On almost self-centered graphs and almost peripheral graphs

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Abstract

An almost self-centered graph is a connected graph of order \( n \) with exactly \( n - 2 \) central vertices, and an almost peripheral graph is a connected graph of order \( n \) with exactly \( n - 1 \) peripheral vertices. We determine (1) the maximum girth of an almost self-centered graph of order \( n \); (2) the maximum independence number of an almost self-centered graph of order \( n \) and radius \( r \); (3) the minimum order of a \( k \)-regular almost self-centered graph and (4) the maximum size of an almost peripheral graph of order \( n \); (5) which numbers are possible for the maximum degree of an almost peripheral graph of order \( n \); (6) the maximum number of vertices of maximum degree in an almost peripheral graph of order \( n \) whose maximum degree is the second largest possible. Whenever the extremal graphs have a neat form, we also describe them.

Key words. Almost self-centered graph; almost peripheral graph; girth; independence number

Mathematics Subject Classification. 05C35, 05C07, 05C69

1 Introduction

We consider finite simple graphs. The order of a graph is its number of vertices, and the size its number of edges. We denote by \( V(G) \) and \( E(G) \) the vertex set and edge set of

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a graph $G$ respectively. Denote by $d_G(u,v)$ the distance between two vertices $u$ and $v$ in $G$. The eccentricity, denoted by $ecc_G(v)$, of a vertex $v$ in a graph $G$ is the distance to a vertex farthest from $v$. Thus $ecc_G(v) = \max\{d_G(v,u) | u \in V(G)\}$. If the graph $G$ is clear from the context, we omit the subscript $G$. If $ecc(v) = d(v,x)$, then the vertex $x$ is called an eccentric vertex of $v$. The radius of a graph $G$, denoted $\text{rad}(G)$, is the minimum eccentricity of all the vertices in $V(G)$, whereas the diameter of $G$, denoted $\text{diam}(G)$, is the maximum eccentricity. A vertex $v$ is a central vertex of $G$ if $ecc(v) = \text{rad}(G)$. The center of a graph $G$, denoted $C(G)$, is the set of all central vertices of $G$. A vertex $u$ is a peripheral vertex of $G$ if $ecc(u) = \text{diam}(G)$. The periphery of $G$ is the set of all peripheral vertices of $G$. A graph with a finite radius or diameter is necessarily connected.

If $\text{rad}(G) = \text{diam}(G)$, then the graph $G$ is called self-centered. Thus, a self-centered graph is a graph in which every vertex is a central vertex. This class of graphs have been extensively studied. See [2] and the references therein. Since a nontrivial graph has at least two peripheral vertices, a connected non-self-centered graph of order $n$ has at most $n-2$ central vertices. The following concept was introduced in [4].

**Definition 1.** A connected graph of order $n$ is called almost self-centered if it has exactly $n-2$ central vertices.

Since every graph has at least one central vertex, a connected graph of order $n$ has at most $n-1$ peripheral vertices. The following concept was introduced in [5].

**Definition 2.** A connected graph of order $n$ is called almost peripheral if it has exactly $n-1$ peripheral vertices.

In this paper we investigate several extremal problems on these two classes of graphs. In particular, we determine (1) the maximum girth of an almost self-centered graph of order $n$; (2) the maximum independence number of an almost self-centered graph of order $n$ and radius $r$; (3) the minimum order of a $k$-regular almost self-centered graph and (4) the maximum size of an almost peripheral graph of order $n$; (5) which numbers are possible for the maximum degree of an almost peripheral graph of order $n$; (6) the maximum number of vertices of maximum degree in an almost peripheral graph of order $n$ whose maximum degree is the second largest possible. Whenever the extremal graphs have a neat form, we also describe them.

In Section 2 we treat almost self-centered graphs, and in Section 3 we treat almost peripheral graphs.
For graphs $G$ and $H$, the notation $G + H$ means the disjoint union of $G$ and $H$. A *dominating vertex* in a graph of order $n$ is a vertex of degree $n - 1$. Two vertices $u$ and $v$ on a cycle $C$ of length $n$ are called *antipodal vertices* if $d_C(u, v) = \lfloor n/2 \rfloor$. An $(x, y)$-*path* is a path with endpoints $x$ and $y$. A *diametral path* in a graph $G$ is a shortest $(x, y)$-path of length $\text{diam}(G)$. We list some notations which will be used:

- $C_n$: the cycle of order $n$, $P_n$: the path of order $n$, $K_n$: the complete graph of order $n$,
- $\overline{G}$: the complement of the graph $G$, $e(G)$: the size of the graph $G$,
- $\delta(G)$: the minimum degree of vertices of the graph $G$, $\Delta(G)$: the maximum degree of vertices of the graph $G$,
- $\alpha(G)$: the independence number of the graph $G$, $g(G)$: the girth of the graph $G$,
- $\text{deg}(v)$: the degree of the vertex $v$, $N(v)$: the neighborhood of the vertex $v$,
- $N[v]$: the closed neighborhood of the vertex $v$; i.e., $N[v] = N(v) \cup \{v\}$,
- $N_i(v)$: the $i$-th neighborhood of the vertex $v$; i.e., $N_i(v) = \{x \in V(G) | d(v, x) = i\}$.

It is known [6, p.288] that if $G$ is a connected graph satisfying $\text{diam}(G) \geq \text{rad}(G) + 2$, then every integer $k$ with $\text{rad}(G) < k < \text{diam}(G)$ is the eccentricity of some vertex. Thus if $G$ is an almost self-centered graph or an almost peripheral graph, then the vertices of $G$ have only two distinct eccentricities and hence $\text{diam}(G) = \text{rad}(G) + 1$.

