Two-Population Dynamics in a Growing Network Model

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Abstract

We introduce a growing network evolution model with nodal attributes. The model describes the interactions between potentially violent $V$ and non-violent $N$ agents who have different affinities in establishing connections within their own population versus between the populations. The model is able to generate all stable triads observed in real social systems. In the framework of rate equations theory, we employ the mean-field approximation to derive analytical expressions of the degree distribution and the local clustering coefficient for each type of nodes. Analytical derivations agree well with numerical simulation results. The assortativity of the potentially violent network qualitatively resembles the connectivity pattern in terrorist networks that was recently reported. The assortativity of the network driven by aggression shows clearly different behavior than the assortativity of the networks with connections of non-aggressive nature in agreement with recent empirical results of an online social system.

Keywords: Networks, Complex systems, Rate equation approach, Two-population dynamics, Stable triads, Terror networks

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1. Introduction

Complex networks have been the focus of the study of dynamical properties of complex systems in nature and society in the last decade \cite{1, 2, 3, 4}. Usually all nodes are assumed to belong to one population or class, and the interactions between two distinct populations have been reported just recently \cite{3, 6}. The dynamical properties of two interacting populations, extraverts and introverts were recently studied using dynamical network evolution model \cite{5}. Buldyrev et al \cite{6} developed a framework for understanding the robustness of interacting networks. Using the generating functions method they present the exact analytical solutions for the critical fraction of nodes, which upon removal, will lead to a complete fragmentation into interdependent networks.

The focus of the current approach in understanding the formation and evolution of terrorist networks is on middle-range perspective as opposed to the micro-level approach that considers individual terrorist and macro-level analysis of the root causes of terrorism \cite{7}. In particular, the interest is in placing the relationships between individuals in the context of (i) their interactions with each other, (ii) how they are influenced by ideas originating from their environment, (iii) their interactions with people and organizations outside of their group \cite{8}. Motivated by this description which conceptually refers to a complex system and because networks provide a fruitful framework to model complex systems \cite{9}, we introduce a network model that aims to describe the interactions between potentially terrorist and non-terrorist populations.

Our goal is to present a model of a social network that contains two types of agents which demonstrate different affinities in establishing connections within their own population versus connections with the other population. We also aim that the model is simple enough to allow the derivation of approximate analytical expressions of the basic characteristics of the network. The latter can be achieved in the general framework of rate equation theory in the mean field approximation which has been introduced to study fundamental characteristics of growing network models \cite{10, 11}. Within this framework we introduce the rate equations that are specific to our model, solve them and obtain analytical expressions that predict the growth dynamics of the degree of individual vertices.

While the probability to create initial connections is easily defined, the probability for the secondary contacts between nodes is difficult to derive
Figure 1: Possible combinations of connections between a new node \( v \), the initial contact \( j \) and the secondary contact \( i \). (a)-(d) show the connections when the new node is an \( N \) node. (e)-(h) show the connections when the new node is a \( V \) node. Empty symbols (○) mark an \( N \) node while full symbols (●) mark a \( V \) node. Full lines represent links between nodes of the same type, e.g. friends, such as \( NN \) or \(VV \). Dashed lines represent links between nodes of different types such as \( NV \), e.g. enemies.

because of the interactions between the two types of nodes. As a first approximation, we assume that the initial and secondary contacts form edges with the same probabilities. To include a more precise contribution of both the initial and the secondary contacts in the rate equations we empirically obtain their functional dependences. This leads to an improved agreement between analytical and numerical simulation results.

From the functional dependence of the degree of a node as a function of time using the mean-field arguments, we derive the degree distribution for each of the two types of nodes. We also derive analytical expressions of the structural, three-point correlations between nodes to study the clustering properties of the networks.

2. Model

The network models are broadly classified into two categories: the network evolution models in which addition of new edges depends on the local structure of the network, and nodal attribute models in which the existence of edges is determined solely by the attributes of the nodes (for review see [12]). The network evolution models can be further categorized into growing network evolution models and dynamical network evolution models. In the former the network growth starts with a small seed network and nodes and
links are added according to specific rules until the network reaches a pre-
determined size. Dynamical network evolution models start with an empty
network and edges are added and deleted according to specific rules until
statistical properties of the network stabilize.

In this paper, we propose a model that incorporates a growing network
evolution process with nodal attributes which could be thought of as a new
class of model. The model has five free parameters, three describing the
growing network evolution process and two describing nodal attributes. The
parameters of the growing network evolution process are the number of nodes
\( N \), the average number of nodes selected at random as initial contacts \( m_r \),
and the average number of nodes selected as secondary contacts \( m_s \) among
the neighbors of each initial contact \[13\]. Two parameters quantify the type
and amount of nodes, non-violent \( N \) or potentially violent \( V \) nodes and the
type of interactions between them which are as follows: (i) nodes are ran-
domly marked as non-violent \( N \) with probability \( p_N \) and potentially violent
\( V \) with probability \( p_V = 1 - p_N \); (ii) when establishing initial contacts, nodes
connect with probability \( p_s \) if the nodes are of the same type, such as \( V \) with
\( V \) or \( N \) with \( N \), or with probability \( p_d = 1 - p_s \) if the nodes are of
different types such as \( N \) with \( V \) nodes. The secondary contacts are estab-
lished with nodes among the neighbors of the initial contacts. The model
combines the random attachment of initial contacts with the implicit prefer-
ential attachment of the secondary contacts. In that, the model represents a
generalization with two types of node attributes of growing models such as
\[14, 15, 13\].

