Central Charges and Extra Dimensions in Supersymmetric Quantum Mechanics

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Abstract
We systematically include central charges into supersymmetric quantum mechanics formulated on curved Euclidean spaces, and explain how the background geometry manifests itself on states of the theory. In particular, we show in detail how, from the point of view of non-relativistic $d = 1$ world-line physics, one can infer the existence of target space dualities typically associated with string theory. We also explain in detail how the presence of a non-trivial supersymmetry central charge restricts the background geometry in which a particle may propagate.

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1 Introduction

As is well appreciated, supersymmetry \cite{1} is a concept which not only provides elegant and useful solutions to interesting problems, such as the hierarchy problem in the standard model, but which also plays a key role in the structure of a variety of theories. For example, it appears as a required ingredient in consistent string theories \cite{2}, and also underlies the presence of shape invariance in exactly solvable systems in ordinary quantum mechanics \cite{3}. As is also well appreciated, attempts to find more fundamental descriptions of nature frequently benefit from the inclusion of extra, less obvious, dimensions as part of our physical space. It is interesting to consider what the two ideas of supersymmetry and extra dimensions imply, at a basic level, when they are imposed simultaneously on ordinary, non-relativistic quantum mechanics.

A conspicuous hallmark of extra dimensions is the appearance of central charges in the symmetry algebras of physical systems. In the context of string theory, and its effective description in terms of supergravity theories, these typically appear as central terms in superalgebras. Although supersymmetry central charges are a relatively mature subject in higher-dimensional field theories \cite{4,5}, relatively little attention has been applied to basic questions regarding similar charges in supersymmetric quantum mechanics. Accordingly, we undertook the seemingly academic exercise of re-visiting the systematic development of supersymmetric quantum mechanics \cite{6}, with a specific intent to methodically build-in a non-trivial central charge.

In this paper we critically examine the algebraic constraints that limit the inclusion of central terms into quantum $d = 1$ superalgebras. We explain in detail how non-relativistic particle models based on supersymmetric sigma models can be extended to admit a non-trivial vector as a background field, in such a way that this vector appears as a central charge in the corresponding superalgebra. We show how this can be done only if the background geometry has an isometry, in which case the central charge vector must be a Killing vector. We explicitly quantize two classes of models that conform to these constraints, namely models constructed on a target-space with topology $\mathbb{R} \times (S^1)^{D-1}$ and others with topology $\mathbb{R} \times T^2$. In the second class of models, we demonstrate the invariance of the quantum theory under $SL(2,\mathbb{Z})$ modular transformations which preserve the size of the $T^2$ factor. In both cases we prove the existence of a $\mathbb{Z}_2$ duality which equates models with “large” compact space with ostensibly distinct models having “small” compact spaces.

We formulate supersymmetric quantum mechanics by canonically quantizing a classical field theory describing the non-relativistic “world-line” description of a point particle.
propagating in a $D$-dimensional Euclidean target space. We allow one or more of the target space dimensions to be compact. In the interest of simplicity, we do not in this paper include a superpotential per se. Instead, all interactions are inherited from the background geometry. The fermionic operators transform non-trivially under “spin” transformations inherited from the structure group on the target space. If the central charge vanishes, then the quantum supercharge organizes as $Q = i \mathcal{D}$, where $\mathcal{D}^m$ is a spin-covariant derivative. Furthermore, Kaluza-Klein interactions appear, owing to the connection pieces in this derivative.

Suppose, for introductory purposes, that we have exactly one non-compact dimension, parameterized by $X^1$, and exactly one circular compact dimension parameterized by an angle $X^2$. Assume that the circular dimension has radius $R(X^1)$, which can depend on $X^1$. The fermionic operators are described by elements of a complex Clifford algebra, with elements $\Gamma^{1,2}$ and $\Gamma^{1,2\dagger}$, subject to $\{ \Gamma^M, \Gamma^N\dagger \} = \delta^{MN}$ and $\{ \Gamma^M, \Gamma^N \} = 0$, where $M$ and $N$ are local frame indices. The complexification is required to accommodate the central charge; in ordinary supersymmetric quantum mechanics a real Clifford algebra is sufficient. If we systematically include a central term in the superalgebra in as minimal a fashion as possible the modified supercharge turns out to be

$$Q = i \mathcal{D} + \mu R(X^1) \Gamma^2\dagger,$$

where $\mu$ is a parameter associated with the central charge, and the slash denotes contraction with $\Gamma^M$, not with $\Gamma^M\dagger$. As a result, $Q$ transforms in a reducible spinor representation of the structure group $SO(D)$, rather than as an irreducible spinor. We can make explicit the dependence of $\mathcal{D}$ on the angular momentum $i \partial_2 \equiv \nu \in \mathbb{Z}$, which is quantized since $X^2$ is an angular variable, by writing

$$Q = i \mathcal{D} + \frac{\nu}{R(X^1)} \Gamma^2 + \mu R(X^1) \Gamma^2\dagger. \quad (1.2)$$

Here the operator $\mathcal{D}$ includes all of the terms in $\mathcal{D}$ which do not depend on $\nu$. By writing the supercharge as in (1.2), one notices an amusing feature. Namely, the Hamiltonian, defined via $H = \frac{1}{2} \{ Q, Q^\dagger \}$, exhibits a duality under the following transformation

$$R(X^1) \rightarrow \frac{1}{R(X^1)} \quad \mu \leftrightarrow \nu.$$

In particular, under (1.3), the Hamiltonian undergoes a unitary transformation $H \rightarrow \Omega^\dagger H \Omega$, where $\Omega$ squares to the identity. Thus, this class of models exhibits a $T$-duality, wherein models constructed with a small compact dimensions are physically identical to
ostensibly distinct models formulated on a relatively large compact dimension \(^1\). This scenario represents the simplest example of a phenomenon which appears generically in supersymmetric quantum mechanics when a central charge is switched on.

Some of the discussion in this paper parallels similar arguments known previously in string theory. Indeed, the dualities which we describe are probably closely related to string theory target-space dualities \(^2\). However, we believe that making firm connections between the string theory phenomenon and the point particle analog is not a trivial exercise, and may include physically relevant subtlety. At the same time, we find it interesting how the existence of target space dualities can be inferred, on basic grounds, using modestly minimalist modification to ordinary quantum mechanics. We find this point of view potentially useful for identifying points of departure from string theory or for ways to connect string theory with other ideas, such as shape invariance or loop quantum gravity. Indeed, owing to a conjectured relationship between string theory and loop quantum gravity \(^3\), it seems that basic quantum mechanics is a natural realm to look for points of connection.

Our motivation for studying centrally extended \(d = 1\) superalgebras stemmed originally from our efforts to understand the deceptively simple algebraic structure of shape invariance \(^4\), found in ordinary quantum mechanics. Although shape invariance is not crucial to the results described in this paper, we feel that it is useful to mention this concept at the outset, since it has been an important motivator, and because we believe there may ultimately be some significant connections between shape invariance and the work in this paper. We find it compelling that centrally extended \(d = 1\) superalgebras appear naturally in a context which has no \textit{a priori} relationship to higher-dimensional quantum field theories.

This paper is organized as follows.

In section 2 we define the algebraic basis for including central terms into the \(d = 1\) \(\mathcal{N} = 1\) superalgebra. We use superspace techniques to determine the transformation rules for the unique multiplet that includes a real commuting field as lowest component, and identify the modifications required to switch on a non-trivial central charge. We show that the central charge can be incorporated as an arbitrary background vector field on the target space.

In section 3 we use superspace techniques to systematically derive an action which is invariant under the modified transformation rules derived in section 2. This action

\(^1\)This observation was made previously by us in \([9]\). In that paper, however, a particular choice of fermion representation was imposed, so that the geometrical significance of the result was somewhat obscured.
incorporates the extended real multiplets as fundamental fields, and describes a supersymmetric sigma model with a target-space metric as a background field. We explain how this is possible only if the background central charge vector field and the background metric field are constrained to obey a system of coupled differential equations. In this way, we show how the background geometry is limited by the requirement of the supersymmetry central charge. We describe a class of solutions to this constraint.

In section 4 we analyze a subset of the sigma models derived in section 3 corresponding to a class of toroidal compactification schemes in which the lattice describing the compact space is orthogonal. We quantize this construction and show how the supercharge organizes to transform as a target space spinor, in such a way that the target space duality structure is manifest.

In section 5 we analyze a class of centrally extended sigma models constructed on target-spaces having topology $\mathbb{R} \times T^2$. In this case we allow an arbitrary constant complex modulus on the $T^2$ factor and also allow a scale factor which can depend on the coordinate of the non-compact dimension. We quantize this model and show how the quantum supercharge organizes into a target-space spinor, the structure of which makes clear the existence of a generalization of the duality explained in section 4.

In section 6 we study the behavior under scale-preserving modular transformations of the quantum supercharge obtained in the context of the $\mathbb{R} \times T^2$ compactifications described in section 5. We demonstrate that the states in this model exhibit an appropriate $SL(2, \mathbb{Z})$ symmetry structure so as to ensure that the scale-preserving modular transformations represent a symmetry. This provides a useful consistency check.

In section 7 we study the behavior under $\mathbb{Z}_2$ transformations that change the scale of the $T^2$ factor in the $\mathbb{R} \times T^2$ compactifications described in section 5. By finding an appropriate $\mathbb{Z}_2$ generator which acts on the states of the model, we show how these transformations describe an interesting generalization of the duality described in section 4, as anticipated by the discussion in section 5.

We conclude by making some comments on possible relationships between the results of this paper with other ideas, including shape invariance and string theory.

