THE INCIDENCE COEFFICIENTS IN THE NOVIKOV COMPLEX ARE GENERICALLY RATIONAL FUNCTIONS

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Abstract. For a Morse map \( f : M \to S^1 \) Novikov [11] has introduced an analog of Morse complex, defined over the ring \( \mathbb{Z}[\![t][t^{-1}] \) of integer Laurent power series. Novikov conjectured, that generically the matrix entries of the differentials in this complex are of the form \( \sum a_i t^i \), where \( a_i \) grow at most exponentially in \( i \). We prove that for any given \( f \) for a \( C^0 \) generic gradient-like vector field all the incidence coefficients above are rational functions in \( t \) (which implies obviously the exponential growth rate estimate).

Introduction

A. Morse-Novikov theory. The classical Morse-Thom-Smale construction associates to a Morse function \( f : M \to \mathbb{R} \) on a closed manifold a free chain complex \( C_*(f) \) where the number \( m(C_p(f)) \) of free generators of \( C_p(f) \) equals the number of the critical points of \( f \) of index \( p \) for each \( p \). The boundary operator in this complex is defined in a geometric way, using the trajectories of a gradient of \( f \), joining critical points of \( f \) (see [8], [21], [23], [25]).

In the early 80s S.P.Novikov generalized this construction to the case of maps \( f : M \to S^1 \) (he was lead to this generalization by considering a problem of motion of a solid in a fluid, see [9], [10]). The corresponding analog of Morse complex is a free chain complex \( C_*(f) \) over \( \mathbb{Z}[\![t][t^{-1}] \). Its number of free generators equals the number of critical points of \( f \) of index \( p \), and the homology of \( C_*(f) \) equals to the completed homology of the cyclic covering. In [11] the analogs of Morse inequalities were extracted. These inequalities quickly became an object of study and applications. Farber [3] obtained an exactness theorem for these inequalities in the case \( \pi_1(M) = \mathbb{Z} \), \( \dim M \geq 6 \). The author [14, 15] obtained an exactness result for these inequalities in the case \( \pi_1 M = \mathbb{Z}^m, \dim M \geq 6 \). These inequalities were applied by J.Cl.Sikorav [22] to the theory of Lagrangian intersections.

The study of the properties of the chain complex itself advanced more slowly. In [21] a refined version of the complex, defined over a completion of the group ring of \( \pi_1(M) \) was defined. In [16, 17] I proved that this complex computes the completed simple homotopy type of \( M \) and in [18, 19] I obtained an exactness result for the case of general fundamental group. This theorem implies the results of [3] and [14]. The algebraic result of Ranicki [20] show that this result implies also the Farrell fibration theorem [4]. (Note that the resulting proof of the Farrell’s result is independant of Farrell’s original proof).

One of the objects here, which behaves very differently from the classical situation is the boundary operator in the Novikov complex. Fix some \( k \). The boundary
remark. conjecture above: note that every rational function of the form \( C \) is residual in the set of all the nice analytic properties. In particular he conjectured that

Generically the coefficients \( a_n \) of \( \partial_{ij} = \sum_{n=-N}^{\infty} a_n t^n \) grow at most exponentially with \( n \).

See [13, p. 229] and [1, p. 83] for (different) published versions of this conjecture. In the present paper we prove that generically the incidence coefficients are rational functions of \( t \) of the form \( \frac{P(t)}{Q(t)} \) where \( P(t), Q(t) \) are polynomials, \( k \in \mathbb{N} \) and \( Q(0) = 1 \).

The methods developed in the present paper allow to make explicit computations of the Novikov complex. These computations and further generalizations are the subject of the second part of the work, to appear.

We would like to conclude this subsection by the following remark. Studying the algebraic properties of the Novikov complex and of the Novikov completion one passes often from localization to completion and vice versa (see [3], [13, 14], [20]). We see now that these passages have a geometric background.

B. Statement of the main theorem. Let \( M \) be a closed connected manifold and \( f : M \to S^1 \) a Morse map, non homotopic to zero. Denote the set of all the \( C^\infty \) gradients, satisfying the transversality assumption (see §1 for terminology), will be denoted by \( \mathcal{G}t(f) \). By Kupka-Smale theorem it is residual in the set of all the \( C^\infty \) gradients. Choose \( v \in \mathcal{G}t(f) \). Denote by \( \tilde{M} \overset{P}{\to} M \) the connected infinite cyclic covering for which \( f \circ P \) is homotopic to zero. Choose a lifting \( F : \tilde{M} \to \mathbb{R} \) of \( f \circ P \) and let \( t \) be the generator of the structure group of \( P \) such that \( F(xt) < F(x) \). The \( t \)-invariant lifting of \( v \) to \( \tilde{M} \) will be denoted by the same letter \( v \). For every critical point \( x \) of \( f \) choose a lifting \( \tilde{x} \) of \( x \) to \( \tilde{M} \). Choose orientations of stable manifolds of critical points. Then for every \( x, y \in S(f) \) with \( \text{ind}x = \text{ind}y + 1 \) and every \( k \in \mathbb{Z} \) the incidence coefficient \( n_k(x, y; v) \) is defined (as the algebraic number of \((-v)\)-trajectories joining \( x \) to \( yt^k \)).

Main Theorem. In the set \( \mathcal{G}t(f) \) there is a subset \( \mathcal{G}t_0(f) \) with the following properties:

1. \( \mathcal{G}t_0(f) \) is open and dense in \( \mathcal{G}t(f) \) with respect to \( C^0 \) topology.
2. If \( v \in \mathcal{G}t_0(f), x, y \in S(f) \) and \( \text{ind}x = \text{ind}y + 1 \), then \( \sum_{k \in \mathbb{Z}} n_k(x, y; v)t^k \) is a rational function of \( t \) of the form \( \frac{P(t)}{Q(t)} \), where \( P(t) \) and \( Q(t) \) are polynomials with integral coefficients, \( m \in \mathbb{N} \), and \( Q(0) = 1 \).
3. Let \( v \in \mathcal{G}t_0(f) \). Let \( U \) be a neighborhood of \( S(f) \). Then for every \( w \in \mathcal{G}t_0(f) \) such that \( w = v \) in \( U \) and \( w \) is sufficiently close to \( v \) in \( C^0 \) topology we have: \( n_k(x, y; v) = n_k(x, y; w) \) for every \( x, y \in S(f) \), \( k \in \mathbb{Z} \).

Remark. This theorem implies easily the exponential estimate from the Novikov conjecture above: note that every rational function of the form \( \frac{P(t)}{Q(t)} \), where \( Q(0) \neq 0 \), has a non zero radius of convergency in \( 0 \).

C. Further generalizations. In the second part of this work, to appear, we develop our methods and generalize them to the incidence coefficients with values in the Novikov completions of the group rings. We prove that for a \( C^0 \) generic Morse map \( f : M \to S^1 \), there exists a finite constant \( N \) such that for every \( x, y \in S(f) \) with \( \text{ind}x = \text{ind}y + 1 \), then \( \sum_{k \in \mathbb{Z}} n_k(x, y; v)t^k \) is a rational function of \( t \) of the form \( \frac{P(t)}{Q(t)} \), where \( P(t) \) and \( Q(t) \) are polynomials with integral coefficients, \( m \in \mathbb{N} \), and \( Q(0) = 1 \).
f-gradient these coefficients belong actually to the image in the Novikov ring of the corresponding localization of the group ring of the fundamental group. For the case of irrational Morse forms we prove the analog of this result, concerning the incidence coefficients related with free abelian coverings. We state and prove the generalization of the exponential growth rate estimate for the case of Morse forms of arbitrary irrationality degree (and the incidence coefficients related to the universal covering). We also give an example of explicit computation of the Novikov complex.

D. Remarks on the contents of the paper. The proof of the main theorem is contained in §4.E. The technical results on Morse functions and their gradients which we use are that of [19,§2]. We reformulate them (in a slightly generalized and modified form) in §1. §5 contains some results on $C^0$ stability of trajectories of $C^1$ vector fields. These results can be of independent interest. §5 is independent of other sections and the results of §5 are used in §1 - 4.

E. Remarks on the terminology. If $W$ is a manifold with boundary, then $W^\circ$ stands for $W \setminus \partial W$. $\gamma(x, t; v)$ stands always for the value at $t$ of the integral curve of the vector field $v$, starting at $x$. If $\gamma(x, \cdot; v)$ is defined on $[\alpha, \beta]$, then \{\gamma(x, t; v) \mid t \in [\alpha, \beta]\} is denoted by $\gamma(x, [\alpha, \beta]; v)$. For a (time dependant) vector field $w$ on a closed manifold $M$ and $t \in \mathbb{R}$ we denote by $\Phi(w, t)$ the diffeomorphism $x \mapsto \gamma(x, t; w)$ of $M$. The overline denotes closure (e.g. $\overline{U}$). The bar is reserved for the objects related to the infinite cyclic covering (e.g. $\overline{P} : \overline{M} \to M$).

A chart $\Phi : U \to V \subset \mathbb{R}^n$ of $M$ we denote by $G(x, \Phi)$ the number $\sup_{h \in T_x M, h \neq 0} \left( \max \left( \frac{|h|_\rho}{|\Phi_* h|_e}, \frac{|\Phi_* h|_e}{|h|_\rho} \right) \right)$. For $K \subset U$ we denote by $G(K, \Phi)$ the number $\sup_x G(x, \Phi)$, where $x$ ranges over points of $K$.

The end of a definition, remark or construction is marked by $\triangle$.

F. Acknowledgements. I am grateful to S.P.Novikov, J.Priszytycki, J.-Cl.Sikorav, C.Simpson, T. tom Dieck, A.N.Tyurin, P.Vogel for valuable discussions.

A part of this work was done during my stay in Göttingen University in the spring 1994. It is a pleasure to express here my gratitude to Sonderforschungsbereich 170 Analysis and Geometry for hospitality and financial support.

§1 Preliminaries on Morse functions and their gradients

Subsections A-D contain generalities. In Subsection G we formulate Theorem 1.17, which is the main aim of §§1 and 3. Subsections E,F contain some techniques, useful for its proof. All the objects are supposed to be of class $C^\infty$.

A. Terminology: functions and gradients. In the present and the following subsections $M$ stands for a closed manifold.

Definition 1.1. Let $f : M \to \mathbb{R}$ be a Morse function on a closed manifold $M$. Denote $\dim M$ by $n$. The set of critical points of $f$ will be denoted by $S(f)$. A chart $\Phi : U \to V \subset \mathbb{R}^n$ of $M$ and $x \in U$ we denote by $G(x, \Phi)$ the number $\sup_{h \in T_x M, h \neq 0} \left( \max \left( \frac{|h|_\rho}{|\Phi_* h|_e}, \frac{|\Phi_* h|_e}{|h|_\rho} \right) \right)$.
Let $\Phi_p : U_p \to B^n(0, r_p)$ (where $p \in S(f), U_p$ is a neighborhood of $p$, $r_p > 0$) be called a **standard chart for** $f$ **around** $p$ **of radius** $r_p$ (or simply $f$-chart) if there is an extension of $\Phi_p$ to a chart $\tilde{\Phi}_p : V_p \to B^n(0, r''_p)$, (where $\overline{U_p} \subset V_p$ and $r''_p > r_p$), such that $(f \circ \tilde{\Phi}^{-1}_p)(x_1, ..., x_n) = f(p) + \sum_{i=1}^{n} \alpha_i x_i^2$, where $\alpha_i < 0$ for $i \leq \text{ind}_fp$ and $\alpha_i > 0$ for $i > \text{ind}_fp$. The domain $U_p$ is called a **standard coordinate neighborhood**. Any such extension $\tilde{\Phi}_p$ of $\Phi_p$ will be called a **standard extension** of $\Phi_p$. The set $\Phi_p^{-1}(\mathbb{R}^k \times \{0\})$, resp. $\Phi^{-1}_p(\{0\} \times \mathbb{R}^{n-k})$, where $k$ stands for ind$_fp$, is called **negative disc**, resp. **positive disc**. If for every $i$ we have $\alpha_i = \pm 1$, we shall say that the coordinate system $\{\Phi_p\}$ is **strongly standard**.

A family $\mathcal{U} = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$ of $f$-charts is called a **$f$-chart-system**, if the family $\{\overline{U_p}\}_{p \in S(f)}$ is disjoint. We denote $\min_p r_p$ by $d(\mathcal{U})$, and $\max_p r_p$ by $D(\mathcal{U})$. If all the $r_p$ are equal to $r$, we shall say that $\mathcal{U}$ is of **radius** $r$. The set $\Phi_p^{-1}(B^n(0, \lambda))$, where $\lambda \leq r_p$ will be denoted by $U_p(\lambda)$. For $\lambda \leq d(\mathcal{U})$ we denote $\cup_{p \in S(f)} U_p(\lambda)$ by $U(\lambda)$.

Let $\mathcal{U} = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}, \mathcal{U}' = \{\Phi'_p : U'_p \to B^n(0, r'_p)\}_{p \in S(f)}$ be two $f$-chart-systems. We say, that $\mathcal{U}'$ is a **restriction of** $\mathcal{U}$, if for every $p \in S(f)$ we have: $r'_p \leq r_p, U'_p \subset U_p, \Phi'_p = \Phi_p|U'_p$.

Given an $f$-chart system $\mathcal{U} = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$ we say, that a vector field $v$ on $M$ is an $f$-**gradient with respect** to $\mathcal{U}$, if

1) $\forall x \in M \setminus S(f)$ we have $df(v)(x) > 0$;

2) $\forall p \in S(f)$ we have $(\tilde{\Phi}_p)_* v = (-x_1, ..., -x_k, x_{k+1}, ..., x_n)$, where $k = \text{ind}_fp$, and $\tilde{\Phi}_p$ is some standard extension of $\Phi_p$.

We say that a vector field $v$ is an $f$-**gradient** if there is an $f$-chart system $\mathcal{U}$, such that $v$ is an $f$-gradient with respect to $\mathcal{U}$. △

**Definition 1.2.** Assume that $M$ is riemannian. Let $\delta_0 > 0$ be the radius of injectivity of $M$. For $0 < r < \delta_0$ and $x \in M$ we denote by $D_r(x)$, resp. $B_r(x)$, the image of $D^n(0, r)$, resp. $B^n(0, r)$, with respect to the exponential map exp$_x$. Let $f : M \to \mathbb{R}$ be a Morse function, $v$ be an $f$-gradient, $p \in S(f)$. Set:

$$B_\delta(p, v) = \{x \in M \mid \exists t \geq 0 : \gamma(x, t; v) \subset B_\delta(p)\}$$

$$D_\delta(p, v) = \{x \in M \mid \exists t \geq 0 : \gamma(x, t; v) \subset D_\delta(p)\}, \quad D(p, v) = \{x \in M \mid \lim_{t \to \infty} \gamma(x, t; v) = p\}$$

For $s \in \mathbb{N}, 0 \leq s \leq n$ we denote by $B_\delta(\text{ind} \leq s; v)$ the union of $B_\delta(p, v)$ where $p$ ranges over critical points of $f$ of index $\leq s$. Similar notations like $D_\delta(\text{ind} \leq s; v)$ or $K(\text{ind} = s; v)$ etc. are now clear without special definition. △

**Remarks.**
1) Our definition of $f$-gradient is wider than that of [18, 19] (and than that of gradient-like vector field in [6]), since we allow $\alpha_i \neq \pm 1$ in 1.1. It is not difficult to show that all the results from [19, §2] rest true with the present definition. The terminology of [19, §2] concerning Morse functions and their gradients is carried over to the present case without changes. For convenience of the reader we include the results and some of the terminology from [19, §2] which we use here, to Subsection C.

2) One can define these sets for any $\delta > 0$, see [19, §2]; they were denoted there by $B_\delta(\leq s; v)$, resp. $D_\delta(\leq s; v)$.

3) Similarly to 1.1 one defines the notion of $f$-gradient for Morse maps $f : M \to \mathbb{R}$.
B. Terminology: \(\mathcal{M}\)-flows.

**Definition 1.3.** Assume that \(M\) is riemannian. We say, that a triple \(\mathcal{V} = (f, v, \mathcal{U})\) is an \(\mathcal{M}\)-flow on \(M\) (\(\mathcal{M}\) for Morse) if

1. \(f : M \to \mathbb{R}\) is a Morse function,
2. \(\mathcal{U}\) is an \(f\)-chart system,
3. \(v\) is an \(f\)-gradient with respect to \(\mathcal{U}\),
4. for each \(f\)-chart \(\Phi_p : U_p \to B^n(0, r_p)\) the coordinate frame in \(p\) is orthonormal with respect to the riemannian metric of \(M\).

\(\triangle\)

Let \(\mathcal{V} = (f, v, \mathcal{U})\) be an \(\mathcal{M}\)-flow on \(M\), where \(\mathcal{U} = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}\).

We denote \(\max_p \mathcal{G}(U_p, \Phi_p)\) by \(\mathcal{G}(\mathcal{V})\). We say that \(\mathcal{V}\) is of radius \(r\), if \(\mathcal{U}\) is of radius \(r\).

Let \(\gamma\) be a \(v\)-trajectory.

A) The number of sets \(U_p = U_p(r_p)\), intersected by \(\gamma\), will be denoted by \(N(\gamma)\). The number \(\max \gamma N(\gamma)\) will be denoted by \(N(\mathcal{V})\). The set \(\{t \in \mathbb{R} | \gamma(t) \notin \cup_{p \in S(f)} U_p\}\) is a finite union of closed intervals; its measure will be denoted by \(T(\gamma)\). Let \(\beta > 0\), \(C > 0\). We say, that \(\mathcal{V}\) is \((C, \beta)\)-quick, if \(\|v\| \leq C\) and for every \(v\)-trajectory \(\gamma\) we have \(T(\gamma) \leq \beta\).

B) Let \(\delta \leq d(\mathcal{U})\). The number of sets \(U_p(\delta)\), intersected by \(\gamma\), will be denoted by \(N(\gamma, \delta)\). The number \(\max \gamma N(\gamma, \delta)\) will be denoted by \(N(\mathcal{V}, \delta)\). The set \(\{t \in \mathbb{R} | \gamma(t) \notin \cup_{p \in S(f)} U_p(\delta)\}\) is a finite union of closed intervals and its measure will be denoted by \(T(\gamma, \delta)\). Let \(\beta > 0\), \(C > 0\). We say, that \(\mathcal{V}\) is \((C, \beta, \delta)\)-quick, if \(\|v\| \leq C\) and for every \(v\)-trajectory \(\gamma\) we have \(T(\gamma, \delta) \leq \beta\).

Let \(\mathcal{V}^{(i)} = (f^{(i)}, v^{(i)}, \mathcal{U}^{(i)})\), where \(i = 1, 2\) be two \(\mathcal{M}\)-flows on \(M\). Set \(\mathcal{U}^{(i)} = \{\Phi_p^{(i)} : U_p^{(i)} \to B^n(0, r_p^{(i)})\}_{p \in S(f^{(i)})}\). We say, that \(\mathcal{V}^{(2)}\) is subordinate to \(\mathcal{V}^{(1)}\), if \(\mathcal{U}^{(2)}\) is a restriction of \(\mathcal{U}^{(1)}\), \(v^{(2)} = \phi \cdot v^{(1)}\), where \(\phi : M \to \mathbb{R}^+\) is a \(C^\infty\) function such that \(\phi(x) = 1\) for \(x\) in a neighborhood of the closure of \(\cup_p U_p^{(2)}\).

**Remark.** There is an obvious analog of the terminology of Subsections A, B for a more general situation of Morse functions on compact cobordisms. We shall make free use of it. \(\triangle\)

C. First properties of \(f\)-gradients. In this subsection \(f : W \to [a, b]\) is a Morse function on a compact cobordism, \(V_0 = f^{-1}(a)\), \(V_1 = f^{-1}(b)\), \(v\) is an \(f\)-gradient. Denote \(\cup_{p \in S(f)} B_\delta(p, v)\) by \(B_\delta(v)\), \(\cup_{p \in S(f)} D_\delta(p, v)\) by \(D_\delta(v)\), \(\cup_{p \in S(f)} D(p, v)\) by \(K(v)\). Denote \(\dim W\) by \(n\).

**Lemma 1.4.** (1) For every \(p \in S(f)\) the set \(B_\delta(v)\) is open.

(2) \(D_\delta(v)\) and \(K(v)\) are compact.

(3) \(D_\delta(v) = \cap_{\theta > 0} B_\theta(v)\) and \(K(v) = \cap_{\theta > 0} B_\theta(v)\).

(4) \(D_\delta(v) = \overline{B_\delta(v)}\).

**Proof.** Proof of (1) - (3) is similar to that of Lemma 2.3 of [19]. (4) follows from (2) and the (obvious) inclusions \(B_\delta(v) \subset D_\delta(v) \subset \overline{B_\delta(v)}\). \(\square\)

We shall accept here the terminology of [19, §2]. We only recall from there that \(v\) is called:

1. perfect, if \((x, y) \in S(f)\) \(\Rightarrow (D(x, v) \cap D(y, -v))\)
2. almost good, if \((x, y) \in S(f)\) and \(\text{ind} x \leq \text{ind} y \Rightarrow (D(x, v) \cap D(y, -v))\).

