PROOF OF A CONGRUENCE CONCERNING TRUNCATED
HYPERGEOMETRIC SERIES $6F_5$

CHEN WANG

Abstract. In this paper, we mainly prove the following congruence conjectured by J.-C. Liu:

$$6F_5\left[\begin{array}{cccccc}
\frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & 1 & 1 & 1 & 1 & 1
\end{array}\right] - 1 \equiv -\frac{p^3}{16} \Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^5},$$

where $p \geq 5$ are primes with $p \equiv 3 \pmod{4}$.

1. Introduction

Define the truncated hypergeometric series

$$nF_{n-1} \left[\begin{array}{cccccc}
x_1 & x_2 & \cdots & x_n \\
y_1 & y_2 & \cdots & y_{n-1}
\end{array}\right] := \sum_{k=0}^{m} \frac{(x_1)_k(x_2)_k \cdots (x_n)_k}{(y_1)_k \cdots (y_{n-1})_k} \frac{z^k}{k!},$$

where

$$(x)_k = \begin{cases} x(x+1) \cdots (x+k-1), & k \geq 1, \\
1, & k = 0. \end{cases}$$

Clearly, the truncated hypergeometric series is a finite analogue of the classical hypergeometric series

$$nF_{n-1} \left[\begin{array}{cccccc}
x_1 & x_2 & \cdots & x_n \\
y_1 & y_2 & \cdots & y_{n-1}
\end{array}\right] := \sum_{k=0}^{\infty} \frac{(x_1)_k(x_2)_k \cdots (x_n)_k}{(y_1)_k \cdots (y_{n-1})_k} \frac{z^k}{k!}.$$ In 1997, van Hamme [12] posed several conjectures involving $p$-adic analogue of Ramanujan type series. These congruences are closely related to truncated hypergeometric series. For example, in his paper van Hamme conjectured that

$$6F_5\left[\begin{array}{cccccc}
\frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & 1 & 1 & 1 & 1 & 1
\end{array}\right] - 1 \equiv \begin{cases}
-p \Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^3}, & p \equiv 1 \pmod{4}, \\
0 \pmod{p^5}, & p \equiv 3 \pmod{4},
\end{cases} (1.1)$$

where $\Gamma_p(\cdot)$ denotes the $p$-adic Gamma function. Note that the above congruence is a $p$-adic analogue of the following hypergeometric identity due to Ramanujan:

$$6F_5\left[\begin{array}{cccccc}
\frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & 1 & 1 & 1 & 1 & 1
\end{array}\right] - 1 = \frac{2}{\Gamma\left(\frac{5}{4}\right)^4}.$$
which was confirmed by Hardy [3] and Watson [14]. The conjectural congruence of van Hamme was later proved by McCarthy and Osburn [7]. Note that a lot of congruences involving truncated hypergeometric series have been studied during the past years. One can refer to [2, 4, 5, 8, 9, 10, 11, 13] for details.

In 2015, H. Swisher [11] showed that the congruence (1.1) also holds modulo $p^5$ for primes $p \equiv 1 \pmod{4}$. Recently, J.-C. Liu [4] investigated (1.1) modulo $p^5$ for $p \equiv 3 \pmod{4}$ and posed the following conjecture.

**Conjecture 1.1.** [4] For primes $p \geq 5$ with $p \equiv 3 \pmod{4}$, we have

$$6 \binom{\frac{5}{4}}{1} \binom{\frac{1}{2}}{1} \binom{\frac{1}{2}}{1} \binom{\frac{1}{2}}{1} \binom{\frac{1}{2}}{1} \binom{\frac{1}{2}}{1} - 1 \equiv -\frac{p^3}{16} \Gamma_p \left( \frac{1}{4} \right)^4 \pmod{p^5}.$$ 

In the same paper, by using the Mathematica package Sigma Liu proved that Conjecture 1.1 holds modulo $p^4$.

In this paper we shall give a complete proof of Conjecture 1.1.

**Theorem 1.2.** Conjecture 1.1 is true.

The organization of this paper is as follows. In next section, we shall give some necessary lemmas in order to show Theorem 1.2. We give the proof of Theorem 1.2 in Section 3.

