Abstract. We investigate global continuation of periodic orbits of a differential equation depending on a parameter, assuming that a closed 1-form satisfying certain properties exists. We begin by extending the global continuation theory of Alexander, Alligood, Mallet-Paret, Yorke, and others to the situation that such a 1-form exists, formulating a new notion of global continuability and a new global continuation theorem tailored for this situation. In particular, we show that the existence of such a 1-form provides a topological obstruction ensuring that local continuability of periodic orbits implies global continuability. Using our general theory, we then develop techniques for proving the existence of periodic orbits. In contrast to previous work, a key feature of our results is that existence of periodic orbits can be proven (i) without finding trapping regions (or, e.g., Conley index pairs) for the dynamics and (ii) without establishing difficult a priori upper bounds on the periods of orbits. As opposed to local results such as the Hopf bifurcation theorem, our existence results are global and can be used to prove existence of periodic orbits on large parameter intervals. We illustrate the theory in examples inspired by the synthetic biology literature, proving existence of periodic orbits on large parameter intervals for (i) the “repressilator” model of a synthetic genetic regulatory network and (ii) an “elegant chaotic” non-monotone system considered by Sprott.

Contents

1. Introduction 2
1.1. Discussion of related continuation results 4
1.2. Main results 6
1.3. Outline of the sequel 7
Acknowledgements 8
2. Generic families 8
2.1. Background on generic families 8
2.2. Global continuation for generic families 12
3. Non-generic families 15
4. Examples 23
4.1. Basic symmetry considerations 23
4.2. A closed 1-form 23
4.3. The repressilator: existence of periodic orbits 24
4.4. The Sprott system: existence of periodic orbits 28
References 37

Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109
E-mail addresses: kvalheim@umich.edu, abloch@umich.edu.

arXiv:1906.03528v3 [math.DS] 17 Jul 2019
1. Introduction

In this paper, we study families of periodic orbits of a $C^1$ autonomous ordinary differential equation (ODE) with one parameter
\begin{equation}
\dot{x} = f(x, \mu) =: f_\mu(x), \quad (x, \mu) \in Q \times \mathbb{R}
\end{equation}
on a smooth manifold $Q$. Our primary contributions are (i) a theorem on the global continuation of periodic orbits as the parameter is varied and (ii) theorems on existence of periodic orbits based on our global continuation theory. A key hypothesis for our theorems is the existence of a closed 1-form $\eta$ on $Q \times \mathbb{R}$ satisfying certain properties.

Several authors have previously studied the global continuation of periodic orbits of (1). Some important early efforts are represented by [Ful67, AY78, CMP78]. These authors study connected components of periodic orbits in $(x, \mu, \tau)$-space, where $\tau$ is the period of a periodic orbit. Subsequently several authors showed that more refined information could be obtained by studying components of periodic orbits in $(x, \mu)$ space using other techniques [AMPY81, MPY82, CMPY83, AMPY83, AY84]. We mention also [Fie88] who refined and extended many of these global continuation results to families of differential equations which are equivariant under certain groups of symmetries.

The motivation for the present paper was to obtain useful techniques for proving existence of periodic orbits for concrete ODEs. In particular, the results in this paper grew out of our attempts to prove existence of periodic orbits for the following ODE
\begin{equation}
\begin{align*}
\dot{x} &= y^2 - z - \mu x \\
\dot{y} &= z^2 - x - \mu y \\
\dot{z} &= x^2 - y - \mu z
\end{align*}
\end{equation}
on $\mathbb{R}^3$ which depend on the parameter $\mu \in \mathbb{R}$. The system without damping ($\mu = 0$) was considered by Sprott [Spr10, Eq. 4.7] as an example of an “elegant chaotic” system, so we refer to (2) as the “Sprott system”; Figure 1 displays some of its intrinsically rich dynamical structure. Our interest in this system was originally inspired by various systems that have been analyzed in the synthetic biology literature such as the repressilator and its generalizations, see e.g. [EL00], [MPS90] [RS17] and [RPM+17]. The repressilator is a model of a synthetic genetic regulatory network consisting of a ring oscillator, and a reduced-order model for this system is given [BKP09, BPK10] by the ODE
\begin{equation}
\begin{align*}
\dot{x} &= \frac{\mu}{1+y^s} - x \\
\dot{y} &= \frac{\mu}{1+z^s} - y \\
\dot{z} &= \frac{\mu}{1+x^s} - z
\end{align*}
\end{equation}
on $\mathbb{R}^3$, where $s > 2$ and $\mu > 0$ are parameters. Both (2) and (3) are symmetric with respect to the cyclic permutation $(x, y, z) \mapsto (y, z, x)$ (see [MdCG06] for other work on cyclic systems). However, in many ways (2) is more subtle to analyze, and many of the standard techniques applied to such systems fail. For example, the periodic orbit existence proof for (3) in [BKP09] does not work for (2); additionally, (3) has the structure of a monotone cyclic feedback system [MPS90] while (2) does not. Using a single technique based on our results we give proofs that both (2) and (3) have periodic orbits for all $\mu \in (-0.25, 0.5)$ and all $\mu \in (\mu_c(s), \infty)$, respectively, where $s > 2$ and $\mu_c(s)$ is a certain parameter value at which a Hopf bifurcation for (3) occurs.

Perhaps the most famous technique to prove that periodic orbits exist is the Poincaré-Bendixson theorem [Poi81, Ben01] for autonomous ODEs on the plane. More recently, some authors have
proven existence theorems for $n$-dimensional ODEs by finding conditions under which an $n$-dimensional system can be projected onto a two-dimensional one so that the Poincaré-Bendixson theorem can be applied [Gra77, Smi80]. Another example of this approach includes a Poincaré-Bendixson theorem for the class of monotone cyclic feedback systems [MPS90] which is relevant for various applications in biology; in particular, this theorem yields an alternative proof that the repressilator (3) has periodic orbits. There is also a rich literature on periodic orbit existence for Hamiltonian systems; we mention [Rab78, Wei79] as notable examples, and also the solution [CZ83, CZ84, Flo89] of the celebrated Arnold conjecture [Zeh86, Zeh19]. For the case of general $n$-dimensional ODEs, the “torus principle” [Li81] based on Brouwer’s fixed point theorem is widely used to prove existence of periodic orbits; application of this principle is made easier by recent work of Brockett and Byrnes [Byr07, Byr10] which utilizes Lyapunov 1-forms [FKLZ03, FKLZ04], results on the topology of Lyapunov function level sets [Wil67], and various advances in topology including the solution of the Poincaré conjecture [MT07]. The torus principle is generalized by periodic orbit existence theorems based on the Conley and Lefschetz indices, which allow the toroidal trapping region to be replaced with an isolating neighborhood having the Conley index of a hyperbolic periodic orbit [MMM95, Con78]; one body of work has focused on rigorous computer-assisted periodic orbit existence proofs based on these topological results [Pil99, BDJ05], with applications including the
In this paper, we are interested in proving existence of periodic orbits for families of ODEs depending on a parameter, but the existence results just mentioned are formulated for a single ODE. Additionally, applying these existence results is often easier said than done, and we were unable to apply any of these results to the Sprott system (2): for example, we were unable to find “by hand” a toroidal trapping region or suitable Conley index pair to prove periodic orbit existence for (2); equation (2) does not satisfy the “point-dissipative” or “ultimately bounded” hypothesis of [Byr10, Thm 4.3]; and as previously mentioned the Poincaré-Bendixson theorem for monotone cyclic feedback systems does not apply. Inspired by a suggestion of Rajapakse and Smale [RS17, p. 1214], we set out to find continuation-based techniques to prove existence results—tailored to families—which do not require finding trapping regions or index pairs, and which therefore might prove easier to apply to systems such as (2). We found that one difficulty in using the previously mentioned continuation results [Ful67, AY78, MPY82, CMPY83, AMPY83, AY84, Fie88] to prove existence is that a priori upper bounds on the periods (or virtual periods, to be defined in §1.1) of periodic orbits of (1) are required, and it seems that there are few general techniques to obtain such bounds. However, we show that the existence of a closed 1-form \( \eta \) on \( Q \times \mathbb{R} \) satisfying certain properties enables a priori period upper bounds to be replaced with conditions such as \( \eta((f,0)) > 0 \) which are in principle computable.\(^1\) Our first such existence result is Theorem 2, stated in §1.2. Using Theorem 2 we also prove a rather specific existence result in Theorem 3, which we use in our applications. These theorems are essentially corollaries of our main result, Theorem 1.

We state our main results in §1.2. In order to motivate the statement of our results, in §1.1 we first discuss in more detail related work of [MPY82, AY83, AY84]. In the sequel, for notational simplicity we often identify the image \( \Gamma \) of a periodic orbit \( \gamma \) of \( f_{\mu} \) with the set \( \Gamma \times \{ \mu \} \subset Q \times \mathbb{R} \) when there is no risk of confusion.

1.1. Discussion of related continuation results. Our first main result (Theorem 1) concerns global continuability. Multiple notions of global continuability have appeared in the literature; the following definition of \( P \)-global continuability (called global continuability in [AMPY81, AY83]) is essentially taken from [AMPY83, AY84].

**Definition 1** (\( P \)-global continuability). Let \( A \subset Q \times \mathbb{R} \) be a connected component of nonstationary periodic orbits of (1), and let \( \gamma \) be a periodic orbit with image \( \Gamma \subset A \). We say that \( \gamma \) is \( P \)-globally continuable if at least one of the following holds.

- \( A \setminus \Gamma \) is connected,
- or each connected component \( A^i \) of \( A \setminus \Gamma \) satisfies one of the following:
  1. \( A^i \) is not contained in any compact subset of \( Q \times \mathbb{R} \),
  2. the closure \( \text{cl}(A^i) \) of \( A^i \) in \( Q \times \mathbb{R} \) contains a generalized center (i.e., a stationary point \((x, \mu) \) such that \( D_{x_0}f_{\mu} \) has some purely imaginary eigenvalues), or
  3. the periods of orbits in \( A^i \) are unbounded.

Mallet-Paret and Yorke considered a certain “generic” subset (i.e., containing a residual subset) \( \mathcal{K} \) of families (1)—discussed in more detail in §2.1—and proved several results involving the continuation of periodic orbits [MPY82]. Necessary for the statement of these results is the concept of a Mōbius orbit, which is a periodic orbit having an odd number of Floquet multipliers in \((-\infty, 1)\) and no multipliers equal to \(-1\). The following result is a special case of [MPY82, Thm 4.2]; a direct proof appears in [AMPY83, Thm 2.2].

\(^1\)Note that \( \eta \) satisfying this last condition can be viewed as a Lyapunov 1-form in the sense of [FKLZ03, FKLZ04].
Proposition 1 (Mallet-Paret and Yorke). Let \( f \in K \) be a generic family of vector fields. Let \( \gamma \) be a periodic orbit of some \( f_{\mu_0} \). Assume that \( \gamma \) is not a Möbius orbit, and assume that \( \pm 1 \) are not Floquet multipliers of \( \gamma \). Then \( \gamma \) is \( P \)-globally continuable.

Although the subset \( K \) is generic, given a specific family (1) it is usually difficult to determine whether this specific family belongs to \( K \) (c.f. [SY12, p. 5]). Therefore, it would be desirable to extend Proposition 1 to a result valid for arbitrary (i.e., “non-generic”) \( C^1 \) families. By extending to periodic orbits the notion of virtual periods, previously defined for stationary points of ODEs [MPY82] and fixed points of maps [CMPY83], Alligood and Yorke introduced a modification of Definition 1 to prove such a generalization in [AY84]; see [AMPY83, Fie88] for more general results. Briefly, if \( \tau \) is the minimal period of \( \gamma \), then \( \bar{\tau} = k\tau \) is a virtual period of order \( k \in \mathbb{N} \geq 1 \) for \( \gamma \) if the linearization of a Poincaré map for \( \gamma \) has a periodic point of minimal period \( k \) [AMPY83, AY84, Fie88]. The following definition is essentially [AY84, Def. 1.3] and is obtained from Definition 1 by simply replacing “periods” with “virtual periods”.

Definition 2 (Global continuability). Let \( A \subset Q \times \mathbb{R} \) be a connected component of nonstationary periodic orbits of (1), and let \( \gamma \) be a periodic orbit with image \( \Gamma \subset A \). We say that \( \gamma \) is \( P \)-globally continuable if at least one of the following holds:

- \( A \setminus \Gamma \) is connected,
- or each connected component \( A^i \) of \( A \setminus \Gamma \) satisfies one of the following:
  1. \( A^i \) is not contained in any compact subset of \( Q \times \mathbb{R} \),
  2. the closure \( \text{cl}(A^i) \) of \( A^i \) in \( Q \times \mathbb{R} \) contains a generalized center (i.e., a stationary point \((x, \mu)\) such that \( D_{xx} f_{\mu_0} \) has some purely imaginary eigenvalues), or
  3. the virtual periods of orbits in \( A^i \) are unbounded.

The following result is [AY84, Thm 3.1]; it generalizes Proposition 1 to the case of arbitrary \( C^1 \) families of vector fields.

Proposition 2 (Alligood and Yorke). Let \( f \in C^1(Q \times \mathbb{R}, TQ) \) be a family of vector fields. Let \( \gamma \) be a periodic orbit of some \( f_{\mu_0} \). Assume that \( \gamma \) is not a Möbius orbit, and assume that \( \gamma \) has no Floquet multipliers which are roots of unity. Then \( \gamma \) is globally continuable.

The assumption that \( \gamma \) is not Möbius in Proposition 1 is important: as shown in [AMPY81], there are examples of hyperbolic Möbius orbits \( \gamma \) whose components \( A \subset Q \times \mathbb{R} \) satisfy none of the conditions of either Definition 1 or 2. In other words, such an orbit \( \gamma \) is not globally continuable even if it is locally continuable (via, say, the implicit function theorem applied to a Poincaré map). The reason is related the possibility that \( A \) can contain branches of periodic orbits emanating from a period-doubling bifurcation at one parameter value which annihilate each other at another parameter value. If \( \gamma \) is Möbius, this possibility implies that the “orbit diagram” of \( A \) (orbit diagrams are discussed in detail in §2.1) can look like that of Figure 4, so that \( A \) satisfies none of the conditions of Definitions 1 or 2.

For families of periodic orbits in \( \mathbb{R}^3 \), however, Alexander and Yorke [AY83] showed that, in the presence of a certain additional assumption, Möbius orbits are globally continuable.\(^2\) The basic idea is that, in three dimensions, linking numbers (and also, e.g., knot types) of periodic orbits provide topological obstructions to various bifurcations [GH93, GHS97], including the phenomenon of orbit annihilation following period-doubling mentioned above. This motivates the following basic observation which generalizes to higher dimensions: the linking number of a periodic orbit with another submanifold of state space also provides an obstruction to the same phenomenon, as long

\(^2\)Without this certain additional assumption, a slightly more complicated variant of the orbit diagram in Figure 4 can still occur; see [AY83, Fig. 2.1].
as periodic orbits do not intersect this submanifold (so that the linking number is defined). Now one way to compute such a linking number is to integrate a certain closed differential 1-form over the periodic orbit [BT91, pp. 227–234], and in fact the preceding observation generalizes to yield an obstruction in the situation that one has any closed 1-form having nonzero integral over the Möbius orbit. This observation led to the formulations of Definition 3 and Theorem 1 below and is crucial to the periodic orbit existence Theorems 2 and 3.

1.2. Main results. In this section we give statements of our main results. In order to state Theorem 1, we first define our own variant of global continuability—$(\eta, \ell)$-continuability—which is motivated by the discussion at the end of §1.1. Definition 3 below should be compared with the very similar Definitions 1 and 2 of P-global continuability and global continuability, respectively.

In Definition 3 and in the rest of the paper, for each $\mu \in \mathbb{R}$ we let $t_\mu : Q \hookrightarrow Q \times \mathbb{R}$ be the inclusion $t_\mu(x) = (x, \mu)$ and $t_\mu^* \eta$ the pullback of the 1-form $\eta$ on $Q \times \mathbb{R}$ by $t_\mu$.

**Definition 3** $(\eta, \ell)$-global continuability. Let $\ell > 0$, $\eta$ be a $C^1$ closed 1-form on an open subset $\text{dom}(\eta) \subset Q \times \mathbb{R}$, and $A \subset \text{dom}(\eta)$ be a connected component of nonstationary periodic orbits of $f|_{\text{dom}(\eta)}$. Define $A_\ell \subset A$ to be the subset of points on periodic orbits $\alpha_\mu$ with $|\int_{\alpha_\mu} t_\mu^* \eta| = \ell$ and $A_{\leq \ell}$ the subset with $|\int_{\alpha_\mu} t_\mu^* \eta| \leq \ell$.

Let $\gamma$ be a periodic orbit with image $\Gamma \subset A_\ell$. Let $\bar{A}_{\leq \ell} \subset A_{\leq \ell}$, $\bar{A}_\ell \subset A_\ell$ be the connected components of $A_{\leq \ell}$, $A_\ell$ containing $\gamma$. We say that $\gamma$ is $(\eta, \ell)$-globally continuable if at least one of the following holds:

- $\bar{A}_{\leq \ell} \setminus \Gamma$ is connected,
- or each connected component $\bar{A}_\ell^\dagger$ of $\bar{A}_{\leq \ell} \setminus \Gamma$ containing a connected component $\bar{A}_\ell^\dagger$ of $\bar{A}_\ell \setminus \Gamma$ satisfies one of the following:
  1. $\bar{A}_\ell^\dagger$ is not contained in any compact subset of $\text{dom}(\eta)$,
  2. the closure $\text{cl}(\bar{A}_\ell^\dagger) \subset \text{dom}(\eta)$ of $\bar{A}_\ell^\dagger$ in $\text{dom}(\eta)$ contains a generalized center (i.e., a stationary point $(x, \mu)$ such that $D_x f_{\mu_0}$ has some purely imaginary eigenvalues),
  3. the periods of $\bar{A}_\ell^\dagger$ are unbounded, or
  4. $\bar{A}_\ell^\dagger \neq \bar{A}_{\leq \ell}^\dagger$.

The following theorem is our most general result and should be compared with Propositions 1 and 2.

**Theorem 1** $(\eta, \ell)$-global continuability for non-generic families. Let $f \in C^1(Q \times \mathbb{R}, TQ)$ be a family of vector fields, and let $\eta$ be a $C^1$ closed 1-form on an open subset $\text{dom}(\eta) \subset Q \times \mathbb{R}$. Let $\gamma$ be a periodic orbit of some $f_{\mu_0}$ with image $\Gamma$ satisfying $\Gamma \times \{\mu_0\} \subset \text{dom}(\eta)$, and assume that $\gamma$ does not have $+1$ as a Floquet multiplier. Define $\ell := |\int_{\gamma} t_{\mu_0}^* \eta|$, and assume $\ell > 0$. Then $\gamma$ is $(\eta, \ell)$-globally continuable.

The following theorem is our most general result for proving existence of periodic orbits and is essentially a straightforward corollary of Theorem 1. Three key points are that the hypotheses of Theorem 2 (i) do not require verification that the family of vector fields belong to $\mathcal{K}$ as in Proposition 1, (ii) do not require any a priori upper bounds on the (virtual) periods of all periodic orbits to be established, as one might hope to do in order to directly apply Propositions 1 or 2, and (iii) do not require a trapping region to be found, as the hypotheses only require that periodic orbits in $\bar{A}_\ell^\dagger$ do not meet the boundary of $\mathcal{C}$.

Given a subset $X \subset Q \times \mathbb{R}$ and any interval $J \subset \mathbb{R}$, in the statement of Theorem 2 we use the notation $X_J := X \cap (Q \times J)$. 

---

**6 FAMILIES OF PERIODIC ORBITS: CLOSED 1-FORMS AND GLOBAL CONTINUABILITY**
Theorem 2 (Global existence of periodic orbits). Assume the hypotheses of Theorem 1 and notation of Definition 3. Assume that $A_{≤0} \setminus \Gamma$ is disconnected, let $A_{≥0}$ be one of its connected components, and assume that $A_{≥0}$ is equal to a connected component $A_{≥0}$ of $A_{≤0} \setminus \Gamma$. Further assume that there exists $C ⊂ \text{dom}(η) ⊂ Q × \mathbb{R}$ and $μ^* < μ_0$ (resp. $μ^* > μ_0$) satisfying the following properties:

1. $A_{≥0} \cap (Q \times \{μ^*\}) = \emptyset$,
2. $A_{≥0} ⊂ C$,
3. $i_μ^*(η(f_μ(x))) > 0$ for all $(x, μ) ∈ C_{[μ^*,∞)}$ (resp. $(x, μ) ∈ C_{(-∞,μ^*)}$), and
4. for every $μ ≥ μ^*$ (resp. $μ ≤ μ^*$), $C_{[μ^*,μ]}$ (resp. $C_{[μ,μ^*)}$) is compact.

Then for all $μ > μ_0$ (resp. $μ < μ_0$), $A_{≥0} \cap (Q \times \{μ\}) ≠ \emptyset$. In particular, $f_μ$ has a periodic orbit for all $μ ≥ μ_0$.

The following result is proven using Theorem 2. Although its statement appears rather specific and complicated, Theorem 3 represents the formalization of a common argument we have used to apply Theorem 2 in multiple concrete examples. Specifically, in §4 we use Theorem 3 to prove global existence results for periodic orbits in both the Sprott system (2) and repressilator (3).

Given a subset $X ⊂ Q × \mathbb{R}$ and any interval $J ⊂ \mathbb{R}$, we again use the notation $X_J := X \cap (Q × J)$ in Theorem 3. By a point $(x, μ) ∈ Q × \mathbb{R}$ of generic Hopf bifurcation, we mean a point satisfying the hypotheses of the standard Hopf bifurcation theorem (see [GH00, Rue89, Rob99, Kuz13]). In Theorem 3 we refer to a closed 1-form representing the Poincaré dual of a submanifold (see [BT91, pp. 50-53]); this construction is typically carried out in the $C^∞$ setting, but since every $C^1$ closed form is cohomologous to a $C^∞$ closed form [dDR84, pp. 61-70], no distinctions need to be made.

