ADJOINT DIVISORS AND FREE DIVISORS

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Abstract. We describe two situations where adding the adjoint divisor to a divisor $D$ with smooth normalization yields a free divisor. Both also involve stability or versality. In the first, $D$ is the image of a corank 1 stable map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, and is not free. In the second, $D$ is the discriminant of a versal deformation of a weighted homogeneous function with isolated critical point (subject to certain numerical conditions on the weights). Here $D$ itself is already free.

We also prove an elementary result, inspired by these first two, from which we obtain a plethora of new examples of free divisors. The presented results seem to scratch the surface of a more general phenomenon that is still to be revealed.

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1. Introduction

Free divisors play an important role in singularity theory. Kyoji Saito first proved ([Sai80a]) that the discriminant in the base of a versal deformation of an isolated function singularity is free. In [Dam98], Damon showed that the discriminant in the base of a $\mathcal{X}_D$-versal deformation of a non-linear section of a free divisor $D$ is free provided certain genericity condition holds, and gave conditions for these to hold (the existence of “Morse-type singularities”). In particular, the bifurcation set in the base of an $\mathcal{X}_S$-versal deformation of a map-germ $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a free divisor, in the range of dimension pairs $(n, p)$ for which the hypothesis on the existence of Morse-type singularities holds ([Dam98, §6]). Van Straten showed in [vS95] that the discriminant in the base of a versal deformation of a space curve singularity is free, and this was extended by Buchweitz, Ebeling and Graf von Bothmer, who in [BEGvB09] gave conditions for the discriminants of curve, surface and threefold singularities to be free. Looijenga generalized Saito’s result to the discriminant of

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an ICIS in [Loo84], and this played a key role in the calculation of the “discriminant Milnor number”, the rank of the vanishing homology in the discriminant of a mapping \((\mathbb{C}^n,0) \to (\mathbb{C}^p,0)\) for \(n \geq p\), in [DM91]. In order to carry out similar calculations for the rank of the vanishing homology in the image of a mapping \((\mathbb{C}^n,0) \to (\mathbb{C}^p,0)\) for \(n \geq p\), in [DM91].

In order to carry out similar calculations for the rank of the vanishing homology in the image of a mapping \((\mathbb{C}^n,0) \to (\mathbb{C}^n+1,0)\) one would like a good description of \(\text{Der}(−\log D)\) where \(D\) is now the image of a stable mapping in these dimensions. Unfortunately such images are not free divisors: their module of logarithmic derivations has projective dimension 1 ([HM99, §3]). It is therefore of interest that if \(D\) is the image of a corank 1 stable map-germ, it is possible to find other divisors whose union with \(D\) is free.

This was first shown by Damon in [Dam98, Ex. 8.4]: \(D\) together with two copies of the \(\mathbb{A}_n\) discriminant is itself the \(\mathcal{K}_V\)-discriminant of a non-linear section of the union of the coordinate axes in the plane, which is free by his general theory.

In this paper, we exhibit a different procedure which adds to \(D\) a divisor in such a way that the union is free. The divisor we add here is the adjoint of \(D\), in the following sense.

**Definition 1.1.** We call a germ of a divisor \(A \subset (\mathbb{C}^{n+1},0)\) an adjoint divisor for a germ of a divisor \(D \subset (\mathbb{C}^{n+1},0)\) if the pull-back of \(A\) by the normalization of \(D\) defines the conductor.

In particular, \(D \cap A = \text{Sing}(D)\) (as sets, though not as schemes) if \(A\) is adjoint to \(D\). In [2] we prove

**Theorem 1.2.** If \(D\) is the image of a stable corank 1 map-germ \((\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0)\) and \(A\) is an adjoint divisor for \(D\) then \(D + A\) is free.

It seems that \(D + A\) does not arise as a discriminant using Damon’s procedure.

In [3] we prove an essentially identical statement for the discriminants of \(\mathcal{R}_e\)-versal deformations of weighted homogeneous function singularities, subject to a numerical condition on the weights.

**Theorem 1.3.** Let \(D \subset (\mathbb{C}^n,0)\) be the discriminant of a versal deformation of a weighted homogeneous function singularity with Milnor number \(\mu\), and let \(A\) be an adjoint divisor for \(D\). Denote by \(d\) the degree of \(f\) and by \(d_1 \geq d_2 \geq \cdots \geq d_\mu\) the degrees of the members of a weighted homogeneous basis of the Jacobian algebra. Then provided \(d - d_1 + 2d_i \neq 0\) for \(i = 2, \ldots, \mu\), \(D + A\) is a free divisor.

Motivated by these theorems, in [4] we describe a general procedure which constructs, from a triple consisting of a linear free divisor \(D\) with \(k\) irreducible components, a partition of \(k\) into \(\ell\) parts, and a free divisor in \((\mathbb{C}^\ell,0)\) containing the coordinate hyperplanes, a new free divisor containing \(D\). By this means we are able to construct a surprisingly large number of new examples of free divisors.

## 2. Images of stable maps

Let \(f : X := (\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0) =: T\) be a finite and generically one-to-one map-germ with image \(D\). By [MP89] Prop. 2.5, the \(\mathcal{O}_T\)-module \(\mathcal{O}_X\) has a free resolution of the form

\[
0 \longrightarrow \mathcal{O}_T^{k} \overset{\lambda}{\longrightarrow} \mathcal{O}_T^{k} \overset{\alpha}{\longrightarrow} \mathcal{O}_X \longrightarrow 0,
\]

in which the matrix \(\lambda\) can be chosen symmetric (we shall recall the proof below). For \(1 \leq i, j \leq k\), we denote by \(m_{ij}\) the minor obtained from \(\lambda\) by deleting the \(i\)th
row and the $j$th column. The morphism $\alpha$ sends the $i$th basis vector $e_i$ to $g_i \in \mathcal{O}_X$, where $g_1, \ldots, g_k$ generate $\mathcal{O}_X$ over $\mathcal{O}_T$. It will be convenient to assume, without loss of generality, that $g_k = 1$.

Note that although $\text{Sing}(D)$ has codimension 2 in $T$, it is far from being a complete intersection. By [MP89, Thm. 3.4, Prop. 3.5.i)], it is the determinantal variety defined by the first Fitting ideal

$$F_1 := F_{\mathcal{O}_T}^1(\mathcal{O}_X) = \langle m^k_j \mid j = 1, \ldots, k \rangle_{\mathcal{O}_T}$$

of $\mathcal{O}_X$ as $\mathcal{O}_T$-module. By Cramer’s rule one finds that in $\mathcal{O}_X$, $g_i m^k_j = \pm g_j m^k_i$ for $1 \leq i, j, s \leq k$ (see [MP89, Lem. 3.3]), and so in particular, and invoking the symmetry of $\lambda$, we have $g_i m^k_j = \pm g_j m^k_i$ for $1 \leq i, j \leq k$. Since $g_k = 1$, we deduce that $m^k_i = \pm g_i m^k_k$, so that $\mathcal{O}_X F_1 = \langle m^k_k \rangle_{\mathcal{O}_X}$ is a principal ideal. It follows that the adjoint divisors are all divisors $A = V(\sum_{j=1}^k c_j m^k_j)$ with the $c_j \in \mathcal{O}_T$ and $c_k$ a unit.

**Example 2.1.**

1. For the Whitney umbrella $D = V(z^2 - x^2 y)$ in $\mathbb{C}^3$ (whose parametrization is the stable germ of type $\Sigma^{1,0}$) the above recipe gives $A = V(x)$ (see Figure 1). One calculates that $\text{Der}(- \log D)$ and $\text{Der}(- \log (D + A))$ are generated, respectively, by the vector fields whose coefficients are displayed as the columns of the matrices

$$\begin{pmatrix} x & x & 0 & z \\ 2y & 0 & 2z & 0 \\ 0 & z & x^2 & xy \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x & x & 0 \\ 2y & 0 & 2z \\ 2z & z & x^2 \end{pmatrix}.$$ 

2. Let $D_0$ be the image of the stable map-germ of type $\Sigma^{2,0}$,

$$(u_1, u_2, u_3, u_4, x, y) \mapsto (u_1, u_2, u_3, u_4, x^2 + u_1 y, xy + u_2 x + u_3 y, y^2 + u_4 x).$$

One calculates that $D_0 + A$ is not free.

3. In the case of stable map-germs of type $\Sigma^k$ for $k > 1$, there are points where $D$ is isomorphic to the product of $D_0$ and a smooth factor. It follows that that $D + A$ also is not free.

