Quantum Bound States on Some Hyperbolic Surfaces

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Abstract. The aim of this work is to highlight results of energy eigenstates on some noncompact finite hyperbolic surfaces. Such systems are known to exhibit both continuous and discrete spectra and are dependent on the subgroups of the modular group that underlie these surfaces. We study explicitly the cases of Maass cusp forms on the singly punctured two-torus and the triply punctured two-sphere for their eigenvalues. The eigenvalues for the torus system are doubly degenerate while for the sphere case, the eigenvalues are nondegenerate. We also note that the lowest eigenvalue of the sphere system is larger than that of the torus system.

1. Introduction
Systems of a particle on hyperbolic surfaces are known to exhibit chaotic properties ever since the late 19th century [1]. This led researchers to study their quantum versions to understand quantum chaos [2]. Mathematically, hyperbolic surfaces exhibit richer family of surfaces with nontrivial topology, characterised by their genus and number of cusps/punctures (points at infinity) with finite or infinite area [3]. In mathematics, there is a vast number of literature connecting the spectra of the hyperbolic laplacian with number theory [3,4].

In physics, however, there is considerably less literature on the nature of quantum states on hyperbolic surfaces and is almost exclusively limited to the discussion of scattering states on noncompact finite area hyperbolic surfaces (see [5–9]). Lesser is known about the bound states on such surfaces. In this note, we will review the results of [10, 11] for the bound states on two hyperbolic surfaces namely singly punctured two-torus and triply punctured two sphere and make comparisons between their spectra.

2. Construction of the Hyperbolic Surfaces
To construct the abovementioned hyperbolic surfaces, we begin with the upper half plane model $H = \{ z = x + iy | \text{Im}(z) > 0 \}$ of the hyperbolic plane i.e. equipped with the hyperbolic metric $ds^2 = y^2(dx^2 + dy^2)$. The geodesics on this plane are given by (portions of) vertical straight lines and semicircles meeting x-axis perpendicularly. The upper half plane can also be written as the homogeneous space $H = \text{PSL}(2, \mathbb{R}) / \text{SO}(2)$ and there is a (projective) action of SL(2, $\mathbb{R}$) on $H$ via fractional linear transformations:
To construct further hyperbolic surfaces out of the hyperbolic plane, we construct tessellating polygons using discrete subgroups of \( \text{PSL}(2, \mathbb{R}) \). A standard example is that from the modular group \( \Gamma(1) \), whose fundamental domain is given by the hyperbolic triangle with vertices \( \left\{ e^{i\pi/3}, i\infty \right\} \). Elements of \( \Gamma(1) \) generated by

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]

will tessellate the whole hyperbolic plane with the equivalent hyperbolic triangles. The surface \( H/\Gamma(1) \) formed by the fundamental domain is a hyperbolic surface with one cusp.

One can build further hyperbolic surfaces by choosing smaller discrete groups like subgroups of the modular group. One particular subgroup is the commutator subgroup \( \Gamma' \) of modular group \( \Gamma(1) \). This subgroup is generated by

\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},
\]

which are themselves commutators of the generator of the modular group i.e. \( A' = ST^{-1}S^{-1}T \) and \( B' = STS^{-1}T^{-1} \). The fundamental domain of this subgroup is a quadrangle whose sides are identified by the generators (3) as shown in Fig. 1. The identification is similar to the one for the parallelogram in Euclidean plane representing a two-torus but it now includes the cusp at infinity. Figure 2 graphically depicts the type of surface \( H/\Gamma' \) being constructed.

**Figure 1.** The fundamental domain of \( \Gamma' \), a parallelogram with sides identified by \( A \) and \( B \).
Next, we consider instead the principal congruence subgroup of level two of the modular group defined by

$$\Gamma(2) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{PSL}(2, \mathbb{Z}) \mid \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \mod 2 \right\}. $$

(4)

This subgroup is generated by the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}. $$

(5)

Its fundamental domain is again a parallelogram with sides identified by generators (5) (see Fig. 3). Figure 4 graphically depicts the surface $H / \Gamma(2)$ formed by the fundamental domain.

**Figure 3.** Fundamental domain of $\Gamma(2)$ with sides identified by $A$ and $B$. 
It is interesting to note that the fundamental domains in Figs. 1 and 3 are identical parallelograms with only different side pairings and as such have the same area of $2\pi$. It is the different pairings that gave the resultant surfaces the different topological signature i.e. genus 1 and one cusp for $H/\Gamma'$ and genus 0 and three cusps for $H/\Gamma(2)$.

