OBSERVATIONS REGARDING THE REPETITION OF THE LAST DIGITS OF A TETRATION OF GENERIC BASE

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Abstract. This paper investigates the behavior of the last digits of a tetration \[1\] of generic base. In fact, last digits of a tetration are the same starting from a certain hyper-exponent and in order to compute them we reduce those expressions \(\mod 10^n\). Very surprisingly (although unproved) I think that the repeating digits depend on the residue \(\mod 10\) of the base and on the exponents of a particular way to express that base. Then I’ll discuss about the results and I’ll show different tables and examples in order to support my conjecture.

1. Definitions

Definition (Tetration). In mathematics, tetration is an operation based on iterated, or repeated, exponentiation. It is the next hyperoperation after exponentiation, but before pentation. The word was coined by Reuben Louis Goodstein from tetra- (four) and iteration. \(a\) \(n\) represent the \(a\)-th tetration of \(n\), or:

\[ n^{n^{n^{\ldots}}a \text{ times}} \]

Definition (Floor and Ceiling function). \[2\] In mathematics and computer science, the floor function is the function that takes as input a real number \(x\), and gives as output the greatest integer less than or equal to \(x\), denoted \([x]\). Similarly, the ceiling function maps \(x\) to the least integer greater than or equal to \(x\), denoted \(\lceil x \rceil\).

2. Main Conjecture

Let \(f_q(x, y, n)\) be a function such that if:

\[ f_q(x, y, n) = u \]

Then:

\[ \propto \left[q^{(2^x\cdot 5^y)\cdot a}\right] \equiv u \left[q^{(2^x\cdot 5^y)\cdot a}\right] \mod (10^n) \]

where \(x, y, q, a \in \mathbb{N}\), \(q \neq 10h\), \(a \neq 2h\) and \(a \neq 5h\) and \(u\) is the minimum value such that this congruence is true.

**Note that these formulas work if** \(x \geq 2\)

At the end of this paper there will be a section for the \(x < 2\) case.

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2.1. $q \equiv 1, 9 \mod 10$. I define $\Delta_2$ and $\Delta_5$ as:
\[
\Delta_2 = \max[v_2(q + 1), v_2(q - 1)] \\
\Delta_5 = \max[v_5(q + 1), v_5(q - 1)]
\]
We’ll have that:
\[
f_{q \equiv 1, 9 \mod 10}(x, y, n) = \max \left[ \left\lceil \frac{n}{x + \Delta_2} \right\rceil, \left\lceil \frac{n}{y + \Delta_5} \right\rceil \right] - 1
\]

2.2. $q \equiv 3, 7 \mod 10$. I define $\Gamma_2$ and $\Gamma_5$ as:
\[
\Gamma_2 = \max[v_2(q + 1), v_2(q - 1)] \\
\Gamma_5 = \max[v_5(q + 1), v_5(q^2 - 1)]
\]
We’ll have that:
\[
f_{q \equiv 3, 7 \mod 10}(x, y, n) = \max \left[ \left\lceil \frac{n}{x + \Gamma_2} \right\rceil, \left\lceil \frac{n}{y + \Gamma_5} \right\rceil \right] - 1
\]

2.3. $q \equiv 5 \mod 10$. $q = 10h + 5$ for some $h \in \mathbb{N}$. It is also possible to write it as $q = 5^k h$ for some $h, k \in \mathbb{N}$. As we know from my previous paper, if the base is a power of 5, the last repeating digits only depend on the 2-adic valuation of the base’s exponent. This is right not only for the power of base 5 but for all multiples of 5. We can write the base as $5^k h$, for some $h, k \in \mathbb{N}$, $k \neq 2a \land h \neq 5b$. So we’ll have:
\[
f_{q \equiv 5 \mod 10}(x, y, n) = \left\lceil \frac{n}{x + \Delta_2} \right\rceil - 1
\]
Note that here $y$ could be every integer number, but the result depends on $x$.