4. If $D$ is the image of an unstable corank 1 germ then in general $D + A$ is not free.

5. For the the normal crossing divisor $N = V(y_1 \cdots y_k) \subset \mathbb{C}^k$, the divisor $A = V(\sum_{j=1}^k y_1 \cdots \hat{y}_j \cdots y_k) \subset \mathbb{C}^k$ is an adjoint, for it is evident that the pullback of $A$ to the normalization defines the conductor. The vector fields $\delta_1, \ldots, \delta_k$ (whose
coefficients are) displayed as columns of the following matrix are all tangent to $D + A$.

\[
\begin{pmatrix}
y_1 & y_1^2 & 0 & \cdots & \cdots & 0 \\
y_2 & -y_2^2 & y_2^2 & \cdots & \cdots & \\
y_3 & 0 & -y_3^2 & y_3^2 & \cdots & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \cdots \\
y_{k-1} & \cdots & -y_{k-1}^2 & y_{k-1}^2 & \cdots & 0 \\
y_k & 0 & \cdots & \cdots & 0 & -y_k^2
\end{pmatrix}
\]

The determinant of the matrix is a reduced equation for $D + A$, and it follows from Saito’s criterion ([Sai80b, Thm. 1.8.(ii)]) that $N + A$ is a free divisor, and that $\delta_1, \ldots, \delta_k$ are a basis for $\text{Der}(-\log(D + A))$.

It is interesting to note that all commutators $[\delta_i, \delta_j]$ for $i, j \geq 2$ are zero. In fact the vector fields $\delta_2, \ldots, \delta_k$ generate a rational (i.e. not regular) non-linear action of the additive group $\mathbb{C}$, whose orbits foliate the complement of $N + E$. The integral flow of $\delta_i$ leaves all coordinates but the $i$th and $(i + 1)$st unchanged and maps $(y_i, y_{i+1})$ to $(y_i/(1 - ty_i), y_{i+1}/(1 + ty_{i+1}))$.

Our proof of Theorem 1.2 is based on Saito’s criterion. Using an explicit list of generators of $\text{Der}(-\log D)$ constructed by Houston and Littlestone in [HL09], and testing them on the equation $m_k$ of $A$, we find a collection of vector fields $\xi_1, \ldots, \xi_{2k-1}$ in $\text{Der}(-\log D)$ which are in $\text{Der}(-\log A)$ “to first order”, in the sense that for $j = 1, \ldots, 2k - 1$, we have $\xi_j \in m_k^{\mathfrak{m}}$ for $j = 1, \ldots, 2k - 1$. As a consequence, the determinant of their Saito matrix must be divisible by the equation of $D + A$. This determinant contains a distinguished monomial also present in the equation of $D + A$, so the quotient of the determinant by the equation of $D + A$ is a unit, the determinant is a reduced equation for $D + A$, and $D + A$ is a free divisor, by Saito’s criterion.

By Mather’s construction of stable map-germs as $\mathcal{K}_c$-versal deformations of germs of rank 0 ([Mat69]), on $X$ and $T$ there are coordinates $(u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-1}, x)$ and $(U_1, \ldots, U_{k-2}, V_1, \ldots, V_{k-1}, W_1, W_2)$ with respect to which $f$ takes the form

\[
f(u, v, x) = \left( u, v, x^k + \sum_{i=1}^{k-2} u_i x^{k-1-i}, \sum_{i=1}^{k-1} v_i x^{k-i} \right).
\]

We will require some slightly more detailed information about the matrix $\lambda$ of (2.1). We make use of a trick previously used in [MPS9]: Embed $X$ as $(\mathbb{C}^n, 0) \times \{0\}$ into $(\mathbb{C}^{n+1}, 0) := S$, and let the additional variable in $S$ be denoted by $t$. Extend $f$ to a map

\[
F: S = \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} = T,
\]

\[
F(u, v, x) := (U, V, W) = (f_1(u, v, x), \ldots, f_n(u, v, x), f_{n+1}(u, v, x) + t).
\]
Then it is obvious from (2.3) that $\mathcal{O}_S/F^*(\mathfrak{m}_T)\mathcal{O}_S$ is generated over $\mathbb{C}$ by the classes of $1, x, \ldots, x^{k-1}$, from which it follows by [Gun74, Cor. 2, p. 137] that $F_*(\mathcal{O}_S)$ is a free $\mathcal{O}_T$-module on the basis $g = g_1, \ldots, g_k$ where $g_i = x^{k-i}$, $i = 1, \ldots, k$. Consider the short exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow \mathcal{O}_S \overset{i}{\longrightarrow} \mathcal{O}_S \overset{g}{\longrightarrow} \mathcal{O}_X \longrightarrow 0$$

defined by multiplication by $t$. This yields a presentation of $\mathcal{O}_X$ as $\mathcal{O}_T$-module

(2.5) $$0 \longrightarrow \mathcal{O}_T^{k+1} \overset{\lambda}{\longrightarrow} \mathcal{O}_T^{k+1} \overset{g}{\longrightarrow} \mathcal{O}_X \longrightarrow 0,$$

where now $\lambda := [t]_g$ denotes the matrix of the $\mathcal{O}_T$-linear map multiplication by $t$ with respect to the basis $g$. This presentation can be improved by a change of basis on the source of $\lambda$, as follows:

Since $\mathcal{O}_S$ is Gorenstein, $\text{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$ is isomorphic to $\mathcal{O}_S$ as $\mathcal{O}_S$-module (the existence of an isomorphism as $\mathcal{O}_T$-module is obvious, but the result here is deeper, see e.g. [SS75]). Let $\Phi$ be an $\mathcal{O}_S$-generator of $\text{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$. Then $\Phi$ induces a symmetric perfect pairing

$$\langle \cdot, \cdot \rangle : \mathcal{O}_S \times \mathcal{O}_S \rightarrow \mathcal{O}_T, \quad \langle a, b \rangle = \Phi(ab).$$

Now choose a basis $\tilde{g} = \tilde{g}_1, \ldots, \tilde{g}_k$ for $\mathcal{O}_S$ over $\mathcal{O}_T$ dual to $g$ with respect to $\langle \cdot, \cdot \rangle$; that is, such that $\langle g_i, \tilde{g}_j \rangle = \delta_{i,j}$. Then the $(i,j)$th entry of $[\phi]_g$ equals $(\tilde{t}^{\tilde{g}_j}, \tilde{g}_i)$, and so redefining

$$\lambda := (\lambda^t_j) = [t]_g^\tilde{g}$$

yields a symmetric presentation matrix in (2.5).

**Lemma 2.2.** With an appropriate choice of generator $\Phi$ of $\text{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$, we have

$$\lambda \equiv \begin{pmatrix}
-V_1 & -V_2 & -V_3 & \cdots & -V_{k-1} & W_2 \\
-V_2 & -V_3 & \ddots & W_2 & 0 \\
-V_3 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-V_{k-1} & W_2 & \ddots & \ddots & \ddots \\
W_2 & 0 & \cdots & \cdots & 0
\end{pmatrix} \mod \langle U, W_1 \rangle.$$

**Proof.** Since $x^{k-1}$ projects to the socle of the 0-dimensional Gorenstein ring $\mathcal{O}_S/F^*(\mathfrak{m}_T)\mathcal{O}_S$, we can define the $\mathcal{O}_S$-generator $\Phi$ of $\text{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$ by taking as $\Phi(h)$, for $h \in \mathcal{O}_S$, the coefficient of $x^{k-1}$ in the representation of $h$ in the basis $g$. First, we use the relation

(2.6) $$W_1 = x^k + \sum_{i=1}^{k-2} U_i x^{k-1-i}$$

from (2.3) to compute

$$\tilde{g}_1 = 1, \tilde{g}_j = \left( W_1 - \sum_{i=j-1}^{k-2} U_i x^{k-1-i} \right) / x^{k+1-j} = x^{j-1} + \sum_{i=1}^{j-2} U_i x^{j-i-2}, \ j = 2, \ldots, k.$$

Note that

(2.7) $$\tilde{g}_2 = x \tilde{g}_1, \ \tilde{g}_j = x \tilde{g}_{j-1} + U_{j-2}, \ j = 3, \ldots, k.$$
Now let us calculate the columns $\lambda_1, \ldots, \lambda_k$ of $\lambda = [t g]$. Using $g_1 = 1$ and the relation $t + \sum_{i=1}^{k-1} V_i g_i = W_2 g_k$ from (2.3), we first compute

$$\lambda_1 = [t g_1]_g = ((t,g)_j) = \left( W_2 g_k - \sum_{i=1}^{k-1} V_i g_i, g_j \right) = (-V_1, \ldots, -V_{k-1}, W_2)^t.$$  

By (2.7), each of the remaining columns $\lambda_j$ is obtained by multiplying by $x$ the vector represented by its predecessor, and, for $j \geq 3$, adding $U_{j-2} \lambda_1$. Thus,

(2.8)  
$$\lambda_2 = [x g]_g \lambda_1, \quad \lambda_j = [x g]_g \lambda_{j-1} + U_{j-2} \lambda_1, \quad j = 3, \ldots, k.$$  

Using (2.6) again, observe that (2.9)

$$[x g]_g = \begin{pmatrix} W_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ -U_1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ -U_{k-2} & \cdots & \cdots & \ddots & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \ddots & 1 \end{pmatrix} \mod (U, W_1).$$  

The result follows. \hfill \square

**Corollary 2.3.** The reduced equation, $h$, of the image of the map $f$ of (2.3) contains the monomial $W_2^k$ with coefficient $\pm 1$. The minor $m^k$ contains the monomial $W_2^{k-2} W_1$ with coefficient $\pm 1$.

