Dispersionless version of the constrained Toda hierarchy and symmetric radial Löwner equation

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Abstract
We study the dispersionless version of the recently introduced constrained Toda hierarchy. Like the Toda lattice itself, it admits three equivalent formulations: the formulation in terms of Lax equations, the formulation of the Zakharov–Shabat type and the formulation through the generating equation for the dispersionless limit of logarithm of the tau-function. We show that the dispersionless constrained Toda hierarchy describes conformal maps of reflection-symmetric planar domains to the exterior of the unit disc. We also find finite-dimensional reductions of the hierarchy and show that they are characterized by a differential equation of the Löwner type which we call the symmetric radial Löwner equation. It is also shown that solutions to the symmetric radial Löwner equation are conformal maps of the exterior of the unit circle with two symmetric slits to the exterior of the unit circle.

Keywords Dispersionless constrained Toda hierarchy · One-variable reduction · Multivariable reduction · Symmetric radial Löwner equation

Mathematics Subject Classification 35Q51 · 30C20

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1 Introduction

The constrained Toda hierarchy was recently introduced in [1]. It is a subhierarchy of the Toda lattice hierarchy [2] defined by the constraint

\[ \bar{L} = L^\dagger \]  

(1.1)

for the two pseudo-difference Lax operators \( L, \bar{L} \) (in the symmetric gauge). The constraint is preserved by the flows \( \partial t_k - \partial t_{-k} \) and is destroyed by the flows \( \partial t_k + \partial t_{-k} \) of the Toda lattice hierarchy, so one has to fix \( t_k + t_{-k} = 0 \). Here \( \{t_k\}_{k \in \mathbb{Z}} \) is the infinite set of independent variables (times) indexed by integer numbers. The times \( t_k \) with \( k \neq 0 \) are in general complex numbers while \( t_0 = n \) is special: It takes integer values.

The hierarchy is defined by the infinite set of Lax equations (evolution equations for the Lax operator in the times \( t_k \)).

An equivalent formulation is via the tau-function \( \tau \) which is a function of the infinite set of independent variables satisfying functional relations which are bilinear in the case of the Toda lattice and have a more complicated form for the constrained Toda hierarchy. In order to perform the dispersionless limit [3, 4], one should substitute

\[ t_k \mapsto \frac{t_k}{\hbar}, \quad \tau(\{t_i/\hbar\}) = e^{F(\{t_i\}, \hbar)/\hbar^2} \]  

(1.2)

and take the \( \hbar \to 0 \) limit which allows one to obtain an equation for the function \( F = \lim_{\hbar \to 0} F(\{t_i\}, \hbar) \) from the equation for the tau-function.

Around 2000 it was observed that the dispersionless hierarchies are deeply related to the theory of univalent functions and in particular to conformal maps of planar domains. This observation was developed in two seemingly different but related directions. One of them treats equations of the dispersionless Toda hierarchy as governing equations for conformal maps of planar simply connected domains with smooth boundary as functions of their harmonic moments which are identified with the hierarchical times \( t_k \) [5, 6]. Another one is related to conformal maps of domains with curved slits. In the seminal papers [7, 8] it was shown that one-variable reductions of the dispersionless...
Kadomtsev–Petviashvili (KP) hierarchy are classified by solutions of a Löwner-type differential equation which characterizes one-parameter families of conformal mappings of domains with a growing slit onto a fixed reference domain [9–12]. Later this important observation was extended to hierarchies of other types and other versions of the Löwner equation [13–19].

The aim of this paper is to suggest the dispersionless version of the constrained Toda hierarchy (Sect. 3) and clarify its relation to conformal maps (Sect. 4). We will show that equations of the dispersionless constrained Toda hierarchy govern conformal maps of domains symmetric under reflection with respect to the real axis (reflection-symmetric domains).

Furthermore, we study finite-dimensional reductions of the dispersionless constrained Toda hierarchy (Sects. 5, 6). The meaning of reduction is as follows. Infinite hierarchies of partial differential equations contain an infinite number of independent variables (“times”) and an infinite number of dependent variables. The simplest possible reduction (one-variable reduction) is a reduction to just one dependent variable which depends on all the times. All other dependent variables become functions of it. We will show that the one-variable reductions of the dispersionless constrained Toda hierarchy are described by solutions of a single ordinary differential equation of Löwner type which we call symmetric Löwner equation. The equation contains an arbitrary function (the “driving function”) which characterizes the reduction. Geometrically, the driving function characterizes the shape of the slit and the single dependent variable is a parameter along the slit. One can also consider multivariable (N-variable) reductions when there are N dependent variables. In this case the reduction is described by a system of N symmetric Löwner equations with N driving functions.

Finally, Sect. 7 is devoted to clarifying the geometric meaning of the symmetric radial Löwner equation. We show that solutions of this equation are conformal maps of domains with two curved slits symmetric under reflection with respect to the real axis. In appendix we give a rigorous proof of the symmetric radial Löwner equation.

2 Constrained Toda hierarchy

We begin with the Toda lattice hierarchy [2] in the symmetric gauge [20, 21]. The two Lax operators are

\begin{align*}
L &= c(n)e^{\partial_n} + \sum_{k \geq 0} U_k(n)e^{-k\partial_n}, \\
\bar{L} &= c(n-1)e^{-\partial_n} + \sum_{k \geq 0} \bar{U}_k(n)e^{k\partial_n},
\end{align*}

(2.1)

where $e^{k\partial_n}$ are shift operators acting on functions of $n$ as $e^{k\partial_n}f(n) = f(n+k)$ and $c(n), U_k(n), \bar{U}_k(n)$ are some functions of $n$. The Lax operators are pseudo-difference operators (infinite Laurent series in the shift operator). Given the Lax operators, one can introduce the difference operators

\begin{align*}
B_m = (L^m)_{>0} + \frac{1}{2}(L^m)_{0}, \quad B_{-m} = (\bar{L}^m)_{<0} + \frac{1}{2}(\bar{L}^m)_{0}, \quad m = 1, 2, 3, \ldots
\end{align*}

(2.2)
where for a subset $S \subset \mathbb{Z}$, we denote \( \sum_{k \in \mathbb{Z}} U_k e^{k\partial_n} \) \( \in \mathbb{Z} \). The Toda lattice hierarchy is given by the Lax equations

\[
\partial_t L = [B_m, L], \quad \partial_t \tilde{L} = [B_m, \tilde{L}]
\]

which define the hierarchical flows parametrized by the times $t_m$ for any non-zero integer $m$. These equations seem to be disconnected, but actually they are connected via the common function $c(n)$. An equivalent formulation is through the zero curvature (Zakharov–Shabat) equations

\[
\partial_t B_m - \partial_t B_k + [B_m, B_k] = 0.
\]

They encode differential difference equations for the coefficient functions $c(n), U_k(n), \tilde{U}_k(n)$.

The constrained Toda hierarchy is obtained by imposing the constraint

\[
\tilde{L} = L^\dagger,
\]

where the $\dagger$-operation is defined as $(f(n) \circ e^{k\partial_n})^\dagger = e^{-k\partial_n} \circ f(n)$. This implies that $\tilde{U}_k(n) = U_k(n + k)$. It is easy to see that the constraint is preserved by the flows $\partial_t - \partial_{-t}$ and is destroyed by the flows $\partial_t + \partial_{-t}$, so one has to fix $t_k + t_{-k} = 0$. In this way all the coefficient functions can be regarded as functions of $t_k$ with $k > 0$ only. Introducing difference operators

\[
A_m = B_m - B_{-m},
\]

we can write the Lax and Zakharov–Shabat equations of the constrained hierarchy in the form

\[
\partial_t L = [A_m, L], \quad [\partial_t - A_m, \partial_t - A_k] = 0, \quad m > 0.
\]

The simplest nontrivial equation of the hierarchy (obtained at $m = 1, k = 2$) is

\[
(\partial_{t_2} - \partial_{t_1}^2)\varphi_{n+1} - (\partial_{t_2} + \partial_{t_1}^2)\varphi_n = 2e^{\varphi_n - \varphi_{n-1}} - 2e^{\varphi_{n+2} - \varphi_{n+1}} + \frac{1}{2}(\partial_{t_1}(\varphi_{n+1}))^2 - \frac{1}{2}(\partial_{t_1}(\varphi_n))^2,
\]

where the function $\varphi_n$ is defined as $c(n) = e^{\frac{i}{2}(\varphi_{n+1} - \varphi_n)}$.

