Electron-electron interactions and charging effects in graphene quantum dots

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We analyze charging effects in graphene quantum dots. Using a simple model, we show that, when the Fermi level is far from the neutrality point, charging effects lead to a shift in the electrostatic potential and the dot shows standard Coulomb blockade features. Near the neutrality point, surface states are partially occupied and the Coulomb interaction leads to a strongly correlated ground state which can be approximated by either a Wigner crystal or a Laughlin like wave function. The existence of strong correlations modify the transport properties which show non equilibrium effects, similar to those predicted for tunneling into other strongly correlated systems.

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I. INTRODUCTION

Graphene has attracted a great deal of attention, because of its novel fundamental properties and its potential applications. The interest on graphene devices has motivated recent research on the transport properties of small devices. Features such as charging effects and quantum confinement are of crucial importance for their understanding. The confinement of electrons, and the observation of Coulomb blockade effects has already been demonstrated experimentally. Note that the confinement of electrons in graphene is not trivial, due to the Klein's paradox, which makes potential barriers transparent for normally incident quasi-particles. Electrons in graphene can be confined, however, by exploiting the angular dependence of scattering at a barrier.

For graphene layers, electron-electron interaction is usually neglected, including works on localization, even though disorder enhances the effect of interaction. The reasoning for this is to assume a “normal” ground-state at zero doping - characterized by a semi-metal. Because the kinetic and interaction energy equally scale with the carrier density, the interaction does not become important at finite doping, either. It is thus well agreed on that at finite doping electron-electron interaction can be treated within the random-phase approximation (RPA). Nevertheless, at the Dirac point RPA seems to fail leading to a novel plasmon mode in graphene quantum dots. Similar studies have been performed previously for semiconducting quantum dots. Also in a quantum dot, we find that electron-electron interaction has to be treated differently for doping regimes close to and away from the Dirac point.

The main part of this work is the prediction and characterization of strongly correlated few-electron states in graphene quantum dots. Similar studies have been performed previously for semiconducting quantum dots. In order to obtain strongly correlated ground states the Coulomb interaction has to dominate over the other energy scale, namely the shell structure of the single particle spectrum determined by the confinement. This is typically achieved by either using strong magnetic fields, so that the single particle levels form highly degenerate Landau levels, or using rather weak confinement. Interestingly, strongly correlated states naturally arise already in small graphene quantum dots even without magnetic field. The reason for that is the appearance of a highly degenerate zero energy band of surface states, which is strongly affected by Coulomb interaction. Close to half filling these states are occupied by few electrons which are strongly correlated and can be approximated by a Laughlin like wave function or alternatively by a quasi one-dimensional Wigner crystal.

The paper is organized as follows. In section II, we present a simple model which allows us to describe qualitatively the charging of a graphene dot. In section III, we show that the charging properties of the graphene dot are in agreement with Coulomb blockade theory when the Fermi energy is far from the neutrality point. Thereafter, we show in section IV that close to the neutrality point charging effects are strongly modified by the presence of midgap states, associated to the edges. We show that electrons occupying these midgap states form a strongly correlated state which is characterized in detail. In section V, we then discuss implications for transport properties. We close with conclusions and outlook.

II. THE MODEL

The linearized tight-binding Hamiltonian for a graphene sheet with circular symmetry is given by

$$H_s = v_F \left( \begin{array}{cc} 0 & e^{i\gamma}\left( -is\partial_r + \frac{s}{r}\partial_{\theta} \right) \\ e^{-i\gamma}\left( -is\partial_r - \frac{s}{r}\partial_{\theta} \right) & 0 \end{array} \right),$$

(1)

where $s = \pm$ determines the valley. We assume that the dot is ballistic, i.e., with no internal disorder. The general solutions with energy $E_k = \pm v_F k$ are of the type

$$\begin{pmatrix} \Psi^A (r, \theta) \\ \Psi^B (r, \theta) \end{pmatrix} = \begin{pmatrix} J_{m+s}(kr)e^{i(m+s)\theta} \\ \mp i J_m(kr)e^{im\theta} \end{pmatrix},$$

(2)

with $J_m(x)$ denoting the $m$-th Bessel function. The dot has a circular shape with radius $R$. The circular symme-
try of the dot allows us to classify the solutions according to their angular momenta.

