Constraints on spacetime manifold in Euclidean supergravity in
terms of Dirac eigenvalues

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Abstract

We generalize previous work on Dirac eigenvalues as dynamical variables of Euclidean supergravity. The most general set of constraints on the curvatures of the tangent bundle and on the spinor bundle of the spacetime manifold, under which spacetime admits Dirac eigenvalues as observables, are derived.
I. INTRODUCTION

The formulation of gravity in terms of noncommutative geometry aims at giving a resolution to the problem of quantum gravity. There are already several approaches to the subject, for instance Connes-Dixmier-Wodzicki formulation\(^1\) based on Dixmier trace\(^2\) and Wodzicki residue formula\(^3\) which was generalized to any dimensional gravity\(^4\), the formulation in terms of linear connections\(^5\) or the theory based on the spectral action principle\(^6\) which also provides the action of the Standard Model\(^7\). All of these theories are formulated on Euclidean spacetime manifolds because the spectral triple necessary to formulate the noncommutative geometry can be defined only in this case. However, there are attempts to formulate spectral triples with Minkowskian signature\(^8\). For fundamental and recent reviews of these topics the reader should see\(^9\). A supersymmetric extension of the action that obeys the spectral principle has been recently reported in\(^10\) while in\(^11\) has been shown that quantum versions of Riemann objects exhibit a noncommutative structure.

In a very recent paper\(^12\) one argues that Dirac eigenvalues are the observables of Euclidean gravity. In a previous work one of us developed the idea that Dirac eigenvalues can play the same role in the case of Euclidean supergravity once some constraints are imposed on the covariant phase space as well as on Dirac eigenspinors\(^13\). Moreover, it has been shown that a spacetime manifold which supports such of a description of supergravity should also satisfy some constraints. Indeed, by applying the parallel transport to the constraints on the phase space and under some simplifying assumptions, a set of equations that should be satisfied by the curvatures of the tangent bundle and of the spinor bundle, respectively, were derived\(^14\).

Here, we extend our understanding of the compatibility between Dirac eigenvalues as observables of Euclidean supergravity and the geometrical structure of spacetime, by investigating the most general set of constraints on the two curvatures resulting from the parallel transport of the constraints on the phase space, or primary constraints, as well as from the parallel transport of the constraints on Dirac eigenspinors, or secondary constraints.

The plan of the paper is as follows. In Sec. II we review the basic results obtained in the case of Euclidean supergravity and present the basic ideas which allow us to derive the constraints on the curvatures. In Sec. III we determine these constraints in the most general form. In Sec. IV we summarize the results and make some concluding remarks. Units have been chosen so that \(8\pi G = 1\).

II. PRIMARY AND SECONDARY CONSTRAINTS OF EUCLIDEAN SUPERGRAVITY

The aim of this section is the brief presentation of the constraints of Euclidean supergravity that should be imposed locally in order to have Dirac eigenvalues as observables of the system. For extended discussions the reader should see\(^13\).

Consider a compact four dimensional (4D) (spin)manifold without boundary \(M\), on which Euclidean minimal supergravity is defined. The \(N = 1\) on-shell supergraviton multiplet contains the graviton, represented by vierbein fields \(e^a_\mu(x)\), where \(\mu = 1, \ldots, 4\) are spacetime indices and \(a = 1, \ldots, 4\) are internal Euclidean indices, and the gravitino fields \(\psi^a_\mu(x)\), which are required to satisfy, instead of the usual Majorana condition, the following one \(\bar{\psi} = \psi^T C\mathcal{O}\). This replacement is necessary because the group of local rotations \(SO(4)\) does not admit a
Majorana representation. Alternatively, one could work with symplectic spinors which could also be more suitable for generalizations to higher dimensional Euclidean supergravity.

The covariant phase space of the theory is defined to be the phase space of the solutions of the equations of motion modulo the gauge transformations, which are 4D diffeomorphisms, local $SO(4)$ rotations and $N = 1$ local supersymmetry. The observables of the theory are functions on the phase space. On $M$, the Dirac operator is defined in the presence of local supersymmetry as

$$D = \hat{D} + K,$$

where

$$\hat{D} = i\gamma^a e^\mu_a (\partial_\mu + \frac{1}{2} \tilde{\omega}_{abc}(e) \sigma^{bc}),$$

is the Dirac operator in the absence of local supersymmetry and

$$K = i\gamma^a e^\mu_a K_{abc}(\psi) \sigma^{bc}.$$ (3)

Here, $\gamma^a$’s form an Euclidean representation of the Clifford algebra $C_4$, i.e. $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$, $\tilde{\omega}_{abc}(e)$ is the usual spin connection in the absence of supersymmetry and $\sigma^{bc} = \frac{1}{4}[\gamma^a, \gamma^b]$.

The contortion term, required by for local supersymmetry, is given by

$$K_{\mu ab}(\psi) = i\frac{1}{4}(\bar{\psi}_\mu \gamma^a \psi_b - \bar{\psi}_\mu \gamma^b \psi_a + \bar{\psi}_b \gamma^a \psi_\mu).$$ (4)

Since we have assumed that $M$ is compact, the Dirac operator which is an elliptic operator, has a discrete spectrum and a complete set of eigenspinors

$$D\chi^n = \lambda^n \chi^n,$$ (5)

where $n = 0, 1, 2, \ldots$. $\lambda^n$’s depend on $(e, \psi)$ and thus define a discrete family of functions on the space of all supermultiplets. If we denote the space of all gravitational supermultiplets by $\mathcal{F}$, then $\lambda^n$’s define a discrete family of functions on $\mathcal{F}$ since these functions depend on $(e, \psi)$. This is a consequence of the dependence of $D$ on $(e, \psi)$. Therefore, $\lambda^n$’s define a discrete family of real valued functions on $\mathcal{F}$ and a function from $\mathcal{F}$ into the space of infinite sequences $R^\infty$

$$\lambda^n : \mathcal{F} \longrightarrow R^\infty, \quad (e, \psi) \rightarrow \lambda^n(e, \psi),$$

Let us note that, in general, Dirac eigenvalues are not invariant under the gauge transformations which represents a different situation from the gravity case. Therefore, they cannot be immediately use as observables and we must identify the circumstances which allow us to treat $\lambda^n$’s as observables.

