The Complexity of Bounded Length Graph Recoloring

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Abstract

We study the following question: Given are two $k$-colorings $\alpha$ and $\beta$ of a graph $G$ on $n$ vertices, and integer $\ell$. The question is whether $\alpha$ can be modified into $\beta$, by recoloring vertices one at a time, while maintaining a $k$-coloring throughout, and using at most $\ell$ such recoloring steps. This problem is weakly PSPACE-hard for every constant $k \geq 4$. We show that it is also strongly NP-hard for every constant $k \geq 4$. On the positive side, we give an $\mathcal{O}(f(k, \ell)n^{O(1)})$ algorithm for the problem, for some computable function $f$. Hence the problem is fixed-parameter tractable when parameterized by $k + \ell$. Finally, we show that the problem is W[1]-hard (but in XP) when parameterized only by $\ell$.

1 Introduction

Given a graph $G$ and a positive integer $k$, the classical NP-complete $k$-Coloring problem asks for an assignment of at most $k$ colors to the vertices of $G$ such that no two adjacent vertices receive the same color. Such a color assignment is called a $k$-coloring or simply a coloring of $G$. Under the reconfiguration framework, we are interested in structural and algorithmic questions related to the solution space of the $k$-Coloring problem. In particular, for any graph $G$ and integer $k$, we can define the $k$-Color Graph $\mathcal{C}_k(G)$ as follows. The vertex set of $\mathcal{C}_k(G)$ corresponds to all $k$-colorings of $G$ and two colorings are adjacent if and only if they differ on exactly one vertex. The integer $k$ is also called the number of admissible colors. Given two $k$-colorings of $G$, $\alpha$ and $\beta$, the $k$-Color Path problem asks if there exists a path in $\mathcal{C}_k(G)$ from $\alpha$ to $\beta$. This is a well-studied problem, which is known to be solvable in polynomial time for $k \leq 3$ [10]. In addition, it is PSPACE-complete for every constant $k \geq 4$, even when restricted to bipartite graphs, and PSPACE-complete for $k = 4$ for planar bipartite graphs [5]. However, it can be solved in polynomial time for any $k$, if

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$G$ is a $(k - 2)$-degenerate graph [8, 13]; see also [5] (planar bipartite graphs are 3-degenerate).

In addition to results on the $k$-Color Path problem, many other results have been obtained in the setting of $k$-coloring reconfiguration. For the aforementioned PSPACE-complete cases, examples have been explicitly constructed where any path from $\alpha$ to $\beta$ has exponential length [5]. On the other hand, for the polynomial case $k \leq 3$, the diameter of components of $C_k(G)$ is known to be polynomial [10]. It has been conjectured that this is also the case for $k$ colors and $k - 2$ degenerate graphs [5]. This conjecture has been answered positively for the subclass of graphs of treewidth $(k - 2)$ [1]. The connectedness of $C_k(G)$ has been studied in [5, 9]. In particular, deciding whether $C_k(G)$ is connected is coNP-complete for $k = 3$ and $G$ bipartite, but can be answered in polynomial time if $G$ is planar and bipartite [9]. One may ask whether a shortest path from $\alpha$ to $\beta$ can be found in $C_k(G)$. In [10], a polynomial time algorithm for $k = 3$ and certain pairs of colorings has been given. Finally, it is known that for any two $k$-colorings $\alpha$ and $\beta$, $2k - 1$ colors suffice to find a path from $\alpha$ to $\beta$, and that this bound is tight [7, 23, 19].

Similar reconfiguration questions can be formulated for almost any search problem, after defining a (symmetric) adjacency relation between solutions. Such questions have received considerable attention in recent literature; see e.g. the survey by Van den Heuvel [17]. In most cases, the complexity behavior of these problems is similar. For instance, deciding whether a path exists between two solutions is often PSPACE-hard in general, although polynomial time solvable restricted cases can be identified. For PSPACE-hard cases, it is not surprising that shortest paths between solutions can have exponential length. More surprisingly, for most known polynomial time solvable cases, shortest paths between solutions have been shown to have polynomial length. Results of this kind have for instance been obtained e.g. for the reconfiguration of independent sets [6, 22], vertex covers [24], shortest paths [3, 2, 21], or boolean assignments [18].

These problems are interesting for a variety of reasons. From an algorithmic standpoint, reconfiguration problems model dynamic situations in which we seek to transform a solution into a more desirable one, maintaining feasibility during the process (see [19] for such an application of $k$-Color Path). Reconfiguration also models questions of evolution; it can represent the evolution of a genotype where only individual mutations are allowed and all genotypes must satisfy a certain fitness threshold, i.e. be feasible. Moreover, the study of reconfiguration yields insights into the structure of the solution space of the underlying problem, crucial for the design of efficient algorithms. In fact, one of the initial motivations behind such questions was to study the performance of heuristics [15] and random sampling methods [5], where connectivity and other properties of the solution space play a crucial role.

In many applications of reconfiguration problems, the existence of a path between two solutions is irrelevant, if every such path has exponential length. So the more important question is in fact: does there exist a path between two solutions of length at most $\ell$, for some integer $\ell$? Such length-bounded reconfig-
uration questions have been considered e.g. in \[3, 4, 16, 21, 24, 25\]. In some cases where the existence of paths between solutions can bedecided efficiently, one can in fact find shortest paths efficiently \[3, 4, 16\]. On the other hand, NP-hard cases have also been identified \[21, 24\]. If we wish toobtain a more detailed picture of the complexity of length bounded reconfiguration, the settingof parameterized complexity \[12, 14\] is very useful, where we choose \(\ell\) as parameter. A systematic study of the parameterized complexity of reconfigurationproblems was initiated by Mouawad et al \[25\]. However, in \[25\], only negative results wereobtained for length-bounded reconfiguration: various problems were identified where the problem was not only NP-hard, but also W[1]-hard, whenparameterized by \(\ell\) (or even when parameterized by \(k + \ell\), where \(k\) is another problemparameter). In this paper, we give an example of a length bounded reconfiguration problem that is NP-hard, but admits an FPT algorithm. Anotherexample, namely vertex cover reconfiguration in graphs of bounded degree, was very recently obtained by Mouawad et al in \[24\].

\(\ell\)-Length \(k\)-Color Path (\(\ell\)-L \(k\)-CP):

**Instance:** A graph \(G\) on \(n\) vertices, nonnegative integers \(k\) and \(\ell\), and two \(k\)-colorings \(\alpha\) and \(\beta\) of \(G\).

