checked by considering the case \( \sigma = 1 \) (i.e. (1.1)) because these systems satisfy the same structural conditions uniformly with respect to the parameter \( \sigma \in [0,1] \).

We present an application of Theorem 2.1. This example concerns cross diffusion systems with polynomial growth data on planar domains. This type of systems occurs in many applications in mathematical biology and ecology. An famous example of such systems is the SKT model (see [12, 20, 22]) for two species with population densities \( \{u,v\} \) for \( \mathbb{R}^n \times \mathbb{R}^n \) respectively. We will assume that \( u,v \) satisfy the following conditions.

\[
\begin{align*}
\lambda_i(u) &\leq \lambda(u), \quad |u||\lambda_u(u)| \leq \lambda(u), \quad |(\lambda_i(u))_{uu}| \leq |\lambda_{uu}(u)|.
\end{align*}
\]
(2.12)

The matrices \( L = \text{diag}[\lambda_1(u), \ldots, \lambda_m(u)] \) and, with a slight abuse of notation, \( L_u = D_u[\lambda_i(u)]_{i=1}^m \) satisfy the following conditions.

\[
\begin{align*}
|L + \text{diag}[u_1, \ldots, u_m]|L_u\zeta, \zeta &\geq \lambda(u)|\zeta|^2, \\
|L_u| &\leq |\lambda_u(u)|, \quad |L_u^{-1}| \leq |\lambda_u(u)|^{-1}.
\end{align*}
\]
(2.13)

(2.14)
As $\Delta(P_t(u)) = \text{div}(A(u)Du)$ with $A(u) = (L + \text{diag}[u_i|L_{ui}])$, the condition \eqref{eq210} is necessary for \eqref{eq210} being elliptic. In fact, if $|\lambda_i(u)||u| \leq c_i\lambda_i(u)$ for some small $c_i$ then it is not difficult to see that \eqref{eq52} holds.

We now embed \eqref{eq210} into the following family of system

$$u_t - \text{div}(A(\sigma u)Du) = B_i(\sigma u, \sigma Du) + f_i(\sigma u), \quad \sigma \in [0, 1], \ i = 1, \ldots, m. \quad \text{(2.15)}$$

As a consequence of Theorem 2.1 we will have the following.

**Theorem 2.2** Assume \( L \) and \( \text{(2.17)} \). Assume further that \( n = 2, \lambda \) satisfies \( AR \) and there is a constant \( C_0 \) such that

$$|\lambda_i(u)|\lambda^{-2}(u) \leq C_0 \text{ for all } u \in \mathbb{R}^m. \quad \text{(2.16)}$$

In addition to the integrability condition \( M.0 \), assume that any strong solution \( u \) to \( \text{(2.15)} \) satisfies

$$\int_0^{T_0} \int_{\Omega} |\lambda(\sigma u)f_i(\sigma u)|^2 \ dx \leq C_0 \text{ for all } i = 1, \ldots, m. \quad \text{(2.17)}$$

Then \( \text{(2.10)} \) has a unique strong solution on \( \Omega \times (0, T_0) \).

If $\lambda(u) \sim (\lambda_0 + |u|)^k$ then \eqref{eq216} holds if $k > -1$. Therefore, this theorem includes the singular case of SKT system when $|u|$ becomes unbounded.

**Remark 2.3** The condition \( \text{(2.14)} \) is inspired by the SKT system \( \text{(2.9)} \). In fact, let $\alpha_i = [\alpha_{ij}]_{j=1}^m$ be $m$ linearly independent vectors in $\mathbb{R}^m$. For some $k > 0$ and $d_i > 0$ we define $\lambda_i(u) = d_i + \langle u, \alpha_i \rangle^k$ and $\alpha = [\alpha_i]_{i=1}^m$. Then $\partial_u \lambda_i(u) = k\langle u, \alpha_i \rangle^{k-1}\alpha_i$ so that $L_u = k\text{diag}[\langle u, \alpha_i \rangle^{k-1}]\alpha$ and $L_{u^{-1}} = k^{-1}\alpha^{-1}\text{diag}[\langle u, \alpha_i \rangle^{-k+1}]$. If $\langle u, \alpha_i \rangle \sim \langle u, \alpha_j \rangle$ for $i \neq j$ then $|L_{u^{-1}}| \sim |\alpha^{-1}||\lambda_u(u)|^{-1}$ with $\lambda(u) = \sum_i (d_i + |\langle u, \alpha_i \rangle|^k)$. The system \( \text{(2.10)} \) is degenerate when $|u| \to \infty$. We see that the SKT system \( \text{(2.9)} \) is included in this case for $m = 2, k = 1$.

On the other hand, we can consider the singular case when $k < 0$. We define $\lambda_i(u) = (d_i + \langle u, \alpha_i \rangle)^k$. Then $\partial_u \lambda_i(u) = k(d_i + \langle u, \alpha_i \rangle)^{k-1}\alpha_i$ so that $L_u = k\text{diag}[(d_i + \langle u, \alpha_i \rangle)^{k-1}]\alpha$ and $L_{u^{-1}} = k^{-1}\alpha^{-1}\text{diag}[(d_i + \langle u, \alpha_i \rangle)]$. We then have $|L_{u^{-1}}| \sim |\alpha^{-1}||\lambda_u(u)|^{-1}$ with $\lambda(u) = \sum_i (d_i + \langle u, \alpha_i \rangle)^k$. In both cases, we see that \( \text{(2.14)} \) holds.

## 3 A general local weighted Gagliardo-Nirenberg inequality

In this section, we present a local weighted Gagliardo-Nirenberg inequality in our recent work \[14\], which will be one of the main ingredients of the proof of our main technical theorem in Section 4. This inequality generalizes \[13\] Lemma 2.4 by replacing the Lebesgue measure with general one and the BMO norm of $u$ with that of $K(u)$ where $K$ is a suitable map on $\mathbb{R}^m$, and so the applications of our main technical theorem in the next section will be much more versatile than those in \[12\] \[13\].

Let us begin by describing the assumptions in \[14\] for this general inequality. We say that $\Omega$ and $\mu$ support a $q$-Poincaré inequality if the following holds.
There exist \( q_0 \in (0, 2] \), \( \tau_0 \geq 1 \) and some constant \( C_P \) such that

\[
\int_{\mathcal{B}} |h - h_B| \, d\mu \leq C_P l(\mathcal{B}) \left( \int_{\tau_0 \mathcal{B}} |Dh|^{q_0} \, d\mu \right)^{\frac{1}{q_0}}
\]

(3.1) for any cube \( \mathcal{B} \subset \Omega \) with side length \( l(\mathcal{B}) \) and any function \( u \in C^1(\mathcal{B}) \).

Here and throughout this section, we denote by \( l(\mathcal{B}) \) the side length of \( \mathcal{B} \) and by \( \tau \mathcal{B} \) the cube which is concentric with \( \mathcal{B} \) and has side length \( \tau l(\mathcal{B}) \). We also write \( B_R(x) \) for a cube centered at \( x \) with side length \( R \) and sides parallel to to standard axes of \( \mathbb{R}^n \). We will omit \( x \) in the notation \( B_R(x) \) if no ambiguity can arise.

We consider the following conditions on the measure \( \mu := \omega(x) dx \) for the validity of (3.1) (see [4]).

**LM.1** For some \( N \in (0, n] \) and any ball \( B_r \), we have \( \mu(B_r) \leq C \mu r^N \). Assume also that \( \mu \) supports the 2-Poincaré inequality (3.1) in \( P \). Furthermore, \( \mu \) is doubling and satisfies the following inequality for some \( s_0 > 0 \)

\[
\left( \frac{r}{r_0} \right)^{s_0} \leq C \frac{\mu(B_r)}{\mu(B_{r_0}(x_0))},
\]

(3.2)

where \( B_r(x), B_{r_0}(x_0) \) are any cubes with \( x \in B_{r_0}(x_0) \).

**LM.2** \( \omega = \omega_0^2 \) for some \( \omega_0 \in C^1(\Omega) \) and \( d\mu = \omega_0^2 dx \) also supports a Hardy type inequality:

There is a constant \( C_H \) such that for any function \( u \in C_0^1(\mathcal{B}) \)

\[
\int_{\Omega} |u|^2 |D\omega_0|^2 \, dx \leq C_H \int_{\Omega} |Du|^2 \omega_0^2 \, dx.
\]

(3.3)

We assume the following hypotheses.

**A.1** Let \( K : \text{dom}(K) \to \mathbb{R}^m \) be a \( C^1 \) map on a domain \( \text{dom}(K) \subset \mathbb{R}^m \) such that \( \mathbb{K}(U) = (K_U(U)^{-1})^T \) exists and \( \mathbb{K}_U \in L^\infty(\text{dom}(K)) \).

