Shape-constrained regularization by statistical multiresolution for inverse problems: asymptotic analysis

Klaus Frick$^{1,3}$, Philipp Marnitz$^1$ and Axel Munk$^{1,2}$

$^1$ Institute for Mathematical Stochastics, University of Göttingen, Goldschmidtstraße 7, 37077 Göttingen, Germany
$^2$ Max Planck Institute for Biophysical Chemistry, Am Faßberg 11, 37077 Göttingen, Germany

E-mail: frick@math.uni-goettingen.de, stochastik@math.uni-goettingen.de and munk@math.uni-goettingen.de

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Abstract
This paper is concerned with a novel regularization technique for solving linear ill-posed operator equations in Hilbert spaces from data that are corrupted by white noise. We combine convex penalty functionals with extreme-value statistics of projections of the residuals on a given set of sub-spaces in the image space of the operator. We prove general consistency and convergence rate results in the framework of Bregman divergences which allows for a vast range of penalty functionals. Various examples that indicate the applicability of our approach will be discussed. We will illustrate in the context of signal and image processing that the presented method constitutes a locally adaptive reconstruction method.

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper, we are concerned with the solution of the equation

\[ Ku = g, \]

where \( K : U \rightarrow V \) is a linear and bounded operator mapping between two Hilbert spaces \( U \) and \( V \). Equations of type (1) are called well-posed if for given \( g \in V \) there exists a unique solution \( u^* \in U \) that depends continuously on the right-hand side \( g \). If one of these conditions is not satisfied, the problem is called ill-posed. In the case of ill-posedness, arbitrary small deviations on the right-hand side \( g \) may lead to useless solutions \( u \) (if solutions exist). These deviations are commonly modelled as random. They are due to indispensable numerical errors.
as well as to the random nature of the measurement process itself. *(Statistical) regularization methods* aim at computing stable approximations of true solutions $u$ from a (statistically) perturbed signal $g$.

In this paper, we assume that we are given the observation

$$Y = Ku + \sigma \varepsilon,$$  \hspace{1cm} (2)

Here, $\sigma > 0$ denotes the noise level and $\varepsilon : V \rightarrow L^2(X, \mathcal{F}, \mathbb{P})$ a Gaussian white noise process, i.e. $\varepsilon$ is linear and continuous and for all $v, w \in V$ one has

$$\varepsilon(v) \sim N(0, \|v\|^2) \quad \text{and} \quad \text{Cov}(\varepsilon(v), \varepsilon(w)) = \langle v, w \rangle,$$

where $N(\mu, \sigma^2)$ denotes the normal distribution with expectation $\mu$ and variance $\sigma^2$. The white noise model (2) is very common in the theory of statistical inverse problems (see e.g. [6, 11, 13, 14, 16, 40, 49]) and it can be regarded as reasonable approximation to models relevant for many areas of application. A statistical regularization method amounts to computing an estimator $\hat{u} = \hat{u}(\sigma)$ given the data $Y$ in (2) such that $\hat{u}(\sigma) \rightarrow u^\dagger$ (in an appropriate sense) as $\sigma \rightarrow 0^+$. The simplest case covered by model (2) is classical nonparametric regression and its amplitude of applications. Here, $U$ and $V$ are suitable function spaces where it is assumed that $U$ can be continuously embedded into $V$. $U$ models the smoothness of the true signal $u^\dagger$ and $K$ is the embedding operator $K : U \hookrightarrow V$ (cf [6]). More sophisticated examples for $K$ arise in imaging, when blurring induced by the recording optical systems is modelled as a convolution with a kernel $k(x-y)$. Beyond convolution, different operators $K$ occur in various other applications (see e.g. [4, 28, 53]).

Due to the broad area of applications, the literature on statistical regularization methods is vast. We only give a few, selective references: penalized least-squares estimation (that includes Tikhonov–Philipps and maximum entropy regularization) [5, 50, 57], wavelet-based methods [21, 23, 38, 39, 41], estimation in Hilbert scales [6, 34, 44–47] and regularization by projection [12, 13, 16, 37, 45] to name but a few.

In this work, we follow a different route and study a variational estimation scheme that defines estimators $\hat{u}$ as solutions of

$$\inf_{u \in U} J(u) \quad \text{subject to} \quad T_N(\sigma^{-1} (Y - Ku)) \leq q_N(\alpha).$$  \hspace{1cm} (3)

Here, $J$ is a convex regularization functional that is supposed to measure the regularity of candidate estimators $u \in U$ and $T_N$ is a data fidelity term on $V$ that measures the deviation of the data $Y$ and the estimated image $Ku$. In this work, we consider fidelity measures $T_N$ of the form

$$T_N(v) = \max_{1 \leq n \leq N} \mu_n(v) \quad \text{for} \quad v \in V.$$  \hspace{1cm} (4)

The functions $\mu_n : V \rightarrow \mathbb{R}$ are designed to be sensitive to non-random structures in $v$. We will refer to (4) as multiresolution statistic (MR-statistic) and to corresponding solutions of the optimization problem (3) as statistical multiresolution estimators (SMRE).

The parameter $q_N(\alpha)$ in (3) is chosen to be the $(1 - \alpha)$-quantile of the statistic $T_N(\varepsilon)$ and governs the trade-off between data fit and regularity. Hence, the *admissible region*

$$A_N(\alpha) = \{u \in U : T_N(\sigma^{-1} (Y - Ku)) \leq q_N(\alpha)\}$$  \hspace{1cm} (5)

constitutes a $(1 - \alpha)$-confidence region for a solution $\hat{u}$ of (3), i.e. a region which covers the true solution $u^\dagger$ with probability $1 - \alpha$ at least. This gives the estimation procedure (3) a precise statistical interpretation. Since for each solution $u^\dagger$ of (1) one has $u^\dagger \in A_N(\alpha)$ with probability at least $1 - \alpha$, it follows from (3) that

$$\mathbb{P}(J(\hat{u}) \leq J(u^\dagger)) \geq 1 - \alpha.$$
Summarizing, regularization methods of type (3) pick among all estimators \( \hat{u} \) for which the distance between \( K\hat{u} \) and the data \( Y \) does not exceed the threshold value \( q_N(\alpha) \) one with largest regularity. The probability that this particular estimator is more regular than any solution of (1) is bounded from below by \( 1 - \alpha \). This is in contrast to many other regularization techniques where regularization parameters merely govern the trade-off between fit-to-data and smoothness and do not allow such an interpretation. (In the case of wavelet-thresholding, this property was studied in [22].)

Whereas most of the literature is concerned with the proper choice of the regularization functional \( J \), in this work we will discuss the issue of the data fidelity term \( T_N \). We claim that from a statistical perspective, the choice of \( T_N \) is of equal importance as the choice of \( J \).

In definition 3.1, we will delimit a class of feasible functions for \( \mu_1, \ldots, \mu_N \) in (4).

However, in order to make ideas clear (and also to justify the notion ‘multiresolution’), we will start with a simple, yet illustrative example. Let \( G \subset [0,1]^d \) be the equispaced grid of points in the unit cube and assume that \( V \) consists of all real-valued functions \( v: G \to \mathbb{R} \). Moreover, let \( \{S_1, S_2, \ldots, S_N\} \) be a sequence of non-empty subsets of \( G \). We define for \( n \in \mathbb{N} \) and \( v \in V \) the local average function \( \mu_n(v) = \frac{1}{\#S_n} \sum_{\nu \in S_n} v_\nu \), where \( \#S_n \) denotes the number of grid points in \( S_n \). Thus, the MR-statistic \( T_N \) reads as

\[
T_N(\sigma^{-1}(Y - Ku)) = \max_{1 \leq n \leq N} \frac{1}{\#S_n} \left| \sum_{\nu \in S_n} \sigma^{-1}(Y - Ku)_\nu \right|.
\]

In other words, the statistic \( T_N \) returns the largest local average of the residuals \( \sigma^{-1}(Y - Ku) \) over the sets \( S_1, \ldots, S_N \). Under the hypothesis that \( u^\dagger \) is the true solution of (1), we have that \( T_N(\sigma^{-1}(Y - Ku^\dagger)) = T_N(\varepsilon) \) does not exceed the threshold \( q_N(\alpha) \) with probability \( 1 - \alpha \) at least. Recall that \( \varepsilon \) is a white noise process and hence ‘oscillates around zero’ as an effect of which the quantile values \( q_N(\alpha) \) are relatively small due to cancellations in the sums in (6). If, however, \( u \) is wrongly specified, the residual \( Y - Ku \) contains a non-random signal which may happen to be covered by a set \( S_{m_n} \). As an effect, the local average over \( S_{m_n} \)—and thus also the statistic \( T_N(\sigma^{-1}(Y - Ku)) \)—becomes relatively large and \( u \) lies outside the admissible domain of the optimization problem (3).

The choice of the system \( \{S_1, \ldots, S_N\} \) is subtle, since it should not miss any non-random information in the residual, if present. Put differently, it encodes \textit{a priori} information on where one expects to encounter non-random behaviour in the residuals of any possible estimator \( \hat{u} \). Thus, \( T_N \) would be most sensible against a large variety of signals \( \hat{u} \), if we employ a large number \( N \) of overlapping sets \( S_n \) that cover \( G \). This approach, however, turns (3) into an optimization problem with a huge number of constraints which is difficult to tackle numerically (this is treated separately in [32]). Besides these numerical difficulties, there is also a statistical limitation which will be a major issue to be discussed in this paper. If the dictionary \( \{S_1, \ldots, S_N\} \) is too large (in the sense of a metric entropy), the asymptotic distribution of \( T_N \) will degenerate. In practical situations, \textit{a priori} knowledge on the true solution of (1) can be used in order to design dictionaries whose entropy guarantees a non-degenerate limit of \( T_N \) and in addition allows us to derive rates of convergence of the SMRE to the true signal. A similar comment applies to the choice of the regularization functional \( J \) which models \textit{a priori} information on the regularity of the true solution.

As a consequence, the MR-statistic \( T_N \) plugged in into (3) plays the role of a \textit{shape constraint} and the resulting estimation method is capable of adapting the amount of regularization in a \textit{locally adaptive} manner. Put differently, our approach offers a general methodology to \textit{localize} any global convex regularization functional in order to obtain spatial adaption. This is in contrast to global data fidelity terms such as the widely used squared
2-norm fidelity (or any other $p$-norm, $p \geq 1$, for that matter) that do not allow for adaptation to local structures. This is illustrated in the following example.

**Example 1.** Assume that $U = V = \mathbb{R}^n$ with $n = 1024$ and let $K : U \rightarrow V$ be the identity operator, i.e. (2) can be rewritten into the simple nonparametric regression model

$$Y_i = u_i^\dagger + \sigma \varepsilon_i, \quad i = 1, \ldots, n,$$

with $\varepsilon_1, \ldots, \varepsilon_n$ i.i.d. standard normal random variables. The signal $u_i^\dagger \in U$ and the data $Y$ according to (2) with $\sigma = 0.05$ are depicted in figure 1.

The signal $u_i^\dagger$ exhibits kinks, jumps, peaks and smooth portions simultaneously which makes estimation a delicate matter. For example, the regularization functional

$$J(u) = \frac{1}{n} \sum_{k=1}^{n-1} |u_{k+1} - u_k|^2$$

(7)

appears to be well suited to recover at least the smooth parts of the signal, however, with a tendency to ‘smear out’ edges, peaks and kinks. In the following, we will show how this deficiency can be repaired by localizing $J$ by means of MR statistics. To this end we will compute SMREs, solutions of (3), that is, with $J$ as in (7).

Before we do so, we start with reconstructing $u_i^\dagger$ by the usual ‘global’ approach for the purpose of comparison. We compute a $J$-penalized least-squares estimator $\hat{u}_2$, i.e. the solution of

$$\min_{u \in \mathbb{R}^n} \frac{1}{n} \sum_{l=1}^{n} |Y_l - u_l|^2 + \frac{1}{n} \sum_{l=1}^{n-1} |u_{l+1} - u_l|^2.$$

(8)

Here, the proper selection of smoothness amounts to a proper choice of the parameter $\lambda > 0$. It is instructive to rewrite (8) in a slightly different form, such that the relationship to (3) becomes obvious: to each $\lambda > 0$, there corresponds a threshold value $q = q(\lambda)$, such that $\hat{u}_2$ is a solution of

$$\min_{u \in \mathbb{R}^n} \frac{1}{n} \sum_{k=1}^{n-1} |u_{k+1} - u_k|^2 \quad \text{s.t.} \quad \frac{1}{n} \sum_{l=1}^{n} |Y_l - u_l|^2 \leq q.$$

(9)

The first three panels in the upper row of figure 2 depict solutions $\hat{u}_2$ for $q = 25, 43$ and 50. The choice $q = 43$ yields the visually best result; however, it becomes immediately clear that there are under- and oversmoothed parts in the reconstruction. The latter becomes undeniably visible in the qq-plot of the residual $Y - \hat{u}_2$ (lower row) which indicates that there is a significant number of outliers. Note that less oversmoothing, i.e. fewer outliers in the residuals,
can only be achieved at the cost of more artefacts in the reconstruction (by decreasing $q$) and vice versa fewer artefacts only by accepting severe oversmoothing (by increasing $q$). This is due to the fact that each residual value $Y_l - u_l$ ($1 \leq l \leq n$) contributes equally to the quadratic fidelity in (8) (or likewise in (9)) independent of its spatial position.

