Fully distributed Nash equilibrium seeking over time-varying communication networks with linear convergence rate

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Abstract—We design a distributed algorithm for learning Nash equilibria over time-varying communication networks in a partial-decision information scenario, where each agent can access its own cost function and local feasible set, but can only observe the actions of some neighbors. Our algorithm is based on projected pseudo-gradient dynamics, augmented with consensual terms. Under strong monotonicity and Lipschitz continuity of the game mapping, we provide a very simple proof of linear convergence, based on a contractivity property of the iterates. Compared to similar solutions proposed in literature, we also allow for a time-varying communication and derive tighter bounds on the step sizes that ensure convergence. In fact, our numerical simulations show that our algorithm outperforms the existing gradient-based methods. Finally, to relax the assumptions on the network structure, we propose a different pseudo-gradient algorithm, which is guaranteed to converge on time-varying balanced directed graphs.

I. INTRODUCTION

Nash equilibrium (NE) problems arise in several network systems, where multiple selfish decision-makers, or agents, aim at optimizing their individual, yet inter-dependent, objective functions. Engineering applications include communication networks [1], demand-side management in the smart grid [2], charging of electric vehicles [3] and demand response in competitive markets [4]. From a game-theoretic perspective, the challenge is to assign the agents behavioral rules that eventually ensure the attainment of a NE, a joint action from which no agent has an incentive to unilaterally deviate.

Literature review: Typically, NE seeking algorithms are designed under the assumption that each agent can access the decisions of all the competitors [5], [6], [7]. Such an hypothesis, referred as full-decision information, requires the presence of a coordinator, that broadcast the data to the network, and it is impractical for some applications [8], [9]. One example is the Nash-Cournot competition model described in [10], where the profit of each of a group of firms depends not only on its own production, but also on the whole amount of sales, a quantity not directly accessible by any of the firms. Therefore, in recent years, there has been an increased attention for fully distributed algorithms that allow to compute NEs relying on local information only. One solution is offered by pay-off based schemes [11], [12], where the agents are not required to communicate between each other, but shall be able to measure their own cost functions. Instead, in this paper, we are interested in a different, model-based approach. Specifically, we consider the so-called partial-decision information scenario, where the agents agree on sharing their strategies with some neighbors on a network; based on the knowledge exchange, they can estimate and eventually reconstruct the actions of all the competitors. This setup has only been introduced very recently. In particular, most of the results available resort to (projected) gradient and consensus dynamics, both in continuous time [13], [14], and discrete time. For the discrete time case, fixed-steps algorithms were proposed in [15], [16], [17] (the latter for generalized games), all exploiting a certain restricted monotonicity property. Alternatively, the authors of [18] developed a gradient-play scheme by leveraging contractivity properties of doubly stochastic matrices. Nevertheless, in all these approaches theoretical guarantees are provided only for step sizes that are typically very small, affecting the speed of convergence. Furthermore, all the methods cited are designed for the case of a time-invariant, undirected network. To the best of our knowledge, switching communication topologies have only been addressed with diminishing step sizes. For instance, the early work [10] considered aggregative games over time-varying undirected graphs. This result was extended by the authors of [19] to games with affine coupling constraints, based on dynamic tracking and on the forward-backward splitting [20, 26.5]. In [21], an asynchronous gossip algorithm was presented to seek a NE over directed graphs. The main drawback is that vanishing step sizes typically result in slow convergence.

Contribution: Motivated by the above, in this paper we present the first fixed-step NE seeking algorithms for strongly monotone games over time-varying communication networks. Our novel contributions are summarized as follows:

- We propose a simple, fully distributed, projected gradient-play algorithm, that is guaranteed to converge with linear rate when the network adjacency matrix is doubly stochastic. With respect to the formulation in [18], we consider a time-varying communication topology and we allow for constrained action sets. Moreover, differently from [18], we provide an upper bound on the step size that is independent of the number of agents.

- We show via numerical simulations that, even in the case of fixed networks, our algorithm outperforms the existing pseudo-gradient based dynamics, when the step sizes are set to their theoretical upper bounds.

- We prove that linear convergence to a NE on time varying weight-balanced directed graphs can be achieved via a forward-backward algorithm [22, 12.7.2], which has already been studied in [17], [16], but only for the case of...
fixed undirected networks \([IV]\).

