THE SNAPBACK REPELLERS FOR CHAOS IN MULTI-DIMENSIONAL MAPS

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Abstract. The key of Marotto’s theorem on chaos for multi-dimensional maps is the existence of snapback repeller. For practical application of the theory, locating a computable repelling neighborhood of the repelling fixed point has thus become the key issue. For some multi-dimensional maps \( F \), basic information of \( F \) is not sufficient to indicate the existence of snapback repeller for \( F \). In this investigation, for a repeller \( \bar{z} \) of \( F \), we start from estimating the repelling neighborhood of \( \bar{z} \) under \( F^k \) for some \( k \geq 2 \), by a theory built on the first or second derivative of \( F^k \). By employing the Interval Arithmetic computation, we locate a snapback point \( z_0 \) in this repelling neighborhood and examine the nonzero determinant condition for the Jacobian of \( F \) along the orbit through \( z_0 \). With this new approach, we are able to conclude the existence of snapback repellers under the valid definition, hence chaotic behaviors, in a discrete-time predator-prey model, a population model, and the FitzHugh nerve model.

1. Introduction. After Li and Yorke [12] proved the celebrated period-three-implies-chaos theorem for one-dimensional maps, extending analytic theory of chaos to multi-dimensional maps has become an interesting research topic. In 1978, Marototto advocated the notion of snapback repeller to generalize Li-Yorke’s theorem on chaos from one-dimension to multi-dimension. Due to a technical flaw on the condition which leads to the repelling neighborhood of a repelling fixed point (repeller), the definition of snapback repeller was modified in 2005 to validate Marotto’s theorem.

Let us first make clear the definition of repelling neighborhood.

Definition 1.1. Consider a \( C^1 \) map \( F : \mathbb{R}^n \to \mathbb{R}^n \) and denote by \( B_r(x) \) the closed ball in \( \mathbb{R}^n \) with center at \( x \) and radius \( r > 0 \) under certain norm on \( \mathbb{R}^n \). A fixed point \( \bar{z} \) of \( F \) is called repelling if all eigenvalues of \( DF(\bar{z}) \) exceed one in magnitude. Moreover, if there exist a norm \( \| \cdot \| \) on \( \mathbb{R}^n \) and a constant \( c > 1 \) such that

\[
\| F(x) - F(y) \| > c \cdot \| x - y \|, \quad \text{for all } x, y \in B_r(\bar{z}), \ x \neq y,
\]

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where $B_r(\bar{z})$ is defined under this norm, then we call $B_r(\bar{z})$ a repelling neighborhood of $\bar{z}$.

It is known that if $\bar{z}$ is a repelling fixed point of $F$, then there exist a norm and an $r > 0$ so that $B_r(\bar{z})$ is a repelling neighborhood of $\bar{z}$. Notice that this property does not necessarily hold for the Euclidean norm in general. Even if $\bar{z}$ is a fixed point and $B_r(\bar{z})$ is a closed ball centered at $\bar{z}$, under some norm, such that

$$|\lambda(z)| > 1, \text{ for all eigenvalues } \lambda(z) \text{ of } DF(z), \text{ for all } z \in B_r(\bar{z}),$$  \hspace{1cm} (1.2)

$B_r(\bar{z})$ still may not be a repelling neighborhood of $\bar{z}$. The reason is that the norm constructed for such a property depends on the matrix $DF(z)$ which varies at different points $z$, as the mean-value inequality is applied, cf. [5].

Now, we recall the modified definition of snapback repeller [15] and state Marotto’s theorem. For any point $z_0 \in \mathbb{R}^n$, we denote $z_j = F^j(z_0)$, for $0 \leq j < \infty$, and $z_{-j} = F(z_{-j-1})$, for $j \geq 0$, if such pre-images of $z_0$ exist.

**Definition 1.2.** Let $\bar{z}$ be a repelling fixed point of the $C^1$ map $F$. If there exist a point $z_0 \neq \bar{z}$ in a repelling neighborhood of $\bar{z}$ and a positive integer $\ell \geq 2$, such that $z_{\ell} = \bar{z}$ and $\det(DF(z_j)) \neq 0$ for $1 \leq j < \ell$, then $\bar{z}$ is called a snapback repeller of $F$.

We call the point $z_0$ in the Definition 1.2 a “snapback point” of $F$. Under Definition 1.2, the following Marotto’s theorem holds [14, 15].

**Theorem 1.3.** (Marotto’s Theorem) If a $C^1$ map $F$ has a snapback repeller, then $F$ is chaotic in the following sense: there exist (i) a positive integer $N$, and $F$ has periodic points of period $p$ for every integer $p \geq N$, (ii) a scrambled set of $F$, i.e., an uncountable set $S$ containing no periodic points of $F$, and

(a) $F(S) \subset S$,

(b) $\limsup_{k \to \infty} \|F^k(x) - F^k(y)\| > 0$, for all $x, y \in S$, with $x \neq y$,

(c) $\limsup_{k \to \infty} \|F^k(x) - F^k(y)\| > 0$, for all $x \in S$ and periodic points $y$ of $F$,

(iii) an uncountable subset $S_0$ of $S$, and $\liminf_{k \to \infty} \|F^k(x) - F^k(y)\| = 0$, for every $x, y \in S_0$.

Marotto’s theorem provides an analytic method to detect chaos, which is effective in applications in finding the chaotic regimes (parameter ranges) for dynamical systems. However, in respecting this new definition, locating the snapback point in a repelling neighborhood of the repeller becomes the key issue in applying this theorem. If the norm in (1.1) is not Euclidean, then computing the repelling neighborhood is quite inaccessible. In addition, because of the unstable nature for chaotic behaviors, confirming the extent of repelling neighborhood and finding a snapback point in this neighborhood have been a nontrivial task, from computational view point. In order to overcome such a difficulty, developing skillful computation technique, combined with analytical properties of the maps, is an important research task. While a general approach to confirming snapback repeller has been lacking, certain piecewise-linear maps or maps in specific forms were considered in [7, 18].

Notably, in [8], the condition $\det(DF(z_j)) \neq 0$, $1 \leq j < \ell$, is not included in the definition of snapback repeller. Instead, a homoclinic orbit to a repeller was called nondegenerate therein if such condition also holds, and the investigation focussed on bifurcation from degenerate homoclinic orbit to nondegenerate homoclinic orbit.

In this paper, we establish a new approach to judge whether a repeller is a snapback repeller. This approach then leads to effective applications of Marotto’s
theorem to conclude chaotic behaviors in multi-dimensional maps. The chaotic behaviors for the maps considered in [4, 6, 9, 17, 19, 20] were studied under the original invalid definition of snapback repeller [14]. With the present approach, we are able to investigate the snapback repellers, under the valid Definition 1.2, and hence chaotic behaviors, for those maps.

