FUNDAMENTAL GROUP OF DESARGUES CONFIGURATION SPACES

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Abstract. We compute the fundamental group of various spaces of Desargues configurations in complex projective spaces: planar and non-planar configurations, with a fixed center and also with an arbitrary center.

1. INTRODUCTION

Let \( M \) be a manifold and \( \mathcal{F}_k(M) \) be its ordered configuration space of \( k \)-tuples \( \{(x_1, \ldots, x_k) \in M^k \mid x_i \neq x_j, i \neq j\} \). The \( k \)th pure braid group of \( M \) is the fundamental group of \( \mathcal{F}_k(M) \). The pure braid group of the plane, denoted by \( \mathcal{PB}_n \), has the presentation \([4]\)

\[
\pi_1(\mathcal{F}_n(\mathbb{C})) = \mathcal{PB}_n \cong \langle \alpha_{ij}, 1 \leq i < j \leq n \mid (YB\ 3)_n, (YB\ 4)_n \rangle
\]

where generators \( \alpha_{ij} \) are represented in the figure and the Yang-Baxter relations

\[
(YB\ 3)_n \quad \text{and} \quad (YB\ 4)_n \quad \text{are, for any} \quad 1 \leq i < j < k \leq n,
\]

\[
(YB\ 3)_n : \alpha_{ij} \alpha_{ik} \alpha_{jk} = \alpha_{ik} \alpha_{jk} \alpha_{ij} = \alpha_{jk} \alpha_{ij} \alpha_{ik}
\]

and, for any \( 1 \leq i < j < k < l \leq n, \)

\[
(YB\ 4)_n : [\alpha_{kl}, \alpha_{ij}] = [\alpha_{jl}, \alpha_{ik}^{-1} \alpha_{jk} \alpha_{ik}] = [\alpha_{il}, \alpha_{jk}] = [\alpha_{jl}, \alpha_{kl}^{-1} \alpha_{ik} \alpha_{kl}] = 1.
\]

The pure braid group of \( S^2 \approx \mathbb{CP}^1 \) have the presentation (see [5] and [4]):

\[
\pi_1(\mathcal{F}_{k+1}(S^2)) \cong \langle \alpha_{ij}, 1 \leq i < j \leq k \mid (YB\ 3)_k, (YB\ 4)_k, D_k^2 = 1 \rangle,
\]

where \( D_k = \alpha_{12}(\alpha_{13} \alpha_{23}) \ldots (\alpha_{1k} \ldots \alpha_{k-1,k}) \) (in \( B_k \), the Artin braid group, \( D_k \) is the square of the Garside element \( \Delta_k \), see [6] and [2]). In [2] we started to study the topology of configuration spaces under simple geometrical restrictions. Using the geometry of the projective space we can stratify the configuration space \( \mathcal{F}_k(\mathbb{CP}^n) \) with complex submanifolds:

\[
\mathcal{F}_k(\mathbb{CP}^n) = \coprod_{i=1}^{n} \mathcal{F}_k^{i,n}.
\]

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where $\mathcal{F}^i_{k,n}$ is the ordered configuration space of all $k$-tuples in $\mathbb{CP}^n$ generating a subspace of dimension $i$. Their fundamental groups are given by (see [2]):

**Theorem 1.1.** The spaces $\mathcal{F}^i_{k,n}$ are simply connected with the following exceptions

(1) for $k \geq 2$,
\[
\pi_1(\mathcal{F}^i_{k+1,n}) \cong \langle \alpha_{ij}, \ 1 \leq i < j \leq k \rangle (YB3)_k, (YB4)_k, D^2_k = 1;
\]

(2) for $k \geq 3$ and $n \geq 2$,
\[
\pi_1(\mathcal{F}^i_{k+1,n}) \cong \langle \alpha_{ij}, \ 1 \leq i < j \leq k \rangle (YB3)_k, (YB4)_k, D^3_k = 1).
\]

In this paper we compute the fundamental groups of various configuration spaces related to projective Desargues configurations. We do not use special notations for the dual projective space: if $P_1, P_2, P_3$ are three points and $d_1, d_2, d_3$ are three lines in $\mathbb{CP}^2$, $(P_1, P_2, P_3) \in \mathcal{F}^1_{3,2}$ is equivalent with the collinearity of these points and $(d_1, d_2, d_3) \in \mathcal{F}^2_{3,2}$ is equivalent with the concurrency of these lines. We define $\mathcal{D}^{2,n}$, the space of planar Desargues configurations in $\mathbb{CP}^n (n \geq 2)$, by
\[
\mathcal{D}^{2,n} = \{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathcal{F}^2_{6,n} \mid (d_1, d_2, d_3) \in \mathcal{F}^1_{3,2}, A_i, B_i \in d_i \setminus \{I\}\}
\]

(here $I = d_1 \cap d_2 \cap d_3$).

We consider also $\mathcal{D}^{2,n}_{I}$, the space of planar Desargues configuration with a fixed intersection point $I \in \mathbb{CP}^n$, defined by
\[
\mathcal{D}^{2,n}_{I} = \{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathcal{D}^{2,n} \mid d_1 \cap d_2 \cap d_3 = I\}.
\]

**Theorem 1.2.** The fundamental group of $\mathcal{D}^{2,n}_{I}$ is given by
\[
\pi_1(\mathcal{D}^{2,n}_{I}) \cong \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 2, \\
\mathbb{Z} \oplus \mathbb{Z} & \text{if } n \geq 3.
\end{cases}
\]

The first group is generated by $[\alpha], [\beta], [\sigma]$ and the second group is generated by $[\alpha]$ and $[\sigma]$. Precise formulae for $\alpha, \beta$ and $\sigma$ are given in section 2; here is a diagram representing these generators (there is a similar picture for $\beta$):

$\alpha : B_1$ is moving on the line $d_1 \setminus \{I^0, A_1^0\}$  \hspace{1cm} $\sigma :$ the lines $d_1$ and $d_2$ are moving
Theorem 1.3. The fundamental group of $\mathcal{D}^{2,n}$ is given by:

$$\pi_1(\mathcal{D}^{2,n}) \cong \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 2, \\
\mathbb{Z} & \text{if } n \geq 3.
\end{cases}$$

The first group is generated by $[\alpha]$ and $[\beta]$ and the second is generated by $[\alpha]$ (or by $[\alpha]$). We will use the same notations for $[\alpha], [\beta], [\sigma]$ and their images through different natural maps: $\mathcal{D}^{1,*} \rightarrow \mathcal{D}^{*,*}, \mathcal{D}^{1,*} \rightarrow \mathcal{D}^{1,**+1}, \mathcal{D}^{*,*} \rightarrow \mathcal{D}^{*,*+1}$.

