Surgery Transformations and Spectral Estimates of $\delta$ Beam Operators

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Abstract
We introduce $\delta$ type vertex conditions for beam operators, the fourth-order differential operator, on finite, compact and connected metric graphs. Our study the effect of certain geometrical alterations (graph surgery) of the graph on their spectra. Results are obtained for a class of vertex conditions which can be seen as an analogue of $\delta$-conditions for graphs Laplacian. There are a number of possible candidates of $\delta$ type conditions for beam operators. We develop surgery principles and record the monotonicity properties of their spectrum, keeping in view the possibility that vertex conditions may change within the same class after certain graph alterations. We also demonstrate the applications of surgery principles by obtaining several lower and upper estimates on the eigenvalues.

Keywords Beam operators · Metric graphs · Surgery principles · Bounds on eigenvalues

Mathematics Subject Classification Primary: 34B45 · 35P15; Secondary: 05C50

1 Introduction

The study of spectral estimates of quantum graphs has attracted significant attention in the last couple of decades. One of the major objectives of these studies is to shed light on the relationships between spectrum and topological properties of quantum networks and, by doing so, develop methods and techniques to estimate the eigenvalues

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in terms of parameters associated with differential operators and graph topology or connectivity. Most of the recent studies have focused on investigating the effect of surgical operations on the spectrum of a given graph and how one can use a set of surgical operations for spectral estimations. This involves transforming a graph into a simpler one by accurately predicting the corresponding changes in the spectrum. This leads to certain spectral estimates on the original graph in terms of quantities such as total length, Betti number and number of edges and vertices. These techniques were applied successfully to derive lower and upper bounds on the spectral gap and higher eigenvalues for graph Laplacian equipped with Kirchhoff, δ and Dirichlet vertex conditions, see [1] and references therein.

In the current paper, motivated by [1] and [2], we discuss the surgery principles and spectral estimates of the fourth-order operator, $\frac{d^4}{dx^4}$, on metric graphs equipped with δ—type conditions. We will refer to this operator as δ beam operator. Several possible vertex conditions for beam operators correspond to analogous δ-type conditions for Laplacian. Among all others, we only consider four of their kinds, see vertex condition (C1), (C2), (C3) and (C4). Some special cases of these conditions have already appeared at several places in connection with the studies of Euler-Bernoulli beam operators. For instance, in [3] Dekoninck and Nicaise study the characteristic equation for the spectrum of the Euler-Bernoulli beam operator on networks equipped with vertex conditions (C1) and (C3) with $\alpha_m = 0$ and showed the dependence of spectrum on the structure of the graph. In [4] they study the exact controllability problem of hyperbolic systems of networks of beams subject to the special case of vertex conditions (C2) with $\alpha_m = 0$. Gregorio and Mungnolo in [5] have discussed the qualitative properties of one-dimensional parabolic evolution equations associated with the fourth-order differential operator, defined on metric graphs equipped with these vertex conditions, and have investigated how different transmission conditions in the vertices of a network can arouse rather different behaviors of solutions to such partial differential equations. In [6], based on physical-geometrical considerations of beams embedded in $\mathbb{R}^2$, self-adjoint vertex conditions were derived, which take into account the geometry of the underlying graph and depend on angles between the edges. The obtained vertex conditions are a particular case of (C3). In three dimension, the vertex conditions and corresponding self-adjoint beam operators for rigid joints of elastic beam frames have recently appeared in [7, 8].

Surgery principles for beam operators were recently studied by Kurasov and Muller in [2]. Vertex conditions (C4) with $\alpha_m = 0$ were considered and certain lower and upper estimates were obtained on the spectral gap. For the Laplacian, on the other hand, there has been significant progress on the topic of spectral estimation using surgery principles and other methods, see [9–18]. In [19] Nicaise showed that among all graphs with a fixed length, the minimal spectral gap is obtained for the single-edge graph. In [13] Friedlander proved a more general result, showing that the minimum of the $k$-th eigenvalue is uniquely obtained for a star graph with $k$ edges. Exner and Jex [20] showed that modifying edge lengths can either increase or decrease the spectral gap, depending on the graph’s topology. In the past few years, a series of works on the subject has emerged. Kurasov and Naboko in [21] treated the spectral gap minimization and together with Malenová they explored how spectral gap changes with
various modifications of graph connectivity [17]. In [22] also studied the dependence of the spectral gap on the choice of edge lengths. Kennedy, Kurasov, Malenová and Mugnolo provided a broad survey on bounding the spectral gap in terms of various geometric quantities of the graph [12]. In [11] Karreskog, Kurasov and Kupersmidt generalized the minimization results, mentioned above, to Schrödinger operators with potentials and δ-type vertex conditions. In [23] Del Pezzo and Rossi proved upper and lower bounds for the spectral gap of the p-Laplacian and derived a formula for the first eigenvalue’s shape derivative when altering an edge’s length in the graph. In [14] Rohleder solved the spectral gap maximization problem for all eigenvalues of tree graphs. Ariturk provides some improved upper bounds for all graph eigenvalues [9]. Berkolaiko, Kennedy, Kurasov and Mugnolo further generalize lower and upper bounds for the spectral gap, considering the edge connectivity [10] and intensively studied the surgery principles for Laplacian in [1].

The paper is organized as follows. In Sect. 2, we introduce the δ beam operators and discuss the quadratic forms and their domains associated with different graphs equipped with different vertex conditions. This is followed by some preliminary results in Sect. 3. Section 4 deals with the eigenvalues dependence on the vertex conditions, these results co-relate the eigenvalues of same underlying metric graph subject to different vertex conditions. Sections 5 and 6 are regarding the surgery principles for the metric graphs with beam operators. In Sect. 5, we consider surgical transformations which changes graph’s connectivity but total length remains the same. These operations include the gluing and splitting of vertices. This surgical operation is classified into three major parts, depending on vertex conditions. Depending on the choice of vertex conditions, the eigenvalues of the glued graph are interlaced by one, two, or more degrees with the eigenvalues of the unglued graph. The operation of splitting a vertex, in general, does not behave as a converse of gluing. Section 6 deals with the surgical operation that increases the total length of a graph. This includes attaching a pendant graph and inserting a graph at some vertex of another graph. Both of these operations lower the eigenvalues of a metric graph. Sections 7 and 8 are dedicated to the application of surgery principles for obtaining eigenvalue estimates. Some of the estimates for the lowest non-zero eigenvalue are obtained using some trial functions from the domain of the quadratic form. The bounds on general eigenvalues are derived by using topological alterations of graphs that influence the eigenvalues in a predictable manner. Using these techniques, certain classes of metric graphs permit stronger upper estimates (bipartite graphs), whilst others permit stronger lower estimates (Eulerian graphs).

To enhance the spectral estimates for beam operators that appear in references [7, 8], we plan to extend the principles of surgery to apply to such formulations of beam operators in our future work.

2 The δ Beam Operators

Let $\Gamma$ be a connected finite compact metric graph with edge set $E = \{e_n\}_{n=1}^{|E|}$ and vertex set $V = \{v_m\}_{m=1}^{|V|}$. Each edge $e_n$ is identified with an interval $[x_{2n-1}, x_{2n}]$ of the
real-line, and each vertex $v_m$ can be considered as a partition of the set of all endpoints \( \{ x_j \}_{j=1}^{2|E|} \), for more details see [24, 25].

We define the $\delta$ beam operators via the closed quadratic form

\[
 h[\varphi] := \int_{\Gamma} |\varphi''(x)|^2 \, dx + \sum_{m=1}^{\lfloor V \rfloor} \alpha_m |\varphi(v_m)|^2
\]

(2.1)

with $\varphi \in W_2^2(\Gamma \setminus V)$ and $\alpha_m \in \mathbb{R}$. It is natural to assume that the beams are connected at vertices. This means that the functions satisfy continuity condition

\[
 \varphi(x_i) = \varphi(x_j) = \varphi(v_m), \quad x_i, x_j \in v_m
\]

(2.2)

at the vertices. If no further conditions are assumed on the domain of the quadratic form $h$, then (2.1) corresponds to the fourth-order differential operator $\frac{d^4}{dx^4}$ defined on the functions $\varphi \in W_2^4(\Gamma \setminus V)$ satisfying vertex conditions

\[
 \begin{align*}
 \varphi(x_i) &= \varphi(x_j) \equiv \varphi(v_m), \quad x_i, x_j \in v_m, \\
 \frac{\partial^2}{\partial x^2} \varphi(x_j) &= 0, \\
 \sum_{x_j \in v_m} \frac{\partial^3}{\partial x^3} \varphi(x_j) &= -\alpha_m \varphi(v_m).
\end{align*}
\]

(C1)

In addition to the continuity condition (2.2), if we further assume that the normal derivatives of functions from the domain of the quadratic form $h$ satisfy the condition

\[
 \sum_{x_j \in v_m} \sigma_{x_j}^{v_m} \frac{\partial \varphi(x_j)}{\partial x} = 0, \quad \sigma_{x_j}^{v_m} \in \mathbb{C} \setminus \{0\},
\]

(2.3)

then the corresponding self-adjoint fourth-order differential operator $\frac{d^4}{dx^4}$ is defined on functions from $W_2^4(\Gamma \setminus V)$ satisfying the following conditions at vertices

\[
 \begin{align*}
 \varphi(x_i) &= \varphi(x_j) \equiv \varphi(v_m), \quad x_i, x_j \in v_m, \\
 \sum_{x_j \in v_m} \frac{\sigma_{x_j}^{v_m} \frac{\partial}{\partial x} \varphi(x_j)}{\sigma_{x_j}^{v_m}} &= 0, \quad \sigma_{x_j}^{v_m} \in \mathbb{C} \setminus \{0\}, \\
 \frac{\sigma_{x_j}^{v_m} \frac{\partial^2}{\partial x^2} \varphi(x_j)}{\sigma_{x_j}^{v_m}} &= \frac{\sigma_{x_j}^{v_m} \frac{\partial^2}{\partial x^2} \varphi(x_j)}{\sigma_{x_j}^{v_m}}, \\
 \sum_{x_j \in v_m} \frac{\partial^3}{\partial x^3} \varphi(x_j) &= -\alpha_m \varphi(v_m).
\end{align*}
\]

(C2)

If $\sigma_{x_j}^{v_m} = 0$, we set $\frac{\partial^2}{\partial x^2} \varphi(x_j) = 0$ and impose no further condition on $\frac{\partial}{\partial x} \varphi(x_j)$. Vertex conditions (C1) correspond to the case when $\sigma_{x_j}^{v_m}$ are equal to zero in (C2). A special set of vertex conditions of the form (C2) for planar graphs was discussed in [6].

**Remark 2.1** For vertex conditions (C2) with $\alpha_m = 0, m = 1, 2, \ldots, \lfloor V \rfloor$, and $\sigma_{x_j}^{v_m} = \sigma_{x_j}^{v_{m}}$, any interior point of an edge can be considered as a vertex $v_m$ of degree two.
equipped with (C2) with $\alpha_m = 0$. Since at any interior point functions and their first, second and third derivatives are continuous. Similarly, any vertex $v_m$ of degree two equipped with vertex condition (C2) with $\alpha_m = 0$ and $\sigma_{x_i}^{v_m} = \sigma_{x_j}^{v_m}$ can be regarded as an interior point, and two edges are replaced by a single long edge.

Note that, changing the role of the first and second normal derivatives in the vertex conditions (C2) does not effect the self-adjointness of the operator. This allows us to define the $\delta$ beam operator $\frac{d^4}{dx^4}$ on the functions from $W^4_2(\Gamma \setminus V)$ satisfying the vertex conditions

$$
\begin{align*}
\phi(x_i) &= \phi(x_j) \equiv \phi(v_m), \quad x_i, x_j \in v_m, \\
\frac{\partial \phi(x_i)}{\sigma_{x_i}^{v_m}} &= \frac{\partial \phi(x_j)}{\sigma_{x_j}^{v_m}}, \quad \sigma_{x_j}^{v_m} \in \mathbb{C} \setminus \{0\}, \\
\sum_{x_j \in v_m} \sigma_{x_j}^{v_m} \partial^2 \phi(x_j) &= 0, \\
\sum_{x_j \in v_m} \partial^3 \phi(x_j) &= -\alpha_m \phi(v_m). 
\end{align*}
$$

(C3)

If $\sigma_{x_j}^{v_m} = 0$, we set $\partial \phi(x_j) = 0$ and do not assume any condition on $\partial^2 \phi(x_j)$. In this case, the vertex conditions (C3) become

$$
\begin{align*}
\phi(x_i) &= \phi(x_j) \equiv \phi(v_m), \quad x_i, x_j \in v_m, \\
\partial \phi(x_j) &= 0, \\
\sum_{x_j \in v_m} \partial^3 \phi(x_j) &= -\alpha_m \phi(v_m). 
\end{align*}
$$

(C4)

When $\alpha_m = \infty$ at a specific vertex $v_m$, we shall call such vertex conditions (C1)-(C4) as the extended vertex conditions, and the functions vanish at all endpoints of intervals incident to $v_m$. In this case, the continuity condition is replaced by $\phi(x_j) = 0$, and the last equation in all four conditions becomes redundant. Note that the assumption $\alpha_m = \infty$ in (C1) and (C4) has the effect of disconnecting edges at vertex $v_m$, and the spectrum of a graph is equal to the union of the spectrum of disjoints intervals. However, we can not decouple a graph into disjoints intervals for conditions (C2) and (C3). This is because functions living on incident edges of the vertex are interconnected at the vertex due to the vanishing of a scalar combination of normalized derivative in (C2) and normalized continuity in the first derivative in (C3). Moreover, for a boundary vertex, a vertex of degree one, the conditions (C2) and (C4) are equivalent, and the conditions (C1) and (C3) also coincide.

