Construction of Fractal Surfaces by Recurrent Fractal
Interpolation Curves

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Abstract

A method to construct fractal surfaces by recurrent fractal curves is provided. First we
construct fractal interpolation curves using a recurrent iterated functions system(RIFS) with
function scaling factors and estimate their box-counting dimension. Then we present a method
of construction of wider class of fractal surfaces by fractal curves and Lipschitz functions and
calculate the box-counting dimension of the constructed surfaces. Finally, we combine both
methods to give more flexible constructions of fractal surfaces.

Keywords: Fractal curve, Fractal surface, Recurrent iterated function system(RIFS), Box-
counting dimension, Fractal interpolation function.

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1 Introduction

A Fractal surface (or fractal curve) is a fractal set which is a graph of some continuous function on
$\mathbb{R}^2$ (or $\mathbb{R}$). The method of construction of fractal surfaces (or fractal curves) are closely related to
the generation of fractal interpolation functions(FIF). The FIFs were introduced by Barnsley [2] in
1986 and after that have been widely studied and used in approximation theory, image compression,
computer graphics and modeling of natural surfaces such as rocks, metals, planets, terrains and so
on. (See [1, 7, 11, 18].)

The constructions of fractal surfaces (such as self-similar, self-affine or non self-affine surfaces) by
IFSs or RIFSs have been studied in many papers (see [5, 6, 8, 10, 12, 13, 17]). These surfaces are
all attractors of some IFSs or RIFSs. In order to ensure the continuity of the surface, some authors
in the early days assumed that the interpolation nodes on the boundary are collinear. Later Malysz
[10], Metzler et al. [13], Feng et al [8] and Yun [17] studied the construction of fractal interpolation
surfaces on arbitrary data sets using IFS. Furthermore, [10], [13] and [17] estimated the dimensions
of the result surfaces. In [6], the authors studied the construction of recurrent fractal interpolation
surfaces. In [6, 10] they use constant contraction factors in construction of IFS or RIFS and in
[13, 17, 8] they use function contraction factors in construction of IFS.

The fractal properties of such rough surfaces as those of metals or rocks may expressed by sectional
profiles of those surfaces. That’s why constructions of fractal surfaces by fractal curves have been
studied in many papers (see [5, 7, 11, 15, 16, 19]). In [11], Mandelbrot suggested that the fractal
dimension of a surfaces constructed by a single curve can be obtained by adding 1 to the fractal
dimension of the curve. In [7], Falconer introduced fractal surfaces constructed by the movement of a
fractal curve along a segment and determined their fractal dimension. Xie et al. [16] proposed the so
called star product fractal surfaces which are constructed by the movement of a fractal curves along another one. After that [19] provided a construction of fractal surfaces by 4 fractal curves which are boundary curves of the constructed surface and in [15] a construction of Bush type fractal surfaces by two Bush curves were studied. [5] provided a construction of fractal surfaces by the fractal interpolation. All these constructions of fractal surfaces by fractal curves have common property that the fractal dimension of the constructed surfaces is determined by the fractal dimensions of the fractal curves constructing them and used the fractal curves constructed by IFS or RIFS with constant contraction factors. Such a constant vertical scaling factors are not consistent with reality [8].

We construct recurrent fractal curves by more general RIFSs with function scaling factors and estimate the box-counting dimension of the constructed curves. And we construct wider class of fractal surfaces by fractal curves and Lipschitz functions, and calculate the box-counting dimension of the constructed surfaces. Finally we combine both constructions to construct fractal surfaces.

The remainder of the article is organized as follows: The section 2 describes some basic notions of RIFS. The section 3 constructs recurrent fractal curves with function vertical scaling factors and estimates their box counting dimensions. The section 4 provides a general method of construction of fractal surfaces combining fractal curves using Lipschitz functions and a formula of the box-counting dimensions. As one application of the results of the section 3 and 4, the section 5 describes a method of constructing fractal surfaces combining the recurrent fractal curves constructed in the section 3 using some Lipschitz functions and provides some examples.

2 Preliminaries on RIFS

In this section we describe some basic notions on recurrent iterated function systems and related lemmas. The class of irreducible matrices are important in RIFS theory.

Definition 1 [9] Let \( Q = (q_{ij}) \) be \( n \times n \) matrix. \( Q \) is called reducible, if the set of indices \( \{1, 2, \ldots, n\} \) can be decomposed into two distinct and complementary subsets \( \{i_1, i_2, \ldots, i_\mu\} \) and \( \{j_1, j_2, \ldots, j_\nu\} \), i.e.,

\[
\{i_1, i_2, \ldots, i_\mu\} \cap \{j_1, j_2, \ldots, j_\nu\} = \emptyset, \\
\{i_1, i_2, \ldots, i_\mu\} \cup \{j_1, j_2, \ldots, j_\nu\} = \{1, 2, \ldots, N\}
\]

such that

\[ q_{ab} = 0, \quad a = 1, 2, \ldots, \mu; \quad b = 1, 2, \ldots, \nu. \]

If this is not possible, then \( Q \) is an irreducible matrix.

The following results on irreducible matrices are useful and readers can refer to [9], [14].

Lemma 1 Consider an \( n \times n \) square matrix \( A \). The following statements are equivalent.

1) \( A \) is irreducible.

2) For every pair \( (i, j) \), \( i, j = 1, 2, \ldots, n \), there is a \( k > 0 \) such that the element of the \( i \)-th row \( j \)-th column of the matrix \( A^k \) is positive.

Lemma 2 \( A \) is a non-negative, reducible \( n \times n \) matrix, iff the matrix \( (I_n + A)^{n-1} \) is positive (where \( I_n \) is the identity matrix).

Lemma 3 (Perron-Frobenius Theorem) Let \( A \geq 0 \) be an irreducible square matrix. Then we have the following two statements.

1) The spectral radius \( \rho(A) \) of \( A \) is an eigenvalue of \( A \) and it has strictly positive eigenvector \( y \) (i.e., \( y_i > 0 \) for all \( i \)).

2) \( \rho(A) \) increases if any element of \( A \) increases.
Definition 2 \[3, 4\] \((X, d_X)\) is a complete metric space and \(X_1, \ldots, X_n\) are subsets of \(X\). Let \(W_i : X_i \to X (i = 1, \ldots, n)\) be contraction maps. \(M = (p_{ij})_{i, j = 1}^n\) is a \(n \times n\) row-stochastic matrix (i.e., such a matrix that \(\sum_{j=1}^n p_{ij} = 1, i = 1, \ldots, n\)) and irreducible. Then \(\{X; M; W_1, \ldots, W_n\}\) is called a \textit{recurrent iterated functions system} (or simply RIFS). The \(n \times n\) matrix \(C = (c_{ij})\) defined by

\[
c_{ij} = \begin{cases} 
1, & p_{ji} > 0, \\
0, & p_{ji} = 0
\end{cases}
\]

is called the \textit{connection matrix} of RIFS.