2 The Centrally Extended Superalgebra

A supercharge $Q$ is, by definition, an operator that obeys $\{ Q, Q^\dagger \} = 2H$, where $H$ is the Hamiltonian. Ordinary supersymmetric quantum mechanics follows from including such operators, subject to the additional requirement that $Q^2 = 0$, into the fundamental symmetry algebra of a physical system. We are interested in extending this algebra by
introducing an additional non-trivial central charge \( Z \), such that \( Q^2 = Z \), and asking what sorts of basic physics follows from this. Thus, we are interested in the centrally extended superalgebra described by

\[
\{ Q, Q^\dagger \} = 2H \quad Q^2 = Z \quad [Q, H] = 0. \tag{2.1}
\]

It follows trivially that \([Q, Z] = 0\). One also computes

\[
[Q^\dagger, Z] = [Q^\dagger, Q^2] = \{Q^\dagger, Q\} Q - Q \{Q^\dagger, Q\} = 2[H, Q] = 0, \tag{2.2}
\]

where we pass to the final line using the third relationship in (2.1). Represent the Hamiltonian as \( H = i \partial_t \), where \( t \) is a “time” coordinate, and represent the central charge by writing \( Z = i \delta Z \) and \( Z^\dagger = i \delta Z^\dagger \), where \( \delta Z \) describes a corresponding central charge transformation. Define a supersymmetry transformation as \( \delta_Q(\epsilon) = \epsilon Q + \epsilon^\dagger Q^\dagger \), where \( \epsilon \) is a complex anti-commuting parameter. Using this, we derive

\[
[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = -4i \epsilon_1^\dagger \epsilon_2^\dagger \partial_t - 2i \epsilon_1 \epsilon_2 \delta Z - 2i \epsilon_1^\dagger \epsilon_2^\dagger \delta Z^\dagger
\]

\[
[\delta_Q(\epsilon), \delta_Z] = 0
\]

\[
[\delta_Q(\epsilon), \delta_Z^\dagger] = 0. \tag{2.3}
\]

It is straightforward to find multiplet structures which represent these relationships. There are a variety of possibilities. Two of these are analogs of the vector multiplets and chiral multiplets familiar from supersymmetric field theories. There also exist related multiplets with the positions of the commuting and anti-commuting fields in the superfield swapped, which we refer to as “flipped” multiplets \(^2\). In this paper we keep things simple by focussing exclusively on real commuting multiplets.

### 2.1 Real Multiplets and Harmonic Supercharges

Construct a \( d = 1 \) \( N = 1 \) superspace by combining our real commuting “time” coordinate \( t \) with one additional complex anti-commuting coordinate \( \theta \). Introduce superspace

\(^2\)The flipped multiplets were discussed originally in [11] where the operation of interchanging commuting fields and anti-commuting fields was referred to as a Klein flip.
operators

\[ Q = \frac{\partial}{\partial \theta} - i \theta \delta Z \]

\[ Q^\dagger = \frac{\partial}{\partial \theta^\dagger} - i \theta^\dagger \delta Z^\dagger, \]

(2.4)

which by-construction satisfy the algebra \([2.1]\) with the signs on \(H\) and \(Z\) reversed. The sign-reversal is necessary, since the superspace coordinates \(\theta\) are anti-commuting, so that the algebra generated on superfield components by these operators respects \([2.1]\). The inclusion of a central charge transformation is reminiscent of a technique used in supersymmetric field theories in the context of so-called Harmonic superspace \([14]\). Accordingly, we refer to the operators in \([2.4]\) as harmonic supercharges. Introduce a set of \(D\) real superfields

\[ V^n = X^n + i \theta \psi^n + i \theta^\dagger \psi^n^\dagger + \theta^\dagger \theta B^n, \]

(2.5)

where \(n = 1, ..., D\). We interpret the lowest components \(X^n\) as the spatial coordinates on a \(D\)-dimensional Euclidean target-space in which a particle, whose physics we wish to study, will propagate. Parameterize the particle trajectory in this space using the time coordinate \(t\). The superfields \(V^n\) are, therefore, functions of \(t\). A world-line supersymmetry transformation is given by \(\delta_Q(\epsilon) = \epsilon Q + \epsilon^\dagger Q^\dagger\), where \(\epsilon\) is a complex anti-commuting parameter. Applying \([2.4]\) to \([2.5]\), we derive

\[ \delta_Q(\epsilon) X^n = i \epsilon \psi^n + i \epsilon^\dagger \psi^n^\dagger \]

\[ \delta_Q(\epsilon) \psi^n = \epsilon^\dagger (\dot{X}^n + i B^n) + \epsilon \delta Z X^n \]

\[ \delta_Q(\epsilon) \psi^n^\dagger = \epsilon (\dot{X}^n - i B^n) + \epsilon^\dagger \delta Z^\dagger X^n \]

\[ \delta_Q(\epsilon) B^n = \epsilon \dot{\psi}^n - \epsilon^\dagger \dot{\psi}^n^\dagger - \epsilon \delta Z \psi^n^\dagger + \epsilon^\dagger \delta Z^\dagger \psi^n. \]

(2.6)

An important question is how \(\delta_Z\) and \(\delta_{Z^\dagger}\) act on the component fields \(X^n, \psi^n\) and \(B^n\). The central charge transformation should commute with complex conjugation. Since \(X^n\) is real, this imposes \(\delta_Z X^n = \delta_{Z^\dagger} X^n\). We need other commutators in the algebra to resolve the consistent possibilities, subject to this constraint. In the simplest class of possibilities, \(\delta_Z X^n\) appears as an arbitrary function of the bosonic fields,

\[ \delta_Z X^n = f^n(X), \]

(2.7)

where \(f^n(X)\) is an unspecified real-valued function of \(X^1, ..., X^D\). It is also possible to include fermion bilinears in \(\delta_Z X^n\). For instance, we could write \(\delta_Z X^n = f^n(X) + \)
$c_{mn}(X) \psi^m \psi^n$, where $c_{mn}(X)$ is an unspecified real-valued symmetric tensor. There are several other ways in which (2.7) could also be modified. However, restricting $\delta_Z X^n$ to depend only on $X^1, ..., X^D$ provides for a tractable and elegant multiplet structure which admits an interesting class of invariant actions. Thus, in the spirit of minimalism, we restrict attention to the possibility described by (2.7). In this case, imposing $[\delta_Q, \delta_Z] = [\delta_Q, \delta_{Z1}] = 0$, requires

\[
\begin{align*}
\delta_Z \psi^n &= \partial_m f^n(X) \psi^m \\
\delta_Z \psi^{n\dagger} &= \partial_m f^n(X) \psi^{m\dagger} \\
\delta_Z B^n &= \partial_m f^n(X) B^m + \partial_m \partial_l f^n(X) \psi^{m\dagger} \psi^l,
\end{align*}
\]

(2.8)

where $\partial_m = \partial/\partial X^m$. One derives (2.8) by using $[\delta_Q, \delta_Z] = 0$ together with (2.7) and (2.6). Together, these imply that the superfields transform as

\[
\delta_Z V^n = f^n(V).
\]

(2.9)

In order that the central charge preserve the reality constraint $V = V^\dagger$, we also require $\delta_Z = \delta_{Z1}$. Using (2.7) and (2.8), the component transformation rules (2.6) are

\[
\begin{align*}
\delta_Q(\epsilon) X^n &= i \epsilon \psi^n + i \epsilon^\dagger \psi^{n\dagger} \\
\delta_Q(\epsilon) \psi^n &= \epsilon^\dagger (\dot{X}^n + i B^n) + \epsilon f^n(X) \\
\delta_Q(\epsilon) \psi^{n\dagger} &= \epsilon (\dot{X}^n - i B^n) + \epsilon^\dagger f^n(X) \\
\delta_Q(\epsilon) B^n &= \epsilon \dot{\psi}^n - \epsilon^\dagger \dot{\psi}^{n\dagger} - \epsilon \partial_m f^n(X) \psi^{m\dagger} + \epsilon^\dagger \partial_m f^n(X) \psi^m.
\end{align*}
\]

(2.10)

In the case where $f^n(X) = 0$ these correspond to the transformation rules for real supermultiplets in ordinary supersymmetric quantum mechanics. The terms involving $f^n(X)$ describe the basic modifications which switch on the central charge.

For the purpose of forming a representation of the superalgebra, the central charge functions $f^n(X)$ can be chosen freely; i.e., the representation of $Z$ is relatively unconstrained by the algebra. However, interesting restrictions on the possible choices for $f^n(X)$ appear if one imposes additional requirements based on physics, such as the existence of an invariant action functional involving only a finite number of time derivatives.

\[\text{In this paper we construct supersymmetric sigma models which have only a target space metric as a background field. This proves possible given the minimal choice given in (2.7). In planned extensions to this work we intend to include additional background fields, such as an antisymmetric tensor. It might be necessary in such cases to include fermions in the transformation } \delta_Z X^n.\]
In the context of supersymmetric sigma models, the possible choices for the functions $f^n(X)$ are correlated with the possible choices of sigma model metric $^4$.

### 3 Invariant Action

In this section we construct invariant actions that incorporate the centrally extended real multiplets derived above as fundamental fields. A logical method is to start with a “lowest-order” functional $S_0$ whose supersymmetry variation vanishes when the functions $f^n(X)$ vanish. This is easily accomplished by writing $S_0$ as an ordinary superspace integral.

As minimalists, we disallow terms in the component Lagrangian involving more than two time derivatives. We also restrict attention to supersymmetric sigma models which include only a target space metric $g_{mn}(V)$ as a background field. Accordingly, we choose as a “lowest order” action,

$$S_0 = \frac{1}{2} \int dt \, d\theta \, d\theta^\dagger \, g_{mn}(V) \, D^V D^n,$$

(3.1)

where $ds^2 = g_{mn}(X) \, dX^m \, dX^n$ describes a line element on the target space and $D$ is a superspace derivative $^5$, defined as $D = \partial/\partial \theta + i \theta^\dagger \partial_t$. Note that $S_0$ is supersymmetric in the case where $f^n(X) = 0$, but requires modifications to restore supersymmetry when $f^n(X) \neq 0 \, ^6$. To systematize the analysis, it is useful to separate the terms in the transformation rules (2.10) into those terms not involving $f^n(X)$ and those which do include these modifications. Accordingly, we write

$$\delta Q(\epsilon) = \delta Q^{(0)}(\epsilon) + \delta Q^{(1)}(\epsilon),$$

(3.2)

where $\delta Q^{(0)}(\epsilon)$ describes all terms in the transformation rules (2.10) which do not include $f^n(X)$. If follows that $\delta Q^{(1)}(\epsilon)$ includes all terms in (2.10) which do include these functions.

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$^4$The technology described in this and in the following section resembles similar technology used in a two-dimensional context in [12]. At the classical level, the constructions in that paper are probably related to ours by dimensional reduction. The techniques described here also usefully generalize some related techniques described in [13], which describes the rudiments of a theory of linear representations of $d = 1$ supersymmetry without central charges.

$^5$We have used the symbol $D$ for the target space dimensionality and also for the superspace derivative. This should not cause any confusion, since the distinction is naturally clear from the context in which this symbol is used. See Appendix A for a more detailed description of the superspace conventions and techniques employed in this section.