**Question:** Does \(C_k(G)\) contain a path from \(\alpha\) to \(\beta\) of length at most \(\ell\)?

In this paper we explore fully how the complexity of the above problem depends on the problem parameters \(k\) and \(\ell\). Firstly, \(\ell\)-L \(k\)-CP is easily observed to be PSPACE-hard in general, for \(k \geq 4\): Since there are at most \(k^n\) different \(k\)-colorings of a graph on \(n\) vertices, a path from \(\alpha\) to \(\beta\) exists if and only if there exists one of length at most \(k^n\). So setting \(\ell = k^n\) yields a trivial reduction from the PSPACE-hard \(k\)-Color Path problem to \(\ell\)-L \(k\)-CP. Nevertheless, this only establishes weak PSPACE-hardness, since the chosen value of \(\ell\) is exponential in the instance size. In other words, if we require that all integers are encoded in unary, then this is not a polynomial reduction. And indeed, the complexity status of the problem changes under that requirement: in that case, \(\ell\)-L \(k\)-CP is easily observed to be in NP. (If \((G, k, \ell, \alpha, \beta)\) is a yes-instance, then a path of length \(\ell\) in \(C_k(G)\) from \(\alpha\) to \(\beta\) is a polynomial certificate.) In Section 4, we show that \(\ell\)-L \(k\)-CP is in fact NP-complete when \(\ell\) is encoded in unary, or in other words: it is strongly NP-hard. On the positive side, in Section 5 we show that the problem can be solved in time \(O(2^{k(\ell+1)} \cdot \ell^\ell \cdot \text{poly}(n))\). This establishes that \(\ell\)-L \(k\)-CP is fixed parameter tractable (FPT), when parameterized by \(k + \ell\). The reader is referred to the excellent books of Downey and Fellows \[12\], Flum and Grohe \[14\], and Niedermeier \[20\] for an introduction to parameterized complexity. One may ask whether the problem is still FPT when only parameterized by \(\ell\). We show that this is not the case (unless W[1]=FPT), by showing that \(\ell\)-L \(k\)-CP is W[1]-hard when only parameterized by \(\ell\). We observe however that a straightforward branching algorithm can solve the problem in time \(n^{O(\ell)}\), so in polynomial time for any constant \(\ell\). In other words, \(\ell\)-L \(k\)-CP is in XP, parameterized by \(\ell\).

Our results are summarized in Table 1. Our main results are marked by (*), and trivial results are marked with (T). Unmarked results follow immediately.
from results in an adjacent row or column. To emphasize the strength of both the negative and positive results, we added rows for both \( k \) constant and \( k \) as parameter, even though the complexity is always the same. The FPT result was also obtained very recently and independently by Johnson et al \cite{20}, although they use a very different algorithm than ours. In addition, in \cite{20}, the case \( k \leq 3 \) is considered, and is shown to be polynomial (as conjectured by Cereceda et al \cite{10}).

### 2 Preliminaries

For general graph theoretic definitions, we refer the reader to the book of Diestel \cite{11}. Unless otherwise stated, we assume that each graph \( G \) is a simple, undirected graph with vertex set \( V(G) \) and edge set \( E(G) \), where \( |V(G)| = n \) and \( |E(G)| = m \). The open neighborhood of a vertex \( v \) is denoted by \( N_G(v) = \{ u \mid uv \in E(G) \} \) and the closed neighborhood by \( N_G[v] = N_G(v) \cup \{ v \} \).

For a set of vertices \( S \subseteq V(G) \), we define \( N_G(S) = \{ v \notin S \mid uv \in E(G), u \in S \} \) and \( N_G[S] = N_G(S) \cup S \). We drop the subscript \( G \) when clear from context. The subgraph of \( G \) induced by \( S \) is denoted by \( G[S] \), where \( G[S] \) has vertex set \( S \) and edge set \( \{ uv \in E(G) \mid u, v \in S \} \). The maximum degree of a graph \( G \) is denoted by \( \Delta(G) \). Let \( u \) and \( v \) be vertices in a graph \( G \). A pseudowalk from \( u \) to \( v \) of length \( \ell \) is a sequence \( w_0, \ldots, w_{\ell} \) of vertices in \( G \) with \( w_0 = u, w_{\ell} = v \), such that for every \( i \in \{ 0, \ldots, \ell - 1 \} \), either \( w_i = w_{i+1} \), or \( w_i w_{i+1} \in E(G) \).

A k-color assignment for a graph \( G \) is a function \( \alpha : V(G) \to \{ 1, \ldots, k \} \) of colors to the vertices of \( G \). It is a k-coloring if there are no edges \( uv \in E(G) \) with \( \alpha(u) = \alpha(v) \). On the other hand, if there exists such an edge \( uv \), then this edge is said to give a color conflict. A graph that admits a k-coloring is called k-colorable. The minimum \( k \) such that \( G \) is k-colorable is called the chromatic number of \( G \), denoted by \( \chi(G) \). For a k-coloring \( \alpha \) of \( G \), the set of colors used by \( \alpha \) is \( \{ \alpha(v) \mid v \in V(G) \} \). Pseudowalks in \( \mathcal{C}_k(G) \) from \( \alpha \) to \( \beta \) are also called k-recoloring sequences from \( \alpha \) to \( \beta \). If there exists an integer \( k \) such that \( \alpha_0, \ldots, \alpha_m \) is a k-recoloring sequence, then this is called a recoloring sequence from \( \alpha_0 \) to \( \alpha_m \). Clearly, if there exists a k-recoloring sequence from \( \alpha \) to \( \beta \) of length \( \ell \), then \( \mathcal{C}_k(G) \) contains a path from \( \alpha \) to \( \beta \) of length at most \( \ell \).

A k-color list assignment for a graph \( G \) is a mapping \( L \) that assigns a color list \( L(v) \subseteq \{ 1, \ldots, k \} \) to each vertex \( v \in V(G) \). A k-coloring \( \alpha \) of \( G \) is an


**3 W[1]-hardness**

We give a reduction from the following problem:

**t-Independent Set (t-IS):**

**Instance:** A graph $G$ and a positive integer $t$.

**Question:** Does $G$ have an independent set of size at least $t$?

The t-IS problem is known to be W[1]-hard \([12, 14]\) when parameterized by $t$. We will also use the following result, which was shown independently by Cereceda [7], Marcotte and Hansen [23] and Jacob [19]: For every pair of $k$-colorings $\alpha$ and $\beta$ of a graph $G$, there exists a path from $\alpha$ to $\beta$ in $C_{2k-1}^k(G)$, and there are examples where at least $2k - 1$ colors are necessary. The graphs constructed in [7, 23, 19] to prove the latter result are in fact very similar. We will use these graphs for our reduction, though we will need a slightly stronger claim, so we need to restate the proof. The next definition and proof follow [7, Section 6.1] closely.

For any integer $k \geq 1$, let $B_k = K_k \times K_k$ (the complement of the Cartesian product graph of two complete graphs on $k$ vertices). More precisely, we let $V(B_k) = \{b_i^j \mid i, j \in \{1, \ldots, k\}\}$, and two vertices $b_i^j$ and $b_i'^j'$ are adjacent if and only if $i \neq i'$ and $j \neq j'$. Define two $k$-colorings $\alpha^k$ and $\beta^k$ for $B_k$ by setting $\alpha^k(b_i^j) = i$ and $\beta^k(b_i^j) = j$ for all vertices $b_i^j$.