Let \(H(X)\) be the set of all nonempty compact subset of \(X\) and \(h\) the Hausdorff distance in \(H(X)\). Then \((H(X), h)\) is a complete metric space \([1]\). In the set \(\tilde{H}(X) = H(X) \times \cdots \times H(X)\), we define a distance \(\tilde{h} : \tilde{H}(X) \times \tilde{H}(X) \to \mathbb{R}\) as follows:

\[
\tilde{h}(\{(A_1, \ldots, A_n), (B_1, \ldots, B_n)\}) := \max_{i = 1, \ldots, n} h(A_i, B_i), \forall (A_1, \ldots, A_n), (B_1, \ldots, B_n) \in \tilde{H}(X).
\]

Then \((\tilde{H}(X), \tilde{h})\) is a complete metric space \([1]\).

For a given recurrent iterated functions system \(\{X; M; W_1, \ldots, W_n\}\), we define the transformation \(W : \tilde{H}(X) \to \tilde{H}(X)\) of \(\tilde{H}(X)\) as follows: let

\[
W(B) := \begin{pmatrix}
c_{11}W_1(B_1) & c_{12}W_1(B_2) & \cdots & c_{1n}W_1(B_n) \\
c_{21}W_2(B_1) & c_{22}W_2(B_2) & \cdots & c_{2n}W_2(B_n) \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1}W_n(B_1) & c_{n2}W_n(B_2) & \cdots & c_{nn}W_n(B_n)
\end{pmatrix}
= \begin{pmatrix}
\bigcup_{j \in \Lambda(1)} W_1(B_j) \\
\bigcup_{j \in \Lambda(2)} W_2(B_j) \\
\vdots \\
\bigcup_{j \in \Lambda(n)} W_n(B_j)
\end{pmatrix}.
\]

for every \(B = (B_1, \ldots, B_n) \in \tilde{H}(X)\). Here \(\Lambda(i) = \{j : c_{ij} = 1\}, i = 1, \ldots, n\). We will denote the transformation \(W\) as a matrix form by

\[
W := \begin{pmatrix}
c_{11}W_1 & c_{12}W_1 & \cdots & c_{1n}W_1 \\
c_{21}W_2 & c_{22}W_2 & \cdots & c_{2n}W_2 \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1}W_n & c_{n2}W_n & \cdots & c_{nn}W_n
\end{pmatrix}.
\]

This transformation \(W\) is a contraction map in \(\tilde{H}(X)\) and the unique fixed point \(A = (A_1, \ldots, A_n)\) is called the \textit{attractor} or \textit{invariant set} of the RIFS, or \textit{recurrent fractal} \([3, 4]\).

Note. The invariant set of RIFS is a \textit{vector} whose elements are nonempty compact sets. If \(A = (A_1, \ldots, A_n)\) is the invariant set of a RIFS \(\{X; M; W_1, \ldots, W_n\}\), then we have \(A_i = \bigcup_{j \in \Lambda(i)} W_i(A_j)\), \(\forall i = 1, \ldots, n\). Usually, making a slight abuse of notation, we often call the union \(A = \bigcup_{i=1}^n A_i\) of all \(A_i\) as the attractor of RIFS, too.

Definition 3. A curve which is the invariant set of a RIFS on \(\mathbb{R}^2\) is called a \textit{recurrent fractal curve} (or RFC).

Definition 4. In general, the \textit{box-counting dimension} \(\dim_B A\) of a fractal set \(A\) is defined by

\[
\dim_B A = \lim_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta}
\]

(if this limit exists), where \(N_\delta(A)\) is any of the followings (see \([7]\):
In this section, we construct recurrent fractal curves with function vertical scaling factors and estimate

3 Construction of Recurrent Fractal Interpolation Curves with Function Vertical Scaling Factors

In this section, we construct recurrent fractal curves with function vertical scaling factors and estimate their box-counting dimensions. Let a data set be

\[ P = \{(x_i, y_i) \in \mathbb{R}^2; i = 1, \ldots, n\}, \quad (x_0 < x_1 < \ldots < x_n) \]

and let

\[ N_n = \{1, \ldots, n\}, \quad I = [x_0, x_n], \quad I_i = [x_{i-1}, x_i], \quad i = 1, \ldots, n. \]

We denote Lipschitz (or contraction) constant of Lipschitz (or contraction) mapping \( f \) by \( L_f \) (or \( c_f \)). Let \( l \geq 2, l \in \mathbb{N} \) and let \( \tilde{I}_k = [x_{s(k)}, x_{e(k)}], \quad x_{s(k)}, x_{e(k)} \in \{x_0, \ldots, x_n\} \), here \( e(k) - s(k) \geq 2, \quad k = 1, \ldots, l. \) as a custom, \( I_i \) is called a region and \( \tilde{I}_k \) a domain.

We collect a map \( \gamma : N_n \rightarrow N_i \). This means that we relate every region to a domain. For each \( i \in N_n \), let \( k = \gamma(i) \). For \( i \in N_n \) (and \( k = \gamma(i) \)), let a mapping \( L_i = L_{i,k} : \tilde{I}_k \rightarrow I_i \) be such a contraction homeomorphism that \( L_{i,k}(\{x_{s(k)}, x_{e(k)}\}) = \{x_{i-1}, x_i\} \). Such a map can be easily constructed in a standard way. Let \( H \subset \mathbb{R} \) be such a sufficiently large interval that \( y_i \in H, \quad i = 1, \ldots, n. \) Let \( F_i = F_{i,k} : I_i \times H \rightarrow \mathbb{R} \) be a function defined by

\[ F_{i,k}(x, y) = s_{i,k}(L_{i,k}(x))a(y) + b_{i,k}(x) \]

and satisfy

\[ F_{i,k}(x, y) = s_{i,k}(L_{i,k}(x))a(y) + b_{i,k}(x) \]

Here \( a(y) \) is a Lipschitz function on \( H \) such that \( a(y) = y, \quad \alpha \in \{s(k), e(k)\} \); here \( \beta \in \{i-1, i\} \) is such that \( L_{i,k}(x_{\alpha}) = x_{\beta}. \)

Example 1. The function \( F_{i,k} \) satisfying \( [\square] \) can be easily constructed. When \( h(x), g(x) \) are Lipschitz mappings on \( I \) and satisfy the conditions

\[ g(x_{\alpha}) = y_{\alpha}, \quad \alpha \in \{s(k), e(k)\}, \quad h(x_i) = y_i, \quad i = 0, 1, \ldots, n \]

and \( s_{i,k}(x) \) are taken as free unknown functions, the functions

\[ F_{i,k}(x, y) = s_{i,k}(L_{i,k}(x))(a(y) - g(x)) + h(L_{i,k}(x)) \]

satisfy \( [\square] \).

