Mutual and coherent informations for infinite-dimensional quantum channels*

A.S. Holevo, M.E. Shirokov
Steklov Mathematical Institute, RAS, Moscow
holevo@mi.ras.ru, msh@mi.ras.ru

Abstract

The work is devoted to study of quantum mutual information and coherent information – the two important characteristics of quantum communication channel. Appropriate definitions of these quantities in the infinite-dimensional case are given and their properties are studied in detail. The basic identities relating quantum mutual information and coherent information of a pair of complementary channels are proved. An unexpected continuity property of quantum mutual information and coherent information, following from the above identities, is observed. The upper bound for the coherent information is obtained.

Contents

1 Introduction 2
2 Finite-dimensional case 3
3 Mutual information 5
4 The relation between mutual informations of complementary channels 13

*The work is partially supported by RFBR grant 09-01-00424 and the scientific program “Mathematical control theory” of RAS.
One of achievements in quantum information theory is discovery of a whole number of important entropic and informational characteristics of quantum systems (see e.g. [8, 13]). Some of them such as the $\chi$-capacity and quantum mutual information have direct classical analogs, others – such as coherent information and various entanglement measures – do not have such analogs or they are trivial.

Until recent time the main attention in quantum information theory was paid to finite-dimensional systems, but recently considerable interest to infinite-dimensional systems appeared: a broad class important for applications in quantum optics constitute Bosonic Gaussian systems [6, 8]. Note that the properties of the entropy and the relative entropy were studied in great detail, including infinite-dimensional case, in connection with quantum statistical mechanics, see e.g. [12, 14, 18]. A study of entropic and informational characteristics of quantum communication channels from the general viewpoint of operator theory in separable Hilbert space was undertaken in [9, 10], where the quantities related to the classical capacity, in the first place – the $\chi$-capacity, were investigated. The present work is devoted to two other characteristics – the quantum mutual information and the coherent information. The first one is closely related to the entanglement-assisted classical capacity while the second – to the quantum capacity of a channel. One of the author’s goal was to give an appropriate definition of these quantities in the infinite-dimensional case, which would not require additional artificial assumptions. The difficulty which was overcome is in the uncertainties in expressions containing differences of entropies, each of which can be infinite in the infinite-dimensional case. A main result is Theorem 1 which implies that these quantities are naturally defined and finite on the set of input states with finite entropy, where they satisfy identities (22) and (24) for complementary channels (the quantum mutual information is defined uniquely for all input states but can be infinite, still satisfying identity (22)).
In the introductory Section 2 a description of the corresponding quantities for a finite quantum system is given. In Section 3 we give the definition and study the properties of the quantum mutual information in the infinite dimensional case. The main identity \((22)\) for complementary channels is proved in Section 4. Section 5 is devoted to the coherent information. In Section 6 we point out somewhat unexpected continuity property of the mutual and coherent informations implied by the identity \((22)\).

2 Finite-dimensional case

Consider the quantum system described by a finite-dimensional Hilbert space \(\mathcal{H}\) and denote by \(\mathcal{S}(\mathcal{H})\) the convex set of quantum states described by density operators in \(\mathcal{H}\), i.e. positive operators with unit trace: \(\rho \geq 0, \text{Tr}\rho = 1\). Entropy of the state \(\rho\) (von Neumann entropy) is defined by the relation

\[
H(\rho) = \text{Tr}\eta(\rho), \quad \eta(x) = \begin{cases} -x \log x, & x > 0, \\ 0, & x = 0. \end{cases}
\tag{1}
\]

Let three systems \(A, B, E\), described by the spaces \(\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_E\), correspondingly, and an isometric operator \(V: \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E\) be given, then the relations

\[
\Phi(\rho) = \text{Tr}_E V\rho V^*, \quad \tilde{\Phi}(\rho) = \text{Tr}_B V\rho V^*, \quad \rho \in \mathcal{S}(\mathcal{H}_A),
\tag{2}
\]

where \(\text{Tr}_X(\cdot) \doteq \text{Tr}_{\mathcal{H}_X}(\cdot)\), define completely positive trace-preserving maps, i.e. quantum channels \(\Phi: \mathcal{S}(\mathcal{H}_A) \rightarrow \mathcal{S}(\mathcal{H}_B)\) and \(\tilde{\Phi}: \mathcal{S}(\mathcal{H}_A) \rightarrow \mathcal{S}(\mathcal{H}_E)\), which are called mutually complementary (this construction is generalized to the infinite dimensional case without changes). The systems \(A, B\) describe, correspondingly, input and output of the channel \(\Phi\), and \(E\) – its “environment” (see details in [8, 13]). The identity operator in a space \(\mathcal{H}_X\) and the identity transformation of the set \(\mathcal{S}(\mathcal{H}_X)\) will be denoted \(I_X\) and \(\text{Id}_X\) correspondingly.

Let \(\rho = \rho_A\) be an input state in the space \(\mathcal{H}_A\), \(\rho_B\) and \(\rho_E\) be the results of action of the channels \(\Phi\) and \(\tilde{\Phi}\) on the state \(\rho_A\) correspondingly. The quantum mutual information is defined as follows

\[
I(\rho, \Phi) = H(A) + H(B) - H(E),
\tag{3}
\]

\(^1\)In the present paper log denotes the natural logarithm.
where the brief notations $H(A) = H(\rho_A)$, etc., are used [1]. By introducing the reference system $\mathcal{H}_R \cong \mathcal{H}_A$ and the purification vector $\psi_{AR} \in \mathcal{H}_A \otimes \mathcal{H}_R$ for the state $\rho_A$, the mutual information can be represented as follows

$$I(\rho, \Phi) = H(R) + H(B) - H(BR),$$

(4)

where $\rho_{BR} = (\Phi \otimes \text{Id}_R)(|\psi_{AR}\rangle\langle \psi_{AR}|)$.

The mutual information $I(\rho, \Phi)$ have the several properties similar to the properties of the Shannon information (see Proposition 1 below). In [3] (see also [8]) it is shown that

$$\max_\rho I(\rho, \Phi) = C_{ea}(\Phi)$$

(5)

is the classical entanglement-assisted capacity of the channel $\Phi$.

By introducing the analogous characteristic for the complementary channel

$$I(\rho, \tilde{\Phi}) = H(A) + H(E) - H(B) = H(R) + H(E) - H(RE),$$

(6)

we have the following basic identity

$$I(\rho, \Phi) + I(\rho, \tilde{\Phi}) = 2H(\rho).$$

(7)

An important component of the quantum mutual information $I(\rho, \Phi)$ is the coherent information (see [2])

$$I_c(\rho, \Phi) = H(B) - H(E) = H(B) - H(RB).$$

(8)

This notion is closely related to the quantum capacity of the channel $\Phi$. In [5] (see also [8]) it is shown that

$$Q(\Phi) = \lim_{n \to +\infty} \frac{1}{n} \max_\rho I_c(\rho, \Phi^{\otimes n})$$

(9)

is the quantum capacity of the channel $\Phi$. Identity (7) is equivalent to the following one

$$I_c(\rho, \Phi) + I_c(\rho, \tilde{\Phi}) = 0.$$  

(10)

The aim of this paper is to explore definitions and properties of the analogs of the values $I(\rho, \Phi)$ and $I_c(\rho, \Phi)$ in an infinite dimensional Hilbert space. In particular, it will be shown that the coherent information is naturally defined on the set of states with finite entropy, where the analog of identity (10) holds. The results of this paper can be used for generalization of relations (5) and (9) to the case of infinite dimensional channels.
3 Mutual information

In what follows \( \mathcal{H} \) is a separable Hilbert space. Let \( \mathfrak{S}(\mathcal{H}) \) be the Banach space of trace class operators, so that \( \mathfrak{G}(\mathcal{H}) \subset \mathfrak{S}(\mathcal{H}) \). Consider the natural extension of the von Neumann entropy \( H(\rho) = \text{Tr} \eta(\rho) \) of a quantum state \( \rho \in \mathfrak{G}(\mathcal{H}) \) to the cone \( \mathfrak{S}_+(\mathcal{H}) \) of all positive trace class operators.

**Definition 1.** The **entropy** of an operator \( A \in \mathfrak{S}_+(\mathcal{H}) \) is defined as follows

\[
H(A) = \text{Tr} AH \left( \frac{A}{\text{Tr}A} \right) = \text{Tr} \eta(A) - \eta(\text{Tr}A). \tag{11}
\]

The entropy is a concave lower semicontinuous function on the cone \( \mathfrak{S}_+(\mathcal{H}) \), taking values in \([0, +\infty]\). By using Definition 1 and the well known properties of the von Neumann entropy (see \[14\]), it is easy to obtain the following relations:

\[
H(\lambda A) = \lambda H(A), \quad \lambda \geq 0, \tag{12}
\]

\[
H(A) + H(B - A) \leq H(B) \leq H(A) + H(B - A) + \text{Tr} B h_2 \left( \frac{\text{Tr}A}{\text{Tr}B} \right), \tag{13}
\]

where \( A, B \in \mathfrak{S}_+(\mathcal{H}) \), \( A \leq B \), and \( h_2(x) = \eta(x) + \eta(1 - x) \).

We will also use the function \( S(A) = \text{Tr} \eta(A) \) on the cone \( \mathfrak{S}_+(\mathcal{H}) \) coinciding with the function \( H(A) \) on the set \( \mathfrak{G}(\mathcal{H}) \).

**Definition 2.** The **relative entropy** of operators \( A, B \in \mathfrak{S}_+(\mathcal{H}) \) is defined as follows

\[
H(A\|B) = \begin{cases} 
\sum_{i=1}^{+\infty} \langle e_i | (A \log A - A \log B + B - A) | e_i \rangle, & \text{supp } A \subseteq \text{supp } B, \\
+\infty, & \text{supp } A \nsubseteq \text{supp } B,
\end{cases}
\]

where \( \{|e_i\}_{i=1}^{+\infty} \) is the orthonormal basis of eigenvalues of the operator \( A \) \[12\].

