Combination with anti-tit-for-tat remedies problems of tit-for-tat

Su Do Yi\textsuperscript{a}, Seung Ki Baek\textsuperscript{b,*}, Jung-Kyoo Choi\textsuperscript{c,**}

\textsuperscript{a}Department of Physics and Astronomy, Seoul National University, Seoul 08826, Korea
\textsuperscript{b}Department of Physics, Pukyong National University, Busan 48513, Korea
\textsuperscript{c}School of Economics and Trade, Kyungpook National University, Daegu 41566, Korea

Abstract

One of the most important questions in game theory concerns how mutual cooperation can be achieved and maintained in a social dilemma. In Axelrod’s tournaments of the iterated prisoner’s dilemma, Tit-for-Tat (TFT) demonstrated the role of reciprocity in the emergence of cooperation. However, the stability of TFT does not hold in the presence of implementation error, and a TFT population is prone to neutral drift to unconditional cooperation, which eventually invites defectors. We argue that a combination of TFT and anti-TFT (ATFT) overcomes these difficulties in a noisy environment, provided that ATFT is defined as choosing the opposite to the opponent’s last move. According to this TFT-ATFT strategy, a player normally uses TFT; turns to ATFT upon recognizing his or her own error; returns to TFT either when mutual cooperation is recovered or when the opponent unilaterally defects twice in a row. The proposed strategy provides simple and deterministic behavioral rules for correcting implementation error in a way that cannot be exploited by the opponent, and suppresses the neutral drift to unconditional cooperation.

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*Principal corresponding author
**Corresponding author

Email addresses: seungki@pknu.ac.kr (Seung Ki Baek), jkchoi@knu.ac.kr (Jung-Kyoo Choi)
1. Introduction

Game theory has provided useful frameworks to understand conflict and cooperation in the absence of a central authority. In many studies, reciprocity is a particularly important idea to explain cooperation between rational players. In the iterated version of the Prisoner’s Dilemma (IPD) game, the repetition makes it profitable for each player to cooperate because one has to take the other’s reaction into account \[1, 2, 3, 4, 5, 6\]. As long as both are sufficiently patient and free from error, therefore, the players cooperate from the first encounter and this continues forever. The notion of reciprocity has been popularized by the success of Tit-for-Tat (TFT) in Axelrod’s tournaments \[7\]. TFT cooperates at the first encounter with the co-player, and replicates the co-player’s previous move in the subsequent rounds. Axelrod has argued that this simple strategy is nice, retaliating, forgiving, and non-envious. However, the situation becomes much more complicated in a noisy environment, where it is possible to make an implementation error: For example, there can be an inefficient alternation of cooperation and defection between two TFT players when one of them makes a mistake \[8, 9\]. Tit-for-Two-Tats (TF2T) is more tolerant, because it retaliates when the co-player defects twice in a row. Although TF2T was proposed as a remedy to avoid the inefficiency of TFT, its predictability opens another possibility of being exploited. Generous TFT avoids this dilemma between generosity and exploitability by introducing randomness in forgiving the co-player’s defection \[10, 11, 12, 13\]. However, its unpredictability might be a double-edged sword when it comes to public policy making, in which such random factors are not popular ideas \[14\]. Moreover, Generous TFT is overwhelmed by a more generous strategy called Win-Stay-Lose-Shift (WSLS), also known as Pavlov \[13\], in the evolution of cooperation \[16\]. WSLS is one of aspiration-based strategies \[17, 18\], which means that it attempts changes when its payoff becomes less than a certain level. For example, when WSLS is in mutual defection with its co-player, it receives a lower payoff than the aspiration level and thus cooperates next time. The success of WSLS originates from the ability to correct error quickly when played against another WSLS player. The price is that WSLS easily falls into a victim of an unconditional defector (AllD). All these difficulties are rooted in the error vulnerability of TFT. Worse is that this is not the only weakness: A TFT population is invaded via neutral drift by unconditional cooperators (AllC), which, in turn, opens the back door to AllD \[11, 12, 13, 19\]. It would thus be desirable if some
modification overcame these shortcomings while preserving the strengths of TFT. This work will show that such a combination is actually possible.

A method of searching for successful cooperating strategies is to perform an evolutionary experiment \textit{in silico} \cite{16,20,21}. The purpose of such an experiment is to simulate natural selection. The experimenter does not have direct control over selection, but only determines evolutionary dynamics and an accessible set of strategies. If this can be called a bottom-up approach, it is also possible to think of a top-down approach; we can impose certain criteria that a successful strategy is expected to satisfy and sort out strategies that meet the criteria. This work follows the latter approach. One of its advantages is that it does not need pairwise comparisons between all the strategies under consideration so that the computational cost scales only linearly with the number of strategies. We do not mean that this approach will find an optimal strategy such that guarantees maximal payoffs. Nor do we mean that it will be equivalent to the result of evolutionary experiments or actual human behavior. Rather, our purpose is to design a workable solution to avoid a series of wasting retaliation from a mistaken move, when players are bound to interact repeatedly.