We define transformations \( W_i = W_{i,k} : \tilde{I}_k \times H \rightarrow I_i \times \mathbb{R}; \quad (i = 1, \ldots, n; \quad k = \gamma(i)) \) by

\[ W_{i,k}(x, y) = (L_{i,k}(x), F_{i,k}(x, y)). \]
Lemma 4 There exists a distance $\rho_0$ equivalent to the Euclidean metric on $\mathbb{R}^2$ such that $W_i = W_{i,k}(i = 1, \ldots, n; k = \gamma(i))$ are contraction transformations with respect to this distance.

Proof: Fix a positive number $\theta$ and define $\rho_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\rho_0((x, y), (x', y')) = |x - x'| + \theta|y - y'|.$$ 

Then $\rho_0$ is evidently equivalent to the Euclidean metric on $\mathbb{R}^2$. In what follows, if we denote maximum value of $|f(x)|$ on its domain by $\bar{f}$, then for $(x, y), (x', y') \in \tilde{I}_k \times H$, we have

$$\rho_0(W_{i,k}(x, y), W_{i,k}(x', y')) = |L_{i,k}(x) - L_{i,k}(x')| + \theta|F_{i,k}(x, y) - F_{i,k}(x', y')|$$

$$\leq c_{L_{i,k}}|x - x'| + \theta|s_{i,k}(L_{i,k}(x))a(y) - s_{i,k}(L_{i,k}(x'))a(y')| + \theta|b_{i,k}(x) - b_{i,k}(x')|$$

$$\leq c_{L_{i,k}}|x - x'| + \theta s_{i,k}L_{a}|y - y'| + \theta c_{s_{i,k}}c_{L_{i,k}}|x - x'|a + \theta L_{b_{i,k}}|x - x'|$$

$$\leq [c_L + \theta(c_{s_{i,k}}L_{a} + L_{b})]|x - x'| + \theta s_{L_{a}}|y - y'|.$$ 

Here $c_L = \max\{c_{L_{i,k}}|i = 1, \ldots, n\}$, and similarly, all $c_s, L_b$ and $s$ (without indices) are defined as the maximum of those with indices. Note that $sL_a < 1$. If we select $\theta$ so that

$$\theta > \frac{1 - c_L}{c_{s_{i,k}}L_{a} + L_{b}},$$

then all the transformations $W_{i,k}$ are contraction maps with contraction constants less than max$\{c_L + \theta(c_{s_{i,k}}L_{a} + L_{b})\}, sL_a\}$. (QED)

We give a row-stochastic matrix $M = (p_{ij})_{i=1}^{n}$ as follows. For every $i = 1, \ldots, n$, let $a_i$ be the number of the domains $\tilde{I}_{\gamma(j)}$ containing the region $I_i$. This means that if we let $I_i \subset \tilde{I}_{\gamma(j)} \iff \tilde{c}_{ij} = 1$ and $I_i \not\subset \tilde{I}_{\gamma(j)} \iff \tilde{c}_{ij} = 0$ for every $i, j = 1, \ldots, n$, then for fixed $i = 1, \ldots, n$, $a_i$ is the number of elements of the set $\{\tilde{c}_{ij}|j = 1, \ldots, n\}$ and $\tilde{c}_{ij} > 0$. Now let

$$p_{ij} = \begin{cases} \frac{1}{a_i}, & I_i \subset \tilde{I}_{\gamma(j)} \\ 0, & I_i \not\subset \tilde{I}_{\gamma(j)} \end{cases}$$

Then the matrix $C = (c_{ij})_{i,j=1}^{n}$ (where $c_{ij} = \tilde{c}_{ij}$, $i, j = 1, \ldots, n$) is the connection matrix.

Let $E = I \times H \subset \mathbb{R}^2$. The attractor of the recurrent iterated functions system (RIFS) $\{E; M; W_{i,\gamma(i)}(1), \ldots, W_{n,\gamma(n)}\}$ is denoted by $A$. Then the following theorem shows that $A$ is a recurrent fractal curve.

Theorem 1 The attractor $A$ constructed above is a graph of some continuous function which interpolates the data set $P$.

Proof Let $C(I) = \{\varphi \in C^0(I); \varphi(x_i) = y_i, i = 0, 1, \ldots, n\}$, then the set $C(I)$ is complete metric space with respect to norm $\| \cdot \|_\infty$. We can easily know that the operator $T : C(I) \rightarrow C(I)$ ; $(T\varphi)(x) = F_{i,k}(L_{i,k}^{-1}(x), \varphi(L_{i,k}^{-1}(x))), x \in I_i$ is well defined and the operator $T$ is a contraction on the complete metric space $C(I)$. Therefore the operator $T$ has a unique fixed point in $C(I)$, which we denote by $f$. Then the $f$ is presented by

$$f(x) = s_{i,k}(x)f(L_{i,k}^{-1}(x) + b_{i,k}(x)), i = 1, \ldots, n (k = \gamma(i)),$$

which means that $Gr(f) = A$, where $Gr(f)$ denote a graph of $f$. (QED)

We estimate box-counting dimensions of the recurrent fractal curves constructed above. We can assume that $I = [0, 1]$, since the box-counting dimension is invariant under Lipschitz homeomorphisms. For a set $D(\subset \mathbb{R}^3$ or $\mathbb{R}^2$) and a function $f$ defined on $D$,

$$R_f[D] = \sup \{|f(x_2) - f(x_1)|; x_1, x_2 \in D\}.$$
is called the maximum variation of $f$ on $D$.