Furthermore, let \( \Phi, \Lambda : \text{dom}(K) \to \mathbb{R}^+ \) be \( C^1 \) positive functions. We assume that for all \( U \in \text{dom}(K) \)

\[
\|\mathbb{K}(U)\| \lesssim \Lambda(U) \Phi^{-1}(U),
\]

(3.4)

\[
|\Phi_U(U)||\mathbb{K}(U)| \lesssim \Phi(U).
\]

(3.5)

Let \( \Omega^* \) be a proper subset of \( \Omega \) and \( \omega^* \) be a function in \( C^1(\Omega) \) satisfying

\[
\omega^* \equiv 1 \text{ in } \Omega^* \text{ and } \omega^* \leq 1 \text{ in } \Omega.
\]

(3.6)

For any \( U \in C^2(\Omega, \text{dom}(K)) \) we denote

\[
I_1 := \int_{\Omega} \Phi^2(U)|DU|^{2p+2} \, d\mu, \quad I_2 := \int_{\Omega} \Lambda^2(U)|DU|^{2p-2}|D^2U|^2 \, d\mu,
\]

(3.7)

\[
\bar{I}_1 := \int_{\Omega} |\Lambda_U(U)|^2|DU|^{2p+2} \, d\mu, \quad I_{1, \ast} := \int_{\Omega^*} \Phi^2(U)|DU|^{2p+2} \, d\mu,
\]

(3.8)
\[ \tilde{I}_{0, *} := \sup_{\Omega} |D\omega_s|^2 \int_{\Omega} \Lambda^2(U)|D\nu|^2 \, d\mu. \] (3.9)

We established the following local weighted Gagliardo-Nirenberg inequality in [14].

**Theorem 3.1** Suppose LM.1)-LM.2), A.1). Let \( U \in C^2(\Omega, \text{dom}(K)) \) and satisfy
\[ \langle \omega_s \omega_0^2 \Phi^2(U) \mathbb{K}(U) DU, \tilde{v} \rangle = 0 \] (3.10)
on \( \partial \Omega \) where \( \tilde{v} \) is the outward normal vector of \( \partial \Omega \). Let \( W := \Lambda^{p+1}(U(x)) \Phi^{-p}(U(x)) \) and assume that \( [W^\alpha]_{\beta+1} \) is finite for some \( \alpha > 2/(p+2) \) and \( \beta < p/(p+2) \).

Then, for any \( \varepsilon > 0 \) there are constants \( C, C([W^\alpha]_{\beta+1}) \) such that
\[ I_{1,*} \leq \varepsilon I_1 + \varepsilon^{-1} C ||K(U)||^2_{BMO(\mu)} [I_2 + \tilde{I}_1 + C([W^\alpha]_{\beta+1})[I_2 + \tilde{I}_1 + \tilde{I}_{0,*}]]. \] (3.11)
Here, \( C \) also depends on \( C_P, C_H \) and \( C_H \).

For our purpose in this paper we need only a special case of Theorem 3.1 where \( \omega \) satisfies AR) so that the Poincaré and Hardy inequalities in LM.1) and LM.2) are verified (\( N, s_* = n \)). In addition, let \( \Omega, \Omega_\varepsilon \) be concentric balls \( B_s, B_t \), \( 0 < s < t \). We let \( \omega_s \) be a cutoff function for \( B_s, B_t \): \( \omega_s \) is a \( C^1 \) function satisfying \( \omega_s \equiv 1 \) in \( B_s \) and \( \omega_s \equiv 0 \) outside \( B_t \) and \( |D\omega_s| \leq 1/(t-s) \). The condition (3.10) of the above theorem is clearly satisfied on the boundary of \( \Omega = B_t \). We also consider only the case \( \Phi(U) \sim |\Lambda(U)| \).

We then have the following corollary.

**Corollary 3.2** Suppose that AR) and A.1) holds for \( \Phi(U) = |\Lambda(U)| \). Accordingly, define \( W_p(x) := \Lambda^{p+1}(U(x))|\Lambda(U(x))|^{-p} \) and let \( B_t(x_0) \) be any ball in \( \Omega \) and assume that
\[ [W^\alpha]_{\beta+1,B_t(x_0)} \] is finite for some \( \alpha > 2/(p+2) \) and \( \beta < p/(p+2) \).

We denote (compare with [3.7]-[3.9])
\[ I_0(t, x_0) := \int_{B_t(x_0)} \Lambda^2(U)|D\nu|^{2p} \, d\mu, \quad I_1(t, x_0) := \int_{B_t(x_0)} |\Lambda(U)|^2|D\nu|^{2p+2} \, d\mu, \] (3.12)
\[ I_2(t, x_0) := \int_{B_t(x_0)} \Lambda^2(U)|D\nu|^{2p-2}|D^2\nu|^2 \, d\mu. \] (3.13)

Then, for any \( \varepsilon > 0 \) and any ball \( B_s(x_0) \), \( 0 < s < t \), there are constants \( C, C([W^\alpha]_{\beta+1,B_t(x_0)}) \) with \( C \) also depending on \( C_P, C_H \) such that for
\[ C_{\varepsilon,U,W} = \varepsilon + \varepsilon^{-1} C ||K(U)||^2_{BMO(\mu)} [1 + C([W^\alpha]_{\beta+1,B_t(x_0)})] \]
we have
\[ I_1(s, x_0) \leq C_{\varepsilon,U,W} [I_1(t, x_0) + I_2(t, x_0) + (t-s)^{-2} I_0(t, x_0)]. \] (3.14)

**Remark 3.3** We can see that the condition H) implies the condition A.1) in Theorem 3.1 and then Corollary 3.2 with \( \Lambda(u) = \lambda(u) \) and \( \Phi(u) = |\Lambda(u)| \), (3.14) is then applicable. Indeed, the assumption (3.4) in this case is (2.3). It is not difficult to see that the assumption in f.2) that \( |\lambda_{u\mu}(u)| |\lambda(u)| \leq |\lambda(u)|^2 \) and (2.3) imply \( |\Phi(u)||\mathbb{K}(u)| \leq \Phi(u) \), which gives (3.5) of A.1). Hence, A.1) holds by H). In particular, if \( \lambda \) has a polynomial growth in \( u \), i.e., \( \lambda(u) \sim (\lambda_0 + |u|)^k \) for some \( k \neq 0 \) and \( \lambda_0 \geq 0 \), then H) reduced to the simple condition \( |\mathbb{K}(u)| \leq |u| \).
4 Proof of The Main Theorem

In this section, we prove Theorem 2.1. We consider the following system

\[
\begin{align*}
&u_t - \text{div}(A(x, u) Du) = \hat{f}(x, u, Du), \quad x \in \Omega, \\
&u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{align*}
\]  

(4.1)

We imbed this system in the following family of systems

\[
\begin{align*}
&u_t - \text{div}(A(x, \sigma u) Du) = \hat{f}(x, \sigma u, \sigma Du), \quad x \in \Omega, \sigma \in [0, 1], \\
&u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{align*}
\]  

(4.2)

The proof of Theorem 2.1, which asserts the existence of strong solutions \(u\) to (4.1), relies on the Leray Schauder fixed point index theorem. Such a strong solution \(u\) of (4.1) is a fixed point of a nonlinear map defined on an appropriate Banach space \(X\). The proof will be based on several lemmas and we will sketch the main steps below.

We will show in Lemma 4.5 that there exist \(p > n/2\) and a constant \(M^*\) depending only on the constants in A) and F) such that any strong solution \(u\) of (4.2) will satisfy

\[
\sup_{\tau \in (0, T_0)} \|Du\|_{L^2(\Omega, \mu)} \leq M^* \|u_t\|_{L^q(0)} \leq M^*.
\]  

(4.3)

We will show that there are positive constants \(\alpha, M_0\) such that

\[
\|u\|_{C^{\alpha, \alpha/2}(Q)} \leq M_0.
\]  

(4.4)

Following [10], for some \(q, r \geq 1\) we denote by \(V_{q,r}(Q)\) the Banach space of vector valued functions on \(Q\) with finite norm

\[
\|u\|_{V_{q,r}(Q)} = \sup_{t \in (0, T_0)} \|u(\cdot, t)\|_{L^2(\Omega)} + \|Du\|_{q,r,Q},
\]

where

\[
\|v\|_{q,r,Q} := \left( \int_0^{T_0} \left( \int_\Omega |v(x, t)|^q \, dx \right)^{\frac{r}{q}} \, dt \right)^{\frac{1}{r}}.
\]

For \(\sigma \in [0, 1]\) and any \(u \in \mathbb{R}^m\) and \(\zeta \in \mathbb{R}^{mn}\) we define the vector valued functions \(F^{(\sigma)}\) and \(f^{(\sigma)}\) by

\[
F^{(\sigma)}(x, u, \zeta) := \int_0^1 \partial_\zeta F(\sigma, u, t\zeta) \, dt, \quad f^{(\sigma)}(x, u) := \int_0^1 \partial_u F(\sigma, x, tu, 0) \, dt.
\]  

(4.5)

For any given \(u, w \in V_{q,r}(Q)\) we write

\[
\hat{f}(\sigma, x, u, w) = F^{(\sigma)}(x, u, Du) Dw + f^{(\sigma)}(x, u) w + \hat{f}(x, 0, 0).
\]  

(4.6)

We will define a suitable Banach space \(X\) and for each \(u \in X\) we consider the following linear systems, noting that \(\hat{f}(\sigma, x, u, w)\) is linear in \(w, Dw\)

\[
\begin{align*}
&w_t - \text{div}(A(x, \sigma u) Dw) = \hat{f}(\sigma, x, u, w) \quad (x, t) \in \Omega \times (0, T_0), \\
&w = 0 \text{ or } \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial \Omega \times (0, T_0), \\
&w(x, 0) = U_0(x) \text{ on } \Omega.
\end{align*}
\]  

(4.7)
We will show that the above system has a unique weak solution \( w \) if \( u \) satisfies (4.2). We then define \( T_\sigma(u) = w \) and apply the Leray-Schauder fixed point theorem to establish the existence of a fixed point of \( T_1 \). It is clear from (4.6) that \( \hat{f}(x,\sigma u,\sigma Du) = \hat{f}(\sigma,x,u,u) \). Therefore, from the definition of \( T_\sigma \) we see that a fixed point of \( T_\sigma \) is a weak solution of (4.2).