The Zakharov–Shabat and Lax equations are compatibility conditions for the linear problems

\[
\partial_t \psi = A_m \psi, \quad L \psi = z \psi
\]

for the wave function $\psi = \psi(n, t, z)$ depending on the spectral parameter $z \in \mathbb{C}$ (and on all the times $t = \{t_1, t_2, t_3, \ldots\}$). The set of linear equations is equivalent to the
bilinear relation [1]
\[
\left( \oint_{C_\infty} - \oint_{C_0} \right) \psi(n, t, z) \psi(n', t', z^{-1}) \frac{dz}{2\pi i z} = 0
\]  
valid for all \( n, n', t, t' \), where \( C_\infty, C_0 \) are small contours around \( \infty \) and \( 0 \), respectively.

It was proven in [1] that there exists a tau-function \( \tau_n(t) \) of the constrained Toda hierarchy, and the wave function can be consistently expressed through the tau-function as follows:
\[
\psi(n, t, z) = z^n e^{\xi(t, z)} G_n^{1/2}(t, z) \frac{\tau_n(t - [z^{-1}])}{\tau_n(t)}, \quad z \to \infty,
\]  
(2.11)
\[
\psi(n, t, z^{-1}) = z^{-n} e^{-\xi(t, z)} \tilde{G}_n^{1/2}(t, z) \frac{\tau_{n+1}(t) \tau_{n+1}(t + [z^{-1}])}{\tau_n(t)}, \quad z \to \infty.
\]  
(2.12)

Here
\[
\xi(t, z) = \sum_{k \geq 1} t_k z^k,
\]  
(2.13)
\[
t \pm [z^{-1}] = \left\{ t_1 \pm z^{-1}, t_2 \pm \frac{1}{2} z^{-2}, t_3 \pm \frac{1}{3} z^{-3}, \ldots \right\},
\]  
(2.14)
and
\[
G_n(t, z) = 1 - z^{-2} \left( \frac{\tau_{n+1}(t) \tau_{n-1}(t - [z^{-1}])}{\tau_n(t) \tau_n(t - [z^{-1}])} \right)^2,
\]  
(2.15)
\[
\tilde{G}_n(t, z) = 1 - z^{-2} \left( \frac{\tau_n(t) \tau_{n+2}(t + [z^{-1}])}{\tau_{n+1}(t) \tau_{n+1}(t + [z^{-1}])} \right)^2.
\]  
(2.16)

Substituting these expressions into (2.10), one obtains an integral equation\(^1\) for the tau-function which is the generating equation of the hierarchy. In particular, setting \( t - t' = [a^{-1}] + [b^{-1}] \), \( n - n' = 2 \), the residue calculus yields:
\[
\frac{a^2 b}{a - b} G_n^{1/2}(t, a) G_{n-2}^{1/2}(t-[a^{-1}]-[b^{-1}], a) \frac{\tau_n(t-[a^{-1}]) \tau_{n-2}(t-[a^{-1}]-[b^{-1}] \tau_{n-1}(t-[b^{-1}])}{\tau_n(t) \tau_{n-2}(t-[a^{-1}]-[b^{-1}] \tau_{n-1}(t-[b^{-1}])} + \frac{a b^2}{a - b} G_n^{1/2}(t, b) G_{n-2}^{1/2}(t-[a^{-1}]-[b^{-1}], b) \frac{\tau_n(t-[b^{-1}]) \tau_{n-2}(t-[a^{-1}]-[b^{-1}] \tau_{n-1}(t-[a^{-1}])}{\tau_n(t) \tau_{n-2}(t-[a^{-1}]-[b^{-1}] \tau_{n-1}(t-[a^{-1}])} = a b \frac{\tau_n^2(t-[a^{-1}]-[b^{-1}])}{\tau_n^2(t-[a^{-1}] - [b^{-1}])} - (ab)^{-1} \frac{\tau_{n+1}^2(t)}{\tau_n(t)}.
\]

\(^1\) Unlike the Toda lattice hierarchy, this equation is not bilinear but has a more complicated form with square roots of (2.15), (2.16).
Finally, let us point out the relation between the tau-function of the constrained Toda hierarchy and the tau-function \( \tau_{Toda} \) of the Toda lattice hierarchy satisfying the constraints

\[
(\partial_{t_k} + \partial_{t_{-k}}) \log \tau_n |_{t_j+t_{-j}=0} = 0 \quad \text{for all } k, j \text{ and } t_0, t_j - t_{-j}
\] (2.18)

which are equivalent to (2.5). We have (see [1]):

\[
\tau_n(t) = \sqrt{\tau_{Toda}n(\ldots - t_2, -t_1; t_1, t_2, \ldots)}.
\] (2.19)

### 3 The dispersionless limit

In the dispersionless limit [3, 4], one should substitute

\[
t_k \longrightarrow \frac{t_k}{\hbar}, \quad \tau_{t_0/\hbar}(t/\hbar) = e^{F(t_0, t, \hbar)}/\hbar^2
\] (3.1)

and take the \( \hbar \to 0 \) limit which allows one to obtain an equation for the function \( F = \lim_{\hbar \to 0} F(t_0, t, \hbar) \) from the equation for the tau-function. In accordance with this prescription, we can write

\[
\tau_{n\pm 1}(t) \longrightarrow \exp \left( \hbar^{-2} e^{\pm \hbar \partial_0} F \right),
\]

\[
\tau_n(t \pm [a^{-1}]) \longrightarrow \exp \left( \hbar^{-2} F(t_0, t \pm \hbar[a^{-1}], \hbar) \right) = \exp \left( \hbar^{-2} e^{\pm \hbar D(a)} F \right),
\]

where \( D(z) \) is the differential operator

\[
D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_k.
\] (3.2)

The dispersionless limit of equation (2.17) is then straightforward. We only note that

\[
\lim_{\hbar \to 0} G_{t_0/\hbar}(t/\hbar, z) = \lim_{\hbar \to 0} \tilde{G}_{t_0/\hbar}(t/\hbar, z) = 1 - z^{-2} e^{2\partial_0(\partial_0 + D(z))} F,
\] (3.3)

so the square roots in (2.18) disappear in the \( \hbar \to 0 \) limit. The final result is

\[
ae^{-\partial_0(D(a))}F - be^{-\partial_0(D(b))}F = \frac{a - b}{abe^{-\partial_0(2\partial_0 + D(a) + D(b))}F - 1} e^{(\partial_0 + D(a))(\partial_0 + D(b))} F.
\] (3.4)
This is the dispersionless analogue of the bilinear equation for the tau-function. In the zero dispersion limit it is not bilinear since it is written for logarithm of the tau-function. Differential equations of the hierarchy are obtained from (3.4) by expanding both sides in inverse powers of $a, b$ and equating the coefficients. For example, the simplest nontrivial equation is

$$F_{01}^2 - F_{02} + 2F_{11} = 2e^{F_{00}},$$

(3.5)

where $F_{jk} := \partial_{ij}\partial_{hk} F$.

Introducing the function

$$w(z) = ze^{-\partial_(\partial_0 + D(z))F},$$

(3.6)

we can write equation (3.4) in the form

$$\frac{w(a) - w(b)}{w(a)w(b) - 1} = (b^{-1} - a^{-1})e^{(\partial_0 + D(a)) (\partial_0 + D(b)) F}.$$  

(3.7)

Taking logarithm of both sides and applying the $t_0$-derivative, we can represent it in the form of a closed equation for the function $w(z)$:

$$(\partial_0 + D(a)) \log w(b) = -\partial_0 \log \frac{w(a) - w(b)}{w(a)w(b) - 1}.$$  

(3.8)

It is instructive to rewrite this equation in terms of the inverse function to $w(z)$ which we denote as $z(w)$. Noting the relation

$$\partial_k w(z) = -\frac{\partial_k z(w)}{\partial_t z(w)}$$

between the partial derivatives, and, as a consequence,

$$w\partial_w z(w) D(a) \log w(z) = -D(a)z(w)$$

(the derivative in the right-hand side is taken at constant $w$), we can rewrite equation (3.8) in the form

$$D(a)z(w) = \left\{ z(w), \log \frac{w(a) - w}{w(a) - w^{-1}} \right\},$$

(3.9)

where

$$\{f, g\} = w \frac{\partial f}{\partial w} \frac{\partial g}{\partial t_0} - w \frac{\partial f}{\partial t_0} \frac{\partial g}{\partial w}$$

(3.10)

is the Poisson bracket. Equation (3.9) is the generating Lax equation for the dispersionless constrained Toda hierarchy. To see this, we introduce the Faber polynomials
$B_k(w)$ according to the expansion

$$-\log\left(w(z) - w\right) + \frac{1}{2} \log \frac{zw(z)}{r} = \sum_{k \geq 1} \frac{z^{-k}}{k} B_k(w), \quad r = (\lim_{z \to \infty} (w(z)/z))^{-1}. $$

(3.11)

Here $w$ is an independent variable ($w \neq w(z)$). Then equation (3.9) implies the Lax equations of the dispersionless constrained Toda hierarchy:

$$\partial_t z(w) = \left\{ B_k(w) - B_k(w^{-1}), \ z(w) \right\}. $$

(3.12)

It is easy to see that

$$B_k(w) = \left( z^k(w) \right)_{z \geq 0} + \frac{1}{2} \left( z^k(w) \right)_0. $$

(3.13)

Comparing with (2.1), (2.6), we see that the function (or rather the Laurent series) $z(w)$ is the dispersionless limit of the Lax operator: $e^{\delta_n} \rightarrow w$, $L \rightarrow z(w)$ and the formulas (3.12), (3.13) are dispersionless limits of (2.7), (2.2).