We will use the following lemma (\[12\] Lemma 4).

**Lemma 1.** Let \( \{P_n\} \) be a nondecreasing sequence of projectors, converging to the identity operator \( I \) in the strong operator topology, and \( A, B \) be arbitrary positive trace class operators. Then the sequences \( \{H(P_n AP_n)\} \) and \( \{H(P_n AP_n\|P_n BP_n)\} \) are nondecreasing,

\[
H(A) = \lim_{n \to +\infty} H(P_n AP_n) \quad \text{and} \quad H(A\|B) = \lim_{n \to +\infty} H(P_n AP_n\|P_n BP_n).
\]
Definition 3. A **quantum channel** is a linear trace-preserving map 
\( \Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B) \) such that the dual map \( \Phi^* \) from the \( C^* \)-algebra \( \mathfrak{B}(\mathcal{H}_B) \) into the \( C^* \)-algebra \( \mathfrak{B}(\mathcal{H}_A) \) is completely positive [4].

In what follows we will use the fundamental **monotonicity** property of the relative entropy established in [12] and expressed by the inequality:

\[
H(\Phi(A)\|\Phi(B)) \leq H(A\|B)
\]

valid for an arbitrary quantum channel \( \Phi \) and arbitrary positive trace class operators \( A \) and \( B \).

**Definition 4.** Let \( \Phi : \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B) \) be a quantum channel and \( \rho \) be an arbitrary quantum state in \( \mathfrak{S}(\mathcal{H}_A) \) with the spectral representation \( \rho = \sum_{i=1}^{+\infty} \lambda_i |e_i\rangle \langle e_i| \). The **mutual information** of the channel \( \Phi \) at the state \( \rho \) is defined as follows

\[
I(\rho, \Phi) = H(\Phi \otimes \text{Id}_R(\varphi_\rho)\langle \varphi_\rho|)\|\Phi(\rho) \otimes \rho),
\]

where

\[
|\varphi_\rho\rangle = \sum_{i=1}^{+\infty} \sqrt{\lambda_i} |e_i\rangle \otimes |e_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_R
\]

is a purification vector\(^2\) for the state \( \rho \).

Note that in the case \( \dim \mathcal{H}_A < +\infty \) and \( \dim \mathcal{H}_B < +\infty \) this definition is equivalent to (3), (4), since

\[
H(\Phi \otimes \text{Id}_R(\varphi_\rho)\langle \varphi_\rho|)\|\Phi(\rho) \otimes \rho) = H(\rho_{BR}\|\rho_B \otimes \rho_R)
\]

\[
= \text{Tr}(\rho_{BR}(\log(\rho_{BR}) - \log(\rho_B \otimes \rho_R)))
\]

\[
= -H(\rho_{BR}) + H(\rho_B) + H(\rho_R) = -H(BR) + H(B) + H(R).
\]

**Remark 1.** The above definition of the value \( I(\rho, \Phi) \) does not depend on the choice of the space \( \mathcal{H}_R \) and of the purification vector \( \varphi_\rho \). This can be shown by using the well known relation between different purification vectors of a given state (see [8, 13]) and properties of the relative entropy. \( \square \)

In the finite dimensional case concavity of the mutual information as a function of \( \rho \) on the set \( \mathfrak{S}(\mathcal{H}_A) \) follows from concavity of the conditional

\(^2\)This means that \( \text{Tr}_R|\varphi_\rho\rangle\langle \varphi_\rho| = \rho \).
entropy $H(EB) - H(E)$ \[1, 8\]. In the case $\dim \mathcal{H}_A = +\infty$ and $\dim \mathcal{H}_B < +\infty$ this implies concavity of the mutual information as a function of $\rho$ on the set $\mathcal{S}_f(\mathcal{H}_A) = \{ \rho \in \mathcal{S}(\mathcal{H}_A) \mid \text{rank} \rho < +\infty \}$.

In what follows convergence of quantum states means convergence of the corresponding density operators to a limit operator in the trace norm, which is equivalent to the weak operator convergence (see \[\text{[1]}\] or \[\text{[9, Appendix A]}\]). Note that the entropy and the relative entropy are lower semicontinuous in their arguments with respect to this convergence \[\text{[18]}\].

Let $\mathcal{F}(A, B)$ be the set of all quantum channels from $\mathcal{S}(\mathcal{H}_A)$ to $\mathcal{S}(\mathcal{H}_B)$ endowed with the strong convergence topology \[\text{[10]}\]. Strong convergence of a sequence $\{ \Phi_n \} \subset \mathcal{F}(A, B)$ to a quantum channel $\Phi_0 \in \mathcal{F}(A, B)$ means that $\lim_{n \to +\infty} \Phi_n(\rho) = \Phi_0(\rho)$ for any state $\rho \in \mathcal{S}(\mathcal{H}_A)$.

The following proposition is devoted to generalization of the observations in \[\text{[1]}\] to the infinite dimensional case.

**Proposition 1.** The function $(\rho, \Phi) \mapsto I(\rho, \Phi)$ is nonnegative and lower semicontinuous on the set $\mathcal{S}(\mathcal{H}_A) \times \mathcal{F}(A, B)$. It has the following properties:

1) concavity in $\rho$: $I(\lambda \rho_1 + (1 - \lambda) \rho_2, \Phi) \geq \lambda I(\rho_1, \Phi) + (1 - \lambda) I(\rho_2, \Phi)$;
2) convexity in $\Phi$: $I(\rho, \lambda \Phi_1 + (1 - \lambda) \Phi_2) \leq \lambda I(\rho, \Phi_1) + (1 - \lambda) I(\rho, \Phi_2)$;
3) the 1-th chain rule: for arbitrary channels $\Phi: \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ and $\Psi: \mathcal{S}(\mathcal{H}_B) \to \mathcal{S}(\mathcal{H}_C)$ the inequality $I(\rho, \Psi \circ \Phi) \leq I(\rho, \Phi)$ holds for any $\rho \in \mathcal{S}(\mathcal{H}_A)$;
4) the 2-th chain rule: for arbitrary channels $\Phi: \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ and $\Psi: \mathcal{S}(\mathcal{H}_B) \to \mathcal{S}(\mathcal{H}_C)$ the inequality $I(\rho, \Psi \circ \Phi) \leq I(\Phi(\rho), \Psi)$ holds for any $\rho \in \mathcal{S}(\mathcal{H}_A)$;
5) subadditivity: for arbitrary channels $\Phi: \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ and $\Psi: \mathcal{S}(\mathcal{H}_C) \to \mathcal{S}(\mathcal{H}_D)$ the inequality

$$I(\omega, \Phi \otimes \Psi) \leq I(\omega_A, \Phi) + I(\omega_C, \Psi) \quad (16)$$

holds for any $\omega \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_C)$.

**Proof.** Nonnegativity of the value $I(\rho, \Phi)$ follows from nonnegativity of the relative entropy. By Lemma \[\text{[2]}\] below and Remark \[\text{[1]}\] lower semicontinuity of the function $(\rho, \Phi) \mapsto I(\rho, \Phi)$ follows from lower semicontinuity of the relative entropy in the both arguments.
To prove concavity of the function $\rho \mapsto I(\rho, \Phi)$ suppose first that $\dim \mathcal{H}_B$ is finite. Let $\rho = \alpha \sigma_1 + (1 - \alpha) \sigma_2$ and $\{P_n\}$ be an increasing sequence of finite rank spectral projectors of the state $\rho$ strongly converging to $I_A$. Let

$$\rho_n = \frac{P_n \rho P_n}{\text{Tr} P_n \rho} = \frac{\alpha P_n \sigma_1 P_n + (1 - \alpha) P_n \sigma_2 P_n}{\alpha \text{Tr} P_n \sigma_1 + (1 - \alpha) \text{Tr} P_n \sigma_2} = \frac{\mu_1^n \sigma_1^n + \mu_2^n \sigma_2^n}{\mu_1^n + \mu_2^n},$$

where

$$\mu_1^n = \alpha \text{Tr} P_n \sigma_1, \quad \sigma_1^n = \frac{P_n \sigma_1 P_n}{\mu_1^n},$$

$$\mu_2^n = (1 - \alpha) \text{Tr} P_n \sigma_2, \quad \sigma_2^n = (1 - \alpha) \frac{P_n \sigma_2 P_n}{\mu_2^n}.$$

By concavity of the function $\rho \mapsto I(\rho, \Phi)$ on the set $\mathcal{S}_1(\mathcal{H}_A)$ mentioned before Proposition 1 we have

$$I(\rho_n, \Phi) \geq \frac{\mu_1^n}{\mu_1^n + \mu_2^n} I(\sigma_1^n, \Phi) + \frac{\mu_2^n}{\mu_1^n + \mu_2^n} I(\sigma_2^n, \Phi).$$

Lemma 3 below implies $\lim_{n \to +\infty} I(\rho_n, \Phi) = I(\rho, \Phi)$. By using lower semicontinuity of the function $\rho \mapsto I(\rho, \Phi)$, we obtain

$$I(\rho, \Phi) \geq \liminf_{n \to +\infty} \frac{\mu_1^n}{\mu_1^n + \mu_2^n} I(\sigma_1^n, \Phi) + \liminf_{n \to +\infty} \frac{\mu_2^n}{\mu_1^n + \mu_2^n} I(\sigma_2^n, \Phi)$$

$$\geq \alpha I(\sigma_1, \Phi) + (1 - \alpha) I(\sigma_2, \Phi).$$

Let $\Phi$ be an arbitrary quantum channel. Consider the sequence of channels $\Phi_n = \Pi_n \circ \Phi$ with finite dimensional output, where

$$\Pi_n(\rho) = P_n \rho P_n + [\text{Tr}((I - P_n)\rho)]|\psi\rangle\langle\psi|$$

is a quantum channel from $\mathcal{S}(\mathcal{H}_B)$ to itself for each $n$, $\{P_n\}$ is an increasing sequence of finite rank projectors strongly converging to $I_B$, $|\psi\rangle\langle\psi|$ is a fixed pure state in $\mathcal{S}(\mathcal{H}_B)$. Then for each $n$ the function $\rho \mapsto I(\rho, \Phi_n)$ is concave by the above observation. Since

$$I(\rho, \Phi_n) \leq I(\rho, \Phi) \quad \forall n \quad \text{and} \quad \liminf_{n \to +\infty} I(\rho, \Phi_n) \geq I(\rho, \Phi)$$

by monotonicity of the relative entropy and lower semicontinuity of the function $\Phi \mapsto I(\rho, \Phi)$, we have

$$I(\rho, \Phi) = \lim_{n \to +\infty} I(\rho, \Phi_n).$$
Hence the function $\rho \mapsto I(\rho, \Phi)$ is concave as a pointwise limit of a sequence of concave functions.

