Dissipation of Oscillation Energy and Distribution of Damping Power in a Multimachine Power System: A Small-signal Analysis

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Abstract—This paper revisits the concept of damping torque in a multimachine power system and its relation to the dissipation of oscillation energy in synchronous machine windings. As a multimachine extension of an existing result on a single-machine-infinite-bus (SMIB) system, we show that the total damping power for a mode stemming from the interaction of electromagnetic torques and rotor speeds is equal to the sum of average power dissipations in the generator windings corresponding to the modal oscillation. Further, counter-intuitive to the SMIB result, we demonstrate that, although the equality holds on an aggregate, such is not the case for individual machines in an interconnected system. To that end, distribution factors are derived for expressing the average damping power of each generator as a linear combination of average powers of modal energy dissipation in the windings of all machines in the system. These factors represent the distribution of damping power in a multimachine system. The results are validated on IEEE 4-machine and 16-machine test systems.

Index Terms—Damping torque, damping power, multimachine system, oscillation energy dissipation, synchronous machines

I. INTRODUCTION

In small-signal analysis, stability of a power system under electromechanical oscillations is determined by studying the eigenvalues of the system linearized around the quasi-steady-state operating point or by performing modal estimation on the response variables. In either case, the damping ratios obtained indicate the margin of system stability, but do not quantify the damping contribution from individual sources. In that regard, the small-signal representation of a generator’s electromagnetic torque as a phasor in its synchronously rotating rotor speed-angle reference frame offers geometric intuition into decomposing the torque into its damping and synchronizing components. Conceptualized by Park in his 1933 paper [1] and furthered by Concordia [2], [3], Shepherd [4], and notable others [5]–[9], the damping and synchronizing torque coefficients contribute towards insightful understanding of the stabilizing contributions coming from the machine and its associated governor and excitation systems. Consequently, some of the early designs of power system stabilizers for damping oscillations have evolved out of these notions. However, historically, these studies on damping torque have been presented either considering a SMIB system or by reducing the system to a linearized single-machine equivalent. Unlike a SMIB system, damping torque of a generator in a multimachine system depends not only on its own speed-deviation but also on that of the other machines, their excitation systems, and the overall network structure and parameters. Also, the analytical modeling of these differ in literature, for instance, authors in [10] model damping torque as a higher degree polynomial in speed-deviations, in contrast to linear terms in [11]. The 1999 IEEE task force report investigating modeling adequacy for representing damping in multimachine stability studies [12] identified eight different sources of damping and recommended abstracting their contributions into a single retarding torque in the swing equation of each generator.

Damping torque, thus, over the years, has largely remained a conceptual tool for analyzing stability in power systems. Although some of the initial works listed before highlighted an intuitive link between damping torque and dissipation of oscillation energy, it is only in the recent works [13] and [14] that a rigorous mathematical connection between the two has been established for a SMIB system. However, for multimachine systems such a connection is yet to be confirmed – in this paper, we make a maiden attempt to fill this gap. To that end, we use a simplified mathematical model for multimachine systems to establish an equivalence between the average power dissipation due to the damping torques on the rotors and the average rates of oscillation energy dissipation in the machine windings. In this context, we make a note of [15], where claims regarding the consistency of damping and dissipation coefficients in multimachine systems are made based on presupposition of this equality without any formal proof. Going ahead, in the paper we also demonstrate that the damping power of each machine stemming from the interaction of its own speed and torque, is derived in parts from the rates of energy dissipation in the windings of all machines, over and above it’s own winding.

At this point, it is important to clarify that the focus of this paper is not on improving the algorithmic tools of [14] or [13] for perfecting the science of locating oscillation sources or to present an alternate path for doing the same, but to derive further analytical insights from the findings of these two seminal papers. Our primary intention is to bridge a mathematical connection between the concept of transient energy dissipation, as introduced in [14], and the notion of damping torque, which is nearly a century old.

The contributions of the paper are as follows: (1) we develop a phasor-based small-signal formulation for calculating the mode-wise average powers (i.e. damping powers) of electromechanical oscillation due to the interaction of damping torque and speed in each machine; (2) using this framework for a simplified system model with lossless transmission network and constant power loads, we extend the SMIB results in [14]
to show that the total damping power for a mode is equal to the sum of average power dissipations in the generator windings corresponding to the modal oscillation; and finally, (3) counterintuitive to the SMIB result in [14], we demonstrate that the aforementioned equality does not hold for individual machines in an interconnected system – in fact the damping power in each machine can be expressed as a weighted linear combination of power dissipation in windings of different generators. These weighing factors (called ‘distribution factors’ in the paper) are analytically derived – which essentially describe the participation of the power dissipation in different machines in constituting the damping power of each generator.

To that end, in the next section, we derive a linearized representation of the simplified system mentioned earlier with a third-order synchronous generator model. Building on this model, we present contributions (1)–(3) in Sections II–IV which are followed by case studies on IEEE 4-machine and 16-machine test systems in Section VI to validate the claims – both for the simplified model used in derivation, and for systems with detailed machine models. Finally, concluding remarks are presented in Section VII.

Notations: Superscripts $T$, $*$, and $H$ are respectively the transpose, conjugate, and Hermitian operators. $R\{\cdot\}$ and $\Im\{\cdot\}$ denote the real and imaginary parts of a complex entity.

II. SIMPLIFIED SYSTEM MODEL: LINEARIZED REPRESENTATION

Consider a $n$–bus transmission system of which, without loss of generality, first $n_g$ are designated as generator buses. The network is lossless and each synchronous generator is described by a third-order machine model capturing the electromechanical dynamics of the rotor and the field flux. In addition, assume manual excitation for the generators and constant power loads at all buses.

The differential equations describing the dynamics of each generator are same as those given in eqns (6.132) – (6.134) of [11]. Further, the stator and network algebraic equations can be obtained from eqns (6.142) – (6.144). Since we assume a lossless network, in eqns (6.143) and (6.144) of [11], $\dot{\omega}_i = \pi/2$ for $i \neq k$, and $\omega_{ik} = -\pi/2$. Unless specified otherwise, all symbols have their usual meanings in [11].