We first propose a methodology to exploit the repelling neighborhood of a repeller. In section 2, we extend a previous result on finding repelling neighborhood of a repeller in [13]. Instead of considering the repelling neighborhood of a repeller \( \bar{z} \) for map \( F \), herein, we locate a repelling neighborhood \( U \) of \( \bar{z} \) for map \( F^k \), for some integer \( k \geq 2 \). Moreover, if there exists a point \( y_0 \in U \), \( y_0 \neq \bar{z} \), such that \( (F^k)^\ell(y_0) = \bar{z} \), for some integer \( \ell \geq 2 \), and \( \det(DF(y_j)) \neq 0 \) for \( -\infty < j < k\ell \), then \( \bar{z} \) is a snapback repeller of \( F^k \). Subsequently, we can further argue that \( \bar{z} \) is also a snapback repeller of \( F \).

In section 3, we introduce our computational approach which is based on the Interval Arithmetic (IA). Interval Arithmetic is an arithmetic defined on sets of intervals, rather than sets of real numbers. Modern development of IA can be traced back to Moore’s dissertation [16]; further notion and details for IA can be found in [1, 2]. Indeed, IA is an arithmetic that contains all numerical errors of each numerical computation step, and thus can provide a rigorous justification via numerical computation. We shall perform IA computation to find a repelling neighborhood \( \tilde{U} \) of repeller \( \bar{z} \) for map \( F^k \), for some \( k \geq 1 \), under the Euclidean norm. The above-mentioned two approaches lead to two neighborhoods \( U \) and \( \tilde{U} \) of \( \bar{z} \) for map \( F^k \). The existence of true pre-images of \( \bar{z} \) under \( F \) can be confirmed by IA computation. By justifying \( \det(DF(z)) \neq 0 \) for \( z \in U \) or \( \tilde{U} \), we can then confirm that \( \bar{z} \) is a snapback repeller of \( F^k \).

Computer-assisted approaches have played crucial roles in investigating complex dynamics in state-of-the-art research. For example, computer-assisted Morse decomposition and Conley index were performed to study the recurrent dynamics, and the global dynamics in [3]. On the other hand, IA computation is a handy tool with convenient programming packages, and is rather powerful for investigating and confirming local dynamical objects.

The rest of this presentation is organized as follows. In section 2, we present an estimation on the radius of repelling neighborhood of a repeller for map \( F^k \), \( k \geq 1 \), under the Euclidean norm. Moreover, we explore the relation of snapback repeller between map \( F \) and map \( F^k \), \( k \geq 2 \). In section 3, we briefly introduce the notion of IA and apply IA computation to find a repelling neighborhood of repeller \( \bar{z} \) for map \( F^k \). We construct in this region a backward orbit of \( \bar{z} \) under map \( F^k \) to confirm that \( \bar{z} \) is a snapback repeller of \( F^k \) for some \( k \geq 2 \), and hence \( F \). We demonstrate the present theory and IA computation technique in a discrete-time predator-prey system in section 3. In section 4, we apply the present methodologies to study snapback repellers and chaotic behaviors in the discrete-time FitzHugh nerve model and a discrete-time population model.

2. Estimating repelling neighborhood. Consider a \( C^1 \) map \( F \) with fixed point \( \bar{z} \). An eigenvalue condition on \( DF(\bar{z}) \), the linearization of \( F \) at \( \bar{z} \), which leads to the existence of a repelling neighborhood of \( \bar{z} \), was given in [10]:

\[
\text{If } \lambda > 1, \text{ for all eigenvalues } \lambda \text{ of } (DF(\bar{z}))^TDF(\bar{z}); \quad (2.1)
\]

then there exist an \( c > 1 \) and an \( r > 0 \) such that all eigenvalues of \( (DF(z))^TDF(z) \) are greater than one for all \( z \in B_r(\bar{z}) \) (under Euclidean norm), and \( \|F(x) - \bar{z}\| < r \).
Consider a repeller $\bar{z}$. The following proposition provides an estimation for the repelling neighborhood of $F$.

For a given map $F$, we consider its $k$-fold composition $G := F^k$, where $k$ is a positive integer. Let $\bar{z}$ be a repelling fixed point of $F$, and hence $G$. We denote

$$s_k := \sqrt{\text{minimal eigenvalue of } (DG(\bar{z}))^T DG(\bar{z})},$$

and the $n \times n$ matrix

$$L_k(w, \bar{z}) := DG(w) - DG(\bar{z}).$$

For an $r > 0$, on the closed ball under Euclidean norm, $B_r(\bar{z})$, we set

$$\eta_{k,r} := \max_{w \in B_r(\bar{z})} \|L_k(w, \bar{z})\|_2 = \max_{w \in B_r(\bar{z})} \sqrt{\text{maximal eigenvalue of } (L_k(w, \bar{z}))^T L_k(w, \bar{z})}.$$  

The following proposition provides an estimation for the repelling neighborhood of repeller $\bar{z}$ under map $G = F^k$.

**Proposition 2.1.** Consider a $C^1$ map $F$ with fixed point $\bar{z}$. Let $s_k, \eta_{k,r}$ be defined as in (2.2)-(2.3). Then $B_r(\bar{z})$ is a repelling neighborhood of $\bar{z}$ for map $F^k$, under the Euclidean norm, provided that there exist an $k \in \mathbb{N}$ and an $r > 0$ such that

$$s_k - \eta_{k,r} > 1.$$  

If $F$ is $C^2$, then through the eigenvalues of $(DG(x))^T DG(x)$ and Hessian matrix $[\partial_k \partial_l G_i(x)]_{k \times l}$, we can also obtain a different estimation for the repelling neighborhood of $\bar{z}$ for map $G$. Such an estimation for the repelling neighborhood has been proposed for a map $F$ in [13]. Herein, we extend it to an estimation for map $F^k$. Notice that when (2.4) fails to hold for map $F$ (i.e., $k = 1$), it may still hold for map $F^k$ for some $k \geq 2$. We plan to use the properties when (2.4) holds for map $F^k$ for some $k \geq 2$, to justify the existence of snapback repeller for map $F$.

Next, we discuss the relation between the existence of snapback repeller for map $F$ and map $F^k$, $k \geq 2$.

