\( \eta \) and \( \lambda \) deformations as \( \mathcal{E} \)-models

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Abstract

We show that the so called \( \lambda \) deformed \( \sigma \)-model as well as the \( \eta \) deformed one belong to a class of the \( \mathcal{E} \)-models introduced in the context of the Poisson-Lie-T-duality. The \( \lambda \) and \( \eta \) theories differ solely by the choice of the Drinfeld double; for the \( \lambda \) model the double is the direct product \( G \times G \) while for the \( \eta \) model it is the complexified group \( G^\mathbb{C} \). As a consequence of this picture, we prove for any \( G \) that the target space geometries of the \( \lambda \)-model and of the Poisson-Lie T-dual of the \( \eta \)-model are related by a simple analytic continuation.

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1 Summary

Consider the actions $S_\eta(g)$ and $S_\lambda(g)$ of the so called $\eta$ and $\lambda$ deformed $\sigma$-models on the target of a simple compact Lie group $G$:

$$S_\eta(g) = \frac{1}{2} \int d\xi^+ d\xi^- (g^{-1} \partial_+ g, (1 - \eta R)^{-1} g^{-1} \partial_- g), \quad (1)$$

$$S_\lambda(g) = S_{WZW}(g) + \lambda \int d\xi^+ d\xi^- ((1 - \lambda \text{Ad}_g)^{-1} \partial_+ gg^{-1}, g^{-1} \partial_- g). \quad (2)$$

Here $g(\xi^+, \xi^-) \in G$, the derivatives $\partial_\pm$ are taken with respect to the light-cone variables $\xi^\pm$, $(.,.)$ is the Killing-Cartan form on the Lie algebra $G^C$ of $G^C$, $R : G \to G$ is the so called Yang-Baxter operator and $S_{WZW}(g)$ is the standard WZW action

$$S_{WZW}(g) := \frac{1}{2} \int d\xi^+ d\xi^- (g^{-1} \partial_+ g, g^{-1} \partial_- g) + \frac{1}{12} \int d^{-1}(dgg^{-1}, [dgg^{-1}, dgg^{-1}]). \quad (3)$$

The models (1) and (2) were respectively introduced in [29, 30] and [38], with the parameters $\eta, \lambda$ real and $|\lambda| < 1$.

It may seem that the expression (2) defines a $\sigma$-model also on the complexified group $G^C$, however, this is a false appearance. The reason is that the action $S_\lambda$ evaluated on $G^C$-valued configurations takes generically complex values. However, if we evaluate $S_\lambda$ exclusively on configurations $p$ with values in the space $P$ of positive definite Hermitian elements of $G^C$ and we take $\lambda$ to be a complex number of modulus 1 then $-iS_\lambda(p)$ is always real and defines some $\sigma$-model on $P$. Our Result 1 (the principal one) then states:

The $\sigma$-model $-iS_\lambda(p)$ on $P$ for $\lambda = \frac{1-i\eta}{1+i\eta}$ is the Poisson-Lie T-dual of the $\eta$-model.

Few remarks are in order:

1) The replacing of the unitary argument $g$ by the positive definite Hermitian one $p$ in (2) can be interpreted as a simple analytic continuation of the coordinates parametrizing the Cartan torus; our result therefore generalizes to any $G$ the $SU(2)$ result of Refs. [18], [39] stating that the $\lambda$-model is related by analytical continuation to the Poisson-Lie T-dual of the $\eta$-model.

2) It is very probable that our purely bosonic result can be generalized to the supergroup context. This would mean that, up to the analytic continuation, the $\lambda$-deformed $(AdS_5 \times S^5)_\lambda$ superstring of Ref. [15] is the Poisson-Lie T-dual of the $\eta$-deformed $(AdS_5 \times S^5)_\eta$ superstring of Ref. [10].
3) For the group $G = SU(N)$, $P$ coincides with the spaces of positive definite Hermitian $N \times N$ matrices.

The results of [18] and [39] on the analytic continuation were obtained by working in appropriate coordinates on the group $SU(2)$ and on its dual Borel group. It appears extremely difficult to generalize that method to higher dimensional groups because the action of the dual $\eta$-model (in its version known before the present paper; cf. Eq. (40) of [29]) becomes prohibitively complicated in any coordinate system. To move forward we have to find a completely coordinate-free framework to work with and this turns out to be possible thanks to our following Result 2:

The $\lambda$-model on any simple Lie group target $G$ belongs to the class of the $\mathcal{E}$-models considered in [25, 26] in the context of the Poisson-Lie T-duality.

The next Result 3 is the consequence of the previous one

For every simple compact Lie group $G$ there exists a manifold $P$, a distinguished function $H$ on $P$ and two compatible Poisson structures $\{.,.\}_0$, $\{.,.\}_1$ on $P$ such that the dynamical system with the phase space $P$, the Hamiltonian $H$ and the Poisson structure $\{.,.\}_0 + \varepsilon \{.,.\}_1$ can be identified with

i) the principal chiral model on $G$, for $\varepsilon = 0$;

ii) the $\lambda$-model on $G$, for $\varepsilon > 0$, where $\lambda = (1 - \varepsilon \frac{1}{2})(1 + \varepsilon \frac{1}{2})$;

iii) the $\eta$-model on $G$, for $\varepsilon < 0$, where $\eta = (-\varepsilon)^{\frac{1}{2}}$.