$^6$The reason for this is the following. The superspace integrand in (3.1) is itself a real superfield. The highest component of $\delta Q \, V$, where $V$ is a real superfield, is a total derivative when $f^n(X) = 0$. However, $\delta Q \, V$ also has terms proportional to $f^n(X)$ which do not describe a total derivative.
whereby

\[ \delta_Q^{(1)}(\epsilon) X^n = 0 \]
\[ \delta_Q^{(1)}(\epsilon) \psi^n = \epsilon f^n(X) \]
\[ \delta_Q^{(1)}(\epsilon) \psi^n^\dagger = \epsilon^\dagger f^n(X) \]
\[ \delta_Q^{(1)}(\epsilon) B^n = -\epsilon \partial_m f^n(X) \psi^m + \epsilon^\dagger \partial_m f^n(X) \psi^m. \] (3.3)

These component transformation rules (3.3) are concisely described by the following superfield transformation,

\[ \delta_Q^{(1)} V^n = \left( i \theta \epsilon + i \theta^\dagger \epsilon^\dagger \right) f^n(V). \] (3.4)

Using the superspace variation (3.4), it is straightforward to compute the supersymmetry variation of (3.1). We find

\[
\delta_Q^{(1)} S_0 = \frac{1}{2} \int dt d\theta d\theta^\dagger \left\{ \partial_t g_{mn}(V) \left( i \theta \epsilon f^j(V) + i \theta^\dagger \epsilon^\dagger f^j(V) \right) D^j V^m D V^n \\
+ g_{mn}(V) D^j \left( i \theta \epsilon f^m(V) + i \theta^\dagger \epsilon^\dagger f^m(V) \right) D V^n \\
+ g_{mn}(V) D^j V^m D \left( i \theta \epsilon f^n(V) + i \theta^\dagger \epsilon^\dagger f^n(V) \right) \right\}. \] (3.5)

In the case where the central charge functions \( f^n(X) \) vanish, we see, naturally, that \( S_0 \) is supersymmetric, i.e., \( \delta_Q^{(1)} S_0 \) vanishes. In the case where \( f^n(X) \) is non-vanishing, \( S_0 \) ceases to be supersymmetric by itself. To restore supersymmetry, we therefore must add to \( S_0 \) new terms whose supersymmetry variation cancels against (3.5).

This process is systematized by the following sequence of operations. First, if possible, construct a superspace functional \( S_1 \) with the property \( \delta_Q^{(0)} S_1 = -\delta_Q^{(1)} S_0 \). The supersymmetry variation of the sum \( S_0 + S_1 \) is then given by \( \delta_Q^{(1)} S_1 \). If this is non-vanishing, then iterate this procedure by constructing another superspace functional \( S_2 \) with the property \( \delta_Q^{(0)} S_2 = -\delta_Q^{(1)} S_1 \). As we will show with explicit calculation, in those cases where one can construct \( S_1 \) and \( S_2 \) according to the above prescription, the superspace integrand in \( S_2 \) turns out to be quadratic in fermionic coordinates, i.e., this expression is proportional to \( \theta^\dagger \theta \). Since the operator (3.4) is itself linear in \( \theta \) and \( \theta^\dagger \), it follows that \( \delta_Q^{(1)} S_2 = 0 \). Therefore, this procedure terminates after two iterations, and the combination \( S_0 + S_1 + S_2 \) is supersymmetric. An important question remains: under what circumstances can one construct \( S_1 \) and \( S_2 \) according to our prescription?
It is useful to re-write equation (3.5) in a more useful form. After a small amount of algebra, one finds
\[ \delta_Q^{(1)} S_0 = \int dtd\theta d\theta^\dagger \left\{ \frac{1}{2} g_{mn}(V) \left( i \epsilon^I f^m(V) DV^n - i \epsilon f^m(V) D^I V^n \right) + \left( i \theta \epsilon + i \theta^\dagger \epsilon^I \right) \Omega_{mn}(V) D^I V^m DV^n \right\}, \] (3.6)
where
\[ \Omega_{mn}(V) = g_{(m}(V) \partial_{n)} f^I(V) + \frac{1}{2} f^I(V) \partial_l g_{mn}(V). \] (3.7)
Using the definition of the affine connection, \( \Gamma^{(m}_{(n}) \partial_{l)} = \frac{1}{2} g^{lr} \left( \partial_m g_{nr} + \partial_n g_{mr} - \partial_r g_{mn} \right), \) it is straightforward to prove that \( \Omega_{mn} = \nabla_{(m} f_{n)} \), where \( \nabla_m f_n = \partial_m f_n - \Gamma^{(m}_{(n}) f_l \) is a derivative covariant with respect to target space coordinate transformations.

The second line in (3.6) has the following special feature. If we replace \( \epsilon \) with \( \theta \) and replace \( \epsilon^I \) with \( \theta^I \), then this line vanishes identically. As explained in detail in Appendix A, this structure tells us that this line cannot represent a basic supersymmetry variation; that is, this line does not represent \( \delta_Q^{(0)} \) of any expression. Therefore, our only hope for finding a supersymmetric extension to \( S_0 \) is if this line vanishes identically. Accordingly, we must insist that the target space metric components and the central charge functions are correlated in such a way that \( \nabla_{(m} f_{n)} \) vanishes. This implies that the system of coupled differential equations defined by \( \nabla_{(m} f_{n)} = 0 \) is satisfied. There is another way to understand this condition. Notice that the transformation of \( S_0 \) under the central charge is
\[ \delta_Z S_0 = \int dtd\theta d\theta^\dagger \nabla_{(m} f_{n)}(V) D^I V^m DV^n. \] (3.8)
Thus, the requirement \( \delta_Z S_0 = 0 \) is equivalent to the requirement that we can find proper counter-terms \( S_1 \) to cancel \( \delta_Q S_0 \). When we impose the condition \( \nabla_{(m} f_{n)} = 0 \), equation (3.6), simplifies to
\[ \delta_Q^{(1)} S_0 = \frac{1}{2} \int dtd\theta d\theta^\dagger g_{mn}(V) \left( i \epsilon^I f^m(V) DV^n - i \epsilon f^m(V) D^I V^n \right). \] (3.9)
Now, following our procedure, we need to find an \( S_1 \) which has the property \( \delta_Q^{(0)} S_1 = -\delta_Q^{(1)} S_0 \). This is achieved by
\[ S_1 = \frac{1}{2} \int dtd\theta d\theta^\dagger g_{mn}(V) \left( i \theta^I f^m(V) DV^n - i \theta f^m(V) D^I V^n \right). \] (3.10)
To obtain this, we simply replace each instance of \( \epsilon \) in (3.9) with \( \theta \) and each instance of \( \epsilon^I \) with \( \theta^I \). Now consider the next order in the supersymmetry variation. After some
algebra, we derive
\[
\delta^{(1)}_{Q} S_1 = - \int dt d\theta d\theta^\dagger \left\{ \frac{1}{2} \left( \theta \epsilon^\dagger - \theta^\dagger \epsilon \right) g_{mn}(V) f^m(V) f^n(V) + i \theta^\dagger \theta \nabla_{(m} f_{n)}(V) f^m(V) \left( \epsilon D V^n + \epsilon^\dagger D^\dagger V^n \right) \right\}. \tag{3.11}
\]
Since \(\nabla_{(m} f_{n)}\) vanishes for any of the allowable backgrounds, equation (3.11) automatically simplifies to
\[
\delta^{(1)}_{Q} S_1 = - \frac{1}{2} \int dt d\theta d\theta^\dagger \left( \theta \epsilon^\dagger - \theta^\dagger \epsilon \right) g_{mn}(V) f^m(V) f^n(V). \tag{3.12}
\]
The variation (3.12) is cancelled by adding terms \(S_2\) having the property \(\delta^{(0)}_{Q} S_2 = -\delta^{(1)}_{Q} S_1\). This is achieved by
\[
S_2 = - \frac{1}{2} \int dt d\theta d\theta^\dagger \theta g_{mn}(V) f^m(V) f^n(V). \tag{3.13}
\]
To obtain this, replace each instance of \(\epsilon\) in (3.12) with \(\theta\) and each instance of \(\epsilon^\dagger\) with \(\theta^\dagger\), and divide by two, since the ultimate result is quadratic in \(\theta\) and \(\theta^\dagger\). We see that \(\delta^{(1)}_{Q} S_2 = 0\), so that the sum \(S = S_0 + S_1 + S_2\) is supersymmetric. Adding up the terms (3.1), (3.10) and (3.13), and then factorizing, we obtain
\[
S = \frac{1}{2} \int dt d\theta d\theta^\dagger g_{mn}(V) \left( D^\dagger V^m + i \theta^\dagger f^m(V) \right) \left( D V^n + i \theta f^n(V) \right) \tag{3.14}
\]
where the metric \(g_{mn}(V)\) and the central charge functions \(f^m(V)\) are constrained by
\[
\nabla_{(m} f_{n)} = 0. \tag{3.15}
\]
Notice that this is Killing’s equation. Thus, the allowed central charge functions \(f^n(X)\) must organize as the components of a Killing vector. This tells us that the background geometry must possess an isometry in order for the sigma model to admit a supersymmetry central charge.

Our goal in this paper is to address the basic features of interest that appear when the supersymmetry central charges are switched on. Therefore, rather than describe general solutions to (3.15), we restrict attention to the simplest class of allowable target space metrics that exhibit novel features related to the central extension. We plan to address more general backgrounds in more comprehensive future work.

### 3.1 Consistent Backgrounds

A simple way to satisfy (3.15) is to consider a target space manifold with topology
\[
\mathcal{M}^D = S^p \times X^{D-p}, \tag{3.16}
\]
where $S^p$ is a $p$-dimensional “space” and $X^{D-p}$ is a $(D-p)$-dimensional “internal” space. The metric decomposes as $g = g_S \otimes g_X$ where $g_S$ is the metric on $S^p$, which we do not let depend on the coordinates on $X^{D-p}$, and $g_X$ is the metric on the internal space. The internal metric $g_X$ may, in general, depend on any of the $D$ coordinates on $M^D$. The coordinates on $M^D$ are given by the bosonic component fields $X^1, ..., X^D$. Correspondingly, the superscript $n$ which appears on the superfields $V_n$ and on the central charge functions $f_n(V)$ describes a contravariant target space vector index. The central charge functions then describe a vector which decomposes as $f = f_S \otimes f_X$, where $f_S \equiv (f^1, ..., f^p)$ is a vector on $S$ and $f_X \equiv (f^{D+1}, ..., f^d)$ is a vector on the internal space $X$. If we choose $f_S = 0$ and restrict the metric to depend only the coordinates on $S^p$; i.e., $g = g(X^1, ..., X^p)$, then (3.15) is satisfied.

