Online Algorithms for Multi-shop Ski Rental with Machine Learned Predictions

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Abstract

We study the problem of augmenting online algorithms with machine learned (ML) predictions. In particular, we consider the multi-shop ski rental (MSSR) problem, which is a generalization of the classical ski rental problem. In MSSR, each shop has different prices for buying and renting a pair of skis, and a skier has to make decisions on when and where to buy. We obtain both deterministic and randomized online algorithms with provably improved performance when either a single or multiple ML predictions are used to make decisions. These online algorithms have no knowledge about the quality or the prediction error type of the ML predictions. The performance of these online algorithms are robust to the poor performance of the predictors, but improve with better predictions. We numerically evaluate the performance of our proposed online algorithms in practice.

1. Introduction

Uncertainty plays a critical role in many real world applications where the decision maker is faced with multiple alternatives with different costs. These decisions arise in our daily lives, such as whether to rent an apartment or buy a house, which cannot be answered reliably without knowledge of the future. In a more general setting with multiple alternatives, such as a large number of files with different execution time in a distributed computing system, it is hard to decide which file should be executed next without knowing which file will arrive in the future. These decision-making problems are usually modeled as online rent-or-buy problems, such as the classical ski rental problem (Karlin, Manasse, McGeoch, & Owicki, 1994; Lotker, Patt-Shamir, & Rawitz, 2008; Khanafer, Kodialam, & Puttaswamy, 2013).

Two paradigms have been widely studied to deal with such uncertainty. On the one hand, online algorithms are designed without prior knowledge to the problem, and competitive ratio (CR) is used to characterize the goodness of the algorithm in lack of the future. On the other hand, machine learning is applied to address uncertainty by making future predictions via building robust models on prior data. Recently, there is a popular trend in the design of online algorithms by incorporating machine learned (ML) predictions to improving their performance (Medina & Vassilvitskii, 2017; Lykouris & Vassilvitskii, 2018; Purohit, Svitkina, & Kumar, 2018; Mitzenmacher, 2018; Gollapudi & Panigrahi, 2019; Kodialam, 2019; Angelopoulos, Dürr, Jin, Kamali, & Renault, 2019; Boyar,
Favrholdt, Kudahl, Larsen, & Mikkelsen, 2016; Lee, Hajiesmaili, & Li, 2019). Two properties are desired in online algorithm design with ML predictions: (i) if the predictor is good, the online algorithm should perform close to the best offline algorithm (a design goal called consistency); and (ii) if the predictor is bad, the online algorithm should not degrade significantly, i.e., its performance should be close to the online algorithm without predictions (a design goal called robustness). Importantly, these properties are achieved under the assumption that the online algorithm has no knowledge about the quality of the predictor or the prediction error types.

While previous studies focused on using ML predictions for a single skier to buy or rent the skis in a single shop, we study the more general setting where the skier has multiple shops to buy or rent the skis with different buying and renting prices. We call this a multi-shop ski rental (MSSR) problem. This is often the case in practice, where the skier not only needs to decide when to buy, but also where to buy, whereas only decision on when to buy is needed in the classical single shop ski rental problem. Furthermore, we consider not only the case of using a single ML prediction, which is inspired by recent work (Lykouris & Vassilvitskii, 2018; Purohit et al., 2018; Kodialam, 2019; Angelopoulos et al., 2019), but also the case of getting predictions from multiple ML models. Closest to ours is the work by (Gollapudi & Panigrahi, 2019), which considered the case where multiple experts provide advice in a single shop, which can be considered as a special case of our problem. However, we incorporate multiple predictions into decision making by comparing the number of predictions to a threshold, which is much easier to implement in real world systems.

1.1 The ski rental problem

In ski rental problem, the skier is going to ski for an unknown number of days, and has to make a decision on either renting skis with a unit cost each day or buying skis at a higher price $b$. It is easy to see that the skier should buy skis on the first day if she is going to ski more than $b$ days, otherwise, rent every day. The number of days is unknown in advance, and only revealed by the end of skiing season. It is well-known that the best deterministic algorithm is to rent for the first $b-1$ days and then buy on day $b$, which achieves a competitive ratio of 2. On the other hand, the best randomized algorithm (Karlin et al., 1994) achieves a competitive ratio of $e/(e-1)$. The ski-rental problem and many of its generalizations such as dynamic TPC acknowledgement (Karlin, Kenyon, & Randall, 2001), the parking permit problem (Meyerson, 2005), snoopy caching (Karlin, Manasse, Rudolph, & Sleator, 1988), renting cloud servers (Khanafer et al., 2013) and others are canonical examples of online rent-or-buy problems, which play a central role in decision making in many different settings, and have continuously been extensively studied in different domains.

1.2 The multi-shop ski rental problem

We consider the multi-shop ski rental problem, in which the skier has multiple shops to buy or rent the skis with different buying and renting prices. In such a MSSR, the skier has to make a two-fold decision, i.e., when and where to buy. Specifically, we consider the case that the skier must choose one shop at the beginning of the skiing season, and must buy or rent the skis at that particular shop since then. In other words, once a shop is chosen by the skier, the only decision variable is when she should buy the skis. The MSSR not only naturally extends the classical ski rental problem, where a single skier rents or buys the skis in a single shop, but also allows heterogeneity in skier’s options. This desirable feature makes the ski rental problem a more general modeling framework for online algorithm design. Here we give a few real world applications that can be modeled with MSSR.
Table 1: Price option for Microsoft Azure basic service.

| Options            | Hourly price ($) |
|--------------------|------------------|
| Pay-as-you-go      | 0.0075           |
| 1 year reserved    | 0.0059           |
| 3 year reserved    | 0.0038           |

Example 1: Cost in Cloud CDN Service. With the advent of cloud computing, the content service provided by content distribution network (CDN) has been offered as managed platforms with a novel pay-as-you-go model for cloud CDNs. For example, cloud providers such as Microsoft Azure and Amazon AWS, now provide different price options to users based on their demand, which is usually unknown in advance. Table 1 lists the price option provided by Microsoft Azure. Each price option can be considered as a shop in the MSSR problem, and the hourly price is the renting price.

Example 2: Caching in Wireless Sensor Networks. A content can be replicated and stored in multiple base stations to serve requests from users. Upon a user request, if the requested content is stored in base stations, the service latency is short, otherwise, it incurs a longer latency to fetch the requested content from remote servers. On the other hand, the content can be prefetched and stored in base stations at the expense of wasting space if the content will not be requested by users. In this application, each base station is considered as a shop, and renting corresponds to serve requests on-demand, and buying refers to prefetch content in advance.

1.3 Consistency and Robustness

The competitive ratio of an online algorithm is defined as the worst-case ratio of the algorithm cost (ALG) to that of the offline optimum (OPT). Inspired by (Lykouris & Vassilvitskii, 2018; Purohit et al., 2018; Gollapudi & Panigrahi, 2019; Angelopoulos et al., 2019), we also use the notions of consistency and robustness to evaluate our algorithms. We denote the prediction error as ζ, which is the absolute difference between the prediction and the actual outcome. We say that an online algorithm is α-consistent if ALG ≤ α · OPT when the prediction is accurate, i.e., ζ = 0, and β-robust if ALG ≤ β · OPT for all ζ and feasible outcomes to the problem. We call α and β the consistency factor and robustness factor, respectively. Thus consistency characterizes how well the algorithm does in case of perfect predictions, and robustness characterizes how well it does in worst-case predictions.