Theorem 3 (Global existence of periodic orbits following a Hopf bifurcation). Assume that $Q$ is orientable, and let $N ⊂ Q × \mathbb{R}$ be a properly embedded, smooth, orientable, codimension-1 submanifold with boundary $M = \partial N$. Let $f ∈ C^1(\mathbb{R} × \mathbb{R}, TQ)$ be a family of vector fields, and let $η \in [η] ∈ H^1(\mathbb{R} × \mathbb{R}) \setminus M; \mathbb{Z}$ be a $C^1$ closed 1-form representing the (closed) Poincaré dual $[η]$ of $N$. Further assume that there exists $C ⊂ Q × \mathbb{R}$, $(x_c, μ_c) ∈ M \cap \text{int}(C)$ and $μ^* < μ_c$ (resp. $μ^* > μ_c$) satisfying the following properties:

1. $f_μ^*$ has no periodic orbits contained in $C_{μ^*}$,
2. no periodic orbits of $f$ intersect $(∂C)_{[μ^*,∞)}$ (resp. $∂C_{(-∞,μ^*)}$),
3. for every $μ_1 > μ^*$ (resp. $μ_1 < μ^*$), there exists $ε > 0$ such that $i_μ^*(η(f_μ(x))) ≥ ε$ for all $(x, μ) ∈ (C \setminus M)_{[μ,μ_1]}$ (resp. $(x, μ) ∈ (C \setminus M)_{[μ_1,μ]}$),
4. for every $μ ≥ μ^*$ (resp. $μ ≤ μ^*$), $C_{[μ^*,μ]}$ (resp. $C_{[μ,μ^*)}$) is compact,
5. $f$ is $C^3$ on a neighborhood of $(x_c, μ_c)$, $(x_c, μ_c) ∈ M \cap \text{int}(C)$ is a point of generic Hopf bifurcation for $f$, and $C_{[μ^*,∞)}$ (resp. $C_{(-∞,μ^*)}$) contains no other generalized centers,
6. no nonstationary periodic orbits of $f$ intersect $(C \cap M)_{μ^*}$ (resp. $(C \cap M)_{(-∞,μ^*)}$), and
7. letting $E^c \subset T_{x_c}Q$ be the two-dimensional center subspace for $D_{x_c}f_{μ_c}$,

$T_{(x_c,μ_c)}(Q × \mathbb{R}) = (D_{(x_c,μ_c)}f_{μ_c})E^c ⊕ T_{(x_c,μ_c)}M$.

Then for all $μ > μ_c$ (resp. $μ < μ_c$), $f_μ$ has a periodic orbit contained in $(C \setminus M)_{μ}$.

1.3. Outline of the sequel. The remainder of the paper is organized as follows.

In §2 we develop the theory for the so-called generic families of vector fields, i.e., those families belonging to a certain generic subset $\mathcal{K} ⊂ C^5(Q × \mathbb{R}, TQ)$ of the $C^5$ one-parameter families. We begin in §2.1 by discussing $\mathcal{K}$ and giving the relevant background on periodic orbits for $f ∈ \mathcal{K}$. Along the way we introduce orbit diagrams, which are very useful in the generic setting. Section 2.2 introduces some of the key ideas and proves Theorem 1 in the special case that the vector field family is generic (Lemma 4).
In §3 we extend the results of §2.2 to prove Theorem 1 for the general case of an arbitrary family $f \in C^1(Q \times \mathbb{R}, TQ)$. The proof is by generic approximation and was inspired by techniques of [AY84]. As a straightforward corollary of Theorem 1 we obtain Theorem 2, which is a fairly general theorem for proving existence of periodic orbits. We then record as Theorem 3 a systematic argument involving Theorem 2 for proving existence of periodic orbits on large parameter intervals following a Hopf bifurcation, in a setting which appears common in certain applications.

In §4 we illustrate the utility of our results in some specific ODEs. In §4.3 we give a periodic orbit existence proof for the repressilator (3). Our proof is distinct from the proof of [BKP09] and does not use techniques of monotone systems [MPS90]. §4.4 is more involved and uses our results to give a periodic orbit existence proof for the Sprott system (2). The proofs in both §4.3 and §4.4 amount to showing that the repressilator and Sprott system satisfy the hypotheses of Theorem 3.

**Acknowledgements.** Kvalheim was supported by ARO grants W911NF-14-1-0573 and W911NF-17-1-0306. Bloch was supported by NSF grant DMS-1613819 and AFOSR grant FA 0550-18-0028. We would like to thank R. W. Brockett and H. L. Smith for valuable comments during the course of this work and J. Guckenheimer, E. Sander, and J. A. Yorke for useful discussions related to large-period phenomena. We would also like to thank S. Revzen for a suggestion regarding a calculation related to the repressilator and J. C. Sprott for information regarding the undamped version of his eponymous system.

2. **Generic families**

2.1. **Background on generic families.** Sotomayor showed that, in the $C^5$ Whitney (or strong $C^5$) topology, there is a residual subset $\mathcal{K}' \subset C^5(Q \times \mathbb{R}, TQ)$ of vector field families such that all periodic orbits of $f \in \mathcal{K}'$ are either hyperbolic or are “quasi-hyperbolic”, meaning that they possess one of three normal forms [Sot73, Thm A]; similar results emphasizing diffeomorphisms rather than flows were obtained by Brunovsky [Bru71a, Bru71b]. Sotomayor’s results in particular imply that every periodic orbit either (i) has no Floquet multipliers which are roots of unity, (ii) is a point of generic saddle-node bifurcation, or (iii) is a point of generic period-doubling bifurcation. An outline of another proof is given in the appendix of [AY84], where the hyperbolic and quasi-hyperbolic

---

3Actually, Sotomayor assumed that $Q$ is compact, replaced the parameter space $\mathbb{R}$ with the circle $S^1$, and considered the weak $C^5$ (or $C^5$ compact-open) topology. However, the same proofs work for noncompact $Q$ and parameter space $\mathbb{R}$ if the $C^5$ Whitney topology is used. Sotomayor does point out that the parameter space can be taken to be $\mathbb{R}$ if the Whitney topology is used in [Sot73, p. 572, Rem. 4].
periodic orbits of [Sot73] are referred to simply as types $0'$, $1'$, and $2'$. Type $1'$ and $2'$ orbits have no multipliers other than $\pm 1$ on the unit circle, but utilizing Lyapunov-Schmidt—rather than center manifold—reduction in our reasoning will enable us to relax this restriction and prove results for a subset $\mathcal{K} \supset \mathcal{K}'$ of families having periodic orbits of three types which are more general than $0'$, $1'$, and $2'$. Following [MPY82, AY83, AMPY83, AY84], we refer to these more general types of orbits as types 0, 1, and 2 (type 0 is actually the same as type $0'$). Note that since the space of $\mathcal{C}^5$ vector field families equipped with the Whitney topology is a Baire space [Hir94, Thm 4.4(b)], it follows that $\mathcal{K}'$—hence also $\mathcal{K}$—is dense in the space of $\mathcal{C}^5$ families. In the sequel, as in the mentioned references we sometimes simply refer to families in $\mathcal{K}$ as “generic”.

In order to provide visual aid for our descriptions of these orbit types, we introduce “orbit diagrams” as in [MPY82, AY83, AMPY83]. We can introduce an equivalence relation $\sim$ on the subset $\mathcal{O} \subset Q \times \mathbb{R}$ of periodic orbits of $f$ so that $(x, \mu) \sim (y, \nu)$ if and only if $\mu = \nu$ and $x, y$ lie on the same periodic orbit. Since the natural projection $\pi_2: \mathcal{O} \rightarrow \mathbb{R}$ descends to a map $\bar{\pi}_2: (\mathcal{O}/ \sim) \rightarrow \mathbb{R}$, we can “plot” $(\mathcal{O}/ \sim)$ as a multi-valued function of $\mu$ with each point representing a periodic orbit of $f$. An example orbit diagram for a generic family is shown in Figure 2. This specific orbit diagram happens to contain orbits of all three types. We now proceed to define orbits of type 0, 1, and 2, which are also illustrated via orbit diagrams in Figure 3.

A type 0 orbit is one which has no Floquet multipliers that are roots of unity. In particular, since $+1$ is not a Floquet multiplier, applying the implicit function theorem to a Poincaré map shows that a type 0 orbit is locally continuable as a function of $\mu$ along a unique branch of orbits on which periods vary continuously.

A type 1 orbit $\gamma$ has a single (algebraically simple) Floquet multiplier equal to $+1$, no other multipliers which are roots of unity, and we require that the eigenvalue $\lambda_1(\mu)$ satisfying $\lambda_1(\mu_0) = 1$ crosses the unit circle with nonzero velocity: $\lambda_1'(\mu_0) \neq 0$. Let $(x_0, \mu_0) \in \Gamma$ be a point on the image $\Gamma$ of $\gamma$ and let $T_0$ be the period of $\gamma$. Letting $S$ be a codimension-1 submanifold intersecting $\Gamma$ transversely at $(x_0, \mu_0)$, $U \subset S$ and $J \subset \mathbb{R}$ sufficiently small neighborhoods of $x_0$ and $\mu_0$, and $P: U \times J \rightarrow S$ a ($\mu$-dependent) Poincaré map, for a type 1 orbit we additionally require that certain generic conditions are satisfied by the partial derivatives of $P$ at $(x_0, \mu_0)$ so that the one-dimensional Lyapunov-Schmidt reduction [GS85, Ch. 1.3] of the equation $(P(x, \mu) - x = 0)$ undergoes a generic saddle node bifurcation at $(x_0, \mu_0)$ (see [Rob99, pp. 241–242]). It follows that there are two unique branches of fixed points of $P$—and hence two branches of periodic orbits for $f$—which approach each other as $\mu$ increases (resp. decreases), coalesce at $\mu = \mu_0$, and disappear for $\mu > \mu_0$ (resp.
\( \mu < \mu_0 \). See Figure 3. Furthermore, it follows from the implicit function theorem that the periods of the orbits corresponding to \( \mu \) in each of these two families asymptotically become equal as \( \mu \to \mu_0 \); additionally, there are no other periodic orbits near \( \Gamma \) having periods near \( T_0 \) except for those orbits on one of the two bifurcating branches described above.

A type 2 orbit \( \gamma \) has a single (algebraically simple) Floquet multiplier equal to \(-1\), no other multipliers which are roots of unity, and we require that the eigenvalue \( \lambda_1(\mu) \) satisfying \( \lambda_1(\mu_0) = -1 \) crosses the unit circle with nonzero velocity: \( \lambda'(\mu_0) \neq 0 \). As above let \( (x_0, \mu_0) \in \Gamma \) and let \( T_0 \) be the period of \( \gamma \). Letting \( P: U \times J \to S \) be a Poincaré map as above, we additionally require that certain generic conditions are satisfied by the partial derivatives of \( P \) at \((x_0, \mu_0)\) so that the one-dimensional Lyapunov-Schmidt reduction of the equation \( (P_\mu \circ P_\mu(x) - x = 0) \) undergoes a standard pitchfork bifurcation at \((x_0, \mu_0)\); this implies that \( P \) undergoes a version of the period-doubling or flip bifurcation at \((x_0, \mu_0)\). The preceding implies that \( \gamma \) is locally continuance as a function of \( \mu \) (since \(+1\) is not a multiplier of \( \gamma \)), and also that there exists an additional branch of periodic orbits bifurcating from \( \gamma \). Furthermore, the (minimal) periods of the orbits on the bifurcating branch at \( \mu \) tend to twice the period of \( \gamma \) as \( \mu \to \mu_0 \), and no orbits on the bifurcating branch sufficiently close to \( \gamma \) have \(+1\) as a multiplier. It follows from the implicit function theorem that the periods of orbits vary continuously when traveling between the two branches of “short” orbits emanating from a type 2 orbit, but that the periods jump by a factor of two when entering the branch of “long” orbits arising from the period-doubling bifurcation. Because the Lyapunov-Schmidt proof of this period-doubling bifurcation is based on the implicit function theorem, it additionally follows that there are no periodic orbits near \( \Gamma \) having periods near \( T_0 \) or \( 2T_0 \) except for those orbits on one of the three branches (two “short” and one “long”) described above.

If \( A \subset Q \times \mathbb{R} \) is a connected component of periodic orbits for a generic family and \( \sim \) is the equivalence relation defined above, it follows from the above discussion that \( A/\sim \) has a fairly simple structure, except possibly for phenomena involving orbits with very large periods; compare with Figure 2. After stating the following definition, we record the properties of \( A/\sim \) we need in Proposition 3.

**Definition 4** (Consistently oriented curves in the Möbius band). Let \( X \) be the Möbius band (with boundary). Let \( \Gamma_1 \) be the middle circle of the Möbius band, \( \Gamma_2 \) be the boundary circle, and let \( \pi: X \to \Gamma_1 \) be the straight-line retraction of \( X \) onto the middle circle. Then (depending on orientations) the degree of \( \pi|_{\Gamma_2}: \Gamma_2 \to \Gamma_1 \) is \( \pm2 \). We say that \( \Gamma_1 \) and \( \Gamma_2 \) are consistently oriented if the degree of \( \pi|_{\Gamma_2} \) is \( +2 \).

**Proposition 3.** Let \( Y \subset Q \times \mathbb{R} \) be an arbitrary subset of nonstationary periodic orbits for a generic family \( f \in K \subset C^5(Q \times \mathbb{R}, TQ) \). Define an equivalence relation \( \sim \) on \( Y \) so that \((x, \mu) \sim (y, \nu)\) if and only if \( \mu = \nu \) and \( x, y \) lie on the same periodic orbit. Let \( \pi: Y \to Y/\sim \) be the quotient map, and let \([x, \mu] := \pi(x, \mu)\) denote the equivalence class of \((x, \mu) \in Y\). If \( \gamma \) is a periodic orbit for \( f_\mu \) with image \( \Gamma \) satisfying \( \Gamma \times \{\mu\} \subset Y \), then by an abuse of notation we let \([\gamma] := [(\gamma(0), \mu)]\).

We have the following.

1. The quotient map \( \pi: Y \to Y/\sim \) is open. If the periods of orbits in \( Y \) are uniformly bounded from above, then \( \pi \) is also closed and \( Y/\sim \) is Hausdorff.

2. Assume that \( Y \) is an open subset of a connected component of nonstationary periodic orbits. If \([\gamma] \in Y/\sim \) is a type 0 or type 1 orbit, then there exists \( \epsilon > 0 \) and a \( C^5 \) homotopy \( H: S^1 \times (-\epsilon, \epsilon) \to Q \times \mathbb{R} \) with the following properties.

   - For each \( s \in (-\epsilon, \epsilon) \), \( H_s := H(\cdot, s) \) is a diffeomorphism onto the image of a periodic orbit in \( Y \).

---

See [Rob99, Thm 7.3.1] for conditions applicable to the one-dimensional case, and [GS85, p. 33, eq. 3.23] for the Lyapunov-Schmidt translation to conditions applicable to the higher-dimensional case.
• For any $z \in S^1$, the map $(-\epsilon, \epsilon) \to Y/ \sim$ given by $s \mapsto \pi \circ H_s(z)$ is a homeomorphism onto a subset $U \subset Y/ \sim$ containing $[\gamma]$, and $\pi \circ H_0(z) = [\gamma]$.

• For every $N > 0$, there exists a neighborhood $V_N \subset Y/ \sim$ of $[\gamma]$ such that $V_N \setminus U$ contains only orbits with periods greater than $N$.

(3) Assume that $Y$ is an open subset of a connected component of nonstationary periodic orbits. If $[\gamma] \in Y/ \sim$ is a type 2 orbit, then there are three disjoint arcs $S_1, S_2, S_3 \subset Y/ \sim$ homeomorphic to open intervals such that

• There exists $\epsilon > 0$ and a $C^5$ homotopy $H : S^1 \times (-\epsilon, \epsilon) \to Q \times \mathbb{R}$ satisfying the same properties as the homotopy in 2, except that the map $s \mapsto \pi \circ H_s(z)$ is a homeomorphism onto $U := S_1 \cup [\gamma] \cup S_2$.

• If $[\alpha] \in S_3$, then there exists a $C^1$ embedded Möbius band $X \subset Q \times \mathbb{R}$ such that, when viewed as subsets of $Q \times \mathbb{R}$, the images of $\gamma$ and $\alpha$ are respectively the middle and boundary circles of $X$, and these images are consistently oriented when given the orientations induced by $\gamma$ and $\alpha$.

• For every $N > 0$, there exists a neighborhood $V_N \subset Y/ \sim$ of $[\gamma]$ such that $V_N \setminus ([\gamma] \cup S_1 \cup S_2 \cup S_3)$ contains only orbits with periods greater than $N$.

(4) Assume that $Y$ is a connected component of periodic orbits. If $(x_n, \mu_n)$ is a sequence of points on the images of periodic orbits $\gamma_n$ with $(x_n, \mu_n) \not\in Y$ but $(x_n, \mu_n) \to (x, \mu) \in Y$ as $n \to \infty$, then the periods $\tau_n$ of the $\gamma_n$ satisfy $\tau_n \to \infty$.

\textbf{Proof.} We begin by proving 1, which is true even without the hypothesis that $f \in K$. Let $\Phi$ be the flow of the vector field $(f, 0)$ on $Q \times \mathbb{R}$. First note that, for any subset $S \subset Q \times \mathbb{R}$, $\pi^{-1}(\pi(S \cap Y))$ is equal to the intersection $Y \cap \bigcup_{t \in T} \Phi^t(S)$, and it additionally equal to the intersection $Y \cap \bigcup_{t \in [0, T]} \Phi^t(S)$ if the periods of orbits through $S$ are bounded above by $T$. Next, note that $\pi$ is an open (closed) map if and only if, for every open (closed) subset $S \subset Q \times \mathbb{R}$, $\pi^{-1}(\pi(S \cap Y))$ is open (closed) in $Y$. It follows that $\pi$ is open since

$$\pi^{-1}(\pi(U \cap Y)) = Y \cap \bigcup_{t \in T} \Phi^t(U)$$

is the intersection of an open subset with $Y$ if $U \subset Q \times \mathbb{R}$ is open. If the periods of $Y$ are bounded above by $T > 0$, then it follows that $\pi$ is closed since

$$\pi^{-1}(\pi(C \cap Y)) = Y \cap \bigcup_{t \in [0, T]} \Phi^t(C)$$

is the intersection of a closed subset with $Y$ if $C \subset Q \times \mathbb{R}$ is closed. To show that $Y/ \sim$ is Hausdorff if the periods of $Y$ have an upper bound $T$, consider any $(x_1, \mu_1), (x_2, \mu_2) \in Y$ belonging to distinct orbits, and let $U_1, U_2 \subset Q \times \mathbb{R}$ be disjoint neighborhoods of the images $\Gamma_1, \Gamma_2 \subset Q \times \mathbb{R}$ of the periodic orbits of $(f, 0)$ through $(x_i, \mu_i)$. For $i \in \{1, 2\}$ we have that $\Phi^{-1}(U_i)$ is an open neighborhood of $\Gamma_i \times \mathbb{R}$ and therefore contains a subset of the form $V_i \times [0, T]$ with $V_i \subset U_i$ an open neighborhood of $\Gamma_i$. Hence $\pi^{-1}(\pi(V_1 \cap Y)) = Y \cap \bigcup_{t \in [0, T]} \Phi^t(V_1) \subset U_1$ and $\pi^{-1}(\pi(V_2 \cap Y)) = Y \cap \bigcup_{t \in [0, T]} \Phi^t(V_2) \subset U_2$ are disjoint open neighborhoods of $\Gamma_1$ and $\Gamma_2$ in $Y$, so it follows that $\pi(Y \cap V_1)$ and $\pi(Y \cap V_2)$ are disjoint neighborhoods of $\pi(\Gamma_1)$ and $\pi(\Gamma_2)$ as desired. This completes the proof of 1.

The existence of homotopies $H$ satisfying the properties claimed in 2 and 3 follows from the discussion preceding Definition 4 and standard techniques from the textbooks cited therein. To show that the neighborhoods $V_N$ of 2 and 3 exist, fix any $N > 0$ and let $\gamma$ be a type 0, 1, or 2 orbit with image $\Gamma$ and period $T_0$. Assume, to obtain a contradiction, that every neighborhood $V \subset Q \times \mathbb{R}$ of $\Gamma$ contains a point $(x, \mu)$ on a periodic orbit having period less than $N$. By the discussion preceding Definition 4 and continuity of the flow, by taking $V$ to be a sufficiently small tubular neighborhood of $\Gamma$ we may assume that the period $T$ of the orbit through any such $(x, \mu)$ satisfies $2T_0 - \epsilon \leq T \leq N$ if $\gamma$ is a type 0 or 1 orbit, and $3T_0 - \epsilon \leq T \leq N$ for the case that $\gamma$ is a
Lemma 2. Let $X$ be a $C^1$ Möbius band (with boundary). Let $\Gamma_1$ be the middle circle of the Möbius band and $\Gamma_2$ the boundary, and assume $\Gamma_1, \Gamma_2$ are consistently oriented. Then if $\eta$ is any $C^1$ closed 1-form on $X$, 

$$\int_{\Gamma_2} \eta = 2 \int_{\Gamma_1} \eta.$$ 

Proof. For $i = 1, 2$, let $\iota_i: \Gamma_i \to X$ be the inclusion. Let $\pi: X \to \Gamma_1$ be the straight line retraction of $X$ onto $\Gamma_1$. Let $h: X \times [0, 1] \to X$ be the straight-line deformation retraction of $X$ onto $\Gamma_1$, with $h_t := h(\cdot, t)$, $h_0 = \text{id}_X$, and $h_1 = \iota_1 \circ \pi$.

