Andreev Tunneling in Strongly Interacting Quantum Dots

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We review recent work on resonant Andreev tunneling through a strongly interacting quantum dot connected to a normal and to a superconducting lead. We derive a general expression for the current flowing in the structure and discuss the linear and non-linear transport in the nonperturbative regime. New effects associated to the Kondo resonance combined with the two-particle tunneling arise. The Kondo anomaly in the $I-V$ characteristics depends on the relative size of the gap energy and the Kondo temperature.

I. INTRODUCTION

In recent years electrical transport through confined regions has seen impressive theoretical and experimental activity. Several well known phenomena have received renewed attention due to the present possibility of studying them in novel and more controllable situations. For instance, Andreev scattering (Andreev 1964), according to which a particle-like excitation impinging on a normal metal-superconductor interface is reflected back as a hole-like excitation, has been shown to be the key mechanism controlling transport in a variety of hybrid mesoscopic superconducting devices. (For a review see, for example, Hekking, Sch"{o}n, and Averin 1994, Beenakker 1995, Lambert and Raimondi 1998). At the same time, electron-electron interaction in small confined regions, or quantum dots (QDs), has been shown to lead to the so-called Coulomb blockade of electrical transport. This occurs at low temperatures, when the Fermi energy of the contacts falls in the gap between the ground state energies of the dot with $N$ and $N+1$ electrons (Kastner 1992). However, a QD attached to metallic leads resembles an impurity level in a metal. As a consequence, even in the Coulomb blockade regime, transport will occur due to the Kondo effect (Glazman and Raikh 1988, Ng 1988). This is due to the formation of a spin singlet between the impurity level and the conduction electrons, which gives rise to a quasiparticle peak at the Fermi energy in the dot spectral function. This suggestion has been explored theoretically by several groups (Meir, Wingreen, and Lee 1991,1993, Ng 1993, Levy Yeyati, Martin-Rodero, and Flores 1993, Schoeller and Sch"{o}n 1994, Hettler and Schoeller 1995). This has lead to the prediction of a zero-bias anomaly in the current voltage characteristics and an increase of the linear conductance in the Coulomb blockade regime for decreasing temperature. Such phenomena have indeed been observed in different QD systems (Ralph and Buhrman 1994, Goldhaber-Gordon et al. 1998, Cronenwett et al. 1998).

What will happen if the QD is coupled to a normal and a superconducting lead as shown schematically in Fig.1? Does the zero-bias anomaly observed in the N-QD-N case survive in the N-QD-S case? Such a problem has been investigated recently by various groups (Fazio and Raimondi 1998, K. Kang 1999, Schwab and Raimondi 1999, Sun, Wang, and Lin 1999, Clerk, Ambegaokar, Hershfield 1999). In this paper we review mainly the work done in Refs. (Fazio and Raimondi 1998) and (Schwab and Raimondi 1999).

The problem one is faced with has all the difficulty of the original Kondo problem. It is well known that in the limits of high and low temperatures (here high and low are with respect to the Kondo temperature), qualitatively correct results are obtained by means of different techniques. Such an attitude we adopt here, having in mind to analyze the interplay between Andreev scattering and Kondo physics. We use a simple equation-of-motion approach in the perturbative regime above the Kondo temperature. This corresponds to taking into account only the leading logarithmic corrections in the renormalization group sense. The cross-over behaviour to low temperature may be rather complicated, but as happens in the standard Kondo problem, we expect that the extreme low-temperature phase can be described in simple terms. Such a description is obtained by means of the slave-boson technique in a mean-field approximation. In the high temperature regime, we find a suppressed Andreev current at low bias voltage due to the competition between the Coulomb energy and the superconducting proximity effect in the QD. In the low-temperature case, in contrast, our analysis predicts that the linear conductance of the N-QD-S system is enhanced as compared to the normal case and may reach the maximum universal value of $G_{NS} = 4e^2/h$ which is twice the maximum for the N-QD-N system.

Before entering the technical details, it is also worthwhile discussing the relevant energy scales. In investigating Andreev scattering at a normal metal-superconductor interface one is often interested in voltages and temperatures well below the energy scale set by the superconducting gap $\Delta$ and one could be tempted to take $\Delta \to \infty$ from the outset. However, the dot charging energy $U$ (see below) introduces another energy scale into the problem, and the physics is different in the two limits $U \gg \Delta$. 

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and $\Delta \gg U$. In order to appreciate this point, let us consider the Hamiltonian of an isolated QD, plus a term which models Andreev scattering in the limit $\Delta \to \infty$ by 

$$H_A = \epsilon_A d_\sigma^\dagger d_\sigma + c.c.$$ 

For this Hamiltonian the off-diagonal element of the inverse dot Green’s function reads 

$$-t_A [1 + U(\epsilon + \epsilon_d + U)/[\epsilon^2 - (\epsilon_d + U)^2 - t_A^2]]$$

and vanishes in the $U \to \infty$ limit, i.e., in the limiting sequence $\Delta \to \infty$ first, $U \to \infty$ after, the induced superconductivity and hence transport in the dot is completely suppressed. In order to have Andreev scattering in the large $U$ limit two electrons have to enter the superconductor without doubly occupying the QD. This can only happen on a time scale of the order $1/\Delta$. We therefore concentrate on the limit $U \gg \Delta$ in the following analysis.