We say that a \(f\)-gradient is \(\delta\)-separated if there is an ordered Morse function \(\phi : W \to [a, b]\), adjusted to \((f, v)\), with an ordering sequence \(a_0, \ldots, a_{n+1}\), such that

1. for every \(p \in S(f)\) we have \(D_\delta(v) \subset \phi^{-1}(\{a_{n-1}\})\), where \(h = \text{ind} \phi\).
Lemma 1.5. If \( v \) is \( \delta \)-separated for some \( \delta > 0 \). □

Proposition 1.6. If \( v \) is \( \delta \)-separated, then \( \forall \delta \in [0, \delta_0] \) and \( \forall s : 0 \leq s \leq n \)

(1) \( D(\text{ind} \leq s ; v) \) is compact ;
(2) \( K(\text{ind} \leq s ; v) \) is compact ;
(3) \( \bigcap_{\theta > 0} B_\theta(\text{ind} \leq s ; v) = K(\text{ind} \leq s ; v) \);
(4) \( \bigcap_{\theta \geq \delta} B_\theta(\text{ind} \leq s ; v) = D(\text{ind} \leq s ; v) \);
(5) \( B(\text{ind} \leq s ; v) = D(\text{ind} \leq s ; v) \).

For two regular values \( \lambda, \mu \) of \( f \) with \( \lambda < \mu \) we denote by \( K_\mu^+ \) the set \( f^{-1}(\mu) \cap D(\text{ind} \leq s ; v) \) by \( K^-_\lambda \) the set \( f^{-1}(\lambda) \cap D(\text{ind} \leq s ; v) \) and by \( v_{[\mu, \lambda]}^{\sim} \) the \( C^\infty \) diffeomorphism \( f^{-1}(\mu) \setminus K^+_\mu \to f^{-1}(\lambda) \setminus K^-_\lambda \), which associates to each point \( x \) the point of intersection of \( \gamma(x, ;; -v) \) with \( f^{-1}(\lambda) \). For \( X \subset f^{-1}(\mu) \) we denote (by abuse of notation) \( v_{[\mu, \lambda]}^{\sim}(X \setminus K^+_\mu) \) by \( v_{[\mu, \lambda]}^{\sim}(X) \). If the values of \( \mu \) and \( \lambda \) are clear from the context we suppress them in the notation.

D. \( C^0 \)-stability properties. In this subsection \( f : W \to [a, b] \) is a Morse function on a compact riemannian cobordism \( W \), \( f^{-1}(a) = V_0, f^{-1}(b) = V_1, v \) is an \( f \)-gradient.

Proposition 1.7. Let \( \delta > 0 \). Let \( K \) be a compact in \( V_0 \cap B_\delta(v), R_1 \) be an open neighborhood of \( D_\delta(v) \cap V_0, R_2 \) be an open neighborhood of \( K(v) \cap V_0 \). Then there is \( \epsilon > 0 \) such that for every \( f \)-gradient \( w \) with \( \|w - v\| < \epsilon \) we have:

(1) \( K \subset B_\delta(w) \),
(2) \( D(\text{ind} \leq s ; v) \cap V_0 \subset R_1 \),
(3) \( K(w) \cap V_0 \subset R_2 \).

Proof. (1) is proved by a compactness argument similar to the proof of 5.6. To prove (2) note that \( V_0 \setminus R_1 \) is a compact such that each \( (-v) \)-trajectory starting at a point of \( V_0 \setminus R_1 \) reaches the boundary and that \( \tau(V_0 \setminus R_1, -v) \subset W \setminus \cup_{p \in S(f)} D_\delta(p) \).

Next lemma is obvious.

Lemma 1.8. Let \( g : W \to [a, b] \) be a Morse function, adjusted to \((f, v)\). Then there is \( \epsilon > 0 \) such that every \( f \)-gradient \( w \) with \( \|w - v\| < \epsilon \) is a \( g \)-gradient. □

Corollary 1.9. If \( v \) is almost good, resp. \( \delta \)-separated, then there is \( \epsilon > 0 \) such that every \( f \)-gradient \( w \) with \( \|w - v\| < \epsilon \) is almost good, resp. \( \delta \)-separated. □

Proposition 1.10. Let \( p \in S(f) \). Let \( U \) be an open set in \( V_0 \). Assume that for every \( x \in D(p, v) \) the trajectory \( \gamma(x, ;; -v) \) reaches the boundary and that \( D(p, v) \cap V_0 \subset U \). Then there is \( \epsilon > 0 \) such that for every \( f \)-gradient \( w \) with \( \|w - v\| < \epsilon \) and
of the form $(x, \gamma)$. Lemma 1.11. If $\gamma$ spends in a region $\gamma(x, \gamma)$ then we must compute, which will be left to the reader. \[ \triangle \]

E. Two lemmas on standard gradients in $\mathbb{R}^n$. During this subsection we refer to $\mathbb{R}^n$ as to the product $\mathbb{R}^k \times \mathbb{R}^{n-k}$; a point $x \in \mathbb{R}^k$ is therefore denoted by $(x, y)$, where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k}$. The vector field with components $(x, -y)$ is denoted by $v_0$; the standard euclidean norm in $\mathbb{R}^n$ will be denoted by $| \cdot |$. The Morse function $(x, y) \mapsto -|x|^2 - |y|^2$ will be denoted by $f_0$. (Then $-2v_0$ is the riemannian gradient of $f_0$ with respect to the euclidean metric.) The $v_0$-trajectories are all of the form $(x_0 e^t, y_0 e^{-t})$. Using this fact, the following two lemmas become the matter of computation, which will be left to the reader.

Lemma 1.11. Let $R > r > 0$ and $\gamma$ be a $v_0$-trajectory. Then the time, which $\gamma$ spends in $B^n(0, R) \setminus B^n(0, r)$ is not more than $\ln \left( \frac{R}{r} \right)^2 + \sqrt{\left( \frac{R}{r} \right)^4 - 1} \right)$ and the length of the corresponding part of $\gamma$ is not more than $2R$. \[ \square \]

Lemma 1.12. Let $r > 0$ and $\gamma$ be a $v_0$-trajectory. Then the time, which $\gamma$ spends in the set $f_0^{-1}([-r^2, r^2]) \setminus B^n(0, r)$ is not more than $2$. \[ \square \]

Corollary 1.13. Let $U = (f, v, U)$ be an $\mathcal{M}$-flow on a closed manifold $M$. Assume, that $U$ is $(C, \beta, \alpha)$-quick. Then $U$ is $(C, \beta + 8N(U), \alpha/2)$-quick.

Proof. By 1.11 the time, which a $v$-trajectory can spend in $\cup_{p \in S(f)} (U_p(\alpha) \setminus U_p(\alpha'/2))$ is not more than $N(U) \ln(2^2 + 2^4 - 1) \leq 8N(U)$. \[ \square \]

Remark 1.14. Corollary 1.13 is true as it stands for Morse functions on cobordisms. \[ \triangle \]

F. A-construction. Let $U = (f, v, U)$ be an $\mathcal{M}$-flow on a closed manifold $M$, where $U = \{ \Phi_p : U_p \to B^n(0, r_p) \}_{p \in S(f)}$ and $n = \dim M$. Let $\mu = \min_{p \in S(f)} r_p$. Denote $U_p(\mu)$ by $U'_p$. Let $\Gamma \geq 1$. We shall construct an $\mathcal{M}$-flow $U' = (f, w, U')$, subordinate to $U$ where $w = \phi \cdot v$, and $\phi : M \to \mathbb{R}^+$, $\phi(x) = 1$ for $x \in \cup_{p \in S(f)} U'_p$ and $\phi(x) = \Gamma$ for $x \notin \cup_{p \in S(f)} U'_p$. The main property of the gradient $w$ is an explicit estimate of the time, which a $w$-trajectory can spend inside each $U_p \setminus U'_p$. This construction will be used in the subsection G. Since we have set $w(x) = \Gamma \cdot v(x)$ for $x \notin \cup_{p \in S(f)} U_p$, we have only to define $w(x)$ inside each $U_p$ for $p \in S(f)$, which will be done in 1.15. Denote $\|v\|$ by $B$, and $G(U)$ by $D$.

Construction 1.15. In 1.15 we deal only with a neighborhood of one critical point (say, $p$), so we drop the index $p$ from the notation. The index of this critical point is denoted by $k$, the standard chart is denoted by $\Phi : U \to B^n(0, r)$ and we are given $\mu > 0$ with $0 < \mu < r$. Denote the function $x \mapsto \ln(x^2 + \sqrt{x^4 - 1})$ by $LN(x)$. Choose $\delta \in (0, \frac{r-\mu}{2}]$ so small that

$$
(1.1) \quad LN(r/(r-\delta)) < \min\left(\frac{Dr}{2B}, 1/2\right), \quad LN((\mu+\delta)/\mu) < \min\left(\frac{Dr}{2B}, 1/2\right)
$$

Let $\theta : \mathbb{R} \to [0, 1]$ be a $C^\infty$ function such that $\text{supp} \theta \subset [\mu, r]$ and $\theta(x) = 1$ for $x \in [\mu + \delta, r - \delta]$. Let $\lambda : [0, r] \to \mathbb{R}^+$ be the $C^\infty$ function, defined by the following
formulas:

\[
\lambda(t) = (1 - \theta(t)) + \theta(t) \frac{B}{Dt} \quad \text{for} \quad t \in [0, \mu + \delta]; \\
\lambda(t) = \frac{B}{Dt} \quad \text{for} \quad t \in [\mu + \delta, r - \delta] \\
\lambda(t) = \Gamma(1 - \theta(t)) + \theta(t) \frac{B}{Dt} \quad \text{for} \quad t \in [r - \delta, r]
\]

Note that \(\lambda(t) = 1\) for \(t \in [0, \mu]\), \(\lambda(t) = \Gamma\) nearby \(r\). Define a vector field \(v_1\) in \(B^n(0, r)\) by \(v_1(x) = \lambda(|x|)v_0(x)\), where \(v_0(x) = (-x_1, ..., -x_k, x_{k+1}, ..., x_n)\). Note that \((\Phi_s^{-1})(v_1)\) equals \(\Gamma v\) in \(\Phi^{-1}(B^n(0, r) \setminus B^n(0, r - \nu))\) for some small \(\nu > 0\). \(\triangle\)

Applied to the neighborhood \(U_p\) of each critical point \(p\) this construction extends \(w\) to the whole of \(M\). Denote by \(\Phi_p\) the restriction of \(\Phi\) to \(U_p\) and set \(U' = \{\Phi_p\}_{p \in S(f)}\). It is obvious from the construction that \(V' = (f, w, U')\) is an \(\mathcal{M}\)-flow on \(M\), subordinate to \(V\). We shall denote it by \((A)(V, \Gamma, \mu)\).

**Lemma 1.16.** (1) \(w(x) = \Gamma v(x)\) for \(x \notin \cup_{p \in S(f)} U_p\) and \(\|w\|_\rho \leq \Gamma \|v\|_\rho\);  
(2) For any \(w\)-trajectory \(\gamma\) and every \(p \in S(f)\) the time, which \(\gamma\) spends inside \(U_p \setminus U'_p\) is not more, than \((3\Gamma(V) r_p) / \|v\|_\rho\).

**Proof.** The first part of (1) is obvious from the construction. The estimate of \(\|w\|_\rho\) is obtained by an explicit computation in the standard coordinate systems which we leave to the reader. To prove (2) note that the time which a \(v_1\)-trajectory can spend in \(B^n(0, r) \setminus B^n(0, r - \delta)\) or in \(B^n(0, \mu + \delta) \setminus B^n(0, \mu)\) is not more than \(\frac{Dr}{2B}\), which follows directly from the definition of \(v_1\), Lemma 1.11 and (1.1). Further, the euclidean length of the part of the \((v_1)\)-trajectory in \((B^n(0, r - \delta) \setminus B^n(0, \mu + \delta))\) is not more than \(2r\) (by Lemma 1.11), and the euclidean norm of the tangent vector is equal to \(B/D\). Therefore the time spent in \(B^n(0, r - \delta) \setminus B^n(0, \mu + \delta)\) is not more than \((2Dr)/B\), and the total time spent in \(B^n(0, r)\) is not more than \((3Dr)/B\). \(\square\)

**G. Statement of the theorem on the existence of quick flows.**

**Theorem 1.17.** Let \(M\) be a closed riemannian manifold, \(\dim M = n\). Let \(C > 0, \beta > 0, A > 1\). Then there is an almost good \(\mathcal{M}\)-flow \(V = (f, v, U)\) of radius \(r\), and for every \(\mu \in [0, r]\) there is an \(\mathcal{M}\)-flow \(W = (f, w, U')\), subordinate to \(V\), and such that

(1) \(W\) is of radius \(\mu\) and is \((C, \beta)\)-quick;  
(2) \(N(W) \leq n2^n\);  
(3) \(\mathcal{G}(W) \leq A\).

This theorem will be proved by induction in \(n\) with the help of the theorem 1.18.

**Theorem 1.18.** Let \(M\) be a closed riemannian manifold, \(\dim M = n\). Let \(B > 0, \mu_0 > 0\). Then there is an almost good \(\mathcal{M}\)-flow \(V = (f, v, U)\) on \(M\) with the following properties:

(1) \(\mathcal{G}(V) \leq 2\) and \(D(V) < \mu_0\);  
(2) \(N(V) \leq n2^n\);  
(3) \(V\) is \((B, R(n))\)-quick, where \(R(n) = 100 + n2^{n+7}\).

The following corollary of 1.17 will be used in §4.

**Corollary 1.19.** Let \(M\) be a closed riemannian manifold, \(\dim M = n\). Let \(C > 0, \beta > 0\). Then there is an almost good \(\mathcal{M}\)-flow \(V = (f, v, U)\) on \(M\), such that for every \(\delta > 0\) sufficiently small there is an \(\mathcal{M}\)-flow \(W = (f, w, U')\), subordinate to \(V\) and such that

(1) \(\|w\| \leq C\) and \(U'\) is of radius \(\delta/2\);
(2) For every $t \geq \beta$ and every $s : 0 \leq s \leq n$ we have

$$\Phi(-w, t)(M \setminus B_\delta(\text{ind} \leq n-s-1 ; -v)) \subset B_\delta(\text{ind} \leq s ; v)$$

$$\Phi(w, t)(M \setminus B_\delta(\text{ind} \leq n-s-1 ; v)) \subset B_\delta(\text{ind} \leq s ; -v)$$

Proof. Let $V = (f, v, U)$ be the $\mathcal{M}$-flow, satisfying 1.17 with respect to $C$, $\beta$, and $A = 2$. Let $\delta > 0$ be so small that $\forall \rho \in S(f)$ the disc $\rho(p, x) \leq \delta$ belongs to $U_p$. Since $A = 2$, that implies, that this disc contains $U_p(\delta/2)$. Let $W = (f, w, U')$ be the $\mathcal{M}$-flow, satisfying (1)-(3) of 1.17 with respect to $\delta/2$. I claim, that $W$ satisfies our conclusion. Indeed, 1) goes by construction. To prove 2), let $y \in M \setminus B_\delta(\text{ind} \leq n-s-1 ; v)$. Since $W$ is $(C, \beta)$-quick, there is $t_0 \in [0, \beta]$, such that $\gamma(y, t_0; -w) \in U_p(\delta/2)$ for some $p \in S(f)$. Since $M \setminus B_\delta(\text{ind} \leq n-s-1 ; -v)$ is $v$- and $w$-invariant, we have $\text{ind} p \leq s$. Since $B_\delta(\text{ind} \leq s ; v)$ is $(-v)$- and $(-w)$-invariant, we have $\gamma(y, \lambda; -w) \in B_\delta(\text{ind} \leq s ; v)$ for all $\lambda \geq t_0$. Proof of (1.3) is similar. □

The scheme of the proof of 1.17 and 1.18 is as follows. The proof of 1.18 $\Rightarrow$ 1.17 is given below. The proof of 1.17(n) $\Rightarrow$ 1.18(n+1) is done in §3 with the help of what we call (S)-construction. This proof also gives the proof of 1.18(1).

Proof of 1.18 $\Rightarrow$ 1.17. We are given $C$, $\beta$ and $A$. Choose $C'$ such that $C' \leq C$ and that $R(n)\frac{C'}{C'} \leq \beta/2$. Then choose $\mu_0 > 0$ such that $6n2^n\frac{C'}{C'} < \beta/2$. Let $V = (f, v, U)$ be an $\mathcal{M}$-flow on $M$, which satisfies the conclusions of 1.18 with respect to $\mu_0$ and $C'$. Here $U = \{\Phi_p : U_p \rightarrow B^n(0, r_p)\}_{p \in S(f)}$. Choose $r > 0$ so small that $r < \min_p r_p$ and $G(p, r) \leq A$ for every $p \in S(f)$. Denote by $U' = \{\Phi'_p : U'_p \rightarrow B^n(0, r)\}_{p \in S(f)}$ the corresponding restriction of $U$. We claim that the $\mathcal{M}$-flow $V' = (f, v, U')$ satisfies the conclusions of 1.17. Indeed, let $0 < \mu < r$. Apply the $A$-construction to $V'$, $\Gamma = C/C'$ and to $\mu$. Denote the resulting flow $(A,V', \Gamma, \mu)$ by $W = (f, w, U'')$. We claim that this flow satisfies (1)-(3) of 1.17. Indeed, $W$ is of radius $\mu$ and $G(W) \leq A$ by definition. Since $W$ is subordinate to $V$ we have $N(W) \leq N(V)$. By 1.16 we have $\|w\| \leq \|v\| = C$. Further, let $\gamma$ be a $w$-trajectory. The time which $\gamma$ can spend in $\cup_{p \in S(f)}(U_p(r_p) \setminus U_p(\mu))$ is not more than $6 \cdot N(V') \cdot r_p/C' \leq 6 \cdot 2^n \cdot n\mu_0/C' \leq \beta/2$ and the time which $\gamma$ can spend in $M \setminus \cup_{p \in S(f)}(U_p(r_p))$ is not more than $R(n) \cdot C'/C$ (since in this set we have $w = C/C' \cdot v$), which does not exceed $\beta/2$. □

§2. Transversality notions

In this section we study the families of submanifolds of certain type. The main example of such families is provided by the union of descending discs of an almost good gradient of a Morse function.

A. Stratified submanifolds. Let $A = \{A_0, ..., A_k\}$ be a finite sequence of subsets of a topological space $X$. For $0 \leq s \leq k$ we denote $A_s$ also by $A_s$. For $0 \leq s \leq k$ we denote $A_0 \cup ... \cup A_s$ by $A_{\leq s}$ and also by $A_s\langle s \rangle$. We say that $A$ is a compact family if $A_s\langle s \rangle$ is compact for every $s : 0 \leq s \leq k$.

Definition 2.1. Let $M$ be a manifold without boundary. A finite sequence $X = \{X_0, ..., X_k\}$ of subsets of $M$ is called $s$-submanifold of $M$ ($s$ for stratified) if

(1) $X$ is disjoint and each $X_i$ is a submanifold of $M$ of dimension $i$ with the trivial normal bundle.*

---

*This restriction is technical; it makes proofs easier.
(2) $X$ is a compact family. △

For an $s$-submanifold $X = \{X_0, ..., X_k\}$ we denote $k$ by $\dim X$. For a diffeomorphism $\Phi : M \to N$ and an $s$-submanifold $X$ of $M$ we denote by $\Phi(X)$ the $s$-submanifold of $N$ defined by $\Phi(X(i)) = \Phi(X(i))$.

If $V$ is a submanifold of $M$ and $X$ is an $s$-submanifold of $M$, then we say that $V$ is transversal to $X$ (notation: $V \pitchfork X$) if $V \pitchfork X(i)$ for each $i$. If $V$ is a compact submanifold of $M$, transversal to an $s$-submanifold $X$, then the family $\{X(i) \cap V\}$ is an $s$-submanifold of $V$ (and of $M$) which will be denoted by $X \pitchfork V$.

If $X, Y$ are two $s$-submanifolds of $M$, we say, that $X$ is transversal to $Y$ (notation: $X \pitchfork Y$) if $X(i) \pitchfork Y(j)$ for every $i, j$; we say that $X$ is almost transversal to $Y$ (notation: $X \downarrow Y$) if $X(i) \pitchfork Y(j)$ for $i + j < \dim M$.

Remark 2.2. $X \pitchfork Y$ if and only if $X(\leq i) \cap Y(\leq j) = \emptyset$ whenever $i + j < \dim M$. △

Lemma 2.3. Let $f : M \to \mathbb{R}$ be a Morse function, where $M$ is a closed manifold, $v$ an almost good $f$-gradient. Denote $\dim M$ by $n$. Then the family $\{K(\text{ind} = i ; v)\}_{0 \leq i \leq n}$ is an $s$-submanifold transversal to $f^{-1}(\lambda)$ for every regular value $\lambda$ of $f$.

Proof. The set $\{K(\text{ind} \leq i ; v)\}$ is compact (by Lemma 1.6) and the normal bundle to $D(p, v)$ is obviously trivial. □

The $s$-submanifold $\{K(\text{ind} = i ; v)\}_{0 \leq i \leq n}$ will be denoted by $\mathbb{K}(v)$, the $s$-submanifold $\mathbb{K}(v) \cap f^{-1}(\lambda)$ by $\mathbb{K}_{\lambda}(v)$.

For a manifold $M$ without boundary we denote by $\text{Vectt}(M)$ the subspace of $\text{Vect}^\infty(M \times [0, 1])$, consisting of all the $C^\infty$ vector fields which have the second coordinate zero and which vanish with all the partial derivatives in $M \times \{0, 1\}$. Assume that $M$ is compact. Then there is a natural topology on the set $\text{Vect}^\infty(M \times [0, 1])$ (see, e.g. [19, §8]), with respect to which $\text{Vectt}(M)$ is a closed subset. Further, for every $v \in \text{Vectt}(M)$ and $t \in [0, 1]$ the map $x \mapsto \gamma(x, t; v)$ is a $C^\infty$ diffeomorphism of $M$, denoted by $\Phi(v, t)$.

The following transversality result is proved by induction in $\dim X + \dim Y$ with the help of standard general position argument like [6, Lem.5.3]. We omit the proof and just indicate that the openness of the set in consideration follows from Remark 2.2.