We finish by two more remarks:

4) The statement of Result 3 could be in principle reconstructed by composing together several facts established already in [9, 38, 43], however, we consider as an independent result the way how we obtain it directly and naturally from the formalism of the $\mathcal{E}$-models.

5) The Poisson structure $\{.,.\}_0 + \varepsilon \{.,.\}_1$ is the (symplectic version of) the current algebra built on a one-parameter family $\mathcal{D}_{\varepsilon}$ of the Drinfeld doubles of the Lie algebra $\mathcal{G} \equiv \text{Lie}(G)$. The Hamiltonian $H$ is a quadratic expression in the currents and it is completely determined by the Hamiltonian of the principal chiral model because it does not depend on $\varepsilon$.

## 2 Introduction

A problem how to deform an integrable non-linear $\sigma$-model on group manifold in a way preserving the integrability was formulated some forty years ago
and it turned out to be a difficult one. Several integrable deformations of the principal chiral model have been found in the eighties and the nineties for the simplest case of the group $SU(2)$ [1, 7, 13, 14] but for long decades no examples were constructed for higher dimensional groups. Some effort (see e.g. [36]) has been made to determine a complete system of conditions which a target geometry on a general Lie group must fulfill in order to guarantee integrability, however, attempts to find solutions of this complicated highly overdetermined system of conditions essentially failed for other groups than $SU(2)$. This situation lasted until 2008 when, in [30], the present author established the integrability of the so-called $\eta$-deformed (or, equivalently, Yang-Baxter) $\sigma$-model [29] for any simple compact Lie group target $G$.

The integrable $\eta$-deformation of the principal chiral model described in [30] was generalised to the context of integrable coset and supercoset targets in [9] and [10], respectively. In particular, the result [10] has triggered an important activity in the field because of its relevance in the AdS/CFT story [1, 2, 3, 5, 8, 12, 17, 19, 22, 20, 32, 34, 40, 42]. In a short period of few years, several new integrable deformations of the integrable nonlinear $\sigma$-models were obtained, some of them multi-parametric [6, 11, 16, 21, 31, 33, 38]. In the present paper, we shall concentrate mainly on the integrable deformation of the WZW model proposed in [38]. It is now called the "$\lambda$-deformation", it belongs to a class of $\sigma$-models introduced in [41] and, similarly as in the $\eta$ case, it was later generalized to the integrable supercoset targets [15].

Three papers [43], [18] and [39] have recently discussed the issue of possible structural relations between the integrable $\eta$- and $\lambda$-deformations and all of them emphasized the relevance of the concept of the Poisson-Lie T-duality [24, 25] in this context. In particular, Vicedo [43] studied extensively the case of the $\lambda$-model on a non-compact simple Lie group admitting the so-called split Yang-Baxter operator on its Lie algebra and pointed out the existence of the Poisson-Lie T-dual theory resembling the variant of the $\eta$-model with real poles of the so-called twist functions (the poles of the twist function of the original $\eta$-model [29, 30] are complex conjugated). On the other hand, Hoare and Tseytlin [18] and Sfetsos, Siampos and Thompson [39] have stuck to the compact case and showed that the $\lambda$-deformation on the $SU(2)$ target is related by an appropriate analytic continuation to the

\footnote{A second-order action of this dual theory is not explicitly given in [43] because of the problems with the factorizability of the underlying Drinfeld double. In this respect, the formula (14) of the present paper includes also the case of the non-factorizable doubles and its usefulness for the further development of the results of [43] looks very probable.}
Poisson-Lie T-dual of the \( \eta \)-deformation. The principal goal of the present paper is to generalize this result of [18, 39] to any target \( G \).

The other goal of the present article is to point out that the structural relation between the \( \eta \)- and \( \lambda \)-deformations is particularly explicit, obvious and neat in the framework of the theory of the \( \mathcal{E} \)-models developed in the context of the Poisson-Lie T-duality in [25, 26]. In this regard, we wish to stress the conceptual and technical utility of several papers on the Poisson-Lie T-duality like [26, 27] which so far remain somewhat in the shadow of the initial works [24, 25]. Indeed, as we shall show, the results of the paper [26] permit to establish that not only the \( \eta \)-deformation but also their \( \lambda \)-counterpart belongs to the class of the \( \mathcal{E} \)-models introduced in [25, 26]. In fact, the difference between \( \eta \)- and \( \lambda \)-deformations turns out to be given solely by the choice of the Drinfeld double encoding the Hamiltonian structure of the integrable \( \sigma \)-model in question. The choice of the complexified group \( G^C \) yields the \( \eta \)-deformation while the double \( G \times G \) corresponds to the \( \lambda \)-deformation.

The paper is organized as follows: In Section 4, we review the notion of the Drinfeld double current algebra as well as that of the \( \mathcal{E} \)-model [25, 26]. In Section 5, we show that the \( \lambda \)-model on arbitrary compact simple group target \( G \) is a particular case of the \( \mathcal{E} \)-model and, for completeness, we review also the result of [29] establishing the same thing for the \( \eta \)-model. In Section 6, we establish the result concerning the analytical continuation relation between the \( \lambda \) and the dual \( \eta \) target geometries for any \( G \) and, finally, we devote Section 7 to a discussion of the results and to an outlook.