We eliminate the stator algebraic equations by substituting $I_{d_i}$ and $I_{q_i}$ obtained from eqn (6.142) into eqns (6.143) and (6.144) of [11] – with stator resistances neglected. This leaves us with $3n_g$ differential equations and $2n$ algebraic equations as functions of state variables $\delta_i$, $\omega_i$, and $E_{q_i}^V$, for $i = 1 \ldots n_g$, and algebraic variables $\theta_i$ and $V_i$, for $i = 1 \ldots n$ as described below

$$\dot{\delta}_i = \omega_i - \omega$$  \hspace{1cm} (1)

$$\frac{\omega_i}{2} = \frac{T_{m_i}}{2H_i} - \frac{E_{q_i}^V V_i \sin(\delta_i - \theta_i)}{2H_i x_{d_i}} + \frac{E_{q_i}^V \sin^2(\delta_i - \theta_i)}{4H_i} \left(\frac{x_{q_i} - x_{d_i}'}{x_{q_i} x_{d_i}'}\right)$$  \hspace{1cm} (2)

$$\dot{E}_{q_i}^V = \frac{E_{d_i}}{T_{do_i}} - \frac{E_{q_i}^V}{T_{do_i}} - \frac{E_{q_i}^V - V_i \cos(\delta_i - \theta_i)}{x_{d_i}} \left(\frac{x_{d_i} - x_{d_i}'}{T_{do_i}}\right)$$  \hspace{1cm} (3)

for $i = 1 \ldots n_g$

$$0 = f_i = \begin{align*}
E_{q_i}^V \frac{\sin(\delta_i - \theta_i)}{2} - V_i^2 \sin^2(\delta_i - \theta_i) & \frac{2 \left(x_{q_i} - x_{d_i}'\right)}{x_{q_i} x_{d_i}'} \\
& + P_{L_i} - \sum_{k=1,k \neq i}^{n_g} V_i V_k Y_{ik} \sin(\delta_i - \theta_k) \\
& \text{for } i = 1, \ldots n_g
\end{align*}$$  \hspace{1cm} (4)

$$0 = g_i = \begin{align*}
E_{q_i}^V \frac{\cos(\delta_i - \theta_i)}{2} - V_i^2 \cos^2(\delta_i - \theta_i) & + Q_{L_i} - V_i^2 Y_{ii} \\
& - \frac{V_i^2 \sin^2(\delta_i - \theta_i)}{x_{q_i}} + \sum_{k=1,k \neq i}^{n_g} V_i V_k Y_{ik} \cos(\delta_i - \theta_k) \\
& \text{for } i = 1, \ldots n_g
\end{align*}$$  \hspace{1cm} (5)

Linearizing [1] – [3] around an operating point, with $V_{i_0}$ as the voltage magnitude of bus $i$ at that point and defining a new variable $\nu_i = V_i/V_{i_0}$, we obtain

$$\begin{bmatrix} \Delta \delta \\ \Delta \omega \\ \Delta E_{q_i}' \\ \Delta E_{q_i}' \end{bmatrix} = M \begin{bmatrix} \Delta \delta \\ \Delta \omega \\ \Delta E_{q_i}' \\ \Delta E_{q_i}' \end{bmatrix} + N \begin{bmatrix} \Delta \theta \\ \Delta \nu \end{bmatrix} + B \begin{bmatrix} \Delta T_m \\ \Delta E_{fd} \end{bmatrix}$$  \hspace{1cm} (6)

where, $\delta$, $\omega$, $E_{q_i}'$, $\theta$, and $\nu$ are the vectorized state and algebraic variables of respective type, for instance, $\delta = [\delta_1 \ldots \delta_{n_g}]^T$ and $\nu = [\nu_1 \ldots \nu_n]^T$. Finally, eliminating the algebraic variables, we get

$$\begin{bmatrix} \Delta \delta \\ \Delta \omega \\ \Delta E_{q_i}' \end{bmatrix} = A \begin{bmatrix} \Delta \delta \\ \Delta \omega \\ \Delta E_{q_i}' \end{bmatrix} + B \begin{bmatrix} \Delta T_m \\ \Delta E_{fd} \end{bmatrix}$$  \hspace{1cm} (7)

where, $A = M - ND^{-1} C$. It follows from the equations above that $A$ is of the form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & A_{21} & 0 & A_{23} \\ A_{31} & 0 & A_{33} \end{bmatrix}$$  \hspace{1cm} (8)

Note, in (8), every $A_{ij}$ block is a submatrix of $A$ whose elements are derived later in the paper (see, Appendix B). Apart from the state variables $\Delta \delta_i$, $\Delta \omega_i$, and $\Delta E_{q_i}'$, the output variable $\Delta T_{e_i}$, which is the electromagnetic torque of generator $i$, is of specific interest to us. From the swing equation in [2] this is expressed as $\Delta T_{e_i} = -\frac{2H_i}{\omega_i} \Delta \omega_i$.

Following any disturbance in the system or perturbation in the inputs, the time-evolution of these state and output variables can be expressed as sum of damped sinusoids with modal frequencies $\omega_{d_i}$s with differing amplitudes and phases. As a result, for each mode $r$, $\Delta T_{e_i,r}(t)$, $\Delta \delta_{i,r}(t)$ and $\Delta \omega_{i,r}(t)$ can be expressed as rotating phasors – for details, see Appendix A. The notions of damping torque and damping power for a given mode originate from the phasor representation of $\Delta T_{e_i,r}$ in the $\Delta T_{e_i,r}$ vs $\Delta \omega_{i,r}$ plane and the power resulting from the interaction of the torque and the speed. This is explained next.
III. DAMPING POWER IN A MULTIMACHINE SYSTEM

In any $i$th machine, for a mode $r$, let the average power of the electromagnetic torque $\Delta T_{e,i}(t)$ over a cycle starting from $t = t_0$ be denoted by $W_{d,i}(t_0)$, as shown below

$$W_{d,i}(t_0) = \frac{\int_{t_0}^{t_0 + \frac{2\pi}{\omega_i}} \Delta T_{e,i}(t) \Delta \omega_{i,r}(t) dt}{\frac{2\pi}{\omega_i}}$$

Using the phasor notation described in Appendix A, let $\Delta T_{e,i,r}(t) = \beta_1 e^{\sigma_r t} \angle \gamma_1$ and $\Delta \omega_{i,r}(t) = \beta_2 e^{\sigma_r t} \angle \gamma_2$. Therefore, $W_{d,i}(t_0) = \frac{\beta_1 \beta_2 \omega_{d,i}}{4\pi} \int_{t_0}^{t_0 + \frac{2\pi}{\omega_i}} e^{2\sigma_r t} \{ \cos(\gamma_1 - \gamma_2) + \cos(2\omega_i t + \gamma_1 + \gamma_2) \} dt$

Now, considering that our mode of interest is poorly-damped (as is the premise of our paper), implying $|\sigma_r| << \omega_d$, we may expand the exponential $e^{2\sigma_r t}$ and neglect the second and higher order terms. On doing so, the first term in (10) reduces to

$$\frac{\beta_1 \beta_2 \omega_{d,i}}{4\pi} e^{2\sigma_r t} \cos(\gamma_1 - \gamma_2) 1 - \frac{1}{2} \frac{\cos(2\omega_i t + \gamma_1 + \gamma_2)}{2\sigma_r} = \frac{\beta_1 \beta_2 \omega_{d,i}}{4\pi} e^{2\sigma_r t} \cos(\gamma_1 - \gamma_2)$$