**Theorem 2.2.** Let $\bar{z}$ be a fixed point of $C^1$ map $F$. (i) If $\bar{z}$ is a snapback repeller of $F$ with snapback point $x_0$, and

$$\det(DF(x_j)) \neq 0, \text{ for } -\infty < j \leq -1,$$

where $F(x_j) = x_{j+1}$, then $\bar{z}$ is a snapback repeller of $F^k$, for each $k \geq 2$. (ii) If $\bar{z}$ is a snapback repeller of $F^{k_0}$ with snapback point $y_0$, for some $k_0 \geq 2$, and

$$\det(DF(y_j)) \neq 0, \text{ for } -\infty < j \leq -1,$$

where $F(y_j) = y_{j+1}$, then $\bar{z}$ is also a snapback repeller of $F$. 


Remark 2.1. (i) If \( \tilde{z} \) is a repeller of \( F \), then \( \tilde{z} \) is also a repeller of \( F^k \) for each \( k \geq 2 \), as \( |\lambda(DF(\tilde{z}))| > 1 \) implies \( |\lambda(DF^k(\tilde{z}))| = |(DF(\tilde{z}))^k| > 1 \). Fix an \( k \geq 2 \). There exist a norm \( \| \cdot \| \) and a \( r_1 > 0 \), such that \( B_{r_1}(\tilde{z}) := \{ x \in \mathbb{R}^n : \| x - \tilde{z} \| \leq r_1 \} \) is a repelling neighborhood of \( \tilde{z} \) for \( F^k \). As \( \tilde{z} \) is a snapback repeller of \( F \), there exist a \( r_0 > 0 \), a norm \( \| \cdot \| \), on \( \mathbb{R}^n \), a repelling neighborhood \( B_{r_0}(\tilde{z}) := \{ x \in \mathbb{R}^n : \| x - \tilde{z} \| \leq r_0 \} \), and a snapback point \( x_0 \in B_{r_0}(\tilde{z}), x_0 \neq \tilde{z} \), such that \( F^k(x_0) = \tilde{z} \), for some \( \ell \geq 2 \), and \( \text{det}(D(F)(x_0)) \neq 0 \), for \( 0 < j < \ell \). Moreover, \( F \) is one-to-one in \( B_{r_0}^s(\tilde{z}) \) and \( F(B_{r_0}^s(\tilde{z})) \supset B_{r_0}^s(\tilde{z}) \). Hence, \( F^{-1} \) is a contraction on \( B_{r_0}^s(\tilde{z}) \) and there exists an integer \( m \) such that \( \tilde{x}_0 := F^{-km+\ell}(x_0) \in B_{r_0}^s(\tilde{z}) \) is a snapback point of \( F^k \), \( \tilde{x}_0 \neq \tilde{z} \), \( (F^k)^m(\tilde{x}_0) = \tilde{z} \), and \( \text{det}(D(F^k)(\tilde{x}_j)) \neq 0 \) with \( \tilde{x}_j := (F^k)^j(\tilde{x}_0) \) for \( j = 1, 2, \cdots, m \), due to (2.5). Hence, \( \tilde{z} \) is also a snapback repeller of map \( F^k \), for each \( k \geq 2 \).

(ii) As \( \tilde{z} \) is a snapback repeller of \( F^{k_0} \), there exists a norm \( \| \cdot \| \), such that \( B_{r_2}^s(\tilde{z}) := \{ x \in \mathbb{R}^n : \| x - \tilde{z} \| \leq r_2 \} \) is a repelling neighborhood of \( \tilde{z} \) under \( F^{k_0} \), and there is a snapback point \( y_0 \in B_{r_2}^s(\tilde{z}) : y_0 \neq \tilde{z} \) and \( (F^{k_0})^m(y_0) = \tilde{z} \), for some \( \ell_0 \geq 2 \). In addition, \( \text{det}(D(F^{k_0})(y_j)) \neq 0 \), for \( 1 < j < \ell_0 \), where \( y_j := (F^{j})^j(y_0) \) for \( j = 1, 2, \cdots, m \), due to (2.5). Hence, \( \tilde{z} \) is also a snapback repeller of map \( F^k \).

\[ \text{det}(D(F)(y_j)) \neq 0, \quad 0 \leq j < k_0\ell_0. \]

Notably, \( \tilde{z} \) is also a repeller of \( F \), due to \( |\lambda(DF^{k_0}(\tilde{z}))| = |(DF(\tilde{z}))^{k_0}| > 1 \), and hence there exist a norm \( \| \cdot \| \) and a \( r_3 > 0 \), such that \( B_{r_3}^s(\tilde{z}) := \{ x \in \mathbb{R}^n : \| x - \tilde{z} \| \leq r_3 \} \) is a repelling neighborhood of \( \tilde{z} \) for \( F \). Since \( F^{-k_0} \) is a contraction on \( B_{r_3}^s(\tilde{z}) \) and there exists an integer \( m_0 \) such that \( z_0 := F^{-k_0m_0}(y_0) \in B_{r_3}^s(\tilde{z}) \), and \( F^{k_0m_0}(z_0) = \tilde{z} \), we see that \( z_0 \) is a snapback point of \( F \). Moreover, \( \text{det}(D(F(z_j))) \neq 0 \) with \( z_j := F^j(z_0) \) and \( z_j = y_{j-m_0}, \) for \( j = 1, 2, \cdots, k_0m_0 - 1 \), by (2.6), and \( \text{det}(D(F(z_j))) \neq 0 \), for \( j = k_0m_0, \cdots, k_0(m_0 + \ell_0) - 1 \), as \( z_j = y_{j-m_0} \), by (2.7). Hence, \( \tilde{z} \) is also a snapback repeller of map \( F \). \( \square \)

Remark 2.1. (i) Theorem 2.2 combined with Proposition 2.1 actually serves as an indicator to whether a repeller is a snapback repeller. In applying Marotto's theorem, this indication prevails over condition (2.1) largely. In practice, to justify that \( \tilde{z} \) is a snapback repeller of map \( F \), we look for a \( k_0 \geq 2 \) and an \( r > 0 \) such that \( B_r(\tilde{z}) \) is a repelling neighborhood of \( \tilde{z} \) under map \( F^{k_0} \) through, say Proposition 2.1, and further locate a point \( y_0 \neq \tilde{z} \) in \( B_r(\tilde{z}) \) such that \( (F^{k_0})^m(y_0) = \tilde{z} \), for some integer \( \ell_0 \geq 2 \). Then \( \tilde{z} \) is a snapback repeller of map \( F^{k_0} \), provided (2.7) holds. Actually, it is more straightforward to examine

\[ \text{det}(DF(y_j)) \neq 0, \quad -\infty < j \leq k_0\ell_0 - 1, \]

which is equivalent to (2.6) with (2.7).

(ii) There is an entropy relationship between maps \( F \) and \( F^k \), namely, \( 0 < h(F^k) \leq k \cdot h(F) \), for each \( k \geq 1 \), where \( h(F^k) \) is the topological entropy for map \( F^k \), see Proposition 2.8 of [11].

