On the QED Radiator at order $\alpha^3$

Guido MONTAGNA$^{a,b}$, Oreste NICROSINI$^{b,a}$ and Fulvio PICCININI$^{b,a}$

$^a$ Dipartimento di Fisica Nucleare e Teorica, Università di Pavia, Italy
$^b$ INFN, Sezione di Pavia, Italy

Abstract

The $O(\alpha^3)$ leading logarithmic contribution to the QED radiator in the additive form is considered. The effect of the correction on two-fermion physics at LEP1 and LEP2 is evaluated and critically compared with the one of next-to-leading $O(\alpha^2)$ corrections. A critical comparison with existing estimates for the LEP1 energy range is also performed. The $O(\alpha^3)$ leading logarithmic corrections turn out to be relevant in view of the experimental precision already reached at LEP1 and foreseen at LEP2.

E-mail:
montagna@pv.infn.it
nicrosini@pv.infn.it, nicrosini@vxcern.cern.ch
piccinini@pv.infn.it
QED corrections are, as well known, an essential ingredient of precision physics at LEP/SLC. In particular, initial-state photonic radiation (ISR) plays a central rôle in the determination of the energy effectively available in the center of mass of the \( e^+e^- \) reaction. Nowadays, this effect is popularly described by the so called Structure Functions (SF) method, pioneered in \([1]\) and subsequently developed in \([2, 3, 4, 5, 6, 7, 8]\). For processes of the kind \( e^+e^- \rightarrow \gamma Z^0 \rightarrow f\bar{f} \) around the \( Z^0 \) resonance, the effect of ISR is crucial in the precise determination of the peak cross section. Going above the \( Z^0 \) peak, ISR plays a new rôle, namely it is responsible for the so called “\( Z^0 \) radiative return”, i.e. the emission of hard photons such that the energy available in the kernel reaction is brought back to the \( Z^0 \) mass. Being the \( Z^0 \) radiative return caused by ISR, the most precise knowledge of the hard tail of the IS photonic spectrum is of utmost importance. It must be noticed that the \( Z^0 \) radiative return enhances the contribution of precisely the hard photons that reduce the centre of mass energy to the \( Z^0 \) mass. This means that most of the events are characterized by the fact that their invariant mass after ISR is very close to the \( Z^0 \) mass, or, in other words, they are “LEP1-like” events. Moreover, once the resonance is produced there is no more particular enhancement of any portion of the spectrum: hence final state radiation (and initial-final state interference) gives contributions substantially equivalent to the ones it already gives at LEP1. Last, but not least, in the case of \( s \)-channel processes ISR represents a gauge invariant subset of the full set of QED corrections, and therefore represents a meaningful subject of investigation.

In the case in which only a cut on the invariant mass of the event after ISR is considered, the SF method provides a very simple recipe for computing the corrected cross section as a one-dimensional integration of the proper kernel cross section times the so called “radiator” (or “flux function”), namely as

\[
\sigma(s) = \int_0^{1-x_{\text{cut}}} dx H(x, s)\sigma_0((1-x)s),
\]

where the radiator is defined as

\[
H(x, s) = \int_{1-x}^1 \frac{dz}{z} D(z, s)D\left(\frac{1-x}{z}, s\right),
\]

\(D(x, s)\) being the electron SF. Equation (1), with the definition (2), is a very useful tool when considering semianalytical calculations \([9]\) devoted to data analysis; actually, the availability of simple and accurate analytical formulae is mandatory for the development of fast fitting programs.

The radiators available in the literature can be classified as belonging to two groups, according to the kind of solution for the electron SF adopted: additive \([1, 2, 3]\) and factorized \([4, 5, 6]\) radiators, respectively. As a matter of
fact, the additive radiators are extensively implemented in standard computational tools used for the two-fermion data analysis at LEP, for instance \[9\]. From now on, we will mainly consider this kind of radiators.