The basic requirement is that Dirac eigenvalues be gauge invariant. The action of gauge transformations on the supergraviton are given by the following equations

$$\delta e^a_\mu = \xi^\nu \partial_\nu e^a_\mu + \theta^{ab} e^b_\mu + \frac{1}{2} \varepsilon^{e\nu} \psi_\mu,$$ (8)

$$\delta \psi^a_\mu = \xi^\nu \partial_\nu \psi^a_\mu + \theta^{ab} (\sigma_{ab})^\beta_\mu \psi^\beta_\mu + \mathcal{D}_\mu e^a.$$ (9)
where $\xi = \xi^\nu \partial_\nu$ is an infinitesimal vector field on $M$, $\theta_{ab} = -\theta_{ba}$ parametrize an infinitesimal rotation and $\epsilon$ is an infinitesimal "Majorana" spinor field. Here, $D_\mu$ is the nonminimal covariant derivative acting on spinors and associated to the minimal one acting on vectors and on graviton according to the usual rules

\[
D_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\sigma} A^\sigma \tag{10}
\]
\[
D_\mu A^a = \partial_\mu A^a + \omega^a_{\mu b} A^b \tag{11}
\]
\[
D_\nu e^a_\mu = \partial_\nu e^a_\mu - \Gamma^a_{\mu \nu} e^\beta_\mu + \omega^a_{\nu b} e^b_\mu, \tag{12}
\]
where $A = A^\mu \partial_\mu$ is an arbitrary spacetime vector, $A = A^a \partial_a$ is an arbitrary $SO(4)$ vector and $\Gamma^a_{\mu \nu}$’s are the Christoffel symbols. Then the nonminimal covariant derivative is given by

\[
D_\mu \phi = \partial_\mu \phi + \omega_{\mu ab} \sigma^{ab} \phi \tag{13}
\]
for an arbitrary spinor $\phi$. Under local supersymmetry $\omega_{\mu ab}$ transforms as

\[
\delta \omega_{\mu ab} = A_{\mu ab} - \frac{1}{2} e^b_\mu A^c + \frac{1}{2} e^a_\mu A^b, \tag{14}
\]
where

\[
A^{\mu \nu}_a = \bar{\epsilon} \gamma_5 \gamma_a D_\lambda \psi_\rho \epsilon^{\mu \lambda \rho}, \tag{15}
\]
while under an $SO(4)$ rotations the coefficients of spin connection transform as

\[
\delta \omega_{\mu ab} = i [\theta_\sigma, \omega_{\mu ab}] - i \partial_\mu \theta_\sigma M_{ab}. \tag{16}
\]

As was shown in Ref. 13, the variations of $\lambda^n$’s under gauge transformations vanish only if the following relations hold:

\[
T^{\mu \rho}_{\nu a} \partial_\rho e^a_\mu - \Gamma^n_{\alpha \mu} \partial_\nu \psi^\alpha_\mu = 0 \tag{17}
\]
as a consequence of diffeomorphisms,

\[
T^{\mu \rho}_{\nu a} e^b_\mu + \Gamma^n_{\mu \sigma a b} \psi^\rho_\mu = 0 \tag{18}
\]
as a consequence of local $SO(4)$ transformations and

\[
T^{\mu \rho}_{\nu a} \epsilon^a_\rho + \Gamma^n_{\mu \rho a} D_\mu \epsilon = 0 \tag{19}
\]
which follows from $N = 1$ local supersymmetry. The less obvious terms in Eqs. (17), (18) and (13) are given by

\[
T^{\mu \rho}_{\nu a}(x) = \frac{\delta \lambda^n}{\delta e^a_\mu} = \langle \chi^n | \frac{\delta D}{\delta e^a_\mu} \chi^n \rangle, \quad \Gamma^{\mu \rho}_{\nu a} = \frac{\delta \lambda^n}{\delta \psi^\rho_\mu} = \langle \chi^n | \frac{\delta D}{\delta \psi^\rho_\mu} \chi^n \rangle, \tag{20}
\]
where the scalar product is defined in the Hilbert space of spinors on $M$ and is given by

\[
\langle \chi, \phi \rangle = \int \sqrt{e} \chi^* \phi. \tag{21}
\]
Now once the Eqs. (17), (18) and (19) (called primary constraints) hold, there is another set of constraints (called secondary constraints) which should be imposed on the Dirac eigenspinors. The reason for that is simply the fact that Dirac eigenvalue problem (5) also transforms under (9) and in its variations we must take into account the fact that the variations of eigenvalues vanish if they are to be considered observables. The equations reflecting this consistency condition are the following ones:

\[
\{[b^\mu (\xi) - c(\lambda, \xi)^\mu] \partial_\mu + f(\xi)\} \chi^n = 0, \tag{22}
\]

which follows from (17), where we have employed the following notations

\[
b_\mu (\xi) = i\gamma^a b_a^\mu (\xi), \quad b_a^\mu (\xi) = \xi^\nu \partial_\nu e_a^\mu \quad c(\lambda, \xi)^\mu = (\lambda^n - D)\xi^\mu, \quad f(\xi) = i\gamma^a \xi^\nu \partial_\nu (e_a^\mu \omega_{abc})\sigma^{bc}. \tag{23}
\]