In this paper we not only restrict our attention to the manifolds described in the previous paragraph, but we further simplify to a case involving a flat target space $M^D = \mathbb{R}^p \times X^{D-p}$ (3.17) where $X^{D-p}$ is a $(D-p)$-dimensional torus. This is done in the interest of stripping down the basic physics implied by supersymmetry central charges to its essence. In future work we intend to study the extra ramifications which follow from more general choices in the class of allowable backgrounds.

4 A class of toroidal compactifications

It is instructive to specialize to the following case. Restrict the target space to have topology $\mathbb{R} \times (S^1)^{D-1}$. Let $X^1 \in \mathbb{R}$ parameterize the non-compact dimension, and let $X^{i \neq 1} \in [0, 2\pi]$ describe one angular coordinate on each compact dimension. The compact dimensions are taken as circles having radii $R_i(X^1)$, which can depend independently on $X^1$. Accordingly, choose the metric

$$ds^2 = (dX^1)^2 + \sum_{i=2}^D R_i(X^1)^2 \left( dX^i \right)^2.$$  (4.1)

By convention, indices $i, j, k$ enumerate compact dimensions, whereas indices $m, n, p$ enumerate all dimensions. Thus, $i = 2, ..., D$, whereas $n = 1, 2, ..., D$. The class of metrics (4.1) describe a restricted class of toroidal compactification schemes in which the lattice describing the torus is orthogonal. (We generalize this to include a slightly more general class of lattices in the following section.) Furthermore, let the central charge functions $f^n(X)$ be constant real numbers defined by

$$( f^1, f^2, ..., f^d ) \equiv (0, \mu^2, \mu^3, ..., \mu^D).$$  (4.2)
Thus, the central charge is parameterized by one real number $\mu^i$ for each compact dimension. Following the procedure described in section 3, the action invariant under centrally extended supersymmetry is

$$ S = \int dt d\theta d\bar{\theta} \left\{ \frac{1}{2} D^{\dagger} V^1 D V^1 + \frac{1}{2} \sum_{i=2}^{D} R_i(V^1)^2 D^{\dagger} V^i D V^i + \frac{1}{2} \sum_{i=2}^{D} \left( i \mu^i (\theta^i D - \theta D^\dagger) V^i - (\mu^i)^2 \theta^i \theta \right) \right\} . \quad (4.3) $$

The first line in (4.3) describes an ordinary supersymmetric sigma model. The second line includes terms which extend the basic supersymmetry so as to switch on the desired central charge. The component Lagrangian corresponding to (4.3) is

$$ L = \frac{1}{2} \dot{X}^1 \dot{X}^1 - \frac{1}{2} i \psi^{1\dagger} \partial_t \psi^1 + \frac{1}{2} B^1 B^1 
+ \sum_{i=2}^{D} \left\{ R_i(X^1)^2 \left( \frac{1}{2} \dot{X}^i \dot{X}^i - \frac{1}{2} i \psi^{i\dagger} \partial_t \psi^i + \frac{1}{2} B^i B^i \right) 
+ 2 i R_i(X^1) R'_i(X^1) \psi^{[1\dagger, \psi^i]} \dot{X}^i 
+ R_i(X^1) R''_i(X^1) \left( \psi^{1\dagger} \psi^i B^i + \psi^i \dot{\psi}^i B^i - \dot{\psi}^{i\dagger} \psi^i B^1 \right) 
- \left( R_i(X^1) R''_i(X^1) + R'_i(X^1)^2 \right) \psi^{i\dagger} \psi^{1\dagger} \psi^1 \psi^1 
- i \mu^i R_i(X^1) R'_i(X^1) \left( \psi^1 \dot{\psi}^i + \psi^{1\dagger} \psi^{i\dagger} \right) - \frac{1}{2} (\mu^i)^2 R_i(X^1)^2 \right\} . \quad (4.4) $$

The action $S = \int dt L$ is invariant under the supersymmetry transformations (2.10) and also under the $(D - 1)$ independent transformations $\delta_Z X^i = \mu^i$.

The classically-conserved charges are obtained as follows. Under a supersymmetry transformation (2.10), we find $\delta Q L = K$, where

$$ K = \frac{1}{2} i \epsilon (\dot{X}^1 - i B^1) \psi^1 + \frac{1}{2} \sum_{i=2}^{D} i \epsilon R_i(X^1) (\dot{X}^i - i B^i) \psi^i 
- \sum_{i=2}^{D} \left( \epsilon R_i(X^1) R'_i(X^1) \psi^{i\dagger} \psi^{1\dagger} \psi^1 + \frac{1}{2} i \epsilon \mu^i R_i(X^1)^2 \psi^{i\dagger} \right) 
+ \text{h.c.} \quad (4.5) $$

The parameter-dependent supercharge, determined by the Noether procedure, is given by

$$ \tilde{Q} = \delta Q X^m P_m + \delta Q \psi^m \Pi_{\psi^m} + \delta Q \psi^{m\dagger} \Pi_{\psi^{m\dagger}} - K, \quad (4.6) $$
where \( P_m = g_{mn}(X) \dot{X}^n - i g_{m[n,
u]}(X) \psi^{n \dag} \psi^\nu \) is the momentum conjugate to \( X^m \), \( \Pi_{\psi^m} = -\frac{1}{2} i g_{mn} \psi^{n \dag} \) is the momentum conjugate to \( \psi^m \), and \( \Pi_{\psi^{m \dag}} = -\frac{1}{2} i g_{mn} \psi^n \) is the momentum conjugate to \( \psi^{m \dag} \). Now write \( \tilde{Q} = i \epsilon Q + i \epsilon^{\dag} Q^{\dag} \), which defines \( Q \) as the parameter-independent supercharge, with phase chosen as a matter of convention. In this way, we determine
\[
Q = P_m \psi^m + \sum_{i=2}^{D} \left( -i R_i(X^1) R^i_1(X^1) \psi^i \psi^{i \dag} + \mu^i R_i(X^1)^2 \psi^{i \dag} \right). \tag{4.7}
\]
The conserved central charge, which is determined similarly, is given by
\[
Z = \sum_{i=2}^{D} \mu^i P_i. \tag{4.8}
\]
In a similar way, one can compute the Noether Hamiltonian, defined as \( H = \dot{X}^m P_m + \dot{\psi}^m \Pi_{\psi^m} + \dot{\psi}^{m \dag} \Pi_{\psi^{m \dag}} - L \). After some algebra, one readily verifies that the expression determined in this way is the same as \( H = \frac{1}{2} \{ Q, Q^{\dag} \} \).

### 4.1 Quantization

The quantum operator algebra, obtained from the Dirac brackets associated with (4.4), is described by
\[
[P_m, X^n] = i \delta_m^n
\]
\[
\{ \psi^i, \psi^{i \dag} \} = 1
\]
\[
\{ \psi^i, \psi^{j \dag} \} = R_i^{-2} \delta^{ij}
\]
\[
[P_1, \psi^i] = -i R'_i R_i^{-1} \psi^i
\]
\[
[P_1, \psi^{i \dag}] = -i R'_i R_i^{-1} \psi^{i \dag}, \tag{4.9}
\]
where \( i = 2, ..., D \). Achieve this by writing \( P_m = i \partial_m \) and
\[
\psi^1 = \Gamma^1
\]
\[
\psi^i = \frac{1}{R_i} \Gamma^i, \tag{4.10}
\]
where \( \Gamma^M = (\Gamma^1, ..., \Gamma^D) \) are elements of a complex Clifford algebra. The coordinate index \( m \) is replaced by a tangent space index \( M \) by writing \( \psi^m = \Gamma^m \), where \( \{ \Gamma^m, \Gamma^{n \dag} \} = g^{mn}(X) \). The coordinate index \( m \) is replaced by a tangent space index \( M \) by writing \( \Gamma^m = \Gamma^M E_M^m \), where \( E_M^m = \text{diag}(1, R_1^{-1}, ..., R_D^{-1}) \) is an inverse vielbein. We have chosen a particular frame in writing (4.9). As a result, the target space transformation properties are not manifest in many of the expressions in this and also in the following section.
Thus, \( P_i \equiv \nu_i \in \mathbb{Z} \). Using these results, and after resolving a few ordering ambiguities, the quantum supercharge corresponding to (4.7) is found to be
\[
Q = i \partial_1 \Gamma^1 + \frac{1}{2} i \sum_{i=2}^{D} \frac{R'_i}{R_i} [\Gamma^i, \Gamma^{i\dagger}] \Gamma^1 + \sum_{i=2}^{D} \left( \frac{\nu_i}{R_i} \Gamma^i + \mu^i R_i \Gamma^{i\dagger} \right). \tag{4.11}
\]
Similarly, the quantum central charge is
\[
Z = \sum_{i=2}^{D} \mu^i \nu_i, \tag{4.12}
\]
and the Hamiltonian is \( H = \frac{1}{2} \{ Q, Q^\dagger \} \). The ordering ambiguities mentioned above are found in the fermion cubic term in \( Q \) and in the fermion quartic term in \( H \). After some determined algebra, one finds that these terms can be ordered so that \( Q^2 = Z \) and \( H = \frac{1}{2} \{ Q, Q^\dagger \} \). The result of this work is reflected in the particular ordering which appears in (4.11).

