Effect of an impulsive force on vortices in a rotating Bose-Einstein condensate

Sadhan K. Adhikari and Paulsamy Muruganandam

Instituto de Física Teórica, Universidade Estadual Paulista, 01.405-900 São Paulo, São Paulo, Brazil

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The effects of a sudden increase and decrease of the interatomic interaction and harmonic-oscillator trapping potential on vortices in a quasi two-dimensional rotating Bose-Einstein condensate are investigated using the mean-field Gross-Pitaevskii equation. Upon increasing the strength of interaction suddenly the condensate enters a nonstationary oscillating phase which starts to develop more vortices. The opposite happens if the strength is reduced suddenly. Eventually, the number of vortices attains a final value at large times. Similarly, the number of vortices increases (decreases) upon a sudden reduction (augmentation) in the trapping potential. We also study the decay of vortices when the rotation of the condensate is suddenly stopped. Upon a free expansion of a rotating BEC with vortices the radius of the vortex core increases more rapidly than the radius of the condensate. This makes the counting and detection of multiple vortex easier after a free expansion.

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I. INTRODUCTION

Since the successful detection \[\text{[1]}\] of Bose-Einstein condensates (BEC) in dilute trapped bosonic atoms at ultralow temperature, one problem of extreme interest is the formation of vortices in a rotating condensate in an axially symmetric trap. The experimental observation \[\text{[2, 3, 4, 5, 6, 7, 8, 9]}\] of vortices in the condensate has intensified theoretical investigations on vortices using the mean-field Gross-Pitaevskii (GP) equation \[\text{[10]}\]. For large frequencies of rotation, a very large number of vortices arranged in a triangular lattice, frequently referred to as a vortex lattice, have been observed. The vortex lattices are highly excited collective states of BECs with an angular momentum of up to 60\(\hbar\) per particle \[\text{[3]}\]. There have been many theoretical studies on different aspects of BEC \[\text{[11, 12, 13]}\] and especially, on the vortices in a rotating BEC in axially symmetric traps \[\text{[4, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]}\].

When an ordinary fluid is placed in a rotating container the fluid rotates with the container because of viscosity. For superfluid \(^4\text{He(II)}\) in a rotating container no motion of the fluid is observed below a critical rotational frequency. Above the critical frequency, lines of singularity appear in its velocity field. These lines of singularity correspond to vortices and appear in the form of a lattice specially for large rotational frequencies. Superfluidity exhibits many peculiar features: absence of viscosity, reduction of moment of inertia, occurrence of persistent current, appearance of quantized vortices among others \[\text{[13]}\]. The formation of vortex lattice in a rotating \(^4\text{He(II)}\) is intimately related to its superfluidity and is a manifestation of quantized vortices \[\text{[2, 3, 4, 5, 6, 7, 8]}\]. However, because of the strong interaction between \(^4\text{He}\) atoms, the theoretical description of this system is not an easy task.

Similar vortices can be generated in theoretical mean-field models of trapped BEC based on the GP equations \[\text{[1, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]}\] and have been observed experimentally \[\text{[2, 3, 4, 5, 6, 8, 9]}\] in BEC with repulsive interaction. In contrast to liquid \(^4\text{He(II)}\), a trapped BEC is a very dilute and weakly interacting system, which makes a mean-field analysis appropriate. This analogy of a trapped BEC with liquid \(^4\text{He(II)}\) suggests the presence of superfluidity in the BEC. Many different ways for generating quantized vortices in a BEC have been suggested \[\text{[10]}\], e.g., via spontaneous vortex formation in evaporative cooling \[\text{[13]}\], via controlled excitation to an excited state of atoms \[\text{[20]}\], by stirring a BEC using an external laser with an angular frequency above a critical value for generating a quantized vortex with a single quantum \[\text{[15]}\], or by the rotation of an axially symmetric trap with an angular frequency above a similar critical value \[\text{[22]}\].