From the second of the primaries, namely (18), the following equation results

\[
[\theta_a^\mu D - g(\theta) + h(\theta)] \chi^n = 0. \tag{26}
\]

where

\[
g(\theta) = [\gamma^c e^\mu_c ([\theta\sigma, \omega_{\mu ab}] - \partial_\mu \theta\sigma M_{ab})]\sigma^{ab} \tag{27}
\]

\[
h(\theta) = i(\lambda^n - D)\theta\sigma. \tag{28}
\]

Finally, \(N = 1\) local supersymmetry implies

\[
[j_a^\mu (\epsilon) \partial_\mu + k_a(\epsilon) + l_a] \chi^n = 0 \tag{29}
\]

where

\[
j_a^\mu (\epsilon) = \frac{1}{2} \gamma_a \bar{\psi}^{\mu}, \quad k_a(\epsilon) = \frac{1}{2} \gamma_a \bar{\psi}^{\mu} \omega_{\mu cd}\sigma^{cd} \tag{30}
\]

\[
l_a = e_a^\mu[A_{\mu cd} - \frac{1}{2} \epsilon_{\mu ad} A_{\sigma cd} + \frac{1}{2} \epsilon_{\mu c d} A_{\sigma cd}]\sigma^{cd}. \tag{31}
\]

Some comments are now in order. The first set of constraints given by eqs. (17), (18) and (19) should be interpreted as constraints on the phase space of the theory. Therefore, the observables of the theory are defined on the intersection of the solutions of the equations of motion with the solutions of the primary constraints. This intersection represents a subset of the set of all supergravitons. The second set of constraints given by eqs. (22), (26) and (29) points out the set of Dirac eigenspinors for which the variations of the corresponding eigenvalues under gauge transformations vanishes. This shows us a second restrictions, this time on the possible observables of the theory.

We must notice that the entire discussion has a local character up to now. For the theory to be fully consistent, we should address the problem of compatibility of the primary and secondary constraints with the global structure of spacetime manifold \(M\). A very general and important class of manifolds can be obtained by considering the parallel transport of the two types of constraints from one point of \(M\) to another one along two different arbitrary
paths. The compatibility of the geometric structure of $M$ with the two sets of constraints can then be rephrased as the condition of obtaining the same result after the transport of any of the constraints along the two paths. This problem was partially addressed in \cite{14} where a set of constraints on the curvatures of the tangent bundle and spinor bundle, respectively, was obtained. However, the equations obtained in \cite{14} were calculated under two restrictions: firstly, there were analysed only the displacements of the primaries. Secondly, there was made the simplifying assumption that Dirac eigenspinors are subject to parallel transport along the two paths.

In the next section we obtain the constraints on the manifold $M$ by relaxing the above assumption, that is by considering an arbitrary transformation on the Dirac eigenspinors. Moreover, the constraints arising from the transport of the secondaries will also be deduced there in the general case. The technique used here is the same as the one employed in \cite{14}, but to make the computations in the following section more transparent, we give a brief account of it before ending this paragraph.

Let us consider two congruences $c(\lambda)$ and $d(\mu)$ on $M$ with $\lambda$ and $\mu$ the parameters of the curves, and let us take a curvilinear rectangle at the intersection of the two congruences, \{Q, P, R, S\} ∈ $c(\lambda) \cap d(\mu)$, where $|QP| \in c$, $|RS| \in c$, $|PR| \in d$, $|QS| \in d$ and the vertical bars denote the curvilinear segment. Assume that the lengths of the sides of the rectangle are $\lambda$ and $\mu$ measured in units of natural parameters of $c(\lambda)$ and $d(\mu)$, respectively. We take two commuting vector fields $\xi$ and $\eta$ defined along $c(\lambda)$ and $d(\mu)$, respectively and we denote the path $Q \rightarrow P \rightarrow R$ by 1 and the path $Q \rightarrow S \rightarrow R$ by 2. Any object carrying the subscript 1 or 2 will be understood as transported along the respective path. Now, since bosons as well as fermions enter the objects to be transported along the two paths, we need two connections, $\nabla$ on the tangent bundle and $\nabla^S$ on the spinor bundle, related to the minimal covariant derivative and to the nonminimal covariant derivative, respectively.

Then a boson transported along path 1 is given by

$$A_1 = e^{\mu \nabla_\eta} e^{\lambda \nabla_\xi} A.$$  \hspace{1cm} (32)

If we consider $\lambda$ and $\mu$ small and power expand (32) up to the second order, we obtain

$$A_1 = A_1^{(0)} + \mu A_1^{(1)} + \lambda A_1^{(2)} + \frac{\mu^2}{2} A_1^{(3)} + \frac{\lambda^2}{2} A_1^{(4)} + \mu \lambda A_1^{(5)}. \hspace{1cm} (33)$$

Since the equality between two objects transported along path 1 and path 2 should hold at all orders in power expansion (33), we expect to obtain some information about the two curvatures from the coefficient of $\mu \lambda$. Notice that for products of two objects transported along the two paths the following relation holds:

$$A_1 B_1 - A_2 B_2 |_{\mu \lambda} = A_1^{(0)} (B_1^{(5)} - B_2^{(5)}) + (A_1^{(5)} - A_2^{(5)}) B_0,$$  \hspace{1cm} (34)

where $NS$ subscript denotes the nonsymmetric part with respect to $\nabla_\eta \nabla_\xi$. Equation (34) can be easily generalized to an arbitrary number of terms entering the product.