**Proof.** The determinant of the matrix $\lambda$ of (2.4) is a reduced equation for the image of $f$ (see e.g. [Tei77]). Both statements then follow from Lemma 2.2. \hfill \square

**Example 2.4.** For the stable map-germs

$$(u_1, v_1, v_2, x) \mapsto (u_1, v_1, v_2, x^4 + u_1 x^2 + v_2 x)$$

and

$$(u_1, v_2, v_1, v_2, v_3, x) \mapsto (u_1, u_2, v_1, v_2, v_3, x^4 + u_1 x^2 + u_2 x, v_1 x^3 + v_2 x^2 + v_3 x)$$

(2.3) with $k = 3$ and $k = 4$ respectively) the matrix $\lambda$ is equal to

$$\begin{pmatrix} -V_1 & -V_2 \\ -V_2 & W_2 + U_1 V_1 \\ W_2 & -V_1 W_1 \end{pmatrix}$$

and to

$$\begin{pmatrix} -V_1 & -V_2 & -V_3 \\ -V_2 & U_1 V_1 - V_3 & W_2 + U_2 V_1 \\ -V_3 & W_2 V_1 - U_1 V_1 & V_2 W_1 - U_1 W_2 \end{pmatrix}$$

respectively.

**Proof of Theorem 1.2.** In [HL09], Thms. 3.1-3.3], Houston and Littlestone give an explicit set of generators for $\text{Der}(- \log D)$. Proof that they lie in $\text{Der}(- \log D)$ is by exhibiting a lift of each. The list consists of the Euler field $\xi_e$ and three families $\xi_i^j, 1 \leq i \leq 3, 1 \leq j \leq k - 1$. 


Denote by $\tilde{\xi}_j$ the linear part of $\xi_j$. After dividing by 1, $k$, $k$ and $k^2$ respectively, these linear parts are

$$
\tilde{\xi}_e = \sum_{i=1}^{k-2} (i+1)U_i \partial V_i + \sum_{i=1}^{k-1} i V_i \partial V_i + kW_1 \partial W_1 + kW_2 \partial W_2,
$$

$$
\tilde{\xi}_1^3 = -W_2 \partial V_1 + \sum_{i<j} V_{i-j+k} \partial V_i, \quad 1 \leq j \leq k-1,
$$

$$
\tilde{\chi} = -\tilde{\xi}_1 = -\sum_{i=1}^{k-2} (i+1)U_i \partial V_i + \sum_{i=1}^{k-1} (k-i)V_i \partial V_i + kW_1 \partial W_1,
$$

$$
\tilde{\xi}_j^2 = \sum_{i<k-j} (i+j)U_{i+j-1} \partial U_i - \sum_{i<k-j+1} (k-i-j+1) V_{i+j-1} \partial V_i, \quad 1 < j \leq k-1,
$$

$$
\tilde{\xi}_j^3 = -(1-\delta_{j,1})W_2 \partial V_{k-j} + \sum_{i<k-j} V_{i+j} \partial U_i + \delta_{j,1} W_2 \partial W_1, \quad 1 \leq j \leq k-1.
$$

Also let

$$
\bar{\sigma} = (\tilde{\xi}_e + \tilde{\chi})/k = \sum_{i=1}^{k-1} V_i \partial V_i + W_2 \partial W_2.
$$

We now test the above vector fields for tangency to $A = V(m_\ell^k)$. Vector fields in $\text{Der}(\log D)$ are preserved modulo $m_T$, so for each $\xi \in \text{Der}(\log D)$ there exist $c_j$, unique modulo $m_T$, such that

$$
\xi(m_\ell^k) = \sum_j c_j m_j^k.
$$

We determine their value modulo $m_T$ with the help of distinguished monomials: Let $\iota$ be the sign of the order-reversing permutation $k-1, k-2, \ldots, 1$. Then, by Lemma 2.2, for $1 < j \leq k$, the monomial $W_2^{k-2} V_{k-j+1}$ appears in the polynomial expansion of $m_j^k$ with coefficient $(-1)^{j-1} \iota$ but does not appear in the polynomial expansion of $m_\ell^k$ for $\ell \neq j$, and the monomial $W_2^{k-1}$ has coefficient $\iota$ in the polynomial expansion of $m_j^k$ but does not appear in that of $m_\ell^k$ for $j \geq 2$. Let $\lambda'_1, \ldots, \lambda'_k$ denote the columns of the matrix $\lambda$ of 2.2 with its last row deleted. For any $\delta \in \Theta_T$, we have

$$
\delta(m_\ell^k) = \sum_{r=1}^{k-1} \det(\lambda'_1, \ldots, \delta(\lambda'_r), \ldots, \lambda'_{k-1}).
$$

For $\delta = \xi_j^2$, the only distinguished monomial to appear in any of the summands in (2.10) is $V_j W_2^{k-2}$, which appears for in the summand with $r = 1$, with coefficient $(k-j)(-1)^{k-1} \iota$. Thus

$$
(2.11) \quad \xi_j^2(m_\ell^k) = (1-j)(k-j)m_{k-j+1}^k \mod m_T \mathcal{F}_1, \quad 1 \leq j \leq k-1.
$$

Similarly we find

$$
\xi_j^2(\lambda'_r) = \begin{cases} 
\lambda'_{k-j+r} & \text{if } r \leq j, \\
0 & \text{else},
\end{cases}
$$

and, using (2.10) with $\delta = \xi_j^2$, it follows that

$$
(2.12) \quad \xi_j^2(m_\ell^k) = (1-j)(k-j)m_{k-j+1}^k \mod m_T \mathcal{F}_1, \quad 1 \leq i \leq k, \ 1 \leq j \leq k-1.
$$

Note that (2.11) and (2.12) with $i = k$ each imply.
Proposition 2.5. The \( \mathcal{O}_T \)-linear map \( \text{Der}(\mathcal{O}_T) \rightarrow \mathcal{F}_1 \) sending \( \xi \in \text{Der}(-\log D) \) to \( \xi \cdot m_k^k \) is surjective. \( \square \)

Combining (2.11) and (2.12) with \( i = k \), we construct vector fields

\[
\eta_j = (j - 1)\xi_1^1 - \xi_{k-j+1}^2, \quad 2 \leq j \leq k - 1
\]

with linear part

\[
\bar{\eta}_j \equiv (1 - j)W_2 \partial V_j + \sum_{i<j} (2j - i - 1)V_{k+i-j} \partial V_i \pmod{\langle \partial U, \partial W_1 \rangle},
\]

which lie in \( \text{Der}(-\log(D + A)) \) to first order, since \( \eta_j(m_k^k) \in m_T \mathcal{F}_1 \).

Both \( \bar{\chi} \) and \( \bar{\sigma} \) are semi-simple, and so by consideration of the distinguished monomials, \( \chi \) and \( \sigma \) must therefore lie in \( \text{Der}(-\log A) \) to first order. The vector fields \( \xi_j^3 \) lie in \( \text{Der}(-\log A) \) to first order, since it is clear by consideration of the distinguished monomials that \( \xi_j^3(m_k^k) \in m_T \mathcal{F}_1 \).

Thus we have \( 2k - 1 \) vector fields \( \eta_2, \ldots, \eta_{k-1}, \chi, \sigma, \xi_1^3, \ldots, \xi_{k-1}^3 \) in \( \text{Der}(-\log A) \) to first order, which are also in \( \text{Der}(-\log(D + A)) \). The determinant of the Saito matrix of the modified vector fields \( \bar{\eta}_2, \ldots, \bar{\eta}_{k-1}, \bar{\chi}, \bar{\sigma}, \xi_1^3, \ldots, \xi_{k-1}^3 \) must be a multiple \( \alpha h m_k^k \) of the equation of \( D + A \). We now show that \( \alpha \) is a unit, from which it follows, by Saito’s criterion, that \( D + A \) is a free divisor.

The modification of the vector fields does not affect the lowest order terms in the determinant of their Saito matrix, and these are the same as the lowest order terms in the determinant of the Saito matrix of their linear parts. With the rows representing the coefficients of \( \partial U, \ldots, \partial U_{k-2}, \partial W, \partial V, \ldots, \partial V_{k-1}, \partial W \) in this order, this matrix is of the form

\[
\begin{pmatrix}
* & B_1 \\
B_2 & 0
\end{pmatrix},
\]

with

\[
B_1 = 
\begin{pmatrix}
V_2 & V_3 & V_4 & \cdots & V_{k-1} & -W_2 \\
V_3 & V_4 & \cdots & -W_2 & 0 \\
V_4 & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots \\
V_{k-1} & -W_2 & \cdots & \cdots \\
W_2 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]
and

\[
\begin{pmatrix}
0 & 0 & 0 & \cdots & \cdots & 0 & 0 & W_2 \\
2V_{k-1} & 4V_{k-2} & 6V_{k-3} & \cdots & \cdots & (2k-4)V_2 & (k-1)V_1 & V_1 \\
-W_2 & 3V_{k-1} & 5V_{k-2} & \cdots & \cdots & (2k-5)V_3 & (k-2)V_2 & V_2 \\
0 & -2W_2 & 4V_{k-1} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & 0 & -3W_2 & \cdots & \cdots & kV_{k-2} & 3V_{k-3} & V_{k-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & (k-2)W_2 & V_{k-1} & V_{k-1}
\end{pmatrix}
\]

In its determinant we find the monomial $W_2^{2k-2}V_1$ with coefficient $\pm (k-1)!$. By Corollary [2.3] this monomial is present in the equation of $D + A$. This proves that $\alpha$ is a unit and completes the proof that $D + V(m_k^k)$ is a free divisor.

