\textbf{σ-FINITE INVARIANT DENSITIES FOR EVENTUALLY CONSERVATIVE MARKOV OPERATORS}

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\textbf{Abstract.} We establish equivalent conditions for the existence of an integrable or locally integrable fixed point for a Markov operator with the maximal support. Maximal support means that almost all initial points will concentrate on the support of the invariant density under the iteration of the process. One of the equivalent conditions for the existence of a locally integrable fixed point is weak almost periodicity of the jump operator with respect to some sweep-out set. This result includes the case of the existence of an absolutely continuous \(σ\)-finite invariant measure when we consider a nonsingular transformation on a probability space. Weak almost periodicity implies the Jacobs-Deleeuw-Glicksberg splitting and we show that constrictive Markov operators which guarantee the spectral decomposition are typical weakly almost periodic operators.

1. \textbf{Introduction.} In this paper, we consider the asymptotic behavior of the iterates of a Markov operator on \(L^1\) space over a probability space. Our main result of this paper is to give equivalent conditions for the existence of a nonnegative (not necessarily strictly positive) fixed point (also called \(σ\)-finite invariant density) of a Markov operator with “nice support property” in locally integrable (not necessarily \(L^1\)) space (Theorem 4.11). “Nice support property” means that the support of the invariant density contains a sweep-out set with respect to the fixed reference measure which has finite measure with respect to the invariant density. Here, a sweep-out set is a set which almost all points will visit sooner or later under the process. We should insist that our main result also gives necessary and sufficient conditions for the existence of an absolutely continuous \(σ\)-finite invariant measure for an “eventually conservative system” (see the definition in §4). One of the equivalent conditions is weak almost periodicity of the jump operator for this Markov operator with respect to some sweep-out set (see precise definitions in §4). Weak almost periodicity is well-known property which admits the Jacobs-Deleeuw-Glicksberg splitting of complex Banach space (see \cite{7, 20}). We also characterize the existence of a \(σ\)-finite invariant density via the induced operator. The method of the induced transformation or the jump transformation for a fibred system (more generally, the induced operator for a Markov operator) to construct an absolutely continuous \(σ\)-finite invariant measure is well-studied, for example in \cite{2, 9, 12, 32, 33, 41, 45, 46}. We generalize the method of the jump transformation by Thaler and Schweiger to the
method of the jump operator for a Markov operator. The jump operator coincides the Perron-Frobenius operator associated to the jump transformation when we consider nonsingular transformations. For proving Theorem 4.11, we prepare Theorem 3.1 which shows that weak almost periodicity of a Markov operator is equivalent to the existence of a nonnegative integrable fixed point of the Markov operator whose support is sweep-out. We apply Theorem 3.1 to the jump operator with respect to some sweep-out set and we obtain equivalence of weak almost periodicity of the jump operator and the existence of a $\sigma$-finite invariant density for the (original) Markov operator. In proving Theorem 3.1, the method of Banach limits plays an important role which originally came from [10]. Theorem 4.11 also can be applicable to the Perron-Frobenius operator (which is known to be a Markov operator, see §2) corresponding to a measurable and nonsingular transformation $T$ on a probability space $(X, \mathcal{F}, m)$ into itself (i.e., $T^{-1} \mathcal{F} \subset \mathcal{F}$ and $m \circ T^{-1} \ll m$). Since an absolutely continuous $\sigma$-finite (infinite) invariant measure corresponds to a fixed point of the Perron-Frobenius operator in nonnegative locally integrable space, the following conditions are equivalent (Corollary 4.12):

1. The existence of an absolutely continuous $\sigma$-finite $T$-invariant measure $\mu$ (i.e. $\mu \circ T^{-1} = \mu$) whose support contains a sweep-out set $A$ (i.e. $\bigcup_{n \geq 0} T^{-n} A = X \mod m$) with $\mu(A) < \infty$;

2. The existence of an absolutely continuous finite $T^*$-invariant measure $\nu$ with the maximal support in the sense that $\bigcup_{n \geq 0} T^{*-n} \left[ \frac{d\nu}{dm} > 0 \right] = X \ (mod \ m)$ for some jump transformation $T^*$ on $X$.

Further, we clarify the relation between weak almost periodicity and mean ergodicity for Markov operators in Proposition 3.9. In particular, when we consider the Perron-Frobenius operator associated to a nonsingular transformation $T$, an absolutely continuous finite $T^*$-invariant measure with the maximal support $\nu$ can be written as

$$\nu(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} m \circ T^{*-i}(A) \quad (A \in \mathcal{F})$$

(Corollary 3.10). Of course, the existence of $\nu$ may not be unique because we do not necessarily assume ergodicity of the system. In order to apply Theorem 4.11 to some examples, we shall consider constrictive Markov operators. Constrictive Markov operators admit the spectral decomposition (see Proposition 5.2 and [15, 18, 19, 23, 24, 30]) and this property is much stronger than the property of weak almost periodicity (Proposition 5.5). Markov operators or the jump operators with respect to a suitable sweep-out set in our examples will be shown to be constrictive and hence we can apply Theorem 4.11. Our advantage is that we can treat not only deterministic dynamical systems but also random dynamical systems which are represented by Markov processes since we consider Markov operators in our setting.

From a historical point of view, for invertible transformations $T$ (i.e., $T$ is bi-measurable and $m \circ T \sim m$), Hajian and Kakutani in 1964 ([10]) established the necessary and sufficient conditions for the existence of an “equivalent” $T$-invariant probability measures by using Banach limits. We also show that each condition in Theorem 3.1 is necessary and sufficient for the set function obtained via Banach limits being an absolutely continuous finite $T$-invariant measure. Sucheston extended their result to non-invertible transformations in [40] and then Dean and Sucheston in [3] and also Ito in [16] obtained equivalent conditions for the existence
of a “strictly positive” fixed point of a Markov operator $P$ over a probability space $(X, \mathcal{F}, m)$ as follows (cf. [3, 8, 10, 16, 27, 35, 37, 38, 40]):

- There exists a strictly positive fixed point of $P$ of an $L^1$ function;
- $\liminf_{n \to \infty} \int_B P^n 1_X dm > 0$ if $m(B) > 0$;
- $\liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_B P^i 1_X dm > 0$ if $m(B) > 0$;
- $\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_B P^i 1_X dm > 0$ if $m(B) > 0$;
- Any set $F \in \mathcal{F}$ with $\sum_{i=1}^{\infty} \int_F P^n 1_X dm < \infty$ for some $\{n_i\}_i \subset \mathbb{N}$ has $m$-measure zero.

The last condition in the above list means that any weakly wandering set in the sense of Hajian and Kakutani has measure zero when $P$ is the Perron-Frobenius operator corresponding to a nonsingular transformation $T$. We also know that, in ergodic measure preserving systems, if the invariant measure is $\sigma$-finite infinite then a weakly wandering set of positive measure always exists (see [8]). For the case of an absolutely continuous $\sigma$-finite (it might be infinite) invariant measure of a nonsingular transformation, many sufficient conditions are known (see [1, 2, 9, 41, 46]). However necessary and sufficient conditions presented in this paper are, as far as we know, new.

The organization of this paper is as follows. We prepare necessary definitions in §2. In §3, we consider absolutely continuous finite invariant measures and we establish Theorem 3.1 and Theorem 3.12. In §4, we consider absolutely continuous $\sigma$-finite invariant measures and we give our main theorem (Theorem 4.11). In §5, we study stochastic properties of constrictive Markov operators which guarantees the well-known spectral decomposition theorem. In particular, we establish the converse of the spectral decomposition theorem in Proposition 5.3. Furthermore, we can give a sufficient condition for a Markov operator being weakly almost periodic. §6 consists of examples to which we can apply our theorems. Examples contains a random dynamical system which is random iteration of a family of intermittent maps on the unit interval.

Throughout this paper, $\ell^\infty$, $L^1(m)$ and $L^\infty(m)$ (or $L^1$ and $L^\infty$ for short), for a measure space $(X, \mathcal{F}, m)$, stand for the space of all real valued bounded sequences, all real valued $m$-integrable functions and all real valued $m$-essentially bounded functions respectively. Then $(\ell^\infty, \| \cdot \|_\infty)$, $(L^1(m), \| \cdot \|_1)$ and $(L^\infty(m), \| \cdot \|_\infty)$ are Banach spaces with norms $\|\{a_n\}_n\|_\infty = \sup_m |a_n|$, $\|f\|_1 = \int_X |f| dm$ and $\|g\|_\infty = \text{ess sup}_x |g(x)|$ respectively. We distinguish $L^1(m)$ from the space of complex valued $m$-integrable functions denoting by $L^1(m)$. Further we will not frequently regard differences of null sets or distinguish between equivalent classes of functions.

2. Preliminaries. In this section, we give necessary definitions including the definition of Banach limits and give some properties of Banach limits.

**Definition 2.1.** Let $(X, \mathcal{F}, m)$ be a measure space and $L^1(m)_+ = \{f \in L^1(m) : f \geq 0\}$.

- We denote the set of all density functions by $D = \{f \in L^1(m)_+ : \|f\|_1 = 1\}$. 
A linear operator \( P : L^1(m) \to L^1(m) \) is called a Markov operator if \( P \) satisfies 
\[ P(D) \subset D. \]
We often use the adjoint operator \( P^* \) which is determined by 
\[ \int_X Pf \cdot g dm = \int_X f \cdot P^*g dm \quad (\forall f \in L^1, \forall g \in L^\infty). \]

Let \( T \) be a measurable and nonsingular transformation on a measure space \((X, \mathcal{F}, m)\). Then from the Radon-Nikodym theorem, we can define the Perron-Frobenius operator which is also a Markov operator on \( L^1 \) given by 
\[ \int_A Pf dm = \int_{T^{-1}A} f dm \quad (\forall f \in L^1(m), \forall A \in \mathcal{F}). \]

The Perron-Frobenius operator is also called the dual or transfer operator and the adjoint operator \( U \) with respect to the Perron-Frobenius operator which is called the Koopman operator, satisfies 
\[ \int_X Pf \cdot g dm = \int_X f \cdot g \circ T dm = \int_X f \cdot Ug dm \]
for \( f \in L^1(m) \) and \( g \in L^\infty(m) \).

A family of functions \( F \subset L^1 \) is called precompact (resp. weakly precompact) if \( \forall \{n\}_n \subset F, \exists \{k\}_k \subset N, \exists f \in L^1 \text{ s.t. } \lim_{k \to \infty} \|f_{n_k} - f\|_1 = 0 \) (resp. \( \forall g \in L^\infty, \lim_{k \to \infty} \int_X f_{n_k} g dm = \int_X f g dm \)). Namely, \( F \) is conditionally compact (resp. conditionally weakly compact).

A Markov operator \( P \) is called almost periodic (resp. weakly almost periodic) if \( \forall f \in L^1(m), \{P^n f\}_n \) is precompact (resp. weakly precompact).

We denote absolute continuity of a measure \( \mu \) with respect to \( m \) by \( \mu \ll m \) i.e., \( \mu(A) = 0 \) if \( m(A) = 0 \) and equivalence of \( m \) and \( \mu \) by \( m \sim \mu \) i.e., \( m \ll \mu \) and \( \mu \ll m \) hold. Note that for probability measures \( m \) and \( \mu \), absolute continuity of \( \mu \) with respect to \( m \) is equivalent to uniform absolute continuity of \( \mu \) with respect to \( m \) (denoted by \( \mu \ll m \) (unif.)) i.e., \( \forall \varepsilon > 0, \exists \delta > 0, [m(A) < \delta \Rightarrow \mu(A) < \varepsilon] \).

A sequence of measures \( \{\nu_n\}_{n \in \Lambda} \) with some index set \( \Lambda \) is called equi-uniformly absolutely continuous with respect to a measure \( \mu \) (which we denote by \( \{\nu_n\}_{n \in \Lambda} \ll \mu \) (equi-unif.)) if \( \forall \varepsilon > 0, \exists \delta > 0, [\mu(A) < \delta \Rightarrow \sup_{n \in \Lambda} \nu_n(A) < \varepsilon] \).

A linear functional \( L : L^\infty \to \mathbb{R} \) is called a Banach limit if \( L \) satisfies the following three conditions:
(1) \( \{x_n\}_{n \in \mathbb{N}} \in L^\infty, x_n \geq 0 \quad (\forall n \in \mathbb{N}) \Rightarrow L(\{x_n\}_{n \in \mathbb{N}}) \geq 0; \)
(2) \( \forall \{x_n\}_{n \in \mathbb{N}} \in L^\infty, L(\{x_n\}_{n \in \mathbb{N}}) = L(\{x_{n+1}\}_{n \in \mathbb{N}}); \)
(3) \( L(\{1, 1, 1, \ldots\}) = 1. \)

Remark 1 (Theorem in [39]). The following statements are true: Banach limits exist and any Banach limit \( L \) satisfies
\[ \lim_{n \to \infty} \left( \inf_{j \geq 0} \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} \right) \leq L(\{x_n\}) \leq \lim_{n \to \infty} \left( \sup_{j \geq 0} \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} \right) \quad (\forall \{x_n\} \in L^\infty). \]

Moreover, left hand side and right hand side of the above inequality are also Banach limits. Since the construction of Banach limits is due to the Hahn-Banach extension theorem, a Banach limit is not necessarily unique.

The following properties about Banach limits are easy to see and we omit the proof.