We define $\mathcal{D}^{3,n}$, the space of non-planar Desargues configurations in $\mathbb{C}P^n$ ($n \geq 3$):

$$\mathcal{D}^{3,n} = \{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathbb{F}^{0,6}_n \mid d_1 \cap d_2 \cap d_3 = I, A_i, B_i \in d_i \setminus \{I\}\}$$

and $\mathcal{D}^{3,n}_I$, the associated space of non-planar Desargues configurations with a fixed intersection point $I \in \mathbb{C}P^n$.

Theorem 1.4. The fundamental group of $\mathcal{D}^{3,n}_I$ is given by:

$$\pi_1(\mathcal{D}^{3,n}_I) \cong \begin{cases} 
\mathbb{Z} & \text{if } n = 3, \\
1 & \text{if } n \geq 4.
\end{cases}$$

Theorem 1.5. The fundamental group of $\mathcal{D}^{3,n}$ is given by:

$$\pi_1(\mathcal{D}^{3,n}) \cong \begin{cases} 
\mathbb{Z}_4 & \text{if } n = 3, \\
1 & \text{if } n \geq 4.
\end{cases}$$

In the last two theorems, in the non-simply connected cases, the fundamental groups are generated by $[\alpha]$.

2. DESARGUES CONFIGURATIONS IN THE PROJECTIVE PLANE

In order to find the fundamental groups of the spaces $\mathcal{D} = \mathcal{D}^{2,2}$ and $\mathcal{D}_I = \mathcal{D}^{2,2}_I$ we use two fibrations and their homotopy exact sequences.

Lemma 2.1. The projection

$$\mu : \mathcal{D} \rightarrow \mathbb{C}P^2, (A_1, B_1, A_2, B_2, A_3, B_3) \mapsto I = d_1 \cap d_2 \cap d_3$$

is a locally trivial fibration with fiber $\mathcal{D}_I$.

Proof. Fix a point $I^0 \in \mathbb{C}P^2$ and choose a line $l \subset \mathbb{C}P^2 \setminus \{I^0\}$ and the neighborhood $\mathcal{U}_l = \mathbb{C}P^2 \setminus l$ of $I^0$. For a point $I$ in this neighborhood and a Desargues configuration $(A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^0)$ on three lines $d_1^0, d_2^0, d_3^0$ containing $I^0$ construct lines $d_1, d_2, d_3$ containing $I$ and the configuration $(A_1, B_1, \ldots, A_3, B_3)$ as follows: consider the points $D_i = l \cap d_i^0$ and $Q = l \cap I^0 I$ and define $d_i = ID_i, A_i = d_i \cap QA_i^0$ and in the same way $B_i (i = 1, 2, 3)$. We describe this construction using coordinates to show that the map

$$(I, (A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^0)) \mapsto (A_1, B_1, A_2, B_2, A_3, B_3)$$

has a continuous extension on the singular locus $(d_1^0 \cup d_2^0 \cup d_3^0 \cup l)$. Choose a projective frame such that $I^0 = [0 : 0 : 1], l : X_2 = 0$. If $I = [s : t : 1]$ and $A_i^0 = [n_i : -m_i : a_i], B_i^0 = [m_i : -n_i : b_i]$ (a_i, b_i are distinct and non zero and also $n_i m_j \neq m_i n_j$ for distinct $i, j = 1, 2, 3$), then we define $A_i = [n_i + sa_i : -m_i + ta_i : a_i]$ and $B_i = [m_i + sb_i : -n_i + tb_i : b_i]$ (i = 1, 2, 3), and these formulae agree with the geometrical construction given for nondegenerate positions of $I \in \mathbb{C}P^2 \setminus (d_1^0 \cup d_2^0 \cup d_3^0 \cup l)$. The trivialization over $\mathcal{U}_l$ is given by

$$\varphi : \mathcal{U}_l \times \mathcal{D}_{I^0} \rightarrow \gamma^{-1}(\mathcal{U}_l), (I, (A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^0)) \mapsto (A_1, B_1, A_2, B_2, A_3, B_3)$$

$\square$
Lemma 2.2. The projection

\[ \lambda: D_1 \to F_3(\mathbb{CP}^1), (A_1, B_1, A_2, B_2, A_3, B_3) \mapsto (d_1, d_2, d_3) \]

is a locally trivial fibration with fiber \( F_2(\mathbb{C}) \times F_2(\mathbb{C}) \times F_2(\mathbb{C}) \).

Proof. Fix a point \( d_i^0 = (d_i^0, d_i^0, d_i^0) \) in \( F_3(\mathbb{CP}^1) \) and choose a point \( Q \) in \( \mathbb{CP}^2 \setminus (d_1^0 \cup d_2^0 \cup d_3^0) \) and the neighborhood \( U_Q = \{(d_1, d_2, d_3) \in F_3(\mathbb{CP}^1)|Q \notin d_1 \cup d_2 \cup d_3\} \). The trivialization over \( U_Q \) is given by

\[
\psi: U_Q \times F_2(d_1^0 \setminus \{I\}) \times F_2(d_2^0 \setminus \{I\}) \times F_2(d_3^0 \setminus \{I\}) \to \lambda^{-1}(U_Q)
\]

where \( A_i = d_i \cap QA_i^0 \) and similarly for \( B_i (i = 1, 2, 3) \). Obviously, \( A_i, B_i \) and \( I \) are three distinct points on \( d_i \).