Convexity of the function $\Phi \mapsto I(\rho, \Phi)$ follows from joint convexity of the relative entropy in their arguments [18].

The 1-th chain rule immediately follows from Definition 4 and monotonicity of the relative entropy.

The 2-th chain rule is also proved by using monotonicity of the relative entropy as follows.

Let $|\phi\rangle\langle\phi|$ be a purification of the state $\rho \in \mathcal{S}(\mathcal{H}_A)$ in the space $\mathcal{H}_A \otimes \mathcal{H}_R$, then $|\psi\rangle\langle\psi| = V \otimes I_R |\phi\rangle\langle\phi| V^* \otimes I_R$ is a purification of the state $\Phi(\rho) \in \mathcal{S}(\mathcal{H}_B)$ in the space $\mathcal{H}_B \otimes \mathcal{H}_E \otimes \mathcal{H}_R$ ($V$ is the isometry from representation (2) of the channel $\Phi$). Hence

$$I(\Phi(\rho), \Psi) = H(\Psi \otimes I_{ER}(|\psi\rangle\langle\psi|) \| \Psi(\text{Tr}_{ER}|\psi\rangle\langle\psi|) \otimes \text{Tr}_B|\psi\rangle\langle\psi|).$$

A direct verification shows that taking the partial trace over the space $\mathcal{H}_E$ on each arguments of the relative entropy in the above expression transforms the right side of this expressions to

$$H((\Psi \circ \Phi) \otimes \text{Id}_R(|\varphi\rangle\langle\varphi|) \| (\Psi \circ \Phi)(\text{Tr}_{R}|\varphi\rangle\langle\varphi|) \otimes \text{Tr}_A|\varphi\rangle\langle\varphi|) = I(\rho, \Psi \circ \Phi).$$

The subadditivity property of the mutual information will be derived from the corresponding property of this characteristics for finite dimensional channels [11, 8].

Let $\{Q^X_n\}$ be an increasing sequence of finite rank projectors in the space $\mathcal{H}_X$, strongly converging to the operator $I_X$, where $X = B, D$. The sequence of channels

$$\Pi^X_n(\rho) = Q^X_n \rho Q^X_n + (\text{Tr}(I_X - Q^X_n)\rho)\tau_X$$
from $\mathcal{S}(\mathcal{H}_X)$ to itself, where $\tau_X$ is an arbitrary pure state in the space $\mathcal{H}_X$, strongly converges to the channel $\text{Id}_X$.

Let $\omega$ be an arbitrary state in $\mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_C)$. Let $\{P^X_n\}$ be an increasing sequence of finite rank spectral projectors of the state $\omega_X$, strongly converging to the operator $I_X$, where $X = A, C$.

Consider the sequence of states

$$\omega^n = (\text{Tr}((P^A_n \otimes P^C_n) \cdot \omega))^{-1}(P^A_n \otimes P^C_n) \cdot \omega \cdot (P^A_n \otimes P^C_n),$$

converging to the state $\omega$. 

9
A direct verification shows that
\[ \lambda_n \omega^n_X \leq \omega_X, \quad X = A, C, \] where \( \lambda_n = \text{Tr}\left( (P_n^A \otimes P_n^C) \cdot \omega \right) \).

By Lemma 4 below we have
\[ \lim_{n \to +\infty} I(\omega^n_A, \Pi_n^B \circ \Phi) = I(\omega_A, \Phi) \quad \text{and} \quad \lim_{n \to +\infty} I(\omega^n_C, \Pi_n^D \circ \Psi) = I(\omega_C, \Psi). \quad (17) \]

Subadditivity of the mutual information for finite dimensional channels implies
\[ I(\omega^n_A, (\Pi_n^B \circ \Phi) \otimes (\Pi_n^D \circ \Psi)) \leq I(\omega_A^n, \Pi_n^B \circ \Phi) + I(\omega_C^n, \Pi_n^D \circ \Psi). \]

By (17) and lower semicontinuity of the mutual information as a function of a pair (state, channel) passing to the limit in this inequality implies (16).

In the proof of Proposition 1 the following lemmas were used.

\textbf{Lemma 2.} Let \( H \) be a separable Hilbert space. For an arbitrary sequence \( \{\rho_n\} \subset \mathcal{S}(H) \), converging to a state \( \rho_0 \), there exists a corresponding purification sequence \( \{\hat{\rho}_n\} \subset \mathcal{S}(H \otimes H) \), converging to a purification \( \hat{\rho}_0 \) of the state \( \rho_0 \).

\textit{Proof.} The assertion of the lemma follows from the inequality
\[ \beta(\rho, \sigma)^2 \leq \|\rho - \sigma\|_1 \]
for the Bures distance \( \beta(\rho, \sigma) = \inf \|\varphi_{\rho} - \varphi_{\sigma}\| \), where the infimum is over all purification vectors \( \varphi_{\rho} \) and \( \varphi_{\sigma} \) of the states \( \rho \) and \( \sigma \). \( \square \)

\textbf{Lemma 3.} Let \( \Phi: \mathcal{S}(H_A) \to \mathcal{S}(H_B) \) be a quantum channel such that \( \dim H_B < +\infty \) and \( \rho_0 \) be a state in \( \mathcal{S}(H_A) \) with the spectral representation \( \rho_0 = \sum_{i=1}^{+\infty} \lambda_i |e_i\rangle \langle e_i| \). Let
\[ \rho_n = \frac{1}{\mu_n} \sum_{i=1}^{n} \lambda_i |e_i\rangle \langle e_i|, \quad \text{where} \quad \mu_n = \sum_{i=1}^{n} \lambda_i, \quad (18) \]
for every \( n \). Then \( \lim_{n \to +\infty} I(\rho_n, \Phi) = I(\rho_0, \Phi) \).
Proof. Let 

\[ P_n = \sum_{i=1}^{n} |e_i\rangle\langle e_i|, \quad n = 1, 2, \ldots \]

Since \( \dim \mathcal{H}_B < \infty \), the following value is finite

\[
I_n = H(\Phi \otimes \text{Id}_R(\hat{\rho}_n))\|\Phi(\rho_0) \otimes \rho_n) \]
\[
= \mu_n^{-1} H(Q_n (\Phi \otimes \text{Id}_R(\hat{\rho}_n)) Q_n \| Q_n (\Phi(\rho_0) \otimes \rho_0) Q_n),
\]

where

\[
\hat{\rho}_0 = \sum_{i,j=1}^{+\infty} \sqrt{\lambda_i \lambda_j} |e_i\rangle \langle e_i| \otimes |e_j\rangle \langle e_j|, \quad \hat{\rho}_n = \mu_n^{-1} \sum_{i,j=1}^{n} \sqrt{\lambda_i \lambda_j} |e_i\rangle \langle e_i| \otimes |e_j\rangle \langle e_j|
\]

and \( Q_n = I_B \otimes P_n \). By Lemma 1 we have

\[
\lim_{n \to +\infty} I_n = H(\Phi \otimes \text{Id}_R(\hat{\rho}_n))\|\Phi(\rho_0) \otimes \rho_0) = I(\rho_0, \Phi) \leq +\infty. \quad (19)
\]

We will prove that \( \lim_{n \to +\infty} I_n = \lim_{n \to +\infty} I(\rho_n, \Phi) \) by considering the difference \( I_n - I(\rho_n, \Phi) \). Since \( H(\Phi \otimes \text{Id}_R(\hat{\rho}_n)) < +\infty \), we have

\[
I_n - I(\rho_n, \Phi) = H(\Phi \otimes \text{Id}_R(\hat{\rho}_n))\|\Phi(\rho_0) \otimes \rho_n) - H(\Phi \otimes \text{Id}_R(\hat{\rho}_n))\|\Phi(\rho_0) \otimes \rho_0) \]
\[
= -H(\Phi \otimes \text{Id}_R(\hat{\rho}_n)) - \text{Tr}(\Phi \otimes \text{Id}_R(\hat{\rho}_n))(\log \Phi(\rho_0) \otimes \rho_n) \]
\[
+ H(\Phi \otimes \text{Id}_R(\hat{\rho}_n)) + \text{Tr}(\Phi \otimes \text{Id}_R(\hat{\rho}_n)) \log(\Phi(\rho_0) \otimes \rho_n) = A - B,
\]

where

\[
A = -\text{Tr}(\Phi \otimes \text{Id}_R(\hat{\rho}_n))(\log \Phi(\rho_0) \otimes \rho_n), \quad B = -\text{Tr}(\Phi \otimes \text{Id}_R(\hat{\rho}_n)) \log(\Phi(\rho_0) \otimes \rho_n).
\]

