ON LIMITING CHARACTERISTICS FOR A NON-STATIONARY TWO-PROCESSOR HETEROGENEOUS SYSTEM WITH CATASTROPHES, SERVER FAILURES AND REPAIRS

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(Communicated by Wuyi Yue)

ABSTRACT. In this paper, we display a method for the computation of convergence bounds for a non-stationary two-processor heterogeneous system with catastrophes, server failures and repairs when all parameters varying with time. Based on the logarithmic norm of linear operators, the bounds on the rate of convergence and the main limiting characteristics of the queue-length process are obtained. Finally a numerical example is presented to show the effect of parameters.

1. Introduction. A multiprocessor system is considered to be one of the most important systems in queueing systems, because it has wide applications in telecommunications, flexible manufacturing systems, reliability and traffic control. Many researchers have studied multiprocessor systems.

In this paper, we study a non-stationary Markovian queueing model of a two-processor heterogeneous system where all parameters are allowed to vary with time, which was firstly investigated in [1], see also time-dependent analysis of this model without catastrophes, server failures and repairs in the recent papers [2, 41]. In general, non-stationary queueing models have been actively studied for decades, see, for instance [1, 12, 13, 14, 18, 27] and the references therein.

2010 Mathematics Subject Classification. Primary: 60J27, 60J28.
Key words and phrases. Two-processor heterogeneous system, catastrophes, repairs, bounds, rate of convergence.

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There are many papers concerning the study of Markovian queueing models with possible catastrophes, see for instance, \([5, 6, 7, 20, 33, 34, 37, 38, 39, 42]\) and the references therein. Such models are widely used in simulations on high-performance computers. In particular, in some recent papers the authors deal with more or less special birth-death processes with additional transitions from and to origin \([5, 6, 7, 20, 37, 38, 39, 42]\).

In the present paper we consider another class of Markovian queueing models with possible catastrophes and obtain key bounds on the rate of convergence, which allow us to compute the limiting characteristics of the corresponding processes. These characteristics are very useful for optimizing service processes in queues. The authors of \([3]\) formulated an optimization problem for a call center with heterogeneous agent pools to minimize expected customer waiting time. They proposed a threshold routing policy which is asymptotically more optimal than the common routing policy used in call centers.

It is well-known that the direct computation of time-dependent limiting characteristics is not a very appealing way to analyze the behavior of system described by processes with time-dependent rates. A good alternative is to have bounds for performance characteristics of interest, which can be computed fast and are tight enough to make results meaningful. With respect to this observation, the analytic method we use here, which is based on the logarithmic norm of linear operators, as indicated by the papers \([36, 37, 38, 40]\), looks promising. It also serves as an alternative to the well-elaborated methods of asymptotic analysis (as in \([21, 22]\)), or point-wise stationary approximations (as in \([16, 30]\)) for the time-dependent queues.

There are many papers dealing with upper bounds for the rate of convergence, see, for instance, \([17, 25, 26, 28]\). Moreover, there are a number of other approaches to obtain upper bounds on the convergence time of Markov chains in different contexts, including the “cut-off phenomenon”, Markov chain Monte Carlo algorithms, and so on, see, for instance, \([9, 10, 29]\) and references therein. Some initial lower bounds for mixing times in for homogeneous discrete-time situation have been discussed in \([19]\). Usually, either the limiting mode is determined \((4)\), or, on the contrary, the state probabilities are determined as functions of time \(t\) and initial conditions \((11)\).

As far back as in 1993 in the series of papers by Meyn and Tweedy (see, e.g., \([23]\)) a general method was developed for the construction of convergence rate estimates in the case of geometric ergodicity. This method can be applied to any homogeneous Markov chain with an arbitrary state space. However, it is quite reasonable to expect that for some special classes of chains more accurate results can be obtained by other specific methods targeting specific features of particular classes of chains. This is the case discussed in this paper. Our approach is based on the notion of logarithmic norm of a linear operator and a special similarity transformation of the matrix of intensities of the Markov chain considered.

