Homogeneous Yang–Baxter deformations as non-abelian duals of the $AdS_5$ $\sigma$-model

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Abstract
We propose that the Yang–Baxter deformation of the symmetric space $\sigma$-model parameterized by an $r$-matrix solving the homogeneous (classical) Yang–Baxter equation is equivalent to the non-abelian dual of the undeformed model with respect to a subgroup determined by the structure of the $r$-matrix. We explicitly demonstrate this on numerous examples in the case of the $AdS_5$ $\sigma$-model. The same should also be true for the full $AdS_5 \times S^5$ supercoset model, providing an explanation for and generalizing several recent observations relating homogeneous Yang–Baxter deformations based on non-abelian $r$-matrices to the undeformed $AdS_5 \times S^5$ model by a combination of $T$-dualities and nonlinear coordinate redefinitions. This also includes the special case of deformations based on abelian $r$-matrices, which correspond to $TsT$ transformations: they are equivalent to non-abelian duals of the original model with respect to a central extension of abelian subalgebras.

Keywords: non-abelian duality, integrable 2D sigma models, string theory, TsT transformations, Yang–Baxter deformed models

1. Introduction
The class of integrable deformations of the $AdS_5 \times S^5$ superstring model based on the ‘Yang–Baxter’ (YB) $\sigma$-model [1–3] has recently been under active investigation. It generalizes the usual group-space or coset-space $\sigma$-model as $\mathcal{L} = \text{Tr}(J J) \rightarrow \text{Tr}(J \mathcal{O} J)$ where $J = g^{-1}dg$ and $\mathcal{O}$ depends on $g$ and a constant antisymmetric operator $R$ acting on the Lie algebra (referred to as the ‘$r$-matrix’) satisfying a (modified) classical Yang–Baxter equation (cYBE), i.e. $[R_{X}, R_{Y}] = R[(R_{X}, Y) + [X, Y]] = c[X, Y]$.

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The two inequivalent cases are $c = 1$ and $c = 0$. The first case represents a non-trivial $q$-deformation of the symmetry algebra of the original symmetric space $\sigma$-model \cite{3, 4}. The second case (based on the homogeneous cYBE) studied in \cite{5–14} appears to be simpler and more closely related to the original coset $\sigma$-model. Indeed, it was observed on particular examples \cite{5–7, 9} and proved in general \cite{14} that for abelian $r$-matrices the resulting deformed model can be obtained from the original undeformed one by abelian $T$-dualities (more precisely, by a TsT transformation combining $T$-duality with a linear coordinate shift which is a special case of the $O(d, d)$ $T$-duality transformation).

Furthermore, for several examples defined by non-abelian $r$-matrices solving the homogeneous cYBE it was recently observed \cite{12, 13} that the resulting deformed $\sigma$-model can be related to the original $AdS_5 \times S^5$ supercoset model by a combination of $T$-dualities along non-commuting directions and non-linear coordinate transformations (required to be able to perform the $T$-dualities).

Here we will generalize (and provide an explanation for) these observations by demonstrating that the homogeneous YB deformations of a symmetric space $\sigma$-model are equivalent to non-abelian duality (NAD) transformations \cite{15–20} of the original model with respect to various (in general, non-semi-simple) subgroups of the global symmetry group. We will focus on the bosonic $AdS_5 \sigma$-model but the same should be true in general and also for the supercoset $AdS_5 \times S^5$ model.

The subgroup in which we will dualize is determined by the structure of the $r$-matrix. Not all subgroups have a corresponding classical $r$-matrix and hence not all NAD transformations correspond to a homogeneous YB deformation. Indeed the corresponding subalgebra should be quasi-Frobenius (or a central extension thereof), which, in particular, implies it is solvable. Therefore, the correspondence is absent, for example, if the subgroup is non-abelian and simple (such as $SO(1, 2) \subset SO(2, 4)$ or $SO(3) \subset SO(6)$). In these cases one would not expect the NAD to be a deformation of the original $\sigma$-model. On the other hand, the case of abelian $r$-matrices is naturally included: as we shall explain below, the abelian TsT transformation may be viewed as a special case of NAD with respect to a central extension of an abelian symmetry group. It is an interesting open question what the criteria is for a subgroup to give NAD model that can be understood as a deformation of the original $\sigma$-model, when one needs to centrally extend the group, and if these cases are in correspondence with the homogeneous YB deformations.

As the NAD should preserve the classical integrability of a symmetric $\sigma$-model \cite{21}, one may then, instead of studying homogeneous YB deformations, directly consider all possible NAD transforms of the $AdS_5 \times S^5\sigma$-model with respect to all possible (centrally extended) subgroups of $PSU(2, 2|4)$. As mentioned above, there will be special NAD models that have a ‘deformation’ interpretation (i.e. depend on free parameters that when taken to zero give back the original model) and other NAD models (corresponding to simple subgroups) that are close cousins of the original model (e.g. sharing the same first-order structure) but not reducing to it in a limit and not allowing one to reverse the NAD transform.\footnote{One may also consider $c = -1$. However, no solutions exist for the real form of the superalgebra $psu(2, 2|4)$, the symmetry algebra of the $AdS_5 \times S^5$ superstring.}

An advantage of NAD over the YB deformation is that it can be performed systematically as a path integral transformation (determining also the dilaton). This allows us, in particular, to answer the question of which NAD transforms of the $AdS_5 \times S^5\sigma$-model will...
still be Weyl-invariant so that the corresponding backgrounds will be solutions of the standard type II supergravity equations (and will thus define critical string models).

In general, the NAD transform of a Weyl-invariant $\sigma$-model is not Weyl but only scale-invariant [20] (in the context of GS superstring $\sigma$-model this means that the corresponding dual background solves only the generalized supergravity equations of [22, 23]). The condition for NAD to preserve Weyl invariance is that the structure constants of the group which is dualized should be traceless, i.e. $n_a \equiv f^b_{\ c a} = 0$ [19, 20]. An equivalent ‘unimodularity’ condition was found also for the homogeneous YB deformations of the $AdS_5 \times S^5$ $\sigma$-model as a condition on the corresponding non-abelian $r$-matrices [13]. If $n_a \neq 0$ the NAD-transformed background does not solve the Weyl-invariance or supergravity equations but it can still be mapped by a formal $T$-duality (along the $n_a$ direction, as discussed in section 2) to a proper supergravity solution, in agreement with the general discussion in [22].

While abelian TsT transformations (and thus equivalent abelian YB deformations) of $AdS_5 \times S^5$ are related, via $AdS$/CFT, to non-commutative gauge theories [9, 24–27] the role of the NADs of $AdS_5 \times S^5$ (beyond a supergravity solution generating technique) is presently unclear. In the TsT case one is guided by the action of $T$-duality on open strings or $D$-branes but similar intuition is absent in the NAD case (see though [28]). Still, the quantum Weyl-invariant and integrable $\sigma$-models obtained by NAD from the $AdS_5 \times S^5$ model may provide new non-trivial examples of solvable string models.

Let us briefly comment on the case of the inhomogeneous YB deformation of the $AdS_5 \times S^5$ $\sigma$-model [3] that non-trivially depends on one ‘quantum’ deformation parameter. The corresponding 2D theory is scale but not Weyl invariant, i.e. the associated background [29, 30] only solves the generalized equations of [22, 23]. At the same time, it is classically related [32, 33], by Poisson–Lie (PL) duality [31], to the ‘$\lambda$-model’ of [34, 35] (generalizing the bosonic model of [21]) which is Weyl-invariant at the quantum level [13, 36–38]. This suggests that as for the NAD in the case of $n_a \neq 0$ the PL duality (which should preserve quantum equivalence on a flat 2D background and thus scale invariance [39]) here should be ‘Weyl-anomalous’ at the quantum level.

A special ‘undeformed’ (level $k \to \infty$ or $q = e^{i\pi} \to 1$) limit of the $\lambda$-model is just the NAD of the $AdS_5 \times S^5$ supercoset model with respect to the full $PSU(2, 2|4)$ which is still Weyl invariant. The $\lambda$-model can thus be interpreted as a Weyl-invariance preserving $q$-deformation of the full NAD of the $AdS_5 \times S^5$ superstring model. It would be of interest to see if one can construct similar non-trivial deformations of NADs of $AdS_5 \times S^5$ with respect to some subgroups of $PSU(2, 2|4)$ (see [40]).