The following consequence of [2.5] is needed to prove that Theorem 1.2 holds for any adjoint divisor of $D$, and not just for the adjoint defined by $m_k^k$.

**Corollary 2.6.** The adjoint divisor $A$ is unique up to isomorphism preserving $D$.

**Proof.** Let $A_0 = V(m_k^k)$. Any adjoint divisor $A_1$ must have equation $m := m_k^k + \sum_{i=1}^{k-1} c_i m_j^k$. Consider the family of divisors

\[
A_s = V \left( m_k^k + s \sum_{j=1}^{k-1} c_j m_j^k \right), \quad s \in \mathbb{C}
\]

and pick a coordinate $S$ in $\mathbb{C}$. Let

\[
M = M(U, V, W, S) = m_k^k + S \sum_{j=1}^{k-1} c_j m_j^k
\]

denote the equation of the total space $\bigcup_s A_s$ in $T \times \mathbb{C}$. We claim that for each $s \in \mathbb{C}$, there exists a germ of vector field $\Xi \in \text{Der}_{T \times \mathbb{C}/\mathbb{C}}(-\log(D \times \mathbb{C}))_{(0, s)}$ such that

\[
\Xi(M) = \partial_S(M).
\]

Then the vector field $\partial_S - \Xi$ is tangent to $D \times \mathbb{C}$, and its integral flow trivializes the family [2.13] in a neighborhood of $(0, s)$ while preserving $D$. A finite number of these neighborhoods cover the interval $\{0\} \times [0, 1] \subset T \times \mathbb{R} \subset T \times \mathbb{C}$, and it follows that $A_0$ and $A_1$ are isomorphic by an isomorphism preserving $D$.

To show that [2.14] has a solution, recall that by Proposition [2.5]

\[
dm_k^k : \text{Der}(-\log D) \to \mathcal{F}_1 \text{ is surjective.}
\]

For $j < k$, $\deg(m_k^k) < \deg(m_{k-1}^k) \leq \cdots \leq \deg(m_1^k)$ (the inequalities here are in fact strict, but we want to use the argument again later in a context where the strict inequalities do not hold). From this it follows that for any adjoint divisor with equation $m = m_k^k + \sum_{i<k} c_j m_j^k$,

\[
dm(D(-\log D)) \to \mathcal{F}_1 \text{ is surjective.}
\]
To make this clear, choose germs of weighted homogeneous vector fields $\delta_1, \ldots, \delta_k \in \text{Der}(-\log D)_0$ such that $dm^k_j(\delta_j) = m^k_j$ for $j = 1, \ldots, k$. Pick $\alpha^\ell_{i,j} \in \mathcal{O}_T$ such that

$$dm^k_j(\delta_i) = \sum_{\ell=1}^k \alpha^\ell_{i,j} m^k_j, \quad 1 \leq i, j \leq k,$$

and set $L_j := (\alpha^\ell_{i,j})_{1 \leq \ell, i, j \leq k}$. The constant parts $\alpha^\ell_{i,j}(0)$ are uniquely defined. By choice of the $\delta_i$, $L_k$ is the identity matrix. If $j < k$ we have

$$\text{deg}(dm^k_j(\delta_i)) > \text{deg}(dm^k_j(\delta_i)) = \text{deg}(m^k_j) \geq \text{deg}(m^k_{j+1}) \geq \cdots \geq \text{deg}(m^k_k),$$

so $dm^k_j(\delta_i) \in \langle m^k_1, \ldots, m^k_{i-1} \rangle + \mathfrak{m}_T \mathcal{F}_1$. The matrices $L_j(0)$ are therefore strictly upper triangular, for $j < k$, and so the matrix $L_k + \sum_{j<k} c_j(0)L_j(0)$ of $dm$ is invertible. This shows that $dm(\text{Der}(-\log D) + \mathfrak{m}_T \mathcal{F}_1 = \mathcal{F}_1$, and (2.16) follows by Nakayama’s Lemma.

The remainder of the argument to deduce the existence of the vector field $\Xi$ solving (2.14) is standard: By applying (2.16) to the restriction of $M$ to $S = s$, we find that

$$dM(\text{Der}_T \otimes _{\mathbb{C}}(\mathcal{O}_X, \mathbb{C}, (0,s)) + \mathfrak{m}_T \mathcal{F}_1 \otimes \mathcal{O}_X, \mathbb{C}, (0,s)) \supset \mathcal{O}_T \otimes _{\mathbb{C}}(\mathcal{O}_X, \mathbb{C}, (0,s)) \mathcal{F}_1,$$

and Nakayama’s Lemma yields

$$dM(\text{Der}_T \otimes _{\mathbb{C}}(\mathcal{O}_X, \mathbb{C}, (0,s)) \supset \mathcal{O}_T \otimes _{\mathbb{C}}(\mathcal{O}_X, \mathbb{C}, (0,s)) \mathcal{F}_1.$$

Then any preimage $\Xi$ of $\partial_S(M)$ in $\mathcal{F}_1$ under $dM$ solves (2.14).

The proof of Theorem 1.2 is now complete. \(\square\)

3. Discriminants of singularities

Let $f : X := (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) := T$ have a quasihomogeneous isolated critical point. We shall use the same notation for a good representative of $f$. Let $\chi_0$ be an Euler vector field for $f$, that is, $\chi(f) = d \cdot f$. Denote by $J_f = \langle \partial_{x_1}(f), \ldots, \partial_{x_n}(f) \rangle$ the Jacobian ideal of $f$. Pick a weighted homogeneous basis $g_1, \ldots, g_\mu = 1$ of the Jacobian algebra

$$M_f := \mathcal{O}_X / J_f$$

of decreasing degrees $d_i = \text{deg}(g_i)$. The highest degree $d_1$ is necessarily that of the Hessian determinant of $f$, which generates the socle of $M_f$. Then

$$F(x, u) = f(x) + g_1(x) u_1 + \cdots + g_\mu(x) u_\mu$$

defines a versal unfolding

$$F \times \pi_S : Y = X \times S \to T \times S$$

with base space $S = (\mathbb{C}^\mu, 0)$, where

$$\pi = \pi_S : Y = X \times S \to S$$

is the natural projection. Again, we shall use the same notation for a good representative of $F$. We denote by $H$ the Hessian determinant of $F$. Setting $\text{wt}(u_i) = w_i = d - d_i$ makes $F$ weighted homogeneous of degree $d = \text{deg}(f)$. We denote by $\chi$ the Euler vector field $\chi_0 + \delta_1$ where $\delta_1 = \sum_{i=1}^\mu w_i u_i \partial_{u_i}$.

Let $\Sigma \subset Y$ be the relative critical locus of $F$, defined by the relative Jacobian ideal $J_F^{\Sigma} = \langle \partial_{x_i}(F), \ldots, \partial_{x_n}(F) \rangle$, and set $\Sigma^0 = \Sigma \cap V(F)$. Then $\mathcal{O}_\Sigma$ is a finite free $\mathcal{O}_S$-module with basis $g = g_1, \ldots, g_\mu$. As $\Sigma$ is smooth and hence Gorenstein,
the Gorenstein pairing. Because of the form of $F_\partial V$ on $S$ and as $\tilde{V}$ field tangent to $\text{Con}$ Conversely, if $dF$ we denote the induced Gorenstein pairing on $\mathcal{O}_\Sigma/\mathcal{O}_\Sigma=\mathcal{O}_X/J_f=M_f$.

Let $\hat{g} = \hat{g}_1, \ldots, \hat{g}_\mu$ denote the dual basis of $g$ with respect to the Gorenstein pairing, and denote by $\hat{d}_i$ the degree of $\hat{g}_i$. We have $d_i + \hat{d}_i = 1$, so $\hat{d}_i = d_{\mu+i-1}$ (recall that we have ordered the $g_i$ by descending degree).

The discriminant $D = \pi_S(\Sigma^\circ) \subset S$ was shown by Kyoji Saito in [Sai00] to be a free divisor. The following argument proves this, and shows also that it is possible to choose a basis for $\text{Der}(-\log D)$ whose Saito matrix (matrix of coefficients) is symmetric.

**Theorem 3.1.** There is a free resolution of $\mathcal{O}_\Sigma$ as $\mathcal{O}_S$-module

\[ 0 \longrightarrow \mathcal{O}_\Sigma^\mu \xrightarrow{\Lambda} \Theta_S \xrightarrow{dF} \mathcal{O}_\Sigma \longrightarrow 0 \]

in which $\Lambda$ is symmetric, and is the Saito matrix of a basis of $\text{Der}(-\log D)$.