Our finding is that the top-down approach singles out a strategy that combines TFT and anti-TFT (ATFT) in a way that it stabilizes mutual cooperation in the presence of implementation error. Note that ATFT does the opposite to the co-player’s last move. According to this proposed strategy, our focal player Alice normally uses TFT but switches to ATFT when she recognizes her own error. Alice returns back to TFT either when mutual cooperation is recovered or when Bob unilaterally defects twice in a row. The strategy says that Alice is responsible only for her own error without any need to judge Bob’s behavior. Our result provides a simple deterministic error-correcting rule, which is secured against repeated exploitation at the same time.

Before proceeding to the next section, let us assume that the players in this work perceive their respective payoffs with no uncertainty as defined by the game, although they do make implementation errors. In other words, their history of moves is given as public information, whose accessibility is limited only by their memory spans. Moreover, we set aside other public information such as social standing, which is required in Contrite TFT \cite{22,23,24,25} as well as in some form of indirect reciprocity \cite{26,27,28,29,30,31,32}. After deriving our main results under these assumptions, we will revisit the problem of perception error.
2. Method and Result

The payoff matrix $M$ of the PD game can be written as follows:

$$
egin{pmatrix}
  C & D \\
  C & M_{CC} & M_{CD} \\
  D & M_{DC} & M_{DD}
\end{pmatrix},
$$

(1)

where $C$ and $D$ denote each player’s possible moves, cooperation and defection, respectively. The matrix elements satisfy two inequalities: First, $M_{DC} > M_{CC} > M_{DD} > M_{CD}$ for mutual defection to be a Nash equilibrium. Second, $M_{CC} > (M_{CD} + M_{DC})/2$ for mutual cooperation to be Pareto optimal.

In the IPD, we define a state as a history of moves chosen by two players Alice and Bob for the past $m$ time steps. The number of states is $n = 2^{2m}$, because each of the two players has $m$ binary choices between $C$ and $D$. Choosing $m = 2$, for example, we work with $n = 16$ states, each of which at time $t$ can be written as $S_t = (A_{t-2}A_{t-1}, B_{t-2}B_{t-1}) = (Alice’s$ moves at $t-2$ and $t-1, Bob’s$ moves at $t-2$ and $t-1)$. Alice’s strategy is represented as a mapping from the state $S_t$ to her move $A_t \in \{C, D\}$ at an arbitrary time step $t$. The total number of Alice’s possible strategies is thus $N = 2^n = 2^{2^{2m}}$. Here, we do not specify initial moves for $t \leq m$ in defining a strategy, because they are irrelevant when we consider long-term averages in the presence of error, as will be explained below.

It is instructive to note that the state at time $t+1$ is written as $S_{t+1} = (A_{t-1}A_{t}, B_{t-1}B_{t})$, which always shares $A_{t-1}$ and $B_{t-1}$ with $S_t = (A_{t-2}A_{t-1}, B_{t-2}B_{t-1})$. In addition, $A_t$ is determined from $S_t$ by our focal player Alice’s strategy. For this reason, when Alice’s strategy and state $S_t$ are given, the only unknown part is $B_t \in \{C, D\}$. We impose no restriction on Bob’s strategy, so each state can generally be followed by one of two different states at the next time step. For example, suppose that Alice finds herself in $S_t = (A_{t-2}A_{t-1}, B_{t-2}B_{t-1}) = (CD, CC)$ at time $t$ and her strategy prescribes $A_t = D$ for this $S_t$. Then, we conclude that her next state $S_{t+1} = (A_{t-1}A_{t}, B_{t-1}B_{t})$ must be either $(DD, CC)$ or $(DD, CD)$. Graphically, this idea can be represented as depicted in Fig. I.

Suppose a sequence of moves in the IPD between two strategies $i$ and $j$. Let $F_{ij}^{(t)} \in \{M_{CC}, M_{CD}, M_{DC}, M_{DD}\}$ denote the payoff of strategy $i$ against
Figure 1: Possible transitions from state $S_t = (A_{t-2}A_{t-1}, B_{t-2}B_{t-1}) = (CD, CC)$ to $S_{t+1} = (A_{t-1}A_t, B_{t-1}B_t)$, which can be either ($DD, CC$) or ($DD, CD$) when Alice’s strategy prescribes $A_t = D$ for that $S_t$.