We conclude this section by noting that the same considerations hold true for the case of fermions transported along path 1 and path 2, with $\nabla$ replaced by $\nabla^S$. For other details the reader should see Sec. V and Appendix B and C of \cite{14}.
III. COMPATIBILITY OF PRIMARY AND SECONDARY CONSTRAINTS WITH SPACETIME GEOMETRY

In this sections we are going to compute the constraints on the curvatures \( R \) and \( R^S \) of the tangent bundle and spinor bundle of the manifold \( M \). We notice that, since the two vector fields \( \xi \) and \( \eta \) commute, the two curvatures can be expressed as:

\[
R(\eta, \xi) = [\nabla_\eta, \nabla_\xi], \quad R^S(\eta, \xi) = [\nabla^S_\eta, \nabla^S_\xi].
\] (35)

Let us see what happens when the first primary (17) is transported along the two paths. The general strategy is to transport each of the fields, e.g. \( e^a_\mu, \psi^\alpha_\mu \), etc. from one point to another and then to reconstruct the whole equation. These terms obey the Eq. (32) with the power expansion truncated up to the second order given by Eq. (33). Then from such of transported objects we form differences like in Eq. (34) which must cancel because we require that the result of the parallel transport be independent of the path. In deducing the corresponding relation in (34) one assumed that the Dirac eigenspinors \( \chi^\alpha \)'s are subject to parallel transport. That implies, of course, that certain simplifications occur.

Consider now that the Dirac eigenspinors transform along path 1 in a general way, i.e.

\[
\chi_1 = e^{\mu} \nabla^S_\eta e^{\nu} \nabla^S_\xi \chi.
\] (36)

and in a similar manner along the path 2. From the power expansion (33) we can see that

\[
\chi_2^{n5} - \chi_1^{n5} = R^S(\eta, \xi) \chi^n.
\] (37)

After some algebra we can extract the coefficient of \( \mu \lambda \) from the power expansion. It can be shown, exactly as in (34), that the rest of the terms vanish. Therefore, we obtain the following equation:

\[
\langle \chi^n \rangle \frac{\delta}{\delta e^\alpha_\mu} (R(\eta, \xi)e^\rho_\alpha) (\partial_\rho + \frac{i}{2} \frac{\partial^{(0)}}{\delta \rho F g} \sigma^f g) - \frac{\delta}{\delta e^\alpha_\mu} [\sum_{\rho, b, c} (R^S(\eta, \xi)(\bar{\psi}_\rho \gamma_b \psi_c) \sigma^{bc} \chi^n) (R^S(\eta, \xi)(\partial_\rho \psi^\alpha_\mu)] - \frac{1}{8} \langle \chi^n \rangle \frac{\delta}{\delta \psi^\alpha_\mu} (R(\eta, \xi)e^\rho_\alpha) (\partial_\rho + \frac{i}{2} \frac{\partial^{(0)}}{\delta \rho F g} \sigma^f g) - \frac{1}{8} \langle \chi^n \rangle \frac{\delta}{\delta \psi^\alpha_\mu} [\sum_{\rho, b, c} (R^S(\eta, \xi)(\bar{\psi}_\rho \gamma_b \psi_c) \sigma^{bc} \chi^n) (R^S(\eta, \xi)(\partial_\rho \psi^\alpha_\mu)] - \frac{1}{8} \langle \chi^n \rangle \frac{\delta}{\delta \psi^\alpha_\mu} \frac{\delta}{\delta \psi^\alpha_\mu} [\sum_{\rho, b, c} (R^S(\eta, \xi)(\bar{\psi}_\rho \gamma_b \psi_c) \sigma^{bc} \chi^n) (R^S(\eta, \xi)(\partial_\rho \psi^\alpha_\mu)] - \frac{1}{8} \langle \chi^n \rangle \frac{\delta}{\delta \psi^\alpha_\mu} \frac{\delta}{\delta \psi^\alpha_\mu} [\sum_{\rho, b, c} (R^S(\eta, \xi)(\bar{\psi}_\rho \gamma_b \psi_c) \sigma^{bc} \chi^n) (R^S(\eta, \xi)(\partial_\rho \psi^\alpha_\mu)] = 0,
\] (38)
Eq. (38) represent the results of transporting the first primary along the two paths together with the condition of vanishing the coefficient of $\mu \lambda$. In a similar manner we can derive the equation resulting from the transport of the second primary which reads, after some lengthy calculations, as follows

\[
\langle (R^S(\eta, \xi)\chi^n|T^a_\mu|\chi^n) + \langle \chi^n|T^a_\mu|R^S(\eta, \xi)\chi^n)\rangle e_{ba} + \\
\langle (R^S(\eta, \xi)\chi^n|\Gamma^a_\mu|\chi^n) + \langle \chi^n|\Gamma^a_\mu|R^S(\eta, \xi)\chi^n)\rangle (\sigma_{ab})^a_\beta \psi^a_\mu + \\
\langle \chi^n|i\gamma^d\delta_{da}\delta^{\mu\nu}(\partial_\nu + \frac{1}{2}\omega^{(0)}_{\mu\rho})\sigma^{fg}\rangle - \\
\frac{1}{8}\gamma_\alpha g^{\mu\nu} \sum_{(\nu,d,c)} [\bar{\psi}_{\nu}\gamma_d\psi_c]\sigma^{dc}|\chi^n|(R(\eta, \xi)e_{ba}) + \langle \chi^n|i\gamma^d\delta_{da}\delta^{\mu\nu}[(R(\eta, \xi)\partial_\nu) + \\
\frac{1}{2}\omega^{(5)}_{\nu fg NS}(R(\eta, \xi))\sigma^{fg}] + i\gamma^d[\frac{\delta}{\delta e^a_\mu}(R(\eta, \xi)e^a_\mu)(\partial_\nu + \frac{1}{2}\omega^{(0)}_{\rho fg})\sigma^{fg}] - \\
\frac{1}{8}\gamma_\alpha g^{\mu\nu} \sum_{(\nu,d,c)} [R^S(\eta, \xi)(\bar{\psi}_{\nu}\gamma_d\psi_c)]\sigma^{dc}|\chi^n|e_{ba} + \\
\langle \chi^n|i\gamma^d e^a_\mu \frac{\delta}{\delta \psi^a_\mu} \frac{1}{2} \sum_{(\rho,t,c)} [R^S(\eta, \xi)(\bar{\psi}_{\rho}\gamma_t\psi_c)] + \delta_{\mu\nu}^a \frac{\delta}{\delta e^a_\mu}(R(\eta, \xi)e^a_\mu) + \\
\frac{1}{2}\frac{\delta}{\delta \psi^a_\mu} (R^S(\eta, \xi)\psi^a_\nu)[\psi^a_\nu] \sum_{(\rho,t,c)} [\bar{\psi}_{\rho}\gamma_t\psi_c])\sigma^{de}|\chi^n|\sigma_{ab})_\gamma \gamma^c_\mu = 0. \tag{39}
\]