From now on, keeping in mind applications to conformal maps of planar domains, we will consider the real form of the Toda lattice hierarchy such that the time variables satisfy the conditions

$$t_{-k} = -\tilde{t}_k, \quad k \geq 1, \quad \text{and} \ t_0 \text{ is real.} $$

(3.14)

Moreover, the tau-function of the real form of the Toda lattice hierarchy is real. For the constrained Toda hierarchy this means that $t_k = \tilde{t}_k$, i.e., all the time variables are real.

The dispersionless limit of the Lax operator $L$ is the Laurent series $z(w)$ while the limit of the second Lax operator, $\tilde{L}$, is $\tilde{z}(w^{-1}) = \frac{1}{\tilde{w}(w^{-1})}$. Since $e^{\delta_n}$ in the dispersionless limit becomes $w$ and $(e^{\delta_n})^\dagger = e^{-\delta_n}$, we have $w^\dagger = w^{-1}$, so the constraint (2.5) in the dispersionless limit implies that $z(w) = \frac{1}{\tilde{w}(w)}$, or $w(z) = \tilde{w}(\tilde{z})$, i.e., all coefficients of the Laurent series $z(w)$ and $w(z)$ are real. The constraints (2.18) for the tau-function then mean that $v_k = \delta_{t_k} F$ are also real.

4 Dispersionless constrained Toda hierarchy and conformal maps

It is known that the real reduction of the dispersionless Toda lattice hierarchy governs conformal maps of planar domains [6, 22, 23]. Let us recall the main facts related to this connection.

Given a simply connected compact planar domain $D$ with a smooth boundary, one can define the harmonic moments

$$v_k = \frac{1}{\pi} \int_D z^k d^2 z, \quad k \geq 1, \quad d^2 z \equiv dx dy $$

(4.1)
together with the complimentary moments of the exterior domain $\mathbb{C} \setminus D$

\begin{equation}
t_k = -\frac{1}{\pi k} \int_{\mathbb{C} \setminus D} z^{-k} \, d^2 z. \tag{4.2}
\end{equation}

Without loss of generality we assume that the domain $D$ contains the origin. The moments $t_k$ and $v_k$ are in general complex numbers. We also denote

\begin{equation}
t_0 = \frac{1}{\pi} \int_D d^2 z = \frac{\text{Area of } D}{\pi} \tag{4.3}
\end{equation}

which is a real number.

Let $z(w)$ be the conformal map from the exterior of the unit circle to the domain $\mathbb{C} \setminus D$ normalized in such a way that $z(\infty) = \infty$ and $z'(\infty) = r > 0$ ($r$ is a real positive number called conformal radius). As it was shown in [6], the function $z(w)$ satisfies the equations

\begin{equation}
\partial_k z(w) = \{B_k(w), z(w)\}, \quad \partial_{\bar{k}} z(w) = -\{\bar{B}_k(w^{-1}), z(w)\} \tag{4.4}
\end{equation}

with $B_k(w)$ given by (3.13). This allows one to identify $z(w)$ with the Lax function of the dispersionless Toda hierarchy (the dispersionless limit of the Lax operator).

The dispersionless limit of logarithm of tau-function (the $F$-function) is given by

\begin{equation}
F_{\text{Toda}} = -\frac{1}{\pi^2} \int_D \int_D d^2 z d^2 \xi \log |z^{-1} - \xi^{-1}| \tag{4.5}
\end{equation}

(see [22]). It is a real-valued function of $t_0$, $t_k$ and $\bar{t}_k$. The first derivatives of $F$ with respect to $t_k$’s yield the complimentary moments $v_k$:

\begin{equation}
v_k = \partial_{t_k} F_{\text{Toda}}. \tag{4.6}
\end{equation}

The second mixed derivatives of $F$ provide full information about the conformal map $w(z)$ from $\mathbb{C} \setminus D$ to the exterior of the unit circle (the inverse function of $z(w)$) and the Green function $G(z, \xi)$ of the Dirichlet boundary problem in $\mathbb{C} \setminus D$. Namely,

\begin{equation}
w(z) = z \exp \left( -\frac{1}{2} \partial_{t_0}^2 F_{\text{Toda}} - \partial_{t_0} D(z) F_{\text{Toda}} \right), \tag{4.7}
\end{equation}

\begin{equation}
G(z, \xi) = \log |z^{-1} - \xi^{-1}| + \frac{1}{2} \nabla(z) \nabla(\xi) F_{\text{Toda}}, \tag{4.8}
\end{equation}

where

$$\nabla(z) = \partial_{t_0} + D(z) + \overline{D(z)}$$
and $D(z)$ is the differential operator (3.2). The well-known formula

$$G(z, \zeta) = \log \frac{|w(z) - w(\zeta)|}{|w(z)w(\zeta) - 1|}$$

(4.9)

which expresses the Green function through the conformal map is equivalent to the dispersionless limit of the Toda equations for the tau-function.

In the case of the constrained Toda hierarchy, the coefficients of the Laurent series $z(w)$ and the moments $t_k$ are real. It is obvious from (4.2) that for domains symmetric under reflection with respect to the real axis (reflection-symmetric domains) all moments are real. The converse statement, i.e. that for real moments the corresponding domain is symmetric, can be proved by means of the Schwarz function.\(^2\) This means that we deal with conformal maps of reflection-symmetric domains. The moments $v_k$ are also real, and formula (4.6) means that the constraint (2.18) is satisfied if we are in the class of reflection-symmetric domains. From (2.19) it follows that the $F$-function of the dispersionless constrained Toda hierarchy is related to $F^{\text{Toda}}$ as

$$F^{\text{Toda}} = 2 F.$$  

(4.10)

Therefore, one can see that formula (3.6) is a specialization of (4.7) for reflection-symmetric domains. In a similar way, equation (3.7) is a specialization of (4.8) for reflection-symmetric domains with $z, \zeta$ lying on the real axis. In the class of reflection-symmetric domains, the function $F$, regarded as a function of real moments $t_k$, satisfies equation (3.4).

As it follows from the result of [22], the function $F$ satisfies the following quasi-homogeneity equation:

$$2F = -\frac{1}{4} t_0^2 + t_0 \partial_{t_0} F + \sum_{k \geq 1} (2 - k) t_k \partial_{t_k} F.$$ 

(4.11)

**Example** For the ellipse with half-axes $a, b$ centered at $x_0 \in \mathbb{R}$ we have:

$$t_0 = ab, \quad t_1 = \frac{2b x_0}{a + b}, \quad 2t_2 = \frac{a - b}{a + b},$$

and all other moments are equal to zero. For the ellipse the function $F$ is

$$F = \frac{1}{4} t_0^2 \log t_0 - \frac{3}{8} t_0^2 - \frac{1}{4} t_0^2 \log(1 - 4t_2^2) + \frac{t_0 t_1^2}{2(1 - 2t_2^2)}$$

(4.12)

(see [6]). One can check that this function does satisfy equation (3.5).

\(^2\) The Schwarz function $S(z)$ [24] is a holomorphic function in a strip-like neighborhood of the boundary of the domain such that $\bar{z} = S(z)$ on the boundary. It is a generating function of the harmonic moments. Therefore, if all harmonic moments are real, $\bar{z} = S(z)$ implies $z = S(\bar{z})$, which means that complex conjugation preserves the boundary.
5 One-variable reductions

In this section we consider reductions of the dispersionless constrained Toda hierarchy. For one-variable reductions, the dependence of \( w(z) \) on the times is implemented by means of a single variable \( \lambda = \lambda(t) \), i.e., \( w(z; t) = w(z, \lambda(t)) \). This means that instead of the function of infinitely many variables \( w(z; t) \) we now deal with a function of two variables \( w(z, \lambda) \). Our goal is to find all possible forms of this function consistent with the infinite hierarchy. The reduction is an exceptional non-generic solution. We will see that such solutions correspond to conformal maps of domains with slits.

