Connecting Solutions of the Lorentz Force Equation do Exist

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Abstract: Recent results on the maximization of the charged-particle action $I_{x_0,x_1}$ in a globally hyperbolic spacetime are discussed and generalized. We focus on the maximization of $I_{x_0,x_1}$ over a given causal homotopy class $C$ of curves connecting two causally related events $x_0 \leq x_1$. Action $I_{x_0,x_1}$ is proved to admit a maximum on $C$, and also one in the adherence of each timelike homotopy class $C$. Moreover, the maximum $\sigma_0$ on $C$ is timelike if $C$ contains a timelike curve (and the degree of differentiability of all the elements is at least $C^2$).

In particular, this last result yields a complete Avez-Seifert type solution to the problem of connectedness through trajectories of charged particles in a globally hyperbolic spacetime endowed with an exact electromagnetic field: fixed any charge-to-mass ratio $q/m$, any two chronologically related events $x_0 \ll x_1$ can be connected by means of a timelike solution of the Lorentz force equation corresponding to $q/m$. The accuracy of the approach is stressed by many examples, including an explicit counterexample (valid for all $q/m \neq 0$) in the non-exact case.

As a relevant previous step, new properties of the causal path space, causal homotopy classes and cut points on lightlike geodesics are studied.

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1. Introduction

Recently, there has been a renewed interest in the existence of solutions to the Lorentz force equation (LFE; Eq. (1) below) connecting two events $x_0 \ll x_1$ of a spacetime $M$, for an (exact) electromagnetic field, $F = d\omega$. Even though many, sometimes competitive, results have been obtained [5, 12, 13, 17, 11, 29, 30], and related mathematical problems studied [6, 16, 31], the full answer to the original problem has remained open:

**Question (Q).** Assume that $M$ is globally hyperbolic, fix a charge-to-mass ratio $q/m$, and let $x_0 \ll x_1$ be any two fixed chronologically related events: must a timelike solution of the corresponding LFE, connecting the two events, exist?

Our main aim is to give a complete (affirmative) answer to this question. Moreover, we will study also some properties of causal homotopy classes not only essential for (Q) but also interesting in their own right.

The existence of a length-maximizing causal geodesic $\sigma_0$ connecting $x_0$ and $x_1$, when $x_0 \leq x_1$, is a well-known property since the works by Avez [3] and Seifert [39] (see Subsect. 3.1).

For the LFE, one must solve two problems:

(A) Prove that the associated action functional $I_{x_0, x_1}$ admits local maximizers (or at least critical points).

(B) Show that one of these maximizers is timelike (at all the points).

As a difference with the Avez-Seifert geodesic case, now this second point is not trivial and becomes essential because, otherwise, the maximizer cannot be interpreted as a solution to the LFE (see Sect. 3). Our progress can be summarized as follows:

(A) The existence of a maximizer of $I_{x_0, x_1}$ on each causal homotopy class $C_{x_0, x_1}$ will be proved. In principle, a possible proof would follow steps analogous to the Avez-Seifert theorem, but new technicalities would appear (to prove the upper semi-continuity of $I_{x_0, x_1}$, or to try to reduce the space of connecting curves to a finite-dimensional one, as in Subsect. 2.3), which propagate to problem (B). Instead, we follow an approach based on Kaluza-Klein metrics as in [17, 29]. This approach reduces problem (A) to a problem on lightlike geodesics, and gives a simple geometrical interpretation for the functional $I_{x_0, x_1}$, which is somewhat reminiscent of the time of the arrival functional in the Fermat principle of General Relativity [25, 35] (see Subsect. 4.2). In the present paper, this approach will be refined and clarified to obtain a maximizer of $I_{x_0, x_1}$ on each causal homotopy class $C_{x_0, x_1}$ (and in the closure of each timelike homotopy class $C_{x_0, x_1}$).

(B) We must emphasize that when the maximizer in $C_{x_0, x_1}$ is not timelike, it becomes a lightlike geodesic (Theorems 3.2, 4.2). This excludes the existence of non-time-like maximizers for generic pairs $x_0 \ll x_1$, but lightlike maximizers can exist for particular $x_0, x_1$ (Example 3.1). Nevertheless, as a main goal in the present paper, we prove that, in this case, the causal homotopy class of a lightlike maximizer only can contain lightlike pregeodesics (some of them, global maximizers in the class), that is, $x_0$ and $x_1$ are only causally but not chronologically related by curves in the class. Thus, as $x_0 \ll x_1$, we can choose a causal homotopy class which contains a timelike curve, and the required timelike maximizer is obtained. As shown by means of an explicit example (Remark 5.2), differentiability $C^2$ will be essential (this is the natural degree of differentiability for LFE, even though the associated variational problem makes sense for $C^1$ differentiability).
Our main result is then the following:

**Theorem 1.1.** Let \((M, g)\) be a \(C^2\) globally hyperbolic spacetime, and \(F\) be an exact electromagnetic field on \(M\) (\(F = d\omega\) for some \(C^2\) differential form). Choose \(x_1 \in J^+(x_0)\), \(x_1 \neq x_0\), and fix any causal homotopy class \(C_{x_0, x_1}\).

For each charge-to-mass ratio \(q/m \in \mathbb{R}\) there exists a future-directed causal curve \(\sigma_0\) which connects \(x_0\) and \(x_1\) and maximizes the corresponding action functional \(I_{x_0, x_1}\) on \(C_{x_0, x_1}\).

This maximizer \(\sigma_0\) is lightlike if and only if \(C_{x_0, x_1}\) only contains lightlike curves (necessarily geodesics, up to reparametrizations). In this case, \(\sigma_0\) is a lightlike geodesic with no conjugate point to \(x_0\) strictly before \(x_1\), and \(x_1\) will be conjugate if there exists a second curve in \(C_{x_0, x_1}\) which is not a reparametrization of \(\sigma_0\). Otherwise, \(\sigma_0\) is timelike, and its reparametrization with respect to proper time becomes a solution of the LFE for the charge-to-mass ratio \(q/m\).

In particular given \(q/m\), if \(x_1 \in I^+(x_0)\) there exists at least one solution to the LFE which connects \(x_0\) and \(x_1\).

This work is organized as follows.

- In Sect. 2, causal homotopy classes are studied, specially in globally hyperbolic spacetimes. In Subsect. 2.1, the general framework is introduced, and some new general properties are given (Theorem 2.1, Corollary 2.1; compare with \[7, Theorem 9.15\]). In Subsect. 2.2, we introduce the notion of homotopic cut point, and prove that, for lightlike geodesics, this point becomes equivalent to the first conjugate point, Theorem 2.2, Remark 2.3. This result will be essential to solve problem (B) above, in Sect. 5. In Subsect. 2.3, first some properties of the standard approach for the timelike path space of a globally hyperbolic spacetime \[41\], \[7, Ch. 10.2\] are shown to be extensible to the causal case. More originally, Theorem 2.3 proves, in particular, that any geodesic limit of curves contained in a single timelike or causal homotopy class, also belongs to this same causal homotopy class; this result will turn out essential to solve problem (A) above, in Sect. 4.

- In Sect. 3, LFE and question (Q) are introduced (Subsect. 3.1), and the variational framework for the LFE is discussed (Subsect. 3.2). Even though this framework is well-known, we discuss it with some detail, because there are some related variational frameworks (widely studied in recent references) which may lead to confusion. In Subsect. 3.3, we summarize the known results, and give a counterexample which shows that the basic question (Q) was still open, in general.

- In Sect. 4, Problem (A) on the existence of local maximizers is solved, as explained above. Even though we follow the approach in \[17, 29\], the proof is rewritten completely. In fact, apart from some simplifications of these references, several new technicalities appear when causal homotopy classes are considered, in both the Kaluza-Klein fiber bundle (Subsect. 4.1) and the limit process on curves of the homotopy class (Subsect. 4.2, Lemma 4.1). The main result, Theorem 4.2, includes a refinement on the existence of local maximizers for the action \(I_{x_0, x_1}\) in the closure of any timelike homotopy class (which remains valid for \(C^1\) elements).

- In Sect. 5, Problem (B) is solved completely. In Subsect. 5.1, we give a result on the impossibility for a lightlike geodesic with conjugate points to be a local maximizer of actions as those for the LFE (Lemma 5.1, Remark 5.1). Then, the timelike character of the maximizer follows from the properties of causal homotopy classes in Sect. 2. In Subsect. 5.2, we give examples which show the accuracy of our results: (i) even though there are maximizers in the closure of any timelike homotopy class, these
maximizers can be lightlike in some of these classes, and one can ensure the existence of a timelike maximizer only in the whole causal homotopy class (which may contain more than one timelike homotopy class), but (ii) if the degree of differentiability were only $C^1$, such a timelike maximizer may not exist.

– In Sect. 6 we provide an example which shows that the results obtained do not admit further generalizations to the non-exact electromagnetic field case. Remarkably, in this example: (a) for a suitably chosen pair $x_0 \ll x_1$, no connecting solution of the LFE exists, whatever the value of $q/m(\neq 0)$ is chosen (in particular, no timelike connecting solution exists for the related Eq. (5) considered in Subsect. 3.2), and (b) even though $F$ is non-exact, it is the curvature of a suitable bundle of fiber $S^1$.

– In Sect. 7 we give the conclusions. Finally, in a short appendix a general result on global hyperbolicity for Kaluza-Klein metrics is given. This result makes our paper self-contained (compare with [14, 42] and [17, Lemma 5]), and it is provided not only for completeness, but also to incorporate the recent progress on the splitting of globally hyperbolic spacetimes in [9, 10]. In particular, this rules out the problem of differentiability in the parametrizations of the curves in the causal path space [7, Lemma 10.34].

2. Causal Homotopy Classes

2.1. General properties. Throughout this paper, $(M, g)$ will denote a $C^r$ spacetime (connected, time-oriented Lorentzian manifold), $r \in \{2, \ldots, \infty\}$ of arbitrary dimension $n_0 \geq 2$ and signature $(-, -\ldots, -)$. Nevertheless, for the maximization result of the action functional in Sect. 4, it is enough $r_0 = 1$. Without loss of generality, causal curves will be regarded as piecewise $C^r$ (piecewise smooth) and future-directed from now on¹. Notice that the curves are regarded as parametrized (we will not be specifically interested in the space of all the unparametrized causal curves, in the spirit of Morse theory), and suitable reparametrizations will be chosen, if necessary. In principle, strong causality is our (minimal) ambient causal assumption for $(M, g)$, always assumed implicitly. But most of the results need global hyperbolicity, and sometimes this assumption will be imposed (explicitly in this section) for simplicity. For background results in this section, see for example [7, Ch. 9, 10] and [33, Ch. 10]; timelike homotopy classes have also been studied in different contexts (see for example, [40, 18, 38] and references therein); some of the difficulties circumvented in our approach are illustrated in the beginning of the proof of [34, Th. 6.5].

