ABSOLUTELY CONTINUOUS INVARIANT MEASURES OF PIECEWISE LINEAR LORENZ MAPS

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Abstract. Consider piecewise linear Lorenz maps on \([0, 1]\) of the following form

\[ f_{a, b, c}(x) = \begin{cases} \frac{ax + 1 - ac}{b(x - c)} & x \in [0, c) \\ b(x - c) & x \in (c, 1] \end{cases} \]

We prove that \(f_{a, b, c}\) admits an absolutely continuous invariant probability measure (acim) \(\mu\) with respect to the Lebesgue measure if and only if \(f_{a, b, c}(0) \leq f_{a, b, c}(1)\), i.e. \(ac + (1 - c)b \geq 1\). The acim is unique and ergodic unless \(f_{a, b, c}\) is conjugate to a rational rotation. The equivalence between the acim and the Lebesgue measure is also fully investigated via the renormalization theory.

1. Introduction

Lorenz maps are one-dimensional maps with a single singularity, which arise as Poincaré return maps for flows on branched manifolds that model the strange attractors of Lorenz systems. A Lorenz map on the interval \(I := [0, 1]\) is a map \(f : I \to I\) such that for some critical point \(c \in (0, 1)\) we have

(i) \(f\) is continuous and strictly increasing on \([0, c)\) and on \((c, 1]\);
(ii) \(\lim_{x \uparrow c} f(x) = 1, \lim_{x \downarrow c} f(x) = 0\).

A Lorenz map \(f\) is said to be piecewise linear if it is linear on both intervals \([0, c)\) and \((c, 1]\). Such a map is of the form

\[(1.1) f_{a, b, c}(x) = \begin{cases} \frac{ax + 1 - ac}{b(x - c)} & x \in [0, c) \\ b(x - c) & x \in (c, 1] \end{cases} ,\]

where \(a > 0, b > 0, 0 < c < 1, ac \leq 1\) and \(b(1 - c) \leq 1\).

Let \(\beta > 1\). The map \(T_{\beta}(x) = \beta x \mod 1\) is the well known \(\beta\)-shift related to \(\beta\)-expansion (25). Assume \(0 \leq \alpha < 1\). The transformation \(T_{\beta, \alpha}\) defined by

\[T_{\beta, \alpha}(x) = \beta x + \alpha \mod 1\]

is a natural generalization of \(\beta\)-shift. There are many works done on \(T_{\beta, \alpha}\) (see 11 13 16 21 22 23 24 25). When \(1 < \beta \leq 2\), \(T_{\beta, \alpha}\) is a piecewise linear Lorenz map. In fact, \(T_{\beta, \alpha} = f_{\beta, \beta, c}\) with \(c = (1 - \alpha)/\beta\). Recently, Dajani et al [5] studied another variation \(S_{\beta, \alpha}\) of \(\beta\)-shift. For \(0 < \alpha < 1\) and \(1 < \beta < 2\),

\[(1.2) S_{\beta, \alpha}(x) = \begin{cases} \beta x & x \in [0, 1/\beta) \\ \alpha(x - 1/\beta) & x \in (1/\beta, 1]\end{cases} ,\]

which is the piecewise linear Lorenz map \(f_{\beta, \alpha, 1/\beta}\). Lorenz maps arise as return maps to a cross-section of a semi-flow on a two dimensional branched manifold (cf. 11, 15, 27). The flow lines starting from \(c\) never return to \(I\). So usually the map is considered not defined at \(c\) (cf. 12). But it is also convenient to regard \(c\) as two points \(c+\) and \(c-\), the right and left of \(c\),
so that the Lorenz map is a continuous map defined on the disconnected compact space $[0, c-] \cup [c+, 1]$. Different dynamical aspects of Lorenz maps are studied in the literatures such as rotation interval, asymptotic periodicity, topological entropy and renormalization etc (see [2, 7, 8, 12]). In this paper we shall study the absolutely continuous invariant probability measures (acim for short) of piecewise linear Lorenz maps. The existence of acim and the equivalence of acim with respect to the Lebesgue measure are studied.

The Lebesgue measure is clearly quasi invariant under $f_{a,b,c}$. Let $P_{a,b,c}$ be the associated Perron-Frobenius operator and let

$$A_n(h) = \frac{1}{n} \sum_{i=0}^{n-1} P_{a,b,c}^i h, \quad h \in L^1([0,1]).$$

Our results are stated in the following two theorems.

**Theorem A.** The piecewise linear Lorenz map $f_{a,b,c}$ admits an absolutely continuous invariant probability measure $\mu$ with respect to the Lebesgue measure if and only if $f_{a,b,c}(0) \leq f_{a,b,c}(1)$, i.e. $ac + b(1-c) \geq 1$. More precisely,

1. If $ac + b(1-c) = 1$ and $\log a/\log b$ is rational, then there exists positive integer $n$ such that $f_{a,b,c}^n(x) = x$ for all $x \in [0,1]$. Consequently, for each density $g$ on $[0,1]$, $A_n(g)$ is the density of an invariant measure of $f_{a,b,c}$.

2. If $ac + b(1-c) = 1$ and $\log a/\log b$ is irrational, then the acim is unique and its density is bounded from below and from above by the two constants $(\frac{a}{b})^n$ and $(\frac{a}{b})^{-n}$.

3. If $ac + b(1-c) > 1$, then the acim is unique and its density is of bounded variation.

4. If $ac + b(1-c) < 1$, then $f_{a,b,c}$ admits no acim.

It is well known to Lasota and Yorke ([19]) that a strongly expanding interval map $f$ (i.e. $|f'(x)| > \lambda > 1$ except finite points) admits an acim with respect to the Lebesgue measure. It is also known that a piecewise linear Lorenz map with a fixed point also admits an acim with respect to the Lebesgue measure (cf. [5, 6]). These results don’t apply to the Lorenz maps defined by (1.1) which, in general, are not strongly expanding and admit no fixed point.

Suppose that $f_{a,b,c}$ admits a unique acim with respect to the Lebesgue measure. If $f_{a,b,c}$ is a homeomorphism (i.e., $ac + b(1-c) = 1$) with irrational rotation number, we shall see from the proof of Theorem A that the acim of $f_{a,b,c}$ is equivalent to the Lebesgue measure. If $ac + b(1-c) > 1$, the acim is not necessarily equivalent to the Lebesgue measure, even if $f_{a,b,c}$ is strongly expanding (i.e. $a > 1$ and $b > 1$). For example, Parry [24] proved that the acim of symmetric piecewise linear Lorenz map $f_{a,a,1/2}$ is not equivalent to the Lebesgue measure if and only if $1 < a < \sqrt{2}$. We point out that the support of the acim of $T_{\beta,a}$ was studied in [11, 10].