Experimentally, quantized vortices have been generated by different groups in a rotating BEC using a variety of techniques. Vortices have been detected in a \(^{87}\text{Rb}\) condensate in a cylindrical trap by Madison et al at ENS \[\text{[2]}\], where angular momentum is generated by a stirring laser beam. At MIT Raman et al \[\text{[2]}\] also studied the nucleation of vortices in a Bose-Einstein condensate stirred by a laser beam. They have been observed by Matthews et al \[\text{[3]}\] at JILA in coupled BEC’s comprised of two spin states of \(^{87}\text{Rb}\) in a spherical trap, where angular momentum is generated by a controlled excitation of the atoms between the two states. The tilting of the vortex axis in a nonspherical trapping potential has been studied at JILA by Haljan et al \[\text{[4]}\]. Vortices have also been generated by evaporatively spinning up a normal gas of \(^{87}\text{Rb}\) and then cooling below quantum degeneracy at JILA by Haljan et al \[\text{[4]}\]. A large number of vortices in the form of a lattice have been observed at MIT by Abo-Shaeer et al \[\text{[5]}\] as well as at ENS by Madison et al \[\text{[2]}\] for sufficiently large rotational frequency. The interesting dynamics of the formation and decay of vortices has also been studied \[\text{[3]}\]. Madison et al \[\text{[3]}\] also confirmed that before a vortex pattern is formed the condensate passes through dynamically unstable distorted configurations. The dynamical instability transforms the distorted condensate to an appropriately symmetric condensate with vortices.
There have been theoretical studies of how vortices are generated in a BEC via these dynamically unstable configurations using the mean-field GP equations in two \[20\] and three \[24\] dimensions. The theoretical study of the dynamical instability using mean-field approach in the generation of vortex is of interest as one can then compare the results of the models with experiments. This will provide a more stringent test of the mean-field theories than the comparison of stable configurations of BEC.

However, there is another type of instability which has been studied in relation to a nonrotating BEC under the application of an impulsive force. One can now suddenly change the trapping potential on a stable BEC. This can be done in experiment by varying the current in the coil responsible for confinement. One can also change the strength of interatomic interaction suddenly by changing the magnetic field near a Feshbach resonance \[23\]. These set in dynamical instabilities in the BEC, which can be studied both experimentally \[29\] and theoretically \[30\].

Similar impulsive force can be applied on a rotating BEC with vortices and the resultant oscillation studied. One can study how the number of vortices change after the application of the impulsive force on a rotating BEC. In this paper we perform such mean-field analyses of the dynamics of a rotating BEC subject to three types of impulsive forces. We consider a sudden change in (i) the strength of interaction near a Feshbach resonance \[23\] induced by a variation of an external magnetic field, (ii) the trapping potential, and (iii) the rotation of the BEC. Specifically, we study the dynamics when the strength of interaction or the trapping potential is suddenly increased or reduced. We also study the decay of vortices once the rotation of the condensate is suddenly stopped. There have been certain experimental measurements on such decay of vortices \[1\]. Finally, the trapping potential can be removed suddenly in a rotating BEC with vortices and the resultant expansion studied.

We base our study on the numerical solution \[12\] \[14\] \[10\] of the nonlinear time-dependent mean-field GP equation \[10\], which should provide a faithful description of the formation and evolution of vortices. The experiments on the vortices used a highly anisotropic trap which may be simulated by a quasi two-dimensional model. Hence as in the recent studies on the subject in Refs. \[23\] \[20\] we employ the GP equation in two dimensions. We use a split-step method where in the time evolution of the GP equation the kinetic energy and angular momentum (derivative) terms in the two directions are dealt with in two independent steps by the Crank-Nicholson rule complimented by the known boundary conditions \[23\] \[21\]. All the harmonic oscillator and the nonlinear terms are treated in a third separate step. We find that this time-dependent split-step approach leads to good convergence.

In Sec. II we describe briefly the time-dependent GP equation for a rotating BEC and a numerical method for its solution. In Sec. III we report the numerical results of the present investigation and finally, in Sec. IV we give a summary of our study.

