On the image of the trivial source ring in the ring of virtual characters of a finite group

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December 25, 2020

Abstract

We examine the cokernel of the canonical homomorphism from the trivial source ring of a finite group to the ring of $p$-rational complex characters. We use Boltje and Coşkun’s theory of fibered biset functors to determine the structure of the cokernel. An essential tool in the determination of this structure is Bouc’s theory of rational $p$-biset functors.

1 Introduction

Associated to a finite group $G$ are various representation rings which encode the actions of the group on sets, vector spaces, or other structures. Typical examples are the virtual character ring $R_C(G)$, the Burnside ring $B(G)$, and the trivial source ring $T_O(G)$ (for $O$ a complete discrete valuation ring). The theory of biset functors, introduced by Bouc in [6] and [7], provides a means of unifying the constructions $G \mapsto R_C(G)$, $G \mapsto B(G)$, etc. by realizing each as an additive functor defined on the biset category $\mathcal{C}$. The objects of $\mathcal{C}$ are finite groups, and $\mathcal{C}$ is generated as a preadditive category by five types of morphisms known as induction, restriction, inflation, deflation, and isomorphism. The images of these morphisms under a given representation ring (viewed now as a functor defined on $\mathcal{C}$) are the usual operations from the representation theory of finite groups. In [3] Boltje and Coşkun expand this framework by defining, for a fixed abelian group $A$, the $A$-fibered biset category $\mathcal{C}^A$ and $A$-fibered biset functors defined on it. Endowing a biset functor with the extra structure of fibered biset functor grants an additional type of operation: informally speaking, that of “multiplication” by a 1-dimensional $A$-character.
Several fundamental theorems have reinterpretations within this framework. For example, recall that for each finite group $G$ there is a linearization morphism $\text{lin}_G : B_{\mathbb{C}^*}(G) \to R_{\mathbb{C}}(G)$ from the $\mathbb{C}^*$-monomial Burnside ring to the virtual character ring. It turns out that the maps $\text{lin}_G$ are the components of a natural transformation $\text{lin} : B_{\mathbb{C}^*} \to R_{\mathbb{C}}$ between $\mathbb{C}^*$-fibered biset functors, and Brauer’s Induction Theorem is equivalent to the statement that this natural transformation is surjective. From this perspective, some questions that arise are: (1) given a pair of fibered biset functors, what are the natural transformations between them? (2) given a natural transformation of fibered biset functors, what is its image/kernel/cokernel, etc.?

In this note we examine the cokernel of a certain natural transformation $\kappa$ from the trivial source ring to a subring of the virtual character ring. Let us briefly recall the definitions. If $\mathcal{O}$ is a complete discrete valuation ring with field of fractions $K$ of characteristic 0 and residue field $k$ of positive characteristic $p$, then the triple $(K, \mathcal{O}, k)$ is called a $p$-modular system. Assume that $k$ is algebraically closed. If $G$ is a finite group, recall that a finitely generated $\mathcal{O}G$-module is a trivial source (or $p$-permutation) module if it is isomorphic to a direct summand of a permutation $\mathcal{O}G$-module. The Grothendieck ring of the category of trivial source modules, with respect to split short exact sequences, is denoted $T_{\mathcal{O}}(G)$ and is called the trivial source ring (see [2, Section 5.5]). We also have $R_K(G)$, the Grothendieck ring of the category of finitely generated $KG$-modules. Extension of scalars from $\mathcal{O}$ to $K$ induces for each finite group $G$ a homomorphism $\kappa_G : T_{\mathcal{O}}(G) \to R_K(G)$, and in fact the maps $\kappa_G$ are the components of a natural transformation $\kappa : T_{\mathcal{O}} \to R_K$ between fibered biset functors. Let $\mu'_p$ denote the group of complex roots of unity whose orders are not divisible by $p$. Then because of the assumption on $k$ we can identify $K = \mathbb{Q}(\mu'_p)$ as a subfield of $K$, and we can identify $R_K(G)$ as a subring of $R_K(G)$. By a theorem of Dress (specifically, [11, Theorem 1]), for each finite group $G$ the image of $\kappa_G$ is contained in $R_K(G)$. Thus we may regard $\kappa$ as a natural transformation from $T_{\mathcal{O}}$ to $R_K$.

Our motivation for studying the cokernel $R_K / \text{im}(\kappa)$ comes from recent work of Boltje and Perpeletisky. Let $G$ and $H$ be finite groups, let $A$ be a block algebra of $\mathcal{O}G$, and let $B$ be a block algebra of $\mathcal{O}H$. In [4] Boltje and Perpeletisky define a $p$-permutation equivalence between $A$ and $B$ to be an element $\gamma \in T^{\Delta}(A, B)$, the subgroup of the Grothendieck group of trivial source $(A, B)$-bimodules spanned by the isomorphism classes of indecomposable trivial source bimodules having twisted diagonal vertices, that satisfies

$$\gamma_H \cdot \gamma^o = [A] \in T^{\Delta}(A, A) \quad \text{and} \quad \gamma^o \cdot \gamma = [B] \in T^{\Delta}(B, B).$$
Here $\gamma^o$ denotes the $O$-dual of $\gamma$ and $\cdot$: is the map induced by the tensor product over $OH$. The authors show (see \[4\] Theorem 1.6 and Theorem 15.4) that a $p$-permutation equivalence $\gamma \in T^\Delta(A, B)$ induces an isotypy between $A$ and $B$. The perfect isometries that make up this induced isotypy are elements of $\mathrm{im}(\kappa_C)$ for various groups $C$. Thus if one wishes to lift a given isotypy to a $p$-permutation equivalence one must first check that the perfect isometries of the isotypy are contained in the image of $\kappa$. In other words, the cokernel $R_K/\mathrm{im}(\kappa)$ measures a first obstruction to the problem of lifting an isotypy to a $p$-permutation equivalence.

It turns out that the image of $\kappa$ almost always coincides with $R_K$ (see Corollary 5.5):

**Theorem 1.1.** If $p \neq 2$ then $\mathrm{im}(\kappa) = R_K$. If $p = 2$ then for any finite group $G$ the quotient $R_K(G)/\mathrm{im}(\kappa_G)$ is a finite-dimensional vector space over $\mathbb{F}_2$.

Thus when $p = 2$ the cokernel $R_K/\mathrm{im}(\kappa)$ is a nonzero fibered biset functor for the fiber group $A = \mu_2'$. In Theorem 5.8 we determine the complete subfunctor structure of this quotient. As part of this classification it is shown that $R_K/\mathrm{im}(\kappa)$ is generated as a fibered biset functor by the unique faithful irreducible character of the quaternion group $Q_8$. If “the problem” with the prime 2 is that $\mathrm{im}(\kappa) \neq R_K$, then in some sense this result identifies $Q_8$ as the source of the problem. We also provide, in Corollary 5.9, a necessary and sufficient condition for a character $\chi \in R_K(G)$ to be an element of $\mathrm{im}(\kappa_G)$.

The characterization of the subfunctors of $R_K/\mathrm{im}(\kappa)$ makes use of Bouc’s theory of rational $p$-biset functors in an essential way. These are functors that are defined on the full subcategory $C_p$ of the biset category whose objects are $p$-groups and behave like the functor $R_Q$ of rational representations (we recall the exact definition in Section 1). The class of rational $p$-biset functors is closed under taking subfunctors and taking quotients, and the subfunctors of a fixed rational $p$-biset functor have been completely classified by Bouc (see \[7\] Section 10.2). In Section 5 we show that the restriction of $R_K/\mathrm{im}(\kappa)$ to $C_p$ is a rational $p$-biset functor; in fact, it is equal to $\overline{R}_Q/R_Q$ (here, if $G$ is a finite group then $\overline{R}_Q(G)$ denotes the ring of $\mathbb{Q}$-valued virtual characters of $G$). In subsection 5.3 we will see that $\overline{R}_Q$ has a biset functor structure, hence restricts to a $p$-biset functor). Using rationality, we determine the subfunctors of $\overline{R}_Q/R_Q$ in Theorem 4.4. We then find, through induction of fibered biset functors, that the subfunctor structure of $R_K/\mathrm{im}(\kappa)$ is identical to that of $\overline{R}_Q/R_Q$. In particular, the nonzero subfunctors (in the case $p = 2$) are generated by characters of the generalized quaternion groups and both functors are uniserial. Moreover, Theorem 1.1 reflects a theorem of Roquette, which is given in \[13\] Corollary 10.14: if $p \neq 2$ then $\overline{R}_Q(P) = R_Q(P)$ for
any finite \( p \)-group \( P \) and if \( p = 2 \) then the quotient \( \overline{R}_Q(P)/R_Q(P) \) is a finite-dimensional vector space over \( \mathbb{F}_2 \).

The author would like to thank his advisor, Robert Boltje, for his constant support and for donating generous amounts of his time toward the completion of this project.

## 2 Characters and Schur Indices

We begin by setting some of the notation that will be used throughout. Let \( G \) be a finite group and let \( K \) be a field of characteristic 0. The category of finitely generated left \( KG \)-modules is denoted \( KG\text{-mod} \). The corresponding Grothendieck ring, with respect to short exact sequences, is denoted \( KG \)-modules is denoted \( R_K(G) \). The corresponding Grothendieck ring, with respect to short exact sequences, is denoted \( R_K(G) \).

If \( V \in KG\text{-mod} \) we write \([V]\) for the isomorphism class of \( V \) and, abusively, the image of \( V \) in \( R_K(G) \). The ring \( R_K(G) \) is identified with the virtual character ring of \( KG \), which is the subring of the ring of \( K \)-valued class functions on \( G \) generated by the characters of the irreducible \( KG \)-modules.

The set \( \text{Irr}_K(G) \) of irreducible characters of \( KG \) is a \( \mathbb{Z} \)-basis of \( R_K(G) \). If \( L \) is an extension of \( K \) then extension of scalars from \( K \) to \( L \) induces an injective ring homomorphism \( R_K(G) \to R_L(G) \). Since this homomorphism is compatible with the identification of \( R_K(G) \) with the virtual character ring of \( KG \) we identify \( R_K(G) \) as a subring of \( R_L(G) \).

Let \( L \supseteq K \) be an extension of fields. If \( \chi \in \text{Irr}_L(G) \) then \( K(\chi) \) denotes the subfield of \( L \) generated by \( K \) and the character values \( \chi(g), g \in G \). If also \( \psi \in \text{Irr}_L(G) \) then \( \chi \) and \( \psi \) are said to be \emph{Galois conjugate over \( K \)} if \( K(\chi) = K(\psi) \) and if there exists an automorphism \( \sigma \in \text{Gal}(K(\chi)/K) \) such that \( \sigma(\chi(g)) = \psi(g) \) for all \( g \in G \). In this case we write \( ^\sigma \chi = \psi \). Keeping \( K \) and \( L \) fixed any \( \chi \in \text{Irr}_L(G) \) determines a \emph{Galois class sum} \( \overline{\chi} \), defined to be the sum of the distinct Galois conjugates of \( \chi \) over \( K \).

When \( L \) is algebraically closed the ring of \( K \)-valued elements of \( R_L(G) \) is denoted \( R_K(G) \). If \( \chi_1, \chi_2, \ldots, \chi_r \) are representatives for the equivalence classes of \( \text{Irr}_L(G) \) with respect to Galois conjugacy over \( K \) then the Galois class sums \( \overline{\chi_i}, 1 \leq i \leq r \), form a \( \mathbb{Z} \)-basis of \( R_K(G) \). Now if \( \chi \in \text{Irr}_L(G) \) then some positive multiple \( m_\chi \) of \( \chi \) is the character of a \( K(\chi)G \)-module. The least \( m \) for which this holds is the \emph{Schur index of} \( \chi \) over \( K \) and is denoted \( m_{K}(\chi) \).

If \( \theta \in \text{Irr}_K(G) \) then \( \theta = m_{K}(\chi_i)\overline{\chi_i} \) for some unique \( \chi_i \) among the representatives above. Furthermore, any character of the form \( m_{K}(\chi_i)\overline{\chi_i} \) is afforded by a (unique, up to isomorphism) irreducible \( KG \)-module. Therefore the characters \( m_{K}(\chi_i)\overline{\chi_i}, 1 \leq i \leq r \), form a \( \mathbb{Z} \)-basis of \( R_K(G) \). Consequently, \footnote{A thorough exposition of the theory of Schur indices may be found in \cite{13} Chapters 9 and 10].}
as abelian groups we have

\[ \mathcal{R}_K(G)/R_K(G) \cong \prod_{i=1}^{r} \mathbb{Z}/m_K(\chi_i)\mathbb{Z}. \]  

(1)

For the remainder of this section we take \( L = \mathbb{C} \). For ease, set \( \text{Irr}(G) = \text{Irr}_\mathbb{C}(G) \).

**Lemma 2.1.** Let \( P \) be a \( p \)-group, where \( p \in \mathbb{N} \) is prime, and let \( \chi \in \text{Irr}(P) \). If \( m_Q(\chi) \neq 1 \) then \( p = 2 \), and if \( p = 2 \) then \( m_Q(\chi) = 1 \) or 2. In particular, \( R_Q(P) \neq \mathcal{R}_Q(P) \) only when \( P \) is a 2-group, and in this case \( \mathcal{R}_Q(P)/R_Q(P) \) is a (finite-dimensional) vector space over \( \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} \).