In practical application, finding a snapback point in the repelling neighborhood and examining condition (2.6) still require some computation technique which can control numerical error and lead to a rigorous justification on the dynamics. We shall employ Interval Arithmetic computation for such a task in the next section.
3. Computation via Interval Arithmetic. We introduce the Interval Arithmetic in subsection 3.1. In subsection 3.2, we demonstrate our approach and computation to a map $F$ arising from predator-prey models. We hope to verify that the map $G := F^2$ (i.e., $k = 2$) admits a snapback repeller, and then deduce that this repeller is also a snapback repeller of map $F$.

3.1. Interval arithmetic. In this subsection, we introduce the chief notion and theory of the interval arithmetic. First, we denote by $\mathbb{IR}$ the collection of all bounded closed intervals in $\mathbb{R}$, and by $\mathbb{IR}^n$ the collection of all bounded closed $n$-dimensional boxes in $\mathbb{R}^n$, namely,

$$
\mathbb{IR} := \{[a, b] | a \leq b, a, b \in \mathbb{R}\},
\mathbb{IR}^n := \{([a_1, b_1], [a_2, b_2], \cdots, [a_n, b_n]) | a_i \leq b_i, a_i, b_i \in \mathbb{R}, \text{ for } 1 \leq i \leq n\}.
$$

A real interval vector $X \in \mathbb{IR}^n$ represents a box in $\mathbb{R}^n$:

$$
X = (X_1, X_2, \cdots, X_n), \quad \text{with } X_i = [a_i, b_i] \in \mathbb{IR},
$$

for some $a_i, b_i, a_i \leq b_i, 1 \leq i \leq n$. Basic arithmetic operations with intervals are defined by

$$
X \circ Y := \{x \circ y | x \in X, y \in Y\},
$$

where operator $\circ \in \{+,-,\times, \div\}$. For division, the assumption $0 \notin Y$ is required. This definition can be easily extended to arithmetic on vectors and matrices with interval entries. Note that a real number $a$ can be treated as a degenerate interval $a := [a, a]$. The interval arithmetic can therefore be regarded as an extension from the usual arithmetic of real numbers. For a function $f = f(x), x \in \mathbb{R}$, its “natural extension” of interval function $f(X)$ is evaluated on interval $X$, and is a closed interval enclosing $\{f(x) | x \in X\}$. Such interval functions admit the inclusion monotonicity, i.e.,

$$
f(X) \subseteq f(Y), \quad \text{for all } X \subseteq Y, \quad (3.1)
$$

where $X$ and $Y$ lie in the domain of $f$. Similar notation and consideration apply to functions $f = f(x), x \in \mathbb{R}^n$ and $f(X)$ on interval vector $X$. One of the most important virtues of IA is that computation is executed in terms of the endpoints of intervals, in the outward rounding mode\(^1\). One can therefore obtain rigorous bounds for complicated inequalities or inclusions with only machine operations. Furthermore, in combining with the following Interval Newton’s method, it is feasible to perform qualitative mathematical proofs through numerical computation.

**Theorem 3.1. (Interval Newton’s Method [2].)** For a $C^1$ function $f : \mathbb{R}^n \to \mathbb{R}^n$, define an operator $I : C^1(\mathbb{R}^n, \mathbb{R}^n) \times \mathbb{IR}^n \to \mathbb{IR}^n$ by

$$
I(f, Z) := \hat{z} - (Df(\hat{Z}))^{-1}f(\hat{z}), \quad (3.2)
$$

where $Z = (Z_1, \cdots, Z_n)$ is an interval vector in $\mathbb{IR}^n$ with $Z_i = [a_i, b_i] \in \mathbb{IR}$ for $i = 1, \cdots, n$, and $\hat{z} = (z_1, \cdots, z_n) \in \mathbb{R}^n$ is the middle point of $Z$, i.e., $z_i = (a_i + b_i)/2$, for $i = 1, \cdots, n$. If there exists an interval vector $Z_0 \in \mathbb{IR}^n$ such that $I(f, Z_0) \subseteq Z_0$, then there exists a unique solution to $f(x) = 0$ in $Z_0$.

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\(^1\) For an interval $[a, b]$, the lower bound $a$ is rounded down to the largest machine-representable number that is less than $a$ and the upper bound $b$ is rounded up to the smallest machine-representable number that is greater than $b$. 
3.2. **Application to a predator-prey system.** Our approach is to locate the repelling neighborhood of a repeller $\bar{z}$ for $F^k$ under the Euclidean norm, for some $k \geq 2$, and then find a snapback point $y_0$ in this neighborhood under $F^k$. We then employ multi-shooting method to locate the homoclinic orbit of $\bar{z}$ under $F$, via IA computation. The last step is to examine condition (2.6) or (2.8) by IA computation and apply Theorem 2.2 to confirm that $\bar{z}$ is a snapback repeller of $F$. This process will conclude the existence of snapback repeller for a $C^1$ map $F$ through rigorous justification via numerical computation based on IA.

In this subsection, we demonstrate our approach by the following discrete-time predator-prey system [4, 17]

$$F(x, y) = (f_1(x, y), f_2(x, y)) = \left( x e^{b(1-x)-ay}, x(1-e^{-ay}) \right).$$  \hspace{1cm} (3.3)

We consider $a = 5$ and $b = 3$ to illustrate the idea. This map has three fixed points

$\bar{z}_1 = (0, 0)$, $\bar{z}_2 \approx (0.42142964483936, 0.34714221309639) =: \bar{z}_{2,q}$, $\bar{z}_3 = (1, 0)$.

In [17], shadowing arguments and computer-assisted techniques were employed to show that $\bar{z}_2$ is a snapback repeller of $F$, under the original (invalid) definition of snapback repeller, and sup-norm on $\mathbb{R}^2$. Herein, we employ the valid Definition 1.2 of snapback repeller to justify that $\bar{z}_2$ is indeed a snapback repeller of $F$. Moreover, the approaches in [10, 13] can not be applied in this case, as the condition (2.1) fails to hold for $\bar{z}_2$. In addition, by using Proposition 2.1, we find that $B_{0,009}(\bar{z}_2)$ is a repelling neighborhood of $\bar{z}_2$ for map $F^2$. Hence, there is a potential that $\bar{z}_2$ is a snapback repeller of $F^2$. Therefore, we plan to verify that $\bar{z}_2$ is a snapback repeller of map $G(x, y) := F^2(x, y)$, and hence of $F$, by Theorem 2.2.