In \( D_{p^e=[0:0:1]} \) we choose the base point \( D^0 = (A_i^0, B_i^0, A_i^0, B_i^0, A_i^0, B_i^0) \) where, for \( k = 1, 2, A_k^0 = [-1 : k : 1], B_k^0 = [-1 : k : 2], A_3^0 = [0 : 1 : 1], B_3^0 = [0 : 1 : 2] \). The corresponding lines are given by the equations \( d_k^0 : kX_0 + X_1 = 0, d_3^0 : X_0 = 0 \) and we identify the affine line \( \mathbb{C} \) with \( d_k^0 \) as follows: for \( k = 1, 2, z \mapsto [-1 : k : z] \), and for \( k = 3, z \mapsto [0 : 1 : z] \) (therefore the intersection point \( I^0 = [0 : 0 : 1] \) is the point at infinity of these lines). We identify the set of three distinct lines through \( I^0 \) with the configuration space \( F_3(\mathbb{CP}^1) \); in this space the base point is \( d_k^0 = (d_k^0, d_k^0, d_k^0) \). In the configuration spaces \( F_2(d_i^0 \setminus \{I^0\}) \) we choose the base points \( (A_i^0, B_i^0), i = 1, 2, 3 \). The homotopy exact sequence from Lemma 2.2 and the triviality of \( \pi_2(F_3(\mathbb{CP}^1)) \) (see [3]) give the short exact sequence

\[ 1 \to \pi_1(F_2(\mathbb{C})) \times \pi_1(F_2(\mathbb{C})) \times \pi_1(F_2(\mathbb{C})) \xrightarrow{\lambda} \pi_1(D_{p^e}) \xrightarrow{\lambda} \pi_1(F_3(\mathbb{CP}^1)) \to 1. \]

Proof of Theorem 1.2 (the case \( n = 2 \)). The first group, isomorphic to \( \mathbb{Z}^3 \), is generated by the pure braids \( a, b, c \), hence their images in \( \pi_1(D_{p^e}) \) are given by the

\[
\text{homotopy classes of the maps } \alpha, \beta, \gamma : (S^1, 1) \to (D_{p^e}, D^0)
\]

\[
\alpha(z) = (A_1^0, B_1^0(z), A_2^0, B_2^0(z), A_3^0, B_3^0), \quad B_1^{\alpha(z)} = [-1 : 1 : 1 + z],
\]

\[
\beta(z) = (A_1^0, B_1^0, A_2^0, B_2^0(z), A_3^0, B_3^0), \quad B_2^{\beta(z)} = [-1 : 2 : 1 + z],
\]

\[
\gamma(z) = (A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^0(z)), \quad B_3^{\gamma(z)} = [0 : 1 : 1 + z].
\]

The third group, \( \pi_1(F_3(\mathbb{CP}^1)) \cong \mathbb{Z}_2 \), is generated by the homotopy class of the map

\[
s: (S^1, 1) \to (F_3(\mathbb{CP}^1), d_k^0), \ z \mapsto (d_k^0(z) : zX_0 + X_1 = 0, d_2^0(z) : 2zX_0 + X_1 = 0, d_3^0),
\]

because this corresponds to the braid \( \alpha_{12} \) in \( \mathbb{CP}^1 \). We lift the map \( s \) to the map

\[
\sigma : (S^1, 1) \to (D_{p^e}^0, D^0), \ z \mapsto (A_1^0(z), B_1^0(z), A_2^0(z), B_2^0(z), A_3^0, B_3^0),
\]

where \( A_k^0(z) = [-1 : kz : 1], B_k^0(z) = [-1 : kz : 2], k = 1, 2. \)
The group \( \pi_1(\mathcal{D}_{P^0}, \mathcal{D}^0) \) is generated by the homotopy classes of \( \alpha, \beta, \gamma \) and \( \sigma \); the defining relations are commutation relations between \([\alpha], [\beta] \) and \([\gamma]\) from \( \pi_1(\mathcal{F}_2(\mathbb{C})^3) \) and the four relations, to be proved in the next two lemmas:

\[
\begin{align*}
\alpha & : [\sigma][\alpha][\sigma]^{-1} = [\alpha], \\
\beta & : [\sigma][\beta][\sigma]^{-1} = [\beta], \\
\gamma & : [\sigma][\gamma][\sigma]^{-1} = [\gamma], \\
\sigma & : [\sigma]^2 = [\alpha]^{-1}[\beta]^{-1}[\gamma].
\end{align*}
\]

The generator \([\gamma]\) can be eliminated, \([\sigma]\) commutes with \([\alpha]\) and \([\beta]\), and the third relation, \(\gamma\), is a consequence of the previous commutation relations. \(\square\)

**Lemma 2.3.** In \( \pi_1(\mathcal{D}_{P^0}, \mathcal{D}^0) \) the next relation holds:

\[
\sigma [\sigma]^2 = [\alpha]^{-1}[\beta]^{-1}[\gamma].
\]