2.4. $q \equiv 0 \mod 2$. $q = 10h + 2$ for some $h \in \mathbb{N}$. It is also possible to write it as $q = 2^k h$ for some $h, k \in \mathbb{N}$. As we know from my previous paper, if the base is a power of 2, the last repeating digits only depend on the 5-adic valuation of the base’s exponent. This is right not only for the power of base 2 but for all multiples of 2. We can write the base as $2^k h$, for some $h, k \in \mathbb{N}$, $k \neq 5a \land h \neq 2b$. So we’ll have:
\[
f_{q \equiv 0 \mod 2}(x, y, n) = \left\lceil \frac{n}{y + \Gamma_5} \right\rceil - 1
\]
Note that here $x$ could be every integer number, but the result depends on $y$.

3. Introduction

When you hear talking about tetrations, you instantly think about a giant number, which is very difficult to compute. One of the first things you could notice when you are studying them is that the last digits of every tetration with a positive integer base begins to repeat after a certain hyper-exponent.

4. Examples and other observations

First of all I would like to mention that the $q \equiv 1, 3, 7, 9 \mod 10$ case is now divided in 2 sub-cases, namely: $q \equiv 1, 9 \mod 10$ and $q \equiv 3, 7 \mod 10$. Then I understood that they can not be grouped together. Furthermore, initially I thought that the last repeating digits were depending on the prime factorization of the base $q$, but then I understood that they only depend on the residue $\mod 10$ of that base and on the relationship of the discriminants $\Delta$ and $\Gamma$. Anyway at the end of this paper it is possible to find a table of some values of $f_p(x, y, n)$ where $p$ is a prime number.
4.1. **Example 1.** Consider the infinite tetration of $4599^{2^8 \cdot 5^4}$, or $4599^{800000}$. We know from our first formula that the last 40 digits are the same starting from the 5-th tetration of that number. Indeed, $4599 \equiv 9 \mod 10$ and $\left\lfloor \frac{n}{9+\Delta_5} \right\rfloor \geq \left\lfloor \frac{n}{x+\Delta_2} \right\rfloor$.

In fact:

\[
\Delta_2 = \max[v_2(4599 + 1), v_2(4599 - 1)] = \max[3, 1] = 3
\]

\[
\Delta_5 = \max[v_5(4599 + 1), v_5(4599 - 1)] = \max[2, 0] = 2
\]

And:

\[
\left\lfloor \frac{40}{5+2} \right\rfloor \geq \left\lfloor \frac{40}{8+3} \right\rfloor
\]

So we’ll have that:

\[
f_{4599}(8, 5, 40) = \left\lfloor \frac{40}{5+\max[v_5(4600), v_5(4598)]} \right\rfloor - 1
\]

\[
f_{4599}(8, 5, 40) = \left\lfloor \frac{40}{5+\max[2, 1]} \right\rfloor - 1
\]

\[
f_{4599}(8, 5, 40) = \left\lfloor \frac{40}{5 + 2} \right\rfloor - 1
\]

\[
f_{4599}(8, 5, 40) = 5
\]

In fact:

\[
5 \left[ 4599^{(2^8 \cdot 5^4)} \right] \equiv 574081590929428693334403581932320000001 \mod (10^{40})
\]

And also:

\[
6 \left[ 4599^{(2^8 \cdot 5^4)} \right] \equiv 574081590929428693334403581932320000001 \mod (10^{40})
\]

And so on for every hyper-exponent greater or equal to 5. So 5 is minimum, in fact if you consider the fourth tetration of that base, it doesn’t work.

\[
4 \left[ 4599^{(2^8 \cdot 5^4)} \right] \equiv 3530881590929428693334403581932320000001 \mod (10^{40})
\]

4.2. **Example 2.** Consider the infinite tetration of $1251^{2^2 \cdot 5^4}$, or $1251^{2500}$. We know from our first formula that the last 30 digits are the same starting from the 7-th tetration of that number. Indeed, $1251 \equiv 1 \mod 10$ and $\left\lfloor \frac{n}{x+\Delta_2} \right\rfloor \geq \left\lfloor \frac{n}{9+\Delta_5} \right\rfloor$.