We will use the following property of logarithm

\[
\log(\rho \otimes \sigma) = \log(\rho) \otimes I + I \otimes \log(\sigma), \quad (20)
\]

where in the case of not-full-rank states \( \rho \) and \( \sigma \) the restrictions to the subspaces \( \text{supp}(\rho) \) and \( \text{supp}(\sigma) \) are kept in mind, that is

\[
P_\rho \otimes P_\sigma(\log(\rho \otimes \sigma)) = (P_\rho \log(\rho) P_\rho) \otimes P_\sigma + P_\rho \otimes (P_\sigma \log(\sigma) P_\sigma), \quad (21)
\]

where \( P_\rho \) and \( P_\sigma \) are respectively the projectors onto \( \text{supp}(\rho) \) and \( \text{supp}(\sigma) \). Since \( P_{\Phi(\rho_n)} \leq P_{\Phi(\rho_0)} \), we have

\[
A = -\text{Tr}(\Phi \otimes \text{Id}_R(\hat{\rho}_n))(\log \Phi(\rho_0) \otimes I_R) - \text{Tr}(\Phi \otimes \text{Id}_R(\hat{\rho}_n))(I_B \otimes \log(\rho_n))
\]
\[
= -\text{Tr}(\Phi(\rho_n)) \log \Phi(\rho_0) + H(\rho_n).
\]
In the similar way we obtain

\[ B = -\text{Tr}(\Phi \otimes \text{Id}_R(\hat{\rho}_n))(\log(\rho_n) \otimes I_R) - \text{Tr}(\Phi \otimes \text{Id}_R(\hat{\rho}_n))(I_B \otimes \log(\rho_n)) = H(\Phi(\rho_n)) + H(\rho_n). \]

Hence,

\[ I_n - I(\rho_n, \Phi) = A - B = -\text{Tr}(\rho_n \log(\rho_n) - H(\rho_n)) = H(\Phi(\rho_n)\|\Phi(\rho_0)). \]

By monotonicity of the relative entropy we have

\[ H(\Phi(\rho_n)\|\Phi(\rho_0)) \leq H(\rho_n\|\rho_0) = -\sum_{i=1}^{n} \lambda_i \log \mu_n = -\log \mu_n. \]

Since \( \mu_n \to 1 \) as \( n \to +\infty \), \( \lim_{n \to +\infty} (I_n - I(\rho_n, \Phi)) = 0. \) This and (19) imply

\[ \lim_{n \to +\infty} I(\rho_n, \Phi) = I(\rho_0, \Phi) \leq +\infty. \]

\[ \square \]

**Lemma 4.** Let \( \Phi \) be an arbitrary channel from \( \mathcal{S}(\mathcal{H}_A) \) to \( \mathcal{S}(\mathcal{H}_B) \) and \( \{\Pi_n\} \) be a sequence of channels from \( \mathcal{S}(\mathcal{H}_B) \) to \( \mathcal{S}(\mathcal{H}_B) \), strongly converging to the identity channel. Let \( \{\rho_n\} \) be a sequence of states in \( \mathcal{S}(\mathcal{H}_A) \) converging to a state \( \rho_0 \) such that \( \lambda_n \rho_n \leq \rho_0 \) for some sequence \( \{\lambda_n\} \) converging to 1. Then

\[ \lim_{n \to +\infty} I(\rho_n, \Pi_n \circ \Phi) = I(\rho_0, \Phi). \]

**Proof.** It follows from the inequality \( \lambda_n \rho_n \leq \rho_0 \) that \( \rho_0 = \lambda_n \rho_n + (1 - \lambda_n)\sigma_n \), where \( \sigma_n \) is a state in \( \mathcal{S}(\mathcal{H}_A) \). Hence concavity and nonnegativity of the mutual information and the 1-th chain rule imply the inequality

\[ \lambda_n I(\rho_n, \Pi_n \circ \Phi) \leq I(\rho_0, \Pi_n \circ \Phi) \leq I(\rho_0, \Phi), \]

showing that \( \limsup_{n \to +\infty} I(\rho_n, \Pi_n \circ \Phi) \leq I(\rho_0, \Phi). \) This and lower semicontinuity of the function \( (\rho, \Phi) \mapsto I(\rho, \Phi) \) imply the assertion of the lemma. \( \square \)
The relation between mutual informations of complementary channels

The main result of this section is an infinite dimensional generalization of relation (7) between mutual informations of a pair of complementary channels (nontriviality of this result is connected with a possible uncertainty \(\infty - \infty\) in expressions (4) and (6)).

Let \(H_A, H_B, H_E\) be separable Hilbert spaces and \(V: H_A \to H_B \otimes H_E\) be an isometry, then relations (2) define a pair of complementary channels \(\Phi, \tilde{\Phi}\) similar to the finite dimensional case.

**Theorem 1.** For an arbitrary state \(\rho \in S(H_A)\) the following relation holds:

\[
I(\rho, \Phi) + I(\rho, \tilde{\Phi}) = 2H(\rho).
\]

**Proof.** Let \(\{|h_i\rangle\}_{i=1}^{+\infty}\) be an orthonormal basis in the space \(H_E\), then

\[
V|\varphi\rangle = \sum_{i=1}^{+\infty} V_i|\varphi\rangle \otimes |h_i\rangle,
\]

where \(V_i: H_A \to H_B\) is a sequence of bounded operators, satisfying the condition \(\sum_{i=1}^{+\infty} V_i^* V_i = I_A\), the channel \(\Phi\) has the Kraus representation \(\Phi(\rho) = \sum_{i=1}^{+\infty} V_i \rho V_i^*\), the complementary channel \(\tilde{\Phi}\) has the representation \(\tilde{\Phi}(\rho) = \sum_{i,j=1}^{+\infty} [\text{Tr}V_i \rho V_j^*] |h_i\rangle \langle h_j|\) (cf. [7]).

Let \(\rho = \sum_{i=1}^{m} \lambda_i |e_i\rangle \langle e_i|\) be a finite rank state in \(S(H_A)\) and \(\hat{\rho}\) be its purification in \(S(H_A \otimes H_R)\). Consider the sequence of quantum operations \(\Phi_n(\rho) = \sum_{i=1}^{n} V_i \rho V_i^*\). The sequence \(\{\Phi_n\}\) strongly and monotonously converges to the channel \(\Phi\) (that is \(\Phi_n(\rho) \leq \Phi_{n+1}(\rho)\) for all \(n\) and \(\rho \in S(H_A)\)).

Since \(\Phi_n \otimes \text{Id}_R(\hat{\rho})\) is a finite rank state, we have

\[
X_n = H(\Phi_n \otimes \text{Id}_R(\hat{\rho})) - S(\Phi_n \otimes \text{Id}_R(\hat{\rho})) - \text{Tr}(\Phi_n \otimes \text{Id}_R(\hat{\rho})) \log(\Phi_n(\rho)) + R_n,
\]

where \(R_n = 1 - \text{Tr}\Phi_n(\rho) \to 0\) as \(n \to +\infty\). By Lemma 7 in the Appendix \(\lim_{n \to +\infty} X_n = I(\rho, \Phi)\). Since

\[
\Phi_n \otimes \text{Id}_R(\hat{\rho}) = \text{Tr}_E (I_B \otimes P_n) \cdot (V \otimes I_R) \cdot \hat{\rho} \cdot (V^* \otimes I_R) \cdot (I_B \otimes P_n \otimes I_R),
\]

A quantum operation is a linear completely positive trace non-increasing map [8, 13].
where \( P_n = \sum_{i=1}^{n} |h_i\rangle\langle h_i| \) is a finite dimensional projector in the space \( \mathcal{H}_E \) and the partial trace is taking in the space \( \mathcal{H}_B \otimes \mathcal{H}_E \otimes \mathcal{H}_R \), the operator \( \Phi_n \otimes \text{Id}_R(\hat{\rho}) \) is isomorphic to the operator 

\[
\tilde{\Phi}_n(\rho) = \text{Tr}_{BR} (I_B \otimes P_n \otimes I_R) \cdot (V \otimes I_R) \cdot (V^* \otimes I_R) \cdot (I_B \otimes P_n \otimes I_R),
\]

where \( \tilde{\Phi}_n(\cdot) = P_n \tilde{\Phi}(\cdot) P_n \) is the quantum operation complementary to the operation \( \Phi_n \). Thus \( S(\Phi_n \otimes \text{Id}_R(\hat{\rho})) = S(\tilde{\Phi}_n(\rho)) \). By using (20) and by noting that \( \Phi_n(\cdot) \leq \Phi(\cdot) \) we obtain

\[
- \text{Tr}(\Phi_n \otimes \text{Id}_R(\hat{\rho})) \log(\Phi(\rho) \otimes \rho) = - \text{Tr}(\Phi_n \otimes \text{Id}_R(\hat{\rho}))(\log(\Phi(\rho)) \otimes I_R) - \text{Tr}(\Phi_n \otimes \text{Id}_R(\hat{\rho}))(I_B \otimes \log(\rho)) = - \text{Tr}\Phi_n(\rho) \log(\Phi(\rho)) - \text{Tr}(\text{Tr}_B \Phi_n \otimes \text{Id}_R(\hat{\rho})) \log(\rho).
\]

Consider the value

\[
Y_n = H(\tilde{\Phi}_n \otimes \text{Id}_R(\hat{\rho})) - S(\tilde{\Phi}_n(\rho) \otimes \rho) = -S(\tilde{\Phi}_n \otimes \text{Id}_R(\hat{\rho}))(\log(\tilde{\Phi}_n(\rho)) \otimes I_R) - \text{Tr}(\tilde{\Phi}_n \otimes \text{Id}_R(\hat{\rho})) \log(\tilde{\Phi}_n(\rho) \otimes \rho).
\]

By Lemma 1 \( \lim_{n \to +\infty} Y_n = I(\rho, \tilde{\Phi}) \). Similar to the calculations of the summands of \( X_n \) we obtain

\[
S(\tilde{\Phi}_n \otimes \text{Id}_R(\hat{\rho})) = S(\Phi_n(\rho))
\]

and

\[
- \text{Tr}(\tilde{\Phi}_n \otimes \text{Id}_R(\hat{\rho})) \log(\tilde{\Phi}_n(\rho) \otimes \rho) = -S(\tilde{\Phi}_n \otimes \text{Id}_R(\hat{\rho}))(\log(\tilde{\Phi}_n(\rho)) \otimes I_R) - \text{Tr}(\tilde{\Phi}_n \otimes \text{Id}_R(\hat{\rho}))(I_E \otimes \log(\rho)) = S(\tilde{\Phi}_n(\rho)) - \text{Tr}(\text{Tr}_E \tilde{\Phi}_n \otimes \text{Id}_R(\hat{\rho})) \log(\rho).
\]