**Proof.** The map

\[
\Lambda : (D^2, S^1) \to (\mathcal{F}_3(\mathbb{C}P^1), d_0^0 = (d_0^1, d_0^2, d_0^3)), \quad z \mapsto (d_1^\Lambda(z), d_2^\Lambda(z), d_3^\Lambda(z)),
\]

where \(d_1^\Lambda(z) = (kz - r)X_0 + (\overline{\tau} + kr)X_1 = 0, (k = 1, 2)\), and \(d_3^\Lambda(z) = zX_0 + rX_1 = 0\) (the notation \(r = 1 - |z|\) will be used in this proof and the next ones), shows that \(s^2 \simeq \text{constant}_{\theta_2}\). We lift this homotopy to

\[
\tilde{\Lambda} : D^2 \to \mathcal{D}_{P^0}, \quad \tilde{\Lambda}(z) = (A_1^\Lambda(z), B_1^\Lambda(z), A_2^\Lambda(z), B_2^\Lambda(z), A_3^\Lambda(z), B_3^\Lambda(z)),
\]

where \(A_k^\Lambda(z) = [-\tau - kr : kz - r : \overline{\tau}], B_k^\Lambda(z) = [-\tau - kr : kz - r : \overline{\tau} + 1], (k = 1, 2)\), and \(A_3^\Lambda(z) = [-r : z : z + 1], B_3^\Lambda(z) = [-r : z : z + 1]\); the map

\[
\tilde{\Lambda}|_{S^1} : S^1 \to \mathcal{D}_{P^0}, \quad z \mapsto (A_1^z, B_1^z, A_2^z, B_2^z, A_3^z, B_3^z)
\]

(with \(A_k^z = [-1 : kz^2 : 1], B_k^z = [-1 : kz^2 : 1 + z], (k = 1, 2)\), and \(B_3^z = [0 : 1 + \overline{\tau}]\) has a trivial homotopy class, therefore we have the relation \([\sigma]^2 = [\sigma*\sigma*(\tilde{\Lambda}|_{S^1})^{-1}]\).

Now we construct a homotopy between \(\sigma*\sigma*(\tilde{\Lambda}|_{S^1})^{-1}\) and \(\alpha^{-1}*\beta^{-1}*\gamma\):

\[
L : S^1 \times I \to \mathcal{D}_{P^0}, (z, t) \mapsto (A_k^{L(z,t)}, B_k^{L(z,t)}, A_2^{L(z,t)}, B_2^{L(z,t)}, A_3^{L(z,t)}, B_3^{L(z,t)}),
\]

where \((k = 1, 2)\):

\[
A_k^{L(z,t)} = [-1 : kL^1(z,t) : 1], B_k^{L(z,t)} = [-1 : kL^1(z,t) : L_k^1(z,t)]
\]

\[
B_3^{L(z,t)} = \begin{cases} 
0 : 1 : 2, & 0 \leq \arg z \leq \pi \\
0 : 1 : 1 + z^2, & \pi \leq \arg z \leq 2\pi,
\end{cases}
\]

and

\[
L_1^1(z,t) = \begin{cases} 
z^4 & 0 \leq \arg z \leq t\pi \\
\exp(4t\pi i) & t\pi \leq \arg z \leq (2 - t)\pi \\
\end{cases}
\]

\[
(2 - t)\pi \leq \arg z \leq 2\pi,
\]

\[
L_2^2(z,t) = \begin{cases} 
2 & 0 \leq \arg z \leq \frac{t + k - 1}{k}\pi \\
1 + \exp\left(4\frac{(2 - k)t\pi - \arg z}{1 + t}\right) & \frac{t + k - 1}{k}\pi \leq \arg z \leq \frac{1 + (5 - 2k)\pi}{3 - k}\pi \\
\frac{1 + (5 - 2k)\pi}{3 - k}\pi \leq \arg z \leq 2\pi.
\end{cases}
\]

It is easy to check that \(L(-, 0) = (\alpha^{-1}*\beta^{-1})*\gamma\) and \(L(-, 1) = (\sigma*\sigma)*(\tilde{\Lambda}|_{S^1})^{-1}\). \(\square\)
Lemma 2.4. In $\pi_1(D_{1\nu}, D^0)$ the next relations hold:

\(\alpha \) \(\sigma[\alpha][\sigma]^{-1} = [\alpha] \);

\(\beta \) \(\sigma[\beta][\sigma]^{-1} = [\beta] \);

\(\gamma \) \(\sigma[\gamma][\sigma]^{-1} = [\gamma] \).

Proof. The loop $\sigma * \alpha * \sigma^{-1}$ in $D_{1\nu}$ is given by $z \mapsto (A^\sigma_1(z), B^\sigma_1(z), A^\sigma_2(z), B^\sigma_2(z), A^\sigma_3, B^\sigma_3)$, where the points $A^\sigma_k$ are given by:

- $A^\sigma_k = [-1 : k z^3 : 1]$ for $k = 1, 2, 3$.

We define two maps

\[
\varepsilon : S^1 \times I \to S^1, \quad \varepsilon(z, t) = \begin{cases} 
  z^3 & \text{if } 0 \leq \arg z \leq \frac{2 \pi t}{3} \\
  \exp(2t \pi i) z^3 & \text{if } \frac{2 \pi t}{3} \leq \arg z \leq \frac{2(3-t)}{3} \pi \\
  z & \text{if } \frac{2(3-t)}{3} \pi \leq \arg z \leq 2 \pi,
\end{cases}
\]

and a new homotopy

\[
\eta : S^1 \to \mathbb{C} \setminus \{1\}, \quad \eta(z) = \begin{cases} 
  2 & \text{if } \arg z \in [0, \frac{2 \pi}{3}] \cup [\frac{4 \pi}{3}, 2 \pi] \\
  1 + z^3 & \text{if } \arg z \in [\frac{2 \pi}{3}, \frac{4 \pi}{3}].
\end{cases}
\]

and a new homotopy

\[
K_{\alpha}(z, t) : S^1 \times I \to D_{1\nu}, \quad K_{\alpha}(z, t) = (A_1(z, t), \tilde{B}_1(z, t), A_2(z, t), B_2(z, t), A^\sigma_3, B^\sigma_3),
\]

where $A_k(z, t) = [-1 : k \varepsilon(z, t) : 1]$, $B_k(z, t) = [-1 : k \varepsilon(z, t) : 2]$, $k = 1, 2$, $\tilde{B}_1(z, t) = [-1 : \varepsilon(z, t) : \eta(z)]$. One can check that $K_{\alpha}|_{t=0} \simeq \alpha$ and $K_{\alpha}|_{t=1} = \sigma * \alpha * \sigma^{-1}$. Similarly we have a homotopy $K_{\beta}$ between $\beta$ and $K_{\beta}|_{t=1} = \sigma * \beta * \sigma^{-1}$.