2. Preliminary results

Recall that for any $z \in \mathbb{C}$ with $\Re z > 0$, the well-known Gamma function is defined as

$$\Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t} dt.$$ 

One may check that

$$\Gamma(z + 1) = z\Gamma(z). \quad (2.2)$$

For Gamma function we have the following remarkable reflection formula and duplication formula

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}, \quad (2.3)$$

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z). \quad (2.4)$$

Now we recall the $p$-adic Gamma function $\Gamma_p$ which was first introduced by Y. Morita [6] in 1975 as a $p$-adic analogue of the classical Gamma function. Suppose that $p$ is an odd prime. Let $\mathbb{Z}_p$ denote the ring of all $p$-adic integers and let $| \cdot |_p$ denote the $p$-adic norm over $\mathbb{Z}_p$. For each integer $n \geq 1$, define

$$\Gamma_p(n) := (-1)^n \prod_{1 \leq k < n, \ (k,p)=1} k.$$
In particular, set \( \Gamma_p(0) = 1 \). For any \( x \in \mathbb{Z}_p \), define
\[
\Gamma_p(x) := \lim_{n \to \infty} \frac{\Gamma_p(n)}{|x-n|_p}.
\]
For \( p \)-adic Gamma function we have the following known results:
\[
\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} 
-x, & x \notin p\mathbb{Z}_p, \\
-1, & x \in p\mathbb{Z}_p,
\end{cases}
\tag{2.5}
\]
and
\[
\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\langle x \rangle_p},
\tag{2.6}
\]
where \( \langle x \rangle_p \) denotes the least nonnegative residue of \( x \) modulo \( p \).

Let \( G_k(x) = \Gamma_p^{(k)}(x)/\Gamma_p(x) \), where \( \Gamma_p^{(k)} \) denotes the \( k \)th derivative of \( \Gamma_p \). By taking Taylor expansion at \( x_0 \), we have
\[
\Gamma_p(x) = \Gamma_p(x_0) \sum_{k \geq 0} \frac{G_k(x_0)}{k!}(x-x_0)^k.
\]

For more properties of \( p \)-adic Gamma functions one may consult [5, 6, 8].

To show Theorem 1.2 we also need the following lemmas.

**Lemma 2.1.** [1, page 147] Let \( a, b, c, d, e \in \mathbb{C} \). Then we have the following identity.
\[
\mathbf{6F_5}\left[ \begin{array}{c}
\frac{a}{2} + 1 \\
\frac{a}{2} \\
1 + a - b \\
1 + a - c \\
1 + a - d \\
1 + a - e
\end{array} \right] = \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)} \cdot \mathbf{3F_2}\left[ \begin{array}{c}
1 + a - b - c \\
1 + a - b \\
1 + a - c
\end{array} \right].
\]

We also need the following identity.

**Lemma 2.2.** [1, page 149] For \( a, b, c, d, e \in \mathbb{C} \) we have
\[
\mathbf{3F_2}\left[ \begin{array}{c}
a \\
b \\
c \\
d \\
e
\end{array} \right] = \frac{\pi\Gamma(d)\Gamma(e)}{2^{2c-1}\Gamma(a+d)\Gamma(a+c)\Gamma(b+d)\Gamma(b+c)},
\]
provided \( a + b = 1 \) and \( d + e = 2c + 1 \).

## 3. Proof of Theorem 1.2

Set
\[
\Psi(x) := \mathbf{6F_5}\left[ \begin{array}{c}
\frac{5}{4} \\
\frac{3}{4} \\
\frac{1}{2} \\
\frac{1}{2} \\
1 + ix \\
1 - ix
\end{array} \right] \pmod{p^5}.
\]

We first show that
\[
\Psi(p) \equiv \mathbf{6F_5}\left[ \begin{array}{c}
\frac{5}{4} \\
\frac{3}{4} \\
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2}
\end{array} \right] \pmod{p^5}.
\]
It is clear that $\Psi(x)$ is a rational function in $x^4$ in $\mathbb{Z}_p[[x^4]]$, so by Taylor expansion we have $\Psi(x) = \sum_{n \geq 0} a_n x^{4n}$, where $a_n \in \mathbb{Z}_p$. Define

$$\Phi(x) := \frac{1}{6} \binom{\frac{5-p}{4}}{\frac{1-p}{2}} \frac{1}{\Gamma(\frac{1}{2}) \Gamma(\frac{1+ix}{2})} 3F_2 \left[ \begin{array}{ccc} \frac{1-p}{2} & \frac{1+ix}{2} & 1 + \frac{ix}{2} \\ \end{array} \right].$$

Obviously, $\Phi(x)$ is also a rational function in $x^4$ in $\mathbb{Z}_p[[x^4]]$. It is easy to see that $\Phi(x)$ and $\Psi(x)$ share the same coefficients in $\mathbb{Z}_p[[x^4]]$ modulo $p$. On the other hand, by Lemma 2.1 we have

$$\Phi(x) = \frac{\Gamma(1 + \frac{x-p}{2}) \Gamma(1 - \frac{x+p}{2})}{\Gamma(\frac{3-p}{2}) \Gamma(\frac{1}{2})} \cdot 3F_2 \left[ \begin{array}{ccc} \frac{1-p}{2} & \frac{1+ix}{2} & 1 + \frac{ix}{2} \\ \end{array} \right].$$

