SEQUENCES OF GLOBALLY REGULAR AND BLACK HOLE SOLUTIONS IN
SU(4) EINSTEIN-YANG-MILLS THEORY

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SU(4) Einstein-Yang-Mills theory possesses sequences of static spherically symmetric globally regular and black hole solutions. Considering solutions with a purely magnetic gauge field, based on the 4-dimensional embedding of $su(2)$ in $su(4)$, these solutions are labelled by the node numbers $(n_1, n_2, n_3)$ of the three gauge field functions $u_1$, $u_2$ and $u_3$. We classify the various types of solutions in sequences and determine their limiting solutions. The limiting solutions of the sequences of neutral solutions carry charge, and the limiting solutions of the sequences of charged solutions carry higher charge. For sequences of black hole solutions with node structure $(n, j, n)$ and $(n, n, n)$, several distinct branches of solutions exist up to critical values of the horizon radius. We determine the critical behaviour for these sequences of solutions. We also consider SU(4) Einstein-Yang-Mills-dilaton theory and show that these sequences of solutions are analogous in most respects to the corresponding SU(4) Einstein-Yang-Mills sequences of solutions.
1. INTRODUCTION

SU(2) Einstein-Yang-Mills (EYM) theory possesses a sequence of asymptotically flat static spherically symmetric solutions, which are globally regular \( \Box \). Based on a purely magnetic gauge field ansatz, these solutions carry no charge. They are characterized by the node number \( n \) of the single magnetic gauge field function \( u \). With increasing node number \( n \), the sequence of neutral solutions tends to a limiting solution with magnetic charge \( P = 1 \), which consists of two parts, an inner oscillating part and an outer Reissner-Nordstrøm (RN) part \( \Box \).

Besides the globally regular solutions, SU(2) EYM theory possesses static spherically symmetric black hole solutions with non-trivial non-abelian gauge field configurations outside their regular event horizon \( \Box \). For any value of the event horizon radius \( x_H \), there exists a corresponding sequence of black hole solutions, characterized by the node number \( n \) of the gauge field function. For fixed horizon radius \( x_H \) and increasing \( n \), the sequence of neutral black hole solutions also tends to a limiting charged solution with magnetic charge \( P = 1 \) \( \Box \). For horizon radius \( x_H > 1 \), this limiting solution corresponds to an embedded RN solution with the same horizon radius \( x_H \). For \( 0 < x_H < 1 \), the limiting solution again consists of an oscillating part and a RN part \( \Box \). Interestingly, the only charged black hole solutions of SU(2) EYM theory are embedded RN solutions \( \Box \).

In SU(3) EYM theory, besides embedded RN solutions \( \Box \) charged static spherically symmetric black hole solutions exist, which possess non-trivial gauge field configurations outside their regular event horizon \( \Box \). These black hole solutions have magnetic charge of norm \( P = \sqrt{3} \) residing in the \( su(3) \) Cartan subalgebra \( \Box \). Like the RN solutions, the magnetically charged non-abelian black hole solutions exist only for horizon radius \( x_H \geq P \), where \( x_H = P \) leads to extremal black hole solutions \( \Box \). These charged non-abelian SU(3) black hole solutions are characterized by the node number \( n \) of a single magnetic gauge field function, similar to the neutral non-abelian SU(2) black hole solutions. With increasing \( n \) and fixed horizon radius \( x_H \), the sequence of non-abelian black hole solutions with charge \( P = \sqrt{3} \) tends to a solution with higher charge, \( P = 2 \) \( \Box \). For horizon radius \( x_H > 2 \), this limiting solution again corresponds to an embedded RN solution with the same horizon radius \( x_H \), whereas for \( \sqrt{3} < x_H < 2 \) the limiting solution again consists of an oscillating part and a RN part \( \Box \).

SU(3) EYM theory also possesses neutral globally regular and black hole solutions. The genuine neutral static spherically symmetric SU(3) EYM solutions are obtained by embedding the 3-dimensional representation of \( su(2) \) in \( su(3) \) \( \Box \). The purely magnetic gauge field ansatz then involves two functions, \( u_1 \) and \( u_2 \), and the solutions are labelled by the corresponding node numbers \( (n_1, n_2) \). For fixed horizon radius \( x_H \), a discrete set of regular neutral SU(3) EYM solutions is obtained \( \Box \). These neutral solutions form sequences with node structure \( (n, n) \) and \( (i, i + n) \), with \( i \) fixed. With increasing \( n \), these sequences of solutions again tend to limiting solutions, carrying magnetic charge of norm \( P = 2 \) and \( P = \sqrt{3} \), respectively \( \Box \). For the \( (i, i + n) \) sequences, the limiting solution of black hole solutions with horizon radius \( x_H > \sqrt{3} \) corresponds to a charged non-abelian SU(3) black hole solution with \( i \) nodes and with the same horizon radius \( x_H \), whereas for black hole solutions with horizon radius \( 0 < x_H < \sqrt{3} \) and for globally regular solutions the limiting solution again consists of two parts \( \Box \). The \( (n, n) \) sequences either represent genuine SU(3) solutions with \( u_1 \neq u_2 \) or scaled SU(2) solutions with \( u_1 = u_2 \). Their limiting solutions carry magnetic charge of norm \( P = 2 \) and represent embedded RN solutions for \( x_H > 2 \). Unlike the \( (i, i + n) \) sequences of black hole solutions, the genuine SU(3) \( (n, n) \) black hole solutions exist only for sufficiently small values of the horizon radius \( x_H \). At critical values of the horizon radius, they merge into the scaled SU(2) solutions. Therefore, of the two types of \( (n, n) \) sequences only the scaled SU(2) solutions persist for large \( x_H \) \( \Box \).

Here we investigate the sequences of SU(4) EYM solutions. The genuine static spherically symmetric SU(4) solutions are obtained by embedding the 4-dimensional representation of \( su(2) \) in \( su(4) \) \( \Box \). The purely magnetic gauge field ansatz then contains three functions, \( u_1, u_2 \) and \( u_3 \). The classification of the discrete set of regular solutions in sequences again involves their node numbers \( (n_1, n_2, n_3) \) \( \Box \). When some of the gauge field functions possess the same number of nodes, several solutions can exist for sufficiently small values of the horizon radius \( x_H \), leading to a complex critical behaviour of the solutions as functions of \( x_H \). When all three gauge field functions are non-trivial, neutral solutions are obtained, whereas when one or more gauge field functions are identically zero, charged solutions arise. We determine the limiting solutions of the sequences of globally regular and black hole solutions and specify their magnetic charge. The extension to SU(N) EYM theory is straightforward.

By including a dilaton field, all these solutions can be generalized to the solutions of Einstein-Yang-Mills-dilaton (EYMD) theory \( \Box \). The classification of the solutions of EYMD theory and EYM theory is identical. However, the charged EYMD black hole solutions exist for arbitrarily small values of the horizon radius \( x_H \), and the limiting solutions of some EYMD sequences of solutions are embedded Einstein-Maxwell-dilaton (EMD) solutions \( \Box \).

In this paper we present in section II the SU(4) EYM action and the static spherically symmetric ansatz for the metric and the purely magnetic gauge field, based on the 4-dimensional embedding of \( su(2) \) in \( su(4) \), and we determine
the boundary conditions for asymptotically flat globally regular and black hole solutions. In section III we discuss the charged SU(4) EYM black hole solutions. We classify them and construct numerically several sequences of charged black hole solutions. In particular, we demonstrate the exponential convergence of some of their properties. In section IV we classify the globally regular SU(4) EYM solutions in sequences. We consider several sequences in detail and present numerical results for them. The sequences of neutral SU(4) EYM black hole solutions are discussed in section V, where we consider in particular the complex pattern of bifurcations, occurring at critical values of the horizon radius. In section VI we briefly generalize our results to SU(4) EYMD theory and present some numerical examples. We give our conclusions in section VII.

II. SU(4) EINSTEIN-YANG-MILLS EQUATIONS OF MOTION

A. SU(4) Einstein-Yang-Mills action

We consider SU(4) EYM theory with action

\[ S = S_G + S_M = \int L_G \sqrt{-g} d^4x + \int L_M \sqrt{-g} d^4x , \]

\[ L_G = \frac{1}{16\pi G} R , \]

gauge field Lagrangian

\[ L_M = -\frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) , \]

field strength tensor

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu] , \]

gauge field

\[ A_\mu = \frac{1}{2} \lambda^a A^a_\mu , \]

and gauge coupling constant e.

Variation of the action eq. (1) with respect to the metric \( g^{\mu\nu} \) leads to the Einstein equations

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} , \]

with stress-energy tensor

\[ T_{\mu\nu} = g_{\mu\nu} L_M - 2\frac{\partial L_M}{\partial g^{\rho\sigma}} = 2\text{Tr}(F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}) , \]

and variation with respect to the gauge field \( A_\mu \) leads to the gauge field equations.

