Inverse resonance problem for $\mathbb{Z}_2$-symmetric analytic obstacles in the plane*

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Abstract

We given an exposition of a proof that a mirror symmetric configuration of two convex analytic obstacles in $(\mathbb{R})^2$ is determined by its Dirichlet resonance poles. It is the analogue for exterior domains of the proof that a mirror symmetric bounded simply connected analytic plane domain is determined by its Dirichlet eigenvalues. The proof uses ‘interior/exterior duality’ to simplify the argument.

1 Introduction

This article is part of a developing series [Z1, Z2] concerned with the inverse spectral problems for analytic plane domains. It is essentially the lecture we presented at the IMA workshop on Inverse Spectral Problems in July, 2001 and represents the state of our knowledge, methods and results at that time. We apply these methods to prove an analogue for exterior domains of the result proved in [Z2] for interior ones, namely that a mirror symmetric configuration of two convex analytic obstacles in $(\mathbb{R})^2$ is determined by its Dirichlet or Neumann resonance poles. The proof combines the known result that wave invariants of an exterior domain are resonance invariants (see §3) with the method of [Z1, Z2] for calculating the wave invariants explicitly in terms of the boundary defining function. In keeping with the expository

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nature of the lecture, we give a detailed exposition of the background results on the Poisson relation for exterior domains and of the main steps in \(E_1 \), \(E_2\) in the calculation of wave invariants (\(\S 6\)) and on determining the domain from its wave invariants (\(\S 7\)).

The motivating problem is whether analytic domains are determined by their spectra for Dirichlet or Neumann boundary conditions. This inverse spectral problem can be posed for both interior and exterior domains. For the interior problem, we assume the domain is a bounded simply connected plane domain and ask whether its Dirichlet or Neumann spectrum determines the boundary up to rigid motion. For the exterior problem, we assume the domain is the complement of two bounded, simply connected obstacles and ask whether we can determine the pair of obstacles from the resonance poles (or scattering phase) of the exterior Dirichlet or Neumann Laplacian. Since the method is similar for both boundary conditions, we assume the boundary conditions are Dirichlet in the exterior domain.

In the previous paper \(Z_2\), the interior inverse spectral problem is studied for analytic domains with a certain mirror symmetry \(\sigma\). The symmetry is assumed to fix a bouncing ball orbit (as a set) and to reverse its orientation; i.e. if the domain is translated and rotated so that the bouncing ball orbit lies along the y-axis with its midpoint at the origin, then \(\sigma(x, y) = (x, -y)\). In addition, we assume the length of the bouncing ball orbit is a fixed number \(L\) and that the orbit satisfies a non-degeneracy condition. The main result (see Theorem (1.4)) is that the interior domain satisfying some generic conditions is determined by its Dirichlet spectrum among other such domains. We note that this result is stronger than the one in \(Z_3\) in having eliminated one of the two symmetries. We should emphasize that the proof does not immediately imply the result for the other mirror symmetry \(\sigma(x, y) = (-x, y)\), which fixes \(\gamma\) pointwise and preserves the orientation.

Our main purpose in this paper is to extend the result to connected exterior domains of the form

\[
\Omega^c := (\mathbb{R})^2 - \Omega, \quad \text{with} \quad \Omega = \{O \cup \tau_{x,L}O\}
\]

where \(O\) is a bounded, simply connected analytic domain and where

\[
\tau_{x,L} = \text{reflection through the line } \langle \nu_x, x + 1/2L\nu_x - y \rangle = 0.
\]

Here, \(x \in O\), \(\nu_x\) is the outward unit normal to \(O\) at \(x\) and \(L > 0\) is a given positive number. Thus, the obstacle consists of two (non-intersecting)
isometric components which are mirror images of each other across a common orthogonal segment of length $L$. Such an obstacle is called a (symmetric) 2-component scatterer. The segment is the projection to $(\mathbb{R})^2$ of a bouncing ball orbit $\gamma$ of the exterior billiard problem. Throughout, the notation $\Omega$ refers to a bounded domain and $\Omega^c$ is the unbounded complement.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{scatterer.png}
\caption{Mirror symmetric 2-obstacle scatterer.}
\end{figure}

The problem is thus to recover the obstacle $\mathcal{O} \cup \tau_{x,L} \mathcal{O}$ from the set of resonance poles $\{\lambda_j\}$ of the exterior Dirichlet Laplacian $\Delta^{\Omega^c}_D$, i.e. the poles of the analytic continuation of its resolvent

$$R^{\Omega^c}_D(k+i\tau) = (\Delta^{\Omega^c}_D + (k+i\tau)^2)^{-1}$$

\begin{align*}
\end{align*}

to the logarithmic plane. To be precise, we consider the class $\mathcal{OBSTA} - \mathcal{CLE}_{\mathbb{Z}_2,L}$ of two-component obstacles satisfying:
• (i) $\mathcal{O}$ is simply connected and real analytic;

• (ii) $\gamma$ is a non-degenerate bouncing ball orbit, whose length $L$ is isolated in $Lsp(\Omega) \cup Lsp(\Omega^c)$.

We denote by $\mathcal{R}ES(\Omega^c)$ the set of resonance poles of the Laplacian $\Delta_{D}^{\Omega^c}$ of the domain $\Omega^c$ with Dirichlet boundary conditions.

We now state the main results.

**Theorem 1.1.** $\mathcal{R}ES : OBSTACLE_{2,2,L} \mapsto \mathbb{C}^N$ is 1-1.

In fact, in combination with a result of M. Zworski [Zw], the proof shows more: the obstacle is determined by the resonances close to the real axis which are associated to the bouncing ball orbit $\gamma$ of the exterior billiard problem. We note that assumption (ii) is stronger than the one in the interior case (cf. Theorem (1.4) in demanding multiplicity one for $L$ in the combined interior and exterior length spectra. By using the original proof in [Z1, Z2], one could remove $Lsp(\Omega)$ from the assumption and just demand multiplicity one in $Lsp(\Omega^c)$. We do not do so, because we wish to present a simpler proof than the one in [Z1, Z2], as we will explain below in the introduction.

![Figure 2: Bouncing ball orbit.](image)

As a corollary, we prove that the obstacle is determined by its scattering phase. We denote by $S_D(\lambda)$ the Dirichlet scattering operator for $\Omega^c$ and by $s_D(\lambda) = \det S_D(\lambda)$ the scattering phase.

**Corollary 1.2.** If two exterior domains $\Omega$ in $OBSTACLE_{2,2,L}$ have the same scattering phase $s_D(\lambda)$, then they are isometric.

We note that the length $L$ of the segment between the components has been marked, since apriori it is not a resonance invariant. Under some additional assumptions, it is a resonance invariant and one does not have to
mark it. For instance, if one assumes that $O$ is convex, then the bouncing ball orbit $\gamma$ is the unique periodic reflecting ray of the exterior domain up to iterates, so its length is a resonance invariant. Thus, let $\text{OBSTACLE}_{Z_2}$ denote the convex component obstacles in $\text{OBSTACLE}_{Z_2}$. We then have:

**Corollary 1.3.** $\text{RES}: \text{OBSTACLE}_{Z_2} \mapsto \mathbb{C}^N$ is 1-1.

For the sake of completeness, we also recall the precise statement of the interior result of [Z2]. Let $\text{Spec}(\Omega)$ denote the spectrum of the interior Laplacian $\Delta^B_{\Omega}$ of the domain $\Omega$ with either Dirichlet or Neumann boundary conditions.

**Theorem 1.4.** Let $D_L$ denote the class of simply connected real-analytic plane domains $\Omega$ satisfying:

- (i) There exists an isometric involution $\sigma$ of $\Omega$;
- (ii) $\sigma$ ‘reverses’ a non-degenerate bouncing ball orbit $\gamma \mapsto \gamma^{-1}$;
- (iii) The lengths $2rL$ of all iterates $\gamma^r$ have multiplicity one in $\text{Lsp}(\Omega)$, and the eigenvalues of the linear Poincare map $P_{\gamma}$ are not roots of unity;

Then: $\text{Spec}: D_{L} \mapsto (\mathbb{R})_+^N$ is 1-1.

To be cautious, we should point out that these results are part of a work-in-progress which has not yet reached its final form. They have been independently verified when there is an additional mirror symmetry (i.e. each obstacle is itself ‘left-right’ mirror symmetric), and in this case the relevant calculations of wave invariants are easy. When there is only one mirror symmetry, the calculations become somewhat messy (see [Z2]) and the proof therefore becomes rather ‘unstable’; at this time of writing, some details have been independently verified (by R. Bacher [B]), but some have not.

Let us briefly describe the proof and the organization of the paper. As emphasized by Zworski, there is nothing really new in Theorem 1.1 beyond the result of [Z1, Z2] in the interior case (i.e. Theorem 1.4). We are just combining the known fact that wave trace invariants of the exterior domain are resonance invariants with the calculation of the wave trace invariants in terms of the germ of the defining function of $\partial \Omega$ at the endpoints of the bouncing ball orbit. This calculation was done for the germ of any kind of bouncing ball orbit in [Z2], developing a method originating in the work of
Balian-Bloch \cite{BB1, BB2}. However, it seems to us worthwhile to collect all the facts one needs for the proof of Theorem (1.1) in one place and to explain the main steps without all the details in \cite{Z1, Z2}.

Moreover, our exposition has one novel point: in \S4 we use the so-called interior/exterior duality to simplify formula for the resolvent trace in \cite{Z1}. Interior/exterior duality is a distributional trace formula, which we informally write for $\tau >> 0$ as

$$
\text{Tr}_X \left[ R_D^{\text{eff}}(k + i\tau) \oplus R_N^{\text{eff}}(k + i\tau) - R_0(k + i\tau) \right] = \frac{d}{dk} \log \det \left( I + N(k + i\tau) \right). \tag{2}
$$

Here, the determinant is the Fredholm determinant, $R_N^X, R_D^X$ denote Neumann (resp. Dirichlet) resolvents on $X$ and the notation $Tr_X$ indicates the space on which the trace is taken. We write $L^2((\mathbb{R})^2) = L^2(\Omega) \oplus L^2(\Omega^c)$ and view $R_D^{\text{eff}}(k + i\tau) \oplus R_N^{\text{eff}}(k + i\tau)$ as an operator on this space. For notational simplicity, we do not put in the explicit projections $1_\Omega, 1_{\Omega^c}$ (i.e. the characteristic functions). Also, $N(k + i\tau)$ is a boundary integral operator which will be defined in \S4. The correct statement and proof of (2) will be given in Proposition (4.1). It is essentially a differentiated version of the formula

$$
\frac{d}{dE} \log \det S_D(E) + \pi \frac{d}{dE} N_N(E) = 3 \frac{d}{dE} \log \zeta_D,N(E + i0) \tag{3}
$$

of Eckmann-Pillet (cf. \cite{EP, EP2}), relating the scattering phase of the exterior Dirichlet problem, the eigenvalue counting function of the interior Neumann problem and a spectral zeta function for the integral operator $N$ along the boundary.

This reduction to the boundary trace on the right side of (2) simplifies some of the technical details of \cite{Z1} and allows one to deduce the inverse spectral result simultaneously for the interior and exterior problems. In addition, it brings our calculations into closer contact with the physics literature, where the Balian-Bloch approach is now almost always applied to the boundary trace (see e.g. \cite{AG, THS, THS2, GP}. The price we pay is that the combination of inside and outside requires us to make the additional multiplicity free assumption on the combined inner and outer length spectra to obtain the inverse results in Theorem 1.1. As mentioned above, this assumption could be eliminated; but we feel that the simplification in the proof is worth the extra assumption. To simplify the exposition, we also assume
in the last step that \( \mathcal{O} \) is convex. Hence, we only complete the proof for hyperbolic bouncing ball orbits. For the general case, we refer to [Z2].

Let us give a brief outline of the method of [Z1, Z2] and this paper for determining an analytic domain from its spectrum. It is based on the use of certain spectral (i.e. resonance) invariants known as the wave trace invariants at the bouncing ball orbit. As the name implies, the wave trace invariants at a periodic reflecting ray \( \gamma \) are coefficients of the singularity expansion of the relative trace of the wave group \( E_D^{\nu}(t) = \cos t \sqrt{\Delta_D^{\nu}} \) at \( t = L_\gamma \) (the length). The inverse results are proved by explicitly calculating the wave trace invariants at a bouncing ball orbit in terms of the Taylor coefficients of the boundary defining function at the endpoints, and then determining the boundary from these Taylor coefficients.