Since $(3 - p)/2$ and $(1 - p)/2$ are all negative integers, $\Phi(x) = 0$, that is, its all coefficients vanish in $\mathbb{Z}_p[[x^4]]$. Thus we have $a_1 \equiv 0 \pmod{p}$. This concludes that

$$\Psi(p) \equiv a_0 = \Psi(0) = \frac{1}{6} \binom{\frac{5-p}{4}}{\frac{1-p}{2}} \frac{1}{\Gamma(\frac{1}{2}) \Gamma(\frac{1+ix}{2})} 3F_2 \left[ \begin{array}{ccc} \frac{1-p}{2} & \frac{1+ix}{2} & 1 + \frac{ix}{2} \\ \end{array} \right] \pmod{p^5}. \quad (3.1)$$

Also, by Lemma 2.1 and Lemma 2.2 we have

$$\Psi(p) = \frac{\Gamma(1 + \frac{p}{2}) \Gamma(1 - \frac{p}{2})}{\Gamma(\frac{3}{2}) \Gamma(\frac{1}{2})} \cdot 3F_2 \left[ \begin{array}{ccc} \frac{1-p}{2} & \frac{1+ix}{2} & 1 + \frac{ix}{2} \\ \end{array} \right].$$

In view of (2.4) we have

$$\Gamma \left( 1 + \frac{ip}{2} \right) \Gamma \left( 1 - \frac{ip}{2} \right) = \frac{1}{\pi} \Gamma \left( \frac{1}{2} + \frac{ip}{4} \right) \Gamma \left( \frac{1}{2} - \frac{ip}{4} \right) \Gamma \left( 1 + \frac{ip}{4} \right) \Gamma \left( 1 - \frac{ip}{4} \right).$$

Thus we deduce that

$$\Psi(p) = \frac{\Gamma \left( 1 + \frac{p}{2} \right) \Gamma \left( 1 - \frac{p}{2} \right) \Gamma \left( \frac{1}{2} + \frac{ip}{4} \right) \Gamma \left( \frac{1}{2} - \frac{ip}{4} \right) \Gamma \left( 1 + \frac{ip}{4} \right) \Gamma \left( 1 - \frac{ip}{4} \right)}{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{3-p+ip}{4} \right) \Gamma \left( \frac{3-p-ip}{4} \right) \Gamma \left( \frac{3+p+ip}{4} \right) \Gamma \left( \frac{3+p-ip}{4} \right)} \cdot \quad (3.2)$$