3 Preliminary Results

A standard method in obtaining eigenvalue estimates is based on variational arguments comparing the Rayleigh quotients on suitable finite-dimensional subspaces. In this section, we collect results, without proofs, that will be needed in subsequent sections. We will follow the strategy of [1], where the Laplacian is considered on finite compact
graphs with $\delta$ and Dirichlet vertex conditions. We will need the following min-max characterisation of eigenvalues.

**Proposition 3.1** Let $\Gamma$ be a connected finite compact metric graph, and $h$ be its quadratic form, which is semi-bounded from below. Then the eigenvalues $\lambda_k(\Gamma)$ satisfy

$$
\lambda_k(\Gamma) = \min_{X_k \subset D(h)} M(h, X_k)
$$

(3.1)

$$
= \max_{X_{k-1} \subset D(h)} m(h, X_{k-1}^\perp)
$$

(3.2)

where, $M(h, X_k) = \max_{0 \neq \varphi \in X_k} \frac{h[\varphi]}{||\varphi||^2}$ and $m(h, X_{k-1}^\perp) = \min_{0 \neq \varphi \in X_{k-1}^\perp} \frac{h[\varphi]}{||\varphi||^2}$.

If a $k$-dimensional subspace realises the minimum $\lambda_k(\Gamma)$ in (3.1), we call it a minimising subspace for $\lambda_k(\Gamma)$. The following Lemma characterizes the equality.

**Lemma 3.2** [1, Lemma 4.1] Any minimising subspace $X_k$ for eigenvalue $\lambda_k(\Gamma)$ contains an eigenvector of $\lambda_k(\Gamma)$. Moreover, if $\lambda_k(\Gamma) < \lambda_{k+1}(\Gamma)$, then the eigenspace of $\lambda_k(\Gamma)$ is the intersection of its all possible $k$-dimensional minimising subspaces.

Let $\Gamma$ and $\tilde{\Gamma}$ be connected finite compact metric graphs, and $h$ and $\tilde{h}$ be their corresponding semi-bounded from below quadratic forms. We say that $\tilde{h}$ is a positive rank-$n$ perturbation of $h$ if $\tilde{h} = h$ on some $Y \subset D(h)$ and either $Y = D(\tilde{h}) \subset_n D(h)$ or $\tilde{h} \geq h$ with $D(\tilde{h}) = D(h)$. Here, the symbol $\subset_n$ denotes the subspace such that the quotient space $D(h)/Y$ is $n$-dimensional.

**Theorem 3.3** [1, Theorem 4.3]. Let $\Gamma$ and $\tilde{\Gamma}$ be two connected finite compact metric graphs, and $h$ and $\tilde{h}$ be their corresponding semi-bounded from below quadratic forms. If form $\tilde{h}$ is a positive rank-1 perturbation of the form $h$, then the eigenvalues of $\Gamma$ and $\tilde{\Gamma}$ satisfy

$$
\lambda_k(\Gamma) \leq \lambda_k(\tilde{\Gamma}) \leq \lambda_{k+1}(\Gamma) \leq \lambda_{k+1}(\tilde{\Gamma}), \quad k \geq 1.
$$

**4 Eigenvalues Dependence on Vertex Conditions**

In this section, we present the dependence of eigenvalues of a metric graph on vertex conditions. We describe the relationship between eigenvalues of graphs equipped with any of the four vertex conditions. The general principle of monotonicity for eigenvalues applies when either the interaction strength at a particular vertex is increased or when the conditions at that vertex are replaced with different vertex conditions. The following proposition, which describes the influence of vertex conditions on a graph’s spectrum, presents a series of interlacing inequalities that connect eigenvalues of the same underlying metric graphs equipped with arbitrary self-adjoint vertex conditions at all vertices except for $v_1$, where any of the four vertex conditions is imposed.
Proposition 4.1 Let $\Gamma^1$ be connected finite compact metric graph, and $v$ in $\Gamma^1$ be equipped with vertex conditions (C1). Let $\Gamma^2$, $\Gamma^3$ and $\Gamma^4$ be graphs obtained from $\Gamma^1$ by imposing vertex conditions (C2), (C3), and (C4) at $v$, respectively. Then the eigenvalues of $\Gamma^1$, $\Gamma^2$, $\Gamma^3$ and $\Gamma^4$ satisfy the following inequalities:

1. $\lambda_k(\Gamma^1) \leq \lambda_k(\Gamma^2) \leq \lambda_{k+1}(\Gamma^1)$.
2. $\lambda_k(\Gamma^1) \leq \lambda_k(\Gamma^3) \leq \lambda_{k+d-1}(\Gamma^1)$.
3. $\lambda_k(\Gamma^1) \leq \lambda_k(\Gamma^4) \leq \lambda_{k+d}(\Gamma^1)$.
4. $\lambda_k(\Gamma^2) \leq \lambda_{k+1}(\Gamma^3) \leq \lambda_{k+d}(\Gamma^2)$.
5. $\lambda_k(\Gamma^2) \leq \lambda_k(\Gamma^4) \leq \lambda_{k+d}(\Gamma^2)$.
6. $\lambda_k(\Gamma^3) \leq \lambda_k(\Gamma^4) \leq \lambda_{k+d}(\Gamma^3)$.

Where $d$ is the degree of vertex $v$.

Proof Consider the quadratic forms $h^i$ and their respective domains $D(h^i)$ associated with metric graphs $\Gamma^i$, where $i = 1, 2, 3, 4$. For functions $\varphi$ from $W^2_2(\Gamma^1\setminus V)$, which are continuous across the entire graph, the domains are defined as follows.

\[
D(h^1) = \{ \varphi \in W^2_2(\Gamma^1\setminus V) : \varphi \text{ is continuous on } \Gamma^1 \},
\]

\[
D(h^2) = \{ \varphi \in D(h^1) : \sum_{x_j \in v} \sigma_{x_j}^v \partial \varphi(x_j) = 0 \},
\]

\[
D(h^3) = \left\{ \varphi \in D(h^1) : \frac{\partial \varphi(x_j)}{\sigma_{x_j}^v} = \frac{\partial \varphi(x_j)}{\sigma_{x_j}^v} \right\}, \quad D(h^4)
\]

\[
= \{ \varphi \in D(h^1) : \partial \varphi(x_j) = 0, x_j \in v \}.
\]

Then the following two inclusions hold.

\[
D(h^4) \subset D(h^3) \subset D(h^1) \quad \text{and} \quad D(h^4) \subset D(h^3) \subset D(h^1). \tag{4.1}
\]

Since quadratic forms $h^i$ and $h^i$ agree on $D(h^4)$. Moreover, the domains $D(h^2)$, $D(h^3)$ and $D(h^4)$ are co-dimension one, $d - 1$ and $d$ subspaces of $D(h^1)$, respectively. Thus, the first three inequalities follow from the conclusion of Theorem (3.3). The remaining inequalities follow directly from the first three inequalities. \hfill \square

The above interlacing inequalities are very useful in obtaining estimates on eigenvalues, particularly when comparing eigenvalues of the same underlying graph equipped with different vertex conditions. If the spectrum of one of the graphs is known, then the spectrum of another graph can be estimated accordingly. To have a clearer view of its application, we provide the following example to illustrate the above inequality.

Example 4.2 Let $C$ be a loop of length $\ell$, parameterized by $[0, \ell]$, and the vertex $v_0$ is equipped with condition (C2) with $\sigma_0 = \sigma_\ell$, or (C3) with $\sigma_0 = -\sigma_\ell$, with strength zero, then the non-zero eigenvalues are $\lambda_{k+1}(C) = \left(\frac{2\pi}{\ell}\right)^4$, each having multiplicity two. Let $C'$ be the loop obtained from $C$ by imposing vertex conditions (C4) at $v_0$ with $\sigma_0 = 0$, then $\left(\frac{2\pi}{\ell}\right)^4$ is the first non-zero eigenvalue. Moreover, other eigenvalues are
zeros of the following equation.

\[
\sin \left( \sqrt[4]{\lambda \ell} \right) \left( 1 - \cosh \left( \sqrt[4]{\lambda \ell} \right) \right) + \sinh \left( \sqrt[4]{\lambda \ell} \right) \left( 1 - \cos \left( \sqrt[4]{\lambda \ell} \right) \right) = 0. \tag{4.2}
\]

Furthermore, one can obtain a subsequence of eigenvalues of \( C' \) by using Proposition (4.1). Since eigenvalues of \( C \) are

\[
0, \left( \frac{2\pi}{\ell} \right)^4, \left( \frac{2\pi}{\ell} \right)^4, \left( \frac{4\pi}{\ell} \right)^4, \left( \frac{4\pi}{\ell} \right)^4, \left( \frac{6\pi}{\ell} \right)^4, \ldots
\]

and \( \lambda_k(C) \leq \lambda_k(C') \leq \lambda_{k+1}(C) \). Therefore,

\[
\left( \frac{2\pi}{\ell} \right)^4 \leq \lambda_1(C') \leq \left( \frac{2\pi}{\ell} \right)^4, \quad \left( \frac{2\pi}{\ell} \right)^4 \leq \lambda_2(C') \leq \left( \frac{4\pi}{\ell} \right)^4,
\]

\[
\left( \frac{4\pi}{\ell} \right)^4 \leq \lambda_3(C') \leq \left( \frac{4\pi}{\ell} \right)^4, \quad \left( \frac{4\pi}{\ell} \right)^4 \leq \lambda_4(C') \leq \left( \frac{6\pi}{\ell} \right)^4,
\]

\[
\left( \frac{6\pi}{\ell} \right)^4 \leq \lambda_5(C') \leq \left( \frac{6\pi}{\ell} \right)^4, \quad \left( \frac{6\pi}{\ell} \right)^4 \leq \lambda_6(C') \leq \left( \frac{8\pi}{\ell} \right)^4.
\]

It is evident from [1, 26] that, for Laplacian, the operation of changing vertex conditions by varying the interaction strengths of \( \delta \)-conditions at a vertex influences graph’s spectrum. Following their ideas, we show that a similar result holds for the fourth-order differential operator.

**Theorem 4.3** If the graph \( \tilde{\Gamma} \) is obtained from \( \Gamma \) by increasing the interaction strength of vertex conditions at a vertex \( v_m \) from \( \alpha_m \) to \( \tilde{\alpha}_m \). If \( \alpha_m < \tilde{\alpha}_m \), then the conclusions of Theorem (3.3) hold for their eigenvalues. Furthermore, if the k-th eigenvalue of \( \tilde{\Gamma} \) is simple and the corresponding eigenfunction \( \varphi_k \) is non-zero at \( v_m \), then

\[
\lambda_k(\Gamma) < \lambda_k(\tilde{\Gamma}) < \lambda_{k+1}(\Gamma).
\]

**Proof** Let \( h \) and \( \tilde{h} \) denote quadratic forms associated with graphs \( \Gamma \) and \( \tilde{\Gamma} \), respectively. Since the interaction strength does not enter into the form’s domain, therefore their respective domains are equal. That is, \( D(h) = D(\tilde{h}) \). The assumption \( \alpha_m < \tilde{\alpha}_m \) implies that the quadratic forms satisfy

\[
h[\varphi] = \int_{\Gamma} |\varphi''(x)|^2 \, dx + \sum_{m=1}^{\lvert V \rvert} \alpha_m |\varphi(v_m)|^2 \leq \int_{\tilde{\Gamma}} |\varphi''(x)|^2 \, dx + \sum_{m=1}^{\lvert V \rvert} \tilde{\alpha}_m |\varphi(v_m)|^2 = \tilde{h}[\varphi].
\]

Moreover, \( \tilde{h} = h \) if \( \varphi(v_m) = 0, \quad m = 1, \ldots, \lvert V \rvert \). This implies that \( \tilde{h} \) is a positive rank-1 perturbation of \( h \) and hence conclusions of Theorem (3.3) hold.

For strict inequality, assume that corresponding to each first \( k \) eigenvalues of a graph \( \tilde{\Gamma} \), there is a \( k \)-dimensional subspace \( X_k \) of \( D(\tilde{h}) \). Since the \( k \)-th eigenvalue

\[\square\] Springer
\( \lambda_k(\tilde{\Gamma}) \) is simple, thus the corresponding eigenfunction \( \varphi_k \) belongs to the subspace \( X_k \). One this subspace, we have

\[
h[\varphi] - \tilde{h}[\varphi] = (\alpha_m - \tilde{\alpha}_m)|\varphi(v_m)|^2.
\]

Since the eigenfunction \( \varphi_k \) is non-zero at vertex \( v_m \), and \( \alpha_m - \tilde{\alpha}_m < 0 \). Therefore, one has \( h[\varphi_k] < \tilde{h}[\varphi_k] \), and the strict inequality follows from the min-max description of eigenvalues. Since there is exactly one eigenvalue of the graph \( \tilde{\Gamma} \) between two consecutive eigenvalues of the graph \( \Gamma \), this establishes the last inequality. \( \square \)

The above theorem shows that whenever \( \alpha < \tilde{\alpha} \), there is exactly one eigenvalue of a graph \( \tilde{\Gamma} \) between two consecutive eigenvalues of a graph \( \Gamma \). This useful observations leads to the following generalization of Theorem (4.3).

