ON NONTRIVIALITY OF HOMOTOPY GROUPS OF SPHERES

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Abstract. For $n \geq 2$, the homotopy groups $\pi_n(S^2)$ are non-zero.

1. Introduction

In [5], E. Curtis proved that $\pi_n(S^4) \neq 0$, for all $n \geq 4$. The main method from [5] of proving that a given element of the homotopy groups of spheres is non-zero is the analysis of Adams' $d$ and $e$-invariants of the stabilization of either that element or its Hopf image. This method allowed E. Curtis to prove that (see [5])

$$\pi_n(S^2) \neq 0, \ n \neq 1 \mod 8$$

The same results on non-vanishing terms of the homotopy groups of spheres were obtained with the help of the composition method by M. Mimura, M.Mori and N. Oda [12]. Using the methods of the stable homotopy theory, the analysis of the image of the $J$-homomorphism and $K$-theory, it was shown by M. Mahowald [10,11] and M. Mori [13] that

$$\pi_n(S^5) \neq 0, \ n \geq 5. $$

From the other hand, since the fourth stable homotopy group of spheres is zero, one can not get such kind of result for higher spheres, indeed $\pi_{n+4}(S^n) = 0, \ n \geq 6$. The only remaining case to consider when such kind of phenomena can happen is the case of $S^2$ and $S^3$. The main result of this paper is the following

Theorem 1. For $n \geq 2$, the homotopy groups $\pi_n(S^2)$ are non-zero.

Since $\pi_n(S^3) = \pi_n(S^2)$, $n \geq 3$, the same result follows for the homotopy groups $\geq 3$ of the 3-sphere.

In the proof of theorem [11] we cover the gaps in dimensions $\equiv 1 \mod 8$ by showing that, for any odd prime $p$ and $n \geq 2$,

$$\mathbb{Z}/p \subseteq \pi(2p-2)_{n+1}(S^3),$$

In particular,

$$\mathbb{Z}/3 \subseteq \pi_{4n+1}(S^3), \ \mathbb{Z}/15 \subseteq \pi_{8n+1}(S^3).$$

Let $\pi_k^2$ denote the 2-component of $\pi_k(S^n)$. According to [5] the Table on page 543], the 2-component $\pi_k^2 \neq 0$ for $k > 4$. M. Mahowald [11] Theorem 1.6] and M. Mori [13 Corollary 5.12 (iv)] also proved the stronger statement that $\pi_k^2 \neq 0$ for $k > 5$. For the 2-component $\pi_k^3$ of $\pi_*(S^3)$, Curtis proved that $\pi_k^3 \neq 0, \ n \neq 1,2 \mod 8$. The non-triviality of these cases can be also read from the fact that the

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2-local $v_1$-periodic homotopy group $v_1^{-1}\pi_n^2 \neq 0$ if and only if $n \not\equiv 1, 2 \pmod{8}$ by [6, Theorem 4.2]. For the remaining cases of $\pi_n(S^3)$ with $n \equiv 1, 2 \pmod{8}$, notice that the 2-components of $\pi_9(S^3)$ and $\pi_{10}(S^3)$ both vanish, and so it is necessary to fulfill odd primes for having the non-triviality. Indicated from [15, Figure 3.3.18], one could have the following conjecture:

**Conjecture.** The 2-component of $\pi_n(S^3)$ is non-trivial for $n > 10$.

*After writing this paper, the authors became aware of the result from the paper [7]. The gaps in dimensions $\equiv 1 \pmod{8}$ have been covered by a result of Brayton Gray [7, Theorem 12(e)] although Theorem 7 was not aware in [7]. We point out that the method of the present paper for proving Theorem 1 is different from that in [7].*

### 2. Lambda-algebra and Toda Elements

Recall that, for any $k \geq 1$ and an odd prime $p$, the homotopy groups $\pi_{2(p-1)k+2}(S^3)$ contain non-trivial elements $\alpha_k(3)$ called the Toda elements. The elements $\alpha_k(3)$ have non-zero stable images in $\pi_{2(p-1)k+1}(S^3)$. We will use the standard notation

$$\alpha_k(m) = \Sigma^{m-3}(\alpha_k(3)) \in \pi_{2(p-1)k+1+m-1}(S^m), \quad m \geq 3.$$  