Among other open questions, it would be useful to give a general proof of the equivalence between the homogeneous YB deformations of the $AdS_5 \times S^5$ supercoset model and its NADs with respect to the corresponding subalgebras. One may be able to establish this relation by identifying the underlying first-order actions which include auxiliary gauge fields.

The structure of this paper is as follows. We shall start in section 2 with a review of the NAD transformation of a bosonic $\sigma$-model, explaining also how one can interpret the abelian TsT transformation as a special case. In section 3, after reviewing the YB deformation and the NAD of the symmetric space $\sigma$-model we shall turn our attention to the $AdS_5 \sigma$-model. For a range of cases of different types of $r$-matrices (abelian and non-abelian, both unimodular and Jordanian) we will explicitly demonstrate that the YB deformation is equivalent to the non-abelian dual of the $AdS_5 \sigma$-model with respect to a specific (centrally extended) subalgebra that can be identified from the $r$-matrix. In the appendix we will present a large number of additional Jordanian examples.

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7 For the bosonic YB deformed model such a first-order action was given in [33].
2. Non-abelian duality

In this section we shall discuss NAD in bosonic string $\sigma$-models [15–20, 41]. NAD relates one bosonic $\sigma$-model with a non-abelian global symmetry group $H$ (i.e. with $H$-invariant metric, $B$-field and dilaton $\phi$) to another one that generically has a smaller symmetry group (and no symmetry in the case when $H$ is non-abelian and simple). The standard abelian (or ‘torus’) $T$-duality [42] is a special case when $H$ is abelian, i.e. $\mathbb{R}^d$ or $U(1)^d$. The $O(d, d)$ generalization of the $T$-duality (and, in particular, the special $T\sigma T$ case) can also be viewed, as we shall explain below, as a particular case of NAD when one considers a central extension of the abelian group.

If the original $\sigma$-model is classically integrable then NAD maps it, as for the standard $T$-duality case [47, 43, 44], to a classically integrable model [21]. If the original $\sigma$-model is quantum Weyl-invariant, its NAD counterpart is also Weyl-invariant unless $H$ is non-semisimple with the generators in the adjoint representation satisfying $\text{Tr} T_a = 0$, i.e. $n_a \equiv f_{\alpha a}^\alpha = 0$ [19, 20]. In the latter case the NAD transformation preserves only the scale invariance of the $\sigma$-model.

It should be possible to generalize NAD to the case of the GS superstring $\sigma$-model and then for $n_a = 0$ NAD should again preserve Weyl invariance, i.e. (viewed as a transformation on the target space couplings of the $\sigma$-model) will map a supergravity solution to a supergravity solution\(^8\). For $n_a \neq 0$ NAD may not preserve Weyl invariance\(^9\), i.e. may transform a Weyl-invariant $\sigma$-model into a scale-invariant one. Then the original target space background that was a supergravity solution will be mapped by NAD to a solution of the generalized supergravity equations or superstring $\sigma$-model scale invariance conditions [22, 23].

As we shall see below, in agreement with the general expectation [22], in that case one can still apply a formal (classical, i.e. ignoring the dilaton transformation) $T$-duality to associate to the resulting generalized background a solution of the standard Weyl-invariance or supergravity equations that has a linear non-isometric term in the dilaton obstructing the reverse $T$-duality at the quantum level [30, 33].

2.1. NAD of bosonic $\sigma$-model in curved 2D background

Let us start with a bosonic string $\sigma$-model depending on group $H$ coordinates $h$ and ‘spectator’ coordinates $x'$ that is invariant under a global $H$-symmetry, $h \rightarrow h_0 h, h_0 \in H$. Its action can be written in conformal gauge $g_{ij} = e^{2\eta} \delta_{ij}$, with $\eta = \text{diag}(-1, 1)$, as

$$
I[h, x] = \frac{1}{4\pi\alpha'} \int d^2z \left[ E_{ab}(x) J_a^r J_b^r + L_{ab}(x) J_a^r J_b^r + M_{ab}(x) J_a^r J_b^r + K_{ab}(x) j^a j^b \right] - 2\alpha'\phi(x) \partial_+ \partial_- \sigma, \quad (2.1)
$$

$$
J_a^r = \text{Tr} (h^{-1} \partial_+ h T_a), \quad j_a^r = \partial_+ \partial_- \sigma, \quad [T_a, T_b] = f_{ab}^c T_c, \quad \text{Tr} (T_a T_b) = \delta^b_a. \quad (2.2)
$$

Here $i = 0, 1, \partial_\pm = \partial_0 \pm \partial_1, a, b = 1, 2, \ldots, d = \dim H, r, s = 1, \ldots, n. T_a$ is a basis for the algebra $\mathfrak{h}$ of $H$ (taken below to be given by matrices in the adjoint representation, $T_a = -f_{ab}^c$) and $T^a$ is a dual algebra basis (assuming $H$ is embedded into a simple group)\(^10\). We have also included a possible $H$-invariant dilaton term $\frac{1}{4\pi} \int d^2z \sqrt{g} R^{(2)} \phi(x) = -\frac{1}{2\pi} \int d^2z \phi(x) \partial^2 \phi$.

\(^8\) A prescription for the NAD transformation of the RR fields has already been proposed (by analogy with $T$-duality) in [45]. It has also been checked on examples that for simple groups $H$, like $SU(2)$, NAD indeed maps supergravity solutions to supergravity solutions and that the transformation is a symmetry of the supersymmetry variations of supergravity [46].

\(^9\) Here we are not considering the possibility of dualizing in fermionic directions.

\(^10\) For example, the case of a coset space may be included by taking $E_{ab}$ to be degenerate.
where \( \sigma \) is the conformal factor of the 2D metric. The path integral over \( h \) with the action (2.1) can be obtained from the path integral over the 2D gauge field \( A^a_\nu \) and ‘Lagrangemultiplier’ field \( \nu_\alpha \) with the following first-order or ‘interpolating’ action

\[
\hat{I}[A, \nu, x] = \frac{1}{4\pi \alpha'} \int d^2z \left[ F_{ab}(x) A^a_\nu A^b_\nu + L_{ar}(x) A^a_\nu \bar{j}^r_\nu A^b_\nu + M_{ab}(x) \bar{j}^r_\nu A^a_\nu + K_{\alpha}(x) \bar{j}^r_\nu j^\alpha \right.
\]
\[+ \nu_\alpha \epsilon^{a[b}(F^c_{cd})^{-1}_e) f^e_{ab} \left( 2\epsilon \sigma \delta_{a}^b \partial_\nu \partial_\sigma + 2\epsilon \phi(x) \partial_\nu \partial_\sigma \right). \]

(2.3)

Integrating over \( \nu_\alpha \) gives \( \nu_\alpha = 0 \) or \( A^a_\nu T_a = \sqrt{-h} \partial_\nu h \) and the first line in (2.3) becomes equivalent to the first line of (2.1) on a flat 2D background.

The \( n_\alpha \)-dependent term [20] in the second line of (2.3) [20], which is non-local in the 2D metric \( \sigma = -\frac{1}{4} \partial^2 \sqrt{h} R^{(2)} \), is important for the quantum equivalence of (2.3) and (2.1) on a curved 2D background. It comes from the Jacobian of the transformation from \( A_\nu = \sqrt{-h} \partial_\nu h, A_\sigma = h^{-1} \partial^2 h \) to \( h, h' \) on a curved 2D background and is related to the ‘mixed’ anomaly [19] \( \left( \partial_{\alpha} \frac{\delta}{\delta A_{\beta}} \sim R^{(2)} \right) \) which is present only if \( H \) is such that \( n_\alpha \neq 0 \).

The dual of the model (2.1) is then found by integrating out the gauge field \( A^a_\nu \) in (2.3)

\[
\hat{I}[\nu, x] = \int d^2z \left[ (\partial_\nu \nu_\alpha + M_{\alpha}(x) \partial_\nu x^\alpha + \epsilon_\nu \epsilon_\alpha \partial\nu h, \partial_\nu \nu_\alpha \partial_\nu x^\alpha \right] N^{ab} (\partial_\nu \nu_\beta - L_{br}(x) \partial_\nu x^\beta + \epsilon_\nu \epsilon_\alpha \partial_\nu h, \partial_\nu \nu_\alpha \partial_\nu x^\alpha \right]
\]
\[+ K_{\alpha}(x) \partial_\nu x^\alpha \partial_\nu x^\alpha = 2\epsilon \left( \phi(x) + \frac{1}{2} \ln \det \nabla \right) \partial_\nu \partial_\sigma \).

