A CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH SINE HYPERBOLIC FUNCTIONS

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ABSTRACT. We introduce a class of analytic functions subordinate to the function $1 + \sinh(z)$ and obtain various necessary and sufficient conditions for functions to be in the class. These conditions mainly comprise of the coefficient inequalities involving convolution. Further, we have obtained sharp five initial coefficients, a conjecture for the general $n$th coefficient and the third Hankel determinant bounds for the functions in this class. Also derived certain differential subordination implication results involving $1 + \sinh(z)$.

1. Introduction and Definitions

Let $A$ be the class of all analytic functions $f(z)$ defined in the open unit disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ with the power series representation as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, $S$ denote the class of functions $f \in A$ that are univalent in $\mathbb{D}$. The function $f_1(z)$ is subordinate to $f_2(z)$, symbolically written as $f_1(z) \prec f_2(z)$, if there exists a Schwarz function $\omega(z)$, $|\omega(z)| \leq |z|$, such that $f_1(z) = f_2(\omega(z))$, $(z \in \mathbb{D})$. Furthermore, if the function $f_2$ belongs to class $S$, then we have following equivalence condition $f_1(z) \prec f_2(z)$, $(z \in \mathbb{D})$ if and only if $f(\mathbb{D}) \subseteq g(\mathbb{D})$ and $f(0) = g(0)$. For function $f$ of the form $1.1$ and $g$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product or convolution of $f$ and $g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Recall that $f(z) = f(z) * \frac{z}{1-z}$ and $z f'(z) = f(z) * \frac{z}{(1-z)^2}$. Let $P$ be the class of analytic functions $k(z)$ with positive real part in $\mathbb{D}$ with the normalization

$$k(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{1.2}$$

Date: 2010 Mathematics Subject Classification. 30C45, 30D30.

Key words and phrases. Analytic functions, Sine hyperbolic function, Subordination, Convolution, Third Hankel determinant.
In 1992, Ma and Minda [13] introduced and studied the following subclass of starlike functions in $A$:

$$S^*(h) = \left\{ f \in A : \frac{zf'(z)}{f(z)} < h(z) < \frac{1 + z}{1 - z}, \ z \in \mathbb{D} \right\},$$

where $h$ has positive real part, $h(\mathbb{D})$ symmetric about the real axis with $h'(0) > 0$ and $h(0) = 1$. Now by changing the function on the right hand side of (1.3), we obtain several subclasses of the class $S$, which were introduced and investigated earlier, for example if we set $h(z) = (1 + Az)/(1 + Bz)$, where $-1 \leq B < A \leq 1$, we obtain Janowski class $S^*[A, B]$, see [3]. If $h(z) = 1 + \sin(z)$, we obtain the class $S^*_{\text{sin}}$ introduced by Cho et al. [5] and also see [2]. By setting $h(z) = \sqrt{1 + z}$ we get the class $S^*_L$, which was introduced and studied by Sokół and Stankiewicz [25] and further studied by authors in [26]. By varying $h(z)$, following classes are obtained:

1. If $h(z) = \cosh(z)$, Alotaibi et al. [1] introduced and discussed class $S^*_{\cosh} = S^*(\cosh(z))$.
2. If $h(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, the class $S^*_{\text{Car}} = S^* \left( 1 + \frac{4}{3}z + \frac{2}{3}z^2 \right)$ associated with cardioid introduced by Sharma et al. [22].
3. If $h(z) = e^z$, the class $S^*_e = S^*(e^z)$ was introduced and studied by Mendi-ratta et al. [17] and further investigated by Shi et al. [24].
4. If $h(z) = z + \sqrt{1 + z^2}$, Raina and Sokol et al. [20] introduced and discussed the class $S^*_{\text{R}} = S^* \left( z + \sqrt{1 + z^2} \right)$.
5. If $h(z) = \frac{2}{1 + z^2}$, recently the class was introduced and discussed by Goel and Kumar [7].
6. If $h(z) = 1 + z - \frac{1}{3}z^3$, more recently Wani and Swaminathan introduced the class $S^*_3$.

Also, several subclasses of starlike functions were recently introduced in [3, 4, 6, 10, 13] by choosing a particular function $h(z)$ such as functions associated with Bell numbers, functions associated with shell-like curve connected with Fibonacci numbers or functions connected with the conic domains.

Kumar and Gangania [12] consider the analytic univalent function $\psi$ in $\mathbb{D}$ such that $\psi(0) = 0$, $\psi(\mathbb{D})$ is starlike with respect to 0 and introduced the following class of analytic functions:

$$\mathcal{F}(\psi) := \left\{ f \in A : \frac{zf'(z)}{f(z)} - 1 \prec \psi(z), \ \psi(0) = 0 \right\}.$$  