By Definition 4, the degree of $\pi|_{\Gamma_2}: \Gamma_2 \to \Gamma_1$ is $+2$. Hence 

$$\int_{\Gamma_2} (\pi|_{\Gamma_2})^* \eta := \int_{\Gamma_2} (\pi|_{\Gamma_2})^* \iota_1^* \eta = 2 \int_{\Gamma_1} \iota_1^* \eta = 2 \int_{\Gamma_1} \eta.$$ 

Since $h_t|_{\Gamma_2}$ yields a homotopy

$$\iota_2 = h_0|_{\Gamma_2} \simeq h_1|_{\Gamma_2} = \iota_1 \circ \pi|_{\Gamma_2},$$

$\iota_2^*: H^1_{\text{dr}}(X) \to H^1_{\text{dr}}(\Gamma_2)$ and $(\iota_1 \circ \pi|_{\Gamma_2})^*: H^1_{\text{dr}}(X) \to H^1_{\text{dr}}(\Gamma_2)$ are the same map on cohomology. It follows that $\iota_2^* \eta = (\iota_1 \circ \pi|_{\Gamma_2})^* \eta + dV$ for some exact 1-form $dV$, so $\iota_2^* \eta$ and $(\iota_1 \circ \pi|_{\Gamma_2})^* \eta$ have the
same integral over $\Gamma_2$. Since $(\nu_1 \circ \pi|_{\Gamma_2}^*) = (\pi|_{\Gamma_2}^*)\nu_1^*$ we obtain
\[\int_{\Gamma_2} \eta := \int_{\Gamma_2} \nu_2^* \eta = \int_{\Gamma_2} (\pi|_{\Gamma_2}^*)\nu_1^* \eta = \int_{\Gamma_2} (\pi|_{\Gamma_2}^*)\eta = 2 \int_{\Gamma_1} \eta,\]
where the last equality follows from (5). This completes the proof.

One of the key ideas needed for Lemma 4 is contained in the following Lemma 3 which shows that, for a generic family, the periodic orbit components $A_\ell^i$ of Definition 3 are topological 1-manifolds if the periods of $A_\ell^i$ are uniformly bounded.

Recall that $\nu_\mu : Q \mapsto Q \times \mathbb{R}$ denotes the inclusion $\nu_\mu(x) = (x, \mu)$ for each $\mu \in \mathbb{R}$.

**Lemma 3.** Let $f \in \mathcal{C}^\infty(Q \times \mathbb{R}, \mathbb{T}Q)$ be a generic family of vector fields and let $\eta$ be a $C^1$ closed 1-form on an open subset $\text{dom}(\eta) \subset Q \times \mathbb{R}$. Let $A \subset \text{dom}(\eta)$ be a connected component of nonstationary periodic orbits of $f|_{\text{dom}(\eta)}$, and let the equivalence relation $\sim$ on $A$ be as in Proposition 3. For any $\ell > 0$, let $A_\ell \subset A$ be the subset of points on periodic orbits $\alpha_\mu$ with $|\int_{\alpha_\mu} \nu_\mu^* \eta| = \ell$ and $A_{\leq \ell}$ the subset with $|\int_{\alpha_\mu} \nu_\mu^* \eta| \leq \ell$.

Fix an open subset $U \subset A/\sim$ and $\ell > 0$, and assume that $U \cap (A_\ell/\sim) \neq \emptyset$. Let $V_\ell$ be a connected component of $U \cap (A_\ell/\sim)$ and let $V_{\leq \ell}$ be the unique component of $U \cap (A_{\leq \ell}/\sim)$ containing $V_\ell$. Assume that the periods of orbits belonging to $V_{\leq \ell}$ have a uniform upper bound. Then

1. $V_\ell$ is a topological 1-manifold (without boundary).
2. If $V_\ell = V_{\leq \ell}$, then $V_\ell$ is also closed as a subset of $U$.

**Proof.** Let $\tilde{V}_{\leq \ell}$ and $\tilde{\nu}_\ell$ be the union of orbits in $A$ with $(\tilde{V}_{\leq \ell}/\sim) = V_{\leq \ell}$ and $(\tilde{\nu}_\ell/\sim) = V_\ell$, respectively. Define the quotient map $\pi : \tilde{V}_{\leq \ell} \to V_{\leq \ell}$. Since the periods of orbits in $\tilde{V}_{\leq \ell}$ are bounded above, part 1 of Proposition 3 implies that $\pi$ is open, $\pi$ is closed, and $\tilde{V}_{\leq \ell}$ is Hausdorff; since $\pi$ is open and $\tilde{V}_{\leq \ell}$ is second countable, so is $V_{\leq \ell}$. It follows that the subspace $\tilde{V}_\ell \subset \tilde{V}_{\leq \ell}$ is also Hausdorff and second countable.

We now show that $V_\ell$ is a topological 1-manifold. Since we have already shown that $V_\ell$ is Hausdorff and second countable, we need only establish that $V_\ell$ is locally Euclidean of dimension 1. If $\alpha$ is a type 0 or type 1 orbit with $[\alpha] \in V_\ell$, then Lemma 1, period-boundedness, and part 2 of Proposition 3 imply that $[\alpha]$ has a neighborhood in $U$ homeomorphic to an open interval and contained in $V_\ell$.

If instead $\alpha$ is a type 2 orbit, then Lemmas 1 and 2, period-boundedness, and part 3 of Proposition 3 imply (since we are assuming $\ell > 0$) that $[\alpha]$ again has a neighborhood in $U$ homeomorphic to an open interval and contained in $V_\ell$. This shows that $V_\ell$ is locally Euclidean and completes the proof that $V_\ell$ is a topological 1-manifold.

We next show that $V_{\leq \ell}$ is closed as a subset of $U$. Being a connected component of $U \cap (A_{\leq \ell}/\sim)$, $V_{\leq \ell}$ is automatically closed in $U \cap (A_{\leq \ell}/\sim)$, so any $[\beta] \in U \cap (A_{\leq \ell}/\sim) \setminus V_{\leq \ell}$ has a neighborhood disjoint from $V_{\leq \ell}$. It remains only to show that any point in $A_\ell$ but not in $A_{\leq \ell}$ has a neighborhood disjoint from $V_{\leq \ell}$. Fix $\tilde{\beta} \in U \setminus (A_{\leq \ell}/\sim)$ with $\beta$ an orbit of $f_\mu$, so that $|\int_{\tilde{\beta}} \nu_\mu^* \eta| = \ell' > \ell$. If $\beta$ is a type 0 or type 1 orbit, then Lemma 1, period-boundedness of $V_{\leq \ell}$, and part 2 of Proposition 3 imply that $[\beta]$ has a neighborhood in $U$ disjoint from $V_{\leq \ell}$. If instead $\beta$ is a type 2 orbit, then part 3 of Proposition 3 implies that, for each $N > 0$, $[\beta]$ has a neighborhood $W \subset U$ such that every $[\gamma] \in W$ having period smaller than $N$ satisfies $|\int_{\gamma} \nu_\mu^* \eta| \in \{\ell', 2\ell'\}$, where $\mu'$ is such that $\gamma$ is an orbit of $f_{\mu'}$. Taking $N$ to be larger than an upper bound for the periods of $V_{\leq \ell}$ and using the fact that $\ell' > \ell$, it follows that $W \cap V_{\leq \ell} = \emptyset$. This completes the proof that $V_{\leq \ell}$ is closed in $U$.

Since $V_{\leq \ell}$ is closed in $U$, the additional assumption that $V_\ell = V_{\leq \ell}$ implies that $V_\ell$ is closed in $U$. This completes the proof.

---

5Strictly speaking, we are also using the fact that every $C^1$ closed form is cohomologous to a $C^\infty$ closed form [dR84, pp. 61–70] since we only assume $\eta \in C^1$. 
The main idea behind Lemma 3 is that its hypotheses imply that portions of orbit diagrams such as the one shown above cannot occur if the corresponding periodic orbits are contained in $\text{dom}(\eta)$ and have uniformly bounded periods. In more detail: if the periodic orbit $\gamma$ represented by the above dot at $\mu_0$ satisfies 

$$\ell := \left\| \iota_\mu^* \mu_0 \eta \right\| > 0,$$

then the orbit diagram above cannot occur for a generic one-parameter family. To see this, let $\beta$ be a periodic orbit represented by a point in the top of the loop at $\mu_1$. There is a homotopy of periodic orbits corresponding to the path indicated by the arrows above, so homotopy invariance (Lemma 1) implies that 

$$\left\| \iota_\mu^* \mu_1 \eta \right\| = \ell.$$ 

On the other hand, applying Lemma 2 to the branch of bifurcating orbits near the type 2 orbit implies that 

$$\left\| \iota_\mu^* \mu_1 \eta \right\| = 2\ell \neq \ell,$$

a contradiction.

We now state the main result of this section. Lemma 4 yields a result for general families slightly stronger than Theorem 1, because it does not require the hypothesis that $+1$ is not a Floquet multiplier of the periodic orbit $\gamma$.

**Lemma 4** ($\eta, \ell$-global continuability for generic families). Let $f \in \mathcal{K} \subset C^5(Q \times \mathbb{R}, TQ)$ be a generic family of vector fields, and let $\eta$ be a $C^1$ closed 1-form on an open subset $\text{dom}(\eta) \subset Q \times \mathbb{R}$. Let $A \subset \text{dom}(\eta)$ be a connected component of nonstationary periodic orbits of $f|_{\text{dom}(\eta)}$. Let $\gamma$ be a periodic orbit for some $f_{\mu_0}$ with image $\Gamma$ satisfying $\Gamma \times \{\mu_0\} \subset A$, define $\ell := \left| \iota_\gamma^* \mu_0 \eta \right|$, and assume $\ell > 0$. Then $\gamma$ is $(\eta, \ell)$-globally continuable.
Remark 1. The following proof is similar in spirit to the proof of [AY84, Thm 2.2] with “orbits \( \alpha_\mu \) satisfying \( |f_{\alpha_\mu} \alpha_\mu' \mu' \eta | = \ell \)” playing the role of “non-Möbius orbits.” (Recall that a Möbius orbit is a periodic orbit which has an odd number of Floquet multipliers in \((-\infty, 1)\) and which additionally has no multiplier equal to \(-1\).)

Proof. We use the notation of Definition 3, and identify \( f \in A \leq \ell \). Assume that \( \Gamma = \{ \gamma \in \{(A/\sim) \}\} / \sim \) is compact and contains no generalized centers, and that the periods of orbits in \( A^1_\ell \) have a uniform upper bound.

Since \( (A/\sim) \sim \) is an open subset of \( (A/\sim) \), Lemma 3 implies that \( (A^1_\ell / \sim) \subset (A/\sim) / \sim \). \( (A^1_\ell / \sim) \) is not compact because it has the sole limit point \( [\gamma] \in (A/\sim) \setminus (A^1_\ell / \sim) \). Therefore, \( (A^1_\ell \cup \Gamma) / \sim \) is closed as a subset of \( A/\sim \), and the classification theorem for topological 1-manifolds implies that \( (A^1_\ell \cup \Gamma) / \sim \) is homeomorphic to \([0, 1]\).

To complete the proof it suffices to show that \( A^1_\ell \cup \Gamma \) is closed in \( \text{dom}(\eta) \) and therefore compact, because this would imply that \( (A^1_\ell \cup \Gamma) / \sim \) is compact, contradicting the fact that \( (A^1_\ell \cup \Gamma) / \sim \) is homeomorphic to \([0, 1]\).

So let \( (x, \mu) \not\in (A^1_\ell \cup \Gamma) \) be a limit point of \( (A^1_\ell \cup \Gamma) \) in \( \text{dom}(\eta) \). Then there is a sequence \( (x_n, \mu_n) \) in \( A^1_\ell \cup \Gamma \) of points on periodic orbits \( \gamma_n \) with \( (x_n, \mu_n) \to (x, \mu) \). Let \( \tau_n \) be the period of \( \gamma_n \). Since we are assuming that the periods of \( A^1_\ell \) are bounded, we may pass to a subsequence and assume that \( \tau_n \to \tau > 0 \). Letting \( \Phi^\tau_\mu \) be the flow of \( f_\mu \), by continuity we have

\[
\Phi^\tau_\mu(x) = \lim_{n \to \infty} \Phi^\tau_{\mu_n}(x_n) = \lim_{n \to \infty} x_n = x.
\]

Since we are assuming that the closure \( \text{cl}(A^1_\ell) \) of \( A^1_\ell \) in \( \text{dom}(\eta) \) contains no generalized centers, it follows that \( (x, \mu) \in \text{cl}(A^1_\ell \cup \Gamma) \) must be a nonstationary periodic orbit for \( f_\mu \).\(^6\) It cannot be the case that \( (x, \mu) \) belongs to a component \( B \subset \text{dom}(\eta) \) of periodic orbits of \( f|_{\text{dom}(\eta)} \) different from \( A \), because this would contradict the fact that \( A \) is closed in the space of periodic orbits of \( f|_{\text{dom}(\eta)} \) (being a connected component). Hence \( (x, \mu) \in A \), and since \( A^1_\ell \cup \Gamma \) is closed in \( A \) it follows that \( (x, \mu) \in A^1_\ell \cup \Gamma \). Hence \( A^1_\ell \cup \Gamma \) is closed in \( \text{dom}(\eta) \). As discussed above, this implies a contradiction and completes the proof. \(\Box\)

3. Non-generic families

In this section, we prove our main theorems on global continuation of periodic orbits for arbitrary \( C^1 \) families of vector fields. Before doing this, we require one additional lemma. Lemma 5 enables us to prove Theorem 1 without the consideration of “virtual periods” as required in [AY84, Thm 3.1, Lem. 3.2].

Lemma 5. Let \( f_n \in C^1(M, TM) \) be a sequence of \( C^1 \) vector fields on a smooth manifold \( M \) which converge in the weak \( C^1 \) topology to a \( C^1 \) vector field \( f \) on \( M \), and let \( \eta \) be a \( C^1 \) closed 1-form on \( M \). For each \( n \) let \( \gamma_n \) be a periodic orbit of \( f_n \) with image \( \Gamma_n \) and (minimal) period \( \tau_n \), and let \( \gamma \) be a periodic orbit of \( f \) with image \( \Gamma \) and (minimal) period \( \tau \). Assume that the periods \( \tau_n \) have a uniform upper bound, and assume that for each \( n \) there exists \( x_n \in \Gamma_n \) such that \( x_n \to x_0 \in \Gamma \). Then

\[
(1) \lim \inf_{n \to \infty} \left| f_{\gamma_n} \eta \right| \geq \left| f_{\gamma} \eta \right| \ \text{and} \ \lim \inf_{n \to \infty} \tau_n \geq \tau;
\]

\(^6\)This is because, if \( x \) were an equilibrium for \( f_\mu \), then the boundedness of the \( \tau_n \) would imply that \( x \) is a generalized center for \( f_\mu \). This is true even if \( f \in C^1 \), and follows from [CMPY83, Prop. 3.2]; see also [Fie88, Cor. 4.6]. A proof for the case \( f \in C^2 \) is given in [MPY82, Prop. 3.1].
(2) \( \lim_{n \to \infty} |f_{\gamma_n} \eta| = |f_\eta \eta| \) if and only if \( \lim_{n \to \infty} \tau_n = \tau \).

**Proof.** We begin with some preparations. Let \( \pi: U \to \Gamma \) be a \( C^1 \) tubular neighborhood of \( \Gamma \) with \( U \) precompact, so in particular \( \pi \) is a submersion and retraction. Since \( \pi|_\Gamma = id_\Gamma \), by shrinking \( U \) we may assume by continuity that \( D_y \pi f(y) \neq 0 \) for all \( y \in U \). Since \( f_n \to f \) uniformly on \( U \), there exists \( N_0 > 0 \) such that the same is true of \( D_y \pi f_n(y) \) for all \( n > N_0 \). Since the periods of the \( \gamma_n \) have a uniform upper bound and since \( f_n \to f \), continuous dependence of a flow on its vector field implies that there exists \( N_1 > N_0 \) such that \( \Gamma_n \subset U \) for all \( n > N_1 \). Hence \( \pi|_{\Gamma_n}: \Gamma_n \to \Gamma \) is well-defined and an orientation-preserving local diffeomorphism for \( n > N_1 \), where \( \Gamma \) and \( \Gamma_n \) are given the orientations induced by \( \gamma \) and \( \gamma_n \).

Next, since \( f \) is transverse to the manifold \( S_0 := \pi^{-1}(x_0) \), the implicit function theorem implies that there is a well-defined \( C^1 \) “first impact time map” \( t_f: S_1 \to S_0 \) from a neighborhood \( S_1 \subset S_0 \) of \( x_0 \) to \( S_0 \), with \( t_f(y) \) defined to be the smallest positive real number such that \( \Phi^{t_f(y)}(y) \in S_0 \), where \( y \in S^1 \) and \( \Phi_f \) is the local flow of \( f \). By the implicit function theorem, \( t_f(y) \) is a fortiori jointly continuous in \( y \) and \( f \) in the \( C^1 \) topology. Let \( N_2 > N_1 \) be such that \( \Gamma_n \cap S_0 \subset S_1 \) for all \( n > N_2 \). In the remainder of the proof, assume \( n > N_2 \).

We now proceed with the proof of 1. First, note that for any \( y_n \in \Gamma_n \cap S_1 \), the definition of the first impact time map implies

\[
(7) \quad t_{f_n}(y_n) \leq \tau_n.
\]

Since the impact time map is continuous and since \( y_n \to x_0 \), the left hand side converges to \( \tau \). This proves the statement about the periods in 1. Next, since \( \pi|_{\Gamma_n} \) is an orientation-preserving local diffeomorphism, it follows that the degree \( d_n \) of \( \pi|_{\Gamma_n} \) satisfies \( d_n \geq 1 \). Since \( U \) deformation retracts onto \( \Gamma \), the inclusion \( \Gamma_n \hookrightarrow U \) is homotopic to the composition of \( \pi|_{\Gamma_n} \) with the inclusion \( \Gamma \hookrightarrow U \). Hence we have

\[
(8) \quad \left| \int_{\gamma_n} \eta \right| = \left| \int_{\Gamma_n} (\pi|_{\Gamma_n})^* \eta \right| = d_n \left| \int_{\gamma} \eta \right| \geq \left| \int_{\gamma} \eta \right|.
\]

This completes the proof of 1.

Next, note that (8) implies that \( \lim_{n \to \infty} |f_{\gamma_n} \eta| = |f_\eta \eta| \) if and only if \( \lim_{n \to \infty} d_n = 1 \). Since \( \pi|_{\Gamma_n}: \Gamma_n \to \Gamma \) is an orientation-preserving local diffeomorphism, this in turn holds if and only if \( \Gamma_n \) intersects \( S_1 \) in a single point for all sufficiently large \( n \). And by the definition of the first impact time map, this latter statement holds if and only if \( t_{f_n}(y_n) = \tau_n \) for all sufficiently large \( n \), where \( y_n \in \Gamma_n \cap S_0 \). So to prove 2, it suffices to prove that this final statement holds if and only if \( \lim_{n \to \infty} \tau_n = \tau \).

Assume that \( t_{f_n}(y_n) = \tau_n \) for all large \( n \). Since \( t_{f_n}(y_n) \to \tau \), it follows that \( \tau_n \to \tau \). Conversely, assume that there exists a subsequence \( n_k \to \infty \) arbitrarily large with \( t_{f_{n_k}}(y_{n_k}) < \tau_n \). Then

\[
(9) \quad t_{f_{n_k}} \left( \Phi^{t_{f_{n_k}}(y_{n_k})}(y_{n_k}) \right) + t_{f_{n_k}}(y_{n_k}) \leq \tau_{n_k}.
\]

By continuity of the impact time map and of \( \Phi \) with respect to all arguments and the fact that \( y_n \to x_0 \), the left hand side converges to \( 2t_f(x_0) = 2\tau \). Hence \( \lim sup_{n \to \infty} \tau_n > \tau \). This proves 2. \( \square \)

**Theorem 1** \((\eta, \ell)\)-global continuability for non-generic families. Let \( f \in C^1(Q \times \mathbb{R}, TQ) \) be a family of vector fields, and let \( \eta \) be a \( C^1 \) closed 1-form on an open subset \( \text{dom}(\eta) \subset Q \times \mathbb{R} \). Let \( \gamma \) be a periodic orbit of some \( f_{\mu_0} \) with image \( \Gamma \) satisfying \( \Gamma \times \{ \mu_0 \} \subset \text{dom}(\eta) \), and assume that \( \gamma \) does not have \(+1\) as a Floquet multiplier. Define \( \ell := |f_\gamma \pi_{\mu_0} \eta| \), and assume \( \ell > 0 \). Then \( \gamma \) is \((\eta, \ell)\)-globally continuable.
Remark 2. Our proof is inspired by the proof of [AY84, Thm 3.1, Lem. 3.2], and we have tried to keep our proof similar to theirs in an effort to make the similarities and differences readily discernible, although we have added some details. One key difference is that there is no mention of “virtual periods” anywhere in our proof; using Lemma 5, their role is instead filled by integrals of the form \([f \tau_\mu^* \eta]\). This difference also explains why [AY84, Lem 3.2] requires the assumption that \(\gamma\) has no Floquet multipliers which are roots of unity, whereas we need only assume that +1 is not a multiplier. Another key difference in our proof is that our definition of the function \(F\) in (9) below differs from the definition of \(F\) in the proof of [AY84, Lem 3.2] in that we have added a second term imposing a “cost” for \([f \tau_\mu^* \eta]\) to deviate from \(\ell\).

Proof. The weak \(C^1\) (or \(C^1\) compact-open) topology on \(C^1(Q \times \mathbb{R}, \mathbb{T}Q)\) is (completely) metrizable, and this induces a metric on the closed subspace of one-parameter families of vector fields [Hir94, p. 62]. Throughout the remainder of this proof, we denote this metric by \(d_{C^1}(\cdot, \cdot)\). In the following, we identify the images of periodic orbits such as \(p\). Throughout the remainder of this proof, we denote this metric by \(d_{C^1}(\cdot, \cdot)\). In the following, we identify the images of periodic orbits such as \(p\).