(2.5)

\[
N^{ab} = \left[ \epsilon_\nu \epsilon_\alpha + \nu_\alpha f^c_{ab} \right]^{-1}.
\]

(2.6)

If \( n_\alpha \neq 0 \) (2.5) does not have the interpretation of a local \( \sigma \)-model action on a curved 2D background. If we formally ignore the \( n_\alpha \)-dependent terms in (2.5) the corresponding dual target space background \( (\hat{G}, \hat{B}, \text{dilaton} \hat{\phi} ) \) will no longer represent a solution of the critical string Weyl-invariance conditions [20], explaining what was observed in [17] for a specific example. Considered on a flat 2D background the dual \( \sigma \)-model (2.5) will still be scale invariant so that the dual metric and \( B \)-field will solve the scale-invariance conditions (i.e., in the superstring context, the generalized supergravity equations [22, 23]).

Let us note that the scale-invariant dual background \( (\hat{G}, \hat{B}) \) has a remarkable property that it can still be naturally associated to a proper solution of the Weyl-invariance conditions by applying a formal (2D flat space) duality transformation in the direction of the isometric part of \( \nu_\alpha \) parallel to \( n_\alpha \). The key observation is that while for \( n_\alpha = 0 \) the model (2.5) can not be interpreted as a local \( \sigma \)-model on a curved 2D background, applying a 2D duality transformation to one scalar field in (2.5) restores such an interpretation. Let us split the Lagrange multiplier field \( \nu_\alpha \) in (2.3) as \( \nu_\alpha = \nu_a + n_a y \). Then \( y \) will appear in (2.3) and thus also in (2.5) only through its derivatives, i.e. shifts of \( y \) will be an abelian isometry of the dual background. Indeed, we will have \( \nu_a f^a_{bc} = f^a_{dc} f^d_{bc} = 0 \) as follows from the Jacobi identity. Then the \( y \)-dependent and related terms in (2.3) will be (up to integration by parts)
Applying $T$-duality in the $y$ direction amounts to replacing $\partial y$ by $B_y$ adding at the same time the term $-2\epsilon^y \xi \partial \bar{y} B_y$. Then integrating over $B_y$ gives $A_y = \partial \bar{y}$ and thus the last term in (2.7) becomes (after integrating by parts) $\epsilon^y \partial \bar{y} A_y \rightarrow -2\alpha^y \partial \bar{y}_s \partial_s$. The latter is the standard $R^{2|\phi}$ dilaton term with $\phi$ linear in the dual coordinate, $\phi \sim \bar{y}_s$. Integrating the rest of $A_y^a$ out will then give a local Weyl-invariant $\sigma$-model with a linear non-isometric dilaton term. The above steps can be carried out explicitly, e.g., on the examples of the models discussed in [20].

We conclude that the scale-invariant $\sigma$-model background found by applying the NAD transformation to the action (2.1), which, taking $n_\sigma = 0$, does not solve the Weyl-invariance equations, is still related by formal $T$-duality to a Weyl-invariant background with a linear non-isometric dilaton. This background cannot then be $T$-dualized back to give a standard Weyl-invariant $\sigma$-model with local couplings to the worldsheet metric. Thus NAD provides a particular example of the general case discussed in [22].

2.2. TsT duality as special case of NAD

Let us now consider the particular case of $O(2, 2)$ $T$-duality where one starts with a metric having two abelian isometries, $ds^2 = dx^2 + f_1(x)dy_1^2 + f_2(x)dy_2^2$, applies $T$-duality $y_1 \rightarrow \bar{y}_1$, then shifts $y_2 \rightarrow y_2 + \gamma y_1$ introducing the parameter $\gamma$, and finally $T$-dualizes back $\bar{y}_1 \rightarrow y_1$. This generates the following non-trivial background depending on $\gamma$

$$
\begin{align*}
\text{d} s^2 &= \text{d} x^2 + U(x)[f_1(x)\text{d} y_1^2 + f_2(x)\text{d} y_2^2], \\
B_{y_1 y_2} &= \gamma f_1 f_2 U, \\
e^{2\phi} &= U(x) \equiv \frac{1}{1 + \gamma^2 f_1(x)f_2(x)}.
\end{align*}
$$

(2.8)

Applications of such TsT transformations were discussed, e.g., in [24–26, 47, 48].

Below we shall show that this transformation of the corresponding 2D $\sigma$-model may be viewed as a special case of NAD with the algebra of $H$ being the centrally extended 2D translation algebra (or Heisenberg algebra)

$$
[P_r, P_s] = \epsilon_{rs} Z, \quad [P_r, Z] = 0, \quad r, s = 1, 2.
$$

(2.9)

Let us start with $L = f_1(\partial y_1)^2 + f_2(\partial y_2)^2$. The ‘interpolating’ action that corresponds to the first two steps of the above TsT transformation may be written as

$$
\tilde{L} = f_1 (A_y)^2 + f_2 (\partial y_2 + \gamma \partial \bar{y}_1)^2 + 2\epsilon^y \partial \bar{y}_1 A_y.
$$

(2.10)

$T$-dualizing again $\bar{y}_1 \rightarrow y_1$ by introducing another abelian gauge field $A'_y$ we find

$$
\begin{align*}
\tilde{L}' &= f_1 (A_y)^2 + f_2 (\partial y_2 + \gamma A'_y)^2 + 2\epsilon^y A'_y A_y + 2\epsilon^y \partial \bar{y}_1 A'_y, \\
\text{redefining } A_1' \equiv A_y, \quad \partial y_2 + \gamma A'_y \equiv A_2', \quad \text{and then sending } y_1 \rightarrow -\gamma y_2, \quad y_2 \rightarrow \gamma y_1 \text{ we arrive at}
\end{align*}
$$

$$
\tilde{L}' = f_1 (A_y)^2 + f_2 (A'_y)^2 + 2\epsilon^y (v_1 \partial A'y_1 + v_2 \partial A'_y) - 2\gamma^{-1} \epsilon^y A'_y A_y^2.
$$

(2.11)

(2.12)

If we first integrate over $v_i$ we have $A'_y = \partial y'_y$ and thus go back to the original model (the last term in (2.12) is then total derivative). Integrating instead over the two gauge fields in (2.12) one ends up with the $\sigma$-model corresponding to the TsT background (2.8) (with $v_i \rightarrow y_i$). Note that without the last $-2\gamma^{-1} \epsilon^y A'_y A_y^2$ term (2.12) is the first-order action for $T$-dualizing on both $y_1$ and $y_2$.

13. Indeed, the $\gamma = \infty$ limit of (2.8) is the double $T$-dual background. This last term in (2.10) which is absent in this limit, is analogous to ‘current–current’ deformation related to $O(2, 2)$ duality (see [48]).
Let us now derive the same Lagrangian (2.12) starting instead with the NAD model (2.3) with \( h \) taken to be the centrally extended algebra (2.9) with \( T_a = (P_1, P_2, Z) \) (for which only \( f_{12}^1 \) is non-zero and thus \( n_a = 0 \)). Here

\[
v_v \epsilon^y F_y = 2 \epsilon^y [v_1 \partial_i A^1_j + v_2 \partial_j A^2_i + w(\partial_i C_j + A^1_i A^2_j)], \tag{2.13}
\]

where \( w \equiv v^3 \) and \( C_i \equiv A_i^3 \) corresponds to the central generator \( Z \) in (2.9). Note that as the original \( \sigma \)-model is assumed to be invariant under the abelian translations, the central generator \( Z \) should act trivially on the coordinates. Gauging it is still possible as integrating out \( v \) and \( w \) brings us back to the original model: we get \( A_i^y = \partial_y y^i \) (and \( dC \) expressed in terms of \( y' \)), so that \( f_v(A'_i)^2 \rightarrow f_v(\partial y')^2 \).

Since the gauge field \( C_i \) does not enter the rest of the \( \sigma \)-model action, we can readily integrate it out getting the condition \( w = w^{(0)} = \text{const} \). Then (2.13) becomes the same as the last three terms in (2.12) with \( w^{(0)} \sim -\gamma^{-1} \).

Thus considering the central extension of the abelian translation group allows one to introduce an extra free parameter \( \gamma \) (absent in the standard first-order abelian T-duality action): the TS-T parameter \( \gamma \) acquires the interpretation of the background value of the dual coordinate corresponding to the central generator \( Z \). Let us note also that in the example we considered above the origin of the \( B \)-field of the resulting background can be traced to the non-abelian nature of the Heisenberg algebra (2.9) (which is also related to the non-commutativity of the dual gauge theory [24–26]).