Theorem 2.4. Let $X, Y$ be $s$-submanifolds of a closed manifold $M$. Then the set of $v \in \text{Vectt}(M)$, such that $\Phi(v, 1)(X) \pitchfork Y$, is open and dense in $\text{Vectt}(M)$. □

B. Good fundamental systems of neighborhoods and $ts$-submanifolds.

Definition 2.5. Let $X$ be a topological space, $A \subset X$ a closed subset, $I$ an open interval $]0, \delta_0[$. A good fundamental system of neighborhoods of $A$ (abbreviation: $gfn$-system for $A$) is a family $\{A(\delta)\}_{\delta \in I}$ of open subsets of $X$, satisfying the following conditions:

(fs1) for each $\delta \in I$ we have $A \subset A(\delta)$ and $\delta_1 < \delta_2 \Rightarrow A(\delta_1) \subset A(\delta_2)$,
(fs2) for each $\delta \in I$ we have $\overline{A(\delta)} = \bigcap_{\theta > \delta} A(\theta)$,
(fs3) $A = \bigcap_{\theta > 0} A(\theta)$. △
Lemma 2.6. Assume that $X$ is compact. Let $\{A(\delta)\}_{\delta \in I}$ be a $gfn$-system for $A$. Then:

1. The family $\{A(\theta)\}_{\theta > 0}$ is a fundamental system of neighborhoods of $A$.
2. $\forall \delta \in I$ the family $\{A(\theta)\}_{\theta > \delta}$ is a fundamental system of neighborhoods of $A(\delta)$.

Proof. (1) Let $U$ be an open neighborhood of $A$. The sets $\{X \setminus A(\theta)\}_{\theta > 0}$ form an open covering of the compact $X \setminus U$. There is a finite subcovering, therefore there is $\theta > 0$, such that $\overline{A(\theta)} \subset U$. (2) is proved similarly. \qed

Definition 2.7. Let $X$ be a topological space, $\mathbb{A} = \{A_0, ..., A_k\}$ be a compact family of subsets of $X$, $I$ an open interval $]0, \delta_0[$. A good fundamental system of neighborhoods of $\mathbb{A}$ (abbreviation: $gfn$-system for $\mathbb{A}$) is a family $\{A_s(\delta)\}_{\delta \in I, 0 \leq s \leq k}$ of open subsets of $X$, satisfying the following conditions:

(FS1) For every $s : 0 \leq s \leq k$ and every $\delta \in I$ we have $A_s \subset A_s(\delta)$
(FS2) For every $s : 0 \leq s \leq k$ we have $\delta_1 < \delta_2 \Rightarrow A_s(\delta_1) \subset A_s(\delta_2)$
(FS3) For every $\delta \in I$ and every $j$ with $0 \leq j \leq k$ we have $A_{\leq j}(\delta) = \bigcap_{\theta > \delta}(A_{\leq j}(\theta))$
(FS4) For every $j$ with $0 \leq j \leq k$ we have $A_{\leq j} = \bigcap_{\theta > 0}(A_{\leq i}(\theta))$.

$I$ is called interval of definition of the system. \triangle

The following lemma follow from 2.6.

Lemma 2.8. Assume that $X$ is compact, and let $\{A_s(\delta)\}_{\delta \in I, 0 \leq s \leq k}$ be a $gfn$-system for a compact family $\mathbb{A}$. Then for every $s : 0 \leq s \leq k$

1. $\forall \delta \in I$ the family $\{A_{\leq s}(\theta)\}_{\theta > 0}$ is a fundamental system of neighborhoods of $\overline{A_{\leq s}(\delta)}$.
2. the family $\{A_{\leq s}(\theta)\}_{\theta > 0}$ is a fundamental system of neighborhoods of $A_{\leq s}$.

Let $I = ]0, \delta_0[$ and $Z = \{Z(\delta)\}_{\delta \in I}$ be a family of subsets of some space $X$. Let $0 < \epsilon_0 < \delta_0$. The family $\{Z(\delta)\}_{\delta \in ]0, \epsilon_0[}$ will be called restriction of $Z$. We adopt the same terminology also for good fundamental systems of neighborhoods of compact families.

The basic example of a $gfn$-system is given by the next lemma, which follows immediately from Proposition 1.6.

Lemma 2.9. Let $M$ be a closed manifold or a compact cobordism. Let $f : M \to \mathbb{R}$ be a Morse function and $v$ be an almost good $f$-gradient. Then for some $\epsilon > 0$ the family $\{B_\delta(\text{ind}=s ; v)\}_{\delta \in ]0, \epsilon[, 0 \leq s \leq n}$ is a $gfn$-system for $\mathbb{K}(v)$. \qed

The $gfn$-system, introduced above, will be denoted by $\mathbb{K}(v)$. Note that there is no canonical choice of the interval of definition for this system. We shall say that $\lambda > 0$ is in the interval of definition of $\mathbb{K}(v)$ if there is $\epsilon > \lambda$ such that $\{B_\delta(\text{ind}=s ; v)\}_{\delta \in ]0, \epsilon[, 0 \leq s \leq n}$ is a $gfn$-system. 1.6 implies that if $v$ is $\lambda$-separated, then $\lambda$ is in the interval of definition of $\mathbb{K}(v)$.

Let $M$ be a manifold without boundary, $X$ be an $s$-submanifold of $M$. A good fundamental system of neighborhoods of $X$ will be called thickened stratified submanifold with the core $X$ (abbreviation: $ts$-submanifold). For a $ts$-submanifold $X = \{X_s(\delta)\}_{\delta \in I, 0 \leq s \leq k}$ we shall denote $X_i(\delta)$ by $X_{(i)}(\delta)$, and $X_{\leq i}(\delta)$ by $X_{(\leq i)}(\delta)$.

Lemma 2.9 implies that if $f : M \to \mathbb{R}$ is a Morse function on a closed manifold and $v$ is an almost good $f$-gradient, then $\mathbb{K}(v)$ is a $ts$ manifold with the core $\mathbb{K}(v)$. 
C. Tracks of subsets, $s$-submanifolds and $ts$-submanifolds. Let $f : W \to [a, b]$ be a Morse function on a compact riemannian cobordism $W$, $f^{-1}(b) = V_1$, $f^{-1}(a) = V_0$, $v$ be an $f$-gradient.

Let $X \subset V_1$. The set $\{\gamma(x, t; -v) \mid t \geq 0, x \in X\}$ will be called track of $X$ (with respect to $v$) and denoted by $T(X, v)$.

**Lemma 2.10.** (1) If $X$ is compact, then $T(X, v) \cup K(v)$ is compact.

(2) If $X$ is compact, and every ($-v$)-trajectory starting from a point of $X$ reaches $V_0$ then $T(X, v)$ is compact.

(3) For any $X$ we have $\overline{T(X, v) \cup K(v)} = T(\overline{X}, v) \cup K(v)$.

(4) For any $X$ and $\delta > 0$ we have $\overline{T(X, v) \cup B_\delta(v)} = T(\overline{X}, v) \cup D_\delta(v)$.

**Proof.** (1) The set $Y = W \setminus (T(X, v) \cup K(v))$ consists of such points $y \in W$ that $\gamma(y, t; v)$ reaches $V_1$ and meets it at a point which is not in $X$. Then 5.4 implies that $Y$ is open.

(2) By an easy compactness argument there is $\delta > 0$ such that $T(X, v) \cap B_\delta(p) = \emptyset$ for every $p \in S(f)$. Therefore $T(X, v) \cap B_\delta(v) = \emptyset$ which together with (1) implies (2).

(3) We have obviously: $T(X, v) \cup K(v) \subset T(\overline{X}, v) \cup K(v) \subset T(\overline{X}, v) \cup K(\overline{v})$, which implies (3) in view of (1). (4) is proved similarly. □

Note that if $X$ is a submanifold of $V_1$ of dimension $k$, then $T(X, v) \cap W^c$ is a submanifold of $W^c$ of dimension $k + 1$. If $\lambda$ is a regular value of $f$, then $T(X, v) \cap f^{-1}(\lambda)$ is a submanifold of dimension $k$ of $f^{-1}(\lambda)$.

From here to the end of the section we assume that $v$ is an almost good $f$-gradient. $A$ stands for an $s$-submanifold $\{A_0, ..., A_k\}$ of $V_1$, such that $A \uparrow \mathbb{K}_b(-v)$.

**Definition 2.11.** For $0 \leq i \leq k + 1$ denote by $TA_i(v)$ the set $T(A_{i-1}, v) \cup \mathbb{K}^{\text{ind}=i} ; v$ (where we set $A_{-1} = \emptyset$). Denote by $\mathbb{T}(A, v)$ the family $\{TA_i(v)\}_{0 \leq i \leq k+1}$ of subsets of $W$ and by $v_{[b, \lambda]}(A)$ the family $\{T(A_i(v)) \cap f^{-1}(\lambda)\}_{0 \leq i \leq k}$ of subsets of $f^{-1}(\lambda)$. If the values $b, \lambda$ are clear from the context we shall abbreviate $v_{[b, \lambda]}(A)$ to $v\hat{}(A)$. The family $\mathbb{T}(A, v)$ will be called track of $A$, and the family $v_{[b, \lambda]}(A)$ will be called $\hat{v}$-image of $A$. □

**Lemma 2.12.** (1) $\mathbb{T}(A, v)$ and $v\hat{}(A)$ are compact families.

(2) If $\lambda$ is a regular value of $f$, then $v_{[b, \lambda]}(A)$ is an $s$-submanifold of $f^{-1}(\lambda)$.

**Proof.** (1) Let $0 \leq s \leq k + 1$. Denote $T(A_{s-1}, v) \cup \mathbb{K}^{\text{ind}=s} ; v$ by $Y(s)$. Let $\phi : W \to [a, b]$ be an ordered Morse function, adjusted to $(f, v)$. Let $\lambda$ be a regular value of $\phi$, such that all the critical points of $\phi$ of indices $s$ are above $\lambda$ and all the critical points of $\phi$ of indices $\leq s$ are below $\lambda$. Since $A \uparrow (\mathbb{K}(-v) \cap V_1)$, every ($-v$)-trajectory, starting at a point of $A_{s-1}$ reaches $\phi^{-1}(\lambda)$. Then $Y(s) \cap \phi^{-1}([a, \lambda])$ is compact by 2.10(2) and $Y(s) \cap \phi^{-1}([a, \lambda])$ is compact by 2.10(1), therefore $Y(s)$ is compact. (2) is obvious. □

We shall now define the notion of the track of a $ts$-submanifold.

**Definition 2.13.** Let $A = \{A_s(\delta)\}_{\delta \in [0, \delta_0]} : 0 \leq s \leq k$ be a $ts$-submanifold of $V_1$ with the core $A$. Assume that $\phi$ is $\delta_1$-separated (where $\delta_1 > 0$). For $0 < \delta < \min(\delta_0, \delta_1)$ and $0 \leq s \leq k + 1$ set $TA_s(\delta, v) = T(A_{s-1}(\delta), v) \cup B_{\delta(\text{ind}=s)} ; v$ (where we set by definition $A_{-1}(\delta) = \emptyset$).
We shall prove that some restriction of \{TA_s(δ, v)\} is a \textit{gfn}-system for \(T(A, v)\).

Up to the end of this section \(Q(s, δ)\) stands for \(T(A_{≤s−1}(δ), v) ∪ D_δ(\text{ind}_{≤s}; v)\).

**Lemma 2.14.** Let \(0 < ε ≤ \min(δ_0, δ_1)\). Then (i) \(⇒\) (ii).

(i) For every \(0 < δ < ε\) and every \(s : 0 ≤ s ≤ k\) the set \(Q(s, δ)\) is compact
(ii) \(TA_s(δ, v)_{0 < δ < ε, 0 ≤ s ≤ k+1}\) is a \textit{gfn}-system for \(T(A, v)\)

**Proof.** It is easy to see that for every \(δ ∈ [0, \min(δ_0, δ_1)]\) we have:

\[
Q(s, δ) = \bigcap_{θ > δ} TA_{≤s}(θ, v), \quad TA_{≤s}(v) = \bigcap_{θ > 0} TA_{≤s}(θ, v)
\]

Now to prove (ii) \(⇒\) (i) note that (FS3) together with (2.1) implies \(Q(s, δ) = TA_{≤s}(δ, v)\).

To prove (i) \(⇒\) (ii) note that (FS1), (FS2), and (FS4) hold for every \(δ ∈ [0, \min(δ_0, δ_1)]\), and (i) implies (FS3) in view of 2.2. □

**Proposition 2.15.** There is \(ε ∈ [0, \min(δ_0, δ_1)]\) such that \(\{TA_s(δ, v)\}_{δ ∈ [0, ε]}\) is a \textit{gfn}-system for \(T(A, v)\).

**Proof.** Let \(φ : W → [a, b]\) be the Morse function with respect to which \(v\) is \(δ_1\)-separated, and let \(μ\) be a regular value of \(φ\) such that for every critical point \(p\) of \(φ\) of indices \(≤ s\) (resp. \(> s\)) the disc \(D_p(δ_1)\) is in \(φ^{-1}([a, μ])\) (resp. in \(φ^{-1}([μ, b])\)). Since \(A \uparrow \mathbb{K}_b(−v)\), every \((−v)\)-trajectory starting at a point of \(A_{≤s−1}\) reaches \(φ^{-1}(μ)\).

\(A_{≤s−1}\) is compact and \(\{A_{≤s−1}(θ)\}_{θ > 0}\) is a \textit{gfn}-system for \(A_{≤s−1};\) therefore an easy compactness argument based on 5.4 shows that there is \(θ > 0\) such that each \((−v)\)-trajectory starting at a point of \(A_{≤s−1}(θ)\) reaches \(φ^{-1}(μ)\).

We claim that \(ε = \min(δ_0, δ_1, θ)\) satisfies the conclusions of our proposition. Indeed, let \(δ < ε\). Then each \((−v)\)-trajectory starting at a point of \(A_{≤s−1}(δ)\) reaches \(φ^{-1}(μ)\), and \(T(A_{≤s−1}(δ), v)\) is compact by 2.10(2). Denote \(v_{[μ, b]}(A_{≤s−1}(δ))\) by \(C(δ)\); it is a compact set. The intersection of the set \(Q(s, δ)\) with \(φ^{-1}([μ, b])\) is compact by the above, and its intersection with \(φ^{-1}([a, μ])\) is compact by 2.10(4), applied to the cobordism \(φ^{-1}([a, μ])\) and the compact \(C(δ) ⊂ φ^{-1}(μ)\). □

We shall denote the \textit{gfn}-system, introduced in 2.15, by \(T(A, v)\) and call it \textit{track} of \(A\). Note that there is no canonical choice of the interval of definition for this system. We shall say that \(λ > 0\) belongs to the interval of definition of \(T(A, v)\) if there is \(ε > λ\) such that \(\{TA_s(δ, v)\}_{δ ∈ [0, ε]}\) is a \textit{gfn}-system. The next two lemmas contain some properties of \(T(A, v)\).

**Lemma 2.16.** Assume that \(δ > 0\) is in the interval of definition of \(T(A, v)\). Then for \(λ = a, b\) we have: \(\text{c}(A)_{(δ)} \cap f^{-1}(λ) = TA_{(δ)} \cap f^{-1}(λ)\).

**Proof.** Obvious. □

Let \(α, β\) be regular values of \(f\), such that \(a < α < β < b\) and there are no critical points of \(f\) in \(f^{-1}([α, β])\). Denote by \(\bar{f}\) the restrictions of \(f, v\) to \(\bar{W} = f^{-1}([α, β])\). Denote \(f^{-1}(α)\) by \(\bar{V}_a\), \(f^{-1}(β)\) by \(\bar{V}_b\). Let \(\bar{A} = \{\bar{A}_s(δ)\}_{δ ∈ [0, δ_0]}\) be a \(ts\)-submanifold of \(\bar{V}_1\) with the core \(\bar{A} = v_{[α, β]}(\bar{A})\), and assume that for every \(s : 0 ≤ s ≤ k\) and for every \(δ ∈ [0, δ_0]\) we have: \(v_{[α, β]}(A_s(δ)) \subset A_s(δ)\). Assume that \(\bar{A}\) is \(\bar{δ}\)-separated.
Lemma 2.17. Let $0 < \nu < \min(\delta_0, \delta_1)$ and assume that $\nu$ is in the interval of definition of $T(\widetilde{A}, \widetilde{v})$. Then $\nu$ is in the interval of definition of $T(A, v)$.

Proof. Choose $\theta$ in $[\lambda, \min(\delta_0, \delta_1)]$. Since $\nu$ is obviously $\delta_1$-separated, it is sufficient to prove that for every $\delta \in [0, \theta]$, $s : 0 \leq s \leq k$ the set $Q(s, \delta)$ is compact. The set $Q(s, \delta) \cap f^{-1}([\beta, b])$ is compact by 2.10(2). The set $Q(s, \delta) \cap f^{-1}([\alpha, \beta])$ equals to $T(v_{[b, \beta]}(A_{\leq s-1}(\delta)), \widetilde{v}) \cup D_s(\text{ind} \leq s ; \widetilde{v})$. If $Z' \subset Z$ are compacts in $V_1$ then $T(Z', \widetilde{v})$ is closed in $T(Z, \widetilde{v})$, therefore $Q(s, \delta) \cap f^{-1}([\alpha, \beta])$ is a closed subset of $T(A_{\leq s-1}(\delta), \widetilde{v}) \cup D_s(\text{ind} \leq s ; \widetilde{v})$, therefore it is compact. Finally, compactness of $Q(s, \delta) \cap f^{-1}([\alpha, \beta])$ follows from the compactness of $Q(s, \delta) \cap f^{-1}([\alpha, \beta])$ and from the absence of critical points of $f$ in $f^{-1}([\alpha, \beta])$. □

§3. S-Construction

In Subsection A we present a construction, which produces from a Morse function without critical points on a compact cobordism another one, having two series of critical points and behaving roughly as "skladka". This new function is equipped with an almost good gradient, and the main property of the construction is an explicit estimate of the quickness of the resulting flow. The properties of S-construction are listed in the theorem 3.1 below. We invite the reader first to have a look at the construction itself (proof of 3.1). In Subsection B we prove 1.17(n) ⇒ 1.18(n + 1) with the help of the S-construction.

Terminology: If $W$ is a compact cobordism, $v \in \text{Vect}^1(W, \bot)$ and $U \subset W$, we say that $U$ is $v$-invariant if for every $x \in U$ the trajectory $\gamma(x, \cdot ; v)$ is defined on $[0, \infty]$ and $\gamma(x, t ; v) \in U$ for all $t \geq 0$. We say that $U$ is weakly $v$-invariant if for every $x \in U$ we have: $\gamma(x, t ; v) \in U$ for all $t$ of the interval of definition of $\gamma(x, \cdot ; v)$.

A. S-construction. Let $W$ be a compact riemannian cobordism. During §3 we denote by $|\cdot|$ the norm on the tangent spaces, induced by the metric and by $\|\cdot\|$ the corresponding $C^0$ norm in the space of vector fields.

Theorem 3.1. Let $g : W \to [a, b]$ be a Morse function without critical points, $g^{-1}(a) = V_0, g^{-1}(b) = V_1$. Let $C > 0$ and let $w$ be a $g$-gradient, such that $\|w\| \leq C$. Denote $g^{-1}(\frac{2a+b}{3})$ by $V_{1/3}, g^{-1}(\frac{2b+a}{3})$ by $V_{2/3}, g^{-1}(\frac{a+2b}{3})$ by $V_{1/2}$. Denote $g^{-1}(\frac{2a+b}{3})$ by $W_0$, $g^{-1}(\frac{2b+a}{3})$ by $W_2$, $g^{-1}(\frac{a+2b}{3})$ by $W_1$. Denote by $\text{grad}(g)$ the riemannian gradient of $g$ and by $\text{grd}(g)$ the vector field $\text{grad}(g)/|\text{grad}(g)|$. Denote by $T$ the maximal length of the domain of $\text{grd}(g)$-trajectory. Then there is $\nu_0 > 0$ such that:

For every $s$-submanifold $X$ of $V_0$, $s$-submanifold $Y$ of $V_1$, every almost good $M$-flow $V_1 = (F_1, u_1, U_1)$ on $V_{1/3}$ and almost good $M$-flow $V_2 = (F_2, u_2, U_2)$ on $V_{2/3}$, and every $\mu \leq \min(\nu_0, d(V_1), d(V_2))$ there is an almost good $M$-flow $V = (F, u, U)$ on $W$, satisfying the following properties:

1. $F : W \to [a, b], V_0 = F^{-1}(a), V_1 = F^{-1}(b), S(F) = S(F_1) \cup S(F_2)$; in a neighborhood of $\partial W$ we have: $F = g, u = w$.
2. $\mathcal{V}$ is of radius $\mu$; $F(\mathcal{V}) \leq \frac{3}{2}\max(G(V_1), G(V_2))$.
3. $V_{1/3}, V_{2/3}, W_1$ are $(\pm u)$-invariant. $W_0$ is $u$-invariant and weakly $(\pm u)$-invariant. $W_2$ is $(\pm u)$-invariant and weakly $u$-invariant.
4. Assume that $V_1$ is $(C_1, \beta_1, \mu/2)$-quick and $V_2$ is $(C_2, \beta_2, \mu/2)$-quick. Then $V$ is $(3/2(C_1 + C_2 + C_3), \beta_1 + \beta_2 + 5 + 4T/C, \mu)$-quick.
(5) Let $\gamma$ be an $u$-trajectory. If $\text{Im } \gamma \subset W_1$, then $N(\gamma, \mu) \leq N(\Psi_1, \mu) + N(\Psi_2, \mu)$. If $\text{Im } \gamma \subset W_0$, then $N(\gamma, \mu) \leq N(\Psi_1, \mu)$. If $\text{Im } \gamma \subset W_2$, then $N(\gamma, \mu) \leq N(\Psi_2, \mu)$.

