Collisionless Magnetic Reconnection in a Five-Moment Two-Fluid Electron-Positron Plasma

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Abstract. We simulate magnetic reconnection in electron-positron (pair) plasma using a collisionless two-fluid model with isotropic pressure. In this model the resistive, Hall, and electrokinetic pressure terms are absent from the curl of Ohm’s law, leaving the inertial term alone to provide for magnetic reconnection. Our simulations suggest that for pair plasma simulated with isotropic pressure fast reconnection does not occur without the aid of sufficient (numerical) diffusion. We contrast this result with simulations and published results showing fast reconnection for collisionless two-fluid plasma with isotropic pressures and non-canceling mass-to-charge ratios, where Hall effects are present and numerical diffusion is small, and with published PIC studies of pair plasma which observe fast reconnection and attribute it to nonisotropic pressure.

1. Introduction

An important issue of controversy in the magnetic reconnection community is the minimal conditions required for fast magnetic reconnection to occur in a plasma and the minimal modeling requirements to resolve it [7]. The first attempt to model reconnection was carried out by Sweet [9] and Parker [6], who used a resistive magnetohydrodynamic (MHD) model. Their approach was only successful in modeling a slow form of reconnection and not the much faster reconnection that is observed in laboratory and space plasma.

The Geospace Environmental Modeling (GEM) Reconnection Challenge problem was introduced in [3] and studied with a variety of models to identify the essential physics required to model collisionless magnetic reconnection. This original GEM article concluded that all models that include the Hall term in the generalized Ohm’s law produced essentially indistinguishable rates of reconnection. The only other model in their study that admitted fast reconnection was MHD with large anomalous (e.g., current-dependent) resistivity, although as expected it did not exhibit the quadrupole out-of-plane magnetic field pattern that appears to characterize models which incorporate the Hall term.

2000 Mathematics Subject Classification. Primary 65M60, 65Z05.
Key words and phrases. Magnetic reconnection, plasma physics, discontinuous Galerkin.
This work is supported in part by NSF Grant DMS-0711885 and a Graduate Fellowship from the Wisconsin Space Grant Consortium.
Since inclusion of Hall effects had been identified as the critical ingredient to admit fast reconnection, Bessho and Bhattacharjee studied electron-positron plasmas, for which the Hall term is zero\(^1, 2\). Their particle-in-cell (PIC) simulations of the GEM problem (with the mass ratio reset from 25 to 1) exhibited fast reconnection for temperature ratios of 5 and 1. They found that the structure of the reconnection region still shows an X-point, although as expected the out-of-plane magnetic field does not show the quadrupole structure that appears in models that incorporate Hall term effects.

This prompted us to ask if it is possible to get fast reconnection if the Hall term is absent and the pressures are modeled as isotropic. Therefore we chose to study reconnection in an electron-positron plasma using a two-fluid model with isotropic pressure.

2. Physical models

2.1. Particle-in-cell description. Plasma consists of charged particles interacting with the electromagnetic field. In the absence of gravity and quantum-mechanical effects the particles of a plasma satisfy Maxwell’s equations and the Lorentz force to govern particle motion:

\[
\begin{align*}
\partial_t \mathbf{B} &= -\nabla \times \mathbf{E}, \\
\partial_t \mathbf{E} &= c^2 \nabla \times \mathbf{B} - \mathbf{J}/\epsilon_0, \\
d_t \mathbf{v}_p &= \frac{q_p}{m_p} \left( \mathbf{E}(x_p) + \mathbf{v}_p \times \mathbf{B}(x_p) \right), \\
\mathbf{J} &= \sum_p S_p(x_p) q_p \mathbf{v}_p, \\
\sigma &= \sum_p S_p(x_p) q_p;
\end{align*}
\]

here \(\mathbf{B}\) is magnetic field, \(\mathbf{E}\) is electric field, \(c\) is the speed of light, \(\epsilon_0\) is electric permittivity, \(p\) is particle index, \(x_p(t)\) is particle position, \(\mathbf{v}_p(t) = \gamma_p \mathbf{v}_p\) is (proper) particle velocity, where \(\gamma = (1 - (\mathbf{v}/c)^2)^{-1/2} \approx 1\) is the Lorentz factor, \(q_p\) is particle charge, \(m_p\) is particle mass, \(\sigma\) is charge density, \(\mathbf{J}\) is current density, and \(S_p(x - x_p)\) is particle charge distribution (e.g., a unit impulse function).

Particle-in-cell (PIC) codes model plasma by attempting to evolve in a mixed Eulerian-Lagrangian framework: particles are treated in a Lagrangian way, while the electromagnetic field sits on an Eulerian computational mesh.