Note that when $1 + \psi(z) \not\subseteq (1 + z)/(1 - z)$, then the functions in the class $\mathcal{F}(\psi)$ may not be univalent in $\mathbb{D}$ which also implies $\mathcal{F}(\psi) \not\subseteq S^*$ in general. Thus in case, when the function $1 + \psi := h$ has positive real part, $h(\mathbb{D})$ symmetric about the real axis with $h'(0) > 0$, then $\mathcal{F}(\psi)$ reduces to the class $S^*(h)$. With the condition that maximum and minimum of the real part of $\psi(z)$ is given by $\psi(\pm r)$, where $r = |z|$, they established growth theorem and obtained the sharp upper bound for distortion theorem for the class $\mathcal{F}(\psi)$. Hence improved the results which was known for $0 \leq \alpha < 3 - 2\sqrt{2}$ and $0 \leq \beta \leq 1/2$ for the following classes:

$$\mathcal{BS}(\alpha) := \left\{ f \in A : \frac{zf'(z)}{f(z)} - 1 \prec \frac{z}{1 - \alpha z^2}, \ \alpha \in [0, 1) \right\},$$

where $z/(1 - \alpha z^2) =: \psi(z)$ is an analytic univalent function (known as Booth Lenniscate function) and symmetric with respect to the real and imaginary axes.
and
\[ S_{cs}(\beta) := \left\{ f \in \mathcal{A} : \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{z}{(1-z)(1+\beta z)} \beta \in [0,1) \right\}, \]
where \( \frac{z}{(1-z)(1+\beta z)} := \psi(z) \) is univalent, analytic, symmetric about the real-axis and maps the unit disk \( \mathbb{D} \) onto the domain bounded by \textit{Cissoid of Diocles}:
\[ CS(\beta) := \left\{ w = u + iv \in \mathbb{C} : \left( u - \frac{1}{2(\beta - 1)} \right) (u^2 + v^2) + \frac{2\beta}{(1+\beta)^2(\beta - 1)} v^2 = 0 \right\}. \]

Motivated from the above, we introduce the subclass \( \mathcal{G}_{sh} \) of \( \mathcal{F}(\psi) \) connected with a sine hyperbolic function as:
\[ \mathcal{G}_{sh} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} - 1 \prec \sinh(z) \right\}. \]

Let \( \phi(z) := 1+\sinh(z) \). Note that \( \phi(z) \) is not a Carathéodory function as \( \Re(\phi(z)) \neq 0 \ \forall \ z \in \mathbb{D} \). A function \( f \in \mathcal{G}_{sh} \) if and only if there exists an analytic function \( q \) satisfying \( q(z) \prec \phi(z) \) such that
\[ f(z) = z \exp \left( \int_0^z \frac{q(t) - 1}{t} dt \right). \]

Thus choosing \( q(z) = \phi(z) \), we have the following function
\[ f_0(z) = z \exp \left( \int_0^z \frac{\sinh(t)}{t} dt \right). \]

As a consequence of [12] Theorem 2.1, Corollary 2.1, 2.2, pg 3], we have the sharp results for the class \( \mathcal{G}_{sh} \):

**Theorem 1.** Let \( f \in \mathcal{G}_{sh} \) and \( f_0 \) be given as in [15]. Then for \( |z| = r \), we have

1. (growth theorem) \(-f_0(-r) \leq |f(z)| \leq f_0(r)\).
2. (covering theorem) either \( f \) is a rotation of \( f_0 \) or
\[ \{w \in \mathbb{C} : |w| \leq -f_0(-1)\} \subset f(\mathbb{D}), \]
where \( -f_0(-1) = \lim_{r \to 1} (-f_0(-r)) \).
3. \( \Re \frac{f(z)}{z} \leq \frac{f_0(r)}{r} \) and \( |f'(z)| \leq \frac{(1+\sinh(r)f_0(r)}{r} \).

In this paper, we consider some important properties like convolution problems, necessary and sufficient conditions, coefficient problems, convex combination, upper bounds for coefficients, Fekete-szegő problems and third Hankel determinant for the class \( \mathcal{G}_{sh} \).

Let \( f \in \mathcal{A} \), then \( q \)th Hankel determinant of \( f \) is defined for \( q \geq 1 \), and \( n \geq 1 \) by
\[ H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \]

Thus second and third Hankel determinants are respectively:
\[ H_{2,2}(f) = a_2a_4 - a_3^2, \]
\[ H_{3,1}(f) = a_3 \left( a_2a_4 - a_3^2 \right) - a_4 (a_4 - a_2a_3) + a_5 \left( a_3 - a_2^2 \right). \]
2. Preliminary

The following lemmas are important for proving our results.

Lemma 1. [18] If \( k \in \mathcal{P} \) and it is of the form (1.2), then for \( \lambda \in \mathbb{C} \)

\[
|c_n| \leq 2 \text{ for } n \geq 1,
\]

and

\[
|c_2 - \lambda c_1^2| \leq 2 \max \{1; |2\lambda - 1|\}.
\]

Lemma 2. [18] If \( k \in \mathcal{P} \) and is represented by (1.2), then

\[
|c_2 - \nu c_1^2| \leq \begin{cases} 
-4\nu + 2 & (\nu \leq 0), \\
2 & (0 \leq \nu \leq 1), \\
4\nu - 2 & (\nu \geq 1).
\end{cases}
\]

Lemma 3. [14, 15] If \( k \in \mathcal{P} \) be expressed in series expansion (1.2), then

\[
2c_2 = c_1^2 + x (4 - c_1^2)
\]

for some \( x \), \( |x| \leq 1 \) and

\[
4c_3 = c_1^4 + 2 (4 - c_1^2) c_1 x - (4 - c_1^2) c_1 x^2 + 2 (4 - c_1^2) \left( 1 - |x|^2 \right) z
\]

for some \( z \), \( |z| \leq 1 \).

