THE ANISOTROPIC TRANSPORT EFFECTS ON DILUTE PLASMAS

EBRU DEVLEN
Department of Astronomy & Space Sciences, Faculty of Science, University of Ege, Bornova 35100, Izmir, Turkey; ebru.devlen@ege.edu.tr

ABSTRACT

We examine the linear stability analysis of a hot, dilute, and differentially rotating plasma by considering anisotropic transport effects. In dilute plasmas, the ion Larmor radius is small compared with its collisional mean free path. In this case, the transport of heat and momentum along the magnetic field lines becomes important. This paper presents a novel linear instability that may be more powerful and greater than ideal magnetorotational instability and ideal magnetorotational instability in the dilute astrophysical plasmas. This type of plasma is believed to be found in the intracluster medium (ICM) of galaxy clusters and radiatively inefficient accretion flows around black holes. We derive the dispersion relation of this instability and obtain the instability condition. There is at least one unstable mode that is independent of the temperature gradient direction for a helical magnetic field geometry. This novel instability is driven by the gyroviscosity coupled with differential rotation. Therefore, we call it gyroviscous-modified magnetorotational instability (GvMRI). We examine how the instability depends on signs of the temperature gradient and the gyroviscosity and also on the magnitude of the thermal frequency and on the values of the pitch angle. We provide a detailed physical interpretation of the obtained results. The GvMRI is applicable not only to the accretion flows and ICM but also to the transition region between cool dense gas and the hot low-density plasma in stellar coronae, accretion disks, and the multiphase interstellar medium because it is independent of the temperature gradient direction.

Key words: accretion, accretion disks – instabilities – magnetohydrodynamics (MHD) – plasmas

Online-only material: color figures

1. INTRODUCTION

In recent years, magnetorotational instability (MRI) has been recognized as a powerful source of angular momentum transport in accretion disks. An accretion disk with angular velocity decreasing outward and threaded by a weak magnetic field is linearly unstable (Balbus & Hawley 1991). Local and global simulations of Keplerian disks showed that the MRI leads to turbulence, transporting energy and angular momentum outward (Balbus 2003). Similarly, the dilute, stratified plasma is buoyantly unstable when the temperature increases in the direction of gravity. Heat is transported mainly along magnetic field lines in such a medium (Balbus 2000, 2001). The magnetothermal instability (MTI) has been studied with nonlinear simulations. Parrish & Stone (2005) investigated the nonlinear evolution of the MTI and showed that the instability causes turbulence and heat transport. They noted that the MTI may explain the almost isothermal temperature profile observed in the outer part of X-ray emitting regions in the intracluster medium (ICM) of galaxy clusters and the structure of radiatively inefficient accretion flows. In the MRI and MTI, the weak magnetic fields turn free-energy gradients into sources of instability.

If the plasma is sufficiently dilute, the viscous stress tensor is also anisotropic (Braginskii 1965). Balbus (2004) showed that the viscous stress tensor can cause a strong instability in the dilute astrophysical disks. He showed that the maximum growth rate of instability exceeds that of the MRI. However, he only took the parallel components of the stress tensor into account. The cause of the magnetoviscous instability is the same as that for MTI: Initially, magnetic field lines are isorotational (isothermal in the case of the MTI). Perturbed magnetic field lines are stretched out in the direction of the angular velocity (temperature) gradient. Thus, angular momentum (heat) is transferred from one fluid element to another. The fluid element at a smaller radius drops down to smaller radii and the other elements move to more distant radii. The field lines therefore become more bent and the process runs away (Islam & Balbus 2005).

Indeed, to understand the true nature of dilute astrophysical plasmas, we consider magnetohydrodynamic (MHD) equations that include terms describing transport of heat and momentum by thermal conduction and viscosity (Braginskii 1965). Dilute means that the ion Larmor radius ($r_{Li}$) is small compared with a mean free path ($\lambda_i$) and any macroscopic length scales in the plasma. This is tantamount to saying that the ion cyclotron frequency ($\omega_c$) is much greater than the ion–ion collision frequency ($\nu_i$). Under this condition, parallel heat conduction of electrons is much larger than that of ions by the factor ($m_i/m_e)^{1/2}$ and the parallel viscosity of ions is much larger than that of electrons by the same factor. Therefore, an anisotropic electron heat conduction and an anisotropic ion viscosity must be taken into account in the MHD equations. Ramos (2003) obtained dynamic evolution equations of the parallel heat fluxes in a collisionless magnetized plasma. He noted that neglecting the parallel heat fluxes in low collisional regimes cannot be justified physically. He emphasized that one must consider the contribution of the gyroviscosity in the stress tensor for consistency in the analysis. Ramos (2005) presented the fluid moment equations with the finite Larmor radius (FLR) effect for collisionless magnetized plasmas. His analysis included gyroviscous stress, pressure anisotropy, and anisotropic heat fluxes. He claimed that his formalism is applicable to arbitrary magnetic field geometry and plasma pressure as well as fully electromagnetic nonlinear dynamics. He extended a previous study considering collisional terms based on full Fokker–Planck operators for non-Maxwellian distribution functions. The low collisional regime of interest is described with two small parameters: The ratio of the electron to the ion masses is comparable to $\delta$ (i.e., $(m_e/m_i)^{1/2} \approx \delta \ll 1$) and the ratio of the ion collision to cyclotron frequencies is smaller than
\[ \delta^2 \text{ (i.e., } v_{i}/\omega_{ci} \lesssim \delta^2 \text{)}, \] where \( \delta \) is the fundamental expansion parameter, which is the ratio of the ion Larmor radius to the shortest macroscopic length scale (Ramos2007).

Recently, Ferraro (2007) examined the FLR effects on the MRI. He showed that the FLR effects are dominant compared with the other effects, which include the Hall effect in the limit of weak magnetic fields. He restricted his analysis to a vertical magnetic field geometry. He found that the growth rate of the unstable mode is around \( \Omega \) when the ratio of the gyroviscous force to the magnetic tension force is greater than zero. When the ratio of the gyroviscous force to the magnetic tension force is smaller than zero, there is no unstable mode. Devlen & Pekünlü (2010, hereafter Paper I) investigated the stability properties of weakly magnetized, dilute plasmas by considering combined effects of gyroviscosity and parallel viscosity, which are components of the stress tensor in the presence of a helical magnetic field geometry. They showed that although the parallel viscosity is greater than gyroviscosity under the condition of dilute plasma, it has no effect on the instability condition and growth rates. They also showed that the powerful instability emerges due to FLR effects. They estimated that the growth rates of this gyroviscous-modified magnetorotational instability (GvMRI) varied in the range of \( 0 \lesssim \Omega \lesssim 3 \Omega \) for the different values of pitch angles, the angle between the magnetic field vector and the \( \phi \) axis of the coordinate system. When the ratio of the gyroviscous force to the magnetic tension force is smaller than zero (i.e., in the case of \( \Omega \uparrow \downarrow B \)), they found that there are unstable modes with growth rates around \( 2 \Omega \). This result was contrary to that of Ferraro (2007).

To clearly comprehend the dynamics of dilute astrophysical plasmas, all the anisotropic transport effects must be taken into account. Therefore, in this work, we extend our previous study (Paper I) by considering the anisotropic electron heat conduction term. We show that weakly magnetized, differentially rotating dilute plasmas are unstable in the presence of the parallel viscosity, gyroviscosity, and thermal conduction. This novel instability is extremely powerful and occurs at all wavenumbers.

This paper is organized as follows. In the next section (Section 2), we give the linearization of the MHD equations used in our analysis, examine the physical structure of modes, and derive the dispersion relation of instability and the instability criterion. In Section 3, we examine the numerical solutions of the dimensionless dispersion relation. Finally, in Section 4, we discuss the physical interpretation of instability and summarize our results.

2. LINEAR STABILITY ANALYSIS

2.1. Dilute Plasma Properties

The plasma fulfilling the conditions \( r_{\text{L}} \ll \lambda \) and \( \epsilon \equiv \omega_{ci} \tau_{i} \gg 1 \) is called dilute plasma, where \( r_{\text{L}} \) is the ion Larmor radius, \( \lambda \) is the ion collision mean free path, \( \omega_{ci} \) is the ion cyclotron frequency, and \( \tau_{i} = 1/\nu_{i} \) is the inverse of the ion–ion collision frequency. The presence of the magnetic field introduces anisotropy to the medium. Cyclotron frequencies of the plasma species and the velocity gradients at macroscopic scales are the sources of anisotropy.

If plasma consists only of hydrogen, then \( \epsilon \) may be taken as (Islam & Balbus 2005)

\[ \epsilon = \left( \frac{1.09 \times 10^5}{n} \right) T_{4}^{3/2} B_{\text{G}} / \ln \Lambda, \tag{1} \]

where \( n \) is the proton number density in \( \text{cm}^{-3} \), \( T_{4} \) is the temperature in units of \( 10^{4} \text{ K} \), \( B_{\text{G}} \) is the magnetic field in units of \( 10^{-6} \), and \( \ln \Lambda \) is the Coulomb logarithm. The condition of \( \epsilon \gg 1 \) is fulfilled even in the presence of a very weak field with \( n \lesssim 1 \) and \( T_{4} \gtrsim 1 \).

Under these conditions, the plasma dynamics described by MHD equations should include the anisotropic terms accounting for the free flow of particles along the magnetic field lines (Braginskii 1965). Ion parallel viscosity is higher by a factor \( (m_{i}/m_{e})^{1/2} \) than that of electrons. So, the viscosity of the dilute plasma is determined mainly by the ions. Even if the ion viscosity is very small, in a rotating system, it may become very important (Balbus 2004). Similarly, since the electron contribution to the heat flux is higher than the ion contribution by a factor of \( (m_{i}/m_{e})^{1/2} \), the ion contribution may be considered as negligible. Since the electrons have mean free paths much longer than their gyro-radii in the dilute plasma, the thermal conductivity is strongly anisotropic. That is, in astrophysical dilute plasma threaded by even a weak magnetic field, the momentum by ions and heat flux by electrons is transported primarily along the magnetic field lines.