A typical additive radiator consists of an exponentiated part, taking into account soft multi-photon emission, plus finite-order leading-logarithmic (LL) corrections accounting for hard collinear bremsstrahlung up to $O(\alpha^2)$, namely \[1, 2\]

\[
H_{NT}(x, s) = \Delta_2 \beta x^{\beta - 1} + h_1(x, s) + h_2(x, s),
\]

\[
h_1(x, s) = -\frac{1}{2} \beta (2 - x),
\]

\[
h_2(x, s) = \frac{1}{8} \beta^2 \left[ (2 - x) (3 \ln(1 - x) - 4 \ln x) - 4 \frac{\ln(1 - x)}{x} + x - 6 \right],
\]

\[
\beta = 2 \left( \frac{\alpha}{\pi} \right) [L - 1],
\]

\[
L = \ln(s/m^2),
\]

\[
\Delta_2 = 1 + \left( \frac{\alpha}{\pi} \right) \delta_1 + \left( \frac{\alpha}{\pi} \right)^2 \delta_2.
\]

The corrections $\delta_{1,2}$, which within the SF formalism are determined at the LL level, can be adjusted to take into account process dependent soft plus virtual next-to-leading (NL) contributions.\[ For processes of the kind $e^+e^- \rightarrow \gamma, Z^0 \rightarrow f\bar{f}$, they are known \[2\] in such a way that the radiator of eq. (3) reproduces the exact $O(\alpha^2)$ soft plus virtual perturbative results \[11\].

For such processes, a more accurate form of the additive radiator has also been derived \[12\], taking into account $O(\alpha^2)$ NL hard-photon corrections, in such a way that the full $O(\alpha^2)$ perturbative calculation \[13\] is reproduced, namely

\[
H_B(x, s) = H_{NT}(x, s) - h_2(x, s) + \delta_2^H (1 - x, s),
\]

where $\delta_2^H$ is defined in eqs. (3.19) and (3.20) of ref. \[12\]. The radiator \[1\] differs from \[3\] by terms which turn out to be numerically negligible at the $Z^0$ peak, but can in principle be important far from the resonance because of the well known relevance of hard photon radiation. Actually, in \[12\] also some ansatz for factorized radiators can be found, which reproduce \[1\] up to the $O(\alpha^2)$ NL corrections and take into account additional higher order contributions (more on this later).

\[1\] Actually, also additional pair radiation gives contributions at the $O(\alpha^2)$; it can be accounted for by properly redefining the constant $\delta_2$ for the soft plus virtual part, and adding a proper contribution to the hard part of the radiator \[10\].
It has to be noticed that the corrections present in eq. (4) and neglected in (3) are dominated by terms of $O(\alpha^2 L)$, $L$ being the collinear logarithm. Being second-order NL corrections, they are in principle of the same order of magnitude as the third-order LL ones. These last corrections are already known in the literature at the SF level [5, 6] and for the factorized radiator [4].

By using the additive solution of the SF of [6], it is possible to derive the $O(\beta^3)$ corrections to the QED additive radiator. By combining these results with the ones already available, we propose the following form for the QED additive radiator, including both NL $O(\alpha^2)$ and $O(\beta^3)$ contributions:

$$H(x, s) = \Delta_3 \beta x^{\beta - 1} + h_1(x, s) + \delta_2^x (1 - x, s) + h_3(x, s),$$

$$h_3(x, s) = \frac{1}{3!} \left( \frac{\beta}{2} \right)^3 \left[ -\frac{27}{2} + \frac{15}{4} x + 4(1 - \frac{1}{2} x) \left( \pi^2 - 6 \ln^2 x + 3 \text{Li}_2(x) \right) 
+ 3 \ln(1 - x) \left( 7 - \frac{6}{x} - \frac{3}{2} x \right) + \ln^2(1 - x) \left( -7 + \frac{4}{x} + \frac{7}{2} x \right) 
- 6 \ln x (6 - x) + 6 \ln x \ln(1 - x) \left( 6 - \frac{4}{x} - 3x \right) \right],$$

