On the Origin of Kaluza’s Idea of Unification

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Abstract

We argue that the starting point of Kaluza’s idea of unifying electrodynamics and gravity was the analogy between gravitation and electromagnetism which was pointed out by Einstein and Thirring. It seems that Kaluza’s attention was turned to this point by the three papers on the Lense-Thirring effect and the analogy between gravitation and electromagnetism which were published a short time before Kaluza’s paper was submitted. We provide here an English translation of the third of these papers.

Keywords: History of physics, Unification, Higher-dimensional theories.

1 Introduction

In most descriptions of the Kaluza-Klein approach to unification \cite{1, 2, 3, 4}, Kaluza’s idea \cite{5} (English translations exist in \cite{6, 7}) to add a fifth dimension to spacetime is considered to be the starting point for the whole structure.

Most authors also mention the previous five-dimensional unifying theory by Nordstrom \cite{8} (English translation in \cite{7}) which was a flat five-dimensional Maxwell theory with the additional requirement that all dynamical variables are independent of the fifth coordinate. Nordstrom’s interpretation of the resulting theory was that of a usual four-dimensional Maxwell system coupled with relativistic scalar gravity thus obtaining a common source for both kinds of forces.

Since it is generally accepted that Kaluza was unaware of Nordstrom’s theory, Kaluza’s idea is regarded as a ”quantum leap” with no direct connection to any previous work except, of course, Einstein’s general relativity. In addition, Weyl’s four-dimensional unifying theory \cite{9} can be considered a source of the ”spirit of unification” of the time.

We wish to suggest a different view, viz. that Kaluza’s starting point for his theory was the analogy between the Maxwell and Einstein equations for fields of slowly varying sources. This analogy was pointed out by Thirring (based on earlier work by Einstein \cite{10} - English translation in \cite{11}) during his work on the Thirring-Lense effect \cite{12, 13} (English translation in \cite{14}). It is explicitly mentioned (although in a footnote) in the first 1918 paper by Thirring \cite{12} and an expanded treatment appears in a 2-page paper \cite{15} in the same year.

Indeed, Einstein was the first, in 1913, to note an electrodynamic-gravitational analogy \cite{10} but it was based on a tentative version of the gravitational field equations. A corrected version which was consistent with the final Einstein equations is Thirring’s 2-page 1918 paper \cite{15}. This is actually the basis for the gravito-electromagnetism formalism \cite{16, 17} but it is rarely acknowledged as such. Usually, the other two Thirring & Lense papers (and their translations) are referred to in this context.

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These three papers \[\text{[12, 13, 15]}\] were published slightly before Kaluza prepared a manuscript of his idea and sent it to Einstein. This must have happened in the first third of 1919 as is evident from Einstein’s first very warm reaction in a letter dated 21 April 1919 \[\text{[18, 2, 3, 4]}\]. Moreover, the 2-page 1918 paper by Thirring is cited by Kaluza in the context of this very same analogy between electrodynamics and gravity at the beginning of his paper and may be regarded a preparation and motivation for all the rest.

Therefore, we suggest viewing Kaluza’s work as being based on two main ingredients: the electrodynamic-gravitational analogy and the addition of a fifth dimension. Kaluza’s starting point was the electrodynamic-gravitational analogy for slowly varying sources. Next came the observation that this is more than an analogy, in that general relativity in some sense contains electromagnetism. Adding a fifth dimension was only the next step which was unavoidable in order to have enough “room” for both fields within the same theory. Going to five dimensions was probably an independent contribution of Kaluza; certainly so was his idea to go to five-dimensional general relativity.

In order to strengthen our suggestion, in the next section we provide an English translation of Thirring’s 2-page 1918 paper \[\text{[15]}\] so that non-German speaking readers can judge for themselves. The last section includes a more detailed discussion about the connection of Thirring’s paper to Kaluza’s.

2 English Translation of the Thirring Paper \[\text{[15]}\]

Translated by N. K. Nielsen.

On the Formal Analogy between the Electromagnetic Fundamental Equations and the Einsteinian Equations of Gravity in the First Approximation

by Hans Thirring.