2.2. Basic Equations

In an attempt to investigate parallel viscosity, gyroviscosity, and heat flux in a dilute plasma, one should consider the two-fluid equations. Below are the standard extended MHD equations that are obtained by using two-fluid equations including stress tensor \( \Pi \) and the heat flux \( Q \) (see Appendix A):

\[
\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0, \tag{2}
\]

\[
\rho \frac{d\mathbf{v}}{dt} = -\nabla P - \rho \mathbf{g} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{c} + \rho \mathbf{g}, \tag{3}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{4}
\]

\[
\frac{dP}{dt} + \frac{5}{3} P (\nabla \cdot \mathbf{v}) = -\frac{2}{3} \nabla \cdot (\mathbf{Q}). \tag{5}
\]

where \( \rho \) is the mass density, \( \mathbf{v} \) is the fluid velocity, \( P \) is the scalar pressure, \( \Pi \) is the stress tensor, \( \mathbf{B} \) is the magnetic field, \( \mathbf{g} \) is the gravitational acceleration, \( \mathbf{Q} \) is the heat flux, and \( d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla \) is a Lagrangian derivative.
Stress tensors have three components: parallel (\(\|\)), perpendicular (\(\perp\)), and the gyroviscous (\(gv\)) (Braginskii 1965). Perpendicular viscosity is greater than parallel viscosity by a factor of \((r_{Li}/l)^2\), where \(r_{Li}\) is the Larmor radius and \(l\) is the mean free path of the particles in a dilute plasma, and therefore it may not be taken into account. We use parallel and gyroviscous components of the stress tensor, which are given by

\[
\Pi^v = 0.96 \frac{P_i}{2v_i} (I - 3\hat{b}\hat{b}) \delta v \cdot \hat{b},
\]

\[
\Pi^{gv} = \frac{P_i}{4\omega_c} (\hat{b} \times W \cdot (I + 3\hat{b}\hat{b}) + \delta W \cdot (\hat{b} \times W \cdot (I + 3\hat{b}\hat{b}))^T),
\]

where \(\hat{b} = B/B, \omega_c = eB/m_i c\) are the unit vectors along the magnetic field and the cyclotron frequency, respectively, \(v_i\) is the ion collision frequency, and \(W = \nabla v + (\nabla v)^T - 2/3I (\nabla \cdot v)\) is the rate of strain tensor.

In a dilute astrophysical plasma, heat flux \(Q\) is dominantly along the magnetic field lines. The parallel heat flux is given by

\[
Q = -\chi_C \delta v \cdot \hat{b} \nabla T,
\]

where \(\chi_C\) is the Coulomb conductivity given by Balbus (2001) referring to Spitzer (1962) as \(\chi_C \approx 6 \times 10^{-7} T^{5/2} \text{ erg cm}^{-1} \text{ K}^{-1}\).

2.3. Linearized Expressions for Perturbed Quantities

We apply a standard Wentzel–Kramers–Brillouin (WKB) perturbation analysis on the equilibrium state. To this analysis, all the variables in the MHD equations are denoted by sums of an equilibrium value (denoted with a “0” subscript) and a small perturbed quantity (denoted with \(\delta\)):

\[
\rho = \rho_0 + \delta \rho,
\]

\[
v = v_0 + \delta v,
\]

\[
P = P_0 + \delta P,
\]

\[
\Pi = \Pi_0 + \delta \Pi,
\]

\[
B = B_0 + \delta B,
\]

\[
Q = Q_0 + \delta Q.
\]

Substituting the formulae in Equation (9) into Equations (2)–(8) and retaining only terms up to linear order in perturbations, the linearized perturbation equations are obtained as

\[
\nabla \cdot \delta v = 0,
\]

\[
\frac{\partial \delta v}{\partial t} + \delta v \cdot \nabla v = \frac{\delta \rho}{\rho} \nabla P - \nabla \cdot \delta \Pi - \frac{1}{\rho} \nabla \left( \delta P + \frac{\delta B \cdot B}{4\pi \rho} \right) + \frac{B \cdot \nabla}{4\pi \rho} \delta B,
\]

\[
\frac{\partial \delta B}{\partial t} = \nabla \times (\delta v \times B) + \nabla \times (v \times \delta B),
\]

\[
5 \frac{\partial}{\partial t} \frac{\delta \rho}{\rho} - \delta v \cdot \nabla \ln P \rho^{-5/3} = \frac{2}{3} \nabla \cdot \delta Q.
\]

The perturbed parallel and gyroviscous components of the stress tensor are

\[
\delta \Pi^v = 0.96 \frac{P_i}{2v_i} \left[ (I - 3\hat{b}\hat{b})(\delta v \cdot \hat{b}) + (I - 3\hat{b}\hat{b})(\delta W \cdot \hat{b}) \right],
\]

\[
\delta \Pi^{gv} = \frac{P_i}{4\omega_c} \left[ \left( \hat{b} \times W \cdot (I + 3\hat{b}\hat{b}) + \delta W \cdot (\hat{b} \times W \cdot (I + 3\hat{b}\hat{b}))^T \right) \right].
\]

The perturbed heat flux is given by

\[
\delta Q = -\chi_C [\hat{b}(\delta v \cdot \hat{b})T - i\delta v(\delta B/B)\delta T],
\]

The perturbed unit vector of the magnetic field is given by \(\delta \hat{b} = \delta (B/B) = \delta B/B - \hat{b}(\delta B/B)\).
2.4. The Physical Structure of Modes

Before we obtain the dispersion relation, which includes all anisotropic transport effects for general axisymmetric disturbances, let us take a glance at the physical meaning of the FLR effect and modes that emerge in the plasma. We consider the local stability of a uniformly rotating dilute plasma including only a vertical magnetic field, \( \mathbf{B} = B\hat{z} \). We ignore parallel viscosity, heat flux, and radial stratification. We restrict our study to plane wave perturbations that depend only on \( z \), i.e., of the form \( \exp(ikz - i\omega t) \). Thus, the radial and azimuthal components of the linearized motion equation are obtained as

\[
-i\omega \delta v_R - (2\Omega - 4V_{gyro})\delta v_\phi - \frac{i k B}{4\pi \rho} \delta B_R = 0,
\]

\[
-i\omega \delta v_\phi + (2\Omega - 4V_{gyro})\delta v_R - \frac{i k B}{4\pi \rho} \delta B_\phi = 0,
\]

where \( V_{gyro} = k^2 P/4\omega_{ci}\rho \) is the inverse timescale of the gyroviscous stress. The same components of the linearized magnetic induction equation are

\[
-i\omega \delta B_R - ikB \delta v_R = 0,
\]

\[
-i\omega \delta B_\phi - ikB \delta v_\phi = 0.
\]

The gyroviscous force introduces a term like a Coriolis term in the equation of motion. The dispersion relation is

\[
\omega^4 - \omega^2 \left[ 2k^2 v_A^2 + 4\Omega^2 \left( 1 - \frac{k^2 v_A^2}{\Omega^2} \right)^2 \right] + k^4 v_A^4 = 0,
\]

where \( v_A^2 = B^2/4\pi \rho \) is the Alfvén velocity and \( v_D^2 = P\Omega/2\omega_{ci}\rho \) is the drift velocity.

FLR stress results from changes in particle drift velocities across a gyro-orbit. This stress gives rise to distortions of particle orbits and guiding-center drift. Kaufman (1960) presented a detailed discussion of this stress. Because ions and electrons have different Larmor radii, they move differently due to FLR effects. This difference in motion gives rise to charge separation, which produces a finite parallel electric field. Physically, the FLR effects introduce a drift wave that convects the perturbations along the velocity gradient.

If the angular velocity and magnetic field vectors are oriented in the same sense, the term including \( V_{gyro} \) is positive. From Equations (17)–(20), one can easily see that the induced drift motion is opposite to the Coriolis force. Thus, the dynamic epicycle is slowed and magnetic tension force is effectively increased. This, in turn, increases the angular momentum transfer. The result is an instability. If the angular velocity and magnetic field vectors are counteraligned, the signs of these effects should reverse.

In the limit \( \Omega \to 0 \), the dispersion relation is obtained as

\[
\omega^4 - \omega^2 \left[ 2k^2 v_A^2 + k^4 \left( \frac{Pmc}{eB\rho} \right)^2 \right] + k^4 v_A^4 = 0.
\]

The solutions of Equation (22) give two roots:

\[
\omega^2 = \frac{1}{2} \left[ 2k^2 v_A^2 + k^4 \left( \frac{Pmc}{eB\rho} \right)^2 \right] \pm \frac{1}{2} \left[ k^4 \left( \frac{Pmc}{eB\rho} \right)^2 \left( 4k^2 v_A^2 + k^4 \left( \frac{Pmc}{eB\rho} \right)^2 \right) \right]^{1/2}.
\]

One of the roots describes pure drift mode at large wavenumbers. For small wavenumbers (low frequencies) the other root is

\[
\omega^2 = k^2 v_A^2 \left( 1 \pm k^2 \frac{Pmc}{eB\rho} \right),
\]

corresponding to Alfvén waves, with the gyroviscous force producing a small frequency-splitting of the Alfvén wave (see Figure 1).