There exists a $p$-local EHP sequence

$$J_{p-1}(S^4) \rightarrow \Omega S^5 \xrightarrow{H_p} \Omega S^{2p+1},$$

where $J_{p-1}(S^4)$ is the $(2p-1)$-skeleton of $\Omega S^5$, which implies the long exact sequence of homotopy groups [17, (2.11), p.103]

$$\ldots \rightarrow \pi_{n+1}(S^{4p+1}) \xrightarrow{P} \pi_{n-1}(J_{p-1}(S^4)) \xrightarrow{E} \pi_n(S^5) \xrightarrow{H_p} \pi_n(S^{4p+1}) \rightarrow \ldots.$$  

The following statement seems to be known. For example, there is a discussion of this result at the end of page 535 in [2]. However, we were not able to find an explicit reference to this statement and give here a proof.

**Proposition 2.** For $k \geq 2$, if the image of the map

$$H_p : \pi_{2k(p-1)+4}(S^5) \rightarrow \pi_{2k(p-1)+4}(S^{4p+1})$$

contains the element $\alpha_{k-2}(4p+1)$, then $k \equiv 0 \pmod{p}$.

Let $p$ be a fixed odd prime number. The mod-$p$ lambda algebra $\Lambda = \Lambda$ is an $\mathbb{F}_p$-algebra generated by elements $\lambda_i$ of degree $2(p-1)i-1$ for $i \geq 1$ and elements $\mu_j$ of degree $2(p-1)j$ for $j \geq 0$. We will use the following notations for $a(k,j), b(k,j) \in \mathbb{F}_p$

$$a(k,j) = (-1)^{j+1} \binom{p-1}{j}(p-j-1),$$

$$b(k,j) = (-1)^j \binom{p-1}{j}(p-j),$$

and for for $N(k), N'(k) \in \mathbb{Z}$:

$$N(k) = \left\lfloor k - \frac{k+1}{p} \right\rfloor, \quad N'(k) = \left\lfloor k - \frac{k}{p} \right\rfloor$$

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*During the private circulation of this article, Doug Ravenel wrote a comment that it could be the case that all other 2-components of $\pi_4(S^3)$ are nontrivial except $\pi_9(S^3)$ and $\pi_{10}(S^3)$. 

[1]: A reference to a previous work.

[2]: Another reference to a previous work.

[3]: A third reference to a previous work.
The ideal of relations in $\Lambda$ is generated by the following relations:

$$\lambda_i \lambda_{pi+k} = \sum_{j=0}^{N(k)} a(k, j) \lambda_{i+k-j} \lambda_{pi+j}, \quad i \geq 1, k \geq 0$$

$$\lambda_i \mu_{pi+k} = \sum_{j=0}^{N(k)} a(k, j) \lambda_{i+k-j} \mu_{pi+j} + \sum_{j=0}^{N'(k)} b(k, j) \mu_{i+k-j} \lambda_{pi+j}, \quad i \geq 1, k \geq 0$$

$$\mu_i \lambda_{pi+k+1} = \sum_{j=0}^{N(k)} a(k, j) \mu_{i+k-j} \lambda_{pi+j+1}, \quad i \geq 0, k \geq 0$$

$$\mu_i \mu_{pi+k+1} = \sum_{j=0}^{N(k)} a(k, j) \mu_{i+k-j} \mu_{pi+j+1}, \quad i \geq 0, k \geq 0.$$ 

The differential $\partial : \Lambda \to \Lambda$ is given by

$$\partial \lambda_k = \sum_{j=1}^{N(k)} a(k, j) \lambda_{k-j} \lambda_j,$$

$$\partial \mu_k = \sum_{j=0}^{N(k)} a(k, j) \lambda_{k-j} \mu_j + \sum_{j=1}^{N'(k)} b(k, j) \mu_{k-j} \lambda_j.$$ 