To overcome this obvious ‘lack of locality’, we compute solutions of (3) where we employ the MR-statistic in (6) as the fidelity measure. To be more precise, we choose the sets $\{S_1, \ldots, S_N\}$ to consist of all discrete intervals of the type $[i, j]/n$ with $1 \leq i < j \leq n$ and $j - i \leq 20$ (i.e. $N = 20.290$). Put differently, the SMRE $\hat{u}_{\text{SMRE}}$ is a solution of the convex optimization problem

$$
\min_{u \in \mathbb{R}^n} \frac{1}{n} \sum_{l=1}^{n-1} |u_{l+1} - u_l|^2 \quad \text{s.t.} \quad \max_{1 \leq i < j \leq n} \left( \frac{1}{\sqrt{j - i + 1}} \sum_{l=i}^{j} Y_l - u_l \right) \leq q.
$$

For the computation of $\hat{u}_{\text{SMRE}}$ in the rightmost panel of figure 2, we set $q = q_N(\alpha) = 2.9$ which corresponds to a small value of $1 - \alpha \approx 0.01$ in order to avoid oversmoothing. The value of $\alpha$ was determined by simulations of the statistic $T_N(\varepsilon)$. Indeed, the result is visually appealing: the kinks, jumps and peaks are strikingly well recovered, both in location and height and the smooth parts of the signal exhibit no artefacts. Also the corresponding qq-plot confirms that there are hardly any outliers in the residuals $Y - \hat{u}_{\text{SMRE}}$, which indicates that oversmoothing is limited to a reasonable amount. Again, this is all the more remarkable as the regularization functional $J$ is known to usually blur edges, peaks and kinks.

Summarizing, it becomes evident that the SMRE approach outperforms the standard method that employs the global quadratic fidelity. In particular, this example shows that plugging in the MR-statistic $T_N(\varepsilon)$ into (3) results in an estimation scheme that regularizes in a locally adaptive manner. Aside to the specific choice (7), any other convex regularization functional $J$ can be ‘localized’ in this way, of course, as for example the total variation semi-norm

$$
J(u) = \frac{1}{n} \sum_{k=1}^{n-1} |u_{k+1} - u_k|.
$$
Figure 3. Global estimators $\hat{u}_2$ for $q = 29, 33$ and $36$ and SMRE $\hat{u}_{\text{SMRE}}$ w.r.t. the total variation penalty.

It has turned out, however, that for the present example (7) is preferable since it accounts well for the smooth parts in the signal, whereas it is well known and also visible that the total variation penalty induces an undesired ‘staircasing’ effect. This is illustrated in figure 3, where global estimators $\hat{u}_2$ and the SMRE $\hat{u}_{\text{SMRE}}$ are depicted.

We finally remark that all estimators in this example were computed by an alternating direction method of multipliers (ADMM) as developed in [32] and its details will not be discussed here. In [32], also simulation studies are performed giving quantitative evidence of the good performance of our method (see also examples 4.15 and 4.16).

The regularization scheme (3) with the MR-statistic $T_N$ as in (6) was studied in [20] for the specific case of non-parametric regression in one space dimension and the total-variation semi-norm as the regularization functional $J$. In this paper, we will show that the general formulation in (3) reveals the SMRE as a powerful regularization method far beyond this situation: it can be extended to space dimensions larger than 1 as well as to inverse problems with general $K$ as in (2) including deconvolution problems. Furthermore, we present very general consistency and convergence rate results for SMRE in the context of statistical inverse problems and discuss their impact on particular applications. To our best knowledge, results of this type have never been obtained before. It is necessary to assume additional regularity of the true solution of (1) in order to come up with convergence rate results. In the context of inverse problems, this is usually done by imposing source conditions. These determine smoothness classes of solutions for (1) that guarantee risk bounds and fast convergence of the estimator to the true signal. In this work, we study the standard source condition [9] used in the framework of Bregman divergences that yield for each penalty functional $J$ in (3) one specific smoothness class. The formulation of conditions that give optimal convergence rates in a scale of smoothness classes for a general but fixed $J$ will not be treated in this work (cf [29] and references therein).

This paper is organized as follows. After reviewing some basic definitions from convex analysis and the theory of inverse problems in section 2, we develop in section 3.1 a general scheme for the estimation of the statistical inverse problem (2) based on the convex optimization problem (3). In section 3.2, we then prove consistency and convergence rate results in terms of the Bregman divergence w.r.t. the regularization functional $J$. In section 4, we study the performance of the so constructed estimators for various examples, as the Gaussian sequence model (section 4.1) and linear inverse regression problems (section 4.2). In section 4.3, we investigate the particular situation when the regularization functional $J$ is chosen to be the total-variation semi-norm, which has a particular appeal for imaging problems. Finally, some examples that illustrate the notions of source condition and Bregman divergence are
given in appendix A and the proofs of the main results as well as some auxiliary lemmata are collected in appendix B.

2. Basic definitions

In this section, we summarize some relevant definitions and assumptions needed throughout the paper.

Assumption 2.1.

(i) \( U \) and \( V \) denote separable Hilbert spaces. The norms on \( U \) and \( V \) are not further specified, and will be always denoted by \( \| \cdot \| \), since the meaning is clear from the context.

(ii) Let \( J : U \to \mathbb{R} \) be a convex functional from \( U \) into the extended real numbers \( \mathbb{R} = \mathbb{R} \cup \{ \infty \} \).

The domain of \( J \) is defined by

\[ D(J) = \{ u \in U : J(u) \neq \infty \}. \]

\( J \) is called proper if \( D(J) \neq \emptyset \) and \( J(u) > -\infty \) for all \( u \in U \). Throughout this paper \( J \) denotes a convex, proper and lower semi-continuous (l.s.c.) functional with dense domain \( D(J) \).

(iii) \( K : U \to V \) is a linear and bounded operator. By \( \text{ran}(K) = K(U) \) we denote the range of \( K \) and by \( K^* : V \to U \) the adjoint operator of \( K \).

In the course of this paper, we will frequently make use of tools from convex analysis. For a standard reference, see [27].

- The sub-differential (or generalized derivative) \( \partial J(u) \) of \( J \) at \( u \) is the set of all elements \( p \in U \) satisfying

\[ J(v) - J(u) - \langle p, v - u \rangle \geq 0 \quad \text{for all } v \in U. \]

The domain \( D(\partial J) \) of the sub-differential consists of all \( u \in U \) for which \( \partial J(u) \neq \emptyset \).

- We will prove the consistency of estimators with respect to the Bregman divergence. For \( u \in D(J) \), the Bregman divergence of \( J \) between \( u \) and \( v \) with respect to \( \xi \in \partial J(u) \) is defined

\[ D_\xi J(v, u) = J(v) - J(u) - \langle \xi, v - u \rangle. \]

The following basic estimates hold:

\[ 0 \leq D_J(v, u) \leq D_\xi J(v, u) \quad \text{for all } \xi \in \partial J(u). \]
Remark 2.1. Clearly, the Bregman divergence does not define a (quasi-)metric on $U$. It is non-negative but in general it is neither symmetric nor satisfies the triangle inequality. The big advantage, however, of formalizing asymptotic results w.r.t. the Bregman divergence (such as consistency or convergence rates) for estimators defined by a variational scheme of type (3), is the fact that the regularizing properties of the used penalty functional $J$ are incorporated automatically. If, for example, the functional $J$ is slightly more than strictly convex, it was shown in [51] that convergence w.r.t. the Bregman divergence already implies convergence in norm. If, however, $J$ fails to be strictly convex (e.g. if it is of linear growth) it is in general difficult to establish norm-convergence results but convergence results w.r.t. the Bregman divergence, though weaker, may still be at hand. In appendix A we compute the Bregman divergence for some particular choices of $J$.

The concept of Bregman divergence in optimization was introduced in [7] and has recently attracted much attention, e.g., in the inverse problems community [see 9, 18, 33] or in statistics and machine learning [17, 42, 58].

Next, we introduce different classes of solutions for equation (1) discussed in this paper.

Definition 2.2.

(i) Let $u \in D(J)$ be a solution of (1). Then, $g$ is called attainable.

(ii) An element $u \in D(J)$ is called the $J$-minimizing solution of (1) if $u$ solves (1) and

$$J(u) = \inf \{ J(\tilde{u}) : K\tilde{u} = g \} .$$

(iii) Let $g \in V$ be attainable. An element $p \in V$ is called a source element if there exists a $J$-minimizing solution $u$ of (1) such that

$$K^*p \in \partial J(u) .$$

Then, we say that $u$ satisfies the source condition (10).

It is well known in the theory of inverse problems with deterministic noise (see [28]) that the source condition (10) is sufficient for establishing convergence rates for regularization methods. It can be understood as a regularity condition for $J$-minimizing solutions of equation (1). Put differently, for each regularization functional $J$ and each operator $K$, the source condition (10) characterizes one particular smoothness class of solutions for (1) for which fast reconstruction is guaranteed. We clarify the notions Bregman divergence and source condition by some examples in appendix A.

Under fairly general conditions, the existence of a $J$-minimizing solution can be guaranteed. We formalize these conditions in the following result; however, we omit the proof since it is standard in convex analysis (see [27, chapter II, proposition 2.1]).

Proposition 2.3. Let $g \in V$ be attainable and assume that for all $c \in \mathbb{R}$ the sets

$$\{ u \in U : \| Ku \| + J(u) \leq c \}$$

are bounded in $U$. Then, there exists a $J$-minimizing solution of (1).

3. A general scheme for estimation

In this section, we construct a family of estimators $\hat{u}$ for $J$-minimizing solutions (cf definition 2.2) of equation (1) from noisy data $Y$ given by the white noise model (2). We define the estimators in a variational framework and prove consistency as well as convergence rates in terms of the Bregman divergence w.r.t. $J$. 

3.1. MR-statistic and SMR-estimation

We introduce a class of similarity measures in order to determine whether the residuals \( Y - K \hat{u} \) for a given estimator \( \hat{u} \in U \) resemble a white noise process or not. To this end, we will consider the extreme-value distribution of projections of the residuals onto a predefined collection of lines in \( V \). To this end, assume that \( \Phi_1 = \{ \phi_1, \phi_2, \ldots \} \subset \text{ran}(K) \setminus \{ 0 \} \) is a fixed dictionary such that \( \| \phi_n \| \leq 1 \) for all \( n \in \mathbb{N} \). For the sake of simplicity, we will frequently make use of the abbreviation \( \phi_n^* = \phi_n / \| \phi_n \| \).

**Definition 3.1.** Let \( \{ t_N : \mathbb{R}^+ \times (0, 1] \to \mathbb{R} \}_{N \in \mathbb{N}} \) be a sequence of functions that satisfy the following conditions.

(i) For all \( r \in (0, 1] \), the function \( s \mapsto t_N(s, r) \) is convex, increasing and Lipschitz-continuous with Lipschitz-constants \( L_{N,r} \) such that \( L_{N,r} \leq L < \infty \) for all \( N \in \mathbb{N} \) and

\[
0 > \lambda_N(r) := \inf_{s \in \mathbb{R}^+} t_N(s, r) > -\infty.
\]

(ii) There exist constants \( c_1, c_2 > 0 \) and \( \sigma_0 \in (0, 1) \) such that for all \( 0 < \sigma < \sigma_0 \)

\[
t_N(s, r) \geq c_1 s + c_2 t_N(\sigma s, r) \quad \text{for } (s, r) \in \mathbb{R}^+ \times (0, 1] \quad \text{and} \quad N \in \mathbb{N}.
\]

Then, for \( N \in \mathbb{N} \), the mapping \( T_N : V \to \mathbb{R} \) defined by

\[
T_N(v) = \max_{1 \leq n \leq N} t_N(\| v, \phi_n^* \|, \| \phi_n \|)
\]

is called a multiresolution statistic (MR-statistic).

**Remark 3.1.** Let \( \{ t_N \}_{N \in \mathbb{N}} \) be a sequence of functions satisfying (i) and (ii) in definition 3.1. For a fixed \( N \in \mathbb{N} \), the mappings \( \mu_n : V \to \mathbb{R} \) defined by

\[
\mu_n(v) = t_N(\| v, \phi_n^* \|, \| \phi_n \|)
\]

can be interpreted as the average of the signal \( v \) restricted to the subspace spanned by \( \phi_n^* \). With \( \mu_n \) as above, the MR-statistic \( T_N(v) \) in definition 3.1 takes the form (4) and hence can be considered to measure the maximal local average of \( v \) w.r.t. the dictionary \( \{ \phi_1, \ldots, \phi_N \} \).

Definition 3.1 allows for a vast class of MR-statistics and the conditions in (i) and (ii) appear rather technical. The following example sheds some light on a special class of MR-statistics that later on will be studied in more detail. We note, however, that our general setting also applies to more involved statistics, as e.g. introduced in [25, 26].

**Example 3.2.** Assume that \( \{ f_N : (0, 1] \to \mathbb{R} \}_{N \in \mathbb{N}} \) is a sequence of positive functions and define

\[
t_N(s, r) := s - f_N(r).
\]

Then, the assumptions in definition 3.1 are satisfied; to be more precise, we can set \( L = 1 \), \( \lambda_N(r) = -f_N(r) \) and \( c_1 = 1 - \sigma_0 \) and \( c_2 = 1 \), where \( \sigma_0 \in (0, 1) \) is arbitrary but fixed. Moreover, for a fixed \( N \in \mathbb{N} \), the average functions \( \mu_n : V \to \mathbb{R} \) in remark 3.1 read as

\[
\mu_n(v) = | v, \phi_n^* | - f_N(\| \phi_n \|).
\]
For a white noise process $\varepsilon : V \to L^2(\Omega, \mathfrak{F}, \mathbb{P})$ and $N \in \mathbb{N}$, consider the random variable

$$ T_N(\varepsilon) = \max_{1 \leq n \leq N} t_n(\{\varepsilon(\phi_n^\ast)\}, \|\phi_n\|). $$

Then, for a level $\alpha \in (0, 1)$ we denote the $(1 - \alpha)$-quantile of $T_N(\varepsilon)$ by $q_N(\alpha)$, that is,

$$ q_N(\alpha) := \inf\{q \in \mathbb{R} : \mathbb{P}(T_N(\varepsilon) \leq q) \geq 1 - \alpha\}. \quad (14) $$

Our key paradigm is that an estimator $\hat{u}$ for a solution of (1) fits the data $Y$ sufficiently well if the statistic $T_N(Y - Ku)$ does not exceed the threshold $q_N(\alpha)$ ($\alpha \in (0, 1)$ and $N \in \mathbb{N}$ fixed). Among all those estimators, we will pick the most parsimonious by minimizing the functional $J$.