**Proof.** The first part of the lemma is just a particular case of a result of Roquette [15] — a proof is also given in [13, Corollary 10.14]. The second part follows from the first and the isomorphism given in (1), above.

**Lemma 2.2.** Let \( P \) be a 2-group. If \( P \) is cyclic, dihedral, or semidihedral then \( m_Q(\chi) = 1 \) for all \( \chi \in \text{Irr}(P) \).

**Proof.** If \( P \) is cyclic this is a consequence of Wedderburn’s Theorem. A proof for the case where \( P \) is dihedral or semidihedral is given in [12, (11.7)].

We now describe a part of the representation theory of the generalized quaternion group \( Q_{2^n} \). Recall that \( Q_{2^n} \) is the group of order \( 2^n \), \( n \geq 3 \), defined by the presentation

\[ Q_{2^n} = \langle a, b | a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, ab = a^{-1} \rangle. \]

The center \( Z(Q_{2^n}) \) of \( Q_{2^n} \) is generated by \( b^2 \) and is the unique minimal nontrivial subgroup of \( Q_{2^n} \). Note that the quotient \( Q_{2^n}/Z(Q_{2^n}) \) is dihedral of order \( 2^{n-1} \). Also note that when \( n \geq 4 \) the subgroup of \( Q_{2^n} \) generated by \( a^2 \) and \( b \) is isomorphic to \( Q_{2^{n-1}} \). In fact, there are exactly 2 subgroups of \( Q_{2^n} \) that are isomorphic to \( Q_{2^{n-1}} \).

**Lemma 2.3.** Fix an integer \( n \geq 3 \) and let \( Q_{2^n} \) be defined as above.

(a) The set of faithful \( \chi \in \text{Irr}(Q_{2^n}) \) is a (nonempty) Galois conjugacy class over \( \mathbb{Q} \). Define

\[ \gamma_n = \sum_{\chi \in \text{Irr}(Q_{2^n}) \text{ faithful}} \chi. \]

(b) Let \( \chi \in \text{Irr}(Q_{2^n}) \). Then \( \chi \) is faithful if and only if \( m_Q(\chi) = 2 \).
(c) The quotient $\overline{\mathbb{Q}}_Q(Q_{2^n})/Q(Q_{2^n})$ is an $F_2$-space of dimension 1, with unique nonzero element (the coset containing) $\gamma_n$.

(d) Let $\lambda : Z(Q_{2^n}) \to \mathbb{Q}^\times$ be the representation defined by $\lambda(b^2) = -1$ and let $Q\lambda$ be a $\mathbb{Q}Z(Q_{2^n})$-module affording $\lambda$. Set $\Phi_{Q_{2^n}} = \text{Ind}_{Z(Q_{2^n})}^{Q_{2^n}}(\mathbb{Q}\lambda)$. Then $\Phi_{Q_{2^n}}$ is a faithful irreducible $\mathbb{Q}Q_{2^n}$-module with character $2\gamma_n$.

(e) If $n \geq 4$ then we have

$$\text{Ind}_{Q_{2^{n-1}}}^{Q_{2^n}}(\gamma_{n-1}) = \gamma_n \quad \text{and} \quad \text{Res}_{Q_{2^{n-1}}}^{Q_{2^n}}(\gamma_n) = 2\gamma_{n-1},$$

where $Q_{2^{n-1}}$ is understood to be either of the 2 subgroups of $Q_{2^n}$ isomorphic to $Q_{2^{n-1}}$.

**Proof.** The faithful irreducible complex characters of the subgroup $\langle a \rangle$ induce to faithful irreducible characters of $Q_{2^n}$, and any faithful irreducible character of $Q_{2^n}$ arises this way. Since Galois conjugation commutes with induction, (a) holds. Let $\chi \in \text{Irr}(Q_{2^n})$ and suppose that $\chi$ is not faithful. Then $\chi$ is an irreducible character of the dihedral group $Q_{2^n}/Z(Q_{2^n})$ so $m_\mathbb{Q}(\chi) = 1$ by Lemma 2.2. If $\chi$ is faithful then $m_\mathbb{Q}(\chi) = 2$: this is shown in [12, (11.8)]. Thus (b) holds and (c) follows.

From (a) and (b) we know that $2\gamma_n$ is the character of a faithful irreducible $\mathbb{Q}Q_{2^n}$-module. Therefore to prove (d) it suffices to prove that the character $\text{Ind}_{Z(Q_{2^n})}^{Q_{2^n}}(\lambda)$ is equal to $2\gamma_n$. This follows from a short argument involving Frobenius reciprocity and the fact that the faithful irreducible complex characters of $Q_{2^n}$ each have degree 2.

It remains to prove (e). Assume that $n \geq 4$ and denote by $Q_{2^{n-1}}$ a subgroup of $Q_{2^n}$ isomorphic to $Q_{2^{n-1}}$. Set $Z = Z(Q_{2^n})$. Then $Z = Z(Q_{2^{n-1}})$. We have

$$2\gamma_n = \text{Ind}_{Z}^{Q_{2^n}}(\lambda) = \text{Ind}_{Q_{2^{n-1}}}^{Q_{2^n}}(2\gamma_{n-1}) = 2 \text{Ind}_{Q_{2^{n-1}}}^{Q_{2^n}}(\gamma_{n-1}),$$

hence $\text{Ind}_{Q_{2^{n-1}}}^{Q_{2^n}}(\gamma_{n-1}) = \gamma_n$. By Mackey’s formula we have

$$\text{Res}_{Q_{2^{n-1}}}^{Q_{2^n}}(2\gamma_n) = \text{Res}_{Q_{2^{n-1}}}^{Q_{2^n}}(\text{Ind}_{Z}^{Q_{2^n}}(\lambda)) = 2 \text{Ind}_{Z}^{Q_{2^{n-1}}}(\lambda) = 4\gamma_{n-1},$$

so $\text{Res}_{Q_{2^{n-1}}}^{Q_{2^n}}(\gamma_n) = 2\gamma_{n-1}$. The proof is complete.

We remark that $\Phi_{Q_{2^n}} = \text{Ind}_{Z(Q_{2^n})}^{Q_{2^n}}(\mathbb{Q}\lambda)$ is the unique faithful irreducible $\mathbb{Q}Q_{2^n}$-module up to isomorphism.

Now let $p \in \mathbb{N}$ be prime. A root of unity is a $p'$-root of unity if it has order relatively prime to $p$. The set of $p'$-roots of unity in $\mathbb{C}$ forms a subgroup of $\mathbb{C}^\times$ and is denoted $\mu_{p'}$. 

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Lemma 2.4. ([8, Lemma 1*]) Let \( G \) be a finite group and let \( K = \mathbb{Q}(\mu_p) \), where \( p \in \mathbb{N} \) is prime. Then \( m_K(\chi) = 1 \) for all \( \chi \in \text{Irr}(G) \). In particular, \( R_K(G) = R_K(G) \).

3 Biset Functors

In this section we provide a review of some of the basic notions from the theories of biset functors, which was introduced by Bouc in [6], and of fibered biset functors, developed by Boltje and Coşkun in [3].

3.1 Bisets, Burnside Groups, and Biset Categories

Let \( G \) and \( H \) be finite groups. The categories of finite left and right \( G \)-sets are denoted \( \mathbb{G}\text{set} \) and \( \text{set}_G \), respectively, and the category of finite \((G, H)\)-bisets is denoted \( \mathbb{G}\text{set}_H \). The Burnside groups \( B(G) \) and \( B(G, H) \) are defined to be the Grothendieck groups, with respect to disjoint unions, of \( \mathbb{G}\text{set} \) and \( \mathbb{G}\text{set}_H \).

Any right \( H \)-set \( X \) may be viewed as a left \( H \)-set via the rule \( h \cdot x = x h^{-1} \) for all \( h \in H \) and \( x \in X \), and this convention induces category isomorphisms \( \mathbb{G}\text{set}_H \cong H\text{set} \) and \( \mathbb{G}\text{set}_H \cong G \times H\text{set} \). In particular, \( B(G, H) \cong B(G \times H) \).

Let \( A \) be an abelian (potentially infinite) group. An \( A \)-fibered \( G \)-set is a (left) \( G \)-set equipped with a free, two-sided action of \( A \) that has finitely many \( A \)-orbits and commutes with the action of \( G \). Any \( A \)-fibered \( G \)-set may be viewed as an \( A \times G \)-set in a natural way. The category of \( A \)-fibered \( G \)-sets is denoted \( A\text{set}_G \).

The \( (A \)-fibered) Burnside group \( B^A(G) \) of \( A \)-fibered \( G \)-sets, which was introduced by Dress in [10], is defined as the Grothendieck group, with respect to disjoint unions, of \( A\text{set} \). Likewise the Burnside group \( B^A(G, H) \) is the Grothendieck group of the category \( A\text{set}_H \). Note that \( B(G) \cong B^{(1)}(G) \) and that \( B^A(G, H) \cong B^A(G \times H) \) — we identify the groups in these pairs. The construction \( X \mapsto A \times X \) produces a split injective homomorphism \( B(G) \to B^A(G) \) with left inverse \( B^A(G) \to B(G) \) induced by \( Y \mapsto Y/A \).

The isomorphism classes of transitive \( A \)-fibered \( G \)-sets form a \( \mathbb{Z} \)-basis of \( B^A(G) \) which we call the standard basis. There is a bijection between the set of isomorphism classes \([X]\) of transitive \( A \)-fibered \( G \)-sets and the set of \( G \)-conjugacy classes of pairs \((U, \phi)\) where \( U \leq G \) and \( \phi : U \to A \) is a group homomorphism: to such a class \([X]\) one associates the \( G \)-conjugacy class of the stabilizing pair \((U_x, \phi_x)\) where \( x \in X \), \( U_x \) is the stabilizer in \( G \) of the
A-orbit of $x$, and $\phi_x : U_x \to A$ is the (well-defined) group homomorphism given by the equation $gx = \phi_x(g)x$ for each $g \in U_x$. If $X$ is a transitive $A$-fibered $G$-set whose isomorphism class $[X]$ corresponds to the conjugacy class of the pair $(U, \phi)$ then we write

$$[X] = \left[ \frac{G}{U, \phi} \right].$$

Similar statements hold for $A$-fibered $(G, H)$-bisets, and in particular for $A$-fibered $G \times H$-sets.

Let $G$, $H$, and $K$ be groups. If $X$ is a $(G, H)$-biset and $Y$ is an $(H, K)$-biset then the Cartesian product $X \times Y$ is a left $H$-set via $h \cdot (x, y) = (xh^{-1}, hy)$ for all $h \in H$, $x \in X$, and $y \in Y$. The set of $H$-orbits is a $(G, K)$-biset denoted $X \times_H Y$. The operation $\times_H$ is associative up to a canonical isomorphism and extends uniquely to a bilinear map $\times_H : B(G, H) \times B(H, K) \to B(G, K)$. If in addition $X$ and $Y$ are $A$-fibered then in a similar manner one may construct an $A$-fibered $(G, K)$-biset denoted $X \otimes_{AH} Y$.

Again the operation $\otimes_{AH}$ is associative up to a canonical isomorphism and extends uniquely to a bilinear map $\otimes_{AH} : B_A(G, H) \times B_A(H, K) \to B_A(G, K)$.

Bouc defines the biset category $C$ as follows (see [7, Section 3.1]): the objects of $C$ are all finite groups. If $G$ and $H$ are finite groups then we set $\text{Hom}_C(H, G) = B(G, H)$. If also $K$ is a finite group then the composition $v \circ u$ of $v \in \text{Hom}_C(H, G)$ and $u \in \text{Hom}_C(K, H)$ is $v \times_H u$. The isomorphism class of the $(G, G)$-biset $G$ — a biset under left and right multiplication — is the identity of $G$. We note that the biset category $C$ is preadditive.

In [3], Boltje and Coşkun define the $A$-fibered biset category $C^A$ to be the preadditive category whose objects are all finite groups and whose morphism sets $\text{Hom}_{C^A}(H, G) = B^A(G, H)$ are the $A$-fibered Burnside groups. If $G$, $H$, and $K$ are finite groups then the composition $v \circ u$ of $v \in \text{Hom}_{C^A}(H, G)$ and $u \in \text{Hom}_{C^A}(K, H)$ is $v \otimes_{AH} u$. The identity of $G$ is the isomorphism class of the $A$-fibered $(G, G)$-biset $A \times G$.

The injective group homomorphisms $B(G, H) \to B^A(G, H)$ described above induce a faithful additive functor $C \to C^A$ that is the identity on objects. We therefore view $C$ as a subcategory of $C^A$ without further comment.

Let $p$ be a prime number. The $p$-biset category $C_p$ is the full subcategory of the biset category $C$ whose objects are the finite $p$-groups. Similarly, one has the $A$-fibered $p$-biset category $C^A_p$.

