The early history of the Vogt-Russell theorem is retraced following its route starting at the realization of a correlation between mass and luminosity of binary and pulsating stars, through the embossing of this observation into a theorem, and finally to the emerging first signs of its failure to serve as a theorem in the strict mathematical sense of the word.

Introduction

Astrophysics is not an exact science in the sense of mathematics. On the observational side, time and again the Universe is considerably more creative than human imagination; on the theoretical side, the equations that are derived to model the intricate processes and phenomena in astrophysics usually lack the simple symmetries or properties to attract mathematicians’ attention for in-depth analyses of their properties. Therefore, proclaiming a theorem in astrophysics – by astrophysicists – was, is, and likely will remain a daring undertaking.

THE theorem in stellar physics, the topic of this marginalia, is known by the name of Vogt-Russell (VR) theorem: It claims, roughly speaking, that the structure of any star is uniquely determined by its mass and its chemical composition alone. Today we know that the theorem does not hold in its original strict formulation; nonetheless, the VR theorem continues to enjoy some popularity and pops up in almost all courses on stellar structure and evolution, in textbooks, and even recently in research papers.  

The following exposition retraces the history of the VR theorem during the (semi-) analytic era of stellar astrophysics. The first statement of what not much later became the theorem appeared essentially en passant in a research note of Heinrich Vogt where he generalized Eddington’s mass-luminosity relation. The ‘proof’ by Henry Norris Russell, which advanced the original claim to a theorem, was enshrined later in Chandrasekhar’s seminal reference work, An Introduction to the Study of Stellar Structure. After that the VR theorem lived an apparently quiet and unquestioned life to the end of the 1950s when first counterexamples, albeit rather un-astrophysical ones, were put forth. The literature of the early 1960s contains more cases of by then more naturalistic star models that seemed to violate the VR theorem. The following exposition covers the history up to the publication the monograph of Cox &Giuli (1968) where the authors devoted a short chapter to the VR
theorem and gave an impression of the thinking on this matter at that time.

A forthcoming second part of the history of the VR theorem will deal with a revived interest in the topic in the 1970s, an epoch when stellar evolution theory had turned into a computation-intensive enterprise with ever more complex input physics and ever more complicated resulting star structures.

First formulations

The 1910s and 1920s were the years when astrophysicists started to understand the stars as long-living self-gravitating fluid spheres. Even though the source of energy was not yet identified, the thermo-mechanical structure of stars was already modeled mathematically. In accordance with this intellectual achievement, the body of history-related literature collecting, scrutinizing the contributions, and cross-linking the players in the field is as huge as authoritative. For those interested in the topic, Hufbauer (2006), Gingerich (1995), Cowling (1966), and references therein serve as fertile starting points.

For this marginalia it is sufficient to realize that in 1924 Arthur Eddington published a paper with the title *On the relation between the masses and the luminosities of the stars* (Eddington 1924) where he used his analytical thermo-mechanical star models to fit the observed correlation of luminosities and masses of stars (see Appendix A for a discussion). Eddington collected the data of 46 stars belonging to various kinds of binary stars for which masses and absolute magnitudes were reported; all of them (plus five pulsating variable stars) obeyed a remarkably smooth relation.

In December 1925, Heinrich Vogt, an astronomer at Heidelberg’s Königstuhl Observatory, submitted a short theoretical paper to the *Astronomische Nachrichten* (Vogt 1926) in which he generalized Eddington’s analytical star models by introducing spatially variable forms of the mass-absorption coefficient and of the energy generation rate. In the last paragraph of the research note, Vogt mentioned laconically, what he later referred to as the *Eindeutigkeitssatz* (uniqueness theorem):

\[
\text{Wir müssen annehmen, daß die mittlere Dichte, die effektive Temperatur und die absolute Leuchtkraft eines Sternes nur von seiner Gesamtmasse abhängen [...]}
\]

Translation: We must assume that the mean density, the effective temperature and the absolute luminosity of a star depend on its total mass only.

