Curve Flows and Solitonic Hierarchies
Generated by (Semi) Riemannian Metrics

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Abstract

We investigate bi–Hamiltonian structures and related mKdV hierarchy of solitonic equations generated by (semi) Riemannian metrics and curve flow of non–stretching curves. The corresponding nonholonomic tangent space geometry is defined by canonically induced nonlinear connections, Sasaki type metrics and linear connections. One yields couples of generalized sine–Gordon equations when the corresponding geometric curve flows result in hierarchies on the tangent bundle described in explicit form by nonholonomic wave map equations and mKdV analogs of the Schrödinger map equation.

Keywords: Curve flow, (semi) Riemannian spaces, nonholonomic manifold, nonlinear connection, bi–Hamiltonian, solitonic equations.

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1 Introduction

In recent years, the differential geometry of plane and space curves is receiving considerable attention in the theory of nonlinear partial differential equations and applications to modern physics [1, 2, 3, 4, 5]. One proved that curve flows on Riemannian spaces of constant curvature are described geometrically by hierarchies defined by wave map equations and mKdV analogs of Schrödinger map equation. The main results on vector generalizations of KdV and mKdV
equations and the geometry of their Hamiltonian structures are summarized in Refs. [6, 7, 8], see also a recent work in [9, 10].

In [11, 12], the flows of non–stretching curves were analyzed using moving parallel frames and associated frame connection 1–forms in a symmetric spaces $M = G/\text{SO}(n)$ and the structure equations for torsion and curvature encoding $O(n – 1)$–invariant bi–Hamiltonian operators.¹ It was shown that the bi–Hamiltonian operators produce hierarchies of integrable flows of curves in which the frame components of the principal normal along the curve satisfy $O(n – 1)$–soliton equations. The crucial condition for such constructions is the fact that the frame curvature matrix is constant on the curved manifolds like $M = G/\text{SO}(n)$. The approach was developed into a geometric formalism mapping regular Lagrange mechanical systems into bi–Hamiltonian structures and related solitonic equations [13], following certain methods elaborated in the geometry of generalized Finsler and Lagrange spaces [14, 15, 16] and nonholonomic manifolds with applications in modern gravity [17, 18, 19].

The aim of this paper is to prove that solitonic hierarchies can be generated by any (semi) Riemannian metric $g_{ij}$ on a manifold $V$ of dimension $\dim V = n \geq 2$ if the the geometrical objects are lifted in the total space of the tangent bundle $TV$, or of a vector bundle $E = (M, \pi, E)$, $\dim E = m \geq n$, by defining such frame transforms when constant matrix curvatures are defined canonically with respect to certain classes of preferred systems of reference.

The paper is organized as follows:

In section 2 we outline the geometry of vector bundles provided with nonlinear connection. We emphasize the possibility to define fundamental geometric objects induced by a (semi) Riemannian metric on the base space when the Riemannian curvature tensor has constant coefficients with respect to a preferred nonholonomic basis.

In section 3 we consider curve flows on nonholonomic vector bundles. We sketch an approach to classification of such spaces defined by conventional horizontal and vertical symmetric (semi) Riemannian subspaces and provided with nonholonomic distributions defined by the nonlinear connection structure. It is constructed a class of nonholonomic Klein spaces for which the bi–Hamiltonian operators are derived for a canonical distinguished connection, adapted to the nonlinear connection structure, for which the distinguished curvature coefficients are constant.

Section 4 is devoted to the formalism of distinguished bi–Hamiltonian operators and vector soliton equations for arbitrary (semi) Riemannian spaces.

¹$G$ is a compact semisimple Lie group with an involutive automorphism that leaves fixed a Lie subgroup $\text{SO}(n) \subset G$, for $n \geq 2$
We define the basic equations for nonholonomic curve flows. Then we consider the properties of cosympletic and sympletic operators adapted to the nonlinear connection structure. Finally, there are constructed solitonic hierarchies of bi–Hamiltonian anholonomic curve flows.

We conclude the results in section 5. The Appendix contains necessary definitions and formulas from the geometry of nonholonomic manifolds.

2 Nonholonomic Structures on Manifolds

In this section, we prove that for any (semi) Riemannian metric $g_{ij}$ on a manifold $V$ it is possible to define lifts to the tangent bundle $TV$ provided with canonical nonlinear connection (in brief, N–connection), Sasaki type metric and canonical linear connection structure. The geometric constructions will be elaborated in general form for vector bundles.

2.1 N–connections induced by Riemannian metrics

Let $E = (E, \pi, F, M)$ be a (smooth) vector bundle of over base manifold $M$, when the dimensions are stated respectively; $\dim M = n$ and $\dim E = (n + m)$, for $n \geq 2$, and $m \geq n$ being the dimension of typical fiber $F$. It is defined a surjective submersion $\pi : E \to M$. In any point $u \in E$, the total space $E$ splits into ”horizontal”, $M_u$, and ”vertical”, $F_u$, subspaces. We denote the local coordinates in the form $u = (x, y)$, or $u^\alpha = (x^i, y^a)$, with horizontal indices $i, j, k, \ldots = 1, 2, \ldots, n$ and vertical indices $a, b, c, \ldots = n+1, n+2, \ldots, n+m$. The summation rule on the same ”up” and ”low” indices will be applied.

The base manifold $M$ is provided with a (semi) Riemannian metric, a second rank tensor of constant signature, $h_{ij} = g_{ij}(x)dx^i \otimes dx^j$. It is possible to introduce a vertical metric structure $v_{ab} = g_{ab}(x)dy^a \otimes dy^b$ by completing the matrix $g_{ij}(x)$ diagonally with $\pm 1$ till any nondegenerate second rank tensor $g_{ab}(x)$ if $m > n$. This defines a metric structure $g_{\alpha \beta} = [h_{ij}, v_{ab}]$ on $E$. We can deform the metric structure, $g_{\alpha \beta} \rightarrow g_{\alpha \beta} = [g_{ij}, g_{ab}]$, by considering a frame (vielbein) transform,

$$g_{\alpha \beta}(x, y) = e^\alpha_{\alpha}(x, y) \ e^\beta_{\beta}(x, y) g_{\alpha \beta}(x),$$

2In a particular case, we have a tangent bundle $E=TM$, when $n = m$; for such bundles both type of indices run the same values but it is convenient to distinguish the horizontal and vertical ones by using different groups of Latin indices.

3In physical literature, one uses the term (pseudo) Riemannian/Euclidean space
where the coefficients $g_{\alpha\beta}(x)$ have been written as $g_{\alpha\beta}(x)$. The coefficients $e_\alpha\alpha(x, y)$ will be defined below (see formula (18)) from the condition of generating curvature tensors with constant coefficients with respect to certain preferred systems of reference.

For any $g_{ab}$ from the set $g_{\alpha\beta}$, we can construct an effective generation function

$$\mathcal{L}(x, y) = g_{ab}(x, y)y^ay^b$$

inducing a vertical metric

$$\tilde{g}_{ab} = \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial y^a \partial y^b}$$

(2)

which is "weakly" regular if $\det|\tilde{g}_{ab}| \neq 0$. 4

By straightforward calculations we can prove this result 5:

**Theorem 2.1** The Euler–Lagrange equations on $TM$,

$$\frac{d}{d\tau} \left( \frac{\partial \mathcal{L}}{\partial y^i} \right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0,$$

for the Lagrangian $\mathcal{L} = \sqrt{|\mathcal{L}|}$, where $y^i = \frac{dx^i}{d\tau}$ for a path $x^i(\tau)$ on $M$, depending on parameter $\tau$, are equivalent to the “nonlinear” geodesic equations

$$\frac{d^2 x^i}{d\tau^2} + 2\tilde{G}^i(x, y)\frac{dx^j}{d\tau} = 0$$

defining paths of a canonical semispray $S = y^i \frac{\partial}{\partial x^i} - 2\tilde{G}^i(x, y)\frac{\partial}{\partial y^i}$, where

$$2\tilde{G}^i(x, y) = \frac{1}{2} \tilde{g}^{ij} \left( \frac{\partial^2 \mathcal{L}}{\partial y^i \partial x^k} y^k - \frac{\partial \mathcal{L}}{\partial x^i} \right)$$

with $\tilde{g}^{ij}$ being inverse to (2).

On holds

**Conclusion 2.1** For any (semi) Riemannian metric $g_{ij}(x)$ on $M$, we can associate canonically an effective regular Lagrange mechanics on $TM$ with the Euler–Lagrange equations transformed into nonlinear (semispray) geodesic equations.

4Similar values, for $e_\alpha\alpha = \delta_\alpha^\alpha$, where $\delta_\alpha^\alpha$ is the Kronecker symbol, were introduced for the so-called generalized Lagrange spaces when $\mathcal{L}$ was called the "absolute energy" [14].

5see Refs. [14, 15] for details of a similar proof; here we note that in our case, in general, $e_\alpha^\alpha \neq \delta_\alpha^\alpha$. 

4
We denote by $\pi^\top : TE \to TM$ the differential of map $\pi : E \to M$ defined by fiber preserving morphisms of the tangent bundles $TE$ and $TM$. The kernel of $\pi^\top$ is just the vertical subspace $vE$ with a related inclusion mapping $i : vE \to TE$.

**Definition 2.1** A nonlinear connection (N–connection) $N$ on a vector bundle $E$ is defined by the splitting on the left of an exact sequence

$$0 \to vE \xrightarrow{i} TE \to TE/vE \to 0,$$

i.e. by a morphism of submanifolds $N : TE \to vE$ such that $N \circ i$ is the unity in $vE$.

In an equivalent form, we can say that a N–connection is defined by a Whitney sum of conventional horizontal (h) subspace, $(hE)$, and vertical (v) subspace, $(vE)$,

$$TE = hE \oplus vE. \quad (3)$$

This sum defines a nonholonomic (equivalently, anholonomic, or nonintegrable) distribution of horizontal and vertical subspaces on $TE$. Locally, a N–connection is defined by its coefficients $N^a_i(u)$,

$$N = N^a_i(u)dx^i \otimes \frac{\partial}{\partial y^a}. \quad (4)$$

The well known class of linear connections consists on a particular subclass with the coefficients being linear on $y^a$, i.e., $N^a_i(u) = \Gamma^a_{bj}(x)y^b$.

**Remark 2.1** A bundle space, or a a manifold, is called nonholonomic if it provided with a nonholonomic distribution (see historical details and summary of results in [17]). In particular case, when the nonholonomic distribution is of type [3], such spaces are called N–anholonomic [19].

Any N–connection $N = \{N^a_i(u)\}$ may be characterized by a N–adapted frame (vielbein) structure $e_\nu = (e_i, e_a)$, where

$$e_i = \frac{\partial}{\partial x^i} - N^a_i(u)\frac{\partial}{\partial y^a} \quad \text{and} \quad e_a = \frac{\partial}{\partial y^a}, \quad (4)$$

and the dual frame (coframe) structure $e^\mu = (e^i, e^a)$, where

$$e^i = dx^i \quad \text{and} \quad e^a = dy^a + N^a_i(u)dx^i. \quad (5)$$

In order to preserve a relation with the previous denotations, we note that $e_\nu = (e_i, e_a)$ and $e^\mu = (e^i, e^a)$ are, respectively, the former ”N–elongated”
partial derivatives $\delta \nu = \delta / \partial u^\nu = (\delta_i, \partial_a)$ and N–elongated differentials $\delta^\mu = \delta u^\mu = (d^i, \delta^a)$ which emphasize that operators (4) and (5) define, correspondingly, certain “N–elongated” partial derivatives and differentials which are more convenient for tensor and integral calculations on such nonholonomic manifolds.\(^6\)

For any N–connection, we can introduce its N–connection curvature

$$\Omega = \frac{1}{2} \Omega_{ij}^a \, d^i \wedge d^j \otimes \partial_a,$$

with the coefficients defined as the Neijenheuse tensor,

$$\Omega_{ij}^a = e^a_{[ij]} N_{ij}^a = e_j N_i^a - e_i N_j^a = \partial N_i^a / \partial x^j - \partial N_j^a / \partial x^i + N_i^b \partial N_j^a / \partial y^b - N_j^b \partial N_i^a / \partial y^b. \quad (6)$$

The vielbeins (5) satisfy the nonholonomy (equivalently, anholonomy) relations

$$[e_{\alpha}, e_{\beta}] = e_{\alpha} e_{\beta} - e_{\beta} e_{\alpha} = W_{\alpha \beta}^\gamma e_{\gamma}, \quad (7)$$

with (antisymmetric) nontrivial anholonomy coefficients $W_{i a}^b = \partial_a N_i^b$ and $W_j^a = \Omega_{ij}^a$.

The geometric objects can be defined in a form adapted to a N–connection structure, following decompositions being invariant under parallel transports preserving the splitting (3). In this case we call them to be distinguished (by the connection structure), i.e. d–objects. For instance, a vector field $X \in TV$ is expressed

$$X = (hX, vX), \quad \text{or} \quad X = X^ae_{a} = X^i e_i + X^a e_a,$$

where $hX = X^i e_i$ and $vX = X^a e_a$ state, respectively, the adapted to the N–connection structure horizontal (h) and vertical (v) components of the vector (which following Refs. [14, 15] is called a distinguished vector, in brief, d–vector). In a similar fashion, the geometric objects on $V$, for instance, tensors, spinors, connections, ... are called respectively d–tensors, d–spinors, d–connections if they are adapted to the N–connection splitting (3).

Theorem 2.2 Any (semi) Riemannian metric $g_{ij}(x)$ on $M$ induces a canonical N–connection structure on $TM$.