**Theorem 1** For every integer $k \geq 1$, let $B_k$, $\alpha^k$ and $\beta^k$ be as defined above. Then

1. every recoloring sequence from $\alpha^k$ to $\beta^k$ contains a coloring that uses at least $2k - 1$ different colors, and

2. there exists a $(2k - 1)$-recoloring sequence of length at most $2k^2$ from $\alpha^k$ to $\beta^k$.

**Proof:** Consider a recoloring sequence $\gamma_0, \ldots, \gamma_m$ from $\alpha^k$ to $\beta^k$. For $i \in \{1, \ldots, k\}$, the vertex set $R_i = \{b_i^j \mid j \in \{1, \ldots, k\}\}$ is called a *row* of $B_k$. For every $i$, the coloring $\alpha^k$ colors the vertices of row $R_i$ all with the same color, and $\beta^k$ colors them all differently. So we may choose the lowest index $j$ such
that \( \gamma_j \) colors all vertices of at least one row \( R_i \) differently. The choice of \( j \) guarantees that for \( \gamma_j \), for all other rows, there exists a color that is used for at least two different vertices in the row, and thus is not used in any other row (this follows easily from the definition of \( B_k \)). We conclude that \( \gamma_j \) uses at least \( 2k - 1 \) different colors in total, which proves the first statement.

To obtain a \((2k + 1)\)-recoloring sequence from \( \alpha^k \) to \( \beta^k \), we can apply the following general method (which applies in fact to any two \( k \)-colorings of any \( k \)-colorable graph): Choose an arbitrary \( k \)-coloring \( \gamma \) of \( B_k \). For every vertex \( v \in V(B_k) \) with \( \gamma(v) < k - 1 \), recolor \( v \) to the color \( k + \gamma(v) \) (this is a color in \( \{k + 1, \ldots, 2k - 1\} \), so it is not used by \( \alpha^k \) or \( \beta^k \)). Next, recolor all vertices \( v \) to their target color \( \beta(v) \), starting with those vertices \( v \) with \( \gamma(v) = k \). It can be verified that this way, a \( k \)-coloring is maintained throughout, and that every vertex is recolored at most twice. \( \square \)

We remark that the distance from \( \alpha^k \) to \( \beta^k \) in \( C_{2k-1}(B_k) \) is in fact strictly smaller than \( 2k^2 \), but we do not need to know or prove the exact distance. We can now state the reduction from \( t \)-IS to \( t \)-L \( k \)-CP that we use to prove \( W[1] \)-hardness.

**Construction** For ease of presentation, we give a reduction from the \((t-1)\)-IS problem, which remains \( W[1] \)-hard. Given an instance \((G, t-1)\) of \((t-1)\)-IS, where \( G = (V, E) \) and \( V = \{v_1, \ldots, v_n\} \), we construct a graph \( G' \) in time polynomial in \( n + m + t \) as follows.

\( G' \) contains a copy of \( G \) and a copy of \( B_t \) with all edges between them. In addition, \( G' \) contains \( n + t + 1 \) independent sets \( C_1, \ldots, C_{n+t+1} \), each of size \( 2t + 2t^2 \). We say that \( C_i \) (for \( 1 \leq i \leq n + t + 1 \)) is a color-guard set, as it will be used to enforce some coloring constraints: in the colorings we define, and all colorings reachable from them using at most \( |C_i| - 1 \) recolorings, \( C_i \) will contain at least one vertex of color \( i \). We let \( V_G = \{g_1, \ldots, g_n\} \), \( V_B = \{b^i_j \mid i, j \in \{1, \ldots, t\}\} \), \( V_C = C_1 \cup \ldots \cup C_{n+t+1} \), and hence \( V(G') = V_G \cup V_B \cup V_C \). The total number of vertices in \( G' \) is therefore \( n + t^2 + (n + t + 1)(2t + 2t^2) \). For every vertex \( g_i \in V_G \), we add all edges between \( g_i \) and the vertices in \( V_C \setminus (C_i \cup C_{n+t+1}) \). Similarly, for every vertex \( b \in V_B \), we add all edges between \( b \) and the vertices in \( C_{n+t+1} \). We define \( \alpha \) as follows. For every vertex \( g_i \in V_G \), \( 1 \leq i \leq n \), we set \( \alpha(g_i) = i \). For every \( i \in \{1, \ldots, n + t + 1\} \) and every vertex \( c \in C_i \), we set \( \alpha(c) = i \). For every vertex \( b^i_j \in V_B \), with \( i, j \in \{1, \ldots, t\} \), we choose \( \alpha(b^i_j) = n + i \).

**Proposition 2** Let \( G' \) and \( \alpha \) be as constructed above, from a graph \( G \) on \( n \) vertices. Let \( k = n + t + 1 \), and let \( \alpha_0, \ldots, \alpha_\ell \) be a \( k \)-recoloring sequence of length at most \( 2t + 2t^2 \) with \( \alpha_0 = \alpha \). Then for all \( g_i \in V_G \) and \( 0 \leq x \leq \ell \), we have \( \alpha_x(g_i) \in \{i, n + t + 1\} \). Moreover, for all \( b \in V_B \) and \( 0 \leq x \leq \ell \), we have \( \alpha_x(b) \neq n + t + 1 \).

**Proof:** Since every vertex \( g_i \in V_G \) is adjacent to all vertices in \( V_C \setminus (C_i \cup C_{n+t+1}) \), and \( |C_j| = 2t + 2t^2 \) for all \( j \), it follows that if \( g_i \) receives a color \( j \not\in \{i, n+t+1\} \),
then all vertices in $C_j$ must have been recolored earlier, which contradicts the fact that we consider a recoloring sequence of length at most $2t + 2t^2$. Similarly, every vertex $b \in V_B$ is adjacent to all vertices in $C_{n+t+1}$. Therefore, if $b$ receives color $n + t + 1$ then all vertices in $C_{n+t+1}$ must have been recolored first. □

Finally, we define the target coloring $\beta$: for every vertex $v \in V_G \cup V_C$ we set $\beta(v) = \alpha(v)$. For every vertex $b_i \in V_B$ (with $i, j \in \{1, \ldots, t\}$), we choose $\beta(b_i) = n + j$. In other words, the goal is to change from a ‘row coloring’ to a ‘column coloring’ for $V_B$, while maintaining the same coloring for vertices in $V_G \cup V_C$. The corresponding $\ell$-L $k$-CP instance is denoted by $(G', k, \ell, \alpha, \beta)$ with $k = n + t + 1$ and $\ell = 2t + 2t^2$.