**Proof.** There is a commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{O}_\Sigma \\
\leftarrow g & \cong & \Theta_S \\
\leftarrow \mathcal{O}_\Sigma^\mu & \cong & dF \\
\leftarrow 0 & \cong & \text{Der}(-\log D)
\end{array}
\]

where $\Lambda = (\Lambda^i_j)_{1 \leq i, j \leq \mu}$ is the matrix of the $\mathcal{O}_S$-linear map multiplication by $F$ with respect to bases $\hat{g}$ in the source and $g$ in the target. As explained before the Lemma 2.2 symmetry of $\Lambda$ follows from self-adjointness of $\cdot F$ with respect to the Gorenstein pairing. Because of the form of $F$, the map $\Theta_S \rightarrow \mathcal{O}_\Sigma$ sending $\partial_{u_j}$ to $g_j$ coincides with evaluation of $dF$ on the trivial lift to $Y$ of vector fields on $S$. The kernel of this map consists of vector fields on $S$ which lift to vector fields on $V(F)$: for if $\eta \in \Theta_S$ lifts to $\tilde{\eta}$ in $\Theta_{V(F)}$ then $dF(\eta) = dF(\tilde{\eta})$ modulo $J_F^{\mu+1}$, and as $\tilde{\eta}$ is tangent to $V(F)$, $dF(\tilde{\eta})$ vanishes on $V(F)$ and in particular on $\Sigma^\circ$. Conversely, if $dF(\eta) = 0 \in \mathcal{O}_\Sigma$, then there is a vector field $\xi = \sum_{i=1}^n \xi_i \partial_x$, such that $dF(\eta) = dF(\xi) \mod (F)$, which means that $\tilde{\eta} = \eta - \xi$ is a lift of $\eta$ to a vector field tangent to $V(F)$. It is well known (see e.g. [Loo84], Lem. 6.14) that the set of liftable vector fields is equal to $\text{Der}(-\log D)$. \hfill $\square$

Denote by $\Lambda^i_j$ the submatrix of $\Lambda$ obtained by deleting the $i$th row and the $j$th column, and by $m^i_j := \det(\Lambda^i_j)$ the corresponding minor. By [MP89] Thm. 3.4, the 1st Fitting ideal of $\mathcal{O}_\Sigma$ equals

\[(3.1) \quad \mathcal{F}_1 := \mathcal{F}_1^{\mathcal{O}_S}(\mathcal{O}_\Sigma) = \langle m^i_j \mid j = 1, \ldots, \mu \rangle_{\mathcal{O}_S}, \]
and \( m_\mu \) is an \( \mathcal{O}_{S^0} \)-generator of the conductor ideal \( \text{Ann}_{\mathcal{O}_S}(\mathcal{O}_{S^0} / \mathcal{O}_S) \). As in \( \S 2 \) the adjoints of \( D \) are divisors

\[
A = V(m_\mu + \sum_{j < \mu} c_j m_j^{\mu}).
\]

Although it is not part of the main thrust of our paper, the following result seems to be new, and is easily proved. It assumes that \( D \) is the discriminant of an \( \mathcal{R}_c \)-versal deformation, but does not require any assumption of weighted homogeneity.

**Theorem 3.2.** Let \( A \) be any adjoint divisor for \( D \). Then

\[
D_0 := \pi^{-1}(A) \cap \Sigma^0
\]

is a free divisor in \( \Sigma^0 \) containing \( V(H) \), with reduced defining equation \( (m_\mu \circ \pi) / H \).

**Proof.** By Corollary \( \S 3.8 \) below, we may assume that \( A = V(m_\mu) \), and hence \( D_0 = V(m_\mu \circ \pi) \cap \Sigma_0 \). First it is necessary to show that \( H^2 \) divides \( m_\mu \circ \pi \) and that \( m_\mu \) is reduced. Since \( \Sigma^0 \) is regular, it is enough to check this at generic points of \( V(H) \). This reduces to checking that it holds at an \( A_2 \)-point. The miniversal deformation of an \( A_2 \)-singularity is given by \( G(x, v_1, v_2) = x^3 + v_1 x + v_2 \). In this case, \( m_\mu \) is, up to multiplication by a unit, simply the coefficient of \( x \) \( i \leq \mu \), since \( 1 \leq \mu \leq 2 \) for \( \mu = 2 \). Then (3.2) gives

\[
d\pi \cdot (\delta_1, \ldots, \delta_\mu) = (\delta_1, \ldots, \delta_\mu) \circ \pi.
\]

Here \( \pi: \Sigma^0 \to S \) is identified with a \( \mu \)-tuple by means of the the coordinates \( u_1, \ldots, u_\mu \) on \( S \), and \( d\pi \) is considered as a matrix using some (any) coordinate system on \( \Sigma^0 \). Let \( \delta_1, \ldots, \delta_\mu \) be the lifts to \( \Sigma^0 \) of the symmetric basis \( \delta_1, \ldots, \delta_\mu \) of \( \text{Der}( - \log D ) \) constructed in Theorem \( \S 3.1 \). Then there is an equality of matrices

\[
(3.2)
d\pi^\mu \cdot \delta_j = (\delta_1, \ldots, \delta_\mu) \circ \pi.
\]

If the coordinate functions \( x_1, \ldots, x_n \) are among the \( \mathcal{O}_S \) basis \( g_1, \ldots, g_\mu \) of \( \mathcal{O}_\Sigma \), then

\[
\pi^\mu = \left( - \frac{\partial f}{\partial x_1} - \sum_{1 \neq i < \mu} \frac{\partial g_i}{\partial x_1} u_i, \ldots, - \frac{\partial f}{\partial x_n} - \sum_{n \neq i < \mu} \frac{\partial g_i}{\partial x_n} u_i + u_{n+1}, \ldots, u_{\mu - 1} \right),
\]

and hence \( \det(d\pi^\mu) = \pm H \). If we do not assume this, then \( \det(d\pi^\mu) = H \) up to multiplication by a unit. It then follows from (3.3) that

\[
\det(\delta_1, \ldots, \delta_{\mu - 1}) = (m_\mu \circ \pi) / H,
\]

and so is a reduced defining equation for \( V(m_\mu \circ \pi) \). Now \( \tilde{\delta}_j \in \text{Der}( - \log V(m_\mu \circ \pi) ) \) for all \( j = 1, \ldots, \mu \), since \( m_\mu \circ \pi \) generates the conductor ideal, whose locus of zeros is invariant under any automorphism of the normalization map \( \pi: \Sigma^0 \to D \). The theorem now follows by Saito’s criterion (\Sai80c Thm. 1.8.(ii)).

\[\Box\]
Remark 3.3. Computation with examples appears to show that closure
\[ C_v := D_0 \setminus V(H) \]
is also a free divisor. Moreover its Saito matrix satisfies the RC condition of [MP89, Def. 3.12]: the ideal of its submaximal minors is equal to the ideal of maximal minors of the Saito matrix with its last (highest weight) column removed. The term “RC” refers to Rouché-Capelli and “Ring Condition”, the latter because it is a necessary and sufficient condition for the cokernel of the Saito matrix to have a ring structure in which \( \mathcal{O}_{D_0} \) is a subring ([MP89, Prop. 3.14]).

We now go on to show that the divisor \( D + V(m^\mu) \) is free. Although the proof of Theorem 3.2 is simple, and one might expect that freeness of \( D_0 \) would be related to freeness of \( D + V(m^\mu) \), we have not succeeded in finding a similarly simple proof for the freeness of \( D + V(m^\mu) \). Just as in §2, our proof makes use of the representation of \( \text{Der}(−\log D) \) on \( F_1 \), and relies on the surjectivity of \( dm^\mu: \text{Der}(−\log D) \to F_1 \).

Proposition 3.4. Assume that \( d − d_1 + 2d_i \neq 0 \) for \( i = 2, \ldots, \mu \). Then
\[ dm^\mu(\text{Der}(−\log D)) = F_1. \]

Inclusion of the left hand side in the right is a consequence of the \( \text{Der}(−\log D) \)-invariance of \( F_1 \). To show equality, it is enough to show that it holds modulo \( m_S F_1 \). This will cover most of the remainder of this section.

Denote by \( \bar{\Lambda} = (\bar{\lambda}_{ij}) \) the linear part of \( \Lambda \), and let \( \bar{\delta}_i = \sum_j \bar{\lambda}_{ij} u_i \partial_u \) be the linear part of \( \delta_i \).

Theorem 3.5. The entries of \( \Lambda \) are given by \( \lambda_{i,j} = \sum_k \langle \hat{g}_i \hat{g}_j, g_k \rangle w_k u_k \). In particular, \( \bar{\lambda}_{i,j} = \sum_k \langle \hat{g}_i \hat{g}_j, g_k \rangle w_k u_k \).