Let us show that \( \lim_{n \to +\infty} (X_n + Y_n) = 2H(\rho) \). By the definition of the relative entropy we have

\[
X_n + Y_n = -\text{Tr}\Phi_n(\rho) \log(\Phi(\rho)) - S(\Phi_n(\rho)) + C_n + D_n + R_n = H(\Phi_n(\rho) \| \Phi(\rho)) + C_n + D_n,
\]

where

\[
C_n = -\text{Tr}(\text{Tr}_B \Phi_n \otimes \text{Id}_R(\hat{\rho})) \log(\rho), \quad D_n = -\text{Tr}(\text{Tr}_E \tilde{\Phi}_n \otimes \text{Id}_R(\hat{\rho})) \log(\rho).
\]
Let us prove that \( \lim_{n \to +\infty} C_n = \lim_{n \to +\infty} D_n = H(\rho) \). By noting that
\[
\text{Tr}_B \Phi_n \otimes \text{Id}_R(\hat{\rho}) = \sum_{i,j=1}^m \sqrt{\lambda_i \lambda_j} \text{Tr} \Phi_n(|e_i\rangle\langle e_j|)|e_i\rangle\langle e_j|,
\]
we obtain
\[
C_n = \sum_{i=1}^m (-\lambda_i \log \lambda_i) \text{Tr} \Phi_n(|e_i\rangle\langle e_i|),
\]
and hence \( \lim_{n \to +\infty} C_n = H(\rho) \), since \( \lim_{n \to +\infty} \text{Tr} \Phi_n(|e_i\rangle\langle e_i|) = 1 \). In the similar way one can prove that \( \lim_{n \to +\infty} D_n = H(\rho) \). Lemma 7 in the Appendix implies
\[
\lim_{n \to +\infty} H(\Phi_n(\rho)\|\Phi(\rho)) = 0.
\]
Thus we have \( \lim_{n \to +\infty} (X_n + Y_n) = 2H(\rho) \). Since \( \lim_{n \to +\infty} X_n = I(\rho, \Phi) \) and \( \lim_{n \to +\infty} Y_n = I(\rho, \tilde{\Phi}) \), the assertion of the theorem is proved for finite rank states. Since the left and the right sides of relation (22) are concave lower semicontinuous nonnegative functions (by Proposition 1), validity of this relation for all states follows from lemma 6 in [17], stated that any concave lower semicontinuous lower bounded function on the set of quantum states is uniquely determined by its restriction to the set of finite rank states. 

5 Coherent information

Since in the infinite dimensional case the right side in definition (8) of the coherent information \( I_c(\rho, \Phi) \) may not be defined even for the state \( \rho \) with finite entropy while the results of Section 4 show finiteness of the mutual information \( I(\rho, \Phi) \) for any such state \( \rho \) and any channel \( \Phi \), it seems natural to use the following definition of the coherent information for an infinite dimensional quantum channel.

**Definition 5.** Let \( \Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B) \) be a quantum channel and \( \rho \) be a state in \( \mathcal{S}(\mathcal{H}_A) \) with finite entropy. The **coherent information** of the channel \( \Phi \) at the state \( \rho \) is defined as follows
\[
I_c(\rho, \Phi) = I(\rho, \Phi) - H(\rho).
\]

In the case \( H(\rho) < +\infty \) and \( H(\Phi(\rho)) < +\infty \) this definition is consistent with the conventional one, since \( I(\rho, \Phi) = H(\rho) + H(\Phi(\rho)) - H(\tilde{\Phi}(\rho)) \) and
hence
\[ I_c(\rho, \Phi) = H(\Phi(\rho)) - H(\tilde{\Phi}(\rho)). \] (23)

The above-defined value inherits properties 2, 3 of the mutual information (see Proposition 1). Theorem 1 implies the inequalities
\[ -H(\rho) \leq I_c(\rho, \Phi) \leq H(\rho) \]
and a generalization of identity (10) to the infinite dimensional case.

**Corollary 1.** Let \( \Phi : \mathcal{G}(H_A) \to \mathcal{G}(H_B) \) be a quantum channel and \( \tilde{\Phi} : \mathcal{G}(H_A) \to \mathcal{G}(H_E) \) be the channel complementary to the channel \( \Phi \). For an arbitrary state \( \rho \in \mathcal{G}(H_A) \) with finite entropy the following relation holds:
\[ I_c(\rho, \Phi) + I_c(\rho, \tilde{\Phi}) = 0. \] (24)

**Remark 2.** An alternative expression for the coherent information of the channel \( \Phi \) at the state \( \rho \) with finite entropy can be obtained by using the relation of this quantity with the secret classical capacity of a channel mentioned in [16]. Consider the \( \chi \)-function of the channel \( \Phi \) defined as follows
\[ \chi_\Phi(\rho) = \sup \sum \pi_i H(\Phi(\rho_i)\|\Phi(\rho)), \quad \rho \in \mathcal{G}(H), \]
where the supremum is taken over all convex decompositions \( \rho = \sum_i \pi_i \rho_i, \rho_i \in \mathcal{G}(H) \). This function is closely connected to the classical capacity of the channel \( \Phi \) (cf. [8]). If \( H(\Phi(\rho)) < +\infty \) then \( \chi_\Phi(\rho) = H(\Phi(\rho)) - \co H_\Phi(\rho) \), where \( \co H_\Phi(\rho) \) is the convex closure of the output entropy of the channel \( \Phi \) (cf. [10]). Since \( \co H_\Phi \equiv \co H_{\tilde{\Phi}} \) and \( |H(\Phi(\rho)) - H(\tilde{\Phi}(\rho))| \leq H(\rho) \) by the triangle inequality [13], for an arbitrary state \( \rho \) such that \( H(\rho) < +\infty \) and \( H(\Phi(\rho)) < +\infty \) we have
\[ I_c(\rho, \Phi) = \chi_\Phi(\rho) - \chi_{\tilde{\Phi}}(\rho). \] (25)

Since \( \max\{\chi_\Phi(\rho), \chi_{\tilde{\Phi}}(\rho)\} \leq H(\rho) \) by monotonicity of the relative entropy, the right side in (25) is a correctly defined value in \([-H(\rho), H(\rho)]\) under the single condition \( H(\rho) < +\infty \) (for arbitrary value of \( H(\Phi(\rho)) \)), which can be used for definition of \( I_c(\rho, \Phi) \).

**Definition** of the coherent information coincides with the definition given by (25) for all states with finite entropy. This assertion is proved in Example 2 after the below Proposition 5. \( \square \)
In finite dimensions the equality $H(\rho) = I_c(\rho, \Phi)$ is a necessary and sufficient condition of perfect reversibility of the channel $\Phi$ on the state $\rho$ (see [13, Theorem 12.10]). Generalize this to the infinite dimensional case.

**Definition 6.** A channel $\Phi$ is called *perfectly reversible on a state* $\rho$ in $\mathcal{S}(\mathcal{H}_A)$ if there exists a channel $D: \mathcal{S}(\mathcal{H}_B) \to \mathcal{S}(\mathcal{H}_A)$ such that

$$D \circ \Phi(\tilde{\rho}) = \tilde{\rho}$$

for all states $\tilde{\rho}$ with $\text{supp} \tilde{\rho} \subset \mathcal{L} \equiv \text{supp} \rho$.

In other words the subspace $\mathcal{L}$ is a quantum code correcting errors of the channel $\Phi$ [13]. Introduce the reference system $\mathcal{H}_R$ and consider a purification $\rho_{AR} = |\phi_{AR}\rangle \langle \phi_{AR}| \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_R)$ of the state $\rho$.

**Lemma 5.** A channel $\Phi$ is perfectly reversible on a state $\rho \in \mathcal{S}(\mathcal{H}_A)$ if and only if there exists a channel $D: \mathcal{S}(\mathcal{H}_B) \to \mathcal{S}(\mathcal{H}_A)$ such that

$$(D \circ \Phi \otimes \text{Id}_R)(\rho_{AR}) = \rho_{AR}.$$  \hfill (26)

The proof of this lemma is presented in the Appendix.

**Proposition 2.** Let $H(\rho) < \infty$. A channel $\Phi: \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ is perfectly reversible on the state $\rho$ if and only if one of following equivalent conditions holds: $I_c(\rho, \Phi) = H(\rho)$; $I(\rho, \tilde{\Phi}) = 0$.

**Proof.** By Theorem 1 and Definition [5] we have

$$H(\rho) - I_c(\rho, \Phi) = I(\rho, \tilde{\Phi}) \geq 0,$$

where the equality holds if and only if $\rho_{RE} = \rho_R \otimes \rho_E$, since $I(\rho, \tilde{\Phi}) = H(\rho_{RE}||\rho_R \otimes \rho_E)$. The following part of the proof is similar to the proof presented in [8] and we give it here for completeness.

Necessity. Let $V: \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ be the isometry from representation [2] of the channel $\Phi$. Consider the pure state $\rho_{BRE} = |\varphi_{BRE}\rangle \langle \varphi_{BRE}|$, where $|\varphi_{BRE}\rangle = (V \otimes I_R)|\varphi_{AR}\rangle$. Since the channel $\Phi$ is perfectly reversible, there exists a channel $D$ such that (26) holds and hence

$$(D \otimes \text{Id}_{RE})(\rho_{BRE}) = \rho_{ARE}.$$  \hfill (26)

Since $\rho_{AR}$ is a pure state, we have $\rho_{ARE} = \rho_{AR} \otimes \rho_E$. By taking partial traces over the space $\mathcal{H}_A$, we obtain $\rho_{RE} = \rho_R \otimes \rho_E$. 

