A second-order accurate scheme for two-dimensional space fractional diffusion equations with time Caputo-Fabrizio fractional derivative

Jiankang Shi · Minghua Chen

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Abstract We provide and analyze a second order scheme for the model describing the functional distributions of particles performing anomalous motion with exponential Debye pattern and no-time-taking jumps eliminated, and power-law jump length. The model is derived in [M. Chen, J. Shi, W. Deng, arXiv:1809.03263], being called the space fractional diffusion equation with the time Caputo-Fabrizio fractional derivative. The designed schemes are unconditionally stable and have the second order global truncation error with the nonzero initial condition, being theoretically proved and numerically verified by two methods (a prior estimate with \(L^2\)-norm and mathematical induction with \(l_\infty\) norm). Moreover, the optimal estimates are obtained.

Keywords Caputo-Fabrizio fractional derivative · Stability and convergence · Optimal estimates

1 Introduction

The Caputo-Fabrizio fractional derivative [9] has been used to model a variety of applied scientific phenomena, such as control systems [4], physics [1, 6, 8], medicine [27], fluid dynamics [2, 5, 22, 26]. It is able to describe the material heterogeneities and the fluctuations of different scales. Based on the continuous time random walk with exponential Debye pattern and no-time-taking jumps eliminated, and power-law jump length, i.e., taking the waiting time distribution is \(\sigma\left(1 + \sigma - \delta(t)\right)\exp(-\sigma t)\), \(\sigma = \gamma/(1 - \gamma)\) and the jump length distribution is \(|x|^{-1-\alpha}\), we derive the following space fractional diffusion equation with the time Caputo-Fabrizio fractional operator [13]

\[
\begin{cases}
\frac{C^\gamma_0 D_i^\alpha u(x,y,t)}{} = \frac{\partial^\alpha u(x,y,t)}{\partial|x|^\alpha} + \frac{\partial^\beta u(x,y,t)}{\partial|y|^\beta} + f(x,y,t), \\
u(x,y,0) = u_0(x,y) \quad \text{for} \quad (x,y) \in \Omega, \\
u(x,y,t) = 0 \quad \text{for} \quad (x,y,t) \in \partial\Omega \times [0,T].
\end{cases}
\]
on a finite rectangular domain $\Omega = (0, x_R) \times (0, y_R)$ and $0 < t \leq T$. The Caputo-Fabrizio fractional derivative, for $0 < \gamma < 1$, is defined by \[9, 13\]

\[
\begin{align*}
\frac{\mathrm{CF}D_t^\gamma u(t)}{\partial t} &= \frac{1}{1-\gamma} \int_0^t u'(s) e^{-\frac{s-t}{\gamma}} ds - \frac{1}{1-\gamma} \int_0^t u'(s) e^{-\frac{s-t}{\gamma^\alpha}} ds, \quad 0 < \gamma < 1,
\end{align*}
\]

The Riesz fractional derivative is given in \[23\]

\[
\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} = \kappa_{\alpha} (\partial D_x^\alpha + \partial D_y^\alpha) u(x,t), \quad \kappa_{\alpha} = \frac{1}{2\cos(\alpha\pi/2)} > 0, \quad 1 < \alpha < 2
\]

In recently years, numerical method for solving the Caputo-Fabrizio fractional derivative \[9\] is experiencing rapid development. For example, the new operational matrix together with Tau method has been used to solve the equation with Caputo-Fabrizio operator \[20\]. Numerical approach of Fokker-Planck equation with Caputo-Fabrizio fractional derivative is discussed in \[16\]. A second-order Crank-Nicolson scheme \[28\] of the time fractional Caputo-Fabrizio derivative with $1 < \gamma < 2$ is proposed in \[19\]; and the stability analysis of the numerical scheme for the groundwater model with Caputo-Fabrizio operator are proven in \[15\]. Using the Lubich’s operator \[21\] and the discretized fractional substantial calculus \[10, 11\], the stability of the second-order scheme for Caputo-Fabrizio fractional equation are proved \[13\] by a priori estimate \[17\] under the zero initial condition. Based on the idea of L1 formula \[18, 23\], a numerical approximation to the Caputo-Fabrizio derivative by using a linear interpolation is provided \[13\]; and the first-order convergence analyse are given in \[17\]. It seems that achieving a second-order accurate scheme for L1 formula is not an easy task. This paper focused on providing effective and a second-order accurate scheme for \[14\]. Under the nonzero initial condition, the numerical stability and convergence of the L1 scheme with second-order accuracy are theoretically proved by two ways (a priori estimate with $L^2$-norm and mathematical induction with $L_{\infty}$-norm); and the optimal estimates are obtained.

The paper is organized as follows. In Section 2, we provide the approximation scheme to the Caputo-Fabrizio fractional derivative, and the full discretization of \[14\]. In Section 3, the unconditionally stability and convergence of the numerical schemes are proved in detail. In Section 4, we use the numerical example to verify the unconditionally stability and the convergence order of the difference schemes. Finally, we conclude the paper with some remarks.

### 2 Discretization schemes

Let the mesh points $x_i = i \Delta x$, $i = 0, 1, 2, \ldots, N$, $y_j = j \Delta y$, $j = 0, 1, 2, \ldots, N$, and $t_n = n \tau$, $n = 0, 1, 2, \ldots, N$, where $\Delta x = \frac{x_R}{N}$, $\Delta y = \frac{y_R}{N}$ and $\tau = \frac{T}{N}$ are the uniform space stepsize and time steplength, respectively. Denote $u^n_{i,j}$ as the numerical approximation to $u(x_i, y_j, t_n)$ and $f^n_{i,j} = f(x_i, y_j, t_n)$. ...
2.1 Discretized Caputo-Fabrizio fractional derivative

In this subsection we provide a second-order discretization L1 formula for the Caputo-Fabrizio fractional derivative, although there is less than the second-order convergence for \( t \geq 0 \).

**Lemma 2.1** Let \( 0 < \gamma < 1 \) with \( \sigma = \frac{1}{\gamma} \). Let \( u(t) \) be sufficiently smooth for \( t \geq 0 \). Then

\[
\frac{\text{CF}}{0} D_t^{\gamma} u(t_h) = \frac{1}{1-\gamma} \sum_{k=1}^{n} \frac{u(t_k) - u(t_{k-1})}{\sigma \tau} e^{-\sigma(n-k)\tau} (1 - e^{-\sigma \tau}) + O(\tau^2).
\]

**Proof** We can rewrite (1.2) as

\[
\frac{\text{CF}}{0} D_t^{\gamma} u(t_h) = \frac{1}{1-\gamma} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} u'(s) e^{-\sigma(n-s)} ds = \frac{1}{1-\gamma} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} u'(s) e^{-\sigma(n-s)} ds + r^0_t
\]

\[
= \frac{1}{1-\gamma} \sum_{k=1}^{n} \frac{u(t_k) - u(t_{k-1})}{\tau} \int_{t_{k-1}}^{t_k} e^{-\sigma(n-s)} ds + r^0_t
\]

Here \( r^0_t = -L_1 - L_2 \) with

\[
I_1 = \frac{1}{1-\gamma} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} u''(s) \left(t_k + t_{k-1} - 2s\right) e^{-\sigma(n-s)} ds
\]

\[
I_2 = \frac{1}{1-\gamma} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} u'''(s) \left(t_k + t_{k-1} - 2s\right) e^{-\sigma(n-s)} ds + r^0_t
\]