In order to analyze the possible role of surface states, we assume that the boundary conditions at the edges are those appropriate for a zig-zag graphene edge ending always on the same lattice site

$$\Psi_s^*(r, \theta) = 0 .$$

The boundary condition is not experimentally realizable for a circular dot, though it enables a detailed analysis of the interplay between Coulomb interaction and surface states. For the chosen boundary condition the wave vector is quantized by \( k = z_{nm}/R \), where \( z_{nm} \) denotes the \( n \)-th root of the \( m \)-th Bessel function, \( J_m(z_{nm}) = 0 \). In addition to the finite energy states given in Eq. (2) the boundary condition allows for surface states which can be written as

$$\Psi_s^*(r, \theta) = \left( \frac{0}{\sqrt{\frac{m+1}{\sqrt{R}^{2m+1}}} e^{ikr} \cos \theta} \right) ,$$

with \( m \geq 0 \) to guarantee normalizability. Note, that for the surface states the angular momentum is given by \( sm \), see Eq. (4) and that these functions have an analytical dependence on either \( z = x + iy \) or \( \tilde{z} = x - iy \). Discrete lattice effects impose a maximal (absolute) value on the angular momentum of order \( m_{max} \sim R/a \), where \( R \) is the radius of the disk and \( a \) is a length comparable to the lattice spacing.

Charging effects arise from electron-electron interaction which is generally described by

$$H_C = \frac{e^2}{4\pi \varepsilon_0 e} \sum_{n<m} \frac{1}{|\mathbf{r}_n - \mathbf{r}_m|} .$$

The total Hamiltonian is given by the sum of Eqs. (1) and (5). We note that both parts scale as \( 1/R \). Furthermore in graphene \( (e^2/4\pi \varepsilon_0 e)/\hbar v_F \approx 1 \), so that both single particle and interaction energies can be expressed in units of \( (e^2/4\pi \varepsilon_0 e R) = \hbar v_F / R \), as will be done throughout this paper.

Charging effects are mostly determined by the overall geometry of the dot, so that the lack of disorder in the model described here does not change qualitatively the main features of Coulomb blockade. Graphene dots have, most likely, rough edges. Hence, the possible surface states are confined to certain regions of the edges. The model overestimates the number of surface states of a given dot. On the other hand, wave functions localized in the angular coordinate \( \theta \) can be built from the wave functions in Eq. (2) or Eq. (4). A dot where the edge has a region of size \( l \) of the zig-zag type has states localized at the edge with an angular width \( \Delta \theta \sim l/R \). These states will be approximately described by superpositions of states with angular momenta \( m \lesssim R/l \). Hence, when \( R/l \gg 1 \) and \( l/a \gg 1 \), these states, which will change over distances larger than the lattice spacing, will be well described by superpositions of the states derived from our continuum model.

III. CHARGING EFFECTS AWAY FROM THE DIRAC ENERGY

We analyze the effects induced by increasing the number of electrons in the dot using the Hartree approximation. The self-consistent Hartree potential describes, within a mean field approximation, the screening of charges within the dot. We assume that a half filled dot is neutral, as the ionic charge compensates the electronic charge in the filled valence band. Away from half filling, the dot is charged. Then, an electrostatic potential is induced in its interior, and there is an inhomogeneous distribution of charges. We describe charged dots by fixing the chemical potential, and obtaining a self-consistent solution where all electronic states with lower energies are filled. The Hartree approximation should give a reasonable description when Coulomb blockade effects can be described as a rigid shift of the electrostatic potential within the dot.

The Hartree potential needs to be calculated self consistently which must be done numerically, despite of the simplicity of the model. The Dirac equation for each angular momentum channel is discretized, and an effective tight binding model is defined for each channel. Details are given in Appendix A.

The conservation of the angular momentum allows for the possibility of solving dots with a large number of electrons. Typical results for dots charged with electrons or holes away from the Dirac energy are shown in Fig. 4. The calculation has been done in a discrete lattice with \( N = 100 \) sites (see the Appendix). The Hartree potential changes little within the dot and, to a first approximation, the deviation from neutrality of the dot can be approximated by a rigid shift of the electrostatic potential.

