SU(N) Lattice Gauge Theory on a Single Cube

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(Dated: November 30, 2021)

Abstract

In this paper we study the viability of pursuing analytic variational techniques for the calculation of glueball masses in 3+1 dimensional Hamiltonian lattice gauge theory (LGT) in the pure gauge sector. We discuss the major problems presented by a move from 2+1 to 3+1 dimensions and develop analytic techniques to approximate the integrals appearing in 3+1 dimensional variational glueball mass calculations. We calculate 0++ and 1+− glueball masses on a lattice consisting of a single cube. Despite the use of a very simplistic model, promising signs of an approach to asymptotic scaling is displayed by the SU(N) 1+− glueball mass as N is increased.

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I. OUTLINE

In this paper we explore the viability of extending the analytic techniques used with success in Refs. 1 and 2 to the calculation of glueball masses in the pure gauge sector in 3+1 dimensions. The primary difficulty lies in the calculation of expectation values in 3+1 dimensions. In Section II, we briefly review what has been achieved in Hamiltonian LGT 3 in 3+1 dimensions. We discuss the difficulties faced in 3+1 dimensions and possible solutions in Section III. In Section IV, we consider the problem of Gauss’ law constraints. This is a topic that has been discussed in the context of Hamiltonian LGT most recently by Ligterink, Walet and Bishop 4 and concerns the constraint equations that appear when non-abelian gauge theories are canonically quantised. In Section V, we move on to the calculation of variational glueball masses on a single cube using the analytic variational technique discussed in Ref. 1. We finish in Section VI with a discussion of the viability of pursuing analytic techniques for pure SU(N) LGT in 3+1 dimensions based on the results of Section V.

II. INTRODUCTION

From a renormalisation point of view, the key difference between 2+1 and 3+1 dimensional gauge theory lies in the units of the coupling constant. 2+1 dimensional gauge theory has a coupling constant, $e^2$, with the dimensions of mass and so the coupling constant explicitly sets a mass scale for calculations on the lattice. In contrast, the 3+1 dimensional coupling constant is dimensionless. This makes the extraction of continuum physics from lattice calculations more subtle in 3+1 dimensions than in 2+1.

On a practical level, there is a more serious problem faced in moving from 2+1 to 3+1 dimensions for Hamiltonian LGT calculations. The analytic techniques that were used with success in Refs. 1 and 2 are no longer applicable. These techniques rely heavily upon the fact that in 2+1 dimensions a change of variables from links to plaquettes has unit Jacobian. The form for the equivalent Jacobian in 3+1 dimensions is considerably more complicated. The most comprehensive study of the change of variables from links to plaquettes is due to Batrouni 5, 6. We discuss this change of variables in more detail in Section III.

The extension of the techniques used in Ref. 1 to 3+1 dimensions is not straightforward. There are a number of immediate problems. Firstly, since plaquettes are not independent
variables in 3+1 dimensions one can not automatically work in the infinite volume limit. In a precise study one would need to calculate identical quantities on different sized lattices and extrapolate to the infinite volume limit. Secondly, in the context of analytic calculations, even on small lattices the integrals involved in the calculation of basic matrix elements are considerably more complicated than those encountered in 2+1 dimensions. Such matrix elements could in principle be carried out analytically on small lattices but since the number of integration variables increases quickly with the volume of the lattice a calculation on even a $5^3$ lattice would seem exceedingly difficult. How quickly the infinite volume limit is reached will therefore determine the worth of pursuing analytic Hamiltonian methods in 3+1 dimensions. Finally, there is the complication of Gauss’ law which we discuss in Section IV.

The only Hamiltonian techniques to have been applied with any success to the case of SU(3) gauge theory in 3+1 dimensions have been strong coupling expansions, the $t$-expansion and exponential wave function methods. Each of which we now summarise.

Strong coupling perturbative techniques were used in the early days of LGT in an attempt to bridge the gap between the strong and weak coupling limits. Strong coupling expansions of the Callan-Symanzik $\beta$ function were calculated to $O(g^{-24})$ \cite{7, 8} and showed signs of interpolating smoothly between the strong and weak coupling limits. This suggested that the $\beta$ function was a smooth function of the coupling with its only zero at $g = 0$, providing a strong argument at the time for the continuum limit of LGT confining quarks. Corresponding strong coupling expressions for glueball masses did not share the same success. Despite strong coupling calculations to $O(g^{-28})$ \cite{9} scaling was not observed in $0^{++}$, $1^{+-}$ or $2^{++}$ glueball masses. It was later realised that a roughening transition prevents a smooth crossover from strong to weak coupling physics (see Ref. \cite{10} and references within).

The $t$-expansion was introduced by Horn and Weinstein \cite{11} as an analytic method suitable for the study of LGT in the Hamiltonian formulation. It has been applied in the calculation of glueball masses in 3+1 dimensions for SU(2) \cite{12} and SU(3) \cite{13, 14, 15} LGT in the pure gauge sector. More recently it has been used in an attempt to calculate the lowest hadron masses \cite{16}. For each case however asymptotic scaling of masses was not directly observed. Extrapolation techniques such as Padé approximants were required to probe the weak coupling region. The extrapolated mass ratio results agreed with Monte Carlo estimates of the time.

The coupled cluster method and related exponential wave function techniques have re-
ceived by far the most attention in Hamiltonian LGT in recent years. Essentially these techniques aim to solve the Kogut-Susskind eigenvalue equation by making a suitable ansatz for the wave function. The coupled cluster method was originally constructed with applications in nuclear physics in mind [17, 18] but has since found the majority of its applications in molecular physics [19]. Its application in the context of Hamiltonian LGT is described in Refs. 20 and 21. The truncated eigenvalue method, developed by Guo, Chen and Li [22], is another exponential wave function technique to have found application in Hamiltonian LGT.

A number of groups have made considerable progress in the application of exponential wave function techniques to Hamiltonian LGT in 3+1 dimensions. While most studies have explored gauge groups other than SU(3) in less than three dimensions, studies of SU(3) glueballs in 3+1 dimensions have commenced. Results tangent to the expected scaling form, indicating an approach to scaling, have been obtained for SU(3) pure gauge theory in a coupled cluster calculation by Leonard [23]. However convergence with increasing orders appears to be slow. Finite order truncation errors appear to be under more control in the truncated eigenvalue method. The first calculations of 3+1 dimensional SU(3) glueball masses with this method [24] gave a 1−− to 0++ mass ratio that was consistent with the Monte Carlo results of the time. The agreement was not as good for the 0−− to 0++ mass ratio. A convincing demonstration of asymptotic scaling has not yet been produced in a calculation of glueball masses for SU(3) in 3+1 dimensions.

