TRANSITION PROBABILITIES BETWEEN QUASIFREE STATES

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We obtain a general formula for the transition probabilities between any state of the \( C^* \)-algebra of the canonical commutation relations (CCR-algebra) and a squeezed quasifree state (Theorem III.1). Applications of this formula are made for the case of multimode thermal squeezed states of quantum optics using a general canonical decomposition of the correlation matrix valid for any quasifree state. In the particular case of a one mode CCR-algebra we show that the transition probability between two quasifree squeezed states is a decreasing function of the geodesic distance between the points of the upper half plane representing these states. In the special case of the purification map it is shown that the transition probability between the state of the enlarged system and the product state of real and fictitious subsystems can be a measure for the entanglement.

I. INTRODUCTION

The notion of the quasifree state has appeared and was developed in the framework of the \( C^* \)-algebra approach to the canonical commutation relations (CCR)\(^1\)–\(^{11}\). The quasifree states are the natural ground states for the Hamiltonians which are at most quadratic in the bosonic creation and annihilation operators. The essential property of the quasifree states is the fact that all their correlations are expressible in terms of the one and two-point functions. The advantage of the \( C^* \)-algebra approach to the theory of coherent and squeezed states (thermal or not) follows from the fact that the results are obtained in a representation independent form. This representation independence allows to elude the hard noncommutative calculation. But a price must be paid for this. A simple result in usual quantum mechanics is obtained, in this case, with the help of slightly difficult results concerning the transition probabilities between the states of \( C^* \)-algebras\(^12\).

We shall illustrate this assertion by giving a representation independent formula for the transition probability between a squeezed state and an arbitrary state of the \( C^* \)-algebra of CCRs (Theorem III.1).

Many interesting states of the radiation field in quantum optics are quasifree states. The factorization of the correlations in coherent states was discovered by Glauber as the essential property of these states of coherent light. The squeezed states are also quasifree states. The most general case is that of multimodal thermal displaced and squeezed states.

The main result of the paper, given in Section III, is the formula for the transition probabilities between any state of the \( C^* \)-algebra of the canonical commutation relations and any squeezed quasifree pure state (Theorem III.1). This result is obtained using the fact that any quasifree squeezed pure state can be obtained from a coherent state by a Bogoliubov automorphism, and the fact that for any coherent state there is an element of the CCR-algebra which is a minimal orthoprojection\(^4\). Then a general result of P. M. Alberti and V. Heinemann\(^12\) is applicable.

A new approach to the theory of multimode squeezed thermal states is described in Section IV, and is based on the combination of the Williamson theorem\(^13\) which gives the most general structure of a positive definite matrix with a theorem of Balian, De Dominicis and Itzykson\(^14\) which gives the most general structure of a symplectic matrix. In this way a general natural parametrization of the correlation matrix of a quasifree state is obtained. This parametrization points out the squeezing and the orthogonal symplectic transformations. The orthogonal transformations mix the different modes.

The geometric interpretation, given in Section V, is valid only for the one mode case and shows that the transition probability between two pure squeezed states decreases with the increase of the geodesic distance between the corresponding points of the Poincaré upper half plane.

In Section VI the relation between the Wigner function of a quasifree state of a system and that of a subsystem is given in most general form. A first application of this relation is given in Section VII where the correlation matrix of a pure quasifree state of an enlarged system (which reduces to the mixed state of the original system) is obtained in an elementary way\(^4\).
In Section VIII it is shown that the transition probability between the above pure state obtained for the enlarged system and the mixed state obtained as the direct product of the state of the original subsystem with the state of the fictitious subsystem can be a measure of the entanglement of these two subsystems reflected in the pure state of the total system.

Finally, in Section IX, it is shown how the correlations between the original subsystem and the fictitious one are created by the Bogoliubov transformations which map the Fock state into the above pure state obtained by the purification procedure.

II. QUASIFREE STATES

The phase space considered in the present paper is a finite dimensional symplectic space \((E, \sigma)\). This is a real vector space endowed with a real, bilinear, antisymmetric form \(\sigma(., .)\) which gives the symplectic structure on \(E\). Then \(E\) is of even real dimension \(2n\) and there exist in \(E\) symplectic bases of vectors \(\{e_j, f_j\}_{j=1, \ldots, n}\), i.e. reference systems such that \(\sigma(e_j, e_k) = \sigma(f_j, f_k) = 0\) and \(\sigma(e_j, f_k) = -\sigma(f_k, e_j) = \delta_{jk}\), \(j, k = 1, \ldots, n\). The coordinates \((\xi^j, \eta^j)\) of a vector \(u \in E\) in a symplectic basis \(u = \sum_{j=1}^n (\xi^j e_j + \eta^j f_j)\) are called symplectic coordinates. The measure \(dm(u) = \prod_{j=1}^n d\xi^j d\eta^j\) is the same for all symplectic coordinates and is called the Liouville measure on \((E, \sigma)\). There is a one-to-one correspondence between the symplectic bases and the linear operators \(J\) on \(E\) defined by \(Je_k = -f_k\) and \(Jf_k = e_k\), \(k = 1, \ldots, n\). The essential properties of these operators are:

\(\sigma(Ju, u) = 0\), \(\sigma(Ju, v) + \sigma(u, Jv) = 0\) \((u, v \in E\) and \(J^2 = -I\), \(I\) denotes the identity operator on \(E)\). Such operators are called complex structures.