3 \( \mathcal{E} \)-models

Let \( \mathcal{D} \) denote a real finite dimensional Lie algebra and let \((.,.)_{\mathcal{D}}\) be an ad-invariant non-degenerate symmetric bilinear form on \( \mathcal{D} \). We then construct an infinite-dimensional Poisson manifold \( P_{\mathcal{D}} \) the coordinates \( j^A(\sigma) \) of which are labeled by one discrete parameter \( A = 1, \ldots, \text{dim}\mathcal{D} \) and one continuous (loop) parameter \( \sigma \), with the defining Poisson brackets given by

\[
\{j^A(\sigma), j^B(\sigma')\} = F^A_{\cdot C} j^C(\sigma) \delta(\sigma - \sigma') + D^{AB} \partial_\sigma \delta(\sigma - \sigma').
\] (4)

Here \( F^A_{\cdot C} \) are the structure constants of \( \mathcal{D} \) in some basis \( T^A \in \mathcal{D} \) and

\[
D^{AB} := (T^A, T^B)_{\mathcal{D}}.
\] (5)
The Poisson manifold $P_D$ is referred to as the (symplectic version of the) current algebra associated to $\mathcal{D}$.

In what follows, we shall study only quadratic Hamiltonians in $j^A(\sigma)$ based on a choice of an $\mathbb{R}$-linear self-adjoint idempotent operator $\mathcal{E}: \mathcal{D} \to \mathcal{D}$ and given by the following formula

$$H_\mathcal{E} := \frac{1}{2} \int d\sigma (j(\sigma), \mathcal{E} j(\sigma))_{\mathcal{D}}.$$  \hspace{1cm} (6)

Here we have used a $\mathcal{D}$-valued coordinates $j(\sigma)$ on $P_D$ defined by

$$(j(\sigma), T^A)_\mathcal{D} := j^A(\sigma).$$ \hspace{1cm} (7)

We also state, for the completeness, that the self-adjointness and the idempotency of $\mathcal{E}$ (which are essential for the world-sheet Lorentz invariance of the Hamiltonian) mean, respectively

$$(\mathcal{E} x, y)_\mathcal{D} = (x, \mathcal{E} y)_\mathcal{D}, \quad \forall x, y \in \mathcal{D}; \quad \mathcal{E}^2 x = x, \quad \forall x \in \mathcal{D}. \hspace{1cm} (8)$$

The dynamical system on the phase space $P_D$ defined by the current algebra Poisson brackets (4) and by the quadratic Hamiltonian (6) is referred to as an $\mathcal{E}$-model. It was originally defined in \cite{25,26} and its equations of motion have the zero-curvature form valued in $\mathcal{D}$, that is

$$\partial_\tau j = \partial_\sigma (\mathcal{E} j) + [\mathcal{E} j, j]. \hspace{1cm} (9)$$

Here $\tau$ stands for the time.

**Remark 1:** In \cite{25,26}, we have been using a parametrization of the phase space $P_D$ in terms of a group-like variable $l(\sigma)$ taking values in the loop group of the Drinfeld double $D$. ($D$ is a Lie group the Lie algebra of which is $\mathcal{D}$.) The relation with the current algebra description $j(\sigma)$ reads

$$j(\sigma) = \partial_\sigma l(\sigma) l(\sigma)^{-1} \hspace{1cm} (10)$$

and the equation of motion (9) takes form

$$\partial_\tau ll^{-1} = \mathcal{E} \partial_\sigma ll^{-1}. \hspace{1cm} (11)$$

\[\text{The invertibility of the Poisson tensor may fail only in a finite-dimensional zero mode sector in the Fourier-transformed current components } j^A(\sigma) \text{ which is determined by boundary conditions imposed on the currents.}\]
The Poisson brackets expressed in terms of the variables \( l(\sigma) \) are more cumbersome than the elegant current algebra formula (4), nevertheless, the expression for the symplectic form on \( P_D \) is simpler in the \( l(\sigma) \) language (see [25, 26] for details).

Remark 2: Suppose that there is a linear one-parameter family of the Lie algebra structures on the vector space \( D \), which means that the structure constants \( F^{AB}_C \) can be written as

\[
F^{AB}_C = F^{AB}_C + \varepsilon F^{AB}_C, \quad \varepsilon \in \mathbb{R}.
\]

Then the current algebra Poisson structure (4) can be represented accordingly as

\[
\{ j^A(\sigma), j^B(\sigma') \} = \{ j^A(\sigma), j^B(\sigma') \}_0 + \varepsilon \{ j^A(\sigma), j^B(\sigma') \}_1.
\]

The Poisson structures \( \{.,.\}_0 \) and \( \{.,.\}_1 \) appearing in this relation can be readily read off from Eq. (4) and they are automatically compatible because the structure constants \( F^{AB}_C \) verify the Lie algebra Jacobi identity for every \( \varepsilon \).