One can obviously consider various generalizations. First, one can readily repeat the above discussion for generic model with two abelian isometries

\[
\mathcal{L} = f_1(\partial y_1)^2 + f_2(\partial y_2)^2 + g_1(\partial y_1)(\partial y_2) \quad \text{or} \quad h_1(\partial y_1) \partial y_2 + h_2(\partial y_2) \partial y_1 \quad \text{and} \quad h_i \quad \text{are functions of the remaining fields, including fermionic degrees of freedom.}
\]

If the rank of the abelian isometry algebra is greater than 2, constructing its central extensions and performing NAD will introduce several continuous parameters \( w^{(0)}_n \) as expected in the general \( O(d, d) \) T-duality case. Instead of an abelian isometry group one may also have a non-abelian one, \( H \), that has an abelian (translational) subgroup. While the direct application of the NAD transformation with respect to \( H \) may give a dual model with no free parameters, starting instead with a centrally extended group and then applying NAD will lead to a model containing several free parameters. In general, the central extension of an algebra satisfying \( f_{12}^1 = 0 \) will also satisfy this property.

For example, one may start with the euclidean \( AdS_3 \) space \( ds^2 = z^{-2}[d{x}^2 + dy_1^2 + dy_2^2] \) and perform NAD with respect to the 2D Euclidean group \( ISO(2) \subset SO(1, 3) \) with the algebra \( [J, P_1] = \epsilon_P P_1, \ [P_1, P_1] = 0 \). Considering its central extension then allows one to introduce a free parameter. For rank four and higher algebras one may be able to introduce several parameters. Such examples will be discussed below for the \( AdS_5 \sigma \)-model.

### 3. Homogeneous YB deformations of the \( AdS_5 \sigma \)-model and NAD

We now turn to demonstrating the conjectured equivalence between homogeneous YB deformations of a coset \( \sigma \)-model and NADs of that same model. We shall first make some general remarks and then focus on the bosonic \( AdS_5 \) model but similar considerations should apply also to the full \( AdS_5 \times S^5 \) supercoset model.

In general, the homogeneous YB deformation of the \( F/G \) symmetric space \( \sigma \)-model is based on a solution to the cYBE for \( f = \text{Lie}(F) \). These solutions are in correspondence with the quasi-Frobenius subalgebras of \( f \).


The homogeneous YB deformations with r-matrices corresponding to abelian subgroups are equivalent to TsT transformations of the original coset model [14]. As we have seen in section 2.2, the TsT transformation can, in fact, be reformulated as the NAD with respect to a centrally extended abelian subgroup.

Furthermore, in [13] it was shown that the YB deformation of the $AdS_5 \times S^5$ supercoset model [3, 7] corresponds to a supergravity solution (i.e. is one-loop Weyl invariant when supplemented with an appropriate dilaton) if the r-matrix $r = r^{\alpha \beta} e_{\alpha} \wedge e_{\beta}$ is unimodular, i.e. satisfies

$$r^{\alpha \beta} [e_{\alpha}, e_{\beta}] = 0.$$  

(3.1)

Here the $e_{\alpha}$ generate the quasi-Frobenius subalgebra $\mathfrak{h}$ of $\mathfrak{f}$. The unimodularity condition (3.1), combined with the quasi-Frobenius property, implies that the structure constants on $\mathfrak{h}$ satisfy

$$f^{\gamma}_{\alpha \beta} = 0.$$  

(3.2)

As discussed in section 2, this is precisely the same as the requirement on $\mathfrak{h}$ for the NAD with respect to an $H$-invariant $\sigma$-model to preserve the one-loop Weyl invariance [19, 20].

Based on these observations we shall make a conjecture that the homogeneous YB deformation based on a classical r-matrix is always equivalent to the NAD transformation of the original coset model with respect to the corresponding (centrally extended) quasi-Frobenius subalgebra. As already mentioned, for abelian r-matrices this follows from the results in section 2.2 and [14]. For non-abelian r-matrices we will not prove this conjecture directly, but will provide a comprehensive range of examples (including those in [11–13]), explicitly demonstrating its validity.

As our primary focus is on deformations of the $AdS_5 \times S^5$ superstring and its lower-dimensional counterparts here we will restrict our attention to the bosonic $AdS_5 \sigma$-model. The isometry algebra $so(2, 4)$ admits a number of different subgroups corresponding to non-abelian r-matrices of various types, both unimodular and Jordanian15. While it is unclear if it is a universal rule, we find that for unimodular r-matrices we need to consider the centrally extended algebra, while for Jordanian r-matrices this appears not to be required.

3.1. YB deformation and NAD for the symmetric space $\sigma$-model

Our starting point for both the homogeneous YB deformation and the NAD transformation will be the symmetric space $\sigma$-model. Here we will define them for a generic symmetric space $F/G$ with both $F$ and $G$ being semi-simple. In the case of $AdS_5$ we have $F = SO(2, 4)$ and $G = SO(1, 4)$.

For $\mathfrak{f} = \text{Lie}(F)$ and $\mathfrak{g} = \text{Lie}(G)$ we use the standard bilinear form on $\mathfrak{f}$ to define $\mathfrak{p}$ as the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{f}$ so that for a symmetric space $\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{p}$, $\text{Tr}[\mathfrak{g}\mathfrak{p}] = 0$, $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$, $[\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g}$.

(3.3)

The Lagrangian for the symmetric space $\sigma$-model is then given by

$$\mathcal{L} = \text{Tr}[J, PJ], \quad J = f^{-1} df, \quad f \in F,$$

(3.4)

where $P$ is the projector onto $\mathfrak{p}$, $\text{Tr}$ is appropriately normalized and $\pm$ are light-cone coordinates on the worldsheet. This action has a global $F$ symmetry $f \rightarrow f_0 f$ and a local $G$ gauge symmetry $f \rightarrow fg$.

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14 Compared to section 2 we now use Greek indices $\alpha, \beta, \ldots$ for the subalgebra $\mathfrak{h}$ in which we will dualize and Latin indices $a, b, \ldots$ for the algebra $\mathfrak{f}$.

15 Due to the compact nature of the isometry algebra $so(6)$ of $S^4$, it only admits abelian solutions of the cYBE.
The homogeneous YB deformation [1, 2, 4, 7] of the symmetric space $\sigma$-model is defined as
\[
\mathcal{L} = \text{Tr} \left[ J_{i} P \frac{1}{1 - R_{j} P J_{j}} \right], \quad R_{j} = \text{Ad}_{j}^{-1} \text{RAd}_{j},
\] (3.5)
where operator $R$ is an antisymmetric solution of the cYBE for the algebra $\mathfrak{f}$
\[
[RX, RY] = R([RX, Y] + [X, RY]), \quad X, Y \in \mathfrak{f}.
\] (3.6)
As this equation is homogeneous in $R$ we can always multiply any solution by an overall constant $\eta$. Then in the limit $\eta \rightarrow 0$ we recover the Lagrangian (3.4) of the symmetric space $\sigma$-model. The deformed action (3.5) preserves the local gauge symmetry $f \rightarrow fg$; however, the global symmetry is broken to a subgroup of $F$ depending on the choice of $R$.

In general, we will write the operator $R$ in terms of an $r$-matrix taking values in $\mathfrak{f} \otimes \mathfrak{f}$
\[
r = T_{0} \wedge T_{2} + T_{3} \wedge T_{4} + \ldots, \quad T_{r} \wedge T_{s} = T_{r} \otimes T_{s} - T_{s} \otimes T_{r},
\] (3.7)
where $T_{a}$ is a basis of $\mathfrak{f}$ and by the use of the wedge product we enforce the antisymmetry of the operator. The operator $R$ is then defined using the bilinear form as
\[
RX = \text{Tr}_{2}(r (1 \otimes X)),
\] (3.8)
where $\text{Tr}_{2}$ denotes that the contraction is taken over the second space in the tensor product $\mathfrak{f} \otimes \mathfrak{f}$.