$$\Delta_3 = \Delta_2 + \left( \frac{\alpha}{\pi} \right)^3 \delta_3,$$

$$\delta_3 = (L - 1)^3 \left( \frac{9}{16} - \frac{1}{2} \pi^2 - \frac{4}{3} \psi^{(2)}(1) \right),$$

where $\psi^{(n)}(z)$ is the $n$-th order polygamma function, $\psi^{(n)}(z) = d^n \psi(z)/dz^n$, $\psi(z) = \Gamma'(z)/\Gamma(z)$. It is worth noting that computing the radiator at $O(\beta^3)$ requires a redefinition of the normalization in front of the exponentiated term, $\Delta_3$, which picks up an $O(\beta^3)$ contribution, $\delta_3$, originating from the Gribov-Lipatov form factor.

In the following, a sample of numerical results will be shown and commented. Only results concerning $\mu$ cross sections will be considered, since the forward-backward asymmetry requires an analysis beyond the aim of the present paper.

In Fig. 1 the separate effects of the NL $O(\alpha^2)$ and $O(\beta^3)$ corrections to the radiator are shown for the QED corrected cross section as a function of the $s'$ cut defined as $s'/s \geq x_{\text{cut}}$. The relative deviations with respect to the cross section computed by means of the radiator of eq. (3) are shown for several centre of mass energies. Both the $O(\beta^3)$ (Fig. 1a) and NL $O(\alpha^2)$ (Fig. 1b) corrections amount to a contribution of several $0.1\%$ when the $Z^0$ radiative return is included, but they tend to compensate one another. When the $Z^0$ radiative return

---

2The $O(\beta^4)$ corrections to the SF are known to be at $1 \times 10^{-4}$ level, according to [4, 13, 16].
radiative return is excluded, or near the $Z^0$ resonance, the NL $\mathcal{O}(\alpha^2)$ corrections are confined at the level of 0.01-0.02%, whereas the $\mathcal{O}(\beta^3)$ ones remain at the level of 0.05-0.1%. It should be noted that in the recent analysis shown in [15], when studying the radiative corrections to two-fermion production at LEP2, only the effect of NL $\mathcal{O}(\alpha^2)$ has been taken into account, leading to results which, in the light of the present study, are incomplete.

Figure 2 shows the relative deviations of the cross section computed with the full radiator of eq. (5) with respect to the one computed by means of the one of eq. (3). In Fig. 2a the effect of the newly proposed additive radiator close to the resonance is quoted. It is worth noting that at the $Z^0$-peak the NL $\mathcal{O}(\alpha^2)$ plus $\mathcal{O}(\beta^3)$ corrections introduce a systematic shift of about $-0.07\%$, dominated by the $\mathcal{O}(\beta^3)$ corrections. This effect has not been included in the analyses of precision calculations performed in [14], but the present experimental accuracy [17] requires that it is carefully taken into account. Going beyond the $Z^0$ peak (Fig. 2b), the effect of the radiator of eq. (5) amounts to about $-0.1\%$ when the $Z^0$ radiative return is excluded, raising to about 0.25% when it is included. By combining the information of Figs. 1 and 2, one concludes that taking into account the NL $\mathcal{O}(\alpha^2)$ corrections but neglecting the $\mathcal{O}(\beta^3)$ ones, leads to theoretical predictions for the two-fermion processes that are underestimated by about 1% in the inclusive cases. Again, in view of the experimental precision foreseen for the inclusive cross sections, this effect has to be taken into account in the theoretical predictions. Moreover, it is worth noting that in the case of the hadronic cross section, the effect of the full additive radiator of eq. (5) as compared to the one of eq. (3), grows up from 0.25% (see Fig. 2b) to around 0.4% when including the $Z^0$ radiative return.

