Infinitesimal Deformations and Obstructions for Toric Singularities

Klaus Altmann∗
Dept. of Mathematics, M.I.T., Cambridge, MA 02139, U.S.A.
E-mail: altmann@math.mit.edu

Abstract
The obstruction space $T^2$ and the cup product $T^1 \times T^1 \rightarrow T^2$ are computed for toric singularities.

Contents

1 Introduction 1

2 $T^1$, $T^2$, and the cup product (in general) 2

3 $T^1$, $T^2$, and the cup product (for toric varieties) 4

4 Proof of the $T^2$-formula 10

5 Proof of the cup product formula 16

6 An alternative to the complex $L(E^R)$ 20

7 3-dimensional Gorenstein singularities 21

1 Introduction

For an affine scheme $Y = \text{Spec } A$, there are two important $A$-modules, $T^1_Y$ and $T^2_Y$, carrying information about its deformation theory: $T^1_Y$ describes the infinitesimal deformations, and $T^2_Y$ contains the obstructions for extending deformations of $Y$ to larger base spaces.

*Die Arbeit wurde mit einem Stipendium des DAAD unterstützt.
In case $Y$ admits a versal deformation, $T^1_Y$ is the tangent space of the versal base space $S$. Moreover, if $J$ denotes the ideal defining $S$ as a closed subscheme of the affine space $T^1_Y$, the module $(J/m_{T^1_Y} J)^*$ can be canonically embedded into $T^2_Y$, i.e. $(T^2_Y)^*$-elements induce the equations defining $S$ in $T^1_Y$.

The vector spaces $T^i_Y$ come with a cup product $T^1_Y \times T^1_Y \rightarrow T^2_Y$. The associated quadratic form $T^1_Y \rightarrow T^2_Y$ describes the quadratic part of the elements of $J$, i.e. it can be used to get a better approximation of the versal base space $S$ as regarding its tangent space only.

(1.2) In [Al 1] we have determined the vector space $T^1_Y$ for affine toric varieties. The present paper can be regarded as its continuation - we will compute $T^2_Y$ and the cup product. These modules $T^i_Y$ are canonically graded (induced from the character group of the torus). We will describe their homogeneous pieces as cohomology groups of certain complexes, that are directly induced from the combinatorial structure of the rational, polyhedral cone defining our variety $Y$. The results can be found in §3.

Switching to another, quasiisomorphic complex provides a second formula for the vector spaces $T^i_Y$ (cf. §3). We will use this particular version for describing these spaces and the cup product in the special case of three-dimensional toric Gorenstein singularities (cf. §7).

2 $T^1$, $T^2$, and the cup product (in general)

In this section we will give a brief reminder to the well known definitions of these objects. Moreover, we will use this opportunity to fix some notations.

(2.1) Let $Y \subseteq \mathcal{O}^{w+1}$ be given by equations $f_1, \ldots, f_m$, i.e. its ring of regular functions equals

$$A = P/I \quad \text{with} \quad P = \mathcal{O}[z_0, \ldots, z_w]$$

$$I = (f_1, \ldots, f_m).$$

Then, using $d : I/I^2 \rightarrow A^{w+1}$ ($d(f_i) := (\frac{\partial f_i}{\partial z_0}, \ldots, \frac{\partial f_i}{\partial z_w})$), the vector space $T^1_Y$ equals

$$T^1_Y = \frac{\text{Hom}_A(I/I^2, A)}{\text{Hom}_A(A^{w+1}, A)}.$$

(2.2) Let $\mathcal{R} \subseteq P^m$ denote the $P$-module of relations between the equations $f_1, \ldots, f_m$. It contains the so-called Koszul relations $\mathcal{R}_0 := \langle f_i e^j - f_j e^i \rangle$ as a

2
submodule. Now, $T^2_Y$ can be obtained as

$$T^2_Y = \frac{\text{Hom}_P(\mathcal{R}/\mathcal{R}_0, A)}{\text{Hom}_P(P^m, A)}.$$  

(2.3) Finally, the cup product $T^1 \times T^1 \to T^2$ can be defined in the following way:

(i) Starting with an $\varphi \in \text{Hom}_A(I/I^2, A)$, we lift the images of the $f_i$ obtaining elements $\tilde{\varphi}(f_i) \in P$.

(ii) Given a relation $r \in \mathcal{R}$, the linear combination $\sum_i r_i \tilde{\varphi}(f_i)$ vanishes in $A$, i.e. it is contained in the ideal $I \subseteq P$. Denote by $\lambda(\varphi) \in P^m$ any set of coefficients such that

$$\sum_i r_i \tilde{\varphi}(f_i) + \sum_i \lambda_i(\varphi) f_i = 0 \quad \text{in } P.$$  

(Of course, $\lambda$ depends on $r$ also.)

(iii) If $\varphi, \psi \in \text{Hom}_A(I/I^2, A)$ represent two elements of $T^1_Y$, then we define for each relation $r \in \mathcal{R}$

$$(\varphi \cup \psi)(r) := \sum_i \lambda_i(\varphi) \psi(f_i) + \sum_i \varphi(f_i) \lambda_i(\psi) \in A.$$  

**Remark:** The definition of the cup product does not depend on the choices we made:

(a) Choosing a $\lambda'(\varphi)$ instead of $\lambda(\varphi)$ yields $\lambda'(\varphi) - \lambda(\varphi) \in \mathcal{R}$, i.e. in $A$ we obtain the same result.

(b) Let $\tilde{\varphi}'(f_i)$ be different liftings to $P$. Then, the difference $\tilde{\varphi}'(f_i) - \tilde{\varphi}(f_i)$ is contained in $I$, i.e. it can be written as some linear combination

$$\tilde{\varphi}'(f_i) - \tilde{\varphi}(f_i) = \sum_j t_{ij} f_j.$$  

Hence,

$$\sum_i r_i \tilde{\varphi}'(f_i) = \sum_i r_i \tilde{\varphi}(f_i) + \sum_i \sum_{ij} t_{ij} r_i f_j,$$

and we can define $\lambda'_j(\varphi) := \lambda_j(\varphi) - \sum_i t_{ij} r_i$ (corresponding to $\tilde{\varphi}'$ instead of $\tilde{\varphi}$). Then, we obtain for the cup product

$$(\varphi \cup \psi)'(r) - (\varphi \cup \psi)(r) = -\sum_i r_i \cdot \left(\sum_j t_{ij} \psi(f_j)\right),$$

but this expression comes from some map $P^m \to A$.  

3
3 \( T^1, T^2 \), and the cup product (for toric varieties)

(3.1) We start with fixing the usual notations when dealing with affine toric varieties (cf. [Ke] or [Od]):

- Let \( M, N \) be mutually dual free Abelian groups, we denote by \( M_{IR}, N_{IR} \) the associated real vector spaces obtained by base change with \( IR \).

- Let \( \sigma = \langle a^1, \ldots, a^N \rangle \subseteq N_{IR} \) be a rational, polyhedral cone with apex - given by its fundamental generators.
  \( \sigma^\vee := \{ r \in M_{IR} \mid \langle \sigma, r \rangle \geq 0 \} \subseteq M_{IR} \) is called the dual cone. It induces a partial order on the lattice \( M \) via \([ a \geq b \text{ iff } a - b \in \sigma^\vee ]\).

- \( A := A[\sigma^\vee \cap M] \) denotes the semigroup algebra. It is the ring of regular functions of the toric variety \( Y = \text{Spec} \ A \) associated to \( \sigma \).

- Denote by \( E \subset \sigma^\vee \cap M \) the minimal set of generators of this semigroup ("the Hilbert basis"). \( E \) equals the set of all primitive (i.e. non-splittable) elements of \( \sigma^\vee \cap M \). In particular, there is a surjection of semigroups \( \pi : N^E \rightarrow \sigma^\vee \cap M \), and this fact translates into a closed embedding \( Y \hookrightarrow I^E \).

To make the notations coherent with §2, assume that \( E = \{ r^0, \ldots, r^w \} \) consists of \( w+1 \) elements.

(3.2) To a fixed degree \( R \in M \) we associate “thick facets” \( K^R_i \) of the dual cone

\[ K^R_i := \{ r \in \sigma^\vee \cap M \mid \langle a^i, r \rangle < \langle a^i, R \rangle \} \quad (i = 1, \ldots, N). \]

Lemma:

(1) \( \cup_i K^R_i = (\sigma^\vee \cap M) \setminus (R + \sigma^\vee) \).

(2) For each \( r, s \in K^R_i \) there exists an \( \ell \in K^R_i \) such that \( \ell \geq r, s \). Moreover, if \( Y \) is smooth in codimension 2, the intersections \( K^R_i \cap K^R_j \) (for 2-faces \( \langle a^i, a^j \rangle < \sigma \)) have the same property.

Proof: Part (i) is trivial; for (ii) cf. (3.7) of [Al1] \( \square \).