To find the non-abelian dual of the symmetric space $\sigma$-model (3.4) with respect to a subgroup $H \subset F$ we first write the group element $f$ as
\[
f = hf', \quad h \in H, \quad f' \in F.
\] (3.9)
Substituting this into the Lagrangian (3.4) and gauging the global $H$ symmetry, i.e. replacing $h^{-1}dh \rightarrow A$, where $A \in \mathfrak{h} = \text{Lie}(H)$, we find as in (2.3)
\[
\mathcal{L} = \text{Tr} \{ f'^{-1}A_{+}f' + f'^{-1}\partial_{-}f'P(f'^{-1}A_{+}f' + f'^{-1}\partial_{-}f') + \nu F_{+}(A) \}.
\] (3.10)
Here $\nu$ is a Lagrange multiplier imposing the flatness of the connection $A$, i.e. $F_{+}(A) \equiv \partial_{-}A_{+} - \partial_{+}A_{-} + [A_{-}, A_{+}] = 0$, and hence the equivalence of (3.10) and (3.4).

In general, the algebra $\mathfrak{h}$ need not be semi-simple. In this case $\nu$ should not be taken in the algebra $\mathfrak{h}$, but rather in its dual $\mathfrak{h}^*$. Introducing $T_{a}$ as a basis for $\mathfrak{f}$ and $e_{a}$ as a basis for $\mathfrak{h}$ we can define a basis of $\mathfrak{h}^*$ to be
\[
\tilde{e}_{a} = \sum_{a=1}^{\text{dim}\mathfrak{f}} \text{Tr}[T_{a} e_{a}] T_{a},
\] (3.11)
as $F$ is assumed to be semi-simple. The NAD model (defined on a flat 2D background, see (2.3) and (2.5)) is then found upon integrating out the gauge field $A$ in (3.10).

The Lagrangian (3.10) still has a local $G$ gauge symmetry $f' \rightarrow hf', A \rightarrow hAh^{-1} - \partial h^{-1}, \nu \rightarrow \nu hf^{-1}$.\footnote{One may be concerned that the action of $H$ on $\nu$ does not preserve $\nu \in \mathfrak{h}$ if $\mathfrak{h} \neq \mathfrak{h}$. Indeed, this may be the case, but one can always write the new terms as the sum of a part valued in $\mathfrak{h}$ and a part that drops out in the bilinear form. To show this we first write an algebra element $X \in \mathfrak{f}$ as $X = X^{I}e_{I} + Y^{I}t_{I}$ where $t_{I}$ is some extension of $e_{a}$ to a basis of $\mathfrak{h}$. Then if $\text{Tr}[e_{a} e_{b}] = \epsilon_{b} = 0$ for some $I$ and $\sigma$, it follows from (3.11) that $e_{a} = ct_{I} + \ldots$. We can then define a new generator $\tilde{t}_{I} = t_{I} - \epsilon_{a} a^{i=1} \text{Tr}[T_{a} e_{I}]$, which does satisfy $\text{Tr}[\tilde{t}_{I} e_{a}] = 0$. Replacing $\tilde{t}_{I} \rightarrow \tilde{t}_{I}$ in the basis of $\mathfrak{f}$ still gives a basis of $\mathfrak{h}$. Therefore, applying this process iteratively, we can construct a basis $(e_{a}, \tilde{t}_{I})$ with the desired property $\text{Tr}[\tilde{t}_{I} e_{a}] = 0$.}

In the case when the NAD is with respect to the full group $H = F$, the $H$ gauge symmetry may be
used to fix $f' = 1$. In this gauge fixing the $G$ gauge symmetry then has a compensating action on $A$ and $v$ which may be used to fix $v$. When $H$ is a subgroup one may no longer be able to fix $f' = 1$. The gauge condition may, in general, involve both $f'$ and $v$. Our approach will be to gauge fix $f'$ as far as possible and use the remaining gauge symmetry to constrain $v$.

As suggested by the example of the TsT transformation discussed in section 2.2, in some cases it will not be enough to perform the NAD in a subgroup $H$ of $F$. Rather we will need to start with a central extension of $H$ (which, in general, need not be admissible in the full group). We shall define the NAD with respect to the centrally extended group $H_{c.e.}$ as (see (2.13))

$$\hat{\mathcal{L}} = \text{Tr}[ (f^{-1}A_m f' + f^{-1} \partial_{\beta} f' ) P (f^{-1}A_m f' + f^{-1} \partial_{\beta} f' ) + v F_{\gamma \delta}(A) ] + w_m (\partial_{\gamma} C^m_{\gamma \delta} - \partial_{\delta} C^m_{\gamma \gamma} ) + w_m [ A_{\gamma \delta} , A_{\gamma \delta} ]_{c.e.}. \quad (3.12)$$

Here the first line is identical to (3.10), i.e. no central extension is present and the Lie brackets remain those of $\mathfrak{f}$ and the subalgebra $\mathfrak{h}$. In the second line the index $m$ labels the central extensions of $\mathfrak{h}$, while $w_m$ and $C^m_{\gamma \delta}$ are the corresponding Lagrange multipliers and gauge fields respectively. The bracket $[,]_{c.e.}$ is then the Lie bracket on the centrally extended algebra $\mathfrak{h}_{c.e.}$.

On integrating out the Lagrange multipliers $v$ and $w_m$ in (3.12) we still recover the symmetric space $\sigma$-model (3.4), demonstrating that, like (3.10) and (3.12) is still equivalent to (3.4). However, integrating out the gauge fields in (3.12) will now lead to a more general model (depending on extra parameters) than that found from (3.10). Indeed, following the discussion in section 2.2, integrating out $C^m_{\gamma \delta}$ implies that $w_m = w_m^{(0)} = \text{const.}$ Substituting this back into (3.12) gives

$$\hat{\mathcal{L}} = \text{Tr}[ (f^{-1}A_m f' + f^{-1} \partial_{\beta} f' ) P (f^{-1}A_m f' + f^{-1} \partial_{\beta} f' ) + v F_{\gamma \delta}(A) ] + w_m^{(0)} [ A_{\gamma \delta} , A_{\gamma \delta} ]_{c.e.}. \quad (3.13)$$

Then integrating out the gauge field $A_{\gamma \delta}$ defines the NAD model, which will now depend on arbitrary constant parameters $w_m^{(0)}$. As was shown in section 2.2, in the case when $\mathfrak{h}$ is abelian these constants can then be interpreted as the parameters of the TsT transformations.

3.2. $AdS_5$ and $so(2,4)$

Let us now turn to the $F/G = SO(2,4)/SO(1,4)$ symmetric space $\sigma$-model for $AdS_5$ and first briefly state our conventions for the algebra $so(2,4)$. Introducing the $\gamma$-matrices (here $\sigma_{1,2,3}$ are the standard Pauli matrices and $\sigma_0$ is the $2\times 2$ identity matrix)

$$\gamma_0 = i \sigma_3 \otimes \sigma_0, \quad \gamma_1 = \sigma_2 \otimes \sigma_2, \quad \gamma_2 = - \sigma_2 \otimes \sigma_1, \quad \gamma_3 = \sigma_1 \otimes \sigma_0, \quad \gamma_4 = \sigma_2 \otimes \sigma_3. \quad (3.14)$$

we define the following basis for $so(2,4)$

$$T_{ij} = \frac{1}{4} [ \gamma_i , \gamma_j ], \quad T_{55} = - T_{55} = \frac{1}{2} \gamma_5, \quad T_{55} = 0, \quad i, j = 0, \ldots, 4. \quad (3.15)$$

We then use the standard matrix trace as our bilinear form. It is this basis that we use to define bases for the duals of non semi-simple subalgebras of $so(2,4)$ as in (3.11).

The deformations based on non-abelian $\tau$-matrices that we consider are more naturally understood in the Poincaré patch of $AdS_5$. Therefore, we will use the corresponding basis of

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17 One may attempt to promote the Lagrange multiplier terms to a gauged WZW model as in [21] in order to construct a $\lambda$-model type theory with an additional deformation parameter based on (3.12). It would be interesting to see if this preserves integrability, or if one also needs to modify the first term. There may also be subtleties for centrally extended and non-semi-simple algebras.
so(2, 4)

\[
D = T_{45}, \quad P_\mu = T_{\mu 5} - T_{\mu 4}, \quad K_\mu = T_{\mu 5} + T_{\mu 4},
\]

\[
M_{\mu \nu} = T_{\mu \nu}, \quad \mu, \nu = 0, \ldots, 3. \quad (3.16)
\]

The symmetric space \( AdS_5 \) can be represented as the coset \( SO(2, 4)/SO(1, 4) \). The subalgebra \( so(1, 4) \) corresponding to the gauge group is spanned by \( T_{ij} \) where \( i, j = 0, \ldots, 4 \) and therefore the projector \( P \) onto the coset part of the algebra is given by

\[
P(X) = -\text{Tr}[XT_{05}]T_{05} + \sum_{i=1}^{4} \text{Tr}[XT_{i5}]T_{i5}. \quad (3.17)
\]

Taking the gauge-fixed field \( f \) to be

\[
f = \exp \left[ -x_0 P_0 + x_1 P_1 + x_2 P_2 + x_3 P_3 \right] \exp \left[ \log z \ D \right], \quad (3.18)
\]

and substituting it into the Lagrangian (3.4) of the symmetric space \( \sigma \)-model we find that it takes the form of the \( \sigma \)-model with the target space metric being the \( AdS_5 \) metric in Poincaré patch

\[
ds^2 = \frac{-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2}{z^2}. \quad (3.19)
\]

### 3.3. Abelian \( r \)-matrices

The first examples of the relation between homogeneous YB deformations and NAD that we will consider are based on abelian \( r \)-matrices. While the equivalence in these cases follows from the results of section 2.2 and [14], it will be instructive to consider two examples explicitly. Prior to the general investigation of [14], deformations based on abelian \( r \)-matrices and their relation to TsT transformations of \( AdS_5 \times S^5 \) have been studied extensively on a case by case basis [5, 6, 8–12].