**Basic notation:** \(\mathbb{N}\) is the set of natural numbers, including 0. \(\mathbb{R} \,(\mathbb{R}_{\geq 0})\) denotes the set of (nonnegative) real numbers. \(\Omega_n (1_n)\) is the vector of dimension \(n\) with all elements equal to 0 (1); \(I_i\) denotes the identity matrix of dimension \(n\); the subscripts might be omitted when there is no ambiguity. For a matrix \(A \in \mathbb{R}^{n \times n}\), its transpose is \(A^T\), \([A]_{i,j}\) denotes the element on the row \(i\) and column \(j\). \(A \succ 0\) stands for symmetric positive definite matrix. \(\sigma_{\min}(A) = \sigma_1(A) \leq \cdots \leq \sigma_n(A) =: \sigma_{\max}(A) = \|A\|\) are the singular values of \(A\); if \(A \in \mathbb{R}^{n \times n}\) is symmetric, \(\lambda_{\min}(A) = \lambda_1(A) \leq \cdots \leq \lambda_n(A) =: \lambda_{\max}(A)\) denote its eigenvalues. \(\otimes\) denotes the Kronecker product. \(\text{diag}(A_1, \ldots, A_N)\) denotes the block diagonal matrix with \(A_1, \ldots, A_N\) on its diagonal. Given \(N\) vectors \(x_1, \ldots, x_N\), \(x := \text{col}(x_1, \ldots, x_N) = [x_1^T \cdots x_N^T]^T\) and \(x_{-i} = \text{col}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)\). \(\|\cdot\|\) denotes the Euclidean vector norm. For a differentiable function \(g : \mathbb{R}^n \rightarrow \mathbb{R}, \nabla g(x)\) denotes its gradient. \(\text{proj}_{\Omega} : \mathbb{R}^n \rightarrow S\) denotes the projection operator onto a closed convex set \(S\). An operator \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is \((\mu,\ell_0)-\text{Lipschitz continuous}, for some } \mu,\ell_0 > 0\); for any \(x, y \in \mathbb{R}^n\), \(\|F(x) - F(y)\| \leq \mu \|x - y\|^2\) and \(\|F(x) - F(y)\| \leq \ell_0 \|x - y\|\). \(\square\)

**II. Mathematical setup**

We consider a set of agents \(\mathcal{I} := \{1, \ldots, N\}\), where each agent \(i \in \mathcal{I}\) shall choose its strategy (i.e., decision variable) \(x_i\) from its local decision set \(\Omega_i \subseteq \mathbb{R}^n\). Let \(x = \text{col}(x_i)_{i \in \mathcal{I}} \in \Omega\) denote the stacked vector of all the agents’ decisions, \(\Omega = \Omega_1 \times \cdots \times \Omega_N \subseteq \mathbb{R}^n\) the overall action space and \(n \equiv \sum_{i=1}^N n_i\). The goal of each agent \(i \in \mathcal{I}\) is to minimize its objective function \(J_i(x_i, x_{-i})\), which depends on both the local variable \(x_i\) and on the decision variables of the other agents \(x_{-i} = \text{col}(x_j)_{j \in \mathcal{I} \backslash \{i\}}\). The game is then represented by the inter-temporal optimization problems:

\[
\forall i \in \mathcal{I} : \quad \arg \min_{y_i \in \Omega_i} J_i(y_i, x_{-i}).
\]  

(1)

The technical problem we consider in this paper is the computation of a NE as defined next.

**Definition 1:** A Nash equilibrium is a set of strategies \(x^* = \text{col}(x_i^*)_{i \in \mathcal{I}} \in \Omega\) such that, for all \(i \in \mathcal{I}\):

\[
J_i(x_i^*, x_{-i}) \leq \inf \{J_i(y_i, x_{-i}) \mid y_i \in \Omega_i\}.
\]

(2)

The following regularity assumptions are common for NE problems, see, e.g., [17, Ass. 1], [16, Ass. 1].

**Standing Assumption 1 (Regularity and convexity):** For each \(i \in \mathcal{I}\), the set \(\Omega_i\) is non-empty, closed and convex; \(J_i\) is continuous and the function \(J_i(\cdot, x_{-i})\) is convex and continuously differentiable for every \(x_{-i}\). \(\square\)

Under Standing Assumption \(\square\) a joint strategy \(x^*\) is a NE of the game in \(\square\) if only if it solves the variational inequality \(\text{VI}(F, \Omega)\) [22, Prop. 1.4.2], or, equivalently, if and only if, for any \(\alpha > 0\) [22, Prop. 1.5.8],

\[
x^* = \text{proj}_\Omega(x^* - \alpha F(x^*)),
\]

\(\square\)

From an operator \(M : \mathbb{R}^n \rightarrow \mathbb{R}^n\) and a set \(\mathcal{S} \subseteq \mathbb{R}^n\), the variational inequality \(\text{VI}(M, \mathcal{S})\) is the problem of finding a vector \(\omega^* \in \mathcal{S}\) such that \(M(\omega^*)(\omega - \omega^*) \geq 0\), for all \(\omega \in \mathcal{S}\) [22, Def. 1.1.1].