**Definition 3.3.** Let $N \in \mathbb{N}$ and $\alpha \in (0, 1)$. Moreover, assume that $T_N$ is an MR-statistic and that $Y$ is given by (2). Then, every element $\hat{u}_N(\alpha) \in U$ solving the convex optimization problem (3) is called a SMRE.

An SMRE $\hat{u}_N(\alpha)$ depends on the regularization parameters $N \in \mathbb{N}$ and $\alpha \in (0, 1)$ that determine the admissible region $A_N(\alpha)$ in (5). In order to guarantee the existence of a solution of the convex problem in definition 3.3, that is, the existence of an SMRE, it is necessary to impose further standard assumptions.

**Assumption 3.4.** There exists $N_0 \in \mathbb{N}$ such that for all $c \in \mathbb{R}$ the sets

$$ \Lambda(c) = \{u \in U : \max_{1 \leq n \leq N_0} \|Ku(\phi_n^\ast)\| + J(u) \leq c\} $$

are bounded in $U$.

Assumption 3.4 guarantees weak compactness of the level sets of the objective functional $J$ restricted to the admissible region $A_N(\alpha)$. We note that if $J$ is strongly coercive (e.g. when $J$ is as in example A.1), then assumption 3.4 is satisfied without any restrictions on the operator $K$. If $J$ lacks strong coercivity (as it is, e.g., the case with the total-variation semi-norm studied in section 4.3) additional properties of $K$ are required in order to meet assumption 3.4.

Application of standard arguments from convex optimization yields

**Proposition 3.5.** Assume that assumption 3.4 holds and let $N \geq N_0$ and $\alpha \in (0, 1]$. Then, an SMRE $\hat{u}_N(\alpha)$ exists.

Finally, we note that assumption 3.4 already implies the requirements in proposition 2.3 and consequently the existence of $J$-minimizing solutions.

### 3.2. Consistency and convergence rates

We investigate the asymptotic behaviour of $\hat{u}_N(\alpha)$ as the noise level $\sigma$ in (2) tends to zero. According to the reasoning following definition 3.3, the parameters $N \in \mathbb{N}$ and $\alpha \in (0, 1)$ can be interpreted as regularization parameters and have to be chosen accordingly. The model parameter $N$ has to be increased in order to guarantee a sufficiently accurate approximation of the image space $V$, whereas the test level $\alpha$ tends to 0 such that the true solution (asymptotically) satisfies the constraints of (3) almost surely. We formulate consistency and convergence rate results by means of the Bregman divergence of the SMRE $\hat{u}_N(\alpha)$ and a true solution $u^\dagger$ in terms of almost sure convergence.

Throughout this section, we will assume that $\{\sigma_k\}_{k \in \mathbb{N}}$ is a sequence of positive noise levels in (2) such that $\sigma_k \to 0^+$ as $k \to \infty$. Moreover, we assume that $\{\sigma_k\}_{k \in \mathbb{N}} \subset (0, 1)$ is a sequence of significance levels and that $N_k \geq N_0$ is such that

$$ \sum_{k=1}^\infty \sigma_k < \infty \quad \text{and} \quad \lim_{k \to \infty} N_k = \infty. \quad (15) $$

Theorem 3.6. Assume that assumptions 2.1 and 3.4 hold. Let further \( u^\dagger \) be a J-minimizing solution of (1) where \( g \in \text{span}\Phi \) and assume that
\[
\sup_{N \in \mathbb{N}} T_N(\varepsilon) < \infty
\]
and
\[
\zeta_k := \sigma_k \max\left( \inf_{1 \leq n \leq N_k} \lambda_n(\|\phi_n\|), \sqrt{-\log \alpha_k} \right) \to 0.
\] (16)

Then, for \( \hat{u}_k := \hat{u}_{N_k}(\alpha_k) \) as in (3), one has
\[
\sup_{k \in \mathbb{N}} \|\hat{u}_k\| < \infty, \quad J(\hat{u}_k) \to J(u^\dagger) \quad \text{and} \quad D_J(u^\dagger, \hat{u}_k) \to 0 \quad \text{a.s.}
\] (17)
as well as
\[
\limsup_{k \to \infty} \max_{1 \leq n \leq N_k} \frac{\|\phi_n^*, K\hat{u}_k - Ku^\dagger\|}{\zeta_k} < \infty \quad \text{a.s.}
\] (18)

Theorem 3.6 states that if for a given vanishing sequence of noise levels \( \sigma_k \), suitable (in the sense of (16)) sequences of regularization parameters \( N_k \) and \( \alpha_k \) can be constructed, then the sequences of corresponding SMRE converges to a true J-minimizing solution \( u^\dagger \) w.r.t. the Bregman divergence. We note that the assumption on the boundedness of the MR-statistic \( T_N(\varepsilon) \) is crucial and in general non-trivial to show.

It is well known that without further regularity restrictions on \( u^\dagger \), the speed of convergence in (17) can be arbitrarily slow. Source conditions as in definition 2.2 (iii) are known to constitute sufficient regularity conditions with quadratic fidelity (cf [6, 43, 44]). In our situation, where the fidelity controls the maximum over all residuals, we additionally have to assume that the source elements exhibit certain approximation properties.

Assumption 3.7. There exists a J-minimizing solution \( u^\dagger \) of (1) that satisfies the source condition (10) with source element \( p^\dagger \). Moreover, for \( n, N \in \mathbb{N} \) there exist \( b_{n,N} \in \mathbb{R} \) such that
\[
\text{err}_N(p^\dagger) := \left\| p^\dagger - \sum_{n=1}^{N} b_{n,N} \phi_n^* \right\| \to 0 \quad \text{and} \quad \sup_{N \in \mathbb{N}} \left\| \sum_{n=1}^{N} b_{n,N} \right\| < \infty.
\] (19)

Remark 3.2.
(i) Assumption 3.7 amounts to saying that there exists a J-minimizing solution \( u^\dagger \) that satisfies the source condition (10) with a source element \( p^\dagger \) that can be approximated sufficiently well by the dictionary \( \Phi \) in use. From (10) it becomes clear that we can always assume that \( p^\dagger \in \text{ran}(K) \), such that the first condition in (19) is not very restrictive, in fact.
(ii) Good estimates of approximation errors for non-orthogonal dictionaries \( \Phi \) are difficult to come up with in general. Examples of non-orthogonal dictionaries where such estimates are available are wavelet- [19] and curvelet-frames [10].
(iii) It is important to note that given prior information on the true solution \( u^\dagger \), the conditions in assumption 3.7 may indicate whether a given dictionary is well suited for the reconstruction of \( u^\dagger \) or not. As we will see in section 4, a priori information on the smoothness of \( u^\dagger \) can typically be employed.

Theorem 3.8. Let the requirements of theorem 3.6 be satisfied and assume further that assumption 3.7 holds with \( g \in \text{span}\Phi \). If \( \eta_k := \max(\zeta_k, \text{err}_N(p^\dagger)) \to 0 \), then
\[
\limsup_{k \to \infty} \frac{D_J^p(\hat{u}_k, u^\dagger)}{\eta_k} < \infty \quad \text{and} \quad \limsup_{k \to \infty} \max_{1 \leq n \leq N_k} \frac{\|\phi_n^*, K\hat{u}_k - Ku^\dagger\|}{\eta_k} < \infty \quad \text{a.s.}
\] (20)
Remark 3.3. The convergence rate result in theorem 3.8 is rather general, in the sense that the rate function $\eta_k$ in (20) has to be determined for each choice of $K$, $J$ and $\Phi$ separately. We outline a general procedure of how this can be done in practice. Assume that $u^*$ is a $J$-minimizing solution of (1) that satisfies assumption 3.7 with a source element $p^*$.

(i) The sequence $\left\{-\inf_{1 \leq n \leq N} \lambda_N(\|\phi_n\|)\right\}_{N \in \mathbb{N}}$ is positive according to (12). Hence, $N_k := \inf \left\{N \in \mathbb{N} : \text{err}_N(p^*) \leq -\sigma_k \inf_{1 \leq n \leq N} \lambda_N(\|\phi_n\|)\right\}$ is well defined and since $\{\sigma_k\}_{k \in \mathbb{N}}$ is non-increasing, one has $N_k \leq N_{k+1}$ and $N_k \to \infty$ as $k \to \infty$.

(ii) After setting $\eta_k = -\sigma_k \inf_{1 \leq n \leq N_k} \lambda_N(\|\phi_n\|)$ it remains to check that the sequence of test levels $\alpha_k = \exp \left(-\left(\kappa \eta_k / \sigma_k\right)^2\right)$ is summable (for some constant $\kappa > 0$).

For the so-constructed sequences $N_k$, $\eta_k$ and $\alpha_k$, the assertions of theorem 3.8 hold.

As we will see in section 4, the procedure in remark 3.3 typically results in convergence rates $\eta \sim \sqrt{-\log \sigma}$. For orthogonal dictionaries $\Phi$ it will turn out in section 4.1 that these rates are nearly optimal for the smoothness class induced by assumption 3.7 (cf example 4.4 below). It is an open question what the optimal rates are for general (non-orthogonal) dictionaries.

4. Applications and examples

In section 3, we developed a general method for the estimation of $J$-minimizing solutions of linear and ill-posed operator equations from noisy data. Our estimation scheme thereby employs the MR-statistic $T_N$ (cf definition 3.1). In this section, we will study particular instances of MR-statistics covered by the general theory in section 3.

- We study the case where $T_N$ constitutes the extreme-value statistic of the coefficients w.r.t. an orthonormal dictionary $\Phi$ (section 4.1). We show how assumption 3.7 in this case reduces to the requirement that the true solution $u^*$ lies in a Sobolev-ellipsoid w.r.t. the system $\Phi$. Moreover, it will turn out that for the case when $\Phi$ denotes the eigensystem of a compact operator, SMRE can be considered as soft-thresholding.

- In section 4.2, we skip the assumption of orthonormality and examine general SMREs w.r.t. (non-orthonormal) dictionaries that satisfy certain entropy conditions. In particular, we will consider the case when $U = V = \mathcal{L}^2([0,1]^d)$ and when $\Phi$ consists of indicator functions w.r.t. a redundant system of subcubes in $[0,1]^d$.

- Finally, we study the case when the penalty functional $J$ is chosen to be the total-variation semi-norm on $U = \mathcal{L}^2(\Omega)$ in section 4.3. We highlight the implications of our general convergence rate results for image deconvolution and complement the theoretical results by some numerical examples. In particular, we compare our approach to the locally adaptive image reconstruction method recently introduced in [35].

Throughout this section, we assume that assumptions 2.1 and 3.4 hold. Moreover, we will agree upon $\{\sigma_k\}_{k \in \mathbb{N}}$ being a sequence of noise levels such that $\sigma_k \to 0^+$ and that for $k \in \mathbb{N}$ there are $\alpha_k \in (0,1)$ and $N_k \in \{N_0, N_0 + 1, \ldots\}$ such that (15) holds.
4.1. Introductory example: Gaussian sequence model

In this section, we will consider the case where the dictionary \( \Phi = \{ \phi_1, \phi_2, \ldots \} \) constitutes an orthonormal basis (ONB) of \( \text{ran}(K) \). Evaluation of equation (2) at the elements \( \phi_n \) hence yields
\[
y_n = \theta_n + \sigma \varepsilon_n,
\]
where \( Y(\phi_n) = y_n, \theta_n = \langle Ku, \phi_n \rangle \) and \( \varepsilon_n = \varepsilon(\phi_n) \). We define the MR-statistic \( T_N \) by setting
\[
t_N(s, r) = s - \sqrt{2 \log N} \text{ in definition 3.1.}
\]
In other words, we consider the maximum of the coefficients w.r.t. the dictionary \( \Phi \), that is
\[
T_N(v) = \max_{1 \leq n \leq N} |\langle v, \phi_n \rangle| - \sqrt{2 \log N}. \tag{21}
\]
Since \( \{ \phi_1, \phi_2, \ldots \} \) are linearly independent and normalized, it follows that the random variables \( \varepsilon_1, \varepsilon_2, \ldots \) are independent and standard normally distributed. This implies that \( T_N(\varepsilon) \) is bounded almost surely.

In what follows, we will apply theorems 3.6 and 3.8 to the present case. To this end, we observe that for \( \sigma > 0 \) and \( N \in \mathbb{N} \),
\[
-\sigma \inf_{\|v\| = 1} \lambda_N(\|v\|) = \sigma \sqrt{2 \log N}.
\]

With the above preparations, we are able to reformulate the consistency result in theorem 3.6.

**Corollary 4.1.** Let \( u^\dagger \in U \) be a J-minimizing solution of (1) where \( g \in \text{span}\Phi \). Moreover, assume that \( \sigma_k^2 \max(\log N_k, -\log \alpha_k) \to 0 \). Then, the SMRE \( \hat{u}_k = \hat{u}_{N_k}(\alpha_k) \) almost surely satisfies (17) and (18).

