Relativistic frequency shifts in a rotating waveguide

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Abstract

A photon source is located in a rotating waveguide. An absorber with a sharp absorbing frequency absorbs some of the emitted photons. This decreases the number of photons which are detected by a detector. The frequency (energy) spectrum measured in the absorber depends on the setup (the positions of the source and the detector), and this spectrum determines the amount of absorption. The shifts are calculated for different configurations. Among other things, it is seen that even if the source and the absorber are at rest (while the waveguide is rotating), there is a nonzero frequency shift.

Keywords: Frequency shifts, Mössbauer effect, Relativity tests, Rotor experiments

1 Introduction

Relativity is one of the cornerstones of today’s theoretical physics. During more than a century, several experiments have been designated to test the theory or determine its domain of validity. Among these are tests based on the frequency shift. First order shifts are easier to measure, but there also cases where there is no first order shift and the leading effect is second order. There are also gravitational shifts, resulting from the difference in the gravitational potential. Among these is the Pound-Rebka experiment of the gravitational frequency shift based on the Mössbauer spectroscopy. The Mössbauer spectroscopy can also be used for shifts which are caused by motion. Important examples of this are the Mössbauer rotor experiments. In these experiments, the effect of frequency shift on the absorption rate of a rotating absorber is measured. In the case that the source is at the center of the rotating disc and the absorber is rotating, the frequency which is measured by the absorber is blue shifted by the Lorentz factor corresponding to the absorber. The Mössbauer rotor experiments have been extensively studied. Some more recent studies are found

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There have also been attempts to interpret such phenomena in terms of general-relativistic effects. Some examples are [10–14]. Motion can affect other properties of photons as well. A review of some classical and quantum aspects of such effects has been presented in [15]. Among these are the frequency (Doppler) shifts caused by linear or rotational motions, as well as the so-called angular momentum Doppler effect. This effect, introduced in [16], involves the change of the angular momentum of a photon as a result of its interaction with a moving medium. This has been further discussed in [17, 18].

Inspired by the Mössbauer rotor experiments performed so far, here the frequency shift is calculated, which resulted from the rotation of the waveguide in which the emitted photons move. The calculation is performed for several possible configurations. In the setup, there is a waveguide which is on a diagonal of a rotating disc, and rotates with it. A photon source is in the waveguide and rotates with the disc. An absorber absorbs some of the photons, and after that there is a detector. The presence of the absorber decreases the number of photons which reach the detector. The absorption probability (fraction) depends on the frequency of the photons as measured by the absorber. The frequency shifts are derived using both ray and particle methods, leading to equivalent results (as expected). In the configurations which are studied here, the source rotates with the waveguide; while the absorber either rotates with the waveguide, or is stationary. A special case of these configurations is when the source is at the center of the rotating disc, hence stationary. It is seen that in that case, when the absorber rotates with the disc it measures a frequency which is blue shifted by the Lorentz factor corresponding to the absorber [5]. But when the absorber is stationary, it still measures a blue shifted frequency, this time corresponding to the square of the Lorentz factor corresponding to the edge of the disc. This is in spite of the fact that in this case both the source and the absorber are at rest.

The scheme of the paper is the following. In section 2, the setup and introduced and the frequency shift (the energy change) is calculated for several possible setups. In section 3, the number of detected particles is calculated for those setups. Section 4 is devoted to the concluding remarks. The relations governing the collision of photons with a wall are presented in the appendix.

