Universal central extensions of superalgebras with bracket

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Abstract. Superalgebras and, in particular, Lie superalgebras and Poisson superalgebras play an important role in theoretical physics. For example, one of the applications of such algebras is in supersymmetric unified field theories for describing bosons and fermions. In this regard, any research in the theory of superalgebras turns out to be significant from the point of view of theoretical physics. The article discusses some generalizations of Poisson superalgebras, namely, a superalgebra with a bracket. For these superalgebras, a universal central extension is constructed and the kernel of such an extension as the second homology group of the superalgebra with coefficients in the trivial module is calculated.

1. Introduction
One of the aspects in the study of groups, algebras, and superalgebras is the study of their central extensions. Important directions of these studies are the formulation of conditions for the existence of a universal central extension of an algebraic structure $G$, the construction of a model of a universal central extension of an algebraic structure and the description of the kernel of this extension in terms of homology groups with coefficients in a trivial module, as well as the study of the possibility of raising automorphisms and differentiations of an algebraic structure $G$ by its central expansion.

For the first time the concept of universal central extension was introduced by J. Schur [14] for finite groups, therefore the core of universal central extension is often called the "Schur’s multiplier". R. Steinberg [15] proposed a description of universal central extensions of algebraic groups and Lie algebras by generators and relations. J. Milnor [13] developed the ideas of the previous work, introduced the term "Steinberg group" for a universal central extension of a group generated by elementary matrices with elements from the associative ring $R$. Milnor defined the Steinberg group by generators and relations imitating the relations between elementary matrices and proved that the center Steinberg group is a functor $K_2$ of the ring $R$. He also proved necessary and sufficient conditions for the existence of universal central extensions of groups.

Further research was devoted to various Lie-type algebras: Lie algebras and Lie superalgebras, conformal Lie algebras and Lie superalgebras, Leibniz algebras and algebras with bracket [4–12, 16, 17]. In these papers, both general questions of extensions are considered, and universal central extensions of concrete classes of algebras are constructed.

In the present paper, we consider Lie superalgebras with a bracket and study their central extensions.

2. Preliminaries
Recall the basic definitions of the theory of central extensions for Lie type algebras (including superalgebras).

Let \( R \) and \( G \) be algebras of the same Lie type.

A pair \((R,\psi)\) consisting of an algebra \( R \) and a surjective homomorphism \( \psi: R \rightarrow G \) called the central extension of algebra \( R \) if the kernel of the homomorphism \( \psi \) lies in the center of the algebra \( R \). Therefore, a short exact sequence of algebras is defined

\[
0 \rightarrow \text{Ker} \psi \rightarrow R \xrightarrow{\psi} G \rightarrow 0.
\]

A central extension \((R,\psi)\) of a algebra \( G \) is called splitting if there exists a homomorphism \( s: G \rightarrow R \) such that \( \psi s = i d_G \), and the homomorphism \( s \) itself is called a section.

A central extension \((U,\psi)\) is called universal if for any other central extension \((R,\psi)\) of algebra \( G \) there exists a unique homomorphism \( h: U \rightarrow R \) such that \( \psi h = \varphi \), i.e. commutative diagram

\[
\begin{array}{c}
0 \rightarrow \text{Ker} \varphi \rightarrow U \xrightarrow{\varphi} G \rightarrow 0 \\
h \downarrow \quad \text{id} \downarrow \\
0 \rightarrow \text{Ker} \psi \rightarrow R \xrightarrow{\psi} G \rightarrow 0
\end{array}
\]

Any central extension of the algebra \( G \) with the kernel \( I \) can be constructed using the 2-cocycle \( f: G \times G \rightarrow I \), where \( I \) is the trivial \( G \)-module. Then

\[
G_f = G \oplus I,
\]

where \( x, y \in G; a,b \in I; [x,y] \) is the commutation operation defined in algebra \( G \).

In this case, the central extension of the algebra \( G \)

\[
0 \rightarrow I \overset{i}{\rightarrow} G_f \overset{\varphi}{\rightarrow} G \rightarrow 0,
\]

where \( i(a) = (0,a); \varphi(x,a) = x \).

A universal central extension of algebra \( G \) can be constructed using free generators. To this end, we consider the free algebra \( F \) generated by the set \( G \), and the homomorphism \( \tau: F \rightarrow G \) with the kernel \( \text{Ker} \tau = C \). Then by a short exact sequence

\[
0 \rightarrow C \rightarrow F \xrightarrow{\tau} G \rightarrow 0
\]

we build a short exact sequence

\[
0 \rightarrow (C \cap [F,F])/[C,F] \rightarrow [F,F]/[C,F] \xrightarrow{\tau^*} G \rightarrow 0,
\]

which defines the universal central extension of algebra \( G \).