Two causal (resp. timelike) curves, $\gamma_i : [\lambda_0, \lambda_1] \to M$, $i = 0, 1$ with fixed extremes $x_j = \gamma_0(\lambda_j) = \gamma_1(\lambda_j), j = 0, 1$, are causally homotopic if there exists a causal (resp. timelike) homotopy (with fixed extremes $x_0, x_1$) connecting $\gamma_0, \gamma_1$, i.e., a continuous map

$$H : [0, 1] \times [\lambda_0, \lambda_1] \to M$$

$$(\epsilon, \lambda) \to \gamma_{\epsilon}(\lambda)$$

¹ The reader can check that no more generality is obtained if the causal curves (in our at least $C^2$ spacetime) are regarded only as piecewise $C^1$ smooth. Even more, essentially the results in the present section can be extended if future-directed causal curves are regarded only as $C^0$ (i.e., a continuous curve $\lambda \to \gamma(\lambda)$ which satisfies: for any open connected subset $U \subset M$, if $\lambda < \lambda'$ and $[\lambda, \lambda'] \subset \gamma^{-1}(U)$ then $\gamma(\lambda)$ and $\gamma(\lambda')$ can be joined by a piecewise smooth future directed causal curve contained in $U$; such causal curves satisfy a Lipschitzian condition, see [34, p. 17]). For example, Lemma 2.1 (and, then, Theorem 2.1 and Corollary 2.1) or Theorem 2.2 hold obviously if the longitudinal curves of the causal homotopies are allowed to be only continuous causal curves.
such that each longitudinal curve $\gamma_\epsilon$ is causal (resp. timelike) for all $\epsilon$. This divides the set of causal curves joining $x_0$ and $x_1$ into causal homotopy classes, each one containing none, one or more classes of timelike homotopy (see Example 2.1). In our notation a generic causal (resp. timelike) homotopy class will be denoted with $C_{x_0,x_1}$ (resp. $C_{C_{x_0,x_1}}$).

The following lemma to Theorem 2.1 solves the technical difficulty associated to our choice of parametrized curves.

**Lemma 2.1.** Consider two causally homotopic lightlike geodesics $\gamma_0, \gamma_1 : [\lambda_0, \lambda_1] \to M$ joining $x_0$ with $x_1$.

If $\gamma_0$ and $\gamma_1$ maximize the time-separation (or “length”) in its causal homotopy class $\mathcal{C}$ (that is, there is no timelike curve in $\mathcal{C}$), then there exist a causal homotopy connecting them through longitudinal geodesics.

**Proof.** By the hypothesis, all the longitudinal curves of the causal homotopy $H(\epsilon, \lambda)$ between $\gamma_0$ and $\gamma_1$ are necessarily lightlike pregeodesics [33, Prop. 10.46]. Let $\gamma_\epsilon : [\lambda_0, \lambda_1] \to M$ be the (unique) reparametrization as a geodesic of the longitudinal curve corresponding to $\epsilon \in [0, 1]$, and put $u_\epsilon = \gamma'_\epsilon(\lambda_0)$. The map

$$h : (\epsilon, \lambda) \to \gamma_\epsilon(\lambda)$$

will be continuous (and, thus, the required causal homotopy) if and only if the map

$$\epsilon \to u_\epsilon, \quad \epsilon \in [0, 1]$$

is continuous. Consider the tangent space at $\gamma(\lambda_0)$ and the canonical projection on the projective space $\pi : TM_{x_0} \to PTM_{x_0}$. The curve of directions $\epsilon \to \pi(u_\epsilon)$ is continuous since $\pi(u_\epsilon) = \pi(\exp_{x_0}^{-1} H(\epsilon, \lambda))$, in any normal neighborhood of $x_0$, independently of $\lambda$. Since $\exp_{x_0}(u_\epsilon(\lambda_1 - \lambda_0)) = x_1$, the continuity of the curve $\epsilon \to u_\epsilon$ follows directly from [7, Lemma 9.25] (the strong causality of $M$ is used there).

**Theorem 2.1.** Let $\gamma_0 : [\lambda_0, \lambda_1] \to M$ be a lightlike geodesic which connects two fixed points $x_0, x_1$ and maximizes the time-separation in its causal homotopy class. If there exists a distinct geodesic $\gamma_1$ in this class then $x_1$ is the first conjugate point of $x_0$ along $\gamma_0$ (and, then, along $\gamma_1$).

**Proof.** By the previous lemma, there exists a causal homotopy from $\gamma_0$ to $\gamma_1$ through lightlike geodesics. Thus, $\exp_{x_0}$ cannot be injective in any neighborhood of $(\lambda_1 - \lambda_0)\gamma'_0(\lambda_0)$, $i = 0, 1$, and $x_1$ is a conjugate point. Even more, it is the first one because, otherwise, $\gamma_0$ would not maximize in its causal homotopy class.

**Remark 2.1.** The converse implication may not hold: even though a variation of $\gamma_0$ through lightlike geodesics with variational vector field zero at the extremes will exist [33, Corollary 10.40], conjugate points are only “almost meeting points” of geodesics, that is, the longitudinal geodesics of the variation may not reach $x_1$.

Cut points on causal geodesics have been widely studied [7, Ch. 9]: recall that a lightlike geodesic ray maximizes the time-separation until its cut point.

**Corollary 2.1.** Let $\gamma : [0, b) \to M$ be a lightlike geodesic with a cut point $\gamma(\lambda_c)$, $\lambda_c \in (0, b)$. If $\gamma(\lambda_c)$ is not a conjugate point, then:
(1) No other lightlike geodesic which connects $\gamma(0)$ and $\gamma(\lambda_c)$ is causally homotopic to $\gamma$.

(2) If $(M, g)$ is globally hyperbolic, there exist at least another lightlike geodesic $\hat{\gamma}$ (necessarily non-causally homotopic to $\gamma$) which connects $\gamma(0)$ and $\gamma(\lambda_c)$.

Proof. Assertion (1) is straightforward from Theorem 2.1. Then, (2) is a consequence of the well-known existence of a second connecting lightlike geodesic for any non-conjugate cut point on a lightlike geodesic (see [7, Theorem 9.15]).

Remark 2.2. The well-known behavior of lightlike geodesics of bidimensional de Sitter spacetime illustrates Corollary 2.1. No lightlike geodesic can have a conjugate point (because of dimension 2) but any such geodesic has a cut point, reached by a non-causally homotopic lightlike geodesic. By removing a point of this second geodesic, the necessity of the assumption of global hyperbolicity for (2) is stressed.

2.2. Homotopic cut points. For the following crucial result, we introduce an auxiliary concept:

Definition 2.1. Let $\gamma : [0, b) \to M$ be a lightlike geodesic. The point $\gamma(\lambda_c)$, $\lambda_c > 0$, is the homotopic cut point along $\gamma$ of $\gamma(0)$, if $\lambda_c$ is the first point such that, for any $\delta > 0$, the restricted curve $\gamma|_{[0, \lambda_c+\delta]}$ does not maximize the length in its causal homotopy class (that is, $\gamma|_{[0, \lambda_c+\delta]}$ is causally homotopic to a timelike curve).

Theorem 2.2. Let $\gamma : [0, b) \to M$ be a lightlike geodesic in a globally hyperbolic spacetime $(M, g)$. The point $\gamma(\lambda_c)$ is the homotopic cut point along $\gamma$ of $\gamma(0)$ if and only if it is the first conjugate point to $\gamma(0)$.

Proof. ($\Rightarrow$). Assume that $\gamma(\lambda_c)$ is not the first conjugate point. As conjugate points on causal geodesics are discrete [7, Th. 10.77], we can choose $\lambda_\delta = \lambda_c + \delta > \lambda_c$ such that no conjugate point $\gamma(\lambda)$ appears for $\lambda \in [0, \lambda_\delta]$. By hypothesis on $\gamma(\lambda_c)$, there exists a timelike geodesic $\rho$ from $\gamma(0)$ to $\gamma(\lambda_\delta)$ which is causally homotopic to $\gamma|_{[0, \lambda_\delta]}$, and maximizes the time-separation in its causal homotopy class. We can assume that, for such a causal homotopy $H(\epsilon, \lambda)$, the longitudinal curves $H_\epsilon$ are not lightlike pregeodesics close to $\gamma$ (neither, in particular, equal to $\gamma_0$); otherwise, $\gamma(\lambda_\delta)$ would be conjugate to $\gamma(0)$ along $\gamma$. We will need a variation $h(\epsilon, \lambda)$ of $\gamma|_{[0, \lambda_\delta]}$ which is piecewise $C^2$ (that is, continuous and $C^2$ on each closed rectangle corresponding to a suitable partition of the domain $(\epsilon, \lambda)$), and satisfies the other technical properties of the following result, to be proved at the end.

Lemma 2.2. Curve $\gamma_0 = \gamma|_{[0, \lambda_\delta]}$ admits a piecewise $C^2$ causal homotopy

$$h : [0, 1] \times [0, \lambda_\delta] \to M, \quad (\epsilon, \lambda) \to \gamma_\epsilon(\lambda)$$

with fixed extremes, such that the longitudinal curves $\gamma_n := \gamma_{\epsilon_n}$ are causal for some sequence $\epsilon_n \searrow 0$, and the variational vector field $V = \partial_\epsilon|_0 \gamma(\lambda)$ is not identically 0.

In particular, $V$ satisfies:

$$V(0) = V(\lambda_\delta) = 0, \quad V \neq 0, \quad g(V, \gamma') \equiv 0.$$

We remark that, a priori, the longitudinal curves are causal only for the sequence $\{\gamma_n\}$, that is, the variation may be “non-admissible”, in the terminology of [7, Sect. 10.3] (recall also Remark 2.1).
Recall that, according to [7, Dfn. 10.47, 10.49, 10.54, 10.57, 10.59], $V$ induces a class $[V]$ which belongs to the domain $X_0(y_0)$ of the quotient index form $\tilde{I}$; i.e., the index form defined on the piecewise smooth sections (vanishing at the extremes) of the quotient bundle $G(y)$ defined by taking vector fields on $y$ modulo $y'$. As each $y_n$ is causal$^2$, then $g(y_n', y_n')$ is non-decreasing at 0, and

$$0 \leq \frac{1}{2} g^{\gamma}(\lambda, \gamma(\lambda))d\lambda = \tilde{I}(V, V) = \tilde{I}([V], [V]),$$

(see also [33, pp. 289-290]), in contradiction with [7, Th. 10.69].