Our paper is organized as follows. In Section II we derive a formula for the current through the dot in the presence of Andreev scattering. Calculating the current explicitly requires the dot Green’s function. In Sections III and IV we present the equation-of-motion and slave-boson approaches, respectively. A few conclusions are drawn in section V.

II. THE ANDREEV CURRENT FORMULA

In this section we define the system under consideration and derive an expression for the Andreev current which holds in the presence of electron-electron interaction.

The model Hamiltonian for the N-QD-S system is

$$H = H_N + H_S + H_D + H_{T,N} + H_{T,S}$$

where $H_N$, $H_S$, are the Hamiltonians of the normal and the superconducting leads (\(\Delta\) being the superconducting gap),

$$H_N = \sum_{\k,\sigma} \epsilon_k c_{N,\k,\sigma}^\dagger c_{N,\k,\sigma}$$

$$H_S = \sum_{\k,\sigma} \epsilon_k c_{S,\k,\sigma}^\dagger c_{S,\k,\sigma} + \sum_{\k} (\Delta c_{S,\k,\uparrow}^\dagger c_{S,-\k,\downarrow} + c.c.) \quad .$$

If we restrict ourselves to temperatures and bias voltages much smaller than the average level spacing in the QD, then transport occurs through a single level. In this case the Hamiltonian of the quantum dot $H_D$ reads

$$H_D = \epsilon_d d_\sigma^\dagger d_\sigma + U n_{d\uparrow} n_{d\downarrow} \quad .$$

The level $\epsilon_d$ is assumed to be spin degenerate and the electron-electron interaction is included through the on-site repulsion $U$ ($\sim 1 - 5 K$ for currently fabricated QDs). Experimentally the position of the dot level can be modulated by an external gate voltage. Tunneling between the leads and the dot is described by $H_{T,N}$ and $H_{T,S}$

$$H_{T,\eta} = \sum_{k\sigma} (V_\eta c_{\eta,\k,\sigma}^\dagger d_\sigma + c.c.) \quad$$

where $\eta = N, S$ and $V_\eta$ is the tunneling amplitude.

The starting point for deriving the current formula is

$$I = -e(d/dt) \langle N_N \rangle = i\epsilon \langle N_N, H \rangle$$

where $N_N$ is the electron number operator in the normal lead. In the case of a hybrid structure, like the one we are considering now, it turns out to be more convenient to evaluate the current in the normal lead. The average current can be rewritten as follows

$$I = 2eIm \sum_{\k,\sigma} V_\eta \langle c_{\eta,\k,\sigma}^\dagger d_\sigma \rangle \quad .$$

Since we deal with a nonequilibrium situation we work in the framework of the Keldysh technique, as employed in the literature (Meir and Wingreen 1992). By introducing the Nambu notation

$$\Psi_{N,\k} = \begin{pmatrix} c_{N,\k,\uparrow}^\dagger & c_{N,\k,\downarrow} \\ c_{N,-\k,\downarrow} & c_{N,-\k,\uparrow}^\dagger \end{pmatrix} \quad ,$$

the average current in eq. (6) requires the evaluation of the lesser Green’s function $G_<^{\alpha\beta,\k}(t, t') = \langle \Psi^{\dagger}_{\beta,\eta,\k}(t') \phi_\alpha(t) \rangle$ which, by means of the Dyson equation, can be expressed in terms of the exact Green’s function $G$ of the QD and the free Green’s function of the normal lead $g_{N,\k}$. The lesser component of the Dyson equation reads

$$\hat{G}^<_k(\epsilon) = \hat{G}^R(\epsilon) \begin{pmatrix} V_N & 0 \\ 0 & V_N^* \end{pmatrix} \hat{g}^<_N(\epsilon)$$

$$+ \hat{G}^<(\epsilon) \begin{pmatrix} V_N & 0 \\ 0 & V_N^* \end{pmatrix} \hat{g}^>_N(\epsilon) \quad .$$

where $\hat{G}^R(\epsilon)$, $\hat{G}^<$ are the retarded (advanced) and the lesser Green’s functions of the dot (for example $\hat{G}^R(t) = -i\theta(t) \langle \{ \phi(t), \phi^\dagger(0) \} \rangle$). Using the relation $\hat{G}^< = \hat{G}^R \hat{\Sigma}^< \hat{G}^A$ and the expression for the Green’s functions in the normal lead (diagonal in Nambu space) $g^R_N(\epsilon) = 1/(\epsilon - \epsilon_k + i\eta)$, $g^<_{N,\k}(\epsilon) = 2\pi i f(\epsilon) \delta(\epsilon - \epsilon_k) (f(x) is the Fermi function), it is possible to rewrite the current in the following form

$$I = ie \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \Gamma_N(\epsilon) \text{Tr} \{ \hat{f}_N \hat{G}^R(\epsilon) \hat{\Sigma}^R(\epsilon) f_N(\epsilon) - f_N(\epsilon) \hat{\Sigma}^A(\epsilon) + \hat{\Sigma}^<(\epsilon) \hat{G}^A(\epsilon) \}$$

In the formula above, we introduced the elastic rate

$$\Gamma_N(\epsilon) = 2\pi \sum_{\k} |V_N|^2\delta(\epsilon - \epsilon_k)$$

The diagonal matrix $f_N$ has elements $f_{N,11} = f(\epsilon - \mu_N)$ and $f_{N,22} = 1 - f(-\epsilon - \mu_N)$. The normal electrode is kept at a chemical potential $\mu_N = -eV$ while that of the superconductor is fixed to zero.