More promising 3+1 dimensional results have been obtained for higher dimensional gauge groups. Ironically these results have been obtained with much simpler methods than either of the exponential wave function methods described above. Asymptotic scaling of the lowest 0++ glueball mass has been demonstrated by Chin, Long and Robson in a variational calculation on a small volume (6^3 sites) lattice for SU(5) and SU(6) [25]. This calculation follows the variational technique described in Ref. 1 but uses Monte Carlo rather than analytic techniques to calculate the required expectation values. It uses only plaquette states in the minimisation basis rather than the large basis of rectangular states used in Refs. 1 and 2. Naively one would hope that the same method could be explored on similar sized lattices for larger \(N\), with additional states in the minimisation basis, using the analytic techniques of Ref. 1. We take this as our motivation for the studies presented in this paper. It will become clear however that attempting a similar calculation to Chin, Long and Robson on a
III. THE MOVE TO 3+1 DIMENSIONS

In LGT one usually works in 2+1 dimensions to test a technique with the intention of later extending it to the physically relevant 3+1 dimensions. This may be justified in the Lagrangian approach where the time coordinate is treated on the same footing as the spatial coordinates and adding another dimension equates to nothing more than an increased load on computer memory. However, in Hamiltonian LGT two serious technical differences exists between 2+1 and 3+1 dimensions. The first is Gauss’ law and the second is related to constructing a Jacobian for transforming from link to plaquette variables.

To understand the first difference one needs to recall that Hamiltonian LGT is formulated in the temporal gauge which sets $A_0 = 0$. If one starts with the Yang-Mills Lagrangian and performs the standard equal-time quantisation, one runs into problems because the time derivative of $A_0$ does not appear in the Lagrangian. The variational principle gives equations of motion for the space-like components resulting in the standard Yang-Mills Hamiltonian. For the time-like component we obtain a set of algebraic constraint equations which are the analogue of Gauss’ law. There is one constraint equation for each colour component of $A_0$, $N^2 - 1$ for $SU(N)$, at each lattice site. As has been pointed out in the context of Hamiltonian LGT by Ligterink, Walet and Bishop [4], it is only when one works with a set of variables whose number of degrees of freedom matches the number of unconstrained degrees of freedom in the theory, that one can avoid the technicalities of constraint equations. For this reason, in 2+1 dimensions we really are quite lucky. The number of plaquette variables on a square two dimensional lattice is precisely equal to the number of unconstrained variables [4]. For a three dimensional cubic lattice this is not the case. One could envisage constructing a polyhedral lattice in $d$ dimensions such that the number of faces (plaquettes) would equal the number of unconstrained variables. However such lattices appear to be prohibited by Euler’s equation which relates the number of vertices, edges and faces of polyhedra.

The other serious technical difference between 2+1 and 3+1 dimensions, that of constructing a Jacobian for transforming from link to plaquette variables, is relevant to both the Hamiltonian and Lagrangian formulations of LGT. It only becomes important when the method of choice relies on plaquette variables for its implementation. The transformation
from link variables to plaquette variables is required in the approach taken in Refs. 1 and 2 to make use of the analytic results available for certain group integrals. Such a transformation does not need to be made if one is happy to use Monte Carlo techniques to handle the integrals as is the case in Ref. 25.

The most complete treatment of the transformation from link to plaquette variables is due to Batrouni [5, 6] who worked in the Lagrangian formulation. His approach was based on the continuum work of Halpern [26] who constructed a field-strength formulation of gauge theory. In Halpern’s construction the definition of the field strength $F_{\mu\nu}$ is inverted to give an expression for $A_\mu$ as a function of the field strength. To do this requires the choice of a suitable gauge. The Jacobian of the transformation is precisely the Bianchi identity; a constraint equation on $F_{\mu\nu}$. For the Abelian case the Bianchi identity is equivalent to the requirement that the total magnetic flux leaving a volume is zero. Batrouni developed an equivalent field strength formulation on the lattice. On the lattice, the link elements, $U_l$, correspond to the vector potentials, $A_\mu$, and the plaquette variables correspond to the field strengths, $F_{\mu\nu}$. Batrouni demonstrated that the Jacobian of the transformation from link variables to plaquette variables was the lattice analogue of the Bianchi identity. For Abelian gauge theories the lattice Bianchi identity can be separated into factors which depend only on the plaquette variables of elementary cubes, with one factor for each cube of the lattice. For non-abelian theories the Bianchi identity has only been found to separate in this way for special types of lattices, the largest volume example being an infinite tower of cubes. For the infinite lattice the Bianchi identity is a complicated nonlinear inseparable function of distant plaquette variables. Interestingly it is the only source of correlations between plaquette variables in LGT. Mean plaquette methods have been developed to deal with the added complications of the non-abelian Bianchi identity but have not progressed far [27].

To summarise, Hamiltonian LGT in 3+1 dimensions faces some serious problems. Care must be taken in choosing appropriate variables to work with if constraints on the lattice electric fields are to be avoided. Additionally, if one wishes to use analytic techniques to calculate matrix elements the complications of the Bianchi identity restricts the calculations to small lattices.
IV. CONSTRAINT EQUATIONS

In this section we focus on one of the problems faced in moving from 2+1 to 3+1 dimensions, that of Gauss’ law. From the discussion of Section III, it would seem that one needs either to construct an appropriate set of variables with precisely the correct number of unconstrained degrees of freedom or be faced with the problem of building constraint equations on the lattice electric fields. As has been pointed out by Ligterink, Walet and Bishop, there is an alternative. The problem of satisfying Gauss’ law can be solved by working with wave functions that are annihilated by the generator of Gauss’ law. We discuss this matter in what follows.

The generator of Gauss’ law on the lattice can be written as

\[ G^a(x) = \sum_i \left[ E^a_i(x) + E^a_{-i}(x) \right]. \]  

(1)

Here \( E^a_i(x) \) is the lattice chromoelectric field on the directed link running from \( x \) to \( x + \hat{i} a \).

In this notation the lattice electric fields satisfy the commutation relations

\[ [E^a_i(x), U_j(y)] = T^a U_j(y) \delta_{ij} \delta_{xy} \]  

(2)

\[ [E^a_{-i}(x + \hat{i} a), U_j(y)] = -U_j(y) T^a \delta_{ij} \delta_{xy}. \]  

(3)

Here \( \hat{i} \) is a unit vector in the \( i \) direction and \( \{ T^a : 1 \leq a \leq N^2 - 1 \} \) is a basis for SU(\( N \)). It is common to use the Gell-Mann basis, in which case \( T^a = \lambda^a/2 \), where \( \{ \lambda^a : 1 \leq a \leq N^2 - 1 \} \) is the set of traceless \( N \times N \) Gell-Mann matrices. It should be pointed out that in this notation we have \( U^\dagger_i(x) = U_{-i}(x + \hat{i} a) \). For physical states we must therefore have \( G^a(x) |\psi\rangle = 0 \) for each lattice site, \( x \), and all \( a = 1, \ldots, N^2 - 1 \). It should be checked that this is the case for the one plaquette exponential trial state,

\[ |\phi_0\rangle = \exp \left( c \sum_{x, i < j} \text{Re} \right) |0\rangle. \]  

(4)

Here, \( |0\rangle \) is the strong coupling vacuum defined by \( E^a_i(x) |0\rangle = 0 \) for all \( i, x \) and \( \alpha = 1, 2, \ldots, N^2 - 1 \). The directed square denotes the traced ordered product of link operators, \( U_i(x) \), around an elementary square, or plaquette, of the lattice,

\[ \square := \text{Tr} \left[ U_i(x) U_j(x + \hat{i} a) U_i^\dagger(x + \hat{j} a) U_j^\dagger(x) \right] \]  

(5)
where \(a\) is the lattice spacing.