Intersecting these sets with \( E \subseteq \sigma^\vee \cap M \), we obtain the basic objects for describing the modules \( T^i_Y \):

\[ E^R_i := K^R_i \cap E = \{ r \in E \mid \langle a^i, r \rangle < \langle a^i, R \rangle \} , \]

\[ E^R_0 := \bigcup_{i=1}^N E^R_i , \text{ and} \]

\[ E^R_\tau := \bigcap_{a^i \in \tau} E^R_i \text{ for faces } \tau < \sigma . \]
We obtain a complex \( L(E^R) \) of free Abelian groups via

\[
L(E^R)_{-k} := \bigoplus_{\tau < \sigma, \dim \tau = k} L(E^R_{\tau})
\]

with the usual differentials. \((L(\ldots)\) denotes the free Abelian group of integral, linear dependencies.)

The most interesting part \((k \leq 2)\) can be written explicitly as

\[
L(E^R)^\bullet : \cdots \rightarrow \oplus_{(a, a') < \sigma} L(E^R_i \cap E^R_j) \rightarrow \oplus_i L(E^R_i) \rightarrow L(E^R_0) \rightarrow 0.
\]

(3.3) **Theorem:**

1. \( T_Y^1(-R) = H^0 \left( L(E^R)^\bullet \otimes_{\mathbb{Z}} A \right) \)
2. \( T_Y^2(-R) \supseteq H^1 \left( L(E^R)^\bullet \otimes_{\mathbb{Z}} A \right) \)

3. Moreover, if \( Y \) is smooth in codimension 2 (i.e. if the 2-faces \((a^i, a^j) < \sigma\) are spanned by a part of a \(\mathbb{Z}\)-basis of the lattice \( N \)), then

\[
T_Y^2(-R) = H^1 \left( L(E^R)^\bullet \otimes_{\mathbb{Z}} A \right).
\]

4. **Module structure:** If \( x^s \in A \) (i.e. \( s \in \sigma^\vee \cap M \)), then \( E^R_{-s} \subseteq R^R \), hence \( L(E^R)^\bullet \subseteq L(E^R_{-s})^\bullet \). The induced map in cohomology corresponds to the multiplication with \( x^s \) in the \( A \)-modules \( T_Y^1 \) and \( T_Y^2 \).

The first part was shown in [Al 1]; the formula for \( T^2 \) will be proved in §4. Then, the claim concerning the module structure will become clear automatically.

**Remark:** The assumption made in (3) can not be dropped:

Taking for \( Y \) a 2-dimensional cyclic quotient singularity given by some 2-dimensional cone \( \sigma \), there are only two different sets \( E^R_1 \) and \( E^R_2 \) (for each \( R \in M \)). In particular,

\[
H^1 \left( L(E^R)^\bullet \otimes_{\mathbb{Z}} A \right) = 0.
\]

(3.4) We want to describe the isomorphisms connecting the general \( T^i \)-formulas of (2.1) and (2.2) with the toric ones given in (3.3).

\( Y \hookrightarrow A^{w+1} \) is given by the equations

\[
f_{ab} := z^a - z^b \quad (a, b \in \{1, \ldots, w+1\} \text{ with } \pi(a) = \pi(b) \in \sigma^\vee \cap M),
\]
and it is easier to deal with this infinite set of equations (which generates the ideal \( I \) as a \( \mathcal{C} \)-vector space) instead of selecting a finite number of them in some non-canonical way. In particular, for \( m \) of (2.1) and (2.2) we take

\[
m := \{(a, b) \in \mathbb{N}_+^{w+1} \times \mathbb{N}_+^{w+1} \mid \pi(a) = \pi(b)\}.
\]

The general \( T^i \)-formulas mentioned in (2.1) and (2.2) remain true.

**Theorem:** For a fixed element \( R \in M \) let \( \varphi : L(E)_{\sigma} \to \mathcal{C} \) induce some element of

\[
\left( L(E_0^R) / \sum_i L(E_i^R) \right)^* \otimes \mathbb{Z} \mathcal{C} \cong T_Y^1(-R) \quad (\text{cf. Theorem (3.3)(1)}).
\]

Then, the \( A \)-linear map

\[
\begin{align*}
I/I^2 & \quad \longrightarrow \quad A \\
\vec{x}^a - \vec{x}^b & \quad \mapsto \quad \varphi(a - b) \cdot x^{\pi(a) - R}
\end{align*}
\]

provides the same element via the formula (2.4).

Again, this Theorem follows from the paper [Al 1] - accompanied by the commutative diagram of (4.3) in the present paper. (Cf. Remark (4.4).)

**Remark:** A simple, but nevertheless important check shows that the map \( (\vec{x}^a - \vec{x}^b) \mapsto \varphi(a - b) \cdot x^{\pi(a) - R} \) goes into \( A \), indeed:

Assume \( \pi(a) - R \notin \sigma' \). Then, there exists an index \( i \) such that \( \langle a^i, \pi(a) - R \rangle < 0 \).

Denoting by "supp \( q \)" (for a \( q \in R_E \)) the set of those \( r \in E \) providing a non-vanishing entry \( q_r \), we obtain

\[
\text{supp} (a - b) \subseteq \text{supp} a \cup \text{supp} b \subseteq E_i^R,
\]

i.e. \( \varphi(a - b) = 0 \).

\[\text{(3.5)}\]

The \( P \)-module \( \mathcal{R} \subseteq P^m \) is generated by relations of two different types:

\[
\begin{align*}
r(a, b; c) & := e^{a+c, b+c} - c e^{a,b} \quad (a, b, c \in \mathbb{N}_+^{w+1}; \pi(a) = \pi(b)) \quad \text{and} \\
s(a, b; c) & := e^{b+c} - e^{a,c} + e^{a,b} \quad (a, b, c \in \mathbb{N}_+^{w+1}; \pi(a) = \pi(b) = \pi(c)).
\end{align*}
\]

\( (e^{**} \) denote the standard basis vectors of \( P^m. \))

**Theorem:** For a fixed element \( R \in M \) let \( \psi_i : L(E_i^R)_{\sigma} \to \mathcal{C} \) induce some element of

\[
\left( \frac{Ker \left( \oplus_i L(E_i^R) \longrightarrow L(E_i^R) \right)}{\text{Im} \left( \oplus_{(a', a_i) < \sigma} L(E_i^R \cap E_j^R) \rightarrow \oplus_i L(E_i^R) \right)} \right)^* \otimes \mathbb{Z} \mathcal{C} \subseteq T_Y^2(-R) \quad (\text{cf. (3.3)(2)}).
\]

6
Then, the $P$-linear map

\[
\begin{align*}
\mathcal{R}/\mathcal{R}_0 & \longrightarrow A \\
r(a,b;c) & \mapsto \begin{cases} 
\psi_i(a-b) x^{\pi(a+c)-R} & \text{for } \pi(a) \in K_i^R; \ \pi(a+c) \geq R \\
0 & \text{for } \pi(a) \geq R \text{ or } \pi(a+c) \in \bigcup_i K_i^R 
\end{cases} \\
s(a,b,c) & \mapsto 0
\end{align*}
\]

is correctly defined, and via the formula (3.4) it induces the same element of $T_2^Y$.

For the proof we refer to §4. Nevertheless, we check the correctness of the definition of the $P$-linear map $\mathcal{R}/\mathcal{R}_0 \rightarrow A$ instantly:

(i) If $\pi(a)$ is contained in two different sets $K_i^R$ and $K_j^R$, then the two fundamental generators $a^i$ and $a^j$ can be connected by a sequence $a^{i_0}, \ldots, a^{i_p}$, such that

- $a^{i_0} = a^i$, $a^{i_p} = a^j$,
- $a^{i_{v-1}}$ and $a^{i_v}$ are the edges of some 2-face of $\sigma$ $(v = 1, \ldots, p)$, and
- $\pi(a) \in K_i^R$ for $v = 0, \ldots, p$.

Hence, supp $(a-b) \subseteq E_{i_{v-1}}^R \cap E_{i_v}^R$ $(v = 1, \ldots, p)$, and we obtain

\[
\psi_i(a-b) = \psi_{i_1}(a-b) = \ldots = \psi_{i_{p-1}}(a-b) = \psi_j(a-b).
\]

(ii) There are three types of $P$-linear relations between the generators $r(\ldots)$ and $s(\ldots)$ of $\mathcal{R}$:

\[
\begin{align*}
0 & = z^d r(a,b;c) - r(a,b;c+d) + r(a+c,b+c;d), \\
0 & = r(b,c;d) - r(a,c;d) + r(a,b;d) - s(a+d,b+d,c+d) + z^d s(a,b,c), \\
0 & = s(b,c,d) - s(a,c,d) + s(a,b,d) - s(a,b,c). 
\end{align*}
\]

Our map respects them all.

(iii) Finally, the typical element $(z^a - z^b)e^{cd} - (z^c - z^d)e^{ab} \in \mathcal{R}_0$ equals

\[
-r(c,d) + r(c,d) + r(a,b;c) - r(a,b;d) - s(a+c,b+c,a+d) - s(a+d,b+c,b+d).
\]

It will be sent to 0, too.

\[\text{(3.6)}\] The cup product $T^1_Y \times T^1_Y \rightarrow T^2_Y$ respects the grading, i.e. it splits into pieces

\[
T^1_Y(-R) \times T^1_Y(-S) \longrightarrow T^2_Y(-R-S) \quad (R,S \in M).
\]
To describe these maps in our combinatorial language, we choose some set-theoretical section \( \Phi : M \to \mathbb{Z}^{w+1} \) of the \( \mathbb{Z} \)-linear map
\[
\pi : \mathbb{Z}^{w+1} \to M
a \mapsto \sum_v a_v r^v
\]
with the additional property \( \Phi(\sigma^\vee \cap M) \subseteq \mathbb{N}^{w+1} \).