Next homotopy (we also use the notation $B_3(z, t) = [0 : 1 : \eta(z)]$

\[
K_{\gamma}(z, t) : S^1 \times I \to D_{1\nu}, \quad (z, t) \mapsto (A_1(z, t), B_1(z, t), A_2(z, t), B_2(z, t), A^\sigma_3, B^\sigma_3),
\]

gives the last relation: $K_{\gamma}|_{t=0} \simeq \gamma$, $K_{\gamma}|_{t=1} = \sigma * \gamma * \sigma^{-1}$.

Proof of Theorem 1.3 (the case $n = 2$). Lemma 2.1 gives the exact sequence

\[
\ldots \to \pi_2(CP^2) \xrightarrow{\delta} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(D) \to 1
\]

where the first group is cyclic generated by the homotopy class of the map

\[
\Phi : (D^2, S^1) \to (CP^2, I^0), \quad z \mapsto [0 : r : z].
\]

We choose the lift

\[
\bar{\Phi} : (D^2, S^1) \to (D, D_{1\nu}), \quad z \mapsto (A^\Phi_1, B^\Phi_1, A^\Phi_2, B^\Phi_2, A^\Phi_3, B^\Phi_3),
\]

where $(k = 1, 2, 3)$

\[
A^\Phi_k(z) = \begin{cases} 
  -1 : (2k+1)r + k\pi : (2k+1)z + k(r - 2), & k = 1, 2 \\
  -1 : (2k+2)r + k\pi : (2k+2)z + k(r - 2), & k = 3
\end{cases},
\]

\[
B^\Phi_k(z) = \begin{cases} 
  -1 : (2k+1)r + k\pi : (2k+1)z + k(r - 2), & k = 1, 2 \\
  -1 : (2k+2)r + k\pi : (2k+2)z + k(r - 2), & k = 3
\end{cases},
\]

\[
A^\Phi_3(z) = \begin{cases} 
  -r : \pi + 4r : 4z - 3(r + 1), & k = 1, 2 \\
  -r : \pi + 5r : 5z - 3(r + 1), & k = 3
\end{cases},
\]

hence $\text{Im } \delta_1$ is generated by the homotopy class of the map

\[
\bar{\Phi} |_{S^1} : S^1 \to D_{1\nu}, \quad z \mapsto (A^\Phi_1, B^\Phi_1, A^\Phi_2, B^\Phi_2, A^\Phi_3, B^\Phi_3),
\]
with \((k = 1, 2)\)
\[
\lambda_k(z) = \begin{cases} 
 1 : k\pi : (2k + 1)z - 2k, & B_k(z) = \begin{cases} 
 1 : k\pi : (2k + 2)z - 2k, \\
 0 : \pi : 4z - 3, & B_3(z) = \begin{cases} 
 0 : \pi : 5z - 3.
\end{cases}
\end{cases}
\end{cases}
\]
The maps \(\lambda \circ \Phi_{s,1}\) and \(s^{-1}\) coincide, therefore the product \([\Phi_{s,1}] \cdot [\sigma]\) belongs to \(\ker \lambda_s = \text{Im } j_s\). We show that \([\Phi_{s,1}] \cdot [\sigma] = [\alpha] \cdot [\beta] \cdot [\gamma]\) and this implies the claim of the theorem. We define the homotopy:
\[
H : S^1 \times I \to \mathcal{D}_{\mathcal{I}}(z, t) \mapsto \left(A_i^{H(z, t)}, B_1^{H(z, t)}, A_2^{H(z, t)}, B_2^{H(z, t)}, A_3^{H(z, t)}, B_3^{H(z, t)}\right),
\]
where \((k = 1, 2)\)
\[
A_i^{H(z, t)} = \begin{cases} 
 1 : H_1^k(z, t) : H_1^k(z, t) & B_k^H(z, t) = \begin{cases} 
 1 : H_1^k(z, t) : H_2^k(z, t) + H_3^k(z, t), \\
 0 : 1 : H_3^k(z, t) + H_5^k(z, t) \end{cases}
\end{cases}
\]
and
\[
\begin{align*}
H_1^k(z, t) &= \begin{cases} 
 k \pi^2 & 0 \leq \arg z \leq t\pi \\
 k \pi^2 & t\pi \leq \arg z \leq (2-t)\pi \\
 k \pi^2 & (2-t)\pi \leq \arg z \leq 2\pi,
\end{cases} \\
H_2^k(z, t) &= \begin{cases} 
 1 + (2k + 1)t(z - 1) & 0 \leq \arg z \leq \pi \\
 1 & \pi \leq \arg z \leq 2\pi,
\end{cases} \\
H^3(z, t) &= \begin{cases} 
 1 + t(4z^4 - 3z^2 - 1) & 0 \leq \arg z \leq \pi \\
 1 & \pi \leq \arg z \leq 2\pi,
\end{cases} \\
H_1^k(z, t) &= \begin{cases} 
 \exp \left(\frac{4\arg z}{1 + t}\right) & 0 \leq \arg z \leq \frac{1-t}{4} \pi \\
 1 & \frac{1-t}{4}\pi \leq \arg z \leq 2\pi,
\end{cases} \\
H_2^k(z, t) &= \begin{cases} 
 \exp \left(\frac{2\arg z - (1-t)\pi}{1 + t}\right) & 0 \leq \arg z \leq \frac{1-t}{4}\pi \\
 1 & \frac{1-t}{4}\pi \leq \arg z \leq \pi,
\end{cases} \\
H^5(z, t) &= \begin{cases} 
 \exp[4(\arg z - (1-t)\pi)i] & 0 \leq \arg z \leq (1-t)\pi \\
 1 & (1-t)\pi \leq \arg z \leq (2-t)\pi,
\end{cases}
\]
These computations give \(\text{Im } \delta_s = \mathbb{Z}(2[\alpha] + 2[\beta] + [\sigma])\), therefore we can choose \([\alpha]\) and \([\beta]\) as generators of the fundamental group of \(\mathcal{D}\).