One may rewrite the dispersion relation (21) in the presence of the uniform rotation as follows:

\[
\omega^2 \pm \omega_{\phi,}\left( 1 - \frac{4\omega_D^2}{\omega_I^2} \right) - \omega_A^2 = 0,
\]

where drift wave frequency is \( \omega_D = Pmc(k \cdot \Omega)^2/2e\rho(\mathbf{\Omega} \cdot \mathbf{B}) \), Alfvén wave frequency is \( \omega_A = (k \cdot \mathbf{B})/4\pi \rho \), and pure inertial wave frequency is \( \omega_I = 2(k \cdot \mathbf{\Omega})/k \) for general magnetic field geometries and wavenumbers (Moffatt 1978). The solution for Equation (25) is

\[
\omega = \pm \frac{1}{2} \omega_I \left( 1 - \frac{4\omega_D^2}{\omega_I^2} \right) \pm \frac{1}{2} \left[ \omega_I^2 \left( 1 - \frac{4\omega_D^2}{\omega_I^2} \right) + 4\omega_A^2 \right]^{1/2}.
\]
For \((\mathbf{k} \cdot \Omega) = 0\), it follows that \(\omega_+ = \omega_- = \omega_A\). If the first term in the brackets in Equation (26) is greater than the second one, then it is possible to carry out a Taylor series expansion of this equation. One then finds a very clear splitting of the fast and slow wave frequencies:

\[
\omega_+ = \pm \omega_I \left( 1 - \frac{4\omega_D^2}{\omega_I^2} \right) \left( 1 + \frac{\omega_A^2}{\omega_I^2 \left( 1 - \frac{4\omega_D^2}{\omega_I^2} \right)^2} \right) \tag{27}
\]

and

\[
\omega_- = \pm \frac{\omega_A^2}{\omega_I \left( 1 - \frac{4\omega_D^2}{\omega_I^2} \right)} \tag{28}
\]

respectively. These waves result from a combination of the inertial and Alfvén waves. They are referred to as magnetocoriolis (MC) waves. When the magnetic field and rotation axis are aligned, the MC wave that imparts a circularly polarized component to the velocity perturbation is in the counterclockwise direction. Therefore, the Coriolis and Lorentz forces are in phase. The resulting force causes inertial acceleration. This mode is known as the fast MC wave (i.e., \(\omega_+\) mode). In Equations (27) and (28), the inertial wave is coupled to the drift mode. This mode has a larger frequency compared with that of the pure fast MC wave. For the \(\omega_-\) mode, the MC wave that imparts a circularly polarized component to the velocity perturbation is in the clockwise direction. Hence, the Coriolis and Lorentz forces are out of phase and the resulting force is weakened. This slow MC mode is sometimes referred to as a magnetostrophic wave or hydromagnetic-inertial wave (see Figure 2). These waves are especially important in the dynamo problem (Moffatt 1978; Acheson & Hide 1973).

When the first term in the brackets in Equation (26) is smaller than the second one, one obtains modified Alfvén waves with frequency splitting after Taylor series expansion:

\[
\omega = \pm \omega_A \left( 1 \pm \frac{\omega_I}{2\omega_A} \left( 1 - \frac{4\omega_D^2}{\omega_I^2} \right) \right). \tag{29}
\]

A wavenumber–frequency diagram of these waves is given in Figure 3. In general, the angular velocity and magnetic field vectors will not be parallel and the situation will be more complex. However, the physical picture remains the same.

### 2.5. Dispersion Relation with All Anisotropic Transport Effects

We now consider the axisymmetric behavior of the instability for the helical magnetic field and more general wavenumbers.

#### 2.5.1. Equilibrium State

We work in a cylindrical coordinate system \((R, \phi, z)\). The plasma is assumed to be thermally stratified in the presence of a uniform gravitational field in the radial direction, \(\mathbf{g} = -g \hat{R}\). The weak magnetic field is taken to be \(\mathbf{B} = (0, B_0 \cos \theta, B_0 \sin \theta)\), where \(\theta = \tan^{-1}(B_z/B_\phi)\), called “pitch angle.” In equilibrium the field lines are assumed to be isothermal and therefore heat flux is negligible. We consider differentially rotating plasma with a Keplerian velocity profile, i.e., \(v_\phi = R \Omega(R)\). For this profile and a weak magnetic field, the stress tensor is negligibly small in the equilibrium, for an arbitrary pitch angle (see Appendix B). Hence, plasma is in hydrostatic equilibrium:

\[
\frac{\nabla P_0}{\rho_0} = \mathbf{g} + R\Omega^2. \tag{30}
\]
2.5.2. General Axisymmetric Disturbances

All the perturbed quantities are assumed to have a space–time dependence \( \exp(i \mathbf{k} \cdot \mathbf{r} + \omega t) \), where \( \mathbf{k} = k_R \hat{R} + k_z \hat{z} \). WKB assumption requires \( k_R \gg 1 \). The time dependence of the perturbations is assumed as \( \exp(\omega t) \). This assumption ensures that all coefficients in the dispersion relation are real. We work in the Boussinesq limit. In this limit, pressure changes are much smaller than temperature and density changes, i.e., \( \delta T = -\frac{T}{\rho} \delta \rho \).

The above equations may be written in explicit component form. The perturbation equation of mass continuity is given by

\[
k_R \delta v_R + k_z \delta v_z = 0. \tag{31}
\]

The radial, azimuthal, and axial components of the linearized momentum conservation equation are given by Equations (32), (33), and (34), respectively:

\[
\begin{align*}
\omega \delta v_R - 2 \Omega \delta v_\phi - \frac{\delta \rho}{\rho^2} \frac{\partial P}{\partial R} + i k_R \frac{\delta P}{\rho} + \frac{1}{4 \pi \rho} i k_R (B_\phi \delta B_\phi + B_z \delta B_z) - \frac{1}{4 \pi \rho} i k_z B_z \delta B_R \\
&= - V_{\text{par}} \left[ \frac{d \Omega}{d \ln R} \frac{k_R}{k_z} \sin \theta \delta v_R + \frac{k_R}{k_z} \sin \theta \delta v_\phi + 2 \frac{k_R}{k_z} \sin^2 \theta \delta v_z \right] \\
&- V_{\text{gyro}} \left[ 2 \sin \theta \frac{d \Omega}{d \ln R} \frac{1}{k_z^2} \left( k_R^2 - 1 \right) \delta v_R + \left( 2 \cos \theta \frac{k_R^2}{k_z^2} + E \right) \delta v_z - \left( \frac{2 k_R^2}{k_z^2} \sin \theta + A \right) \delta v_\phi \right] = 0, \tag{32}
\end{align*}
\]
\[ \frac{\omega \delta v_R + \kappa^2}{2\Omega} \delta v_R - \frac{1}{4\pi \rho} ik_z (B, \delta B) + V_{\text{par}} 2D \left[ \frac{d \Omega}{d ln R} \frac{1}{\omega} \cos \theta \delta v_R + \cos \theta \delta v_\rho + \sin \theta \delta v_\varphi \right] \]

\[ + V_{\text{gyro}} \left[ -2D \frac{d \Omega}{d ln R} \frac{1}{\omega} \delta v_R + \frac{d \Omega}{d ln R} + B \delta v_\varphi + \left( \frac{d \Omega}{d ln R} \right)^2 \frac{1}{\omega^2} B - A + 2 \sin \theta \frac{k^2}{k^2} \right] \delta v_\varphi = 0, \quad (33) \]

\[ \omega \delta v_R + ik_z \frac{\delta P}{\rho} + \frac{1}{4\pi \rho} ik_z B \delta B + V_{\text{par}} \left[ \frac{F}{\rho} \frac{d \Omega}{d ln R} \frac{1}{\omega} \delta v_R + F \delta v_\varphi + C \delta v_\varphi \right] \]

\[ + V_{\text{gyro}} \left[ \left( \frac{4k^2}{k^2} \cos \theta + E + 4D \left( \frac{d \Omega}{d ln R} \right)^2 \frac{1}{\omega^2} \right) \delta v_R + \left( \frac{4k^2}{k^2} \sin \theta + 4D \frac{d \Omega}{d ln R} \frac{1}{\omega} \right) \delta v_\varphi \right] = 0. \quad (34) \]

Similarly, the radial, azimuthal, and axial components of the linearized magnetic induction equation are given by Equations (35), (36), and (37), respectively:

\[ \omega \delta B_R - ik_z B \delta v_R = 0, \quad (35) \]

\[ \omega \delta B_\theta - ik_z B \delta v_\theta - \frac{d \Omega}{d ln R} \delta B_R = 0, \quad (36) \]

\[ \omega \delta B_\varphi - ik_z B \delta v_\varphi = 0. \quad (37) \]

Finally, the linearized energy equation is

\[ \frac{\delta \rho}{\rho} \left( \omega + V_{\text{ther}} \sin^2 \theta \right) - \delta v_R \left( \frac{3}{5} \frac{d ln R \rho^{-5/3}}{d R} + V_{\text{ther}} \frac{1}{\omega} \sin^2 \theta \frac{d ln T}{d R} \right) = 0, \quad (38) \]

where \( V_{\text{par}} = 0.96k^2/2\nu \) is the inverse timescale of the dissipation due to parallel viscosity, \( V_{\text{gyro}} = k^2 P/4\omega_{ci} \rho \) is the inverse timescale of the gyroviscous stress, and \( V_{\text{ther}} = 2k^2 \chi T/5P \) is the inverse timescale of the dissipation due to thermal conductivity or thermal frequency. \( \kappa^2 \) is epicyclic frequency. Other constants which depend on pitch angle \( \theta \) are \( A = \sin \theta(1 - 3 \cos 2\theta) \), \( B = \sin \theta(1 + 9 \cos^2 \theta - 3 \sin^2 \theta) \), \( C = 2 \sin^2 \theta(1 + 3 \sin^2 \theta) \), \( D = 3 \sin^2 \theta \cos \theta \), \( E = 2 \cos \theta(1 + 3 \sin^2 \theta) \), \( F = 2 \sin \theta D - \sin 2\theta \), \( G = \sin \theta(1 + 3 \cos 2\theta) \), and \( H = 3 \sin \theta \cos^2 \theta \).

