Generic initial ideals of some monomial complete intersections in four variables

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Abstract. Let $\mathbb{K}[x_1, x_2, x_3, x_4]$ be the polynomial ring over a field of characteristic zero. For the ideal $(x_1^{a}, x_2^{b}, x_3^{c}, x_4^{d}) \subset \mathbb{K}$, where at least one of $a$, $b$, $c$ and $d$ is equal to two, we prove that its generic initial ideal with respect to the reverse lexicographic order is the almost revlex ideal corresponding to the same Hilbert function.

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1. Introduction

Generic initial ideals play an important role in commutative ring theory. But it is very difficult to determine them, and there are only a few results which determine generic initial ideals. Even in the case of monomial complete intersections, their generic initial ideals are not determined in general. Our result is a starting point of this problem. In the polynomial rings with one or two variables, generic initial ideals are trivially determined, since Borel-fixed ideals are unique for Hilbert functions. In the case of three variables, due to the result of Ahn-Cho-Park [ACP07] or Cimpoea¸s [Cim07], the generic initial ideals of Artinian monomial complete intersections are determined. In this note we focus on the case of four variables. For the monomial complete intersections $(x_1^{a}, x_2^{b}, x_3^{c}, x_4^{d})$, where at least one of $a$, $b$, $c$ and $d$ is equal to two, we prove that their generic initial ideals are the almost revlex ideals (Theorem 9).

Throughout this note, $\mathbb{K}$ denotes a field of characteristic zero, and $\mathbb{K}$ the polynomial ring over $\mathbb{K}$. The only term order on $\mathbb{K}$ used in this note is the reverse lexicographic order with $x_1 > x_2 > \cdots$, and $\text{gin}(I)$ (resp. $\text{in}(I)$) denotes the generic initial ideal (resp. initial ideal) with respect to the reverse lexicographic order.
2. The $k$-strong Lefschetz property

In this section we review the definition of $k$-strong Lefschetz property and results needed for our main theorem.

**Definition 1 (the SLP and the $k$-SLP).** Let $A$ be a graded Artinian algebra over a field $K$, and $A = \bigoplus_{i=0}^{\infty} A_i$ its decomposition into graded components.

1. The algebra $A$ is said to have the *strong Lefschetz property* (SLP for short), if there exists an element $\ell \in A_1$ such that the multiplication map $x \mapsto \ell x : A_i \to A_{i+s}$ is full-rank for every $i \geq 0$ and $s > 0$. In this case, $\ell$ is called a *Lefschetz element*, and we also say that $(A, \ell)$ has the SLP.

2. Let $k$ be a positive integer. The algebra $A$ is said to have the *$k$-strong Lefschetz property* ($k$-SLP for short), if there exist linear elements $g_1, g_2, \ldots, g_k \in A_1$ satisfying the following two conditions.

   (i) $(A, g_1)$ has the SLP,
   (ii) $(A/(g_1, \ldots, g_{i-1}), g_i)$ has the SLP for all $i = 2, 3, \ldots, k$.

In this case, we say that $(A, g_1, \ldots, g_k)$ has the $k$-SLP. In other words, $A$ is said to have the $k$-SLP, if $A$ has the SLP with a Lefschetz element $g_1$, and $A/(g_1)$ has the $(k-1)$-SLP.

Note that the 1-SLP is nothing but the SLP, and that if a graded algebra has the $k$-SLP, then it has the $(k-1)$-SLP. Note also that the $n$-SLP is equivalent to the $(n-2)$-SLP for the quotient rings $K[x_1, x_2, \ldots, x_n]/I$, since all graded $K$-algebras $K[x_1]/J$ and $K[x_1, x_2]/J$ have the SLP [HMNW03 Theorem 4.4].

**Definition 2 (almost revlex ideals).** A monomial ideal $I$ is called an *almost revlex ideal*, if the following condition holds: for each monomial $u$ in the minimal generating set of $I$, every monomial $v$ with $\deg v = \deg u$ and $v >_{\text{revlex}} u$ belongs to $I$.