Rather than studying the wave trace per se in [Z1, Z2] we follow [BB1, BB2] in studying an essentially equivalent expansion involving the resolvent. Potential theory gives an exact formula for the Dirichlet (or Neumann) resolvent of a domain in terms of the free resolvent of \((\mathbb{R})^2\) and of the layer potentials associated to the domain. Formally, one may derive an infinite series expansion known as the ‘multiple reflection expansion’ of [BB1, BB2], whose \( M \)th term corresponds intuitively to \( M \) reflections on the boundary. The trace of each term is, again formally, an oscillatory integral corresponding to \( M \)-fold periodic reflecting rays.

The advantage of this approach is that the formula is exact and does not require a microlocal paramatrix construction, which is messy and complicated for bounded domains and therefore very hard to use in inverse spectral theory. Moreover, the terms of the multiple reflection expansion are ‘canonical’ in that the integrands are the same for all domains; the only difference lies in the domain one integrates over. Once it is ‘legalized’, it gives an explicit and even routine algorithm for calculating wave invariants. At this time of writing, no other approach seems to provide a workable algorithm for doing the calculations. The disadvantages are that the legalization has two complicated parts: one needs to estimate the remainder in the infinite series and to regularize the oscillatory integral defined by the \( M \) term.

Our formulae for the wave invariants come from applying stationary phase to the regularized traces. One of the principal results is that, for each iterate \( \gamma^r \), only the term \( M = rm \) of the multiple reflection expansion contributes important inverse spectral data, because only this term contains the maximum number of derivatives of the defining function of \( \Omega \) at a given order of \( k^{-j} \). The heart of the matter is then the calculation and analysis of the
wave invariants. Since one needs wave invariants of all orders, one seemingly runs into an infinite jungle of complicated expressions. To tame the jungle, we enumerate and evaluate the terms of the wave trace invariants using Feynman diagram techniques. It turns out that only five diagrams play an important role (this has been confirmed by R. Bacher). The remaining issue is the calculation of the amplitudes and their dependence on $r$. It is this latter dependence that is crucial in allowing one to remove a symmetry of the domain.

The calculation of wave trace invariants is the same for periodic reflecting rays of interior and exterior domains since it depends only on the germ of the boundary at the reflection points. The material in §5 - §7 summarizes the calculation in [Z1, Z2]. We hope that this guide clarifies the calculation and makes it easier to check the details.

We close the introduction with some remarks on open problems in the inverse spectral problem for analytic plane domains. The most obvious one is whether one can eliminate the remaining symmetry assumption. It may be that one can recover the domain from wave invariants at one closed billiard orbit, or that one has to combine information from several orbits. A key problem is, how much of the Taylor expansion of the boundary defining function one can recover from wave invariants at one orbit?

The method of this paper and [Z2] gives detailed but somewhat undigested information about wave invariants in terms of the Taylor coefficients of the boundary defining function at endpoints of a bouncing ball orbit. Another approach, suggested originally by Colin de Verdiere [CdV], is to focus on the Birkhoff normal form of the billiard map at the closed orbit. As was proved by V. Guillemin in the boundaryless case [G], the Birkhoff normal form at a closed orbit is a spectral invariant. In [Z3] the author generalized this to the boundary case. Colin de Verdiere’s observation that the Birkhoff normal form at a bouncing ball orbit determines an analytic plane domain with two symmetries then gave the solution of the inverse spectral problem for analytic plane domains with these symmetries [Z3]. We found it difficult however to calculate the normal form coefficients by this method without assuming two symmetries. That motivated us to try the Balian-Bloch approach.

In the recent paper [ISZ], Iantchenko-Sjostrand-Zworski give another proof of the inverse result with two symmetries using Birkhoff normal forms of the billiard map and quantum monodromy operator rather than the Laplacian; the method is quite elegant and flexible. However, we believe that their nor-ti-
mal form of the monodromy operator is the same as the normal form in \([Z3]\) of what is called there the semiclassical wave group. We changed to the present approach because we found it too difficult to calculate the latter without the two symmetry assumption. However, the possibility remains that the Birkhoff normal form could be a better way to organize the inverse spectral data than the wave invariants themselves. The question is, how much of the boundary defining function is determined by the Birkhoff normal form of the wave group or the monodromy operator at a periodic orbit? It is known that the classical Birkhoff normal form by itself will not determine the Taylor coefficients of the domain unless there are two symmetries.

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2 Billiards and the length functional

Throughout this paper, we will assume that our obstacle is up-down symmetric in the sense that there is an isometric involution \(\sigma\) of \(\Omega\) which interchanges the endpoints of a bouncing ball orbit (extremal diameter). However, we only use the assumption in the last step of calculating wave invariants and determining the domain. Elsewhere, we make it for notational convenience. We now extend the notation and terminology of \([Z1]\) so that it applies to exterior domains which are complements of obstacles with two components.

We align the obstacles so that the bouncing ball orbit \(\gamma\) projects to the line segment \(\overline{a_-a_+}\) along the \(y\)-axis and so that its midpoint lies at the origin \(0 \in (\mathbb{R})^2\) and its endpoints are \(a_\pm = (0, \pm L/2)\). We refer to the top component of \(\Omega\) as the + component \(\mathcal{O}_+\) and the bottom one as the − component \(\mathcal{O}_-\).

There are two convenient parametrizations of \(\partial\Omega\). First, we denote by \(q_\pm(\theta)\) the arc-length parameterization of \(\mathcal{O}_\pm\) with \(q_\pm(0) = a_\pm\). At the end we will also use graph parametrizations: In a small strip \(T_\varepsilon(\gamma)\) around \(\overline{a_-a_+}\), the boundary consists of two components which are symmetric graphs over the \(x\)-axis. We write the graphs in the form \(y = \pm f(x)\) near \(a_\pm\).

In the multiple reflection expansion of the exterior resolvent \(R_D^{\Omega^c}(k +\)
\( i \tau \), we will encounter multiple integrals \( \int_{(\partial \Omega)M} \). Since \( \partial \Omega \) consists of two components, each such term breaks up into \( 2^M \) multiple integrals over the circle \( T \). The terms can be enumerated by maps \( \sigma : \{1, \ldots, M\} \to \{\pm\} \). We needed the same enumeration in [22] to denote the two local components of \( \Omega \) at the ends of the bouncing ball orbit.

### 2.1 Length functional

For each map \( \sigma : \{1, \ldots, M\} \to \{\pm\} \) as above, we define a length functional on \( T^M \) by:

\[
L_{\sigma}(\varphi_1, \ldots, \varphi_M) = |q_{\sigma_1}(\varphi_1) - q_{\sigma_2}(\varphi_2)| + \cdots + |q_{\sigma_{M-1}}(\varphi_{M-1}) - q_{\sigma_M}(\varphi_M)|. 
\]

(4)

It is clear that \( L_{\sigma} \) is a smooth function if \( \sigma(j) \neq \sigma(j+1) \) for any \( j \). When there do exist such \( j \), then \( L_{\sigma} \) is singular on the ‘large diagonal’ \( \Delta_{j,j+1} := \{\varphi_j = \varphi_{j+1}\} \), where it has \(|x|\) singularities. A standard (and easy) calculation shows that

\[
\frac{\partial}{\partial \varphi_j} L_{\sigma} = \sin \angle(\nu_{q_{\sigma_{j+1}}(\varphi_{j+1})} - q_{\sigma_j}(\varphi_j), q_{\sigma_{j+1}}(\varphi_{j+1}), q_{\sigma_j}(\varphi_j)) - \sin \angle(q_{\sigma_j}(\varphi_j) - q_{\sigma_{j+1}}(\varphi_{j+1}), q_{\sigma_j}(\varphi_j), \nu_{q_{\sigma_{j+1}}(\varphi_{j+1})}). 
\]

(5)

Here we denote the acute angle between the link \( q_{\sigma_{j+1}}(\varphi_{j+1}) - q_{\sigma_j}(\varphi_j) \) and the inward unit normal \( \nu_{q_{\sigma_{j+1}}(\varphi_{j+1})} \) by \( \angle(\nu_{q_{\sigma_{j+1}}(\varphi_{j+1})} - q_{\sigma_j}(\varphi_j), q_{\sigma_{j+1}}(\varphi_{j+1}), q_{\sigma_j}(\varphi_j)) \). The condition that \( \frac{\partial}{\partial \varphi_j} L = 0 \) is thus that the 2-link defined by the triplet \( (q_{\sigma_{j-1}}(\varphi_{j-1}), q_{\sigma_{j}}(\varphi_{j}), q_{\sigma_{j+1}}(\varphi_{j+1})) \) is Snell at \( \varphi_j \), i.e. satisfies the law of equal angles at this point. A smooth critical point of \( L \) on \( T^M \) is thus the same as an \( M \)-link Snell polygon. Note that Snell polygons include polygons which have links inside the obstacles. They are known as ‘ghost orbits’ in the physics literature, since they cross the boundary between the interior and exterior domains. One may also include singular critical points of \( L_{\sigma} \) which correspond to polygons in which an edge is collapsed.

### 2.2 Length spectrum

We will denote by \( Lsp(\Omega^c) \), resp. \( Lsp(\Omega) \), the lengths of periodic billiard trajectories in \( \Omega^c \), resp. \( \Omega \). We do not include ghost orbits. By periodic
trajectory, we mean closed bicharacteristic curve; for the definition see [GM, PS]. Special periodic trajectories are $m$-link periodic reflecting ray, i.e. $m$-link Snell polygons. Their lengths are isolated in the length spectrum.

When $\Omega^c$ is the complement of a pair of disjoint, strictly convex obstacles, the only exterior transversal reflecting rays are iterates $\gamma^r$ of the bouncing ball orbit $\gamma$ between the components, i.e. the line segment between the points $a_\pm \in O_\pm$. When $\mathcal{O}$ is not assumed convex, there may be additional periodic reflecting rays and the shortest periodic orbit of $\Omega^c$ need not be the invariant bouncing ball orbit. Recall that we assume throughout that $L_\gamma$ is isolated in $Lsp(\Omega) \cup Lsp(\Omega^c)$.

It is part of the folklore of the Balian-Bloch approach that contributions to the trace formula due to ghost orbits are exactly cancelled by other ghost orbits. We will not prove that this cancellation occurs since the rigorous Poisson relation already implies that only $\gamma^r$ contribute to the wave trace. Additionally, there are so-called diffraction orbits, which have points of tangential intersection with the boundary and may glide along the boundary. They are not critical points of the length functionals, and correspond not to smooth singularities in the wave trace, but to analytic singularities.

3 Poisson relation for the exterior problem

We now recall the Poisson relation for exterior domains in dimension 2.

The exterior Green’s kernel $G_{\Omega^c}^{\Omega^c}(k+i\tau, x, y) \in \mathcal{D}'(\Omega^c \times \Omega^c)$ with boundary condition $B = D$, or $N$ (Dirichlet or Neumann) is the kernel of the exterior resolvent $R_{\Omega^c}^{\Omega^c}(k+i\tau) = -(\Delta_{\Omega^c}^{\Omega^c}+(k+i\tau)^2)^{-1} : H^s(\Omega^c) \rightarrow H^{s+2}(\Omega^c)$. Here, $k \in R_D^\Omega$ and $\tau > 0$. Its kernel (the exterior Green’s function) may be characterized as the unique solution of the boundary problem:

$$\begin{cases} 
-\Delta_{\Omega^c}^{\Omega^c}(k+i\tau)^2G_{\Omega^c}^{\Omega^c}(k+i\tau, x, y) = \delta_y(x), & (x, y \in \Omega) \\
BG_{\Omega^c}^{\Omega^c}(k+i\tau, x, y) = 0, & x \in \partial \Omega \\
\partial G_{\Omega^c}^{\Omega^c}(k+i\tau, x, y) \over \partial r - i(k+i\tau)G_{\Omega^c}^{\Omega^c}(k+i\tau, x, y) = o(1), & \text{as } r \rightarrow \infty. 
\end{cases}$$

(6)

Here, the boundary operator could be either $Bu = u|_{\partial \Omega^c} (D)$ or $Bu = \partial_r u|_{\partial \Omega^c} (N)$. We use a similar notation for the interior resolvent and boundary conditions. The boundary conditions $D$ and $N$ are in a certain sense comple-
mentary, and we will write $B'$ for the complementary boundary condition to $B$.