(6) $K_a(u) \not\!\subset X; \; K_b(-u) \not\!\subset Y$.

Proof. The proof occupy the rest of Subsection A. We shall assume that $a = 0, b = 1$, since the general case is easily reduced to this one by an affine transformation of $R$.

1. Function $f$, its gradient $v$, and the choice of $\mu_0$.

Lemma 3.2. There is a Morse function $f : W \rightarrow [0, 1]$ without critical points and an $f$-gradient $v$, such that: (1) $\|v\| \leq C$;

(2) for $x$ in a neighborhood of $V_0 \cup V_1$ we have: $f(x) = g(x)$ and $v(x) = w(x)$;

(3) $f^{-1}(1/3) = V_{1/3}$, $f^{-1}(2/3) = V_{2/3}$, $f^{-1}(1/2) = V_{1/2}$;

(4) for $x$ in a neighborhood of $V_{1/3} \cup V_{1/2} \cup V_{2/3}$ we have:

\[ |v(x)| = C \quad \text{and } df(v)(x) = C; \]

(5) for $\lambda = 1/3, 1/2, 2/3$ and $x \in g^{-1}(\lambda)$ we have: $v(x) \perp g^{-1}(\lambda)$;

(6) The maximal length of the domain of a $v$-trajectory is not more than $2T/C$.

Proof. Let $U$ be an open neighborhood of $V_0 \cup V_1$, such that $U \cap (V_{1/3} \cup V_{2/3} \cup V_{1/2}) = \emptyset$ and let $h : W \rightarrow [0, 1]$ be a $C^\infty$ function such that $\text{supp } h \subset U$ and for $x$ in a neighborhood of $\partial W$ we have $h(x) = 1$. Set $v(x) = h(x)w(x) + (1 - h(x))C\text{grd}(g)$.

It is obvious that $v$ is a $g$-gradient, satisfying 1) and 5). We have also $v(x) = w(x)$ nearby $\partial W$, as well as $\|v\| = C$ nearby $V_{1/3} \cup V_{2/3} \cup V_{1/2}$. 6) holds also, if only $U$ was chosen sufficiently small. Applying now Corollary 8.14 of [19] to the cobordisms $W_0$, $g^{-1}([1/3, 1/2]), g^{-1}([1/2, 2/3])$, $W_2$ and the restrictions of $g$ and $v$ to these cobordisms and gluing the results together we obtain a Morse function $f : W \rightarrow [0, 1]$, satisfying (together with $v$) all of our conclusions. \(\square\)

For $\lambda \in [0, 1]$ we denote $f^{-1}(\lambda)$ by $V_\lambda$. Fix some $\epsilon \in (0, 1/12]$. For $\nu > 0$ sufficiently small the map $(x, \tau) \mapsto \gamma(x, \tau; v/C)$ is defined on $V_i \times [-2\nu, 2\nu]$, where $i = 1/3, 1/2, 2/3$, on $V_0 \times [0, 2\nu]$ and on $V_1 \times [-2\nu, 0]$. The corresponding embeddings will be denoted by

\[ \Psi_0(\nu) : V_0 \times [0, 2\nu] \rightarrow W, \quad \Psi_{3/2}(\nu) : V_{1/2} \times [-2\nu, 2\nu] \rightarrow W, \quad \Psi_3(\nu) : V_1 \times [-2\nu, 0] \rightarrow W, \quad \Psi_1(\nu) : V_{1/3} \times [-2\nu, 2\nu] \rightarrow W, \quad \Psi_2(\nu) : V_{2/3} \times [-2\nu, 2\nu] \rightarrow W. \]

Terminology: Two riemannian metrics on a manifold $N$ are called $C$-equivalent, if for every tangent vector $h$ to $M$ we have: $|h|_{g_1}/|h|_{g_2} \leq C, \quad |h|_{g_2}/|h|_{g_1} \leq C$.

Let $v_0$ satisfy the following restriction:

\[ \text{(R): } 2v_0 < C. \]

For $i = 0, 1/3, 1/2, 2/3, 1$ the riemannian metric induced by $\Psi_{3i}(v_0)$ from $W$ and the product metric on the domain of $\Psi_{3i}(v_0)$ are $3/2$-equivalent; further, we have: $f(\text{Im } \Psi_{3i}(v_0)) \subset [i - \epsilon, i + \epsilon]$. For $i = 1/3, 1/2, 2/3$ we have: $|v(x)| = df(v(x)) = C$ for $x$ in a neighborhood of $\text{Im } \Psi_{3i}(v_0)$.

We shall prove that this $v_0$ satisfy the conclusions of our theorem. So let $X, Y, \Psi_1, \Psi_2$ be as in the statement of the theorem. Denote $\text{dim } W = n + 1$. Set $U_1 = \{\Phi_{1p} : U_{1p} \rightarrow B^n(0, r_{1p})\}$ and $U_2 = \{\Phi_{2p} : U_{2p} \rightarrow B^n(0, r_{2p})\}$. Let $\mu \leq \min(v_0, d(V_1), d(V_2))$. We shall denote $\Psi_{3i}(\mu)$ by $\Psi_{3i}$.

For $\theta \in [0, 2\mu]$ and $i = 1, 3/2, 2$ we denote by $T_{b1}(\theta)$ the set $\Psi_1(V_{i/3} \times [-\theta, \theta])$. For $\theta \in [0, 2\mu]$ we denote by $T_{b0}(\theta)$ the set $\Psi_0(V_0 \times [0, \theta])$ and by $T_{b3}(\theta)$ the set $\Psi_3(V_1 \times [-\theta, \theta])$.\]
For \( \lambda \in [0, 1/3] \) we denote by \( \mathbb{L}_\lambda(u_1) \) the \( s \)-submanifold \( v_{[1/3, \lambda]}^\sim(K(u_1)) \) of \( V_\lambda \). For \( \lambda \in [1/3, 1] \) we denote by \( \mathbb{L}_\lambda(u_1) \) the \( s \)-submanifold \( (-v)_{[1/3, \lambda]}^\sim(K(u_1)) \) of \( V_\lambda \). For \( \lambda \in [2/3, 1] \) we denote by \( \mathbb{L}_\lambda(-u_2) \) the \( s \)-submanifold \( (-v)_{[2/3, \lambda]}^\sim(K(-u_2)) \) and for \( \lambda \in [0, 2/3] \) we denote by \( \mathbb{L}_\lambda(-u_2) \) the \( s \)-submanifold \( v_{[2/3, \lambda]}^\sim(K(-u_2)) \). Note that since \( v \) have no zeros, \( v_{[\alpha, \beta]}^\sim \) is a diffeomorphism of \( V_\alpha \) onto \( V_\beta \) for any \( \beta < \alpha \). Note also that \( \mathbb{L}_{2/3}(-u_2) \) equals to \( K(-u_2) \) and \( \mathbb{L}_{1/3}(u_1) \) equals to \( K(u_1) \).

2. Auxiliary functions and vector fields.

Let \( \chi : [0, 1] \rightarrow [0, 1] \) be a \( C^\infty \) function, with the following properties:

\[
\begin{align*}
\chi(x) &= x \quad \text{for} \quad x \in [0, \varepsilon) \cup [1 - \varepsilon, 1]; \\
\chi(x) &= 1 - x \quad \text{for} \quad x \in [1/2 - \varepsilon, 1/2 + \varepsilon]; \\
\chi(x) &= 2/3 - (x - 1/3)^2 \quad \text{for} \quad x \in [1/3 - \varepsilon, 1/3 + \varepsilon]; \\
\chi'(x) &= \frac{1}{3} \quad \text{for} \quad x \in [0, 1/3] \cup [2/3, 1]; \\
\chi'(x) &= \frac{1}{3} + (x - 2/3)^2 \quad \text{for} \quad x \in [2/3 - \varepsilon, 2/3 + \varepsilon]; \\
\chi'(x) &= 0 \quad \text{for} \quad x \in [1/2, 2/3].
\end{align*}
\]

Lemma 3.3. \( (1) (\Psi_i^{-1})_*(v) = (0, C) \) for \( i = 0, 1, 2, 3, 2, 3 \).

(2) \( (f \circ \Psi_i)(x, \tau) = \tau + i/3 \), where \( i = 1, 2, 3/2 \).

(3) \( (\chi \circ f \circ \Psi_1)(x, \tau) = 2/3 - \tau^2 \), \( (\chi \circ f \circ \Psi_2)(x, \tau) = 1/3 + \tau^2 \).

(4) For \( \lambda \in [-2\mu, 2\mu] \) and \( i = 1, 3/2, 2 \) we have \( f^{-1}(i/3 + \lambda) = \Psi_i(V_{i/3} \times \{\lambda\}) \).

Proof. (1) - (3) follow immediately from (9). To prove (4) let \( i = 1 \) and \( \lambda \in [-2\mu, 2\mu] \). Each \( v \)-trajectory intersects each level surface of \( f \) at exactly one point. Therefore, on each \( v \)-trajectory starting at a point \( x \in V_{1/3} \) (and, hence, on every \( v \)-trajectory), the point \( \Psi_1(x, \lambda) \) is the only point in \( f^{-1}(1/3 + \lambda) \). The cases \( i = 3/2, 2 \) are similar.

Let \( B : \mathbb{R} \rightarrow [0, 1] \) be a \( C^\infty \) function such that \( B(t) = 0 \) for \( |t| \geq 5\mu/3 \) and \( B(t) = 1 \) for \( |t| \leq 4\mu/3 \). Let \( B_1 : \mathbb{R} \rightarrow \mathbb{R}^+ \) be a \( C^\infty \) function such that \( \operatorname{supp} B_1 \subset [\mu, 2\mu] \) and that \( \int_0^\infty B_1(t)dt = C \).

Let \( z_0 \) be a \( C^\infty \) vector field on \( V_0 \) such that \( \Phi(z_0, 1)(\mathbb{L}_0(u_1)) \nsubseteq \mathbb{X} \). Let \( z_1 \) be a \( C^\infty \) vector field on \( V_1 \) such that \( \Phi(z_1, 1)(\mathbb{L}_1(-u_2)) \nsubseteq \mathbb{Y} \). Let \( z_{1/2} \) be a \( C^\infty \) vector field on \( V_{1/2} \) such that \( \Phi(z_{1/2}, 1)(\mathbb{L}_{1/2}(u_1)) \nsubseteq \mathbb{L}_{1/2}(-u_2) \). We shall assume that \( z_i \) (where \( i = 0, 1/2, 1 \)) are chosen so small, that \( \sup_{\tau} |B_1(\tau)| \cdot \|z_i\| < C/9 \).

3. Morse function \( F \).

Let \( a_1, a_2 > 0 \). Set: \( F(y) = \psi(y) \) for \( y \in W \setminus ( T_{b_1}(2\mu) \cup T_{b_2}(2\mu) ) \);

\[
(F \circ \Psi_1)(x, \tau) = a_1 B(\tau) F_1(x) + 2/3 - \tau^2 \quad \text{for} \quad (x, \tau) \in V_{1/3} \times [-2\mu, 2\mu];
\]

\[
(F \circ \Psi_2)(x, \tau) = a_2 B(\tau) F_2(x) + 1/3 + \tau^2 \quad \text{for} \quad (x, \tau) \in V_{2/3} \times [-2\mu, 2\mu].
\]

Since \( \psi \circ \Psi_1(x, \tau) = 2/3 - \tau^2 \) and \( \psi \circ \Psi_2(x, \tau) = 1/3 + \tau^2 \), these formulas define correctly a smooth function \( F : W \rightarrow \mathbb{R} \), which equals to \( f \) nearly \( \partial W \). To find critical points of \( F \) note that \( S(F) \) is contained obviously in \( T_{b_1}(2\mu) \cup T_{b_2}(2\mu) \).

For \( (x, \tau) \in V_{1/3} \times [-2\mu, 2\mu] \) we have \( d(F \circ \Psi_1)(x, \tau) = \frac{a_1 b'(\tau)}{2} F_1(x) - 2\tau d\tau + a_1 b(\tau) dF_1(x) \). For \( a_1 \) small enough this can vanish only for \( \tau = 0 \). We conclude therefore (applying the same reasoning to \( T_{b_2}(2\mu) \)), that \( S(F) = S(F_1) \cup S(F_2) \) if only \( a_1 \) are small enough (and we make this assumption from now on). To prove that \( F \) is a Morse function we shall explicit the standard charts for \( F \). Let \( p \in S(F_1) \) and write \( F_1 \circ \Phi_1^{-1}(x) = F_1(x) + \sum_{i} \alpha_i x_i^2 \). Consider the chart \( \Phi_1 \circ \id : U \rightarrow \text{chart} \).
\[ B^n(0, r_{1p}) \times [ -\mu, \mu ] \text{ of the manifold } V_{1/3} \times [ -\mu, \mu ] \text{ around the point } p \times 0. \] We have \( F \circ \Psi_1 \circ (\Phi_{1p} \times \id)^{-1}(x, \tau) = F_1(p) + 2/3 + a_1 \sum_i \alpha_i x_i^2 - \tau^2 \), therefore the chart \(((\Phi_{1p} \times \id) | (\Phi_{1p} \times \id)^{-1}(B^{n+1}(0, \mu)) \circ (\Psi_1 | \Im \Psi_1)^{-1}) \) is a standard chart of radius \( \mu \) for \( F \) at \( p \in S(F_1) \). These charts together with the similar ones for \( q \in S(F_2) \) give an \( F \)-chart-system of radius \( \mu \). We shall denote this system by \( \mathcal{U} \). Note also that \( \text{ind}_{F \mathcal{P}} = \text{ind}_{F \mathcal{P}2} + 1 \) for \( p \in S(F_1) \) and that \( \text{ind}_{F \mathcal{Q}} = \text{ind}_{F \mathcal{Q}2} \) for \( q \in S(F_2) \). Note that if \( a_1 \) and \( a_2 \) are chosen sufficiently small, then \( F(\text{Tb}_1(2\mu)) \subseteq [2/3 - \epsilon, 2/3 + \epsilon], \) and \( F(\text{Tb}_2(2\mu)) \subseteq [1/3 - \epsilon, 1/3 + \epsilon] \). In this case also \( F^{-1}(1/2) = V_{1/2} \cup V_0 \cup V_{\theta'}, \) where \( \epsilon < \theta < 1/3 - \epsilon, \ 2/3 + \epsilon < \theta' < 1 - \epsilon. \)

4. \( F \)-gradient \( u \).

Define a vector field \( u \) on \( W \) as follows:

\[
\begin{align*}
  u(y) &= v(y) \quad \text{for} \quad y \in (W_0 \setminus (\text{Tb}_1(2\mu) \cup \text{Tb}_0(2\mu))) \cup (W_2 \setminus (\text{Tb}_2(2\mu) \cup \text{Tb}_3(2\mu))); \\
  u(y) &= -v(y) \quad \text{for} \quad y \in W_1 \setminus (\text{Tb}_1(2\mu) \cup \text{Tb}_2(2\mu) \cup \text{Tb}_3(2\mu)); \\
  ((\Psi_0^{-1})_*u)(x, \tau) &= (-B_1(\tau)z_0(x), C); \quad \text{ } ((\Psi_3^{-1})_*u)(x, \tau) = (B_1(-\tau)z_1(x), C); \\
  ((\Psi_1^{-1})_*u)(x, \tau) &= (B(\tau)u_1(x), -B(\tau)\tau - (1 - B(\tau)) \cdot C \cdot \text{sgn}\tau); \\
  ((\Psi_3^{-1})_*u)(x, \tau) &= (-B_1(-\tau)z_1(2x), -C); \\
  ((\Psi_2^{-1})_*u)(x, \tau) &= (B(\tau)u_2(x), B(\tau)\tau + (1 - B(\tau)) \cdot C \cdot \text{sgn}\tau).
\end{align*}
\]

An easy computation using the definition of \( F \) and \( u \) shows that these formulas define correctly a \( C^\infty \) vector field on \( W \), and that \( (F, u, \mathcal{U}) \) is an \( \mathcal{M} \)-flow on \( W \) of radius \( \mu \), if only \( z_0, z_1, z_{1/2} \) are small enough (which assumption we make from now on). It is also easy to see that \( (u - v) \subseteq (W \cup \text{Tb}_1(2\mu)) \), and that \( u \) is an \( f \)-gradient in \( W_0 \setminus V_{1/3} \) and \( W_2 \setminus V_{2/3}; \) \( -u \) is an \( f \)-gradient in \( W_1 \setminus (V_{1/3} \cup V_{2/3}). \)

We claim that \( (F, u, \mathcal{U}) \) satisfies all the conclusions of 3.1. (1) and (2) follow immediately from the construction. To prove (3) note that \( V_{1/3} \) is a closed submanifold of \( W^\circ \) and \( u \) is tangent to \( V_{1/3}; \) therefore \( V_{1/3} \) is (\( \pm u \))-invariant (same for \( V_{2/3} \)). To prove that \( W_1 \) is \( u \)-invariant, let \( x \in W_1^\circ \) and assume that for some \( t \) we have \( \gamma(x, t; u) \not\subseteq W_1^\circ. \) The trajectory \( \gamma(x, \cdot; u) \) does not intersect \( V_{1/3} \cup V_{2/3}. \) Consider a continuous function \( \phi : \tau \mapsto f(\gamma(x, t; u)). \) The domain of definition of \( \phi \) is an interval (finite or infinite) of \( \mathbb{R}, \) \( \phi \) never takes values \( 1/3, 2/3, \) and \( \phi_0 \in ]1/3, 2/3[. \) Therefore \( \text{Im } \phi \subseteq ]1/3, 2/3[ \) and \( W_1 \) is \( u \)-invariant. The other assertions of (3) are proved similarly.

5. Estimate of the quickness of \( V. \)

To obtain the estimate \( ||u|| \leq C \) note that the inequality \( ||u(x)|| \leq 3/2(C + C_1 + C_2) \) is to be checked only for \( x \in \cup \text{Tb}_1(2\mu), \) where it is a matter of a simple computation; we leave it to the reader. To estimate the time, which an \( u \)-trajectory spends outside \( U(\mu), \) note first that for a trajectory, starting at a point of \( V_{1/3} \) (resp. \( V_{2/3} \)), this time is not more than \( \beta_1 \) (resp. \( \beta_2 \)), since it is actually a \( u_{1/3} \) (resp. \( u_{2/3} \))-trajectory. Now let \( \gamma \) be an \( u \)-trajectory, passing by a point of \( W_1^\circ. \) By (4) and (5) it stays in \( W_1^\circ \) forever. Since \( u \) is a \((f)\)-gradient in \( W_1^\circ, \) the function \( t \mapsto f(\gamma(t)) \) is strictly decreasing and \( \lim_{t \to -\infty} f(\gamma(t)) = 2/3, \lim_{t \to \infty} f(\gamma(t)) = 1/3. \) This implies \( \lim_{t \to -\infty} \gamma(t) = q \in S(F_2) \) and \( \lim_{t \to \infty} \gamma(t) = p \in S(F_1). \) We shall estimate the time which \( \gamma \) spends outside \( U(\mu) \) between the various level surfaces of \( f. \)
1) \( f(\gamma(t)) \in [2/3 - \mu/2, 2/3] \). By Lemma 3.3(4) this condition is equivalent to:
\( \gamma(t) \in \Psi_2(V_{2/3} \times [-\mu/2, 0]) \). The curve \( \Psi_2^{-1}(\gamma(t)) \) is a product of an \( u_2 \)-trajectory and the curve \( \tau \mapsto \alpha e^\tau \) with \( \alpha < 0 \). Since \( \Psi_2^{-1}(\mathcal{U}(\mu)) \) contains \( U_p(\mu/2) \times [-\mu/2, \mu/2] \) for every \( p \in S(F_2) \) the time which \( \gamma(t) \) spends in \( \Psi_2(V_{2/3} \times [-\mu/2, 0]) \) outside \( \Psi_2^{-1}(\mathcal{U}(\mu)) \) is not more than \( \beta_2 \).

2) \( f(\gamma(t)) \in [2/3 - 2\mu, 2/3 - \mu/2] \), in other words, \( \gamma(t) \in \Psi_2(V_{2/3} \times [-2\mu, -\mu/2]) \). The vector field \( (\Psi_2^{-1})_*(u) \) equals to \( (B(\tau)u_2(x), \alpha(\tau)) \), where \( \alpha(\tau) \leq \tau \). Therefore the total time which \( \gamma \) can spend in this domain is not more than \( \ln (2\mu/(\mu/2)) \) < 2.

3) \( f(\gamma(t)) \in [1/2, 2/3 - 2\mu] \). In the domain \( f^{-1}([1/2, 2/3 - 2\mu]) \) we have \( u = v \). Therefore the time is \( \leq 2T/C \).

4) \( f(\gamma(t)) \in [1/2 - 2\mu, 1/2] \). Here \( \gamma(t) \in \Psi_{3/2}(V_{1/2} \times [-2\mu, 0]) \). The second coordinate of \( (\Psi_{3/2}_1)^{-1}(u) \) is equal to \(-C \), therefore the total time spent here is not more than \( 2\mu/C \leq 1 \).

Similarly to the cases 1) - 3) above one shows that the time which \( \gamma \) spends in \( f^{-1}([1/3, 1/2 - 2\mu]) \setminus \mathcal{U}(\mu) \) is not more than \( 2T/C + 2 + \beta_1 \). Summing up, we obtain that the time which \( \gamma \) can spend in \( W_1 \setminus \mathcal{U}(\mu) \) is not more than \( \beta_1 + \beta_2 + 4T/C + 5 \).

Similar analysis of behaviour of \( u \)-trajectories in \( W_0 \) and \( W_2 \) shows that the same estimate holds in these cases also.

