KMS WEIGHTS ON GROUPOID AND GRAPH C*-ALGEBRAS

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Abstract. The paper contains a description of the KMS weights for the one-parameter action on the reduced C*-algebra of a second countable locally compact Hausdorff étale groupoid, arising from a continuous real valued homomorphism satisfying two conditions. The result is subsequently applied to identify the KMS weights for the gauge action on a simple graph algebra. The von Neumann algebra generated by the GNS-representation of an extremal β-KMS weight is a factor, and tools are developed to determine its type. The paper concludes with three examples to illustrate the results.

1. Introduction

The existence and uniqueness of the KMS state for the gauge action on the C*-algebra of a finite irreducible graph was proved by Enomoto, Fujii and Watatani, [EFW], following the result of Olesen and Pedersen concerning the Cuntz algebra $O_n$, cf. [OP]. More recently, the KMS states for the gauge action on the graph algebra of a general finite graph, as well as on its Toeplitz extension, has been studied in [EL], [KW] and [aHLRS]. The present work started from the wish to extend these results to the gauge action on the C*-algebra of an infinite graph, but it soon became clear that there are typically many more KMS weights than KMS states since the C*-algebra of an infinite graph is not unital, and often stable. It is therefore more natural to look for weights rather than states, and to consider KMS states, when they exist, as special KMS weights. This is the point of view taken here, and we obtain a complete description of the KMS weights for the gauge action on a simple graph C*-algebra of a row-finite graph without sinks. To obtain this we consider the graph algebra as a groupoid C*-algebra as in the paper by Kumjian, Pask, Raeburn and Renault, [KPRR], where C*-algebras of infinite graphs were first introduced.

The KMS states for quite general cocycle actions on the C*-algebra of an étale groupoid were described by Neshveyev in [N], extending the work by Renault in [Re1], and the key result in the present work is a partial extension to weights of the results of Neshveyev. Specifically, we consider a second countable locally compact Hausdorff étale groupoid $\mathcal{G}$ and two continuous homomorphisms (sometimes called cocycles), $c : \mathcal{G} \to \mathbb{R}$ and $c_0 : \mathcal{G} \to \mathbb{R}$. Such homomorphisms induce continuous one-parameter groups, $\sigma^c$ and $\sigma^{c_0}$, of automorphisms on the reduced groupoid C*-algebra $C_\text{r}(\mathcal{G})$ by a canonical construction, [Re1]. Under the assumption that $\ker c_0$ has trivial isotropy groups and is open in $\mathcal{G}$, it is shown in Theorem [2.2] below that there is a bijective correspondence between proper $\sigma^{c_0}$-invariant KMS weights for $\sigma^c$ and regular Borel measures on the unit space $\mathcal{G}(0)$ of $\mathcal{G}$ that satisfy a certain conformality condition introduced by Renault in [Re1]. The KMS weight $\varphi_m$ corresponding to...
such a measure \( m \) is defined by the expression

\[
\varphi_m(a) = \int_{G(0)} P(a) \, dm,
\]

where \( P : C^*_r(G) \to C_0(G(0)) \) is the canonical conditional expectation. The proof of this, as well as the entire approach to KMS weights, builds on the theory developed with locally compact quantum groups in mind, by Kustermans and Vaes in [Ku], [KV2] and [KV3].

The result is then applied in a relatively straightforward way to the Renault, Deaconu, Anantharaman-Delaroche groupoid, [Re1], [De], [An], or RDA-groupoid for short, arising from a local homeomorphism on a locally compact second countable Hausdorff space, giving a bijective correspondence between measures and gauge-invariant KMS weights for a general cocycle action. Since the graph algebra of a countable row-finite graph without sinks is the RDA-groupoid of the shift on the locally compact Hausdorff space of infinite paths in the graph, we can then subsequently specialize to graph algebras.

Let \( G \) be a countable row-finite graph without sinks. We show in Theorem 4.6 below that for any \( \beta \in \mathbb{R} \) there is a bijective correspondence between the \( \beta \)-KMS weights on \( C^*(G) \) and positive \( e^\beta \)-eigenvectors for the adjacency matrix of \( G \). The \( \beta \)-KMS states correspond to such eigenvectors whose coordinates sum to 1; a conclusion which has also recently been obtained by Toke M. Carlsen and Nadia Larsen, [Ca], by a different method. Thus the search for the KMS weights boils down to an interesting eigenvalue problem for possibly infinite non-negative matrices, the solution of which is described for cofinal matrices in the accompanying paper [Th3]. For irreducible matrices much of the story on the solutions was already known or could be derived from known results, some parts from the results of Pruitt, [P], and Vere-Jones, [V], and other parts from the theory of countable state Markov chains, cf. e.g. [Wo].

It turns out that the set of \( \beta \)-values for which there is a \( \beta \)-KMS weight for the gauge action first of all depends on what we here call the non-wandering part of \( G \), by which we mean the set \( NW_G \) of vertexes that are contained in a loop in the graph. When the graph is cofinal these vertexes and the edges they emit constitute an irreducible sub-graph of \( G \), and we let \( \beta_0 \) be the exponential growth rate of the number of loops based at a vertex in \( NW_G \). This does not depend on the vertex and it can be any element of \([0, \infty)\]. Our main result concerning KMS weights on graph algebras is the following.

**Theorem 1.1.** Assume that \( G \) is cofinal and let \( \beta \in \mathbb{R} \).

1) Assume that the non-wandering part \( NW_G \) is empty. There is a \( \beta \)-KMS weight for the gauge action on \( C^*(G) \) for all \( \beta \in \mathbb{R} \).

2) Assume that the non-wandering part \( NW_G \) is non-empty and finite. There is a \( \beta \)-KMS weight for the gauge action on \( C^*(G) \) if and only if \( \beta = \beta_0 \).

3) Assume that the non-wandering part \( NW_G \) is non-empty and infinite. There is a \( \beta \)-KMS weight for the gauge action on \( C^*(G) \) if and only if \( \beta \geq \beta_0 \).

By combining the results from [Th3] with those of this paper, it is also possible to obtain a description of the corresponding KMS weights, at least in principle.

In Section 4.4 we study the \( \Gamma \)-invariant of Connes, [CT], for the factor \( \pi_\psi(C^*(G))'' \) generated by the GNS-representation \( \pi_\psi \) of an extremal \( \beta \)-KMS weight \( \psi \). When \( G \) is cofinal with uniformly bounded out-degree we show that there are natural numbers,
$d'_G$ and $d_G$, both generalizing the period of a finite irreducible graph, such that
\[ \mathbb{Z}d'_G \beta \subseteq \Gamma (\pi_\psi(C^*_\alpha(G))') \subseteq \mathbb{Z}d_G \beta. \]
In some cases the numbers $d'_G$ and $d_G$ are the same, but not in general. We leave it as an open problem to find a method to determine the $\Gamma$-invariant when they differ.

We conclude the paper with three examples to illustrate the results, and in particular demonstrate the significance of considering weights, and not only states, in relation to the KMS condition.

2. Measures and KMS weights on groupoid $C^*$-algebras

Let $A$ be a $C^*$-algebra and $A_+$ the convex cone of positive elements in $A$. A weight on $A$ is map $\psi : A_+ \to [0, \infty]$ with the properties that $\psi(a+b) = \psi(a) + \psi(b)$ and $\psi(\lambda a) = \lambda \psi(a)$ for all $a, b \in A_+$ and all $\lambda \in \mathbb{R}$, $\lambda > 0$. By definition $\psi$ is densely defined when $\{a \in A_+ : \psi(a) < \infty\}$ is dense in $A_+$ and lower semi-continuous when $\{a \in A_+ : \psi(a) \leq \alpha\}$ is closed for all $\alpha \geq 0$. We will use [Ku], [KV1] and [KV2] as our source for information on weights, and as in [KV2] we say that a weight is proper when it is non-zero, densely defined and lower semi-continuous.

Let $\psi$ be a proper weight on $A$. Set $\mathcal{N}_\psi = \{a \in A : \psi(a^* a) < \infty\}$ and note that
\[ \mathcal{N}_\psi \mathcal{N}_\psi = \text{Span} \{a^* b : a, b \in \mathcal{N}_\psi\} \]
is a dense $*$-subalgebra of $A$, and that there is a unique well-defined linear map $\mathcal{N}_\psi \mathcal{N}_\psi \to \mathbb{C}$ which extends $\psi : \mathcal{N}_\psi \mathcal{N}_\psi \cap A_+ \to [0, \infty)$. We denote also this densely defined linear map by $\psi$.

Let $\alpha : \mathbb{R} \to \text{Aut} A$ be a point-wise norm-continuous one-parameter group of automorphisms on $A$. Let $\beta \in \mathbb{R}$. Following [C] we say that a proper weight $\psi$ on $A$ is a $\beta$-KMS weight for $\alpha$ when
i) $\psi \circ \alpha_t = \psi$ for all $t \in \mathbb{R}$, and
ii) for every pair $a, b \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*$ there is a continuous and bounded function $F$ defined on the closed strip $D_\beta$ in $\mathbb{C}$ consisting of the numbers $z \in \mathbb{C}$ whose imaginary part lies between $0$ and $\beta$, and is holomorphic in the interior of the strip and satisfies that
\[ F(t) = \psi(\alpha_t(b)), F(t + i \beta) = \psi(\alpha_t(b)a) \]
for all $t \in \mathbb{R}$. \[ \]
A $\beta$-KMS weight $\psi$ with the property that
\[ \sup \{\psi(a) : 0 \leq a \leq 1\} = 1 \]
will be called a $\beta$-KMS state. This is consistent with the standard definition of KMS states, [BR], except when $\beta = 0$ in which case our definition requires also that a $0$-KMS state, which is also a trace state, is $\alpha$-invariant.

In this section we investigate KMS weights for a specific class of one-parameter groups of automorphisms on the $C^*$-algebra of an étale groupoid. To introduce these algebras, let $\mathcal{G}$ be an étale second countable locally compact Hausdorff groupoid with unit space $\mathcal{G}^{(0)}$. Let $r : \mathcal{G} \to \mathcal{G}^{(0)}$ and $s : \mathcal{G} \to \mathcal{G}^{(0)}$ be the range and source maps, respectively. For $x \in \mathcal{G}^{(0)}$ put $\mathcal{G}^x = r^{-1}(x)$, $\mathcal{G}_x = s^{-1}(x)$ and $\mathcal{G}^e_x = s^{-1}(x) \cap r^{-1}(x)$.

\[注\]\footnote{Note that we apply the definition from [C] for the action $\alpha_{-t}$ in order to use the same sign convention as in [BR], for example.}
Note that $G^r_x$ is a group, the \textit{isotropy group} at $x$. The space $C_c(G)$ of continuous compactly supported functions is a $*$-algebra when the product is defined by
\[(f_1 * f_2)(g) = \sum_{h \in G^{(o)}} f_1(h)f_2(h^{-1}g)\]
and the involution by $f^*(g) = \overline{f(g^{-1})}$. To define the \textit{reduced groupoid $C^*$-algebra} $C^*_r(G)$, let $x \in G^{(0)}$. There is a representation $\pi_x$ of $C_c(G)$ on the Hilbert space $l^2(G_x)$ of square-summable functions on $G_x$ given by
\[\pi_x(f)\psi(g) = \sum_{h \in G^{(o)}} f(h)\psi(h^{-1}g).\]
$C^*_r(G)$ is the completion of $C_c(G)$ with respect to the norm
\[\|f\|_r = \sup_{x \in G^{(0)}} \|\pi_x(f)\|.
\]
Note that $C^*_r(G)$ is separable since we assume that the topology of $G$ is second countable.