## 2 Almost self-centered graphs

A *binocle* is a graph that consists of two cycles $C, D$ and a $(u, v)$-path $P$ such that $V(P) \cap V(C) = \{u\}$, $V(P) \cap V(D) = \{v\}$ and $V(C) \cap V(D) \subseteq V(P)$. Here we allow the possibility that $P$ has length 0; i.e., $P$ is a vertex. Note also that if $P$ is nontrivial, then $C$ and $D$ are vertex-disjoint. A *theta* (or *theta graph*) is a graph that consists of three internally vertex-disjoint paths sharing the same two endpoints. $\theta_{a,b,c}$ will denote the theta consisting of three paths with lengths $a, b$ and $c$ respectively. A binocle and a theta is depicted in Figure 1.

![Fig. 1. A binocle and $\theta_{4,4,5}$](image-url)
We make the convention that the girth of an acyclic graph is undefined. Thus whenever we talk about the girth of a graph, the graph is not acyclic. A connected graph is said to be unicyclic if it contains exactly one cycle. Recall that a connected graph of order \( n \) is unicyclic if and only if it has size \( n \) [7, p.77].

**Lemma 1.** Let \( G \) be a unicyclic graph of order \( n \geq 6 \). Then \( G \) is almost self-centered if and only if \( n \) is odd and \( G \) is the graph obtained from \( C_{n-1} \) by attaching one edge.

**Proof.** Suppose \( G \) is almost self-centered. We have \( e(G) = n \) and \( G \neq C_n \). It is known [3] that the center of any connected graph lies within one block. Let \( B \) be the block of \( G \) in which \( C(G) \) lies. Then \( B \) is unicyclic and \( \delta(B) = 2 \). Thus \( B \) is a cycle of order \( n - 2 \) or \( n - 1 \). If \( B = C_{n-2} \), let \( V(G) \setminus V(B) = \{x, y\} \). Then \( x \) and \( y \) are leaves. Since \( x \) and \( y \) are the only two peripheral vertices, their neighbors are a pair of antipodal vertices of the cycle \( B \). But then \( B \) contains a vertex whose eccentricity in \( G \) is \( \text{diam}(G) - 2 \), contradicting the assumption that \( G \) is almost self-centered. Hence \( B = C_{n-1} \) and \( G \) is the graph obtained from \( C_{n-1} \) by attaching one edge. Since \( G \) is almost self-centered, \( n \) is odd.

Conversely, it is easy to verify that this graph is almost self-centered. \( \square \)

**Lemma 2.** If \( G \) is a connected graph of order \( n \) and size \( n + 1 \) with \( \delta(G) = 2 \), then \( G \) is either a binocle or a theta.

**Proof.** Since \( \sum_{x \in V(G)} \deg(x) = 2n + 2 \) and \( \delta(G) = 2 \), the degree sequence of \( G \) is \((2, 2, \ldots, 2, 4)\) or \((2, 2, \ldots, 2, 3, 3)\). In the former case, \( G \) is a graph consisting of two cycles sharing a common vertex, which is a binocle, while in the latter case, \( G \) is a theta. \( \square \)

Lemma 2 can also be proved easily using induction on the order.

**Lemma 3.** Let \( a, b, c \) be positive integers with \( a \leq b \leq c \). Then \( \text{rad}(\theta_{a,b,c}) = \lfloor (a+c)/2 \rfloor \) and \( \text{diam}(\theta_{a,b,c}) = \lfloor (b+c)/2 \rfloor \). Consequently \( \theta_{a,b,c} \) is self-centered if and only if \( b = a \) if \( a + c \) is odd and \( b \leq a + 1 \) if \( a + c \) is even. Also \( \text{diam}(\theta_{a,b,c}) = \text{rad}(\theta_{a,b,c}) + 1 \) if and only if \( a + 1 \leq b \leq a + 2 \) if \( a + c \) is odd and \( a + 2 \leq b \leq a + 3 \) if \( a + c \) is even.

**Proof.** Easy verification. \( \square \)

**Lemma 4.** Let \( G \) be a connected graph of order \( n \) and size \( n + 1 \) with \( \delta(G) = 2 \). Then \( G \) is almost self-centered if and only if \( n \) is even and \( G = \theta_{1,2,n-2} \).

**Proof.** By Lemma 2, \( G \) is either a binocle or a theta. Suppose that \( G \) is almost self-centered. It is easy to see that an almost self-centered graph with minimum degree
2 is 2-connected. Since a binocle has connectivity 1, we deduce that \( G \) is a theta. Let \( G = \theta_{a,b,c} \) with \( a \leq b \leq c \). Since \( G \) is almost self-centered, \( \text{diam}(G) = \text{rad}(G) + 1 \). By Lemma 3, \( a + 1 \leq b \leq a + 2 \) if \( a + c \) is odd and \( a + 2 \leq b \leq a + 3 \) if \( a + c \) is even. First suppose that \( a + c \) is odd. We assert that \( a = 1 \). To the contrary, assume \( a \geq 2 \). Then \( b \geq a + 1 \geq 3 \). Let \( G = \theta_{a,b,c} \) consist of the three \((x,y)\)-paths \( P_1, P_2, P_3 \) of lengths \( a, b, c \) respectively. Denote \( r = \text{rad}(G) \) and \( d = r + 1 = \text{diam}(G) \). Note that \( x \) and \( y \) are central vertices of \( G \); i.e., \( \text{ecc}(x) = \text{ecc}(y) = r \). Let \( w_1 \) be the neighbor of \( x \) on \( P_2 \) and let \( w_2 \) be the neighbor of \( y \) on \( P_2 \). Let \( x_1 \) and \( x_2 \) be the two antipodal vertices of \( x \) on the odd cycle \( C = P_1 \cup P_3 \) where \( d_C(x_2,y) = d_C(x_1,y) + 1 \). Then \( d_G(w_1,x_2) \geq r + 1 = d \). Thus both \( w_1 \) and \( x_2 \) are peripheral vertices. Similarly, \( w_2 \) is a peripheral vertex. But then \( G \) contains at least three peripheral vertices, a contradiction.

