EXTENDING HOPF INVARIANT

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Abstract. We give here a version of some extended Hopf invariant in ‘Ganea-Whithehead’ style.

1. The Hopf category

The background for this paper is to be found in [1].

Recall the following construction:

Definition 1. For any map \( \iota : A \rightarrow X \), the Ganea construction of \( \iota \) is the following sequence of homotopy commutative diagrams (\( i \geq 0 \)):

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_i} & G_i(\iota_X) \\
\downarrow{\alpha_{i+1}} & & \downarrow{\gamma_i} \\
F_i(\iota_X) & \xrightarrow{\beta_i} & G_{i+1}(\iota_X) \\
\end{array}
\]

where the outside square is a homotopy pullback, the inside square is a homotopy pushout and the map \( g_{i+1} = (g_i, \iota_X) : G_{i+1} \rightarrow X \) is the whisker map induced by this homotopy pushout. The iteration starts with \( g_0 = \iota_X : A \rightarrow X \).

Let \( \alpha_0 = \text{id}_A \). For any \( i \geq 0 \), there is a whisker map \( \theta_i = (\text{id}_A, \alpha_i) : A \rightarrow F_i(\iota_X) \) induced by the homotopy pullback. Thus we have the sequence of maps \( A \xrightarrow{\theta_i} F_i(\iota_X) \xrightarrow{\eta_i} A \) and \( \theta_i \) is a homotopy section of \( \eta_i \). Moreover we have \( \gamma_i \circ \alpha_i \simeq \alpha_{i+1} \).

Definition 2. Let \( \iota_X : A \rightarrow X \) be any map.

1) The sectional category of \( \iota_X \) is the least integer \( n \) such that the map \( g_n : G_n(\iota_X) \rightarrow X \) has a homotopy section, i.e. there exists a map \( \sigma : X \rightarrow G_n(\iota_X) \) such that \( g_n \circ \sigma \simeq \text{id}_X \).

2) The relative category of \( \iota_X \) is the least integer \( n \) such that the map \( g_n : G_n(\iota_X) \rightarrow X \) has a homotopy section \( \sigma \) and \( \sigma \circ \iota_X \simeq \alpha_n \).

We denote the sectional category by \( \text{secat}(\iota_X) \), and the relative category by \( \text{relcat}(\iota_X) \). If \( A = * \), \( \text{secat}(\iota_X) = \text{relcat}(\iota_X) \) and is denoted simply by \( \text{cat}(X) \); this is the ‘normalized’ version of the Lusternik-Schnirelmann category.

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In the sequel, we consider an homotopy epimorphism $\eta: W \to A$ and the homotopy pushout:

\[
\begin{array}{ccc}
W & \xrightarrow{\eta} & A \\
\downarrow & & \downarrow \\
A & \xrightarrow{\theta} & \Sigma_A W
\end{array}
\]

and any given map $f: \Sigma_A W \to X$. Let $\iota_X \simeq f \circ \theta$ and let $\omega_0 \simeq (\text{id}_A, \text{id}_A): \Sigma_A W \to A$.

For $n \geq 1$, consider the following homotopy commutative diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{\eta} & A \\
\downarrow & & \downarrow \\
\Sigma_A W & \xrightarrow{\theta} & \Sigma_A W \\
\downarrow & & \downarrow \\
\Sigma_A W & \xrightarrow{\theta} & \Sigma_A W \\
\downarrow & & \downarrow \\
F_{n-1}(\iota_X) & \xrightarrow{\omega_n} & G_n(\iota_X) \\
\downarrow & & \downarrow \\
A & \xrightarrow{\alpha_n} & G_n(\iota_X) \\
\downarrow & & \downarrow \\
A & \xrightarrow{\alpha_n} & G_n(\iota_X) \\
\downarrow & & \downarrow \\
A & \xrightarrow{\alpha_n} & G_n(\iota_X) \\
\downarrow & & \downarrow \\
A & \xrightarrow{\alpha_n} & G_n(\iota_X) \\
\downarrow & & \downarrow \\
\Sigma_A W & \xrightarrow{\theta} & \Sigma_A W \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & X \\
\end{array}
\]

where the map $W \to F_{n-1}$ is induced by the bottom outer homotopy pullback and the map $\omega_n: \Sigma_A W \to G_n$ is induced by the top inner homotopy pushout. We have $f \simeq g_n \circ \omega_n$ by the ‘Whiskers maps inside a cube’ lemma. (Warning: This is not true for $n = 0$.) Also notice that $\omega_n \circ \theta \simeq \alpha_n$. Finally we can see that $\omega_n \simeq (\alpha_n, \alpha_n)$ is the whisker map of $\alpha_n$ and $\alpha_n$ induced by the homotopy pushout $\Sigma_A W$.