$\text{}$

$j$ at time step $t$. We are interested in its long-term average, i.e.,

$$\overline{F}_{ij}(T) = \frac{1}{T} \sum_{t=1}^{T} F_{ij}^{(t)} \quad (2)$$

with $T \gg 1$. If both $i$ and $j$ have finite memories, the calculation of $\overline{F}_{ij}(T \to \infty)$ can be simplified as follows: Any pair of strategies with finite memories constitute a Markov process described by a stochastic matrix. When error is absent, the moves of two deterministic strategies eventually form a loop. By a loop, we mean a sequence of consecutive states $S_t \to S_{t+1} \to \ldots \to S_{t+\nu}$ allowed by the strategies, in which $S_{t+\nu} = S_t$ with a positive integer $\nu$. The integer $\nu$ is called the length of the loop. If the number of possible loops is greater than one, which loop the players will be trapped in will be determined by an initial state, which corresponds to their first moves. However, when implementation error occurs with probability $e \ll 1$ so that intended cooperation or defection can fail, they can escape from the loop with a time scale of $O(1/e)$. The probabilities assigned to states thus converge to a unique stationary distribution in the long run, regardless of the initial state [33, 34]. For example, when AllC meets TFT, the interaction can be described by the following $n \times n$ stochastic matrix $U$ with $n = 2^{2m} = 4$:

$$i = \begin{pmatrix} (C, C) & (C, D) & (D, C) & (D, D) \\ (C, D) & (1-e)^2 & (1-e)^2 & (1-e)e & (1-e)e \\ (D, C) & (1-e)e & (1-e)e & (1-e)^2 & (1-e)^2 \\ (D, D) & e(1-e) & e(1-e) & e^2 & e^2 \end{pmatrix}, \quad (3)$$
where its element $U_{ij}$ is the probability to observe state $i$ given the previous state $j$. We calculate the stationary distribution over $n$ states from the principal eigenvector $\vec{v}$ satisfying $U\vec{v} = \vec{v}$. For notational convenience, we simply identify $\vec{v}$ with the stationary distribution, assuming that it is normalized as a probability distribution. Let $f_{ij}$ denote the inner product between $\vec{v}$ and $\vec{m} \equiv (M_{CC}, M_{CD}, M_{DC}, M_{DD})$ with $i=$AllC and $j=$TFT. The Perron-Frobenius theorem tells us that $F_{ij}(T)$ converges to $f_{ij}$ as $T \to \infty$. Generalization to other strategies is straightforward.

We require that a successful strategy $k$ should satisfy the following criteria:

1. Efficiency: It must achieve mutual cooperation if the co-player employs the same strategy. It means that $f_{kk}$ should approach $M_{CC}$ as $e \to 0$.
2. Distinguishability: It must be able to exploit AllC to avoid the neutral drift. To be more precise, when $j=$AllC, the payoff difference $f_{kj} - f_{jk}$ should remain positive finite as $e \to 0$.
3. Defensibility: It must not be exploited repeatedly by the co-player. In other words, for strategy $k$ to be defensible, it should satisfy the following inequality:

$$\sum_{\tau=0}^{\nu-1} \left[ F_{kj}^{(t+\tau)} - F_{jk}^{(t+\tau)} \right] \geq 0$$

against any finite-memory strategy $j$, where the payoffs are evaluated along every possible loop $S_t \to S_{t+1} \to \ldots \to S_{t+\nu}$ allowed by $k$ and $j$.

Let us explain those criteria in more detail. The first and second criteria are relatively easy to check, because we only have to match each of the $N = 2^{2m}$ strategies against itself and AllC, respectively. For each of such pairs, we carry out the linear-algebraic calculation of the principal eigenvector as in Eq. (3) to obtain the stationary distribution $\vec{v}$. By looking at which states occupy the most probabilities as $e \to 0$, we can tell whether each given strategy meets the criteria.

As for the defensibility criterion, let us begin by assuming that both Alice and Bob use deterministic strategies with memory length $m$. As mentioned above, the players can be trapped in a loop for a period of $O(1/e) \gg 1$. If the loop gives a lower payoff to Alice than to Bob in one cycle, we say that she is exposed to the risk of repeated exploitation. The defensibility criterion means that the player’s strategy should not allow such a risky loop.
example, TF2T is not defensible, because it can be trapped in the following sequence of moves when it meets WSLS:

\[
\begin{align*}
\text{TF2T} & \quad CCDD \quad CCDD \quad \cdots \\
\text{WSLS} & \quad DDDC \quad DDDC \quad \cdots
\end{align*}
\]

which is clearly disadvantageous to TF2T at every repetition. In fact, the defensibility criterion does not restrict the co-player’s strategy to deterministic ones. Now suppose that Bob has adopted a stochastic strategy with memory length \( m \). For every possible loop with nonzero probability, we can find a deterministic memory-\( m \) strategy for Bob to reproduce the loop. For this reason, if a given strategy does not contain risky loops against any deterministic memory-\( m \) strategy, it is also unexploitable by stochastic memory-\( m \) strategies. Even if Bob uses a longer memory than \( m \), Alice can marginalize his strategy to reconstruct an effective stochastic one with memory length \( m \).