Suppose now that there is a Lie subalgebra \( \mathcal{G} \subset D \) isotropic with respect to the bilinear form \( (.,.)_D \) and such that \( \dim \mathcal{G} = \frac{1}{2} \dim D \) (the isotropy means \( (x,x)_D = 0, \forall x \in \mathcal{G} \)). Then it was shown in [26] that there is a non-linear \( \sigma \)-model on the target \( D/G \) which can be identified with the \( \mathcal{E} \)-model \( (P_D, H_{\mathcal{E}}) \). Here \( G \) is the subgroup of \( D \) corresponding to the subalgebra \( \mathcal{G} \) and "can be identified" means the existence of a symplectomorphism (i.e. a canonical transformation) taking the phase space and the Hamiltonian of the \( D/G \) \( \sigma \)-model onto \( P_D \) and \( H_{\mathcal{E}} \), respectively. The target space geometry of the \( D/G \) model was worked out in detail in [26, 27, 29] and it is encoded in the following action:

\[
S_{\mathcal{E}}(f) = S_{WZW,D}(f) - \int d\xi^+ d\xi^- (P_f(\mathcal{E}) f^{-1} \partial_+ f, f^{-1} \partial_- f)_D.
\]

Here the action \( S_{WZW,D}(f) \) is given by

\[
S_{WZW,D}(f) := \frac{1}{2} \int d\xi^+ d\xi^- (f^{-1} \partial_+ f, f^{-1} \partial_- f)_D + \frac{1}{12} \int d^1(df f^{-1}, [df f^{-1}, df f^{-1}])_D,
\]

the usual light-cone variables \( \xi^\pm \) and derivatives \( \partial_\pm \) read

\[
\xi^\pm := \frac{1}{2}(\tau \pm \sigma), \quad \partial_\pm := \partial_\tau \pm \partial_\sigma,
\]
f stands for the parametrization of the right coset $D/G$ by elements $f$ of $D$ (if there exists no global section of this fibration, we can choose several local sections covering the whole base space $D/G$) and, finally, $P_f(\mathcal{E})$ is a projection from $\mathcal{D}$ into $\mathcal{D}$ defined by the relations

$$\text{Im} P_f(\mathcal{E}) = \mathcal{G}, \quad \text{Ker} P_f(\mathcal{E}) = (1 + \text{Ad}_f^{-1} \mathcal{E} \text{Ad}_f) \mathcal{G}.$$

**Remark 3**: The use of the projection $P_f(\mathcal{E})$ in the formula (14) is a new result (a by-line one) of the present paper which encompasses the results of [26, 27, 29] (e.g. the formula (12) of [26]) in a basis independent way.

We do not repeat here the derivation of the formula (14) for the $\sigma$-model action from the $\mathcal{E}$-model data $(P_D, H_\mathcal{E})$ as it is presented in [26, 27, 29] but we do write down the symplectomorphism associating to every solution of the equation of motion of the $\sigma$-model (14) the solution of the equation of motion (9) because this result is not contained in [26, 27, 29]:

$$j = \partial_\sigma f f^{-1} - \frac{1}{2} f (P_f(\mathcal{E}) f^{-1} \partial_+ f - P_f(\mathcal{E}) f^{-1} \partial_- f) f^{-1}. \quad (17)$$

### 4 Current algebras of $\eta$ and $\lambda$ deformations

Consider a simple compact real Lie algebra $\mathcal{G}$ equipped with its standard Killing-Cartan form $(\cdot, \cdot)$. We introduce one-parameter family of real Lie-algebras $\mathcal{D}_\varepsilon$ which all have the property of being the Drinfeld doubles of $\mathcal{G}$. As the vector space, $\mathcal{D}_\varepsilon$ is just the direct sum of the vector space $\mathcal{G}$ with itself:

$$\mathcal{D}_\varepsilon := \mathcal{G} + \mathcal{G}, \quad (18)$$

the Lie algebra bracket $[\cdot, \cdot]_\varepsilon$ on $\mathcal{D}_\varepsilon$ is defined in terms of the commutator $[\cdot, \cdot]$ in $\mathcal{G}$ as follows

$$[x_1 + x_2, y_1 + y_2]_\varepsilon := ([x_1, y_1] + \varepsilon [x_2, y_2]) + ([x_1, y_2] + [x_2, y_1]), \quad x_i, y_i \in \mathcal{G}, \quad (19)$$

and, finally, the ad-invariant non-degenerate symmetric bilinear form $(\cdot, \cdot)_\mathcal{D}$ does not depend on $\varepsilon$ and it is given by

$$(x_1 + x_2, y_1 + y_2)_\mathcal{D} := (x_2, y_1) + (x_1, y_2). \quad (20)$$

Note that $\mathcal{G}$ is embedded in $\mathcal{D}_\varepsilon$ as $\mathcal{G} \oplus 0$, or, said in other words, $\mathcal{D}_\varepsilon$ is the Drinfeld double of its subalgebra $\mathcal{G} \oplus 0 \simeq \mathcal{G}$. 

7
We now introduce a one-parameter family of $\mathcal{E}$-models $(P_{D_\varepsilon}, H)$ based on the current algebra (4) for the Drinfeld double $D_\varepsilon$ and equipped with the quadratic Hamiltonian (6) given by the following choice of the self-adjoint idempotent operator $\mathcal{E}$:

$$\mathcal{E}(x_1 + x_2) := (x_2 + x_1). \quad (21)$$

Because here we speak about the particular operator $\mathcal{E}$ given by Eq. (21), we denote just by $H$ the Hamiltonian associated to it via (6), reserving the notation $H(\mathcal{E})$ to situations when a generic operator $\mathcal{E}$ occurs.