It is clear that if two almost revlex ideals have the same Hilbert function, then they are equal. In addition, it is easy to see that almost revlex ideals are Borel-fixed [HW08 Remark 11].

We write Hilbert functions of graded algebras as $h$-vectors $h = (h_0, h_1, \ldots, h_s)$. A Hilbert function $h$ is said to be *unimodal*, if there exist an integer $a$ such that $h_0 \leq h_1 \leq \cdots \leq h_a \geq h_{a+1} \geq \cdots \geq h_s$. A Hilbert function $h = (h_0, h_1, \ldots, h_s)$ ($h_s \neq 0$) is said to be *symmetric*, if $h_i = h_{s-i}$ for every $i \geq 0$. The *difference* $\Delta h$ of $h$ is defined by

$$\Delta h_i = \max\{h_i - h_{i-1}, 0\} \quad (i = 0, 1, 2, \ldots),$$

where $h_{-1}$ is defined as zero. We define the $k$th difference $\Delta^k h$ by applying $\Delta$ to $h$ $k$-times. The following is a direct consequence of [HW08 Corollary 27]

**Proposition 3.** Let $I \subset K[x_1, x_2, x_3, x_4]$ be a graded Artinian ideal whose quotient ring has the 2-SLP. Suppose that the Hilbert function $h$ of $K[x_1, x_2, x_3, x_4]/I$ is symmetric. Then the generic initial ideal $\text{gin}(I)$ is the unique almost revlex ideal for the Hilbert function $h$. \hspace{1cm} \square
We conclude this section by an analogue of Wiebe’s result [Wie04, Proposition 2.9].

**Proposition 4.** Let $I$ be a graded Artinian ideal of $R = K[x_1, x_2, \ldots, x_n]$, and let $1 \leq k \leq n$. If $R/\text{in}(I)$ has the $k$-SLP, then $R/I$ has the $k$-SLP.

**Proof.** Let $H_A(t)$ denotes the Hilbert function of a graded algebra $A$. From the proof of [Wie04, Proposition 2.9], we have

$$H_{R/(I+(g_1, \ldots, g_{i-1}, g_i'))}(t) \leq H_{R/(\text{in}(I)+(g_1, \ldots, g_{i-1}, g_i'))}(t)$$

for generic linear forms $g_1, \ldots, g_i$, $s \geq 1$ and all $t \geq 0$. In order to prove our claim, it is enough to show that the Hilbert function of $R/(I+(g_1, \ldots, g_{i-1}, g_i'))$ coincides with that of $R/(\text{in}(I)+(g_1, \ldots, g_{i-1}, g_i'))$ for every $i = 1, 2, \ldots, k$ under the assumption that the Hilbert function of $R/(I+(g_1, \ldots, g_{i-1}))$ is equal to that of $R/(\text{in}(I)+(g_1, \ldots, g_{i-1}))$. Set $h = H_{R/(I+(g_1, \ldots, g_{i-1}))}$. By our assumption, $R/(\text{in}(I)+(g_1, \ldots, g_{i-1}))$ has the SLP. Hence, it follows that the Hilbert function of $R/(\text{in}(I)+(g_1, \ldots, g_{i-1}, g_i'))$ is equal to the sequence $(b_t)_{t \geq 0}$:

$$b_t = \max\{h_t - h_{t-s}, 0\},$$

where $h_t = 0$ for $t < 0$. Furthermore one can easily check that

$$b_t \leq H_{R/(I+(g_1, \ldots, g_i'))}(t)$$

for all $t \geq 0$. Hence it follows from (1) that

$$H_{R/(I+(g_1, \ldots, g_i'))}(t) = H_{R/(\text{in}(I)+(g_1, \ldots, g_i'))}(t)$$

for all $t \geq 0$. □

3. Main theorem

In this section we prove the main theorem (Theorem 9). For two graded Artinian $K$-algebras $A$ and $B$ having the SLP, $A \otimes_K B$ also has the SLP if their Hilbert functions are symmetric [Wat87]. But this is not the case unless both Hilbert functions are symmetric (see [HW03, Example 5], e.g.). The following lemma gives a necessary and sufficient condition for $A \otimes_K K[y]/(y^2)$ to have the SLP, when the Hilbert function of $A$ is not necessarily symmetric.