2.2. Boltzmann description. The Boltzmann model replaces the particles with particle (probability) density functions \(f_s(x, \mathbf{v}, t)\) of space, velocity, and time, for each species \(s\). The Boltzmann equation asserts conservation (or balance) of particles in phase space:

\[
\begin{align*}
\partial_t f_s + \nabla_x \cdot (\mathbf{v} f_s) + \frac{1}{r_g} \nabla_\mathbf{v} \cdot \left( \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right) &= C_s; \\
\end{align*}
\]

here \(\mathbf{v} = \gamma \mathbf{v} \approx \mathbf{v}\) is (proper) velocity in phase space, and \(C_s\) is a collision operator which is a function of \(\{\mathbf{v} \mapsto f_p(t, x, \mathbf{v})\}_p\), where \(p\) ranges over all species. The collisionless Boltzmann equation (alias Vlasov equation) asserts that \(C_s = 0\). Maxwell’s equations are coupled to the Boltzmann equation by the relations

\[
\begin{align*}
\mathbf{J} &= \sum_s \int_{\mathbf{v}} f_s q_s \mathbf{v}, \\
\sigma &= \sum_s \int_{\mathbf{v}} f_s q_s;
\end{align*}
\]
2.3. Two-fluid model. Multiplying the Boltzmann equation by powers of velocity and integrating over velocity space yields fluid equations. Generic two-fluid equations for a two-species plasma are:

\[
\begin{align*}
\partial_t & \begin{bmatrix} \rho_i \\ \rho_e \\ \rho_i u_i \\ \rho_e u_e \\ \mathcal{E}_i \\ \mathcal{E}_e \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho_i u_i \\ \rho_e u_e + P_i \\ \rho_i u_i u_i + P_i \\ \rho_e u_e u_e + P_e \\ u_i \mathcal{E}_i + u_i \cdot \mathcal{P}_i + q_i \\ u_e \mathcal{E}_e + u_e \cdot \mathcal{P}_e + q_e \end{bmatrix} = \\
&= \begin{bmatrix} 0 \\ 0 \\ \sigma_i E + J_i \times B \\ \sigma_e E + J_e \times B \\ J_i \cdot E \\ J_e \cdot E \end{bmatrix} + \begin{bmatrix} R_i \\ R_e \\ R_i \cdot u_i + Q_{R,i} + Q_i \\ R_e \cdot u_e + Q_{R,e} + Q_e \end{bmatrix},
\end{align*}
\]

The variables are defined as follows: \(i\) and \(e\) are ion and electron species indices; \(q_s = \pm e\) is particle charge, \(m_s\) is particle mass, \(n_s\) is particle number density, \(\rho_s = m_sn_s\) is mass density, \(\sigma_s = q_sn_s\) is charge density, \(J_s = u_s\sigma_s\) is current density, \(\mathcal{P}_s\) is the pressure tensor, \(\mathcal{E}_s\) is gas-dynamic energy, \(q_s\) is the heat flux, \(R_s = -R_e\) denotes the interspecies drag force on the ions, \(Q_{R,s}\) denotes heating due to friction (drag), and \(Q_i = -Q_e\) denotes the interspecies thermal heat transfer to the ions.

2.4. Collisionless isotropic closure. To close the system we must posit constitutive relations for the nonevolved quantities. In a collisionless model we neglect the terms that come from the collision operator: \(q_s, R_s, Q_{R,s}\), and \(Q_s\). In an isotropic model we assume that the pressure tensor is a scalar pressure times the identity tensor: \(P_s = p_s I\); this leads to the constitutive relation \(\mathcal{E}_s = (3/2)p_s + \rho_s u_s^2 / 2\).

2.5. Ohm’s law. Multiplying the momentum equations of each species by its charge to mass ratio and summing gives a balance law for net current. Invoking the assumption of quasineutrality (\(\sigma \approx 0\)) and solving this law for electric field gives the generalized Ohm’s law,

\[
E = B \times u + E',
\]

where \(u\) is the mass-averaged fluid velocity and where the electric field in the frame of reference of the fluid is the sum of four terms:

\[
E' = \eta \cdot J + \frac{\bar{m}_i - \bar{m}_e}{\rho} J \times B + \frac{1}{\rho} \nabla \cdot (\bar{m}_e \mathcal{P}_i - \bar{m}_i \mathcal{P}_e) + \frac{\bar{m}_i \bar{m}_e}{\rho} \left( \partial_t J + \nabla \cdot (uJ + J\left(\frac{\bar{m}_e - \bar{m}_i}{\rho}Ju\right) \right)
\]