**Remark 4:** We note that the structure constants of the Lie algebra $D_\varepsilon$ have precisely the structure (12) of Remark 2 which means that the symplectic structure of the $\mathcal{E}$-model $(P_{D_\varepsilon}, H)$ has the form of the linear combination $\{., .\}_0 + \varepsilon \{., .\}_1$ of two compatible Poisson structures as mentioned in the Result 3 of the section Summary.

We now evaluate, for every $\varepsilon$, the second order $\sigma$-model action (14) of the $\mathcal{E}$-model $(P_{D_\varepsilon}, H)$. We start with the simplest case $\varepsilon = 0$ where it turns out to hold:

*The $\mathcal{E}$-model $(P_{D_0}, H)$ can be identified with the principal chiral model on $G$.*

Let us demonstrate this statement:

We first remark, that the Drinfeld double $D_0$ is the semi-direct product of manifolds $G$ and $\tilde{G}$, i.e. the group law reads

$$(g_1, x_1)(g_2, x_2) = (g_1g_2, x_1 + g_1x_2g_1^{-1}), \quad g_1, g_2 \in G, \quad x_1, x_2 \in \tilde{G}. \quad (22)$$

It can be easily checked that, indeed, the law (22) gives rise to the Lie algebra commutator (19) for $\varepsilon = 0$. Now note that the commutation relation (19) implies

$$[0 + x_2, 0 + y_2]_0 = 0. \quad (23)$$

Denote the Abelian Lie algebra $0 + \tilde{G}$ by the symbol $\tilde{G}$ and the corresponding Lie group by $\tilde{G}$. (The elements of $\tilde{G}$ are therefore $(e, x) \in D_0$, $e$ being the unit element of $G$.)

Consider now the $\sigma$-model (14) on the target $D_0/\tilde{G}$. This coset can be obviously identified with the subgroup $G$ of $D_0$, the elements of which are $f = (g, 0) \in D_0$. Thus the field $f$ featuring in (14) can be chosen to take values $(g, 0) \in D_0$. In this case the part $S_{WZW_D}(f)$ of the action
vanishes because the Lie algebra \( \mathcal{G} \) of \( G \) is maximally isotropic (i.e. \( (\mathcal{G} + 0, \mathcal{G} + 0)_D = 0 \)). Since the operator \( \mathcal{E} \) given by (21) evidently commutes with \( \text{Ad}_{(g,0)} \), the projection \( P_{(g,0)}(\mathcal{E}) \) does not depend on \( g \) and it is easily found to be given by

\[
P_{(g,0)}(\mathcal{E})(x_1 + x_2) = (0 + (x_2 - x_1)),
\]

hence

\[
P_{(g,0)}f^{-1}\partial_+ f = P_{(g,0)}(g^{-1}\partial_+ g + 0) = (0 + -g^{-1}\partial_+ g).
\]

Combining (14), (20) and (25) we find the following action of the \( \sigma \)-model on \( D_0/G \):

\[
S_{\xi,0}(g) = \int d\xi^+ d\xi^- (g^{-1}\partial_+ gg, g^{-1}\partial_- gg).
\]

This is indeed the action of the principal chiral model on the group \( G \).

Now we show that the evaluation of the second order \( \sigma \)-model action (14) of the \( \mathcal{E} \)-models \( (P_{D_\epsilon}, H) \) for \( \epsilon > 0 \) gives the \( \lambda \)-model of [38]. More precisely, it holds

For \( \epsilon > 0 \), the \( \mathcal{E} \)-model \( (P_{D_\epsilon}, H) \) can be identified with the \( \lambda \)-model on \( G \) characterized by the action

\[
S_{\lambda}(g) = \frac{1}{2} \int d\xi^+ d\xi^- (g^{-1}\partial_+ gg, g^{-1}\partial_- gg) + \frac{1}{12} \int d^- (dg^{-1}, [dg^{-1}, dg^{-1}])
\]

\[
+ \lambda \int d\xi^+ d\xi^- (\partial_+ gg, (1 - \lambda \text{Ad}_{g^{-1}})^{-1} g^{-1}\partial_- g),
\]

where

\[
\lambda = \frac{1 - \epsilon^{\frac{1}{2}}}{1 + \epsilon^{\frac{1}{2}}},
\]

We start the argument by considering the Lie algebra \( \mathcal{G} \oplus \mathcal{G} \) (i.e. the direct sum of the Lie algebra \( \mathcal{G} \) with itself), the elements of which will be typically denoted \((\alpha_1, \alpha_2)\). There is an ad-invariant non-degenerate symmetric bilinear form on \( \mathcal{G} \oplus \mathcal{G} \) given by the formula

\[
((\alpha_1, \alpha_2), (\beta_1, \beta_2))_{\mathcal{G} \oplus \mathcal{G}} := (\alpha_1, \beta_1) - (\alpha_2, \beta_2).
\]

For each \( \epsilon \) positive there is an isomorphism of Lie algebras \( \Phi_{\epsilon} : D_\epsilon \rightarrow \mathcal{G} \oplus \mathcal{G} \) given by

\[
\Phi_{\epsilon}(x_1 + x_2) = (x_1 + \epsilon^{\frac{1}{4}} x_2, x_1 - \epsilon^{\frac{1}{4}} x_2).
\]
This isomorphism preserves the bilinear forms (20) and (29) up to normalization, that is

\[(\Phi_\varepsilon(x), \Phi_\varepsilon(y))_{G\oplus G} = 2\varepsilon\frac{1}{2}(x, y)_D, \quad x, y \in D_\varepsilon.\] \hfill (31)