Vogt allowed, additionally, for a small variation of the magnitudes of the global stellar quantities at fixed mass because they might be influenced by the nature of the stellar material, i.e. by the star’s chemical composition. In other words, Vogt claimed that the global stellar quantities \( L_*(M, \mu) \), \( T_{\text{eff}}(M, \mu) \), \( \rho(M, \mu) \) are uniquely determined by the value of stellar mass and the star’s composition alone. As it seems, the uniqueness claim was not particularly important to Vogt: About one year after the first note on the subject, in
a long review of the theory of stellar structure and evolution, Vogt (1927) did not even touch the aspect of uniqueness of the solutions to the equations describing the structure of the stars. Another year later, when Vogt (1928) devoted a more extensive paper particularly to the mass – luminosity law to emphasize once again that a form very close to that of Eddington can be recovered even if the internal structure of the stars differs from what Eddington assumed. And again, the paper of 1928 does not mention the Eindeutigkeits-property. Only in 1930, in the longest treatise (Vogt 1930) concerning the relation of mass, luminosity, and effective temperature of the stars – essentially an attempt to understand the distribution of the stars in the Hertzsprung-Russell (HR) diagram – did Vogt refer briefly and informally to the dependence of the global stellar quantities on mass and chemical composition; however, without mentioning his 1926 paper. Even after the 1930 publication, one wonders how much importance Vogt actually attributed to the uniqueness conjecture and if he realized its consequences.

On the other side of the Atlantic, Henry Norris Russell studied the physical basis of stellar evolution since the earliest days of modern astrophysics (cf. DeVorkin 2000). In the mid 1920s, he and colleagues at Princeton Observatory overhauled the textbook Astronomy - A revision of Young’s Manual of Astronomy, updating it also with the latest ideas and results from the thriving field of stellar astronomy (Russell et al. 1927). In the second volume, in Section 975, the authors state:



The above statement was made without further elucidation and without going into any technical details. In any case, the uniqueness claim of Russell et al. appears more deliberately formulated and in particular physically more coherent than Vogt’s statement from a year earlier. In the same Section 975 of the textbook, the uniqueness property was then applied to the interpretation of the structure of what later became the HR diagram and of Eddington’s mass – luminosity diagram. Any scatter to the observed distribution of stars in these diagrams that goes beyond the observational uncertainties was interpreted by the authors to mean that [...] generation of heat [...] is different in different stars. The stars cannot therefore all contain the same proportion of ‘active matter’ [...] in other words [...] they [the stars] must differ in composition. Hence, in the textbook of Russell et al. (1927), the uniqueness statement of the structure of the stars is of auxiliary use only, namely as a supporting argument in the interpretation of the distribution of the stars in the HR diagram. The stars were thought to populate the HR plane as a function of
mass and of varying chemical composition. The mass was believed to diminish with age so that stars evolve across the HR plane from high to low mass during their life. At the end of Section 975 of Astronomy, where contributors to the content of the stellar-evolution discussion, including the uniqueness conjecture, were referred to, only Russell himself and Eddington appeared – Vogt’s research note was not mentioned.

**The path to theorem**

In a review, which discussed the state of the theory of the constitution of the stars, Russell (1931a) clarified that Vogt and, independently, he himself formulated a few years earlier a theorem on the uniqueness of stellar structure. When referring to Vogt, Russell cited the extensive paper of Vogt (1930) on the nature of the correlations of global stellar observables and their relation to their internal structure rather than the short note of 1926. One volume of the MNRAS later, Russell (1931b) eventually set the records straight: In an *addendum*, he reported that Vogt brought to his attention the proper reference containing the *first* statement of the uniqueness claim, which goes back to the year 1926. In any case, as early as 1931, Russell regarded the uniqueness statement as a *theorem* (of deep insight and as a contribution to the field of stellar astrophysics which he considered to have remained much undervalued in the community). Not much later, Russell (1933) reiterated his opinion in a non-technical survey on stellar astrophysics in general, and stellar structure and evolution in particular; he referred again to the *Vogt theorem* as the [...] most important general proposition regarding stellar constitution [...] As of then, Russell apparently attributed to the uniqueness theorem much more importance than Vogt ever did in any of his writing; this might be connected with Russell’s interest to thoroughly interpret the distribution of stars across the spectral class – luminosity diagram, on which he actively worked with the goal to pack all the stars into a coherent story in the framework of the theory of stellar evolution as it stood then (cf. Gingerich (1995); Hufbauer (2006)).

Russell, in contrast to Vogt, resorted already early on to a mathematical argumentation and *proofed* the uniqueness theorem (cf. Russell (1931a)). The stellar structure problem was considered as the solution to a system of four differential equations with distributed boundary conditions. Since the number of boundary conditions was counted to be three, compared with four differential equations, Russell concluded after some meandering that unique one-parameter sequences of solutions must exist for a prescribed chemical composition, and that without loss of generality, the star’s mass can be chosen as this parameter.