With the same assumption that $\sigma_r$ is small, the second term in (10) becomes negligible, and can be ignored for mathematical tractability. This is because, with $|\sigma_r| << \omega_d$, for a complete cycle of $\cos(2\omega_d t)$, the $e^{2\sigma_r t}$ term remains almost constant, and therefore, the positive and negative half cycles almost add to zero. Therefore,

$$W_{d,i}(t_0) \approx \frac{1}{2} \left[ \frac{\beta_1 \beta_2}{\omega_d,i} e^{2\sigma_r t} \cos(\gamma_1 - \gamma_2) \right]$$

Next, we express this sum of damping powers in terms of system matrices by using the definition of electromagnetic torque (from the swing equation) and the linearized system description obtained in [7] and [8], as shown below.

$$\Delta T_e(s) = -\frac{2H}{\omega_s} \Delta \omega(s) - \frac{2H}{\omega_s} \{ A_{21} \Delta \delta(s) + A_{23} \Delta E'(s) \}$$

where, $K_{i,j,r}$ is the $(i, j)$th element of $K_r$. We call $\bar{W}_d$, the 'total damping power' of the system for mode $r$.

Next, we simplify the expression in eqn (15) using the set of claims (1) - (4) below. Claims:

1. $P_i A_{33}^T = A_{33}$
2. $A_{31}^T P = \frac{2H}{\omega_s} A_{23}$
3. $2A_{21}^T H = 2H A_{21}$
4. $\forall x \in C^{n_x}, \Re(x^H K_r x) = x^H \Re(K_r) x$

where, $P$ is a diagonal matrix of machine parameters with $P(i, i) = \frac{T_{qe,i}}{\omega_d,i}$. Claims (1) - (3) are derived using the differential and algebraic equations of the system modeled in Section [II]. These claims are then used to establish the symmetry of $K_r$, which is in-turn used in proving claim (4). Detailed proof of these claims are outlined in Appendix B.

Using claim (4) we further reduce eqn (15) as follows

$$\bar{W}_d = \frac{1}{2} \Delta \omega_r^H \Re(K_r) \Delta \omega_r$$

where, $\Re(K_r) =$

$$= -\Re \left\{ \frac{2H}{\omega_s} \left( A_{21} + A_{23} \left( j \omega_d, I - A_{33} \right)^{-1} A_{31} \right) \right\}$$

$$= -\frac{2H}{\omega_s} A_{23} \Re \left\{ \left( -j \omega_d, I - A_{33} \right) \left( -j \omega_d, I - A_{33} \right)^{-1} \right\} \left( j \omega_d, I - A_{33} \right)^{-1} A_{31}$$

$$= -\frac{2H}{\omega_s} A_{23} \Re \left\{ \left( -j \omega_d, I - A_{33} \right) \left( j \omega_d, I + A_{33} \right)^{-1} \right\} A_{31}$$

$$= \frac{2H}{\omega_s} A_{23} \left( \omega_d^2, I + A_{33} \right)^{-1} A_{31}$$

This, along with claim (2) when substituted in eqn (16) gives

$$\bar{W}_d = \frac{1}{2} \Delta \omega_r^H A_{31}^T P \left( \omega_d^2, I + A_{33} \right)^{-1} A_{31} \Delta \omega_r$$
IV. CONSISTENCY OF DAMPING POWER WITH POWER DISSIPATION IN SYNCHRONOUS MACHINE WINDINGS

Considering the third-order system model in Section III the only source of power dissipation in the machines is in the field windings. For any mode $r$, we denote the average power dissipation in the winding of machine $i$ by $\dot{W}_{fr,i}$. Following the phasor notation as discussed, this is expressed as

$$\dot{W}_{fr,i} = \frac{1}{2} \Re \left( (\Delta I_{fr,i}^r, R_{fr,i}) \Delta I_{fr,i}^r \right) = \frac{1}{2} R_{fr,i} |\Delta I_{fr,i}^r|^2$$  \hspace{1cm} (18)

where, $\Delta I_{fr,i}^r$ are the phasors of the field current and derivative of transient e.m.f. due to field flux linkage, respectively. Further, using notations from (16) (Appendix A) we may write

$$\Delta E_{q_{ij},r} = j\omega_d \Delta E_{q_{ij},r}^f = j\omega_d 2 c_r e^{\sigma r} \psi_{E_{q_{ij},r}}.$$  \hspace{1cm} (19)

We define, $\tilde{c}_r = c_r e^{\sigma r}$. Next, substituting (19) in (18) we get

$$\dot{W}_{fr,i} = 2 |\tilde{c}_r|^2 \frac{T_{do}}{x_{d,i} - x_{q,i}} \omega_d^2 \psi_{E_{q_{ij},r}}^2.$$  \hspace{1cm} (20)

We obtain the total power dissipation for the mode by adding the dissipation in individual machines. This is shown below.

$$\dot{W}_{fr} = \sum_{i=1}^{n_g} \dot{W}_{fr,i} = 2 |\tilde{c}_r|^2 \frac{T_{do}}{x_{d,i} - x_{q,i}} \omega_d^2 \psi_{E_{q_{ij},r}}^2.$$  \hspace{1cm} (21)

where, $\psi_{E_{q_{ij},r}} = \left[ \psi_{E_{q_{ij},r}}^r, \psi_{E_{q_{ij},r}}^q, \psi_{E_{q_{ij},r}}^{r+q} \psi_{E_{q_{ij},r}}^{r-q} \right]^T$.

To show that our notion of total damping power, as defined in Section III, is consistent with the power dissipated in the system, we need to prove that for any mode $r$, $\dot{W}_{fr}$ is equal to $\dot{W}_{d,r}$. To do so, we make the following algebraic manipulations.