The case when \( a + c \) is even can be treated similarly. Hence \( a = 1 \). Lemma 3 implies that \( b \geq 2 \) if \( a + c \) is odd and \( b \geq 3 \) if \( a + c \) is even. If \( b \geq 3 \), using the above argument we obtain contradictions. Thus \( a + c = 1 + c \) is odd and \( b = 2 \). It follows that \( n = a + b + c - 1 = 1 + (1 + c) \) is even and \( G = \theta_{1,2,n-2} \).

Conversely, it is easy to verify that if \( n \) is even then the theta \( \theta_{1,2,n-2} \) is almost self-centered.

Now we are ready to state and prove the first main result.

**Theorem 5.** Let \( g(n) \) denote the maximum girth of an almost self-centered graph of order \( n \) with \( n \geq 5 \). Then

\[
g(n) = \begin{cases} 
  n - 1 & \text{if } n \text{ is odd}, \\
  4\lfloor n/6 \rfloor & \text{if } n \text{ is even and } n \neq 10, \\
  5 & \text{if } n = 10.
\end{cases}
\]

Furthermore, if \( n \geq 12 \) and 6 divides \( n \), then \( g(n) \) is attained uniquely by the graph obtained from \( \theta_{n/3,n/3,n/3} \) by attaching an edge to a vertex of degree three.

**Proof.** Let \( G \) be an almost self-centered graph of order \( n \geq 5 \). Clearly \( G \neq C_n \). Hence \( g(n) \leq n - 1 \). On the other hand, if \( n \) is odd, then the graph obtained from \( C_{n-1} \) by attaching an edge is almost self-centered and has girth \( n - 1 \). Hence \( g(n) = n - 1 \) if \( n \) is odd.

Now suppose that \( n \) is even. Note that adding edges to a graph does not increase its girth. The cases \( n \leq 16 \) can be verified by a computer search. Using Lemma 1 and the fact [1, p.195] that a graph of order \( n \) and size \( n + 3 \) has girth at most \( \lfloor 4(n+3)/9 \rfloor \), we
need only check the sizes \(n + 1\) and \(n + 2\) for a graph of order \(n \leq 16\).

Next suppose that \(n\) is even and \(n \geq 18\). We first show that \(g(G) \leq 4\lfloor n/6 \rfloor\). It is known [1, p.195] that a graph of order \(n\) and size \(n + 2\) has girth at most \(\lfloor n/2 \rfloor + 1\). The inequality \(\lfloor n/2 \rfloor + 1 \leq 4\lfloor n/6 \rfloor\) for \(n \geq 18\) implies that if \(e(G) \geq n + 2\), then \(g(G) \leq 4\lfloor n/6 \rfloor\). Also, Lemma 1 excludes the possibility that \(e(G) = n\). It remains to consider the case when \(e(G) = n + 1\), and from now on we make this assumption. It is known [3] that the center of any connected graph lies within one block. Let \(B\) be the block of \(G\) in which \(C(G)\) lies. Since \(|C(G)| = n - 2\) and \(e(G) = n + 1\), the size of \(B\) equals its order plus one. Since \(n \geq 18\), \(B\) is 2-connected and \(\delta(B) = 2\). By Lemma 2, \(B\) is a theta. Let \(B = \theta_{a,b,c}\), which consists of three \((x,y)\)-paths \(P_1, P_2, P_3\) whose lengths are \(a, b, c\) respectively with \(a \leq b \leq c\).

Since the eccentricities of two adjacent vertices differ by at most one, every leaf of \(G\) is a peripheral vertex. Hence \(G\) has at most two leaves. We first exclude the possibility of two leaves. To the contrary, assume that \(G\) has two distinct leaves \(u\) and \(v\) whose neighbors are \(s\) and \(t\) respectively.

Denote \(d = \text{diam}(B)\) and \(f = \text{diam}(G)\). Then \(d + 1 \leq f \leq d + 2\). Clearly it is impossible that \(f < d\). It is also impossible that \(f = d\), since otherwise \(G\) would have at least four peripheral vertices, a contradiction. Hence \(f \geq d + 1\). The inequality \(f \leq d + 2\) follows from the fact that adding two leaves to \(B\) can increase its diameter by at most 2. We distinguish two cases.

Case 1. \(f = d + 2\). In this case \(\text{rad}(G) = d + 1\). Since adding leaves to \(B\) can increase the eccentricity of any vertex of \(B\) by at most 1, we deduce that \(B\) is self-centered. Clearly \(d_B(s,t) = d \geq 2\). Let \(w\) be an internal vertex on a shortest \((s,t)\)-path in \(B\). Then \(\text{ecc}_G(w) = d\), contradicting \(\text{rad}(G) = d + 1\).

Case 2. \(f = d + 1\). Since \(\text{rad}(G) = d\), we have \(\text{rad}(B) \geq d - 1\). We further consider two subcases.

Subcase 2.1. \(\text{rad}(B) = d\); i.e., \(B\) is self-centered. Let \(s'\) be an eccentric vertex of \(s\) in \(B\). Then \(d_B(s',s) = d\), implying that \(d_G(s',u) = d + 1 = f\). But then \(G\) has at least three peripheral vertices \(u, v, s'\), a contradiction.