**Lemma 3** Let $G'$, $k$, $\ell$, $\alpha$, and $\beta$ be as constructed above, from a graph $G$ on $n$ vertices. If $G$ has an independent set of size at least $t - 1$, then $C_k(G')$ contains a path from $\alpha$ to $\beta$, of length at most $\ell = 2t + 2t^2$.

**Proof:** We let $S$ be an independent set of size $t - 1$ in $G$. The following recoloring sequence is an $(n + t + 1)$-recoloring sequence from $\alpha$ to $\beta$ of length at most $2t + 2t^2$:

- Assign the vertices in $G'$ corresponding to the vertices in $S$ color $n + t + 1$ (one by one). By our construction of $G'$, none of these steps will introduce any color conflicts since vertices in $V_G$ are not connected to vertices in $C_{n+t+1}$.
- From Theorem 1 we know that we can recolor the vertices in $V_B$ using $2t - 1$ colors and at most $2t^2$ recoloring steps. Since $t - 1$ vertices have been recolored in the previous step, we can now use the corresponding $t - 1$ colors to apply the $(2t - 1)$-recoloring sequence to $V_B$.
- Recolor the $t - 1$ vertices that were initially recolored by assigning them their original color again (which is also their target color in $\beta$). None of these steps will introduce any color conflicts since at this point every vertex in $V_B$ is assigned a color greater than $n$ again.

Clearly, the described recoloring sequence consists of at most $2(t - 1) + 2t^2 < 2t + 2t^2$ recoloring steps. □

**Lemma 4** Let $G'$, $k$, $\ell$, $\alpha$, and $\beta$ be as constructed above, from a graph $G$ on $n$ vertices. If $C_k(G')$ contains a path from $\alpha$ to $\beta$ of length at most $\ell = 2t + 2t^2$, then $G$ has an independent set of size at least $t - 1$.

**Proof:** Assume that there exists some $k$-recoloring sequence from $\alpha$ to $\beta$ of length at most $\ell = 2t + 2t^2$. Theorem 1 shows that this sequence contains a coloring $\gamma$ that assigns at least $2t - 1$ different colors to $V_B$. Let $U = \{\gamma(b) \mid b \in V_B\}$ denote this color set with $|U| \geq 2t - 1$. Vertices in $V_B$ cannot be assigned color $n + t + 1$ in a sequence of this length (Proposition 2), so $|U \cap \{1, \ldots, n\}| \geq t - 1$. Since $G'$ contains all edges between $V_B$ and $V_G$, we conclude that all vertices
in \( S = \{g_i \in V_G \mid i \in U\} \) must have \( \gamma(g_i) \neq i \), and thus \( \gamma(g_i) = n + t + 1 \) (Proposition 2). It follows that \( S \) is an independent set of \( G \) of size at least \( t - 1 \). \( \square \)

Combining Lemmas 3 and 4 with the fact that \((t - 1)\)-IS is \(W[1]\)-hard yields:

**Theorem 5** \( \ell \)-Length \( k \)-Color Path is \( W[1]\)-hard when parameterized by \( \ell \).

### 4 NP-hardness

In this section we show that the \( \ell \)-L \( k \)-CP problem is strongly NP-hard for every fixed constant \( k \geq 4 \). We give a reduction from the following problem which was shown to be NP-complete by Garey, Johnson, and Stockmeyer [15].

**Planar Graph 3-Colorability (P3C):**

**Instance:** A planar graph \( G \).

**Question:** Is \( G \) 3-colorable?

In contrast to the previous section, here we construct an instance of the \( \ell \)-L \( k \)-CP problem. An instance \( G, L, \alpha, \beta, \ell \) of the \( \ell \)-L \( k \)-CP problem with \( L(v) \subseteq [4] \) for all \( v \) is easily transformed to an instance of \( \ell \)-L \( k \)-CP, by adding one complete graph on four vertices \( x_i \) with \( i \in [4] \) and \( \alpha(x_i) = \beta(x_i) = i \), and edges \( vx_i \) for every vertex \( v \in V(G) \) and \( i \notin L(v) \). This yields:

**Lemma 6 ([5])** For any \( k \geq 4 \), in polynomial time an \( \ell \)-L \( k \)-CP instance \( (G, L, \ell, \alpha, \beta) \), with color lists \( L(v) \subseteq [4] \), can be transformed into an \( \ell \)-L \( k \)-CP instance \( (G', k, \ell, \alpha', \beta') \), such that the distance between \( \alpha \) and \( \beta \) in \( C(G, L) \) (possibly infinite) is the same as the distance between \( \alpha' \) and \( \beta' \) in \( C_k(G') \).

We will also make heavy use of the notion of \((a, b)\)-forbidding paths and their properties, which were introduced in [5]. Informally, these are paths that can be added between any pair of vertices \( u \) and \( v \) (provided that \( L(u), L(v) \neq [4] \)), that function as a special type of edge, which only excludes the color combination \((a, b)\) for \( u \) and \( v \) respectively, but allows (recoloring to) any other color combination. More precisely, for \( a, b \in [4] \), a path \( P \) with color lists \( L(x) \subseteq [4] \) for all \( x \in V(P) \) and end vertices \( u \) and \( v \) is an \((a, b)\)-forbidding path if the following two properties hold:

- For any combination of colors \( x \in L(u) \) and \( y \in L(v) \), there exists an \( L \)-coloring \( \gamma \) for \( P \) with \( \gamma(u) = x \) and \( \gamma(v) = y \) if and only if \( x \neq a \) or \( y \neq b \). Such a pair \((x, y)\) is called admissible for \( P \).
- For any \( L \)-coloring \( \gamma \) of \( P \) and any admissible pair \((x, y)\) with \( x = \gamma(u) \) or \( y = \gamma(v) \): there exists an \( L \)-recoloring sequence from \( \gamma \) to an \( L \)-coloring \( \delta \) with \( \delta(u) = x \) and \( \delta(v) = y \), in which every internal vertex is recolored at most once, and \( u \) and \( v \) are not recolored until the last step.
(Note that an \((a, b)\)-forbidding path from \(u\) to \(v\) is a \((b, a)\)-forbidding path from \(v\) to \(u\).) The second property implies that such an \(L\)-recoloring sequence has length at most \(|V(P)| - 1\). For instance, a \((1, 4)\)-forbidding path from \(u\) to \(v\) with \(L(u) = \{1, 2, 3\}\) and \(L(v) = \{2, 3, 4\}\) can be obtained by assigning the following lists along a path of length 6 (starting with \(L(u)\) and ending with \(L(v)\)): \(\{1, 2, 3\}, \{1, 4\}, \{2, 4\}, \{2, 3\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\). It can be verified that the two properties above hold. In [5], it was observed that for any combination of \(L(u), L(v)\) and \((a, b)\) with \(L(u), L(v) \neq [4]\), an \((a, b)\)-forbidding path can be constructed, of length six. We repeat the construction here for completeness.