**Example 1.** The first case we shall consider corresponds to the rank 2 abelian \( r \)-matrix

\[
\eta \quad P_2 \wedge P_3, \quad (3.20)
\]

where \( P_\mu \) are translation generators and \( \eta \) is a free deformation parameter. This \( r \)-matrix and also the rank 4 one below (3.25) were first discussed in [6] where it was shown that the metrics and \( B \)-fields of corresponding YB deformed models are those of the ‘non-commutative dual’ backgrounds of [24, 25]. Using the gauge-fixed field \( f \) (3.18), the Lagrangian of the corresponding YB deformed model (3.5) with \( r \)-matrix given by (3.20) is found to be that of the \( \sigma \)-model with the following target space metric and \( B \)-field (see (2.8))

\[
ds^2 = \frac{-dx_0^2 + dx_1^2 + dx_2^2}{z^2} + \frac{z^2}{z^4 + \eta^2} (dx_2^2 + dx_3^2),
\]

\[
B = \frac{\eta}{z^4 + \eta^2} dx_2 \wedge dx_3. \quad (3.21)
\]

On the other hand, let us consider the NAD of the \( AdS_5 \) \( \sigma \)-model with respect to the central extension of the algebra \( \mathfrak{h} = \{ P_2, P_3 \} \) (equivalent to (2.9)). In this case the dual algebra is given by \( \mathfrak{h} = \{ K_2, K_3 \} \). We use the local \( H \) gauge symmetry of the NAD model (3.13) to fix
f' = \exp[-x_0P_0 + x_1P_1] \exp[\log z] ,
\text{and also parametrize the gauge field and Lagrange multiplier as}
\begin{align*}
A_\pm &= A_{1\pm}P_2 + A_{2\pm}P_3, \\
v &= \frac{1}{2\eta}(x_2K_0 + x_3K_3).
\end{align*}
Substituting these expressions into the Lagrangian (3.13) where we take the explicit form of the central extension term to be (see (2.13))
\begin{equation}
w^{(0)}_m[A_-, A_+]_{\mu e} = \frac{1}{\eta}(A_{1-}A_{2-} - A_{2-}A_{1-}),
\end{equation}
and integrating out the gauge field we find that the NAD of AdS5 with respect to the central extension of \( h = \{P_2, P_3\} \) again gives the \( \sigma \)-model based on the background metric and \( B \)-field (3.21) (up to a total derivative term in the \( B \)-field)\(^\ddagger\).

**Example 2.** Our second example is defined by the rank 4 abelian \( r \)-matrix
\begin{equation}
r = \eta P_0 \wedge P_1 + \zeta P_2 \wedge P_3,
\end{equation}
where \( \eta \) and \( \zeta \) are independent parameters. Using again the gauge-fixed field \( f \) (3.18), the deformed metric and \( B \)-field corresponding to the YB deformed model (3.5) are found to be
\begin{align*}
ds^2 &= \frac{z^2}{z^4 - \eta^2}(-dx_0^2 + dx_1^2) + \frac{z^2}{z^4 + \zeta^2}(dx_2^2 + dx_3^2) + \frac{dz^2}{z^2}, \\
B &= \frac{\eta}{z^4 - \eta^2}dx_0 \wedge dx_1 + \frac{\zeta}{z^4 + \zeta^2}dx_2 \wedge dx_3.
\end{align*}
Next, we construct the NAD of the AdS5 \( \sigma \)-model with respect to central extension of the algebra \( h = \{P_0, P_1, P_2, P_3\} \) which is implied by the form of the \( r \)-matrix (3.25)
\begin{equation}
[P_0, P_1] = Z_1, \quad [P_2, P_3] = Z_2.
\end{equation}
The dual algebra is given by \( \tilde{h} = \{K_0, K_1, K_2, K_3\} \). We now use the local \( H \) gauge symmetry of (3.13) to fix
\begin{equation}
f' = \exp[\log z] ,
\end{equation}
and parametrize the gauge field and the Lagrange multiplier in (3.13) as
\begin{align*}
A_\pm &= A_{1\pm}P_0 + A_{2\pm}P_1 + A_{3\pm}P_2 + A_{4\pm}P_3, \\
v &= \frac{1}{2\eta}(x_1K_0 - x_0K_1) + \frac{1}{2\zeta}(x_2K_2 + x_3K_3).
\end{align*}
We then substitute these expressions into the Lagrangian (3.13) where, according to (3.27) the explicit form of the central extension term is now
\begin{equation}
w^{(0)}_m[A_-, A_+]_{\mu e} = \frac{1}{\eta}(A_{1-}A_{2-} - A_{2-}A_{1-}) + \frac{1}{\zeta}(A_{3-}A_{4-} - A_{4-}A_{3-}).
\end{equation}
Integrating out the gauge field \( A \) we conclude that the NAD transform of AdS5 \( \sigma \)-model with respect to the central extension of \( h = \{P_0, P_1, P_2, P_3\} \) gives the \( \sigma \)-model based again on the metric and \( B \)-field in (3.26). In addition, in both examples the NAD procedure determines also the dilaton field given by the standard \( T \)-duality expression [42] (see (2.5) and (2.8)).

\( \ddagger \) In what follows we will always ignore total derivative terms when comparing the \( B \)-fields.
3.4. Unimodular non-abelian r-matrices

Let us now turn our attention to homogeneous YB deformations based on non-abelian r-matrices. We start with unimodular examples (3.1) for which the YB deformation of the supercoset $\sigma$-model background \[3, 7\] preserves the satisfaction of the supergravity equations or one-loop Weyl invariance (with an appropriate dilaton) \[13\].

All abelian r-matrices, defining deformations which are equivalent to sequences of TsT transformations of $AdS_5 \times S^5$ \[14\], are unimodular. Furthermore, at rank 2 all unimodular r-matrices are abelian. However, for the algebra $so(2, 4)$ there are rank 4 and rank 6 non-abelian unimodular r-matrices. Those of rank 4 were classified in \[13\] and fall into three classes characterized by the algebra of the corresponding generators. Defining the r-matrix as

$$r = e_1 \wedge e_2 + e_3 \wedge e_4,$$  \hspace{1cm} (3.31)

the non-vanishing commutation relations determining the three classes are$^{19}$

class 1: $[e_1, e_2] = e_3$,  
class 2: $[e_1, e_3] = e_4$,  
class 3: $[e_1, e_4] = e_3$,  \hspace{1cm} (3.32)

In \[13\] the deformations corresponding to examples from the first two classes were observed to be equivalent to a sequence of two TsT transformations (with a non-linear coordinate redefinition in between) of the $AdS_5 \times S^5$ model; however a similar result for the last class was not found.

Let us now consider one example from each class demonstrating that the corresponding YB deformation is equivalent to the NAD of the $AdS_5$ $\sigma$-model with respect to the following central extension of the algebra$^{20}$ $h = \{e_1, e_2; e_3, e_4\}$

$$[e_1, e_2] = Z_1, \quad [e_3, e_4] = Z_2.$$  \hspace{1cm} (3.33)

As mentioned in section 2.2, centrally extending a unimodular algebra preserves this property. At the end of this section we will also consider one rank 6 example.