The idea of our approach is very simple and is based on special properties of linear systems of differential equations with non-diagonally nonnegative matrices. If the column-wise sums of the elements of this matrix are identical and equal to, say, \(-\alpha^*(t)\), then the exact upper bound of order \(\exp\left\{ -\int_0^t \alpha^*(u) \, du \right\}\) can be obtained for the rate of convergence (rapprochement) of the solutions of the system in the corresponding metric. Moreover, if the column-wise sums of the absolute values of the elements of this matrix are identical and equal to, say, \(-\chi^*(t)\), then
the exact lower bound of order $\exp \left\{ - \int_0^t \chi^*(u) \, du \right\}$ can be obtained for the rate of convergence as well.

This approach can be applied to continuous-time Markov chains in the following way. First, the equation for the zero state is excluded from the forward Kolmogorov system so that the “reduced” system is obtained whose matrix is obviously, in general, not non-diagonally nonnegative. Further, the main problem should be solved, namely, that of finding the transformation of the “reduced” system into a new system with non-diagonally nonnegative matrix.

This method was first developed only for birth-death processes. It was developed in a series of papers, see [15].

In particular, within this approach it became possible to rather easily prove the existence of the transformations yielding the exact upper and lower bounds for the case of finite time-homogeneous birth-death processes and point out their explicit form, see [28].

The advantages of our approach are, first of all, in that

(i) it does not matter whether the process is time-homogeneous or time-inhomogeneous;

(ii) it gives exact both upper and lower bounds for the convergence rate.

In general, the approach consists of two steps: firstly we find bounds on the rate of convergence to the limiting characteristics and choose the moment from which the distance between an arbitrary and the limiting regimes is sufficiently small; hence the solution of forward Kolmogorov equation with initial condition $X(0) = 0$ gives us periodic limiting behavior with the required accuracy on interval $[t, t + T]$ (and all characteristics you are interested in).

The article is organized as follows. In Section 2, the model of a non-homogeneous two-processor heterogeneous system with catastrophes, server failures and repairs is described. Section 3 contains the main result of the paper i.e. the theorem which specifies the convergence bounds. Finally, in the last section an extensive numerical example is presented.

2. Model description. We consider a multiprocessor system consisting of two types of processors, which for convenience will be referred to as the main and backup processors [1]. Each job requires exactly one processor for its execution. When both processors are idle, the main processor is scheduled for service before the backup processor. A computer system consists of two processors, a main processor, and a backup processor. A description of the model is as follows:

(a) jobs arrive at the system according to the Poisson process with an arrival rate $\lambda(t)$. Service is exponentially distributed, and two servers provide heterogeneous service rates $\mu_1(t), \mu_2(t)$ such that $\mu_2(t) \leq \mu_1(t)$;

(b) each job needs only one server to be served and the jobs select the servers on the basis of fastest server first (FSF);

(c) when the system is idle or busy, catastrophes occur at the service station according to Poisson process of rate $\gamma(t)$;

(d) whenever a catastrophe occurs at the system, all the jobs there are destroyed immediately, both the servers get inactivated, i.e., the servers are subject to catastrophe failure;

(e) the repair time of failed servers is i.i.d, according to an exponential distribution with mean $\xi^{-1}(t)$;

(f) let $Q(t)$ be the probability that the servers are under repair at the instant $t$. 
3. **Bounds on the rate of convergence.** Consider again the queue-length process \( \{X(t), t \geq 0\} \), this is an inhomogeneous continuous-time Markov chain. All possible transition intensities, say \( q_{ij}(t) \), are non-random functions of time and may depend on the state of the process. We suppose that all intensity functions are nonnegative and locally integrable on \([0, \infty)\).