In fact:

\[
\Delta_2 = \max[v_2(1251 + 1), v_2(1251 - 1)] = \max[2, 1] = 2
\]

\[
\Delta_5 = \max[v_5(1251 + 1), v_5(1251 - 1)] = \max[0, 4] = 4
\]

And:

\[
\left\lfloor \frac{30}{2+2} \right\rfloor \geq \left\lfloor \frac{30}{4+4} \right\rfloor
\]

So we’ll have that:

\[
f_{1251}(2, 4, 30) = \left\lfloor \frac{30}{2+\max[v_2(1252), v_2(1250)]} \right\rfloor - 1
\]

\[
f_{1251}(2, 4, 30) = \left\lfloor \frac{30}{2 + 2} \right\rfloor - 1 = 7
\]

In fact:

\[
7 \left[ 1251^{(2^2 \cdot 5^4)} \right] \equiv 297934155568039465330081250001 \mod (10^{30})
\]

And also:

\[
8 \left[ 1251^{(2^2 \cdot 5^4)} \right] \equiv 297934155568039465330081250001 \mod (10^{30})
\]
And so on for every hyper-exponent greater or equal to 7. So 7 is minimum, in fact if you consider the sixth tetration of that base, it doesn’t work.

\[
\left[ 1251^{(2^5 \cdot 5^3)} \right] \equiv 479341555680394653300812500001 \mod (10^{30})
\]

4.3. **Example 3.** Consider the infinite tetration of \(17^{2^3 \cdot 5^6}\), or \(17^{125000}\). We know from our second formula that the last 53 digits are the same starting from the 7-th tetration of that number. Indeed, \(17 \equiv 7 \mod 10\) and \(\left\lceil \frac{n}{7} \right\rceil \geq \left\lceil \frac{n}{9} \right\rceil\).

In fact:

\[
\Gamma_2 = \max\{v_2(17 + 1), v_2(17 - 1)\} = \max\{1, 4\} = 4
\]

\[
\Gamma_5 = v_5(17^2 + 1) = 1
\]

And:

\[
\left\lceil \frac{53}{6 + 1} \right\rceil \geq \left\lceil \frac{53}{3 + 4} \right\rceil
\]

So we’ll have that:

\[
f_{17}(3, 6, 53) = \left\lceil \frac{53}{6 + v_5(290)} \right\rceil - 1
\]

\[
f_{17}(3, 6, 53) = \left\lceil \frac{53}{7} \right\rceil - 1 = 7
\]

So we’ll have that:

\[
7\left[ 17^{(2^3 \cdot 5^6)} \right] \equiv 52737008157199929548933683973150858896289457010000001 \mod (10^{53})
\]

And so on for every hyper-exponent greater or equal to 7.

4.4. **Example 4.** Consider the infinite tetration of \(63^{2^3 \cdot 5^7 \cdot 3}\), or \(63^{2400}\). We know from our second formula that the last 15 digits are the same starting from the 4-th tetration of that number. Indeed, \(63 \equiv 3 \mod 10\) and \(\left\lceil \frac{n}{9} \right\rceil \geq \left\lceil \frac{n}{7} \right\rceil\).

In fact:

\[
\Gamma_2 = \max\{v_2(63 + 1), v_2(63 - 1)\} = \max\{6, 1\} = 6
\]

\[
\Gamma_5 = v_5(63^2 + 1) = 1
\]

And:

\[
\left\lceil \frac{15}{2 + 1} \right\rceil \geq \left\lceil \frac{15}{5 + 6} \right\rceil
\]

So we’ll have that:

\[
f_{63}(5, 2, 15) = \left\lceil \frac{15}{2 + v_5(3970)} \right\rceil - 1
\]

\[
f_{63}(5, 2, 15) = \left\lceil \frac{15}{3} \right\rceil - 1 = 4
\]

So we’ll have that:

\[
4\left[ 63^{(2^1 \cdot 5^2 \cdot 3)} \right] \equiv 547909642496001 \mod (10^{15})
\]

And so on for every hyper-exponent greater or equal to 4.
4.5. **Example 5.** Consider the infinite tetration of $255^{2^4 \cdot 5^3}$, or $255^{2000}$. We know from our third formula that the last 34 digits are the same starting from the 3-th tetration of that number. Indeed, $255 \equiv 5 \mod 10$.