In the following some formal developments will be carried out that in an earlier article \[\text{[1]}\] only found room in a footnote. The matter under consideration is the analogy between the Maxwell-Lorentz equations on one hand, and those equations that determine the motion of a point particle in a weak gravitational field in the first approximation, on the other. Einstein himself already referred to this analogy in his speech at the Wiener Naturforschertag 1913 \[\text{[2]}\]; however, since his field equations have been subject to quite an important modification, it appears not improper to develop the formulas in question for the final version of the theory.

We remark in advance that in the following, we always use a coordinate system where the velocity of light is 1, and as coordinates we choose:

\[x_1 = x, x_2 = y, x_3 = z, x_4 = it.\]

As is well known, the equations of motion are

\[
\frac{d^2x_\tau}{ds^2} = \Gamma_{\nu}^\tau \frac{dx_\mu}{ds} \frac{dx_\nu}{ds}, \quad \tau = 1 \cdots 4. \tag{1}
\]

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1H. Thirring, this journal \textbf{19}, 33, 1918.
2A. Einstein, this journal \textbf{14}, 1261, 1913.
* Translator’s note: literally: Scientist Congress in Vienna 1913. Actually the reference should be \textbf{10}; A. Einstein, this journal \textbf{14}, 1249, 1913. The analogy is indeed mentioned on p. 1261.
On the right-hand side the velocity components of the point mass occur; if we neglect, according to our assumptions, their squares and products the equations become:

\[
\frac{d^2 x_\tau}{dt^2} = 2i(\Gamma_{14} \frac{dx_\tau}{dt} + \Gamma_{24} \frac{dx_2}{dt} + \Gamma_{34} \frac{dx_3}{dt}) - \Gamma_{44}.
\] (2)

In the following we consider the three spatial components of the equations of motion, and thus \(\tau = 1, 2, 3\). For weak fields, the three-index symbols are:

\[
\Gamma_{\sigma 4}^\tau = - \left\{ \begin{array}{c} \sigma 4 \\ \tau \end{array} \right\} = \left[ \begin{array}{c} \sigma 4 \\ \tau \end{array} \right] = \frac{1}{2} \left( \frac{\partial g_{\sigma \tau}}{\partial x_\sigma} + \frac{\partial g_{\tau 4}}{\partial x_{\sigma 4}} - \frac{\partial g_{\sigma 4}}{\partial x_\tau} \right).
\] (3)

The derivatives \(\frac{\partial g_{\sigma \tau}}{\partial x_\sigma}\) (\(\sigma \neq 4, \tau \neq 4\)) are, as will turn out immediately, of order of magnitude \(\tilde{v}^2\); in \(\mathbb{E}\) they are multiplied by \(\frac{dx_\tau}{dt}\) (order of magnitude \(|\tilde{v}|\)) and are hence negligible in our approximation. In \(\Gamma_{44}\) only the derivatives of \(g_{\tau 4}\) and \(g_{44}\) occur; hence for the following we need only those coefficients \(g_{\mu \nu}\) that contain the index 4 at least once. In order to compute them we use the approximate solution of Einstein's:

\[
g_{\mu \nu} = -\delta_{\mu \nu} + \gamma_{\mu \nu}, \quad \delta_{\mu \nu} = \left\{ \begin{array}{c} 1, \quad \mu = \nu, \\ 0, \quad \mu \neq \nu, \end{array} \right.
\]

\[
\gamma_{\mu \nu} = \gamma'_{\mu \nu} - \frac{1}{2}\delta_{\mu \nu} \sum_\alpha \gamma'_{\alpha \alpha},
\]

\[
\gamma'_{\mu \nu} = -\frac{\kappa}{2\pi} \int \frac{T_{\mu \nu}(x', y', \tilde{v}, t' - R)}{R} dV_0,
\] (4)

where \(T_{\mu \nu}\) denotes the energy tensor, \(x', y', \tilde{v}\) the coordinates of the integration space, \(R\) is the distance from the integration element to position of the point mass, and \(dV_0\) the naturally measured volume element. The energy tensor for incoherent matter is given by

\[
T_{\mu \nu} = T^{\mu \nu} = \rho_0 \frac{dx_\mu}{ds} \frac{dx_\nu}{ds}.
\] (5)