IV. CHARGING EFFECTS NEAR THE NEUTRALITY POINT

The Hartree calculations mentioned above fail to give a self consistent solution when the surface band is partially occupied and a more advanced treatment of the interaction has to be applied. This also implies that deviations from conventional Coulomb blockade can be expected in this regime.

Instead of treating the interactions within a mean field approach we therefore employ the method of configuration interaction to fully take into account all correlations within the truncated Hilbert space of surface states. The truncation of the Hilbert space can be justified by the energy gap to extended states of finite energy which in our model is given by \( 24\hbar v_F / R \). In principle, the effect of the extended states can be added to the following analysis as a perturbation, but we do not expect qualitative changes of our main conclusions.

In the following we deal, therefore, with a few-electron problem and consider the eigenspectrum of \( N \) interact-
The chemical potential considered in the top part: Black (so lid (center right), and 6038, right. Bottom part: Hartree potential), \( I \) function can be characterized by the valley-polarization addition to the particle number, the few-electron wave action rather than a (point-like) Hubbard interaction. In filling, it seems sensible to consider a long-ranged interaction as opposed to next-nearest neighbor hopping. For one valley in the Brillouin Zone, and different total angular momentum, for one valley in the Brillouin Zone, and different electron-electron interactions.

Figure 1: (Color online). Electronic structure of a quantum dot in the Hartree approximation. All energies are in units of \( \hbar v_F N/R \) with \( N = 100 \) and the position in units of \( r/R \).

Top part: Electron energies as a function of the angular momentum, for one valley in the Brillouin Zone, and different values of the chemical potential. Left: \( \epsilon_F = 0.5 \). Center left: \( \epsilon_F = 0.25 \). Center right: \( \epsilon_F = -0.25 \). Right: \( \epsilon_F = -0.5 \). The total number of states per valley is \( 12000 \). The number of occupied states per valley is \( 6155 \) (left), \( 6122 \) (center left), \( 6066 \) (center right), and \( 6038 \), right. Bottom part: Hartree potential (top), and charge density (bottom) for the four values of the chemical potential considered in the top part: Black (solid line), \( \epsilon_F = 0.5 \). Red (dashed line), \( \epsilon_F = 0.25 \). Green (dash-dotted line), \( \epsilon_F = -0.25 \). Blue (dotted line), \( \epsilon_F = -0.5 \).

Surface states are characterized by only populating one of the two sub-lattices and thus avoiding the kinetic energy due to nearest-neighbor hopping \( t \). However, next-nearest neighbor hopping \( t' \approx t/10 \approx 0.3 \text{eV} \) connects sites within the same sub-lattice so that surface states gain some finite kinetic energy and the zero energy band becomes dispersive. This kinetic term delocalizes the wave-function of the surface states and leads to a stable few-electron ground-state with finite angular momentum \( M_0 \). From Ref.\textsuperscript{24}, the kinetic energy due to next-nearest neighbor hopping reads \( t' a^z p^2 / (2 R) \), where \( a \) the lattice spacing and \( p \) the momentum operator. As shown in appendix\textsuperscript{13} the Hamiltonian for next nearest neighbor hopping \( H_{\text{kin}} \) can be written to lowest order perturbation in \( t' \) as

\[
H_{\text{kin}} = \frac{\hbar v_F}{R} \frac{3a}{10 R} \sum_m m (m + 1) c_m^\dagger c_m.
\]

This kinetic term competes with the Coulomb interaction, since it reduces the Coulomb correlations of the ground state. This competition is also visible in the dependence of the few-electron energy on the total angular momentum as shown in Fig.\textsuperscript{2}. In the absence of next-nearest neighbor hopping (solid line) the energy decreases with increasing angular momentum (except for oscillations discussed below), since states of higher angular momentum have lower Coulomb energy. However, when next-nearest neighbor hopping is included then the occupation of states with large angular momentum is hindered and the energy as function of angular momentum shows a well defined minimum. We note that the ratio between kinetic energy and Coulomb energy increases with decreasing dot size (the numerical calculations are done for \( R = 22 \text{nm} \)). Consequently for smaller dots the angular momentum of the ground-state decreases. In the studied subspace of valley and spin polarized electrons, the minimal angular momentum is given by \( M_{\text{min}} = N(N - 1)/2 \).