It proves illuminating to compute the components of the spin-connection on the target space, as explained in Appendix B. In doing so, one finds that the terms in (4.11) that are cubic in the \( \Gamma^M \)'s organize into spin connection pieces which, when combined with the ordinary derivatives appearing in \( Q \), form a spin covariant derivative. In this way, one finds that the expression for \( Q \) given in (4.11) organizes as
\[
Q = i \bar{\mathcal{D}} + \sum_{i=2}^{D} \mu^i R_i \Gamma^{i\dagger}, \tag{4.13}
\]
where \( \mathcal{D} \) is the spin-covariant derivative \(^8\). It is instructive to separate out the terms in this derivative that depend on \( \nu_i \), by re-writing (4.13) as
\[
Q = i \tilde{\mathcal{D}} + \sum_{i=2}^{D} \left( \frac{\nu_i}{R_i} \Gamma_i + \mu^i R_i \Gamma^{i\dagger} \right), \tag{4.14}
\]
where \( \tilde{\mathcal{D}} \) is the spin covariant derivative minus all terms which depend on \( \nu_i \). Written this way, a certain duality structure becomes manifest. Specifically, under the transformation
\[
\begin{align*}
\mu^i &\leftrightarrow \nu_i \\
R_i &\leftrightarrow \frac{1}{R_i}, \tag{4.15}
\end{align*}
\]
one finds \( Q \to \Omega^{\dagger} Q^{\dagger} \Omega \), where \( \Omega \) is a unitary operator which generates a \( \mathbb{Z}_2 \) parity operation as follows,
\[
\begin{align*}
\Omega \Gamma^i \Omega^{\dagger} &= \Gamma^{i\dagger} \\
\Omega \Gamma^{i\dagger} \Omega^{\dagger} &= \Gamma^i. \tag{4.16}
\end{align*}
\]
\(^8\)This is explained in detail in Appendix B where the computation is done quite explicitly in the case involving one compact dimension.
The Hamiltonian is given by $H = \frac{1}{2} \{ Q, Q^\dagger \}$. Under the above transformation, we have $H \to \tilde{H} = \Omega^\dagger H \Omega$. Since $H$ and $\tilde{H}$ are related by a unitary transformation, it follows that $H$ and $\tilde{H}$ are iso-spectral. Thus, the transformation (4.15) represents a duality.

5 SQM on $\mathbb{R} \times T^2$

In this section we generalize the results of the previous section to include a twist angle into the internal metric corresponding to compactification on a two-torus. Thus, we consider a target space having topology $\mathbb{R} \times T^2$. Parameterize the noncompact dimension using $X^1 \in \mathbb{R}$, and parameterize the $T^2$ factor using two angular variables $X^{2,3} \in [1, 2\pi]$. Characterize the two-torus using modular parameter

$$\tau = \frac{R_3}{R_2} e^{i\alpha},$$

(5.1)

where $R_2$ and $R_3$ are the radii of the circles corresponding to the respective coordinates $X^2$ and $X^3$, and $\alpha$ is an arbitrary phase. In this case, the target space metric is

$$ds^2 = (dX^1)^2 + R_2^2 (dX^2)^2 + 2 R_2 R_3 \cos \alpha dX^2 dX^3 + R_3^2 (dX^3)^2.$$  

(5.2)

This is the same as (4.1) in the case $D = 3$ except for the new off-diagonal term which manifests a non-trivial twist. It is convenient to define a complex coordinate $Y = X^2 + \tau X^3$ and also a scale factor $\phi(X^1)$ according to

$$R_2(X^1) \equiv e^{\phi(X^1)}.$$  

(5.3)

In the case where the modulus $\tau$ does not depend on $X^1$, the metric (5.2) is more concisely expressed as $ds^2 = (dX^1)^2 + e^{2\phi(X^1)} |dY|^2$. For the computational purposes used in this paper, we find (5.2) more convenient, however.

The supersymmetric action is the same as that given in section 4 plus new terms which correspond to the cross terms in the metric. Thus, the action is given by $S_{\text{old}} + S_{\text{new}}$ where $S_{\text{old}}$ is given in (4.3) and

$$S_{\text{new}} = \text{Re} \tau \int dt d\theta d\theta^\dagger e^{2\phi(V^1)} \left( D^\dagger V^2 D V^3 + i \mu^2 (\theta D - D^\dagger ) V^3 - \mu^2 \mu^3 \theta^\dagger \theta \right).$$

(5.4)

The component lagrangian is $L_{\text{old}} + L_{\text{new}}$, where $L_{\text{old}}$ is the lagrangian given in (4.4),
restricted to the case $D = 3$, and

$$L_{\text{new}} = 2 \Re \tau e^{2\phi(X^1)} \left\{ \frac{1}{2} \left( \dot{X}^2 \dot{X}^3 + B^2 B^3 - i \psi^{(2\dagger)} \psi^{(3)} + \dot{X}^{(2\dagger)} \psi^{(3)} \right) \right.$$

$$-i \phi'(X^1) \left( \dot{X}^{(2\dagger)} \psi^{(3)} \right) + \dot{X}^{(2\dagger)} \psi^{(3)} \right)$$

$$+\phi'(X^1) \left( B^{(2\dagger)} \psi^{(3)} + B^{(2\dagger)} \psi^{(3)} \right)$$

$$- \left( \phi''(X^1) + 3 \phi'(X^1)^2 \right) \psi^{(3)} \psi^{(2\dagger)} \psi^{(3)}$$

$$+i \phi'(X^1) \left( \mu^{(2\dagger)} \psi^{(2\dagger)} \psi^{(3)} \right)$$

$$- \mu^{(2\dagger)} \mu^{(3)} \right\}. \quad (5.5)$$

Since $L_{\text{new}}$ is supersymmetric, it follows that the supersymmetric variation of $(5.5)$ is a total derivative, i.e., $\delta Q L_{\text{new}} = \dot{K}_{\text{new}}$. Determined calculation yields

$$K_{\text{new}} = \Re \tau e^{2\phi(X^1)} \left( i \dot{X}^{(2\dagger)} \psi^{(3)} + B^{(2\dagger)} \psi^{(3)} - 2 \phi'(X^1) \psi^{(2\dagger)} \psi^{(3)} - i \mu^{(2\dagger)} \psi^{(3)} \right) + \text{h.c.} \quad (5.6)$$

The “new” contributions to the parameter-dependent supercharge $(4.6)$ are

$$\tilde{Q}_{\text{new}} = \delta Q \psi^{(\dagger)} \Pi^{(\text{new})}_{\psi^{(\dagger)}} + \delta Q \psi^{(\dagger)} \Pi^{(\text{new})}_{\psi^{(\dagger)}} - K_{\text{new}}. \quad (5.7)$$

where

$$\Pi^{(\text{new})}_{\psi^{(2\dagger)}} = \frac{1}{2} i \Re \tau e^{2\phi(X^1)} \psi^{(3\dagger)}$$

$$\Pi^{(\text{new})}_{\psi^{(2\dagger)} \psi^{(3\dagger)}} = \frac{1}{2} i \Re \tau e^{2\phi(X^1)} \psi^{(3\dagger)} \psi^{(2\dagger)}. \quad (5.8)$$

The parameter-independent supercharge $Q$ is defined via $\tilde{Q} = i \epsilon Q + i \epsilon^\dagger Q^\dagger$. In this way, we determine

$$Q_{\text{new}} = 2 \Re \tau e^{2\phi(X^1)} \left( -i \phi'(X^1) \psi^{(2\dagger)} \psi^{(3)} + \mu^{(2\dagger)} \psi^{(3)} \right). \quad (5.9)$$

The full supercharge is obtained by re-writing $Q_{\text{old}}$, given in $(4.7)$, in terms of the re-defined parameters $\tau$ and $\phi(X^1)$, and then adding the result to $Q_{\text{new}}$. Thus, the classically-conserved Noether supercharge is

$$Q = P_1 \psi^1 + P_2 \psi^2 + P_3 \psi^3$$

$$+ e^{2\phi} \left( -i \phi' \psi^{(2\dagger)} \psi^2 \psi^1 + \mu^2 \psi^{(2\dagger)} \right)$$

$$+ e^{2\phi} \left| \tau \right|^2 \left( -i \phi' \psi^{(3\dagger)} \psi^3 \psi^1 + \mu^3 \psi^{(3\dagger)} \right)$$

$$+ 2 e^{2\phi} \Re \tau \left( -i \phi' \psi^{(2\dagger)} \psi^3 \psi^1 + \mu^2 \psi^{(3\dagger)} \right). \quad (5.10)$$
In the quantum supercharge, the operator $P_1$ is replaced with $i\partial_1$, and the fermion cubic terms organize into spin connection pieces which, when combined with the ordinary derivatives appearing in $Q$, form a spin covariant derivative \(^9\). Furthermore, since $X^{2,3}$ are angular variables, it follows that the momenta $P_{2,3}$ are quantized as integers. Thus, we write $P_{2,3} \equiv \nu_{2,3} \in \mathbb{Z}$. Accordingly, the quantum supercharge is

$$Q = i\tilde{D} + \nu_2 \psi^2 + \nu_3 \psi^3$$

$$+ e^2(\mu^2 \psi^2 + |\tau|^2 \mu^3 \psi^3 + 2 \text{Re}\tau \mu^{(2)} \psi^3)$$ \hspace{1cm} (5.11)

where $\tilde{D}$ is the spin-covariant derivative minus the terms that include $\nu_{2,3}$. This is explained more completely in the following subsection.

### 5.1 Quantization

The quantum operator algebra, obtained from the Dirac brackets, is described by

$$[P_m, X^n] = i\delta_m^n$$

$$\{\psi^m, \psi^n\} = g^{mn}(X)$$

$$[P_m, P_n] = \frac{1}{4} i g^{pq} \partial_m g_{pq} \partial_n g_{qr} \psi^q \psi^r$$

$$[P_m, \psi^n] = -\frac{1}{2} i g^{np} \partial_m g_{pq} \psi^q, \hspace{1cm} (5.12)$$

where $g_{mn}(X)$ is the target space metric and $g^{mn}(X)$ is its inverse. This result is valid for any model described by (3.14). For the case at hand, the metric is

$$g_{mn} = e^{2\phi} \begin{pmatrix} e^{-2\phi} & 1 & \text{Re}\tau \\ 1 & \text{Re}\tau & |\tau|^2 \\ \text{Re}\tau & |\tau|^2 & 1 \end{pmatrix}.$$ \hspace{1cm} (5.13)

This can be written in terms of a dreibein $E_m^M$, defined by $g_{mn} = E_m^M E_n^N \delta_{MN}$. We can choose

$$E_m^M = e^\phi \begin{pmatrix} e^{-\phi} & 1 & 0 \\ 1 & 0 & \text{Re}\tau \\ 0 & \text{Re}\tau & \text{Im}\tau \end{pmatrix}, \hspace{1cm} (5.14)$$

in which case the inverse dreibein is

$$\tilde{E}_M^m = e^{-\phi} \begin{pmatrix} e^\phi & 1 & 0 \\ 1 & 0 & \text{Re}\tau (\text{Im}\tau)^{-1} \\ 0 & \text{Re}\tau (\text{Im}\tau)^{-1} & (\text{Im}\tau)^{-1} \end{pmatrix}.$$ \hspace{1cm} (5.15)