Proposition 2.2. Banach limits \( L \) satisfy the following properties:
(1) \( \forall \{x_n\}, \forall \{y_n\} \in L^\infty \text{ with } x_n \geq y_n \quad (\forall n \in \mathbb{N}), \quad L(\{x_n\}) \geq L(\{y_n\}) \);
(2) Any Banach limit is continuous.

Remark 2. Let \((X, \mathcal{F}, m)\) be a probability space and \(P : L^1 \to L^1\) be a Markov operator. By definition, for any \(f \in D\), \(Pf\) is an element of \(D\) so that we can define for each \(n \in \mathbb{N}\) a probability measure \(m_{f,n}(\cdot) = \int P^n f \, dm\) which satisfies that \(m_{f,n} \ll m\) (unif.). Then the Dunford-Pettis theorem (see [5] or [29]) implies that for any \(f \in D\), \(\{P^n f\}_n \subset D\) is weakly precompact if and only if \(\{m_{f,n}\}_n \ll m\) (equi-unif.) holds. In terms of densities of \(\{m_{f,n}\}_n\), if \(\{m_{f,n}\}_n \ll m\) (equi-unif.) then the sequence of densities \(\{\frac{dm_{f,n}}{dm}\}_n\) is called uniformly integrable.

3. Finite invariant measures for Markov operators. In this section, we consider necessary and sufficient conditions for the existence of fixed points of Markov operators. Weak almost periodicity of a Markov operator plays an important role in considering invariant measures. In the latter of this section, we consider necessary and sufficient conditions for the existence of equivalent finite invariant measures for nonsingular transformations which is the generalization of the result of [10] to the case of not necessarily invertible transformations.

The following theorem gives equivalent conditions for the existence of a fixed point of a density function with the maximal support.

Theorem 3.1. Let \((X, \mathcal{F}, m)\) be a probability space and \(P : L^1(m) \to L^1(m)\) be a Markov operator. The following are equivalent.

(A) There exists a fixed point of \(P\), \(f_0 \in D\) s.t. \(\lim_{n \to \infty} P^n 1_{[f_0 > 0]}(x) = 1\) m-a.e.;

(B) \(P\) is weakly almost periodic;

(C) \(\{P^n 1_X\}_n\) is weakly precompact;

(D) For any Banach limit \(L\), a set function \(\mu(\cdot) = L(\{\int P^n 1_X dm\}_n)\) is an absolutely continuous probability measure with respect to \(m\) s.t.

\[
P \frac{d\mu}{dm} = \frac{d\mu}{dm} \quad \text{m-a.e.} \quad \text{and} \quad \lim_{n \to \infty} P^n 1_{|dm| > 0}(x) = 1 \quad \text{m-a.e.}
\]

Remark 3. As long as we can construct an absolutely continuous finite invariant measure via Banach limits method which is originally due to [10], the resulting measure should satisfy condition (D).

A Markov operator \(P\) on \(L^1\) over \(\mathbb{R}\) can be extended to a contraction on \(L^1\) over \(\mathbb{C}\) (denoted by \(\hat{L}\)) by \(P(f + ig) = Pf + iPg\). \(P\) on \(\hat{L}\) is linear and satisfies \(\|P(f + ig)\| = \|Pf + iPg\| = \int_X \sqrt{(Pf)^2 + (Pg)^2} \, dm \leq \int_X \sqrt{f^2 + g^2} \, dm = \|f + ig\|\). Note that \(P\) on \(\hat{L}\) is weakly almost periodic if and only if \(P\) on \(L^1\) is weakly almost periodic and so one of the conditions (and equivalently, all conditions) in Theorem 3.1 implies weak almost periodicity of \(P\) on \(\hat{L}\). The Jacobs-Deleeuw-Glicksberg splitting theorem (see Theorem 1.1.4 in [7] or [20]) allows a Banach space \(X\) over \(\mathbb{C}\) to be decomposed into direct sum corresponding to an Abelian weakly almost periodic operator \(P\):

\[
X = \mathcal{X}_{uds}(P) \oplus \mathcal{X}_{fl}(P)
\]

where \(\mathcal{X}_u(P) = \overline{\text{span}}\{v \in X : \exists \lambda \in \mathbb{C} \text{ with } |\lambda| = 1 \text{ s.t. } P^n v = \lambda^n v\}\) and \(\mathcal{X}_{fl}(P) = \{v \in X : 0 \in w - c\{P^n v\}_n\}\) (uds and fl stand for unimodular discrete spectrum and flight respectively). Moreover, the restriction of the closure of \(\{P^n\}_n\) in the weak operator topology to \(\mathcal{X}_u(P)\) is a group. From these observations, we obtain the following corollary.

\[
\text{(2) Any Banach limit is continuous.}
\]

Remark 2. Let \((X, \mathcal{F}, m)\) be a probability space and \(P : L^1 \to L^1\) be a Markov operator. By definition, for any \(f \in D\), \(Pf\) is an element of \(D\) so that we can define for each \(n \in \mathbb{N}\) a probability measure \(m_{f,n}(\cdot) = \int P^n f \, dm\) which satisfies that \(m_{f,n} \ll m\) (unif.). Then the Dunford-Pettis theorem (see [5] or [29]) implies that for any \(f \in D\), \(\{P^n f\}_n \subset D\) is weakly precompact if and only if \(\{m_{f,n}\}_n \ll m\) (equi-unif.) holds. In terms of densities of \(\{m_{f,n}\}_n\), if \(\{m_{f,n}\}_n \ll m\) (equi-unif.) then the sequence of densities \(\{\frac{dm_{f,n}}{dm}\}_n\) is called uniformly integrable.

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(A) There exists a fixed point of \(P\), \(f_0 \in D\) s.t. \(\lim_{n \to \infty} P^n 1_{[f_0 > 0]}(x) = 1\) m-a.e.;

(B) \(P\) is weakly almost periodic;

(C) \(\{P^n 1_X\}_n\) is weakly precompact;

(D) For any Banach limit \(L\), a set function \(\mu(\cdot) = L(\{\int P^n 1_X dm\}_n)\) is an absolutely continuous probability measure with respect to \(m\) s.t.

\[
P \frac{d\mu}{dm} = \frac{d\mu}{dm} \quad \text{m-a.e.} \quad \text{and} \quad \lim_{n \to \infty} P^n 1_{|dm| > 0}(x) = 1 \quad \text{m-a.e.}
\]

Remark 3. As long as we can construct an absolutely continuous finite invariant measure via Banach limits method which is originally due to [10], the resulting measure should satisfy condition (D).

A Markov operator \(P\) on \(L^1\) over \(\mathbb{R}\) can be extended to a contraction on \(L^1\) over \(\mathbb{C}\) (denoted by \(\hat{L}\)) by \(P(f + ig) = Pf + iPg\). \(P\) on \(\hat{L}\) is linear and satisfies \(\|P(f + ig)\| = \|Pf + iPg\| = \int_X \sqrt{(Pf)^2 + (Pg)^2} \, dm \leq \int_X \sqrt{f^2 + g^2} \, dm = \|f + ig\|\). Note that \(P\) on \(\hat{L}\) is weakly almost periodic if and only if \(P\) on \(L^1\) is weakly almost periodic and so one of the conditions (and equivalently, all conditions) in Theorem 3.1 implies weak almost periodicity of \(P\) on \(\hat{L}\). The Jacobs-Deleeuw-Glicksberg splitting theorem (see Theorem 1.1.4 in [7] or [20]) allows a Banach space \(X\) over \(\mathbb{C}\) to be decomposed into direct sum corresponding to an Abelian weakly almost periodic operator \(P\):

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where \(\mathcal{X}_u(P) = \overline{\text{span}}\{v \in X : \exists \lambda \in \mathbb{C} \text{ with } |\lambda| = 1 \text{ s.t. } P^n v = \lambda^n v\}\) and \(\mathcal{X}_{fl}(P) = \{v \in X : 0 \in w - c\{P^n v\}_n\}\) (uds and fl stand for unimodular discrete spectrum and flight respectively). Moreover, the restriction of the closure of \(\{P^n\}_n\) in the weak operator topology to \(\mathcal{X}_u(P)\) is a group. From these observations, we obtain the following corollary.
Lemma 3.3. Assume that \( \varepsilon \) is unif. and we have
\[
\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \|P^n\| < \varepsilon\}
\]
for any mutually disjoint sets \( \{A_i\}_i \). For showing this property, we prepare the following assertion:

\[
\lim_{N \to \infty} \left\{ \sum_{i=1}^{N} \int_{A_i} P^n_X dm \right\}_n = \left\{ \sum_{i=1}^{\infty} \int_{A_i} P^n_X dm \right\}_n
\]

in \( \ell^\infty \) norm where \( \{A_i\}_i \) are mutually disjoint and \( A_i \in \mathcal{F} \) (\( \forall i \in \mathbb{N} \)). Since \( \{A_i\}_i \) are mutually disjoint and \( \int P^n_X dm \) is a probability measure, \( \sum_{i=1}^{\infty} \int_{A_i} P^n_X dm \) exists (\( \forall n \in \mathbb{N} \)). In particular, \( \exists \lim_{N \to \infty} \sum_{i=1}^{N} m(A_i) \) i.e., \( \forall \delta > 0, \exists N_0 \in \mathbb{N}, \sum_{i=1}^{N_0} m(A_i) < \delta \). Hence, we have \( m(\bigcup_{i=N_0}^{\infty} A_i) < \delta \) for sufficiently large \( N_0 \).

Let \( \hat{P} = \{ \hat{P}^n \}_{n=1}^{\infty} \) be a Markov operator. If one of the conditions in Theorem 3.1 holds, then the following assertions are true.

(i) \( P \) on \( \hat{L}^1 \) is weakly almost periodic;
(ii) \( \hat{L}^1 \) is the direct sum of the closed invariant subspaces:

\[
\hat{L}^1 = \hat{L}^1_{uds}(P) \oplus \hat{L}^1_{j1}(P)
\]

where \( \hat{L}^1_{uds}(P) \) is the set of eigenfunctions of unimodular eigenvalue and \( \hat{L}^1_{j1}(P) \) is the set of right functions.

We prepare a sequence of lemmas to prove the above theorem.

**Corollary 3.2.** Let \( (X, \mathcal{F}, m) \) be a probability space and \( P : L^1 \to L^1 \) be a Markov operator. If one of the conditions in Theorem 3.1 holds, then the following assertions are true.

(ii) \( \hat{L}^1 \) is the direct sum of the closed invariant subspaces:

\[
\hat{L}^1 = \hat{L}^1_{uds}(P) \oplus \hat{L}^1_{j1}(P)
\]

where \( \hat{L}^1_{uds}(P) \) is the set of eigenfunctions of unimodular eigenvalue and \( \hat{L}^1_{j1}(P) \) is the set of right functions.

Lemma 3.3. Assume that \( \{P^n_X\}_n \) is weakly precompact. Then for any Banach limit \( L, \mu(\cdot) = L(\{\int P^n_X dm\}_n) \) is an absolutely continuous probability measure with respect to \( m \) satisfying \( \frac{d\mu}{dm} = \frac{d\mu}{dm} \).

Proof. \( \mu : \mathcal{F} \to [0,1] \) and \( \mu \ll m \) is obvious. Further, \( \mu(X) = L(\{1, 1, \ldots\}) = 1 \) holds. Hence, we have to show countable additivity of \( \mu : \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) \) for any mutually disjoint sets \( \{A_i\}_i \). For showing this property, we prepare the following assertion:

\[
\lim_{N \to \infty} \left\{ \sum_{i=1}^{N} \int_{A_i} P^n_X dm \right\}_n = \left\{ \sum_{i=1}^{\infty} \int_{A_i} P^n_X dm \right\}_n
\]

in \( \ell^\infty \) norm where \( \{A_i\}_i \) are mutually disjoint and \( A_i \in \mathcal{F} \) (\( \forall i \in \mathbb{N} \)). Since \( \{A_i\}_i \) are mutually disjoint and \( \int P^n_X dm \) is a probability measure, \( \sum_{i=1}^{\infty} \int_{A_i} P^n_X dm \) exists (\( \forall n \in \mathbb{N} \)). In particular, \( \exists \lim_{N \to \infty} \sum_{i=1}^{N} m(A_i) \) i.e., \( \forall \delta > 0, \exists N_0 \in \mathbb{N}, \sum_{i=1}^{N_0} m(A_i) < \delta \). Hence, we have \( m(\bigcup_{i=N_0}^{\infty} A_i) < \delta \) for sufficiently large \( N_0 \).