**Class 1 example:** An r-matrix from class 1 that we shall consider is \[13\]

$$r = \eta M_{+3} \wedge P_+ + \zeta P_2 \wedge P_3,$$  \hspace{1cm} (3.34)

where $\eta$ and $\zeta$ are two free parameters. Here we have introduced light-cone indices in the $so(1, 3)$ Lorentz subalgebra spanned by $M_{\mu\nu}$ (not be confused with the light-cone coordinates on the worldsheet) defined as $\Lambda_\pm = \Lambda_0 \pm \Lambda_i$, i.e. $M_{i+} = M_{0i} + M_{i3}$. Fixing the group-valued field $f$ as

$$f = \exp \left[ -\frac{1}{2}(x_+ P_+ + x_3 P_3) + x_2 P_2 + x_3 P_3 \right] \exp \left[ \log z \right] D,$$  \hspace{1cm} (3.35)

we find the YB deformed model (3.5) corresponds to following metric and B-field

$$dz^2 = \frac{-dx_+ dx_+ + dz^2}{z^2} - \frac{-\eta^2 x_+^2 dx_+^2 - 2\eta x_2 dx_+ dx_2 + z^4(dx_2^2 + dx_3^2)}{z^2(z^4 + \zeta^2)},$$  \hspace{1cm} (3.36)

$$B = \frac{\eta x_+ dx_+ \wedge dx_3 + \zeta dx_2 \wedge dx_3}{z^4 + \zeta^2}.$$  \hspace{1cm} (3.36)

$^{19}$ In \[13\] the r-matrices are grouped into four classes, with the non-vanishing commutation relations $[e_1, e_3] = e_4$, $[e_1, e_4] = e_3$ determining the additional class. Here we note that these commutation relations are related to those of class 2 by analytic continuation and hence for our purposes do not need to be considered independently.

$^{20}$ One can check that the Jacobi identity is satisfied and this central extension is consistent for all three classes.
Let us now compare this to the NAD of the $AdS_5$ $\sigma$-model with respect to the central extension of the algebra $\mathfrak{h} = \{M_{i3}, P_1; P_2, P_3\}$. The dual algebra is given by $\mathfrak{h} = \{M_{i3}, K_1; K_2, K_3\}$. We first partially use the local $H$ gauge symmetry of (3.13) to fix $f'$ to be

$$f' = \exp \left[ -\frac{1}{2} x_1 P_1 \right] \exp \left[ \log z D \right].$$

This leaves one gauge freedom corresponding to $M_{i3}$ that we cannot use to fix $f'$ further. We therefore use it to fix the $K_1$ component of the Lagrange multiplier $\nu$ to zero, so that it can be parametrized as

$$\nu = -\left( \frac{x_1}{2\eta} + \frac{x_2}{2\zeta} \right) M_{i3} + \frac{1}{2\zeta} \sqrt{-x_2^2 + \eta x_1} \left( \frac{x_2}{2\zeta} + \frac{\eta x_1}{\zeta} \sqrt{-x_2^2 + \eta x_1} \right) K_2.$$  

The reason for this choice is to make manifest the comparison with (3.36). Parametrizing the gauge field as

$$A_{\pm} = A_{1\pm} M_{i3} + A_{2\pm} P_1 + A_{3\pm} P_2 + A_{4\pm} P_3,$$  

we again take the explicit form of the central extension term in (3.13) to be given by (3.30). Integrating out the gauge field we finally find that the NAD of the $AdS_5$ $\sigma$-model with respect to the central extension of $\mathfrak{h} = \{M_{i3}, P_1; P_2, P_3\}$ gives exactly the same $\sigma$-model as defined by the metric and $B$-field in (3.36).

**Class 2 example:** An example of $r$-matrix from class 2 is [13]

$$r = \eta M_{23} \wedge P_1 + \zeta P_2 \wedge P_3.$$  

Fixing the group-valued field $f$ as in (3.18) with

$$x_2 = r \cos \theta, \quad x_3 = r \sin \theta,$$  

we find the following metric and $B$-field of the YB deformed model (3.5)

$$d\mathbf{s}^2 = \frac{-dx_0^2 + dz^2}{z^2} + \frac{(z^4 + \xi^2) dx_1^2 - 2\eta \xi dx_1 dr + (z^4 + \eta^2 r^2) dr^2 + z^4 r^2 d\theta^2}{z^2 (z^4 + \eta^2 r^2 + \xi^2)},$$  

$$B = \frac{\eta r^2 dx_1 \wedge d\theta + \xi r dr \wedge d\theta}{z^4 + \eta^2 r^2 + \xi^2}.$$  

Next, we compare this to the NAD of $AdS_5$ with respect to the central extension of the algebra $\mathfrak{h} = \{M_{23}, P_1; P_2, P_3\}$. The dual algebra is given by $\mathfrak{h} = \{M_{23}, K_1; K_2, K_3\}$. We partially use the local $H$ gauge symmetry of (3.13) to fix

$$f' = \exp \left[ -\frac{x_1}{2\eta} P_1 \right] \exp \left[ \log z D \right].$$  

This leaves one gauge freedom corresponding to $M_{23}$ that we can use to set the $K_3$ component of the Lagrange multiplier $\nu$ to zero, which we then parametrize as

$$\nu = -\left( \frac{x_1}{\eta} + \frac{r^2}{2\zeta} \right) M_{23} + \frac{\theta}{2\eta} K_1 + \frac{r}{2\zeta} K_2.$$  

Parametrizing the gauge field as

$$A_{\pm} = A_{1\pm} M_{23} + A_{2\pm} P_1 + A_{3\pm} P_2 + A_{4\pm} P_3,$$  

we again take the explicit form of the central extension term in (3.13) to be given by (3.30). Integrating out the gauge field we find that the NAD of the $AdS_5$ $\sigma$-model with respect to the
central extension of $\mathfrak{h} = \{M_{23}, P_1; P_2, P_3\}$ gives the $\sigma$-model defined by the metric and $B$-field (3.42).

Let us note that one can also consider the $r$-matrix

$$r = \eta M_{01} \wedge P_3 + \zeta P_0 \wedge P_3,$$

(3.46)

which can be understood as an analytic continuation of (3.40). The corresponding quasi-Frobenius algebra satisfies the commutation relations given in footnote 18. By repeating the above discussion using the obvious analytic continuation it is clear that the corresponding YB deformed model is equivalent to the NAD of $AdS_5$ with respect to the central extension of $\mathfrak{h} = \{M_{01}, P_3; P_2, P_2\}$.

**Class 3 example:** Let us now consider the following $r$-matrix from class 3 $^{21}$

$$r = \eta M_{13} \wedge P_+ + \zeta P_1 \wedge P_3,$$

(3.47)

Fixing the group-valued field $f$ as in (3.18), i.e.

$$f = \exp \left[ -\frac{1}{2} (x_2 P_e + x_e P_2) + x_3 P_2 + x_3 P_3 \right] \exp \left[ \log z D \right],$$

(3.48)

we find the following metric and $B$-field of the YB deformed model (3.5)

$$\begin{align*}
\text{d}x^2 &= \frac{\text{d}x_+^2 + \text{d}z^2}{z^2} \\
&\quad + \frac{-2(2\zeta^4 + 2\eta\zeta x_+ + \zeta^2)\text{d}x_+\text{d}x_+ - \zeta^2 \text{d}x_+^2 - (2\eta x_+ + \zeta)\text{d}x_+^2}{\zeta^2(2\eta x_+ + \zeta^2)} \text{d}x_+^2 + 4\zeta^4 \text{d}x_+^2/
2(\zeta^4 + 2\eta\zeta x_+ + \zeta^2)
\end{align*}$$

(3.49)

$$B = \frac{-\zeta \text{d}x_+ \wedge \text{d}x_3 + (2\eta x_+ + \zeta)\text{d}x_+ \wedge \text{d}x_3}{2(\zeta^4 + 2\eta\zeta x_+ + \zeta^2)}.$$

We are now to compare this to the NAD of $AdS_5$ with respect to the central extension of $\mathfrak{h} = \{M_{13}, P_1; P_2, P_2\}$ with the dual algebra being $\mathfrak{h} = \{M_{13}, K_+; K_1, K_3\}$. We partially use the local $H$ gauge symmetry of (3.13) to fix

$$f' = \exp [x_2 P_2] \exp [\log z D].$$

(3.50)

This leaves one gauge freedom corresponding to $M_{13}$ that we can use this to fix the $K_3$ component of the Lagrange multiplier $\nu$ to zero, which we then parametrize as

$$\nu = -\left( \frac{x_- + x_+}{4\eta} + \frac{x_+^2}{4\zeta} \right) M_{13} + \left( \frac{x_3}{2\zeta} + \sqrt{\frac{-x_+}{2\eta\zeta}} \left( 1 + \frac{2\eta x_+}{3\zeta} \right) \right) K_+$$

$$+ \left( \frac{x_3}{\zeta} + \sqrt{\frac{-x_+}{2\eta\zeta}} \left( 1 + \frac{4\eta x_+}{3\zeta} \right) \right) K_1.$$

(3.51)

Parametrizing the gauge field as

$$A_{\pm} = A_{1\pm} M_{13} + A_{2\pm} P_+ + A_{3\pm} P_1 + A_{4\pm} P_3,$$

(3.52)

choosing the central extension term in (3.13) to be (3.30) and integrating out the gauge field we finally conclude that the NAD of the $AdS_5$ $\sigma$-model with respect to the central extension of $\mathfrak{h} = \{M_{13}, P_1; P_2, P_3\}$ is again equivalent to the YB deformed $\sigma$-model corresponding to the metric and $B$-field in (3.49).