### 2.5.3. Dispersion Relation

The set of Equations (31)–(38) are reduced to form \( M \cdot \delta v = 0 \), where \( M \) is a \( 3 \times 3 \) matrix. \( |M| = 0 \) gives nontrivial solution. Thus, the dispersion relation is obtained as

\[ \omega^5 + a_4 \omega^4 + a_3 \omega^3 + a_2 \omega^2 + a_1 \omega + a_0 = 0, \quad (39) \]

where

\[ a_4 = 6V_{\text{par}} s^2 \frac{k^2}{k^2} + V_{\text{ther}} s^2, \quad (40) \]

\[ a_3 = 2k^2 v^2_{v_A} + k^2 s^2 \frac{k^2}{k^2} + k^2 V_{\text{gyro}} \Omega^2(2s^2 - A^2) + V_{\text{gyro}} \Omega(G + H) + 4\Omega^2 \frac{k^2}{k^2} V_{\text{gyro}}(2s^2 - A) + N^2 \frac{k^2}{k^2} + 6V_{\text{par}} V_{\text{ther}} s^2 \frac{k^2}{k^2}, \quad (41) \]

\[ a_2 = 6V_{\text{par}} s^2 k^2 v^2_{v_A} + 2k^2 V_{\text{gyro}} s^2 G \Omega^2 \frac{k^2}{k^2} + k^2 V_{\text{par}} \Omega^2 c^2 D + k^2 V_{\text{gyro}} 6s \Omega D + (V_{\text{gyro}})^2 3y \Omega D(2s^2 - A) \Omega^2 \frac{k^2}{k^2} \]

\[ + \frac{k^2}{k^2} V_{\text{par}} N^2 2c D - \frac{k^2}{k^2} V_{\text{ther}} s^2 \frac{1}{\rho} \frac{d P}{d R} \frac{d ln T}{d R} + V_{\text{ther}} s^2 \left[ \frac{2k^2}{k^2} V_{v_A} + \frac{k^2}{k^2} \frac{k^2}{k^2} V_{\text{gyro}} \Omega^2(2s^2 - A) + V_{\text{gyro}} \Omega(G + H) \right], \quad (42) \]

\[ a_1 = k^2 V_{\text{gyro}} 2D^2(\Omega)^2 + \left[ k^2 v^2_{v_A} + V_{\text{gyro}} \Omega(G/2 + H) \right] \left[ k^2 v^2_{v_A} + V_{\text{gyro}} \Omega(G/2 + H) \right] + k^2 N^2 \left[ k^2 v^2_{v_A} + V_{\text{gyro}} \Omega \left( \frac{G}{2} + H \right) \right] \]

\[ - \frac{k^2}{k^2} V_{\text{ther}} s^2 V_{\text{par}} 2c D \frac{1}{\rho} \frac{d P}{d R} \frac{d ln T}{d R} + V_{\text{ther}} s^2 \left[ 6V_{\text{par}} s^2 k^2 v^2_{v_A} + V_{\text{par}} \Omega^2 s^2 G \Omega^2 \frac{k^2}{k^2} + \frac{k^2}{k^2} V_{\text{par}} \Omega^2 c^2 D + k^2 V_{\text{gyro}} 6s \Omega D \right] \]

\[ + (V_{\text{gyro}})^2 3y \Omega D(2s^2 - A) \Omega^2 \frac{k^2}{k^2}, \quad (43) \]
in the cooling flow clusters, GvMRI may operate. dwarfs and neutron stars, hot accretion flows on compact objects release the gravitational potential energy and cause currents produced by

\[ \gamma = \frac{k_R}{k_o}, \quad \mathcal{Q} = d\Omega^2/d\ln R, \quad s = \sin \theta, \quad c = \cos \theta, \quad k_0^2 = k_R^2 + k_o^2 \cos^2 \theta, \quad \text{and} \quad \tilde{V}_{\text{gyro}} = k_R^2 P_{\text{f}}/4\Omega_{\text{osc}}. \quad N^2 = (3/5\rho) (\partial P/\partial R)(\partial \ln P \rho^{-5/3}/\partial R) \text{ is the Brunt–Vaisala frequency.} 

The above dispersion relation is reduced to several previously obtained relations in the appropriate limits. Taking \( V_{\text{par}} = \tilde{V}_{\text{gyro}} = V_{\text{ther}} = \theta = N^2 = 0 \), one recovers the result of Balbus \& Hawley (1991). Taking \( V_{\text{gyro}} = V_{\text{ther}} = \theta = N^2 = 0 \), one recovers the result of Islam \& Balbus (2005). Taking \( N^2 = 0 \) and \( V_{\text{par}} = 0 \), one recovers the dispersion relation given in Paper I. If one sets \( V_{\text{par}} = \tilde{V}_{\text{gyro}} = \theta = 0 \), after some algebra, one recovers the dispersion relations given by Balbus (2001).

2.6. Instability Criterion

The solution of Equation (39) gives five modes that exist in weakly magnetized, dilute plasmas. By analysis of the Routh–Hurwitz criterion, the instability criterion is given by

\[
a_0 = V_{\text{ther}}^2 \left[ k_o^2 v_A^2 + \tilde{V}_{\text{gyro}} \mathcal{Q} \left( \frac{G}{2} + H \right) \right] \left[ k_o^2 v_A^2 + \tilde{V}_{\text{gyro}} \mathcal{Q} \left( \frac{G}{2} + H \right) \right] < 0. \tag{44}
\]

This complex criterion is simplified if one considers a vertical magnetic field. In this case, \( \theta = 90^\circ \), \( D = 0 \), \( H = 0 \), \( G = -2 \), and the instability criterion is reduced to

\[
a_0 = \left[ k_o^2 v_A^2 - \tilde{V}_{\text{gyro}} \frac{d\Omega^2}{d\ln R} \right] \left[ k_o^2 v_A^2 - \tilde{V}_{\text{gyro}} \frac{d\Omega^2}{d\ln R} \right] < 0. \tag{45}
\]

In the absence of gyroviscosity and heat conduction, this criterion is reduced to the ideal MRI one; in the absence of gyroviscosity only, it is reduced to the ideal MTI one. As seen from the instability criterion in Inequality (46), any dynamic instability appears to be dependent on the signs of the angular velocity, temperature gradient, and gyroviscous force, which are the free energy sources. Departures from uniform rotation and isothermality are indeed sources of dynamic instability. Gyroviscous force is coupled with an angular velocity gradient. If one refers to the definition of \( \tilde{V}_{\text{gyro}} \), one clearly sees that gyroviscous force stabilizes or destabilizes the modes depending on the sign of \( \mathbf{\Omega} \cdot \mathbf{B} \).

We consider the case of astrophysical interest, \( d\Omega^2/d\ln R < 0 \). If \( \mathbf{\Omega} \) and \( \mathbf{B} \) are aligned in the same direction, i.e., \( \mathbf{\Omega} \cdot \mathbf{B} > 0 \), the torque term that is the first factor of Inequality (46) is positive. The instability condition is determined by the sign of the radial force term (second factor) in Inequality (46); that is, if the second factor is negative then instability arises:

\[
k_o^2 v_A^2 - \tilde{V}_{\text{gyro}} \frac{d\Omega^2}{d\ln R} = -k_o^2 \frac{1}{k_o^2} \frac{\partial P}{\partial R} \ln T - \tilde{V}_{\text{gyro}} \frac{d\Omega^2}{d\ln R} < 0. \tag{47}
\]

This condition differs from the ideal MRI only by additional terms, i.e., gyroviscous and temperature gradient on the left-hand side of Inequality (47). To the ideal MRI, as long as \( d\Omega^2/d\ln R < 0 \) there will be an instability for a small enough \( k \) (Balbus \& Hawley 1998).

The gyroviscous force acts in the same direction with the magnetic tension force because of \( \mathbf{\Omega} \cdot \mathbf{B} > 0 \). Therefore, the instability is suppressed because the currents produced by \( \mathbf{E} \times \mathbf{B} \) drift are out of phase with the current of MRI eigenmode. In the astrophysical situation \( \partial \ln T/\partial R < 0 \) because of the hydrostatic equilibrium. If one assumes that the temperature decreases in the direction of gravity, i.e., \( \partial \ln T/\partial R > 0 \), then the dilute plasma is completely stable. If one assumes that the temperature increases in the direction of gravity, i.e., \( \partial \ln T/\partial R < 0 \), then the dilute plasma may be unstable only if the temperature gradient is very steep.

But in the situation where \( \mathbf{\Omega} \) and \( \mathbf{B} \), are counteraligned, i.e., \( \mathbf{\Omega} \cdot \mathbf{B} < 0 \), the gyroviscous force acts in the opposite direction of the magnetic tension force and enhances instability. Thus, one may find unstable modes even if \( \partial \ln T/\partial R > 0 \); this is because the currents produced by \( \mathbf{E} \times \mathbf{B} \) drift are in phase with the current of MRI eigenmode. If one assumes that the temperature increases in the direction of gravity, i.e., \( \partial \ln T/\partial R < 0 \), then MTI arises. In this case, GvMRI and MTI have equal weights for dominance. It is expected that, especially at higher \( k \) values, the former is the dominant one. In many astrophysical plasmas, like cooling white dwarfs and neutron stars, hot accretion flows on compact objects release the gravitational potential energy and cause \( \partial \ln T/\partial R < 0 \).

In plasmas having a temperature profile in which there are no unstable modes to prompt ideal MTI, i.e., \( \partial \ln T/\partial R > 0 \), for example, in the cooling flow clusters, GvMRI may operate.

By referring to the general instability criterion shown in Inequality (45), one can argue that perturbations with \( k_R \) always stabilize because of the last term in Inequality (45). Interpretation of the criterion becomes difficult if we take into account pitch angles different from \( 90^\circ \). In such cases, the signs and the ratios of the different terms come into play. Therefore, it is more instructive to look at the numerical solutions of the dispersion relation.

