New category of the Fuzzy $d$-algebras

Shuker Mahmood Khalil

Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq

1. Introduction

In 1966, two classes of abstract algebras were introduced – the first one was $BCK$-algebra, which was introduced by Imai and Iseki [1] and another was $BCI$-algebra, which was introduced by Iseki [2]. After that, a wide class of abstract algebras ($BCH$-algebras) were introduced by Hu [3] in 1983 and Li [4] in 1985. Further, they showed that the classes of $BCK$-algebras were proper subclasses of the classes of $BCI$-algebras and the classes of $BCI$-algebras were proper subclasses of the classes of $BCH$-algebras. Next, some new classes of algebras are given ([5–8]). A Fuzzy set was a class of objects with a continuum of grades of membership was a concept proposed by Zadeh [9] in 1965. After the introduction of Fuzzy topology by Chang [10] in 1968, there have been several generalizations of notions of Fuzzy set and Fuzzy topology.

By adding the degree of non-membership to fuzzy set, Atanassov [11] proposed intuitionistic fuzzy set (IFS) in 1986 which looks more accurate to uncertainty quantification and provides the opportunity to precisely model the problem based on the existing knowledge and observations. In recent years, some interesting studies on these sets have been discussed by Shuker [12–15] and others.

In 1993, Jun [16] combined the structure of Fuzzy topological spaces with that of a Fuzzy $BCK$-algebras to formulate the elements of a theory of Fuzzy topological $BCK$-algebras. In 1999, the concept of $d$-algebra, which is another generalization of $BCK$-algebras, is introduced by Neggers and Kim [17]. Also, the notion of $d$-ideal in $d$-algebra is discussed by Jun, Neggers and Kim [18]. After that, they introduced the notions of fuzzy $d$-subalgebra, fuzzy $d$-ideal, fuzzy $d^*$-ideal, fuzzy $d^*$-ideal, fuzzy $B$-algebras, fuzzy $BCI$-algebras* and the relations among them are shown, see [19–21].

In this paper, new construction of Fuzzy $d$-algebra is called $\Gamma$-Fuzzy $d$-algebra, this notation is more useful than Fuzzy $d$-algebra because the degree of membership of any $x \in X$ in the Fuzzy set does not depend on the structure of $d$-algebra $X$. Therefore, in this work, we introduced a new construction of Fuzzy $d$-algebra that is dependent on the structure of $d$-algebra. Moreover, in this paper, we show that for any $d$-algebra there is a $\Gamma$-Fuzzy $d$-algebra. Further, the interesting relation between $\Gamma$-Fuzzy $d$-algebra and edge $d$-algebra are shown. For any $d$-algebra, we determined unique $\Gamma$-Fuzzy topological space $(X, \Gamma^d)$ using $\Gamma$-Fuzzy $d$-algebra. Also, in this work, for any $f \in X$ a new class of $d$-algebra $(\Psi(X), \tau, (X, \otimes_I, f)^{\infty})$ is introduced. Finally, a new class of IFS called intuitionistic $\Gamma$-Fuzzy $d$-algebra is investigated and discussed.

2. Definitions and notations

The following definitions have been used to obtain the results and properties developed in this paper.

**Definition 2.1**: [17] A $d$-algebra is a nonempty set $X$ with a constant 0 and a binary operation* satisfying the following axioms:
be defined as follows

\[ a \ast x = \{ ax | x \in X \} \]

Then \( X \) is said to be edge if \( a \ast X = \{ a, a \} \), for all \( a \in X \).

**Definition 2.3:** [9] For a set \( X \), we define a Fuzzy set in \( X \) to be a function \( \mu : X \to [0, 1] \). Here, \( \mu(x) \) "represents the degree of membership of \( x \) in the Fuzzy set \( A \).

[Note]: Any subset \( A \) of a set \( X \) can be identified with its characteristic function \( \chi_A : X \to \{0, 1\} \) defined by

\[ \chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \]

and such characteristic functions are Fuzzy sets in \( X \).

**Definition 2.4:** [9] The characteristic functions of subsets of a set \( X \) are referred to as the crisp Fuzzy sets in \( X \).

**Definition 2.5:** [9] The union (respectively, intersection) of the Fuzzy sets \( \mu_i \) \( i \in I \) is defined by

\[ \bigvee_{i \in I} \mu_i(x) = \sup_{i \in I} \mu_i(x) \]

(respectively, \( \bigwedge_{i \in I} \mu_i(x) = \inf_{i \in I} \mu_i(x) \)).