\(^6\)We shall use “boldface” symbols if it would be necessary to emphasize that any space and/or geometrical objects are provided/adapted to a N–connection structure, or with the coefficients computed with respect to N–adapted frames.
**Proof.** We sketch a proof by defining the coefficients of \( N \)-connection

\[
\tilde{N}^i_j(x, y) = \frac{\partial \tilde{G}^i_j}{\partial y^j} \tag{8}
\]

where

\[
\tilde{G}^i_j = \frac{1}{4} \tilde{g}^{ij} \left( \frac{\partial^2 \mathcal{L}}{\partial y^i \partial x^j} - \frac{\partial \mathcal{L}}{\partial x^j} \right) = \frac{1}{4} \tilde{g}^{ij} g_{jm} y^j y^m, \tag{9}
\]

\[
\tilde{\gamma}^i_{jm} = \frac{1}{2} g^{ih} (\partial_m g_{ih} + \partial_h g_{mh} - \partial_h g_{im}), \quad \partial_h = \partial/\partial x^h,
\]

with \( g_{ab} \) and \( \tilde{g}_{ij} \) defined respectively by formulas (1) and (2). □

The \( N \)-adapted operators (1) and (2) defined by the \( N \)-connection coefficients \( \tilde{\mathcal{E}} \) are denoted respectively \( \tilde{\mathcal{E}}^i = (\tilde{\mathcal{E}}^i_x, e_a) \) and \( \tilde{\mathcal{E}}^\mu = (e^i, \tilde{\mathcal{E}}^a) \).

### 2.2 Canonical linear connection and metric structures

The constructions will be performed on a vector bundle \( \mathbf{E} \) provided with \( N \)-connection structure. We shall emphasize the special properties of a tangent bundle \( (TM, \pi, M) \) when the linear connection and metric are induced by a (semi) Riemannian metric on \( M \).

**Definition 2.2** A distinguished connection (in brief, \( d \)-connection) \( \mathcal{D} = (h\mathcal{D}, v\mathcal{D}) \) is a linear connection preserving under parallel transports the non-holonomic decomposition (1).

The \( N \)-adapted components \( \Gamma^\alpha_{\beta\gamma} \) of a \( d \)-connection \( \mathcal{D}_\alpha = (e_a | \mathcal{D}) \) are defined by the equations

\[
\mathcal{D}_\alpha e_\beta = \Gamma^\gamma_{\alpha\beta} e_\gamma, \quad \text{or} \quad \Gamma^\gamma_{\alpha\beta} (u) = (\mathcal{D}_\alpha e_\beta) \parallel e^\gamma. \tag{10}
\]

The \( N \)-adapted splitting into \( h \)- and \( v \)-covariant derivatives is stated by

\[
h\mathcal{D} = \{ D_k = (L^i_{jk}, L^a_{bk}) \}, \quad \text{and} \quad v\mathcal{D} = \{ D_c = (C^i_{jk}, C^a_{bc}) \},
\]

where, by definition, \( L^i_{jk} = (D_k e_j) \parallel e^i, L^a_{bk} = (D_k e_b) \parallel e^a, C^i_{jc} = (D_c e_j) \parallel e^i, C^a_{bc} = (D_c e_b) \parallel e^a \). The components \( \Gamma^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}) \) completely define a \( d \)-connection \( \mathcal{D} \) on \( \mathbf{E} \).

The simplest way to perform \( N \)-adapted computations is to use differential forms. For instance, starting with the \( d \)-connection 1–form,

\[
\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} e^\gamma, \tag{11}
\]
with the coefficients defined with respect to N–elongated frames \( (5) \) and \( (4) \), the torsion of a d–connection,

\[
\mathcal{T}^\alpha \triangleq \text{De}^\alpha = \text{de}^\alpha + \Gamma^\alpha_{\beta \gamma} \wedge e^\beta, \tag{12}
\]
is characterized by (N–adapted) d–torsion components,

\[
\begin{align*}
T^i_{jk} &= L^i_{jk} - L^i_{kj}, \\
T^a_{bi} &= -T^a_{ib} = \frac{\partial N^a_i}{\partial y^b} - L^a_{bi}, \\
T^a_{ji} &= \Omega^a_{ji}, \\
T^a_{bc} &= C^a_{bc} - C^a_{cb}.
\end{align*}
\tag{13}
\]

For d–connection structures on \( TM \), we have to identify indices in the form \( i \mapsto a, j \mapsto b, ... \) and the components of N– and d–connections, for instance, \( N^a_i \mapsto N^3_i \) and \( L^i_{jk} \mapsto L^a_{bk} ; C^a_{ja} \mapsto C^b_{ca} \mapsto C^i_{jk} \).

**Definition 2.3** A distinguished metric (in brief, d–metric) on a vector bundle \( E \) is a usual second rank metric tensor \( g = g^{ij}(x, y) e_i \otimes e^j + h^{ab}(x, y) e^a \otimes e^b \), adapted to the N–connection decomposition \( (6) \).

From the class of arbitrary d–connections \( D \) on \( V \), one distinguishes those which are metric compatible (metrical) satisfying the condition

\[
Dg = 0 \tag{15}
\]
including all h- and v-projections \( D_j g_{kl} = 0, \ D_a g_{kl} = 0, \ D_j h_{ab} = 0, \ D_a h_{bc} = 0 \). For d–metric structures on \( V \simeq TM \), with \( g_{ij} = h_{ab} \), the condition of vanishing ”nonmetricity” \( (15) \) transform into

\[
hD(g) = 0 \text{ and } vD(h) = 0, \tag{16}
\]
i.e. \( D_j g_{kl} = 0 \) and \( D_a g_{kl} = 0 \).

For any metric structure \( g \) on a manifold, there is the unique metric compatible and torsionless Levi Civita connection \( \nabla \) for which \( \nabla \mathcal{T}^\alpha = 0 \) and \( \nabla g = 0 \). This connection is not a d–connection because it does not preserve under parallelism the N–connection splitting \( (3) \). One has to consider less constrained cases, admitting nonzero torsion coefficients, when a d–connection is constructed canonically for a d–metric structure. A simple minimal metric compatible extension of \( \nabla \) is that of canonical d–connection \( \hat{D} \) which is metric compatible, with \( T^a_{ji} = 0 \) and \( T^a_{bc} = 0 \) but \( T^a_{ij}, T^a_{ji} \) and \( T^a_{bi} \) are not zero, see \( (13) \). The coefficient formulas for such connections are given in Appendix, see \( (61) \) and related discussion.
Lemma 2.1 Any (semi) Riemannian metric \( g_{ij}(x) \) on a manifold \( M \) induces a canonical d–metric structure on \( TM \),
\[
\tilde{g} = \tilde{g}_{ij}(x, y) \tilde{e}^i \otimes \tilde{e}^j + \tilde{g}_{ij}(x, y) \tilde{\tilde{e}}^i \otimes \tilde{\tilde{e}}^j,
\]
where \( \tilde{e}^i \) are elongated as in (5), but with \( \tilde{N}_i^j \) from (8).

Proof. This construction is similar to that of lifting of the so–called Sasaki metric [20], but using the coefficients \( \tilde{g}_{ij} \).□

Proposition 2.1 There are canonical d–connections on \( TM \) induced by a (semi) Riemannian metric \( g_{ij}(x) \) on \( M \).

Proof. We can construct an example in explicit form by introducing \( \tilde{g}_{ij} \) and \( \tilde{g}_{ab} \) in formulas (62), see Appendix, in order to compute the coefficients \( \tilde{\Gamma}^\alpha_{\beta\gamma} = (\tilde{L}_i^j\,\tilde{C}_a^b) \).□

From the above Lemma and Proposition, one follows the proof of

Theorem 2.3 Any (semi) Riemannian metric \( g_{ij}(x) \) on \( M \) induces a nonholonomic (semi) Riemannian structure on \( TM \).

We note that the induced Riemannian structure is nonholonomic because there is also a nonholonomic distribution \( \tilde{N}_j \) defining \( \tilde{N}_j \). The corresponding curvature tensor \( \tilde{R}^\alpha_{\beta\gamma\tau} = \{\tilde{R}_i^j\,\tilde{C}_a^b, \tilde{P}_j^k, \tilde{S}_b^a\} \) can be computed by introducing respectively the values \( \tilde{g}_{ij}, \tilde{\tilde{N}}_i^j \) and \( \tilde{e}_k \) into formulas (67) from Appendix, for defined \( \tilde{\Gamma}^\alpha_{\beta\gamma} = (\tilde{L}_i^j\,\tilde{C}_a^b) \). Here one should be noted that the constructions on \( TM \) depend on arbitrary vielbein coefficients \( e^\alpha_{\beta}(x, y) \) in (1). We can restrict such sets of coefficients in order to generate various particular classes of (semi) Riemannian geometries on \( TM \), for instance, in order to generate symmetric Riemannian spaces with constant curvature, see Refs. [21, 22, 23].

Corollary 2.1 There are lifts of a (semi) Riemannian metric \( g_{ij}(x) \) on \( M \), \( \text{dim } M = n \), generating a Riemannian structure on \( TM \) with the curvature coefficients of the canonical d–connection coinciding (with respect to \( N \)–adapted bases) to those for a Riemannian space of constant curvature of dimension \( n + n \).

Proof. For a given set \( g_{ij}(x) \) on \( M \), we chose such coefficients \( e^\alpha_{\beta}(x, y) = \{e^a_b(x, y)\} \) in (1) that
\[
g_{ab}(x, y) = e^a_{\alpha}(x, y) e^b_{\beta}(x, y) g_{ab}(x)
\]
results in (2) of type
\[ \bar{\gamma}_{ef} = \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial y^e \partial y^f} = \frac{1}{2} \frac{\partial^2 (e_{a}^{b} e_{b}^{a} y^{a} y^{b})}{\partial y^e \partial y^f} \hat{g}_{ab}(x) = \bar{\gamma}_{ef}, \]

where \( \hat{g}_{ab} \) is the metric of a symmetric Riemannian space (of constant curvature). Considering a prescribed \( \hat{g}_{ab} \), we have to integrate two times on \( y^e \) in order to find any solution for \( e_{a}^{b} \) defining a frame structure in the vertical subspace. The next step is to construct the d–metric \( \hat{\gamma}_{\alpha\beta} = [ \hat{\gamma}_{ij}, \hat{\gamma}_{ab} ] \) of type (17), in our case, with respect to a nonholonomic base elongated by \( \tilde{N}_{ij} \), generated by \( g_{ij}(x) \) and \( \bar{\gamma}_{ef} = \hat{\gamma}_{ab} \), like in (8) and (9). This defines a constant curvature Riemannian space of dimension \( n + n \). The coefficients of the canonical d–connection, which in this case coincide with those for the Levi Civita connection, and the coefficients of the Riemannian curvature can be computed respectively by introducing \( \bar{\gamma}_{ef} = \hat{\gamma}_{ab} \) in formulas (62) and (67), see Appendix. Finally, we note that the induced symmetric Riemannian space contains additional geometric structures like the N–connection and anholonomy coefficients \( W_{\gamma\alpha\beta} \), see (7).

Example 2.1 The simplest example when a Riemannian structure with constant matrix curvature coefficients is generated on \( TM \) is to consider a d–metric induced by \( \tilde{g}_{ij} = \delta_{ij} \), i.e.
\[ \tilde{g}_{[E]} = \delta_{ij} e^{i} \otimes e^{j} + \delta_{ij} \tilde{e}^{i} \otimes \tilde{e}^{j}, \]
with \( \tilde{e}^{i} \) defined by \( \tilde{N}^{i}_{j} \) in their turn defined by a given set \( g_{ij}(x) \) on \( M \).

It should be noted that the metric (19) is generic off–diagonal with respect to a coordinate bases because, in general, the anholonomy coefficients from (7) are not zero. This way, we model on \( TM \) a nonholonomic Euclidean space with vanishing curvature coefficients of the canonical d–connection (it can be verified by introducing respectively the constant coefficients of metric (19) into formulas (62) and (67)). We note that the conditions of Theorem 2.1 are not satisfied by the d–metric (19) (the coefficients \( \tilde{g}_{ij} = \delta_{ij} \) are not defined as in (2)), so we can not relate directly a geometrical mechanics model for such constructions.

There is an important generalization:
Example 2.2 We can consider $\mathcal{L}$ as a hypersurface in $TM$ for which the matrix $\partial^2 \mathcal{L}/\partial y^a \partial y^b$ (i.e. the Hessian, following the analogy with Lagrange mechanics and field theory) is constant and nondegenerate. This states that $\tilde{g}_{ij} = \text{const}$ which results in zero curvature coefficients for the canonical d–connection induced by $\tilde{g}_{ij}(x)$ on $M$.

Finally, in this section, we note that a number of geometric ideas and methods applied in this section were considered in the approaches to the geometry of nonholonomic spaces and generalized Finsler–Lagrangian geometry elaborated by the schools of G. Vranceanu and R. Miron and by A. Bejancu in Romania [21, 25, 14, 15, 16, 17]. We emphasize that this way it is possible to construct geometric models with metric compatible linear connections which is important for elaborating standard approaches compatible with modern (non)commutative gravity and string theory [18, 19]. For Finsler spaces with nontrivial metricity, for instance, for those defined by the the Berwald and Chern connections, see details in [26], the physical theories with local anisotropy are not imbedded in the class of standard models.

3 Curve Flows and Anholonomic Constraints

We formulate the geometry of curve flows adapted to the nonlinear connection structure.

3.1 Non–stretching and N–adapted curve flows

Let us consider a vector bundle $\mathcal{E} = (E, \pi, F, M)$, dim $E = n + m$ (in a particular case, $E = TM$, when $m = n$) provided with d–metric $g = [g, h]$ (14) and N–connection $N^a_i$ (3) structures. A non–stretching curve $\gamma(\tau, l)$ on $V$, where $\tau$ is a parameter and $l$ is the arclength of the curve on $V$, is defined with such evolution d–vector $Y = \gamma_{\tau}$ and tangent d–vector $X = \gamma_{l}$ that $g(X, X) = 1$. A such curve $\gamma(\tau, l)$ swept out a two–dimensional surface in $T_{\gamma(\tau, l)}V \subset TV$.