Lemma 4. If \( k \in \mathcal{P} \) be expressed in series expansion (1.2), then

\[
|ac_3 - bc_1c_2 + dc_3| \leq 2 |a| + 2 |b - 2a| + 2 |a - b + d|
\]

Lemma 5. [21] Let \( m, n, l \) and \( r \) satisfy the inequalities \( 0 < m < 1 \), \( 0 < r < 1 \) and

\[
8r (1 - r) \left( (mn - 2l)^2 + (m (r + m) - n)^2 \right) + m (1 - m) (n - 2rm)^2
\]

\[
\leq 4m^2 (1 - m)^2 r (1 - r).
\]

If \( k \in \mathcal{P} \) and has power series (1.2) then

\[
\left| lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3}{2} \nu c_1^2 c_2 - c_4 \right| \leq 2.
\]

Lemma 6. [8] Let \( w(z) \) be analytic in \( \mathbb{D} \) with \( w(0) = 0 \). If \( |w(z)| \) attains its maximum value on the circle \( |z| = r \) at a point \( z_0 = re^{i\theta} \), for \( \theta \in [-\pi, \pi] \), we can write that

\[
z_0 w'(z_0) = mw(z_0),
\]

where \( m \) is real and \( m \geq 1 \).

3. Main Results

We begin with the following result:

Theorem 2. Let \( f \in \mathcal{A} \) be of the form (1.1). Then \( f \in \mathcal{G}_{sh} \), if and only if

\[
\frac{1}{z} \left( f(z) \ast \frac{z - \beta z^2}{(1 - z)^2} \right) \neq 0,
\]

where \( \beta = \beta_0 = \frac{1 + \sinh(e^{i\theta})}{\sinh(e^{i\theta})} \).
Proof. Let \( f \in G_{sh} \), if and only if
\[
\frac{zf'(z)}{f(z)} < 1 + \sinh(z).
\]
if and only if there exist a Schwartz function \( s(z) \) such that
\[
\frac{zf'(z)}{f(z)} = 1 + \sinh(s(z)) \quad (z \in D)
\]
\[
\Leftrightarrow \frac{zf'(z)}{f(z)} \neq 1 + \sinh(e^{i\theta}) \quad (z \in D; \theta \in [0, 2\pi])
\]
\[
\Leftrightarrow \frac{1}{z} (zf'(z) - f(z) (1 + \sinh(e^{i\theta}))) \neq 0
\]
\[
\Leftrightarrow \frac{1}{z} \left( f(z) \ast \frac{z - \beta z^2}{(1 - z)^2} \right) \neq 0,
\]
where \( \beta \) is as given above and that completes the proof. \( \square \)

Note that the forward part of Theorem 2 also holds for \( \beta = 1 \). As if \( f \in G_{sh} \), then \( f \) is analytic in \( D \) and thus \( f(z)/z \neq 0 \).

**Theorem 3.** Let \( f \in A \) be of the form (1.1). Then necessary and sufficient condition for function \( f(z) \) belong to class \( G_{sh} \) is that
\[
1 - \sum_{n=2}^{\infty} n - \left( 1 + \sinh(e^{i\theta}) \right) \frac{a_n z^{n-1}}{\sinh(e^{i\theta})} \neq 0.
\]

**Proof.** In the light of above Theorem 2 we show that \( G_{sh} \) if and only if
\[
0 \neq \frac{1}{z} \left[ f(z) \ast \frac{z - \beta z^2}{(1 - z)^2} \right]
\]
\[
= \frac{1}{z} (zf'(z) - \beta (zf'(z) - f(z)))
\]
\[
= 1 - \sum_{n=2}^{\infty} \left( (\beta - 1) n - \beta \right) a_n z^{n-1}
\]
\[
= 1 - \sum_{n=2}^{\infty} \frac{n - \left( 1 + \sinh(e^{i\theta}) \right) a_n}{\sinh(e^{i\theta})} z^{n-1}.
\]
Hence the proof completes. \( \square \)

**Theorem 4.** Let \( f \in A \) and satisfies
\[
\sum_{n=2}^{\infty} \left| n - \left( 1 + \sinh(e^{i\theta}) \right) \right| \frac{|a_n|}{\sinh(e^{i\theta})} < 1,
\]
then \( f \in G_{sh} \).
Proof. To show $f \in G_{sh}$, we need to show (3.2). Consider

$$1 - \sum_{n=2}^{\infty} ((\beta - 1) n - \beta) a_n z^{n-1} > 1 - \sum_{n=2}^{\infty} |((\beta - 1) n - \beta) a_n z^{n-1}|$$

$$= 1 - \sum_{n=2}^{\infty} |((\beta - 1) n - \beta)| |a_n| |z|^{n-1}$$

$$> 1 - \sum_{n=2}^{\infty} |((\beta - 1) n - \beta)| |a_n|$$

$$= 1 - \sum_{n=2}^{\infty} \frac{|n - (1 + \sinh (e^{i\theta}))|}{\sinh (e^{i\theta})} |a_n| > 0,$$

so by Theorem 3, $f \in G_{sh}$. □

**Theorem 5.** The class $G_{sh}$ is convex.

Proof. Let

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} z^n, \text{ for } i = \{1, 2\}.$$

