Dynamical truncation of the string spectrum at finite $N$

Gilad Lifschytz and Vipul Periwal

Department of Physics
Joseph Henry Laboratories
Princeton University
Princeton, NJ 08544, U.S.A.

gilad@viper.princeton.edu, vipul@mail.princeton.edu

Abstract

We exhibit a nonperturbative background independent dynamical truncation of the string spectrum and a quantization of the string coupling constant directly from the Hamiltonian governing the dynamics of strings constructed from Yang-Mills theories.
1 Introduction

The relation between Yang-Mills theories and string theories has been a leitmotif in particle physics ever since Mandelstam[1] first noted the significance of Wilson loops as observables in Yang-Mills theories[2, 3]. Schwinger-Dyson equations satisfied by Wilson loop observables in the gauge theory were found to have natural interpretations in terms of geometric operations on strings, such as string splitting and joining[3]. With $N$ the rank of the gauge group $U(N)$ in the Yang-Mills theory, the topological classification of Feynman diagrams in the 't Hooft large $N$ limit[4] in Yang-Mills theory gave a further reason for expecting that strings should have some significance in gauge theories. In this limit the power of $N^{-2}$ accompanying a given Feynman diagram depends only on the topology of the surface defined by the double line representation of that diagram. The Schwinger-Dyson loop equations in the large $N$ limit also simplify—strings can no longer join, and the equations are satisfied by the expectation values which factorize in this limit.

The AdS↔CFT conjecture[7] gives a geometric picture of the large $N$ limit of maximally supersymmetric Yang-Mills theory in four dimensions. In a sense made precise in [8, 9] it is equal to the classical type IIB supergravity theory compactified on $AdS_5 \times S^5$. For finite $N$ the correspondence is rather unclear. A natural extension of the large $N$ conjecture to finite values of $N$ would be that the supergravity theory should be replaced by string theory on a background with the appropriate isometry group at finite $N$. Given that the Yang-Mills theory is perfectly well-defined at any value of $N$ and string theory on $AdS_5 \times S^5$ seems quite difficult to construct, it has been suggested that the Yang-Mills theory is as good a definition of string theory on $AdS_5 \times S^5$ as one is likely to obtain.

What of the Wilson loops that initiated the quest for a string description of Yang-Mills theory in the first instance? There is a natural string theory canonically defined by the Yang-Mills theory as follows[10]: Define the loop operator

$$H \equiv \int e^{i S} \frac{\delta}{\delta A_{\mu}} e^{-S} \frac{\delta}{\delta A_{\mu}}$$

(1)

which acts on products of Wilson loops. The loop equations (which are the Schwinger-Dyson equations for Wilson loops) of the Yang-Mills theory are
then
\[ \langle H \prod_i W(C_i) \rangle = 0. \]  \hfill (2)

Here \( \langle \ldots \rangle \) denotes correlation functions in the Yang-Mills theory defined by a functional integral weighted by the action of the Yang-Mills theory, \( S \). This operator \( H \) turns out to be the Fokker-Planck Hamiltonian\(^\text{[1]}\) in Parisi-Wu stochastic quantization\(^\text{[2]}\) of the Yang-Mills theory. This means introducing an additional Euclidean dimension with a coordinate \( \tau \) (called the stochastic time). In this enlarged coordinate space, an auxiliary classical stochastic theory is defined. The Hamiltonian for evolution in the \( \tau \) direction of this auxiliary theory is just the operator \( H \). Correlation functions of Wilson loops in the Yang-Mills theory are given by boundary correlations of the auxiliary theory:
\[ \langle \prod_i W(C_i) \rangle = \lim_{\tau \uparrow \infty} \langle 0 | \exp(-\tau H) \prod_i W(C_i) | 0 \rangle , \]  \hfill (3)

where \( |0\rangle \rangle \) denotes the ground state of the Hamiltonian \( H \). One now identifies the Fokker-Planck Hamiltonian\(^\text{[3]}\) with a Hamiltonian of a noncritical string field theory in a very special gauge\(^\text{[4]}\)—temporal gauge—where the time \( (\tau) \) is identified with the Liouville degree of freedom. We say more about the string interpretation in our concluding comments.

We see therefore that the Wilson loops and their Schwinger-Dyson equations of motion responsible for motivating a string interpretation of Yang-Mills theory are also at the root of a direct identification of a string field theory from the gauge theory.