B. Static spherically symmetric ansätze

To construct static spherically symmetric solutions we employ Schwarzschild-like coordinates and adopt the static spherically symmetric metric

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -A^2 N dt^2 + N^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \]

with the metric functions \( A(r) \) and

\[ N(r) = 1 - \frac{2m(r)}{r} . \]
The static spherically symmetric ansätze for the gauge field $A_\mu$ of SU(4) EYM theory are based on the $su(2)$ subalgebras of $su(4)$. Here we consider the gauge field ansatz corresponding to the 4-dimensional embedding of $su(2)$ in $su(4)$ [10]

$$A_\mu dx^\mu = \frac{1}{2e} \begin{pmatrix} 3 \cos \theta \phi & \omega_1 \Theta & 0 & 0 \\ \omega_1 \Theta & \cos \theta \phi & \omega_2 \Theta & 0 \\ 0 & \omega_2 \Theta & -\cos \theta \phi & \omega_3 \Theta \\ 0 & 0 & \omega_3 \Theta & -3 \cos \theta \phi \end{pmatrix},$$  \hspace{1cm} \text{(10)}

with

$$\Theta = i \phi + \sin \theta d\phi ,$$  \hspace{1cm} \text{(11)}

i. e.

$$A_0 = A_r = 0 .$$  \hspace{1cm} \text{(12)}

The ansatz contains three gauge field functions $\omega_j(r), j = 1, 2, 3$, and leads to the field strength tensor components

$$F_{\theta\phi} = \partial_r A_\theta ,$$  \hspace{1cm} \text{(13)}

$$F_{r\phi} = \partial_r A_\phi ,$$  \hspace{1cm} \text{(14)}

and

$$F_{\theta\phi} = (1/2e)\text{diag}(f_1, \ldots, f_4) \sin \theta ,$$  \hspace{1cm} \text{(15)}

with

$$f_j = \omega_j^2 - \omega_{j-1}^2 + \delta_j , \quad \delta_j = 2j - 5 , \quad j = 1, \ldots, 4 \quad (\omega_0 = \omega_4 = 0) .$$  \hspace{1cm} \text{(16)}

Another parametrization of the gauge field ansatz is given in [14].

**C. Field equations**

With the above ansätze we derive the set of EYM equations. The metric [8] yields for the $tt$ and $rr$ components of the Einstein equations

$$G_{tt} = \frac{2m'A^2N}{r^2} = 8\pi GT_{tt} ,$$  \hspace{1cm} \text{(17)}

and

$$G_{rr} = -\frac{G_{tt}}{A^2N^2} + \frac{2}{r} \frac{A'}{A} = 8\pi GT_{rr} ,$$  \hspace{1cm} \text{(18)}

where the prime indicates the derivative with respect to $r$. The static spherically symmetric ansatz for the fields [10] yields for the $tt$ component of the stress-energy tensor $T_{tt} = -A^2N L_M$, 

$$T_{tt} = \frac{1}{e^2r^2} A^2N (N\mathcal{G} + \mathcal{P}) ,$$  \hspace{1cm} \text{(19)}

and for the $rr$ component

$$T_{rr} = \frac{1}{e^2r^2} A^2N (N\mathcal{G} - \mathcal{P}) ,$$  \hspace{1cm} \text{(20)}

with

$$\mathcal{G} = \sum_{j=1}^{3} \omega_j^2 , \quad \mathcal{P} = \frac{1}{4r^2} \sum_{j=1}^{4} f_j^2 .$$  \hspace{1cm} \text{(21)}
We introduce the dimensionless coordinate
\[ x = \frac{er}{\sqrt{4\pi G}}, \]  
the dimensionless mass function
\[ \mu = \frac{em}{\sqrt{4\pi G}} , \]  
and the scaled gauge field functions \[ u_j = \frac{\omega_j}{\sqrt{\gamma_j}} , \quad \gamma_j = j(4-j) , \quad j = 1, \ldots, 3 \].

The above Einstein equations then yield for the metric functions the equations
\[ \mu' = NG + P , \]  
\[ \frac{A'}{A} = \frac{2G}{x} , \]

where the prime now indicates the derivative with respect to \( x \) and
\[ G = \sum_{j=1}^{3} \gamma_j u_j'^2 , \quad P = \frac{1}{4x^2} \sum_{j=1}^{4} f_j^2 , \]

with \( f_j = \gamma_j u_j^2 - \gamma_{j-1} u_{j-1}^2 + \delta_j \) (see eq. (16)).

For the gauge field functions we obtain the equations
\[ (\mathcal{A}u_j')' + \frac{1}{2x^2} \mathcal{A}(f_{j+1} - f_j)u_j = 0 , \]  
where the metric function \( \mathcal{A} \) can be eliminated by means of eq. (26) to yield
\[ x^2 \mathcal{A} u_j'' + 2(\mu - xP)u_j' + \frac{1}{2}(f_{j+1} - f_j)u_j = 0 . \]

We note the symmetry of the equations with respect to the transformation
\[ u_1 \rightarrow u_3 , \quad u_3 \rightarrow u_1 . \]

\section{D. Boundary conditions}

We now consider the boundary conditions for the SU(4) EYM solutions. Globally regular solutions must satisfy boundary conditions at the origin. These are \[ \mu(0) = 0 , \]

and
\[ u_j(0) = \pm 1 , \quad j = 1, 2, 3 . \]

For black hole solutions with a regular horizon with radius \( x_H \), boundary conditions are imposed at \( x_H \). These read
\[ \mathcal{N}(x_H) = 0 \quad \text{or} \quad \mu(x_H) = \frac{x_H}{2} , \]

and
\[ \mathcal{N}' u_j' + \frac{1}{2x^2}(f_{j+1} - f_j)u_j \bigg|_{x_H} = 0 . \]
For charged black hole solutions the coefficient of \( u_j' \) in eq. (34) may vanish, i.e. \( N' = 0 \). Then extremal black hole solutions are obtained, which satisfy
\[
(f_{j+1} - f_{j})u_j|_{x = H} = 0.
\] (35)

At infinity black hole solutions and globally regular solutions satisfy the same set of boundary conditions. Asymptotic flatness implies that the metric functions \( A \) and \( \mu \) both approach a constant at infinity. We here adopt
\[
A(\infty) = 1,
\] (36)
thus fixing the time coordinate. The mass of the solutions is given by \( \mu(\infty) \). Magnetically neutral solutions are obtained, when all gauge field functions are non-trivial and satisfy the boundary conditions
\[
u_j(\infty) = \pm 1, \quad j = 1, 2, 3,
\] (37)
yielding an asymptotically vanishing field strength tensor component \( F_{\theta\phi} = 0 \). No globally regular magnetically charged solutions exist. Magnetically charged black hole solutions are obtained, when one or more gauge field functions are identically zero, \( u_i \equiv 0 \). Their non-vanishing gauge field functions \( u_j \neq u_i \) then approach constants \( c_j \neq \pm 1 \) at infinity [8],
\[
u_j(\infty) = c_j,
\] (38)
determined in section III.A.

### III. Magnetically Charged SU(4) EYM Black Hole Solutions

When one or more gauge field functions are identically zero, \( u_j(x) \equiv 0 \), magnetically charged SU(4) EYM black hole solutions are obtained. These solutions are important in classifying the neutral SU(4) EYM solutions in sequences and in constructing their limiting solutions. Therefore we here consider these charged SU(4) EYM black hole solutions first. A general discussion of the magnetically charged SU(N) EYM solutions was given previously [8].

#### A. Classification of the magnetically charged SU(4) EYM black hole solutions

We now classify the magnetically charged SU(4) EYM black hole solutions, obtained within the ansatz (10) [8]. When one gauge field function is identically zero, \( \omega_k \equiv 0 \), the ansatz reduces to
\[
A^{(4)}_{\mu} dx^\mu = \begin{pmatrix} A^{(k)}_{\mu} dx^\mu \\ A^{(4-k)}_{\mu} dx^\mu \end{pmatrix} + \mathcal{H}_k,
\] (39)
with \( \mathcal{H}_k = \frac{\cos \theta d\phi}{2e} h_k \) and
\[
h_k = \begin{pmatrix} (4-k)1(k) \\ -k1(N-k) \end{pmatrix}.
\] (40)
Here \( A^{(N)}_{\mu} \) \((\tilde{N} = k, 4-k)\) denotes the non-abelian static spherically symmetric ansatz for the \( su(\tilde{N}) \) subalgebra of \( su(4) \) (based on the \( \tilde{N}\)-dimensional embedding of \( su(2) \)) and \( \mathcal{H}_k \) represents the ansatz for the element \( h_k \) of the Cartan subalgebra of \( su(4) \).