We shall work with N–adapted bases (4) and (5) and the connection 1–form $\Gamma^a_{\beta\gamma} = \Gamma^a_{\beta\gamma} e^\gamma$ with the coefficients $\Gamma^a_{\beta\gamma}$ for the canonical d–connection operator $D$ (61) (see Appendix) acting in the form

$$D_X e_\alpha = (X | \Gamma^\gamma_\alpha) e_\gamma \text{ and } D_Y e_\alpha = (Y | \Gamma^\gamma_\alpha) e_\gamma,$$

(20)

where $"\mid"$ denotes the interior product and the indices are lowered and raised respectively by the d–metric $g_{\alpha\beta} = [g_{ij}, h_{ab}]$ and its inverse $g^{\alpha\beta} = [g^{ij}, h^{ab}]$. We note that $D_X = X^\alpha D_\alpha$ is the covariant derivation operator along curve
\( \gamma(\tau, t) \). It is convenient to fix the N–adapted frame to be parallel to curve \( \gamma(t) \) adapted in the form

\[
e^1 \equiv hX, \text{ for } i = 1, \text{ and } e^i, \text{ where } h g(hX, e^i) = 0, \tag{21}
e^{a+1} \equiv vX, \text{ for } a = n + 1, \text{ and } e^a, \text{ where } v g(vX, e^a) = 0,
\]

for \( i = 2, 3, \ldots n \) and \( a = n + 2, n + 3, \ldots, n + m \). For such frames, the covariant derivative of each "normal" d–vectors \( e^a \) results into the d–vectors adapted to \( \gamma(t) \),

\[
D_X e^i = -\hat{\rho}^i(u) X \text{ and } D_{hX} hX = \hat{\rho}^i(u) e_i, \tag{22}
D_X e^a = -\hat{\rho}^a(u) X \text{ and } D_{vX} vX = \hat{\rho}^a(u) e_a,
\]

which holds for certain classes of functions \( \hat{\rho}^i(u) \) and \( \hat{\rho}^a(u) \). The formulas (20) and (22) are distinguished into h– and v–components for \( X = hX + vX \) and \( D = (hD, vD) \) for \( D = \{\Gamma^\gamma_{\alpha\beta}\}, hD = \{L_{jk}, L_{kk}\} \) and \( vD = \{C_{jc}, C_{kc}\} \).

Along \( \gamma(1) \), we can move differential forms in a parallel N–adapted form. For instance, \( \Gamma^\alpha_\beta \mid X = X \Gamma^\alpha_\beta \). The algebraic characterization of such spaces, can be obtained if we perform a frame transform preserving the decomposition (3) to an orthonormalized basis \( e_\alpha \), when

\[
e_\alpha \rightarrow A^\alpha_{\alpha'}(u) e_{\alpha'}, \tag{23}
\]

called orthonormal d–basis. In this case, the coefficients of the d–metric transform into the Euclidean ones \( g_{\alpha'\beta'} = \delta_{\alpha'\beta'} \). In distinguished form, we obtain two skew matrices

\[
\Gamma^i_\alpha \mid X = X \Gamma^i_\alpha = 2 e^i_{hX} \rho^\beta_\gamma \text{ and } \Gamma^a_\beta \mid X = X \Gamma^a_\beta = 2 e^a_{vX} \rho^\gamma_\beta
\]

where

\[
e^i_{hX} \equiv g(hX, e^i) = [1, 0, \ldots, 0] \quad \text{and} \quad e^a_{vX} \equiv h(vX, e^a) = [1, 0, \ldots, 0]
\]

and

\[
\Gamma^i_\alpha \mid v = \begin{bmatrix} 0 & \rho^\beta_\gamma \\ -\rho^\beta_\gamma & 0 \end{bmatrix} \quad \text{and} \quad \Gamma^a_\beta \mid v = \begin{bmatrix} 0 & \rho^\gamma_\beta \\ -\rho^\gamma_\beta & 0 \end{bmatrix}
\]

with \( 0_{[h]} \) and \( 0_{[v]} \) being respectively \((n - 1) \times (n - 1)\) and \((m - 1) \times (m - 1)\) matrices. The above presented row–matrices and skew–matrices show that locally an N–anholonomic manifold \( V \) of dimension \( n + m \), with respect to distinguished orthonormalized frames are characterized algebraically by couples of unit vectors in \( \mathbb{R}^n \) and \( \mathbb{R}^m \) preserved respectively by the \( SO(n - 1) \)
and $SO(m-1)$ rotation subgroups of the local $N$–adapted frame structure group $SO(n) \oplus SO(m)$. The connection matrices $\Gamma^j_{iX} \alpha' \beta'$ and $\Gamma^i_{Xa} \alpha' \beta'$ belong to the orthogonal complements of the corresponding Lie subalgebras and algebras, $so(n-1) \subset so(n)$ and $so(m-1) \subset so(m)$.

The torsion (12) and curvature (63) (see Appendix) tensors can be in orthonormalized component form with respect to (21) mapped into a distinguished orthonormalized dual frame (23),

$$\mathcal{T}^\alpha_\beta \doteq D_X e^\alpha_\beta - D_Y e^\alpha_\beta + e^\beta_\gamma \Gamma_{X}^{\alpha'}_\gamma - e^\beta_\gamma \Gamma_{Y}^{\alpha'}_\gamma,$$  

and

$$R^\alpha_\beta(X, Y) = D_Y G_{X}^{\alpha'}_\beta - D_X G_{Y}^{\alpha'}_\beta + G_{Y}^{\gamma'}_\beta \Gamma_{X}^{\alpha'}_\gamma - G_{X}^{\gamma'}_\beta \Gamma_{Y}^{\alpha'}_\gamma,$$  

where $e^\alpha_\beta \doteq g(Y, e^\alpha \gamma)$ and $G_{Y}^{\beta'}_\gamma \doteq Y \Gamma_{Y}^{\alpha'}_\beta = g(e^\alpha \alpha', D_Y e^\beta \beta')$ define respectively the $N$–adapted orthonormalized frame row–matrix and the canonical $d$–connection skew–matrix in the flow directs, and $R^\alpha_\beta(X, Y) = g(e^\alpha \alpha', D_X, D_Y) \Gamma_{Y}^{\beta'}_\beta$ is the curvature matrix. Both torsion and curvature components can be distinguished in $h$– and $v$–components like (13) and (64), by considering $N$–adapted decompositions of type

$$g = [g, h], e^\beta \doteq (e^\beta_\gamma, e^\beta_\beta), e^\alpha \doteq (e^\alpha \gamma, e^\alpha_\beta), X = hX + vX, D = (hD, vD).$$

Finally, we note that the matrices for torsion (24) and curvature (25) can be computed for any metric compatible linear connection like the Levi Civita and the canonical $d$–connection. For our purposes, in this work, we are interested to define such a frame of reference with respect to which the curvature tensor has constant coefficients and the torsion tensor vanishes.

### 3.2 On anholonomic bundles with constant matrix curvature

For vanishing $N$–connection curvature and torsion and constant matrix curvature, we get a holonomic Riemannian manifold and the equations (24) and (25) directly encode a bi–Hamiltonian structure, see details in Refs. [7, 12].

A well known class of Riemannian manifolds for which the frame curvature matrix constant consists of the symmetric spaces $M = G/H$ for compact semisimple Lie groups $G \supset H$. A complete classification and summary of main results on such spaces are given in Refs. [21, 22]. The Riemannian curvature and the metric tensors for $M = G/H$ are covariantly constant and $G$–invariant resulting in constant curvature matrix. In [11, 12], the bi–Hamiltonian operators were investigated for the symmetric spaces with
$M = G/\text{SO}(n)$ with $H = \text{SO}(n) \supset \text{O}(n-1)$ and two examples when $G = \text{SO}(n+1), \text{SU}(n)$. Then it was exploited the existing canonical soldering of Klein and Riemannian symmetric–space geometries [23].

3.2.1 Symmetric nonholonomic tangent bundles

We suppose that the base manifold is a symmetric space $M = hG/\text{SO}(n)$ with the isotropy subgroup $hH = \text{SO}(n) \supset \text{O}(n)$ and the typical fiber space to be a symmetric space $F = vG/\text{SO}(m)$ with the isotropy subgroup $vH = \text{SO}(m) \supset \text{O}(m)$. This means that $hG = \text{SO}(n+1)$ and $vG = \text{SO}(m+1)$ which is enough for a study of real holonomic and nonholonomic manifolds and geometric mechanics models.

Our aim is to solder in a canonic way (like in the N–connection geometry) the horizontal and vertical symmetric Riemannian spaces of dimension $n$ and $m$ with a (total) symmetric Riemannian space $V$ of dimension $n+m$, when $V = G/\text{SO}(n+m)$ with the isotropy group $H = \text{SO}(n+m) \supset \text{O}(n+m)$ and $G = \text{SO}(n+m+1)$. First, we note that for the just mentioned horizontal, vertical and total symmetric Riemannian spaces one exists natural settings to Klein geometry. For instance, the metric tensor $hg = \{g_{ij}\}$ on $M$ is defined by the Cartan–Killing inner product $\langle \cdot, \cdot \rangle_v$ on $T_x hG \simeq hg$ restricted to the Lie algebra quotient spaces $hp = hg/h$, with $T_x hH \simeq h$, where $hg = h \oplus hp$ is stated such that there is an involutive automorphism of $hG$ under $hH$ is fixed, i.e. $[h,h] \subseteq hp$ and $[hp,h] \subseteq h$. In a similar form, we can define the group spaces and related inner products and Lie algebras,

$$
\begin{align*}
\text{for } \mathfrak{v} \mathfrak{g} & = \{\hat{h}_{ab}\}, \langle \cdot, \cdot \rangle_v, T_y \mathfrak{v} \mathfrak{g} \simeq \mathfrak{v} \mathfrak{g}, \mathfrak{v} \mathfrak{p} = \mathfrak{v} \mathfrak{g} / \mathfrak{h}, \text{with} \\
T_y \mathfrak{v} \mathfrak{H} & \simeq \mathfrak{v} \mathfrak{h}, \mathfrak{v} \mathfrak{g} = \mathfrak{v} \mathfrak{h} \oplus \mathfrak{v} \mathfrak{p}, \text{where} \ [\mathfrak{v} \mathfrak{h}, \mathfrak{v} \mathfrak{p}] \subseteq \mathfrak{v} \mathfrak{p}, [\mathfrak{v} \mathfrak{p}, \mathfrak{v} \mathfrak{p}] \subseteq \mathfrak{v} \mathfrak{h}; \\
\text{for } \mathfrak{g} & = \{\hat{g}_{\alpha\beta}\}, \langle \cdot, \cdot \rangle_\mathfrak{g}, T_{(x,y)}G \simeq \mathfrak{g}, \mathfrak{p} = \mathfrak{g} / \mathfrak{h}, \text{with} \\
T_{(x,y)}H & \simeq \mathfrak{h}, \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}, \text{where} \ [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}.
\end{align*}
$$

(26)

We parametrize the metric structure with constant coefficients on $V = G/\text{SO}(n+m)$ in the form

$$
\hat{g} = \hat{g}_{\alpha\beta} du^\alpha \otimes du^\beta,
$$

where $u^\alpha$ are local coordinates and

$$
\hat{g}_{\alpha\beta} = \begin{bmatrix} \hat{g}_{ij} + \hat{N}_{ij}^a \hat{h}_{ab} & \hat{N}_{ij}^e \hat{h}_{ae} \\ \hat{N}_{ij}^e \hat{h}_{be} & \hat{h}_{ab} \end{bmatrix}
$$

(27)

7it is necessary to consider $hG = \text{SU}(n)$ and $vG = \text{SU}(m)$ for the geometric models with spinor and gauge fields
when trivial, constant, N–connection coefficients are computed \( \hat{N}_j^e = \hat{h}^{eb} \hat{g}_{jb} \) for any given sets \( \hat{h}^{eb} \) and \( \hat{g}_{jb} \), i.e. from the inverse metrics defined respectively on \( hG = SO(n+1) \) and by off–blocks \( (n \times n) \)– and \( (m \times m) \)–terms of the metric \( \hat{g}_{\alpha\beta} \). As a result, we define an equivalent d–metric structure of type (14)

\[
\hat{g} = \hat{g}_{ij} e^i \otimes e^j + \hat{h}_{ab} \hat{e}^a \otimes \hat{e}^b, \\
e^i = dx^i, \quad \hat{e}^a = dy^a + \hat{N}_i^a dx^i
\]

defining a trivial \((n + m)\)–splitting \( \hat{g} = \hat{g} \oplus \hat{N} \hat{h} \) because all nonholonomy coefficients \( \hat{W}_{\gamma}^{\alpha\beta} \) and N–connection curvature coefficients \( \hat{\Omega}_{ij}^a \) are zero. In more general form, we can consider any covariant coordinate transforms of (28) preserving the \((n + m)\)–splitting resulting in any \( W_{\alpha\beta}^\gamma = 0 \) (7) and \( \Omega_{ij}^a = 0 \) (6).

It should be noted that even such trivial parametrizations define algebraic classifications of symmetric Riemannian spaces of dimension \( n + m \) with constant matrix curvature admitting splitting (by certain algebraic constraints) into symmetric Riemannian subspaces of dimension \( n \) and \( m \), also both with constant matrix curvature and introducing the concept of N–anholonomic Riemannian space of type \( \hat{V} = [hG = SO(n + 1), vG = SO(m + 1), \hat{N}_i^a] \).

One can be considered that such trivially N–anholonomic group spaces have possess a Lie d–algebra symmetry \( so_{\hat{N}}(n + m) \equiv so(n) \oplus so(m) \).

