Smooth models of motivic spheres

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Abstract

We study the representability of motivic spheres by smooth varieties. We show that certain explicit “split” quadric hypersurfaces have the $\mathbb{A}^1$-homotopy type of motivic spheres over the integers and that the $\mathbb{A}^1$-homotopy types of other motivic spheres do not contain smooth schemes as representatives. We then study some applications of these representability/non-representability results to the construction of new exotic $\mathbb{A}^1$-contractible smooth schemes.

Contents

1 Introduction.........................................................1
2 Geometric models of motivic spheres.........................3
3 $\mathbb{A}^1$-contractible subvarieties of quadrics..............8

1 Introduction

In this note, we make precise the idea that (split) smooth affine quadric hypersurfaces are “spheres” from the standpoint of the Morel-Voevodsky $\mathbb{A}^1$-homotopy theory [MV99]. Given a commutative Noetherian ring $k$ (in the sequel, $k$ will usually be $\mathbb{Z}$ or a field), the Morel-Voevodsky $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ is constructed by first enlarging the category $\text{Sm}_k$ of schemes smooth over $\text{Spec} \ k$ to a category $\text{Spc}_k$ of spaces over $\text{Spec} \ k$ and then performing a categorical localization. The category $\text{Spc}_k$ has all small limits and colimits, but the functor $\text{Sm}_k \rightarrow \text{Spc}_k$ does not preserve all colimits.

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that exist in $Sm_k$, e.g., quotients in $Sm_k$ that exist need not coincide with quotients computed in $Spc_k$.

The resulting theory bears a number of similarities to the classical homotopy category. There are two objects of $Spc_k$ that play a role analogous to that played by the circle in classical homotopy theory. On the one hand, one can view the circle as $I/\{0, 1\}$ where $I$ is the unit interval $[0, 1]$. If we view $A_1$ as analogous to the unit interval, we can form the quotient space $A_1/\{0, 1\}$ (pointed by the image of $\{0, 1\}$ in the quotient), and we call this space $S^1_\ast$. This quotient exists as a singular scheme (i.e., a nodal curve). However, it is not a priori clear whether $S^1_\ast$ is isomorphic in $H(k)$ to a smooth scheme. On the other hand, if we emphasize the fact that $S^1$ has a group structure, then we could also think of $G_\mathbb{m}$ (pointed by 1) as a version of the circle. Of course, $G_\mathbb{m}$ is a smooth $k$-scheme.

General motivic spheres are obtained by taking smash products (this notion makes sense in $Spc_k$ because of existence of colimits) of copies of $S^1_\ast$ and $G_\mathbb{m}$ [MV99, §3.2]; by convention, we set $S^0_\ast$ to be the disjoint union of two copies of $Spec_k$. While in classical topology spheres are by construction smooth manifolds, the example of $S^1_\ast$ shows that, in stark contrast, the following question in $A_1$-homotopy theory has no obvious answer.

**Question 1.** Which motivic spheres $S^1_\ast \land G_{\mathbb{m}}^j$ have the $A_1$-homotopy type of a smooth scheme?

Throughout this paper, we consider the smooth affine quadric hypersurfaces

$$Q_{2m-1} := Spec\, k[x_1, \ldots, x_m, y_1, \ldots, y_m]/\left(\sum_i x_iy_i - 1\right),$$

$$Q_{2m} := Spec\, k[x_1, \ldots, x_m, y_1, \ldots, y_m, z]/\left(\sum_i x_iy_i - z(1 + z)\right).$$

One immediately verifies that both of these hypersurfaces are in fact smooth over $Spec\, k$. The hypersurface $Q_{2m-1}$ is well-known to be $A_1$-weakly equivalent to $A^m_1 \setminus 0$ (by projection onto $x_1, \ldots, x_m$) and the latter is $A_1$-weakly equivalent to the motivic sphere $S^m_{\ast} \land G^m_\mathbb{m}$ [MV99, Example 2.20]. Our main result provides an explicit description of the $A_1$-homotopy type of the other family of quadrics, thus answering Question 1 in a collection of cases.

**Theorem 2** (See Theorem 2.2.5). Let $k$ be a (Noetherian) commutative unital ring and $n \geq 0$ an integer. There are explicit $A_1$-weak equivalences

$$Q_n \sim_{A_1} \begin{cases} S^m_{\ast} \land G^m_\mathbb{m} & \text{if } n = 2m - 1 \text{ and} \\ S^m_{\ast} \land G^m_\mathbb{m} & \text{if } n = 2m \end{cases}$$

of spaces over $Spec\, k$.

**Remark 3.** Results along the lines of Theorem 2 were known stably over fields having characteristic unequal to 2 by unpublished work of Morel [Mor08], and Dugger and Isaksen [DI08] (the latter uses the Nisnevich topology in an essential way). We emphasize that Theorem 2 is an unstable result, but it also uses the Nisnevich topology in an essential way, via the Morel-Voevodsky homotopy purity theorem.

While Theorem 2 tells us that some motivic spheres do have the $A_1$-homotopy type of smooth schemes, the following result shows that not all motivic spheres have this property.
**Proposition 4** (See Proposition 2.3.1). If $i, j$ are integers with $i > j$, the spheres $S^i_s \wedge G^j_m$ do not have the $\mathbb{A}^1$-homotopy type of smooth (affine) schemes.