Consider first a state consisting of a single plaquette acting on the strong coupling vacuum,

\[
|p_{ij}(x)\rangle = \text{Tr} \left[ U_i(x) U_j(x + \hat{i}a) U_{-i}(x + \hat{j}a) U_{-j}(x) \right] |0\rangle.
\]  

(6)

Since \(|0\rangle\) is annihilated by the electric field, using Eqs. (2) and (3) we immediately have

\[
G^a(y)|p_{ij}(x)\rangle = 0,
\]  

(7)

for all lattice sites, \(y\), not lying on the corners of the plaquette, \(p_{ij}(x)\). Consider now sites lying on the corners of the plaquette in question. In particular consider \(y = x\). Making use of the commutation relations of Eqs. (2) and (3) we have

\[
G^a(x)|p_{ij}(x)\rangle = \text{Tr} \left[ T^a U_i(x) U_j(x + \hat{i}a) U_{-i}(x + \hat{j}a) U_{-j}(x) \right] |0\rangle \\
- \text{Tr} \left[ U_i(x) U_j(x + \hat{i}a) U_{-i}(x + \hat{j}a) U_{-j}(x) T^a \right] |0\rangle \\
= 0.
\]  

(8)

The same applies for other sites on the plaquette. This argument is not specific to plaquettes. Gauss’ law is found to be satisfied locally by any closed Wilson loop on the lattice, traced over colour indices, acting on the strong coupling vacuum. It is easy to extend this result to products of such loops acting on \(|0\rangle\). To see this we consider how Gauss’ law applies to the product of two plaquettes

\[
|p_{ij}(x)p_{ij}(x + a\hat{i})\rangle = p_{ij}(x)p_{ij}(x + a\hat{i})|0\rangle.
\]  

(9)

Only at the sites \(x + a\hat{i}\) and \(x + a(\hat{i} + \hat{j})\) does this case differ from the single plaquette example. Let us consider Gauss’ law at \(x_+ = x + a\hat{i}\). Once again, using the fact that the strong coupling vacuum is annihilated by the lattice chromoelectric field, and Eqs. (2) and (3) we have

\[
G^a(x_+)|p_{ij}(x)p_{ij}(x_+)\rangle = \sum_k \left[ E_k^a(x_+) + E_{-k}^a(x_+), p_{ij}(x)p_{ij}(x_+) \right] |0\rangle \\
= \sum_k \left\{ \left[ E_k^a(x_+) + E_{-k}^a(x_+), p_{ij}(x) \right] p_{ij}(x_+)|0\rangle \\
+ p_{ij}(x) \left[ E_k^a(x_+) + E_{-k}^a(x_+), p_{ij}(x_+) \right] |0\rangle \right\} \\
= 0.
\]  

(10)
This result is easily extended to arbitrary products of closed Wilson loops, traced over colour indices, acting on $|0\rangle$. Consequently, any function of such loops acting on $|0\rangle$ also satisfies Gauss’ law provided it admits a Taylor series expansion. The one plaquette trial state of Eq. (4) thus obeys Gauss’ law and is therefore suitable for use in simulating physical states.

V. THE ONE CUBE UNIVERSE

In this section we adapt the analytic techniques used with success in Refs. 1 and 2 for use in 3+1 dimensions. As a starting point we consider the case of a single cube. This will serve as a test case to assess the feasibility of a larger volume study. Our aim is to use a small basis of states to calculate 3+1 dimensional SU($N$) glueball masses variationally, and check if an approach to the correct scaling form is observed. We do not expect to achieve scaling with such a small lattice. However, an approach to scaling would warrant an extended study on larger lattices. We start with a general description of the approach. The starting point is the choice of trial state. As in 2+1 dimensions [1, 2] we choose to work with the one plaquette trial state of Eq. (4).

Many Hamiltonians are available for LGT calculations [28]. In Ref. 1 we used Kogut-Susskind [3], improved and tadpole improved Hamiltonians to calculate glueball masses for $N \leq 5$. Here we employ the simplest of these, the Kogut-Susskind Hamiltonian, which is defined for pure SU($N$) gauge theory with coupling, $g^2$, on a lattice with spacing, $a$, by,

\[ H = \frac{g^2}{2a} \sum_{x,i} \mathcal{E}^a_i(x)^2 + \frac{2N}{ag^2} \sum_{x,i<j} P_{ij}(x), \]  

(11)

where the plaquette operator is given by,

\[ P_{ij}(x) = 1 - \frac{1}{N} \text{Re} \]

(12)

The variational parameter, $c$, is fixed by minimising the vacuum energy density,

\[ \epsilon_0 = \frac{a}{N_p} \langle \mathcal{H} \rangle = \left( \frac{N^2-1}{2\beta} \right) c - \frac{2\beta}{N} \langle \right) + 2\beta, \]

(13)

as was done for the case of 2+1 dimensions in Ref. 1. Here $\beta = N/g^2$ and $N_p$ is the number of plaquettes on the lattice. In 3+1 dimensions, however, the expression for the plaquette
expectation value in terms of the variational parameter is significantly more complicated than for the case of 2+1 dimensions. Having fixed the variational parameter we then construct a small basis of states, with each state fitting on a single cube, and minimise the glueball mass over this basis. The process of minimising the massgap follows precisely Ref. 1. The key difficulty of working in 3+1 dimensions is the calculation of the required integrals. In the next section we explain how an analytic approximation to these integrals can be obtained.

A. SU($N$) integrals in 3+1 dimensions

The analytic techniques used in Ref. 1 rely on the fact that the transformation from link to plaquette variables has unit Jacobian in 2+1 dimensions. Batrouni has calculated the Jacobian for arbitrary numbers of dimensions [5, 6]. For a lattice consisting of a single cube, the result of Ref. 5 is

$$J = \frac{1}{d_r} \prod_{i=1}^{6} \chi_r(P_i),$$

where $P_1, \ldots, P_6$ are the six plaquette variables on the single cube and the sum is over all characters, $\chi_r$, of SU($N$). $d_r$ denotes the dimension of the character, $\chi_r$. The second line is simply a character expansion of the first line. In the variational study of glueball masses, we need to calculate the integrals of overlapping trace variables on a single cube. It is always possible to reduce these integrals to integrals involving non-overlapping trace variables using the orthogonality properties of the characters. For example, consider the expectation value, on a single cube, of a twice covered bent rectangle, with respect to the one-plaquette trial state of Eq. (4),

$$\langle \phi_0 | \phi_0 \rangle = \frac{1}{d_r} \int \prod_{i=1}^{4} dP_i \chi_r(P_i) e^{i\text{Tr}(P_i + P_i')}$$

$$\times \int dP_5 dP_6 \chi_r(P_5) \chi_r(P_6) [\chi_2(P_5P_6) + \chi_{11}(P_5P_6)] e^{i\text{Tr}(P_5P_6 + P_5'P_6')}.$$