A sufficient condition for the existence of a unique NE is the strong monotonicity of the pseudo-gradient [22, Th. 2.3.3], as postulated next. This assumption is always used for GNE seeking under partial-decision information with fixed step sizes, e.g., in [16, Ass. 2], [17, Ass. 3] (while it is sometimes replaced by strict monotonicity and compactness of \(X\) when allowing for vanishing step sizes [10, Ass. 2]). It implies strong convexity of the functions \(J_i(\cdot, x_{-i})\) for every \(x_{-i}\), but not necessarily (strong) convexity of \(J_i\) in the full argument.

**Standing Assumption 2:** The pseudo-gradient mapping in \(\square\) is \(\mu\)-strongly monotone and \(\ell_0\)-Lipschitz continuous, for some \(\mu, \ell_0 > 0\); for any \(x, y \in \mathbb{R}^n\), \(\|F(x) - F(y)\| \geq \mu \|x - y\|^2\) and \(\|F(x) - F(y)\| \leq \ell_0 \|x - y\|\). \(\square\)

In our setup, each agent \(i\) can only access its own cost function \(J_i\) and feasible set \(\Omega_i\). Moreover, agent \(i\) does not have full knowledge of \(x_{-i}\), and only relies on the information exchanged locally with neighbors over a time-varying directed communication network \(G_k(\mathcal{I}, \mathcal{E}_k)\). The ordered pair \((i, j)\) belongs to the set of edges, \(\mathcal{E}_k\), if and only if agent \(i\) can receive information from agent \(j\) at time \(k\).

We denote \(W_k = [w_{i,j}^k]_{i,j \in \mathcal{I}} \in \mathbb{R}^{N \times N}\) the weighted adjacency matrix of \(G_k\), with \(w_{i,j}^k > 0\) if \((i, j) \in \mathcal{E}_k\), \(w_{i,j}^k = 0\) otherwise; \(D^k = \text{diag}(d_{i}^k)_{i \in \mathcal{I}}\) and \(L^k = D^k - W^k\) the indegree and Laplacian matrices of \(G^k\), with \(d_i^k = \sum_{j=1}^N w_{i,j}^k\); \(\Lambda_i^k = \{j \mid (i, j) \in \mathcal{E}\}\) the set of in-neighbors of agent \(i\).

**Standing Assumption 3:** For each \(k \in \mathbb{N}\), the graph \(G_k\) is strongly connected.

**Assumption 1:** For all \(k \in \mathbb{N}\), the following hold:

(i) **Self-loops:** \(w_{i,i}^k > 0\) for all \(i \in \mathcal{I}\);

(ii) **Double stochasticity:** \(W_k 1_N = 1_N, 1_N W_k = 1_N^T\). \(\square\)

**Remark 1:** Assumption \(\square\) is intended just to ease the notation. Instead, Assumption \(\square\) is stronger. It is typically used for networked problems on undirected symmetric graphs, e.g., in [10, Ass. 6], [19, Ass. 3], [18, Ass. 3], justified by the fact that it can be satisfied by assigning the following Metropolis weights to the communication:

\[
\hat{w}_{i,j}^k = \begin{cases} 
\frac{w_{i,j}^k}{\max(d_i^k, d_j^k)} + 1 & \text{if } j \in \mathcal{N}_i \backslash \{i\}; \\
0 & \text{if } j \notin \mathcal{N}_i; \\
1 - \sum_{j \in \mathcal{N}_i \backslash \{i\}} \hat{w}_{i,j}^k & \text{if } i = j.
\end{cases}
\]

In practice, in the case of symmetric communication, to satisfy Assumption \(\square\), even in the case of time-varying topology, it suffices for the agents to exchange their indegree with their neighbors at every time step. Therefore, Standing Assumption \(\square\) and Assumption \(\square\) can be easily fulfilled for undirected graphs that are connected at each time step. For directed graphs, given any strongly connected topology, weights can be assigned such that the resulting adjacency matrix (with self-loops) is doubly stochastic, via an iterative distributed process [23]. However this can be impractical, especially if the network is time-varying. \(\square\)
Under Assumption [1] it holds that $\sigma_{N-1}(W_k) < 1$, for all $k$, where $\sigma_{N-1}(W_k)$ denotes the second largest singular value of $W_k$. Moreover, for any $y \in \mathbb{R}^N$,

$$\|W_k(y - 1_N y)\| \leq \sigma_{N-1}(W_k)\|y - 1_N y\|,$$  \hspace{1cm} (4)

where $\bar{y} = \frac{1}{N}1_N y$ is the average of $y$. We will further assume that $\sigma_{N-1}(W_k)$ is bounded away from 1; this automatically holds if the networks $G_k$ are chosen from a finite family.