Next we shall estimate \( r^0_t = O(\tau^2) \). According to the first equation of (2.1), the first mean value theorem for definite integrals and Taylor series expansion, we have

\[
I_1 = \frac{1}{1-\gamma} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} u''(s) \left(t_k + t_{k-1} - 2s\right) e^{-\sigma(n-s)} ds
\]

\[
+ \frac{1}{1-\gamma} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} u''(s) \cdot \left(t_k + t_{k-1} - 2s\right) e^{-\sigma(n-s)} ds
\]

\[
= \frac{1}{1-\gamma} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} u''(s) \cdot \left(t_k + t_{k-1} - 2s\right) e^{-\sigma(n-s)} ds
\]

\[
= I_{11} + I_{12},
\]

where

\[
I_{11} = \frac{1}{2(1-\gamma)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_k + t_{k-1} - 2s) e^{-\sigma(n-s)} ds
\]

\[
I_{12} = \frac{1}{2(1-\gamma)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_k + t_{k-1} - 2s) e^{-\sigma(n-s)} ds
\]
with $\eta_{ik}$, $\eta_{2k}$, $\eta_{3k}$ $\in (t_{k-1}, t_k)$. From the definite integrals of \[\text{(4.32)}\], there exists
\[
I_{3k} = \int_{t_{k-1}}^{t_k} (t_k + t_{k-1} - 2s) e^{-\sigma (t_k - s)} ds
\]
\[
= e^{-\sigma (n-k) \tau} \left[ - \frac{\tau}{\sigma} (1 - e^{-\sigma \tau}) + \frac{2}{\sigma^2} (1 - e^{-\sigma \tau}) \right] = e^{-\sigma (n-k) \tau} \left[ - \frac{\sigma}{\sigma^2} + O (\tau^3) \right];
\]
and
\[
I_{4k} = \int_{t_k}^{t_{k+1}} (t_k + t_{k-1} - 2s) e^{-\sigma (t_k - s)} ds
\]
\[
= e^{-\sigma (n-k) \tau} \left[ \frac{2}{\sigma^2} (1 - e^{-\sigma \tau}) - \frac{\tau}{\sigma} \right] = e^{-\sigma (n-k) \tau} \left[ - \frac{1}{4} \tau^2 + O (\tau^3) \right].
\]
Therefore, we have
\[
I_{11} = \frac{1}{2(1 - \gamma)} \sum_{k=1}^{n} u''(\eta_{ik}) I_{3k} = O (\tau^2),
\]
\[
I_{12} = \frac{1}{2(1 - \gamma)} \sum_{k=1}^{n} (\eta_{2k} - \eta_{ik}) u'''(\eta_{3k}) I_{4k} = O (\tau^2),
\]
where we use $\eta_{2k} - \eta_{ik} = O (\tau)$. From the above equations, we get
\[
I_1 = I_{11} + I_{12} = O (\tau^2).
\]
Since
\[
I_2 = I_{21} + I_{22}
\]
with
\[
I_{21} = \frac{1}{6\tau(1 - \gamma)} \sum_{k=1}^{n} u'''(\eta_{2k}) \int_{t_{k-1}}^{t_k} (t_k - s)^3 e^{-\sigma (t_k - s)} ds,
\]
\[
I_{22} = \frac{1}{6\tau(1 - \gamma)} \sum_{k=1}^{n} u'''(\eta_{3k}) \int_{t_{k-1}}^{t_k} (t_k - s)^3 e^{-\sigma (t_k - s)} ds.
\]
It is easy to check that
\[
I_{3k} = \int_{t_{k-1}}^{t_k} (t_k - s)^3 e^{-\sigma (t_k - s)} ds = e^{-\sigma (n-k+1) \tau} \left[ \frac{6}{\sigma^3} e^{\sigma \tau} - \frac{6\tau}{\sigma^3} - \frac{3\tau^2}{\sigma^2} - \frac{\tau^3}{\sigma} \right]
\]
\[
= e^{-\sigma (n-k+1) \tau} \left[ \frac{1}{4} \tau^2 + O (\tau^3) \right];
\]
and
\[
I_{4k} = \int_{t_k}^{t_{k+1}} (t_k - s)^3 e^{-\sigma (t_k - s)} ds = e^{-\sigma (n-k) \tau} \left[ \frac{6}{\sigma^3} (1 - e^{-\sigma \tau}) - \frac{6\tau}{\sigma^3} + \frac{3\tau^2}{\sigma^2} - \frac{\tau^3}{\sigma} \right]
\]
\[
= e^{-\sigma (n-k) \tau} \left[ - \frac{1}{4} \tau^2 + O (\tau^3) \right].
\]
It yields
\[
I_2 = I_{21} + I_{22} = O (\tau^2).
\]
According to the above equations, we have
\[
|r_{n1}^2| \leq |I_1| + |I_2| = O (\tau^2).
\]
The proof is completed.
2.2 Derivation of numerical schemes for 1D

Consider the one-dimensional time-space Caputo-Riesz fractional diffusion equation

\[ \frac{\partial^\alpha}{\partial |x|^\alpha} D^\alpha_t u(x,t) = \frac{\partial^\alpha}{\partial |x|^\alpha} u(x,t) + f(x,t). \]  

(2.3)

To discrete the Riesz fractional derivative for \(1 < \alpha < 2\), we notice that the approximation operator of (1.3) is given in (14)

\[ \delta_{\alpha,+} u^i := \frac{1}{\Gamma(4-\alpha)(\Delta x)^\alpha} \sum_{m=0}^{i+1} g_{m,n}^\alpha u_{i-m+1}^m; \]

\[ \delta_{\alpha,-} u^i := \frac{1}{\Gamma(4-\alpha)(\Delta x)^\alpha} \sum_{m=0}^{i+1} g_{m,n}^\alpha u_{i-m-1}^m; \]

and there exists

\[ a D^\alpha_t u(x,t) \big|_{x=x_i} = \delta_{\alpha,+} u^i + O\left((\Delta x)^2\right) \]

and

\[ \delta_{\alpha,-} u^i = \delta_{\alpha,-} u^i + O\left((\Delta x)^2\right) \]

with

\[ g_{m,n}^\alpha = \begin{cases} 1, & m = 0, \\ -4 + 2^{3-\alpha}, & m = 1, \\ 6 - 2^{5-\alpha} + 3^\alpha, & m = 2, \\ (m+1)^3 - 4m^{3-\alpha} - 6(m-1)^3 + 4(m-2)^3 - (m-3)^3, & m \geq 3. \end{cases} \]

(2.4)

Hence, the discrete scheme of the Riesz fractional derivative is

\[ \frac{\partial^\alpha}{\partial |x|^\alpha} u(x,t) \big|_{x=x_i} = \kappa_\alpha \left(a D^\alpha_t u(x,t) \big|_{x=x_i} - \delta_{\alpha,+} u^i + O\left((\Delta x)^2\right)\right) \]

(2.5)

\[ = \frac{\kappa_\alpha}{\Gamma(4-\alpha)(\Delta x)^\alpha} \sum_{m=0}^{N_x} \tilde{s}_{i,m}^\alpha u_{i-m}^m + O\left((\Delta x)^2\right), \]

where

\[ \tilde{s}_{i,m}^\alpha = \begin{cases} g_{m+1,i}^\alpha, & m < i-1, \\ g_{i-1,i}^\alpha, & m = i-1, \\ \frac{2g_{i-1,i}}{2g_{i-1,i}^\alpha}, & m = i, \\ g_{i,i}^\alpha + g_{i+1,i+1}^\alpha, & m = i+1, \\ \frac{2g_{i,i}^\alpha}{2g_{i,i}^\alpha}, & m > i+1, \end{cases} \]

(2.6)

with \(i = 1,2,\cdots,N_x-1\), \(u_0^i\) and \(u_{N_x}^i\) are the boundary conditions.

Taking \(u = [u(x_1), u(x_2), \cdots, u(x_{N_x-1})]^T\), then

\[ \sum_{m=0}^{N_x} \tilde{s}_{1,m}^\alpha u(x_m), \sum_{m=0}^{N_x} \tilde{s}_{2,m}^\alpha u(x_m), \cdots, \sum_{m=0}^{N_x} \tilde{s}_{N_x-1,m}^\alpha u(x_m) = A^\alpha u, \]

where

\[ A^\alpha = B^\alpha + B^\alpha_T \text{ with } B^\alpha = \begin{bmatrix} g_1^\alpha & \cdots & g_{N_x}^\alpha \\ g_2^\alpha & \cdots & g_{N_x-1}^\alpha \\ \vdots & \vdots & \vdots \\ g_{N_x}^\alpha & \cdots & g_1^\alpha \end{bmatrix}_{(N_x-1)\times(N_x-1)}. \]

(2.7)
According to (2.3) and Lemma 2.1, we can rewrite (2.3) as
\[
\frac{1}{1 - \gamma} \sum_{k=1}^{n} \frac{u_i^k - u_i^{k-1}}{\sigma \tau} e^{-\sigma(n-k)\tau} (1 - e^{-\sigma \tau}) = \frac{K_\alpha}{\Gamma(4 - \alpha)(\Delta x)^\alpha} \sum_{m=0}^{N_x} \gamma^m u_m^0 + f_i^n + r_i^n \quad (2.8)
\]
with the local truncation error
\[
|r_i^n| \leq C_u \left( \tau^2 + (\Delta x)^2 \right), \quad (2.9)
\]
where the positive constant $C_u$, independent of $\tau$ and $h$.