The derivation below is parallel to that for reductions of the dispersionless Toda hierarchy \([19]\) (see also \([16, 25]\)). Assuming that \( w(z) = w(z, \lambda) \) and using the chain rule of differentiating, we have:

\[
(\partial_{t_0} + D(\zeta)) \log w(z) = \partial_\lambda \log w(z) \cdot (\partial_{t_0} + D(\zeta)) \lambda
\]

and

\[
(\partial_{t_0} + D(z)) \lambda = \frac{(\partial_{t_0} + D(z)) \log r}{\partial_\lambda \log r}. \tag{5.1}
\]

We recall that \( \log r = -\log \lim_{z \to \infty} (w(z)/z) \). Tending \( b \to \infty \) in equation (3.8), we get:

\[
\partial_{t_0} \log w(z) = -(\partial_{t_0} + D(z)) \log r. \tag{5.2}
\]

Therefore, equation (5.1) can be written as

\[
(\partial_{t_0} + D(z)) \lambda = -\frac{\partial_\lambda \log w(z)}{\partial_\lambda \log r} \partial_{t_0} \lambda. \tag{5.3}
\]

Substituting this into (3.8) and assuming that \( \partial_{t_0} \lambda \) is not identically zero, we get:

\[
\partial_\lambda \log w(z) \partial_\lambda \log w(\zeta) = \partial_\lambda \log r \left[ \partial_\lambda \log \left( 1 + \frac{w(z)}{w(z) - w(\zeta)} - \frac{w(\zeta)}{w(z) - w(\zeta)} \right) \right]
\]

\[
- \partial_\lambda w(\zeta) \left( \frac{1}{w(z) - w(\zeta)} + \frac{w(\zeta)}{w(z) - w(\zeta)} - 1 \right)
\]

or, after rearranging,

\[
w(z) + w^{-1}(z) + \partial_\lambda \log r \frac{w(z) - w^{-1}(z)}{\partial_\lambda \log w(z)}
\]

\[
= w(\zeta) + w^{-1}(\zeta) + \partial_\lambda \log r \frac{w(\zeta) - w^{-1}(\zeta)}{\partial_\lambda \log w(\zeta)}.
\]
The left-hand side depends only on $z$ while the right-hand side depends only on $\zeta$. It then follows from this equation that

$$
\eta(\lambda) := w(z) + w^{-1}(z) + \partial_\lambda \log r \frac{w(z) - w^{-1}(z)}{\partial_\lambda \log w(z)} \tag{5.4}
$$
does not depend on $z$. Equation (5.4) can be read as the differential equation for the function $w(z) = w(z, \lambda)$:

$$
\partial_\lambda \log w(z) = - \frac{w(z) - w^{-1}(z)}{w(z) + w^{-1}(z) - \eta(\lambda)} \partial_\lambda \log r, \tag{5.5}
$$

where $\eta(\lambda)$ is a real-valued function of $\lambda$. Setting

$$
\eta(\lambda) = e^{i\xi(\lambda)} + e^{-i\xi(\lambda)}, \tag{5.6}
$$

where $\xi(\lambda)$ is another real-valued function of $\lambda$, we can represent equation (5.5) in the form

$$
\partial_\lambda \log w(z) = - \frac{1}{2} \left( \frac{w(z) + e^{i\xi(\lambda)}}{w(z) - e^{i\xi(\lambda)}} + \frac{w(z) + e^{-i\xi(\lambda)}}{w(z) - e^{-i\xi(\lambda)}} \right) \partial_\lambda \log r. \tag{5.7}
$$

In particular, one can choose $\lambda = \log r$, then the equation simplifies:

$$
\frac{\partial \log w(z)}{\partial \log r} = - \frac{1}{2} \left( \frac{w(z) + e^{i\xi}}{w(z) - e^{i\xi}} + \frac{w(z) + e^{-i\xi}}{w(z) - e^{-i\xi}} \right). \tag{5.8}
$$

We call equation (5.7) (or (5.8)) symmetric radial Löwner equation. Its right-hand side is the half-sum of the right-hand sides of the radial Löwner equations with the driving functions $\xi(\lambda)$ and $-\xi(\lambda)$. This is similar to the quadrant Löwner equation [17] which appears as a reduction condition in the dispersionless BKP (or CKP) hierarchy\(^3\) and whose right-hand side is the half-sum of the right-hand sides of the chordal Löwner equations with the driving functions $\xi(\lambda)$ and $-\xi(\lambda)$. Moreover, in a certain scaling limit the symmetric radial Löwner equation becomes the quadrant Löwner equation.

To wit, setting $w(z) = e^{i\epsilon p(z)}$, $r = e^{-\epsilon^2 u/2}$, $\xi(\lambda) \rightarrow \epsilon \xi(\lambda)$ and taking the limit $\epsilon \rightarrow 0$, we obtain the quadrant Löwner equation

$$
\partial_\lambda p(z) = - \frac{1}{2} \left( \frac{1}{p(z) - \xi(\lambda)} + \frac{1}{p(z) + \xi(\lambda)} \right) \partial_\lambda u. \tag{5.9}
$$

As is shown in Sect. 7, solutions of the symmetric radial Löwner equation are functions which define conformal mappings from the exterior of the unit circle with two symmetric slits to the exterior of the unit circle.

\(^3\) As is shown in [26], the dispersionless limits of the BKP and CKP hierarchies are the same.
The dependence of $\lambda$ on the times $t_k$ is determined by a system of equations of the hydrodynamic type. They follow from equation (5.2) which can be written as

$$(\partial_0 + D(z))\lambda = \frac{(\partial_0 + D(z)) \log r}{\partial_\lambda \log r} = -\frac{\partial_0 \log w(z)}{\partial_\lambda \log r} = -\frac{\partial_\lambda \log w(z)}{\partial_\lambda \log r} \partial_0 \lambda.$$  

Plugging here the symmetric Löwner equation (5.7), we obtain

$$D(z)\lambda = \left( \frac{e^{i\xi(\lambda)}}{w(z)} + \frac{e^{-i\xi(\lambda)}}{w(z)} \right) \partial_0 \lambda.$$  

(5.10)

Taking the $w$-derivative of equation (3.11) defining the Faber polynomials, we get

$$e^{\pm i\xi(\lambda)} = \sum_{k \geq 1} \frac{z^{-k}}{k} \phi_k(e^{\pm i\xi}), \quad \phi_k(w) \equiv w \partial_w B_k(w).$$  

(5.11)

Therefore, equation (5.10) is equivalent to the following infinite system of equations of hydrodynamic type:

$$\partial t_k \lambda = \left( \phi_k(e^{i\xi(\lambda)}) + \phi_k(e^{-i\xi(\lambda)}) \right) \partial_0 \lambda.$$  

(5.12)

The hodograph equation

$$t_0 + \sum_{k \geq 1} t_k \left( \phi_k(e^{i\xi(\lambda)}) + \phi_k(e^{-i\xi(\lambda)}) \right) = R(\lambda)$$  

(5.13)

gives a general solution in an implicit form. Here $R(\lambda)$ is an arbitrary function of $\lambda$.

6 Multivariable reductions

A multivariable ($N$-variable) reduction of the dispersionless constrained Toda hierarchy means that the function $w(z)$ depends on the times $t$ through $N$ functions $\lambda_j = \lambda_j(t)$, i.e., $w(z) = w(z; t) = w(z; \lambda_1(t), \ldots, \lambda_N(t))$. Below we prove that solutions of a system of symmetric radial Löwner equations give solutions to the hierarchy.