If we consider now the third primary and transport it along the two paths, we obtain the following result

\[
\langle (R^S(\eta, \xi)\chi^n|T^a_\mu|\chi^n) + \langle \chi^n|T^a_\mu|R^S(\eta, \xi)\chi^n)\rangle e_{ba} + \\
\langle (R^S(\eta, \xi)\chi^n|\Gamma^a_\mu|\chi^n) + \langle \chi^n|\Gamma^a_\mu|R^S(\eta, \xi)\chi^n)\rangle D_{\mu}^a e^a + \\
\langle \chi^n|i\gamma^d\delta_{da}\delta^{\mu\nu}(\partial_\nu + \frac{1}{2}\omega^{(0)}_{\mu\rho})\sigma^{fg}\rangle - \\
\frac{1}{8}\gamma_\alpha g^{\mu\nu} \sum_{(\nu,b,c)} [\bar{\psi}_{\nu}\gamma_b\psi_c]\sigma^{bc}|\chi^n|(R^S(\eta, \xi)\bar{\psi}_{\nu}\gamma_b\psi_c) + \epsilon_{\gamma}^a(\epsilon_{\gamma}^a(R^S(\eta, \xi)\psi_{\mu})) + \\
\langle \chi^n|i\gamma^d\delta_{da}\delta^{\mu\nu}[(R(\eta, \xi)\partial_\nu) + \frac{1}{2}\omega^{(5)}_{\nu fg NS}(R(\eta, \xi))\sigma^{fg}] + i\gamma^d[\frac{\delta}{\delta e^a_\mu}(R(\eta, \xi)e^a_\mu)(\partial_\nu + \frac{1}{2}\omega^{(0)}_{\rho fg})\sigma^{fg}] - \\
\frac{1}{8}\gamma_\alpha g^{\mu\nu} \sum_{(\nu,b,c)} [\bar{\psi}_{\nu}\gamma_b(R^S(\eta, \xi)\psi_c) + (R^S(\eta, \xi)\bar{\psi}_{\nu})\gamma_b\psi_c]\sigma^{bc}|\chi^n|\epsilon_{\gamma}^a\psi_{\mu} + \\
\frac{1}{8}(\chi^n|i\gamma^a e^a_\mu \frac{\delta}{\delta \psi^a_\mu} \sum_{(\rho,b,c)} [\bar{\psi}_{\rho}\gamma_b\psi_c])\sigma^{bc}|\chi^n|(R(\eta, \xi)\partial_\mu)e^a + \\
\]

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\[ \partial_\mu (R^S(\eta, \xi) e^\alpha) + \frac{1}{2} [\tilde{\omega}_{\mu abNS}(R(\eta, \xi)) [\sigma^{a\beta}]_\beta e^\beta + \omega_{\mu ab}^{(0)} [\sigma^{a\beta}]_\beta (R^S(\eta, \xi) e^\beta)] + \frac{1}{8} \sum_{(\mu,b,c)} [\bar{\psi}_\mu \gamma_\nu \psi_\nu] [\sigma^{bc}]_\beta (R^S(\eta, \xi) e^\beta) + \sum_{(\mu,b,c)} [\bar{\psi}_\mu \gamma_\nu (R^S(\eta, \xi) \psi_\nu) + (R^S(\eta, \xi) \bar{\psi}_\mu) \gamma_\nu \psi_\nu] [\sigma^{bc}]_\beta e^\beta + \chi^n i \gamma^a \{ e^\rho_a \frac{\delta}{\delta \psi_\rho^a} [\frac{1}{8} \sum_{(\rho,b,c)} [(R^S(\eta, \xi) \bar{\psi}_\rho) \gamma_\nu \psi_\nu] + [\delta^\nu_\rho \delta^\beta_\alpha (R(\eta, \xi) e^\rho_a)] + \frac{\delta}{\delta \psi_\rho^a} (R^S(\eta, \xi) \psi_\rho^a e^\rho_a) \times \frac{\delta}{\delta \psi_\rho^a} [\bar{\psi}_\rho \gamma_\nu \psi_\nu] \} [\sigma^{bc}]_\beta \chi^n] D_\mu e^\alpha = 0. \tag{40} \]

Equations (38), (39) and (40) represent the constraints imposed on the two curvatures \( R(\eta, \xi) \) and \( R^S(\eta, \xi) \) of the tangent bundle and of the spinor bundle, respectively. They are nonlinear equations and, as in the particular case previously studied in \( 14 \), they admit as solutions manifolds with both curvatures vanishing

\[ R(\eta, \xi) = R^S(\eta, \xi) = 0. \tag{41} \]

Let us notice that in the case when Dirac eigenspinors are subject to the parallel transport, these equations reduce to the ones in \( 14 \). However, the fact that Dirac eigenspinors are no longer subject to the parallel transport has some nontrivial consequences which is in place to be discussed here.