Remark 2. Our proof is inspired by the proof of [AY84, Thm 3.1, Lem. 3.2], and we have tried to keep our proof similar to theirs in an effort to make the similarities and differences readily discernible, although we have added some details. One key difference is that there is no mention of “virtual periods” anywhere in our proof; using Lemma 5, their role is instead filled by integrals of the form \([f \tau_\mu^* \eta]\). This difference also explains why [AY84, Lem 3.2] requires the assumption that \(\gamma\) has no Floquet multipliers which are roots of unity, whereas we need only assume that +1 is not a multiplier. Another key difference in our proof is that our definition of the function \(F\) in (9) below differs from the definition of \(F\) in the proof of [AY84, Lem 3.2] in that we have added a second term imposing a “cost” for \([f \tau_\mu^* \eta]\) to deviate from \(\ell\).

Assume that \(\gamma\) is not \((\eta, \ell)\)-globally continuable. Using the notation of Definition 3 (with \(B \subset \text{dom}(\eta)\) replacing \(A\)), it follows that \(\overline{B}_{\leq \ell} \setminus \Gamma\) is disconnected and has a component \(\overline{B}_{\leq \ell}^1\) which contains a component \(\overline{B}_{\leq \ell}^1\) of \(\overline{B}_{\leq \ell} \setminus \Gamma\) such that \(\overline{B}_{\leq \ell}^1\) and \(\overline{B}_{\leq \ell}^1\) satisfy none of the conditions of Definition 3. In particular, \(\overline{B}_{\ell}^1 = \overline{B}_{\leq \ell}^1\), and \(f_\mu(x) \neq 0\) for all \((x, \mu) \in \text{cl}(\overline{B}_{\ell}^1)\). Since we assume that +1 is not a Floquet multiplier of \(\gamma\), it follows from the implicit function theorem applied to a Poincaré map and Lemma 5 that there is a relatively open neighborhood \(W \subset Q \times \{\mu_0\}\) of \(\Gamma\) with \(W \subset \text{dom}(\eta)\) and such that \(\gamma\) is the only periodic orbit in \(W\) on which \([f \tau_\mu^* \eta]\) is \(\leq \ell\) except, perhaps, for orbits of very long period.

Let

\[
p_0 := \inf_{(x, \mu) \in \overline{B}_{\leq \ell}^1} \{\tau: \tau \text{ is the period of the orbit through } (x, \mu)\}
\]

and

\[
p_1 := \sup_{(x, \mu) \in \overline{B}_{\leq \ell}^1} \{\tau: \tau \text{ is the period of the orbit through } (x, \mu)\}.
\]

Note that \(p_0 > 0\) [Yor69] and \(p_1 < \infty\) by our assumptions. If \(t \mapsto \Phi_\mu^t(x)\) is the trajectory of \(f\) through \((x, \mu)\), we define the function

\[
F(x, \mu) := \min_{\frac{1}{2}p_0 \leq t \leq 2p_1} d(\Phi_\mu^t(x), x) + \|\ell - \int_{\Phi_\mu^{[0,\tau]}(x)} \tau_\mu^* \eta\|,
\]

on \(\text{dom}(\eta)\), where \(d(\cdot, \cdot)\) is the distance associated to some Riemannian metric on \(Q\). The set of zeros of \(F\) are points on the images of periodic orbits \(\alpha\) of \(f\) such that \(\ell\) is an integer multiple of \([f \tau_\mu^* \eta]\). Loosely speaking, \(F\) measures how close the trajectory through \((x, \mu)\) is to being periodic and satisfying \(\ell/\|f \tau_\mu^* \eta\| \in \mathbb{N}\), for periods between \(\frac{1}{2}p_0\) and \(2p_1\). Since \(\Phi\) is continuous in \((x, \mu, t)\), \(F\) is continuous in \((x, \mu)\).

For \(\epsilon > 0\) let \(N_\epsilon := \{(x, \mu) \in \text{dom}(\eta): F(x, \mu) \leq \epsilon\}\), and let \(N_\epsilon^0\) be the component of \(N_\epsilon\) containing \(\overline{B}_{\leq \ell}^1\). Choose \(\epsilon\) small enough so that

1. the component \(M_\epsilon\) of \(N_\epsilon^0 \cap \{Q \times \{\mu_0\}\}\) containing \(\Gamma\) is a subset of \(W\);
2. \(N_\epsilon^0 \setminus W\) is disconnected, and we denote by \(N_\epsilon^1\) the component containing \(\overline{B}_{\ell}^1\);
3. there are no zeros of \(f\) in the closure \(\text{cl}(N_\epsilon^1)\) of \(N_\epsilon^1\) in \(\text{dom}(\eta)\);
(4) The closure \( \text{cl}(N^1_\ell) \) of \( N^1_\ell \) in \( \text{dom}(\eta) \) is compact;

(5) there exists \( \rho_1 > 0 \) such that, when \( d_{C^1}(f, g) < \rho_1 \), the system \( \dot{x} = g(x, \mu_0) \) will have exactly one periodic orbit \( \gamma_g \) in \( M_\ell \) having period \( \leq 2p_1 \) and satisfying \( \left| \int_{\gamma_g} \iota_{\mu_0}^* \eta \right| \leq \ell \), and this orbit satisfies \( \left| \int_{\gamma_g} \iota_{\mu_0}^* \eta \right| = \ell \) and does not have +1 as a Floquet multiplier;

(6) there exists \( \rho_2 > 0 \) such that, when \( d_{C^1}(f, g) < \rho_2 \), the system \( \dot{x} = g(x, \mu) \) will have no orbits \( \alpha \) contained in \( N^\ell_\ell \) satisfying either (i) the period of \( \alpha \) is \( \leq 2p_1 \) and \( \left| \int_{\alpha} \iota_{\mu}^* \eta \right| < \ell \), or (ii) the period of \( \alpha \) belongs to \( J = (-\infty, \frac{2}{3}p_0) \cup [\frac{2}{3}p_1, \frac{2}{3}p_1] \) and \( \left| \int_{\alpha} \iota_{\mu}^* \eta \right| = \ell \).

By the definition of \( F \) and the sentence preceding the definition of \( p_0 \), it follows that \( \Gamma = F^{-1}(0) \cap W \). It follows that \( F \) attains a minimum \( m > 0 \) on the compact boundary of a tubular neighborhood of \( \Gamma \) in \( W \). Taking \( \epsilon < m \) ensures that 1 is satisfied.

We now argue that conditions 2, 3, and 4 can be satisfied by taking \( \epsilon \) sufficiently small. Let \( U \subset \text{dom}(\eta) \) be an arbitrary precompact open neighborhood of \( \tilde{B}^1_\ell = \tilde{B}^1_{\ell, \ell} \) such that (i) \( \Gamma \cup \tilde{B}^1_\ell \) is a connected component of \( \text{cl}(U) \cap F^{-1}(0) \) and (ii) \( \Gamma \cap \text{cl}((\partial U) \setminus W) = \emptyset \).\(^7\) We claim that, for all \( \epsilon > 0 \) sufficiently small, the component \( \tilde{N}^0_\ell \) of \( (N^0_\ell \cap \text{cl}(U)) \setminus W \) containing \( \tilde{B}^1_\ell \) is contained in \( \text{int}(U) = U \).

If not, there is a sequence \( (\epsilon_i)_{i \in \mathbb{N}} \) decreasing to zero with \( \tilde{N}^0_{\epsilon_i} \cap (\partial U) \setminus W \neq \emptyset \) for all \( i \in \mathbb{N} \). Since \( \text{cl}(\tilde{N}^0_{\epsilon_i}) \supset \text{cl}(N^0_{\epsilon_i}) \supset \cdots \) is a decreasing sequence of compact sets having nonempty intersection with \( (\partial U) \setminus W \), it follows that \( \text{cl}((\partial U) \setminus W) \cap \text{cl}(N^0_{\epsilon_i}) \neq \emptyset \). Since the intersection of any decreasing sequence of compact connected subsets of a Hausdorff space is always connected, it follows that \( \cap_{i \in \mathbb{N}} \text{cl}(N^0_{\epsilon_i}) \) is a connected subset of \( \text{cl}(U) \cap F^{-1}(0) \) containing both \( \Gamma \cup \tilde{B}^1_\ell \) and some point in \( \text{cl}((\partial U) \setminus W) \). \((\Gamma \cup \tilde{B}^1_\ell) \cap \text{cl}((\partial U) \setminus W) = \emptyset \) by (ii) and the fact that \( U \) is a neighborhood of \( \tilde{B}^1_\ell \), so \( \Gamma \cup \tilde{B}^1_\ell \) is a proper subset of the connected set \( \cap_{i \in \mathbb{N}} \text{cl}(N^0_{\epsilon_i}) \subset (\text{cl}(U) \cap F^{-1}(0)) \); this contradicts (i), so the claim that \( N^0_\ell \subset \text{int}(U) \) for all sufficiently small \( \epsilon \) is proved. From this claim it follows that \( N^0_\ell \setminus W \) is disconnected for sufficiently small \( \epsilon \) (with the component \( N^1_\ell \) in \( N^0_\ell \setminus W \) containing \( \tilde{B}^1_\ell \) being equal to \( \tilde{N}^0_\ell \)), proving that we may choose \( \epsilon \) small enough so that 2 is satisfied; 4 is automatically satisfied since \( U \) is precompact and \( \text{cl}(N^1_\ell) \subset \text{cl}(U) \). Since \( f \) is bounded away from zero on the compact set \( \text{cl}(\tilde{B}^1_\ell) \subset \text{cl}(U) \), and since the precompact neighborhood \( U \) satisfying (i-ii) was arbitrary, we may ensure that 3 is satisfied by shrinking \( U \) so that \( f \) is nonzero on \( \text{cl}(U) \) and choosing \( \epsilon \) sufficiently small so that \( N^0_\ell \subset \text{int}(U) \) as above. Since \( U \) satisfying (i-ii) was arbitrary, the above discussion also implies the following fact which we will use:

\[
\bigcap_{\epsilon \in (0, \epsilon_0)} N^1_\ell = \tilde{B}^1_\ell, \tag{10}
\]

where \( \epsilon_0 \) is sufficiently small so that \( N^0_\ell \setminus W \) is disconnected and \( N^1_\ell \) is well-defined.

Let \( \tau_0 \) be the period of \( \beta \). To show that condition 5 can be satisfied by taking \( \epsilon \) sufficiently small, we argue as follows. First, note that the implicit function theorem applied to a Poincaré map and Lemma 5 imply that there are \( \epsilon_1, \rho_1 \) such that \( \dot{x} = g(x, \mu) \) will have only one orbit \( \gamma_g \) in \( M_\ell \) satisfying \( \left| \int_{\gamma_g} \iota_{\mu}^* \eta \right| = \ell \) and having period \( \leq 2p_1 \) when \( d_{C^1}(f, g) < \rho_1 \), and by choosing \( \epsilon_1 \)

\(^7\)Here is an explicit construction of such a \( U \). Let \( d(\cdot, \cdot) \) be the distance induced by any complete Riemannian metric on \( Q \times \mathbb{R} \). Since \( +1 \) is not a multiplier of \( \Gamma \), we may choose \( \delta > 0 \) small enough so that the \( \delta \)-neighborhood \( V_\delta := \{(x, \mu) \in Q \times \mathbb{R} : d((x, \mu), \Gamma) < \delta \} \) of \( \Gamma \) contains at most one periodic orbit of \( F^{-1}(0) \cap (Q \times \{\mu_0\}) \) for each \( \mu \) and satisfies \( V_\delta \cap (Q \times \{\mu_0\}) \subseteq W \). It follows that of \( V_\delta \setminus W \) is disconnected, and the triangle inequality further implies that the \( \frac{\epsilon}{2} \)-neighborhood \( \{(x, \mu) \in (Q \times \mathbb{R}) \setminus \text{cl}(W) : d((x, \mu), \tilde{B}^1_\ell) < \frac{\epsilon}{2} \} \) of \( \tilde{B}^1_\ell \) consists of precisely two connected components, one of which contains \( \tilde{B}^1_\ell \) and is disjoint from \( \tilde{B}^2_{\ell, \ell} \cap V_\delta \); define \( U \) to be this component. Property (i) follows since, e.g., \( \tilde{B}^1_\ell \) is a connected component of \( F^{-1}(0) \cap U \) and (ii) follows since \( d((x, \mu), \Gamma) = \frac{\epsilon}{2} \) for all \( (x, \mu) \in \text{cl}((\partial U) \setminus W) \). Note that \( U \) is precompact since it is bounded and the metric inducing \( d(\cdot, \cdot) \) is complete.
small enough we can ensure that \( \gamma_g \) has no Floquet multiplier equal to +1. Suppose that there exist sequences \((\epsilon_i)_{i \in \mathbb{N}}\) and \((\rho_i)_{i \in \mathbb{N}}\) decreasing to zero and \((g_i)_{i \in \mathbb{N}}\) satisfying \( d_{C^1}(f, g_i) < \rho_i \) and with each \( g_i \) having a periodic orbit \( \gamma_i \) in \( M_{\epsilon_i} \) having period \( \leq 2p_1 \) and satisfying \( \int_{\gamma_i} \nu_{\mu_i}^* \eta < \ell \).

The images of \( \gamma_i \) converge uniformly to the image of \( \gamma \) since \( \epsilon_i \to 0 \) and since the periods of the \( \gamma_i \) are bounded, so Lemma 5 implies that \( \int_{\gamma} \nu_{\mu}^* \eta < \ell \), a contradiction. Hence we may ensure the satisfaction of condition 5 by choosing \( \epsilon \) sufficiently small.

To show that condition 6 may be satisfied, we argue as follows. Suppose that, for sequences \((\epsilon_i)_{i \in \mathbb{N}}\) and \((\rho_i)_{i \in \mathbb{N}}\) decreasing to zero, there exist a sequence of functions \((g_i)_{i \in \mathbb{N}}\) with \( d_{C^1}(f, g_i) < \rho_i \) and a corresponding sequence of orbits \((\alpha_i)_{i \in \mathbb{N}}\) such that \( \alpha_i \) is a periodic orbit of \( \dot{x} = g_i(x, \mu) \) satisfying at least one of (i, ii) of condition 6. Choose a point \((x_i, \mu_i) \in N_{\epsilon_i}^1 \) on the image of \( \alpha_i \) for each \( i \). Since \( N_{\epsilon_i} \) is precompact for \( i \) large, the \((x_i, \mu_i)\) will have an accumulation point, and (10) implies that this accumulation point belongs to the image of a periodic orbit \( \alpha \) of \( f \) contained in \( \bar{B}_2^1 \cup \Gamma \). The property (ii) cannot be satisfied, since if the \( \alpha_i \) satisfy \( \int_{\alpha_i} \nu_{\mu_i}^* \eta = \ell \) for all \( i \), then Lemma 5 implies that the periods \( \tau_i \) of the \( \alpha_i \) satisfy \( \tau_i \to \tau \in [p_0, p_1] \), so \( \tau_i \not\in J \) for all sufficiently large \( i \). And if the property (i) is satisfied, so that \( \tau_i \leq 2p_1 \) and \( \int_{\alpha_i} \nu_{\mu_i}^* \eta < \ell \) for all \( i \), then Lemma 5 implies that \( \alpha \) satisfies \( \int_{\alpha} \nu_{\mu}^* \eta < \ell \), contradicting \( \int_{\alpha} \nu_{\mu}^* \eta = \ell \). Hence we may ensure the satisfaction of condition 6 by choosing \( \epsilon \) sufficiently small.

Following the choice of \( \epsilon \), we let \( g_\epsilon \in K \) be a generic family sufficiently close to \( f \) in the weak \( C^1 \) topology so that

1. \( d_{C^1}(f, g_\epsilon) < \min\{\rho_1, \rho_2\} \);
2. \( \min_{(x, \mu) \in cl(N_{\epsilon}^1)} \|g(x, \mu)\| > \frac{1}{2} \min_{(x, \mu) \in cl(N_{\epsilon}^1)} \|f(x, \mu)\| \); and
3. \( \max_{(x, \mu) \in cl(N_{\epsilon}^1)} |F_G| < \frac{\epsilon}{2} \), where \( G_\epsilon \) is defined analogously to \( F \) for solutions of \( \dot{x} = g_\epsilon(x, \mu) \) (again using \( \ell \) and \( \frac{1}{4} p_0 \leq t < 2p_1 \)).

Condition 1 can be satisfied since the set of \( C^5 \) vector field families is dense in the space of \( C^1 \) vector field families [Hir94, Ch. 2.2, Ex. 3], and \( K \) is dense in the space of \( C^5 \) families [Sot73, Thm A] as discussed in §2.1. Similar reasoning implies that condition 2 can be satisfied, using also the fact that \( f \) has no zeros in the compact set \( cl(N_{\epsilon}^1) \). To show that \( g_\epsilon \) can be chosen to satisfy condition 3, we argue as follows. Let \( \Psi_{\mu}^t(x) \) be the solution of

\[
\dot{x} = g_\epsilon(x, \mu)
\]

through \((x, \mu)\). Since \( cl(N_{\epsilon}^1) \) is compact and since flows depend continuously on their vector fields, \( g_\epsilon \) can be chosen so that

\[
\left| \left( d(\Phi_{\mu}^t(x), x) + \ell - \int_{\Psi_{\mu}^t(x)} \nu_{\mu}^* \eta \right) - \left( d(\Psi_{\mu}^t(x), x) + \ell - \int_{\Psi_{\mu}^t(x)} \nu_{\mu}^* \eta \right) \right| < \frac{\epsilon}{2}
\]

for all \((x, \mu) \in cl(N_{\epsilon}^1)\) and \( \frac{1}{2} p_0 \leq t < 2p_1 \). Since in general two real-valued functions \( P, Q \) uniformly satisfying \( |P - Q| < \frac{\epsilon}{2} \) must also satisfy \( \min P - \min Q < \frac{\epsilon}{2} \), if follows that \( |F - G_\epsilon| < \frac{\epsilon}{2} \) uniformly on \( cl(N_{\epsilon}^1) \).

Let \( \gamma_\epsilon \) be the unique solution of (11) in \( M_\epsilon \) having period \( \leq 2p_1 \) and satisfying \( \int_{\gamma_\epsilon} \nu_{\mu}^* \eta = \ell \), let \( \Gamma_\epsilon \) be the image of \( \gamma_\epsilon \), and define the sets \( \bar{A}_{\epsilon, \ell}, \bar{A}_{\epsilon, \leq \ell} \) as in Definition 3 (with \( A \) replacing \( A \)). Define \( Y : = (\bar{A}_{\epsilon, \ell} \setminus \Gamma_\epsilon) \cap cl(N_{\epsilon}^1) \) and \( Z : = (\bar{A}_{\epsilon, \leq \ell} \setminus \Gamma_\epsilon) \cap cl(N_{\epsilon}^1) \). Because +1 is not a Floquet multiplier of \( \gamma_\epsilon \), \( Y \) and \( Z \) are not empty. We want to show that \( Y \) is contained in \( int(N_{\epsilon}^1) \) and that \( Y = Z \). We begin with the first statement. Now \( Y \) can only escape from \( int(N_{\epsilon}^1) \) through \( \partial N_{\epsilon}^1 \subset dom(\eta) \) i.e., (i) through \( X : = \partial N_{\epsilon}^1 \cap N_{\epsilon}^1 \) or (ii) through \( M_\epsilon \). We discuss each case separately.

Suppose \((x, \mu) \in Y \cap X \). Then by condition 3 on \( g_\epsilon \), \((x, \mu)\) must be on an orbit with period \( \tau \), where \( \tau \in (-\infty, \frac{1}{2} p_0) \cup (2p_1, \infty) \). By taking \( \epsilon \) smaller, we may assume that \( \gamma_\epsilon \) is sufficiently near \( \gamma \).
that the period of $\gamma_\epsilon$ belongs to $(\frac{3}{2}p_0, \frac{4}{3}p_1)$. Suppose $\tau > 2p_1$. Then $Y$ contains orbits with periods less than $\frac{2}{3}p_1$ and orbits with periods greater than $2p_1$. Since periods on a branch of $(f_\epsilon^*p_\epsilon)$-constant orbits of a generic family change continuously, there must be an orbit $\alpha$ with image in $Y$ and with period in $[\frac{2}{3}p_1, \frac{4}{3}p_1]$, and no orbit on the “path” from $\gamma_\epsilon$ to $\alpha$ with period greater than $\frac{5}{3}p_1$. But then it is easily seen that the image of $\alpha$ is contained in $N^1_\epsilon$, contradicting condition 6 on the choice of $\epsilon$. A similar argument shows that $\tau < \frac{1}{2}p_0$ would also contradict 6. Thus $Y$ and $X$ are disjoint. By condition 5 on the choice of $\epsilon$, $\gamma_\epsilon$ is the only periodic orbit of $g_\epsilon$ with image in $M_\epsilon$ having period less than or equal to $2p_1$ and satisfying $|f_\epsilon^*t^*p_\epsilon| = \ell$. By condition 6 on the choice of $\epsilon$, $Y$ contains no orbits with periods greater than $\frac{4}{3}p_1$. Thus $Y$ and $M_\epsilon$ are also disjoint. It follows that $\tilde{A}_{\epsilon, \ell} \setminus \Gamma_\epsilon$ (and hence also $\tilde{A}_{\epsilon, \ell}$) is disconnected, with two components $\tilde{A}_{\epsilon, \ell}^1$ and $Y = \tilde{A}_{\epsilon, \ell}^2$.

The fact that $Y \subset \text{int}(N^1_\epsilon)$ contains no periods larger than $\frac{2}{3}p_1$ also implies that $Y = Z$, for the orbit $\alpha$ through any limit point of $Y$ in $Z \setminus Y$ would be contained in $\text{cl}(N^1_\epsilon) = N^1_\epsilon \cup M_\epsilon$, have period less than or equal to $\frac{2}{3}p_1$, and satisfy $|f_\epsilon^*t^*p_\epsilon| < \ell$, contradicting either condition 5 or condition 6 on the choice of $\epsilon$. It follows that the connected component $\tilde{A}_{\epsilon, \ell}^1$ of $\tilde{A}_{\epsilon, \ell} \setminus \Gamma_\epsilon$ satisfies $\tilde{A}_{\epsilon, \ell}^1 = \tilde{A}_{\epsilon, \ell}^2 = Y = Z$.