Under the spell of the success of quantum theory, the young Danish astrophysicist Bengt Strömgren set out, in the early 1930s, to improve the understanding of the HR diagram adopting hydrogen-
rich stellar models with the use of more elaborate microphysics. In long review paper, Strömgren (1937) presented a comprehensive view of his understanding of the theory of the stellar interiors and of stellar evolution. A whole section 4 of the exposition was dedicated to the uniqueness conjecture of Vogt and Russell. Strömgren referred to it as the [...] Satz von Vogt und Russell [...] (p. 477) and he also outlined its proof, following the line of argumentation of Russell (1931a). Not much later and apparently influenced by Strömgrens work, the authoritative monograph Stellar Structure of Chandrasekhar (1939) consolidated the theorem status of what, at best, should still have been considered the conjecture of Vogt and Russell. In Section VII.1 of his book, Chandrasekhar contemplated, nonetheless, circumstances which could void the VR theorem. He hypothesized material functions as sources of trouble; e.g. a nuclear energy generation rate, which does not depend on the local values of ρ and T. Macroscopic counterexamples, however, were yet beyond the intellectual horizon.

Kurth (1953) studied homology transformations of the stellar structure equations and their properties (in a remarkably modern formulation). In this context, Kurth formulated an aggravated version of the VR theorem in that he progressed from if to iff: In the framework of homologous, chemically homogeneous star models in complete equilibrium he concluded that the stars have nearly the same internal structure iff they have nearly the same mass and nearly the same chemical composition. Only a few sentences after this conclusion, Kurth cautioned the reader that [...] Bewiesen ist nichts, es handelt sich nur um Plausibilitätsbetrachtungen [...], all statements were indeed clearly declared as plausibility considerations; in particular, he explicitly assumed that solutions to the stellar structure equations exist. Finally, Kurth also distinguished between the pure stellar structure problem (i.e. stationary solutions to the structure equations), for which the VR theorem was formulated, and the full stellar-evolution problem which, by its very nature, is a time-dependent problem. Therefore, he warned that any extrapolation of conclusions from the application of the original VR theorem to the realistic stellar-evolution problem has to be treated with utmost caution.

Only a few years later, Odgers (1957) reported his attempts to construct homologous series of chemically homogeneous star models in complete equilibrium. The aim was to derive a simple analytical formulation of the mass – luminosity law of main-sequence stars. Already early on in the paper, Odgers criticized the mathematical assumptions that entered the uniqueness statement of the VR theorem and he even deduced an explicit homologous series which violated it. Because the inferred energy-generation law in the conflicting model series was unphysical in the stellar context the counterexample to the VR theorem was considered at best of formal interest. Even though Odgers’ paper was an actual mathematical blow for the VR theorem, the paper made no impact in the
astrophysical community: The circumstances under which multiple solutions occurred were either stellar-physically unacceptable or they were too restrictive that their realization in nature seemed unlikely. A further handicap of the Odgers paper was that it was published and circulated as an observatory report only and as such must have had a diminished audience. Last but not least, the report’s content is very formal and likely was too tough to digest for many in the astronomical community.

In the second edition of his textbook, Aufbau und Entwicklung der Sterne, Vogt (1957) came back to the Eindeutigkeitssatz der Theorie des Sternaufbaus and devoted a whole chapter to it. As in earlier publications, Vogt did not adopt a particularly mathematical point of view to discuss the problem. Instead, he merely insisted that the structure equations admit of unique solutions as long as the material properties (such equation of state, opacity, or nuclear energy generation) are well defined functions of thermodynamic state variables. As means to destroy the one-parameter families (characterized by different chemical composition) of evolutionary tracks on the HR plane, Vogt contemplated physical effects such as rotation, electromagnetic braking, or tidal effects in binaries.