We recall the definitions of some important bisets. Let $f : H \to G$ be a

\[\text{Some extra care is needed here to ensure that the resulting biset has a free } A\text{-action. The precise definition is given in [5, Section 2.1].}\]
homomorphism of finite groups. Set
\[ f \Delta(H) = \{(f(h), h) : h \in H\} \quad \text{and} \quad \Delta_f(H) = \{(h, f(h)) : h \in H\}. \]
If \( H \) is a subgroup of \( G \) and \( f \) is the inclusion map we write \( \Delta(H) \) for \( f \Delta(H) \) or \( \Delta_f(H) \). If \( f \) is an isomorphism set
\[ \text{Iso}(f) = \left[ \frac{G \times H}{f \Delta(H), 1} \right]. \]
Notice that \( \text{Iso}(f) \) is an isomorphism between \( H \) and \( G \) and that \( \text{Iso}(\text{id}_G) \) is the identity of \( G \) in \( \mathcal{C}^A \). If \( H \leq G \) set
\[ \text{Ind}_H^G = \left[ \frac{G \times H}{\Delta(H), 1} \right] \quad \text{and} \quad \text{Res}_H^G = \left[ \frac{H \times G}{\Delta(H), 1} \right]. \]
These morphisms are called induction and restriction, respectively. Now let \( N \trianglelefteq G \) and let \( \pi : G \rightarrow G/N \) denote the canonical projection. The morphisms
\[ \text{Inf}_{G/N}^G = \left[ \frac{G \times G/N}{\Delta_\pi(G), 1} \right] \quad \text{and} \quad \text{Def}_{G/N}^G = \left[ \frac{G/N \times G}{\pi\Delta(G), 1} \right] \]
are called inflation and deflation, respectively. If \( (H, N) \) is a section of \( G \) — that is, if \( N \trianglelefteq H \leq G \) — then set
\[ \text{Indinf}_{H/N}^G = \text{Ind}_H^G \circ \text{Inf}_{H/N}^G \quad \text{and} \quad \text{Defres}_{H/N}^G = \text{Def}_{H/N}^G \circ \text{Res}_H^G. \]
Finally, let \( \phi : G \rightarrow A \) be a group homomorphism. Precomposition with the canonical group isomorphism \( \Delta(G) \cong G \) produces a group homomorphism \( \Delta(G) \rightarrow A \) which we denote, abusively, by \( \phi \). Set
\[ \text{Mult}(\phi) = \left[ \frac{G \times G}{\Delta(G), \phi} \right]. \]
Note that \( \text{Mult}(\phi) \), which we call multiplication by \( \phi \), is an endomorphism of \( G \) in \( \mathcal{C}^A \) (but is not an endomorphism in \( \mathcal{C} \), in general). We remark that multiplication by \( \phi \) is called a “twist with \( \phi \)” by Boltje and Yılmaz in [5, Section 3.7(d)].

### 3.2 Biset Functors

We continue with the notation of the previous subsection: \( G \) and \( H \) stand for finite groups and \( A \) is a fixed abelian group. Let \( \mathcal{D} \) be a preadditive
subcategory of the A-fibered biset category $\mathcal{C}^A$. An A-fibered biset functor defined on $\mathcal{D}$ is an additive functor from $\mathcal{D}$ to $\underline{z\text{Mod}}$, the category of left $\mathbb{Z}$-modules. If $\mathcal{D} \subseteq \mathcal{C}$ then an A-fibered biset functor defined on $\mathcal{D}$ is simply called a biset functor, and if $\mathcal{D} = \mathcal{C}_p$ for some prime $p$ — that is, if $\mathcal{D}$ is the full subcategory of $\mathcal{C}$ whose objects are the finite $p$-groups — then a biset functor defined on $\mathcal{D}$ is called a $p$-biset functor.

When $\mathcal{D}$ is essentially small the A-fibered biset functors defined on $\mathcal{D}$ are the objects of a category $\mathcal{F}_D^A$ whose morphisms are natural transformations. Since $\underline{z\text{Mod}}$ is abelian, so is $\mathcal{F}_D^A$: kernels and cokernels are constructed “pointwise” at each group $G \in \mathcal{D}$. Moreover, a morphism $\eta$ is monic/epic if and only if $\eta_G$ is injective/surjective for all $G \in \mathcal{D}$. For simplicity we write $\mathcal{F}^A$ in place of $\mathcal{F}_D^A$ and we write $\mathcal{F}$ in place of $\mathcal{F}^A$. In addition, if $\mathcal{D} = \mathcal{C}_p$ for some prime $p$ then $\mathcal{F}_p$, denotes the category of $p$-biset functors.

When $F \in \mathcal{F}_D^A$, $\theta \in F(H)$ for some $H \in \mathcal{D}$, and $u \in \text{Hom}_p(H,G)$ we sometimes write $u(\theta)$ instead of $F(u)\theta)$. Thus we may write $\text{Ind}_{G}^{H}(\theta)$ rather than $F(\text{Ind}_{G}^{H}(\theta))$, for example.

A homomorphism $A' \rightarrow A$ of abelian groups induces homomorphisms $B^A(G,H) \rightarrow B^{A'}(G,H)$ for all finite groups $G$ and $H$. These maps in turn induce an additive functor $\mathcal{C}^{A'} \rightarrow \mathcal{C}^A$, and restriction along this functor yields an additive functor $\mathcal{F}^A \rightarrow \mathcal{F}^{A'}$. In particular, if $A'$ is a subgroup of $A$ then any $A$-fibered biset functor restricts to an $A'$-fibered biset functor. The inclusion Tor$(A) \subseteq A$ induces isomorphisms $B^{\text{Tor}(A)}(G,H) \cong B^{A}(G,H)$, so the resulting functors $C^{\text{Tor}(A)} \rightarrow \mathcal{C}^A$ and $\mathcal{F}^A \rightarrow \mathcal{F}^{\text{Tor}(A)}$ are category isomorphisms. In particular, no structure is lost when restricting an $A$-fibered biset functor to a Tor$(A)$-fibered biset functor.

If $F'$ and $F$ are objects in $\mathcal{F}_D^A$ then we say that $F'$ is a subfunctor of $F$ and we write $F' \subseteq F$ if $F'(G)$ is a subgroup of $F(G)$ for all $G \in \mathcal{D}$ and if the inclusion maps $\iota_G : F'(G) \rightarrow F(G)$ define a natural transformation $\iota : F' \rightarrow F$. Note that a subfunctor $F'$ of $F$ is uniquely determined by the data of a subgroup $F'(G)$ of $F(G)$, for each $G \in \mathcal{D}$, such that $F'(u)$ restricts to a homomorphism $F'(H) \rightarrow F'(G)$ for all $u \in \text{Hom}_p(H,G)$. If $F''$ and $F'''$ are subfunctors of $F \in \mathcal{F}_D^A$ then $F'' = F'''$ if and only if $F''(G) = F'''(G)$ for all groups $G \in \mathcal{D}$. Given a set of subfunctors $\{F_j\}$ of an $A$-fibered biset functor $F$ defined on $\mathcal{D}$ the intersection $\cap F_j$ is the subfunctor of $F$ defined by $(\cap F_j)(G) = \cap F_j(G)$ for all $G \in \mathcal{D}$. One also has the notion of a subfunctor generated by elements: let $S$ be a set of objects of $\mathcal{D}$ and for each $H \in S$ let $S_H$ be a subset of $F(H)$. The subfunctor of $F$ generated by $\cup_{H \in S} S_H$ is the intersection of all subfunctors $F'$ of $F$ such that $F'(H) \supseteq S_H$ for all $H \in S$. When $\cup_{H \in S} S_H$ consists of a single element $\gamma$ then the subfunctor generated by $\cup_{H \in S} S_H$ will instead be called the subfunctor generated by $\gamma$. Note that
if $F'$ is the subfunctor of $F$ generated by $\gamma \in F(H)$ then we have

$$F'(G) = \text{Hom}_D(H, G)(\gamma) = \{u(\gamma) : u \in \text{Hom}_D(H, G)\}$$

for all $G \in \mathcal{D}$.

### 3.3 Examples

**Example 3.1.** (The virtual character ring) Let $K$ be a field of characteristic 0 and let $G$ be a finite group. As in [3, Section 11B] the construction $R_K : G \mapsto R_K(G)$ may be extended to a $K^\times$-fibered biset functor defined on the full $K^\times$-fibered biset category $\mathcal{C}^{K^\times}$ by setting $R_K \left( \left[ \frac{G \times H}{U, \phi} \right] \right)$, for any finite groups $G, H$ and any standard basis element $\left[ \frac{G \times H}{U, \phi} \right] \in B^{K^\times}(G, H)$, equal to the map

$$R_K \left( \left[ \frac{G \times H}{U, \phi} \right] \right) : R_K(H) \to R_K(G)$$

$$[V] \mapsto [\text{Ind}^{G \times H}_U(K \phi) \otimes_{KH} V]$$

where $V$ is any $KH$-module. The images under $R_K$ of the morphisms $\text{Ind}^G_H$, $\text{Res}^G_H$, $\text{Inf}^{G/N}_H$, etc. which were introduced in Subsection 3.1 coincide with the familiar operations from representation theory. For example, if $H \leq G$ then $R_K(\text{Ind}^G_H)$ is the map induced by extension of scalars $KG \otimes_{KH} \cdot$, which justifies the abuse of notation $\text{Ind}^G_H = R_K(\text{Ind}^G_H)$. Also, if $\phi : G \to K^\times$ is a homomorphism and $\chi$ is a $K$-character of $G$ then $R_K(\text{Mult}(\phi))(\chi) = \phi \chi$.

Let $L$ be an extension of $K$. Since $K^\times \leq L^\times$ we may view $R_L$ as a $K^\times$-fibered biset functor. The maps $\iota_G : R_K(G) \to R_L(G)$ induced by extension of scalars from $K$ to $L$ are the components of an injective (monic) natural transformation $\iota : R_K \hookrightarrow R_L$. We identify $R_K$ with its image under $\iota$ so that $R_K$ is a $K^\times$-fibered subfunctor of $R_L$.

Now let $L$ be an algebraically closed extension of $K$. Recall that $\overline{R_K}(G)$ denotes the subring of $K$-valued virtual characters in $R_L(G)$. The character formula in the first part of [4, Lemma 7.1.3] can be used to show that $\overline{R_K}$ is a $K^\times$-fibered subfunctor of $R_L$. Since $R_K(G) \subseteq \overline{R_K}(G)$ for all finite groups $G$ it follows that $R_K$ is a subfunctor of $\overline{R_K}$. In particular one may form the quotient $\overline{R_K}/R_K \in \mathcal{F}^{K^\times}$, whose evaluation at a group $G$ is given by

$$\left( \overline{R_K}/R_K \right)(G) = \overline{R_K}(G)/R_K(G).$$

**Example 3.2.** (The trivial source ring) Let $(\mathbb{K}, \mathcal{O}, k)$ be a $p$-modular system (see Section 1) and assume that $k$ is algebraically closed. Then $\mathcal{O}$ contains
all $p'$-roots of unity. Fix an embedding $\mathbb{Q}(\mu_{p'}) \to \mathbb{K}$, where as before $\mu_{p'}$ denotes the group of complex roots of unity having order prime to $p$. Then $\mu_{p'} \leq \mathcal{O}^\times$ and $\mu_{p'}$ maps bijectively onto the group of roots of unity in $k$ via the canonical projection $\mathcal{O} \to k$. We identify $\mu_{p'}$ with its image under this projection.

Recall that an $\mathcal{O}_G$-module is a trivial source (or $p$-permutation) module if it is isomorphic to a direct summand of a permutation $\mathcal{O}_G$-module. The Grothendieck ring of the category of trivial source $\mathcal{O}_G$-modules, with respect to split short exact sequences, is denoted $T^G\mathcal{O}$ and is called the trivial source ring (see [2, Section 5.5]). Following the procedure of the previous example the construction $T^G\mathcal{O} \mapsto T^G\mathcal{O}$ may be given the structure of a $\mu_{p'}$-fibered biset functor.

For each finite group $G$ there is a homomorphism $\kappa_G : T^G\mathcal{O}(G) \to R^G\mathcal{K}(G)$ induced by extension of scalars from $\mathcal{O}$ to $\mathcal{K}$. Since $\mu_{p'} \leq \mathcal{K}^\times$ we may regard $R^G\mathcal{K}$ as a $\mu_{p'}$-fibered biset functor. Then $\kappa : T^G \to R^G\mathcal{K}$ is a morphism in $\mathcal{F}^\mu_{p'}$. Now the image $\im(\kappa)$ of $\kappa$ is a subfunctor of $R^G\mathcal{K}$, and by [11, Theorem 1] the evaluation of $\im(\kappa)$ at a group $G$ is equal to

$$\im(\kappa_G) = \langle [\Ind_H^G(G) \phi] : H \leq G, \phi : H \to \mu_{p'} \rangle.$$

Observe that $\im(\kappa_G)$ is a subgroup (in fact, a subring) of $R^G\mathcal{K}(G)$ for any group $G$. Thus $\im(\kappa)$ is a $\mu_{p'}$-fibered subfunctor of $R^G\mathcal{K}(\mu_{p'})$. In Section 5 we determine the subfunctor structure of the quotient $R^G\mathcal{K}(\mu_{p'})/\im(\kappa)$.

### 3.4 Induction of Fibered Biset Functors

Fix an abelian group $A$ and an essentially small preadditive subcategory $\mathcal{D}$ of $\mathcal{C}^A$. Restriction of $A$-fibered biset functors from $\mathcal{C}^A$ to $\mathcal{D}$ may be realized as an additive and exact functor $\text{Res} : \mathcal{F}^A \to \mathcal{F}^\mathcal{D}$, called restriction. The purpose of this subsection is to introduce the left adjoint $\text{Ind} : \mathcal{F}^\mathcal{D} \to \mathcal{F}^A$, called induction.