**Definition 2.6:** [10] A Fuzzy topology on a set \( X \) is a collection \( \delta \) of Fuzzy sets in \( X \) satisfying:

(i) \( 0 \in \delta \) and \( 1 \in \delta \),
(ii) if \( \mu \) and \( \nu \) belong to \( \delta \), then so does \( \mu \land \nu \), and
(iii) if \( \mu \) belongs to \( \delta \) for each \( i \in I \), then so does \( \bigvee_{i \in I} \mu_i \).

If \( \delta \) is a Fuzzy topology on \( X \), then the pair \((X, \delta)\) is called a Fuzzy topological space. Members of \( \delta \) are called open Fuzzy sets. Fuzzy sets of the form \( 1 - \mu \), where \( \mu \) is an open Fuzzy set, are called closed Fuzzy sets.

**Definition 2.7:** [11] An IFS \( A \) over the universe \( X \) can be defined as follows \( A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X \} \) where \( \mu_A(x) : X \to [0, 1] \), \( \nu_A(x) : X \to [0, 1] \) with the property \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \), \( \forall x \in X \). The values \( \mu_A(x) \) and \( \nu_A(x) \) represent the degree of membership and non-membership of \( x \) to \( A \), respectively.

**Definition 2.8:** [11] Let \( \pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \) be the IFS index or hesitation margin of \( x \) in \( A \) is the degree of indeterminacy of \( x \in X \) to the IFS \( A \) and \( \pi_A(x) \in [0, 1] \), i.e. \( \pi_A(x) : X \to [0, 1] \) and \( 0 \leq \pi_A \leq 1 \) for every \( x \in X \).

**Remark 2.9:** [11] \( \pi_A(x) \) expresses the lack of knowledge of whether \( x \) belongs to IFS \( A \) or not. For example, let \( A \) be an IFS with \( \mu_A(x) = 0.5 \) and \( \nu_A(x) = 0.3 \) \( \Rightarrow \pi_A(x) = 1 - (0.5 + 0.3) = 0.2 \). It can be interpreted as "the degree that the object \( x \) belongs to IFS \( A \) is 0.5, the degree that the object \( x \) does not belong to IFS \( A \) is 0.3 and the degree of hesitancy is 0.2".

### 3. Note on fuzzy \( d \)-algebras

In this section, we will explain that in the fuzzy \( d \)-algebra \( X \) the degree of membership of any \( x \in X \) in the fuzzy set does not depend on structure of \( d \)-algebra \( X \).

**Definition 3.1:** [19] A Fuzzy set \( A \) in a \( d \)-algebra \( X \) with membership function \( \mu_A \) is called a Fuzzy \( d \)-algebra of \( X \) if \( \mu_A(x \ast y) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x, y \in X \).

**Example 3.2:** Let \( X = \{0, a, b, c\} \) be a set with binary operation \( \ast \) defined in Table 1. Then \((X, \ast, 0)\) is a \( d \)-algebra. Define a Fuzzy set \( A \) in \( X \) with membership function \( \mu_A \) by function \( \mu_A(0) = \mu_A(a) = \mu_A(c) = 1 = \mu_A(b) \) and \( \mu_A(b) = 0, \forall \lambda > 0 \). Then \( A \) is a Fuzzy \( d \)-algebra of \( X \). From definition (3.1) and Table 1, we consider that the values \( \lambda \) and \( \rho \) have already existed and hence these values do not depend on the structure of \( d \)-algebra \( X \). On the other hand, let \((X, \ast, 0)\) be \( d \)-algebra with Table 1 and define a Fuzzy set \( B \) in \( X \) with membership function \( \mu_B \) by function \( \mu_B(0) = \mu_B(a) = \lambda \) and \( \mu_B(c) = \mu_B(b) = \rho \). Then \( B \) is a Fuzzy set in \( d \)-algebra \( X \), but \( \mu_B(0) < \min \{ \mu_B(x), \mu_B(y) \} \) and hence \( B \) is not Fuzzy \( d \)-algebra. However, \( X \) is \( d \)-algebra. Further, if \( X = \{0, a, b, c\} \) is a set with binary operation \( \ast \) defined in Table 2. Then \((X, \ast, 0)\) is not \( d \)-algebra. However, for Table 2, we consider that \( \mu_A(x \ast y) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x, y \in X \). That means there is no relation between Fuzzy set and the structure of \( d \)-algebra. This fact is also true for the structure of edge \( d \)-algebra. Therefore, in this paper, we will introduce new constructions of Fuzzy \( d \)-algebra that is dependent on the structure of \( d \)-algebra is called \( \Gamma \)-Fuzzy \( d \)-algebra and explain its application on edge \( d \)-algebra.