2 The frequency shift (the energy change)

A source is located at a disc of radius $r$, in a radial waveguide. The disc and the radial waveguide rotate (according to the rest frame) with the angular velocity $\omega$. The distance of the source from the disc center is $\rho_s$. An absorber; either is in the same waveguide, rotating with the disc and at a distance $\rho_a$ from the disc center; or is outside the disc and stationary (according to the rest frame). A detector detects the particles (photons) which have not been absorbed by the absorber. $(\rho, \phi)$ denote the polar coordinates, with the origin at the disc center. Unprimed (primed) quantities are those which are measured according
to the rest frame (the absorber). The red shift from the source to the absorber is denoted by \( z \):

\[
z = \frac{\nu'_s}{\tilde{\nu}} - 1. \quad (1)
\]

Where \( \nu'_s \) and \( \tilde{\nu} \) are the frequencies measured by the source and the absorber, respectively. Of course this red shift is also the ratio of the particle energies measured by the source and the absorber, respectively:

\[
z = \frac{E'_s}{E} - 1. \quad (2)
\]

The red shift is actually a blue shift, when \( z \) is negative.

### 2.1 The rotating absorber

Consider a light signal originating at the source at the time \( t_s \). According to the rest frame, this signal arrives at the detector at the time \( t_a \). The motion is inside the waveguide. So the radial speed \( \dot{\rho} \) satisfies

\[
(\dot{\rho})^2 + \omega^2 \rho^2 = c^2. \quad (3)
\]

Integrating this, one finds the difference between the arrival time and the emission time:

\[
t_a - t_s = \int_{\rho_s}^{\rho_a} \frac{d\rho}{\sqrt{c^2 - \omega^2 \rho^2}}. \quad (4)
\]

In the above, \( \zeta \) is 1 (−1) if the source and the absorber are on the same (opposite) sides of the disc center. One arrives at,

\[
t_a - t_s = T, \quad (5)
\]

where

\[
T = \frac{|\sin^{-1} \beta_a - \zeta \sin^{-1} \beta_s|}{\omega}, \quad (6)
\]

\[
\beta = \frac{\omega \rho}{c}. \quad (7)
\]

It is seen that \( T \) itself is independent of time. So, if a tap is emitted at the source from \( t_{s1} \) to \( t_{s2} \), and arrives at the absorber from \( t_{a1} \) to \( t_{a2} \),

\[
t_{a2} - t_{a1} = t_{s2} - t_{s1}. \quad (8)
\]
According to the source, the signal is emitted from $t'_{s1}$ to $t'_{s2}$. The corresponding time difference (according to the source) is the proper time corresponding to the world-line of the source between these two events. The source is moving at a constant speed ($\omega_{s}$). So,

$$t'_{s2} - t'_{s1} = \int_{t'_{s1}}^{t'_{s2}} dt \sqrt{1 - \beta_{s}^2}. \tag{9}$$

That is

$$t'_{s2} - t'_{s1} = \frac{t_{s2} - t_{s1}}{\gamma_{s}}, \tag{10}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}. \tag{11}$$

Similarly, according to the absorber, the signal is arrived from $t'_{a1}$ to $t'_{a2}$. The corresponding time difference (according to the absorber) is the proper time corresponding to the world-line of the absorber between these two events. The absorber is moving at a constant speed ($\omega_{a}$). So,

$$t'_{a2} - t'_{a1} = \frac{t_{a2} - t_{a1}}{\gamma_{a}}. \tag{12}$$

One then arrives at

$$\frac{\Delta t'_{a}}{\Delta t'_{s}} = \frac{\gamma_{a}}{\gamma_{s}}. \tag{13}$$

The relation between $\nu'_{a}$ (the frequency that is received by the absorber) and $\nu'_{s}$ (the frequency that is emitted by the source) is

$$\frac{\nu'_{a}}{\nu'_{s}} = \frac{\Delta t'_{s}}{\Delta t'_{a}}. \tag{14}$$

So,

$$\frac{\nu'_{a}}{\nu'_{s}} = \frac{\gamma_{a}}{\gamma_{s}} \tag{15}$$

It is seen that the absorber receives a blue shifted (red shifted) frequency, if its distance from the disc center is larger (smaller) than the distance of the source from the disc center. It is to be noted that the time delay $T$ between the emission and absorption is irrelevant in this. Specifically, this time delay is larger when the source and the absorber are on the opposite sides of the disc.
center, compared to the case they are on the same side (but with the same distances from the disc center). But the frequency shift is the same.