**Assumption 2**: There exists $\bar{\sigma} \in (0, 1)$ such that $\sigma_{N-1}(W_k) \leq \bar{\sigma}$, for all $k \in \mathbb{N}$. \hfill \blacksquare

### III. DISTRIBUTED NASH EQUILIBRIUM SEEKING

In this section, we present a pseudo-gradient algorithm to seek [NE] of the game [1] in a fully distributed way. To cope with partial-decision information, each agent keeps an estimate of all other agents’ actions. Let $x_i = \text{col}((x_{ij})_{j \in I}) \in \mathbb{R}^{Nn_i}$, where $x_{i,j} = x_i$ and $x_{i,j}$ is agent $j$’s action for all $j \neq i$, also, $x_{j,-i} = \text{col}((x_{ij})_{i \in I \setminus \{i\}})$. The agents aim at asymptotically reconstructing the true value of the opponents’ actions, based on the data received by their neighbors. The procedure is summarized in Algorithm 1.

Each agent updates its estimates according to consensus dynamics, then its strategy via a gradient step. We remark that each agent computes the partial gradient of its cost in its local estimates $x_i$, not on the actual joint strategy $x$.

To write the algorithm in compact form, let $x = \text{col}((x_{i,j})_{i \in I})$; as in [17, Eq. 13-14], let, for all $i \in I$,

$$R_i := \left[ \begin{array}{cc} 0_{n_i \times n_{i\leq}} & J_{n_i} \end{array} \right] \in \mathbb{R}^{n_i \times n},$$

(5)

where $n_{<i} := \sum_{j<i,j \in I} n_{ij}$, $n_{>i} := \sum_{j>i,j \in I} n_{ij}$, and $\mathcal{R} := \text{diag}((R_i)_{i \in I}) \in \mathbb{R}^{Nn \times Nn}$. In simple terms, $R_i$ selects the $i$-th $n_i$ dimensional component from an $n$-dimensional vector. Thus, $R_i x_i = x_{i,i} = x_i$, and $x = \mathcal{R} x$. We define the extended pseudo-gradient mapping $F$ as

$$F(x) := \text{col}((\nabla_s J_i(x_i, x_{i,-i}))_{i \in I}).$$

(6)

Therefore, Algorithm 1 reads in compact form as:

$$x^{k+1} = \text{proj}_{\Omega}(W_k x^k - \alpha \mathcal{R}^T F(W_k x^k)),$$

(7)

where $\Omega := \{x \in \mathbb{R}^{Nn} | \mathcal{R} x \in \Omega\}$ and $W_k := W_k \otimes I_{n_i}$.

**Lemma 1**: The mapping $F$ in (6) is $\ell$-Lipschitz continuous, for some $\mu \leq \ell \leq \ell_0$. \hfill \blacksquare

We are now ready to prove the main result of this section.

**Theorem 1**: Let Assumptions [12] hold and let

$$M_\alpha := \begin{bmatrix} 1 - 2\frac{\mu}{N} + \frac{\sigma^2}{N} & \frac{\sigma}{\sqrt{N}} \left( \frac{\alpha(\ell+\mu) + \alpha^2 \ell^2}{\sqrt{N}} \right) \\ \frac{\sigma}{\sqrt{N}} \left( \frac{\alpha(\ell+\mu) + \alpha^2 \ell^2}{\sqrt{N}} \right) & \left( 1 + 2\alpha \ell + \alpha^2 \ell^2 \right) \sigma^2 \end{bmatrix}.$$ 

(8)

If the step size $\alpha > 0$ is chosen such that

$$\rho_\alpha := \sigma_{\text{max}}(M_\alpha) = \|M_\alpha\| < 1,$$ 

(9)

then, for any initial condition, Algorithm 1 converges to the point $x^* = 1_N \otimes x^*$, where $x^*$ is the unique NE of the game in [1], with linear rate: for all $k \in \mathbb{N}$,

$$\|x^k - x^*\| \leq (\sqrt{\rho_\alpha})^k \|x^0 - x^*\|.$$ 

**Proof**: See Appendix A. \hfill \blacksquare

**Algorithm 1** Fully distributed NE seeking

**Initialization**: for all $i \in I$, set $x_i^0 \in \Omega$, $x_{i,-i}^0 \in \mathbb{R}^{n-n_i}$.

**Iterate until convergence**: for all $i \in I$.

Distributed averaging: $\bar{x}_i^k = \sum_{j=1}^N w_{ij}^k x_j^k$

Local variables update: $x_{i,-i}^{k+1} = \text{proj}_{\Omega}(\bar{x}_i^k - \alpha \nabla_{x_i} J_i(x_i^k))$.