Further by $\nu_i$ we denote an element of $\{\lambda_i, \mu_i\}$. A monomial $\nu_{i_1} \ldots \nu_{i_l}$ is said to be admissible if $i_{k+1} \leq pi_k - 1$ whenever $\nu_{i_k} = \lambda_i$ and if $i_{k+1} \leq pi_k$ whenever $\nu_{i_k} = \mu_i$. The set of admissible monomials is a basis of $\Lambda$. The unstable lambda algebra $\Lambda(n)$ is a dg-subalgebra of $\Lambda$ generated by admissible elements $\nu_{i_1} \ldots \nu_{i_l}$ such that $i_1 \leq n$. We denote by $\Lambda(n)_m$ the subspace generated by monomials of degree $m$ in $\Lambda(n)$ and by $\Lambda(n)_{m,l}$ the vector space generated by monomials of length $l$ in $\Lambda(n)_m$. Then

$$\Lambda(n) = \bigoplus_{m,l} \Lambda(n)_{m,l}, \quad \Lambda(n)_m = \bigoplus_l \Lambda(n)_{m,l}.$$ 

Consider the left ideal of $\Lambda$:

$$(1) \quad \Lambda \Lambda = \sum_i \Lambda \lambda_i.$$ 

The set of all admissible monomials $\nu_{i_1} \ldots \nu_{i_l}$ such that $\nu_{i_l} = \lambda_i$ forms a basis of $\Lambda \Lambda$. Further we put

$$\Lambda \lambda(n) = \Lambda \lambda \cap \Lambda(n).$$

There exists a spectral sequence which converges to the $p$-primary components of the homotopy groups of spheres, whose $E^1$-page is the lambda-algebra and $d^1$-differential is the differential in the lambda algebra:

$$E^1(n) = \Lambda \lambda(n) \Rightarrow (p)_* \pi_*(S^{2n+1}).$$

This is an integral version of the well-known lower central series spectral sequence of six authors [3]. This spectral sequence was considered in details in the thesis of D. Leibowitz [9].

In the language of lambda-algebra, the elements $\alpha_k$ can be presented as (see, for example 2.9 [10]) $\mu_1^{k-1} \lambda_1$. The map $H_p : \Omega S^5 \to \Omega S^{4p+1}$ induces the map $h_p$ on
the level of $E^1$-terms of the spectral sequence (see page 23 in [16], and also [19] and [8]) with the short exact sequence
\[
0 \to \Lambda(1) \oplus \lambda_2 \Lambda(5) \to \Lambda(2) \xrightarrow{h_p} \Lambda(2p) \to 0,
\]
for any $\alpha$ and
\[
h_p(\mu_2 \alpha) = h_p(\lambda_1 \alpha) = h_p(\lambda_2 \alpha) = 0.
\]

**Lemma 3.** The linear map
\[
d_1 : \text{span}(\mu_1^k \lambda_2, \{\mu_1^{k-i} \mu_1^{i-1} \lambda_1\}_{i=1}^k) \to \text{span}(\{\mu_1^{k-i} \lambda_1 \mu_1^i \lambda_1\}_{i=0}^k)
\]
is an isomorphism if and only if $k + 2 \not\equiv 0 \pmod{p}$.

**Proof.** Using the definition of $d_1 = : d$ we get
\[
d(\lambda_1) = 0, \quad d(\mu_1) = -\lambda_1 \mu_0, \quad d(\mu_2) = -2\lambda_1^2, \quad d(\mu_2) = -\lambda_2 \mu_0 - 2\lambda_1 \mu_1 + \mu_1 \lambda_1.
\]

Using the relations $\mu_0 \mu_1 = 0 = \mu_0 \lambda_1$ and $\mu_0 \mu_2 = -\mu_1 \lambda_1$, $\mu_0 \mu_2 = -\mu_1 \mu_1$, it is easy to compute that
\[
d(\mu_2 \lambda_1) = -2\lambda_1 \mu_1 \lambda_1 + \mu_1 \lambda_1^2,
\]
\[
d(\mu_1 \lambda_2) = \lambda_1 \mu_1 \lambda_1 - 2\mu_1 \lambda_1^2,
\]
\[
d(\mu_2 \lambda_1) = \lambda_1 \mu_1 \lambda_1 - 2\mu_1 \mu_1 \lambda_1 + \mu_1 \lambda_1^2.
\]