The conclusion is the following: If a given strategy has no risky loops against an arbitrary deterministic memory-\( m \) strategy, it is a sufficient condition for defensibility. Therefore, for each strategy that pass the efficiency and distinguishability criteria, we examine all its loops by matching it against \( N = 2^{2m} \) deterministic strategies one by one with probing every possible initial state. The computational cost is bigger than those of the other two criteria. Still, it is far less than \( N^2 \) needed for direct pairwise comparison, because only a small number of strategies pass the efficiency and distinguishability criteria. In practice, it is convenient to break up the defensibility test into two parts: We first check if a given strategy leads to mutual defection against AllD. If it does, we then proceed to check all its loops.

It turns out that no strategy with \( m \leq 1 \) satisfies these criteria together. We thus proceed to \( m = 2 \) to consider \( N = 2^{16} \) strategies. Then, our numerical calculation shows that the joint application of all these criteria singles out only four strategies, which are identical except for moves at \((CD, CD)\) and \((DC, DC)\).

To resolve this four-fold degeneracy, we have to consider a stronger form of the efficiency criterion: For this criterion to hold true in general, mutual cooperation should be restored even if the players, adopting the same successful strategy, go from \((CC, CC)\) to \((CD, CD)\) via simultaneous mistakes with probability \( e^2 \). This argument determines the moves at \((CD, CD)\) and \((DC, DC)\) as follows: Suppose that both the players choose \( D \) at \( S_t = (CD, CD) \). It results in mutual defection \((DD, DD)\) and one must keep defecting at \((DD, DD)\) to be defensible against AllD. In short, defection at
(CD, CD) is not the correct choice to recover cooperation. It is therefore clear that the successful strategy must prescribe C at $S_t = (CD, CD)$, which leads to $S_{t+1} = (DC, DC)$. Once again, unless the players choose C here, they will take the following undesired path:

$$(CC, CC) \rightarrow [(CD, CD) \rightarrow (DC, DC)] \rightarrow [(CD, CD) \rightarrow (DC, DC)] \rightarrow \ldots,$$

where the square brackets denote a repeating loop of length two. To sum up, C is the correct choice both at (CD, CD) and (DC, DC), and the recovery path from simultaneous mistakes goes as follows:

$$(CC, CC) \rightarrow (CD, CD) \rightarrow (DC, DC) \rightarrow (CC, CC).$$

Now, we have fully determined the move at every possible state. The resulting strategy is tabulated in Table 1. Surprisingly, it can be understood as a simple combination of TFT and ATFT. Let us therefore call it TFT-ATFT and explain the reason in the next section.

| state         | move | state         | move |
|---------------|------|---------------|------|
| (CC,CC)       | C    | (DC,CC)      | C    |
| (CC,CD)       | D    | (DC,CD)      | D    |
| (CC,DC)       | C    | (DC,DC)$^\dagger$ | C    |
| (CC,DD)       | D    | (DC,DD)      | C    |
| (CD,CC)       | D    | (DD,CC)      | D    |
| (CD,CD)$^\dagger$ | C    | (DD,CD)      | C    |
| (CD,DC)       | C    | (DD,DC)      | C    |
| (CD,DD)       | D    | (DD,DD)      | D    |

3. How TFT-ATFT stabilizes cooperation

We have seen that a player’s history of the IPD can be understood as a series of transitions in a space of $n = 2^{2m}$ states. If Alice has memory length $m = 2$, her strategy has prescribed her move $A_t$ at state $S_t =$
\((A_{t-2}, A_{t-1}, B_{t-2}, B_{t-1})\), by which it puts a restriction on possibilities of the next state \(S_{t+1} = (A_{t-1}A_t, B_{t-1}B_t)\) as illustrated in Fig. 1. It is convenient to visualize a strategy as a transition graph \(^2\) by collecting all the possible transitions between states. Suppose that Alice is using TFT-ATFT against Bob. Figure 2 represents her strategy as a transition graph, in which each node corresponds to a state \(S_t\). Recall that each state can be followed by two different states depending on Bob’s move \(B_t \in \{C, D\}\). For this reason, every node in this graph has outgoing links to two different nodes. We stress that the resulting graph specifies all the possible transitions allowed by Alice, regardless of Bob’s strategy.