This novel analytical framework can bring the gap between two radically online algorithm design methodologies. On the one hand, the worst-case analysis framework always assumes that the future is unpredictable, and try to design online algorithms with a bounded competitive ratio. On the other hand, historical data are usually used to make predictions for decision making in real-world systems. However, this approach results in poor performance if the future inputs look different to the past ones. In this framework, a hyperparameter λ ∈ (0, 1) is leveraged to determine the trust on ML predictions, where λ = 0 indicates fully trust on ML predictions and λ = 1 indicates no trust on ML predictions. To that end, such online algorithm design with ML predictions can provide a full spectrum coverage from pure worst-case to fully prediction-based decision making. In this paper, our goal is to design online algorithms for MSSR that improve consistency factor without degrading robustness factor significantly, compared to algorithms for MSSR without predictions.
1.4 Main Results

Our main contribution is to develop online algorithms for MSSR with consistency and robustness properties in presence of ML predictions. We develop new analysis techniques for online algorithms with ML predictions via the hyperparameter. We first define a few notions before presenting our main results. We assume there are \( n \) shops with buying prices \( b_1 > \cdots > b_n \) and renting prices \( r_1 < \cdots < r_n \). We develop several online algorithms for MSSR with a single ML prediction or multiple predictions as highlighted below:

- We first present a best deterministic algorithm (achieving minimal competitive ratio) for MSSR without ML predictions. It turns out that the algorithm chooses exactly one shop \( i \) with the minimal value of \( r_i + (b_i - r_i)/b_n \), and buy on the start day \( b_n \) at shop \( i \).
- Next, we consider MSSR with a single ML prediction. We show that if this ML prediction is naively used in algorithm design, the proposed algorithm cannot ensure robustness (Section 3.1). We then incorporate ML prediction in a judicious manner by first proposing a deterministic online algorithm that is \( ((\lambda + 1)r_n + b_1/b_n)\)-consistent, and \( (\max\{r_n, b_1/b_n\} + 1/\lambda)\)-robust (Section 3.2). We further propose a randomized algorithm with consistency and robustness guarantees (Section 3.3). We numerically evaluate the performance of our online algorithms (Section 3.4). We show that with a natural prediction error model, our algorithms are practical, and achieve better performance than the ones without ML predictions. We also investigate impacts of several parameters and provide insights on the benefits of using ML predictions. It turns out that the predictions need to be carefully incorporated in online algorithm design.
- We then study a more general setting where we get \( m \) ML predictions from some ML models. We redefine the prediction error to incorporate the average ML predictions’ impact into our algorithms. We slightly modify the algorithms and show that similar techniques lead to tight results for online algorithms of MSSR with multiple ML predictions. In particular, we propose both deterministic algorithm (Section 4.1) and randomized algorithm (Section 4.2) with consistency and robustness guarantees. Finally, numerical results are given to demonstrate the impact of multiple ML predictions.

1.5 Related Work

Our work is inspired by the aforementioned recent trend of incorporating ML predictions into online algorithms design (Medina & Vassilvitskii, 2017; Lykouris & Vassilvitskii, 2018; Purohit et al., 2018; Mitzenmacher, 2018; Gollapudi & Panigrahi, 2019; Kodialam, 2019; Angelopoulos et al., 2019; Boyar et al., 2016; Lee et al., 2019). In particular, we use the concepts of consistency and robustness from (Lykouris & Vassilvitskii, 2018). For example, (Lykouris & Vassilvitskii, 2018) incorporates ML predictions into the classical Marker algorithm ensuring both robustness and consistency for caching. (Purohit et al., 2018) and (Gollapudi & Panigrahi, 2019) extend the models for a comprehensive understanding of the classical single shop ski rental problem with a single bit advice and multiple bits advice, respectively. (Angelopoulos et al., 2019) further quantifies the impact of advice quality and proposed Pareto-optimal algorithm for ski rental problem. While we operate in the same framework, none of previous results can be directly applied to our setting, as our work significantly differs from previous studies in the sense that we consider a multi-shop ski rental problem with multiple ML predictions, where the skier has to make a two-fold decision on when and where to buy. This makes the problem considerably more challenging but more practical.
Closest to our model is that multiple options in one shop (Lotker et al., 2008) or multiple shops (Ai, Wu, Huang, Huang, Tang, & Li, 2014), however, no ML prediction is incorporated in their online algorithms design. On the other hand, there is an extensive study for online optimization with advice model, in particular, multiple predictions has been studied in the context of online learning, however, existing techniques are not applicable to our multi-shop setting. We refer interested readers to the surveys (Boyar et al., 2016; Masoudnia & Ebrahimpour, 2014) for a comprehensive discussion.

2. Preliminaries

We consider the multi-shop ski rental (MSSR) problem, where a skier goes to ski for an unknown number of days. The skier can buy or rent skis from multiple shops with different buying and renting prices. The skier must choose one shop as soon as she starts the skiing.

More precisely, we assume that there are totally \( n \) shops and denote the set of shops as \( \mathcal{N} = \{1, \cdots, n\} \). Each shop \( i \) offers a renting price of \( r_i \) dollars per day, and a buying price \( b_i \) dollars, where \( r_i, b_i > 0, \forall i \in \mathcal{N} \). In particular, our model reduces to the classical ski rental problem when \( n = 1 \). In a MSSR problem, it is obvious that if one shop has higher prices for both renting and buying than another shop, it is suboptimal to choose this shop. To that end, we assume \( r_1 < \cdots < r_n \) and \( b_1 > \cdots > b_n \). For the ease of exposition, we set \( r_1 = 1 \), which is the same with that used in classical ski rental problem. Let \( x \) be the actual number of skiing days which is unknown to the algorithm.

We first consider the offline optimal algorithm where \( x \) is known. It is easy to see that the skier should rent at shop 1 if \( x \leq b_n \) and buy on day 1 at shop \( n \) if \( x > b_n \).

2.1 Best Deterministic Online Algorithm for MSSR

It is well-known that the best deterministic algorithm for the classical ski rental problem is the break-even algorithm: rent until day \( b - 1 \) and buy on day \( b \). The corresponding competitive ratio is 2 and no other deterministic algorithm can do better. Now we consider the best deterministic online algorithm (BDOA) that obtains a minimal competitive ratio for MSSR without any ML prediction. We make the following assumption.

**Assumption 1.** The skier cannot change the shop once she chooses it, but she can decide to buy or continue to rent the skis in that particular shop at any time.

**Lemma 1.** The best deterministic algorithm for MSSR is that the skier rents for the first \( b_n - 1 \) days and buys on day \( b_n \) at shop \( i \), where \( i = \arg\min (r_i + \frac{b_i - r_i}{b_n}) \). The corresponding competitive ratio is \( r_i + (b_i - r_i)/b_n \).