To proceed with this integral we need to perform character expansions. The orthogonality of the characters can then be used to write characters over two plaquettes in terms of one plaquette characters. To demonstrate how this is done we consider the character expansion
of one of the integrals in Eq. (15),
\[ \int dP \chi_{r}(P) \chi_{2}(PP') e^{cTr(P+P')} = \sum_{r'} c_{r} \int dP \chi_{r'}(P) \chi_{2}(PP'). \]  
(16)

Here the \( c_{r} \) are given by
\[ c_{r} = \int dP \chi_{r}(P) \chi_{r}(P) e^{cTr(P+P')} \]  
(17)

Making use of the orthogonality property of characters given by,
\[ \int dU \chi_{r'}(U)V\chi_{r}(U) = \frac{1}{d_{r}} \delta_{r'r} \chi_{r}(V), \]  
(18)

we obtain
\[ \int dP \chi_{r}(P) \chi_{2}(PP') e^{cTr(P+P')} = \frac{c_{2}}{d_{2}} \chi_{2}(P) \chi_{2}(PP'). \]  
(19)

Making use of this, and proceeding similarly for the analogous integral involving \( \chi_{11}(P_{5}P_{6}) \), we can reduce Eq. (15) to
\[ \langle \phi_{0} | \mathcal{P} | \phi_{0} \rangle = \sum_{r} \frac{1}{d_{r}^{4}} \left[ \int dP \chi_{r}(P) e^{cTr(P+P')} \right]^{4} \times \left\{ \frac{1}{d_{2}} \left[ \int dQ \chi_{2}(Q) \chi_{r}(Q) e^{cTr(Q+Q')} \right]^{2} + \frac{1}{d_{11}} \left[ \int dQ \chi_{11}(Q) \chi_{r}(Q) e^{cTr(Q+Q')} \right]^{2} \right\}. \]  
(20)

The plaquette matrix element provides a more straightforward example;
\[ \langle \phi_{0} | \mathcal{P} | \phi_{0} \rangle = \sum_{r} \frac{1}{d_{r}^{4}} \left[ \int dP \chi_{r}(P) e^{cTr(P+P')} \right]^{4} \int dQ Tr \chi_{r}(Q) e^{cTr(Q+Q')} \]  
(21)

To proceed further we need expressions for the integrals over characters that appear in each matrix element of interest. We define these character integrals generically by
\[ \mathcal{C}_{r_{1}r_{2}...r_{n}}(c) = \int dP \chi_{r_{1}}(P) \chi_{r_{2}}(P) \cdots \chi_{r_{n}}(P) e^{cTr(P+P')} \]  
(22)

All SU(\( N \)) integrals encountered in the calculation of glueball masses in 3+1 dimensions can be expressed in terms of character integrals. It is possible to calculate them generally, however three points need to be considered. Firstly, if we are to use the machinery of Ref. 1, we need to know how to express a given character in terms of trace variables. Secondly, as the dimension of the gauge group increases the number of independent trace variables
increases rapidly. In order to contain the number of integrals required for a given order of approximation, it is necessary to express all high order trace variables in terms of a basis of lower order ones. Finally, the collection of SU($N$) integrals presented in Ref. 1 must be extended. We now address each of these points in turn.

The problem of expressing the characters of SU($N$) in terms of trace variables was solved by Bars [29] who showed that for SU($N$),

$$\chi_n(U) = \sum_{k_1,\ldots,k_n} \delta \left( \sum_{i=1}^{n} i k_i - n \right) \prod_{j=1}^{n} \frac{1}{k_j!} (\text{Tr} U^j)^{k_j}. \quad (23)$$

General characters can then be expressed in terms of $\chi_n(U)$ using standard techniques from group theory [30],

$$\chi_{r_1 r_2 \cdots r_{N-1}}(U) = \det \left[ \chi_{r_j+i-j}(U) \right]_{1 \leq i,j \leq N-1}. \quad (24)$$

Here the quantities inside the determinant are to be interpreted as the $(i,j)$-th entry of an $N \times N$ matrix. For a given character, Eq. (23) produces a multinomial of traces of different powers of $U$. However, not all of these trace variables are independent. The so-called Mandelstam constraints define the relationship between dependent trace variables. For SU(2) and SU(3) we have

$$\text{Tr} U = \text{Tr} U^\dagger \quad \forall U \in \text{SU}(2) \quad \text{and}$$

$$\text{Tr}(U^2) = (\text{Tr} U)^2 - 2\text{Tr}(U U^\dagger) \quad \forall U \in \text{SU}(3). \quad (25)$$

Similar relations can be obtained to express all trace variables in terms of $\text{Tr} U$ for SU(2), and for SU(3) in terms of $\text{Tr} U$ and $\text{Tr} U^\dagger$. To reduce the calculation of character integrals to a manageable size, we require expressions for high power trace variables in terms of a minimal set of lower order trace variables. To do this we proceed as follows. We start with an alternative expression of $\det U = 1$, satisfied for all SU($N$) matrices $U$,

$$\varepsilon_{i_1 i_2 \cdots i_N} U_{i_1 j_1} U_{i_2 j_2} \ldots U_{i_N j_N} = \varepsilon_{j_1 j_2 \cdots j_N}. \quad (26)$$

Here the colour indices of the group elements have been made explicit and all repeated indices are summed over. $\varepsilon_{i_1 \ldots i_n}$ is the totally antisymmetric Levi-Civita symbol defined to be 1 if $\{i_1,\ldots,i_n\}$ is an even permutation of $\{1,2,\ldots,n\}$, $-1$ if it is an odd permutation and 0 otherwise (i.e. if an index is repeated). Multiplying Levi-Civita symbols produces
multiplying both sides of Eq. (26) by $U^\dagger_{JN_i}$ and contracting over repeated indices gives

$$\varepsilon_{i_1i_2\ldots i_{N-1}J_{N+1}} U_{i_1j_1} U_{i_2j_2} \cdots U_{i_{N-1}j_{N-1}} = \varepsilon_{j_1j_2\ldots j_N} U^\dagger_{JN_j},$$

(27)

In the last line we have renamed dummy indices and multiplied both sides by a Levi-Civita symbol. Making use of Eq. (27) in Eq. (28) produces a product of delta functions on each side of the equation. By introducing appropriate combinations of delta functions and contracting over repeated indices we can construct trace identities as we please. We find that identities useful in our context are produced by the introduction of delta functions and an additional matrix, $A$, which is not necessarily an element of SU($N$), into Eq. (28) as follows:

$$\varepsilon_{k_1\ldots k_{N-1}i_1} A_{j_1k_1} U_{i_1j_1} \prod_{m=2}^{N-1} \delta_{km} U_{im} = \varepsilon_{k_1\ldots k_{N-1}i_1} A_{j_1k_1} U^\dagger_{JN_i} \prod_{m=2}^{N-1} \delta_{km}.$$