First, since $\lambda_r$ is an eigenvalue of the system, we may write $A \Psi_r = \lambda_r \Psi_r$, where the right eigenvector $\Psi_r = \left[ \psi_{\delta_r}, \psi_{\omega_r}, \psi_{E_{q_{ij},r}} \right]^T$. Next, using the structure of matrix $A$ as in (9), we can split this into the following equations

$$\psi_{E_{q_{ij},r}} = (\lambda_r I - A_{33})^{-1} A_{31} \psi_{\delta_r} \quad \text{and} \quad \psi_{\delta_r} = \frac{1}{\lambda_r} \psi_{\omega_r}.$$  \hspace{1cm} (22)

Using these, along with (34) describing $\Delta \omega_r = 2 \tilde{c}_r \psi_{\omega_r}$, we may re-write (21) as follows

$$\dot{W}_{fr} = 2 |\tilde{c}_r|^2 \frac{T_{do}}{x_{d,i} - x_{q,i}} \omega_d^2 \left( (\lambda_r I - A_{33})^{-1} A_{31} \Delta \omega_r \right) \frac{\Delta \omega_r}{2 c_r \lambda_r}.$$  \hspace{1cm} (23)

Finally, considering that our mode of interest is poorly-damped $|\sigma_r| << \omega_d$, as-is the premise of [14], and for consistency this paper’s too, we substitute $\lambda_r = j\omega_d$. This, along with use of claim (1) reduces (23) as follows

$$\dot{W}_{fr} = \frac{1}{2} \frac{\Delta \omega_r^H A_{31} \Delta \omega_r}{2 c_r \lambda_r} = \frac{1}{2} \frac{\Delta \omega_r^H A_{31} \Delta \omega_r}{2 c_r \lambda_r} = \frac{1}{2} \frac{(j\omega_d I - A_{33})^{-1} A_{31} \Delta \omega_r}{2 c_r \lambda_r}.$$  \hspace{1cm} (24)

Thus, for any given mode, the equivalence of the total damping power and the sum of average rate of change of energy dissipations in the machine windings is established.

Remarks: (1) While the concepts of damping torque and damping power are derived using the linearized system models, the average rate of energy dissipation in windings are more fundamental and does not limit itself to small-signal analysis.

(2) For machines with detailed models, the power dissipation in damper windings should be added to $\dot{W}_{fr}$ to obtain the total dissipation in the system. Average power dissipation in the damper winding is expressed as [13]

$$\dot{W}_{d,r} = 2 |\tilde{c}_r|^2 \frac{T_{do}}{x_{d,i} - x_{q,i}} \omega_d^2 \psi_{E_{q_{ij},r}}^2.$$  \hspace{1cm} (25)

$E_{d_{ij},r}$ is the state variable describing the dynamics of the damper winding transient e.m.f. Further, with dynamics of the excitation systems modeled, the expression of $\dot{W}_{fr}$ in (21) would get modified to include the effect of $\Delta E_{f_d}$ as

$$\dot{W}_{fr} = 2 |\tilde{c}_r|^2 \left\{ \frac{T_{do}}{x_{d,i} - x_{q,i}} \omega_d^2 \psi_{E_{q_{ij},r}}^2 - \frac{1}{x_{d,i} - x_{q,i}} \psi_{E_{q_{ij},r}}^2 \right\}.$$  \hspace{1cm} (26)

However, the expression for $\dot{W}_{d,r}$ in (15) would remain the same (with block matrices $A_{23}, A_{31},$ and $A_{33}$ larger in dimensions to account for the additional state variables like $\Delta E_{d'}$, $\Delta E_{f_d}$, etc. now concatenated to the vector $\Delta E'$ and the equality of total damping power and total power dissipation would still be true. This will be demonstrated in Section VI.

(3) Additionally, with a power system stabilizer (PSS) and an IEEE ST1A exciter modeled, as shown in Fig. 1 three new states will be added — one each due to the washout block, the lead-lag compensator, and the time constant of the transducer. Concatenating these to the existing state variables, the block matrices $A_{23}, A_{31},$ and $A_{33}$ would be further expanded. Additionally, due to the speed feedback to the washout block, $A_{32}$ term would now be non-zero, implying $K_r = 2 I_{\omega_d} \left( \frac{A_{23}}{j\omega_d} + A_{23} (j\omega_d I - A_{33})^{-1} (A_{31} \frac{1}{j\omega_d} + A_{32}) \right)$.

The equality of total damping power $\dot{W}_{d,r}$ (as calculated with the modified $K_r$) and that of the sum of $\dot{W}_{fr}$ and $\dot{W}_{gr}$ will be demonstrated in Section VI.

Fig. 1: PSS and IEEE ST1A excitation system.

(4) Next, let us explore the importance of our deductions in the context of stability monitoring. From [13], we know that, the relative damping contribution of individual generators can be inferred from the power dissipations in their windings.
Further, our paper establishes \( \sum \bar{W}_{dl,r} = \sum (W_{f,l,r} + \bar{W}_{d,r}) \), which building on derivations in [13] leads to the claim that the stability margin of the \( r^{th} \) mode \( \sigma_r \propto -\sum \bar{W}_{dl,r} \). Note that, estimating the variables \( \Delta T_{ei,r} \) and \( \Delta \omega \) are relatively simpler. Because, either they can be directly measured, as with rotor speed, or estimated from measurable outputs, like torque from power. And therefore, damping power \( \bar{W}_{dl,r} \) of individual machines can be easily calculated from their terminal measurements upon filtering for the mode of interest. This has the potential for future monitoring applications, like determining stability margin of a mode by measuring total damping power contribution from all generators. To this end, we propose \( \sum \bar{W}_{dl,r} \) as a complementary measure of stability margin for the mode, with individual \( \bar{W}_{dl,r} \)-s as indices for identifying the prospective generator locations for damping enhancement.

V. DISTRIBUTION FACTORS: EXPRESSING DAMPING POWER OF EACH MACHINE AS WEIGHTED SUM OF DISSIPATIONS IN ALL MACHINES

With the equality in [24] now proved, it might be tempting to draw an intuitive conclusion that such a power balance should hold for individual generators. However, this is not the case. This is because, the modeshape of \( E'_q \) of each generator is a function of modeshapes of speed-deviation of all other machines and vice-versa (see, [22]). Therefore, the mathematical representation of damping power in each machine is in reality an abstraction of dissipative effects coming from the windings of all machines in the system including it’s own. To that end, we next derive the distribution factors describing the fractional contribution of windings of different machines in constituting the damping power of a single machine.