Finally, the paper closes with some extensions of the results of [AD07]. These results are closely connected with those mentioned above; see Subsection 2.1 and Remark 3.1.2 for detailed explanations of the connections.

**Theorem 5** (See Theorem 3.1.1 and Corollary 3.2.3). Suppose $k$ is a (Noetherian) commutative unital ring. If $X_{2m}$ is the open subscheme of $Q_{2m}$ defined as the complement of $x_1 = \cdots = x_m = 0, z = -1$, then $X_{2m}$ is $\mathbb{A}^1$-contractible over $\text{Spec } k$. Moreover, for $m \geq 3$, $X_{2m}$ cannot be realized as the quotient of an affine space by the free action of a unipotent group.

The proofs of Theorem 2 and (the first part of) Theorem 5 are written to be accessible to readers with knowledge of the basic aspects of [MV99] and some standard results about pointed model categories [Hov99, Chapter 6]. The proof of the second part of Theorem 5 is more algebro-geometrically involved.

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The results claimed in this paper were announced in talks by first and second named authors as early as 2007. The original proof of Theorem 2 we envisioned, specifically the description of $\mathbb{A}^1$-homotopy types of $Q_n$ for $n$ even $\geq 6$, was an elementary geometric argument that was supposed to work over $\text{Spec } \mathbb{Z}$; this argument contained a gap. The authors would also like to thank Paul Balmer, Dan Isaksen, Christian Haesemeyer, and Fabien Morel, for useful discussions over the course of the project and Matthias Wendt for a number of useful comments on a first draft of this note.

**Preliminaries/Notation**

Throughout the paper, we continue with the notation of the introduction, i.e., $k$ is a (Noetherian) commutative unital ring, $\text{Sm}_k$ is the category of schemes that are separated, finite type and smooth over $\text{Spec } k$, the category $\text{Spc}_k$ of “spaces over $k$” is the category of simplicial Nisnevich sheaves on $\text{Sm}_k$, $\mathcal{H}(k)$ (resp. $\mathcal{H}_a(k)$) is the (pointed) Morel-Voevodsky $\mathbb{A}^1$-homotopy category [MV99]. If $X \in \text{Spc}_k$, we write $X_+$ for the pointed space $X \bigsqcup \text{Spec } k$, pointed by the disjoint copy of $\text{Spec } k$. If $Y \to X$ is a closed immersion of smooth scheme, we write $\nu_{Y/X}$ for the normal bundle to $Y$ in $X$. In that case, $Th(\nu_{Y/X})$ is the Thom space of the vector bundle $\nu_{Y/X}$ as in [MV99, §3 Definition 2.16]. Finally, if $(X, x)$ is a pointed space, we write $\pi_{i,j}^{\mathbb{A}^1}(X, x)$ for the Nisnevich sheaf associated with the presheaf $U \mapsto \text{Hom}_{\mathcal{H}(k)}(S^i_s \wedge G^j_m \wedge U_+, (X, x))$.

## 2 Geometric models of motivic spheres

This section is devoted to establishing Theorem 2 from the introduction. The argument proceeds by induction. Subsection 2.1 gives an elementary geometric argument showing that $Q_2$ is $\mathbb{A}^1$-weakly equivalent to $\mathbb{P}^1$, and a related (mostly elementary) argument showing that $Q_4$ is $\mathbb{A}^1$-weakly equivalent to $(\mathbb{P}^1)^{\wedge 2}$; the first computation is the base-case for the induction argument. Subsection 2.2 contains the proof of the main result; here it is Theorem 2.2.5. Finally, Subsection 2.3 establishes some non-geometrizability results for motivic spheres.
2.1 On the $\mathbb{A}^1$-homotopy types of $Q_2$ and $Q_4$

Proposition 2.1.1. There is a (pointed) $\mathbb{A}^1$-weak equivalence $Q_2 \xrightarrow{\sim} \mathbb{P}^1$.

Proof. We can identify $Q_2$ as the quotient of $SL_2$ by its maximal torus $T := \text{diag}(t, t^{-1}) \cong G_m$ acting by right multiplication; this follows from a straightforward computation of invariants. If $B$ is the linear algebraic group of upper triangular matrices with determinant 1, then the closed immersion group homomorphism $T \hookrightarrow B$ induces a morphism of homogeneous spaces $SL_2/T \to SL_2/B$. This morphism of homogeneous spaces is Zariski locally trivial with fibers isomorphic to $\mathbb{A}^1$ (use the standard open cover of $\mathbb{P}^1$ by two copies of $\mathbb{A}^1$), in particular an $\mathbb{A}^1$-weak equivalence. To conclude, we observe that the quotient $SL_2/B$ is isomorphic to $\mathbb{P}^1$, and we can make this morphism a pointed morphism by picking any $k$-point in $SL_2/T$ and looking at the image of this $k$-point in $\mathbb{P}^1$. \hfill \Box

There are two “natural” (pointed) $\mathbb{A}^1$-weak equivalences of $Q_2 \to \mathbb{P}^1$ that we know. In the previous proof, the $\mathbb{A}^1$-weak equivalence we wrote down arose from the structure of $Q_2$ as a homogeneous space, but as we now explain this seems to be an “exceptional” low-dimensional phenomenon. If $k$ is a ring in which 2 is invertible, the varieties $Q_{2n}$ are all isomorphic to homogeneous spaces for suitable orthogonal groups, but this seems false if 2 is not invertible in $k$.