Alternatively, one could calculate the energy of a photon according to the absorber. A photon of the energy $E_s$ (according to the rest frame) is emitted at the source, and is received at the absorber. When the photon arrives at the absorber, its energy is $E_a$ (again according to the rest frame). The reason that $E_a$ is different from $E_s$, is that the path of the photon (according to the rest frame) is not a straight line. The momentum $P$ of the photon changes during its travel, and this changes the energy of the photon as well: the photon is continuously bouncing back from the wall of the waveguide. In the collision of a particle with a wall moving with the velocity ($c\beta$) the change in the energy and momentum of the particle are (as explained in the appendix) related through

$$\delta E = c\beta \cdot \delta P.$$  \hfill (16)

For the motion of the photon inside the waveguide, the energy and momentum of the photon vary continuously: collisions produce infinitesimal changes in the energy and momentum of the photon, and that the above becomes a relation between the differentials of the energy and momentum:

$$dE = c\beta \hat{\phi} \cdot dP,$$  \hfill (17)

where $\hat{\rho}$ and $\hat{\phi}$ are the unit vectors in the radial and angular directions, respectively. The above equation results in

$$dE = c\beta [d(\hat{\phi} \cdot P) - P \cdot d\hat{\phi}],$$

$$= c\beta [d(\hat{\phi} \cdot P) + (\hat{\rho} \cdot P) d\phi].$$  \hfill (18)

One has,

$$c\hat{\phi} \cdot P = \beta E.$$  \hfill (19)

$$c\hat{\rho} \cdot P = \sqrt{1 - \beta^2} E.$$  \hfill (20)

Also, equation (18) can be rewritten as

$$\left(\frac{d\beta}{d\phi}\right)^2 + \beta^2 = 1,$$  \hfill (21)

which results in

$$d\phi = \frac{d\beta}{\sqrt{1 - \beta^2}}.$$  \hfill (22)

One then arrives at

$$dE = \beta [d (\beta E) + E d\beta].$$  \hfill (23)
That is,

\[(1 - \beta^2) dE = E d(\beta^2).\]  \hspace{1cm} (24)

So,

\[\frac{E_a}{E_s} = \frac{1 - \beta_a^2}{1 - \beta_s^2}.\]  \hspace{1cm} (25)

Or,

\[\frac{E_a}{E_s} = \frac{\gamma_a^2}{\gamma_s^2}.\]  \hspace{1cm} (26)

The source and the absorber are moving with the velocities \((c \beta_s)\) and \((c \beta_a)\), respectively. The Lorentz transformation relating the energy of the photon according to the rest frame, to the energy of the photon according to the source or the absorber, is

\[E = \gamma (E' + c \beta \cdot P').\]  \hspace{1cm} (27)

According to the source (absorber), the photon is moving radially when it is at the location of the source (absorber). So \(P'_s\) is normal to \(\beta_s\), and \(P'_a\) is normal to \(\beta_a\). And one arrives at

\[E_s = \gamma_s E'_s,\]  \hspace{1cm} (28)

\[E_a = \gamma_a E'_a.\]  \hspace{1cm} (29)

So,

\[\frac{E'_a}{E'_s} = \frac{\gamma_a}{\gamma_s}.\]  \hspace{1cm} (30)

This is the same shift \((15)\), which was found for the frequency. For a rotating absorber,

\[\tilde{\nu} = \nu'_a.\]  \hspace{1cm} (31)

\[\tilde{E} = E'_a.\]  \hspace{1cm} (32)

\[z = \frac{\gamma_s}{\gamma_a} - 1.\]  \hspace{1cm} (33)

### 2.2 The stationary absorber

The quantities measured at the edge of the waveguide (a distance \(r\) from the disc center), are denoted by the subscript \(e\). The frequency \(\nu_e\) (measured according to the rest frame) is related to \(\nu'_c\), through the following Lorentz transformation.