(29)

Here we have again renamed dummy indices. Some examples of SU($N$) identities obtained from Eq. (29) valid for all $N \times N$ matrices $A$ are as follows:

$$- \text{Tr} U \text{Tr}(UA) + \text{Tr}(U^2A) = -\text{Tr} A U^\dagger + \text{Tr}(U^\dagger A) \quad \forall U \in SU(3)$$

$$-(\text{Tr} U)^2 \text{Tr}(UA) + \text{Tr}(UA) \text{Tr}(U^2) + 2 \text{Tr} U \text{Tr}(U^2A) - 2 \text{Tr}(U^3A)$$

$$= -2 \text{Tr} A U^\dagger + 2 \text{Tr}(U^\dagger A) \quad \forall U \in SU(4).$$

(30)

We require formulae for $\text{Tr}(U^n)$ in terms of lower order trace variables for SU($N$). Such formulae, known commonly as Mandelstam constraints, are obtained by setting $A = U^{n-N+1}$ in Eq. (29). In Appendix A we list the Mandelstam constraints obtained in this way up to SU($8$). We see that for SU($N$) it is possible to express all characters in terms of $N - 1$ trace variables. In what follows we will choose to express the general SU($N$) characters in terms of the set of trace variables \{Tr$U^\dagger$, Tr$U$, Tr$U^2$, $\ldots$, Tr$U^{N-2}$\}. It may well be possible to reduce the size of this set. This would indeed improve the efficiency of our technique. However, no effort has been made to do this at this stage.

The procedure for calculating the general character integral $C_{r_1\ldots r_n}(\epsilon)$ is then as follows. We first express the characters $\chi_{r_1}, \ldots, \chi_{r_n}$ in terms of trace variables using Eqs. (23) and
We then simplify these expressions using the Mandelstam constraints of Appendix A. The final step is to perform the integrals over trace variables, which is an increasingly non-trivial task as one increases the dimension of the gauge group. For instance, the general character integral for SU($N$) involves integrals over powers of the trace variables $\text{Tr}U^\dagger$, $\text{Tr}U$, $\text{Tr}U^2, \ldots \text{Tr}U^{N-2}$. To proceed we need the following integral,

$$
T_{p_1 \ldots p_k}^{q_1 \ldots q_k}(c) = \int_{\text{SU}(N)} dU (\text{Tr}U^{q_1})^{p_1} (\text{Tr}U^{q_2})^{p_2} \cdots (\text{Tr}U^{q_k})^{p_k} e^{c \text{Tr}(U+U^\dagger)}.
$$

(31)

The cases of interest to us here for SU($N$) are described by,

$$
G_{q_1 \ldots q_k}(c, \gamma_1, \ldots, \gamma_k) = \int_{\text{SU}(N)} dU e^{c \text{Tr}(U+U^\dagger)+\sum_{i=1}^k \gamma_i \text{Tr}(U^{q_i})}.
$$

(32)

Following the procedure of Ref. 1 we obtain,

$$
G_{q_1 \ldots q_k}(c, \gamma_1, \ldots, \gamma_k) = \sum_{l=-\infty}^{\infty} \det [\lambda_{l+i,q_1,\ldots,q_k}(c,\gamma_1,\ldots,\gamma_k)]_{1 \leq i,j \leq N};
$$

(33)

with,

$$
\lambda_{m,q_1,\ldots,q_k}(c,\gamma_1,\ldots,\gamma_k) = \sum_{s_1,s_2,\ldots,s_k=0}^{\infty} \frac{\gamma_1^{s_1} \cdots \gamma_k^{s_k}}{s_1! \cdots s_k!} I_{m+s_1q_1+\cdots+s_kq_k}(2c).
$$

(34)

We then obtain an expression for $T_{q_1 \ldots q_k}^{p_1 \ldots p_k}(c)$ by differentiating $G_{q_1 \ldots q_k}$ appropriately;

$$
T_{q_1 \ldots q_k}^{p_1 \ldots p_k}(c) = \left. \frac{\partial^{p_1+\cdots+p_k}}{\partial \gamma_1^{p_1} \ldots \partial \gamma_k^{p_k}} G_{q_1 \ldots q_k}(c,\gamma_1,\ldots,\gamma_k) \right|_{\gamma_1=\cdots=\gamma_k=0}.
$$

(35)

**B. The variational ground state**

In this section we fix the variational ground state following the usual procedure of minimising the unimproved vacuum energy density given by Eq. (13). This equation is independent of the number of dimensions. The dimensionality of the lattice arises at the stage of calculating the plaquette expectation value.

Making use of Eq. (21), the variational parameter can be fixed as a function of $\beta$ for the one-cube lattice. In practice, the character sum in Eq. (21) and the infinite $l$-sum in Eq. (33) need to be truncated. We truncate the infinite $l$-sum at $\pm l_{\text{max}}$ and instead of summing over all SU($N$) characters, we sum over only those characters, $r = (r_1, r_2, \ldots, r_{N-1})$, with
\[
\begin{align*}
\text{FIG. 1: The one cube SU}(N) \text{ variational parameter as a function of } \beta \text{ for various } N \text{ showing the dependence on the character sum truncation.}
\end{align*}
\]

\[
\begin{align*}
r_{\text{max}} \geq r_1 \geq r_2 \geq \cdots \geq r_{N-1}. \text{ With this truncation scheme, memory constraints restrict calculations of the variational SU}(N) \text{ ground state to } N \leq 7, \text{ when working with } r_{\text{max}} = 2 \text{ and } l_{\text{max}} = 2 \text{ on a desktop computer.}
\end{align*}
\]

The dependence of the variational parameter on \( r_{\text{max}} \) with \( l_{\text{max}} = 2 \) is shown for various gauge groups in Fig. 1. As \( r_{\text{max}} \) increases the variational parameter appears to converge for each \( N \) considered. Moreover the convergence appears to improve as the dimension of the gauge group is increased. With the exception of SU(3), the \( r_{\text{max}} = 1 \) and 2 results are indistinguishable on the range \( 0 \leq \xi \leq 0.7 \), where \( \xi = 1/(Ng^2) \). As this is the range of interest to us later in the paper we restrict further calculations to \( r_{\text{max}} = 1 \).

The SU(\( N \)) variational parameters on the one-cube lattice with \( r_{\text{max}} = 1 \) and \( l_{\text{max}} = 2 \) are shown for various \( N \) in Fig. 2. The results do not differ greatly. The corresponding variational energy densities are shown in Fig. 3.
FIG. 2: The one cube SU($N$) variational parameter in units of $N$ as a function of $1/(Ng^2)$ for various $N$.