2 Truncation of the spectrum

Kaluza-Klein(KK) modes on \( S^5 \) are associated with chiral primary operators in the supersymmetric Yang-Mills theory, which are finite in number at finite \( N \), due to simple matrix identities. One particularly interesting aspect of the AdS↔CFT correspondence is that it therefore implies that \textit{a priori} independent excitations of the dual string theory cannot be independent. However, the perturbative fundamental string does not know about the integrality of \( N \).

The aim of this paper is to exhibit the corresponding dynamical truncation phenomenon directly in the string theories constructed as above from
Yang-Mills theories. We shall show that the string Hamiltonian exhibits a quantization of the string coupling (in our case it is just $1/N^2$) and a coupling constant dependent truncation of the spectrum of a single string, phenomena that are invisible in string perturbation theory to any order.

The simplest model of the noncritical string theory ↔ Yang-Mills theory correspondence is the one-plaquette model

$$Z = \int dU \exp \left( (N/\lambda) tr \left[ U + U^\dagger \right] \right). \tag{4}$$

In our notation $tr$ is the usual trace and $Tr \equiv tr/N$. Let \{ $T_\alpha$ \} be a normalized anti-Hermitian orthonormal basis for the Lie algebra of $U(N)$ and $h \equiv h^a T_\alpha$. As usual from the translation invariance of the $U(N)$ Haar measure, we can derive Schwinger-Dyson equations for Wilson loops

$$\sum_\alpha \frac{\partial}{\partial h^\alpha} \int dU e^{(N/\lambda) tr[Ue^h + e^{-h} U^\dagger]} \sum_{j=1}^m n_j Tr \left( T_\alpha (U e^h)^{n_j} \right) \prod_{i \neq j} Tr \left( (U e^h)^{n_i} \right) \bigg|_{h=0} = 0. \tag{5}$$

Interpreting this equation as the condition for equilibrium in stochastic quantization

$$\lim_{\tau \to \infty} \frac{d}{d\tau} \langle 0 \mid \exp(-\tau \hat{H}) \prod_i Tr U^{n_i} \mid 0 \rangle = \lim_{\tau \to \infty} \langle 0 \mid \exp(-\tau \hat{H}) [\hat{H}, \prod_i Tr U^{n_i}] \mid 0 \rangle = 0, \tag{6}$$

one finds the Hamiltonian\[13\]

$$\hat{H} = \sum_{n>0} n \left[ \frac{1}{\lambda} (a_{n+1}^\dagger - (1 - \delta_{n,1}) a_{n-1}^\dagger - \delta_{n,1}) + a_n^\dagger + \sum_{p=1}^{n-1} a_p^\dagger a_{n-p}^\dagger \right] a_n$$

$$+ (n < 0 \text{ terms}) + \frac{1}{N^2} \sum_{i,j} i j a_i^\dagger a_j a_i a_j. \tag{7}$$

Here $a_i^\dagger$ creates $Tr U^i$ in the correlation function, $[a_j, a_i^\dagger] = \delta_{i,j}, a_0^\dagger \equiv 1$. Thus we see explicitly the weighting of joining interactions with a factor of $1/N^2$, while splitting interactions are $O(1)$, keeping fixed the expectation value $\langle W(C) \rangle = O(1)$ in the large $N$ limit. While $\hat{H}$ is not Hermitian, it is related to a Hermitian Hamiltonian by a similarity transformation as usual\[12].

The raison d’etre of $\hat{H}$ is to reproduce the Schwinger-Dyson equations. In deriving $\hat{H}$ there is an implicit assumption that the $1/N^2$ is treated perturbatively, since no relationship between traces of matrices is taken into account.
Of course, at finite $N$ there are simple identities relating traces. If one wants to take these identities into account the derivation of the appropriate Hamiltonian is much more difficult even though the identities in eq. 5 are still valid. *A priori* it is not clear that $\hat{H}$ is suitable for finite $N$.

While $\hat{H}$ certainly does reproduce formally the Schwinger-Dyson equations, it is also crucial for the existence of the stochastic quantization that $\hat{H}$ should have a unique ground state. In perturbation theory in $1/N^2$ this is obvious. Our aim is now to consider what happens at finite values of $N$. We will show that requiring a unique vacuum and no negative energy states implies relationships among the *a priori* independent creation operators, and that $N$ be an integer. These relations exactly enforce the known identities between traces of powers of matrices.