Therefore the resulting discretization of (2.3) is
\[
\frac{1}{1 - \gamma} \sum_{k=1}^{n} \frac{u_i^k - u_i^{k-1}}{\sigma \tau} e^{-\sigma(n-k)\tau} (1 - e^{-\sigma \tau}) = \frac{K_\alpha}{\Gamma(4 - \alpha)(\Delta x)^\alpha} \sum_{m=0}^{N_x} \gamma^m u_m^0 + f_i^n \quad (2.10)
\]
i.e.,
\[
\frac{1}{(1 - \gamma)\sigma \tau} \left[ u_i^0 (1 - e^{-\sigma \tau}) - \sum_{k=1}^{n-1} u_i^k e^{-\sigma(n-k)\tau} (1 - e^{-\sigma \tau}) \right] = \frac{K_\alpha}{\Gamma(4 - \alpha)(\Delta x)^\alpha} \sum_{m=0}^{N_x} \gamma^m u_m^0 + f_i^n + \frac{1}{(1 - \gamma)\sigma \tau} u_i^0 (1 - e^{-\sigma \tau}) e^{-\sigma(n-1)\tau}, \quad (2.11)
\]
which is equivalent to
\[
u_i^n - K_{\Delta x, \tau} \sum_{m=0}^{N_x} \gamma^m u_m^0 = \sum_{k=1}^{n-1} u_i^k e^{-\sigma(n-k)\tau} (1 - e^{-\sigma \tau}) + u_i^0 e^{-\sigma(n-1)\tau} + \frac{(1 - \gamma)\sigma \tau}{1 - e^{-\sigma \tau}} f_i^n \quad (2.12)
\]
with $K_{\Delta x, \tau} = \frac{(1 - \gamma)\sigma \tau}{1 - e^{-\sigma \tau}} \frac{K_\alpha}{\Gamma(4 - \alpha)(\Delta x)^\alpha}$.

2.3 Derivation of numerical schemes for 2D

In the same way, we can rewrite (2.1) as
\[
\frac{1}{1 - \gamma} \sum_{k=1}^{n} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\sigma \tau} e^{-\sigma(n-k)\tau} (1 - e^{-\sigma \tau}) = \frac{K_\alpha}{\Gamma(4 - \alpha)(\Delta x)^\alpha} \sum_{m=0}^{N_x} \gamma^m u_{m,j}^0 + f_{i,j}^n + r_{i,j}^n \quad (2.13)
\]
with the local truncation error
\[
|r_{i,j}^n| \leq C_u (\tau^2 + (\Delta x)^2 + (\Delta y)^2). \quad (2.14)
\]

Therefore the resulting discretization of (2.1) is
\[
\frac{1}{1 - \gamma} \sum_{k=1}^{n} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\sigma \tau} e^{-\sigma(n-k)\tau} (1 - e^{-\sigma \tau}) = \frac{K_\alpha}{\Gamma(4 - \alpha)(\Delta x)^\alpha} \sum_{m=0}^{N_x} \gamma^m u_{m,j}^0 + f_{i,j}^n + r_{i,j}^n \quad (2.15)
\]
Lemma 3.1 A real matrix $A$ of order $n$ is positive definite iff its symmetric part $H$ is positive definite and $k_n$ is positive.

Lemma 3.2 A strictly diagonally dominant matrix that is symmetric with positive diagonal entries is also positive definite.

Lemma 3.3 The coefficients $g_{ij}$ and $g_{im}$ defined in (2.8) satisfy

1. $g_{ij} < 0$, $g_{im} > 0$ $(m \neq i)$;
2. $\sum_{m=0}^{N_i} g_{im} < 0$ and $g_{ii} > \sum_{m=0, m \neq i}^{N_i} g_{im}$;
3. $g_{0} > 0$, $g_{1} < 0$, $g_{2} > 0$, $g_{k} > 0$ $\forall k \geq 3$.

Lemma 3.4 Let $1 < \alpha < 2$ and $g_{0}$ given by (2.8). Then

$$\sum_{m=0}^{i+1} \frac{1}{\Gamma(1-\alpha)} < 0.$$
Proof Using (2.13) and (2.4) with \( u(x,t) = 1 \), we obtain

\[
\sum_{m=0}^{i+1} g_m = \frac{\Gamma(4 - \alpha) \alpha}{(\Delta x)^2 \Gamma(2 - \alpha)} \int_0^{r_i} (x_{i-1} - \xi)^{1-\alpha} d\xi - 2 \int_0^{r_i} (x_i - \xi)^{1-\alpha} d\xi + \int_0^{r_{i+1}} (x_{i+1} - \xi)^{1-\alpha} d\xi
\]

\[
= (3 - \alpha)^{2 - \alpha} \left[ \left( 1 - \frac{1}{7} \right)^{2-\alpha} - 2 + \left( 1 + \frac{1}{7} \right)^{2-\alpha} \right]
\]

\[
\leq \frac{(3 - \alpha)(2 - \alpha)(1 - \alpha)}{\rho^2} \leq \frac{1}{\rho^2 \Gamma(1 - \alpha)} < 0.
\]

The proof is completed.

**Lemma 3.5** Let \( 1 < \alpha < 2 \) and \( A_\alpha \) be given in (2.7). Then

\[
-\frac{1}{(\Delta x)^2} (A_\alpha v, v) \geq -\frac{2}{\Gamma(1 - \alpha) |x_\Omega|^2} ||v||^2 > 0 \text{ with } v \in \mathbb{R}^{M-1}, \Omega = (0, x_R).
\]

**Proof** Let the vector \( v = (v_1, v_2, \ldots, v_{N-1})^T \) with \( v_0 = v_{N-1} = 0 \). From (2.7) and Lemma 3.3 there exists

\[
(B_\alpha v, v) = \Delta x \sum_{i=1}^{N-2} \left( \sum_{k=0}^{N-2} g_k^2 v_{i+k-1} \right) v_i = \Delta x \sum_{i=1}^{N-2} \left( \sum_{k=0}^{N-2} g_k^2 \left( \sum_{j=1}^{N-2} v_{i+j-1} \right) \right)
\]

\[
= g_0^2 \Delta x \sum_{i=1}^{N-2} v_i^2 + g_0^2 \Delta x \sum_{i=1}^{N-2} v_{i+1} + \Delta x \sum_{i=1}^{N-2} \left( \sum_{k=0}^{N-2} g_k^2 \left( \sum_{j=1}^{N-2} v_{i+j-1} \right) \right)
\]

\[
\leq g_0^2 ||v||^2 + g_0^2 \Delta x \sum_{i=1}^{N-2} \frac{v_i^2 + v_{i+1}^2}{2} + \Delta x \sum_{i=1}^{N-2} \left( \sum_{k=0}^{N-2} g_k^2 \left( \sum_{j=1}^{N-2} v_{i+j-1} \right) \right)
\]

\[
\leq \left( \sum_{k=0}^{N-2} g_k^2 \right) ||v||^2 \leq \left( \sum_{k=0}^{N-2} g_k^2 \right) ||v||^2.
\]

Since \( (B_\alpha v, v) = (B_\alpha v, v) \) and \( A_\alpha = B_\alpha^* + B_\alpha \), we have

\[
(A_\alpha v, v) \leq 2 \left( \sum_{k=0}^{N-2} g_k^2 \right) ||v||^2 < 0.
\]

Using the above inequality and Lemma 3.3 we obtain

\[
-\frac{1}{(\Delta x)^2} (A_\alpha v, v) \geq -\frac{2}{\Gamma(1 - \alpha) |x_\Omega|^2} ||v||^2 \geq \frac{2}{\Gamma(1 - \alpha)} \left( 1 + \frac{1}{7} \right)^{2-\alpha} ||v||^2
\]

\[
\geq -\frac{2}{\Gamma(1 - \alpha)} |x_\Omega|^2 ||v||^2 > 0 \text{ with } \Omega = (0, x_R), v \in \mathbb{R}^{N-1}.
\]

The proof is completed.