The definition of establishing edges can be thought of as creating links
between friends (solid lines in Fig. 1) or between enemies (dashed lines). By
varying the value of \( p_s \), we can generate different strengths of friendliness or
animosity. The possible configurations of triads that arise in social networks
in such a context are: (a) three friendly interactions; (b) one friendly and
two unfriendly connections; (c) two friendly interactions and one unfriendly;
(d) three unfriendly interactions \[3\]. According to the strong formulation of
structural balance theory in social sciences, configurations (a) and (b) are
considered stable while (c) and (d) are unstable and likely to break apart
\[16\]. In their empirical large-scale verification of the long-standing structural
balance theory, the authors of \[21\] find that the unstable triads, especially
formation (c) are extremely underrepresented in an online social system in
comparison to a null model. Our model produces correctly the stable configu-
res (a) (Fig. 1a,e) and (b) (Fig. 1b-d,f-h) but cannot produce the unstable
configurations (c) and (d) in accordance with the strong formulation of the structural balance theory [16].

The model algorithm consists of the following steps: (1) start with a seed network of \( n_0 \) connected nodes among which some are \( N \) and some are \( V \), depending on \( p_N \); (2) at each time step add a new node, which has probability \( p_N \) to be a non-violent and \( p_V = 1 - p_N \) to be a violent node; (3) select on average \( m_r \geq 1 \) random nodes as initial contacts. The probability to connect the same type of nodes is \( p_s \) while \( p_d = 1 - p_s \) is the probability for initial contacts if they are of different types. (4) select on average \( m_s \geq 0 \) nodes among the neighbors of each initial contact as secondary contacts. Connecting the new node with the secondary contacts is done without checking if it is the same type of node or not. There are two reasons for this choice. (i) Because the probability to establish inter-population connections \( (p_s) \) is higher than the probability to connect nodes intra-population \( (p_d) \), it is more likely that the first contact and its neighbors (potential secondary contacts) are of the same type than of different types. Therefore the secondary contacts will be more likely to be intra-population contacts even without explicitly modifying their probability to connect based on the type. (ii) The secondary contacts are meant to mimic the ‘friend-of-a-friend’ type of contacts in the real world, and we think that the implicit preferences given by the existing network connections would more accurately describe the nature of such contacts without including an explicit separate probability. Apply steps (2) to (4) until the network reaches the necessary size.

2.1. Rate Equations

We start with constructing the rate equations that describe the change of the degree of a node on average during one time step of the network growth process for each of the non-violent \( N \) and violent \( V \) nodes. The degree of a node grows via two processes. One is the random attachment of connecting a new node to \( m_r \) nodes that are its initial contacts. The second process is when the new node is further connected to the \( m_s \) nodes among the neighbors of the initial contacts. In the following we assume that the probability of this second process is linear with respect to the degree of the node which leads
to implicit preferential attachment. The rate equations are:

\[
\frac{\partial k_{i}(N)}{\partial t} = \frac{m_{r}}{t_{p_{N}}} \left( p_{N}p_{s} + p_{V}p_{d} \right)
+ m_{r}m_{s}p_{N} \left( \frac{p_{s}k_{is}}{\sum_{j,NN} k_{j} + \sum_{j,NN} k_{j}} + \frac{p_{d}k_{id}}{\sum_{j,NN} k_{j} + \sum_{j,NN} k_{j}} \right)
\]

\[
+ m_{r}m_{s}p_{v} \left( \frac{p_{s}k_{is}}{\sum_{j,NN} k_{j} + \sum_{j,NN} k_{j}} + \frac{p_{d}k_{id}}{\sum_{j,NN} k_{j} + \sum_{j,NN} k_{j}} \right)
\]  \hspace{1cm} (1)

\[
\frac{\partial k_{i}(V)}{\partial t} = \frac{m_{r}}{t_{p_{V}}} \left( p_{V}p_{s} + p_{N}p_{d} \right)
+ m_{r}m_{s}p_{V} \left( \frac{p_{s}k_{is}}{\sum_{j,NN} k_{j} + \sum_{j,NN} k_{j}} + \frac{p_{d}k_{id}}{\sum_{j,NN} k_{j} + \sum_{j,NN} k_{j}} \right)
\]

\[
+ m_{r}m_{s}p_{N} \left( \frac{p_{s}k_{is}}{\sum_{j,NN} k_{j} + \sum_{j,NN} k_{j}} + \frac{p_{d}k_{id}}{\sum_{j,NN} k_{j} + \sum_{j,NN} k_{j}} \right)
\]  \hspace{1cm} (2)

where \( k_{i} \) is the degree of node \( i \) and we assumed that \( k_{is} = k_{i}p_{s} \) and \( k_{id} = k_{i}p_{d} \).

All possible combinations of triads of a new node \( v \), the initial contact \( j \), and the secondary contact \( i \) are schematically shown in Fig. 1(a-h) and presented by the third through the sixth terms in Eq. (1) and Eq. (2) for an \( N \) and \( V \) node, respectively.