**Theorem 4.4** Let \( \tilde{\Gamma} \) be connected finite compact graph obtained from \( \Gamma \) by replacing strengths from \( \alpha_m \) to \( \tilde{\alpha}_m \) at each vertex \( v_m \) of \( \Gamma \). If \( \alpha_m \leq \tilde{\alpha}_m \) and \( |V| \) denotes the number of vertices in \( \Gamma \), then

\[
\lambda_k(\Gamma) \leq \lambda_k(\tilde{\Gamma}) \leq \lambda_{k+|V|}(\Gamma). \tag{4.3}
\]

Let \( \Gamma^i_\alpha \) for \( i = 1, 2, 3, 4 \) be the same underlying metric graphs with vertex \( v \) equipped with vertex conditions (C1), (C2), (C3) and (C4), respectively, with interaction strength \( \alpha \). Let \( \Gamma^{i}_\infty \) be graphs obtained from \( \Gamma^{i}_\alpha \) by assigning \( \alpha = \infty \) at the vertex \( v \). Let \( h^{i}_\alpha \) and \( h^{i}_\infty \) represent their quadratic forms and \( D(h^{i}_\alpha) \) and \( D(h^{i}_\infty) \) be their respective domains. Then the following inclusions hold.

\[
D(h^{4}_\infty) \subset D(h^{2}_\infty) \subset D(h^{1}_\infty) \quad \text{and} \quad D(h^{4}_\infty) \subset D(h^{3}_\infty) \subset D(h^{1}_\infty). \tag{4.4}
\]

Since the quadratic forms \( h^{1}_\infty \) and \( h^{i}_\infty \) agree on \( D(h^{i}_\infty) \) for \( i = 2, 3, 4 \), and the domains \( D(h^{i}_\infty) \) are co-dimension one, \( d-1 \) and \( d \) subspaces of \( D(h^{1}_\infty) \), respectively, for \( i = 2, 3, 4 \). Where \( d \) is the degree of vertex \( v \). Hence, the conclusion of Theorem (3.3) leads to the following inequalities.

\[
\begin{align*}
\lambda_k(\Gamma^{1}_\infty) & \leq \lambda_k(\Gamma^{2}_\infty) \leq \lambda_{k+1}(\Gamma^{1}_\infty), \\
\lambda_k(\Gamma^{1}_\infty) & \leq \lambda_k(\Gamma^{3}_\infty) \leq \lambda_{k+d-1}(\Gamma^{1}_\infty), \\
\lambda_k(\Gamma^{1}_\infty) & \leq \lambda_k(\Gamma^{4}_\infty) \leq \lambda_{k+d}(\Gamma^{1}_\infty),
\end{align*}
\]

Moreover, for the functions \( \varphi \) from \( W^{2}_2(\Gamma\setminus V) \), which are also continuous on the whole graph, we have

\[
h^{i}_\alpha[\varphi] = h^{i}_\infty[\varphi] + \alpha|\varphi(v)|^2, \quad D(h^{i}_\infty) = \{ \varphi \in D(h^{i}_\alpha) : \varphi(v) = 0 \}.
\]

Since the quadratic forms \( h^{i}_\alpha \) and \( h^{i}_\infty \) agree on \( D(h^{i}_\infty) \) for \( i = 1, 2, 3, 4 \), and the domains \( D(h^{i}_\infty) \) are co-dimension one subspaces of \( D(h^{i}_\alpha) \). Thus, the conclusion of Theorem (3.3) leads to the following inequality.

\[
\lambda_k(\Gamma^{i}_\alpha) \leq \lambda_k(\Gamma^{i}_\infty) \leq \lambda_{k+1}(\Gamma^{i}_\alpha) \quad i = 1, 2, 3, 4.
\]
5 Joining and Splitting of Vertices

In this section, our objective is to explore how changes in the connectivity of the underlying graph impact its spectrum. We will examine how the spectrum of a graph behaves when two of its vertices, which may not be of the same type, are connected, and their coupling conditions are integrated appropriately. This process does not change the overall number of edges and, consequently, maintains the total length of the graph. However, it increases the graph’s connectivity.

Let $\Gamma$ be a connected finite compact metric graph equipped with any self-adjoint vertex conditions at all vertices except at two distinct vertices $v_1$ and $v_2$. Assume that the vertices $v_1$ and $v_2$ are equipped with conditions either (C1), (C2), (C3), or (C4) (not necessarily of the same type at $v_1$ and $v_2$) with interaction strengths $\alpha_1$ and $\alpha_2$, respectively. Let $\tilde{\Gamma}$ be obtained by identifying $v_1$ and $v_2$ to obtain a new vertex $v_0$ with strength $\alpha_0 = \alpha_1 + \alpha_2$.

Let $D(h)$ and $D(\tilde{h})$ denote the domains of the quadratic forms associated with the graphs $\Gamma$ and $\tilde{\Gamma}$, respectively. When $v_1, v_2$ and $v_0$ are equipped with (C1),

$$D(\tilde{h}) = \{ \varphi \in D(h) : \varphi(v_1) = \varphi(v_2) \}.$$ 

Therefore, $D(\tilde{h})$ is co-dimension one subspace of $D(h)$.

$D(\tilde{h}) \subset_1 D(h)$.

When $v_1, v_2$ are equipped with (C2) and (C1), respectively. And $v_0$ is equipped with (C2). Then besides continuity condition at $v_0$, we also require functions to satisfy

$$\sum_{x_j \in v_2} \sigma_{x_j} \partial \varphi(x_j) = 0.$$ 

Thus,

$$D(\tilde{h}) = \left\{ \varphi \in D(h) : \varphi(v_1) = \varphi(v_2), \sum_{x_j \in v_2} \sigma_{x_j} \partial \varphi(x_j) = 0 \right\}.$$ 

These are two conditions therefore $D(\tilde{h})$ is co-dimension two subspace of $D(h)$.

$D(\tilde{h}) \subset_2 D(h)$.

When $v_1, v_2$ are equipped with (C4) and (C1), respectively. And $v_0$ is equipped with (C4). Then besides continuity condition at $v_0$ we also require functions to satisfy

$$\partial \varphi(x_j) = 0, x_j \in v_2.$$ 

Thus,

$$D(\tilde{h}) = \{ \varphi \in D(h) : \varphi(v_1) = \varphi(v_2), \varphi(x_j) = 0 \}.$$ 

These are $d + 1$ conditions, where $d$ is the degree of $v_2$, therefore $D(\tilde{h})$ is co-dimension $d + 1$ subspace of $D(h)$.

$D(\tilde{h}) \subset_{d+1} D(h)$.
Based on the types of conditions imposed at $v_1$ and $v_2$, these two vertices can be glued together in several different ways. Below, we list all the possibilities and divide them into three classes. The first two correspond to the rank one and two perturbations, and the third corresponds to the rank $d_2 + 1$ perturbation, where $d_2$ is the degree of vertex $v_2$.

I 1. Conditions (C1) are imposed at $v_1$ and $v_2$. The glued vertex $v_0$ is also equipped with conditions (C1).
2. Conditions (C1) and (C2) are imposed at $v_1$ and $v_2$, respectively. The glued vertex $v_0$ is equipped with conditions (C1).
3. Conditions (C1) and (C3) are imposed at $v_1$ and $v_2$, respectively. The glued vertex $v_0$ is equipped with conditions (C1).
4. Conditions (C1) and (C4) are imposed at $v_1$ and $v_2$, respectively. The glued vertex $v_0$ is equipped with conditions (C1).
5. Conditions (C2) are imposed at $v_1$ and $v_2$. The glued vertex $v_0$ is also equipped with conditions (C2).
6. Conditions (C2) and (C4) are imposed at $v_1$ and $v_2$, respectively. The glued vertex $v_0$ is equipped with conditions (C2).
7. Conditions (C4) are imposed at $v_1$ and $v_2$. The glued vertex $v_0$ is also equipped with conditions (C4).

II 1. Conditions (C2) and (C1) are imposed at $v_1$ and $v_2$, respectively. The glued vertex $v_0$ is equipped with conditions (C2).
2. Conditions (C3) are imposed at $v_1$ and $v_2$. The glued vertex $v_0$ is also equipped with conditions (C3).
3. Conditions (C3) and (C4) are imposed at $v_1$ and $v_2$, respectively. The glued vertex $v_0$ is equipped with conditions (C3).
4. Conditions (C4) and (C3) are imposed at $v_1$ and $v_2$, respectively. The glued vertex $v_0$ is equipped with conditions (C4).

III 1. Conditions (C3) and (C1) are imposed at $v_1$ and $v_2$, respectively. The glued vertex $v_0$ is equipped with conditions (C3).
2. Conditions (C4) and (C1) are imposed at $v_1$ and $v_2$, respectively. The glued vertex $v_0$ is equipped with conditions (C4).
3. Conditions (C4) and (C2) are imposed at $v_1$ and $v_2$, respectively. The glued vertex $v_0$ is equipped with conditions (C4).

Since $\tilde{\Gamma}$ is equipped with the same vertex conditions as $\Gamma$ at all vertices $v \in V \setminus \{v_1, v_2\}$ and based on conditions at $v_1$ and $v_2$, the glued vertex $v_0$ satisfy vertex conditions according to any one of the choices mentioned in the above three cases. The following theorem inspired by [1] shows that the gluing of vertices results in a non-
decrease of eigenvalues and specific interlacing properties between the eigenvalues of $\Gamma$ and $\tilde{\Gamma}$.

**Theorem 5.1**

1. If the vertices $v_1$ and $v_2$ of $\Gamma$ and the glued vertex $v_0$ of $\tilde{\Gamma}$ are equipped with vertex conditions according to the classifications I. Then the conclusions of Theorem (3.3) hold for the eigenvalues of operators $\Gamma$ and $\tilde{\Gamma}$.

2. If the vertices $v_1$ and $v_2$ of $\Gamma$ and the glued vertex $v_0$ of $\tilde{\Gamma}$ are equipped with vertex conditions according to the classifications II. Then,

$$
\lambda_k(\Gamma) \leq \lambda_k(\tilde{\Gamma}) \leq \lambda_{k+2}(\Gamma), \quad k \geq 1.
$$

3. If the vertices $v_1$ and $v_2$ of $\Gamma$ and the glued vertex $v_0$ of $\tilde{\Gamma}$ are equipped with vertex conditions according to the classifications III. Then,

$$
\lambda_k(\Gamma) \leq \lambda_k(\tilde{\Gamma}) \leq \lambda_{k+d_2+1}(\Gamma) \leq \lambda_{k+d_2+1}(\tilde{\Gamma}), \quad k \geq 1.
$$

Where $d_2$ is the degree of vertex $v_2$.

**Proof** We have provided a proof for part (2) only; a similar argument holds for the remaining parts. Let $D(h)$ and $D(\tilde{h})$ denote the domains of the quadratic forms $h$ and $\tilde{h}$ corresponding to the metric graphs $\Gamma$ and $\tilde{\Gamma}$, respectively. Since the gluing of two vertices imposes conditions on the domain $D(\tilde{h})$, the domain is restricted, and we have $D(\tilde{h}) \subset 2 D(h)$. The inequality $\lambda_k(\Gamma) \leq \lambda_k(\tilde{\Gamma})$ follows from the fact that minimizing over the smaller set $D(\tilde{h})$ results in larger eigenvalues. The inequality $\lambda_k(\tilde{\Gamma}) \leq \lambda_{k+2}(\Gamma)$ follows from the rank-two nature of the perturbation. $\Box$

**Corollary 5.2** Let the vertices $v_1$ and $v_2$ of the graph $\Gamma$ are glued together to obtain a new graph $\tilde{\Gamma}$. Denote the glued vertex by $v_0$. Suppose that for some $k \geq 1$ there exist eigenfunctions $\varphi_1, \ldots, \varphi_k$ corresponding to eigenvalues $\lambda_1(\Gamma), \ldots, \lambda_k(\Gamma)$, respectively, such that

$$
\varphi_1(v_1) = \varphi_1(v_2), \ldots, \varphi_k(v_1) = \varphi_k(v_2). \quad (5.1)
$$

1. If gluing is performed according to the classifications I, then

$$
\lambda_1(\tilde{\Gamma}) = \lambda_1(\Gamma), \ldots, \lambda_k(\tilde{\Gamma}) = \lambda_k(\Gamma). \quad (5.2)
$$

2. If gluing is performed according to the classification II(1) and, in addition to (5.1), eigenfunctions also satisfy

$$
\sum_{x_j \in v_2} \sigma_{x_j} \varphi(x_j) = 0, \quad (5.3)
$$

then (5.2) holds.
(3) If gluing is performed according to the classification II(2) and, in addition to (5.1), eigenfunctions also satisfy
\[
\frac{\partial \varphi_1(x_i)}{\sigma_{x_i}^v} = \frac{\partial \varphi_1(x_j)}{\sigma_{x_j}^v}, \ldots, \frac{\partial \varphi_k(x_i)}{\sigma_{x_i}^v} = \frac{\partial \varphi_k(x_j)}{\sigma_{x_j}^v}, \quad x_i \in v_1, x_j \in v_2.
\]  
then (5.2) holds.

(4) If gluing is performed according to the classification II(3) and, in addition to (5.1), eigenfunctions also satisfy
\[
\varphi(x_1) = 0, \quad x_1 \in v_1
\]
then (5.2) holds.

(5) If gluing is performed according to the classification III(I, 2, 3, 4) and, in addition to (5.1), eigenfunctions also satisfy
\[
\varphi(x_j) = 0, \quad x_j \in v_2
\]
then (5.2) holds.

