RELATIVE WEAK MIXING IS GENERIC

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ABSTRACT. A classical result of Halmos asserts that among measure preserving transformations the weak mixing property is generic. We extend Halmos’ result to the collection of ergodic extensions of a fixed, but arbitrary, ergodic transformation $T_0$. We then use a result of Connes, Feldman and Weiss to extend this relative theorem to the general (countable) amenable group.

INTRODUCTION

In chapter 19 of Halmos’ seminal book [Ha-56], entitled “Category”, he proves what he calls the “second category theorem” (the original proof was published in [Ha-44]):

0.1. Theorem. In the weak topology the set of all weakly mixing transformations is an everywhere dense $G_δ$.

Only later researchers in ergodic theory got interested in properties of extensions of measure preserving transformations and their structure (see e.g. [Th-75], [Z-76a, Z-76b] and [Fur-77], [Ru-79]). Surprisingly, as far as we know, the natural question, whether Halmos’ theorem holds for extensions, did not receive any consideration. The recent result of M. Schnurr [Sch-17] caught our attention and in the present note we improve his result.

In section 1 we extend Halmos’ result to the collection of ergodic extensions of a fixed, but arbitrary, ergodic transformation $T_0$. In section 2 we apply a general argument regarding extensions, originated in [RW-00], and then use a result of Connes, Feldman and Weiss [CFW-81], to extend the relative theorem to the general (countable) amenable group.

1. THE RELATIVE THEOREM FOR $\mathbb{Z}$-ACTIONS

Let $\nu$ denote the Lebesgue measure on the unit interval $I$. Let Aut $(\nu)$ denote the Polish group of measure preserving automorphisms of the standard Lebesgue space $(I, \nu)$. We let $\mu = \nu \times \nu$ be the product measure on the space $X = I \times I$. Sometimes we also write $(X, \mu) = (Z, \nu) \times (Y, \nu)$, where $Z = I = Y$.

Let $T_0 \in \text{Aut} (\nu)$ be a fixed ergodic transformation. On $(X, \mu)$ let $\mathcal{A}$ denote the $\sigma$-algebra of sets of the form $A \times I$. Let

$$\mathcal{M} = \{ T \in \text{Aut} (\mu) : T \upharpoonright \mathcal{A} = T_0 \}.$$
a closed subset of the Polish group Aut (μ). Denote by G the closed subgroup of Aut (μ) consisting of transformations S ∈ Aut (μ) such that S | A = Id. Clearly M is invariant under conjugation by elements of G.

1.1. Theorem. The collection W of measure preserving transformations in M which are weakly mixing relative to T₀ is a dense Gδ subset of M.

Proof. Let W₀ ∈ Aut (ν) be a fixed weakly mixing transformation and set W = T₀ × W₀ on I × I. Note that W is ergodic and a weakly mixing extension of T₀.

We want to show that the G-orbit of W is dense in M. Since every element of this orbit is weakly mixing relative to T₀ (being a conjugate of W = T₀ × W₀), and as the collection of all the transformations which are weakly mixing relative to T₀ is a Gδ subset of M (see Lemma 1.2 below), this claim will complete the proof.

Given a measurable partition \( \{ A_0, A_1, \ldots, A_k \} \) of I and a finite set \( \{ R_0, R_1, \ldots, R_k \} \) of elements of Aut (ν), define a piecewise constant skew product over T₀, \( S ∈ M ⊂ Aut (μ) \) by the formula

\[
S(z, y) = (T₀z, ρ_z(y)),
\]

where for each \( j \), \( ρ_z \) is the constant transformation \( R_j \) on the cell \( A_j \). It is not hard to see that the collection of piecewise constant skew products over T₀ is dense in M (see Lemma 1.3 below).

So let \( S = S_{\{ A_j, R_j \}} \), a piecewise constant skew product over T₀, and \( ε > 0 \) be given. Let \( \{ B, T₀B, \ldots, T^{n-1}B \} \) be a Rohlin tower for T₀ with \( ν(\bigcup_{i=0}^{n-1} T₀^i B) < ε \). (see e.g. [Ha-56]).