**Lemma 5.** Let $A$ be a graded Artinian $K$-algebra having the SLP, and $B = K[y]/(y^2)$. The tensor product $A \otimes_K B$ has the SLP, if and only if the Hilbert function $h = h_A$ of $A$ satisfies the following two conditions:

(C1) For any $h_i < h_{i+1}$, there exist at most one $j$ such that $h_i < h_j < h_{i+1}$.

(C2) For any $h_i > h_{i+1}$, there exist at most one $j$ such that $h_i > h_j > h_{i+1}$.
Proof. First we recall that, given a graded Artinian $K$-algebra $A$ and a linear form $g \in A$, we may consider $A$ a $K[x]$-module via $x \ast a = ga$. Then, by the structure theorem of finitely generated module over PID, $A$ decomposes uniquely as direct sum of $K[x]/(x^i)\{-b_i\}$ where the shift $[-b_i]$ indicates that the generators sits in degree $b_i$. So the Hilbert function of $A$ is $\sum \lambda^b_i (1 - \lambda^{a_i})/(1 - \lambda)$. Which $b_i$ actually occur is given by the Hilbert function of $A/(g)$ which is $\sum \lambda^b_i$. Call the pairs $(b_i, a_i)$ the basic invariants of $(A, g)$ and we can easily prove that $y$ is a SL element for $A$ if and only if the basic invariants have the following properties:

\[
\text{if } (b_i, a_i) \text{ and } (b_j, a_j) \text{ are basic invariants with } b_i < b_j \text{ then } b_i + a_i \geq b_j + a_j,
\]

(2)

Next we consider the tensor product $C = K[x]/(x^a) \otimes_K B \simeq K[x, y]/(x^a, y^2)$ as a $K[z]$-module, where the action of $K[z]$ induced on $C$ is given by $z \ast f(x, y) = (x + y)f(x, y)$. Then, thanks to [HW03 Proposition 8], the tensor product $D = K[x]/(x^a)[[z]] \otimes_K B$ decomposes into two modules, $D = K[z]/(z^{|a|})\{-b_i\} \oplus K[z]/(z^{|b|})\{-b_i + 1\}$. If $a_i = 1$, then the second component does not appear in $D$.

Thus $A \otimes_K B$ has the SLP, if and only if every combination of two basic invariants of $A \otimes_K B$ satisfies Condition (2).

Furthermore we consider the following conditions:

(C1)$'$ If $b_i = b_j$ then $|a_i - a_j| \leq 1$.

(C2)$'$ If $b_i + a_i = b_j + a_j$ then $|b_i - b_j| \leq 1$.

Note that these conditions are equivalent to our conditions, that is, (C1) is equivalent to (C1)$'$, and (C2) is equivalent to (C2)$'$.

Suppose that $A \otimes_K B$ has the SLP. Hence the basic invariants $\{(b_i, a_i+1), (b_i+1, a_i-1)\}$ of $A \otimes_K B$ satisfy Condition (2). First assume that there exist $i$ and $j$ such that $b_i = b_j$ and $a_i \geq a_j + 2$. Then $b_i + 1 > b_j$ and $(b_i + 1) + (a_i - 1) > b_j + (a_j + 1)$. This means that two basic invariants $(b_i + 1, a_i - 1)$ and $(b_j, a_j + 1)$ do not satisfy Condition (2). Next assume that there exist $i$ and $j$ such that $b_i + a_i = b_j + a_j$ and $b_i + 2 \leq b_j$. Then $b_i + 1 < b_j$ and $(b_i + 1) + (a_i - 1) < b_j + (a_j + 1)$. This also means that two basic invariants $(b_i + 1, a_i - 1)$ and $(b_j, a_j + 1)$ do not satisfy Condition (2). Thus the Hilbert function of $A$ satisfies Conditions (C1) and (C2).