3. NUMERICAL SOLUTIONS OF THE DIMENSIONLESS DISPERSION RELATION

The dimensionless dispersion relation is obtained when all the terms of Equation (39) are divided by \( \Omega^5 \):

\[
\gamma^5 + b_4 \gamma^4 + b_3 \gamma^3 + b_2 \gamma^2 + b_1 \gamma + b_0 = 0, \tag{48}
\]
where

\[
b_4 = 6 \tilde{v}_n \nu_i X s^2 \frac{k_f^2}{k^2} + \tilde{v}_n X s^2,
\]

\[
b_3 = \tilde{v}_n \nu_i X d \ln \Omega^2 \left( \frac{G + H}{d \ln R} \right) + \frac{k_f^2}{k^2} \frac{\tilde{v}_n \nu_i}{k^2} X (2s y^2 - A) + 2X + \frac{k_f^2}{k^2} \frac{\tilde{v}_n \nu_i}{k^2} (\tilde{v}_n \nu_i)^2 X^2 (2s y^2 - A)^2 + \tilde{v}_n \nu_i X^2 s^4 \frac{k_f^2}{k^2},
\]

\[
b_2 = 6 \tilde{v}_n \nu_i \nu_{gyro} X^3 \frac{k_f^2}{k^2} + \tilde{v}_n \nu_{gyro} X^2 3s^2 G \frac{d \ln \Omega^2}{d \ln R} \frac{k_f^2}{k^2} + \tilde{v}_n \nu_i X \frac{d \ln \Omega^2}{d \ln R} 2c D + \frac{k_f^2}{k^2} \tilde{v}_n \nu_{gyro} 6y XD \frac{d \ln \Omega^2}{d \ln R}
\]

\[
+ (\tilde{v}_n \nu_{gyro})^2 X^2 3y D (2s y^2 - A) \frac{d \ln \Omega^2}{d \ln R} \frac{k_f^2}{k^2} \tilde{v}_n \nu_i X s^2 P_M T
\]

\[
+ \tilde{v}_n \nu_i X s^2 \left[ 2X + \frac{k_f^2}{k^2} \frac{\tilde{v}_n \nu_i}{k^2} (\tilde{v}_n \nu_i)^2 X^2 (2s y^2 - A)^2 + \tilde{v}_n \nu_i X^2 d \ln \Omega^2 \left( \frac{G + H}{d \ln R} \right) \right]
\]

\[
+ 4 \frac{k_f^2}{k^2} \tilde{v}_n \nu_{gyro} (2s y^2 - A)
\]

\[
b_1 = \left[ X \left( 1 + \tilde{v}_n \nu_i \frac{d \ln \Omega^2}{d \ln R} \left( \frac{G}{d \ln R} \right) + \frac{k_f^2}{k^2} \frac{\tilde{v}_n \nu_i}{k^2} \frac{d \ln \Omega^2}{d \ln R} \right) \right]
\]

\[
+ \frac{k_f^2}{k^2} \frac{\tilde{v}_n \nu_i}{k^2} \frac{d \ln \Omega^2}{d \ln R} \left[ X + \tilde{v}_n \nu_i \nu_{gyro} X^2 2D \frac{d \ln \Omega^2}{d \ln R} \frac{k_f^2}{k^2} \frac{\tilde{v}_n \nu_i}{k^2} \frac{d \ln \Omega^2}{d \ln R} \left( \frac{G}{d \ln R} \right) + \tilde{v}_n \nu_i X \frac{d \ln \Omega^2}{d \ln R} \right]
\]

\[
+ \frac{k_f^2}{k^2} \frac{\tilde{v}_n \nu_i}{k^2} \frac{d \ln \Omega^2}{d \ln R} \left[ \tilde{v}_n \nu_i \frac{d \ln \Omega^2}{d \ln R} \right]
\]

\[
b_0 = \tilde{v}_n \nu_i X s^2 \left[ X + \tilde{v}_n \nu_i \frac{d \ln \Omega^2}{d \ln R} \left( \frac{G}{d \ln R} \right) \right]
\]

\[
+ \frac{k_f^2}{k^2} \frac{\tilde{v}_n \nu_i}{k^2} \frac{d \ln \Omega^2}{d \ln R} \left[ \tilde{v}_n \nu_i \frac{d \ln \Omega^2}{d \ln R} \right]
\]

\[
\gamma = \omega / \Omega, \quad X = \frac{k_f^2}{k^2} \frac{\tilde{v}_n \nu_i}{k^2} \frac{\tilde{v}_n \nu_{gyro}}{X}, \quad \tilde{v}_n \nu_i = \tilde{v}_n \nu_{gyro} / X, \quad \tilde{v}_n \nu_i = \tilde{v}_n \nu_{gyro} / X, \quad \tilde{v}_n \nu_i = \tilde{v}_n \nu_{gyro} / X, \quad \tilde{v}_n \nu_i = \tilde{v}_n \nu_{gyro} / X, \quad \tilde{v}_n \nu_i = \tilde{v}_n \nu_{gyro} / X.
\]

The dilute plasma condition can be expressed as \( \epsilon \equiv \omega \alpha \nu_i \gg 1 \). From this condition one can easily derive the inequality \( \tilde{v}_n \nu_i \gg \tilde{v}_n \nu_{gyro} \). Therefore, we assume \( \tilde{v}_n \nu_i = 1000 \). We consider the convectively stable plasmas, i.e., \( \tilde{N}^2 > 0 \) and suppose \( P_M = -1 \).

Figure 4 shows that there is an instability in small wavenumbers for all pitch angle values when only heat conduction is considered. Maximum growth rate of the instability is smaller than growth rate of an ideal MRI (0.75 \( \Omega \)). Indeed, this instability is an MTI that emerges in the presence of the helical magnetic geometry. Although the plasma with a temperature gradient increasing outward is stable to the ideal MTI, it turns out to be unstable when threaded by a helical magnetic field.
Figure 5. Growth rates of GvMRI. Figure 5(a) is drawn for \( k_R = 0 \). Figure 5(b) shows growth rates vs. pitch angle and \( k_R/k_z \) for \( X = (15/16)^{1/2} \). The mode with any wavenumber is unstable for the pitch angles \( \theta < 50^\circ \).

Figure 6. Growth rates of the instability in the presence of anisotropic transport effects. Figure 6(a) is drawn for \( k_R = 0 \). The mode with large wavenumbers is unstable for the pitch angles \( 10^\circ < \theta < 50^\circ \). Figure 6(b) shows growth rates vs. pitch angle and \( k_R/k_z \) for \( X = (15/16)^{1/2} \). The radial wavenumber reduces the growth rates.

When the angular velocity vector and the \( B_z \) component of the magnetic field are parallel (\( \Omega \uparrow \uparrow B_z \)), gyroviscosity assumes positive values. Figure 5(a) is drawn only for \( \tilde{V}^{n}_{\text{gyro}} = 1 \) and \( k_R = 0 \) for the case without heat conduction. This instability is a gyroviscous instability, which is mentioned in Paper I. The mode with a small wavenumber \(< 0.5\) is unstable for all values of pitch angles. But, for all the wavenumbers, the unstable mode emerges only when \( \theta < 50^\circ \). The maximum growth rate of instability is about \( 3\Omega \). Figures 4 and 5(a) clearly show that GvMRI is more powerful and greater an instability than a magnetothermal one.

Figure 5(b) demonstrates dimensionless growth rates versus pitch angle and \( k_R/k_z \). In this figure, the normalized wavenumber of the fastest growing mode of MRI is adopted as \( X = (15/16)^{1/2} \). For the pitch angles \( \theta < 50^\circ \), the mode with any wavenumber is unstable, but even if the maximum growth rate is smaller than in the case when \( k_R = 0 \), it is still greater than its correspondent in the ideal MRI case.

The combined effects of the gyroviscosity and the heat conduction on the instability are seen in Figure 6(a). The cases of \( \Omega \uparrow \uparrow B_z \) and \( (d\ln T/d\ln R) > 0 \) are considered together. While the mode with small wavenumbers \((\sim <0.5)\) for all the pitch angle values is stable, the one with large wavenumbers becomes unstable only when the pitch angle is within the range of \( 10^\circ < \theta < 50^\circ \). Comparison of Figures 5(a) and 6(a) shows that the temperature gradient acts to reduce the growth rate of the instability. Also, it suppresses the instability that appears at small wavenumbers \((\sim <0.5)\) and small pitch angles \((\theta < 10^\circ)\).

Figure 6(b) shows the growth rates versus pitch angle and \( k_R/k_z \) for \( X = (15/16)^{1/2} \). The radial wavenumber of the unstable mode reduces the growth rates.

When the angular velocity vector and the \( B_z \) component of the magnetic field are antiparallel (\( \Omega \uparrow \downarrow B_z \)), gyroviscosity assumes negative values. When \( \tilde{V}^{n}_{\text{gyro}} < 0 \), an interesting situation arises. Instability sets in for the larger pitch angles \((\theta = 60^\circ–90^\circ)\). This figure is not included here for spatial economy, because it looks like a mirror image of the one with \( \Omega \uparrow \uparrow B_z \), in Figure 6(a), but the growth rates of instability are larger, i.e., \( \sim 3\Omega \). When we increase the value of \( \tilde{V}^{n}_{\text{gyro}} \), irrespective of its sign, the maximum growth rates of the instability become larger; for example, for \( \tilde{V}^{n}_{\text{gyro}} = 5 \), the maximum growth rate is \( \sim 6\Omega \).
Figure 7. Growth rates of the instability in the presence of the steep temperature gradient. Instability with higher growth rates arises for $\theta < 40^\circ$. There is another rather weak unstable region revealing itself as a mild crest for $\theta > 40^\circ$.

Figure 8. Growth rates of the instability for the situation of $\omega_{\text{cond}} \gg \omega_{\text{dyn}} \gg k \cdot v_A$. Figures 8(a) and 8(b) are drawn for $kR = 0$. The instability is suppressed when $T = 5$, but when $T = -5$ instability arises at almost all the wavenumbers and pitch angles.

Now, let us investigate the situation wherein the temperature gradient decreases outward. Instability behaves in the same manner, that is, the appearance of figures is the same but the maximum growth rate is comparatively smaller.