Proof. Since \( \chi_0(F) \in J_{F}^{\text{rel}} \), we have
\[ F = \chi(F) \equiv \delta_1(F) = \sum_{k=1}^\mu w_k u_k g_k \mod J_{F}^{\text{rel}}, \]
and hence
\[ \bar{\lambda}_j = \langle \hat{g}_i, F \hat{g}_j \rangle = \langle \hat{g}_i \hat{g}_j, F \rangle = \langle \hat{g}_i \hat{g}_j, \sum_k w_k u_k g_k \rangle = \sum_k \langle \hat{g}_i \hat{g}_j, g_k \rangle w_k u_k. \]
\[ \square \]

We call a homogeneous basis \( g \) of \( M_f \) self-dual if
\[ (3.4) \quad \hat{g}_i = g_{\mu+1-i}. \]

Lemma 3.6. \( M_f \) admits self-dual bases.

Proof. Denote by \( W_j \subset M_f \) the weight space of weight \( d_j \). The highest weight space \( W_1 \) is 1-dimensional generated by the Hessian of \( f \). Therefore two weight spaces \( W_j \) and \( W_k \) are orthogonal unless \( d_j + d_k = d_1 \), in which case \( \langle \cdot, \cdot \rangle_0 \) induces a non-degenerate pairing \( W_j \otimes \mathbb{C} W_k \to \mathbb{C} \). If \( j \neq k \), one can choose the basis of
Let $\bar{W}_j$ be the reverse dual basis of a basis of $W_k$. Otherwise, $\bar{W}_j = W_k$ and (since quadratic forms are diagonalizable) there is a basis of $\bar{W}_j$ for which the matrix of $\langle \cdot, \cdot \rangle_0$ is diagonal. Self-duality on $\bar{W}_j$ is then obtained by joining the bases of $\bar{W}_j$.

**Lemma 3.7.** Suppose $\delta$ is an $O_S$-basis for $\bar{O}_S$ whose restriction to $\bar{M}_j$ is self-dual. Then the following equalities hold true:

(a) $m_S \mathcal{F}_1 = \mathcal{F}_1 \cap m_S^* \bar{O}_S$ and $\mathcal{F}_1$ is minimally generated by $m_\mu^1, \ldots, m_\mu^\mu$. In particular, $\bar{m}_\mu = m_\mu^\mu \mod m_S \mathcal{F}_1$ for $i = 1, \ldots, \mu$.

(b) $\bar{\delta}(m_\mu^\mu) = (-1)^{i+1} (d - d_1 + 2d_i) m_{\mu-1}^\mu$ for $i = 2, \ldots, \mu$.

(c) $\bar{\delta}(m_\mu^\mu) = m_\mu^\mu \mod C^*$.

**Proof.** We introduce new variables $v_i = w_j u_i$ for $i = 1, \ldots, \mu$. Under the self-duality hypothesis, Theorem 3.3 implies that the matrix $\bar{\Lambda}$ has the form

$$
\bar{\Lambda} = \begin{pmatrix}
    v_1 & v_2 & \cdots & v_{\mu-1} & v_\mu \\
    v_2 & * & \cdots & * & v_\mu \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    v_{\mu-1} & * & \cdots & * & v_\mu \\
    v_\mu & 0 & \cdots & \cdots & 0
\end{pmatrix}
$$

where $*$ entries do not involve the variable $v_\mu$. For the first row and column, this is clear. For the remaining entries, we note that, by Theorem 3.3, $v_\mu$ appears in $\bar{\lambda}_j^i$ if and only if $0 \neq \langle g_i \bar{g}_j, g_\mu \rangle_0 = \langle \bar{g}_i, g_j \rangle_0$. By the self-duality assumption, this is equivalent to $i + j = \mu + 1$, in which case $\langle \bar{g}_i g_j, g_\mu \rangle = 1$ and $\bar{\lambda}_\mu^{\mu-1+1} = v_\mu$.

As in the proof of Theorem 1.2, it is convenient to use $i$ to denote the sign of the order-reversing permutation $\mu - 1, \mu - 2, \ldots, 2, 1$. From (3.5) it follows that $\bar{m}_{\mu+1-i}^\mu$ involves a distinguished monomial $v_i v_\mu^{\mu-2}$, with coefficient $(-1)^{\mu-i}$, that does not appear in any other minor $\bar{m}_{\mu+1-j}^\mu$ for $i \neq j$. This implies (3.4).

In order to prove (3.5), assume, for simplicity of notation, that $\Lambda$ and $\delta$ are linear, and fix $i \in \{2, \ldots, \mu\}$. We know that $\delta_i(m_\mu^\mu)$ is a linear combination of $m_1^i, \ldots, m_\mu^\mu$. We will show that

$$
\delta_i(m_\mu^\mu) = (-1)^{i}(w_1 - 2w_{\mu-i+1})m_{\mu-i+1}^\mu
$$

where $i = \sqrt{-1}$, for $\dim_{\mathbb{C}}(\bar{W}_j)$ even or odd, respectively. A self-dual basis of $\bar{M}_j$ is then obtained by joining the bases of the $\bar{W}_j$ constructed above.

Let $\bar{m}_j^\mu$ be the $(\mu-1)$-jet of $m_j^\mu$, that is, the corresponding minor of $\bar{\Lambda}$.
by showing that the coefficient \( c_{i,j} \) of the distinguished monomial \( v_j v_\mu^{\mu-2} \) in \( \delta_i(m_\mu^j) \) satisfies
\[
(3.7) \quad c_{i,j} = (-1)^j (w_1 - 2w_{\mu-i+1})\delta_{i,j}.
\]

The self-duality assumption \((3.4)\) implies that \( d_1 - d_\ell = d_\ell - d_{\mu-\ell+1} \). Using \( w_\ell = d - d_\ell \), this gives
\[
w_1 - 2w_{\mu-i+1} = d - d_1 + 2d_{\mu-i+1} - 2d = d - d_1 + 2d_1 - 2d - d = d - d_1 + 2d_1.
\]

So \((3.6)\) will follow from \((3.6)\).

By linearity of \( \delta_i \), the only monomials in the expansion of \( m_\mu^j \) that could conceivably contribute to a non-zero \( c_{i,j} \) are of the following three forms:
\[
(3.8) \quad v_\mu^{\mu-1}, \quad v_j v_\mu^{\mu-2}, \quad v_j v_k v_\mu^{\mu-3}.
\]

The first monomial does not figure in the expansion of \( m_\mu^j \). Monomials of the other two types do appear. The second type of monomial \((3.8)\) must satisfy \( j = 1 \) and arises as the product
\[
(3.9) \quad (1)^{\mu-2} v_1 v_\mu^{\mu-2} = (1)^{\mu-2} \lambda_1^1 \lambda_{\mu-1}^2 \lambda_{\mu-2}^3 \cdots \lambda_2^{\mu-1}.
\]

Monomials of the third type \((3.8)\) must satisfy \( k = \mu - j + 1 \). Each such monomial arises in two ways:
\[
(3.10) \quad (1)^{\mu-3} v_j v_k v_\mu^{\mu-3} = (1)^{\mu-3} \lambda_1^1 \lambda_{\mu-1}^2 \cdots \lambda_{\mu-j}^{j-1} \lambda_{\mu-j+1}^{j-1} \cdots \lambda_2^{\mu-1},
\]
\[
(3.11) \quad = (1)^{\mu-3} \lambda_1^{j-1} \lambda_{\mu-j+1}^{j-1} \cdots \lambda_{\mu-j+2}^1 \lambda_{\mu-j}^1 \cdots \lambda_2^{\mu-1}.
\]

If \( j = \mu - j + 1 = k \) then the expressions on the right hand sides of \((3.10)\) and \((3.11)\) coincide and the monomial appears only once, otherwise it appears twice.

In terms of the coordinates \( v_1, \ldots, v_\mu, \delta_i \) contains monomials
\[
(3.12) \quad w_j u_j \delta_{\mu_i} = v_j v_\mu \delta_{\mu_i},
\]
\[
(3.13) \quad w_\mu u_\mu \delta_{\mu-j+1} = v_\mu v_{\mu-j+1} \delta_{\mu-j+1}.
\]

Now \((3.12)\) applied to \((3.9)\) contributes \( w_1 (1)^{\mu-2} \) to \( c_{i,j} \), \((3.13)\) applied to one copy of \((3.10)\) for \( j = k \), or to two copies otherwise, contributes \( 2(1)^{\mu-3} w_\mu v_{\mu-j+1} \) in both cases. There are no contributions to the coefficient of any other distinguished monomial.

We have proved \((3.6)\), from which \((3.6)\) follows; \((3.6)\) is clear, since \( \delta_1 \) is the Euler vector field. \( \Box \)

Proposition \((3.4)\) is an immediate consequence of \((3.1)\) and Lemma \((3.7)\). The next result, closely analogous to Corollary \((2.6)\), follows from \((3.4)\) by the same argument by which \((2.6)\) is deduced from Proposition \((2.6)\).