*Proof sketch of Lemma 1:* It is obvious that \( \text{OPT} = \min\{x, b_n\} \). Under Assumption 1, we can consider the competitive ratio of shop \( \forall i \in \mathcal{N} \). Let \( d_i \) be the buying day. Then \( \text{ALG}_i = xr_i \) if \( x < d_i \), otherwise \( \text{ALG}_i = (d_i - 1)r_i + b_i \). It is easy to argue that the worst case happens when \( x = d_i \). We have \( \text{CR}_i = \text{ALG}_i / \text{OPT} \). Inspired by the classical ski rental problem, we can show that the best \( \text{CR}_i = r_i + (b_i - r_i)/b_n \) is achieved when \( d_i = b_n \). Thus, we have \( \text{CR} = \min_i \text{CR}_i \). The proof details are available in Appendix A.
3. Online Algorithms for MSSR with a single ML prediction

In this section, we consider MSSR with a single ML prediction. Let $y$ be the predicted number of skiing days. Then $\zeta = |y - x|$ is the prediction error. For the ease of exposition, we use the two-shop ski rental problem as a motivating example, and then generalize the results to the general MSSR with $n$ shops.

3.1 A Simple Algorithm with ML prediction

**Algorithm 1** A simple learning-aided algorithm

if $y \geq b_2$ then
  Buy on day 1 at shop 2
else
  Rent at shop 1

**Lemma 2.** The cost of Algorithm 1 satisfies $\text{ALG} \leq \text{OPT} + \zeta$.

**Proof.** Since there is only one break-even point $b_2$, we consider four cases based on the relations of $x$ and $y$ with $b_2$.

(i) $y \geq b_2, x \geq b_2$: $\text{ALG} = b_2, \text{OPT} = b_2$, i.e., $\text{CR} = 1$;
(ii) $y \geq b_2, x < b_2$: $\text{ALG} = b_2, \text{OPT} = x$, i.e., $\text{CR} = b_2/x$;
(iii) $y < b_2, x \geq b_2$: $\text{ALG} = x, \text{OPT} = b_2$, i.e., $\text{CR} = x/b_2$;
(iv) $y < b_2, x < b_2$: $\text{ALG} = x, \text{OPT} = x$, i.e., $\text{CR} = 1$.

Combining (i)-(iv), $\text{CR} = \max\{b_2/x, x/b_2\}$, which is unbounded.

Furthermore, we can rewrite (ii), $\text{ALG} = b_2 = x + b_2 - x \leq \text{OPT} + y - x = \text{OPT} + \zeta$.

Similarly, by rewriting (iv), we also have $\text{ALG} = x = b_2 + x - b_2 < \text{OPT} + x - y = \text{OPT} + \zeta$. 

We now generalize Algorithm 1 and Lemma 2 to the general MSSR with $n$ shops. Inspired by Lemma 1, it is easy to check that it is suboptimal to buy at shop $i$ with $b_i \geq b_n$, and rent at shop $j$ with $r_j > r_1$.

**Corollary 1.** The simple algorithm with ML prediction for the general MSSR with $n$ shops follows that the skier buys on day 1 at shop $n$ if $y \geq b_n$, otherwise it rents at shop 1. The corresponding cost satisfies $\text{ALG} \leq \text{OPT} + \zeta$.

We note that by simply following the ML prediction, the competitive ratio of Algorithm 1 is unbounded (e.g., $x \gg b_2$) even when the prediction $y$ is small (due to case (iii)). Furthermore, Algorithm 1 has no robustness guarantee. In the following, we show how to properly integrate the ML prediction into online algorithm design to achieve both consistency and robustness.

3.2 A Deterministic Algorithm with Consistency and Robustness Guarantee

We develop a new deterministic algorithm by introducing a hyperparameter $\lambda \in (0, 1)$, which gives us a smooth tradeoff between the consistency and robustness of the algorithm.

**Theorem 1.** The competitive ratio of Algorithm 2 is at most $\min\{\langle \lambda + 1 \rangle r_2 + b_1, b_2 + \max\{\lambda r_2 + 1, b_1 b_2^\lambda, \frac{1}{1-\lambda} \text{OPT}, \max\{r_2 + \frac{1}{\lambda}, b_1 b_2^\lambda (1 + \frac{1}{\lambda})\}\}, where $\lambda \in (0, 1)$ is a parameter. In particular, Algorithm 2 is $((\lambda + 1) r_2 + b_1 b_2^\lambda)$-consistent and $(\max\{r_2, \frac{b_1}{b_2^\lambda}\} + \frac{1}{\lambda})$-robust.
Algorithm 2 A deterministic algorithm with consistency and robustness guarantee

if $y \geq b_2$ then
  Rent until day $\lceil \lambda b_2 \rceil - 1$ at shop 2, then buy on day $\lceil \lambda b_2 \rceil$ at shop 2
else
  Rent until day $\left\lfloor \frac{b_1}{\lambda} \right\rfloor - 1$ at shop 1, then buy on day $\left\lfloor \frac{b_1}{\lambda} \right\rfloor$ at shop 1

Proof. We first prove the first bound. When $y \geq b_2$, we consider two cases.

First, if $x < \lfloor \lambda b_2 \rfloor$, then $\text{OPT} = x$, i.e., rent at shop 1 since $r_1 = 1 < r_2$. Hence we have

$$\text{ALG} = r_2 x = r_2 \text{OPT},$$
i.e., $\text{CR}_1 = r_2$.

Second, if $x \geq \lfloor \lambda b_2 \rfloor$, we have

$$\text{ALG} = r_2 (\lfloor \lambda b_2 \rfloor - 1) + b_2 \leq (\lambda r_2 + 1) b_2.$$

When $x \geq b_2$, we have $\text{OPT} = b_2$, i.e., buy at shop 2 on day 1 as $b_2 < b_1$, then $\text{ALG} \leq (\lambda r_2 + 1) b_2 \leq (\lambda r_2 + 1) (\text{OPT} + \zeta)$. When $\lfloor \lambda b_2 \rfloor \leq x < b_2$, we have $\text{OPT} = x$, then $b_2 \leq y = x + \zeta = \text{OPT} + \zeta$, thus, $\text{ALG} \leq (\lambda r_2 + 1) b_2 \leq (\lambda r_2 + 1) (\text{OPT} + \zeta)$. Combining these two cases, we have $\text{CR}_2 \leq (\lambda r_2 + 1) (1 + \frac{\zeta}{\text{OPT}})$.

Similarly, when $y < b_2$, we consider the following three cases.

First, if $x < b_2$, we have $\text{ALG} = x$. It is clear that $\text{OPT} = x$, i.e., $\text{CR} = 1$.

Second, if $x \in \left( b_2, \left\lfloor \frac{b_1}{\lambda} \right\rfloor \right)$, we have $\text{OPT} = b_2$, i.e., buy at shop 2 on day 1, and

$$\text{ALG} \stackrel{(a)}{=} x \stackrel{(b)}{=} y + \zeta < \text{OPT} + \zeta,$$

where (a) is obtained by following Algorithm 2, i.e., rent at shop 1 with $r_1 = 1$, and (b) holds true due to the predictor error definition. Therefore, we have $\text{CR}_3 < 1 + \frac{\zeta}{\text{OPT}}$.

Finally, if $x \geq \left\lfloor \frac{b_1}{\lambda} \right\rfloor$, we have $\text{OPT} = b_2$, and

$$\text{ALG} = \left\lfloor \frac{b_1}{\lambda} \right\rfloor - 1 + b_1 \leq \frac{b_1}{\lambda} + b_1 \leq \frac{1}{2} \left( \frac{1}{\lambda - \frac{\lambda}{1-\lambda}} \right),$$

where (c) follows $\zeta = x - y > \frac{b_2}{\lambda} - b_2$, i.e., $b_2 < \frac{\lambda}{1-\lambda} \zeta$, then $b_1 < \frac{b_1}{b_2} \left( 1 + \frac{1}{\lambda - \frac{\lambda}{1-\lambda}} \right)$.