Assume that $ac + b(1-c) > 1$. As we shall see in Lemma [4], the acim of $f_{a,b,c}$ is equivalent to the Lebesgue measure if and only if $f_{a,b,c}$ is transitive, i.e. $\bigcup_{n \geq 0} f_{a,b,c}^n(U)$ is dense in $I$ for each non-empty open set $U \subset I$. In general, the transitivity of a Lorenz map is not easy to check. Palmer [21] studied the transitivity of $T_{\beta,a}$ by using so-called primary cycle (see also [11]). Alves et al introduced a topological invariant to study the transitivity of $T_{\beta,a}$ ([3]). The conditions of both primary cycle and the topological invariant of Alves et al are difficult to check too. We will provide a rather simple criterion of the transitivity for the piecewise linear Lorenz maps $f_{a,b,c}$ with $ac + b(1-c) > 1$. 

Let us describe our criterion. Assume \( ac + b(1 - c) > 1 \). Then \( f_{a,b,c} \) admits periodic points, because it admits positive topological entropy ([2]). Let \( \kappa \) be the minimal period of the periodic points of \( f_{a,b,c} \). Assume \( 2 \leq \kappa < \infty \). Then \( f_{a,b,c} \) admits a unique \( \kappa \)-periodic orbit. Let \( P_L \) and \( P_R \) be adjacent \( \kappa \)-periodic points such that \( c \in [P_L, P_R] \). It can be proved that \( f_{a,b,c}^\kappa \) is linear on \([P_L, c)\) and on \((c, P_R]\) (Lemma 2.6). Write

\[
A := f_{a,b,c}^\kappa (c +), \quad B := f_{a,b,c}^\kappa (c -).
\]

That \( \kappa = 2 \) means \( f_{a,b,c} \) has no fixed point, \( f_{a,b,c}(0) = A < B = f_{a,b,c}(1) \) and \( c \in [A, B] \). When \( \kappa = 2 \), let

\[
M := \min \left\{ \frac{c - A}{B - c}, \frac{B - c}{c - A} \right\}.
\]

**Theorem B.** Suppose \( ac + b(1 - c) > 1 \). If \( \kappa = 1 \), then the acim of \( f_{a,b,c} \) is equivalent to the Lebesgue measure. If \( 2 \leq \kappa < \infty \), then the acim of \( f_{a,b,c} \) is equivalent to the Lebesgue measure if and only if

\[
[A, B]\backslash[P_L, P_R] \neq \emptyset \quad \text{or} \quad [A, B] = [P_L, P_R].
\]

In particular, when \( \kappa = 2 \), the acim of \( f_{a,b,c} \) is equivalent to the Lebesgue measure if and only if

\[
\begin{cases} 
ab > 1 + M & \text{if } M < 1 \\
ab \geq 2 & \text{if } M = 1.
\end{cases}
\]

See Figure 1 for a piecewise linear map whose acim is not equivalent to the Lebesgue measure.

Parry [24] proved that the acim of \( f_{a,a,1/2} \) \((1 < a \leq 2)\) is not equivalent to the Lebesgue measure if and only if \( 1 < a < \sqrt{2} \). This may be obtained as a special case of Theorem B.

We shall collect some basic useful facts in §2, including rotation number, Lyapunov exponent, Frobenius-Perron operator and renormalization. Theorem A is proved in §3 and Theorem B in §4. Densities of some piecewise linear Lorenz maps will be presented in §5.

2. Preliminaries

In this section, we present some facts concerning the rotation number, Lyapunov exponent, Frobenius-Perron operator and the renormalization of Lorenz maps, which will be useful later.

2.1. Rotation number and Lyapunov exponent.

We denote by \( e : \mathbb{R} \to S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \) the natural covering map \( e(x) = \exp(2\pi ix) \). Let \( f \) be a Lorenz map, not necessarily linear. There exists a map \( F : \mathbb{R} \to \mathbb{R} \) such that \( e \circ F = f \circ e \) and \( F(x + 1) = F(x) + 1 \). \( F \) is called a degree one lifting of \( f \). Furthermore, if \( F(0) = f(0) \), then there exists a unique such lifting (cf. [2]).

The rotation number of \( f \) at \( x \) is defined by

\[
\rho(x) = \limsup_{n \to \infty} \frac{F^n(x) - x}{n}.
\]
It is known that the set of all rotation numbers \( \rho(x) \) of \( f \) is an interval and that this interval is reduced to a singleton when \( f(0) = f(1) \) ([17]). The rotation number is tightly relate to the number of returns of \( x \) into the interval \( (c, 1] \), defined by

\[
m_n(x) = \# \{ 0 \leq i < n : f^i(x) \in (c, 1] \}
\]

**Lemma 2.1.** Let \( f \) be a Lorenz map (not necessarily piecewise linear). Let \( F \) be the unique degree one lifting map of \( f \) such that \( F(0) = f(0) \). Then for any \( n \geq 1 \) and any \( x \in [0, 1] \) we have

\[
F^n(x) = m_n(x) + f^n(x).
\]

**Proof.** The proof of this Lemma would be found in the literatures, we give a proof for completeness. We prove it by induction. First note that

\[
f(x) = \begin{cases} 
F(x) & x \in [0, c) \\
F(x) - 1 & x \in (c, 1]
\end{cases}
\]

which implies the desired equality for \( n = 1 \):

\[
F(x) = 1_{[c, 1]}(x) + f(x).
\]

Suppose now that the equality is true for an arbitrary \( n \). Since \( F(x+k) = F(x) + k \) for all positive integers \( k \), by the hypothesis of induction we have

\[
F^{n+1}(x) = F(F^n(x)) = \sum_{i=0}^{n-1} 1_{[c, 1]}(f^i(x)) + F(f^n(x)).
\]
According to what we have proved for \( n = 1 \), we get
\[
F^{n+1}(x) = \sum_{i=0}^{n} 1_{\{x, 1\}}(f^i(x)) + f^{n+1}(x) = m_{n+1}(x) + f^{n+1}(x).
\]
\[\square\]

Write
\[
C_f = \bigcup_{n \geq 0} f^{-n}(c).
\]

When \( f = f_{a,b,c} \) we write \( C_{a,b,c} := C_{f_{a,b,c}} \). The Lyapunov exponent of \( f \) at \( x \notin C_f \) is defined by
\[
\lambda(f, x) = \limsup_{n \to \infty} \frac{1}{n} \log(f^n)'(x).
\]

For piecewise linear Lorenz map \( f_{a,b,c} \),
\[\text{(2.2)}\]
\[
\frac{df_{a,b,c}^n}{dx}(x) = a^{n-m_n(x)} b^{m_n(x)}, \quad \forall x \in C_{a,b,c}.
\]

For any linear Lorenz map such that \( f_{a,b,c}(0) = f_{a,b,c}(1) \), its rotation number and its Lyapunov exponent are determined in the following way.