II. NONLINEAR GROSS-PITAEVSKII EQUATION

At zero temperature, the time-dependent Bose-Einstein condensate wave function $\Psi(r, \tau)$ in two dimensions at position $r$ and time $\tau$ may be described by the following mean-field nonlinear GP equation \[10\] \[13\]

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) + gN|\Psi(r, \tau)|^2 - \Omega L_z \right. \\
- i\hbar \frac{\partial}{\partial \tau} \right] \Psi(r, \tau) = 0. \quad (2.1)$$

Here $m$ is the mass and $N$ the number of atoms in the condensate, $g$ the strength of interatomic interaction. The trapping potential in two dimensions may be written as $V(r) = \frac{\hbar}{2m \omega^2} (x^2 + y^2)$ where $\omega$ is the angular frequency. The rotational term $-\Omega L_z = i\hbar \hat{\Omega}(\hat{x} \partial_y - \hat{y} \partial_x)$ where $\hat{\Omega}$ is the rotational frequency along the $\hat{z}$ direction with $L_z$ the $\hat{z}$ component of angular momentum. The normalization condition of the wave function is $\int d\tau |\Psi(r, \tau)|^2 = 1$. The GP equation \[2.1\] can accommodate quantized vortex states with rotational motion of the condensate around the $\hat{z}$ axis.

It is convenient to use the dimensionless variables defined by $x = \hat{x}/l$, $y = \hat{y}/l$, $t = \tau \omega/2$, $l \equiv \sqrt{\hbar/(m\omega)}$, $\Omega = 2\Omega/\omega$, and $\varphi(x, y; t) = l\Psi(r; t)$. In these units Eq. \[2.1\] becomes

$$\left[ -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + (x^2 + y^2) + i\Omega \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \right. \\
+ \mathcal{N} \varphi(x, y; t)^2 - i\frac{\partial}{\partial t} \left] \varphi(x, y; t) = 0. \quad (2.2)$$

where $\mathcal{N} = 2gN\hbar^2$. The normalization condition of the wave function is

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\varphi(x, y; t)|^2 = 1. \quad (2.3)$$

The explicit inclusion of the rotational energy in the GP hamiltonian above simulates the time evolution of a rotational BEC which permits multiple-vortex formation. The initial input solution to the GP equation is a circularly symmetric state obtained with $\Omega = 0$. The method of solution with $\Omega \neq 0$ is time iteration using an appropriate algorithm. If one iterates the GP equation above in time using the full angular momentum term the circular symmetry is maintained always and a solution with multiple vortices is generated. It is essential that the circular symmetry of the solution is broken at some stage before multiple vortices are generated during time iteration.

The circular symmetry of the solution can be broken in the process of time evolution of the GP equation above
by introducing \( \delta \) a phase in the initial wave function for \( \Omega = 0 \) of the form

\[
\delta = \sum_i \arctan((y_i - y_i)/(x_i - x_i)),
\]

where \( x_i \) and \( y_i \) are a set of parameters such that the positions \((x_i, y_i)\) fall inside the boundary of the BEC. This initial phase preserves the norm of the wave function but accelerates the formation of vortex nucleation after time evolution of the GP equation. However, there is no direct relation between the parameters \((x_i, y_i)\) and the positions or the numbers of the vortices in the final BEC. Another way of facilitating vortex formation is to include a small dissipative term in the initial wave function \( \delta \) given by Eq. (2.4). In the beginning the system passes through deformed configurations with unstable oscillating boundaries. The vortices are generated in the peripheral region via the oscillation of the boundary. More and more vortices are generated which move towards the center and which arrange themselves in a regular form in the condensate. For a sufficiently large number of vortices they are arranged as in a lattice and maintain that pattern during rotation. When such a regular equilibrated final pattern is formed the condensate attains the symmetry of the problem — circular in the present case.

III. NUMERICAL RESULTS

In the present numerical study on vortices we use an equilibrated condensate with \( N = 500 \) trapped in the stationary potential as the initial state. Figure 1 shows the dynamics of the condensate as it starts to rotate suddenly with an angular frequency \( \Omega = 1.5 \). In the simulation we use as input the wave function with phase \( \delta \) given by Eq. (2.4). In the beginning the system passes through deformed configurations with unstable oscillating boundaries. The vortices are generated in the peripheral region via the oscillation of the boundary. More and more vortices are generated which move towards the center and which arrange themselves in a regular form in the condensate. For a sufficiently large number of vortices they are arranged as in a lattice and maintain that pattern during rotation. When such a regular equilibrated final pattern is formed the condensate attains the symmetry of the problem — circular in the present case.