### 4. \( \Gamma \)-Fuzzy \( d \)-algebras

In this section, we will introduce a new concept called \( \Gamma \)-Fuzzy \( d \)-algebra and show that for any \( d \)-algebra
there is a $\Gamma$-Fuzzy $d$-algebra. Further, this new construction helps us to find the interesting relation between $\Gamma$-Fuzzy $d$-algebra and edge $d$-algebra.

**Definition 4.1:** Let $(X, \ast, 0)$ be a $d$-algebra. Then for any $a \in X - \{0\}$, defined the following $\Gamma(a) = \{x \in X \mid a \ast x = a\}$ and $\Gamma^*(a) = \{x \in X \mid ax = 0\}$. Now, let $|X|$, $|\Gamma(a)|$ and $|\Gamma^*(a)|$ be referred to the cardinalities of their sets. Define $\Gamma(X) \to \{0, 1\}$ as follows: $\Gamma(x) = \begin{cases} \frac{|\Gamma(x) + |\Gamma^*(x)|}{|X|}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0 \end{cases}$ $x \in X$. Then $\Gamma$ is said to be $\Gamma$-Fuzzy $d$-algebra.

**Remarks 4.2:**

(1) Every $d$-algebra determines unique $\Gamma$-Fuzzy $d$-algebra. Also, for any $(X, \ast, 0)$ $d$-algebra, its $\Gamma$-Fuzzy $d$-algebra depends on its constructs.

(2) If $X$ is an infinite set, then $\Gamma(x)$ is defined by $\Gamma(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0 \end{cases}$ for all $x \in X$.

**Example 4.3:** For any positive integer $n > 1$. Let $\Omega = \{0, 1, 2, \ldots, p\}$ be a finite set and $|\Omega| = p + 1$. Define binary operation ($\ast$) on $\Omega$ as follows: $X \ast y = \begin{cases} y, & \text{if } x = y \text{ or } x = 0, \\ 0, & \text{if } x \neq y \text{ and } x = 0. \end{cases}$ Then $(\Omega, \ast, 0)$ is a $d$-algebra with Table 3.

Then $\Gamma$-Fuzzy $d$-algebra of $X$ is $\Gamma(x) = \begin{cases} x + 1, & \text{if } x \neq 0, \\ p + 1, & \text{if } x = 0 \end{cases}$ for all $x \in X$.

**Definition 4.5:** Let $\mu$ be a $\Gamma$-Fuzzy $d$-algebra of $X$. Then $\mu$ is called an absolutely $\Gamma$-Fuzzy $d$-algebra of $X$ and its complement $\Gamma^*(x)$ is called null $\Gamma$-Fuzzy $d$-algebra of $X$ if and only if $\mu(x) = 1$, for all $x \in X$.

**Table 3.** $(\Omega, \ast, 0)$ is a $d$-algebra.

| $\ast$ | 0 | 1 | 2 | 3 | 4 | 5 | p-3 | p-2 | p-1 | P |
|-------|---|---|---|---|---|---|-----|-----|-----|---|
| 0     | 0 | 0 | 0 | 0 | 0 | 0 | 0   | 0   | 0   | P |
| 1     | 1 | 0 | 2 | 3 | 4 | 5 | 0   | 0   | 0   | P |
| 2     | 2 | 2 | 0 | 3 | 4 | 5 | 0   | 0   | 0   | P |
| 3     | 3 | 3 | 3 | 0 | 4 | 5 | 0   | 0   | 0   | P |
| 4     | 4 | 4 | 4 | 4 | 0 | 5 | 0   | 0   | 0   | P |
| 5     | 5 | 5 | 5 | 5 | 5 | 5 | 0   | 0   | 0   | P |
| ...   | ...| ...| ...| ...| ...| ...| ... | ... | ... | ... |
| p-3   | p-3| p-3| p-3| p-3| p-3| p-3| p-3 | p-3 | p-3 | P |
| p-2   | p-2| p-2| p-2| p-2| p-2| p-2| p-2 | p-2 | p-2 | P |
| p-1   | p-1| p-1| p-1| p-1| p-1| p-1| p-1 | p-1 | p-1 | P |
| p     | p  | p  | p  | p  | p  | p  | p   | p   | p   | P |

5. $\Gamma$-Fuzzy $d$-algebras and their applications

Suppose that $\mu$ is a $\Gamma$-Fuzzy $d$-algebra of $X$. Then we will consider a new method to check that if $X$ is edge or not edge. In this case, for any $(X, \ast, 0)$ $d$-algebra, we can use the form of its $\Gamma$-Fuzzy (see Theorem (5.1)). Also, for any $(X, \ast, 0)$ $d$-algebra, a new class in Fuzzy topological spaces called $\Gamma$-Fuzzy topological space is investigated. Moreover, we will consider that $X$ is edge $d$-algebra if and only if its $\Gamma$-Fuzzy topological space is $\Gamma$-Fuzzy indiscrete topological space. Further, for any $(X, \ast, 0)$ $d$-algebra, a new class of IFS is called intuitionistic $\Gamma$-Fuzzy $d$-algebra is investigated and its applications are discussed.