The map $C_c(G) \to C_c(G^{(0)})$ which restricts functions to $G^{(0)}$ extends to a conditional expectation $P : C^*_r(G) \to C_0(G^{(0)})$. Via $P$ a regular Borel measure $m$ on $G^{(0)}$ gives rise to a weight $\varphi_m : C^*_r(G) \to [0, \infty]$ defined by the formula
\[\varphi_m(a) = \int_{G^{(0)}} P(a) \, dm.\]
It follows from Fatou’s lemma that $\varphi_m$ is lower semi-continuous. Since $\varphi_m(faf) < \infty$ for every non-negative function $f$ in $C_c(G^{(0)})$, it follows that $\varphi_m$ is also densely defined, i.e. $\varphi_m$ is a proper weight on $C^*_r(G)$.\(^2\)

Let $c : G \to \mathbb{R}$ be a continuous homomorphism, i.e. $c$ is continuous and $c(gh) = c(g) + c(h)$ when $s(g) = r(h)$. For each $t \in \mathbb{R}$ we can then define an automorphism $\sigma^c_t$ of $C_c(G)$ such that
\[\sigma^c_t(f)(g) = e^{itc(g)} f(g).\] (2.1)
For each $x \in G^{(0)}$ the same expression defines a unitary $u_t$ on $l^2(G_x)$ such that $u_t \pi_x(f)u^*_t = \pi_x(\sigma^c_t(f))$ and it follows therefore that $\sigma^c_t$ extends by continuity to an automorphism $\sigma^c_t$ of $C^*_r(G)$. It is easy to see that $\sigma^c = (\sigma^c_t)_{t \in \mathbb{R}}$ is a continuous one-parameter group of automorphisms on $C^*_r(G)$. The $*$-subalgebra $C_c(G)$ of $C^*_r(G)$ consists of elements that are analytic for $\sigma^c$, cf. [BR]. Since $P(C_c(G)) \subseteq C_c(G^{(0)})$ we see that $C_c(G) \subseteq \mathcal{N}_m \cap \mathcal{N}_\varphi$ for every regular Borel measure $m$ on $G^{(0)}$. Let $\beta \in \mathbb{R}$.

As in [Th3] we say that $m$ is $(G, c)$-\textit{conformal with exponent} $\beta$ when
\[m(s(W)) = \int_{r(W)} e^{\beta c(r^{-1}(x))} \, dm(x)\] (2.2)
for every open bi-section $W \subseteq G$, where $r^{-1}_W$ denotes the inverse of $r : W \to r(W)$. This terminology is motivated by the resemblance with the notion of conformality for measures used for dynamical systems, cf. [DU]. In certain cases the notions actually coincide, cf. Lemma 3.2 below.

The relation between $(G, c)$-conformality and KMS weights is given by the following

\(^2\)We consider only non-zero measures.
Proposition 2.1. Let $m$ be a regular Borel measure on $\mathcal{G}^{(0)}$. The following are equivalent:

1) $m$ is $(\mathcal{G}, c)$-conformal with exponent $\beta$.

2) $\varphi_m(fg) = \varphi_m\left(g\sigma_{i\beta}^r(f)\right)$
   for all $f, g \in C_c(\mathcal{G})$.

3) $\varphi_m$ is a $\beta$-KMS weight for $\sigma^c$.

Proof. The equivalence of 1) and 2) follows from a calculation first performed by Renault on page 114 in [Re1] and later extended by Neshveyev in the proof of Theorem 1.3 in [N]. We will not repeat it here.

2) $\Rightarrow$ 3): Consider elements $a, b \in \mathcal{N}_{\varphi_m} \cap \mathcal{N}_{\varphi_m^*}$. Let $\{g_k\}$ be an approximate unit for $C^*_r(\mathcal{G})$ consisting of elements from $C_c(\mathcal{G}^{(0)})$ and choose sequences $\{a_n\}, \{b_n\}$ in $C_c(\mathcal{G})$ converging to $a$ and $b$, respectively. For $k, n \in \mathbb{N}$ consider the entire function

$$F_{k,n}(z) = \varphi_m\left(g_k a_n g_k \sigma^c_{i\beta}(b_n)\right).$$

Note that by assumption $F_{k,n}(t+i\beta) = \varphi_m\left(g_k a_n \sigma_{i\beta}^c (\sigma^r_{i\beta}(g_k b_n))\right) = \varphi_m(g_k \sigma^c_{i\beta}(b_n) g_k a_n)$. Since $P(ad) \leq P(cc)^{\frac{1}{2}} P(d^*d)^{\frac{1}{2}}$ for all $c, d \in C^*_r(\mathcal{G})$ and $P \circ \sigma^c_{i\beta} = P$, it follows that

$$\sup_{t \in \mathbb{R}} |F_{k,n}(t) - \varphi_m(g_k a_n g_k \sigma^c_{i\beta}(b))|$$

$$\leq \sup_{t \in \mathbb{R}} (|\varphi_m(g_k (a_n-a) g_k \sigma^c_{i\beta}(b_n))| + |\varphi_m(g_k a_n g_k \sigma^c_{i\beta}(b_n-b))|)$$

$$\leq \int_{\mathcal{G}^{(0)}} g_k \left(P((a_n-a)(a_n-a)^*)^{\frac{1}{2}} P(b_n^* b_n) + P(aa^*)^{\frac{1}{2}} P((b_n-b)^* g_k^2(b_n-b))^{\frac{1}{2}}\right) dm.$$ 

It follows that

$$\lim_{n \to \infty} \sup_{t \in \mathbb{R}} |F_{k,n}(t) - \varphi_m(g_k a_n g_k \sigma^c_{i\beta}(b))| = 0. \quad (2.3)$$

Similar estimates show that

$$\lim_{n \to \infty} \sup_{t \in \mathbb{R}} |F_{k,n}(t+i\beta) - \varphi_m(g_k \sigma^c_{i\beta}(b) g_k a)| = 0. \quad (2.4)$$

For each $k, n$ the function $F_{k,n}$ is bounded by

$$\sup_{|s| \leq |\beta|} \sqrt{\varphi_m(g_k a_n a^*_k g_k) \varphi_m(\sigma^c_{i\beta}(b_n)^* \sigma^c_{i\beta}(b_n))}$$

on the closed strip $D_\beta$ in $\mathbb{C}$ where the imaginary part is between 0 and $\beta$. We can therefore use Hadamard’s three-lines theorem, Proposition 5.3.5 in [BR], to conclude from (2.3) and (2.4) that the sequence $\{F_{k,n}\}_{n=1}^{\infty}$ is Cauchy in the supremum norm on $D_\beta$ and therefore converges uniformly, as $n$ tends to $\infty$, to a continuous and bounded function $F_k : D_\beta \to \mathbb{C}$ which is holomorphic in the interior of $D_\beta$ and satisfies that

$$F_k(t) = \varphi_m(g_k a_n g_k \sigma^c_{i\beta}(b))$$

and $F_k(t+i\beta) = \varphi_m(g_k \sigma^c_{i\beta}(b) g_k a)$

for all $t \in \mathbb{R}$. Now note that

$$|F_k(t) - \varphi_m(a \sigma^c_{i\beta}(b))| \leq |\varphi_m((g_k a - a) g_k \sigma^c_{i\beta}(b))| + |\varphi_m(a (g_k \sigma^c_{i\beta}(b) - \sigma^c_{i\beta}(b)))|$$

$$\leq \int_{\mathcal{G}^{(0)}} |g_k - 1| P(aa^*)^{\frac{1}{2}} P(b^*b)^{\frac{1}{2}} + P(aa^*)^{\frac{1}{2}} P(b^* (g_k - 1)^2 b)^{\frac{1}{2}} dm.$$
The integrand is dominated by 
\[
2P(aa^*)^{1/2}P(bb^*)^{1/2}
\]
which is in \(L^1(m)\) because \(a\) and \(b\) are both in \(N_{\varphi_m} \cap N_{\varphi_m}^*\). We can therefore apply Lebesgue’s dominated convergence theorem to conclude that
\[
\lim_{k \to \infty} \sup_{t \in \mathbb{R}} |F_k(t) - \varphi_m(a\sigma_t^c(b))| = 0.
\]

Similar arguments show that
\[
\lim_{k \to \infty} \sup_{t \in \mathbb{R}} |F_k(t + i\beta) - \varphi_m(\sigma_t^c(b)a)| = 0,
\]
and 3) follows then from Hadamard’s three-lines theorem as above.

3) \(\Rightarrow\) 2): Since \(C_c(\mathcal{G}) \subseteq N_{\varphi_m} \cap N_{\varphi_m}^*\) this implication is obtained exactly as for states. See the proof of (3) \(\Rightarrow\) (1) in Proposition 5.3.7 of [BR].

The key result of the paper is the following theorem. It has a predecessor in Proposition 5.4 of [Re1], but the two results are not directly comparable because the definitions of KMS weights are not the same. In particular, a major problem overcome in the following proof is that with the present definition, KMS weights are not a priori finite on \(C_c(\mathcal{G})\).

**Theorem 2.2.** Let \(c_0 : \mathcal{G} \to \mathbb{R}\) and \(c : \mathcal{G} \to \mathbb{R}\) be continuous homomorphisms. Assume that

a) ker \(c_0\) = \(\{g \in \mathcal{G} : c_0(g) = 0\}\) is an equivalence relation, i.e. \(\mathcal{G}^*_x \cap \text{ker } c_0 = \{x\}\) for all \(x \in \mathcal{G}^{(0)}\), and

b) ker \(c_0\) is open in \(\mathcal{G}\).

Let \(\psi\) be a \(\beta\)-KMS weight for the action \(\sigma^c\) on \(C^*_c(\mathcal{G})\). Assume that \(\psi\) is invariant under \(\sigma^c_0\) in the sense that \(\psi \circ \sigma^c_0 = \psi\) for all \(t \in \mathbb{R}\).

There is a regular Borel measure \(m\) on \(\mathcal{G}^{(0)}\) such that \(\psi = \varphi_m\).

**Proof.** By the Riesz representation theorem it suffices to show that \(\psi\) takes finite values on non-negative elements from \(C_c(\mathcal{G}^{(0)})\) and that \(\psi \circ P = \psi\).

By assumption ker \(c_0\) is an open (and closed) sub-groupoid of \(\mathcal{G}\) and hence \(C^*_c(\text{ker } c_0) \subseteq C^*_c(\mathcal{G})\). For \(R > 0\), set
\[
Q_R(a) = \frac{1}{R} \int_0^R \sigma_t^c(a) \, dt.
\]

(2.5)

It follows easily from the formula defining the action \(\sigma^c\), cf. (2.1), that
\[
\lim_{R \to \infty} Q_R(f)(g) = \begin{cases} f(g) & \text{when } g \in \text{ker } c_0 \\ 0 & \text{otherwise} \end{cases}
\]
for every \(f \in C_c(\mathcal{G})\). Since \(\|Q_R\| \leq 1\) it follows therefore that the limit
\[
Q(a) = \lim_{R \to \infty} Q_R(a)
\]
exists for all \(a \in C^*_c(\mathcal{G})\) and that \(Q(a) \in C^*_c(\text{ker } c_0)\). We aim to prove that \(\psi \circ Q_R = \psi\) and \(\psi \circ Q \leq \psi\). To this end observe that there is a directed subset \(\Lambda\) of
\[
\{\omega \in C^*_c(\mathcal{G})^* : 0 \leq \omega \leq \psi\}
\]
such that \(\psi(a) = \lim_{\omega \in \Lambda} \omega(a)\) for all \(a \geq 0\), cf. [Ku]. Fix \(R > 0\) and let \(a \geq 0\). Assume first that \(\psi(a) = \infty\). For any \(r > 0\) and any \(t \in [0, R]\) there is an \(\omega_t \in \Lambda\) such
that \( \omega(\sigma_t^n(a)) > r \) for all \( s \) in an open neighborhood of \( t \). Thanks to the directedness of \( \Lambda \) there is therefore an \( \omega \in \Lambda \) such that \( \omega(\sigma_t^n(a)) \geq r \) for all \( t \in [0, R] \). Then

\[
\psi(Q_R(a)) \geq \omega(Q_R(a)) = \frac{1}{R} \int_0^R \omega(\sigma_t^n(a)) \, dt \geq r,
\]

proving that \( \psi(Q_R(a)) = \infty \). If instead \( \psi(a) < \infty \) we let \( \epsilon > 0 \) and choose in the same way an \( \omega \in \Lambda \) such that \( \omega(\sigma_t^n(a)) \geq \psi(a) - \epsilon \) for all \( t \in [0, R] \). Then

\[
\psi(Q_R(a)) \geq \omega(Q_R(a)) = \frac{1}{R} \int_0^R \omega(\sigma_t^n(a)) \, dt \geq \psi(a) - \epsilon. \tag{2.6}
\]

On the other hand the set \( \{ b \in C^*_r(G) : b \geq 0, \psi(b) \leq \psi(a) \} \) is closed, convex and contains all Riemann sums for the integral in (2.5), and it follows therefore that \( \psi(Q_R(a)) \leq \psi(a) \). Combined with (2.6) this shows that \( \psi(Q_R(a)) = \psi(a) \) also when \( \psi(a) < \infty \). Thus

\[
\psi \circ Q_R = \psi. \tag{2.7}
\]