Subcase 2.2. \(\text{rad}(B) = d - 1\). By Lemma 3, \(a + 1 \leq b \leq a + 2\) if \(a + c\) is odd and \(a + 2 \leq b \leq a + 3\) if \(a + c\) is even. It suffices to consider the two cases: \(b \geq a + 2\); \(b = a + 1\) and \(a + c\) is odd.
First suppose \( b \geq a + 2 \). Since \( \text{ecc}_B(x) = \text{rad}(B) = d - 1 \) and \( \text{rad}(G) = d \), one of the two leaves, say \( u \), must be an eccentric vertex of \( x \) in \( G \). Let \( p \) be the neighbor of \( x \) on \( P_2 \). Using the structure of the theta \( B \) and the condition \( b \geq a + 2 \), we deduce that \( d_G(p, u) \geq d + 1 \). Consequently \( G \) has at least three peripheral vertices \( u, v, p \), which is a contradiction.

Next suppose that \( b = a + 1 \) and \( a + c \) is odd. If in \( G \), \( x \) and \( y \) have a common eccentric vertex (which must be one of the two leaves), say \( u \), then \( s \) is a common eccentric vertex of \( x \) and \( y \) in \( B \). Note that now \( s \) lies in \( P_3 \). Such a situation occurs only if \( a = 1 \), and hence \( b = 2 \). Let \( q \) be the internal vertex of \( P_2 \). Then \( d_G(q, u) = d + 1 \). Thus \( G \) has at least three peripheral vertices \( u, v, q \), which is a contradiction.

If in \( G \), \( x \) and \( y \) do not have a common eccentric vertex, then one of \( u \) and \( v \) is an eccentric vertex of \( x \) and the other is an eccentric vertex of \( y \). The conditions \( a + b + c = n - 1 \), \( b = a + 1 \), and \( n \geq 18 \) imply that \( a + c \geq 8 \). Hence \( d - 1 = \lfloor (a + c)/2 \rfloor \geq 4 \). Since \( d_G(u, v) = d + 1 \), we have \( d_B(s, t) = d - 1 \geq 4 \). Note that \( s \) and \( t \) lie in \( P_3 \). Choose two adjacent vertices \( v_1 \) and \( v_2 \) on \( P_3 \) between \( s \) and \( t \). Since the cycle \( P_1 \cup P_3 \) is odd, \( v_1 \) and \( v_2 \) have a common antipodal vertex \( z \) on \( P_1 \). It is easy to verify that \( \text{ecc}_G(z) = d - 1 \), contradicting the fact that \( \text{rad}(G) = d \).

If \( G \) has no leaf, by Lemma 4 \( G = \theta_{1,2,n-2} \). Thus \( g(G) = 3 < 4\lfloor n/6 \rfloor \).

Finally we consider the case when \( G \) has exactly one leaf. Let \( u \) be the leaf and let \( s \) be its neighbor. Note that \( s \in V(B) \), since otherwise the vertices of \( G \) would have at least three distinct eccentricities, contradicting the assumption that \( G \) is almost self-centered. We continue using the notations \( d = \text{diam}(B) \) and \( f = \text{diam}(G) \). If \( f = d \), then \( G \) would have at least three peripheral vertices, a contradiction. It is also impossible that \( f \geq d + 2 \), since adding a leaf increases the eccentricity of any vertex by at most 1. Hence \( f = d + 1 \). Clearly \( \text{rad}(B) \geq d - 1 \).

We assert that \( B \) is self-centered; i.e., \( \text{rad}(B) = d \). To the contrary, suppose \( \text{rad}(B) = d - 1 \). Then \( \text{ecc}_B(x) = \text{ecc}_B(y) = d - 1 \). Since \( \text{rad}(G) = d \), we deduce that \( u \) must be the common eccentric vertex of \( x \) and \( y \), implying that \( s \) is a common antipodal vertex of \( x \) and \( y \) on the cycle \( P_1 \cup P_3 \). As argued above, \( a + c \) is odd and \( a = 1 \). By Lemma 3, \( b = 2 \) or \( b = 3 \). If \( b = 2 \), let \( z \) be a neighbor of \( s \) on \( P_3 \). Then \( \text{ecc}_G(z) = d - 1 \), contradicting the fact that \( \text{rad}(G) = d \). If \( b = 3 \), let \( w_1 \) be the neighbor of \( x \) on \( P_2 \) and let \( w_2 \) be the neighbor of \( y \) on \( P_2 \). Then it is easy to check that \( G \) has at least three peripheral vertices \( u, w_1, w_2 \), which is a contradiction again. Thus \( B \) is self-centered.
By Lemma 3, \( b = a \) if \( a + c \) is odd and \( b \leq a + 1 \) if \( a + c \) is even. We have \( a + b + c = n \), and clearly \( g(G) = a + b \). There are two possibilities: (1) \( b = a \); (2) \( b = a + 1 \) and \( a + c \) is even. We have \( a + b + c = n \), and clearly \( g(G) = a + b \).

Suppose \( b = a \). We have \( 3a \leq n \), implying that \( a \leq n/3 \). If \( n = 6k \) or \( n = 6k + 2 \), we obtain \( g(G) = 2a \leq 4k \). If \( n = 6k + 4 \), we have \( a \leq 2k + 1 \). The case \( a = 2k + 1 \) will be excluded. Assume \( a = 2k + 1 \). Then \( b = 2k + 1 \) and \( c = 2k + 2 \). Thus \( a + c = b + c \) is odd. But then \( G \) has at least three peripheral vertices, a contradiction. Hence \( a \leq 2k \) and \( g(G) = 2a \leq 4k \).

Suppose \( b = a + 1 \) and \( a + c \) is even. We have \( a \leq (n - 1)/3 \leq 2k + 1 \), where in the second inequality we have used \( n \leq 6k + 4 \). But it is impossible that \( a = 2k + 1 \), since otherwise \( c = 2k + 1 < 2k + 2 = b \), contradicting our assumption that \( b \leq c \). The conditions \( a + b + c = n, b = a + 1 \) and that both \( n \) and \( a + c \) are even imply that \( a \) is odd. Thus \( a = 2k \) is also impossible. It follows that \( a \leq 2k - 1 \) and consequently \( g(G) = a + b = 2a + 1 \leq 4k - 1 \).