The gauge field functions of the \( su(\tilde{N}) \) parts of the solutions satisfy the boundary conditions (eq. (37))
\[
\tilde{u}_i(\infty) = \pm 1,
\] (41)
corresponding to neutral \( su(\tilde{N}) \) solutions [8]. Identifying the non-vanishing functions \( \omega_j \) of the \( su(4) \) ansatz with the corresponding functions \( \tilde{\omega}_j \) of the non-abelian \( su(\tilde{N}) \) ansätze,
\[ \gamma_i u_i^2 = \gamma_j \bar{u}_j^2 , \]
then yields the asymptotic boundary conditions for the functions \( u_j \).

\[ u_j(\infty) = c_j = \pm \sqrt{\gamma_i / \gamma_j} . \]

The charge of the solutions is carried by the Cartan subalgebra part of the gauge field. A solution based on the element \( h_k \) of the Cartan subalgebra carries magnetic charge of norm \( P \),

\[ P^2 = \frac{1}{2} \text{Tr} \ h_k^2 . \]

Expanding the element \( h_k \) in terms of the basis \( \{ \lambda_{n^2-1} \mid n = 2, 3, 4 \} \), the charge can also be directly read off the expansion coefficients,

\[ h_k = \sum_{n=2}^{N} d_k^n P_{n^2-1} \lambda_{n^2-1} , \]

where \( P_{n^2-1} = \sqrt{n(n-1)/2} \). The expansion coefficients, corresponding to cases 2a-c, are shown in Table 1.

By applying these considerations again to the subalgebras \( su(\tilde{N}) \) of eq. (39), we obtain cases 1a-c of Table 1, where two gauge field functions are identically zero. In the special case (case 0 of Table 1) where all gauge field functions are identically zero, an embedded RN solution is obtained with charge of norm \( P \).

RN black hole solutions exist only for horizon radius \( x_H \geq P \), and the extremal RN solution has \( x_H = P \). As first observed for SU(3) EYM theory [8], the same is true for charged non-abelian black hole solutions. Non-abelian black hole solutions with charge of norm \( P \) exist only for horizon radii \( x_H \geq P \). For extremal black hole solutions \( N = 0 \), so the coefficient of \( u_j \) in eq. (34) vanishes. This yields the boundary conditions

\[ u_j(x_H) = c_j , \]
corresponding to \( \bar{u}_j(x_H) = \pm 1 \).

**B. Numerical solutions**

All possible cases of magnetically charged SU(4) EYM black hole solutions are classified in Table 1. Since several cases are equivalent, 1a \( \sim \) 1b \( \sim \) 1c and 2a \( \sim \) 2c, three non-equivalent non-abelian cases remain, which are discussed below.

The magnetically charged non-abelian static spherically symmetric solutions of SU(4) EYM theory are labelled by the nodes of their non-vanishing gauge field functions, i. e. by the nodes of the subsets of gauge field functions \( \bar{u}_i \), belonging to the subalgebras \( su(2) \) or \( su(3) \). These solutions form sequences, completely analogously to the sequences of neutral solutions in SU(2) and SU(3) EYM theory, which are classified in sequences by their node structure, \( n \) for SU(2) and \( (j, j+n) \) and \( (n, n) \) for SU(3). With increasing \( n \), the sequences of neutral solutions converge to charged solutions. Similarly with increasing \( n \), the sequences of charged solutions converge to limiting solutions with higher charge. When the limiting solution has charge of norm \( P \), then one of the gauge field functions \( \bar{u}_i \) becomes identically zero for \( x > P \) in the limiting solution. In particular, the limiting solutions of case 1a are embedded RN solutions with charge of norm \( P = \sqrt{10} \) for \( x > P \). In contrast, the limiting solutions of cases 2a and 2b are non-abelian solutions, which carry charge of norm \( P = 3 \) and whose \( su(2) \) gauge field function has \( j \) nodes. Here only for \( j = 0 \) embedded RN solutions with charge of norm \( P = 3 \) are obtained for \( x > P \).

In Table 2 we show the mass \( \mu(\infty) \) of the first few solutions of several sequences together with the mass of their limiting solutions. For every case of Table 1, two sequences are shown: one sequence with extremal horizon corresponding to \( x_H = \sqrt{9} \), \( x_H = \sqrt{6} \) and \( x_H = \sqrt{5} \) for cases 1a, 2a, and 2b, respectively, and one sequence with horizon radius \( x_H = \sqrt{10} \). For cases 2a and 2b, the second non-vanishing function of the sequences shown has no node. Therefore, in all three cases, the limiting solutions are embedded RN solutions for \( x > P \), where \( P = \sqrt{10} \) for case 1a, and \( P = 3 \) for cases 2a and 2b.

We demonstrate the convergence of the functions for the extremal solutions of case 1a in Figs. 1. Fig. 1a shows the gauge field function \( \bar{u}_1 \) for \( n = 1 \ldots 7 \) with odd \( n \). Since the limiting solution has charge of norm \( P = \sqrt{10} \), we need
to distinguish two regions, an inner region $3 < x < \sqrt{10}$, beginning at the horizon, and an outer region $x > \sqrt{10}$. Satisfying the boundary condition $\bar{u}_1(x_{11}) = 1$, the function $\bar{u}_1$ becomes increasingly steep with increasing $n$ in the inner region, reaching a limiting solution whose first node resides just before $x = \sqrt{10}$, the norm of the charge of the limiting solution. Beyond $x = \sqrt{10}$ the function approaches zero in an exponentially increasing region. In Fig. 1b we show the metric function $N$. With increasing node number $n$ the function $N$ approaches a second zero, located at $x = \sqrt{10}$, the charge of the limiting solution. Beyond $x = \sqrt{10}$ the function $N$ approaches the metric function of the extremal RN solution with charge $P = \sqrt{10}$. In Fig. 1c we show the charge function $P(x)$,

$$P^2(x) = 2x (\mu(\infty) - \mu(x)) \ .$$

(47)

With increasing $n$ the charge function $P(x)$ tends to the charge of the limiting solution in an exponentially increasing region.

All this is strongly reminiscent of the behaviour of the neutral globally regular SU(2) EYM solutions and their limiting solution [78]. The limiting solution of the neutral globally regular solutions has charge of norm $P = 1$. It also consists of two regions, an inner region $0 < x < 1$, beginning at the origin, and an outer region $x > 1$. Satisfying the boundary condition $u(0) = 1$, the function $u$ becomes increasingly steep with increasing $n$ in the inner region, reaching a limiting solution whose first node resides just before $x = 1$, the charge of the limiting solution. Beyond $x = 1$, the function $u$ approaches zero in an exponentially increasing region. Similarly, with increasing node number $n$, the function $N$ approaches a zero located at $x = 1$, the charge of the limiting solution, whereas beyond $x = 1$ it approaches the metric function of the extremal RN solution with charge $P = 1$.

From the above similarity between the neutral globally regular solutions and their limiting solution with the charged extremal solutions and their limiting solution, we conjecture, that the charged extremal black hole solutions here play the role of the neutral globally regular solutions in determining the convergence properties of the functions. Considering neutral black hole solutions, we observed previously, that the location of the innermost node of the corresponding globally regular solutions with respect to the black hole event horizon is strongly indicative about the degree of convergence of the black hole solutions to their corresponding limiting solutions [6].

As for the globally regular solutions, the convergence of the location of the innermost node of the charged extremal black hole solutions is exponential. This is seen in Fig. 2, where we present $\Delta z^1_n$ as a function of the node number $n$, with $\Delta z^1_n$ being defined as the deviation of the location of the innermost node of the $n$-th extremal black hole solution from the location of the innermost node of the limiting solution,

$$\Delta z^1_n = |z^1_n - z^1_\infty| \ .$$

(48)

Considering the logarithm of $\Delta z^1_n$, we observe, that its slope is the same for the solutions of case 1a and case 2b. In contrast, the slope is different for the solutions of case 2a. Interestingly, for the globally regular neutral SU(2) EYM solutions the logarithm of $\Delta z^1_n$ also has the same slope as for the solutions of cases 1a and 2b. We conclude, that it is only the (sub)algebra of the non-abelian solutions, which determines the slope.

Similarly, the deviation of the mass of the $n$-th extremal black hole solution from the mass of the corresponding limiting solution,

$$\Delta \mu_n = \mu_\infty(\infty) - \mu_n(\infty) \ ,$$

(49)

decays with the same exponent, when the extremal solutions correspond to the same non-abelian (sub)algebra, as seen in Fig. 3, where in addition to cases 1a and 2b also the extremal charged SU(5) EYM solutions (with $su(2)$ subalgebra) and the globally regular SU(2) EYM solutions are shown.

IV. REGULAR SU(4) EINSTEIN-YANG-MILLS SOLUTIONS

Here we consider the static spherically symmetric globally regular solutions of SU(4) EYM theory, based on the ansatz [14]. These discrete [12] globally regular SU(4) EYM solutions are magnetically neutral. They are analogous to the globally regular SU(2) and SU(3) EYM solutions, obtained previously [7]. The SU(4) EYM solutions can be labelled by the node numbers $(n_1, n_2, n_3)$ of the gauge field functions $u_1$, $u_2$ and $u_3$. The solutions can then be classified into sequences. We first discuss several sequences which have embedded abelian limiting solutions with magnetic charge of norm $P$ for $x > P$. Then we discuss the general case and classify the general solutions into sequences, presenting some numerical examples.
A. Sequences with embedded abelian limiting solutions

Here we discuss those sequences of globally regular solutions, which for \( x > P \) tend to embedded abelian limiting solutions with magnetic charge of norm \( P \). In these sequences a single index \( n \) characterizes the number of nodes of one, two or all three gauge field functions, while the remaining gauge field functions have zero nodes. With increasing node number \( n \) the gauge field functions with zero nodes tend towards some finite constant value for \( x > P \), whereas the gauge field functions with \( n \) nodes tend to limiting functions, which vanish for \( x > P \). Consequently the limiting solutions represent embedded charged abelian black holes for \( x > P \), whose charge may be read off Table 1.