The lower semi-continuity of \( \psi \) now implies that

\[
\psi \circ Q \leq \psi. \tag{2.8}
\]

We aim next to show that \( \psi(g) < \infty \) for all non-negative \( g \in C_r(G^{(0)}) \). Let therefore \( f \) be such a function. Since \( \psi \) is densely defined there is a sequence \( \{ a_n \} \) of non-negative elements in \( C^*_r(G) \) such that \( \lim_n a_n = \sqrt{f} \) and \( \psi(a_n) < \infty \) for all \( n \). It follows that \( \lim_n Q(a_n) = \sqrt{f} \). Set \( b_n = Q(a_n) \) and note that \( \psi(b_n) \leq \psi(a_n) < \infty \) by (2.8). Let \( k \in \mathbb{N} \) and consider

\[
c_n = \sqrt{k} \frac{1}{\pi} \int_R \sigma^*_t(b_n) e^{-kt^2} \, dt. \tag{2.9}
\]

Then \( c_n \in C^*_r(\text{ker } c_0) \) is analytic for \( \sigma_e \) and if \( k \) is large enough \( \|c_n - b_n\| \leq \frac{1}{n} \), cf. [BR]. By approximating the integral in (2.9) by convex combinations of elements of the form \( \sigma^*_t(b_n) \) and by using that \( \psi \circ \sigma^*_e = \psi \), it follows from the lower semi-continuity of \( \psi \) that \( \psi(c_n) \leq \psi(b_n) < \infty \). Since \( \text{ker } c_0 \) has trivial isotropy by the first condition a) on \( c_0 \), it follows from (the proof of) Lemma 2.24 in [Th1] that there is a sequence

\[
\{ d_j^n : j = 1, 2, \ldots, N_n \}, \quad n = 1, 2, 3, \ldots,
\]

of non-negative elements in \( C_0(G^{(0)}) \) such that

\[
\sum_{j=1}^{N_n} d_j^n = 1 \tag{2.10}
\]

for all \( n \), and

\[
P(b) = \lim_{n \to \infty} \sum_{j=1}^{N_n} d_j^n b d_j^n \tag{2.11}
\]

for all \( b \in C^*_r(\text{ker } c_0) \). By using that \( \psi \) is densely defined in combination with (2.8), we can choose, for each \( d_j^n \) and each \( k \in \mathbb{N} \), a positive element \( c(n, j, k) \in C^*_r(\text{ker } c_0) \) such that \( \|c(n, j, k) d_j^n - d_j^n\| \leq \frac{1}{k} \) and \( \psi(c(n, j, k)) < \infty \). Set

\[
c'(n, j, k) = \frac{1}{\sqrt{\pi}} \int_R e^{-t^2} \sigma^*_t(c(n, j, k)) \, dt.
\]
Then \( c'(n, j, k) \) is analytic for \( \sigma^c \), \( \psi(c'(n, j, k)) \leq \psi(c(n, j, k)) < \infty \), and

\[
\left\| c'(n, j, k)d^n_j - d^n_j \right\| = \left\| \frac{1}{\sqrt{\pi}} \int_\mathbb{R} \sigma^c_1(c(n, j, k)d^n_j - d^n_j)e^{t^2} dt \right\| \leq \frac{1}{k}.
\]

Note that it follows from the definition of a \( \beta \)-KMS weight that when \( a \in \mathcal{N}_\psi \cap \mathcal{N}^*_\psi \) is analytic for \( \sigma^c \) the following identity holds:

\[
\psi(a^*a) = \psi(\sigma^c_{\frac{t}{\sqrt{\beta}}} (a) \sigma^c_{\frac{t}{\sqrt{\beta}}} (a)^* ).
\]  
(2.12)

(See also Proposition 1.11 in [KV2].) Since \( c'(n, j, k)d^n_jc_m \in \mathcal{N}_\psi \cap \mathcal{N}^*_\psi \) is analytic for \( \sigma^c \) it follows from (2.12) that

\[
\psi \left( \sum_{j=1}^{N_n} c'(n, j, k)d^n_jc_m^2d^n_jc'(n, j, k) \right) = \sum_{j=1}^{N_n} \psi (c'(n, j, k)d^n_jc_m^2d^n_jc'(n, j, k)) = \sum_{j=1}^{N_n} \psi (\sigma^c_{\frac{t}{\sqrt{\beta}}} (c_m)d^n_j\sigma^c_{\frac{t}{\sqrt{\beta}}} (c'(n, j, k))\sigma^c_{\frac{t}{\sqrt{\beta}}} (c'(n, j, k))^* d^n_j\sigma^c_{\frac{t}{\sqrt{\beta}}} (c_m)^*) .
\]

Because

\[
\lim_{k \to \infty} \frac{d^n_j\sigma^c_{\frac{t}{\sqrt{\beta}}} (c'(n, j, k)) - d^n_j}{\sqrt{\pi}} = \lim_{k \to \infty} \frac{1}{\sqrt{\pi}} \int_\mathbb{R} \sigma^c_1(d^n_jc(n, j, k) - d^n_j)e^{-(t^2 \frac{t}{\sqrt{\beta}})^2} dt = 0
\]

and \( \sigma^c_{\frac{t}{\sqrt{\beta}}} (c_m) \in \mathcal{N}^*_\psi \), it follows that

\[
\lim_{k \to \infty} \sum_{j=1}^{N_n} \psi \left( \sigma^c_{\frac{t}{\sqrt{\beta}}} (c_m)d^n_j\sigma^c_{\frac{t}{\sqrt{\beta}}} (c'(n, j, k))\sigma^c_{\frac{t}{\sqrt{\beta}}} (c'(n, j, k))^* d^n_j\sigma^c_{\frac{t}{\sqrt{\beta}}} (c_m)^* \right)
\]

\[
= \sum_{j=1}^{N_n} \psi \left( \sigma^c_{\frac{t}{\sqrt{\beta}}} (c_m)d^n_j\sigma^c_{\frac{t}{\sqrt{\beta}}} (c_m)^* \right).
\]

We can therefore let \( k \) tend to infinity in (2.13) and use the lower semi-continuity of \( \psi \) in combination with (2.10) and (2.12) to conclude that

\[
\psi \left( \sum_{j=1}^{N_n} d^n_jc_m^2d^n_j \right) \leq \sum_{j=1}^{N_n} \psi \left( \sigma^c_{\frac{t}{\sqrt{\beta}}} (c_m)d^n_j d^n_j\sigma^c_{\frac{t}{\sqrt{\beta}}} (c_m)^* \right) \leq \psi (e^2_m) \leq \|c_m\| \|\psi(c_m)\).
\]

Let then \( n \) tend to infinity to conclude that \( \psi (P(e^2_m)) \leq \|c_m\| \|\psi(c_m)\). \) Since \( \psi(c_m) < \infty \) and \( \lim_{m \to \infty} P(e^2_m) = f \), we conclude that \( f \) is the limit under uniform convergence of the sequence \( g_m = P(e^2_m) \) of non-negative functions from \( C_0(G^{(0)}) \) such that \( \psi(g_m) < \infty \) for all \( m \).

Consider then an arbitrary non-negative \( g \in C_c(G^{(0)}) \) and let \( f \) be a non-negative function \( f \in C_c(G^{(0)}) \) such that \( f(x) = 2 \) for all \( x \) in the support of \( g \). As we have just shown there is a non-negative function \( h \in C_0(G^{(0)}) \) such that \( |h(y) - f(y)| \leq 1 \) for all \( y \in G^{(0)} \) and \( \psi(h) < \infty \). Then \( 0 \leq g \leq gh \) and \( \psi(g) \leq \psi(gh) < \infty \).

Let then \( a \geq 0 \) in \( C_c^*(G) \) and \( 0 \leq a \leq 1 \) in \( C_c(G^{(0)}) \). By the same methods as above we construct a sequence \( \{c_k\} \) of positive \( \sigma^c \)-analytic elements in \( C_c^*(\ker c_0) \)
converging to $\sqrt{Q(a)}$ with $\psi(c_k) < \infty$ for all $k$. Using that $\psi(g^2) < \infty$ we find that

$$\psi(gP(c_k)g) = \lim_{n \to \infty} \sum_{j=1}^{N_n} \psi(d_j^n g c_k^j d_j^n)$$

$$= \lim_{n \to \infty} \sum_{j=1}^{N_n} \psi\left(\frac{\sigma_{-\beta}^j(c_k)g d_j^n g \sigma_{-\beta}^j(c_k)}{2}\right) \leq \psi\left(\frac{\sigma_{-\beta}^j(c_k)g^2 \sigma_{-\beta}^j(c_k)}{2}\right)$$

$$= \psi\left(\frac{\sigma_{-\beta}^j(c_k)g^2 \sigma_{-\beta}^j(c_k)}{2}\right) = \psi(g c_k^2 g)$$

for all $k$, and when we let $k$ tend to infinity we deduce that $\psi(gP(a)g) = \psi(gQ(a)g)$ since $P \circ Q = P$. But $\psi(gQ(a)g) = \lim_{R \to \infty} \psi(gQ_R(a)g) = \lim_{R \to \infty} \psi((Q_R(gag)) = \psi(gag)$, thanks to (2.7). Thus

$$\psi(gP(a)g) = \psi(gag). \quad (2.14)$$

Since $gP(a)g \leq P(a)$ it follows that

$$\psi(gag) \leq \psi(P(a)). \quad (2.15)$$

Let $\{g_k\}$ be a sequence from $C_c(G^{(0)})$ which constitutes an approximate unit in $C^*_c(G)$. Inserting $g_k$ for $g$ in (2.15), the lower semi-continuity of $\psi$ yields that

$$\psi(a) \leq \psi(P(a)). \quad (2.16)$$

Let now $a \geq 0$ be a positive element in $C^*_c(G)$ such that $\psi(P(a)) < \infty$. Then $\psi(a) < \infty$ and we can combine the Cauchy-Schwarz inequality with (2.16) to get

$$|\psi(g_k a g_k) - \psi(a)| \leq |\psi(g_k a(g_k - 1))| + |\psi((g_k - 1)a)|$$

$$\leq \psi((g_k a g_k)^\frac{1}{2}) \psi(((g_k - 1)a(g_k - 1))\frac{1}{2}) + \psi(((g_k - 1)a g_k - 1))\frac{1}{2} \psi(a)^\frac{1}{2}$$

$$\leq \psi(g_k P(a) g_k) \frac{1}{2} \psi((g_k - 1)P(a)(g_k - 1))^\frac{1}{2} \psi((g_k - 1)P(a)(g_k - 1))^\frac{1}{2} \psi(a)^\frac{1}{2}.$$ 

Note that $\lim_{k \to \infty} \psi(g_k P(a)) = \lim_{k \to \infty} \psi(g_k^2 P(a)) = \psi(P(a))$ since the sequences $\{g_k P(a)\}$ and $\{g_k^2 P(a)\}$ both converge increasingly to $P(a)$. It follows that

$$\lim_{k \to \infty} \psi((g_k - 1)P(a)(g_k - 1)) = 0$$

and the estimate above then shows that $\lim_{k \to \infty} \psi(g_k a g_k) = \psi(a)$. By using this conclusion in combination with (2.14) it follows that $\psi(a) = \psi \circ P(a)$ when $a \geq 0$ and $\psi \circ P(a) < \infty$. Since $\psi \circ P$ is a proper weight we can now use Corollary 1.15 in [K] to conclude that $\psi = \psi \circ P$.

By taking $c = c_0$ in the theorem we obtain the following corollary.

**Corollary 2.3.** Let $c : G \to \mathbb{R}$ be a homomorphism such that $\ker c \cap G^*_x = \{x\}$ for all $x \in G^{(0)}$ and $c(x)$ is open in $G$.

The map $m \mapsto \varphi_m$ is a bijection from the $(G,c)$-conformal measures $m$ on $G^{(0)}$ with exponent $\beta$ onto the set of $\beta$-KMS weights for the action $\sigma^t$ on $C^*_c(G)$.

The bijection in Corollary 2.3 between measures and weights restricts of course to a bijective correspondence between KMS states and $(G,c)$-conformal Borel probability measures. For those the corollary is a consequence of Theorem 1.3 in [N], and the condition that $c(x)$ is open is not necessary for states. But without the condition that $\ker c \cap G^*_x$ is trivial there can be KMS states that do not factor through the conditional expectation $P$ and are not determined by their restriction to $C_0(G^{(0)})$. 