In order to apply the convergence rate result in theorem 3.8, assumption 3.7 has to be verified. We set \( b_{n,N} = \langle p^\dagger, \phi_n \rangle \) in assumption 3.7. Note that the expression \( \text{err}_N(p) \) denotes the approximation error of the \( N \)th partial Fourier series w.r.t. \( \Phi \). Thus, assumption 3.7 is linked to absolute summability of the Fourier coefficients w.r.t. the basis \( \Phi \), i.e.
\[
\sum_{n=1}^{\infty} |\langle p^\dagger, \phi_n \rangle| < \infty. \tag{22}
\]

The Bernstein–Stechkin criterion is a classical method for testing the absolute summability. We present a version suitable for our purpose in the following.

**Proposition 4.2.** Let \( p^\dagger \in V \). Then, (22) is satisfied if \( \sum_{n=1}^{\infty} \text{err}_N(p^\dagger)/\sqrt{N} < \infty \).

**Proof.** The classical version of the Bernstein–Stechkin theorem (see, e.g., [48, theorem 7.4]) states that for each \( f \in L^2([0, 1]) \) and each ON-basis \( v = \{ v_1, v_2, \ldots \} \) of \( L^2([0, 1]) \), the Fourier coefficients of \( f \) are absolutely summable if \( \sum_{n=1}^{\infty} \text{err}_N(p^\dagger)/\sqrt{N} < \infty \). Since each separable Hilbert space is isometrically isomorphic to \( L^2([0, 1]) \), the assertion finally follows.

Following the procedure outlined in remark 3.3 (section 3), we define
\[
N_k := \inf\{ N \in \mathbb{N} : \text{err}_N(p^\dagger) \leq \sigma_k \sqrt{2 \log N} \} \quad \text{and} \quad \eta_k := \sigma_k \sqrt{2 \log N_k}. \tag{23}
\]

**Corollary 4.3.** Let \( g \in V \) be attainable and \( u^\dagger \in U \) be a J-minimizing solution of (1) that satisfies the source condition with a source element \( p^\dagger \) such that the condition in proposition 4.2 holds. Moreover, let \( N_k \) and \( \eta_k \) be defined as in (23). If
\[
\alpha_k := e^{-\left(\frac{\eta_k}{\sigma_k^2}\right)} = N_k^{-2\kappa^2} \in \ell^1(0, 1)
\]
for a constant \( \kappa > 0 \), then the SMRE \( \hat{u}_k = \hat{u}_{N_k}(\alpha_k) \) almost surely satisfies (20).
The problem of characterizing those elements $p^i \in V$ that satisfy the assumption of proposition 4.2 is a classical issue in Fourier analysis and approximation theory. Sufficient conditions are usually formalized by characterizing the decay properties of the Fourier coefficients. In a function space setting, this leads to particular smoothness classes of functions and in the general situation can be given in terms of Sobolev ellipsoids. For constants $\beta, Q > 0$, we define $\Theta(\beta, Q)$ as the infinite-dimensional ellipsoid

$$\Theta(\beta, Q) = \left\{ \theta \in \ell^2 : \sum_{n \in \mathbb{N}} n^{2\beta} \theta_n^2 \leq Q^2 \right\}.$$  

(24)

The Sobolev class $W(\beta, Q) \subset V$ is then defined as consisting of all $v \in V$ such that $\langle \langle v, \phi_n \rangle \rangle_{n \in \mathbb{N}} \subset \Theta(\beta, Q)$ (see [55, section 1.10.1]). For $v \in W(\beta, Q)$, we have that proposition 4.2 is applicable if $\beta > 1/2$.

**Example 4.4.** Assume that $J(u) = \frac{1}{2} \|u\|^2$ and let $K$ be a compact operator with singular value decomposition (SVD) $\{\psi_n, \phi_n, s_n\}_{n \in \mathbb{N}}$: $\{\psi_n\}_{n \in \mathbb{N}}$ is an ONB of $\ker(K)^\perp$, $\{\phi_n\}_{n \in \mathbb{N}}$ is an ONB of $\text{ran}(K)$ and the singular values $\{s_n\}_{n \in \mathbb{N}}$ are positive and $s_n \to 0$ as $n \to \infty$. Moreover, for all $n \in \mathbb{N}$. For $N \in \mathbb{N}$ and $\alpha \in (0, 1]$ it turns out (e.g. by applying the method of Lagrangian multipliers) that the SMRE $\hat{u}_N(\alpha)$ with $T_N$ as in (21) is a shrinkage estimator given by

$$\hat{u}_N(\alpha) = \sum_{n=1}^N s_n^{-1} y_n \left(1 - \frac{q_N(\alpha) + \sqrt{2 \log N}}{|y_n|}\right) \psi_n.$$  

We note that $\hat{u}_N(\alpha)$ is a particular instance of a soft thresholding estimator.

Now, let $u^1 \in U$ be a minimum-norm solution of (1) that satisfies the source condition $K^*p^i = u^1$ (cf example A.1) with the source element $p^i \in W(\beta, Q)$ for $Q > 0$ and $\beta > 1/2$. Then, $\text{err}_N(p^i) \leq QN^{-\beta}$ and it follows from (23) that

$$N_k \sim \left(\frac{Q}{\alpha_k}\right)^2$$  

and $\eta_k \sim \alpha_k \sqrt{-\log \sigma_k}$.  

If $\sigma_k$ has a polynomial decay, we can choose a constant $\kappa > 0$ such that $\sigma_k = \exp(-(\kappa \eta_k^2/\sigma_k)^2) = \sigma_k^\kappa$ is summable and it follows from corollary 4.3 and example A.1 that

$$\limsup_{k \to \infty} \frac{1}{\sigma_k \sqrt{-\log \sigma_k}} \|u^1 - \hat{u}_N(\alpha_k)\|^2 < \infty \quad \text{a.s.}$$

If the operator equation $Ku = g$ is mildly ill-posed, i.e. $s_n \sim n^{-\gamma}$ for some $\gamma > 0$, then the equation $K^*p^i = u^1$ together with $p^i \in W(\beta, Q)$ implies that $u^1 \in W(\beta + \gamma, Q)$. The optimal rates (w.r.t. the quadratic risk) are known to be of order $\psi(\beta) = \sigma_k^{4(\beta + \gamma)/(4\beta + 2\beta + 1)}$ (cf [11, theorem 1]). Since $\psi(\beta) \to \sigma_k$ for $\beta \to 1/2$ the convergence rate implied by theorem 3.8 is optimal (up to a log-factor).

As mentioned above, sufficient conditions for the Bernstein–Stechkin criterion (cf proposition 4.2) in a function space setting are usually formalized in characterizing smoothness properties. The following example shows how this applies to Hölder continuity.

**Example 4.5.** Let $V = L^2_{\text{per}}([0, 1])$ be the Hilbert space of all square integrable and periodic functions on the unit interval. Moreover, we assume that $\text{ran}(K) = L^2([0, 1])$ and consider the trigonometric basis

$$\phi_{2n} = \sqrt{2} \cos(n\pi x) \quad \text{and} \quad \phi_{2n+1} = \sqrt{2} \sin(n\pi x).$$

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Assume that \( p^1 \in \mathcal{H}_B([0,1]) \cap V \) (cf definition B.4) with \( \beta > 1/2 \). Then, we have that \( \text{err}_{N_t}(p^1) \leq QN^{-\beta} \log N \) for a suitable constant \( Q > 0 \) and therefore it follows from proposition 4.2 that (22) holds.

Hence, if \( u^1 \) is a \( J \)-minimizing solution of (1) that satisfies the source condition (10) with the source element \( p^1 \in \mathcal{H}_B([0,1]) \) and if the sequences \( N_k, n_k \) and \( \alpha_k \) are chosen as in example 4.4, then \( \hat{u}_k = \hat{u}_{N_k}(\alpha_k) \) almost surely satisfies (20).

**Remark 4.1.**

(i) The assertions of example 4.5 still hold if the trigonometric basis is replaced by any other ONB \( \{\phi_n\}_{n \in \mathbb{N}} \) of \( \text{ran}(K) \) such that the Bernstein–Stechkin criterion in proposition 4.2 is satisfied. This holds for example for a vast class of orthonormal wavelet bases of \( L^2([0,1]) \) as studied in [15].

(ii) For the trigonometric basis in example 4.5, the Bernstein–Stechkin criterion 4.2 can be replaced by the requirement that \( p^1 \in \mathcal{H}_B([0,1]) \) for any \( \beta > 0 \) is additionally of bounded variation (see [60, Vol 1 theorem 3.6]).

### 4.2. Non-orthogonal models

In contrast to section 4.1, where we considered orthonormal dictionaries, we will now focus on more general (non-orthonormal) systems. In other words, we consider sequences \( \gamma > 0 \) where \( \lambda_N(r) = -\sqrt{-2\gamma \log r} > -\infty \) for all \( r \in (0,1) \). The parameter \( \gamma \) that appears in (26) has to be chosen appropriately depending on \( \Phi \) in order to guarantee that the MR-statistic \( T_N(\varepsilon) \) is bounded almost surely. A sufficient condition on \( \gamma \) has for example been given in [26, theorem 7.1].

**Proposition 4.6.** If there exists constants \( A, B > 0 \) such that

\[
D(u, \{\phi \in \Phi : \|\phi\| \leq \delta\}) \leq Au^{-B} \delta^{-\gamma} \quad \text{for all } u, \delta \in (0,1),
\]

then almost surely \( \sup_{N \in \mathbb{N}} T_N(\varepsilon) < \infty \). Here, \( D \) denotes the capacity number (cf definition B.6).

**Corollary 4.7.** Let \( u^1 \in U \) be a \( J \)-minimizing solution of (1) where \( g \in \text{span}\Phi \) and \( \gamma > 0 \) be chosen such that the assumption of proposition 4.6 is satisfied. Moreover, assume that

\[
\sigma_k^2 \min_{1 \leq n \leq N_k} \log(\|\phi_n\|, \log \alpha_k) \to 0.
\]

Then, the SMRE \( \hat{u}_k = \hat{u}_{N_k}(\alpha_k) \) almost surely satisfies (17).

In order to apply the convergence rate results in theorem 3.8, it is necessary that a \( J \)-minimizing solution \( u^1 \) of (1) satisfies the source condition (10) with a source element \( p^1 \) that can be approximated by the dictionary \( \Phi \) sufficiently well (cf assumption 3.7). We illustrate the assertion of theorem 3.8 when \( U = V = L^2([0,1]^d) \) (\( d \geq 1 \)) and when \( \Phi \) consists of a countable selection of indicator functions on cubes in \( [0,1]^d \) (cf example 1.1).

First, we will examine when proposition 4.6 holds. To this end, we will focus first on the (uncountable) collection \( \Phi_s \) of indicator functions on cubes in \( [0,1]^d \). Then, according...
to proposition B.8, the assumptions of proposition 4.6 are satisfied for \( \Phi = \Phi_2 \) and \( \gamma = d \). Particularly, it follows that the assertion of proposition 4.6 also holds for arbitrary (countable) sub-systems \( \Phi \subset \Phi_2 \), that is, the statistic
\[
T_{\nu} (s) = \max_{1 \leq n \leq N} \{ s(\chi_{Q_n}) | - \sqrt{d \log |Q_n|} \}
\]
where \( \chi_{Q_n} \in \Phi \) stays bounded a.s. as \( N \to \infty \) (note here that \( \| \chi_{Q_n} \| = \sqrt{|Q_n|} \).

Next, we study assumption 3.7 in the present setting. Let \( \mathcal{P} = \{ Q_1, Q_2, \ldots \} \) be a countable system of cubes and set \( \Phi = \{ \chi_{Q_n} : n \in \mathbb{N} \} \). We will assume that \( \mathcal{P} \) satisfies the conditions of lemma B.5 (where \( \Omega = [0,1]^d \) and \( A_i = Q_i \) for \( i \in \mathbb{N} \)). Let \( \{ n_i \}_{i \in \mathbb{N}} \) and \( \{ \delta_i \}_{i \in \mathbb{N}} \) be defined accordingly. Moreover, we define
\[
e_{\delta_i} = \inf_{n_i < j \leq n_{i+1}} \sqrt{|Q_j|} = \inf_{n_i < j \leq n_{i+1}} \| \chi_{Q_j} \|,
\]
where we assume that \( \{ n_i \}_{i \in \mathbb{N}} \) is non-increasing. This means that we partition the set \([0,1]^d \) into disjoint sub-cubes \( \{ Q_j \}_{n_i < j \leq n_{i+1}} \) whose size (or scale) is bounded by \( \{ \delta_i, \delta_i \} \). It is more natural to formulate convergence rate results in terms of the total number \( m \) of used scales rather than in the total number of sub-cubes \( N = N(m) = n_{m+1} \). Following remark 3.3 and applying lemma B.5, we therefore define for a given continuous function \( p^i : [0,1]^d \to \mathbb{R} \)
\[
m_k := \inf \left\{ m \in \mathbb{N} : \frac{m + 1}{\sum_{i=0}^{\infty} \omega^{-2}(\delta_i, p^i)} \leq -2\sigma_k^2 \log \varepsilon_m \right\} \quad \text{and} \quad \eta_k := \sigma_k \sqrt{-2 \log \varepsilon_m}.
\]
Here, \( \omega(\cdot, p^i) \) denotes the modulus of continuity of \( p^i \) (cf definition B.4). With this and the general convergence rate result in theorem 3.8, we obtain

**Corollary 4.8.** Let \( u^i \in L^2([0,1]^d) \) be a J-minimizing solution of (1) where \( g \in \text{span} \Phi \) and that satisfies the source condition (10) with the source element \( p^i \in C([0,1]^d) \). Moreover, let \( m_k \) and \( \eta_k \) be defined as in (29). If
\[
\lim_{k \to \infty} \eta_k = 0 \quad \text{and} \quad \alpha_k := e^{-\left(\frac{\varepsilon_m}{\sigma_k^2}\right)^2} = e^{-2\varepsilon_m^2} \in \ell^1(0,1)
\]
for a constant \( \kappa > 0 \), then the SMRE \( \hat{u}_k = \hat{u}_{N(m_k)}(\alpha_k) \) almost surely satisfy (20).