Proof We will prove this for case (1) only, but the same reasoning applies to the other cases as well. Since, the domain of quadratic form for \(\tilde{\Gamma}^1\) only involves continuity of functions at vertices, so \(\varphi_k(x)\) is also in domain of quadratic form for \(\tilde{\Gamma}\). Furthermore, if \(\varphi_k(x)\) additionally fulfills (5.1), it becomes a minimizer for \(\tilde{\Gamma}\), resulting in identical eigenvalues. □

Remark 5.3 In the previous theorem, we have only considered the case in which glued vertex \(v_0\) is equipped with conditions that were either imposed on \(v_1\) or \(v_2\). However, one can consider the case when glued vertex \(v_0\) is equipped with conditions, which is neither imposed on \(v_1\) nor \(v_2\). For demonstration, we describe the relation between eigenvalues of graphs \(\Gamma\) and \(\tilde{\Gamma}\) when \(v_1\) and \(v_2\) are equipped with (C1), and \(v_0\) is equipped with (C2). Let \(\Gamma'\) be another graph obtained from \(\Gamma\) by gluing \(v_1\) and \(v_2\), in accordance to Theorem (5.1), and the glued vertex \(v_0^*\) be equipped with conditions (C1). Then
\[
\lambda_k(\Gamma) \leq \lambda_k(\Gamma') \leq \lambda_{k+1}(\Gamma).
\]
Now, create a graph \(\tilde{\Gamma}\) from \(\Gamma'\) by replacing the conditions at \(v_0^*\) from (C1) to (C2) using Proposition (4.1). Thus, we get
\[
\lambda_k(\Gamma') \leq \lambda_k(\tilde{\Gamma}) \leq \lambda_{k+1}(\Gamma').
\]
By combining these two sets of inequalities, we derive:
\[
\lambda_k(\Gamma) \leq \lambda_k(\tilde{\Gamma}) \leq \lambda_{k+2}(\Gamma).
\]
Analogously, similar interlacing inequalities can be established for other cases as well.
One can observe that if the form \( \tilde{h} \) is a positive rank-\( t \) perturbation of the form \( h \), that it, if \( D(\tilde{h}) \) is co-dimension \( t \) subspace of \( D(h) \), then we obtain following inequality by applying the same arguments.

\[
\lambda_k(\Gamma) \leq \lambda_k(\tilde{\Gamma}) \leq \lambda_{k+t}(\Gamma).
\]

This observation gives rise to the following generalization of Theorem (5.1).

**Theorem 5.4**

(1) If the vertices \( v_1, v_2, \ldots, v_t, v_{t+1} \) of graph \( \Gamma \) and the glued vertex \( v_0 \) of graph \( \tilde{\Gamma} \) are equipped with vertex conditions according to the classification I. Then

\[
\lambda_k(\Gamma) \leq \lambda_k(\tilde{\Gamma}) \leq \lambda_{k+t}(\Gamma). \tag{5.7}
\]

(2) If the vertices \( v_1, v_2, \ldots, v_t, v_{t+1} \) of graph \( \Gamma \) and the glued vertex \( v_0 \) of graph \( \tilde{\Gamma} \) are equipped with vertex conditions according to the classification II. Then

\[
\lambda_k(\Gamma) \leq \lambda_k(\tilde{\Gamma}) \leq \lambda_{k+2t}(\Gamma). \tag{5.8}
\]

**Proof** The gluing of \( t + 1 \) vertices at one vertex is the same as pairwise gluing of \( t \) pair of vertices. At each step, we identify two vertices and add parameters \( \alpha_m \). By the repeated application of Theorem (5.1), we can achieve the desired inequality. \( \square \)

The converse of gluing operation is splitting a vertex into two or more vertices. Let \( \Gamma \) and \( \tilde{\Gamma} \) be two finite and compact metric graphs, as shown in figure 2. Let \( v_0, v_1 \) be vertices equipped with vertex conditions \((C1)\), and \( v_2 \) be equipped with conditions \((C2)\). Since the expressions for quadratic form for both graphs are the same. Therefore, we need to compare their respective domains. A graph with a larger domain will have smaller eigenvalues. We can view figure in two ways. First, the graph \( \Gamma \) is obtained from \( \tilde{\Gamma} \) by gluing vertices \( v_1 \) and \( v_2 \). Second, the graph \( \tilde{\Gamma} \) is obtained from \( \Gamma \) by splitting vertex \( v_0 \), producing two vertices \( v_1 \) and \( v_2 \). In the former case, when gluing vertices together, the domain is restricted, and eigenvalues become large. Since splitting is converse of gluing, in the latter case, splitting should make the domain of quadratic form for \( \tilde{\Gamma} \) larger and hence smaller eigenvalues. However, any continuous function on \( \Gamma \) is also continuous on \( \tilde{\Gamma} \), but in addition to continuity, we require this function to satisfy second equation in \((C2)\) at \( v_2 \).

We compare Laplacian to \( \delta \) beam operator to have a more precise idea of why splitting a vertex for \( \delta \) beam operator with specific conditions does not behave as a

---

**Fig. 2** Splitting the vertex \( v_0 \) into two vertices \( v_1 \) and \( v_2 \), we obtain \( \tilde{\Gamma} \). Gluing the vertices \( v_1 \) and \( v_2 \) to form single vertex \( v_0 \), we obtain \( \Gamma \)
converse of gluing of vertices. Let $\Gamma$ be a finite and compact metric graph obtained from $\tilde{\Gamma}$ by gluing $v_1$ and $v_2$, and conversely, let $\tilde{\Gamma}$ be retrieved from $\Gamma$ by splitting vertex $v_0$ (as shown in the Fig. 3). Assume that the vertices $v_0$, $v_1$ and $v_2$ are equipped with $\delta$ conditions, and the Laplace operator acts on each edge of $\Gamma$ and $\tilde{\Gamma}$. The quadratic form’s domain contains continuous functions on the whole graph. Therefore, any function from $D(\tilde{h})$ with condition $\varphi(v_1) = \varphi(v_2)$ is suitable trial function on $\Gamma$. Conversely, every eigenfunction on $\Gamma$ can be lifted on $\tilde{\Gamma}$, and either this function will be an eigenfunction for $\tilde{\Gamma}$, or there will be some other function in $D(\tilde{h})$ which will be an eigenfunction for $\tilde{\Gamma}$. In both cases, we have following inequality and corresponding domains.

$$\lambda_k(\tilde{\Gamma}) \leq \lambda_k(\Gamma),$$

$$D(h) = \{ \varphi \in D(\tilde{h}) : \varphi(v_1) = \varphi(v_2) \}.$$

Now assume that vertices $v_0$, $v_1$ and $v_2$ are equipped with conditions (C2), and $\delta$ beam operator acts on each edge. The form domain consists of continuous functions on the whole graph and satisfies condition $\sum_{x_j \in v_0} \sigma_{x_j}^{v_0} \partial \varphi(x_j) = 0$. Therefore, any function on $\tilde{\Gamma}$ satisfying $\varphi(v_1) = \varphi(v_2)$ can be used as a test function on $\Gamma$, and the domain $D(h)$ is a co-dimension one subspace of $D(\tilde{h})$. Now, consider the case when splitting is performed. Every eigenfunction on $\Gamma$ is also continuous on $\tilde{\Gamma}$. But to be a test function on $\tilde{\Gamma}$, we require this function to satisfy second equation in (C2) at both vertices $v_1$ and $v_2$. However, the eigenfunction satisfying the equation $\sum_{x_j \in v_0} \sigma_{x_j}^{v_0} \partial \varphi(x_j) = 0$ at $v_0$ will not necessarily satisfy $\sum_{x_j \in v_1} \sigma_{x_j}^{v_1} \partial \varphi(x_j) = 0$ at $v_1$ and $\sum_{x_j \in v_2} \sigma_{x_j}^{v_2} \partial \varphi(x_j) = 0$ at $v_2$. Therefore, we can not lift any function from $\Gamma$ to $\tilde{\Gamma}$. The converse of Theorem (5.1) is in general not true. Instead, we have the following result:

**Theorem 5.5** Consider a vertex $v_0$ of a connected finite compact metric graph $\Gamma$ with delta interaction of type either (C1), (C2), (C3), or (C4) with strength $\alpha_0$. Let the graph $\tilde{\Gamma}$ be obtained by splitting the vertex $v_0$ into its descendant vertices $v_1$ and $v_2$. The two new vertices can be equipped with either of the vertex conditions (C1), (C2), (C3), or (C4) (not necessarily of the same type as $v_0$) with corresponding strengths $\alpha_1$ and $\alpha_2$ such that $\alpha_0 = \alpha_1 + \alpha_2$. Let the vertex $v_0$ be splitted into $v_1$ and $v_2$ in any one of the following ways:

1. Conditions (C1) are imposed at $v_0$, and its descendants $v_1$ and $v_2$ are also equipped with conditions (C1).
(2) Conditions (C3) are imposed at \(v_0\), and its descendant vertices \(v_1\) and \(v_2\) are also equipped with conditions (C3).

(3) Conditions (C4) are imposed at \(v_0\), and its descendant vertices \(v_1\) and \(v_2\) are also equipped with conditions (C4).

(4) Conditions (C4) are imposed at \(v_0\), and its descendant vertices \(v_1\) and \(v_2\) are equipped with conditions (C1) and (C4), respectively.

(5) Conditions (C3) are imposed at \(v_0\), and its descendant vertices \(v_1\) and \(v_2\) are equipped with conditions (C1) and (C3), respectively.

(6) Conditions (C4) are imposed at \(v_0\), and its descendant vertices \(v_1\) and \(v_2\) are equipped with conditions (C2) and (C4), respectively.

(7) Conditions (C4) are imposed at \(v_0\), and its descendant vertices \(v_1\) and \(v_2\) are equipped with conditions (C3) and (C4), respectively.

Then the eigenvalues of \(\Gamma\) and \(\tilde{\Gamma}\) satisfy

\[
\lambda_k(\Gamma) \geq \lambda_k(\tilde{\Gamma}). \quad (5.9)
\]

**Proof** We will prove the part (7) only, remaining parts can be proved similarly. Consider the quadratic forms \(h\) and \(\tilde{h}\) associated with the graphs \(\Gamma\) and \(\tilde{\Gamma}\). These forms share the same expression but have different domains, denoted as \(D(h)\) and \(D(\tilde{h})\). Consider the case when \(v_0\) and \(v_2\) are equipped with (C4) and \(v_1\) is equipped with (C3). When the vertex \(v_0\) is split into its descendant vertices, \(v_1\) and \(v_0\), the operation of splitting relaxes the vertex conditions, resulting in \(D(\tilde{h})\) being larger than \(D(h)\).

\[
D(h) = \{\varphi \in D(\tilde{h}) : \varphi(v_1) = \varphi(v_2), \partial \varphi(x_j) = 0, x_j \in v_1\}
\]

Therefore, \(D(h)\) is a subspace of \(D(\tilde{h})\). Hence, the conclusion follows from min-max argument. \(\square\)

### 6 Increasing Volume of a Graph

In this section, we focus our study on understanding the precise nature of the relationships between eigenvalues of a graph and the operations that increase its volume either by attaching a new subgraph to it or by scaling a part of a graph. These basic changes help to transform a graph into another graph such that the effect on one or several eigenvalues is predictable. The influence of surgical operations, such as increasing the length of an edge, attaching pendant edges, or inserting a new graph, on eigenvalues varies depending on the specific set of vertex conditions. Furthermore, the results and proof ideas have been drawn from [1].

#### 6.1 Attaching a Pendant Graph

Consider a vertex \(v\) of a graph \(\Gamma\) and a vertex \(w\) of a graph \(\tilde{\Gamma}\), both are equipped with any of the vertex conditions (C1), (C2), (C3) or (C4) with interaction strengths \(\alpha_v\) and \(\alpha_w\), respectively. All other vertices of \(\Gamma\) and \(\tilde{\Gamma}\) are equipped with arbitrary self-adjoint
Fig. 4 Identifying $v, w$ together, we can glue the graph $\hat{\Gamma}$ to $\Gamma$, thus obtaining the graph $\tilde{\Gamma}$.

vertex conditions. Let the graph $\tilde{\Gamma}$ be obtained by gluing together $v$ and $w$, according to either of the classifications $I$, $II$ or $III$ of section 4. The new vertex, say $v_0$, is equipped with any of the vertex conditions (C1), (C2), (C3) or (C4) with interaction strength $\alpha_0 = \alpha_v + \alpha_w$. We now say that the graph $\tilde{\Gamma}$ is obtained by attaching a pendant graph $\hat{\Gamma}$ to the graph $\Gamma$.

**Theorem 6.1** Let $\tilde{\Gamma}$ be formed from $\Gamma$ by attaching a vertex $w$ of a pendant graph $\hat{\Gamma}$ at a vertex $v$ of $\Gamma$ and assume that for some $r$ and $k_0$

$$\lambda_r(\hat{\Gamma}) \leq \lambda_{k_0}(\Gamma).$$

(1) If the vertices $v$ and $w$ are glued according to classifications $I$, then

$$\lambda_{k+r-1}(\tilde{\Gamma}) \leq \lambda_k(\Gamma), \quad k \geq k_0.$$  

(2) If the vertices $v$ and $w$ are glued according to classifications $II$, then

$$\lambda_{k+r-2}(\tilde{\Gamma}) \leq \lambda_k(\Gamma), \quad k \geq k_0.$$  

(3) If the vertices $v$ and $w$ are glued according to classifications $III$, then

$$\lambda_{k+r-d-1}(\tilde{\Gamma}) \leq \lambda_k(\Gamma), \quad k \geq k_0.$$  

Where $d$ is the degree of vertex $v$.