Let $F : \mathcal{D} \to \text{zMod}$ be an $A$-fibered biset functor defined on $\mathcal{D}$. Define the additive functor $\text{Ind}(F) : \mathcal{C}^A \to \text{zMod}$ as follows: for any $G \in \mathcal{C}^A$,

$$\text{Ind}(F)(G) = \left( \bigoplus_{P \in \mathcal{S}} \text{Hom}_{\mathcal{C}^A}(P, G) \otimes_{\mathbb{Z}} F(P) \right)/\mathcal{R}_G$$

where $\mathcal{S}$ is a fixed skeleton of $\mathcal{D}$ and $\mathcal{R}_G$ is the subgroup of the direct sum generated by all elements of the form

$$(v \circ w) \otimes \theta - v \otimes w(\theta),$$

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where \( w \in \text{Hom}_\mathcal{D}(P,Q) \) for \( P, Q \in \mathcal{S} \), \( \theta \in F(P) \), and \( v \in \text{Hom}_{\mathcal{C}A}(Q,G) \). If \( u \in \text{Hom}_{\mathcal{C}A}(H,G) \) then \( \text{Ind}(F)(u) \) is the group homomorphism satisfying

\[
\text{Ind}(F)(u)(v \otimes \theta) = (u \circ v) \otimes \theta.
\]

Here \( v \otimes \theta \) is a typical “simple tensor” in \( \text{Ind}(F)(H) \); \( v \in \text{Hom}_{\mathcal{C}A}(P,H) \), \( P \in \mathcal{S} \), and \( \theta \in F(P) \).

The construction \( \text{Ind} : F \mapsto \text{Ind}(F) \) extends to a functor from \( \mathcal{F}_\mathcal{A}^\mathcal{D} \) to \( \mathcal{F}_\mathcal{A} \); let \( F \) and \( F' \) be objects of \( \mathcal{F}_\mathcal{A}^\mathcal{D} \) and let \( \eta \in \text{Hom}_{\mathcal{F}_\mathcal{A}^\mathcal{D}}(F,F') \) be a natural transformation. For each \( H \in \mathcal{C}_A \) set \( \text{Ind}(\eta)_H : \text{Ind}(F)(H) \to \text{Ind}(F')(H) \) equal to the group homomorphism defined by

\[
\text{Ind}(\eta)_H(v \otimes \theta) = v \otimes \eta_P(\theta),
\]

where \( v \otimes \theta \) is a simple tensor of \( \text{Ind}(F)(H) \), as above. Then \( \text{Ind}(\eta) : \text{Ind}(F) \to \text{Ind}(F') \) is a natural transformation (i.e., a morphism in \( \mathcal{F}_\mathcal{A} \)). With these definitions in hand, checking that \( \text{Ind} \) is a functor from \( \mathcal{F}_\mathcal{A}^\mathcal{D} \) to \( \mathcal{F}_\mathcal{A} \) is straightforward.

**Proposition 3.3.** Let \( A \) be an abelian group and let \( \mathcal{D} \) be an essentially small preadditive subcategory of \( \mathcal{C}_A \). Then induction \( \text{Ind} : \mathcal{F}_\mathcal{A}^\mathcal{D} \to \mathcal{F}_\mathcal{A} \) is left adjoint to restriction \( \text{Res} : \mathcal{F}_\mathcal{A} \to \mathcal{F}_\mathcal{A}^\mathcal{D} \).

**Proof.** Fix a skeleton \( \mathcal{S} \) of \( \mathcal{D} \), let \( F \in \mathcal{F}_\mathcal{A}^\mathcal{D} \), and let \( F' \in \mathcal{F}_\mathcal{A} \). Define a natural bijection

\[
\psi = \psi_{F,F'} : \text{Hom}_{\mathcal{F}_\mathcal{A}^\mathcal{D}}(F,\text{Res}(F')) \to \text{Hom}_{\mathcal{F}_\mathcal{A}}(\text{Ind}(F),F')
\]

as follows: if \( \zeta : F \to \text{Res}(F') \) is a natural transformation then let \( \psi(\zeta) : \text{Ind}(F) \to F' \) be the natural transformation that satisfies

\[
\psi(\zeta)_H(v \otimes \theta) = F'(v)(\zeta_P(\theta)).
\]

In the above, \( H \in \mathcal{C}_A \), \( P \in \mathcal{S} \), \( v \in \text{Hom}_{\mathcal{C}_A}(P,H) \), and \( \theta \in F(P) \). \( \square \)

We remark that \( \text{Ind}(F) \) can alternatively be defined as a coend of the functor \( \mathcal{S} : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{F}_\mathcal{A} \) that maps an object \((P,Q)\) to the \( A \)-fibered biset functor \( \text{Hom}_{\mathcal{C}_A}(P,\cdot) \otimes_{\mathbb{Z}} F(Q) \). Using \emph{ends} one may also define the right adjoint of \( \text{Res} \). We refer to [7, Section 3.3] for more information.
4 Rational $p$-Biset Functors

Fix a prime number $p$. Throughout this section $R_Q$, $\overline{R}_Q$, and $\overline{R}_Q/R_Q$ are regarded only as $p$-biset functors, i.e., as functors defined on $C_p$, the full subcategory of the biset category $C$ with objects all finite $p$-groups. We provide a review of Bouc’s theory of rational $p$-biset functors, focusing only on the results that will be needed later (the reader should consult [7, Chapters 9 and 10] for a more in-depth discussion of this material). We then apply this theory to determine the subfunctor structure of the $p$-biset functor $\overline{R}_Q/R_Q$. Many of the results on rational representations of $p$-groups in this section go back to Roquette; see [15].

Let $P$ be a $p$-group. If $P$ does not contain any normal subgroup isomorphic to $C_p \times C_p$ then $P$ is said to have normal $p$-rank 1. If $p$ is odd then the $p$-groups of normal $p$-rank 1 are the cyclic groups, generalized quaternion groups, dihedral groups of order at least 16, and the semidihedral groups. If $P$ has normal $p$-rank 1 then there is a unique (up to isomorphism) faithful irreducible $\mathbb{Q}P$-module, denoted $\Phi_P$.

If $P$ is a $p$-group and $S \leq P$ is such that $N_P(S)/S$ has normal $p$-rank 1 then set $V(S) = \text{Ind}_{N_P(S)/S}^{P}(\Phi_{N_P(S)/S})$.

The functor $\text{Ind}_{N_P(S)/S}^{P} : \mathbb{Q}N_P(S)/S\text{mod} \rightarrow \mathbb{Q}P\text{mod}$ induces an injective $\mathbb{Q}$-algebra homomorphism

$$\text{End}_{\mathbb{Q}N_P(S)/S}(\Phi_{N_P(S)/S}) \hookrightarrow \text{End}_{\mathbb{Q}P}(V(S)).$$

Following Bouc, we say that $S$ is a genetic subgroup of $P$ if this map is an isomorphism of $\mathbb{Q}$-algebras. A group-theoretic characterization of genetic subgroups is given in [7, Theorem 9.5.6].

If $S$ is a genetic subgroup of $P$ then $\text{End}_{\mathbb{Q}P}(V(S))$ is a division $\mathbb{Q}$-algebra. It follows that $V(S)$ is an indecomposable, hence irreducible $\mathbb{Q}P$-module. In this way, genetic subgroups of $P$ give rise to irreducible $\mathbb{Q}P$-modules. It turns out ([7, Corollary 9.4.5]) that any irreducible $\mathbb{Q}P$-module $V$ arises in this fashion, up to an isomorphism: there exists a genetic subgroup $S$ of $P$ such that $V \cong V(S)$ (we remark that Lemma 2.1 follows from these observations). Define an equivalence relation $\sim$ on the set of genetic subgroups of $P$ by setting $S \sim T$ if and only if there is a $\mathbb{Q}P$-isomorphism $V(S) \cong V(T)$. Bouc calls a set of representatives for the equivalence classes of $\sim$ a genetic basis of $P$. Of course, if $\mathcal{G}$ is a genetic basis of $P$ then $|\mathcal{G}| = |\text{Irr}_{\mathbb{Q}}(P)|$.

We remark that [7, Theorem 9.6.1] shows that the group structure of $P$ determines precisely when two genetic subgroups $S$ and $T$ of $P$ satisfy $V(S) \cong V(T)$.
V(T). In particular, the relation \( \sim \) can be redefined solely in terms of the group structure of \( P \). Furthermore, it can be shown that if \( S \) and \( T \) are genetic subgroups of \( P \) satisfying \( S \sim T \) then the groups \( N_P(S)/S \) and \( N_P(T)/T \) are isomorphic.

Now let \( F \in \mathcal{F}_p \) be a \( p \)-biset functor and let \( P \) be a \( p \)-group. Following Bouc (see [7, Definition 6.3.1 and Lemma 6.3.2]), set

\[
\partial F(P) = \bigcap_{\{1\} \neq N \leq P} \ker(\text{Def}^P_{P/N} : F(P) \to F(P/N)),
\]

the subgroup of faithful elements of \( F(P) \).

**Proposition 4.1.** Let \( P \) be a \( p \)-group and let \( \chi_1, \chi_2, \ldots, \chi_r \) denote representatives for the Galois conjugacy classes of \( \text{Irr}(P) \) over \( \mathbb{Q} \). Recall that the Galois class sums \( \chi_i \) form a \( \mathbb{Z} \)-basis of \( \mathbb{Q}(P) \). The subgroup \( \partial \mathbb{Q}(P) \) of faithful elements of \( \mathbb{Q}(P) \) is generated by the \( \chi_i \) for which \( \chi_i \) is faithful. In particular, \( \partial \mathbb{Q}(P) \) is a free \( \mathbb{Z} \)-module and if \( P \) has normal \( p \)-rank 1 then \( \partial \mathbb{Q}(P) \) has \( \mathbb{Z} \)-rank 1.

**Proof.** Notice that if \( \chi, \psi \in \text{Irr}(P) \) are Galois conjugate over \( \mathbb{Q} \) then \( \ker(\chi) = \ker(\psi) \). If \( N \leq P \) recall that \( \text{Irr}(P/N) \) can be identified with the set \( \{\chi \in \text{Irr}(P) : N \leq \ker(\chi)\} \). Under this identification, a Galois conjugacy class of irreducible characters of \( P/N \) is equal to a Galois conjugacy class of irreducible characters of \( P \). For any Galois class sum \( \chi \) we have

\[
\text{Def}^P_{P/N}(\chi) = \begin{cases} 
\chi & \text{if } N \leq \ker(\chi), \\
0 & \text{else.}
\end{cases}
\]

It follows from this formula that \( \partial \mathbb{Q}(P) \) is generated by the \( \chi_i \) for which \( \chi_i \) is faithful. In particular, \( \partial \mathbb{Q}(P) \) is a free \( \mathbb{Z} \)-module.

Now let \( P \) be a \( p \)-group of normal \( p \)-rank 1. Recall that \( \Phi_P \) is the unique faithful irreducible \( \mathbb{Q}P \)-module, up to isomorphism. The character of any irreducible constituent of \( \mathbb{C} \otimes \Phi_P \) is faithful, so the set of faithful \( \chi \in \text{Irr}(P) \) is nonempty. If we show that this set forms a full Galois conjugacy class over \( \mathbb{Q} \) then the proof will be complete. So let \( \chi, \psi \in \text{Irr}(P) \) be faithful. The characters \( m_\mathbb{Q}(\chi)i \) and \( m_\mathbb{Q}(\psi)i \) are afforded by faithful irreducible \( \mathbb{Q}P \)-modules, thus both are afforded by \( \Phi_P \). But if \( m_\mathbb{Q}(\chi)i = m_\mathbb{Q}(\psi)i \) then \( \chi \) and \( \psi \) must be Galois conjugate over \( \mathbb{Q} \).

**Theorem 4.2.** ([7, Theorem 10.1.1]) Let \( P \) be a \( p \)-group, \( G \) a genetic basis of \( P \), and let \( F \) be a \( p \)-biset functor. Then the homomorphism

\[
\mathfrak{D}_G = \bigoplus_{S \in G} \text{Indinf}^P_{N_P(S)/S} : \bigoplus_{S \in G} \partial F(N_P(S)/S) \to F(P)
\]

is split injective.
Definition 4.3. (Bouc) A $p$-biset functor $F$ is rational if for any $p$-group $P$ and any genetic basis $G$ of $P$ the map $J_G$ is an isomorphism.

It is immediate that the $p$-biset functor $R_Q$ is rational and in fact $R_Q$ is the prototypical example of a rational $p$-biset functor. As Barker observed in [1, Example 3.A] the same is true if $Q$ is replaced by any field of characteristic 0. In particular, $R_C$ is a rational $p$-biset functor. Since by [7, Theorem 10.1.5] the class of rational $p$-biset functors is closed under taking subfunctors and quotients it follows that $R_Q$ and $R_Q/R_Q$ are both rational.