**Example 4.9.** We consider the system of all dyadic partitions \( \mathcal{P} = \mathcal{P}_2 \) of \([0,1]^d \) as in example B.9. In particular, we note that the assumptions of lemma B.5 are fulfilled with \( n_l = \left( \frac{1}{2^{l+1}} - 1 \right) / (2^l - 1), \delta_l = 2^{-l} \sqrt{d} \) and \( \varepsilon_l = 2^{-l+d/2} \).

If \( p^i \in \mathcal{H}_\beta([0,1]^d) \) for \( 0 < \beta \leq 1 \), then there exists a constant \( Q = Q(p^i) > 0 \) such that \( \omega(\delta_l, p^i) \leq Q\delta_l^\beta \). This shows that
\[
\frac{m + 1}{\sum_{i=0}^{\infty} \omega^{-2}(\delta_i, p^i)} \leq Q^2 d^\beta (2^{2\beta} - 1) \frac{m + 1}{2^{2\beta(m+1)} \sigma_k^2}
\]
for \( m \in \mathbb{N} \) large enough. From this and (29), it is easy to see that
\[
m_k + 1 \sim \frac{1}{2\beta \log 2} \log \left( \frac{Q^2 d^\beta (2^{2\beta} - 1)}{d\sigma_k^2 \log 2} + 1 \right) \quad \text{and} \quad \eta_k \sim \sigma_k \sqrt{-\log \sigma_k}.
\]
Thus, if there exists a constant \( \kappa > 0 \) such that
\[
\alpha_k = e^{-\left(\frac{\varepsilon_m}{\sigma_k^2}\right)^2} = \sigma_k^2
\]
is summable and if the true J-minimizing solution \( u^i \) satisfies the source condition (10) with the source element \( p^i \in \mathcal{H}_\beta([0,1]^d) \), it follows that the SMRE \( \hat{u}_k = \hat{u}_{N(m_k)}(\alpha_k) \) almost surely satisfy (20) with \( \eta_k = \sigma_k \sqrt{-\log \sigma_k} \).
4.3. TV-regularization for imaging

In this section, we will study the theoretical properties of SMRE for the special case where $J$ denotes the total-variation semi-norm of measurable, bi-variate functions. It has been argued (e.g. in \cite{52}) that this has a particular appeal for linear inverse problems arising in imaging (such as deconvolution), since discontinuities along curves (i.e. edges) are not smoothed by minimizing $J$.

We assume henceforth that $\Omega \subset \mathbb{R}^2$ is an open and bounded domain with Lipschitz boundary $\partial \Omega$ and outer unit normal $\nu$. Moreover, we set $U = L^2(\Omega)$ and define $BV(\Omega)$ to be the collection of $u \in U$ whose derivative $Du$ (in the sense of distributions) is a signed $\mathbb{R}^2$-valued Radon measure with finite total variation $|Du|$, that is,

$$|Du|(\Omega) = \sup_{\phi \in C_c^1(\Omega, \mathbb{R}^2), \|\phi\|_{L^1} \leq 1} \int_{\Omega} \text{div}(\phi) u \, dx < \infty.$$  

We note that the norm $\|u\|_{BV} := |u|_{L^1} + |Du|(\Omega)$ turns $BV(\Omega)$ into a Banach space and that with this norm $BV(\Omega)$ is continuously embedded into $L^2(\Omega)$. The embedding is even compact if $L^2(\Omega)$ is replaced by $L^p(\Omega)$ with $p < 2$ (a proof of these embedding results can be found in \cite[theorem 2.5]{1}). For an exhaustive treatment of $BV(\Omega)$, see \cite{59}). With this, we define

$$J(\gamma) = \begin{cases} |Du|(\Omega) & \text{if } u \in BV(\Omega) \\ +\infty & \text{else.} \end{cases} \quad (30)$$

The functional $J$ is convex and proper and, as was shown, e.g., in \cite[theorem 2.3]{1}, $J$ is lower semi-continuous on $L^2(\Omega)$. This shows that $J$ satisfies assumption 2.1 (ii). Next, we examine assumption 3.4.

**Lemma 4.10.** If there exists $n_0 \in \mathbb{N}$ such that $|\langle K\mathbf{1}, \phi_{n_0} \rangle| > 0$, then assumption 3.4 holds. Here, $\mathbf{1}$ denotes the constant 1-function on $\Omega$.

**Proof.** Let $c \in \mathbb{R}$ and $\{u_k\}_{k \in \mathbb{N}} \subset A(c)$. Then, in particular it follows that $\sup_{k \in \mathbb{N}} J(u_k) \leq c < \infty$ and thus we find with Poincaré’s inequality (see \cite[theorem 5.11.1]{59})

$$\|u_k - \bar{u}_k\|_{L^2} \leq c_1 J(u_k) \leq c_2 < \infty$$

for suitable constants $c_1, c_2 \in \mathbb{R}$, where $\bar{u}_k = |\Omega|^{-1} \int_\Omega u_k(\tau) \, d\tau$. Now choose $\phi \in \{\phi_1, \ldots, \phi_N\}$ and observe that

$$\frac{|\bar{u}_k|(|\phi, K\mathbf{1})|}{\|\phi\|} = \frac{|\langle \phi, K\bar{u}_k \rangle|}{\|\phi\|} \leq \frac{|\langle \phi, K(u_k - \bar{u}_k) \rangle|}{\|\phi\|} + \frac{|\langle \phi, Ku_k \rangle|}{\|\phi\|} \leq K\|u_k - \bar{u}_k\|_2 + \max_{1 \leq n \leq N} \frac{|\langle Ku_k, \phi_n \rangle|}{\|\phi_n\|} \leq K\|c_2 + c\|.$$

Let $1 \leq n_0 \leq N$ be such that $|\langle K\mathbf{1}, \phi_{n_0} \rangle| =: \gamma > 0$. Then, $|\bar{u}_n| \leq (\|K\|c_2 + c)\|\phi_{n_0}\|/\gamma =: c_3$ and we find

$$\|u_n\|_{L^2} \leq (\|u_n - \bar{u}_n\|_2 + |\bar{u}_n|) \leq c_2 + c_3|\Omega|. \quad \square$$

We note that the assumptions in lemma 4.10 already imply the weak compactness of the sets (11) and thus guarantee existence of a $J$-minimizing solution of (1). From the above cited embedding properties of the space $BV(\Omega)$, it is easy to derive an improved version of the consistency result in theorem 3.6.
**Corollary 4.11.** Let \( g \in \text{span} \Phi \) and assume that \( u^\dagger \in \text{BV} (\Omega) \) is the unique \( J \)-minimizing solution of (1). Moreover, let \( \{a_k\}_{k \in \mathbb{N}} \) and \( \{N_k\}_{k \in \mathbb{N}} \) be as in theorem 3.6 and define \( \hat{u}_k = \hat{u}_{N_k} (a_k) \). Then, additionally to the assertions in theorem 3.6, we have that

\[
\lim_{k \to \infty} \| \hat{u}_k - u^\dagger \|_{L^p} = 0 \quad \text{a.s.}
\]

for every \( 1 \leq p < 2 \).

**Proof.** From theorem 3.6, it follows that \( \{\hat{u}_k\}_{k \in \mathbb{N}} \) is bounded a.s. in \( L^2 (\Omega) \) and that each weak cluster point is a \( J \)-minimizing solution of (1). Since we assumed that \( u^\dagger \) is the unique \( J \)-minimizing solution of (1), it follows that \( \hat{u}_k \rightharpoonup u^\dagger \) in \( L^2 (\Omega) \) a.s. and therefore also in \( L^p (\Omega) \) for each \( 1 \leq p < 2 \).

Since \( \Omega \) is assumed to be bounded, it follows that \( L^2 (\Omega) \) is continuously embedded into \( L^1 (\Omega) \). Thus, it follows from theorem 3.6 that almost surely \( \sup_{k \in \mathbb{N}} \| \hat{u}_k \|_{\text{BV}} < \infty \). From the compact embedding \( \text{BV} (\Omega) \hookrightarrow L^p (\Omega) \) for \( 1 \leq p < 2 \), it hence follows that \( \{\hat{u}_k\}_{k \in \mathbb{N}} \) is compact in \( L^p (\Omega) \). Thus, the assertion follows, since weak and strong limits coincide. \( \square \)

Unfortunately, the above embedding technique cannot be used in order to improve the convergence rate result in theorem 3.8 to strong \( L^p \)-convergence and thus we have to settle for the general results in theorem 3.8.

We recall that a function \( u \in \text{BV} (\Omega) \) satisfies the source condition if there exists \( \xi \in \text{ran}(K^*) \) such that \( \xi \in \partial J (u) \). It is important to note that in many applications the elements in \( \text{ran}(K^*) \) exhibit high regularity such as continuity or smoothness. Thus, it is of particular interest if such regular elements in \( \partial J (u) \) exist. If \( u \) is itself a smooth function, the application of Green’s formula and example A.4 yield (see also [53, lemma 3.71]).

**Lemma 4.12.** Let \( u \in \mathcal{C}_0^1 (\Omega, \mathbb{R}^2) \) and set \( E [u] = \{ x \in \Omega : \nabla u (x) \neq 0 \} \). Assume that there exists \( z \in \mathcal{C}_0^1 (\Omega, \mathbb{R}^2) \) with \( |z| \leq 1 \) and

\[
z (x) = - \frac{\nabla u (x)}{|\nabla u (x)|} \quad \text{for } x \in E [u].
\]

Then, \( \xi := \text{div} (z) \in \partial J (u) \).

In many applications (such as imaging), the true solution \( u \in \text{BV} (\Omega) \) is not continuous, as, e.g., if \( u \) is the indicator function of a smooth set \( D \subset \Omega \). The following examples show that in this case we still have \( \partial J (u) \cap \mathcal{C}_0^\infty (\Omega) \neq \emptyset \). For the analytical details, we refer to [53, example 3.74].

**Example 4.13.** Assume that \( D \subset \Omega \) is a closed and bounded set with \( \mathcal{C}^\infty \)-boundary \( \partial D \) and set \( u = \chi_D \). The outward unit normal \( n \) of \( D \) can then be extended to a compactly supported \( \mathcal{C}^\infty \)-vector field \( z \) with \( |z| \leq 1 \). Independent of the choice of the extension, we then have \( \xi := \text{div} (z) \in \partial J (u) \) and \( \xi \in \mathcal{C}_c^\infty (\Omega) \).

**Example 4.14.** We consider \( \Omega = [0, 1]^2 \) and \( V = L^2 (\Omega) \). Moreover, we assume that \( \mathcal{P}_2 \) denotes the set of all dyadic partitions of \( \Omega \) (cf example B.9) and that \( \Phi \) is the collection of indicator functions w.r.t. elements in \( \mathcal{P}_2 \).

For a function \( k : \mathbb{R}^2 \to \mathbb{R} \), we consider the *convolution operator* on \( U \) defined by

\[
(K u)(x) = \int_{\mathbb{R}^2} k (x - y) \tilde{u} (y) \, dx \quad \text{for } x \in \Omega,
\]

where \( \tilde{u} \) denotes the extension of \( u \) on \( \mathbb{R}^2 \) by zero-padding. Assume further that \( u^\dagger \) is the indicator function on a closed and bounded set \( D \subset \Omega \) with \( \mathcal{C}^\infty \)-boundary \( \partial D \) and that
\( \xi \in \mathcal{H}(u^t) \) is as in example 4.13. If the Fourier transform \( \mathcal{F}(k) =: \hat{k} \) of \( k \) is non-zero a.e. in \( \mathbb{R}^2 \) and if there exists \( \beta \in (1, 2) \) such that
\[
(1 + |\cdot|^2)^{-\beta/2}(\xi/\hat{k}) \in L^2(\mathbb{R}^2) \quad \text{and} \quad \text{supp}(p^\dagger := \mathcal{F}^{-1}(\xi/\hat{k})) \subset \Omega,
\]
then assumption 3.7 is satisfied. To be more precise, we have that \( p^\dagger \in \mathcal{H}_{\beta-1}(\Omega) \) (see [2, theorem 7.63]) and if there exists a constant \( k > 0 \) such that \( \alpha_k := \sigma_k^{2k} \) is summable it follows from example 4.9 and example A.4 that
\[
\lim sup_{k \to \infty} \frac{|\hat{u}_k|_2}{\alpha_k \sqrt{-\log \sigma_k}} = \lim sup_{k \to \infty} \frac{\int_{\Omega} \frac{1}{\sigma_k \sqrt{-\log \sigma_k}} |\mathcal{D}\hat{u}_k|_1}{\sum_{i,j \in S} u_{ij}} < \infty \quad \text{a.s.}
\]
for the SMRE \( \hat{u}_k = \hat{u}_{N_k}(\alpha_k) \) (where \( N_k \) is as in example 4.9).

We close this section by two numerical examples that indicate the applicability of SMRE for total variation-based imaging. All SMREs are computed by an ADMM. For implementation details and further numerical comparisons, see [32]. We note that, in contrast to the theoretical considerations in section 3, the dictionary \( \Phi \) is usually fixed when computing SMRE for specific applications. Further, the variance \( \sigma^2 \) is estimated from the data. Thus, the probability \( \alpha \) remains the only parameter to be chosen in the definition of the SMRE.