Let $I$ be a interval in $\mathbb{R}$, $L : I \rightarrow I$ a contraction homeomorphism, $a, b : \mathbb{R} \rightarrow \mathbb{R}$ Lipschitz mappings and $s : I \rightarrow \mathbb{R}$ contraction mappings with $|s(x)L_0| < 1$. We define a mapping $F : I \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x, y) = s(L(x))a(y) + b(x), (x, y) \in I \times \mathbb{R}.$$ 

**Theorem 2** Let $\lambda = \max_{\gamma} |s(x)|$. Then

$$F(x, y) \leq \bar{a}(x)L_0 + |s(x)L_0| \lambda + b(x), (x, y) \in I \times \mathbb{R}.$$ 

Proof of 1) If $f \in C(I)$, we have $\bar{a}(x)L_0 + |s(x)L_0| \lambda + b(x)$ is continuous functions in $I \times \mathbb{R}$ and $\bar{a}(x)L_0 + |s(x)L_0| \lambda + b(x)$ is irreducible and therefore the region $I_0$ is mapped into arbitrary regions $I$ by applying the appropriately selected transformations from $\{W_{u, \gamma(i)} : u = 1, \ldots, n\}$ several times. So in each region $I_i$ there exist the $(a + 1)$ image points of interpolation points which are not collinear. Thus for every interval $I_i$ there

**Lemma 5** Let $f : I \rightarrow \mathbb{R}$ be continuous function. Then

$$R_{F(L^{-1}, f \circ L^{-1})}[L(I)] \leq sL_0R_f[I] + |I|(\bar{a}_f + L_0),$$

where $|I|$ is a length of the interval $I$, $\bar{a} = \max_{\gamma} |a(x)|$ and $\bar{a}_f = \max |a(f(x))|$. 

Proof For any $x(\in L(I))$, let denote $\bar{x} = L^{-1}(x)(\in I)$. Then

$$|F(L^{-1}, f \circ L^{-1})| = |F(L^{-1}, f \circ L^{-1})| - |F(L^{-1}, f \circ L^{-1})| = |F(L^{-1}, f \circ L^{-1})| - |F(L^{-1}, f \circ L^{-1})|$$

We assume that the row-stochastic matrix $M$ is irreducible and the mapping $\nu(x)$ is identity. Note that in what follows the latter ‘$a$’ is used in another meaning. Let $x_{i+1} - x_i = \frac{1}{n}(i = 0, 1, \ldots, n - 1), x_{c(k)} - x_{a(k)} = \frac{1}{n}(k = 1, \ldots, l; a \in \mathbb{N})$. Let $L_{a,k}(i \in N_n, k = \gamma(i))$ be similitude contractions. Then the number of $I_I$ contained in $I_0$ is $a$. Let $\bar{S}$ and $\bar{S}$ be diagonal matrices

$$\bar{S} = \text{diag}(\bar{s}_1, \ldots, \bar{s}_n), \quad \bar{S} = \text{diag}(\bar{s}_1, \ldots, \bar{s}_n),$$

respectively, where $\bar{s}_i = \max_{\gamma} |s_{i, \gamma}(x)|$, $\bar{s}_i = \min_{\gamma} |s_{i, \gamma}(x)|$.

**Theorem 2** Let $\mathcal{A}$ be the recurrent fractal curve in Theorem 1. If there is some interval $I_{\lambda_0}$ such that the points of $P \cap (I_{\lambda_0} \times \mathbb{R})$ are not collinear, then the box-counting dimension $\dim_B \mathcal{A}$ of $\mathcal{A}$ has the following lower and upper bounds:

1) If $\lambda_0 \leq 1$, then

$$1 + \log_\bar{\alpha} \lambda \leq \dim_B \mathcal{A} \leq 1 + \log_\bar{\lambda} \lambda,$$

2) If $\lambda_0 \leq 1$, then

$$\dim_B \mathcal{A} = 1,$$

where $\bar{\alpha} = \rho(S\mathcal{C}), \bar{\lambda} = \rho(S\mathcal{C})$ are respectively spectral radii of the irreducible matrices $S\mathcal{C}, \mathcal{C}$. 

**Proof** Proof of 1) Let $f$ be a contraction function whose graph is $\mathcal{A}$. We denote $R_f[I_0]$ by $R_k$ and $\frac{1}{\bar{a}}$ by $\varepsilon_r$ for simplicity. Then $r \rightarrow \infty \iff \varepsilon_r \rightarrow 0$.

After applying once each $W_{a,k}$ to the interpolation points in the interval $I_k$, we have $(a + 1)$ new image points of the interpolation points in every interval $I_i (i \in N_n)$. According to the hypothesis, the interpolation points lying inside $I_{\lambda_0}$ are not collinear and the $(a + 1)$ image points in the region $I_{\lambda_0} (k_0 = \gamma(i_0))$ are not collinear. On the other hand the connection matrix $C$ is irreducible and therefore the region $I_{\lambda_0}$ is mapped into arbitrary regions $I$ by applying the appropriately selected transformations from $\{W_{u, \gamma(u)} : u = 1, \ldots, n\}$ several times. So in each region $I_i$ there exist the $(a + 1)$ image points of interpolation points which are not collinear. Thus for every interval $I_i$ there
are at least three points which are not collinear, the maximum vertical distance (computed only with respect to the y-axis \([6]\)) from one of the three points to the line through other two interpolation points is greater than 0. The maximum value is called a height and denoted by \(H_i\).

By Lemma \([5]\) on each region \(I_i\) we have

\[
R_f[I_i] \leq \bar{s}_i R_{k} + \frac{a}{n}e,
\]

where \(e = c_s \bar{f} + L_0\).

We define non negative vectors \(h_1, r, u_1\) and \(i\) by

\[
\begin{align*}
\mathbf{h}_1 &= \begin{bmatrix} H_1 \\ \vdots \\ H_n \end{bmatrix}, & \mathbf{r} &= \begin{bmatrix} \bar{s}_1 R_1 \\ \vdots \\ \bar{s}_n R_n \end{bmatrix}, & \mathbf{i} &= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, & \mathbf{u}_1 &= \mathbf{r} + \frac{a}{n} \mathbf{i}.
\end{align*}
\]

Since \(A\) is the graph of a continuous function defined on \(I\), the smallest number of \(\varepsilon, n\)-mesh squares necessary to cover \(I_i \times R \cap A\) is greater than the smallest number of \(\varepsilon, n\)-mesh squares necessary to cover vertical line with the length \(H_i\) and less than the smallest number of \(\varepsilon, n\)-mesh squares necessary to cover a rectangle \(I_i \times [\bar{f}_i, \bar{f}],\) where

\[
f_{i \downarrow} = \min_{I_i} f(x, y), \quad \bar{f}_i = \max_{I_i} f(x, y).
\]

Therefore

\[
\sum_{i=1}^{n} (H_i \varepsilon^{-1}) - n \leq N(\varepsilon) \leq \sum_{i=1}^{n} \left( \bar{s}_i R_i + \frac{a}{n} \varepsilon^{-1} + 1 \right) \left( \left\lceil \frac{\varepsilon^{-1}}{n} \right\rceil + 1 \right),
\]

i.e.