Up to now no approximations were involved. In order to determine the current an expression for the self-energy of the QD should be found. A determination of
\[ I \text{ requires both the lesser and retarded parts of the self-}
\text{energy. We formulate an ansatz for the lesser Green’s}
\text{function which is expressed solely in terms of the retarded}
\text{one. This ansatz automatically guarantees current}
\text{conservation. This generalizes an ansatz put forward by (Ng}
1996) for two normal leads to the calculation of \( \Sigma^< (\epsilon) \)
\text{in the presence of a superconducting lead.}

\[ \Sigma^< (\epsilon) = - \sum_{\eta=N,S} \left[ \Sigma^R_{0,\eta}(\epsilon) f_{\eta}(\epsilon) - \hat{f}_{\eta}(\epsilon) \Sigma^A_{0,\eta}(\epsilon) \right] \] (10)

\[ \Sigma^\gamma (\epsilon) = - \sum_{\eta=N,S} \left\{ \Sigma^R_{0,\eta}(\epsilon) [\hat{1} - \hat{f}_{\eta}(\epsilon)] - \left[ \hat{1} - \hat{f}_{\eta}(\epsilon) \right] \Sigma^A_{0,\eta}(\epsilon) \right\}. \] (11)

In this case (with no interaction) the nonequilibrium self-energy has the same form as in equilibrium but with the
Fermi functions of the two leads kept at different chemical
potentials. The idea is to assume that even in the
presence of interaction the dependence on the Fermi distribution is the same and that both the lesser and the
greater functions depend on a single function \( \hat{A} \) such that
\[ \Sigma^< = \Sigma^< A, \quad \Sigma^\gamma = \Sigma^\gamma A. \] (12)

The matrix \( \hat{A} \) is determined by the condition
\[ \Sigma^< - \Sigma^\gamma = \hat{A}^R - \hat{A}^A, \] (13)

which is a general property of the Keldysh Green’s
functions. As already mentioned, this ansatz leads to
current conservation. Moreover it is exact both in the
non-interacting limit, \( U = 0 \), and in absence of superconductivity, \( \Delta = 0 \). As a result
\[ \Sigma^< = \Sigma^< A (\Sigma^R - \Sigma^A)^{-1} (\Sigma^R - \Sigma^A). \] (14)

Eq. (14) allows us to evaluate the expression of the current \( I \), once we know the retarded Green’s function of the dot.

The expression for the current can be greatly simplified in the relevant limit \( U, \Delta \gg k_B T, eV \). In this case
the non-interacting self-energy due to the superconducting
lead \( \Sigma^R \) is real and purely off-diagonal whereas that due to the normal lead, \( \Sigma^R(A) \), is diagonal. Using
these properties of the self-energy and substituting eq. (14) in eq. (15) we obtain the following form for the
Andreev current through a QD.
\[ I = i e \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \Gamma N \text{Tr} \left\{ \Sigma^R (\epsilon) \right\}. \] (16)

This equation generalizes the formula obtained by (Beenakker 1992, Clauthon, Leadbeater, and Lambert
1995) for the non-interacting case to the case of a strongly
interacting dot.

In the following sections we apply this formula to the
study of the \( I - V \) curves of a N-QD-S system.

III. EQUATION OF MOTION

Our aim is to find an equation for the dot’s Green’s
function \( \hat{G}(\epsilon) = -i \langle T [\hat{\phi}(t) \hat{\phi}(0)] \rangle \). The retarded and
advanced Green’s functions necessary for the evaluation of the
current can easily be obtained from the time-ordered
one. In this section we will describe the equation-of-
motion approach. It is useful to introduce quasiparticle operators by performing a Bogolubov transformation
\[ \gamma_{\eta,k} = u_{\eta,k} c_{\eta,k} + \text{sign}(\sigma) e^{-i\phi} v_{\eta,k} \epsilon^\dagger \eta, -k \sigma \] (16)

\[ \gamma_{\eta,-k} = -\text{sign}(\sigma) e^{i\phi} v_{\eta,k} c_{\eta,k} + u_{\eta,k} \epsilon^\dagger \eta, -k \sigma \] (17)

where \( u_{\eta,k}^2 = (1 \pm \epsilon_k / E_{\eta,k}) / 2 \), \( E_{\eta,k} = \sqrt{\epsilon_k^2 + |\Delta_{\eta}|^2} \), and \( \Delta_{\eta} = e^{i\phi} |\Delta_{\eta}| \). In the normal lead we have \( \Delta_N = 0 \) and the Bogolubov transformation reduces to the identity transformation. The hopping term
becomes in the quasiparticle basis
\[ \hat{T}_{\eta,k} = \left( \begin{array}{cc} u_{\eta,k} V_{\eta} & -e^{i\phi} v_{\eta,k} V_{\eta}^* \\ -e^{-i\phi} v_{\eta,k} V_{\eta} & -u_{\eta,k} V_{\eta}^* \end{array} \right). \] (18)