Since \( \{P^n_X\}_n \) is weakly precompact, it holds that \( \{\int P^n_X dm\}_n \ll m \) (equi-unif.) and we have \( \forall \varepsilon > 0, \exists \delta > 0, m(\bigcup_{i=N_0}^{\infty} A_i) < \delta \) so that \( \int_{\bigcup_{i=N_0}^{\infty} A_i} P^n_X dm < \varepsilon \) (\( \forall n \in \mathbb{N} \)). Therefore, \( \forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall N \geq N_0 \), it follows

\[
\left\| \sum_{i=1}^{N} \int_{A_i} P^n_X dm \right\|_n - \left\| \sum_{i=1}^{\infty} \int_{A_i} P^n_X dm \right\|_n \leq \sum_{i=1}^{\infty} \int_{A_i} P^n_X dm
\]

So, we have the equation (3.1). Now, we show countable additivity of \( \mu \). Since \( m \) is a probability measure, for any mutually disjoint sets \( \{A_i\}_i \) (\( i \in \mathbb{N} \)),

\[
\mu \left( \bigcup_{i=1}^{\infty} A_i \right) = L \left( \left\{ \int_{\bigcup_{i=1}^{\infty} A_i} P^n_X dm \right\}_n \right) = L \left( \left\{ \sum_{i=1}^{\infty} \int_{A_i} P^n_X dm \right\}_n \right).
\]

From the above assertion (3.1), we have

\[
L \left( \left\{ \sum_{i=1}^{\infty} \int_{A_i} P^n_X dm \right\}_n \right) = L \left( \left\{ \sum_{i=1}^{\infty} \int_{A_i} P^n_X dm \right\}_n \right).
\]
By Proposition 2.2, it follows
\[
L \left( \sum_{i=1}^{\infty} \left\{ \int_{A_i} P^n 1_X dm \right\} \right) = \sum_{i=1}^{\infty} L \left( \left\{ \int_{A_i} P^n 1_X dm \right\} \right) = \sum_{i=1}^{\infty} \mu(A_i).
\]

Therefore, \( \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) \) for any mutually disjoint sets \( \{A_i\}_i \) and \( \mu \) is an absolutely continuous probability measure with respect to \( m \) on \( \mathcal{F} \). \( P \)-invariance of \( \frac{dm}{\mu} \) follows from shift invariant property of Banach limits.

Lemma 3.4. If \( f_0 \in D \) is a fixed of \( P \), then \( \{P^n 1_{[f_0,0]}\}_n \) is monotonic increasing.

Proof. We show that \((P^1 M - 1_M)(x) \geq 0 \) m-a.e. \( x \in X \) where \( M = [f_0 > 0] \). This inequality is obvious if \( x \in M^c \) and we assume \( A \subset M \). Then
\[
\int_A (P^1 M - 1_M) dm = \int_A P^1 M dm - m(A) = \int_A (P^1 X - P^1 M^c) dm - m(A)
\]
\[
= \int_X P^1_A dm - \int_A P^1 M dm - m(A) = \int_A P^1 M dm
\]

since \( \int_X P^1 M dm = 0 \) implies \( P^1 M^c(x) = 0 \) m-a.e. \( x \in M \).

The following Lemma 3.5 is obtained in [3] and it plays an important role in proving weak almost periodicity of a Markov operator.

Lemma 3.5 (Proposition 3 in [3]). If there exists an a.e. positive fixed point of \( P \) in \( L^1(m) \), then \( \{P^n g\}_n \) is weakly precompact for any \( g \in L^1(m)_+ \).

Proof of Theorem 3.1. (A) \( \Rightarrow \) (B): Recall that in a probability space, for any \( f \in L^1(m) \), \( \{P^n f\}_n \) is weakly precompact if and only if \( \{\int |P^n f| dm\}_n \ll m \) (equi-unif.) holds. Since \( \int f \cdot |P^n f| dm \leq \int f \cdot P^n |f| dm \) (\( \forall f \in L^1(m) \)), we only have to show that \( \forall g \in L^1(m)_+ \), it holds that \( \{\int f \cdot P^n g dm\}_n \ll m \) (equi-unif.). We consider the following measure \( \nu \) and operator \( \tilde{P} : \nu(\cdot) = m(\cdot \cap M) \) where \( M = [f_0 > 0] \) and \( \tilde{P} : L^1(\nu) \rightarrow L^1(\nu), \tilde{P} f = P(f 1_M) \). Then \( \tilde{P} \) is a Markov operator on \( L^1(\nu) \) and \( \tilde{P} f_0 = f_0 \). Hence by Lemma 3.5 we have that \( \forall g \in L^1(\nu)_+ \),
\[
\forall \varepsilon > 0, \exists \delta > 0, \left[ \nu(A) < \delta \Rightarrow \sup_{n \geq 0} \int_A \tilde{P}^n g dm < \varepsilon \right].
\]

By virtue of Lemma 4.5 in [24], one can see that \( \tilde{P} [P^0 g > 0] \subset M \) (\( \forall g \in L^1(\nu)_+ \)) and we have \( \forall g \in L^1(m)_+ \),
\[
\forall \varepsilon > 0, \exists \delta > 0, \left[ A \in \mathcal{F} \cap M, \ m(A) < \delta \Rightarrow \sup_{n \geq 0} \int_A P^n(g 1_M) dm < \varepsilon \right].
\]

Since \( \sup_{n \geq 0} \int_A P^n(g 1_M) dm = \sup_{n \geq 0} (\int_A P^n g dm - \int_A P^n g 1_M dm) \), it holds that for any \( g \in L^1(m)_+ \), \( \forall \varepsilon > 0, \exists \delta > 0, \)
\[
\left[ A \in \mathcal{F} \cap M, \ m(A) < \delta \Rightarrow \sup_{n \geq 0} \int_A P^n g dm < \sup_{n \geq 0} \int_A P^n(g 1_M) dm + \varepsilon \right].
\]

Hence, we have to show \( \sup_{n \geq 0} \int_A P^n(g 1_M) dm \) is dominated by an arbitrary small \( \varepsilon \). By assumption that \( f_0 \) has the maximal support, \( \lim_{n \to \infty} \int_X P^m 1_{M^c} dm = 0 \) and by the fact that \( [f_1 > 0] \subset [f_2 > 0] \) implies \( [Pf_1 > 0] \subset [Pf_2 > 0] \) for any \( f_1, f_2 \in \)
Thus, one can see that for any nonnegative measurable sets with \( m \), it holds that

\[
\forall A \in \mathcal{F} \cap M, \quad \sup_{n \geq 0} \int_A P^n (g1_{M^e}) dm = \sup_{n \geq 0} \int_A P^{n+N_0} (g1_{M^e}) dm \\
\leq \sup_{n \geq 0} \int_A P^n (P^{N_0}(g1_{M^e})1_{M^e}) dm + \sup_{n \geq 0} \int_A P^n (P^{N_0}(g1_{M^e})1_{M^e}) dm \\
\leq \sup_{n \geq 0} \int_A \tilde{P}^n (P^{N_0}(g1_{M^e})1_{M^e}) d\nu + \|P^{N_0}(g1_{M^e})1_{M^e}\|_1.
\]

Then it holds \( \forall g \in L^1(m)_+ \),

\[
\forall \varepsilon > 0, \exists \delta > 0, \left[ A \in \mathcal{F} \cap M, \ m(A) < \delta \Rightarrow \sup_{n \geq 0} \int_A P^n g dm < \varepsilon \right].
\]

On the other hand, by Lemma 3.4 and assumption \( \lim_{n \to \infty} \int_M P^n 1_X dm = 1 \), \( \{P^n 1_{M^e}\}_n \) is monotonic decreasing to 0 \( m \)-a.e. and from Lebesgue convergence theorem, it holds that

\[
\lim_{n \to \infty} \int_M P^n g dm = \int_X g \lim_{n \to \infty} P^n 1_{M^e} dm = 0.
\]

This implies that \( \forall g \in L^1(m)_+ \),

\[
\forall \varepsilon > 0, \exists \delta > 0, \left[ A \in \mathcal{F} \cap M, \ m(A) < \delta \Rightarrow \sup_{n \geq 0} \int_A P^n g dm < \varepsilon \right].
\]

Therefore, \( P \) is weakly almost periodic.

(C)⇒(D): Take a set function \( \mu \) as in the condition (C). Then \( \mu \) is an absolutely continuous probability by equi-uniform absolute continuity of \( \{ \int P^n 1_X dm \}_n \) and Lemma 3.3. Using \( P^* : L^\infty \to L^\infty \) the adjoint operator of \( P \), it holds that

\[
\mu(A) = L \left( \left\{ \int_X P^n 1_A dm \right\}_n \right).
\]

Note that for any nonnegative function \( f \in L^\infty(m) \), it follows that \( f \in L^1(\mu) \) since \( \mu \ll m \) and \( \mu \) is a probability. Firstly, we show that for any nonnegative \( L^\infty(m) \) function \( f \), it holds that \( \int_X f dm = L(\int_X P^n f dm)_n \). For any nonnegative simple function \( \sum_{i=1}^k a_i 1_{A_i} \),

\[
\int_X \sum_{i=1}^k a_i 1_{A_i} dm = \sum_{i=1}^k a_i \mu(A_i) = \sum_{i=1}^k a_i L \left( \left\{ \int_X P^n 1_{A_i} dm \right\}_n \right) \\
= L \left( \left\{ \int_X P^n \left( \sum_{i=1}^k a_i 1_{A_i} \right) dm \right\}_n \right).
\]

For \( \sum_{i=1}^\infty a_i 1_{A_i} \) satisfying \( 0 < a_i < C (\forall i \in \mathbb{N}) \) for some \( C > 0 \) and \( \{A_i\} \) are disjoint measurable sets with \( m(A_i) > 0 \),

\[
\int_X \sum_{i=1}^\infty a_i 1_{A_i} dm = \lim_{N \to \infty} L \left( \left\{ \int_X P^n \left( \sum_{i=1}^N a_i 1_{A_i} \right) dm \right\}_n \right) \\
= L \left( \lim_{N \to \infty} \left\{ \int_X P^n \left( \sum_{i=1}^N a_i 1_{A_i} \right) dm \right\}_n \right).
\]
Equi-uniform absolute continuity of \( \{ \int X P^n dm \} \) with respect to \( m \) implies that
\[
\left\{ \int X P^n \left( \sum_{i=1}^N a_i 1_{A_i} \right) dm \right\}_n \rightarrow \left\{ \int X P^n \left( \sum_{i=1}^\infty a_i 1_{A_i} \right) dm \right\}_n
\]
as \( N \rightarrow \infty \) in \( \ell^\infty \)-norm. Indeed,
\[
\left\| \left\{ \int X P^n \left( \sum_{i=1}^\infty a_i 1_{A_i} \right) dm \right\}_n - \left\{ \int X P^n \left( \sum_{i=1}^N a_i 1_{A_i} \right) dm \right\}_n \right\|_\infty
= \sup_n \int X P^n \left( \sum_{i=N+1}^\infty a_i 1_{A_i} \right) dm = \sup_n \sum_{i=N+1}^\infty a_i \int X P^n dm
\leq C \sup_n \int_{\cup_{i>N} A_i} P^n dm
\]
and equi-uniform absolute continuity implies that \( \forall \varepsilon > 0, \exists \delta > 0, \exists N_0 \in \mathbb{N} \), for any \( N \geq N_0 \), \( m(\cup_{i>N} A_i) < \delta \) holds so that \( \sup_n \int_{\cup_{i>N} A_i} P^n dm < \varepsilon / C \). Therefore, it holds that
\[
\int X \sum_{i=1}^\infty a_i 1_{A_i} dm = L \left( \left\{ \int X P^n \left( \sum_{i=1}^\infty a_i 1_{A_i} \right) dm \right\}_n \right).
\]
Hence, by simple function approximation, \( \int X f dm = L(\{ \int X P^n f dm \} ) \) holds (\( \forall f \in L^\infty(m) \)) and for \( P^1 A (\forall A \in \mathcal{F}) \),
\[
\int X P^n 1_A dm = L \left( \left\{ \int X P^n (P^1 A) dm \right\}_n \right) = L \left( \left\{ \int X P^n dm \right\}_n \right) = \int X 1_A dm.
\]
Therefore, we have that
\[
P \frac{d\mu}{dm}(x) = \frac{d\mu}{dm}(x)
\]
m-a.e. \( x \in X \).

Finally, we show \( \lim_{n \rightarrow \infty} \int_M P^n 1_X dm = 1 \) where \( M = [\frac{d\mu}{dm} > 0] \). To show this we prepare the following inequality:
\[
\int_M P^n 1_X dm \leq \int_M P^n+1 1_X dm \quad (\forall n \in \mathbb{N}).
\]
This inequality is true from Lemma 3.4 and \( \int_X P^n (P^1 M - 1_M) dm \geq 0 \). Therefore, the Banach limit of \( \{ \int_M P^n 1_X dm \} \) coincides \( \lim_{n \rightarrow \infty} \int_M P^n 1_X dm \) and it holds that
\[
\lim_{n \rightarrow \infty} \int_{[\frac{d\mu}{dm} > 0]} P^n 1_X dm = 1
\]
since \( \mu(M) = 1 \).

If we consider the Perron-Frobenius operator corresponding to a nonsingular transformation \( T \), the following corollary of Theorem 3.1 is immediately obtained since weak precompactness of \( \{ P^n X \} \) is equivalent to equi-uniform absolute continuity of \( \{ m \circ T^{-n} \} \) with respect to \( m \).

**Corollary 3.6.** Let \( (X, \mathcal{F}, m) \) be a probability measure space and \( T : X \rightarrow X \) be a measurable and nonsingular transformation. The followings are equivalent.
(1) There exists an absolutely continuous finite invariant measure $\mu$ with respect to $m$ s.t.
\[ \bigcup_{n=0}^{\infty} T^{-n} \left[ \frac{d\mu}{dm} > 0 \right] = X \mod m; \]

(2) $\{m \circ T^{-n}\}_{n \in \mathbb{N}} \ll m$ (equi-unif.) holds;

(3) For any Banach limit $L$, a set function $\mu(\cdot) = L(\{m \circ T^{-n}(\cdot)\})$ is an absolutely continuous $T$-invariant probability measure with respect to $m$ s.t.
\[ \bigcup_{n=0}^{\infty} T^{-n} \left[ \frac{d\mu}{dm} > 0 \right] = X \mod m. \]

**Remark 4.** Let us look at the following condition:

(2') $\exists \delta > 0$, and $\exists \alpha \in (0, 1)$, s.t. $m(A) < \delta \Rightarrow \sup_{n \geq 0} m \circ T^{-n}(A) < \alpha$.