FIG. 3: The one cube SU($N$) energy density in units of $N$ as a function of $1/g^2$ for various $N$. 
C. Expressions for the glueball mass

We follow precisely the method described in Ref. 1 for the calculation of 3+1 dimensional glueball masses on a single cube. Here, however, we choose a different basis of states to minimise over. Instead of rectangular loops, we use states which fit in a single cube. We start with a basis of two states, the plaquettes and the bent rectangles,

$$B = \{|1\rangle, |2\rangle\},$$ (36)

with

$$|\hat{i}\rangle = \sum_x [F_i(x) - \langle F_i(x) \rangle] |\phi_0\rangle$$ (37)

and

$$F_1(x) = \begin{array}{c} \square \pm \square \end{array},$$

$$F_2(x) = \begin{array}{c} \square \pm \square \end{array}.$$ (38)

In 3+1 dimensions, the “+” sign corresponds to the $0^{++}$ state and the “−” sign corresponds to the $1^{+-}$ state [9]. In order to calculate the glueball masses, following Arisue [31], we need expressions for the matrix elements $N^C_{ii'}$ and $D^C_{ii'}$ defined by,

$$N^C_{ii'} = \frac{1}{N_p} \langle \hat{\mathbb{H}} - E_0 | l' \rangle$$ (39)

and

$$D^C_{ii'} = \frac{1}{N_p} \langle l | l' \rangle = \sum_x \left[ \langle F_i^\dagger(x) F_{i'}(0) \rangle - \langle F_i(x) \rangle^* \langle F_{i'}(0) \rangle \right].$$ (40)

Here the superscript, $C$, denotes the charge conjugation eigenvalue, $C = \pm 1$, of the state in question. Following Arisue [31], we have the following reduction (which was generalised to improved Hamiltonians in Ref. 1):

$$N^C_{ii'} = -\frac{g^2}{2a} \sum_{i,x} \sum_{i',x'} \left[ \langle E_\alpha^\dagger(x), F_i^\dagger(x') \rangle [E_\alpha^\dagger(x), F_{i'}(0)] \right].$$ (41)
After carefully counting the number of possible overlaps between different loops we arrive at:

\[
\frac{2aN_{11}^C}{N_p g^2} = 4 \left[ \frac{1}{N} \langle \hat{O} \rangle - \frac{1}{N} \langle \hat{O} \rangle + C \left( -N + \frac{1}{N} \langle \hat{O} \rangle \right) \right]
\]

\[
\frac{D_{11}^C}{N_p} = 2 \langle \hat{O} + C \hat{O} \rangle + 10 \langle \hat{O} + C \hat{O} \rangle - 24 \left( \frac{C+1}{2} \right) \langle \hat{O} \rangle^2
\]

\[
\frac{2aN_{22}^C}{N_p g^2} = 12 \left[ \frac{1}{N} \langle \hat{O} \rangle - \frac{1}{N^2} \langle \hat{O} \rangle + C \left( - \frac{1}{N^2} \langle \hat{O} \rangle \right) \right]
\]

\[
\frac{D_{22}^C}{N_p} = \frac{8}{N} \langle \hat{O} + C \hat{O} \rangle + \frac{16}{N} \langle \hat{O} + C \hat{O} \rangle - \frac{48}{N} \left( \frac{C+1}{2} \right) \langle \hat{O} \rangle \langle \hat{O} \rangle
\]

Here we have introduced the notation

\[
\langle \langle O \rangle \rangle = \langle \phi_0 | O | \phi_0 \rangle.
\]

The combinatorics which lead to the coefficients in the matrix elements in Eq. (42) are a result of counting the possible overlaps within a single cube. The remaining matrix elements \( N_{22} \) and \( D_{22} \) can be calculated similarly, resulting in

\[
\frac{2aN_{22}^C}{N_p g^2} = 12 \left[ \frac{1}{N} \langle \hat{O} \rangle - \frac{1}{N} \langle \hat{O} \rangle + C \left( -N + \frac{1}{N} \langle \hat{O} \rangle \right) \right]
\]

\[
\frac{D_{22}^C}{N_p} = 4 \langle \hat{O} + C \hat{O} \rangle + 24 \frac{N}{N^2} \langle \hat{O} + C \hat{O} \rangle - \frac{96}{N^2} \left( \frac{C+1}{2} \right) \langle \hat{O} \rangle \langle \hat{O} \rangle
\]

Having calculated \( N_{ii'}^C \) and \( D_{ii'}^C \), we follow the minimisation procedure described in Ref. 1 to arrive at the glueball mass, \( \Delta M^{PC} \).
D. Results

In this section we present calculations of the $0^{++}$ (symmetric) and $1^{+-}$ (antisymmetric) glueball masses on the one-cube lattice for SU($N$) with $4 \leq N \leq 7$. We first define the rescaled glueball mass, $\mu^{PC}$, corresponding to $\Delta M^{PC}$ as follows:

$$
\mu^{PC} = \log(a\Delta M^{PC}\xi^{-51/121}) - \frac{51}{121} \log\left(\frac{48\pi^2}{11}\right) + \frac{24\pi^2}{11}\xi.
$$

Here, as usual, $P$ and $C$ denote the parity and charge conjugation eigenvalues of the state in question. As a result of standard renormalisation group arguments \[32, 33\], asymptotic scaling of a glueball mass is observed if the corresponding rescaled glueball mass becomes constant for some range of couplings.

The results for the rescaled symmetric glueball mass are shown in Fig. 4. Fig. 4(a) shows the rescaled glueball mass calculated with only plaquettes in the minimisation basis. Fig. 4(b) shows the same quantity calculated with the minimisation basis of Eq. (36). The aim of this exploratory study is to observe whether or not a move towards scaling is apparent as the number of states in the minimisation basis is increased. This is clearly not the case for the symmetric glueball mass on the range of couplings explored here. There is no sign of $\mu^{++}$ becoming constant on any range of couplings within $0 \leq \xi \leq 0.55$.

The prospects are marginally better for the antisymmetric case. The results for the rescaled antisymmetric glueball mass are displayed in Fig. 5. A move towards scaling is observed, most clearly for the $N = 7$ case, near $\xi = 0.38$. This matches the region of couplings for which Chin, Long and Robson observed scaling of the $0^{++}$ glueball mass for $N = 5$ and 6 on a $6^3$ lattice using only plaquettes in their minimisation basis \[25\].

E. String tension

If the exploratory study presented here was to be extended to larger lattices, at some stage it would be useful to compute the string tension, $\sigma$. It is common for calculations in Lagrangian LGT to express results for masses in units of $\sqrt{\sigma}$. This allows masses calculated on the lattice to be expressed in $MeV$ since the string tension can be calculated in $MeV$ from the decay of heavy quarkonia for example. In the Hamiltonian formulation, precise calculations of the string tension have only been performed in the strong coupling regime. Variational estimates, at least for SU(2) \[31\], have not exhibited asymptotic scaling when
making use of the one plaquette ground state of Eq. (4). It is possible that the situation may be improved for higher dimensional gauge groups but this has not yet been tested.