Finally we prove that the upper bound \( 4\lfloor n/6 \rfloor \) can be attained and when 6 divides \( n \), the extremal graph is unique. Denote \( k = \lfloor n/6 \rfloor \). Let \( G \) be the graph obtained from \( \theta_{2k,2k,n-4k} \) by attaching an edge to one of the two vertices of degree three. Then \( G \) is an almost self-centered graph of order \( n \) with girth \( 4\lfloor n/6 \rfloor \).

Suppose \( G \) is an almost self-centered graph of order \( n = 6k \geq 18 \) with girth \( 4k \). Then the above analysis shows that \( G \) is a graph obtained from \( \theta_{a,b,c} \) by attaching an edge where \( a = b \). Since \( g(G) = a + b = 2a = 4k \), we have \( a = b = 2k \). The condition \( a + b + c = n \) further implies \( c = 2k \). Thus the theta is \( \theta_{2k,2k,2k} \). There is only one way to attach an edge to this theta so that the resulting graph is almost self-centered; i.e., attach the edge to a vertex of degree three. This shows that the extremal graph is unique. The proof is complete. \( \Box \)

One conclusion in Theorem 5 states that if \( n \geq 12 \) and 6 divides \( n \), then the extremal graph for \( g(n) \) is unique. We remark that if \( n \) is even with \( n \geq 14 \) and 6 does not divide \( n \), then there are at least three extremal graphs for \( g(n) \). This can be seen as follows. Using the notations in the proof of Theorem 5, we may attach an edge to any vertex on \( P_1 \) of the theta \( \theta_{2k,2k,n-2k} \) to obtain an extremal graph.

Next we consider the independence number. There is only one almost self-centered graph of order \( n \) and radius 1; i.e., the graph obtained from \( K_n \) by deleting an edge.
Theorem 6. The maximum independence number of an almost self-centered graph of order \( n \) and radius \( r \) with \( r \geq 2 \) is \( n - r \).

Proof. Let \( G \) be an almost self-centered graph of order \( n \) and radius \( r \). First recall that \( \text{diam}(G) = \text{rad}(G) + 1 = r + 1 \). Let \( P \) be a diametral path of \( G \). If \( r = 2 \), \( P \) has order 4. Any independent set can contain at most 2 of the four vertices on \( P \). Thus \( \alpha(G) \leq n - 2 \).

Suppose \( r \geq 3 \). Let \( x \) be a central vertex of the path \( P \). Now \( P \) has order at least 5 and any vertex on \( P \) is not an eccentric vertex of \( x \). Let \( y \) be an eccentric vertex of \( x \). It is known [3] that the center of any connected graph lies within one block. Let \( B \) be the block of \( G \) in which \( C(G) \) lies. Then \( x, y \in V(B) \). By Menger’s theorem [7, p.167], there are two internally disjoint \((x, y)\)-paths \( Q_1 \) and \( Q_2 \). Denote by \( k \) the length of the cycle \( D = Q_1 \cup Q_2 \). Then \( k \geq 2r \). Any independent set can contain at most \( \lfloor k/2 \rfloor \) vertices on \( D \). Thus \( \alpha(G) \leq \lfloor k/2 \rfloor + (n - k) = n - \lfloor k/2 \rfloor \leq n - r \).

Conversely we construct a graph to show that the upper bound \( n - r \) can be attained. Attaching an edge to the cycle \( v_1, v_2, \ldots, v_{2r} \) at the vertex \( v_1 \) we obtain a graph \( H \). Adding \( n - 2r - 1 \) new vertices to \( H \) such that each of them has \( v_1 \) and \( v_3 \) as neighbors, we obtain the graph \( Z(n, r) \). It is easy to see that \( Z(n, r) \) is an almost self-centered graph of order \( n \) and radius \( r \) with independence number \( n - r \). The graph \( Z(12, 4) \) is depicted in Figure 2.

\[ \text{Fig. 2. The graph } Z(12, 4) \]

Corollary 7. The maximum independence number of an almost self-centered graph of order \( n \) with \( n \geq 5 \) is \( n - 2 \), and there are exactly two extremal graphs.

Proof. By Theorem 6 and the fact that the almost self-centered graph of order \( n \) and radius 1 has independence number 2, we deduce that the maximum independence number is \( n - 2 \).

Suppose \( G \) is an almost self-centered graph of order \( n \) whose independence number is \( n - 2 \). By Theorem 6, \( \text{rad}(G) = 2 \) and consequently \( \text{diam}(G) = 3 \). Let \( P : x_1, x_2, x_3, x_4 \)
be a diametral path of $G$. Then $x_1$ and $x_4$ are the two peripheral vertices of $G$. Denote $S = V(G) \setminus \{x_1, x_2, x_3, x_4\}$. $G$ has only one maximum independent set; i.e, $S \cup T$ where $T$ consists of two vertices from $P$. There are three possible choices for $T : \{x_1, x_3\}, \{x_2, x_4\}$ and $\{x_1, x_4\}$, the first two of which will yield isomorphic graphs. Since every leaf of an almost self-centered graph is a peripheral vertex, every vertex in $S$ has degree at least 2. If $T = \{x_1, x_3\}$, then every vertex in $S$ has $x_2$ and $x_4$ as neighbors; if $T = \{x_1, x_4\}$, then every vertex in $S$ has $x_2$ and $x_3$ as neighbors. Conversely, it is easy to see that these two graphs satisfy all the requirements. 

Now we consider regular almost self-centered graphs.