By an appropriate choice of \( X \), we will show that these fixed points are strong solutions of (4.2), and so a fixed point of \( T_1 \) is a strong solution of (4.1).

From the proof of Leray-Schauder fixed point theorem in \([5, Theorem 11.3]\), we need to find some ball \( B_M \) of radius \( M \) and centered at 0 of \( X \) such that \( T_\sigma : B_M \to X \) is compact and that \( T_\sigma \) has no fixed point on the boundary of \( B_M \). The topological degree \( \text{ind}(T_\sigma,B_M) \) is then well defined and invariant by homotopy so that \( \text{ind}(T_1,B_M) = \text{ind}(T_0,B_M) \). It is easy to see that the latter is nonzero because the linear system

\[
\left\{ \begin{array}{l}
    u_t - \text{div}(A(x,0)Du) = \hat{f}(x,0,0) \quad x \in \Omega \times (0,T_0), \\
    u = 0 \quad \text{or} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T_0), \\
    u(x,0) = U_0(x) \quad \text{on} \quad \Omega,
\end{array} \right.
\]

has a unique solution in \( B_M \). Hence, \( T_1 \) has a fixed point in \( B_M \).

Therefore, the theorem is proved as we will establish the following claims.

**Claim 1** There exist a Banach space \( X \) and \( M > 0 \) such that the map \( T_\sigma : \overline{B}_M \to X \) is well defined and compact.

**Claim 2** \( T_\sigma \) has no fixed point on the boundary of \( B_M \). That is, \( \|u\|_X < M \) for any fixed points of \( u = T_\sigma(u) \).

The following lemma defines the space \( X \), the map \( T_\sigma \) and establishes Claim 1.

**Lemma 4.1** Suppose that there exist \( p > n/2 \), \( q_0 > 1 \) and a constant \( M_* \) such that any strong solution \( u \) of (4.2) satisfies

\[
\sup_{\tau \in (0,T_0)} \|Du\|_{W^{1,2p}(\Omega,\mu)} \leq M_* \quad \text{and} \quad \|u_t\|_{L^{q_0}(Q)} \leq M_*.
\]

Then, there exist \( M,\beta > 0 \) and \( q,r \geq 1 \) such that for \( X = C^{\beta,\beta/2}(Q,\mathbb{R}^m) \cap V_q,r(Q) \) the map \( T_\sigma : \overline{B}_M \to X \) is well defined and compact for all \( \sigma \in [0,1] \). Moreover, \( T_\sigma \) has no fixed points on \( \partial B_M \).

**Proof:** For some constant \( M_0 > 0 \) we consider \( u : Q \to \mathbb{R}^m \) satisfying

\[
\sup_{\tau \in (0,T_0)} \|u\|_{C(\Omega)} \leq M_0 \quad \text{and} \quad \int_Q |Du|^2 d\mu d\tau \leq M_0,
\]

and write the system (4.7) as a linear parabolic system for \( w \)

\[
w_t = \text{div}(a(u)Dw) + b(u)Dw + g(u)w + f,
\]

where \( a(x,t) = A(x,\sigma u) \), \( b(x,t) = F^{(\sigma)}(x,u,Du) \), \( g(x,t) = f^{(\sigma)}(x,u) \), and \( f(x) = \hat{f}(x,0,0) \). The matrix \( a(u) \) being regular elliptic with uniform ellipticity constants by A), AR) if \( u \)}
is bounded. We recall the following well known result in \[10\] Chapter VII. If there exist positive constants \(m\) and \(q, r\) such that (see the condition (1.5) in \[10\] Chapter VII)

\[
\|b(u)\|_{q,r,Q}, \|g(u)\|_{q,r,Q}, \|f\|_{q,r,Q} \leq m, \quad 1/r + n/(2q) = 1, \quad q \geq n/2 \quad \text{and} \quad r \geq 1,
\]

(4.11)

then the system (4.10) satisfies the assumptions of Theorem 1.1 in \[10\] Chapter VII which asserts that (4.7) has a unique weak solution \(w\).

Moreover, as the initial condition \(w(\cdot, 0) = U_0(x)\) belongs to \(W^{1,p_0}(\Omega)\) and then \(C^{\beta_0}(\Omega)\) for \(\beta_0 = 1 - n/p_0 > 0\), a combination of Theorems 2.1 and 3.1 in \[10\] Chapter VII shows that \(w\) belongs to \(C^{\alpha_0,\alpha_0/2}(Q, \mathbb{R}^m)\) for some \(\alpha_0 > 0\) depending only on \(\beta_0, \|u\|_\infty\) and \(m\).

Next, we will show that (4.11) holds by F) and (4.9). We consider the two cases f.1) and f.2). If f.1) holds then from the definition (4.5) there is a constant \(C(|u|)\) such that

\[
|b(x, t)| = |F^{(\sigma)}(x, u, \zeta)| \leq C(|u|), \quad |g(x, t)| = |f^{(\sigma)}(x, u)| \leq C(|u|).
\]

From (4.9), we see that \(\sup_{x \in (0, T_0)} \|u\|_\infty \leq M_0\) and so there is a constant \(m\) depending on \(M_0\) such that (4.11) holds for any \(q\) and \(n\).

If f.2) holds then

\[
|F^{(\sigma)}(x, u, \zeta)| \leq C(|u||\zeta|), \quad |f^{(\sigma)}(x, u)| \leq C(|u|).
\]

(4.12)

Therefore, \(\|b\|_{L^2(Q)}\) is bounded by \(C\|Du\|_{L^2(Q)}\). Again, if \(n \leq 3\) then (4.9) implies the condition (4.11) for \(q = 2\).

In both cases, (4.10) (or 4.7) has a unique weak solution \(w\). We then define \(T_\sigma(u) = w\). Moreover, as we explained earlier, \(w \in C^{\alpha_0,\alpha_0/2}(Q)\) for some \(\alpha_0 > 0\) depending on \(M_0\).

We now consider a fixed point \(u\) of \(T_\sigma\). By Lemma 4.2 following this proof we see that \(u\) is a strong solution and we can use the assumption (4.8). The first bound in the assumption (4.8) implies \(u\) is Hölder continuous in \(x\). This and the integrability of \(u_t\) in the second bound of the assumption and \([17\) Lemma 4] provide positive constants \(\alpha, M_1\) such that any strong solution \(u\) of (4.2) satisfies \(\|u\|_{C^{\alpha,\alpha/2}(\Omega)} \leq M_1\). Also, the assumption AR) implies that \(\lambda(u), \omega\) are bounded from below, yield that \(\|Du\|_{L^2(Q)} \leq C(C_0)\). Thus, there is a constant \(M_1\), depending on \(M_0, C_0\) such that any strong solution \(u\) of (4.2) satisfies

\[
\|u\|_{C^{\alpha,\alpha/2}(\Omega)} \leq M_1, \quad \|Du\|_{L^2(Q)} \leq M_1.
\]

(4.13)

It is well known that there is a constant \(c_0 > 1\), depending on \(\alpha, T_0\) and the diameter of \(\Omega\), such that \(\|u\|_{C^{\beta,\beta/2}(\Omega)} \leq c_0 \|u\|_{C^{\alpha,\alpha/2}(\Omega)}\) for all \(\beta \in (0, \alpha)\). We now let \(M_0\), the constant in (4.9), be \(M = (c_0 + 1)M_1\).

Define \(X = C^{\beta,\beta/2}(Q) \cap V^{1,0}(Q)\) for some positive \(\beta < \min\{\alpha, \alpha_0\}\), where \(\alpha_0\) is the Hölder continuity exponent for solutions of (4.10), and

\[
V^{1,0}(Q) := \{u : Du \in L^2(Q)\}.
\]

The space \(X\) is equipped with the norm \(\|u\|_X = \max\{\|u\|_{C^{\beta,\beta/2}(\Omega)}, \|Du\|_{L^2(Q)}\}\) and consider the ball \(B_M\) in \(X\) centered at 0 with radius \(M\).