\[
\Phi(\mathbf{h}_1 \varepsilon^{-1}) - n \leq N(\varepsilon) \leq \Phi(\mathbf{u}_1 \varepsilon^{-1} + \mathbf{i} \left\lceil \frac{\varepsilon^{-1}}{n} \right\rceil + 1),
\]

where \(\frac{1}{n} > \varepsilon,\) and

\[
\Phi(\mathbf{a}) = a_1 + \cdots + a_n, \quad \mathbf{a} = (a_1, \ldots, a_n).
\]

After applying \(W_{i, k}\) twice, we have \(a\) subintervals of length \(\frac{1}{m_n}\) in each \(I_i (i \in N_n)\). Those subintervals are \(L_{i,h}(I_j)\) mapped by the transformation \(W_{i,k}\) from the subintervals (regions) \(I_j\) lying inside the interval (domain) \(I_{i(i)}\) corresponding to the interval \(I_i\) containing theirselves, and thus for each subinterval \(I_i\) the height on the each subinterval \(L_{i,k}(I_j)\) produced in \(I_i\) is not less than \(\bar{s}_i h\), where \(h\) is the height on original subinterval \(I_j\) contained in the interval \(I_{i(i)}\). Therefore, the sum of maximum variance of \(f\) on a subintervals of the length \(\frac{1}{m_n}\) contained in the interval \(I_i (i \in N_n)\) is not greater than \(i\)-th coordinate of vector \(\mathbf{u}_2 = SC\mathbf{u}_1 + \frac{a}{n} \mathbf{i}\) and the sum of the heights is not less than \(i\)-th coordinate of vector \(\mathbf{h}_2 = \mathbb{S}Ch_1\). Therefore

\[
\Phi(\mathbf{h}_2 \varepsilon^{-1}) - an \leq N(\varepsilon) \leq \Phi(\mathbf{u}_2 \varepsilon^{-1} + ai) \left( \left\lceil \frac{\varepsilon^{-1}}{an} \right\rceil + 1 \right),
\]

where \(\frac{1}{m_n} > \varepsilon,\)

By induction we know that after taking \(k\) such that \(a \varepsilon \geq \frac{1}{m^{k-1} n} \geq \varepsilon,\) that is, \(r - \log_a n + 1 > k \geq r - \log_a n\) and applying \(W_{j, \Lambda(i')}\) \(k\) times, we get \(a^{k-1}\) subintervals of the length \(\frac{1}{m^{k-1} n}\) contained in each interval \(I_i\) and

\[
\Phi(\mathbf{h}_k \varepsilon^{-1}) - a^{k-1} n \leq N(\varepsilon) \leq \Phi(\mathbf{u}_k \varepsilon^{-1} + a^{k-1}i) \left( \left\lceil \frac{\varepsilon^{-1}}{a} \right\rceil + 1 \right),
\]

(2)
where \( \mathbf{u}_k = \mathbf{S} \mathbf{u}_{k-1} + \frac{\nu}{n} \mathbf{b}_i \), \( \mathbf{h}_k = \mathbf{S} \mathbf{h}_{k-1} \). Then we have
\[
\mathbf{u}_k = (\mathbf{S} \mathbf{C})^{k-1} \mathbf{r} + (\mathbf{S} \mathbf{C})^{k-2} \frac{\nu}{n} \mathbf{e}_i + \ldots + (\mathbf{S} \mathbf{C})^{k-n} \frac{\nu}{n} \mathbf{e}_i + \frac{\nu}{n} \mathbf{e}_i,
\]
\[
\mathbf{h}_k = (\mathbf{S} \mathbf{C})^{(k-1)} \mathbf{h}_1.
\]
Since \( \mathbf{S} \mathbf{C} \) and \( \mathbf{S} \mathbf{C} \) are non-negative irreducible matrix, from Perron-Frobenius’s theorem (Lemma 3) there exist strictly positive eigenvectors of \( \mathbf{S} \mathbf{C} \) and \( \mathbf{S} \mathbf{C} \) (which correspond to eigenvalues \( \lambda = \rho(\mathbf{S} \mathbf{C}) \) and \( \bar{\lambda} = \rho(\mathbf{S} \mathbf{C}) \)) and we can choose such strictly positive eigenvectors \( \mathbf{e}, \mathbf{e} \) (which correspond to eigenvalues \( \lambda, \bar{\lambda} \) respectively) that
\[
r \leq \mathbf{e}, \quad a\mathbf{e} < \mathbf{n}, \quad 0 < \mathbf{e} < \mathbf{h}_1.
\]
Then
\[
N(\varepsilon_r) \leq \Phi(\mathbf{u}_k \varepsilon_r^{-1} + a^{k-1} \mathbf{1}) \left( \frac{\varepsilon_r^{-1}}{a^{k-1} \mathbf{1}} + 1 \right)
\leq \Phi(\mathbf{n} \varepsilon_r^{-1} + a^{k-1} \mathbf{1})(a + 1)
\leq \Phi((\mathbf{S} \mathbf{C})^{k-1} \mathbf{e}_r^{-1} + (\mathbf{S} \mathbf{C})^{k-2} \frac{\nu}{n} \mathbf{e}_i \varepsilon_r^{-1} + \ldots + (\mathbf{S} \mathbf{C})^{k-n} \frac{\nu}{n} \mathbf{e}_i \varepsilon_r^{-1} + \frac{\nu}{n} \mathbf{e}_i \varepsilon_r^{-1}) (a + 1)
\leq \lambda^k \Phi(\mathbf{e}) \varepsilon_r^{-1} + \lambda^{k-1} \Phi(\mathbf{e}) \varepsilon_r^{-1} + \ldots + \lambda \Phi(\mathbf{e}) \varepsilon_r^{-1} + \Phi(\mathbf{e}) \varepsilon_r^{-1} + a^{k-1} \Phi(\mathbf{i})(a + 1)
\leq \lambda^k \Phi(\mathbf{e}) \varepsilon_r^{-1} + \lambda^{k-1} \Phi(\mathbf{e}) \varepsilon_r^{-1} + \ldots + \lambda \Phi(\mathbf{e}) \varepsilon_r^{-1} + \Phi(\mathbf{e}) \varepsilon_r^{-1} + a^{k-1} \Phi(\mathbf{i})(a + 1),
\]
where \( \nu := \log_a n \).