\(^9\)See Appendix B for details.
Using the metric (5.13), the quantum algebra (5.12) is given by

\[ [P_m, X^n] = i \delta_m^n \quad \{ \psi^1, \psi^{1\dagger} \} = 1 \]

\[ [P_m, P_n] = 0 \quad \{ \psi^2, \psi^{2\dagger} \} = (\text{Im} \tau)^{-2} |\tau|^2 e^{-2\phi} \]

\[ [P_1, \psi^2] = -i \phi'(X^1) \psi^2 \quad \{ \psi^3, \psi^{3\dagger} \} = (\text{Im} \tau)^{-2} e^{-2\phi} \]

\[ [P_1, \psi^3] = -i \phi'(X^1) \psi^3 \quad \{ \psi^3, \psi^{2\dagger} \} = -\text{Re} \tau (\text{Im} \tau)^{-2} e^{-2\phi}. \] (5.16)

In general, we can represent (5.12) by writing

\[ P_n = i \partial_n \quad \psi^m = \Gamma^M \tilde{E}_M{}^m, \]

where \( \Gamma^{1,2,3} \) are elements of a complex Clifford algebra \( \{ \Gamma^M, \Gamma^N\dagger \} = \delta^{MN} \), and \( \tilde{E}_M{}^m \) is the inverse vielbein. For the case at hand, \( \tilde{E}_M{}^m \) is given by (5.15), using which we determine

\[ \psi^1 = \Gamma^1 \]

\[ \psi^2 = e^{-\phi} \left( \Gamma^2 - \frac{\text{Re} \tau}{\text{Im} \tau} \Gamma^3 \right) \]

\[ \psi^3 = \frac{e^{-\phi}}{\text{Im} \tau} \Gamma^3. \] (5.17)

Substituting (5.17) into (5.11) we obtain after a small amount of algebra,

\[ Q = i \tilde{\Phi} + e^{-\phi} \left( \nu_2 \Gamma^2 + \frac{1}{\text{Im} \tau} (\nu_3 \text{Re} \tau - \nu_2) \Gamma^3 \right) + e^{\phi} \left( \mu^2 \text{Re} \tau \mu^3 \Gamma^2\dagger + \text{Im} \tau \mu^3 \Gamma^3\dagger \right). \] (5.18)

By using (5.14) and (5.15), we can re-write (5.18) as

\[ Q = i \tilde{\Phi} + \Gamma^M \tilde{E}_M{}^m \nu_m + \mu^m E_m^M \Gamma^1_M \]

\[ = i \tilde{\Phi} + \Gamma^M \nu_M + \mu^M \Gamma^1_M. \] (5.19)

It is gratifying that the quantum supercharge organizes into an object with manifest target-space transformation properties. The structure of (5.19) also suggests that there is quite likely a non-trivial generalization of the \( R_i \leftrightarrow 1/R_i \) duality encountered in the case of the \( (S^1)^{D-1} \) compactification described above. To investigate this, we will, in the next two sections, look at two classes of transformations which one can make in the case of the \( T^2 \) compactifications, each of which is codified as a transformation of the torus modulus \( \tau \). The first class of transformations describes the \( SL(2,\mathbb{Z}) \) modular group describing re-parameterizations of the torus. This set of transformations describes an expected symmetry group. The second class of transformations includes scale transformations. It is less clear from the basic considerations described in this paper that these should
comprise a symmetry although, as we will show, these do in fact describe a verifiable duality relationship.

Using the Clifford algebra, it is easy to show that the central charge operator, defined as \( Z = Q^2 \), is given by

\[
Z = \nu_m \mu^m. \tag{5.20}
\]

Note that this is proportional to the unit operator, and is therefore diagonal in any basis. Note that, based on developments to this point, the central charge \( Z \) is not subject to a quantization condition. This is because although the \( \nu_i \) are integers, there is no \textit{a priori} quantization condition on the permitted values of \( \mu^i \). However, as explained below in section 7 a \( \phi \to -\phi \) duality exists when \( \mu^i \) are quantized in units of \( |\tau|/\text{Im} \tau = 1/\sin \alpha \), where \( \alpha \) is the phase of \( \tau \).

6 Modular Transformations

It is interesting to consider the invariance properties of the \( \mathbb{R} \times T^2 \) model by computing what happens to the supercharge \( Q \) when the parameters describing the torus are modified. As is well known, these transformations are described by an \( SL(2, \mathbb{Z}) \) group of transformations which acts on the modular parameter \( \tau \). In our analysis we will also keep careful track of the overall size of our torus. This is facilitated by the real parameter \( \phi \), which may be chosen independently of the complex modulus \( \tau \). In this section we consider only transformations that preserve the scale of the torus. Consistency requires that the quantum theory is invariant under these. One purpose of this section is to demonstrate that this is so for size-preserving modular transformations on the \( T^2 \) factor in these compactification schemes. In the following section we will consider certain transformations which do change the size of the torus.

Consider, for example, the re-parametrization \( R_2 \leftrightarrow R_3 \), taken along with \( \alpha \to \pi - \alpha \). In terms of \( \phi, \text{Re} \tau \) and \( \text{Im} \tau \), this transformation is described by

\[
T : \quad \begin{align*}
\text{Re} \tau & \to -\frac{1}{|\tau|^2} \text{Re} \tau \\
\text{Im} \tau & \to \frac{1}{|\tau|^2} \text{Im} \tau \\
\phi & \to \phi + \ln |\tau|.
\end{align*} \tag{6.1}
\]

The transformation of \( \phi \) compensates for the scale change inherent in the \( \tau \) transformations, in such a way that the overall size of the torus is maintained. We then find that \( Q \),
given in (5.18), is invariant if we also take

\[ T : \begin{pmatrix} \nu_2 \\ \nu_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} \nu_2 \\ \nu_3 \end{pmatrix} \]

\[ \begin{pmatrix} \mu^2 \\ \mu^3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} \mu^2 \\ \mu^3 \end{pmatrix} \]

\[ \begin{pmatrix} \Gamma^2 \\ \Gamma^3 \end{pmatrix} \rightarrow \frac{1}{|\tau|} \begin{pmatrix} \text{Re} \tau & \text{Im} \tau \\ -\text{Im} \tau & \text{Re} \tau \end{pmatrix} \begin{pmatrix} \Gamma^2 \\ \Gamma^3 \end{pmatrix}. \] (6.2)

It is reassuring that we can find a transformation on \( \mu^i, \nu_i \) and \( \Gamma^M \) which, in conjunction with \( R_2 \leftrightarrow R_3, \alpha \rightarrow \pi - \alpha \) leaves \( H \) invariant, since this describes nothing more than a re-labelling of the coordinates on the \( T^2 \).

Next consider the transformation obtained by simply adding \( 2\pi \) to the twist angle. This is given by

\[ S : \begin{array}{ccc}
\text{Re} \tau & \rightarrow & \text{Re} \tau + 1 \\
\text{Im} \tau & \rightarrow & \text{Im} \tau \\
\phi & \rightarrow & \phi
\end{array} \] (6.3)

Then \( Q \) is invariant if we also take

\[ S : \begin{pmatrix} \nu_2 \\ \nu_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \nu_2 \\ \nu_3 \end{pmatrix} \]

\[ \begin{pmatrix} \mu^2 \\ \mu^3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu^2 \\ \mu^3 \end{pmatrix} \]

\[ \begin{pmatrix} \Gamma^2 \\ \Gamma^3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma^2 \\ \Gamma^3 \end{pmatrix}. \] (6.4)

Again, it is reassuring that we can find a transformation on \( \mu^i, \nu_i \) and \( \Gamma^M \) which, in conjunction with (6.3) leaves \( H \) invariant, since this latter transformation is nothing more than a re-parametrization of the \( T^2 \).

Taken together, the \( T \) and \( S \) transformations described above generate the group \( SL(2, \mathbb{Z}) \). The generating transformations on the complex modulus and on the scale factor are

\[ T : \ \tau \rightarrow -\frac{1}{\tau} \quad \phi \rightarrow \phi + \ln |\tau| \]

\[ S(n) : \ \tau \rightarrow \tau + n \quad \phi \rightarrow \phi. \] (6.5)
where \( n \in \mathbb{Z} \). A generic action is obtained by considering
\[
S(\frac{b+1}{d})T S(d) T S(\frac{1-c}{d})T,
\]
where \( b, c, d \in \mathbb{Z} \). Applying these operations right to left on \( \tau \) we obtain
\[
\tau \to \frac{a \tau + b}{c \tau + d},
\]
where \( ad - bc = 1 \). Now applying using the same sequence of transformations, using the matrices appearing in (6.4), we obtain
\[
\nu_m \to \nu_n (M^{-1})^n m
\]
\[
\mu^m \to (M)^m n \mu^n
\]
where
\[
(M)^m n = \begin{pmatrix}
-a & b \\
c & -d
\end{pmatrix}.
\]
Notice that the central charge \( Z = \nu_m \mu^n \) is \( SL(2, \mathbb{Z}) \) invariant.

We have shown that when these transformations arise from a re-parametrization of the \( T^2 \), they do not alter the theory. This provides a useful consistency check, since a mere re-parametrization cannot change the physics.

## 7 Scale Transformations

Consider the scale transformation, \( R_i \to R_i^{-1} \), taken along with \( \alpha \to \pi - \alpha \). In terms of the complex modulus and the scale factor, this transformation is described by
\[
T : \begin{align*}
\text{Re} \, \tau & \to -\frac{1}{|\tau|^2} \text{Re} \, \tau \\
\text{Im} \, \tau & \to \frac{1}{|\tau|^2} \text{Im} \, \tau \\
\phi & \to -\phi.
\end{align*}
\]
This transformation acts the same way on \( \tau \) as the \( T \) transformation given in (6.1), but acts differently on \( \phi \). This transformation is more interesting than the \( T \) transformation, however, since it exchanges a “small” torus with a “large” torus, rather than merely re-parameterizing the same torus. If we apply the transformations to the supercharge \( Q \), given in (5.18), we find that the supercharge is mapped to its Hermitian conjugate \( Q \to Q^\dagger \), provided we simultaneously transform the parameters \((\nu_i, \mu^i)\) and the elements
of the Clifford algebra according to
\[
\begin{pmatrix}
\nu_{2,3} \\
\mu^{2,3}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
|\tau|^{-1} \text{Im} \tau
\end{pmatrix}
\begin{pmatrix}
\nu_{2,3} \\
\mu^{2,3}
\end{pmatrix}
\]
\[
\begin{pmatrix}
\Gamma^2 \\
\Gamma^3
\end{pmatrix}
\rightarrow
\frac{1}{|\tau|}
\begin{pmatrix}
\text{Im} \tau & -\text{Re} \tau \\
\text{Re} \tau & \text{Im} \tau
\end{pmatrix}
\begin{pmatrix}
\Gamma^2 \\
\Gamma^3
\end{pmatrix},
\tag{7.2}
\]
along with \(\Gamma^1 \rightarrow \Gamma^{1\dagger}\). In the case of an orthogonal lattice, where the torus modulus is purely imaginary \(\tau = i |\tau|\), the transformation (7.2) is the same as (4.15).