Denote by \( p(t) = (Q(t), p_{00}(t), p_{01}(t), p_{10}(t), \ldots)^T \) the vector of state probabilities at the moment \( t \). Put \( a_{ij}(t) = q_{ji}(t) \) for \( j \neq i \) and

\[
a_{ii}(t) = - \sum_{j \neq i} a_{ji}(t) = - \sum_{j \neq i} q_{ij}(t),
\]

(3.1)

As in our previous papers [15, 36], we have

\[
\sup_i |a_{ii}(t)| = L(t) < \infty,
\]

(3.2)

for almost all \( t \geq 0 \).

The probabilistic dynamics of the process is presented by the forward Kolmogorov system of differential equations:

\[
\frac{dp(t)}{dt} = A(t)p(t),
\]

(3.3)

where \( A(t) \) is a transposed intensity matrix \( A(t) = (a_{ij}(t)) \):

\[
a_{11}(t) = - \xi(t),
\]

\[
a_{1j}(t) = \gamma(t), \text{ for } j \geq 2,
\]

\[
a_{21}(t) = \xi(t), \ a_{22}(t) = - (\lambda(t) + \gamma(t)), \ a_{23}(t) = \mu_1(t), \ a_{24}(t) = \mu_2(t),
\]

\[
a_{32}(t) = \lambda(t), \ a_{33}(t) = - (\lambda(t) + \mu_1(t) + \gamma(t)), \ a_{35}(t) = \mu_2(t),
\]

\[
a_{44}(t) = - (\lambda(t) + \mu_2(t) + \gamma(t)), \ a_{45}(t) = \mu_1(t), \ a_{53}(t) = \lambda(t),
\]

\[
a_{i,i+1}(t) = \mu(t), \text{ for } i \geq 5,
\]

\[
a_{i,i}(t) = - (\lambda(t) + \mu(t) + \gamma(t)), \text{ for } i \geq 5,
\]

\[
a_{i+1,i}(t) = \lambda(t), \text{ for } i \geq 4.
\]

Denote by \( \| \cdot \| \) the \( l_1 \)-norm of vector, \( \|x\| = \sum_i |x_i|, \|B\| = \sup_i \sum_j |b_{ij}|, \) if \( B = (b_{ij})_{i,j=0}^\infty \), and denote by \( \Omega \) the set of all vectors from \( l_1 \) with nonnegative coordinates and unit norm.

We have \( \|A(t)\| = 2 \sup_k |a_{kk}(t)| \leq 2L(t) \) for almost all \( t \geq 0 \). Hence the operator function \( A(t) \) from \( l_1 \) to itself is bounded for almost all \( t \geq 0 \) and locally integrable on interval \([0; \infty)\).

Consider the relation (3.3) as a differential equation in the space \( l_1 \), then one can apply to (3.3) the approach of [8].

Rewrite the forward Kolmogorov system (3.3) as

\[
\frac{dp}{dt} = A^*(t)p + g(t), \quad t \geq 0.
\]

(3.4)

Here \( g(t) = (\gamma(t), 0, 0, \ldots)^T, \ A^*(t) = (a^*_{ij}(t))_{i,j=0}^\infty, \) and

\[
a^*_{ij}(t) = \begin{cases} 
  a_{0j}(t) - \gamma(t), & \text{if } i = 0, \\
  a_{ij}(t), & \text{otherwise}.
\end{cases}
\]

(3.5)

Then \( A^*(t) = (a^*_{ij}(t)) \):

\[
a_{11}(t) = - (\xi(t) + \gamma(t)),
\]
\[ a_{21}(t) = \xi(t), \quad a_{22}(t) = - (\lambda(t) + \gamma(t)), \quad a_{23}(t) = \mu_1(t), \quad a_{24}(t) = \mu_2(t), \]
\[ a_{32}(t) = \lambda(t), \quad a_{33}(t) = - (\lambda(t) + \mu_1(t) + \gamma(t)), \quad a_{35}(t) = \mu_2(t), \]
\[ a_{44}(t) = - (\lambda(t) + \mu_2(t) + \gamma(t)), \quad a_{45}(t) = \mu_1(t), \quad a_{53}(t) = \lambda(t), \]
\[ a_{i,i+1}(t) = \mu(t), \quad \text{for } i \geq 5, \]
\[ a_{i,k}(t) = - (\lambda(t) + \mu(t) + \gamma(t)), \quad \text{for } i \geq 5, \]
\[ a_{i+1,i}(t) = \lambda(t), \quad \text{for } i \geq 4. \]