6. Estimate of \( N(\gamma, \mu) \).

Let \( \gamma(\cdot) \) be an \( u \)-trajectory in \( W_1 \). If \( \gamma \) is in \( V_{1/3} \), (resp. in \( V_{2/3} \)), then obviously \( N(\gamma, \mu) \leq N(V_{1, \mu}) \) (resp. \( N(\gamma, \mu) \leq N(V_{2, \mu}) \)). Assume that \( \text{Im} \gamma \subset W_1 \). Let \( \alpha \in \mathbb{R} \) (resp. \( \beta \in \mathbb{R} \)) be the unique number, such that \( f(\gamma(\alpha)) = 2/3 - \mu \) (resp. \( f(\gamma(\beta)) = 1/3 + \mu \)). We have \( \cup_{p \in S(F_1)} U_p(\mu) \subset \mathbb{T}_i(\mu) \) (for \( i = 1, 2 \)). Therefore \( \gamma(t) \in \mathcal{U}(\mu) \) can occur only if \( t \geq \beta \) or \( t \leq \alpha \). For \( t \leq \alpha \) (resp. \( t \geq \beta \)) the curve \( \Psi_2^{-1}(\gamma(t)) \) (resp. \( \Psi_1^{-1}((\gamma(t))) \)) is an integral curve of the vector field \( (u_2(x), \alpha(\tau)) \) (resp. \( (u_1(x), -\tau)) \). Since an integral curve of \( u_2 \) (resp. \( u_1 \)) can intersect no more than \( N(V_{2, \mu}) \) (resp. \( N(V_{1, \mu}) \)) standard coordinate neighborhoods of radius \( \mu \), the first part of (5) follows. The case of curves in \( W_0 \) and \( W_2 \) is considered similarly.

7. Transversality conditions.

The last lemma implies that \( \mathcal{V} \) is almost good and the (6) of our conclusions.

Lemma 3.4. (1) The family \( \{ \cup_{p \in S(F_1)} D_p(-u) \}_{0 \leq i \leq n} \) equals to \( \mathbb{K}(-u_1) \).

The family \( \{ \cup_{q \in S(F_2)} D_q(u) \}_{0 \leq i \leq n} \) equals to \( \mathbb{K}(u_2) \).

(2) For each \( \lambda \in [0, 1] \) the family \( \{ \cup_{q \in S(F_2)} D_q(-u) \cap V_{\lambda} \}_{0 \leq i \leq n} \) is an \( s \)-submanifold of \( V_{\lambda} \), which is equal to:

(1) \( \Phi(z_1, 1)(\mathbb{L}_0(-u_2)) \) if \( \lambda = 1 \), (2) \( \mathbb{L}_\lambda(-u_2) \) if \( \lambda \in [1/2, 1 - \epsilon] \), (3) \( \emptyset \) if \( \lambda \leq 1/3 \).

(3) For each \( \lambda \in [0, 1] \) the family \( \{ \cup_{p \in S(F_1)} D_p(u) \cap V_{\lambda} \}_{0 \leq i \leq n} \) is an \( s \)-submanifold of \( V_{\lambda} \), which is equal to:

(1) \( \Phi(z_0, 1)(\mathbb{L}_0(u_1)) \) for \( \lambda = 0 \); (2) \( \mathbb{L}_\lambda(u_1) \) for \( \lambda \in [\epsilon, 1/2 - 2\mu] \); (3) \( \Phi(z_{1/2}, 1)(\mathbb{L}_{1/2}(u_1)) \) for \( \lambda = 1/2 \); (4) \( \emptyset \) for \( \lambda \geq 2/3 \);

(4) \( u \) is an almost good \( F \)-gradient, the \( s \)-submanifold \( \mathbb{K}(u) \cap V_0 \) is almost transversal to \( \mathbb{K} \), and the \( s \)-submanifold \( \mathbb{K}(-u) \cap V_1 \) is almost transversal to \( \mathbb{K} \).
Proof. (1) For every \( p \in S_{i}(F_{1}) \) the positive disc of \( f \) belongs to \( V_{1/3} \), which implies immediately the first assertion; the second is proved similarly.

(2) It is not difficult to see that it suffices to prove the assertion for \( \lambda \in [-2 \mu + 2/3, 2 \mu + 2/3] \). For these values of \( \lambda \) it follows from the analysis of the behaviour of \( u \)-trajectories in \( \text{Tb}_{2}(2 \mu) \), carried out in the proof of Subsection 5 above. (3) is proved similarly. (4): To prove that \( u \) is almost good let \( p, q \in S(F) \), \( \text{ind} p \leq \text{ind} q \). Let \( \gamma \) be an \((-u)\)-trajectory joining \( p \) with \( q \). The case \( p, q \in S(F_{1}) \) or \( p, q \in S(F_{2}) \) follows easily from (1). Let \( p \in S(F_{1}), q \in S(F_{2}) \). Denote by \( z \) the (unique) point of intersection of \( \gamma \) with \( V_{1/2} \), \( \text{ind} p \) by \( k \), \( \text{ind} q \) by \( r \). Then \( z \in (\Phi(z_{1/2}, 1)(\mathbb{L}_{1/2}(u_{1})))_{k-1} \) and \( z \in \mathbb{L}(-u_{2})_{n-r} \), which is impossible by the choice of \( z_{1/2} \). The last point is already proved. \( \square \)

B. Proof of 1.17(\( n \)) \( \Rightarrow \) 1.18(\( n + 1 \)) . We shall first establish the existence of \( \mathcal{M} \)-flows with the estimate of their quickness similar to that of 1.18 for functions on cobordisms without critical points, then for functions having one critical point. The proof will be finished by the usual induction procedure. For the rest of §3 we fix a natural number \( n \) and we assume that 1.17(\( n \)) is true. We shall need only the following lemma, which is an obvious corollary of 1.17(\( n \)).

Lemma 3.5. Let \( M \) be a closed riemannian manifold, \( \text{dim} M = n \). Let \( D > 0 \), \( \beta > 0 \), \( A > 1 \). Then for every \( \mu > 0 \) sufficiently small there is an almost good \( \mathcal{M} \)-flow \( W \) such that: (1) \( W \) is of radius \( \mu \) and is \((D, \beta)\)-quick; (2) \( N(W) \leq n2^{n} \); (3) \( \mathcal{G}(W) \leq A \). \( \square \)

1. Functions without critical points.

Lemma 3.6. Let \( W \) be an \((n + 1)\)-dimensional compact cobordism, endowed with a riemannian metric. Let \( g : W \to [a, b] \) be a Morse function without critical points. Let \( C > 0 \) and let \( w \) be an \( g \)-gradient such that \( ||w|| \leq C \). Let \( X \) be an \( s \)-submanifold of \( V_{0} \), \( Y \) be an \( s \)-submanifold of \( V_{1} \). Let \( \mu_{0} > 0 \). Then there is an almost good \( \mathcal{M} \)-flow \( V = (F, u, U) \) on \( W \) with the following properties.

(1) In a neighborhood of \( \partial W \) we have \( u = w \) and \( F = g \). (2) \( D(V) \leq \mu_{0} \), \( \mathcal{G}(V) \leq 2 \).

(3) \( V \) is \((2C, 16 + n2^{n+5})\)-quick, and \( N(V) \leq n2^{n+1} \). (4) \( \mathbb{K}_{a}(u) \vdash X \), \( \mathbb{K}_{b}(-u) \vdash Y \).

(5) There are \( \alpha, \beta \in \mathbb{R} \), with \( a < \alpha < \beta < b \), such that: \( g^{-1}((\alpha, \beta)) \) are \((\pm u)\)-invariant, \( g^{-1}([\alpha, \beta]) \) is \( u \)-invariant and \( g^{-1}([\beta, b]) \) is \((-u)\)-invariant. For any \( u \)-trajectory \( \gamma \) in \( g^{-1}([\alpha, \beta]) \) or in \( g^{-1}([\beta, b]) \) we have \( N(\gamma) \leq n2^{n} \).

Proof. Let \( N > 0 \) be a natural number. For \( 0 \leq s \leq N \) denote \( a + \frac{b-a}{N} s \) by \( a_{s} \), \( \frac{2a + s a_{s+1}}{3} \) by \( b_{s} \), \( \frac{a_{s} + 2a_{s+1}}{3} \) by \( c_{s} \). Denote the cobordism \( g^{-1}([a_{s}, a_{s+1}]) \) by \( W_{s} \). Choose \( N \) so large that for each \( s \) the time which a \( \text{grd}(g) \)-trajectory spends in \( W_{s} \) is not more than \( C/4 \). For \( 0 \leq i \leq N - 1 \) we shall define by induction in \( i \) a sequence of \( \mathcal{M} \)-flows \( V_{i} = (F_{i}, u_{i}, U_{i}) \) of radius \( \kappa_{i} \leq \mu_{0} \) on \( W_{i} \).

Let \( 0 \leq i \leq N - 1 \). For \( i > 0 \) assume that \( V_{i-1} = (F_{i-1}, u_{i-1}, U_{i-1}) \) is already constructed. Denote \( g|W_{1} \) by \( g_{1} : W_{i} \to [a_{i}, a_{i+1}] \) and \( w|W_{i} \) by \( w_{i} \). Apply 3.1 to \( g_{i}, w_{i}, C \) and get the corresponding number \( \nu_{0} > 0 \). Choose \( (3.5) \) an \( \mathcal{M} \)-flow \( W_{1} = (f_{1}, v_{1}, T_{1}) \) on \( g^{-1}(b_{1}) \) and an \( \mathcal{M} \)-flow \( W_{2} = (f_{2}, v_{2}, T_{2}) \) on \( g^{-1}(c_{1}) \), such that the flow \( W_{1} \) is of radius \( \kappa_{i} \leq \min(\nu_{0}, \mu_{0}) \) and \( N(W_{s}) \leq n2^{n}, \mathcal{G}(W_{s}) \leq 4/3, \) and \( W_{2} \) is \((C/6, 1)\)-quick (where \( s = 1, 2 \)). Choose the \( s \)-submanifold of \( g^{-1}(a_{i}) \) as follows: \( X_{i} \) for \( i = 0 \) and \( \mathbb{K}_{a}(-u_{i} \cap q^{-1}(\alpha_{i})) \) for \( i > 0 \). Choose the \( s \)-submanifold
of $g^{-1}(a_{i+1})$ as follows: empty for $i < N - 1$ and $\mathbb{Y}$ for $i = N - 1$. Theorem 3.1 then provides an almost good $\mathcal{M}$-flow $\mathcal{V}_i = (F_i, u_i, U_i)$ of radius $\varkappa_i$. Since $\mathcal{W}_s$ are $(C/6, 1, \varkappa_i)$-quick, they are $(C/6, 1 + n2^{n+3}, \varkappa_i/2)$-quick (by 1.13), therefore $\mathcal{V}_s$ is $(2C, 8 + n2^{n+4}, \varkappa_i)$-quick.

The property (1) of 3.1 imply that $\mathcal{V}_i$ glue together to obtain an $\mathcal{M}$-flow $\mathcal{V} = (F, u, U)$ on $W$. I claim that it satisfies the conclusions of our Lemma. (1), (2) are immediate from (1), (2) of 3.1. An easy induction argument using (±)-invariance of $g^{-1}(c_i)$ shows that $\mathcal{V}$ is almost good. Proceeding to the estimate of $N(\gamma)$ and $I(\gamma)$ demanded by (3) (where $\gamma$ is a $v$-trajectory) note that the (±)-invariance of $g^{-1}(b_i)$ and $g^{-1}(c_i)$ implies easily that $N(\gamma) \leq n2^{n+1}$ and $T(\gamma) \leq 8 + n2^{n+4}$ for any $u$-trajectory $\gamma$, passing by a point of $g^{-1}(b_i, c_i)$ for some $i$. Assume that $\gamma$ passes by a point of $g^{-1}(c_{i-1}, b_i)$ for some $i$. It is easy to show that $\gamma$ is defined on $\mathbb{R}$. If $\gamma$ does not intersect $g^{-1}(a_i)$, then it stays in $W_i$ or in $W_{i-1}$ and the required estimates follow (actually this case does not occur). If $g(\gamma(t_0)) = a_i$, then the $u_i$-(resp. $-u_{i-1}$)-invariance of $g^{-1}([a_i, b_i])$ (resp. $g^{-1}(c_{i-1}, a_i)$) imply that $g(\gamma(t)) \geq a_i$ for $t \geq t_0$ (resp. $t \leq t_0$) and $N(\gamma) \leq n2^{n+1}$, $T(\gamma) \leq 2(8 + n2^{n+4})$, and we obtain (3). Set $a = b_0$, $\beta = c_{N-1}$; then (5) follows from (3) and (5) of 3.1. Now (4) is easy to prove. □

2. Functions with one critical point.

**Lemma 3.7.** Let $W$ be an $(n + 1)$-dimensional compact riemannian cobordism. Let $g : W \to [a, b]$ be a Morse function with one critical point $p$ and a strongly standard chart $\Phi_p : U_p \to B^{n+1}(0, r_p)$ around $p$. Let $w$ be a $g$-gradient with respect to $\{\Phi_p\}$. Then for every $\delta > 0$ sufficiently small the time which a $w$-trajectory can spend in $g^{-1}([g(p) - \delta^2, g(p) + \delta^2]) \setminus U_p(\delta/2)$ is not more than 6.

**Proof.** Denote $g^{-1}([g(p) - \delta^2, g(p) + \delta^2])$ by $S_\delta$. Let $\gamma$ be a $w$-trajectory. The function $dg(w)$ is bounded from below in $S_\delta \setminus U_p$, therefore for $\delta > 0$ sufficiently small the time which $\gamma$ spends in $S_\delta \setminus U_p$ is less than 1. The time, which $\gamma$ spends in $U_p \cap (S_\delta \setminus U_p(\delta))$ is not more than 2 (by 1.12), and the time which $\gamma$ spends in $U_p(\delta) \setminus U_p(\delta/2)$ is not more than $\ln 8 < 3$ (by 1.11). □

**Lemma 3.8.** Let $W$ be an $(n + 1)$-dimensional compact cobordism endowed with a riemannian metric. Let $(g, w, \mathcal{U})$ be an $\mathcal{M}$-flow on $W$, where $g : W \to [a, b]$ is a Morse function with one critical point $p$, $g(p) = c$, and the $g$-chart around $p$ is strongly standard. Let $C > 0$, and assume that $\|w\| \leq C$. Let $\mu_0 > 0$. Let $\mathbb{X}$ be an $s$-submanifold of $V_0$, and $\mathbb{Y}$ be an $s$-submanifold of $V_1$. Then there is an almost good $\mathcal{M}$-flow $\mathcal{V} = (F, u, \mathcal{U})$ on $W$, with the following properties.

1. In a neighborhood of $\partial W$ we have $u = w$ and $F = g$.
2. $D(\mathcal{V}) \leq \mu_0$; $\mathcal{G}(\mathcal{V}) \leq 2$.
3. $\mathcal{V}$ is $(2C, 40 + n2^{n+6})$-quick, and $N(\mathcal{V}) \leq 1 + n2^{n+1}$.

**Proof.** Let $\varepsilon \in \{0, \mu_0\}$ be so small that it is less than the radius of the standard chart $\Phi_p$ of $g$ around $p$, and that the conclusions of 3.7 hold and that $\mathcal{G}(U_p(\delta), \Phi_p) \leq 2$. Denote $g^{-1}(c - \delta^2)$ by $V_-$, $g^{-1}((a, c - \delta^2])$ by $V_-$, $g^{-1}(c + \delta^2)$ by $V_+$, $g^{-1}((a + \delta^2, b))$ by $V_+$, $g^{-1}((c - \delta^2, c + \delta^2))$ by $W$. Apply Lemma 3.6 to the
function $g|W_+$, $(g|W_+)$-gradient $w$, the number $\mu_0$, the $s$-submanifold $\mathcal{Y}$ of $V_1$, and the $s$-submanifold $D(p, -w) \cap V_+ \cap V_+$ (consisting of one compact manifold). Denote the resulting flow by $\mathcal{V}_+ = (F_+, u_+, \mathcal{U}_+)$, The corresponding values $\alpha, \beta$, satisfying (5) of 3.6 with respect to $\mathcal{V}_+$ will be denoted by $b''$, $b'$; we have $b'' < b'$.

Denote by $\Phi'_i : U'_p \to B'^{n+1}(0, \delta/2)$ the restriction of the $g$-chart around $p$, and by $\mathcal{V}'$ the $\mathcal{M}$-flow $(g|W'_w, w|W'_w)$, (1) of 3.6 implies that we can glue $\mathcal{V}_+$ to $\mathcal{V}'$. Denote the resulting $\mathcal{M}$-flow by $\mathcal{V}_+ = (F_1, u_1, \mathcal{U}_1)$. It is almost good since $\mathbb{K}(u_+) \cap V_+ \cap D(p, -w) \cap V_+$. Apply Lemma 3.6 to $g|W_-$, the $(g|W_-)$-gradient $w$, the number $\mu_0$, the $s$-submanifold $\mathbb{K}(u_1) \cap V_-$ and to the $s$-submanifold $\mathbb{K}$ of $V_0$. Denote the resulting $\mathcal{M}$-flow by $\mathcal{V}_- = (F_-, u_-, \mathcal{U}_-)$. The corresponding values $\alpha, \beta$ satisfying (5) of 3.6 with respect to $\mathcal{V}_-$ will be denoted by $a', a''$, so that $a' < a''$. (1) of 3.6 implies that we can glue $\mathcal{V}_-$ to $\mathcal{V}_1$. Denote the resulting $\mathcal{M}$-flow on $W$ by $\mathcal{V} = (F, u, \mathcal{U})$. It is almost good, since $\mathbb{K}(u_1) \cap V_+ \cap \mathbb{K}(-u_-) \cap V_-$.

We claim that the $\mathcal{M}$-flow $\mathcal{V}$ together with the numbers $a', b'$ satisfy the conclusions of our Lemma. Indeed, (1), (2) and (5) follow from the construction. (4) follows since $g^{-1}([b', b])$ (resp. $g^{-1}([a, a'])$) is $u$-invariant and from the corresponding transversality property of $\mathcal{V}_+$ (resp. of $\mathcal{V}_-$). To prove (3) note that $(\pm u)$-invariance of $g^{-1}(a'\prime)$ and of $g^{-1}(b'\prime)$ together with (3) of 3.6 imply already the required estimates of $N(\gamma)$ and of $T(\gamma)$ for all the $u$-trajectories $\gamma$ passing by a point of $g^{-1}([a, a'])$ or of $g^{-1}([b', b])$. Let now $\gamma$ be an $u$-trajectory, passing by a point of $g^{-1}([a', b'])$. Since $g^{-1}(a'\prime)$ and $g^{-1}(b'\prime)$ are $(\pm u)$-invariant, $\gamma$ is defined on $\mathbb{R}$. If $\gamma$ does not intersect $W'$, then $\text{Im} \gamma \subset W_+$ or $\text{Im} \gamma \subset W_-$ and the required estimates of $T(\gamma)$ and $N(\gamma)$ again follow from those of 3.6 (actually this case does not occur). If there is $t_0 \in \mathbb{R}$ such that $\gamma(t_0) = g^{-1}([c - \delta^2, c + \delta^2])$, then it is not difficult to show (using the fact that $u$ is a $g$-gradient in $W'$, $u$-invariance of $g^{-1}([c + \delta^2, b'])$ and $(\pm u)$-invariance of $g^{-1}([a'', c - \delta^2])$) that three possibilities can occur for $\gamma$:

1. $\exists \alpha, \beta \in \mathbb{R} : \gamma([\alpha, \beta]) \subset W_+$, $\gamma([\alpha, \beta]) \subset W'_+$, $\gamma([\alpha, \beta]) \subset W'$;
2. $\exists \beta \in \mathbb{R} : \gamma([\beta, \alpha]) \subset W_+$, $\gamma([\beta, \alpha]) \subset g^{-1}([c, c + \delta^2])$, and $\lim_{t \to -\infty} \gamma(t) = p$;
3. $\exists \alpha \in \mathbb{R} : \gamma([\alpha, \beta]) \subset W_-$, $\gamma([\alpha, \beta]) \subset g^{-1}([c - \delta^2, c])$, and $\lim_{t \to \infty} \gamma(t) = p$.

It is easy to check the required estimates for $T(\gamma)$ and $N(\gamma)$ in each of these cases. □

3. End of the proof of 1.17(n) $\Rightarrow$ 1.18(n+1).

We are given $B > 0, \mu_0 > 0$. Let $\mathcal{V}_0 = (g, w, \mathcal{U})$ be an $\mathcal{M}$-flow on $\mathcal{M}$, such that $\|w\| \leq B/2$, all the charts of $\mathcal{U}$ are strongly standard, and if $p, q \in S(g)$ and $p \neq q$, then $g(p) \neq g(q)$ (existence of such a flow is an easy exercise in Morse theory). Let $p_1, \ldots, p_r$ be the critical points of $g$, so that $g(p_i) < g(p_{i+1})$. Denote $g(p_i)$ by $a_i$, and set $b_i = \frac{a_i + a_{i+1}}{2}$ for $1 \leq i \leq r - 1$, $b_0 = a_1 - 1$, $b_r = a_r + 1$. Denote $g^{-1}([b_i, b_{i+1}])$ by $W_i$. We shall construct by induction in $i$ a sequence of $\mathcal{M}$-flows $\mathcal{V}_i = (F_i, v_i, \mathcal{U}_i)$ on $W_i$.