The four relevant coefficients \(g_{\mu 4}\) are accordingly:

\[
g_{14} = -i\frac{\kappa}{2\pi} \int \frac{\rho_0 v'_\mu}{R} \frac{dt'}{ds}^2 dV_0,
\]

\[
g_{24} = -i\frac{\kappa}{2\pi} \int \frac{\rho_0 v'_\mu}{R} \frac{dt'}{ds}^2 dV_0,
\]

\[
g_{34} = -i\frac{\kappa}{2\pi} \int \frac{\rho_0 v'_\mu}{R} \frac{dt'}{ds}^2 dV_0,
\]

\[
g_{44} = -1 + \frac{\kappa}{4\pi} \int \frac{\rho_0 dV_0}{(dt/ds)^2}.
\] (6)

The field components \(\Gamma_{\sigma 4}^\tau\) entering \(\mathbb{E}\) are now computed from the \(g_{\mu \nu}\) with the applied approximations as follows:

\[
\Gamma_{14}^1 = 0 \quad \Gamma_{24}^1 = \frac{1}{2} \left( \frac{\partial g_{14}}{\partial x_1} + \frac{\partial g_{24}}{\partial x_2} \right) \quad \Gamma_{34}^1 = \frac{1}{2} \left( \frac{\partial g_{34}}{\partial x_3} + \frac{\partial g_{44}}{\partial x_4} \right) \quad \Gamma_{44}^1 = -\frac{1}{2} \left( \frac{\partial g_{44}}{\partial x_1} + \frac{\partial g_{14}}{\partial x_4} \right)
\]

\[
\Gamma_{14}^2 = \frac{1}{2} \left( \frac{\partial g_{24}}{\partial x_1} - \frac{\partial g_{14}}{\partial x_2} \right) \quad \Gamma_{24}^2 = 0 \quad \Gamma_{34}^2 = \frac{1}{2} \left( \frac{\partial g_{34}}{\partial x_3} - \frac{\partial g_{44}}{\partial x_2} \right) \quad \Gamma_{44}^2 = -\frac{1}{2} \left( \frac{\partial g_{44}}{\partial x_3} + \frac{\partial g_{24}}{\partial x_4} \right)
\]

\[
\Gamma_{14}^3 = \frac{1}{2} \left( \frac{\partial g_{34}}{\partial x_1} - \frac{\partial g_{14}}{\partial x_3} \right) \quad \Gamma_{24}^3 = \frac{1}{2} \left( \frac{\partial g_{44}}{\partial x_2} - \frac{\partial g_{24}}{\partial x_3} \right) \quad \Gamma_{34}^3 = 0 \quad \Gamma_{44}^3 = -\frac{1}{2} \left( \frac{\partial g_{44}}{\partial x_3} + \frac{\partial g_{34}}{\partial x_4} \right)
\]

\[
\Gamma_{14}^4 = \frac{1}{2} \left( \frac{\partial g_{44}}{\partial x_1} - \frac{\partial g_{14}}{\partial x_4} \right) \quad \Gamma_{24}^4 = \frac{1}{2} \left( \frac{\partial g_{44}}{\partial x_2} - \frac{\partial g_{24}}{\partial x_4} \right) \quad \Gamma_{34}^4 = \frac{1}{2} \left( \frac{\partial g_{44}}{\partial x_3} - \frac{\partial g_{34}}{\partial x_4} \right) \quad \Gamma_{44}^4 = 0.
\] (7)

\[3\text{A. Einstein, Berl. Ber. 1916, p.688.}
\]

\[\ast \text{Translator’s note: more accurately: Sitzungsber. Konigl. Preuss. Akad. Wiss. (Berlin) 1916, 688 (1916). English translation in [19].}\]
The equations (2), (6) and (7) now correspond, disregarding some numerical factors, completely to the fundamental electrodynamic equations. In order to make this similarity more obvious, we set

$$A_x = ig_{14}, \quad A_y = ig_{24}, \quad A_z = ig_{34}, \quad \Phi = g_{44} + \frac{1}{2}.$$  