So we’ll have that:

$$f_{255}(4, 3, 34) = \left\lceil \frac{34}{3 + \max[v_2(256), v_2(254)]]} \right\rceil - 1$$

$$f_{255}(4, 3, 34) = \left\lceil \frac{34}{11} \right\rceil - 1 = 3$$

So:

$$3\left[255^{(2^4 \cdot 5^3)}\right] \equiv 6154363253735937178134918212890625 \mod (10^{34})$$

And so on for every hyper-exponent greater or equal to 3.

4.6. **Example 6.** Consider the infinite tetration of $192^{2^2 \cdot 5^3}$, or $192^{500}$. We know from our fourth formula that the last 20 digits are the same starting from the 4-th tetration of that number. Indeed, $192 \equiv 0 \mod 2$ and:

$$\Gamma_5 = \max[v_5(192^2 + 1), v_5(192^2 - 1)] = \max[1, 0] = 1$$

So we’ll have that:

$$f_{192}(2, 3, 20) = \left\lceil \frac{20}{3 + 1} \right\rceil - 1$$

$$f_{192}(2, 3, 20) = \left\lceil \frac{20}{4} \right\rceil - 1 = 4$$

So:

$$4\left[255^{(2^2 \cdot 5^3)}\right] \equiv 14517958004101349376 \mod (10^{20})$$

And so on for every hyper-exponent greater or equal to 4.

5. **$x < 2$ Case**

I think that our formula in order to work must respect the condition: $x \geq 2$. In fact if $x = 0$ or $x = 1$, some of the results seem to be different from the main formula.

5.1. **When $q \equiv 1, 9 \mod 10$.** We are now considering the case where the base $q$ is a number congruent to $1, 9 \mod 10$. Consider a function $f_q(n)$ such that if:

$$f_{q \equiv 1, 9 \mod 10}(x, 0, n) = u$$

Then:

$$q^x \equiv u \left[q^{2^x}\right] \mod (10^n)$$
When $q \equiv 1 \mod 50$ and when $q \equiv 49 \mod 50$:

| $q \equiv 1 \mod 50$ | $f_q(0, 0, n)$ | $f_q(1, 0, n)$ | $q \equiv 49 \mod 50$ | $f_q(0, 0, n)$ | $f_q(1, 0, n)$ |
|----------------------|----------------|----------------|----------------------|----------------|----------------|
| 1                    | 0              | 1              | 9                    | 0              | 1              |
| 11                   | $n - 1$        | $n - 1$        | 19                   | $n$            | $n - 1$        |
| 21                   | $n - 1$        | $n - 1$        | 29                   | $n$            | $n - 1$        |
| 31                   | $n - 1$        | $n - 1$        | 39                   | $n$            | $n - 1$        |
| 41                   | $n - 1$        | $n - 1$        | 49                   | $\left\lceil \frac{n}{4} \right\rceil - 1$ | $\left\lceil \frac{n}{4} \right\rceil - 1$ |
| 51                   | $\left\lceil \frac{n}{5} \right\rceil - 1$ | $\left\lceil \frac{n}{5} \right\rceil - 1$ | 59                   | $n$            | $n - 1$        |
| 61                   | $n - 1$        | $n - 1$        | 69                   | $n$            | $n - 1$        |
| 71                   | $n - 1$        | $n - 1$        | 79                   | $n$            | $n - 1$        |
| 81                   | $n - 1$        | $n - 1$        | 89                   | $n$            | $n - 1$        |
| 91                   | $n - 1$        | $n - 1$        | 99                   | $\left\lceil \frac{n}{5} \right\rceil - 1$ | $\left\lceil \frac{n}{5} \right\rceil - 1$ |
| 101                  | $\left\lceil \frac{n}{10} \right\rceil - 1$ | $\left\lceil \frac{n}{10} \right\rceil - 1$ | 109                  | $n$            | $n - 1$        |