Let $0 \leq i \leq r - 1$. For $i > 0$ assume that the $\mathcal{M}$-flow $\mathcal{V}_{i-1}$ is already constructed. Denote $g|W_i$ by $g_i : W_i \to [b_i, b_{i+1}]$, $w|W_i$ by $w_i$. Choose some restriction $\mathcal{U}_i'$ of the standard chart for $g_i$ to obtain an $\mathcal{M}$-flow $W_i = (g_i, w_i, \mathcal{U}_i')$ on $W_i$; we have $\|w_i\| \leq B/2$. For $i = 0$ choose the empty $s$-submanifold of $f^{-1}(b_0) = \emptyset$ and the empty $s$-submanifold of $f^{-1}(b_1)$. For $0 < i$ choose the $s$-submanifold $\mathbb{K}(-u_{i-1}) \cap f^{-1}(b_i)$ of $f^{-1}(b_i)$ and the empty $s$-submanifold of $f^{-1}(b_{i+1})$. Applying 3.8, we obtain an
The corresponding numbers \( a', b' \) will be denoted by \( c_i, d_i \); then 
\( b_i < c_i < a_{i+1} < d_i < b_{i+1} \). (1) of 3.8 implies that \( \mathcal{V}_i \) glue together to an \( \mathcal{M} \)-flow on \( M \), which will be denoted by \( \mathcal{V} = (f, v, U) \). An easy induction argument using the \((\pm v)\)-invariance of \( g^{-1}(c_i) \) and the almost transversality of \( \mathbb{K}(-v_i) \cap g^{-1}(b_{i+1}) \) to \( \mathbb{K}(v_i) \cap g^{-1}(b_{i+1}) \) shows that \( \mathcal{V} \) is almost good. The property (1) of 1.18 follows immediately from the theorem cited above. The estimates of \( T(\gamma) \) and \( N(\gamma) \) demanded by (2) and (3) of 1.18 follow from the \((\pm v)\)-invariance of \( g^{-1}(c_i), g^{-1}(d_i) \) similarly to the estimates of \( T(\gamma) \) and \( N(\gamma) \) of Lemma 3.6. □

§4. Ranging systems and proof of Main Theorem

In §4 the excision isomorphisms \( H_*(X \setminus B, A \setminus B) \to H_*(X, A) \) are denoted by \( \text{Exc} \).

A. Generalities on intersection indices. Let \( M \) be a manifold without boundary, \( \dim M = m \). Let \( X \subseteq M \) and \( N \) be an oriented submanifold of \( M \), such that \( N \setminus \text{Int} X \) is compact. Then the orientation class \( \mu_{N \setminus \text{Int} X} \in H_n(N, N \cap \text{Int} X) \) is defined, where \( n = \dim N \) (see [7, Th.A8]). Its image in \( H_n(M, X) \) will be denoted by \([N]_{M, X}\) (or simply by \([N]\) if there is no possibility of confusion). The next lemma follows immediately from the theorem cited above.

Lemma 4.1. 1. Let \( Y \) be a closed subset of \( \text{Int} X \). Then \([N \setminus Y]_{M \setminus Y, X \setminus Y}\) equals to the image of \([N]_{M, X}\) with respect to \( \text{Exc}^{-1}: H_*(M, X) \xrightarrow{\cong} H_*(M \setminus Y, X \setminus Y) \).

2. Let \( M' \) be a manifold, \( U \) be an open subset of \( M' \), \( X' \subseteq U \). Let \( \phi: M \to U \) be a diffeomorphism such that \( \phi(X) \subseteq X' \). Denote \( \phi(N) \) by \( N' \). Denote by \( \Phi : (M, X) \to (M', X') \) the resulting map of pairs. Then \([N']_{M', X'} = \Phi_*([N]_{M, X})\). □

Let \( L \) be a compact cooriented submanifold without boundary of \( M \). Then the canonical coorientation class \( [L] \in H^{m-1}(M, M \setminus L) \) is defined, where \( l = \dim L \). Assume that \( X \cap L = \emptyset \), \( N \pitchfork L \) and \( n + l = m \). Denote by \( j \) the embedding \( (M, X) \hookrightarrow (M, M \setminus L) \). The image of the class \([N]\) in \( H_n(M, M \setminus L) \) will be denoted again by \([N]\). The following lemma is standard.

Lemma 4.2. In the above assumptions the set \( N \cap L \) is finite and the intersection index \( N \pitchfork L \) equals \( \langle j^*([L]), [N] \rangle \). □

B. Ranging systems and \( \tilde{v} \)-images of the fundamental classes. Let \( f : W \to [a, b] \) be a Morse function on a compact riemannian cobordism \( W \), \( v \) be an \( f \)-gradient, and denote \( f^{-1}(a) \) by \( V_0 \), \( f^{-1}(b) \) by \( V_1 \). This terminology is valid for Subsections B and C.

Definition 4.3. Let \( \Lambda = \{\lambda_0, \ldots, \lambda_k\} \) be a finite set of regular values of \( f \), such that \( \lambda_0 = a, \lambda_k = b \), and for each \( 0 \leq i \leq k - 1 \) we have \( \lambda_i < \lambda_{i+1} \) and there is exactly one critical value of \( f \) in \( [\lambda_i, \lambda_{i+1}] \). The values \( \lambda_i, \lambda_{i+1} \) will be called adjacent. The set of pairs \( \{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda} \) is called ranging system for \((f, v)\) if

\begin{align*}
&\text{(RS1) } \forall \lambda \in \Lambda, A_\lambda \text{ and } B_\lambda \text{ are disjoint compacts in } f^{-1}(\lambda). \\
&\text{(RS2) } \text{Let } \lambda, \mu \in \Lambda \text{ be adjacent. Then for every } p \in S(f) \cap f^{-1}(\lambda, \mu) \text{ either } \\
&\quad \text{i) } D(p, v) \cap f^{-1}(\lambda) \subset \text{Int } A_\lambda \quad \text{or } \quad \text{ii) } D(p, -v) \cap f^{-1}(\mu) \subset \text{Int } B_\mu. \\
&\text{(RS3) } \text{Let } \lambda, \mu \in \Lambda \text{ be adjacent. Then } v^{-\infty}_{[\lambda, \mu]}(A_\mu) \subset \text{Int } A_\lambda \text{ and } (-v)^\infty_{[\lambda, \mu]}(B_\lambda) \subset \text{Int } B_\mu. \\
&\end{align*}
The following lemma is obvious.

**Lemma 4.4.**  
(1) Let \( S = \{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda} \) be a ranging system for \((f, v)\). Let \( \mu, \nu \in \Lambda \).  
Then \( S' = \{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda, \mu \leq \lambda \leq \nu} \) is a ranging system for \((f \circ f^{-1}([\mu, \nu]), v)\).

(2) Let \( c \in [a, b[ \) be a regular value of \( f \). Let \( \{(A'_\lambda, B'_\lambda)\}_{\lambda' \in \Lambda'} \) be ranging systems for \((f \circ f^{-1}([a, c]), v)\), and, respectively, for \((f \circ f^{-1}([c, b]), v)\). Assume that \( A'_c \subset A'_c \), \( B'_c \subset B''_c \) and \( A'_c \cap B''_c = \emptyset \). Then the system \( \{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda} \), defined by \((C)\) is a ranging system for \((f, v)\).

\[ \Lambda = \Lambda' \cup \Lambda''; \quad A_\lambda = A'_\lambda \quad \text{and} \quad B_\lambda = B'_\lambda \quad \text{for} \quad \lambda \in \Lambda' \setminus \{c\}; \quad A_\lambda = A''_\lambda \quad \text{and} \quad B_\lambda = B''_\lambda \quad \text{for} \quad \lambda \in \Lambda'' \setminus \{c\}; \quad A_c = A'_c, \quad B_c = B''_c \quad \text{ (C) } \]

\( \square \)

In 4.5-4.7 \( \{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda} \) stands for a ranging system for \((f, v)\).

**Lemma 4.5.** There is \( \epsilon > 0 \) such that for any \( f \)-gradient \( w \) with \( \|v - w\| < \epsilon \) the ranging system \( \{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda} \) is also a ranging system for \((f, w)\).

**Proposition 4.6.** Let \( N \) be a submanifold of \( V_1 \setminus B_b \) such that \( N \setminus \text{Int} \, A_b \) is compact.

Then \( N' = v^\ast_{[a, b]}(N) \) is a submanifold of \( V_0 \setminus B_a \) such that \( N' \setminus \text{Int} \, A_a \) is compact.

**Proposition 4.7.** There is a homomorphism \( H(v) : H_\ast(V_1 \setminus B_b, A_b) \longrightarrow H_\ast(V_0 \setminus B_a, A_a) \), such that

1. If \( N \) is an oriented submanifold of \( V_1 \), satisfying the hypotheses of 4.6, then \( H(v)([N]) = [v^\ast_{[a, b]}(N)] \).
2. There is an \( \epsilon > 0 \) such that for every \( f \)-gradient \( w \) with \( \|w - v\| < \epsilon \) we have \( H(v) = H(w) \).

The proof of 4.5 - 4.7 occupies the rest of Subsection B. An easy induction argument shows that it is sufficient to prove each of them in the case card \( \Lambda = 1 \). Let \( S(f) = S_1(f) \sqcup S_2(f) \), where for every \( p \in S_1(f) \), resp. \( p \in S_2(f) \) the i), resp. ii) of (RS2) holds. Pick Morse functions \( \phi_1, \phi_2 : W \to [a, b], \) adjusted to \((f, v)\), such that there are regular values \( \mu_1, \mu_2 \) of \( \phi_1, \mu_2 \) of \( \phi_2 \) satisfying: (1) for every \( p \in S_1(f) \) we have: \( \phi_1(p) < \mu_1 \) and \( \phi_2(p) > \mu_2 \). (2) for every \( p \in S_2(f) \) we have: \( \phi_1(p) > \mu_1 \) and \( \phi_2(p) < \mu_2 \).

**Proof of 4.5.** Proposition 1.10 implies that (RS2) holds for all \( f \)-gradients \( w \), sufficiently close to \( v \). Passing to (RS3), consider the cobordism \( \phi_1^{-1}([\mu_1, b]) \) and the \( f \)-gradient \( v \). Denote \( T(A_b, v) \cap \phi_1^{-1}(\mu_1) \) by \( Z_+ \). Since every \(-v\)-trajectory starting in \( A_b \) reaches \( \phi_1^{-1}(\mu_1) \), \( Z_+ \) is compact. Consider the cobordism \( \phi_1^{-1}([a, \mu_1]) \) and the compact \( V_0 \setminus \text{Int} \, A_c \subset V_0 \). Every \( v \)-trajectory starting in \( V_0 \setminus \text{Int} \, A_c \) reaches \( \phi_1^{-1}(\mu_1) \) and the compacts \( Z_+ \) and \( Z_- = \phi_1^{-1}(\mu_1) \cap T(V_0 \setminus \text{Int} \, A_c, -v) \) are disjoint.

Choose disjoint open neighborhoods: \( U_+ \) of \( Z_+ \) and \( U_- \) of \( Z_- \) in \( \phi_1^{-1}(\mu_1) \). It follows from 5.6 that there is \( \epsilon'' > 0 \) such that for every \( w \in \text{Vect}_1(W, \perp) \) with \( \|w - v\| < \epsilon'' \) we have:

\[ T(A_b, w) \cap \phi_1^{-1}(\mu_1) \subset U_+, \quad \text{and} \quad T(V_0 \setminus \text{Int} \, A_c, -w) \cap \phi_1^{-1}(\mu_1) \subset U_- \]

therefore for all \( f \)-gradients \( w \), sufficiently \( C^0 \) close to \( v \), we have: \( \widetilde{w}(A_b) \subset \text{Int} \, A_c \).

The second part of (RS2) is considered in the same way. \( \square \)
For $\delta > 0$ denote by $D1_\delta(v)$, resp. by $D1_\delta(-v)$, the intersection with $V_0$, resp. with $V_1$, of $\cup_{p \in S1(f)} D_\delta(p, v)$, resp. of $\cup_{p \in S1(f)} D_\delta(p, -v)$. By abuse of notation the intersection of $\cup_{p \in S1(f)} D(p, v)$ with $V_0$ will be denoted by $D1_0(v)$. Denote $D2_\delta(-v) \cup (-v) \sim (B_a)$ by $\Delta(\delta, -v)$ and $D1_\delta(v) \cup \sim v(A_b)$ by $\nabla(\delta, v)$. The similar notation like $D2_\delta(-v)\Delta(0, -v)$, $D2_0(v)$ etc. are now clear without special definition. For $\delta > 0$ sufficiently small we have

$$\forall p \in S1(f) : D_\delta(p) \in \phi_1^{-1}([a, \mu_1]) \quad \text{and} \quad D_\delta(p) \subset \phi_2^{-1}([\mu_2, b])$$

(D1)

$$\forall p \in S2(f) : D_\delta(p) \in \phi_1^{-1}([\mu_2, b]) \quad \text{and} \quad D_\delta(p) \subset \phi_2^{-1}([a, \mu_2])$$

Applying 2.9 and 2.10 to functions $\phi_1, \phi_2$ and their restrictions it is easy to prove that for $\delta > 0$ sufficiently small we have:

(D2) $\nabla(\delta, v) \subset \text{Int } A_a, \quad \Delta(\delta, -v) \subset \text{Int } B_b, \quad \Delta(\delta, -v) \cap D1_\delta(-v) = \emptyset$

Fix some $\delta > 0$ satisfying (D1) and (D2).

Proof of 4.6. The set $N \setminus \text{Int } A_a$ is a closed subset of $\sim v(N \setminus (\text{Int } A_b \cup B_\delta(-v)))$. The set $N \setminus (\text{Int } A_b \cup B_\delta(-v))$ is a compact subset of the domain of $\sim v$ and 4.6 follows. □

Homomorphism $H(v; \mu', \mu; U) : H_*(V_1 \setminus B_b, A_b) \to H_*(V_0 \setminus B_a, A_a)$.

Let $0 \leq \mu' < \mu \leq \delta$. Let $U$ be any subset of $V_1$ such that

$$\Delta(0, -v) \subset U \subset B_b \quad \text{and} \quad U \cap D1_\delta(-v) = \emptyset$$

(4.1) (for example $U = \Delta(\delta, -v)$ will do). Denote by $H(v; \mu', \mu; U)$ the following sequence of homomorphisms

$$H_*(V_1 \setminus B_b, A_b) \overset{I}{\to} H_*(V_1 \setminus U, A_b \cup D1_\mu(-v)) \overset{\text{Exc}^{-1}}{\to}$$

$$H_*(V_1 \setminus (U \cup D1_{\mu'}(-v)), (A_b \cup D1_{\mu'}(-v)) \setminus D1_{\mu'}(-v)) \overset{\sim v}{\to} H_*(V_0 \setminus B_a, A_a).$$

(Here $I$ is the corresponding inclusion arrow. Note that the last arrow is well defined since $(-v) \sim (B_a) \subset U$ and $D0(-v) \cap V_1 \subset D2_0(-v) \cup D1_{\mu'}(-v)$.) The composition $\sim v \circ \text{Exc}^{-1} \circ I_*$ of this sequence will be denoted by $H(v; \mu', \mu; U)$.

Homomorphism $H(v) : H_*(V_1 \setminus B_b, A_b) \to H_*(V_0 \setminus B_a, A_a)$.

Suppose that $U'$ is another subset of $B_b$ satisfying (4.1) and $U \subset U'$. Then the inclusion $V \setminus U' \subset V_1 \setminus U$ induces a map of the sequence $H(v; \mu', \mu; U')$ to the sequence $H(v; \mu', \mu; U)$, which is identity on the first term $H_*(V_1 \setminus B_b, A_b)$ and on the last term $H_*(V_0 \setminus B_a, A_a).$ This implies easily that $H(v; \mu', \mu; U)$ does not depend on the choice of $U$. Similarly one checks that $H(v; \mu', \mu; U)$ does not depend on the choice of $\mu'$ and $\mu$, neither on the choice of $\delta$, or on the choice of presentation $S(f) = S1(f) \cup S2(f)$ (if there is more than one such presentation). Therefore this homomorphism is well determined by $v$ and the ranging system $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$. We shall denote it by $H(v)$.

Proof of (1) of 4.7. Consider the sequence $H(v; 0, \delta; U)$ with any $U$ compact and apply Lemma 4.1. □

Passing to the proof of 4.7 (2) choose and fix some $\mu', \mu$ with $0 < \mu' < \mu < \delta$ and an open subset $U$, such that $\Delta(0, -v) \subset U \subset \text{Int } B_b$, and $D1_\delta(-v) \subset V_1 \setminus U$. The proof of the next lemma is similar to the proof of 4.5 above and will be omitted.
Lemma 4.8. There is $\epsilon > 0$ such that for every $f$-gradient $w$ with $\|w - v\| < \epsilon$ we have:

\[
\Delta(0, -w) \subset U; \quad \nabla(\delta, w) \subset \text{Int } A_a
\]

(4.3)

\[
D_{1}(0) \subset B_{1}(v) \subset B_{1}(w) \subset D_{1}(w) \subset V_1 \setminus \overline{U}. \quad \Box
\]

Proof of (2) of 4.7. Denote $V_1 \setminus (U \cup B_{1}(v))$ by $R$ and $(A_b \cup D_{1}(v)) \setminus B_{1}(v)$ by $Q$. Then $R$ is in the domain of $\tilde{w}$, and $\tilde{v}$ maps the pair of compacts $(R, Q)$ to the pair of open sets $(V_0 \setminus B_a, \text{Int } A_a)$. By 5.6 there is $\kappa > 0$ such that for every $u \in \text{Vect}^1(W, \bot)$ with $\|u - v\| < \kappa$ each $(-u)$-trajectory starting at a point $x \in R$ reaches $V_0$ and intersects it at a point $L(u, x) \in V_0 \setminus B_a$, and for $x \in Q$ we have $L(u, x) \in \text{Int } A_a$. By [19, 8.10] the map $(u, x) \mapsto L(u, x)$ is continuous with respect to $C^1$ topology in $\text{Vect}^1(W, \bot)$. Therefore $H(v)$ equals to the composition of the following sequence:

\[
H_*(V_1 \setminus B_b, A_b) \xrightarrow{I} H_*(V_1 \setminus U, A_b \cup D_1(v)) \xrightarrow{\Delta} H_*(V_1 \setminus (U \cup D_1(v)), (A_b \cup D_1(v)) \setminus D_1(v)) \xrightarrow{\tilde{w}} H_*(V_0 \setminus B_a, A_a).
\]

Let $\epsilon > 0$ satisfy the conclusions of 4.8, and set $\epsilon_0 = \min(\epsilon, \kappa)$. Let $w$ be an $f$-gradient with $\|w - v\| < \epsilon_0$. Then (4.2), (4.3) imply that there is a morphism of the sequence (4.4) to the sequence $H(w; 0; \delta, U)$ such that the homomorphisms on the first term and on the last term are identical (the two homomorphisms in the middle are induced by inclusions). This implies that the composition of (4.4) equals to $H(w)$ and 4.7 follows. $\Box$

C. Constructing ranging systems. In this subsection we construct ranging systems from $\mathcal{M}$-flows on $V_0$ and $V_1$. Denote $\text{dim } W$ by $n$. We assume that $\text{S}(f) \neq \emptyset$.

Definition 4.9. Let $v$ be an almost good $f$-gradient. Let $\mathcal{V}_0 = (\phi_0, u_0, U_0), \mathcal{V}_1 = (\phi_1, u_1, U_1)$ be almost good $\mathcal{M}$-flows on $V_0$, resp. $V_1$. We shall say that $(\mathcal{V}_0, \mathcal{V}_1)$ is a ranging pair for $(f, v)$ if $\mathbb{K} \ni \mathbb{K}(u_0), \mathbb{K}(u_1)$, and there is a $\delta > 0$, such that for any $0 \leq s \leq n - 1$ the following conditions hold:

(RP1) for some $\delta_1 > \delta$ the gradients $v, u_0$ and $u_1$ are $\delta_1$-separated

(RP2) $\overline{T(\mathbb{K}(u_1), v)_{\leq s+1}(\delta)} \cap V_0 \subset \mathbb{K}(u_0)_{\leq s}(\delta)$

(RP3) $\overline{T(\mathbb{K}(u_0), v)_{\leq s+1}(\delta)} \cap V_1 \subset \mathbb{K}(u_1)_{\leq s}(\delta)$

Remark. Conditions (RP2) and (RP3) demand in particular that $\delta$ is in the interval of definition of $\mathbb{T(\mathbb{K}(u_1), v)}$, resp. $\mathbb{T(\mathbb{K}(u_0), v)}$.

We shall show that every ranging pair $(\mathcal{V}_0, \mathcal{V}_1)$ generates a bunch of ranging systems.
Construction 4.10. Let $\Lambda = \{\lambda_0, \ldots, \lambda_k\}$ be a set of regular values of $f$, such that $\lambda_i < \lambda_{i+1}$, $\lambda_0 = a$, $\lambda_k = b$ and in each $[\lambda_i, \lambda_{i+1}]$ there is only one critical value of $f$. Choose some $\delta' \in ]\delta, \delta_1[ \text{ in such a way, that } \forall s : 0 \leq s \leq n - 1 \text{ we have}$

(RP4) \[
\begin{align*}
\overline{T(K(u_1), v)_{(\leq s+1)}(\delta')} \cap V_0 & \subset K(u_0)_{(\leq s)}(\delta) \\
\overline{T(K(-u_0), -v)_{(\leq s+1)}(\delta')} \cap V_1 & \subset K(-u_1)_{(\leq s)}(\delta)
\end{align*}
\]

(such $\delta'$ exists since $\delta$ is in the interval of definition of $T(K(u_1), v)$, resp. $T(K(-u_0), -v)$.)