The simplest generalization on a vector bundle \( \hat{E} \) is to consider nonholonomic distributions on \( V = G/SO(n + m) \) defined locally by arbitrary N–connection coefficients \( N_i^a(x, y) \) with nonvanishing \( W_{\alpha\beta}^\gamma \) and \( \Omega_{ij}^a \) but with constant d–metric coefficients when

\[
\hat{g} = \hat{g}_{ij} e^i \otimes e^j + \hat{h}_{ab} \hat{e}^a \otimes \hat{e}^b, \\
e^i = dx^i, \quad \hat{e}^a = dy^a + N_i^a(x, y) dx^i
\]

This metric is very similar to (19) but with the coefficients \( \hat{g}_{ij} \) and \( \hat{h}_{ab} \) induced by the corresponding Lie d–algebra structure \( so_{\hat{N}}(n + m) \). Such spaces transform into N–anholonomic Riemann–Cartan manifolds \( \hat{V}_N = [hG = SO(n + 1), vG = SO(m + 1), N_i^a] \) with nontrivial N–connection curvature and induced d–torsion coefficients of the canonical d–connection (see formulas (13) computed for constant d–metric coefficients and the canonical d–connection coefficients in (51)). One has zero curvature for the canonical d–connection (in general, such spaces are curved ones with generic off–diagonal metric (29) and nonzero curvature tensor for the Levi Civita connection).\(^8\) This allows us to classify the N–anholonomic manifolds (and vector bundles)

\(^8\)Introducing, constant values for the d–metric coefficients we get zero coefficients for the canonical d–connection which in its turn results in zero values of (64).
as having the same group and algebraic structures of couples of symmetric Riemannian spaces of dimension \( n \) and \( m \) but nonholonomically soldered to the symmetric Riemannian space of dimension \( n + m \). With respect to N–adapted orthonormal bases \((23)\), with distinguished h– and v–subspaces, we obtain the same inner products and group and Lie algebra spaces as in \((26)\).

The classification of N–anholonomic vector bundles is almost similar to that for symmetric Riemannian spaces if we consider that \( n = m \) and try to model tangent bundles of such spaces, provided with N–connection structure. For instance, we can take a (semi) Riemannian structure with the N–connection induced by a absolute energy structure like in \((8)\) and with the canonical d–connection structure \((71)\), for \( \tilde{g}_{ef} = \tilde{g}_{ab} \), like in \((18)\). A straightforward computation of the canonical d–connection coefficients\(^9\) and of d–curvatures for \( ^\circ\tilde{g}_{ij} \) and \( ^\circ\tilde{N}_{ij} \) proves that the nonholonomic Riemannian manifold \((M = SO(n + 1)/SO(n), ^\circ\mathcal{L})\) possess constant both zero canonical d–connection curvature and torsion but with induced nontrivial N–connection curvature \( ^\circ\tilde{\Omega}_{ij} \). Such spaces, being tangent to symmetric Riemannian spaces, are classified similarly to the Riemannian ones with constant matrix curvature, see \((26)\) for \( n = m \) but provided with a nonholonomic structure induced by generating function \( ^\circ\mathcal{L} \).

### 3.2.2 N–anholonomic Klein spaces

The bi–Hamiltonian and solitonic constructions \([12, 11, 5]\) are based on an extrinsic approach soldering the Riemannian symmetric–space geometry to the Klein geometry \([23]\). For the N–anholonomic spaces of dimension \( n + n \), with constant d–curvatures, similar constructions hold true but we have to adapt them to the N–connection structure.

There are two Hamiltonian variables given by the principal normals \( ^h\nu \) and \( ^v\nu \), respectively, in the horizontal and vertical subspaces, defined by the canonical d–connection structure \( D = (hD, vD) \), see formulas \((21)\) and \((22)\),

\[
^h\nu = D_h x hX = \nu^i e^i \quad \text{and} \quad ^v\nu = D_v x vX = \nu^\alpha e^\alpha.
\]

This normal d–vector \( v = (^h\nu, ^v\nu) \), with components of type \( \nu^\alpha = (\nu^i, \nu^\alpha) = (\nu^1, \nu^\nu, \nu^{n+1}, \nu^\alpha) \), is in the tangent direction of curve \( \gamma \). There is also the principal normal d–vector \( \varpi = (^h\varpi, ^v\varpi) \) with components of type \( \varpi^\alpha = (\varpi^i, \varpi^\alpha) = (\varpi^1, \varpi^\nu, \varpi^{n+1}, \varpi^\alpha) \) in the flow direction, with

\[
^h\varpi = D_h x hX = \varpi^i e^i, \quad ^v\varpi = D_v x vX = \varpi^\alpha e^\alpha.
\]

\(^9\)on tangent bundles, such d–connections can be defined to be torsionless
representing a Hamiltonian d–covector field. We can consider that the normal part of the flow d–vector
\[ h_\perp \overset{\perp}{\equiv} Y_\perp = h^\perp e_i + h^\perp e_a \]
represents a Hamiltonian d–vector field. For such configurations, we can consider parallel N–adapted frames \( e_{\alpha'} = (e_i', e_{\alpha'}) \) when the h–variables \( \nu^{\beta'} \), \( \nu^{\alpha'} \) are respectively encoded in the top row of the horizontal canonical d–connection matrices \( \Gamma_{h_i X, \alpha'} \) and \( \Gamma_{h Y, \alpha'} \) and in the row matrix \( (e_Y')_\perp \overset{\perp}{\equiv} e_Y' - g_{\parallel} e_X' \) where \( g_{\parallel} \overset{\parallel}{\equiv} g(h Y, h X) \) is the tangential h–part of the flow d–vector. A similar encoding holds for v–variables \( \nu^{\alpha'}, \nu^{\alpha'}, \nu^{\alpha'}, \nu^{\alpha'} \) in the top row of the vertical canonical d–connection matrices \( \Gamma_{v_i X, \alpha'} \) and \( \Gamma_{v Y, \alpha'} \) and in the row matrix \( (e_Y')_\perp \overset{\perp}{\equiv} e_Y' - h_{\parallel} e_X' \) where \( h_{\parallel} \overset{\parallel}{\equiv} h(v Y, v X) \) is the tangential v–part of the flow d–vector. In a compact form of notations, we shall write \( v^\alpha' \) and \( v^\alpha' \) where the primed small Greek indices \( \alpha', \beta', \ldots \) will denote both N–adapted and then orthonormalized components of geometric objects (like d–vectors, d–covectors, d–tensors, d–groups, d–algebras, d–matrices) admitting further decompositions into h– and v–components defined as nonintegrable distributions of such objects.

With respect to N–adapted orthonormalized frames, the geometry of N–anholonomic manifolds is defined algebraically, on their tangent bundles, by couples of horizontal and vertical Klein geometries considered in [23] and for bi–Hamiltonian soliton constructions in [11]. The N–connection structure induces a N–anholonomic Klein space stated by two left–invariant \( h G \)– and \( v G \)–valued Maurer–Cartan form on the Lie d–group \( G = (h G, v G) \) is identified with the zero–curvature canonical d–connection 1–form \( G^\Gamma = \{ G^\Gamma_{\alpha'}^{\beta'} \} \), where
\[ G^\Gamma_{\alpha'}^{\beta'} = G^\Gamma_{\alpha'}^{\beta'} e^{\alpha'} = h G L_{j'k'}^i e^{i'} + v G C_{j'k'}^i e^{i'} . \]
For trivial N–connection structure in vector bundles with the base and typical fiber spaces being symmetric Riemannian spaces, we can consider that \( h G L_{j'k'}^i \) and \( v G C_{j'k'}^i \) are the coefficients of the Cartan connections \( h G L \) and \( v G C \), respectively for the \( h G \) and \( v G \), both with vanishing curvatures, i.e. with
\[ d G^\Gamma + \frac{1}{2} [ G^\Gamma, G^\Gamma] = 0 \]
and h– and v–components, \( d h G L + \frac{1}{2} [ h G L, h G L] = 0 \) and \( d v G C + \frac{1}{2} [ v G C, v G C] = 0 \), where \( d \) denotes the total derivatives on the d–group manifold \( G = h G \oplus v G \) or their restrictions on \( h G \) or \( v G \). We can consider that \( G^\Gamma \) defines the so–called Cartan d–connection for nonintegrable N–connection structures, see details and supersymmetric/ noncommutative developments in [18, 19].
Through the Lie d–algebra decompositions $\mathfrak{g} = h\mathfrak{g} \oplus v\mathfrak{g}$, for the horizontal splitting: $h\mathfrak{g} = \mathfrak{so}(n) \oplus h\mathfrak{p}$, when $[h\mathfrak{p}, h\mathfrak{p}] \subset \mathfrak{so}(n)$ and $[\mathfrak{so}(n), h\mathfrak{p}] \subset h\mathfrak{p}$; for the vertical splitting $v\mathfrak{g} = \mathfrak{so}(m) \oplus v\mathfrak{p}$, when $[v\mathfrak{p}, v\mathfrak{p}] \subset \mathfrak{so}(m)$ and $[\mathfrak{so}(m), v\mathfrak{p}] \subset v\mathfrak{p}$, the Cartan d–connection determines an N–anholonomic Riemannian structure on the nonholonomic bundle $\tilde{E} = [hG = SO(n+1), vG = SO(m+1), N\mathfrak{e}]$. For $n = m$, and canonical d–objects (N–connection, d–metric, d–connection, ...) derived from (29), or any N–anholonomic space with constant d–curvatures, the Cartan d–connection transforms just in the canonical d–connection (32). It is possible to consider a quotient space with distinguished structure group $V_N = G/\mathfrak{so}(n) \oplus \mathfrak{so}(m)$ regarding $G$ as a principal $(\mathfrak{so}(n) \oplus \mathfrak{so}(m))$–bundle over $\tilde{E}$, which is a N–anholonomic bundle. In this case, we can always fix a local section of this bundle and pull–back $G\mathfrak{g}$ to give a $(h\mathfrak{g} \oplus v\mathfrak{g})$–valued 1–form $\mathfrak{g}\Gamma$ in a point $u \in \tilde{E}$. Any change of local sections define $SO(n) \oplus SO(m)$ gauge transforms of the canonical d–connection $\mathfrak{g}\Gamma$, all preserving the nonholonomic decomposition (33).

There are involutive automorphisms $h\sigma = \pm 1$ and $v\sigma = \pm 1$, respectively, of $h\mathfrak{g}$ and $v\mathfrak{g}$, defined that $\mathfrak{so}(n)$ (or $\mathfrak{so}(m)$) is eigenspace $h\sigma = +1$ (or $v\sigma = +1$) and $h\mathfrak{p}$ (or $v\mathfrak{p}$) is eigenspace $h\sigma = -1$ (or $v\sigma = -1$). It is possible both a N–adapted decomposition and taking into account the existing eigenspaces, when the symmetric parts

$$\Gamma \triangleq \frac{1}{2} (\mathfrak{g}\Gamma + \sigma (\mathfrak{g}\Gamma)),$$

with respective h– and v–splitting $L \triangleq \frac{1}{2} (h\mathfrak{g}L + h\sigma (h\mathfrak{g}L))$ and $C \triangleq \frac{1}{2} (v\mathfrak{g}C + h\sigma (v\mathfrak{g}C))$, defines a $(\mathfrak{so}(n) \oplus \mathfrak{so}(m))$–valued d–connection 1–form. Under such conditions, the antisymmetric part

$$\mathbf{e} \triangleq \frac{1}{2} (\mathfrak{g}\Gamma - \sigma (\mathfrak{g}\Gamma)),$$

with respective h– and v–splitting $he \triangleq \frac{1}{2} (h\mathfrak{g}L - h\sigma (h\mathfrak{g}L))$ and $ve \triangleq \frac{1}{2} (v\mathfrak{g}C - h\sigma (v\mathfrak{g}C))$, defines a $(h\mathfrak{p} \oplus v\mathfrak{p})$–valued N–adapted coframe for the Cartan–Killing inner product $\langle \cdot, \cdot \rangle_p$ on $T_u \mathbf{G} \simeq h\mathfrak{g} \oplus v\mathfrak{g}$ restricted to $T_u \mathbf{V}_N \simeq \mathfrak{p}$. This inner product, distinguished into h– and v–components, provides a d–metric structure of type $\mathfrak{g} = [g, h]$ (14), where $g = \langle he \otimes he \rangle_{h\mathfrak{p}}$ and $h = \langle ve \otimes ve \rangle_{v\mathfrak{p}}$ on $\mathbf{V}_N = \mathbf{G}/SO(n) \oplus SO(m)$.

We generate a $\mathbf{G}(= h\mathbf{G} \oplus v\mathbf{G})$–invariant d–derivative $\mathbf{D}$ whose restriction to the tangent space $T\mathbf{V}_N$ for any N–anholonomic curve flow $\gamma(\tau, l)$ in $\mathbf{V}_N = \mathbf{G}/SO(n) \oplus SO(m)$ is defined via

$$\mathbf{D}_X \mathbf{e} = [\mathbf{e}, \gamma_1 \mathbf{\Gamma}] \quad \text{and} \quad \mathbf{D}_Y \mathbf{e} = [\mathbf{e}, \gamma_2 \mathbf{\Gamma}], \quad (30)$$

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admitting further h- and v–decompositions. The derivatives $D_X$ and $D_Y$ are equivalent to those considered in (20) and obey the Cartan structure equations (21) and (25). For the canonical d–connections, a large class of N–anholonomic spaces of dimension $n = m$, the d–torsions are zero and the d–curvatures are with constant coefficients.