**Lemma 7** ([5]) For any \(L_u \subset [4], L_v \subset [4], a \in L_u\) and \(b \in L_v\), there exists an \((a, b)\)-forbidding \((u, v)\)-path \(P\) of length six with \(L(u) = L_u\), \(L(v) = L_v\) and all other color lists \(L(w) \subseteq [4]\).

**Proof:** We can choose colors \(c \in [4] \setminus L_u\) and \(d \in [4] \setminus L_v\). Next, we can choose colors \(e \in [4] \setminus \{a, c\}\) and \(f \in [4] \setminus \{b, d\}\) with \(e \neq f\). Assign the following colors lists along the path: \(L_u, \{a, c\}, \{c, e\}, \{e, f\}, \{f, d\}, \{d, b\}, L_v\). It can be verified that this path satisfies the desired properties. \(\square\)

The reader is referred to [5] for more details on the list recoloring version of the problem and the use of \((a, b)\)-forbidding paths.

**Construction** Given an instance \(G\) of P3C, we construct an instance \((G', L, \ell, \alpha, \beta)\) of \(\ell\)-\(L\)-P3C as follows. Start with the vertex set \(V(G)\). All of these vertices \(u \in V(G)\) receive color \(\alpha(u) = 1\) and \(L(u) = \{1, 2, 3\}\). For every edge \(uv \in E(G)\), add three vertices \(x_{uv}, y_{uv}\) and \(z_{uv}\), with \(\alpha(x_{uv}) = 4\), \(\alpha(y_{uv}) = 4\), \(\alpha(z_{uv}) = 4\), \(L(x_{uv}) = \{1, 2, 4\}\), \(L(y_{uv}) = \{3, 4\}\), and \(L(z_{uv}) = \{1, 2, 4\}\). We add the following edges and paths between these vertices.

- Add edges \(ux_{uv}\) and \(uy_{uv}\).
- Add a \((1, 2)\)-forbidding path from \(u\) to \(x_{uv}\).
- Add a \((1, 3)\)-forbidding path from \(u\) to \(x_{uv}\).
- Add a \((2, 3)\)-forbidding path from \(u\) to \(y_{uv}\).
- Add a \((2, 1)\)-forbidding path from \(v\) to \(x_{uv}\).
- Add a \((3, 2)\)-forbidding path from \(v\) to \(x_{uv}\).
- Add a \((1, 3)\)-forbidding path from \(v\) to \(y_{uv}\).
- Add a \((4, 1)\)-forbidding path from \(x_{uv}\) to \(z_{uv}\).
- Add a \((4, 2)\)-forbidding path from \(y_{uv}\) to \(z_{uv}\).
In all cases, we choose an \((a, b)\)-forbidding path of length six (Lemma 7). We denote \(Z = \{z_{uv} \mid uv \in E(G)\}\). Informally, these edges and forbidding paths ensure the following property (see Lemma 8 below for details): whenever \(u\) and \(v\) have the same color, both \(x_{uv}\) and \(y_{uv}\) must have color 4, and therefore \(z_{uv}\) must have color 4. On the other hand, when \(u\) and \(v\) currently have different colors, then either \(x_{uv}\) or \(y_{uv}\) can be recolored to a different color, and thus \(z_{uv}\) can either be recolored to 1 or 2.

Next, we add four vertices \(a, b, c, d\) to \(G'\), with the following colors and lists:
\[
\alpha(a) = 1, \quad \alpha(b) = 2, \quad \alpha(c) = 3, \quad \alpha(d) = 4, \quad L(a) = \{1, 2, 3\}, \quad L(b) = \{1, 2\}, \\
L(c) = \{3, 4\}, \quad L(d) = \{4\}.
\]
Add all edges between these vertices, except for \(cd\). Add edges from all vertices in \(Z\) to \(c\). This yields the graph \(G'\). Note that for all \((a, b)\)-forbidding paths that we added, we chose the colors \(\alpha\) of the end vertices \(p\) and \(q\) such that \(\alpha(p) \neq a\) or \(\alpha(q) \neq b\). So we can extend \(\alpha\) to an \(L\)-coloring of the entire graph (by the definition of \((a, b)\)-forbidding paths); we do this in an arbitrary way.

Finally, we define the target coloring \(\beta\): for all vertices \(v \in V(G') \setminus \{a, b\}\), set \(\beta(v) = \alpha(v)\). We set \(\beta(a) = 2\) and \(\beta(b) = 1\), so the goal is to reverse the colors of these two vertices, while keeping all other colors the same.

**Lemma 8** Let \(\gamma\) be an \(L\)-coloring of \(G'\), and let \(uv \in E(G)\). If \(\gamma(u) = \gamma(v)\), then \(\gamma(z_{uv}) = 4\). On the other hand, if \(\gamma(u) \neq \gamma(v)\), then there exists an \(L\)-recoloring sequence of length at most \(O(1)\) from \(\gamma\) to an \(L\)-coloring \(\gamma'\) with \(\gamma'(z_{uv}) \neq 4\) and with \(\gamma(w) = \gamma'(w)\) for all \(w \in V(G) \cup Z\).

**Proof:** We first show that if \(\gamma(z_{uv}) \neq 4\), then \(\gamma(u) \neq \gamma(v)\). Suppose that \(\gamma(z_{uv}) = 1\). Then the \((4, 1)\)-forbidding path from \(x_{uv}\) to \(z_{uv}\) ensures that \(\gamma(x_{uv}) \in \{1, 2\}\). If \(\gamma(x_{uv}) = 1\), then the edge \(ux_{uv}\) and \((3, 1)\)-forbidding path from \(u\) ensure that \(\gamma(u) \notin \{1, 3\}\), so \(\gamma(u) = 2\). In addition, the \((2, 1)\)-forbidding path from \(v\) to \(x_{uv}\) forces \(\gamma(v) = 4\), so in this case, \(\gamma(u) = \gamma(v)\). Similarly, if \(\gamma(x_{uv}) = 2\) then considering the incident edge \(ux_{uv}\) and incident forbidding paths, we conclude that \(\gamma(u) = 3\) and \(\gamma(v) \neq 3\), so again \(\gamma(u) \neq \gamma(v)\).

Next, suppose that \(\gamma(z_{uv}) = 2\). Then the \((4, 2)\)-forbidding path from \(y_{uv}\) to \(z_{uv}\) ensures that \(\gamma(y_{uv}) = 3\). Similar to before (considering the remaining forbidding paths), this implies that \(\gamma(u) = 1\) and \(\gamma(v) \neq 1\), so \(\gamma(u) \neq \gamma(v)\). We have now proved the statement in all cases. Hence if \(\gamma(u) = \gamma(v)\), then \(\gamma(z_{uv}) = 4\).