We shall now define for each integer $s : 0 \leq s \leq n$ compact subsets $A^{(s)}_\lambda$ and $B^{(s)}_\lambda$ of $f^{-1}(\lambda)$. Set $A^{(0)}_\lambda = B^{(0)}_\lambda = \emptyset$. Let $\delta = \delta_k < \delta_{k+1} < \ldots < \delta_1 < \delta_0 = \delta'$ be a sequence of real numbers. For $1 \leq s \leq n$ set

\[
\begin{align*}
A^{(s)}_\lambda &= \overline{T(K(u_1), v)_{(\leq s)}(\delta_l) \cap f^{-1}(\lambda_l)} \text{ for } 0 < l \leq k \\
A^{(s)}_\lambda &= K(u_0)_{(\leq s-1)}(\delta) = D_\delta(\text{ind} \leq s-1 ; u_0) \\
B^{(s)}_\lambda &= \overline{T(K(-u_0), -v)_{(\leq s)}(\delta_{k-l}) \cap f^{-1}(\lambda_l)} \text{ for } 0 \leq l < k \\
B^{(s)}_\lambda &= K(-u_1)_{(\leq s-1)}(\delta) = D_\delta(\text{ind} \leq s-1 ; -u_1)
\end{align*}
\]

Lemma 2.16 imply that $A^{(s)}_b = D_\delta(\text{ind} \leq s-1 ; u_1)$ and $B^{(s)}_a = D_\delta(\text{ind} \leq s-1 ; -u_0)$. $\triangle$

Lemma 4.11. (1) $A^{(r)}_\lambda \cap B^{(s)}_\lambda = \emptyset$ for $r + s \leq n$.

(2) If $0 \leq s \leq n - 1$, then $\{(A^{(s)}_\lambda, B^{(n-s-1)}_\lambda)\}_{\lambda \in \Lambda}$ is a ranging system for $(f, v)$.

(3) For every $0 \leq s \leq n - 1$ the pair $(V_1 \setminus B^{(n-s-1)}_b, A^{(s)}_b)$ is homotopy equivalent to a finite CW-pair, having only cells of dimension $s$.

Proof. (1) For $\lambda = a, \lambda = b$ it follows directly from (RP1). Let $0 < l < k$ and set $\lambda = \lambda_l$. Then $\delta_l$, $\delta_{k-l} < \delta'$ and $A^{(r)}_\lambda \subset T(K(u_1)_{(\leq r-1)}(\delta'), v) \cup B^{(r)}_\delta(\text{ind} \leq r ; v)$, and $B^{(s)}_\lambda \subset T(K(-u_0)_{(\leq s)}(\delta'), -v) \cup B^{(s)}_\delta(\text{ind} \leq s , -v)$. Therefore if $A^{(r)}_\lambda \cap B^{(s)}_\lambda \neq \emptyset$, there is a $(-v)$-trajectory joining $1$ a point of $K(u_1)_{(\leq r-1)}(\delta')$ with a point of $K(-u_0)_{(\leq s)}(\delta')$ or $2$ a point of $K(u_1)_{(\leq r-1)}(\delta')$ with a point of $B^{(r)}_\delta(p)$, where $p \in S(f)$, $\text{ind}p \geq n - s$ or $3$ a point of $K(-u_0)_{(\leq s)}(\delta')$ with a point of $B^{(r)}_\delta(q)$, where $q \in S(f), \text{ind}q \leq r$ or $4$ a point of $B^{(r)}_\delta(p)$ with a point of $B^{(r)}_\delta(q)$, where $p, q \in S(f), \text{ind}q \geq n - s, \text{ind}p \leq r$.

The last case is impossible, since $v$ is $\delta'$-separated, the first is impossible because of (RP4) and since $u_0$ is $\delta'$-separated. $2$ and $3$) are considered similarly.

(2) $(RS1)$ follows from $(1)$. $(RS2)$ follows, since for all $p \in S(f)$ we have $\text{ind}p \leq s$ or $\text{ind}p \geq s + 1$. $(RS3)$ follows from the construction.

(3) Let $\phi : V_1 \to \mathbb{R}$ be a Morse function, adjusted to $(\phi_1, u_1)$, and such that if $p \in S(\phi)$ then $D_\delta(p) \subset \phi^{-1}([a_k, a_{k+1}])$ where $k = \text{ind}p$. Using deformations along $(-u_1)$-trajectories it is easy to see that $(V_1 \setminus B^{(n-s-1)}_b, A^{(s)}_b) \sim (\phi^{-1}([a_0, a_{s+1}]), \phi^{-1}([a_0, a_s]))$. $\square$

We proceed to construction of ranging pairs. Let $v$ be an almost good $f$-gradient. Choose $\alpha, \beta \in ]a, b[$ such that $a \leq \beta$ and that the intervals $[a, \alpha]$, $[\beta, b]$ are regular.
Let \( \mu > 0 \) be so small that the map \((x, \tau) \mapsto \gamma(x, \tau; v)\) is defined on \( V_0 \times [0, 3\mu] \) and on \( V_1 \times [-3\mu, 0] \). We denote these maps by \( \Psi_0 : V_0 \times [0, 3\mu] \to W \), and, resp. by \( \Psi_1 : V_1 \times [-3\mu, 0] \to W \). We assume further that \( \mu \) is so small that \( \operatorname{Im} \Psi_0 \subset f^{-1}([a, \alpha]) \), \( \operatorname{Im} \Psi_1 \subset f^{-1}([\beta, b]) \), and that \( \Psi_0(\Psi_1) \)-induced metric is \( 2 \)-equivalent to the product metric. The \( \Psi_i \)-image of \( V_i \times [\alpha, \beta](i = 0, 1) \) will be denoted by \( U_i(\alpha, \beta) \).

**Lemma 4.12.** Let \( A, B \) be \( s \)-submanifolds of \( V_1 \), resp. \( V_0 \). Then there is an almost good \( f \)-gradient \( v_1 \), \( C^\infty \)-close to \( v \) and such that

1. \( \operatorname{supp} (v - v_1) \subset (U_0(2\mu, 3\mu) \cup U_1(3\mu, -2\mu)) \)
2. \( \mathbb{K}_b(-v_1) \uparrow A; \quad \mathbb{V}_1(A) \uparrow B \), \( (-v_1) \uparrow (B) \uparrow A \)

**Proof.** Note first that any \( f \)-gradient \( v_1 \), satisfying (1) is almost good. Note also that the first property of (3) implies the second. Now let \( \xi \in \text{Vectt}(V_0) \) be a small vector field, such that \( \Phi(\xi, \mu)(\mathbb{K}_b(-v_1)) \uparrow A \) (such vector field exists by 2.4). Then the vector field \( v_0 \) which equals \( v \) everywhere except \( U_1(-3\mu, -2\mu) \) and in this neighborhood equals \( (\Psi_0)_*(\xi \times \frac{1}{\mu}) \), satisfies (1) and (2). Therefore the \( s \)-submanifold \( \mathbb{V}_0(\mathbb{V}_0) \) of \( V_0 \) is defined. Applying to \( v_0 \) the same procedure as above (nearby \( V_0 \)) we get an \( f \)-gradient \( v_1 \), satisfying (1), (2) and the first part of (3). \( \Box \)

**Theorem 4.13.** Let \( \varepsilon > 0 \). Then there is an almost good \( f \)-gradient \( w \), and a ranging pair \((V_0, V_1)\) for \((f, w)\), such that \( \|w - v\| \leq \varepsilon \) and \( \operatorname{supp} (w - v) \subset U_0(\mu, 3\mu) \cup U_1(-3\mu, -\mu) \).

**Proof.** Choose \( \varepsilon' > 0 \) so small that \( \varepsilon' \leq \varepsilon \) and \( \varepsilon' < \frac{1}{\|\text{grad} f\|} \cdot \min_{x \in \mathbb{V}_0 \cup \mathbb{V}_1} df(v)(x) \).

Let \( V_i = (\phi_i, u_i, U_i)(i = 0, 1) \) be an almost good \( M \)-flow on \( V_i \), satisfying the conclusions of 1.19 with respect to \( C = \varepsilon'/4, \beta = \mu/3 \). Let \( \delta_0 > 0 \) be so small that \( u_1, u_0 \) are \( \delta_0 \)-separated; then \( \delta_0 \) is in the interval of definition of \( K(-u_0), K(u_1) \).

Let \( u \) be an almost good \( f \)-gradient, satisfying the conclusions of the preceding lemma with respect to \( A = K(u_1), B = K(-u_0) \), and such that \( \|u - v\| < \varepsilon'/2 \). Denote by \( \mathbb{W} \) the cobordism \( f^{-1}([\alpha, \beta]) \) and by \( f, \widetilde{u} \) the restrictions of \( f \), resp. \( u \) to \( \mathbb{W} \). Denote \( f^{-1}(\alpha) \) by \( V_1 \) and \( f^{-1}(\beta) \) by \( V_0 \). Denote by \( \mathbb{A}, \mathbb{B} \), the image of \( K(u_1), K(-u_0) \), with respect to the diffeomorphism \( v_{[\alpha, \beta]} \), resp. \( (-u)_{[a, a]} \).

Fix \( \theta \) as the interval of definition of \( \mathbb{A}, \mathbb{B} \). The cores of \( \mathbb{A}, \mathbb{B} \) will be denoted by \( \mathbb{A}, \mathbb{B} \). Choose \( \delta_1 > 0 \) such that \( \widetilde{u} \) is \( \delta_1 \)-separated. Choose \( \theta > 0 \) so small that \( \theta < \delta_0, \theta < \delta_1, \theta \) is in the interval of definition of \( T(\mathbb{A}, \mathbb{A}) \) and of \( T(\mathbb{B}, \mathbb{B}) \) and for every \( 0 \leq s \leq n - 1 \) we have:

\[
T(\mathbb{A}, \mathbb{A})_{[s]}(\theta) \cap T(\mathbb{B}, \mathbb{B})_{[s+n-s-2]}(\theta) = \emptyset; \quad T(\mathbb{B}, \mathbb{A})_{[s]}(\theta) \cap T(\mathbb{A}, \mathbb{B})_{[s+n-s-2]}(\theta) = \emptyset
\]

Let \( W_i = (\phi_i, u'_i, U'_i)(i = 0, 1) \) (where \( i = 0, 1 \) be an \( M \)-flow on \( V_i \), subordinate to \( V_i \) and satisfying the condition 2) of 1.19 with respect to some \( \delta \in [0, \theta] \). Let \( \varkappa : [0, 2\mu] \to [0, 1] \) be a \( C^\infty \) function, such that \( \operatorname{supp} \varkappa \subset [\mu, 2\mu] \) and \( \varkappa(t) = 1 \) for \( t \in \left[ \frac{5}{3}, \frac{5}{3} \right] \).

Define a vector field \( w \) on \( W \), setting

\[
w(x) = u(x) \quad \text{for} \quad x \notin U_0(\mu, 2\mu) \cup U_1(-2\mu, -\mu)
\]

\[
((\Psi_0^{-1})_* w)(y, t) = (\varkappa(t) u'_0(y), 1) \quad \text{for} \quad y \in V_0, t \in [\mu, 2\mu]
\]

\[
((\Psi_1^{-1})_* w)(y, t) = (\varkappa(t) u'_1(y), 1) \quad \text{for} \quad y \in V_1, t \in [-2\mu, -\mu]
\]
We claim that \( w \) and \( (V_0, \mathcal{V}_1) \) satisfy the conclusions of the theorem. Our choice of \( \epsilon' \) implies that \( w \) is an \( f \)-gradient. By the construction \( w \) is almost good and \( \|w - v\| < \epsilon \). The condition \( \text{supp}(w - v) \subset U_0(\mu, 3\mu) \cup U_1(-3\mu, -\mu) \) follows from (1) of 4.12 and from (4) above. To prove that \( (V_0, \mathcal{V}_1) \) is a ranging pair for \( (f, w) \) note first that \( 0 < \delta < \theta \), the gradients \( u_0, u_1 \) are \( \theta \)-separated. Since \( \tilde{u} \) is \( \theta \)-separated, \( w \) is also \( \theta \)-separated; (RP1) follows. The formulas (4.6) allow to give the following description of the \((w)\)-trajectories, starting at a point \( x \in V_1 \).

(1) \( \gamma(x, \cdot, -w) \) reaches \( \tilde{V}_1 \) and intersects it at a point \( x_1 = u^\gamma_{[\beta, \beta]}(\gamma(x, \tau(x); -u') \) where \( \tau(x) \geq \mu/3 \).

(2A) \( \gamma(x_1, \cdot, -\tilde{u}) \) converges to some \( p \in S(f) \). Then the same is true for \( \gamma(x, \cdot, -w) \).

(2B) \( \gamma(x_1, \cdot, -\tilde{u}) \) reaches \( \tilde{V}_0 \) and intersects it at a point \( x_2 \). Then the same is true for \( \gamma(x, \cdot, -w) \) and we have:

(3) \( \gamma(x, \cdot, -w) \) reaches \( V_0 \) and intersects it at a point \( x_3 = \gamma(u^\gamma_{[\alpha, \alpha]}(x_2), \tau(x_2); -u', \) \( \tau(x_2) \geq \mu/3 \).

(Similarly for \( w \)-trajectories starting at a point of \( V_0 \).) This implies that \( u^\gamma_{[\beta, \beta]}(\mathbb{K}(u_1)) = \tilde{A} \) and \( (-\tilde{w})^\gamma_{[\alpha, \alpha]}(\mathbb{K}(-u_0)) = \tilde{B} \), from that one deduces easily that \( \mathbb{K}_\beta(-w) \upharpoonright \mathbb{K}(u_1) \), and \( \mathbb{K}_\alpha(w) \upharpoonright \mathbb{K}(-u_0) \). Further, the point (1) of the above description implies that

\( (\star) \quad \forall \lambda < \delta_0 \) and \( \forall s : 0 \leq s \leq n - 1 \) we have \( u^\gamma_{[\beta, \beta]}(\mathbb{K}(u_1)(s))(\lambda)) \subset \tilde{A}(s)(\lambda) \)

Therefore 2.17 applies to show that \( \theta \) is in the interval of definition of \( T(\mathbb{K}(u_1), w) \).

Similarly, \( \theta \) is in the interval of definition of \( T(\mathbb{K}(-u_0), -w) \). Further, \((\star)\) and (4.5) imply that for every \( z \in T(\mathbb{K}(u_1), u)(\ell(s))(\theta) \cap \tilde{V}_0 \) we have: \( u^\gamma_{[\alpha, \alpha]}(z) \in \\tilde{V}_0 \setminus \mathbb{B}_{\tilde{f}}(\text{ind} \leq \kappa - s - 2, -u_0) \). Therefore the third formula of (4.6) together with (1.2) imply that \( T(\mathbb{K}(u_1), u)(\ell(s))(\theta) \cap V_0 \subset \mathbb{B}_{\tilde{f}}(\text{ind} \leq \kappa - s - 2, -u_0) \) which implies (RP1) since \( \delta < \theta \).

(RP2) is similar. \( \square \)

D. Equivariant ranging systems and the proof of the main theorem. We assume here the terminology of Subsection B of Introduction. It is easy to see that it suffices to prove the Main Theorem in the case when \( |f| \in H^1(M, \mathbb{Z}) \) is indivisible, and we make this assumption from now on. This assumption implies that \( \forall z \in M : F(z) - F(zt) = 1 \). We assume also \( S(f) \neq \emptyset \). Choose a riemannian metric on \( M \). Then \( M \) obtains a \( t \)-invariant riemannian metric.

**Definition 4.14.** Let \( u \) be any \( f \)-gradient. Let \( \Sigma \) be a non-empty set of regular values of \( F \), satisfying (S) below.

(S): for every \( A, B \in \mathbb{R} \) the set \( \Sigma \cap F^{-1}([A, B]) \) is finite; if \( \sigma \in \Sigma \) then \( \forall n \in \mathbb{Z} \): \( \sigma + n \in \Sigma \); if \( \lambda, \mu \in \Sigma \) are adjacent, then there is only one critical value of \( F \) between \( \lambda \) and \( \mu \).

A set \( \{ (A_\sigma, B_\sigma) \}_{\sigma \in \Sigma} \) is called \( t \)-equivariant ranging system for \( (F, u) \) if

(ERS1) For every \( \mu, \nu \in \Sigma, \mu < \nu \) we have: \( \{ (A_\sigma, B_\sigma) \}_{\sigma \in \Sigma, \nu \leq \sigma \leq \nu} \) is a ranging system for \( (F|F^{-1}([\mu, \nu]), u) \).

(ERS2) \( A_{\sigma - n} = A_\sigma \cdot t^n \), \( B_{\sigma - n} = B_\sigma \cdot t^n \) for every \( n \in \mathbb{Z} \). \( \triangle \)

Let \( \{ (A_\sigma, B_\sigma) \}_{\sigma \in \Sigma} \) be a \( t \)-equivariant ranging system for \( (F, u) \). For \( \nu, \mu \in \Sigma, \nu < \mu \), denote by \( H_{[\mu, \nu]}(u) \) the homomorphism \( H(u) \), associated (by (4.7)) to the system \( \{ (A_{\sigma - n}, B_{\sigma - n}) \} \) for \( \mu \in \Sigma \) set by definition \( H_{\mu - 1} = \text{id} : H(F^{-1}(u)) \).
\[ B_\mu, A_\mu \to H_\ast (F^{-1}(\mu) \setminus B_\mu, A_\mu) \]. It follows from the construction that \[ H_{[\mu, \nu-1]}(u) = t \circ H_{[\mu, \nu]}(u) \circ t^{-1} \] and that \[ H_{[\nu, \theta]}(u) \circ H_{[\mu, \nu]}(u) = H_{[\mu, \theta]}(u) \]. For \( \nu \in \Sigma \) denote \( t^{-1} \circ H_{[\nu, \theta]}(u) \circ H_{(\nu)}(u) \). It is an endomorphism of \( H_\ast (F^{-1}(\nu) \setminus B_\nu, A_\nu) \). We have obviously \( H_{[\nu, \nu-k]}(u) = t^k \circ (H_\nu(u))^k \). The next lemma follows directly from 4.5 - 4.7.

**Lemma 4.15.** Let \( \{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma} \) be a \( t \)-equivariant ranging system for \((F, u)\). Let \( \nu, \mu \in \Sigma, \nu \leq \mu \) and \( k \in \mathbb{N} \). Let \( N \) be an oriented submanifold of \( F^{-1}(\mu) \setminus B_\mu \) such that \( N \setminus \text{Int} A_\mu \) is compact. Let \( L \) be a compact cooriented submanifold of \( F^{-1}(\nu) \setminus A_\nu \). Assume that \( \text{dim} N + \text{dim} L = \text{dim} M - 1 \). Then:

1. \( N' = u_{[\mu, \nu-k]}(N) \) is an oriented submanifold of \( F^{-1}(\nu-k) \setminus B_{\nu-k} \) such that \( N' \setminus \text{Int} A_{\nu-k} \) is compact. If \( N' \cap L^k \), then \( N' \cap L^k \) is finite and

\[
N' \cap L^k = \left\{ i_* \cdot \left( H_{(\nu)}(u) \right)^k \left( H_{[\mu, \nu]}(u)([N]) \right) \right\},
\]

where \( i : (F^{-1}(\nu) \setminus B_\nu, A_\nu) \to (F^{-1}(\nu), F^{-1}(\nu) \setminus L) \) is the inclusion map.

2. For every \( f \)-gradient \( w \), sufficiently close to \( u \) in \( C^0 \)-topology, \( \{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma} \) is also a \( t \)-equivariant ranging system for \((F, w)\) and \( H_{(\nu)}(u) = H_{(\nu)}(w), H_{[\mu, \nu]}(u) = H_{[\mu, \nu]}(w) \). \( \Box \)

Passing to the proof of the Main Theorem fix first two points \( x, y \in S(f) \), \( \text{ind}_x = \text{ind}_y + 1 \), and assume that \( F(\bar{y}) < F(\bar{x}) \leq F(\bar{y}) + 1 \). Denote \( \text{dim} M \) by \( n \); denote \( \text{ind} x \) by \( l + 1 \), then \( \text{ind} y = l \). Choose some set \( \Sigma \) of regular values of \( F \), satisfying (S) of Definition 4.14.

Denote by \( \theta \) the maximal element of \( \Sigma \) with \( \theta < F(\bar{x}) \) and by \( N(\nu) \) the intersection \( D(\bar{x}, v) \cap F^{-1}(\theta) \); \( N(\nu) \) is an oriented submanifold of \( F^{-1}(\theta) \), diffeomorphic to \( S^l \). Denote by \( \eta \) the minimal element of \( \Sigma \), satisfying \( \eta > F(\bar{y}) \); then \( \eta \leq \theta < \eta + 1 \). Denote by \( L(\nu) \) the intersection \( D(\bar{y}, -v) \cap F^{-1}(\eta) \); \( L(\nu) \) is a cooriented submanifold of \( F^{-1}(\eta) \), diffeomorphic to \( S^{n-1-1} \). Denote by \( W \) the cobordism \( F^{-1}([\eta, \eta + 1]) \). Note that \( \bar{x} \in W^0 \). Denote \( F^{-1}(\eta) \) by \( V_0, F^{-1}(\eta + 1) \) by \( V_1, \Sigma \cap [\eta, \eta + 1] \) by \( \Lambda \).

Denote by \( \mathcal{G}_t_0(f; x, y) \) the subset of \( \mathcal{G}_t(f) \), consisting of all \( f \)-gradients \( v \), such that there is an equivariant ranging system \( \{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma} \) for \((F, v)\) satisfying

\[
(4.7) \quad N(v) \cap B_\theta = \emptyset, \quad L(\nu) \cap A_\eta = \emptyset,
\]

\[
(4.8) \quad H_\ast (F^{-1}(\eta) \setminus B_\eta, A_\eta) \text{ is a finitely generated abelian group.}
\]

Now we shall prove 4 properties of the set \( \mathcal{G}_t_0(f; x, y) \).

1. \( \mathcal{G}_t_0(f; x, y) \) is open in \( C^0 \)-topology.
   This follows immediately from 4.15(2) and 1.10.