To summarize, we have shown that $\tilde{A}_{\epsilon, \ell} \setminus \Gamma_\epsilon = \tilde{A}_{\epsilon, \ell}^1 \cup \tilde{A}_{\epsilon, \ell}^2$ is disconnected, that $\tilde{A}_{\epsilon, \ell}^1 = \tilde{A}_{\epsilon, \ell}^2$, and that $\tilde{A}_{\epsilon, \ell}^1$ is contained in the compact set $\text{cl}(N^1_\epsilon) \subset \text{dom}(\eta)$ which contains no generalized centers. Additionally, the periods $\tilde{A}_{\epsilon, \ell}$ have the uniform upper bound $\frac{2}{3}p_1$. But this implies that $\gamma_\epsilon$ is not $(\eta, \ell)$-globally continuable, contradicting Lemma 4. This contradiction implies that $\gamma$ must in fact be $(\eta, \ell)$-globally continuable and completes the proof.

Armed with Theorem 1, we now proceed to prove our main results on existence of periodic orbits. We will use the following lemma to convert data from a closed 1-form and a vector field into a priori bounds on the periods of periodic orbits.

**Lemma 6.** Let $M$ be a smooth manifold and let $\eta$ be a $C^1$ closed 1-form on $M$. Let $\gamma$ be a periodic orbit of a $C^1$ vector field $f : M \to TM$. Assume that there exists $\epsilon > 0$ such that $\eta(\gamma(t)) \geq \epsilon$ for all $t$. Then $\ell := \int_0^\tau \eta > 0$, and the period $\tau$ of $\gamma$ satisfies

$$\tau \leq \frac{\ell}{\epsilon}.$$  

**Proof.** We have

$$\ell = \int_0^\tau \eta = \int_0^\tau \eta(\gamma(t)) dt \geq \epsilon \tau > 0,$$

with the first inequality following since $\eta(\gamma(t)) \geq \epsilon$. This completes the proof. \qed

We now prove Theorem 2, our first periodic orbit existence result. Theorem 2 is fairly general, and it follows straightforwardly from Theorem 1 and Lemma 6. We continue to identify $\Gamma$ with $\Gamma \times \{\mu_0\}$ when there is no risk of confusion in the following.

Given a subset $X \subset Q \times \mathbb{R}$ and any interval $J \subset \mathbb{R}$, we use the notation $X_J := X \cap (Q \times J)$ in Theorems 2 and 3 below.

**Theorem 2** (Global existence of periodic orbits). Assume the hypotheses of Theorem 1 and notation of Definition 3. Assume that $\tilde{A}_{\epsilon, \ell} \setminus \Gamma_\epsilon$ is disconnected, let $\tilde{A}_{\epsilon, \ell}^1$ be one of its connected components, and assume that $\tilde{A}_{\epsilon, \ell}^1$ is equal to a connected component $\tilde{A}_{\ell}^1$ of $\tilde{A}_{\ell} \setminus \Gamma$. Further assume that there exists $C \subset \text{dom}(\eta) \subset Q \times \mathbb{R}$ and $\mu^* < \mu_0$ (resp. $\mu^* > \mu_0$) satisfying the following properties:

1. $\tilde{A}_{\ell}^1 \cap (Q \times \{\mu^*\}) = \emptyset$,
2. $\tilde{A}_{\ell}^1 \subset C$, 

...
(3) \( t^*_{\mu} \eta(f_\mu(x)) > 0 \) for all \((x, \mu) \in C_{[\mu^*, \infty)} \) (resp. \((x, \mu) \in C_{(-\infty, \mu]} \)), and
(4) for every \( \mu \geq \mu^* \) (resp. \( \mu \leq \mu^* \)), \( C_{[\mu^*, \mu]} \) (resp. \( C_{[\mu, \mu^*]} \)) is compact.

Then for all \( \mu > \mu_0 \) (resp. \( \mu < \mu_0 \)), \( \mathcal{A}_1 \) \( \cap (Q \times \{ \mu \}) \neq \emptyset \). In particular, \( f_\mu \) has a periodic orbit for all \( \mu \geq \mu_0 \).

**Proof.** We prove the result in the case that \( \mu^* < \mu_0 \), with the proof for the case \( \mu^* > \mu_0 \) being similar.

Assume, to obtain a contradiction, that there exists \( \mu_1 > \mu_0 \) such that \( \mathcal{A}_1 \) \( \cap (Q \times \{ \mu_1 \}) = \emptyset \). Then connectedness of \( \mathcal{A}_1 \) and hypotheses 1 and 2 imply that \( \mathcal{A}_1 \) is contained in the set \( C_{[\mu^*, \mu_1]} \). Since \( C_{[\mu^*, \mu_1]} \) is compact by hypothesis 4, hypothesis 3 implies that there is \( \epsilon > 0 \) such that \( t^*_{\mu} \eta(f_\mu(x)) \geq \epsilon \) for all \((x, \mu) \in C_{[\mu^*, \mu_1]} \). Hence Lemma 6 implies that the periods of orbits in \( \mathcal{A}_1 \) are uniformly bounded above by \( \frac{\ell}{\epsilon} \). By assumption we also have \( \mathcal{A}_{1 \leq \ell} = \mathcal{A}_1 \), and \( \text{cl}(\mathcal{A}_1) \) contains no equilibria and hence no generalized centers by hypotheses 2 and 3. But Theorem 1 implies that \( \gamma \) is \((\eta, \ell)\)-globally continuable, so we have a contradiction. This completes the proof. \( \square \)

We now use Theorem 2 to formalize a rather specific argument involving Theorem 2 and a Hopf bifurcation, which we have used to prove the existence of periodic orbits in applications (see §4). While the statement appears rather complicated, the upshot is that we do not have to repeat this argument in each of our individual examples.

Given a subset \( X \subset Q \times \mathbb{R} \) and any interval \( J \subset \mathbb{R} \), we again use the notation \( X_J := X \cap (Q \times J) \) in Theorem 3 below.

**Theorem 3** (Global existence of periodic orbits following a Hopf bifurcation). Assume that \( Q \) is orientable, and let \( N \subset Q \times \mathbb{R} \) be a properly embedded, smooth, orientable, codimension-1 submanifold with boundary \( M = \partial N \). Let \( f \in C^1(Q \times \mathbb{R}, TQ) \) be a family of vector fields, and let \( \eta \in [\eta] \in H^1((Q \times \mathbb{R}) \setminus M; \mathbb{Z}) \) be a \( C^1 \) closed 1-form representing the (closed) Poincaré dual \([\eta]\) of \( N \). Further assume that there exists \( C \subset Q \times \mathbb{R} \), \((x_c, \mu_c) \in M \cap \text{int}(C) \) and \( \mu^* < \mu_c \) (resp. \( \mu^* > \mu_c \)) satisfying the following properties:

(1) \( f_{\mu^*} \) has no periodic orbits contained in \( C_{\mu^*} \),
(2) no periodic orbits of \( f \) intersect \((\partial C)_{[\mu^*, \infty)} \) (resp. \( \partial C_{(-\infty, \mu^*]} \)),
(3) for every \( \mu_1 > \mu^* \) (resp. \( \mu_1 < \mu^* \)), there exists \( \epsilon > 0 \) such that \( t^*_{\mu} \eta(f_\mu(x)) \geq \epsilon \) for all \((x, \mu) \in (C \setminus M)_{[\mu^*, \mu_1]} \) (resp. \((x, \mu) \in (C \setminus M)_{[\mu_1, \mu^*]} \)),
(4) for every \( \mu \geq \mu^* \) (resp. \( \mu \leq \mu^* \)), \( C_{[\mu^*, \mu]} \) (resp. \( C_{[\mu, \mu^*]} \)) is compact,
(5) \( f \) is \( C^3 \) on a neighborhood of \((x_c, \mu_c) \), \((x_c, \mu_c) \in M \cap \text{int}(C) \) is a point of generic Hopf bifurcation for \( f \), and \( C_{[\mu^*, \infty)} \) (resp. \( C_{(-\infty, \mu^*]} \)) contains no other generalized centers,
(6) no nonstationary periodic orbits of \( f \) intersect \((C \cap M)_{[\mu^*, \infty)} \) (resp. \((C \cap M)_{(-\infty, \mu^*]} \)), and
(7) letting \( E^c \subset T_{x_c}Q \) be the two-dimensional center subspace for \( D_{x_c}f_{\mu_c} \),
\[
T_{(x_c, \mu_c)}(Q \times \mathbb{R}) = (D_{(x_c, \mu_c)}t^*_{\mu}E^c) \oplus T_{(x_c, \mu_c)}M.
\]

Then for all \( \mu > \mu_c \) (resp. \( \mu < \mu_c \)), \( f_\mu \) has a periodic orbit contained in \((C \setminus M)_{[\mu]} \).

**Proof.** We prove the theorem for the case that the Hopf bifurcation is supercritical and \( \mu^* \leq \mu_c \), with the other three cases being similar.

The proof of the Hopf bifurcation theorem [GH00, Thm 3.4.2] implies that there exists \( \delta > 0 \) and a \( \mu \)-dependent two-dimensional center manifold \( W^c_\mu \) for \( \mu \in (\mu_c - \delta, \mu_c + \delta) \) such that (i) \((x_c, \mu_c) \in t^*_{\mu_c}(W^c_\mu) \cap M \), (ii) the orbit at each \( \mu \) on the bifurcating branch of periodic orbits is contained in \( W^c_\mu \), and (iii) the image of the periodic orbit at \( \mu \) on the bifurcating branch tends to \( x_c \) uniformly as \( \mu \rightarrow \mu_c \). Since \( T_{x_c}W^c_\mu = E^c \), after shrinking \( W^c_\mu \) if necessary it follows from hypothesis 7 that \( t^*_{\mu_c}(W^c_\mu) \) intersects \( M \) transversely. Hence (by an implicit function theorem argument) there
exist local coordinates \((y, z, \mu)\) on a neighborhood of \((x_c, \mu_c) \subset Q \times \mathbb{R}\) in which \(W^c_\mu \times \{\mu\} = \{(y, 0, \mu)\}\) and \(M = \{(0, z, \mu)\}\). This fact, (ii–iii) above, and hypothesis 6 imply that, for \(\mu > \mu_c\) sufficiently close to \(\mu_c\), the disk \(D_\mu \subset W^c_\mu\) bounded by the image of the bifurcating periodic orbit \(\gamma_\mu\) intersects \(\iota_\mu^{-1}(M)\) exactly once (and this intersection is transverse by hypothesis 7). Fixing such a \(\mu_0 > \mu_c\) sufficiently close to \(\mu_c\) and defining \(\gamma := \gamma_{\mu_0}\), it follows that the intersection number of \(\iota_{\mu_0}(\text{int}(D_\mu))\) with \(M\) is \(\pm 1\). Since \(\eta\) is Poincaré dual to \(N\) in \((Q \times \mathbb{R}) \setminus M\), it follows that \(\int_\gamma \iota^{*}_{\mu_0} \eta = \int_{\iota_{\mu_0} \circ \gamma} \eta\) is actually equal to this intersection number \([BT91, \text{pp. 50–52, 229–234}]^8\).

\[
\left| \int_\gamma \iota^{*}_{\mu_0} \eta \right| = 1.
\]

Note that, since \((x_c, \mu_c) \in \text{int}(C)\) by hypothesis 5, we may assume that \(\mu_0\) is chosen sufficiently close to \(\mu_c\) that \(\Gamma \subset \text{int}(C)\). By the proof of the Hopf bifurcation theorem we may furthermore assume that \(\mu_0\) is chosen sufficiently close to \(\mu_c\) that \(\gamma\) is hyperbolic, so in particular \(+1\) is not a Floquet multiplier of \(\gamma\).

Let \(\text{dom}(\eta) := (Q \times \mathbb{R}) \setminus M\), and let \(A, \widetilde{A}_1, \widetilde{A}_{\leq 1}\) be the components containing \(\Gamma\) as in Definition 3 with \(\ell = 1\) (identifying \(\Gamma\) with \(\Gamma \times \{\mu_0\}\)). Note that the periodic orbit component \(A\) of \(f|_{\text{dom}(\eta)}\) is also a periodic orbit component of \(f\) due to hypotheses 1, 2, and 6. The proof of the Hopf bifurcation theorem implies the existence of a compact neighborhood \(K_0 \subset Q \times \mathbb{R}\) of \((x_c, \mu_c)\) containing a connected subset \(B \subset \widetilde{A}_1 \cap K_0\) of nonstationary periodic orbits of \(f\) such that (i) \(\Gamma \subset B\), (ii) all periods of \(B\) are close to the period of \(\Gamma\), (iii) any other periodic orbits in \(K_0 \setminus B\) have very large period, and (iv) \(B \setminus \Gamma\) consists of two connected components \(B^1, B^2\) with \(B^2 \subset \text{int}(K_0)\) and the closure of \(B^2\) containing \((x_c, \mu_c)\). Due to (iii) above, hypothesis 3, and Lemma 6, it follows that \(B = \widetilde{A}_1 \cap K_0 = \widetilde{A}_{\leq 1} \cap K_0\). Hence both \(\widetilde{A}_1 \setminus \Gamma\) and \(\widetilde{A}_{\leq 1} \setminus \Gamma\) are disconnected with two connected components. As in Definition 3, let \(\widetilde{A}^1_1, \widetilde{A}^2_1\) and \(\widetilde{A}^1_{\leq 1}, \widetilde{A}^2_{\leq 1}\) denote the connected components of \(\widetilde{A}_{\leq 1} \setminus \Gamma\) and \(\widetilde{A}_{\leq 2} \setminus \Gamma\), labeled so that \(B^i \subset \widetilde{A}^i_1 \subset \widetilde{A}^i_{\leq 1}\). Since \(\eta\) is the Poincaré dual of a submanifold, its integral around any closed curve is an integer (which is positive by Lemma 6), so we automatically have

\[
1 = \min \left\{ \left| \int_{\alpha_\mu} \iota^{*}_\mu \eta \right| : \alpha_\mu \text{ is a periodic orbit with image in } A \right\}.
\]

In particular, it follows that \(\widetilde{A}_1^i = \widetilde{A}_{\leq 1}^i\) for both \(i = 1, 2\).

By the above paragraph, there exists a neighborhood \(U_0 \subset K_0\) of \((x_c, \mu_c)\) such that \(\widetilde{A}_1^1 \cap U_0 = \emptyset\). Hypothesis 2 implies that \(\widetilde{A}_1^1 \subset \text{int}(C)_{(\mu, \infty)}\), so we have \(\widetilde{A}_1^1 \subset \text{int}(C) \setminus U_0(\mu, \infty)\). Since \(\text{dom}(\eta) = (Q \times \mathbb{R}) \setminus M\), we have \(\widetilde{A}_1^1 \cap M = \emptyset\) by definition. We now show that there is furthermore a neighborhood \(U_1 \subset Q \times \mathbb{R}\) of \(M\) such that \(\widetilde{A}_1^1 \cap U_1 = \emptyset\). If this were not true, then there is a sequence \((x_i, \mu_i)\) in \(\widetilde{A}_1^1\) with \((x_i, \mu_i) \to (x, \mu) \in M \setminus U_0\). This sequence must be contained in \((C \setminus M)_{(\mu, \infty)}\) for some \(\mu_0\), so hypothesis 3 and Lemma 6 imply that the periods of the orbits through \((x_i, \mu_i)\) are uniformly bounded above by \(\frac{\ell}{2}\). This implies that \((x, \mu)\) is either a generalized center or lies on a nonstationary periodic orbit; hypothesis 6 rules out the latter option, so \((x, \mu)\) is a generalized center. But hypothesis 2 further implies that \((x, \mu) \in C_{(\mu, \infty)}\), and this contradicts hypothesis 5. Hence there exists a neighborhood \(U_1 \subset Q \times \mathbb{R}\) of \(M\) with \(\widetilde{A}_1^1 \cap U_1 = \emptyset\), as desired.

Define the set \(\widetilde{C} := C \setminus U_1\). By the last paragraph, \(\widetilde{A}_1^1 \subset \widetilde{C}\). Additionally, \(\widetilde{A}_1^1 \cap (Q \times \{\mu^*\}) = \emptyset\) by hypothesis 1, \(\iota^{*}_\mu \eta(f_\mu(x)) > 0\) for all \((x, \mu) \in \widetilde{C}\) by hypothesis 3, and \(\widetilde{C}\) contains no generalized

---

8In [BT91] the authors work with \(C^\infty\) forms, whereas we assume \(\eta \in C^1\), but there is no issue since every \(C^1\) closed form is cohomologous to a \(C^\infty\) closed form [dR84, pp. 61–70].

9If not, then (since \(Q \times \mathbb{R}\) is first countable) each \((x, \mu) \in M\) has an open neighborhood \(U_{x, \mu}\) satisfying \(U_{x, \mu} \cap \widetilde{A}_1^1 = \emptyset\). But then \(\bigcup_{(x, \mu) \in M} U_{x, \mu}\) is an open neighborhood of \(M\) having empty intersection with \(\widetilde{A}_1^1\), a contradiction.
centers for $f$ by 5. Finally, for every $\mu \geq \mu^*$,

$$\bar{C}_{[\mu^*, \mu]} = C_{[\mu^*, \mu]} \setminus U_1$$

is compact by hypothesis 4. Since we have already shown that $\bar{A}_1 = \bar{A}_{\leq 1}$ and that $+1$ is not a Floquet multiplier of $\gamma$, it follows that the hypotheses of Theorem 2 are satisfied with $\ell = 1$ and $\bar{C}$ playing the role of $C$. Hence $\bar{A}_1 \cap (Q \times \{\mu\}) \neq \emptyset$ for all $\mu \geq \mu_0$. In particular, $f_\mu$ has a periodic orbit contained in $(C \setminus M)_{(\mu)}$ for all $\mu \geq \mu_0$. Since $\mu_0 > \mu_c$ was arbitrary, it follows that $f_\mu$ has a periodic orbit for all $\mu > \mu_c$ as well. This completes the proof. $\square$

4. EXAMPLES

In this section, we illustrate our results by proving periodic orbit existence results for the represilator (3) in §4.3 and for the Sprott system (2) in §4.4. We begin with some preliminary discussion relevant for both systems. Both systems admit the symmetry $(x, y, z) \mapsto (y, z, x)$, and we discuss some consequences of this fact in §4.1. In §4.2 we define a 1-form $d\theta$—to be used in the proofs for both systems—and briefly discuss some of its properties.

4.1. Basic symmetry considerations. In the remainder of §4, we use the notation $1 := (1, 1, 1)$ and $x := (x, y, z)$.

Define the linear permutation symmetry $\sigma: \mathbb{R}^3 \to \mathbb{R}^3$ via $\sigma(x, y, z) := (y, z, x)$. Letting $f_\mu: \mathbb{R}^3 \to \mathbb{R}^3$ denote either the represilator (3) or Sprott (2) vector fields, we see that $\sigma \circ f_\mu := D\sigma \circ f_\mu \circ \sigma^{-1} = f_\mu$. It follows that $\sigma$ commutes with the (local) flow $\Phi_\mu$ of $f_\mu$ and therefore maps solution curves to solution curves. Since the diagonal $\Delta := \text{span}\{1\}$ is the fixed point set of $\sigma$, it follows that $\Delta$ is $\Phi_\mu$-invariant since, for all $p \in \Delta$, $\sigma \circ \Phi_\mu(p) = \Phi_\mu^t \circ \sigma(p) = \Phi_\mu^t(p)$. Since $\sigma^3 = \text{id}_{\mathbb{R}^3}$, the dynamics have a $Z_3$ symmetry group whose action on $\mathbb{R}^3$ is generated by $\sigma$. It follows that any invariant set is either fixed by $\sigma$ or is one member of a family of three distinct invariant sets permuted by $\sigma$.

The linear map $\sigma \in \text{SO}(3) \subset \text{GL}(3, \mathbb{R})$ is a rotation having the unique finest $\sigma$-invariant splitting $\Delta \oplus \Delta^\perp = \mathbb{R}^3$. Identifying $\sigma$ with $D\sigma$, it follows that, for any $x \in \Delta$, the matrix representing $\sigma$ commutes with the matrix $D_x f_\mu$; assume $x \in \Delta$ in the following. $\sigma$-invariance of the splitting $\Delta \oplus \Delta^\perp$ implies that $D_x f_\mu \Delta$ and $D_x f_\mu \Delta^\perp$ are $\sigma$-invariant subspaces, since

$$\sigma \circ D_x f_\mu(\Delta) = D_x f_\mu \circ \sigma(\Delta) = D_x f_\mu \Delta$$

$$\sigma \circ D_x f_\mu(\Delta^\perp) = D_x f_\mu \circ \sigma(\Delta^\perp) = D_x f_\mu \Delta^\perp.$$

In particular, if $D_x f_\mu$ is invertible then $D_x f_\mu \Delta \oplus D_x f_\mu \Delta^\perp = \mathbb{R}^3$ is a $\sigma$-invariant splitting of $\mathbb{R}^3$ into one and two-dimensional subspaces, so uniqueness of the finest $\sigma$-invariant splitting $\Delta \oplus \Delta^\perp = \mathbb{R}^3$ implies that

$$D_x f_\mu \Delta = \Delta, \quad D_x f_\mu \Delta^\perp = \Delta^\perp.$$  \hspace{1cm} (14)

If $(x, \mu)$ is a point of generic Hopf bifurcation for $f_\mu$, then $D_x f_\mu$ is invertible and there is a unique finest $D_x f_\mu$-invariant splitting $E \oplus E^c = \mathbb{R}^3$ into one and two-dimensional subspaces, with $E^c$ the two-dimensional center subspace. Uniqueness of the finest $D_x f_\mu$-invariant splitting and (14) therefore imply that

$$E^c = \Delta^\perp.$$  \hspace{1cm} (15)

4.2. A closed 1-form. With respect to the orthogonal splitting

$$\mathbb{R}^3 = \Delta \oplus \Delta^\perp,$$

we may write any $x \in \mathbb{R}^3$ uniquely as

$$x = x_\Vert + x_\perp.$$
with \( x_0 \in \Delta \) and \( x_\perp \in \Delta^\perp \). A direct computation shows that \( \|x_\perp\|^2 = \frac{2}{3}(\|x\|^2 - \langle x, \sigma(x) \rangle) \), where \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) are the Euclidean norm and inner product.