3. Planar Desargues configuration in \(\mathbb{C}P^n\)

First we reduce the computations of \(\pi_1(\mathcal{D}_{\mathcal{I}}^{2,n})\) and of \(\pi_1(\mathcal{D}^{2,n})\) to the case \(n = 3\).

Lemma 3.1. The following projections are locally trivial fibrations:

a) \(\mathcal{D}_{\mathcal{I}}^{2,2} \to \mathcal{D}_{\mathcal{I}}^{2,n} \to \mathbb{C}P(C\mathbb{P}^{n-1})\), \((A_1, B_1, A_2, B_2, A_3, B_3) \mapsto \text{line } (d_1, d_2, d_3)\);

b) \(\mathcal{D}^{2,2} \to \mathcal{D}^{2,n} \to \mathbb{C}P(n)\), \((A_1, B_1, A_2, B_2, A_3, B_3) \to 2\text{-plane } (d_1, d_2, d_3)\).

Proof. a) Fix a 2-plane \(P_0\) through \(I\) and choose a hyperplane \(H \subset \mathbb{C}P^n\) such that \(I \notin H\) and an \((n-3)\) dimensional subspace \(Q \subset H\) such that \(Q \cap \mathcal{I} = \emptyset\). Take as a neighborhood of \(P_0\) the set \{\(P\) a 2-plane in \(\mathbb{C}P^n\) \(\mid I \in P, P \cap Q = \emptyset\)\} and associate to a Desargues configuration in \(\mathcal{D}_I(P)\) the projection from \(Q\), an element in \(\mathcal{D}I(P)\): \(C_i = d_i \cap \mathcal{I}, \quad l = P \cap H, \quad C_i = (Q \cap C_i) \cap l, \quad Q_i = Q \cap (C_i \cap C_i), \quad d_i = IC_i, \quad A_i = Q_i A_i \cap d, \quad B_i = Q_i B_i \cap d_i \) (for \(i = 1, 2, 3\)). Using projective coordinates one can show that this trivialization is well defined on the
singular locus \( P = P_0 \): if \( I = [0 : \ldots : 0 : 1] \), \( P_0 : X_0 = \ldots = X_{n-3} = 0 \), \( A^0_0 = [0 : \ldots : a^0_{n-2, i} : a^0_{n-1, i}] \), \( B^0_1 = [0 : \ldots : b^0_{n-2, i} : b^0_{n-1, i}] \), and \( P \) is defined by the equations \( X_k = p_k,1X_{n-2} + p_k,2X_{n-1} + p_k,3X_n \) \((k = 0, \ldots, n - 3)\), then \( A_i = [p_0,0a^0_{n-2, i} + p_0,1a^0_{n-1, i} : \ldots : p_{n-3,0}a^0_{n-2, i} + p_{n-3,1}a^0_{n-1, i} : a^0_{n-2, i} : a^0_{n-1, i}] \), \( B_i = [p_0,0b^0_{n-2, i} + p_0,1b^0_{n-1, i} : \ldots : p_{n-3,0}b^0_{n-2, i} + p_{n-3,1}b^0_{n-1, i} : b^0_{n-2, i} : b^0_{n-1, i}] \).

b) Fix a 2-plane \( P_0 \) and choose as center of projection a disjoint \( n - 3 \) dimensional subspace \( Q \). Take as a neighborhood of \( P_0 \) the set of 2-planes disjoint from \( Q \). The projection from \( Q \) associate to a Desargues configuration in \( D^2(P_0) \) a Desargues configuration in \( D^2(P) \): \( P \cap (Q \setminus P_0) = I, P \cap (Q \setminus A^0_0) = A_i, d_i \cap (Q \setminus B^0_i) = B_i \). □

**Corollary 3.2.** For \( n \geq 3 \) we have

a) \( \pi_1(D^{2,3}) \cong \pi_1(D^{2,n}) \);

b) \( \pi_1(D^{2,3}) \cong \pi_1(D^{2,n}) \).

**Proof.** This is a consequence of the stability of the second homotopy group of the complex Grassmannians:

\[
\begin{align*}
\pi_2(\text{Gr}^1(\mathbb{C}P^2)) & \rightarrow \pi_2(D^{2,2}) \rightarrow \pi_1(D^{2,3}) \rightarrow 1 \\
\pi_2(\text{Gr}^1(\mathbb{C}P^{n-1})) & \rightarrow \pi_1(D^{2,2}) \rightarrow \pi_1(D^{2,n}) \rightarrow 1
\end{align*}
\]

and also

\[
\begin{align*}
\pi_2(\text{Gr}^2(\mathbb{C}P^3)) & \rightarrow \pi_1(D^{2,2}) \rightarrow \pi_1(D^{2,3}) \rightarrow 1 \\
\pi_2(\text{Gr}^2(\mathbb{C}P^n)) & \rightarrow \pi_1(D^{2,2}) \rightarrow \pi_1(D^{2,n}) \rightarrow 1
\end{align*}
\]

□

Using the fibration of Lemma 3.1 a) for \( n = 3 \) we have the exact sequence

\[
\ldots \rightarrow \pi_2(\mathbb{C}P^2) \xrightarrow{\delta} \pi_1(D^{2,2}) \rightarrow \pi_1(D^{2,3}) \rightarrow 1.
\]

We choose the base point in \( D^{2,3} \) the image of the base point in \( D_I \) through the embedding \([x_0 : x_1 : x_2] \mapsto [x_0 : x_1 : x_2 : 0]\) and we denote the compositions \( \alpha, \beta : S^1 \rightarrow D^{2,2} \rightarrow D^{2,3} \) with the same letters.