![Figure 1: Formation of the vortices for \( N = 500 \) and \( \Omega = 1.5 \) at different times; size of each square is 10 x 10.](image)

In actual experiment the size of the condensate reduces with time due to inelastic atomic collisions \( \delta \). However, throughout the present study we renormalize the number of particles during time iteration so that the size is conserved. This should not affect the generation of vortices.

For \( N = 500 \) one has a hexagonal lattice structure with seven vortices at large times. In recent theoretical studies \( \delta \), some general features of vortex formation have been established which are confirmed in the present study. For a fixed large \( N \) there is a critical \( \Omega \) for the appearance of a certain number of vortices \( \delta \). The critical value of \( \Omega \) increases as the number of vortices increases. For a fixed \( \Omega \) the number of vortices increases as \( N \) increases. It is interesting to recall that for \( N = 0 \) only states with a single vortex at the center can be formed.

An interesting set of experiments can be performed where on a stable rotating BEC with vortices the strength of atomic interaction or the harmonic oscillator trapping
potential is suddenly changed. Once either of these parameters are changed suddenly on a vortex lattice with hexagonal structure, the system enters again a dynamically unstable asymmetric configuration with highly oscillating deformed boundary. Through the oscillating deformed boundary new vortices can enter the condensate or some of the existing vortices can get out. Eventually, a new equilibrated BEC with vortices is formed in the condensate with circular symmetry. However, as the condensate oscillates after the application of the impulsive force it is more difficult to obtain perfect symmetry numerically in the final state. Small imperfections remain near the boundary and in the arrangement of the vortices in the final state.

FIG. 2: Dynamics of the condensate of Fig. 1 at different times $t$ upon sudden increase of $N$ from 500 to 1200 at $t = 0$; size of each square is $10 \times 10$.

FIG. 3: Dynamics of the condensate of Fig. 1 at different times $t$ upon sudden decrease of $N$ from 500 to 100 at $t = 0$; size of each square is $10 \times 10$.

In the simulation presented in Fig. 2 the nonlinear parameter $N$ on the hexagonal vortex lattice of Fig. 1 is suddenly changed to 1200 from 500 at $t = 0$ and the evolution of the resultant condensate studied. This corresponds to an increase of the strength of interatomic interaction by a factor 2.4. The size of the condensate increases and it accommodates more vortices. The system seems to keep on oscillating with $(12 \pm 1)$ vortices. This oscillation seems to be a natural consequence of the sudden change of the strength of interaction. Next we consider in Fig. 3 the sudden change in the nonlinear parameter $N$ from 500 to 100 at $t = 0$ on the final configuration of Fig. 1. With the reduction of the strength of interaction by a factor 0.2, the condensate shrinks in size by passing through dynamically unstable and asymmetric configurations and starts to oscillate. The condensate tries to reach an equilibrated stage by getting rid of vortices. However, at large times the system continues to oscillate with $(6 \pm 1)$ vortices.

FIG. 4: Dynamics of the condensate of Fig. 1 at different times upon sudden increase of the harmonic oscillator trapping potential by a factor of 1.5 at $t = 0$; size of each square is $10 \times 10$.

FIG. 5: Dynamics of the condensate of Fig. 1 at different times upon sudden decrease of the harmonic oscillator trapping potential by a factor of 0.75 at $t = 0$; size of each square is $10 \times 10$.