If the temperature gradient assumes a value of 5, and $\tilde{V}_{\text{gyro}} > 0$ then instability arises for $\theta < 40^\circ$. Another unstable but rather weak region reveals itself as a mild crest for $\theta > 40^\circ$. For $k v_A/\Omega = 2$, the growth rate is 0.45 at the pitch angle 50$^\circ$ and 0.13 at 80$^\circ$. The growth rates have bigger values at smaller wavenumbers (see Figure 7). If the temperature gradient assumes a value of $-5$, instability is seen for $\theta < 40^\circ$, but there is no mild crest for $\theta > 40^\circ$. Growth rates remain the same. When $\tilde{V}_{\text{gyro}} < 0$ and $\partial \ln T/\partial \ln R = 5$, there is instability with the same growth rates at pitch angles $\theta > 60^\circ$. For $\theta < 60^\circ$ a mild crest again appears as a second unstable region.

In the weak magnetic field limit, $\omega_{\text{cond}} \gg \omega_{\text{dyn}} \gg k \cdot v_A$ ordering holds true. Here, $\omega_{\text{dyn}} \sim (g/H)^{1/2}$ is the local dynamic frequency (Quataert 2008). If $\tilde{V}_{\text{gyro}} = V_{\text{ther}}/X \Omega = (2k^2 \chi T/5P)/(k v_A) \approx \omega_{\text{cond}}/k v_A = 1000$, and $\tilde{V}_{\text{gyro}} = 1$, and $\partial \ln T/\partial \ln R = 1$, then the figure of the instability looks exactly like Figure 5(a); that is, only the mode with wavenumbers smaller than 0.5 is unstable for all the possible pitch angles, and the maximum growth rate is not altered (see Figure 5(a)). If this situation is compared with Figure 6(a), which assumes $V_{\text{ther}} = 1$, we see that only the mode with very small wavenumbers is unstable for a very large characteristic conduction frequency.

A steep temperature gradient and a very large characteristic frequency for conduction assumption reveal a very interesting situation. While instability is suppressed for $T = 5$, when $T = -5$, it arises at almost all the wavenumbers and pitch angles (see Figures 8(a) and 8(b)).

### 3.1. Summary

From numerical solutions of the dimensionless dispersion relation we have deduced the effects of various parameters on the behavior of instability. These are as follows.

1. The maximum growth rate of the GvMRI ($2.5–3 \Omega$) is greater than the growth rates of ideal MRI and MTI.
2. The growth rate of unstable mode depends sensitively on the pitch angle and the gyroviscosity parameter $V_{\text{gyro}}/\Omega X = \tilde{V}_{\text{gyro}}$. 
3. The gyroviscosity parameter is positive or negative if the angular velocity vector and \( B \) component of the magnetic field are oriented in the same or opposite sense. For these situations, instability regions in the figures are seen as mirror images of each other. In other words, the instability sets in at the smaller pitch angles (\( \theta < 50' \)) for the positive gyroviscosity parameter, but when this parameter is negative instability sets in at the larger pitch angles (\( 50' < \theta < 90' \)).

4. For ideal MTI, plasma is unstable when the temperature increases in the direction of gravity, i.e., \( \partial \ln T / \partial \ln R < 0 \). We showed that there is at least one unstable mode for plasma with \( \partial \ln T / \partial \ln R < 0 \) or \( \partial \ln T / \partial \ln R > 0 \) in the presence of a helical magnetic field. Instability occurs due to gyroviscous force.

5. We determined the effects of temperature gradient on the GvMRI. First, the maximum growth rate is reduced. Second, the modes with the very small wavenumbers (\( k < 0.5 \)) remain formally stable (see Figures 5(a) and 6(a)).

6. Only in the presence of the steep temperature gradient does instability occur at almost all the wavenumbers and pitch angles (see Figure 7).

7. In the presence of the steep temperature gradient and very large characteristic frequency for conduction, the temperature gradient term stabilizes or destabilizes depending on whether it is positive or negative (see Figure 8).

4. DISCUSSION AND CONCLUSION

In a dilute plasma, ion cyclotron frequency greatly exceeds ion–ion collision frequency and the electron mean free path is also much larger than the gyroradius. Therefore, transport of momentum and heat by viscosity and thermal conduction is highly anisotropic with respect to magnetic field orientation. In this regime, for a more accurate plasma model the anisotropic transport terms, i.e., parallel viscosity, gyroviscosity, and thermal conduction, must be taken into consideration in the MHD equations. Many astrophysical plasmas display characteristics of dilute plasma. For example, physical parameters and the conditions of the ICM of the galaxy clusters are revealed by telescopes with high resolving powers. Chandra X-ray Observatory measured X-ray luminosity (\( 10^{35–10^{46}} \text{erg s}^{-1} \)) emitted by the hot plasma in the ICM; based on this measurement the density distribution of ICM as a function of radius was determined (Parrish et al. 2008). Peterson & Fabian (2006) report that typical densities are in the range of \( 10^{-3} \) to \( 10^{-2} \text{cm}^{-3} \); temperatures are \( 1–15 \text{keV} \). Carilli & Taylor (2002) estimated the magnetic field strength in the center of ICM about \( 1–19 \mu \text{G} \) and \( 0.1–1.0 \mu \text{G} \) at the radius of \( 1 \text{Mpc} \). With the above quoted values, plasma beta is \( \beta = 8 \pi P / B^2 \sim 200–2000 \). This value implies that the ICM plasma is dilute and the mean free path of electrons is much longer than their gyroradius (Narayan & Medvedev 2001).

Under these physical conditions, dilute and hot plasma in a differentially rotating disk is open to a blend of instabilities such as MRI, MTI, and GvMRI. The analysis proceeds from simple to complex. Let us start with a useful mechanical model developed by Balbus & Hawley (1992) for the MRI. This model consists of two fluid elements that are tethered to each other with a vertical magnetic field. They are also embedded in a radial angular velocity gradient, so that the element orbiting at a smaller radius rotates more rapidly than the other element orbiting at a larger radius. We suppose that these elements reside at different vertical locations, but at the same radial locations initially. When these elements are radially displaced, the magnetic field will force them to return to their original locations. The outward element acquires angular momentum because it has a smaller velocity in its new radial location. The inward element loses angular momentum because it has a greater velocity in its new radial location. When the field lines become more stretched, the inner element continues to lose its angular momentum and will fall farther inward; the other element moves farther out and gains higher angular momentum. Thus, the process runs away and instability occurs. This is a classical MRI picture (Balbus & Hawley 1992, 1998). Because ion viscosity is higher than that of electrons in the dilute plasma, the ions in the fluid elements gyrate around the magnetic field lines and will be under the influence of the spatially varying electric field that arises from FLR effects. In the shear flow there is no rest frame. Therefore, the elements in different locations are exposed to different electric fields (\( \mathbf{E} = -\nabla \times \mathbf{B}/c = -R \Omega(R) B_0 \sin \theta \mathbf{R} \)). Along a gyro-orbit, the length scale of the electric field is comparable to the length scale of the velocity gradient (Williams & Jokipii 1991). If the initial positions of elements are selected as a rest frame, the fluid velocity increases toward smaller \( R \) values. The inward element will therefore see a larger electric field because it has greater velocity than the outward element. Since the drift is controlled by the magnitude of the electric field, the fluid element with a larger relative velocity will drift more rapidly. Because the magnetic field that tethers fluid elements to each other acts as a spring-like force, the rapidly increasing element separation gives rise to growing tension. Thus, the process runs away and instability occurs quickly. This is a GvMRI picture.

The modified Hill equations including gyroviscosity, thermal conduction, and parallel viscosity can help one to get a better physical understanding of instability. In the absence of all three dynamic effects, one recovers the original set of equations describing the MRI (Balbus & Hawley 1992, 1998). In Equations (54) and (55) below, \( \xi_R \) and \( \xi_\phi \) are the radial and azimuthal displacements of fluid elements, respectively:

\[
\frac{d^2 \xi_R}{dt^2} - 2\Omega \left( 1 - \frac{1}{2} V_{\text{gyro}} A \right) \frac{\partial \xi_R}{\partial t} = - \left[ \frac{(k_z V_A)^2}{\partial^2 ln R} + \frac{\partial_\gamma^2}{\partial^2 ln R} \left( 1 + \frac{V_{\text{gyro}}}{V_\gamma} \right) \right] \xi_R, \tag{54}
\]

\[
\frac{d^2 \xi_\phi}{dt^2} + 2\Omega \left( 1 - \frac{1}{2} V_{\text{gyro}} A \right) \frac{\partial \xi_R}{\partial t} = - \left[ \frac{(k_z V_A)^2}{\partial^2 ln R} + \frac{V_{\text{gyro}}}{V_\gamma} \left( \frac{V^2}{V_\gamma} + H \right) \right] \xi_\phi; \tag{55}
\]

equations describing the MRI (Balbus & Hawley 1992, 1998). In Equations (54) and (55) below, \( \xi_R \) and \( \xi_\phi \) are the radial and azimuthal displacements of fluid elements, respectively:

In the right-hand sides of Equations (54) and (55) represent the torque applied to the fluid elements that correspond to “spring” constants in radial and azimuthal directions. Comparison of the right-hand side of Equations (54) and (55) shows that while the
gyroviscosity couples to the differential rotation, the thermal conduction couples to the Coriolis force, and the radial gradients of the temperature and the pressure. These complex couplings make the roles of the anisotropic forces intangible. We solve Equation (55) for $\xi_\phi$ and then substitute it into Equation (54) to find the acceleration of the perturbed fluid element in the radial direction. In order for the hot, dilute, and differentially rotating disk to be unstable there should be a net outward acceleration:

$$\frac{\partial^2 \xi_R}{\partial t^2} = -(k_z v_A)^2 \xi_R - \left[ \kappa^2 + (N^2 - \Omega^2 \omega - \Omega^2 V_{\text{ther}} \sin^2 \theta P_M) \frac{T}{\omega + V_{\text{ther}} \sin^2 \theta} \right] \xi_R$$

$$- \left[ \frac{d \Omega^2}{d \ln R} \sqrt{2} + 4 \Omega^2 \left( -\tilde{V}_{\text{gyro}} A + \frac{1}{4} \tilde{V}_{\text{gyro}}^2 A^2 \right) - 8 \Omega^2 \left( 1 - \frac{1}{2} \tilde{V}_{\text{gyro}} A \right)^2 \frac{Y}{(\Omega^2 + Y)} \right] \xi_R,$$

where $Y = [(k_z v_A)^2 + \tilde{V}_{\text{gyro}} \frac{d \Omega^2}{d \ln R} (G + H) + 2 D \omega V_{\text{par}} \cos \theta]$. Acceleration given by Equation (56) can be written as the sum of three terms, i.e., $\frac{\partial^2 \xi_R}{\partial t^2} = a_T = a_M + a_{H,\text{th}} + a_{M,\text{gyr}}$.