We have to show that $\mu f_1(z) + (1 - \mu) f_2(z) \in G_{sh}$. As

$$\mu f_1(z) + (1 - \mu) f_2(z)$$

$$= z + \sum_{n=2}^{\infty} (\mu a_{n,1} + (1 - \mu) a_{n,2}) z^n$$

Consider

$$\sum_{n=2}^{\infty} \frac{|n - (1 + \sinh (e^{i\theta}))|}{\sinh (e^{i\theta})} |a_{n,1} + (1 - \mu) a_{n,2}|$$

$$\leq \mu \sum_{n=2}^{\infty} \frac{|n - (1 + \sinh (e^{i\theta}))|}{\sinh (e^{i\theta})} |a_{n,1}| + (1 - \mu) \sum_{n=2}^{\infty} \frac{|n - (1 + \sinh (e^{i\theta}))|}{\sinh (e^{i\theta})} |a_{n,2}|$$

$$< \mu + (1 - \mu) = 1.$$

Thus by virtue of Theorem 4, $\mu f_1(z) + (1 - \mu) f_2(z) \in G_{sh}$. □

**Theorem 6.** Let $f \in G_{sh}$ be of the form (1.1). Then

$$|a_2| \leq 1,$$

$$|a_3| \leq \frac{2}{3},$$

$$|a_4| \leq \frac{1}{3},$$

$$|a_5| \leq \frac{1}{4}.$$

These inequalities are sharp respectively for

$$f(z) = z \exp \int_0^z \frac{\sinh (t^{n-1})}{t} dt \text{ for } n = 2, 3, 4, 5.$$
Proof. Since \( f \in \mathcal{G}_{sh} \), then there exists an analytic function \( s(z) \), \(|s(z)| < 1\) and \( s(0) = 0\), such that

\[
(zf'(z)) = 1 + \sinh (s(z)).
\]

Denote

\[
\Psi (s(z)) = 1 + \sinh (s(z))
\]

and

\[
k(z) = 1 + c_1 z + c_2 z^2 + \cdots \equiv 1 + \frac{s(z)}{1-s(z)}.
\]

Obviously, the function \( k \in \mathcal{P} \) and \( s(z) = \frac{k(z)-1}{k(z)+1} \). This gives

\[
1 + \sinh \left( \frac{k(z)-1}{k(z)+1} \right) = 1 + \frac{1}{2} c_1 z + \left( \frac{1}{2} c_2 - \frac{1}{4} c_1^2 \right) z^2 + \left( \frac{7}{48} c_1^3 - \frac{1}{2} c_2 c_1 + \frac{1}{2} c_3 \right) z^3
\]

\[
+ \left( -\frac{3}{32} c_1^4 + \frac{7}{16} c_1^2 c_2 - \frac{1}{2} c_3 c_1 - \frac{1}{4} c_2^2 + \frac{1}{2} c_4 \right) z^4 + \cdots .\]  

And other side,

\[
zf'(z) = 1 + a_2 z + \left( 2a_3 - a_2^2 \right) z^2 + \left( 3a_4 - 3a_2 a_3 + a_2^3 \right) z^3
\]

\[
+ \left( 4a_5 - 2a_2^2 - 4a_2 a_4 + 4a_2^2 a_3 - a_2^4 \right) z^4 + \cdots .\]

On equating coefficients of (3.6) and (3.7), we get

\[
a_2 = \frac{1}{2} c_1,
\]

\[
a_3 = \frac{1}{4} c_2,
\]

\[
a_4 = \frac{1}{144} c_1^3 - \frac{1}{24} c_1^2 c_2 + \frac{1}{6} c_3,
\]

\[
a_5 = -\frac{5}{1152} c_1^4 + \frac{5}{192} c_1^2 c_2 - \frac{1}{24} c_1 c_3 - \frac{1}{32} c_2^2 + \frac{1}{8} c_4.
\]

Using (2.1) with equations (3.8) and (3.9), we get

\[
|a_2| \leq 1 \quad \text{and} \quad |a_3| \leq \frac{1}{2}.
\]

Application of Lemma \( \square \) to equation (3.10), we get

\[
|a_4| \leq \frac{1}{3}.
\]

Application of Lemma \( \square \) to equation (3.11), we get

\[
|a_5| \leq \frac{1}{4}.
\]

\[
\square
\]

Conjecture 1. Let \( f \in \mathcal{G}_{sh} \) be of the form (1.1). Then

\[
|a_n| \leq \frac{1}{n-1} \quad \text{for} \quad n \geq 2.
\]

As a consequence of above theorem, we have following remark:
Remark 1. Let \( f \in G_{sh} \) be of the form \((1.1)\). Then for \( \lambda \in \mathbb{C} \)
\[
|a_3 - \lambda a_2| \leq \frac{1}{2} \max \{1, |2\lambda - 1|\}
\]
and
\[
|a_4 - a_2 a_3| \leq \frac{1}{3}.
\]
For \( \lambda = 1 \), the above remark gave:

Corollary 1. Let \( f \in G_{sh} \) be of the form \((1.1)\). Then
\[
|a_3 - a_2| \leq \frac{1}{2}
\]
The result is sharp.

Theorem 7. Let \( f \in G_{sh} \) be of the form \((1.1)\). Then
\[
|a_2 a_4 - a_3| \leq \frac{1}{36}.
\]
The result is sharp.