**Proposition 3.3.** In the exact sequence of the fibration \( D^{2,3} \rightarrow \mathbb{C}P^2 \) we have:

a) \( \text{Im} \delta_* = \mathbb{Z}([\alpha] + [\beta] + [\sigma]) \);

b) \( \pi_1(D^{2,3}) \cong \mathbb{Z} \oplus \mathbb{Z} \) is generated by \([\alpha]\) and \([\beta]\).

**Proof.** a) The base point in \( \text{Gr}^1(\mathbb{C}P^2) \approx \mathbb{C}P^2 \) is the line \( X_3 = 0 \) (in the dual space of lines through \( I^0 = [0 : 0 : 1 : 0] \)) and we choose the generator of \( \pi_2(\mathbb{C}P^2) \) the homotopy class of the map

\[ \Pi : (D^2, S^1) \rightarrow \text{Gr}^1(\mathbb{C}P^2), z \mapsto (1 - |z|)X_1 + zX_3 = 0. \]
The homotopy $A$ where

\[ A(z) = [2r|z| - 1 : kz : 1 : -kr], \quad A_3(z) = [0 : z : z : -r], \]

\[ B(z) = [2r|z| - 1 : kz : 2 : -kr], \quad B_3(z) = [0 : z : z + 1 : -r], \]

where the corresponding lines are

\[ d_k(z) : kX_0 + \Xi X_1 - rX_3 = 0, \quad rX_1 + zX_3 = 0, \quad d_3(z) : X_0 = 0, \quad rX_1 + zX_3 = 0. \]

The homotopy

\[ M : S^1 \times I \to D_{2,2}^2, \quad (z, t) \mapsto (A_1^{M(z,t)}, B_1^{M(z,t)}, A_2^{M(z,t)}, B_2^{M(z,t)}, A_3^{M(z,t)}, B_3^{M(z,t)}), \]

where $A_k^{M(z,t)} = [-1 : km_1(z, t) : 1], B_k^{M(z,t)} = [-1 : km_1(z, t) : 2], B_3^{M(z,t)} = [0 : 1 : 1 + m_2(z, t)]$ are defined by:

\[
m_1(z, t) = \begin{cases} 
\exp \left( \frac{2 \arg z}{2 - t} \right) & 0 \leq \arg z \leq (2 - t)\pi \\
1 & (2 - t)\pi \leq \arg z \leq 2\pi,
\end{cases} \]

\[
m_2(z, t) = \begin{cases} 
\exp \left( \frac{2 t\pi - \arg z}{2 - t} \right) & 0 \leq \arg z \leq t\pi \\
1 & t\pi \leq \arg z \leq 2\pi,
\end{cases} \]

shows that the restriction $\tilde{\Pi}|_{D^2}$ and the loop $\sigma \ast \gamma^{-1}$ are homotopic. Using this and the relation $[\gamma] = [\alpha] + [\beta] + 2[\sigma]$ we find $\delta_*([\Pi]) = [\tilde{\Pi}|_{D^2}] = [-\alpha] - [\beta] - [\sigma]$.

b) The second part is a consequence of part a). \qed

**Proposition 3.4.** The fundamental group of $D_{2,3}^2$ is isomorphic to $\mathbb{Z}$ and it is generated by $[\alpha]$ (or by $[\beta]$).

**Proof.** This is a consequence of Proposition 3.3 and the computations in section 2:

\[
\pi_2(\mathbb{CP}^2) \cong \mathbb{Z}([\Phi]) \xrightarrow{\delta^3_2} \pi_1(D_{2,2}^2) = \mathbb{Z}([\alpha], [\beta], [\sigma]) \xrightarrow{i_*} \pi_1(D_{2,2}^2) \xrightarrow{i_*} 1 \\
\pi_2(\mathbb{CP}^3) \cong \mathbb{Z}([\Phi^3]) \xrightarrow{\delta^3_2} \pi_1(D_{2,3}^2) = \mathbb{Z}([\alpha], [\beta]) \xrightarrow{i_*} \pi_1(D_{2,3}^2) \xrightarrow{i_*} 1
\]

hence $\delta_*([\Phi^3]) = i_* \delta^3_2([\Phi]) = i_*([\Phi^3|_{D^2}]) = i_*([\alpha] + 2[\beta] + [\sigma]) = [\alpha] + [\beta].$ \qed

4. **Non planar Desargues Configurations**

First we analyze the fundamental group of two three-dimensional configuration spaces $D^3_1 = D^3_{1,3}$ and $D^3_3 = D^3_{3,3}$.

**Lemma 4.1.** The following projections are locally trivial fibrations:

a) $F_2(\mathbb{C}) \times F_2(\mathbb{C}) \times F_2(\mathbb{C}) \xrightarrow{\pi_2} D^3_{1,3} \to F_{2,3}^2, \quad (A_1, B_1, A_2, B_2, A_3, B_3) \mapsto (d_1, d_2, d_3)$

b) $D^3_{1,3} \mapsto D^3 \to \mathbb{CP}^3, \quad (A_1, B_1, A_2, B_2, A_3, B_3) \mapsto I = d_1 \cap d_2 \cap d_3$.

**Proof.** The proofs are similar to those of Lemmas 2.1 and 2.2. \qed
Proof of Theorem 1.4 (the case \( n = 3 \)). We modify a little the previous notations: the base point in these solid Desargues configurations are related to the center \( I^0 = [0 : 0 : 1 : 0] \) and to the points:

\[
A_1^0 = [0 : 0 : 0 : 1], \quad B_1^0 = [0 : 0 : 1 : 1], \quad d_1^0 : X_0 = X_1 = 0,
\]

\[
A_2^0 = [0 : 1 : 0 : 0], \quad B_2^0 = [0 : 1 : 1 : 0], \quad d_2^0 : X_0 = X_3 = 0,
\]

\[
A_3^0 = [1 : 0 : 0 : 0], \quad B_3^0 = [1 : 0 : 1 : 0], \quad d_3^0 : X_1 = X_3 = 0.
\]