In the following we shall use the matricial notations with \(J\) as column vectors. Then \(\sigma(u, v) = u^T J v\) and the scalar product is given by \(\sigma(Ju, v) = u^T J v, u, v \in E\). A linear operator \(S\) on \(E\) is called a symplectic operator if \(S^T J S = J\). When \(S\) is a symplectic operator then \(S^T\) and \(S^{-1}\) are also symplectic operators. The group of all symplectic operators \(Sp(E, \sigma)\) is called the symplectic group of \((E, \sigma)\). The Lie algebra of \(Sp(E, \sigma)\) is denoted by \(sp(E, \sigma)\) and its elements are operators \(R\) on \(E\) with the property: \((JR)^T = JR\). If \(J\) and \(K\) are two complex structures, there exists a symplectic transformation \(S\) such that \(S = S^{-1} KS\).

**Definition II.1**\(^\square\) The \(C^*-\)algebra \(A\) of the canonical commutation relations (CCR) is obtained by the completion of the \(^*\)-algebra \(\text{Span}\{ \delta_u : u \in E\}\), the elements of which satisfy the Weyl relations:

\[
\delta_u \delta_v = exp(-i\sigma(u, v)/2)\delta_{u+v}, \quad \delta_u^* = \delta_{-u}, \quad u, v \in E
\]

A state on \(A\) is a positive linear functional \(\omega\) normalized by \(\omega(\delta_0) = 1\)\(^\square\). The complex valued function on the phase space \(\omega(\delta_u)\) is called the characteristic or the generating function of the state \(\omega\). A continuous complex valued function \(f\) on \(E\), with \(f(0) = 1\), is the characteristic function of a state on \(A\) if and only if:

\[
\sum_{j,k=1}^N \bar{a}_j a_k \exp(i\sigma(u_j, u_k)/2)f(u_k - u_j) \geq 0
\]

for all \(a_k \in \mathbb{C}, u_k \in E, k = 1, \ldots, N,\) and \(N \in \mathbb{N}\).

For any \(v \in E\) an automorphism \(\tau_v\) of \(A\) can be defined by:

\[
\tau_v(x) = \delta_v^* x \delta_v
\]

A Bogoliubov transformation of \(A\) is a \(^*\)-automorphism of \(A\) defined for any \(S \in Sp(E, \sigma)\) by

\[
\alpha_S(\delta_u) = \delta_{Su}, \quad u \in E.
\]

The quasi-free automorphisms of \(A\) are those of the form \(\tau_v \circ \alpha_S\) where \(v \in E\) and \(S \in Sp(E, \sigma)\).

Let \(A\) be a symmetric \((A^T = A)\) and positive \((u^T Au \geq 0\) for all \(u \in E\)) matrix.

**Definition II.2**\(^\square\) A quasi-free state \(\omega(A, v)\) on \(A\) is a state whose characteristic function is given by the formula:

\[
\omega(A, v)(\delta_u) = exp(-u^T Au/4 + iu^T v)
\]

with the restriction:

\[
-JAJ \geq A^{-1}
\]
Evidently, we have:

\[ \omega(A,v) \circ \alpha_S = \omega(S^r A, S^r v) \]  

(2.7)

and \( \omega(A,v) = \omega(A,0) \circ \tau_v \). The state \( \omega(A,v) \) is pure iff \( \omega(A,0) \) is pure and this is the case iff \(- (JA)^2 = I \) (this is equivalent with \( JA \in Sp(E, \sigma) \) ) and, in particular, when \( A = I \) i.e. when the state is a coherent state. The pure squeezed states are the quasi-free states of the form \( \omega(I,v) \circ \alpha_S = \omega(S^r S, S^r v) \).

III. TRANSITION PROBABILITIES

The transition probability \( P(\omega_1; \omega_2) \) between two arbitrary states \( \omega_1 \) and \( \omega_2 \) on \( A \) can be defined in many ways. \[ \]  

Theorem III.1 If the state \( \omega_1 \) is a squeezed state the transition probability to an arbitrary state \( \omega_2 \) can be computed by the following representation independent formula:

\[ P(\omega_1; \omega_2) = (2\pi)^{-n} \int_E \omega_1(\delta_u) \omega_2(\delta_u) dm(u) \]  

(3.1)

where \( dm(u) \) is the Liouville measure defined above.

A proof of this result can be obtained by combining the construction of a projection as in Lemma 3 from \[ \]  

Lemma III.1 For any coherent state \( \omega(I,v) \) we can define an element \( p_v \in A \) such that it is a projector (i.e. \( p_v^2 = p_v \) and \( p_v^* = p_v \)) and has the property

\[ p_v x p_v = \omega(I,v)(x) p_v \]  

(3.2)

for any \( x \in A \).