Despite the lack of evidence of any impact in stellar astrophysics of the report of Odgers (1957) his paper marks the begin of the era in which multiple solutions began to pop up in the ever more detailed stellar-model computations. For example, Cox & Salpeter (1964) calculated low-mass pure helium star models in complete equilibrium. Below a critical mass, $M_{\text{min}}$, helium stars cannot maintain steady helium burning. Around this $M_{\text{min}} \approx 0.305 M_\odot$, a low- and a high-density solution with the same total mass and with identical composition were revealed. Not much later, for pure carbon stars too, double solutions for equal-mass model stars were encountered (Deinzer & Salpeter 1965). For carbon stars, the minimum mass is larger than that of the helium stars. Neither in Cox & Salpeter (1964) nor in Deinzer & Salpeter (1965) is there any indication that the authors connected the double solutions with a failure of the VR theorem. Bodenheimer (1966) on the other hand questioned the validity of the VR theorem upon realizing that his pre – main-sequence model stars all converged essentially to the same evolutionary locus along the Hayashi line, independent of the initial conditions he prescribed for his model sequences. Even though Bodenheimer was the most attentive author back then, his models do not serve as counterexamples to the VR theorem because they are not in complete equilibrium as they need to be for the orginal VR theorem to be applicable.

A next higher level of complexity in stellar modeling was reached with composite models that consisted of a core and of a grafted envelope, both in complete equilibrium but both with differing chemical composition. Adopting the mass of the core as the control parameter allows to study the dependence of the physical properties of a series of star models under continuous variation.
of the control parameter, such model sequences are known as linear series. To investigate the onset of the Schönberg-Chandrasekhar instability, Gabriel & Ledoux (1967) chose the relative mass, $q_{\text{He}}$, of the inert helium core below a hydrogen-burning shell and a hydrogen-rich envelope as the control parameter of their linear series of model stars with constant total mass. Gabriel and Ledoux concluded that the instability develops at a turning point of their linear series. In the neighborhood of this turning point, the stellar structure equations were observed to admit of double solutions at constant $q_{\text{He}}$. Investigations of the stability of these double solutions revealed then that one branch was secularly unstable. Even though double-solutions for the same stellar mass and the same chemical-composition profile were encountered, the result was not yet discussed in the context of the VR theorem. The situation changed when Gabriel & Noëls-Grötsch (1968) studied pure carbon stars in the neighborhood of the respective $M_{\text{min}}$, again they found that only one branch, the low-degeneracy one, of the double-solution region was secularly stable. Eventually, the authors concluded that turning points of linear series signal violations of the classical VR theorem. Resorting to a more restricted formulation, Gabriel & Noëls-Grötsch (1968) tried to save the VR theorem by adding the aspect of secular stability: [...] For a given mass and chemical composition there exists only one secularly stable configuration. [...] 

Consulting Cox & Giuli (1968), who published a comprehensive textbook which details knowledge and understanding of structure and evolution of simple single stars by the mid 1960s, one finds it to offer indeed an appropriate endpoint to the first part of this review of the history of the VR theorem. In PSS, a whole – albeit short – chapter is devoted to the VR theorem; that choice met criticism already early on by one of the reviewers of the books (Sweet 1969) and it likely sheds light on the importance attributed to the VR theorem at that time. Be it as it may, Chapter 18 of PSS is very useful here because it offers a glimpse at the perception of the VR theorem in the mid 1960s. Early on in the discussion, the authors emphasized that the VR theorem is not a theorem in a strict sense because cases of multiple solutions had been encountered and that a watertight proof had never been put forth. Nonetheless, Cox and Giuli could not resist the temptation to give a kind of a plausibility-’proof’ of the VR theorem, following the line of thought already present in Russell (1931a). The system of equations that entered the proof remained those of a stellar configuration in complete equilibrium so that the problem reduced to system of ordinary differential equation. The separated boundary conditions were introduced and it was argued, without going into any mathematical detail, that the implied algebraic constraints of the boundary conditions, being of lower dimensionality than the dimension of the solution space, ensure [...] under ordinary conditions [...] unique solutions. More originally, Cox and Giuli offered also a physical interpreta-

N.B. Increasing the magnitude of $q_{\text{He}}$ can be understood as an emulation of stellar evolution through a sequence of equilibrium states.

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9 The monograph, Principles of Stellar Structure, is referred to as PSS subsequently.

10 Simple in the sense of radial symmetry of the star models, devoid of rotation and magnetic fields.
tion of the VR theorem, they resorted to an order-of-magnitude discussion of the stellar structure problem (PSS, chapter 18.2) and thereby collapsed the differential equations to a set of algebraic relations. They showed that if pressure, density, temperature, and radiated luminosity were prescribed, all emerging relations can be expressed as functions of mass, radius, and chemical composition. Upon prescribing additionally also thermal equilibrium, the radius dependence of the set of algebraic equations can be eliminated so that eventually the order-of-magnitude approximations of the physical quantities of a star in hydrostatic and thermal equilibrium are found to depend on mass and chemical composition only. Although intuitively attractive, the method applied only to a model star in a coarse integral sense and failed to be mathematically rigorous (cf. Appendix B).