$$x^{k+1} = \text{proj}_{\Omega}(x^k - \tau(F_\alpha(x))).$$

(11)
We also recover this iteration when considering [17, Alg. 1] in the absence of coupling constraints. The drawback is that exploiting the monotonicity of $F_k$ results in conservative theoretical upper bounds on the parameters $\tau$ and $\gamma$, and consequently in slow convergence (see [14, Th. 1]). More recently, the authors of [18] studied the convergence of (11) based on contrivactivity properties of the iterates, in the case of a fixed undirected network with doubly stochastic adjacency matrix $W$, unconstrained action sets (i.e., $\Omega = \mathbb{R}^n$), and by fixing $\tau = 1$, which results in the algorithm:

$$x^{k+1} = (W \otimes I_N)x - \alpha \nabla^\top F(x^k).$$

(12)

While this algorithm requires only one parameter to be set, the upper bound on $\alpha$ provided in [18, Th. 1] is decreasing to zero when the number of agents $N$ grows unbounded (in contrast with the one in our Theorem 1, see Remark 2).

IV. BALANCED DIRECTED GRAPHS

In this section, we relax Assumption [1] to the following.

Assumption 3: For all $k \in \mathbb{N}$, the communication graph is weight balanced: $(I_N^k W_k)^{-1} = W_k I_N$. □

For weight-balanced digraphs, in-degree and out-degree of each node coincide. Therefore, the matrix $L_k := (L_k + L_k^\top)/2 = D_k - (W_k + W_k^\top)/2$ is itself the symmetric Laplacian of an undirected graph. Besides, such a graph is connected by Standing Assumption [5] hence $L_k$ has a simple eigenvalue in 0, and the others are positive, i.e., $\lambda_2(L_k) > 0$.

Assumption 4: There exist $\sigma, \tilde{\lambda} > 0$ such that $\sigma \lambda_{\max}(L_k) \leq \sigma$ and $\lambda_2(L_k) \geq \tilde{\lambda}$, for all $k \in \mathbb{N}$. □

Remark 3: Like Assumption 2 Assumption 4 always holds if the communication networks switches among a finite family. However, $\sigma, \tilde{\lambda}$ and $\tilde{\lambda}$ are global parameters, that could be difficult to compute in a distributed way; upper/lower bounds might be available for special classes of networks, e.g., unweighted graphs.

To seek a NE over switching balanced digraphs, we propose the iteration in Algorithm 2 in compact form, it reads as

$$x^{k+1} = \text{proj}_{\Omega}(x^k - \tau (\nabla^\top F(x^k) + L_k x^k))$$

(13)

where $L_k = L_k \otimes I_n$. Clearly, (13) is the same scheme of (11), just adapted to take the switching topology into account. In fact, the proof of convergence of Algorithm 2 is based on a restricted strong monotonicity property of the operator

$$F_a^k(x) := \gamma \nabla^\top F(x) + L_k x,$$

(14)

that still holds for balanced directed digraphs, as we show next.

Theorem 2: Let Assumptions [5][4] hold, and let

$$M := \left[ \begin{array}{c} \frac{\mu}{N} + \frac{\mu}{2\sqrt{N}} - \frac{\mu}{\gamma} \\
\frac{\mu}{N} + \frac{\mu}{2\sqrt{N}} \end{array} \right],
\gamma_{\max} := \frac{4\mu\tilde{\lambda}}{(\ell_0 + \theta)^2 + 4\mu\theta},
\ell := \ell - \tilde{\lambda} - \gamma_{\max}(M),
\ell_{\max} := 2\mu/\ell^2,
\rho_{\gamma, \tau} := 1 - 2\tau \bar{\mu} + \tau^2 \bar{\ell}^2.
$$

If $\gamma \in (0, \gamma_{\max})$, then $M > 0$ and, for any $\tau \in (0, \tau_{\max})$, for any initial condition, Algorithm 2 converges to the point $x^* = 1_N \otimes x^*$, where $x^*$ is the unique NE of the game in $[1]$, with linear rate: for all $k \in \mathbb{N}$,

$$||x^k - x^*|| \leq \left(\sqrt{\frac{\rho_{\gamma, \tau}}{\gamma}}\right)^k ||x^0 - x^*||.$$

Proof: See Appendix [3].