$(\Leftarrow)$. Obviously, the homotopic cut point of $y(0)$ must appear not beyond the first conjugate point, but from the proved implication, it can neither appear before this point. \(\square\)

**Proof of Lemma 2.2.** Fix a finite covering of convex neighborhoods of $M$ which cover the image of $y_0$, and choose $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k = \lambda_\delta$ such that $\gamma([\lambda_1, \lambda_{i+1}])$ is included in one of such neighborhoods, $U_i$, for all $i = 0, \ldots, k$. Notice that, taken normal coordinates on each $U_i$, the tangent space to $y_0(\lambda_i)$ is identifiable to $\mathbb{R}^n$. Even more, as

$$\lim_{\epsilon\to 0} H_{\epsilon}(\lambda_i) = y_0(\lambda_i)$$

and the set of directions in $\mathbb{R}^n$ is compact, there exists a sequence $\{\epsilon_n\} \searrow 0$ and a $C^1$-curve $\alpha_\epsilon$ such that $\alpha_\epsilon(\epsilon_n) = H_{\epsilon_n}(\lambda_i)$ for all $n$. By taking each sequence $\{\epsilon_n\}$ as a subsequence of $\{\epsilon_{n'}\}$, we can assume that all the sequences are equal $\epsilon_n \equiv \epsilon_1$. Iterating the process, $\alpha_\epsilon$ can be chosen $C^r$, for any finite $r > 1$. Let us show that $\alpha_\epsilon(0) \neq 0$ can also be assumed for some $\epsilon_0$. Indeed, otherwise, as the curves $H_{\epsilon}$ are different to $y_0$ for small $\epsilon$, we can find another $C^2$ reparametrization of some $\alpha_{\epsilon_0}$ such that its velocity at 0 does not vanish, and consider this new parameter as the original transversal parameter of the homotopy $H$.

Now, choose as variation $y_\epsilon(\lambda)$ of $y_0$ (for small $\epsilon$) the homotopy $h$ defined as: $y_\epsilon$ is the unique broken geodesic which joins $\alpha_\epsilon(\epsilon)$ and $\alpha_{\epsilon+1}(\epsilon)$ in $U_i$ when $\lambda$ varies between $\lambda_i$ and $\lambda_{i+1}$. Let $v_\epsilon(\lambda)$ be the tangent vector at $\lambda_i$ of such a $y_\epsilon$ restricted to $[\lambda_i, \lambda_{i+1}]$. As each pair of curves $\alpha_\epsilon, \alpha_{\epsilon+1}$ are $C^2$, the curve in $TM$ which maps each $\epsilon$ to $v_\epsilon(\lambda)$ is $C^2$ too (use [33, Lemma 5.9]). Thus, the homotopy $h$ can be written as

$$h(\epsilon, \lambda)|_{\lambda\in[\lambda_i, \lambda_{i+1}]} = y_\epsilon(\lambda)|_{[\lambda_i, \lambda_{i+1}]} = \exp_{\alpha_\epsilon(\epsilon)}((\lambda - \lambda_i)v_\epsilon(\epsilon))$$

for small $\epsilon$, and all the required properties follow. \(\square\)

**Remark 2.3.** The definition of homotopic cut point is obviously extendible to timelike geodesics and to geodesics in Riemannian manifolds, but the analogous of Theorem 2.2 would not hold, in general. In fact, it is easy to construct a Riemannian manifold $(S, d\ell^2)$ with a closed geodesic $c: [0, 1] \to S$, $c(0) = c(1)$, without conjugate points, such that its middle point $c(1)$ is the cut point, and the two pieces of the geodesic $c_0 = c|_{[0, 1]}$, and $c_1(1) = c(2 - \lambda), \forall \lambda \in [0, 1]$ are homotopic with fixed extremes $p_0 = c(0), p_1 = c(1)$.

Concretely, let $S$ be the following surface embedded in Euclidean space $\mathbb{R}^3$ (with natural coordinates $(x, y, z)$) and induced metric $d\ell^2$. $S$ is obtained by gluing the semi-cylinder $x^2 + y^2 = r^2, z \leq 0$, with a spherical cap of radius $r, x^2 + y^2 + z^2 = r^2, z \geq 0$. The metric $d\ell^2$ is $C^1$ but can even be made smooth by suitably redefining the cap near

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$^2$ With our sign convention $(+, -, \ldots, -)$, different from [7, 33].
the equator (this manifold will be used in the next examples; it can be also replaced by a paraboloid, with straightforward modifications). The required geodesic then would be $c(\lambda) = (r \cos \pi \lambda, r \sin \pi \lambda, 1)$, with $p_0 = (r, 0, -1), p_1 = (-r, 0, -1)$.

This counterexample for Riemannian geodesics is extended to timelike geodesics in the following example, which also shows the possible existence of different timelike homotopy classes in a single causal one.

**Example 2.1.** Consider the Riemannian surface $S \ni p_0, p_1$ as above, and define the spacetime $M = \mathbb{R} \times S$, $g = dt^2 - dl^2$. Choose $T > \pi r$, and take $z_0 = (0, p_0), z_1 = (T, p_1) \in M$. The timelike geodesics $\sigma_i(\lambda) = (T \lambda, c_1(\pi \lambda))$, $i = 0, 1$ maximize the time-separation between $z_0$ and $z_1$, and do not have conjugate points. If $T > 2 + \pi r$ then they are timelike homotopic, and if $T = 2 + \pi r$ then they are causally homotopic, but not timelike homotopic (if $S$ is smoothed as suggested above, the critical value $T = 2 + \pi r$ must be replaced by $T = L$ where $L$ satisfies: (i) any smooth homotopy in $S$ between $c_0$ and $c_1$ contains a curve of length $L$, (ii) no $L' > L$ satisfies (i)).

2.3. Arc-connectedness of the closure of the classes. Now, let us remark some properties of causal homotopy classes in globally hyperbolic spacetimes, in relation to Uhlenbeck’s study of the timelike case [41], carefully developed further by Beem et al. [7]. Our main aim is to show an appropriate sense of compactness of the piecewise geodesics in timelike homotopy classes (Theorem 2.3, Remark 2.4), which will be directly extendible to the fibered classes in Sect. 4. Along this subsection, $(M, g)$ will be globally hyperbolic, and a temporal Cauchy function $t : M \rightarrow \mathbb{R}$ (see the Appendix) is chosen to parametrize all the causal curves. Two points $x_0 < x_1$ will be also fixed and, as it is not restrictive to assume $t(x_i) = i$, $i = 0, 1$, all the connecting causal curves will be parametrized in $[0, 1]$.

Following [7, Sect. 10.2] (our basic reference throughout this subsection), there exist a $N > 0$ and a partition $0 = t_0 < t_1 < \cdots < t_N < 1$ of $[0, 1]$ such that for any causal chain $(z_0, \ldots, z_N)$, $z_i \in M$, $z_i \prec z_{i+1}$ with $t(z_i) = i$ and $z_0 = x_0, z_N = x_1$, there exists one and only one maximal causal geodesic connecting $z_i$ with $z_{i+1}$ (thus, $(z_0, \ldots, z_N)$ can be identified with the piecewise causal geodesic obtained connecting $z_i$ with $z_{i+1}$). Let $M_{t_0, x_1}$ be the space of such causal chains and $M_{t_0, x_1}$ the subset containing all the chronologically related chains (i.e., $z_i \prec z_{i+1}$). Clearly, $M_{t_0, x_1}$ and $M_{t_0, x_1}$ are subsets of a product of $N - 1$ Cauchy hypersurfaces $S_i$ (the fixed extremes of the chain can be disregarded) and, then, inherit a topology. By using that the relation $\leq$ is closed on convex neighborhoods, it is straightforward to check:

**Proposition 2.1.**

1. The space of causal chains $M_{t_0, x_1}$ is compact.
2. The space of chronological chains $M_{t_0, x_1}$ is open in $S_1 \times \cdots \times S_{N-1}$, and its closure is included in $M_{t_0, x_1}$.

One can check, reasoning for $M_{t_0, x_1}$ as in [7, Prop. 10.36] for $M_{t_0, x_1}$, that there exists a well defined length non-decreasing retraction from the set $\mathcal{N}_{t_0, x_1}$ of all the $t$--parametrized causal curves connecting $x_0$ and $x_1$ to $M_{t_0, x_1}$ (the topology of $\mathcal{N}_{t_0, x_1}$ can be chosen as the topology associated to the uniform distance $d$, i.e.: $d(\gamma_1, \gamma_2) = \text{Max}(d_R(\gamma_1(t), \gamma_2(t)) : t \in [0, 1])$, where $d_R$ is the distance associated to any auxiliary Riemannian metric).

Given an arc-connected component $U_{t_0, x_1}$ (resp. $\mathcal{U}_{t_0, x_1}$) of $M_{t_0, x_1}$ (resp. $M_{t_0, x_1}$), its piecewise smooth geodesics are timelike (resp. causally) homotopic. In fact, any
continuous curve \((z_0(\epsilon), \ldots, z_N(\epsilon))\) in \(U_{x_0, x_1}\) (resp. \(U_{\bar{x}_0, \bar{x}_1}\)) joining two given chains, also yields the required homotopy between the associated piecewise geodesics. We will be interested just in the problem of the invariance of the arc-connected components under limits. More precisely, recall that, as \(M_{x_0, x_1}\) is an open subset of a manifold, its connected and arc-connected components are equal. Even more, the following property (which is false for open subsets of \(\mathbb{R}^N\), in general) holds:

**Lemma 2.3.** Any chain \((z_{0, \infty}, \ldots, z_{N, \infty})\) in the closure of a (arc-)connected component \(U_{x_0, x_1}\) (resp. \(\overline{U}_{x_0, x_1}\)) of \(M_{x_0, x_1}\) (resp. \(\overline{M}_{x_0, x_1}\)) can be connected to any element of \(U_{x_0, x_1}\) (resp. \(\overline{U}_{x_0, x_1}\)) by means of an arc totally included in \(U_{x_0, x_1}\) (resp. \(\overline{U}_{x_0, x_1}\)), except one extreme.