Proof. We take

\[ p_v = (2\pi)^{-n} \int_E \omega(I,v)(\delta_u) \delta_u dm(u) \]  

(3.3)

The fact that \( p_v \) is a selfadjoint element of \( A \) is evident. The fact that it is an idempotent element of \( A \) results from

\[ p_v^2 = (2\pi)^{-2n} \int_E \int_E \omega(I,v)(\delta_u) \omega(I,v)(\delta_w) \delta_u \delta_w dm(u) dm(w) = \]

\[ (2\pi)^{-2n} \int_E \omega(I,z)(\delta_z) I(z) dm(z) \]  

(3.4)

where \( I(z) = \int_E \frac{\omega(I,v)(\delta_u) \omega(I,v)(\delta_z)}{\omega(I,v)(\delta_z)} exp(-\frac{i}{2} \sigma(u,z)) dm(u) \). Thus it is sufficient to prove that \( I(z) = (2\pi)^n \) and this follows by direct computations. In this way we have associated to any coherent state \( \omega(I,v) \) a projector from \( A \). In order to prove the last assertion it is sufficient to prove that:

\[ \int_E \omega(I,v)(\delta_u) \omega(I,v)(\delta_w) \exp\{-\frac{1}{2} i (\sigma(w,u) + \sigma(w,y))\} \]

\[ dm(w) = (2\pi)^n \omega(I,v)(\delta_u) \omega(I,v)(\delta_y) \exp\frac{1}{2} i \sigma(y) \]

(3.5)

As it results from \[ \]  

the property (3.5) is valid for any \( x \in A \) and (3.2) follows.

Then we can apply the following result of P.M. Alberti and V.Heinemann given in the Section II of the paper at the point (vi):
Lemma III.2 Let $A$ a $C^*$-algebra and $p \in A$ be a minimal orthoprojection, i.e., $pAp = Cp$. Let $pxp = \nu(x)p$, with $\nu(x) \in C$. Then the map $\nu : x \to \nu(x) \in C$ defines a state $\nu \in S(A)$ (the set of states on $A$) with $\nu(p) = 1$ and we have

$$P(\omega; \nu) = \omega(p)$$

for any state $\omega \in S(A)$.

**Proof of the Theorem.** We must verify the fact that

$$\omega_{(I,v)}(p_v) = 1.$$  

We have

$$\omega_{(I,v)}(p_v) = (2\pi)^{-n} \int_E \exp\left(-\frac{1}{2}w^T w\right)dm(x) = 1$$

Thus we have proved that for any state $\omega$ and coherent state $\omega_{(I,v)}$ the transition probability is given by :

$$P(\omega; \omega_{(I,v)}) = \omega(p_v) = (2\pi)^{-n} \int_E \omega_{(I,v)}(\delta_w)\omega(\delta_v)dm(v)$$

**Corollary III.1** The general formula for the transition probability between any quasifree state and a pure quasifree state is given by:

$$P(\omega_A, \omega_{(B,w)}) = \left(\det\left(\frac{A + B}{2}\right)\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(w - v)^T(I + A)^{-1}(w - v)\right)$$

**Proof.** It is well known that for any automorphism $\alpha$ of $A$ we have :

$$P(\omega_1 \circ \alpha; \omega_2 \circ \alpha) = P(\omega_1; \omega_2)$$

When the state $\omega$ is a quasi-free state $\omega_A$ we have

$$P(\omega_A; \omega_{(I,v)}) = \det\left(\frac{I + A}{2}\right)^{-1/2} \exp\left(-\frac{1}{2}(I + A)^{-1}v\right)$$

The transition probability between two squeezed states can be also given if we take into account the fact that any squeezed state is obtained by a Bogoliubov automorphism from a coherent state and the invariance of the transition probabilities with respect to automorphisms.

**Remark.** When both states are squeezed pure states we have:

$$P(\omega_{(A,v)}; \omega_{(B,w)}) = \det\left(\frac{I + S^T S}{2}\right)^{-1/2} \exp\left(-\frac{1}{2}(w - v)^T(I + A)^{-1}(w - v)\right)$$

where $S$ is the symplectic matrix such that $A = S^T BS$.

**IV. APPLICATIONS IN QUANTUM OPTICS**

**Theorem IV.1** The most general form of a correlation matrix $A$ is given by:

$$A = O^T \left( \begin{array}{cc} M & 0 \\ 0 & M^{-1} \end{array} \right) O^T \left( \begin{array}{cc} D & 0 \\ 0 & D \end{array} \right) O \left( \begin{array}{cc} M & 0 \\ 0 & M^{-1} \end{array} \right) O'$$

where $O$ and $O'$ are symplectic and orthogonal ($O^T O = I$) operators and $M$ is a diagonal $n \times n$ matrix.
The thermal squeezed states considered in [18] are obtained when 
\[ M = mI \]
and 
\[ D = dI \]
where \( D \) is a diagonal \( n \times n \) matrix.

The most general real symplectic transformation \( S \in Sp(E, \sigma) \) has the following structure:
\[ S = O \left( \begin{array}{cc} M & 0 \\ 0 & M^{-1} \end{array} \right) O' \]  
(4.3)

where \( O \) and \( O' \) are symplectic and orthogonal \( (O^T O = I) \) operators and \( M \) is a diagonal \( n \times n \) matrix.