For any transition graph, we can classify the nodes into recurrent and transient ones \(^3\): Bob can visit any of recurrent nodes repeatedly by choosing \(C\) and \(D\) in an appropriate order. For example, \((CC, CD)\) is recurrent for Alice’s strategy in Table 1 and Fig. 2. It means that Bob, starting from \((CC, CD)\), can return to it by choosing \(C\), \(C\), and \(D\) in sequence, generating the following path:

\[
(CC, CD) \rightarrow (CD, DC) \rightarrow (DC, CC) \rightarrow (CC, CD).
\]

On the other hand, transient nodes can be visited again only when Alice deviates from her strategy. An example is \((CD, CC)\), which has no incoming links in Fig. 2.

Now, let us look into the three criteria from a different perspective than in the previous section. First of all, if we wish to enforce a certain relationship between Alice’s and Bob’s payoffs, as required by the defensibility criterion, we have to consider zero-determinant (ZD) strategies \(^3\). In the Appendix, we argue that TFT is indeed the only deterministic case of the generous ZD strategies, which guarantees equal payoffs for both the players. For this reason, the defensibility criterion essentially requires that Alice should normally use TFT, which does not allow repeated exploitation no matter what Bob does. Based on this observation, we enumerate accessible states from \((CC, CC)\) under the assumption that Alice is using TFT:

\[
(CC, CC) \xrightarrow{} (CC, CC) \\
(CC, CD) \xrightarrow{} \begin{cases}
(CC, CC) \rightarrow \ldots \\
(DC, CC) \rightarrow \ldots \\
(DC, CD) \rightarrow \ldots \\
(DD, CC) \rightarrow \ldots \\
(DD, CD) \rightarrow \ldots \\
(DD, DD) \rightarrow \ldots
\end{cases}
\]

(4)
Figure 2: Visualization of the proposed strategy, TFT-ATFT. We represent each state as a node and connect possible transitions between them. For example, \((CD, CC)\) in the top-left corner is connected to \((DD, CC)\) and \((DD, CD)\), because TFT-ATFT prescribes \(D\) at \(S_t = (CD, CC)\) so that the next state should be either \(S_{t+1} = (DD, CC)\) or \((DD, CD)\) depending on the co-player’s move. The nodes drawn in thick black lines are recurrent and the others are transient. The square cluster of eight recurrent nodes constitutes the pure TFT to meet the defensibility criterion, and the other recurrent node at the top, \((DD, CC)\), guarantees the distinguishability criterion by exploiting AllC. The red dotted lines indicate how a player’s implementation error is corrected when both the players use this strategy. The blue dashed lines show another path to recover cooperation when both defect by mistake. The efficiency criterion is fulfilled by these two error-correcting paths. Note that they make use of transient nodes, so that the co-player cannot activate the error-correcting paths at will.
One can readily check that this sequence is eventually closed with visiting eight different states, which are underlined in Table 1. The use of TFT as the default mode thereby determines moves at the eight different states. All these states are recurrent.

However, pure TFT does not meet the efficiency criterion, and we need to accommodate it in the following way: If Alice made a mistake last time, she should go to ‘Plan B’ to correct it, which will work under the assumption that Bob has adopted the same strategy. At the same time, this process must be secured against Bob’s exploitation, because Bob may well become nasty to Alice. These two requirements appear to be contradictory to each other. Our point is that the security is ensured by making use of transient nodes, which are out of Bob’s control. More specifically, the most probable scenario in Plan B is that Alice visits \((CD, CC)\) by mistake with probability \(e \ll 1\). Let us suppose that Bob uses the same strategy and thus normally behaves as a TFT strategist as shown above. Alice will choose \(D\) at \((CD, CC)\) for the following reason: If she did not, the next state would be \((DC, CD)\), one of the recurrent states for which Alice’s move is prescribed by the TFT part [Eq. (4)]. According to the prescription, however, they end up with a series of TFT retaliation which Alice wants to avoid:

\[
(CD, CC) \to (DC, CD) \to (CD, DC) \to (DC, CD) \to \ldots \text{ (wrong)}.
\]