The other “natural” (pointed) $\mathbb{A}^1$-weak equivalence arises as follows. The locus of points where $x_1 = 0$ and $z = -1$ is a closed subscheme of $Q_2$ isomorphic to $\mathbb{A}^1$; the open complement of this closed subscheme is isomorphic to $\mathbb{A}^2$. The normal bundle to this embedding of $\mathbb{A}^1$ comes equipped with a prescribed trivialization, and we can use homotopy purity together with $\mathbb{A}^1$-contractibility of $\mathbb{A}^2$ to construct another $\mathbb{A}^1$-weak equivalence of $Q_2$ with $\mathbb{P}^1$. We now explain how to identify the $\mathbb{A}^1$-homotopy type of $Q_4$ using the method just sketched.

Proposition 2.1.2. There is a (pointed) $\mathbb{A}^1$-weak equivalence $Q_4 \xrightarrow{\sim} \mathbb{P}^1 \wedge 2$.

Proof. Let $E_2$ be the closed subscheme of $Q_4$ defined by $x_1 = x_2 = 0$ and $z = -1$. Observe that $E_2$ is isomorphic to $\mathbb{A}^2$. The normal bundle to $E_2$ is trivial, and we fix a choice of trivialization. Let $X_4 := Q_4 \setminus E_2$. The homotopy purity theorem gives a cofiber sequence of the form

$$X_4 \rightarrow Q_4 \rightarrow T h(\nu_{E_2/Q_4}) \rightarrow \cdots.$$  

The choice of trivialization determines an isomorphism $T h(\nu_{E_2/Q_4}) \cong \mathbb{P}^1 \wedge 2 \wedge (E_2)_+$. The map $E_2 \to \text{Spec } k$ is an $\mathbb{A}^1$-weak equivalence, it follows that $T h(\nu_{E_2/Q_4}) \cong \mathbb{P}^1 \wedge 2$ in $\mathcal{H}_4(k)$. We showed in [AD08, Corollary 3.1 and Remark 3.3] that $X_4$ is $\mathbb{A}^1$-contractible over $\text{Spec } \mathbb{Z}$ (indeed, it is the base of a Zariski locally trivial morphism with total space $\mathbb{A}^5$ and fibers isomorphic to $\mathbb{A}^1$). Properness of the $\mathbb{A}^1$-local model structure [MV99, §2 Theorem 3.2] guarantees that pushouts of $\mathbb{A}^1$-weak equivalences along cofibrations are $\mathbb{A}^1$-weak equivalences. Applying this fact to the diagram $\ast \leftarrow X_4 \rightarrow Q_4$, one concludes that the map $Q_4 \to T h(\nu_{E_2/Q_4})$ is an $\mathbb{A}^1$-weak equivalence and the proposition follows by combining the stated isomorphisms. \hfill \Box
2.2 On the $\mathbb{A}^1$-homotopy type of $Q_{2n}, n \geq 3$

The “octahedral” axiom in a pretriangulated category

If $C$ is a pointed model category, then recall that one can understand the homotopy cofiber of a composite in terms of the homotopy cofibers of the components by means of the following result, which is sometimes called the “octahedral” axiom, by analogy with a corresponding result for stable model categories [Hov99, Proposition 6.3.6].

**Proposition 2.2.1.** Suppose given a sequence of maps

$$X \xrightarrow{u} Y \xrightarrow{v} Z.$$ 

If $U = \operatorname{hocofib}(u)$, $V = \operatorname{hocofib}(uv)$, $W = \operatorname{hocofib}(v)$, then there is a commutative diagram of the form

$$
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{id} & & \downarrow{v} \\
X & \xrightarrow{uv} & Z \\
\downarrow{u} & & \downarrow{v} \\
Y & \xrightarrow{v} & W \\
\end{array}
$$

where the rows and the third column are cofiber sequences, and the map $r$ and $s$ are the maps induced by functoriality of the homotopy cofiber construction.

The induction step

Consider the quadric $Q_{2n}$ introduced in the introduction. The open subscheme $U_n$ defined by the non-vanishing of $x_n$ is isomorphic to $\mathbb{A}^{2n-1} \times \mathbb{G}_m$ (with coordinates $(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}, z)$ on the first factor and $x_n$ on the second factor). The closed complement of this open subscheme, which we will call $Z_n$, is isomorphic to $Q_{2(n-2)} \times \mathbb{A}^1$, with coordinates $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}, z$ on the first factor and $y_n$ on the second factor. The subvariety $Z_n$ has a distinguished point “0” corresponding to $x_1 = \cdots = x_n = y_1 = \cdots = y_{n-1} = z = 0$. Moreover, the normal bundle to $Z_n$ in $Q_{2n}$ is a line bundle equipped with a chosen trivialization (coming from $x_n$).