For example, the fifth term in Eq. (1) describes the rate of change of the degree of vertex \( i \) (which is an \( N \) node) due to establishing the configuration of contacts shown in Fig. 1c. The fifth term contains four factors. The first factor is the average number of secondary contacts which is \( m_{r}m_{s} \). \( p_{V} \) is the probability that the new node created at time step \( t \) is a \( V \) node. \( p_{s} \) is the probability that the newly created node connects to the initial contact which is a node of the same type. Finally, \( k_{id}/(\sum_{j,NN} k_{j} + \sum_{j,NN} k_{j}) \) is the probability that the node selected for initial contact (node \( j \) in Fig. 1c).
shares an edge with the node \( i \). This is a standard expression for preferential attachment except for the complications induced by having two distinct populations. The nominator \( k_{id} \) is the degree of the \( i \) node if we count only the links to different types of nodes (in this case \( V \) nodes). The denominator is the sum of all possible links which the type of node selected for initial contact (in this case \( V \)) could have. Assuming that the initial and secondary contacts are created with the same probabilities the nominator would be equal to \( k_{id} = k_{i}p_{d} \) and we will approximate it in this way and the denominator would be equal to \((p_{V}p_{s} + p_{d})2m_{r}(1 + m_{s})t \). We will use this expression as a first approximation and we will also empirically derive functional dependences of the denominator and compare the results.

Note that the number of edges, multiplied by two, that exist between \( N \) and \( N \) nodes is equal to \( \sum_{j,NN} k_{j} \) and include both edges created as initial contacts and edges created as secondary contacts. The same is true for the number of edges between \( V \) and \( V \) nodes which is \( \sum_{j,VV} k_{j} \), and the number of edges between \( N \) and \( V \) nodes which is \( \sum_{j,NV} k_{j} \) in Eqs. (1) and (2). We know that the probabilities for creating edges as initial contacts are \( p_{s} \) or \( p_{d} \) if between same type of nodes or different types of nodes, respectively. We do not know, however, what these probabilities are when the edges represent connections to secondary contacts. This is the reason to express the respective summations by the following relations:

\[
\sum_{j,NN} k_{j} = g(\rho_{N}(p_{N}, p_{s}))2m_{r}(1 + m_{s})t \quad (3a)
\]
\[
\sum_{j,VV} k_{j} = h(\rho_{V}(p_{V}, p_{s}))2m_{r}(1 + m_{s})t \quad (3b)
\]
\[
\sum_{j,NV} k_{j} = q(\rho_{d}(p_{d}))2m_{r}(1 + m_{s})t. \quad (3c)
\]

Eqs. (3) contain the term \( 2m_{r}(1 + m_{s})t \) because there are \( \sim t \) vertices at time \( t \) and \( m_{r}(1 + m_{s}) \) is the average initial degree of a vertex. \( g(\rho_{N}(p_{N}, p_{s})) \) in Eq. (3a) is the probability that the edge is between \( N \) nodes. \( h(\rho_{V}(p_{V}, p_{s})) \) in Eq. (3b) is the probability that the edge is between \( V \) nodes, and \( q(\rho_{d}(p_{d})) \) in Eq. (3c) is the probability that the edge is between nodes of different types. Functions \( g, h, \) and \( q \) contain edges established due to both initial and secondary contacts, whose contributions cannot be separated and derived analytically. Therefore, we will obtain these functional dependences through empirical considerations. If we assume that the edges to secondary contacts
are established with the same probabilities as the edges to initial contacts,
then the relations would have been:

\[
g(\rho_N(p_N, p_s)) = p_N p_s \quad (4a)
\]
\[
h(\rho_V(p_V, p_s)) = p_V p_s \quad (4b)
\]
\[
q(\rho_d(p_d)) = p_d. \quad (4c)
\]

2.2. Solutions of the Rate Equations

After separating the variables and integrating the rate equation of the
degree \(k_{i(N)}\) of \(N\) node Eq.\(1\) from \(k_{init}\) to \(k_i\) and from \(t_i\) to \(t\) we obtain the
following expression for the degree as a function of time \(k_i(t)\)

\[
k_{i(N)}(t) = G_3 \left( \frac{t}{t_i} \right)^{1/G_1} - G_2, \quad (5)
\]

where

\[
G_1 = C \left( \frac{p_N p_s^2 + p_V p_d p_s + p_N p_s p_d}{g(\rho_N) + q(\rho_d)} \right)^{-1} \quad (6a)
\]
\[
G_2 = A_N G_1 \quad (6b)
\]
\[
G_3 = G_2 + k_{init} \quad (6c)
\]
\[
C = 2(1 + m_s)/m_s \quad (6d)
\]
\[
A_N = m_r (p_N p_s + p_V p_d) / p_N \quad (6e)
\]
\[
k_{init} = m_r (1 + m_s). \quad (6f)
\]