We now see that \(T_\sigma\) is well defined and maps the ball \(\bar{B}_M\) of \(X\) into \(X\). Moreover, from the definition \(M = (c_0 + 1)M_1\), it is clear that \(T_\sigma\) has no fixed point on the boundary of \(B_M\) because such a fixed points \(u\) satisfies (4.13) which implies \(\|u\|_X \leq c_0 M_1 < M\).
Finally, we need only show that $T_\sigma$ is compact. If $u$ belongs to a bounded set $K$ of $B_M$ then $\|u\|_{\bf X} \leq C(K)$ for some constant $C(K)$ and there is a constant $C_1(K)$ such that $\|T_\sigma(u)\|_{C^{\alpha_0-0.2}(Q)} = \|w\|_{C^{\alpha_0-0.2}(Q)} \leq C_1(K)$. Thus $T_\sigma(K)$ is compact in $C^{3,3/2}(Q)$ because $\beta < \alpha_0$. So, we need only show that $T(K)$ is precompact in $V^{1,0}(Q)$. We will discuss only the quadratic growth case where (4.12) holds because the case $f$ has linear growth is similar and easier.

First of all, for $u \in K$ we easily see that $\|Dw\|_{L^2(Q)}$ is uniformly bounded by a constant depending on $K$. The argument is standard by testing the linear system (4.7) by $w$ and using the boundedness of $\|w\|_{L^\infty}$ and $\|u\|_{L^\infty}$, (4.12), AR and Young’s inequality.

Let $\{u_n\}$ be a sequence in $K$ and $w_n = T_\sigma(u_n)$. We have, writing $W = w_n - w_m$

$$W_t - \text{div}(A(x, \sigma u_n)DW) = \text{div}(\alpha_{m,n}Dw_m) + \Psi_{m,n},$$

where $\alpha_{m,n} = (A(x, \sigma u_n) - A(x, \sigma u_m)$ and $\Psi_{m,n}$ is defined by

$$F^\sigma(x, u_n, Du_m)Dw_n - F^\sigma(x, u_m, Du_m)Dw_m + f^\sigma(x, u_n)u_n - f^\sigma(x, u_m)u_m.$$

Testing the above system with $W$ and using AR) and the fact that $W(x,0) = 0$, we have for $dz = dxdt$

$$\lambda_s(K)\mu_s \iint_Q |DW|^2 \, dz \leq \iint_Q \left[\|\alpha_{m,n}\|Dw_m\|DW\| + \|\Psi_{m,n}\|W\right] \, dz.$$

By Young’s inequality, we find a constant $C$ depending on $K$ and $\mu_s$ such that

$$\iint_Q |DW|^2 \, dz \leq C \iint_Q \left[\|\alpha_{m,n}\|Dw_m\|^2 \, dz + \sup_Q |W|\|\Psi_{m,n}\|_{L^1(Q)}\right].$$

By (4.12), it is clear that $|\Psi_{m,n}| \leq C(K)[(|Du_n| + |Du_m|)(|Dw_n| + |Dw_m|) + 1]$. Using the fact that $|Dw_n|_{L^2(Q)}$ and $|Du_n|_{L^2(Q)}$ are uniformly bounded, we see that $\|\Psi_{m,n}\|_{L^1(Q)}$ is bounded. Hence,

$$\iint_Q |Dw_n - Dw_m|^2 \, dz \leq C(K)\max\left\{\sup_{\Omega} |A(x, \sigma u_n) - A(x, \sigma u_m)|, \sup_{\Omega} |w_n - w_m|\right\}.$$

Since $u_n, w_n$ are bounded in $C^{3,3/2}(Q)$, passing to subsequences we can assume that $u_n, w_n$ converge in $C^0(Q)$. Thus, $\|A(x, \sigma u_n) - A(x, \sigma u_m)\|_{L^\infty}, \|w_n - w_m\|_{L^\infty} \to 0$. We then see from the above estimate that $Dw_n$ converges in $L^2(\Omega)$. Thus, $T_\sigma(K)$ is precompact in $V^{1,0}(Q)$.

Hence, $T_\sigma : {\bf X} \to {\bf X}$ is a compact map. The proof is complete. 

We now turn to Claim 2, the hardest part of the proof, and provide a uniform estimate for the fixed points of $T_\sigma$ and justify the key assumption (4.8) of Lemma 4.1. The proof is complicated and will be devided into many lemmas described as follows.

- Lemma 4.2 is quite standard and shows that the fixed points of $T_\sigma$ are strong solutions.
- Lemma 4.3 follows [13] Lemma 3.2 and establishes an energy estimate of $Du$. In Lemma 4.4 the assumptions H) and M.1) then allow us to apply the local Gagliardo-Nirenberg inequality (3.14) to obtain a better estimate.
Lemma 4.2 and Lemma 4.6 then show that the estimate in Lemma 4.4 is self-improving to obtain the key estimate (4.8).

Hence, we first have the following lemma.

Lemma 4.2 A fixed point of $T_\sigma$ is also a strong solution of (4.2).

Proof: If $u$ is a fixed point of $T_\sigma$ in $X$ then it solves (4.2) weakly and is continuous. Thus, $u$ is bounded and belongs to $VMO(Q)$. By AR), the system (4.2) is regular elliptic. We can adapt the proof in [8]. If $\hat{f}$ satisfies a quadratic growth in $Du$ then, because $u$ is bounded, the condition (S) (0.4) that $|\hat{f}| \leq a|Du|^2 + b$ is satisfied here. The proof of (S) Theorems 2.1 and 3.2] assumed the 'smallness condition' (see [S] (0.6)) $2aM < \lambda_0$, where $M = \sup |u|$. This 'smallness condition' was needed because only weak bounded solutions, which are not necessarily continuous, were considered in [S]. In our case, $u$ is continuous so that we do not require this 'smallness condition'. Indeed, a careful checking of the arguments of the proof in [S] Lemma 2.1 and page 445] shows that if $R$ is small and one knows that the solution $u$ is continuous then these arguments still hold as long we can absorb the integrals involving $|Du|^2, |Dw|^2$ (see the estimate after [S] (3.7)) on the right hand side of the estimates. Thus, (S) Theorems 2.1 and 3.2 apply to our case and yield that $u \in C^{a,a/2}(Q)$ for all $a \in (0,1)$ and that, since $A(x,u)$ is differentiable, $Du$ is locally H"older continuous in $Q$. Therefore, $u$ is also a strong solution.

Thanks to Lemma 4.2 we need only consider a strong solution $u$ of (4.2) and establish (4.8) for some $p > n/2$. Because the data of (4.7) satisfy the structural conditions A), F) with the same set of constants and the assumptions of the theorem are assumed to be uniform for all $\sigma \in [0,1]$, we will only present the proof for the case $\sigma = 1$ in the sequel.

Let $u$ be a strong solution of (1.1) on $\Omega$. We begin with an energy estimate for $Du$. For $p \geq 1$ and any ball $B_s$ with center $x_0 \in \Omega$ we denote $\Omega_s = B_s \cap \Omega$, $Q_s = \Omega_s \times (0,T_0)$ and

$$A_p(s) = \sup_{\tau \in (0,T_0)} \int_{\Omega_s \times \tau} |Du|^{2p} \, dx, \quad (4.14)$$

$$H_p(s) := \int\int_{Q_s} \lambda(u)|Du|^{2p-2}|D^2u|^2 \, d\mu d\tau, \quad (4.15)$$

$$B_p(s) := \int\int_{Q_s} \frac{\lambda(u)^2}{\lambda(u)} |Du|^{2p+2} \, d\mu d\tau, \quad (4.16)$$

$$C_p(s) := \int\int_{Q_s} \left((|f(u)| + \lambda(u))|Du|^{2p} \right) \, d\mu d\tau, \quad (4.17)$$

and

$$F_{\omega,p}(s) := \int\int_{Q_s} \left(\lambda(u)|Du|^{2p}|D\omega_0|^2 + |f(u)||Du|^{2p-1}|D\omega_0|\omega_0 \right) \, d\mu d\tau. \quad (4.18)$$

The following lemma establishes an energy estimate for $Du$.

Lemma 4.3 Assume A), F). Let $u$ be any strong solution of (1.1) on $\Omega$ and $p$ be any number in $[1,\max\{1,n/2\}]$. 