Astrophysics is not mathematics; in the latter, the VR theorem would have met its fate once one counterexample popped up – independent of how academic it were. The astrophysical community, on the other hand, put up with the dilemma of the VR theorem and its counterexamples. After all, in astronomically relevant cases it seemed to remain predictive and explanations sounded plausible. Nonetheless, the beginning era of ever faster and easier accessible electronic computers at the end of the 1960s allowed to compute physically complex models in large numbers and the rapidly growing repository of stellar models had intriguing challenges in store.

The forthcoming second part of this marginalia on the history of the VR theorem will focus on the developments in the 1970s when a few astrophysicists set out to look more closely into matter of the VR theorem after ever more complicated star models could be computed and some of them exposed violations of the VR theorem. The field benefited from fresh insights imported by people who applied to stellar astrophysics heavier mathematical machinery than usual.

Appendix A: Eddington’s mass – luminosity relation

A correlation of stars’ masses and luminosities was hinted at as early as 1911; it was first mentioned in a clause of a paper on the motion of the stars in the Galaxy (Halm 1911). Analyzing an appropriately chosen sample of 14 binary stars, Hertzsprung (1923) was able to report quantitatively on a relation of mass and brightness of his sample stars, finding clear evidence that more massive ones were consistently brighter than the less massive brethren. It was Eddington, however, who pushed the issue of the mass – luminosity relation (\(M - L\) relation in the following) further because he relied on it as an observational foundation on which he could rest his mathematical modeling of the internal structure of the stars (Eddington 1924).
Figure 3 displays the same data as were shown in Fig. 1 of Eddington (1924). The different markers in the plot identify different classes of calibrated stars; filled circles show the pb stars, the primary binary stars, triangles are the s_b, the secondary binaries, squares show the e_b, the eclipsing binaries, and finally, the asterisks stand for cep, the pulsating variables. Eddington referred to them collectively as Cepheïds; this despite the fact that the star RR Lyr was also in the sample. The physical differences between Cepheïds and RR Lyrae variables were not known at the time.

From the present point of view, Eddington’s adding pulsating stars, the cep group, to the graph to make the case of an $M - L$ relation is a dubious undertaking. It was essentially pure luck that made the outcome to look so seemingly convincing. At the time of Eddington’s article, no direct determinations of masses of Cepheïds were available. For the stars to find their place in the figure, Eddington resorted to his theoretical $M - L$ relation and applied it ad hoc to the Cepheïds too. The computed stellar mass was then iterated until the pulsation period of the modeled Cepheid eventually converged to the observed period. Therefore, the pulsating stars in Fig. 3 are no fundamental indicators of the existence an $M - L$ law but the result of an already plugged-in $M - L$ relation coupled with the pulsation theory of Eddington (1919). In contrast, the derivation of the masses of the members of the binary-star sample relies on Newton’s laws acting in a $1/r$-gravitational potential only. Therefore, only the binary stars serve as legitimate indicators of the correlation between mass and luminosity.

An inspection of the right panel of Fig. 3 makes clear that correlation of $M - L$ relation of the binary stars persists also with modern physical calibrations of the binary stars, although the scattering increases at very low and very high luminosities. Furthermore, the relation based on modern data has a steeper slope plus a few outliers in the graph, the Cepheïds in particular scatter. Concen-
trating on the arguably small number of Cepheids with the equally small mass spread, the modern data do not really call for the same $M - L$ relation as it is suggested by the mostly main-sequence binary stars. Nonetheless, the overall relation is still impressively tight, particularly when accounting for the fact that very different kinds (evolutionary stages) of stars convene in the graph.

Because the stars spend most of their lifetime burning hydrogen most stars observed in the sky are therefore likely in their main-sequence phase. Along the main sequence, the stars’ luminosities grow with increasing mass. One representative form of an empirically calibrated main-sequence $M - L$ relation is e.g. from Smith (1983):

$$\log \frac{L}{L_\odot} = 4.0 \cdot \log \frac{M}{M_\odot} \text{ for } M/M_\odot > 0.43.$$ 

The binary-star data entering Eddington’s $M - L$ relation fit the above relation quite well so that we can assume that the respective stars are indeed likely in their main-sequence phase of their life.