We first consider a limit where this analysis simplifies. A very important feature of this string field theory Hamiltonian is that the string splitting and joining terms are independent of the choice of the gauge theory action. Only the $\lambda$ dependent terms, which generate deformations of loops and annihilation of small loops, depend on $S$. In a precise sense, the choice of $S$ is equivalent to the choice of a background in which the strings propagate. Let us define $H \equiv \lim_{\lambda \to \infty} \hat{H}$. Then

$$\exp \left( -\frac{N^2}{\lambda} (a_1^\dagger + a_{-1}^\dagger) \right) H \exp \left( \frac{N^2}{\lambda} (a_1^\dagger + a_{-1}^\dagger) \right) = \hat{H}. \quad (8)$$

Thus a background, in this model, amounts to a coherent state of Wilson loops. On this background other Wilson loops ‘propagate’ using the Hamiltonian $H$.

Let us start with the case $\lambda = \infty$. The total winding number of the Wilson loops in the correlation function is preserved by $H$. Acting with $H$ on a state with only positive windings will only produce states with positive windings, and the same is true for states with purely negative windings. We shall begin with this case. Later we shall consider states with positive and negative winding strings. We compute first in the winding number $w = 2$ sector. There are two states $a_2^\dagger |0\rangle, (a_1^\dagger)^2 |0\rangle$ corresponding to one doubly wound Wilson line and a product of two singly wound Wilson lines. In this sector

$$H_2 = \begin{pmatrix} 2 & 2/N^2 \\ 2 & 2 \end{pmatrix}$$

which has a vanishing eigenvalue if $N = 1$. 

4
In the $w = 3$ sector, the states are $a_3^\dagger |0\rangle$, $a_2^\dagger a_1^\dagger |0\rangle$, and $a_1^{13} |0\rangle$. Here we find

$$
H_3 = \begin{pmatrix}
3 & 4/N^2 & 0 \\
6 & 3 & 6/N^2 \\
0 & 2 & 3
\end{pmatrix}.
$$

For $N = 2$ this has a vanishing eigenvalue with eigenvector

$$
V_3 \equiv (a_3^\dagger - 3a_2^\dagger a_1^\dagger + 2a_1^{13}) |0\rangle.
$$

For $N > 2$ there are no zero eigenvalues.

In the $w = 4$ sector, the states are $a_4^\dagger |0\rangle$, $a_3^\dagger a_1^\dagger |0\rangle$, $a_2^\dagger a_1^{12} |0\rangle$, $a_2^{12} |0\rangle$, and $a_1^{14} |0\rangle$. Now

$$
H_4 = \begin{pmatrix}
4 & 6/N^2 & 8/N^2 & 0 & 0 \\
8 & 4 & 0 & 8/N^2 & 0 \\
4 & 0 & 4 & 2/N^2 & 0 \\
0 & 6 & 4 & 4 & 12/N^2 \\
0 & 0 & 0 & 2 & 4
\end{pmatrix}.
$$

This has a negative eigenvalue for $N = 2$, a zero eigenvalue for $N = 3$ and only positive eigenvalues for $N > 3$. The eigenvalues in a sector with winding number $w$ are of the form $w \pm (m/N)$ where $m$ is an integer. For example, for $w = 2$, $m = 2$, for $w = 3$, $m = 0,6$ and for $w = 4$, $m = 0, 4, 12$. The eigenvalue of interest for a given value of $N$ is the largest value of $m$, which is $w(w - 1)$.

The pattern evident in these examples continues: For a given value of $N$, $H_w$ has a vanishing eigenvalue when $w = N + 1$, and negative eigenvalues if $w > N + 1$. We see therefore that the stochastic quantization of $H$ is ill-defined as it stands. At the first level we demand that the vacuum is unique so we impose (for $N = 2$) $V_3 = 0$, i.e. we demand the identity

$$
a_3^\dagger = 3a_2^\dagger a_1^\dagger - 2a_1^{13}.
$$

This is just the $2 \times 2$ matrix identity

$$
\text{Tr}U^3 = 3\text{Tr}U^2\text{Tr}U - 2(\text{Tr}U)^3.
$$

We see that this relation is encoded in the Hamiltonian automatically.