Combining $\text{CR}_1, \text{CR}_2, \text{CR}_3$ and $\text{CR}_4$, we get the first bound.

Now we prove the second bound. According to Algorithm 2, the skier rents the skis at shop 2 until day $\lceil \lambda b_2 \rceil - 1$ and then buys on day $\lfloor \lambda b_2 \rfloor$ at shop 2, when the predicted day satisfies $y \geq b_2$, we have

$$\text{ALG} = r_2 (\lceil \lambda b_2 \rceil - 1) + b_2,$$
if $x \geq \lfloor \lambda b_2 \rfloor$. It is easy to see that the worst CR is obtained when $x = \lfloor \lambda b_2 \rfloor$, for which $\text{OPT} = \lfloor \lambda b_2 \rfloor$. Therefore,

$$\text{ALG} \leq (\lambda r_2 + 1) b_2 \leq \frac{\lambda r_2 + 1}{\lambda} \lfloor \lambda b_2 \rfloor = \left( r_2 + \frac{1}{\lambda} \right) \text{OPT}. $$
Similarly, the skier rents the skis at shop 1 until day \( \lceil \frac{b_1}{\lambda} \rceil - 1 \) and then buys on day \( \lceil \frac{b_1}{\lambda} \rceil \) at shop 1, when \( y < b_2 \), the worst CR is obtained when \( x = \lceil \frac{b_1}{\lambda} \rceil \), for which \( \text{OPT} = b_2 \), and

\[
\text{ALG} = \lceil \frac{b_1}{\lambda} \rceil - 1 + b_1 \leq \frac{b_1}{\lambda} + b_1 = \frac{b_1}{b_2} \left( 1 + \frac{1}{\lambda} \right) \text{OPT}.
\]

Similarly, we can generalize the above results to the general MSSR with \( n \) shops.

**Corollary 2.** The deterministic algorithm with a single-bit ML prediction for the general MSSR with \( n \) shops follows that the skier buys on day \( \lceil \lambda b_n \rceil \) at shop \( n \) if \( y \geq b_n \), otherwise it buys on day \( \lceil \frac{b_1}{\lambda} \rceil \) at shop 1. The corresponding competitive ratio is at most \( \min(\lambda + 1) \text{OPT} + b_1/\text{OPT} \). In particular, the deterministic algorithm is \((\lambda + 1)\text{OPT} + b_1/\text{OPT}\)-consistent and \( \max\{r_n + b_1/(\lambda \text{OPT})\} \)-robust.

**Remark 1.** The competitive ratio is a function of hyperparameter \( \lambda \) and prediction error \( \zeta \), which is different from the conventional competitive design. By tuning the \( \lambda \) value, one can achieve different values for competitive ratio. The competitive ratio might be even worse than the BDOA for some cases (e.g., prediction error is large). We will show this in Section 3.4. This shows that decision making based on ML predictions comes at the cost of lower worst-case performance guarantee. Finally, it is possible to find the optimal \( \lambda \) to minimize the worst-case competitive ratio if the prediction error \( \zeta \) is known (e.g., from historically observed error values).

### 3.3 A Randomized Algorithm with Consistency and Robustness Guarantee

We consider a class of randomized algorithms for MSSR in this section. Similarly, we consider a hyperparameter \( \lambda \) satisfying \( \lambda \in (1/b_2, 1) \). First, we emphasize that a randomized algorithm that naively modifies the distribution used for randomized algorithm design for the classical ski rental algorithm with or without predictions fail to achieve a better consistency and robustness at the same time. We customize the distribution functions carefully by incorporating different renting and buying prices from different shops into the distributions, as summarized in Algorithm 3.

**Algorithm 3** A randomized algorithm with consistency and robustness guarantee

if \( y \geq b_2 \) then

Let \( k = \lfloor \lambda b_2 \rfloor \)

Define \( p_i = \left( \frac{b_2 - r_2}{b_2} \right)^{k-i} \cdot \frac{r_2}{b_2(1-(1-\frac{r_2}{b_2})^k)} \), for \( i = 1, \ldots, k \)

Choose \( j \in \{ 1, 2, \ldots, k \} \) randomly from the distribution defined by \( p_i \)

Rent till day \( j - 1 \) at shop 2, then buy on day \( j \) at shop 2

else

Let \( l = \lceil \frac{b_1}{\lambda} \rceil \)

Define \( q_i = \left( \frac{b_1 - 1}{b_1} \right)^{l-i} \cdot \frac{1}{b_1(1-(1-\frac{1}{b_1})^l)} \), for \( i = 1, \ldots, l \)

Choose \( j \in \{ 1, 2, \ldots, l \} \) randomly from the distribution defined by \( q_i \)

Rent till day \( j - 1 \) at shop 1, then buy on day \( j \) at shop 1
Theorem 2. The competitive ratio of Algorithm 3 is at most \( \min \left\{ \frac{r_2\lambda}{1-e^{-r_2}} (1 + \frac{\zeta}{\text{OPT}}) \right\} \).

Proof. We compute the competitive ratio of Algorithm 3 under four cases.

**Case 1.** \( y \geq b_2 \) and \( x \geq k \). It is clear that \( \text{OPT} = \min\{b_2, x\} \). According to Algorithm 3, the skier should rent at shop 2 until day \( j - 1 \) and buy on day \( j \). This happens with probability \( p_i \), for \( i = 1, \cdots, k \), and incurs a cost \( (b_2 + (i - 1)r_2) \). Therefore, we have

\[
\mathbb{E}[\text{ALG}] = \sum_{i=1}^{k} (b_2 + (i - 1)r_2)p_i = \sum_{i=1}^{k} (b_2 + (i - 1)r_2) \left( \frac{b_2 - r_2}{b_2} \right)^{k-i} \cdot \frac{r_2}{b_2} \left( 1 - (1 - \frac{r_2}{b_2})^k \right)
\]

where (a) holds since \((1 + x)^k \leq e^{kx}\), for \( 0 \leq x < 1 \), and (b) follows that \( k = \lfloor \lambda b_2 \rfloor \leq \lambda b_2 \), i.e., \( k/b_2 \leq \lambda \) and \( \frac{x}{1-e^{-x}} \) increases in \( x \geq 0 \).

**Case 2.** \( y \geq b_2 \) and \( x < k \). Since \( x < k = \lfloor \lambda b_2 \rfloor < b_2 \), we have \( \text{OPT} = x \). If the skier buys the skis on day \( i \leq x \), then it incurs a cost \( (b_2 + (i - 1)r_2) \), otherwise, the cost is \( xr_2 \). Therefore, we obtain the robustness through the following

\[
\mathbb{E}[\text{ALG}] = \sum_{i=1}^{x} (b_2 + (i - 1)r_2)p_i + \sum_{i=x+1}^{k} xr_2p_i
\]

\[
= \frac{r_2}{b_2} \left( 1 - (1 - \frac{r_2}{b_2})^k \right) \left[ \sum_{i=1}^{x} (b_2 + (i - 1)r_2) \left( \frac{b_2 - r_2}{b_2} \right)^{k-i} + \sum_{i=x+1}^{k} xr_2 \left( \frac{b_2 - r_2}{b_2} \right)^{k-i} \right]
\]

where (c) holds true since \( \lambda b_2 - 1 \leq k = \lfloor \lambda b_2 \rfloor < b_2 \), i.e., \( k/b_2 \geq \lambda - 1/b_2 \), and \( b_1/b_2 > 1 \). To get the consistency, we can rewrite the above inequality

\[
\mathbb{E}[\text{ALG}] \leq \frac{r_2}{1-e^{-r_2/b_2}} \text{OPT} \quad \leq \frac{r_2 \cdot k/b_2}{1-e^{-r_2/k/b_2}} \text{OPT} + \frac{r_2 \cdot (b_2 - k)/b_2}{1-e^{-r_2/b_2}} x
\]