We refine the tower with respect to the partition \( \{ A_j : 1 ≤ j ≤ k \} \). This partitions B into subsets \( \{ B_l : 1 ≤ l ≤ L \} \), such that for each \( l \) and each \( i \) there is some \( α(l, i) \) with \( T^α B_l ⊂ A_{α(l, i)} \).

Next, define \( Q ∈ Aut (μ) \), a piecewise constant skew product over Id, as follows. For each \( l \), set \( Q_{l,0} = Id \). For \( i = 0, 1, \ldots, n - 2 \), let \( Q_{l,i} ∈ Aut (ν) \) satisfy \( Q_{l,i+1} W₀ Q_{l,i}⁻¹ = R_{α(l,i)} \), and \( Q_{l,n} W₀ Q_{l,n-1}⁻¹ = R₀ \). Then, let \( Q ∈ G ⊂ Aut (μ) \) be defined by

\[
Q(z, y) = (z, Q_{l,i}y) \quad \text{for} \quad z ∈ T₀^i B_l, \ i = 0, 1, \ldots, n - 1,
\]

\[
Q(z, y) = (z, y), \quad \text{for} \quad z ∈ I \ \bigcup_{i=0}^{n-1} T₀^i B,
\]

and set \( V = QWQ⁻¹ \). It is now easy to check that \( V ^ {ε + \frac{1}{n}} \sim S \), and we are done. □

1.2. Lemma. The collection \( \mathcal{W} \) consisting of transformations which are relatively weakly mixing over T₀ is a Gδ subset of M.

Proof. Let

\[
X × X = \{ ((z, y), (z, y')) : z, y, y' ∈ I \}
\]
and let
\[ \lambda = \int_{z \in I} (\delta_z \times \nu) \times (\delta_z \times \nu) \, d\nu(z), \]
denote the relatively independent product measure \( \mu \times \mu \) which is supported on \( X \times X \). An element \( T \in \mathcal{M} \) is relatively weakly mixing over \( T_0 \) when (and only when) the measure \( \lambda \) is \( T \times T \)-ergodic.

Given \( A, B \in \mathcal{X} \), the \( \sigma \)-algebra of measurable subsets on \( X \), \( \epsilon > 0 \) and \( N \in \mathbb{N} \), set
\[ \mathcal{E}_{A,B,\epsilon,N} = \left\{ T \in \mathcal{M} : \left\| \frac{1}{N} \sum_{n=0}^{N-1} (T^m \times T^m)(1_A \otimes 1_B) - E(1_A|Z)E(1_B|Z) \right\|_{L^2(\lambda)} < \epsilon \right\}. \]
Clearly each \( \mathcal{E}_{A,B,\epsilon,N} \) is an open set and
\[ W = \bigcap_{i,j,k,N} \bigcup \mathcal{E}_{A_i,A_j,1_k,N}, \]
where \( \{A_i\}_{i\in\mathbb{N}} \) is a countable dense collection in \( \mathcal{X} \).

1.3. Lemma. The collection of piecewise constant skew products over \( T_0 \) is a dense subset of \( \mathcal{M} \).

Proof. Let \( d \) be a compatible metric on \( \text{Aut}(\nu) \). An arbitrary element \( T \in \mathcal{M} \) has the form
\[ T(z,y) = (T_0z, \rho_z(y)), \]
where \( \rho : [0,1] \to \text{Aut}(\nu) \) is a measurable mapping. Denote by \( \tilde{\nu} \) the push-forward measure \( \tilde{\nu} = \rho_*\nu = \nu \circ \rho^{-1} \). This is a regular measure on \( \text{Aut}(\nu) \) hence, given \( \epsilon > 0 \), there is a compact set \( K \subset \text{Aut}(\nu) \) with \( \tilde{\nu}(K) > 1 - \epsilon \). Let \( \{C_i\}_{i=1}^k \) be a partition of \( K \) into sets with diameter \( < \epsilon \) and set \( C_0 = \text{Aut}(\nu) \setminus K \). Let \( A_i = \rho^{-1}(C_i), \) \( 0 \leq i \leq k \). Define \( R_0 = \text{Id} \) and for each \( 1 \leq i \leq k \) choose some \( R_i \in C_i \). Finally let \( S = S_{\{A_i,R_i\}} \) be defined by
\[ S(z,y) = (T_0z, \sigma_z(y)), \]
where for each \( i \), \( \sigma_z \) is the constant transformation \( R_i \) on the cell \( A_i \). Clearly \( T \mathbin{\sim} S \).