The existence of the isomorphism $\Phi_\varepsilon$ means that we can work with the double $G\oplus G$ instead of $D_\varepsilon$, if we translate by $\Phi_\varepsilon$ to the $G\oplus G$ context also the operator $\mathcal{E} : D_\varepsilon \to D_\varepsilon$ given by (21). The translated operator $\mathcal{E}_\varepsilon : G \oplus G \to G \oplus G$ is defined by the requirement

\[\mathcal{E}_\varepsilon \circ \Phi_\varepsilon = \Phi_\varepsilon \circ \mathcal{E},\] \hfill (32)

which gives

\[\mathcal{E}_\varepsilon(\alpha, \beta) = \frac{1}{2}(\varepsilon^{\frac{1}{2}} + \varepsilon^{-\frac{1}{2}})(\alpha, -\beta) + \frac{1}{2}(\varepsilon^{\frac{1}{2}} - \varepsilon^{-\frac{1}{2}})(\beta, -\alpha).\] \hfill (33)

The group Drinfeld double of the Lie algebra $G \oplus G$ is evidently $G \times G$ (i.e. the direct product of $G$ with itself) and its elements will be typically denoted as $(a_1, a_2)$. The diagonal subgroup of $G \times G$ generated by the elements of the form $(a, a)$ will be denoted as $G^\delta$. The corresponding Lie algebra $G^\delta$ is maximally isotropic (it is the image of the subalgebra $G^+ + 0 \subset D_\varepsilon$ under the isomorphism $\Phi_\varepsilon$) and its elements are $(\alpha, \alpha)$. In order to apply to the present situation the general formula (14), there remains to parametrize the cosets $D/G^\delta$ by the elements of $D$ and to identify the projection $P_f(\mathcal{E}_\varepsilon)$. Obviously, the coset $D/G^\delta$ can be identified with the first copy $G$ in the direct product $G \times G$ which gives the parametrization $f = (g, e)$. $P_{(g, e)}(\mathcal{E}_\varepsilon)$ is then straightforwardly found to be equal to

\[P_{(g, e)}(\mathcal{E}_\varepsilon)(\alpha, \beta) = \left(\frac{\lambda}{\lambda - \text{Ad}_g^{-1}}\alpha + \frac{1}{1 - \lambda\text{Ad}_g}\beta, \frac{\lambda}{\lambda - \text{Ad}_g^{-1}}\alpha + \frac{1}{1 - \lambda\text{Ad}_g}\beta\right),\] \hfill (34)

where $\lambda$ is given by the formula (28).

Finally, taking into account that $f^{-1}\partial_+ f = (g^{-1}\partial_+ g, 0)$, the wanted formula (27) follows directly (up to an overall normalisation) from Eqs. (14), (29) and (34).

Remark 5: Note that when the parameter $\varepsilon$ ranges from 0 to $+\infty$, the parameter $\lambda$ given by (28) ranges from $-1$ to 1. This is to be compared with the original paper [38] where the way of obtaining the action (27) (by a gauging procedure)
leads to the interval of the values of $\lambda$ between 0 and 1. Thus the vantage point based on the $\mathcal{E}$-models "sees" more possible values of $\lambda$.

The fact that for $\varepsilon < 0$ the evaluation of the second order $\sigma$-model action (14) of the $\mathcal{E}$-models $(P_{D_\varepsilon}, H)$ gives the $\eta$-model of [29] was proven already in [29]. However, to keep the exposition self-contained we outline here the argument:

Consider the Lie algebra $G^C$ (i.e., the complexification of $G$) the elements of which will be typically denoted as $z$. There is an ad-invariant non-degenerate symmetric bilinear form on $G^C$ given by the formula

$$(z_1, z_2)^G_C := -i(z_1, z_2) + i(z_1, z_2),$$

where $(.,.)$ is the Killing-Cartan form on $G^C$ and number stands for the complex conjugation of the number.

For each $\varepsilon$ negative, there is an isomorphism of Lie algebras $\Psi_\varepsilon : D_\varepsilon \rightarrow G^C$ given by

$$\Psi_\varepsilon(x_1 + x_2) = x_1 + |\varepsilon|^{\frac{1}{2}}i x_2.$$

This isomorphism relates the bilinear forms (20) and (35) up to normalization, that is

$$(\Psi_\varepsilon(x), \Psi_\varepsilon(y))^G_C = 2|\varepsilon|^{\frac{1}{2}}(x, y)_{D}, \quad x, y \in D_\varepsilon.$$