Considering the system model described in Section [1] in this section, we express the damping power of each generator \( \bar{W}_{dl,r} \) as a linear combination of individual \( W_{f,l,r} \)-s. Recall, from [12]

\[
\bar{W}_{dl,r} = \frac{1}{2} \sum_{j=1}^{n_g} k_{d,j,r} |\Delta \omega_{ij,r}|^2 = 2 |\bar{c}_r|^2 k_{d,r} \Psi_{\omega_r}^H \mathbf{I}_i \Psi_{\omega_r},
\]

where, \( \mathbf{I}_i \) is a \( n_g \)-dimensional square matrix with the \((i,i)\)th entry as 1 and remaining all entries as zeros. Next, from the eigen decomposition of \( \mathbf{A} \) as before, we may write \( \Psi_{\omega_r} = \lambda_r (\mathbf{A})^{-1} \mathbf{A}_{21} \Delta Q \Psi_{E_q'} \). Substituting this, \( \bar{W}_{dl,r} \) may be expressed as

\[
\bar{W}_{dl,r} = 2 |\bar{c}_r|^2 k_{d,r} \Psi_{E_q'}^H \mathbf{Q}^H \mathbf{I}_i \mathbf{Q} \Psi_{E_q'}
\]

\[
\quad = 2 |\bar{c}_r|^2 k_{d,r} \sum_{\ell=1}^{n_g} \sum_{j=1}^{n_g} Q_{\ell j} \psi_{E'_{\ell j,r}} \psi_{\omega}^* \psi_{E_{\ell j,r}}
\]

\[
\quad = 2 |\bar{c}_r|^2 k_{d,r} \sum_{j=1}^{n_g} \frac{\sum_{\ell=1}^{n_g} Q_{\ell j} \psi_{E_{\ell j,r}}^* \psi_{\omega} \psi_{E_q'}^* \psi_{E_q'}^*}{\psi_{E_{\ell j,r}}^*} \psi_{E_{\ell j,r}}
\]

\[
\quad = \sum_{j=1}^{n_g} \frac{k_{d,j,r}}{P_j \omega_{d,j}} Q_{ij} \psi_{\omega} \psi_{E_{ij,r}} \bar{W}_{f_{ij,r}}
\]

where, \( P_j \) is the diagonal element \( \mathbf{P}(j,j) = \frac{T_{d,j}}{x_{d,j} - \omega_{d,j}} \). Next, we define \( \beta_{ij} \) as the component of \( \Delta \omega_{ij,r} \) due to \( \Delta E_{ij,r}^H \)

\[
\beta_{ij} = \frac{Q_{ij} \Delta E_{ij,r}^*}{\Delta \omega_{ij,r}} = \frac{Q_{ij} \psi_{E_{ij,r}}^*}{\psi_{\omega}^*}.
\]

Since, \( \psi_{E_{ij,r}}^* = \sum_{j=1}^{n_g} Q_{ij} \psi_{E_{ij,r}}^* \), we may say \( Q_{ij} \psi_{E_{ij,r}}^* \) is the contribution of \( \psi_{E_{ij,r}}^* \) in the modeshape \( \psi_{\omega},r \). Therefore,

\[
\bar{W}_{dl,r} = \sum_{j=1}^{n_g} k_{d,j,r} |Q_{ij}|^2 \frac{\beta_{ij}}{P_j \omega_{d,j}^2} \bar{W}_{f_{ij,r}}
\]

Since, \( \bar{W}_{dl,r} \) and \( \bar{W}_{f_{ij,r}} \)-s are real quantities, the imaginary part of \( \bar{W}_{dl,r} \) is zero. Hence,

\[
\bar{W}_{dl,r} = \sum_{j=1}^{n_g} k_{d,j,r} |Q_{ij}|^2 \frac{\beta_{ij}}{P_j \omega_{d,j}^2} \bar{W}_{f_{ij,r}}
\]

We call \( \alpha_{ij} \)-s the ‘distribution factors,’ because, for a fixed \( i \), the ratios \( \alpha_{ij} \frac{\bar{W}_{f_{ij,r}}}{\bar{W}_{dl,r}} \) for \( j = 1 \) to \( n_g \), are the fractions in which the damping power of generator \( i \) is derived from the power dissipation in the windings of the machines 1 to \( n_g \). Further, looking from the other side, fixing a machine \( j \), the factors \( \alpha_{ij} \)-s describe the ratios in which the power dissipation in that machine winding is distributed in the ‘abstract’ damping power of all other machines.

A. Connection to the Heffron-Phillips Model

We know from the Heffron-Phillips model [17] of a SMIB system that, for a third-order machine model \( (K1 - K4 \text{ model}) \), the angle between the \( \Delta T_{ei,r}^H \) and \( \Delta \omega_{ij,r}^H \) phasors is determined by the field circuit time-constant. Higher the resistance of the field circuit, smaller is the angle, and therefore, higher is the damping power of the generator. This is consistent with the results in [14] that for a SMIB system, the damping power of the machine is derived exclusively from the power dissipation in its field circuit. Extending the same to the Heffron-Phillips model for multimachine systems, we see (from Fig. 6 in [18]) that the damping power of each generator has contributions from the field circuit dissipations in multiple other machines in the system – depending on the relative participation of those machines in the mode of interest. However, given the complexity in calculating the mode-wise \( K1 - K4 \) constants in a multimachine system, we, through the distribution factors derived in this section, offer an alternative path to express the damping powers of each generator as a weighted sum of field winding dissipations of all machines in the system, including it’s own.

B. Potential Application in Understanding Dissipative Contribution from PSS and Other Controllers

The lead-lag compensators in a \( \Delta \omega \)-PSS are designed knowing the angular relationship between the \( \Delta T_{ei,r}^H \) and \( \Delta \omega_{ij,r}^H \) phasors, and the phase-shift required to rotate \( \Delta T_{ei,r}^H \) further in the direction of \( \Delta \omega_{ij,r}^H \). This phase-shift introduces additional damping for the mode by increasing the total power loss in the system for the mode for which the PSS is designed. However, since a PSS in one location might negatively affect
the damping in some other location, dissipation in some individual generators may be reduced. While, methods like the one in [18] quantify the effect of PSS on damping powers of individual machines, they do not describe how the dissipations in their windings would change. To this end, the distribution factors connecting damping and dissipative powers become useful. Also, this is not limited to PSSs in generators, if derived for higher-order models, the analysis can be extended to other types of controllers.

While distribution factors may not have direct usefulness in terms of monitoring or controlling a mode, we believe they serve as an important tool that bridge an insightful link between two apparently different frameworks.

VI. CASE STUDIES

We now verify the aforementioned claims on the fundamental frequency phasor models of IEEE 4-machine [17] and 16-machine [19] test systems. In each case we consider two types of models: (a) Simplified model with assumptions described in Section II and (b) Detailed model considering 4th-order synchronous machine dynamics (including 1 damper winding) along with exciters, where the network is still assumed to be lossless and loads are of constant power nature.

A. IEEE 2-area 4-machine Kundur Test System

Consider the 4-machine system [17] shown in Fig. 2 with a total load of 2,734 MW under nominal condition.

![Fig. 2: Single-line diagram of 2-area 4-machine test system.](image)

(a) Simplified model: Under nominal loading, there are three poorly-damped modes, see Table I. Given modeling assumptions, the only source of damping is in the field windings of the generators. Therefore, following our proposition, we need to show that for small perturbations in the system, for each of these oscillatory modes, the sum of average damping powers is numerically equal to the sum of power dissipations in the field windings, across the operating points. This is demonstrated in the Figs. 3 (a) and (b) for two of the three modes. The operating point is varied by progressively reducing the tie flow between buses 7 and 9 from 433 MW under nominal condition to −400 MW while maintaining the total load of the system constant.