Let \( q \in L(E) \subseteq \mathbb{Z}^{w+1} \) be an integral relation between the generators of the semigroup \( \sigma^\vee \cap M \). We introduce the following notations:

- \( q^+, q^- \in \mathbb{N}^{w+1} \) denote the positive and the negative part of \( q \), respectively.
  (With other words: \( q = q^+ - q^- \) and \( \sum_v q_v^+ q_v^- = 0 \).)
- \( \bar{q} := \pi(q^+) = \sum_v q_v^+ r^v = \pi(q^-) = \sum_v q_v^- r^v \in M \).
- If \( \varphi, \psi : L(E) \to \mathbb{Z} \) are linear maps and \( R, S \in M \), then we define
  \[
t_{\varphi, \psi, R, S}(q) := \varphi(q) \cdot \psi \left( \Phi(\bar{q} - R) + \Phi(R) - q^- \right) + \psi(q) \cdot \varphi \left( \Phi(\bar{q} - S) + \Phi(S) - q^+ \right).
\]

**Theorem:** Assume that \( Y \) is smooth in codimension 2.
Let \( R, S \in M \), and let \( \varphi, \psi : L(E) \to \mathcal{A} \) be linear maps vanishing on \( \sum_i L(E^R_i) \sigma \) and \( \sum_i L(E^S_i) \sigma \), respectively. In particular, they define elements \( \varphi \in T_Y^1(-R) \), \( \psi \in T_Y^1(-S) \) (which involves a slight abuse of notations).
Then, the cup product \( \varphi \cup \psi \in T_Y^2(-R - S) \) is given (via (3.3)(3)) by the linear maps \( (\varphi \cup \psi)_i : L(E^{R+S}_i) \to \mathcal{A} \) defined as follows:

(i) If \( q \in L(E^{R+S}_i) \) (i.e. \( \langle a^i, \text{supp } q \rangle < \langle a^i, R + S \rangle \)) is an integral relation, then there exists a decomposition \( q = \sum_k q^k \) such that
- \( q^k \in L(E^{R+S}_i) \), and moreover
- \( \langle a^i, q^k \rangle < \langle a^i, R + S \rangle \).

(ii) \( (\varphi \cup \psi)_i \left( q \in L(E^{R+S}_i) \right) := \sum_k t_{\varphi, \psi, R, S}(q^k) \).

It is even not obvious that the map \( q \mapsto \sum_k t(q^k) \)

- does not depend on the representation of \( q \) as a particular sum of \( q_k \)'s (which would instantly imply linearity on \( L(E^{R+S}_i) \)), and
- yields the same result on \( L(E^{R+S}_i \cap E^{R+S}_j) \) for \( i, j \) corresponding to edges \( a^i, a^j \) of some 2-face of \( \sigma \).
The proof of these facts (cf. (5.4)) and of the entire theorem is contained in §5.

**Remark 1:** Replacing all the terms $\Phi(\bullet)$ in the $t$’s of the previous formula for $(\varphi \cup \psi)_i(q)$ by arbitrary liftings from $M$ to $\mathbb{Z}^{w+1}$, the result in $T^2_Y(-R-S)$ will be unchanged as long as we obey the following two rules:

(i) Use always (for all $q, q^k$, and $i$) the *same liftings* of $R$ and $S$ to $\mathbb{Z}^{w+1}$ (at the places of $\Phi(R)$ and $\Phi(S)$, respectively).

(ii) Elements of $\sigma^\vee \cap M$ always have to be lifted to $\mathbb{N}^{w+1}$.

**Proof:** Replacing $\Phi(R)$ by $\Phi(R) + d (d \in L(E))$ at each occurrence changes all maps $(\varphi \cup \psi)_i$ by the summand $\psi(d) \cdot \varphi(\bullet)$. However, this additional linear map comes from $L(E)^*$, hence it is trivial on $\text{Ker} \left( \oplus_i L(E_i^{R+S}) \to L(E_0^{R+S}) \subseteq L(E) \right)$.

Let us look at the terms $\Phi(\bar{q} - R)$ in $t(q)$ now: Unless $\bar{q} \geq R$, the factor $\varphi(q) = 0$. On the other hand, the expression $t(q)$ is never used for those relations $q$ satisfying $\bar{q} \geq R + S$ (cf. conditions for the $q^k$’s). Hence, we may assume that

$$(\bar{q} - R) \geq 0 \quad \text{and, moreover,} \quad (\bar{q} - R) \in \bigcup_i K_i^S.$$  

Now, each two liftings of $(\bar{q} - R)$ to $\mathbb{N}^{w+1}$ differ by an element of $\text{Ker} \psi$ only (apply the method of Remark (3.4) again), in particular, they cause the same result for $t(q)$.

**Remark 2:** In the special case of $R \geq S \geq 0$ we can choose liftings $\Phi(R) \geq \Phi(S) \geq 0$ in $\mathbb{N}^{w+1}$. Then, there exists an easier description for $t(q)$:

(i) Unless $\bar{q} \geq R$, we have $t(q) = 0$.

(ii) In case of $\bar{q} \geq R$ we may assume that $q^+ \geq \Phi(R)$ is true in $\mathbb{N}^{w+1}$. (The general $q$’s are differences of those ones.) Then, $t$ can be computed as the product $t(q) = \varphi(q) \psi(q)$.

**Proof:** (i) As used many times, the property $\bar{q} \in \bigcup_i E_i^R$ implies $\varphi(q) = 0$. Now, we can distinguish between two cases:

**Case 1:** $\bar{q} \in \bigcup_i E_i^S$. We obtain $\psi(q) = 0$, in particular, both summands of $t(q)$ vanish.

**Case 2:** $\bar{q} \geq S$. Then, $\bar{q} - S, S \in \sigma^\vee \cap M$, and $\Phi$ lifts these elements to $\mathbb{N}^{w+1}$. Now, the condition $\bar{q} \in \bigcup_i E_i^R$ implies that $\varphi(\Phi(\bar{q} - S) + \Phi(S) - q^+) = 0$.

(ii) We can choose $\Phi(\bar{q} - R) := q^+ - \Phi(R)$ and $\Phi(\bar{q} - S) := q^+ - \Phi(S)$. Then, the claim follows straightforward.

□
4 Proof of the $T^2$-formula

(4.1) We will use the sheaf $\Omega^1_Y = \Omega^1_{A|y}$ of Kähler differentials for computing the modules $T^i_y$. The maps

$$\alpha_i : \text{Ext}^i_A \left( \Omega^1_Y / \text{tors} (\Omega^1_Y), A \right) \mapsto \text{Ext}^i_A (\Omega^1_Y, A) \cong T^i_y \quad (i = 1, 2)$$

are injective. Moreover, they are isomorphisms for

- $i = 1$, since $Y$ is normal, and for
- $i = 2$, if $Y$ is smooth in codimension 2.

(4.2) As in [AL], we build a special $A$-free resolution (one step further now)

$$\mathcal{E} \xrightarrow{d_E} \mathcal{D} \xrightarrow{d_D} \mathcal{C} \xrightarrow{d_C} \mathcal{B} \xrightarrow{d_B} \Omega^1_Y / \text{tors} (\Omega^1_Y) \to 0.$$ 

With $L^2(E) := L(L(E))$, $L^3(E) := L(L^2(E))$, and

$$\text{supp}^2 \xi := \bigcup_{q \in \text{supp} \xi} (\xi \in L^2(E)), \quad \text{supp}^3 \omega := \bigcup_{\xi \in \text{supp} \omega} \text{supp}^2 \xi \quad (\omega \in L^3(E)),$$

the $A$-modules involved in this resolution are defined as follows:

$$\mathcal{B} := \bigoplus_{r \in E} A \cdot B(r), \quad \mathcal{C} := \bigoplus_{q \in L(E)} A \cdot C(q; \ell),$$

$$\mathcal{D} := \left( \bigoplus_{q \in L(E)} A \cdot D(q; \ell, \eta) \right) \oplus \left( \bigoplus_{\xi \in \text{supp}^2 \eta} A \cdot D(q, \eta) \right), \quad \text{and}$$

$$\mathcal{E} := \left( \bigoplus_{q \in L(E)} A \cdot E(q; \ell, \eta, \mu) \right) \oplus \left( \bigoplus_{\xi \in \text{supp}^2 \omega} A \cdot E(\xi; \eta, \mu) \right) \oplus \left( \bigoplus_{\omega \in L^3(E)} A \cdot E(\omega; \mu) \right).$$

($B, C, D,$ and $E$ are just symbols). The differentials equal

- $d_B : B(r) \mapsto dx^r$
- $d_C : C(q; \ell) \mapsto \sum_{r \in E} q_r x^{r - \ell} \cdot B(r)$
- $d_D : D(q; \ell, \eta) \mapsto C(q; \eta) - x^{\eta - \ell} \cdot C(q, \ell)$
- $d_D : D(\xi; \eta) \mapsto \sum_{q \in L(E)} \xi_q \cdot C(q, \eta)$
- $d_E : E(q; \ell, \eta, \mu) \mapsto D(q; \eta, \mu) - D(q; \ell, \mu) + x^{\mu - \eta} \cdot D(q; \ell, \eta)$
- $d_E : E(\xi; \eta, \mu) \mapsto D(\xi; \mu) - x^{\mu - \eta} \cdot D(\xi; \eta) - \sum_{q \in L(E)} \xi_q \cdot D(q; \eta, \mu)$
- $d_E : E(\omega; \mu) \mapsto \sum_{\xi \in L^3(E)} \omega_{\xi} \cdot D(\xi; \mu).$
Looking at these maps, we see that the complex is $M$-graded: The degree of each of the elements $B$, $C$, $D$, or $E$ can be obtained by taking the last of its parameters ($r$, $\ell$, $\eta$, or $\mu$, respectively).