The closed subscheme of $U_n$ defined by $x_1 = \cdots = x_{n-1} = y_1 = \cdots = y_{n-1} = z = 0$ is isomorphic to $\mathbb{G}_m$ with coordinate $x_n$ and the inclusion map

$$\mathbb{G}_m \rightarrow U_n$$

is a cofibration and $\mathbb{A}^1$-weak equivalence (it even admits a retraction). The closed subscheme of $Q_{2n}$ defined by $x_1 = \cdots = x_{n-1} = y_1 = \cdots = y_{n-1}$ is isomorphic to $Q_2$. Imposing further the condition $y_n = 0$, the resulting variety is isomorphic to two copies of $\mathbb{A}^1$ (corresponding to $z = 0$ and $z = -1$). Imposing yet further the condition $z = 0$ gives a subvariety of $Q_{2n}$ isomorphic to $\mathbb{A}^1$ with coordinate $x_n$. Note that, by construction, the intersection of this copy of the affine line with
$Z_n$ is precisely the point “0”, i.e., there is a pullback square (of smooth schemes) of the form:

\[
\begin{array}{ccc}
0 & \to & A^1 \\
\downarrow & & \downarrow \\
Z_n & \to & Q_{2n}.
\end{array}
\]

Moreover, the natural map from the pullback of the normal bundle to $Z_n$ in $Q_{2n}$ to the normal bundle of 0 in $A^1$ is an isomorphism compatible with the specified trivializations.

It follows from the construction of the purity isomorphism [MV99, §3 Theorem 2.23] that, given a pullback square as in the previous paragraph, the diagram

\[
\begin{array}{ccc}
A^1/G_m & \to & Th(\nu_{0/A^1}) \\
\downarrow & & \downarrow \\
Q_{2n}/U_n & \to & Th(\nu_{Z_n/Q_{2n}})
\end{array}
\]

is commutative in $H^*(k)$ (cf. [Voe03a, Lemma 2.1]). Moreover, the specified trivializations yield $A^1$-weak equivalences of the form $Th(\nu_{0/A^1}) \cong \mathbb{P}^1$ and $Th(\nu_{Z_n/Q_{2n}}) \cong \mathbb{P}^1\wedge(Z_n)_+$ by [MV99, §3 Proposition 2.17.2] and their compatibility ensures that the diagram

\[
\begin{array}{ccc}
Th(\nu_{0/A^1}) & \to & \mathbb{P}^1 \\
\downarrow & & \downarrow \\
Th(\nu_{Z_n/Q_{2n}}) & \to & \mathbb{P}^1\wedge(Z_n)_+
\end{array}
\]

commutes (this time in the category of pointed spaces). Furthermore, under these identifications, we can make the right vertical map in the diagram very explicit. Indeed, consider the map $S^0 \to (Z_n)_+$ sending the base-point of $S^0$ to the base-point of $(Z_n)_+$ and the point 1 of $S^0$ to the point “0” in $Z_n$ described above; the right hand vertical map is the $\mathbb{P}^1$-suspension of this map.

Now, we apply Proposition 2.2.1 to the composition

\[
(G_m)_+ \xrightarrow{u} (A^1)_+ \xrightarrow{v} (Q_{2n})_+.
\]

Observe that the cofiber of $v$ is $Q_{2n}$ pointed with the image of “0”, while the cofiber of $u$ is, by means of the purity isomorphism, $\mathbb{P}^1$. It remains to identify the cofiber of $(G_m)_+ \to (Q_{2n})_+$, which is the content of the next lemma.

**Lemma 2.2.2.** The cofiber of the map $(G_m)_+ \to (Q_{2n})_+$ is $A^1$-weakly equivalent to $\mathbb{P}^1\wedge(Q_{2n-2})_+$.

**Proof.** The projection map $Z_n \to Q_{2n-2}$ is an $A^1$-weak equivalence, it follows that it induces an $A^1$-weak equivalence $\mathbb{P}^1\wedge(Z_n)_+ \to \mathbb{P}^1\wedge(Q_{2n-2})_+$. On the other hand, by the purity isomorphism, the space $\mathbb{P}^1\wedge(Z_n)_+$ is $A^1$-weakly equivalent to $Q_{2n}/U_n$. Since the map $G_m \to U_n$ described above is an $A^1$-weak equivalence and cofibration and since the induced map $G_m \to Q_{2n}$ is a cofibration, it follows that the induced map of cofibers

\[
Q_{2n}/G_m \to Q_{2n}/U_n
\]
is an $\mathbb{A}^1$-weak equivalence. Moreover, the same statement remains true for the cofibers of the pointed morphisms $(G_m)_+ \to (Q_{2n})_+$ and $(U_n)_+ \to (Q_{2n})_+$. \hfill $\square$

Combining Proposition 2.2.1 and Lemma 2.2.2, we deduce the following result.