This is the reason that Alice’s mistake must be followed by another \(D\). In general, we can construct a decision tree, which starts from \((CD, CC)\) and branches off depending on Alice’s choice, while Bob’s moves are prescribed by TFT. By pruning away ‘wrong’ branches leading to inefficiency, we determine Alice’s correct moves. The result is as follows:

\[
\begin{align*}
(CD, CC) & \rightarrow (DC, CD) \rightarrow (CD, DC) \rightarrow (DC, CD) \rightarrow \ldots \text{ : TFT retaliation (wrong)} \\
\downarrow & \\
(DD, CD) & \rightarrow (DD, DD) \rightarrow (DD, DD) \rightarrow \ldots \text{ : mutual defection (wrong)} \\
\downarrow & \\
(DC, DD) & \rightarrow (CD, DC) \rightarrow (DC, CD) \rightarrow \ldots \text{ : TFT retaliation (wrong)} \\
\downarrow & \\
(CC, DC) & \rightarrow (CD, CC) \text{ : back to the starting point (wrong)} \\
\downarrow & \\
(CC, CC). & \text{(5)}
\end{align*}
\]
The underlined states thus describe the correct path to recover mutual cooperation in the first scenario of Plan B. It is identical to the red path in Fig. 2.

There also exists another scenario that both Alice and Bob mistakenly choose $D$ with probability $e^2$. The starting point will be $(CD, CD)$ this time, and the previous section has already shown that mutual cooperation should be recovered in the following way:

$$(CD, CD) \rightarrow (DD, DD) \rightarrow (DD, DD) \rightarrow \ldots : \text{mutual defection (wrong)}$$

$$(DC, DC) \rightarrow (CD, CD) : \text{back to the starting point (wrong)}$$

$$(CC, CC),$$

provided that Alice and Bob have adopted the same strategy. This corresponds to the blue path in Fig. 2, and completes the second scenario of Plan B. In total, the efficiency criterion determines moves at six different states, underlined in Eqs. (5) and (6).

We point out that Alice has chosen the opposite to Bob’s previous move every time until reaching mutual cooperation at $(CC, DC)$ in Eq. (5) or $(DC, DC)$ in Eq. (6). In other words, the strategy means the following: Play TFT, but turn to ATFT if you made a mistake last time, and return back to TFT when mutual cooperation is recovered. This interpretation is actually consistent with the use of $m = 2$, because your memory should be as long as two time steps at least, to tell if you made a mistake last time. That is, to make a decision between TFT and ATFT at time step $t$, you have to compare what you were supposed to do as a TFT-strategist and what you actually did at time $t - 1$, the former of which is encoded in the moves at $t - 2$. Given the last time step only, you would have no way to judge whether your move was right.

Among the $n = 16$ states, the remaining ones are $(DD, CC)$ and $(CC, DD)$. The former state is accessed when Bob does not react to Alice’s erroneous defection, e.g., when Bob is using AllC. If that is the case, Alice as an ATFT strategist will maintain $D$ at $(DD, CC)$ to gain a higher payoff than Bob on average:

$$(CD, CC) \rightarrow (DD, CC) \rightarrow (DD, CC) \rightarrow \ldots . \ldots$$

Due to this property, this strategy is able to exploit AllC, satisfying the distinguishability criterion. Now, the last state to consider is $(CC, DD)$. It
is reached when Bob defects at \((DC, DD)\). Then, Alice cannot maintain her ATFT strategy, because it would make \((CC, DD)\) recurrent and thus violate the defensibility criterion. For this reason, Alice must return back to TFT at \((CC, DD)\) by choosing \(D\):

\[
(CC, DD) \rightarrow (CC, DD) \rightarrow (CC, DD) \rightarrow \ldots : \text{indefensible (wrong)}
\]

\[
\downarrow
\]

\[(CD, DD)\]. (8)

This completes the derivation of TFT-ATFT.

4. Evolutionary dynamics

Although we have been mainly concerned about the game between two players Alice and Bob, let us consider an evolving population in this section. The purpose is to see how our designed TFT-ATFT performs in the presence of other strategies such as TFT and WSLS in an evolutionary framework. Unfortunately, an investigation on the full set of strategies with \(m = 2\) is unfeasible due to our limited computing resources at the moment. As a preliminary test, this section checks the performance of TFT-ATFT against the 16 strategies with \(m = 1\).

By adding TFT-ATFT, we have \(N = 17\) strategies in total. The fraction of strategy \(i\) is denoted as \(x_i\). The normalization condition gives \(\sum_i x_i = 1\). If TFT-ATFT is successful, its fraction will be of \(O(1)\) in the long run. In this evolutionary setting, our assumption is that every player plays the IPD against each other for a long time to obtain an error-average payoff. To be more specific, we will construct the payoff matrix \(M\) by choosing \(M_{DC} = b = 1\), \(M_{CC} = b - c = 1/2\), \(M_{DD} = 0\), and \(M_{CD} = -c = -1/2\) for our calculation. The error probability is set to be \(e = 10^{-2}\) in calculating error-averaged payoffs between every pair of strategies. As in Sec. 2, let \(f_{ij}\) denote the payoff that strategy \(i\) earns against strategy \(j\) on average. In a well-mixed population, the expected payoff of strategy \(i\) is calculated as \(f_i = \sum_j f_{ij}x_j\). The replicator equation [42, 43, 44] then describes the time evolution of \(x_i\) in the following way:

\[
\dot{x}_i = (1 - \mu)f_i x_i - \langle f \rangle x_i + \frac{\mu}{N - 1} \sum_{j \neq i} f_j x_j,
\]
Figure 3: Numerical integration of the replicator equation. TFT-ATFT is included in addition to the 16 strategies with $m = 1$. The payoff matrix $M$ [Eq. (1)] is given by $M_{DC} = 1$, $M_{CC} = 1/2$, $M_{DD} = 0$, and $M_{CD} = -1/2$. For every pair of strategies, we consider average payoffs over a long time with error probability $e = 10^{-2}$. The mutation rate is set as $\mu = 10^{-4}$.

where the left-hand side means the time derivative of $x_i$ and $\langle f \rangle \equiv \sum_j f_j x_j = \sum_{ij} f_{ij} x_i x_j$ is the mean payoff of the population. The last term on the right-hand side describes mutation with rate $\mu$, which is set to be $10^{-4}$ in our numerical calculation. The replicator equation expresses an idea that a successful strategy, relative to the population mean, will increase its fraction in a well-mixed infinite population. Equation (9) generates a trajectory on the 16-dimensional space of $(x_0, \ldots, x_{15})$, and $x_{16}$ is constrained by the normalization condition. Under the deterministic dynamics of Eq. (9), the trajectory generally depends on the initial condition. For this reason, we have tested various initial conditions by taking 4,854 grid points with $x_i = 1/33, 5/33, \ldots, 17/33$ in this 16-dimensional space. Our numerical calculation shows that TFT-ATFT occupies virtually the entire population for 99.3% of the initial conditions. Figure 3 depicts one such example.

Our simulation has shown that TFT-ATFT performs well even when many strategies compete simultaneously. One might say that the success of TFT-ATFT is not unexpected, because it has a longer memory than anybody else. However, a longer memory does not necessarily lead to better performance. The polymorphic population of memory-one strategies can be viewed as a mixed-strategy player with $m = 1$ from the viewpoint of TFT-ATFT, and longer memory does not bring an advantage over a memory-one
stochastic strategy [36]. In this respect, it is not completely trivial that
the acquisition of additional memory bits makes such a huge impact on the
ecosystem of memory-one strategies.

5. Discussion and Summary

It is well known from the notion of direct reciprocity that cooperation
can constitute an equilibrium in the repeated prisoner’s dilemma. However,
when players are in conflict with each other, as we are witnessing all around
the world, we have to instead ask ourselves how to lead them back to the
cooperative equilibrium. One may introduce population dynamics such as
Eq. (9) to achieve this goal. As members of a WSLS population, for ex-
ample, Alice and Bob can keep cooperating with each other, resisting the
temptation of defection with the aid of other members. However, we do not
know whether such a population always happens to exist around the players.
Moreover, even if it does exist, population dynamics can take place over a
much longer time scale than that of individual interactions. In this work, we
have tried to propose a solution such that recovers mutual cooperation from
implementation error within a time scale of individual interactions. The so-
lution turns out to be a simple combination of TFT and ATFT. If Alice uses
this strategy against Bob, she normally uses TFT but shifts to ATFT upon
recognizing her own error. Alice begins to behave as a TFT strategis
t when mutual cooperation is restored at \((CC, DC)\) or \((DC, DC)\) [Eqs. (5) and
(6)], or when Bob keeps defecting after her cooperation [Eq. (8)]. It is worth
noting that Alice does not experience any security risk by announcin
g what she plans to do. Predictability does not necessarily imply exploitability. The
point is that when designing a strategy, one should make use of transient
nodes, which the co-player cannot visit at will.

Knowing that Alice employs TFT-ATFT, Bob can also safely choose the
same strategy, considering that it is efficient and defensible. Having adopted
TFT-ATFT, he will find the following: The efficiency criterion means that
the average payoff \(f_{ii}\) approaches \(M_{CC}\) as \(\epsilon \to 0\) for \(i =\text{TFT-ATFT}\). Due
to the defensibility criterion, furthermore, the average payoff \(f_{ij}\) satisfies
\(f_{ij} \geq f_{ji}\) against an arbitrary strategy \(j\). By definition of the PD game,
mutual cooperation is Pareto optimal, which means that \(f_{ii} \geq (f_{ij} + f_{ji})/2\).
An immediate consequence is that \(f_{ii} \geq (f_{ij} + f_{ji})/2 \geq (f_{ji} + f_{ji})/2 = f_{ji}\).
For Bob, therefore, switching to another strategy \(j \neq i\) does not bring him
any advantage. In other words, the strategy is the best response to itself and thus constitutes a Nash equilibrium in the limit of \( e \to 0 \).