We are interested in the regularized distribution trace of the combined operator $R^{\Omega^c}_{\Omega}(k + i\tau) \oplus R^{\Omega^c}_{\Omega'}(k + i\tau)$, more usually regarded as a distribution trace of the combined wave group $E^{\Omega^c}_{\Omega}(t) \oplus E^{\Omega^c}_{\Omega'}(t)$, where $E^{\Omega^c}_{\Omega}(t) = \cos t\sqrt{-\Delta^{\Omega^c}_{\Omega}}$. Let $\hat{\rho} \in C_0^\infty((\mathbb{R})^+)$. Then:

$$R^{\Omega^c}_{\rho B}(k + i\tau) \oplus R^{\Omega^c}_{\rho B'}(k + i\tau)$$

$$\quad = \int_{R^{\Omega^c}_{\rho B'}} \rho(k - \mu)(\mu + i\tau)R^{\Omega^c}_{\rho B'}(\mu + i\tau) \oplus R^{\Omega^c}_{\rho B'}(\mu + i\tau)d\mu. \quad (7)$$

From the resolvent identity (e.g.)

$$R^{\Omega^c}_{\Omega}(\mu + i\tau) = \frac{1}{\mu + i\tau} \int_0^\infty e^{i(\mu + i\tau)t} E^{\Omega^c}_{\Omega}(t)dt,$$

it follows that

$$R^{\Omega^c}_{\rho B}(k + i\tau) \oplus R^{\Omega^c}_{\rho B'}(k + i\tau) = \int_0^\infty \hat{\rho}(t)e^{i(k + i\tau)t} E^{\Omega^c}_{B}(t) \oplus E^{\Omega^c}_{B'}(t)dt. \quad (8)$$

### 3.1 Poisson relation and scattering phase

The following is the Birman-Krein formula for the relative trace of the wave group of an exterior problem, as generalized by Lax-Phillips and Bardos-Guillot-Ralston (see [BGR], Theoreme 3) : Let $\hat{\varphi} \in C_0^\infty(R^{\Omega^c}_{\rho D})$. Then:

$$Tr \int_{R^{\Omega^c}_{\rho B'}} \hat{\varphi}(t)[E^{\Omega^c}_{\Omega}(t) \oplus E^{\Omega^c}_{\Omega'}(t) - E_0(t)]dt = \frac{1}{2\pi} \int_{R^{\Omega^c}_{\rho B'}} \varphi(\lambda)[ds_B(\lambda) + dN_{B'}(\lambda)].$$

Putting $\hat{\varphi}(t) = \hat{\rho}(t)e^{i(k + i\tau)t}$ and rewriting in terms of the resolvent, we get:

$$Tr[R^{\Omega^c}_{\rho B}(k + i\tau) \oplus R^{\Omega^c}_{\rho B'}(k + i\tau) - R_0\rho(k + i\tau)]$$

$$\quad = \frac{1}{2\pi} \int_{R^{\Omega^c}_{\rho B}} \rho(k + i\tau - \lambda)[ds_B(\lambda) + dN_{B'}(\lambda)]. \quad (9)$$

Here, $s_B(\lambda) = \log \det S_B(\lambda)$ where $S_B(E)$ is the scattering operator and $N_{B'}$ is the interior Weyl counting function with the indicated boundary conditions.
The singular support of the regularized wave trace $Tr[E_B^{\text{fr}}(t) - E_0(t)]$ is contained in the set of lengths of exterior periodic generalized billiard trajectories. That is, by [BGR], Theoreme 5, we have

$$\text{singsupp}Tr[E_B^{\text{fr}}(t) - E_0(t)] \subset Lsp(\Omega^c).$$

Similarly for the interior wave group with $Lsp(\Omega)$ replacing $Lsp(\Omega^c)$. When $L_\gamma$ is the length of a non-degenerate periodic reflecting ray $\gamma$, and when $L_\gamma$ is not the length of any other generalized periodic orbit, then $Tr[E^{\text{fr}}_B(k+i\tau) - E_0(t)]$ has a complete asymptotic expansion in powers of $k^{-1}$. Let us recall the statement in the exterior case (see [GM], Theorem 1, and [PS] Theorem 6.3.1 for the interior case, and [BGR], §6 for the exterior case). Let $\gamma$ be a non-degenerate billiard trajectory whose length $L_\gamma$ is isolated and of multiplicity one in $Lsp(\Omega)$. Let $\Gamma_L$ be a sufficiently small conic neighborhood of $(\mathbb{R})^+\gamma$ and let $\chi$ be a microlocal cutoff to $\Gamma_L$. Then for $t$ near $L_\gamma$, the trace of the wave group has the singularity expansion

$$Tr\chi[E_B^{\text{fr}}(t) - E_0(t)] \sim a_\gamma(t-L_\gamma+i0)^{-1} + a_{\gamma 0} \log(t-L_\gamma+i0)$$

$$+ \sum_{k=1}^{\infty} a_{\gamma k}(t-L_\gamma+i0)^k \log(t-L_\gamma+i0)$$

where the coefficients $a_{\gamma k}$ are calculated by the stationary phase method from a Lagrangian parametrix. In the interior case, one of course omits the term $E_0(t)$.

The result may be re-stated as follows: Let $\hat{\rho} \in C_0^\infty(L_\gamma - \epsilon, L_\gamma + \epsilon)$, equal to one on $(L_\gamma - \epsilon/2, L_\gamma + \epsilon/2)$ and with no other lengths in its support. Then the interior trace $TrR_B^\Omega(k + i\tau)$ and the exterior trace $Tr[R_B^\Omega(k + i\tau) - R_0(k + i\tau)]$ admit complete asymptotic expansions of the form

$$Tr[R_B^\Omega(k + i\tau) - R_0(k + i\tau)] \sim \sum_{j=0}^{\infty} B_{\gamma,j} k^{-j}$$

$$TrR_B^\Omega(k + i\tau) \sim e^{i(k+i\tau)\tau L_\gamma} \sum_{j=0}^{\infty} B_{\gamma,j} k^{-j},$$

whose coefficients $B_{\gamma,j}$ are canonically related to the wave invariants $a_{\gamma,j}$ of periodic (internal, resp. external) billiard orbits. We have removed the cutoff operator $\chi$ since there are no singularities of the trace at $t = L_\gamma$ in the microsupport of $I - \chi$. The coefficients depend on the choice of boundary condition but we do not indicate this in the notation.
3.2 Poisson relation and resonances

We now recall the Poisson relation in dimension 2. Let \( \{\lambda_j\} \) denote the resonances of \( \Delta^\Omega_{\mathcal{BC}} \), i.e. the poles of the analytic continuation of \( R^\Omega_B(k + i\tau) \) from \( \{\tau > 0\} \) to the logarithmic plane \( \Lambda \). Let \( \theta \) denote a (small) angle, and let \( \Lambda_\theta \) denote the conic neighborhood of the real axis in \( \Lambda \) consisting of points with angular coordinates in \((-\theta, \theta)\). Further, let \( \hat{\varphi} \in C^\infty((\mathbb{R})^+) \) and let \( E_0(t) \) denote the free even wave kernel. The global Poisson formula (asserts the following:

**Proposition 3.1.** (see Zworski \[Zw\], Theorem 1)

\[
\text{Tr}\left( \int_{R^\Omega_D} \hat{\varphi}(t)(E^\Omega_B(t) - E_0(t))dt \right) = \sum_{\lambda_j \in \Lambda_\theta} m(\lambda_j)\varphi(\lambda_j) + m(0)\varphi(0) \\
+ 2 \int_0^\infty \psi(\lambda)\varphi(\lambda) \frac{ds_B}{d\lambda} d\lambda + \int_0^\infty \hat{\varphi}(t)\nu_{\theta,\psi}(t)dt,
\]

with \( \nu_{\theta,\psi} \in C^\infty(R^\Omega_D \setminus \{0\}) \), \( \partial_t^k \nu_{\theta,\psi} = O(t^{-N}) \), \( |t| \to \infty \),

where \( s_B \) denotes the scattering phase of \( \Delta^\Omega_{\mathcal{BC}} \), and where \( m(\lambda) \) denotes the multiplicity of the resonance \( \lambda \). Also, \( \psi \in C^\infty_0(R^\Omega_D) \) is a cutoff which equals 1 for \( t \) near 0.

(We note that the sum over eigenvalues term in the formula stated in \[Zw\] (Theorem 1) is absent in the case of an exterior domain.)

Substituting \( \hat{\varphi}(t) = \hat{\rho}(t)e^{i(k + i\tau)t} \), with \( \hat{\rho} \in C^\infty((\mathbb{R})^+) \) with \( \text{supp}\hat{\rho} \) sufficiently close to \( rL \) for some \( r \), into (3.1) and using (8), we obtain:

\[
\text{Tr}[R^\Omega_{\rho B}(k + i\tau) - R_{0\rho}(k + i\tau)] = \sum_{\lambda_j \in \Lambda_\theta} m(\lambda_j)\rho(\lambda + k + i\tau) + O(|k|^{-M}), \quad (k \to \infty).
\]

(12)

In the remainder estimate we use that

\[
m(0)\rho(k + i\tau), \quad \int_0^\infty \psi(\lambda)\rho(\lambda + k + i\tau) \frac{ds_B}{d\lambda} d\lambda, \quad \langle \nu_{\theta,\psi}, \hat{\rho}(t)e^{i(k + i\tau)t} \rangle = O(k^{-N}).
\]

In the case of the first two terms, this follows immediately from the fact that \( \varphi(\lambda) = \rho(k + i\tau - \lambda) \) and that both terms are integrals over a compact set of values of \( \lambda \). For the third term we additionally use that \( \hat{\rho} \) vanishes in a neighborhood of \( t = 0 \); integration by parts and the estimate on on \( \nu_{\theta,\psi} \) then
gives the rapid decay in $k$ of the term. It follows that the asymptotics of the regularized resolvent trace $\text{Tr}[R_{\rho B}^{\Omega}(k + i\tau) - R_{0\rho}(k + i\tau)]$ are an invariant of the resonance poles.

The following well-known proposition, explained to the author by M. Zworski, implies more: only poles in a logarithmic neighborhood $\{|\Re \lambda_j| > N \log |\lambda_j|\}$ of the real axis cause singularities in the wave trace:

**Proposition 3.2.** For any $N \geq 0$, there exists $k(N)$ with $k(N) \to \infty$ as $N \to \infty$, such that

$$\sum_{\lambda_j: |\Im \lambda_j| < \rho |\lambda_j|, |\Re \lambda_j| > N \log |\lambda_j|} e^{i\lambda_j t} \in C^k(N)(R^{\Omega}_D - \{0\}).$$

**Proof.** We have

$$|\hat{\partial}_t^k \sum_{\lambda_j: |\Re \lambda_j| < \rho |\lambda_j|, |\Re \lambda_j| > N \log |\lambda_j|} e^{i\lambda_j t}|$$

$$\leq \sum_{\lambda_j: |\Im \lambda_j| < \rho |\lambda_j|, |\Re \lambda_j| > N |\lambda_j|} |\lambda_j|^k |\lambda_j|^{-Nt}$$

$$\leq \int_{(\mathbb{R})^+} r^{k-tN} dN(r) \leq C \int_{(\mathbb{R})^+} r^{n+k-1-tN} dr. \quad (13)$$

In the last line we used a polynomial bound on the number of resonance poles (cf. [Zw] (2.1) for the estimate). Thus, the portion of the wave trace coming from poles outside of sufficiently large logarithmic neighborhoods of the real axis is as smooth as desired. \qed

### 4 Neumann expansion and Poisson relation

In the Balian-Bloch approach to the Poisson relation, one uses the classical theory of layer potentials ([TII], §7.11) to express $G_D^{\Omega}(k + i\tau, x, y)$ in terms of the ‘free’ Green’s function $G_0(k + i\tau, x, y)$ of $(\mathbb{R})^2$, i.e. the kernel of the free resolvent $(\Delta_0 - k + i\tau^2)^{-1}$ of the Laplacian $\Delta_0$ on $(\mathbb{R})^2$. A classical reference for this approach is the paper of Pleijel [P]. This approach is also used in [M] to make a reduction to the boundary in studying resonance poles and in the duality result of (3). We review the necessary background and notation.
4.1 Potential theory

We recall that the free Green’s function in dimension two is given by: \( G_0(k + \imath \tau, x, y) = H_0^{(1)}(k + \imath \tau|x - y|) \), where \( H_0^{(1)}(z) \) is the Hankel function of index 0. We recall its small and large distance asymptotics ([TI], Chapter 3, §6):

\[
H_\nu^{(1)}(r) = \begin{cases}
-\frac{1}{2\pi} \ln(r) & \text{as } r \to 0, \text{ if } \nu = 0 \\
-\frac{i\Gamma(\nu)}{\pi} \left(\frac{2}{r}\right)^\nu & \text{as } r \to 0, \text{ if } \nu > 0 \\
e^{i(k + \imath \tau r/2 - \pi/4)} \frac{1 + O\left(\frac{1}{|r|^3}\right)}{r^{1/2}} & \text{as } |k + \imath \tau| r \to \infty. 
\end{cases}
\] (13)

Here it is assumed that \( \tau > 0 \). Thus, \( G_0(k + \imath \tau, x, y) \) has two kinds of asymptotics: a semiclassical asymptotics for large \( |k + \imath \tau| |x - y| \) and a singularity asymptotics along the diagonal. It is the first kind of asymptotics which accounts for the connection to billiard trajectories. The singularity along the diagonal will have to be regularized.