**Corollary 2.2.3.** For any integer $n \geq 1$, there is a cofiber sequence of the form

$$\mathbb{P}^1 \to \mathbb{P}^1 \wedge (Q_{2n-2})_+ \to Q_{2n} \to \cdots.$$ 

To finish, we provide an alternative identification of the cofiber of $\mathbb{P}^1 \to \mathbb{P}^1 \wedge (Q_{2n-2})_+$. Recall that this map is induced by $\mathbb{P}^1$-suspension of a map $S^0_s \to (Z_n)_+$. We prove a more general statement about smashing this map with the identity map.

**Proposition 2.2.4.** Assume $(\mathcal{X}, x)$ is a pointed space, and $\mathcal{X}_+$ is $\mathcal{X}$ with a disjoint base-point attached. Let $i : S^0_s \to \mathcal{X}_+$ be the map that sends the base-point of $S^0_s$ to the (disjoint) base-point of $\mathcal{X}_+$ and the non-base-point to $x \in \mathcal{X}(k)$. If $(\mathcal{Y}, y)$ is a pointed space, then the map $\mathcal{Y} \cong S^0_s \wedge \mathcal{Y} \xrightarrow{i \wedge id} \mathcal{X}_+ \wedge \mathcal{Y}$ fits into a split cofiber sequence of the form

$$\mathcal{Y} \xrightarrow{i \wedge id} \mathcal{X}_+ \wedge \mathcal{Y} \to \mathcal{X} \wedge \mathcal{Y}.$$ 

**Proof.** The splitting is given by the map $\mathcal{X}_+ \to S^0_s$ that collapses the non-base-point component of $\mathcal{X}_+$ to the non-base-point of $S^0_s$. In particular, the map $i \wedge id$ is a cofibration. The homotopy cofiber of this map is then computed by the actual quotient.

The quotient $\mathcal{X}_+ \wedge \mathcal{Y}$ is, by definition, the quotient $\mathcal{X}_+ \times \mathcal{Y}$ by $\mathcal{X}_+ \vee \mathcal{Y}$, where the latter is the disjoint union of $\mathcal{X}_+ \times y$ and $\text{Spec } k \times \mathcal{Y}$. Identify $\mathcal{X}_+ \times \mathcal{Y}$ as $(\mathcal{X} \times \mathcal{Y}) \amalg (\text{Spec } k \times \mathcal{Y})$. We describe the quotient in two steps. First, we collapse $\text{Spec } k \times \mathcal{Y}$ to the base-point and then the image of $\mathcal{X}_+ \times y$ to the base-point. We conclude that there is an isomorphism of spaces $\mathcal{X}_+ \wedge \mathcal{Y} \cong (\mathcal{X} \times \mathcal{Y}) / (\mathcal{X} \times y)$. To conclude, we simply observe that $\mathcal{X} \times \mathcal{Y} / (\mathcal{X} \times y)$ modulo the image of $x \times \mathcal{Y}$ is $\mathcal{X} \wedge \mathcal{Y}$. \hfill $\square$

**Theorem 2.2.5.** There is an $\mathbb{A}^1$-weak equivalence $Q_{2n} \xrightarrow{\sim} \mathbb{P}^1 \wedge n$.

**Proof.** By Proposition 2.1.1, we know that $Q_2 \xrightarrow{\sim} \mathbb{P}^1$. Combining Proposition 2.2.4 and Corollary 2.2.3, we conclude that for every $n \geq 2$ there is an $\mathbb{A}^1$-weak equivalence $Q_{2n} \cong Q_{2n-2} \wedge \mathbb{P}^1$. The result follows by induction. \hfill $\square$

### 2.3 Non-geometric motivic spheres

**Proposition 2.3.1.** If $k$ is an arbitrary (Noetherian) commutative unital ring, and if $i > j$ are integers, then $S^i_s \wedge G_m^{\wedge j}$ does not have the $\mathbb{A}^1$-homotopy type of a smooth scheme.

**Proof.** Assume to the contrary that $X$ is a smooth scheme having the $\mathbb{A}^1$-homotopy type of $S^i_s \wedge G_m^{\wedge j}$. By picking a suitable point of $\text{Spec } k$, we can, without loss of generality, assume $k$ is an algebraically closed field, in particular perfect. Then, by $\mathbb{A}^1$-representability of motivic cohomology (see, e.g., [Voe03b, §2]), we know that for arbitrary integers $p, q$

$$H^{p,q}(X, \mathbb{Z}) = [X, K(\mathbb{Z}(q), p)]_{\mathbb{A}^1}.$$
Next, observe that by the cancellation theorem [Voe10, Corollary 4.10] and the (simplicial) suspension isomorphism, there is a non-trivial morphism $S^n_i \wedge \mathbb{G}^m_{n,j} \to K(\mathbb{Z}(j), i + j)$, giving a non-trivial class in $H^{i+j}(X, \mathbb{Z})$. However, if $X$ is a smooth scheme, and $i > j$, then this latter group must vanish by [MVW06, Theorem 19.3].