**Examples.** Various particular kinds of such matrices are obtained taking \( O, O', D \) or \( M \) to be equal or proportional to the corresponding identity operator. A pure squeezed state is obtained when \( M = I \). If this condition is not satisfied, the state is a mixed state called thermal squeezed state. When \( M = I \) there is no squeezing and the correspondig states are pure coherent states or thermal coherent states. All these states have correlations between the different modes produced by the orthogonal symplectic operators \( O \) and \( O' \). Combining Eq. (3.2) and Eq. (3.12) we obtain that:
\[ P(\omega_A \circ \alpha_S; \omega(I,0)) = P(\omega_A; \omega(I,0)) \]  
(4.4)

iff \( S^T A S = A \) or iff \( S^T S = I \) i.e. iff \( S \) is orthogonal.

The first situation appears for example in the case of a parametric amplifier and this property explains the fact that the nonentanglement of states is preserved by such a transformation. Hence the parametric amplifier is a transition probability preserving device. The second situation appears if there is no squeezing i.e. \( M = I \). As it was shown in [4] for any quasifree state \( \omega_A \) we have:
\[ \omega_A = (\omega_{d_1} \otimes \omega_{d_2} \otimes ... \otimes \omega_{d_n}) \circ \alpha_S \]  
(4.5)

where \( d_1, ..., d_n \) are the diagonal elements of the matrix \( D \), and where \( \omega_{d_i}, i = 1, ..., n \) are the one mode quasi-free states defined on the phase spaces \( E_i \) generated by the vectors \( \{e_i, f_i\} \) (on which the operator \( A \) acts by \( Ae_i = d_ie_i \) and \( Af_i = d_if_i \)) by the formula:
\[ \omega_{d_i}(\delta_{\xi e_i + \eta f_i}) = \exp(-d_i((\xi)^2 + (\eta)^2)/4) \]  
(4.6)

with \( d_i \geq 1 \) as it follows from Eq. (2.6). It is clear that
\[ P(\omega_{d_1} \otimes \omega_{d_2} \otimes ... \otimes \omega_{d_n}; \omega(I,0)) = \prod_{i=1}^{n} P(\omega_{d_i}; \omega(I,0)) \]  
(4.7)

Combining Eqs. (4.5) and (4.7) we obtain that the equality
\[ P(\omega_A; \omega_I) = \prod_{i}^{n} P(\omega_{d_i}; \omega_I) \]  
(4.8)

is valid only for nonsqueezed states (i.e. when \( M = I \)).

For pure squeezed states \( (D = I) \) we still have such a decomposition of the transition probability as a product of transition probabilities for the individual modes. Moreover, there is another class of quasi-free states defined by the condition \( O = I \) in Eq. (4.3), and for which such a decomposition is valid. The formula which describes these two situations is:
\[ P(\omega_A; \omega_I) = \prod_{i}^{n} \frac{2}{\sqrt{(m_i^2 + d_i)(m_i^{-2} + d_i)}} \]  
(4.9)

The thermal squeezed states considered in [18] are obtained when \( M = mI \) and \( D = dI \). In this case the correlation matrix \( A \) takes the following form:
\[ A = O'T \begin{pmatrix} m^2dI & 0 \\ 0 & m^{-2}dI \end{pmatrix} O' \]  
(4.10)

and

\[ P(\omega_A; \omega_I) = \left( \frac{2}{\sqrt{(m^2 + d)(m^{-2} + d)}} \right)^n \]  
(4.11)

We shall take into account the fact that the most general form of an orthogonal symplectic matrix is

\[ O' = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \]  
(4.12)

where \( X \) and \( Y \) are \( n \times n \) matrices which satisfy the conditions: \( X^TX + Y^TY = I \) and \( X^TY = Y^TX \).

When \( Y = 0 \) the corresponding quasifree state is nonentangled and remains nonentangled in all symplectic frames obtained by such Bogoliubov transformations (i.e. orthogonal with \( Y = 0 \)). The ideal (lossless) beam splitter effects a transformation of such kind and the nonentanglement is preserved only when all modes are with the same temperature and equally squeezed. It follows from the above discussion that the ideal beam splitter is a transition probability preserving device.

V. A GEOMETRIC INTERPRETATION

The squeezed states are in a one-to-one correspondence with the elements of \( \mathbb{R} = \text{sp}(E, \sigma) \cap \text{Sp}(E, \sigma) \). This correspondence is covariant with respect to the adjoint action of the symplectic group \( \text{Sp}(E, \sigma) \) on \( \mathbb{R} \) and the action of this group on the squeezed states given by the corresponding Bogoliubov automorphisms. Because the symplectic group acts transitively on \( \mathbb{R} \) it follows that this is a coadjoint orbit of \( \text{Sp}(E, \sigma) \). In fact, this is the Hermitian symmetric space \( \text{Sp}(E, \sigma)/U(n) \), where \( U(n) \) is the subgroup of \( \text{Sp}(E, \sigma) \) the elements of which are also orthogonal. Hence the squeezed states are labeled by the elements of this Hermitian symmetric space. In the one mode case \( (n = 1) \) the Hermitian symmetric space \( \text{Sp}(2, R)/U(1) \) is the Poincaré upper half plane \( \mathbb{H} \). The group \( \text{Sp}(2, R) \) acts on \( \mathbb{H} \) in the usual way:

\[ \gamma : z \rightarrow \frac{az + b}{cz + d} \]  
(5.1)

where \( z = x + \sqrt{-1}y \) and where

\[ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]  
(5.2)

with \( \text{det}(\gamma) = ad - bc = 1 \). The function

\[ u(z, z') = \frac{|z - z'|^2}{4yy'} \]  
(5.3)

is invariant to the action of the group \( \text{Sp}(2, R) \):

\[ u(\gamma z, \gamma z') = u(z, z'). \]  
(5.4)

The Poincaré metrics on \( \mathbb{H} \) is also invariant and is given by:

\[ ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{dzd\bar{z}}{y^2} \]  
(5.5)

The corresponding distance function \( \rho(z, z') \) is equal with the length of the geodesic between \( z \) and \( z' \).

**Proposition V.1** For any pair of squeezed states \( \omega_z \) and \( \omega_{z'} \):

\[ P(\omega_z; \omega_{z'}) = \frac{1}{ch(s/2)} \]  
(5.6)

where \( s \) is the geodesic distance between \( z \) and \( z' \).
Proof. For any two points $z, z' \in \mathbf{H}$ there exists $\gamma \in \text{Sp}(2, R)$ such that $\gamma z = \sqrt{-1}$ and $\gamma z' = \sqrt{-1}y_0$ for a real $y_0 \geq 1$. Evidently $\rho(z, z') \equiv \rho(\sqrt{-1}, \sqrt{-1}y_0)$. The geodesic between $\sqrt{-1}$ and $\sqrt{-1}y_0$ is the vertical straight line connecting these two points. Then

$$\rho(\sqrt{-1}, \sqrt{-1}y_0) = \int_1^t dy y = \ln(t) = s$$

(5.7)

and $1 + u(\sqrt{-1}, \sqrt{-1}t) = ch^2(s/2)$. From $\gamma \sqrt{-1} = \frac{a_0 + ib}{c_0 + id} = x + \sqrt{-1}y$ we obtain $x = \frac{bd+ac}{c^2+d^2}$ and $y = \frac{ad-bc}{c^2+d^2}$ and

$$1 + u(z, \sqrt{-1}) = \frac{a^2 + b^2 + c^2 + d^2 + 2}{4}$$

(5.8)

The correspondence between the points of the Poincaré half plane $\mathbf{H}$ and the squeezed states is given by

$$\sqrt{-1} \leftrightarrow J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(5.9)

and $z \leftrightarrow J\gamma$. Then

$$P(\omega_z, \omega_{\sqrt{-1}}) = \det\left(\frac{1 + \gamma^T \gamma}{2}\right)^{-1/2} = \frac{1}{ch(s/2)}$$

(5.10)

Evidently, this result is valid for any pair of squeezed states.

VI. THE REDUCTION OF THE WIGNER FUNCTIONS TO THE SUBSYSTEMS

In the following we shall use the fact that the Wigner functions of the states on the algebra of the bosonic commutation relations give a complete description of these states.

Definition VI.1 The Wigner function of the state $\omega$ is defined as the Fourier symplectic transform of the corresponding characteristic function $\omega(\delta_u)$:

$$W_\omega(u) = (2\pi)^{-2n} \int_E \exp[i\sigma(u, v)]\omega(\delta_v)dm(v)$$

(6.1)

Proposition VI.1 The Wigner function of a quasi-free state is given by

$$W_\omega(u) = \pi^{-n}(\det G)^{1/2} \exp(-u^T Gu)$$

(6.2)

where $G = -JA^{-1}J$ is a $2n \times 2n$ real, symmetric and positive definite matrix which satisfies a restriction which is equivalent with the restriction (2.6):

$$-(JG)^2 \leq I$$

(6.3)

Remarks. We have also $JG \in \text{sp}(E, \sigma)$. If $\omega$ is a pure quasi-free state then $G = A$ and $\det G = \det A = 1$ (because $G$ and $A$ belong to $\text{Sp}(E, \sigma)$).

Corollary VI.1 The most general form of the matrix $G$ which defines the Wigner function of a quasi-free state is the following:

$$G = O^T \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} O^T \begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix} O \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} O'$$

(6.4)

where $O$ and $O'$ are symplectic and orthogonal ($O^T O = I$) matrices and $M$ is a diagonal $n \times n$ matrix.
Proof Since $S^TJS = (S^{-1})^TJS^{-1} = SJST = J$ for any symplectic matrix $S$ it follows that

$$G = -JA^{-1}J = -JS^{-1}\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} (S^T)^{-1}J = S^T \begin{pmatrix} D^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} S$$ (6.5)

The most general real symplectic transformation $S \in Sp(E, \sigma)$ has [17] the following structure:

$$S = O \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} O'$$ (6.6)

where $O$ and $O'$ are symplectic and orthogonal ($O^T O = I$) operators and $M$ is a diagonal $n \times n$ matrix.