2. \( \mathcal{G}_t_0(f; x, y) \) is dense in \( C^0 \)-topology.
   By Th. 4.13 there is an almost good \( F \)-gradient \( w \) with \( ||w - v|| < \epsilon/2 \) and a ranging pair \((\mathcal{V}_0, \mathcal{V}_1)\) for \((F|W, w|W)\). By 4.11 this ranging pair generates the ranging system \( \{(A^{(i)}_\lambda, B^{(n-1)}_\lambda)\}_{\lambda \in \Lambda} \). Since supp \((w - v) \subset W^\circ \), we can extend \( w \) to a \( t \)-invariant \( F \)-gradient on \( \bar{M} \) (it will be denoted by the same letter \( w \)). Checking through the proof of 4.13 shows that we can choose \( \mathcal{V}_1 = \mathcal{V}_0 \cdot t \). In this case for every \( i \) we have: \( A^{(i)}_\eta = A^0_{(i+1)} \cdot t \) and \( B^{(i)}_\eta = B^0_{(i+1)} \cdot t \); therefore we can extend this ranging system to a \( t \)-equivariant ranging system on \( \bar{M} \). The property

\[
(4.8) \quad H_\ast (F^{-1}(\eta) \setminus B_\eta, A_\eta) \text{ is a finitely generated abelian group.}
\]
Proof. It is sufficient to consider the case of free f.g. polynomials and Lemma 4.16.

Lemma 4.16. Let \( G \) be a finitely generated abelian group, \( A \) be an endomorphism of \( G \) and \( \lambda : G \to Z \) be a homomorphism. Then for every \( p \in G \) the series \( \sum_{k \geq 0} \lambda(A^k p) t^k \in Z[[t]] \) is a rational function of \( t \) of the form \( \frac{P(t)}{Q(t)} \), where \( P, Q \) are polynomials and \( Q(0) = 1 \).

Proof. It is sufficient to consider the case of free f.g. abelian group \( G \). Consider a free f.g. \( Z[[t]] \)-module \( R = G[[t]] \) and a homomorphism \( \phi : R \to R \), given by \( \phi = 1 - At \). Then \( \phi \) is invertible, the inverse homomorphism given by the formula \( \phi^{-1} = \sum_{k \geq 0} A^k t^k \). On the other hand the inverse of \( \phi \) is given by Cramer formulas, which are rational functions with denominator \( Q(t) = \det (1 - At) \). \( \square \)

(4.8) follows from 4.11 (3). By definition \( N(w) = D(\tilde{x}, w) \cap F^{-1}(\theta) \) is in \( A^{(l+1)} \), and \( L(-w) = D(\tilde{y}, -w) \cap F^{-1}(\eta) \) is in \( B^n_{(n-1)} \), which (in view of 4.11(1)) implies (4.7) with respect to \( w \).

Choose an \( f \)-gradient \( \tilde{w} \), with \( \|w - \tilde{w}\| < \epsilon/2 \) satisfying the transversality assumption. If only \( \tilde{w} \) is close enough to \( w \), the system \( \{(A^{(l)}(\lambda), B^{(n-1)(l)}_\lambda)\}_{\lambda \in \Lambda} \) is still a \( t \)-equivariant ranging system for \( \tilde{w} \) (by 4.15(2)), and (4.7) still hold (by 1.10).

3. For every \( v \in Gt_0(f; x, y) \) we have: \( n(x, y; v) = \frac{P(t)}{Q(t)} \), where \( P, Q \in Z[t] \) and \( Q(0) = 1 \).

Denote \( H_1(F^{-1}(\eta) \setminus B_n, A_n) \) by \( H \). Denote by \([x]\) the element \([v_{[\theta, \eta]}(N(v))]\) of \( H \). Denote by \([y]\) the element \([v_{[\theta, \eta]}(N(v))]\) of \( H \). Denote the homomorphism \( H_1(\lambda)(v) \) of \( H \). Denote by \( h \) the endomorphism \( H_1(\lambda)(v) \) of \( H \). By definition we have \( n_k(x, y; v) = (D(\tilde{x}, v) \cap F^{-1}(\eta - k)) \subset (D(\tilde{y}^k, -v) \cap F^{-1}(\eta - k)) \). Therefore 4.15 implies that \( n_k(x, y; v) = [y](h^k([x])) \) if \( k \geq 0 \). Since \( F(\tilde{x}) \leq F(\tilde{y}) + 1 \), we have \( n_k(x, y; v) = 0 \) for \( k < 0 \). Therefore the demanded formula for \( n(x, y; v) \) follows immediately from the next lemma.
Set $\mathcal{G} t_0(f)$ to be the intersection of $\mathcal{G} t_0(f; x, y)$ over all pairs $x, y \in S(f)$ with $\text{ind} x = \text{ind} y + 1$ and the proof of the Main Theorem is over. □

§5. Appendix. $C^0$ Perturbations of Vector Fields and Their Integral Curves

In this appendix we prove some technical results on integral curves of vector fields. The main technical results are 5.1, 5.2, which state that the trajectories of a $C^1$ vector field are in a sense stable under small $C^0$ perturbations of the vector field. For a manifold $M$ (without boundary) we denote by $\text{Vect}^1(M)$ (resp. $\text{Vect}^1_1(M)$) the vector space of $C^1$ vector fields on $M$ (resp. the vector space of $C^1$ vector fields on $M$ with compact support).

A. Manifolds without boundary. In this subsection $M$ is a riemannian manifold without boundary, $v \in \text{Vect}^1(M)$, $n = \dim M$.

**Proposition 5.1.** Let $a, b \in M$, $t_0 \geq 0$ and $\gamma(a, t_0; v) = b$.

Then for every open neighborhood $U$ of $\gamma(x, [0, t_0]; v)$ and every open neighborhood $R$ of $b$ there exist $\delta > 0$ and an open neighborhood $S \subset U$ of $a$ such that $\forall x \in S$ and $\forall w \in \text{Vect}^1_0(M)$ with $\|w - v\| < \delta$ we have: $\gamma(x, t_0; w) \in R$ and $\gamma(x, [0, t_0]; w) \subset U$.

**Proposition 5.2.** Let $a, b \in M$, $t_0 \geq 0$ and $\gamma(a, t_0; v) = b$. Let $E$ be a submanifold without boundary of $M$ of codimension 1, such that $b \in E$ and $v(b) \notin T_b E$.

Let $U$ be an open neighborhood of $\gamma(a, [0, t_0]; v)$ and $R$ be an open neighborhood of $b$ Then for every $\theta > 0$ sufficiently small there exist $\delta > 0$ and an open neighborhood $S \subset U$ of $a$, such that $\forall x \in S$ and $\forall w \in \text{Vect}^1_0(M)$ with $\|w - v\| < \delta$ we have:

1. $\gamma(x, [-\theta, t_0 + \theta]; w) \subset U$, and $\gamma(x, [t_0 - \theta, t_0 + \theta]; w) \subset R$.
2. There is a unique $\tau_0 = \tau_0(w, x) \in [t_0 - \theta, t_0 + \theta]$, such that $\gamma(x, \tau_0; w) \in E$.
3. If $E$ is compact and $t_0$ is the unique $t$ from $[0, t_0]$ such that $\gamma(x, t_0; v) \in E$, then $\forall y \in S$ the number $\tau_0(w, y)$ is the unique $\tau$ from $[-\theta, \theta + \tau]$ such that $\gamma(y, \tau; w) \in E$.

**Proof of 5.1.** The case $M = \mathbb{R}^n$ (with the euclidean metric) is obtained immediately from the next lemma. For $v \in \text{Vect}^1_0(\mathbb{R}^n)$ we denote by $\|v\|$ the norm of the derivative $dv : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^n)$.

**Lemma 5.3.** Let $u, w \in \text{Vect}^1_0(\mathbb{R}^n)$, $\|u - w\| < \alpha$, $\|u\|_1 \leq D$, where $D > 0$.

Let $\gamma, \eta$ be trajectories of, respectively, $u, w$, and assume that $|\gamma(0) - \eta(0)| \leq \epsilon$. Then for every $t \geq 0$ we have: $|\gamma(t) - \eta(t)| \leq \epsilon e^{Dt} + \frac{\alpha}{D}(e^{Dt} - 1)$.

**Proof.** $|\eta'(t) - \gamma'(t)| = |w(\eta(t)) - u(\gamma(t))| \leq |w(\eta(t)) - u(\eta(t))| + |u(\eta(t)) - u(\gamma(t))| \leq \alpha + D|\eta(t) - \gamma(t)|$. Set $s(t) = \eta(t) - \gamma(t)$. Then $|s'(t)| \leq \alpha + D|s(t)|$, and the standard argument (see, for example, [2, p.117]), shows that $|s(t)| \leq |s(0)| e^{Dt} + \frac{\alpha}{D}(e^{Dt} - 1)$. □

Passing to the general case, note that if $0 \leq t_1 \leq t_0$ and 5.1 is true for the curve $\gamma(a, \cdot; v)|[0, t_1]$ and for the curve $\gamma(a', \cdot; v)|[0, t_0 - t_1]$, where $a' = \gamma(a, t_0; v)$ then it is true for $\gamma(a, \cdot; v)|[0, t_0]$. Therefore, applying successive subdivisions, we can assume that $\gamma(a, [0, t_0]; v)$ belongs to the domain of a chart $\phi : W \to \mathbb{R}^n$, and that $W \subset U$. Choose open neighborhoods $W'' \subset \overline{W'} \subset W' \subset \overline{W} \subset W$ of $\gamma(a, [0, t_0]; v)$, such that $\overline{W}$ is compact, and a $C^\infty$ function $h : M \to [0, 1]$ such that supp $h \subset W'$. Then $h \circ \gamma(a, \cdot; v)$ is in the intersection of the curves $G t_0(f; x, y)$ over all pairs $x, y \in S(f)$ with $\text{ind} x = \text{ind} y + 1$ and the proof of the Main Theorem is over. □
and \( h(x) = 1 \) for \( x \in W'' \). Choose \( C > 0 \) so that the metric, induced by \( \phi \) from \( M \) on \( \mathbb{R}^n \) and the euclidean one are \( C \)-equivalent in \( \overline{W} \). For every \( w \in \text{Vect}_0^1(M) \) denote the vector field \( \phi_*(h \cdot w) \) by \( \tilde{w} \in \text{Vect}_0^1(\mathbb{R}^n) \). Note that \( \|\tilde{w}\|_e \leq C \|w\|_\rho \). Let \( R' \subset \phi(R \cap W'') \) be an open neighborhood of \( \phi(b) \). By 5.3 there is \( \delta_0 > 0 \) and an open neighborhood \( S' \subset \phi(W'') \) of \( \phi(a) \) such that for every \( u \in \text{Vect}_0^1(\mathbb{R}^n) \) with \( \|\tilde{v} - u\|_e < \delta_0 \) and \( x \in S' \) the \( u \)-trajectory \( \gamma(x, t; u) \) stays in \( \phi(W'') \) for \( t \in [0, t_0] \) and \( \gamma(x, t_0; u) \subset R' \). It is easy to see that the neighborhood \( \phi^{-1}(S') \) of \( a \) and the number \( \delta_0/C > 0 \) satisfy the conclusions of 5.1. \( \Box \)

**Proof of 5.2. 1) The case \( t_0 = 0 \).**

We represent \( \mathbb{R}^n \) as the product \( \mathbb{R}^1 \times \mathbb{R}^{n-1} \); the elements \( z \in \mathbb{R}^n \) will be therefore referred to as pairs \( z = (x, y) \), where \( x \in \mathbb{R}, y \in \mathbb{R}^{n-1} \); \( x \) is called first coordinate of \( z \). Set \( \mathbb{R}^n_+ = \{(x, y) \mid x > 0 \}, \mathbb{R}^n_- = \{(x, y) \mid x < 0 \}. \) For \( r > 0 \) denote \( J = \{ -r, r \} \times B_1(0, r) \) by \( W_r \). Let \( h : \mathbb{R}^n \to [0, 1] \) be a \( C^\infty \) function such that \( h \in C^\infty \) and \( h(x) = 1 \) for \( x \in W_1 \). Choose a chart \( \phi : W \to \mathbb{R}^n \) of \( M \), such that \( \phi(a) = 0 \); \( \phi(W \cap E) \) is \( \{0\} \times \mathbb{R}^{n-1} \) and the first coordinate of \( \phi_*(v)(a) \) is 1. For a vector field \( w \) on \( M \) we denote by \( \tilde{w} \) the vector field \( h \cdot \phi_* w \) on \( \mathbb{R}^n \). Choose \( C > 0 \) so that the metric, induced by \( \phi \) from \( M \) on \( \mathbb{R}^n \) and the euclidean one are \( C \)-equivalent in \( \overline{W}_2 \). Then for every \( w \in \text{Vect}_0^1(M) \) we have \( \|\tilde{w}\|_e \leq C \|w\|_\rho \). Let \( r \in (0, 1/k) \) be so small that the first coordinate of \( \phi_*(v)(x) \) is not less than \( 1/k \) for \( x \in W_r \). Let \( \theta > 0 \) be so small that

\[
\gamma(0, [-\theta, \theta]; \tilde{v}) \subset W_r, \quad \gamma(0, \theta; \tilde{v}) \in \mathbb{R}^n_+ \quad \text{and} \quad \gamma(0, -\theta; \tilde{v}) \in \mathbb{R}^n_-.
\]

By 5.3 there is \( \delta_0 > 0 \) and a neighborhood \( S_0 \subset W_r \) such that

\[
\gamma(x, [-\theta, \theta]; u) \subset W_r, \quad \gamma(x, \theta; u) \in \mathbb{R}^n_+ \quad \text{and} \quad \gamma(x, -\theta; u) \in \mathbb{R}^n_-
\]

whenever \( x \in S_0, u \in \text{Vect}_0^1(\mathbb{R}^n), \|u - \tilde{v}\|_e < \delta_0 \). Therefore for \( x \in S_0, u \in \text{Vect}_0^1(\mathbb{R}^n), \|u - \tilde{v}\|_e < \delta_0 \) there is \( \tau_0 = \tau_0(x, u) \in [-\theta, \theta] \) such that \( \gamma(x, \tau_0; u) \in \{0\} \times \mathbb{R}^{n-1} \). Further, if \( \|u - \tilde{v}\|_e < \delta_1 = \min(\delta_0, 1/2) \) the first coordinate of \( u \) is positive in \( W_r \), and therefore for every \( x \in S_0 \), there is one and only one \( \tau_0 = \tau_0(x, u) \in [-\theta, \theta] \) with \( \gamma(x, \tau_0; u) \in \{0\} \times \mathbb{R}^{n-1} \). It is easy to see that (1) and (2) of the conclusions of our proposition hold for \( \delta = \delta_1/C \) and \( S = \phi^{-1}(S_0) \). (3) is an immediate consequence of (2).

2) **General case.**

The part 1) of the present proof applied to the point \( b \) and \( t_0 = 0 \), implies that there is \( \theta_0 > 0 \) such that for every \( \theta \in [0, \theta_0] \) there is \( \delta_0(\theta) > 0 \) and a neighborhood \( R_0(\theta) \subset U \cap R \) such that for every \( x \in R_0(\theta) \) and for every \( w \in \text{Vect}_0^1(M) \) with \( \|w - v\| < \delta_0(\theta) \) we have:

\[
\gamma(x, [-\theta, \theta]; w) \subset U
\]

(5.1)

\[
\exists! \tau_0 = \tau_0(w, x) \in [-\theta, \theta], \text{such that} \quad \gamma(x, \tau_0; w) \in E
\]

Choose \( \theta_1 > 0 \) such that \( \gamma(x, [-\theta_1, t_0 + \theta_1]; v) \subset U \), and that \( \gamma(x, [t_0 - \theta_1, t_0 + \theta_1]; v) \subset R \). Applying 5.1 one obtains easily that for \( \theta \in [0, \theta_0] \) there exist \( \delta_1(\theta) > 0 \) and an open neighborhood \( S(\theta) \subset U \) of \( a \), such that for every \( y \in S(\theta) \) and every \( w \in \text{Vect}_0^1(M) \) with \( \|w - v\| < \delta_1(\theta) \) we have

\[
\gamma(y, [-\theta_1, t_0 + \theta_1]; w) \subset U \quad \text{and} \quad \gamma(y, [t_0 - \theta_1, t_0 + \theta_1]; w) \subset R
\]

(5.3)

\[
\gamma(y, [t_0 - \theta_1, t_0 + \theta_1]; w) \subset \gamma(y, t_0; \phi(w)) \subset \gamma(y, t_0; v) \subset R
gamma(y, t_0; \phi(w)) \subset \gamma(y, t_0; v) \subset R
\]

(5.4)
We claim that for every $\theta \in ]0, \min(\theta_0, \theta_1)]$ the number $\delta_2(\theta) = \min(\delta_1(\theta), \delta_3(\theta))$ and the neighborhood $S(\theta)$ of $a$ satisfy (1) and (2) of conclusions of 5.2. Indeed, (1) is immediate from (5.3). To prove (2), note that the existence of the unique $\tau_0 \in [t_0 - \theta, t_0 + \theta]$ such that $\gamma(\gamma(t_0; w), \tau_0; w) \in E$ is equivalent to the existence of the unique $\tau_1 \in [\theta, \theta] \implies [\theta, \theta]$ such that $\gamma(\gamma(t_0; w), \tau_1; w) \in E$ which is guaranteed for $y \in S(\theta)$ and $\|w_v\| \leq \delta_2(\theta)$ by (5.4) and (5.2). Therefore for non compact $E$ the proof is over. For the case of compact $E$ choose $\theta_3 > 0$ such that $\gamma(a, [-\theta_3, t_0], v) \cap E = \emptyset$. For every $\theta \in ]0, \theta_3[$ choose $\delta_3(\theta) > 0$ and a neighborhood $S_3(\theta)$ of $a$, such that $\gamma(x, [-\theta_3, t_0]; w) \subset M \setminus E$ for $x \in S_3(\theta)$ and $w \in \text{Vect}_1(M)$ with $\|w_v\| \leq \delta_3(\theta)$. Then for every $\theta \in ]0, \min(\theta_1, \theta_2, \theta_3)$ the number $\min(\delta_1(\theta), \delta_2(\theta), \delta_3(\theta))$, and the neighborhood $S_3(\theta) \cap S(\theta)$ of $a$ satisfy (1) - (3) of 5.2. □

**B. Manifolds with boundary.** Let $W$ be a compact riemannian manifold with boundary. Recall from [19, §8C] that we denote by $\text{Vect}^1(W, \bot)$ the space of $C^1$ vector fields $v$ on $W$, such that $v(x) \not\in T_x(\partial W)$ for $x \in \partial W$. Choose an embedding of $W$ into a closed manifold $M$ without boundary, $\dim M = \dim W$ (for example, one can take the double of $W$). Pick a riemannian metric on $M$ extending that of $W$ (the existence of such a metric is easily proved using the standard partition of unity argument). The same argument proves that every $C^1$ vector field on $W$ can be extended to a vector field $v_0 \in \text{Vect}^1(M)$ such that $\|v_0\| \leq 2\|v\|$. The following corollaries are "$C^0$-analogs" of Propositions 8.10, 8.11 of [19]. They are proved similarly to 8.10, 8.11 of [19] using Propositions 5.1, 5.2 of the present appendix instead of [19, Prop. 8.2].

**Corollary 5.4.** Let $v \in \text{Vect}^1(W, \bot), x \in W$. Assume that the $v$-trajectory $\gamma(x, \cdot; v)$ reaches the boundary at a moment $T \geq 0$. Let $U$ be an open neighborhood of $\gamma(x, [0, T]; v)$, and $R$ be an open neighborhood of $\gamma(x, T; v)$. Then there is $\delta > 0$ and a neighborhood $S$ of $x$, such that for every $y \in S$ and for every $w \in \text{Vect}^1(W, \bot)$ with $\|w_v\| < \delta$ the trajectory $\gamma(y, \cdot; v)$ reaches the boundary at a moment $T(y, w) \geq 0$, and $\gamma(y, [0, T(y, w)]; w) \subset U$ and $\gamma(y, T(y, w); w) \subset R$. □

**Corollary 5.5.** Let $v \in \text{Vect}^1(W, \bot), x \in W, T > 0$. Assume, that $\gamma(x, T; v) \in \partial W$. Let $U$ be an open neighborhood of $\gamma(x, [0, T]; v)$, and $R$ be an open neighborhood of $\gamma(x, T; v)$. Then there is $\delta > 0$ and a neighborhood $S$ of $x$, such that for every $y \in S$ and for every $w \in \text{Vect}^1(W, \bot)$ with $\|w_v\| < \delta$ the trajectory $\gamma(y, \cdot; v)$ is defined on $[0, T]$ and $\gamma(y, [0, T]; w) \subset U$ and $\gamma(y, T(y, w); w) \subset R$. □

Let $v \in \text{Vect}^1(W, \bot)$, $x \in W$, $T > 0$. Assume that $\gamma(x, T; v) \in \partial W$. Denote by $V_0$, resp. by $V_1$ the set of $x \in \partial W$, where $v(x)$ points inward $W$, resp. outward $W$. For a subset $Z \subset W$ denote by $\tau(Z, v)$ the set $\{\gamma(z, t; v) | z \in Z, t \geq 0\}$. The next corollary is deduced from 5.4 by an easy compactness argument.

**Corollary 5.6.** Let $K \subset W$ be a compact such that every $v$-trajectory starting at a point of $K$ reaches the boundary. Let $U$ be an open neighborhood of $\tau(K, v)$. Let $R \subset V_1$ be an open neighborhood of $\tau(K, v) \cap V_1$. Then there is $\delta > 0$ such that for every $w \in \text{Vect}^1(W, \bot)$ with $\|w_v\| < \delta$ each $w$-trajectory starting at a point of $K$ reaches the boundary and we have: $\tau(K, w) \subset U, \tau(K, w) \cap V_1 \subset R$. □

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