Let us assume that even if the Dirac eigenspinors change when they are transported along the two paths, the result of the transport is still a Dirac eigenspinor at the new point. In other words, we assume that Dirac eigenspinors do not drop out the set of eigenspinors after parallel transport from one point to another, which, by itself, is not an obvious thing in the general case because it would imply that the Dirac operator defines some global sections of the spinor bundle on \( M \) in general. Let us now consider eq. (33) transported along the two paths and apply eq. (34) for each term. We obtain the following equation

\[ D(R^S(\eta, \xi) \chi^n) - \chi^n (R^S(\eta, \xi) \chi^n) + i \gamma^a \{(R(\eta, \xi) e^\rho_a \partial_\mu) + (e^\mu_a \omega_{\mu bc}^{(5)} (R(\eta, \xi)) + (R(\eta, \xi) e^\mu_a) \omega_{\mu bc} + (e^\mu_a K_{\mu bc}^{(5)} (R^S(\eta, \xi)) + (R(\eta, \xi) e^\mu_a K_{\mu bc}) [\sigma^{bc}]_\beta \chi^n \} = 0, \tag{42} \]

where

\[ \omega_{\nu bc1NS} = (\nabla_\eta \nabla_\xi e^\mu_a) \partial_\nu e^\eta_{b\mu} + \cdots + \frac{1}{2} (\nabla_\eta \nabla_\xi e^\rho_a) e^\sigma_b \partial_\sigma e^\rho_c + \frac{1}{2} (\nabla_\xi \nabla_\eta e^\rho_a) e^\eta_b \partial_\sigma e^\rho_c + \cdots \]

\[ K_{\nu ab1NS} = \frac{1}{8} \sum_{(\nu,b,c)} [\bar{\psi}_\nu \gamma_\beta (\nabla_\eta \nabla_\xi \psi_\nu^S) + (\nabla_\eta \nabla_\xi \bar{\psi}_\nu) \gamma_\beta \psi_\nu]. \tag{43} \]

Here, \( \partial_\nu e^\eta_{b\mu} \) refers to antisymmetrization with respect to the indices \( \nu \) and \( \mu \) only and the sums on fermion products mean

\[ \sum_{(\nu,b,c)} [A_{\nu} \gamma_b B_c] = A_{\nu} \gamma_b B_c - A_{\nu} \gamma_c B_b + A_b \gamma_\nu B_c. \tag{44} \]
Also, the index $NS$ indicates the nonsymmetric part in $\nabla_\eta$ and $\nabla_\xi$ and similarly for spinors with the appropriate spin connection.

Equation (12) expresses a relationship between the geometrical structure of $M$, represented by the two curvatures, and the eigenspinors of $D$. It represents a particular case of the previous assumption which in its most general form should state that an eigenspinor which does not leave the set of eigenspinors, can become a linear combination of them. In this case we should write

$$\chi^n \xrightarrow{i} \chi^i_1 = \sum_k e^{n}_{k1} \chi^i_k$$

(45)

where $i = 1, 2$ is the index for the two paths and $e^{n}_{k1}$ are real coefficients which may differ for the two paths. The equation that describes this situation is

$$\sum_k \left((e^{\mu} \bar{\nabla}_\eta e^{\lambda} \nabla_\xi D)(e^{\mu} \nabla_\eta e^{\lambda} \nabla_\xi \chi^k)\right) e^m_{2k} - \left((e^{\mu} \bar{\nabla}_\eta e^{\lambda} \nabla_\xi D)(e^{\mu} \nabla_\eta e^{\lambda} \nabla_\xi \chi^k)\right) e^m_{1k}$$

$$= \lambda^n \sum_k \left( e^{n}_{2k} (e^{\mu} \nabla_\eta e^{\lambda} \nabla_\xi - e^{n}_{1k} (e^{\mu} \nabla_\eta e^{\lambda} \nabla_\xi) \chi^k), \right)$$

(46)

where $\bar{\nabla}$ stands for either $\nabla$ or $\nabla^S$ according to the type of the object entering $D$ to which is applied.

Notice that in Eq. (46) the terms in different powers of $\mu$ and $\lambda$ are present and the situation remains the same for higher order terms. That shows that we cannot obtain a relationship among the two curvatures and the eigenspinors only, because the connections in the tangent bundle and the spinor bundle should be constrained, too.

Let us go on and see what happens when the secondaries are transported along path 1 and path 2. In this case we should compute the transformation of the terms entering (22), (26) and (29). We should also keep in mind that $\chi^n$ transforms, too.

In order to compute the constraint on the spacetime manifold $M$ that arises from the first secondary, it is worth noting that (22) already encodes the behaviour of the eigenspinors under a diffeomorphism. Therefore we should consider an arbitrary infinitesimal vector field $\zeta$ defined on $M$ with respect to which a general diffeomorphism is defined. Then $\zeta$ replaces $\xi$ in (22). Now since $\zeta$ is defined everywhere on $M$, we can consider the diffeomorphisms generated by $\zeta$ around the points $Q, P, R, S$. Since, in general, $\zeta$ does not have to be subject under the parallel transport, it changes when it is transported along path 1 and along path 2 and this change should be taken into account when the first secondary is transported along the two paths. After performing some tedious algebra we obtain the following result

$$i \gamma^a \left\{ [R(\eta, \xi)] [e^a_q + \zeta [R(\eta, \xi)] e^a_q] - R(\eta, \xi) (e^a_q \partial_a \zeta^\mu) - \right.$$  

$$2[R(\eta, \xi) (e^a_q \zeta^\nu \omega_{\nu bc})] \partial_b \chi^n +$$

$$+ b^\mu(\zeta) [R(\eta, \xi) \partial_a] \chi^n + b(\zeta) [R^S(\eta, \xi) \chi^n] +$$

$$+ [R(\eta, \xi) (\lambda x^n)] - D[R(\eta, \xi) \chi^n] -$$

$$- i \gamma^a [R(\eta, \xi) (e_a) + (R(\eta, \xi) e^a_q) \omega_{\nu bc} \sigma^b] +$$

$$+ [e^a_q (\omega_{\nu bc} (R(\eta, \xi)) + K_{\nu bc} (R^S(\eta, \xi)))] \partial_b \chi^n +$$

$$+ c(\lambda, \zeta) ^\mu (R(\eta, \xi) \partial_a] \chi^n + c(\lambda, \zeta) ^\mu \partial_a [R^S(\eta, \xi) \chi^n] +$$

$$+ i \gamma^a \left\{ [R(\eta, \xi)] [e^a_q \omega_{\nu bc}] + \zeta [R(\eta, \xi) (e^a_q \omega_{\nu bc})] \sigma^b \right\} \chi^n +$$

$$+ f(\zeta) [R^S(\eta, \xi) \chi^n] = 0,$$

(47)
where $\omega_{\mu bc} = \hat{\omega}_{\mu bc} + K_{\mu bc}$.