\[\text{(inertial term).}\]

\[\text{We remark that our collisionless model with isotropic pressure seems not to correspond to any general physical regime of plasma (although it may apply to particular configurations). A model is considered to be physical if it agrees with a physical regime in some physical limit. We assume an isotropic pressure tensor yet no resistivity, but for an electron-positron plasma the time scale over which particles thermalize is the same as the time scale over which resistive drag force seeks to equilibrate the velocities of the two species.}\]
Here we adopt the convention that \( \tilde{m}_s := m_s/e \), and we have assumed that the electrical resistance \( R_i \), \( \tilde{m}_s + \tilde{m}_e \) equals \( \eta \cdot \mathbf{J} \) (where \( \eta \) is the resistivity), a simple function of the drift velocity, i.e., of the current.

Ideal MHD assumes that \( \mathbf{E}' = 0 \), and resistive MHD assumes that \( \mathbf{E}' = \eta \cdot \mathbf{J} \).

Substituting Ohm’s law into Faraday’s law \( \partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0 \) yields \( \partial_t \mathbf{B} + \nabla \cdot (\mathbf{uB} + \mathbf{Bu}) = \nabla \times \mathbf{E'} \), which implies that the flux of \( \mathbf{B} \) through a surface convected by \( \mathbf{u} \) can only change if the curl of \( \mathbf{E'} \) is nonzero.\(^2\)

In an electron-positron plasma the masses of ions and electrons are identical and the Hall term vanishes. Also, if pressure is isotropic and density varies slowly, then the curl of the pressure term is zero. In the collisionless two-fluid model the resistivity is zero. So for our two-fluid model reconnection could only happen by means of the inertial term (or numerical diffusion).

3. The five-moment two-fluid model

The collisionless two-fluid equations we solved were

\[
\partial_t \begin{bmatrix} \rho_i \\
\rho_i \mathbf{u}_i \\
\rho_e \mathbf{u}_e \\
\rho_e \mathbf{e}_e \\
\mathbf{E} \\
\end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho_i \mathbf{u}_i \\
\rho_i \mathbf{u}_i + \rho_i \mathbf{I} \\
\rho_e \mathbf{u}_e + \rho_e \mathbf{I} \\
\mathbf{E}_e^2 + \rho_e \mathbf{e}_e \\
\n \end{bmatrix} = \begin{bmatrix} 0 \\
\sigma_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) \\
\sigma_i \mathbf{u}_i \cdot \mathbf{E} \\
0 \\
\sigma_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) \\
\sigma_e \mathbf{u}_e \cdot \mathbf{E} \\
\end{bmatrix},
\]

These equations were studied extensively by Shumlak and Loverich\(^3\) and Hakim, Shumlak, and Loverich\(^4\). A version of this model with anisotropic pressure was also considered by Hakim\(^4\).

This system is identical in appearance with the two-fluid system\(^2\) \((2.1)–(2.2)\) if the correction potentials \( \psi \) and \( \phi \), which we have added for numerical divergence cleaning purposes, are zero. These equations imply a wave equation that propagates the divergence constraint error at the speed \( c \sqrt{\chi} \). We select \( \chi = 1.05 \).

We nondimensionalized this system by choosing typical values of magnetic field \( B_0 \), (ion) number density \( n_0 \), particle charge \( q_0 = e \), and combined particle mass \( m_0 = m_i + m_e \). This implies a choice (1) of characteristic time scale \( \omega_g^{-1} := \frac{m_0}{q_0 B_0} \), where \( \omega_g \) is the cyclotron frequency of a typical particle, (2) of characteristic velocity \( v_A := \frac{B_0}{\sqrt{m_0 n_0 q_0}} \), a typical Alfvén speed, where \( \mu_0 := (c^2 e_0)^{-1} \) is the permeability of magnetic field and (3) straightforwardly of all other quantities. Replacing every

\(^2\) We remark that if there exists a velocity field \( \mathbf{v} \) for which \( \partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{v}) = 0 \), then magnetic flux is convected by \( \mathbf{v} \) and the topology of magnetic field lines cannot change. In particular, if we merely add the Hall term to the ideal Ohm’s law, then \( \partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times (\mathbf{u} + \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{J} \times \mathbf{B})) \), i.e., the magnetic field is essentially carried by the electrons. Hall-mediated fast reconnection requires a small amount of resistivity as well.