### 3. Universal central extensions of superalgebras with bracket

Algebras with a bracket were introduced in [3] as a generalization of Poisson algebra. We give the definition of a superalgebra with a bracket and some other necessary definitions.

An associative superalgebra \( A = A_1 \oplus A_2 \) on which a bilinear map is defined

\[
[-,-]: A \otimes A \rightarrow A, \quad (a \otimes b) \rightarrow [a,b],
\]

satisfying the relation

\[
[a \cdot b,c] = (-1)^{|b||c|}[a,c] \cdot b + a \cdot [b,c]
\]

for all \( a,b,c \in A; |a| \in \mathbb{Z}_2 \) is called a superalgebra with a bracket.

The commutator ideal or commutant \( [A,A] \) of a superalgebra with bracket \( A \) is the ideal generated by all products \( a \cdot b \) and \([a,b]\) for any \( a,b \in A \).

A superalgebra with bracket \( A \) is called perfect if it coincides with its commutant \( A = [A,A] \).

The central extension \((R,\psi)\) of a superalgebra with bracket \( G \) in which the superalgebra \( R \) is perfect is called a covering. In this case, obviously, the superalgebra \( G \) is also perfect.

We state the lemmas characterizing the central extensions of superalgebras with a bracket. These lemmas are similar to the results obtained earlier in [7, 9, 11].
Lemma 1. If \((U, \varphi)\) and \((R, \psi)\) are two central extensions of a superalgebra with a bracket \(G\), and each central extension of a superalgebra with a bracket \(U\) splits, then there is a homomorphism \(h : U \rightarrow R\) such that \(\psi h = \varphi\).

Lemma 2. If \((U, \varphi)\) and \((R, \psi)\) are two central extensions of a superalgebra with a bracket \(G\), and \(U\) is a perfect superalgebra with bracket, then there is at most one homomorphism \(h : U \rightarrow R\) such that \(\psi h = \varphi\).

Lemma 3. If in the central extension \((U, \varphi)\) of a superalgebra with a bracket \(G\) the superalgebra \(U\) is not perfect, then there is a central extension \((R, \psi)\) of the superalgebra with a bracket \(G\), for which there are at least two homomorphisms \(f_i : U \rightarrow R\) satisfying the condition \(\psi f_i = \varphi\).

Lemma 4. If \((W', \nu)\) is the central extension of a perfect superalgebra with a bracket \(U\), then the commutant \([W', W'] = W\) is a perfect superalgebra with a bracket that maps to the entire conformal algebra \(U\).

Lemma 5. If \((U, \varphi)\) is a universal central extension of a superalgebra with a bracket \(G\), then any central extension \((W', \nu)\) of a superalgebra with a bracket \(U\) split.

Lemma 6. Any superalgebra with bracket \(G\) has a central extension.

Based on these lemmas, we formulate a criterion for the existence of a universal central extension of a superalgebra with a bracket.

Theorem 7. (Criterion for the existence of a universal central extension of a superalgebra with a bracket)

A) The central extension \((U, \varphi)\) of a superalgebra with a bracket \(G\) is universal if and only if \(U\) is a perfect superalgebra with a bracket and any central extension of the superalgebra \(U\) splits.

B) A superalgebra with bracket \(G\) has a universal central extension if and only if the superalgebra \(G\) is perfect.

To prove the first statement, we note first that if \((U, \varphi)\) is a central extension of a superalgebra with bracket \(G\), the superalgebra \(U\) is perfect and each of its central extensions splits. It follows from Lemmas 1 and 2 that for any central extension \((R, \psi)\) of a superalgebra with bracket \(G\) there exists a unique homomorphism \(h : U \rightarrow R\) such that \(\psi h = \varphi\), which means the universality of the extension \((U, \varphi)\).

Conversely, if \((U, \varphi)\) is a universal central extension of a superalgebra with bracket \(G\), then its universality implies the existence of a unique homomorphism \(h : U \rightarrow R\) for any other central extension \((R, \psi)\) of a superalgebra with a bracket \(G\) with the property \(\psi h = \varphi\). Hence, by Lemma 2, the superalgebra \(U\) is perfect, and by Lemma 5, any central extension of it splits.