V. NUMERICAL EXAMPLE: A NASH-COURNOT GAME

We consider the Nash-Cournot game in [17, 6]. $N$ firms produces a commodity that is sold to $m$ markets. Each firm $i \in I = \{1, \ldots, N\}$ is only allowed to participate in $n_i$ of the markets. The decision variables of each firm are the quantities $x_i \in \mathbb{R}^{n_i}$ of commodity to be delivered to these $n_i$ markets, bounded by the local constraints $0 \leq x_i \leq X_i$. Let $A := [A_1, \ldots, A_N]$, where $A_i \in \mathbb{R}^{m_i \times n_i}$ is the matrix that expresses which markets firm $i$ participates in. Specifically, the $j$-th column of $A_j$ has its $k$-th element equal to 1 if $[x_{i,j}]$ is the amount of product sent to the $k$-th market by agent $i$, for all $j = 1, \ldots, n_i$; all the other elements are 0. Therefore, $Ax := \sum_{i=1}^N A_i x_i \in \mathbb{R}^m$ is the vector of the quantities of total product delivered to each market. Firm $i$ aims at maximizing its profit, i.e., minimizing the cost function $J_i(x_i, x_{-i}) = c_i(x_i) - p(Ax)^\top A_i x_i$. Here, $c_i(x_i) = x_i^\top Q_i x_i + q_i^\top x_i$ is firm $i$'s production cost, with $Q_i \in \mathbb{R}^{n_i \times n_i}$, $q_i \geq 0$, $q_i \in \mathbb{R}^n$. Instead, $p : \mathbb{R}^m \rightarrow \mathbb{R}$ associates to each market a price that depends on the amount of product delivered to that market. Specifically, the price for the market $k$, for $k = 1, \ldots, m$, is $p_k(x) = \bar{p}_k - \chi_k |Ax_k|$, where $\bar{p}_k$, $\chi_k > 0$. We set $N = 20$, $m = 7$. The market structure is defined as in [17, Fig. 1], that defines which firms are allowed to participate in which market. Therefore, $x = \text{col}([x_i]) \in \mathbb{R}^n$ and $n = 32$. We select randomly with uniform distribution $r_k$ in $[1, 2]$, $Q_i$ diagonal with diagonal elements in $[14, 16]$, $q_i$ in $[1, 2]$, $\bar{p}_k$ in $[10, 20]$, $\chi_k$ in $[1, 3]$, $X_i$ in $[5, 10]$, for all $i \in I$, $k = 1, \ldots, m$. The resulting setup satisfies Standing Assumptions 1[12][17, VI]. The firms cannot access the production of all the competitors, but can communicate with some neighbors on a network.

We first consider the case of a fixed, undirected graph, under Assumption 1 Algorithm 2 in this case reduces to [16, Alg. 1] or, in the absence of coupling constraints, to [17, Alg. 1]. We compare Algorithms 1[2] with the inexact ADMM in [15] and the accelerated gradient method in [16], for the step sizes that ensure convergence. Specifically, we set $\alpha$ as in Theorem 1 for Algorithm 1. The convergence of all the other Algorithms is based on the monotonicity of $F_a$ in (14); hence we set $\gamma$ as in Theorem 2. Instead of using the conservative bounds in [14] for the strong monotonicity and Lipschitz constants of $F_a$, $\bar{\mu}$ and $\bar{\ell}$, we obtain a better
result by computing the exact values numerically. \( F_i \) is (non-
restricted) strongly monotone for our parameters, hence also the
convergence result for [16, Alg. 2] holds. Figure 1 shows
that Algorithm 1 outperforms all the other methods (we also
remark that the accelerated gradient in [16, Alg. 2] requires
two projections and two communications per iterations). As
a numerical example, we also compare Algorithm 1 with the
scheme in (12) by removing the local constraints, in Figure 2.

For the case of doubly stochastic time-varying networks,
we randomly generate 5 connected graphs and for each
iteration we pick one with uniform distribution. In Figure 3
we compare the performance of Algorithms 1 and 2 for step
sizes set to their best theoretical values as in Theorems 1 and 2.

Finally, in Figure 4 we test Algorithm 2 when the
communication topology is switching between two balanced directed
graphs: the unweighted directed ring, where each agent \( i \)
can send information to the agent \( i+1 \) (with the convention
\( N+1 = 1 \)), and a second graph, where agent \( i \) is also
allowed to transmit to agent \( i+2 \), for all \( i \in T \).

VI. CONCLUSION

Nash equilibrium problems on time-varying graphs can be
solved with linear rate via fixed-step pseudo-gradient
algorithms, if the network is connected at every iteration
and the game mapping is Lipschitz continuous and strongly
monotone. Our algorithm proved much faster than the ex-
isting gradient-based methods, even in the case of a fixed
communication topology, at least in our numerical experience.
The extension of our results to games with coupling

\[ x_{k+1} = \text{proj}_{\Omega}(x_k - \alpha R^T F \hat{x}_k), \quad \hat{x}_k = W_k x_k. \]