17
Sufficiency. Consider the vector \( |\varphi_{BRE} \rangle = (V \otimes I_R) |\varphi_{AR} \rangle \). Then \( |\varphi_{BRE} \rangle \) is a purification vector for the state \( \rho_{RE} \). Since \( \rho_{RE} = \rho_R \otimes \rho_E, |\varphi_{AR} \rangle \otimes |\varphi_{EE'} \rangle \) is a purification vector for the state \( \rho_{RE} \), where \( E' \) is a reference system for the system \( E \).

Without loss of generality we can assume that the Hilbert spaces of both purifications are infinite dimensional, so that there exists an isometry \( W : \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_{E'} \) such that

\[
(I_{RE} \otimes W) |\varphi_{BRE} \rangle = |\varphi_{AR} \rangle \otimes |\varphi_{EE'} \rangle,
\]

and respectively

\[
(I_{RE} \otimes W) |\varphi_{BRE} \rangle \langle \varphi_{BRE} | (I_{RE} \otimes W^*) = |\varphi_{AR} \rangle \langle \varphi_{AR} | \otimes |\varphi_{EE'} \rangle \langle \varphi_{EE'} |.
\]

By taking partial traces over the spaces \( \mathcal{H}_E \) and \( \mathcal{H}_{E'} \), we obtain perfect reversibility condition (26), where

\[
D(\sigma) = \text{Tr}_{E'} W \sigma W^*, \quad \sigma \in \mathfrak{S}(\mathcal{H}_B).
\]

As mentioned before, the entropy \( H(\rho) \) of a state \( \rho \) is the upper bound for the coherent information \( I_c(\rho, \Phi) \) of an arbitrary channel \( \Phi \) at this state. The following proposition gives the more precise upper bound for \( I_c(\rho, \Phi) \), expressed via the Kraus operators of the channel \( \Phi \).

**Proposition 3.** Let \( \Phi(\cdot) = \sum_{i=1}^{+\infty} V_i(\cdot)V_i^* \) be a quantum channel. Then for an arbitrary state \( \rho \) with finite entropy the following inequality holds

\[
I_c(\rho, \Phi) \leq \sum_{i=1}^{+\infty} H(V_i \rho V_i^*) = \sum_{i=1}^{+\infty} \text{Tr}V_i \rho V_i^* H\left( \frac{V_i \rho V_i^*}{\text{Tr}V_i \rho V_i^*} \right). \tag{27}
\]

The equality holds in (27) if \( \text{Ran} V_i \perp \text{Ran} V_j \) for all \( i \neq j \).

The expression in the right side of (27) can be considered as the mean entropy of a posteriori state in quantum measurement, described by the collection of operators \( \{V_i\}_{i=1}^{+\infty} \), at a priori state \( \rho \) \cite{8,13}. By the Groenevold-Lindblad-Ozawa inequality this value does not exceed \( H(\rho) \) \cite{15}.

**Proof.** Show first that the equality holds in (27) if \( \text{Ran} V_i \perp \text{Ran} V_j \) for all \( i \neq j \).
Let $\rho$ be a state in $\mathcal{S}(\mathcal{H}_A)$ with finite entropy and $|\varphi\rangle\langle\varphi|$ be its purification in $\mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_R)$. By using the well known properties of the relative entropy (see [12]) and by noting that $\sum_{i=1}^{+\infty} V_i^* V_i = I_A$ we obtain

$$I(\rho, \Phi) = H(\Phi \otimes \text{Id}_R(|\varphi\rangle\langle\varphi|)\| \Phi \otimes \text{Id}_R(\rho \otimes \rho))$$

$$= H\left(\sum_{i=1}^{+\infty} V_i \otimes I_R |\varphi\rangle \langle V_i^* \otimes I_R \right) - H(\sum_{i=1}^{+\infty} (V_i \otimes I_R)(\rho \otimes \rho)(V_i^* \otimes I_R))$$

$$= H(\rho) + \sum_{i=1}^{+\infty} [S(V_i \rho V_i^*) - \eta(\text{Tr}V_i \rho V_i^*)] = H(\rho) + \sum_{i=1}^{+\infty} H(V_i \rho V_i^*).$$

Let $\Phi(\cdot) = \sum_{i=1}^{+\infty} V_i(\cdot)V_i^*$ be an arbitrary channel and $\mathcal{H}_C = \bigoplus_{i=1}^{+\infty} \mathcal{H}_B^i$, where $\mathcal{H}_B^i \cong \mathcal{H}_B$. Let $U_i$ be an isometrical embedding of $\mathcal{H}_B$ in $\mathcal{H}_C$ such that $U_i \mathcal{H}_B = \mathcal{H}_B^i$ for each $i$.

As proved before, for the quantum channel $\hat{\Phi}(\cdot) = \sum_{i=1}^{+\infty} U_i V_i(\cdot)V_i^* U_i^*$ the following equality holds

$$I_c(\rho, \hat{\Phi}) = \sum_{i=1}^{+\infty} H(U_i V_i \rho V_i^* U_i^*) = \sum_{i=1}^{+\infty} H(V_i \rho V_i^*),$$

By applying the 1-th chain rule for the coherent information to the composition $\Psi \circ \hat{\Phi} = \Phi$, where $\Psi(\cdot) = \sum_{i=1}^{+\infty} U_i^*(\cdot)U_i$ is a channel from $\mathcal{S}(\mathcal{H}_C)$ to $\mathcal{S}(\mathcal{H}_B)$, we obtain (27) from the above equality.

6 On continuity of mutual and coherent informations

Proposition [11] and Theorem 1 provide the following continuity condition for mutual and coherent informations.

**Proposition 4.** For an arbitrary quantum channel $\Phi: \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ the functions

$$\rho \mapsto I(\rho, \Phi) \quad \text{and} \quad \rho \mapsto I_c(\rho, \Phi)$$

are continuous on any subset $A \subset \mathcal{S}(\mathcal{H}_A)$, on which the von Neumann entropy is continuous.
Proof. By Proposition 4 the functions $\rho \mapsto I(\rho, \Phi)$ and $\rho \mapsto I(\rho, \tilde{\Phi})$ are lower semicontinuous while by Theorem 1 their sum coincides with the double von Neumann entropy, which is continuous on the set $A$ by the condition. Hence these functions are continuous on the set $A$. The function $\rho \mapsto I_c(\rho, \Phi)$ is continuous on the set $A$ as a difference between two functions continuous on this set.

**Example 1.** Let $H$ be a Hamiltonian of quantum system $A$. Then the subset $K_{H,h}$ of $S(H_A)$, consisting of states $\rho$ such that $\text{Tr}H \rho \leq h$, can be treated as a set of states with the mean energy not exceeding $h$. If the operator $H$ is such that $\text{Tr} e^{-\lambda H} < +\infty$ for all $\lambda > 0$ then the von Neumann entropy is continuous on $K_{H,h}$ [14, 18]. This holds, for example, for the Hamiltonian of the system of quantum oscillators [18]. By Proposition 4 for an arbitrary quantum channel $\Phi$ the mutual information $I(\rho, \Phi)$ and the coherent information $I_c(\rho, \Phi)$ are continuous functions of a state $\rho \in K_{H,h}$ for each finite $h > 0$. Hence these functions are bounded and achieve their supremum on the set $K_{H,h}$ (by compactness of this set [9]).

**Remark 3.** Proposition 4 and identity (22) show that for an arbitrary channel $\Phi$ the function $\rho \mapsto I(\rho, \Phi)$ is continuous and bounded on the set $S_k(H_A) = \{\rho \in S(H_A) \mid \text{rank} \rho \leq k\}$ for each $k = 1, 2, ...$ Hence the properties of the function $\rho \mapsto I(\rho, \Phi)$ can be explored by using the approximation method considered in [17, Section 4]. This method makes it possible to clarify the sense of the continuity condition for the function $\rho \mapsto I(\rho, \Phi)$ in Proposition 4 and to show its necessity for the particular class of channels.

By Proposition 3 in [17] the function $\rho \mapsto I(\rho, \Phi)$ is a pointwise limit of the increasing sequence of concave continuous on the set $S(H_A)$ functions $\rho \mapsto I_k(\rho, \Phi) = \sup_{\{\pi_i, \rho_i\} \in \mathcal{E}_\rho^k} \sum_i \pi_i I(\rho_i, \Phi)$, \hfill (28)

where $\mathcal{E}_\rho^k$ is the set of all ensembles $\{\pi_i, \rho_i\}$ such that $\sum_i \pi_i \rho_i = \rho$ and $\text{rank} \rho_i \leq k$ for all $i$. Hence a necessary and sufficient condition of continuity of the function $\rho \mapsto I(\rho, \Phi)$ on a set $A \subset S(H_A)$ can be expressed as follows

$$\lim_{k \to +\infty} \sup_{\rho \in A_c} \Delta_k^\rho(\rho, \Phi) = 0 \quad \text{for any compact set } A_c \subseteq A,$$

(29)

---

4 An ensemble $\{\pi_i, \rho_i\}$ is a collection of states $\{\rho_i\}$ with the corresponding probability distribution $\{\pi_i\}$. 

---
where \( \Delta^I_k(\rho, \Phi) = I(\rho, \Phi) - I_k(\rho, \Phi) \). It is possible to show that

\[
\Delta^I_k(\rho, \Phi) = \inf_{\{\pi, \rho_i\} \in \mathcal{E}^k} \sum_i \pi_i \left[ H(\rho_i \| \rho) + H(\Phi(\rho_i) \| \Phi(\rho)) - H(\tilde{\Phi}(\rho_i) \| \tilde{\Phi}(\rho)) \right]
\]

for any state \( \rho \) with finite entropy. By monotonicity and nonnegativity of the relative entropy the expression in the square brackets does not exceed \( 2H(\rho_i \| \rho) \). Thus (29) holds if

\[
\lim_{k \to +\infty} \sup_{\rho \in \mathcal{A}_c} \inf_{\{\pi, \rho_i\} \in \mathcal{E}_k} \sum_i \pi_i H(\rho_i \| \rho) = 0 \quad \text{for any compact set} \quad \mathcal{A}_c \subseteq \mathcal{A},
\]

which is equivalent to continuity of the entropy on the set \( \mathcal{A} \), since it means uniform convergence of the sequence \( \{H_k\} \) of continuous approximators of the entropy (defined by formula (28) with \( H(\rho_i) \) instead of \( I(\rho_i, \Phi) \) in the right side) on compact subsets of \( \mathcal{A} \) [17].