**Proof.** We will reason for \(U_{\bar{x}_0, \bar{x}_1}\), being analogous for \(U_{x_0, x_1}\). Consider any converging sequence of chronological chains in \(U_{x_0, x_1}\), \\(\{[z_{0,k}, \ldots, z_{N,k}]_k \rightarrow [z_{0,\infty}, \ldots, z_{N,\infty}]\}\). Let \(N_i\) be the convex neighborhood which contains \(z_{i,\infty}, z_{i+1,\infty}\) and make lighter the notation putting \(x_k = z_{i,k}, y_k = z_{i+1,k}\), for \(k = 1, 2, \ldots, \infty\). The result follows immediately by applying the following claim recurrently for \(i = 0, \ldots, N-1\):

**Claim.** Let \(x_k \ll y_k, k \in \mathbb{N}\), and \(x_\infty < y_\infty\) be as above. If there exists a continuous curve \(\alpha : [0, 1] \rightarrow N_i\) and a decreasing sequence \(\lambda_k \rightarrow 0\) such that \(x_k = \alpha(\lambda_k)\) for large \(k\), then there exists a continuous curve \(\beta : [0, 1] \rightarrow N_i\) such that, for some \(\epsilon \in (0, 1]\):

\[\alpha(\lambda) \ll \beta(\lambda), \quad \beta(\lambda_k) = y_k, \quad \forall \lambda, \lambda_k \in (0, \epsilon).\]

Notice that this claim is obvious in the particular case \(x_\infty \ll y_\infty\). For the general case, let \(v_k\) be the velocity at zero of the unique timelike geodesic (causal, when \(k = \infty\)), defined on \([0, 1]\), from \(x_k\) to \(y_k\). As the bundle of the future timelike cones on \(\alpha\) is arc-connected, we can find a continuous timelike vector field \(V\) on \(\alpha\) such that \(V(\lambda_k) = v_k\).

The required curve is then

\[\beta(\lambda) = \exp_{\alpha(\lambda)}(V(\lambda)).\]

Recall that, in the case of \(U_{\bar{x}_0, \bar{x}_1}\), the a priori excluded extreme will also belong to \(U_{\bar{x}_0, \bar{x}_1}\). Summing up:

**Theorem 2.3.** Let \((M, g)\) be globally hyperbolic.

1. The closure \(\overline{U}_{x_0, x_1}\) of any connected component \(U_{x_0, x_1}\) of \(M_{x_0, x_1}\) is arc-connected.
2. Any arc-connected component \(U_{x_0, x_1}\) of \(M_{x_0, x_1}\) is closed.

**Remark 2.4.** In particular, when \([y_k]_{k \in \mathbb{N}}\) is a sequence of causal geodesics in the same causal (resp. timelike) homotopy class, such that its initial velocities converge to the velocity of a geodesic \(y_0\), then \(y_0\) belongs to the same causal homotopy class (resp. to the adherence of the timelike homotopy class – which is included in the same causal homotopy class of the sequence).

### 3. Connectedness Through Solutions to the LFE

In the following sections we shall consider three kinds of manifolds: the spacetime \(M\) of dimension \(n_0 \geq 2\), the Kaluza-Klein spacetime \(P\) of dimension \(n_0 + 1\) and (in some examples) a spacelike hypersurface \(S\) of dimension \(n_0 - 1\). We adopt the convention of denoting curves belonging to \(P\) with \(\gamma\), curves belonging to \(M\) with \(\sigma\) or \(x\), and curves belonging to \(S\) with \(c\).
3.1. Avez-Seifert type problem for the LFE. Consider on $M$ a fixed (exact) electromagnetic field $F = d\omega$, where $\omega$ is any differential 1-form. A point particle of rest mass $m > 0$ and electric charge $q \in \mathbb{R}$, moving under $F$, has a timelike worldline which satisfies the Lorentz force equation (LFE) (cf. [32, Sect. 3.1], [20, Sect. 11.9] or [26, Sect. 23])

$$D_s \left( \frac{dx}{ds} \right) = \frac{q}{m} \hat{F}(x) \left[ \frac{dx}{ds} \right].$$

Here the units are such that $c = 1$, $x = x(s)$ is the world line of the particle parametrized with proper time, $\frac{dx}{ds}$ is the velocity, $D_s \left( \frac{dx}{ds} \right)$ is the covariant derivative of $\frac{dx}{ds}$ along $x(s)$ associated to the Levi-Civita connection of $g$, and $\hat{F}(x)[\cdot]$ is the linear map on $T_xM$ defined by

$$g(x)[v, \hat{F}(x)[w]] = F(x)[v, w],$$

for any $v, w \in T_xM$.

We remark that $s$ must be the proper time parametrization, otherwise the constant $q/m$ in front of $\hat{F}$ cannot be interpreted as the charge-to-mass ratio of the particle. Recall that, for the LFE, the ratio $q/m$ is fixed, but the individual values of $q$ and $m$ become irrelevant.

As commented in the introduction, question (Q) becomes natural now. For the case $q/m = 0$, the LFE is just the geodesic equation, and the solution to question (Q) is well-known [7, Theorem 3.18, Prop. 10.39], [33, Prop. 14.19], [19, Prop. 6.7.1]:

**Theorem 3.1.** (Avez [3], Seifert [39]). Let $(M, g)$ be a globally hyperbolic spacetime, and $x_0 \leq x_1$ two causally related events. Then, in each causal homotopy class, $c_{x_0,x_1}$ there exists a causal geodesic $\sigma$ which maximizes the length among the causal curves in the class.

In particular, if $x_0 \ll x_1$ the two points can be connected by means of a timelike geodesic (in fact, by one for each time like homotopy class in $\mathcal{C} = \{X = 0, X = 1\}$, as will be apparent below). If $x_1 \in E^+(x_0) = J^+(x_0) \setminus I^+(x_0)$ then $x_0$ and $x_1$ can still be joined by a lightlike geodesic, but this case does not make sense for the LFE. One can also wonder for the connectedness of $x_0, x_1$ by means of a geodesic even if they are not causally related, as in variational frameworks described below. Although this question has a geometrical interest (see for instance the survey [37]), it does not have a direct physical interpretation, nor equivalence for LFE.

3.2. Related variational problems. Question (Q) can be approached as a variational one [26, Sect. 16]. Indeed, let $N_{x_0,x_1}$ be the set of all (piecewise) $C^1$ causal curves $\sigma : [0, 1] \to M$ from $x_0$ to $x_1 > x_0$. Using $F = d\omega$, consider the functional $I_{x_0,x_1}$ on $N_{x_0,x_1}$,

$$I_{x_0,x_1}[\sigma] = \int_{\sigma} (dx + \frac{q}{m} \omega) = \int_0^1 \left( \sqrt{g(\sigma'(\lambda), \sigma'(\lambda))} + \frac{q}{m} \omega(\sigma'(\lambda)) \right) d\lambda,$$

for any $\sigma \in N_{x_0,x_1}$.

\footnote{Sometimes the mass of the particle is assumed to be known, and the curve $x$ is parametrized with $r = s/m$. In this case the LFE becomes equivalent to the system [36, Defs. 3.1.1 and 3.8.1] $D_x \left( \frac{dx}{dr} \right) = q\hat{F}(x) \left[ \frac{dx}{dr} \right], \left| \frac{dx}{dr} \right| = m$.}
for all $\sigma \in \mathcal{N}_{x_0, x_1}$. The functional is invariant under monotonic reparametrizations of $\gamma$; in this sense, when talking about critical points, we may refer to non-parametrized curves. The connecting timelike solutions of the LFE (1), if they exist, are critical points of this functional, as it follows from a computation of the Euler-Lagrange equation. Conversely, every timelike extremal of this functional, once parametrized with respect to proper time, is a solution of the LFE.

In the geodesic case $q/m = 0$, it is convenient to replace functional $I_{x_0, x_1}$ by the “energy” functional

$$E[\sigma] = \frac{1}{2} \int_0^1 g(\sigma'(\lambda), \sigma'(\lambda)) d\lambda,$$

(3)

because of several reasons: (i) the critical curves of this functional are parametrized directly as geodesics, and (ii) the domain of the functional can be enlarged to include non-causal curves (making sense also for non-causally related $x_0, x_1$) and, then, non-causal connecting geodesics also become critical points. Moreover, the choice of extremes $\lambda_0 = 0, \lambda_1 = 1$ simplifies the domain of curves without loss of generality. Nevertheless, if one only knows that there exists a critical curve $\sigma_0$ of (3), neither the causal character of $\sigma_0$ nor (if the curve were timelike) the time of arrival, would be known a priori.

In the case of the LFE, one can also consider a functional, introduced in [8], which is related to the action functional and closer to (3):

$$J_{x_0, x_1}[\sigma] = \int_0^1 \left( \frac{1}{2} g(\sigma'(\lambda), \sigma'(\lambda)) + b \omega(\sigma'(\lambda)) \right) d\lambda,$$

(4)

on the space of all the (absolutely continuous) curves, non necessarily causal, which connect $x_0$ and $x_1$ in the interval $[0, 1]$. Concretely, Bartolo and Antonacci et al. [2, 5] studied the connectedness of the whole spacetime by means of critical points (non-necessarily causal) of this functional, and further results were obtained in posterior references (see, for example, [6, 13, 11, 16, 31] or the detailed account in [30]). Remarkably, in [14, 15] the authors were able to prove, under global hyperbolicity, the existence of at least one (uncontrolled) value of $q/m$ such that a timelike connecting solution of the associated LFE does exist. Indeed, a timelike critical point $\sigma_0$ (if it exists) is a solution of the Lorentz force equation for some (uncontrolled) ratio $q/m$. This follows from the Euler-Lagrange equation for $J_{x_0, x_1}$,

$$D_\lambda (\sigma') = b \hat{F}(x) [\sigma'].$$

(5)

In particular, $ds/d\lambda = C$ is a constant and, thus, $\sigma_0$ would satisfy the LFE with charge-to-mass ratio $q/m = b/C$. However, $C$ depends on the critical curve $\sigma_0$ ($C = \int_{\sigma_0} ds$), which in turn depends on the coefficient $b$ in an uncontrolled way.

Summing up, the variational approach for functional $J_{x_0, x_1}$, even though mathematically appropriate to study non-timelike curves, presents the following two limitations for our question (Q):

(a) For chronologically related points, one cannot control easily the causal character of the critical points.
(b) Even if the critical point is proved to be timelike, one cannot know a priori its charge-to-mass ratio and indeed it could in the end correspond to an unphysical value.
3.3. Known results on (Q), and a counterexample. An approach conceived to study directly the physical question (Q) started in [17, 29], where the authors considered the solutions to the LFE as projections of suitable lightlike geodesics for a Kaluza-Klein metric. The relation between the LFE and the projections of timelike geodesics for a higher dimensional Kaluza-Klein spacetime is well known [21–24, 27, 28]. However, it proved more useful for question (Q) to consider the solutions of the LFE as projections of lightlike geodesics in a Kaluza-Klein spacetime with a scale factor (for the additional dimension) proportional to the charge-to-mass ratio. Indeed this allowed to prove the existence of solutions having an a priori fixed charge-to-mass ratio in most cases.