Moreover, we obtain $d(\mu_1) \mu_1 = 0$ and $d(\mu_1) \lambda_1 = 0$. It follows that
\[
d(\mu_1^k \lambda_2) = \mu_1^{k-1} d(\mu_1 \lambda_2) = \mu_1^{k-1} \lambda_1 \mu_1 \lambda_1 - 2\mu_1^k \lambda_1^2,
\]
\[
d(\mu_1^{k-1} \mu_2 \lambda_1) = \mu_1^{k-2} d(\mu_1 \mu_2 \lambda_1) = \mu_1^{k-2} \lambda_1 \mu_1 \lambda_1 - 2\mu_1^{k-1} \lambda_1 \mu_1 \lambda_1 + \mu_1^k \lambda_1^2.
\]
\[
d(\mu_1^{k-i} \mu_2 \mu_1^i \lambda_1) = \mu_1^{k-i-2} d(\mu_1) \mu_2 \mu_1^i \lambda_1 + \mu_1^{k-i-1} d(\mu_2) \mu_1^i \lambda_1 = \mu_1^{k-i-2} \lambda_1 \mu_1 \lambda_1 - 2\mu_1^{k-i-1} \lambda_1 \mu_1^i \lambda_1 + \mu_1^{k-i} \lambda_1 \mu_1^i \lambda_1
\]
for $1 \leq i \leq k - 2$ and
\[
d(\mu_2 \mu_1^{k-1} \lambda_1) = d(\mu_2) \mu_1^{k-1} \lambda_1 = -2\lambda_1 \mu_1 \lambda_1 + \mu_1 \lambda_1 \mu_1^{k-1} \lambda_1.
\]

If we denote $v_i := \mu_1^{k-i} \lambda_1 \mu_1^i \lambda_1$ for $0 \leq i \leq k$, and $u_i = \mu_1^{k-i} \mu_2 \mu_1^{i-1} \lambda_1$ for $1 \leq i \leq k$ and $u_0 = \mu_1^k \lambda_2$, then
\[
d(u_0) = v_1 - 2v_0, \quad d(u_i) = v_{i+1} - 2v_i + v_{i-1}, \quad d(u_k) = -2v_k + v_{k-1}.
\]

The matrix corresponding to this linear map is the following matrix
\[
\begin{pmatrix}
-2 & 1 & 0 & 0 & \ldots & 0 \\
1 & -2 & 1 & 0 & \ldots & 0 \\
0 & 1 & -2 & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 0 & 1 & -2
\end{pmatrix}.
\]

It is easy to check by induction that its determinant is equal to $(-1)^{k+1}(k + 2)$. It follows that $d : \text{span}(u_0, \ldots, u_k) \to \text{span}(v_0, \ldots, v_k)$ is an isomorphism if and only if $k + 2 \not\equiv 0 \pmod{p}$.

Now we are ready to prove proposition [2]
Proof of proposition 2. Indeed, if the Toda element $\alpha_{k-2} = \mu_1^{k-3}\lambda_1$ lies in the $H_p$ image, then there must be some term on $E^2$-page like $C\mu_2\mu_1^{k-3}\lambda_1 + \sum \ldots$. $C \neq 0 \mod p$, which maps onto $\alpha_{k-2}$ by $h_p$. However, by lemma 2, this is possible only in the case $k \equiv 0 \mod p$, in all other cases the corresponding $E^2$-term of the spectral sequence for $S^k$ is zero. \hfill \Box

3. Proof of theorem 4

For the proof of theorem 4 we will use the following classical results in homotopy theory.

(1) [1] or [17] (4.3), p.112. The element $\alpha_k \in \pi_{2(p-1)k-1}^S$ is not divisible by $p$ for $k \neq 0 \mod p$.