Thus the assertion of Proposition 4 is explained by the implication

\[
H_k(\rho) \xrightarrow{\mathcal{A}} H(\rho) < +\infty \Rightarrow I_k(\rho, \Phi) \xrightarrow{\mathcal{A}} I(\rho, \Phi) < +\infty \quad \forall \mathcal{A} \subset \mathcal{G}(\mathcal{H}_A) \quad (30)
\]

valid for any channel \( \Phi \) by monotonicity of the relative entropy.

If \( \Phi \) is a degradable channel, that is \( \tilde{\Phi} = \Lambda \circ \Phi \) for some channel \( \Lambda \), then \( I(\rho, \Phi) < +\infty \Rightarrow H(\rho) < +\infty \) by Theorem 1 and the 1-th chain rule from Proposition 1 while the expression in the square brackets in the above formula for \( \Delta^I_k(\rho, \Phi) \) is not less than \( H(\rho_i \| \rho) \) by monotonicity of the relative entropy. Thus for degradable channel \( \Phi \) ” \( \Leftrightarrow \) ” holds in (30) and hence the continuity condition for the function \( \rho \mapsto I(\rho, \Phi) \) in Proposition 4 is necessary and sufficient:

\[
\lim_{n \to +\infty} H(\rho_n) = H(\rho_0) < +\infty \quad \Leftrightarrow \quad \lim_{n \to +\infty} I(\rho_n, \Phi) = I(\rho_0, \Phi) < +\infty
\]

for any sequence \( \{\rho_n\} \) converging to a state \( \rho_0 \).

To explore continuity of capacities as functions of a channel it is necessary to consider the corresponding entropic characteristics as functions of a pair (input state, channel), that is as functions on the Cartesian product of the set of all input states \( \mathcal{G}(\mathcal{H}_A) \) and the set of all channels \( \mathcal{F}(A, B) \) from \( A \) to \( B \) endowed with the appropriate (sufficiently weak) topology. As shown in [10], for this purpose it is reasonable to use the strong convergence topology on the set \( \mathcal{F}(A, B) \) described before Proposition 1 in Section 3. By using the
Proposition 5 implies a complementary pair for each $n$.

$$\lim_{n \to +\infty} I(\rho_n, \Phi_n) = I(\rho_0, \Phi_0) \quad \text{and} \quad \lim_{n \to +\infty} I_c(\rho_n, \Phi_n) = I_c(\rho_0, \Phi_0) \quad (31)$$

hold for any sequence $\{\rho_n\}$ of states in $\mathcal{S}(\mathcal{H}_A)$ converging to the state $\rho_0$ such that $\lim_{n \to +\infty} H(\rho_n) = H(\rho_0) < +\infty$.

Example 2. By using Proposition 5 one can prove representation (25) for any state $\rho$ with finite entropy as follows. Let $\Phi(\cdot) = \sum_{i=1}^{+\infty} V_i(\cdot)V_i^*$. Consider the sequence of channels $\Phi_n(\cdot) = \sum_{i=1}^{n} V_i(\cdot)V_i^* + W_n(\cdot)W_n$, where $W_n = \sqrt{I_A - \sum_{i=1}^{n} V_i^*V_i}$. By noting that the sequence $\{W_n\}$ converges to the zero operator in the strong operator topology it is easy to show that the sequences $\{\Phi_n\}$ and $\{\tilde{\Phi}_n\}$ strongly converges to the channels $\Phi$ and $\Phi$.

Let $\Psi(\cdot) = \sum_{i=1}^{n} V_i(\cdot)V_i^*$ and $\Theta_n(\cdot) = W_n(\cdot)W_n$ be quantum operations. By Proposition 6B in [10] we have $\lim_{n \to +\infty} \chi_{\Psi_n}(\rho) = \chi_{\Phi}(\rho)$ while Corollary 8 in [17] implies $\lim_{n \to +\infty} \chi_{\Theta_n}(\rho) = \lim_{n \to +\infty} H(W_n\rho W_n) = 0$. By using Corollary 3 in [10] we conclude that $\lim_{n \to +\infty} \chi_{\Psi_n}(\rho) = \chi_{\Phi}(\rho)$. Since $H(\rho) < +\infty$ implies $H(\Phi_n(\rho)) < +\infty$ for all $n$ we have (see Remark 2)

$$I_c(\rho, \Phi_n) = \chi_{\Psi_n}(\rho) - \chi_{\tilde{\Phi}_n}(\rho).$$

By the above observations and Corollary 3 in [10] passing to the limit in this equality leads to the inequality

$$I_c(\rho, \Phi) \leq \chi_{\Phi}(\rho) - \chi_{\tilde{\Phi}}(\rho).$$

By using the same approximation for the channel $\tilde{\Phi}$ instead of $\Phi$ and repeating the above arguments we obtain the converse inequality. □

Let $\mathcal{W}_1(A, B)$ be the set of all sequences $\{V_i\}_{i=1}^{+\infty}$ of operators from $\mathcal{H}_A$ into $\mathcal{H}_B$ such that $\sum_{i=1}^{+\infty} V_i^*V_i = I_A$ endowed with the topology of coordinate-wise strong operator convergence.
Corollary 2. For an arbitrary subset \( A \subset \mathcal{S}(\mathcal{H}_A) \), on which the von Neumann entropy is continuous, the functions

\[
(\rho, \mathcal{V}) \mapsto I(\rho, \Phi[\mathcal{V}]), \quad (\rho, \mathcal{V}) \mapsto I_c(\rho, \Phi[\mathcal{V}]), \quad (\rho, \mathcal{V}) \mapsto \sum_{i=1}^{+\infty} H(V_i \rho V_i^*),
\]

where \( \Phi[\mathcal{V}](\cdot) = \sum_{i=1}^{+\infty} V_i(\cdot)V_i^* \), are continuous on the set \( A \times \mathcal{V}_1(A, B) \).

Proof. By Proposition 5 continuity of the first two functions follows from continuity of the maps

\[
\mathcal{V}_1(A, B) \ni \mathcal{V} \mapsto \Phi[\mathcal{V}] \in \mathcal{F}(A, B) \quad \text{and} \quad \mathcal{V}_1(A, B) \ni \mathcal{V} \mapsto \tilde{\Phi}[\mathcal{V}] \in \mathcal{F}(A, E),
\]

where \( \tilde{\Phi}[\mathcal{V}](\cdot) = \sum_{i,j=1}^{+\infty} [\operatorname{Tr} V_i(\cdot)V_j^*]|h_i\rangle\langle h_j| \) and \( \{|h_i\rangle\} \) is a particular orthonormal basis in \( \mathcal{H}_E \).

To prove continuity of the above maps it suffices to show that

\[ \lim_{n \to +\infty} \Phi[\mathcal{V}_n](|\varphi\rangle\langle \varphi|) = \Phi[\mathcal{V}_0](|\varphi\rangle\langle \varphi|) \] (32)

and

\[ \lim_{n \to +\infty} \tilde{\Phi}[\mathcal{V}_n](|\varphi\rangle\langle \varphi|) = \tilde{\Phi}[\mathcal{V}_0](|\varphi\rangle\langle \varphi|) \] (33)

for any sequence \( \{\mathcal{V}_n\} \subset \mathcal{V}_1(A, B) \), converging to a vector \( \mathcal{V}_0 \in \mathcal{V}_1(A, B) \), and for any unit vector \( \varphi \in \mathcal{H}_A \).

Let \( \mathcal{V}_n = \{V^n_i\}_{i=1}^{+\infty} \) for each \( n \geq 0 \). Relation (32) can be proved by noting that the condition \( \sum_{i=1}^{+\infty} \|V^n_i|\varphi\rangle\| = 1 \) for all \( n \geq 0 \) implies

\[ \lim_{m \to +\infty} \sup_{n \geq 0} \| \sum_{i=1}^{+\infty} V^n_i |\varphi\rangle\langle \varphi| V^n_i \| = 0. \]

Relation (33) is easily proved by using the result from [4], mentioned before Proposition 1.

To prove continuity of the third function consider the following construction. Let \( \mathcal{H}_C = \bigoplus_{i=1}^{+\infty} \mathcal{H}_B \), where \( \mathcal{H}_B \cong \mathcal{H}_B \), and \( U_i \) be an isometrical embedding of \( \mathcal{H}_B \) into \( \mathcal{H}_C \) such that \( U_i\mathcal{H}_B = \mathcal{H}_B^i \) for each \( i \).

For an arbitrary sequence \( \{V_i\}_{i=1}^{+\infty} \) in \( \mathcal{V}_1(A, B) \) one can take the sequence \( \{\hat{V}_i = U_i V_i\}_{i=1}^{+\infty} \) in \( \mathcal{V}_1(A, C) \) such that \( \operatorname{Ran} \hat{V}_i \perp \operatorname{Ran} \hat{V}_j \) for all \( i \neq j \). Since the above correspondence is continuous (as a map from \( \mathcal{V}_1(A, B) \) into
the above observation shows continuity on the set $A \times \mathfrak{V}_1(A, B)$ of the function

$$(\rho, \nabla) \mapsto I_c(\rho, \hat{\Phi}[\nabla]) = \sum_{i=1}^{+\infty} H(\hat{V}_i \rho \hat{V}_i^*) = \sum_{i=1}^{+\infty} H(V_i \rho V_i^*),$$

where $\hat{\Phi}[\nabla](\cdot) = \sum_{i=1}^{+\infty} \hat{V}_i(\cdot) \hat{V}_i^*$ and the first equality follows from the last assertion of Proposition 3.