Next we consider a sudden change in the harmonic oscillator potential. First, the harmonic oscillator potential is increased by a factor of 1.5 on the rotating BEC of Fig. 1 at $t = 0$. The time evolution of the resultant condensate is shown in Fig. 4. With the sudden increase in the trapping potential the condensate starts to shrink and oscillate. It passes through dynamically unstable asymmetric configurations and reaches an equilibrium stage.
We study the decay of the vortices of Fig. 1 once the rotation of the BEC is suddenly stopped at \( t = 0 \). The dynamics of the decay of vortices has been studied experimentally in Refs. [2, 9]. The numerical simulation of the decay of vortices is shown in Fig. 6. The vortices do not decay immediately after stopping the rotation. Rather they stay for a long time after the rotation stops. As time passes vortices get out of the condensate until one with a single vortex results. This BEC with a single vortex survives for a long time. If we consider \( \omega = 2\pi \times 219 \text{ s}^{-1} \) as in the experiment of Madison et al [2], one unit of present dimensionless time is 0.000726 s. The system gets rid of six vortices in about 600 units of time or in 0.4 s. However, the state with one vortex stays up to \( t \approx 3000 \), which gives a lifetime of about 2400 units of time or about 1.6 s. Madison et al [2] measured the half life of a single vortex state and found it to be about 1 s, which is of the same order of magnitude as in the present simulation. However, a quantitative comparison between the two is not to the point because of approximate nature of the present simulation compared to experiment. The present calculation used a quasi two-dimensional model opposed to a full three-dimensional one. Also, the nonlinearity of the present model is not the same as in the experiment. However, in a rotating BEC with many vortices, most of the vortices decay within one second whereas a small number of vortices survive for a long time. This was also experimentally confirmed by Abo-Shaeer et al [9] who used \( \omega \approx 2\pi \times 85 \text{ s}^{-1} \).

Finally, we consider the free expansion of the vortices in Fig. 1 when the harmonic oscillator trap is suddenly removed. The dynamics of free expansion is shown in Fig. 7. It is interesting that the vortex pattern is not destroyed immediately after the removal of the trap. On the other hand, upon expansion the radius of the vortex core is found to increase more rapidly than the radius of the condensate. Due to the very small radius of the vortex core it is difficult to observe the vortices and count them properly in an experiment [2]. However, because of the increase in the radius of the vortex core after a free expansion the detection and counting of vortices will be easier from a picture of the condensate after expansion [2]. In Fig. 7 the vortices are more prominently visible at \( t = 1.6 \) than at \( t = 0.2 \). Using the radial trapping frequency \( \omega = 2\pi \times 85 \text{ s}^{-1} \) as in the experiment of Abo-Shaeer et al [9], \( t = 1.6 \) of Fig. 7 corresponds to about 3 ms. In the actual experiment [9] after 42.5 ms of expansion the vortex cores were magnified 20 times their size in the trap. The present simulation has again led to the right order of magnitude.

**IV. SUMMARY**

We have investigated the effect of an impulsive force on a rotating BEC with vortices using the mean-field GP equation. We have considered the effect of suddenly increasing or decreasing the strength of atomic interaction and the harmonic oscillator trapping potential. Upon application of such an impulsive force the condensate passes through highly unstable distorted configurations with oscillating boundary. New vortices can enter the condensate via the oscillating boundary. Also the condensate can get rid of some of the vortices in a similar fashion. Eventually, a (more) symmetric condensate with a different number of vortices result. When the atomic interaction is suddenly increased (decreased) the final condensate grows (shrinks) in size and accommodates more (less) vortices. When the trapping potential is suddenly increased (decreased) the final condensate shrinks (grows) in size and accommodates less (more) vortices. Experiments can be performed when similar
impulsive force in mind on a rotating BEC with vortices and compared with the mean-field prediction.

We also study the decay of vortices in a rotating BEC when the rotation is suddenly stopped. Most of the vortices decay quickly (in a fraction of a second), but a few vortices survive during a relatively long time before they decay. In a typical experimental situation, the single vortex at the end survives for an unusually long interval of time (a couple of seconds).

In another simulation of a free expansion of a rotating BEC we find that the vortices are not immediately lost during expansion. Rather, upon expansion the radius of the vortex core increases at a faster rate than that of the condensate. Hence the vortices are more prominently visible in a picture of the condensate during the free expansion of the condensate. Experimentally, it is difficult to observe and count the vortices correctly as the vortex cores have usually small radii. The present simulation suggests that the free expansion could aid in the observation and counting of vortices in a rotating BEC, specially, when there is a large number of vortices. This was noted in the experiment of Abo-Shaeer et al. In a typical experimental situation, on free expansion the vortices may be observed during few tens of milliseconds.

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