The results in [29] improve those in [17]; the best achieved result is then:

**Theorem 3.2.** Let \((M, g)\) be a globally hyperbolic spacetime, and \(F = d\omega\) be an electromagnetic field on \(M\). Let \(x_0\) be an event in the chronological future of \(x_0\) and \(q/m \in \mathbb{R} - \{0\}\) any charge-to-mass ratio. Then there exists a future-directed causal curve \(\sigma_0\) which connects \(x_0\) and \(x_1\) and maximizes the functional \(I_{x_0, x_1}\) on \(N_{x_0, x_1}\).

Moreover, \(\sigma_0\) is everywhere timelike or lightlike. In the former case, the reparametrization of \(\sigma_0\) with respect to proper time becomes a solution of the LFE (1); in the latter case, \(\sigma_0\) is a lightlike geodesic.

Even though the hypotheses of this theorem are optimal, it is not completely satisfactory for our question (Q) because Theorem 3.2 does not forbid the maximizing curve \(\sigma_0\) to be a lightlike geodesic. We present below an example of such a situation, even in a simply connected (and contractible) spacetime. Summing up: if a connecting lightlike geodesic exists, **Theorem 3.2 does not answer our question (Q).**

**Example 3.1.** Consider the 3-dimensional spacetime \(M = \mathbb{R} \times S, ds^2 = dr^2 - dz^2\), \(t \in \mathbb{R}\), in Example 2.1, Remark 2.3. Let \(F = d\omega\) independent of \(t\), with \(\omega \equiv 0\) in the cap of \(S\) and, on the cylinder: \(\omega = Br\mu(z)d\theta\), where \(\theta\) is the angle, \(B \in \mathbb{R}\) and \(\mu(z)\) is a smooth monotone decreasing function such that \(\mu(-2\pi r) = 1, \mu(-\pi r) = 0\). Notice that \(F\) is different from zero only in the set \(\mathbb{R} \times R\), where the ribbon \(R \subset S\) is defined by \(R = \{q \in S : -2\pi r < z < -\pi r\}\).

Let \(p \in S\) be defined by the \(\mathbb{R}^3\) coordinates \((r, 0, -3\pi r)\), and let \(x_0 = (0, p)\), \(x_1 = (2\pi r, p)\). The events \(x_0\) and \(x_1\) are connected by two lightlike geodesics \(\sigma_1, \sigma_2\). Their projections \(\tilde{\sigma}_1, \tilde{\sigma}_2\) differ only on the orientation of their parametrization as their image is the circle \(x^2 + y^2 = r^2, z = -3\pi r\). Let \(\sigma_L\) be a generic timelike connecting curve such that its projection \(\tilde{c}_L\) on \(S\) has length \(L\). Since \(\sigma_L\) is timelike, \(dl/dt < 1\), and since it is also connecting we have \(L < 2\pi r\). This equation implies that any timelike connecting curve has a projection \(c\) completely contained in the region \(z \leq -2\pi r\) (\(c\) belongs to the identity of the homotopy group of the cylindric part of \(S\), with base point \(p\)). In particular timelike connecting curves cannot enter the region of the non-vanishing electromagnetic field.

Thus, there are three causal homotopy classes. Two classes \(\mathcal{C}_i\) for the two isolated lightlike geodesics \(\sigma_i, i = 1, 2\) and a further causal class \(\mathcal{C}\) containing all the timelike connecting curves (this last class contains lightlike curves, which are not geodesics). Roughly speaking the curves in \(\mathcal{C}\) cannot be deformed to the lightlike geodesics \(\sigma_i\) since the projections would reach the cap, and hence the deformed curves would become non-causal.

We shall now show that, for any \(|\frac{q}{m}B| > 1\), the absolute maximum and minimum of \(I_{x_0, x_1}[c]\) are reached in the geodesics \(\sigma_i\). First, notice that the electromagnetic term of
the action can be rewritten
\[ \frac{q}{m} \int_\sigma \omega = \frac{q}{m} \int C \mu(z) \, \mathrm{d}z, \]
which vanishes on \( C \), and is equal to \( \pm 2\pi \frac{q}{m} B \) \( r \) on the geodesics \( \sigma_i \). Thus, on the class \( C \) the action functional is equivalent to the length functional, which is bounded by the length \( 2\pi r \) of the maximizing geodesic (the particle at rest at \( p \)), as required.

4. Maximization of \( I_{x_0, x_1} \) over Homotopy Classes

First, let us introduce a trivial principal bundle \( P = M \times \mathbb{R}, \Pi : P \rightarrow M \) with structural group \((\mathbb{R}, +)\); \( b \in \mathbb{R}, \; p = (x, y), \; p' = pb = (x, y + b) \). Let \( y \) be the fiber coordinate and \( b \) be a dimensional positive constant. Given a potential 1-form \( \omega \) on \( M \) and a connection \( \tilde{\omega} = dy + \beta \omega \) on \( P \), consider the Kaluza-Klein metric
\[ \tilde{g} = g - \alpha^2 \tilde{\omega}^2, \]
and choose the scale factor \( \alpha \) as \( \alpha = \beta^{-1} \left| \frac{q}{m} \right| \). The actual value of the dimensional constant \( \beta \) will have no role in our work\footnote{It should be said that in the physical Kaluza-Klein theory one usually chooses \( \alpha = \beta^{-1} \sqrt{\frac{G}{16\pi}} \), where \( G \) is the Newton constant, to obtain the correct coupling between gravity and electromagnetism. This choice is obviously incompatible with our constraint (even approximately, because both imply \( \beta^2 \frac{q}{m} - 1 \) and for realistic particles this coefficient is huge). However, note that the Kaluza-Klein spacetime is used by us only as a technical tool: given \( q/m \) we define \( \alpha \), and so \( \alpha \) changes with the particle considered - the spacetime \( P \) is not pretending to be a physical spacetime.}

Fix \( x_0 \leq x_1 \) and a causal homotopy class \( \mathcal{C}_{x_0, x_1} \) of \( \mathcal{N}_{x_0, x_1} \). We are looking for critical curves of (2) on \( \mathcal{C}_{x_0, x_1} \) (and, thus, on \( \mathcal{N}_{x_0, x_1} \)).

4.1. Causal homotopy classes in a K-K bundle.

Let \( \mathcal{C}_{x_0, x_1} \) denote a (continuous) causal homotopy class of (piecewise \( C^1 \)) curves on \( P \), starting at some \( p_0 \in \Pi^{-1}(x_0) \) and ending in \( \Pi^{-1}(x_1) \) where, now, the homotopy does not keep fixed the second endpoint.

**Proposition 4.1.** Fixed \( x_0 \leq x_1 \) and \( p_0 \in \Pi^{-1}(x_0) \):

1. \( \gamma_1, \gamma_2 \in \mathcal{C}_{p_0, x_1} \) then \( \sigma_1 = \Pi \circ \gamma_1 \text{ and } \sigma_2 = \Pi \circ \gamma_2 \) belong to the same class \( \mathcal{C}_{x_0, x_1} \).
2. \( \gamma_1 \) and \( \gamma_2 \) projects on curves \( \sigma_1 = \Pi \circ \gamma_1 \text{ and } \sigma_2 = \Pi \circ \gamma_2 \) which are causally homotopic, then \( \gamma_1 \text{ and } \gamma_2 \) belong to the same \( \mathcal{C}_{p_0, x_1} \).

Thus, the projection \( \pi : P \rightarrow M \) sends homotopy classes of type \( \mathcal{C}_{p_0, x_1} \) to homotopy classes of type \( \mathcal{C}_{x_0, x_1} \), and induces a bijective map between classes of type \( \mathcal{C}_{p_0, x_1} \) and \( \mathcal{C}_{x_0, x_1} \).

**Proof.** Assertion (1) is obvious. Recall also that it ensures the induction of a map between homotopy classes. The surjectivity of this map is ensured because, for any future-directed causal curve \( \sigma \) in \( M \) which connects \( x_0 \) and \( x_1 \), there exists a future directed causal curve \( \tilde{\sigma} \) in \( P \) (i.e., the horizontal lift) starting at \( p_0 \) and projecting on \( \sigma \).

For (2) (which ensures injectivity), consider first the case when the \( \gamma_1 \)'s project on the same curve \( \sigma \), that is,
\[ \gamma_1(\lambda) = (\sigma(\lambda), y_1(\lambda)), \quad \gamma_2(\lambda) = (\sigma(\lambda), y_2(\lambda)), \quad \forall \lambda \in [0, 1]. \]
The map
\[ H(\epsilon, \lambda) = (\sigma(\lambda), \epsilon y_1(\lambda) + (1 - \epsilon)y_2(\lambda)) \] (8)
is a causal homotopy without the second point fixed, as required.

If the projections \( \sigma_i \) are different, consider their horizontal lifts \( \tilde{\sigma}_i \) starting at \( p_0 \). Then each \( \tilde{\sigma}_i \) is continuously causally homotopic to \( \gamma_t \) (from the previous case), and the lifting of the causal homotopy between \( \sigma_1 \) and \( \sigma_2 \) is clearly a causal homotopy between \( \tilde{\sigma}_1 \) and \( \tilde{\sigma}_2 \). \( \square \)

Remark 4.1. An analogous result holds for timelike classes type \( C_{p_0, x_1} \). In general, all the study of Subsect. 2.3 for the spaces containing piecewise geodesics, \( M_{\tilde{x}_0, x_1}, M_{x_0, x_1} \), and its arc-connected components \( U_{\tilde{x}_0, x_1}, U_{x_0, x_1} \) (constructed from causal and time path spaces from \( x_0 \) to \( x_1 \)), can be extended to analogous spaces containing piecewise geodesics from \( p_0 \) to \( \Pi^{-1}(x_1) \), namely, \( M_{p_0, x_1}, M_{p_1, x_1} \), and its arc-connected components \( U_{p_0, x_1}, U_{p_1, x_1} \). Just take into account:

1. If \( \sigma \) is a Cauchy temporal function on \( M \) then so is \( \tilde{\sigma} = \sigma \circ \Pi \) on \( P \) (see the Appendix).

In particular, \( J^+(p_0) \cap \Pi^{-1}(x_1) \) is compact because, any chosen Cauchy hypersurface \( S \) of \( M \) through \( x_1 \), it can be written as \( (J^+(p_0) \cap \Pi^{-1}(S)) \cap \Pi^{-1}(x_1) \) (the intersection between a compact and a closed subset).

2. Analogous to Theorem 2.3, Remark 2.4 can be stated and, in particular, the limit of geodesics in a class \( C_{p_0, x_1} \) (resp. \( C_{p_0, x_1} \)) also belong to this class (resp. to the closure of this class).