Let us abbreviate $\bar{z}_2$ as $\bar{z}$ and adopt the Euclidean norm on $\mathbb{R}^2$. We shall use the built-in IA tools in Mathematica to perform the computation and rigorous justification. For convenience, definitions of all functions below will be naturally extended to their interval versions without further noting. Denote by $N_r(z)$ the box

$$N_r(z) := ([x-r, x+r], [y-r, y+r]), \text{ with } z = (x, y) \in \mathbb{R}^2.$$  \hspace{1cm} (2.1)

Let us carry out the following tasks (I)-(IV):

(I) Locating the true fixed point $\bar{z}$: Since $\bar{z}_{2,q}$ is a fixed point of map (3.3) in numerical expression, let us locate the rigorous interval vector of the true fixed point $\bar{z}$. Denote $\bar{z}_{2,q} = (\bar{x}_q, \bar{y}_q)$ and choose an interval vector

$$\mathbf{Z}_0 = ([\bar{x}_q - 10^{-14}, \bar{x}_q + 10^{-14}], [\bar{y}_q - 10^{-14}, \bar{y}_q + 10^{-14}]).$$

**Proposition 3.2.** There exists a true fixed point $\bar{z}$ of $F$ in $\mathbf{Z}_0$.

**Proof.** Define $H(z) := F(z) - z$, for $z \in \mathbb{R}^2$. From (3.2), we compute

$$I(H, \mathbf{Z}_0) = ([0.4214296448393566, 0.4214296448393568], [0.34714221309638, 0.3471422130963862]) \subset \mathbf{Z}_0.$$

Therefore, there exists a unique $\bar{z} \in \mathbf{Z}_0$ such that $H(\bar{z}) = 0$, and thus a true fixed point $\bar{z}$ of $F$ in $\mathbf{Z}_0$, by Theorem 3.1.

(II) Finding a repelling neighborhood of $\bar{z}$ under $G$:

**Proposition 3.3.** $B_{0.036}(\bar{z})$ is repelling neighborhood of $\bar{z}$ under $G$.

**Proof.** Write $G = (g_1, g_2)$. For any $x, y \in B_{0.036}(\bar{z})$,

$$G(x) - G(y) = (\nabla g_1(c_1) \cdot (x-y), \nabla g_2(c_2) \cdot (x-y)).$$
for some \( c_1, c_2 \in B_{0.036}(z) \), by the mean value theorem. We shall justify that
\[
\|G(x) - G(y)\| \geq 1.00465|x - y|, \text{ for all } x, y \in B_{0.036}(z), x \neq y.
\]
(3.4)
For convenience of IA computation, we partition the ball \( B_{0.036}(z) \) into a collection of 10000 interval vectors \( I_i \subseteq \mathbb{R}^2 \), each of dimension \((7.2 \times 10^{-4}) \times (7.2 \times 10^{-4})\), i.e., \( B_{0.036}(z) \subseteq \bigcup_{i=1}^{10000} I_i \). Now \( (\nabla g_1(c_1), \nabla g_2(c_2)) \in DG(B_{0.036}(z)) \subseteq \bigcup_i DG(I_i) \).
Thus, by IA computation,
\[
\|G(x) - G(y)\| = ||(\nabla g_1(c_1), \frac{(x-y)}{||x-y||} ||x - y||, \nabla g_2(c_2), \frac{(x-y)}{||x-y||} ||x - y||)||
\leq \|DG(B_{0.036}(z))\| \frac{||x-y||}{||x-y||} \|x - y\| \leq [1.00465, 2.28372]\|x - y\|,
\]
for all \( x, y \in B_{0.036}(z), x \neq y, c_1, c_2 \in B_{0.036}(z) \). We have thus justified (3.4). \( \square \)

Remark 3.1. Proposition 2.1 allows us to find a preliminary repelling neighborhood of \( z \) under \( G(B_{0.009}(z)) \) for this predator-prey system. By employing IA computing, it is possible to locate a larger repelling neighborhood \( B_{0.036}(z) \) in this case. Basically, it takes less computation to find a repelling neighborhood of \( z \) by using Proposition 2.1, whereas larger repelling neighborhood is more advantageous to finding the snapback point.

(III) Finding a snapback point \( z_0 \) in the repelling neighborhood \( B_{0.036}(z) \) and examining (2.7): We shall construct successive pre-images of \( z \) under \( G \) to locate a point \( y_{0,q} \in B_{0.036}(z) \) with \( G^\ell(y_{0,q}) \) lies in the region \( B_{10^{-11}}(z) \), for some \( \ell \geq 2 \). We then apply Theorem 3.1 to confirm the true homoclinic orbit of \( z \) under \( G \).
We use the forward iteration to exhaustively sift the possible points of \( y_{0,q} \) by robust numerical computation. First, we check the forward iterations of numerous sample points in the region \( B_{0.036}(z) \). If there is a point which is mapped back into the region \( B_{10^{-11}}(z) \) after some iterations, then we pick it as a candidate for the snapback point. Second, we use numerical Newton’s method to refine the locations of those candidate points and find the numerical snapback points. Herein, through computation, we choose one of the numerical snapback points:
\[
y_{0,q} = (0.3958825716962441, 0.3712349439188911) \in B_{0.036}(z).
\]
Let us denote by \( \{y_j\}_{j = -\infty}^{\ell} := \{y_{0,q} : -\infty < j \leq \ell\} \) the numerical homoclinic orbit for \( z \) under \( G \), where \( y\) \( y_{0,q} = y_0 \), and by \( \{y_j\}_{j = -\infty}^{\ell} := \{y_j : -\infty < j \leq \ell\} \) the true homoclinic orbit for \( z \) under \( G \), should it exist, where \( y_0 = y_0 \) with \( G^\ell(y_0) = z \).

Since IA must enclose all numerical errors in the iterations of functions, overestimation is inevitable. On the other hand, a little deviation of the initial point in the beginning may result in very different terminal point by iterations, and the IA computing will break down due to the enormously accumulated overestimation. To prevent this, we employ the multi-shooting method which decomposes the whole iteration into several shorter sections and restrains the overestimation of IA.
More precisely, instead of solving a single equation \( G^\ell(y) = z \), with one node \( \{y\} \), to determine the entire long orbit, \( \{y = y_0, G(y_0), \ldots, G^{\ell}(y_0) = z\} \), we consider a system of \( m \) connecting equations, \( S_m(y, v_2, \ldots, v_m) = 0 \), i.e.,
\[
G^p(y) - v_2 = 0, G^p(v_2) - v_3 = 0, \ldots, G^{\ell-(m-1)p}(v_m) - z = 0,
\]
with \( m \) independent nodes \( \{y, v_2, \ldots, v_m\} \), whose shorter sections of orbits can be lumped together into
\[
\{y = y_0, G(y), \ldots, G^{p-1}(y), G^p(y) = v_2, G(v_2), \ldots, G^p(v_{m-1}) = v_m, G(v_m), \ldots, G^{\ell-(m-1)p}(v_m) = z\}.
\]
With such a treatment, the number of iterations for each node is \( p \) which is far less than \( \ell \), and therefore the numerical errors induced by iterations can be controlled. Based on this multi-shooting method, we obtain the following theorem.