The Cepheids, on the other hand, are as we know today radially pulsating intermediate-mass supergiants. Under favorable circumstances intermediate-mass stars loop across the HR diagram during their central helium-burning stage and some of these looped stars migrate through the classical instability strip to become then observable as Cepheids. From stellar-evolution modeling we learned that close to the instability strip the luminosities of the same branches of these blue loops tend to be ordered in mass; i.e. Cepheids of different mass but a comparable evolutionary stage tend to obey an $M - L$ relation too. The 2nd crossing of the instability strip during the early core helium-burning phase is usually the slowest and therefore the favored one to observe Cepheids in. Adopting hence this second crossing as the relevant one here, a fit to the computed intersections of evolutionary tracks with the instability strip (Chiosi et al. 1993) reads as

$$\log \frac{L}{L_\odot} = 3.57 \cdot \log \frac{M}{M_\odot} + 0.54.$$ 

Interestingly enough, the slopes of the Cepheids’ and the main-sequence stars’ $M - L$ relation happen to be quite similar. From all we know, this is an accident of nature. The vertical displacement, $\Delta \log(L/L_\odot)$, of the two relations say at $M_\ast = 5 M_\odot$ is only 0.23 so that the composite nature of the observed $M - L$ relation in Fig. 3 is hardly discernible, in particular in the presence of unavoidable scattering of observational data.

Apart from some increased scattering in the modern version of Eddington’s $M - L$ relation, introduced by the pulsating variables, one obvious disagreement is apparent in comparison with the original one: The data point of the star RR Lyr lies far off the general trend. This is no surprise: Today we know that the family of RR Lyr variables is made up of low-mass ($\approx 0.6 M_\odot$) population II stars living on the horizontal branch. Rather than following an $M - L$ relation of the above kind, the spread in mass and in luminosity is
small so that the class of RR Lyrae variables would form kind of a clump around the isolated prototype RR Lyr in Fig. 3 (right panel).

Appendix B: The nature of the equations

To properly state what astrophysicists mean if they talk stars on the theory level, the set of the governing equations and assumptions are laid out in the following. In each case of the following collection of formulae, the first line contains, as the starting point, the general fluid-dynamical equations in their Lagrangian form. We follow mostly the notation of Mihalas & Mihalas-Weibel (1984) (only if not self-evident, deviations therefrom are explained). The second line specializes then on spherical symmetry\footnote{Spherical symmetry is appropriate for non-rotating, non-magnetic star. Even though it seems intuitively obvious that a static self-gravitating fluid configuration assumes the form of a sphere, the proof that the sphere is the only equilibrium figure came relatively late, see e.g. Poincaré & Dreyfus (1902) with a proof which relied on Lyapunov’s master thesis of 1884 or consult Carleman (1919) who used a more geometrical ansatz.} and on mass as the independent variable – choices usually adopted in stellar structure/evolution modeling:

\[ D_t \rho = -\rho \left( \nabla \cdot \vec{v} \right), \]  
\[ D_m r = \frac{1}{4\pi r^2} \rho. \]  

\[ \Delta \Phi = 4\pi \rho G, \]  
\[ g = -\frac{Gm}{r^2}, \]  

In spherical symmetry, eq. 3 can be integrated once; defining \( g = -d_r \Phi \) leads then to eq. 4.

\[ \rho D_t \vec{v} = \vec{f} + \nabla \cdot \vec{T}, \]  
\[ D_m P = -\frac{Gm}{4\pi r^4} - \frac{1}{4\pi r^2} D_t^2 r. \]  

The stress tensor \( \vec{T} \) is made up of the components of viscous stress and it harbors on its diagonal the isotropic hydrostatic pressure. Neglecting viscosity turns Cauchy’s equation into Euler’s equation. Physically, the stress tensor is relatively unimportant in stellar matter; for numerical reasons (in ‘hydrodynamical’ computations) an artificial stress tensor might become instrumental to smear out shocks and to stabilize the computational scheme.