A. Trial functions for the correlated ground-state

The lack of well converged Hartree solutions, which are given by Slater determinants, imply that the wavefunction which describes the surface states in the presence of charging effects is strongly correlated. We have...
chosen two ansätze which are compared to the numerically exact solution.

1. Laughlin wave function

The appearance of a partially filled degenerate energy band separated by an energy gap from the rest of the spectrum strongly resembles Fractional Quantum Hall physics. However, now the zero energy band is caused by the boundary condition and the gap is due to the confinement rather than due to high magnetic fields. Not only the band structure is similar in both systems, but also the form of the one-particle states is similar. Both the surface states as well as the orbitals of the lowest Landau level (in symmetric gauge) depend on $z^m$.

This analogy can be used to propose a trial wave function much like Laughlin’s original wave function for the ground-state in the Fractional Quantum Hall regime. The Laughlin like wavefunction in Eq. (6) is the exact eigenstate. With increasing value of $p$, the correlations increase in the presence of superposed Slater determinants, much like Laughlin’s original wave function for the Fractional Quantum Hall state. The Laughlin like wavefunction in Eq. (6) is a parameter free trial wavefunction that conserves the Coulomb energy and explains the good agreement between both systems in the excitation spectrum. We note however, that we use the similarity between the studied system and the Fractional Quantum Hall state to superpose the wave functions in such a way that electrons are maximally separated in angle. We write such a trial function as

$$|\Psi_{WC}\rangle = \sum_{m_1, m_2, \ldots, m_N} e^{-i \sum_{n} \frac{2\pi m_n}{N} m_n} \prod_{n}(m_n + 1)^w e^{\dagger}_{m_n} |0\rangle$$

(7)

where the operator $e^{\dagger}_{m_n}$ creates a state with momentum $m_n$. Note that due to the constraint imposed by the antisymmetry requirement the wave function can only be defined for total angular momenta of the form $M = N(N - 1)/2 + jN$ where $j$ is a positive integer. While the phase factor guarantees for the angular correlations in the wave function, we use the factor $(m_n + 1)^w$ to optimize radial correlations (note that the normalization constant of a surface state is proportional to $\sqrt{m + 1}$). For a strictly one dimensional system the usual definition of a quasi-classical Wigner crystal implies $w = 0$. In the numerical calculations, we choose $w$ such that the wave function optimizes the ground-state energy ($w \approx 2$ for low $M$ and $t'$).

B. Correlations

For $M \rightarrow \infty$ and without next-nearest neighbor hopping, the quantum mechanical configuration approaches the classical one, which minimizes the Coulomb energy by pinning the electrons at $r = R$ and $\varphi_n = 2\pi n/N$. However, due to the rotational symmetry of our problem, the ground-state is a superposition of all orientations such that there is a constant density distribution around the circumference. To characterize a Wigner crystal or more generally a density correlated system, one thus has to look at the density-density correlation function

$$C_M^N(r_0, r) = \langle N, M; 0 \mid \sum_{i \neq j} \delta(r_0 - r_i), \delta(r - r_j) \rangle \frac{N(N - 1)}{N} |N, M; 0\rangle$$

(8)

where $|N, M; 0\rangle$ is the ground state of the $N$ particle system to fixed angular momentum $M$ and $i, j \in 1, \ldots, N$.

C. Effect of Disorder

We can use the same truncated basis to study disorder due to the roughness of the edges. Due to the flat dispersion in the absence of disorder, single particle states tend to localize near imperfections of the edge, however one can show that the degeneracy of the zero energy states is only reduced by the number of impurities, which can be assumed to be much smaller than the number of surface states. The correlated state found in the presence of interactions will also be pinned by disorder, leading to glassy features.
FIG. 2: (color online). Ground-state energy (in units of $\frac{e^2}{4\pi\epsilon_0 R}$) for $N = 2, 3, 4, 5$ electrons occupying surface states as function of the total angular momentum $M$ with (red solid line) and without (blue dotted line) next-nearest neighbor hopping $t'$ (assuming $R = 22\text{nm}$). For $t' = 0$ and $M \rightarrow \infty$ the energy approaches the classical energy of $N$ point charges on a disc. The classical configuration is shown in the inset and the classical energy is indicated by the constant dashed line.