**Theorem 3.4.** \( \tilde{z} \) is a snapback repeller of map \( G \).

**Proof.** For \( \ell = 11 \), if we set \( m = 4 \), \( p = 3 \) and take

\[
U = N_{10^{-11}/\sqrt{2}}(\tilde{y}_0, q) \times N_{10^{-11}/\sqrt{2}}(\tilde{y}_3, q) \times N_{10^{-11}/\sqrt{2}}(\tilde{y}_6, q) \times N_{10^{-11}/\sqrt{2}}(\tilde{y}_9, q),
\]

then we can justify \( (DS)_j \neq 0 \) on \( U \), by IA computation. Next, define \( I(S_4, U) : = \tilde{z} - (DS_4(U))^{-1}S_4(\tilde{z}) \), where \( \tilde{z} = (\tilde{y}_0, q, \tilde{y}_3, q, \tilde{y}_6, q, \tilde{y}_9, q) \in \mathbb{R}^8 \) is the middle point of \( U \). We can then justify \( I(S_4, U) \subseteq U \), by IA computation. Therefore, in the interval vector \( U \), there exists a root \( (y_0, v_{0,2}, v_{0,3}, v_{0,4}) \) of \( S_4 \). Accordingly, \( \tilde{z} = G^{11}(y_0) \), and thus \( y_0 \in B_{0,036}(\tilde{z}) \) is a snapback point of \( G \). We can further use IA to verify \( \det(DG(x)) \neq 0 \), for any \( x \in \cup_{j=1}^8 N_{10^{-11}/\sqrt{2}}(\tilde{y}_j, q) \). Hence, \( \det(DG(y_j)) \neq 0 \), for \( 1 \leq j \leq 10 \), and thus \( \tilde{z} \) is a snapback repeller of \( G \). \( \square \)

**Remark 3.2.** For the original map \( F \), there does not exist a constant \( c > 1 \) such that

\[
\|F(x) - F(y)\| > c \cdot \|x - y\|, \text{ for any } x, y \in B_{0,036}(\tilde{z}), \ x \neq y.
\]

Hence, even though \( B_{0,036}(\tilde{z}) \) is a repelling neighborhood of \( \tilde{z} \) under \( G \), it is not a repelling neighborhood under \( F \). This reveals the reason we consider the map \( G = F^2 \), rather than \( F \) directly.

(IV) Justifying that \( \tilde{z} \) is a snapback repeller of map \( F \): We denote by \( \{y_j\}_{-\infty}^{22} := \{y_j : -\infty < j \leq 22\} \) the targeted homoclinic orbit of \( F \), with \( F(y_j) = y_{j+1} \). Proposition 3.3 indicates that \( B_{0,036}(\tilde{z}) \) is a repelling neighborhood of \( \tilde{z} \) under \( G \). But this does not provide enough information on the orbit \( \{y_j\}_{-\infty}^{22} \) under \( F \). We need to trace the pre-image of \( y_0 \) under \( F \) and examine the deterministic condition (2.6) at \( y_j \), for \( -\infty < j \leq -1 \), to apply Theorem 2.2.

**Proposition 3.5.** For any \( v \in N_{0,036}(\tilde{z}) \), there exists an \( u \in N_{0,072}(\tilde{z}) \), such that \( F(u) = v \).

**Proof.** First, in order to prevent overestimation, we divide the region \( N_{0,036}(\tilde{z}) \) into finitely many small parts \( V_i \), namely, \( \cup_i V_i = N_{0,036}(\tilde{z}) \). For each \( V_i \), we firstly use numerical Newton’s method to find a region \( U_i \subseteq N_{0,072}(\tilde{z}) \) such that for any \( v \in V_i \subseteq N_{0,036}(\tilde{z}) \), there exists a numerical pre-image \( u_i \in U_i \), of \( v \) under map \( F \). Next, we use IA to prove that there exists a true preimage \( u \in U_i \) of \( v \) such that \( F(u) = v \). To this end, we need to examine \( I(h_X^i, U_i) \subseteq U_i \), for each \( v \in V_i \subseteq N_{0,036}(\tilde{z}) \), where function \( I \) is defined as (3.2) and \( h_X^i(z) := F(z) - v \). We define an interval function \( H_F(z) := F(z) - v \). It can be computed by IA that \( I(H_F, U_i) \subseteq U_i \subseteq N_{0,072}(\tilde{z}) \), and thus \( I(h_X^i, U_i) \subseteq I(H_F, U_i) \), for each \( v \in V_i \), due to \( h_X^i(z) = F(z) - v \in F(z) - V_i \), and inclusion monotonicity (3.1). Subsequently, we obtain \( I(h_X^i, U_i) \subseteq U_i \), for each \( v \in V_i \subseteq N_{0,036}(\tilde{z}) \). Therefore, there exists a unique \( u \in U_i \subseteq N_{0,072}(\tilde{z}) \) such that \( h_X^i(u) = 0 \), i.e., \( F(u) = v \), by Theorem 3.1. \( \square \)

Proposition 3.5 indicates that \( N_{0,036}(\tilde{z}) \subseteq F(N_{0,072}(\tilde{z})) \). In addition, \( G^{-1} \) is a contraction on \( B_{0,036}(\tilde{z}) \), as \( B_{0,036}(\tilde{z}) \) is a repelling neighborhood of \( \tilde{z} \). Therefore, \( y_j \in N_{0,072}(\tilde{z}) \), for \( -\infty < j \leq -1 \), since \( y_0 \in B_{0,036}(\tilde{z}) \). Moreover, by using IA computation, we can justify \( \det(DF(x)) \neq 0 \), for all \( x \in N_{0,072}(\tilde{z}) \). Therefore, (2.6) is met, and \( \tilde{z} \) is a snapback repeller of map \( F \).
4. Further applications. In this section, we further apply our approach and IA computation to study the existence of snapback repellers in two two-dimensional maps.