**Definition 3.** The Hopf category of $f$ is the least integer $n \geq 0$ such that $g_n: G_n(\iota_X) \to X$ has a homotopy section $\sigma: X \to G_n(\iota_X)$ such that $\sigma \circ f \simeq \omega_n$.

We denote this integer by $\text{hcat}(f)$.

Notice that $\text{relcat}$ is a particular case of $\text{hcat}$: When $W = A$, $\eta \simeq \text{id}_A$, then $\iota_X \simeq f$, $\omega_n \simeq \alpha_n$ and $\text{hcat}(f) = \text{relcat}(\iota_X)$.

**Proposition 4.** Whatever can be $f$ (and $\iota_X \simeq f \circ \theta$), we have

\[
\text{relcat}(\iota_X) \leq \text{hcat}(f)
\]

**Proof.** If $\text{hcat}(f) \leq n$, then $\sigma \circ \iota_X \simeq \sigma \circ f \circ \theta \simeq \omega_n \circ \theta \simeq \alpha_n$, so $\text{relcat}(\iota_X) \leq n$. □

2. THE STRONG HOPF CATEGORY

**Definition 5.** The strong Hopf category of a map $f: \Sigma_A W \to X$ is the least integer $n \geq 0$ such that:

- there are maps $\iota_0: A \to X_0$ and a homotopy inverse $\lambda: X_0 \to A$, i.e. $\iota_0 \circ \lambda \simeq \text{id}_{X_0}$ and $\lambda \circ \iota_0 \simeq \text{id}_A$;
– for each $i$, $0 \leq i < n$, there is a homotopy commutative cube:

\[
\begin{array}{c}
\begin{array}{c}
W \\
\downarrow \eta \\
\Sigma_A W \\
\downarrow \omega_{i+1} \\
Z_i \\
\downarrow \chi_i \\
A \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\downarrow \iota_i \\
A \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\downarrow \iota_{i+1} \\
X_{i+1} \\
\end{array}
\end{array}

- $X_n = X$ and $\omega_n \simeq f$.

We denote this integer by $\text{Hcat}(f)$.

Notice that, as $\eta$ is a homotopy epimorphism, we have $\iota_{i+1} \simeq \chi_i \circ \iota_i$.

**Proposition 6.** A map $g : \Sigma_A W \to Y$ has $\text{hcat}(g) \leq n$ iff $g$ is relatively dominated by a map $f : \Sigma_A W \to X$ with $\text{Hcat}(f) \leq n$.

**Proof.**

3. Taking a relative cofibre

Consider the following homotopy commutative diagram where the square is a homotopy pushout:

\[
\begin{array}{c}
\begin{array}{c}
A \\
\downarrow \theta \\
\Sigma_A W \\
\downarrow \omega_0 \\
A \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\downarrow \iota_X \\
X \\
\downarrow \chi \\
C \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\downarrow f \\
\Sigma_A W \\
\downarrow \omega_0 \\
A \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\downarrow \iota_C \\
C \\
\end{array}
\end{array}
\]

**Proposition 7.** The homotopy pushout above can be decomposed into the following homotopy commutative diagram:

\[
\begin{array}{c}
\begin{array}{c}
\Sigma_A W \\
\downarrow \omega_0 \\
A \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\downarrow \iota_C \\
C \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
G_1(\iota_X) \\
\downarrow \chi \\
X \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
G_n(\iota_X) \\
\end{array}
\end{array}
\]