Let us suppose that the state of the quantum system $S$ is described by the Wigner function $W(u)$ defined on the phase space $(E, \omega)$ which can be considered as being composed from two subsystems $S_1$ and $S_2$ with the phase spaces $(E_1, \omega_1)$ and $(E_2, \omega_2)$ respectively, where $E = E_1 \bigoplus E_2$ and $\omega_1$ and $\omega_2$ are the restrictions of the state $\omega$ to $E_1$ and to $E_2$ respectively. In this case the Wigner function $W_\omega(u) = \pi^{-n}(detG)^{\frac{1}{2}} \exp(-u^T Gu)$ can be written as

$$W(x, y) = \pi^{-n} \left( det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \right)^{\frac{1}{2}} \exp(- (x^T y^T) \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix})$$ (6.7)

Theorem VI.1 If $detC \neq 0$ then the Wigner function of the subsystem $S_1$ is given by

$$W_1(x) = \pi^{-\frac{1}{2}} (det(A - BC^{-1}B^T))^{\frac{1}{2}} \exp \left[ -x^T (A - BC^{-1}B^T)x \right]$$ (6.8)

and if $detA \neq 0$ the Wigner function of the subsystem $S_2$ is given by

$$W_2(y) = \pi^{-\frac{1}{2}} (det(C - B^T A^{-1}B))^{\frac{1}{2}} \exp \left[ -y^T (C - B^T A^{-1}B)y \right]$$ (6.9)

Proof. The Wigner function $W_1(x)$ of the subsystem $S_1$ is defined by the integration on the coordinates of the subsystem $S_2$

$$W_1(x) = \int_{E_2} W(x, y) dm(y)$$ (6.10)

If $detC \neq 0$ then

$$W_1(x) = \pi^{-\frac{1}{2}} (det(A - BC^{-1}B^T))^{\frac{1}{2}} \exp \left[ -x^T (A - BC^{-1}B^T)x \right]$$ (6.11)

We have used Schur’s formula

$$det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = detC det(A - BC^{-1}B^T)$$ (6.12)

Analogously, if $detA \neq 0$ then

$$W_2(y) = \pi^{-\frac{1}{2}} (det(C - B^T A^{-1}B))^{\frac{1}{2}} \exp \left[ -y^T (C - B^T A^{-1}B)y \right]$$ (6.13)

Also we have used the fact that

$$det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = detAdet(C - B^T A^{-1}B)$$ (6.14)
THE PURIFICATION OF A MIXED STATE

Theorem VII.1 For any quasifree mixed state $\omega_A$ on the $C^*$-algebra of commutation relations defined on the phase space $E$ there exists a pure quasifree state $\tilde{\omega}_A$ on the $C^*$-algebra of commutation relations defined on the phase space $E \oplus E$ endowed with the symplectic structure $\begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}$. This pure quasi-free state $\tilde{\omega}_A$ is defined by the correlation matrix $\tilde{A}$ (which is a $4n \times 4n$ real, symmetric and positive definite matrix)

$$\tilde{A} = \begin{pmatrix} A & A_\sqrt{I+(JA)^{-2}} \\ A_\sqrt{I+(JA)^{-2}} & A \end{pmatrix} \quad \text{(7.1)}$$

Proof. This pure quasi-free state $\tilde{\omega}_A$ is defined by the correlation matrix $\tilde{A}$ (which is a $4n \times 4n$ real, symmetric and positive definite matrix)

$$\tilde{A} = \begin{pmatrix} U & V \\ V & U \end{pmatrix} \quad \text{(7.2)}$$

which satisfies the restriction

$$-(\begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \begin{pmatrix} U & V \\ V & U \end{pmatrix})^2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad \text{(7.3)}$$

i.e.

$$(JU)^2 - (JV)^2 = -I, \quad JUJV = JVJU \quad \text{(7.4)}$$

The map which associates a pure quasifree state $\tilde{\omega}_A$ to any mixed quasi-free state $\omega_A$ is called the purification map. Because $\tilde{\omega}_A$ is a pure quasi-free state the corresponding Wigner function is defined by the same matrix:

$$\tilde{G} = \begin{pmatrix} U & V \\ V & U \end{pmatrix} \quad \text{(7.5)}$$

We remark that the original system can be considered as the reduction of the extended one. Then one must have the relation

$$\int_E W_{\omega_A}((u, \tilde{u})) dm(\tilde{u}) = W_{\omega_A}(u) \quad \text{(7.6)}$$

Hence we can use the result of the preceding section i.e. we have

$$G_1 = U - VU^{-1}V = G = -JA^{-1}J \quad \text{(7.7)}$$

From the Schur formula it follows that

$$\text{det} G = \text{det} U \text{det}(U - VU^{-1}V) = \text{det} A \text{det} A^{-1} = 1 \quad \text{(7.8)}$$

From the equations (7.3) and (7.6) we have

$$JG = JU - (JV)^2(JU)^{-1}. \quad \text{Hence } JGJU = (JU)^2 - (JV)^2 = -I, \quad \text{from which it follows that } U = A \text{ and } (JV)^2 = I + (JA)^2. \quad \text{It is easy to prove that } V = A_\sqrt{I+(JA)^{-2}} \text{ is a solution of the last equation.}$$