In this section we calculate the symmetric and antisymmetric glueball masses in units of \( \sqrt{\sigma} \). For the string tension, since reliable variational results are not available, we use the strong coupling expansions of Kogut and Shigemitsu [34]. These are available for SU(5) and SU(6), as well as SU(2) and SU(3) which we do not consider here.

Since the square root of the string tension has units of mass, the ratio of a mass to the string tension is constant in a scaling region. Again, with the crude model presented here, we do not expect to observe scaling. We seek only an indication of an approach to scaling. Such behaviour would warrant further study.

The results for the symmetric states are shown in Fig. 6. Since we perform calculations with two states in the minimisation basis, the two lowest mass states are accessible. In Fig. 6(a) the results for the lowest mass state are shown. Calculations with different minimisation bases are shown, with the “SU(N)-plaquettes” label indicating that only plaquette states are used. For each case the glueball mass has a local minimum in the range \( 0.33 \leq \xi \leq 0.37 \). The various results do not differ greatly. There is no sign of improved scaling behaviour when either bent rectangles are included or \( N \) is increased. The effect of
FIG. 5: The one cube SU(\(N\)) \(1^{++}\) rescaled glueball mass as a function of \(1/(Ng^2)\) for various \(N\) obtained with different minimisation bases.

including bent rectangles is to lower the local minimum.

The second lowest glueball mass is shown in Fig. 6(b). Again, no improvement in scaling behaviour is seen as \(N\) is increased. In Fig. 6 the horizontal lines indicate the 3+1 dimensional SU(5) calculations (and error bars) of Lucini and Teper [35]. Interestingly, both masses calculated here with a simplistic model, have minima that lie within the error bars.

We now move on to the antisymmetric states. The results for the antisymmetric glueball masses in units of \(\sqrt{\sigma}\) are shown in Fig. 7. The results are more promising than the symmetric case. We see an improved approach to scaling when bent rectangles are included in the minimisation basis for the lowest glueball mass. For this case the glueball mass shows promising signs of becoming constant in the ranges 0.32 \(\leq \xi \leq 0.39\) for SU(6) and 0.3 \(\leq \xi \leq 0.37\) for SU(5). Only marginal improvement in scaling behaviour is evident as \(N\) is increased from 5 to 6. For each of these cases no data is available for comparison to our knowledge. Such promising signs are not apparent in the second lowest glueball mass in this sector as seen in Fig. 7(b).

It would be interesting to perform analogous calculations for SU(7), for which the most promising results were displayed in Section V D. However strong coupling expansions for the SU(7) string tension are not available in Ref. 34 and have not been published elsewhere.
FIG. 6: The one cube SU(5) and SU(6) $0^{++}$ glueball masses in units of $\sqrt{\sigma}$ as functions of $1/(Ng^2)$. The horizontal lines indicate the result and error bars of the SU(5) $0^{++}$ calculation of Lucini and Teper [35].

FIG. 7: The one cube SU(5) and SU(6) $1^{+-}$ glueball masses in units of $\sqrt{\sigma}$ as functions of $1/(Ng^2)$. to our knowledge.
VI. FUTURE WORK

In this paper we have studied the variational $0^{++}$ and $1^{+-}$ glueball masses on a single cube in 3+1 dimensions. Our intention was to determine the viability of using analytic Hamiltonian methods in 3+1 dimensional glueball mass calculations. With such a crude small volume approximation and a minimisation basis containing only two states, the observation of asymptotic scaling was not expected. Promising signs of an approach to asymptotic scaling was displayed by the $1^{+-}$ glueball mass, as $N$ was increased, at couplings close to the scaling window observed by a comparable, but larger volume, study by Chin, Long and Robson [25]. Interesting results were also observed for calculations of the glueball masses in units of the string tension. With no variational results for the string tension available, the strong coupling results of Kogut and Shigemitsu [34] were used. Interestingly, the mass of both SU(5) $0^{++}$ glueball states calculated had minima which were consistent with the Lagrangian calculation of Lucini and Teper [35]. Better scaling behaviour was exhibited by the SU(5) and SU(6) $1^{+-}$ states, although no alternative calculations are available for comparison to our knowledge. For the lowest mass $1^{+-}$ states calculated, the scaling behaviour was improved by increasing the number of states in the minimisation basis and the dimension of the gauge group. The promising results observed in this paper warrant further study on larger lattices, with additional states in the minimisation basis.

To extend this calculation to larger lattices and minimisation bases, two challenges will be faced. Firstly, the correct implementation of the Bianchi identity, within our plaquette based analytic approach, forces the number of integration variables to grow quickly with the volume. New approaches to handling the general character integrals may need to be developed in order to avoid the inevitable memory restrictions. Secondly, great care will need to be taken in the counting of overlaps between different states. This process could, in principle, be automated using techniques from graph theory and symbolic programming.

It would also be of interest to extend the calculation presented here to larger $N$. To do this a more efficient technique for handling the general character integrals would need to be developed to minimise the memory demands of the calculation. The following is just one possibility. It is likely, that by expressing the products of characters appearing in the character integrals as sums of characters, one could reduce the memory requirements of the calculation drastically. The general character integrals could then be expressed in terms of
integrals of the form,

\[ G_r(c) = \int_{\text{SU}(N)} dU \chi_r(U^\dagger)e^{c\text{Tr}(U+U^\dagger)}, \] (46)

where \( r = (r_1, r_2, \ldots r_{N-1}) \) labels a representation of SU\((N)\). Having completed the calculations presented in this paper, it was discovered that \( G_r(c) \) can be handled using the techniques of Ref. 1 with the result,

\[ G_r(c) = \sum_{l=-\infty}^{\infty} \det [I_{r_i+l+i}(2c)]_{1 \leq i,j \leq N}. \] (47)

The final stage of this improvement would be to find a convenient way to express the products of characters appearing in the calculation as a sum of characters. This would be possible with the symbolic manipulation of Young tableaux.

**Acknowledgments**

We wish to acknowledge useful and interesting discussions with J. A. L. McIntosh and L. C. L. Hollenberg. We also thank R. F. Bishop for bringing the problem of Gauss’ law in 3+1 dimensions to our attention.