We now define a 1-form \( d\theta \) on \( \mathbb{R}^3 \setminus \Delta \):

\[
d\theta := \frac{\sqrt{3}}{2} \frac{(z - y)dx + (x - z)dy + (y - x)dz}{\|x\|^2 - \langle x, \sigma(x) \rangle} = \frac{1}{\sqrt{3}} \frac{(z - y)dx + (x - z)dy + (y - x)dz}{\|x_\perp\|^2}.
\]

It can be shown that \( d\theta \) is closed. In fact, choose orthogonal coordinates \( (u, v, w) \) adapted to the splitting \( \mathbb{R}^3 = \Delta^\perp \oplus \Delta \) so that \( (u, v) \) are coordinates for \( \Delta^\perp \) and \( w \) is a coordinate for \( \Delta \). Then it can be shown that \( d\theta \) is equal to the standard “angle 1-form” about the \( w \)-axis in these coordinates:

\[
d\theta = \frac{uv + vu}{u^2 + v^2}.
\]

Defining \( N := \{x: x = y \text{ and } z \leq x\} \) and \( M := \Delta = \partial N \), note that \( \frac{d\theta}{2\pi} \) is Poincaré dual to \( N \) in \( \mathbb{R}^3 \setminus M \) (see [BT91, Ex. 5.16(a)]).

### 4.3. The repressilator: existence of periodic orbits.

In this section we apply our theory to the repressilator (see [EL00, BKP09]) which models a synthetic genetic regulatory network consisting of a ring oscillator. We consider here the three-dimensional reduced-order model studied in [BKP09, BPK10] and prove existence of nonstationary periodic orbits. Our proof is different from that of [BKP09], and does not use techniques specific to monotone cyclic feedback systems [MPS90].

Fix \( s > 0 \) and consider the one-parameter family of ODEs on \( \mathbb{R}^3 \) given by

\[
\begin{align*}
\dot{x} &= \frac{\mu}{1 + y^s} - x \\
\dot{y} &= \frac{\mu}{1 + z^s} - y \\
\dot{z} &= \frac{\mu}{1 + x^s} - z,
\end{align*}
\]

with parameter \( \mu \in \mathbb{R} \). Let \( \mathbb{R}^3_+ := \{(x, y, z) \in \mathbb{R}^3: x, y, z \geq 0\} \) be the closed positive orthant. Notice that, for any \( s, \mu > 0 \), \( \mathbb{R}^3_+ \) is positively invariant for the flow \( f_{s,\mu} \) of (17). Furthermore, since \( 0 < \frac{\mu}{1 + r^s} < \mu \) whenever \( s, \mu > 0 \) and \( r \geq 0 \), it follows that the interior of the cube \( \{(x, y, z): 0 \leq x, y, z \leq \mu\} \) is positively invariant and attracts every initial condition \( x \in \mathbb{R}^3 \). It follows that the same is true of the interior of the smaller cube

\[
K_\mu := \left\{(x, y, z): \frac{\mu}{2 + \mu^s} \leq x, y, z \leq \mu \right\}
\]

since \( \frac{\mu}{1 + r^s} \geq \frac{\mu}{1 + \mu^s} > \frac{\mu}{2 + \mu^s} \) whenever \( r \leq \mu \); in particular, \( \partial K_\mu \) immediately flows into \( \text{int}(K_\mu) \).

We now prove that (17) has a periodic orbit for all \( s > 2 \) and \( \mu > \mu_c \), where \( \mu_c \) is defined below. To do this, we simply verify that (17) satisfies all of the hypotheses of Theorem 3. We delay the (slightly lengthier) verification of hypothesis 3 of Theorem 3 to §4.3.1 below.

**Theorem 4.** Let \( f_{s,\mu} = f(\cdot, s, \mu) \) be the repressilator vector field (17). Fix \( s > 2 \) and define

\[
\mu_c := \left( \frac{2}{s - 2} \right)^{\frac{s + 1}{2}} + \left( \frac{2}{s - 2} \right)^{\frac{1}{2}}.
\]

Then for all \( \mu > \mu_c \), \( f_{s,\mu} \) has a periodic orbit contained in the cube \( K_\mu \).

**Proof.** Define \( C := \{(x, \mu) \in \mathbb{R}^3 \times \mathbb{R}: \mu \geq 0 \text{ and } x \in K_\mu\} \), \( N := \{x: x = y \text{ and } z \leq x\} \times \mathbb{R} \), and \( M := \Delta \times \mathbb{R} = \partial N \).
Since the origin is exponentially stable for $f_{s_0}$, there exists $\mu^* > 0$ such that, for all $0 < \mu \leq \mu^*$, $f_{s,\mu}$ has no periodic orbits whose images intersect$^{10}$ $K_\mu$, so in particular hypothesis 1 of Theorem 3 is satisfied. If $\mu > 0$ and $\Phi_{s,\mu}$ is the flow of $f_{s,\mu}$, then every initial condition $x \in \partial K_\mu$ satisfies $\Phi_{s,\mu}(x) \in \text{int}(K_\mu)$ for all $t > 0$, so no periodic orbits of $f_{s,\mu}$ intersect $\partial K_\mu$; hence hypothesis 2 of Theorem 3 is satisfied. The compactness hypothesis 4 is satisfied since any set of the form $C_{[\mu^*, \mu]} := C \cap (\mathbb{R}^3 \times [\mu^*, \mu])$ is a closed subset of the compact set $\{x : 0 \leq x, y, z \leq \mu\} \times [\mu^*, \mu]$.

In [BKP09, Sec.2, Appendix] it is shown that there is exactly one generalized center $(x_c, \mu_c)$ for $f_s$, that $x_c \in \Delta \cap \text{int}(K_\mu)$, that $\mu_c > \mu^*$, and that $f_s$ undergoes a supercritical generic Hopf bifurcation at $(x_c, \mu_c)$. Hence hypothesis 5 of Theorem 3 is satisfied. Hypothesis 6 is satisfied because $\Delta$ is an invariant manifold for $f_{s,\mu}$ by symmetry (see §4.1) and $\Delta$ is diffeomorphic to $\mathbb{R}$, so no nonstationary periodic orbits can intersect $\Delta$. Finally, the center subspace $E^c$ of $D_{x_c} f_{s,\mu}$ is orthogonal to $\Delta$ by Equation (15), so hypothesis 7 is satisfied.

In §4.2 we defined a closed 1-form $\frac{\partial}{\partial t}$ on $\mathbb{R}^3 \setminus \Delta$ such that $\frac{\partial}{\partial t}$ is Poincaré dual to $\{x = y$ and $z \leq x\}$ on $\mathbb{R}^3 \setminus \Delta$. In §4.3.1 below, in Proposition 4 we prove that, for every $s, \mu_1 > 0$, there exists $\epsilon > 0$ such that $\frac{\partial}{\partial t} f_{s,\mu} \geq \epsilon$ on $K_\mu \setminus \Delta$ for all $\mu \in [\mu^*, \mu_1]$. Let $\pi_2 : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ denote the projection onto the second coordinate, and for any $\mu \in \mathbb{R}$ let $\iota_\mu : \mathbb{R}^3 \hookrightarrow \mathbb{R}^3 \setminus \mathbb{R}$ be the inclusion $\iota_\mu(x) = (x, \mu)$. Defining $\eta := \pi_2(\frac{\partial}{\partial t})$, noting that $\pi_2(\frac{\partial}{\partial t})$ is Poincaré dual to $N$ in $(\mathbb{R}^3 \setminus \mathbb{R}) \setminus M$ [BT91, p. 69], and noting that $\iota_\mu^* \eta = \frac{\partial}{\partial t}$ for any $\mu \in \mathbb{R}$, it follows that the lone remaining hypothesis 3 of Theorem 3 is also satisfied. This completes the proof.

4.3.1. Rotational rate of the flow. In this section, we complete the proof of Theorem 4 by showing that $\frac{\partial}{\partial t}$ satisfies the remaining hypothesis 3 of Theorem 3.

Lemma 7. Fix $s, \mu > 0$ and let $f_{s,\mu}$ be the repressilator vector field (17) on $\mathbb{R}^3$. Let $\mathbb{R}^3_+ \subset \mathbb{R}^3 \setminus \Delta$ be the closed positive orthant and $\Delta \subset \mathbb{R}^3$ be the diagonal. Then $\frac{\partial}{\partial t} f_{s,\mu} > 0$ on $\mathbb{R}^3_+ \setminus \Delta$.

Proof. Define the 1-form $\omega := (z-y)dx + (x-z)dy + (y-x)dz$ to be the “numerator” of $d\theta$. It suffices to show that $\omega(f_{s,\mu}) > 0$ on $\mathbb{R}^3_+ \setminus \Delta$. We compute

$$\omega(f_{s,\mu}) = \frac{z-y}{p(y)} + \frac{x-z}{p(z)} + \frac{y-x}{p(x)},$$

where the positive function $p$ is defined as $p(r) := \mu \cdot (1 + r^s)$. Define the function

$$N(x, y, z) := (z-y)p(z)p(x) + (x-z)p(x)p(y) + (y-x)p(y)p(z).$$

Writing $\omega \cdot f_{s,\mu} := \omega(f_{s,\mu})$, note that $(\omega \cdot f_{s,\mu})(x, y, z) = \frac{N(x, y, z)}{p(x)p(y)p(z)}$, so that $\omega(f_{s,\mu}) > 0$ if and only if $N > 0$.

Let $x = (x, y, z) \in \mathbb{R}^3_+ \setminus \Delta$ and consider the terms $(z-y), (x-z), (y-x)$. Since these terms sum to zero and since $x \not\in \Delta$, it must be the case that there is one nonzero term which has a different

$^{10}$Proof: fix $s > 2$. It is shown in [BKP09] that (17) has a unique equilibrium $x_{s,\mu} \in K_\mu$ for all $\mu \geq 0$ which depends continuously on $\mu$; define $V_{s,\mu}(x) := \frac{1}{2}\|x - x_{s,\mu}\|^2$. Applying Taylor’s theorem to $f_{s,\mu}$ about the point $x_{s,\mu}$ shows that the derivative of $V_{s,\mu}$ along the flow of $f_{s,\mu}$ is $V_{s,\mu}'(x) = (x - x_{s,\mu}, f_{s,\mu}(x)) = -\|x - x_{s,\mu}\|^2 + R_{s,\mu}(x, \mu)\|x - x_{s,\mu}\|^2$, where $R_{s,\mu}$ is continuous and satisfies $R_{s,\mu}(\cdot, 0) = 0$. Hence $\|R_{s,\mu}\| < \frac{1}{2}$ on some neighborhood $U$ of $\mathbb{R}^3 \times \{0\}$, so $V_{s,\mu}'(x) \leq -\frac{1}{2}\|x - x_{s,\mu}\|^2$ for all $(x, \mu) \in U$. Continuity of $\mu \mapsto x_{s,\mu}$ therefore implies that there are $\mu_0, \epsilon_0 > 0$ such that, for all $0 < \epsilon \leq \epsilon_0$ and $0 < \mu \leq \mu_0$, $V_{s,\mu} \leq -\frac{1}{2}\|x - x_{s,\mu}\|^2$ on the closed ball $B_\epsilon(x_{s,\mu})$ of radius $\epsilon$ centered at $x_{s,\mu}$. For such values of $\epsilon, \mu$ it follows that $B_{\epsilon}(x_{s,\mu})$ is contained in the stability basin of $x_{s,\mu}$ and so does not meet the images of any nonstationary periodic orbits. Finally, defining $\mu^* := \min\{\frac{\mu_0}{\sqrt{2s}}, \mu_0\}$ suffices to prove the claim since $K_\mu \subset \{0 \leq x, y, z \leq \mu\} \subset B_{2\sqrt{2s}}(x_{s,\mu}) \subset B_{\epsilon_0}(x_{s,\mu})$ for $0 < \mu < \mu^*$, with the second inclusion following since $x_{s,\mu} \in K_\mu$. (Something stronger is actually true: $x_{s,\mu}$ is globally asymptotically stable for $|\mu|$ sufficiently small, but we will not need this. This fact follows from [SW99, Cor. 2.3].)
sign than both of the other two terms. Divide the term which has sign different from the other two by the pair of functions that multiply it. Without loss of generality, assume that \((z - y)\) is nonzero and has sign different from \((x - z), (y - x)\). We obtain

\[
N(x, y, z) = (z - y) + (x - z) \frac{p(y)}{p(z)} + (y - x) \frac{p(y)}{p(x)}.
\]

Since \(r \mapsto p(r)\) is strictly increasing, in the case that \((z - y) > 0\) we obtain \(\frac{p(y)}{p(z)} < 1\) and \(\frac{p(y)}{p(x)} \leq 1\), with \(\frac{p(y)}{p(x)} = 1\) if and only if \((y - x) = 0\). It is clear that \(\frac{N(x, y, z)}{p(z)p(x)}\) and hence \(N(x, y, z)\) is positive in this case. Similarly, in the case that \((z - y) < 0\) we obtain \(\frac{p(y)}{p(z)} > 1\) and \(\frac{p(y)}{p(x)} \geq 1\), with \(\frac{p(y)}{p(x)} = 1\) if and only if \((y - x) = 0\), so again \(N(x, y, z)\) is positive. As discussed it follows that, in both cases, we have \((\omega \cdot f_{s,\mu})(x) > 0\) and hence also \((d\theta \cdot f_{s,\mu})(x) > 0\), completing the proof.

\[
\square
\]

For use in Lemma 8 and Proposition 4 below, we recall the definition of the set

\[
\mathcal{C} := \{(x, \mu) \in \mathbb{R}^3 \times \mathbb{R}; \mu \geq 0 \text{ and } x \in K_\mu\},
\]

where again \(K_\mu\) is defined for \(\mu \geq 0\) as

\[
K_\mu := \{(x, y, z): \frac{\mu}{2 + \mu} \leq x, y, z \leq \mu\}.
\]

For any interval \(J \subset \mathbb{R}\), we also define \(\mathcal{C}_J := \mathcal{C} \cap (\mathbb{R}^3 \times J)\).

**Lemma 8.** Fix \(s > 2\) and \(\mu^* > 0\). Then for every \(\mu_1 > \mu^*\), there exists \(\delta > 0\) and a relatively open neighborhood \(U \subset \mathcal{C}_{[\mu^*, \mu_1]} \cap (\Delta \times [\mu^*, \mu_1])\) such that, for all \((x, \mu) \in U \setminus (\Delta \times [\mu^*, \mu_1])\),

\[
d\theta(f_{s,\mu}(x)) \geq \delta.
\]

**Proof.** Define the 1-form \(\omega := (z - y)dx + (x - z)dy + (y - x)dz\) to be the “numerator” of \(d\theta\). Writing \(\omega \cdot f_{s,\mu} := \omega(f_{s,\mu})\) and defining \(q_\mu(r) := \frac{\mu}{1 + r}\) for \(\mu > 0\), we have

\[
(21) \quad \omega \cdot f_{s,\mu} = (z - y)q_\mu(y) + (x - z)q_\mu(z) + (y - x)q_\mu(x),
\]

so

\[
(22) \quad (\omega \cdot f_{s,\mu})|_{\Delta} \equiv 0.
\]

Note that \(q_\mu\) is \(C^\infty\) on \(\mathbb{R} \setminus \{0\}\). From (21) we compute the first derivative \(D_x(\omega \cdot f_{s,\mu})\) at \(x = (x, y, z) \neq 0\) to be

\[
(23)
\]

\[
D_x(\omega \cdot f_{s,\mu}) = \left[q_\mu(z) - q_\mu(x) + (y - x)q_\mu'(x), \quad q_\mu(x) - q_\mu(y) + (z - y)q_\mu'(y), \quad q_\mu(y) - q_\mu(z) + (x - z)q_\mu'(z)\right]
\]

from which it follows that

\[
(24) \quad D(\omega \cdot f_{s,\mu})|_{\Delta \setminus \{0\}} \equiv 0.
\]

From (23) we compute the second derivative at \((r, r, r) \in \Delta \setminus \{0\}\) to be

\[
D^2(\omega \cdot f_{s,\mu})(r) = q_\mu'(r) \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix},
\]

\[
\begin{aligned}
11\text{Note that this term need not be unique, since one of the terms may be zero.}
\end{aligned}
\]
so for any $v \in \mathbb{R}^3$ we have

$$
D^2_{(r,r)}(\omega \cdot f_{s,\mu})(v, v) = -2d'_{\mu}(r) \left( \|v\|^2 - \langle v, \sigma(v) \rangle \right) = -3d'_{\mu}(r)\|v\|^2,
$$

where $\sigma$ is the cyclic permutation $\sigma(x, y, z) = (y, z, x)$ and the notation $v = v_\parallel + v_\perp \in \Delta \oplus \Delta^\perp = \mathbb{R}^3$ is defined preceding (16). Writing $x = x_\parallel + x_\perp$ and using $x_\parallel = \frac{x + y + z}{3}1$, equations (22), (24), and (26) together with Taylor’s theorem imply that, for all $x \in \mathbb{R}^3$,

$$
(\omega \cdot f_{s,\mu})(x) = \frac{1}{2}D^2_{x_\parallel}(\omega \cdot f_{s,\mu}) \cdot (x_\perp, x_\perp) + R_{\mu}(x_\parallel)x_\perp^3 \\
= -\frac{3}{2}d'_{\mu} \left( \frac{x + y + z}{3} \right) \|x_\perp\|^2 + R_{\mu}(x_\parallel)x_\perp^3,
$$

where $(x_\parallel, \mu) \mapsto R_{\mu}(x_\parallel)$ is smooth on $12(\Delta \setminus \{0\}) \times \mathbb{R}$. Since the function $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R} \to \mathbb{R}$ given by $(x, \mu) \mapsto \min\{1, \|R_{\mu}(x_\parallel)\|\|x_\perp\|\}$ is continuous and since $C_{[\nu^*, \mu_1]}$ is disjoint from $(\Delta \perp \times \mathbb{R})$, for each $0 < \epsilon < 1$ the set

$$
U_\epsilon := \{(x, \mu) \in C_{[\nu^*, \mu_1]} : \|R_{\mu}(x_\parallel)\|\|x_\perp\| < \epsilon \}
$$

is a relatively open neighborhood of $(\Delta \times [\nu^*, \mu_1]) \cap C_{[\nu^*, \mu_1]}$ in $C_{[\nu^*, \mu_1]}$. Since $s > 2$ and $\mu > 0$, $-\sqrt{3}d'_{\mu}(r) = 3s\nu^{s-1} > 0$ is jointly continuous in $(r, \mu)$ and hence attains a minimum $m > 0$ on the compact set $[0, \mu_1] \times [\nu^*, \mu_1]$. Choose $\epsilon \leq \frac{m}{2}$. Using (27) and the fact that $U_\epsilon \subset [0, \mu_1]^3 \times [\nu^*, \mu_1]$, it follows that for all $(x, \mu) \in U_\epsilon \setminus (\Delta \times [\nu^*, \mu_1])$:

$$
(d\theta \cdot f_{s,\mu})(x) = \frac{1}{\sqrt{3}} \frac{(\omega \cdot f_{s,\mu})(x)}{\|x_\perp\|} \\
= -\frac{3}{2}d'_{\mu} \left( \frac{x + y + z}{3} \right) + R(x_\parallel) \frac{x_\perp^3}{\|x_\perp\|^2} \\
\geq -\frac{3}{2}d'_{\mu} \left( \frac{x + y + z}{3} \right) - \|R(x_\parallel)\|\|x_\perp\| \\
\geq -\frac{3}{2}d'_{\mu} \left( \frac{x + y + z}{3} \right) - \epsilon \\
\geq \frac{m}{2}.
$$

Taking $U := U_\epsilon$ and $\delta := \frac{m}{2}$ completes the proof. \hfill \Box

**Proposition 4.** Fix $s > 2$ and $\nu^* > 0$. Then for every $\mu_1 > \nu^*$, there exists $\epsilon > 0$ such that, for all $\mu \in [\nu^*, \mu_1]$, $d\theta(f_{s,\mu}) \geq \epsilon$ on $K_{\mu} \setminus \Delta$.

**Proof.** By Lemma 8, there exists $\delta_1 > 0$ and a relatively open neighborhood $U \subset C_{[\nu^*, \mu_1]}$ of $(\Delta \times [\nu^*, \mu_1]) \cap C_{[\nu^*, \mu_1]}$ in $C_{[\nu^*, \mu_1]}$ such that, for all $(x, \mu) \in U \setminus (\Delta \times [\nu^*, \mu_1])$,

$$
d\theta(f_{s,\mu}(x)) \geq \delta_1.
$$

Here $C$ and $C_{[\nu^*, \mu_1]}$ are as defined preceding Lemma 8. By Lemma 7, $d\theta(f_{s,\mu}(x)) > 0$ for all $(x, \mu)$ in the compact set $C_{[\nu^*, \mu_1]} \setminus U$ and therefore attains a minimum $\delta_2$ on this set. Defining $\epsilon := \min\{\delta_1, \delta_2\}$, it follows that

$$
\forall (x, \mu) \in C_{[\nu^*, \mu_1]} \setminus (\Delta \times [\nu^*, \mu_1]) : d\theta(f_{s,\mu}(x)) \geq \epsilon.
$$

From the definition of $C_{[\nu^*, \mu_1]}$ we can write $C_{[\nu^*, \mu_1]} = \{(x, \mu) \in \mathbb{R}^3 \times \mathbb{R} : \nu^* \leq \mu \leq \mu_1 \text{ and } x \in K_{\mu}\}$, so it follows that $d\theta(f_{s,\mu}(x)) \geq \epsilon$ whenever $\mu \in [\nu^*, \mu_1]$ and $x \in K_{\mu} \setminus \Delta$. This completes the proof. \hfill \Box

---

\[12\] We restrict attention to $x_\parallel \in \Delta \setminus \{0\}$ since $d'_{\mu}$ is not differentiable at zero if $2 < s < 3$. This poses no problem for us since $K_{\mu} \cap \Delta^\perp = \emptyset$ for $\mu > 0$, i.e., $x \in K_{\mu}$ implies $x_\parallel \neq 0$. 
28 FAMILIES OF PERIODIC ORBITS: CLOSED 1-FORMS AND GLOBAL CONTINUABILITY

Figure 6. Shown here is a trajectory segment of (29) with \( \mu = 0 \) and initial condition \((x_0, y_0, z_0) = (0.3, 0.2, -0.3)\). Also shown is the sphere \( \dot{V}^{-1}(0) \) (c.f. Equation (30) and Figure 9).