**Lemma 2.2.** Let \( f_{a,b,c} \) be a piecewise linear Lorenz map such that \( f_{a,b,c}(0) = f_{a,b,c}(1) \). Let \( \rho \) be the rotation number of \( f_{a,b,c} \). Then we have
\[\text{(2.3)}\]
\[
\lim_{n \to \infty} \sup_{x \in [0,1]} \left| \frac{F^n(x) - x}{n} - \rho \right| = 0.
\]
The rotational number \( \rho \) is the solution of the equation
\[
a^{1-\rho} b^\rho = 1.
\]
The number \( \rho \) is rational if and only if \( \log a / \log b \) is rational. Furthermore, we have
\[
\lambda(f_{a,b,c}, x) = 0 \quad (\forall x \notin C_{a,b,c}).
\]

**Proof.** The uniform convergence follows from the observation
\[
F^n(0) < F^n(x) < F^{n+1}(0) \quad (\forall x \in [0, 1])
\]
and the fact that \( \rho = \lim_{n \to \infty} \frac{F^n(0)}{n} \).

According to Lemma 2.1, the rotation number \( \rho \) of \( f_{a,b,c} \) is nothing but the frequency of visits to \((c, 1]\) of any given point of \([0, 1]\). Since \( f_{a,b,c} \) is piecewise linear, for \( x \notin C_{a,b,c} \) we have
\[\text{(2.4)}\]
\[
(f^n_{a,b,c})'(x) = \prod_{i=0}^{n-1} f'_i(a,b,c)(f'_i(a,b,c)(x)) = a^{n-m_n(x)} b^{m_n(x)}
\]
where \( m_n(x) \) is defined by (2.1). It follows that
\[
\lambda(f_{a,b,c}, x) = \lim_{n \to \infty} \frac{1}{n} \log a^{n-m_n(x)} b^{m_n(x)} = \log a^{1-\rho} b^\rho.
\]

Let \( \lambda = \log a^{1-\rho} b^\rho \). We will prove \( a^{1-\rho} b^\rho = 1 \) by showing \( \lambda = 0 \), which implies that
\[
\rho = 1 + \frac{\log b}{\log a - \log b}.
\]
So $\rho$ is rational if and only if $\log a / \log b$ is rational. Suppose $\lambda > 0$. According to Lemma 2.1 [2.3] and [2.4], there exists a positive integer $N$ such that

$$(f_{a,b,c}^N)'(x) > 1 \quad (\forall x \notin C_{a,b,c}).$$

This and the piecewise linearity of $f_{a,b,c}$ imply that $f_{a,b,c}^N$ is piecewise expanding. However, it is not possible because $f_{a,b,c}^N$ is a homeomorphism. Thus $\lambda \leq 0$. In the same way, one proves $\lambda \geq 0$ by considering $f_{a,b,c}^{-1}$. Hence we get $\lambda = 0$. \hfill \Box

Lemma 2.3. Let $f_{a,b,c}$ be a piecewise linear Lorenz map such that $f_{a,b,c}(0) = f_{a,b,c}(1)$. Let $\rho$ be the rotation number of $f_{a,b,c}$. If $\rho$ is irrational, then

$$(2.5) \quad |m_n(x) - n\rho| \leq 4 \quad (\forall x \in [0, 1], \ \forall n \geq 1).$$

Proof. Since $\rho$ is irrational, $f_{a,b,c}$ is topologically conjugate to the rigid irrational rotation $R_\rho$, i.e. $R_\rho(x) = \rho + x$ (cf. [20], p. 38-39). In other words, there exists a continuous strictly increasing function $\pi$ on $[0, 1]$ onto $[0, 1]$ such that

$$(2.6) \quad \pi \circ f_{a,b,c} \circ \pi^{-1} = R_\rho.$$

Let $F$, $G$ and $G^{-1}$ be the degree one lifting map of $f_{a,b,c}$, $\pi$ and $\pi^{-1}$ respectively. The lifted form of (2.6) is

$$G \circ F \circ G^{-1}(x) = x + \rho.$$  

By induction, we have

$$(2.7) \quad G \circ F^n \circ G^{-1}(x) = x + n\rho.$$  

Write

$$(2.8) \quad |m_n(x) - n\rho| \leq |m_n(x) - F^n(x)| + |F^n(x) - G \circ F^n(x)| + \left| G \circ F^n(x) - G \circ F^n \circ G^{-1}(x) \right| + |G \circ F^n \circ G^{-1}(x) - n\rho|.$$

Now we estimate the four terms on the right hand side. Notice first that $F$, $G$ and $G^{-1}$ are increasing functions on $\mathbb{R}$ and that $F(x) - x$, $G(x) - x$ and $G^{-1}(x) - x$ are 1-periodic functions on $\mathbb{R}$ taking with values in $[0, 1]$. So when $|x - y| \leq 1$ we have

$$(2.9) \quad |F(x) - F(y)| \leq 1, \quad |G(x) - G(y)| \leq 1, \quad |G^{-1}(x) - G^{-1}(y)| \leq 1.$$  

According to Lemma 2.1 we have the following estimate for the first term:

$$|m_n(x) - F^n(x)| \leq 1.$$  

The fact $0 \leq G(x) - x \leq 1$ implies immediately an estimate for the second term:

$$|F^n(x) - G \circ F^n(x)| \leq 1.$$  

Repeating the first inequality in (2.9) we get $|F^n(x) - F^n(y)| \leq 1$ when $|x - y| \leq 1$. This and the fact $|G^{-1}(x) - x| \leq 1$ imply an estimate for the third term:

$$|G \circ F^n(x) - G \circ F^n \circ G^{-1}(x)| \leq 1.$$  

A direct consequence of (2.7) is the following estimate for the fourth term:

$$|G \circ F^n \circ G^{-1}(x) - n\rho| \leq 1.$$  

The estimation (2.5) is thus proved. \hfill \Box
2.2. Comparison of rotation numbers in different maps. Let \( f_{a,b,c} \) be a piecewise linear Lorenz map. If \( a > 1 \) and \( b > 1 \), it is well known that \( f_{a,b,c} \) is expanding and then admits a unique acim. If \( a < 1 \) and \( b < 1 \), \( f_{a,b,c} \) is contracting and then admits no acim. So we may assume that \( a > 1 \geq b \) or \( b > 1 \geq a \). In these cases, we will compare \( f_{a,b,c} \) with homeomorphic piecewise linear Lorenz maps \( f_{a_0,b,c} \) and \( f_{a,b_0,c} \), where \( a_0 = \frac{1-b(1-c)}{c} \) and \( b_0 = \frac{1-ac}{1-c} \). See Figure 2 for the pictures of \( f_{a_0,b,c} \) and \( f_{a,b_0,c} \). More precisely, in the case \( a > 1 \geq b \), we compare \( f_{a,b,c} \) with

\[
(2.10) \quad f_{a_0,b,c}(x) = \begin{cases} 
  a_0x + 1 - ac & x \in [0, c) \\
  b(x - c) & x \in (c, 1] 
\end{cases}
\]

if \( f_{a,b,c}(0) < f_{a,b,c}(1) \), and compare \( f_{a,b,c} \) with

\[
(2.11) \quad f_{a,b_0,c}(x) = \begin{cases} 
  ax + 1 - ac & x \in [0, c) \\
  b_0(x - c) & x \in (c, 1] 
\end{cases}
\]

if \( f_{a,b,c}(0) > f_{a,b,c}(1) \).