The existence of the isomorphism $\Psi_\varepsilon$ means that we can work with the double $G^C$ instead of $D_\varepsilon$, if we translate to the $G^C$ context also the operator $E : D_\varepsilon \rightarrow D_\varepsilon$ given by (21). The translated operator $E_\varepsilon : G^C \rightarrow G^C$ is defined by the requirement

$$E_\varepsilon \circ \Psi_\varepsilon = \Psi_\varepsilon \circ E,$$

which gives

$$E_\varepsilon z = i \frac{1}{2}(|\varepsilon|^{\frac{1}{2}} - |\varepsilon|^{-\frac{1}{2}})z - i \frac{1}{2}(|\varepsilon|^{\frac{1}{2}} + |\varepsilon|^{-\frac{1}{2}})z^*.$$
the operator $\mathcal{E}_\varepsilon$ as given by (39) obviously commutes with $\text{Ad}_g$ therefore the projection $\tilde{P}_{f=g}(\mathcal{E}_\varepsilon)$ does not depend on $f$ (we put tilde over $P(\mathcal{E}_\varepsilon)$ in order to indicate that the image of this projection is $\tilde{G}$ and, in what follows, we suppress the subscript $f$). In order to find $\tilde{P}(\mathcal{E}_\varepsilon)$ explicitly, we note that the elements of $\tilde{G}$ can be parametrized by the elements of $G$ by using the so-called Yang-Baxter operator $R : G \rightarrow G$ (the explicit formula for $R$ can be found in [29, 30]). Explicitly, every $\zeta \in \tilde{G}$ can be uniquely written as $\zeta = (R - i)u$ (40) for some $u \in G$. With this insight, we find straightforwardly

$$\tilde{P}(\mathcal{E}_\varepsilon)z = \frac{1}{2} \frac{R - i}{1 + \sqrt{\varepsilon}R} \left( (i + \sqrt{\varepsilon})z + (i - \sqrt{\varepsilon})z^* \right).$$

(41)

Taking into account the isotropy of the group $G$ (which eliminates the $S_{WZW,D}(f)$ term from the action (14)), applying $\tilde{P}(\mathcal{E}_\varepsilon)$ on $g^{-1}\partial_+ g$ and inserting the result in the general formula (14) we find

$$S_\eta(g) = \frac{1}{2} \int d\xi^+ d\xi^- (g^{-1}\partial_+ g, (1 - \eta R)^{-1}g^{-1}\partial_- g),$$

(42)

where $\eta = \sqrt{\varepsilon}$. This coincides with the action of the $\eta$-model of Ref. [29, 30].

We note finally, that in the present Section 4 we have established the Results 2 and 3 as stated in the Section Summary.

5 T-duality and analytic continuation

By the Poisson-Lie T-dual of the $\eta$-model (42) we shall mean the model (14) based on the same $\mathcal{E}_\varepsilon$ operator (39) as the original model (12) but with the target space being $D/G$ instead of $D/\tilde{G}$. As in [29, 30], we can identify the coset $D/G$ with the group $\tilde{G} = AN$ and, by setting $f = b \in AN$ and realizing that $S_{WZW,D}(b) = 0$, we trivially obtain from the basic formula (14) the action of the dual model in the following form

$$\tilde{S}_\eta(b) = \frac{1}{2} \int d\xi^+ d\xi^- (\partial_+ bb^{-1}, \tilde{O}(b)^{-1}\partial_- bb^{-1})_D.$$

(43)
We do not specify further \[^3\] the \(b\)-dependent linear operator \(\tilde{O} : \mathcal{G} \to \tilde{\mathcal{G}}\) because it is not the form \((43)\) of the dual action that we are going to compare with the \(\lambda\)-model action \((27)\). Indeed, in trying to do so we would hurt on a very complicated dependence of \(\tilde{O}(b)\) on \(b\). Fortunately, we find in this paper a way out of these technical difficulties by identifying the coset \(D/G\) not with the group \(AN\) but with the space \(P\) of all positive definite Hermitian elements of the group \(G^C\). This new identification is based on the well-known fact that every element of \(D = G^C\) admits a unique polar decomposition as the product of a positive definite Hermitian element with an unitary element. From this statement it can be easily derived that the \(AN\)-parametrization and the \(P\)-parametrization of the coset \(D/G\) is related by the diffeomorphism \(\Upsilon : AN \to P\):

\[
\Upsilon(b) = \sqrt{bb^*}. \tag{44}
\]

To obtain the action of the dual model in the \(P\)-parametrization, it is now sufficient to set \(f = \Upsilon(b)\) and to identify the projection \(P_{\Upsilon(b)}(E)\):

\[
P_{\Upsilon(b)}(E)z = \left(\sqrt{|\epsilon|} - i + (\sqrt{|\epsilon|} + i)\text{Ad}_{bb^*}z - (\sqrt{|\epsilon|} - i)z^*\right)^{-1} \left((\sqrt{|\epsilon|} + i)\text{Ad}_{bb^*}z - (\sqrt{|\epsilon|} - i)z^*\right). \tag{45}
\]

Here \(z^*\) means the Hermitian conjugation of the element \(z\).

Inserting \((45)\) and \((35)\) into the basic formula \((14)\) and taking into account that \(\Upsilon(b)\) is Hermitian (this gives e.g. \((\Upsilon(b)^{-1}\partial_+ \Upsilon(b), \Upsilon(b)^{-1}\partial_- \Upsilon(b))_{G^C} = 0\)) we obtain for the action of the dual \(\eta\)-model

\[
\tilde{S}_\eta(b) = -2iS_{WZW}(\Upsilon(b)) + 2i \int d\xi^+d\xi^- \left(\frac{i + (\eta + i)\text{Ad}_{\Upsilon(b)}}{i + (\eta + i)\text{Ad}_{\Upsilon(b)^{-1}} - (\eta - i)\text{Ad}_{\Upsilon(b)^{-1}}} \Upsilon(b)^{-1}\partial_+ \Upsilon(b), \Upsilon(b)^{-1}\partial_- \Upsilon(b)\right). \tag{46}
\]