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There is a constant $C$, which depends only on the parameters in $A$) and $F$), such that for any two concentric balls $B_s, B_t$ with center $x_0 \in \Omega$ and $s < t$

$$\mathcal{A}_p(s) + \mathcal{H}_p(s) \leq CB_p(t) + C(1 + (t - s)^{-2})[\mathcal{C}_p(t) + F_{\omega_p}(t)] + \|DU_0\|_{L^1(\Omega_t)}^{2p}.$$  \hspace{1cm} (4.19)

**Proof:** The proof is similar to the energy estimate of $Du$ for the parabolic case in [13] Lemma 3.2. Roughly speaking, we differentiated the system in $x$ to obtain

$$(Du)_t - \text{div}(A(x,u)D^2u + A_u(x,u)DuDu + A_x(x,u)Du) = D\hat{f}(x,u,Du).$$  \hspace{1cm} (4.20)

For any two concentric balls $B_s, B_t$, with $s < t$, let $\psi$ be a cutoff function for $B_s, B_t$. That is, $\psi$ is a $C^1$ function satisfying $\psi \equiv 1$ in $B_s$ and $\psi \equiv 0$ outside $B_t$ and $|D\psi| \leq 1/(t-s)$. Consider any given triple $t_0, T, T'$ satisfying $0 < t_0 < T < T' < T_0$ and $\eta$ being a cutoff function for $(T-t_0, T'), (T, T')$. We then test \((4.20)\) with $|Du|^{2p-2}Du\psi^2\eta$ and obtain, using integration by parts and Young's inequality

$$\sup_{t \in (T,t')} \int_{\Omega_s} |Du|^{2p} dx + \int_{T-t_0}^{T'} \int_{\Omega_t} \lambda(u)|Du|^{2p-2}|D^2u|^{2p} \psi^2 \eta dxd\tau \leq$$

$$C \int_{T-t_0}^{T'} \int_{\Omega_t} \frac{|\lambda(u)|^2}{\lambda(u)} |Du|^{2p+2} + |D\psi|^2 \lambda(u)|Du|^{2p} |\eta| dxd\tau$$

$$+ C \int_{T-t_0}^{T'} \int_{\Omega_t} |A(x,u)||Du|^{2p-1}|D^2u|^{2p} \psi^2 + |D\hat{f}(x,u,Du)||Du|^{2p-1} \psi^2 dxd\tau$$

$$+ C t_0^{-1} \int_{T-t_0}^{T'} \int_{\Omega_t} |Du|^{2p} dxd\tau.$$  \hspace{1cm} (4.21)

Here, integrals in the first line of \((4.21)\) result from the same argument in the proof of [13] Lemma 3.2 using the spectral gap condition SG) we are assuming here (see also [15], Lemma 6.5]). The integrals in the second and third lines can be estimated by simple uses of Young's inequality and the condition $F$) as in [13] [15]. Finally, we formally let $T, t_0 \to 0$ in the last integral, which will be justified below, to obtain \((4.19)\).

Using the difference quotient operator $\delta_h$ instead of $D$ in \((4.20)\), we obtain

$$(\delta_hu)_t = \text{div}(A(x,u)D(\delta_hu) + \delta_h(A(x,u))Du) + \delta_h\hat{f}(x,u,Du).$$  \hspace{1cm} (4.22)

We test this with $|\delta_hu|^{2p-2}\delta_hu\psi^2\eta$ to obtain a similar version of \((4.21)\) with the operator $D$ being replaced by $\delta_h$. We can integrate the result over $(0,T_0)$ and obtain

$$\sup_{t \in (0,T_0)} \int_{\Omega_s} |\delta_hu|^{2p} dx + \int_{Q_s} \lambda(u)|\delta_hu|^{2p-2}|D\delta_hu|^{2} dz \leq$$

$$C \int_{Q_t} \frac{|\lambda(u)|^2}{\lambda(u)} |Du|^{2p} |\delta_hu|^{2p} + |D\psi|^2 \lambda(u)|\delta_hu|^{2p} dz + \cdots + C \int_{\Omega_t} |\delta_hu(x,0)|^{2p} dx.$$  \hspace{1cm} (4.19)

Since $u \in C([0,T'], L^{2p}(\Omega))$, we can let $h$ tend to 0 and obtain a similar energy estimate \((4.19)\) for $Du$ with $T = t_0 = 0$ and $\eta \equiv 1$. We complete the proof. \[ \square \]

Next, under the condition AR), the density $\omega$ supports the Poincaré-Sobolev inequality with $\pi_\omega = 2n/(n-2)$. By Remark 3.3, we can apply the local Gagliardo-Nirenberg inequality \((3.14)\) here. Thus, if the condition \((2.8)\) of M.1) holds then we combine the energy estimate and \((3.14)\) to have the following stronger estimate.
Lemma 4.4 In addition to the assumptions of Lemma 4.3, we suppose that $H$ and $M.1$ hold for some $p$. That is, for any given $\mu_0 > 0$ there exist a constant $C_0$ and a positive $R_{\mu_0}$ sufficiently small in terms of the constants in $A)$ and $F$) such that

\[
\sup_{x_0 \in \Omega, \tau \in (0, T_0)} [W_p^{\alpha}]_{\beta+1, \Omega_R(x_0)} \leq C_0, \quad \sup_{x_0 \in \Omega, \tau \in (0, T_0)} \|K(u)\|_{BMO(\Omega_R(x_0), \mu)}^2 \leq \mu_0. \tag{4.23}
\]

Then for sufficiently small $\mu_0$ there is a constant $C$ depending only on the parameters of $A)$ and $F$) such that for $2R < R_{\mu_0}$ we have

\[
\mathcal{A}_p(R) + \mathcal{B}_p(R) + \mathcal{H}_p(R) \leq C(1 + R^{-2})[\mathcal{C}_p(2R) + \mathcal{F}_{\omega,p}(2R)] + \|DU_0\|_{L^1(\Omega_2R)}. \tag{4.24}
\]

**Proof:** Recall the energy estimate (4.19) in Lemma 4.3.

\[
\mathcal{A}_p(s) + \mathcal{H}_p(s) \leq C\mathcal{B}_p(t) + C(1 + (t-s)^{-2})[\mathcal{C}_p(t) + \mathcal{F}_{\omega,p}(t)] + \|DU_0\|_{L^1(\Omega_t)}^2, \quad 0 < s < t. \tag{4.25}
\]

We apply Corollary 3.2 to estimate $\mathcal{B}_p(t)$, the integral on the right hand side of (4.25). We let $\Lambda(u) = \lambda^\pm(u)\omega.p$ in Corollary 3.2 and note that $W_p$ defined there is now comparable to the $W_p = \lambda^{p+\frac{1}{2}}(u)|\lambda_p(u)|^p$ in M.1). We compare the definitions (3.12) and (3.13) with those in (4.15)-(4.17) to see that for $U(x) = u(x, \tau)$ with $\tau \in (0, T_0)$

\[
\mathcal{B}_p(t) = \int_0^T I_1(t, x_0) d\tau, \quad \mathcal{C}_p(t) = \int_0^T I_0(t, x_0) d\tau, \quad \mathcal{H}_p(t) = \int_0^T I_2(t, x_0) d\tau.
\]

Hence, for any $\varepsilon > 0$ we can use (3.14) obtain a constant $C$ such that (using the bound $[W_p^{\alpha}]_{\beta+1, \Omega_R(x_0) \cap \Omega} \leq C_0$ and the definitions of $\mu_0$ in (4.23) and $C(\varepsilon, U, \mathbf{W})$ in Corollary 3.2)

\[
I_1(s, x_0) \leq C_(\varepsilon, U, \mathbf{W})[I_1(t, x_0) + I_2(t, x_0) + (t-s)^{-2}I_0(t, x_0)].
\]

Integrating the above over $(0, T_0)$ to get

\[
\mathcal{B}_p(s) \leq \varepsilon \mathcal{B}_p(t) + C \varepsilon^{-1} \mu_0 \mathcal{H}_p(t) + C \varepsilon^{-1} \mu_0 (t-s)^{-2} \mathcal{C}_p(t) \quad 0 < s < t \leq R_{\mu_0}.
\]

Define $F(t) := \mathcal{B}_p(t)$, $G(t) := \mathcal{H}_p(t)$, $g(t) := \mathcal{C}_p(t)$ and $\varepsilon_0 = \varepsilon + C \varepsilon^{-1} \mu_0$. The above yields

\[
F(s) \leq \varepsilon_0 [F(t) + G(t)] + C(t-s)^{-2} g(t). \tag{4.26}
\]

Now, for $h(t) := \mathcal{F}_{\omega,p}(t) + \|DU_0\|_{L^1(\Omega_t)}^2$ the energy estimate (4.25) implies

\[
G(s) \leq C [F(t) + (1 + (t-s)^{-2}) (g(t) + h(t))]. \tag{4.27}
\]

As $\varepsilon_0 = \varepsilon + C \varepsilon^{-1} \mu_0$, it is clear that we can choose and fix some $\varepsilon$ sufficiently small and then $\mu_0$ small in terms of $C, \varepsilon$ to have $2C \varepsilon_0 < 1$. Thus, if $\mu_0$ is sufficiently small in terms of the constants in $A), F)$, then we can apply a simple iteration argument [13] Lemma 3.11] to the two inequalities (4.26) and (4.27) and obtain for $0 < s < t \leq R_{\mu_0}$

\[
F(s) + G(s) \leq C(1 + (t-s)^{-2}) [g(t) + h(t)].
\]
For any \( R < R_{\mu_0}/2 \) we take \( t = 2R \) and \( s = \frac{3}{2}R \) in the above to obtain
\[
\mathcal{B}_p(\frac{3}{2}R) + \mathcal{H}_p(\frac{3}{2}R) \leq C(1 + R^{-2})|C_p(2R)| + \mathcal{F}_{\omega,p}(2R) + \|DU_0\|^2_{L^1(\Omega)}.
\]
Combining this and (4.25) with \( s = R \) and \( t = \frac{3}{2}R \), we see that
\[
A_p(R) + B_p(R) + H_p(R) \leq C(1 + R^{-2})|C_p(2R) + \mathcal{F}_{\omega,p}(2R) + \|DU_0\|^2_{L^1(\Omega)}.
\]
This is (4.24) and the proof is complete. \( \blacksquare \)

Finally, we have the following lemma giving a uniform bound for strong solutions.