Now suppose \(\gamma(u) \neq \gamma(v)\). Then considering the same color combinations and forbidding paths as above, we conclude that in every case, \(x_{uv}\) can be recolored to 1 or 2, or \(y_{uv}\) can be recolored to 3, possibly after first recoloring some internal vertices of incident forbidding paths, but without recoloring any other vertices. Next, \(z_{uv}\) can be recolored to 1 or 2, again after possibly recoloring some internal vertices of the incident forbidding paths. This shows that \(z_{uv}\) can be recolored to a color different from 4 in \(O(1)\) recoloring steps, without changing the color of any other vertex in \(V(G) \cup Z\).

**Lemma 9** If \(G\) is \(3\)-colorable, then there exists an \(L\)-recoloring sequence for \(G'\) from \(\alpha\) to \(\beta\), of length \(O(m)\), where \(m = |E(G)|\).
Proof: The vertices of $G$ form an independent set in $G'$. Therefore, if $G$ is 3-colorable we can start by recoloring the corresponding vertices in $G'$ to such a 3-coloring, using the colors 1, 2, 3. Since all of the $(a, b)$-forbidding paths in $G'$ from a vertex $u \in V(G)$ to another vertex $v$ satisfy $\alpha(v) \neq b$ and $v \notin V(G)$, it follows from the definition that this can be done while recoloring all vertices from $V(G)$ and all internal vertices of $(a, b)$-forbidding paths at most once, and recoloring no other vertices of $G'$. So this can be done using $O(m)$ recoloring steps.

Now by Lemma 8 we can subsequently recolor the vertices of these $(a, b)$-forbidding paths and all vertices in $Z$, such that all vertices in $Z$ receive color 1 or 2. For every vertex in $Z$, this can be done in $O(1)$ recoloring steps, so this gives $O(m)$ steps in total.

Next, we can recolor $c$ to color 4. At this point, $a$ can temporarily be recolored to 3, $b$ to 1, and next $a$ to 2. Then we can reverse all previous vertex recolorings, except for $a$ and $b$, and end up with the coloring $\beta$. The total length of this $L$-recoloring sequence is in $O(m)$. □

Lemma 10 If there exists an $L$-recoloring sequence from $\alpha$ to $\beta$ for $G'$, then $G$ is 3-colorable.

Proof: By considering the vertices $a, b, c, d$, we see that any such $L$-recoloring sequence must contain a coloring $\gamma$ with $\gamma(c) = 4$. This implies that for every $z \in Z$, $\gamma(z) \in \{1, 2\}$. Then Lemma 8 shows that for every $uv \in E(G)$, $\gamma(u) \neq \gamma(v)$. Hence $\gamma$ restricted to $V(G)$ is a 3-coloring of $G$. □

Combining Lemmas 8 and 10 with the fact that we can easily transform the $\ell$-$L$-$k$-$CP$ instance to an $\ell$-$L$-$k$-$CP$ instance (Lemma 6), and the NP-hardness of $P3C$, shows that $\ell$-$L$-$k$-$CP$ is strongly NP-hard. For this it is essential that the parameter $\ell$ in the reduction is polynomial in the size of the $P3C$-instance. More precisely, we have the following result:

Theorem 11 For any constant $k \geq 4$, the problem $\ell$-$L$-$k$-$CP$, with $\ell$ encoded in unary, is NP-complete.

5 FPT algorithm

For every constant $\ell$, the $\ell$-$L$-$k$-$CP$ problem can be solved in polynomial time, using the following simple branching algorithm. Denote the given instance by $(G, k, \ell, \alpha, \beta)$, with $|V(G)| = n$. Start with with the initial $k$-coloring $\alpha$. For every coloring generated by the algorithm, consider all possible $k$-colorings that can be obtained from it using one recoloring step. Recurse on these choices, up to a recursion depth of at most $\ell$. Return yes if and only if in one of the recursion branches, the target coloring $\beta$ is obtained. Clearly, this algorithm yields the correct answer. For one coloring, there are at most $kn$ possible recoloring steps, so branching with depth $\ell$ shows that at most $O((kn)^\ell)$ colorings will be considered. This shows that for parameter $\ell$, the problem is in XP.
Because \( \ell \)-L \( k \)-CP is NP-hard for every constant \( k \geq 4 \), a similar result cannot be obtained for the parameter \( k \), unless \( P = NP \). Since the \( \ell \)-L \( k \)-CP problem is \( W[1] \)-hard when parameterized by \( \ell \), we also do not expect any algorithms solving the problem in \( O(h(\ell)(kn)^{O(1)}) \) time (for any computable function \( h \)). However, we shall see in this section that the problem is in fact fixed-parameter tractable when parameterized by \( k + \ell \): it can be solved in \( O(f(k,\ell)n^{O(1)}) \) time, for some computable function \( f \).

We first give a high-level description of the algorithm. Denote \( S = \{ v \in V(G) \mid \alpha(v) \neq \beta(v) \} \). Clearly, when \(|S| > \ell\) we have a no-instance and when \(|S| = 0\) we have a trivial yes-instance. In what follows, we assume \( 0 < |S| \leq \ell \).

The main challenge that we need to overcome is that the number of vertices that potentially need to be recolored cannot easily be bounded by a function of \( \ell \); in particular, there may be too many vertices at distance at most \( \ell \) from \( S \). However, once we know which vertices will be recolored, the problem can be solved using a branching algorithm similar to the one above.

Let \( R = \alpha_0, \ldots, \alpha_\ell \) be a recoloring sequence for a graph \( G \). For every vertex \( v \in V(G) \), the used-color lists for \( R \) are defined as \( U(v) = \{ \alpha_i(v) \mid i \in \{0, \ldots, \ell\} \} \). Note that a vertex \( v \) is recolored at least once in \( R \) if and only if \(|U(v)| \geq 2\). In addition we have the following simple but useful observation.

**Proposition 12** Let \( R \) be a recoloring sequence for \( G \) of length \( \ell \), and let \( U \) denote the used-color lists for \( R \). Then \( \sum_{v \in V(G)}(|U(v)| - 1) \leq \ell \).

Our main algorithm to solve the \( \ell \)-L \( k \)-CP problem is Algorithm 2, which uses the subroutine given in Algorithm 1. The RECOLOR algorithm (Algorithm 2) consists of a two-stage branching algorithm. The first stage of the algorithm ignores the ordering of recoloring steps and simply tries to “guess” the used-color lists for each vertex, assuming that a recoloring sequence \( R \) from \( \alpha \) to \( \beta \) of length at most \( \ell \) exists. These guesses are stored in the lists \( L(v) \). Clearly, \( \{\alpha(v), \beta(v)\} \subseteq L(v) \) should hold. To construct these lists \( L \), the algorithm maintains two disjoint sets of vertices \( A \) and \( B \) as follows. All vertices in \( A \cup B \) will be recolored at least once, according to our current guess. Vertices are in \( B \) if we have already guessed a used-color list for them. Initially, we have \( A = S \) and \( B = \emptyset \). While \( A \) is not empty, we pick a vertex \( v \in A \) and branch on all possible lists \( L(v) \). We then delete \( v \) from \( A \) and add it to \( B \). Before continuing with the next vertex from \( A \), we inspect the neighbors of \( v \). If there exists \( u \in N_G(v) \setminus (A \cup B) \) such that \( \alpha(u) \in L(v) \), we add \( u \) to \( A \) since \( u \) must also be recolored at least once, assuming that \( v \) will indeed receive all colors in \( L(v) \).