Recall that, by Lemma 2.1, the $p$-biset functor $R_Q/R_Q$ is trivial if $p \neq 2$. We turn our attention now to the nontrivial case $p = 2$. By Lemma 2.2 if $P$ is a cyclic, dihedral, or semidihedral group then $(R_Q/R_Q)(P) = \{0\}$. Since $\partial(R_Q/R_Q)(Q_{2^n}) = (R_Q/R_Q)(Q_{2^n})$ for any $n \geq 3$, if $P$ is a 2-group with genetic basis $G$ then the map $J_G$ can alternatively be defined:

$$J_G = \bigoplus_{S \in Q} \text{Indinf}_{N_P(S)/S}^P (R_Q/R_Q)(N_P(S)/S) \sim (R_Q/R_Q)(P),$$

where

$$Q = \{S \in G : N_P(S)/S \cong Q_{2^n}, \text{some } n \geq 3\}.$$

In this case the inverse $D_G$ of $J_G$ can be defined as below (see Corollary 6.4.5, Lemma 9.5.2, and Theorem 9.6.1 of [7]):

$$D_G = \bigoplus_{S \in Q} \text{Defres}_{N_P(S)/S}^P (R_Q/R_Q)(P) \sim \bigoplus_{S \in Q} (R_Q/R_Q)(N_P(S)/S).$$

In the following theorem we make use of the notation set up in Lemma 2.3. In particular, recall that

$$\gamma_n = \sum_{\chi \in \text{Irr}(Q_{2^n}) \text{ faithful}} \chi$$

is a representative for the unique nonzero coset in $(R_Q/R_Q)(Q_{2^n})$. For ease, we write $\gamma_n$ for this coset.

Theorem 4.4. For each integer $n \geq 3$ let $F_n$ denote the subfunctor of the 2-biset functor $R_Q/R_Q$ generated by $\gamma_n$. Then the subfunctors $F_n$ are rational 2-biset functors and are precisely the nonzero subfunctors of $R_Q/R_Q$. We have

$$R_Q/R_Q = F_3 \supseteq F_4 \supseteq F_5 \supseteq \cdots.$$

In particular, $R_Q/R_Q$ is generated by $\gamma_3$.

---

3The fact that $R_Q$ is a rational $p$-biset functor also follows directly from the definitions and the well-known interpretation of Schur indices as indices of certain endomorphism algebras; see [17, Section 12.2].
Proof. Since subfunctors of rational $p$-biset functors are rational, each $F_n$, $n \geq 3$, is a rational 2-biset functor. Let $n \geq 4$ and view $Q_{2n-1}$ as a subgroup of $Q_{2^n}$. Recall from Lemma 2.3 that $\text{Ind}^{Q_{2^n}}_{Q_{2n-1}}(\gamma_{n-1}) = \gamma_n$. For any 2-group $P$ we have

$$F_n(P) = \text{Hom}_C(Q_{2^n}, P)(\gamma_n) = \text{Hom}_C(Q_{2^n}, P)(\text{Ind}^{Q_{2^n}}_{Q_{2n-1}}(\gamma_{n-1})) \subseteq \text{Hom}_C(Q_{2n-1}, P)(\gamma_{n-1}) = F_{n-1}(P).$$

Thus $F_n$ is a subfunctor of $F_{n-1}$.

Keeping $n \geq 4$ we now show that $F_n$ is a proper subfunctor of $F_{n-1}$. Since $F_{n-1}(Q_{2n-1}) \neq \{0\}$ it suffices to show that $F_n(Q_{2n-1}) = \{0\}$. Suppose that this is not the case. Then there must exist a transitive $(Q_{2n-1}, Q_{2^n})$-biset $X$ such that $[X]$ does not annihilate $\gamma_n$, i.e., such that $[(\overline{T}_Q/R_Q)([X])] (\gamma_n) \neq 0$. By [7, Lemma 2.3.26] there exists a section $(P_1, K_1)$ of $Q_{2n-1}$, a section $(P_2, K_2)$ of $Q_{2^n}$, and a group isomorphism $f : P_2/K_2 \rightarrow P_1/K_1$ such that

$$[X] = \text{Ind}^{P_2}_P \circ \text{Iso}(f) \circ \text{Defres}^{Q_{2^n}}_{P_2/K_2}.$$

Since $[X]$ does not annihilate $\gamma_n$, neither does $\text{Defres}^{Q_{2^n}}_{P_2/K_2}$. Now, if $P_2$ is a proper subgroup of $Q_{2^n}$ then $\text{Res}^{Q_{2^n}}_{P_2}(\gamma_n) = 0$ — this follows from Lemma 2.3 and the fact that the non-cyclic maximal subgroups of $Q_{2^n}$ are isomorphic to $Q_{2n-1}$. Therefore we must have $P_2 = Q_{2^n}$. If $K_2$ is a nontrivial normal subgroup of $Q_{2^n}$ then we have already observed that $\text{Defres}^{Q_{2^n}}_{Q_{2n-1}}(\gamma_n) = 0$. Thus we must also have $K_2 = \{1\}$. In particular, $P_2/K_2 \cong Q_{2^n}$. But $P_2/K_2 \cong P_1/K_1$ and the latter group is a subquotient of $Q_{2n-1}$. This contradiction establishes that $F_n(Q_{2n-1}) = \{0\}$, from which it follows that $F_n$ is a proper subfunctor of $F_{n-1}$.

To complete the proof it is enough to show that any nonzero subfunctor $F$ of $\overline{T}_Q/R_Q$ is equal to $F_n$ for some $n \geq 3$. So let $F$ be such a subfunctor. Then there exists a 2-group $P$ such that $F(P) \neq \{0\}$. By rationality of $F$ there must exist a genetic subgroup $S$ of $P$ such that $F(N_P(S)/S) \neq \{0\}$ and $N_P(S)/S$ is generalized quaternion. In particular, there exists an integer $n \geq 3$ such that $F(Q_{2^n}) \neq \{0\}$. Let $n$ be the smallest such integer. We show that $F = F_n$. Since $\gamma_n \in F(Q_{2^n})$ it is clear that $F_n \subseteq F$. Now let $P$ be an arbitrary 2-group and let $\mathcal{G}$ be a genetic basis of $P$. Then, as in the remarks preceding the statement of the theorem, we have an isomorphism

$$\bigoplus_{s \in \mathcal{Q}} \text{Ind}^{P}_{N_P(S)/S} : \bigoplus_{s \in \mathcal{Q}} F(N_P(S)/S) \cong F(P),$$

where $\mathcal{Q} = \{s \in \mathcal{G} : N_P(S)/S \cong Q_{2^m}, \text{some } m \geq 3\}$. Observe that if $m$ is an integer in the range $3 \leq m < n$ then $F_n(Q_{2^m}) = \{0\}$ by the previous
paragraph and \( F(Q_{2m}) = \{0\} \) by minimality of \( n \). If \( m \geq n \) then \( F_n(Q_{2m}) \neq \{0\} \), hence also \( F(Q_{2m}) \neq \{0\} \). So we see that \( F_n(Q_{2m}) = F(Q_{2m}) \) for all integers \( m \geq 3 \). It follows that the image of the map above is contained in \( F_n(P) \) — in other words, we have \( F(P) \subseteq F_n(P) \). Since \( P \) was an arbitrary 2-group we conclude that \( F \subseteq F_n \), hence \( F = F_n \) as desired.

Now for each integer \( n \geq 3 \) the quotient \( F_n/F_{n+1} \) is a simple 2-biset functor. In terms of Bouc’s parametrization of simple biset functors (see [7, Section 4.3]) we have:

**Corollary 4.5.** For each integer \( n \geq 3 \) let \( F_n \) denote the subfunctor of the 2-biset functor \( \mathcal{R}_Q/R_Q \) generated by \( \gamma_n \). Then \( F_n/F_{n+1} = S_{Q_{2m}, \mathbb{F}_2} \) where \( \mathbb{F}_2 \) is equipped with the trivial action of \( \text{Out}(Q_{2m}) \).

We also remark that Theorem 4.4 can alternatively be obtained via Bouc’s characterization of the subfunctors of rational \( p \)-biset functors, which is described in [7, Section 10.2].

We conclude this section with a lemma that will be used in the proof of Theorem 5.8.

**Lemma 4.6.** Let \( P, \mathcal{G}, \) and \( Q \) be as in the remarks preceding Theorem 4.4. For each \( S \in Q \) let \( \gamma_S \) denote the unique nonzero element in \( (\mathcal{R}_Q/R_Q)(N_P(S)/S) \). The set \( \{ \text{Ind}_{N_P(S)/S}(\gamma_S) : S \in Q \} \) is an \( \mathbb{F}_2 \)-basis of \( (\mathcal{R}_Q/R_Q)(P) \). For each integer \( n \geq 3 \) set

\[
Q_n = \{ S \in \mathcal{G} : N_P(S)/S \cong Q_{2m} \text{ for some } m \geq n \}.
\]

Then

\[
F_n(P) = \text{span}(\text{Ind}_{N_P(S)/S}(\gamma_S) : S \in Q_n).
\]

**Proof.** Let \( \sum_{S \in Q} a_S \text{Ind}_{N_P(S)/S}(\gamma_S) \) be an arbitrary element of \( F_n(P) \). If some coefficient \( a_T \neq 0 \) then

\[
\text{Defres}_{N_P(T)/T} \left( \sum_{S \in Q} a_S \text{Ind}_{N_P(S)/S}(\gamma_S) \right) = a_T \gamma_T \in F_n(N_P(T)/T).
\]

Therefore \( F_n(N_P(T)/T) \neq \{0\} \). Now if \( N_P(T)/T \cong Q_{2m} \) then the proof of Theorem 4.4 shows that \( n \leq m \). Since clearly \( \text{Ind}_{N_P(S)/S}(\gamma_S) \in F_n(P) \) for all \( S \in Q_n \) the equality holds.
5 The Subfunctors of $R_K/\text{im}(\kappa)$

Throughout this section $p$ denotes a fixed prime number and $K = \mathbb{Q}(\mu_p)$, where as always $\mu_p$ denotes the group of complex roots of unity that have order relatively prime to $p$. Recall that $R_K(G) = R_K(G)$ for any finite group $G$ — this is Lemma 2.4.

Let $(K, \mathcal{O}, k)$ be a $p$-modular system such that $k$ is algebraically closed. Fix an embedding $K \hookrightarrow \mathbb{K}$. Recall the morphism of $\mu_p$-fibered biset functors $\kappa : T_\mathcal{O} \rightarrow R_\mathbb{K}$ introduced in Subsection 3.3. Recall also that the image of $\kappa$ is a $\mu_p$-fibered subfunctor of $R_K$. In this section we determine the subfunctor structure of the quotient $R_K/\text{im}(\kappa)$.

We begin by recalling some well-known definitions from finite group theory. Let $q$ be a prime number and let $H = \langle x \rangle \rtimes Q$ be a semidirect product of a cyclic group $\langle x \rangle$ of order prime to $q$ by a $q$-group $Q$. Recall that such a group $H$ is called quasi-elementary for $q$. If $Q$ centralizes $\langle x \rangle$, i.e., if $H = \langle x \rangle \times Q$, then $H$ is elementary for $q$. We say that $H$ is $\mathbb{Q}(\mu_p)$-elementary for $q$ if $Q$ centralizes the $p$-complement of $\langle x \rangle$ (this definition coincides with the standard — and more general — notion of $K$-elementary groups given, for example, in [9, Chapter 2, Section 21] in the case where $K = \mathbb{Q}(\mu_p)$, which is the only case of importance in the sequel). Note that if $p$ does not divide the order of $H$, if $p = q$, or if $p = 2$ then $H$ is $\mathbb{Q}(\mu_p)$-elementary for $q$ and only if it is elementary for $q$. Finally, recall that a group $H$ is quasi-elementary if it is quasi-elementary for some prime $q$. The classes of $\mathbb{Q}(\mu_p)$-elementary and elementary groups are defined analogously.

The proof of Theorem 5.4 requires a special case of the Witt-Berman Theorem, which is stated below. We remark that the Witt-Berman Theorem holds over arbitrary fields $K$ of characteristic 0, but we shall not make use of this fact.

**Theorem 5.1.** (Witt-Berman; [4, Theorem 21.6]) Let $p$ be a prime, let $K = \mathbb{Q}(\mu_p)$, and let $G$ be a finite group. If $\mathcal{H}^p$ denotes the set of $\mathbb{Q}(\mu_p)$-elementary subgroups of $G$ then the homomorphism

$$\bigoplus_{H \in \mathcal{H}^p} \text{Ind}_H^G : \bigoplus_{H \in \mathcal{H}^p} R_K(H) \rightarrow R_K(G)$$

is surjective.