$a_{M,\text{gyr}}$, given by Equation (56), is the acceleration term arising from the magnetic tension force, which is always stabilizing. The second acceleration term $a_{H,\text{th}}$ is due to the radial buoyancy force and anisotropic thermal conduction. It is a pure hydrodynamic term. This term is related to the convective instability in which the source of free energy is the temperature gradient. In the absence of the viscous force, when the temperature increases in the direction of gravity, this term has a destabilizing effect. When the temperature decreases in the direction of gravity, it has a stabilizing effect. The third acceleration term $a_{M,\text{gyr}}$ is due to the radial gyroviscous force. It is related to the GvMRI. Depending on the pitch angle, the wavenumbers, gyroviscous force, and the parallel viscosity, this term may be either positive or negative, which corresponds to either the destabilizing or stabilizing effects, respectively.

The relative importance of the two terms that can be stabilizing or destabilizing are depicted in Figure 9. The figure is drawn as the logarithmic normalized wavenumber versus the logarithmic ratio of accelerations, $\log(a_{M,\text{gyr}}/a_{H,\text{th}})$, for convectively stable plasma, i.e., $N^2 = 1$. If $a_{M,\text{gyr}} \geq a_{H,\text{th}}$, all the unstable modes are GvMRI modes. If $a_{M,\text{gyr}} < a_{H,\text{th}}$, the unstable modes are called thermal modes. As shown in Figure 9, MTI by driven heat conduction is dominant in very small wavenumbers only for the $\theta = 90^\circ$, i.e., in considering only the vertical magnetic field. In all the possible situations throughout the whole wavelength range, the acceleration term $a_{M,\text{gyr}}$ is dominant due to the gyroviscous force. Moreover, this acceleration increases rapidly in the large wavenumbers. The $(a_{H,\text{th}}) \xi_R$ term may be positive for the given values. Still, this term does not make a contribution to instability even if $\tilde{V}_{\text{ther}} = 1000$.

The above analysis shows that, regardless of the sign of the temperature gradient, a weakly magnetized and convectively stable dilute plasma harboring the combined effects of gyroviscosity, parallel viscosity, and thermal conduction is unstable due to the gyroviscous force. Extension of the unstable regions and the growth rates of GvMRI depends sensitively on the pitch angle and the gyroviscosity parameter. When the angular velocity vector and $B_z$ component of the magnetic field are parallel ($\Omega \uparrow \uparrow B_z$), gyroviscosity assumes positive values. When the angular velocity vector and $B_z$ component of the magnetic field are antiparallel ($\Omega \uparrow \downarrow B_z$), gyroviscosity assumes negative values. In the case of $\Omega \uparrow \uparrow B_z$, while instability first sets in for the smaller pitch angles ($\theta < 50^\circ$), in the case of $\Omega \uparrow \downarrow B_z$, it sets in for the larger pitch angles ($\theta = 60^\circ - 90^\circ$). Maximum growth rates of the unstable mode are approximately 2.5 $\Omega$ and 3 $\Omega$, respectively. These values are higher than the ones for MRI (0.75 $\Omega$) and the ones for MTI (0.5 $\Omega$, see Figure 4). If the $k_R$ wavenumber is taken into consideration, the instability region becomes narrower and the growth rate of the instability is reduced.

GvMRI is dominant in the blend of pure MRI, MTI, and gyroviscous-modified MRI. Because the magnitude and the direction of the temperature gradient do not have a great effect on the unstable mode, this instability may work for all the astrophysical media, including dilute plasma.
I thank E. Rennan Pekünlü for helpful discussions. I am particularly grateful to the referee for the constructive comments that led to the substantial improvement of this manuscript. This research was supported by the Scientific & Technological Research Council of Turkey (TÜBİTAK).

APPENDIX A

TWO-FLUID EQUATIONS

The two-fluid equations together with the Faraday and Ampère laws, respectively, are given by (Braginskii 1965) as

\[
\frac{dn_i}{dt} = -n_i \nabla \cdot \mathbf{v}_i, \tag{A1}
\]

\[
m_i n_i \frac{d\mathbf{v}_i}{dt} = -\nabla P_i - \nabla \cdot \mathbf{\Pi}_i + q_i n_i \left( \mathbf{E} + \frac{\mathbf{v}_i \times \mathbf{B}}{c} \right) + m_i n_i \mathbf{g}, \tag{A2}
\]

\[
\frac{3}{2} n_s \frac{dT_s}{dt} = -P_i (\nabla \cdot \mathbf{v}_i) - \nabla \cdot \mathbf{Q}_s, \tag{A3}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}, \tag{A4}
\]

where the subscript “s” stands for electrons and ions, \( m \) is the mass, \( n \) is the number density, \( \mathbf{v} \) is the velocity of the plasma components, \( P \) is the scalar pressure, \( \mathbf{\Pi} \) is the stress tensor, \( q \) is the particle charge, \( \mathbf{E} \) and \( \mathbf{B} \) are the electric and magnetic fields, respectively, \( \mathbf{g} \) is the gravitational acceleration, \( \mathbf{Q} \) is the heat flux, \( c \) is the speed of the light, \( e \) is the electric charge (\( q = e \)), \( Z \) is the charge state, and \( d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla \) is a Lagrangian derivative.

Most plasmas of interest are electrically neutral over sufficiently long distance and timescales. One assumes that the Debye length (\( \lambda_D = (kT/4\pi n e^2)^{1/2} \)) is smaller than all the relevant spatial scales \( \lambda_D \ll L \). Hence, ion and electron densities are essentially equal, i.e., \( n_e = Z n_i \), with quasi-neutrality. Then the mass continuity equation is written only for ion number density as

\[
\frac{dn_i}{dt} = -n_i \nabla \cdot \mathbf{v}_i. \tag{A6}
\]

Let us multiply Equation (A6) with \( m_i \). Since \( \rho = m_i n_i \), Equation (A6) now reads

\[
\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0. \tag{A7}
\]

Let us first write \( \mathbf{E} \) by using from the electron momentum equation:

\[
\mathbf{E} = \frac{1}{en_e} \left[ -m_e n_e \frac{d\mathbf{v}_e}{dt} - \nabla P_e - \nabla \cdot \mathbf{\Pi}_e + m_e n_e \mathbf{g} \right] - \frac{\mathbf{v}_e \times \mathbf{B}}{c}. \tag{A8}
\]

Let us eliminate \( \mathbf{E} \) (Equation (A8)) by substituting from the electron momentum equation into the ion momentum equation:

\[
m_i n_i \frac{d\mathbf{v}_i}{dt} = -\nabla P_i - \nabla \cdot \mathbf{\Pi}_i + Z n_i \left( \frac{m_e n_e}{en_e} \right) \frac{d\mathbf{v}_e}{dt} - \nabla P_e - \nabla \cdot \mathbf{\Pi}_e + m_e n_e \mathbf{g} \right] - Z n_i \frac{\mathbf{v}_e \times \mathbf{B}}{c} + Z n_i \frac{\mathbf{v}_i \times \mathbf{B}}{c} + m_i n_i \mathbf{g}. \tag{A9}
\]

Let us put \( \rho = m_i n_i \), \( P = P_i + P_e \), \( \mathbf{\Pi} = \mathbf{\Pi}_i + \mathbf{\Pi}_e \), and assume that \( m_e/m_i \sim 0 \) because \( m_e \ll m_i \). With the substitutions and using Ampère laws, the ion momentum equation becomes

\[
\frac{d\mathbf{v}_i}{dt} = -\nabla P - \nabla \cdot \mathbf{\Pi} + \frac{\mathbf{J} \times \mathbf{B}}{c} + \rho \mathbf{g}. \tag{A10}
\]

Let us substitute \( \mathbf{E} \) (Equation (A8)) into the magnetic induction equation as given in Equation (A4):

\[
\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \left[ \frac{cm_e Z}{en_e m_i} \rho \left( \mathbf{g} - \frac{d\mathbf{v}_e}{dt} \right) - \frac{1}{en_e} \nabla P_e - \frac{1}{en_e} \nabla \cdot \mathbf{\Pi}_e - \mathbf{v}_e \times \mathbf{B} \right]. \tag{A11}
\]

Since it is assumed that \( n_e = Z n_i \), then \( \mathbf{J} = (c/4\pi) \nabla \times \mathbf{B} = en_e (\mathbf{v}_i - \mathbf{v}_e) \) may be written in this form. From this equation one obtains \( \mathbf{v}_e = \mathbf{v}_i - \mathbf{J}/en_e \). After substituting \( \mathbf{v}_e \) and using \( m_e/m_i \sim 0 \), one obtains a magnetic induction equation as given below:

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{v}_i \times \mathbf{B} - \frac{1}{en_e} \mathbf{J} \times \mathbf{B} + \frac{c}{en_e} \nabla P_e + \frac{c}{en_e} \nabla \cdot \mathbf{\Pi}_e \right). \tag{A12}
\]
The second term on the right-hand side of Equation (A12) is the Hall effect. This term is negligible because in the present investigation a $\beta \gg 1$ limit is considered ($\beta$ is the ratio of the gas pressure to the magnetic pressure). The third term on the right-hand side of Equation (A12) is the thermodiffusion term. The ions carry most of the momentum due to their higher masses. Therefore, the third and the last term on the right-hand side of Equation (A12) are negligible. The resulting magnetic induction equation is given by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).$$  \hspace{1cm} (A13)