Integrating the rate equation of the degree \(k_i\) of \(V\) nodes, Eq.\(2\) produces
the time dependence of the degree of any \(V\) node

\[
k_{i(V)}(t) = H_3 \left( \frac{t}{t_i} \right)^{1/H_1} - H_2, \quad (7)
\]

where

\[
H_1 = C \left( \frac{p_V p_s^2 + p_N p_d p_s + p_V p_d p_s}{h(\rho_V) + q(\rho_d)} \right)^{-1} \quad (8a)
\]
\[
H_2 = A_V H_1 \quad (8b)
\]
\[
H_3 = H_2 + k_{init} \quad (8c)
\]
\[
A_V = m_r (p_V p_s + p_N p_d) / p_V. \quad (8d)
\]
If we assume that the edges to secondary contacts are established with the same probabilities as the edges to initial contacts, then the solution of the rate equation Eq. (1) will be the following for an \( N \) node

\[
k_{i(N)}(t) = D_3 \left( \frac{t}{t_i} \right)^{1/D_1} - D_2,
\]

where

\[
D_1 = C \left( \frac{p_N p_s^2 + p_V p_s p_d}{p_n p_s + p_d} + \frac{p_N p_d^2 + p_V p_s p_d}{p_v p_s + p_d} \right)^{-1} \tag{10a}
\]

\[
D_2 = A_N D_1 \tag{10b}
\]

\[
D_3 = D_2 + k_{\text{init}}, \tag{10c}
\]

and for a \( V \) node:

\[
k_{i(V)}(t) = E_3 \left( \frac{t}{t_i} \right)^{1/E_1} - E_2, \tag{11}
\]

where

\[
E_1 = C \left( \frac{p_V p_s^2 + p_N p_s p_d}{p_v p_s + p_d} + \frac{p_V p_d^2 + p_N p_s p_d}{p_n p_s + p_d} \right)^{-1} \tag{12a}
\]

\[
E_2 = A_V E_1 \tag{12b}
\]

\[
E_3 = E_2 + k_{\text{init}}, \tag{12c}
\]

after making use of Eqs. \( \text{(4)} \).

2.3. Degree distribution

In the mean field approximation, the degree \( k_i(t) \) of a node \( i \) evolves with time \( t \) strictly monotonically after the node was added to the network at time \( t_i \). Therefore, the nodes added to the network more recently will have on average lower degree than those added to the network earlier. Assuming that we add nodes to the network at equal intervals, the probability density of \( t_i \) is \( 1/t \). Using the properties of cumulative probability distribution function, we can write that the probability of a node to have degree \( \tilde{k} \leq k_i \) is equal to the probability that the node has been added to the network at time \( \tilde{t} \geq t_i \)

\[
F(k) = P(\tilde{k} \leq k_i) = P(\tilde{t} \geq t_i) = \frac{t - t_i}{t}. \tag{13}
\]
We can derive the probability density function of \( N \) nodes, \( P_N(k) \) by obtaining an expression for \( t_i^{(N)} \) from Eq. (5), then replacing it in Eq. (13) and differentiating the resultant equation with respect to \( k_i \), which is \( P(k) = \frac{\partial F}{\partial k} \). The result is:

\[
P_N(k) = G_1(G_2 + k)^{-G_1-1}G_3^{G_1}.
\]

(14)

Similarly, we can derive the probability density function of \( V \) nodes, \( P_V(k) \) by obtaining an expression for \( t_i^{(V)} \) from Eq. (7), then replacing it in Eq. (13) and differentiating the resultant equation with respect to \( k_i \) to obtain:

\[
P_V(k) = H_1(H_2 + k)^{-H_1-1}H_3^{H_1}.
\]

(15)

If we use Eqs. (9) and (11) to derive expressions for \( t_i^{(N)} \) and \( t_i^{(V)} \), respectively, then the degree distributions are presented by

\[
P_N(k) = D_1(D_2 + k)^{-D_1-1}D_3^{D_1};
\]

\[
P_V(k) = E_1(E_2 + k)^{-E_1-1}E_3^{E_1}.
\]

(16)

(17)

It should be noted that \( k_i \) and all quantities are expectation values and can be compared to simulation results which are assemble averages. The analytical results converge to those reported in Ref. [13] in the limit of one population which means \( p_N = 1, p_V = 0, p_s = 1, \) and \( p_d = 0 \).

2.4. Clustering Characteristics

The dependence of the clustering coefficient as a function of the degree of a node can be derived using the rate equation method [10, 11]. The number of triangles \( E_{i(N)}(E_{i(V)}) \) around a node if \( i \) is an \( N \) (\( V \)) node is changing with time following two processes. The first process is when node \( i \) is selected as one of the initial contacts with probability \( A_N/t (A_V/t) \) and the new node links to some of its neighbors which are \( m_s \) on average. The second process is when node \( i \) is selected as a secondary contact and a triangle is formed between the new node, the initial contact, and the secondary contact. It is possible that two neighboring initial contacts and the new node form a triangle, but the contribution of this process is negligible. The rate equation for the number of connections between the nearest neighbors of a node of
degree $k_{iN}$ ($k_{iV}$) is given by

$$
\frac{\partial E_{i(N)}}{\partial t} = \frac{m_r m_s}{t p_N} (p_N p_s + p_V p_d)
$$

$$
+ m_r m_s p_N \left( \sum_{j,NN} k_j + \sum_{j,NV} k_j + \sum_{j,NN} k_j + \sum_{j,NV} k_j \right)
$$

$$
+ m_r m_s p_V \left( \sum_{j,NV} k_j + \sum_{j,NN} k_j + \sum_{j,NV} k_j + \sum_{j,NN} k_j \right)
$$

$$
\left( \sum_{j,NN} k_j + \sum_{j,NV} k_j + \sum_{j,NV} k_j + \sum_{j,NN} k_j \right)
$$

$$
+ m_r m_s p_N \left( \sum_{j,NN} k_j + \sum_{j,NV} k_j + \sum_{j,NV} k_j + \sum_{j,NN} k_j \right)
$$

$$
+ m_r m_s p_V \left( \sum_{j,NV} k_j + \sum_{j,NN} k_j + \sum_{j,NV} k_j + \sum_{j,NN} k_j \right)
$$