We consider the system of $N$ symmetric radial Löwner equations of the form (5.7) which characterize the dependence of $w(z) = w(z; \lambda_1, \ldots, \lambda_N)$ on the variables $\lambda_j$:

$$\frac{\partial \log w(z)}{\partial \lambda_j} = -\frac{1}{2} \left( \frac{w(z) + v_j}{w(z) - v_j} + \frac{w(z) + \bar{v}_j}{w(z) - \bar{v}_j} \right) \frac{\partial \log r}{\partial \lambda_j}, \quad v_j := e^{i\xi_j}. $$  

(6.1)
The driving functions $\xi_j = \xi_j(\{\lambda_i\})$ are real-valued, so $|\nu_j| = 1$, i.e., $\bar{\nu}_j = \nu_j^{-1}$. The compatibility conditions of this system are

$$
\frac{\partial}{\partial \lambda_k} \frac{\partial \log w(z)}{\partial \lambda_j} = \frac{\partial}{\partial \lambda_j} \frac{\partial \log w(z)}{\partial \lambda_k}.
$$

In order to resolve these conditions, one should substitute equations (6.1) and cancel all poles. A long but straightforward calculation shows that the compatibility conditions are equivalent to the following system of the Gibbons–Tsarev type:

\[
\begin{align*}
\frac{\partial \nu_j}{\partial \lambda_k} &= \frac{v_j}{2} \left( \frac{v_k + v_j}{v_k - v_j} + \frac{\bar{v}_k + \bar{v}_j}{\bar{v}_k - \bar{v}_j} \right) \frac{\partial \log r}{\partial \lambda_k}, \\
\frac{\partial^2 \log r}{\partial \lambda_j \partial \lambda_k} &= 2 \left( \frac{v_j v_k}{(v_j - v_k)^2} + \frac{v_j \bar{v}_k}{(v_j - \bar{v}_k)^2} \right) \frac{\partial \log r}{\partial \lambda_j} \frac{\partial \log r}{\partial \lambda_k}. 
\end{align*}
\] (6.2)

Now we are going to use the fact that each $\lambda_j$ is a function of the times: $w(z; t) = w(z; \{\lambda_j(t)\})$. Applying the chain rule of differentiating, in the case of $N$-variable reduction we can rewrite equation (3.8) of the dispersionless constrained Toda hierarchy in the form

$$
\sum_{j=1}^{N} D(\xi) \lambda_j \cdot \partial_{\lambda_j} \log w(z) = \sum_{j=1}^{N} \partial_{t_0} \lambda_j \cdot \partial_{\lambda_j} \log \left( \frac{w(\xi) - w^{-1}(z)}{w(\xi) - w(z)} \right). 
$$ (6.3)

A rather long but straightforward calculation using the symmetric Löwner equation (6.1) yields

$$
\partial_{\lambda_j} \log \left( \frac{w(\xi) - w^{-1}(z)}{w(\xi) - w(z)} \right) = -\frac{1}{2} \left( \frac{v_j}{w(\xi) - v_j} + \frac{\bar{v}_j}{w(\xi) - \bar{v}_j} \right) \left( \frac{w(z) + v_j}{w(z) - v_j} + \frac{w(z) + \bar{v}_j}{w(z) - \bar{v}_j} \right) \partial_{\lambda_j} \log r.
$$

Therefore, if we introduce the dependence of the $\lambda_j$’s on the times by means of the relations

$$
D(z) \lambda_j = \left( \frac{v_j}{w(z) - v_j} + \frac{\bar{v}_j}{w(z) - \bar{v}_j} \right) \partial_{t_0} \lambda_j,
$$ (6.4)

the equation (6.3) will be satisfied identically. As it follows from (5.11), equation (6.4) is equivalent to an infinite system of partial differential equations of hydrodynamic type:

$$
\frac{\partial \lambda_j}{\partial t_k} = \left( \phi_{j,k}(\{\lambda_i\}) + \phi_{j,k}(\{\lambda_i\}) \right) \frac{\partial \lambda_j}{\partial t_0}, \quad \phi_{j,k} = v_j B_k'(v_j),
$$ (6.5)
where $B_k'(w) = \partial_w B_k(w)$. The generating function of $\phi_{j,k}$'s is

$$
\sum_{k \geq 1} \left( \phi_{j,k}(\{\lambda_i\}) + \overline{\phi_{j,k}(\{\lambda_i\})} \right) \frac{z^{-k}}{k} = \frac{v_j}{w(z) - v_j} + \frac{\bar{v}_j}{w(z) - \bar{v}_j}
$$

$$
:= Q\left(w(z, \{\lambda_i\}), v_j(\{\lambda_i\})\right).
$$

(6.6)

Now we are going to show that the system (6.5) is consistent and can be solved by Tsarev’s generalized hodograph method [27]. It can be directly verified that the compatibility condition of the system (6.5) is

$$
\frac{\partial \lambda_j \text{Re} \phi_{i,n}}{\text{Re} \phi_{j,n} - \text{Re} \phi_{i,n}} = \frac{\partial \lambda_j \text{Re} \phi_{i,n'}}{\text{Re} \phi_{j,n'} - \text{Re} \phi_{i,n'}} \quad \text{for all } i \neq j, n, n'.
$$

(6.7)

In other words, the condition is that

$$
\Gamma_{ij} := \frac{\partial \lambda_j \text{Re} \phi_{i,n}}{\text{Re} \phi_{j,n} - \text{Re} \phi_{i,n}}
$$

(6.8)

does not depend on $n$. It is easy to see that this is equivalent to the $z$-independence of the ratio

$$
\frac{\partial \lambda_j Q(w(z), v_i)}{Q(w(z), v_j) - Q(w(z), v_i)}
$$

where $Q$ is the generating function (6.6). If this holds, then

$$
\Gamma_{ij} = \frac{\partial \lambda_j Q(w(z), v_i)}{Q(w(z), v_j) - Q(w(z), v_i)}.
$$

(6.9)

A direct calculation which makes use of the symmetric Löwner equation (6.1) and the Gibbons–Tsarev equations (6.2) gives

$$
\Gamma_{ij} = \left( \frac{v_i v_j}{(v_i - v_j)^2} + \frac{v_i v_j}{(v_i v_j - 1)^2} \right) \partial \lambda_j \log r.
$$

(6.10)

Let $R_i = R_i(\{\lambda_i\})$ ($i = 1, \ldots, N$) satisfy the system of equations

$$
\frac{\partial R_i}{\partial \lambda_j} = \Gamma_{ij}(R_j - R_i), \quad i, j = 1, \ldots, N, \quad i \neq j,
$$

(6.11)

where $\Gamma_{ij}$ is defined in (6.10) (for $N = 1$ this condition is void). We claim that the system (6.11) is compatible in the sense of Tsarev [27]. To see this, we note that $\Gamma_{ij}$ can be expressed as

$$
\Gamma_{ij} = \frac{1}{2} \partial \lambda_j \log g_i, \quad g_i = \frac{\partial \log r}{\partial \lambda_i}.
$$

(6.12)
This directly follows from the Gibbons–Tsarev equations (6.2). It is then obvious that

$$\frac{\partial \Gamma_{ij}}{\partial \lambda_k} = \frac{\partial \Gamma_{ik}}{\partial \lambda_j}, \quad i \neq j \neq k,$$

which are Tsarev’s compatibility conditions. This means that the system (6.5) is semi-Hamiltonian. The main geometric object associated with a semi-Hamiltonian system is a diagonal metric. The quantities $g_i = g_{ii}$ are components of this metric while $\Gamma_{ij} = \Gamma_{ij}^i$ are the corresponding Christoffel symbols. Moreover, from (6.12) it is clear that the metric $g_i$ is of Egorov type, i.e.,

$$\frac{\partial g_i}{\partial \lambda_k} = \frac{\partial g_k}{\partial \lambda_i}.$$

Assume that $R_i$ satisfy the system (6.11). Then the same argument as in the proof of Theorem 10 of Tsarev’s paper [27] shows that if $\lambda_i$ is defined implicitly by the hodograph relations

$$t_0 + 2\text{Re} \sum_{k \geq 1} t_k \phi_i,k(\{\lambda_j\}) = R_i(\{\lambda_j\}),$$

then $\lambda_j$ satisfy (6.5).

We have found sufficient conditions for $N$-variable diagonal reductions of the dispersionless constrained Toda hierarchy. The reduction is given by a system of $N$ symmetric Löwner equations (6.1) for a function $w(z, \lambda_1, \ldots, \lambda_N)$ supplemented by a diagonal system of hydrodynamic type (6.5) for the variables $\lambda_j, j = 1, \ldots, N$.

7 The symmetric radial Löwner equation

In this section we clarify the geometric meaning of the symmetric radial Löwner equation (5.7). We will sketch the derivation of a differential equation obeyed by the conformal map of the unit disk with two symmetric slits to the exterior of the unit disk which turns out to be the symmetric radial Löwner equation. Rigorous proof is in Appendix 1.