Therefore, we have

\[
\mathbb{E}[\text{ALG}] = \sum_{i=1}^{x} (b_1 + (i - 1) \cdot 1)p_i + \sum_{i=x+1}^{l} x \cdot 1 \cdot p_i = \frac{x}{1 - (1 - 1/b_1)^l}
\]

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\[ \frac{x}{1 - e^{-l/b_1}} \leq \frac{x}{1 - e^{-1/\lambda}} \leq \frac{\lambda}{1 - e^{-1/\lambda}} (\text{OPT} + \zeta) \leq \frac{r_2 \lambda}{1 - e^{-r_2 \lambda}} (\text{OPT} + \zeta), \]

where (g) follows that \( l = \lceil b_1 / \lambda \rceil \geq b_1 / \lambda \), i.e., \( 1 / \lambda \leq l / b_1 \), (h) follows from two cases i) when \( x < b_2 \), we have \( \text{OPT} = x \geq x - \zeta \); and ii) when \( x \geq b_2 \), we have \( x < x + b_2 - y = b_2 + \zeta \) as \( y < b_2 \), thus \( b_2 > x - \zeta \). Hence, \( \text{OPT} = b_2 \geq x - \zeta \). (i) holds since \( r_2 > 1 \) and \( \frac{x}{1 - e^{-x}} \) increases in \( x \geq 0 \) as mentioned earlier.

**Case 4.** \( y < b_2 \) and \( x \geq l \). As \( x \geq l \), we have \( \text{OPT} = b_2 \). Similar to Case 1, we have the robustness as

\[
\mathbb{E}[\text{ALG}] = \sum_{i=1}^{l} (b_1 + (i - 1) \cdot 1)p_i = \frac{l}{1 - (1 - 1/b_1)^l} \leq \frac{l}{1 - e^{-l/b_1}} \\
= \frac{[b_1 / \lambda]}{1 - e^{-l/b_1}} \leq \frac{b_2 \cdot \frac{b_1}{b_2} (\frac{1}{\lambda} + \frac{1}{r_2 \lambda})}{1 - e^{-1/\lambda}} = \frac{b_2 \cdot \frac{b_1}{b_2} (\frac{1}{\lambda} + \frac{1}{r_2 \lambda})}{1 - e^{-1/\lambda}} \text{OPT},
\]

where (j) follows that \( \lceil b_1 / \lambda \rceil \leq b_1 / \lambda + 1 = b_1 (1 / \lambda + 1 / b_1) \), and \( l = \lceil b_1 / \lambda \rceil \geq b_1 / \lambda \), i.e., \( l / b_1 \geq 1 / \lambda \). Again, we rewrite the above inequality to get the consistency

\[
\mathbb{E}[\text{ALG}] \leq \frac{l}{1 - e^{-l/b_1}} \leq \frac{l}{1 - e^{-1/\lambda}} \leq \frac{b_2 + l - b_2}{1 - e^{-1/\lambda}} \leq \frac{1}{1 - e^{-1/\lambda}} (\text{OPT} + \zeta) \leq \frac{r_2 \lambda}{1 - e^{-r_2 \lambda}} (\text{OPT} + \zeta),
\]

where (k) follows that \( \text{OPT} = b_2 \) and \( \zeta = x - y > l - b_2 \).

Again, we can generalize Algorithm 3 to the general MSSR problem with \( n \) shops. As it is suboptimal to rent at any shop besides shop 1 and buy at any shop besides shop \( n \). The randomized algorithm for the general MSSR simply replaces shop 2 by shop \( n \) with the corresponding \( b_n \) and \( r_n \) in Algorithm 3. Similarly, the corresponding competitive ratio can be achieved by replacing \( b_2 \) and \( r_2 \) in Theorem 2 by \( b_n \) and \( r_n \) of shop \( n \). The hyperparameter should satisfy \( \lambda \in (1/b_n, 1) \).

### 3.4 Model Validation and Insights

In this section, we numerically evaluate the performance of our algorithms. For all our experiments, we set the number of shops \( n = 6 \), the buying costs are 100, 95, 90, 85, 80, 75 dollars with \( b_1 = 100 \) and \( b_6 = 75 \), and the renting costs 1, 1.05, 1.10, 1.15, 1.20, 1.25 dollars with \( r_1 = 1 \) and \( r_6 = 1.25 \). Note that the actual values of \( b_i \) and \( r_i \) are not important as we can scale all these values by some constant factors. The actual number of skiing days \( x \) is a random variable uniformly drawn from \([1, \Gamma]\), where \( \Gamma < \infty \) is a constant. The predicted number of skiing days \( y \) is set to \( x + \epsilon \) where \( \epsilon \) is drawn from a normal distribution with mean \( \delta \) and standard variation \( \sigma \). We vary either the value of \( \sigma \) from 0 to \( \Gamma \), or the value of \( \delta \) to verify the consistency and robustness of our algorithms. To characterize the impact of the hyperparameter \( \lambda \) on the performance of deterministic and randomized algorithms, we consider the values of 0.25, 0.5, 0.75 and 1 for \( \lambda \). Note \( \lambda = 1 \) means that our algorithms ignore the ML prediction, and reduce to the algorithms without predictions. For each value of \( \sigma \), we plot the average competitive ratio by running the corresponding algorithm over 10,000 independent trials. We consider both unbiased and biased prediction errors in our experiments.
We first consider unbiased prediction errors, i.e., $\delta = 0$, to characterize the impact of $\Gamma$ and $\lambda$.

**The impact of $\Gamma$.** As $x$ is uniformly drawn from $[1, \Gamma]$, $\Gamma$ is an important parameter that can impact the competitive ratio. We consider two possible values of $\Gamma$: $\Gamma = 3b_1$ and $\Gamma = b_1$. Since $b_6 = 75$, $\Gamma = 3b_1$ means that it is highly possible the actual number of skiing days $x$ is larger than $b_6$. Thus according to Algorithm 2, buying as early as possible will be a better choice, i.e., small $\lambda$ results in better competitive ratio as shown in Figure 1a. On the other hand, with $\Gamma = b_1$, it is highly possible that $x$ is smaller than $b_6$. Therefore, if the prediction is more accurate (small $\sigma$), smaller $\lambda$ (i.e., more trust on ML predictions) achieves smaller competitive ratio, while the prediction is inaccurate (with large $\sigma$), larger $\lambda$ achieves smaller competitive ratio. This can be observed from Figure 1b. In particular, with the values of $b$’s and $r$’s in our setting, $\lambda = 1$, i.e., do not trust the prediction achieves the best competitive ratio when the prediction error is large. We can observe a similar trend for the randomized algorithm (Algorithm 3) as shown in Figure 2.
We further compare the performance of the deterministic algorithm (Algorithm 2) and the randomized algorithms (Algorithm 3), as shown in Figure 3 with $\Gamma = 3b_1$. We make the following observations: (i) with the same prediction errors (e.g., $\lambda = 0.5$), the randomized algorithm always performs better than the deterministic algorithm. Similar trends are observed for other $\lambda$ values and hence are omitted due to space constraints. (ii) our deterministic algorithm with ML prediction can beat the performance of classical randomized algorithm without ML predictions when the standard deviation of prediction error is smaller than $2.5b_1 = 250$.