\[2 \pi \nu_e = \gamma_c (2 \pi \nu'_c + c \beta_e \cdot k'_c).\]  \hspace{1cm} (34)
Where \( k'_e \) is the wave vector received by the point at the edge of the waveguide. This wave vector is radial, hence normal to \( \beta_e \). So,

\[
\nu_r = \gamma_e \nu'_e. \tag{35}
\]

The frequency received by the absorber is \( \nu_r \). And, using (15), one arrives at

\[
\frac{\nu_r}{\nu'_s} = \frac{\gamma_e^2}{\gamma_s}. \tag{36}
\]

Of course

\[
\gamma_e = \frac{1}{\sqrt{1 - \beta_e^2}}. \tag{37}
\]

\[
\beta_e = \frac{\omega r}{c}. \tag{38}
\]

Again, one could arrive at this, using the energy of a photon emerging from the waveguide. The energy of the photon at the detector is \( E_r \), and the following Lorentz transformation relates this to \( E'_e \):

\[
E_r = \gamma_e (E'_e + c \beta_e \cdot P'_e). \tag{39}
\]

According to edge of the waveguide (which is rotating with the disc), the photon is moving radially when it is at the edge. So \( P'_e \) is normal to \( \beta_e \). Hence,

\[
E_r = \gamma_e E'_e. \tag{40}
\]

And, using (30), one arrives at

\[
\frac{E'_r}{E'_s} = \frac{\gamma_e^2}{\gamma_s}. \tag{41}
\]

It is seen that this is consistent with (36). For a stationary absorber,

\[
\tilde{\nu} = \nu_r. \tag{42}
\]

\[
\tilde{E} = E_r. \tag{43}
\]

\[
z = \frac{\gamma_s}{\gamma_e^2} - 1. \tag{44}
\]

### 3 The detected particles

In the absence of the absorber, it is expected that \( N_s \) particles be detected at the detector. The emitted particles have an energy spectrum \( \Upsilon \): The number of emitted particles with energy around \( E \) and in an interval of \( (\Delta E) \), is \([\langle N_s \Delta E \rangle \Upsilon(E)]\). Obviously,

\[
1 = \int dE \Upsilon(E). \tag{45}
\]
The probability that a particle with the energy \( E \) is absorbed, is \( A(E) \). For a particle emitted with the energy \( E'_s \) (as measured by the source), the energy that is measured by the absorber is \( \tilde{E} \). The number of particles which are detected is denoted by \( N_d \). Assuming no loss, apart from the absorption, one has

\[
\frac{N_d}{N_s} = 1 - \int dE' \, \Upsilon(E'_s) \, A \left( \frac{E'_s}{1 + z} \right). 
\] (46)

Or,

\[
\frac{N_d}{N_s} = 1 - (1 + z) \int d\tilde{E} \, \Upsilon(1 + z) \tilde{E} A(\tilde{E}). 
\] (47)

A special case is when the absorption function \( A \) is sharply peaked around some \( E_m \). In that case, (47) can be approximated by

\[
\frac{N_d}{N_s} = 1 - (1 + z) \Upsilon(1 + z) E_m \int d\tilde{E} \, A(\tilde{E}). 
\] (48)

The integral on the right-hand side is independent of the state of the absorber. Denoting that by \( B \), one arrives at

\[
\frac{N_d}{N_s} = 1 - (1 + z) B \Upsilon(1 + z) E_m]. 
\] (49)

In almost all practical situations, \(|z| \) is much smaller than 1. So,

\[
\frac{N_d}{N_s} = 1 - B \Upsilon(E_m) - z B \{ \Upsilon(E_m) + E_m (D \Upsilon)(E_m) \} + \cdots, 
\] (50)

where \((D \Upsilon)\) is the derivative of \( \Upsilon \). Also,

\[
\sigma = \frac{\sigma \omega^2}{c^2} + O(c^{-4}), 
\] (51)

where \( \sigma \) is a constant of the dimension length squared. The final result is then

\[
\frac{N_d}{N_s} = 1 - B \Upsilon(E_m) - \{ B [\Upsilon(E_m) + E_m (D \Upsilon)(E_m)] \} \frac{\sigma \omega^2}{c^2} + O(c^{-4}). 
\] (52)

Some special cases are discussed below.