Remark: If one preferred a resolution with free $A$-modules of finite rank (as it was used in [Al 1]), the following replacements would be necessary:

(i) Define successively $F \subseteq L(E)$, $G \subseteq L(F) \subseteq L^2(E)$, and $H \subseteq L(G) \subseteq L^2(F) \subseteq L^3(E)$ as the finite sets of normalized, minimal relations between elements of $E$, $F$, or $G$, respectively. Then, use them instead of $L^i(E)$ ($i = 1, 2, 3$).

(ii) Let $\ell$, $\eta$, and $\mu$ run through finite generating (under $(\sigma^Y \cap M)$-action) systems of all possible elements meeting the desired inequalities.

The disadvantages of those treatment are a more complicated description of the resolution, on the one hand, and difficulties to obtain the commutative diagram (4.3), on the other hand.

Combining the two exact sequences

$$\mathcal{R}/I_1^2 \rightarrow A^m \rightarrow I_2^2 \rightarrow 0$$

and

$$I/I_2^2 \rightarrow \Omega_{\sigma^{w+1}}^1 \otimes A \rightarrow \Omega_Y^1 \rightarrow 0,$$

we get a complex (not exact at the place of $A^m$) involving $\Omega_Y^1$. We will compare in the following commutative diagram this complex with the previous resolution of $\Omega_Y^1/_{tors}(\Omega_Y^1)$:

Let us explain the three labeled vertical maps:

(B) $p_B : dz_r \mapsto B(r)$ is an isomorphism between two free $A$-modules of rank $w + 1$.

(C) $p_C : c^{ab} \mapsto C(a - b; \pi(a))$. In particular, the image of this map is spanned by those $C(q, \ell)$ meeting $\ell \geq \bar{q}$ (which is stronger than just $\ell \geq \text{supp } q$).

(D) Finally, $p_D$ arises as pull back of $p_C$ to $\mathcal{R}/I_1^2$. It can be described by $r(a, b; c) \mapsto D(a - b; \pi(a), \pi(a + c))$ and $s(a, b, c) \mapsto D(\xi; \pi(a))$ ($\xi$ denotes the relation $\xi = [(b - c) - (a - c) + (a - b) = 0]$).
**Remark:** Starting with the typical $\mathcal{R}_0$-element mentioned in (3.3)(iii), the previous description of the map $p_D$ yields 0 (even in $\mathcal{D}$).

(4.4) By (4.1) we get the $A$-modules $T_Y^i$ by computing the cohomology of the complex dual to those of (4.2).

As in [Al 1], denote by $G$ one of the capital letters $B$, $C$, $D$, or $E$. Then, an element $\psi$ of the dual free module $(\bigoplus_G \mathcal{E}[\sigma \cap M] \cdot G)^*$ can be described by giving elements $g(x) \in \mathcal{E}[\sigma \cap M]$ to be the images of the generators $G$ ($g$ stands for $b$, $c$, $d$, or $e$, respectively).

For $\psi$ to be homogeneous of degree $-R \in M$, $g(x)$ has to be a monomial of degree

$$\deg g(x) = -R + \deg G.$$ 

In particular, the corresponding complex coefficient $g \in \mathcal{E}$ (i.e. $g(x) = g \cdot x^{-R + \deg G}$) admits the property that

$$g \neq 0 \implies -R + \deg G \geq 0 \quad \text{(i.e. } -R + \deg G \in \tilde{\sigma}).$$

**Remark:** Using these notations, Theorem (3.3)(1) was proved in [Al 1] by showing that

$$\left( \frac{L(E_0^R)}{\sum_i L(E_i^R)} \right)^* \otimes_{\mathbb{Z}} \mathcal{E} \rightarrow \text{Ker}(C^*_{-R} \rightarrow D^*_{-R})/\text{Im}(B^*_{-R} \rightarrow C^*_{-R})$$

$$\varphi \mapsto [\ldots, c(q; \ell) := \varphi(q), \ldots]$$

is an isomorphism of vector spaces.

Moreover, looking at the diagram of (4.3), $e^{ab} \in A^m$ maps to both $\tilde{z}^a - \tilde{z}^b \in \bar{I}/I^2$ and $C(a-b; \pi(a)) \in \mathcal{C}$. In particular, we can verify Theorem (3.4): Each $\varphi : L(E)_{\tilde{\sigma}} \rightarrow \mathcal{E}$, on the one hand, and its associated $A$-linear map

$$\frac{I/I^2}{\tilde{z}^a - \tilde{z}^b} \rightarrow A$$

$$\varphi(a-b) \cdot x^{\pi(a)-R},$$

on the other hand, induce the same element of $T_Y^1(-R)$.

(4.5) For computing $T_Y^2(-R)$, the interesting part of the dualized complex $\mathcal{E}^*$ in degree $-R$ equals the complex of $\mathcal{E}$-vector spaces

$$C^*_{-R} \xrightarrow{d_{d_{-R}}} D^*_{-R} \xrightarrow{d_{d_{-R}}} E^*_{-R}.$$
with coordinates \( c \), \( d \), and \( \xi \), respectively:

\[
\mathcal{C}_R = \{ (c(q;\ell), c(q;\ell) = 0 \text{ for } \ell - R \notin \sigma) \}
\]

\[
\mathcal{D}_R = \{ (d(q;\ell,\eta), d(\xi;\eta) \mid d(q;\ell,\eta) = 0 \text{ for } \eta - R \notin \sigma, \text{ and } d(\xi;\eta) = 0 \text{ for } \eta - R \notin \bar{\sigma}) \}
\]

\[
\mathcal{E}_R = \{ (\varepsilon(q;\ell,\eta,\mu), \varepsilon(\xi;\eta,\mu), \varepsilon(\omega;\mu) \mid \text{each coordinate vanishes for } \mu - R \notin \bar{\sigma}) \}
\]

The differentials \( d_D \) and \( d_E \) can be described by

\[
\begin{align*}
d(q;\ell,\eta) &= c(q;\eta) - c(q;\ell), \\
d(\xi;\eta) &= \sum_{q \in F} \xi_q \cdot c(q;\eta), \\
e(q;\ell,\eta,\mu) &= d(q;\eta,\mu) - d(q;\ell,\mu) + d(q;\ell,\eta), \\
e(\xi;\eta,\mu) &= d(\xi;\eta) - d(\xi;\eta) - \sum_{q \in F} \xi_q \cdot d(q;\eta,\mu), \\
e(\omega;\mu) &= \sum_{\xi \in G} \omega_\xi \cdot d(\xi;\mu).
\end{align*}
\]

Denote \( V := \text{Ker } d_E^* \subseteq \mathcal{D}_R^* \) and \( W := \text{Im } d_D^* \subseteq V \), i.e.

\[
V = \{ (d(q;\ell,\eta), d(\xi;\eta) \mid q \in L(E), \eta \geq \ell \geq \supp q \text{ in } M; \\
\xi \in L^2(E), \eta \geq \supp^2 \xi; \\
d(q;\ell,\eta) = d(\xi;\eta) = 0 \text{ for } \eta - R \notin \bar{\sigma}, \\
d(q;\ell,\mu) = d(q;\ell,\eta) + d(q;\eta,\mu) (\mu \geq \eta \geq \ell \geq \supp q), \\
d(\xi;\eta) = d(\xi;\eta) + \sum_{q \in F} \xi_q \cdot d(q;\eta,\mu) (\mu \geq \eta \geq \supp^2 \xi), \\
\sum_{\xi \in G} \omega_\xi \cdot d(\xi;\mu) = 0 \text{ for } \omega \in L^3(E) \text{ with } \omega \geq \supp^3 \omega \}.
\]

\[
W = \{ (d(q;\ell,\eta), d(\xi;\eta) \mid \text{there are } c(q;\ell) \text{'s with } c(q,\ell) = 0 \text{ for } \ell - R \notin \bar{\sigma}, \text{ and} \\
d(q;\ell,\eta) = c(q;\eta) - c(q;\ell), \\
d(\xi;\eta) = \sum_{q} \xi_q \cdot c(q;\eta) \}.
\]

By construction, we obtain

\[
V/W = \text{Ext}_A^1 \left( \Omega_Y^1 / \text{tors } (\Omega_Y^1), A \right) (-R) \subseteq T_Y^2 (-R)
\]

(which is an isomorphism, if \( Y \) is smooth in codimension 2).

\[ (4.6) \]

Let us define the much easier vector spaces

\[
V_1 := \{ [x_i(q)_{q \in L(E^R)}] \mid x_i(q) = x_j(q) \text{ for } \bullet \langle a_i, a_j \rangle < \sigma \text{ is a 2-face and} \\
\xi \in L^2(E^R) \text{ implies } \sum_{q} \xi_q \cdot x_i(q) = 0 \} \text{ and}
\]

\[
W_1 := \{ [x(q)_{q \in \cup_i L(E^R)}] \mid \xi \in L(\cup_i L(E^R)) \text{ implies } \sum_{q} \xi_q \cdot x(q) = 0 \}.
\]

**Lemma:** The linear map \( V_1 \to V \) defined by

\[
\begin{align*}
d(q;\ell,\eta) &= \begin{cases} x_i(q) & \text{for } \ell \in K^R, \eta \geq R \\
0 & \text{for } \ell \geq R \text{ or } \eta \in \bigcup_i K^R \end{cases} \\
d(\xi;\eta) &= 0
\end{align*}
\]
induces an injective map

\[ V_1/W_1 \rightarrow V/W. \]

If \( Y \) is smooth in codimension 2, it will be an isomorphism.