We see something particular when $q \equiv 1 \mod 50$ and when $q \equiv 49 \mod 50$:

| $q \equiv 1 \mod 50$ | $f_q(0, 0, n)$ | $f_q(1, 0, n)$ | $q \equiv 49 \mod 50$ | $f_q(0, 0, n)$ | $f_q(1, 0, n)$ |
|----------------------|----------------|----------------|----------------------|----------------|----------------|
| 1                    | $\left\lceil \frac{n}{2} \right\rceil$ | $\left\lceil \frac{n}{2} \right\rceil - 1$ | 49                   | $\left\lceil \frac{n}{2} \right\rceil$ | $\left\lceil \frac{n}{2} \right\rceil - 1$ |
| 51                   | $\left\lceil \frac{n}{5} \right\rceil - 1$ | $\left\lceil \frac{n}{5} \right\rceil - 1$ | 99                   | $\left\lceil \frac{n}{5} \right\rceil$ | $\left\lceil \frac{n}{5} \right\rceil - 1$ |
| 101                  | $\left\lceil \frac{n}{10} \right\rceil - 1$ | $\left\lceil \frac{n}{10} \right\rceil - 1$ | 149                  | $\left\lceil \frac{n}{10} \right\rceil$ | $\left\lceil \frac{n}{10} \right\rceil - 1$ |
| 151                  | $\left\lceil \frac{n}{5} \right\rceil - 1$ | $\left\lceil \frac{n}{5} \right\rceil - 1$ | 199                  | $\left\lceil \frac{n}{5} \right\rceil$ | $\left\lceil \frac{n}{5} \right\rceil - 1$ |
| 201                  | $\left\lceil \frac{n}{10} \right\rceil - 1$ | $\left\lceil \frac{n}{10} \right\rceil - 1$ | 249                  | $\left\lceil \frac{n}{10} \right\rceil$ | $\left\lceil \frac{n}{10} \right\rceil - 1$ |
| 251                  | $\left\lceil \frac{n}{25} \right\rceil - 1$ | $\left\lceil \frac{n}{25} \right\rceil - 1$ | 299                  | $\left\lceil \frac{n}{25} \right\rceil$ | $\left\lceil \frac{n}{25} \right\rceil - 1$ |

Using our formula we see that the last $-1$ term is fundamental, until $x \geq 1$. But if $x, y = 0$, as in the first column of the table, something strange happens.

When $q \equiv 9 \mod 10$:

$$f_{q\equiv9 \mod 10}(0, 0, n) = \max \left( \left\lceil \frac{n}{\Delta_2} \right\rceil, \left\lceil \frac{n}{\Delta_5} \right\rceil \right) - 1$$

When $q \equiv 1 \mod 10$:

$$f_{q\equiv1 \mod 10}(0, 0, n) = \begin{cases} \max \left( \left\lceil \frac{n}{\Delta_2} \right\rceil, \left\lceil \frac{n}{\Delta_5} \right\rceil \right) & \text{if } q \equiv 51 \mod 200 \\ \max \left( \left\lceil \frac{n}{\Delta_2} \right\rceil, \left\lceil \frac{n}{\Delta_5} \right\rceil \right) - 1 & \text{otherwise} \end{cases}$$

5.2. When $q \equiv 3, 7 \mod 10$. We are now considering the case where the base $q$ is a number congruent to $3, 7 \mod 10$. Consider a function $f_q(n)$ such that if:

$$f_{q\equiv1, 9 \mod 10}(x, 0, n) = u$$

Then:

$$\lfloor q^{u^n} \rfloor \mod (10^n)$$
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\[ q \equiv 3 \mod 10 \]