Here \(\eta = \sqrt{|\epsilon|}\) and the action \(S_{WZW}(\Upsilon(b))\) appearing in \((46)\) is based on the ordinary Killing-Cartan form \((\ldots)\) and not on \((\ldots)_{G^C}\). Explicitly,

\[
S_{WZW}(\Upsilon(b)) := \frac{1}{2} \int d\xi^+d\xi^- (\Upsilon(b)^{-1}\partial_+ \Upsilon(b), \Upsilon(b)^{-1}\partial_- \Upsilon(b)) + \frac{1}{12} \int d^{-1}(d\Upsilon(b)\Upsilon(b)^{-1}, [d\Upsilon(b)\Upsilon(b)^{-1}, d\Upsilon(b)\Upsilon(b)^{-1}]). \tag{47}
\]

\[^3\]The interested reader can find the explicit expression for \(\tilde{O}(b)\) in \([29]\) where \(\tilde{O}(b)\) is related to the well-known Poisson-Lie structure \(\tilde{\Pi}(b)\) on the group \(AN\) via the formula \(\tilde{\Pi}(b) = \tilde{O}(b) - \tilde{O}(1)\).
Note that the hermiticity of $\Upsilon(b)$ implies that the dual action $\tilde{S}_\eta(b)$ is \textit{real} inspite of the factor $i$ standing in front of the r.h.s. of (46). In particular, the WZW term in the r.h.s. of (47) is purely imaginary. Finally, we use the Polyakov-Wiegmann formula [37]

$$S_{WZW}(bb^*) = 2S_{WZW}(\Upsilon(b)) + \int d\xi^+ d\xi^- (\Upsilon(b)^{-1}\partial_- \Upsilon(b), \partial_+ \Upsilon(b) \Upsilon(b)^{-1}),$$

and the identity

$$(bb^*)^{-1}\partial_\pm (bb^*) = \Upsilon(b)^{-1}(\Upsilon(b)^{-1}\partial_\pm \Upsilon(b))(\Upsilon(b)) + \Upsilon(b)^{-1}\partial_\pm \Upsilon(b),$$

which gives together

$$\tilde{S}_\eta(b) =$$

$$= -iS_{WZW}(bb^*) - i\lambda \int d\xi^+ d\xi^-((1-\lambda Ad_{bb^*})^{-1}\partial_+(bb^*) (bb^*)^{-1}, (bb^*)^{-1}\partial_- (bb^*))$$

with

$$\lambda = \frac{1 - i\eta}{1 + i\eta}.$$

Comparing the resulting expression (50) with the $\lambda$-model action $S_\lambda$ given by the formula (2) or (27), we conclude

$$\tilde{S}_\eta(b) = -iS_\lambda(bb^*), \quad \lambda = \frac{1 - i\eta}{1 + i\eta}.$$

Of course, the replacing the unitary argument $g$ by the positive definite Hermitian argument $bb^*$ in the $\lambda$-model action (2) can be interpreted as a simple analytic continuation of the coordinates parametrizing the Cartan torus. This is because both $g$ and $bb^*$ can be parametrized in the Cartan way:

$$g = hth^{-1}, \quad bb^* = hah^{-1},$$

where $h$ is in $G$, $t$ is in the compact Cartan torus $T$ of $G$ and $a$ is in the noncompact part $A$ of the complex Cartan torus $T^C$ of $G^C$. Note in this respect that here $A$ is the same $A$ which appears in the Iwasawa decomposition $G^C = GAN$. 

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As an example, let us explicitly describe the analytic continuation from the non-compact to the compact Cartan torus in the case of the group $SU(N)$ in which $A$ is formed by the real diagonal matrices of the form

$$a_{ij} = e^{\psi_i} \delta_{ij}, \quad \sum_j \psi_j = 0, \quad i, j = 1, \ldots N.$$  \hfill (54)

The analytic continuation of the real Cartan coordinates $\psi_j$ to the strictly imaginary values $i\psi_j$ obviously transforms $a_{ij}$ into an element of the compact Cartan torus $T$ hence it switches from the positive definite Hermitian $bb^*$ to the unitary $g$.

We note finally, that in the present Section 5 we have established the Result 1 as stated in the Section Summary with the notation $p = bb^*$.

\section{Conclusions and outlook}

We have identified the $\lambda$-model on a simple compact Lie group $G$ as a particular case of the $\mathcal{E}$-model and we have used this result to relate the $\lambda$-model to the Poisson-Lie T-dual of the $\eta$-model by the analytic continuation for any simple compact Lie group $G$. We have also interpreted the $\lambda$-model and the $\eta$-model as two branches of a single one-parameter family of dynamical systems characterized by the same Hamiltonian but by the varying Poisson brackets.

It is probable that the framework of the $\mathcal{E}$-models will be useful to establish, for general $G$, the analytic continuation relating the two-parametric $\lambda$ models of Ref. [39] with the duals of the bi-Yang-Baxter models of Ref. [31]. It is also plausible that the dressing cosets generalization of the $\mathcal{E}$-models of Ref. [28] will represent a suitable framework for establishing the analytic continuation relation between the $\eta$ and the $\lambda$ deformations of the $\sigma$-models living on cosets of $G$.

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