**Lemma 3.6** Let \( 0 < \gamma < 1 \) and \( \sigma = \frac{1}{1 - \gamma} \). Then for any vector \( V_i = (v_1^i, v_2^i, \ldots, v_n^i) \in \mathbb{R}^N \), we have

\[
\sum_{n=1}^{N} v_n^i (1 - e^{-\sigma \gamma}) - \sum_{k=1}^{n-1} v_k^i e^{-\sigma(n-1-k)\gamma} (1 - e^{-\sigma \gamma})^2 \geq 0.
\]
Proof By the mathematical induction method, we have
\[ \sum_{n=1}^{N} v_i^n (1 - e^{-\sigma \tau}) - \sum_{k=1}^{n-1} v_k^n e^{-\sigma(n-1-k)\tau} (1 - e^{-\sigma \tau})^2 \] \[ v_i^n = (AV_i, V_i), \]
where
\[ A = \begin{bmatrix} b & 0 & 0 & \ldots & 0 \\ a_1 & b & 0 & \ldots & 0 \\ a_2 & a_1 & b & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N-2} & a_{N-3} & a_{N-4} & \ldots & b \\ a_{N-1} & a_{N-2} & a_{N-3} & \ldots & a_1 & b \end{bmatrix}_{N \times N} \]
with \( b = 1 - e^{-\sigma \tau} \) and \( a_l = -e^{-\sigma(l-1)\tau} (1 - e^{-\sigma \tau})^2 < 0, l = 1, 2, \ldots, N - 1 \). We next prove the matrix \( A \) is positive definite. Since
\[ H = \frac{A + A^T}{2} = \begin{bmatrix} b & a_1 & a_2 & \ldots & a_{N-1} \\ a_1 & b & a_2 & \ldots & a_{N-2} \\ a_2 & a_1 & b & \ldots & a_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N-2} & a_{N-3} & a_{N-4} & \ldots & b \\ a_{N-1} & a_{N-2} & a_{N-3} & \ldots & a_1 & b \end{bmatrix}_{N \times N} \]
and
\[ \sum_{j=1}^{N} |h_{i,j}| \leq 2 \sum_{l=1}^{N-1} |a_l| = -2 \sum_{l=1}^{N-1} a_l = \sum_{l=1}^{N-1} e^{-\sigma(l-1)\tau} (1 - e^{-\sigma \tau})^2 \]
\[ = (1 - e^{-\sigma \tau}) (1 - e^{-\sigma(N-1)\tau}) < 1 - e^{-\sigma \tau} = b, \]
it yields the matrix \( H \) is strictly diagonally dominant. Form Lemmas 3.1 and 3.2, we know that the matrix \( A \) is positive definite. The proof is completed.

For simplifying the proof of the stability and convergence, we first provide the following a priori estimate.

Lemma 3.7 Suppose \( v_i^n \) is the solution of the difference scheme (2.11), i.e.,
\[ \frac{1}{(1 - \gamma)\sigma \tau} \left[ v_i^n (1 - e^{-\sigma \tau}) - \sum_{k=1}^{n-1} v_k^n e^{-\sigma(n-1-k)\tau} (1 - e^{-\sigma \tau})^2 \right] \]
\[ = \frac{\kappa_\alpha}{\Gamma(4 - \alpha) (\Delta x)\alpha} \int_{s_{i,m}} \int_{s_{m}} \int_{s_{m}} f_i^n + 1}{(1 - \gamma)\sigma \tau} v_i^n (1 - e^{-\sigma \tau})e^{-\sigma(n-1)\tau}, \] \( (3.3) \)
\[ v_i^0 = \phi_i, \quad 1 \leq i \leq N_k - 1, \]
\[ v_i^0 = v_i^0 = 0, \quad 0 \leq n \leq N. \]

Then for any positive integer \( N \) with \( N \tau \leq T \), we have
\[ \tau \sum_{n=1}^{N} ||v_i^n||^2 \]
\[ \leq \frac{(\Gamma(1 - \gamma)\Gamma(4 - \alpha) (s_k)\alpha)^2}{2 \kappa_\alpha} \cdot \tau \sum_{n=1}^{N} ||f_i^n||^2 + \frac{\Gamma(1 - \alpha)\Gamma(4 - \alpha) (s_k)\alpha^2 T}{(1 - \gamma)^2 \kappa_\alpha^2} ||v_i^0||^2. \]
Proof. Multiplying (3.3) by $(\Delta x)v^\alpha_i$ and summing up for $i$ from 1 to $N_\tau - 1$, we get

$$
\frac{1}{(1-\gamma)\sigma \tau} \sum_{i=1}^{N-1} \left[ v_i^0 (1 - e^{-\sigma \tau}) - \sum_{k=1}^{n-1} v_k^0 e^{-\sigma(n-1-k)\tau} \left( 1 - e^{-\sigma \tau} \right)^2 \right] \Delta v_i^0
$$

$$
= \sum_{i=1}^{N-1} \left[ \frac{\kappa_a}{(4-\alpha)(\Delta x)^2} \sum_{m=0}^{N_\tau} \eta_m x_m + f_i^\alpha \left( 1 - e^{-\sigma \tau} \right) e^{-\sigma(n-1)\tau} \right] \Delta x v_i^0
$$

$$
= \frac{\kappa_a}{(4-\alpha)(\Delta x)^2} \sum_{n=1}^{N} \left( A_{\alpha} v^n, v^n + (f^n, v^n) + \frac{1}{(1-\gamma)\sigma \tau} \Delta x v_i^0 \right)
$$

Multiplying the above equation by $\tau$ and summing up for $n$ from 1 to $N$, we obtain

$$
\tau \frac{2}{(1-\gamma)\sigma \tau} \sum_{i=1}^{N-1} \sum_{n=1}^{N} \left[ v_i^0 (1 - e^{-\sigma \tau}) - \sum_{k=1}^{n-1} v_k^0 e^{-\sigma(n-1-k)\tau} \left( 1 - e^{-\sigma \tau} \right)^2 \right] (\Delta x) v_i^0
$$

$$
= \tau \sum_{n=1}^{N} \frac{\kappa_a}{(4-\alpha)(\Delta x)^2} (A_{\alpha} v^n, v^n) + \tau \sum_{n=1}^{N} (f^n, v^n) + \frac{\tau (1 - e^{-\sigma \tau})}{(1-\gamma)\sigma \tau} \tau \sum_{n=1}^{N} e^{-(n-1)\tau} (v^0, v^n)
$$

From Lemma 3.5 and Lemma 3.6, we have

$$
- \frac{2}{(1-\gamma)\sigma \tau} \sum_{n=1}^{N} \left\| v^n \right\|^2 \leq - \tau \sum_{n=1}^{N} \frac{\kappa_a}{(4-\alpha)(\Delta x)^2} (A_{\alpha} v^n, v^n)
$$

$$
\leq \tau \sum_{n=1}^{N} (f^n, v^n) + \frac{1}{(1-\gamma)\sigma \tau} \tau \sum_{n=1}^{N} e^{-(n-1)\tau} (v^0, v^n)
$$

$$
\leq \tau \sum_{n=1}^{N} \left( \| f^n \| \| v^n \| + \frac{1}{(1-\gamma)\sigma \tau} \sum_{n=1}^{N} \left\| v^0 \right\| \| v^n \| \right)
$$

$$
\leq \tau \sum_{n=1}^{N} \left( \varepsilon \| v^n \|^2 + \frac{\| f^n \|^2}{4\varepsilon} \right) + \frac{1}{(1-\gamma)} \sum_{n=1}^{N} \left( \eta \| v^0 \|^2 + \frac{\| v^0 \|^2}{4\eta} \right),
$$

where $\varepsilon, \eta > 0$ and $(f^n, v^n) \leq \| f^n \| \| v^n \|$.

Taking $\varepsilon = \frac{2}{(1-\gamma)\sigma \tau} \sum_{n=1}^{N} \left\| v^0 \right\|^2$, and $\eta = \frac{1}{(1-\gamma)\sigma \tau} \sum_{n=1}^{N} \left\| v^0 \right\|^2$, and using the above inequality, there exists

$$
- \frac{1}{2\Gamma(1-\alpha)\Gamma(4-\alpha)(\Delta x)^2} \sum_{n=1}^{N} \left\| v^n \right\|^2 \leq \frac{1}{4\varepsilon \eta} \tau \sum_{n=1}^{N} \left\| f^n \right\|^2 + \frac{T}{4\eta^2} \left\| v^0 \right\|^2
$$

and

$$
\tau \sum_{n=1}^{N} \left\| v^n \right\|^2 \leq \frac{1}{4\varepsilon \eta} \tau \sum_{n=1}^{N} \left\| f^n \right\|^2 + \frac{T}{4\eta^2} \left\| v^0 \right\|^2
$$

$$
= \left( \frac{(1-\alpha)\Gamma(4-\alpha)(\Delta x)^2}{2\kappa_a^2} \right) \tau \sum_{n=1}^{N} \left\| f^n \right\|^2 + \left( \frac{(1-\alpha)\Gamma(4-\alpha)(\Delta x)^2}{(1-\gamma)^2}\right) \tau \sum_{n=1}^{N} \left\| v^0 \right\|^2.
$$

The proof is completed.
3.2 Convergence and stability for 1D

In this subsection, we prove that the scheme (2.11) is unconditionally stable and convergence by two methods, i.e., a prior estimate and mathematical induction which correspond to the discrete $L^2$-norm and $L_\infty$ norm.