Moreover, if $\lambda_{k-1}(\Gamma) < \lambda_k(\Gamma)$ and $\lambda_r(\hat{\Gamma}) < \lambda_{k_0}(\Gamma)$ and the eigenvalue $\lambda_k(\Gamma)$ has an eigenfunction which is non-zero at $v$, then the above three inequalities are strict.

**Proof** (1) Assume that for some $r$ and $k_0$, we have, $\lambda_r(\hat{\Gamma}) \leq \lambda_{k_0}(\Gamma)$. Since the spectrum of the union of two graphs is equal to the union of their spectrum. Therefore, $\lambda_{k_0}(\Gamma) = \lambda_m(\Gamma \cup \hat{\Gamma})$ for some $m \geq k_0 + r$. Now, attach the vertex $w$ of $\hat{\Gamma}$ with the vertex $v$ of $\Gamma$ to obtain a new graph $\tilde{\Gamma}$. By Theorem (5.1), we obtain the following interlacing inequalities.

$$\lambda_{m-1}(\Gamma \cup \hat{\Gamma}) \leq \lambda_{m-1}(\tilde{\Gamma}) \leq \lambda_{m}(\Gamma \cup \hat{\Gamma}) \leq \lambda_{m}(\tilde{\Gamma}),$$

combining the estimate $m \geq k_0 + r$ with $\lambda_{m-1}(\tilde{\Gamma}) \leq \lambda_{m}(\Gamma \cup \hat{\Gamma})$, we get

$$\lambda_{k_0+r-1}(\tilde{\Gamma}) \leq \lambda_{k_0}(\Gamma).$$
Since \( \lambda_r(\hat{\Gamma}) \leq \lambda_{k_0+i}(\Gamma) \), for \( i = 0, 1, 2, \ldots \), therefore, repeating the same process we obtain

\[
\lambda_{k_0+i+r-1}(\hat{\Gamma}) \leq \lambda_{k_0+i}(\Gamma).
\]

Thus,

\[
\lambda_{k+r-1}(\hat{\Gamma}) \leq \lambda_k(\Gamma), \quad k \geq k_0.
\]

Let \( \lambda_{m-1}(\Gamma \cup \hat{\Gamma}) < \lambda_m(\Gamma \cup \hat{\Gamma}) = \lambda_k(\Gamma) \) be the first occurrence of \( \lambda_k(\Gamma) \) in the spectrum of \( \Gamma \cup \hat{\Gamma} \). Since \( \lambda_{k-1}(\Gamma) < \lambda_k(\Gamma) \) and \( \lambda_r(\hat{\Gamma}) < \lambda_{k_0}(\Gamma) \) we still have \( m \geq k + r \). Suppose that \( \lambda_{m-1}(\Gamma \cup \hat{\Gamma}) \leq \lambda_{m-1}(\hat{\Gamma}) = \lambda_m(\Gamma \cup \hat{\Gamma}) \), this shows that the eigenfunction corresponding to the eigenvalue \( \lambda_m(\Gamma \cup \hat{\Gamma}) \) is also an eigenfunction of \( \hat{\Gamma} \). But the eigenspace of \( \lambda_m(\Gamma \cup \hat{\Gamma}) \) contains a function which vanishes identically on \( \hat{\Gamma} \) and is non-zero at \( v \). But this eigenfunction can not be contained in the eigenspace of \( \hat{\Gamma} \) because it does not belong to its form’s domain. Thus, \( \lambda_{m-1}(\hat{\Gamma}) < \lambda_m(\Gamma \cup \hat{\Gamma}) = \lambda_k(\Gamma) \) and \( \lambda_{k+r-1}(\hat{\Gamma}) \leq \lambda_{m-1}(\hat{\Gamma}) < \lambda_k(\Gamma) \).

The same arguments can be applied to prove the remaining parts. \( \square \)

### 6.2 Inserting a Graph at a Vertex

Consider a vertex \( v_1 \), equipped with any of the vertex conditions (C1), (C2), (C3) or (C4) with interaction strength \( \alpha_1 \), of a metric graph \( \Gamma \) having set of incident edges \( \{e_1, e_2, \ldots, e_t\} \). Let another metric graph \( \hat{\Gamma} \) with a subset of vertices \( \{w_1, \ldots, w_s\} \subseteq V(\hat{\Gamma}), s \leq t \). Let \( \tilde{\Gamma} \) be a new graph formed from \( \Gamma \) by first removing the vertex \( v_1 \) of \( \Gamma \) and then attaching the incident edges \( e_n, n = 1, 2, \ldots, t \) to the vertices \( w_m, m = 1, \ldots, s \) of \( \hat{\Gamma} \) in such a way that the newly formed vertices, say \( \tilde{w}_m, m = 1, 2, \ldots, s \), are equipped with either of the vertex conditions (C1), (C2), (C3) or (C4) with total interaction strength equal to \( \alpha_1 \). We call \( \tilde{w}_m \) the post insertion descendents (PIDs) of \( v_1 \) and say that \( \tilde{\Gamma} \) is formed by inserting \( \hat{\Gamma} \) into \( \Gamma \) at \( v_1 \).

**Theorem 6.2** Let conditions (C4) be imposed on all vertices of \( \hat{\Gamma} \) with interaction strengths equal to zero before insertion. If the graph \( \tilde{\Gamma} \) is obtained by inserting \( \hat{\Gamma} \) at a vertex \( v_1 \) of \( \Gamma \), then for all \( k \) such that \( \lambda_k(\Gamma) \geq 0 \) and

![Fig. 5 Inserting \( \hat{\Gamma} \) into \( \Gamma \) at \( v_1 \), we obtain the graph \( \tilde{\Gamma} \).](image-url)
(1) for the cases described in classification I, we have
\[ \lambda_k(\tilde{\Gamma}) \leq \lambda_k(\Gamma). \]

(2) for the cases described in classification II, we have
\[ \lambda_k(\tilde{\Gamma}) \leq \lambda_{k+1}(\Gamma). \]

(3) for the cases described in classification III, we have
\[ \lambda_k(\tilde{\Gamma}) \leq \lambda_{k+d}(\Gamma). \]

where \( d \) is the degree of vertex \( v_1 \).

Moreover, if the eigenfunction in non zero at \( v_1 \) and \( \lambda_k(\Gamma) > \max(0, \lambda_{k-1}(\Gamma)) \), then above three inequalities are strict.

**Proof**  (1) Let \( \tilde{\Gamma} \) be obtained by gluing the vertices \( w_1 \) and \( w_2 \) of \( \tilde{\Gamma} \) in such a way that the glued vertex, \( w^* \), is also equipped with conditions (C4). As all the vertices of \( \tilde{\Gamma} \) are equipped with vertex conditions (C4) with zero delta interactions, therefore \( \lambda_1(\tilde{\Gamma}) = 0 \). Assumption \( \lambda_k(\Gamma) \geq 0 \) implies \( \lambda_k(\Gamma) \geq \lambda_{r-1}(\tilde{\Gamma}) \). We obtain a new graph \( \Gamma' \) by gluing vertex \( v_1 \) of \( \Gamma \), which is equipped with conditions (C1) with delta interaction strength \( \alpha_1 \), and \( w^* \) of \( \tilde{\Gamma} \) such that the new vertex \( v_1^* \) is equipped with (C1) with delta interaction strength \( \alpha_1 \). Theorem 6.1(1) with \( r = 1 \) gives \( \lambda_k(\Gamma') \leq \lambda_k(\Gamma) \). Graph \( \tilde{\Gamma} \) is created by splitting the vertex \( v_1^* \) of \( \Gamma' \) and restoring the vertices \( w_1 \) and \( w_2 \) which, now, are denoted by \( \tilde{w}_1 \) and \( \tilde{w}_2 \) and are equipped...
with conditions (C1) and total delta interaction strength equal to $\alpha_1$. Theorem (5.5) implies $\lambda_k(\tilde{\Gamma}) \leq \lambda_k(\Gamma')$. The case of strict inequality is obtained by using the strict version of Theorem (6.1).

The same arguments can be applied to prove the remaining parts. \qed

**Corollary 6.3** Suppose there exist two vertices $v_1$ and $v_2$ of $\Gamma$ and first $n$ eigenfunctions $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that

$$\varphi_1(v_1) = \varphi_1(v_2), \ldots, \varphi_n(v_1) = \varphi_n(v_2). \tag{6.1}$$

Let both vertices be equipped with either vertex conditions (C1) or (C4) (of the same type on both $v_1$ and $v_2$). If $\lambda_k(\Gamma) \geq 0$ for all $k = 1, \ldots, n$, then the graph $\tilde{\Gamma}$ obtained by inserting an edge of arbitrary length between $v_1$ and $v_2$, with preserved type and interaction strength of vertex conditions at both vertices, satisfies

$$\lambda_k(\tilde{\Gamma}) \leq \lambda_k(\Gamma), \quad k = 1, \ldots, n.$$  

If $v_1$ and $v_2$ are both equipped with vertex conditions (C3), then we require that the first $n$ eigenfunctions, in addition to (6.1), also satisfy

$$\frac{\partial \varphi_1(x_i)}{\sigma_{x_i}^1} = \frac{\partial \varphi_1(x_j)}{\sigma_{x_j}^1}, \ldots, \frac{\partial \varphi_n(x_i)}{\sigma_{x_i}^1} = \frac{\partial \varphi_n(x_j)}{\sigma_{x_j}^1}, \quad x_i \in v_1, x_j \in v_2.$$

In that case, if $\lambda_k(\Gamma) \geq 0$ for all $k = 1, \ldots, n$, then the eigenvalues of the graph $\tilde{\Gamma}$ which is obtained by inserting an edge of arbitrary length between $v_1$ and $v_2$, with preserved type and interaction strength of vertex conditions at both vertices, satisfy

$$\lambda_k(\tilde{\Gamma}) \leq \lambda_{k+1}(\Gamma), \quad k = 1, \ldots, n.$$  

**Proof** Let both vertices $v_1$ and $v_2$ are equipped with vertex conditions (C1). Let $\Gamma'$ be a graph obtained from $\Gamma$ by gluing the vertices $v_1$ and $v_2$ in accordance with the definition of gluing of vertices; let $v^*$ be the glued vertex. Then by Corollary (5.2) (1),

$$\lambda_k(\Gamma') = \lambda_k(\Gamma), \quad k = 1, 2, \ldots n.$$  

Now, insert an edge graph $\hat{\Gamma}$ satisfying the conditions mentioned in Theorem (6.2)(1) at vertex $v^*$; this graph coincides with $\tilde{\Gamma}$, and we get, $\lambda_k(\hat{\Gamma}) \leq \lambda_k(\Gamma') = \lambda_k(\Gamma), \quad k = 1, 2, \ldots n$.

When $v_1$ and $v_2$ are both equipped with vertex conditions (C3), the proof is identical to the above proof; however, we use corollary (5.4)(2) and Theorem (6.2)(2). \qed

The reason for imposing conditions (C4) on vertices of the graph $\tilde{\Gamma}$ before insertion is twofold. First, during the proof, at one time, we split the vertex $v^*$, and in general, we know that splitting does not behave exactly as a converse of gluing. However, the two are compatible when conditions (C4) are specified. Second, at every step of the surgery of a graph, the proof does not involve the choices for more conditions.
understand the second reason, we assume that \( v_1 \) is equipped with condition (C1), and vertices \( w_1 \) and \( w_2 \) are endowed with conditions (C1) and (C2), respectively. We have two choices for \( w^* \) either it is equipped with condition (C1) or (C2). If the vertex \( w^* \) is equipped with (C1), then the vertex \( v_1^* \) and the vertices \( \tilde{w}_1 \) and \( \tilde{w}_2 \) after the insertions are also equipped with condition (C1), and we have the following inequality.

\[
\lambda_{k+r-1}(\tilde{\Gamma}) \leq \lambda_k(\Gamma).
\]

When \( w^* \) is equipped with condition (C2), now, we have two choices for \( v_1^* \). It is either equipped with condition (C1) or (C2). In first choice, \( \tilde{w}_1 \) and \( \tilde{w}_2 \) after the insertion are also endowed with condition (C1), and we obtain the same inequality as above. In second choice, when \( v_1^* \) is equipped with (C2), the splitting cannot be performed, and we cannot obtain the desired inequality.

**Theorem 6.4** Suppose \( \hat{\Gamma} \) is formed by adding an edge of length \( \ell \) between two vertices \( v_1 \) and \( v_2 \) of \( \Gamma \). Type and interaction strength of vertex conditions at both vertices \( v_1 \) and \( v_2 \) are preserved after adding an extra edge. Let \( \lambda_{k_0}(\Gamma) \geq (\pi/\ell)^4 \) for some integer \( k_0 \).

If

(1) vertices \( v_1 \) and \( v_2 \) are equipped with conditions (C1) or
(2) vertices \( v_1 \) and \( v_2 \) are equipped with conditions (C2) or
(3) vertices \( v_1 \) and \( v_2 \) are equipped with conditions (C4) or
(4) vertices \( v_1 \) and \( v_2 \) are equipped with conditions (C1) and (C2), respectively, or
(5) vertices \( v_1 \) and \( v_2 \) are equipped with conditions (C1) and (C4), respectively, or
(6) vertices \( v_1 \) and \( v_2 \) are equipped with conditions (C2) and (C4), respectively.

Then

\[
\lambda_k(\hat{\Gamma}) \leq \lambda_k(\Gamma), \quad k \geq k_0.
\]

If

(7) vertices \( v_1 \) and \( v_2 \) are equipped with conditions (C1) and (C3), respectively, or
(8) vertices \( v_1 \) and \( v_2 \) are equipped with conditions (C2) and (C3), respectively, or
(9) vertices \( v_1 \) and \( v_2 \) are equipped with conditions (C3) and (C4), respectively.