1. \((n, 0, 0)\) sequence

In Figs. 4a-c we present the first few globally regular solutions of the sequence with node structure \((n, 0, 0)\), \( n = 1−7 \), with odd \( n \). The first member of the \((1, 0, 0)\) sequence is the solution with the lowest mass of all SU(4) EYM solutions (within the ansatz \((\text{I})\)), \( \mu(\infty) = 1.723995 \) (see Table 3).

The limiting solution of this sequence is classified by case 2c of Table 1. For \( n \to \infty \), the limiting solutions corresponding to case 2c of Table 1 are EYM solutions based on the su(3) subalgebra of su(4), which carry magnetic charge of norm \( P = \sqrt{6} \). For \( x > P \) the gauge field function \( u_1 \) of the solutions of this sequence tends to zero, while the functions \( u_2 \) and \( u_3 \) approach the SU(3) vacuum solution. Therefore for \( x > P \) in this case the limiting solution corresponds to an embedded abelian solution, an extremal RN solution with the same charge (of norm \( P = \sqrt{6} \)).

In Fig. 4a the gauge functions \( u_1, u_2 \) and \( u_3 \) are presented. For \( x > P \) the limiting gauge field functions are \( u_1 \equiv 0 \), \( u_2 \equiv \sqrt{1/2} \) and \( u_3 \equiv \sqrt{2/3} \), whereas for \( x < P \) they are non-trivial functions. With increasing node number \( n \), the location of the innermost node of the function \( u_1 \) tends exponentially to a finite limiting value and an accumulation point appears at \( x = P \).

In Fig. 4b we present the metric function \( N \). For small node number \( n \) \((n \leq 3)\) \( N \) possesses only one minimum, while for larger \( n \) several local minima of \( N \) develop. For \( n \to \infty \), the global minimum \( N|_{\text{min}} = 0 \) occurs at \( x(N|_{\text{min}}) = P \). It represents the event horizon of the embedded extremal RN solution, which corresponds to the limiting solution for \( x > P \).

In Fig. 4c we present the charge function \( P(x) \). For finite node number \( n \) all solutions are neutral, and \( P(x) \) decays asymptotically to zero. However, with increasing \( n \) the charge function \( P(x) \) tends to the charge of the limiting solution \( P = \sqrt{6} \) in an exponentially increasing region.

Since the equations of motion are symmetric under the interchange of \( u_1 \) and \( u_3 \), the solutions with node structure \((0, 0, n)\) form an equivalent degenerate sequence.

2. Other sequences

We now consider the other sequences of globally regular solutions which tend to embedded abelian solutions with charge of norm \( P \) for \( x > P \). Turning to the sequence with node structure \((0, n, 0)\), the limiting solution of this sequence is classified by case 2b of Table 1. The limiting solutions corresponding to case 2b of Table 1 carry magnetic charge of norm \( P = \sqrt{8} \). For \( x > P \) they represent EYM solutions based on the \( su(2) \oplus su(2) \) subalgebra of \( su(4) \). Since the gauge field functions \( u_1 \) and \( u_3 \) of the solutions of this sequence approach the SU(2) vacuum solution, in this case the limiting solution corresponds to an embedded abelian solution for \( x > P \), an extremal RN solution with charge \( P = \sqrt{8} \). The masses \( \mu_n(\infty) \) of this sequence are shown in Table 3. In addition to this sequence with \( u_1 = u_3 \), a second sequence with node structure \((0, n, 0)\) might exist with \( u_1 \neq u_3 \). However, we do not find such solutions numerically.

Next we consider the cases where two gauge field functions have a finite number of nodes and the third function has zero nodes. The first such case is the sequence with node structure \((n, n, 0)\). The limiting solution of this sequence is classified by case 1c of Table 1. The limiting solutions corresponding to case 1c of Table 1 carry magnetic charge of norm \( P = \sqrt{6} \). For \( x > P \) they represent EYM solutions based on the \( su(2) \) subalgebra of \( su(4) \). Since the gauge field function \( u_3 \) of the solutions of the \((n, n, 0)\) sequence approaches the SU(2) vacuum solution, in this case the limiting solution corresponds to an embedded extremal RN solution with charge \( P = \sqrt{6} \) for \( x > P \). The masses \( \mu_n(\infty) \) of this sequence are shown in Table 3 \([9]\). An equivalent degenerate sequence is \((0, n, n)\).

Considering solutions with node structure \((n, 0, n)\), we find two types of solutions leading to two distinct sequences. In the solutions of the first type the gauge functions \( u_1 \) and \( u_3 \) are identical, whereas in the second type \( u_1 \neq u_3 \).
The limiting solutions of both sequences are identical and classified by case 1b of Table 1. Analogously to case 1c, the limiting solutions corresponding to case 1b of Table 1 carry magnetic charge of norm $P = \sqrt{9}$ and represent EYM solutions for $x > P$, based on the $su(2)$ subalgebra of $su(4)$. Only for the $(n,0,n)$ sequences the limiting solution corresponds to an embedded extremal RN solution with $P = \sqrt{9}$ for $x > P$. The masses $\mu_n(\infty)$ of both of these sequences are shown in Table 3.

Last we consider the solutions with node structure $(n,n,n)$. Here three distinct types of solutions exist. In the solutions of the first type all three gauge field functions are identical, $u_1 = u_2 = u_3$. These simply represent scaled SU(2) solutions, with scaling factor $\sqrt{10}$. In the solutions of the second type only two gauge field functions are identical, $u_1 = u_3$, and in the solutions of the third type all three gauge field functions differ from each other. For the second and third type of solutions, for a given $n$ several distinct non-degenerate solutions may exist. These solutions are discussed in detail in section V.B. The limiting solutions of the $(n,n,n)$ sequences are classified by case 0 of Table 1. Thus for $n \to \infty$, the sequences tend to an embedded extremal RN solution with magnetic charge of norm $P = \sqrt{10}$ for $x > P$. The masses $\mu_n(\infty)$ for several $(n,n,n)$ sequences are shown in Table 4.

B. General sequences

All sequences of neutral globally regular solutions not included in the previous subsection tend to non-abelian limiting solutions for $x > P$, which carry magnetic charge of norm $P$. The norm of the charge of the limiting solutions is the same as in the corresponding abelian cases. In this subsection we first present a classification scheme for the general globally regular solutions of SU(4) EYM theory (based on the ansatz (10)) and then demonstrate the main features of these solutions by constructing several sequences numerically.

1. Classification of the general solutions

The general globally regular solutions are labelled by the node numbers $(n_1, n_2, n_3)$ of the corresponding gauge field functions $u_1, u_2, u_3$. The case, where all node numbers are identical, has already been considered in the previous subsection. Assuming next, that the node numbers of any two gauge field functions ($u_j$) are identical, we obtain four types of sequences, labelled by the fixed index $k$ and the running index $n$. For fixed $k$, $k = 1, 2, 3,...$, and running $n$, $n = 1, 2, 3,...$, these four types of sequences are $(k,k,n+k)$, $(k,n+k,k)$, $(k,n+k,n+k)$ and $(n+k,k,n+k)$. For $k = 0$ the sequences of the previous subsection are obtained. In this scheme any solution with two identical node numbers is included exactly once, apart from equivalent degenerate solutions and the cases, where solutions with both $u_1 \neq u_3$ and $u_1 = u_3$ exist. The limiting solutions of the sequences $(k,n+k,n+k)$ and $(n+k,k,n+k)$ correspond to cases 1a and 1b, respectively, and the limiting solutions of the sequences $(k,n+k,k)$ and $(k,k,n+k)$ correspond to cases 2b and 2a, respectively (Table 1).

When all three node numbers differ from each other, we obtain a similar classification, which now requires three distinct indices, $l$, $k$, and $n$. For fixed indices $l$ and $k$ with $l = 0, 1, 2,...$, and $k = 1, 2, 3,...$, and running index $n$, there are three types of sequences, $(l,l+k,l+k+n)$, $(l+k,l,l+k+n)$ and $(l,l+k+n,l+k)$. Any solution with three different node numbers is thus included exactly once in this scheme, apart from equivalent degenerate solutions. The limiting solutions of the $(l,l+k,l+k+n)$ and $(l+k,l,l+k+n)$ sequences correspond to case 2a of Table 1 and the limiting solutions of the $(l,l+k+n,l+k)$ sequence corresponds to case 2b.

It is straightforward to extend this classification to the globally regular solutions of SU(N) EYM theory.

2. $(n+1,1,0)$ sequence and $(n+1,0,1)$ sequence

We now discuss the properties of these general sequences. Their qualitative features are analogous to those of the sequences discussed in the previous subsection. In order to demonstrate this, we here present two numerically constructed sequences: $(n+1,1,0)$ and $(n+1,0,1)$.