**Theorem 5.1:** Let $(X, \ast, 0)$ be $d$-algebra. Then $X$ is edge $d$-algebra if and only if its $\Gamma$-Fuzzy $d$-algebra is an absolutely $\Gamma$-Fuzzy $d$-algebra.

**Proof:** Let $(X, \ast, 0)$ be $d$-algebra and $\mu$ be an absolutely $\Gamma$-Fuzzy $d$-algebra of $X$. If $X$ is not edge. Then $a \ast X \neq \{0, a\}$ for some $a \neq a$ in $X$, thus there is $b \in X$ such that $0 \neq b \neq a$ and $b \in a \ast X$. Hence $|\Gamma(a) + |\Gamma^*(a)| < |X|$. This implies that at $a \in X$ we have $\mu(a) < 1$ and that means $\mu$ is not an absolutely, but this contradiction. Hence $X$ is edge. Conversely, if $X$ is edge and $\mu$ is not an absolutely $\Gamma$-Fuzzy $d$-algebra of $X$. Then $\mu(a) \neq 1$ for some $a \in X$. However, $\mu(a) = 1$, if $x = 0$, thus $\mu(a) = 1$ and hence $\mu(a) = \frac{|\Gamma(a) + |\Gamma^*(a)|}{|X|}$. This implies that $|\Gamma(a) + |\Gamma^*(a)| < |X|$. (since $|\Gamma(a) + |\Gamma^*(a)| < |X|$ for all $x \in X$). Then there is $b \in X$ such that $0 \neq b \neq a$ and $b \in a \ast X$, thus $a \ast X \neq \{0, a\}$ for some $a \neq a$. Thus $X$ is not edge, but this contradiction. Then $\mu$ is an absolutely $\Gamma$-Fuzzy $d$-algebra of $X$.

**Definition 5.2:** Let $(X, \ast, 0)$ be a $d$-algebra and $\mu$ be a $\Gamma$-Fuzzy $d$-algebra of $X$. Then $(X, \Gamma^*)$ is called $\Gamma$-Fuzzy topological space, where $\Gamma^* = \{x, 1x, \mu\}$.

**Remark 5.3:** Every $d$-algebra $(X, \ast, 0)$ determine unique $\Gamma$-Fuzzy topological space $(X, \Gamma^*)$ and it is...
called $\Gamma$-Fuzzy indiscrete topological space, if $\Gamma' = \{0_x, 1_x, \mu\}$. That means $1_x = \mu$ or $0_x = \mu$.

**Example 2.4:** Let $(X, \ast, 1)$ be $d$-algebra where $X = \{1, 2, 3, 4, 5, 6\}$ with the binary operation $\ast$ defined in Table 4.

Then $(X, \Gamma')$ is $\Gamma$-Fuzzy topological space, $\Gamma' = \{0_x, 1_x, \mu\}$ where $0_x(x) = 0, \forall x \in X$, $1_x(x) = 1, \forall x \in X$, $\mu(1) = \mu(2)$

$$\mu(3) = 1, \mu(4) = \frac{\| \Gamma' + \Gamma'' \|}{X} = \frac{(2 + 3)/6}{2} = 0.833, \mu(5) = \mu(6) = 0.5.$$

**Corollary 5.5:** Let $(X, \ast, 0)$ be $d$-algebra. Then $X$ is edge $d$-algebra if and only if its $\Gamma$-Fuzzy topological space $(X, \Gamma')$ is an indiscrete.

**Proof:** Let $(X, \ast, 0)$ be $d$-algebra and $\mu$ be $\Gamma$-Fuzzy $d$-algebra of $X$. If $X$ is edge $d$-algebra, then by Theorem (5.1) we have $\mu$, an absolutely $\Gamma$-Fuzzy $d$-algebra of $X$ and hence $1_x = \mu$. Hence, $(X, \Gamma')$ is $\Gamma$-Fuzzy indiscrete topological space. Conversely, $(X, \Gamma')$ is $\Gamma$-Fuzzy indiscrete topological space. Then $\Gamma' = \{0_x, 1_x\}$ and hence $1_x = \mu$ or $0_x = \mu$. Suppose that $1_x \neq \mu$ hence $0_x = \mu$. This implies that $\mu(x) = 0, \forall x \in X$, but this is a contradiction since $0 \in X$ and $\mu(0) = 1$. Hence, $\mu$ is an absolutely $\Gamma$-Fuzzy $d$-algebra of $X$. Then by Theorem (5.1), we consider that $X$ is edge $d$-algebra.