**Proof:** 1) The map \( V_1 \rightarrow V \) is correct defined: On the one hand, an argument as used in [3][3] shows that \( \ell \in K^R_i \cap K^R_j \) would imply \( x_i(q) = x_j(q) \). On the other hand, the image of \([x_i(q)]_{q \in L(E_i^R)}\) meets all conditions in the definition of \( V \).

2) \( W_1 \) maps to \( W \) (take \( c(q, \ell) := x(q) \) for \( \ell \geq R \) and \( c(q, \ell) := 0 \) otherwise).

3) The map between the two factor spaces is injective: Assume for \([x_i(q)]_{q \in L(E_i^R)}\) that there exist elements \( c(q, \ell) \), such that

\[
\begin{align*}
\sum_{q} \xi_q \cdot c(q; \eta) &= 0 \quad \text{for} \ \eta \geq \text{supp}^2 \xi. \\
\end{align*}
\]

In particular, \( x_i(q) \) do not depend on \( i \), and these elements meet the property

\[
\sum_{q} \xi_q \cdot \mathbf{x}(q) = 0 \quad \text{for} \ \xi \in L(\bigcup_i L(E_i^R)).
\]

4) If \( Y \) is smooth in codimension 2, the map is surjective:

Given an element \([d(q; \ell, \eta), d(\xi; \eta)] \in V\), there exist complex numbers \( c(q; \eta) \) such that:

(i) \( d(\xi; \eta) = \sum_q \xi_q \cdot c(q; \eta) \),

(ii) \( c(q; \eta) = 0 \) for \( \eta /\in R + \sigma^\vee \) (i.e. \( \eta \notin \bigcup_i K^R_i \)).

(Do this separately for each \( \eta \) and distinguish between the cases \( \eta \in R + \sigma^\vee \) and \( \eta /\in R + \sigma^\vee \).)

In particular, \([c(q; \eta) - c(q; \ell), d(\xi; \eta)] \in W\). Hence, we have seen that we may assume \( d(\xi; \eta) = 0 \).

Let us choose some sufficiently high degree \( \ell^* \geq E \). Then,

\[ x_i(q) := d(q; \ell, \eta) - d(q; \ell^*, \eta) \]

(with \( \ell \in K^R_i \), \( \ell \geq \text{supp} q \) (cf. Lemma [3][2](2)), and \( \eta \geq \ell, \ell^*, R \) defines some preimage:

(i) It is independent from the choice of \( \eta \): Using a different \( \eta' \) generates the difference \( d(q; \eta, \eta') - d(q; \eta, \eta') \).
(ii) It is independent from $\ell \in K^R_i$: Choosing another $\ell' \in K^R_i$ with $\ell' \geq \ell$ would add the summand $d(q; \ell, \ell')$, which is 0; for the general case use Lemma (3.2)(2).

(iii) If $\langle a^i, a^j \rangle < \sigma$ is a 2-face with $\text{supp} q \subseteq L(E^R_i) \cap L(E^R_j)$, then by Lemma (3.2)(2) we can choose an $\ell \in K^R_i \cap K^R_j$ achieving $x_i(q) = x_j(q)$.

(iv) For $\xi \in L^2(E^R_i)$ we have

$$\sum_q \xi_q \cdot d(q; \ell, \eta) = \sum_q \xi_q \cdot d(q; \ell^*, \eta) = 0,$$

and this gives the corresponding relation for the $x_i(q)$’s.

(v) Finally, if we apply to $[x_i(q)] \in V_1$ the linear map $V_1 \rightarrow V$, the result differs from $[d(q; \ell, \eta), 0] \in V$ by the $W$-element built from

$$c(q; \ell) := \begin{cases} \ d(q; \ell, \eta) - d(q; \ell^*, \eta) & \text{if } \ell \geq R \\ 0 & \text{otherwise} \end{cases}.$$ 

$\blacksquare$

(4.7) Now, it is easy to complete the proofs for Theorem (3.3) (part 2 and 3) and Theorem (3.5):

First, for a tuple $[x_i(q)]_{q \in L(E^R_i)}$, the condition

$$\xi \in L^2(E^R_i) \implies \sum_q \xi_q \cdot x_i(q) = 0$$

is equivalent to the fact the components $x_i(q)$ are induced by elements $x_i \in (L(E^R_i)_\sigma)^*$. The other condition for elements of $V_1$ just says that for 2-faces $\langle a^i, a^j \rangle < \sigma$ there is $x_i = x_j$ on $L(E^R_i \cap E^R_j)_\sigma = L(E^R_i)_\sigma \cap L(E^R_j)_\sigma$. In particular, we obtain

$$V_1 = \text{Ker} \left( \bigoplus_i L(E^R_i)^*_\sigma \rightarrow \bigoplus_{\langle a^i, a^j \rangle < \sigma} L(E^R_i \cap E^R_j)_\sigma \right).$$

In the same way we get

$$W_1 = \left( \bigoplus_i L(E^R_i)_\sigma \right)^*,$$

and our $T^2$-formula is proven.

Finally, if $\psi_i : L(E^R_i)_\sigma \rightarrow \mathcal{C}$ are linear maps defining an element of $V_1$, they induce the following $A$-linear map on $D$ (even on $\text{Im } d_D$):

$$D(q; \ell, \eta) \mapsto \begin{cases} \psi_i(q) \cdot x^{\eta-R} & \text{for } \ell \in K^R_i, \ \eta \geq R \\ 0 & \text{for } \ell \geq R \ \text{or } \eta \in \bigcup_i K^R_i \end{cases}.$$ 

$$D(\xi; \eta) \mapsto 0.$$ 

Now, looking at the diagram of (4.3), this translates exactly into the claim of Theorem (3.4).
5 Proof of the cup product formula

(5.1) Fix an \( R \in M \), and let \( \varphi \in L(E)_q^* \) induce some element (also denoted by \( \varphi \)) of \( T^1_q(-R) \). Using the notations of (2.3), (3.4), and (3.6) we can take

\[
\bar{\varphi}(f_{\alpha \beta}) := \varphi(\alpha - \beta) \cdot \bar{\varphi}^{\Phi(\pi(a)-R)}
\]

for the auxiliary \( P \)-elements needed to compute the \( \lambda(\varphi) \)'s (cf. Theorem (3.4)).

Now, we have to distinguish between the two several types of relations generating the \( P \)-module \( R \subseteq P^m \):

(r) Regarding the relation \( r(a, b; c) \) we obtain

\[
\sum_{(\alpha, \beta) \in m} r(a, b; c)_{\alpha \beta} \cdot \bar{\varphi}(f_{\alpha \beta}) = \varphi(a - b) \cdot \left( \bar{\varphi}^{\Phi(\pi(a+c)-R)} - \bar{\varphi}^{\Phi(\pi(a)-R)} \right)
\]

In particular,

\[
\lambda^{r(a,b,c)}(\varphi)(\alpha, \beta) = \begin{cases} 
\varphi(a - b) & \text{for } [\alpha, \beta] = [c + \Phi(\pi(a) - R), \Phi(\pi(a+c) - R)] \\
0 & \text{otherwise}.
\end{cases}
\]

(s) The corresponding result for the relation \( s(a, b, c) \) is much nicer:

\[
\sum_{(\alpha, \beta) \in m} s(a, b; c)_{\alpha \beta} \cdot \bar{\varphi}(f_{\alpha \beta}) = \varphi(a - b) \cdot \left( \varphi(c + \Phi(\pi(a) - R) - \Phi(\pi(a+c) - R)) \cdot x^{\pi(\pi(a+c)+R)-S} \\
\right)
\]

In particular, \( \lambda^{s(a,b,c)}(\varphi) = 0 \).