In these additional KMS states are given a general description in terms of fields of traces.

It is not inconceivable that Corollary 2.3 remains true without the condition that \( \ker c \) is open; this is left as an unsolved problem. Theorem 2.2 and Corollary 2.3 suffice for the actions on graph algebras we study in this paper.

3. KMS weights on the \( C^* \)-algebra of a local homeomorphism

3.1. The RDA-groupoid of a local homeomorphism. Now we turn to the case where \( G \) is the RDA-groupoid \( \Gamma_\phi \) of a local homeomorphism \( \phi \) on a locally compact second countable Hausdorff space \( X \). To introduce this construction set

\[
\Gamma_\phi = \{ (x, k, y) \in X \times \mathbb{Z} \times X : \exists n, m \in \mathbb{N}, k = n - m, \phi^n(x) = \phi^m(y) \}.
\]

This is a groupoid with the set of composable pairs being

\[
\Gamma^{(2)}_\phi = \{ ((x, k, y), (x', k', y')) \in \Gamma_\phi \times \Gamma_\phi : y = x' \}.
\]

The multiplication and inversion are given by

\[
(x, k, y)(y, k', y') = (x, k + k', y') \quad \text{and} \quad (x, k, y)^{-1} = (y, -k, x).
\]

Note that the unit space of \( \Gamma_\phi \) can be identified with \( X \) via the map \( x \mapsto (x, 0, x) \).

Under this identification the range map \( r : \Gamma_\phi \to X \) is the projection \( r(x, k, y) = x \) and the source map the projection \( s(x, k, y) = y \).

To turn \( \Gamma_\phi \) into a locally compact topological groupoid, fix \( k \in \mathbb{Z} \). For each \( n \in \mathbb{N} \) such that \( n + k \geq 0 \), set

\[
\Gamma_\phi(k, n) = \{ (x, l, y) \in X \times \mathbb{Z} \times X : l = k, \phi^{k+n}(x) = \phi^n(y) \}.
\]

This is a closed subset of the topological product \( X \times \mathbb{Z} \times X \) and hence a locally compact Hausdorff space in the relative topology. Since \( \phi \) is locally injective \( \Gamma_\phi(k, n) \) is an open subset of \( \Gamma_\phi(k, n + 1) \) and the union

\[
\Gamma_\phi(k) = \bigcup_{n \geq -k} \Gamma_\phi(k, n)
\]

is a locally compact Hausdorff space in the inductive limit topology. The disjoint union

\[
\Gamma_\phi = \bigcup_{k \in \mathbb{Z}} \Gamma_\phi(k)
\]

is then a locally compact Hausdorff space in the topology where each \( \Gamma_\phi(k) \) is an open and closed set. In fact, as is easily verified, \( \Gamma_\phi \) is a locally compact groupoid in the sense of [Re1] and an étale groupoid, i.e. the range and source maps are local homeomorphisms.

By construction there is a canonical periodic one-parameter group acting on \( C^*_r(\Gamma_\phi) \), given by the homomorphism \( c_0 \) on \( \Gamma_\phi \) defined such that \( c_0(x, k, y) = k \).

The gauge action \( \gamma \) on \( C^*_r(\Gamma_\phi) \) is the corresponding one-parameter group \( \gamma = \sigma^{c_0} \).

That is,

\[
\gamma_t(f)(x, k, y) = e^{ikt}f(x, k, y)
\]

when \( f \in C_b(\Gamma_\phi) \). Since \( c_0 \) clearly satisfies the two conditions required to apply Theorem 2.2 we obtain the following.
Proposition 3.1. Let $c : \Gamma_\phi \to \mathbb{R}$ be a continuous homomorphism. There is a bijective correspondence between the $(\Gamma_\phi, c)$-conformal measures $m$ on $X$ with exponent $\beta$ and the gauge-invariant $\beta$-KMS weights for the action $\sigma^c$ on $C^*_r(\Gamma_\phi)$. The gauge-invariant $\beta$-KMS weight $\varphi_m$ associated to $m$ is

$$\varphi_m(\cdot) = \int_X P(\cdot) \, dm.$$  

Let $F : X \to \mathbb{R}$ be a continuous function and define $c_F : \Gamma_\phi \to \mathbb{R}$ such that

$$c_F(x, k, y) = \lim_{n \to \infty} \left( \frac{1}{n+k} \sum_{i=0}^{n+k} F(\phi^i(x)) - \frac{1}{n} \sum_{i=0}^n F(\phi^i(y)) \right).$$

Then $c_F$ is a continuous homomorphism. Following a terminology first introduced in dynamical systems by Denker and Urbanski in [DU] we say that a Borel measure $m$ on $X$ is $e^{\beta F}$-conformal when

$$m(\phi(A)) = \int_A e^{\beta F(x)} \, dm(x)$$

for every Borel set $A \subseteq X$ such that $\phi : A \to X$ is injective.

Lemma 3.2. Let $m$ be a regular Borel measure on $X$. Then $m$ is $(\Gamma_\phi, c_F)$-conformal with exponent $\beta$ if and only if $m$ is $e^{\beta F}$-conformal.

Proof. Assume first that $m$ is $(\Gamma_\phi, c_F)$-conformal with exponent $\beta$ and consider a Borel subset $A \subseteq X$ such that $\phi$ is injective on $A$. Since $\phi$ is a local homeomorphism and $X$ is second countable we can write $A$ as a countable disjoint union

$$A = \bigsqcup_{i \in I} A_i$$

of Borel sets with the property that for each $i$ there is an open set $U_i$ such that $A_i \subseteq U_i$ and $\phi : U_i \to \phi(U_i)$ is a homeomorphism. For each $i$ and each open subset $V \subseteq U_i$ the set $\{(x, 1, \phi(x)) : x \in V\}$ is an open bi-section $W$ in $\Gamma_\phi$ with $s(W) = \phi(V)$, $r(W) = V$, and it follows therefore that

$$m(\phi(V)) = \int_V e^{\beta c_F(x, 1, \phi(x))} \, dm(x) = \int_V e^{\beta F(x)} \, dm(x).$$

By additivity and regularity of $m$,

$$m(\phi(A)) = \sum_{i \in I} m(\phi(A_i)) = \sum_i \int_{A_i} e^{\beta F(x)} \, dm(x) = \int_A e^{\beta F(x)} \, dm(x).$$

Assume next that (3.1) holds whenever $A$ is a Borel subset on which $\phi$ is injective. It follows by iteration that

$$m(\phi^n(A)) = \int_A e^{\beta \sum_{i=0}^{n-1} F(\phi^i(x))} \, dm(x)$$  \hspace{1cm} (3.2)

for every Borel set $A \subseteq X$ such that $\phi^n : A \to X$ is injective, $n = 1, 2, 3, \ldots$, and then by 'inversion' that

$$m(B) = \int_A e^{\beta \left( \sum_{i=1}^{n-1} F(\phi^i(x)) - \sum_{j=1}^n F(\phi^{-j}(\phi^n(x))) \right)} \, dm(x)$$  \hspace{1cm} (3.3)

when $A$ and $B$ are Borel subsets of $X$ such that $\phi^n$ is injective on $A$, $\phi^m$ is injective on $B$ and $\phi^n(A) = \phi^m(B)$, where $\phi^{-j}$ denotes the inverse of $\phi^j : \phi^{-j}(B) \to \phi^m(B)$. 

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Consider then an open bi-section \(W\) of \(\Gamma_\phi\). For \(k \in \mathbb{Z}\) set \(W_k = \Gamma_\phi(k) \cap W\). By additivity of measures, to establish (2.2) it suffices to establish it for \(W_k\). As a further reduction note that by monotone continuity of measures and integrals it suffices to establish (2.2) with \(W_k\) replaced by the set
\[
W(k, n) = W \cap \Gamma_\phi(k, n).
\]
Since \(W_{k, n}\) is a bi-section in \(\Gamma_\phi\) it follows that \(\phi^{n+k}\) is injective on \(r(W(k, n))\) and \(\phi^n\) is injective on \(s(W(k, n))\). In addition \(\phi^{n+k}(r(W(k, n))) = \phi^n(s(W(k, n)))\) and we can apply (3.3). This establishes (2.2) because
\[
c_F \left( r_{W(k, n)}^{-1}(x) \right) = \sum_{i=0}^{n+k-1} F(\phi^i(x)) - \sum_{j=1}^{n} F(\phi^{-j}(\phi^n(x)))
\]
when \(x \in r(W(k, n))\).

□

For homomorphisms \(\Gamma_\phi \to \mathbb{R}\) of the form \(c_F\) we have now proved the following reformulation of Proposition 3.1.

**Proposition 3.3.** There is a bijective correspondence between the regular \(e^{\beta F}\)-conformal measures \(m\) on \(X\) and the gauge-invariant \(\beta\)-KMS weights for the action \(\sigma^{c_F}\) on \(C^*_r(\Gamma_\phi)\). The gauge-invariant \(\beta\)-KMS weight \(\varphi_m\) associated to \(m\) is
\[
\varphi_m(\cdot) = \int_X P(\cdot) \, dm.
\]

If \(F\) is either strictly positive or strictly negative, the equality \(\ker c_F \cap (\Gamma_\phi)_x^x = \{x\}\) holds for all \(x \in X\), and it follows then from \([N]\) that the \(\beta\)-KMS states are all gauge-invariant. When \(F\) is also locally constant it follows from Corollary 2.5 applied with \(c = c_F\), that the same is true for \(\beta\)-KMS weights, and Proposition 3.3 is then true with the word ‘gauge-invariant’ deleted.

**Remark 3.4.** Since \(\phi\) is a local homeomorphism the Ruelle operator \(L_g : C_c(X) \to C_c(X)\) is defined for every continuous and bounded real-valued function \(g\) on \(X\); viz.
\[
L_g(f)(x) = \sum_{y \in \phi^{-1}(x)} e^{g(y)} f(y).
\]
The dual operator \(L^*_g\) acts on the regular measures in the natural way:
\[
L^*_g(m)(f) = \int_X L_g(f) \, dm.
\]
The Ruelle operator makes it possible to present an integrated version of the identity (3.1) defining conformality of a measure. Specifically, it is not difficult to see that a regular Borel measure \(m\) on \(X\) is \(e^{\beta F}\)-conformal if and only if \(L^*_g(m) = m\), cf. \([Re2]\). Therefore Proposition 3.3 is a generalization to weights of Theorem 6.2 in \([Th2]\).
4. KMS weights on graph $C^*$-algebras

4.1. KMS weights for the generalized gauge actions on graph algebras.

Graph algebras were introduced in [KPRR] and we shall adopt notation and terminology from [KPRR]. In particular, we assume that the directed graph $G$ under consideration has at most countably many vertexes and edges, and that it is ‘row-finite’, meaning that the number of edges emitted from any vertex is finite. Furthermore, we assume that there are no sinks, i.e. every vertex emits an edge.

Let $V_G$ and $E_G$ denote the set of vertexes and edges in $G$, respectively. An infinite path in $G$ is an element $p = (E_G)^\mathbb{N}$ such that $r(p_i) = s(p_{i+1})$ for all $i$, where we let $r(e)$ and $s(e)$ denote the range and source of an edge $e \in E_G$, respectively. A finite path $\mu = e_1e_2\ldots e_n$ is defined similarly, and we extend the range and source maps to finite paths such that $s(\mu) = s(e_1)$ and $r(\mu) = r(e_n)$. The number of edges in $\mu$ is its length and we denote it by $|\mu|$. We let $P(G)$ denote the set of infinite paths in $G$ and extend the source map to $P(G)$ such that $s(p) = s(p_1)$ when $p = (p_i)^\infty_{i=1}$.