4.4. The Sprott system: existence of periodic orbits. In this section we apply our theory to prove existence of periodic orbits for the Sprott system discussed in §1. The equations are given on \( \mathbb{R}^3 \) by

\[
\begin{align*}
\dot{x} &= y^2 - z - \mu x \\
\dot{y} &= z^2 - x - \mu y \\
\dot{z} &= x^2 - y - \mu z,
\end{align*}
\]

(29)

and depend on the parameter \( \mu \in \mathbb{R} \). We note that, unlike the repressilator (17), the Sprott system is not a monotone cyclic feedback system [MPS90]. Some trajectory segments of the dynamics for \( \mu = 0 \) are shown in Figures 1 and 6, and for other values of \( \mu \) in Figures 7 and 8. The sphere shown is defined in §4.4.1. In the sequel, we let \( f_\mu \) denote the vector field defined by (29).

At the end of §4.4 we will prove that (29) has a periodic orbit for all \( \mu \in (-0.25, 0.5) \). Just like for the repressilator, the proof will amount to showing that (29) satisfies the hypotheses of Theorem 3. In the intervening sections, we will construct the ingredients required to do this. First, in §4.4.1 we find a certain compact set \( K_\mu \) which contains all bounded trajectories of (29); we will define the set \( C \) of Theorem 3 in terms of \( K_\mu \). Unlike the sets \( K_\mu \) defined for the repressilator, in this section \( K_\mu \) is not a trapping region and is not even invariant; this illustrates the flexibility allowed by the hypotheses of Theorem 3. §4.4.2 consists of deriving estimates involving \( d\theta(f_\mu) \) (where \( d\theta \) is defined in §4.2) used to establish hypothesis 3 of Theorem 3. In §4.4.3 we determine the equilibria and associated eigenvalues of \( Df_\mu \). In §4.4.4 we show that (29) exhibits Hopf bifurcations, needed in particular to verify hypothesis 5 of Theorem 3. Finally, §4.4.5 combines these ingredients to prove the periodic orbit existence theorem.

4.4.1. A compact set containing all bounded trajectories. Define the function \( V: \mathbb{R}^3 \to \mathbb{R} \) via \( V(x) := x + y + z \). A computation shows that the Lie derivative \( \dot{V} \) of \( V \) is

\[
\dot{V}(x, y, z) = \|x\|^2 - (\mu + 1)(x + y + z) = \langle 1, f_\mu(x) \rangle.
\]

(30)
For any $c \geq -\frac{3}{4}(\mu + 1)^2$, the sublevel set $B_{\mu, c} := \mathcal{V}^{-1}(-\infty, c]$ is the closed ball of radius $\sqrt{\frac{3(\mu+1)^2+4c}{2}}$ centered at $(\frac{\mu+1}{2})\mathbf{1}$. In particular, the zero sublevel set of $\mathcal{V}$ is centered at the midpoint of the two equilibria on the diagonal, with the two equilibria being antipodal points on the bounding sphere. Furthermore, the planes $\mathcal{V}^{-1}(0)$ and $\mathcal{V}^{-1}(3(1 + \mu))$ are tangent to the sphere at these antipodal points. See Figure 9.

This geometry implies that the subsets $\mathcal{V}^{-1}(-\infty, 0)$ and $\mathcal{V}^{-1}(3(1 + \mu), \infty)$ are respectively negatively and positively invariant for $\mu \geq -1$. Furthermore, trajectories in these regions tend to $\infty$ in negative and positive time, respectively. It follows that any bounded trajectory must be contained

Figure 7. Shown here is a trajectory segment of (29) with $\mu = 0.15$ and initial condition $(x_0, y_0, z_0) = (0.3, 0.2, -0.3)$. Also shown is the sphere $\mathcal{V}^{-1}(0)$ (c.f. Equation (30) and Figure 9).

Figure 8. Shown here is a trajectory segment of (29) with $\mu = 0.3$ and initial condition $(x_0, y_0, z_0) = (0.3, 0.2, -0.3)$. Also shown is the sphere $\mathcal{V}^{-1}(0)$ (c.f. Equation (30) and Figure 9).
in $V^{-1}[0, 3(1 + \mu)]$ when $\mu \geq -1$. We will further refine these considerations to produce a certain compact set containing all bounded trajectories.

Define translated coordinates $x_{\mu} := (x_{\mu}, y_{\mu}, z_{\mu}) := x - \frac{(1+\mu)}{2}1$ and define $r_{\mu} := \|x_{\mu}\|$.

**Theorem 5.** For $\mu > -1$, every bounded trajectory is contained in the compact set $K_{\mu}$ defined by

$$K_{\mu} := \left\{ x \in \mathbb{R}^3 : V(x) \geq r_{\mu} - \frac{3^3 + 3^4}{2} (1 + \mu) \arctan \left( \frac{2r_{\mu}}{(3)^{1/4}(1 + \mu)} \right) - \frac{\sqrt{3}}{2} (1 + \mu) + \frac{3^3 + 3^4}{2} (1 + \mu) \arctan \left( \frac{3^{1/4}}{1 + \mu} \right) \right\} \cap V^{-1}[0, 3(1 + \mu)].$$

For $\mu = -1$, the only bounded trajectory is the equilibrium at the origin; we define $K_{-1} := \{0\}$.

For a visual depiction of $K_{\mu}$, see Figure 10. Note that the sphere $\hat{V}^{-1}(0)$ shown in Figure 9 is contained in $K_{\mu}$.

**Proof.** For the case that $\mu = -1$, positive invariance of $V^{-1}(0, \infty)$, negative invariance of $V^{-1}(-\infty, 0)$, and the fact that $\hat{V}^{-1}(0) = \{0\}$ implies that the equilibrium at the origin is the only bounded trajectory of $f_{-1}$. For the remainder of the proof, we consider the case $\mu > -1$.

Let $t \mapsto x(t)$ be a trajectory of (29). If $V(x(0)) \not\subseteq [0, 3(1 + \mu)]$, then $\|x(t)\| \to \infty$ in either positive or negative time, so every bounded trajectory is contained in $V^{-1}[0, 3(1 + \mu)]$. Hence it suffices to restrict our attention to trajectories satisfying $x(0) \in V^{-1}[0, 3(1 + \mu)]$. Since any trajectory in $V^{-1}(-\infty, 0)$ tends to $\infty$ in negative time, to prove the theorem it suffices to show that for all $x(0) \in V^{-1}[0, 3(1 + \mu)]$, if $x(0) \not\in K_{\mu}$ then there exists a time $t_f < 0$ such that $V(x(t_f)) < 0$.

Define the shifted function $V_{\mu} := V - \frac{3(1+\mu)}{2}$. We compute

$$\dot{r}_{\mu} = \frac{x_{\mu} \cdot f_{\mu}(x_{\mu} + \frac{(1+\mu)}{2}1)}{r_{\mu}} = \frac{(x_{\mu}y_{\mu}^2 + y_{\mu}z_{\mu}^2 + z_{\mu}x_{\mu}^2) + \mu(x_{\mu}y_{\mu} + y_{\mu}z_{\mu} + z_{\mu}x_{\mu}) - \mu r_{\mu}^2 - \frac{1}{2}(1 + \mu)^2 V_{\mu}}{r_{\mu}}.$$
Figure 10. The compact set $K_\mu$ of Theorem 5 is the region bounded by the blue surface, red plane $V^{-1}(3(1+\mu))$, and green plane $V^{-1}(0)$. Note that the sphere $\hat{V}^{-1}(0)$ shown in Figure 9 is contained in $K_\mu$. This figure was generated using $\mu = 0.4$.

The Cauchy-Schwarz inequality and subadditivity of $\sqrt{\cdot}$ applied to the first and second numerator terms yields

$$\dot{r}_\mu \leq \frac{r_\mu^3 - \frac{1}{4}(1+\mu)^2 V_\mu}{r_\mu}. \quad (32)$$

Additionally, we have

$$\dot{V}_\mu = r_\mu^2 - \frac{3}{4}(1+\mu)^2. \quad (33)$$

Consider now a trajectory $x_\mu(t) = x(t) - \frac{(1+\mu)}{2} \mathbf{1}$ with initial condition $x_\mu(0) \in \{r_\mu^2 \geq \frac{3}{4}(1+\mu)^2\}$. $V_\mu$ increases monotonically along $x_\mu(t)$ as long as $x_\mu(t) \in \{r_\mu \geq \frac{\sqrt{3}}{2}(1+\mu)\}$, so time can be written as a function $t(V_\mu)$ of $V_\mu$, and we may therefore parametrize $r_\mu$ as a function of $V_\mu$. Using the chain rule, we compute

$$\frac{dr_\mu}{dV_\mu} = \frac{\dot{r}_\mu}{\dot{V}_\mu} \leq \frac{r_\mu^2 - \frac{1}{4}(1+\mu)^2 V_\mu}{r_\mu^2 - \frac{3}{4}(1+\mu)^2}. \quad (34)$$

We now further restrict our attention to a trajectory segment satisfying $0 \leq V \leq 3(1+\mu)$, or $-\frac{3}{2}(1+\mu) \leq V_\mu \leq \frac{3}{2}(1+\mu)$. We continue to assume that $r_\mu \geq \frac{\sqrt{3}}{2}(1+\mu)$ along this trajectory segment. It follows that

$$\frac{dr_\mu}{dV_\mu} \leq \frac{r_\mu^2 + \frac{\sqrt{3}}{4}(1+\mu)^2}{r_\mu^2 - \frac{3}{4}(1+\mu)^2}. \quad (34)$$

Let $\tilde{r}_\mu(V_\mu)$ denote a solution to the ODE defined by replacing the inequality in (34) with equality. This ODE is separable and admits the implicit solution family

$$c + V_\mu = \tilde{r}_\mu - \frac{3^{\frac{3}{4}} + 3^{\frac{1}{4}}}{2}(1+\mu) \arctan\left(\frac{2\tilde{r}_\mu}{(3)^{1/4}(1+\mu)}\right), \quad (35)$$
where $c$ is an arbitrary constant of integration. Considering (35) for different values $V_{\mu,0} := V_{\mu}(x(0))$ and $V_{\mu,tf} := V_{\mu}(x(tf))$ and subtracting the resulting two equations, we obtain
\[
V_{\mu}(x(0)) - V_{\mu}(x(tf)) = \tilde{r}_{\mu}(V_{\mu,0}) - \frac{3^3 + 3^\frac{3}{2}}{2}(1 + \mu) \arctan \left( \frac{2\tilde{r}_{\mu}(V_{\mu,0})}{(3)^{1/4}(1 + \mu)} \right) - \tilde{r}_{\mu}(V_{\mu,tf}) + \frac{3^3 + 3^\frac{3}{2}}{2}(1 + \mu) \arctan \left( \frac{2\tilde{r}_{\mu}(V_{\mu,tf})}{(3)^{1/4}(1 + \mu)} \right).
\]
(36)

Positivity of the right-hand side of (34) implies that $V_{\mu} \mapsto \tilde{r}_{\mu}(V_{\mu})$ is strictly increasing, which in turn implies that the right-hand side of (35) is a strictly increasing function of $\tilde{r}_{\mu}$ for $\tilde{r}_{\mu} \geq \frac{\sqrt{3}}{2}(1 + \mu)$. If we assume that $\tilde{r}_{\mu}(V_{\mu,0}) = r_{\mu}(0)$ (viewing $r_{\mu}$ as a function of $t$) and stipulate that $tf \leq 0$, then the comparison lemma [Arn73, Sec. 2.7] and (34) imply that $r(tf) \geq \tilde{r}_{\mu}(V_{\mu,tf})$, so it follows from (36) and the preceding sentence that
\[
V(x(0)) - V(x(tf)) \geq r_{\mu}(0) - \frac{3^3 + 3^\frac{3}{2}}{2}(1 + \mu) \arctan \left( \frac{2r_{\mu}(0)}{(3)^{1/4}(1 + \mu)} \right) - r_{\mu}(tf) + \frac{3^3 + 3^\frac{3}{2}}{2}(1 + \mu) \arctan \left( \frac{2r_{\mu}(tf)}{(3)^{1/4}(1 + \mu)} \right),
\]
(37)

where we have used the fact that $V(x(tf)) - V(x(0)) = V_{\mu}(x(tf)) - V_{\mu}(x(0))$.

Fix $\epsilon > 0$. Assume that there exists $t \leq 0$ such that $r_{\mu}^2(t) = \tilde{r}_{\mu}(V_{\mu,0}) = \frac{3}{4}(1 + \mu)^2 + \epsilon$ and $V(x(t)) > 0$, and let $tf \leq 0$ be the largest such time. Since $V(x(tf)) > 0$ and since the right-hand side of (35) is a strictly increasing function of $\tilde{r}_{\mu}$, we obtain from (37)
\[
V(x(0)) \geq r_{\mu}(0) - \frac{3^3 + 3^\frac{3}{2}}{2}(1 + \mu) \arctan \left( \frac{2r_{\mu}(0)}{(3)^{1/4}(1 + \mu)} \right) - \sqrt{\frac{3}{2}}(1 + \mu)\ldots
\]
(38)

\[\ldots + \frac{3^3 + 3^\frac{3}{2}}{2}(1 + \mu) \arctan \left( \frac{2r_{\mu}(tf)}{(3)^{1/4}(1 + \mu)} \right) - \frac{3}{2}(1 + \mu)\]

where $\sigma(0) = 0, \sigma(\epsilon) > 0$ for $\epsilon > 0$, and $\epsilon \mapsto \sigma(\epsilon)$ is strictly increasing. Note that the right hand side of (38) is independent of $tf$. Define $K_{\mu,\epsilon}$ to be the set of points $x(0) \in V^{-1}[0, 3(1 + \mu)]$ which satisfy (38), and note that $K_{\mu,\epsilon}$ is compact since it is clearly closed and bounded.

It follows that, if $x(0) \in V^{-1}[0, 3(1 + \mu)] \setminus K_{\mu,\epsilon}$, then $r_{\mu}^2(t) > \frac{3}{4}(1 + \mu)^2 + \epsilon$ for all $t \leq 0$, and hence $V(x(t)) > \epsilon$ for all $t \leq 0$ by (33). This uniform lower bound on $V$ implies that there exists $tf < 0$ with $V(x(tf)) < 0$. Finally, given any $x(0) \in V^{-1}[0, 3(1 + \mu)] \setminus K_{\mu}$, there exists $\epsilon > 0$ such that $x(0) \in V^{-1}[0, 3(1 + \mu)] \setminus K_{\mu,\epsilon}$ since
\[
V^{-1}[0, 3(1 + \mu)] \setminus K_{\mu} = \bigcup_{\epsilon > 0} V^{-1}[0, 3(1 + \mu)] \setminus K_{\mu,\epsilon}.
\]
(The last equation follows from the fact that $K_{\mu} = \bigcap_{\epsilon > 0} K_{\mu,\epsilon}$.) Therefore, for every $x(0) \in V^{-1}[0, 3(1 + \mu)] \setminus K_{\mu}$ there exists $tf < 0$ such that $V(x(tf)) < 0$. By the second paragraph of the proof, this completes the proof. \qed

4.4.2. Cylindrical coordinates and rotation of the flow. Define an orthogonal matrix $M$ via
\[
M = \begin{bmatrix}
\frac{\sqrt{6}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\
\frac{\sqrt{6}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\
-\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3}
\end{bmatrix}
\]
and define coordinates $[u, v, w]^T := M^{-1}[x, y, z]^T = M^T[x, y, z]^T$. The $w$-axis corresponds to $\Delta$ in the original coordinates, and the $u$ and $v$ axes determine an orthonormal coordinate system for $\Delta^\perp$. 

We further define cylindrical coordinates \((ρ, θ, w)\) via

\[
\begin{align*}
    u &= ρ \cos θ \\
    v &= ρ \sin θ.
\end{align*}
\]

Using the symbolic package SymPy, we obtain the equations of motion in these new coordinates in closed form:

\[
\begin{align*}
    \dot{ρ} &= ρ \left(-\sqrt{2}\rho \sin^3(θ) + \frac{\sqrt{6}}{3} ρ \cos^3(θ) - \frac{\sqrt{6}}{2} ρ \cos(θ + \frac{π}{3}) - \frac{1}{\sqrt{3}} w - μ + \frac{1}{2}\right) \\
    \dot{θ} &= ρ \left(\frac{\sqrt{6}}{3} \sin^3(θ) - \frac{\sqrt{6}}{2} \sin(θ + \frac{π}{3}) + \sqrt{2} \cos^3(θ)\right) - w - \frac{\sqrt{3}}{2} \\
    \dot{w} &= \frac{1}{\sqrt{3}} (ρ^2 + w^2) - (μ + 1)w.
\end{align*}
\]

(41)

Now

\[\dot{θ} = \left(\frac{uv - vu}{u^2 + v^2}\right)(f_μ),\]

and \((u, v, w)\) are orthogonal coordinates adapted to the splitting \(\mathbb{R}^3 = Δ^⊥ ⊕ Δ\) with \((u, v)\) coordinates for \(Δ^⊥\) and with \(w\) a coordinate for \(Δ\). It follows from the discussion in §4.2 that \(θ = dθ(f_μ)\), where \(dθ\) is defined in §4.2. Because the hypotheses of Theorem 3 are stated in terms of a closed 1-form, we will write \(dθ(f_μ)\) instead of \(θ\) in the following results.

**Lemma 9.** The following estimate holds on \(\mathbb{R}^3 \setminus Δ\) or, equivalently, whenever \(ρ > 0\):

\[
-0.41ρ - w - \frac{\sqrt{3}}{2} \leq dθ(f_μ) \leq 0.41ρ - w - \frac{\sqrt{3}}{2}.
\]

(42)

**Proof.** The sinusoidal function \(θ \mapsto (\frac{\sqrt{6}}{3} \sin^3(θ) - \frac{\sqrt{6}}{2} \sin(θ + \frac{π}{3}) + \sqrt{2} \cos^3(θ))\) has zero mean and amplitude smaller than 0.41. The result now follows from (41). \(\square\)

The following result concerns the rotation rate \(\dot{θ} = dθ(f_μ)\) on the compact set \(K_μ\) (defined in Theorem 5) which contains all bounded trajectories.

**Theorem 6.** There exists \(ε > 0\) such that, for all \(-1 ≤ μ ≤ 0.8\), \(dθ(f_μ) < -ε\) on \(K_μ \setminus Δ\).

**Proof.** We have \(K_{-1} \setminus Δ = ∅\) since \(K_{-1} = \{0\}\), so the statement holds vacuously for \(μ = -1\). For the remainder of the proof we assume that \(μ ∈ (-1, 0.8]\).

It follows from Lemma 9 that \(dθ(f_μ) < -ε\) whenever \(ρ ≤ \frac{1}{0.41} w + \frac{\sqrt{3}}{0.82} - \frac{ε}{0.41} \approx 2.439w + 2.11 - 2.438ε\). Since \(w ≥ 0\) on \(K_μ\) it follows that, for any sufficiently small \(ε > 0\), the \(ρ, w\) coordinates of every point in \(K_μ\) satisfy either the preceding inequality or the inequality \(ρ ≥ 2.1\). Hence it suffices to find an \(ε > 0\) such that \(dθ(f_μ) < -ε\) whenever \(x ∈ K_μ\) and \(ρ > 2.1\).