Conversely suppose that the basic invariants $(b_i, a_i)$ satisfy Conditions (C1)$'$ and (C2)$'$. We can check that the basic invariants $\{(b_i, a_i+1), (b_i+1, a_i-1)\}$ of $A \otimes_K B$ satisfy Condition (2) as follows. Take two basic invariants $(b_i, a_i + 1)$ and $(b_i + 1, a_i - 1)$ for example. If $b_i < b_i + 1$, then there are two possibilities. (i) When $b_i = b_i$, we have $|a_i - a_j| \leq 1$ from (C1)$'$. Hence $b_i + (a_i + 1) = (b_i + 1) + (a_i - 1) + (a_i - a_j + 1) \geq (b_i + 1) + (a_j - 1)$. (ii) When $b_i < b_i$, we have $b_i + a_i \geq b_i + a_i$ from Condition (2) for $A$. Hence $b_i + (a_i + 1) > (b_i + 1) + (a_j - 1)$, and Condition (2) for $A \otimes_K B$ is satisfied. If $b_i > b_i + 1$, then we have $b_i + a_i \neq b_i + a_i$ from the contraposition of (C2)$'$ and $b_i + a_i \leq b_i + a_i$ from Condition (2) for $A$. Hence $b_i + (a_i + 1) \leq (b_i + 1) + (a_j - 1)$. Thus Condition (2) for $A \otimes_K B$ is satisfied.
Calculations are similar for other choices \((b_i, a_i + 1)\) and \((b_j, a_j + 1)\), or \((b_i + 1, a_i - 1)\) and \((b_j + 1, a_j - 1)\) of basic invariants, and thus \(A \otimes_K B\) has the SLP. \(\square\)

The following two lemmas give sufficient conditions for the tensor product \(A \otimes_K A[y]/(y^2)\) to have the \(k\)-SLP. In Lemma 6, \(A\) is a quotient ring by a Borel-fixed ideal having the \(k\)-SLP. In Lemma 7, \(A\) is any graded algebra having the \(k\)-SLP.

**Lemma 6.** Let \(A = R/I\) be a graded Artinian \(K\)-algebra having the \(k\)-SLP, where \(R = K[x_1, x_2, \ldots, x_n]\), and \(I\) is a Borel-fixed ideal of \(R\). Let \(h_A\) be the Hilbert function of \(A\). Let \(B = K[y]/(y^2)\).

Suppose that every \(s\)-th difference \(\Delta^s h_A\) \((0 \leq s \leq k - 1)\) satisfies Conditions (C1) and (C2) of Lemma 5. Suppose also that every \(s\)-th difference \(\Delta^s h_A\) \((0 \leq s \leq k - 2)\) satisfies the following condition:

(C3) For \(h = (h_0, h_1, \ldots, h_c)\), there are two or more \(i\) such that \(h_i = \max_j \{h_j\}\), or there is only one \(i\) such that \(h_i = \max_j \{h_j\}\), and \(h_{i-1} \geq h_{i+1}\). Then the tensor product \(A \otimes_K B\) has the \(k\)-SLP.

**Proof.** We prove the lemma by induction on \(k\). For \(k = 1\), the lemma follows from Lemma 5.

Let \(k > 1\), and assume that the lemma holds up to \(k - 1\). Since \(I\) is Borel-fixed, \(x_n\) is a Lefschetz element of \(A\) [Wie04, Lemma 2.7]. Hence by the assumption of induction, \(A \otimes_K B\) has the \(k\)-SLP, and \(x_n + y \in A \otimes_K B \cong A[y]/(y^2)\) is a Lefschetz element [Wat67]. Thus it suffices to show that \(A \otimes_K B/(x_n + y)\) has the \((k-1)\)-SLP.