Example 4.1. Consider the two-dimensional map \( F = (f_1, f_2) \) defined by
\[
\begin{align*}
  f_1(x, y) &= x(a + 1 - bx - y), \\
  f_2(x, y) &= xy.
\end{align*}
\]
(4.1)
The map (4.1) arises from a population model, see [20]. Existence of Marotto’s chaos was concluded by employing the original definition of snapback repeller in [14], which is invalid. Let us consider specifically the parameter \( a = 7, b = 1.3, \) as an illustration. For system (4.1), \( \bar{z} := (\bar{x}, \bar{y}) = (1, 5.7) \) is a repelling fixed point. We compute the eigenvalues of \( (DF(\bar{z}))^T \) and find \( \lambda_1^{(1)} \approx 33.7151 \) and \( \lambda_2^{(1)} \approx 0.864894 \) so that condition (2.1) is not met. However, we compute the eigenvalues of \( (DF^2(\bar{z}))^T \) and find \( \lambda_1^{(2)} \approx 54.318 \) and \( \lambda_2^{(2)} \approx 15.6542 \). Therefore, Proposition 2.1 can be applied with \( k = 2 \). We compute to find \( s_2 \approx 3.95654 \), and \( r = 0.08 \), and hence \( B_{0.08}(\bar{z}) \) is a repelling neighborhood of \( \bar{z} \) for \( G := F^2 \). In addition, the multi-valued inverse of map (4.1) can be expressed explicitly as
\[
\begin{align*}
  x_{k-1} &= \frac{1}{2b} \left[ a + 1 \pm \sqrt{(a+1)^2 - 4b(x_k+y_k)} \right], \\
  y_{k-1} &= \frac{y_k}{2(x_k+y_k)} \left[ a + 1 \pm \sqrt{(a+1)^2 - 4b(x_k+y_k)} \right].
\end{align*}
\]
We compute to find numerical pre-images of \( \bar{z} \) under \( G \):
\[
\begin{align*}
  \bar{z}_{-1, q} &= (0.920024, 1.20211), \\
  \bar{z}_{-2, q} &= (0.64275, 6.73221), \\
  \bar{z}_{-3, q} &= (1.07391, 5.55262), \\
  \bar{z}_{-4, q} &= (0.984392, 5.71821) \\
  &\in B_{0.08}(\bar{z}).
\end{align*}
\]
Next, we use IA to show the existence of the true pre-images of \( \bar{z} \). Define a function \( \tilde{G}(z) := G^4(z) - \bar{z} \), and take \( Z_4 := N_{10^{-8}}(\bar{z}_{-4, q}) = ([0.984392 - 10^{-8}, 0.984392 + 10^{-8}], [5.71821 - 10^{-8}, 5.71821 + 10^{-8}]) \subseteq B_{0.08}(\bar{z}). \) Computation shows that \( \emptyset \not\in \det(\tilde{D}G(Z_4)) \) and \( I(\tilde{G}, Z_4) \subseteq Z_4 \) where \( I \) is defined as (3.2).
Hence, by Theorem 3.1, there exists an unique \( y_0 \in Z_4 \) such that \( \tilde{G}(y_0) = 0 \), i.e., \( G^4(y_0) = \bar{z} \).

Returning to map \( F \), we denote \( y_{0,q} := \bar{z}_{-4, q} \) and \( y_{j,q} := F^j(y_{0,q}) \), for \( -\infty < j < 8 \). We use IA to justify \( y_j := F^j(y_0) \in N_{10^{-11}/\sqrt{2}}(y_{j,q}) \), for \( 1 \leq j \leq 8 \). In addition, by IA computation, we confirm that \( B_{0.08}(\bar{z}) \subseteq F(N_{0.0922}(\bar{z})) \). By further IA computation, we obtain \( \det(DF(x)) \neq 0 \), for all \( x \in N_{0.0922}(\bar{z}) \). In addition, since \( G^{-1} \) is a contraction on \( B_{0.08}(\bar{z}) \), we have \( \det(DF(x)) \neq 0 \), for \( -\infty < j < 7 \). Therefore, \( \bar{z} \) is a snapback repeller of \( F \), according to Theorem 2.2.

Example 4.2. Consider the two-dimensional FitzHugh nerve map \( F = (f_1, f_2) \) defined by
\[
\begin{align*}
  f_1(x, y) &= x + \delta(y - x^3/3 - x^2), \\
  f_2(x, y) &= y + \delta\rho(-x - by),
\end{align*}
\]
(4.2)cf. [9, 19]. Let us consider the parameter values \( \delta = 0.8, \rho = 0.98 \), and \( b = 0.4 \), as an illustration. For system (4.2), \( \bar{z} := (\bar{x}, \bar{y}) = (0, 0) \) is a repelling fixed point and the eigenvalues of \( (DF(\bar{z}))^T \) are \( \lambda_1^{(1)} \approx 1.72615 \) and \( \lambda_2^{(1)} \approx 0.999647 \), and the eigenvalues of \( (DF^2(\bar{z}))^T \) are \( \lambda_1^{(2)} \approx 2.57535 \) and \( \lambda_2^{(2)} \approx 1.15616 \). We apply Proposition 2.1 and confirm that \( B_{0.014}(\bar{z}) \) is a repelling neighborhood of \( G := F^2 \), with \( s_2 \approx 1.07525, r = 0.014 \).
Now, consider the point
\[ y_{0,q} = (-0.00781980404126274, -0.00477060931584242) \in B_{0.014}(\bar{z}). \]
We compute to see that \( G^{20}(y_{0,q}) \) is very close to the repelling fixed point \( \bar{z} \). Since the targeted homoclinic orbit is quite long, we shall use IA and multi-shooting method to justify the existence of a true homoclinic orbit and locate a point \( y_0 \) which is very close to \( y_{0,q} \) such that \( G^{20}(y_0) = \bar{z} \). First, we define a multi-shooting map with \( p = 3, m = 7 \) and \( \ell = 20 \). Next, we take an interval vector
\[
U := N_{10^{-11}/\sqrt{2}}(\hat{y}_{0,q}) \times N_{10^{-11}/\sqrt{2}}(\hat{y}_{3,q}) \times N_{10^{-11}/\sqrt{2}}(\hat{y}_{6,q}) \times N_{10^{-11}/\sqrt{2}}(\hat{y}_{9,q}) \times N_{10^{-11}/\sqrt{2}}(\hat{y}_{12,q}) \times N_{10^{-11}/\sqrt{2}}(\hat{y}_{15,q}) \times N_{10^{-11}/\sqrt{2}}(\hat{y}_{18,q})
\]
where \( \hat{y}_{3k,q} := G^k(\hat{y}_{3k-3,q}), k = 1, \cdots, 6 \). Then, by IA computation, we obtain
\[
I(S_7, U) := \bar{z} - (DS_7(U))^{-1} S_7(\bar{z}) \subseteq U, \quad \bar{z} = (\hat{y}_{0,q}, \hat{y}_{3,q}, \hat{y}_{6,q}, \hat{y}_{9,q}, \hat{y}_{12,q}, \hat{y}_{15,q}, \hat{y}_{18,q}) \in \mathbb{R}^{14} \text{ is the middle point of } U \text{ and } I \text{ is defined as (3.2). By using IA and Theorem 3.1, we confirm the existence of solution } (y_0, v_{0,2}, v_{0,3}, v_{0,4}, v_{0,5}, v_{0,6}, v_{0,7}) \text{ satisfying}
\]
\[
S_7(y_0, v_{0,2}, v_{0,3}, v_{0,4}, v_{0,5}, v_{0,6}, v_{0,7}) = 0,
\]
and thus \( G^{20}(y_0) = \bar{z} \). Denote \( y_{j,q} := F^j(y_{0,q}) \) for \( 1 \leq j \leq 40 \). We perform IA computation to conclude that there exist \( y_j \in N_{10^{-11}/\sqrt{2}}(y_{j,q}), \text{ such that } F(y_j) = y_{j+1}, 0 \leq j \leq 39, \text{ and } \bar{z} = F^{40}(y_0). \) Next, we confirm via IA computation that \( B_{0.014}(\bar{z}) \subseteq F(N_{0.095}(\bar{z})) \). Moreover, IA computation is employed to justify
\[
det(DF(x)) \neq 0, \text{ for all } x \in N_{0.095}(\bar{z}) \cup (\bigcup_{j=1}^{39} B_{10^{-11}/\sqrt{2}}(y_{j,q})).
\]
In addition, since \( G^{-1} \) is a contraction on \( B_{0.014}(\bar{z}) \), we have \( det(DF(y_j)) \neq 0, \text{ for } -\infty < j \leq 39 \). Thus, \( \bar{z} \) is a snapback repeller of \( F \).