The Hamiltonian is given by \(H = \frac{1}{2} \{ Q, Q^\dagger \}\). Therefore, under the transformation given by (7.1) and (7.2), which induces \(Q \rightarrow \Omega^\dagger Q^\dagger \Omega\), we have \(H \rightarrow \tilde{H} = \Omega^\dagger H \Omega\), where \(\Omega\) generates the \(\mathbb{Z}_2\) parity automorphism of the Clifford algebra described by the transformations of \(\Gamma^1, \Gamma^2, \Gamma^3\). Since \(H\) and \(\tilde{H}\) are related by a unitary transformation, it follows that \(H\) and \(\tilde{H}\) are iso-spectral, and that the transformation represents a duality. Notice also that the supersymmetry central charge \(Z = \nu_2 \mu^2 + \nu_3 \mu^3\) is invariant under this duality transformation.

### 7.1 Central Charge Quantization

Since \(\nu_{1,2} \in \mathbb{Z}\) it follows from (7.2) that the existence of a \(\phi(X^1) \rightarrow -\phi(X^1)\) duality is contingent upon a quantization of \(\mu^{1,2}\) as well. In particular, the duality requires
\[
\frac{\text{Im} \tau}{|\tau|} \mu^{2,3} \in \mathbb{Z}.
\tag{7.3}
\]
Thus, \(\mu^{2,3}\) are quantized in units of \(1/\sin \alpha\), where \(\alpha\) is the phase of the complex modulus \(\tau\). In the case described in section 4 where \(\alpha = \pi/2\), the duality is present only if \(\mu^{2,3}\) are integers. In more general \(T^2\) compactifications, the presence of our \(\phi \rightarrow -\phi\) duality implies that \(Z\) quantization is correlated with the phase of the complex modulus \(\tau\).

In summary, provided the parameters \(\mu^{2,3}\) are quantized according to (7.3), it follows that under the “large” \(\leftrightarrow\) “small” torus transformation given by
\[
\begin{align*}
\tau & \rightarrow -\frac{1}{\tau} \\
\phi & \rightarrow -\phi,
\end{align*}
\tag{7.4}
\]
the quantized charge operators transform according to
\[
\begin{align*}
H & \rightarrow \Omega^\dagger H \Omega \\
Q & \rightarrow \Omega^\dagger Q^\dagger \Omega \\
Z & \rightarrow \Omega^\dagger Z \Omega.
\end{align*}
\tag{7.5}
\]
where $\Omega$ generates a $Z_2$ parity. We are certain that this structure generalizes to much more general compactification schemes. The examples described in this paper provide the simplest examples of a more pervasive phenomenon which we hope to address more fully in the near future.

8 Conclusions

We have shown explicitly how non-trivial supersymmetry central charges are naturally incorporated into quantum mechanical sigma models as background vector fields. We have explained how these vector fields are constrained along with the target space metric so as to satisfy a particular set of coupled differential equations. We have explicitly quantized models having target-space topology $\mathbb{R} \times (S^1)^{D-1}$ and others with topology $\mathbb{R} \times T^2$. In the second class of models we have proven the quantum invariance under $SL(2, \mathbb{Z})$ modular transformations that preserve the size of the $T^2$ factor. In both cases, we have shown the existence of a $Z_2$ duality that equates models with “large” compact space with ostensibly distinct models having “small” compact spaces.

The emergence of $T$-duality in the manner demonstrated in this paper might be construed as an obvious manifestation of known dualities in string theory. Although we are fairly certain that the two classes of phenomena are intimately related, we also believe that making a firm connection between $T$-duality in string theory and $T$-duality in quantum mechanics is not as trivial an exercise as it might superficially seem. For instance, by dimensionally-reducing a two-dimensional sigma model, one degenerates the length of the string to zero size. This operation requires that the size of any internal cycle which the string wraps also degenerates. However, the appearance of $T$-dualities in supersymmetric quantum mechanics is insensitive to the size of these cycles. We think it would be interesting to explain the quantization of the parameters $\mu^i$ in terms of the topological quantization of winding modes in string theory, and plan to address this in a future paper.

As mentioned in the introduction, we hope, among other things, to use the constructions in this paper as a basis for further elucidating the geometric or topological meaning of shape invariance. Typically, shape invariance is explained in terms of an algebraic relationship connecting superpotentials in otherwise distinct sectors of extended models. It is possible to use the sigma models described in this paper to describe precisely these sorts of extended models. One way to do this is to choose a particular matrix representation for the $\Gamma^M$ operators which appear in our models. If one diagonalizes the Hamiltonian, then this delineates a multiplicity of sectors, each of which has its own superpotential. This is readily accomplished for the $\mathbb{R} \times (S^1)^{D-1}$ models and $\mathbb{R} \times T^2$ models which we
have presented. The form of these superpotentials is determined by the choice of the function $R_i(X^1)$. The functions $R_i(X^1)$ can be tuned to provide shape invariant quantum mechanics as an effective theory. In these constructions, the shape transformation is realized geometrically. But it is not known how the requirement of shape invariance is realized as a specific geometric or topological restriction on the background. We think this is an interesting problem, and feel that our sigma model constructions should provide a powerful context for probing a more fundamental explanation for shape invariance.

Shape invariance is but one application we see for the ideas in this paper. Indeed, the constructions developed in this paper are sufficiently basic that we anticipate that they might prove useful in a variety of problems in physics. For instance, in [13] an operation called automorphic duality is introduced which appropriates the notion of Hodge duality into the context of quantum mechanics. It is found that this operation can be performed only on models which exhibit target space isometries. We have shown in this paper that this is precisely the condition needed to include a supersymmetry central charge vector into the background. Since we have also shown that these background fields imply interesting target-space dualities, our work implies a basic connection between worldline automorphic duality and nontrivial target space dualities. We think that this, and related issues, are worthy of further study.

A Superfield Conventions

In this paper we have used a $d = 1$ $N = 1$ superspace $^{10}$, where the $N = 1$ implies that there is one complex anti-commuting coordinate $\theta$. A general, unconstrained superfield is therefore described by

$$S = A + i \theta \psi + i \theta^\dagger \lambda + \theta^\dagger \theta C$$  \hspace{1cm} (A.1)

where $A$ and $C$ are independent complex commuting component fields, and $\psi$ and $\lambda$ are independent complex anti-commuting component fields. Thus, this superfield describes $4+4$ off-shell degrees of freedom. The basic supercharge operator $Q_0$ and the superspace

---

$^{10}$We refer to the smallest $d = 1$ superspace, having one real anti-commuting coordinate as an $N = 1/2$ superspace.
derivatives $D$ are described by

$$
Q_0 = \frac{\partial}{\partial \theta} - i \theta^\dagger \partial_t \quad D = \frac{\partial}{\partial \theta} + i \theta^\dagger \partial_t
$$

$$
Q_0^\dagger = \frac{\partial}{\partial \theta^\dagger} - i \theta \partial_t \quad D^\dagger = \frac{\partial}{\partial \theta^\dagger} + i \theta \partial_t.
$$

We distinguish the basic supercharge operator $Q_0$ from the harmonic supercharge operator $Q$ defined in (2.4) by the subscript 0. The operator $Q_0$ generates a basic supersymmetry transformation on component fields via the superspace operation

$$
\delta^{(0)}_Q (\epsilon) = \epsilon Q_0 + \epsilon \dagger Q_0^\dagger.
$$

In the main text, we have constructed sigma models involving $D$ real superfields $V^1, ..., V^D$, where the reality of the superfield implies $V^n = V^n^\dagger$. These are defined by

$$
V^n = X^n + i \theta \psi^n + i \theta^\dagger \psi^{n^\dagger} + \theta^\dagger \theta B^n,
$$

where $X^n$ and $B^n$ are real-valued component fields and $\psi^n$ are complex anti-commuting component fields. Thus, each real multiplet has $2 + 2$ off-shell degrees of freedom, and describes one type of irreducible multiplet. An arbitrary differentiable function involving the $V^n$ is given by

$$
F(V) = F(X) + i \theta F_n(X) \psi^n + i \theta^\dagger F_n(X) \psi^{n^\dagger} + \theta^\dagger \theta \left( F_n(X) B^n + F_{mn}(X) \psi^{m^\dagger} \psi^n \right)
$$

where subscripts on $F_n(V)$ on $F_{mn}(V)$ denote derivatives, e.g. $F_n(X) = \partial F(X)/\partial X^n$. Using the operators $D$ and $D^\dagger$ defined in (A.2), it is simple to compute

$$
DV^n = i \psi^n + i \theta^\dagger \left( \dot{X}^n + i B^n \right) - \theta^\dagger \theta \dot{\psi}^n
$$

$$
D^\dagger V^n = i \psi^{n^\dagger} + i \theta \left( \dot{X}^n - i B^n \right) + \theta^\dagger \theta \dot{\psi}^{n^\dagger}
$$

Note that $(DV^n)^\dagger = - (D^\dagger V^n)$. This explains, for instance, why certain superspace expressions appearing in the main text, such as (3.1), are real.

---

11 It should not cause a problem that we have used $D$ for the target space dimensionality and also for the superspace derivatives; the distinction is naturally clear from the contexts in which it is used!