In the case \( b > 1 \geq a \), we compare \( f_{a,b,c} \) with \( f_{a,b_0,c} \) when \( f_{a,b,c}(0) > f_{a,b,c}(1) \), and compare \( f_{a,b,c} \) with \( f_{a_0,b,c} \) when \( f_{a,b,c}(0) < f_{a,b,c}(1) \).

**Lemma 2.4.** Let \( f_{a,b,c} \) and \( f_{a,b_0,c} \) be defined as above. We have the following conclusions:

1. If \( f_{a,b,c}(0) < f_{a,b,c}(1) \), then for all \( x \in [0, 1] \),
   \[
   \begin{cases} 
   \rho(f_{a,b,c}, x) \leq \rho(f_{a_0,b,c}) & \text{if } a > 1 \geq b \\
   \rho(f_{a,b,c}, x) \geq \rho(f_{a,b_0,c}) & \text{if } a \leq 1 < b. 
   \end{cases}
   \]

2. If \( f_{a,b,c}(0) > f_{a,b,c}(1) \), then for all \( x \in [0, 1] \),
   \[
   \begin{cases} 
   \rho(f_{a,b,c}, x) \geq \rho(f_{a_0,b,c}) & \text{if } a \geq 1 > b \\
   \rho(f_{a,b,c}, x) \leq \rho(f_{a,b_0,c}) & \text{if } a < 1 \leq b. 
   \end{cases}
   \]

**Proof.** Let \( F \) be the degree one lifting of \( f_{a,b,c} \) and \( F_0 \) be the degree one lifting of \( f_{a_0,b,c} \). For the case \( f_{a,b,c}(0) < f_{a,b,c}(1) \) and \( a > 1 \geq b \), since \( f_{a,b,c}(x) \leq f_{a_0,b,c}(x) \) for all \( x \in [0, 1] \), we have \( F(y) \leq F_0(y) \) for all \( y \in \mathbb{R} \). It follows that \( F^n(y) \leq F_0^n(y) \) for \( y \in \mathbb{R} \) and for every positive integer \( n \). Therefore \( \rho(f_{a,b,c}, x) \leq \rho(f_{a,b_0,c}) \). The other inequalities can be similarly proved. \( \square \)
2.3. Frobenius-Perron operator and invariant density. The Frobenius-Perron operator associated with $f_{a,b,c}$ is defined as follows: for any $h \in L^1(I)$,

$$P_{a,b,c}h(x) = \frac{1}{a}(1-ac)h\left(x - \frac{(1-ac)l}{a}\right) + \frac{1}{b}(b(1-c))h\left(x + \frac{bc}{b}\right).$$

The invariant density $h_\ast$ of the Frobenius-Perron operator corresponds to an acim \(\mu\) of $f_{a,b,c}$: $\int_A h_\ast dm$, where $A \in \mathcal{B}$ is a Borel set and $m$ is the Lebesgue measure on $I$.

Lemma 2.5. Let $P$ be the Frobenius-Perron operator associated with $f_{a,b,c}$. Then we have the following statements:

1. If there exists a positive integer $n \geq 1$ such that $f_{a,b,c}^n(x) = x$ for all $x \in I$, then for each density $g$ on $[0, 1]$, $A_n(g)$ is an invariant density of $P_{a,b,c}$.
2. If there exists a constant $r > 0$ such that for all positive integer $n$,

$$rm(A) \leq m(f_{a,b,c}^{-n}(A)) \leq m(A)/r, \quad \forall \ A \in \mathcal{B},$$

then $f_{a,b,c}$ admits a unique acim whose density is bounded by $r$ and $1/r$.
3. If there exists a positive integer $n$ such that $f_{a,b,c}^n$ is strongly expanding, i.e.,

$$(f_{a,b,c}^n)'(x) > \lambda > 1 \text{ for } x \in I \text{ except finite points},$$

then $f_{a,b,c}$ admits an acim whose density is of bounded variation.
4. If there exists a positive integer $n$ such that $f_{a,b,c}^n(x) < \lambda < 1 \text{ for all } x \in I \text{ except finite points},$ then $f_{a,b,c}$ admits no acim.

Proof. The assertions (1) and (4) are obvious. The assertion (3) is a direct consequence of Lasota and Yorke’s Theorem ([18] [19]). Now we prove (2). The assumption in this case means that $r \leq P_{a,b,c}^n \leq 1/r$. So \(\{A_n(1)\}_{n \geq 0}\) is weakly precompact in $L^1(I)$. From the weakly compactness of \(\{A_n(1)\}_{n \geq 0}\) we can extract a subsequence $A_{n_k}(1)$ that converges weakly to $g$ and $P_{a,b,c}g = g$. By the abstract ergodic Theorem of Kakutani and Yosida ([18]), $A_n(1)$ converges strongly to $g$. This implies that $g$ is an invariant density of $P_{a,b,c}$ and $r \leq g(x) \leq 1/r$. \(\square\)

In the third case, $f_{a,b,c}$ is said to be eventually piecewise expanding [14].

2.4. Renormalization of Lorenz map. Let $f_{a,b,c}$ be a piecewise linear Lorenz map satisfying $ac + b(1-c) > 1$. The equivalence between the acim of $f_{a,b,c}$ and the Lebesgue measure is a question of transitivity of $f_{a,b,c}$ (see Lemma [11]). One can describe the transitivity of $f_{a,b,c}$ by using the device of renormalization.

A Lorenz map $f : I \to I$ is said to be renormalizable if there is a proper subinterval $[u, v]$ which contains the critical point $c$, and integers $\ell, r > 1$ such that the map $g : [u, v] \to [u, v]$ defined by

$$(2.12)\quad g(x) = \begin{cases} f^\ell(x) & x \in [u, c), \\ f^r(x) & x \in (c, v]. \end{cases}$$

is itself a Lorenz map on $[u, v]$.

A Lorenz map $f$ is said to be expanding if the preimages of the critical point is dense in $I$. The renormalization theory of expanding Lorenz map is well understood (see [7] [12]). The transitivity of an expanding Lorenz map can be characterized by its renormalization. For example, $f$ is transitive if it is not renormalizable (7).

Let $f$ be an expanding Lorenz map. The renormalizability of $f$ is closely related to the periodic orbit with minimal period. Denote $\kappa$ the smallest period of the periodic points of $f$. If $\kappa = 1$ (i.e., $f$ admits a fixed point), we must have $f(0) = 0$ or $f(1) = 1$ because $f$ is expanding. It follows that $f$ is transitive (7). If $\kappa = \infty$, 

i.e. \( f \) admits no periodic point, then \( f \) is topologically conjugates to an irrational rotation on the circle because \( f \) is expanding ([12]). For the case \( 1 < \kappa < \infty \), we have the following Lemma.