**The impact of hyperparameter $\lambda$.** Hyperparameter $\lambda$ incorporates the trust of ML predictions in online algorithm design. In particular, $\lambda$ close to 0 means more trust on predictions while $\lambda$ close to 1 means less trust. We investigate the impact of $\lambda$ on the deterministic algorithm (Algorithm 2) by considering a perfect prediction and an extremely erroneous prediction. From Figure 4 with $\Gamma = 3b_1$, we observe (i) With an extremely erroneous prediction, blinding trust the prediction (smaller $\lambda$) leads to even worse performance than the BDOA without ML predictions; (ii) By prop-
erly choosing $\lambda$, our algorithm achieves better performance than the BDOA even with extremely erroneous prediction. This demonstrates the importance of hyperparameter $\lambda$.

### 3.4.2 Biased Prediction Errors

Next we consider the impact of biases on prediction errors. We consider three possible values of 10, 20, 50 for $\delta$. The performance of deterministic algorithm (Algorithm 2), and randomized algorithm (Algorithm 3) with $\Gamma = 3b_1$ and $\Gamma = b_1$ are shown in Figure 5 and Figure 6, respectively. With the above analysis of $\Gamma$’s impact and the same trust on ML predictions ($\lambda = 0.5$), a smaller bias benefits the competitive ratio when the variance is small, however, when the variance is large, the impact of bias is negligible.

### 4. Online Algorithms for MSSR with Multiple ML Predictions

Now we consider a more general case that there are $m$ ML predictions, and denote them as $y_1, \cdots, y_m$. Without loss of generality, we assume $y_1 < y_2 < \cdots < y_m$. We define an indicator function $f(i)$ to represent the relation between $y_i$ and $b_n$, satisfying

$$f(i) = \begin{cases} 
1, & y_i \geq b_n, \\
0, & \text{otherwise}.
\end{cases}$$

Let $z = \sum_{i=1}^{m} f(i)$, which indicates the number of predictions that is greater than $b_n$. We redefine the prediction error under the multiple predictions case as $\zeta = \max_{i} |y_i - x|$.

In this section, we design deterministic and randomized algorithms for MSSR with multiple ML predictions. We slightly modify the algorithms proposed in Section 3, and show that similar techniques lead to tight results for online algorithms of MSSR with multiple ML predictions. Again for the ease of exposition, we take the two-shop ski rental problem as a motivating example, and the results can be easily generalized to the general $n$-shop MSSR.
4.1 A Deterministic Algorithm with Multiple ML Predictions
We first design a deterministic algorithm tuned by a hyperparameter $\lambda \in (0, 1)$ to achieve a tradeoff between consistency and robustness.

**Algorithm 4** A deterministic algorithm with multiple ML predictions

```
if $z \geq m/2$ then
    Rent at shop 2 until buying on day $\left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor$ at shop 2
else
    Rent at shop 1 until buying on day $\left\lfloor \frac{(m-2z+1)b_2}{\lambda} \right\rfloor$ at shop 1
```

**Theorem 3.** The competitive ratio of Algorithm 4 is at most

$$\min\{\left(1+\frac{1}{\lambda}\right)r_2 + \frac{b_1}{b_2} + \max\{\lambda r_2 + \frac{1}{1-\lambda}, \frac{1}{\lambda}\} \cdot \text{OPT}, \max\{r_2, \frac{b_1}{b_2}\} + \frac{m+1}{\lambda}\},$$

where $\lambda \in (0, 1)$ is a parameter. In particular, Algorithm 4 is $(\left(1+\frac{1}{\lambda}\right)r_2 + \frac{b_1}{b_2})$-consistent and $(\max\{\lambda r_2 + \frac{1}{1-\lambda}, 1\} + \frac{m+1}{\lambda})$-robust.

**Proof sketch of Theorem 3:** Denote $\Psi = \left\lceil \frac{\lambda b_2}{2z-m+1} \right\rceil$ and $\Xi = \left\lceil (m-2z+1)b_2/\lambda \right\rceil$. For the first bound, when $z \geq m/2$, we consider two cases $x < \Psi$ and $x \geq \Psi$; and when $z < m/2$, we consider three cases, $x < b_2$, $x \in [b_2, \Xi)$ and $x \geq \Xi$. We can compute the corresponding ALG and OPT to achieve the results. Due to space constraints, we omit the details here. Similar to the proof of Theorem 1, we can achieve the worst case CR when $x = \Psi$ if $z \geq m/2$, and when $x = \Xi$ if $z < m/2$. The proof details are available in Appendix B.

**Remark 2.** We add a term $+1$ into the break-even point in Algorithm 4 as $2z - m$ or $m - 2z$ may equal 0 when $z$ is an even number. We numerically evaluate its impact in Section 4.3.

4.2 A Randomized Algorithm with Multiple ML Predictions

In this section, we propose a randomized algorithm with multiple ML predictions that achieves a better tradeoff between consistency and robustness than the deterministic algorithm.

**Algorithm 5** A randomized algorithm with multiple ML predictions

```
if $z \geq m/2$ then
    Let $k = \left\lceil \frac{\lambda b_2}{2z-m+1} \right\rceil$
    Define $p_i = \left(\frac{b_2-r_2}{b_2}\right)^{k-i} \cdot \frac{r_2}{b_2(1-(1-\frac{r_2}{b_2})^k)},$ for $i = 1, \cdots, k$
    Choose $j \in \{1, 2, \ldots, k\}$ randomly from the distribution defined by $p_i$
    Rent till day $j-1$ and then buy on day $j$ at shop 2
else
    Let $l = \left\lfloor \frac{m-2z+1}{\lambda}b_1 \right\rfloor$
    Define $q_i = \left(\frac{b_1-1}{b_1}\right)^{l-i} \cdot \frac{1}{b_1(1-(1-\frac{1}{b_1})^l)},$ for $i = 1, \cdots, l$
    Choose $j \in \{1, 2, \ldots, l\}$ randomly from the distribution defined by $q_i$
    Rent till day $j-1$ and then buy on day $j$ at shop 1
```
Theorem 4. The competitive ratio of Algorithm 5 is at most \( \min \{ \frac{b_1}{b_2} \max \left\{ \frac{r_2 \lambda}{1 - e^{-r_2 (\lambda/(m+1)) - 1/b_2}}, \frac{m+1/\lambda + 1/b_1}{1 - e^{-r_2 (\lambda/(m+1)) - 1/b_2}} \right\}, \frac{r_2 \lambda}{1 - e^{-r_2 (\lambda/(m+1)) - 1/b_2}} \right\} \). In particular, Algorithm 5 is \( \left( \frac{r_2 \lambda}{1 - e^{-r_2 (\lambda/(m+1)) - 1/b_2}} \right) \)-consistent and \( \left( \frac{b_1}{b_2} \max \left\{ \frac{r_2 \lambda}{1 - e^{-r_2 (\lambda/(m+1)) - 1/b_2}}, \frac{m+1/\lambda + 1/b_1}{1 - e^{-r_2 (\lambda/(m+1)) - 1/b_2}} \right\} \)-robust.