**Remark 2.3.2.** One can obtain another more “homotopic” proof of the proposition at the expense of introducing slightly different terminology; we freely use the conventions of [AF14] and some terminology from [Mor12]. Consider an Eilenberg-Mac Lane space $K(K^M_j, i)$, i.e., a space with exactly 1 non-vanishing $\mathbb{A}^i$-homotopy sheaf in degree $j$, in which degree it is isomorphic to the unramified Milnor $K$-theory sheaf $K^M_j$. If $X$ is a smooth scheme, the explicit Gersten resolution of $K^M_j$ shows that $H^i(X, K^M_j)$ vanishes for $i > j$. On the other hand, there is a non-trivial (pointed) morphism $S^i_n \wedge \mathbb{G}^m_{n,j} \to K(K^M_j, i)$. Indeed, by [Mor12, Theorem 6.13], we have $\pi_j^{\mathbb{A}_i}(K(K^M_j, i)) \cong (K^M_j)_{-j}$ and by applying, e.g., [AF14, Lemma 2.7] we deduce $(K^M_j)_{-j} \cong \mathbb{Z}$.

The following conjecture summarizes our expectations regarding representability of the remaining motivic spheres by smooth schemes.

**Conjecture 2.3.3.** If $k$ is an arbitrary (Noetherian) commutative ring, and if $i, j \in \mathbb{N}$ with $i < j - 1$, the sphere $S^n_i \wedge \mathbb{G}^m_{n,j}$ does not have the $\mathbb{A}^i$-homotopy type of a smooth $k$-scheme.

## 3 $\mathbb{A}^1$-contractible subvarieties of quadrics

A smooth $k$-scheme $X$ is $\mathbb{A}^1$-contractible if the structure map $X \to \text{Spec } k$ is an isomorphism in $\mathcal{H}(k)$, i.e., an $\mathbb{A}^1$-weak equivalence. The affine space $\mathbb{A}^n_k$ is $\mathbb{A}^1$-contractible by construction of the $\mathbb{A}^1$-homotopy category. An exotic $\mathbb{A}^1$-contractible smooth scheme is an $\mathbb{A}^1$-contractible smooth scheme that is not isomorphic to an affine space. In [AD07] the first two authors constructed many exotic $\mathbb{A}^1$-contractible smooth schemes as quotients of an affine space by a free action of the additive group $\mathbb{G}_a$ (or, more generally, free actions of unipotent groups on affine spaces). All of the examples constructed in [AD07] could be realized as open subschemes of affine schemes with complement of codimension $\leq 2$.

Let $E_n \subset Q_{2n}$ be the closed subscheme defined by $x_1 = \cdots = x_n = 0$ and $z = -1$. Observe that $E_n \cong \mathbb{A}^n$ and has codimension $n$ in $Q_{2n}$. Set

$$X_{2n} := Q_{2n} \setminus E_n.$$ 

It is straightforward to check that $X_2 \cong \mathbb{A}^2$, and we recalled in the proof of Proposition 2.1.2 that $X_4$ is $\mathbb{A}^1$-contractible. Subsection 3.1 contains the proof that, more generally, $X_{2n}$ is $\mathbb{A}^1$-contractible for any $n \geq 1$. Subsection 3.2 contains the proof that, for an integer $n \geq 3$, the variety $X_{2n}$ cannot be realized as a quotient of an affine space by the free action of a unipotent group.

### 3.1 On the $\mathbb{A}^1$-contractibility of $X_{2n}$

If $k = \mathbb{C}$, one can check that $X_{2m}(\mathbb{C})$ is a contractible complex manifold. Over $\mathbb{C}$, the variety $Q_{2n}$ is isomorphic to the variety defined by $\sum_{i=1}^{2n+1} x_i^2 = 1$. Wood explains [Woo93, §2] how to identify the last variety with the tangent bundle of a sphere. Under this isomorphism, the variety
3.1 On the $\mathbb{A}^1$-contractibility of $X_{2n}$

$E_m$ can be identified with the tangent space at a point. The complement of $E_m$ is then a contractible topological space diffeomorphic to $\mathbb{R}^{4m}$. Here is the generalization of this result to $\mathbb{A}^1$-homotopy theory.

**Theorem 3.1.1.** If $k$ is any (Noetherian) commutative unital ring, then the variety $X_{2n}$ is $\mathbb{A}^1$-contractible over $\text{Spec} \, k$.

**Proof.** We continue with the notation of Subsection 2.2. The proof is essentially a repeat of the Proof of Theorem 2.2.5, so we will explain only the changes required. Consider the closed subvariety $Z_n \subset Q_{2n}$ defined by $x_n = 0$. We saw that $Z_n \cong Q_{2n-2} \times \mathbb{A}^1$ and that the normal bundle to $Z_n$ comes equipped with a specified trivialization. Observe that, by construction $E_n \subset Z_n$ and therefore we see that $X_{2n} := Q_{2n} \setminus E_n$ contains $U_n := Q_{2n} \setminus Z_n$ as an open subscheme. Moreover, the subvariety $Z_n \setminus E_n$ is isomorphic to $X_{2n-2} \times \mathbb{A}^1$.

Recall that $U_n \cong \mathbb{A}^{2n-1} \times G_m$ with coordinates $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}, z$ on the first factor and $x_n$ on the last factor. The closed subscheme $G_m \hookrightarrow U_n$ is defined by $x_1 = \cdots = x_{n-1} = y_1 = \cdots y_{n-1} = 0$ and $z = 0$. The intersection of $X_{2n}$ and $Q_2$, viewed as a subvariety of $Q_{2n}$ by imposing $x_1 = \cdots = x_{n-1} = y_1 = \cdots = y_{n-1} = 0$ is isomorphic to $\mathbb{A}^2$. The copy of $\mathbb{A}^1 \subset Q_2$ identified by further imposing the conditions $y_n = z = 0$ is disjoint from $E_n$ and therefore contained in $X_{2n}$. Note that the point “0” of $Q_{2n}$ is contained in $X_{2n}$.