### 3.1 The rotating absorber

The absorber is attached to the disc, and rotates with the disc:

\[
\sigma = \frac{\rho_s^2 - \rho_a^2}{2}. 
\] (53)
In particular, if the disc center is half way between the source and the absorber:

\[ \rho_s = \rho_a, \quad (54) \]

then

\[ \sigma = 0. \quad (55) \]

And if the source is on the disc center:

\[ \rho_s = 0, \quad (56) \]

then

\[ \sigma = -\frac{\rho_e^2}{2}. \quad (57) \]

### 3.2 The stationary absorber

The absorber is just outside the disc, stationary according to the rest frame:

\[ \sigma = \frac{\rho_s^2 - 2 \rho_e^2}{2}. \quad (58) \]

In particular, if the source is on the disc center:

\[ \rho_s = 0, \quad (59) \]

then

\[ \sigma = -\rho_e^2. \quad (60) \]

### 4 Concluding remarks

The relativistic frequency shift caused by rotating waveguide was studied. In the setup, a radiation source is in a waveguide, which is on a diagonal of a rotating disc. Both the waveguide and the source rotate with the disc. An absorber absorbs some of the photons, decreasing the number of the photons which reach a detector. The absorption fraction depends on the frequency of the photons according to the absorber, and this can be used as a method to measure possible frequency shifts from the source to the detector. Among other things, it was shown even if both the source and the absorber are stationary, there is still a nonzero shift from the source to the absorber. The reason is that according to the rest frame, in which the disc is rotating, the photon is moving on a curved path, hence its energy (frequency) is constantly changing.

Based on the results, various Mössbauer rotor experiments can be designated as further tests of special relativity.
5 Appendix: Relativistic collision with a wall

Consider a two body collision. The conservation relations are

\[ E + E = E_{f} + E_{i}, \quad (61) \]
\[ P + P = P_{f} + P_{i}. \quad (62) \]

Quantities corresponding to the second (later large) body are denoted by calligraphic letters, and the subscript f refers to the quantities after the collision. The above equations result in

\[ E_{f}^{2} - c^{2} P_{f} \cdot P_{f} = (E + E - E_{i})^{2} - c^{2} (P + P - P_{i}) \cdot (P + P - P_{i}). \quad (63) \]

That is,

\[ E_{f}^{2} - c^{2} P_{f} \cdot P_{f} = E^{2} - c^{2} P \cdot P \]
\[ + 2 [E (E - E_{i}) - c^{2} P \cdot (P - P_{i})] \]
\[ + (E - E_{i})^{2} - c^{2} (P - P_{i}) \cdot (P - P_{i}). \quad (64) \]

Using the mass shell relation for the second body, one arrives at

\[ 2 (E \delta E - c^{2} P \cdot \delta P) = (\delta E)^{2} - c^{2} (\delta P) \cdot (\delta P), \quad (65) \]

where

\[ \delta \mathbf{x} = \mathbf{x}_{f} - \mathbf{x}. \quad (66) \]

One has

\[ c \mathbf{P} = \beta E. \quad (67) \]

Where \((c \beta)\) is the velocity of the second body (before the collision). So,

\[ \delta E - c \beta \cdot \delta P = \frac{(\delta E)^{2} - c^{2} (\delta P) \cdot (\delta P)}{2 E}. \quad (68) \]

The collision with a \textit{wall} corresponds to an infinite \(E\). In that case,

\[ \delta E - c \beta \cdot \delta P = 0. \quad (69) \]

This is the same as \((10)\).

\textbf{Acknowledgement:} This work was supported by the Research Council of the Alzahra University.
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