4.2. A Fermat-type equivalent problem. Given \( \sigma : [0, 1] \to M \) in \( C_{x_0, x_1} \), consider the two lightlike lifts
\[ \tilde{\sigma}^\pm(\lambda) = (\sigma(\lambda), y^\pm(\lambda)), \quad \lambda \in [0, 1] \]
of \( \sigma \) in \( P \) starting at a fixed \( p_0 = (x_0, y_0) \in \Pi^{-1}(x_0) \). Explicitly, the requirement to be lightlike implies
\[ (y^\pm)'(\lambda) = \pm a \left[ |\sigma'(\lambda)| - \beta \omega(\sigma'(\lambda)) \right] \] (9)and, thus:
\[ \tilde{\sigma}^\pm(\lambda) = \left( \sigma(\lambda), y_0 + \frac{1}{a} \int_0^\lambda \sqrt{g(\sigma'(\lambda), \sigma'(\lambda))} \, d\lambda - \beta \int_0^\lambda \omega(\sigma'(\lambda)) \, d\lambda \right) \] (10)
Then, the fiber coordinate \( Y_1^\pm[\sigma] \) of the final point of this curve is, essentially, the action functional:
\[ Y_1^\pm[\sigma] = y^\pm(1) = y_0 + \frac{1}{a} \left( \int_\sigma ds + (\pm q/m) \int_\sigma \omega \right) \] (11)
Comparing this expression with the one of \( I_{x_0, x_1} \), it is obvious that a maximization on \( C_{x_0, x_1} \) of \( I_{x_0, x_1} \) relative to the ratio \(+ q/m\) (resp. \(- q/m\)), corresponds to a minimization (resp. maximization) of \( Y_1^+ [\sigma] \) (resp. \( Y_1^- [\sigma] \)). Summing up:

**Theorem 4.1.** The curve \( \sigma_0 \) is a maximum of \( I_{x_0, x_1} \) on \( C_{x_0, x_1} \) and \( q/m > 0 \) (resp. \( q/m < 0 \)) if and only if \( \sigma_0 \) is a minimum (resp. maximum) of the arrival coordinate
\[ Y_1^+: C_{x_0, x_1} \to \mathbb{R}, \quad \sigma \mapsto Y_1^+(\sigma) \]
(resp. \( Y_1^- : C_{x_0, x_1} \to \mathbb{R}, \quad \sigma \mapsto Y_1^-(\sigma) \)).
4.3. The maximization result. The above variational principle reduces our problem to ensure the existence of maxima or minima for $Y_1^\pm$. The following result yields two candidates.

**Lemma 4.1.** The set $K$ containing the points in $\Pi^{-1}(x_1)$ reached by means of a causal curve in $C_{p_0, x_1}$, is a compact interval $K = \{x_1\} \times [y_1^-, y_1^+], y_1^- \leq y_1^+$.

Moreover, $p_0$ will be connectable with any of the two extremes of $K$ by means of a lightlike geodesic $\gamma^\pm$ (necessarily without conjugate points before the endpoint) in $C_{p_0, x_1}$.

**Proof.** The arc-connectedness of $K$ is straightforward from the existence of a causal homotopy between any two curves in $C_{p_0, x_1}$. For the compactness of $K$, recall that as $J^+(p_0) \cap \Pi^{-1}(x_1)$ is compact (Remark 4.1, Item (1)), any Cauchy sequence $\{(x_1, y_k)\}$ in $K$ will have a limit $(x_1, y_\infty)$ in $\Pi^{-1}(x_1)$. By the Avez-Seifert theorem, there exists a maximizing causal geodesic $\gamma_k \in U_{p_0, x_1} \subset C_{p_0, x_1}$ connecting $p_0$ and each $y_k$. Then, the limit curve $\gamma_0$ of the sequence $[y_k]$ will be a causal geodesic in $C_{p_0, x_1}$ too, and it will cross the Cauchy hypersurface $\Pi^{-1}(S)$ at some point, necessarily $(x_1, y_\infty)$. Thus, $\gamma_0$ belongs to the same causal homotopy class (Remark 4.1, Item (2)).

For the last assertion, notice that the extreme $y^-_1$ (resp. $y^+_1$) is connectable with $p_0$ by means of a length-maximizing causal geodesic $\gamma^+$ (resp. $\gamma^-$). Even more, $\gamma^\pm$ must be lightlike because, otherwise, an open neighborhood of the final point of $\gamma^\pm$ in $\Pi^{-1}(x_1)$ would lie in $K$. Finally, a conjugate point cannot exist because $\gamma^\pm$ is maximizing (see Theorem 2.2). □

**Lemma 4.2.** Assume $q / m > 0$ (resp. $< 0$), and let $\gamma(\lambda) = (\sigma(\lambda), y(\lambda)), \lambda \in [0, 1]$ be the lightlike geodesic in $C_{p_0, x_1}$ which connects $p_0$ and $(x_1, y^-_1)$ (resp. $(x_1, y^+_1)$). Then, $\sigma$ is a maximum of $I_{x_0, \gamma_1}$ on $C_{x_0, \gamma_1}$.

**Proof.** From Theorem 4.1, we only have to prove $y = \tilde{\sigma}^+$ (resp. $\tilde{\sigma}^-$) because in this case $\sigma$ is obviously a maximum (resp. minimum) of $Y^+$ (resp. $Y^-$). As $\gamma$ is a geodesic and $\partial_\tau$, a Killing field on $P$, we have the constant $\nu \equiv \tilde{g}(\gamma', \partial_\tau) = -a^2 (\gamma' + \beta \omega(\sigma'))$, that is:

$$\gamma' = -\frac{\nu}{a^2} - \beta \omega(\sigma'). \quad (12)$$

Even more, as the extreme $(x_1, y^-_1)$ is minimum (resp. $(x_1, y^+_1)$ maximum) in $K$, then $\nu \geq 0$ (resp. $\leq 0$) – otherwise, the horizontal lift of $\sigma$ would end beyond the extreme. As $\gamma$ is lightlike, $\tilde{g}(\gamma', \gamma') \equiv 0$ thus

$$\gamma' = \varepsilon \frac{|\sigma'|}{a} - \beta \omega(\sigma'), \quad (13)$$

where $\varepsilon$ equals 1, or $-1$. To specify $\varepsilon$, notice from $(12)$, $(13)$ and the sign of $\nu$:

$$|\sigma'| = \frac{\nu}{a} \quad \text{(resp. } |\sigma'| = -\frac{\nu}{a}). \quad (14)$$

Now, use $(14)$ to check that the expression for $(\gamma^+)'$ (resp. $(\gamma^-)')$ in $(9)$ coincides with the expression of $\gamma'$ in $(12)$, and the result follows. □

Notice also that, in the proof, $\nu = 0$ if and only if $\sigma$ is a lightlike geodesic and $\gamma$ is its horizontal lift [29]; otherwise $\sigma$ is timelike.
Remark 4.2. Analogous results hold if $K$ in Lemma 4.1 is taken as the adherence of the points in $\Pi^{-1}(x_1)$ reachable by curves in a timelike homotopy class $C_{p_0,x_1}$. The lightlike geodesics $\gamma^{\pm}$ will lie in $\overline{C}_{p_0,x_1}$, and will be causally homotopic to the curves in $C_{p_0,x_1}$. The analog of Lemma 4.2 states that if $q/m > 0$ (resp. $q/m < 0$), $\Pi(\gamma^+)$ (resp. $\Pi(\gamma^-)$) maximizes $I_{x_0,x_1}$ on $\overline{C}_{x_0,x_1}$. This projection is either a null geodesic belonging to the boundary $\partial C_{x_0,x_1}$ or a timelike curve belonging to $C_{x_0,x_1}$.

Summing up, the following generalization of Theorem 3.2 to causal homotopy classes is obtained:

**Theorem 4.2.** Let $(M, g)$ be a globally hyperbolic spacetime, and $F = \omega$ be an electromagnetic field on $M$. Let $x_1$ be an event in the causal future of $x_0$ and fix any causal homotopy class $C_{x_0,x_1}$.

For each $q/m \in \mathbb{R} - \{0\}$ there exists a future-directed causal curve $\sigma_0$ which connects $x_0$ and $x_1$ and maximizes the functional $I_{x_0,x_1}$ on $C_{x_0,x_1}$.

Moreover, $\sigma_0$ is everywhere timelike or lightlike. In the former case, the parameterization of $\sigma_0$ with respect to proper time becomes a solution of the LFE (1) for the charge-to-mass ratio $q/m$; in the latter case, $\sigma_0$ is a lightlike geodesic.

Even more, for any timelike homotopy class $C_{x_0,x_1} \subset N_{x_0,x_1}$ there exists a maximizer in $C_{x_0,x_1}$ which is either a timelike curve in $C_{x_0,x_1}$ or a lightlike geodesic in the boundary $\partial C_{x_0,x_1}$.

5. Existence of Timelike Local Maximizers

5.1. The existence result for causal homotopy classes. In the previous section the existence of maximizers, either timelike curves or lightlike geodesics, in each $C_{x_0,x_1}$ ($\overline{C}_{x_0,x_1}$) has been ensured. Here, we will prove that the maximizer on $C_{x_0,x_1}$ cannot be a lightlike geodesic if there exists a timelike curve in $C_{x_0,x_1}$. The proof is carried out in two steps. In the first one (Lemma 5.1), a maximizing lightlike geodesic $\sigma_0$ is shown to be free of conjugate points (except at most the two extremes). The second step is to check that, if such a $\sigma_0$ exists, all the other curves in $C_{x_0,x_1}$ must be lightlike.

**Lemma 5.1.** Let $\sigma : [0, 1] \to M$ be a lightlike geodesic such that $\sigma(r), 0 < r < 1$ is the first conjugate point to $\sigma(0)$. Then there exists a smooth ($C^0$) variation of $\sigma$ through causal curves such that the functional $I_{x_0,x_1}$ is strictly bigger on the variated longitudinal curves.

Therefore, the maximum of $I_{x_0,x_1}$ on a causal homotopy class $C_{x_0,x_1}$ cannot be attained at a lightlike geodesic with a conjugate point to $\sigma(0) = x_0$ before $\sigma(1) = x_1$.