If this is a constraint it has to be compatible with time evolution. For $
\lambda = \infty$ this is trivial in the $w = 3$ sector. However, if we want now to go to
the $w = 4$ sector (keeping $N = 2$) we must check the consistency of imposing this constraint on those states. Looking at states with $w = 4$ at $N = 2$ the consistency condition is

\[ H a_3^\dagger a_1^\dagger |0\rangle = H(3a_2^\dagger a_1^{i2} - 2a_1^{i4})|0\rangle \] (10)

but since we know

\[
H a_3^\dagger a_1^\dagger |0\rangle = \left(6a_1^{i2}a_2^\dagger + \frac{3}{2}a_1^{i4}\right)|0\rangle \\
H a_1^{i2} a_2^\dagger |0\rangle = \left(2a_1^{i4} + \frac{1}{2}a_2^{i2} + 2a_3^\dagger a_1^\dagger\right)|0\rangle \\
H a_1^{i4} |0\rangle = 3a_1^{i2} a_2^\dagger |0\rangle,
\]

(11)

we must have a new constraint

\[ V_4 \equiv \left(a_4^\dagger - a_2^{i2} - 4a_1^{i2} a_2^\dagger + 4a_1^{i4}\right)|0\rangle = 0. \] (12)

This is precisely the identity

\[ \text{Tr}U^4 - (\text{Tr}U^2)^2 - 4(\text{Tr}U)^2\text{Tr}U^2 + 4(\text{Tr}U)^4 = 0 \]

valid for $2 \times 2$ matrices.

To systematize what we have found, let us rephrase the manner in which $V_4 = 0$ was derived above. Let $\mathcal{O}_3 \equiv a_3^\dagger - 3a_2^\dagger a_1^\dagger + 2a_1^{i3}$. We compute the commutator

\[ [H, a_1^\dagger \mathcal{O}_3]_+ = \frac{3}{2}\mathcal{O}_4 - 2a_1^\dagger \mathcal{O}_3, \]

where $[,]_+$ means we normal order the terms and keep only terms with creation operators, and $\mathcal{O}_4 \equiv a_4^\dagger - a_2^{i2} - 4a_1^{i2} a_2^\dagger + 4a_1^{i4}$. We found a vanishing eigenvalue at $w = 3$ when $N = 2$. For a unique vacuum, we must impose the constraint $V_3 \equiv \mathcal{O}_3 |0\rangle = 0$, and since $HV_3 = [H, \mathcal{O}_3] |0\rangle = 0$, we have $\exp(-H\tau)V_3 = 0$ for any $\tau$. However, this is not enough for consistency. We must also ensure that

\[ \exp(-H\tau)\mathcal{O}_3 \prod_i a_{n_i}^\dagger |0\rangle = 0 \]

for all choices of $n_i$ and any $\tau$. For example, this implies $\exp(-H\tau)\mathcal{O}_3 a_1^\dagger |0\rangle = 0$ for all $\tau$. We must have

\[ [H, \mathcal{O}_3 a_1^\dagger]_+ |0\rangle = [H, \mathcal{O}_3 a_1^\dagger] |0\rangle = 0, \]
which is precisely a linear combination of the new constraint $\mathcal{O}_4 |0\rangle = 0$ and the initial constraint $\mathcal{O}_3 a_1^\dagger |0\rangle = 0$. We cannot stop here. Clearly, \( \exp(-\tau H)\mathcal{O}_4 |0\rangle = 0 \) must be true for all \( \tau \) as well so we compute

\[
[H, \mathcal{O}_4] |0\rangle = 2 \mathcal{O}_4 |0\rangle.
\] (13)

Thus, as desired, $\mathcal{O}_4 |0\rangle = 0$ can be maintained without further constraints. This procedure proceeds recursively. It is important that at each winding number $w$ we get only one constraint.

One can work out the constraints for positive and negative windings independently. It appears to be possible to implement all these constraints for any value of $N$ (not necessarily an integer) as long as we do not consider states with positive and negative windings. However, explicit calculations show that the constraints are only valid on all states when $N$ is an integer.