| \( q \) | \( f_q(0,0,n) \) | \( f_q(1,0,n) \) | \( q \equiv 7 \mod 10 \) | \( f_q(0,0,n) \) | \( f_q(1,0,n) \) |
|-------|----------------|----------------|----------------|----------------|----------------|
| 3     | \( n + 1 \)   | \( n \)        | 7              | \( \frac{n}{1} + 1 \) | \( \frac{n}{1} \) |
| 13    | \( n \)       | 17             | \( n, n \)     | \( n, n \)     | \( n, n \)     |
| 23    | \( n + 1 \)   | \( n \)        | 27             | \( n + 1 \)    | \( n \)        |
| 33    | \( n \)       | 57             | \( n, n \)     | \( n, n \)     | \( n, n \)     |
| 43    | \( \frac{n}{1} + 1 \) | \( \frac{n}{1} \) | 47             | \( n + 1 \)    | \( n \)        |
| 53    | \( n \)       | 57             | \( \frac{n}{1} + 1 \) | \( \frac{n}{1} \) | \( \frac{n}{1} \) |
| 63    | \( n + 1 \)   | \( n \)        | 67             | \( n + 1 \)    | \( n \)        |
| 73    | \( n \)       | 77             | \( n, n \)     | \( n, n \)     | \( n, n \)     |
| 83    | \( n + 1 \)   | \( n \)        | 87             | \( n + 1 \)    | \( n \)        |
| 93    | \( \frac{n}{1} \) | \( \frac{n}{1} \) | 97             | \( n \)        | \( n \)        |
| 103   | \( n + 1 \)   | \( n \)        | 107            | \( \frac{n}{1} + 1 \) | \( \frac{n}{1} \) |

Table 3. Values of \( f_q(x,0,n) \) by varying \( q \) and \( x \)

We see something particular when \( q \equiv 43 \mod 50 \) and when \( q \equiv 7 \mod 50 \):

\[ q \equiv 43 \mod 50 \]

| \( q \) | \( f_q(0,0,n) \) | \( f_q(1,0,n) \) | \( q \equiv 7 \mod 50 \) | \( f_q(0,0,n) \) | \( f_q(1,0,n) \) |
|-------|----------------|----------------|----------------|----------------|----------------|
| 43    | \( \frac{n}{1} + 1 \) | \( \frac{n}{1} \) | 7              | \( \frac{n}{1} + 1 \) | \( \frac{n}{1} \) |
| 93    | \( \frac{n}{1} \) | \( \frac{n}{1} \) | 57             | \( \frac{n}{1} \) | \( \frac{n}{1} \) |
| 143   | \( \frac{n}{1} + 1 \) | \( \frac{n}{1} \) | 107            | \( \frac{n}{1} + 1 \) | \( \frac{n}{1} \) |
| 193   | \( \frac{n}{1} \) | \( \frac{n}{1} \) | 157            | \( \frac{n}{1} \) | \( \frac{n}{1} \) |
| 243   | \( \frac{n}{1} + 1 \) | \( \frac{n}{1} \) | 207            | \( \frac{n}{1} + 1 \) | \( \frac{n}{1} \) |
| 293   | \( \frac{n}{1} \) | \( \frac{n}{1} \) | 257            | \( \frac{n}{1} \) | \( \frac{n}{1} \) |
| 343   | \( \frac{n}{1} + 1 \) | \( \frac{n}{1} \) | 307            | \( \frac{n}{1} \) | \( \frac{n}{1} \) |

Table 4. Values of \( f_q(x,0,n) \) by varying \( q \) and \( x \)

We see something particular when \( q \equiv 193 \mod 250 \) and when \( q \equiv 57 \mod 250 \). This remind us some kind of \( p \)-adic valuation structure.

\[ q \equiv 193 \mod 250 \]

| \( q \) | \( f_q(0,0,n) \) | \( f_q(1,0,n) \) | \( q \equiv 57 \mod 250 \) | \( f_q(0,0,n) \) | \( f_q(1,0,n) \) |
|-------|----------------|----------------|----------------|----------------|----------------|
| 193   | \( \frac{n}{1} \) | \( \frac{n}{1} \) | 57             | \( \frac{n}{1} \) | \( \frac{n}{1} \) |
| 443   | \( \frac{n}{1} \) | \( \frac{n}{1} \) | 307            | \( \frac{n}{1} \) | \( \frac{n}{1} \) |

Table 5. Values of \( f_q(x,0,n) \) by varying \( q \) and \( x \)

It is like terms are alternating; a term with +1 and the other without the +1, looking at \( f_q(0,0,n) \). Even though there are some exceptions; for example, very surprisingly, \( f_{307}(0,0,n) = \left\lfloor \frac{n}{7} \right\rfloor \).