**Theorem 3.1** The difference scheme (2.11) is unconditionally stable.

**Proof** From Lemma 3.7, the desired results is obtained.

**Theorem 3.2** Let $\mathbf{u}_i^h$ be the approximate solution of $u(x_i,t_n)$ computed by the difference scheme (2.11). Let $\mathbf{e}_i^h = u(x_i,t_n) - \mathbf{u}_i^h$. Then

$$
\tau \sum_{n=1}^{N} \| e_i^n \| = \left[ \Gamma(1-\alpha) \Gamma(4-\alpha) (x_k)^{\alpha-1/2} \right] \frac{T}{\sqrt{2K_{12}}} C_{u} \cdot (\tau^2 + (\Delta x)^2),
$$

where $C_{u}$ is defined by (2.9) and $(x_i,t_n) \in (0,x_R) \times (0,T]$ with $N \tau \leq T$.

**Proof** Let $u(x_i,t_n)$ be the exact solution of (2.11) at the mesh point $(x_i,t_n)$, and $e_i^n = u(x_i,t_n) - u_i^n$. Subtracting (2.8) from (2.11) with $\mathbf{e}_i^0 = 0$, we obtain

$$
\frac{1}{(1-\gamma)\sigma^2} \left[ e_k^n (1-e^{-\sigma t}) - \sum_{l=0}^{n-1} e_l^i e^{-\sigma(n-1-k)} (1-e^{-\sigma t}) \right] = \frac{\kappa_{12}}{T} \sum_{m=0}^{N} \Gamma(4-\alpha) (\Delta x)^{\alpha} e_i^{m} + r_i^n.
$$

From Lemma 3.7 and (2.9), it holds

$$
\tau \sum_{n=1}^{N} \| e_i^n \|^2 \leq \frac{(\Gamma(1-\alpha) \Gamma(4-\alpha) (x_k)^{\alpha})^2}{2K_{12}^2} \cdot \tau \sum_{n=1}^{N} \| r_i^n \|^2 \leq \frac{(\Gamma(1-\alpha) \Gamma(4-\alpha) (x_k)^{\alpha})^2}{2K_{12}^2} \cdot x_R T C_{u}^2 \cdot (\tau^2 + (\Delta x)^2)^2.
$$

Using Cauchy-Schwarz inequality for the above inequality, we have

$$
\left( \tau \sum_{n=1}^{N} \| e_i^n \| \right)^2 \leq \left( \tau \sum_{n=1}^{N} 1 \right) \left( \tau \sum_{n=1}^{N} \| e_i^n \|^2 \right) \leq \frac{(\Gamma(1-\alpha) \Gamma(4-\alpha) (x_k)^{\alpha})^2}{2K_{12}^2} \cdot x_R T^2 C_{u}^2 \cdot (\tau^2 + (\Delta x)^2)^2,
$$

and

$$
\tau \sum_{n=1}^{N} \| e_i^n \| \leq \frac{(\Gamma(1-\alpha) \Gamma(4-\alpha) (x_k)^{\alpha-1/2} T}{\sqrt{2K_{12}}} C_{u} \cdot (\tau^2 + (\Delta x)^2).
$$

The proof is completed.

Besides the discrete $L^2$-norm, the stability and convergence can also be obtained in $L_\infty$ norm by the following mathematical induction.

**Theorem 3.3** The difference scheme (2.12) is unconditionally stable.
Proof Let \( \tilde{u}_i^n \) be the approximate solution of \( u_i^n \), which is the exact solution of the difference scheme (2.12). Denoting \( e_i^n = \tilde{u}_i^n - u_i^n \), there exists

\[
(1 - \kappa_{\Delta x, \tau} g_{\alpha}^{\alpha}) e_i^1 - \kappa_{\Delta x, \tau} \sum_{m=0, m \neq i}^{N} g_{\alpha,m}^{\alpha} e_m = e_i^0 e^{-\sigma(n-1) \tau}, \quad n = 1,
\]

\[
(1 - \kappa_{\Delta x, \tau} g_{\alpha}^{\alpha}) e_i^n - \kappa_{\Delta x, \tau} \sum_{m=0, m \neq i}^{N} g_{\alpha,m}^{\alpha} e_m = \sum_{k=1}^{n-1} \epsilon_k e^{-\sigma(n-k) \tau} (1 - e^{-\sigma \tau}) + e_i^0 e^{-\sigma(n-1) \tau}, \quad n > 1.
\]

We next prove the scheme is unconditionally stable by the mathematical induction.

Let \( E^n = [e_{i_0}^n, e_{i_1}^n, \cdots, e_{i_N}^n] \) and \( |E_i^n| := ||E_i^n||_\infty = \max_{0 \leq i \leq N} |e_i^n| \). From Lemma 5.3, we obtain

\[
||E^1||_\infty = ||e_{i_0}^1|| \leq ||e_{i_0}^1|| - \kappa_{\Delta x, \tau} \sum_{m=0}^{N} g_{\alpha,m}^{\alpha} |e_{i_0}^1| = ||e_{i_0}^1|| - \kappa_{\Delta x, \tau} \sum_{m=0, m \neq i_0}^{N} g_{\alpha,m}^{\alpha} |e_{i_0}^1|
\]

\[
\leq (1 - \kappa_{\Delta x, \tau} g_{\alpha}^{\alpha}) ||e_{i_0}^1|| - \kappa_{\Delta x, \tau} \sum_{m=0, m \neq i_0}^{N} g_{\alpha,m}^{\alpha} |e_{i_0}^1|
\]

\[
\leq (1 - \kappa_{\Delta x, \tau} g_{\alpha}^{\alpha}) e_{i_0}^1 - \kappa_{\Delta x, \tau} \sum_{m=0, m \neq i_0}^{N} g_{\alpha,m}^{\alpha} e_m^n = ||e_{i_0}^1|| e^{-\sigma(n-1) \tau} \leq ||E^0||_\infty.
\]

Assuming \( ||E^s||_\infty \leq ||E^0||_\infty, s = 1, 2, 3, \cdots, n - 1 \), we have

\[
||E^n||_\infty = ||e_{i_0}^n|| \leq ||e_{i_0}^n|| - \kappa_{\Delta x, \tau} \sum_{m=0}^{N} g_{\alpha,m}^{\alpha} |e_{i_0}^n| = ||e_{i_0}^n|| - \kappa_{\Delta x, \tau} \sum_{m=0, m \neq i_0}^{N} g_{\alpha,m}^{\alpha} |e_{i_0}^n|
\]

\[
\leq (1 - \kappa_{\Delta x, \tau} g_{\alpha}^{\alpha}) ||e_{i_0}^n|| - \kappa_{\Delta x, \tau} \sum_{m=0, m \neq i_0}^{N} g_{\alpha,m}^{\alpha} |e_{i_0}^n|
\]

\[
\leq (1 - \kappa_{\Delta x, \tau} g_{\alpha}^{\alpha}) e_{i_0}^n - \kappa_{\Delta x, \tau} \sum_{m=0, m \neq i_0}^{N} g_{\alpha,m}^{\alpha} e_m^n
\]

\[
\leq \sum_{k=1}^{n-1} \epsilon_k e^{-\sigma(n-k) \tau} (1 - e^{-\sigma \tau}) + e_i^0 e^{-\sigma(n-1) \tau} \left( ||e_{i_0}^1|| \right)
\]

\[
\leq \sum_{k=1}^{n-1} e^{-\sigma(n-k) \tau} (1 - e^{-\sigma \tau}) + e_i^0 e^{-\sigma(n-1) \tau} \left( ||E^0||_\infty \right) = ||E^0||_\infty.
\]

The proof is completed.