The Nambu formalism is introduced for the \( \gamma \) operator via \( \Psi_{\eta,k} = (\gamma_{\eta,k}, \gamma_{\eta,-k}) \). We start by writing down the equation of motion for the operators \( \phi \) and \( \Psi \):
\[ i \frac{\partial}{\partial \tau} \phi = c_d \hat{\sigma}_z + U \hat{\sigma}_z \Phi + \sum_{\eta,k} \hat{T}_{\eta,k}^\dagger \Psi_{\eta,k} \] (19)

\[ i \frac{\partial}{\partial \tau} \Psi_{\eta,k} = E_{\eta,k} \hat{\sigma}_z \Psi_{\eta,k} + \hat{T}_{\eta,k} \phi. \] (20)

In the equation for \( \phi \), the new operator \( \Phi = (d^\dagger_{\eta} d_{\eta}, d^\dagger_{\eta} d_{\eta}) \), appears because of the interaction. It is straightforward to iterate once the equation of motion and get an equation for \( \Phi \)
\[ i \frac{\partial}{\partial \tau} \Phi = (c_d + U) \hat{\sigma}_z \Phi + \sum_{\eta,k} \left[ \hat{N} \hat{T}_{\eta,k}^\dagger \Psi_{\eta,k} \right] \]
\[ + d^\dagger_{\eta} d_{\eta} \left( \hat{T}_{\eta,k} \right)^t \left( \Psi_{\eta,k} \right)^t \] (21)

where
\[ \hat{N} = \left( \begin{array}{cc} d^\dagger_{\eta} d_{\eta} & d^\dagger_{\eta} d_{\eta} \\ d^\dagger_{\eta} d_{\eta} & d^\dagger_{\eta} d_{\eta} \end{array} \right). \]
hierarchy at some point. Following the decoupling procedure used in the absence of superconducting leads (Meir, Wingreen and Lee 1991), we neglect correlations in the leads. To see how this is achieved, consider the general expression for the single particle Green’s function obtained as 

\[ \hat{G} = \frac{1}{\hat{\Sigma} + \hat{G}_0} \]

and the equation for the single particle Green’s function motion for \( \hat{\Sigma} \).

A further approximation is done by considering the limit of large Coulomb interaction, i.e., \( U \to \infty \). In this case one can safely neglect, in the third order of iteration of the equation of motion, all operators having two creation or two annihilation \( d \) operators, because they will give rise to terms of the order \( 1/U \). We write here the resulting equations of motion for \( \hat{G} \) and \( \hat{G}^{II} = -i \langle T [\Phi(t)\phi(0)] \rangle \).

\[
(\omega \hat{\sigma}_0 - \epsilon_d \hat{\sigma}_d) \hat{G}(\omega) = \hat{\sigma}_0 + \hat{\Sigma}_0(\omega) \hat{G}(\omega) + U \hat{\sigma}_d \hat{G}^{II}(\omega) \tag{22}
\]

\[
U \hat{\sigma}_d \hat{G}^{II}(\omega) = -\langle \hat{N} \rangle + \hat{\Sigma}_I(\omega) \hat{G}(\omega) \tag{23}
\]

and the equation for the single particle Green’s function becomes

\[
(\omega \hat{\sigma}_0 - \epsilon_d \hat{\sigma}_d - \hat{\sigma}(\omega)) \hat{G}(\omega) = \hat{\sigma}_0 - \langle \hat{N} \rangle, \tag{24}
\]

where \( \hat{\sigma} = \hat{\Sigma}_0 + \hat{\Sigma}_I \). The noninteracting self-energy \( \hat{\Sigma}_0 \) is obtained as

\[
\hat{\Sigma}_0(\omega) = -\frac{i}{2} \sum_\eta \frac{\Gamma_\eta}{\sqrt{\omega^2 - \Delta_\eta^2}} \begin{pmatrix} \omega & \Delta_\eta \\ -\Delta_\eta & -\omega \end{pmatrix} \tag{25}
\]

with \( \Gamma_\eta = 2\pi \sum_k |V_{\eta k}|^2 \delta(\epsilon - \epsilon_k) \). For the calculation of the interacting self-energy \( \hat{\Sigma}_I(\omega) \) we assume that in the superconducting reservoir there are no quasiparticles present (we consider energies much smaller than the gap \( \Delta \)).