Then

\[
\lambda_k(\hat{\Gamma}) \leq \lambda_{k+1}(\Gamma), \quad k + 1 \geq k_0.
\]

(10) Finally, if both vertices \( v_1 \) and \( v_2 \) of \( \Gamma \) are equipped with conditions (C3) then

\[
\lambda_k(\hat{\Gamma}) \leq \lambda_{k+2}(\Gamma), \quad k + 2 \geq k_0.
\]

**Proof** 1. Suppose the vertices \( v_1 \) and \( v_2 \) of \( \Gamma \) are equipped with conditions (C1), and consider the graph \( \hat{\Gamma} \) of single edge of length \( \ell \), satisfying conditions (C4) with zero interaction strengths at both ends. The lowest eigenvalue \( \lambda_1(\hat{\Gamma}) \) is equal to zero and \( \lambda_2(\hat{\Gamma}) = (\pi/\ell)^4 \). By gluing one end point of \( \hat{\Gamma} \) with vertex \( v_1 \) of \( \Gamma \) such that the new vertex also satisfy conditions (C1) with the same delta interaction strength as \( v_1 \). We denote this new graph by \( \hat{\Gamma} \). Theorem (6.1) with \( r = 2 \) implies \( \lambda_{k+1}(\hat{\Gamma}) \leq \lambda_k(\Gamma) \), for all \( k \geq k_0 \). Now we glue \( v_2 \) and the other endpoint of \( \hat{\Gamma} \) to
obtain $\tilde{\Gamma}$. Then the Part (1) of Theorem (5.1) implies $\lambda_k(\tilde{\Gamma}) \leq \lambda_{k+1}(\tilde{\Gamma})$ for all $k$. This proves the claim. Parts (2 – 6) of this theorem can be proved using the same argument.

7. Suppose the vertices $v_1$ and $v_2$ of $\Gamma$ are equipped with conditions (C1) and (C3), respectively. As before, we glue one endpoint of $\hat{\Gamma}$ and vertex $v_1$ of $\Gamma$ to obtain $\tilde{\Gamma}$. Theorem (6.1) implies $\lambda_{k+2}(\tilde{\Gamma}) \leq \lambda_{k+1}(\Gamma)$, for all $k \geq k_0$. This can be rewritten as $\lambda_{k+2}(\tilde{\Gamma}) \leq \lambda_{k+1}(\Gamma)$, for all $k + 1 \geq k_0$. Gluing vertex $v_2$ and the other end point of $\hat{\Gamma}$ and applying part (2) of Theorem (5.1) we get $\lambda_k(\tilde{\Gamma}) \leq \lambda_{k+2}(\tilde{\Gamma})$ for all $k$. Proofs of parts (8) and (9) are similar.

10. Suppose both vertices $v_1$ and $v_2$ of $\Gamma$ are equipped with conditions (C3), and let us glue $v_1$ and one endpoint of $\hat{\Gamma}$. Theorem (6.1) yields $\lambda_{k+2}(\tilde{\Gamma}) \leq \lambda_{k+2}(\Gamma)$, for all $k + 2 \geq k_0$. Finally, gluing vertex $v_2$ and the other endpoint of $\hat{\Gamma}$ and applying part (2) of Theorem (5.1) we obtain $\lambda_k(\tilde{\Gamma}) \leq \lambda_{k+2}(\tilde{\Gamma})$ for all $k$.

\[ \square \]

7 Bounds on the Lowest Non-zero Eigenvalue

In this section, we will present some upper and lower bounds on the non-zero lowest eigenvalue. First, we obtained a few upper bounds using trial functions from the domain of the quadratic form. For a star graph, a simple lower estimate is expressed involving the maximal length of an edge. And then, a lower bound is also provided using the Eulerian cycle approach.

7.1 Upper Bounds

Let $\ell_n$ denotes the length of each edge $e_n$ and $L = \sum_{n=1}^{\mid E \mid} \ell_n$ be the total length of a graph $\Gamma$. Since $\varphi \equiv 1$ is in the domain of the quadratic form of a graph whose vertices are equipped with vertex conditions either (C1), (C2), (C3), or (C4), therefore, we can derive a simple upper estimate on the first eigenvalue given by the following inequality.

\[ \lambda_1(\Gamma) \leq \frac{\sum_{m=1}^{\mid V \mid} \alpha_m}{L}. \]  

(7.1)
Similarly, the function $\varphi(x) = \cos\left(\frac{2\pi x}{\ell_n}\right)$ on each edge $e_n = [0, \ell_n]$ is also in the domain, so we can use this function to get another upper bound on the lowest eigenvalue.

$$\lambda_1(\Gamma) < \frac{2}{L} \left( \frac{\sum_{n=1}^{\left|E\right|} 8\pi^4}{\ell_n^3} + \sum_{m=1}^{\left|V\right|} \alpha_m \right). \quad (7.2)$$

Clearly, it can be seen that the upper bound in (7.1) is a better estimate than (7.2). We obtain equality in (7.1) if and only if $\alpha_m = 0$, and the function $\varphi(x) \equiv 1$ is an eigenfunction corresponding eigenvalue zero. Assume that $\varphi(x)$ is an eigenfunction corresponding to an eigenvalue $\lambda_1(\Gamma) = 0$. Then,

$$\varphi^{(4)}(x) = 0 \implies \varphi(x) = a_n + b_n x + c_n x^2 + d_n x^3, \quad x \in e_n,$$

along each edge $e_n$. Since this function also minimizes Rayleigh quotient, this implies that $\int_{\Gamma} |\varphi''(x)|^2 dx = 0$, and thus $\varphi''(x) = 0$ implies $c_n = d_n = 0$. The eigenfunction is reduced to $\varphi(x) = a_n + b_n x$, $x \in e_n$. When the conditions (C4) are imposed at each vertex of $\Gamma$, the function $\varphi$ satisfies second equation in (C4) if and only if $b_n = 0$. Additionally, the first equation in (C4) implies that all $a_n$ are equal and function is constant throughout the entire graph. Furthermore, in a connected graph, the eigenfunction $\varphi(x) = 1$ is unique for eigenvalue $\lambda_1(\Gamma) = 0$ when vertices have conditions (C2) or (C4). However, for conditions (C1) or (C3) with $\alpha_m = 0$, uniqueness may not hold, and it can depend on edge geometry and parameterization. For example, in special cases like a tree or a star graph, where all edges have the same length and appropriate parameterization ensures $\varphi$’s continuity on $\Gamma$, and vertices have condition (C3) with $\sigma_{x_i}^{1m} = \sigma_{x_j}^{1m}$, the function $\varphi(x) = a + bx$ is an eigenfunction with eigenvalue $\lambda_1(\Gamma) = 0$.

By using the variational principle, we present upper bounds for the spectral gap. From now onward, unless specified, we assume that the interaction strengths $\alpha_m$ for all conditions at each vertex are zero. The following upper bound for the conditions (C4) with zero delta potential at all vertices was given in [2], but the same result can be obtained for the other three vertex conditions.

**Proposition 7.1** Let $\Gamma$ be a connected finite compact metric graph equipped with vertex conditions (C1), (C2), (C3), or (C4) at each vertex, and let $\alpha_m = 0$ for all $m$. Let $L$ be the total length of $\Gamma$, and $\ell_n$ be the length of each edge $e_n = [0, \ell_n]$. Then

$$\lambda_2(\Gamma) \leq 16 \left(\frac{\pi}{L}\right)^4 \sum_{n=1}^{\left|E\right|} \left(\frac{L}{\ell_n}\right)^3.$$

If $\Gamma$ is a bipartite graph, then

$$\lambda_2(\Gamma) \leq \left(\frac{\pi}{L}\right)^4 \sum_{n=1}^{\left|E\right|} \left(\frac{L}{\ell_n}\right)^3.$$
Proof Since $\varphi(x) = \cos\left(\frac{2\pi x}{\ell_n}\right)$, $x \in e_n = [0, \ell_n]$ belongs to the domain of quadratic form; moreover, this function is also orthogonal to constant function. Using it as test function and with the help of a min-max description of eigenvalues, we arrive at the required result. For second inequality, we consider a test function defined by $\varphi(x) = \cos\left(\frac{\pi x}{\ell_n}\right)$ on $e_n$, parameterised in a way that $\varphi$ is continuous on $\Gamma$, using the same technique, we obtain the desired inequality.

If the graph $\Gamma$ has certain edges having small lengths, then the bounds from above proposition are reasonably bad. In that case, one can find better bounds by comparing with eigenvalues of other graphs, as introduced in [2].

### 7.2 Lower Bounds

Generally, the Rayleigh quotient is used to obtain upper estimates on eigenvalues of a metric graph. However, we now provide a lower bound on the lowest non-zero eigenvalue of a star graph with the help of the Rayleigh quotient. The following estimate on the lowest non-zero eigenvalue is derived by following the idea presented in [27].

Let $I$ be an interval of length $\ell$, parameterized by $[0, \ell]$, and the endpoints of $I$ are equipped with following conditions:

$$\varphi(0) = \varphi''(0) = 0, \quad \varphi'(\ell) = \varphi'''(\ell) = 0.$$  

Then the eigenvalues are $\lambda_{k+1}(I) = (2k + 1)^4 \left(\frac{\pi}{2\ell}\right)^4$, $k \geq 0$.

Consider a star graph $\Gamma$ and let the internal vertex of star graph be equipped with (C4) with $\alpha_m = 0$, and the boundary vertices are equipped with the extended form of vertex conditions (C1). That is,

$$\varphi(x_j) = \varphi''(x_j) = 0, x_j \in v_m.$$  

Cleary, zero is not an eigenvalue of $\Gamma$. Assume that $\psi_1$ is an eigenfunction corresponding to eigenvalue $\lambda_1(\Gamma)$. Since $\left(\frac{\pi}{2\ell_n}\right)^4$ is the first eigenvalue of an edge $e_n$ of length $\ell_n$.

Therefore we have

$$\int_{e_n} |\psi_1''(x)|^2 dx \geq \left(\frac{\pi}{2\ell_n}\right)^4 \int_{e_n} |\psi_1(x)|^2 dx \geq \left(\frac{\pi}{2\ell_{\text{max}}}\right)^4 \int_{e_n} \psi_1(x)^2 dx,$$

$$\lambda_1(\Gamma) = \frac{\int_{\Gamma} |\psi_1''(x)|^2 dx}{\int_{\Gamma} |\psi_1(x)|^2 dx} \geq \left(\frac{\pi}{2\ell_{\text{max}}}\right)^4.$$  

The estimate is sharp for the equilateral star graph.

Now, we present a lower bound on the spectral gap in terms of the total length of a graph. For standard Laplacian, in [13] L. Friedlander uses the symmetrization
technique, which involves the co-area formula, and showed that the spectral gap is minimum for an interval among all the graphs with the same total length. To the best of our knowledge, we believe the co-area formula does not hold for higher-order derivatives and hence, the symmetrization technique can’t be easily generalized for higher-order differential operators. On the other hand, Nicaise in [19] and Kurasov and Naboko in [16] provided the same estimates for the spectral gap of a given graph in terms of the spectral gap for a single interval (of the same length). They achieved this using the Eulerian cycle approach, which enables the deformation of the original graph through successive steps, resulting in graphs with progressively smaller spectral gaps. We will use the Eulerian cycle approach for fourth-order differential operator to obtain a lower bound on the spectral gap. Moreover, in [11], it was shown that the lowest eigenvalue of Schrödinger operator on compact metric graphs with $\delta$-conditions

\[ \phi'(0) = I_+ \phi(0), \quad \phi'(L) = I_- \phi(L). \]

Where $I_+$ and $I_-$ are the total positive and total negative strengths. The proof was based on the Eulerian cycle approach presented in [16], in which the following two bounds were established for standard Laplacian on a metric graph $\Gamma$,

\[ \lambda_2(\Gamma) \geq \left( \frac{\pi}{L} \right)^2. \]

If the degree of each vertex is even, then the above estimate can be improved as,

\[ \lambda_2(\Gamma) \geq 4 \left( \frac{\pi}{L} \right)^2. \]

Following the Eulerian cycle approach presented in [16], a similar bounds were proved in [2] for the lowest non-zero eigenvalue of the beam operator on a metric graph equipped with condition (C4) with zero strengths. We use the same Eulerian cycle approach, as in [11], to obtain a similar estimate for lowest non-zero eigenvalue of a metric graph $\Gamma$ equipped with conditions (C3) with $\sigma_{x_i}^{v_m} = \sigma_{x_j}^{v_m}$.

We recall vertex conditions (C3) such that

\[ \sigma_{x_i}^{v_m} = \sigma_{x_j}^{v_m}, \quad x_i, x_j \in v_m \quad \text{and} \quad \sigma_{x_i}^{v_m} \in \mathbb{C}, \quad (7.3) \]

can be written as

\[
\begin{cases}
\phi(x_i) = \phi(x_j) \equiv \phi(v_m), & x_i, x_j \in v_m, \\
\partial \phi(x_i) = \partial \phi(x_j), \\
\sum_{x_j \in v_m} \partial^2 \phi(x_j) = 0, \\
\sum_{x_j \in v_m} \partial^3 \phi(x_j) = -\alpha_m \phi(v_m).
\end{cases} \quad (7.4)
\]
Theorem 7.2 Let $\Gamma$ be a connected finite compact metric graph of length $L$ equipped with conditions (7.4) at all vertices. Also assume that the strengths of delta interactions, $\alpha_m$, at all vertices have the same sign. Then, the lowest non-zero eigenvalue $\lambda_1(\Gamma)$ is bounded from below by the lowest non-zero eigenvalue $\lambda_1(C^v_{2L})$ of loop $C^v_{2L}$, which is parameterized by the interval $[0, 2L]$, of total length $2L$ and single vertex $v = \{0, 2L\}$ satisfying the conditions

\[
\begin{align*}
\varphi(0) &= \varphi(2L), \\
\varphi'(0) &= \varphi'(2L), \\
\varphi''(0) &= \varphi''(2L), \\
\varphi'''(0) - \varphi'''(2L) &= -2\alpha \varphi(0).
\end{align*}
\]

Here, $\alpha := \sum_{m=1}^{\vert V \vert} \alpha_m$ denotes the total strength of delta interaction.