Let
\[ N = \{ T \in \text{Aut}(\mu) : T \text{ leaves the } \sigma \text{-algebra } \mathcal{A} \text{ invariant} \}, \]
and let
\[ \tilde{W} = \{ T \in N : T \text{ is a weakly mixing extension of } T' = T \upharpoonright A \}. \]
As the collection of ergodic transformations is a dense \( G_\delta \) subset of \( \text{Aut}(\nu) \), Theorem 1.1, combined with the Kuratowskii-Ulam theorem [Ox-80, Theorem 15.4], immediately yield the following result of M. Schnurr [Sch-17]:

1.4. Corollary. The collection \( \tilde{W} \) forms a residual subset of \( N \).
2. The relative theorem for amenable groups

The result of Halmos, that weak mixing is generic in $\text{Aut}(\nu)$, can be extended to any countable amenable group as soon as one establishes that orbits under conjugation by $\text{Aut}(\nu)$ of free actions are dense. In turn, this relies on the Rohlin lemma (in fact Halmos’ original proof used a weaker version of the lemma which was first stated by Rohlin [Ro-48]). Such a result for amenable groups was established in [OW-80]. It is therefore natural to ask about the analogue of theorem 1.1 for actions of an amenable group. The fact that, for a given amenable group $G$, the collection of the relatively weakly mixing extensions of a fixed action forms a $G_\delta$ can be proved just like we proved lemma 1.2. It thus remains to show that this collection is dense. In order to do this we proceed as follows.

2.1. Definition. If $G$ and $H$ are two groups acting as measure preserving transformations $\{T_g\}_{g \in G}, \{S_h\}_{h \in H}$ on the measure space $(Z, \nu)$ we say that the actions are orbit equivalent if for $\nu$-a.e. $z \in Z$, $Gz = Hz$.

In [CFW-81] is is shown that any ergodic measure preserving action of an amenable group is orbit equivalent to an action of $Z$.

We fix an arbitrary countable amenable group $G$. As in section 1 we let $\nu$ denote the Lebesgue measure on the unit interval $I$, and let $\mu = \nu \times \nu$ be the product measure on the space $X = I \times I$. Again we write $(X, \mu) = (Z, \nu) \times (Y, \nu)$, where $Z = I = Y$.

Let $A(\nu)$ denote the Polish space of measure preserving actions $\{T_g\}_{g \in G}$ of $G$ on $(Z, \nu)$. Let $\{T_0\}_{g \in G} \in A(\nu)$ be a fixed ergodic $G$-action. On $(X, \mu)$ let $A(\nu)$ denote the $\sigma$-algebra of sets of the form $A \times I$. Let

$$M = \{T \in A(\mu) : T|A = T_0\},$$

a closed subset of the Polish space $A(\mu)$.

Next we remark that if two actions $\{T_g\}$ of $G$ and $\{S_h\}$ of $H$ are orbit equivalent, the $T$ action is ergodic if and only if the $S$ action is ergodic. The acting groups may be quite different. The same holds for relative weak mixing, which is equivalent to the ergodicity of the relative product over the base (see the proof of lemma 1.2).

2.2. Theorem. Let $G$ be a countable amenable group. The collection $W$ of measure preserving actions in $M$ which are weakly mixing relative to a given ergodic action $\{(T_0)_g\}_{g \in G}$ is a dense $G_\delta$ subset of $M$.

Proof. Apply the theorem from [CFW-81] to get a measure preserving transformation $R \in \text{Aut}(\nu)$ that is orbit equivalent to the $G$-action $\{(T_0)_g\}_{g \in G}$. Now check the relative weak mixing using this transformation.

More explicitly, when the equivalence relation is fixed the extensions of $R$ are in one to one correspondence with cocycles of the $T_0$ $G$-action with values in the group of measure
preserving transformations of the fiber $(Y, \nu)$. Thus theorem 1.1 when applied to this $\mathbb{Z}$-action gives the desired result. □

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