We have

\[
A \otimes_K B/(x_n + y) \cong K[x_1, x_2, \ldots, x_n, y]/(I) + (x_n + y) + (y^2) \\
\cong K[x_1, x_2, \ldots, x_n]/I + (x_n^2) \\
\cong A/(x_n^2).
\]

[Case 1. There are two or more \(i\) such that \(h_i = \max_j \{h_j\}\)] In this case, if a monomial \(u \in R\) not divisible by \(x_n\) is a standard monomial (i.e., a monomial not belonging to \(I\)), then \(ux_n\) is also a standard monomial, since \(x_n\) is a Lefschetz element. Therefore we have an algebra isomorphism \(A/(x_n^2) \cong A/(x_n) \otimes_K K[z]/(z^2)\), since \(A\) is a quotient of a Borel-fixed ideal. Here \(A/(x_n)\) has the \((k-1)\)-SLP, and has the Hilbert function \(\Delta h_A\). Therefore it follows from the assumption of induction that \(A/(x_n^2) \cong A/(x_n) \otimes_K K[z]/(z^2)\) has the \((k-1)\)-SLP.

[Case 2. There is only one \(i\) such that \(h_i = \max_j \{h_j\}\), and \(h_{i-1} \geq h_{i+1}\)] In this case, for a standard monomial \(u \in R\) not divisible by \(x_n\), \(ux_n\) is also standard if \(\deg(u) < i\), and is not standard if \(\deg(u) = i\). Therefore we have an algebra isomorphism

\[
A/(x_n^2) \cong [A/(x_n) \otimes_K K[z]/(z^2)]/[m^{i+1}],
\]

where \(m\) is the graded maximal ideal of \(A/(x_n) \otimes_K K[z]/(z^2)\). Namely \(A/(x_n^2)\) is isomorphic to the algebra obtained by dropping the homogeneous component of
the socle degree of \( A/(x_n) \otimes_K K[z]/(z^2) \). In general, for an algebra having the k-SLP, the algebra obtained by dropping the homogeneous component of the socle degree again has the k-SLP. Hence \( A/(x_n^2) \) has the \((k - 1)-\text{SLP}\).

In both cases we have proved that \( A/(x_n^2) \) has the \((k - 1)-\text{SLP}\), and by induction we have proved the lemma.

\[ \square \]

**Lemma 7.** Let \( A \) be a graded Artinian \( K \)-algebra having the \( k \)-SLP, and \( h_A \) its Hilbert function. Let \( B = K[y]/(y^2) \). Suppose that every \( s \)-th difference \( \Delta^s h_A \) \( (0 \leq s \leq k - 1) \) satisfies Conditions \((C1) \) and \((C2) \) of Lemma 5, and suppose also that every \( s \)-th difference \( \Delta^s h_A \) \( (0 \leq s \leq k - 2) \) satisfies Condition \((C3) \) of Lemma 6. Then the tensor product \( A \otimes_K B \) has the \( k \)-SLP.

**Proof.** Let \( A = R/I \), where \( R = K[x_1, x_2, \ldots, x_n] \), and \( I \) a graded ideal of \( R \). Let \( g \in GL_n(K) \) be an element for which \( \deg(g) = \deg(I) \), and let \( \tilde{g} \in GL_{n+1}(K) \) be the element given by embedding \( g \) into first \( n \) dimensions. Then we have

\[
R/\deg(g) \otimes_K B = R/\deg(I) \otimes_K B \\
\simeq R[y]/(\deg(gI) + (y^2)) \\
= R[y]/(\deg(I) + (y^2)) \\
\simeq K[x_1, x_2, \ldots, x_n, y]/\deg(I + (y^2)),
\]