The change of the specific heat content of the stellar matter, \( q \), remodeled with the help of the 1st law of thermodynamics, can be written as:

\[ \rho D_t q = \rho \left[ D_t e + P D_t \left( \frac{1}{\rho} \right) \right] = \Psi_{\text{visc}} - \nabla \cdot \vec{F} + \rho s_{\text{nuc}}, \]  
\[ D_t q = -D_m L + \epsilon. \]  

As in the momentum equation, the dissipation term – here the viscous energy dissipation function \( \Psi_{\text{visc}} \) – is physically not important in stellar interiors except if sharp fronts develop and then in connection with the necessity to stabilize the numerical treatment. In
canonical quasi-static stellar-evolution computations, $\Psi_{\text{visc}}$ can be neglected.

The energy flux is denoted by $\vec{F}$. Astronomers prefer to work with the local radially streaming luminosity, which is defined as $L = 4\pi r^2 F_r$, with $F_r$ being the radial component of the local energy flux. Furthermore, the energy source, $s$, is attributed to nuclear burning in the star, therefore it is subscripted with ‘nuc’. In the stellar astrophysical form of the equation, as shown on the second line, the nuclear energy input rate, measured as energy per unit time and unit mass, is referred to as $\epsilon$. It is important to keep in mind that energy gain by nuclear burning contributes positively and possible energy loss by neutrino production is to be subtracted because neutrinos do not contribute to the heat content of the stellar matter.

To get a handle on the physics of the energy flux, stellar astrophysicists adopt Fourier’s law to model the flow of photons as a diffusion process driven by the spatial temperature gradient. Even energy transport by material motion can be appropriately accommodated.

$$\vec{F} = -K \cdot \vec{\nabla} T, \quad (9)$$
$$D_m T = -\frac{Gm}{4\pi r^4} \cdot \nabla_0. \quad (10)$$

The quantity $K$ denotes the coefficient of thermal conductivity; in radiative regions it can be written as $K = \frac{acT^3}{(3\kappa \rho)}$ with $\kappa(\rho, T, \vec{\chi})$ being the Rosseland opacity. The quantity $\nabla_0$ measures the temperature stratification:

$$\nabla_0 = \frac{d \ln T}{d \ln P} = \begin{cases} 
\nabla_{\text{rad}} & \text{radiative region}, \\
\nabla_{\text{c}} & \text{convective region}.
\end{cases}$$

with $\nabla_{\text{rad}} = \frac{3kLP}{(16\pi acGmT^4)}$ being a purely local function of stellar quantities. In the easiest case of a stellar-convection description, such as in elementary mixing-length models for example, also $\nabla_{\text{c}}$ is a function of local variables alone. More elaborate treatments of convection can, however, introduce non-local contributions.

Nuclear burning is the source of a star’s evolution; the resulting spatio-temporal change of nuclear species $X_i$ in a star can formally be written as

$$D_t X_i = Q_i - S_i + \vec{\nabla} \left( \sigma_D \vec{\nabla} X_i \right). \quad (11)$$

Apart from the source-, $Q_i$, and the sink-term, $S_i$, both determined by the type and complexity of the nuclear burning network, nuclear species can be smeared out spatially by a multitude of physical processes (such as convection, thermohaline mixing, semi-convection, settling, levitation); these transport processes are hidden away in a diffusion-type term in the equation with the particular physical process manifesting itself in the specification of the diffusion coefficient $\sigma_D$. In the majority of the numerical realizations, the

Extreme conditions such as encountered e.g. during a stellar core collapse can trap even neutrinos and hence require then a careful treatment of the energy budget including the neutrinos.

Fourier’s law for the flux (parabolic equation)

$X_i$ be the mass fraction of nuclear species $i$; $\sum X_i = 1$.

$i \in \{1, \ldots, N_{\text{spec}}\}$, with $N_{\text{spec}}$ the number of species accounted for in the stellar matter.
computation of the nuclear evolution is decoupled from the stellar structure problem.\textsuperscript{12} We presume that at each epoch $t$, the vector $\vec{\chi} = (X_1(m,t), \ldots, X_{N_{\text{spec}}}(m,t))$ is known via some suitable computational procedure.