Now the full subcategory of the $\mu_p$-fibered biset category whose objects are the $p$-groups, $\mathcal{C}_p^\mu$, is a subcategory of the biset category $\mathcal{C}$. Thus we have $\mathcal{C}_p^\mu = \mathcal{C}_p$. Restriction from $\mathcal{C}_p^\mu$ to $\mathcal{C}_p$ induces a functor $\text{Res} : \mathcal{F}_{\mu} \rightarrow \mathcal{F}_p$ with left adjoint $\text{Ind} : \mathcal{F}_p \rightarrow \mathcal{F}_{\mu}$, as discussed in Subsection 3.4. For any
finite group $G$ we have

$$\text{Ind}(\text{Res}(R_K))(G) = \left( \bigoplus_{P \in S} B^{p'}(G, P) \otimes_{\mathbb{Z}} R_K(P) \right) / \mathcal{R}_G$$

where $S$ is a fixed set of representatives for the isomorphism classes of $p$-groups and where $\mathcal{R}_G$ is the subgroup generated by all elements of the form

$$\left[ \frac{G \times Q}{U, \phi} \right] \circ \left[ \frac{Q \times P}{T, 1} \right] \otimes \chi = \left[ \frac{G \times Q}{U, \phi} \right] \otimes R_K \left( \left[ \frac{Q \times P}{T, 1} \right] \right)(\chi).$$

Here $P, Q \in S$, $\left[ \frac{G \times Q}{U, \phi} \right] \in B^{p'}(G, Q)$ and $\left[ \frac{Q \times P}{T, \theta} \right] \in B^{p'}(Q, P)$ are standard basis elements, and $\chi$ is a $K$-character of $P$. By Proposition 3.3 there is a bijection

$$\psi : \text{Hom}_{\mathcal{F}_p}(\text{Res}(R_K), \text{Res}(R_K)) \sim \rightarrow \text{Hom}_{\mathcal{F}_{p'}}(\text{Ind}(\text{Res}(R_K)), R_K).$$

We set

$$\alpha = \psi(\text{id}_{\text{Res}(R_K)}).$$

At a finite group $G$ the natural transformation $\alpha$ has component

$$\alpha_G : \text{Ind}(\text{Res}(R_K))(G) \rightarrow R_K(G)$$

$$\left[ \frac{G \times P}{U, \phi} \right] \otimes \chi \mapsto R_K \left( \left[ \frac{G \times P}{U, \phi} \right] \right)(\chi).$$

**Lemma 5.2.** Let $G$ be a finite group. Then $\text{im}(\alpha_G)$ is a unital subring of $R_K(G)$.

**Proof.** It is clear that $\text{im}(\alpha_G)$ is a subgroup of $R_K(G)$ containing the principal character $1_G = \alpha_G(1_{G/G} \otimes 1_{G/G})$, so we only need to show that $\text{im}(\alpha_G)$ is closed under multiplication. In fact, we need only show that

$$\alpha_G \left( \left[ \frac{G \times P}{U, \phi} \right] \otimes \chi \right) \cdot \alpha_G \left( \left[ \frac{G \times Q}{T, \theta} \right] \otimes \psi \right)$$

is contained in $\text{im}(\alpha_G)$ for any $p$-groups $P, Q \in S$, standard basis elements $\left[ \frac{G \times P}{U, \phi} \right] \in B^{p'}(G, P)$, $\left[ \frac{G \times Q}{T, \theta} \right] \in B^{p'}(Q, G)$, and characters $\chi \in R_K(P)$, $\psi \in \hat{R}_K(Q)$. By the formula given in part 1 of [7, Lemma 7.1.3], the product in

$$g \mapsto \frac{1}{|P||Q|} \sum_{x \in P, \ y \in Q} \text{Ind}_{U}^{G \times P}(\phi)(g, x) \text{Ind}_{T}^{G \times Q}(\theta)(g, y)\chi(x)\psi(y), \quad g \in G.$$
We may assume that $P \times Q \in S$. If $\sigma : (G \times P) \times (G \times Q) \xrightarrow{\sim} (G \times G) \times (P \times Q)$ and $\delta : \Delta(G) \xrightarrow{\sim} G$ denote the obvious group isomorphisms then

\[
\text{Iso}(\delta) \text{ Res}_{G \times G} \Delta(G) \left( \alpha_{G \times G} \left( \left[ (G \times G) \times (P \times Q) \right] \otimes (\chi \otimes \psi) \right) \right),
\]

which is an element of $\text{im}(\alpha_G)$, has character

\[
g \mapsto \frac{1}{|P||Q|} \sum_{x \in P} \sum_{y \in Q} \text{Ind}_{\sigma(U \times T)}^{G \times G \times (P \times Q)} \left( ((\phi \times \theta) \circ \sigma^{-1})(g, g, x, y) \chi(x) \psi(y) \right), \quad g \in G.
\]

But since

\[
\text{Ind}_{\sigma(U \times T)}^{G \times G \times (P \times Q)} \left( ((\phi \times \theta) \circ \sigma^{-1})(g, g, x, y) \right)
= \text{Ind}_{U \times P}^G (\phi)(g, x) \text{Ind}_{T \times Q}^G (\theta)(g, y),
\]

the characters of (2) and (3) are seen to be equal, and the proof is complete. \hfill \Box

**Lemma 5.3.** If $G$ and $H$ are finite groups whose orders are relatively prime then $R_K(G \times H) \cong R_K(G) \otimes_{Z} R_K(H)$.

**Proof.** We may assume that $p$ divides one of $|G|$ or $|H|$. Say $p$ divides $|G|$. If $\chi \in \text{Irr}(G)$, $\psi \in \text{Irr}(H)$ then $K(\chi \otimes \psi) = K(\chi)$ and the Galois class sum $\chi \otimes \psi$, taken with respect to $K$, is equal to $\chi \otimes \psi$. It follows that

\[
R_K(G \times H) = \overline{R}_K(G \times H) \cong \overline{R}_K(G) \otimes_{Z} R_K(H) = R_K(G) \otimes_{Z} R_K(H).
\]

\hfill \Box

**Theorem 5.4.** The natural transformation $\alpha : \text{Ind}(\text{Res}(R_K)) \rightarrow R_K$, defined above, is surjective.

**Proof.** Fix a finite group $G$. We must show that $\alpha_G$ is surjective. From the definition of $\alpha$ it is clear that $\alpha_G$ is surjective if $G$ is a $p$-group. On the other hand, if $G$ has order prime to $p$ then since $\text{im}(\alpha_G)$ contains all of the monomial characters $\text{Ind}_H^G(\phi)$ for $H \leq G$ and $\phi : H \rightarrow \mu_p'$, Brauer's Induction Theorem implies that $\alpha_G$ is surjective. Thus we may assume that the order of $G$ is divisible by $p$.

Let $H^p$ denote the set of $\mu_p'$-elementary subgroups of $G$. Then by naturality of $\alpha$ there is a commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{H \in H^p} \text{Ind}(\text{Res}(R_K))(H) & \xrightarrow{\oplus \text{Ind}_H^G} & \text{Ind}(\text{Res}(R_K))(G) \\
\oplus \alpha_H & & \alpha_G \\
\bigoplus_{H \in H^p} R_K(H) & \xrightarrow{\oplus \text{Ind}_H^G} & R_K(G).
\end{array}
\]
The lower horizontal map is surjective by the Witt-Berman Theorem (see Theorem 5.1). Observe that if $\alpha_H$ is surjective for each $H \in \mathcal{H}^q$, then $\alpha_G$ is surjective. Therefore we may assume that $G$ is $\mathbb{Q}(\mu_{p^q})$-elementary for some prime number $q$. Then $G = \langle x \rangle \times Q$ for some $q$-group $Q$ and an element $x$ of order prime to $q$ such that $Q$ centralizes the $p$-complement of $\langle x \rangle$.

Suppose that $q = p$. Then $Q$ centralizes $\langle x \rangle$, so $G = \langle x \rangle \times Q$ and $R_K(G) \cong R_K(\langle x \rangle) \otimes \mathbb{Z} R_K(Q)$ by Lemma 5.3. A $\mathbb{Z}$-basis for $R_K(G)$ is formed by the set of characters $\lambda \otimes \overline{x}$, where $\lambda \in \text{Irr}(\langle x \rangle)$ and $\overline{x} \in R_K(Q)$ is a Galois class sum of irreducible complex characters of $Q$. Fix such a character $\lambda \otimes \overline{x}$. After identifying $Q$ with the quotient $G/\langle x \rangle$ we have

$$(\text{Mult}(\lambda \otimes 1_Q) \circ \text{Inf}^G_Q)(\overline{x}) = \text{Mult}(\lambda \otimes 1_Q)(1_{\langle x \rangle} \otimes \overline{x}) = \lambda \otimes \overline{x},$$

where $1_Q$ and $1_{\langle x \rangle}$ denote the principal characters of $Q$ and $\langle x \rangle$, respectively. Since $\overline{x} \in \text{im}(\alpha_Q)$, the computation above shows that $\lambda \otimes \overline{x} \in \text{im}(\alpha_G)$. So $\alpha_G$ is surjective in this case.

Suppose instead that $q \neq p$. Arguing as in the previous paragraph, we may assume without loss of generality that $\langle x \rangle$ is a $p$-group. Now if $p = 2$ then $\text{Aut}(\langle x \rangle)$ is a 2-group, so $G = \langle x \rangle \times Q$ and the characters $\overline{x} \otimes \chi$, for $\overline{x} \in R_K(\langle x \rangle)$ a Galois class sum and $\chi \in \text{Irr}(Q)$, form a $\mathbb{Z}$-basis of $R_K(G)$. Since $\alpha_{\langle x \rangle}$ and $\alpha_G$ are surjective (as noted above) the inflated characters $\overline{x} \otimes 1_Q$ and $1_{\langle x \rangle} \otimes \chi$ are both contained in $\text{im}(\alpha_G)$. Lemma 5.2 then implies that $\overline{x} \otimes \chi \in \text{im}(\alpha_G)$, so $\alpha_G$ is surjective. Thus we may assume that $p \neq 2$.

With this additional assumption, notice that the group $\text{Aut}(\langle x \rangle)$ is cyclic.

Let $\chi \in \text{Irr}(G)$ and let $\overline{x} \in R_K(G) = R_K(G)$ be the corresponding Galois class sum. If we show that $\overline{x} \in \text{im}(\alpha_G)$ then, since $\chi$ was chosen arbitrarily, it will follow that $\alpha_G$ is surjective. Toward this end we may assume without loss of generality that $\chi$ is faithful. Let $\lambda \in \text{Irr}(\langle x \rangle)$ be an irreducible constituent of $\text{Res}_{\langle x \rangle}^G(\chi)$. Then $\ker(\lambda) \trianglelefteq G$ (in fact, every subgroup of $\langle x \rangle$ is normal in $G$). It follows that the kernel of $\lambda$ coincides with the kernel of $\text{Res}_{\langle x \rangle}^G(\chi)$, hence $\ker(\lambda) = \{1\}$.

Set

$$T = I_G(\lambda) = \{ g \in G : g \lambda = \lambda \},$$

the inertia group of $\lambda$ in $G$, and set $C = C_Q(x)$. Since $\lambda$ is faithful we have $T = C_G(x) = \langle x \rangle \times C$, hence $T \trianglelefteq G$.

Observe that the Galois conjugacy class of $\lambda$ over $K$ is equal to the set of faithful irreducible characters of $\langle x \rangle$, and that this set is stable under the action of $Q$ by conjugation. Let $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$ denote representatives for the $Q$-orbits, and note that $I_G(\lambda_i) = T$ for each $\lambda_i$. For each integer $i = 1, 2, \ldots, n$ set

$$\text{Irr}(G|\lambda_i) = \{ \psi \in \text{Irr}(G) : \lambda_i \text{ is a constituent of } \text{Res}_{\langle x \rangle}^G(\psi) \}.$$
Define $\text{Irr}(T|\lambda_i)$ analogously. Recall ([13] Theorem 6.11)] that induction from $T$ to $G$ yields a bijection

$$\text{Ind}_T^G : \text{Irr}(T|\lambda_i) \xrightarrow{\sim} \text{Irr}(G|\lambda_i).$$

Since $\text{Irr}(T|\lambda) = \{\lambda \otimes \beta : \beta \in \text{Irr}(C)\}$, it follows that there is a unique $\beta \in \text{Irr}(C)$ such that $\chi = \text{Ind}_T^G(\lambda \otimes \beta)$. Let $\beta = \beta_1, \beta_2, \ldots, \beta_m$ denote the distinct $Q$-conjugates of $\beta$, and note that $K(\chi) \subseteq K(\lambda \otimes \beta) = K(\lambda)$.

Fix a pair $\lambda_i, \beta_j$. Then $\beta_j = u \beta$ for some $u \in Q$ and $u^{-1} \lambda_i = \sigma \lambda$ for some $\sigma \in \text{Gal}(K(\lambda)/K)$. We compute:

$$\text{Ind}_T^G(\lambda_i \otimes \beta_j) = \text{Ind}_T^G(\lambda_i \otimes u \beta) = \text{Ind}_T^G(u^{-1} \lambda_i \otimes \beta) = \text{Ind}_T^G(\sigma \lambda \otimes \beta)$$

$$= \text{Ind}_T^G(\sigma(\lambda \otimes \beta)) = \sigma \text{Ind}_T^G(\lambda \otimes \beta)$$

$$= \sigma \chi.$$

Thus $\text{Ind}_T^G(\lambda_i \otimes \beta_j)$ is Galois conjugate to $\chi$ over $K$. In particular, $\text{Ind}_T^G(\lambda_i \otimes \beta_j) \in \text{Irr}(G)$. Since each $\sigma \in \text{Gal}(K(\chi)/K)$ extends to an automorphism of $K(\lambda)$ the equalities above also show that every Galois conjugate of $\chi$ over $K$ is equal to $\text{Ind}_T^G(\lambda_i \otimes \beta_j)$ for some $\lambda_i, \beta_j$.