Proof. Since from (3.8), (3.9) and (3.10), we have
\[
|a_2 a_4 - a_3| = \left| \frac{1}{288} c_1^4 - \frac{1}{48} c_2 c_1^2 + \frac{1}{12} c_3 c_1 - \frac{1}{16} c_2^2 \right|.
\]
Using Lemma 3 also put \( c_1 = c \in [0, 2] \), without loss of generality assume \( |x| = y \in [0, 1] \) and eliminating \( z \) using triangular inequality, we get
\[
|a_2 a_4 - a_3| \leq \psi(c, y),
\]
where
\[
\psi(c, y) := \frac{1}{288} \left( \frac{1}{2} c^4 + 6c^2 y (4 - c^2) + 6c^2 y^2 (4 - c^2) + 12c (4 - c^2) (1 - y^2) + \frac{9}{2} y^2 (4 - c^2)^2 \right).
\]
Now differentiating \( \psi(c, y) \) with respect to \( y \), we have
\[
\frac{\partial \psi(c, y)}{\partial y} = \frac{1}{288} \left( 6c^2 (4 - c^2) + 12c y (4 - c^2) - 24c (4 - c^2) y + 9y (4 - c^2)^2 \right),
\]
since \( \frac{\partial \psi(c, y)}{\partial y} > 0 \), thus \( \psi(c, y) \) is an increasing function and maximum occur at \( y = 1 \), so
\[
\psi(c, 1) = \chi(c) = \frac{1}{288} \left( \frac{1}{2} c^4 + 12c^2 (4 - c^2) + \frac{9}{2} (4 - c^2)^2 \right),
\]
differentiating \( \chi(c) \) with respect to \( c \), we have
\[
\chi'(c) = \frac{1}{288} \left( 2c^3 + 24c (4 - c^2) - 24c^3 - 18c (4 - c^2) \right)
\]
\[
\chi''(c) = \frac{1}{288} \left( 24 - 84c^2 \right),
\]
then \( \chi''(c) < 0 \) for \( c = 2 \), maxima exists at \( c = 2 \), we have
\[
|a_2 a_4 - a_3| \leq \frac{1}{36}.
\]
This bound is sharp for function defined as follow:

\[ f(z) = z \exp \left( \int_0^z \frac{\sinh(t)}{t} \, dt \right) = z + z^2 + \frac{1}{2} z^3 + \frac{2}{9} z^4 + \cdots , \]

which concludes the proof. \(\square\)

**Theorem 8.** Let \( f \in \mathcal{G}_{sh} \) be of the form \((1.1)\). Then

\[ |H_{3,1}(f)| \leq \frac{1}{4} \approx 0.25. \]

**Proof.** Since we know that

\[ H_{3,1}(f) = a_3 (a_4 a_2 - a_3^3) - a_4 (a_2 a_3 - a_4) + a_5 (a_3 - a_2^2), \]

using Theorem \(6\) and \(7\) and corollary \(1\) along with triangular inequality, we get the desired result. \(\square\)

4. Differential Subordination

**Theorem 9.** For \(-1 \leq B < A \leq 1\) and \( f \in \mathcal{A} \), if the conditions

\[(4.1) \quad |\alpha| \geq \frac{A - B}{1 + \cos 1 - \sin 1 - |B| (1 + \sinh 1 + \cosh 1)} \]

and

\[ 1 + \alpha z f'(z) < \frac{1 + Az}{1 + Bz}, \]

holds. Then

\[ \frac{f(z)}{z} < 1 + \sinh z. \]

**Proof.** Let us define a function

\[(4.2) \quad p(z) = 1 + \alpha z f'(z), \]

also we consider

\[(4.3) \quad \frac{f(z)}{z} = 1 + \sinh w(z). \]

Now to prove our result, we only need to prove that \( w(z) \) is a Schwarz function, that is \(|w(z)| < 1 \) for \(|z| < 1\). Upon logarithmic differentiation of \((4.3)\) and using \((4.2)\) we obtain

\[ p(z) = 1 + \alpha z (1 + \sinh w(z) + zw'(z) \cosh w(z)) \]

and so

\[ \left| \frac{p(z) - 1}{A - B p(z)} \right| = \left| \frac{\alpha (1 + \sinh w(z) + zw'(z) \cosh w(z))}{A - B (1 + \alpha (1 + \sinh w(z) + zw'(z) \cosh w(z)))} \right| \]

\[ = \left| \frac{\alpha (1 + \sinh w(z) + zw'(z) \cosh w(z))}{(A - B) - \alpha B (1 + \sinh w(z) + zw'(z) \cosh w(z))} \right|. \]

Now if \( w(z) \) attains its maximum value at some \( z = z_0 \) and \(|w(z_0)| = 1\). Then by Lemma \(3\) for \( m \geq 1 \), we have, \( z_0 w'(z_0) = mw(z_0) \) and \( w(z_0) = e^{i\theta} \), for
\[
\theta \in [-\pi, \pi], \text{ we get } \\
\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| = \left| \frac{\alpha (1 + \sinh e^{i\theta} + mw(z_0) \cosh e^{i\theta})}{(A - B) - \alpha B (1 + \sinh e^{i\theta} + mw(z_0) \cosh e^{i\theta})} \right| \\
\geq \frac{|\alpha| (1 + m|\cosh e^{i\theta}| - |\sinh e^{i\theta}|)}{(A - B) + |\alpha||B|(1 + |\sinh e^{i\theta}| + m|\cosh e^{i\theta}|)}. \\
\]

Since
\[
|\cosh e^{i\theta}|^2 = \cosh^2(\cos \theta) \cos^2(\sin \theta) + \sinh^2(\cos \theta) \sin^2(\sin \theta) = \Psi(\theta) \\
|\sinh e^{i\theta}|^2 = \sinh^2(\cos \theta) \cos^2(\sin \theta) + \cosh^2(\cos \theta) \sin^2(\sin \theta) = \Theta(\theta),
\]
once we see that, if we let \(\Psi'(\theta) = 0\) and \(\Theta'(\theta) = 0\) has the roots \(\theta = 0, \pm \pi, \pm \frac{\pi}{2}\) in \([-\pi, \pi]\), also \(\Psi(\theta)\) and \(\Theta(\theta)\) are even functions in this interval so
\[
\max \{\Psi(\theta)\} = \Psi(0) = \Psi(\pi) = \cosh^2(1), \\
\min \{\Psi(\theta)\} = \Psi\left(\frac{\pi}{2}\right) = \cos^2(1), \\
\max \{\Theta(\theta)\} = \Theta(0) = \Theta(\pi) = \sinh^2(1), \\
\min \{\Theta(\theta)\} = \Theta\left(\frac{\pi}{2}\right) = \sin^2(1).
\]