**Proof.** Consider the following homotopy commutative cube:
Using the original square

\[
\begin{array}{ccc}
\Sigma A W & \xrightarrow{\omega_0} & A \\
\downarrow f & & \downarrow \iota_C \\
X & \xleftarrow{x} & C
\end{array}
\]

we can link diagrams (†) and (‡) edge by edge, getting first a homotopy commutative square:

\[
\begin{array}{ccc}
W & \xrightarrow{\eta} & A \\
\downarrow & & \downarrow \\
F_{n-1}(\iota_X) & \xrightarrow{=} & F_{n-1}(\iota_C)
\end{array}
\]

where the horizontal maps are induced by the homotopy pullbacks, and next a homotopy commutative square:

\[
\begin{array}{ccc}
\Sigma A W & \xrightarrow{\omega_0} & A \\
\downarrow \omega_n & & \downarrow \\
G_n(\iota_X) & \xrightarrow{=} & G_n(\iota_C)
\end{array}
\]

where the horizontal maps are induced by the homotopy pushouts, and moreover a homotopy commutative cube:

\[
\begin{array}{ccc}
\Sigma A W & \xrightarrow{\omega_n} & \Sigma A W \\
\downarrow & & \downarrow \\
A & \xleftarrow{=} & A \\
\downarrow & & \downarrow \\
G_n(\iota_X) & \xrightarrow{=} & X
\end{array}
\]

\[
\begin{array}{ccc}
\Sigma A W & \xrightarrow{\omega_n} & \Sigma A W \\
\downarrow & & \downarrow \\
A & \xleftarrow{=} & A \\
\downarrow & & \downarrow \\
G_n(\iota_C) & \xrightarrow{=} & C
\end{array}
\]

and this gives the required splitting of the righter square which is the original pushout. \qed

**Proposition 8.** Whatever can be \(f\), we have

\[
\text{secat } (\iota_C) \leq \text{relcat } (\iota_C) \leq \text{hcat} (f).
\]

**Proof.** Let \(\text{hcat} (f) \leq n\). First apply the ‘Whiskers maps inside a cube’ lemma to the outer part of the following homotopy commutative diagram:

\[
\begin{array}{ccc}
\Sigma A W & \xrightarrow{\omega_n} & A \\
\downarrow & & \downarrow a \\
G_n(\iota_X) & \xrightarrow{=} & G_n(\iota_C) \\
\downarrow & & \downarrow b \\
\Sigma A W & \xrightarrow{=} & A \\
\downarrow & & \downarrow g_n \\
X & \xrightarrow{=} & C
\end{array}
\]
where the inner horizontal squares are homotopy pushouts. Next apply the ‘Prism’ lemma to get the dotted map in the following homotopy commutative diagram:

\[
\begin{array}{c}
\Sigma A W \\
X \\
\Sigma A W \\
G_n(\iota_X)
\end{array} \xrightarrow{f} \begin{array}{c}
A \\
C \\
A \\
S
\end{array} \xrightarrow{\iota_C} \begin{array}{c}
A \\
A \\
G_n(\iota_C)
\end{array}
\]

Let \( \sigma' = b \circ d \). We have \( g_n \circ \sigma' \simeq g_n \circ b \circ d \simeq c \circ d \simeq \text{id}_C \) and \( \sigma' \circ \iota_C \simeq b \circ d \circ \iota_C \simeq b \circ a \simeq \alpha_n \).

4. WHITEHEAD STYLE VERSION OF THE HOPF CATEGORY

Recall the following:

**Proposition 9.**

\[
\begin{array}{c}
A \\
\delta_n \\
A \times X^n
\end{array} \xrightarrow{\alpha_n} \begin{array}{c}
G_n(\iota_X) \\
\tau_n \\
T_n(\iota_X)
\end{array} \xrightarrow{g_n} \begin{array}{c}
X \\
\Delta \\
X^{n+1}
\end{array}
\]

where \( \delta_n \) is the whisker map of \( \text{id}_A \) and \( n \) copies of \( \iota_X \).

**Theorem 10.** The left side of the diagram of Proposition \( \Box \) splits into two homotopy pullbacks:

\[
\begin{array}{c}
A \\
\delta_n \\
A \times X^n
\end{array} \xrightarrow{\theta} \begin{array}{c}
\Sigma A W \\
\delta^n_W \\
\Sigma A W \times X^n
\end{array} \xrightarrow{\omega_n} \begin{array}{c}
G_n(\iota_X) \\
\tau_n \\
T_n(\iota_X)
\end{array}
\]

**Proof.** Consider the following homotopy commutative diagram:

\[
\begin{array}{c}
A \\
\delta_n \\
A \times X^n
\end{array} \xrightarrow{\eta} \begin{array}{c}
W \\
\delta^n_W \\
W \times X^n
\end{array} \xrightarrow{\eta} \begin{array}{c}
A \\
A \\
A \times X^n
\end{array}
\]