**Lemma 4.5** Assume as in Lemma 4.4 and AR. We assume also the integrability condition \( M.0 \). Then there exist \( p > n/2, q_0 > 1 \) and a constant \( M_* \) depending only on the parameters of \( A \) and \( F \), \( \mu_0, R_{\mu_0}, C_0 \) and the geometry of \( \Omega \) such that
\[
\sup_{\tau \in (0, T_0)} \int_{\Omega} |Du(\cdot, \tau)|^{2p} \, d\mu \leq M_*, \tag{4.28}
\]
\[
\|u_t\|_{L^{q_0}(Q)} \leq M_* \tag{4.29}
\]

**Proof:** First of all, by the condition AR, there is a constant \( C_\omega \) such that \( |D\omega_0| \leq C_\omega \omega_0 \) and therefore we have from the the definition (4.13) that
\[
\mathcal{F}_{\omega,p}(s) \leq C_\omega \int_{Q_s} (\lambda(u)|Du|^{2p} + f(u)|Du|^{2p-1})\omega_\tau^2 \, d\mu d\tau.
\]
By Young’s inequality, \( f(u)|Du|^{2p-1} \lesssim |f_u(u)||Du|^{2p} + (f(u)|f_u(u)|^{-1})^{2p}|f_u(u)| \). It follows from the assumption (4.7) that \( (f(u)|f_u(u)|^{-1})^{2p}f_u(u) \lesssim (|u| + 1)^{2p}f_u(u) \). We then have from (4.24) that
\[
A_p(R) + B_p(R) + H_p(R) \leq C(1 + R^{-2})|C_p(2R) + \mathcal{F}_{\omega,p}(2R) + C_0|, \tag{4.30}
\]
\[
\mathcal{F}_{\omega,p}(s) := \int_{Q_s} |u|^{2p}|f_u(u)| \, d\mu d\tau. \tag{4.31}
\]

The main idea of the proof is to show that (4.30) is self-improving in the sense that if it is true for some exponent \( p \geq 1 \) then it is also true for \( \gamma_* p \) with some fixed \( \gamma_* > 1 \) and \( R \) being replaced by \( R/2 \). To this end, assume that for some \( p \geq 1 \) we can find a constant \( C(C_0, R, p) \) such that
\[
C_p(2R) + \mathcal{F}_{\omega,p}(2R) \leq C(C_0, R, p), \tag{4.32}
\]
which and (4.30) and the definitions of \( B_p(R), H_p(R), C_p(R) \) yield that
\[
A_p(R) + \int_{Q_R} [\lambda(u)|Du|^{2p} + \lambda(u)|Du|^{2p-2}|D^2u|^2 + \Phi^2(u)|Du|^{2p+2}] \, d\mu d\tau \leq C(C_0, R, p),
\]
where \( \Phi(u) = |(\sqrt[\gamma_\ast]{u}(u))_\alpha| \). The above two estimates yield for \( V = \lambda^{\frac{1}{\gamma_\ast}}(u)|Du|^p \)
\[
\sup_{\tau \in (0, T_0)} \int_{\Omega_R} |Du|^{2p} \, dx + \int_{Q_R} [V^2 + |DV|^2] \, d\mu d\tau \leq C(C_0, R, p). \tag{4.33}
\]
In the technical Lemma 4.6 following this proof, we will show that if M.0) and (4.32) hold for some \( p \geq 1 \) then together with its consequence (4.33) provide some \( \gamma_* > 1 \) such that (4.32) holds again for the new exponent \( \gamma_* p \) and \( R/2 \).

By the assumption (2.7), (4.32) holds for \( p = 1 \). It is now clear that, as long as the energy estimate (4.19) is valid by Lemma 4.3, we can repeat the argument \( k_0 \) times to find a number \( p > n/2 \) such that (4.32) and then its consequence (4.33) hold. It follows that there is a constant \( C \) depending only on the parameters of \( A \) and \( F \), \( \mu_0 \), \( R \mu_0 \) and \( k_0 \) such that for some \( p > n/2 \) we obtain from (4.33) that

\[
\sup_{(0,T_0)} \int_{\Omega_{R_0}} |Du|^{2p} \, d\mu \leq C \text{ for } R_0 = 2^{-k_0} R_{\mu_0}. \tag{4.34}
\]

Summing the above inequalities over a finite covering of balls \( B_{R_0} \) for \( \Omega \), we find a constant \( C \), depending also on the geometry of \( \Omega \), and obtain the desired estimate (4.28).

Similarly, we obtain from (4.33) with \( p = 1 \) that

\[
\iint_Q \lambda(u)|D^2u|^2 \, d\mu d\tau \leq C. \tag{4.35}
\]

As \( u \) is a strong solution, we have \( |u_t| \leq |\text{div}(A(x,u)Du)| + |\tilde{f}| \) a.e. in \( Q \). Therefore,

\[
\|u_t\|_{L^{q_0}(Q)} \lesssim \|\lambda(u)|D^2u|\|_{L^{q_0}(Q)} + \|\lambda_\mu(u)|Du|^2\|_{L^{q_0}(Q)} + \|f(u)\|_{L^{q_0}(Q)}.
\]

If \( q_0 \in (1, 2) \) then the first and third norms on the right can be treated by Hölder’s inequality and (4.35) and the boundedness of \( u \), thanks to (4.28). For \( q_0 = p > n/2 \geq 1 \), the second norm is also bounded by (4.28). Thus, there is \( q_0 > 1 \) such that (4.29) holds.

The lemma is proved. \( \blacksquare \)

Thus, we need to show that (4.32) is self improving in the following lemma.

**Lemma 4.6** Assume as in Lemma 4.5. Suppose that for some \( p \geq 1 \) we can find a constant \( C(C_0, R, p) \) such that

\[
C_p(2R) + F_{*,\omega, p}(2R) \leq C(C_0, R, p), \tag{4.36}
\]

then there exists a fixed \( \gamma_* > 1 \) such that

\[
C_{\gamma*, p}(R) + F_{*,\omega, \gamma_* p}(R) \leq C(C_0, R, p). \tag{4.37}
\]

In the sequel, we will repeatedly make use of the following parabolic Sobolev inequality

\[
\iint_{Q_R} v^{2q_*}|V|^2 \, d\mu d\tau \lesssim \sup_t \left( \int_{\Omega_R} v^2 \, d\mu \right)^{q_*} \int_{Q_R} \|DV|^2 + V^2 \| \, d\mu d\tau, \quad q_* = 1 - \frac{2}{\pi \ast}. \tag{4.38}
\]

To see this, we recall the inequality

\[
\left( \int_\Omega |V|^2 \, d\mu \right)^{\frac{1}{2q_*}} \lesssim \left( \int_\Omega |DV|^2 \, d\mu \right)^{\frac{1}{2}} + \left( \int_\Omega |V|^2 \, d\mu \right)^{\frac{1}{2}}.
\]

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which is just a simple consequence of the Poincaré–Sobolev inequality PS). For \( q_* = (1 - \frac{2}{\alpha_*}) \) we use Hölder’s inequality and the above inequality to have

\[
\iint_{\Omega \times I} v^{2p} |V|^2 \, d\mu d\tau \leq \int_I \left( \int_\Omega v^2 \, d\mu \right)^{1 - \frac{2}{\alpha_*}} \left( \int_\Omega |V|^{\alpha_*} \, d\mu \right)^{\frac{2}{\alpha_*}} d\tau 
\]

\[
\lesssim \sup_I \left( \int_\Omega v^2 \, d\mu \right)^{q_*} \int_I \left( \int_\Omega |DV|^2 \, d\mu + \int_\Omega |V|^2 \, d\mu \right) d\tau.
\]

This is (4.38).