From these, we obtain
\[
(4.4) \quad \cos(1) \leq |\cosh e^{i\theta}| \leq \cosh(1), \\
(4.5) \quad \sin(1) \leq |\sinh e^{i\theta}| \leq \sinh(1).
\]

Now we use (4.4) and (4.5) to obtain
\[
\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \leq \frac{|\alpha|(1 + m \cos(1) - \sin(1))}{(A - B) + |\alpha||B|(1 + \sinh(1) + m \cosh(1))}.
\]

Let
\[
\phi(m) = \frac{|\alpha|(1 + m \cos(1) - \sin(1))}{(A - B) + |\alpha||B|(1 + \sinh(1) + m \cosh(1))},
\]
which implies
\[
\phi'(m) = \frac{|\alpha| \cos(1)(A - B) + |\alpha|^2 B \cos(1)(1 + \sinh(1)) - \cosh(1)(1 - \sin(1))}{((A - B) + |\alpha||B|(1 + \sinh(1) + m \cosh(1)))^2} > 0,
\]
which shows that \(\phi(m)\) is an increasing function and hence it will have its minimum value at \(m = 1\) and so
\[
\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \leq \frac{|\alpha|(1 + \cos(1) - \sin(1))}{(A - B) + |\alpha||B|(1 + \sinh(1) + \cosh(1))}
\]
now by (4.1), we have
\[
\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \geq 1,
\]
which contradicts the fact that \(p(z) \propto (1 + Az)/(1 + Bz)\) and hence we get the desired result. \(\square\)
Corollary 2. For $-1 \leq B < A \leq 1$, and $g \in \mathcal{A}$, then if the following conditions

$$|\alpha| \geq \frac{A - B}{1 + \cos 1 - \sin 1 - |B|(1 + \sinh 1 + \cosh 1)}$$

and

$$1 + \frac{\alpha^2 g''(z)}{g(z)} \left( 2 + \frac{zg''(z)}{g'(z)} - \frac{zg'(z)}{g(z)} \right) \leq \frac{1 + Az}{1 + Bz},$$

(4.6)

holds, then $g \in \mathcal{G}_{sh}$.

Proof. Take

$$l(z) = \frac{z^2 g'(z)}{g(z)},$$

then we have

$$zl''(z) = \frac{z^2 g'(z)}{g(z)} \left( 2 + \frac{zg''(z)}{g'(z)} - \frac{zg'(z)}{g(z)} \right)$$

and so by (4.6), we get

$$1 + \alpha zl''(z) \leq \frac{1 + Az}{1 + Bz}$$

and hence by Theorem 9, we get

$$\frac{l(z)}{z} = \frac{zg'(z)}{g(z)} < 1 + \sinh z,$$

thus $g \in \mathcal{G}_{sh}$. \hfill \Box

Theorem 10. For $-1 \leq B < A \leq 1$ and $f \in \mathcal{A}$, then if the conditions

$$|\alpha| \geq \frac{(A - B)(1 + \sinh 1)}{1 + \cos 1 - \sin 1 - B(1 + \cosh 1 + \sinh 1)}$$

and

$$1 + \alpha \frac{zf''(z)}{f(z)} \leq \frac{1 + Az}{1 + Bz},$$

(4.8)

holds, then

$$\frac{f(z)}{z} \times 1 + \sinh z.$$

Proof. Let

$$p(z) = 1 + \alpha \frac{zf'(z)}{f(z)}.$$

and

$$\frac{f(z)}{z} = 1 + \sinh w(z),$$

then we have to show that $|w(z)| < 1$ for $|z| < 1$. Now using simple calculations, we obtain that

$$p(z) = 1 + \alpha \frac{1 + zw'(z) \cosh w(z) + \sinh w(z)}{1 + \sinh w(z)}$$

and so

$$\frac{|p(z) - 1|}{|A - Bp(z)|} = \frac{\alpha (1 + zw'(z) \cosh w(z) + \sinh w(z))}{(A - B)(1 + \sinh w(z)) - Ba (1 + zw'(z) \cosh w(z) + \sinh w(z))}.$$
On the contrary if $w(z)$ attains its maximum value at some $z = z_0$ and $|w(z_0)| = 1$. Then by Lemma 6 for $m \geq 1$, we have, $z_0w'(z_0) = mw(z_0)$, so we have

$$\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| = \frac{\alpha (1 + mw(z_0) \cosh e^{i\theta} + \sinh e^{i\theta})}{(A - B) (1 + \sinh e^{i\theta}) - B\alpha (1 + mw(z_0) \cosh e^{i\theta} + \sinh e^{i\theta})} \geq \frac{|\alpha| (1 + m |\cosh e^{i\theta}| - |\sinh e^{i\theta}|)}{(A - B) (1 + |\sinh e^{i\theta}|) + |B| |\alpha| (1 + m |\cosh e^{i\theta}| + |\sinh e^{i\theta}|)}.$$