**Lemma 2.6.** ([12]) Let \( f : [0, 1] \to [0, 1] \) be an expanding Lorenz map with \( 1 < \kappa < \infty \).

1. The minimal period of \( f \) is equal to \( \kappa = m + 2 \), where 
   \[ m = \min\{i \geq 0 : f^{-i}(c) \in \{f(0), f(1)\}\}. \]
2. \( f \) admits a unique \( \kappa \)-periodic orbit \( O \).
3. Let \( P_L \) and \( P_R \) be adjacent points in \( O \) such that \( c \in [P_L, P_R] \). Then \( f^k \) is continuous on \( [P_L, c) \) and on \( (c, P_R] \). Moreover, we have
   \[ \bigcup_{i=0}^{\kappa-1} f^i([P_L, P_R]) = I. \]

For general expanding Lorenz map \( f \), it is difficult to check whether \( f \) is renormalizable or not. However, for piecewise linear Lorenz map \( f_{a,b,c} \) satisfying \( ac + b(1-c) > 1 \), one can check the renormalizability easily. According to the proof of Theorem A, \( f_{a,b,c} \) is expanding. Denote \( O \) as the \( \kappa \)-periodic orbit, and
\[ D := \bigcup_{n \geq 0} f_{a,b,c}^{-n}(O). \]

**Lemma 2.7.** ([4]) If \( D \neq O \), then \( f_{a,b,c} \) is not renormalizable.

### 3. Existence of Absolutely Continuous Invariant Measure

Now we prove Theorem A by distinguishing four cases: \( f_{a,b,c}(0) = f_{a,b,c}(1) \) and \( \log a/\log b \) is rational, \( f_{a,b,c}(0) = f_{a,b,c}(1) \) and \( \log a/\log b \) is irrational, \( f_{a,b,c}(0) < f_{a,b,c}(1) \) and \( f_{a,b,c}(0) > f_{a,b,c}(1) \). In the first case, we will show that some power of \( f_{a,b,c} \) is identity, i.e., there exists \( n > 0 \) such that \( f_{a,b,c}^{n}(x) = x \) for all \( x \in I \). In the second case we will prove
\[ rm(A) \leq m(f_{a,b,c}^{-n}(A)) \leq m(A)/r, \quad \forall A \in \mathcal{B}, \]
for some constant \( r \) and \( n \geq 0 \). In the third case we will show that some power of \( f_{a,b,c} \) is expanding. In the forth case, we will compare \( f_{a,b,c} \) with a suitable homeomorphic piecewise linear Lorenz map and prove that some power \( f_{a,b,c}^{n} \) is contracting.

#### 3.1. Proof of Theorem A when \( f_{a,b,c}(0) = f_{a,b,c}(1) \) and \( \log a/\log b \) is rational.

According to Lemma 2.6, it suffice to prove the following proposition.

**Proposition 3.1.** Suppose \( f_{a,b,c}(0) = f_{a,b,c}(1) \) and \( \log a/\log b \) is rational. Then there exists positive integer \( n \) such that \( f_{a,b,c}^{n}(x) = x \) for all \( x \in I \).

**Proof.** In this case, \( f_{a,b,c} \) can be regarded as a homeomorphism on the unit circle. Since \( \log a/\log b \) is rational, the rotation number of \( f_{a,b,c} \) is also rational (Lemma 2.2). Write \( \rho(f_{a,b,c}) = \frac{m}{n} \) with \( (m, n) = 1 \). We shall prove that \( f_{a,b,c}^{n}(x) = x \) for all \( x \in [0, 1] \).
Since \( \rho(f_{a,b,c}) = \frac{m}{n} \), \( f_{a,b,c} \) admits an \( n \)-periodic orbit (2). Let \( p_1 < p_2 < \cdots < p_n \) be an \( n \)-periodic orbit. The orbit forms a partition of \( I \):
\[
[p_1, p_2), \ldots, [p_{n-1}, p_n), \ [p_n, 1] \cup [0, p_1),
\]
and \( f_{a,b,c} \) maps one subinterval onto the next one in the partition. Each subinterval in the partition contains only one point in \( C_{a,b,c} \). Since \( \rho(f_{a,b,c}) = \frac{m}{n} \), \( c \in [p_{n-m}, p_{n-m+1}) \).

If \( c \) doesn’t belong to the periodic orbit, \( c \in (p_{n-m}, p_{n-m+1}) \). Consider the interval \([p_{n-m}, c)\), it follows that \( f_{a,b,c}^{p_{n-m}} \) is continuous and linear on \([p_{n-m}, c)\) because \( f_{a,b,c}^{p_{n-m}} \) has only one discontinuity \( c \) in \([p_{n-m}, p_{n-m+1})\). Notice that \( f_{a,b,c}^{p_{n-m}}(p_{n-m}) = p_{n-m} \) and \( (f_{a,b,c}^{p_{n-m}})'(p_{n-m}) = 1 \), we obtain that \( f_{a,b,c}^{n}(x) = x \) on \([p_{n-m}, c)\), which implies that \( c \) is an \( n \)-periodic point. So 1 is also an \( n \)-periodic point.

We denote the \( n \)-periodic orbit of 1 as \( 0 < q_1 < q_2 < \cdots < q_{n-m-1} < q_{n-m} = c < q_{n-m+1} < \cdots < q_n = 1 \). Since \( f_{a,b,c}^{p_{n}} \) is linear on \([q_i, q_{i+1})\), it follows that \( f_{a,b,c}^{p_{n}}(x) = x \) on \([q_i, q_{i+1})\), \( i = 1, 2, \ldots, n \). So \( f_{a,b,c}^{n}(x) = x \) on \( I \). □

3.2. Proof of Theorem A when \( f_{a,b,c}(0) = f_{a,b,c}(1) \) and \( \log a / \log b \) is irrational.

In this case, according to Lemma 2.4, we have only to prove the following proposition.

**Proposition 3.2.** Suppose \( f_{a,b,c}(0) = f_{a,b,c}(1) \) and \( \log a / \log b \) is irrational, there exists a constant \( r > 0 \) such that
\[
rm(A) \leq m(f_{a,b,c}^{-n}(A)) \leq m(A)/r, \quad \forall A \in \mathcal{B}.
\]

**Proof.** The condition \( f_{a,b,c}(0) = f_{a,b,c}(1) \) means \( c = \frac{1-b}{a-b} \). Consider \( f_{a_1,b_1,c_1} := f_{a,b,c}^{-1} \), the inverse map of \( f_{a,b,c} \). It is also a piecewise linear Lorenz map such that \( f_{a_1,b_1,c_1}(0) = f_{a_1,b_1,c_1}(1) \). In fact, we have
\[
(3.1) \quad a_1 = \frac{1}{b}, \quad b_1 = \frac{1}{a}, \quad c_1 = \frac{(a-1)b}{a-b}.
\]

Let \( \rho := \rho(f_{a_1,b_1,c_1}) \) be its rotation number. Write
\[
m_n^*(x) = \# \{ 0 \leq i < n : f_{a_1,b_1,c_1}^i(x) \in (c_1, 1) \}.
\]

By Lemma 2.2 we have \( a_1^{1-r}b_1^r = 1 \). Thus for all \( x \not\in C_{a_1,b_1,c_1} \), we have
\[
(f_{a_1,b_1,c_1})(x) = a_1^{n-m_n^*(x)}b_1^{n^*_{n}(x)}
\]
\[
= a_1^{n(1-r)}b_1^{n^*_{n}(x)-n}\left(\frac{b_1}{a_1}\right)^{m_n^*(x)-n}. \quad (**)
\]