In Fig. 5a we present the gauge field functions of the globally regular solutions of the sequence with node structure $(n+1,1,0)$ for $n = 1 – 5$. The limiting solution of this sequence is an EYM solution based on the $su(3)$ subalgebra of $su(4)$, with magnetic charge of norm $P = \sqrt{6}$, corresponding to case 2c of Table 1. For $x > P$ the limiting solution is the extremal SU(3) EYM black hole solution with node structure $(1,0)$. In Fig. 5b the metric function $N$ is shown. With increasing node number $n$ the metric function $N$ tends to zero at $x = \sqrt{6}$, the charge of the limiting solution. The metric function $N$ has the same qualitative behaviour for the $(n+1,1,0)$ sequence as for the $(n,0,0)$ sequence.
The charge function $P(x)$ is shown in Fig. 5c. Again, with increasing $n$ it approaches $P = \sqrt{6}$ in an exponentially increasing region.

The mass $\mu(\infty)$ of the solutions of the $(n+1, 1, 0)$ sequence again converges exponentially to the mass of the limiting solution. This is seen in Fig. 6, where $\Delta \mu_n$ (eq. (49)) is presented. Also shown in Fig. 6 is the mass of the solutions of the $(n+1, 0, 1)$ sequence. For a given $n$, the solutions of the $(n+1, 0, 1)$ sequence have a lower mass than the solutions of the $(n+1, 1, 0)$ sequence, but both sequences tend to the same limiting solution. For comparison, also the mass of the solutions of the $(n, 0, 0)$ sequence is shown. All three sequences are based on the $su(3)$ subalgebra of $su(4)$, and the function $\ln \Delta \mu_n$ has the same slope for all three sequences. This is analogous to the case of the solutions based on the $su(2)$ subalgebra of $su(4)$, discussed in section III.B. The masses of the solutions of the $(n+1, 1, 0)$ and $(n+1, 0, 1)$ sequences and of their limiting solution are shown in Table 3.

V. NEUTRAL SU(4) EINSTEIN-YANG-MILLS BLACK HOLES

The SU(4) EYM black hole solutions are obtained analogously to the globally regular solutions, but with boundary conditions imposed at the event horizon (see section II). As for the SU(3) EYM black hole solutions, in general sequences of neutral SU(4) EYM black hole solutions exist for all values of the horizon radius. However, for the SU(3) EYM solutions with node structure $(n, n)$ there are two distinct types of solutions, the scaled SU(2) solutions with $u_1 = u_2$ and the genuine SU(3) solutions with $u_1 \neq u_2$. The genuine SU(3) solutions exist only up to a critical value of the horizon radius $x_{Hn}^{cr}$, where they merge into the scaled SU(2) solutions. Analogously, in SU(4) EYM theory critical values of the horizon radius occur for the various types of solutions with node structure $(n, j, n)$ and $(n, n, n)$. Their complex critical behaviour is analyzed below [19].

A. General sequences

To all of the globally regular solutions discussed above, the corresponding neutral black hole solutions exist at least for sufficiently small values of the horizon radius $x_{H}$. For fixed horizon radius $x_{H}$, the sequences of neutral black hole solutions tend to limiting solutions with charge of norm $P$. For $x_{H} < P$ the sequences of black hole solutions tend to a non-trivial limiting solution for $x_{H} < x < P$ and to a charged non-abelian black hole solution or an embedded extremal RN solution for $x > P$. For $x_{H} > P$ the sequences of black hole solutions tend to a charged non-abelian black hole solution or an embedded RN solution with charge $P$ and the same value of the horizon radius. Since the other qualitative features of the black hole solutions are completely analogous to those of the corresponding globally regular solutions, we do not discuss them here further. Instead we turn to the solutions with node structure $(n, j, n)$ and $(n, n, n)$, whose complex critical behaviour shows interesting novel features.

B. Bifurcations

We recall that for the globally regular solutions with node structure $(n, 0, n)$, two types of solutions exist, one type with $u_1 = u_3$ and a second type with $u_1 \neq u_3$, whereas for the globally regular solutions with node structure $(n, n, n)$, three types of solutions exist, one type with $u_1 = u_2 = u_3$, representing scaled SU(2) solutions, a second type with $u_1 = u_3 \neq u_2$, and a third type with $u_1 \neq u_3 \neq u_2$.

Similarly, for the black hole solutions with node structure $(n, 0, n)$, there are two distinct types of black hole solutions for sufficiently small values of the event horizon. At a critical value of the horizon radius $x_{Hn}^{cr}$, the $u_1 \neq u_3$ solutions with node number $n$ merge into the $u_1 = u_3$ solutions. Beyond this critical value only the $u_1 = u_3$ solutions persist. With increasing node number $n$ the critical value of the horizon radius $x_{Hn}^{cr}$ decreases. For the black hole solutions with $n = 1, 3$ and $5$ the critical values of the horizon radius are $x_{H1}^{cr} = 1.972$, $x_{H3}^{cr} = 1.244$ and $x_{H5}^{cr} = 1.180$ (see Table 5).

In Fig. 7a the value of the gauge field functions at the horizon of the black hole solutions with node structure $(n, 0, n)$ is shown for $n = 1, 3$ and $5$, together with the corresponding three critical values of the horizon radius. For $n = 3$ and $5$ the $u_1 = u_3$ solutions are only shown in the inset, in order not to cover part of the corresponding $u_1 \neq u_3$ solutions. For the $u_1 \neq u_3$ solutions, we obtain two degenerate branches of solutions. They are degenerate because of the symmetry with respect to the interchange of $u_1$ and $u_3$, i. e. if in a given solution $u_1(x_{H})$ lies on the upper branch and $u_3(x_{H})$ lies on the lower branch, then in the degenerate solution $u_1(x_{H})$ and $u_3(x_{H})$ are interchanged. In
Fig. 7b we present the corresponding masses. The mass of the \( u_1 \neq u_3 \) solutions is always larger than the mass of the \( u_1 = u_3 \) solutions. In contrast, the temperature is lower for the \( u_1 \neq u_3 \) solutions, as seen in Fig. 7c.

We now consider the black hole solutions with node structure \((n, n, n)\), whose critical behaviour is more complex. As for the globally regular solutions, three distinct types of black hole solutions exist for sufficiently small values of the horizon radius. There is one solution of the first type with \( u_1 = u_3 = u_2 \), we find two non-degenerate solutions of the second type with \( u_1 = u_3 \neq u_2 \), and we find up to four non-degenerate solutions of the third type with \( u_1 \neq u_3 \neq u_2 \).

At the largest critical value of the horizon radius \( x_{H_1}^{cr,1} \), the two non-degenerate branches of \( u_1 = u_3 \neq u_2 \) solutions merge into each other. Notably, this critical value of the horizon radius \( x_{H_1}^{cr,1} \) is slightly beyond the value of the horizon radius, where the branches intersect the branch of scaled SU(2) solutions. Beyond \( x_{H_1}^{cr,1} \) only the scaled SU(2) solutions persist. At the second largest critical value of the horizon radius \( x_{H_1}^{cr,2} \), two degenerate branches of \( u_1 \neq u_3 \neq u_2 \) solutions merge into the branch of scaled SU(2) solutions. At the third critical value \( x_{H_1}^{cr,3} \) another two degenerate branches of \( u_1 \neq u_3 \neq u_2 \) solutions merge into the lower branch of the \( u_1 = u_3 \neq u_2 \) solutions. At the smallest fourth critical value \( x_{H_1}^{cr,4} \), two non-degenerate branches of \( u_1 \neq u_3 \neq u_2 \) solutions merge into each other, being neither close to the branch of scaled SU(2) solutions nor one of the \( u_1 = u_3 \neq u_2 \) branches. They cease to exist beyond \( x_{H_1}^{cr,4} \).

We illustrate this complex critical behaviour for the black hole solutions with node structure \((1,1,1)\) in Figs. 8. The value of the gauge field functions at the horizon radius \( x_H \) is shown in Fig. 8a for all solutions found, together with the four critical values of the horizon radius. The scaled SU(2) solutions, having \( u_1 = u_3 = u_2 \), exist for all values of the horizon radius. We see two distinct non-degenerate branches of \( u_1 = u_3 \neq u_2 \) solutions, up to the critical value of the horizon radius \( x_{H_1}^{cr,1} = 4.158 \). Of the six branches of \( u_1 \neq u_3 \neq u_2 \) solutions two degenerate branches of solutions merge into the branch of scaled SU(2) solutions at the critical value \( x_{H_1}^{cr,2} = 2.714 \), another two degenerate branches merge into the lower branch of the \( u_1 = u_3 \neq u_2 \) solutions at the critical value \( x_{H_1}^{cr,3} = 1.025 \), as seen in the upper right inset of Fig. 8a, and two non-degenerate branches merge into each other at \( x_{H_1}^{cr,4} = 0.445 \), as seen in the lower left inset of Fig. 8a. In Fig. 8b we present the mass of all branches of solutions as a function of the horizon radius. Of all seven distinct branches of solutions, the scaled SU(2) solutions have the lowest mass, and first one and then the other of the two branches of \( u_1 = u_3 \neq u_2 \) solutions has the highest mass, except very close to \( x_{H_1}^{cr,1} \). The intersection of the two \( u_1 = u_3 \neq u_2 \) branches is seen clearly in Fig. 8c, where the temperature of all branches is shown.