By $\mathbb{U}$ we denote the unit disk $|z| < 1$ and $\mathbb{U}^c = \mathbb{C} \setminus \mathbb{U}$ its compliment. Let $\Gamma : [0, \infty) \to \mathbb{U}^c$ be a simple curve with no self-intersections in the upper half plane with the condition $|\Gamma(0)| = 1$ and $\bar{\Gamma}$ be the mirror symmetric curve with respect to the real axis: $\bar{\Gamma}(t) = \Gamma(-t)$. The curves $\Gamma$ and $\bar{\Gamma}$ do not intersect, as $\Gamma(t)$ is in the upper half plane. Let $\Gamma_t$ be the arc of the curve $\Gamma$ corresponding to the values of the parameter from $0$ to $t$: $\Gamma_t : [0, t] \to \mathbb{U}^c$, and $\bar{\Gamma}_t$ be the symmetric arc in the lower half plane.

Let $g(z, t)$ be the univalent conformal map from the domain $\mathbb{D}_t = (\text{interior of } \mathbb{U}^c) \setminus (\Gamma_t \cup \bar{\Gamma}_t)$ to $\mathbb{U}^c = (\text{interior of } \mathbb{U}^c)$ normalized by the condition

$$g(z, t) = z/r(t) + O(1), \quad z \to \infty, \quad r(t) \in \mathbb{R}_+.$$
The Riemann mapping theorem ensures that such a map exists and is unique. The quantity \( r(t) \) is called the conformal radius of the domain \( \mathbb{D}_t \) relative to infinity (Fig. 1).

Our goal is to derive a differential equation for the function \( g(z, t) \) in the variable \( t \). According to our convention, \( g(z, 0) = z \), which is going to be the initial condition for the differential equation. Note that since the domain \( \mathbb{D}_t \) is symmetric and \( g(z, t) \) is normalized as \( 7.1 \) \( (r(t) \in \mathbb{R}_+) \), the function \( g(z, t) \) enjoys the property

\[
g(z, t) = \overline{g(\overline{z}, t)}.
\]  
(7.2)

(See Lemma A.1.)

The main technical tool is the complex Poisson formula (or the Schwarz integral formula) which is a version of the Cauchy integral formula. Given a bounded continuous function \( f(z) \) in \( \mathbb{U}_c \), holomorphic in its interior \( \mathbb{U}_c \) (thus holomorphic also at \( z = \infty \) by Riemann’s removable singularity theorem), the complex Poisson formula reads

\[
f(z) = {1 \over 2\pi} \int_{-\pi}^{\pi} \text{Re} \ f(e^{i\theta}) \left( z + e^{i\theta} \over z - e^{i\theta} \right) \ d\theta + i \text{Im} \ f(\infty).
\]  
(7.3)

We define a map \( h(z; s, t) \) (\( 0 < s < t \)) for \( z \in \mathbb{U}_c \) by

\[
h(z; s, t) = g(g^{-1}(z, t), s) = {r(t) \over r(s)} z + O(1),
\]  
(7.4)

where \( g^{-1} \) is the function inverse to \( g \). (See Fig. 2.)

We want to apply the complex Poisson formula to the function \( \log(h(z; s, t)/z) \). Hereafter we assume that \( g^{-1}(z, t) \) and hence \( h(z; s, t) \) are continuously extended to \( \mathbb{U}_c \), the closure of the original domain of definition \( \mathbb{U}_c \). (In Appendix 1 we shall see that such extension is really possible for certain class of functions.)
Fig. 2 Conformal map \( h(z; s, t) = g(g^{-1}(z, t), s) \) mapping the outside of the unit disk to a symmetric slit domain

We get:

\[
\log \frac{h(z; s, t)}{z} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |h(e^{i\theta}); s, t)| \frac{z + e^{i\theta}}{z - e^{i\theta}} d\theta. \tag{7.5}
\]

In particular, tending \( z \rightarrow \infty \), we obtain the following corollary of (7.5):

\[
\log r(t) - \log r(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |h(e^{i\theta}); s, t)| d\theta. \tag{7.6}
\]

Substituting \( z \mapsto g(z, t) \) in (7.5), we can write:

\[
\log \frac{g(z, s)}{g(z, t)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |h(e^{i\theta}); s, t)| \frac{g(z, t) + e^{i\theta}}{g(z, t) - e^{i\theta}} d\theta. \tag{7.7}
\]
The property (7.2) implies the similar property of the function \( h(z; s, t) \): \( h(z; s, t) = h(\tilde{z}; s, t) \) which allows one to rewrite the relations (7.6), (7.7) as follows:

\[
\log r(t) - \log r(s) = \frac{1}{\pi} \int_{0}^{\pi} \log |h(e^{i\theta}; s, t)| \, d\theta,
\]

\[
\log \frac{g(z, s)}{g(z, t)} = \frac{1}{2\pi} \int_{0}^{\pi} \log |h(e^{i\theta}; s, t)| \left( \frac{g(z, t) + e^{i\theta}}{g(z, t) - e^{i\theta}} + \frac{g(z, t) + e^{-i\theta}}{g(z, t) - e^{-i\theta}} \right) \, d\theta.
\]

(7.8)

(7.9)

Let the image of the tip of the curve \( \Gamma_t \) be \( e^{i\xi(t)} \): \( g(\Gamma_t, t) = e^{i\xi(t)} \). Then the image of the tip of the curve \( \tilde{\Gamma}_t \) is \( e^{-i\tilde{\xi}(t)} \). Note that \( \log |h(e^{i\theta}; s, t)| = 0 \) if \( e^{i\theta} \) is mapped by \( h \) to the boundary of \( U^c \), i.e. to a point on the unit circle. Accordingly, \( \log |h(e^{i\theta}; s, t)| \neq 0 \) if the point \( e^{i\theta} \) lies in a neighborhood of \( e^{i\xi(t)} \) (in an arc of the unit circle containing the point \( e^{i\xi(t)} \)) or in a neighborhood of \( e^{-i\tilde{\xi}(t)} \). If \( s \to t \) from below, this neighborhood shrinks to a point \( e^{i\xi(t)} \) or \( e^{-i\tilde{\xi}(t)} \). Therefore, equations (7.8), (7.9) imply the following differential equation by the mean value theorem\(^4\):

\[
\partial_t \log g(z, t) = -\frac{1}{2} \left( \frac{g(z, t) + e^{i\xi(t)}}{g(z, t) - e^{i\xi(t)}} + \frac{g(z, t) + e^{-i\tilde{\xi}(t)}}{g(z, t) - e^{-i\tilde{\xi}(t)}} \right) \partial_t \log r,
\]

(7.10)

which is the symmetric radial Löwner equation (5.7) \((t \to \lambda, g(z, t) \to w(z))\). Thus we have shown that a family of conformal mappings \( g(z, t) \) \((g(z, 0) = z)\) which conformally map the exterior of the unit circle with two non-intersecting symmetric slits \( \Gamma_t \) (in the upper half plane) and \( \tilde{\Gamma}_t \) (in the lower half plane) to the exterior of the unit circle gives a solution to this equation.

8 Conclusion

In this paper we have introduced the dispersionless version of the constrained Toda hierarchy. It was formulated in two equivalent ways: a) by means of the Lax formalism and b) by means of the Hirota-like equation for the \( F \)-function (the dispersionless limit of logarithm of the tau-function). The geometric meaning of the dispersionless constrained Toda hierarchy was clarified. We have shown that conformal maps of domains symmetric under reflection with respect to the real axis to the reference domain (the unit disk) give solutions to the hierarchy.

We have also studied finite-dimensional reductions of the dispersionless constrained Toda hierarchy. It was shown that the consistency condition for one-variable reduction is a differential equation of the Löwner type which we call the symmetric radial Löwner equation. The geometric meaning of the latter was clarified. We have shown that solutions to this equation are conformal maps of the exterior of the unit circle with two symmetric curved slits to the exterior of the unit circle.

\(^4\) Of course in this argument derivative \( \partial_t \) in (7.10) should be interpreted as the left derivative, but we can show the same equation for the right derivative, too. See Appendix 1.
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Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix A: Proof of the symmetric radial Löwner equation

In this appendix we derive the symmetric radial Löwner equation for an evolution family, a two-parameter family of conformal mappings, rigorously. In particular, extendability of the maps is discussed here. The last part of the proof is almost the same as the arguments in Sect. 7 but with some omitted details.