$$H_x = 2\Gamma^3_{24} = -2\Gamma^2_{34}, \quad H_y = 2\Gamma^3_{14} = -2\Gamma^2_{44}, \quad H_z = 2\Gamma^3_{14} = -2\Gamma^2_{34},$$

$$\mathcal{E}_x = \Gamma^1_{44} = -\Gamma^4_{14}, \quad \mathcal{E}_y = \Gamma^2_{44} = -\Gamma^4_{24}, \quad \mathcal{E}_z = \Gamma^3_{44} = -\Gamma^4_{34},$$

$$k = \frac{\kappa}{8\pi}.$$  

(8)

In terms of these quantities, equations (2), (6) and (7) become:

$$\vec{A} = 4k \int \frac{\rho \vec{v}'}{R} \left( \frac{dt'}{ds} \right)^2 dV_0,$$

$$\Phi = k \int \frac{\rho \vec{v}'}{R} \left( \frac{dt'}{ds} \right)^2 dV_0,$$

(6a)

$$\vec{H} = \text{curl} \vec{A}, \quad \vec{E} = -\text{grad} \Phi - \frac{\partial \vec{A}}{\partial t},$$

(7a)

$$\vec{S} = -\vec{E} - [\vec{v} \vec{H}].$$  

(2a)

Apart from the factor $\left( \frac{dt'}{ds} \right)^2$ that only deviates from unity by quantities of order $\vec{v}'^2$, equations (6a), (7a) and (2a) only differ from the corresponding electrodynamic equations in the wrong sign on the right-hand side of (2a) and in the emergence of a factor 4 in (6a). Thus the analog of the magnetic force in the theory of gravitation is four times larger than in electrodynamics.

To the derivation of this formal analogy, a remark of a principal nature is added. It seems a priori very unlikely that mathematical laws that in one area of phenomena are approximated formulas for certain special cases, provide an exact description of the phenomena in another area. Thus, the conjecture arises (apart from the physical necessity, for formal reasons as well) that the Maxwell-Lorentz equations are also approximate formulas that, even though they are sufficiently precise for the fields generated electrotechnically, need a corresponding generalization for the far stronger fields that occur at the dimensions of atoms and electrons, to which Hilbert and Mie (who have a far more general starting point) have already provided suggestions.

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3 Discussion

First we correct an error which has no effect on the final conclusion: In eq. for the Christoffel symbols (see e.g. Wald [20] p. 36), there is a mistake in the last line where the terms $-\frac{2\rho}{\rho_0}$, $i = 1, 2, 3$, should be left out. This also means that the third line of should be corrected since the equality between the Christoffel symbols does not hold.

It is obvious that the Thirring paper had a strong influence on Kaluza. Both papers are limited to the weak field limit, they both use similar methods to isolate and identify the electromagnetic components and they use an identical matter source - pressureless dust.

Eqs. and could be Kaluza’s starting point. Here Thirring records the gravitational field in the weak-field limit according to the final version of Einstein’s general theory of relativity of a dust cloud with slow but otherwise arbitrary motion. It is very likely that Kaluza had these equations and in mind, when making the conjecture in his paper (second page in both translations or in the original
version) that the electromagnetic field strength should be “equal to somehow amputated three-index symbols”.

There are of course differences due to the giant (if not “quantum”) leap Kaluza made. A small one is the condition $|g| = -1$ used by Kaluza but not by Thirring. A more significant difference is that Kaluza’s $x^5$ (actually, he used $x^0$ for the fourth spacelike coordinate and $x^4$ was $i\times$time) takes the role of time in the Thirring-Lense papers, so the interpretation is, accordingly, different. The four-dimensional analogue of Kaluza’s work is the analysis of the gravitational field of a source which has a weak dependence on one spatial coordinate. The major difference is, of course, the extra dimension which Kaluza added. Possibly Kaluza, aware of the “spirit of unification” of the time, realized that the $D = 4$ Maxwell-Einstein analogy appearing in the Thirring-Lense papers can be turned into a unifying scheme if a fifth dimension is added.

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