We notice that, in the case of LEP1 energies, the results of the present analysis concerning NL $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(\beta^3)$ corrections confirm the estimates obtained in [14], with the conclusion that, whereas at that time such effects were marginal, at present, and in the light of the continuous progress in the reduction of the experimental errors, they are significant. As far as the LEP2 regime is concerned, the present analysis points out the relevance of IS higher order hard photon effects, and completes a first attempt to quantify such contributions recently performed in [15].

As already pointed out above, in [12] a catalog of the radiators available at the time of the LEP1 workshop is given, and several options are offered, according to the particular form of the radiators and the contributions taken into account. In particular, the options can be grouped as follows: options (A) and (B) refer to the radiators reported in the present paper in eqs. (4) and (3) respectively, and are in the additive form; options (C), (D) and (E), on the
contrary, describe radiators in the factorized form, and are superseded by the radiator described in [4, 7]. The radiator presented here, eq. (5), is intended to supersede options (A) and (B), i.e. the additive radiators at present used in some computational tools for the LEP analysis. Given this situation, it is worth at this point to examine carefully in which relation the best additive and factorized radiator are. To this aim, in Fig. 3 a detailed comparison of the radiators in themselves as functions of their argument (Fig. 3a) and of the corresponding cross sections (Fig. 3b) as functions of the invariant mass cut after ISR is shown. The comparison of the radiators shows that their relative difference is within $3 \times 10^{-4}$ for most of the spectrum, raising up to $3 \times 10^{-3}$ in the very hard tail. Analogously, the comparison of the cross sections shows that their relative difference is contained within $2.5 \times 10^{-4}$ for centre of mass energies ranging from LEP1 to LEP2, and for $0.01 \leq x_{\text{cut}} \leq 0.99$, i.e. for every realistic situation. From the present analysis, one can conclude that the two radiators are equivalent from the phenomenological point of view.

The numerical results shown in the present paper point out that the additive radiators implemented in standard computational tools used for the two-fermion data analysis at LEP have to be upgraded both for LEP1 physics, where they do not fulfil the present precision requirements, and for LEP2 physics, where they do not describe appropriately the $Z^0$ radiative return. This is a first result of the present analysis. Moreover, given the results shown in Fig. 3, one can conclude that the upgrade can equivalently be performed either by definitely substituting the additive radiators by means of the factorized one of ref. [4, 7], or by simply modifying the ones already implemented with the addition of a few FORTRAN lines, according to the newly proposed radiator of eq. (5). This is a second result of the present analysis. At this point, the choice is a matter of taste/convenience.

Summarizing, the $O(\beta^3)$ contribution to the QED additive radiator has been considered, and a new additive radiator, accounting for NL $O(\alpha^2)$ and $O(\beta^3)$ corrections, is proposed. The effect of the $O(\beta^3)$ correction on two-fermion cross sections at LEP1 and LEP2 has been investigated and critically compared with the effect of NL $O(\alpha^2)$ corrections. The $O(\beta^3)$ corrections turn out to be relevant in view of the experimental precision already reached at LEP1; moreover, together with the NL $O(\alpha^2)$ terms, they represent a non-negligible contribution to ISR because of the $Z^0$ radiative return and in view of

\[^3\] In the approach of ref. [8], the “soft photon singularities” are regularized by means of the variance-reduction technique known as “control variates”; for its actual implementation, besides the radiator/structure function one needs also their primitives, possibly in analytical form in order to save CPU time; hence, for practical reasons, the implementation of an additive radiator is preferable.
the foreseen experimental precision at LEP2. The newly proposed additive radiator improves considerably the ones implemented in standard computational tools for two-fermion processes at LEP.

References

[1] E.A. Kuraev and V.S. Fadin, Sov. J. Nucl. Phys. 41 (1985) 466; G. Altarelli and G. Martinelli, in Physics at LEP, J. Ellis and R. Peccei, eds., CERN Report 86-02 (Geneva, 1986), vol. 1, p. 47.