5.3. When \( q \equiv 5 \mod 10 \). When \( q \equiv 5 \mod 10 \) our third formula seems to work fine for all integer \( x \geq 1 \). For \( x = 0 \), in some terms don’t appear the -1 part of the main formula.
5.4. **Prime numbers.** Initially I was thinking that the formulas had to depend on the prime factorization of the base of the tetration, but then I have understood that our formulas only depend on the residue $n \mod 10$ of the base $q$. Here below is a table for the first 60 prime numbers:

| $p$ | $f_p(0, 0, n)$ | $f_p(1, 0, n)$ | $f_p(x, 0, n)$ | $p$ | $f_p(0, 0, n)$ | $f_p(1, 0, n)$ | $f_p(x, 0, n)$ |
|-----|----------------|----------------|-----------------|-----|----------------|----------------|----------------|
| 2   | $n + 2$        | $n + 1$        | $n - 1$         | 127 | $n + 1$        | $n$            | $n - 1$        |
| 3   | $n + 1$        | $n$            | $n - 1$         | 131 | $n - 1$        | $n - 1$        | $n - 1$        |
| 5   | $\left\lfloor \frac{n}{5} \right\rfloor - 1$ | $\left\lfloor \frac{n}{5} \right\rfloor - 1$ | $\left\lfloor \frac{n}{5^2} \right\rfloor - 1$ | 137 | $n$            | $n$            | $n - 1$        |
| 7   | $\left\lfloor \frac{n}{7} \right\rfloor + 1$ | $\left\lfloor \frac{n}{7} \right\rfloor - 1$ | $\left\lfloor \frac{n}{7^2} \right\rfloor - 1$ | 139 | $n$            | $n - 1$        | $n - 1$        |
| 11  | $n - 1$        | $n - 1$        | $n - 1$         | 149 | $\left\lfloor \frac{n}{11} \right\rfloor$ | $\left\lfloor \frac{n}{11} \right\rfloor - 1$ | $\left\lfloor \frac{n}{11^2} \right\rfloor - 1$ |
| 13  | $n$            | $n - 1$        | $n - 1$         | 151 | $\left\lfloor \frac{n}{13} \right\rfloor - 1$ | $\left\lfloor \frac{n}{13} \right\rfloor - 1$ | $\left\lfloor \frac{n}{13^2} \right\rfloor - 1$ |
| 17  | $n$            | $n - 1$        | $n - 1$         | 157 | $\left\lfloor \frac{n}{17} \right\rfloor$ | $\left\lfloor \frac{n}{17} \right\rfloor - 1$ | $\left\lfloor \frac{n}{17^2} \right\rfloor - 1$ |
| 19  | $n$            | $n - 1$        | $n - 1$         | 163 | $n + 1$        | $n$            | $n - 1$        |
| 23  | $n + 1$        | $n$            | $n - 1$         | 167 | $n + 1$        | $n$            | $n - 1$        |
| 29  | $n$            | $n - 1$        | $n - 1$         | 173 | $n$            | $n$            | $n - 1$        |
| 31  | $n - 1$        | $n - 1$        | $n - 1$         | 179 | $n$            | $n - 1$        | $n - 1$        |
| 37  | $n$            | $n - 1$        | $n - 1$         | 181 | $n - 1$        | $n - 1$        | $n - 1$        |
| 41  | $n - 1$        | $n - 1$        | $n - 1$         | 191 | $n - 1$        | $n - 1$        | $n - 1$        |
| 43  | $\left\lfloor \frac{n}{43} \right\rfloor + 1$ | $\left\lfloor \frac{n}{43} \right\rfloor$ | $\left\lfloor \frac{n}{43^2} \right\rfloor - 1$ | 193 | $\left\lfloor \frac{n}{43} \right\rfloor - 1$ | $\left\lfloor \frac{n}{43^2} \right\rfloor - 1$ | $\left\lfloor \frac{n}{43^3} \right\rfloor - 1$ |