Now by (2.5),

$$\frac{\Gamma \left( 1 + \frac{p}{2} \right) \Gamma \left( 1 - \frac{p}{2} \right)}{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{1}{2} \right)} = \frac{p}{2} \frac{\Gamma_p \left( 1 + \frac{p}{2} \right) \Gamma_p \left( 1 - \frac{p}{2} \right)}{\Gamma_p \left( \frac{3}{2} \right) \Gamma_p \left( \frac{1}{2} \right)} \cdot \quad (3.3)$$
Hence substituting (3.3)–(3.5) into (3.2) and noting that
\[ \Gamma \equiv \Gamma \pmod{p^3}, \]
we have
\[ \frac{\Gamma \left(1 + \frac{ip}{4}\right) \Gamma \left(1 - \frac{ip}{4}\right)}{\Gamma \left(\frac{3-p+ip}{4}\right) \Gamma \left(\frac{3-p-ip}{4}\right)} = \prod_{k=(3-p)/4}^{0} \left( k + \frac{ip}{4} \right) \left( k - \frac{ip}{4} \right) = \prod_{k=(3-p)/4}^{0} \left( k^2 + \frac{p^2}{16} \right), \tag{3.4} \]
\[ \frac{\Gamma \left(\frac{1}{2} + \frac{ip}{4}\right) \Gamma \left(\frac{1}{2} - \frac{ip}{4}\right)}{\Gamma \left(\frac{3+p+ip}{4}\right) \Gamma \left(\frac{3+p-ip}{4}\right)} = \frac{1}{\prod_{k=0}^{(p-3)/4} \left( k + \frac{1}{2} + \frac{ip}{4} \right) \left( k + \frac{1}{2} - \frac{ip}{4} \right)} \]
\[ = \frac{1}{\prod_{k=0}^{(p-3)/4} \left( k + \frac{1}{2} \right)^2 + \frac{p^2}{16}}. \tag{3.5} \]
Hence substituting (3.3)–(3.5) into (3.2) and noting that
\[ \Gamma_p \left(1 + \frac{p}{2}\right) \Gamma_p \left(1 - \frac{p}{2}\right) \equiv \Gamma_p(1)^2 \left( 1 + \frac{p}{2} G_1(1) \right) \left( 1 - \frac{p}{2} G_1(1) \right) \equiv \Gamma_p(1)^2 \pmod{p^2} \]
and
\[ \Gamma_p \left(\frac{p+1}{4}\right) \Gamma_p \left(\frac{1-p}{4}\right) \equiv \Gamma_p \left( \frac{1}{4} \right)^2 \left( 1 + \frac{p}{4} G_1 \left( \frac{1}{4} \right) \right) \left( 1 - \frac{p}{4} G_1 \left( \frac{1}{4} \right) \right) \equiv \Gamma_p \left( \frac{1}{4} \right)^2 \pmod{p^2}, \]
we have
\[ \Psi(p) = \frac{p^3}{32} \cdot \frac{\Gamma_p \left(1 + \frac{p}{2}\right) \Gamma_p \left(1 - \frac{p}{2}\right)}{\Gamma_p \left(\frac{3}{2}\right) \Gamma_p \left(\frac{1}{2}\right)} \cdot \frac{\prod_{k=(3-p)/4}^{-1} (k^2 + \frac{p^2}{16})}{\prod_{k=0}^{(p-3)/4} ((k + 1/2)^2 + p^2/16)} \]
\[ \equiv -\frac{p^3}{16} \cdot \frac{\prod_{k=(3-p)/4}^{-1} k^2}{\prod_{k=0}^{(p-3)/4} (k + 1/2)^2} = -\frac{p^3}{16} \cdot \frac{\Gamma_p((p+1)/4)^2 \Gamma_p((1/4)^2)}{\Gamma_p((p+3)/4)^2} \]
\[ \equiv -\frac{p^3}{16} \cdot \Gamma_p \left( \frac{p+1}{4} \right)^2 \Gamma_p \left( \frac{1-p}{4} \right)^2 \equiv -\frac{p^3}{16} \cdot \Gamma_p \left( \frac{1}{4} \right)^4 \pmod{p^5}. \]
Combining this with (3.1) we immediately obtain the desired theorem. □

Acknowledgments. The author would like to thank Prof. Hao Pan at Nanjing University of Finance and Economics and Dr. Hai-Liang Wu at Nanjing University for their helpful comments.

References

[1] G. E. Andrews, R. Askey and R. Roy, Special Functions, Encyclopedia of Mathematics and its Applications 71, Cambridge University Press, Cambridge.
[2] A. Deines, J. G. Fuselier, L. Long, H. Swisher and F.-T. Tu, Hypergeometric series, truncated hypergeometric series, and Gaussian hypergeometric functions, Directions in number theory, 125C159, Assoc. Women Math. Ser., 3, Springer, 2016.
[3] G. H. Hardy, Some formulas of Ramanujan, Proc. London Math. Soc. 22 (1924), 12-13.
[4] J.-C. Liu, On Van Hamme’s (A.2) and (H.2) supercongruences, J. Math. Anal. Appl. 471 (2019), 613-622.
[5] L. Long, R. Ramakrishna, Some supercongruences occurring in truncated hypergeometric series, Adv. Math. 290 (2016), 773-808.
[6] Y. Morita, A $p$-adic analogue of the $\Gamma$-function, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975), 255-266.
[7] D. MaCarthy, R. Osburn, A $p$-adic analogue of a formula of Ramanujan, Arch. Math.(Basel) 91 (2008), 492-504.
[8] G.-S. Mao, H. Pan, $p$-adic analogues of Hypergeometric identities, preprint, arXiv: 1703.01215.
[9] Z.-W. Sun, On congruences related to central binomial coefficients, J. Number Theory 131 (2011), 2219-2238.
[10] Z.-W. Sun, Super congruences and Euler numbers, Sci. China Math. 54 (2011), 2509-2535.
[11] H. Swisher, On the supercongruence conjectures of van Hamme, Res. Math. Sci. 2 (2015), Art. 18, 21 pp.
[12] L. van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, $p$-adic functional analysis(Nijmegen, 1996), Lecture Notes in Pure and Appl. Math., vol. 192, Dekker, New York, 1997, 223-236.
[13] C. Wang, H. Pan, Supercongruences concerning truncated hypergeometric series, preprint, arXiv: 1806.02735.
[14] G. N. Watson, Theorems stated by Ramanujan (XI), J. London Math. Soc. 6 (1931), 59-65.

Department of Mathematics, Nanjing University, Nanjing 210093, People’s Republic of China
E-mail address: cwang@smail.nju.edu.cn