$$
\left( \sum_{j,NN} k_j + \sum_{j,NV} k_j + \sum_{j,NV} k_j + \sum_{j,NN} k_j \right)
$$

$$
\right)
$$

$$
(18)
$$

respectively. After some algebra and using Eq. (1) if $i$ is an $N$ node and Eq. (2) if $i$ is a $V$ node we obtain

$$
\frac{\partial E_{i(N)}}{\partial t} = \frac{\partial k_{i(N)}}{\partial t} + \frac{(m_s - 1) A_N}{t}
$$

$$
(20)
$$

and

$$
\frac{\partial E_{i(V)}}{\partial t} = \frac{\partial k_{i(V)}}{\partial t} + \frac{(m_s - 1) A_V}{t}.
$$

$$
(21)
$$

After integrating both sides of Eqs. (20) and (21) with respect to $t$, using the initial condition $E_{i(N)}(k_{init}, t_i) = E_{i(V)}(k_{init}, t_i) = m_r m_s$, and $k_{init}$ given by Eq. (6f), we obtain the expressions for the change with time of the number of connections between the nearest neighbors of a node of degree $k_{iN}$

$$
E_{i(N)} = k_i - m_r + (m_s - 1) A_N \ln \left( \frac{t}{t_i} \right)
$$

$$
(22)
$$
if $i$ is an $N$ node and of a node of degree $k_i$:

$$E_{i(V)} = k_i - m_r + (m_s - 1)A_V \ln \left( \frac{t}{t_i} \right)$$  \hspace{1cm} (23)

if $i$ is a $V$ node. We use Eqs. (14) and (15) to obtain expressions for $\ln(t/t_i)$ and insert them in Eqs. (22) and (23), respectively. Finally, the degree-dependent clustering coefficient $C(k)$ as a function of the degree $k$ of the node which is also referred as the clustering spectrum, is given by:

$$C_N(k) = 2 \frac{k - m_r + (m_s - 1)A_NG_1 \ln [(k + G_2)/G_3]}{k(k - 1)}$$  \hspace{1cm} (24)

for an $N$ node and by:

$$C_V(k) = 2 \frac{k - m_r + (m_s - 1)A_VH_1 \ln [(k + H_2)/H_3]}{k(k - 1)}$$  \hspace{1cm} (25)

for a $V$ node, where we make use of $C(k) = 2E(k)/[k(k - 1)]$, which defines the clustering coefficient of a vertex as the ratio of the total number of existing connections between all $k$ of its neighbors and the number $k(k - 1)/2$ of all possible connections between them. The degree-dependent clustering coefficient $C(k)$ defines the local clustering properties of the network.

The global clustering characteristics of a network include the mean clustering coefficient $\bar{C}$ as averaged over the vertex degree, the mean clustering $< C > = 1/n \sum_i C_i$ as averaged over the nodes of the network (where $C_i$ is the clustering coefficient of node $i$), and the so-called transitivity $C$. Making use of degree-dependent local clustering coefficient (Eqs. (24) and (25)) and the degree distribution (Eqs. (16) and (17)) for $N$ or $V$ node one can define the respective mean clustering coefficient as:

$$\bar{C} = \sum_k P(k)C(k).$$  \hspace{1cm} (26)

Transitivity is a measure of the ratio of the total number of loops of length three in a graph to the total number of connected triples and is defined as $[3, 17]$

$$C = \frac{\sum_k P(k)E(k)}{\sum_k P(k)k(k - 1)/2}.$$  \hspace{1cm} (27)

The mean clustering coefficient $\bar{C}$ and the transitivity $C$ assess in a different manner the clustering properties of a graph. In real networks they could have very different values for the same network $[18]$.
3. Comparison between Theory and Simulation Results

We compare three outputs: the analytical derivation of degree distribution \( P(k) \) (Eqs. (16) and (17)) that was obtained assuming that the links to secondary contacts are established with the same probabilities as the links to initial contacts Eq. (1), the derivation of \( P(k) \) (Eqs. (14) and (15)) obtained by using functional dependences Eq. (3), and the numerical simulation results.

3.1. Functional dependence

First, let us focus on the functional dependence of the probability that the edge is between \( N \) nodes \( g(\rho_N) \) on \( \rho_N \) (Eq. (3a)), which represents the combined probabilities that the edge is between same type of nodes \( \rho_N \) and the probability that they are \( N \) nodes. We empirically estimate \( g(\rho_N) \) by calculating the number of \( NN \) edges, multiplying it by two and dividing it by the average number of nodes at time \( t \) which is \( 2m_r(1 + m_s)t \). We obtain the functional dependence of empirically estimated \( g(\rho_N) \) by a matrix multiplication of two vectors; one is the vector of \( g(\rho_N) \) values as a function of \( p_N \) at fixed \( p_s = 0.7 \) and the other is the vector of \( g(\rho_N) \) values as a function of \( p_s \) at fixed \( p_N = 0.8 \). Next, we aim to construct a function of \( p_N \) and \( p_s \), \( \rho_N(p_N, p_s) \) such that the empirically estimated \( g(\rho_N(p_N, p_s)) \) which express the probabilities for both initial and secondary contacts is a linear function of \( \rho_N \). We plot the result for Case I (one initial contact and two secondary contacts) in Fig. 2a for \( p_N \in [0.5, 0.9] \) at fixed values of \( p_s = 0.7 \) and for \( p_s \in [0.5, 0.9] \) at fixed values of \( p_N = 0.8 \) (circles ○). Squares (□) mark results for \( p_N \in [0.5, 0.9] \) at fixed values of \( p_s = 0.5 \) and for \( p_s \in [0.5, 0.9] \) at fixed values of \( p_N = 0.5 \). We obtain that the combined probability \( \rho_N(p_N, p_s) \) of the form