As an example consider $a_1^\dagger \mathcal{O}_3 |0\rangle$. For general $N$

\[
\mathcal{O}_3(N) = a_3^\dagger - (3N/2)a_1^\dagger a_2^\dagger + (N^2/2)a_4^{13}
\]

satisfies $H \mathcal{O}_3(N) |0\rangle = (3 - 6/N) \mathcal{O}_3(N) |0\rangle$. We then find

\[
[H, a_1^\dagger \mathcal{O}_3(N)] |0\rangle = (4 - 6/N)a_1^\dagger \mathcal{O}_3(N) |0\rangle + \frac{3}{N^2}(N - 2)(a_2^\dagger + Na_1^{12}) |0\rangle.
\] (14)

Thus, only when $N = 2$ is the constraint $a_1^\dagger \mathcal{O}_3(N) |0\rangle = 0$ consistent with the time evolution defined by $H$.

We now take $\lambda$ to be finite. $\hat{H}$ does not preserve winding number so time evolution takes us out of the winding sector we started with, so we need to check that the constraints are satisfied under this different time evolution. This can be seen to follow from the relationship in equation (8). As a check we can compute

\[
[H, \mathcal{O}_N \prod_i a_{n_i}^\dagger] |0\rangle = 0.
\] (15)

This is not surprising from the gauge theory perspective since the relations between traces are independent of the gauge theory action $S$.

Summarizing these calculations, we find that the Hamiltonian $\hat{H}$ when treated non-perturbatively at finite $N$ requires the integrality of $N$. For a given value of $N$, we find an infinite set of $N$-dependent constraints on physical states $\{ \mathcal{O}_j(N), j = N + 1, \ldots \}$ which are consistently propagated by $\hat{H}$.
These constraints ensure that the vacuum is unique and that no negative energy states appear, and are exactly the known relationships among traces of powers of $N \times N$ matrices.

We have taken some care in this analysis to avoid any use of the gauge theory origins of $\hat{H}$. Indeed in the gauge theory the integrality of $N$, the appearance of constraints, their form, and the fact that the constraints are background independent, are all completely obvious. From the string Hamiltonian point of view, we know of no obvious reason why the string field theory Hamiltonian derived without regard to finite $N$ niceties exhibits all the required phenomena when treated carefully at finite values of the coupling, but in fact we have shown that it does!

The Hamiltonian $H$ is universal: it is part of any string theory derived from a Yang-Mills theory. Consider a given Wilson loop specified as the holonomy of the gauge field around some curve $C$. There are Wilson loops entirely analogous to the multiple winding states we considered above obtained by computing the holonomy for multiple circuits around $C$. The joining and splitting of these loops is described precisely by $H$. Thus the truncation of the string spectrum is universal in such Yang-Mills strings. To be explicit, the point is that oscillator states of a multiply-wound string are identified with oscillator states of two (or more) strings at finite coupling. The stronger the coupling, the fewer the independent states of a single string—this is a fascinating dynamical phenomenon from the string point of view. It is difficult to see how such identifications of the string spectrum would be visible in first-quantized string theory, though we cannot exclude this definitively since we have only a gauge-fixed framework[10].

3 Remarks and speculations

In what follows, we list some remarks and speculations.

- In this paper we started from a Hamiltonian designed to reproduce the Schwinger-Dyson equations, without explicitly taking matrix identities into account. This resulted in a simple Hamiltonian[15], eq. 4, as compared to the complicated Hamiltonian which arises if one attempts to take the matrix identities into account explicitly[4]. What we found is that the simple Hamiltonian ‘dynamically’ generates constraints that
exactly implement the matrix identities. This of course results in a reduction of single string states above winding $N \equiv 1/g_{\text{st}}$.

- The analogues of Wilson loops for theories with adjoint scalars are obvious and satisfy the same constraints. The naive counting of dimensions suggests that the string theory constructed from $\mathcal{N} = 4, d = 4$ Yang-Mills theory is associated with a string theory in 11 dimensions. However, given a string field theory Hamiltonian in a particular gauge one has to explicitly calculate to deduce the approximate background geometry on which the string propagates, if such a classical geometry exists. In stochastic quantization only the infinite time correlation functions are uniquely defined\cite{12}. This corresponds in the string theory to boundary correlation functions. We speculate that in the $\mathcal{N} = 4, d = 4$ Yang-Mills theory the stochastic time will be identified with the radial coordinate of the $AdS_5$, and that this connection is at the root of holography\cite{17}. It is by no means clear that a holographic relationship exists in every case of the suggested association\cite{10} of a string theory with a Yang-Mills theory even though the boundary correlation functions will always be reproduced.