| 47  | $n + 1$        | $n$            | $n - 1$         | 197 | $n$            | $n$            | $n - 1$        |
| 53  | $n$            | $n - 1$        | $n - 1$         | 199 | $\left\lfloor \frac{n}{53} \right\rfloor$ | $\left\lfloor \frac{n}{53} \right\rfloor - 1$ | $\left\lfloor \frac{n}{53^2} \right\rfloor - 1$ |
| 59  | $n$            | $n - 1$        | $n - 1$         | 211 | $n - 1$        | $n - 1$        | $n - 1$        |
| 61  | $n - 1$        | $n - 1$        | $n - 1$         | 223 | $n + 1$        | $n$            | $n - 1$        |
| 67  | $n + 1$        | $n$            | $n - 1$         | 227 | $n + 1$        | $n$            | $n - 1$        |
| 71  | $n - 1$        | $n - 1$        | $n - 1$         | 229 | $n$            | $n - 1$        | $n - 1$        |
| 73  | $n$            | $n - 1$        | $n - 1$         | 233 | $n$            | $n$            | $n - 1$        |
| 79  | $n$            | $n - 1$        | $n - 1$         | 239 | $n$            | $n - 1$        | $n - 1$        |
| 83  | $n + 1$        | $n$            | $n - 1$         | 241 | $n - 1$        | $n - 1$        | $n - 1$        |
| 89  | $n$            | $n - 1$        | $n - 1$         | 251 | $\left\lfloor \frac{n}{89} \right\rfloor$ | $\left\lfloor \frac{n}{89} \right\rfloor - 1$ | $\left\lfloor \frac{n}{89^2} \right\rfloor - 1$ |
| 97  | $n$            | $n - 1$        | $n - 1$         | 257 | $\left\lfloor \frac{n}{97} \right\rfloor$ | $\left\lfloor \frac{n}{97} \right\rfloor - 1$ | $\left\lfloor \frac{n}{97^2} \right\rfloor - 1$ |
| 101 | $\left\lfloor \frac{n}{101} \right\rfloor - 1$ | $\left\lfloor \frac{n}{101} \right\rfloor - 1$ | $\left\lfloor \frac{n}{101^2} \right\rfloor - 1$ | 263 | $n + 1$        | $n$            | $n - 1$        |
| 103 | $n + 1$        | $n$            | $n - 1$         | 269 | $n$            | $n - 1$        | $n - 1$        |
| 107 | $\left\lfloor \frac{n}{107} \right\rfloor + 1$ | $\left\lfloor \frac{n}{107} \right\rfloor$ | $\left\lfloor \frac{n}{107^2} \right\rfloor - 1$ | 271 | $n - 1$        | $n - 1$        | $n - 1$        |
| 109 | $n$            | $n - 1$        | $n - 1$         | 277 | $n$            | $n$            | $n - 1$        |
| 113 | $n$            | $n - 1$        | $n - 1$         | 281 | $n - 1$        | $n - 1$        | $n - 1$        |

**Table 6.** Values of $f_p(x, 0, n)$ by varying $p$ and $x$

Where in the last column $x \geq 2$.

5.5. **When $q \equiv 0 \mod 2$.** When $q \equiv 0 \mod 2$ and $x < 2$ it could be the most anomalous case. Because as we have seen in my last papers, the repeating last digits of a tetration which base is a power of 2 has a lot of sub-cases. So it may be complicated to find a generic "nice" formula when $x < 2$. 

6. Conclusions

We are very near to a proof for a formula which finds the minimum hyper-exponent $u$ of a tetration with a generic base $q$ such that the last $n$ digits of the tetration after the $u$-th one are the same.

References

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