\[
\rho_N(p_N, p_s) = p_N(c_{pN} + p_s(1 - c_{pN})) \tag{28}
\]

produces the following least-squares linear fit

\[
g(\rho_N) = 0.54\rho_N - 0.10 \pm \Delta_N \quad (\bigcirc) \tag{29a}
g(\rho_N) = 0.29\rho_N - 0.04 \pm \Delta_V \quad (\bigbox) \tag{29b}
\]

for \( c_{pN} = 0.8 \) (○) and \( c_{pN} = 0.5 \) (□), respectively.

The prediction error estimate was generated for \( g(\rho_N) \) and found to be \( \Delta_N = 0.0221 \) (for ○) and \( \Delta_N = 0.0095 \) and (for □) which allows us to obtain
Figure 2: (a) Probability to establish both initial and secondary contacts between $N$ nodes $g(\rho_N(p_N, p_s))$ as a function of $\rho_N(p_N, p_s) = p_N(c_{p_N} + p_s(1 - c_{p_N}))$, where $c_{p_N} = 0.8$ for $p_N = 0.8$ (○) and $c_{p_N} = 0.5$ for $p_N = 0.5$ (□). (b) Probability to establish both initial and secondary contacts between $V$ nodes $h(\rho_V(p_V, p_s))$ as a function of $\rho_V(p_V, p_s) = p_V(c_{p_V} + p_s(1 - c_{p_V}))$, where $c_{p_V} = 1 - c_{p_N} = 0.2$ for $p_V = 0.2$ (○) and $c_{p_V} = 0.5$ for $p_V = 0.5$ (□). (c) Probability to establish both initial and secondary contacts between different types of nodes as a function of $\rho_d(p_d) = p_d/(p_d + 2)$ for $p_d \in [0.1, 0.5]$ at fixed $p_N = 0.8$ (○), and fixed $p_N = 0.5$ (□), respectively. Values on y-axis in (a) represent matrix multiplication of the vector of $g(\rho_N(p_N, p_s))$ as a function of $p_N \in [0.5, 0.9]$ at fixed $p_s = 0.7$ (○) ($p_s = 0.5$ (□)) multiplied by the vector of normalized number of $NN$ edges as a function of $p_s \in [0.5, 0.9]$ at fixed $p_N = 0.8$ (○) ($p_N = 0.5$ (□)). Values on y-axis in (b) represent matrix multiplication of the vector of $h(\rho_V(p_V, p_s))$ as a function of $p_V \in [0.1, 0.5]$ at fixed $p_s = 0.7$ (○) ($p_s = 0.5$ (□)) multiplied by the vector of normalized number of $VV$ edges as a function of $p_s \in [0.5, 0.9]$ at fixed $p_V = 0.2$ (○) ($p_V = 0.5$ (□)). Case I: one node as initial contact and two nodes as secondary contacts.
a range of values for \(g(\rho_N)\) limited by \(g(\rho_N) \pm \Delta_N\). We use these functional dependences within their range to express \(g(\rho_N(p_N, p_s))\) in the solutions of the rate equations Eqs. (5) and (7) and in the expression of respective degree distributions and clustering coefficients.

Applying the same reasoning, we obtain the combined probability \(h(\rho_V(p_V, p_s))\) as a function of \(\rho_V\) which represents both the initial and secondary contacts. Results are shown in Fig. 2b for \(p_V \in [0.1, 0.5]\) at fixed \(p_s = 0.7\) and \(p_s \in [0.5, 0.9]\) at fixed \(p_V = 0.2\) (○). Squares (□) mark results for \(p_V \in [0.1, 0.5]\) at fixed \(p_s = 0.5\) and \(p_s \in [0.5, 0.9]\) at fixed \(p_V = 0.5\). For an argument of the form

\[
\rho_V(p_V, p_s) = p_V(c_{p_V} + p_s(1 - c_{p_V})), \tag{30}
\]

the least-square fit produces

\[
h(\rho_V) = 0.11\rho_V \pm \Delta_V \quad (\bigcirc) \tag{31a}
\]

\[
h(\rho_V) = 0.20\rho_V - 0.01 \pm \Delta_V \quad (\square) \tag{31b}
\]

for \(c_{p_V} = 1 - c_{p_N} = 0.2\) (○) and \(c_{p_V} = 0.5\) (□), respectively. The prediction error estimate defines the range of values for \(h(\rho_V) \pm \Delta_V\), where \(\Delta_V = 0.0056\) (for ○) and \(\Delta_V = 0.0039\) (for □).