In addition, we remark the following three properties of TFT-ATFT: First, the defensibility criterion holds true regardless of the complexity of Bob’s strategy, because this criterion solely depends on the transitions allowed by Alice (Fig. 2). Second, in terms of the stationary probability distribution \( \vec{v} \), the efficiency criterion implies that \((DD, DD)\) has vanishingly small probability between two TFT-ATFT players when \( e \ll 1 \). Indeed, players can escape from mutual defection via erroneous cooperation (Fig. 2):

\[
(DD, DD) \xrightarrow{\text{error}} (DC, DD) \rightarrow (CC, DC) \rightarrow (CC, CC),
\]

with a time scale of \( O(1/e) \). Third, we may relax one of our assumptions that each player correctly perceives her or his own payoff: Suppose that Alice sometimes miscounts her payoff, by which she erroneously perceives Bob’s cooperation as defection. However, the discrepancy in their memories lasts only for \( m \) time steps, after which they remember the same history. They have a different state than \((CC, CC)\) at this point, but we have already seen that two TFT-ATFT players, from an arbitrary initial condition, end up with mutual cooperation after \( O(1/e) \) time steps. We thus conclude that cooperation based on TFT-ATFT is resilient to perception error, if it occurs with a sufficiently longer time scale than \( O(1/e) \).

It is an interesting question if nature has already discovered TFT-ATFT, as has been claimed in case of TFT \([45, 46, 47]\). It is not unreasonable, because this strategy has such a large basin of attraction under the replicator dynamics as shown in the previous section. Even if that is not the case, we believe that this strategy will find its own use in applied game theory. Of course, one should be careful at this point, because our results have been obtained under highly ideal conditions. For example, we have assumed that error occurs with probability \( e \ll 1 \) and that two players interact sufficiently many times compared to \( O(1/e) \), and any of these assumptions can put serious limitation on the applicability of the proposed strategy.

One may also think of designing more complex strategies with \( m > 2 \). If we take, say, \( m = 3 \), we have \( n = 64 \) states and the number of possibilities amounts to \( N = 2^{64} \approx 2 \times 10^{19} \). It is beyond our ability to check this larger set as we have done in this work. Still, one may attempt to modify TFT-ATFT in several directions by utilizing the extra bits of memory: For example, the recovery path from erroneous defection may be shortened,
and the possibility of two successive errors may also be taken into account. Considering that TFT-ATFT mainly refers to only the co-player’s last move, however, we believe that the modifications will have minor effects on the overall performance, as long as they are based on the three criteria of efficiency, defensibility, and distinguishability.

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Appendix A. TFT as a deterministic generous ZD strategy

Following the common reformulation of the PD game in terms of donation, let us parametrize the payoff matrix [Eq. (1)] by setting $M_{DC} = b$, $M_{CC} = b - c$, $M_{DD} = 0$, and $M_{CD} = -c$, where $b$ and $c$ are the benefit and cost of cooperation, respectively, with $b > c > 0$. Then, a generous ZD strategy is represented by the following four probabilities [38]:

\[
\begin{align*}
    p_{CC} &= 1 \\
    p_{CD} &= 1 - \phi(\chi b + c) \\
    p_{DC} &= \phi(b + \chi c) \\
    p_{DD} &= \phi(1 - \chi)(b - c),
\end{align*}
\]

where $p_{XY}$ denotes Alice’s probability of cooperation when Alice and Bob did $X$ and $Y$, respectively, at the last time step. The parameter $\chi$ must satisfy $0 < \chi \leq 1$ to produce a feasible generous strategy. For this generous ZD strategy to be deterministic, $p_{DD}$ must be either zero or one. If $p_{DD} = 1$, we would get

\[
p_{DC} = \frac{b + \chi c}{(1 - \chi)(b - c)} > \frac{b}{(1 - \chi)(b - c)} > \frac{b}{b - c} > 1,
\]

which does not make sense. Therefore, we conclude that $p_{DD} = \phi(1 - \chi)(b - c) = 0$, which implies that $\phi = 0$ or $\chi = 1$. The former option should be discarded, however, because it gives us a singular strategy $p = (p_{CC}, p_{CD}, p_{DC}, p_{DD}) = (1, 1, 0, 0)$, i.e., “always cooperate or never cooperate” [36]. The other option, $\chi = 1$, yields $p = (1, 0, 1, 0)$, which is identical to TFT. Clearly, Alice and Bob then gain equal payoffs [38].
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