Remarks. The map defined by (7.6) which acts on the Wigner functions is the inverse map of the purification map. From the formula:

$$W_{\omega_A}((u, \tilde{u})) = \pi^{-2n} \exp\{-u^T A u - \tilde{u}^T A \tilde{u} - V^T \tilde{u} - \tilde{u}^T V u\} \quad \text{(7.9)}$$

it follows that the correlations between the variables $u$ of the real system and the variables $\tilde{u}$ of the fictitious one are given by the terms $(A_\sqrt{I+(JA)^{-2}})u + \tilde{u}(A_\sqrt{I+(JA)^{-2}})u$ which appear at the exponent.

In the general case

$$A = S^T \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} S \quad \text{(7.10)}$$
and \( JS^T = S^{-1}J \) and \( J \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} J \). From these relations it follows that

\[
(JA)^{-2} = A^{-1}JA^{-1}J = \quad S^{-1} \begin{pmatrix} D^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} (S^T)^{-1}JS^{-1} \begin{pmatrix} D^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} (S^T)^{-1}J =
\]

\[
-S^{-1} \begin{pmatrix} D^{-2} & 0 \\ 0 & D^{-2} \end{pmatrix} S
\]

Hence

\[
I + (JA)^{-2} = I - S^{-1} \begin{pmatrix} D^{-2} & 0 \\ 0 & D^{-2} \end{pmatrix} S =
\]

\[
S^{-1} \begin{pmatrix} I - D^{-2} & 0 \\ 0 & I - D^{-2} \end{pmatrix} S
\]

and

\[
A\sqrt{I + (JA)^{-2}} = S^T \begin{pmatrix} \sqrt{D^2 - I} & 0 \\ 0 & \sqrt{D^2 - I} \end{pmatrix} S
\]  

(7.11)

VIII. A MEASURE OF THE ENTANGLEMENT

A characteristic feature of quantum mechanics is the presence of the correlations between the subsystems of a quantum systems described by a pure state. In this situation the states of the subsystems are mixed. Any quantum system can be considered to be a subsystem of a larger quantum system which is in a pure state. The enlarged system can contain real or fictitious additional subsystems. The appearance of the quantum correlations between these auxiliary subsystems and the initial one is the price which must be payed for the purity of the state of the whole system.

For any mixed quasi-free state on a n-mode bosonic system there exists a standard procedure which defines a compound 2n-mode bosonic system in a pure quasi-free state such that the restriction of this state to the initial subsystem is the initial mixed quasi-free state. This procedure is called the purification operation.

The corresponding compound system has two subsystems which are identical with the initial one and which are strongly correlated. These correlations are explicitly defined by the purification operation.

In Section III we have obtained the following general formula for the transition probability between any quasi-free state and a pure quasi-free state:

\[
P(\omega_A, \omega_B) = \left( \det \left( \frac{A + B}{2} \right) \right)^{-\frac{1}{2}} \]  

(8.1)

Because the state obtained by purification is a pure state we can apply this formula in the case in which the mixed quasi-free state is that given by the direct product of the mixed states of the real and fictitious systems. The last one is a mixed quasi-free state with the correlation matrix given by:

\[
\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}
\]  

(8.2)

Hence we must find

\[
\det \left( \frac{A}{2} \sqrt{I + (JA)^{-2}} \right)^{\frac{1}{2}} \quad \left( \sqrt{D^2 - I} \right)
\]

(8.3)

One obtains

\[
P(\omega_A \bigoplus A, \tilde{\omega}_A) = \left( \det \left( \frac{3}{4} D^2 + \frac{1}{4} I \right) \right)^{-1}
\]  

(8.4)

This transition probability can be considered as a measure of the correlation between the real and fictitious subsystems. One can remark that this transition probability is a decreasing function of the temperature of each mode i.e. the correlation is increasing with the temperature.
Another useful result of the theory of bosonic commutation relations which can be applied to the pure state which appears from the purification operation is the fact that all pure quasifree states can be obtained from the Fock state by a Bogoliubov transformation. In other words the pure quasifree states are squeezed states in a generalized meaning. Hence the strong correlations between the real modes and the fictitious ones are produced by the action of the Bogoliubov automorphisms which are of two kinds: squeezing transformations of each mode and transformations which mix different modes.