We use the notation from the statement and proof of Theorem 5. From the definition of \(K_μ\), we have

\[
V ≥ r_μ - \frac{3^\frac{3}{4} + 3^\frac{1}{4}}{2}(1 + μ) \arctan \left(\frac{2r_μ}{(3)^{1/4}(1 + μ)}\right) - \frac{\sqrt{3}}{2}(1 + μ) + \frac{3^\frac{3}{4} + 3^\frac{1}{4}}{2}(1 + μ) \arctan \left(3^{1/4}\right)
\]

\[
≥ ρ - \frac{3^\frac{3}{4} + 3^\frac{1}{4}}{2}(1 + μ) \arctan \left(\frac{2ρ}{(3)^{1/4}(1 + μ)}\right) - \frac{\sqrt{3}}{2}(1 + μ) + \frac{3^\frac{3}{4} + 3^\frac{1}{4}}{2}(1 + μ) \arctan \left(3^{1/4}\right)
\]

\[
≥ ρ - 1.8(1 + μ) \arctan \left(\frac{1.52ρ}{1 + μ}\right) + 0.78(1 + μ)
\]

(43)
Figure 11. The blue surface is the same portion of the boundary of $K_\mu$ depicted in Figure 10. The dark surface is the region of space where $d\theta(f_\mu) = 0$ (not including $\Delta$), with $d\theta(f_\mu) < 0$ in the region of space containing the blue surface. The green surface is the boundary of the conservative inner approximation of the region of space where $d\theta(f_\mu) < 0$ obtained in Lemma 9. This figure was generated using $\mu = 0.4$.

on $K_\mu$. The second inequality follows from the following three observations: (i) we showed in the proof of Theorem 5 that the first two terms on the right side of (43) constitute an increasing function of $\rho$ if $\rho \geq \sqrt{3}/2(1 + \mu)$, (ii) $2.1 > \sqrt{3}/2(1 + \mu)$ if $\mu \leq 1.42$, and (iii) $\rho \leq r_\mu$ (the distance to the diagonal $\Delta$ is at most the distance to any individual point on $\Delta$). Since $V = \sqrt{3}w$, it now follows that

$$-w \leq -\frac{1}{\sqrt{3}} \left( \rho - 1.8(1 + \mu) \arctan \left( \frac{1.52\rho}{1 + \mu} \right) + 0.78(1 + \mu) \right)$$

(44)

$$\leq -0.57\rho + 1.04(1 + \mu) \arctan \left( \frac{1.52\rho}{1 + \mu} \right) - 0.45(1 + \mu).$$

Substituting this into Lemma 9, we find that, when $x \in K_\mu$ and $\rho \geq 2.1$,

$$d\theta(f_\mu) \leq -0.15\rho + 1.04(1 + \mu) \arctan \left( \frac{1.52\rho}{1 + \mu} \right) - 0.45(1 + \mu) - 0.86.$$  

(45)

From the same reasoning above, the right-hand side constitutes a decreasing function of $\rho$ if $\rho \geq 2.1$. Hence

$$d\theta(f_\mu) \leq -0.315 + 1.04(1 + \mu) \arctan \left( \frac{3.192}{1 + \mu} \right) - 0.45(1 + \mu) - 0.86.$$  

(46)

The second derivative with respect to $\mu$ of the right-hand side is $-1.04 \frac{2(3.192)^2}{(3.192)^2 + (1 + \mu)^2} < 0$, so $\mu = 0.8$ minimizes the first derivative with respect to $\mu$ on $(-1, 0.8]$. The first derivative of the right-hand side is given by

$$1.04 \left( \arctan \left( \frac{3.192}{1 + \mu} \right) - \frac{3.192/(1 + \mu)}{1 + \left( \frac{3.192}{1 + \mu} \right)^2} \right) - 0.45,$$
so evaluating at $\mu = 0.8$ yields the minimum derivative

$$1.04 \left( \arctan \left( \frac{3.192}{1.8} \right) - \frac{3.192/(1.8)}{1 + \left( \frac{3.192}{1.8} \right)^2} \right) - 0.45 \approx 0.2 > 0.$$ 

Therefore, the first derivative of the right hand side of (46) with respect to $\mu$ is strictly positive for all $-1 < \mu \leq 0.8$. It follows that the right hand side of (46) attains its maximum on $(-1, 0.8]$ at $\mu = 0.8$, so that

$$d\theta(f_\mu) \leq -0.315 + 1.04(1.8) \arctan \left( \frac{3.192}{1.8} \right) - 0.45(1.8) - 0.86$$

whenever $x \in K_\mu$, $\rho \geq 2.1$, and $-1 < \mu \leq 0.8$. By the discussion in the second paragraph of the proof, taking $0 < \epsilon < 0.005$ sufficiently small completes the proof. \hfill \Box

**Corollary 1.** For $\mu \in [-1, 0.8]$, all equilibria of $f_\mu$ belong to the diagonal $\Delta$.

**Proof.** For $\mu \in [-1, 0.8]$, Theorem 5 implies that all equilibria lie in $K_\mu$, and Theorem 6 implies that $d\theta(f_\mu) < 0$ on $K_\mu \setminus \Delta$, so in particular $f_\mu \neq 0$ on $K_\mu \setminus \Delta$. \hfill \Box

**Corollary 2.** For $\mu \in (-1, 0.8]$, all periodic orbits of $f_\mu$ are contained in $K_\mu$, and the winding number $\frac{1}{2\pi} \int_\gamma d\theta$ of any nonstationary periodic orbit $\gamma$ around $\Delta$ satisfies $\frac{1}{2\pi} \int_\gamma d\theta \leq -1$. For the case $\mu = -1$, $f_{-1}$ has no nonstationary periodic orbits.

### 4.4.3. Equilibria

By §4.1, $\Delta$ is invariant and the dynamics restricted to $\Delta$ are given by

$$\dot{x} = x^2 - x - \mu x = x(x - 1 - \mu).$$

**Theorem 7.** For all $\mu \in \mathbb{R}$, the vector field $f_\mu$ has the equilibria $0$ and $(1 + \mu)1$. For $-1 \leq \mu \leq 0.8$, these are the only equilibria.

**Proof.** The first statement follows directly from (47). The second statement follows from Corollary 1. \hfill \Box

We compute

$$D_0 f_\mu = \begin{bmatrix} -\mu & 0 & -1 \\ -1 & -\mu & 0 \\ 0 & -1 & -\mu \end{bmatrix}$$

and

$$D_{(1+\mu)}1 f_\mu = \begin{bmatrix} -\mu & 2(1 + \mu) & -1 \\ -1 & -\mu & 2(1 + \mu) \\ 2(1 + \mu) & -1 & -\mu \end{bmatrix}.$$ 

A symbolic eigenvalue computation using SymPy shows that

$$\text{spec}(D_0 f_\mu) = \left\{ -\mu + \frac{1}{2} \pm \frac{\sqrt{3}}{2}, -1 - \mu \right\}$$

and

$$\text{spec}(D_{(1+\mu)}1 f_\mu) = \left\{ -2\mu - \frac{1}{2} \pm \frac{\sqrt{3\sqrt{4\mu^2 + 12\mu + 9}}}{2}, 1 + \mu \right\}.$$ 

The quadratic $4\mu^2 + 12\mu + 9$ is positive except for a single zero at $\mu = -\frac{3}{2}$. It follows in particular that $D f_\mu$ evaluated at both of these equilibria is always invertible except when $\mu = -1$, which is the value of $\mu$ at which these equilibria coalesce. Additionally, the eigenvalues $\pm (1 + \mu)$ for the two zeros both correspond to the eigenvector $1$. 

4.4.4. Two Hopf bifurcations. Given an equilibrium \( x \) for \( f_\mu \) at a given value of \( \mu \), define the matrix 
\[
A := D_x f_\mu
\]
and the \((1,2)\) tensor \( B := D^2_x f_\mu \). Since \( f_\mu \) is a quadratic vector field, all of its third partial derivatives vanish, and therefore the first Lyapunov coefficient \( \ell_1(0) \) at an equilibrium \((x,\mu)\) having a single pair of purely imaginary eigenvalues is given by [Kuz13, Eq. 5.39]:
\[
\ell_1(0) = \frac{1}{2\omega_0} \text{Re} \left[ \left\langle p, B(q, (2i\omega_0 I_n - A)^{-1}B(q,q)) - 2B(q, A^{-1}B(q,q)) \right\rangle \right],
\]
where \( \pm i\omega_0 \) are the imaginary eigenvalues of \( A \) and \( p, q \in \mathbb{C} \) satisfy \( Aq = i\omega_0q, A^T p = -i\omega_0 p \), and \( \left\langle p, q \right\rangle := p \cdot \bar{q} = 1 \). We numerically compute \( \ell_1(0) \approx -0.808 \) for the equilibrium \( 0 \) at \( \mu = 0.5 \), and \( \ell_1(0) \approx 0.514 \) for the equilibrium \((1 + \mu)1\) at \( \mu = -0.25 \). Additionally, we see from (50) and (51) that the derivatives with respect to \( \mu \) of the real part of the complex eigenvalues is negative for the origin at \( \mu = 0.5 \) and also negative for \((1 + \mu)1\) at \( \mu = -0.25 \). Using this fact, the value of \( \ell_1(0) \), and Theorem 3.3 (p. 98) and Equation 5.39 (p. 180) of [Kuz13], it follows that:

**Theorem 8.** The equilibrium \( 0 \) undergoes a subcritical generic Hopf bifurcation at \( \mu = 0.5 \), and the equilibrium \((1 + \mu)1\) undergoes a supercritical generic Hopf bifurcation at \( \mu = -0.25 \). The first bifurcation produces an exponentially stable limit cycle near \( 0 \) for \( 0 < 0.5 - \mu \ll 1 \), and the second bifurcation produces an exponentially unstable limit cycle near \((1 + \mu)1\) for \( 0 < \mu - (-0.25) \ll 1 \).

4.4.5. Existence of periodic orbits. We now put together the preceding results to obtain a periodic orbit existence result for the Sprott vector field (29). To do this, we show that the restriction \( f|_{(-\infty,0.5)} \) satisfies the hypotheses of Theorem 3 after a (nonlinear) parameter rescaling.

**Theorem 9.** Let \( f_\mu \) be the Sprott vector field (29) and let \( K_\mu \) be defined as in Theorem 5. For all \( \mu \in (-0.25,0.5) \), \( f_\mu \) has a periodic orbit contained in \( K_\mu \).

**Proof.** Let \( \varphi : \mathbb{R} \to (-\infty,0.5) \) be an increasing diffeomorphism satisfying \( \varphi|_{(-\infty,0)} = \text{id}|_{(-\infty,0)} \) and \( \lim_{s \to \infty} \varphi(s) = 0.5 \). Letting \( K_\mu \) be as in Theorem 5 and defining the family \( g := \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \) via \( g_\mu := f_{\varphi(\mu)} \), we will apply Theorem 3 to show that \( g_\mu \) has a periodic orbit contained in \( K_{\varphi(\mu)} \) for all \( \mu \in (-0.25,\infty) \).

Define \( \tilde{C} := \{(x,\mu) \in \mathbb{R}^3 \times \mathbb{R} : \mu \geq -1 \text{ and } x \in K_{\varphi(\mu)} \} \). Using the definition of \( K_\mu \), it is easily seen that any set of the form \( \tilde{C}_{[a,b]} := \tilde{C} \cap (\mathbb{R}^3 \times [a,b]) \subset \mathbb{R}^4 \) is closed and bounded, hence compact. Letting the closed 1-form \( d\theta \) be as defined in §4.2, Theorem 6 implies that there exists \( \epsilon > 0 \) such that \( -d\theta(g_\mu) > \epsilon \) on \( \tilde{C} \setminus (\Delta \times \mathbb{R}) \). By continuity, there exists \( \epsilon > 0 \) and a set \( \tilde{C} \subset \mathbb{R}^3 \times \mathbb{R} \) slightly larger than \( \tilde{C} \) satisfying \( \tilde{C} \subset \text{int}(\tilde{C}), -d\theta(g_\mu) > \epsilon \) on \( \tilde{C} \setminus (\Delta \times \mathbb{R}) \), and with each set of the form \( \tilde{C}_{[a,b]} \cap (\mathbb{R}^3 \times [a,b]) \) compact. In particular, hypotheses 3 and 4 Theorem 3 are satisfied.

Corollary 2 implies that \( g_{-1} = f_{-1} \) has no nonstationary periodic orbits, so hypothesis 1 of Theorem 3 is satisfied with \( \mu^* := -1 \). It follows from Theorem 5 that every periodic orbit of \( g|_{(-1,\infty)} \) is contained in \( \tilde{C} \subset \text{int}(\tilde{C}) \), so in particular no periodic orbits of \( g|_{[1,\infty)} \) intersect \( \partial \tilde{C} \); hence hypothesis 2 of Theorem 3 is satisfied. We showed in §4.4.3 and Theorem 8 that \( g \) has exactly one generalized center \((x_c,\mu_c) := (-0.75, 1, -0.25) \in \text{int}(\tilde{C}) \) at which \( g \) undergoes a supercritical generic Hopf bifurcation. Hence hypothesis 5 of Theorem 3 is satisfied. Hypothesis 6 is satisfied because \( \Delta \) is an invariant manifold for each \( g_\mu \) by symmetry (§4.1), and \( \Delta \) is diffeomorphic to \( \mathbb{R} \), so no periodic orbits can intersect \( \Delta \). Finally, the center subspace \( E_c \) of \( D_{x_c} g_{\mu_c} = D_{x_c} f_{\mu_c} \) is orthogonal to \( \Delta \) by Equation (15), so hypothesis 7 is satisfied.

Theorem 3 now implies that \( g_\mu \) has a periodic orbit contained in \( K_{\varphi(\mu)} \) for all \( \mu \in (-0.25,\infty) \). Since \( g_\mu = f_{\varphi(\mu)} \) by definition, it follows that \( f_\mu \) has a periodic orbit contained in \( K_\mu \) for all \( \mu \in (-0.25,0.5) \). This completes the proof.

---

13That \( \frac{d\theta}{\pi} \) satisfies the relevant Poincaré duality hypotheses of Theorem 3 follows exactly as in the proof of Theorem 4, using the discussion in §4.2 (after flipping the orientations of the submanifolds \( M, N \) due to the minus sign).
References

[AMPY81] K T Alligood, J Mallet-Paret, and J A Yorke, Families of periodic orbits: local continuability does not imply global continuability, Journal of Differential Geometry 16 (1981), no. 3, 483–492.

[AMPY83] ______, An index for the global continuation of relatively isolated sets of periodic orbits, Geometric dynamics, Springer, 1983, pp. 1–21.

[Arn73] V I Arnold, Ordinary differential equations, MIT Press, 1973.

[AY78] J C Alexander and J A Yorke, Global bifurcations of periodic orbits, American Journal of Mathematics 100 (1978), no. 2, 263–292.

[AY83] ______, On the continuability of periodic orbits of parametrized three-dimensional differential equations, Journal of differential equations 49 (1983), no. 2, 171–184.

[AY84] K T Alligood and J A Yorke, Families of periodic orbits: virtual periods and global continuability, Journal of differential equations 55 (1983), no. 2, 171–184.

[AY84] K T Alligood and J A Yorke, Families of periodic orbits: virtual periods and global continuability, Journal of differential equations 55 (1983), no. 2, 171–184.

[BDJ05] A Baker, M Dellnitz, and O Junge, Topological method for rigorously computing periodic orbits using fourier modes, Discrete and Continuous Dynamical Systems 13 (2005), no. 4, 901–920.

[Ben01] Ivar Bendixson, Sur les courbes définies par des équations différentielles, Acta Mathematica 24 (1901), no. 1, 1–88.

[BKP09] O Buşe, A Kuznetsov, and R A Pérez, Existence of limit cycles in the repressilator equations, International Journal of Bifurcation and Chaos 19 (2009), no. 12, 4097–4106.

[BPK10] O Buşe, R Pérez, and A Kuznetsov, Dynamical properties of the repressilator model, Physical Review E 81 (2010), no. 6, 066206.

[Bru71] P Brunovský, One-parameter families of diffeomorphisms, Proceedings of the Symposium on Differential Equations and Dynamical Systems, Springer, 1971, pp. 29–33.

[Bru71b] P Brunovský, On one-parameter families of diffeomorphisms. ii: Generic branching in higher dimensions, Commentationes Mathematicae Universitatis Carolinae 12 (1971), no. 4, 765–784.

[BT91] R Bott and L W Tu, Differential forms in algebraic topology, vol. 82, Springer-Verlag, New York, 1991, Revised third printing.

[Byr07] C I Byrnes, Differential forms and dynamical systems, Modeling, Estimation and Control, Springer, 2007, pp. 35–44.

[Byr10] ______, Topological methods for nonlinear oscillations, Notices of the AMS 57 (2010), no. 9, 1080–1091.

[CMP78] S-N Chow and J Mallet-Paret, The Fuller index and global Hopf bifurcation, Journal of Differential Equations 29 (1978), no. 1, 66–85.

[CMPY83] S-N Chow, J Mallet-Paret, and J A Yorke, A periodic orbit index which is a bifurcation invariant, Geometric dynamics, Springer, 1983, pp. 109–131.

[Con78] C C Conley, Isolated invariant sets and the Morse index, no. 38, American Mathematical Society, 1978.

[CZ83] C C Conley and E Zehnder, The Birkhoff-Lewis fixed point theorem and a conjecture of V I Arnold, Inventiones mathematicae 73 (1983), no. 1, 33–49.

[CZ84] ______, Morse-type index theory for flows and periodic solutions for Hamiltonian equations, Communications on pure and applied mathematics 37 (1984), no. 2, 207–253.

[EL00] M B Elowitz and S Leibler, A synthetic oscillatory network of transcriptional regulators, Nature 403 (2000), no. 6767, 335–338.

[Fie88] B Fiedler, Global bifurcation of periodic solutions with symmetry, Springer, 1988.

[FKLZ03] M Farber, T Kappeler, J Latschev, and E Zehnder, Smooth Lyapunov 1-forms, arXiv preprint math/0304137 (2003), 1–29.

[FKLZ04] ______, Lyapunov 1-forms for flows, Ergodic theory and dynamical systems 24 (2004), no. 5, 1451–1475.

[Flo89] A Floer, Symplectic fixed points and holomorphic spheres, Communications in Mathematical Physics 120 (1989), no. 4, 575–611.

[Ful67] F B Fuller, An index of fixed point type for periodic orbits, American Journal of Mathematics 89 (1967), no. 1, 133–148.

[GH93] R Ghrist and P Holmes, Knots and orbit genealogies in three dimensional flows, Bifurcations and periodic orbits of vector fields, Springer, 1993, pp. 185–239.

[GH00] J Guckenheimer and P Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields, vol. 42, Springer-Verlag, New York, 2000, Corrected reprint of the 1983 original.

[GHS97] R Ghrist, P Holmes, and M Sullivan, Knots and links in three-dimensional flows, Springer-Verlag, Berlin Heidelberg, 1997.
38 FAMILIES OF PERIODIC ORBITS: CLOSED 1-FORMS AND GLOBAL CONTINUABILITY

[GM95] T Gedeon and K Mischaikow, *Structure of the global attractor of cyclic feedback systems*, Journal of Dynamics and Differential Equations 7 (1995), no. 1, 141–190.

[Gra77] W Grasman, *Periodic solutions of autonomous differential equations in higher dimensional spaces*, The Rocky Mountain Journal of Mathematics 7 (1977), no. 3, 457–466.

[GS85] M Golubitsky and D Schaeffer, *Singularities and groups in bifurcation theory*, Springer-Verlag, New York, 1985.

[Hir94] M W Hirsch, *Differential topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, 1994, Corrected reprint of the 1976 original. MR 1336822

[Kuz13] Y A Kuznetsov, *Elements of applied bifurcation theory*, vol. 112, Springer Science & Business Media, 2013.

[Li81] B Li, *Periodic orbits of autonomous ordinary differential equations: theory and applications*, Nonlinear Analysis: Theory, Methods & Applications 5 (1981), no. 9, 931–958.

[MdCG06] A L Maria da Conceiçao and M Golubitsky, *Homogeneous three-cell networks*, Nonlinearity 19 (2006), no. 10, 2313–2364.

[MMM95] C McCord, K Mischaikow, and M Mrozek, *Zeta functions, periodic trajectories, and the Conley index*, Journal of differential equations 121 (1995), no. 2, 258–292.

[MPS90] J Mallet-Paret and H L Smith, *The Poincaré-Bendixson theorem for monotone cyclic feedback systems*, Journal of Dynamics and Differential Equations 2 (1990), no. 4, 367–421.

[MPY82] J Mallet-Paret and J A Yorke, *Snakes: oriented families of periodic orbits, their sources, sinks, and continuation*, Journal of Differential Equations 43 (1982), no. 3, 419–450.

[MT07] J W Morgan and G Tian, *Ricci flow and the Poincaré conjecture*, vol. 3, American Mathematical Society, 2007.

[Pil99] P Pilarczyk, *Computer assisted method for proving existence of periodic orbits*, Topological Methods in Nonlinear Analysis 13 (1999), no. 2, 365–377.

[Poi81] H Poincaré, *Mémoire sur les courbes définies par une équation différentielle (i)*, Journal de mathématiques pures et appliquées 7 (1881), 375–422.

[Rab78] P H Rabinowitz, *Periodic solutions of Hamiltonian systems*, Communications on Pure and Applied Mathematics 31 (1978), no. 2, 157–184.

[Rab99] C Robinson, *Dynamical systems: Stability, symbolic dynamics, and chaos*, 2 ed., Taylor & Francis, 1999.

[RPM+17] S Ronquist, G Patterson, L A Muir, S Lindsly, H Chen, M Brown, M S Wicha, A Bloch, R Brockett, and I Rajapakse, *Algorithm for cellular reprogramming*, Proceedings of the National Academy of Sciences 114 (2017), no. 45, 11832–11837.

[RS17] I Rajapakse and S Smale, *Mathematics of the genome*, Foundations of Computational Mathematics 17 (2017), no. 5, 1195–1217.

[Rue89] D Ruelle, *Elements of differentiable dynamics and bifurcation theory*, Elsevier, 1989.

[Smi80] R A Smith, *Existence of periodic orbits of autonomous ordinary differential equations*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics 85 (1980), no. 1-2, 153–172.

[Sot73] J Sotomayor, *Generic bifurcations of dynamical systems*, Dynamical systems, Elsevier, 1973, pp. 561–582.

[Spr10] J C Sprott, *Elegant chaos: algebraically simple chaotic flows*, World Scientific, 2010.

[SW99] H L Smith and P Waltman, *Perturbation of a globally stable steady state*, Proceedings of the American Mathematical Society 127 (1999), no. 2, 447–453.

[SY12] E Sander and J A Yorke, *Connecting period-doubling cascades to chaos*, Journal of Differential Equations and Chaos 22 (2012), no. 2, 1250022.

[Wei79] A Weinstein, *On the hypotheses of Rabinowitz’ periodic orbit theorems*, Journal of differential equations 33 (1979), no. 3, 353–358.

[Wil67] F W Wilson, Jr., *The structure of the level surfaces of a Lyapunov function*, J. Differential Equations 3 (1967), 323–329. MR 0231409

[Yor69] J A Yorke, *Periods of periodic solutions and the Lipschitz constant*, Proceedings of the American Mathematical Society 22 (1969), no. 2, 509–512.

[Zeh86] E Zehnder, *The Arnold conjecture for fixed points of symplectic mappings and periodic solutions of Hamiltonian systems*, Proceedings of the International Congress of Mathematicians, vol. 1, 1986, p. 2.

[Zeh19] ______, *The beginnings of symplectic topology in Bochum in the early eighties*, Jahresbericht der Deutschen Mathematiker-Vereinigung 121 (2019), no. 2, 71–90.