by use of [Eis95, Proposition 15.15] for example. Here \( R/\deg(I) \) has the \( k \)-SLP, since \( R/I \) has the \( k \)-SLP if and only if \( R/\deg(I) \) has the \( k \)-SLP [HW08, Proposition 18]. It follows from Lemma 8 that \( R/\deg(g) \otimes_K B \) has the \( k \)-SLP. Hence \( K[x_1, x_2, \ldots, x_n, y]/\deg(I + (y^2)) \) has the \( k \)-SLP by Proposition 3. Thus \( A \otimes_K B \simeq K[x_1, x_2, \ldots, x_n, y]/\deg(I + (y^2)) \) has the \( k \)-SLP.

\[ \square \]

We need the following property on Hilbert functions of monomial complete intersections of three variables for the proof of the main theorem.

**Lemma 8.** Let \( a, b \) and \( c \) be positive integers. Define the \( h \)-vector \( h = (h_0, h_1, \ldots) \) by

\[
\sum_{j=0}^{a+b+c-3} h_j t^j = (1 + t + \cdots + t^{a-1})(1 + t + \cdots + t^{b-1})(1 + t + \cdots + t^{c-1}).
\]

(4)

Then its difference \( h' = \Delta h = (h'_0, h'_1, \ldots, h'_j, \ldots) \) is a piecewise linear function in \( j \), and the coefficient of \( j \) is at least \(-2 \) in each linear piece.

**Proof.** The right-hand side of Equation 4 is equal to

\[
\sum_{j=0}^{a+b+c-3} \sum_{0 \leq k \leq a-1, 0 \leq l \leq b-1, 0 \leq m \leq c-1} t^{k+l+m},
\]

and hence \( h_j \) is equal to the number of lattice points on the plane \( P_j = \{v = (k, l, m) \in \mathbb{Z}^3 : k + l + m = j \text{ and } k, l, m \geq 0 \} \) satisfying \( 0 \leq k \leq a-1, 0 \leq l \leq b-1 \leq c-1 \).
and $0 \leq m \leq c - 1$. Thus we have

$$h_j = \#P_j - \# \{ v \in P_j : k \geq a \} - \# \{ v \in P_j : l \geq b \} - \# \{ v \in P_j : m \geq c \} + \# \{ v \in P_j : k \geq a, l \geq b \} + \# \{ v \in P_j : k \geq a, m \geq c \} + \# \{ v \in P_j : l \geq b, m \geq c \}
\[
= \binom{j + 2}{2} - \binom{j - a + 2}{2} - \binom{j - b + 2}{2} - \binom{j - c + 2}{2} + \binom{j - a - b + 2}{2} + \binom{j - b - c + 2}{2} + \binom{j - a - c + 2}{2}
\]$$

for $0 \leq j \leq a + b + c - 3$, where the binomial coefficients $\binom{n}{m}$ are defined as zero if $n < 0$. Note that the formula $(\binom{n-1}{m-1}) + (\binom{n-1}{m})$ still holds unless $(n, m) = (0, 0)$, and that $(\binom{n}{m}) = \max \{0, n\}$. Hence we have

$$h_j - h_{j-1} =
\begin{align*}
&j + 1 - \max \{0, j - a + 1\} - \max \{0, j - b + 1\} - \max \{0, j - c + 1\} \\
&+ \max \{0, j - a - b + 1\} + \max \{0, j - b - c + 1\} + \max \{0, j - a - c + 1\}.
\end{align*}
$$

Therefore $h_j' = \max \{0, h_j - h_{j-1}\}$ is a piecewise linear function in $j$, in which the coefficients of $j$ are at least $-2$ (and at most 1).

Finally we have the main theorem.

**Theorem 9.** Let $R = K[x_1, x_2, x_3, x_4]$, and $I = (x_1^a, x_2^b, x_3^c, x_4^d)$, where at least one of $a, b, c$ and $d$ is equal to two. Then the generic initial ideal of $I$ is equal to the almost revlex ideal corresponding to the same Hilbert function.