**Theorem 8.** Let $r(k)$ denote the minimum order of a $k$-regular almost self-centered graph. Then

$$r(k) = \begin{cases} 
12 & \text{if } k = 3, \\
2k + 2 & \text{if } k \geq 4.
\end{cases}$$

**Proof.** Let $G$ be a $k$-regular almost self-centered graph of order $n$, and let $x$ and $y$ be the two peripheral vertices of $G$. There is only one almost self-centered graph of order $n$ and diameter at most 2; i.e., the graph obtained from $K_n$ by deleting an edge. Thus $\text{diam}(G) \geq 3$, implying that $N[x] \cap N[y] = \phi$. It follows that

$$n \geq |N[x]| + |N[y]| = (k + 1) + (k + 1) = 2k + 2.$$ 

We first show $r(3) = 12$. Suppose $k = 3$. Then $n$ is even and $n \geq 2 \times 3 + 2 = 8$. We will exclude the two orders 8 and 10. If $n = 8$, then $\text{diam}(G) = 3$ and $G - \{x, y\}$ is a 2-regular graph of order 6, which must be $C_6$ or $2C_3$. In each case, $G$ has at least four peripheral vertices, a contradiction.

If $n = 10$, we deduce that $\text{diam}(G) = 3$, since otherwise either $G$ has a vertex of degree at least 4 or $G$ has three peripheral vertices. Recall that $N_i(x) = \{v \in V(G) | d(x, v) = i\}$. We have $|N_1(x)| = 3$ and $|N_3(x)| = 1$, implying $|N_2(x)| = 5$. Here we have used the fact that $G$ has exactly two peripheral vertices. Note that each vertex in $N_2(x)$ has at least one neighbor in $N_1(x)$. Analyzing possible adjacency relations in $G - \{x, y\}$, we deduce that $G$ has at least four peripheral vertices, a contradiction.

Thus we have proved that $n \geq 12$. On the other hand, the graph depicted in Figure 3 is a 3-regular almost self-centered graph of order 12. This shows $r(3) = 12$. 

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Next suppose $k \geq 4$. We have proved above that any $k$-regular almost self-centered graph has order at least $2k + 2$. To show $r(k) = 2k + 2$, it suffices to construct such a graph $R$ of order $2k + 2$. Let $V(R) = \{x_0, y_0\} \cup A \cup B$ where $A = \{x_1, x_2, \ldots, x_k\}$ and $B = \{y_1, y_2, \ldots, y_k\}$. We use the notation $u \leftrightarrow v$ to mean that the two vertices $u$ and $v$ are adjacent. If $k$ is even, the adjacency of $R$ is defined as follows:

$$N(x_0) = A, \quad N(y_0) = B, \quad N(x_i) = \{x_{i+\frac{k}{2}}, y_i, y_{i+1}, \ldots, y_{i+k-3}\} \text{ if } 1 \leq i \leq k/2,$$

$$N(x_j) = \{x_{j-\frac{k}{2}}, y_j, y_{j+1}, \ldots, y_{j+k-3}\} \text{ if } k/2 + 1 \leq j \leq k, \quad y_i \leftrightarrow y_{i+\frac{k}{2}}, \text{ if } 1 \leq i \leq k/2.$$

If $k$ is odd, the adjacency of $R$ is defined as follows:

$$N(x_0) = A, \quad N(y_0) = B, \quad N(x_1) = \{x_2, x_3, \ldots, x_{k+1}\} \cup \{y_1, y_2, \ldots, y_{k+1}\} ,$$

$$N(x_i) = \{x_1\} \cup (B \setminus \{y_{i-1}, y_k\}) \text{ if } 2 \leq i \leq (k+1)/2,$$

$$N(x_j) = \{x_1\} \cup (B \setminus \{y_{j-\frac{k+1}{2}}\}) \text{ if } (k+3)/2 \leq j \leq k,$$

$$N(y_k) = \{x_{k+2}, \ldots, x_k\} \cup \{y_1, y_2, \ldots, y_{k+1}\}.$$  

Here the subscripts of the vertices are taken modulo $k$. It is easy to verify that $R$ is a $k$-regular almost self-centered graph of order $2k + 2$ with periphery $\{x_0, y_0\}$ and center $A \cup B$.

\[\Box\]

### 3 Almost peripheral graphs

**Theorem 9.** The maximum size of an almost peripheral graph of order $n$ is $\lfloor (n-1)^2/2 \rfloor$. If $n$ is odd, this maximum size is attained uniquely by the graph $K_1 + ((n-1)/2)K_2$; if $n$ is even, this maximum size is attained uniquely by the graph $K_1 + ((n-4)/2)K_2 + P_3$.

**Proof.** Use the fact that an almost peripheral graph can have at most one dominating vertex and the degree sum formula. \[\Box\]
In the following result we determine which numbers are possible for the maximum degree of an almost peripheral graph with a given order.

**Theorem 10.** There exists an almost peripheral graph of order \( n \geq 7 \) with maximum degree \( \Delta \) if and only if \( \Delta \in \{3, 4, \ldots, n-4, n-1\} \).

**Proof.** Suppose that \( G \) is an almost peripheral graph of order \( n \geq 7 \) with maximum degree \( \Delta \). Clearly \( 3 \leq \Delta \leq n-1 \). We first exclude the two values \( n-2 \) and \( n-3 \) for \( \Delta \). To the contrary suppose \( \Delta = n-2 \) or \( n-3 \). Note that \( \text{rad}(G) \geq 2 \) and \( \text{diam}(G) = \text{rad}(G) + 1 \geq 3 \). Let \( x \in V(G) \) with \( \deg(x) = \Delta \). There exists a vertex \( y \) with \( y \notin N[x] \) such that \( y \) and \( x \) have a common neighbor \( w \).

If \( \Delta = n-2 \), then both \( x \) and \( w \) have eccentricity at most 2, implying that they are central vertices, a contradiction.

Suppose \( \Delta = n-3 \). Let \( z \) be the vertex outside \( N[x] \cup \{y\} \). We always have \( \text{rad}(G) = 2 \) and hence \( \text{diam}(G) = \text{rad}(G) + 1 = 3 \).