**Proof of Lemma 4.6.** We recall the integrability condition M.0). Namely, there exists \( C_0 \) and \( r_0 > 1, \beta_0 \in (0, 1) \) such that

\[
\sup_{\tau \in (0, T_0)} \| f_u(u)\lambda^{-1}(u) \|_{L^0(\Omega, \mu)}, \sup_{\tau \in (0, T_0)} \| u^{\beta_0} \|_{L^1(\Omega, \mu)} \leq C_0, \quad (4.39)
\]

\[
\iint_Q (|f_u(u)| + \lambda(u))(|Du|^2 + |u|^2) \, d\mu d\tau \leq C_0, \quad (4.40)
\]

We established in the proof of Lemma 4.5 that for \( V = \lambda^\frac{1}{2}(u)|Du|^p \) [4.36] yields (4.30), which and the fact that \( \omega \) is bounded from above imply

\[
\sup_{\tau \in (0, T_0)} \int_{\Omega_R} |Du|^{2p} \, d\mu + \iint_{Q_R} [V^2 + |DV|^2] \, dz \leq C(C_0, R, p). \quad (4.41)
\]

Let \( \gamma_1 = 1 + q_* \). We write \( \lambda(u)|Du|^{\gamma_2 p} = v^{2q_*} V^2 \) with \( v = |Du|^p, V = \lambda^\frac{1}{2}(u)|Du|^p \) and apply (4.38) to get

\[
\iint_{Q_R} \lambda(u)|Du|^{\gamma_2 p} \, d\mu d\tau \lesssim \sup_{(0, T_0)} \left( \int_{\Omega_R} |Du|^{2p} \, d\mu \right)^{q_*} \iint_{Q_R} [|DV|^2 + V^2] \, d\mu d\tau.
\]

Therefore, (4.41) implies

\[
\iint_{Q_R} \lambda(u)|Du|^{\gamma_2 p} \, d\mu d\tau \leq C(C_0, R, p).
\]

Similarly, we write \( |f_u(u)||Du|^{\gamma_2 p} = v^{2q_*} V^2 \) with \( v = (|f_u(u)|\lambda^{-1}(u)|Du|^{2p(\gamma_2 - 1)})^\frac{1}{2q_*} \) and \( V = \lambda^\frac{1}{2}(u)|Du|^p \). In order to apply (4.38) here, we need to estimate the integral of \( v^2 \) over \( \Omega_R \). Assuming \( \gamma_2 \in (1, q_*) \) and using Hölder’s inequality with the exponent \( q_1 = \frac{q_*}{q_* - \gamma_2 + 1} \), the integral of \( v^2 = (|f_u(u)|\lambda^{-1}(u)|Du|^{2p(\gamma_2 - 1)})^\frac{1}{2q_*} \) is bounded by

\[
\left( \int_{\Omega_R} (|f_u(u)|\lambda^{-1}(u))^q \, d\mu \right)^\frac{1}{q_1} \left( \int_{\Omega_R} |Du|^{2p} \, d\mu \right)^\frac{1}{q_*}
\]

We can find \( \gamma_2 \) close to 1 such that \( q_1 \leq r_0 \), which is greater than 1, so that the first integral is bounded by the assumption (4.39). The second integral is bounded because of (4.41).

We now turn to \( F_{*,\omega,p}(s) \) defined by (4.31) and write

\[
I_p(s) := F_{*,\omega,p}(s) = \iint_{\Omega_s} |f_u(u)||u|^{2p} \, d\mu d\tau, \quad J_p(s) = \iint_{\Omega_s} \lambda(u)|u|^{2p} \, d\mu d\tau.
\]

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We will prove that $I_p(R), J_p(R)$ are self improving. We assume first that

$$I_p(R), J_p(R) \leq C(C_0, R, p).$$

(4.42)

The argument is very similar to the above treatment of the integral of $|f_u(u)||Du|^{\gamma_2 p}$ with $Du$ being replaced by $|u|$. In fact, the proof for $I_p, J_p$ are almost identical so that we will denote $g(u) = |f_u(u)|$ and consider $I_p$ first. We write $g(u)|u|^{\gamma_2 p} = v^{2q - 1} V^2$ with $v = (g(u)\lambda^{-1}(u)|u|^{2p(\gamma_2 - 1)})^\frac{1}{2q}$ and $V = \lambda^\frac{1}{p}(u)|u|^p$. We use (4.38) to have

$$I_p(R) \leq \sup R \left( \int_{\Omega_R} v^2 \, d\mu \right)^{1 - \frac{q}{2q}} \int_{Q_R} [|DV|^2 + V^2] \, d\mu d\tau. \quad (4.43)$$

The integral of $v^2 = (g(u)\lambda^{-1}(u)|u|^{2p(\gamma_2 - 1)})^\frac{1}{2q}$ over $\Omega_R$ is estimated by Hölder’s inequality as before by

$$\left( \int_{\Omega_R} (g(u)\lambda^{-1}(u)|u|^{2p(\gamma_2 - 1)})^\frac{1}{2q} \, d\mu \right)^{\frac{1}{2q}} \left( \int_{\Omega_R} |u|^{2p} \, d\mu \right)^{\frac{1}{2q}}.$$ 

Again, the first integral is bounded by (4.39) as $g(u) = |f_u(u)|$. We consider the second integral and use Sobolev’s inequality to have

$$\int_{\Omega_R} |u|^{2p} \, d\mu \leq \int_{\Omega_R} |D(|u|^p)|^2 \, d\mu + \left( \int_{\Omega_R} |u|^{p\beta} \, d\mu \right)^\frac{2}{\beta}.$$ 

Because $|D(|u|^p)|^2 \sim |u|^{2p - 2} |Du|^2 \leq \varepsilon |u|^{2p} + C(\varepsilon) |Du|^{2p}$, we conclude that

$$\int_{\Omega_R} |u|^{2p} \, d\mu \leq \int_{\Omega_R} |Du|^{2p} \, d\mu + \left( \int_{\Omega_R} |u|^{p\beta} \, d\mu \right)^{\frac{2}{\beta}}.$$ 

The first integral on the right hand side is bounded by (4.41). Taking $\beta = \beta_0$, the second integral is bounded by the assumption (4.39).

Finally, for the last integral in (4.43) with $V = \lambda^\frac{1}{p}(u)|u|^p$ we use the fact that $|\lambda_u(u)||u| \lesssim \lambda(u)$ and Young’s inequality to see that

$$|DV|^2 \lesssim \lambda(u) |D(|u|^p)|^2 + |\lambda_u(u)|^2 \lambda^{-1}(u) |Du|^2 |u|^{2p} \lesssim \lambda(u) |Du|^{2p} + \lambda(u) |Du|^{2p} + \lambda(u)|u|^{2p} \lesssim \lambda(u) |Du|^{2p} + \lambda(u)|u|^{2p}.$$ 

Therefore, by the assumptions (4.36) and (4.42), the last integral in (4.43) is bounded by a constant $C(C_0, R, p)$. We conclude that $I_{\gamma_2 p}(R) \leq C(C_0, R, p)$. We repeat the argument with $g(u) = \lambda(u)$ to see that $J_p(R)$ is also self improving. In this case $g(u)\lambda^{-1}(u) \in L^\infty(Q)$ so that we can take $\gamma_2$ to be any number in $(1, \gamma_1)$. We let $\gamma_* = \min\{\gamma_1, \gamma_2\}$ and complete the proof of the lemma. ■

**Remark 4.7** It is also important to note that the estimate of Lemma (4.5) based on those in Lemma (3.3) Lemma (4.4) is *independent* of lower/upper bounds of the function $\lambda_\ast$ in AR but the integrals in M.0. The assumption AR) was used only in Lemma (4.1) to define the map $T_\sigma$ and Lemma (4.2) to show that fixed points of $T_\sigma$ are strong solutions.
We are ready to provide the proof of the main theorem of this section.

**Proof of Theorem 2.2.** It is now clear that the assumptions M.0) and M.1) of our theorem allow us to apply Lemma 4.5 and obtain an a priori uniform bound for any continuous strong solution $u$ of (4.2). The uniform estimate (4.28) shows that the assumption (4.8) of Lemma 4.1 holds true so that the map $T_\sigma$ is well defined and compact on a ball $B_M$ of $X$ for some $M$ depending on the bound $M_\sigma$ provided by Lemma 4.3. Combining with Lemma 4.2, the fixed points of $T_\sigma$ are strong solutions of the system (4.1) so that $T_\sigma$ does not have a fixed point on the boundary of $B_M$. Thus, by the Leray-Schauder fixed point theorem, $T_1$ has a fixed point in $B_M$ which is a strong solution to (4.1), which is unique because $u$, $Du$ are bounded and (4.1) is now regular parabolic. The proof is complete. □

## 5 Proof of the theorem on the general SKT system

We conclude this paper by giving the proof of Theorem 2.1: an application of our main theorem to this end, we need only check the conditions A), H) and M.1) because the condition F) is obvious and M.0) is already assumed.