- $q$-deformations have been suggested\cite{18} as explanations for the truncation of the spectrum of chiral primaries. It would be interesting to see if they are related to our calculations. In a different vein, 't Hooft has suggested that quantum gravity is a dissipative deterministic system\cite{19}. There may be a connection with the approach described in this paper.

- Finally multiply wound strings appear in perturbative calculations of high energy string scattering\cite{20}. It would be interesting to reconsider the true high-energy behaviour in Yang-Mills strings taking into account a truncation of the number of windings as we have found.

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References

[1] S. Mandelstam, Ann. of Phys. 19, 25(1962); Phys. Rev. 175, 1580 (1968)

[2] J.-L. Gervais and A. Neveu, Phys. Lett. B80, 255 (1979); Y. Nambu, Phys. Lett. B80, 372 (1979); A. Polyakov, Phys. Lett. B82, 247 (1979)

[3] G. De Angelis, D. De Falco and F. Guerra, Nuovo Cim. Lett. 19, 55 (1977); F. Guerra, R. Marra and G. Immirzi, Nuovo Cim. Lett. 23, 237 (1978); E. Corrigan and B. Hasslacher, Phys. Lett. B81, 181 (1979); L. Durand and E. Mendel, Phys. Lett. B85, 241 (1979); D. Foerster, Phys. Lett. 87B, 83 (1979); T. Eguchi, Phys. Lett. 87B, 91 (1979); Yu. Makeenko and A. Migdal, Phys. Lett. 88B, 135 (1979)

[4] A. Jevicki and B. Sakita, Nucl. Phys. B185, 89 (1981)

[5] A. Polyakov, Nucl. Phys. Proc. Supp. 68, 1 (1998); Int. J. Mod. Phys. A14, 645 (1999)

[6] G. ’t Hooft, Nucl. Phys. B72, 461 (1974)

[7] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231

[8] S. Gubser, I. Klebanov and A. Polyakov, Phys. Lett. B428, 105 (1998)

[9] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253

[10] V. Periwal, String field theory Hamiltonians from Yang–Mills theories, hep-th/9906032

[11] G. Marchesini, Nucl. Phys. B191, 214 (1981); B239, 135 (1984)

[12] G. Parisi and Y.-S. Wu, Sci. Sin. 24, 484 (1981)

[13] A. Jevicki and J. Rodrigues, Nucl. Phys. B421, 1994 (278); see also S. Das and A. Jevicki, Mod. Phys. Lett. A5 (1990) 1639; G. Moore, N. Seiberg and M. Staudacher, Nucl. Phys. B362, 6 (1991)65

[14] N. Ishibashi and H. Kawai, Phys. Lett. B314, 190 (1993); M. Fukuma, N. Ishibashi, H. Kawai and M. Ninomiya, Nucl. Phys. B427, 139 (1994); M. Ikehara, N. Ishibashi, H. Kawai, T. Mogami, R. Nakayama and N.
Sasakura, Phys. Rev. D50, 7467 (1994); N. Ishibashi and H. Kawai, Phys. Lett. B322, 67 (1994); Phys. Lett. B352, 75 (1995)

[15] V. Periwal, A toy model of Polyakov duality, hep-th/9908203

[16] J. Rodrigues, Nucl. Phys. B260, 350 (1985)

[17] G. ’t Hooft, in Salamfestschrift: a collection of talks, World Scientific Series in 20th century physics, v. 4, eds. A. Ali et al. (World Sci., 1993) and in Proc. Symp. The Oskar Klein Centenary, ed. U. Lindström (World Sci., 1995); L. Susskind, J. Math. Phys. 36, 6377 (1995)

[18] A. Jevicki and S. Ramgoolam, J. High Energy Phys. 9904, 032 (1999); P.-M. Ho, S. Ramgoolam and R. Tatar, Quantum spacetimes and finite $N$ effects in 4d super Yang-Mills theories, hep-th/9907145

[19] G. ’t Hooft, Quantum gravity as a dissipative deterministic system, gr-qc/9903084

[20] D. Gross and P. Mende, Phys. Lett. B197, 129 (1987); Nucl. Phys. B303, 407 (1988)