If \( z \) and \( y \) are adjacent, then \( w \) is the central vertex. Since \( \text{ecc}(x) = 3 \) and \( z \) is the only possible eccentric vertex of \( x \), we deduce that \( z \) is nonadjacent to any vertex in \( N[x] \). It follows that \( z \) is a leaf. Since \( \text{ecc}(z) = 3 \), \( y \) is another central vertex, a contradiction.

If \( z \) and \( y \) are nonadjacent, then \( N(x) \cap N(z) \neq \emptyset \). In this case, \( x \) is the central vertex. Since \( \text{diam}(G) = 3 \), there exists a \((y, z)\)-path \( P \) of length 2 or 3. Then any internal vertex of \( P \) is a central vertex different from \( x \), a contradiction.

Conversely we will show that every number in \( \{3, 4, \ldots, n-4, n-1\} \) can be attained. The star of order \( n \) is an almost peripheral graph with maximum degree \( n-1 \). Next, for each \( \Delta \) with \( 3 \leq \Delta \leq n-4 \) we construct an almost peripheral graph \( G(n, \Delta) \) of order \( n \) with maximum degree \( \Delta \). We will first construct all \( G(n, 3) \) for \( n = 7, 8, \ldots \), and then inductively construct the remaining \( G(n, \Delta) \) with \( \Delta \geq 4 \).

\( G(7, 3), G(8, 3), G(9, 3), \) and \( G(10, 3) \) are depicted in Figure 4.

![Fig. 4. G(n, 3) with n=7, 8, 9, 10](image_url)

We will need the four preliminary graphs in Figure 5.
Now let \( n \geq 11 \) and denote \( k = \lfloor (n+5)/4 \rfloor \). If \( n \equiv 3 \mod 4 \), \( G(n, 3) \) is obtained from the graph in Figure 5 (1) by replacing the edges \( ab \) and \( bd \) by a path of length \( k - 1 \), and replacing the edges \( ac \) and \( cd \) by a path of length \( k - 2 \); if \( n \equiv 0 \mod 4 \), \( G(n, 3) \) is obtained from the graph in Figure 5 (2) by replacing the edges \( ef \) and \( fh \) by a path of length \( k - 1 \), and replacing the edges \( eg \) and \( gh \) by a path of length \( k - 2 \); if \( n \equiv 1 \mod 4 \), \( G(n, 3) \) is obtained from the graph in Figure 5 (3) by replacing the edges \( ik, kp, sq \) and \( qp \) by a path of length \( k - 2 \), \( k - 1 \), \( k - 3 \) and \( k - 2 \) respectively; if \( n \equiv 2 \mod 4 \), \( G(n, 3) \) is obtained from the graph in Figure 5 (4) by replacing the edges \( tu, uy, zw \) and \( wy \) by a path of length \( k - 3 \), \( k - 1 \), \( k - 3 \) and \( k - 2 \) respectively.

For a vertex \( v \) in a graph, the operation duplicating \( v \) means that adding a new vertex \( x \) and adding edges incident to \( x \) such that \( N(x) = N(v) \).

Note that for \( n = 7, n - 4 = 3 \) and that every \( G(n, 3) \) constructed above contains a vertex of degree 3 that has a non-central neighbor of degree 2. Now suppose that we have constructed \( G(n, 3), G(n, 4), \ldots, G(n, n - 4) \) where \( G(n, \Delta) \) contains a vertex of degree \( \Delta \) that has a non-central neighbor \( x_{\Delta} \) of degree 2, \( \Delta = 3, \ldots, n - 4 \), Then in \( G(n, \Delta) \), duplicate the vertex \( x_{\Delta} \) to obtain a new graph which we denote by \( G(n + 1, \Delta + 1) \). Thus we can construct \( G(n + 1, 3), G(n + 1, 4), \ldots, G(n + 1, n - 3) \) which satisfy all the requirements and the additional condition of containing a vertex of maximum degree that has a non-central neighbor of degree 2. Thus the inductive steps can continue. \( \square \)

Finally we consider the maximum number of vertices of maximum degree in an almost peripheral graph. Blowing up a vertex \( v \) in a graph into the complete graph \( K_t \) is the operation of replacing \( v \) by \( K_t \) and adding edges joining each vertex in \( N(v) \) to each vertex in \( K_t \).

**Definition 3.** A vertex \( v \) in a graph \( G \) is called a top vertex if \( \deg(v) = \Delta(G) \).

**Theorem 11.** The maximum number of top vertices in an almost peripheral graph of order \( n \geq 8 \) with maximum degree \( n - 4 \) is \( n - 5 \) and this maximum number is uniquely
attained by the graph obtained from the graph of order 7 in Figure 4 by blowing up a non-central vertex of degree 3 into $K_{n-6}$.

**Proof.** First, it is easy to verify that the extremal graph given in Theorem 11 is an almost peripheral graph of order $n$ with maximum degree $n - 4$ that has $n - 5$ top vertices. Let $G$ be an almost peripheral graph of order $n \geq 8$ with maximum degree $n - 4$. We may suppose that $G$ has at least three top vertices, since otherwise the number of top vertices in $G$ is less than $n - 5$. Recall that $\text{diam}(G) = \text{rad}(G) + 1 \geq 2$.

Let $x$ be a peripheral vertex of degree $n - 4$. Then there are only three vertices outside $N[x]$. We will use the fact that every vertex in $N_i(x)$ has at least one neighbor in $N_{i-1}(x)$ for $1 \leq i \leq \text{diam}(G)$. The proof consists of a series of claims.