(5.2) Now, let \( R, S, \varphi, \) and \( \psi \) as in the assumption of Theorem (3.4). Using formula (2.3)(iii), our previous computations yield \((\varphi \cup \psi)(s(a, b, c)) = 0\) and

\[
(\varphi \cup \psi)(r(a, b; c)) = \sum_{\alpha, \beta} \lambda^{r(a,b,c)}(\varphi) \cdot \psi(f_{\alpha \beta}) + \sum_{\alpha, \beta} \varphi(f_{\alpha \beta}) \cdot \lambda^{r(a,b,c)}(\psi)
\]

\[
= \varphi(a - b) \cdot \psi(c + \Phi(\pi(a) - R) - \Phi(\pi(a+c) - R)) \cdot x^{\pi(\pi(a+c)+R)-S} \\
+ \psi(a - b) \cdot \varphi(c + \Phi(\pi(a) - S) - \Phi(\pi(a+c) - S)) \cdot x^{\pi(\pi(a+c)+S)-R}
\]

\[
= [\varphi(a - b) \cdot \psi(c + \Phi(\pi(a) - R) - \Phi(\pi(a+c) - R)) \\
+ \psi(a - b) \cdot \varphi(c + \Phi(\pi(a) - S) - \Phi(\pi(a+c) - S))] \cdot x^{\pi(a+c)-R-S}.
\]
Now, we obtain a $P$-linear map $T \in \text{Hom}(P^m, A)$ by

$$T : e^{\alpha \beta} \mapsto \begin{cases} t(\alpha, \beta) x^{\pi(\alpha) - R - S} & \text{for } \pi(\alpha) \geq R + S \\ 0 & \text{otherwise} \end{cases}.$$ 

Pulling back $T$ to $\mathcal{R} \subseteq P^m$ yields (in case of $\pi(a + c) \geq R + S$)

$$T(r(a, b; c)) = \begin{cases} [t(a + c, b + c) - t(a, b)] \cdot x^{\pi(a + c) - R - S} & \text{for } \pi(a) \geq R + S \\ t(a + c, b + c) \cdot x^{\pi(a + c) - R - S} & \text{otherwise} \end{cases}$$

Now, the previous lemma does not imply that $t(q)$ is $\mathbb{Z}$-linear in $q$. The assumption for $\pi(\alpha)$ made in (1) is really essential.

Remark: The previous lemma does not imply that $t(q)$ is $\mathbb{Z}$-linear in $q$. The assumption for $\pi(\alpha)$ made in (1) is really essential.
and $T(s(a, b, c)) = 0$ (by (2) of the previous lemma).

On the other hand, $T$ yields a trivial element of $T^2_i(-R - S)$, i.e. inside this group we may replace $\varphi \cup \psi$ by $(\varphi \cup \psi) + T$ to obtain

$$(\varphi \cup \psi)(r(a, b; c)) = \begin{cases} t(a, b) \cdot x^{\pi(a+c)-R-S} & \text{for } \pi(a) \in \bigcup_i K_i^{R+S}; \pi(a + c) \geq R + S \\ 0 & \text{otherwise}, \end{cases}$$

$$(\varphi \cup \psi)(s(a, b; c)) = 0.$$  

Having Theorem (3.5) in mind, this formula for $\varphi \cup \psi$ is exactly what we were looking for:

Given an $r(a, b; c)$ with $\pi(a) \in K_i^{R+S}$, let us compute $(\varphi \cup \psi)_{\varphi \cup \psi}(q := a - b)$ following the recipe of (i), (ii) of Theorem (3.6). We do not need to split $q = a - b$ into a sum $q = \sum_k q^k$ - the element $q$ itself already satisfies the condition

$$\langle a^i, q \rangle \leq \langle a^i, \pi(a) \rangle < \langle a^i, R + S \rangle.$$  

In particular, with $(\varphi \cup \psi)_{\varphi \cup \psi}(a - b) = t(a - b) = t(a, b)$ we will obtain the right result - if the recipe is assumed to be correct.

(5.4) We will fill those remaining gap now, i.e. we will show that

(a) each $q \in L(E_i^{R+S})$ admits a decomposition $q = \sum_k q^k$ with the desired properties,

(b) $\sum_k q^k = 0$ (with $q^k \in K_i^{R+S}$) implies $\sum_k t(q^k) = 0$, and

(c) for adjacent $a^i, a^j$ the relations $q \in L(E_i^{R+S} \cap E_j^{R+S})$ admit a decomposition $q = \sum_k q^k$ that works for both $i$ and $j$.

(In particular, this answers the questions arised right after stating the theorem in (3.6).)

Let us fix an element $i \in \{1, \ldots, N\}$. Since $\sigma^i \cap M$ contains elements $r$ with $\langle a^i, r \rangle = 1$, some of them must be contained in the generating set $E$, too. We choose one of these elements and call it $r(i)$.

Now, to each $r \in E$ we associate some relation $p(r) \in L(E)$ by

$$p(r) := e^r - \langle a^i, r \rangle \cdot e^{r(i)} + [\text{suitable element of } \mathbb{Z}^{E \cap (a^i)^\perp}].$$

The two essential properties of these special relations are

(i) $\langle a^i, p(r) \rangle = \langle a^i, r \rangle$, and

(ii) if $q \in L(E)$ is any relation, then $q$ and $\sum_{r \in E} q_{r^i} \cdot p(r)$ differ by some element of $L(E \cap (a^i)^\perp)$ only.
In particular, this proves (a). For (b) we start with the following

**Claim:** Let \( q^k \in L(E) \) be relations such that \( \sum_k \langle a^i, q^k \rangle < \langle a^i, R + S \rangle \). Then, \( \sum_k t(q^k) = t(\sum_k q^k) \).

**Proof:** We can restrict ourselves to the case of two summands, \( q^1 \) and \( q^2 \). Then, by Lemma (5.3),

\[
\begin{align*}
t(q^1) + t(q^2) &= t\left((q^1)^+, (q^1)^-\right) + t\left((q^2)^+, (q^2)^-\right) \\
&= t\left((q^1)^+ + (q^2)^+, (q^1)^- + (q^2)^-\right) + t\left((q^2)^+ + (q^1)^- + (q^2)^- + (q^1)^-\right) \\
&= t\left((q^1)^+ + (q^2)^+, (q^2)^- + (q^1)^-\right) \\
&= t(q^1 + q^2). \quad \Box
\end{align*}
\]

In particular, if \( \sum_k q^k = 0 \) (with \( q^k \in K_i^{R+S} \)), then this applies for the special decompositions

\[
q^k = \sum_r q^k_r \cdot p(r) + q^{0,k} \quad (q^{0,k} \in L(E \cap (a^i)^-))
\]

of the summands \( q^k \) themselves. We obtain

\[
\sum_{q^k_r > 0} q^k_r \cdot t(p(r)) + t(q^{0,k}) = t\left(\sum_{q^k_r > 0} q^k_r p(r) + q^{0,k}\right) =: t(q^{1,k})
\]

and

\[
\sum_{q^k_r < 0} q^k_r \cdot t(p(r)) = t\left(\sum_{q^k_r < 0} q^k_r p(r)\right) =: t(q^{2,k}).
\]

Up to elements of \( E \cap (a^i)^- \), the relations \( q^{1,k} \) and \( q^{2,k} \) are connected by the common

\[
(q^{1,k})^- = -q^{1,k}_i \cdot e^{r(i)} = \langle a^i, q^k \rangle^+ \cdot e^{r(i)} = q^{2,k}_i \cdot e^{r(i)} = (q^{2,k})^+.
\]

Hence, Lemma (5.3) yields

\[
\sum_r q^k_r \cdot t(p(r)) + t(q^{0,k}) = t(q^{1,k}) + t(q^{2,k}) = t(q^{1,k} + q^{2,k}) = t(q^k),
\]

and we conclude

\[
\sum_k t(q^k) = \sum_k \left(\sum_r q^k_r \cdot t(p(r)) + t(q^{0,k})\right) \\
= \sum_r \left(\sum_k q^k_r\right) t(p(r)) + t\left(\sum_k q^{0,k}\right) \quad \text{(cf. previous claim)} \\
= 0 + t\left(\sum_k q^k - \sum_{k,r} q^k_r p(r)\right) \\
= 0.
\]

19
Finally, only (c) is left. Let $a^i$, $a^j$ be two adjacent edges of $\sigma$. We adapt the construction of the elementary relations $p(r)$. Instead of the $r(i)$’s, we will use elements $r(i,j) \in E$ characterized by the property

$$\langle a^i, r(i,j) \rangle = 1, \langle a^j, r(i,j) \rangle = 0.$$

(Those elements exist, since $Y$ is assumed to be smooth in codimension 2.) Now, we define

$$p(r) := e^r - \langle a^i, r \rangle \cdot e^{r(i,j)} - \langle a^j, r \rangle \cdot e^{r(j,i)} + \text{[suitable element of $Z \cap (a^i)^\perp \cap (a^j)^\perp$]}.$$

These special $p(r)$’s meet the usual properties (i) and (ii) - but for the two different indices $i$ and $j$ at the same time. In particular, if $q \in L(E)$ is any relation, then $q$ and $\sum_{r \in E} q_r \cdot p(r)$ differ by some element of $L(E \cap (a^i)^\perp \cap (a^j)^\perp)$ only.