On the other hands, since \( s_i \leq \bar{s}_i \) for \( i = 1, \ldots, n \), from Perron-Frobenius’s theorem (Lemma 3) we have \( \lambda < \bar{\lambda} < \lambda \). Therefore if \( \lambda > 1 \), then \( 1 > \frac{1}{\lambda} > \frac{1}{\bar{\lambda}} \), we obtain
\[
N(\varepsilon_r) \leq \lambda^r \Phi(\mathbf{e}) \varepsilon_r^{-1} \left( 1 + \frac{1}{\lambda} + \ldots + \frac{1}{\lambda^{r-1}} + \frac{n}{\lambda^r \Phi(\mathbf{e}) a^r} \right) (a + 1)
= \lambda^r \varepsilon_r^{-1} \lambda^r \Phi(\mathbf{e}) \left( 1 + \frac{1 - \frac{1}{\lambda^{r-1}}}{1 - \frac{1}{\lambda}} + \frac{n}{\lambda^r \Phi(\mathbf{e}) a^r} \right) (a + 1).
\]
Here let \( \delta(r) := \lambda - \nu \Phi(\mathbf{e}) \left( 1 + \frac{1 - \frac{1}{\lambda^{r-1}}}{1 - \frac{1}{\lambda}} + \frac{n}{\lambda^r \Phi(\mathbf{e}) a^r} \right) (a + 1) \), then \( \delta(r) > 0 \) and
\[
\dim_B A = \lim_{\varepsilon_r \to 0} \frac{\log N(\varepsilon_r)}{-\log \varepsilon_r} \leq 1 + \log_a \bar{\lambda}.
\]
By (2), we have
\[
N(\varepsilon_r) \geq \Phi(\mathbf{h}_k \varepsilon_r^{-1}) - a^{k-1} n \geq \Phi((\mathbf{S} \mathbf{C})^{k-1} \mathbf{h}_1 \varepsilon_r^{-1}) - a^{k-1} n \geq \lambda^{k-1} \Phi(\mathbf{e}) \varepsilon_r^{-1} - a^{k-1} n
\geq \lambda^r \Phi(\mathbf{e}) \varepsilon_r^{-1} - a^{r-2} n \varepsilon_r^{-1}
\geq \varepsilon_r^{-1} \lambda^r \left( \frac{\lambda - \nu \Phi(\mathbf{e})}{\lambda^r} \right).
\]
Since $\lambda > 1$, there is $r_0$ such that $\eta(r) := \lambda^{-\nu} - \frac{a^{-\nu} n}{\lambda} > 0$ for any $r > r_0$ and therefore we have
\[
\frac{\log N(\varepsilon_r)}{-\log \varepsilon_r} \geq 1 + \log_a \lambda + \frac{1}{r} \log_a \eta(r), \quad r > r_0
\] (4)

By (3), (4) if $\lambda > 1$, then we have
\[
1 + \log_a \lambda \leq \dim_B A \leq 1 + \log_a \bar{\lambda}.
\]

**Proof of 2):** If $\bar{\lambda} \leq 1$, we have
\[
N(\varepsilon_r) \leq \bar{\lambda}^{r - \nu} \Phi(\bar{\varepsilon}) \varepsilon_r^{-1} + \bar{\lambda}^{r - \nu} \Phi(\bar{\varepsilon}) \varepsilon_r^{-1} + \ldots + \bar{\lambda} \Phi(\bar{\varepsilon}) \varepsilon_r^{-1} + \Phi(\bar{\varepsilon}) \varepsilon_r^{-1} + a^{-\nu} n(a + 1)
\]
\[
\leq \varepsilon_r^{-1} [\Phi(\bar{\varepsilon})(r - \nu + 2) + a^{-\nu} n(a + 1)].
\]

Hence, we have
\[
\dim_B A = \lim_{\varepsilon_r \to 0} \frac{\log N(\varepsilon_r)}{-\log \varepsilon_r} \leq 1 + \frac{1}{r} \log_a \Phi(\bar{\varepsilon})(r - \nu + 2) + a^{-\nu} n(a + 1).
\]

On the other hands $A$ is a curve in $\mathbb{R}^2$ and therefore $\dim_B A \geq 1$. Hence $\dim_B A = 1$. (QED)

**Remark 1.** In the case that $s_{i,k}(x) = s_{i,k}(\text{constant})$, if $\lambda = \bar{\lambda} > 1$, then $\dim_B A = 1 + \log_a \lambda$. This is the estimation of box-counting dimension of RFISs in [3].

## 4 Construction of Fractal Surfaces by General Fractal Curves

In this section we provide a *general method* of construction of fractal surfaces combining fractal curves using Lipschitz functions and a formula of the box-counting dimensions.

We consider fractal surfaces only on the unit square $[0, 1] \times [0, 1]$ because our results can easily be generalized to any square $[a, b] \times [c, d]$.

Let denote $I = [0, 1]$ and $E = I \times I$. Let $f, g : I \to \mathbb{R}$ be fractal curves (i.e. $f$ and $g$ are continuous and their graphs are fractal sets). Let $\lambda, \mu : I \to \mathbb{R}$ be continuous Lipschitz functions with Lipschitz constants $L_\lambda$ and $L_\mu$ respectively. For $(x, y) \in E$, we define a continuous function $F : E \to \mathbb{R}$ by
\[
F(x, y) = \lambda(x) f(x) + \mu(y) g(y).
\] (5)

The following theorem shows that the graph of this function $F$ is a fractal set.

**Theorem 3** Let the function $F$ be given by (5). Then the Box-counting dimension $\dim_B \text{Gr}(F)$ of its graph is
\[
\dim_B \text{Gr}(F) = 1 + \max \{ \dim_B \text{Gr}(f), \dim_B \text{Gr}(g) \}.
\]

To prove the theorem, we need some lemmas on box-counting dimension. The following lemma is easily proved from the definition of the Box-counting dimension.
Lemma 6 Let $A$ and $B$ be fractal sets and $\dim_B A \geq \dim_B B$. Then

\[
\lim_{\delta \to 0} \frac{\log [N_\delta (A) + N_\delta (B)]}{-\log \delta} = \dim_B A,
\]

\[
\lim_{\delta \to 0} \frac{\log [N_\delta (A) + N_\delta (B) + \delta^{-1}]}{-\log \delta} = \dim_B A,
\]

\[
\lim_{\delta \to 0} \frac{\log [N_\delta (A) + \delta^{-1}]}{-\log \delta} = \dim_B A.
\]

Lemma 7 Let the functions $f$, $g$ be fractal curves. If we define a function $F' : E \to R$ by

\[
F'(x, y) = f(x) + g(y), \quad (x, y) \in E,
\]

then the Box-counting dimension $\dim_B \text{Gr}(F')$ is as follows:

\[
\dim_B \text{Gr}(F') = 1 + \max \{ \dim_B \text{Gr}(f), \ dim_B \text{Gr}(g) \}.
\]

(Proof) Divide the interval $I$ into $n$ subintervals with the same length, denote the $i$-th subinterval by $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right]$ and denote $E_{ij} = I_i \times I_j$.