According to Lemma 2.3 \( |m_n^*(x)-n| \leq 4 \). It follows that for all \( x \not\in C_{a_1,b_1,c_1} \) and all \( n \geq 0 \) we have
\[
r \leq (f_{a_1,b_1,c_1})(x) \leq 1/r,
\]
where \( r = \min\{b_1a_1^{-4}, b_1^{-4}a_1^4\} = \min\{b^4a^{-4}, b^{-4}a^4\} \). Consequently, by making a change of variables we get
\[
rm(A) \leq m(f_{a,b,c}^{-n}(A)) = \int_A (f_{a,b,c}^n(x))dx
\]
\[
= \int_A (y) \cdot (f_{a_1,b_1,c_1})(y)dy \leq m(A)/r.
\]

□
3.3. Proof of Theorem A when $f_{a,b,c}(0) < f_{a,b,c}(1)$.

In this case, we are going to show

**Proposition 3.3.** Suppose $f_{a,b,c}(0) < f_{a,b,c}(1)$. There exists positive integer $n$ such that

$$(f_{a,b,c}^n)'(x) > \lambda > 1$$

for all $x \in I$ except finite points.

**Proof.** The condition $f_{a,b,c}(0) < f_{a,b,c}(1)$ means $ac + (1-c)b > 1$. So we must have $a > 1$ or $b > 1$. If we have both $a > 1$ and $b > 1$, we take $n = 1$. It remains to consider two cases: $a > 1 \geq b$ and $b > 1 \geq a$.

First we assume $a > 1 \geq b$. Let $f_{a,b,c}$ with $a_0 := \frac{1-b(1-a)}{a} < a$, which is the homeomorphism defined by (2.10) (see Figure 2). We denote by $\rho_0$ the rotation number of $f_{a_0,b,c}$. Notice that $0 < \rho_0 < 1$.

For $x \in [0, 1]$, denote

$$m_n(x) = \sum_{i=0}^{n-1} 1_{(c, 1]}(f_{a,b,c}^i(x)), \quad \tilde{m}_n(x) = \sum_{i=0}^{n-1} 1_{(c, 1]}(f_{a_0,b,c}^i(x)).$$

According to Lemma 2.1, $\tilde{m}_n(x)/n$ converges uniformly to $\rho_0$ as $n \to \infty$. So $a_0^{-m_n(x)/n} b \tilde{m}_n(x)/n$ converges uniformly to 1 as $n \to \infty$ because $a_0^{-m_0} b_0^0 = 1$ (Lemma 2.2). Choose a sufficiently small $\varepsilon_0 > 0$ such that

$$(1 - \varepsilon_0) \left( \frac{a}{a_0} \right)^{(1-\rho_0)/2} > 1.$$ 

There exists a positive integer $N_0 \geq 1$ such that for all $x \in [0, 1] \setminus C_{a_0,b,c}$ and $\forall n \geq N_0$, we have

$$((f_{a_0,b,c}^n)'(x))^{\frac{1}{n}} = a_0^{1-\tilde{m}_n(x)/n} b \tilde{m}_n(x)/n > 1 - \varepsilon_0.$$ 

Since $f_{a_0,b,c} \geq f_{a,b,c}(x)$, by Lemma 2.1 and Lemma 2.2 it is easy to see that $m_n(x) \leq \tilde{m}_n(x)$ for $x \notin C_*(n) := \bigcup_{i=0}^{n-1} (f_{a_0,b,c}^i(c) \cup f_{a_0,b,c}^{i+1}(c))$. So, for large $n$ and $x \notin C_*(n)$, we have

$$((f_{a,b,c}^n)'(x))^{\frac{1}{n}} = a \left( \frac{b}{a} \right)^{m_n(x)/n} \geq a \left( \frac{b}{a} \right)^{\tilde{m}_n(x)/n} = \left( (f_{a_0,b,c}^n)'(x) \right)^{\frac{1}{n}} \left( \frac{a}{a_0} \right)^{(1-\rho_n)/(2n)} > (1 - \varepsilon_0) \left( \frac{a}{a_0} \right)^{(1-\rho_0)/2} > 1.$$ 

This implies that for $n$ large we have $(f_{a,b,c}^n)'(x) > 1$ for all $x \in I \setminus C_*(n)$. Obviously, $C_*(n)$ is consists of finite points. So $f_{a,b,c}^n$ is linear with slope greater than 1 on each component of $I \setminus C_*(n)$. It follows that there exists positive integer $n$ and $\lambda > 1$ such that $(f_{a,b,c}^n)'(x) > \lambda > 1$ for all $x \in I \setminus C_*(n)$.

The proof for the case $b > 1 \geq a$ is similar. We consider $f_{a,b_0,c}$ with $b_0 = \frac{1-ac}{1-c} < b$, which is the homeomorphism defined by (2.11). \[ \square \]
3.4. Proof of Theorem A when $f_{a,b,c}(0) > f_{a,b,c}(1)$.

We finish the proof of Theorem A by showing the following proposition.

**Proposition 3.4.** Suppose $f_{a,b,c}(0) > f_{a,b,c}(1)$. There exists positive integer $n$ such that

$$
(f^n_{a,b,c})'(x) < \lambda < 1
$$

for all $x \in I$ except finite points.

**Proof.** The condition $f_{a,b,c}(0) > f_{a,b,c}(1)$ means $ac + (1-c)b < 1$. So we must have $a < 1$ or $b < 1$. If we have both $a < 1$ and $b < 1$, we take $n = 1$. It remains to consider two cases: $a < 1 \leq b$ and $a \geq 1 > b$.

First we assume $a < 1 \leq b$. Consider $f_{a,b_0,c}$ with $b_0 = \frac{1-ac}{c} > b$, which is the homeomorphism defined by (2.10) (see Figure 2). We denote by $\rho_1$ the rotation number of $f_{a,b_0,c}$. Obviously, $0 < \rho_1 < 1$.