3. Schrödinger wave mechanics on hyperbolic surfaces

Quantum mechanics on these hyperbolic surfaces are described by writing the Schrödinger equation with a hyperbolic Laplacian. Letting $h = 1 = 2m$, the time independent Schrödinger equation is

$$H\psi = \Delta\psi = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)\psi = \lambda\psi, \quad (6)$$

where $\lambda = \frac{1}{4} + r^2$ is the energy eigenvalue. Following the convention in mathematics literature, we will call $r$ as the eigenvalue. To reflect that the wavefunctions are defined on the fundamental domain of the specified discrete group $\Gamma$, we require that the wavefunction to satisfy the automorphy condition:

$$\psi(yz) = \psi(z), \quad y \in \Gamma, \quad z \in H. \quad (7)$$

Note that this can be relaxed to include a phase factor, which will not be considered here. For bound states, we add a further cusp boundary condition i.e. $\psi \to 0$ at the cusps, which physically says that the quantum particle remains on the surface not shooting off to infinity. Such solutions are called Maass cusp forms in the literature [13].

The Maass cusp forms have been computed extensively for the modular group $\Gamma(1)$ initially by Hejhal [14] who found a linear stable algorithm that uses the periodicity under $z \to z + 1$ on the upper half plane to compute the energy eigenfunctions, together with automorphy condition under $\Gamma(1)$. The wavefunctions are expanded as the following Fourier series:
where $K_\nu$ is the K-Bessel function, ensuring the cusp boundary condition is obeyed. This is further refined by Then [15] using adaptive search method to compute large eigenvalues $r > 40000$. To make such computations more accessible, Siddig & Zainuddin [16] have ported the calculations in Mathematica, from which the present calculations are based.

The algorithm for the modular group can be extended to the case of the commutator subgroup by first modifying its fundamental region in a way that its apparent cusp appears only at $i\infty$ as it should. This can be done using (combinations of) the generators of $\Gamma'$ to transport subregions away from the x-axis. As it is done in [1, 5], the parallelogram is transformed to an 8-gon for which each side is identified with another, giving the 4-gon needed (see Fig. 5).

![Figure 5. Transformed fundamental domain of commutator subgroup.](image)

There are two advantages of this transformed domain, namely that there are no identified cusps appearing on the x-axis and that the domain is essentially six copies of the domain of the modular group (cusp width six). Hence one can now apply the same Hejhal-Then algorithm without much modifications apart from the periodicity in $z$ is now $z \rightarrow z + 6$. The new expansion of the wavefunction will be

$$\psi(x + iy) = \sum_{m=-\infty, m \neq 0} a_m y^{1/2} K_\nu(2\pi m \mid y \mid) e^{2\pi i m/6}.$$  (9)

For the triply punctured two-sphere, the computation is much harder since there are now three cusps to consider and one cannot simply apply the Hejhal-Then algorithm directly here. Instead, one need to divide the fundamental domain into subregions individually for each cusp as shown in Figure 6. Each subregion is further transformed in a way that maps each cusp to $i\infty$ and the linearized equations for all the three cusps are solved simultaneously, employing symmetry between the subregions of each cusp. For further details, see [17, 11]. The wavefunction expansion in this case is given by

$$\psi(x + iy) = \sum_{m=-\infty, m \neq 0} a_m y^{1/2} K_\nu(2\pi m \mid y \mid) e^{2\pi i m/2}.$$  (10)

where the cusp width is now two for each cusp.

4. Computational results
In [10, 11], we have managed to find quantum bound states on both singly punctured two-torus and triply punctured two-sphere respectively by computing their eigenvalues and eigenfunctions. We reproduce some of the results of [10, 11] and compare some of their characteristics.