In the calculation of the box-counting dimension, we use $1/n$-mesh cubes of $R^3$ for $n \geq 2$. In a fixed $1/n$-mesh of $R^3$, let denote the set of all $1/n$-mesh cubes that intersect a set $A$ by $\Omega_{1/n}(A)$.

Let $N_{1/n}^{ij}(\text{Gr}(F'))$ be the number of elements of $\{ C \in \Omega_{1/n}(\text{Gr}(F')) \mid C \subset E_{ij} \times R \}$. Similarly, in the $1/n$-mesh of $E \subset R^3$ consisting of $E_{ij}$s, let denote the set of all $1/n$-mesh squares that intersect the graph $\text{Gr}(f)$ of $f$ by $\Omega_{1/n}(\text{Gr}(f))$ and let $N_{1/n}^j(\text{Gr}(f))$ be the number of elements of $\{ S \in \Omega_{1/n}(\text{Gr}(f)) \mid S \subset I_i \times I \}$.

Evidently for any $i, j$ we have

\[
R_{F'}[E_{ij}] = R_f[I_i] + R_g[I_j].
\]  

On the other hand, let $v := R_{F'}[E_{ij}] \cdot \left(\frac{1}{n}\right)^{-1} \left[ R_{F'}[E_{ij}] \cdot \left(\frac{1}{n}\right)^{-1} \right]$. Here $[d]$ is the integer part of $d \in R$ and $0 \leq v < 1$. If $v = 0$, then

\[
R_{F'}[E_{ij}] \cdot \left(\frac{1}{n}\right)^{-1} = \left[ R_{F'}[E_{ij}] \cdot \left(\frac{1}{n}\right)^{-1} \right] \leq N_{1/n}^{ij}(\text{Gr}(F')) \leq \left[ R_{F'}[E_{ij}] \cdot \left(\frac{1}{n}\right)^{-1} \right] + 1
\]

and if $0 < v < 1$, then

\[
R_{F'}[E_{ij}] \cdot \left(\frac{1}{n}\right)^{-1} = \left[ R_{F'}[E_{ij}] \cdot \left(\frac{1}{n}\right)^{-1} \right] + v \leq N_{1/n}^{ij}(\text{Gr}(F')) \leq \left[ R_{F'}[E_{ij}] \cdot \left(\frac{1}{n}\right)^{-1} \right] + 2.
\]

Therefore we have

\[
\left[ R_{F'}[E_{ij}] \cdot \left(\frac{1}{n}\right)^{-1} \right] \leq N_{1/n}^{ij}(\text{Gr}(F')) \leq \left[ R_{F'}[E_{ij}] \cdot \left(\frac{1}{n}\right)^{-1} \right] + 2. \tag{7}
\]

Similarly we have

\[
\left[ R_f[I_i] \cdot \left(\frac{1}{n}\right)^{-1} \right] \leq N_{1/n}^i(\text{Gr}(f)) \leq \left[ R_f[I_i] \cdot \left(\frac{1}{n}\right)^{-1} \right] + 2. \tag{8}
\]

\[
\left[ R_g[I_j] \cdot \left(\frac{1}{n}\right)^{-1} \right] \leq N_{1/n}^j(\text{Gr}(g)) \leq \left[ R_g[I_j] \cdot \left(\frac{1}{n}\right)^{-1} \right] + 2. \tag{9}
\]
On the other hand, noting that for any real numbers \( m, n \) we have \( |m| + |n| \leq |m + n| + 1 \), from (6), (7), (8) and (9), we have
\[
\left[ R_f[I_i] \cdot \left( \frac{1}{n} \right)^{-1} \right] + \left[ R_g[I_j] \cdot \left( \frac{1}{n} \right)^{-1} \right] \leq N_{ij}^{(Gr(F'))} \leq \left[ R_f[I_i] \cdot \left( \frac{1}{n} \right)^{-1} \right] + \left[ R_g[I_j] \cdot \left( \frac{1}{n} \right)^{-1} \right] + 3.
\]
Therefore
\[
N_{ij}^{(Gr(f))} + N_{ij}^{(Gr(g))} - 4 \leq N_{ij}^{(Gr(F'))} \leq N_{ij}^{(Gr(f))} + N_{ij}^{(Gr(g))} + 3.
\]
Thus we have
\[
nN^{(Gr(f))} + nN^{(Gr(g))} - 4n^2 \leq N^{(Gr(F'))} \leq nN^{(Gr(f))} + nN^{(Gr(g))} + 3n^2.
\]
From Lemma 6 and the definition of the box-counting dimension we get the result. (QED)

**Lemma 8** Let the functions \( f, \lambda \) be the same as the above ones. If we define the function \( \lambda f : I \to \mathbb{R} \) by \( (\lambda f)(x) = \lambda(x)f(x), x \in I \), then the Box-counting dimension \( \dim_{\mathbb{B}} Gr(\lambda f) \) is the same as that of the function \( f \).

**Proof** For any \( x, x' \in I \),
\[
| (\lambda f) (x) - (\lambda f) (x') | = | \lambda (x) f (x) - \lambda (x') f (x') |
\]
\[
= | \lambda (x) f (x) - \lambda (x) f (x') + \lambda (x) f (x') - \lambda (x') f (x') |
\]
\[
= | \lambda (x) (f (x) - f (x')) + (\lambda (x) - \lambda (x')) f (x') |
\]

Let denote \( M_f = \max_{x \in I} |f(x)| \), \( c_{\lambda_1} = \min_{x \in I} |\lambda (x)| \) and \( c_{\lambda_2} = \max_{x \in I} |\lambda (x)| \). We can suppose \( c_{\lambda_1} > 0 \) (see Remark 2). Then, for \( n \) large enough,
\[
|\lambda (x)||f(x) - f(x')| - L_\lambda \frac{1}{n}|f(x')| \leq |(\lambda f)(x) - (\lambda f)(x')| \leq |\lambda (x)||f(x) - f(x')| + L_\lambda \frac{1}{n}|f(x')|
\]
\[
c_{\lambda_1} R_f[I_i] - L_\lambda \frac{1}{n} M_f \leq R_{\lambda f}[I_i] \leq c_{\lambda_2} R_f[I_i] + L_\lambda \frac{1}{n} M_f.
\]

Thus
\[
c_{\lambda_1} N_{ij}^{(Gr(f))} - m_1 \leq N_{ij}^{(Gr(\lambda f))} \leq c_{\lambda_2} N_{ij}^{(Gr(f))} + m_2,
\]
\[
c_{\lambda_1} N_{ij}^{(Gr(f))} - n \cdot m_1 \leq N_{ij}^{(Gr(\lambda f))} \leq c_{\lambda_2} N_{ij}^{(Gr(f))} + n \cdot m_2,
\]
where \( m_1 = 2c_{\lambda_1} + L_\lambda M_f, m_2 = 2 + L_\lambda M_f \). Taking logarithms, the definition of the Box-counting dimension and Lemma 1 give the result. (QED)

**Remark 2** In the case where \( c_{\lambda_1} = 0 \), we choose \( \lambda' (x) = \lambda(x) + c \) such that \( c_{\lambda_1'} > 0 \) and define a function \( \lambda' f : I \to \mathbb{R} \) by \( (\lambda' f)(x) = (\lambda f)(x) + cf(x) \) for \( x \in I \). Then \( (\lambda f) (x) = (\lambda' f)(x) + (-cf(x)) \). Therefore, from Lemma 5 and Lemma 7 we have \( \dim_{\mathbb{B}} Gr(\lambda f) = \dim_{\mathbb{B}} Gr(f) \).