We can check that the left and back squares are homotopy pullbacks using the ‘Prism’ lemma in the following (homotopy) commutative diagram:

\[
\begin{array}{c}
W \\
\eta \\
A
\end{array} \xrightarrow{\delta^n_W} \begin{array}{c}
W \times X^n \\
\eta \times \text{id}
\end{array} \xrightarrow{pr_1} \begin{array}{c}
W \\
W
\end{array} \xrightarrow{\eta} \begin{array}{c}
A \\
A
\end{array}
\]
where $\delta^W_n$ is the whisker map of $\id_W$ and $n$ copies of $\iota_X \eta$. So the four vertical squares are homotopy pullbacks.

Now extend this diagram to this one:

\[
\begin{array}{ccc}
\Sigma_A W \times X^n & \xrightarrow{\nu_n} & T_n \\
\downarrow & & \downarrow \\
\Sigma A W & \xrightarrow{\epsilon_n} & \Sigma A W \times X^n \\
\end{array}
\]

where the left bottom square is a homotopy pushout, $\nu_n$ is the whisker map induced by this pushout, and $P$ is the homotopy pullback of $\nu_n$ and $\epsilon_n$. By the ‘Prism’ lemma, all vertical squares of the diagram are homotopy pullbacks. So, by the ‘Cube’ axiom, the left top square is a homotopy pushout, and $P \simeq \Sigma A W$. □

We will denote $\tau^W_n \simeq \nu_n \circ \delta^W_n \simeq \epsilon_n \circ \omega_n$.

The fact that $\sigma \circ f \simeq \omega_n$ can be considered as the ‘Ganea interpretation’ of a ‘null Hopf invariant’ for $f : \Sigma A W \to X$. The next proposition gives the ‘Whitehead interpretation’ of this.

**Proposition 11.** The following conditions are equivalent:

(i) there is a homotopy section $\sigma : X \to G_n(\iota_X)$ of $g_n$ such that $\sigma \circ f \simeq \omega_n$;

(ii) there is a map $\rho : X \to T_n(\iota_X)$ such that $\Delta \simeq t_n \circ \rho$ and $\rho \circ f \simeq \tau^W_n$.

**Proof.** If $\sigma$ is a homotopy section of $g_n$, then consider the map $\rho = \epsilon_n \circ \sigma$. We have $t_n \circ \rho \simeq t_n \circ \epsilon_n \circ \sigma \simeq \Delta \circ g_n \circ \sigma \simeq \Delta$. If, moreover, $\sigma \circ f \simeq \omega_n$, then $\rho \circ f \simeq \epsilon_n \circ \sigma \circ f \simeq \epsilon_n \circ \omega_n$.

For the reverse direction, let $\sigma = (\rho, \id_X)$ be the whisker map induced by the right homotopy pullback of Proposition 9. It is a homotopy section for $g_n$. If, moreover $\rho \circ f \simeq \tau^W_n$, we can build the following homotopy diagram:

\[
\begin{array}{ccc}
\Sigma A W & \xrightarrow{\delta^W_n} & \Sigma A W \times X^n \\
\downarrow & & \downarrow \\
\Sigma A W & \xrightarrow{\delta^W_n} & \Sigma A W \times X^n \\
\end{array}
\]

where the two squares are homotopy pullbacks and $\hat{\sigma}$ is the whisker map of $\delta^W_n$ and $f$. We have $\delta^W_n \simeq \delta^W_n \circ \pi \circ \hat{\sigma}$, and, as $\delta^W_n$ has an obvious (homotopy) retraction, $\id_{\Sigma A W} \simeq \pi \circ \hat{\sigma}$. Finally $\sigma \circ f \simeq \omega_n \circ \pi \circ \hat{\sigma} \simeq \omega_n$. □
Let be given any map $f: \Sigma_A W \to X$ with $\text{secat}(i_X) \leq n$ and any homotopy section $\sigma: X \to G_n$ of $g_n: G_n \to X$. Consider the following homotopy pullbacks:

$$
\begin{array}{c}
Q \xrightarrow{\pi} \Sigma_A W \\
\downarrow \quad \quad \quad \downarrow \theta_n^W \quad \quad \quad \downarrow \eta_n^W \\
\Sigma_A W \xrightarrow{\sigma} H_n \xrightarrow{\eta_n^W} \Sigma A W \\
\downarrow f \quad \quad \quad \quad \quad \downarrow f \quad \quad \quad \quad \quad \downarrow f \\
X \xrightarrow{\sigma} G_n \xrightarrow{g_n} X
\end{array}
$$

where $\theta_n^W = (\omega_n, f)$ is the whisker map induced by the homotopy pullback $H_n$. By the Prism lemma, we know that the homotopy pullback of $\sigma$ and $f_n$ is indeed $\Sigma_A W$, and that $\eta_n^W \circ \sigma \simeq \text{id}_{\Sigma_A W}$. Also notice that $\pi \simeq \pi'$ since $\pi \simeq \eta_n^W \circ \theta_n^W \circ \pi \simeq \eta_n^W \circ \sigma \circ \pi' \simeq \pi'$.

**Proposition 12.** Let be given any map $f: \Sigma_A W \to X$ with $\text{secat}(i_X) \leq n$ and any homotopy section $\sigma: X \to G_n(i_X)$ of $g_n: G_n(i_X) \to X$. With the same definitions and notations as above, the following conditions are equivalent:

(i) $\sigma \circ f \simeq \omega_n$.
(ii) $\pi$ has a homotopy section.
(iii) $\pi$ is a homotopy epimorphism.
(iv) $\theta_n^W \simeq \sigma$.

**Proof.** We have the following sequence of implications:

(i) $\implies$ (ii): Since $\sigma \circ f \simeq \omega_n \simeq f_n \circ \theta_n^W \circ \text{id}_{\Sigma_A W}$, we have a whisker map $(f, \text{id}_{\Sigma_A W}): \Sigma_A W \to Q$ induced by the homotopy pullback $P$ which is a homotopy section of $\pi$.

(ii) $\implies$ (iii): Obvious.

(iii) $\implies$ (iv): We have $\theta_n^W \circ \pi \simeq \sigma \circ \pi$ since $\pi \simeq \pi'$. Thus $\theta_n^W \simeq \sigma$ since $\pi$ is a homotopy epimorphism.

(iv) $\implies$ (i): We have $\sigma \circ f \simeq f_n \circ \sigma = f_n \circ \theta_n^W \simeq \omega_n$.

**Theorem 13.** Let be given a $CW$-complex $A$ and a $(q-1)$-connected map $\iota_X: A \to X$ with $\text{secat} i_X = n$. If $\dim \Sigma A W < (n+1)q - 1$ then $\sigma \circ f \simeq \omega_n$ for any homotopy section $\sigma$ of $g_n$.

**Proof.** Recall that $g_i$ is the $(i+1)$-fold join of $i_X$. Thus by [2], Theorem 47, we obtain that, for each $i \geq 0$, $g_i : G_i \to X$ is $(i+1)q - 1$-connected. As $g_i$ and $\eta_i^W$ have the same homotopy fibre, the Five lemma implies that $\eta_i^W : H_i \to A$ is $(i+1)q - 1$-connected, too. By [3], Theorem IV.7.16, this means that for every CW-complex $K$ with $\dim K < (i+1)q - 1$, $\eta_i^W$ induces a one-to-one correspondence $[K, H_i] \to [K, A]$. Since $\theta_n^W$ and $\sigma$ are both homotopy sections of $\eta_n^W$, we obtain $\theta_n^W \simeq \sigma$, and Proposition [12] implies the desired result.

**Example 14.** Let $A = *$ and $W = S^{r-1}$, so $\Sigma_A W = S^r$, and $X = S^m$. In this case $\text{secat} i_X = \text{cat} S^m = 1$. Hence Theorem [13] means that if $r < 2m - 1$, we have $\sigma \circ f \simeq \omega_1$, whatever can be $f$ and $\sigma$, and so we get by Proposition [8] that the homotopy cofibre $C$ of $f$ has $\text{cat} C \leq 1$. (Notice that if $r < m$ then $f$ is a nullhomotopic, so $C$ is simply $S^m \vee S^{r+1}$.) If $r \geq 2m - 1$ then Theorem [13] doesn’t apply and we can have $\text{cat} C = 2$. This is the case for the Hopf maps $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$. 

References

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