Proof Let $\vert E \vert$ and $\vert V \vert$, respectively, be the total number of edges and vertices of the metric graph $\Gamma$ and denote by $e_n$ its $n^{th}$ edge between vertices $v_m$ and $v'_m$. We construct a graph $\Gamma_2$ from $\Gamma$ by attaching a new edge $e'_n$ of the same length as $e_n$ between vertices $v_m$ and $v'_m$ for all $n = 1, \ldots, \vert E \vert$. Assume that the graph $\Gamma_2$ is equipped with same vertex conditions, (7.4), but with strengths of delta interactions equal to $2\alpha_m$ in all vertices. Without loss of generality we can assume that $\alpha_m \geq 0$, $m = 1, \ldots, \vert V \vert$.

It is easy to observe that each eigenvalue of $\Gamma$ is also an eigenvalue of $\Gamma_2$. Let $\varphi$ be an eigenfunction corresponding to eigenvalue $\lambda_1(\Gamma)$. The eigenfunction $\varphi$ can be extended to $\Gamma_2$ by letting it assume the same values on the new edges $e'_n$ as $e_n$. This newly constructed function $\varphi_2$ on $\Gamma_2$ satisfy the eigenvalue equation on $\Gamma_2$ for the same eigenvalue and the vertex conditions. This, in particular, implies

$$\lambda_1(\Gamma) \geq \lambda_1(\Gamma_2).$$ \hfill (7.5)

Every vertex in $\Gamma_2$ has an even degree; therefore, there exists a closed Eulerian path $T_{2L}$, of length $2L$, that traverses each edge precisely once. We can obtain this closed path by cutting through certain vertices in $\Gamma_2$. We assume that the cutting or splitting of vertices is performed in a way that the nature of vertex conditions and sign of delta interaction strengths are preserved. The closed path $T_{2L}$ has $2\vert E \vert$ number of edges and vertices and each vertex is of degree two. As (7.4) represent conditions (C3) when

$$\sigma_{x_i}^{v_m} = \sigma_{x_j}^{v_m}, \quad x_i, x_j \in v_m \quad \text{and} \quad \sigma_{x_i}^{v_m} \in \mathbb{C}\{0\}$$

and conditions (C4) when

$$\sigma_{x_i}^{v_m} = \sigma_{x_j}^{v_m} = 0, \quad x_i, x_j \in v_m$$

therefore, Theorem (5.5) gives

$$\lambda_1(\Gamma_2) \geq \lambda_1(T_{2L}).$$ \hfill (7.6)
Let \( \varphi_1 \) be the eigenfunction corresponding to the eigenvalue \( \lambda_1(\mathcal{T}_2L) \). There exists a vertex \( v_{\text{min}} \) of \( \mathcal{T}_2L \) such that \( |\varphi_1(v_{\text{min}})| \leq |\varphi_1(v)| \). Here, \( v \) represents an arbitrary vertex of \( \mathcal{T}_2L \). Let \( \tilde{T}_2L \) is obtained from \( T_2L \) by assuming zero delta interaction at all vertices of \( T_2L \) except at \( v_{\text{min}} \), where we assume that the strength of delta interaction is equal to \( 2\alpha \). The eigenfunction \( \varphi_1 \) belongs to the domain of the quadratic form of \( \tilde{T}_2L \) and therefore it can be used to estimate \( \lambda_1(\tilde{T}_2L) \):

\[
\lambda_1(\mathcal{T}_2L) \geq \frac{\int_{\mathcal{T}_2L} |\varphi''(x)|^2 \, dx + 2\alpha |\varphi_1(v_{\text{min}})|^2}{\int_{\mathcal{T}_2L} |\varphi_1(x)|^2 \, dx} \geq \lambda_1(\tilde{T}_2L).
\] (7.7)

The domain of the quadratic form associated with \( \tilde{T}_2L \) contains all the functions from the space \( W_2^2(\tilde{T}_2L \setminus V) \) satisfying the following two conditions at all vertices

\[
\begin{align*}
\varphi(x_i) &= \varphi(x_j), \\
\partial_\nu \varphi(x_i) &= \partial_\nu \varphi(x_j).
\end{align*}
\]

Number of edges and vertices of \( \tilde{T}_2L \) are \( 2|E| \) and each vertex is of degree two. Therefore, we can identify \( n^{th} \) edge by the interval \([0, \ell_n]\) in such a way that any two consecutive vertices are given by the set of end points \([0, 0]\) and \([\ell_n, \ell_{n+1}]\). This arrangement implies that the condition \( \partial_\nu \varphi(x_i) = \partial_\nu \varphi(x_j) \) is equivalent to the condition \( \varphi'(x_i) = \varphi'(x_j) \).

Consider the loop \( C_v^u \) of total length \( 2L \) and a single vertex \( v \). Identify the loop with the interval \([0, 2L]\) and consider the following conditions at the single vertex \( v \),

\[
\begin{align*}
\varphi(0) &= \varphi(2L), \\
\varphi'(0) &= \varphi'(2L), \\
\varphi''(0) &= \varphi''(2L), \\
\varphi'''(0) - \varphi'''(2L) &= -2\alpha \varphi(0).
\end{align*}
\]

The quadratic form, and its domain, associated with the graph \( C_v^u \) coincides with the quadratic form, and its domain, associated with the graph \( \tilde{T}_2L \). Therefore,

\[
\lambda_1(\tilde{T}_2L) = \lambda_1(C_v^u). \tag{7.8}
\]

Equation (7.8) along with inequalities (7.5), (7.6) and (7.7) imply the result. \( \Box \)

In Theorem (7.2) if the original graph \( \Gamma \) is Eulerian, i.e. all vertices are of even degree then we can skip the step of constructing \( \Gamma_2 \) in the proof and can choose an Eulerian path \( \mathcal{T}_L \) on the original graph \( \Gamma \) of length \( L \). The lowest non-zero eigenvalue \( \lambda_1(\Gamma) \) is then bounded from below by the lowest non-zero eigenvalue \( \lambda_1(C_v^u) \).

**Corollary 7.3** Let degrees of all vertices of \( \Gamma \) are even and all other assumptions of Theorem (7.2) be satisfied. Then the lowest non-zero eigenvalue of \( \Gamma \) is bounded from
below by the lowest non-zero eigenvalue of the loop $C^v_L$ of length $L$ and a single vertex $v$ equipped with conditions

$$
\begin{align*}
\varphi(0) &= \varphi(L), \\
\varphi'(0) &= \varphi'(L), \\
\varphi''(0) &= \varphi''(L), \\
\varphi'''(0) - \varphi'''(L) &= -\alpha \varphi(0).
\end{align*}
$$

**Corollary 7.4** If $\alpha_m = 0$ for $m = 1, \ldots, |V|$, then

$$
\lambda_2(\Gamma) \geq \left(\frac{\pi}{L}\right)^4. \tag{7.9}
$$

In addition, if $\Gamma$ is Eulerian, then

$$
\lambda_2(\Gamma) \geq 16 \left(\frac{\pi}{L}\right)^4. \tag{7.10}
$$

**Remark.** Inequalities (7.9) and (7.10) were proved in Theorem (4) of [2] for the vertex conditions (C4) (with $\alpha_m = 0$), which require functions first derivative to be equal to zero at vertices. Corollary (7.4), on the other hand, does not require that the first derivative of functions from the domain are equal to zero at vertices. Instead, we assume a general condition of continuity of the normal first derivative at the vertices, i.e.,

$$
\partial \varphi(x_i) = \partial \varphi(x_j), \quad x_i, x_j \in v_m.
$$

## 8 Bounds on Higher Eigenvalues

In this section, we discuss bounds on higher eigenvalues values of the connected finite compact metric graphs. These bounds are expressed in terms of total length, number of edges $|E|$, number of vertices $|V|$, and the first Betti number $\beta$ of the graph. The first Betti number is the minimum number of edges required to remove from $\Gamma$ to obtain a tree. Equivalently, it is the total number of independent cycles in $\Gamma$ and given by

$$
\beta = |E| - |V| + 1.
$$

Most of these bounds are applicable for the eigenvalues of a graph with zero interaction strengths at all vertices.

The following Theorem presents an upper bound on eigenvalues in terms of the length of a single edge. The length of other edges can also be used to obtain a bound. But, we will use the longest edge with a maximum length $\ell_{max}$ to get a better estimate.
Theorem 8.1  Let $\Gamma$ be a metric graph, and let $\ell_{\text{max}}$ and $\beta$ represent the length of the longest edge in $\Gamma$ and the Betti number of the graph, respectively. If each vertex of $\Gamma$ is equipped with vertex conditions either (C2) or (C4) with $\alpha_m = 0, m = 1, 2, \ldots, |V|$. Then,

$$\lambda_{k+1}(\Gamma) \leq (k + \beta)^4 \left( \frac{\pi}{\ell_{\text{max}}} \right)^4, \quad k \geq 0.$$  \hspace{1cm} (8.1)

**Proof**  First, we select the longest edge $e_1$, having length $\ell_{\text{max}}$. We then add pendants edges to the vertices of $e_1$, producing new boundary vertices. Subsequently, additional pendant edges are added to these new vertices. This process is repeated until we achieve a tree $T$, which has equal number of edges as $\Gamma$. By Theorem (6.1) this surgical transformations, at each step, lowers all eigenvalues. Second, we join the vertices together to obtain the graph $\Gamma$, and each gluing produces a cycle in the graph. By Theorem (5.1). The bound is obtained by the following inequalities

$$\left( \frac{k\pi}{\ell_{\text{max}}} \right)^4 = \lambda_{k+1}(e_1) \geq \lambda_{k+1}(T) \geq \lambda_{k+1-\beta}(\Gamma). \quad \square$$

Lemma 8.2  Let $T$ be a metric tree, and $\Gamma$ be a graph obtained by attaching pendant graphs at some vertices of $T$. Let $e_1$ in $T$ has a maximum length, say $\ell_{\text{max}}$. If the vertices of $\Gamma$ are equipped with conditions (C2) or (C4) with $\alpha_m = 0$ at all vertices, then

$$\lambda_{k+1}(\Gamma) \leq \left( \frac{k\pi}{\ell_{\text{max}}} \right)^4. \quad (8.2)$$

**Proof**  Suppose we want to find an upper estimate for the graph $\Gamma$, as shown in the figure below.

The following figure illustrates the idea of the proof, and at each step, we use a surgical toolkit to obtain the inequality.

First, we choose an edge $e_1$ from $T$ with length $\ell_{\text{max}}$. Now, we want to reconstruct the graphs $T$ and $\Gamma$ from $e_1$, using the surgery toolkit step by step. At second step, we add pendant edges to vertices $v_1$ and $v_2$ producing new vertices $v^*_m$. Third, we add pendant edges at $v^*_m$, if any. Thus, we have recreated the tree $T$. Finally, we attach pendant graphs at some vertices of $T$ to obtain $\Gamma$. At each step, the surgical transformation lowers all eigenvalues, and we get that the eigenvalues of a graph $\Gamma$ are bounded above by the eigenvalues of a maximal edge $e_1$ of $T. \quad \square$
The upper bound (8.1) in the above theorem can also be used to estimate eigenvalues for the class of graph described in the previous lemma. However, this estimate can be applied to more general graphs compared to (8.2), but (8.2) gives better estimate for graphs with large number of cycles such that the maximum length is not a part of any cycle. The following two theorems give an estimate of the eigenvalues in terms of the total length of a graph, in contrast to length of a single long edge.

**Theorem 8.3** Let $\Gamma$ be a metric graph with the total length $L$ and $\beta$ be the first Betti number of the graph $\Gamma$. If each vertex of $\Gamma$ is equipped with vertex condition either (C2) or (C4) with $\alpha_m = 0$, $m = 1, 2, \ldots, |V|$. Then,

$$\lambda_{k+1}(\Gamma) \leq (k - 3 + 3|E| + \beta)^4 \left( \frac{\pi}{L} \right)^4.$$  

(8.3)

**Proof** Consider an interval $I$ of total length $L$ with endpoints equipped with condition (C4) with strengths zero. Obtain a path graph $P$ by creating the vertex of degree two at interior points of $I$, in accordance with remark (2.1), in such a way that the total number of edges in $P$ along with their length coincides with edges in $\Gamma$. Now, we replace the vertex conditions at interior vertices from (C2) to (C4) using Proposition (4.1), call this graph $P'$, and then apply Theorem (5.5) to split the path graph $P'$ into disjoint sub-intervals $I_n$ for $n = 1, 2, \ldots, |E|$. Let $\Gamma'$ be the graph with disconnected components $I_n$, and glue all the intervals $I_n$ together using Theorem (5.1) to obtain the tree graph $T$. The gluing is performed in such a way that the graph $\Gamma$ can be obtained by pairwise gluing of $\beta$ pairs of vertices of $T$.

$$\left( \frac{k\pi}{L} \right)^4 = \lambda_{k+1}(I) = \lambda_{k+1}(P) \geq \lambda_{k-2|E|+3}(P') \geq \lambda_{k-2|E|+3}(\Gamma') \geq \lambda_{k-3|E|+4}(T) \geq \lambda_{k-3|E|+4-\beta}(\Gamma).$$

For any fixed real number $\lambda$, the eigenvalue counting function $N_\Gamma(\lambda)$ is defined as the number of eigenvalues of a graph $\Gamma$ smaller than $\lambda$. Since the quantum graph $\Gamma$ is finite compact, and the operator is self-adjoint with a discrete spectrum bounded from below. Therefore the value of function $N_\Gamma(\lambda)$ is finite.

$$N(\lambda) = \# \{ \lambda_k \in \sigma(\Gamma) : \lambda_k \leq \lambda \}.$$

The following theorem provides upper and lower estimates on general eigenvalues of a metric graph $\Gamma$ with vertices equipped with condition (C1). A similar result for
Neumann Laplacian acting on the edges of a metric graph $\Gamma$ was proved in [28]. In their work, they used the rank of the resolvent difference of Dirichlet and Neumann Laplacian to obtain bounds on the corresponding eigenvalue counting function. In contrast, we have used the interlacing inequalities of eigenvalues to obtain the bounds on the corresponding eigenvalue counting function.