**Definition 5.6:** Let $(X, \ast, 0)$ be $d$-algebra and $\mu$ be an absolutely $\Gamma$-Fuzzy $d$-algebra of $X$. Then $A = (\mu_A, \eta_A)$ is called an intuitionistic $\Gamma$-Fuzzy $d$-algebra of $X$, where $\mu_A(x) = \mu(x)$, and $\eta_A(x) = 1 - \mu(x), \forall x \in X$.

**Example 5.7:** Let $(X, \ast, f)$ be $d$-algebra where $X = \{f, a, b, c\}$ with the binary operation $\ast$ defined by Table 5.

Then $A = (\mu_A, \eta_A)$ is an intuitionistic $\Gamma$-Fuzzy $d$-algebra of $X$ with Table 6.

**Remark 5.8:** Every $d$-algebra determines unique intuitionistic $\Gamma$-Fuzzy $d$-algebra. Moreover, it is clear from Theorem (5.1), for any $(X, \ast, 0)$ $d$-algebra, we consider that $X$ is edge $d$-algebra if and only if its intuitionistic $\Gamma$-Fuzzy $d$-algebra $A = (\mu_A, \eta_A)$ satisfies $\eta_A$ is null. Further, for any intuitionistic $\Gamma$-Fuzzy $d$-algebra $A = (\mu_A, \eta_A)$ the hesitation margin $\pi_A(x) : [0, 1] \rightarrow [0, 1]$ such that $\pi_A(x) = 0, \forall x \in X$.

**Definition 5.9:** Let $(X, \ast, f)$ and $(X, \bullet, h)$ be two $d$-algebras. Then $(X, \ast, f)$ and $(X, \bullet, h)$ are said to have the same structure and denoted by $(X, \ast, f) \approx (X, \bullet, h)$ if and only if their $\Gamma$-Fuzzy $d$-algebras are equal.

**Definition 5.10:** Let $X$ be any nonempty set, then we denote to the collection of all $d$-algebras of $X$ by $\xi(X)$ (i.e. $\xi(X) = \{(X, \ast, f) | (X, \ast, f) \ is \ d-\ algebra\}$.

**Example 5.11:** Let $(X, \ast, f)$, $(X, \bullet, h) \in \xi(X)$ be two $d$-algebras, where $X = \{f, a, b, c\}$ with two binary operations $\ast$ and $\bullet$ defined on $X$ by Tables 7 and 8, respectively. Then $(X, \ast, f)$ has $\Gamma$-Fuzzy $d$-algebra say $\mu$. Thus we consider that

$$\mu(f) = \mu(h) = 1, \mu(a) = \mu(b) = (4/5) = 0.8, \mu(c) = (1/5) = 0.2.$$

Moreover, $(X, \bullet, h)$ has $\Gamma$-Fuzzy $d$-algebra say $\psi$. Hence

$$\psi(h) = \psi(f) = 1, \psi(a) = \psi(b) = 0.8, \psi(c) = (1/5) = 0.2.$$

This implies that $\mu(x) = \psi(x), \forall x \in X$. Then $(X, \ast, f) \approx (X, \bullet, h)$.

**Theorem 5.12:** The relation $\approx$ is an equivalence relation on $\xi(X)$.

**Proof:** The proof is obvious.

**Definition 5.13:** Let $X = \{x_1, x_2, x_3, \ldots\}$ be nonempty set, define $\otimes_{x_0} X \times X \rightarrow Y$ by

| $\ast$ | $t$ | $h$ | $a$ | $b$ | $c$ |
|-------|-----|-----|-----|-----|-----|
| $t$   | $f$ | $h$ | $a$ | $b$ | $c$ |
| $h$   | $f$ | $h$ | $f$ | $f$ | $f$ |
| $a$   | $a$ | $a$ | $f$ | $b$ | $f$ |
| $b$   | $b$ | $b$ | $a$ | $f$ | $b$ |
| $c$   | $c$ | $c$ | $a$ | $f$ | $f$ |

| $\otimes_{x_0}$ | $X \times X$ |
|------------------|---------------|
| $h$              | $f$           |
| $h$              | $h$           |
| $h$              | $h$           |
| $h$              | $h$           |
| $h$              | $h$           |
| $h$              | $h$           |
| $h$              | $h$           |
| $h$              | $h$           |

| $(X, \ast, f)$ is $d$-algebra. |
|------------------|---------------|
| $(X, \ast, f)$ is $d$-algebra. |