12 As a simple example of (A.4), it might be useful to exhibit a quadratic expression,

$$
V^m V^n = X^m X^n + 2 \theta X^{(m} \psi^{n)} + 2 \theta^\dagger X^{(m} \psi^{n^\dagger)} + \theta^\dagger \theta \left( X^{(m} B^{n)} + \psi^{(m^\dagger} \psi^{n)} \right).
$$
A.1 Technique

Under a basic supersymmetry transformation, i.e., one which does not include the central modifications, a superfield $S$ transforms as

$$\delta^{(0)}_Q (\epsilon) S = (\epsilon Q_0 + \epsilon^\dagger Q_0^\dagger) S,$$

where $Q_0 = \partial/\partial \theta - i \theta^\dagger \partial_\theta$. It is straightforward to prove

$$\delta^{(0)}_Q \int dt d\theta d\theta^\dagger (-\theta S) = \int dt d\theta d\theta^\dagger (\epsilon S), \tag{A.6}$$

$$\delta^{(0)}_Q \int dt d\theta d\theta^\dagger (-\theta^\dagger S) = \int dt d\theta d\theta^\dagger (\epsilon^\dagger S).$$

This is easily checked in terms of components. These expressions are valid for any superfield $S$, irrespective of whether $S$ satisfies a reality constraint or any other constraint. It is useful to reverse this argument. If we are looking for a particular superspace expression $\Gamma$ with the property $\delta^{(0)}_Q \Gamma = \int dt d\theta d\theta^\dagger \epsilon \theta S$, then (A.6) solves this problem for us. We see that $\Gamma$ is obtained by simply replacing $\epsilon$ with $-\theta$. In other words, $\Gamma = \int dt d\theta d\theta^\dagger (-\theta S)$. Similarly, if we seek a superfield expression $\Gamma'$ with the property $\delta^{(0)}_Q \Gamma' = \int dt d\theta d\theta^\dagger \epsilon^\dagger S'$, then a similar argument tells us $\Gamma' = \int dt d\theta d\theta^\dagger (-\theta^\dagger S')$. As a general rule, given a superspace integral which is linear in $\epsilon$ or $\epsilon^\dagger$, we obtain an expression whose basic supersymmetry variation produces this expression by replacing $\epsilon$ with $-\theta$ or $\epsilon^\dagger$ with $-\theta^\dagger$.

There is an interesting corollary to this method. Suppose we seek superspace expression $\Gamma''$ with the property $\delta^{(0)}_Q \Gamma'' = \int dt d\theta d\theta^\dagger \theta \epsilon S''$. Using our technique, we obtain, by replacing $\epsilon$ with $\theta$, an expression proportional to $\theta^2$, which vanishes identically since $\theta$ is anti-commuting. We conclude that there is no solution to this particular problem; that is, the expression $\int dt d\theta d\theta^\dagger \theta \epsilon S''$ does not describe a basic supersymmetry variation. Of course, not every superspace expression linear in $\epsilon$ and $\epsilon^\dagger$ is itself a supersymmetry variation of another; this is one example. Using a similar argument, we see that $\int dt d\theta d\theta^\dagger (\epsilon \theta^\dagger + \epsilon^\dagger \theta) S$ is not a supersymmetry variation, since replacing $\epsilon$ with $\theta$ and $\epsilon^\dagger$ with $\theta^\dagger$ produces an expression proportional to the combination $(\theta \theta^\dagger + \theta^\dagger \theta)$, which also vanishes identically! These comments explain some of the technique used in section 3.

B Target-space spin structure

In this appendix we assemble some useful relationships pertaining to the target space geometry associated with the states in the models described in the main text. We have considered particles propagating in $D$-dimensional Euclidean space. Accordingly, the local structure group is $SO(D)$. Denote coordinate indices using small Latin letters ($m, n, ...$), structure group indices using capital Latin letters ($M, N, ...$), and spin indices using small Greek letters ($\alpha, \beta, ...$).
Derivatives covariant with respect to coordinate transformations are given by $\nabla_m V_n = \partial_m V_n - \Gamma_m{}^{l} V_l$, where $\Gamma_m{}^{l}$ is the affine connection. Derivatives covariant with respect to structure group transformations are $D_m = \partial_m + \frac{1}{2} \omega_m{}^{MN} \mathcal{O}_{MN}$, where $\omega_m{}^{MN}$ is the spin connection and $\mathcal{O}_{MN}$ are $SO(D)$ generators. The spin connection is given by

$$\omega_m{}^{MN} = E_m{}^P \left( \Omega_{MN,P} - 2 \Omega_{PMN} \right) \tag{B.1}$$

where $E_m{}^M$ is the vielbein, with inverse $\tilde{E}^M_m$, and $\Omega_{MN,P}$ is the object of holonomy, given by

$$\Omega_{MN,P} = -\tilde{E}^m_{[M} \tilde{E}^n_{N]} \partial_m E_n{}^P. \tag{B.2}$$

In particular, a spin-covariant derivative is given by

$$D_m \psi_\alpha = \partial_m \psi_\alpha + \frac{1}{2} \omega_m{}^{MN} (\Sigma_{MN})_{\alpha}{}^{\beta} \psi_\beta, \tag{B.3}$$

where $\Sigma_{MN}$ generates $SO(D)$ in a spinor representation.

Complex world-line fermions take values in a complex Clifford algebra, and transform according to a reducible spinor representation of the target space structure group $SO(D)$. Accordingly, under an $SO(D)$ transformation these transform according to

$$\Psi_{\alpha} \rightarrow \left( e^{\frac{i}{2} \theta^{MN} \Delta_{MN}} \right)_{\alpha}{}^{\beta} \Psi_{\beta}, \tag{B.4}$$

where $\theta^{MN}$ is a real antisymmetric matrix of parameters and $\Delta_{MN}$ are the $SO(D)$ generators in the particular reducible spinor representation given by

$$\Delta_{MN} = \Gamma_M \Gamma_N - \Gamma_N \Gamma_M, \tag{B.5}$$

where $\Gamma_M$ are elements of the complex Clifford algebra defined by $\{ \Gamma_M, \Gamma_N \} = \delta_{MN}$ and by $\{ \Gamma_M, \Gamma_N \} = \{ \Gamma^\dagger_M, \Gamma^\dagger_N \} = 0$. It is straightforward to prove, using the Clifford algebra, that the generators properly represent the $SO(D)$ algebra,

$$[\Delta_{MN}, \Delta^{OP}] = -\delta_{MO} \Lambda_{NP} + \delta_{MP} \Lambda_{NO} - \delta_{NO} \Lambda_{MP} + \delta_{NP} \Lambda_{MO}. \tag{B.6}$$

One can easily show that $-\Lambda^\dagger_{MN}$ form another reducible representation the same algebra.\footnote{By way of comparison, real worldline spinors would transform according to a smaller $SO(D)$ representation described by the generators $\Sigma_{MN} = \frac{1}{4} [\Gamma_M, \Gamma_N]$, where $\Gamma_M = \Gamma^\dagger_M$ are elements of a real Clifford algebra $\{ \Gamma_M, \Gamma_N \} = 2 \delta_{MN}$.}
B.1 A Simple Example

Consider a two-dimensional manifold with topology $\mathbb{R} \times S^1$. Parameterize the non-compact dimension using $X^1 \in \mathbb{R}$ and the compact dimension using an angular variable $X^2 \in [0, 2\pi]$. Choose metric $ds^2 = (dX^1)^2 + R(X^1)^2 (dX^2)^2$, where $R(X^1)$ describes the radius of the compact dimension. In this case, a possible zweibein is

$$E_m^M = \begin{pmatrix} 1 \\ R(X^1) \end{pmatrix}.$$  \hspace{1cm} (B.7)

One then computes the spin connection

$$\omega_{12}^1 = 0$$
$$\omega_{12}^2 = -R'.$$ \hspace{1cm} (B.8)

Since $X^2$ is an angle, it follows that $\partial_2$ is quantized according to $i \partial_2 = \nu_2 \in \mathbb{Z}$. In this case, the spin covariant derivatives are

$$D_1 = \partial_1$$
$$D_2 = \partial_2 + \frac{1}{2} \omega_2^{MN} \Sigma_{MN}$$
$$= \partial_2 + \omega_2^{12} \Sigma_{12}$$
$$= -i \nu_2 - R' \left( \Gamma_1 \Gamma_2^\dagger - \Gamma_2 \Gamma_1^\dagger \right)$$
$$= -i \nu_2 - R' \Gamma_1 \Gamma_2^\dagger + R' \Gamma_2 \Gamma_1^\dagger,$$ \hspace{1cm} (B.9)
where spin indices have been suppressed. Thus,

\[ \mathcal{P} = g^{mn} \mathcal{D}_m \Gamma_n \]

\[ = g^{mn} E_n \mathcal{D}_m \Gamma_N \]

\[ = g^{11} E_1 \mathcal{D}_1 \Gamma_1 + g^{22} E_2 \mathcal{D}_2 \Gamma_2 \]

\[ = \mathcal{D}_1 \Gamma_1 + \frac{1}{R^2} R \mathcal{D}_2 \Gamma_2 \]

\[ = \mathcal{D}_1 \Gamma_1 + \frac{1}{R} \mathcal{D}_2 \Gamma_2 \]

\[ = \partial_1 \Gamma_1 + \frac{1}{R} \left( -i \nu_2 - R' \Gamma_1 \Gamma_2^\dagger + R' \Gamma_2 \Gamma_1^\dagger \right) \Gamma_2 \]

\[ = \partial_1 \Gamma_1 + \frac{R'}{R} \left( - \Gamma_1 \Gamma_2^\dagger \Gamma_2 - \Gamma_2 \Gamma_1^\dagger \Gamma_2 \right) - i \frac{\nu_2}{R} \Gamma_2 \]

\[ = \partial_1 \Gamma_1 - \frac{R'}{R} \Gamma_1 \Gamma_2^\dagger \Gamma_2 - i \frac{\nu_2}{R} \Gamma_2 \]

\[ = \partial_1 \Gamma_1 + \frac{1}{2} \frac{R'}{R} [\Gamma_2, \Gamma_2^\dagger] \Gamma_1 - i \frac{\nu_2}{R} \Gamma_2, \quad (B.10) \]

where we have used the Clifford algebra, including the relationship \( \Gamma_2^2 = 0 \). Thus,

\[ i \mathcal{P} = i \partial_1 \Gamma_1 + \frac{1}{2} i \frac{R'}{R} [\Gamma_2, \Gamma_2^\dagger] \Gamma_1 + \frac{\nu_2}{R} \Gamma_2. \quad (B.11) \]

This is precisely the relationship which allows us to re-write the expression for \( Q \) appearing in (4.11) in the manner shown in (4.13).

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