D. Numerical results

Fig. 2 shows the energy of the lowest lying spin- and valley-polarized eigenstate of each total angular momentum $M$ for $N = 2, 3, 4, 5$ electrons occupying surface states. The energies are obtained by numerically diagonalizing the few-particle Hamiltonian in this subspace. The dotted line in Fig. 2 shows the results if next nearest neighbor interaction is neglected, so that the total Hamiltonian consists of the Coulomb interaction, only. We note two main features in that case. First, the energy oscillates as function of the angular momentum with local minima at $M = M_{\text{min}}(N) + jN$, where $M_{\text{min}}(N) = N(N - 1)/2$ denotes the minimal angular momentum of $N$- spin and valley-polarized electrons and $j$ is a positive integer. Only at these angular momenta the angular correlations between the electrons can be fully developed. This can be seen in the correlation functions discussed below and in the fact that only for these distinct angular momenta a Wigner trial function can be constructed. The second feature visible in Fig. 2 is that the energy generally decreases with increasing angular momentum for $t' = 0$ and it finally reaches the classical limit corresponding to $N$ point charges on the dot. The classical configurations are shown in the insets, and their energies are indicated by the constant dashed lines.

For finite $t'$ (see solid line in Fig. 2), the Hamiltonian is supplemented by a kinetic term given in Eq. (6) that competes with the Coulomb interaction. Since the kinetic energy of surface states increases quadratically with their angular momentum the cost in kinetic energy exceeds for large total angular momenta the gain in Coulomb energy connected with an increase in angular momentum. Thus the $N$-electron system now has a ground-state with well defined angular momentum $M_0$. The ratio between the kinetic term and the Coulomb energy grows for decreasing dot sizes, which also leads to a decrease in $M_0$.

In Fig. 3 we compare the numerically obtained energies for $N = 3$ electrons with that of the two trial functions described above. The data for the Laughlin-like wave function (defined in Eq. (3)) is indicated by squares while the data for a Wigner-crystal-like wave function (defined in Eq. (4)) is labeled by filled circles. First, we note that the energies of both trial wave functions differ by less than 1% from the numerical data. As noted above the Wigner-crystal-like wave function can be constructed for each angular momentum where the few-electron energy shows a minimum. In contrast a Laughlin-like wave function only exists for each $N - 1$-th minimum. It is interesting to note that the Laughlin-like wave function becomes better for finite $t'$ than for $t' = 0$.

In Fig. 3 we optimized the free parameter $w$ in the Wigner wave function for each $M$ separately, which leads to this extremely good agreement with the exact data for both zero and finite $t'$. We note, however, that the optimal value was $w \approx 2$ for all $M$ in the case of $t' = 0$, while we strongly increased $w$ with increasing $M$ for finite $t'$.

Fig. 4 shows the density plot of the exact, symmetrized density-density correlation function $C^N_M(r) = \sum_{i=1}^{N} C^N_M(R, r; 2\pi/N; r)$ for $N = 2, 3, 4, 5$ electrons and for $M = N + M_{\text{min}}$. The $N$-fold symmetry, which is typical of a 1-d Wigner crystal is clearly seen. We note that also the trial wave function show these correlations, which explains the good agreement of its energies with the exact one.

In Fig. 5 the angular correlations along the perimeter of the dot is shown for $N = 3$. An electron is fixed at $\theta = 0$ and $r = R$ and the probability of finding another electron at a given angle is plotted. The left hand side of Fig. 5 illustrates that correlations are maximally developed at the distinguished angular momentum.
FIG. 4: (color online) Density plot of the exact, symmetrized density-density correlation function $C_M^N(r)$ for $N = 2, 3, 4, 5$ particles and total angular momentum $M = N + M_{\text{min}}$.

FIG. 5: (color online) Angular correlations $C_M^N(R, 0; R, \theta)$ for $N = 3$ along the perimeter of the dot for various total angular momentum $M$ (left hand side) and various next-nearest neighbor hopping $t'$ (right hand side).