\[
\hat{\Sigma}_I(\omega) = V^2_S \sum_k \begin{pmatrix} |v_{\kappa k}|^2 & 2v_{\kappa k} v_{\kappa k}^* \\ 2v_{\kappa k} v_{\kappa k}^* & |v_{\kappa k}|^2 \end{pmatrix} \frac{1}{\omega - \epsilon_k - \mu_N} \tag{26}
\]

\[
\hat{\Sigma}_I(\omega) = -V^2_S \sum_k \begin{pmatrix} f(\epsilon_k) & 0 \\ 0 & f(\epsilon_k) \end{pmatrix} \frac{1}{\omega - \epsilon_k} \tag{27}
\]

In what follows, we consider \( W \gg \Delta \gg T, \omega \) with \( W \) the bandwidth. The self-energy \( \hat{\Sigma}_{I,S,11} \) is weakly energy dependent since \( \omega \ll \Delta < E_{k_S} \). The imaginary part of the diagonal elements of the self-energy \( \hat{\Sigma}_S \) vanishes, so quasiparticles present in the dot cannot decay by tunneling into the superconductor. As a result the contribution to the self-energy due to the superconducting lead simply shifts the dot level to the new value \( \epsilon_d = \epsilon_d + (\Gamma_S/2\pi) \) in \( W/\Delta \). The divergence of the level energy renormalization with \( \Delta \) reveals that the process occurs via a virtual state in which a quasiparticle is created in the superconductor. Substituting the expression of the QD’s Green’s function in eq. \( (13) \) we get the result

\[
I(V) = \int_{-\infty}^{\infty} d\epsilon \frac{f(\epsilon - eV) - f(\epsilon + eV)}{2e} G_{NS}(\epsilon) \tag{28}
\]

with

\[
G_{NS}(\epsilon) = \frac{4e^2}{h} \frac{2(\Gamma_N \Gamma_S(\epsilon))^2 [\Gamma_{1,N} + \Gamma_{2,N}/2\Gamma_N]}{[4(\epsilon - \epsilon_1)(\epsilon + \epsilon_2) - \Gamma_{1,N} \Gamma_{2,N} - \Gamma_S(\epsilon)]^2 + 4 [\Gamma_{1,N}(\epsilon + \epsilon_2) + \Gamma_{2,N}(\epsilon - \epsilon_1)]^2} \tag{29}
\]

where \( \epsilon_{1(2)} = \epsilon_d \pm \text{Re} \hat{\sigma}_{1(2)} \), \( \Gamma_{1(2),N} = -2\text{Im} \hat{\sigma}_{1(2)} \), and \( \Gamma_S(\epsilon) = 2\text{Re} \hat{\sigma}_{12} = \Gamma_S 2e/(\pi \Delta) \). Notice that the anomalous non-interacting \( \hat{\Sigma}_{0,12} \) and interacting \( \hat{\Sigma}_{1,12} \) self-energies exactly cancel in the zero energy limit, resulting in a linear behaviour for energies smaller than the gap \( \Delta \). The spectral function \( G_{NS}(\epsilon) \) associated with the resonant Andreev tunneling is plotted in Fig. 2 for various bias voltages. Several features are worth noticing. First, two peaks at \( \pm \Delta \) are due to particle and hole bare levels. Note that in the interacting case the bare level energy includes the renormalization due to the superconducting electrode self-energy as discussed above. Second, at low temperatures the spectral function at the Fermi energy is completely suppressed, in contrast to what happens in N-QD-N case. Quite remarkably, at finite positive (negative) voltages a Kondo peak develops pinned to the Fermi level of the normal metal while a small kink develops at negative (positive) voltages. At finite voltages hole and particle energies differ by \( 2eV \), and while the electron (hole) is on resonance for positive (negative) voltage, the Andreev reflected hole (electron) is off resonance with respect to the shifted Fermi level. The differential conductance, for various temperatures, is shown in Fig. 3. Lowering the temperature a zero-bias anomaly starts to develop where the conductance is strongly suppressed at low voltages. From eqs. (28), (29) we conclude that the linear conductance is roughly proportional to \( T^2/\Delta^2 \), so that it seems to be completely suppressed in the zero temperature limit. However, the equation-of-motion approach is quantitatively reliable only above the Kondo temperature. Hence the analysis carried out here applies to the regime \( \Delta > T > T_K \). An interesting question to ask is what happens in the opposite regime when the Kondo temperature dominates,
i.e., when \( T_K > \Delta > T \). This regime cannot be explored by the equation-of-motion approach. For this reason in the next section we will investigate the low temperature regime by means of the slave-boson technique.

IV. SLAVE BOSON MEAN FIELD APPROXIMATION

In this section we extend the analysis of the previous section to the extreme low temperature regime. To this end we use slave-boson mean field theory. This approach has been successfully applied to the low temperature properties of a Kondo impurity in the presence of normal (Barnes 1976, Coleman 1984, Read and Newns 1983, Read 1985) as well as for superconducting conduction electrons (Borkowski and Hirschfeld 1994). Despite its simplicity, this method captures the main physical aspects of the Fermi liquid regime at low temperatures, i.e., the formation of a many-body resonance at the Fermi energy. For this reason it presents a convenient framework in which to study the interplay between Andreev scattering and Coulomb interactions.