**Proof.** The integrand in the right-hand side of (2) will be written, for a general smooth variation through causal curves $\sigma_v(\lambda)$, $v \in [0, \epsilon]$, $\epsilon > 0$, as

$$I \equiv I_v(\lambda) = \sqrt{\langle \sigma_v(\lambda), \sigma_v(\lambda) \rangle} + \frac{q}{m} \omega(\sigma_v(\lambda)).$$

(15)

Let $V(\lambda)$ be the variational field. Recall that for this variation, at $v = 0$:

$$\langle \sigma_v(\lambda), \sigma_v(\lambda) \rangle \equiv 0 \quad \partial_v \langle \sigma_v(\lambda), \sigma_v(\lambda) \rangle = 2\langle V(\lambda), \sigma_v(\lambda) \rangle \equiv 0.$$

But the second derivative of $\langle \sigma_v(\lambda), \sigma_v(\lambda) \rangle$ at $v = 0$ can be chosen nonnegative on $[0, 1]$ and strictly positive in some interval $(0, r + \delta)$ as in [33, Prop. 10.48]. Moreover, this
second derivative is equal for the associated variation \( \sigma_{-\epsilon}(\lambda) \) with variational field \(-V\).

Then, by using a Taylor expansion of (15):

\[
\frac{d \hat{I}(\lambda)}{d \nu} \bigg|_{\nu=0} = \sqrt{\frac{\partial^2}{2 \partial \nu^2} (\sigma'_\nu(\lambda), \sigma'_\nu(\lambda))} \bigg|_{\nu=0} + \frac{2}{m} \hat{d}_\nu (\omega (\sigma'_\nu(\lambda))) \bigg|_{\nu=0},
\]

\( \forall \lambda \in (0, r + \delta), \) (16)

where the integral \( \int_0^1 \) of the first term is strictly positive and equal for the two variations \( \sigma_\epsilon(\lambda) \) and \( \sigma_{-\epsilon}(\lambda) \). As the integral of the last term changes with the sign of \( \nu \), the integral of (16) will be strictly positive for at least one of the two variations, as required. \( \square \)

**Remark 5.1.** Even though in Lemma 4.1 the obtained lightlike geodesic in the total space \( P \) cannot have a conjugate point, we have proved here directly the inexistence of conjugate points for its projection on \( M \). In fact, the proof shows that this is a general property for actions of type \( I_{x_0, x_1} \), which contain a free particle term plus lower order terms in \( |\sigma'| \).

With Lemma 5.1 at hand, the last step follows just applying the studied properties of causal homotopy classes.

**Theorem 5.1.** Under the hypotheses of Theorem 4.2, the maximizer \( \sigma_0 \) of \( I_{x_0, x_1} \) on \( C_{x_0, x_1} \) is timelike if \( C_{x_0, x_1} \) contains a timelike curve.

**Proof.** Assume by contradiction that \( \sigma_0 \) is not timelike and, thus, it is a lightlike geodesic with no conjugate points before \( x_1 \). By Theorem 2.2, \( \sigma_0 \) must maximize the time separation in \( C_{x_0, x_1} \), in contradiction with the existence of a timelike curve in \( C_{x_0, x_1} \). \( \square \)

Theorems 4.2 and 5.1 prove directly our main result, Theorem 1.1.

### 5.2. A remarkable example

Lemma 5.1 does not forbid the existence of a lightlike geodesic \( \sigma \) which maximizes the functional on the closure of a timelike class \( \overline{C}_{x_0, x_1} \). However, in that case the maximizer on \( C_{x_0, x_1} \supset \overline{C}_{x_0, x_1} \) does not coincide with \( \sigma \), as the following example shows.

In this way the example will be automatically numbered with a title “Example . . .” before them as previous examples.

Let \( \Sigma \) be a surface embedded in \( \mathbb{R}^3 \) obtained by gluing the spherical cap \( x^2 + y^2 + z^2 = r^2, z > -\sqrt{\frac{3}{2}} r + \epsilon_z \), with a cylinder \( x^2 + y^2 = r^2/4, z < -\sqrt{\frac{5}{2}} r - \epsilon_z \), by making a smooth transition in the points with coordinate \( z \in [-\sqrt{\frac{3}{2}} r - \epsilon_z, -\sqrt{\frac{5}{2}} r + \epsilon_z] \), for some positive \( \epsilon_z < \frac{\sqrt{3}}{2} r \). Notice that this transition can be made smooth and depending only on the azimuthal angle \( \theta \) in a small interval \( (\frac{\pi}{2} - \epsilon_\theta, \frac{3\pi}{2} + \epsilon_\theta) \), \( \epsilon_\theta < \pi/6 \). Only the details of this surface included in the spherical cap with \( \theta \leq \pi/2 + \epsilon \), for some small positive \( \epsilon < \pi/6 \), will be relevant.

Let \( dt^2 \) be the induced Riemannian metric on \( \Sigma \), and fix \( q = (r, 0, 0) \in \Sigma \). Consider the natural product (globally hyperbolic) spacetime \( M = \mathbb{R} \times \Sigma, g = dt^2 - dt'^2 \), with natural projection \( \pi : M \rightarrow \Sigma \), and the fixed events \( x_0 = (0, q), x_1 = (2\pi r, q) \). The timelike curve \( \lambda \mapsto (2\pi r \lambda, q) \) fix a timelike homotopy class \( C_1 := C_{x_0, x_1}^{(1)} \). The connecting lightlike geodesic

\[
\sigma_0(\lambda) = (2\pi r \lambda, c_0(\lambda)), \quad c_0(\lambda) = (r \cos 2\pi \lambda, r \sin 2\pi \lambda, 0), \quad \lambda \in [0, 1],
\]

\[
\sigma_0(\lambda) \neq (2\pi r \lambda, c_0(\lambda)),
\]

where the integral \( \int_0^1 \) of the first term is strictly positive and equal for the two variations \( \sigma_\epsilon(\lambda) \) and \( \sigma_{-\epsilon}(\lambda) \). As the integral of the last term changes with the sign of \( \nu \), the integral of (16) will be strictly positive for at least one of the two variations, as required. \( \square \)

**Remark 5.1.** Even though in Lemma 4.1 the obtained lightlike geodesic in the total space \( P \) cannot have a conjugate point, we have proved here directly the inexistence of conjugate points for its projection on \( M \). In fact, the proof shows that this is a general property for actions of type \( I_{x_0, x_1} \), which contain a free particle term plus lower order terms in \( |\sigma'| \).

With Lemma 5.1 at hand, the last step follows just applying the studied properties of causal homotopy classes.

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Let \( dt^2 \) be the induced Riemannian metric on \( \Sigma \), and fix \( q = (r, 0, 0) \in \Sigma \). Consider the natural product (globally hyperbolic) spacetime \( M = \mathbb{R} \times \Sigma, g = dt^2 - dt'^2 \), with natural projection \( \pi : M \rightarrow \Sigma \), and the fixed events \( x_0 = (0, q), x_1 = (2\pi r, q) \). The timelike curve \( \lambda \mapsto (2\pi r \lambda, q) \) fix a timelike homotopy class \( C_1 := C_{x_0, x_1}^{(1)} \). The connecting lightlike geodesic

\[
\sigma_0(\lambda) = (2\pi r \lambda, c_0(\lambda)), \quad c_0(\lambda) = (r \cos 2\pi \lambda, r \sin 2\pi \lambda, 0), \quad \lambda \in [0, 1],
\]
lies in the boundary $\hat{C}_1$. In fact, $c_0$ can be reached by approximating the part $c_0$ with a constant-speed parametrization $c_\alpha$ of $\Sigma \cap \Pi_q$, where $\Pi_q \subset \mathbb{R}^3$ is the plane through $q$, orthogonal to the plane $z = 0$, which makes an oriented positive angle $\alpha < \pi/2$ with the plane $z = 0$ ($c_\alpha$ is contained in the region $z > 0$ except in the tangent point $q$). However, by letting $\alpha < 0$ we can find a second timelike homotopy class $C_2$ such that $c_0 \in C_2$; of course, $C_1$ and $C_2$ are contained in the same causal homotopy class $\mathcal{C}$. Notice that $c_0$ passes through the antipodal point $-q = (-r, 0, 0)$, which is also a conjugate point of $q$; thus, $c_0$ also contains a conjugate point.

Fix $q/m > 0$ (resp. $q/m < 0$), and let $F = B\pi^*\Omega = d\omega$ be on $M$, where $\Omega$ is the volume 2-form of $\Sigma$ (with the orientation induced by the outer normal in the spherical cap), and where $B : \Sigma \to \mathbb{R}$ is a non-negative (resp. non-positive) function, with $B \equiv B > 0$ (resp. $B < 0$) constant for $\theta < \pi/2$, and monotonically decreasing (resp. increasing) to 0 for $\theta \in (\pi/2, \pi/2 + \epsilon)$. The charged-particle action $I_{q_0,x_1}$ is given by two contributions. The electromagnetic term reads

$$\frac{q}{m} \int_\sigma \omega = \frac{q}{m} \int_R B\Omega$$

where, without loss of generality, $\sigma(\lambda) = (2\pi r \lambda, c(\lambda))$ and $\partial R = c$. For a given length $L \leq 2\pi r$ of $c$ this integral is maximized in $C_1$ by the circle $c_\alpha$ with length $L$, namely $c_L$. Indeed, the maximizer must be a circle in order to maximize the area, and it is tangent to $c_0$ since, otherwise, its enclosed surface $R$ would include regions where $B < B$ (resp. $B > B$). Thus

$$\left| \frac{q}{m} \int_\sigma \omega \right| \leq \frac{q}{m} BA[c^L],$$

where $A[c^L]$ is the area contained in $c^L$. And the equality holds iff $c = c^L$ (up to a reparametrization with the same winding number).

The contribution of the length of $\sigma$ in $I_{q_0,x_1}$ is:

$$\int_\sigma ds = \int_0^{2\pi r} \sqrt{1 - \left( \frac{dl}{dr} \right)^2} dr \leq 2\pi r \sqrt{1 - \left( \frac{l[c]}{2\pi r} \right)^2},$$

where $l[c]$ is the length of $c = \pi \circ \sigma$, and the equality holds when the speed of $c$ is constant. We have then

$$I_{q_0,x_1}[\sigma] \leq 2\pi r \sqrt{1 - \left( \frac{l[c]}{2\pi r} \right)^2} + \frac{q}{m} BA[l[c]],$$

where the equality holds iff $\pi \circ \sigma = c^{l[c]}$. But in terms of the angle $0 \leq \alpha \leq \pi/2$ with $c_\alpha = c'$, we have $l = 2\pi r \cos \alpha$ and $A[c'] = 2\pi r^2 (1 - \sin \alpha)$. Hence if $\frac{q}{m} Br > 1$, 

$$I_{q_0,x_1}[\sigma] \leq 2\pi r^2 \frac{q}{m} B + 2\pi r (1 - \frac{q}{m} Br) \sin \alpha \leq 2\pi r^2 \frac{q}{m} B = I_{q_0,x_1}[\sigma_0],$$

and the equality holds iff $\alpha = 0$ and the projection of $\sigma$ is $c_0(= c^{2\pi r})$, i.e. iff $\sigma = \sigma_0$.