We now validate our claim that although the total power dissipation is equal to the total damping power at the system level, this is not necessarily true for individual machines. In Figs 4(a) and (b), the ratios of dissipation power to damping power is plotted for each of the 4 machines for the 0.69 Hz and 1.04 Hz modes, respectively. Observe that the ratios show strict monotonicity with change in operating points and under no circumstance they are equal to 1 all at once.

Next, in Fig. 5 the relative contributions from the power dissipations in the windings of G1 to G4 in constituting the damping power of G1, as discussed in Section V, are plotted for the 0.69 Hz mode. Observe that, at a given operating point, the fractions \( \alpha_{1j} \frac{W_{d_{r,j}}}{W_{r_{d,1}}} \) for \( j = 1 \) to 4 add to 1. In Fig. 6 the distribution factors \( \alpha_{1i} \)'s for \( i = 1 \) to 4 are shown. These describe fractions in which the power of oscillatory energy dissipation in G1 is distributed in the damping powers of G1 to G4. It can be seen that at any operating point, \( \sum_{j=1}^{4} \alpha_{1j} = 1 \). Since the machines are nearly identical and the network is symmetric with respect to the generators in this system, the nature of the distribution factors are similar for other generators, and thus are not repeated here.

(b) Detailed model: The detailed model considers DC1A excitation system [17] for each generator over and above the

| Table I: Poorly-Damped Modes in IEEE 4-Machine System |
|------------------------------------------|-----------------|-----------------|-----------------|
| Machine Model                           | Eigenvalues     | Modal freq.     | Damp. ratio     |
|------------------------------------------|-----------------|-----------------|-----------------|
| Simplified model                         | \( -0.1183 \pm j4.3816 \) | 0.69            | 0.027           |
|                                          | \( -0.1444 \pm j6.2779 \) | 1.00            | 0.023           |
|                                          | \( -0.1544 \pm j6.5330 \) | 1.04            | 0.024           |
| Detailed model (with DC1A exciters)     | \( -0.1231 \pm j4.2130 \) | 0.67            | 0.029           |
| Detailed model (with ST1A exciters & PSS)| \( -0.1477 \pm j4.8649 \) | 0.77            | 0.030           |
assumptions mentioned earlier—the poorly-damped mode is shown in Table I. For any mode, the power dissipation in the system is the summation of total power dissipations in field winding \( W_f \) and in damper winding \( W_g \), (see, (26) and (25), respectively), where \( W_f = \sum_{i=1}^{n_g} W_{f,i} \) and \( W_g = \sum_{i=1}^{n_g} W_{g,i} \). For the detailed model, the consistency of total power dissipation with the total damping power is illustrated in Fig. 7.

(c) Validation under large disturbances: Next, we consider the 4th-order machine model with IEEE ST1A excitation system for validation under large disturbances. Additionally, G1 is equipped with a PSS. We simulate a 5-cycle three-phase self-clearing fault at \( t = 1 \) s near bus 8. The detrended post-fault time-domain plots of the state and output variables of the two generators G1 and G3 are shown in Fig. 8. We obtain the relative modeshapes of these signals using the approach described in [20].

We use \( \Delta \omega_1 \) as the reference signal and compute the modeshapes for all \( \Delta \omega_i, \Delta T_{e,i}, \Delta E_{e,i} \), and \( \Delta E_{f,i} \) for the critical mode in Table I. The damping and dissipative powers of all 4 generators as computed using these modeshapes are shown in Table II. It can be seen that, although for individual machines the values of \( \hat{W}_{d,i} \)-s are different from that of \( \hat{W}_{f,i} \)-s, their sum totals are almost equal. Also, these values nearly match those calculated from the small-signal model.

### Table II: Damping and Dissipative Powers in Detailed 4–Machine System Model for 0.77 Hz Mode.

| Machine Model | Using the eigenvectors obtained from the small-signal model | Using the modeshapes estimated from the time-domain responses |
|---------------|------------------------------------------------------------|-------------------------------------------------------------|
|               | \( W_{d,1} \) | \( W_{f,1} \) | \( W_{s,1} \) | \( W_{g,1} \) |
| G1            | 0.0382         | 0.2294         | 0.0301         | 0.2274         |
| G2            | 0.0321         | −0.1830        | 0.0340         | −0.1877        |
| G3            | 0.0435         | 0.0794         | 0.0405         | 0.0754         |
| G4            | 0.0521         | 0.0401         | 0.0472         | 0.0390         |
| Sum           | 0.1659         | 0.1659         | 0.1578         | 0.1541         |

### Table III: Poorly-Damped Modes in 16-Machine System

| Machine Model | Eigenvalue \( \lambda = \sigma + j\omega \) | Modal freq. \( f_r \) (Hz) | Damping ratio \( \zeta \) |
|---------------|-------------------------------------------|---------------------------|-------------------------|
| Simplified model | \( −0.0336 \pm j2.1031 \) | 0.34 | 0.016 |
|               | \( −0.0338 \pm j3.1288 \) | 0.50 | 0.011 |
|               | \( −0.1062 \pm j3.7500 \) | 0.60 | 0.028 |
|               | \( −0.0264 \pm j4.1031 \) | 0.65 | 0.006 |
| Detailed model | \( −0.0656 \pm j3.1137 \) | 0.49 | 0.021 |
|               | \( −0.0981 \pm j3.5169 \) | 0.56 | 0.028 |
|               | \( −0.1827 \pm j4.9627 \) | 0.79 | 0.037 |

B. IEEE 5–area 16–machine NY–NE Test System

Next, consider the 16–machine New York–New England test system shown in Fig. 9. In the detailed model, G1–G8 have DC1A exciters, G9 is equipped with a ST1A exciter and a power system stabilizer (PSS), and the remaining generators have manual excitation. The machine and the network data can be obtained from [19]. The poorly damped modes of the system under nominal loading, for both simplified and detailed models, are shown in Table III. Different operating points are obtained by uniformly changing the total system load. As before, in Figs 10 and 11 the consistency of total damping power and sum of power dissipation in machine windings is

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**Figures and Tables**

1. **Fig. 5**: Fractions in which the damping power of G1 is distributed as power dissipation in the windings of G1 – G4 for 0.69 Hz mode.

2. **Fig. 6**: Distribution of the power dissipation in G1 in the damping powers of G1 – G4 for 0.69 Hz mode.

3. **Fig. 7**: Equality of total damping power with sum of average power dissipation in windings of all generators across different operating points in detailed 4–machine system model for the 0.67 Hz mode.