The solution of this equation can be written in the form
\[ \mathbf{p}(t) = U^*(t,0) \mathbf{p}(0) + \int_0^t U^*(t,\tau) \mathbf{g}(\tau) \, d\tau, \quad (3.6) \]
where \( U^*(t,s) \) is the Cauchy operator of the corresponding homogeneous equation
\[ \frac{dz}{dt} = A^*(t) z. \quad (3.7) \]

Denote by \( E(t,k) = E\{X(t)|X(0) = k\} \) the mathematical expectation (the mean) of \( X(t) \) at the moment \( t \) if \( X(0) = k \).

**Definition 3.1.** Markov chain \( X(t) \) is called weakly ergodic, if \( \lim_{t \to \infty} \| \mathbf{p}^1(t) - \mathbf{p}^2(t) \| = 0 \) for any initial conditions \( \mathbf{p}^1(0) = \mathbf{p}^1 \in \Omega, \quad \mathbf{p}^2(0) = \mathbf{p}^2 \in \Omega \). In this situation any \( \mathbf{p}^1 \) can be considered as a quasi-stationary distribution of the chain \( X(t) \).

**Definition 3.2.** A Markov chain \( X(t) \) has the limiting mean \( \phi(t) \), if \( |E(t;k) - \phi(t)| \to 0 \) as \( t \to \infty \) for any \( k \).

We use the main approach for studying the rate of convergence of continuous-time Markov chains. It is related to the notion of the logarithmic norm of a linear operator function and the corresponding bounds of Cauchy operator, see [15, 28, 32].

That is, if \( B(t), \ t \geq 0 \) is a one-parameter family of bounded linear operators on a Banach space \( \mathcal{B} \), then
\[ \gamma(B(t)) = \lim_{h \to +0} \frac{\| I + hB(t) \| - 1}{h} \quad (3.8) \]
is called the logarithmic norm of the operator \( B(t) \). If \( \mathcal{B} = l_1 \) then the operator \( B(t) \) is given by the matrix \( B(t) = (b_{ij}(t))_{i,j=0}^\infty, \ t \geq 0 \), and the logarithmic norm of \( B(t) \) can be found explicitly:
\[ \gamma(B(t)) = \sup_j \left( b_{jj}(t) + \sum_{i \neq j} |b_{ij}(t)| \right), \quad t \geq 0. \quad (3.9) \]

Moreover, the following bound holds:
\[ \| U^*(t,s) \| \leq e^{\int_s^t \gamma(B(\tau)) \, d\tau}. \quad (3.10) \]

Let \( \{d_i\}, \ i \geq 0 \) be a sequence of positive numbers such that \( \inf_i d_i = d > 0 \). Let \( D = \text{diag}(d_0, d_1, d_2, \ldots) \) be the corresponding diagonal matrix and \( l_{1D} \) be a space of vectors
\[ l_{1D} = \{ x = (x_0, x_1, x_2, \ldots)|\|x\|_{1D} = \| Dx \| < \infty \}. \quad (3.11) \]