If we replace the condition (2) in Corollary 3.6 by the above weaker condition (2'), we may not obtain an absolutely continuous finite invariant measure with the maximal support in the sense that $\bigcup_{n=0}^{\infty} T^{-n} \left[ \frac{d\mu}{dm} > 0 \right] = X \mod m$. We recall that Straube showed in [36] that the condition (2') is equivalent to the existence of an absolutely continuous finite invariant measure (which does not necessarily have the maximal support).

**Definition 3.7.** A Markov operator $P$ is called mean ergodic if for any $f \in L^1$, the limit $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} P^i f$ exists in strong.

Yosida and Kakutani showed in [44] the following assertion.

**Lemma 3.8 (Theorem 1 in [44]).** Let $(X, \mathcal{F}, m)$ be a probability space and $P : L^1 \to L^1$ be a Markov operator. If $P$ is weakly almost periodic, then $P$ is mean ergodic.

Lemma 3.8 gives a sufficient condition for all Banach limits coinciding. Moreover, it plays an important role in establishing the following result.

**Proposition 3.9.** Let $(X, \mathcal{F}, m)$ be a probability space and $P$ be a Markov operator. Then each of the following conditions is equivalent to the existence of a fixed point of $P$ in $D$ with the maximal support (and hence all of the conditions in Theorem 3.1 hold):

(E) $P$ is mean ergodic;

(F) For any $A \in \mathcal{F}$, the following limit exists:
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_A P^i 1_X \, dm. \]

**Remark 5.** As a related result to Proposition 3.9, in [6], equivalent conditions for Perron-Frobenius operators being weakly almost periodic are given. It reads that the followings are equivalent for the Perron-Frobenius operator $P$,

- $P$ is weakly almost periodic;
- $P$ is mean ergodic;
- $\exists f_0 \in D$ s.t. $\limsup_{n \to \infty} \|P^n f - f_0\|_1 < 2$ (\(\forall f \in D\)).

Further, in [16], Ito showed the equivalence of mean ergodicity of a Markov operator $P$ and weak precompactness of $\{\frac{1}{n} \sum_{i=0}^{n-1} P^i 1_X\}$. Therefore, our result is the extension of their result.
Proof of Proposition 3.9. We show mean ergodicity implies for any $A \in \mathcal{F}$ the convergence of $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_A P^i f dm$. Mean ergodicity of $P$ is that $\forall f \in L^1$, $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} P^i f$ exists in strong. Hence it follows that $\forall A \in \mathcal{F}$,

$$\left| \int_A \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} P^i f - \frac{1}{K} \sum_{i=0}^{K-1} \int_A P^i f dm \right| \leq \int_A \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} P^i f - \frac{1}{K} \sum_{i=0}^{K-1} P^i f \ dm \to 0 \ (\text{as } K \to \infty).$$

Hence $\forall f \in L^1$, $\forall A \in \mathcal{F}$, $\exists \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_A P^i f dm$. Then the rest of the proof is only to show the implication of setwise convergence of Cesàro average of $\int P^n 1_X dm$ to the existence of an absolutely continuous probability measure $\mu$ which satisfies that $\frac{d\mu}{dm}$ is a fixed point of $P$ and $\lim_{n \to \infty} \int_{\{\frac{d\mu}{dm} > \delta\}} P^n 1_X dm = 1$. Set $\mu(\cdot) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \int P^i 1_X dm$ and the Vitali-Hahn-Saks theorem implies that $\mu$ is an absolutely continuous probability measure with respect to $m$ such that $P \frac{d\mu}{dm} = \frac{d\mu}{dm}$. Further, the fact that $P^{n+1} \frac{d\mu}{dm} > \delta$ is monotonic increasing by Lemma 3.4 implies that

$$\mu \left( \frac{d\mu}{dm} > \delta \right) = \lim_{n \to \infty} \int_{\{\frac{d\mu}{dm} > \delta\}} P^n 1_X dm.$$ 

Therefore, there exists an absolutely continuous finite invariant measure with the maximal support and the proof is completed.

In particular, if $P$ is the Perron-Frobenius operator corresponding to a nonsingular transformation $T : X \to X$, the following corollary is valid.

**Corollary 3.10.** The equivalent condition for all of the conditions in Corollary 3.6 is that for any $A \in \mathcal{F}$, the following limit exists:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} m \circ T^{-i}(A).$$

In the rest of this section, we consider necessary and sufficient conditions for the existence of an equivalent finite invariant measure for a nonsingular transformation on a probability space.

**Definition 3.11.** Let $(X, \mathcal{F}, m)$ be a measure space and $T$ be a measurable transformation.

- A set $W \subset X$ is called a weakly wandering set for $T$ (with a weakly wandering sequence $\{n_i\}$) if $\exists \{n_i\}_{i=1}^{\infty} \subset \mathbb{N}$, s.t. $T^{-n_i}W \cap T^{-n_j}W = \emptyset \ (\text{mod } m)$, $\forall i, \forall j \in \mathbb{N}$ with $i \neq j$.
- For measures $\mu$ and $\{\nu_n\}_{n \in \Lambda}$ on $\mathcal{F}$ with some index set $\Lambda$, $\mu$ is called uniformly absolutely continuous with respect to $\{\nu_n\}_{n \in \Lambda}$ (which we denote by $\{\nu_n\}_{n \in \Lambda} \gg \mu$ (unif.)) if $\forall \varepsilon > 0, \exists \delta > 0, \inf_{n \in \Lambda} \nu_n(A) < \delta \Rightarrow \mu(A) < \varepsilon$.
- $\mu$ and $\{\nu_n\}_{n \in \Lambda}$ are called uniformly equivalent (denoted by $\{\nu_n\}_{n \in \Lambda} \sim \mu$ (unif.)) if $\exists \{\nu_n\}_{n \in \Lambda} \ll \mu$ (equi-unif.) and $\{\nu_n\}_{n \in \Lambda} \gg \mu$ (unif.) hold.

In [8] and [10], for an invertible transformation, one can see an equivalent condition for the existence of an equivalent finite invariant measure (1) as the conditions
(IV) in the next Theorem 3.12. Further, they proved the equivalence of (I) and \{m \circ T^{-n}\}_{n \in \mathbb{Z}} \ll m \text{ (equi-unif.)} instead of (II) for invertible case. We extend their results to not necessarily invertible case.

**Theorem 3.12.** Let \((X, \mathcal{F}, m)\) be a probability space and \(T : X \to X\) be measurable and nonsingular with respect to \(m\). Then the followings are equivalent.

(I) There exists an equivalent \(T\)-invariant probability measure \(\mu\) with respect to \(m\);

(II) \(\{m \circ T^{-n}\}_{n} \sim m \text{ (unif.)}\) holds;

(III) For any Banach limit \(L\), a set function \(\mu(\cdot) = L(\{m \circ T^{-n}(\cdot)\}_{n})\) is an equivalent \(T\)-invariant probability measure with respect to \(m\);

(IV) Let \(A, B \in \mathcal{F}\) such that there exist \(\{A_{i}\}_{i \in \mathbb{N}}\) and \(\{B_{i}\}_{i \in \mathbb{N}}\) the partitions of \(A\) and \(B\), and exist \(\{n_{i}\}_{i \in \mathbb{N}}\) satisfying \(T^{-n_{i}}A_{i} = B_{i}\) for any \(i \in \mathbb{N}\). Assume that \(A \supset B\) and \(m(A) > 0\) holds. Then \(m(A \setminus B) = 0\).

**Remark 6** (The Hopf decomposition). For a measurable and nonsingular transformation \(T\) on a \(\sigma\)-finite measure space \((X, \mathcal{F}, m)\), it is well-known that \(X\) can be decomposed into the conservative part \(\mathcal{C}\) and the dissipative part \(\mathcal{D}\) of \(X\) uniquely (mod \(m\)) where

(Cons.) : \(\forall f \in L^{1}_{+}, \sum_{n=0}^{\infty} P^{n}f = \infty \text{ on } \mathcal{C} \cap \left[ \sum_{n=0}^{\infty} P^{n}f > 0 \right] \), and

(Diss.) : \(\forall f \in L^{1}_{+}, \sum_{n=0}^{\infty} P^{n}f < \infty \text{ on } \mathcal{D}\).

The existence of an equivalent finite invariant measure \(\mu\) implies the system \((T, m)\) is conservative and for this system, \(X = \mathcal{C}\) holds. That is, for any non-negative integrable function \(f\), it holds

\[
\sum_{n=0}^{\infty} P^{n}f = \infty \quad \text{or} \quad \sum_{n=0}^{\infty} P^{n}f = 0.
\]

The converse (i.e., conservative systems have equivalent finite invariant measures) may not be true. The answer is obtained by Ito in [16].

To prove Theorem 3.12, we prepare the following two lemmas.

**Lemma 3.13.** Assume that there exists a \(T\)-invariant probability measure \(\mu\) s.t. \(\mu \sim m\). Then, \(\{m \circ T^{-n}\}_{n} \ll m \text{ (equi-unif.)}\) holds.

**Proof.** We assume that \(\{m \circ T^{-n}\}_{n} \ll m \text{ (equi-unif.)}\) does not hold i.e., \(\exists \varepsilon_{0} > 0, \exists \{n_{k}\}_{k} \subset \mathbb{N}, \exists \{A_{k}\}_{k} \subset \mathcal{F}, \text{ s.t. } m(A_{k}) < 1/2^{k} \) and \(m \circ T^{-n_{k}}(A_{k}) \geq \varepsilon_{0} \) (\(\forall k \in \mathbb{N}\)).

We set \(B_{k} = \bigcup_{i=k}^{\infty} A_{i}\). Then

\[
\lim_{k \to \infty} m(B_{k}) = 0 \quad \text{(3.2)}
\]

\[
m(T^{-n_{k}}B_{k}) \geq \varepsilon_{0} \text{ (}\forall k \in \mathbb{N}\). \quad \text{(3.3)}
\]

By the equation (3.2) and \(\mu \ll m\), we have that \(\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, \exists N_{0} = N_{0}(\delta(\varepsilon)) \in \mathbb{N}, \text{ s.t. } \forall k \geq N_{0}, \ m(B_{k}) < \delta \text{ so that } \mu(B_{k}) < \varepsilon\). Since \(\mu\) is \(T\)-invariant, \(\forall \varepsilon > 0, \exists N_{0} \in \mathbb{N}, \forall k \geq N_{0}, \mu(T^{-n_{k}}B_{k}) < \varepsilon\). By \(\mu \gg m\), the above assertion \(\mu(T^{-n_{k}}B_{k}) < \varepsilon\) contradicts to the inequality (3.3). \(\square\)

The following result was proven in [40] for not necessarily invertible case.
Lemma 3.14 (Theorem 1 in [40]). There exists an equivalent finite invariant measure if and only if there is no weakly wandering set with positive m-measure.

Proof of Theorem 3.12. (II)⇒(III): From Lemma 3.3, we only have to show \( \{ m \circ T^{-n} \}_{n} \) does not hold i.e., \( \exists A \in \mathcal{F} \), s.t. \( \mu(A) = 0 \) and \( m(A) = c \) for some \( c > 0 \). By the property of Banach limits (see Remark 1),

\[
\lim_{n \to \infty} \left( \inf_{j \geq 0} \frac{1}{n} \sum_{i=j}^{n-1+j} m \circ T^{-i}(A) \right) = 0.
\]

Thus, \( \forall \varepsilon > 0 \) fixed, \( \exists N_{0} \in \mathbb{N} \), \( \forall n > N_{0} \), \( \inf_{j \geq 0} \frac{1}{n} \sum_{i=j}^{n-1+j} m \circ T^{-i}(A) < \varepsilon \). It follows that for this \( \varepsilon \), \( \exists j_{0} \geq 0 \), s.t.

\[
\frac{1}{n} \sum_{i=j_{0}}^{n-1+j_{0}} m \circ T^{-i}(A) - \varepsilon < \inf_{j \geq 0} \frac{1}{n} \sum_{i=j}^{n-1+j} m \circ T^{-i}(A).
\]

Since \( \frac{1}{n} \sum_{i=j_{0}}^{n-1+j_{0}} m \circ T^{-i}(A) < 2\varepsilon \), \( \exists n_{0} \in \{ j_{0}, j_{0} + 1, \ldots, j_{0} + n - 1 \} \), s.t.

\[ m \circ T^{-n_{0}}(A) < 2\varepsilon. \tag{3.4} \]

On the other hand, by \( \{ m \circ T^{-n} \}_{n} \) does not hold i.e., \( \exists \delta > 0 \), \( \inf_{n \geq 1} m \circ T^{-n}(A_{k}) < 1/2^{k} \) and \( m(A_{k}) \geq \varepsilon_{0} \) (\( \forall k \in \mathbb{N} \)). Then we have \( \exists \varepsilon_{0} > 0 \), \( \exists \{ A_{k} \}_{k} \subset \mathcal{F} \), \( \exists \{ n_{k} \}_{k} \) s.t. \( m \circ T^{-n_{k}}(A_{k}) < 1/2^{k-1} \) and \( m(A_{k}) \geq \varepsilon_{0} \) (\( \forall k \in \mathbb{N} \)). We set \( B_{k} = \bigcup_{i=k}^{\infty} A_{i} \). Then

\[
\lim_{k \to \infty} m \circ T^{-n_k}(B_k) = 0 \tag{3.5}
\]

\[ m(B_k) \geq \varepsilon_{0} \quad (\forall k \in \mathbb{N}). \tag{3.6} \]

By the equation (3.5) and µ ≪ m (unif.), \( \forall \varepsilon > 0 \), \( \exists \delta > 0 \), \( \exists N_{0} \in \mathbb{N} \), \( \forall k \geq N_{0} \), \( m(T^{-n_k}B_k) < \delta \) so that \( m(T^{-n_k}B_k) < \varepsilon \). Since \( m \ll \mu \) (unif.), for \( \varepsilon_{0} > 0 \), \( \exists \delta_{0} > 0 \), s.t. \( \mu(B_k) < \delta_{0} \) implies \( m(B_k) < \varepsilon_{0} \). It contradicts to the inequality (3.6) and \( \{ m \circ T^{-n} \}_{n} \) does not hold (unif.) holds.