If the ions and electrons are in thermal equilibrium, then $T_e \simeq T_i \simeq T$. Let us add ion energy equation and electron energy equation:

$$(n_e + n_i) \frac{dT}{dt} = (Z + 1)n_i \frac{dT}{dt} = -\frac{2}{3} \nabla \cdot \mathbf{v}_i (P_e + P_i) + \frac{2}{3} P_e \nabla \cdot \frac{\mathbf{J}}{en_e} - \frac{2}{3} \nabla \cdot \mathbf{Q}. \hspace{1cm} (A14)$$

One assumes an ideal gas equation of state. Therefore, one may write $P = (n_e + n_i)T$. One obtains the resulting energy equation after some algebraic operations:

$$\frac{dP}{dt} + \frac{5}{3} \cdot P \cdot (\nabla \cdot \mathbf{v}) = -\frac{2}{3} \nabla \cdot \mathbf{Q}. \hspace{1cm} (A15)$$

**APPENDIX B**

THE NEGLIGIBLE EFFECT OF THE STRESS TENSOR ON THE EQUILIBRIUM STATE

In the equilibrium state, the contribution from parallel viscosity components of the stress tensor is

$$\nabla \cdot \mathbf{\Pi}_0^v = 0.96 \frac{P_i}{2v_i^2} \frac{\partial}{\partial R} \left( \mathbf{b} \times \mathbf{W} \cdot (\mathbf{i} - 3\mathbf{b}\mathbf{b})(\mathbf{b} \cdot \mathbf{W} \cdot \mathbf{b}) \right). \hspace{1cm} (B1)$$

The rate of the strain tensor has two components in the equilibrium state, $W_{R\theta} = W_{\phi R} = d\Omega / d\ln R$. The unit vector along the magnetic field is $\mathbf{b} = \phi b_{\phi} + \hat{z} b_z$. For this case,

$$(\mathbf{b} \cdot \mathbf{W} \cdot \mathbf{b}) = (\hat{R} b_{\phi} W_{\phi R}) \cdot (\phi b_{\phi} + \hat{z} b_z) = 0. \hspace{1cm} (B2)$$

Accordingly, there is no contribution of the parallel viscosity to the equilibrium state.

In the equilibrium state, the contribution from gyroviscosity components of the stress tensor is

$$\nabla \cdot \mathbf{\Pi}_0^{gv} = \frac{P_i}{4\omega_{ci}} \frac{\partial}{\partial R} \left[ (\mathbf{b} \times \mathbf{W} \cdot (\mathbf{I} + 3\mathbf{b}\mathbf{b}) + [\mathbf{b} \times \mathbf{W} \cdot (\mathbf{I} + 3\mathbf{b}\mathbf{b})]^T \right]. \hspace{1cm} (B3)$$

Since $\mathbf{b} \times \mathbf{W} \cdot 3\mathbf{b}\mathbf{b} = 0$, the gyroviscosity component is given by

$$\nabla \cdot \mathbf{\Pi}_0^{gv} = \frac{P_i}{4\omega_{ci}} \frac{\partial}{\partial R} \left[ \mathbf{b} \times \mathbf{W} \cdot (\mathbf{I} + [\mathbf{b} \times \mathbf{W} \cdot (\mathbf{I} + 3\mathbf{b}\mathbf{b})]^T \right]. \hspace{1cm} (B4)$$

Using vector and dyadic relationships, one obtains

$$\nabla \cdot \mathbf{\Pi}_0^{gv} = -\frac{P_i}{2\omega_{ci}} \sin \theta \left[ \frac{\partial}{\partial R} \left( \frac{d\Omega}{d\ln R} \right) + \frac{2}{R} \frac{d\Omega}{d\ln R} \right]. \hspace{1cm} (B5)$$

The hydrostatic equilibrium Equation (30) should be written as

$$\nabla \frac{P_0}{\rho_0} = \mathbf{g} + R\Omega^2 + \frac{P_i}{2\omega_{ci}} \sin \theta \left[ \frac{\partial}{\partial R} \left( \frac{d\Omega}{d\ln R} \right) + \frac{2}{R} \frac{d\Omega}{d\ln R} \right]. \hspace{1cm} (B6)$$

Now, Equation (B6) may be rewritten for the gyroviscosity parameter $\tilde{V}_{gyro} = \Omega P_i / 4\omega_{ci} \rho v_A^2$:

$$\nabla \frac{P_0}{\rho_0} = \mathbf{g} + R\Omega^2 + \tilde{V}_{gyro} 2v_A^2 \sin \theta \left[ \frac{1}{\Omega} \frac{\partial}{\partial R} \left( \frac{d\Omega}{d\ln R} \right) + \frac{1}{2} \frac{d\Omega}{R^2 \Omega \ln R} \right]. \hspace{1cm} (B7)$$

The first term in the bracket in Equation (B7) is

$$I = \frac{1}{\Omega} \frac{\partial}{\partial R} \left( \frac{d\Omega}{d\ln R} \right) = \frac{2\Omega}{2\Omega^2} \frac{\partial}{\partial R} \left( \frac{d\Omega}{d\ln R} \right) = \frac{1}{2\Omega^2} \left[ \frac{\partial}{\partial \Omega} \left( 2\Omega \frac{d\Omega}{d\ln R} \right) - \frac{2}{R^2} \frac{\partial}{\partial R} \frac{d\Omega}{d\ln R} \right]. \hspace{1cm} (B8)$$

because

$$\frac{\partial}{\partial R} \left( 2\Omega \frac{d\Omega}{d\ln R} \right) = 2\Omega \frac{\partial}{\partial R} \left( \frac{d\Omega}{d\ln R} \right) + 2 \frac{\partial}{\partial R} \frac{d\Omega}{d\ln R}. \hspace{1cm} (B9)$$
After some algebraic manipulation, Equation (B8) becomes

\[ I = \frac{1}{2\Omega^2} \left[ \frac{\partial}{\partial R} \left( \Omega^2 d \ln \Omega^2 \right) d \ln R \right] - \Omega \frac{\partial \Omega}{\partial R} d \ln \Omega^2, \] (B10)

then

\[ I = \frac{1}{2\Omega^2} \left[ \frac{\partial \Omega}{\partial R} d \ln \Omega^2 + \Omega^2 \frac{\partial}{\partial R} \left( \frac{d \ln \Omega^2}{d \ln R} \right) \right], \] (B11)

and finally

\[ I = \left[ \frac{1}{4R} \frac{\partial \ln \Omega^2}{\partial \ln R} d \ln \Omega^2 + \frac{\partial}{\partial R} \left( \frac{d \ln \Omega^2}{d \ln R} \right) \right]. \] (B12)

The second term in the bracket in Equation (B7) is

\[ II = \frac{1}{R} \frac{d \Omega}{d \ln R} = \frac{1}{\Omega^2 R} d \ln \Omega^2. \] (B13)

Now, substituting Equations (B12) and (B13) into the equilibrium state equation, we find

\[ \nabla P_0 = g + R\Omega^2 + \frac{\tilde{V}_n}{\gamma} \sin \theta \left[ \frac{1}{4R} \frac{\partial \ln \Omega^2}{\partial \ln R} d \ln \Omega^2 + \frac{\partial}{\partial R} \left( \frac{d \ln \Omega^2}{d \ln R} \right) + \frac{1}{R} \frac{d \ln \Omega^2}{d \ln R} \right]. \] (B14)

In a Keplerian disk, \( \frac{\partial \ln \Omega^2}{\partial \ln R} = -3 \). Thus, the equilibrium state equation is obtained as

\[ \nabla P_0 = g + R\Omega^2 - \frac{6}{4} \tilde{V}_n \frac{v_A^2}{R} \sin \theta. \] (B15)

One may consider \( \frac{v_A^2}{R} \propto \frac{B^2}{R} \). In a dilute plasma, especially in the ICM of galaxy clusters (see Section 4), the magnetic field is extremely weak and \( R \) is relatively very large. Therefore, the contribution of the stress tensor to the equilibrium state is negligibly small. The equilibrium state is given by

\[ \nabla P_0 = g + R\Omega^2. \] (B16)

REFERENCES

Acheson, D. J., & Hide, R. 1973, Rep. Prog. Phys., 36, 159
Balbus, S. A. 2000, ApJ, 534, 420
Balbus, S. A. 2001, ApJ, 562, 909
Balbus, S. A. 2003, ARA&A, 41, 555
Balbus, S. A. 2004, ApJ, 616, 857
Balbus, S. A., & Hawley, J. F. 1991, ApJ, 376, 214
Balbus, S. A., & Hawley, J. F. 1992, ApJ, 392, 322
Balbus, S. A., & Hawley, J. F. 1998, Rev. Mod. Phys., 70, 1
Braginskii, S. I. 1965, in Reviews of Plasma Physics, Vol. 1, ed. M. A. Leon-
tovich (New York: Consultants Bureau), 205
Carilli, C. L., & Taylor, G. B. 2002, ARA&A, 40, 319
Devlen, E., & Pekünlü, E. R. 2010, MNRAS, 404, 830 (Paper I)
Ferraro, N. M. 2007, ApJ, 662, 512
Islam, T., & Balbus, S. 2005, ApJ, 633, 328
Kaufman, A. N. 1960, Phys. Fluids, 3, 610
Moffatt, H. K. 1978, Magnetic Field Generation in Electrically Conducting
Fluids (Cambridge: Cambridge Univ. Press)
Narayan, R., & Medvedev, M. V. 2001, ApJ, 562, L129
Parrish, I. J., & Stone, J. M. 2008, ApJ, 688, 905
Peterson, J. R., & Fabian, A. C. 2006, Phys. Rep., 427, 1
Quataert, E. 2008, ApJ, 673, 758
Ramos, J. J. 2003, Phys. Plasmas, 10, 3601
Ramos, J. J. 2005, Phys. Plasmas, 12, 052102
Ramos, J. J. 2007, Phys. Plasmas, 14, 052506
Spitzer, L. 1962, Physics of Fully Ionized Gases (2nd ed.; New York: Wiley)
Williams, L. L., & Jokipii, J. R. 1991, ApJ, 371, 639