**Corollary 3.8.** Assume the hypotheses of Proposition \((3.4)\). Then any two adjoint divisors of \( D \) are isomorphic by an isomorphism preserving \( D \). \( \Box \)

Consider the matrix
\[
(3.14) \quad C := (c_{i,j})_{1 \leq i, j \leq \mu}
\]

using a choice of coefficients \( c_{i,j} \in \mathcal{O}_S \) for which \( \delta_i(m_\mu^j) = \sum_{j=1}^\mu c_{i,j} m_\mu^j \). We assume that \( c_{1,\mu} = \text{wt}(m_\mu^\mu) \) and \( c_{1,j} = 0 \) for \( j < \mu \). This, together with Proposition \((3.4)\) and Lemma \((3.7)\) \((x)\), implies
Corollary 3.9. Assume the hypothesis of Proposition 3.4. Then the matrix \( C \) in (3.14) is invertible, and its first column consists of zeros except for its last entry, which is a non-zero scalar.

Proposition 3.10. If \( dm^\mu_j(\text{Der}(-\log D)) = \mathcal{F}_1 \), then \( D + V(m^\mu_j) \) is a free divisor.

Proof. Recall that \( \Lambda \) denotes the Saito matrix of \( D \), with the coefficients of the Euler field as first column. Let \( \Lambda_\mu \) denote \( \Lambda \) minus its last column. Each column of \( \Lambda_\mu \) gives a relation among the \( m^\mu_j \), by Cramer’s rule, and therefore among the \( m^\mu_j \), by the symmetry of \( \Lambda \). As linear combinations of \( \delta_1, \ldots, \delta_\mu \), the columns of the matrix \( AC^{-1} \) determine vector fields \( \eta_1, \ldots, \eta_\mu \) in \( \text{Der}(-\log D) \). By construction \( dm^\mu_j(\eta_j) = m^\mu_j \) for \( j = 1, \ldots, \mu \). Therefore the columns of the matrix \( AC^{-1}\Lambda_\mu \), determine vector fields in \( \text{Der}(\log D) \) which annihilate \( m^\mu_j \). Denote them by \( \xi_2, \ldots, \xi_\mu \). We claim that these, together with the Euler field \( \delta_1 \), form a basis for \( \text{Der}(-\log(D + V(m^\mu_j))) \).

By Saito’s criterion ([Sai80b, Thm. 1.8.(iii)]), this will be proved if we can check that the determinant of their matrix of coefficients is a reduced equation for \( D + V(m^\mu_j) \).

To see this, we argue as follows. By Lemma 3.7(c), the first column of \( C \) consists of zeros followed by a non-zero scalar in the last place. Therefore the last column of \( C^{-1} \) consists of zeros except for a unit in its first place. It follows that, as a column of coefficients, and up to multiplication by a unit in \( \mathcal{O}_S \), \( \delta_1 = AC^{-1}e_\mu \) where \( e_\mu \) is the unit vector \((0, \ldots, 0, 1)^t \). Denote by \( (e_\mu|\Lambda_\mu) \) the \( \mu \times \mu \)-matrix obtained by appending \( e_\mu \) to \( \Lambda_\mu \) as new first column. Then the matrix of coefficients of the vector fields \( \delta_1, \xi_2, \ldots, \xi_\mu \) is equal to \( AC^{-1}(e_1|\Lambda_\mu) \). Its determinant is equal to \( \det(\Lambda)(m^\mu) \), up to multiplication by a unit in \( \mathcal{O}_S \).

It remains only to show that \( m^\mu_j \) is reduced. For this, we consider generic points on the singular set of \( D \), where \( D \) is either a normal crossing of two branches or isomorphic to the product of the discriminant of an \( A_2 \) singularity and a smooth factor.

Consider first an \( A_2 \)-point. As we saw in the proof of Theorem 3.2, \( m^\mu_\mu \circ \pi \) generates the conductor, and is equal to a unit times \( H^2 \). If \( h \) is any function on \( S \) vanishing on the \( A_2 \)-stratum, a local computation in normal form shows that \( h \circ \pi \) is divisible by \( H^2 \). So if at such a point \( m^\mu_\mu \) had more than one factor vanishing on the \( A_2 \)-stratum, \( m^\mu_\mu \circ \pi \) would be divisible by \( H^2 \).

Consider next a normal crossing of two smooth branches with equations \( h_1 \) and \( h_2 \). We know \( m^\mu_\mu \circ \pi \) generates the conductor in each branch, and vanishes on the intersection. So \( m^\mu_\mu \in \langle h_1, h_2 \rangle \). On \( V(h_1) \), the conductor is generated by the restriction of \( h_2 \), and vice versa. Because the crossing is transverse, each of these restrictions is reduced. The restriction of \( m^\mu_\mu \) to \( V(h_1) \) is therefore a unit times the restriction of \( h_2 \), and vice versa. This implies that at such a point, \( m^\mu_\mu = u_1h_1 + u_2h_2 \), where \( u_1 \) and \( u_2 \) are units. So \( m^\mu_\mu \) is reduced, which finishes the proof. \( \square \)

Theorem 1.3 is an immediate consequence of Propositions 3.4 and 3.10 and Corollary 3.8.

Remark 3.11. Theorem 1.3 evidently applies to the simple singularities, since for these \( d_1 < d \). It also applies in many other cases. For example, it is easily checked...
that the hypotheses on the weights hold for plane curve singularities of the form $x^p + y^q$ with $p$ and $q$ coprime.

We conclude this section by a description of the relation with the bifurcation set. For $u \in S$, we set $X_u := \pi_S^{-1}(u)$ and define $f_u : X_u \to T$ by $f_u(x) := F(x, u)$. We consider $S' := (\mathbb{C}^p, 0) \subset S$ with coordinates $u' = u_1, \ldots, u_{\mu-1}$, and we denote by

$$
(3.15) \quad \rho : S \to S' = (\mathbb{C}^{\mu-1}, 0), \quad u \mapsto u',
$$

the natural projection. Recall that the bifurcation set is the set $B \subset S'$ of parameter values $u'$ such that $f_{u'} := f_{(u', 0)}$ has fewer than $\mu$ distinct critical values. The coefficient $u_\mu$ of $g_\mu = 1$ is set to 0 since it has no bearing on the number of critical values. The bifurcation set consists of two parts: the level bifurcation set $B_v$ consisting of parameter values $u'$ for which $f_{u'}$ has distinct critical points with the same critical value, and the local bifurcation set $B_t$ where $f_{u'}$ has a degenerate critical point. H. Terao, in [Ter83], and J.W. Bruce in [Bru85] proved that $B$ is a free divisor and gave algorithms for constructing a basis for $\text{Der}(-\log B)$. The free divisor $B$ is of course singular in codimension 1. The topological double points (points at which $B$ is reducible) are of four generic types:

- Type 1: $f_{u'}$ has two distinct degenerate critical points, $x_1$ and $x_2$.
- Type 2: $f_{u'}$ has two distinct pairs of critical, $x_1, x_2$ and $x_3, x_4$, such that $f_{u'}(x_1) = f_{u'}(x_2)$ and $f_{u'}(x_3) = f_{u'}(x_4)$.
- Type 3: $f_{u'}$ has a pair of critical points $x_1$ and $x_2$ with the same critical value, and also a degenerate critical point $x_3$.
- Type 4: $f_{u'}$ has three critical points $x_1, x_2$ and $x_3$ with the same critical value.

In the neighborhood of a double point of type 1, 2 or 3, $B$ is a normal crossing of two smooth sheets. In the neighborhood of a double point of type 4, $B$ is a union of three smooth sheets which meet along a common codimension 2 stratum.

**Proposition 3.12.** For any adjoint divisor $A$ for $D$, $(3.15)$ induces a surjection

$$
(3.16) \quad \rho : D \cap A \twoheadrightarrow B.
$$

**Proof.** We have $u \in D \cap A$ if the sum of the lengths of the Jacobian algebras of $f_u$ at points $x \in f_u^{-1}(0)$ is greater than 1. The sum may be greater than 1 because for some $x$ the dimension of the Jacobian algebra is greater than 1 − in which case $f_u$ has a degenerate critical point at $x$ – or because $f_u$ has two or more critical points with critical value 0. In either case, it is clear that $\rho(u) \in B$. If $u' \in B$, then $f_{u'}$ has either a degenerate critical point or a repeated critical value (or both). In both cases let $v$ be the corresponding critical value. Then $(u', -v) \in D \cap A$ which proves the claimed surjectivity.

**Remark 3.13.** The projection $(3.16)$ is a partial normalization, in the sense that topological double points of $u' \in B$ of types 1, 2 and 3 are separated. Indeed, in each such case there are two critical points of $f_{u'}$ has two critical points with different critical values, and hence with different preimages under $\rho$. However, a general point $u'$ of type 4 has only one preimage, $(u', -f_{(u', 0)}(x_i))$, in $D \cap A$. Generically, at such a point $D$ is a normal crossing of three smooth divisors, and $D \cap A$ is the union of their pairwise intersections.

Finally, our free divisors $D + A$ and $D_0$ of Theorems 1.3 and 2.2 and the conjecturally free divisor $C_v$ of Remark 3.3 fit into the following commutative diagram,
in which $A$ is any adjoint divisor for $D$, free divisors are doubly underlined, and conjecturally free divisors are simply underlined:

![Diagram](image)

4. LINEAR FREE DIVISORS

Linear free divisors and free hyperplane arrangements are at opposite corners in the field of free divisors: in the former case there is a basis of the module of logarithmic vector fields in which all coefficients are linear, while in the latter, the irreducible factors of the equation of the divisor are linear. The intersection of the two classes consists only of normal crossing divisors. In this section, we describe a construction of new free divisors from pairs consisting of a linear free divisor (with an ordering of its irreducible components), and a free hyperplane arrangement.