Now suppose that $\text{Ind}_T^G(\lambda_i \otimes \beta_j) = \text{Ind}_T^G(\lambda_k \otimes \beta_\ell)$ for some $1 \leq i, k \leq n$, $1 \leq j, \ell \leq m$. For ease, set $\psi = \text{Ind}_T^G(\lambda_i \otimes \beta_j)$. Then $\lambda_i$ and $\lambda_k$ are both constituents of $\text{Res}^G_{\lambda_i}(\psi)$. Since $\psi$ is irreducible it follows that $\lambda_i$ and $\lambda_k$ are conjugate, hence $\lambda_i = \lambda_k$. In particular, $\lambda_i \otimes \beta_j, \lambda_k \otimes \beta_\ell \in \text{Irr}(T|\lambda_i)$. The bijection (4) above then implies that $\lambda_i \otimes \beta_j = \lambda_k \otimes \beta_\ell$.

We have shown that the characters $\text{Ind}_T^G(\lambda_i \otimes \beta_j), 1 \leq i \leq n, 1 \leq j \leq m$, are distinct and are precisely the Galois conjugates of $\chi$ over $K$. Therefore

$$\bar{\chi} = \sum_{i=1}^n \sum_{j=1}^m \text{Ind}_T^G(\lambda_i \otimes \beta_j).$$

Now because the quotient $Q/C$ is cyclic there exists a character $\theta \in R_K(Q)$ that extends $\sum_{j=1}^m \beta_j$. We compute that

$$\bar{\chi} = \sum_{i=1}^n \text{Ind}_T^G(\lambda_i \otimes \text{Res}^G_{\lambda_i}(\theta)) = \sum_{i=1}^n \text{Ind}_T^G(\lambda_i \otimes 1_C) \text{Inf}^G_Q(\theta)$$

$$= \text{Ind}_T^G \left( \sum_{i=1}^n \lambda_i \otimes 1_C \right) \text{Inf}^G_Q(\theta).$$

Notice that $\text{Inf}^G_Q(\theta) \in \text{im}(\alpha_G)$. If $g \in G$ then

$$\text{Ind}_T^G \left( \sum_{i=1}^n \lambda_i \otimes 1_C \right)(g) = \begin{cases} \bar{\chi} \otimes 1)(g) & \text{if } g \in T, \\
0 & \text{if } g \notin T. \end{cases}$$
In the above, $\overline{\lambda}$ denotes the Galois class sum corresponding to $\lambda$, taken with respect to $K$. In other words, $\overline{\lambda}$ is the sum of the faithful irreducible complex characters of $\langle x \rangle$. The formula shows that $\text{Ind}_T^G(\sum_{i=1}^n \lambda_i \otimes 1_C)$ is a $K$-valued character of $G$ with kernel $C$, i.e., is an element of $R_K(G)$ that is inflated from $R_K(G/C)$. It follows that if $\alpha_{G/C}$ is surjective then $\text{Ind}_T^G(\sum_{i=1}^n \lambda_i \otimes 1_C) \in \text{im}(\alpha_G)$. Since $\text{im}(\alpha_G)$ is closed under multiplication by Lemma 5.2, this in turn implies that $\overline{\chi} \in \text{im}(\alpha_G)$, as needed.

By the argument just given we may assume without loss of generality that $C = C_Q(x) = \{1\}$. Then $T = \langle x \rangle$. Let us say that $|x| = p^a$, where $a \geq 1$. We have $\overline{\chi} = \sum_{i=1}^n \text{Ind}_T^G(\lambda_i)$, so $\overline{\chi}(1) = n|Q| = \varphi(p^a)$ because $Q$ acts freely on the set of faithful irreducible complex characters of $\langle x \rangle$ (here $\varphi$ denotes Euler’s totient function). Notice that $\text{Res}_T^G(\chi) = \text{Res}_T^G(\text{Ind}_T^G(\lambda))$ is equal to the regular character of $Q$, so $\chi$ is a constituent of $\text{Ind}_T^G(1_Q)$. Since $\text{Ind}_T^G(1_Q)$ is $K$-valued, it follows that $\overline{\chi}$ is a constituent of $\text{Ind}_T^G(1_Q)$. Further, since the kernel of $\text{Ind}_T^G(1_Q) \cdot (x^{p^a-1})_Q$ contains the nontrivial subgroup $\langle x^{p^a-1} \rangle$ and since $\chi$, hence each Galois conjugate of $\chi$, is faithful, $\overline{\chi}$ is a constituent of the character

$$\text{Ind}_T^G(1_Q) - \text{Ind}_T^G(1_Q \cdot (x^{p^a-1})_Q).$$

In fact, comparing degrees shows that $\overline{\chi}$ is equal to the character above. That $\overline{\chi} \in \text{im}(\alpha_G)$ now follows from Lemma 5.2, which states in particular that $1_H \in \text{im}(\alpha_H)$ for all finite groups $H$. The proof is complete. \qed

We remark that the transformation $\alpha$ is not injective: if $G$ is a group whose order is not divisible by $p$ then $R_K(G) = R_C(G)$ and there is an isomorphism $\text{Ind}(\text{Res}(R_K))(G) \cong B^{C\times}(G)$ that makes the diagram below commute:

$$\text{Ind}(\text{Res}(R_K))(G) \xrightarrow{\alpha_G} R_C(G) \xrightarrow{\text{lin}_G} B^{C\times}(G).$$

Here $\text{lin}_G$ denotes the linearization morphism which maps $[\phi]_{H,\phi} \mapsto \text{Ind}_H^G(\phi)$; see [3, Section 11B].

Next we claim that there are equalities of $p$-biset functors

$$\text{Res}(R_K) = \overline{\mathcal{R}}_Q \quad \text{and} \quad \text{Res}(\text{im}(\kappa)) = R_Q.$$

Since $\text{im}(\kappa) \subseteq R_K$ and $R_Q \subseteq \overline{\mathcal{R}}_Q$ are subfunctors of $R_C$, to prove the claim it suffices to show that $R_K(P) = \overline{\mathcal{R}}_Q(P)$ and $\text{im}(\kappa_P) = R_Q(P)$ for any $p$-group.
P. So fix a $p$-group $P$ and recall that $R_C(P) = \overline{R}_{Q(\zeta_p)}(P)$ for some primitive $p^a$-th root of unity $\zeta_p$. By Lemma 2.3 we have $R_K(P) = \overline{R}_K(P)$, so

$$R_K(P) = \overline{R}_K(P) = \overline{R}_{Q(\zeta_p)}(P) \cap \overline{R}_K(P) = \overline{R}_Q(P).$$

On the other hand, from the description of $\text{im}(\kappa)$ given in Example 3.2 of Subsection 3.3 we can compute that

$$\text{im}(\kappa) = \langle \text{Ind}_{Q}^{G}(1_{Q}) : Q \leq P \rangle = \overline{R}_Q(P).$$

The last equality follows from the Ritter-Segal Theorem; see [14] and [16].

Since $\text{Res}(R_K/\text{im}(\kappa)) = \overline{R}_Q/R_Q$, by the adjunction of Proposition 3.3 there is a natural transformation $\pi : \text{Ind}(\overline{R}_Q/R_Q) \to R_K/\text{im}(\kappa)$ such that the diagram

$$\begin{array}{ccc}
\text{Ind}(\overline{R}_Q) & \longrightarrow & \text{Ind}(\overline{R}_Q/R_Q) \\
\alpha \downarrow & & \downarrow \pi \\
R_K & \longrightarrow & R_K/\text{im}(\kappa)
\end{array}$$

commutes. Because $\alpha$ is surjective by Theorem 5.4, $\pi$ is surjective as well. Combining this observation with Lemma 2.1 gives the following result.

**Corollary 5.5.** We have $\text{im}(\kappa) \neq R_K$ only when $p = 2$, and in this case $R_K(G)/\text{im}(\kappa_G)$ is a (finite-dimensional) vector space over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ for every finite group $G$.

Our next aim is to classify the subfunctors of $R_K/\text{im}(\kappa)$ in the case $p = 2$. We require two lemmas.

**Lemma 5.6.** Let $G$ be a finite group, let $\mathcal{H}$ be the set of quasi-elementary subgroups of $G$, and let

$$\mathcal{U} = \{ U \leq G : U \text{ has a normal Sylow } p\text{-subgroup} \}.$$ 

Then the principal character $1_G$ of $G$ is a $\mathbb{Z}$-linear combination of characters of the form $\text{Ind}_{H_i}^{G}(\phi)$ where $H \in \mathcal{H} \cap \mathcal{U}$ and $\phi : H \to \mu_{p'}$. In particular, if $\theta \in R_K(G)$ then

$$\theta = \sum a_i \text{Ind}_{H_i}^{G}(\phi_i \text{Res}_{H_i}^{G}(\theta))$$

for some $a_i \in \mathbb{Z}$, $H_i \in \mathcal{H} \cap \mathcal{U}$, and $\phi_i : H_i \to \mu_{p'}$.

**Proof.** By [11] Corollary 2 the isomorphism class of the trivial $\mathcal{O}G$-module $\mathcal{O}$ is a $\mathbb{Z}$-linear combination in $T_{\mathcal{O}}(G)$ of (isomorphism classes of) modules of the form $\text{Ind}_{U}^{G}(\mathcal{O}_{\phi})$, where $U \in \mathcal{U}$ and $\phi : U \to \mu_{p'}$. Taking characters, we
Notice that the final group in the sequence above is equal to $R_\mathcal{G} R_\mathcal{G} R_\mathcal{G}$. We compute

$$1_G = 1_G \cdot 1_G = \sum_{i,j} b_i c_j \text{Ind}_{U_i}^G (\phi_i) \text{Ind}_{H_j}^G (1_{H_j})$$

and, by applying Mackey’s Theorem,

$$\text{Ind}_{U_i}^G (\phi_i) \text{Ind}_{H_j}^G (1_{H_j}) = \sum_{g \in U_i \cap G/H_j} \text{Ind}_{g H_j \cap U_i}^G (\text{Res}_{g H_j \cap U_i}^G (\phi_i)).$$

Since $\mathcal{H}$ is stable under conjugation and since both $\mathcal{H}$ and $\mathcal{U}$ are closed under taking subgroups, $g H_j \cap U_i \in \mathcal{H} \cap \mathcal{U}$. The first assertion has been proved, and the second follows from a short computation beginning with the equality $\theta = \theta \cdot 1_G$. \hfill \Box

**Lemma 5.7.** Suppose $p = 2$ and let $H = \langle x \rangle \times P$ be elementary for 2 (so $x$ has odd order and $P$ is a 2-group). Then there is an $\mathbb{F}_2$-isomorphism

$$R_K(H)/\text{im}(\kappa_H) \cong R_C(\langle x \rangle) \otimes_{\mathbb{Z}} (R_Q/R_Q)(P).$$

**Proof.** Applying $R_K(\langle x \rangle) \otimes_{\mathbb{Z}}$ to the short exact sequence $\text{im}(\kappa_P) \hookrightarrow R_K(P) \rightarrow R_K(H)/\text{im}(\kappa_P)$ produces a short exact sequence

$$R_K(\langle x \rangle) \otimes_{\mathbb{Z}} \text{im}(\kappa_P) \hookrightarrow R_K(\langle x \rangle) \otimes_{\mathbb{Z}} R_K(P) \rightarrow R_K(\langle x \rangle) \otimes_{\mathbb{Z}} (R_K(P)/\text{im}(\kappa_P)).$$

Notice that the final group in the sequence above is equal to $R_C(\langle x \rangle) \otimes_{\mathbb{Z}} (R_Q/R_Q)(P)$. Now by Lemma 5.3 there is a group isomorphism $R_K(\langle x \rangle) \otimes_{\mathbb{Z}} R_K(P) \cong R_K(H)$. This isomorphism is part of a commutative diagram

$$\begin{array}{ccc}
T_\mathcal{O}(\langle x \rangle) \otimes_{\mathbb{Z}} T_\mathcal{O}(P) & \longrightarrow & T_\mathcal{O}(H) \\
\kappa(\langle x \rangle \otimes \kappa_P) \downarrow & & \downarrow \kappa_H \\
R_K(\langle x \rangle) \otimes_{\mathbb{Z}} R_K(P) & \cong & R_K(H)
\end{array}$$

where the top horizontal map is induced by $[M] \otimes [N] \mapsto [M \otimes_\mathcal{O} N]$. Surjectivity of this map follows from [11 Theorem 1] and the assumption that $|x|$ and $|P|$ are coprime. In fact, the top horizontal map is an isomorphism: indeed, the Green Correspondence implies that the $\mathbb{Z}$-rank of $T_\mathcal{O}(H)$ is equal to $|x| \cdot \text{rank}_\mathbb{Z} T_\mathcal{O}(P)$. By Brauer’s Induction Theorem $\kappa(\langle x \rangle)$ is surjective, hence bijective after comparing $\mathbb{Z}$-ranks. It follows that the $\mathbb{Z}$-ranks of $T_\mathcal{O}(\langle x \rangle) \otimes_{\mathbb{Z}} T_\mathcal{O}(P)$ and $T_\mathcal{O}(H)$ coincide and the top horizontal map is an isomorphism. In particular, the cokernels of the vertical maps above are isomorphic. The result follows after observing that $\text{im}(\kappa(\langle x \rangle) \otimes \kappa_P) = R_K(\langle x \rangle) \otimes_{\mathbb{Z}} \text{im}(\kappa_P)$. \hfill \Box
We now determine the subfunctor structure of $R_K / \text{im}(\kappa)$ in the nontrivial case $p = 2$. We continue to make use of the notation set in Lemma 2.3. In particular,

$$\gamma_n = \sum_{\chi \in \text{Irr}(Q_{2^n})} \chi$$

is a representative for the unique nonzero coset in $(R_K / \text{im}(\kappa))(Q_{2^n})$. We denote this coset, abusively, by $\gamma_n$.