Proceeding along the same line as in the previous case, after somewhat simpler computations, we are going to compute now the transport of the second secondary. Here, we must pay attention to some other subleties. Namely, when $[20]$ is transported along any of the two paths, the parameter $\theta_{ab} = -\theta_{ba}$ can change, because this is the parameter of the local SO(4) group. Similarly, $M_{ab}$ can in principle change, too, to another SO(4) matrix. We can make the assumption which do not affect the generality of the result, that $M_{ab}$ belongs to a basis of SO(4) and that this basis do not change. Then we can write:

$$M_{ab2} - M_{ab1} = \sum_k (d_{k2} - d_{k1}) M_{ab},$$

where the sum is over all of the elements of the basis of SO(4). Then performing the same steps as in the case of the first secondary we obtain the following result:

$$\left(R(\eta, \xi) \theta^a_0\right)D \chi^n + \theta^a_0 D\left(R^S(\eta, \xi) \chi^n\right) +$$

$$i \theta^a_0 \gamma^d \left[\left(R(\eta, \xi) e_a\right) + \left(\omega_{\mu bc}^{(5)}(R(\eta, \xi)) + K_{\nu bc}^{(5)}(R^S(\eta, \xi))\right)\sigma^{bc}\right] \chi^n +$$

$$\left\{\gamma^e\left[R(\eta, \xi)e^\mu_c\left[\left[\theta\sigma, \omega_{\mu de}\right]\sigma^{de}\right] + \gamma^e e^\mu_c\left[\left[R(\eta, \xi)\theta\sigma, \omega_{\mu de}\right]\sigma^{de}\right] - \right.\right.$$

$$\gamma^e \left\{\sum_{k=1}^4 d_{2k} e^{\nu_b \eta \lambda \nu_d} \left(d_{1k} e^{\lambda \nu_b \eta \nu_d}\right) \left(\partial_\mu \theta \sigma M^j\right)\right\} + g(\theta) \left[R^S(\eta, \xi) \chi^n\right] +$$

$$i \left\{\left[R(\eta, \xi)\left(\lambda^n \theta\sigma\right) - D\left[R(\eta, \xi)\theta\sigma\right] - i \gamma^d \left(R(\eta, \xi)e_a\right) + \left(R(\eta, \xi)e_a^\mu\right)\omega_{\nu bc} + e_a^\nu \left(\omega_{\nu bc}(R(\eta, \xi)) + K_{\nu bc}(R^S(\eta, \xi))\right)\sigma^{bc}\theta\right] \chi^n +$$

$$i(\lambda^n - D)\left[R(\eta, \xi)\theta\sigma\right] \chi^n + h(\theta) \left[R^S(\eta, \xi) \chi^n\right] = 0.$$

We can now work out the constraint that arises from the transport of the third secondary. Proceeding along the same line as in the previous case, after somewhat simpler computations, we obtain the following equation

$$\frac{1}{2} \gamma^a \left[R^S(\eta, \xi)(\bar{e} \psi^{\mu})\right]\partial_\mu \chi^n + j^a_\mu \left[R(\eta, \xi)\partial_\mu\right] \chi^n + j^a_\mu \partial_\mu \left[R^S(\eta, \xi) \chi^n\right] +$$

$$\frac{1}{2} \gamma^a \left[R^S(\eta, \xi)(\bar{e} \psi^{\mu})\right]\omega_{\mu cd} \sigma^{cd} \chi^n + \frac{1}{2} \gamma^a \left[R^S(\eta, \xi)(\bar{e} \psi^{\mu})\right]\omega_{\mu cd} \sigma^{cd} \chi^n +$$

$$K_a \left[R^S(\eta, \xi) \chi^n\right] + \left\{\left[R(\eta, \xi)e^\mu_a\right]A_{\mu cd} \sigma^{cd} + e^\mu_a \left[\left[R(\eta, \xi)(e_\sigma e_\rho e_\sigma e_\rho)\right]A_{\rho \sigma}^\mu + \right.\right.$$

$$\left.\left(e_\rho e_\sigma e_\rho e_\sigma\right)\left\{\left[R^S(\eta, \xi)e\right] R\gamma_5 f D_\lambda \psi_\rho e^\rho \lambda \theta + \right.\right.$$

$$\left.\bar{\psi}\gamma_5 f \left\{\left[R(\eta, \xi)\partial_\lambda\psi_\rho + \partial_\lambda \left[R^S(\eta, \xi) \psi_\rho\right] + \right.\right.$$

$$\frac{1}{2} \left(\omega_{\lambda mn}^{(5)}(R(\eta, \xi))\sigma^{mn} \psi_\theta + \omega_{\lambda mn}^{(0)}(R^S(\eta, \xi) \psi_\theta)\right) +$$

$$\frac{1}{8} \sum_{(\lambda, m, n)} \bar{\psi}\gamma_\lambda \gamma_\mu \psi_\nu \sigma^{mn} \left[R^S(\eta, \xi) \psi_\theta\right] + \sum_{(\lambda, m, n)} \left[R^S(\eta, \xi) \left(\bar{\psi}\gamma_\mu \psi_\nu\right)\sigma^{mn} \psi_\theta\right] e^{\rho \lambda \theta} \sigma^{cd} -$$