Now with the help of (4.4) and (4.5)

$$\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \geq \frac{|\alpha| (1 + m \cos (1) - \sin (1))}{(A - B) (1 + \sinh (1)) + |B| |\alpha| (1 + m \cosh (1) + \sinh (1))}.$$

Now if

$$\phi(m) = \frac{|\alpha| (1 + m \cos (1) - \sin (1))}{(A - B) (1 + \sinh (1)) + |B| |\alpha| (1 + m \cosh (1) + \sinh (1))}$$

then

$$\phi'(m) = \frac{2 (A - B) |\alpha| \cos 1 (\sin (1) + 1) + |B| |\alpha|^2 (\cos 1 (1 + \sin 1) + \cosh 1 (\sin 1 - 1))}{(A - B) (1 + \sinh (1)) + |B| |\alpha| (1 + m \cosh (1) + \sinh (1))^2} > 0,$$

which shows that $\phi(m)$ is an increasing function and hence it will have its minimum value at $m = 1$ and so

$$\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \geq \frac{|\alpha| (1 + \cos (1) - \sin (1))}{(A - B) (1 + \sinh (1)) + B |\alpha| (1 + \cosh (1) + \sinh (1))}.$$

Now by (4.7), we have

$$\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \geq 1,$$

which contradicts (4.8) and so $|w(z)| < 1$ for $|z| < 1$, which completes the proof. □

We obtain the following result using Theorem 10.

**Corollary 3.** For $-1 \leq B < A \leq 1$, and $f \in A_p$ then if the condition

$$|\alpha| \geq \frac{(A - B) (1 + \sinh 1)}{1 + \cos 1 - \sin 1 - B (1 + \cosh 1 + \sinh 1)},$$

and

$$1 + \alpha \left(2 + \frac{z g''(z)}{g'(z)} - \frac{z g'(z)}{g(z)} - 1 + \frac{1 + A z}{1 + B z} \right) \geq 1 + \frac{A z}{1 + B z},$$

holds then $g \in \mathcal{G}_{sh}$.

**Theorem 11.** For $-1 \leq B < A \leq 1$ and $f \in \mathcal{M}_p$ then if the condition

$$|\alpha| \geq \frac{(A - B) (1 + \sinh (1))^2}{1 + \cos (1) - \sin (1) - B (1 + \cosh (1) + \sinh (1))}$$

is true and

$$1 + \alpha \frac{z^2 f'(z)}{(f(z))^2} \geq \frac{1 + A z}{1 + B z},$$

then

$$\frac{f(z)}{z} \leq \sqrt{1 + z}.$$
Proof. Let

\[ p(z) = 1 + \alpha \frac{z^2 f'(z)}{(f(z))^2} \]

and

\[ \frac{f(z)}{z} = 1 + \sinh w(z), \]

then it is to show that \(|w| < 1\) for \(|z| < 1\). Also by simplification, we have

\[ p(z) = 1 + \alpha \frac{1 + zw'(z) \cosh w(z) + \sinh w(z)}{(1 + \sinh w(z))^2} \]

and so

\[ \frac{|p(z) - 1|}{A - Bp(z)} = \frac{\alpha (1 + zw'(z) \cosh w(z) + \sinh w(z))}{(A - B)(1 + \sinh w(z))^2 - B\alpha (1 + zw'(z) \cosh w(z) + \sinh w(z))}. \]

Now if \(w(z)\) attains its maximum value at some \(z = z_0\) and \(|w(z_0)| = 1\). Then by Lemma \([8]\) for \(m \geq 1\), we have, \(z_0 w'(z_0) = mw(z_0)\). So we have

\[ \frac{|p(z_0) - 1|}{A - Bp(z_0)} \geq \frac{\alpha (1 + mw(z_0) \cosh e^{i\theta} + \sinh e^{i\theta})}{(A - B)(1 + \sinh e^{i\theta})^2 - B\alpha (1 + mw(z_0) \cosh e^{i\theta} + \sinh e^{i\theta})}. \]

Now using \((4.4)\) and \((4.5)\)

\[ \frac{|p(z_0) - 1|}{A - Bp(z_0)} \geq \frac{|\alpha| (1 + m \cos (1) - \sin (1))}{(A - B)(1 + \sinh (1))^2 + |B| |\alpha| (1 + m \cosh (1) + \sinh (1))}. \]

Let

\[ \phi(m) = \frac{|\alpha| (1 + m \cos (1) - \sin (1))}{(A - B)(1 + \sinh (1))^2 + |B| |\alpha| (1 + m \cosh (1) + \sinh (1))} \]

which implies

\[ \phi'(m) = \frac{(A - B) |\alpha| \cos 1 (1 + \sinh (1))^2 + |B| |\alpha|^2 (\cos 1 (1 + \sinh 1) + \sin 1 \cosh 1 - \cosh 1)}{(A - B)(1 + \sinh (1))^2 + |B| |\alpha| (1 + m \cosh (1) + \sinh (1)))^2} > 0, \]

which shows that \(\phi(m)\) is an increasing function and hence it will have its minimum value at \(m = 1\) and so

\[ \frac{|p(z_0) - 1|}{A - Bp(z_0)} \geq \frac{|\alpha| (1 + \cos (1) - \sin (1))}{(A - B)(1 + \sinh (1))^2 + |B| |\alpha| (1 + \cosh (1) + \sinh (1))}. \]

Now by \((4.9)\) we have

\[ \frac{|p(z_0) - 1|}{A - Bp(z_0)} \geq 1, \]

which contradicts \((4.10)\), thus \(|w(z)| < 1\) for \(|z| < 1\), which yields the desired result. \(\square\)
Corollary 4. For $-1 \leq B < A \leq 1$, and $g \in \mathcal{A}_p$ then if the condition

$$|\alpha| \geq \frac{(A - B) (1 + \sinh (1))}{1 + \cos (1) - \sin (1) - B (1 + \cosh (1) + \sinh (1))},$$

and

$$1 + \frac{\alpha g(z)}{g'(z)} \left(2 + \frac{z g''(z)}{g'(z)} - \frac{z g'(z)}{g(z)}\right) < \frac{1 + Az}{1 + Bz},$$

holds then $g \in \mathcal{G}_{sh}$.