(2). Let $p > 2$. By the classical work of Cohen, Moore and Neisendorfer [14], there exists a map $\pi: \Omega^2S^{2n+1} \to S^{2n-1}$ such that the composite

$$\Omega^2S^{2n+1} \xrightarrow{\pi} S^{2n-1} \xrightarrow{\Sigma^2} \Omega^2S^{2n+1}$$

is homotopic to the $p$-th power map $p: \Omega^2S^{2n+1} \to \Omega^2S^{2n+1}$, where the case $p = 3$ is given in [13] Theorem 4.1. Following the notation in [14], let $D(n)$ be the homotopy fibre of $\pi: \Omega^2S^{2n+1} \to S^{2n-1}$. According to [14] Section 6, $D(p) \simeq \Omega^pS^3\langle 3 \rangle$ and so there is a fibre sequence

$$\Omega^2S^{2p-1} \xrightarrow{\pi} \Omega^2S^3\langle 3 \rangle \xrightarrow{\theta} \Omega^2S^{2p+1} \xrightarrow{\pi} S^{2p-1}$$

that implies a long exact sequence

$$\ldots \longrightarrow \pi_{n+1}(S^{2p+1}) \longrightarrow \pi_n(S^{2p-1}) \longrightarrow \pi_n(S^3) \longrightarrow \pi_n(S^{2p+1}) \longrightarrow \ldots$$

with the property that, for every $i$, the composition

$$\pi_{i+2}(S^{2p+1}) \longrightarrow \pi_i(S^{2p-1}) \longrightarrow \pi_i(S^3) \longrightarrow \pi_i(S^{2p+1})$$

is the multiplication by $p$.

(3) [17] (2.12), p. 104. For $m \geq 2$, denote by $Q_2^{2m-1}$, the homotopy fibre of the double suspension map $S^{2m-1} \to \Omega^2S^{2m+1}$. We will use the notation from [17]. The natural map $Q_2^{2m-1} \to S^{2m-1}$ induces the map on homotopy groups $p_*$. There is a natural map

$$I: \pi_i(Q_2^{2m-1}) \longrightarrow \pi_{i+3}(S^{2mp+1}),$$

such that the composition

$$\pi_{i+3}(S^{2m+1}) \longrightarrow \pi_i(Q_2^{2m-1}) \xrightarrow{I} \pi_{i+3}(S^{2mp+1})$$

is the Hopf map $H_p$.

For a given $k \neq 1 \mod p$, consider the element

$$\alpha_{k-1} \in \pi_{2(p-1)(k-1)+2p-1}(S^{2p-1}) = \pi_{2(p-1)k}(S^{2p-1}).$$
Suppose that \( \alpha_{k-1}(2p-1) \in im\{\pi_* : \pi_{2(p-1)k+2}(S^{2p+1}) \to \pi_{2(p-1)k}(S^{2p-1})\}. \) Then the element \( \Sigma^2\alpha_{k-1}(2p-1) = \alpha_{k-1}(2p+1) \) is \( p \)-divisible by (2), hence its stable image is \( p \)-divisible. But this is not possible by (1). We conclude that \( \tau_*(\alpha_{k-1}(2p-1)) \neq 0 \)

by the long exact sequence in (2), and so

\[
(A) \quad \mathbb{Z}/p \subseteq \pi_{2(p-1)k+1}(S^3), \ k \neq 1 \mod p
\]

Now we recall the following statement of Toda (Theorem 5.2 (ii) [17], case \( m = 1 \)). For \( k \geq 2 \), there exist an element

\[
\gamma' \in \pi_{2p+2k(p-1)-1}(Q_2^3) = \pi_{2(p-1)(k+1)+1}(Q_2^3)
\]

such that

\[
I(\gamma') = \alpha_{k-1}(4p + 1) \in \pi_{4p+2k(1-p)}(S^{4p+1}).
\]