We therefore have a diagram of the form

$$
G_m \xrightarrow{u} \mathbb{A}^1 \xrightarrow{v} X_{2n}.
$$

We understand the cofiber of this composite map using Proposition 2.2.1 again. In particular, repeating the proofs of Lemma 2.2.2 and Corollary 2.2.3 with all instances of $Q_{2i}$ replaced by $X_{2i}$, we deduce the existence of a cofiber sequence of the form

$$
\mathbb{P}^1 \xrightarrow{\iota} (X_{2n-2})_+ \wedge \mathbb{P}^1 \longrightarrow X_{2n} \longrightarrow \cdots
$$

Now, applying Proposition 2.2.4, we conclude that the cofiber of $\iota$ is $\mathbb{A}^1$-weakly equivalent to $X_{2n-2} \wedge \mathbb{P}^1$. The induction hypothesis guarantees that $X_{2n-2}$ is $\mathbb{A}^1$-contractible, so the smash product $X_{2n-2} \wedge \mathbb{P}^1$ is $\mathbb{A}^1$-contractible as well. Since $X_{2n-2} \wedge \mathbb{P}^1$ is $\mathbb{A}^1$-weakly equivalent to $X_{2n}$, it follows that $X_{2n}$ is $\mathbb{A}^1$-contractible. \qed

**Remark 3.1.2.** Theorem 3.1.1 gives an alternative proof of Theorem 2.2.5, this time repeating the argument of Proposition 2.1.2. In this case, the map $Q_{2n} \to \text{Th}(\nu_{E_n}/Q_{2n})$ obtained by collapsing $X_{2n}$ to a point and then using the homotopy purity theorem is an $\mathbb{A}^1$-weak equivalence by properness of the $\mathbb{A}^1$-local model structure, as explained in the proof of Proposition 2.1.2. The explicit trivialization of the normal bundle to $E_n$ that arises by its presentation as a (connected component of a) complete intersection defined by a regular sequence of $Q_{2n}$ yields an isomorphism $\text{Th}(\nu_{E_n}/Q_{2n}) \cong \mathbb{P}^1 \wedge (E_n)_+$, and the projection map $E_n \to \text{Spec} \, k$ then yields an $\mathbb{A}^1$-weak equivalence $Q_{2n} \xrightarrow{\nu} \mathbb{P}^1 \wedge \mathbb{P}^1$ that is perhaps slightly more explicit than the one arising in the proof of Theorem 2.2.5.
3.2 \(A^1\)-contractibles that are not unipotent quotients

We will now see that for \(m \geq 3\), the varieties \(X_{2m}\) studied in Theorem 3.1.1 cannot be realized as quotients of affine space by free actions of unipotent groups \(U\). We begin by proving a general “excision” style result (Theorem 3.2.1) for (Zariski) cohomology with coefficients in a unipotent group; the main result then follows from this excision result when applied in the special case of degree 1 cohomology (Corollary 3.2.2). Indeed, the degree 1 cohomology group in question can be identified as the (pointed) set of \(U\)-torsors on the scheme \(X\). We warn the reader that, contrary to “understood” prior conventions, the letter \(U\) in this section stands for a unipotent group, as opposed to an open subscheme; we refer the reader to [Bor91, §15 (Definition 15.1)] for general information on split unipotent groups.

**Theorem 3.2.1** (Excision for unipotent groups). Let \(d \geq 3\) be an integer. Suppose \(X\) is a regular scheme over a field and \(j : W \hookrightarrow X\) is an open immersion whose closed complement has codimension \(\geq d\). Suppose \(U\) is a (not necessarily commutative) split unipotent group. The restriction map

\[
  j^* : H^1_{Zar}(X, U) \rightarrow H^1_{Zar}(W, U)
\]

is a pointed bijection, i.e., every \(U\)-torsor on \(W\) extends uniquely to a \(U\)-torsor on \(X\).

**Proof.** Since \(U\) is split, by definition it admits an increasing filtration by normal subgroups \(1 = U_0 \subset U_1 \subset \cdots U_n = U\) with subquotients \(U_{i+1}/U_i\) isomorphic to \(\mathbb{G}_a\). We first show that, working inductively with respect to the dimension of \(U\), it suffices to prove the result in the case \(U = \mathbb{G}_a\).