$$\frac{1}{2} \left[R(\eta, \xi) e_{\mu d}\right] A^e_{\mu c} \sigma^{cd} - \frac{1}{2} e_{\mu d} \left[\left[R(\eta, \xi)(e_\sigma e_\rho e_\chi)\right] A_{\rho \sigma}^e \right.\right.$$

$$e_\rho e_\sigma e_\chi \left[\left[R^S(\eta, \xi)e\right] R\gamma_5 e D_\lambda \psi_\rho e^\rho \lambda \theta + \bar{\psi}\gamma_5 e \left\{\left[R(\eta, \xi)\partial_\lambda\psi_\rho + \partial_\lambda \left[R^S(\eta, \xi) \psi_\rho\right] + \right.\right.$$

$$\frac{1}{2} \left(\omega_{\lambda mn}^{(5)}(R(\eta, \xi))\sigma^{mn} \psi_\theta + \omega_{\lambda mn}^{(0)}(R^S(\eta, \xi) \psi_\theta)\right) +$$

$$\frac{1}{8} \sum_{(\lambda, m, n)} \bar{\psi}\gamma_\lambda \gamma_\mu \psi_\nu \sigma^{mn} \left[R^S(\eta, \xi) \psi_\theta\right] +$$

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\[
\sum_{(\lambda,m,n)} \left[ R^S(\eta,\xi)(\bar{\psi}_\lambda \gamma_m \psi_n)] \sigma^{mn} \psi_\mu \right] e^{\lambda \xi \mu} \right\} \sigma^{cd} + \\
\frac{1}{2}(- \to +, d \to c) \chi^n + +l_a[R^S(\eta,\xi)\chi^n] = 0, 
\]

where \(\frac{1}{2}(- \to +, d \to c)\) means that the terms with \(\frac{1}{2}\) change sign and the indices \(d\) and \(c\) get interchanged in these terms. Some remarks regarding our notations are in order here. For all of the equations (47), (49) and (50) \(R\) acts on the product \(AB\) according to the Leibniz rule. The same for \(R^S\), with no change of sign.

\[\omega^{(5)}_{\mu bc}(R(\eta,\xi))\] and \(K^{(5)}_{\mu bc}(R^S(\eta,\xi))\) mean that only the nonsymmetric part of \(\omega^{(5)}_{\mu bc}\) and \(K^{(5)}_{\mu bc}\) in \(\nabla_\eta \nabla_\xi\) are considered and that these parts, by subtracting the expression for the first path from the expression for the second path, depend on the corresponding curvatures. \(\theta \sigma\) stands for \(\theta_{ab} \sigma^{ab}\) and \(M^k = M^k_{cd} \sigma^{cd}\).

We cannot comment here on the general solution of equations (38), (39), (40), (46), (47), (49) and (50). These equations are nonlinear and highly nontrivial. We see that apparently they do not admit as a solution the spacetime manifolds with both curvatures of the tangent bundle and of the spinor bundle vanishing, as was the case studied in [14]. The main obstructions for that seem to be the equations (46) and (49). Should we have imposed that "totally flat" spacetime satisfy all of the equations, we would have easily checked out that (46) and (49) reduce to the following equations

\[
\sum_k \left(c^n_{2k} - c^n_{1k}\right)(e^{\mu \nabla_\eta} e^{\lambda \nabla_\xi} D)(e^{\mu \nabla_\eta} e^{\lambda \nabla_\xi} \chi^k) = \\
\quad = \chi^n \sum_k \left(c^n_{2k} - c^n_{1k}\right)(e^{\mu \nabla_\eta} e^{\lambda \nabla_\xi} \chi^k) = \lambda^n \sum_k \left(c^n_{2k} - c^n_{1k}\right)(e^{\mu \nabla_\eta} e^{\lambda \nabla_\xi} \chi^k) 
\]

and

\[
\gamma^c \sum_{\alpha=1}^4 \left[(d_{2\alpha} - d_{1\alpha})e^{\mu \nabla_\eta} e^{\lambda \nabla_\xi} (e^\mu_{c} \partial_\mu \theta \sigma M^\alpha)\right]\chi^n = 0. 
\]

Thus we can see that even if the two curvatures vanish, there remain two equations that must be satisfied by \(\chi^k\)'s and the two connections. They are automatically satisfied for \(c^n_{2k} = c^n_{1k}\) and \(d_{2\alpha} = d_{1\alpha}\), that is when the gauge transformations do not mix either the eigenspinors or the matrices \(M\). The first assumption is almost trivial since we do not expect that diffeomorphisms, local rotations or local supersymmetry exchange the spinors corresponding to different Dirac eigenvalues, e.g. to different masses. The second assumption is again natural once we do not expect that \(SO(4)\) matrices be affected by diffeomorphisms, i.e. by the parallel transport.

The conclusion is that, with these two natural assumptions at hand, the manifolds with vanishing tangent curvature and spinor curvature, admit Dirac eigenvalues as observables in the most general case.

IV. SUMMARY AND CONCLUDING REMARKS

In this paper we have derived the most general set of constraints on the curvature of tangent bundle and the curvature of spinor bundle, respectively, under which the spacetime
manifold admits Dirac eigenvalues as observables of Euclidean supergravity. A solution of these constraints has been shown to be given by spacetime manifolds with both curvatures vanishing. If Dirac eigenspinors remain in the set of eigenspinors after being subject to parallel transport, the connections in the tangent bundle are constrained, too.

The equations derived above are simpler if the spacetime has a vanishing bosonic torsion. In this case, owing to Ricci identity, the covariant derivatives of the functions on $M$ that appear in these equations, as $\theta_{ab}$, commute, but some attention should be paid to the components of different tensor and spinor objects. However, it is not obvious at this stage that the bosonic torsion should vanish. Usually, the constraints on the supertorsion appear in a natural way in the superfield approaches to supergravity, when some symmetries are imposed on the action. However, this is not the case here and finding the constraints on the torsions of $\nabla$ and $\nabla^S$ is an interesting problem.

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