Assume that a Jordan curve $\gamma$ is in the upper half plane and tends to $\infty$:

$$
\gamma : [0, +\infty) \to \mathbb{C}, \quad \text{Im} \gamma(t) > 0, \quad \lim_{t \to \infty} \gamma(t) = \infty.
$$

We denote its complex conjugate by $\bar{\gamma}$:

$$
\bar{\gamma}(t) := \overline{\gamma(t)}.
$$

Let $D_t \ (t \geq 0)$ be the following domain, which is symmetric with respect to the real axis:

$$
D_t := \mathbb{C} \setminus (\gamma([t, +\infty)) \cup \bar{\gamma}([t, +\infty))).
$$

As in Sect. 7, the following lemma is essential.

Lemma A.1 Let $D$ be a simply connected domain in $\mathbb{C}$ symmetric with respect to the real axis: $z \in D \iff \bar{z} \in D$. For simplicity we assume that $0 \in D$. Then the holomorphic bijection $f : \mathbb{U} \to D$ normalized by $f(0) = 0$ and $f'(0) > 0$, the existence of which is guaranteed by Riemann’s mapping theorem, satisfies

$$
f(\bar{z}) = \overline{f(z)} \tag{A.4}
$$

for $z \in D$.

Proof Note that $\bar{f}(z) := \overline{f(z)}$ is also a holomorphic bijection between $\mathbb{U}$ and $D$, satisfying the normalization conditions $\bar{f}(0) = 0$, $\bar{f}'(0) > 0$. Since the Riemann
mapping theorem claims that such a mapping is unique, \( \tilde{f} = f \), which proves (A.4).

We denote the normalized conformal mapping from the unit open disk \( U \) to \( D_t \) by \( f_t \):

\[
    f_t : U \to D_t, \quad f_t(0) = 0, \quad f_t'(0) > 0. \tag{A.5}
\]

Because of Lemma A.1 this mapping function satisfies

\[
    f_t(\bar{z}) = \overline{f_t(z)}. \tag{A.6}
\]

The argument in §3.3 [10] or §3.2, Chapter 3 of [28] can be applied to the family of domains \( \{ D_t \} \) and we can reparametrize it so that \( f_t \) satisfies \( f_t'(0) = e^t \):

\[
    f_t(z) = e^t z + a_2(t) z^2 + a_3(t) z^3 + \cdots. \tag{A.7}
\]

(The choice \( f_t'(0) = e^t \) is just for simplicity. We can take any smooth positive increasing function \( r(t) \) as in Sect. 7 instead of \( e^t \).) We call the family \( \{ f_t(z) \}_{t \geq 0} \) the symmetric Löwner chain.

For nonnegative real numbers \( s \) and \( t \) satisfying \( 0 \leq s < t < +\infty \) we define the symmetric evolution family \( \omega_{s,t} : U \to U \) by \( \omega_{s,t} := f_t^{-1} \circ f_s \), which satisfies \( \omega_{s,t}(0) = 0, \ f_s = f_t \circ \omega_{s,t} \),

\[
    \omega_{s,t}(z) = e^{s-t} z + b_2(t) z^2 + b_3(z) z^3 + \cdots, \tag{A.8}
\]

and is univalent. Each \( \omega_{s,t} \) also satisfies the conditions of Lemma A.1:

\[
    \omega_{s,t}(\bar{z}) = \overline{\omega_{s,t}(z)}. \tag{A.9}
\]

**Theorem A.1** For a fixed \( s \geq 0 \) and \( z \in U \) the function \( \omega(t) = \omega_{s,t}(z) \) satisfies an ordinary differential equation

\[
    \frac{d\omega}{dt} = -\frac{\omega}{2} \left( \frac{e^{i\theta(t)} + \omega}{e^{i\theta(t)} - \omega} + \frac{e^{-i\theta(t)} + \omega}{e^{-i\theta(t)} - \omega} \right) \tag{A.10}
\]

on \( t \in [s, \infty) \) with the initial value condition \( w(s) = z \). Here \( \theta(t) \) is a real continuous function on \([0, +\infty)\).

By changing the variables \( t, z \) and \( \omega_{s,t} \) by \( \lambda := -t, \ \zeta := z^{-1}, \ w(\zeta, \lambda) := 1/\omega_{s,t}(1/\zeta) \), we obtain a conformal mapping from \( C \setminus U \) to \( C \setminus \overline{U} \), which satisfies

\[
    \frac{dw}{d\lambda} = -\frac{w}{2} \left( \frac{w + e^{-i\theta(\lambda)}}{w - e^{-i\theta(\lambda)}} + \frac{w + e^{i\theta(\lambda)}}{w - e^{i\theta(\lambda)}} \right). \tag{A.11}
\]

This is nothing but (5.7).
A.1 Behavior of functions on the boundaries

In this subsection we prove extendability of $\omega_{s,t}$ to the boundary of $U$. The variable $t$ is fixed. The argument almost follows that in §3.2, Chapter 3 of [28], which is an improved version of the argument in [10].

If $0 \leq s < t$, $D_t \setminus D_s = \gamma([s, t)) \cup \bar{\gamma}([s, t))$. Hence by (A.9) the image of $\omega_{s,t}$ is

$$\omega_{s,t}(U) = U \setminus (\alpha_{s,t} \cup \bar{\alpha}_{s,t}),$$

where $\alpha_{s,t} := f_t^{-1}(\gamma([s, t)))$ and $\bar{\alpha}_{s,t}$ is its complex conjugate.

Let us show that each arc $\alpha_{s,t}$ has an endpoint $\lambda(t)$ on $\partial U$, $\lambda(t) = \lim_{s \to t} f_t^{-1}(\gamma(s))$, which does not depend on $s \in [0, t)$.

We apply the Carathéodory extension theorem by opening the slit $\gamma([t, \infty)) \cup \bar{\gamma}([t, \infty]) \cup \{ \infty \}$ in the Riemann sphere $\hat{\mathbb{C}}$ by square root.\(^\ast\) We take a branch of

$$\zeta = F(w) := \sqrt{1 - w/\gamma(t)} \sqrt{1 - w/\bar{\gamma}(t)}$$

(A.12)

satisfying $F(0) = 1$. Its inverse map is

$$w = G(\zeta) := \frac{1 - \zeta^2}{\bar{\gamma}(t) - \gamma(t) \bar{\zeta}^2}.$$  (A.13)

Endpoints of the slit are mapped by $F$ to $F(\gamma(t)) = 0$ and $F(\bar{\gamma}(t)) = \infty$. The image of the slit is a Jordan closed curve $\Gamma^+ \cup [0] \cup \Gamma^- \cup \{ \infty \}$ in $\hat{\mathbb{C}}$, where $\Gamma^\pm$ are two connected components of $F(\gamma((t, +\infty)))$ (Fig. 3).

Applying the Carathéodory extension theorem to $F \circ f_t$, we obtain a homeomorphism $h : \bar{\mathbb{U}} \to \bar{F}(D_t)$ which is holomorphic in $\mathbb{U}$. Thus we obtain an extension of $f_t$ on $\bar{\mathbb{U}}$: $f_t = G \circ h : \bar{\mathbb{U}} \to \bar{D}_t = \hat{\mathbb{C}}$. (We use the same notation for the extended map as the original one.)

Since $f_t = F^{-1} \circ h$ on $\mathbb{U}$, we have

$$f_t^{-1}(\gamma(s)) = h^{-1}(F(\gamma(s))), \quad f_t^{-1}(\bar{\gamma}(s)) = h^{-1}(F(\bar{\gamma}(s)))$$

for $0 \leq s < t$. Hence, when $s$ approaches to $t$ from below, there exists limits,

$$\lambda(t) := \lim_{s \to t} f_t^{-1}(\gamma(s)) = \lim_{s \to t} h^{-1}(F(\gamma(s))) = h^{-1}(F(\gamma(t))) = h^{-1}(0), \quad (A.14)$$

\(^\ast\) In the derivation of Lönwre’s equation in §3.3 of [10] Carathéodory’s theorem is applied to a domain whose boundary is not a Jordan curve. In [28] this gap is filled by opening the slit by square root. We use this idea here.
\( \bar{\lambda}(t) := \lim_{s \nearrow t} f_t^{-1}(\bar{\gamma}(s)) = \lim_{s \nearrow t} h^{-1}(F(\bar{\gamma}(s))) = h^{-1}(F(\bar{\gamma}(t))) = h^{-1}(\infty), \)

(A.15)

and, due to (A.2) and (A.6), they are complex conjugate to each other,

\( \bar{\lambda}(t) = \lambda(t). \)

(A.16)

The proof of (right/left) continuity of \( \lambda(t) \) is the same as that in §3.3 of [10].