Put \( \mathbf{w}(t) = D\mathbf{z}(t) \), then (3.7) implies the equation
\[ \frac{d\mathbf{w}}{dt} = A_{D}^*(t)\mathbf{w}(t), \quad (3.12) \]
where \( A_{D}^*(t) = DA^*(t)D^{-1} \) with entries \( a_{ij}^*_{D}(t) = \frac{d}{dt}a_{ij}^*(t) \).
Then one has instead of (3.9):

\[
\gamma(A^* (t))_{1D} = \sup_j \left( a_{jj}^*(t) + \sum_{i \neq j} \frac{d_i}{d_j} |a_{ij}^*(t)| \right), \quad t \geq 0,
\]

and the following bound:

\[
\|U^*(t,s)\|_{1D} \leq e^{\int_0^s \gamma(A^*(\tau))_{1D} d\tau}.
\]

**Theorem 3.3.** Let catastrophe rate be essential, i.e. let

\[
\int_0^\infty \gamma(t) \, dt = +\infty.
\]

Then the queue-length process \(X(t)\) is weakly ergodic in the uniform operator topology and the following bound hold

\[
\|p^*(t) - p^{**}(t)\| \leq e^{-\int_0^t \gamma(\tau) \, d\tau} \|p^*(0) - p^{**}(0)\| \leq 2 e^{-\int_0^t \gamma(\tau) \, d\tau},
\]

for any initial conditions \(p^*(0), p^{**}(0)\) and any \(t \geq 0\).

**Proof.** We have

\[
\gamma(A^* (t)) = \sup_i \left( a_{ii}^*(t) + \sum_{j \neq i} a_{ij}^*(t) \right) = -\gamma(t),
\]

hence, \(\|U^*(t,s)\| \leq e^{-\int_0^s \gamma(\tau) \, d\tau}\), and we obtain

\[
\|p^*(t) - p^{**}(t)\| \leq \|U^*(t,0)\| \|p^*(0) - p^{**}(0)\| \leq e^{-\int_0^t \gamma(\tau) \, d\tau} \|p^*(0) - p^{**}(0)\| \leq 2 e^{-\int_0^t \gamma(\tau) \, d\tau},
\]

for any initial conditions \(p^*(0), p^{**}(0)\) and any \(t \geq 0\). \(\Box\)

**Remark 1.** First results in this direction were obtained in [31]. The corresponding bounds for homogeneous situation follow by the application of well-known Doeblin condition, see the detailed consideration for discrete-time Markov chains in [24].

Let us consider bounds in “weighted” norms. Let \(\{d_i\}, \, 1 = d_0 \leq d_1 \leq \ldots\) be a non-decreasing sequence, and \(D = \text{diag}(d_0, d_1, d_2, \ldots)\) be the corresponding diagonal matrix. Let \(l_1D\) be the space of vectors such that (3.11) holds. Note that all diagonal elements of \(A^*(t)\) are negative and all off-diagonal elements of \(A^*(t)\) are nonnegative.

Put

\[
\gamma_{**}(t) = \inf_i \left( |a_{ii}^*(t)| - \sum_{j \neq i} \frac{d_j}{d_i} a_{ji}^*(t) \right).
\]

Acknowledgments
Consider (3.4) as a differential equation in the space of sequences \( l_{1D} \). We have

\[
\|A^*(t)\|_{1D} = \|DA^*(t)D^{-1}\| = \sup_i \left| a^*_i(t) + \sum_{j \neq i} \frac{d_j}{d_i} a^*_j(t) \right|
\]

\[
= \sup_i \left( 2|a^*_i(t)| + \sum_{j \neq i} \frac{d_j}{d_i} a^*_j(t) - |a^*_i(t)| \right) \leq 2\sup_i |a^*_i(t)| - \gamma_*(t) \leq 2L(t) - \gamma_*(t),
\]

hence the operator function \( A^*(t) \) is bounded on the space \( D \) and we can apply the same approach to equation (3.4) in the space \( l_{1D} \). Now the equality

\[
\gamma(A^*(t))_{1D} = \gamma(DA^*(t)D^{-1}) = \sup_i \left( a^*_i(t) + \sum_{j \neq i} \frac{d_j}{d_i} a^*_j(t) \right) = -\gamma_*(t),
\]

implies the following statement.