For the punctured torus, we computed 104 eigenvalues (see Table 1) and they are doubly degenerate, shared between the odd and even cusp forms. Computationally, this degeneracy may be attributed to the arithmetical symmetries found in the Fourier coefficients as explained in [10], though it is desirable to seek a more physical or geometrical explanation. We also found that the lowest eigenvalue is lower than that of the modular group which could be explained to the larger area accessible to the particle in the singly punctured two-torus.

| r-values               | 2.956458939 | 4.513710759 | 5.815015474 | 6.783467732 | 8.315501837 | 9.275857326 | 9.465901775 | 10.239041998 | 10.694231142 | 11.427555185 | 11.874857830 | 13.420771791 | 13.544228169 |
|------------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 14.233387724           | 15.792883185 | 16.321150693 | 16.428861681 | 16.924117490 | 17.202297470 | 17.314337681 | 17.997724548 | 18.701198322 | 18.901261587 | 19.591854998 | 19.791709677 | 19.977886869 |
| 20.726567345           | 20.941713201 | 21.313410193 | 22.196935326 | 22.473313041 | 23.10712833 | 23.198611455 | 23.677587275 | 24.023489099 | 24.447714998 | 24.677812458 | 25.657928181 | 25.688294998 |
| 25.704632818           | 26.191978900 | 26.24329248 | 26.676103685 | 27.167179212 | 27.193225814 | 27.554581248 | 27.923882677 | 28.116403998 | 28.869327572 | 29.039011365 | 29.739983568 | 29.924027400 |

In Figures 6 and 7, we show the contour and density plots for the odd eigenfunction corresponding to the lowest eigenvalue \( r = 2.956458939 \) and a higher eigenvalue \( r = 29.924027400 \). They clearly show that as the energy increases, the particle’s wavefunction gets more delocalised.

Figure 6. The contour plot (a) and density plot (b) for odd eigenfunction corresponding to \( r = 2.956458939 \) of singly punctured two-torus
Figure 7. The contour plot (a) and density plot (b) for odd eigenfunction corresponding to $r = 29.924027400$ of singly punctured two-torus

Next, we show the case of the triply-punctured two-sphere for which we managed only to compute fewer eigenvalues due to harder computations (more simultaneous equations involved) and computational constraints. Unlike the case of the torus, the eigenvalues found are non-degenerate (see Table 2). Again, the lowest eigenvalue for the sphere is lower than that of the modular group, attributable to the larger area available for the quantum particle as in the torus case. Rather surprisingly, this lowest eigenvalue for the sphere ($r = 3.7033078$) is larger than that of the torus ($r = 2.956458939$), despite that there are three scattering channels for the sphere. This is also reflected in the plots of the eigenfunction corresponding to the lowest eigenvalue of the sphere case in Fig. 8 where one can see the eigenfunction is more localised (in comparison to Fig. 6). In Fig. 9, is the less localised higher energy eigenfunction for $r = 13.7797514$.

Table 2. The $r$-values for the odd and even eigenfunctions on the triply punctured two-sphere.

|      | Odd       | Even       |
|------|-----------|------------|
| 3.7033078 | 9.5336951 | 5.8793541  |
| 5.4273348 | 9.9349198 | 8.0424776  |
| 6.6204223 | 11.3176796 | 9.8598964  |
| 7.2208719 | 11.9727767 | 10.9203917 |
| 8.2736658 | 12.1730084 | 11.4930046 |
| 8.5225029 | 12.0929949 |            |
Figure 8. The contour plot (a) and density plot (b) for the eigenfunction corresponding to the lowest eigenvalue $r = 3.7033078$ of triply punctured two-sphere.

Figure 9. The contour plot (a) and density plot (b) for the eigenfunction corresponding to the eigenvalue $r = 13.7797514$ of triply punctured two-sphere.

5. Conclusion and outlook
We have found bound state solutions for both the singly punctured two-torus and the triply punctured two-sphere. The former has doubly degenerate eigenvalue spectra while the latter has nondegenerate spectra. It is interesting to see whether the degeneracy for the two-torus can be explained physically say through its geometry. This can be further investigated by deforming the torus in a way that the two 1-cycles that define the torus are no longer the same size (see Fig. 10) and studying its spectra.
We have also found the fact that the two-sphere case is more bound than that of the torus from their respective lowest eigenvalues. It is interesting to further examine eigenfunctions with comparable eigenvalues. In Figures 11 and 12, we show respectively the eigenfunctions for the torus for \( r = 13.420771791 \) and for the sphere for \( r = 13.1720749 \). We can again see the eigenfunction is more delocalised for the torus, while for the sphere, there is some form of directional alignments of the higher probability regions. We hope to study further the influence of the presence of cusps in these surfaces and others.

**Figure 10.** Deforming the torus via changing the size of the cycles.

**Figure 11.** The contour and density plots for the even eigenfunction corresponding to \( r = 13.420771791 \) for singly punctured two-torus.

**Figure 12.** The contour and density plots for the eigenfunction corresponding to \( r = 13.1720749 \) for triply punctured two-sphere.

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