To describe the topology of $P(G)$, let $\mu = e_1e_2\ldots e_n$ be a finite path in $G$. We can then consider the cylinder

$$Z(\mu) = \{p \in P(G) : p_i = e_i, \ i = 1, 2, \ldots, n\}.$$ 

$P(G)$ is a totally disconnected second countable locally compact Hausdorff space in the topology for which the collection of cylinders is a base, [KPRR]. The graph $C^*$-algebra $C^*(G)$ is then the (reduced) groupoid $C^*$-algebra $C^*_r(\Gamma, \sigma)$ of the RDA-groupoid $\Gamma, \sigma$ corresponding to the local homeomorphism $\sigma$ on the path space $P(G)$, defined such that

$$\sigma(e_0e_1e_2e_3\ldots) = e_1e_2e_3\ldots.$$ 

Let $F : P(G) \to \mathbb{R}$ be a continuous function. The corresponding homomorphism $c_F : \Gamma, \sigma \to \mathbb{R}$ is then given by

$$c_F(p, k, q) = \lim_{n \to \infty} \left( \sum_{i=0}^{n+k} F(\sigma^i(p)) - \sum_{i=0}^{n} F(\sigma^i(q)) \right),$$

and we can consider the continuous one-parameter action $\alpha^F = \sigma^{c_F}$ on $C^*(G)$ defined such that

$$\alpha^F_\tau(f)(z) = e^{itc_F(z)} f(z)$$

when $f \in C_c(\Gamma, \sigma)$. It follows from Proposition 3.3 that the gauge-invariant $\beta$-KMS weights for $\alpha^F$ are in one-to-one correspondence with the $e^{\beta F}$-conformal Borel measures on $P(G)$. In order to obtain a complete description of these measures we assume now that $F$ is given by a map $F_0 : V_G \to \mathbb{R}$ through the formula

$$F(p) = F_0(s(p)), $$

where $s(p) \in V_G$ is the source of the path $p \in P(G)$. It follows in particular that $F$ and hence also $c_F$ is locally constant.

Remark 4.1. The assumption that $F$ only depends on the initial vertex of a path is not as restrictive as it may appear at first sight. If instead $F$ depends on an initial part of the paths, say on the first $k$ edges, it would still be possible to apply the results we obtain here. To see how, define a new graph $H$ whose vertexes are the paths of length $k$ in $G$ and with an edge from the path $\mu = e_1e_2\ldots e_k$ to $\mu' = e_1' e_2' \ldots e'_k$ when $e_2e_3e_4\ldots e_k = e_1'e_2'e_3'\ldots e'_{k-1}$. The resulting path-space $P(H)$
is then homeomorphic to $P(G)$ under a shift-commuting homeomorphism and the function $F$ would on $P(H)$ only depend on the initial vertex of the paths.

We extend $F_0$ to a map on finite paths such that
\[ F_0(\mu) = \sum_{i=1}^n F_0(s(e_i)), \]
when $\mu = e_1e_2 \cdots e_n$. For any vertex $v \in V_G$ we set
\[ C_v = \{ p \in P(G) : s(p) = v \}. \]

**Lemma 4.2.** Let $m$ be a regular Borel measure on $P(G)$. Then $m$ is $(\Gamma_\sigma, c_F)$-conformal with exponent $\beta$ if and only if
\[ m(Z(\mu)) = e^{-\beta F_0(\mu)} m(C(\mu)) \]
for every cylinder $Z(\mu)$.

**Proof.** To prove that condition (4.2) in the present setting implies (4.2), consider first an edge $e$ in $G$ and apply (4.2) to the bi-section
\[ W = \{ (p, 1, \sigma(p)) : p \in Z(e) \} \]
to conclude that (4.2) holds when $\mu = e$. We can then prove (4.2) by induction in the length $|\mu|$ of $\mu$: Assume that (4.2) holds when $|\mu| = n$. Let $\mu = e_1e_2 \cdots e_{n+1}$ be a path of length $n + 1$. The set
\[ U = \{ (p, 1, \sigma(p)) : p \in Z(\mu) \} \]
is a bi-section in $\Gamma_\sigma$ such that $s(U) = Z(e_2e_3 \cdots e_{n+1})$ and $r(U) = Z(\mu)$. It follows therefore from (4.2) that
\[ m(Z(\mu)) e^{\beta F_0(s(e_1))} = m(Z(e_2e_3 \cdots e_{n+1})). \]
By induction hypothesis $m(Z(e_2e_3 \cdots e_{n+1})) = \exp(-\beta \sum_{i=2}^{n+1} F_0(s(e_i))) m(C(e_{n+1}))$, and it follows therefore that $m(Z(\mu)) = e^{-\beta F_0(\mu)} m(C(\mu))$, as desired.

Conversely, assume that (4.2) holds. It follows straightforwardly that
\[ m(\sigma(Z(\mu))) = e^{\beta F_0(s(\mu))} m(Z(\mu)) \]
for every cylinder $Z(\mu)$, which means that (3.1) holds when $A$ is a cylinder. Consider then an open subset $V$ of $P(G)$ such that $\sigma : V \to P(G)$ is injective. Define Borel measures $\nu_1$ and $\nu_2$ on $V$ such that
\[ \nu_1(B) = m(\sigma(B)), \quad \nu_2(B) = \int_B e^{\beta F(x)} \, dm(x). \]
Since $\nu_1$ and $\nu_2$ agree on cylinder sets it follows from regularity that $\nu_1$ and $\nu_2$ agree on all Borel subsets of $V$. In particular, (3.1) holds for $V$, and it follows as in the first part of the proof of Lemma 3.2 that $m$ is $e^{\beta F}$-conformal. The same lemma then says that $m$ is $(\Gamma_\sigma, c_F)$-conformal.

\[ \square \]

Let $A = (A_{vw})$ be the adjacency matrix of $G$, i.e. for $v, w \in V_G$ we set
\[ A_{vw} = \# \{ e \in E_G : s(e) = v, \ r(e) = w \}. \]
In the following we let $\mathbb{R}^{V_G}$ denote the vector space of all functions $\xi : V_G \to \mathbb{R}$. Since $G$ is row-finite the adjacency matrix defines a linear map $A : \mathbb{R}^{V_G} \to \mathbb{R}^{V_G}$ in the usual way.
Proposition 4.3. The association \( m \mapsto m(C_v) \) is a bijective correspondence between \((\Gamma_\sigma, cF)\)-conformal Borel measures with exponent \( \beta \) and the non-zero elements \( \xi \in \mathbb{R}^{V_G} \) which satisfy the conditions
\[ \begin{align*}
& a) \quad \xi_v \geq 0 \text{ and } \\
& b) \quad \sum_{w \in V} A_{vw} \xi_w = e^{\beta F_0(v)} \xi_v
\end{align*} \]
for all \( v \in V_G \).

Proof. For any measure \( m \) on \( P(G) \) the vector \( \xi_v = m(C_v) \) trivially satisfies a). Property b) follows from (4.2) since it implies that \( m(C_v) = \sum_{\{ e \in E_G : s(e) = v \}} m(Z(\mu e)) = \sum_{w \in V} A_{vw} e^{-\beta F_0(v)} m(C_w) \).

If \( m_1 \) and \( m_2 \) are \((\Gamma_\sigma, cF)\)-conformal measures such that \( m_1(C_v) = m_2(C_v) \) for all \( v \in V_G \) it follows from Lemma 4.2 that \( m_1 \) and \( m_2 \) agree on cylinder sets, and then by regularity on every Borel subset. To complete the proof we must show that any element \( \xi \in \mathbb{R}^{V_G} \) which satisfies a) and b) comes from a \((\Gamma_\sigma, cF)\)-conformal measure. To see this set
\[ m(Z(\mu)) = e^{-\beta F_0(\mu)} \xi_r(\mu). \]
It follows from condition b) that
\[ \sum_{\{ e \in E_G : s(e) = r(\mu) \}} m(Z(\mu e)) = m(Z(\mu)). \]
Thanks to this relation we can for each fixed vertex \( v \) define a measure \( m \) on the algebra of sets generated by the cylinder sets in \( C_v \) such that \( m(C_v) = \xi_v \), and then standard measure theory methods applies to show that \( m \) extends to a regular Borel measure \( m \) on \( P(G) \), cf. e.g. [Pa]. □

Theorem 4.4. There is a bijective correspondence between the gauge-invariant \( \beta \)-KMS weights for \( \alpha^F \) and non-zero elements \( \xi \in \mathbb{R}^{V_G} \) satisfying conditions a) and b) of Proposition 4.3.

Proof. Combine Proposition 3.1 with Proposition 4.3. □

Corollary 4.5. There is a bijective correspondence between the gauge-invariant \( \beta \)-KMS states for \( \alpha^F \) and non-zero elements \( \xi \in \mathbb{R}^{V_G} \) satisfying conditions a) and b) of Proposition 4.3 plus the additional condition that
\[ c) \quad \sum_{v \in V_G} \xi_v = 1. \]

When \( F_0 \) is either strictly positive everywhere or strictly negative everywhere, both Theorem 4.4 and its corollary remain true with the word 'gauge-invariant' deleted, simply because all KMS weights are then automatically gauge-invariant. In that case Corollary 4.5 has been obtained by Carlsen and Larsen by different methods, [Ca].

4.2. KMS weights for the gauge action on graph algebras. In this section we specialize to the case where \( F_0 \) is constant 1 which means that the action \( \alpha^F \) becomes the gauge action \( \gamma \), defined such that \( \gamma_t(f)(p,k,q) = e^{ikt} f(p,k,q) \) when \( f \in C_c(\Gamma_\sigma) \). In this case the previous results take the following form.

Theorem 4.6. There is a bijective correspondence between the \( \beta \)-KMS weights for the gauge action on \( C^* (G) \) and non-zero elements \( \xi \in \mathbb{R}^{V_G} \) satisfying
a’) \( \xi_v \geq 0 \) and
b’) \( \sum_{w \in V_G} A_{vw} \xi_w = e^\beta \xi_v \)
for all \( v \in V_G \).

**Corollary 4.7.** There is a bijective correspondence between the \( \beta \)-KMS states for the gauge action on \( C^*(G) \) and non-zero elements \( \xi \in \mathbb{R}^{V_G} \) satisfying conditions a’) and b’) of Theorem 4.6 plus the additional condition c) of Corollary 4.5.

### 4.3. KMS weights for the gauge action when the graph is cofinal

Recall that \( G \) is cofinal when every vertex can reach every infinite path, i.e. for all \( v \in V_G \) and all \( p \in P(G) \) there is a finite path \( \mu \) in \( G \) such that \( s(\mu) = v \) and \( r(\mu) = s(p_k) \) for some \( k \in \mathbb{N} \). Recall also that simplicity of \( C^*(G) \) implies cofinality of \( G \), and that the converse is almost also true, cf. Proposition 5.1 in [BPRS]. As shown in [Th3] the theory of positive eigenvalues and eigenvectors of non-negative matrices, which is well known in the irreducible row-finite case, has a natural generalization to a class of matrices which comprises the adjacency matrix \( A \) of \( G \), provided \( G \) is cofinal. This allows us here to obtain a complete description of the possible \( \beta \)-values for which there is a \( \beta \)-KMS weight for the gauge action on \( C^*(G) \) when \( G \) is cofinal, and also to obtain a description of the corresponding eigenvectors of \( A \) and in this way the regular measures on \( P(G) \) which correspond to such \( \beta \)-KMS weights, at least in principle.

To formulate the results, let \( NW_G \) be the (possibly empty) set of vertexes in \( G \) that support a loop; i.e. \( v \in NW_G \) if and only if there is a finite path \( \mu \) in \( G \) such that \( s(\mu) = r(\mu) = v \). Assuming that \( G \) is cofinal, the set \( NW_G \) and the edges emitted from any element of \( NW_G \) constitute a (still possibly empty) irreducible subgraph of \( G \) which we call the non-wandering part of \( G \). It follows that if \( v \in NW_G \), the number

\[
\beta_0 = \log \left( \limsup_{n \to \infty} (A^n_{vv})^{\frac{1}{n}} \right),
\]
defined with the convention that \( \log \infty = \infty \), is independent of the choice of \( v \).

**Theorem 4.8.** Assume that \( G \) is cofinal and let \( \beta \in \mathbb{R} \).

1) Assume that the non-wandering part \( NW_G \) is empty. There is a \( \beta \)-KMS weight for the gauge action on \( C^*(G) \) for all \( \beta \in \mathbb{R} \).

2) Assume that the non-wandering part \( NW_G \) is non-empty and finite. There is a \( \beta \)-KMS weight for the gauge action on \( C^*(G) \) if and only if \( \beta = \beta_0 \).

3) Assume that the non-wandering part \( NW_G \) is non-empty and infinite. There is a \( \beta \)-KMS weight for the gauge action on \( C^*(G) \) if and only if \( \beta \geq \beta_0 \).