Let $e^{\alpha'} = (e^{\alpha'}, e^{\alpha'})$ be a N–adapted orthonormalized coframe being identified with the $(h p \oplus v p)$–valued coframe $e$ in a fixed orthonormal basis for $p = h p \oplus v p \subset h g \oplus v g$. Considering the kernel/ cokernel of Lie algebra multiplications in the h- and v–subspaces, respectively, $[e_{hX}, h g]$ and $[e_{vX}, v g]$, we can decompose the coframes into parallel and perpendicular parts with respect to $e_X$. We write

$$e = (e_C = h e_C + v e_C, e_{C\perp} = h e_{C\perp} + v e_{C\perp}),$$

for $p( = h p \oplus v p)$–valued mutually orthogonal d–vectors $e_C$ and $e_{C\perp}$, when there are satisfied the conditions $[e_X, e_C]_g = 0$ but $[e_X, e_{C\perp}]_g \neq 0$; such conditions can be stated in h- and v–component form, respectively, $[h e_X, h e_C]_{h g} = 0, [h e_X, h e_{C\perp}]_{h g} \neq 0$ and $[v e_X, v e_C]_{v g} = 0, [v e_X, v e_{C\perp}]_{v g} \neq 0$. One holds also the algebraic decompositions

$$T_u V_N \simeq p = h p \oplus v p = g = h g \oplus v g / so(n) \oplus so(m)$$

and

$$p = p_C \oplus p_{C\perp} = (h p_C \oplus v p_C) \oplus (h p_{C\perp} \oplus v p_{C\perp}),$$

with $p_\parallel \subseteq p_C$ and $p_{C\perp} \subseteq p_\perp$, where $[p_\parallel, p_C] = 0, < p_{C\perp}, p_C > = 0$, but $[p_\parallel, p_{C\perp}] \neq 0$ (i.e. $p_C$ is the centralizer of $e_X$ in $p = h p \oplus v p \subset h g \oplus v g$); in h- and v–components, one have $h p_\parallel \subseteq h p_C$ and $h p_{C\perp} \subseteq h p_\perp$, where $[h p_\parallel, h p_C] = 0, < h p_{C\perp}, h p_C > = 0$, but $[h p_\parallel, h p_{C\perp}] \neq 0$ (i.e. $h p_C$ is the centralizer of $e_{hX}$ in $h p \subset h g$) and $v p_\parallel \subseteq v p_C$ and $v p_{C\perp} \subseteq v p_\perp$, where $[v p_\parallel, v p_C] = 0, < v p_{C\perp}, v p_C > = 0$, but $[v p_\parallel, v p_{C\perp}] \neq 0$ (i.e. $v p_C$ is the centralizer of $e_{vX}$ in $v p \subset v g$). Using the canonical d–connection derivative $D_X$ of a d–covector perpendicular (or parallel) to $e_X$, we get a new d–vector which is parallel (or perpendicular) to $e_X$, i.e. $D_X e_C \in p_{C\perp}$ (or $D_X e_{C\perp} \in p_C$); in h- and v–components such formulas are written $D_{hX} h e_C \in h p_{C\perp}$ (or $D_{hX} h e_{C\perp} \in h p_C$) and $D_{vX} v e_C \in v p_{C\perp}$ (or $D_{vX} v e_{C\perp} \in v p_C$). All such d–algebraic relations can be written in N–anholonomic manifolds and canonical d–connection settings, for instance, using certain relations of type

$$D_X (e^{\alpha'})_C = v^{\alpha'} (e^{\beta'})_{C\perp} \quad \text{and} \quad D_X (e^{\alpha'})_{C\perp} = -v^{\alpha'} (e^{\beta'})_C,$$

for some antisymmetric d–tensors $v^{\alpha'} = -v^{\beta'}$. We get a N–adapted $(SO(n) \oplus SO(m))$–parallel frame defining a generalization of the concept of
Riemannian parallel frame on N–adapted manifolds whenever \( p_C \) is larger than \( p_\parallel \). Substituting \( e^{a'} = (e', \tilde{e}') \) into the last formulas and considering h– and v–components, we define \( SO(n) \)–parallel and \( SO(m) \)–parallel frames (for simplicity we omit these formulas when the Greek small letter indices are split into Latin small letter h– and v–indices).

The final conclusion of this section is that the Cartan structure equations on hypersurfaces swept out by nonholonomic curve flows on N–anholonomic spaces with constant matrix curvature for the canonical d–connection geometrically encode two \( O(n – 1) \)– and \( O(m – 1) \)–invariant, respectively, horizontal and vertical bi–Hamiltonian operators. This holds true if the distinguished by N–connection freedom of the d–group action \( SO(n) \oplus SO(m) \) on \( e \) and \( \Gamma \) is used to fix them to be a N–adapted parallel coframe and its associated canonical d–connection 1–form is related to the canonical covariant derivative on N–anholonomic manifolds.

4 Anholonomic bi–Hamiltonians and Vector Solitons

Introducing N–adapted orthonormalized bases, for N–anholonomic spaces of dimension \( n + n \), with constant curvatures of the canonical d–connection, we can derive bi–Hamiltonian and vector soliton structures similarly to \([12, 11, 5]\). In symbolic, abstract index form, the constructions for nonholonomic vector bundles are similar to those for the Riemannian symmetric–spaces soldered to Klein geometry. We have to distinguish the horizontal and vertical components of geometric objects and related equations.

4.1 Basic equations for N–anholonomic curve flows

In this section, we shall prove the results for the h–components of certain N–anholonomic manifolds with constant d–curvature and then dub the formulas for the v–components omitting similar details.

There is an isomorphism between the real space \( \mathfrak{so}(n) \) and the Lie algebra of \( n \times n \) skew–symmetric matrices. This allows to establish an isomorphism between \( h\mathfrak{p} \simeq \mathbb{R}^n \) and the tangent spaces \( T_xM = \mathfrak{so}(n + 1)/ \mathfrak{so}(n) \) of the Riemannian manifold \( M = SO(n + 1)/ SO(n) \) as described by the following canonical decomposition

\[
h\mathfrak{g} = \mathfrak{so}(n + 1) \supset h\mathfrak{p} \in \left[ \begin{array}{cc} 0 & h\mathfrak{p} \\ -h\mathfrak{p}^T & h\mathfrak{0} \end{array} \right] \text{ for } h\mathfrak{0} \in h\mathfrak{h} = \mathfrak{so}(n)\]
with \( h^p = \{p'\} \subseteq \mathbb{R}^n \) being the \( h \)-component of the \( d \)-vector \( p = (p', p'') \) and \( h^p^T \) mean the transposition of the row \( h^p \). The Cartan–Killing inner product on \( h^g \) is stated following the rule

\[
\langle h^p, h^p \rangle = \frac{1}{2} \text{tr}\left\{ \left[ \begin{array}{ccc} 0 & h^p & \hbox{ } \\ -h^p^T & h^0 & \hbox{ } \\ \hbox{ } & \hbox{ } & \hbox{ } \end{array} \right] \right\}
\]

where \( \text{tr} \) denotes the trace of the corresponding product of matrices. This product identifies canonically \( h^p \simeq \mathbb{R}^n \) with its dual \( h^p^* \simeq \mathbb{R}^n \). In a similar form, we can consider

\[
v^g = so(m+1) \supset v^p \in \left[ \begin{array}{ccc} 0 & v^p & \hbox{ } \\ -v^p^T & v^0 & \hbox{ } \end{array} \right]
\]

for \( v^0 \in v^h = so(m) \)

with \( v^p = \{p''\} \subseteq \mathbb{R}^m \) being the \( v \)-component of the \( d \)-vector \( p = (p', p'') \) and define the Cartan–Killing inner product \( v^p \cdot v^p = \frac{1}{2} \text{tr}\{ \ldots \} \). In general, in the tangent bundle of a \( N \)-anholonomic manifold, we can consider the Cartan–Killing \( N \)-adapted inner product

\[
 p \cdot p = h^p \cdot h^p + v^p \cdot v^p
\]

Following the introduced Cartan–Killing parametrizations, we analyze the flow \( \gamma(\tau, l) \) of a non–stretching curve in \( V_N = G/\text{SO}(n) \oplus \text{SO}(m) \). Let us introduce a coframe \( e \in T^*_\gamma V_N \otimes (h^p \oplus v^p) \), which is a \( N \)-adapted \((\text{SO}(n) \oplus \text{SO}(m))\)-parallel basis along \( \gamma \), and its associated canonical \( d \)-connection 1–form \( \Gamma \in T^*_\gamma V_N \otimes (so(n) \oplus so(m)) \). Such \( d \)-objects are respectively parametrized:

\[
e^x = e^{hx} + e^{vx},
\]

for

\[
e^{hx} = \gamma^{hx} \mid h^e = \left[ \begin{array}{ccc} 0 & (1, 0) \hbox{ } \\ -(0, 0)^T & h^0 \end{array} \right],
\]

and

\[
e^{vx} = \gamma^{vx} \mid v^e = \left[ \begin{array}{ccc} 0 & (1, 0) \hbox{ } \\ -(0, 0)^T & v^0 \end{array} \right],
\]

where we write \((1, 0) \in \mathbb{R}^n, 0 \in \mathbb{R}^{n-1}\) and \((1, 0) \in \mathbb{R}^m, 0 \in \mathbb{R}^{m-1}\);

\[
\Gamma = [\Gamma_{hx}, \Gamma_{vx}],
\]

for

\[
\Gamma_{hx} = \gamma_{hx} \mid L = \left[ \begin{array}{ccc} 0 & (0, 0) \hbox{ } \\ -(0, 0)^T & L \end{array} \right] \in so(n + 1),
\]

\]

}\]

\]

\]
where
\[
L = \begin{bmatrix} 0 & \nabla \gamma \\ - \nabla v^T & h0 \end{bmatrix} \in \mathfrak{so}(n), \ n \in \mathbb{R}^{n-1}, \ h0 \in \mathfrak{so}(n-1),
\]
and
\[
\Gamma_{\nu X} = \gamma_{\nu X} C = \begin{bmatrix} 0 & 0 \\ - (0, h0)^T & C \end{bmatrix} \in \mathfrak{so}(m+1),
\]
where
\[
C = \begin{bmatrix} 0 & \nabla v \\ - \nabla v^T & v0 \end{bmatrix} \in \mathfrak{so}(m), \ n \in \mathbb{R}^{m-1}, \ v0 \in \mathfrak{so}(m-1).
\]
The above parametrizations are fixed in order to preserve the \(SO(n)\) and \(SO(m)\) rotation gauge freedoms on the \(N\)-adapted coframe and canonical d–connection 1–form, distinguished in \(h\)- and \(v\)-components.

There are defined decompositions of horizontal \(SO(n+1)/SO(n)\) matrices like
\[
hp \ni \begin{bmatrix} 0 & hp \\ - hp^T & h0 \end{bmatrix} = \begin{bmatrix} 0 & \begin{pmatrix} hp_\parallel, \nabla \gamma \end{pmatrix} \\ - \begin{pmatrix} hp_\parallel, \nabla \gamma \end{pmatrix}^T & h0 \end{bmatrix}
\]
\[
+ \begin{bmatrix} 0 & \begin{pmatrix} 0, h0 \overrightarrow{p} \end{pmatrix} \\ - \begin{pmatrix} 0, h0 \overrightarrow{p} \end{pmatrix}^T & h0 \end{bmatrix},
\]
into tangential and normal parts relative to \(e_{hX}\) via corresponding decompositions of \(h\)-vectors \(hp = (hp_\parallel, hp_\perp) \in \mathbb{R}^n\) relative to \((1, \nabla \gamma)\), when \(hp_\parallel\) is identified with \(hp_C\) and \(hp_\perp\) is identified with \(hp_\perp = hp_{C\perp}\). In a similar form, it is possible to decompose vertical \(SO(m+1)/SO(m)\) matrices,
\[
v \ni \begin{bmatrix} 0 & vp \\ - vp^T & v0 \end{bmatrix} = \begin{bmatrix} 0 & \begin{pmatrix} vp_\parallel, \nabla \gamma \end{pmatrix} \\ - \begin{pmatrix} vp_\parallel, \nabla \gamma \end{pmatrix}^T & v0 \end{bmatrix}
\]
\[
+ \begin{bmatrix} 0 & \begin{pmatrix} 0, v0 \overrightarrow{p} \end{pmatrix} \\ - \begin{pmatrix} 0, v0 \overrightarrow{p} \end{pmatrix}^T & v0 \end{bmatrix},
\]
into tangential and normal parts relative to \(e_{vX}\) via corresponding decompositions of \(v\)-vectors \(vp = (vp_\parallel, vp_\perp) \in \mathbb{R}^m\) relative to \((1, \nabla \gamma)\), when \(vp_\parallel\) is identified with \(vp_C\) and \(vp_\perp\) is identified with \(vp_\perp = vp_{C\perp}\).

The canonical d–connection induces matrices decomposed with respect to the flow direction. In the \(h\)-direction, we parametrize
\[
e_{hY} = \gamma_{\nu e} he = \begin{bmatrix} 0 & \begin{pmatrix} he_\parallel, h0 \overrightarrow{e} \perp \end{pmatrix} \\ - \begin{pmatrix} he_\parallel, h0 \overrightarrow{e} \perp \end{pmatrix}^T & h0 \end{bmatrix},
\]
into tangential and normal parts relative to \(e_{hX}\) via corresponding decompositions of \(h\)-vectors \(he = (he_\parallel, he_\perp) \in \mathbb{R}^n\) relative to \((1, \nabla \gamma)\), when \(he_\parallel\) is identified with \(he_C\) and \(he_\perp\) is identified with \(he_\perp = he_{C\perp}\).
when \( e_hY \in h.p, (he_\parallel, h \overrightarrow{e}_\perp) \in \mathbb{R}^n \) and \( h \overrightarrow{e}_\perp \in \mathbb{R}^{n-1} \), and
\[
\begin{bmatrix}
0 \\
-(0, \overrightarrow{0})^T \cdot h\overrightarrow{w}_r
\end{bmatrix} \in \mathfrak{so}(n+1),
\]
where
\[
h\overrightarrow{w}_r = \begin{bmatrix}
0 \\
-(\overrightarrow{w})^T \cdot h\Theta
\end{bmatrix} \in \mathfrak{so}(n), \overrightarrow{w} \in \mathbb{R}^{n-1}, h\Theta \in \mathfrak{so}(n-1).
\]

In the \( v \)-direction, we parametrize
\[
\begin{bmatrix}
0 \\
-(0, \overrightarrow{0})^T \cdot v\overrightarrow{w}_r
\end{bmatrix} \in \mathfrak{so}(n+1),
\]
where
\[
v\overrightarrow{w}_r = \begin{bmatrix}
0 \\
-(\overrightarrow{w})^T \cdot v\Theta
\end{bmatrix} \in \mathfrak{so}(m), \overrightarrow{w} \in \mathbb{R}^{m-1}, v\Theta \in \mathfrak{so}(m-1).
\]