[2] O. Nicrosini and L. Trentadue, Phys. Lett. B196 (1987) 551; Z. Phys. C39 (1988) 479; for a review see also in Radiative Corrections for $e^+e^-$ Collisions, J.H. Kühn ed., (Springer, Berlin, 1989), p. 25; in QED Structure Functions, Ann Arbor, MI, 1989, G. Bonvicini ed., AIP Conf. Proc. No. 201 (AIP, New York, 1990), p. 12; O. Nicrosini, ibid., p. 73.

[3] W. Beenakker, F.A. Berends and W.L. van Neerven, in Radiative Corrections for $e^+e^-$ Collisions, J.H. Kühn ed., (Springer, Berlin, 1989), p. 3.

[4] S. Jadach, M. Skrzypek and B.F.L. Ward, Phys. Lett. B257 (1991) 173.

[5] M. Skrzypek and S. Jadach, Z. Phys. C49 (1991) 577.

[6] M. Cacciari, A. Deandrea, G. Montagna and O. Nicrosini, Europhys. Lett. 17 (1992) 123.

[7] M. Skrzypek, Acta Phys. Pol. B23 (1992) 135.

[8] G. Montagna, O. Nicrosini and F. Piccinini, Phys. Rev. D48 (1993) 1021.

[9] G. Montagna, O. Nicrosini, G. Passarino and F. Piccinini, program TOPAZO 2.0, Comput. Phys. Commun. 93 (1996) 120; G. Montagna, O. Nicrosini, G. Passarino, F. Piccinini and R. Pittau, Nucl. Phys. B401 (1993) 3; Comput. Phys. Commun. 76 (1993) 328; D. Bardin et al., program ZFITTER 4.9, Nucl. Phys. B351 (1991) 1; Z. Phys. C44 (1989) 493; Phys. Lett. B255 (1991) 290; CERN-TH.6443/1992; hep-ph/9412201.

[10] B.A. Kniehl, M. Krawczyk, J.H. Kühn and R.G. Stuart, Phys. Lett. B209 (1988) 337.

[11] G.J.H. Burgers, Phys. Lett. B164 (1985) 167.
[12] F. Berends et al., “Z line shape”, in *Z Physics at LEP 1*, G. Altarelli, R. Kleiss and C. Verzegnassi, eds., CERN Report **89-08** (Geneva, 1989), vol. 1, p. 89.

[13] F. Berends, G.J.H. Burgers and W.L. van Neerven, Nucl. Phys. **B297** (1988) 429.

[14] M. Przybycien, DESY-92-182 (hep-th/9511029).

[15] F. Boudjema, B. Mele et al., “Standard Model Processes”, in *Physics at LEP2*, G. Altarelli, T. Sjöstrand and F. Zwirner, eds., CERN Report **96-01** (Geneva, 1996), vol. 1, p. 207.

[16] D. Bardin et al., “Electroweak Working Group Report”, in *Reports of the Working Group on Precision Calculations for the Z Resonance*, D. Bardin, W. Hollik and G. Passarino, eds., CERN Report **95-03** (Geneva, 1995), p. 7.

[17] A. Blondel, proceedings of ICHEP ’96, Warsaw, 1996; G. Altarelli, “Status of precision tests of the Standard Model”, CERN-TH/96-265, hep-ph/9611239.
Figure 1: The relative deviation between the cross sections computed by means of the radiators of eq. (5) with the substitution $\delta_2^H (1 - x, s) \to h_2(x, s)$ ($\sigma_3$ in a) and of eq. (4) ($\sigma_{2nl}$ in b), and the radiator of eq. (3) ($\sigma_2$), as a function of the invariant mass cut.
Figure 2: The relative deviation between the cross sections computed by means of the radiators of eq. (5) ($\sigma_f$) and the radiator of eq. (3) ($\sigma_2$), as a function of the invariant mass cut. The LEP1 (a) and LEP2 (b) cases.
Figure 3: The relative deviation between the radiators of eq. (5) $H_3^{ad}$ and of [4] $H_3^{YFS}$ (Fig. 3a), and the corresponding cross sections (Fig. 3b).