**Proof.** If one of $a, b, c$ and $d$ is equal to one, then the theorem is reduced to the case of three variables, and follows from [ACP07] or [Cim07]. We consider the case where $a, b, c, d \geq 2$. First we show that $R/I$ has the 3-SLP for $a, b, c \geq 2$ and $d = 2$. Let $A = K[x_1, x_2, x_3]/(x_1^a, x_2^b, x_3^c)$, and $h_A$ the Hilbert function of $A$. Then $A$ has the 3-SLP, and $h_A$ satisfies Conditions (C1), (C2) and (C3), since $h_A$ is unimodal and symmetric. The difference $\Delta h_A$ is of the form $(1, 2, \ldots, k, h'_k, h'_{k+1}, \ldots, h'_m)$, where $k \geq h'_k \geq h'_{k+1} \geq \cdots \geq h'_m \geq 0$. Hence $\Delta h_A$ satisfies Condition (C1). Condition (C2) for $\Delta h_A$ also holds, since $h'_j - h'_{j+1} \leq 2$ for $j = k - 1, k, \ldots, m$ ($h'_{k-1} = k$ and $h'_{m+1} = 0$) by Lemma 8. Therefore it follows from Lemma 7 that $R/I \simeq A \otimes_K K[x_4]/(x_4^2)$ has the 3-SLP.

Since the $k$-SLP is independent of the permutation of the variables, $R/I$ has the 3-SLP, if at least one of $a, b, c$ and $d$ is equal to two. As noted in Definition 11 the 2-SLP, 3-SLP and 4-SLP are equivalent for quotient rings of $R$, it follows from Proposition 3 that $\text{gin}(I)$ is almost revlex. \qed
We conclude this note with the smallest example which does not fit into our theorem.

**Remark 10.** Let \( R = K[x_1, x_2, x_3, x_4] \) and \( I = (x_1^3, x_2^3, x_3^3, x_4^3) \). We can show that the quotient ring \( R/I \) does not have the 2-SLP as follows.

\( R/I \) has the SLP, and we can fix a Lefschetz element \( x_1 + x_2 + x_3 + x_4 \) by changing the coordinate if needed, since Lefschetz elements of \( R/I \) are of the form \( ax_1 + bx_2 + cx_3 + dx_4 \) \((abcd \neq 0)\). Put

\[
A = K[x_1, x_2, x_3]/(x_1^3, x_2^3, x_3^3, (x_1 + x_2 + x_3)^3) \simeq R/I + (x_1 + x_2 + x_3 + x_4).
\]

The Hilbert function of \( A \) is \((1, 3, 6, 3)\). Let \( \ell = ax_1 + bx_2 + cx_3 \) be a general linear form of \( A \), and we look at the rank of the linear mapping \( \times \ell^3 : A_1 \rightarrow A_4 \). Using a computer we have

\[
\begin{align*}
x_1 \ell^3 &= 3a(b-c)^2x_1^2x_3^2 + 3b(2a-c)(b-c)x_1x_2x_3 + 3b^2(a-c)x_2^2x_3^2, \\
x_2 \ell^3 &= 3a^2(b-c)x_1^2x_3^2 + 3a(a-c)(2b-c)x_1x_2x_3^2 + 3b(a-c)^2x_2^2x_3^2, \\
x_3 \ell^3 &= -3a^2(b-c)x_1^2x_3^2 - 3ab(a+b-2c)x_1x_2x_3^2 - 3b^2(a-c)x_2^2x_3^2,
\end{align*}
\]

which are equations in \( A \), and \( \{x_1^2x_3^2, x_1x_2x_3^2, x_2^2x_3^2\} \) is a basis of \( A_4 \). We can show that the determinant of \( \times \ell^3 : A_1 \rightarrow A_4 \) is equal to zero independent of \( a, b \) and \( c \), and therefore \( A \) does not have the SLP. Hence \( R/I \) does not have the 2-SLP.

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