Table 6. $A$ is an intuitionistic $\Gamma$-Fuzzy $d$-algebra.

| $\mu_A$ | $\eta_A$ |
|---------|----------|
| $0.75$  | $0.25$   |
| $0.75$  | $0.25$   |

| $(X, \bullet, h)$ is $d$-algebra. |
|------------------|---------------|
| $(X, \bullet, h)$ is $d$-algebra. |
\[ \otimes_{\alpha}(x_i, x_j) = \begin{cases} x_k, & \text{if } x_i = x_k \text{ or } x_i = x_j, \forall x_i, x_j \in X. \\ x_i, & \text{Otherwise.} \end{cases} \]

Then \((X, \otimes_{\alpha}, x_k)\) is \(d\)-algebra and its \(\Gamma\)-Fuzzy \(d\)-algebra is denoted by \(\mu^{\otimes_{\alpha}}\).

**Remarks 5.14:**

1. Since \(\approx\) is an equivalence relation on \(\xi(X)\), then the equivalence classes form a partition of \(\xi(X)\).
2. For any nonempty set \(X = \{x_1, x_2, x_3, \ldots\}\), we consider that \((X, \otimes_{\alpha}, x_k) \approx (X, \otimes_{\alpha}, x_k), \forall x_i, x_j \in X.\)
3. \(\mu^{\otimes_{\alpha}}(x) = 1_X(x), \forall x \in X.\) Then \(\mu^{\otimes_{\alpha}}\) is an absolutely \(\Gamma\)-Fuzzy \(d\)-algebra and hence \((X, \otimes_{\alpha}, x_k)\) is edge \(d\)-algebra.

**Example 5.15:** Let \(X = \{a, b, c, d\}.\) Hence \((X, \otimes_{a}, a), (X, \otimes_{b}, b), (X, \otimes_{c}, c)\) and \((X, \otimes_{d}, d)\) are \(d\)-algebras, which are defined in Tables 9–12, respectively.

**Definition 5.16:** Let \((X, \ast, f)\) be a \(d\)-algebra and \(\lambda\) be a \(\Gamma\)-Fuzzy \(d\)-algebra of \(X\). Denote the equivalence class of 

\[(X, \ast, f)^\lambda = \{(X, \circ, h) | (X, \ast, f) \approx (X, \circ, h)\}.

**Example 5.17:** See example (5.15). Then we consider that the equivalence classes of 

\[(X, \otimes_{\alpha}, a), (X, \otimes_{b}, b), (X, \otimes_{c}, c)\] and 

\[(X, \otimes_{d}, d)\] are equal, where

\[(X, \otimes_{\alpha}, a)^{\mu^{\otimes_{\alpha}}} = (X, \otimes_{b}, b)^{\mu^{\otimes_{\alpha}}} = (X, \otimes_{c}, c)^{\mu^{\otimes_{\alpha}}} = (X, \otimes_{d}, d)^{\mu^{\otimes_{\alpha}}} = \{(X, \otimes_{\alpha}, a), (X, \otimes_{b}, b), (X, \otimes_{c}, c), (X, \otimes_{d}, d)\}

**Definition 5.18:** Define \(\Psi(X)\) as the set of all equivalence classes \((X, \ast, f)^\lambda\), and equip \(\Psi(X)\) with the following binary operation 

\[\oplus : \Psi(X) \times \Psi(X) \rightarrow \Psi(X)\] that is defined as follows:

\[\oplus((X, \ast, f)^\lambda, (X, \ast, f)^\eta) = \begin{cases} (X, \ast, f)^\lambda & \text{if } \lambda = 1_X \text{ or } \lambda = \eta, \\ (X, \ast, f)^\eta & \text{if } \lambda < \eta, \\ (X, \ast, f)^\lambda & \text{otherwise.} \end{cases}\]

\[\forall (X, \ast, f)^\lambda, (X, \ast, f)^\eta \in \Psi(X).\]

**Theorem 5.19:** If \(f \in X\), then 

\[\Psi(X), (X, \otimes_{\alpha}, f)^{\mu^{\otimes_{\alpha}}}\) is \(d\)-algebra.