In the following we shall discuss the case of pure quasi-free states in detail. If \( \omega_A \) is a pure quasi-free state then \(- (JA)^2 = I\). From the Williamson theorem \(\mathbb{A}\) for any positive definite symmetric matrix \(A\) there exists an element \(S \in Sp(E, \sigma)\) and a diagonal matrix with positive entries \(D\) and with the property \(DJ = JD\) and such that \(A = S^T DS\). Then from the restriction \(- (JA)^2 = I\) one obtains \(D^2 = I\) i.e. \(D = I\). Hence for any pure quasi-free state \(\omega_A\) the correlation matrix \(A\) is of the form \(S^T S\) with \(S \in Sp(E, \sigma)\) i.e. \(\omega_A = \omega_J \circ \alpha_S\) where \(\omega_J\) is called the Fock state. In other words any pure quasi-free state can be obtained from the Fock state by a Bogoliubov transformation. Because the Fock state is a state without correlations between different modes it follows that in the state with the correlation matrix \(\omega_A\) the correlations are produced under the action of the Bogoliubov automorphism \(\alpha_S\), \(S \in Sp(E, \sigma)\). We can apply these considerations to the quasi-free pure state \(\omega_A\) which arises by purification of the quasi-free state \(\omega_A\). It follows that \(\omega_A\) can be obtained from the Fock state by the Bogoliubov automorphism \(\alpha_S\). As we have seen in the preceding section the correlation matrix of the state obtained by purification is given by:

\[
\begin{pmatrix}
S^T & 0
\end{pmatrix}
\begin{pmatrix}
D & 0 \\
0 & D
\end{pmatrix}
\begin{pmatrix}
\sqrt{D^2 - I} & 0 \\
0 & \sqrt{D^2 - I}
\end{pmatrix}
\begin{pmatrix}
S^T & 0
\end{pmatrix}
\]

(9.1)

Hence the most general pure quasifree state is obtained from the mixed quasifree state described by the following correlation matrix:

\[
D = \begin{pmatrix}
(D^2 - I)^{1/2} & 0 \\
0 & (D^2 - I)^{1/2}
\end{pmatrix}
\]

(9.2)

using a specific Bogoliubov transform which does not couple the real and the fictitious systems:

\[
\begin{pmatrix}
S^T & 0
\end{pmatrix}
\begin{pmatrix}
J & 0 \\
0 & -J
\end{pmatrix}
\begin{pmatrix}
S & 0 \\
0 & S
\end{pmatrix} = \begin{pmatrix}
J & 0 \\
0 & -J
\end{pmatrix}
\]

(9.3)

In the enlarged phase space there exists a Bogoliubov transform \(S\) such that \(S^T S = D\). If we denote by \(J = \begin{pmatrix}
J & 0 \\
0 & -J
\end{pmatrix}\)
then \(S^T JS = J\). The solution is

\[
S = \begin{pmatrix}
\sqrt{D + I/2} & 0 \\
0 & \sqrt{D + I/2}
\end{pmatrix}
\begin{pmatrix}
\sqrt{D - I/2} & 0 \\
0 & \sqrt{D - I/2}
\end{pmatrix}
\]

(9.4)

If we denote by \(D\) the matrix

\[
\begin{pmatrix}
D & 0 \\
0 & D
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & D
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
D & 0
\end{pmatrix}
\]

(9.5)
then the transition probability is given by

\[
P(\omega_A, \omega_B) = (\det(\frac{I + SDS^T}{2}))^{-\frac{1}{2}}
\]

and we reobtain the result (8.4).

**X. CONCLUSIONS**

The quasifree states are among the most simple and interesting, from the physical point of view, because they are states on which the quantum correlations (the entanglement) between systems can be relatively easily described. We have applied to the finite dimensional systems the ideas and the methods developed for the infinite dimensional systems by D. Kastler, J. Manuceau, A. Van Daele, A. Verbeure, M. Fannes and A. S. Holevo to the case of finite dimensional systems. For such systems, which are interesting from the physical point of view, the theory was developed (with rare exceptions) in an (apparently) independent way which is based on noncommutative calculations. In comparison, the methods developed for infinite dimensional systems are more simple and more adequate in the finite dimensional case.

The fundamental object which contains the entire information about a quasifree state is its correlation matrix. We have shown in the present paper how we can extract this information by elementary means. For this purpose we have used results obtained many years ago. Of great importance is a result of R. Balian, C. De Dominicis and C. Itzykson concerning the structure of covariance matrices. According to this result all covariance matrices can be obtained from the covariance matrix of a thermal state by orthogonal symplectic transformations and squeezing symplectic transformations. This structure theorem is of fundamental importance and explains all results on the structure of correlations matrices obtained until now. All operations of restriction to subsystems and of extension from a subsystem to the whole system can also be described in terms of correlation matrices. The transition probability from an arbitrary state to a squeezed coherent state can be obtained as the value of the arbitrary state on an element (a projector) of the CCR-algebra which is associated in a natural way to any squeezed coherent state. In the present paper we have shown only that the methods developed for squeezed states can be applied also to the squeezed coherent states. We have constructed explicitly the projector associated to any squeezed coherent state. The transition probability is an immediate result.

The ultimate goal of these researches is the description of the entanglement in the case of quasifree states. We have shown that the most entangled quasifree states are those obtained by the purification of quasifree states. In the case of pure states all correlations are obtained by Bogoliubov transformations. For the states obtained by purifications we have determined explicitly the Bogoliubov transformation by which these states can be obtained from the Fock state. We have also determined explicitly the transition probability from the state obtained by purification to the state of the system obtained by direct product of the original system to the twin fictitious one. This transition probability is a decreasing function of the temperatures of the original system. This means that a state obtained by the purification is most correlated when the original state is most mixed. Hence we have obtained that the probability transition can be considered as a measure of the entanglement.

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