The necessity for the second statement of the theorem is obvious. To prove sufficiency, we note that if a superalgebra with bracket \(G\) is perfect, then by Lemma 6 it has a central extension \((F, \varphi)\), where \(F\) is a free superalgebra with bracket generated by \(G\). Since the superalgebra with the bracket \(G\) is perfect, then by Lemma 4 the commutant of the superalgebra with the bracket \(F / [R, F]\) is a perfect superalgebra with the bracket and maps to the whole superalgebra \(G\). It is also clear that

\[ F / [R, F], F / [R, F] ] \cong [F, F] / [R, F]. \]

It remains to be shown by direct calculations that the following central extension \(([F, F] / [R, F]) \varphi\) is a universal central extension of a superalgebra with bracket \(G\).

This completes the proof of the theorem.

Corollary 8. The kernel of the universal central extension \((F, \varphi) / [R, F]) \varphi\) is canonically isomorphic to the two-dimensional homology group \(H_2G\) of a superalgebra with bracket \(G\) with coefficients in the trivial module \(k\).

To prove the corollary by the short exact sequence

\[ 0 \rightarrow R \rightarrow F \rightarrow G \rightarrow 0 \]

build a short exact sequence

\[ H_2F \rightarrow H_2G \rightarrow H_0(G; H_1R) \rightarrow H_2F \rightarrow H_2G \rightarrow 0. \]

We calculate the obtained homology groups and obtain the statement of the corollary.
Now we construct a model of the universal central extension of a superalgebra with bracket $G$ using the construction described above. Consider the $k$-module $F(G) = G \otimes G + G \otimes [\], G$ and the submodule $R$ in $F(G)$ generated by elements of the form

$$(a_1 \cdot a_2) \otimes a_3 - a_3 \otimes (a_2 \cdot a_1),$$

$$(1)_{[a_2][a_3]}[a_1, a_3] \otimes a_2 + a_1 \otimes [a_2, a_3] - (a_1 \cdot a_2) \otimes [a_3]$$

for all $a_1, a_2, a_3 \in G$; $|a_i| \in \mathbb{Z}_2$. Then the factor module is defined

$$uce(G) = ((G \otimes G + G \otimes [\], G))/R,$$

whose elements are classes

$$\langle a, b \rangle = a \otimes b + R;$$

$$\langle a, b \rangle = a \otimes [a, b] + R.$$

$uce(G)$ is a superalgebra with a bracket; an epimorphism $\varphi : uce(G) \rightarrow G$ is defined by the conditions

$$\varphi(\langle a, b \rangle) = a \cdot b;$$

$$\varphi(\langle a, b \rangle) = [a, b].$$

Theorem 9. The pair $(uce(G), \varphi)$ is a universal central extension of a superalgebra with bracket $G$.

Theorem 10. If $f : G' \rightarrow G$ is a homomorphism of superalgebras with a bracket, then induced homomorphism is defined

$$uce(f) : uce(G') \rightarrow uce(G),$$

for which the following diagram

$$\begin{array}{ccc}
uce(G') & \xrightarrow{uce(f)} & uce(G) \\
\varphi' \downarrow & & \downarrow \varphi \\
G' & \xrightarrow{f} & G
\end{array}$$

is commutative.

This construction of universal central extension allows us to solve the problem of raising automorphisms and differentiations of a superalgebra with bracket $G$ onto its covering.

Let $(G', f)$ be the covering of a superalgebra with bracket $G$ and $K = Ker (\varphi' \circ uce(f)^{-1})$.

Theorem 11. Let $h$ be an automorphism of a superalgebra with a bracket $G$. Then there exists an automorphism $h'$ of a superalgebra with a bracket $G'$, preserving the following diagram

$$\begin{array}{ccc}
G' & \xrightarrow{f} & G \\
h' \downarrow & & \downarrow h \\
G' & \xrightarrow{f} & G
\end{array}$$

commutative if and only if the automorphism $uce(h)$ of the superalgebra with the bracket $uce(G)$ satisfies the condition $uce(h)(K) = K$.

Theorem 12. The differentiation $d$ of a superalgebra with a bracket $G$ rises to the differentiation $d'$ of a superalgebra with a bracket $G'$, which preserves the following diagram

$$\begin{array}{ccc}
G' & \xrightarrow{f} & G \\
h' \downarrow & & \downarrow h \\
G' & \xrightarrow{f} & G
\end{array}$$

commutative if and only if $uce(d)$ of a superalgebra with the bracket $uce(G)$ satisfies the condition $uce(d)(K) \subseteq K$. In this case, $d'$ is uniquely determined and leaves $Ker f$ invariant.

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