(II)⇒(IV): Assume that there exists such \( A \) and \( B \) satisfying \( \bigcup_{i=1}^{\infty} T^{-n_i}A_i = \bigcup_{i=1}^{\infty} B_i \) (disj.). Then, by assumption of the existence of finite invariant measure \( \mu \),

\[
\mu(A \setminus B) = \mu \left( \bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{\infty} T^{-n_i}A_i \right) = \sum_{i=1}^{\infty} (\mu(A_i) - \mu(T^{-n_i}A_i)) = 0
\]

\[ m \ll \mu \text{ implies } m(A \setminus B) = 0. \]

(IV)⇒(I): From Lemma 3.14, we have to show the condition (IV) implies that there is no weakly wandering set with positive measure. We suppose there exists a weakly wandering set \( W \) with a weakly wandering sequence \( \{ n_{i} \}_{i} \) with \( n_{1} = 0 \) s.t. \( m(W) > 0 \). We set \( A_{i} = T^{-n_{i}}W, \ B_{i} = T^{-n_{i+1}}W, A = \bigcup_{i=1}^{\infty} A_{i} \) (disj.), and \( B = \bigcup_{i=1}^{\infty} B_{i} \) (disj.). Clearly it holds that \( T^{-n_{i+1}}A_{i} = B_{i} (\forall i \geq 1) \) and \( A \supset B \). By assumption (F), we have that \( m(A \setminus B) = 0 \). It is contradiction since \( m(A \setminus B) = m(T^{-n_{i}}W) = m(W) \).
4. **σ-finite invariant measures for Markov operators.** In this section, we consider σ-finite invariant densities for Markov operators. We use the induced operator and the jump operator for a Markov operator which are the generalization of the induced transformation and the jump transformation respectively. The induced operator and its use for finding a σ-finite invariant measure were suggested first by Halmos in 1947 ([11]) and matured by Foguel in 1969 ([9]). The method of the jump transformation is given in [32] or [41]. We generalize the jump transformation to the jump operator for a Markov operator. We use these methods for giving equivalent conditions for the existence of a σ-finite invariant density for an eventually conservative Markov operator with nice support condition.

Let $\mathcal{M}_\sigma^+$ denote the space of all non-negative measurable functions $f$ over $(X, \mathcal{F}, m)$ such that the measure $m_f(\cdot) = \int f dm$ is σ-finite. Then a Markov operator on $L^1$ space can be extended to a positive linear operator on the space $\mathcal{M}_\sigma^+$. Further, we extend a Markov operator to a positive linear operator on $\mathcal{M}_\sigma$ where $\mathcal{M}_\sigma$ stands for the space of all non-negative measurable functions on $(X, \mathcal{F}, m)$ (see [20] or [34] for more information). Therefore, we can consider a fixed point of a Markov operator in $\mathcal{M}_\sigma$ as the density of a σ-finite (infinite) invariant measure.

**Definition 4.1.** Let $(X, \mathcal{F}, m)$ be a probability space and $T : X \to X$ be a nonsingular transformation. If $\varphi_E(x) := \min\{n \in \mathbb{N} : T^n x \in E\}$ the hitting time of $A \in \mathcal{F}$ is defined a.e., then we define the induced transformation on $E$, $T_E : X \to E$, by $T_E x := T^{\varphi_E(x)} x$.

**Remark 7.** The conventional definition of the induced transformation is given by $T_E |_{E'} : E' \to E$. But for our purpose of having nice support property of the invariant density, the induced transformation should be defined a.e.

**Definition 4.2.** Let $(X, \mathcal{F}, m)$ be a probability space and $P : L^1 \to L^1$ be a Markov operator. We define the induced operator (by $P$) on $E \in \mathcal{F}$ by

$$P_E = (I_E P) \sum_{n=0}^{\infty} (I_{E^n} P)^n \quad \text{or} \quad P_E^* = \sum_{n=0}^{\infty} (P^* I_{E^n})^n (P^* I_E)$$

where $I_E$ is the restriction operator on $E$: $I_E f = 1_E f$ for any measurable function $f$.

**Definition 4.3.** Let $T$ be a nonsingular transformation on a probability space $(X, \mathcal{F}, m)$. Assume that there exists a set $E$ of positive measure such that $\bigcup_{n \geq 0} T^{-n} E = X$. Then the first entry time $e(x)$ is defined m-a.e. $x \in X$ by $e(x) = \min\{n \geq 0 : T^n x \in E\}$ and we define $T^*$ the jump transformation with respect to $E$ by $T^* x = T^{e(x)+1} x$.

**Definition 4.4.** Let $(X, \mathcal{F}, m)$ be a probability space and $P : L^1 \to L^1$ be a Markov operator. We define the jump operator (by $P$ with respect to $E \in \mathcal{F}$) by

$$\hat{P} = \hat{P}_E = (PI_E) \sum_{n=0}^{\infty} (PI_{E^n})^n \quad \text{or} \quad \hat{P}_E^* = \sum_{n=0}^{\infty} (I_{E^n} P^*)^n (I_E P^*)$$

**Proposition 4.5.** Let $P$ be the Perron-Frobenius operator corresponding to a nonsingular transformation on a probability space $(X, \mathcal{F}, m)$. Then the Perron-Frobenius operator corresponding to the induced transformation $T_E$ is the induced operator $P_E$. 
Proof. Note that $T^{-1}A = \bigcup_{n=1}^{\infty}([\varphi_{E} = n] \cap T^{-n}A)$ and the Perron-Frobenius operator $P_{E}$ corresponding to $T_{E}$ satisfies $\forall A \in \mathcal{F}$ and $\forall f \in L^{1}$,

$$\int_{A} P_{E}f dm = \int_{T^{-1}A} f dm = \sum_{n=1}^{\infty} \int_{[\varphi_{E} = n] \cap T^{-n}A} f dm.$$ 

Since

\[
[\varphi_{E} = n] = \begin{cases} 
T^{-1}E & (n = 1) \\
T^{-n}E \cap \bigcap_{i=1}^{n-1} T^{-i}E^{c} & (n \geq 2),
\end{cases}
\]

it holds that

\[
\int_{A} P_{E}f dm = \int_{T^{-1}(A \cap E)} f dm + \sum_{n=2}^{\infty} \int_{T^{-n}(A \cap E) \cap \bigcap_{i=1}^{n-1} T^{-i}E^{c}} f dm.
\]

One can see that $\forall n \geq 2,$

\[
\int_{T^{-n}(A \cap E) \cap \bigcap_{i=1}^{n-1} T^{-i}E^{c}} f dm = \int_{T^{-n+1}(A \cap E) \cap \bigcap_{i=1}^{n-2} T^{-i}E^{c}} I_{E^{c}} P f dm = \cdots = \int_{T^{-1}(A \cap E)} (I_{E^{c}} P)^{n-1} f dm = \int_{A} I_{E} P (I_{E^{c}} P)^{n-1} f dm
\]

and

\[
\int_{A} P_{E}f dm = \int_{A} I_{E} P f dm + \sum_{n=2}^{\infty} \int_{A} I_{E} P (I_{E^{c}} P)^{n-1} f dm = \sum_{n=0}^{\infty} \int_{A} (I_{E} P)(I_{E^{c}} P)^{n} f dm.
\]

This equality and monotone convergence theorem imply our assertion. \qed

By the same way of proving the above proposition, we have the following proposition about the relation of the jump operator and the jump transformation.

**Proposition 4.6.** Let $P$ be the Perron-Frobenius operator corresponding to a nonsingular transformation $T$ on a probability space $(X, \mathcal{F}, m)$. Then the Perron-Frobenius operator corresponding to the jump transformation $T^{*}$ with respect to $E$ is the jump operator with respect to $E$.

Even if we do not have any nonsingular transformation, under proper assumptions, we may still obtain a well-defined induced operator or jump operator as a Markov operator on $L^{1}(m)$. The following lemma is given in [9] (Lemma B in Chapter VI) only for the induced operator but obviously we have the same statement for the jump operator and we omit the proof.

**Lemma 4.7.** Let $(X, \mathcal{F}, m)$ be a probability space and $P : L^{1} \rightarrow L^{1}$ be a Markov operator. For $E \in \mathcal{F}$, if $\lim_{n \rightarrow \infty} (P^{*} I_{E^{c}})^{n} 1_{X} = 0$, then $P_{E}$ the induced operator and $\hat{P}$ the jump operator with respect to $E$ are well-defined Markov operators on $L^{1}(m)$.

From the above lemma, we naturally give the following definition of sweep-out sets. See [34] for more precise properties of sweep-out sets.
Definition 4.8. A set $E \in \mathcal{F}$ is called a $(P)$-sweep-out set if $P^n_E 1_X(x) = 1$ m.a.e. $x \in X$ or equivalently, $P_E$ and $P$ are well-defined Markov operators on $L^1(m)$.

Remark 8. A set $E$ being a sweep-out set means that almost all points will visit $E$ sooner or later under the process $P$. It is obvious that $E$ is a sweep-out set if and only if it holds

$$\lim_{n \to \infty} (P^* I_E)^n 1_X = 0$$

or equivalently

$$\lim_{n \to \infty} (I_E P^*)^n 1_X = 0.$$

Proposition 4.9 (The Hopf decomposition [9, 20]). Let $P$ be a Markov operator over a probability space $(X, \mathcal{F}, m)$. Then $X$ can be decomposed into two sets $\mathcal{C}$ the conservative part and $\mathcal{D}$ the dissipative part uniquely modulo $m$:

(C) For any $f \in L^1_+, S_{\infty} f = \infty$ on $\mathcal{C} \cap \{S_{\infty} f > 0\}$, and

(D) for any $f \in L^1_+$, $S_{\infty} f < \infty$ on $\mathcal{D}$.

where $S_n f = \sum_{i=0}^{n-1} P_i f$. If $P$ is the Perron-Frobenius operator corresponding to a nonsingular transformation $T$, the dissipative part is the measurable union of all wandering sets for $T$.

Remark 9. The definition of the conservative part and the dissipative part coincides with the definition for nonsingular transformation given in section 3. One can see other characterization of the conservative part and the dissipative part in [9] or [20], that is for any $h \in L^\infty_+$, $\sum_{n=0}^\infty P^n h = \infty$ or $0$ on $\mathcal{C}$ and there exists an $h_D \in L^\infty_+$ s.t. $[h_D > 0] = \mathcal{D}$ and $\sum_{n=0}^\infty P^n h_D \leq 1$. Further it holds that $P^* 1_\mathcal{C} \geq 1_\mathcal{C}$ and $P^* 1_\mathcal{D} \leq 1_\mathcal{D}$. Therefore, we have the following decomposition of $X$ (mod $m$):

$$X = \left\{ x : \lim_{n \to \infty} P^n 1_\mathcal{C}(x) > 0 \right\} \cup \left\{ x : \lim_{n \to \infty} P^n 1_\mathcal{D}(x) > 0 \right\}.$$

Definition 4.10. If $X = \mathcal{C}$ we call the system conservative and if $X = \mathcal{D}$ we call the system totally dissipative. Furthermore, we call the system eventually conservative if $\lim_{n \to \infty} P^n 1_\mathcal{C}(x) = 1$ m.a.e. $x \in X$.

Remark 10. It is obvious by the definition that if a system is ergodic and not totally dissipative, then the system is eventually conservative.

Now, we establish our main result in this paper on the existence of a $\sigma$-finite invariant density for a Markov operator.

Theorem 4.11. Let $(X, \mathcal{F}, m)$ be a probability space and $P : L^1 \to L^1$ be a Markov operator. Then the followings are equivalent.

1. There exists $h \in \mathcal{M}_+$ a fixed point of $P$ and there exists a sweep-out set $A \subset \{h > 0\} \cap \mathcal{C}$ such that $\int_A h dm < \infty$;
2. There exists a sweep-out set $E$ such that $P_E$ the induced operator on $E$ admits a fixed point $h_0 \in L^1_+$ with $[h_0 > 0]$ being a sweep-out set;
3. There exists a sweep-out set $E$ such that $P_E$ admits a fixed point $h_1 \in L^1_+$ with $[h_1 > 0] = E$;
4. There exists a sweep-out set $E$ such that $\hat{P}$ the jump operator with respect to $E$ is weakly almost periodic.
Remark 11. We can apply Theorem 3.1 to the condition (4) in Theorem 4.11, and if \( E = X \) in the condition (4), then \( P \) is weakly almost periodic and \( h \in L^1_+ \) in the condition (1). Hence Theorem 3.1 is a special case of Theorem 4.11 with \( A = \{ h > 0 \} \) in the condition (1).