**Proof:** We need to prove the following:

1. Since \(\mu^{\otimes_{\alpha}}(x) = 1_X(x) = 1, \forall x \in X.\) Then 
   \[\oplus((X, \otimes_{\alpha}, f)^{\mu^{\otimes_{\alpha}}}, (X, \ast, f)^\lambda) = (X, \otimes_{\alpha}, f)^{\mu^{\otimes_{\alpha}}},\]

2. Since \(\lambda(x) = \lambda(x).\) Then 
   \[\oplus((X, \ast, f)^\lambda, (X, \ast, f)^\eta) = (X, \otimes_{\alpha}, f)^{\mu^{\otimes_{\alpha}}},\]

3. If \(\oplus((X, \ast, f)^\lambda, (X, \ast, f)^\eta) = (X, \otimes_{\alpha}, f)^{\mu^{\otimes_{\alpha}}}\) and \(\oplus((X, \ast, f)^\lambda, (X, \ast, f)^\eta) = (X, \otimes_{\alpha}, f)^{\mu^{\otimes_{\alpha}}},\) then this implies that 
   \[\lambda = 1_X \text{ or } \lambda = \eta\] and \(\eta = 1_X \text{ or } \eta = \lambda.\) Hence, there are four cases that cover all probabilities, which are holed as follows:

1. \(\lambda = 1_X \text{ and } \eta = 1_X, \text{ thus } \eta = \lambda = 1_X \text{ and hence } (X, \ast, f)^\lambda = (X, \ast, f)^\eta\)

2. \(\lambda = 1_X \text{ and } \eta = \lambda, \text{ thus } \eta = \lambda = 1_X \text{ and hence } (X, \ast, f)^\lambda = (X, \ast, f)^\eta\)

3. \(\lambda = \eta \text{ and } \eta = 1_X, \text{ thus } \lambda = \eta = 1_X \text{ and hence } (X, \ast, f)^\lambda = (X, \ast, f)^\eta\)

4. \(\lambda = \eta \text{ and } \eta = \lambda, \text{ thus } (X, \ast, f)^\lambda = (X, \ast, f)^\eta.\)

Hence \((\Psi(X), \oplus, (X, \otimes_{\alpha}, f)^{\mu^{\otimes_{\alpha}}})\) is \(d\)-algebra.
**Definition 5.20:** Let \((X, * , f)\) and \((X, \bullet , h)\) be two \(d\)-algebras. Then the normalized Euclidean distance between their intuitionistic \(\Gamma\)-Fuzzy \(d\)-algebras \(A = (\mu_A, \eta_A)\) for \((X, * , f)\) and \(B = (\mu_B, \eta_B)\) for \((X, \bullet , h)\)

is defined as \(d(A, B) = \frac{1}{2n} \sum_{i=1}^{n} \left[ (\mu_A(x_i) - \mu_B(x_i))^2 + (\eta_A(x_i) - \eta_B(x_i))^2 \right] \), where \(x_i \in X\) for all \(i = 1, 2, \ldots, n\).

**Example 5.21:** Let \((X, * , f)\) and \((X, \bullet , c)\) be two \(d\)-algebras, where \(X = \{f, a, b, c\}\) with two binary operations \(*\) and \(\bullet\) defined on \(X\) by Tables 13–16.

Then \((X, * , f)\) has intuitionistic \(\Gamma\)-Fuzzy \(d\)-algebra say \(A = (\mu_A, \eta_A)\) with Table 15.

And \((X, \bullet , c)\) has intuitionistic \(\Gamma\)-Fuzzy \(d\)-algebra say \(B = (\mu_B, \eta_B)\) with Table 16.

Therefore, \((X, \bullet , c)\) is edge, but \((X, * , f)\) is not edge.

That means they have different structures. Further, the normalized Euclidean distance between their intuitionistic \(\Gamma\)-Fuzzy \(d\)-algebras \(A = (\mu_A, \eta_A)\) and \(B = (\mu_B, \eta_B)\)

\[ d(A, B) = \frac{1}{2n} \sum_{i=1}^{n} \left[ (\mu_A(x_i) - \mu_B(x_i))^2 + (\eta_A(x_i) - \eta_B(x_i))^2 \right] \]

\[ = 0.03125. \]

**Theorem 5.22:** Let \((X, * , f)\) and \((X, \bullet , h)\) be two \(d\)-algebras with their intuitionistic \(\Gamma\)-Fuzzy \(d\)-algebras \(A = (\mu_A, \eta_A)\) and \(B = (\mu_B, \eta_B)\), respectively. Then \((X, * , f) \approx (X, \bullet , h)\) if and only if \(d(A, B) = 0\).

**Proof:** Let \((X, * , f)\) and \((X, \bullet , h)\) be two \(d\)-algebras. Assume \(d(A, B) = 0\), then

\[ \frac{1}{2n} \sum_{i=1}^{n} \left[ (\mu_A(x_i) - \mu_B(x_i))^2 + (\eta_A(x_i) - \eta_B(x_i))^2 \right] = 0. \]

Therefore, we consider that \(\sum_{i=1}^{n} \left[ (\mu_A(x_i) - \mu_B(x_i))^2 + (\eta_A(x_i) - \eta_B(x_i))^2 \right] = 0.\) This implies that \((\mu_A(x_i) - \mu_B(x_i)) = 0, x_i \in X\) for all \(i = 1, 2, \ldots, n\). Then \(\mu_A = \mu_B\) and hence \((X, * , f) \approx (X, \bullet , h)\) by similarity, the converse is obviously.