The following theorem (and its proof) should be compared with Theorem 4.4.

**Theorem 5.8.** Set $p = 2$. For each integer $n \geq 3$ let $F_n$ denote the subfunctor of the $\mu_2$-fibered biset functor $R_K / \text{im}(\kappa)$ generated by $\gamma_n$. Then the subfunctors $F_n$ are precisely the nonzero subfunctors of $R_K / \text{im}(\kappa)$. We have

$$R_K / \text{im}(\kappa) = F_3 \supseteq F_4 \supseteq F_5 \supseteq \cdots.$$ 

In particular, $R_K / \text{im}(\kappa)$ is generated by $\alpha$ and if $\text{im}(\kappa_G) \neq R_K(G)$ for some finite group $G$ then a Sylow 2-subgroup of $G$ has a subquotient isomorphic to the quaternion group $Q_8$.

**Proof.** By an argument similar to one employed in the proof of Theorem 4.4 we have $F_n \subseteq F_{n-1}$ for each integer $n \geq 4$, and since $\text{Res}(F_n)$ is the subfunctor of the 2-biset functor $\mathcal{R}_Q/R_Q$ generated by $\gamma_n$ each containment is proper: $F_n \nsubseteq F_{n-1}$. Now $\mathcal{R}_Q/R_Q$ is generated (as a 2-biset functor) by $\gamma_3$, so it is clear from the definition that $\text{Ind}(\mathcal{R}_Q/R_Q)$ is generated (as a $\mu_2$-fibered biset functor) by $\text{id}_{Q_8} \otimes \gamma_3$. It follows that $R_K / \text{im}(\kappa)$ is generated by $\text{Res}(\mathcal{R}_Q/R_Q) = \gamma_3$, i.e., $R_K / \text{im}(\kappa) = F_3$.

Suppose that $\text{im}(\kappa_G) \neq R_K(G)$ for some finite group $G$. Then, by what we have just shown, there must exist a transitive $\mu_2$-fibered $(G, Q_8)$-biset $X$ such that

$$(R_K / \text{im}(\kappa))([X])(\gamma_3) \neq 0.$$ 

By [3] Propositions 1.3 and 2.8 there exist sections $(P_1, K_1)$ of $G$ and $(P_2, K_2)$ of $Q_8$ such that $P_1/K_1 \cong P_2/K_2$ and $[X]$ factors through $\text{Defres}_{P_2/K_2}$ (note: this requires the fact that any homomorphism from a subgroup of $Q_8$ to $\mu_2$ is trivial). Since $[X]$ does not annihilate $\gamma_3$ neither does $\text{Defres}_{P_2/K_2}$. Arguing as in the proof of Theorem 4.4, we find that $P_2 = Q_8$ and $K_2 = \{1\}$. It follows that $G$, hence a Sylow 2-subgroup of $G$, has a subquotient isomorphic to $Q_8$. A similar argument shows that if $F_n(G) \neq \{0\}$ then $G$ has a subquotient isomorphic to $Q_{2^n}$. This implies in particular that $\bigcap_{n \geq 3} F_n = 0$.

Next we show that if $F$ is a subfunctor of $R_K / \text{im}(\kappa)$ satisfying $F \subsetneq F_{n-1}$, some $n \geq 4$, then $F = F_n$. Let $F$ be such a subfunctor and suppose,
by way of contradiction, that \( F_n \subseteq F \). Then there exists a group \( G \) such that \( F_n(G) \subseteq F(G) \). Let \( \theta \in F(G) \setminus F_n(G) \). Then by Lemma 4.6 and by naturality of the projection \( R_K \to R_K/\text{im}(\kappa) \) we may write

\[
\theta = \sum a_i (\text{Ind}_{H_i}^G \circ \text{Mult}(\phi_i) \circ \text{Res}_{H_i}^G)(\theta)
\]

(5)

for some integers \( a_i \), subgroups \( H_i \) of \( G \), and homomorphisms \( \phi_i : H_i \to \mu_2^e \). Moreover, we may assume that the \( H_i \) are quasi-elementary groups that normalize their Sylow 2-subgroups. In fact, we may assume that each \( H_i \) is quasi-elementary for 2 since if some \( H_i \) is quasi-elementary for an odd prime \( p \) then \( H_i \) has no subquotient isomorphic to \( Q_8 \), hence \( \text{im}(\kappa_{H_i}) = R_K(H_i) \) and \( \text{Res}_{H_i}^G(\theta) = 0 \). But any subgroup \( H_i \) that is quasi-elementary for 2 and has a normal Sylow 2-subgroup is elementary for 2, so we may even assume that each subgroup \( H_i \) is elementary for 2.

Now suppose that \( \text{Res}_{H_i}^G(\theta) \in F_n(\theta) \) for all subgroups \( H \) of \( G \) that are elementary for 2. Then in particular \( \text{Res}_{H_i}^G(\theta) \in F_n(H_i) \) for each subgroup \( H_i \) of the previous paragraph, so \( \theta \in F_n(G) \) by equation (5). Since we are assuming \( \theta \notin F_n(G) \) there must exist a subgroup \( H \) of \( G \) that is elementary for 2 and such that \( \text{Res}_{H_i}^G(\theta) \in F(H) \setminus F_n(\theta) \). Write \( H = \langle x \rangle \times P \) where \( |x| \) is odd and \( P \) is a 2-group. Let \( G \) be a genetic basis of \( P \) and for each \( S \in G \) such that \( N_P(S)/S \) is generalized quaternion let \( \gamma_S \) denote the unique nonzero element in \( (\overline{R}_Q/R_Q)(N_P(S)/S) \). Finally, for each integer \( k \geq 3 \) set

\[
Q_k = \{ S \in G : N_P(S)/S \cong Q_{2m} \text{ for some } m \geq k \}.
\]

By Lemma 4.6 the set \( \{ \text{Ind}_{N_P(S)/S}(\gamma_S) : S \in Q_k \} \) is an \( F_2 \)-basis of \( F_k(P) \).

For simplicity, write \( \overline{\chi}_S \) in place of \( \overline{\text{Ind}_{N_P(S)/S}(\gamma_S)} \). By Lemma 5.7, an \( F_2 \)-basis of \( R_K(H)/\text{im}(\kappa_H) \) is given by the set \( \{ \lambda \otimes \overline{\chi}_S : \lambda \in \text{Irr}((x)), S \in Q_3 \} \).

We claim that

\[
F_k(H) = \text{span}(\lambda \otimes \overline{\chi}_S : \lambda \in \text{Irr}((x)), S \in Q_k).
\]

To see why, first note that for any \( \lambda \in \text{Irr}((x)) \) and \( S \in Q_k \) we have

\[
\lambda \otimes \overline{\chi}_S = (\text{Mult}(\lambda \otimes 1_P) \circ \text{Ind}_P^H)(\overline{\chi}_S) \in F_k(H).
\]

On the other hand if \( \sum a_{\lambda,S} \lambda \otimes \overline{\chi}_S \in F_k(H) \) — where the sum is taken over all \( \lambda \in \text{Irr}((x)) \) and \( S \in Q_k \) — then for any \( \mu \in \text{Irr}((x)) \) we have

\[
(\text{Def}_P^H \circ \text{Mult}(\mu^{-1} \otimes 1_P))(\sum_{\lambda \in \text{Irr}((x))} a_{\lambda,S} \lambda \otimes \overline{\chi}_S) = \sum_{S \in Q_3} a_{\mu,S} \overline{\chi}_S \in F_k(P).
\]
Thus if $a_{\mu,S} \neq 0$ for some $S \in \mathcal{Q}_3$ we must have $S \in \mathcal{Q}_k$. This completes the proof of the claim. Now write

$$\text{Res}_H^G(\theta) = \sum_{\lambda \in \text{Irr}(\langle x \rangle)} a_{\lambda,S} \lambda \otimes \chi_S$$

for some $a_{\lambda,S} \in \mathbb{F}_2$. Since $\text{Res}_H^G(\theta) \notin F_n(H)$ there must exist a character $\mu \in \text{Irr}(\langle x \rangle)$ and a genetic subgroup $T \in \mathcal{Q}_{n-1} \setminus \mathcal{Q}_n$ such that $a_{\mu,T} \neq 0$. Notice that $N_P(T)/T \cong Q_{2n-1}$. We have

$$(\text{Defres}_{N_P(T)/T}^P \circ \text{Def}_P^H \circ \text{Mult}(\mu^{-1} \otimes 1_P) \circ \text{Res}_H^G)(\theta) = a_{\mu,T}\gamma_T,$$

so $F(N_P(T)/T) \cong F(Q_{2n-1})$ is nonzero. But then $\gamma_{n-1} \in F(Q_{2n-1})$, i.e., $F_{n-1} \subseteq F$, which contradicts our assumption that $F \nsubseteq F_{n-1}$. We conclude that $F_n$ is a maximal subfunctor of $F_{n-1}$ for all $n \geq 4$.

Let $F$ be a nonzero subfunctor of $R_K/\text{im}(\kappa)$. We complete the proof by showing that $F = F_n$ for some $n \geq 3$. Suppose first that $F(Q_n) = \{0\}$ for all $n \geq 3$. Then $F \subseteq \cap_{n \geq 3} F_n$: if not, then there exists an integer $n \geq 4$ such that $F \subseteq F_{n-1}$ and $F \nsubseteq F_n$. We have $F_n \subseteq F_n + F \subseteq F_{n-1}$, so $F_n + F = F_{n-1}$ since $F_n$ is a maximal subfunctor of $F_{n-1}$. But now

$$\gamma_{n-1} \in F_{n-1}(Q_{2n-1}) = F_n(Q_{2n-1}) + F(Q_{2n-1}) = \{0\},$$

a contradiction. Thus $F \subseteq \cap_{n \geq 3} F_n$. Since $\cap_{n \geq 3} F_n = 0$ (as noted above) we must have $F = 0$, another contradiction. Therefore $F(Q_n) \neq \{0\}$ for some integer $n$. Assume that $n$ is the minimum such integer. We show that $F = F_n$. It is clear that $F_n \subseteq F$ so we need only show that $F \subseteq F_n$. To this end, we prove by induction that $F \subseteq F_k$ for all integers $k$ in the range $3 \leq k \leq n$. The base case $k = 3$ is obvious: $F \subseteq R_K/\text{im}(\kappa) = F_3$. Now if $F \subseteq F_k$ for some $k$, $3 \leq k < n$, but $F \nsubseteq F_{k+1}$ then $F_k = F_{k+1} + F$, hence

$$\gamma_k \in F_k(Q_{2^k}) = F_{k+1}(Q_{2^k}) + F(Q_{2^k}) = F(Q_{2^k}).$$

But this contradicts the minimality of $n$. Thus if $F \subseteq F_k$ for some integer $k$ in the range $3 \leq k < n$ then necessarily $F \subseteq F_{k+1}$. In particular $F \subseteq F_n$, as needed to complete the proof.

In [3, Theorem 9.2], Boltje and Coşkun parameterize the simple $A$-fibered biset functors $S_{(G,L,\alpha,V)}$ in terms of quadruples $(G,L,\alpha,V)$ where $G$ is a finite group, $(L,\alpha)$ is a reduced pair, and $V$ is an irreducible module for a group ring $\mathbb{Z}[G,L,\alpha]$ (see [3, Sections 8.1, 6.1] for the respective definitions). Since, in the notation of the previous theorem, $F_n/F_{n+1}$ is a simple $\mu_2$-fibered biset.
functor we have $F_n/F_{n+1} \cong S_{\{G,L,\lambda,V\}}$ for some such quadruple. Now the proof of \cite[Theorem 9.2]{BoltjeCoskun18} shows that $G$ may be taken to be a group of smallest order for which $(F_n/F_{n+1})(G) \neq \{0\}$. Thus it is clear that we may take $G = Q_{2^n}$. By \cite[Corollary 10.13]{BoltjeCoskun18}, the only reduced pair in this case is $(\{1\}, 1)$. Arguing as in the proof of \cite[Proposition 4.3.2]{Bouc10}, the group $\Gamma_{\{Q_{2^n}, \{1\}, 1\}}$ is isomorphic to $\text{Out}(Q_{2^n})$. The proof of \cite[Theorem 9.2]{BoltjeCoskun18} then makes clear that we may take $V = F_2$, where $V$ is given the unique $\mathbb{Z}\text{Out}(Q_{2^n})$-module structure. Thus:

$$F_n/F_{n+1} \cong S_{\{Q_{2^n}, \{1\}, 1, F_2\}} \quad n \geq 3.$$ 

The corollary below follows directly from Lemma \ref{lem:subcategory} and the arguments used in the proof of Theorem \ref{thm:main}.

**Corollary 5.9.** Let $G$ be a finite group and let $\chi \in R_K(G)$. Then $\chi \in \text{im}(\kappa_G)$ if and only if $\text{Res}_H^G(\chi) \in \text{im}(\kappa_H)$ for all subgroups $H$ of $G$ that are elementary for 2 and possess a subquotient isomorphic to the quaternion group $Q_8$.

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