**Example 4.15.** We first study the case of image denoising. We set \( U = V = \mathbb{R}^{n \times n} \) with \( n = 256 \) equipped with the standard Euclidean inner product and induced norm and study the model
\[
Y_{ij} = u_{ij} + \sigma \varepsilon_{ij}, \quad 1 \leq i, j \leq n,
\]
where \( \varepsilon = \{ \varepsilon_{ij} \} \) is a lattice of independent standard normal random variables. We choose \( u^t \) to be the ‘cameraman’ image (with values in \([0, 255]\)). In the first column of figure 4, the corresponding noisy images \( Y \) are depicted with \( \sigma = 30 \) (upper row) and \( \sigma = 50 \) (lower row).

Let \( S \) be the collection of all discrete squares in \([1, \ldots, 256]^2\) up to a maximal side length of 15. Then, \( S \) consists of \( N = 930 \) 295 elements and we choose the dictionary \( \Phi \) to contain all the scaled indicator functions \( \phi_S = \chi_S/n \) for \( S \in S \). Note that
\[
\|\phi_S\| = \sqrt{\frac{\#S}{n^2}} \leq 1 \quad \text{and} \quad \phi_S^* = \frac{\phi_S}{\|\phi_S\|} = \frac{\chi_S}{\sqrt{\#S}},
\]
where \( \#S \) stands for the number of grid points in \( S \). We choose the function \( t_N(s, r) \) as in section 4.2 (with \( \gamma = d = 2 \)) such that the MR-statistic \( T_N \) (definition 3.1) takes the form
\[
T_N(v) = \max_{S \in S} \left( \left\| \phi_S^* \right\| - \sqrt{-4 \log \|\phi_S\|} \right) = \max_{S \in S} \left( \frac{1}{\sqrt{\#S}} \sum_{(i,j) \in S} v_{ij} \right) - \sqrt{2 \log \left( \frac{n^2}{\#S} \right)}.
\]
Observe that this is the discrete version of the statistic (28). Summarizing, for \( \alpha \in [0, 1] \), the SMRE \( \hat{u}_N(\alpha) \) is a solution of
\[
\inf_{u \in \mathbb{R}^{n \times n}} J(u) \quad \text{s.t.} \quad T_N(\hat{\sigma}^{-1}(Y - u)) \leq q_N(\alpha),
\]
where \( J \) denotes the discrete total variation functional and \( q_N(\alpha) \) the \( 1 - \alpha \) quantile of \( T_N(\varepsilon) \). For the (presumably) unknown \( \sigma \) we use the estimator \( \hat{\sigma} = 1.4826 \text{MAD} \), where MAD denotes the median absolute deviation computed from the data \( Y \). The solutions \( \hat{u}(\alpha) \) are depicted in the middle column of figure 4 together with the normalized residuals \( (Y - \hat{u}(\alpha))/\hat{\sigma} \) (last column).

For all computations, we choose the quantile \( q_N(\alpha) = -2 \) in (31) that corresponds to a value \( \alpha \) close to 1. For both noise levels, the residuals hardly reveal any non-random structure which
Figure 4. First column: data \( Y \) with \( \sigma = 30 \) (top) and \( \sigma = 50 \) (bottom). Middle column: SMREs \( \hat{u}(\alpha) \). Last column: residuals corresponding to the reconstructions in the middle column.

confirms that the MR-statistic constitutes a well-suited measure for data fidelity. Moreover, the reconstructions are reasonably smooth while preserving local details.

To complement the visual impression, we compare our result with the recently established locally adaptive image denoising method [35]. The method requires a user-defined smoothing parameter \( \theta \in [0, 1] \) where we use the value \( \theta = 0.6 \) as suggested in [35]. We note that the procedure in [35] does not require any \textit{a priori} knowledge on the variance \( \sigma^2 \) (in fact, it also applies to heteroscedastic noise). In order to gain a balanced assessment, we compute three different types of distance measures between the estimator and the true signal: the signal-to-noise ratio (SNR) and the integrated absolute error (IAE)

\[
\text{SNR}(u) = 10 \log_{10} \left( \frac{\sum_{i,j} |u_{ij}^\dagger - \bar{u}_{ij}|^2}{\sum_{i,j} |u_{ij}^\dagger - u_{ij}|^2} \right) \quad \text{and} \quad \text{IAE}(u) = \frac{1}{n^2} \sum_{i,j} |u_{ij} - u_{ij}^\dagger|,
\]

where \( \bar{u}^\dagger \) denotes the mean value of \( u^\dagger \). SNR and IAE basically measure the quality of the reconstruction in terms of the image intensity. Additionally, we compute the Bregman distance \( D_\xi^J(\cdot, u^\dagger) \) (where we use a subgradient \( \xi \) as in example 4.12) that measures the mean deviation between the unit normals at the level lines of the reconstruction and the true image (cf example A.4). The Bregman distance hence measures how well the smoothness of the reconstruction matches the smoothness of the true image.

In table 1, the averaged values of 100 simulation runs for the Bregman distance, SNR and IAE, are listed for \( \sigma = 30 \) and \( \sigma = 50 \). As can be seen from table 1, our approach outperforms the method in [35] with respect to all three distance measures. We mention that SNR values corresponding to other reconstruction methods can be found in [35] for a further comparison.
Finally, we stress that computation of SMRE is numerically demanding: whereas the estimators in [35] can be computed roughly in 10 s, the computation of $\hat{u}(\alpha)$ takes up to 15 min (both Matlab implementations on a dual-core (2.4 GHz) computer). In the latter case, the computation time strongly depends on the tolerance for numerical solutions of (31) and on the number of elements in the dictionary $\Phi$. We mention, though, that the algorithmic methodology used in this example (see [32] for details) permits efficient parallelization which is not exploited in the current implementation.

Example 4.16. Finally, we study the performance of the SMRE approach for image deconvolution, i.e. we consider with $U$ and $V$ as in example 4.15 the model

$$Y_{ij} = (K u^\dagger)_{ij} + \sigma \epsilon_{ij}, \quad 1 \leq i, j \leq n,$$

where $K$ is a convolution operator inducing motion blur and where $\sigma = 13$. In figure 5, the data (left image) and the SMRE reconstruction $\hat{u}(\alpha)$ (middle image) are depicted, where $\hat{u}(\alpha)$ solves

$$\inf_{u \in \mathbb{R}^{n \times n}} J(u) \quad \text{s.t.} \quad T_N(\hat{\sigma}^{-1}(Y - Ku)) \leq q_N(\alpha).$$

The statistic $T_N$, the functional $J$, $q_N(\alpha)$ and $\hat{\sigma}$ are chosen as in example 4.15.

The right image in figure 5 shows the standardized residuals. Similar to the denoising case, the non-random structures are reduced to a reasonable amount where at the same time the result $\hat{u}(\alpha)$ does not seem to be under-regularized. This gives numerical evidence that SMREs are a promising approach for image deconvolution.

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Moreover, the source condition (10) can be rewritten as

Example A.3.

Example A.2.

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Let $J$ be a Bregman divergence for some frequently used regularization functionals. We will summarize the meaning of the source-condition (10) and the Bregman divergence for some examples.

Appendix A. Source-condition and Bregman divergence: some examples

The notions of source-condition and Bregman divergence are very common in the field of inverse problems. We will summarize the meaning of the source-condition (10) and the Bregman divergence for some examples.

Example A.1. Let $J(u) = \frac{1}{2} \| u \|^2$. Then, $J$ is differentiable on $U$ and for all $u \in U$ the set $\partial J(u)$ consists of the single element $\{u\}$. We have that $J'(u)(w) = (v, w)$ and consequently

$$D_J(v, u) = D_J^*(v, u) = \frac{1}{2} \| v - u \|^2 \quad \text{for } \xi = u \in \partial J(u).$$

Moreover, the source condition (10) can be rewritten as

$$u^t \in \text{ran}(K^*_+) \text{.}$$

Since $\text{ran}(K^*_+) = \text{ran}(K^* K)^{1/2}$, this shows that the source condition (10) corresponds to the Hölder source condition $u^t \in \text{ran}(K^* K)^{\beta}$ for $\beta = 1/2$ (see [28]). In [6, section 5.3], the Hölder-source condition w.r.t. a smoothing operator $K$ on Hilbert scales has been discussed.

To be more precise, assume that $\{H_\mu\}_{\mu \in \mathbb{R}}$ is a scale of Hilbert spaces and that $K$ is a-smoothing, i.e. $K : H_\mu \to H_\mu$ is continuous with a continuous inverse. Then, the condition $u^t = (K^* K)^{\beta/2} u$ implies that $u^t \in H_{2\beta}$. A prototype for Hilbert scales are Sobolev spaces. Here the index $\mu$ corresponds to the Sobolev index.

Example A.2. Let $\{\psi_n\}_{n \in \mathbb{N}}$ be a ONB of $U$ and define

$$J(u) = \|u\|_1 := \sum_{j \in \mathbb{N}} |\langle u, \psi_n \rangle|.$$ 

In applications, this functional promotes sparse solutions, that is, solutions that have only few non-zero coefficients w.r.t. the basis $\{\psi_n\}_{n \in \mathbb{N}}$. As was argued in [36, Rem. 17] the source-condition (10) holds if and only if there exist constants $a, b, \gamma > 0$ such that $\|u^t\|_1 < a$ and

$$\|u\|_1 - \|u^t\|_1 \geq -\gamma \|K(u - u^t)\|$$

for all $u \in U$ such that $\|u\|_1 < a$ and $\|K(u - u^t)\| < b$. If additionally for every finite set $J \subset \mathbb{N}$ the restriction of $K$ to the set $\{\psi_n : n \in J\}$ is injective, there exist constants $\beta_1, \beta_2 > 0$ such that

$$\|u - u^t\|_1 \leq \beta_1 D_J^* \beta (u, u^t) + \beta_2 \|K(u - u^t)\|$$

for all $u \in U$ (see the proof of [36, theorem 15] and [31, theorem 6.4]).

Example A.3. Assume that $U = L^2(\Omega)$ for an open and bounded set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary $\partial \Omega$ and outer unit normal $\nu$ and let $H^\beta(\Omega)$ denote the Sobolev space of order $\beta \in \mathbb{R}$. We define

$$J(u) = \begin{cases} \int_\Omega |\nabla u|^2 \, dx & \text{if } u \in H^1(\Omega) \\ +\infty & \text{else.} \end{cases}$$

Then (see [3, p 63]), the set $D(\partial J)$ consists of all elements $u \in H^2(\Omega)$ that have the vanishing normal derivative $\langle \nabla u, \nu \rangle$ on $\partial \Omega$ and if $u \in D(\partial J)$, then $\partial J(u) = \{-\Delta u\}$. With this, it follows that $J'(u)(w) = \langle \nabla v, \nabla w \rangle$ and

$$D_J(v, u) = D_J^*(v, u) = \frac{1}{2} \| \nabla (v - u) \|^2 \quad \text{for } \xi = -\Delta u \in \partial J(u).$$
Moreover, \( u^1 \) satisfies the source condition (10) with the source element \( p^1 \in V \) if and only if

\[
-(K^* p^1)(x) = \Delta u^1(x) \quad \text{in } \Omega
\]

\[
\nabla u^1 \cdot v = 0 \text{ } \mathcal{H}^{n-1} \text{-a.e. on } \partial \Omega
\]

(here \( \mathcal{H}^{n-1} \) stands for the \((n-1)\)-dimensional Hausdorff measure on \( \partial \Omega \)).

**Example A.4.** Assume that \( U \) is as in example appendix A.3 with \( \Omega \subset \mathbb{R}^2 \) and let \( J \) be the total variation semi-norm as defined in (30). As was for example proved in [30, theorem 4.4.2], one has \( \xi \in \partial J(\alpha) \) if and only if there exists \( z \in L^\infty(\Omega, \mathbb{R}^2) \) with \( \|z\|_{L^\infty} \leq 1 \) such that \( \langle z, v \rangle = 0 \) on \( \partial \Omega \):

\[
\text{div}(z) = \xi \quad \text{and} \quad \int_\Omega \xi \, u \, \mathrm{d}x = |D\!u|(\Omega).
\]

If \( \xi \in \partial J(\alpha) \), it thus follows that

\[
D^T_j(v, u) = |D\!v|(\Omega) - \int_\Omega \xi \, u \, \mathrm{d}x.
\]

One can show that

\[
D^T_j(v, u) = \int_\Omega (1 - \cos(\gamma(v, u, x))) \mathrm{d}|D\!v|(x),
\]

where \( \gamma(v, u, x) \) denotes the angle between the unit normals of the sub-level sets of \( u \) and \( v \) at the point \( x \in \Omega \).

**Appendix B. Proofs**

**B.1. Proofs of the main results**

In this section, the proofs of the main results, that is existence, consistency and convergence rates for SMRE, are collected. We start with a basic estimate for the quantile function \( q_N(\cdot) \) of the MR statistic as defined in (14). We will assume that assumptions 2.1 and 3.4 hold.