**APPENDIX A: MANDELSTAM CONSTRAINTS**

\[ \text{Tr} U^n = \text{Tr} U^{n-3} - \text{Tr} U^\dagger \text{Tr} U^{n-2} + \text{Tr} U \text{Tr} U^{n-1} \quad \forall U \in \text{SU}(3) \quad (A1) \]

\[ \text{Tr} U^n = -\text{Tr} U^{n-4} + \text{Tr} U^\dagger \text{Tr} U^{n-3} - \frac{1}{2}(\text{Tr} U)^2 \text{Tr} U^{n-2} + \frac{1}{2} \text{Tr} U^2 \text{Tr} U^{n-2} + \text{Tr} U \text{Tr} U^{n-1} \quad \forall U \in \text{SU}(4) \quad (A2) \]

\[ \text{Tr} U^n = \text{Tr} U^{n-5} - \text{Tr} U^\dagger \text{Tr} U^{n-4} + \frac{1}{6}(\text{Tr} U)^3 \text{Tr} U^{n-3} - \frac{1}{2} \text{Tr} U^3 \text{Tr} U^{n-3} + \frac{1}{2} \text{Tr} U^2 \text{Tr} U^{n-3} + \frac{1}{3} \text{Tr} U^3 \text{Tr} U^{n-2} - \frac{1}{2}(\text{Tr} U)^2 \text{Tr} U^{n-2} + \frac{1}{2} \text{Tr} U^2 \text{Tr} U^{n-2} + \text{Tr} U \text{Tr} U^{n-1} \quad \forall U \in \text{SU}(5) \quad (A3) \]
\[ \text{Tr} U^n = -\text{Tr} U^{n-6} + \text{Tr} U^{n-5} - \frac{1}{24} (\text{Tr} U)^4 \text{Tr} U^{n-4} + \frac{1}{4} (\text{Tr} U)^2 \text{Tr} U^2 \text{Tr} U^{n-4} \]
\[-\frac{1}{8} (\text{Tr} U^2)^2 \text{Tr} U^{n-4} - \frac{1}{3} \text{Tr} U U^3 \text{Tr} U^{n-4} + \frac{1}{4} \text{Tr} U^4 \text{Tr} U^{n-4} + \frac{1}{6} (\text{Tr} U)^3 \text{Tr} U^{n-3} \]
\[-\frac{1}{2} \text{Tr} U U^2 \text{Tr} U^{n-3} + \frac{1}{3} \text{Tr} U^3 \text{Tr} U^{n-3} - \frac{1}{2} \text{Tr} U^2 \text{Tr} U^{n-2} + \frac{1}{2} \text{Tr} U^2 \text{Tr} U^{n-2} \]
\[+ \text{Tr} U \text{Tr} U^{n-1} \quad \forall U \in \text{SU}(6) \quad (A4) \]

\[ \text{Tr} U^n = \text{Tr} U^{n-7} - \text{Tr} U^{n-6} + \frac{1}{120} (\text{Tr} U)^5 \text{Tr} U^{n-5} - \frac{1}{12} (\text{Tr} U)^3 \text{Tr} U^2 \text{Tr} U^{n-5} \]
\[+ \frac{1}{8} \text{Tr} U (\text{Tr} U^2)^2 \text{Tr} U^{n-5} + \frac{1}{6} (\text{Tr} U)^2 \text{Tr} U^3 \text{Tr} U^{n-5} - \frac{1}{6} \text{Tr} U^2 \text{Tr} U^3 \text{Tr} U^{n-5} \]
\[-\frac{1}{4} \text{Tr} U U^4 \text{Tr} U^{n-5} + \frac{1}{5} \text{Tr} U^5 \text{Tr} U^{n-5} - \frac{1}{24} (\text{Tr} U)^4 \text{Tr} U^{n-4} + \frac{1}{4} (\text{Tr} U)^2 \text{Tr} U^2 \text{Tr} U^{n-4} \]
\[-\frac{1}{8} (\text{Tr} U)^2 \text{Tr} U^4 \text{Tr} U^{n-4} - \frac{1}{3} \text{Tr} U U^3 \text{Tr} U^{n-4} + \frac{1}{4} \text{Tr} U^4 \text{Tr} U^{n-4} + \frac{1}{6} (\text{Tr} U)^3 \text{Tr} U^{n-3} \]
\[-\frac{1}{2} \text{Tr} U U^2 \text{Tr} U^{n-3} + \frac{1}{3} \text{Tr} U^3 \text{Tr} U^{n-3} - \frac{1}{2} (\text{Tr} U)^2 \text{Tr} U^{n-2} + \frac{1}{2} \text{Tr} U^2 \text{Tr} U^{n-2} \]
\[+ \text{Tr} U \text{Tr} U^{n-1} \quad \forall U \in \text{SU}(7) \quad (A5) \]

\[ \text{Tr} U^n = -\text{Tr} U^{n-8} + \text{Tr} U^{n-7} - \frac{1}{720} (\text{Tr} U)^6 \text{Tr} U^{n-6} + \frac{1}{48} (\text{Tr} U)^4 \text{Tr} U^2 \text{Tr} U^{n-6} \]
\[-\frac{1}{16} (\text{Tr} U)^2 (\text{Tr} U^2)^2 \text{Tr} U^{n-6} + \frac{1}{48} (\text{Tr} U^2)^3 \text{Tr} U^{n-6} - \frac{1}{18} (\text{Tr} U)^3 \text{Tr} U^3 \text{Tr} U^{n-6} \]
\[+ \frac{1}{6} \text{Tr} U U^2 \text{Tr} U^3 \text{Tr} U^{n-5} - \frac{1}{18} (\text{Tr} U^3)^2 \text{Tr} U^{n-6} + \frac{1}{8} (\text{Tr} U)^2 \text{Tr} U^4 \text{Tr} U^{n-6} \]
\[-\frac{1}{8} \text{Tr} U^2 \text{Tr} U^4 \text{Tr} U^{n-6} - \frac{1}{5} \text{Tr} U U^5 \text{Tr} U^{n-6} + \frac{1}{6} \text{Tr} U^6 \text{Tr} U^{n-6} + \frac{1}{120} (\text{Tr} U)^5 \text{Tr} U^{n-5} \]
\[-\frac{1}{12} (\text{Tr} U)^3 \text{Tr} U^2 \text{Tr} U^{n-5} + \frac{1}{8} \text{Tr} U (\text{Tr} U^2)^2 \text{Tr} U^{n-5} + \frac{1}{6} (\text{Tr} U)^2 \text{Tr} U^3 \text{Tr} U^{n-5} \]
\[-\frac{1}{6} \text{Tr} U^2 \text{Tr} U^3 \text{Tr} U^{n-5} - \frac{1}{4} \text{Tr} U U^4 \text{Tr} U^{n-5} + \frac{1}{5} \text{Tr} U^5 \text{Tr} U^{n-5} - \frac{1}{24} (\text{Tr} U)^4 \text{Tr} U^{n-4} \]
\[+ \frac{1}{4} (\text{Tr} U)^2 \text{Tr} U^2 \text{Tr} U^{n-4} - \frac{1}{8} (\text{Tr} U^2)^2 \text{Tr} U^{n-4} - \frac{1}{3} \text{Tr} U U^3 \text{Tr} U^{n-4} + \frac{1}{4} \text{Tr} U^4 \text{Tr} U^{n-4} \]
\[+ \frac{1}{6} (\text{Tr} U)^3 \text{Tr} U^{n-3} + \frac{1}{2} \text{Tr} U U^2 \text{Tr} U^{n-3} + \frac{1}{3} \text{Tr} U^3 \text{Tr} U^{n-3} - \frac{1}{2} (\text{Tr} U)^2 \text{Tr} U^{n-2} \]
\[+ \frac{1}{2} \text{Tr} U^2 \text{Tr} U^{n-2} + \text{Tr} U \text{Tr} U^{n-1} \quad \forall U \in \text{SU}(8) \quad (A6) \]

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