Boundary conditions for the stellar structure equations are distributed ones with the natural choices in the center: $r = 0$ and $L = 0$ at $m = 0$. The surface is, by its very stellar nature, ill-defined and requires suitably chosen physical approximations: Traditionally popular is the assumption of thermal equilibrium of radiation and matter fields at the photosphere leading to: $L = 4\pi r^2 \sigma T^4$ at $m = M_\ast$. The radius at the photosphere is then set equal to star’s radius $r = R_\ast$, and the temperature at the photosphere corresponds to the so called effective temperature $T_{\text{eff}}$. The second outer boundary condition, a mechanical one, determines e.g. the pressure at the photosphere:

$$P = f(\rho, T, \kappa(\rho, T, \vec{\chi})),$$

with some suitable function $f$, which approximates the type of atmosphere which exerts its pressure, $P$, on the photosphere. For simplicity’s sake, and likely sufficient for pure mathematical considerations, it is sufficient to assume some ad hoc constant pressure at the photosphere:

$$P = P_{\text{phot}} = \text{const.} \ll P_{\text{center}}.$$

Finally, initial data that specify the state of the star’s structure are required to get a model sequence started in time. Typically, such a time evolution is initialized with some simplified star model in hydrostatic and if possible also in thermal equilibrium. Both assumptions ensure that pressure and temperature are continuously differentiable in space. Density, on the other hand, can have discontinuities, depending on the spatial structure of the composition vector; luminosity will also react accordingly. Think, for example, of an initial model with a pure helium core and a pure hydrogen envelope: Across the H/He interface pressure and temperature are continuous whereas density and, at sufficiently high temperature, also luminosity develop finite jumps.

Where astronomers were apparently too light-hearted in ‘proofing’ the VR theorem, mathematicians, on the other hand, in particular those rising a warning finger, such as Kurth \textsuperscript{(1953)}, were essentially absent. The asterophobia of the mathematicians is likely caused by the fact that the equations that model the structure of the stars cannot be pigeon-holed: Depending on the specific simplifications introduced to the system of structure and evolution equations, they can change the mathematical character so that different mathematical tools must be applied to the formal study the problem; on the other hand also different numerical methods must be implemented to tackle the computational problem. The simplest approach to model stars, which likely has canalized early thinking of proofing the VR theorem, namely the one used to compute...
polytropes is shortly touched upon in the following.

Separating mechanical and thermal parts of the stellar structure problem was the first ansatz to come to grips with understanding the interior conditions of stars. To establish the necessary barotropic conditions, frequently a polytropic\(^{13}\) relation was postulated. In the static case, i.e. in absence of a velocity field, eqs. 2, 4, and 6 morph into the venerable Lane-Ritter-Emden equation. In the formative years of theoretical astrophysics, this equation was solved as an initial-value problem (IVP): The computation started in the regularly singular center of the model with prescribed values of the dependent variable and its derivative. The integration was followed out to the first root of the dependent variable; its location was then adopted as a measure of the radius of the model star. Looking at the problem as an IVP ensured existence and uniqueness of the solution by the sufficiently smooth character of the right-hand side of the ODE via Picard-Lindelöf’s theorem.

More generally, in particular with a non-vanishing velocity term, the original mechanical fluid-dynamical equations (eqs. 1, 3, and 5, closed with a polytropic relation between \(\rho\) and \(P\)) constitute the so-called Euler-Poisson problem. Assuming a compact support for the density and hence for the whole problem, i.e. \(\rho > 0\) obtains for a finite volume only and defining the outer boundary of the gravitationally-bound fluid sphere by \(\rho(R) = 0\) bring about considerable mathematical complications; a substantial body of literature exists on uniqueness and evolution of such boundary-value problems (BVPs). For some recent advances, consult e.g. Deng & Guo (2003) who proofed uniqueness theorems for the static case, i.e. the BVP which must be solved for the structure of polytropic spheres.

Even though the proof of the VR theorem in Russell (1931) (and all later repetitions thereof) refers to the stellar-structure problem as a BVP, the presentation of how the equations are solved is reminiscent of the direct integration of the Lane-Ritter-Emden equation. In other words, from reading Russell (1931), one comes away with the impression that existence and uniqueness properties were implicitly influenced by the IVP experience gained with polytropes. However, existence and uniqueness statements for higher-order BVPs are mathematically formidable; we superficially referred to the easiest case, that of the Euler-Poisson system, just before.

Properties of the equations of more realistic approximations to the stars’ structure and their evolution will be a topic in the second part of this essay on the history of the Vogt-Russell theorem.

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\(^{13}\) The relation \(P \propto \rho^{1+1/n}\), with \(n\) being the polytropic index, constitutes a stratification relation rather than a state relation within a prescribed fluid element.
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