Proof sketch of Theorem 4: We need to consider four cases, (i) \( z \geq m/2, x \geq k \); (ii) \( z \geq m/2, x < k \); (iii) \( z < m/2, x < l \); and (iv) \( z < m/2, x \geq l \). Similar to the proof of Theorem 2, we can compute the corresponding \( E[\text{ALG}] \) and \( \text{OPT} \) to obtain the results for competitive ratio, consistency and robustness. The proof details are available in Appendix C.

4.3 Model Validation and Insights

We consider the same setting as that in Section 3.4. We vary the number of ML predictions from 1 to 8, and set the associated predictions to \( x + \epsilon \), where \( \epsilon \) is drawn from a normal distribution with mean \( \delta \) and standard variation \( \sigma \), and \( \Gamma = b_1 \). We investigate the impacts of \( m, \lambda \) and \( \delta \) on the performance and make the following observations: (i) For unbiased prediction errors and fixed \( \lambda \), if the prediction is accurate (small \( \sigma \)), increasing \( m \) improves the competitive ratio, however, more predictions hurt the competitive ratio when prediction error is large, see Figure 7; (ii) For \( \delta = 0 \) with fixed \( m = 5 \), if the prediction is accurate, more trust (small \( \lambda \)) benefits the algorithm. On the other hand, less trust achieves better competitive ratio when the prediction error is large. See Figure 8. (iii) For fixed \( m \) and \( \lambda \), a smaller bias benefits the competitive ratio when the variance is small, while a larger bias achieves a smaller competitive ratio when the variance is large. See Figure 9. We also characterize the impact of the \( \text{“} + 1 \text{”} \) term in Algorithm 4, and compare the algorithms with \( \text{“} + 1 \text{”} \) and without it in the break-even points, see Figure 10. We observe that the \( \text{“} + 1 \text{”} \) can improve the competitive ratio as it suggests the skier to buy earlier when more predictions are above \( b_2 \), and rent longer when more predictions are smaller than \( b_2 \), i.e., making decisions more cautious.

![Figure 7: Competitive ratio of deterministic algorithm (Algorithm 4) under unbiased errors (\( \delta = 0 \)) and \( \lambda = 0.5 \) with different \( m \).](image)

![Figure 8: Competitive ratio of deterministic algorithm (Algorithm 4) under unbiased errors (\( \delta = 0 \)) and \( m = 5 \) with different \( \lambda \).](image)
5. Conclusions

In this paper, we investigate how to improve the worst-case performance of online algorithms with (multiple) ML predictions. In particular, we consider the general multi-shop ski rental problem. We develop both deterministic and randomized algorithms when there are either a single or multiple ML predictions. Our online algorithms achieve a smooth tradeoff between consistency and robustness, and can significantly outperform the ones without ML predictions. Going further, we will study extensions of MSSR. e.g., the skier is allowed to switch shops, in which she can simultaneously decide where to buy or rent the skis. We will also consider to integrate prediction costs into the online algorithm design.

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Appendix A. Detailed Proof of Lemma 1

It is obvious that \( \text{OPT} = \min\{x, b_n\} \). Under Assumption 1, we can consider the competitive ratio of shop \( \forall i \in \mathcal{N} \). Let \( d_i \) be the buying day. Then \( \text{ALG}_i = x r_i \) if \( x < d_i \), otherwise \( \text{ALG}_i = (d_i - 1)r_i + b_i \). It is easy to argue that the worst case happens when \( x = d_i \). We have

\[
\text{CR}_i = \frac{\text{ALG}_i}{\text{OPT}} = \frac{(d_i - 1)r_i + b_i}{\min\{x, b_n\}} = \frac{(d_i - 1)r_i + b_i}{\min\{d_i, b_n\}}
\]

\[
= \frac{(d_i + b_n)r_i + b_i - r_i - b_n r_n}{\min\{d_i, b_n\}}
\]

\[
= \frac{(\min\{d_i, b_n\} + \max\{d_i, b_n\} - r_i - b_n r_n)}{\min\{d_i, b_n\}}
\]

\[
= r_i + \frac{\max\{d_i, b_n\} - r_i - b_n r_n}{\min\{d_i, b_n\}}.
\]

Hence, the competitive ratio is minimized when \( d_i = b_n \), i.e., the best competitive ratio satisfies \( \text{CR}_i = r_i + (b_i - r_i)/b_n \). Thus, we have \( \text{CR} = \min_i \text{CR}_i \).
Appendix B. Detailed Proof of Theorem 3

We first prove the first bound. When \( z \geq m/2 \), we consider two cases.

First, if \( x < \left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor \), then \( \text{OPT} = x \), i.e., rent at shop 1 since \( r_1 = 1 < r_2 \). Hence,

\[
\text{ALG} = r_2 x = r_2 \text{OPT},
\]
i.e., \( \text{CR}_1 = r_2 \).

Second, if \( x \geq \left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor \), then \( \text{OPT} = \min\{b_2, x\} \) and

\[
\text{ALG} = r_2 \left( \left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor - 1 \right) + b_2 \leq \left( \frac{\lambda}{2z-m+1} r_2 + 1 \right) b_2 \overset{(a)}{\leq} (\lambda r_2 + 1) (\text{OPT} + \zeta),
\]

where (a) follows from two cases i) when \( \left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor \leq x < b_2 \), we have \( \text{OPT} = x \), and \( \zeta \geq y_m - x > b_2 - x \), i.e., \( b_2 \leq \text{OPT} + \zeta \); ii) \( x \geq b_2 \), we have \( \text{OPT} = b_2 \), then \( b_2 \leq \text{OPT} + \zeta \).

Furthermore, we have \( 2z - m + 1 \geq 1 \). Hence \( \text{CR}_2 = (\lambda r_2 + 1)(1 + \frac{\zeta}{\text{OPT}}) \).

Similarly, when \( z < m/2 \), we consider the following three cases.

First, if \( x < b_2 \), we have \( \text{ALG} = x \). It is clear that \( \text{OPT} = x \), i.e., rent at shop 1. Therefore, we have \( \text{CR}_1 = 1 \).

Second, if \( x \in \left[ b_2, \left\lceil \frac{(m-2z+1)b_2}{\lambda} \right\rceil \right) \), we have \( \text{OPT} = b_2 \), i.e., buy at shop 2 on day 1, and

\[
\text{ALG} = x + b = \text{OPT} + \zeta,
\]

where (b) is obtained by following Algorithm 4, i.e., rent at shop 1 with \( r_1 = 1 \), and (c) follows that \( \zeta \geq x - y_1 \), i.e., \( x \leq \zeta + y_1 \leq \zeta + b_2 \). Therefore, we have \( \text{CR}_3 < 1 + \frac{\zeta}{\text{OPT}} \).

Finally, if \( x \geq \left\lceil \frac{(m-2z+1)b_2}{\lambda} \right\rceil \), we have \( \text{OPT} = b_2 \), and

\[
\text{ALG} = \left\lceil \frac{(m-2z+1)b_2}{\lambda} \right\rceil - 1 + b_1 \leq \frac{(m-2z+1)b_2}{\lambda} + b_1 \overset{(d)}{\leq} b_1 + \frac{m-2z+1}{m-2z+1-\zeta} \overset{(e)}{\leq} b_1 + \frac{1}{1-\zeta},
\]

where (d) follows \( \zeta \geq x - y_1 > \frac{(m-2z+1)b_2}{\lambda} - b_2 \), i.e., \( b_2 \leq \frac{\zeta}{(m-2z+1)/\lambda-1} \), and (e) follows \( m - 2z + 1 \geq 1 \). Thus \( \text{CR}_4 < \frac{b_1}{b_2} + \frac{1}{1-\zeta} \).