**Theorem 3.4** Let \( u_i^n \) be the approximate solution of \( u(x_i, t_n) \) computed by the difference scheme (2.12). Let \( e_i^n = u(x_i, t_n) - u_i^n \). Then

\[
||e^n||_\infty \leq (1 - \gamma) \left( \sigma T + \sigma^2 \right) C_u (\tau^2 + (\Delta x)^2),
\]

where \( C_u \) is defined by (2.12) and \((x_i, t_n) \in (0, b) \times (0, T) \) with \( N \tau \leq T \).
Proof Let \( u(x_i, t_n) \) be the exact solution of (2.8) at the mesh point \((x_i, t_n)\). Defined \(e_j^n = u(x_i, t_n) - u_j^n\) and \(e^n = [e_0^n, e_1^n, \ldots, e_N^n]\). Subtracting (2.8) from (2.12) with \(e_0^0 = 0\), we obtain

\[
(1 - \kappa_{\Delta x, \tau} g_{ij}^\alpha) e_i^1 - \kappa_{\Delta x, \tau} \sum_{m=0, m \neq i}^{N_x} g_{ij,m}^\alpha e_m^1 = \frac{(1 - \gamma) \sigma \tau}{1 - e^{-\sigma \tau}} e_i^1, \quad n = 1,
\]

\[
(1 - \kappa_{\Delta x, \tau} g_{ij}^\alpha) e_i^n - \kappa_{\Delta x, \tau} \sum_{m=0, m \neq i}^{N_x} g_{ij,m}^\alpha e_m^n = \sum_{k=1}^{n-1} e_i^k e^{-\sigma(n-k)\tau} \left( 1 - e^{-\sigma \tau} \right) + \frac{(1 - \gamma) \sigma \tau}{1 - e^{-\sigma \tau}} e_i^n, \quad n > 1.
\]

We next prove the desired results by the mathematical induction. Let \(||e_i^0|| = \max_{0 \leq t \leq N_x} |e_i^0|\) and \(r_{\max} = \max_{0 \leq t \leq N_x} |r_i^0|\). Using Lemma 3.3, we have

\[
||e_1^1|| \leq ||e_i^1|| \leq \kappa_{\Delta x, \tau} \sum_{m=0, m \neq i}^{N_x} g_{ij,m}^\alpha |e_m^1| = |e_i^1| - \kappa_{\Delta x, \tau} \max_{0 \leq t \leq N_x} |r_i^0| = \frac{(1 - \gamma) \sigma \tau}{1 - e^{-\sigma \tau}} r_{\max}.
\]

Supposing \(||e^n|| = ||e_i^n|| = \max_{0 \leq t \leq N_x} |e_i^n|\) and

\[
||e_1^s|| \leq \frac{(s-1) \left( 1 - e^{-\sigma \tau} \right) + 1}{1 - e^{-\sigma \tau}} (1 - \gamma) \sigma \tau r_{\max}, \quad s = 1, 2, \ldots, n - 1,
\]

we have

\[
||e^n|| = ||e_i^n|| \leq ||e_i^n|| - \kappa_{\Delta x, \tau} \sum_{m=0, m \neq i}^{N_x} g_{ij,m}^\alpha |e_m^n| = |e_i^n| - \kappa_{\Delta x, \tau} \max_{0 \leq t \leq N_x} |r_i^0| = \frac{(1 - \gamma) \sigma \tau}{1 - e^{-\sigma \tau}} r_{\max}
\]

with \(\Phi_{\tau} = \sum_{k=1}^{n-1} k \left( 1 - e^{-\sigma \tau} \right) \left( 1 - \gamma \right) \sigma \tau e^{-\sigma(n-k)\tau} \left( 1 - e^{-\sigma \tau} \right) + \frac{(1 - \gamma) \sigma \tau}{1 - e^{-\sigma \tau}} r_{\max} \)

\[
= \frac{(n-1)(1 - e^{-\sigma \tau}) + 1}{1 - e^{-\sigma \tau}} (1 - \gamma) \sigma \tau < (1 - \gamma) \sigma T + e^{\theta \sigma \tau} (1 - \gamma),
\]

where we use \(e^\epsilon = 1 + e^{\theta \Delta x}, 0 < \theta < 1\).

According to (2.29), we can get

\[
||e^n|| \leq (1 - \gamma) \left( \sigma T + e^{\theta \sigma \tau} \right) \max_{0 \leq t \leq N_x} (1 - \gamma) \sigma T + e^{\theta \sigma \tau} (1 - \gamma).
\]

The proof is completed.
3.3 Convergence and stability for 2D

In this subsection, the stability and convergence are obtained by mathematical induction with $L_\infty$ norm.

**Theorem 3.5** The difference scheme (2.16) is unconditionally stable.

**Proof** Let $\tilde{u}_{j}^n$ be the approximate solution of $u_j^n$, which is the exact solution of the difference scheme (2.16). Denoting $\epsilon_{j}^n = \tilde{u}_{j}^n - u_j^n$, there exists

$$
(1 - \kappa_\Delta \tau S_{j}^{\alpha} - \kappa_{\Delta \tau}^{\beta} S_{j}^{\beta}) \epsilon_{j}^1 - \kappa_\Delta \tau \sum_{m=0,m \neq j}^{N_j} \epsilon_{m,j}^1 - \kappa_{\Delta \tau}^{\beta} \sum_{m=0,m \neq j}^{N_j} \epsilon_{j,m}^1 \leq \epsilon_{j}^0 e^{-\sigma(n-1)\tau}, \quad n = 1,
$$

$$
(1 - \kappa_\Delta \tau S_{j}^{\alpha} - \kappa_{\Delta \tau}^{\beta} S_{j}^{\beta}) \epsilon_{j}^n - \kappa_\Delta \tau \sum_{m=0,m \neq j}^{N_j} \epsilon_{m,j}^n - \kappa_{\Delta \tau}^{\beta} \sum_{m=0,m \neq j}^{N_j} \epsilon_{j,m}^n = \sum_{k=1}^{n-1} \epsilon_{j}^{k+1} e^{-\sigma(n-1-k)\tau} \left(1 - e^{-\sigma \tau} \right) + \epsilon_{j}^0 e^{-\sigma(n-1)\tau}, \quad n > 1.
$$

We next prove the scheme is unconditionally stable by the mathematical induction. Let $\|\epsilon_{j}^0\| = \|E_0\| = \max_{0 \leq j \leq N_j} \|u_j^n\|$. From Lemma 3.3 we obtain

$$
\|E_1\|_\infty = \|\epsilon_{j}^1\|_\infty \leq \|\epsilon_{j}^0\|_\infty - \kappa_\Delta \tau \sum_{m=0}^{N_j} S_{j,m}^{\alpha} \|\epsilon_{m,j}^0\| - \kappa_{\Delta \tau}^{\beta} \sum_{m=0}^{N_j} \|\epsilon_{j,m}^0\|.
$$

$$
\leq \left(1 - \kappa_\Delta \tau S_{j}^{\alpha} - \kappa_{\Delta \tau}^{\beta} S_{j}^{\beta}\right) \|\epsilon_{j}^0\| - \kappa_\Delta \tau \sum_{m=0,m \neq j}^{N_j} \|\epsilon_{m,j}^0\| - \kappa_{\Delta \tau}^{\beta} \sum_{m=0,m \neq j}^{N_j} \|\epsilon_{j,m}^0\|,
$$

Assuming $\|E_s\|_\infty \leq \|E_{s-1}\|_\infty$, $s = 1, 2, 3, \cdots, n - 1$, we have

$$
\|E_{n}\|_\infty = \|\epsilon_{j}^{n}\|_\infty \leq \|\epsilon_{j}^{n-1}\| - \kappa_\Delta \tau \sum_{m=0}^{N_j} S_{j,m}^{\alpha} \|\epsilon_{m,j}^{n-1}\| - \kappa_{\Delta \tau}^{\beta} \sum_{m=0}^{N_j} \|\epsilon_{j,m}^{n-1}\|.
$$

$$
\leq \left(1 - \kappa_\Delta \tau S_{j}^{\alpha} - \kappa_{\Delta \tau}^{\beta} S_{j}^{\beta}\right) \|\epsilon_{j}^{n-1}\| - \kappa_\Delta \tau \sum_{m=0,m \neq j}^{N_j} \|\epsilon_{m,j}^{n-1}\| - \kappa_{\Delta \tau}^{\beta} \sum_{m=0,m \neq j}^{N_j} \|\epsilon_{j,m}^{n-1}\|,
$$

$$
= \sum_{k=1}^{n-1} \epsilon_{j}^{k+1} e^{-\sigma(n-1-k)\tau} \left(1 - e^{-\sigma \tau} \right) + \epsilon_{j}^0 e^{-\sigma(n-1)\tau} \leq \sum_{k=1}^{n-1} e^{-\sigma(n-1-k)\tau} \left(1 - e^{-\sigma \tau} \right) + e^{-\sigma(n-1)\tau} \|E_0\|_\infty = \|E_0\|_\infty.
$$

The proof is completed.
Theorem 3.6 Let $u_{i,j}^0$ be the approximate solution of $u(x,y,t_0)$ computed by the difference scheme (2.16). Let $e_{i,j} = u_i(x_i,y_j,t_0) - u_{i,j}^0$. Then

$$||e^0||_w \leq (1 - \gamma) (\sigma T + e^{\sigma \tau}) C_\alpha \left( \tau^2 + (\Delta x)^2 + (\Delta y)^2 \right),$$

where $C_\alpha$ is defined by (2.14) and $(x_i, y_j, t_0) \in \Omega \times (0, T]$ with $N \tau \leq T$.