Remark 12. The condition (1) implies that the system is eventually conservative. Indeed \( \mathcal{D} \subset \mathcal{A}^c \) and \( \lim_{n \to \infty} P^{*n}1_{\mathcal{D}}(x) \leq \lim_{n \to \infty}(P^*1_{\mathcal{A}})^n1_X(x) = 0 \). Further, from Lemma 3.4, the condition (1) implies that \( \lim_{n \to \infty} P^{*n}1_{\{h=0\}} = 0 \) holds so that \( P \) has a fixed point in \( \mathcal{M}^\sigma_\mathcal{E} \) with the maximal support.

In particular, if we consider the Perron-Frobenius operator corresponding to a nonsingular transformation then the following corollary is valid.

Corollary 4.12. Let \( (X, \mathcal{F}, m) \) be a probability space and \( T : X \to X \) be a measurable and nonsingular transformation. Then the followings are equivalent.

1. There exist an absolutely continuous \( \sigma \)-finite \( T \)-invariant measure \( \mu \) and \( A \subset [\frac{d\mu}{dm} > 0] \cap \mathcal{C} \) with \( \mu(A) < \infty \) and \( \bigcup_{n \geq 1} T^{-n}A = X \) (mod \( m \));
2. There exists a \( T \)-finite invariant measure \( \mu_E \) with \( \mu_E \sim m \mid_E \);
3. There exists a \( T \)-finite invariant measure \( \nu \) such that \( \bigcup_{n \geq 0}(T^n)^{-n}\frac{1}{\frac{d\nu}{dm}} > 0 = X \) (mod \( m \)).

Remark 13. If one would like to use usual definition of the induced transformation \( T_E \mid_E \), one has an equivalent condition for Corollary 4.12 as

\( (E, \mathcal{F} \cap E, m \mid_E, T_E \mid_E) \) for some \( E \in \mathcal{F} \) with \( \bigcup_{n \geq 0} T^{-n}E = X \) (mod \( m \)).

To prove Theorem 4.11, we prepare a sequence of lemmas.

Lemma 4.13. If there exists \( h^* \in L^1_+ \) a fixed point of \( P \), then \( \{ h^* > 0 \} \subset \mathcal{C} \).

Proof. The conservative part \( \mathcal{C} \) is characterized as \( \mathcal{C} = \{ x \in X \mid \sum_{n=0}^{\infty} P^nu(x) = \infty \} \) where \( u \) is an arbitrary strictly positive \( L^1 \) function. Take \( u = h^* + 1_{\{h^* = 0\}} \) and for any \( x \in \{ h^* > 0 \} \) with \( \alpha := h^*(x) > 0 \), it holds that \( P^nu(x) = h^*(x) + P^n1_{\{h^* = 0\}}(x) \geq \alpha > 0 \). Hence, \( \sum_{n=0}^{\infty} P^nu(x) = \infty \) and \( x \in \mathcal{C} \). This implies \( \{ h^* > 0 \} \subset \mathcal{C} \) mod \( m \).

The following key lemma for the Theorem 4.11 was proven in [9] with the assumption that the whole space is equal to the conservative part \( \mathcal{C} \) modulo \( m \) and a Markov operator is ergodic. We replace this assumption by weaker one but the proof is essentially same as in [9] and we omit it.

Lemma 4.14. Assume that there exists \( h^* \in L^1_+ \) such that \( P_Eh^* = h^* \) for \( P_E \) the induced operator for some sweep-out set \( E \). Then there exists \( h \in \mathcal{M}^\sigma_\mathcal{E} \) such that \( Ph = h \) given by

\[
\sum_{n=0}^{\infty}(I_E \cdot P)^n h^*. \tag{4.1}
\]

We also have the following formula of the invariant density via the jump operator. The proof is almost same as Lemma 4.14 and omitted.
Lemma 4.15. Assume that there exists \( h^* \in L^1_+ \) such that \( \hat{P}h^* = h^* \) for \( \hat{P} \) the jump operator for some sweep-out set \( E \). Then there exists \( h \in \mathcal{M}_+^\sigma \) such that \( Ph = h \) given by
\[
h = \sum_{n=0}^{\infty} (PI_E^n)^n h^*.
\] (4.2)

The following lemma is generalization of Lemma 3.4.

Lemma 4.16. For any \( f \in \mathcal{M}_+^\sigma \), it holds that \( P^*1_{[f = 0]} \leq 1_{[f = 0]} \).

Proof. We firstly show \( \int_{[f > 0]} P^*1_{[f = 0]} dm = 0 \). Note that for any \( \varepsilon \), there exists \( C > 0 \) s.t. \( m(N_C) < \varepsilon \) where \( N_C = \{ x \in [f > 0] : f(x) < C \} \). Since \( 1_{[f > 0]}(x) \leq f(x)/C \) for \( x \in [f > 0] \setminus N_C \), we have
\[
\int_{[f > 0]} P^*1_{[f = 0]} dm = \int_{[f = 0]} P1_{[f > 0]} dm = \int_{[f = 0]} P(1_{N_C} + 1_{[f > 0] \setminus N_C}) dm
\leq \int_X P1_{N_C} dm + \int_{[f = 0]} \frac{P f}{C} dm
< \varepsilon.
\]
Now \( \varepsilon \) is arbitrary small and we get \( \int_{[f > 0]} P^*1_{[f = 0]} dm = 0 \). This implies \( [P^*1_{[f = 0]} > 0] \subset [f = 0] \) and by \( P^*1_X = 1_X \) we obtain
\[
P^*1_{[f = 0]} \leq 1_{[P^*1_{[f = 0]} > 0]} \leq 1_{[f = 0]}
\]
as desired. \( \square \)

Remark 14. Lemma 4.16 implies that the support of a \( \sigma \)-finite invariant density for a Markov operator \( P \) spreads under the process by \( P \). That is if \( h \in \mathcal{M}_+^\sigma \) is a fixed point of \( P \) then \( P^{*n}1_{[h > 0]} \) is monotonic increasing. Furthermore, if the system is ergodic then the invariant density of \( P \) has the maximal support in the sense that \( \lim_{n \to \infty} P^{*n}1_{[h > 0]} = 1 \) m-a.e. (if we consider a nonsingular transformation \( T \) it means that \( \bigcup_{n \geq 0} T^{-n}[h > 0] = X \) mod \( m \)).

Proof of Theorem 4.11. (1)\( \Rightarrow \)(3): Suppose (1) holds and let \( E \subset [h > 0] \) be as the set in the condition (1). Since \( \lim_{n \to \infty} \| (P^*I_E^n)1_X \|_1 = 0 \) \( P_E \) is a Markov operator. Then we show that \( P_E \) has a fixed point \( h1_E \in L^1_+ \) with the maximal support. Write for any \( B \in \mathcal{F} \) with \( \int_B hdm < \infty \),
\[
\int_B P_E(h1_E) dm = \sum_{n=0}^{\infty} \int_B I_E P(I_E P)^n I_E hdm
= \sum_{n=0}^{\infty} \left( \int_B I_E (PI_E^n) hdm - \int_B I_E (PI_E^{n+1}) hdm \right)
= \int_B I_E hdm - \lim_{n \to \infty} \int_B I_E (PI_E^n) hdm
= \int_B h1_E dm - \lim_{n \to \infty} \int_X h I_E (P^*I_E)^{n-1} P^*1_{E \cap B} dm
= \int_B h1_E dm
\]
since $\int_B I_E(P I_{E^c})^n dm \leq \int_E h dm < \infty$ and $\lim_{n \to \infty} (P^* I_{E^c})^n 1_X = 0$. Therefore, we have $P_E(h 1_E) = 1_E$ and the support of this invariant density for $P_E$ equals to $E$.

(3) $\Rightarrow$ (2): It is obvious.

(2) $\Rightarrow$ (1): Assume that $P_E$ the induced operator is a Markov operator for some $E \in \mathcal{F}$ and $P_E$ has an invariant density $h^*$ with $\lim_{n \to \infty} (P^* I_{h^* = 0})^n 1_X = 0$. From the equation (4.1) in Lemma 4.14, $P$ has a fixed point $h \in \mathcal{M}_+^\sigma$. Set $A = [h^* > 0]$. Then $A \subset [h > 0] \cap \mathcal{C}$ (see the conservative part of the induced operator for $[34]$) and it follows from $A \subset E$ that

$$\int_A h dm = \int_A Ph dm = \int_A P \sum_{n=0}^\infty (I_E P)^n h^* dm$$

$$\leq \int_X I_E P \sum_{n=0}^\infty (I_E P)^n h^* dm = \int_X h^* dm < \infty.$$ 

(3) $\Rightarrow$ (4): We recall that the condition

(4') For $\hat{P}$, there exists $h_1 \in L^1_+ s.t. \hat{P} h_1 = h_1$ and $\lim_{n \to \infty} \hat{P}^n 1_{[h_1 = 0]}(x) = 0$ m.a.e. $x \in X$

is one of the equivalent condition for the condition (4) by Theorem 3.1 and so we prove the implication (3) $\Rightarrow$ (4'). We note that by definition of the induced operator and the jump operator it holds $\hat{P} P = PP_E$. Then if $P_E h_0 = h_0$ for some $h_0 \in L^1_+$ we have also $\hat{P} (Ph_0) = Ph_0$. Now we assume $[h_0 > 0] = E$ as the condition (3) and write

$$\hat{P}^n 1_{[h_0 = 0]} = \sum_{n=0}^\infty (I_{E^c} P^*)_n I_E \hat{P}^n 1_{[h_0 = 0]} \leq \sum_{n=0}^\infty (I_{E^c} P^*)_n I_E 1_{[h_0 = 0]} = 0$$

by using Lemma 4.16. This means weak almost periodicity of $\hat{P}$.

(4) $\Rightarrow$ (1): Since $\hat{P}$ is weakly almost periodic there exists $h_1 \in L^1_+$ a fixed point of $\hat{P}$ such that $\lim_{n \to \infty} \hat{P}^n 1_{[h_1 = 0]} = 0$. Put

$$h = \sum_{n=0}^\infty (PI_{E^c})^n h_1$$

and this is a fixed point of $P$ in $\mathcal{M}_+^\sigma$ as Lemma 4.15 and the above proof of the implication (1) $\Rightarrow$ (2). Hence the rest of the proof is to show $E \cap [h > 0]$ is a sweep-out set, namely, $P^*_{E \cap [h > 0]} 1_X = 1_X$. Proposition 2.2 in [34] implies that

$$P^*_{E \cap [h > 0]} 1_X = \sum_{n=0}^\infty (P^* I_{(E \cap [h > 0])^c})^n P^* I_{E \cap [h > 0]} 1_X$$

$$= \sum_{k=0}^\infty \left( \sum_{l=0}^{\infty} (P^* I_{E^c})^l P^* I_{E \cap [h = 0]} \right)^k \sum_{n=0}^\infty (P^* I_{E^c})^n P^* I_{E \cap [h > 0]} 1_X$$

$$= \sum_{k=0}^\infty (P^* I_{[h = 0]})^k P^* I_{[h > 0]} 1_X.$$ 

The last term equals to $1_X$ if and only if $\lim_{n \to \infty} (P^* I_{[h = 0]})^n 1_X = 0$. From weak almost periodicity of $\hat{P}$, $P^* \hat{P}^* = P^*_E P^*$, and $P^* 1_{[h = 0]} \leq 1_{[h = 0]}$, we have

$$\left\| (P^*_E I_{[h = 0]})^{n+1} 1_X \right\| \leq \left\| (P^*_E)^n P^* 1_{[h = 0]} \right\| \leq \left\| P^* \hat{P}^* 1_{[h = 0]} \right\| \to 0$$
Remark 15. By the proof of implication (1) to (2), we can also add the following equivalent condition to Theorem 4.11:

\( (1') \) There exist \( h \in \mathcal{M}_+^\sigma \) a fixed point of \( P \) and a sweep-out set \( A \subset [h > 0] \) with \( \int_A h dm < \infty \).

This means that for a \( \sigma \)-finite invariant density \( h \), if there exists a sweep-out set of finite measure with respect to this invariant density \( h \) contained in the support of \( h \), then the system \((X, \mathcal{F}, m, P)\) is eventually conservative. Hence it holds

\[
\int_D h dm = \int_X h \cdot P^* n_1 dm \to 0
\]
as \( n \to \infty \) and \([h > 0] \subset C \) (mod \( m \)). That is, the existence of a sweep-out set of finite measure contained in the support of a \( \sigma \)-finite invariant density is necessary for the system being eventually conservative. Equivalently, for a non-eventually conservative system (i.e., the dissipative part remains of positive measure asymptotically) have no sweep-out set of finite measure with respect to a \( \sigma \)-finite invariant density.

On non-eventually conservative (the dissipative part remaining of positive measure) systems, we may not hope that the method of induced operator or the jump operator still works. Therefore, we will have to construct new method to get \( \sigma \)-finite (infinite!) invariant densities for dissipative systems.

5. Constrictive Markov operators. In this section, we study constrictive Markov operators, which always have invariant densities of the form \( \frac{1}{r} \sum_{i=1}^r g_i \) (see Proposition 5.2 below for the representation of \( r \) and \( g_1, \ldots, g_r \)). Many important examples of the Perron-Frobenius operator or Markov operators including examples in section 6 are constrictive. Otherwise, jump operators for Markov operators with respect to suitable sweep-out sets are constrictive. We refer to [18, 19, 22, 23, 24, 30, 31] for interesting properties of asymptotic behavior of constrictive Markov operators.

Definition 5.1. Let \((X, \mathcal{F}, m)\) be a probability space.