**Proof.** Combine Theorem 4.6 with Theorem 2.7 in [Th2].

When \( \beta = \beta_0 \) and \( \sum_{n=0}^{\infty} A^n_{vv} e^{-n\beta_0} = \infty \), which is automatic in case 2), the \( \beta_0 \)-KMS weight is unique up to scalar multiplication. In all other cases there can be more than one extremal \( \beta \)-KMS weight, reflecting that the positive \( e^\beta \)-eigenvectors of \( A \) are not generally a scalar multiply of each other, cf. [Th3]. In [Th3] it is shown how to generalize the well-known Poisson-Martin integral representation of the harmonic functions of a countable state Markov chain to give an integral representation of the positive \( e^\beta \)-eigenvectors of \( A \). This leads in particular to a description of the extremal rays in the set of positive \( e^\beta \)-eigenvectors for \( A \), and hence also, at least in principle, a description of the extremal \( \beta \)-KMS weights. We refer to Theorem 3.9 and Corollary 3.10 in [Th3].
In many cases the summability condition c) of Corollary 4.5 prevents the existence of $\beta$-KMS states, although there may be many $\beta$-KMS weights. See the last section for three different examples of this.

4.4. On the factor type of an extremal KMS weight for the gauge action. Given a weight $\psi$ on a $C^*$-algebra $A$ there is a GNS-type construction consisting of a Hilbert space $H_\psi$, a linear map $\Lambda_\psi : N_\psi \rightarrow H_\psi$ with dense range and a non-degenerate representation $\pi_\psi$ of $A$ on $H_\psi$ such that

- $\psi(b^* a) = \langle \Lambda_\psi(a), \Lambda_\psi(b) \rangle$, $a, b \in N_\psi$, and
- $\pi_\psi(a)\Lambda_\psi(b) = \Lambda_\psi(ab)$, $a \in A$, $b \in N_\psi$,

cf. [Ku], [KV1], [KV2]. A $\beta$-KMS weight $\psi$ on $A$ is extremal when the only $\beta$-KMS weights $\varphi$ on $A$ with the property that $\varphi(a) \leq \psi(a)$ for all $a \in A_+$ are scalar multiples of $\psi$, viz. $\varphi = s\psi$ for some $s > 0$.

**Lemma 4.9.** Let $A$ be a separable $C^*$-algebra and $\alpha$ a continuous one-parameter group of automorphisms on $A$. Let $\psi$ be an extremal $\beta$-KMS weight for $\alpha$. Then $\pi_\psi(A)^\sigma$ is a factor.

**Proof.** It follows from [KV1] that $\psi$ extends to normal faithful semi-finite weight $\tilde{\psi}$ on $\pi_\psi(A)^\sigma$ such that $\tilde{\psi} \circ \pi_\psi = \psi$, and that $t \mapsto \alpha_{-t} \beta$ extends to a $\sigma$-weakly continuous action $\theta$ on $\pi_\psi(A)^\sigma$ which is the modular automorphism group on $\pi_\psi(A)^\sigma$ associated to $\tilde{\psi}$. Let $e$ be a non-zero central projection in $\pi_\psi(A)^\sigma$. Since $\theta$ acts trivially on the center it follows that $\theta_t(e) = e$ and it is then straightforward to verify that

$$A_+ \ni a \mapsto \tilde{\psi}(e\pi_\psi(a))$$

is $\beta$-KMS weight on $A$. Since $\psi$ is extremal there is an $s > 0$ such that $\tilde{\psi}(e\pi_\psi(a)) = s \psi(a) = \tilde{\psi}(s \pi_\psi(a))$ for all $a \in A_+$. This implies that $(e - s)\Lambda_\psi(a) = \Lambda_\psi(a)$ for all $a \in N_\psi$, and hence that $e = s = 1$. \hfill $\square$

The next aim will be to determine the Connes invariant $\Gamma(\pi_\psi(C^*(G))^\sigma)$, assuming that $\psi$ is an extremal $\beta$-KMS weight for the gauge action on $C^*(G)$. For this note that it follows from Section 2.2 in [KV1] that $\psi$ extends to a normal semi-finite faithful weight $\tilde{\psi}$ on $\pi_\psi(C^*(G))^\sigma$ such that $\psi = \tilde{\psi} \circ \pi_\psi$, and that the modular group on $\pi_\psi(C^*(G))^\sigma$ corresponding to $\tilde{\psi}$ is the one-parameter group $\theta$ on $\pi_\psi(C^*(G))^\sigma$ given by

$$\theta_t = \tilde{\gamma}_{-\beta t},$$

where $\tilde{\gamma}$ is the $\sigma$-weakly continuous extension of $\gamma$ defined such that $\tilde{\gamma}_t \circ \pi_\psi = \pi_\psi \circ \tilde{\gamma}_t$. To simplify the notation in the following, we set $M = \pi_\psi(C^*(G))^\sigma$ and let $N \subseteq M$ be the fixed point algebra of $\theta$, viz. $N = M^\theta$. Since $\theta_t = \theta_{t + 2\pi R}$ we can define an action $\tilde{\theta}$ by the circle such that

$$\tilde{\theta}_{t+2\pi} = \theta_{t}.$$  

for all $t \in \mathbb{R}$. Note that $\tilde{\theta}(C^*(G)) = C^*(G)$ and that the Arveson spectrum $\text{Sp}(qMq)$ of the restriction of $\tilde{\theta}$ to $qMq$ is a subset of $\mathbb{Z}$ for all projections $q \in N$. Let $v$ be a vertex in $G$ and $1_v \in C_c(P(G))$ the characteristic function of the set $C_v$ from (4.1). For simplicity of notation we also write $1_v$ for the image $\pi_\psi(1_v) \in N$. By combining
Definition 2.2.1 in [C1] with Lemma 3.4.3, Proposition 2.2.2 and Lemme 2.3.3 in 
[C1] it follows that
\[ \Gamma(M) = \beta \bigcap_{e} \text{Sp}(e1_{v}M1_{v}e) \]  
(4.3)
where \( v \in V_{G} \) can be any vertex and where we take the intersection over all non-zero central projections \( e \) in \( 1_{v}N1_{v} \).

Let \( \mathcal{P}(G) \) be the set of finite paths in \( G \). For every \( v \in V \) we set
\[ \Delta_v = \{|\mu| - |\nu| : \mu, \nu \in \mathcal{P}(G) \ s(\mu) = s(\nu) = v, \ r(\mu) = r(\nu)\}. \]
Then \( \bigcap_{v \in V} \Delta_v \) is a subgroup of \( \mathbb{Z} \) and there is a unique natural number \( d_G \) such that \( \bigcap_{v \in V} \Delta_v = d_G \mathbb{Z} \).

**Proposition 4.10.** Let \( \psi \) be an extremal \( \beta \)-KMS weight for the gauge action on \( C^{*}(G) \). Then \( \pi_{\psi}(C^{*}(G))'' \) is a hyper-finite factor such that
\[ \Gamma(\pi_{\psi}(C^{*}(G))'') \subseteq \mathbb{Z}d_G \beta. \]

**Proof.** That \( \pi_{\psi}(C^{*}(G))'' \) is hyper-finite follows from the nuclearity of \( C^{*}(G) \), cf. [KPRR]. Consider a vertex \( v \in V_{G} \) and let \( k \in \text{Sp}(1_{v}M1_{v}) \). There is an element \( a \in 1_{v}M1_{v} \) such that
\[ \int_{T} \lambda^{-k} \tilde{\theta}_{\lambda}(a) \ d\lambda = a \neq 0. \]
The density of \( 1_{v}C_{c}(\Gamma_{\sigma})1_{v} \) in \( 1_{v}M1_{v} \) for the \( \sigma \)-weak topology implies that there is an element \( b \in 1_{v}C_{c}(\Gamma_{\sigma})1_{v} \) such that
\[ \int_{T} \lambda^{-k} \tilde{\theta}_{\lambda}(b) \ d\lambda \neq 0. \]
Then \( f = \int_{T} \lambda^{-k} \tilde{\theta}_{\lambda}(b) \ d\lambda \in 1_{v}C_{c}(\Gamma_{\sigma})1_{v} \) is an element such that \( \theta_{\lambda}(f) = \lambda^{k}f \) for all \( \lambda \in T \). It follows that \( f \) is supported in
\[ \Gamma_{\sigma}(-k) \cap s^{-1}(C_{v}) \cap r^{-1}(C_{v}), \]
which must therefore be non-empty. Let \( (p, -k, p') \) be an element in this set. Then \( \sigma^{n-k}(p) = \sigma^{n}(p') \) for all large \( n \). In particular, for \( n \) large enough \( \mu = p'_{1}p'_{2} \cdots p'_{n} \) and \( \nu = p_{1}p_{2} \cdots p_{n-k} \) are paths in \( G \) such that \( s(\mu) = s(\nu) = v, \ r(\nu) = r(\mu) \), and \( |\mu| - |\nu| = k \). Since \( \nu \) was arbitrary, \( k \in d_G \mathbb{Z} \). This gives the stated inclusion, thanks to [LK]. \( \square \)

Let \( \mathbb{P} \) be set of integers \( d \in \mathbb{Z} \) with the property that there is a non-empty hereditary set \( H \subseteq V_{G} \) and natural numbers \( M, L \in \mathbb{N} \) such that for every path \( \mu \) in \( H \) of length \( M \) there are paths \( l_{+} \) and \( l_{-} \) with lengths \( |l_{+}| \leq L, \ |l_{-}| \leq L \), and \( s(l_{+}) = s(l_{-}) = s(\mu), \ r(l_{+}) = r(l_{-}) = r(\mu) \), such that
\[ d = |l_{+}| - |l_{-}|. \]

If \( G \) is cofinal the intersection of two non-empty hereditary subsets of \( V_{G} \) is again non-empty and hereditary. Therefore the set \( \mathbb{P} \) is a subgroup of \( \mathbb{Z} \) in this case, and we can define \( d'_{G} \in \mathbb{N} \) as the unique natural number such that \( \mathbb{P} = d'_{G} \mathbb{Z} \).

**Proposition 4.11.** Let \( \psi \) be an extremal \( \beta \)-KMS weight for the gauge action on \( C^{*}(G) \). Assume that \( G \) is cofinal and that \( G \) has uniformly bounded out-degree, i.e. \( \sup_{v \in V_{G}} \#s^{-1}(v) < \infty \). Then \( \pi_{\psi}(C^{*}(G))'' \) is a hyper-finite factor such that
\[ \mathbb{Z}d'_{G} \beta \subseteq \Gamma(\pi_{\psi}(C^{*}(G))'') \subseteq \mathbb{Z}d_{G} \beta. \]
Proof. Consider a vertex $v$ and a non-zero central projection $q$ in $1_vN_1v$. In view of Proposition 1.10 and 1.3, it suffices to show that $d'_G \in \text{Sp}(qMq)$. Since $d'_G \in \mathbb{P}$ there is a non-empty hereditary set $H \subseteq V_G$ and natural numbers $M, L \in \mathbb{N}$ such that for every path $\mu$ in $H$ of length $M$ there are paths $l_{\pm}$ with lengths $|l_{\pm}| \leq L$, $s(l_{\pm}) = s(\mu)$, $r(l_{\pm}) = r(\mu)$ such that $d'_G = |l_+| - |l_-|$. It follows from Lemma 3.7 in [Th3] that because $G$ is cofinal and $H$ hereditary there is an $N \in \mathbb{N}$ such that every path of length $\geq N$ emitted from $v$ terminates in $H$.