Proof Let $u(x_i, y_j, t_0)$ be the exact solution of (1.1) at the mesh point $(x_i, y_j, t_0)$. Defined $e_{i,j} = u_i(x_i, y_j, t_0) - u_{i,j}^0$. Subtracting (2.13) from (2.16) with $e_{i,j}^0 = 0$, we obtain

$$\left( 1 - \kappa_{\Delta x, \tau}^\alpha - s_{\Delta y, \tau}^\beta \right) e_{i,j}^1 - \kappa_{\Delta x, \tau}^\alpha \sum_{m=0, m \neq i}^N \frac{g_{i,m} e_{m,j}^1}{\Delta x^i} - \kappa_{\Delta y, \tau}^\beta \sum_{m=0, m \neq j}^N \frac{g_{j,m} e_{i,m}^1}{\Delta y^j}$$

$$= \frac{(1 - \gamma) \sigma \tau}{1 - e^{-\sigma \tau}} e_{i,j}^0, \quad n = 1,$$

$$\left( 1 - \kappa_{\Delta x, \tau}^\alpha - s_{\Delta y, \tau}^\beta \right) e_{i,j}^n - \kappa_{\Delta x, \tau}^\alpha \sum_{m=0, m \neq i}^N \frac{g_{i,m} e_{m,j}^n}{\Delta x^i} - \kappa_{\Delta y, \tau}^\beta \sum_{m=0, m \neq j}^N \frac{g_{j,m} e_{i,m}^n}{\Delta y^j}$$

$$= \sum_{k=1}^{n-1} e_{i,j}^k e^{-\sigma (n-1-k) \tau} \left( 1 - e^{-\sigma \tau} \right) + \frac{(1 - \gamma) \sigma \tau}{1 - e^{-\sigma \tau}} e_{i,j}^0, \quad n > 1.$$

We next prove the desired results by the mathematical induction.

Let $|e_{i,j}^0| := ||e^0||_w = \max_{0 \leq i \leq N_\alpha, 0 \leq j \leq N_\beta} |e_{i,j}^0|$ and $r_{\max} = \max_{0 \leq i \leq N_\alpha, 0 \leq j \leq N_\beta} |e_{i,j}^0|$. Using Lemma 3.3, we have

$$||e^1||_w = |e_{i,j}^1| \leq \left| |e_{i,j}^0| - \kappa_{\Delta x, \tau}^\alpha \sum_{m=0}^{N_\alpha} g_{i,m}^\alpha |e_{m,j}^0| - \kappa_{\Delta y, \tau}^\beta \sum_{m=0}^{N_\beta} g_{j,m}^\beta |e_{i,m}^0| \right|$$

$$= \left| \left( 1 - \kappa_{\Delta x, \tau}^\alpha \sum_{m=0}^{N_\alpha} g_{i,m}^\alpha \right) e_{i,j}^1 - \kappa_{\Delta y, \tau}^\beta \sum_{m=0}^{N_\beta} g_{j,m}^\beta |e_{i,m}^0| \right|$$

$$\leq \left( 1 - \kappa_{\Delta x, \tau}^\alpha \sum_{m=0}^{N_\alpha} g_{i,m}^\alpha \right) \left| e_{i,j}^1 \right| - \kappa_{\Delta y, \tau}^\beta \sum_{m=0}^{N_\beta} g_{j,m}^\beta |e_{i,m}^0|$$

$$\leq \left( 1 - \gamma \right) \sigma \tau \left| e_{i,j}^0 \right| \frac{1 - e^{-\sigma \tau}}{1 - e^{-\sigma \tau}} r_{\max}.$$

Supposing $||e^0||_w = \max_{0 \leq i \leq N_\alpha, 0 \leq j \leq N_\beta} |e_{i,j}^0|$ and $s = 1, 2, 3, \cdots, n - 1$

$$\left| e_{i,j}^s \right| \leq \frac{(s - 1) (1 - e^{-\sigma \tau}) + 1}{1 - e^{-\sigma \tau}} \left( 1 - \gamma \right) \sigma \tau r_{\max}, \quad s = 1, 2, 3, \cdots, n - 1.$$
we have

\[ ||e^n|| = ||e^\tau_{0,j0}|| \leq ||e_{0,j0}^\tau|| - \kappa_{\Delta x, \tau}^\alpha \sum_{m=0}^{N_x} g_{0,m}^\tau ||e_{0,j0}^\tau|| - \kappa_{\Delta x, \tau}^\beta \sum_{m=0}^{N_x} s_{0,m}^\tau ||e_{0,j0}^\tau|| - \kappa_{\Delta y, \tau}^\beta \sum_{m=0,m \neq j_0}^{N_x} s_{m,j_0}^\tau ||e_{0,j0}^\tau|| \]

\[ = ||e_{0,j0}^\tau|| - \kappa_{\Delta x, \tau}^\alpha \sum_{m=0}^{N_x} g_{0,m}^\tau ||e_{0,j0}^\tau|| - \kappa_{\Delta y, \tau}^\beta \sum_{m=0,m \neq j_0}^{N_x} s_{m,j_0}^\tau ||e_{0,j0}^\tau|| \]

\[ \leq \left( 1 - \kappa_{\Delta x, \tau}^\alpha \sum_{m=0}^{N_x} g_{0,m}^\tau - \kappa_{\Delta y, \tau}^\beta \sum_{m=0,m \neq j_0}^{N_x} s_{m,j_0}^\tau \right) ||e_{0,j0}^\tau|| \leq \left( 1 - \kappa_{\Delta x, \tau}^\alpha \sum_{m=0}^{N_x} g_{0,m}^\tau - \kappa_{\Delta y, \tau}^\beta \sum_{m=0,m \neq j_0}^{N_x} s_{m,j_0}^\tau \right) ||e_{0,j0}^\tau|| \]

\[ \leq \left( 1 - \kappa_{\Delta x, \tau}^\alpha \sum_{m=0}^{N_x} g_{0,m}^\tau - \kappa_{\Delta y, \tau}^\beta \sum_{m=0,m \neq j_0}^{N_x} s_{m,j_0}^\tau \right) ||e_{0,j0}^\tau|| = \sum_{i=1}^{N_x} c^\tau_{i,j0} e^{-\sigma(\sigma-1)\tau} (1 - e^{-\sigma \tau}) + \sum_{i=1}^{N_x} c^\tau_{i,j0} e^{-\sigma(\sigma-1)\tau} \leq \Phi \cdot r_{\text{max}}. \]

where \( \Phi \) is given by (3.4).

According to (2.14), we can get

\[ ||e^n|| \leq (1 - \gamma) \left( \sigma T + e^{\beta \sigma \tau} \right) r_{\text{max}} \leq (1 - \gamma) (\sigma T + e^{\sigma \tau}) C_u (T^2 + (\Delta x)^2 + (\Delta y)^2). \]

The proof is completed.

4 Numerical results

In the section, we numerically verify the above theoretical results including convergence orders and numerical stability. And the \( l_m \) norm and the discrete \( L^2 \)-norm, respectively, are used to measure the numerical errors.