As mentioned in Section 5, the value $\sum_{i=1}^{+\infty} H(V_i \rho V_i^*)$ can be considered as the mean entropy of a posteriori state in the quantum measurement, described by the collection of operators $\{V_i\}_{i=1}^{+\infty}$. Corollary 2 shows that continuity of the entropy $H(\rho)$ of a priori state $\rho$ implies continuity of the mean entropy of a posteriori state as a function of a pair (a priori state, measurement) provided the strong operator topology is used in the definition of convergence of a sequence of measurements. This assertion strengthens the analogous assertion in Example 3 in [17], in which the stronger topology (so called the *-strong operator topology) is used in definition of convergence of a sequence of measurements. Hence by means of Corollary 2 one can strengthen all the assertions in Example 3 in [17] by inserting the strong operator topology in definition of convergence of a sequence of measurements, which seems more natural in this context.

7 Appendix

Lemma 6. Let $\rho$ and $\sigma$ be states in $\mathfrak{S}(\mathcal{H})$ and $C$ be an operator in $\mathfrak{S}_+(\mathcal{H})$. Then

$$H(\lambda \rho + (1 - \lambda)\sigma \| C) \geq \lambda H(\rho \| C) + (1 - \lambda)H(\sigma \| C) - h_2(\lambda), \quad \forall \lambda \in [0, 1],$$

where $h_2(\lambda) = \eta(\lambda) + \eta(1 - \lambda)$.

Proof. Let $\{P_n\}$ be an increasing sequence of finite rank projectors strongly converging to the identity operator. Then $A_n = P_n \rho P_n$, $B_n = P_n \sigma P_n$ and

\footnote{Note that this stronger version can not be proved by means of the method used in [17].}
$C_n = P_n C P_n$ are finite rank operators for each $n$ and hence

\[
H(\lambda A_n + (1 - \lambda) B_n \| C_n) = \operatorname{Tr}(\lambda A_n + (1 - \lambda) B_n)(-\log C_n) \\
- S(\lambda A_n + (1 - \lambda) B_n) + \operatorname{Tr} C_n - \operatorname{Tr}(\lambda A_n + (1 - \lambda) B_n) \\
\geq \lambda \operatorname{Tr} A_n ( - \log C_n ) + (1 - \lambda) \operatorname{Tr} B_n ( - \log C_n ) + \operatorname{Tr} C_n - \lambda \operatorname{Tr} A_n - (1 - \lambda) \operatorname{Tr} B_n \\
- \lambda S(A_n) - (1 - \lambda) S(B_n) - \eta(\operatorname{Tr}(\lambda A_n + (1 - \lambda) B_n)) + \lambda \eta(\operatorname{Tr} A_n) \\
+ (1 - \lambda) \eta(\operatorname{Tr} B_n) - x_n h_2(x_n^{-1} \lambda \operatorname{Tr} A_n) = \lambda H(A_n \| C_n) + (1 - \lambda) H(B_n \| C_n) \\
- \eta(\operatorname{Tr}(\lambda A_n + (1 - \lambda) B_n)) + \lambda \eta(\operatorname{Tr} A_n) + (1 - \lambda) \eta(\operatorname{Tr} B_n) - x_n h_2(x_n^{-1} \lambda \operatorname{Tr} A_n),
\]

where $x_n = \operatorname{Tr}(\lambda A_n + (1 - \lambda) B_n)$ and the inequality

\[
H(\lambda A_n + (1 - \lambda) B_n) \leq \lambda H(A_n) + (1 - \lambda) H(B_n) + x_n h_2(x_n^{-1} \lambda \operatorname{Tr} A_n),
\]

following from (12) and (13) was used. By Lemma 1 passing to the limit $n \to +\infty$ implies the desired inequality. \hfill \Box

**Lemma 7.** Let $\{A_n\}$ be a sequence of operators in $\Sigma_+(\mathcal{H})$ converging in the trace norm to an operator $A_0$ such that $A_n \leq A_0$ for all $n$. Then

\[
\lim_{n \to +\infty} H(A_n \| B) = H(A_0 \| B) \quad \text{for any operator } B \in \Sigma_+(\mathcal{H}).
\]

**Proof.** We can assume that $A_0$ is a state. It can be represented as follows

\[
A_0 = \lambda_n \rho_n + (1 - \lambda_n) \sigma_n,
\]

where

\[
\lambda_n = \operatorname{Tr} A_n, \quad \rho_n = \frac{A_n}{\operatorname{Tr} A_n}, \quad \sigma_n = \frac{A - A_n}{1 - \lambda_n}.
\]

By Lemma 6 and nonnegativity of the relative entropy we have

\[
H(A_0 \| B) \geq \lambda_n H(\rho_n \| B) + (1 - \lambda_n) H(\sigma_n \| B) - h_2(\lambda_n) \\
\geq H(A_n \| \lambda_n B) - h_2(\lambda_n) \\
= H(A_n \| B) - \operatorname{Tr} B(1 - \lambda_n) - \lambda_n \log(\lambda_n) - h_2(\lambda_n),
\]

and hence $\limsup_{n \to +\infty} H(A_n \| B) \leq H(A_0 \| B)$. By lower semicontinuity of the relative entropy this implies the assertion of the lemma. \hfill \Box
Proof of Lemma 5. Let $T = D \circ \Phi$. Consider the set of conditions

$$T(|\psi\rangle \langle \psi|) = |\psi\rangle \langle \psi|, \quad \forall |\psi\rangle \in \text{supp } \rho,$$

(34)

$$T(|\psi\rangle \langle \phi|) = |\psi\rangle \langle \phi|, \quad \forall |\psi\rangle, |\phi\rangle \in \text{supp } \rho,$$

(35)

$$T(|e_i\rangle \langle e_j|) = |e_i\rangle \langle e_j|, \quad \forall i, j,$$

(36)

where $|e_i\rangle$ is the set of eigenvectors of the state $\rho$ corresponding to nonzero eigenvalues. Then Definition 6 $\iff$ (34) follows from the spectral representation, (34) $\iff$ (35) follows from the polarization identity, (35) $\iff$ (36) is obvious, (36) $\iff$ (26) follows from formula (15)).

The authors are grateful to the participants of the seminar ”Quantum probability, statistics, information” (MIAN) for the useful discussion. The authors are also grateful to A.A. Kuznetsova for the discussion and the help in preparing the manuscript.

References

[1] Adami, C., Cerf, N.J., "Von Neumann Capacity of Noisy Quantum Channels", Phys. Rev. A, 1997, vol. 56, no. 5, pp. 3470–3483; arXiv: quant-ph/9609024.

[2] Barnum, H., Nielsen, M.A., Schumacher, B.W., "Information Transmission through a Noisy Quantum Channel", Phys. Rev. A, 1998, vol. 57, no. 6, pp. 4153-4175.

[3] Bennett, C.H., Shor, P.W., Smolin, J.A., Thapliyal, A.V., "Entanglement-Assisted Capacity of a Quantum Channel and the Reverse Shannon Theorem", IEEE Trans. Inform. Theory, 2002, vol. 48, no. 10, pp. 2637–2655; arXiv: quant-ph/0106052.

[4] Davies, E.B., Quantum Theory of Open Systems, London: Academic, 1976.

[5] Devetak, I., "The Private Classical Capacity and Quantum Capacity of a Quantum Channel", IEEE Trans. Inform. Theory, 2005, vol. 51, no. 1, pp. 44–55; arXiv: quant-ph/0304127.

[6] Eisert J., Wolf M.M., "Gaussian quantum channels", arXiv:quant-ph/0505151.
[7] Holevo, A.S., ”Complementary Channels and the Additivity Problem”, *Theory Probab. Appl.*, 2007, vol. 51, no. 1, pp. 92–100; arXiv:quant-ph/0509101.

[8] Holevo, A.S., *Kvantovye sistemy, kanaly, informatsiya* (Quantum Systems, Channels, and Information), Moscow: MCCME, 2010.

[9] Holevo, A.S., Shirokov, M.E., ”Continuous Ensembles and the Capacity of Infinite-Dimensional Quantum Channels”, *Theory Probab. Appl.*, 2006, vol. 50, no. 1, pp. 86–98; arXiv:quant-ph/0408176.

[10] Holevo, A.S., Shirokov, M.E., ”On Approximation of Infinite-Dimensional Quantum Channels”, *Probl. Inf. Trans.*, 2008, vol. 44, no. 2, pp. 73–90; arXiv:0711.2245 [quant-ph].

[11] King C., Matsumoto K., Nathanson M., Ruskai M.B., ”Properties of Conjugate Channels with Applications to Additivity and Multiplicativity”, arXiv:quant-ph/0509126.

[12] Lindblad, G., ”Expectations and Entropy Inequalities for Finite Quantum Systems”, *Comm. Math. Phys.*, 1974, vol. 39, no. 2, pp. 111–119.

[13] Nielsen, M.A., Chuang, I.L., *Quantum Computation and Quantum Information*, Cambridge: Cambridge Univ. Press, 2000.

[14] Ohya, M., Petz, D., *Quantum Entropy and Its Use*, Berlin: Springer, 2004, 2nd ed.

[15] Ozawa, M., ”On Information Gain by Quantum Measurements of Continuous Observables”, *J. Math. Phys.*, 1986, vol. 27, no. 3, pp. 759–763.

[16] Schumacher, B., Westmoreland, M.D., ”Quantum Privacy and Quantum Coherence”, *Phys. Rev. Lett.*, 1998, vol. 80, no. 25, pp. 5695–5697.

[17] Shirokov, M.E., ”Continuity of the von Neumann Entropy”, *Commun. Math. Phys.*, 2010, vol. 296, no. 3, pp. 625–654.

[18] Wehrl, A., ”General Properties of Entropy”, *Rev. Modern Phys.*, 1978, vol. 50, no. 2, pp. 221–260.