4.1.1 Layer potentials and boundary integral operators

The double layer potential is the operator

\[
\mathcal{D}\ell(k + \imath \tau)f(x) = \int_{\partial \Omega} \frac{\partial}{\partial \nu_y} G_0(k + \imath \tau, x, q) f(q) ds(q),
\]

from \( H^s(\partial \Omega) \to H^{s+1/2}_{loc}(\Omega) \), where \( ds(q) \) is the arc-length measure on \( \partial \Omega \), where \( \nu \) is the interior unit normal to \( \Omega \), and where \( \partial \nu = \nu \cdot \nabla \). It induces the boundary operator

\[
N(k + \imath \tau)f(q) = 2 \int_{\partial \Omega} \frac{\partial}{\partial \nu_y} G_0(k + \imath \tau, q, q') f(q') ds(q')
\]

which map \( H^s(\partial \Omega) \to H^{s+1}(\partial \Omega) \).

By the explicit formula we have:

\[
\frac{1}{2} N(k + \imath \tau, q(\varphi), q(\varphi')) = \partial_{\nu_y} G_0(\mu, q(\varphi), q(\varphi'))
= - (k + \imath \tau) H_1^{(1)}(k + \imath \tau|q(\varphi) - q(\varphi')|)
\times \cos \angle(q(\varphi) - q(\varphi'), \nu_{q(\varphi)}).
\]
We will need the ‘jump’ formula of potential theory (\cite[Chapter 7.11, (11.7)]{TI}),
\[ (D\ell(k + i\tau)u)_{\pm}(q) = \frac{1}{2}(N(k + i\tau) \mp I)u(q), \]
where
\[
\begin{align*}
  f_+(x) := \lim_{x \to q, x \in \partial} f(x), \\
  f_-(x) = \lim_{x \to q, x \in \Omega} f(x).
\end{align*}
\]
Note that the sign of each term in (16) depends on the choice of the interior/exterior: in \(N(k + i\tau)\) it depends on the choice of interior/exterior unit normal and in the identity operator term it depends on whether the limit is taken from the interior or exterior. In the interior case we have \((D\ell(k + i\tau)u)_{\pm}(q) = \frac{1}{2}(N(k + i\tau) \pm I)u\) where \(N(k + i\tau)\) is defined using the interior normal and where \(\pm\) have the same meanings as in the interior case.

4.2 Interior/Exterior duality

In \cite{Z1, Z2} we used the classical reduction of the Dirichlet problem to the boundary to study the Dirichlet resolvent (as in \cite{BB1, BB2}). We now use a similar method to obtain a convenient formula for the Fredholm determinant, \(\det((I + N(k)))\). Similar ideas can be found in \cite{THS, THS2, AG, GP}.

The following formula is sometimes referred to as interior/exterior duality. Combined with the Birman-Krein formula (9) it expresses the growth rate of the inside plus outside spectra in terms of the determinant of a boundary integral operator:

**Proposition 4.1.** For any \(\tau \geq 0\), the operator \((I + N(k + i\tau))\) is of trace class and has a well-defined Fredholm determinant, and we have:

\[
Tr_{(\mathbb{R})^2}[R_{\rho_D}^{\Omega e}(k + i\tau) \oplus R_{N,\rho}^{\Omega} - R_{0\rho}(k + i\tau)]
= \int_{R_{\rho_D}^{\Omega}} \rho(k - \lambda) \frac{d}{d\lambda} \log \det(I + N(\lambda + i\tau))d\lambda.,
\]

where \(\det(I + N(\lambda + i\tau))\) is the Fredholm determinant.

**Proof.** We first argue formally. The interior/exterior resolvent kernels can be constructed in the classical way, as follows (see e.g. \cite{P}). For simplicity, we consider the interior Dirichlet resolvent, but the construction for the exterior
resolvent or with a change to Neumann boundary conditions is almost the same. We have:

\[ R^{\Omega}_{\Omega} (k + i\tau) = 1_{\Omega^c} [R_0(k + i\tau) - \mathcal{D} \ell(k + i\tau)(I + N(k + i\tau))^{-1} \gamma R_0(k + i\tau)] 1_{\Omega^c}, \]

\[ (R^{N}_N (k + i\tau))^\tau = 1_\Omega [R_0(k + i\tau) - \mathcal{D} \ell(k + i\tau)(I + N(k + i\tau))^{-1} \gamma R_0(k + i\tau)] 1_\Omega, \]

(18)

where \( \gamma \) denotes the restriction to the boundary taken from within the relevant domain. When we take the regularized trace, we first subtract from \( R^{\Omega}_{\Omega} (k + i\tau) \oplus R^{N}_N (k + i\tau) \) the free operator \( R_0(k + i\tau) \), which removes the first terms on the right. We then cycle the factor \( \gamma R_0(k + i\tau) \) from the right to the left side, obtaining an operator on \( \partial \Omega \). The inside and outside terms add up to the kernel

\[ \int_{(\mathbb{R})^2} G_0(k + i\tau, q', y) \partial_{\nu_q} G_0(k + i\tau, y, q) dy. \]

(19)

This indeed is why the interior Dirichlet and exterior Neumann problems were combined and explains the sense in which they are complementary. We now claim that the resulting kernel equals \( \frac{1}{2k} \frac{d}{dk} N(k + i\tau, q', q) \). To prove this we note that

\[ \frac{d}{d\lambda} R_0(\lambda) = 2\lambda (\Delta + \lambda^2)^{-2} = 2\lambda R_0(\lambda)^2. \]

Hence,

\[ \frac{d}{d\lambda} N(\lambda) = \frac{d}{d\lambda} \gamma \partial_{\nu_q} R_0(\lambda) \gamma = 2\lambda \gamma \partial_{\nu_q} R_0(\lambda) \circ R_0(\lambda) \gamma \]

which is precisely the kernel (19). Hence the right side of (18) equals \( \frac{1}{k + i\tau} \frac{d}{dk} \log(I + N(k + i\tau)) \). The final formula follows by putting \( k = \lambda \), integrating against \( \rho(k - \lambda)(\lambda + i\tau) \) and noting the cancellation of the second factor.

To justify the formal manipulations, we need to show that the relevant traces and determinant are well defined. This will be done in a sequence of Lemmas.

**Lemma 4.2.** For any \( \tau \), \( N(k + i\tau) \in \mathcal{L}_1(\partial \Omega) \), the ideal of trace class operators on \( L^2(\partial \Omega) \). Hence, \( \det(I + N(k + i\tau)) \) is well-defined as a Fredholm determinant.

**Proof.** In [Z1] it is proved that \( N(k + i\tau) \in \Psi^{-2}(\partial \Omega) \) and this immediately implies that it is of trace class. In fact, \( N(k + i\tau, q, q') \) has just a \( |q - q'| \log |q -
singularity on the diagonal. That it is trace class also follows from the Hille-Tamarkin theorem (see [Z1] for references).

It is then a classical remark that the Fredholm determinant \( \det(I + N(k + i\tau)) \) is well-defined, in fact \( |\det(I + N(k + i\tau))| \leq \exp(||N(k + i\tau)||_1) \), where \( || \cdot ||_1 \) is the trace norm. See [S], Lemma 3.3.

**Remark 4.3.** 1 Note that the usual statement (cf. [III]) is that \( N(k + i\tau) \) is of order \(-1\). However, the principal symbol vanishes in dimension 2.

2 We recall (see [S], Theorem 3.10) that the Fredholm determinant is given by:

\[
\det(I + N(k + i\tau)) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\partial\Omega^n} \det[N(k + i\tau, q_i, q_j)]_{1 \leq i, j \leq n} ds(q_1) \cdots ds(q_n).
\]

Again combining classically known facts, we have:

**Lemma 4.4.** For any \( \tau > 0 \), \( \log \det N(k + i\tau) \) is well-defined and differentiable in \( k \), \( (I + N(k + i\tau))^{-1}N'(k + i\tau) \) is of trace class and we have:

\[
\frac{d}{dk} \log \det N(k + i\tau) = Tr_{\partial\Omega}(I + N(k + i\tau))^{-1}N'(k + i\tau).
\]

**Proof.** First we note that \( (I + N(k + i\tau)) \) is invertible on \( L^2(\partial\Omega) \) if \( \tau > 0 \), so \( \det(I + N(k + i\tau)) \neq 0 \). Hence its logarithm is well-defined. Differentiability of \( \log \det(I + N(k + i\tau)) \) and the formula for the derivative follows from the general fact that \( \det(I + A) \) is Frechet differentiable on \( \mathcal{I}_1 \) with derivative equal to \( (I + A)^{-1} \) if \(-1 \notin \sigma(A)\) (here, \( \sigma(A) \) is its spectrum; see [S], Corollary 5.2).

The singularity on the diagonal of \( N(k + i\tau, q, q') \) is independent of \( k + i\tau \), so the derivative is at least as regular. Hence, \( N'(k + i\tau) \in \mathcal{I}_1 \). The statement that \( (I + N(k + i\tau))^{-1}N'(k + i\tau) \in \mathcal{I}_1 \) follows from the fact that \( (I + N(k + i\tau))^{-1} \in \Psi^0(\partial\Omega) \).

**Lemma 4.5.** The operators

\[
\begin{align*}
(i) \quad & 1_\Omega \int_{\partial\Omega} \rho(k - \lambda)D\ell(\lambda + i\tau)(I + N(\lambda + i\tau))^{-1}\gamma R_0(\lambda + i\tau)]d\lambda 1_\Omega, \\
(ii) \quad & 1_{\Omega^c} \int_{\partial\Omega} \rho(k - \lambda)D\ell(\lambda + i\tau)(I + N(\lambda + i\tau))^{-1}\gamma R_0(\lambda + i\tau)]d\lambda 1_{\Omega^c},
\end{align*}
\]
are of trace class and the sum of their traces equals \( \text{Tr}_{\Omega} (I + N(k+i\tau))^{-1} N'(k+i\tau) \).

**Proof.** We expand \( (I + N(\lambda+i\tau))^{-1} = I - N(\lambda)(I + N(\lambda+i\tau))^{-1} \). The \( I \) term gives us \( \mathcal{D}_\ell(\lambda+i\tau)\gamma R_0(\lambda+i\tau) \). We break up each of \( \mathcal{D}_\ell(\lambda+i\tau) \) and \( \gamma R_0(\lambda+i\tau) \) into regular (continuous) and singular parts using the small \( z \) expansion (13) of the Hankel functions \( H_0^1(z), H_1^1(z) \). The singular parts of the kernels are independent of \( (k+i\tau) \). Hence the product of the singular parts gets multiplied by ˆ\( \rho \)(0) = 0 upon integration against \( \rho(k-\lambda) \). Therefore only the regular parts survive the convolution, and these are of trace class. The remaining term is a bounded operator composed with trace class operator for each \( \lambda \), so it is of trace class for each \( \lambda \).

We may then cycle the factor of \( \mathcal{D}_\ell(\lambda+i\tau) \) (or each of its regular and singular parts) to the right side of the traces. After doing so, we reassemble the regular and singular parts into \( \gamma R_0(\lambda+i\tau) \mathcal{D}_\ell(\lambda+i\tau) \) and we reassemble \( I - N(\lambda)(I + N(\lambda+i\tau))^{-1} \) into \( (I + N(\lambda+i\tau))^{-1} \). The calculation at the beginning of the proof then gives the stated formula.

\[ \square \]

This concludes the proof of Proposition (4.1)

\[ \square \]

**Corollary 4.6.** Suppose that \( L_\gamma \) is the only length in the support of \( \hat{\rho} \).

Then,

\[ \int_{R_0^D} \rho(k-\lambda) \frac{d}{d\lambda} \log \det(I + N(\lambda+i\tau))d\lambda \sim \sum_{j=0}^{\infty} B_{\gamma;j} k^{-j} \]

where \( B_{\gamma;j} \) are the wave invariants of \( \gamma \) in (11).