4. **Fig. 8**: Detrended post-fault time-domain plots of the state and output variables of G1 and G3.

### Tables

- **Table I**: Poorly-damped modes in 16-machine system.

- **Table II**: Damping and dissipative powers in detailed 4-machine system model for 0.77 Hz mode.

- **Table III**: Poorly-damped modes in 16-machine system.
shown respectively for the simplified and the detailed model—each corresponding to a particular mode. The negative values of $\omega_{ij}$ in Fig. 11 indicate that the excitation systems are contributing towards negative damping for higher loadings.

Next, we present the following case study for the simplified model to validate our claims regarding the distribution of damping power in windings of all generators across different operating points in the 16-machine system model for 0.50 Hz mode.

For the 0.56 Hz mode, the generators of NETS oscillate against those in NYPS. Next, using the estimated modeshapes, we compute the damping and dissipative powers of the selected generators as shown in Table [IV]. Finally, using the $\alpha_{ij}$-s from the expression in (30) and the values estimated in Table [IV] we compute the fractional contributions from these selected generators towards the damping power of G9. These are shown in Table [V]. As seen, the estimated values match those calculated from small-signal model. The variation in these fractions for change in system loading are shown in Fig. [13].

Now, we consider the other poorly-damped mode at 0.5 Hz, for which generators outside NETS—G14 and G16 oscillate against each other with marginal participation from generators in other areas. The damping and dissipative powers of these two generators, for the mode, are shown in Table [VI]. Note that, since other generators do not participate in the mode, the summation of damping powers of G14 and G16 is approximately equal to the summation of their dissipative powers. Finally, in Fig. [14] the relative dissipative contributions from the generators in different areas in constituting the damping power are shown.
power of G16 are shown. For G1 − G9 and G10 − G13, their aggregates
\[ \sum_{j=1}^{9} \alpha_{16,j} \hat{W}_{f,16,r} \text{ and } \sum_{j=10}^{13} \alpha_{16,j} \hat{W}_{f,16,r} \]
are shown as contributions from NETS and NYPS, respectively. We note that G14 and G16 have the highest participation in the mode, which is aligned with the observation that these generators have the highest dissipative contribution.

**TABLE VI: DAMPING AND DISSIPATIVE POWERS OF G14 AND G16 FOR THE 0.5 Hz MODE**

|        | Calculated from small-signal model | Estimated from time-domain responses |
|--------|-----------------------------------|-------------------------------------|
|        | \( \{ \hat{W}_{f,16,r} \} \)          | \( \{ \hat{W}_{f,16,r} \} \)          |
|        | \( \{ \hat{W}_{d,16,r} \} \)          | \( \{ \hat{W}_{d,16,r} \} \)          |
| G14    | 0.2150                             | 0.2070                              |
| G16    | 0.2665                             | 0.2904                              |
| Sum    | 0.4815                             | 0.4974                              |

VII. CONCLUSIONS

A mathematical proof for the equality of total damping power of the system and the total power dissipation in generators was presented for multimachine systems. It was demonstrated that the equality while true when added over all generators, does not hold for individual machines. Thereafter, distribution factors were derived representing the dissipative contributions from different generators in constituting the damping power of each machine.
Similarly, it can be shown that,

\[
D_{12}(i, i) = V_{io} \frac{\partial f_i}{\partial \theta_j} \bigg|_{T, \omega} = \frac{\partial g_{ii}}{\partial \theta_j} \bigg|_{T, \omega} = D_{21}(i, i). \tag{39}
\]

Therefore, from eqns \((38a), (38b), (39)\), it can be inferred that \(D_{12} = D_{21}\). Additionally, from eqns \((38c) - (38d)\), \(D_{11} = D_{11}\) and \(D_{22} = D_{22}\). Thus,

\[
D^T = \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{bmatrix} = \begin{bmatrix} D_{11}^T & D_{12}^T \\ D_{12}^T & D_{22}^T \end{bmatrix} = D. \tag{40}
\]

Further, \(D\) being real and symmetric implies \(D^{-1}\) is also real and symmetric.

\[
\Rightarrow D^{-T} = D^{-1}. \tag{41}
\]

**Proof of Claim (1):** Recall, \(A = M - ND^{-1}C\). Therefore,

\[
A_{33} = M_{33} - \begin{bmatrix} N_{31} & N_{32} \end{bmatrix} D^{-1} \begin{bmatrix} C_{13} \\ C_{23} \end{bmatrix} \tag{42}
\]

From eqns \((3)\) and \((4)\) observe that, \(\forall j \neq i,\)

\[
N_{31}(i, j) = \left. \frac{\partial E_{i}'}{\partial \theta_j} \right|_{T, \omega} = 0; \quad C_{13}(i, j) = \left. \frac{\partial f_i}{\partial E_{i}'} \right|_{T, \omega} = 0; \tag{43}
\]

and for elements on the principal diagonal,

\[
N_{31}(i, i) = \left. \frac{\partial E_{i}'}{\partial \theta_i} \right|_{0} = \frac{V_{io} \sin(\delta_i - \theta_0)}{x_{di} - x_{d'j}} \left( \frac{x_{d'} - x_{d'}}{T_{do}} \right) \tag{44a}
\]

\[
C_{13}(i, i) = \left. \frac{\partial f_i}{\partial E_{i}'} \right|_{0} = \frac{V_{io} \sin(\delta_i - \theta_0)}{x_{di}'} \tag{44b}
\]

Further note, \(N_{31}\) and \(C_{13}\) are rectangular matrices of dimensions \(R_{n_1 \times n}\) and \(R_{n_1 \times n_2}\) respectively. Therefore, combining eqns \((43)\) and \((44)\) we get

\[
P^{-1}C_{13}^T = N_{31} \tag{45}
\]

where, \(P\) is a diagonal matrix with \(P_{i, i} = \frac{T_{do}}{x_{di} - x_{d'i}}\).

Similarly, from eqns \((3)\) and \((5)\), \(\forall j \neq i,\)

\[
N_{32}(i, j) = \left. \frac{\partial E_{j}'}{\partial \theta_i} \right|_{0} = 0; \quad C_{23}(i, j) = \left. \frac{\partial g_{i}}{\partial E_{j}'} \right|_{0} = 0; \quad \text{and}
\]

\[
N_{32}(i, i) = \left. \frac{\partial E_{i}'}{\partial \theta_i} \right|_{0} = \frac{V_{io} \cos(\delta_i - \theta_0)}{x_{di}} \left( \frac{x_{d'} - x_{d'}'}{T_{do}} \right) \tag{46a}
\]

\[
C_{23}(i, i) = \left. \frac{\partial g_{i}}{\partial E_{i}'} \right|_{0} = \frac{V_{io} \cos(\delta_i - \theta_0)}{x_{di}'} \tag{46b}
\]

Therefore, following arguments as before,

\[
P^{-1}C_{23}^T = N_{32} \tag{46}
\]

Finally, observe that \(M_{33} \in R_{n_1 \times n_2}\) with \(M_{33}(i, j) = \left. \frac{\partial E_{j}'}{\partial \theta_i} \right|_{0} = 0 \quad \forall j \neq i.\) This implies \(M_{33}\) is diagonal.