The critical behaviour of the black hole solutions with node structure \((3,3,3)\) is shown in Fig. 9. The two non-degenerate branches of \( u_1 = u_3 \neq u_2 \) solutions cease to exist at the critical value \( x_{H_3}^{cr,1} = 2.836 \), as seen in the upper right inset, two degenerate branches of \( u_1 \neq u_3 \neq u_2 \) solutions merge into the branch of scaled SU(2) solutions at the critical value \( x_{H_3}^{cr,2} = 2.090 \), as seen in the lower left inset, and two degenerate branches of \( u_1 \neq u_3 \neq u_2 \) solutions merge into the lower branch of \( u_1 = u_3 \neq u_2 \) solutions at \( x_{H_3}^{cr,3} = 0.474 \). The critical values of the horizon radius and the corresponding masses of the \((n,n,n)\) sequences are presented in Table 4 for \( n = 1 \sim 5 \).

VI. SU(4) EINSTEIN-YANG-MILLS-DILATON SOLUTIONS

We here briefly consider the solutions of SU(4) EYMD theory. In EYMD theory the dilaton coupling constant \( \gamma \) represents an additional parameter. For \( \gamma = 0 \) EYM theory is recovered, whereas for \( \gamma = 1 \) contact with the low-energy effective action of string theory is made. Most of the qualitative features of the static spherically symmetric EYMD solutions agree with those of the static spherically symmetric EYM solutions. This was observed previously for the solutions of SU(2) and SU(3) EYMD theory, and it also holds for the solutions of SU(4) EYMD theory. In particular, the solutions of SU(4) EYMD theory can be classified in the same way as the SU(4) EYM solutions. Qualitative differences concern the charged black hole solutions and the limiting solutions of the sequences of solutions. As in EMD theory, where only the extremal solution with \( x_H = 0 \) has a naked singularity at the origin, in EYMD theory charged black hole solutions with a regular horizon exist for any value of the horizon radius \( x_H > 0 \). The limiting solutions of the sequences of EYMD black hole solutions are EYMD and embedded EMD black hole solutions with the same horizon radius and the same dilaton coupling constant, which carry the same charge \( P \) as the limiting solutions of the corresponding sequences of EYM solutions, as demonstrated in detail for SU(3) EYMD theory. The limiting solutions of the sequences of globally regular EYMD solutions are extremal EYMD and embedded EMD solutions.
A. Action and Equations

We briefly discuss the action, the equations of motion and the boundary conditions for the static spherically symmetric solutions of SU(4) EYMD theory. The SU(4) EYMD action is

\[ S = S_G + S_M = \int L_G \sqrt{-g} d^4x + \int L_M^D \sqrt{-g} d^4x , \]  

(50)

with \( L_G \) given in eq. (2) and

\[ L_M^D = -\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \exp(2\kappa \Phi) \frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) , \]  

(51)

with field strength tensor \( F_{\mu\nu} \) given in eq. (4), and dilaton coupling constant \( \kappa \).

In EYMD theory we proceed analogously to EYM theory. We employ Schwarzschild-like coordinates and adopt the static spherically symmetric metric (8). The ansatz for the static spherically symmetric gauge field \( A_\mu \) is given by (10). The corresponding ansatz for the static spherically symmetric dilaton field is \( \Phi = \Phi(r) \). With these ansätze we obtain for the matter Lagrangian

\[ L_M^D = -\frac{1}{2} N \phi'^2 - \frac{\exp(2\gamma \phi)}{e^{2\gamma r^2}} (N G + P) , \]  

(52)

with \( N, G \) and \( P \) given in (9) and (21).

Changing to the dimensionless coordinate \( x \) given in (22) and the dimensionless mass function \( \mu(x) \) given in (23), we further introduce the dimensionless dilaton field \( \phi \),

\[ \phi = \sqrt{4\pi G} \Phi , \]  

(53)

and the dimensionless coupling constant \( \gamma \),

\[ \gamma = \kappa / \sqrt{4\pi G} . \]  

(54)

The choice \( \gamma = 1 \) corresponds to string theory, while 4 + \( n \) dimensional Kaluza-Klein theory has \( \gamma^2 = (2 + n)/n \) [18]. From the Einstein equations we obtain the field equations for \( \mu(x) \) and \( A(x) \),

\[ \mu' = \frac{1}{2} N x^2 \phi'^2 + \exp(2\gamma \phi) (N G + P) , \]  

(55)

\[ \frac{A'}{A} = 2 \frac{x}{x} \left( \frac{1}{2} x^2 \phi'^2 + \exp(2\gamma \phi) G \right) , \]  

(56)

where the prime now indicates the derivative with respect to \( x \).

For the gauge field functions \( u_j(x) \), \( j = 1 - 3 \), and for the dilaton function \( \phi(x) \) we obtain the field equations

\[ (\exp(2\gamma \phi) A N u_j)' = -\frac{\exp(2\gamma \phi)}{2x^2} A (f_{j+1} - f_j) u_j , \]  

(57)

\[ (A N x^2 \phi')' = 2\gamma A \exp(2\gamma \phi) (N G + P) . \]  

(58)

The metric and gauge field functions satisfy the same boundary conditions as in EYM theory, given in section II.D. These are supplemented with the boundary conditions for the dilaton function. At infinity the dilaton function satisfies

\[ \phi(\infty) = 0 , \]  

(59)

for the globally regular solutions the boundary condition at the origin is

\[ \phi'(0) = 0 , \]  

(60)

and for the black hole solutions regularity at the horizon requires

\[ N' \phi' = \frac{2 \gamma \exp(2\gamma \phi)}{x^2} (N G + P) . \]  

(61)
B. Numerical solutions

In SU(4) EYMD theory, sequences of neutral static spherically symmetric globally regular solutions can be constructed for arbitrary dilaton coupling constant γ. Parameterized by the node numbers \((n_1, n_2, n_3)\) of the gauge field functions \((u_1, u_2, u_3)\), these sequences converge to charged static spherically symmetric extremal EYMD solutions, based on the subalgebras of su(4) as classified in Table 1 for EYM theory.

We present some examples of globally regular solutions in the upper part of Table 5. Shown is the mass of the first few solutions (with odd \(n\)) of the sequences with node structure \((n, 0, n)\) with \(u_1 = u_3\) and \(u_1 \neq u_3\) for EYM theory with \(\gamma = 1\) and for EYM theory (\(\gamma = 0\)). Also shown is the mass of the corresponding limiting solutions, representing an embedded extremal EMD solution for EYMD theory and an embedded extremal RN solution for \(x > P\) for EYM theory.

Analogously, sequences of neutral static spherically symmetric black hole solutions can be constructed for arbitrary dilaton coupling constant γ. These sequences converge to the corresponding charged static spherically symmetric EYMD and embedded EMD black hole solutions. The temperature shown in the upper part of Table 5 for the globally regular solutions should be interpreted as the temperature of the corresponding black hole solutions in the limit \(x_H \to 0\).

The lower part of Table 5 shows the mass and temperature of the black hole solutions of the \((n, 0, n)\) sequences at the critical values of the horizon radius \(x_H^{cr}\), where the \(u_1 \neq u_3\) solutions merge into the \(u_1 = u_3\) solutions, for EYM theory with \(\gamma = 1\) and for EYM theory (\(\gamma = 0\)). Also shown are the critical values themselves and the value of the function \(u_1\) at \(x_H^{cr}\). For \(\gamma = 1\) the critical values are smaller than for \(\gamma = 0\). This is in agreement with the critical values of the SU(3) EYMD black hole solutions \([\text{13}]\), which with increasing \(\gamma\) first decrease and then increase again.

The charged EYMD black hole solutions also follow the classification scheme of the corresponding charged EYM black hole solutions. Since these were discussed at length for SU(3) EYMD theory in \([\text{14}]\), we do not present further results here.

VII. CONCLUSIONS

We have investigated the various sequences of globally regular and black hole solutions of SU(4) EYM theory, based on the static spherically symmetric ansatz for the purely magnetic gauge field, obtained by embedding the 4-dimensional representation of su(2) in su(4) \([\text{15}]\). We have classified the discrete set of solutions in sequences by means of the node numbers \((n_1, n_2, n_3)\) of their gauge field functions \(u_1, u_2\) and \(u_3\). We have determined the limiting solutions of the sequences of solutions, and we have constructed numerous sequences and their limiting solutions numerically. In particular, we have shown, that the exponential convergence of the mass of the globally regular and extremal black hole solutions depends on the non-abelian (sub)algebra of the solutions.