Using the fibrations of Lemma 4.1 we find

\[
\pi_2(\mathcal{F}_3^{2,2}) \xrightarrow{\delta_*} \pi_1(\mathcal{F}_2(\mathbb{C}^3)) \cong \mathbb{Z}^3 \to \pi_1(D_f^2) \to 1,
\]

where the first group is isomorphic with \( \pi_2(\mathcal{F}_2(\mathbb{C}P^2)) \cong \mathbb{Z}^2 = Z(\{F \}, \{B \}) \) (use the fibration \( * \simeq \mathbb{C}P^2 \setminus \mathbb{C}P^1 \to \mathcal{F}_3^{2,2} \to \mathcal{F}_2(\mathbb{C}P^2) \)); the homotopy classes \([F]\) and \([B]\) correspond to the free generators of the second homotopy groups of the fiber and of the basis respectively, in the fibration (see [3]) \( \mathbb{C}P^1 \simeq (\mathbb{C}P^2 \setminus \{\ast\}) \to \mathcal{F}_2(\mathbb{C}P^2) \to \mathbb{C}P^2 \):

\[
F : (D^2, S^1) \to (\mathcal{F}_3^{2,2}, d_2^0), \quad z \mapsto (d_1^0, d_2^0, d_3^0),
\]

where \( d_2^0(z) : zX_0 - rX_1 = 0 = X_3 \) and \( d_3^0(z) : rX_0 + \overline{\alpha}X_1 = 0 = X_3 \), and also

\[
B : (D^2, S^1) \to (\mathcal{F}_3^{2,2}, \ast), \quad z \mapsto (d_1^B(z), d_2^B(z), d_3^B(z)),
\]

where \( d_3^B(z) : zX_0 - rX_3 = 0 = X_1, d_3^B(z) : rX_0 + \overline{\alpha}X_3 = 0 = X_1 \). Choosing the lifts \( \tilde{F}, \tilde{B} : (D^2, S^1) \to (D_f^2, F_2(d_2^0)) \times F_2(d_2^0) \times F_2(d_3^0)) :\)

\[
\tilde{F}(z) = (A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^0)
\]

with

\[
A_1^0 = [r : z : 0 : 0], \quad B_2^0 = [r : z : 1 : 0],
\]

\[
A_3^0 = [\overline{\alpha} : -r : 0 : 0], \quad B_3^0 = [\overline{\alpha} : -r : 1 : 0],
\]

respectively

\[
\tilde{B}(z) = (A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^0)
\]

with

\[
A_1^0 = [r : 0 : 0 : 0], \quad B_1^0 = [r : 0 : 1 : 0]
\]

\[
A_3^0 = [\overline{\alpha} : 0 : 0 : -r], \quad B_3^0 = [\overline{\alpha} : 0 : 1 : -r],
\]

we obtain the equalities \( \delta_*([F]) = -[a] + [c], \delta_*([B]) = -[a] + [c] \). Therefore we proved that

**Corollary 4.2.** The fundamental group of the space \( D_1^3 \) is infinite cyclic generated by \([\alpha]\).

Using the second fibration of Lemma 4.1, we find the exact sequence

\[
\to \pi_2(\mathbb{C}P^3) \xrightarrow{\delta_*} \pi_1(D_f^3) \to \pi_1(D^3) \to 1
\]

where the generator \( \Psi : (D^2, S^1) \to (\mathbb{C}P^3, I^0) \), \( z \mapsto [r : 0 : z : 0] \) has the lift

\[
\tilde{\Psi} : (D^2, S^1) \to D^3, \quad z \mapsto (A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^0),
\]

with

\[
B_2^\tilde{\Psi}(z) = [r : 0 : z : 1], \quad B_3^\tilde{\Psi}(z) = [r : 1 : z : 0]
\]

\[
A_2^\tilde{\Psi}(z) = [\overline{\alpha} : 0 : -r : 0], \quad B_3^\tilde{\Psi}(z) = [r + \overline{\alpha} : 0 : z - r : 0].
\]

Therefore \( \delta_*([\tilde{\Psi}]) = [\tilde{\Psi}]^S = [\alpha] + [\beta] + 2[\gamma] = 4[\alpha] \), and we proved:
Corollary 4.3. The fundamental group of the space $D^3$ is cyclic of order four and it is generated by $[\alpha]$.

Proposition 4.4.

$$\pi_1(D^3,4) \cong \pi_1(D^3,n) \quad (n \geq 4);$$

$$\pi_1(D^3,4) \cong \pi_1(D^3,n) \quad (n \geq 4).$$

Proof. This is like in 3.2. □

Proof of Theorem 1.4 and of Theorem 1.5. We show that $\pi_1(D^3,4) = 1$; this implies that $\pi_1(D^3,4) = 1$. Choose as a generator for the fundamental group of the space of 3-planes in $CP^4$ containing the fixed point $I = [0 : 0 : 1 : 0 : 0]$ the class of the map

$$\Sigma : (D^2,S^1) \rightarrow (Gr^2(CP^3), X_4 = 0), \ z \mapsto rX_1 - zX_4 = 0.$$ 

The lift

$$\tilde{\Sigma} : (D^2,S^1) \rightarrow (D^3,4, D^3,3, D^3,0), \ z \mapsto (A_{10}^{00}, B_{10}^{00}, A_{2}^{\tilde{\Sigma}(z)}, B_{2}^{\tilde{\Sigma}(z)}, A_{3}^{00}, B_{3}^{00}),$$

where $A_{10}^{00} = [0 : 0 : 0 : 1 : 0]$, ..., $B_{3}^{00} = [1 : 0 : 1 : 0 : 0]$ are fixed points and

$$A_{2}^{\tilde{\Sigma}(z)} = [0 : z : 0 : 0 : r], \ B_{2}^{\tilde{\Sigma}(z)} = [0 : 0 : 1 : 0 : r],$$

shows that $\delta_* : \pi_2(Gr^2(CP^3)) \rightarrow \pi_1(D^3,0)$ is an isomorphism. □

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