Indeed, suppose we know the result holds true for all unipotent groups of dimension \(n\) and all open immersions of regular schemes with closed complement of codimension \(d \geq 3\). In that case, suppose \(U\) has dimension \(n + 1\) and we are given a \(U\)-torsor \(\varphi : P \rightarrow W\). We know that there is a normal subgroup \(U'\) of \(U\) such that \(U/U' \cong \mathbb{G}_a\). In that case, the quotient map \(U \rightarrow \mathbb{G}_a\) gives rise to a \(\mathbb{G}_a\)-torsor \(W' := P \times_U \mathbb{G}_a \rightarrow W\); note that \(W'\) is again regular. The pullback of \(\varphi\) along \(W' \rightarrow W\) gives a \(U'\)-torsor on \(W'\). Now, again by induction, we also know that the \(\mathbb{G}_a\)-torsor \(W' \rightarrow W\) extends (uniquely) to a \(\mathbb{G}_a\)-torsor \(X' \rightarrow X\); again, \(X'\) is regular. Moreover, working over a Zariski trivialization of \(X' \rightarrow X\), we conclude that \(W' \subset X'\) is open with closed complement having codimension \(\geq 3\) as well. However, the induction hypothesis guarantees that the \(U'\)-torsor over \(W'\) extends uniquely to a \(U'\)-torsor on \(X'\), and this extension provides the required extension of \(\varphi\) over \(W\).

Now, assume \(U = \mathbb{G}_a\). In this case, one knows that by definition \(H^q(X, \mathbb{G}_a) = H^q(X, O_X)\). We will show that the map

\[
  j^* : H^q(X, O_X) \rightarrow H^q(W, O_W)
\]

induced by pull-back along \(j\) is an isomorphism for \(q \leq d - 2\).

To prove the last fact, we use the Cousin complex for \(O_X\) as discussed in, e.g., [Har66, IV.2]. Under the assumption that \(X\) is regular, the Cousin complex provides an injective resolution of \(O_X\) (see especially *ibid.*, p. 239). If \(X^{(p)}\) denote the set of codimension \(p\) points in \(X\) (recall this means \(\dim O_{X,x} = p\)), the \(p\)-th term of the Cousin complex is

\[
  \prod_{x \in X^{(p)}} (i_x)^*(H^p_x(O_X)).
\]
Since $j$ is an open immersion, it is an affine morphism and the Leray spectral sequence for $j$ degenerates to yield isomorphisms $j^*: H^q(W, O_W) \to H^q(X, j_* O_W)$. Adjunction gives rise to a morphism $O_X \to j_* O_W$. The push-forward of the Cousin complex for $O_W$ to $X$ by $j$ remains a flasque resolution of $j_* O_W$ on $X$. Since the inclusion $W \hookrightarrow X$ is, by definition, an isomorphism on points of codimension $d - 1$, it follows that the cokernel of the induced morphism of Cousin complexes, which provides a flasque resolution of the cone of the map $O_X \to j_* O_X$, only depends on points of codimension $\geq d$ and the isomorphism of the theorem statement follows immediately from the long exact sequence in cohomology.

**Corollary 3.2.2.** If $X$ is a regular affine scheme and $W \subset X$ is an open subscheme whose complement has codimension $d \geq 3$, then every torsor under a split unipotent group over $W$ is trivial. In particular, the varieties $X_{2m}$ cannot be realized as quotients of affine space by free actions of split unipotent groups.

**Proof.** By Serre’s vanishing theorem [Gro61, Théorème 1.3.1], we know that all higher cohomology of a quasi-coherent sheaf on an affine scheme vanishes. By Theorem 3.2.1, we conclude that if $W$ is as in the theorem statement, and if $U$ is a (split) unipotent group, then $H^1(W, U) = \ast$, so every $U$-torsor over $W$ is trivial. In particular, the total space of any $U$-torsor over $W$ is of the form $W \times U$, which is itself a quasi-affine variety that is not affine.

**Corollary 3.2.3.** If $k$ is a commutative (Noetherian) ring, and $m \geq 3$ is an integer, then $X_{2m}$ is not a quotient of an affine space by the free action of a unipotent group.

**Proof.** Pick an algebraically closed field $F$ such that $\text{Spec } k$ has a $\text{Spec } F$-point. Assume that $X_{2m}$ is a quotient of $\mathbb{A}^n$ by the free action of a unipotent group. By base change, we conclude that the unipotent group is necessarily smooth. Base-change to $\text{Spec } F$ then allows us to assume that the unipotent group in question is also split. For $m \geq 3$, $X_{2m}$ has codimension $m \geq 3$ in $Q_{2m}$, so the result then follows immediately from Corollary 3.2.2.

**Question 3.2.4.** If $Z \to X_{2n}$ is a Jouanolou device, then $Z$ is an affine $\mathbb{A}^1$-contractible variety. Is $Z \cong \mathbb{A}^m$ for some integer $m$?

**Remark 3.2.5.** If $k$ is a field having characteristic $p > 0$, work of N. Gupta [Gup14, Gup13] shows that there exist smooth affine $k$-varieties of every dimension $d \geq 3$ that are stably isomorphic to affine $k$-space, yet which are not isomorphic to affine $k$-space. Such varieties are necessarily exotic smooth affine $\mathbb{A}^1$-contractible varieties. At the moment, we do not have any examples of exotic smooth affine $\mathbb{A}^1$-contractible varieties over a fields having characteristic $0$. Nevertheless, it seems reasonable to ask: if $k$ is an arbitrary field, is every smooth affine $\mathbb{A}^1$-contractible $k$-variety a retract of an affine $k$-space?

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