The components \( he_\parallel \) and \( h \overrightarrow{e}_\perp \) correspond to the decomposition
\[
e_hY = hg(\gamma_\tau, \gamma_1)e_hX + (\gamma_\tau)_\perp | he_\perp
\]
into tangential and normal parts relative to \( e_hX \). In a similar form, one considers \( ve_\parallel \) and \( v \overrightarrow{e}_\perp \) corresponding to the decomposition
\[
e_vY = vg(\gamma_\tau, \gamma_1)e_vX + (\gamma_\tau)_\perp | ve_\perp.
\]

Using the above stated matrix parametrizations, we get
\[
[e_hX, e_hY] = -\begin{bmatrix}
0 & 0 \\
0 & he_\perp
\end{bmatrix} \in \mathfrak{so}(n+1),
\]
for \( he_\perp = \begin{bmatrix}
0 \\
-(h \overrightarrow{e}_\perp)^T \cdot h\overrightarrow{0}
\end{bmatrix} \in \mathfrak{so}(n);
\]
\[
[\Gamma_hY, e_hY] = -\begin{bmatrix}
0 \\
-(0, \overrightarrow{w})^T \cdot 0
\end{bmatrix} \in h.p;
\]
\[
[\Gamma_hX, e_hY] = -\begin{bmatrix}
0 \\
-(\overrightarrow{w} \cdot h \overrightarrow{e}_\perp, he_\parallel)^T \cdot h\overrightarrow{0}
\end{bmatrix} \in h.p;\]
respectively the geometry Klein N–anholonomic spaces using the relations (30). One obtains (24) and (25) in terms of N–adapted curve flow operators so ordered to the where

\[ R_0 = 0 \]

\( n = \] respectively, and (35) are equivalent, respectively, to (13) and (64). In general, \( T(\gamma_r, \gamma_1) \neq 0 \) and \( R(\gamma_r, \gamma_1) \) can not be defined to have constant matrix coefficients with respect to a N–adapted basis. For N–anholonomic spaces with dimensions \( n = m \), we have \( ^e T(\gamma_r, \gamma_1) = 0 \) and \( ^e R(\gamma_r, \gamma_1) \) defined by constant, or vanishing, d–curvature coefficients (see discussions related to formulas (67) and (62)). For such cases, we can consider the h– and v–components of (34) and (35) in a similar manner as for symmetric Riemannian spaces but with the canonical d–connection instead of the Levi Civita one. One obtains, respectively,

\[ 0 = (D_{hX} \gamma_r - D_{hY} \gamma_1) | h e \]  
\[ = D_{hX} e_Y - D_{hY} e_h + [L_{hX}, e_h Y] - [L_{hY}, e_h X]; \]

and

\[ h R(\gamma_r, \gamma_1) | h e = [D_{hX}, D_{hY}] | h e = D_{hX} L_{hY} - D_{hY} L_{hX} + [L_{hX}, L_{hY}]; \]  
\[ v R(\gamma_r, \gamma_1) | v e = [D_{vX}, D_{vY}] | v e = D_{vX} C_{vY} - D_{vY} C_{vX} + [C_{vX}, C_{vY}]. \]
Following $N$–adapted curve flow parametrizations (32) and (33), the equations (36) and (37) are written

\[ 0 = D_{hX}he_\parallel + \vec{v} \cdot h \vec{e}_\perp, \quad 0 = D_{vX}ve_\parallel + \overrightarrow{v} \cdot v \vec{e}_\perp; \quad (38) \]

\[ 0 = \vec{\omega} - he_\parallel \vec{v} + D_{hX}h \vec{e}_\perp, \quad 0 = \overleftarrow{v}e_\parallel \vec{v} + D_{vX}v \vec{e}_\perp; \quad (39) \]

and

\[ \begin{align*}
D_{hX} \vec{\omega} - D_{hY} \vec{v} + \vec{v} \cdot h \vec{\Theta} & = h \vec{e}_\perp, \quad D_{vX} \overleftarrow{v} - D_{vY} \overrightarrow{v} + \overrightarrow{v} \cdot v \vec{\Theta} = \overleftarrow{v} \vec{e}_\perp; \\
D_{hX} \vec{\Theta} - \overleftarrow{v} \otimes \vec{\omega} + \vec{\omega} \otimes \overleftarrow{v} & = 0, \quad D_{vX} \overleftarrow{v} \vec{\Theta} - \overleftarrow{v} \otimes \overleftarrow{v} + \overleftarrow{v} \otimes \overleftarrow{v} = 0. \quad (39)
\end{align*} \]

The tensor and interior products, for instance, for the $h$–components, are defined in the form: $\otimes$ denotes the outer product of pairs of vectors ($1 \times n$ row matrices), producing $n \times n$ matrices $\vec{A} \otimes \vec{B} = \vec{A}^T \vec{B}$, and $\cdot$ denotes multiplication of $n \times n$ matrices on vectors ($1 \times n$ row matrices); one holds the properties $\vec{A} \cdot (\vec{B} \otimes \vec{C}) = (\vec{A} \cdot \vec{B}) \vec{C}$ which is the transpose of the standard matrix product on column vectors, and $(\vec{B} \otimes \vec{C}) \vec{A} = (\vec{C} \cdot \vec{A}) \vec{B}$. Here we note that similar formulas hold for the $v$–components but, for instance, we have to change, correspondingly, $n \to m$ and $\vec{A} \to \vec{A}$.

The variables $e_\parallel$ and $\Theta$, written in $h$– and $v$–components, can be expressed corresponding in terms of variables $\vec{v}, \vec{\omega}, h \vec{e}_\perp$ and $\overrightarrow{v}, \overleftarrow{v}, v \vec{e}_\perp$ (see respectively the first two equations in (38) and the last two equations in (39)),

\[ \begin{align*}
he_\parallel & = -D_{hX}^{-1}(\vec{v} \cdot h \vec{e}_\perp), \quad ve_\parallel = -D_{vX}^{-1}(\overrightarrow{v} \cdot v \vec{e}_\perp), \\
\end{align*} \]

and

\[ \begin{align*}
h \Theta & = D_{hX}^{-1}(\vec{v} \otimes \vec{\omega} - \vec{\omega} \otimes \vec{v}), \quad v \Theta = D_{vX}^{-1}(\overrightarrow{v} \otimes \overleftarrow{v} - \overleftarrow{v} \otimes \overrightarrow{v}).
\end{align*} \]

Substituting these values, correspondingly, in the last two equations in (38) and in the first two equations in (39), we express

\[ \begin{align*}
\vec{\omega} = -D_{hX}h \vec{e}_\perp - D_{hX}^{-1}(\vec{v} \cdot h \vec{e}_\perp) \vec{v}, \quad \overleftarrow{v} = -D_{vX}v \vec{e}_\perp - D_{vX}^{-1}(\overrightarrow{v} \cdot v \vec{e}_\perp) \overrightarrow{v},
\end{align*} \]

contained in the $h$– and $v$–flow equations respectively on $\vec{v}$ and $\overrightarrow{v}$, considered as scalar components when $D_{hY} \vec{v} = \vec{v}_\tau$ and $D_{hY} \overrightarrow{v} = \overrightarrow{v}_\tau$,

\[ \begin{align*}
\vec{v}_\tau & = D_{hX} \vec{\omega} - \vec{v} \cdot D_{hX}^{-1}(\vec{v} \otimes \vec{\omega} - \vec{\omega} \otimes \vec{v}) - \overrightarrow{R} h \vec{e}_\perp, \\
\overrightarrow{v}_\tau & = D_{vX} \overleftarrow{v} - \overrightarrow{v} \cdot D_{vX}^{-1}(\overrightarrow{v} \otimes \overleftarrow{v} - \overleftarrow{v} \otimes \overrightarrow{v}) - \overleftarrow{S} v \vec{e}_\perp,
\end{align*} \]

where the scalar curvatures of the canonical $d$–connection, $\overleftarrow{R}$ and $\overleftarrow{S}$ are defined by formulas (63) in Appendix. For symmetric Riemannian spaces
like $SO(n + 1)/SO(n) \simeq S^n$, the value $\vec{R}$ is just the scalar curvature $\chi = 1$, see [12]. On N–anholonomic manifolds, it is possible that $\vec{R}$ and $\vec{S}$ are certain zero or nonzero constants.

The above presented considerations consist the proof of

**Lemma 4.1** On N–anholonomic spaces with constant curvature matrix coefficients for the canonical d–connection, there are N–adapted Hamiltonian sympletic operators,

\[ hJ = D_{hX} + D_{hX}^{-1} (\vec{v} \cdot \vec{v}) \] and \[ vJ = D_{vX} + D_{vX}^{-1} (\vec{v} \cdot \vec{v}), \] (41)

and cosympletic operators

\[ hH = D_{hX} + \vec{v} | D_{hX} (\vec{v} \wedge) \] and \[ vH = D_{vX} + \vec{v} | D_{vX} (\vec{v} \wedge), \] (42)

where, for instance, $\vec{A} \wedge \vec{B} = \vec{A} \otimes \vec{B} - \vec{B} \otimes \vec{A}$.

The properties of operators (41) and (42) are defined by

**Theorem 4.1** The d–operators $J = (hJ, vJ)$ and $H = (hH, vH)$ are respectively $(O(n - 1), O(m - 1))$–invariant Hamiltonian sympletic and cosympletic d–operators with respect to the Hamiltonian d–variables $(\vec{v}, \vec{v})$. Such d–operators defines the Hamiltonian form for the curve flow equations on N–anholonomic manifolds with constant d–connection curvature: the h–flows are given by

\[ \vec{v}_h = hH (\vec{v}) - \vec{R} h\vec{e}_\perp = hR (h\vec{e}_\perp) - \vec{R} h\vec{e}_\perp, \]
\[ \vec{v} = hJ (h\vec{e}_\perp); \] (43)

the v–flows are given by

\[ \vec{v}_v = vH (\vec{v}) - \vec{S} v\vec{e}_\perp = vR (v\vec{e}_\perp) - \vec{S} v\vec{e}_\perp, \]
\[ \vec{v} = vJ (v\vec{e}_\perp), \] (44)

where the so–called heriditary recursion d–operator has the respective h– and v–components

\[ hR = hH \circ hJ \] and \[ vR = vH \circ vJ. \] (45)

**Proof.** One follows from the Lemma and (40). In a detailed form, for holonomic structures, it is given in Ref. [7] and discussed in [12]. The above considerations, in this section, consist a soldering of certain classes of generalized Lagrange spaces with $(O(n - 1), O(m - 1))$–gauge symmetry to the geometry of Klein N–anholonomic spaces.\[ \Box \]
4.2 Bi–Hamiltonian anholonomic curve flows and solitonic hierarchies

Following a usual solitonic techniques, see details in Ref. [11, 12], the recursion $h$–operator from (45),

$$h\mathcal{R} = D_hX \left( D_hX + D_{\Delta}^{-1} (\overrightarrow{v} \cdot \overrightarrow{v}) + \overrightarrow{v} \right) \left( \Delta^{-1} (\overrightarrow{v} \wedge D_hX) \right)$$

$$= D^2_hX + |D_hX|^2 + D_{\Delta}^{-1} (\overrightarrow{v} \cdot \overrightarrow{v}) - \overrightarrow{v} \right) \left( \Delta^{-1} (\overrightarrow{v} \wedge D_hX) \right),$$

(46)

generates a horizontal hierarchy of commuting Hamiltonian vector fields $h\overrightarrow{e}_0^{(k)}$ starting from $h\overrightarrow{e}_0^{(0)} = \overrightarrow{v}_1$ given by the infinitesimal generator of 1–translations in terms of arclength $l$ along the curve (we use a boldface $l$ in order to emphasized that the curve is on a N–anholonomic manifold). A vertical hierarchy of commuting vector fields $v\overrightarrow{e}_0^{(k)}$ starting from $v\overrightarrow{e}_0^{(0)} = \overrightarrow{v}_1$ is generated by the recursion $v$–operator

$$v\mathcal{R} = D_vX \left( D_vX + D_{\Delta}^{-1} (\overrightarrow{v} \cdot \overrightarrow{v}) + \overrightarrow{v} \right) \left( \Delta^{-1} (\overrightarrow{v} \wedge D_vX) \right)$$

$$= D^2_vX + |D_vX|^2 + D_{\Delta}^{-1} (\overrightarrow{v} \cdot \overrightarrow{v}) - \overrightarrow{v} \right) \left( \Delta^{-1} (\overrightarrow{v} \wedge D_vX) \right).$$

(47)

There are related hierarchies, generated by adjoint operators $\mathcal{R}^* = (h\mathcal{R}^*, v\mathcal{R}^*)$, of involutive Hamiltonian $h$–covector fields $\overrightarrow{e}_0^{(k)} = \delta(hH^{(k)})/\delta\overrightarrow{v}$ in terms of Hamiltonians $hH = hH^{(k)}(\overrightarrow{v}, \overrightarrow{v}_1, \overrightarrow{v}_{21}, ...)$. Starting from $\overrightarrow{e}_0^{(0)} = \overrightarrow{v}$, $hH^{(0)} = \frac{1}{2} |\overrightarrow{v}|^2$ and of involutive Hamiltonian $v$–covector fields $\overrightarrow{e}_0^{(k)} = \delta(vH^{(k)})/\delta\overrightarrow{v}$ in terms of Hamiltonians $vH = vH^{(k)}(\overrightarrow{v}, \overrightarrow{v}_1, \overrightarrow{v}_{21}, ...)$. Starting from $\overrightarrow{e}_0^{(0)} = \overrightarrow{v}$, $vH^{(0)} = \frac{1}{2} |\overrightarrow{v}|^2$. The relations between hierarchies are established correspondingly by formulas

$$h\overrightarrow{e}_0^{(k)} = h\mathcal{H} \left( \overrightarrow{e}_0^{(k)}, \overrightarrow{e}_0^{(k+1)} \right) = h\mathcal{J} \left( h\overrightarrow{e}_0^{(k)} \right)$$

and

$$v\overrightarrow{e}_0^{(k)} = v\mathcal{H} \left( \overrightarrow{e}_0^{(k)}, \overrightarrow{e}_0^{(k+1)} \right) = v\mathcal{J} \left( v\overrightarrow{e}_0^{(k)} \right),$$

where $k = 0, 1, 2, ...$. All hierarchies (horizontal, vertical and their adjoint ones) have a typical mKdV scaling symmetry, for instance, $l \rightarrow \lambda l$ and $\overrightarrow{v} \rightarrow \lambda^{-1} \overrightarrow{v}$ under which the values $h\overrightarrow{e}_0^{(k)}$ and $hH^{(k)}$ have scaling weight $2 + 2k$, while $\overrightarrow{e}_0^{(k)}$ has scaling weight $1 + 2k$.