For $C^2$ positive scalar functions $\lambda_i$, $i = 1, \ldots, m$, and $\lambda$ on $\mathbb{R}^m$ we recall the notations in $L$: $L = \text{diag}[\lambda_1(u), \ldots, \lambda_m(u)]$, $L_u = D_u[\lambda_i(u)]_{i=1}^m$, and its assumptions

$$\lambda_i(u) \lesssim \lambda(u), \quad |u| |\lambda_i(u)| \lesssim \lambda(u), \quad |(\lambda_i(u))_{uu}| \lesssim |\lambda_{uu}(u)|. \tag{5.1}$$

$$\langle (L + \text{diag}[u_1, \ldots, u_m]L_u)\zeta, \zeta \rangle \geq \lambda(u)|\zeta|^2, \tag{5.2}$$

$$|L_u| \lesssim |\lambda_u(u)|, \quad |L_u^{-1}| \lesssim |\lambda_u(u)|^{-1}. \tag{5.3}$$

Recall that $P_i(u) = u_i\lambda_i(u)$ so that $\partial_{u_i} P_i(u) = \delta_{ij} \lambda_i(u) + u_i \partial_{u_j} \lambda_i(u)$, where $\delta_{ij}$ is the Kronecker delta. Writing $P(u) = [P_i(u)]_{i=1}^m$ and $U = \text{diag}[u_1, \ldots, u_m]$, then we have

$$A(u) := P_u(u) = [\partial_{u_j} P_i(u)] = L + UL_u.$$

We define

$$K(u) = [K_i(u)]_{i=1}^m, \quad K_i(u) = \log(\lambda_i(u)).$$

We first have the following lemma.

**Lemma 5.1** The matrix $A$ and the map $K$ satisfy the conditions A), H) respectively.

**Proof:** It is clear that (5.2) yields $\langle A(u)\zeta, \zeta \rangle \geq \lambda(u)|\zeta|^2$. The conditions (5.1) and (5.3) imply easily that $|A(u)| \lesssim \lambda(u)$. Furthermore, simple calculation shows that they also give that

$$|A_u(u)| \lesssim |L_u| + |u| |(L_u)_{uu}| \lesssim |\lambda_u(u)| + |u| \max_i |(\lambda_i(u))_{uu}| \lesssim |\lambda_u(u)| + |u| |\lambda_u(u)^2| \lesssim |\lambda_u(u)|,$$

because $|u| |\lambda_u(u)| \lesssim \lambda(u)$. Thus, the condition A) is verified.

We turn to the map $K$. Because $\partial_{u_i} K_i(u) = \lambda_i^{-1}(u)\partial_{u_j} \lambda_i(u)$, we have $K_u(u) = L^{-1}L_u$. For $K(u) = (K^{-1}_i(u))^T$ we have $|K_u(u)| = |K^{-1}_u(u)| = |L_u^{-1}L| \lesssim \lambda(u)|\lambda_u(u)|^{-1}$ thanks to (5.3). On the other hand,

$$|K_{uu}(u)| = |K_{uu}^{-1}(u)| = |(L_u^{-1}L)_{uu}| \lesssim |L_u^{-1}| \langle (L_u)_{uu} \rangle + L_u ||(L_u)_{uu}|| L_u^{-1}|^2.$$
Using the facts that \(|(L_u)_u| \leq |\lambda_{uu}(u)|\) and \(\lambda(u)|\lambda_{uu}(u)| \leq |\lambda_u(u)|^2\), we see that \(|K_u(u)|\) is bounded by a constant. Thus, H) is verified and the lemma is proved. ■

We now establish the first part of M.1) by showing that \(K(u)\) is VMO.

**Lemma 5.2** Assume that there is a constant \(C_0\) such that

\[
|\lambda_u(u)|\lambda^{-2}(u) \leq C_0, \quad (5.4)
\]

\[
\int_0^{T_0} \int_\Omega |\lambda(u)f(u)|^2 \, dx \leq C_0, \text{ where } f(u) := [f_i(u)]_{i=1}^m. \quad (5.5)
\]

Then there is a constant \(C(C_0)\) such that

\[
\sup_{\tau \in (0,T_0)} \int_{\Omega \times \{\tau\}} |D(K(u))|^2 \, dx \leq C(C_0). \quad (5.6)
\]

**Proof:** First of all, we test the \(i\)-th equation of the system with \((P_i(u))_t\) and sum the results to have, denoting \(B(u, Du) := [B_i(u, Du)]_{i=1}^m\)

\[
\int_\Omega \langle u_t, P_i(u) \rangle \, dx + \int_\Omega \langle D(u), D(P_i(u)) \rangle \, dx = \int_\Omega \langle B(u, Du) + \tilde{f}(u), P_i(u) \rangle \, dx. \quad (5.7)
\]

Because \(P_u(u) = A(u), |A(u)| \leq \lambda(u)\) and \(|B(u, Du)| \leq \lambda^2(u)|Du|\), we have by Young's inequality

\[
\langle B(u, Du) + \tilde{f}(u), P_i(u) \rangle \leq \varepsilon \lambda(u)|u_t|^2 + C(\varepsilon)\lambda^2(u)|Du|^2 + C(\varepsilon)\lambda(u)|\tilde{f}(u)|^2.
\]

As \(\langle u_t, P_i(u) \rangle \geq \lambda(u)|u_t|^2\) and \(\frac{d|D(u)|^2}{dt} = \langle D(u), D(P_i(u)) \rangle\) (because \(u\) is a strong solution), we can choose \(\varepsilon\) small in the above and derive from (5.7) that

\[
\frac{d}{dt} \int_\Omega |D(u)|^2 \, dx \leq C \int_\Omega \lambda^2(u)|Du|^2 \, dx + C \int_\Omega \lambda(u)|\tilde{f}(u)|^2 \, dx. \quad (5.8)
\]

By the ellipticity condition, \(\lambda(u)|Du|^2 \leq \langle A(u), Du \rangle \leq \langle D(u), Du \rangle\) so that a simple use of Young’s inequality implies \(\lambda(u)|Du|^2 \leq \frac{1}{2}\lambda^{-1}(u)|DP(u)|^2 + \frac{1}{2}\lambda(u)|Du|^2\). This shows that \(D(P(u)) \sim \lambda(u)|Du|\). We then have the following.

\[
y'(t) \leq C(y(t) + \alpha(t)), \text{ where } y(t) = \int_{\Omega \times \{t\}} |D(u)|^2 \, dx, \quad \alpha(t) = \int_{\Omega \times \{t\}} |\lambda(u)\tilde{f}(u)|^2 \, dx.
\]

This is a simple Gronwall inequality for \(y(t)\) and we have

\[
\sup_{\tau \in (0,T_0)} \int_{\Omega \times \{\tau\}} |\lambda(u)Du|^2 \, dx \sim \sup_{\tau \in (0,T_0)} \int_{\Omega \times \{\tau\}} |D(u)|^2 \, dx \leq C \int_Q |\lambda(u)\tilde{f}(u)|^2 \, dz.
\]

On the other hand, \(|D(K(u))| = |K_u(u)Du| \lesssim |L^{-1}u|\lambda^{-1}(u)|\lambda(u)Du|\) so that if (5.4) holds then the above and (5.5) imply (5.6) and conclude the proof. ■
Remark 5.3 The above lemma also shows that $K(u)$ is VMO($\Omega$). We simply apply Poincaré’s inequality for $n = 2$ and use (5.6). Also, the growth in $Du$ of $B_i(u, Du)$ is a bit different from f.1) in this paper but it was considered in $^\text{[13]}$ that $\hat{f}(u, Du) \leq \lambda^2(u)|Du| + f(u)$. $^\text{[13]Lemma 3.2}$ still provides the same energy estimate for $Du$ as (4.19). Thus, the proof of our main theorem can continue.

Proof of Theorem [2.2] By Lemma 5.1 the assumption A), F) and H) of the main theorem are satisfied. As we already assumed M.0), the theorem will follow if M.1) is verified. The first part of M.1) requires that $K(u)$ has small BMO norm in small balls is given by Lemma 5.2 and Remark 5.3. We need only to check the second part by showing $W_p(x, \tau) := \lambda^{p+1} |\lambda(u)|^{-p}$ is a weight. To this end, we will show that $\lambda(u)$ and $|\lambda_u(u)|$ are $A_1$ weights, $A_1 \cap \gamma > 1 A_\gamma$. For $w_1 = \log($\lambda(u)$)$ and $w_2 = \log(|\lambda_u(u)|^{-1})$ we have

$$|Dw_1| \leq \frac{|\lambda_u(u)|}{\lambda(u)} |Du| \leq \frac{|\lambda_u(u)|}{\lambda^2(u)} |\lambda(u)Du|,$$

$$|Dw_2| \leq \frac{|\lambda_{uu}(u)|}{|\lambda_u(u)|} |Du| \leq \frac{|\lambda_{uu}(u)|}{|\lambda_u(u)|^2} |\lambda(u)Du| \leq \frac{|\lambda_u(u)|}{\lambda^2(u)} |\lambda(u)Du|.$$

Since $|\lambda_u(u)|/\lambda^2(u)$ is bounded and $\lambda(u)Du \in L^2(\Omega)$, we see that $ Dw_i \in L^2(\Omega)$ and $w_i$’s have small BMO norm in small balls. It is wellknown that this implies $e^{cw_i}$, and therefore $\lambda^c(u)$ and $|\lambda_u(u)|^{-c}$, are $A_1$ weights for any $c > 0$ (see $^\text{[3]}$ or $^\text{[15]Lemma 5.1}$). Hence, for each $\tau \in (0, T_0)$ and any power of $W_p(x, \tau)$ is also an $A_1$ weight and the last condition in H) is then verified. The proof is complete.

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