**Lemma B.1.** Assume that \( T_N \) is an MR-statistic and let \( \alpha \in (0, 1) \) and \( N \in \mathbb{N} \). Then,

\[
q_N(\alpha) \leq \text{med}(T_N(\epsilon)) + L\sqrt{-2\log(2\alpha)}.
\]

**Proof.** First, we introduce the function \( f(x_1, \ldots, x_N) = \max_{1 \leq n \leq N} T_N(x_n, \|\phi_n\|) \). Then, \( f \) is Lipschitz continuous with \( \|f\|_{Lip} \leq L \). Next, define for \( 1 \leq n \leq N \) the random variables \( \epsilon_n := \epsilon(\phi_n^*). \) Then, \((\epsilon_1, \ldots, \epsilon_N) \sim N(0, \Sigma) \) for a symmetric and positive matrix \( \Sigma \in \mathbb{R}^{N \times N} \) with \( \|\Sigma\|_2 = 1 \). Hence,

\[
T_N(\epsilon) = \max_{1 \leq n \leq N} T_N(\epsilon(\phi_n^*), \|\phi_n\|) = f(\epsilon_1, \ldots, \epsilon_N) = f(\Sigma^{1/2}Z),
\]

where \( Z \) is an \( N \)-dimensional random vector with independent standard normal components. In other words, the statistic \( T_N(\epsilon) \) can be written as the image of \( Z \) under the Lipschitz function \( f(\Sigma^{1/2}) \). Applying Borel’s inequality (see [56, lemma A.2.2]) we find that

\[
2\mathbb{P}(T_N(\epsilon) - \text{med}(T_N(\epsilon)) > L\eta) \leq \exp(-\eta^2/2) \quad \text{for all } \eta \in \mathbb{R}.
\]

Now let \( \alpha \in (0, 1) \), choose \( q < q_N(\alpha) \) and set \( \eta = (q - \text{med}(T_N(\epsilon)))/L \). Then, \( \mathbb{P}(T_N(\epsilon) \leq q) < 1 - \alpha \) and hence

\[
\alpha = 1 - (1 - \alpha) < 1 - \mathbb{P}(T_N(\epsilon) < q) = \mathbb{P}(T_N(\epsilon) \geq q) \leq \frac{1}{2} \times \exp\left(-\frac{1}{2}\left(\frac{q - \text{med}(T_N(\epsilon))}{L}\right)^2\right).
\]

Rearranging the above inequality yields

\[
q < \text{med}(T_N(\epsilon)) + L\sqrt{-2\log(2\alpha)} \quad \text{for all } q < q_N(\alpha).
\]

The assertion follows for \( q \rightarrow q_N(\alpha) \). \( \square \)
We proceed with the proof of the existence result in theorem 3.5. To this end, we use a standard compactness argument from convex optimization. For the sake of completeness, however, we will present the proof.

**Proof of proposition 3.5.** Let \( N \geq N_0 \) and \( y \in V \) be arbitrary. Due to assumption 2.1 (ii), \( D(J) \subset U \) is dense and hence there exists for all given \( \delta > 0 \) an element \( u_0 \in D(J) \) such that \( \| Ku_0 - y \| \leq \delta \), where \( y \) denotes the orthonormal projection of \( y \) onto \( \text{ran}(K) \). Since \( \phi_n \in \text{ran}(K) \) and \( \phi_n^* = 1 \) for all \( n \in \mathbb{N} \), this implies that \( |\langle Ku_0 - y, \phi_n^* \rangle| = |\langle Ku_0 - y, \phi_n \rangle| \leq \delta \) for all \( n \in \mathbb{N} \).

Now let \( \sigma > 0 \) and \( \alpha \in (0, 1) \). Since \( T_N \) is an MR-statistic (cf definition 3.1) we find that \( t_N(0, r) < 0 \) for all \( r \in (0, 1] \). Thus, according to the reasoning above, there exists \( u_0 \in D(J) \) such that for \( 1 \leq n \leq N \)

\[
L\sigma^{-1}|y_n - \langle Ku_0, \phi_n^* \rangle| \leq q_N(\alpha) - \max_{1 \leq n \leq N} \lambda_N(\|\phi_n\|), \tag{B.1}
\]

if the right-hand side is positive. To see this, assume that \( q_N(\alpha) \leq \max_{1 \leq n \leq N} \lambda_N(\|\phi_n\|) \). Since for \( 1 \leq n \leq N \) we have that \( t_N(\|\phi_n\|) \geq \lambda_N(\|\phi_n\|) \) almost surely according to (13), it then follows that

\[
\mathbb{P}(T_N(\varepsilon) \geq q_N(\alpha)) \geq \mathbb{P}(T_N(\varepsilon) \geq \max_{1 \leq n \leq N} \lambda_N(\|\phi_n\|)) = 1.
\]

This is a contradiction to the definition of \( q_N(\alpha) \) in (14) and thus \( u_0 \in D(J) \) as in (B.1) can be chosen. Since \( s \mapsto t_N(s, r) \) is Lipschitz continuous with constant \( L \) and increasing for all \( r \in (0, 1] \), we find \( t_N(\sigma^{-1}|y_n - \langle Ku_0, \phi_n^* \rangle|, \|\phi_n\|) \leq t_N(0, \|\phi_n\|) + L\sigma^{-1}|y_n - \langle Ku_0, \phi_n^* \rangle| \leq q_N(\alpha) \) for \( 1 \leq n \leq N \). In other words, there exists at least one element \( u_0 \in D(J) \) such that

\[
u_0 \in S := \{ u \in U : \max_{1 \leq n \leq N} t_N(\sigma^{-1}|y_n - \langle Ku, \phi_n^* \rangle|, \|\phi_n\|) \leq q_N(\alpha) \}.
\]

Now, choose a sequence \( \{ u_k \}_{k \in \mathbb{N}} \subset S \) such that \( J(u_k) \to \inf_{u \in S} J(u) \). This shows that \( \sup_{k \in \mathbb{N}} J(u_k) := a < \infty \). Moreover, we find from (13) that there exist constants \( c_1, c_2 > 0 \) such that for all \( 1 \leq n \leq N \)

\[
c_1\sigma^{-1}|y_n - \langle Ku_k, \phi_n^* \rangle| \leq c_2 t_N(\|\phi_n\|) \leq q_N(\alpha).
\]

Together with (12), this shows \( c_1\sigma^{-1}|y_n - \langle Ku_k, \phi_n^* \rangle| + c_2\lambda_N(\|\phi_n\|) \leq q_N(\alpha) \). Rearranging the inequality above yields

\[
\max_{1 \leq n \leq N} |\langle Ku_k, \phi_n^* \rangle| \leq \max_{1 \leq n \leq N} |y_n| + \frac{\sigma}{c_1} (q_N(\alpha) - c_2 \inf_{1 \leq n \leq N} \lambda_N(\|\phi_n\|)) =: b < \infty.
\]

Summarizing, we find that \( u_k \in A(a + b) \) for all \( k \in \mathbb{N} \), as a consequence of which we can drop a weakly convergent sub-sequence (indexed by \( r(k) \) say) with the weak limit \( \hat{u} \). Since we assumed that \( t_N(\cdot, r) \) is convex for all \( r \in (0, 1] \), it follows that the admissible region \( S \) is convex and closed and therefore weakly closed. This shows that \( \hat{u} \in S \). Moreover, the weak lower semi-continuity of \( J \) (cf assumption 2.1 (iii)) implies that

\[
J(\hat{u}) \leq \liminf_{k \to \infty} J(u_{r(k)}) = \inf_{u \in S} J(u)
\]

and the assertion follows with \( \hat{u}_N(\alpha) = \hat{u} \).

\[ \square \]

In order to prove Bregman-consistency of SMR-estimation in theorem 3.6, we first establish a basic estimate for the data error.

**Lemma B.2.** Let \( N \geq N_0 \) and \( \alpha \in (0, 1) \). Moreover, assume that \( u^* \) is a solution of (1) and that \( \hat{u}_N(\alpha) \) is an SMRE. Then, for \( 1 \leq n \leq N \)

\[
c_1\sigma^{-1}|(Ku^* - \hat{Ku}_N(\alpha), \phi_n^*)| \leq T_N(\varepsilon) - 2c_2\lambda_N(\|\phi_n\|) + \text{med}(T_N(\varepsilon)) + L\sqrt{-2\log(2\alpha)}.
\]

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Proof. From definition 3.3, it follows that \( t_N(\sigma^{-1} |(Ku^t - K\hat{u}_N(\alpha) + \sigma \epsilon, \phi^*_n)|, \|\phi_n\|) \leq q_N(\alpha) \) for \( 1 \leq n \leq N \). The convexity of \( t_N \) hence implies that

\[
\begin{align*}
t_N((2\sigma)^{-1}|(Ku^t - K\hat{u}_N(\alpha), \phi^*_n)|, \|\phi_n\|) & \leq \frac{1}{2} \left( t_N(\sigma^{-1}|(Y - K\hat{u}_N(\alpha), \phi^*_n)|, \|\phi_n\|) + t_N(\sigma \epsilon, \|\phi_n\|) \right) \\
& \leq \frac{1}{2} \left( q_N(\alpha) + T_N(\epsilon) \right).
\end{align*}
\]

By setting \( v = (2\sigma)^{-1}|(Ku^t - K\hat{u}_N(\alpha), \phi^*_n)| \) and \( r = \|\phi_n\| \) in (13), the above estimate shows that

\[
\frac{c_1}{2\sigma} |(Ku^t - K\hat{u}_N(\alpha), \phi^*_n)| + c_2 t_N \left( \frac{1}{2} |(Ku^t - K\hat{u}_N(\alpha), \phi_n)|, \|\phi_n\| \right) \leq q_N(\alpha) + T_N(\epsilon) - 2c_2 \lambda_N(\|\phi_n\|).
\]

Finally, the assertion follows from lemma B.1.

With these preparations, we are now able to prove Bregman consistency.

Proof of theorem 3.6. By the definition of the SMRE \( \hat{u}_k = \hat{u}_k(\alpha_k) \), it follows that

\[
P(J(\hat{u}_k) > J(u^*) < \alpha_k < \infty), \text{ if follows from the Borel–Cantelli lemma (see [54, p 255]) that} \]

\[
P(J(\hat{u}_k) > J(u^*) \text{ i.o.}) = q_N(\alpha_k) \text{ i.o.} = 0.
\]

In particular, it follows that \( \sup_{k \in \mathbb{N}} J(\hat{u}_k) =: a < \infty \text{ a.s.} \).

Next, we note that \( \sup_{k \in \mathbb{N}} T_N(\epsilon) < \infty \text{ a.s.} \) implies that \( \sup_{N \in \mathbb{N}} \text{med}(T_N(\epsilon)) < \infty \). Hence, it follows from lemma B.2 and (16) that \( \max_{1 \leq n \leq N} |(Ku^t - K\hat{u}_k, \phi^*_n)| = \mathcal{O}(\xi_k) \) almost surely as \( k \to \infty \) which proves (18). In particular, (18) and the fact that \( N_k \to \infty \) imply \( \sup_{k \in \mathbb{N}} \max_{1 \leq n \leq N} |(Ku^t - K\hat{u}_k, \phi^*_n)| =: b < \infty \text{ a.s.} \). Summarizing, we find that \( \hat{u}_k \in \Lambda(a + b) \) which is sequentially weakly precompact according to assumption 3.4 (ii). Choose a sub-sequence indexed by \( \rho(k) \) with the weak limit \( \hat{u} \in U \). Since \( N_k \to \infty \) as \( k \to \infty \), it follows from (18) and (16) that

\[
|g - K\hat{u}, \phi^*_n)| = \lim_{k \to \infty} |(Ku^t - K\hat{u}_{\rho(k)}, \phi^*_n)| = 0 \quad \text{for all } n \in \mathbb{N}.
\]

Since we assumed that \( g \in \text{span} \Phi \), this shows that \( K\hat{u} = g \). Furthermore, according to (B.2) there exists (almost surely) an index \( k_0 \) such that \( J(\hat{u}_k) \) does not exceed \( J(u^*) \) for all \( k \geq k_0 \). Together with the weak lower semi-continuity of \( J \) this shows \( J(\hat{u}) \leq \lim \inf_{k \to \infty} J(u^*_{\rho(k)}) \leq \lim \sup_{k \to \infty} J(u^*_k) \leq J(u^*) \). Since \( u^* \) is a \( J \)-minimizing solution of (1) we conclude that the same holds for \( \hat{u} \) and that \( J(\hat{u}) = J(u^*) = \lim_{k \to \infty} J(\hat{u}_{\rho(k)}) \). In particular, for each sub-sequence \( J(u_{\rho(k)}) \), there exists a further sub-sequence that converges to \( J(u^*) \). This already shows that \( \lim_{k \to \infty} J(\hat{u}_k) = J(u^*) \text{ a.s.} \).

We next prove that \( D_f(u^*, \hat{u}_k) \to 0 \). To this end, recall that there almost surely exists an index \( k_0 \) such that for \( k \geq k_0 \) one has \( T_N(\epsilon) \leq q_N(\alpha_k) \). In order to exploit strong duality arguments, however, we have to make sure that the interior of the admissible region is non-empty (Slater’s constraint qualification). But since we assumed that \( s \mapsto t_N(s, r) \) is (strictly) increasing for each fixed \( r \in (0, 1) \) it follows that \( \mathbb{P}(t_N(|\epsilon \phi^*_n|), \|\phi_n\|) = q_N(\alpha_k) = 0 \) for all \( n \in \mathbb{N} \) and thus

\[
P(\exists k \in \mathbb{N} : T_N(\epsilon) < q_N(\alpha_k) = 0 \text{ for all } k \geq k_0) = 1.
\]
By introducing the functional
\[ G_k(v) = \begin{cases} 0 & \text{if } T_N((\varphi_k(v Y - v)) \leq q_N(\alpha_k), \\ +\infty & \text{else,} \end{cases} \]
we can rewrite (3) into \( \hat{u}_k \in \arg\min_{u \in U} J(u) + G_k(Ku) \). From (B.3), it follows that \( u^1 \) lies in the interior of the admissible set of the convex problem (3). In other words, the functionals \( G_k \) are continuous at \( Ku^1 \) for \( k \) large enough. Therefore, we can apply [27, chapter II proposition 4.1] (cf. also chapter II, remark 4.2 therein) and choose an element \( \xi_k \in V \) such that \( K^* \xi_k \in \partial J(\hat{u}_k) \) and \( -\xi_k \in \partial G_k(K\hat{u}_k) \). The second inclusion and the definition of the sub-gradient show that \( G_k(Ku) \geq G_k(\hat{u}_k) + \langle \xi_k, Ku - K\hat{u}_k \rangle = \langle K^* \xi_k, \hat{u}_k - u \rangle \) for all \( u \in U \). In particular, \( u^1 \) satisfies \( T_N((\varphi_k(v Y - u^1))) = T_N(\varepsilon) < q_N(\alpha_k) \) and thus \( G_k(Ku^1) = 0 \). This shows \( 0 \geq \langle K^* \xi_k, \hat{u}_k - u^1 \rangle \). Since \( J(\hat{u}_k) \rightarrow J(u^1) \) we find
\[
0 \leq \limsup_{k \to \infty} D_J(u^1, \hat{u}_k) \leq \limsup_{k \to \infty} D_J^{K^*}(u^1, \hat{u}_k)
= \limsup_{k \to \infty} J(u^1) - J(\hat{u}_k) - \langle \xi_k, u^1 - \hat{u}_k \rangle \leq \limsup_{k \to \infty} J(u^1) - J(\hat{u}_k) = 0.
\]
This proves (17). \( \square \)

It remains to prove the convergence rate results in theorem 3.8. To this end, additional regularity of the true \( J \)-minimizing solutions \( u^1 \) of (1) has to be taken into account. This is formulated in assumption 3.7. With this we get the following basic estimate.