- A set \( F \subset L^1 \) is called a constrictor (or an attractor) for a Markov operator \( P \) if for any \( h \in D \), \( \lim_{n \to \infty} \inf_{f \in F} \|P^n h - f\|_1 = 0 \).
- A Markov operator \( P \) is called constrictive (resp. weakly constrictive) if there exists a compact (resp. weakly compact) constrictor \( F \subset L^1 \).
- A Markov operator \( P \) is called smoothing if there exists \( \delta > 0 \) s.t. for any \( A \) with \( m(A) < \delta \) and \( f \in D \) it holds \( \limsup_{n \to \infty} \int_A P^n f dm < 1 \).
- A Markov operator \( P \) is called mixing (resp. exact) if \( \forall h \in L^1 \) with \( \int h dm = 0 \), it holds \( \lim_{n \to \infty} \int_X P^n h \cdot g dm = 0 \) for any \( g \in L^\infty \) (resp. it holds \( \lim_{n \to \infty} \int_X P^n h dm = 0 \)).

The next result is well-known spectral decomposition theorem for constrictive Markov operators and given in [18, 19, 24].

Proposition 5.2 (The spectral decomposition theorem). Let \((X, \mathcal{F}, m)\) be a probability space and \( P : L^1 \to L^1 \) be a weakly constrictive or smoothing Markov operator. Then there exist \( r \in \mathbb{N}, g_i \in D, k_i \in L^\infty : \) two sequences of nonnegative functions
(i = 1, 2, ..., r), and \( Q : L^1 \to L^1 \): an operator such that for any \( f \in L^1 \), \( Pf \) is written in the form
\[
Pf(x) = \sum_{i=1}^{r} \lambda_i(f) g_i(x) + Qf(x)
\]
where \( \lambda_i(f) = \int_X f(x) k_i(x) dm(x) \). The functions \( g_i \) and operator \( Q \) have the following properties:

1. Functions \( g_i \) have disjoint supports (i.e., \( g_i(x) g_j(x) = 0 \) for all \( i \neq j \));
2. For any \( i \in \{1, \ldots, r\} \), there exists \( \alpha(i) \in \{1, \ldots, r\} \), such that \( Pf_i = g_{\alpha(i)} \).
   Further \( \alpha(i) \neq \alpha(j) \) for \( i \neq j \) and thus operator \( P \) just serves to permute the function \( g_i \);
3. \( \|P^n Qf\|_1 \to 0 \) as \( n \to \infty \) (for any \( f \in L^1 \)).

The following result gives the converse of Proposition 5.2.

**Proposition 5.3.** Let \( (X, \mathcal{F}, m) \) be a probability space and \( P : L^1 \to L^1 \) be a Markov operator. Then, the followings are equivalent.

1. \( P \) is weakly constrictive;
2. \( P \) allows the spectral decomposition, namely the assertion of proposition 5.2 holds;
3. \( P \) is constrictive;
4. \( P \) is smoothing.

**Proof.**

1\( \Rightarrow \) 2, 4\( \Rightarrow \) 2: It follows from Proposition 5.2.

2\( \Rightarrow \) 3: We show that there exists a compact constrictor for \( P \) if the spectral decomposition theorem holds. We will show that the subset of \( L^1 \),
\[
\mathcal{F} = \left\{ \sum_{i=1}^{r} a_i g_i : |a_i| \leq \max_j k_j(x) (\forall i = 1, \ldots, r) \right\}
\]
is a compact constrictor for \( P \) where \( \{g_i\}_{i=1}^{r} \) and \( \{k_i\}_{i=1}^{r} \) are given in Proposition 5.2. First of all, we show that \( \forall h \in D \), \( P^n h \) belongs to \( \mathcal{F} \) asymptotically. Indeed, by assumption 2, \( \forall h \in D \),
\[
P^n h = \sum_{i=1}^{r} \left( \int_X h \cdot k_i dm \right) g_{\alpha(n)(i)} + P^n Qh
\]
and \( \|hk_i\|_1 = \|k_i\|_\infty \) hold. Thus it follows that \( \sum_{i=1}^{r} \left( \int_X h \cdot k_i dm \right) g_{\alpha(n)(i)} \in \mathcal{F} \) for any \( n \in \mathbb{N} \) and \( \|P^n Qh\|_1 \) tends to 0 as \( n \to \infty \) so that
\[
\inf_{f \in \mathcal{F}} \left\| P^n h - f \right\|_1 \leq \inf_{f \in \mathcal{F}} \left\| \sum_{i=1}^{r} \left( \int_X h \cdot k_i dm \right) g_{\alpha(n)(i)} - f \right\|_1 + \|P^n Qh\|_1
\]
\[
\Rightarrow \|P^n Qh\|_1 \to 0 \quad \text{as} \quad n \to \infty.
\]
Next we show that the set \( \mathcal{F} \) is compact. \( \forall \{f_n\}_n \) a sequence in \( \mathcal{F} \), \( f_n \) can be written in the form \( f_n = \sum_{i=1}^{r} a_i^{(n)} g_i \) where \( |a_i^{(n)}| \leq \max_j ||k_j||_\infty \). Then \( \exists \{n_k\}_k \) s.t.
\[
\exists a_* : \lim_{k \to \infty} a_i^{(n_k)} (\forall i \in \{1, \ldots, r\}) \quad \text{and the function} \quad f^* = \sum_{i=1}^{r} a_i^* g_i \quad \text{satisfies} \quad f^* = \lim_{k \to \infty} f_{n_k}.
\]
3\( \Rightarrow \) 4: We assume that there exists a compact constrictor \( \mathcal{F} \). First of all, we show \( \mathcal{F}_D = \mathcal{F} \cap D \) is also a compact constrictor. \( \mathcal{F}_D \) is closed and hence compact. Since an element of a constrictor belongs to closure of \( \{P^n h\}_n \) and \( D \) is closed, \( \mathcal{F}_D \) the
restriction of the constrictor \( F \) on \( D \) is also a constrictor. Then the rest of the proof is to show \( P \) is smoothing. Since \( \mathcal{F}_D \) is compact, \( \delta := \inf_{f \in \mathcal{F}_D} m([f > 0]) / 2 > 0 \). Hence, \( d := \sup_{A \in \mathcal{F}, m(A) < \delta} \int_A f \, dm < 1 \) holds for any \( f \in \mathcal{F}_D \). Therefore, for any \( A \in \mathcal{F} \) with \( m(A) < \delta \), it holds that for any \( h \in D \)

\[
\int_A P^n h \, dm \leq \int_A f \, dm + \|P^n h - f\|_1 \\
\leq d + \|P^n h - f\|_1 \quad (\forall f \in \mathcal{F}_D)
\]

and \( \limsup_{n \to \infty} \int_A P^n h \, dm < 1 \) (\( \forall h \in D \)). That is \( P \) is smoothing. \( \square \)

The following proposition gives a simple sufficient condition for a Markov operator to be constrictive via exactness (which is generalization of Example 1.3.10 in [7]).

**Proposition 5.4.** Let \((X, \mathcal{F}, m)\) be a probability space and \( P : L^1 \to L^1 \) be a Markov operator. If there exists \( k \in \mathbb{N} \) such that \( P^k \) has an invariant density and exact, then \( P \) is constrictive. Conversely, if \( P \) is constrictive then \( P \) admits an invariant density and there exists \( k \in \mathbb{N} \) such that \( P^k_{|\text{supp}g_i} \) is exact for \( i = 1, \ldots, r \).

**Proof.** Assume there is \( h_0 \in D \) such that \( P^k h_0 = h_0 \) and \( P^k \) is exact. Then \( h = \frac{1}{k} \sum_{i=0}^{k-1} P^i h_0 \) is an invariant density for \( P \) and we will show \( P \) is smoothing (which is equivalent to constrictive according to Proposition 5.3). For each \( f \in D \), \( A \in \mathcal{F} \) and \( N \in \mathbb{N} \),

\[
\sup_{n \geq kN} \int_A P^n f \, dm = \sup_{i \geq N} \left\{ \int_A P^i f \, dm, \int_A P^{i+1} f \, dm, \ldots, \int_A P^{i+k-1} f \, dm \right\} \\
\leq \sup_{i \geq N} \left\{ \|P^i f - h\|_1 + \int_A h \, dm, \ldots, \|P^{i+k-1} f - h\|_1 + \int_A h \, dm \right\} ,
\]

since \( \{n : n \geq kN\} = \{kl+j : j = 0, \ldots, k-1, l \geq N\} \). Note that \( \|P^{kn+j} f - h\|_1 \to 0 \) as \( n \) tends to \( \infty \) for \( j = 0, \ldots, k-1 \) since \( P^j f - h = P^j (f - h) \) is zero-average and \( P^k \) is exact. This implies the above supremum is bounded by 1 for \( A \) small enough (with respect to \( m \)) and \( P \) is smoothing.

Next, conversely, we suppose \( P \) is constrictive. We only have to show \( P^k_{|\text{supp}g_i} \) is exact for some \( k \), \( (i = 1, \ldots, r) \). It follows from \( P \) admitting spectral decomposition and \( P g_i = g_{\alpha(i)} \) that, for any \( f \in D \) supported on \( \text{supp}g_i \), \( \lambda_{\alpha(i)}(f) = 1 \) and \( \lambda_j(f) = 0 \) for any \( j \neq \alpha(i) \). Now let \( k \) be the smallest number with \( P^k g_i = g_i \) for all \( i = 1, \ldots, r \). Then we have \( P^k f = P^{k-1} (g_{\alpha(i)} + Qf) = g_i + P^{k-1} Qf \) and

\[
\lim_{n \to \infty} \|P^{kn} f - g_i\|_1 \leq \lim_{n \to \infty} \|P^{kn-1} Qf\|_1 = 0.
\]

This means \( P^k_{|\text{supp}g_i} \) is exact and completes the proof. \( \square \)

**Remark 16.** From the above Proposition 5.4, we can say the difference between the decomposition of a constrictive Markov operator and that of a quasi-compact operator. Recall that quasi-compact bounded linear operators on complex Banach spaces (see definition in [4, 5]) also admits the decomposition as in Proposition 5.2. Hence it seems that the spectral decomposition for constrictive Markov operators is just “real” version of the decomposition for quasi-compact operators on complex Banach spaces. However we remark the decomposition of a constrictive Markov operator and that of a quasi-compact operator are essentially different. Namely, the decay of \( \{P^n Qf\}_n \) in Proposition 5.2 is not always exponentially fast but the
decay for a quasi-compact operator is always exponentially fast. Indeed, for an intermittent map \( T : [0, 1] \to [0, 1] \)
\[
Tx = \begin{cases} 
  x + 2^\alpha x^{1+\alpha} & x \in [0, 1/2) \\
  2x - 1 & x \in [1/2, 1]
\end{cases}
\]
with \( 0 < \alpha < 1 \), because \( T \) admits a Lebesgue-absolutely continuous finite invariant measure for which \( T \) is exact (see \([41, 42]\)), the corresponding Perron-Frobenius operator is constrictive by Proposition 5.4. But the decay of correlation for this \( T \) is not exponential as the speed of convergence of the iterated Perron-Frobenius operator is not exponentially fast (see \([21, 25, 26]\)) whereas the decay for quasi-compact operators is always exponentially fast.

The next result gives a relation between Theorem 3.1 (weak almost periodicity) and Proposition 5.2 (the spectral decomposition theorem).

**Proposition 5.5.** Let \((X, \mathcal{F}, m)\) be a probability space and \( P : L^1 \to L^1 \) be a weakly constrictive Markov operator. Then \( P \) is almost periodic so that each condition of Theorem 3.1 is valid.

**Proof.** We will show that \( \{P^n f\}_n \) is precompact for any fixed \( f \in L^1 \). By Proposition 5.2, for \( f \in L^1 \),
\[
P^{n+1} f = \sum_{i=0}^r \lambda_i(f) g_{\alpha(i)} + P^n Q f
\]
and note that \( \alpha^r(i) = i \) (\( \forall i \in \{1, \cdots, r\} \)) since \( \alpha \) is permutation. Then it holds that \( \forall \{n_k\}_k \subset \mathbb{N}, \exists \{k_j\}_j \subset \mathbb{N}, \) s.t. \( n_{k_j} = N_0 + m_j r! \) for some \( N_0 \in \mathbb{N} \), and \( \{m_j\}_j \subset \mathbb{N} \). Since
\[
P^{n_{k_j}} f = P^{n_0 + m_j r! - 1} \left( \sum_{i=1}^r \lambda_i(f) g_{\alpha(i)} + Q f \right)
= \sum_{i=1}^r \lambda_i(f) g_{\alpha^{r-1}(i)} + P^{n_{k_j} - 1} Q f
\]
and the operator \( Q \) satisfies that \( \lim_{n \to \infty} \|P^n Q f\|_1 = 0 \), \( P^{n_{k_j}} f \to \sum_{i=1}^r \lambda_i(f) g_{\alpha^{r-1}(i)} \) strongly as \( j \to \infty \). Therefore, \( P \) is almost periodic. \( \Box \)

The following proposition shows that in the class of constrictive Markov operator, mixing property is equivalent to exactness. From this result, in case of an invertible nonsingular transformation, the Perron-Frobenius operator associated to a mixing probability measure cannot be constrictive. The result of Inoue and Ishitani in [15] also implies that the Perron-Frobenius operators corresponding to invertible transformations cannot be constrictive. The following proposition was shown in [22] under the assumption that \( P1_X = 1_X \). We prove the following assertion without this assumption.