$M = jN + M_{\text{min}}$ (here $j = 3$), while for other angular momenta the correlations are washed out. On the right hand side of Fig. 5, we see that the density-density correlations are more pronounced for higher angular momentum (here $j = 15$) while the kinetic energy $t'$ reduces these correlations, which again is a manifestation of the competition between Coulomb interaction and next-nearest neighbor hopping.

V. TRANSPORT PROPERTIES

The addition of one electron to the dot, in the regime where the surface states are partially occupied, not only charges the dot and shifts the electrostatic potential, but changes the correlated wave function as well. Hence, one expects a correction to the local density of states in the dot, which is energy dependent, in a similar way to Anderson’s orthogonality catastrophe, or the singularity in the X-ray core level photoemission. Such Fermi edge singularities have also been discussed in relation to transport in quantum dots and nanotubes.

The correlated state which describes the surface states of the graphene quantum dot resembles a one dimensional system localized along the surface. In this respect, the tunneling into this state can also be analyzed within the related framework of tunneling into correlated one dimensional metals. In this case, and in those described before, one expects that the tunneling density of states of the dot will be described by a power law. We have computed numerically the spectral function

$$A_N(\omega) \propto \sum_{M,n} |(N-1, M_0; 0)\sum_{m=0}^{m_{\text{max}}} c_m |N, M; n|^2 \times \delta(E_n^{N,M} - E_0^{N-1,M_0} - \omega)$$

where $c_m$ annihilates a particle with angular momentum $m$. Next nearest neighbor hopping $t'$ causes a finite total angular momentum $M_0$ of the $N-1$-electron ground state and due to momentum conservation the angular momentum of the N-electron state is given by $M = M_0 + m$. We restrict the $m$-summation by an upper angular momentum.

Results for the spectral function are shown in Fig. 6 which are characterized by a sharp peak, reminiscent to the delta peak of the non-interacting system. Due to the electron-electron interaction, this peak is smeared out and decays as a power law decay, in qualitative agreement with the arguments mentioned above. There is a clear convergence for low energies as function of the maximal angular momentum.

VI. SUMMARY AND OUTLOOK

We have presented a simple model of a graphene quantum dot, suitable for the analysis of interaction effects.
We show that Coulomb blockade effects are similar to those in other systems when the chemical potential is far from the neutrality point.

The Dirac equation which describes the electronic states of graphene allows for the existence of midgap states, near defects or surfaces. The presence of these states changes qualitatively the properties of the dot in the Coulomb blockade regime. As the kinetic energy of these states is nearly zero, the resulting wave function is mostly determined by the interaction, and deviates significantly from a single Slater determinant. In order to describe correlations beyond mean field we employed the method of configuration interaction within the subspace of surface states. Since it is known that screening is weak for these states, near defects or surfaces. The presence of these states, near defects or surfaces. The presence of these states, near defects or surfaces, allows for the existence of midgap states from the neutrality point.

We note that the correlations present in the spin valley- and spin-degree of freedom which will be addressed in a subsequent publication.

VII. ACKNOWLEDGMENTS

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APPENDIX A: DISCRETIZATION OF THE DIRAC EQUATION

The Dirac equation for angular momentum \( l \) can be written as two coupled one dimensional differential equations:

\[
V(r)\psi_A(r) + v_F \left( i \partial_r \pm \frac{l+1}{2r} \right) \psi_B(r) = \epsilon \psi_A(r)
\]

\[
v_F \left( i \partial_r \mp \frac{l}{2r} \right) \psi_A(r) + V(r)\psi_B(r) = \epsilon \psi_B(r)
\]

(A1)

where the two signs correspond to the two Dirac points. We now analyze a given Dirac equation. Extension to the other valley is straightforward. Equation (A1) can be written as:

\[
V(r) \psi_A(r) + \psi_B(r) + v_F \left( i \partial_r \pm \frac{l+1}{2r} \right) [\psi_A(r) + \psi_B(r)] - iv_F \frac{2l+1}{2r} [\psi_A(r) - \psi_B(r)] = \epsilon [\psi_A(r) + \psi_B(r)]
\]

\[
V(r) \psi_A(r) - \psi_B(r) - v_F \left( i \partial_r \mp \frac{l}{2r} \right) [\psi_A(r) - \psi_B(r)] + iv_F \frac{2l+1}{2r} [\psi_A(r) + \psi_B(r)] = \epsilon [\psi_A(r) - \psi_B(r)]
\]