To prove Theorems (1.1) and (1.4), it thus suffices to determine \( \Omega \) from the integrals in Corollary (4.6). Since we have combined the interior/exterior, we emphasize that our inverse result only assumes knowledge of the resonances: Indeed, suppose the resonance poles of \( \Omega^c \) are known. Then by Proposition (3.1), the asymptotics of the exterior resolvent trace in terms of exterior periodic orbits are known. But by Proposition (4.1), these are the same as the asymptotics of the integrals in Corollary (4.6).
5 Trace asymptotics

We now explain how to use Corollary (4.6) to calculate the coefficients $B_{\gamma,j}$. We write

$$\frac{d}{d\lambda} \log \det(I + N(\lambda + i\tau)) = Tr_{\partial \Omega} (I + N(\lambda + i\tau))^{-1} N'(k + i\tau),$$

and then expand $(I + N(k + i\tau))^{-1}$ in a finite geometric series plus remainder:

$$(I + N(\lambda + i\tau))^{-1} = \sum_{M=0}^{M_0} (-1)^M N(\lambda)^M + (-1)^{M_0+1} N(\lambda)^{M_0+1} (I + N(\lambda + i\tau))^{-1}.$$

(20)

We now argue that for each order $k^{-J}$ in the trace expansion of Corollary (4.6) there exists $M_0(J)$ such that

$$(i) \quad \sum_{M=0}^{M_0} (-1)^M Tr \int_{\partial \Omega} \rho(k - \lambda) N(\lambda)^M N'(k + i\tau) d\lambda = \sum_{j=0}^{J} B_{\gamma;j} k^{-j} + O(k^{-J-1}),$$

and

$$(ii) \quad Tr \int_{\partial \Omega} \rho(k - \lambda) N(\lambda)^{M_0+1} (I + N(\lambda + i\tau))^{-1} N'(k + i\tau) d\lambda = O(k^{-J-1}).$$

(21)

We will sketch the proof of (i) in some detail since it explains how to calculate the coefficients $B_{\gamma,j}$. For a discussion of the remainder (in an analogous but not identical calculation) we refer to [Z2].

To simplify the notation, we integrate by parts in (i) to throw the derivative onto $\rho$, and then the issue is to analyse the traces:

$$Tr \int_{\partial \Omega} \rho(k - \lambda) N(\lambda + i\tau)^M d\lambda$$

$$= \int_{\partial \Omega} \int (\partial \Omega)^M \rho(k - \lambda) [\Pi_{j=1}^{M} N(\lambda + i\tau, q_j, q_{j+1}) ds(q_j)] d\lambda \quad \text{(where } q_{M+1} = q_1).$$

(23)

Since

$$N(k + i\tau) : L^2(\mathcal{O}_+) \oplus L^2(\mathcal{O}_-) \rightarrow L^2(\mathcal{O}_+) \oplus L^2(\mathcal{O}_-)$$

we write it as

$$N(k) = \begin{pmatrix} N_{++}(k) & N_{+-}(k) \\ N_{-+}(k) & N_{--}(k) \end{pmatrix}.$$
Then (23) is a sum of terms $\sum_{\sigma: \{1, \ldots, M\} \rightarrow \{\pm 1\}} I_{M,\rho}^\sigma$ with

$$I_{M,\rho}^\sigma = \int_{R_D} \int_{T^M} \rho(k - \lambda) [\prod_{j=1}^{M} N_{\sigma_j,\sigma_{j+1}} (\lambda + i\tau, q_{\sigma_j}(\varphi_j), q_{\sigma_{j+1}}(\varphi_{j+1})) d\varphi_j] d\lambda.$$  (24)

Here and hereafter, indices are understood modulo $M$. Using the asymptotics of the free Green’s function in (13)-(27), the $\sigma$th term is formally an oscillatory integral with phases given by the length functional [(4)]. We would like to calculate the trace asymptotically by the stationary phase method.

As discussed extensively in [Z1], we cannot immediately do so because of the singularities along the diagonals of the integrand. We therefore first need to de-singularize the integrals. One of the main advantages of the present inside/outside reduction to the boundary is that it simplifies the regularization procedure by eliminating the integral over $\Omega$ in [Z1]. We briefly outline the method and refer the reader to [Z1] for further details.

### 5.1 Boundary integral operators as quantized billiard maps

The operators $N_{-,+}(k + i\tau), N_{+,-}(k + i\tau)$ are semiclassical Fourier integral operators with phases $|q_+(\varphi) - q_- (\varphi')|$. The phase (and amplitude) is non-singular since the boundary components are disjoint. The phase in the $+-$ case parametrizes the graph of the following multi-valued billiard map from $B^*\mathcal{O}_+\rightarrow B^*\mathcal{O}_-$: given $(q_+,v_+) \in T_q\mathcal{O}_+$ with $|v| < 1$, add a multiple of the unit outward normal to turn $v$ into an outward unit vector and proceed along the straight line in that direction and let $q_-$ be a point of intersection of this line with $\mathcal{O}_-$. Let $v_-$ be the tangential projection of the terminal velocity vector at $q_-$, and put $\beta_+(q_+,v_+) = (q_-,v_-)$. We have (deliberately) described $\beta_+$ in an ambiguous way: when $q_-$ is the first intersection point, we have the usual exterior billiard map; but the phase actually parametrizes the canonical relation which includes intersection points which occur after the line enters the interior of $\mathcal{O}_-$. As mentioned above, these are ‘ghost orbits’ of the billiard flow and they cancel out of the trace formula and do not contribute to the wave invariants we are calculating.

The operators $N_{+,+}(k + i\tau)$ and $N_{-,+}(k + i\tau)$ are more complicated, and also, as it turns out, less important. Each is a combination of a homogeneous pseudodifferential operator of order $-1$ with singularity on the diagonal and...
a Fourier integral operator of order 0 which ‘quantizes’ the billiard map of the interiors of $\Omega$ and $\Omega^c$. They are exactly the kind of operators discussed extensively in [Z1, HZ]. There is a further transition region between these two regimes in which it behaves like an Airy operator, but this region will not be important for our problem.

To separate the two basic regions (tangential and transverse) of $N_{++}$ and $N_{--}$, we introduce a cutoff of the form $\chi(k^1 - \delta(\varphi - \varphi'))$ in terms of the arc-length coordinate $\varphi$ used in the parameterization $q(\varphi)$. Here, $\chi$ a smooth bump function which is supported on $[-1, 1]$ and equals 1 on $[-1/2, 1/2]$. We then write (dropping the subscripts)

$$\begin{cases} 
N(k + i\tau) = N_0(k + i\tau) + N_1(k + i\tau), \\
N_0(k + i\tau, q(\varphi), q(\varphi')) = \chi(k^{1+\delta}(\varphi - \varphi'))N(k + i\tau, q(\varphi), q(\varphi')), \\
N_1(k + i\tau, q(\varphi), q(\varphi')) = (1 - \chi(k^{1+\delta}(\varphi - \varphi'))N(k + i\tau, q(\varphi), q(\varphi')).
\end{cases}$$

(25)

Each term is a (non-standard) Fourier integral operator with amplitudes belonging to the following symbol class: We denote by $S^{p}_{\delta}(T_{m})$ the class of symbols $a(k, \varphi_1, \ldots, \varphi_m)$ satisfying:

$$|(k^{-1}D_{\varphi})^a a(k, \varphi)| \leq C_{a}|k|^{p-\delta[a]}, \quad (|k| \geq 1).$$

(26)

This follows from the asymptotics of Hankel functions (13), which may be described as follows: there exist amplitudes $a_0, a_1$ satisfying

$$(1 - \chi(k^{1-\delta}z))a_1((k + i\tau)z) \in S^{0}_{\delta}(R^{\Omega}_{D}),$$

$$(1 - \chi(k^{1-\delta}z))a_0((k + i\tau)z) \in S^{-1/2}_{\delta}(R^{\Omega}_{D})$$

and such that

$$\begin{cases} 
(i) \quad H^{(1)}_0((k + i\tau)z) = e^{i((k+i\tau)z}a_0((k + i\tau)z), \\
(ii) \quad (k+i\tau)H^{(1)}_1((k + i\tau)z) = (k+i\tau)^{1/2}e^{-i((k+i\tau)z}a_1((k + i\tau)z),
\end{cases}$$

(27)

The $N_0$ term has a singularity on the diagonal of a pseudodifferential operator of order $-2$. The $N_1$ term is manifestly an oscillatory integral operator of order 0 with phase $|q(\varphi) - q(\varphi')|$. As discussed in the $-+$ case, it generates the interior billiard map (interpreted in the multivalued sense of the $-+$ case).
5.2 Microlocalization to $\gamma$

So far, our considerations have been global. But our goal is to calculate wave invariants at a bouncing ball orbit, and for this purpose we can microlocalize the trace to a neighborhood of $\gamma$. This obvious sounding statement is somewhat problematic in the present approach since the relevant operators are not standard pseudodifferential or Fourier integral operators.

The billiard map $\beta_{+-}$ is a cross section of the billiard flow, and in this cross section $\gamma$ corresponds to the periodic points $(0, 0) \in B^*O_+$ and $(0, 0) \in B^*(O_-)$ of period 2 for $\beta_{+-}$. Here, we choose the parametrizations of $O_{\pm}$ such that $\phi = 0$ at each endpoint of the bouncing ball orbit; it is normal to $O_{\pm}$, so its tangential part is the vector $(0, 0)$. To microlocalize to this orbit we introduce a (block diagonal) semiclassical pseudodifferential cutoff operator

$$\chi(\varphi, k^{-1}D_\varphi) = \begin{pmatrix} \chi^{++} & 0 \\ 0 & \chi^{--} \end{pmatrix},$$

with complete symbol $\chi(\varphi, \eta)$ supported in $V_\epsilon := \{(\varphi, \eta) : |\varphi|, |\eta| \leq \epsilon\}$. We also write

$$\rho \ast \frac{d}{d\lambda} \log \det(I + N(k + i\tau)) = Tr \rho \ast (I + N(k + i\tau))^{-1} \circ \frac{d}{dk} N(k + i\tau).$$

Recall that to test whether $(L, \tau) \in WF(u)$ (wave front set) for $u(t) \in S'(\mathbb{R}^3)$, one checks whether $\mathcal{F}(\hat{\rho}u)(k) \sim 0$ for all $\hat{\rho} \in C_0^\infty(R^3)$ satisfying $\hat{\rho} \equiv 1$ in some interval $(L - \epsilon, L + \epsilon)$. Here, $\mathcal{F}u = \hat{u}$ denotes the Fourier transform. Equivalently, one checks whether $\rho \ast \hat{u} \sim 0$. We now show that in calculating wave invariants we can microlocalize our non-standard operators to $\gamma$ in the standard way.

As a preliminary step, we prove the simpler relation

$$\rho \ast Tr[D(k + i\tau)(I + N(k))^{-1}\mathcal{S}(k + i\tau)^{tr} \circ (1 - \tilde{\chi}_{\gamma}(k))] \sim 0. \quad (28)$$

where $\tilde{\chi}_{\gamma}(k)$ is a semiclassical cutoff in $\Omega^c$ to a microlocal neighborhood of $\gamma$ which has the form

$$\tilde{\chi}_{\gamma}(k) = \tilde{\chi}(r, y, k^{-1}D_y)$$

near the boundary. Here, $(r, y)$ are Fermi normal coordinates in $\Omega^c$ near the endpoints of $\gamma$, with $r$ the distance to the boundary. Thus, differentiations are only in tangential directions. Note that (28) is equivalent to saying that

$$Tr[R^\Omega_{\mu\nu} - 1_{\Omega^c} \circ (I - \tilde{\chi}_{\gamma}(k))] \sim 0, \quad (29)$$

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In view of (18), (29)-(28) follow from standard wave front set (WF) considerations for the exterior even wave group $E_t^{\Omega_c D}(t,x,y)$, and from our assumption that $\gamma^r$ is the only exterior periodic orbit of its length. Recall that $WF(E_t^{\Omega_c D}(t,x,y))$ is the space-time graph

$$\Gamma = \{(t,\tau,x,\xi,x',\xi') \in T^*(R^{\Omega_c D}(t,x,y) \times \Omega_c \times \Omega_c) : \tau = -|\xi|, \quad G^t(x,\xi) = (x',\xi')\}, \quad (30)$$

of the generalized billiard flow in the exterior. The only unusual feature is our particular choice of cutoff operator to microlocalize to $\gamma$. To verify that it behaves correctly like a microlocal cutoff to $\gamma$ we write

$$R_{\rho D}(k) \tilde{\chi}_\gamma(k) = \int_0^{\infty} \hat{\rho}(t) E_t^{D\Omega_c}(t) \tilde{\chi}_\gamma(r,y,|D_t|^{-1}D_y)e^{i(k+i\tau)t} dt. \quad (31)$$

Now in Fermi coordinates $(r,\rho dr, y, \eta dy), (\mathbb{R})^+ \gamma$ is ray in the direction of $dr$. Since the space-time graph of the cotangent bundle along $\gamma$ may be described in normal coordinates as a neighborhood of

$$\{(t, \tau, t, \tau, 0, 0)\}$$

a conic neighborhood may be described in these coordinates by $|y| \leq \epsilon, |\eta/\tau| \leq \epsilon$. This is precisely the set to which $\tilde{\chi}_\gamma(r,y,|D_t|^{-1}D_y)$ microlocalizes. Thus, (29) and hence (28) are correct.