\(^3\) To see this first take the divergence of Maxwell’s evolutions equations. Then either (1) take the time derivative of Maxwell’s constraint equations to eliminate the electromagnetic field and get a wave equation for the correction potentials, or (2) eliminate the correction potentials by taking the time derivative of the divergence of Maxwell’s evolution equations and the Laplacian of the constraint equations to eliminate the correction potentials and get a wave equation for the divergence constraint error.
quantity $X$ with nondimensional representation $\hat{X}X_0$ gives a system governing the $\hat{X}$ values with exactly the same appearance, except that $1/\hat{\epsilon} = \hat{c}^2$. We drop hats.

4. GEM magnetic reconnection challenge problem

With the exception that our nondimensionalized light speed is 10 rather than their value of 20, our settings are equivalent to those of [11], which reflect the settings and conventions of the original GEM problem [3]. To map our SI-like nondimensionalization onto their Gaussian-like nondimensionalization, rescale the electromagnetic field by $B_{\text{GEM}} = \sqrt{4\pi}B$ and $E_{\text{GEM}} = \sqrt{\frac{4\pi}{c}}E$.

4.1. Computational domain. The computational domain is the rectangular domain $[-L_x/2, L_x/2] \times [-L_y/2, L_y/2]$, where $L_x = 8\pi$ and $L_y = 4\pi$. The problem is symmetric under reflection across either the horizontal or vertical axis.

4.2. Boundary conditions. The domain is periodic in the $x$-axis. The boundaries perpendicular to the $y$-axis are thermally insulating conducting wall boundaries. A conducting wall boundary is a solid wall boundary (with slip boundary conditions in the case of ideal plasma) for the fluid variables, and the electric field at the boundary has no component parallel to the boundary. We also assume that magnetic field runs parallel to and so does not penetrate the boundary (this follows from Ohm’s law of ideal MHD, but we assume it holds generally) [3].

4.3. Model Parameters. We carried out simulations for the following choices of the GEM model parameters:

(1) $m_i/m_e = 25$, $T_i/T_e = 5$ (original GEM),
(2) $m_i/m_e = 1$, $T_i/T_e = 5$, and
(3) $m_i/m_e = 1$, $T_i/T_e = 1$.

4.4. Initial conditions. The initial conditions are a perturbed Harris sheet equilibrium. The unperturbed equilibrium is given by

\begin{align*}
B(y) &= B_0 \tanh(y/\lambda) e_x, \\
p(y) &= \frac{B_0^2}{2n_0} n(y), \\
n_i(y) &= n_e(y) = n_0(1/5 + \text{sech}^2(y/\lambda)), \\
p_e(y) &= \frac{T_e}{T_i + T_e} p(y), \\
p_i(y) &= \frac{T_i}{T_i + T_e} p(y).
\end{align*}

On top of this the magnetic field is perturbed by

\[ \delta B = -e_z \times \nabla(\psi), \]

where

\[ \psi(x, y) = \psi_0 \cos(2\pi x/L_x) \cos(\pi y/L_y). \]

In the GEM problem the initial condition constants are

\[ \lambda = 0.5, \quad B_0 = 1, \quad n_0 = 1, \quad \psi_0 = B_0/10. \]

We remark that with the symmetries of the GEM problem at the conducting wall boundary may also be regarded as a symmetry conditions for a periodic boundary if the solution on the entire domain is reflected across its bottom boundary and negated, allowing the GEM problem to be solved on a doubled domain with periodic boundaries and infinitely smooth initial conditions.
5. Properties of the GEM problem

5.1. Reconnected flux. In defining and discussing magnetic flux we restrict ourselves to the first quadrant of the domain, referring to it as if it were the entire domain.

The ideal MHD model implies the frozen-in flux condition, which says that magnetic flux is convected with the fluid. The boundary conditions and symmetries of the problem dictate that at the boundaries fluid can only move parallel to the boundaries and that the in-plane fluid velocity at the corners must be zero. Therefore, the frozen-in flux condition would say that the flux through any boundary must remain constant.

In fact, for models which permit reconnection, the frozen-in flux condition breaks down near the X-point, allowing magnetic field to diffuse and allowing field lines to break and reconnect so that they pass through the horizontal axis. We therefore define magnetic reconnection to be the loss of magnetic flux through the vertical axis into the first quadrant:

**Definition 5.1.** The reconnected flux \( F_{\text{recon}} \) is defined by

\[
F_{\text{left}}(t) := \int_0^{y_{\text{max}}} B_1 \, dy, \quad F_{\text{recon}}(t) := F_{\text{left}}(0) - F_{\text{left}}(t).
\]

**Proposition 5.1.** The rate of reconnection is minus the value of the out-of-plane component of the electric field at the origin (i.e. the X-point).