If we reach a state where \( \sum_{v \in B}(|L(v)| - 1) > \ell \), then the current branch is ignored since these lists cannot correspond to used-color lists of a recoloring sequence of length at most \( \ell \). On the other hand, when \( A = \emptyset \) and \( \sum_{v \in B}(|L(v)| - 1) \leq \ell \), we have a “possible solution”. That is, we still need to make sure that there exists a feasible ordering of the recoloring steps that transforms \( \alpha \) to \( \beta \). This is handled by the LISTRECOLOR subroutine (Algorithm 3), which is a branching algorithm similar to one sketched in the beginning of this section, although we only assign colors from the lists \( L(v) \). Since \( \sum_{v \in B}(|L(v)| - 1) \leq \ell \),
Algorithm 1  \textsc{ListRecolor}(G, L, \alpha, \beta, \ell)

\textbf{Input:} A graph $G$, nonnegative integer $\ell$, color lists $L(v)$ for all $v \in V(G)$, and two $L$-colorings $\alpha$ and $\beta$ of $G$.

\textbf{Output:} “YES” if and only if there exists an $L$-recoloring sequence from $\alpha$ to $\beta$ of length at most $\ell$.

1: Return $\text{Recurse}(\alpha, \ell)$

Subroutine $\text{Recurse}(\gamma, \ell')$:

2: if $\gamma = \beta$ and $\ell' \geq 0$ then return YES
3: if $\ell' \leq 0$ then return NO
4: for all $L$-colorings $\delta$ that can be obtained from $\gamma$ by changing the color of a single vertex $x$ to a different color in $L(x)$:
5: if $\text{Recurse}(\delta, \ell' - 1) = \text{YES}$ then return YES
6: return NO

at any time there are at most $\ell$ different ways to recolor a vertex, according to the lists. So, branching up to a depth of $\ell$, this yields an FPT algorithm for (such instances of) the $\ell$-L $L$-CP problem, parameterized by $\ell$. Combined with Algorithm 2, which generates all relevant guesses for the used-color lists, this yields an FPT algorithm for the $\ell$-L $k$-CP problem, parameterized by $k + \ell$. We now present the details of these algorithms, starting with the \textsc{ListRecolor} subroutine (Algorithm 1).

Lemma 13 Let $(G, L, \alpha, \beta, \ell)$ be an input for Algorithm 1 with $n = |V(G)|$, and let $p = \sum_{x \in V(G)} (|L(x)| - 1)$. In time $O(p^\ell \cdot \text{poly}(n))$, Algorithm 1 decides whether there exists an $L$-recoloring sequence for $G$ from $\alpha$ to $\beta$, of length at most $\ell$.

Proof: An easy induction proof shows that a recursive call $\text{Recurse}(\gamma, \ell')$ in Algorithm 1 returns YES if and only if there exists an $L$-recoloring sequence from $\gamma$ to $\beta$ of length at most $\ell'$: Line 4 guarantees that every new $\delta$ that is generated is again an $L$-coloring, which is adjacent to $\gamma$ in $\mathcal{C}(G, L)$. This shows that the algorithm is correct.

For every $L$-coloring $\gamma$, there are $p$ ways to change the color of some vertex $x \in V(G)$ to a different color in $L(x)$. So at most $p$ new $L$-colorings are generated in one recursive call. Obviously, the recursion depth is at most $\ell$, so this shows that at most $O(p^\ell)$ recursive calls are made in total. One recursive call takes time $\text{poly}(n)$, so this yields the stated complexity bound. \hfill $\square$

Lemma 14 Let $\alpha$ and $\beta$ be two $k$-colorings for a graph $G$, and $\ell \in \mathbb{N}$. For every recursive call $\text{Recurse}(A, B, L)$ made by Algorithm 2 on this input, the following conditions hold:
Algorithm 2 \textsc{Recolor}(G, k, ℓ, α, β)

\textbf{Input:} A graph \(G\), nonnegative integers \(k\) and \(ℓ\), and two \(k\)-colorings \(α\) and \(β\).

\textbf{Output:} “YES” if and only if there exists a \(k\)-recoloring sequence from \(α\) to \(β\) of length at most \(ℓ\).

7: \(S := \{v \in V(G) \mid α(v) \neq β(v)\}\)
8: if |\(S\)| > \(ℓ\) then return “NO”
9: if |\(S\)| = 0 then return “YES”
10: Return \textsc{Recurse}(\(S\), \(∅\), \(\{f^{∅}\}\))

Subroutine \textsc{Recurse}(\(A\), \(B\), \(L\)):

11: if \(\sum_{v \in B}(|L(v)| - 1) > ℓ\) then return “NO”
12: if \(A = \emptyset\) then return \textsc{ListRecolor}(\(G[B]\), \(L|_{B}\), \(α|_{B}\), \(β|_{B}\))
13: Choose \(v \in A\).
14: \(B' := B \cup \{v\}\)
15: for each \(U \subseteq [k]\) with \(2 \leq |U| \leq ℓ\) and \(\{α(v), β(v)\} \subseteq U\):
16: \(L'(v) := U\)
17: for each \(u \in B\):
18: \(L'(u) := L(u)\)
19: \(A' := A \setminus \{v\}\)
20: for each \(u \in N(v) \setminus (A \cup B)\) with \(α(u) \in U\):
21: \(A' := A' \cup \{u\}\)
22: if \textsc{Recurse}(\(A'\), \(B'\), \(L'\)) = “YES” then return “YES”
23: return “NO”

(1) \(A\) and \(B\) are disjoint subsets of \(V(G)\).

(2) For every \(u \in V(G) \setminus B\): \(u \in A\) if and only if
   (a) \(α(u) \neq β(u)\), or
   (b) there exists an edge \(uv \in E(G)\) with \(v \in B\) and \(α(u) \in L(v)\).

Proof: The first time the subroutine \textsc{Recurse} is called (in Line 10), \(B = ∅\), and \(A\) contains exactly those vertices that have different colors in \(α\) and \(β\), so clearly the above conditions are satisfied.