As in Fig. 4 we denote the counterclockwise arc from \( \lambda(t) \) to \( \bar{\lambda}(t) \) by \( I^+ = I_t^+ \) and the other arc by \( I^- = I_t^- : h(I^\pm) = \Gamma^\pm \). Restrictions of \( f_t : \hat{U} \to \hat{C} \) on \( I^\pm, f_t^\pm : I^\pm \to \gamma((t, +\infty)) \cup (\infty) \cup \bar{\gamma}((t, +\infty)), \) are homeomorphisms. Therefore for any \( u \in (t, +\infty) \) the subset \( J_{t,u} := J_{t,u}^+ \cup J_{t,u}^- \) of \( \partial \hat{U} \), where \( J_{t,u}^\pm := (f_t^\pm)^{-1}(\gamma([t, u])) \), is an open arc of \( \partial \hat{U} \).

Similarly, the subset \( \bar{J}_{t,u} := \bar{J}_{t,u}^+ \cup \bar{J}_{t,u}^- \) of \( \partial \hat{U} \), where \( \bar{J}_{t,u}^\pm := (f_t^\pm)^{-1}(\bar{\gamma}([t, u])) \), is an open arc of \( \partial \hat{U} \) and, because of (A.6), \( \bar{J}_{t,u}^\pm = \bar{J}_{t,u}^\pm : \bar{J}_{t,u} = \bar{J}_{t,u} = \bar{J}_{t,u} \).

Then \( f_t(J_{t,u}) = \gamma([t, u]) \) and \( f_t(\bar{J}_{t,u}) = \bar{\gamma}([t, u]) \) and, if \( 0 \leq t < u, \)

\[ I_t^\pm \setminus (J_{t,u}^\pm \cup \bar{J}_{t,u}^\pm) = f_t^{-1}(\gamma([u, \infty)) \cup \bar{\gamma}([u, \infty))) \cap I_t^\pm. \]

Therefore, if we define \( \omega_{t,u} \) on the boundary \( \partial \hat{U} \) as

\[
\omega_{t,u}(z) := \begin{cases} 
  f_u^{-1}(f_t(z)), & (z \in \hat{U} \cup J_{t,u} \cup \bar{J}_{t,u}), \\
  (f_u^\pm)^{-1}(f_t(z)), & (z \in I_t^\pm \setminus (J_{t,u} \cup \bar{J}_{t,u})),
\end{cases}
\]

(A.17)
Fig. 4 $f_t$ gives maps from arcs $J_{t,u}^±$ and $\bar{J}_{t,u}^±$ to $\gamma([t, u])$ and $\bar{\gamma}([t, u])$

it is a continuous map $\omega_{t,u} : \bar{\mathbb{U}} \to \mathbb{U}$. This is the desired extension of the symmetric evolution family $\omega_{t,u}(z)$.

The images of subsets $\mathbb{U}$, $J_{t,u}$ and $\bar{J}_{t,u}$ of $\mathbb{U}$ are

\[
\omega_{t,u}(\mathbb{U}) = \mathbb{U} \setminus (\alpha_{t,u} \cup \bar{\alpha}_{t,u}), \quad \omega_{t,u}(J_{t,u}) = \alpha_{t,u}, \quad \omega_{t,u}(\bar{J}_{t,u}) = \bar{\alpha}_{t,u},
\]

respectively, as shown in Fig. 5.

### A.2 Symmetric radial Löwner equation for $\{\omega_{s,t}(z)\}$.

We add some details in derivation of the symmetric radial Löwner equation discussed in Sect. 7. (Recall that in this appendix $r(t)$ is normalized as $r(t) = e^t$ for simplicity.)

We follow the argument in §3.2, Chapter 3 of [28].

As $\omega_{t,u}(z) = e^{t-u}z + O(z^2)$, we can take a branch of

\[
h(z) = h(z; t, u) := \log \frac{\omega_{t,u}(z)}{z}, \tag{A.18}
\]

such that $h(0) = t - u$. This function is holomorphic in $\mathbb{U}$ and continuous on its closure $\mathbb{U}$. Since $\omega_{t,u}(\partial \mathbb{U} \setminus (J_{t,u} \cup \bar{J}_{t,u})) \subset \partial \mathbb{U}$, $\text{Re}h(z) = 0$, if $z \in \partial \mathbb{U} \setminus (J_{t,u} \cup \bar{J}_{t,u})$, and since $\omega_{t,u}(J_{t,u} \cup \bar{J}_{t,u}) \subset \mathbb{U}$, $\text{Re}h(z) < 0$, if $z \in J_{t,u} \cup \bar{J}_{t,u}$. Hence, applying the
Schwarz integral formula for the unit disk,\(^6\)

\[
f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Re} f(e^{i\theta}) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta + i\text{Im} f(0), \quad (z \in \mathbb{U}),
\]

to \(h(z)\), we have

\[
h(z) = \frac{1}{2\pi} \left( \int_{a}^{b} + \int_{-b}^{-a} \text{Re} h(e^{i\theta}) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta \right), \quad (A.19)
\]

where \(a\) and \(b\) are real numbers such that \(e^{ia}\) and \(e^{ib}\) are endpoints of \(J_{t,u}\) and, correspondingly, \(e^{-ib}\) and \(e^{-ia}\) are endpoints of \(\bar{J}_{t,u}\). Note that \(h(0) = t - u \in \mathbb{R}\) and therefore \(\text{Im} h(0) = 0\).

As in Sect. 7, because of the symmetry \(h(\bar{z}) = \overline{h(z)}\) the above formula is rewritten as

\[
h(z) = \frac{1}{\pi} \int_{a}^{b} \text{Re} h(e^{i\theta}) K(e^{i\theta}, z) d\theta, \quad (A.20)
\]

\(^6\) Recall that formula (7.3) is for functions outside of the unit disk.
where $K(\xi, z)$ is the function defined by $K(z, \xi) := \frac{1}{2} \left( \frac{\xi + z}{\xi - z} + \frac{\xi + \bar{z}}{\bar{\xi} - z} \right)$. Substituting $\omega_{s,t}(z)$ for $z$ in (A.20) and using $\omega_{t,u}(\omega_{s,t}(z)) = \omega_{s,u}(z)$, we have

$$
\log \frac{\omega_{s,u}(z)}{\omega_{s,t}(z)} = \frac{1}{\pi} \int_{a}^{b} \text{Reh}(e^{i\theta})K_R(e^{i\theta}, \omega_{s,t}(z))d\theta + \frac{i}{\pi} \int_{a}^{b} \text{Reh}(e^{i\theta})K_I(e^{i\theta}, \omega_{s,t}(z))d\theta,
$$

(A.21)

where $K_R(\xi, z) := \text{Re}K(\xi, z)$ and $K_I(\xi, z) := \text{Im}K(\xi, z)$. From the mean value theorem of integrals it follows that there exist $\xi$ and $\eta$ in the interval $(a, b)$ such that

$$
\int_{a}^{b} \text{Reh}(e^{i\theta})K_R(e^{i\theta}, \omega_{s,t}(z))d\theta = K_R(e^{i\xi}, \omega_{s,t}(z)) \int_{a}^{b} \text{Reh}(e^{i\theta})d\theta,
$$

$$
\int_{a}^{b} \text{Reh}(e^{i\theta})K_I(e^{i\theta}, \omega_{s,t}(z))d\theta = K_I(e^{i\eta}, \omega_{s,t}(z)) \int_{a}^{b} \text{Reh}(e^{i\theta})d\theta.
$$

Hence by dividing (A.21) by $t - u = h(0) = \frac{1}{\pi} \int_{a}^{b} \text{Reh}(e^{i\theta})d\theta$, we obtain

$$
\frac{\log \omega_{s,u}(z) - \log \omega_{s,t}(z)}{u - t} = -(K_R(e^{i\xi}, \omega_{s,t}(z)) + iK_I(e^{i\eta}, \omega_{s,t}(z))).
$$

When $u \searrow t$, $J_{t,u}$ contracts to $\lambda(t)$ (cf. Fig. 4) and therefore both of $e^{i\xi}$ and $e^{i\eta}$ tend to $\lambda(t)$. Thus

$$
\frac{\partial}{\partial t} \log \omega_{s,t}(z) = -K(\lambda(t), \omega_{s,t}(z)),
$$

(A.22)

where the left-hand side is regarded as the right derivative. Similarly the limit $t \nearrow u$ gives (A.22) as the left derivative, because $J_{t,u}$ contracts to $\lambda(u)$ in this limit. Thus we have proved the symmetric radial Löwner equation (A.10) for $\omega(t) = \omega_{s,t}(z)$, $e^{i\theta(t)} = \lambda(t)$.

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