We now show that this empty WF relation remains true after moving the cutoff to the left of $S\ell(k)^{tr}$ as a cutoff pseudodifferential operator on the boundary to a neighborhood of the billiard map orbit. As mentioned above, this would be straightforward if $D\ell(k)$ and the other operators in the composition were standard FIO’s. However in addition to their semiclassical canonical relations they have singularities on the diagonal and both contribute to their WF. We will shorten the notation for the cutoff operator to $\chi_0$.

**Lemma 5.1.** We have:

$$Tr \rho * (I + N(k + i\tau))^{-1} \circ \frac{d}{dk} N(k + i\tau)$$

$$\sim Tr \rho * (I + N(k + i\tau))^{-1} \circ \chi_0(k) \circ \frac{d}{dk} N(k + i\tau).$$

**Proof.** We first express the various layer potentials in terms of the wave
group:

\[
\begin{cases}
S\ell(k + i\tau)^{tr} = \gamma G_0(k + i\tau) = \int_0^\infty e^{it(k + i\tau)}\gamma E_0(t)dt,
\end{cases}
\]

\[1_{\Omega^c}(D\ell(k + i\tau) \circ (I + N(k + i\tau))^{-1}(x, q) = P_N^{\Omega^c}(k + i\tau, x, q)
\]

\[= \gamma_{\nu_2}G_N^{\Omega^c}(k + i\tau, x, q) = \int_0^\infty e^{it(k + i\tau)}\gamma_{\nu_2}E_N^{\Omega^c}(t)dt,
\]

where \(P_N^{\Omega^c}\) is the Neumann Poisson kernel of the exterior domain and where \(\gamma_{\nu}u = \partial_{\nu}u|_{\partial\Omega^c}, \gamma u = u|_{\partial\Omega^c}\). We have subscripted the restriction operators to clarify which variables they operate on. The composition \(D\ell(k) \circ (I + N(k))^{-1} \circ (1 - \chi_{\nu}(k)) \circ S\ell(k)^{tr}\) may be written (with the relevant value of \(k\)) as

\[
\int_0^\infty e^{it(k + i\tau)}\{\int_0^t \gamma_{\nu_2}E_N^{\Omega^c}(t - s) \circ (I - \chi_{\nu})(y, |D|^{-1}D_y)\gamma \circ E_0(s)ds\}dt. \tag{32}
\]

We therefore have:

\[
\rho \ast S\ell(k) \circ (I + N(k))^{-1}\circ (1 - \chi_{\nu}(k)) \circ D\ell(k)^{tr}
\]

\[= \int_0^\infty \hat{\rho}(t)\left[\int_0^t \gamma_{\nu_2}E_N^{\Omega^c}(t - s) \circ (I - \chi_{\nu})(y, |D|^{-1}D_y)\gamma_{\nu_1} \circ E_0(s)ds\right]e^{i(k + i\tau)t}dt. \tag{33}
\]

Our goal is to show that

\[
WF[Tr \int_0^t \gamma_{\nu_2}E_N^{\Omega^c}(t - s) \circ (I - \chi_{\nu})(y, |D|^{-1}D_y)\gamma \circ E_0(s)ds]\cap[rL_\gamma - \epsilon, rL_\gamma + \epsilon] = \emptyset. \tag{34}
\]

This follows as long as

\[
V(s, t) = Tr\gamma_{\nu_2}E_N^{\Omega^c}(t - s) \circ (I - \chi_{\nu})(y, |D|^{-1}D_y)\gamma \circ E_0(s)
\]

is a smooth function for \(t \in (rL_\gamma - \epsilon, rL_\gamma + \epsilon)\) and for \(s \in (0, rL_\gamma + \epsilon)\). To prove that it is smooth, we observe that the singular support consists of \((s, t)\) such that there exists a closed billiard orbit of length \(t\) outside a phase space neighborhood of \(\gamma^r\) which consists of a straight line segment of length \(s\) from a point \(x \in \Omega^c\) to a boundary point \(q\), followed by a generalized billiard orbit of length \(t - s\) from \(q\) back to \(x\). By our assumption on the length spectrum, the only possible orbit with length \(t \in (rL_\gamma - \epsilon, rL_\gamma + \epsilon)\) is \(\gamma^r\) of length \(r\); but the cutoff has removed this orbit. This completes the proof.

\[\square\]
5.3 Regularization

In the preceding section we explained that the trace can be microlocalized to $\gamma$. However, it is not true that the operator $N$ can be microlcalized to $\gamma$, that is, we cannot replace $N$ by $\chi_{\gamma}N\chi_{\gamma}$. This is due to the singularities on the diagonal. We now explain how to regularize them.

When we expand $(N_{++0} + N_{++1} + N_{+-} + N_{--0} + N_{--1} + N_{--})^M$, (35)
we get a sum of ‘strings’ of factors. The factors of $N_{++0}$ or $N_{--0}$ forbid the immediate application of the stationary phase method due to the singularities of its phase and amplitude on the diagonal. We now explain how to remove such factors and regularize the term except for presence of $N_{++0}^M$ or $N_{--0}^M$, which has to be handled separately. We will only explain how to remove one factor of $N_{++0}$ next to a smooth factor such as $N_{++1}$. In general, one just repeats the procedure one pair at a time.

5.4 The compositions $N_0 \circ N_1$

We begin by characterizing the composition $N_0 \circ N_1$. Essentially the same result holds for $N_1 \circ N_0$ by the same argument.

The composed kernel equals

$$N_0 \circ N_1(k + i\tau, \varphi_1, \varphi_2) := (k + i\tau)^2 \int_T \chi (k^{-1+\delta} (\varphi_1 - \varphi_3))(1 - \chi (k^{-1+\delta} (\varphi_2 - \varphi_3)))$$

$$H_1^{(1)}((k\mu + i\tau)|q(\varphi_3) - q(\varphi_1)|) \cos \angle(q(\varphi_3) - q(\varphi_1), \nu_{q(\varphi_3)})$$

$$H_1^{(1)}((k\mu + i\tau)|q(\varphi_3) - q(\varphi_2)|) \cos \angle(q(\varphi_2) - q(\varphi_3), \nu_{q(\varphi_2)}) d\varphi_3.$$  

(36)

The next proposition shows that these compositions are semiclassical Fourier integral operators. Thus, the singularity on the diagonal is ‘regularized’. The crucial point is that the composition with the singular factor $N_0$ decreased the order by one unit. We state in a somewhat simplified form and refer to [Z1] for a more detailed statement.

**Proposition 5.2.** $N_0 \circ N_1 \circ \chi_0(k+i\tau, \varphi_1, \varphi_2)$ defines a semiclassical Fourier integral operator on $\partial\Omega$ of order $-1$ associated to the billiard map.
Proof. We only sketch the proof and refer to [Z1] for further details.

The proof is an explicit calculation. Following [AG], we change variables

\[ \varphi_3 \to \vartheta = \varphi_1 - \varphi_3, \]

and then to change variables \( \vartheta \to u \), with:

\[
\begin{align*}
  u := \left\{ \begin{array}{ll}
  |q(\varphi_3) - q(\varphi_1)|, & \varphi_1 \geq \varphi_3 \\
  -|q(\varphi_3) - q(\varphi_1)|, & \varphi_1 \leq \varphi_3
  \end{array} \right.
\end{align*}
\]

(37)

In other words, we change from the intrinsic distance along \( \partial \Omega \) to chordal distance. The purpose of this change of variables is to simplify the difficult factor in (36):

\[ H_1^{(1)}((k\mu + i\tau)|q(\varphi_3) - q(\varphi_1)|) \to H_1^{(1)}((k\mu + i\tau)|u|). \]

We then substitute the asymptotic WKB formula for the ‘easy’ factor

\[ H_1^{(1)}((k\mu + i\tau)|q(\varphi_1) - q(\varphi_3)|) \]

which is valid on the support of the cutoff. The key points are then that

\[
\begin{align*}
  (i) \cos(\angle q(\varphi_2) - q(\varphi_2 + \vartheta), \nu_{q(\varphi_2)}) & \to |u|K(\varphi_2, u), \text{ with } K \text{ smooth in } u; \\
  (ii) H_1^{(1)}((k\mu + i\tau)|q(\varphi_1) - q(\varphi_2 - \vartheta)|) & \to e^{ik|q(\varphi_1) - q(\varphi_2)|}e^{ikau}A(k + i\tau, \varphi_1, \varphi_2, u),
\end{align*}
\]

where \( A_k \) is a symbol in \( k \) of order \(-1/2\) and smooth in \( u \).

Here,

\[ a = \sin \vartheta_{1,2}, \text{ with } \vartheta_{1,2} = \angle (q(\varphi_2) - q(\varphi_1), \nu_{q(\varphi_2)}). \]

(38)

It follows that the composed kernel (36) can be expressed in the form

\[ Ae^{i(k + i\tau)|q(\varphi_1) - \varphi_3|} \]

(further composed with \( \chi_0 \)) with

\[
\begin{align*}
  A(k + i\tau, \varphi_1, \varphi_2) & = \int_{-\infty}^{\infty} \tilde{\chi}(k^{1-\delta}u)(1 - \chi(k^{1-\delta}(\varphi_2 - \varphi_1 - u)) \\
  \times |u|e^{ikau}H_1^{(1)}((k + i\tau)|u|)G((k + i\tau), u, \varphi_1, \varphi_2)du,
\end{align*}
\]

(39)

where \( G((k + i\tau), u, \varphi_1, \varphi_2) \) is a symbol in \( k \) of order \(-1/2\) and smooth in \( u \), and where \( \tilde{\chi}(k^{1-\delta}u) = \chi(k^{1-\delta}(\varphi_1 - \varphi_3)). \)
We now change variables again, \( u' = ku \) (and then drop the prime), to get

\[
A(k + i\tau, \varphi_1, \varphi_2) = k^{-2} \int_{-\infty}^{\infty} \tilde{\chi}(k^{-\delta}u)(1 - \chi(k^{1-\delta}(\varphi_2 - \varphi_1 - k^{-1}u))]|u|e^{iau} \\
\cdot H_1^{(1)}(b|u|)G(k + i\tau, \frac{u}{k}, \varphi_1, \varphi_2)du,
\]

with \( b = 1 + i(\tau/k) \). Since \(|u| \leq k^{\delta} \) on the support of the cutoff, we have \(|\frac{u}{k|} \leq k^{-1+\delta} \). The Taylor expansion of \( G(k + i\tau, u, \varphi_1, \varphi_2) \) at \( u = 0 \) produces an asymptotic series in \( k \). The main point is that the change of variables introduced a factor of \( k^{-2} \), which cancels the original factor of \( k^2 \) in (36). We are left with an amplitude of order \(-\frac{1}{2}\), which defines a Fourier integral operator of order \(-1\) with phase \(|q(\varphi_1 - \varphi_2)|\) as long as the integral (40) has the same order as \( G \).

After Taylor expanding, we end up with terms of the form

\[
\int_{-\infty}^{\infty} \tilde{\chi}(k^{-\delta}u)|u|u^n e^{iau}H_1^{(1)}(b|u|)du
\]

times simple functions of \((\varphi_1, \varphi_2)\). The integral may be expressed in the form

\[
\frac{\partial}{\partial b} \frac{\partial}{\partial a^n} \int_{-\infty}^{\infty} \tilde{\chi}(k^{-\delta}u)e^{iau}H_0^{(1)}(b|u|)du|_{a = \sin \vartheta_1, 2, b = (1+i\tau/k)}.
\]

Using the cosine transform of the Hankel function, one can explicitly evaluate (41) as \( \frac{\partial^n}{\partial a^n}(1 - a^2)^{-3/2} \), modulo lower order terms. After composition with the cutoff \( \chi_0 \) this factor is bounded above, so one gets that \( A \) has order \(-1/2\).

We iterate this proposition to deal with the full strings in (35). The only cases where this is not possible are the ones which have no factors of \( N_{+/-}(k) \). However, they are still composed with the cutoff operator \( \chi_0 \) and a somewhat similar calculation shows that this composition produces a semiclassical pseudodifferential operator, whose order is decreased from 0 by one unit of \( k \) for each factor of \( N_0 \).