Observe that the fixed point algebra $C^*(G)^{\gamma}$ of the gauge action is the reduced groupoid $C^*$-algebra of the closed and open sub-groupoid $\Gamma_{\sigma}(0)$ of $\Gamma_{\sigma}$. The corner $1_vC^*(G)^{\gamma}1_v$ is the reduced groupoid $C^*$-algebra of

$$R_v = \bigcup_{n \in \mathbb{N}} \{(p, p') \in P(G) \times P(G) : s(p) = s(p') = v, \ p_i = p'_i, \ i \geq n\}.$$  

For $n \in \mathbb{N}$, let $P_v(n)$ be the set of paths $\mu$ in $G$ of length $n$ such that $s(\mu) = v$. For every pair $(\mu, \mu') \in P_v(n) \times P_v(n)$, let $e_{\mu, \mu'}^n \in C_c(R_v)$ be the characteristic function of the set

$$\{(p, p') \in Z(\mu) \times Z(\mu') : p_i = p'_i, \ i \geq n + 1\}.$$  

Then $\{e_{\mu, \mu'}^n : \mu, \mu' \in P_v(n)\}$ generates a finite dimensional $\ast$-subalgebra $A_n$ of $C_c(R_v)$ such that

$$1_vC^*(G)^{\gamma}1_v = \bigcup_{n \in \mathbb{N}} A_n.$$  

It follows that $\bigcup_{n \in \mathbb{N}} A_n$ is dense in $1_vN_1v$ for the strong operator topology. (Here and in the following we suppress the representation $\pi_\psi$, and regard $C^*(G)$ as acting on $H_\psi$.) Since $0 < \psi(1_v) < \infty$ there is a normal state $\omega_v$ on $1_vM_1v$ such that

$$\omega_v(a) = \psi(1_v)^{-1} \tilde{\psi}(a).$$  

Note that $\omega_v$ is a trace on $1_vN_1v$. Let $\| \cdot \|_v$ be the corresponding 2-norm,

$$\|a\|_v = \sqrt{\omega_v(a^*a)},$$  

$a \in 1_vM_1v$. For any $\epsilon > 0$ there is an $n \in \mathbb{N}$ and an element $a \in A_n$ such that $\|a\| \leq 1$ and $\|q - a\|_v \leq \epsilon$. We shall require, as we can, that $n \geq N$, and to specify how small an $\epsilon$ we want, let $B$ be a uniform upper bound for the out-degree in $G$, i.e. $\#s^{-1}(v) \leq B$ for all $v \in V_G$, and set

$$K_1 = \frac{\min\{e^{(M-L)\beta}, e^{M\beta}\}}{MB}, \ K_2 = \min\{e^{-d_0\beta}, 1\}.$$  

We choose $\epsilon > 0$ so small that $\epsilon < 10^{-2}$ and

$$4e^{\frac{1}{4}} + 2e^{\frac{1}{2}} K_2 + 2e^{\frac{1}{4}} K_1 K_2 < K_1 K_2 \omega_v(q).$$  

Let $U(A_n)$ be the unitary group of $A_n$ and set

$$b = \int_{U(A_n)} uau^* \ du,$$  

where we integrate with respect to the Haar measure $du$ on $U(A_n)$. Then $b$ is in the center of $A_n$ and we have the estimate

$$\|b - q\|_v \leq \int_{U(A_n)} \|uau^* - uqu^*\|_v \ du \leq \epsilon.$$
Standard arguments, e.g. as in the proof of Lemma 12.2.3 in [KR], shows that there is a central projection \( p \) in \( A_n \) such that
\[
\|p - q\| \leq 2e^{\frac{1}{N}}. \tag{4.4}
\]
By definition of \( A_n \) there is a finite set \( F \) of vertexes in \( G \) such that
\[
p = \sum_{w \in F} p_w,
\]
where \( p_w \) is the characteristic function of the set consisting of the elements \((u, v)\).

Note that \( F \subseteq H \) since \( n \geq N \). For each \( w \in F \) we can therefore choose a path \( \mu(w) \) in \( H \) such that \( s(\mu(w)) = w \), \( |\mu(w)| = M \) and
\[
\omega_v(p_{\mu(w)}) \geq \frac{\omega_v(p_w)}{MB}, \tag{4.5}
\]
where \( p_{\mu(w)} \) is the characteristic function of the set
\[
\{ p \in P(G) : s(p) = v, r(p_n) = w, p_{n+1}p_{n+2} \cdots p_{n+M} = \mu(w) \}.
\]

Note that \( p_{\mu(w)} \leq p_w \). For each \( w \) there are paths \( l^w_\pm \) such that \( s(l^w_\pm) = w \), \( r(l^w_\pm) = r(\mu(w)) \), \( |l^w_\pm| \leq L \) and \( d^w_G = |l^w_\pm| - |l^w| \). Let \( u^\pm(w) \in C_c(\Gamma_\sigma) \) be the characteristic function of the set consisting of the elements \((p, M - |l^w_\pm|, p') \in \Gamma_\sigma \) with the properties:
- \( s(p) = s(p') = v \),
- \( r(p_n) = r(p'_n) = w \),
- \( p_i = p'_i, i = 1, 2, \ldots, n \),
- \( p_{n+1}p_{n+2} \cdots p_{n+M} = \mu(w) \),
- \( p'_{n+1}p'_{n+2} \cdots p'_{n+M} = l^w_\pm \),
- \( p_{n+M+i} = p'_{n+M+i} = l^w_\pm |i|, i \geq 1 \).

Then \( u^\pm(w) \) are partial isometries such that

a) \( u^\pm(w)u\mp(w) = p_{\mu(w)} \), \( u^\pm(w)u\mp(w) \leq p_w \),

b) \( \gamma_t(u^\pm(w)) = e^{(M-|l^w_\pm|)h}u^\pm(w) \), \( t \in \mathbb{R} \).

By combining b) with the fact that \( \psi \) is a \( \beta \)-KMS weight for \( \gamma \), we find that
\[
\omega_v(u^\pm(w)^*u_\mp(w)qu_\mp(w)^*u_\pm(w)) = e^{-d^\beta} \omega_v(qu_\mp(w)^*u_\mp(w))u_\pm(w)u^\pm(w). \tag{4.6}
\]

It follows from \( a \) that \( u^\pm(w)u_\pm(w)u_\mp(w) = u_\pm(w) \) and we get then the identity
\[
\omega_v(u^\pm(w)^*u_\mp(w)qu_\mp(w)^*u_\pm(w)) = e^{-d^\beta} \omega_v(qu_\mp(w)^*u_\mp(w)). \tag{4.6}
\]

Set \( u_\pm = \sum_{w \in F} u_\pm(w) \). It follows from \( a \) that \( u_\pm \) are partial isometries such that
\[
u_\pm^*u_\pm = \sum_{w \in F} u_\pm(w)^*u_\pm(w) \leq p. \tag{4.7}
\]

By using that \( \psi \) is a KMS weight it follows that \( \omega_v(u_\pm(w)u_\mp(w)qu_\mp(w)^*u_\pm(w)) = 0 \) when \( w \neq w' \), and by combining with \( (4.6) \) we find that
\[
\omega_v(u_\pm^*u_\mp qu_\mp^*u_\pm) = e^{-d^\beta} \omega_v(qu_\mp^*u_\pm). \tag{4.8}
\]

By a similar reasoning we find by use of \( a \) and \( (4.5) \) that
\[
\omega_v(u_\mp(w)^*u_\pm(w)) = e^{(M-|l^w_\pm|)h} \omega_v(u_\pm(w)u_\mp(w)^*) \geq K_1 \omega_v(p_w).
\]
Summing over $w \in F$ gives that

$$\omega_v(u^*_+ u^-) \geq K_1 \omega_v(p).$$  \hspace{1cm} (4.9)

We can now conclude that

$$\omega_v(qu^*_+ u^-qu^*_+ u^+) \geq \omega_v(u^*_+ u^-qu^*_+ u^+) - 4\epsilon^\frac{1}{4} \quad \text{(using (4.4))}$$

$$= -4\epsilon^\frac{1}{4} + e^{-d_G \beta} \omega_v(qu^*_+ u^-) \quad \text{(using (4.3))}$$

$$\geq -4\epsilon^\frac{1}{4} - 2\epsilon^\frac{1}{4} K_2 + K_2 \omega_v(u^*_+ u^-) \quad \text{(using (4.4) and (4.7))}$$

$$\geq -4\epsilon^\frac{1}{4} - 2\epsilon^\frac{1}{4} K_2 + K_1 K_2 \omega_v(p) \quad \text{(using (4.9))}$$

$$\geq -4\epsilon^\frac{1}{4} - 2\epsilon^\frac{1}{4} K_2 - 2\epsilon^\frac{1}{4} K_1 K_2 + K_1 K_2 \omega_v(q) \quad \text{(using (4.4)).}$$

Thanks to the choice of $\epsilon$ this implies that $qu^*_+ u^-q \neq 0$. Since $\tilde{\theta}_\lambda(qu^*_+ u^-q) = \lambda^{-d_G} qu^*_+ u^-q$, we conclude that $d'_G \in \text{Sp}(qMq)$, as desired. \hfill \square

When $G$ is a finite irreducible graph, and more generally when $G$ is cofinal and $\text{NW}_G$ is finite, the two numbers $d'_G$ and $d_G$ are both equal to the global period of $\text{NW}_G$. For a finite irreducible graph Proposition 4.11 recovers that part of the results from [O] which deals with the gauge action. As will be shown in Example 5.2 and Example 5.3 below, there are other cases where the two numbers $d'_G$ and $d_G$ agree and where Proposition 4.11 therefore determines the $\Gamma$-invariant of $\pi_v(C^*(G))''$. However, it is also easy to construct examples of infinite irreducible graphs, as the one presented in Example 5.1 below, where they differ. I have no idea what the $\Gamma$-invariant is in such cases.

5. Examples

The first two examples are intended to show that in some cases it is quite easy to find all the solutions to the equations and hence identify all KMS weights for the gauge action. The third example is meant to show how methods and results from the theory of countable state Markov chains in some cases can be used for the same purpose. A common feature is the scarcity of the KMS states compared to KMS weights.

Example 5.1. Consider the following graph with labeled vertexes:
For this graph it is quite easy to see that a map $\xi : V_G \to [0, \infty)$ which is normalized such that $\xi_1 = 1$, is a solution to the equation $b')$ of Theorem 4.6 exactly when

i) $\xi_a^1 + \xi_b^1 + \xi_c^1 = e^\beta$,

ii) $\xi_{a-n} = \xi_{b-n} = \xi_{c-n} = e^{-\beta n}$, $n = 1, 2, 3, \ldots$, and

iii) $\xi_{a_{n+1}} + e^{-n\beta} = e^{\beta} \xi_{a_n}$, $\xi_{b_{n+1}} + e^{-n\beta} = e^{\beta} \xi_{b_n}$, $\xi_{c_{n+1}} + e^{-n\beta} = e^{\beta} \xi_{c_n}$, $n \geq 1$

It follows that

$$\xi_{a_{n+1}} = e^{n\beta} \left( \xi_{a_1} - \sum_{j=1}^{n} (e^{-2\beta})^j \right), n \geq 1,$$

combined with similar formulas involving the $b_n$'s and $c_n$'s. The positivity requirement on $\xi$ implies that $\beta > 0$ and that

$$\min \{\xi_{a_1}, \xi_{b_1}, \xi_{c_1} \} \geq \sum_{j=1}^{\infty} (e^{-2\beta})^j = \frac{e^{-2\beta}}{1 - e^{-2\beta}}.$$

Combined with condition i) it follows that $3 \frac{e^{-2\beta}}{1 - e^{-2\beta}} \leq e^\beta$, which means that $\beta \geq \log \alpha \sim 0.5138$, where $\alpha$ is the real root of the polynomial $x^3 - x - 3$. For $\beta = \log \alpha$ there is a unique solution, and hence there is a unique $\beta$-KMS weight for the gauge action in this case, up to scalar multiplication. For all values of $\beta > \log \alpha$ the set of $\beta$-KMS weights form a cone with a triangle as base. The extreme rays of the cone correspond to the three cases where

$$\{\xi_{a_1}, \xi_{b_1}, \xi_{c_1} \} = \left\{ \frac{e^{-2\beta}}{1 - e^{-2\beta}}, e^\beta - \frac{2e^{-2\beta}}{1 - e^{-2\beta}} \right\}.$$
Note that only for the unique solution with $\beta = \log \alpha$ is the sum $\sum_{v \in V} \xi_v$ finite. It follows that only for this value of $\beta$ is there a KMS state, and it is then unique.

Concerning the $\Gamma$-invariant of $\pi_\psi(C^*(G))''$ when $\psi$ is an extremal $\beta$-KMS weight, we observe that in the present example, $d_G = 1$ while $d_G' = 0$. Hence the results from Section 4.4 tell us only that $\Gamma(\pi_\psi(C^*(G))'' \subseteq \mathbb{Z}\beta$.

**Example 5.2.** Consider the following graph, again with a convenient labeling of the vertexes:

![Graph Image]

The solution $\xi$ to the eigenvalue equations a’) and b’) of Theorem 4.6 which is normalized to take the value 1 at the vertex in the lower left hand corner, is unique for all $\beta \in \mathbb{R}$ and is given by

$$\xi_{x_n} = \frac{e^{(2n+1)\beta}}{(1 + e^\beta)^{n+1}}, \quad \xi_{y_n} = \left(\frac{e^{2\beta}}{1 + e^\beta}\right)^{n+1}, \quad n = 0, 1, 2, \ldots$$

Note that the corresponding $\beta$-KMS weight can be normalized to be a state (i.e. equation c) of Theorem 4.5 can be made to hold) if and only if $\beta < \log \left(\frac{1 + \sqrt{5}}{2}\right)$. Thus a $\beta$-KMS state exists only for these values of $\beta$.