Theorem 12. For $-1 \leq B < A \leq 1$ and $f \in \mathcal{M}_p$ then if the condition

$$(4.11) \quad |\alpha| \geq \frac{(A - B) (1 + \sinh (1))^3}{1 + \cos (1) - \sin (1) - B (1 + \cosh (1) + \sinh (1))},$$

holds and

$$(4.12) \quad 1 + \frac{z^3 f'(z)}{(f(z))^3} \times \frac{1 + Az}{1 + Bz},$$

then

$$\frac{f(z)}{z} < 1 + \sinh w(z).$$

Proof. Let us assume

$$p(z) = 1 + \alpha \frac{z^3 f'(z)}{(f(z))^3}$$

and

$$\frac{f(z)}{z} = 1 + \sinh w(z),$$

then, we need to show that $|w| < 1$ for $|z| < 1$. Also by rearrangement, we get

$$p(z) = 1 + \alpha \frac{1 + zw'(z) \cosh w(z) + \sinh w(z)}{(1 + \sinh w(z))^3}$$

and so

$$\frac{p(z) - 1}{A - B p(z)} = \frac{\alpha (1 + zw'(z) \cosh w(z) + \sinh w(z))}{(A - B) (1 + \sinh w(z))^3 - B \alpha (1 + zw'(z) \cosh w(z) + \sinh w(z))}.$$

Now if $w(z)$ attains its maximum value at some $z = z_0$ and $|w(z_0)| = 1$. Then by Lemma 11 for $m \geq 1$, we have, $z_0 w'(z_0) = mw(z_0)$. So we have

$$\frac{p(z_0) - 1}{A - B p(z_0)} \geq \frac{\alpha (1 + mw(z_0) \cosh e^{i\theta} + \sinh e^{i\theta})}{(A - B) (1 + \sinh e^{i\theta})^3 - B \alpha (1 + mw(z_0) \cosh e^{i\theta} + \sinh e^{i\theta})} \geq \frac{|\alpha| (1 + m |\cosh e^{i\theta}| - |\sinh e^{i\theta}|)}{(A - B) (1 + |\sinh e^{i\theta}|)^3 + |B| |\alpha| (1 + m |\cosh e^{i\theta}| + |\sinh e^{i\theta}|)}.$$

Now with the help of (4.11) and (4.12)

$$\frac{p(z_0) - 1}{A - B p(z_0)} \geq \frac{|\alpha| (1 + m \cos (1) - \sin (1))}{(A - B) (1 + \sinh (1))^3 + |B| |\alpha| (1 + m \cosh (1) + \sinh (1))}.$$

Suppose

$$\phi(m) = \frac{|\alpha| (1 + m \cos (1) - \sin (1))}{(A - B) (1 + \sinh (1))^3 + |B| |\alpha| (1 + m \cosh (1) + \sinh (1))},$$
that implies
\[
\phi'(m) = \frac{(A - B) |\alpha| \cos 1 (1 + \sinh (1))^2 + |B| |\alpha|^2 (\cos 1 + \cos 1 \sinh 1 + \sin 1 \cosh 1 - \cosh 1)}{(A - B) (1 + \sinh (1))^3 + |B| |\alpha| (1 + m \cosh 1 + \sinh 1))^2} > 0,
\]
which shows that \( \phi(m) \) is an increasing function and hence it will have its minimum value at \( m = 1 \), thus
\[
\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \geq \frac{|\alpha| (1 + \cos (1) - \sin (1))}{(A - B) (1 + \sinh (1))^3 + |B| |\alpha| (1 + \cosh (1) + \sinh (1))}.
\]
Now by (4.11) we have
\[
\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \geq 1,
\]
which contradicts (4.12), therefore \( |w(z)| < 1 \) for \( |z| < 1 \), which gives the desired result.

**Corollary 5.** For \(-1 \leq B < A \leq 1\), and \( g \in \mathcal{A}_p \), then if the condition
\[
|\alpha| \geq \left( A - B \right) (1 + \sinh (1))^3 \frac{1 + \cos (1) - \sin (1)}{1 + \sinh (1) - B (1 + \cosh (1) + \sinh (1))}
\]
and
\[
1 + \alpha \frac{g(z)^2}{z^2 (g'(z))^2} \left( 2 + \frac{zg''(z)}{g'(z)} - \frac{zg'(z)}{g(z)} \right) < \frac{1 + Az}{1 + Bz},
\]
holds, then \( g \in \mathcal{G}_{sh} \).

**Funding**

Not applicable.

**Availability of data and materials**

Not applicable.

**Competing interests**

The authors declare that they have no competing interests.

**Authors contributions**

All authors jointly worked on the results and they read and approved the final manuscript.

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