**Proof of Theorem 3** Lemma 7 and Lemma 8 give the result of the theorem. (QED)

**Corollary 1** Let \( f_i, g_j \) be fractal curves and \( \lambda_i, \mu_j \) be (one-variable) Lipschitz functions for \( i = 1, \ldots, N, j = 1, \ldots, M \). Then the Box-counting dimension \( \dim_{\mathbb{B}} Gr(F) \) of the function \( F : E \to \mathbb{R} \) defined for \( (x, y) \in E \) by
\[
F(x, y) = \sum_{i=1}^{N} \lambda_i (x) f_i (x) + \sum_{j=1}^{M} \mu_j (y) g_j (y)
\]

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is as follows:
\[
\dim_{B} \text{Gr} (F) = 1 + \max_{i,j} \{ \dim_{B} \text{Gr} (f_i), \dim_{B} \text{Gr} (g_j) \}.
\]

The following theorem can be proved by the same method with the Theorem 3

**Theorem 4** Let \( f_i, g_j \) be the same as in Theorem 3 and \( \lambda_i, \mu_j : E \to \mathbb{R} \) bivariate Lipschitz functions for \( i = 1, \ldots, N, \ j = 1, \ldots, M \). Then the function \( F : E \to \mathbb{R} \) defined for \( (x, y) \in E \) by
\[
F(x, y) = \sum_{i=1}^{N} \lambda_i(x, y)f_i(x) + \sum_{j=1}^{M} \mu_j(x, y)g_j(y)
\]
has the Box-counting dimension of
\[
\dim_{B} \text{Gr} (F) = 1 + \max_{i,j} \{ \dim_{B} \text{Gr} (f_i), \dim_{B} \text{Gr} (g_j) \}.
\]

**Remark 3** The fractal surfaces provided in [7, 15, 16, 19] are contained in the family of fractal surfaces defined by [3] and [10].

## 5 Constructions of Fractal Surfaces using Recurrent Fractal Interpolation Curves

As a simple application of the results of the preceding 2 sections, we can easily construct fractal surfaces if we use at least 2 recurrent fractal interpolation curves.

This methods can have wide applications. For one example, if data sets are given along the (x- and y-) boundaries (or several parallel lines to the boundaries) of a square, we can construct the fractal surfaces interpolating the data sets on the boundaries (or the parallel lines to the boundaries) by control of \( \lambda_i(x, y) \) and \( \mu_j(x, y) \) in (10).

The results of the calculations of the Box-counting dimension in Section 4 show that the complexity of the fractal surfaces defined above are dominated by the fractal curves generating them. Thus, the more flexible a construction of fractal curves is, the more natural the fractal surface constructed by them is.

We can control the complexity and shape of the fractal surfaces constructed in Section 4 in our own way by the vertical contractive function \( s(x) \), the stochastic matrix \( P \) and Lipschitz functions \( g(x) \), \( h(x) \) used in construction of recurrent fractal curves in section 3 and the control lipschitz functions \( \lambda(x) \) and \( \mu(x) \) used in construction of fractal surfaces by combining fractal curves in section 4. So, the fractal surfaces presented in this paper could be more appropriate to model natural objects.

**Example 2.** Construction of RFC and FS by RFC.

The figure 1 and figure 2 show the recurrent fractal curves generated from data set \( P = \{(0, 20), (0.25, 30), (0.5, 10), (0.75, 50), (1.0, 10)\} \) using the method of the example 1 on page 4 with different vertical scaling factor functions, where we use linear transformations \( L_i \) and interpolation polynomials \( g \) and \( h \).

The figure 3 and figure 4 show the fractal surfaces constructed using recurrent fractal curves and different Lipschitz functions. The recurrent fractal curves \( f(x) \) and \( g(y) \) are constructed by the method of the section 3. Here the vertical scaling factor of \( f(x) \) is \( s(x) = \cos(x) \) and the vertical scaling factor of \( g(y) \) is \( d(y) = |(\cos(y) + \sin(y))| \cdot 0.5 \). In the figure 3 we used the Lipschitz functions \( \lambda(x) = 0.5, \mu(y) = 0.8 \) in a and \( \lambda(x) = \cos(14\pi x), \mu(y) = \sin(12\pi x) \) in b. In the figure 5 we used the Lipschitz functions \( \lambda(x, y) = (x - 0.5)^2 + (y - 0.5)^2, \mu(x, y) = -(x - 0.5)^2 - (y - 0.5)^2 \) in c) and \( \lambda(x, y) = (1 - x)y, \mu(x, y) = x(1 - y) \) in d).
Figure 1: Recurrent Fractal Interpolation Curves, a) vertical scaling factor $s(x) = 0.9$, b) $s(x)$: Lagrangean polynomial.
Figure 2: Recurrent Fractal Interpolation Curves, c) vertical scaling factor $s(x) = \cos(x)$, d) $s(x) = \cos(8\pi x)$. 
Figure 3: Fractal Surfaces produced by RFIC $f(x)$ with vertical factor $s(x) = \cos(x)$ and RFIC $g(y)$ with $s(y) = [(\cos(y) + \sin(y)) \cdot 0.5$. 

a) Lipschitz functions $\lambda(x) = 0.5, \mu(y) = 0.8$;  

b) $\lambda(x) = \cos(14\pi x), \mu(y) = \sin(12\pi x)$. 
Figure 4: Fractal Surfaces produced by RFIC $f(x)$ with vertical factor $s(x) = \cos(x)$ and RFIC $g(y)$ with $s(y) = [(\cos(y) + \sin(y)) \cdot 0.5$. c) Lipschitz functions $\lambda(x, y) = (x - 0.5)^2 + (y - 0.5)^2$, $\mu(x, y) = -(x - 0.5)^2 - (y - 0.5)^2$; d) $\lambda(x, y) = (1 - x)y$, $\mu(x, y) = x(1 - y)$
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