5. Conclusion. We have presented an approach which combines analytic property and computation technique to verify the existence of snapback repeller for multi-dimensional maps. For a given map \( F \), we established two methodologies to locate the repelling neighborhood of a repeller of map \( F^k \), for some \( k \geq 2 \), under the Euclidean norm. The first one is to estimate the radius of the repelling neighborhood through computing the first or second derivative of \( F^k \), whereas the second is to employ the Interval Arithmetic computation. We have further employed the IA computation to locate the snapback point in the repelling neighborhood and examine the nonzero determinant condition on the Jacobian of \( F \). By exploring and utilizing the relation between the existence of the snapback repeller for map \( F \) and map \( F^k, k \geq 2 \), we confirmed the existence of snapback repeller for map \( F \). This new finding also serves as an indicator to whether a repeller of \( F \) is indeed a snapback repeller.

The present approach has relaxed the restriction of the methods reported in [10, 13], and provided concrete examples connected to the studies in [11]. Not only that the conditions for the present theory are all computable numerically, but the numerical results through IA computing also provide a rigorous confirmation to the existence of snapback repeller. We have employed the present approach to explore the existence of snapback repellers under the valid definition, hence chaotic behaviors, in a predator-prey model, a discrete-time population model, and the FitzHugh nerve model.
This paper has demonstrated that IA computation is effective and handy (with convenient packages) in observing and concluding chaotic dynamics in multidimensional maps, as incorporated with pertinent analytic theory.

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REFERENCES

[1] G. Alefeld and J. Herzberger, *Introduction to Interval Computations*, in Academic Press, NY, 1983.
[2] G. Alefeld, Inclusion methods for systems of nonlinear equations—the interval Newton method and modifications, *Topics in Validated Computations*, Elsevier, Amsterdam, 5 (1994), 7–26.
[3] Z. Arai, W. Kalies, H. Kokubu, K. Mischaikow, H. Oka and P. Pilarczyk, A database schema for the analysis of global dynamics of multi parameter systems, *SIAM J. Appl. Dyn. Syst.*, 8 (2009), 757–789.
[4] J. R. Beddington, C. A. Free and J. H. Lawton, Dynamic complexity in predator-prey models framed in difference equations, *Nature*, 255 (1975), 58–60.
[5] G. Chen, S.-B. Hsu and J. Zhou, Snapback repellers as a cause of chaotic vibration of the wave equation with a van der Pol boundary condition and energy injection at the middle of the span, *J. Math. Phys.*, 39 (1998), 6459–6480.
[6] S. S. Chen and C. W. Shih, Transversal homoclinic orbits in a transiently chaotic neural network, *Chaos*, 12 (2002), 654–671.
[7] L. Gardini and F. Tramontana, Snapback repellers and chaotic attractors, *Physical Rev. E*, 81 (2010), 046202, 5pp.
[8] L. Gardini, I. Sushko, V. Avrutin and M. Schanz, Critical homoclinic orbits lead to snapback repellers, *Chaos, Solitons, and Fractals*, 44 (2011), 433–449.
[9] Z. Jing, Z. Jia and Y. Chang, Chaos behavior in the discrete FitzHugh nerve system, *China Set. A-Math*, 44 (2001), 1571–1578.
[10] C. Li and G. Chen, An improved version of the Marotto theorem, *Chaos, Solitons, and Fractals*, 18 (2003), 69–77.
[11] M.-C. Li, M.-J. Lyu and P. Zgliczynski, Topological entropy for multidimensional perturbations of snapback repellers and one-dimensional maps, *Nonlinearity*, 21 (2008), 2555–2567.
[12] T.-Y. Li and J. A. Yorke, Period three implies chaos, *Amer. Math. Monthly*, 82 (1975), 985–992.
[13] K.-L. Liao and C.-W. Shih, Snapback repellers and homoclinic orbits for multi-dimensional maps, *J. Math. Anal. Appl.*, 386 (2012), 387–400.
[14] F. R. Marotto, Snapback repellers imply chaos in $R^n$, *J. Math. Anal. Appl.*, 63 (1978), 199–223.
[15] F. R. Marotto, On redefining a snapback repeller, *Chaos, Solitons, and Fractals*, 25 (2005), 25–28.
[16] R. E. Moore and F. Bierbaum, *Methods and Applications of Interval Analysis*, SIAM, Philadelphia, 1979.
[17] C.-C. Peng, Numerical computation of orbits and rigorous verification of existence of snapback repellers, *Chaos*, 17 (2007), 013107, 8pp.
[18] Y. Shi and P. Yu, Chaos induced by regular snapback repellers, *J. Math. Anal. Appl.*, 337 (2008), 1480–1494.
[19] J. Sugie, Nonexistence of periodic solutions for the FitzHugh nerve system, *Quart. Appl. Math.*, 49 (1991), 543–554.
[20] Y. Zhang, Q. Zhang, L. Zhao and C. Yang, Dynamical behaviors and chaos control in a discrete functional response model, *Chaos, Solitons, and Fractals*, 34 (2007), 1318–1327.

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