5.5 Final remarks and remainder estimate

After removing the \( N_0 \) factors from the terms (35) of (20), we end up with a finite sum of ordinary oscillatory integrals plus a complicated remainder. In
the next section, we will describe the result of applying stationary phase to the finite sum. We now make a few comments on the remainder, referring to [Z1] for further discussion.

The estimate of the remainder is complicated because the operator norm $||N(k+i\tau)||$ does not decrease with increasing $\tau$. To obtain a small remainder, we redefine $\tau \rightarrow \tau \log k$. This changes the wave trace expansions by $k^{-CrL_\gamma}$ and hence for $R$ sufficiently large a remainder estimate of $O(k^{-R})$ is sufficient to separate a finite part of the wave trace from the remainder. We then estimate the remainder

$$Tr \int_{R^3} \rho(k-\lambda)N(\lambda)^{M_0+1}(I+N(\lambda+i\tau))^{-1} \chi_0 N'(k+i\tau)d\lambda$$

by applying the Schwarz inequality for the Hilbert-Schmidt inner product. Further, we use standard estimates on the Poisson kernel to remove the factor of $(I+N(\lambda+i\tau))^{-1}$. Unfortunately, this estimate will also remove the cutoff $\rho$ and replaces $N^M$ by $N^M N^\ast M$. If one regularizes these products as above, we find that the critical points correspond to closed circuits of $M$-links which begin at some point $x$, end at some point $x'$ and then return to $x$ by traversing the links in reverse order. The cutoffs $\chi_0, \chi_\gamma$ prohibit a proliferation of small links (grazing rays) and force the links in critical paths to point in the direction of $\gamma$ and hence to be of length roughly $ML_\gamma$. The imaginary part $i\tau \log k$ of the semiclassical parameter then contributes a damping factor of $e^{-\tau ML_\gamma \log k}$ for each link. The links correspond to the $N_1$ factors. Thus, for each string, we have one $k^{-1}$ for each $N_0$ factor and one $e^{-\tau ML_\gamma \log k}$ for each $N_1$ factor. For sufficiently large $\tau$ these combine to give a factor of $k^{-R}$ for any prescribed $R$ for each term of (35).

6 Calculating coefficients

We now apply the stationary phase method directly to each regularized integrals. Additionally, we have to prove that the remainder is as small as claimed. The result is the following explicit formula for the wave invariants:
Proposition 6.1. We have:

\[ B_{\gamma r, j} + B_{\gamma r, j-1} = b_{r,j,2j}f^{2j}(0) + b_{r,j,2j-1}f^{2j-1}(0) + \tilde{B}_{\gamma r, j}, \quad \text{with} \]

\[ b_{r,j,2j}f^{2j}(0) + b_{r,j,2j-1}f^{2j-1}(0) = r\left\{ 2(h^{11})^j f^{(2j)}(0) \right\} \]

\[ + \left\{ 2(h^{11})^j \frac{1}{2 - 2 \cos \alpha/2} + (h^{11})^{j-2} \sum_{q=1}^{2r} (h^{1q})^3 f^{(3)}(0) f^{(2j-1)}(0) \right\} \]

where \( \tilde{B}_{\gamma r, j} \) depends only on the \( 2j - 2 \) Taylor polynomial of \( f \) at \( x = 0 \).

We review the calculation of the coefficients from [Z2] and adapt it to the exterior case. We also take advantage of the reduction to the boundary to simplify the arguments.

6.1 Setting things up

We recall that, in a small strip \( T_\epsilon(\gamma) \) around \( \alpha_+\alpha_- \), the boundary \( \partial\Omega \) consists of two components which are symmetric graphs over the \( x \)-axis. We write the graphs in the form \( y = \pm f(x) \) near \( a_\pm \).

As in (23), integrals over \( (\Omega \cap T_\epsilon(\gamma))^M \) consist of \( 2^M \) terms, corresponding to a choice of an element \( \sigma \) of

\[ \{\pm\}^M := \{\sigma : \mathbb{Z}_M \to \{\pm\} \}. \]

The length functional in Cartesian coordinates for a given assignment \( \sigma \) of signs is given by

\[ L_\sigma(x_1, \ldots, x_{2r}) = \sum_{j=1}^{2r-1} \sqrt{(x_{j+1} - x_j)^2 + (f_{\sigma(j+1)}(x_{j+1}) - f_{\sigma(j)}(x_j))^2}. \]

(36)

If we write out the integrals \( I_{M,\rho}^\sigma \) of (24) in Cartesian coordinates, we obtain

\[ I_{M,\rho}^\sigma(k + i\tau) = \int_{(-\epsilon,\epsilon)^M} \int_0^\infty \int_{-\infty}^\infty \hat{\rho}(t) \left\{ \Pi_{p=1}^{M-1} H_1^{(1)} \left( (k\lambda + i\tau \right) \times \left| (x_{p+1} - x_p, f_{\sigma(p+1)}(x_{p+1}) - f_{\sigma(p)}(x_p)) \right| \right\} \]

\[ \times \frac{(x_{p+1} - x_p) - f'_{\sigma(p+1)}(x_{p+1})(f_{\sigma(p+1)}(x_{p+1}) - f_{\sigma(p)}(x_p))}{\sqrt{(x_{p+1} - x_p)^2 + (f_{\sigma(p+1)}(x_{p+1}) + f_{\sigma(p)}(x_{p+1})(x_p) + L)^2}} \]

\[ \times \chi(1 - \lambda)e^{ik(1-\lambda)t}d\lambda dt dx_1 \ldots dx_M \].

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We now regularize the integrals \(37\) as above and obtain classical oscillatory integrals in \(x - y\) coordinates. We also eliminate the \((t, \lambda)\) variables by stationary phase. The Hessian in these variables is easily seen to be non-degenerate, and the Hessian operator equals \(-\frac{\partial^2}{\partial t \partial \lambda}\). Since the amplitude depends on \(t\) only in the factor \(\hat{\rho}(t)\), which is constant in a neighborhood of the critical point, only the zeroth order term in the Hessian operator survives the method of stationary phase. We may therefore eliminate the \(dsd\mu\) integrals and replace the amplitude and phase by their evaluations at \(\mu = 1, t = rL_\gamma\). This simplifies the above to

\[
k^{-\nu} \int_{[-\epsilon,\epsilon]^r} e^{ikL_\pm (x_1, \ldots, x_{2m})} \hat{\rho}(\mathcal{L}_\pm (x_1, \ldots, x_{2m})) a(k, x_1, x_2, \ldots, x_{2r}) dx_1 \cdots dx_{2r},
\]

(38)

An examination of the amplitudes and of the regularization procedure leads to the following conclusions:

- (i) The regularized integral \(I_{M,\rho}^\sigma((k + i\tau))\) is negligible as \(k \to \infty\) unless \(M = 2r\) (where \(rL_\gamma\) is the unique length in the support of \(\hat{\rho}\)), and where only strings of \(N_{-+}\) and \(N_{+-}\) are left upon regularization. Otherwise there are no critical points.

- (ii) The amplitudes of these oscillatory integrals have the form

\[
\sum_{n=0}^{\infty} k^{-n} A_n(x),
\]

where \(A_n\) depends only on the first \(n + 2\) derivatives of \(f_\sigma\).

- There is a special term, namely the initially regular one in which \(2r\) factors of \(N_{-+}\) and \(N_{+-}\) alternate, corresponding to \(2r\) bounces of the bouncing ball orbit. The phase is simply \(\mathcal{L}_\pm\) and the amplitude is

\[
a^0_\pm(k; x_1, \ldots, x_{2r}) = \Pi_{k=1}^{2\gamma} a_1((k + i\tau) \mu \sqrt{(x_p - x_{p+1})^2 + (f_{\sigma_\pm}(p)(x_p) - f_{\sigma_\pm}(p+1)(x_{p+1}))^2} \\
\times \frac{(x_p - x_{p+1})f'_{\sigma_\pm}(p)(x_p) - (f_{\sigma_\pm}(p)(x_p) - f_{\sigma_\pm}(p+1)(x_{p+1}))}{\sqrt{(x_p - x_{p+1})^2 + (f_{\sigma_\pm}(p)(x_p) - f_{\sigma_\pm}(p+1)(x_{p+1}))^2}}.
\]

(39)

We only need the principal terms in the symbols \(a_1\), but to simplify notation we omit this step.
This amplitude and phase of the special term have certain key attributes which will be used extensively below.
(i) In its dependence on \( f \), the amplitude has the form 
\[ A(x, y, f, f'). \]

(ii) In its dependence on the variables \( x_p \), the amplitude has the form:
\[ a^0(k, x_1, \ldots, x_{2r}) = \Pi_{p=1}^{2r} A_p(x_p, x_{p+1}) \quad (2p + 1 \equiv 1) \]

(iii) \( D^{(2j-1)}_{x_p} \mathcal{L} \equiv ((x_p - x_{p+1})^2 + (f_{x_p}(x_p) - f_{x_{p+1}}(x_{p+1}))^{2\ell})^{-1/2} \)
\[ - f_{x_{p+1}}(x_{p+1}) f^{(2j-1)}_{x_p}(x_p) \]
\[ \implies D^{(2j-1)}_{x_p} \mathcal{L} |_{x=0} = \epsilon_p f^{(2j-1)}(0); \]

(iv) \( \nabla a^0(k, x_1, \ldots, x_{2r}) |_{x=0} = 0. \)

Here, the notation \( \equiv \) means equivalence modulo terms with fewer derivatives of \( f \). The last statement follows from the values
\[ f(0) = L, f'(0) = x|_{x=0} = 0, \angle(A - B, \nu_B) = \pi/2, \]
where \( A, B \) denote the endpoints.

We now use this information to determine where the data \( f^{2j}(0), f^{(2j-1)} \) first appears in the stationary phase expansion for \( I_k(a_{S_0}, \mathcal{L}_\pm) \).

### 6.2 Stationary phase expansion

We now apply the method of stationary phase to these integrals and keep careful track of the dependence on the number of derivatives of \( f \). To analyze the jungle of terms in the stationary phase expansion, we use the diagrammatic approach (see e.g. [AG]). We give a brief review.

Consider a general oscillatory integral \( Z_k = \int_{(\mathbb{R})^n} a(x) e^{ikS(x)} dx \) where \( a \in C_0^\infty((\mathbb{R})^n) \) and where \( S \) has a unique critical point in \( \text{supp} a \) at 0. Let us write \( H \) for the Hessian of \( S \) at 0. The stationary phase expansion takes the form:

\[
Z_k = \left( \frac{2\pi}{k} \right)^{n/2} \frac{e^{i\pi \text{sgn}(H)/4}}{\sqrt{|\det H|}} e^{ikS(0)} Z_k^{h\ell},
\]

where \( Z_k^{h\ell} = \sum_{j=0}^{\infty} k^{-j} \left\{ \sum_{(\Gamma, \ell): \chi_{\Gamma, \ell} = j} \frac{I_k(\Gamma)}{S(\Gamma)} \right\}. \)
Here, the sum runs over the set $G_{V,I}$ of labelled graphs $(\Gamma, \ell)$ with $V$ closed vertices of valency $\geq 3$ (each corresponding to the phase), with one open vertex (corresponding to the amplitude), and with $I$ edges. Further, the graph $\Gamma'$ is defined to be $\Gamma$ minus the open vertex, and $\chi_{\Gamma'} = V - I$ equals its Euler characteristic. We note that there are only finitely many graphs for each $\chi$ because the valency condition forces $I \geq 3/2V$. Thus, $V \leq 2j, I \leq 3j$.

The function $\ell$ ‘labels’ each end of each edge of $\Gamma$ with an index $j \in \{1, \ldots, n\}$. Also, $S(\Gamma)$ denotes the order of the automorphism group of $\Gamma$, and $I_\ell(\Gamma)$ denotes the ‘Feynman amplitude’ associated to $(\Gamma, \ell)$. By definition, $I_\ell(\Gamma)$ is obtained by the following rule: To each edge with end labels $j,k$ one assigns a factor of $-1^{ik}h_{jk}$ where $H^{-1} = (h_{ik})$. To each closed vertex one assigns a factor of $i_\nu \frac{\partial^\nu S(0)}{\partial x_{i_1} \cdots \partial x_{i_\nu}}$ where $\nu$ is the valency of the vertex and $i_1, \ldots, i_\nu$ at the index labels of the edge ends incident on the vertex. To the open vertex, one assigns the factor $\partial \chi(0)$ where $\chi$ is its valence. Then $I_\ell(\Gamma)$ is the product of all these factors. To the empty graph one assigns the amplitude 1. In summing over $(\Gamma, \ell)$ with a fixed graph $\Gamma$, one sums the product of all the factors as the indices run over $\{1, \ldots, n\}$.