We obtain the probability \(q(\rho_d(p_d))\) for creating a link between different type of nodes both as initial and as secondary contacts to be

\[
q(\rho_d) = 1.60\rho_d + 0.1 \pm \Delta_d \quad (\bigcirc) \tag{32a}
\]

\[
q(\rho_d) = 1.86\rho_d + 0.15 \pm \Delta_d \quad (\square). \tag{32b}
\]

The dependence of \(q(\rho_d(p_d))\) as a function of \(\rho_d\) is shown in Fig. 2c: for \(p_d \in [0.5, 0.9]\) at fixed \(p_N = 0.8\) (○) and \(p_N = 0.5\) (□). We applied the linear least-square fit for an argument of the form

\[
\rho_d(p_d) = p_d/(p_d + 2). \tag{33}
\]

The prediction error estimate is obtained to be \(\Delta_d = 0.0207\) (for ○) and \(\Delta_d = 0.0265\) (for □).

We use the above parameterization procedures of the probabilities to establish both initial and secondary contacts between \(N\) nodes \(g(\rho_N(p_N, p_s))\), between \(V\) nodes \(h(\rho_V(p_V, p_s))\), and between different types of nodes \(q(\rho_d(p_d))\) in obtaining the degree distribution (Eqs. (14) and (15)) and clustering coefficient (Eqs. (24) and (25)) for each of the cases considered below.
Figure 3: Degree distribution for two populations consisting of 80% $N$ nodes (◦) $p_N = 0.8$ and 20% $V$ nodes $p_V = 0.2$ (●). Edges are formed with probability $p_s = 0.7$ if they connect $N$ with $N$ or $V$ with $V$ nodes and with probability $p_d = 0.3$ if they connect $N$ with $V$ nodes. Symbols represent results from numerical simulations done on a network with $10^5$ agents and averaged over 100 runs. Lines represent the results of degree distribution $P(k)$ given by Eq. (14) for $N$ nodes (solid) and by Eq. (15) for $V$ nodes (dashed) using functional dependences shown in Fig. 2 to obtain functions $\rho_N \pm \Delta_N$, $\rho_V \pm \Delta_V$, and $\rho_d \pm \Delta_d$. The set of two lines correspond to using $+\Delta_\nu$ or $-\Delta_\nu$ for each of $\rho_\nu$, where $\nu = N, V, d$. The dash-dotted line represents analytical distribution $P(k)$ obtained using Eq. (16) for $N$ nodes. The dotted line represents analytical distribution $P(k)$ obtained using Eq. (17) for $V$ nodes. (a) Case I: one node as initial contact and two nodes as secondary contacts; (b) Case II: one node as initial contact with probability 0.9 and two nodes as initial contacts with probability 0.1; the number of nodes as secondary contacts is from uniform distribution $U[0, 3]$; (c) Case III: two nodes as initial contacts; the number of nodes as secondary contacts is from uniform distribution $U[0, 2]$. 
3.2. Degree distribution

The analytical distribution \( P(k) \) is given by Eq. (16) for \( N \) nodes and by Eq. (17) for \( V \) nodes and plotted in Fig. 3 with dash-dotted and dotted lines, respectively. The degree distribution \( P(k) \) obtained using functional dependences is given by Eq. (14) for \( N \) nodes and by Eq. (15) for \( V \) nodes and plotted in Fig. 3 with solid and dashed lines, respectively. Simulations are conducted on a network with \( 10^5 \) nodes starting with a seed network of 8 nodes and are averaged over a 100 runs. All three cases considered are for value of the probability to create an \( N \) node \( p_N = 0.8 \) and for the value of the probability to establish a link between the same type of nodes \( p_s = 0.7 \).

To touch upon the versatility of the model we consider three different cases. They are Case I: one node as initial contact \( m_r = 1 \) and two nodes as secondary contacts \( m_s = 2 \) (Fig. 3a); Case II: one node as initial contact with probability 0.9 and two nodes as initial contacts with probability 0.1, which gives \( m_r = 1.1 \); the number of nodes as secondary contacts is from uniform distribution \( U[0,3] \) and therefore, \( m_s = 1.5 \) (Fig. 3b); Case III: two nodes as initial contacts \( m_r = 2 \); the number of nodes as secondary contacts is from uniform distribution \( U[0,2] \), \( m_s = 1 \) (Fig. 3c). Results demonstrate that for all cases considered the simulations compare relatively well with the analytical derivation of the degree distribution even though using functional dependences in \( P(k) \) derivation improves the agreement within the limits of the simulations.

3.3. Clustering

The simulation results for the clustering coefficient as a function of the degree of the node \( C(k) \) for \( N \) and \( V \) nodes for Case III are plotted in Fig. 4 with empty and full circles, respectively. The analytical solution for clustering coefficient \( C(k) \) using Eq. (24) for \( N \) nodes and Eq. (25) for \( V \) nodes are plotted with lines which coincide with each other. A clear \( C(k) \sim 1/k \) trend is observed which indicates the hierarchy in the system.

The global clustering properties of the network are assessed by the mean clustering coefficient \( \bar{C} \) (Eq. (26)) which is averaged over vertex degree, and the transitivity \( C \) (Eq. (27)). We study how \( \bar{C} \) and \( C \) change as a function of \( p_N \) for a fixed value of \( p_s \) (plotted in Fig. 5a,c) and as a function of \( p_s \) for a fixed value of \( p_N \) (plotted in Fig. 5b,d). The mean clustering coefficient \( \bar{C} \) for Case I (circles in Fig. 5a) has values in the range between 0.41 and 0.43 as a function of \( p_N \) for both \( N \) and \( V \) nodes for fixed value of \( p_s = 0.7 \). In both Case II (squares) and Case III (diamonds), where the number of secondary
contacts is drawn from a uniform distribution, the $\bar{C}(p_N)$ for $N$ and $V$ nodes is symmetrical with respect to its value at $p_N = 0.5$. Values of $\bar{C}$ for $N$ and $V$ nodes as a function of $p_s$ (Fig. 5b) demonstrate a tendency to converge for $p_s$ approaching one.