**Lemma B.3.** Assume that assumption 3.7 holds and let \( N \geq N_0 \) and \( \alpha \in (0, 1) \). Then,

\[
|\langle K^* p^1, \hat{u}_N(\alpha) - u^1 \rangle| \leq \frac{\sigma}{c_1} (\bar{T}_N(\varepsilon) - 2c_2 \inf_{1 \leq n \leq N} \lambda_N(\| \phi_n \|) + L \sqrt{-2 \log(2\alpha)}) \sum_{n=1}^{N} |b_n| + \rho_N \| K\hat{u}_N(\alpha) - Ku^1 \|
\]
where \( \bar{T}_N(\varepsilon) = T_N(\varepsilon) + \text{med}(T_N(\varepsilon)) \).

**Proof.** From assumption 3.7, we find that
\[
|\langle K^* p^1, \hat{u}_N(\alpha) - u^1 \rangle| \leq \left| \left\langle \sum_{n=1}^{N} b_{n,N} \phi_n^*, K\hat{u}_N(\alpha) - Ku^1 \right\rangle \right| + \rho_N \| K\hat{u}_N(\alpha) - Ku^1 \| 
\leq \sum_{n=1}^{N} |b_{n,N}| \max_{1 \leq n \leq N} |\langle \phi_n^*, K\hat{u}_N(\alpha) - Ku^1 \rangle| + \rho_N \| K\hat{u}_N(\alpha) - Ku^1 \|.
\]
From lemma B.2, it follows that
\[
\max_{1 \leq n \leq N} |\langle \phi_n^*, K\hat{u}_N(\alpha) - Ku^1 \rangle| \leq \frac{\sigma}{c_1} (\bar{T}_N(\varepsilon) - 2c_2 \inf_{1 \leq n \leq N} \lambda_N(\| \phi_n \|) + L \sqrt{-2 \log(2\alpha)})
\]
which shows the assertion. \( \square \)

Combination of the auxiliary result in lemma B.3 with theorem 3.6 paves the way to the proof of theorem 3.8.

**Proof of theorem 3.8.** First, observe that assumption 3.7 and the definition of \( \eta_k \) imply (16), that is, all assumptions in theorem 3.6 are satisfied. Therefore, \( \{\hat{u}_k\}_{k \in \mathbb{N}} \) is bounded almost surely and due to the continuity of \( K \) we find that \( \sup_{k \in \mathbb{N}} \| K\hat{u}_k - Ku^1 \| < \infty \) a.s. After setting
Lemma B.5. Let \( |m_i| \) for all \( i \). Then, for systems of piecewise constant functions defined on a convex and compact set \( \Omega \), in this section, we collect some results on the approximation properties and entropy estimates.

B.2. Approximation of continuous functions and entropy estimates

In this section, we collect some results on the approximation properties and entropy estimates for systems of piecewise constant functions defined on a convex and compact set \( \Omega \subset \mathbb{R}^d \) (\( d \geq 1 \)). We start with the following basic.

Definition B.4. Let \( \Omega \subset \mathbb{R}^d \) be compact and convex.

(i) For a function \( g : \Omega \to \mathbb{R} \), the modulus of continuity is defined by

\[
\omega(\delta, g) = \sup_{s, t \in \Omega, |s - t| \leq \delta} |g(s) - g(t)| \quad \text{for} \quad \delta > 0.
\]

(ii) A function \( g : \Omega \to \mathbb{R} \) is called Hölder continuous with the exponent \( \beta \in (0, 1) \) if \( \omega(\delta, g) = O(\delta^\beta) \). The collection of all functions on \( \Omega \) that are Hölder-continuous with exponent \( \beta \) is denoted by \( \mathcal{H}_\beta(\Omega) \).

The following lemma provides an error estimate for the approximation of a continuous function \( g : \Omega \subset \mathbb{R}^d \to \mathbb{R} \) by piecewise constant functions in terms of the modulus of continuity.

Lemma B.5. Let \( \Omega \subset \mathbb{R}^d \) be a compact and convex set and \( \{A_1, A_2, \ldots\} \) be a collection of measurable sub-sets of \( \Omega \). Assume that there exists an increasing sequence \( \{n_l\}_{l \in \mathbb{N}} \subset \mathbb{N} \) with \( n_0 = 0 \) such that

(i) for all \( n_l + 1 \leq i < j \leq n_{l+1} \) one has \( |A_i \cap A_j| = 0 \)

(ii) and \( \Omega = A_{n_l+1} \cup \ldots \cup A_{n_{l+1}} \)

for all \( l \in \mathbb{N} \). Then, for all continuous \( g : \Omega \to \mathbb{R} \), there exist coefficients \( b_{j,l}^m \) such that

\[
\sup_{m} \sum_{j=0}^{m} \sum_{l= n_l+1}^{n_{l+1}} |b_{j,l}^m| \leq \|g\|_\infty \quad \text{and} \quad \left\| g - \sum_{m=0}^{m} \sum_{j= n_l+1}^{n_{l+1}} b_{j,l}^m \mathcal{A}_j \right\|_2 \leq \frac{m + 1}{\sum_{l=0}^{m} \omega^{-2}(\delta_l, g)},
\]

where \( \delta_l := \max_{n_l < j \leq n_{l+1}} \text{diam}(A_j) \).
Proof. Let \( g : \Omega \to \mathbb{R} \) be continuous. For \( l \in \mathbb{N} \), we define
\[
g_l = \sum_{j=n_l+1}^{n_{l+1}} |A_j|^{-1} \int_{A_j} g(\tau) \, d\tau \cdot \chi_j.
\]
Next, we introduce \( a_{lm} = (\omega^{-2}(\delta_l, g)) / (\sum_{l=0}^{m} \omega^{-2}(\delta_l, g)) \) for \( m \in \mathbb{N} \) and \( 1 \leq l \leq m \). Note, that \( a_{lm} \in (0, 1) \) and \( \sum_{0 \leq l \leq m} a_{lm} = 1 \). With this, we define for \( 0 \leq l \leq m \) and \( n_l < j \leq n_{l+1} \) the coefficients \( b_{jl} = (\int_{A_j} g(\tau) \, d\tau) / |A_j| \). Since we assumed that \( g \) is continuous on the compact set \( \Omega \), it follows that \( |b_{jl}| \leq \|g\|_{\infty} a_{lm} \) and hence \( \sum_{l=0}^{m} \sum_{j=n_l+1}^{n_{l+1}} |b_{jl}| \leq \|g\|_{\infty} \) for all \( m \in \mathbb{N} \). Moreover, we have for all \( s \in \Omega \) that
\[
\left| \sum_{l=0}^{m} a_{lm} g_l(s) - g(s) \right| \leq \sum_{l=0}^{m} a_{lm} \left( \sum_{j=n_l+1}^{n_{l+1}} \int_{A_j} |g(\tau) - g(s)| \, d\tau \cdot \chi_j(s) \right).
\]
After applying Jensen’s inequality and keeping in mind that \( |s - t| \leq \delta_l \) for \( s, t \in A_j \) and \( n_l < j \leq n_{l+1} \), it follows that
\[
\int_{\Omega} \left| \sum_{l=0}^{m} a_{lm} g_l(s) - g(s) \right|^2 \, ds \leq \sum_{l=0}^{m} a_{lm} \int_{\Omega} \left( \sum_{j=n_l+1}^{n_{l+1}} \frac{1}{|A_j|} \int_{A_j} |g(\tau) - g(s)|^2 \, d\tau \cdot \chi_j(s) \right) \, ds
\]
\[
= \sum_{l=0}^{m} a_{lm} \sum_{j=n_l+1}^{n_{l+1}} \frac{1}{|A_j|} \int_{A_j} \int_{\Omega} |g(\tau) - g(s)|^2 \, d\tau \, ds
\]
\[
\leq \sum_{l=0}^{m} a_{lm} \omega^2(\delta_l, g) \sum_{j=n_l+1}^{n_{l+1}} |A_j|.
\]
assumptions (i) and (ii) together with the definition of the coefficients \( a_{lm} \) eventually yield
\[
\int_{\Omega} \left| \sum_{l=0}^{m} a_{lm} g_l(s) - g(s) \right|^2 \, ds \leq \frac{m + 1}{\sum_{l=0}^{m} \omega^2(\delta_l, g)}.
\]
\[\square\]

For the remainder of this section, we collect some results concerning the capacity number of (subsystems of) the set \( \Phi_d \) of indicator functions on convex and closed sets in \( [0, 1]^d \) with \( d \geq 1 \). We first recall the basic definition

**Definition B.6.** Let \((T, d)\) be a semi-metric space, \( T' \subset T \) and \( \varepsilon > 0 \). The capacity number is defined by
\[
D(\varepsilon, T') := \sup_{T'' \subset T} (\#T'' : d(a, b) \geq \varepsilon \text{ for all } a \neq b \in T'').
\]

From a practical point of view, it is often more convenient to express (27) in terms of the \( \varepsilon \)-covering number \( N(\varepsilon, T') \) of \( T' \) which is defined as the smallest number of \( \varepsilon \)-balls in \( T \) needed to cover \( T' \) (the centre points need not be elements of \( T' \), though). It is a common knowledge (see [56, p 98]) that for all \( \varepsilon > 0 \)
\[
N(\varepsilon, T) \leq D(\varepsilon, T) \leq N(\varepsilon/2, T).
\]

We consider \( \Phi_d \subset L^2([0, 1]^d) \) as a metric space with the induced \( L^2 \)-metric, i.e. for \( \chi_P, \chi_Q \in \Phi_d \) we have
\[
d(\chi_Q, \chi_P)^2 = \|\chi_P - \chi_Q\|^2 = \int_{[0,1]^d} (\chi_Q - \chi_P)^2 \, dx = |Q \Delta P|.
\]
The entire set $\Phi_d$ is too large in order to render the test-statistic $T_d$ in (26) finite. It was shown in [8] (see also [24, chapter 8.4]) that the $\epsilon$-covering number of $\Phi_d$ of all nonempty, closed and convex sets contained in the unit ball $\{x \in \mathbb{R}^d : |x| \leq 1\}$ is of the same order as $\exp(\epsilon^{(1-d)/2})$ (for $d \geq 2$) as $\epsilon \to 0^+$. This proves that there cannot exist any constants $A, B$ and $\gamma$ such that (27) holds with $\Phi = \Phi_d$.

For particular classes of convex sets, however, entropy estimates as in (27) are at hand. The collection $\Phi_d$ of indicator functions on $d$-dimensional rectangles in $[0, 1]^d$ constitutes such an example.

**Proposition B.7.** There exists a constant $A = A(d) > 0$ such that

$$D(u, \delta, \{\phi \in \Phi : \|\phi\| \leq \delta\}) \leq A(u\delta)^{-4d}$$

for all $u, \delta \in (0, 1]$.

**Proof.** From [56, theorem 2.6.7] it follows that the $\epsilon$-covering number of $\Phi_d$ can be estimated by $A\epsilon^{-2(\gamma-1)}$ where $\gamma$ denotes the VC index of the set of subgraphs $\{(x, t) : t < \phi(x)\}$ for $\phi \in \Phi_d$. This in turn is equal to the VC index of the collections of all rectangles in $[0, 1]^d$ which is $2d + 1$ (see [56, example 2.6.1]). $\square$

For certain subsets of $\Phi_d$ better estimates can be derived. We close this section with results for the system $\Phi_2$ and $\Phi_3$ of indicator functions on all squares and dyadic partitions in $[0, 1]^d$, respectively. We skip the proofs for they are elementary but rather tedious.

**Proposition B.8.** There exists a constant $A = A(d) > 0$ such that

$$D(u, \delta, \{\phi \in \Phi_2 : \|\phi\| \leq \delta\}) \leq A\epsilon^{-2(d+1)}\delta^{-d}$$

for all $u, \delta \in (0, 1]$.

**Proposition B.9.** Let $d \geq 2$ and consider the system of all dyadic partitions in $[0, 1]^d$, that is, $\mathcal{P}_2 := \{Q \subset [0, 1]^d : Q = 2^{-k}(i + [0, 1]^d), \ k \in \mathbb{N}, i = (i_1, \ldots, i_d) \in \mathbb{N}^d\}$. Let $\Phi_2$ be the set of all indicator functions on elements in $\mathcal{P}_2$. Then, there exists a constant $A = A(d) > 0$ such that

$$A^{-1}u^{-2}\delta^{-2} \leq D(u, \delta, \{\phi \in \Phi_2 : \|\phi\| \leq \delta\}) \leq A\epsilon^{-2}\delta^{-2}$$

for all $u, \delta \in (0, 1]$.

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