A. Proof of Theorem 1

We define the estimate consensus subspace \( E := \{ y \in
\mathbb{R}^N \mid y = 1_N \otimes y, \ y \in \mathbb{R}^n \} \) and its orthogonal complement
\( E_\perp = \{ y \in \mathbb{R}^N \mid (1_N \otimes I_n)^T y = 0_n \} \). Thus, any
vector \( x \in \mathbb{R}^N \) can be written as \( x = x_{\|} + x_{\perp} \), where
\( x_{\|} = \text{proj}_E(x) = \frac{1}{N}(1_N \otimes I_n)x \), \( x_{\perp} = \text{proj}_{E_\perp}(x) \), and
\( x_{\|} x_{\perp} = 0 \). Also, we use the shorthand notation \( F^\perp \) and \( F^\| \)
in place of \( F(x) \) and \( F(x) \). We recast the iteration in (7) as

\[ x_{k+1} = \text{proj}_{\Omega}(x_k - \alpha R^T F^\| x_k), \quad \hat{x}_k = W_k x_k. \]
\[ \begin{align*}
&\leq \|\bar{x}_1 - x^*\|^2 + \|\bar{x}_2\|^2 + 2\alpha^2 \ell (\|\bar{x}_1\|^2 + \frac{\ell}{\alpha^2} \|\bar{x}_2 - x^*\|^2 \\
&\quad + \frac{2\alpha \ell}{\sqrt{N}} \|\bar{x}_1 - x^*\|^2 + \|\bar{x}_2\|^2) + \frac{2\alpha \ell}{\sqrt{N}} \|\bar{x}_1 - x^*\|^2 \|\bar{x}_2\| \\
&\quad - \frac{2\alpha \ell}{\sqrt{N}} \|\bar{x}_1 - x^*\|^2 + 2\alpha \ell \|\bar{x}_2\|^2 + \frac{2\alpha \ell}{\sqrt{N}} \|\bar{x}_2\|^2 \|\bar{x}_1 - x^*\| \\
&= \max (\mathcal{M}(\|x^k - x^*\|^2 + \|x^k\|^2)) \\
&= \max (\mathcal{M}(\|x^k - x^*\|^2 + \|x^k\|^2)).
\end{align*} \]

where the symmetric matrix \(\mathcal{M}\) is as in (8).}

B. Proof of Theorem 2

Let \(x^*\) be the unique NE of the game in (1), and \(x^* = 1_N \otimes x^*\). We recall that the null space \(\text{null}(L_k) = E = \{y \in \mathbb{R}^n \mid y = 1_N y, y \in \mathbb{R}^n\}\) by Standing Assumption 3 and property of the Kronecker product. Therefore, \(L_k x^* = 0_N\) and \(x^*\) is a fixed point of the iteration in (13) by (2). With \(F^k_N\) as in (14), for all \(k \in \mathbb{N}\), for any \(x \in \mathbb{R}^{nN}\), it holds that:

\[
(x - x^*)^\top (F^k_N x - F^k_N x^*) = (x - x^*)^\top (F^k x - F^k x^*) + (x - x^*)^\top L_k (x - x^*)
\]

\[
= (x - x^*)^\top (F^k x - F^k x^*) + (x - x^*)^\top L_k (x - x^*)
\]

where \(L_k = (L_k + L_k^\top)/2 = (L_k + L_k^\top) \otimes I_n/2 = L_k \otimes I_n\), and \(L_k\) is the Laplacian of a connected graph (see [IV] and \(\lambda_2(L_k) > \lambda\) by Assumption 4). Therefore, we can apply [17, Lemma 3] to conclude that \((x - x^*)^\top (F^k x - F^k x^*) \geq \tilde{\mu} \|x - x^*\|^2\), with \(\tilde{\mu} > 0\) as in (15). Also, \(F^k_N\) is Lipschitz continuous with constant \(\tilde{\ell} = \ell + \tilde{\sigma}\), \(\tilde{\sigma}\) as in Assumption 4.

Therefore we can write

\[
\|x^{k+1} - x^*\|^2 \\
= \|\text{proj}_{\Omega}(x^k - \tau F^k_N x^*) - \text{proj}_{\Omega}(x^k - \tau F^k_N x^*)\|^2 \\
\leq \|\langle x^k - \tau F^k_N x^k, -(x^k - \tau F^k_N x^k)\rangle\|^2 \\
= \|x^k - x^*\|^2 - 2\tau \langle x^k - x^*\rangle^\top (F^k_N x - F^k_N x^*) + \tau^2 \|F^k_N x^k - F^k_N x^k\|^2 \\
\leq (1 - 2\tau \tilde{\mu} + \tau^2 (\tilde{\ell} + \tilde{\sigma})^2) \|x^k - x^*\|^2 = \rho_\gamma \tau \|x^k - x^*\|^2,
\]

where in the first inequality we used [20, Prop. 4.16], and \(\rho_\gamma \tau \in (0, 1)\) if \(\tau\) is chosen as in Theorem 2.

References

[1] F. Facchinei and J. Pang, “Nash equilibria: the variational approach,” in Convex Optimization in Signal Processing and Communications, D. P. Palomar and Y. C. Eldar, Eds. Cambridge University Press, 2009, p. 443-493.

[2] S. Saad, Z. Han, H. V. Poor, and T. Basar, “Game-theoretic methods for the smart grid: An overview of microgrid systems, demand-side management, and smart grid communications,” IEEE Signal Processing Magazine, vol. 29, no. 5, pp. 86–105, 2012.