Again we consider an infinite on-site repulsion \( U \), so processes where the dot level is doubly occupied are excluded. The dot level is represented as a fermion \( \hat{b}^\dagger \hat{b} \) or singly occupied, the constraint \( \hat{b}^\dagger \hat{b} + \sum_{\sigma} f_\sigma^\dagger f_\sigma = 1 \) has to be fulfilled.

In mean field approximation, the operator \( \hat{b} \) is replaced by a \( c \)-number \( b_0 \), and the constraint is fulfilled only on average. This is achieved by introducing a chemical potential \( \lambda_0 \) for the pseudo particles. Notice that one ends up with a non-interacting-like problem with renormalized parameters, i.e., an energy shift for the dot level \( \epsilon_d \rightarrow \epsilon_d + \lambda_0 = \epsilon_d \) and a multiplicatively renormalized tunneling amplitude \( V_\eta \rightarrow b_0 V_\eta \).

We discuss the mean field equations and their solution first in equilibrium and then generalize to non-equilibrium. We start from the impurity part of the free energy, which in the presence of both normal and superconducting leads is given by

\[
F = -T \sum_{\epsilon_n} \text{Tr} \ln[i\epsilon_n \sigma_0 - \epsilon_d \sigma_z - \frac{\lambda_0}{2} \hat{\Gamma}(\epsilon_n)] + \lambda_0 b_0^2 + \epsilon_d - \mu, \tag{30}
\]

where \( \epsilon_n \) is a fermionic Matsubara frequency, \( \sigma^i \) are the Pauli matrices, and

\[
\hat{\Gamma}(\epsilon_n) = \sum_{\kappa, \eta} |V_\eta|^2 \hat{\sigma}_z \hat{g}_{\eta, \kappa}(i\epsilon_n) \hat{\sigma}_z \tag{31}
\]

with \( \hat{g}_{\eta, \kappa} \) being the Green’s function of the lead \( \eta \).

By minimizing the free energy with respect to \( \lambda_0 \) and \( b_0 \) we find the equations

\[
b_0^2 + T \sum_{\epsilon_n} \text{Tr} \left[ \hat{g}(i\epsilon_n) \hat{\sigma}_z \right] = 0, \tag{32}
\]

\[
b_0 \lambda_0 + b_0 T \sum_{\epsilon_n} \text{Tr} \left[ \hat{g}(i\epsilon_n) \hat{\Gamma}(i\epsilon_n) \right] = 0, \tag{33}
\]

which have to be solved self-consistently. \( \hat{g}(i\epsilon_n) \) is the pseudo fermion Green’s function given by \( \hat{g}(i\epsilon_n) = [i\epsilon_n \sigma_0 - \epsilon_d \sigma_z - \frac{\lambda_0}{2} \hat{\Gamma}(i\epsilon_n)]^{-1} \). Both in the limit of small and large superconducting gap, we are able to solve the mean field equations analytically as demonstrated here below. The first equation, eq.(32), is the constraint. Since the pseudo fermion level is at maximum singly occupied, the renormalized level is above the Fermi energy.

In the Kondo limit, where the occupancy is nearly one, we find that \( 0 < \epsilon_d < \frac{\lambda_0}{2} \), i.e. \( \lambda_0 \approx |\epsilon_d| \) and \( \epsilon_d \approx 0 \). The renormalization of the tunneling amplitude is determined from eq.(33). A trivial solution \( b_0 = 0 \) always exists. The solutions which minimize the free energy, however, are those with \( b_0 \neq 0 \). By introducing a flat density of states in the leads and the tunneling rates \( \gamma = 2\pi N_0 |V_\eta|^2 \), the elements of the matrix \( \hat{\Gamma}(i\epsilon_n) \) are \( \Gamma_11 = \Gamma_22 = -i\gamma_1 \) and \( \Gamma_{12} = \Gamma_{21} = \gamma_2 \), where

\[
\gamma_1 = \text{sign}(\epsilon_n) \frac{\Gamma_N}{2} + \frac{\Gamma_S}{2 \sqrt{\epsilon_n^2 + |\Delta|^2}} \gamma_2 = \frac{\Gamma_S}{2} \frac{\Delta}{\sqrt{\epsilon_n^2 + |\Delta|^2}} \tag{34}
\]

Restricting ourselves to zero temperature, we replace the Matsubara sum in eq.(33) by an integral and obtain

\[
|\epsilon_d| = 4 \int_0^W \frac{d\epsilon}{2\pi} \frac{\gamma_1(\epsilon + b_0^2 \gamma_1) + b_0^2 \gamma_2}{(\epsilon + b_0^2 \gamma_1)^2 + b_0^4 \gamma_2^2}, \tag{35}
\]

where \( W \) is a cut-off of order the band-width. We simplify the integral by approximating \( \gamma_1 \) and \( \gamma_2 \) as

\[
\gamma_1 = \begin{cases} \Gamma_N/2 & \text{for } \epsilon < \Delta \\ (\Gamma_N + \Gamma_S)/2 & \text{for } \epsilon > \Delta \end{cases}, \quad \gamma_2 = \begin{cases} \Gamma_S/2 & \text{for } \epsilon < \Delta \\ 0 & \text{for } \epsilon > \Delta \end{cases}. \tag{36}
\]