6.2.1 The special term: The data $f^{2j}_\pm(0)$

We first claim that $f^{(2j)}_\pm(0)$ appears first in the $k^{-j+1}$ term. This is because any labelled graph $(\Gamma, \ell)$ for which $I_\ell(\Gamma)$ contains the factor $f^{(2j)}_\pm(0)$ must have a closed vertex of valency $\geq 2j$, or the open vertex must have valency $\geq 2j - 1$. The minimal absolute Euler characteristic $|\chi(\Gamma')|$ in the first case is $1 - j$. Since the Euler characteristic is calculated after the open vertex is removed, the minimal absolute Euler characteristic in the second case is $-j$ (there must be at least $j$ edges.) Hence such graphs do not have minimal absolute Euler characteristic. It follows that the only labelled graph $(\Gamma, \ell)$ with $-\chi(\Gamma') = j - 1$ with $I_\ell(\Gamma)$ containing $f^{(2j)}_\pm(0)$ is given by:

- $G^{2j,0}_{\ell,j} \subset G_{\ell,j}$; with $V = 1, I = j$; the unique graph has no open vertex, one closed vertex and $j$ loops at the closed vertex. The only labels producing the desired data are those $\ell_p$ which assign all endpoints of all edges labelled the same index $p$.

- Assuming the up/down symmetry, the corresponding Feynman amplitude $I_{\ell_p}(\Gamma)$ has the form $2rL(h^{11})^j f^{(2j)}_\pm(0) + \cdots$, where again $\cdots$ refers to terms with $\leq 2j - 1$ derivatives.
To prove the last statement about the Feynman amplitude, we first observe that $f(2j)^{(0)}\pm$ appears linearly in $I_\ell(\Gamma)$. Indeed, the only labelled graphs which produce this datum give the same label to all endpoints of all edges, corresponding to applying only derivatives in a single variable $\frac{\partial}{\partial x_k}$.

Using (41), an examination of (39) shows that the coefficient of $f(2j)^{(0)}$ of $I_k(a_{S_0}, L_\pm)$ equals

$$2r \sum_{p=1, p\equiv 1}^{2r} (h_{pp}^+)^j \left( \frac{\partial}{\partial x_p} \right)^{2j} \mathcal{L}_+(\ldots, x_k, \ldots x_{2r}) = L \left[ \sum_{p=1, p\equiv 1}^{2r} (h_{pp}^+)^j \right] f^{(2j)}_\pm(0).$$

To complete the proof it suffices to observe that the diagonal matrix elements $h_{pp}^\pm$ are constant in $p$ and in the sign $\pm$.

Had we assumed that the obstacles were both up-down and right-left symmetric as in [Z3], then we would already have solved the inverse spectral problem in this step, since all odd Taylor coefficients of $f$ vanish at $x = 0$ and we will be able to recover all even ones.

### 6.2.2 The special term: The data $f^{(2j-1)}_\pm(0)$

We now consider the more difficult odd coefficients $f^{(2j-1)}_\pm(0)$, which will require the attributes of the amplitude (39) detailed in (40).

We again claim that the Taylor coefficients $f^{(2j-1)}_\pm(0)$ appear first in the term of order $k^{-j+1}$, and we enumerate the labelled graphs which have a combinatorial structure capable of producing $f^{(2j-1)}_\pm(0)$ as a factor in $I_\ell(\Gamma)$ in the $k^{-j+1}$ term. In fact, only the main term $I_{\ell, p, q, C}^{2j_\pm}$ will produce such a term. The following is proved in [Z2]:

**Lemma 6.2.** We have:

(i) There are no labelled graphs with $-\chi':= -\chi(\Gamma') \leq j - 1$ for which $I_\ell(\Gamma)$ contains the factor $f^{(2j-1)}_\pm(0)$.

(ii) There are exactly two types of labelled diagrams $(\Gamma, \ell)$ with $\chi(\Gamma') = -j + 1$ such that $I(\Gamma)$ contains the factor $f^{(2j-1)}_\pm(0)$. They are given by:

- $G^{2j-1,3,0}_{2j+1} \subset G_{2j+1}$ with $V = 2, I = j + 1$: Two closed vertices, $j - 1$ loops at one closed vertex, 1 loop at the second closed vertex, one edge between the closed vertices; no open vertex. Labels $\ell_{p,q}$: All labels at the closed vertex with valency $2j - 1$ must be the same index $p$ and all at the closed vertex must be the same index $q$. Feynman amplitude: $(h_{pp}^+)^{j-1} h_{qq}^\mp D_{x_p}^{2j-1} L_\pm D_{x_q}^3 \mathcal{L} \sim (h_{pp}^+)^{j-1} h_{qq}^\mp f^{(2j-1)}_\pm(0) f^{(3)}(0).$
• $G_{2, j+1}^{2j-1, 0} \subset G_{2, j+1}$ with $V = 2, I = j + 1$: Two closed vertices, with $j - 2$ loops at one closed vertex, and with three edges between the two closed vertices; no open vertex. Labels $\ell_{p,q}$: All labels at the closed vertex with valency $2j - 1$ must be the same index $p$ and all at the closed vertex must be the same index $q$; $(\mathcal{h}_{pp}^{q} )^{j-2} (\mathcal{h}_{pq}^{q} )^{3} D_{x_p}^{2j-1} \mathcal{L}_{\pm} D_{x_q}^{3} \mathcal{L}_{\pm} \sim (\mathcal{h}_{pp}^{q} )^{j-2} (\mathcal{h}_{pq}^{q} )^{3} f^{(2j-1)}(0) f^{(3)}(0)$.

This Lemma requires a careful search through the diagrams and an explicit calculation of the amplitudes. It is the most ‘unstable’ step of the proof, in which any calculational error could ruin the result. For instance, it turns out that in addition to the two diagrams which contribute, there are three more which have a combinatorial structure capable of contributing the datum $f^{(2j-1)}(0)$; but the corresponding amplitudes turn out to vanish due to special attributes (40) of the Taylor expansion of the amplitude at the critical point. We should note other attributes in addition to (40) are needed to show that the Feynman amplitude for these diagrams vanishes.

6.2.3 Other terms

We also have:

**Lemma 6.3.** Other terms do not contribute the data $f^{(2j)}(0), f^{(2j-1)}(0)$ in the term of order $k^{-j+1}$.

The proof of this is easy. It merely requires checking how the regularization compares to the stationary phase expansion. As noted above, the regularized integrals are lower order yet do not involve higher derivatives of the amplitude or phase.

7 Recovering the domain

In this section, we sketch the proof of the main result, Theorem [11]. The proof is virtually identical to the proof that one can determine a $\mathbb{Z}_2$-symmetric analytic domain from the wave invariants of a $\mathbb{Z}_2$-symmetric hyperbolic bouncing ball orbit. In fact, the proof requires fewer assumptions, since $L_\gamma$ is a resonance invariant (being the least non-zero singular time in the wave trace.) We may then inspect the wave invariants (i.e. the Balian-Bloch trace invariants) at the iterates $\gamma^r$ of $\gamma$. Give the expressions above for the wave invariants, it suffices to analyse the Hessian coefficients.
7.1 Poincare map and Hessian of the length functional at $\gamma$

We will need to recollect certain facts about the Hessian of the length function at critical points corresponding to the iterates $\gamma^r$. In most respects, the discussion is identical to the interior case, so we refer the reader to [Z1, Z2] for background on the linear Poincare map $P_\gamma$ and its relation to the Hessian of the length function. We will only consider the $\mathbb{Z}_2$-symmetric case since it is the one relevant to our result.

Henceforth we assume $\mathcal{O}$ is convex, so that the bouncing $\gamma$ is necessarily hyperbolic and the eigenvalues of its Poincare map $P_\gamma$ are of the form $\{e^{\pm \lambda}\}$. They are related to the geometry by

$$\cosh \alpha/2 = \left(1 - \frac{L}{R}\right),$$

where $R$ is the common radius of curvature at the endpoints $a_j$. For details, see [PS][KT]. When $\mathcal{O}$ is not convex, $\gamma$ could be elliptic. We refer to [Z2] for the calculation of the wave invariants in that case.

We will need formulae involving matrix elements of the inverse Hessian matrix $H_{2r}^{-1} = (h^{pq})$ at the $\mathbb{Z}_2$-symmetric bouncing ball orbit. As in [Z2], the Hessian in $x - y$ coordinates is the circulant matrix

$$H_{2r} = C(2 \cos \alpha/2, 1, 0, \ldots, 0, 1),$$

that is, the matrix whose rows (or columns) are obtained by cycling the first row (or column) (see [Z1, Z2] for background). Therefore $H_{2r}$ is diagonalized by the finite Fourier matrix $F$ of rank $2r$ (see [Z1, Z2] for its definition and background). Exactly as in [Z1, Z2], we have

$$H_{2r} = F^* \text{diag} \left(2 \cos \alpha/2 + 2, \ldots, 2 \cos \alpha/2 + 2 \cos \frac{(2r - 1)\pi}{r}\right) F. \quad (45)$$

An important role in the inverse problem is played by sums of powers of elements of columns of $H_{2r}^{-1}$. From (45), it follows that

$$H_{2r}^{-1} = F^* \left(\text{diag} \left(\frac{1}{2 \cos \alpha/2 + 2}, \ldots, \frac{1}{2 \cos \alpha/2 + 2 \cos \frac{(2r-1)\pi}{r}}\right)\right) F. \quad (46)$$
Hence, the first row $[H^{-1}]_1 = (h^{11}, \ldots, h^{12r})$ (or column) of the inverse is given by:

$$h^{1q} = \sum_{k=0}^{2r-1} \frac{w^{(q-1)k}}{p_{\alpha,r}(w^k)},$$

(47)

where $w = e^{2\pi i r}$ and where $p_{\alpha,r}(z) = 2 \cosh \alpha/2 + z + z^{-1}$, in [Z1] [Z2] we will need that, for any $p$,

$$\sum_{q=1}^{2r} h^{pq} = \frac{1}{2 + \cosh \alpha/2}.$$

(48)

7.2 Domain recovery

We now prove by induction on $j$ that the Taylor coefficients $f^{2j-1}(0), f^{2j}(0)$ can be determined from $B_{\gamma,r,j}$ as $r$ varies over $r = 1, 2, 3, \ldots$. At first, we assume $f^3(0) \neq 0$.

It suffices to separately determine the two terms

$$2(h^{11}_{2r})^2 \left\{ f^{(2j)}(0) + \frac{1}{2 - 2 \cos \alpha/2} f^{(3)}(0) f^{(2j-1)}(0) \right\},$$

and

$$\left\{ \sum_{q=1}^{2r} (h^{1q}_{2r})^3 \right\} f^{(3)}(0) f^{(2j-1)}(0).$$

(49)

As discussed in [Z2], the terms decouple as $r$ varies if and only if $F_r(\cos \alpha/2) := \sum_{q=1}^{2r} (h^{1q}_{2r})^3$ is non-constant in $r = 1, 2, 3, \ldots$.

By the explicit calculation in [Z2], we have:

$$\sum_{q=1}^{2r} (h^{pq})^3 =$$

$$2r \sum_{k_1,k_2=0}^{2r} \frac{1}{(\cosh \alpha/2 + \cos \frac{k_1\pi}{r})(\cosh \alpha/2 + \cos \frac{k_2\pi}{r})(\cos \alpha/2 + \cosh \frac{(k_1+k_2)\pi}{r})},$$

It is obvious that the sum is strictly increasing as $r$ varies over even integers.

We now begin the inductive argument. From the $j = 0$ term we determine $f''(0)$. Indeed, $(1 - Lf^{(2)}(0) = \cos(h) \alpha/2$ and $\alpha$ is a wave trace invariant. From the $j = 2$ term we recover $f^3(0), f^4(0)$. The induction hypothesis is then that the Taylor polynomial of $f$ of degree $2j - 2$ has been recovered by the
1st stage. By the decoupling argument we can determine \( f^{2j}(0), f^{2j-1}(0) \) as long as \( f^2(0) \neq 0 \).

As discussed in [Z2], the domain can still be recovered if \( f^3(0) = 0 \), but \( f^5(0) \neq 0 \). The argument is essentially the same. Similarly if \( f^3(0) = f^5(0) = 0 \), but \( f^7(0) \neq 0 \), and so on. If all odd derivatives vanish, then one still recovers the domain as noted above.

This completes the sketch of the proof of Theorem (1.1).

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