**Claim 1.** $\text{diam}(G) = 3$.

Clearly $\text{diam}(G) = \text{ecc}(x) \leq 4$. If $\text{ecc}(x) = 4$, let $x, r, s, p, q$ be a diametral path. Then $\text{ecc}(r) \leq 3$ and $\text{ecc}(s) \leq 3$, implying that both $r$ and $s$ are central vertices, a contradiction. Thus $\text{diam}(G) \leq 3$. On the other hand, it is impossible that $\text{diam}(G) = 2$, since otherwise $\text{rad}(G) = 1$, implying that $\Delta(G) = n - 1$, a contradiction. Hence $\text{diam}(G) = 3$.

**Claim 2.** The vertex $x$ has only one eccentric vertex, which is not a leaf.

If $|N_3(x)| = 2$, then $|N_2(x)| = 1$. Now the vertex in $N_2(x)$ and its neighbors in $N(x)$ are central vertices, a contradiction. Thus $x$ has only one eccentric vertex, which we denote by $w$. Let $N_2(x) = \{u, v\}$. If $w$ is a leaf, without loss of generality, suppose $u$ is the neighbor of $w$. Since $\text{ecc}(w) \leq 3$, we deduce that $\text{ecc}(u) \leq 2$; i.e., $u$ is a central vertex. If $u$ and $v$ are adjacent, then every neighbor of $u$ in $N(x)$ is also a central vertex, a contradiction; if $u$ and $v$ are nonadjacent, then $d(u, v) = 2$, implying that $u$ and $v$ have a common neighbor $y$ in $N(x)$. But then $y$ is also a central vertex, a contradiction again.

Claim 2 shows that $N(w) = \{u, v\}$.

**Claim 3.** $u$ and $v$ have at most one common neighbor in $N(x)$.

This holds since every common neighbor of $u$ and $v$ in $N(x)$ is a central vertex.

**Claim 4.** $u$ and $v$ are nonadjacent.

To the contrary, assume that $u$ and $v$ are adjacent. Then any neighbor of either $u$ or $v$ in $N(x)$ is a central vertex. It follows that $u$ and $v$ have a common neighbor $y$ in $N(x)$ and $y$ is their only neighbor in $N(x)$. Now any vertex $z \in N(x) \setminus \{y\}$ must be adjacent to $y$, since $d(z, w) \leq 3$. Consequently $\text{deg}(y) = n - 2 > n - 4 = \Delta(G)$, a contradiction.
Claim 5. Neither \( u \) nor \( v \) is a top vertex.

To the contrary, assume \( \deg(u) = n - 4 \). By Claim 4, \( |N(u) \cap N(x)| = n - 5 \). Let \( z \in N(x) \) be the nonneighbor of \( u \). If \( d(u, z) = 2 \), then \( u \) is a central vertex and \( u, v \) have no common neighbor in \( N(x) \). Hence \( v \) is adjacent to \( z \). Let \( y \) be a common neighbor of \( u \) and \( z \). Then \( y \) is also a central vertex, a contradiction. Hence \( d(u, z) = 3 \). The condition \( d(z, w) \leq 3 \) implies that \( z \) is adjacent to \( v \). If \( u \) and \( v \) have no common neighbor in \( N(x) \), then \( x \) and \( u \) are the only two vertices with degree \( n - 4 \), contradicting our assumption that \( G \) has at least three top vertices.

Similarly we can prove that \( \deg(v) < n - 4 \).

Claim 6. Each of \( u \) and \( v \) has at least two neighbors in \( N(x) \).

To the contrary, assume that \( N(u) \cap N(x) = \{y\} \). Then for any vertex \( z \in N(x) \setminus \{y\} \), \( z \) cannot be adjacent to both \( y \) and \( v \), since otherwise \( y \) and \( z \) are central vertices. Considering \( d(z, u) \) and \( d(z, w) \) we deduce that \( z \) is a peripheral vertex. If \( y \) and \( v \) are nonadjacent, then \( G \) is self-centered, a contradiction. If \( y \) and \( v \) are adjacent, then \( y \) is the central vertex. Since \( d(z, w) \leq 3 \), we obtain \( d(z, v) \leq 2 \). It follows that \( v \) is also a central vertex, a contradiction.

Similarly we can prove that \( v \) has at least two neighbors in \( N(x) \).

Claim 7. \( G \) has at most \( n - 5 \) top vertices and the extremal graph is unique.

By Claim 3 and Claim 6, \( u \) has a neighbor \( y \) in \( N(x) \) that is nonadjacent to \( v \), and \( v \) has a neighbor \( z \) in \( N(x) \) that is nonadjacent to \( u \). Note that if \( f \) is a neighbor of \( u \) in \( N(x) \) and \( g \) is a neighbor of \( v \) in \( N(x) \) with \( f \neq g \), then \( f \) and \( g \) are nonadjacent, since otherwise \( f \) and \( g \) are central vertices. Using Claim 6 again we deduce that neither \( y \) nor \( z \) has maximum degree. Thus \( G \) has at least the five vertices \( y, z, u, v, w \) with degrees less than \( n - 4 \). It follows that \( G \) has at most \( n - 5 \) top vertices.

Conversely, suppose \( G \) has \( n - 5 \) top vertices. Then the above analysis shows that (1) each of \( u \) and \( v \) has exactly two neighbors in \( N(x) \); (2) \( u \) and \( v \) have exactly one common neighbor \( h \) in \( N(x) \); and (3) the closed neighborhood of every vertex in \( N(x) \setminus \{y, z, h\} \) is equal to \( N[x] \). Consequently \( G \) is the graph obtained from the graph of order 7 in Figure 4 by blowing up a non-central vertex of degree 3 into \( K_{n-6} \). This completes the proof. \( \square \)

The extremal graph of order 10 in Theorem 11 is depicted in Figure 6.
Acknowledgement. This research was supported by the NSFC grants 11671148 and 11771148 and Science and Technology Commission of Shanghai Municipality (STCSM) grant 18dz2271000.

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