The result is

\[
|\epsilon_d| = \frac{\Gamma_N}{2\pi} \ln \left( \frac{2\Delta + b_0^2 \Gamma_N}{b_0^2 (\Gamma_N + \Gamma_S)} \right) + \frac{\Gamma_N + \Gamma_S}{2\pi} \ln \left( \frac{4W^2}{(2\Delta + b_0^2 \Gamma_N + b_0^2 \Gamma_S)} \right), \tag{37}
\]

where we neglect a term proportional to \( \Gamma_S \), but without any logarithmic factor. If \( \Delta \) is much smaller than the Kondo temperature which is given by \( T_K = \beta_0^2 \Gamma_N + b_0^2 \Gamma_S \), \( \Delta \) is negligible. One can then easily solve eq.(32) for \( b_0^2 \) and obtain the result for two normal leads with total tunneling rate \( \Gamma_N + \Gamma_S \):

\[
b_0^2 (\Gamma_N + \Gamma_S) = 2W \exp \left( -\frac{\pi |\epsilon_d|}{\Gamma_N + \Gamma_S} \right). \tag{38}
\]
In the opposite limit, where \( \Delta \) is much larger than \( T_K \), we find
\[
b_0^2 \sqrt{\Gamma_N^2 + \Gamma_S^2} = 2W \exp \left( -\pi \frac{|\epsilon_d| - (\Gamma_S/\pi \ln(W/\Delta))}{\Gamma_N} \right). \tag{39}\]

The results agree qualitatively with what we expect from scaling arguments for the Anderson model. In the perturbative regime, a logarithmic correction to \( \epsilon_d \) has been found (Haldane 1978). This applies to the case of a large gap, since scaling due to the superconducting electrons stops at energies of the order \( \Delta \), giving rise to a logarithmic renormalization of \( \epsilon_d \), as seen in eq.(39). In the case of a small gap, the superconducting lead contributes to scaling down to low energies, where one enters the strong coupling regime. Presumably, the fixed point is still reached for energies of the order of \( T_K \), much greater than \( \Delta \), so that the Kondo temperature does not depend on \( \Delta \), as indeed found in eq.(33). Notice that in the presence of normal electrons, we always find a non-trivial solution of the mean field equations. This is to be contrasted with the case of superconducting electrons only, \( \Gamma_N = 0 \), where for large gap only the solution \( b_0 = 0 \) exists, and there is no Kondo effect (Borkowski and Hirschfeld 1994).

In a non-equilibrium situation, when a voltage is applied between the leads, the mean field parameters cannot be obtained by minimizing the free energy. However the mean field equations (32,33) can also be derived using a self-consistent diagrammatic method (Millis and Lee 1987). Then it is straightforward to generalize to non-equilibrium. The equations read
\[
b_0^2 - i \int \frac{d\epsilon}{2\pi} \text{Tr} \left[ \hat{G}^< (\epsilon) \hat{\sigma}_z \right] = 0 \tag{40}\]
\[\lambda_0 b_0 - ib_0 \int \frac{d\epsilon}{2\pi} \text{Tr} \left[ \hat{G}^R (\epsilon) \hat{\Gamma}^< (\epsilon) + \hat{G}^< (\epsilon) \hat{\Gamma}^A (\epsilon) \right] = 0, \tag{41}\]
where the lesser Green’s function \( \hat{G}^< (t, t’) = i\langle \phi(t’) \phi(t) \rangle \) has been introduced, with \( \phi = (f_\uparrow, f_\downarrow) \). The lesser and advanced matrix \( \hat{\Gamma} \) is defined in analogy to its equilibrium version in eq.(31). To obtain \( \hat{G}^< \), we use the general relation \( \hat{G}^< = \hat{G}^R \hat{\Sigma}^< \hat{G}^A \), where at mean-field level \( \hat{\Sigma}^< = b_0^2 \hat{\Gamma} \) and
\[
\hat{\Gamma}^< (\epsilon) = -\sum_{\eta, k} |V_{\eta k}|^2 \hat{\sigma}_z \left[ \hat{g}^R_{\eta, k} (\epsilon) \hat{f}_\eta (\epsilon) - \hat{\bar{f}}_\eta (\epsilon) \hat{\bar{g}}^A_{\eta, k} (\epsilon) \right] \hat{\sigma}_z. \tag{42}\]

Note that the superconducting lead does not contribute to \( \Sigma^< (\epsilon) \) for \( |\epsilon| < \Delta \).

We have solved the mean-field equations in the presence of an external voltage numerically. As long as \( |eV| < T_K \) the solution is almost independent of the voltage. For large voltage, \( |eV| \gg T_K \), we have found that the Kondo peak is pinned to the chemical potential in the normal lead, i.e. \( \hat{\epsilon}_d \to \hat{\epsilon}_d - eV \), and the peak width is decreased.