**Proposition 5.6.** Let \((X, \mathcal{F}, m)\) be a \( \sigma \)-finite measure space and \( P : L^1 \to L^1 \) be a constrictive Markov operators. Then the followings are equivalent.

1. \( P \) is exact;
2. \( P \) is mixing;
3. \( r = 1 \) in representation of Proposition 5.2.

**Proof.** 1\( \Rightarrow \)2: It is obvious.
2⇒3: We assume that \( r > 1 \) and \( f_0 = \frac{1}{r} \sum_{i=1}^{r} g_i \) is fixed point of \( P \). Set \( f = g_1 \) and \( A_i = \{ g_i > 0 \} \), then

\[
\int_X P^n f 1_{A_i} dm = \int_X g_{\alpha^n(1)} 1_{A_i} dm
\]

\[
= \begin{cases} 1 & (\alpha^n(1) = 1) \\ 0 & (\alpha^n(1) \neq 1). \end{cases}
\]

Hence,

\[
\left| \int_X (P^n f - f_0) 1_{A_i} dm \right| = \left| \int_X P^n f 1_{A_i} dm - \int_{A_i} f_0 dm \right|
\]

\[
= \begin{cases} |1 - \int_{A_i} f_0 dm| & (\alpha^n(1) = 1) \\ |\int_{A_i} f_0 dm| & (\alpha^n(1) \neq 1) \end{cases}
\]

\[
> 0
\]

and this contradicts to mixing property of \( P \) since \( \int_X (f - f_0) dm = 0 \).

3⇒1: Assume that \( r = 1 \) i.e., \( \forall f \in L^1, Pf = \lambda(f)g_0 + Qf \). Since \( \lambda(f) = 1 (\forall f \in D) \), \( \int_X f k_0 dm = 1 (\forall f \in D) \). Hence, take \( f = 1_A/m(A) (\forall A \in \mathcal{F} \) with \( m(A) > 0) \), then it holds that \( k_0(x) = 1_X(x) \) a.e. \( x \in X \). Therefore, \( \forall f \in L^1 \) with \( \int_X f dm = 0 \), it holds that \( \lambda(f + g_0) = \int_X (f + g_0) k_0 dm = 1 \) and

\[
\|P^n f\|_1 = \|P^n (f + g_0) - g_0\|_1
\]

\[
= \|\lambda(f + g_0)g_0 + P^{n-1} Q(f + g_0) - g_0\|_1
\]

\[
= \|P^{n-1} Q(f + g_0)\|_1 \to 0
\]

as \( n \to \infty \). \( \square \)

We recall the Jacobs-Deleeuw-Glicksberg splitting theorem and we consider the converse of Proposition 5.5. A Markov operator \( P \) is called quasi-constrictive if the closed subspace

\[
X_0(P) := \{ f \in L^1 : \lim_{n \to \infty} \|P^n f\| = 0 \}
\]

has finite codimension. It is easy to verify that a constrictive Markov operator is also quasi-constrictive. We remark relation between constrictive operators and weakly almost periodic operators. It is known that (see [7]) if a quasi-constrictive Markov operator is weakly almost periodic, it is constrictive. For our future work, we ask further equivalent conditions for a Markov operator being constrictive by using Theorem 3.1 and Corollary 3.2.

6. Applications. In this section, we give some examples to which we can apply our results in §3 and §4. We consider deterministic dynamical systems and random dynamical systems. Here we mean that a deterministic dynamical system is a non-singular transformation on a probability space and a random dynamical system is determined by a Markov process. In particular, we focus on the process such as systems of “nonsingular transformation+noise” and “random iteration of intermittent maps” in this section.

The first example is a family of generalized tent maps to be non-surjective and we can apply Corollary 3.6 to this.
Example 1 (Generalized tent map). We consider a generalized tent map $T_a$ ($a \in [1, 2]$) on $I = [0, 1]$ with the Lebesgue measure $\lambda$ which is defined in [43] by

$$T_a(x) = \begin{cases} 
ax & x \in [0, 1/2] \\
ax(1-x) & x \in (1/2, 1].
\end{cases}$$

For $a \in (1, 2)$, In [28, 43], the Perron-Frobenius operator corresponding $T_a$ and $\lambda$ is shown to be constrictive. Furthermore, explicit formula of invariant densities for specific parameters $a$ are given and exactness of $T_a$ for $a \in (2^{1/2}, 2]$ are shown in [43]. Hence by Proposition 5.5 and Corollary 3.6, for any $a \in (1, 2]$, the $\lambda$-absolutely continuous finite invariant measure for $T_a$ has the maximal support. Meanwhile, $T_a$ has no equivalent finite invariant measure for $a < 2$. If $a = 1$, since $T_a |_{[0,1/2]}$ is the identity map, we can see that the corresponding Perron-Frobenius operator is not constrictive but weakly almost periodic. Hence, Corollary 3.6 is applicable to this case $a = 1$. Further, we can see that any $\lambda$-absolutely continuous $\sigma$-finite measure fully supported on $[0, 1/2]$ is invariant for $T_a$ with the maximal support condition. In total, for any $a \in [1, 2]$, $T_a$ admits an absolutely continuous finite invariant measure with the maximal support and consequently the system is eventually conservative as defined in Definition 4.10.

In the above Example 1, if $a \in (0, 1)$, obviously $T_a$ is totally dissipative with respect to $\lambda$. Hence, we cannot apply Corollary 3.6 or Corollary 4.12. However if we consider the system of $T_a$ with a randomly applied stochastic perturbation as in the following Example 2, then our new observation shows that the system admits an absolutely continuous finite invariant measure with the maximal support even when $a \in (0, 1)$.

Example 2 (Additive noisy type random dynamics). Let $T$ be a nonsingular transformation on the Lebesgue space $([0, 1], \mathcal{B}, \lambda)$. We consider the process defined by $x_{n+1} = T(x_n) + \xi_n \pmod{1}$ where $\xi_0, \xi_1, \ldots$ are independent random variables each having the same density $g$. Let $f_n$ denote the density of $x_n$ and the relation connecting $f_{n+1}$ and $f_n$ is given (see [17, 22]) by

$$f_{n+1}(x) = \int_{[0,1]} f_n(y) \sum_{i=0}^1 g(x - T(y) + i) d\lambda(y).$$

Then we can define the Markov operator $P$ representing the evaluation of the density corresponding to this stochastic process by

$$Pf = \int_{[0,1]} f(y) \sum_{i=0}^1 g(x - T(y) + i) d\lambda(y).$$

Iwata and Ogiha in [17] proved that this Markov operator is constrictive. Hence we can apply Theorem 3.1 for this case. That is, this Markov operator $P$ has a fixed point $f_0 \in D$ with maximal support (i.e., $\lim_{n \to \infty} P^\ast n [f_0 > 0] = 1$) so that the system is eventually conservative.

Now, in order to introduce our examples arising from intermittent maps, we recall interval maps with an indifferent fixed point which is also known as intermittency. Those maps admit absolutely continuous finite invariant measures or $\sigma$-finite infinite invariant measures. Let $S_\alpha : [0, 1] \to [0, 1]$ ($\alpha > 0$) be defined in [25] by

$$S_\alpha x = \begin{cases} 
2x - 1 & x \in [1/2, 1] 
\end{cases}$$

(6.1)
One can see in [41] that this transformation has an invariant density 
\( h \in M_\sigma \) as 
\[ h(x) \sim \frac{1}{x^\alpha} h^*(x) \]
where \( h^*(x) > 0 \) is bounded function on \([0,1]\). Then, for \( 0 < \alpha < 1 \), \( S_\alpha \) admits an equivalent finite invariant measure and for \( \alpha \geq 1 \), \( S_\alpha \) has an equivalent \( \sigma \)-finite infinite invariant measure. In usual way (see [32, 41]), we choose \([1/2,1]\) (on which \( T \) is expanding) as a sweep-out set defining the induced transformation or the jump transformation. By [21, 25, 42], exactness and mixing rate for \( S_\alpha \) with respect to the invariant measure are also known.

The next two examples are modified intermittent maps of (6.1) which admit a \( \sigma \)-finite invariant measure which is absolutely continuous with respect to Lebesgue measure but not equivalent to it. Exactness for both Example 3 and 4 with respect to the invariant measure can be shown by the same manner in [42]. The following Example 3 is not surjective so that there is no equivalent \( \sigma \)-finite invariant measure. However Corollary 4.12 tells us that for Example 3 there exists an absolutely continuous \( \sigma \)-finite invariant measure with the maximal support.

**Example 3** (Non-surjective intermittent map). Fix \( \alpha > 0 \). Let \((X_1 = [0,2], \mathcal{B}, \lambda)\) be the Lebesgue space with normalized Lebesgue probability measure and we consider, using \( S_\alpha \) as in (6.1),
\[ T_1 x = \begin{cases} S_\alpha x & x \in [0,1] \\ x - 1 & x \in (1, 2]. \end{cases} \]
The jump transformation \( T_1^* \) with respect to \([1/2,1]\) as considered in [32, 41] admits an absolutely continuous finite invariant measure supported on \([0,1]\) which has also maximal support. This fact is one of the equivalent conditions in Corollary 4.12 and \( T_1 \) has an absolutely continuous \( \sigma \)-finite invariant measure supported on \([0,1]\) which has the maximal support. Consequently the system is eventually conservative according to Corollary 4.12.

The following example is an intermittent map which is surjective and “bi-nonsingular” (i.e., \( \lambda \circ T^{-1} \sim \lambda \)) but admits no equivalent \( \sigma \)-finite invariant measure with respect to Lebesgue.

**Example 4** (Bi-nonsingular intermittent map with no equivalent invariant measure). Fix \( \alpha > 0 \). Let \((X_2 = [-1,1], \mathcal{B}, \lambda)\) be the Lebesgue space with normalized Lebesgue probability measure on \(X_2\) and we consider the following transformation on it:
\[ T_2 x = \begin{cases} 2x + 1 & x \in [-1,0] \\ S_\alpha x & x \in [0,1]. \end{cases} \]
Then the dissipative part is \([-1,0]\) and the conservative part is \([0,1]\). Consider the jump transformation with respect to \([1/2,1]\) as in Example 3. This map enjoys an absolutely continuous \( \sigma \)-finite invariant measure supported on \([0,1]\) with the maximal support. Consequently, this system is eventually conservative with respect to \( \lambda \).

The last example is a random iteration of a family of (same order of tangency) intermittent maps. We refer to [13] for sufficient conditions for the existence of an absolutely continuous finite invariant measure for certain type of random iteration. Further, Inoue recently showed in [14] a sufficient condition for the existence of an
absolutely continuous \(\sigma\)-finite invariant measure by using the method of inducing for one-dimensional random dynamics. We note that, apart from [14], Theorem 4.11 gives an alternative approach for the random non-uniformly expanding maps via the method of jump operators.

**Example 5** (Random iteration of intermittent maps). Fix \(\alpha > 0\). Let \(W\) be an at most countable set and \((I, \mathcal{B}, \lambda)\) be the Lebesgue space of the unit interval and set transformations parametrized \(w \in W\):

\[
T_w x = \begin{cases} 
    x + 2^\alpha x 1 + \alpha & x \in [0, 1/2) \\
    f_w & x \in [1/2, 1]
  \end{cases}
\]

where \(f_w : [1/2, 1] \to I\) is \(C^1\)-invertible with \(|f'_w| > 1\) uniformly on \([1/2, 1]\) and \(w \in W\). Then we consider the random dynamics defined by the transition probability (with probabilities \(p_w \geq 0\) with \(\sum_{w \in W} p_w = 1\))

\[
P(x, A) = \sum_{w \in W} p_w 1_{T^{-1}_w A}(x) \quad (x \in I, A \in \mathcal{B}).
\]

This random dynamical system is that we choose a transformation \(T_w\) with probability \(p_w\) and apply it on each iteration. Then the Markov operator representing this random dynamics can be written as

\[
P = \sum_{w \in W} p_w P_w
\]

where \(P_w\) is the Perron-Frobenius operator corresponding to \(T_w\). Since \(f_w\) is \(C^1\)-invertible from \([1/2, 1]\) to \(I\), we have

\[
P_w g = g \circ V \cdot |V'| + g \circ V_w \cdot |V'_w| \quad (g \in L^1)
\]

where \(V := (x + 2^\alpha x 1 + \alpha)^{-1} : I \to [0, 1/2)\) and \(V_w := (f_w)^{-1} : I \to [1/2, 1]\) the local inverses. Set \(Q_g := g \circ V \cdot |V'|\) and \(Q_w g := g \circ V_w \cdot |V'_w|\). For the jump operator \(\hat{P}\) with respect to \(E = [1/2, 1]\) we have

\[
\hat{P} = PI_E \sum_{n \geq 0} (PI_{E^n})^n = \left( \sum_{w \in W} p_w Q_w \right) \sum_{n \geq 0} Q^n = \sum_{w \in W} p_w \hat{P}_w
\]

where \(\hat{P}_w\) is the Perron-Frobenius operator corresponding to \(T^*_w\) the jump transformation on \(T_w\) with respect to \(E\). The Markov operator \(\hat{P}\) represents the random dynamical system with choosing the family of piecewise expanding maps \(T^*_w\) with probability \(p_w\) (\(w \in W\)). By the result of [13], we have that \(\hat{P}\) is constrictive and so \(\hat{P}\) is weakly almost periodic. Therefore we can apply Theorem 4.11 to this example. This means the random dynamical system admits an absolutely continuous \(\sigma\)-finite invariant measure with the maximal support and consequently the system is eventually conservative.

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