**Theorem 3.4.** Let \( \{d_i\}, 1 = d_0 \leq d_1 \leq \ldots \) be a non-decreasing sequence such that,

\[
\int_0^\infty \gamma_*(t) \, dt = +\infty.
\]

Then the following bound on the rate of convergence holds:

\[
\|p^*(t) - p^{**}(t)\|_{1D} \leq e^{-\int_0^t \gamma_*(\tau) \, d\tau} \|p^*(0) - p^{**}(0)\|_{1D},
\]

for any initial conditions \( p^*(0), p^{**}(0) \) and any \( t \geq 0 \).

Let \( l_{1E} = \{z = (p_1, p_2, \ldots)\} \) be a space of sequences such that

\[
\|z\|_{1E} = \sum_{k \geq 1} k|p_k| < \infty.
\]

Put \( W = \inf_{k \geq 1} \frac{d_k}{k} \). Then \( W\|z\|_{1E} \leq \|z\|_{1D} \).

**Corollary 1.** Let a sequence \( \{d_i\} \) be such that (3.22) holds, and, let moreover \( W > 0 \). Then \( X(t) \) has the limiting mean, say \( \phi(t) = E(t, 0) \), and the following bound holds:

\[
|E(t, j) - E(t, 0)| \leq \frac{1 + d_j}{W} e^{-\int_0^t \gamma_*(\tau) \, d\tau},
\]

for any \( j \) and any \( t \geq 0 \).

We can obtain more explicit bounds for the model in question. Namely, put \( d_0 = 1 \) and \( d_{k+1} = (1 + \epsilon)d_k \) for \( k \geq 0 \).

Then we have:

\[
|a^*_{00}(t)| - \sum_{j \neq 0} \frac{d_j}{d_0} a^*_j(t) = \gamma(t) - \epsilon\xi(t),
\]

\[
|a^*_{11}(t)| - \sum_{j \neq 1} \frac{d_j}{d_1} a^*_j(t) = \gamma(t) - \epsilon\lambda(t),
\]

\[
|a^*_{22}(t)| - \sum_{j \neq 1} \frac{d_j}{d_2} a^*_j(t) = \gamma(t) + \frac{\epsilon}{1 + \epsilon}\mu_1(t) - (2\epsilon + \epsilon^2)\lambda(t),
\]
for any initial conditions

\[ \gamma^{**}(t) = \inf_i \left( |a^{*}_{ii}(t)| - \sum_{j \neq 1} \frac{d_j}{d_i} a^{*}_{ji}(t) \right) \leq \gamma(t) - \epsilon \nu(t), \]

where \( \nu(t) = \max(\xi, 2\lambda(t)) \), and we obtain the following statement.

**Theorem 3.5.** Let \( 0 < \epsilon << 1 \).

Then the following bound on the rate of convergence holds:

\[ \|p^*(t) - p^{**}(t)\|_{1D} \leq e^{-\int_0^t (\gamma(\tau) - \epsilon \nu(\tau)) d\tau} \|p^*(0) - p^{**}(0)\|_{1D}, \tag{3.25} \]

for any initial conditions \( p^*(0), p^{**}(0) \) and any \( t \geq 0 \).

Moreover,

\[ |E(t, j) - E(t, 0)| \leq \frac{1 + d_j}{W} e^{-\int_0^t (\gamma(\tau) - \epsilon \nu(\tau)) d\tau}, \tag{3.26} \]

for any \( j \) and any \( t \geq 0 \), where \( W = \inf_{k \geq 1} \frac{(1+\epsilon)^k}{k} \).

One can use this approach and formula (3.6) for obtaining the bounds of state probabilities in the other way. Let us consider the space of sequences \( l_{1D} \), and put \( X(0) = 0 \).