**Theorem 8.4** Let $\Gamma$ be a metric graph of total length $L$ with $|V|$ number of vertices and $|E|$ number of edges. Let each vertex of $\Gamma$ be equipped with condition (C1) with $\alpha_m = 0, m = 1, 2, \ldots, |V|$. Then

$$
\left( \frac{\pi}{L} \right)^4 (k - |V|)^4 \leq \lambda_k(\Gamma) \leq \left( \frac{\pi}{L} \right)^4 (k + |E| - 1)^4.
$$

(8.4)

**Proof** Since the interaction strengths are zero at each vertex, therefore the quadratic form is non-negative, and thus the eigenvalues $\lambda_k(\Gamma) \geq 0$. Hence the lower estimate in (8.4) is interesting only if $k > |V|$.

Consider the operator $\frac{d^4}{dx^4}$ acting on an interval $I$ of length $\ell$, and assume that the endpoints are equipped with conditions (C1) with $\alpha_m = \infty, m = 1, 2$. The eigenvalues are $\lambda_k(I) = \left( \frac{k\pi}{\ell} \right)^4$, for $\lambda \geq 0$ the value of eigencounting function is $N_{[0,\ell]}(\lambda) = \left[ \frac{4\sqrt{\lambda}}{\pi \ell} \right]$. Where square brackets mean to take the integer part of the argument. Let $\Gamma_\infty$ be graph obtained from $\Gamma$ by imposing the conditions (C1) with $\alpha_m = \infty$ at each vertex of $\Gamma$, we shall call them extended conditions. Since these conditions imposed at a vertex of degree two or more does not connect the individual function living on the incident edge in any way. The extended conditions have the effect of disconnecting the vertex $v_m$ of degree $d_m$ into $d_m$ vertices of degree one, so the graph $\Gamma_\infty$ is now decoupled into a set of intervals, and the set of eigenvalues is just the union of eigenvalues of each interval (counting multiplicities). Let for some $\lambda \in \mathbb{R}$, $N_{\Gamma}(\lambda)$ and $N_{\Gamma_\infty}(\lambda)$ denote the eigenvalue counting functions for the graphs $\Gamma$ and $\Gamma_\infty$, respectively. The counting function is given by

$$
N_{\Gamma_\infty}(\lambda) = \sum_{n=1}^{|E|} N_{[0,\ell_n]}(\Gamma) = \left[ \frac{4\sqrt{\lambda}}{\pi \ell_1} \right] + \left[ \frac{4\sqrt{\lambda}}{\pi \ell_2} \right] + \ldots + \left[ \frac{4\sqrt{\lambda}}{\pi \ell_{|E|}} \right] \leq \left[ \frac{4\sqrt{\lambda}}{\pi L} \right].
$$

(8.5)

Since for any $a$ and $b$, $[a] + [b] \leq [a + b]$, therefore taking integer part of the sum of the terms is increased at most by the number of terms minus one as compared to adding integer parts only. As the number of terms in (8.5) are equal to number of edges $|E|$, thus

$$
\left[ \frac{4\sqrt{\lambda}}{\pi L} \right] = \left[ \frac{4\sqrt{\lambda}}{\pi} (\ell_1 + \ell_2 + \ldots + \ell_{|E|}) \right] \leq \left[ \frac{4\sqrt{\lambda}}{\pi \ell_1} \right]
$$
The formula (8.5) and (8.6) give bounds for the eigenvalues of the graph $\Gamma_\infty$. Let $h$ and $h_\infty$ denote the quadratic forms of the graphs $\Gamma$ and $\Gamma_\infty$ with domains $D(h)$ and $D(h_\infty)$, respectively. Since the expression of the quadratic forms $h$ and $h_\infty$ are same; moreover, if a function $\varphi$ satisfies the extended conditions at some vertex $v_m$, then it also satisfies (C1) at $v_m$. Therefore, the domain $D(h_\infty)$ is a subspace of $D(h)$ and the quadratic forms $h$ and $h_\infty$ agree on $D(h_\infty)$, thus minimizing over a smaller domain results in large eigenvalues. Furthermore, the domain $D(h_\infty)$ is a co-dimension $|V|$ subspace of $D(h)$. Thus, by the rank $|V|$ nature perturbation, the following interlacing inequalities hold.

\[ \lambda_k(\Gamma) \leq \lambda_k(\Gamma_\infty) \leq \lambda_{k+|V|}(\Gamma) \]

Then the above interlacing inequalities imply the following inequalities between the eigenvalue counting functions.

\[ N_{\Gamma_\infty}(\lambda) \leq N_{\Gamma}(\lambda) \leq N_{\Gamma_\infty}(\lambda) + |V|. \]  

Thus,

\[ \left[ \frac{4\sqrt{\lambda}}{\pi} L \right] - |E| + 1 \leq N_{\Gamma_\infty}(\lambda) \leq N_{\Gamma}(\lambda) \leq N_{\Gamma_\infty}(\lambda) + |V| \leq \left[ \frac{4\sqrt{\lambda}}{\pi} L \right] + |V|. \]

Setting $\lambda = \left( \frac{\pi k}{L^2} \right)^4$ we get

\[ k - |E| + 1 \leq N_{\Gamma} \left( \frac{\pi^4 k^4}{L^4} \right) \leq k + |V|, \]

so

\[ \lambda_{k-|E|+1} \leq \frac{\pi^4}{L^4} k^4 \leq \lambda_{k+|V|}. \]

This estimate implies that the multiplicity of the eigenvalues is uniformly bounded by $|E| + |V|$. Setting $\tilde{k} = k - |E| + 1$ we get $\lambda_{\tilde{k}} \leq \frac{\pi^4}{L^4} (\tilde{k} + |E| - 1)^4$ and similarly setting $\tilde{k} = k + |V|$ we get $\frac{\pi^4}{L^4} (\tilde{k} - |V|)^4 \leq \lambda_{\tilde{k}}$. \qed
Based on the estimates derived in the previous theorem, we can now establish bounds on the eigenvalues of the same underlying metric graph when equipped with the remaining three conditions. While we have demonstrated this for (C2), it’s important to note that similar proofs can be applied to conditions (C3) and (C4), each leading to distinct estimates.

**Theorem 8.5** Let $\Gamma$ be a connected finite compact metric graph equipped with conditions (C2) with $\alpha_m = 0$ at all vertices. Then,

$$\left( \frac{\pi}{L} \right)^4 (k - |V|)^4 \leq \lambda_k(\Gamma) \leq \left( \frac{\pi}{L} \right)^4 (k + |E| + |V| - 1)^4.$$  

(8.8)

**Proof** Consider the metric graph $\tilde{\Gamma}$, which is identical to $\Gamma$ but equipped with vertex conditions (C1) where $\alpha_m = 0$ for all vertices. We can apply Theorem (8.4) to the eigenvalues of $\tilde{\Gamma}$. Additionally, Proposition (4.1) establishes a link between the eigenvalues of graphs $\Gamma$ and $\tilde{\Gamma}$. Using these two results, we can deduce the following inequalities.

$$\left( \frac{\pi}{L} \right)^4 (k - |V|)^4 \leq \lambda_k(\tilde{\Gamma}) \leq \lambda_k(\Gamma) \leq \lambda_{k+|V|}(\tilde{\Gamma}) \leq \left( \frac{\pi}{L} \right)^4 (k + |E| + |V| - 1)^4.$$

\[\square\]

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**Declarations**

**Competing interests** The authors have no conflicts of interest to declare that are relevant to the content of this article.

**References**

1. Berkolaiko, G., Kennedy, J., Kurasov, P., Mugnolo, D.: Surgery principles for the spectral analysis of quantum graphs. Trans. Am. Math. Soc. **372**(7), 5153–5197 (2019)
2. Kurasov, P., Muller, J.: On the spectral gap for networks of beams. In: Schrödinger Operators, Spectral Analysis and Number Theory, pp. 169–179. Springer, Berlin (2021)
3. Dekoninck, B., Nicaise, S.: The eigenvalue problem for networks of beams. Linear Algebra Appl. **314**(1–3), 165–189 (2000)
4. Dekoninck, B., Nicaise, S.: Control of networks of Euler-Bernoulli beams. ESAIM **4**, 57–81 (1999)
5. Gregorio, F., Mugnolo, D.: Bi-Laplacians on graphs and networks. J. Evol. Equ. **20**(1), 191–232 (2020)
6. Kiik, J.-C., Kurasov, P., Usman, M.: On vertex conditions for elastic systems. Phys. Lett. A **379**(34–35), 1871–1876 (2015)
7. Bae, S., Ettehad, M.: On vertex conditions in elastic beam frames: analysis on compact graphs. arXiv preprint arXiv:2112.01466 (2021)
8. Berkolaiko, G., Ettehad, M.: Three-dimensional elastic beam frames: rigid joint conditions in variational and differential formulation. Stud. Appl. Math. 148, 1586 (2022)
9. Ariturk, S.: Eigenvalue estimates on quantum graphs. arXiv preprint arXiv:1609.07471 (2016)
10. Berkolaiko, G., Kennedy, J.B., Kurasov, P., Mugnolo, D.: Edge connectivity and the spectral gap of combinatorial and quantum graphs. J. Phys. A 50(36), 365201 (2017)
11. Karreskog, G., Kurasov, P., Trygg Kupersmidt, I.: Schrödinger operators on graphs: symmetrization and Eulerian cycles. Proc. Am. Math. Soc. USA 144(3), 1197–1207 (2016)
12. Kennedy, J.B., Kurasov, P., Malenová, G., Mugnolo, D.: On the Spectral Gap of a Quantum Gap. Annales Henri Poincare, vol. 17, pp. 2439–2473. Springer, Berlin (2016)
13. Friedlander, L.: Extremal properties of eigenvalues for a metric graph. Ann. Inst. Fourier 55, 199–211 (2005)
14. Rohleder, J.: Eigenvalue estimates for the laplacian on a metric tree. Proc. Am. Math. Soc. USA 145(5), 2119–2129 (2017)
15. Rohleder, J., Seifert, C.: Spectral monotonicity for Schrödinger operators on metric graphs. In: Discrete and Continuous Models in the Theory of Networks, pp. 291–310. Springer, Berlin (2020)
16. Kurasov, P., Naboko, S.: Rayleigh estimates for differential operators on graphs. J. Spectr. Theory 4(2), 211–219 (2014)
17. Kurasov, P., Malenová, G., Naboko, S.: Spectral gap for quantum graphs and their edge connectivity. Journal of Physics A: Mathematical and Theoretical 46(27), 275309 (2013)
18. Berkolaiko, G., Kuchment, P.: Introduction to quantum graphs. Number186. American Mathematical Society, New York (2013)
19. Nicaise, S.: Spectre des réseaux topologiques finis. Bull. Sci. Math. 111(4), 1 (1987)
20. Exner, P., Jex, M.: On the ground state of quantum graphs with attractive δ-coupling. Phys. Lett. A 376(5), 713–717 (2012)
21. Kurasov, P.: On the spectral gap for Laplacians on metric graphs. Acta Phys. Pol. A 124(27), 1060 (2013)
22. Band, R., Lévy, G.: Quantum graphs which optimize the spectral gap. In: Annales Henri Poincaré, vol. 18, pp. 3269–3323. Springer, Berlin (2017)
23. Del Pezzo, L.M., Rossi, J.D.: The first eigenvalue of the p-Laplacian on quantum graphs. Anal. Math. Phys. 6(4), 365–391 (2016)
24. Kurasov, P.: Spectral geometry of graphs. Birkhäuser 1(5), 5–30 (2021)
25. Kurasov, P., Muller, J.: n-Laplacians on metric graphs and almost periodic functions: I. In Annales Henri Poincaré, vol. 22, pp. 121–169. Springer, Berlin (2021)
26. Berkolaiko, G., Kuchment, P.: Dependence of the spectrum of a quantum graph on vertex conditions and edge lengths. Spectr. Geom. 84, 117–137 (2012)
27. Zhao, J., Shi, G.: Eigenvalue estimates for the Laplacian with anti-Kirchhoff conditions on a metric tree. J. Math. Anal. Appl. 477(1), 670–684 (2019)
28. Boman, J., Kurasov, P., Suhr, R.: Schrödinger operators on graphs and geometry ii. spectral estimates for l1-potentials and an Ambartsumian theorem. Integr. Equ. Oper. Theory 90(3), 1–24 (2018)

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