[3] S. Grammatico, “Dynamic control of agents playing aggregative games with coupling constraints,” IEEE Transactions on Automatic Control, vol. 62, no. 9, pp. 4537–4548, 2017.

[4] N. Li, L. Chen, and M. A. Dahleh, “Demand response using linear supply function bidding,” IEEE Transactions on Smart Grid, vol. 6, no. 4, pp. 1827–1838, 2015.

[5] C. Yu, M. van der Schaar, and A. H. Sayed, “Distributed learning for stochastic generalized Nash equilibrium problems,” IEEE Transactions on Signal Processing, vol. 65, no. 15, pp. 3893–3908, 2017.

[6] G. Belgioioso and S. Grammatico, “Projected-gradient algorithms for generalized equilibrium seeking in aggregative games are preconditioned forward-backward methods,” 2018 European Control Conference (ECC), pp. 2188–2193, 2018.

[7] J. S. Shamma and G. Arslan, “Dynamic fictitious play, dynamic gradient play, and distributed convergence to Nash equilibria,” IEEE Transactions on Automatic Control, vol. 50, no. 3, pp. 312–327, 2005.

[8] J. Ghaderi and R. Srikant, “Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate,” Automatica, vol. 50, no. 12, pp. 3209 – 3215, 2014.

[9] K. Bimpikis, S. Ehsani, and R. Ilikic, “Cournot competition in networked markets,” in 15th ACM Conference on Economics and Computation, ser. EC ’14. ACM, 2014, p. 733.

[10] J. Koshal, A. Nedi, and U. V. Shanbhag, “Distributed algorithms for aggregative games on graphs,” Operations Research, vol. 64, pp. 680–704, 2016.

[11] M. S. Stankovic, K. H. Johansson, and D. M. Stipanovic, “Distributed seeking of Nash equilibria with applications to mobile sensor networks,” IEEE Transactions on Automatic Control, vol. 57, no. 4, pp. 904–919, 2012.

[12] P. Frihauf, M. Kristic, and T. Basar, “Nash equilibrium seeking in noncooperative games,” IEEE Transactions on Automatic Control, vol. 57, no. 5, pp. 1192–1207, 2012.

[13] M. Ye and G. Hu, “Distributed Nash equilibrium seeking by a consensus based approach,” IEEE Transactions on Automatic Control, vol. 62, no. 9, pp. 4811–4818, 2017.

[14] D. Gadjov and L. Pavel, “A passivity-based approach to Nash equilibrium seeking over networks,” IEEE Transactions on Automatic Control, vol. 64, no. 3, pp. 1077–1092, 2019.

[15] F. Salehisadaghiani, W. Shi, and L. Pavel, “Distributed Nash equilibrium seeking under partial-decision information via the alternating direction method of multipliers,” Automatica, vol. 103, pp. 27 – 35, 2019.

[16] T. Tatarenko, W. Shi, and A. Nedi, “Geometric convergence of gradient play algorithms for distributed Nash equilibrium seeking,” arXiv preprint arXiv:1909.07383, 2019.

[17] L. Pavel, “Distributed GNE seeking under partial-decision information over networks via a doubly-augmented operator splitting approach,” IEEE Transactions on Automatic Control, DOI: 10.1109/TAC.2019.2922953, 2019.

[18] T. Tatarenko and A. Nedi, “Geometric convergence of distributed gradient play in games with unconstrained action sets,” arXiv preprint arXiv:1907.07144, 2019.

[19] G. Belgioioso, A. Nedic, and S. Grammatico, “Distributed generalized Nash equilibrium seeking in aggregative games on time-varying networks,” arXiv preprint arXiv:1907.00191, 2019.

[20] H. H. Bauschke and P. L. Combettes, Convex analysis and monotone operator theory in Hilbert spaces. Springer, 2017, vol. 2011.

[21] F. Salehisadaghiani and L. Pavel, “Nash equilibrium seeking with non-doubly stochastic communication weight matrix,” EAI Endorsed Transactions on Collaborative Computing, DOI: 10.4108/eai.13-7-2018.158526, 2019.

[22] F. Facchinei and J. Pang, Finite-dimensional variational inequalities and complementarity problems. Springer Science & Business Media, 2007.
[23] B. Gharesifard and J. Corts, “Distributed strategies for generating weight-balanced and doubly stochastic digraphs,” European Journal of Control, vol. 18, no. 6, pp. 539 – 557, 2012.

[24] M. Bianchi and S. Grammatico, “A continuous-time distributed generalized Nash equilibrium seeking algorithm over networks for double-integrator agents,” arXiv preprint arXiv:1910.111608, 2019, (Accepted to the European Control Conference 2020).