The transitivity $C$ (Fig. 5c) as a function of $p_N$ shows a symmetrical pattern for $N$ and $V$ nodes with respect to its value at $p_N = 0.5$ similar to $\bar{C}$ behavior but with a wider difference between the results for $N$ and $V$ nodes and different values. As the probability to establish a link between same type of nodes $p_s$ increases the values of transitivity $C$ (Fig. 5d) for $N$ and $V$ nodes converge. Higher values of both $\bar{C}$ and $C$ among the three cases are obtained for Case II when there is an option to create either one or two initial contacts and the number of secondary contacts vary as well, e.g. in $U[0,3]$.

3.4. Assortativity

The assortative properties of the network describe the degree correlation of the nodes at the ends of an edge and are quantified by the Pearson correlation coefficient $r$. The network is said to be assortative when high degree nodes tend to connect to other high-degree nodes and $r \in (0,1]$ \cite{19}. The network is characterized by disassortative mixing when high degree nodes tend to connect to low degree nodes and $r \in [-1,0)$. We calculate $r$ of each of $NN$, $VV$, and $NV$ networks obtained for a fixed value of $p_N = 0.8$ and
Figure 5: Values of the mean clustering coefficient $\bar{C}$ (Eq. (26)) (a) as a function of the probability to create an $N$ node $p_N$ for a fixed value of the probability to create a link between same type of nodes $p_s = 0.7$ and (b) as a function of $p_s$ for a fixed value of $p_N = 0.8$ for $N$ nodes (empty symbols) and $V$ nodes (full symbols). (c,d) Same as (a,b) but for the values of transitivity $C$ (Eq. (27)). Circles ($N \circ / V \bullet$) represent Case I. Squares ($N \square / V \blacksquare$) represent Case II and diamonds ($N \lozenge / V \biglozenge$) mark Case III. Cases I, II, and III are as defined in Fig. 3.
Figure 6: Values of assortativity coefficient $r$ as a function of the probability to create a link between same type of nodes $p_s$ for a fixed value of $p_N = 0.8$ of (a) $NN$ network; (b) $VV$ network; (c) $NV$ network. Circles (●) represent Case I. Squares (■) represent Case II and diamonds (♦) mark Case III. Cases I, II, and III are as defined in Fig. 3.
plot the $r$-dependence as a function of $p_s$ in Fig. 6a-c for the three cases considered. $NN$ and $NV$ networks of Case I (●) show slight disassortative mixing owing to the two initial contacts and one secondary contact. Increasing the probability $p_s$ to create an edge between the same type of nodes for Case I (● in Fig. 6b) results in changing from disassortative for $p_s < 0.5$ to slightly assortative for $p_s > 0.5$. Varying the number of initial and secondary contacts in Case II (■) and III (♦) produces assortatively mixed $NN$ and $VV$ networks (Fig. 6a,b). The assortativity coefficient $r_{VV}$ of the $VV$ networks (■ and ♦ in Fig. 6b) increases linearly with increasing the probability to create a link between the same type of nodes reaching a plateau around $p_s \sim 0.7$ and then decreasing for $p_s = 0.9$ probably because all available nodes are already connected. The values of $r_{VV}$ are larger versus $r_{NN}$ because of the smaller number ($p_V = 1 - p_N = 0.2$) of $V$ nodes available for contacts. A recent visualization of the connections in a terrorist network such as the Global Salafi Jihad depicts them to form an assortative network [20]. For all three cases the $NV$ networks (Fig. 6c) show disassortative mixing of their degree only except for very high values of $p_s = 0.8$ and 0.9 for Case III (♦). A recent empirical study of an online social system reports that relationships driven by aggression lead to markedly different systemic characteristics than relations of a non-aggressive nature [21]. Assortativity is a characteristic of global properties of the system. In agreement with the empirical findings the assortativity of $NV$ network (Fig. 6c) produced by our model which represents relationships driven by aggression is clearly different from the assortativity of $NN$ and $VV$ networks (Fig. 6a,b) which are driven by non-aggressive relationships.

4. Conclusions

We introduced a model intended to characterize the interactions between two distinct populations, which form links more easily within their group than between groups. We aim to describe the interactions of potentially violent terrorist groups within the context of a largely non-violent population, although the same model could, in principle, be applied to other non-mainstream social groups. The model is kept simple enough so that analytical solutions could be derived and compared with empirical parameterizations and numerical simulation results.

The model produces networks with relatively high mean clustering coefficient $\bar{C}$ and transitivity $C$. Their values vary with the balance between
the initial and secondary contacts. This is expected because of the inter-
play between the random and preferential attachments of the initial and 
secondary connections, respectively. The assortativity pattern of modeled 
networks show that the potentially violent $VV$ network qualitatively resem-
bles the connectivity pattern in terrorist networks reported in [20]. The 
assortativity behavior of $NV$ network which is driven by aggression is clearly 
different than the assortativity pattern of $NN$ and $VV$ networks which are 
non-aggressive relationships; a finding which is in agreement with the results 
of recent empirical study of an online social system [21].

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