Combining \( \text{CR}_1, \text{CR}_2, \text{CR}_3 \) and \( \text{CR}_4 \), we have the first bound.

Now we prove the second bound. According to Algorithm 4, the skier rents the skis at shop 2 until day \( \left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor - 1 \) and then buys on day \( \left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor \) at shop 2, when the predictions satisfy \( z \geq m/2 \). The corresponding cost is \( \text{ALG} = r_2 \left( \left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor - 1 \right) + b_2 \) when \( x \geq \left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor \). It is easy to see that the worst competitive ratio is obtained when \( x = \left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor \), for which we have \( \text{OPT} = \left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor \).

Therefore, we have

\[
\text{ALG} = r_2 \left( \left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor - 1 \right) + b_2 \leq \left( \frac{\lambda r_2}{2z-m+1} + 1 \right) b_2 \leq \left( \frac{\lambda r_2}{2z-m+1} + 1 \right) \frac{2z-m+1}{\lambda} \left\lfloor \frac{\lambda b_2}{2z-m+1} \right\rfloor
\]

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\[
= \left( r_2 + \frac{2z - m + 1}{\lambda} \right) \text{OPT} \leq \left( r_2 + \frac{m + 1}{\lambda} \right) \text{OPT},
\]

where the last inequality follows \(2z - m \leq m\).

Similarly, the skier rents the skis at shop 1 until day \(\left\lceil \frac{(m - 2z + 1)b_2}{\lambda} \right\rceil - 1\) and then buys on day \(x = \left\lceil \frac{(m - 2z + 1)b_2}{\lambda} \right\rceil\) at shop 1, when \(z < m/2\). The worst competitive ratio is obtained when for which we have \(\text{OPT} = b_2\), and

\[
\text{ALG} = \left( \frac{b_1}{b_2} + \frac{m - 2z + 1}{\lambda} \right) \leq \left( \frac{b_1}{b_2} + \frac{m + 1}{\lambda} \right) \text{OPT},
\]

where the last inequality holds since \(m - 2z \leq m\).

**Appendix C. Detailed Proof of Theorem 4**

Here we consider four different cases.

(1): \(z \geq m/2\) and \(x \geq k\). It is clear that \(\text{OPT} = \min\{b_2, x\}\). According to Algorithm 5, the skier should rent at shop 2 until day \(j - 1\) and buy on day \(j\). This happens with probability \(p_i\), for \(i = 1, \cdots, k\), and incurs a cost is \((b_2 + (i - 1)r_2)\). We have

\[
\mathbb{E}[\text{ALG}] = \sum_{i=1}^{k} (b_2 + (i - 1)r_2)p_i = \frac{r_2k}{1 - (1 - \frac{r_2}{b_2})} \leq \frac{r_2k/b_2}{1 - e^{-r_2k/b_2}}b_2
\]

\[
\leq \frac{r_2 \lambda}{1 - e^{-r_2 \lambda}} b_2 \leq \frac{r_2 \lambda}{1 - e^{-r_2 \lambda}} b_2 \leq \frac{r_2 \lambda}{1 - e^{-r_2 \lambda}} (\text{OPT} + \zeta).
\]

where (a) follows that \(k \leq \frac{\lambda b_2}{2x - m + 1}\), i.e., \(k/b_2 \leq \lambda/(2z - m + 1)\) and \(x/(1 - e^{-r_2 \lambda})\) increases in \(x \geq 0\).

(2): \(y \geq m/2\) and \(x \leq k\). We have \(\text{OPT} = x\). If the skier buys the skis on day \(i \leq x\), then it incurs a cost \((b_2 + (i - 1)r_2)\), otherwise, the cost is \(xr_2\). Therefore, we obtain the robustness through the following

\[
\mathbb{E}[\text{ALG}] = \sum_{i=1}^{x} (b_2 + (i - 1)r_2)p_i + \sum_{i=x+1}^{k} x r_2 p_i = \frac{r_2 x}{1 - (1 - \frac{r_2}{b_2})^k}
\]

\[
\leq \frac{r_2}{1 - e^{-r_2k/b_2}} \text{OPT} \leq \frac{b_1}{b_2} \cdot \frac{r_2}{1 - e^{-r_2(\frac{\lambda}{m+1} - \frac{1}{b_2})}} \text{OPT}.
\]

where (b) holds true since \(k = \left\lceil \frac{\lambda b_2}{2z - m + 1} \right\rceil \geq \frac{\lambda b_2}{2z - m + 1} - 1\), i.e., \(k/b_2 \geq \lambda/(m + 1) - 1/b_2\), and \(b_1/b_2 > 1\). To get the consistency, we can rewrite the above inequality

\[
\mathbb{E}[\text{ALG}] \leq \frac{r_2 \cdot k/b_2}{1 - e^{-r_2k/b_2}} \text{OPT} + \frac{r_2 \cdot \zeta/b_2}{1 - e^{-r_2k/b_2}} k \leq \frac{r_2 \lambda}{1 - e^{-r_2 \lambda}} (\text{OPT} + \zeta).
\]
**Case 3.** $z < m/2$ and $x < l$. $\text{OPT} = \min \{b_2, x\}$. Similar to Case 2, we have

$$\mathbb{E}[\text{ALG}] = \sum_{i=1}^{x} (b_1 + (i - 1) \cdot 1)p_i + \sum_{i=x+1}^{l} x \cdot 1 \cdot p_i = \frac{x}{1 - (1 - 1/b_1)^l} \leq \frac{x}{1 - e^{-l/b_1}} \leq \frac{r_2 \lambda b_1}{1 - e^{-r_2 \lambda/(m+1)}} (\text{OPT} + \zeta).$$

**Case 4.** $z < m/2$ and $x \geq l$. $\text{OPT} = b_2$. Similar to Case 1, we have the robustness as

$$\mathbb{E}[\text{ALG}] = \sum_{i=1}^{l} (b_1 + (i - 1) \cdot 1)p_i = \frac{l}{1 - (1 - 1/b_1)^l} \leq \frac{l}{1 - e^{-l/b_1}} \leq \frac{\lceil \frac{z-2m+1}{\lambda} b_1 \rceil}{1 - e^{-l/b_1}} \leq \frac{\frac{z-2m+1}{\lambda} b_1 + 1}{1 - e^{-(z-2m+1)/\lambda}} \leq \frac{b_1 \frac{m+1}{\lambda} + \frac{1}{b_2}}{1 - e^{-1/\lambda}} \text{OPT},$$

where (i) follows that $l = \lceil \frac{m-2z+1}{\lambda} \rceil \geq \frac{m-2z+1}{\lambda}$, i.e., $\frac{l}{b_1} \geq \frac{z-2m+1}{\lambda}$. Again, we rewrite the above inequality to get the consistency

$$\mathbb{E}[\text{ALG}] \leq \frac{l}{1 - e^{-l/b_1}} \leq \frac{r_2 \lambda}{1 - e^{-r_2 \lambda/(m+1)}} (\text{OPT} + \zeta).$$