Here \( I : \pi_{2p+2k(p-1)-1}(Q_2^3) \to \pi_{2p+2k(p-1)+2}(S^{4p+1}) \). Suppose that \( p_*(\gamma') = 0 \), then

\[
\gamma' \in im\{H^2 : \pi_{2(p-1)(k+1)+4}(S^5) \to \pi_{2(p-1)(k+1)+1}(Q_2^3)\}.
\]

In this case, we get

\[
\alpha_{k-1}(4p + 1) \in im\{H : \pi_{2(k+1)(p-1)+4}(S^5) \to \pi_{2(k+1)(p-1)+1}(S^3)\}.
\]

This is possible only for \( k+1 \equiv 0 \mod p \) by proposition 2. For \( k+1 \neq 0 \mod p \), we get \( 0 \neq p_*(\gamma') \in \pi_{2(p-1)(k+1)+1}(S^3) \). Therefore,

\[
(B) \quad \mathbb{Z}/p \subseteq \pi_{2(p-1)k+1}(S^3), \ k \neq 0 \mod p.
\]

The statements (A) and (B) together give the needed statement:

\[
\mathbb{Z}/p \subseteq \pi_{2(p-1)k+1}(S^3), \ k \geq 1.
\]

Theorem 1 now follows, since all dimensions \( \equiv 1 \mod 8 \) are covered, moreover there is a \( \mathbb{Z}/15 \)-summand in homotopy groups \( \pi_{8l+1}(S^2), \ l \geq 2 \). \( \square \)

As a final remark we observe that homotopy groups of \( S^2 \) in certain dimension \( \equiv 1 \mod 8 \) can be covered in another way. For that, we recall the results from [17] and [13].

(iv) (Lemma 15.3 (i), [18]) Let \( y \in \pi_i(S^{2p-1}) \) be an element of order \( p \). There exists an element \( a \in \pi_{i+2}(S^3) \), such that

\[
H_p(a) = x \Sigma^2 y \in \pi_{i+2}(S^{2p+1})
\]

for some \( x \neq 0 \mod p \).

(v) For \( f \geq 0 \), there is a family of elements \( \alpha_{i}^{(f)} \in \pi_{2i(p-1)p^f+2f+2}(S^{2^jf+3}) \) of order \( p^f \), which have non-zero stable image in \( \pi_{2i(p-1)p^f-1}^S \). The \( e \)-invariants of these elements are the following: \( e_C(\alpha_{i}^{(f)}) = -p^{-f-1} \).

(vi) (Lemma 4.1, [13]) Let \( f, g \geq 0, \ i, j \geq 1 \) and

\[
\alpha : S^{2n+2i(p-1)p^f-1} \to S^{2n},
\]

\[
\beta : S^{2n+2i(p-1)p^f+2j(p-1)p^g-2} \to S^{2n+2i(p-1)p^f-1}.
\]
Assume that $e C(\alpha) e C(\beta) = p^{-u}$ and
\[ \nu_p(j) + g + 1 < u \leq \nu_p(i) + f + 1 + i(p-1)p^f, \]
\[ u + \nu_p(ip^f + jp^g) - \nu_p(i) - f - i(p-1)p^f \leq n < u + \nu_p(ip^f + jp^g) - \nu_p(j) - g, \]
then $\alpha \circ \beta$ non zero.

Now we will show that, for any $k \geq 1$, there is a non-zero $p$-torsion element in $\pi_{2(p-1)(p^k+1)}(S^3)$. For that, consider the case $g = 0, f = p - 2, i = p^2 k - 1, j = p^{p-2}$. By (vi), we see that, $\alpha_i^{(p-2)} \circ \alpha_j^{(0)}$ is a non-zero element in homotopy group $\pi_{2(p-1)p^k+1}(S^{2p+1})$ which equals to the image of the double suspension of an element of order $p$ from $\pi_{2(p-1)p^k+2p-3}(S^{2p-1})$. Hence, by (iv), there is an element in $\pi_{2(p-1)p^k+1}(S^3) = \pi_{2(p-1)(p^k+1)}(S^3)$ whose $H_p$-image gives a non-zero multiple of this element.

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