Nevertheless, even though $\sigma_0$ maximizes in $\mathcal{C}_1$, it does not maximize in $\mathcal{C}_2$ (nor in the causal homotopy class $\mathcal{C}$), in agreement with our results the example environment.
Remark 5.2. Take the surface $\Sigma = S$ obtained by gluing the semisphere $x^2 + y^2 + z^2 = r^2, z \leq 0$ with the semicylinder $x^2 + y^2 = r^2, z \leq 0$. By using an appropriate differentiable structure, $S$ can be regarded as a $C^2$ manifold endowed with a $C^1$ metric. Thus, Christoffel symbols and geodesics make sense, but not conjugate points or Jacobi fields. Repeating the above procedure, the timelike homotopy class $C^1$ is defined as above, but $C^2$ will not exist and, then, $C = \overline{C}_1$. Analogous $F, \omega$ makes sense on $z \geq 0$ (and can be extended to $S$), thus, the action $I_{x_0, x_1}$ can be defined on $C$. Then, the lightlike geodesic $\sigma$ maximizes in $C$, and no timelike maximizer in $C$ exists, even though $C$ contains timelike curves. We conclude that the assumption made at the beginning of Section 2.1, that the degree of differentiability of $(M, g)$ is at least $C^2$, is needed for Theorem 5.1 to hold.

6. The Non-Exact Electromagnetic Field Case

Throughout the paper, $F$ has been not only a closed skew-symmetric 2-covariant vector field (i.e., an 'electromagnetic field'), but also exact. This condition is scarcely restrictive: from the mathematical viewpoint, it is fulfilled in any contractible spacetime and, from the physical one, no known experimental evidence of non–exact electromagnetic fields on spacetime exists (magnetic monopoles have not been found). Nevertheless, it is interesting to study this case, in order to understand better our approach and the limits of the expected results. Recall:

(1) If $F$ is not exact the problem of maximizing the action becomes ill posed, since the electromagnetic potential is not globally defined. Thus, the stated variational problem of finding maximizing curves for the action does not make sense, but the existence problem for connecting solutions of the LFE is still perfectly meaningful.

(2) There are non-exact $F$ which can be studied by means of Kaluza-Klein metrics. For such a $F$, the total space $P$, whenever it exists, is necessarily a non-trivial principle bundle with fiber $U(1) \equiv S^1$. In fact, the necessary and sufficient condition for $\beta F$ to be the curvature 2-form of a real-valued connection $\tilde{\omega} = dy + \beta \omega$ over a $U(1)$ bundle $P$ on $M$ of fiber angle $y$ (and, introducing a metric (7), of fiber circumference $2\pi a$), is that the cohomology class $[\frac{\beta}{2\pi} F] \in H^2(M, \mathbb{R})$ is integer (more precisely, the Čech cohomology class canonically associated with $[\frac{\beta}{2\pi} F]$ belongs to the image of the morphism $\epsilon^2 : \hat{H}^2(M, \mathbb{Z}) \to \hat{H}^2(M, \mathbb{R})$ induced by the inclusion $\epsilon : \mathbb{Z} \to \mathbb{R}$).

The example below shows that Theorem 1.1 does not admit a generalization to the non-exact case, and becomes definitive for several reasons: the spacetime and the field $F$ are very simple, $F$ can be described with a principal bundle, and the found events $x_0, x_1$ cannot be connected by a solution of the LFE, whatever value of $q/m$ is chosen.

Example 6.1. Let the 3-dimensional spacetime $M$ be the product $M = \mathbb{R} \times S^2$, $t \in \mathbb{R}$, with natural projection $\pi : M \to S^2$, and metric $ds^2 = dt^2 - dl^2$ where $dl^2$ is the usual Riemannian metric of a sphere $S^2$ of radius $r$. Let $\Omega$ be one of the two associated volume 2-forms on $S^2$, with associated endomorphism field $\overline{\Omega} \equiv (\overline{\Omega}_i)$, and consider on $M$ the

\[ e^{i\frac{\pi}{4}} \]

Remarkably, in this case the complex exponential associated to the charged particle action $e^{i\frac{\pi}{4}}$, can be well defined. The problems for $I$ can therefore be circumvented in the quantum case [1].
electromagnetic field\(^6\) \(F = B\pi^*\Omega\). The LFE for a curve \(\sigma(t) = (t, c(t))\) becomes

\[
D_t \frac{c'}{\sqrt{1 - |c'|^2}} = \frac{q}{m} B \hat{\Omega}[c'],
\]

(22)

where \(D\) is the Levi-Civita connection of \(S^2\). Multiplying by \(c'/\sqrt{1 - |c'|^2}\) one finds \(|c'|^2 = \text{const}\). Thus, from the above equation:

\[
D_t c' = \frac{q}{m} B \sqrt{1 - |c'|^2} \hat{\Omega}[c'],
\]

(23)

from which it follows that \(c\) is a curve having constant velocity and constant curvature. Indeed, the equation above implies that the velocity \(c'(t)\) rotates with angular velocity \(\omega = \frac{q}{m} B \sqrt{1 - |c'|^2}\) with respect to a parallel transported frame along \(c\). But the only constant curvature curves on \(S^2\) are the circles, and \(c\) cannot be a maximal circle (in this case \(c\) would be a geodesic and, hence, the left-hand side of (23) would vanish, whereas the right-hand side would not). As a consequence, there is no solution \(c\) of (23) which connects two opposite points \(p, q\) on \(S^2\), although for \(t > \pi r\), \(x_1 = (q, t)\) lies in the chronological future of \(x_0 = (p, 0)\).

7. Conclusions

A full Avez-Seifert type result on the spacetime connectedness through solutions to the LFE has been proved. The hypothesis of this result (Theorem 1.1) are optimal because, on one hand, no more general hypothesis than \(x_0 \ll x_1\) makes sense and, on the other, the generalization to non-exact electromagnetic fields is not possible.

The proof is based in a purely geometric technique on an auxiliary Kaluza-Klein spacetime, and the consistency of the technique is proved by using the variational interpretation of the solutions to the LFE for exact electromagnetic fields. However, the proof itself is not variational, and the results previously obtained by means of variational methods do not have the accuracy and natural physical interpretation of those studied here.

By the way, new properties of timelike and causal homotopy classes, interesting on their own right, have been obtained. The careful study of the lightlike geodesics in such classes have become essential for our proof, and results as Theorems 2.1, 2.2 do not have analogues in the Riemannian case. Thus, the applications of timelike and causal classes in the present and previous papers, as [18, 38, 40], would justify a further study, as a separate field.

Appendix: Causality of Kaluza-Klein Metrics

Let \((M, g)\) be a spacetime and \(G\) a Lie group endowed with a positive definite Ad-invariant metric \(h\). Let \(\Pi : P \to M\) be a principal fiber bundle with structural group \(G\) endowed with a Kaluza-Klein metric

\[
\tilde{g} = \Pi^* g - \tilde{\omega}^* h,
\]

where \(\tilde{\omega}\) is some fixed 1-form connection on \(P\).

\(^6\) Notice that the electric part of \(F\) with respect to \(\partial_t\) is 0, and the magnetic part corresponds, at a prerelativistic level, with a uniform magnetic field on \(S\), see [4, Sect. 3] and references therein.
Recall that, when \((M, g)\) is globally hyperbolic then it admits a Cauchy temporal function \(t\); that is, \(t\) is smooth with future-directed timelike gradient\(^7\) and each level \(S_0 = t^{-1}(t_0)\) is a Cauchy hypersurface \([10]\). Moreover, \(M\) splits smoothly as a product \(\mathbb{R} \times S\) where \(S\) is a Cauchy hypersurface and the metric has no crossed terms between \(\mathbb{R}\) and \(S\).

**Theorem 7.1.** If \((M, g)\) is globally hyperbolic then \((P, \tilde{g})\) is globally hyperbolic too and, for any Cauchy temporal function \(t : M \to \mathbb{R}\) the composition \(\tilde{t} = t \circ \Pi : P \to \mathbb{R}\) is Cauchy temporal on \(P\).

Therefore, \(P\) splits smoothly as a product \(\Pi^{-1}(S) \times \mathbb{R}\) where the metric \(\tilde{g}\) has no crossed terms and \(\Pi^{-1}(S)\) is a principal fiber bundle on \(S\) with structural group \(G\).

**Proof.** All the conclusions follow easily by proving that \(\tilde{t}\) is a Cauchy temporal function. Clearly

\[
\tilde{g}(\nabla \tilde{t}, \tilde{V}) = d\tilde{t}(\tilde{V}) = dt(d\Pi(\tilde{V})) = g(\nabla t, d\Pi(\tilde{V})).
\]

Thus, \(\nabla \tilde{t}\) is the horizontal lifting of \(\nabla t\) and, in particular, a temporal function.

To check that each hypersurface \(\tilde{S}_t = \tilde{t}^{-1}(t) = \Pi^{-1}(S_t)\) is Cauchy, let \(\gamma\) be an inextendible timelike curve, which can be assumed to be reparametrized with \(\tilde{t}\) without loss of generality. Assume by contradiction that \(\gamma\) crosses the hypersurfaces \(\tilde{S}_t\) for \(t < t_0\) but not \(\tilde{S}_t\) (an analogous reasoning holds for \(t_0 < t\)). The projection \(\sigma = \Pi \circ \gamma\) is also timelike, but it is extendible through \(t_0\) because \(S_0\) is Cauchy. Then, consider a local trivialization \(U \times G\) of the bundle \(P\) with \(\sigma(t) \in U\). Then we can write in this trivialization \(\gamma'(t) = (\sigma(t), \eta(t))\) and

\[
0 < \tilde{g}(\gamma'(t), \gamma'(t)) = g(\sigma'(t), \sigma'(t)) - h(\tilde{\omega}(\eta'(t)), \tilde{\omega}(\eta'(t))). \tag{24}
\]

That is, the \(h\)-length of \(\eta(t)\) is bounded in \([t_0 - \epsilon, t_0]\) by the \(g\)-length of the extension of \(\sigma\) to \([t_0 - \epsilon, t_0]\). Thus, as \(h\) is complete, \(\eta(t)\) is continuously extendible to \(t_0\), and so is \(\gamma\), a contradiction. \(\square\)

**Remark 7.1.** There are other causality properties of \(M\) (as being chronological, causal, strongly causal or stably causal) which are transferred to \(P\).

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\(^7\) This means that \(t\) is temporal; in particular, \(t\) is a time function, that is, a continuous function which grows on any future-directed causal curve.
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