**Remark 5.23:** Let \((X, * , f), (X, \circ , h), (X, \cdot , k) \in \mathcal{E}(X)\) with their intuitionistic \(\Gamma\)-Fuzzy \(d\)-algebras \(A, B\) and \(C\), respectively. Then we consider that \((X, * , f) \approx (X, \circ , k)\) if \(d(A, B) = 0\) and \(d(B, C) = 0\).

### 6. Conclusions

In the present paper, we have introduced the concept of \(\Gamma\)-Fuzzy \(d\)-algebra and investigated some of its applications and essential properties. We think this work would enhance the scope for further study in this field of Fuzzy \(d\)-algebra. It is our hope that this work is going to impact the upcoming research works in this field of algebras with a new horizon of interest and innovation such as \(\Gamma\)-Fuzzy \(d\)-subalgebra, \(\Gamma\)-Fuzzy \(d\)-ideal, \(\Gamma\)-Fuzzy \(d^*\)-ideal and \(\Gamma\)-Fuzzy \(d^*\)-ideal. These new notions will depend on the structure of \(d\)-algebra and hence we can study their properties using the structure of \(d\)-algebra.

### Disclosure statement

No potential conflict of interest was reported by the author.

### References

[1] Imai Y, Iseki K. On axiom systems of propositional calculi XIV. Proc Jpn Acad. 1966;42:19-22.

[2] Iseki K. An algebra related with a propositional calculus. Proc Jpn Acad. 1966;42:26-29.

[3] Hu QP, Li X. BCH-algebras. Math Jpn. 1986;20:87–96.

[4] Hu QP, Li X. BCH-algebras. Math Jpn. 1985;30:659–661.

[5] Mahmood S, Alradha MA. Soft edge \(p\)-algebras of the power sets. Int J Appl Fuzzy Sets Artif Intell. 2017;7:231–243.

[6] Mahmood S, Muhamad A. Soft BCL-algebras of the power sets. Int J of Algebra. 2017;7:329–341.

[7] Mahmood S, Alradha M. Characterizations of \(p\)-algebra and generation permutation topological \(p\)-algebra using permutation in symmetric group, Am J Appl Math Stat. 2017;7(4):152–159.

[8] Mahmood S, Abd Ulrazaq M. Soft \(BCH\)-algebras of the power sets, Am J Appl Math Stat. 2018;8(1):1–7. DOI:10.5923/j.ajm.20180801.01

[9] Zadeh LA. Fuzzy sets. Inform Control. 1965;8:338–353.

[10] Chang CL. Fuzzy topological spaces. J Math Anal Appl. 1968;24:182–190.

[11] Atanassov KT. Intuitionistic fuzzy sets. Fuzzy Sets Syst. 1986;20:87–96.

[12] Mahmood S. Fuzzy Lindelof closed spaces and their operations. Gen Math Notes. 2014;21(1):128–136.
[13] Mahmood S. Dissimilarity fuzzy soft points and their applications. Fuzzy Inform Eng. 2016;8:281–294.
[14] Mahmood S. On intuitionistic fuzzy soft b-closed sets in intuitionistic fuzzy soft topological spaces. Anna Fuzzy Math Inform. 2015;10(2):221–233.
[15] Mahmood S, Al-Batat Z. Intuitionistic fuzzy soft LA-semigroups and intuitionistic fuzzy soft ideals. Int J Appl Fuzzy Sets Artif Intell. 2016;6:119–132.
[16] Jun YB. Fuzzy topological BCK-algebras. Math Japon. 1993;38:1059–1063.
[17] Neggers J, Kim HS. On d-algebras. Math Slovaca. 1999;49:19–26.
[18] Neggers J, Jun YB, Kim HS. On ideals in algebras. Math Slovaca. 1999;49(3):243–251.
[19] Jun YB, Neggers J, Kim HS. Fuzzy d-ideals of d-algebras. J Fuzzy Math. 2000;8:123–130.
[20] Zarandi A, Borumand S. Redefined fuzzy B-algebras. Fuzzy Optim Decis Mak. 2008;7(4):373–386.
[21] Zixin L, Guangji Z, Cheng Z, et al. New kinds of Fuzzy ideal in BCI-algebras*. Fuzzy Optim Decis Mak. 2006;5(2):177–186.