6 An alternative to the complex $L(E^R)$.

(6.1) Let $R \in M$ be fixed for the whole $\S$. The complex $L(E^R)$ introduced in (3.2) fits naturally into the exact sequence

$$0 \rightarrow L(E^R) \rightarrow (Z \cap E^R) \rightarrow \text{span}(E^R) \rightarrow 0$$

of complexes built in the same way as $L(E^R)$, i.e.

$$(Z \cap E^R)_{-k} := \bigoplus_{\dim \tau = k} Z \cap E^R \quad \text{and} \quad \text{span}(E^R)_{-k} := \bigoplus_{\dim \tau = k} \text{span}(E^R).$$

Lemma: The complex $(Z \cap E^R)$ is exact.

Proof: The complex $(Z \cap E^R)$ can be decomposed into a direct sum

$$(Z \cap E^R) = \bigoplus_{r \in M} (Z \cap E^R)(r),$$

showing the contribution of each $r \in M$. The complexes occurring as summands are defined as

$$(Z \cap E^R)(r)_{-k} := \bigoplus_{\dim \tau = k} \left\{ \begin{array}{ll}
Z &= Z^\{r\} \quad \text{for } r \in E^R \tau \\
0 &= \text{otherwise}
\end{array} \right\}$$

$$= \bigoplus_{\dim \tau = k} \{ \tau \mid r \in E^R \}. $$

Denote by $H^+$ the halfspace

$$H^+ := \{ a \in N_R \mid \langle a, r \rangle < \langle a, R \rangle \} \subseteq N_R.$$
Then, for $\tau \neq 0$, the fact that $r \in E^R_\tau$ is equivalent to $\tau \setminus \{0\} \subseteq H^+$. On the other hand, $r \in E^R_0$ corresponds to the condition $\sigma \cap H^+ \neq \emptyset$.

In particular, $(Z^E_R)(r)_*$, shifted by one place, equals the complex for computing the reduced homolgy of the topological space $\cup\{\tau \mid \tau \setminus \{0\} \subseteq H^+\} \subseteq \sigma$ cut by some affine hyperplane. Since this space is contractable, the complex is exact. \[\square\]

**Corollary:** The complexes $L(E^R)_*$ and $\text{span}(E^R)_*[1]$ are quasiisomorphic. In particular, under the usual assumptions (cf. Theorem (3.3)), we obtain

$$T^i_Y(-R) = H^i\left(\text{span}(E^R)_* \otimes_{\mathbb{Z}} \mathcal{F}\right).$$

(6.2) We define the $\mathbb{R}$-vector spaces

$$V^R_i := \text{span}_{\mathbb{R}}(E^R_i) = \begin{cases} 0 & \text{for } \langle a^i, R \rangle \leq 0 \\ [a^i = 0] & \text{for } \langle a^i, R \rangle = 1 \\ M_\mathbb{R} = \mathbb{R}^n & \text{for } \langle a^i, R \rangle \geq 2 \end{cases}$$

and

$$V^R_\tau := \bigcap_{a^i \in \tau} V^R_i \supseteq \text{span}_{\mathbb{R}}(E^R_\tau) \quad \text{(for faces } \tau < \sigma).$$

**Proposition:** With $\mathcal{V}^R_{-k} := \bigoplus_{\dim \tau = k} V^R_\tau$ we obtain a complex $\mathcal{V}^R_* \supseteq \text{span}_{\mathbb{R}}(E^R)_*$. Moreover, if $Y$ is smooth in codimension $k$, then both complexes are equal at $\geq (-k)$.

**Proof:** $V^R_\tau = \text{span}_{\mathbb{R}}(E^R_\tau)$ is true for smooth cones $\tau < \sigma$ (cf.(3.7) of [AI]). \[\square\]

**Corollary:**

1. If $Y$ is smooth in codimension 2, then $T^1_Y(-R) = H^1\left((\mathcal{V}^R_*)^* \otimes_{\mathbb{R}} \mathcal{F}\right)$.

2. If $Y$ is smooth in codimension 3, then $T^2_Y(-R) = H^2\left((\mathcal{V}^R_*)^* \otimes_{\mathbb{R}} \mathcal{F}\right)$.

The formula (1) for $T^1_Y$ (with a more boring proof) was already obtained in (4.4) of [AI].

## 7 3-dimensional Gorenstein singularities

(7.1) We want to apply the previous results for the special case of an isolated, 3-dimensional, toric Gorenstein singularity. We start with fixing the notations.
Let \( Q = \text{conv}(a^1, \ldots, a^N) \subseteq \mathbb{R}^2 \) be a lattice polygon with primitive edges
\[
d^i := a^{i+1} - a^i \in \mathbb{Z}^2.
\]
Embedding \( \mathbb{R}^2 \) as the affine hyperplane \([a_3 = 1]\) into \( N_{\mathbb{R}} := \mathbb{R}^3 \), we can define the cone
\[
\sigma := \text{Cone}(Q) \subseteq N_{\mathbb{R}}.
\]
The fundamental generators of \( \sigma \) equal the vectors \((a^1, 1), \ldots, (a^N, 1)\), which we will also denote by \( a^1, \ldots, a^N \), respectively.

The vector space \( M_{\mathbb{R}} \) contains a special element \( R^* := [0, 0; 1] \):
- \( \langle \cdot, R^* \rangle = 1 \) defines the affine hyperplane containing \( Q \),
- \( \langle \cdot, R^* \rangle = 0 \) describes the vectorspace containing the edges \( d^i \) of \( Q \).

The structure of the dual cone \( \sigma^\vee \) can be described as follows:
- \([c; \eta] \in M_{\mathbb{R}} \) is contained in \( \sigma^\vee \), iff \( \langle Q, -c \rangle \leq \eta \).
- \([c; \eta] \in \partial \sigma^\vee \) iff there exists some \( i \) with \( \langle a^i, -c \rangle = \eta \).
- The set \( E \) contains \( R^* \). However, \( E \setminus \{ R^* \} \subseteq \partial \sigma^\vee \).

**Remark:** The toric variety \( Y \) built by the cone \( \sigma \) is 3-dimensional, Gorenstein, and regular outside its 0-dimensional orbit. Moreover, all those singularities can be obtained in this way.

(7.2) Let \( V \) denote the \((N - 2)\)-dimensional \( \mathbb{R} \)-vector space
\[
V := \{(t_1, \ldots, t_N) \mid \sum_i t_i d^i = 0\} \subseteq \mathbb{R}^N.
\]
The non-negative tuples among the \( t \in V \) describe the set of Minkowski summands \( Q_t \) of positive multiples of the polygon \( Q \). \((t_i \) is the scalar by which \( d^i \) has to be multiplied to get the \( i \)-th edge of \( Q_t \)).

We consider the bilinear map
\[
V \times \mathbb{R}^E \xrightarrow{\Psi} \mathbb{R}
\]
\[
v, \ [c; \eta] \in E \mapsto \begin{cases} 0 & \text{if } c = 0 \ (\text{i.e. } [c; \eta] = R^*) \\ \sum_{v=1}^{i-1} t_v \cdot \langle d^v, -c \rangle & \text{if } \langle a^i, -c \rangle = \eta. \end{cases}
\]
Assuming both \( a^1 \) and the associated vertices of all Minkowski summands \( Q_t \) to coincide with \( 0 \in \mathbb{R}^2 \), the map \( \Psi \) detects the maximal values of the linear functions \( c \) on these summands
\[
\Psi(v, [c; \eta]) = \text{Max}(\langle a, -c \rangle \mid a \in Q_t).
\]
In particular, \( \Psi(\mathbb{L}[1;c;\eta]) = \eta \), i.e. \( \Psi \) induces a map
\[
\Psi : \frac{V}{\mathbb{IR} \cdot 1} \times L_{\mathbb{R}}(E) \longrightarrow \mathbb{IR}.
\]

The results of \([\text{Al 2}]\) and \([\text{Sm}]\) imply that \( \Psi \) provides an isomorphism
\[
V_{\mathcal{G}}/\mathcal{G} \cdot 1 \stackrel{\sim}{\longrightarrow} \left( L(E_0^{R^*}) / \sum_i L(E_i^{R^*}) \right)^* \otimes \mathbb{Z} \mathcal{G} \cong T_Y^1(-R^*) = T_Y^1.
\]

In particular, \( \dim T_Y^1 = N - 3 \).