Example 1

Consider (2.3) on a finite domain with \( 0 < x < 1, 0 < t \leq 1 \), and the forcing function is

\[ f(x,t) = -\frac{\sigma}{1 - \gamma} e^{-\sigma \gamma} x^2 (1 - x)^2 + \frac{e^{-\sigma \gamma}}{2\cos(\alpha \pi/2)} \frac{24}{\Gamma(5 - \alpha)} (x^{4 - \alpha} + (1 - x)^{4 - \alpha}) \]

\[ - \frac{e^{-\sigma \gamma}}{2\cos(\alpha \pi/2)} \frac{12}{\Gamma(4 - \alpha)} (x^{3 - \alpha} + (1 - x)^{3 - \alpha}) \]

\[ + \frac{e^{-\sigma \gamma}}{2\cos(\alpha \pi/2)} \frac{2}{\Gamma(3 - \alpha)} (x^{2 - \alpha} + (1 - x)^{2 - \alpha}) \]

with the nonzero initial condition \( u(x,0) = x^2 (1 - x)^2 \) and the homogeneous Dirichlet boundary conditions. The exact solution of the fractional PDEs is

\[ u(x,t) = e^{-\sigma \gamma} x^2 (1 - x)^2. \]

Tables [12] and [13] show that the schemes (2.11) have the global truncation errors \( O(T^2 + (\Delta x)^2) \) at time \( T = 1 \). Here the \( l_m \) norm and the discrete \( L^2 \)-norm, respectively, are used to measure the numerical errors for (2.11) with \( \tau = \Delta x \).
Table 1 The maximum errors and convergence orders for (2.11) with $\tau = \Delta x$.

| $l_{\infty}$ norm | $\tau$ | $\alpha = 1.2$ | Rate | $\alpha = 1.8$ | Rate |
|-------------------|--------|----------------|------|----------------|------|
| $\gamma = 0.1$    | 1/40   | 1.0686e-04     |      | 1.3426e-04     |      |
|                   | 1/80   | 2.9917e-05     | 1.8367| 3.3543e-05     | 2.0009|
|                   | 1/160  | 7.9022e-06     | 1.9206| 8.3559e-06     | 2.0051|
|                   | 1/320  | 2.0766e-06     | 1.9281| 2.0766e-06     | 2.0086|
| $\gamma = 0.5$    | 1/40   | 4.4671e-05     | 6.0820e-05| 6.0820e-05     |      |
|                   | 1/80   | 1.2415e-05     | 1.8473| 1.5196e-05     | 2.0009|
|                   | 1/160  | 3.2730e-06     | 1.9234| 3.7834e-06     | 2.0059|
|                   | 1/320  | 8.6348e-07     | 1.9224| 9.3952e-07     | 2.0097|
| $\gamma = 0.9$    | 1/40   | 2.9977e-05     | 6.0820e-05| 6.0820e-05     |      |
|                   | 1/80   | 7.4790e-06     | 2.0029| 1.7237e-06     | 1.9973|
|                   | 1/160  | 1.8634e-06     | 2.0049| 4.3057e-07     | 2.0012|
|                   | 1/320  | 4.6419e-07     | 2.0097| 1.0735e-07     | 2.0040|

Table 2 The discrete $L^2$-norm errors and convergence orders for (2.11) with $\tau = \Delta x$.

| $L^2$-norm | $\tau$ | $\alpha = 1.2$ | Rate | $\alpha = 1.8$ | Rate |
|------------|--------|----------------|------|----------------|------|
| $\gamma = 0.1$    | 1/40   | 6.6304e-05     |      | 8.9805e-05     |      |
|                   | 1/80   | 1.6925e-05     | 1.9700| 2.2274e-05     | 2.0114|
|                   | 1/160  | 4.3290e-06     | 1.9670| 5.5195e-06     | 2.0128|
|                   | 1/320  | 1.1060e-06     | 1.9687| 1.3670e-06     | 2.0135|
| $\gamma = 0.5$    | 1/40   | 3.0386e-05     |      | 4.0823e-05     |      |
|                   | 1/80   | 7.6617e-06     | 1.9877| 1.0123e-05     | 2.0118|
|                   | 1/160  | 1.9408e-06     | 2.0128| 2.5056e-06     | 2.0144|
|                   | 1/320  | 4.9221e-07     | 2.0159| 6.1953e-07     | 2.0159|
| $\gamma = 0.9$    | 1/40   | 2.2186e-05     |      | 4.9221e-06     |      |
|                   | 1/80   | 5.5359e-06     | 2.0027| 1.2327e-06     | 1.9975|
|                   | 1/160  | 1.3791e-06     | 2.0050| 3.0788e-06     | 2.0014|
|                   | 1/320  | 3.4350e-07     | 2.0042| 7.6749e-08     | 2.0042|

Example 2

Consider (1.1) on a finite domain with $0 < x < 1, 0 < y < 1, 0 < t \leq 1$, and the forcing function is

\[
f(x, y, t) = -\frac{\sigma}{1 - \gamma} e^{-\sigma \gamma} x^2 (1 - x)^2 \gamma^2 (1 - y)^2 + e^{-\sigma \gamma} x^2 (1 - y)^2 \frac{24(x^{1-\alpha} + (1-x)^{1-\alpha})}{\Gamma(5-\alpha)} + \frac{e^{-\sigma \gamma} y^2 (1-y)^2}{2\cos(\alpha \pi/2)} \left( \frac{2(x^{2-\alpha} + (1-x)^{2-\alpha})}{\Gamma(3-\alpha)} - \frac{12(x^{3-\alpha} + (1-x)^{3-\alpha})}{\Gamma(4-\alpha)} \right) + \frac{e^{-\sigma \gamma} x^2 (1-x)^2}{2\cos(\alpha \pi/2)} \left( \frac{24(y^{4-\alpha} + (1-y)^{4-\alpha})}{\Gamma(5-\alpha)} - \frac{12(y^{3-\alpha} + (1-y)^{3-\alpha})}{\Gamma(4-\alpha)} \right) + \frac{e^{-\sigma \gamma} y^2 (1-y)^2}{2\cos(\alpha \pi/2)} \left( \frac{2(y^{2-\alpha} + (1-y)^{2-\alpha})}{\Gamma(3-\alpha)} \right)
\]

$\gamma = \gamma_1 - \gamma_2$, $\sigma = \frac{\gamma}{1 - \gamma}$.
with the nonzero initial condition 
\[ u(x, y, 0) = x^2(1 - x)^2 y^2(1 - y)^2 \]
and the homogeneous Dirichlet boundary conditions. The exact solution of the (1.1) is
\[ u(x, y, t) = e^{-\frac{\sigma t}{2}} x^2(1 - x)^2 y^2(1 - y)^2. \]

### Table 3: The maximum errors and convergence orders for (2.16) with \( \tau = \Delta x = \Delta y \).

| \( l_\infty \) norm | \( \tau = 1/10 \) | \( \alpha = 1.2, \beta = 1.3 \) Rate | \( \alpha = 1.8, \beta = 1.7 \) Rate |
|---------------------|----------------|-----------------------------------|-----------------------------------|
| \( \gamma = 0.3 \)  | 2.8759e-05    | 8.7959e-05                        | 1.0126e-04                        |
| \( \gamma = 0.7 \)  | 2.733e-05     | 1.8114e-05                        | 1.9828                            |
| \( \gamma = 0.3 \)  | 1/20           | 2.0296                            | 2.5618e-05                        |
| \( \gamma = 0.7 \)  | 1/40           | 2.0282                            | 6.4708e-06                        |
| \( \gamma = 0.3 \)  | 1/80           | 2.0207                            | 1.6713e-06                        |
| \( \gamma = 0.7 \)  | 1/20           | 2.0262                            | 4.5869e-06                        |
| \( \gamma = 0.3 \)  | 1/40           | 2.0284                            | 1.1586e-06                        |
| \( \gamma = 0.7 \)  | 1/80           | 2.0045                            | 2.480e-07                         |

Table 3 shows that the maximum error, at time \( T = 1 \) and \( \tau = \Delta x = \Delta y \), between the exact analytical value and the numerical value. The scheme (2.16) is second-order convergence and this is in agreement with the order of the truncation error.

### 5 Conclusions

As is well known, there is less than the second-order convergence for the Caputo fractional derivative \([18, 23]\) with L1 formula. We notice that there are already some theoretical convergence results for Caputo-Fabrizio fractional derivative \([3, 7]\) with L1 formula. However, it seems that achieving a second-order accurate scheme (optimal estimates) is not an easy task. To our knowledge, this is the first published finite difference method to consider the space fractional diffusion equations with the time Caputo-Fabrizio fractional derivative. The optimal estimates with the second-order convergence for L1 scheme are given by two methods. We remark that the corresponding theoretical including a prior estimate can also be extended to the nonzero initial values \([13, 17]\).

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