The SU(4) EYM black hole solutions with node structure \((n, j, n)\) and \((n, n, n)\) show a complex critical behaviour. For such sequences several branches of black hole solutions exist up to critical values of the horizon radius. For sufficiently small values of the horizon radius two types of solutions with node structure \((n_1, 0, n_2)\) exist. The \(u_1 \neq u_3\) solutions merge into the \(u_1 = u_3\) solutions at a critical value of the horizon radius \(x_H^{cr}\). Beyond \(x_H^{cr}\) only the \(u_1 = u_3\) solutions persist. Again for sufficiently small values of the horizon radius there are three types of solutions with node structure \((n, n, n)\), scaled SU(2) solutions with \(u_1 = u_2 = u_3\), solutions with \(u_1 = u_3 \neq u_2\), and solutions with \(u_1 \neq u_3 \neq u_2\). The two non-degenerate branches of \(u_1 = u_3 \neq u_2\) solutions merge into each other and cease to exist beyond \(x_H^{cr}\). At \(x_H^{cr,2}\), two degenerate branches of \(u_1 \neq u_3 \neq u_2\) solutions merge into the branch of scaled SU(2) solutions. Another two degenerate branches of \(u_1 \neq u_3 \neq u_2\) solutions merge into the lower branch of the \(u_1 = u_3 \neq u_2\) solutions at \(x_H^{cr,3}\), and two non-degenerate branches of \(u_1 \neq u_3 \neq u_2\) solutions merge into each other at \(x_H^{cr,4}\).\([\text{21}]\).

For SU(4) EYMD theory, the classification of the solutions in sequences represents a generalization with respect to SU(3) EYM theory, since three node numbers are needed for the classification instead of two. It is straightforward to generalize the classification of the static spherically symmetric globally regular and black hole solutions further to SU(N) EYMD theory, where \((N - 1)\) node numbers are needed for the classification since \((N - 1)\) gauge field functions are present. For the charged SU(N) EYM black hole solutions, the general classification is given in \([\text{16}]\). An existence proof for regular SU(N) EYMD solutions has been proposed recently \([\text{13}]\), which follows closely the existence proof for regular SU(2) EYMD solutions \([\text{21}]\).

For EYMD theory the same classification of the solutions holds. In EYMD theory charged black hole solutions exist only for horizon radius \(x_H \geq P\), where \(P\) is the norm of the charge, whereas in EYMD theory charged black hole
solutions exist for any value of the horizon radius $x_H > 0$ as in EMD theory. The limiting solutions of the sequences of EYMD black hole solutions are charged EYMD and embedded EMD black hole solutions with the same horizon radius, whereas the limiting solutions of sequences of EYM black hole solutions are charged EYM and embedded RN black hole solutions with the same horizon radius for $x_H > P$. The norm of the charge of the limiting solutions is determined by their $su(4)$ subalgebras and is the same for EYM and EYMD theory.

Recently, new types of solutions were found in SU(2) EYM and EYMD theory. These are static globally regular and black hole solutions with only axial symmetry and non-static non-rotational black hole solutions. Such solutions as well as solutions with only discrete symmetries are also expected in SU(N) EYM and EYMD theory with $N > 2$.

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15
Solutions of the \((n, n, n)\) sequences with \(u_1 \neq u_3 \neq u_2\) which merge into the lower branch of solutions with \(u_1 = u_3 \neq u_2\) at \(x^{\text{cr}, n}_{H}\) have not been numerically constructed for \(n = 4\) or \(5\). Solutions corresponding to \(x^{\text{cr}, n}_{H}\) have not been constructed for \(n > 1\). Since the critical values of the horizon radius decrease with increasing \(n\), these may either be harder to obtain or they might not exist any more.

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Table 1

The classification of the charged black hole solutions of SU(4) EYM theory is presented. Shown are the non-vanishing gauge field functions (denoted by $u_i$) and the identically vanishing gauge field functions (denoted by zero), the norm squared of the charge of the black hole solutions, $P^2$, and the subalgebra of the solutions including the non-abelian subalgebra and the coefficients* (in the given basis) of the corresponding element of the Cartan subalgebra.

| # | $u_1$ | $u_2$ | $u_3$ | $c_1$ | $c_2$ | $c_3$ | $P^2$ | non-abelian subalgebra | Cartan subalgebra * |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10 | $P_3$ | $P_8$ | $P_{15}$ |
| 1a | $u_1$ | 0 | 0 | $\frac{1}{3}$ | 0 | 0 | 9 | $su(2)$ | $\frac{3}{2}P_3$ | $P_8$ | $P_{15}$ |
| 1b | 0 | $u_2$ | 0 | 0 | $\frac{1}{2}$ | 0 | 9 | $su(2)$ | $\frac{3}{2}P_3$ | $P_8$ | $P_{15}$ |
| 1c | 0 | 0 | $u_3$ | 0 | 0 | $\frac{1}{2}$ | 9 | $su(2)$ | $P_3$ | $P_8$ | $\frac{3}{2}P_{15}$ |
| 2a | $u_1$ | $u_2$ | 0 | $\frac{1}{3}$ | $\frac{1}{4}$ | 0 | 6 | $su(3)$ | $P_3$ | $P_8$ | $P_{15}$ |
| 2b | $u_1$ | 0 | $u_3$ | $\frac{1}{3}$ | 0 | $\frac{1}{4}$ | 8 | $su(2) \oplus su(2)$ | $\frac{3}{2}P_3$ | $P_8$ | $\frac{3}{2}P_{15}$ |
| 2c | 0 | $u_2$ | $u_3$ | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | 6 | $su(3)$ | $2P_3$ | $\frac{3}{2}P_8$ | $\frac{3}{2}P_{15}$ |
| n/x_{H} | 1a | 2a | 2b |
|---------|----|----|----|
| √γ     | 3.12998 | 2.83874 | 2.96587 |
| 1       | 3.15685 | 2.95799 | 2.99427 |
| 3       | 3.16139 | 2.98951 | 2.99906 |
| 4       | 3.16213 | 2.99741 | 2.99985 |
| 5       | 3.16225 | 2.99936 | 2.99997 |
| 7       | 3.16228 | 2.99996 | 3.00000 |
| ∞       | 3.16228 | 3.0     | 3.0     |

| x_{H} = \sqrt{10} |
|-------------------|
| 1                 | 3.13343 | 2.89417 | 2.98031 |
| 2                 | 3.15864 | 2.98734 | 2.98734 |
| 3                 | 3.16191 | 3.00255 | 3.00412 |
| 4                 | 3.16224 | 3.00406 | 3.00416 |
| 5                 | 3.16226 | 3.00415 | 3.00416 |
| 7                 | 3.16228 | 3.00416 | 3.00416 |
| ∞                 | 3.16228 | 3.00416 | 3.00416 |

**Table 2**
The dimensionless mass \(\mu(\infty)\) of the first few SU(4) EYM black hole solutions of the sequences corresponding to case 1a of Table 1, case 2a with node structure \((n, 0)\) and case 2b with node structure \(n\) and 0 is presented for their respective extremal horizons and for the horizon radius \(x_{H} = \sqrt{10}\). For each sequence the corresponding limiting value of the mass is shown in the last row (denoted by \(\infty\)).
Table 3
The dimensionless mass $\mu(\infty)$ of the first few solutions of several globally regular SU(4) EYM sequences is presented. For each sequence the corresponding limiting value of the mass is shown in the last row (denoted by $\infty$).
The bifurcation behaviour of the SU(4) EYM \((n, n, n)\) sequences is presented for \(n = 1 - 5\). The mass of the regular solutions is shown together with the corresponding values of the critical horizon and the mass of the black hole solutions at the critical horizon [20]. For the solutions with \(u_1 = u_3 \neq u_2\) (first set), there are two non-degenerate branches. For comparison, the mass of the scaled SU(2) solutions with \(u_1 = u_3 = u_2\) is also presented.
Table 5

Upper part: The dimensionless mass $\mu(\infty)$ of the first few $u_1 \neq u_3$ and $u_1 = u_3$ solutions with odd $n$ is shown for the two globally regular sequences with node structure $(n, 0, n)$ for SU(4) EYM theory and SU(4) EYMD theory with $\gamma = 1$. Also shown is the temperature $T/T_S$ ($T_S = (4\pi x_H)^{-1}$), to be interpreted as the $x_H \to 0$ limit of the black hole solutions. The corresponding limiting values are shown in the last row (denoted by $\infty$).