The Andreev current can now be determined using the current formula eq.(17), with the dot Green’s function \( \hat{G} = b_0^2 \hat{\Gamma} \). The spectral function, which is defined as in eq.(28), is determined as
\[
G_{NS}(\epsilon) = \frac{4e^2}{h} \left( \frac{\Gamma_S}{\Delta} \right)^2 \frac{4(\Gamma_S \hat{\Gamma}_S)^2}{(4\epsilon^2 - 4\Gamma_S^2 - \Gamma_N^2 - \Gamma_S^2)^2 + 16\Gamma_N^2 \epsilon^2} \tag{43}\]

Here the renormalized tunneling rates \( \Gamma_{S, N} = b_0^2 \Gamma_{S, N} \), and \( \hat{\epsilon} = \epsilon(1 + b_0^2 \Gamma_S/2\Delta) \) are introduced. One recovers the current formula for a non-interacting quantum dot (Beenakker 1992), with renormalized parameters which are voltage dependent. On resonance, when \( \hat{\epsilon}_d \approx 0 \) and \( \epsilon = 0 \), the small renormalization factor \( b_0 \) drops out. The differential conductance becomes maximal when \( \hat{\Gamma}_N = \Gamma_S \) with \( G_{NS, max} = 4e^2/h \), twice the maximum for a N-QD-N system. For large voltage \( G_{NS} \) drops quickly, since the resonance moves away from zero energy, \( \hat{\epsilon}_d \approx |eV| \). \( G_{NS} \) as a function of energy is proportional to \( (eV)^{-2} \) near \( \epsilon \approx \pm \hat{\epsilon}_d \) and proportional to \( (eV)^{-4} \) near \( \epsilon = 0 \). As a consequence the current decreases with increasing voltage, leading to a negative differential conductance. This is also demonstrated in Fig. 5, where the current as a function of voltage obtained using \( G_{NS} \) of eq.(43) with the numerically determined, voltage dependent mean field parameters. The results were obtained with \( \hat{\epsilon}_d = W/3 \), \( \Gamma_N = \Gamma_S = 0.14W \), and \( \Delta = 0.2W \). For low voltages the current is, to good approximation, given by \( I = 4e^2V/h \), whereas the current drops when the voltage exceeds the Kondo temperature.

Finally, we want to comment on the reliability of our results. The success of slave-boson mean field theory stems from the fact that it captures the Fermi-liquid regime at low temperature. If the N-QD-S system scales to a Fermi liquid at low temperature, \( G_{NS} \) as given in eq.(43) is exact in the low temperature, low voltage limit. Since it is known that slave boson mean field theory has problems in describing dynamical properties, the results far away from equilibrium need to be treated with caution.

Within the Fermi-liquid point of view, the present mean field approach allows us to estimate the parameters entering eq.(13). In particular, we found that \( \Gamma_N \) and \( \Gamma_S \) renormalize equally, but this may no longer be the case when considering higher order corrections. For illustration, we estimate the effect of residual quasiparticle interaction in the limit \( \Delta \ll T_K \). By assuming an effective quasiparticle interaction of the form \( H_{\text{int}} = U n_i n_{\bar{i}} \), we find to first order in \( \hat{U} \) no corrections to \( \hat{\Gamma}_N \), while, as one could have expected, repulsive quasiparticle interaction suppresses the renormalized coupling to the superconductor, \( \hat{\Gamma}_S = b_0^2 \Gamma_S |1 - 4(\hat{U}/\pi T_K)(\Delta/T_K)\ln(T_K/2\Delta)| \).
V. CONCLUSIONS

We have studied Andreev tunneling in a normal metal-quantum dot-superconductor device, and obtained a general formula for the current through the device. We calculated the current voltage characteristics within the equation-of-motion approach, where we found a suppression of the conductance at low temperature. However it is known that this approach does not provide quantitative results in the Kondo regime. Therefore we extended the analysis to the extreme low temperature regime using slave boson mean field theory. In the regime when the superconducting gap is smaller than the Kondo temperature, we found an enhanced Andreev current at low bias voltage due to the Kondo effect. The zero bias conductance maximum condition may be achieved in an asymmetric QD where $\Gamma_S > \Gamma_N$.

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FIG. 2. The spectral density for the two particle tunneling $G_{NS}(\epsilon)$ is plotted for various bias voltages ($V = 0$ solid line, $V = -0.015$ dotted line, $V = 0.015$ dashed line, $\tilde{\epsilon}_d = -0.07$, $\Gamma_S = \Gamma_N = 0.02$ and $T = 0.0001$, $\Delta = 0.1$, in units of the bandwidth $W$). In the inset the same curves are shown in an extended scale.

FIG. 3. The differential conductance of the N-QD-S device, in units of $4e^2/h$, is plotted for different temperatures ($T = 0.0001$ solid line, $T = 0.001$ dotted line, $T = 0.01$ dot-dashed line, $\epsilon_d = -0.04$, $\Gamma_S = \Gamma_N = 0.02$, $\Delta = 0.1$, in units of the bandwidth $W$).

FIG. 4. Current voltage characteristics of a quantum dot at zero temperature as obtained within slave boson mean field theory. $\epsilon_d = 1/3$, $\Gamma_N = \Gamma_S = 0.14$, and $\Delta = 0.2$ in units of the bandwidth. The Kondo temperature in equilibrium is $T_K \approx b_0^2 \Gamma_N = 0.014$. 