\(7.3\) Let \( R \in M \). Combining the general results of \( \S \) with the fact
\[
\bigcap_i V_i^R = \text{Ker} \left[ \bigoplus_i (V_i^R \cap V_{i+1}^R) \longrightarrow \bigoplus_i V_i^R \right]
\]
coming from the special situation we are in, we obtain the handsome formula
\[
T_Y^2(-R) = \left[ \bigcap_i (\text{span}_\mathcal{G} E_i^R) / \text{span}_\mathcal{G} (\bigcap_i E_i^R) \right]^*.
\]

\( T_Y^1 \) is concentrated in the degree \(-R^*\). Hence, for computing \( T_Y^2 \), the degrees \(-kR^* \) \( (k \geq 2) \) are the most interesting (but not only) ones. In this special case, the vector spaces \( V_i^{kR^*} \) equal \( M_{\mathbb{IR}} \), i.e.
\[
T_Y^2(-kR^*) = \left[ \frac{M_{\mathcal{G}} / \text{span}_\mathcal{G} (\bigcap_i E_i^{kR^*})}{\text{span}_\mathcal{G} (\bigcap_i E_i^{kR^*})} \right]^* = \text{span}_\mathcal{G} (d^1, \ldots, d^N) \subseteq N_{\mathcal{G}}.
\]

**Proposition:** For \( c \in \mathbb{IR}^2 \) denote by
\[
d(c) := \text{Max}(\langle a^i, c \rangle \mid i = 1, \ldots, N) - \text{Min}(\langle a^i, c \rangle \mid i = 1, \ldots, N)
\]
the diameter of \( Q \) in \( c \)-direction. If
\[
k_1 := \text{Min}_{c \in \mathbb{Z}^2 \setminus \emptyset} d(c) \quad \text{and} \quad k_2 := \text{Min}_{c,c' \in \mathbb{Z}^2 \text{lin.indep.}} \text{Max}[d(c), d(c')],
\]
then
\[
\dim T_Y^2(-kR^*) = 2 \quad \text{for} \quad 2 \leq k \leq k_1,
\]
\[
\dim T_Y^2(-kR^*) = 1 \quad \text{for} \quad k_1 + 1 \leq k \leq k_2, \quad \text{and}
\]
\[
\dim T_Y^2(-kR^*) = 0 \quad \text{for} \quad k_2 + 1 \leq k.
\]
Proof: We have to determine the dimension of \( \text{span}_{\mathcal{C}} \left( \cap_i E_i^{kR^*} \right) / \mathcal{Q} \cdot R^* \). Computing modulo \( R^* \) simply means to forget the \( \eta \) in \( [c; \eta] \in M \). Hence, we are done by the following observation for each \( c \in \mathbb{Z}^2 \setminus 0 \):

\[ \exists \eta \in \mathbb{Z} : [c, \eta] \in \cap_i K_i^{kR^*} \iff \exists \eta \in \mathbb{Z} : (k - 1)R^* \geq [c; \eta] \geq 0 \iff d(c) \leq k - 1. \]

\( \blacksquare \)

Corollary: Unless \( Y = \mathcal{C}^3 \) or \( Y = \text{cone over } \mathbb{P}^1 \times \mathbb{P}^1 \), we have

\( T_Y^2(-2R^*) = \text{span}_{\mathcal{C}}(d_1, \ldots, d^N) \),

i.e. \( \dim T_Y^2(-2R^*) = 2 \).

(7.4) Proposition: Using both the isomorphism \( V_{\mathcal{C}}/\mathcal{Q} \cdot \frac{1}{\mathbb{Z}} \xrightarrow{\sim} T_Y^1 \) and the injection \( T_Y^2(-2R^*) \hookrightarrow \text{span}_{\mathcal{C}}(d_1, \ldots, d^N) \), the cup product \( T_Y^1 \times T_Y^1 \rightarrow T_Y^2 \) equals the bilinear map

\[ V_{\mathcal{C}}/\mathcal{Q} \cdot \frac{1}{\mathbb{Z}} \times V_{\mathcal{C}}/\mathcal{Q} \cdot \frac{1}{\mathbb{Z}} \rightarrow \text{span}_{\mathcal{C}}(d_1, \ldots, d^N) \mapsto \sum_i s_i t_i d^i. \]

Proof: Step 1: To apply Theorem (3.6) we will combine the isomorphisms for \( T_Y^2 \) presented in §3 and (5.3). Actually, we will describe the dual map by associating to each \( r \in M \) an element \( [q^1(r), \ldots, q^N(r)] \in \oplus_i L(E_i^{2R^*}) \).

First, for every \( i = 1, \ldots, N \), we have to write \( r \in M = (\text{span } E_i^{2R^*}) \cap (\text{span } E_i^{2R^*}) \) as a linear combination of elements from \( E_i^{2R^*} \cap E_i^{2R^*} \). This set contains a \( \mathbb{Z} \)-basis for \( M \) consisting of

- \( r^i := \text{primitive element of } \sigma^i \cap (a^i)^\perp \cap (a^{i+1})^\perp \),
- \( R^* \), and
- \( r(i) := r(i, i + 1) \) (cf. notation at the end of (3.3)), i.e. \( \langle a^i, r(i) \rangle = 1 \) and \( \langle a^{i+1}, r(i) \rangle = 0 \).

In particular, we can write

\[ r = g^i(r) \cdot r^i + \langle a^{i+1}, r \rangle \cdot R^* + \left( \langle a^i, r \rangle - \langle a^{i+1}, r \rangle \right) \cdot r(i) \]

with some integer \( g^i(r) \in \mathbb{Z} \).

Now, we have to apply the differential in the complex \( (\mathbb{Z}^{E_{2R^*}})_* \), i.e. we map the previous expression via the map

\[ \oplus_i \mathbb{Z}^{E_i^{2R^*}} \cap E_i^{2R^*} \rightarrow \oplus_i \mathbb{Z}^{E_i^{2R^*}}. \]
The result is (for every \(i\)) the element of \(L(E_i^{2R^*})\)
\[
g^i(r) e^{r^i} - g^{i-1}(r) e^{r^{i-1}} + \langle a^i - a^{i+1}, r \rangle \cdot e^{r(i)} - \langle a^{i-1} - a^i, r \rangle \cdot e^{r(i-1)} + \langle a^{i+1} - a^i, r \rangle \cdot e^{R^*}
= \langle d^i, r \rangle \cdot (e^{R^*} - e^{r(i)}) + \langle (a^i)^{\perp}\text{-summands} \rangle = q^i(r).
\]

**Step 2:** Defining
\[
q^i := e^{R^*} - e^{r(i)} + \langle (a^i)^{\perp}\text{-summands} \rangle \in L(E_i^{2R^*}) \quad (i = 1, \ldots, N),
\]
we use Theorem (3.6) and the second remark of (3.6) to obtain
\[
(s \cup t)_i \left(q^i(r)\right) = \langle d^i, r \rangle \cdot t_{\Psi(s \bullet), \Psi(t \bullet), R^*, R^*(q^i)} = \Psi(s, q^i) \cdot \Psi(t, q^i).
\]

To compute those two factors, we take a closer look at the \(q^i\)'s. Let
\[
q^i = e^{R^*} - e^{r(i)} + \sum_v \lambda^i_v e^{[c^v, \eta^v]},
\]
and the sum is taken over those \(v\)'s meeting the property \(\langle a^i, -c^v \rangle = \eta^v\). Then, by definition of \(\Psi\) in (3.2),
\[
\Psi(s, q^i) = \sum_{j=1}^{(i+1)-1} s_j \langle d^j, r(i) \rangle - \sum_v \lambda^i_v \cdot \left(\sum_{j=1}^{i-1} s_j \langle d^j, c^v \rangle\right).
\]

On the other hand, we know that \(q^i\) is a relation, i.e. the equation
\[
R^* - r(i) + \sum_v \lambda^i_v [c^v, \eta^v] = 0
\]
is true in \(M\). Hence,
\[
\Psi(s, q^i) = \sum_{j=1}^{i} s_j \langle d^j, r(i) \rangle - \sum_{j=1}^{i-1} s_j \langle d^j, r(i) \rangle
= s_i \cdot \langle d^i, r(i) \rangle
= -s_i.
\]

\(T^1_Y \subseteq Q^N\) is the tangent space of the versal base space \(S\) of our singularity \(Y\). It is given by the linear equation \(\sum_i t_i \cdot d^i = 0\).

On the other hand, the cup product \(T^1_Y \times T^1_Y \to T^2_Y\) shows the quadratic part of the equations defining \(S \subseteq Q^N\). By the previous proposition, it equals \(\sum_i t_i^2 \cdot d^i\).

These facts suggest an idea how the equations of \(S \subseteq Q^N\) could look like. In [Al 3] we have proved this conjecture; \(S\) is indeed given by the equations
\[
\sum_{i=1}^N t_i^k \cdot d^i = 0 \quad (k \geq 1).
\]
References

[Al 1] Altmann, K.: Computation of the vector space $T^1$ for affine toric varieties. J. Pure Appl. Algebra (to appear).

[Al 2] Altmann, K.: Toric $Q$-Gorenstein Singularities.
Preprint 9/1993 Humboldt-Universität Berlin; e-print alg-geom/9403003.

[Al 3] Altmann, K.: The versal Deformation of an isolated, toric Gorenstein Singularity.
Preprint M.I.T., February 1994; preprint 59 “Europäisches Singularitätenprojekt”; e-print alg-geom/9403004.

[Ke] Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B.: Toroidal Embeddings I.
Lecture Notes in Mathematics 339, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[KPR] Kurke, H., Pfister, G. Roczen, M.: Henselsche Ringe und algebraische Geometrie.
Berlin: Deutscher Verlag der Wissenschaften 1975.

[Od] Oda, T.: Convex bodies and algebraic geometry.
Ergebnisse der Mathematik und ihrer Grenzgebiete (3/15), Springer-Verlag, 1988.

[Sm] Smilansky, Z.: Decomposability of Polytopes and Polyhedra.
Geometriae Dedicata. 24 (1987), 29-49.