Lower part: The critical values of the horizon radius $x_{H1}^c$, where the $u_1 \neq u_3$ solutions with node structure $(n, 0, n)$ merge into the corresponding $u_1 = u_3$ solutions are shown for SU(4) EYM theory and SU(4) EYMD theory with $\gamma = 1$. Also shown are the dimensionless mass $\mu(\infty)$, the temperature $T/T_S$ and $u_1(x_{H1}^c)$. 

| $(n, 0, n)$ | $\mu(\infty)$ | $x_H = 0$ | $T/T_S$ |
|-------------|----------------|-----------|---------|
| $n$ | $\gamma$ | $\infty$ | $1$ | $0$ | $1$ | $0$ | $1$ | $0$ | $1$ |
| 1 | $u_1 \neq u_3$ | 2.3282 | 1.6269 | 0.1031 | 0.3445 |
| | $u_1 = u_3$ | 2.2910 | 1.5919 | 0.1559 | 0.4296 |
| 3 | $u_1 \neq u_3$ | 2.9567 | 2.0886 | 0.0075 | 0.0054 |
| | $u_1 = u_3$ | 2.9559 | 2.0875 | 0.0094 | 0.1104 |
| 5 | $u_1 \neq u_3$ | 2.9974 | 2.1193 | 0.0005 | 0.0237 |
| | $u_1 = u_3$ | 2.9973 | 2.1193 | 0.0006 | 0.0272 |
| $\infty$ | | 3.0 | 2.1215 | 0.0 | 0.0 |

| $(n, 0, n)$ | $x_{H1}^c$ | $\mu(\infty)$ | $T/T_S$ | $u_1(x_{H1}^c)$ |
|-------------|------------|----------------|--------|----------------|
| $n$ | $\gamma$ | $\infty$ | $1$ | $0$ | $1$ | $0$ | $1$ | $0$ | $1$ |
| 1 | | 1.9723 | 1.7742 | 2.5065 | 2.0424 | 0.3038 | 0.5860 | 0.8632 | 0.8552 |
| 3 | | 1.2440 | 0.2927 | 2.9633 | 2.1058 | 0.0154 | 0.1411 | 0.9063 | 0.8812 |
| 5 | | 1.1803 | 0.0690 | 2.9978 | 2.1203 | 0.0009 | 0.0345 | 0.9130 | 0.8860 |
Fig. 1a: The gauge field function $\bar{u}_1(x)$ is shown as a function of the dimensionless coordinate $x$ for the charged SU(4) EYM black hole solutions of case 1a of Table 1 with extremal event horizon $x_H = 3$ and node numbers $n = 1, 3, 5, 7$. The inset illustrates the convergence of the innermost node to a value just below $x = \sqrt{10}$ (thin vertical line) in the limit $n \to \infty$. The thin horizontal line represents the limiting function for $x > \sqrt{10}$. 
Fig. 1b: Same as Fig. 1a for the metric function $N(x)$. The inset illustrates the emergence of a second zero at $x = \sqrt{10}$ in the limit $n \to \infty$. 
Fig. 1c: Same as Fig. 1a for the charge function $P(x)$. The thin horizontal line represents the norm of the charge of the limiting solution, $P = \sqrt{10}$. 
Fig. 2: The logarithm of the absolute deviation from the limiting solution \( \Delta z^1_n = |z^1_\infty - z^1_n| \) is shown as a function of the node number \( n \) for the location of the innermost node of the extremal SU(4) EYM black hole solutions of cases 1a, 2a and 2b of Table 1.
Fig. 3: The logarithm of the absolute deviation from the limiting solution $\Delta \mu_n = \mu_\infty(\infty) - \mu_n(\infty)$ is shown as a function of the node number $n$ for the mass of the extremal SU(4) EYM black hole solutions of cases 1a and 2b of Table 1, together with case 1a of ref. [8], representing extremal SU(5) EYM black hole solutions, as well as for the globally regular SU(2) EYM solutions.
Fig. 4a: The gauge field functions $u_j(x)$, $j = 1 - 3$, are shown as functions of the dimensionless coordinate $x$ for the globally regular SU(4) EYM solutions with node structure $(n,0,0)$ and node numbers $n = 1, 3, 5, 7$. The inset illustrates the convergence of the innermost node to a value just below $x = \sqrt{6}$ (thin vertical line) in the limit $n \to \infty$. The thin horizontal line represents the limiting function for $x > \sqrt{6}$. 
Fig. 4b: Same as Fig. 4a for the metric function $N(x)$. The inset illustrates the emergence of a second zero at $x = \sqrt{6}$ in the limit $n \to \infty$. 
Fig. 4c: Same as Fig. 4a for the charge function $P(x)$. The thin horizontal line represents the norm of the charge of the limiting solution, $P = \sqrt{6}$.
Fig. 5a: The gauge field functions $u_j(x)$, $j = 1 - 3$, are shown as functions of the dimensionless coordinate $x$ for the globally regular SU(4) EYM solutions with node structure $(n + 1, 1, 0)$ and node numbers $n = 1, 3, 5$. The thin horizontal line represents the limiting function for $x > \sqrt{6}$. 
Fig. 5b: Same as Fig. 5a for the metric function $N(x)$ and node numbers $n = 1 - 5$. The inset illustrates the emergence of a second zero at $x = \sqrt{6}$ (thin vertical line) in the limit $n \to \infty$. 

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Fig. 5b: Same as Fig. 5a for the metric function $N(x)$ and node numbers $n = 1 - 5$. The inset illustrates the emergence of a second zero at $x = \sqrt{6}$ (thin vertical line) in the limit $n \to \infty$. 

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Fig. 5c: Same as Fig. 5a for the charge function $P(x)$ and node numbers $n = 1 - 5$. The thin horizontal line represents the norm of the charge of the limiting solution, $P = \sqrt{6}$. 
Fig. 6: The logarithm of the absolute deviation from the limiting solution $\Delta \mu_n = \mu_\infty(\infty) - \mu_n(\infty)$ is shown as a function of the node number $n$ for the mass of the sequences of globally regular SU(4) EYM solutions with node structure $(n,0,0)$, $(n+1,1,0)$ and $(n+1,0,1)$. 
Fig. 7a: The value of the gauge field functions $u_1$ and $u_3$ at the horizon is shown as a function of the horizon radius $x_H$ for the SU(4) EYM black hole solutions with node structure $(n,0,n)$ and node numbers $n = 1, 3, 5$. For $n = 1$ both $u_1 = u_3$ and $u_1 \neq u_3$ solutions are shown. For $n = 3$ and $n = 5$ the $u_1 = u_3$ solutions are only shown in the inset ($\Delta x_H = 0.1$, $\Delta u = 0.02$). The critical values of the horizon radius $x_H^{cr}$ are also indicated.

Fig. 7a: The value of the gauge field functions $u_1$ and $u_3$ at the horizon is shown as a function of the horizon radius $x_H$ for the SU(4) EYM black hole solutions with node structure $(n,0,n)$ and node numbers $n = 1, 3, 5$. For $n = 1$ both $u_1 = u_3$ and $u_1 \neq u_3$ solutions are shown. For $n = 3$ and $n = 5$ the $u_1 = u_3$ solutions are only shown in the inset ($\Delta x_H = 0.1$, $\Delta u = 0.02$). The critical values of the horizon radius $x_H^{cr}$ are also indicated.
Fig. 7b: Same as Fig. 7a for the dimensionless mass $\mu(\infty)$. The inset ($\Delta x_H = 0.7, \Delta \mu = 0.0003$) only shows $n = 5$. 
Fig. 7c: Same as Fig. 7a for the temperature $T/T_S$ ($T_S = 4\pi x_H^{-1}$). The inset ($\Delta x_H = 0.5$, $\Delta T/T_S = 0.0003$) only shows $n = 5$. 
Fig. 8a: The value of the gauge field functions $u_1$, $u_2$ and $u_3$ at the horizon is shown as a function of the horizon radius $x_H$ for the SU(4) EYM black hole solutions with node structure $(1, 1, 1)$. The upper right inset ($\Delta x_H = 0.03$, $\Delta u = 0.004$) shows the merging of the degenerate branches of $u_1 \neq u_3 \neq u_2$ solutions (dotted lines) into the lower branch of $u_1 = u_3 \neq u_2$ solutions at $x_{H1}^{cr,3}$. The lower left inset ($\Delta x_H = 2 \times 10^{-4}$, $\Delta u = 9 \times 10^{-6}$) shows two non-degenerate branches of $u_1 \neq u_3 \neq u_3$ solutions close to their critical value $x_{H1}^{cr,4}$. 
Fig. 8b: Same as Fig. 8a for the dimensionless mass $\mu(\infty)$. The inset ($\Delta x_H = 0.45, \Delta \mu = 0.012$) shows two non-degenerate branches of $u_1 \neq u_3 \neq u_2$ solutions close to their critical value $x_{H1}^{cr}$. 
Fig. 8c: Same as Fig. 8a for the temperature $T/T_S$. The inset ($\Delta x_H = 0.3, \Delta T/T_S = 0.02$) shows two non-degenerate branches of $u_1 \neq u_3 \neq u_2$ solutions close to their critical value $x_{H1}^{cr,4}$. 
Fig. 9: Same as Fig. 8a for the solution with node structure \((3, 3, 3)\). The upper right inset \((\Delta x_H = 0.05, \Delta u = 0.07)\) shows the two non-degenerate branches of \(u_1 = u_3 \neq u_2\) solutions close to their critical value \(x_{H,1}^{cr}\) as well as the branch of scaled SU(2) solutions. The lower left inset \((\Delta x_H = 0.1, \Delta u = 0.05)\) shows two degenerate branches of \(u_1 \neq u_3 \neq u_2\) solutions close to their critical value \(x_{H,3}^{cr}\) as well as the branch of scaled SU(2) solutions.