Concerning the $\Gamma$-invariant of $\pi_\psi(C^*(G))''$ when $\psi$ is an extremal $\beta$-KMS weight, we observe that in the present example, $d_G' = d_G = 1$. It follows therefore from Proposition 4.11 that $\pi_\psi(C^*(G))''$ is the hyper-finite type $\text{III}_{\lambda}$-factor where $\lambda = e^{-|\beta|}$, when $\beta \neq 0$. It is not difficult to see that for $\beta = 0$ it is the hyper-finite $\text{II}_\infty$ factor.

**Example 5.3.** There are many results in the literature which in specific cases can be used to find the solutions of the eigenvalue equations a’) and b’) in Theorem 4.6. Most results about positive eigenvalues and eigenvectors are motivated by applications to Markov chains and therefore concerned with the case where the matrix is stochastic in the sense that $\sum_{w \in V} A_{vw} = 1$ for all $v \in V$, and with the case $\beta = 0$. The transition matrix of a directed graph is rarely stochastic, but nonetheless these results are highly relevant here since a given solution $\xi$ to a’) and b’) give rise
to the stochastic matrix
\[ B_{vw} = e^{-\beta} \xi_{vw}^{-1} \xi_w A_{vw}, \]
and the probabilistic methods and results for stochastic matrices can then be used to find all other solutions. The following is an example of this.

Let \( \mu : \mathbb{Z}^d \to \mathbb{N} \) be a finitely supported map. We define a matrix \( A_{vw}, v, w \in \mathbb{Z}^d \), such that
\[ A_{vw} = \mu(w - v). \]
\( A \) is the adjacency matrix of a graph \( G \) with \( \mathbb{Z}^d \) as vertex set. Assume that the semi-group generated by the support of \( \mu \) is all of \( \mathbb{Z}^d \). Then \( A \) is irreducible (i.e. for all \( v, w \in \mathbb{Z}^d \) there is an \( n \in \mathbb{N} \) such that \( A_{vw}^n > 0 \)) and the graph \( G \) satisfies conditions in 3) of Theorem \( 4.8 \). We seek therefore first the value \( \beta_0 \); the smallest \( \beta \)-value for which there is a positive \( e^\beta \)-eigenvector for \( A \), and for this the methods and results from Chapter 8 in \( [Wo] \) can be adopted.

For every \( c \in \mathbb{R}^d \), define \( f_c : \mathbb{Z}^d \to [0, \infty[ \) such that
\[ f_c(v) = \exp(<c, v>), \ v \in \mathbb{Z}^d, \]
where \( <\cdot, \cdot> \) is the inner product in \( \mathbb{R}^d \). It is straightforward to check that \( f_c \) is a positive \( e^\beta \)-eigenvector for \( A \), where
\[ \beta = \log \left( \sum_{w \in \mathbb{Z}^d} \mu(w) \exp(<c, w>) \right). \]  
(5.2)

To see that the \( f_c \)'s generate all positive eigenvectors of \( A \), consider an arbitrary \( \beta \in \mathbb{R} \) and let \( \psi \) be a positive \( e^\beta \)-eigenvector for \( A \), extremal among those that take the value 1 at 0. For \( u \in \mathbb{Z}^d \), set \( \psi_u(v) = \psi(u)^{-1} \psi(u + v) \). Then \( \psi_u \) is also a positive \( e^\beta \)-eigenvector. Note that
\[ \psi(v) = \sum_{w \in \mathbb{Z}^d} e^{-\beta} \mu(w - v) \psi(w) = \sum_{w \in \mathbb{Z}^d} e^{-\beta} \mu(w) \psi(w) \psi_w(v). \]

Since \( \sum_{w \in \mathbb{Z}^d} e^{-\beta} \mu(w) \psi(w) = \psi(0) = 1 \), the extremality of \( \psi \) implies that \( \psi_w = \psi \) for all \( w \) in the support of \( \mu \). That is, \( \psi(v + w) = \psi(v) \psi(w) \) for all \( v \in \mathbb{Z}^d \) and all \( w \) in the support of \( \mu \). Since this support generates \( \mathbb{Z}^d \) by assumption, it follows that \( \psi(v + w) = \psi(v) \psi(w), v, w \in \mathbb{Z}^d \), which means that there is a vector \( c \in \mathbb{R}^d \) such that \( \psi = f_c \). The relation between \( c \) and \( \beta \) is given by (5.2). As shown in [Th3] the set of positive \( e^\beta \)-eigenvectors that are normalized to take the value 1 at 0 constitute a compact Choquet simplex, so it follows now that they are all contained in the closed convex hull of a set of \( f_c \)'s. It follows in particular from this that \( e^{\beta_0} \) is the minimal value of the strictly convex function
\[ c \mapsto \sum_{w \in \mathbb{Z}^d} \mu(w) \exp(<c, w>) \]
on \( \mathbb{R}^d \). Taking the gradient of this function we see that
\[ \beta_0 = \log \left( \sum_{w \in \mathbb{Z}^d} \mu(w) \exp(<c_{\text{min}}, w>) \right) \]
where \( c_{\text{min}} \) is the unique solution to the equation
\[ \sum_{w \in \mathbb{Z}^d} \mu(w) \exp(<c, w>)w = 0. \]
By passing from $A$ to the matrix $B$ as in (5.1) we can apply a result of Ney and Spitzer, stated as Theorem 8.15 in [Wo], to conclude that the extremal rays in the cone of positive $e^\beta$-eigenvectors are in one-to-one correspondence with the points on the unit sphere $S^{d-1}$ in $\mathbb{R}^d$ for all $\beta \geq \beta_0$, except when $\sum_{w \in \mathbb{Z}^d} \mu(w)w = 0$, in which case there is only one ray, i.e. the positive $e^{\beta_0}$-eigenvector is unique up to scalar multiplication. In particular, the present example includes cases where the base simplex for the cone of $\beta$-KMS weights of the gauge action is the same for all $\beta \geq \beta_0$, namely the simplex of Borel probability measures on $S^{d-1}$, as well as examples where this is only the case for $\beta > \beta_0$ and the simplex collapses to a point when $\beta$ hits its lowest possible value, as it was the case in the first example.

Concerning KMS states we observe that in the present example there are no $\beta$-KMS states at all. To see this consider a positive $e^\beta$-eigenvector $\xi$ for some $\beta \geq \beta_0$. Then $e^\beta = \sum_{w \in \mathbb{Z}^d} \mu(w) \exp(<c',w>)$ for some $c' \in \mathbb{R}^d$ and it follows from Corollary 8.10 in [Wo] that

$$\xi_v = \int_{\mathbb{R}^d} \exp(<c + c',v>) \, d\mu(c)$$

for some Borel measure $\mu$ on $\mathbb{R}^d$. Since $\sum_{v \in \mathbb{Z}^d} \exp(<c+c',v>) = \infty$ for all $c \in \mathbb{R}^d$, it follows that $\sum_{v \in \mathbb{Z}^d} \xi_v = \infty$, i.e. there are no positive $e^\beta$-eigenvectors for $A$ satisfying condition c) in Corollary 1.7.

Concerning the $\Gamma$-invariant of $\pi_\psi(C^*(G))''$ when $\psi$ is an extremal $\beta$-KMS weight, we observe that because $A$ is translation invariant, in the sense that $A_{uv} = A_{u+uv+u}$ for all $u, v, w \in \mathbb{Z}^d$, the two numbers $d_G'$ and $d_G$ from Section 4.4 agree. It follows therefore from Proposition 4.11 that $\pi_\psi(C^*(G))''$ is the hyper-finite type $III_\lambda$-factor where $\lambda = e^{-d_G'}$ for all $\beta \geq \beta_0$, cf. [C2].

References

[An] C. Anantharaman-Delaroche, Purely infinite $C^*$-algebras arising from dynamical systems, Bull. Soc. Math. France 125 (1997), 199–225.

[BPRS] T. Bates, D. Pask, I. Raeburn and W. Szymanski, The $C^*$-algebra of Row-Finite Graphs, New York Jour. of Math. 6 (2000), 307-324.

[BR] O. Bratteli and D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics I + II, Texts and Monographs in Physics, Springer Verlag, New York, Heidelberg, Berlin, 1979 and 1981.

[Ca] T.M. Carlsen, Talk at the Operator Algebra and Dynamics Conference, Faroe Islands, May 2012.

[C] F. Combes, Poids associé à une algèbre hilbertienne à gauche, Compos.Math. 23 (1971), 49-77.

[C1] A. Connes, Une classification des facteurs de type III, Ann. Sci. Ecole Norm. Sup. 6 (1973), 133-252.

[C2] A. Connes, Classification of injective factors, Ann. Math. 104 (1976), 73-115.

[De] V. Deaconu, Groupoids associated with endomorphisms, Trans. Amer. Math. Soc. 347 (1995), 1779-1786.

[DU] M. Denker and M. Urbanski, On the Existence of Conformal Measures, Trans. Amer. Math. Soc. 328 (1991), 563-587.

[EFW] M. Enomoto, M. Fujii and Y. Watatani, KMS states for gauge action on $O_A$, Math. Japon. 29 (1984), 607-619.

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3 When $d \in \{1,2\}$ and $\sum_w \mu(w)w = 0$, Theorem 8.15 in [Wo] does not apply because of the transience condition. In these cases the essential uniqueness of the positive $e^{\beta_0}$-eigenvector must be derived from the results of Vere-Jones, [V], dealing with the recurrent case.
[EL] R. Exel and M. Laca, *Partial dynamical systems and the KMS condition*, Comm. Math. Phys. **232** (2003), 223-277.

[aHLRS] A. an Huef, M. Laca, I. Raeburn and A. Sims, *KMS states on the C*-algebras of a finite graph*, J. Math. Anal. Appl. **405** (2013), 388-399.

[KR] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras II*, Academic Press, London 1986.

[KW] T. Kajiwara and Y. Watatani, *KMS states on finite-graph C*-algebras*, arXiv:1007.4248.

[KPRR] A. Kumjian, D. Pask, I. Raeburn and J. Renault, *Graphs, Groupoids, and Cuntz-Krieger algebras*, J. Math. Anal. Appl. **405** (2013), 388-399.

[KR] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras II*, Academic Press, London 1986.

[KP] J. Kustermans, *KMS-weights on C*-algebras*, arXiv: 9704008v1.

[KV1] J. Kustermans and S. Vaes, *Weight theory for C*-algebraic quantum groups*, arXiv:990163.

[KV2] J. Kustermans and S. Vaes, *Locally compact quantum groups*, Ann. Scient. Éc. Norm. Sup. **33**, 2000, 837-934.

[N] S. Neshveyev, *KMS states on the C*-algebras of non-principal groupoids*, J. Operator Theory, to appear, arXiv:1106.5912v1.

[O] R. Okayasu, *Type III factors arising from Cuntz-Krieger algebras*, Proc. Amer. Math. Soc. **131** (2003), 2145-2153.

[OP] D. Olesen and G.K. Pedersen, *Some C*-dynamical systems with a single KMS-state*, Math. Scand. **42** (1978), 111-118.

[Pa] K. R. Parthasarathy, *Introduction to Probability and Measure*, Macmillan India Press, 1977.

[P] W.E. Pruitt, *Eigenvalues of non-negative matrices*, Ann. Math. Statist. **35** (1964), 1797-1800.

[Re1] J. Renault, *A Groupoid Approach to C*-algebras*, LNM 793, Springer Verlag, Berlin, Heidelberg, New York, 1980.

[Re2] J. Renault, *AF equivalence relations and their cocycles*, Operator algebras and mathematical physics (Constanta, 2001), 365-377, Theta, Bucharest, 2003.

[Th1] K. Thomsen, *Semi-étale groupoids and applications*, Annales de l’Institute Fourier **60** (2010), 759-800.

[Th2] K. Thomsen, *On the C*-algebra of a Locally Injective Surjection and its KMS states*, Comm. Math. Phys. **302** (2011), 403-423.

[Th3] K. Thomsen, *On the positive eigenvalues and eigenvectors of a non-negative matrix*, Preprint, June 2013.

[V] D. Vere-Jones, *Ergodic properties of non-negative matrices I*, Pacific J. Math. **22** (1967), 361-386.

[Wo] W. Woess, *Denumerable Markov Chains*, EMS Textbooks in Mathematics, 2009.

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