We recall from [GMNRS09] that a free divisor $D$ in the vector space $V$ is linear if $\text{Der}(-\log D) = \mathcal{O}_V \otimes \mathbb{C} \text{Der}(-\log D)_0$ where the index 0 denotes the degree 0 part with respect to the usual grading on $\Theta_V$. Since the Lie bracket is additive on weights, $\text{Der}(-\log D)_0$ is a complex Lie algebra of vector fields. If $D \subset V \cong \mathbb{C}^n$ is a linear free divisor then $\text{Der}(-\log D)_0$ must have dimension $n$. This algebra is naturally identified with the Lie algebra of the algebraic subgroup $\iota: G_D \hookrightarrow \text{GL}(V)$ consisting of the identity component of the set of automorphisms preserving $D$. It follows that $(V, G_D, \iota)$ is a prehomogeneous vector space (see [SK77]) with discriminant $D$.

Let $D \subset V = \mathbb{C}^n$ be a linear free divisor and $D = \bigcup_{i=1}^k D_i$ a decomposition into (not necessarily irreducible) components. The corresponding defining equations $f_1, \ldots, f_k$ are polynomial relative invariants of $(V, G_D, \iota)$ with associated characters $\chi_1, \ldots, \chi_k$; that is, for $g \in G_D$ and $x \in V$, $f_j(gx) = \chi_j(g)x$. Let $\mathfrak{g}_D$ denote the Lie algebra of $G_D$. By differentiating the character map $\chi = (\chi_1, \ldots, \chi_k): G \to T = (\mathbb{C}^*)^k$ we obtain an epimorphism of Lie algebras $d\chi: \mathfrak{g}_D \to \mathbb{C}^k$. This yields a decomposition

$$\mathfrak{g}_D = \ker d\chi \oplus \bigoplus_{i=1}^k \mathbb{C} \varepsilon_i, \quad d\chi(\varepsilon_i) = \delta_{i,j}.$$  

For $\delta \in \mathfrak{g}_D$, the equality $f_i(gx) = \chi_i(g) \cdot f_i(x)$ differentiates to $\delta(f_i) = d\chi(\delta) \cdot f_i$, and hence

$$\varepsilon_i(f_j) = \delta_{i,j} \cdot f_i, \quad \text{and} \quad [\varepsilon_i, \varepsilon_j] \subseteq \ker d\chi.$$  

This observation is the starting point for the following more general result.

**Theorem 4.1.** Suppose that $D = \bigcup_{i=1}^k D_i \subset (\mathbb{C}^n, 0) := X$ is a germ of a free divisor without smooth factor and for $i = 1, \ldots, k$ let $f_i \in \mathcal{O}_X$ be a reduced equation for $D_i$. Suppose that for $j = 1, \ldots, k$, there exist vector fields $\varepsilon_j \in \Theta_X$ such that $d\varepsilon_i(\varepsilon_j) = \delta_{i,j} \cdot f_i$. Let $N := V(y_1 \cdots y_k) \subset (\mathbb{C}^k, 0) := Y$ be the normal crossing
divisor, let $E \subset Y$ be a divisor such that $N + E$ is free, and consider the map $f = (f_1, \ldots, f_k) : X \to Y$. Then $D + f^{-1}(E) = f^{-1}(N + E)$ is free.

Proof. Since $D$ has no smooth factor, the vector fields $\varepsilon_1, \ldots, \varepsilon_k$ can be incorporated into a basis $\varepsilon_1, \ldots, \varepsilon_n$ for $\text{Der}(-\log D)$ such that $df_i(\varepsilon_j) = \delta_{ij} \cdot f_i$ and hence

$$tf(\varepsilon_j) = \sum_{i=1}^k df_i(\varepsilon_j) \partial_{y_i} = \begin{cases} y_j \partial_{y_j}, & \text{if } j \leq k, \\ 0, & \text{if } j > k. \end{cases}$$

The Saito matrix $S_{N+E}$ of a basis of $\text{Der}(-\log(N + E))$ can be written in the form $S_E = S_N \cdot A$, where $S_N = \text{diag}(y_1, \ldots, y_k)$ is the standard Saito matrix of $N$ and $A = (a_{i,j}) \in O_Y^{k \times k}$. By Saito’s criterion, $h := \det A$ is a reduced equation for $E$, and $g = y_1 \cdots y_nh$ for $N + E$. For $j = 1, \ldots, k$, consider the vector fields $v_j = \sum_{i=1}^k a_{i,j}y_i \partial_{y_i} \in \text{Der}(-\log(N + E))$ and $\tilde{v}_j = \sum_{i=1}^k (a_{i,j} \circ f) \varepsilon_i \in \Theta_X$. By (4.2), we have $tf(\tilde{v}_j) = \omega f(v_j)$ and so

$$d(g \circ f)(\tilde{v}_j) = (dg(v_j)) \circ f$$

shows that $\tilde{v}_j \in \text{Der}(-\log f^{-1}(N + E))$, and also $\varepsilon_j \in \text{Der}(-\log f^{-1}(N + E))$ for $j > k$. The Saito matrix of the vector fields $\tilde{v}_1, \ldots, \tilde{v}_k, \varepsilon_{k+1}, \ldots, \varepsilon_n$ is equal to

$$(\varepsilon_1, \ldots, \varepsilon_n) \cdot \begin{pmatrix} A \circ f & 0 \\ 0 & I_{n-k} \end{pmatrix}$$

and thus its determinant $\det(\varepsilon_1, \ldots, \varepsilon_n) \det(A \circ f)$ defines $f^{-1}(N + E)$. By Saito’s criterion, $f^{-1}(N+E)$ is a free divisor, provided $\det(A \circ f)$ is reduced. This is the case if no component of $f^{-1}(E)$ lies in the critical space $\Sigma_f$ of $f$. In fact $\Sigma_f$ is contained in $D$. To see this, consider the “logarithmic Jacobian matrix” $(\varepsilon_i(f_j))_{1 \leq i \leq n, 1 \leq j \leq k}$ of $f$. The determinant of the first $k$ columns is equal to $f_1 \cdots f_k$, which shows that $f_1 \cdots f_k$ is in the Jacobian ideal of $f$. This shows that $\Sigma_f \subset D$, and thus the reducedness of all components of $f^{-1}(N + E)$.

Now we return to the motivating case of linear free divisors. This is a graded algebraic instead of a local analytic situation. But the arguments of proof of Theorem 4.1 combined with (4.1) prove the following

**Corollary 4.2.** Let $D$ be a linear free divisor in $V = \mathbb{C}^n$, and let $f = (f_1, \ldots, f_k) : V \to W = \mathbb{C}^k$ be a map whose components are polynomial relative invariants defining a partition of the components of $D$. Let $N = V(y_1 \cdots y_k) \subset W$ be the normal crossing divisor, and let $E \subset W$ be a divisor such that $N + E$ is free. Then $D + f^{-1}(E) = f^{-1}(N + D)$ is a free divisor.

**Example 4.3.**

1. Every plane curve is a free divisor. It follows that if $g \in \mathcal{O}_{\mathbb{C}^2,0}$ is any germ not divisible by either of the variables, then for any $n > 1$ and any $k$ with $1 \leq k < n$,

$$x_1 \cdots x_n \cdot g(x_1 \cdots x_k, x_{k+1} \cdots x_n) = 0$$

defines a free divisor.

2. Let $\sigma_i(y)$ be the $i$th symmetric function of $y = y_1, \ldots, y_k$ and set $N = V(\sigma_k(y))$ and $E = V(\sigma_{k-1}(y))$. As seen in Example 2.1(4), the divisor $N + E$ is free. So for any linear free divisor $D = f^{-1}(N) = V(\sigma_k \circ f) \subset V$, also $D + f^{-1}(E) = V((\sigma_k \cdot \sigma_{k-1}) \circ f)$ is a free divisor. If the components of $D$ are regular in codimension one, then each is normal, so the normalization of $D$ is simply the disjoint union of its components. As the singular locus of any free divisor has pure codimension
1, the singular locus of $D$ is equal to its non-normal locus. As a module over $\mathcal{O}_V$, the ring of functions on $\prod_i V(f_i)$ has presentation matrix $\text{diag}(f_1, \ldots, f_k)$. Thus $V(\sigma_{k-1} \circ f)$ is an adjoint divisor of $D$. Consideration of this construction, in the light of the results of [2] and [3] led us to the more general results of this section.

(3) If $A = \{H_1, \ldots, H_k\}$, where $H_i = \ell_i^{-1}(0)$ with $\ell_i \in V^*$, is an essential central free $n$-arrangement of hyperplanes, then for any linear free divisor $D = V(f_1 \cdot \cdots \cdot f_k)$ also $\tilde{D} = V(\ell_1 \circ f \cdot \cdots \cdot \ell_k \circ f)$ is a free divisor.

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