(A2)

We define

\[
\tilde{\psi}_1(r) = \frac{\psi_A(r) + \psi_B(r)}{\sqrt{r}}
\]

\[
\tilde{\psi}_2(r) = \frac{\psi_A(r) - \psi_B(r)}{\sqrt{r}}
\]

(A3)

and we obtain:

\[
V(r)\tilde{\psi}_1(r) + iv_F \partial_r \tilde{\psi}_1(r) - iv_F \frac{2l+1}{2r} \tilde{\psi}_2(r) = \epsilon \tilde{\psi}_1(r)
\]

\[
V(r)\tilde{\psi}_2(r) - iv_F \partial_r \tilde{\psi}_2(r) + iv_F \frac{2l+1}{2r} \tilde{\psi}_1(r) = \epsilon \tilde{\psi}_2(r)
\]

(A4)
A set of discrete equations which, taking the continuum limit, lead to Eq. (A1) is:

\[
\begin{align*}
(1 - \frac{2l + 1}{4n}) a'_n + (1 + \frac{2l + 1}{4n}) a'_{n+1} + v_n b'_n &= \epsilon b'_n \\
(1 + \frac{2l + 1}{4n}) b'_{n-1} - (1 - \frac{2l + 1}{4n}) b''_n + v_n a'_n &= \epsilon a'_n
\end{align*}
\]

This set of equations is formally equivalent to a dimerized tight binding chain, as schematically shown in Fig. 7. These chains admit zero energy states localized at the ends, when the last hopping is smaller than the previous one. In order to avoid the formation of a spurious level at the center of the dot, \( n = 1 \), the chain is doubled, as also shown in Fig. 7. The Coulomb potential is discretized as:

\[
v_n = \sum_{m=1}^{N} v_{nm} \sum_l a_m^l b_m^l
\]

and:

\[
v_{nm} = v_0 \int_0^{2\pi} d\theta \frac{m}{\sqrt{m^2 + n^2 + 2mn \cos(\theta)}}
\]

In terms of the original Dirac equation, the energies are expressed in units of \( \hbar v_F/R \) and the parameter \( v_0 \) is given by \( v_0 = (e^2/4\pi\epsilon_0) / \hbar v_F \approx 1 \).

**APPENDIX B: KINETIC ENERGY DUE TO NEXT-NEAREST NEIGHBOR HOPPING**

Due to next-nearest neighbor hopping \( t'' \sim 0.1t \), the initially flat band of surface states becomes dispersive. From Ref. 23, the kinetic term due to \( t'' \) is given by \( T = \frac{1}{4} t'' a^2 r^2 - \frac{1}{4} t'' a^2 \Delta \). As for the Coulomb interaction we restrict our Hilbert space to the surface states \( \psi_m(r, \theta) = \Psi_{s+\pi}(r, \theta) \) defined in Eq. (4):

\[
\langle m|T|n \rangle = -\delta_{nm} \frac{9}{4} t'' a^2 \int d^2 r \psi_m^* \tilde{r} \Delta \psi_n \tilde{r}
\]

\[
\int d^2 r \psi_m^* \Delta \psi_m = - \int d^2 r \nabla \psi_m^* \nabla \psi_m + 2\pi [\psi_m^* r \partial_r \psi_m]_{r=R}
\]

In the second row we used partial integration leading to the boundary term (second term on right hand side). Including next nearest neighbor hopping this boundary term has to vanish, while the general form of the wavefunction is assumed to change only close to the boundary. We thus only keep the first term which results in

\[
\langle m|T|n \rangle = \delta_{nm} \frac{9a^2 t''}{2R^2} m(m+1) = \frac{\hbar v_F^2}{R} \frac{3a^2}{10R} m(m+1).
\]
dimensions lead to a charge distribution which varies smoothly over the dot, but has a $\rho(r) \propto 1/\sqrt{R-r}$ divergence at the edges.

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