**Proof.**

\[
d_t F_{\text{recon}}(t) = -d_t F_{\text{left}}(t) = - \int_0^{y_{\text{max}}} \partial_y B_1 \, dy = \int_0^{y_{\text{max}}} \partial_y E_3 \, dy = -E_3(0),
\]

since \( E_3 \) is zero at the conducting wall.

5.2. Reflectional symmetries. The GEM problem has reflectional symmetry across the horizontal and vertical axes. We impose this symmetry by restricting our computations to the first quadrant. At the origin, symmetries across the horizontal and vertical axes mean that vectors have only an out-of-plane component and pseudovectors must be zero.

5.3. Reductions in Ohm’s law at the origin. Symmetries at the origin reduce Ohm’s law to:

\[
E_3 = \eta J_3 + \frac{1}{\rho} (\hat{m}_e (\partial_x P_{r,1,3} + \partial_x P_{i,2,3}) - \hat{m}_i (\partial_x P_{e,1,3} + \partial_x P_{e,2,3})) + \frac{\hat{m}_i \hat{m}_e}{\rho} (\partial_3 J_3 + J_3 \nabla \cdot u + u_3 \nabla \cdot J + \frac{\hat{m}_e - \hat{m}_i}{\rho} J_3 \nabla \cdot J)
\]

If we neglect resistivity, off-diagonal pressure components, and assume that the flow of charge and mass toward or away from the origin is small, then this becomes

\[
-d_t F_{\text{recon}} = E_3 \approx \frac{\hat{m}_i \hat{m}_e}{\rho} \partial_3 J_3 \approx \hat{m}_i \hat{m}_e \partial_3 (J_3 / \rho),
\]

i.e., the value of \(-\hat{m}_i \hat{m}_e J_3 / \rho\) at the origin will tend to track with \(F_{\text{recon}}\).

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5. This confirms the theoretical fact that an MHD model which only includes the \(B \times u\) and Hall terms in Ohm’s law cannot give fast reconnection, since both these terms must vanish at the origin.

6. In simulations which incorporate the entire domain symmetry is often lost due to computational noise and the inherent instability of the problem.
6. Results

Following the precedent of [8], [4], and [5], we carried out simulations of the GEM problem using a third-order shock-capturing Runge-Kutta Discontinuous Galerkin solver for a collisionless two-fluid model with isotropic pressure for each species. (We enforced the divergence cleaning for the magnetic field but not for the electric field.) We carried out simulations on a quarter domain (hence enforcing symmetry) for mesh sizes of $32 \times 16$, $64 \times 32$, and $128 \times 64$.

For the low mesh resolution we seemed to observe fast reconnection in the electron-positron plasma (based on the pattern of magnetic field lines and the rates of reconnection), but not for high resolutions. We hypothesize that the electron inertial term coupled with sufficient (numerical) resistivity is sufficient to yield fast reconnection. In future studies we hope to explicitly introduce collisional (diffusive) terms (rather than relying on numerical diffusion) and explore whether we can show convergence to fast reconnection for resistive isotropic two-fluid plasma.

We plotted reconnected flux, $-\bar{\eta}_m \bar{\eta}_e J_3/\rho$, and a crude calculation of $-\int_0^t E_3$ (the cumulative sum of $-E_3$ evaluated at integer times) for all three combinations of mass ratio and temperature ratio. Our plots indicate that $-\bar{\eta}_m \bar{\eta}_e J_3/\rho$ only tracked with reconnected flux for the high-resolution simulations of electron-positron plasma up until the time when a large magnetic island formed around the origin. (These were the plots where fast reconnection did not occur.)

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Figure 1. Three plots attempting to show convergence of reconnection data for electron-positron plasma with equal initial temperatures. Reconnection is suppressed for finer mesh resolution but fast reconnection occurs for coarse mesh resolution. We conjecture that numerical diffusion coupled with the ion inertial term is sufficient to admit fast reconnection. The appearance of a large central magnetic island beginning around time 15 is curious.
Figure 2. Convergence study of reconnection for electron-positron plasma with initial temperature ratio of 5. The reconnection rate would appear to decrease with decreasing numerical diffusion. Only for the finest mesh does a central magnetic island form.
Figure 3. Convergence study of reconnection for electron-positron plasma with mass ratio of 25 and initial temperature ratio of 5. Fast reconnection occurs for both coarse and fine mesh resolutions. The anomalously early cessation of reconnection in the simulation at intermediate resolution is evidently related to the appearance of a large central magnetic island beginning around time 20.