For $x \in [0, 1]$, put

$$m_n(x) = \sum_{i=0}^{n-1} 1_{[c, 1]}(f^i_{a,b_0,c}(x)), \quad \tilde{m}_n(x) = \sum_{i=0}^{n-1} 1_{(c, 1]}(f^i_{a,b_0,c}(x)).$$

According to Lemma 2.1, $\tilde{m}_n(x)/n$ converges uniformly to $\rho_0$ as $n \to \infty$. So $a^{1-n\rho_0}(a^{\tilde{m}_n(x)/n})$ converges uniformly to 1 as $n \to \infty$ because $a^{1-n\rho_0} = 1$ (Lemma 2.3). Choose a sufficiently small $\varepsilon_1 > 0$ such that

$$(1 + \varepsilon_1) \left(\frac{b}{b_0}\right)^{\rho_1/2} < 1.$$

There exists a positive integer $N_0 \geq 1$ such that for all $x \in [0,1] \setminus C_{a,b_0,c}$ and $\forall n \geq N_0$, we have

$$
(f^n_{a,b_0,c})'(x) < a^{1-\tilde{m}_n(x)/b_0^\rho_0} < 1 + \varepsilon_1.
$$

Since $f_{a,b,c} \leq f_{a,b_0,c}(x)$, by Lemma 2.1 and Lemma 2.3 it is easy to see that $m_n(x) \leq \tilde{m}_n(x)$ for $x \notin C^*(n) := \bigcup_{i=0}^{n-1}(f_{a,b_0,c}^{-1}(c) \cup f_{a,b_0,c}^{-1}(c))$. So, for large $n$ and $x \notin C^*(n)$, we have

$$
(f^n_{a,b,c})'(x) < a \left(\frac{b}{a}\right)^{\tilde{m}_n(x)/n} < a \left(\frac{b}{a}\right)^{m_n(x)/n} \leq a \left(\frac{b}{a}\right)^{\tilde{m}_n(x)/n} < (1 + \varepsilon_1) \left(\frac{b}{b_0}\right)^{\rho_1/2} < 1.
$$

(3.3)

This implies that for $n$ large we have $(f^n_{a,b,c})'(x) < 1$ for all $x \in I \setminus C^*(n)$. Since $f^n_{a,b,c}$ is a piecewise contract linear map with at most finite pieces, there exists $\lambda < 1$ such that $(f^n_{a,b,c})'(x) < \lambda < 1$ for all $x \in I$ except at most finite points.

The proof for the case $a \geq 1 > b$ is similar. We consider $f_{a_0,b,c}$ with $a_0 = \frac{1-(1-c)b}{c} > a$, which is the homeomorphism defined by (2.10).
3.5. Diffeomorphic conjugacy. A partial result of Theorem A may be obtained in a different way. If

\[ c\sqrt{a} + (1 - c)\sqrt{b} > 1, \]

which is stronger than \( f_{a,b,c}(0) \geq f_{a,b,c}(1) \), it is possible to find some diffeomorphism \( h \) such that \( h \circ f_{a,b,c} \circ h^{-1} \) is piecewise expanding. We do find such a diffeomorphism among the one-parameter group of transformations \( h_s : [0, 1] \rightarrow [0, 1] \ (s \in \mathbb{R}_+) \) defined by

\[ h_s(x) = \frac{sx}{1 + (s - 1)x}. \]

We can also prove that the above condition (3. 4) is actually necessary for the existence of such a diffeomorphism \( h_s \). This was one starting point of our study on acim of piecewise linear Lorenz map.

4. Equivalence

Let \( f := f_{a,b,c} \) be a piecewise linear Lorenz map with \( ac + b(1 - c) > 1 \). The acim \( \mu \) is not necessarily equivalent to the Lebesgue measure \( m \), even if \( f \) is strongly expanding. Parry [24] proved that the acim of \( f_{a,a,1/2} \) is not equivalent to the Lebesgue measure if and only if \( 1 < a < \sqrt{2} \). We first show that the equivalence between its acim and the Lebesgue measure is nothing but the transitivity of \( f \).

4.1. Equivalence and transitivity.

**Lemma 4.1.** Let \( f := f_{a,b,c} \) be a piecewise linear Lorenz map with \( ac + b(1 - c) > 1 \). Then the acim of \( f \) is equivalent to the Lebesgue measure if and only if \( f \) is transitive.

**Proof.** Let \( h \) be the density of the acim \( \mu \), i.e.

\[ \mu(A) = \int_A h(x)dx, \quad \forall A \in \mathcal{B}, \]

and let \( \text{supp} (\mu) \) be the support of \( \mu \). Obviously, \( \text{supp}(\mu) \) is an invariant closed set of \( f \).

The measure \( \mu \) is equivalent to the Lebesgue measure \( m \) if and only if \( \text{supp}(\mu) = I \). In fact, \( \text{supp}(\mu) = I \) means \( h(x) > 0 \) for \( m \)-a.e. \( x \in I \). By (4. 1), \( \mu(A) = 0 \) implies \( m(A) = 0 \).

Now we show that \( \text{supp}(\mu) = I \) if and only if \( f \) is transitive. At first, it is easy to see the non transitivity of \( f \) implies \( I \setminus \text{supp} (\mu) \) is nonempty. On the other hand, notice that \( \text{supp}(\mu) \) contains some interval \( J \) because \( h \) is of bounded variation. So, the transitivity of \( f \) implies

\[ I = \bigcup_{n=0}^{\infty} f^n(J) \subseteq \text{supp} (\mu). \]

\( \square \)

Now we are going to discuss the transitivity of piecewise linear Lorenz maps by using the renormalization theory of expanding Lorenz map (cf. [7, 12]).
4.2. Proof of Theorem B.

Let \( f := f_{a, b, c} \) be a piecewise linear Lorenz map with \( ac + b(1-c) > 1 \), \( \kappa \) be the minimal period.

If \( \kappa = 1 \), then \( f \) is not renormalizable (17), which implies that \( f \) is transitive.

In what follows we assume \( \kappa > 1 \). Let \( O \) be the unique \( \kappa \)-periodic orbit, and \( P_L \) and \( P_R \) be adjacent \( \kappa \)-periodic points so that \([P_L, P_R]\) contains the critical point \( c \). By Lemma 2.6 \( f^\kappa \) is continuous and linear on \([P_L, c]\) and on \((c, P_R]\). Put

\[
A := f^\kappa(c+), \quad B := f^\kappa(c-).
\]

We discuss the transitivity of \( f \) by distinguish the following three cases:

1. \([A, B] \not= [P_L, P_R]\);
2. \([A, B] = [P_L, P_R];
3. \([A, B] \subseteq [P_L, P_R].

Case (1). In this case, we have \( D := \bigcup_{n \geq 0} f^{-n}(O) \not= O \). It follows from Lemma 2.7 that \( f \) is not renormalizable. So \( f \) is transitive.

If \([A, B] \subseteq [P_L, P_R]\), \( f \) admits a renormalization

\[
Rf(x) = \begin{cases} 
  f^\kappa(x) & x \in [A, c) \\
  f^\kappa(x) & x \in (c, B].
\end{cases}
\]

Case (2). In this case, \( Rf \) is not renormalizable because it admits fixed point. So \( Rf \) is a transitive Lorenz map on \([P_L, P_R]\). By equation (2.13) in Lemma 2.6 \( f \) is also transitive.

Case (3). In this case, we have \( P_L < A \) or \( B < P_R \). Assume that \( P_L < A \). Since \([A, B] \subseteq [P_L, P_R]\), it follows \( (P_L, A) \cap (\bigcup_{n \geq 0} f^n([A, B])) = \emptyset \), which indicates \( f \) is not transitive. Similarly, \( B < P_R \) implies \( f \) is not transitive.

Now we consider the special case \( \kappa = 2 \). According to Lemma 2.6 \( \kappa = 2 \) implies \( f \) admits no fixed point, \( A = f(0) = 1 - ac < b(1-c) = f(1) = B \) and \( c \in [A, B]\).