Then \( p(0) = 0 \) and we have

\[ \sum_i d_i p_i(t) = \|p(t)\| \leq \int_0^t ||U^t(\tau) g(\tau)|| d\tau \leq \int_0^t \gamma^*(\tau) e^{-\int_0^\tau \gamma^{**}(\tau') d\tau'}, \tag{3.27} \]

in the 1D-norm.

Therefore

\[ d_N \sum_{i \geq N} p_i(t) \leq \|p(t)\| \leq \int_0^t \gamma^*(\tau) e^{-\int_0^\tau \gamma^{**}(\tau') d\tau'}, \tag{3.28} \]

\[ \sum_{i \geq N} p_i(t) \leq d_N^{-1} \int_0^t \gamma^*(\tau) e^{-\int_0^\tau \gamma^{**}(\tau') d\tau'}, \tag{3.29} \]

and the following statement is correct.

**Corollary 2.** Let sequence \( \{d_i\} \) be such that (3.22) holds. Then the following bound holds:

\[ \sum_{i < N} p_i(t) \geq 1 - d_N^{-1} \int_0^t \gamma^*(\tau) e^{-\int_0^\tau \gamma^{**}(\tau') d\tau'}, \tag{3.30} \]
if $X(0) = 0$ and any $t \geq 0$.

4. **Numerical example.** In this section, based on the obtained results in the above sections, we study the general behavior of the proposed system. We take the following rates through the numerical example

$$
\lambda(t) = 3(1 + \sin 2\pi t), \quad \mu_1(t) = 5(1 + \cos 2\pi t), \quad \mu_2(t) = 4(1 + \cos 2\pi t), \quad \gamma(t) = 0.3(1 + \sin 2\pi t), \quad \xi(t) = 0.2(1 + \sin 2\pi t).
$$

Let $\epsilon = 0.025$. Put $d_0 = 1$ and $d_{k+1} = (1 + \epsilon)d_k$ for $k \geq 0$.

Then we obtain $\gamma^*(t) = 0.15(1 + \sin 2\pi t)$, and Theorem 3 gives us the following bounds on the rate of convergence:

$$
\|p^*(t) - p^{**}(t)\|_{1D} \leq e^{-\int_0^t (\gamma(\tau) - \epsilon\nu(\tau)) d\tau} \|p^*(0) - p^{**}(0)\|_{1D} \leq Me^{-0.15t} \|p^*(0) - p^{**}(0)\|_{1D},
$$

$$
|E(t, j) - E(t, 0)| \leq \frac{M(1 + d_j)}{W} e^{-0.1t},
$$

for any $j$ and any $t \geq 0$, where $M < 2$ and $W = \inf_{k \geq 1} \frac{(1+0.025)^k}{k}$. Then our standard approach (see detailed consideration in [36, 40]) can be applied and the corresponding time-dependent and limiting probabilities and mathematical expectation for the queue-length process (namely, $E = p_{10} + p_{01} + \sum_{k \geq 1}(k+1)p_{k1}$) can be computed by truncations of original process $X(t)$.

It is interesting to note that fairly accurate estimates are obtained already while considering an approximating process with states up to $(96.1)$. The corresponding plots for respective initial conditions $P_{10} = 1$ (which corresponds to $X(0) = 2$) are shown in figures 1–5.

![Figure 1. Probability of repair $Q(t)$](image-url)

**Acknowledgments.** Theorem 3.5 was formulated, proved and verified respectively by A. Zeifman and V. Korolev who were supported by the Russian Science Foundation (grant 18-11-00155).
Figure 2. Probabilities $P_{00}(t)$, $P_{10}(t)$, $P_{01}(t)$ blue, green, red respectively.

Figure 3. Probabilities $P_{11}(t)$, $P_{21}(t)$ blue and green respectively.

Figure 4. Probabilities $P_{31}(t)$, $P_{41}(t)$, $P_{51}(t)$ blue, green, red respectively.

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Figure 5. The mean value $E(t)$.

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Received November 2018; revised May 2019.

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