The above presented considerations prove

**Corollary 4.1** There are $N$–adapted hierarchies of distinguished horizontal and vertical commuting bi–Hamiltonian flows, correspondingly, on $\overrightarrow{v}$ and $\overrightarrow{v}$.
associated to the recursion d–operator \([45]\) given by \(O(n - 1) \oplus O(m - 1)\) –invariant d–vector evolution equations,
\[
\vec{v}_\tau = h \vec{e}_{(k+1)} - \vec{R} h \vec{e}_{(k)} = h \mathcal{H} \left( \Phi \left( h \mathcal{H}^{(k+1)} \right) / \delta \vec{v} \right)
\]
with horizontal Hamiltonians \(h \mathcal{H}^{(k+1)} = h \mathcal{H}^{(k+1)} - \vec{R} h \mathcal{H}^{(k)}\) and
\[
\vec{v}_\tau = v \vec{e}_{(k+1)} - \vec{S} v \vec{e}_{(k)} = v \mathcal{H} \left( \Phi \left( v \mathcal{H}^{(k+1)} \right) / \delta \vec{v} \right)
\]
with vertical Hamiltonians \(v \mathcal{H}^{(k+1)} = v \mathcal{H}^{(k+1)} - \vec{S} v \mathcal{H}^{(k)}\), for \(k = 0, 1, 2, \ldots\). The d–operators \(\mathcal{H}\) and \(\mathcal{J}\) are \(N\)–adapted and mutually compatible from which one can be constructed an alternative (explicit) Hamilton d–operator \(\mathcal{A} = \mathcal{H} \circ \mathcal{J} \circ \mathcal{H} = \mathcal{R} \circ \mathcal{H}\).

### 4.2.1 Formulation of the main theorem

The main goal of this paper is to prove that for any regular Lagrange system one can be defined naturally a \(N\)–adapted bi–Hamiltonian flow hierarchy inducing anholonomic solitonic configurations.

**Theorem 4.2** For any vector bundle with prescribed d–metric structure, one can be defined a hierarchy of bi-Hamiltonian \(N\)–adapted flows of curves \(\gamma(\tau, l) = h \gamma(\tau, l) + v \gamma(\tau, l)\) described by geometric nonholonomic map equations. The 0 flows are defined as convective (travelling wave) maps
\[
\gamma_\tau = \gamma_1, \text{ distinguished } (h \gamma)_\tau = (h \gamma)_{h^X} \text{ and } (v \gamma)_\tau = (v \gamma)_{v^X}. \tag{48}
\]
There are +1 flows defined as non–stretching mKdV maps
\[
- (h \gamma)_\tau = D_{h^X} (h \gamma)_{h^X} + \frac{3}{2} |D_{h^X} (h \gamma)_{h^X}|_{h^g}^2 (h \gamma)_{h^X}, \tag{49}
\]
\[
- (v \gamma)_\tau = D_{v^X} (v \gamma)_{v^X} + \frac{3}{2} |D_{v^X} (v \gamma)_{v^X}|_{v^g}^2 (v \gamma)_{v^X},
\]
and the +2,\ldots flows as higher order analogs. Finally, the -1 flows are defined by the kernels of recursion operators \([46]\) and \([47]\) inducing non–stretching maps
\[
D_{h^Y} (h \gamma)_{h^X} = 0 \text{ and } D_{v^Y} (v \gamma)_{v^X} = 0. \tag{50}
\]
Proof. It is given in the next section 4.2.2.

For similar constructions in gravity models with nontrivial torsion and nonholonomic structure and related geometry of noncommutative/ super–
spaces and anholonomic spinors, it is important [18, 19]

Remark 4.1 N–adapted hierarchies of bi–Hamiltonian operators and related solitonic equations can be defined for $SU(n) \oplus SU(m) / SO(n) \oplus SO(m)$ symmetries like it was constructed in Ref. [17] for the Riemannian symmetric spaces. In this paper, we restrict our considerations only for real nonholonomic models. Similar results, to those from the Theorem 4.2, can be reformulated for unitary groups which may be very important in modern quantum / (non)commutative gravity.

Finally, it should be emphasized that a number of exact solutions in gravity can be nonholonomically deformed in order to generate nonholonomic hierarchies of gravitational solitons of type (48), (49) or (50), which will be consider in our further publications.

4.2.2 Proof of the main theorem

We provide a proof of Theorem 4.2 for the horizontal flows. The approach is based on the method provided in Section 3 of Ref. [11] but in this work the Levi Civita connection on symmetric Riemannian spaces is substituted by the horizontal components of the canonical d–connection in a generalized Lagrange space with constant d–curvature coefficients. The vertical constructions are similar but with respective changing of h– variables / objects into v– variables/ objects.

One obtains a vector mKdV equation up to a convective term (can be absorbed by redefinition of coordinates) defining the +1 flow for $h e_{\perp} = \bar{v}_1$,

$$\bar{v}_\tau = \bar{v}_{31} + \frac{3}{2} |\bar{v}|^2 - \bar{R} \cdot \bar{v}_1,$$

when the $(k+1)$ flow gives a vector mKdV equation of higher order $3 + 2k$ on $\bar{v}$ and there is a 0 h–flow $\bar{v}_\tau = \bar{v}_{1}$ arising from $h e_{\perp} = 0$ and $h e_{\parallel} = 1$ belonging outside the hierarchy generated by $hR$. Such flows correspond to N–adapted horizontal motions of the curve $\gamma(\tau, l) = h \gamma(\tau, l) + v \gamma(\tau, l)$, given by

$$(h \gamma)_\tau = f ( (h \gamma)_h, D_h X (h \gamma)_h, D_h X (h \gamma)_h, ... )$$

subject to the non–stretching condition $| (h \gamma)_h |_{hS} = 1$, when the equation of motion is to be derived from the identifications

$$(h \gamma)_\tau \leftrightarrow e_h X, D_h X (h \gamma)_h \leftrightarrow D_h X e_h X = [L_h X, e_h X]$$
and so on, which maps the constructions from the tangent space of the curve to the space $h \mathfrak{p}$. For such identifications, we have

$$[L_{hX}, e_{hX}] = -\begin{bmatrix} 0 & (0, \vec{v}) \hfill \\
-(0, \vec{v})^T & h0 \hfill 
\end{bmatrix} \in h \mathfrak{p},$$

$$[L_{hX}, [L_{hX}, e_{hX}]] = -\begin{bmatrix} 0 & (|\vec{v}|^2, \vec{0}) \hfill \\
-(|\vec{v}|^2, \vec{0})^T & h0 \hfill 
\end{bmatrix}$$

and so on, see similar calculus in (32). At the next step, stating for the +1 $h$–flow

$$h \vec{e}_\perp = \vec{v}_1$$ and $h \vec{e}_\parallel = -D_{hX}^{-1} (\vec{v} \cdot \vec{v}_1) = -\frac{1}{2} |\vec{v}|^2,$

we compute

$$e_{hY} = \begin{bmatrix} 0 & (h e_\parallel, h \vec{e}_\perp) \hfill \\
-(h e_\parallel, h \vec{e}_\perp)^T & h0 \hfill 
\end{bmatrix}$$

$$= -\frac{1}{2} |\vec{v}|^2 \begin{bmatrix} 0 & (1, \vec{0}) \hfill \\
-(0, \vec{0})^T & h0 \hfill 
\end{bmatrix} + \begin{bmatrix} 0 & (0, \vec{v}_{hx}) \hfill \\
-(0, \vec{v}_{hx})^T & h0 \hfill 
\end{bmatrix}$$

$$= D_{hX} [L_{hX}, e_{hX}] + \frac{1}{2} [L_{hX}, [L_{hX}, e_{hX}]]$$

$$= -D_{hX} [L_{hX}, e_{hX}] - \frac{3}{2} |\vec{v}|^2 e_{hX}.$$
where

\[ \vec{v} = -\left[ \begin{array}{cc} 0 & \vec{v}^T \\ \vec{0}^T & h0 \end{array} \right] \in \mathfrak{so}(n), \]

and the derived (applying \( ad([L_{hX}, e_{hX}]) \) again )

\[ ad([L_{hX}, e_{hX}])^2 e_{hX} = -|\vec{v}|^2 \begin{bmatrix} 0 & 1, 0 \\ -1, 0 & 0 \end{bmatrix} = -|\vec{v}|^2 e_{hX}, \]

the equation (49) can be represented in alternative form

\[-(h\gamma)_\tau = D^2_{hX} (h\gamma)_{hX} - \frac{3}{2} \tilde{R}^{-1} ad (D_{hX} (h\gamma)_{hX})^2 (h\gamma)_{hX},\]

which is more convenient for analysis of higher order flows on \( \vec{v} \) subjected to higher-order geometric partial differential equations. Here we note that the 0 flow one \( \vec{v} \) corresponds to just a convective (linear travelling \( h \)-wave but subjected to certain nonholonomic constraints) map equation (48).

Now we consider a -1 flow contained in the \( h \)-hierarchy derived from the property that \( h \vec{e}_\perp \) is annihilated by the \( h \)-operator \( hJ \) and mapped into \( hR(h \vec{e}_\perp) = 0 \). This means that \( hJ(h \vec{e}_\perp) = \vec{v} = 0 \). Such properties together with (31) and equations (10) imply \( L_r = 0 \) and hence \( hD_r e_{hX} = [L_r, e_{hX}] = 0 \) for \( hD_r = hD_r + [L_r, \cdot] \). We obtain the equation of motion for the \( h \)-component of curve, \( h\gamma(\tau, 1) \), following the correspondences \( D_{hY} \leftrightarrow hD_\tau \) and \( h\gamma_1 \leftrightarrow e_{hX} \),

\[ D_{hY} (h\gamma(\tau, 1)) = 0, \]

which is just the first equation in (50).

Finally, we note that the formulas for the \( v \)-components, stated by Theorem 4.2 can be derived in a similar form by respective substitution in the the above proof of the \( h \)-operators and \( h \)-variables into \( v \)-ones, for instance, \( h\gamma \to v\gamma, h \vec{e}_\perp \to v \vec{e}_\perp, \vec{v} \to \vec{v}, \vec{v} \to \vec{v}, D_{hX} \to D_{eX}, D_{hY} \to D_{eY}, L \to C, R \to S, hD \to vD, hR \to vR, hJ \to vJ, ... \)

4.3 Nonholonomic mKdV and SG hierarchies

We consider explicit constructions when solitonic hierarchies are derived following the conditions of Theorem 4.2.

The \( h \)-flow and \( v \)-flow equations resulting from (50) are

\[ \vec{v}_\tau = -\tilde{R} h \vec{e}_\perp \text{ and } \vec{v}_\tau = -\tilde{S} v \vec{e}_\perp, \] (51)
when, respectively,

\[ 0 = \overline{\omega} = -D_{h}X_h \overline{e}_{\perp} + he_{\parallel} \overline{v}, \quad D_{h}X_h e_{\parallel} = h \overline{e}_{\perp} \cdot \overline{v} \]

and

\[ 0 = \overline{\omega} = -D_{v}X_v \overline{e}_{\perp} + ve_{\parallel} \overline{v}, \quad D_{v}X_v e_{\parallel} = v \overline{e}_{\perp} \cdot \overline{v}. \]

The d–flow equations possess horizontal and vertical conservation laws

\[ D_{h}X_h ((h e_{\parallel})^2 + |h \overline{e}_{\perp}|^2) = 0, \]

for \((h e_{\parallel})^2 + |h \overline{e}_{\perp}|^2 = < h e_{\tau}, h e_{\tau} >_{h g} = |(h \gamma)_{\tau}|^2_{h g}, \) and

\[ D_{v}X_v ((v e_{\parallel})^2 + |v \overline{e}_{\perp}|^2) = 0, \]

for \((v e_{\parallel})^2 + |v \overline{e}_{\perp}|^2 = < v e_{\tau}, v e_{\tau} >_{v g} = |(v \gamma)_{\tau}|^2_{v g}. \) This corresponds to

\[ D_{h}X |(h \gamma)_{\tau}|^2_{h g} = 0 \quad \text{and} \quad D_{v}X |(v \gamma)_{\tau}|^2_{v g} = 0. \]

We note that the problem of formulating conservation laws on N–anholonomic spaces (in particular, on nonholonomic vector bundles) in analyzed in Ref. [19]. In general, such laws are more sophisticated than those on (semi) Riemannian spaces because of nonholonomic constraints resulting in non–symmetric Ricci tensors and different types of identities. But for the geometries modelled for dimensions \( n = m \) with canonical d–connections, we get similar h– and v–components of the conservation law equations as on symmetric Riemannian spaces.