A simple computation shows

\[
f^2(x) = \begin{cases} 
  a^2x + (1-ac)(a+1) & x \in [0, \frac{ac+c-1}{a}], \\
  abx - abc + b - bc & x \in \left(\frac{ac+c-1}{a}, c\right), \\
  abx - abc - ac + 1 & x \in (c, \frac{ac+c-1}{b}], \\
  b^2x - b^2c - bc & x \in \left(\frac{ac+c-1}{b}, 1]\right.
\end{cases}
\]

It follows that the map \( f \) admits two 2-periodic points:

\[
P_L = \frac{abc + bc - b}{ab - 1}, \quad P_R = \frac{abc + ac - 1}{ab - 1}.
\]

Thus, \([A, B] \subseteq [P_L, P_R]\) is equivalent to

\[
A \geq \frac{abc + bc - b}{ab - 1} \quad \text{and} \quad B \leq \frac{abc + ac - 1}{ab - 1}
\]
or equivalently

\[
ab \leq 1 + \frac{B - c}{c - A} \quad \text{and} \quad ab \leq 1 + \frac{c - A}{B - c}.
\]

Recall \( M := \min \left\{ \frac{B-c}{c-A}, \frac{c-A}{B-c} \right\} \). We have \([A, B] \subseteq [P_L, P_R]\) if and only if

\[
\begin{cases} 
  1 < ab \leq 1 + M & \text{if } M < 1 \\
  1 < ab \leq 2 & \text{if } M = 1.
\end{cases}
\]
In other words, $f$ is transitive if and only if
\[
\begin{cases}
ab > 1 + M & \text{if } M < 1 \\
ab \geq 2 & \text{if } M = 1.
\end{cases}
\]

5. The densities of the acims

We finish the paper by pointing out how to obtain the density of the acim in some special cases.

5.1. $\beta$-transformation.

The first case is the special Lorenz maps $f_{a,a,c}$ ($a > 1$). It was known that they admit their acims (see also Theorem A). Gelfond [10] and Parry [22, 23] had determined the density of the acim of $f_{a,a,c}$, which is up to a multiplicative constant equal to
\[
(5.1) \quad g(x) = \sum_{\alpha_n, a,c(0) < x} \frac{1}{\alpha_n} - \sum_{\alpha_n, a,c(1) > x} \frac{1}{\alpha_n}.
\]

Suppose that the acim of $f_{a,b,c}$ exists but is not equivalent to the Lebesgue measure. From the proof of Theorem B (see Section 3) we have seen that the restriction of $f_{a,b,c}$ on $[f_{a,b,c}(c+), f_{a,b,c}(c-)]$, where $\kappa$ is the minimal period of periodic points of $f_{a,b,c}$, is a piecewise Linear Lorenz map of the form $f_{a,b,c}$, on the renormalization interval $[f_{a,b,c}(c+), f_{a,b,c}(c-)]$. Thus, we can obtain the density $g_*(x)$ of the acim of $f_{a,b,c}$ by using (5.1). Then we get the density of the acim of $f_{a,b,c}$:
\[
(5.2) \quad g_{a,b,c}(x) = \frac{1}{\kappa} \left[ g_*(x) + P_{a,b,c} g_*(x) + \cdots + P_{a,b,c}^{\kappa-1} g_*(x) \right],
\]
where $P_{a,b,c}$ is the Frobenius-Perron operator associated to $f_{a,b,c}$. Actually we can easily check that
\[
P_{a,b,c}^\kappa g_*(x) = g_*(x).
\]

5.2. Piecewise linear Markov map.

The second case is $f_{\beta,\alpha,1/\beta}$ ($\beta > 1$, $0 < \alpha \leq \frac{1}{\beta-1}$), which is the piecewise linear Lorenz map $S_{\beta,\alpha}$ studied by Dajani et al in [5]. Remember that $S_{\beta,\alpha}$ is defined by equation (1.2), and we only assume $\beta > 1$ rather than $1 < \beta < 2$ in [5]. If $\alpha = \frac{1}{\beta-1}(\beta-1)^k$ for some integer $k \geq 1$, then $f_{\beta,\alpha,1/\beta}$ is a piecewise linear Markov map.

Proposition 5.1. Assume $\beta > 1$ and $\alpha = \frac{1}{\beta-1}(\beta-1)^k$ for some integer $k \geq 1$. Then the density of the acim of $f_{\beta,\alpha,1/\beta}$ is up to a multiplicative constant equal to
\[
g_{\beta,k}(x) = \frac{1}{\beta - 1} 1_{[0,1]}(x) + \sum_{i=1}^{k} \beta^{-i-1} 1_{[\beta^{-i}, \beta^{-i+1}]}(x).
\]

Proof. Let $C$ be the partition of $[0,1]$ given by $0 < \beta^{-k} < \beta^{-(k-1)} < \ldots < \beta^{-2} < \beta^{-1} < 1$. One can easily check that $f_{\beta,\alpha,1/\beta}$ is a piecewise linear Markov map with respect to the partition $C$. Let $P$ be the Perron-Frobenius operator of $f_{\beta,\alpha,1/\beta}$ (see Section 2.3).
Note that
\[ P_1(x) = \frac{1}{\beta} \mathbb{1}_{[0,1]}(x) + (\beta - 1)\beta^{k-1} \mathbb{1}_{[0,\beta^{-k}]}(x) \]
\[ P_1_{[0, \beta^{-k}]}(x) = \frac{1}{\beta} \mathbb{1}_{[0, \beta^{-(k-1)}]}(x) \]
\[ \vdots \]
\[ P_1_{[0, \beta^{-2}]}(x) = \frac{1}{\beta} \mathbb{1}_{[0, \beta^{-1}]}(x) \]
\[ P_1_{[0, \beta^{-1}]}(x) = \frac{1}{\beta} \mathbb{1}_{[0, 1]}(x). \]

We obtain
\[
P g_{\beta,k}(x) = \frac{1}{\beta - 1} P_1_{[0, 1]}(x) + \sum_{i=1}^{k} \beta^{i-1} P_1_{[0, \beta^{-i}]}(x) \]
\[
= \frac{1}{\beta(\beta - 1)} 1_{[0, 1]}(x) + \beta^{k-1} 1_{[0, \beta^{-k}]}(x) + \sum_{i=2}^{k} \beta^{i-2} 1_{[0, \beta^{-(i-1)}]}(x) + \frac{1}{\beta} 1_{[0, 1]}(x) \]
\[
= \frac{1}{\beta - 1} 1_{[0, 1]}(x) + \sum_{i=1}^{k} \beta^{i-1} 1_{[0, \beta^{-i}]}(x) \]
\[
= g_{\beta,k}(x). \]

The special map of the form \( f_{1,n,1-1/n} \) (\( n \geq 2 \) being an integer) is also a piecewise linear Markov map. It was proved in [9] that its density is equal to
\[
g_n(x) = \frac{2}{n+1} \sum_{i=0}^{n-1} 1_{[\frac{i}{n}, 1]}(x). \]

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