Now consider a call \textsc{Recurse}(\(A\), \(B\), \(L\)) where the arguments satisfy the given conditions, and an iteration of the for-loop in Line 15 where a subsequent call \textsc{Recurse}(\(A'\), \(B'\), \(L'\)) is made. Lines 14, 19 and 21 show that \(A'\) and \(B'\) are again disjoint (in Line 21 only vertices outside of \(B'\) are added), so Condition (1) is again satisfied. For vertices in \(A \setminus \{v\}\), it still holds that at least one of the Conditions (2a) and (2b) is satisfied (also with respect to the new lists \(L'\)).

Lines 20 and 21 show that the newly added vertices in \(A'\) are exactly those that
now satisfy Condition (2b), since \( L'(v) = U \) and \( v \) is the only new vertex in \( B' \). We conclude that both Conditions (1) and (2) are maintained for \( A', B', L' \). By induction, it follows that for every recursive call \( \text{RECURSE}(A, B, L) \), the above conditions hold for \( A, B \) and \( L \). □

**Lemma 15** Let \( \alpha \) and \( \beta \) be two \( k \)-colorings for a graph \( G \), and \( \ell \in \mathbb{N} \). If Algorithm 2 returns “YES” then there exists a \( k \)-recoloring sequence for \( G \) from \( \alpha \) to \( \beta \) of length at most \( \ell \).

*Proof:* Consider a recursive call \( \text{RECURSE}(A, B, L) \). If “YES” is returned in Line 12 then \( A = \emptyset \), and there exists an \( L \)-recoloring sequence for \( G[B] \) from \( \alpha|_B \) to \( \beta|_B \) of length at most \( \ell \). Since \( A = \emptyset \), this also yields a valid recoloring sequence for the entire graph \( G \), starting from \( \alpha \), of the same length. This is because any color that is assigned to a vertex \( v \in B \) is chosen from \( L(v) \subseteq [k] \), and therefore does not conflict with the color \( \alpha(w) \) of any vertex \( w \in V(G) \setminus B \) (by Condition (2b) from Lemma 14). In addition, Condition (2a) shows that the recoloring sequence that we obtain for \( G \) this way ends with \( \beta \). We conclude that there exists a \( k \)-recoloring sequence from \( \alpha \) to \( \beta \) for \( G \), of length at most \( \ell \). If YES is returned by a subsequent recursive call in Line 22, then the claim follows by induction. □

**Lemma 16** Let \( \alpha \) and \( \beta \) be two \( k \)-colorings for a graph \( G \), and \( \ell \in \mathbb{N} \). If there exists a \( k \)-recoloring sequence for \( G \) from \( \alpha \) to \( \beta \) of length at most \( \ell \) then Algorithm 2 returns “YES”.

*Proof:* We prove by induction that for every call \( \text{RECURSE}(A, B, L) \), “YES” is returned if

(3) there exists a \( k \)-recoloring sequence \( R \) from \( \alpha \) to \( \beta \) for \( G \), of length at most \( \ell \), such that for every vertex \( v \in B \), \( L(v) \) is (exactly) the set of colors used by \( R \) for \( v \).

Applying this statement to the initial call \( \text{RECURSE}(S, \emptyset, \{ f^0 \}) \) in Line 10 proves the lemma.

Suppose Condition (3) is satisfied for \( A, B, L \). We show that “YES” is returned. Let \( R \) be a corresponding recoloring sequence from \( \alpha \) to \( \beta \), of length at most \( \ell \), which uses exactly the colors \( L(v) \) for each \( v \in B \). First, Proposition 12 shows that “NO” is not returned in Line 11 and the algorithm continues. If there are no vertices \( u \in V(G) \setminus B \) that satisfy Condition (2a) or (2b), then \( A = \emptyset \) (Lemma 14), so a call to \( \text{LISTRECOLOR} \) is made in Line 12. Restricting all colorings in \( R \) to \( B \) yields a \( k \)-recoloring sequence for \( G[B] \) from \( \alpha|_B \) to \( \beta|_B \) that uses exactly the colors in \( L \) for each vertex, of length at most \( \ell \), so in this case, “YES” is returned.

In the remaining case, we may assume that \( A \) is nonempty, and a vertex \( v \in A \) is chosen in Line 13. Let \( U \) be the set of colors used for \( v \) by the sequence \( R \). So \( \{ \alpha(v), \beta(v) \} \subseteq U \). In addition, we argue that \( |U(v)| \geq 2 \). If \( \alpha(v) \neq \beta(v) \),
this is obvious. Otherwise, by Condition (2) from Lemma 14, \( v \) has a neighbor \( u \in B \) with \( \alpha(v) \in L(u) \). Since \( R \) uses exactly the colors \( L(u) \) for \( u, v \) must be recolored at least once, and thus \( |U| \geq 2 \). We conclude that \( U \) will be considered in an iteration of the for-loop in Line 14. Let \( L', A', B' \) be the lists and sets as constructed in this iteration. We observe that \( R \) again satisfies the properties from Condition (3), also with respect to this \( L' \) and \( B' \). Indeed, \( B' = B \cup \{v\} \), and \( L'(v) = U \), which is exactly the set of colors used by \( R \) for \( v \). For all other vertices \( w \in B' \), \( L(w) = L'(w) \). So we may use induction to conclude that “YES” is returned by the call \( \text{Recurse}(A', B', L') \), and thus in Line 22, “YES” is returned.

\[\square\]

**Lemma 17** The Recolor algorithm (Algorithm 3) runs in \( \mathcal{O}(2^{k(\ell+1)} \cdot \ell^k \cdot \text{poly}(n)) \) time, where \( n = |V(G)| \).

**Proof:** The search-tree produced by the Recolor algorithm has depth at most \( \ell + 1 \), since every branch increases the quantity \( \sum_{v \in B}(|L(v)| - 1) \) by at least 1 (Line 15) and the base case is reached whenever \( \sum_{v \in B}(|L(v)| - 1) > \ell \) (Line 11). The number of sets considered in the for-loop in Line 15 is at most \( 2^k \), so every node in the search-tree has at most \( 2^k \) children. We conclude that at most \( \mathcal{O}(2^{k(\ell+1)}) \) recursive calls are made in total.

We now argue that for every recursive call, we spend at most \( \mathcal{O}(\ell^{\ell} \cdot \text{poly}(n)) \) time in total. For Line 12 this follows from Lemma 16 noting that whenever this line is reached, \( p = \sum_{v \in B}(|L(v)| - 1) \leq \ell \) holds (Line 11). All other lines can easily be implemented to run in time \( \text{poly}(n) \). (Note that we may assume w.l.o.g. that \( k \leq n + 1 \).) We attribute the time spent in Lines 16-22 to the resulting recursive call in Line 22. This shows that the entire complexity can be bounded by \( \mathcal{O}(2^{k(\ell+1)} \cdot \ell^k \cdot \text{poly}(n)) \).

Combining Lemmas 15, 16 and 17 yields the main result of this section:

**Theorem 18** \( \ell \)-Length \( k \)-Color Path is fixed-parameter tractable when parameterized by \( k + \ell \).

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