Using the results \((41), (45)\) and \((46)\), eqn \((42)\) can be rewritten as

\[
P^{-1} A_{33} P = P^{-1} M_{33} P - P^{-1} \begin{bmatrix} C_{13}^T \\ C_{23}^T \end{bmatrix} D^{-T} \begin{bmatrix} M_{31}^T \\ M_{32}^T \end{bmatrix} P
\]

\[
= M_{33} - \begin{bmatrix} N_{31} & N_{32} \end{bmatrix} D^{-1} \begin{bmatrix} C_{13} \\ C_{23} \end{bmatrix} = A_{33}. \tag{47}
\]

This concludes the proof. \(\square\)

**Proof of Claim (2):** It can be seen from eqns \((3)\) and \((4)\) that blocks \(M_{31}\) and \(M_{23}\) are diagonal. Also,

\[
M_{31}(i, i) = \left. \frac{\partial E_{i}'}{\partial \theta_i} \right|_{0} = \frac{V_{io} \sin(\delta_i - \theta_0)}{x_{di} - x_{d'i}} \left( \frac{x_{d'} - x_{d'}}{T_{do}} \right) \tag{47a}
\]

\[
M_{23}(i, i) = \left. \frac{\partial g_{i}}{\partial E_{i}'} \right|_{0} = \frac{V_{io} \sin(\delta_i - \theta_0)}{2H x_{di}'} \omega_s \tag{47b}
\]

Therefore, we may write

\[
M_{31}^T P = \frac{2H}{\omega_s} M_{23} \tag{48}
\]

where \(H\) is diagonal with \(H(i, i) = H_i\).

Now, as before, for \(N\) and \(C\) matrices,

\[
N_{21}(i, j) = \left. \frac{\partial \omega_s}{\partial \theta_j} \right|_{0} = 0; \quad C_{11}(i, j) = \left. \frac{\partial f_{i}}{\partial \omega} \right|_{0} = 0. \quad \text{Also,}
\]

\[
N_{21}(i, i) = \left. \frac{\partial \omega_s}{\partial \theta_i} \right|_{0} = \frac{\omega_s E_{i}'}{2H x_{di}'} \sin 2(\delta_i - \theta_0) \left( \frac{x_{d} - x_{d'}}{2H x_{di}'} \omega_s \right) \tag{47c}
\]

\[
C_{11}(i, i) = \left. \frac{\partial f_{i}}{\partial \omega} \right|_{0} = \frac{E_{i}'}{2H} \sin 2(\delta_i - \theta_0) \left( \frac{x_{d} - x_{d'}}{x_{d'} x_{d'}} \right) \tag{47d}
\]

Combining these with the fact that, \(N_{21} \in R_{n_1 \times n}\) and \(C_{11} \in R_{n_1 \times n_2}\) we may write,

\[
C_{11} = \frac{2H}{\omega_s} N_{21}. \tag{49}
\]

Similarly, it can be shown that

\[
C_{21} = \frac{2H}{\omega_s} N_{22}. \tag{50}
\]

Now, recall \(A = M - ND^{-1}C\). Therefore,

\[
A_{31} = M_{31} - \begin{bmatrix} N_{31} & N_{32} \end{bmatrix} D^{-1} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} \tag{51}
\]

Next, substituting eqns \((41)\) and \((45)\) - \((50)\) in \((51)\)

\[
A_{31}^T = \frac{2H}{\omega_s} M_{23} - \frac{2H}{\omega_s} \begin{bmatrix} N_{21} & N_{22} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} \\ C_{23} \end{bmatrix} \tag{52}
\]

This concludes the proof. \(\square\)

**Proof of Claim (3):** As before, observe from eqn \((2)\) that \(M_{21}\) is diagonal. Therefore, we may write

\[
H M_{21} = M_{21}^T H. \tag{53}
\]

Also,

\[
A_{21} = M_{21} - \begin{bmatrix} N_{21} & N_{22} \end{bmatrix} D^{-1} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} \tag{54}
\]

\[
H A_{21} = A_{21}^T H. \tag{55}
\]

This concludes the proof. \(\square\)
Finally, substituting eqns (41), (49), (50), and (53) in (54)

\[ 2 A_{21}^T H = 2 H M_{21} - 2 H \left[ N_{21} N_{22} \right] D^{-1} \left[ C_{11} \ C_{21} \right] \]

\[ = 2 H A_{21}. \]

This concludes the proof. □

**Proof of Claim (4)**: From the definition of \( K_r \) and (14),

\[ K_r^T = -\left\{ \frac{2}{\omega_d} A_{21}^T H + A_{31}^T (j \omega_d I - A_{33}^{-1}) \left( \frac{2}{\omega_d} A_{23}^T H \right) \right\} \frac{1}{j \omega_d} = K_r \]

Now, \( \forall \ x \in \mathbb{C}^{n_r} \), let us decompose it into its real and imaginary parts as shown: \( x = x_1 + j x_2 \). Therefore,

\[ \mathbf{x}^H \mathbf{K}_r \mathbf{x} = (x_1^2 - j x_2^2) \left( \mathbb{R}(K_r) + j \mathbb{I}(K_r) \right) (x_1 + j x_2) \]

Further, using eqn (56), we can infer on the symmetry of both the real and imaginary parts of \( K_r \). Hence,

\[ x_1^2 \mathbb{R}(K_r) x_2 = x_2^2 \mathbb{R}(K_r) x_1 \]

This reduces the real part of eqn (57) as follows

\[ \mathbb{R}(\mathbf{x}^H \mathbf{K}_r \mathbf{x}) = x_1^2 \mathbb{R}(K_r) x_1 + x_2^2 \mathbb{R}(K_r) x_2. \]

Finally, using the real part of eqn (58)

\[ \mathbb{R}(\mathbf{x}^H \mathbf{K}_r \mathbf{x}) = x_1^2 \mathbb{R}(K_r) x_1 + x_2^2 \mathbb{R}(K_r) x_2 + j x_1^2 \mathbb{I}(K_r) x_2 - j x_2^2 \mathbb{I}(K_r) x_1 \]

\[ = x^H \mathbb{R}(K_r) x. \]

This concludes the proof. □

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