It is possible to rescale conformally the variable \( \tau \) in order to get \(|(h \gamma)_{\tau}|^2_{h g} = 1 \) and (it could be for other rescaling) \(|(v \gamma)_{\tau}|^2_{v g} = 1 \), i.e. to have

\((h e_{\parallel})^2 + |h \overline{e}_{\perp}|^2 = 1 \quad \text{and} \quad (v e_{\parallel})^2 + |v \overline{e}_{\perp}|^2 = 1. \)

In this case, we can express \( h e_{\parallel} \) and \( h \overline{e}_{\perp} \) in terms of \( \overline{v} \) and its derivatives and, similarly, we can express \( v e_{\parallel} \) and \( v \overline{e}_{\perp} \) in terms of \( \overline{v} \) and its derivatives, which follows from (51). The N–adapted wave map equations describing the -1 flows reduce to a system of two independent nonlocal evolution equations for the h– and v–components,

\[ \overline{v}_{\tau} = -D_{h}^{-1}X ((\sqrt{R^2} - |\overline{v}_{\tau}|^2) \overline{v}) \quad \text{and} \quad \overline{v}_{\tau} = -D_{v}^{-1}X ((\sqrt{S^2} - |\overline{v}_{\tau}|^2) \overline{v}). \]

For N–anholonomic spaces of constant scalar d–curvatures, we can rescale the equations on \( \tau \) to the case when the terms \( R^2, S^2 = 1 \), and the evolution equations transform into a system of hyperbolic d–vector equations,

\[ D_{h}X(\overline{v}_{\tau}) = -\sqrt{1 - |\overline{v}_{\tau}|^2} \overline{v} \quad \text{and} \quad D_{v}X(\overline{v}_{\tau}) = -\sqrt{1 - |\overline{v}_{\tau}|^2} \overline{v}, \quad (52) \]

32
where $D_{hX} = \partial_{h} l$ and $D_{vX} = \partial_{v} l$ are usual partial derivatives on direction $l = h l + v l$ with $\overrightarrow{v} \tau$ and $\overleftarrow{v} \tau$ considered as scalar functions for the covariant derivatives $D_{hX}$ and $D_{vX}$ defined by the canonical $d$–connection. It also follows that $h \overrightarrow{e}_\bot$ and $v \overleftarrow{e}_\bot$ obey corresponding vector sine–Gordon (SG) equations

$$\left(\sqrt{(1 - |h \overrightarrow{e}_\bot|^2)^{-1}} \partial_{hl}(h \overrightarrow{e}_\bot)\right)_\tau = - h \overrightarrow{e}_\bot \tag{53}$$

and

$$\left(\sqrt{(1 - |v \overleftarrow{e}_\bot|^2)^{-1}} \partial_{vl}(v \overleftarrow{e}_\bot)\right)_\tau = - v \overleftarrow{e}_\bot. \tag{54}$$

The above presented formulas and Corollary 4.1 imply

Conclusion 4.1 The recursion $d$–operator $R = (hR, hR)$ \textcolor{blue}{[43]}, see \textcolor{blue}{[40]} and \textcolor{blue}{[47]}, generates two hierarchies of vector mKdV symmetries: the first one is horizontal,

$$\overrightarrow{v}^{(0)}_\tau = \overrightarrow{v} h l, \quad \overrightarrow{v}^{(1)}_\tau = hR(\overrightarrow{v} h l) = \overrightarrow{v} 3 hl + \frac{3}{2} |\overrightarrow{v}|^2 \overrightarrow{v} h l, \tag{55}$$

$$\overrightarrow{v}^{(2)}_\tau = hR^2(\overrightarrow{v} h l) = \overrightarrow{v} 5 hl + \frac{5}{2} (|\overrightarrow{v}|^2 \overrightarrow{v} 2 hl) h l$$

$$+ \frac{5}{2} \left((|\overrightarrow{v}|^2) h l h l + |\overrightarrow{v} h l|^2 + \frac{3}{4} |\overrightarrow{v}|^4\right) \overrightarrow{v} h l - \frac{1}{2} |\overrightarrow{v} h l|^2 \overrightarrow{v},$$

with all such terms commuting with the $-1$ flow

$$\left(\overrightarrow{v} \tau\right)^{-1} = h \overrightarrow{e}_\bot \tag{56}$$

associated to the vector SG equation \textcolor{blue}{[53]}; the second one is vertical,

$$\overleftarrow{v}^{(0)}_\tau = \overleftarrow{v} v l, \quad \overleftarrow{v}^{(1)}_\tau = vR(\overleftarrow{v} v l) = \overleftarrow{v} 3 vl + \frac{3}{2} |\overleftarrow{v}|^2 \overleftarrow{v} v l, \tag{57}$$

$$\overleftarrow{v}^{(2)}_\tau = vR^2(\overleftarrow{v} v l) = \overleftarrow{v} 5 vl + \frac{5}{2} (|\overleftarrow{v}|^2 \overleftarrow{v} 2 vl) v l$$

$$+ \frac{5}{2} \left((|\overleftarrow{v}|^2) v l v l + |\overleftarrow{v} v l|^2 + \frac{3}{4} |\overleftarrow{v}|^4\right) \overleftarrow{v} v l - \frac{1}{2} |\overleftarrow{v} v l|^2 \overleftarrow{v},$$

with all such terms commuting with the $-1$ flow

$$\left(\overleftarrow{v} \tau\right)^{-1} = v \overleftarrow{e}_\bot \tag{58}$$

associated to the vector SG equation \textcolor{blue}{[53]}. 33
In its turn, using the above Conclusion, we derive that the adjoint d-operator $R^* = J \circ H$ generates a horizontal hierarchy of Hamiltonians,

$$hH^{(0)} = \frac{1}{2} |\vec{v}|^2, \quad hH^{(1)} = -\frac{1}{2} |\vec{v}_h|^{2} + \frac{1}{8} |\vec{v}|^{4},$$

$$hH^{(2)} = \frac{1}{2} |\vec{v}_2|^{2} - \frac{3}{4} |\vec{v}|^{2} |\vec{v}_h|^{2} - \frac{1}{2} (\vec{v} \cdot \vec{v}_h) + \frac{1}{16} |\vec{v}|^{6}, ..., \tag{59}$$

and vertical hierarchy of Hamiltonians

$$vH^{(0)} = \frac{1}{2} |\vec{v}|^2, \quad vH^{(1)} = -\frac{1}{2} |\vec{v}_v|^{2} + \frac{1}{8} |\vec{v}|^{4},$$

$$vH^{(2)} = \frac{1}{2} |\vec{v}_2|^{2} - \frac{3}{4} |\vec{v}|^{2} |\vec{v}_v|^{2} - \frac{1}{2} (\vec{v} \cdot \vec{v}_v) + \frac{1}{16} |\vec{v}|^{6}, ..., \tag{60}$$

all of which are conserved densities for respective horizontal and vertical flows and determining higher conservation laws for the corresponding hyperbolic equations (53) and (54).

The above presented horizontal equations (53), (55), (56) and (59) and of vertical equations (54), (57), (58) and (60) have similar mKdV scaling symmetries but on different parameters $\lambda_h$ and $\lambda_v$ because, in general, there are two independent values of scalar curvatures $\vec{R}$ and $\vec{S}$, see (66). The horizontal scaling symmetries are $h1 \to \lambda_h h1, \vec{v} \to (\lambda_h)^{-1} \vec{v}$ and $\tau \to (\lambda_h)^{1+2k}$, for $k = -1,0,1,2,...$ For the vertical scaling symmetries, one has $v1 \to \lambda_v v1, \vec{v} \to (\lambda_v)^{-1} \vec{v}$ and $\tau \to (\lambda_v)^{1+2k}$, for $k = -1,0,1,2,...$

Finally, we consider again the Remark 4.1 stating that similar results (proved in Section 4) can be alternatively derived for unitary groups with complex variables. It is really so, but the generated bi–Hamiltonian and solitonic horizontal and vertical hierarchies with unitary gauge symmetry are different from those defined for real orthogonal groups; for holonomic spaces this is demonstrated in Section 4 of Ref. [11]. This distinguishes substantially the models of gauge gravity with structure groups like the unitary one from those with orthogonal groups.

5 Conclusion

In this paper, the geometry of (semi) Riemannian spaces was encoded in nonholonomic hierarchies of bi–Hamiltonian structures and related solitonic equations derived for curve flows on tangent spaces. The local algebraic structure of modelled nonholonomic spaces is defined by the dimensions of the base and typical fiber subspaces. If such subspaces are Riemannian symmetric manifolds, respectively, of dimensions $n$ and $m$, their geometric properties
are exhausted by the geometry of distinguished Lie groups $G = GO(n) \oplus GO(m)$ and $G = SU(n) \oplus SU(m)$ and the geometry of nonlinear connections on such vector bundles. This can be formulated equivalently in terms of geometric objects on couples of Klein spaces. The bi–Hamiltonian and related solitonic (of type mKdV and SG) hierarchies are generated naturally by wave map equations and recursion operators associated to the horizontal and vertical flows of curves on such spaces.

We proved that for any (semi) Riemanninan metric on a base manifold $M$ it is possible to define canonical geometric object and their nonholonomic deformations on tangent bundles. The curvature matrix, with respect to the correspondingly adapted frames, can be constructed to posses constant coefficients. For such configurations, we can apply the former methods elaborated for symmetric Riemannian spaces in order to generate curve flow – solitonic hierarchies.

Finally, we note that curve flow – solitonic hierarchies can be constructed in a similar manner for exact solutions of Einstein–Yang–Mills–Dirac equations, derived following the anholonomic frame method, in noncommutative generalizations of gravity and geometry and possible quantum models based on nonholonomic Lagrange–Fedosov manifolds. We are continuing to work in such directions.

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A Some Local Formulas

There are outlined some local results from geometry of nonlinear connections (see Refs. [14, 15, 18, 19] for proofs and details). There are two types of preferred linear connections uniquely determined by a generic off–diagonal metric structure with $n + m$ splitting, see $g = g \oplus_N h$:

1. The Levi Civita connection $\nabla = \{\Gamma^\alpha_{\beta\gamma}\}$ is by definition torsionless, $\mathcal{T} = 0$, and satisfies the metric compatibility condition, $\nabla g = 0$.

2. The canonical d–connection $\hat{\Gamma}^\gamma_{\alpha\beta} = (\hat{L}^i_{jk}, \hat{L}^a_{bk}, \hat{C}^i_{jc}, \hat{C}^a_{bc})$ is also metric compatible, i. e. $\hat{D}g = 0$, but the torsion vanishes only on h– and v–subspaces, i.e. $\hat{T}^i_{jk} = 0$ and $\hat{T}^a_{bc} = 0$, for certain nontrivial values of $\hat{T}^i_{ja}, \hat{T}^a_{ja}, \hat{T}^i_{ji}$. For simplicity, we omit hats on symbols and write, for simplicity, $L^i_{jk}$ instead of $\hat{L}^i_{jk}, T^i_{ja}$ instead of $\hat{T}^i_{ja}$ and so on...but preserve the general symbols $\hat{D}$ and $\hat{\Gamma}^\gamma_{\alpha\beta}$.
By a straightforward calculus with respect to \(N\)-adapted frames \([14]\) and \([15]\), one can verify that the requested properties for \(\hat{D}\) on \(E\) are satisfied if

\[
L^i_{jk} = \frac{1}{2} g^{ir} (e_k g_{jr} + e_j g_{kr} - e_r g_{jk}),
\]

(61)

\[
L^a_{bk} = e_b (N^a_k) + \frac{1}{2} h^{ac} (e_k h_{bc} - h_{dc} e_b N^d_k - h_{db} e_c N^d_k),
\]

\[
C^i_{jc} = \frac{1}{2} g^{ik} e_j g_{jk}, \quad C^a_{bc} = \frac{1}{2} h^{ad} (e_c h_{bd} + e_b h_{cd} - e_d h_{bc}).
\]

For \(E = TM\), the canonical \(d\)-connection \(\hat{D} = (h\hat{D}, v\hat{D})\) can be defined in torsionless form\(^{10}\) with the coefficients \(\Gamma^\alpha_{\beta\gamma} = (L^i_{jk}, L^a_{bk})\),

\[
L^i_{jk} = \frac{1}{2} g^{ih} (e_k g_{jh} + e_j g_{kh} - e_h g_{jk}),
\]

(62)

\[
C^a_{bc} = \frac{1}{2} h^{ae} (e_c h_{be} + e_b h_{ce} - e_e h_{bc}).
\]

The curvature of a \(d\)-connection, \(D\),

\[
\mathcal{R}^\alpha_{\beta\gamma\delta} = d \Gamma^\alpha_{\beta\gamma} - \Gamma^\gamma_{\beta\delta} \wedge \Gamma^\alpha_{\gamma\delta},
\]

(63)

splits into six types of \(N\)-adapted components with respect to \([14]\) and \([15]\),

\[
\mathcal{R}^\alpha_{\beta\gamma\delta} = (R^i_{hjk}, R^a_{bjk}, P^i_{hja}, P^c_{bja}, S^i_{jbc}, S^a_{bdc}),
\]

(64)

Contracting respectively the components, \(R_{\alpha\beta} \doteq R^r_{\alpha\beta r}\), one computes the \(h\)-\(v\)-components of the Ricci \(d\)-tensor (there are four \(N\)-adapted components)

\[
R_{ij} \doteq R^k_{ijk}, \quad R_{ia} \doteq -P^k_{ika}, \quad R_{ai} \doteq P^b_{ai b}, \quad S_{ab} \doteq S^r_{abc}.
\]

(65)

The scalar curvature is defined by contracting the Ricci \(d\)-tensor with the inverse metric \(g^{\alpha\beta}\),

\[
\hat{R} \doteq g^{\alpha\beta} R_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} S_{ab} = \hat{R} + \hat{S}.
\]

\(^{10}\)i.e. it has the same coefficients as the Levi Civita connection with respect to \(N\)-elongated bases \([14]\) and \([15]\)
If $E = TM$, there are only three classes of $d$-curvatures,

\[
R^i_{hjk} = e_k L^i_{hj} - e_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{ha} \Omega^a_{kj},
\]

\[
P^i_{ja} = e_a L^i_{jk} - D_k C^i_{ja} + C^i_{jb} T^b_{ka},
\]

\[
S^a_{bcd} = e_d C^a_{bc} - e_c C^a_{bd} + C^a_{bc} C^a_{ed} - C^a_{bd} C^a_{ec},
\]

where all indices $a, b, ..., i, j, ...$ run the same values and, for instance, $C^e_{bc} \rightarrow C^i_{jk}, ...$

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