$^{21}$ This is the fourth example in [13] for which a TsT interpretation has thus far not been found.
**Rank 6 example:** Finally, we consider the following example of a rank 6 unimodular \( r \)-matrix [13]

\[
r = \eta M_{01} \wedge M_{23} + \zeta P_0 \wedge P_1 + \kappa P_2 \wedge P_3,
\]

where \( \eta, \zeta \) and \( \kappa \) are free parameters. Fixing the group-valued field \( f \) as in (3.18) with

\[
x_0 = t \cosh \chi, \quad x_1 = t \sinh \chi, \quad x_2 = r \cos \theta, \quad x_3 = r \sin \theta,
\]

we find the following metric and \( B \)-field of the YB deformed model (3.5)

\[
ds^2 = \frac{dz^2}{z^2} + \frac{-z^2(z^4 + \eta^2 z^2 + \kappa^2)dr^2 + 2\eta\zeta z^2 r^2 d\theta d\theta}{z^8 + z^4(\eta^2 z^2 - \zeta^2 + \kappa^2) - \zeta^2 \kappa^2} + z^2(z^4 + \eta^2 z^2 - \zeta^2)dr^2 - 2\eta \zeta z^2 r^2 d\theta d\chi + (z^4 + \kappa^2)z^2 r^2 d\chi^2
\]

\[B = \frac{-\eta \zeta \cosh dt \wedge dr + \zeta(z^4 + \kappa^2)tdt \wedge d\chi - \kappa(z^4 - \zeta^2)rd\theta \wedge dr - \eta \zeta z^2 r^2 d\theta d\chi}{z^8 + z^4(\eta^2 z^2 - \zeta^2 + \kappa^2) - \zeta^2 \kappa^2}.
\]

Now we compare this to the NAD of \( AdS_5 \) with respect to the central extension of the algebra \( h = \{ M_{01}, M_{23}; P_0, P_1, P_2, P_3 \} \)\textsuperscript{22}. The dual algebra is given by \( \tilde{h} = \{ M_{01}, M_{23}; K_0, K_1; K_2, K_3 \} \). We partially use the local \( H \) gauge symmetry of (3.13) to fix

\[
f' = \exp \{ \log z \cdot D \}.
\]

This leaves two free gauge transformations corresponding to \( M_{01} \) and \( M_{23} \) that we can use to set the \( K_0, K_2 \) and \( K_1, K_3 \) components of the Lagrange multiplier \( v \) to zero, which we then parametrize as

\[
v = - \left( \frac{\theta}{\eta} - \frac{r^2}{2\zeta} \right) M_{01} - \left( \frac{\chi}{\eta} - \frac{r^2}{2\zeta} \right) M_{23} + \frac{r}{2\zeta} K_1 + \frac{r^2}{2\zeta} K_3.
\]

Parametrizing the gauge field as

\[
A_\pm = A_{1\pm} M_{01} + A_{2\pm} M_{23} + A_{3\pm} P_0 + A_{4\pm} P_1 + A_{5\pm} P_2 + A_{6\pm} P_3,
\]

we again take the explicit form of the central extension term in (3.13) to be given by

\[
w^{(0)}_{m}[A_-, A_+]_c = \frac{1}{\eta} (A_{1+} A_{2-} - A_{2+} A_{1-}) + \frac{1}{\zeta} (A_{3+} A_{4-} - A_{4+} A_{3-}) + \frac{1}{\kappa} (A_{5+} A_{6-} - A_{6+} A_{5-}).
\]

Integrating out the gauge field we find that the NAD of the \( AdS_5 \) \( \sigma \)-model with respect to the central extension of \( h = \{ M_{01}, M_{23}; P_0, P_1, P_2, P_3 \} \) gives the \( \sigma \)-model defined by metric and \( B \)-field (3.55).

### 3.5. Jordanian \( r \)-matrices

We now turn to our final group of examples, the Jordanian \( r \)-matrices. Jordanian \( r \)-matrices have the form (see, e.g., [50] and references therein)

\[
r = T_1 \wedge T_2 + \ldots, \quad [T_i, T_j] = T_k,
\]

\[\text{where } T_i = v_i K_0 + v_i K_1 + \ldots, \text{ this is a valid gauge fixing in the 'patch' in which } v_i^2 > v_0^2.\]

\textsuperscript{22} The central extension is given by \( [M_{01}, M_{23}] = Z_5, [P_0, P_1] = Z_3, [P_2, P_3] = Z_3 \) as follows from the structure of the \( r \)-matrix (3.53). Again, one can check that the Jacobi identity is satisfied.

\textsuperscript{23} Writing \( v = v_0 K_0 + v_1 K_1 + \ldots, \text{ this is a valid gauge fixing in the 'patch' in which } v_i^2 > v_0^2.\)
and the corresponding deformations of $\text{AdS}_5$ have been extensively studied on a case by case basis in [7–12]. When built out of bosonic generators the Jordanian $r$-matrices do not satisfy the unimodularity property (3.1) [13]. Indeed, the backgrounds corresponding to the Jordanian deformations of the supercoset model [3, 7] do not solve the supergravity equations [10–12], solving instead the generalized equations of [22, 23].

In the following we will compare the YB deformations arising from Jordanian $r$-matrices to the NAD of $\text{AdS}_5$ $\sigma$-model with respect to the corresponding quasi-Frobenius subalgebra itself (i.e. without central extension). A possible reason why we do not need to consider central extensions is that, unlike for the unimodular $r$-matrices, the central extension of interest turns out to be trivial. Indeed, let us consider the simplest case of the $r$-matrix $D \wedge P_0$. The two generators here have the commutation relation $[D, P_0] = P_0$. If we try to centrally extend this 2D algebra we get $[D, P_0] = P_0 + Z$, but now defining $P_0' = P_0 + Z$ we see that this extension is trivial. For the extended $r$-matrix $D \wedge P_0 + [M_{01} \wedge P_1 + M_{12} \wedge P_2 + M_{3} \wedge P_3$ we consider the central extension $[D, P_0] = P_0 + Z, [M_{01}, P_1] = P_0 + Z, [M_{12}, P_2] = P_0 + Z$ and $[M_{3}, P_3] = P_0 + Z$. Here we only consider a single extension as there is only a single free parameter scaling the whole $r$-matrix. Again by shifting $P_0$ we see that this extension is trivial. Similar statements hold for the remaining examples that we consider.

As for the abelian $r$-matrices and unimodular non-abelian $r$-matrices in sections 3.3 and 3.4 we will again provide exhaustive evidence for the equivalence of the YB and NAD constructions. In this section we will discuss two special cases in detail, while in appendix we will summarize the key information for twenty additional examples.

**Example 1.** The first example is the case of the rank 2 Jordanian $r$-matrix

$$r = \eta D \wedge P_0, \quad (3.61)$$

where $D$ is the dilatation operator. Parametrizing the group-valued field $f$ as in (3.18) with

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta \cos \phi, \quad x_3 = r \sin \theta \sin \phi, \quad (3.62)$$

we find the following metric and $B$-field of the corresponding YB deformed model (3.5)

$$\begin{align*}
\text{d}s^2 &= -z^4 \text{d}x_0^2 + z^2 (z^2 - \eta^2) \text{d}r^2 + 2\eta^2 z r \text{d}z \text{d}r + (z^4 - \eta^2 r^2) \text{d}z^2 \\
&\quad + \frac{z^2}{z^2} (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2), \\
B &= \frac{\eta^2 r \text{d}r \wedge \text{d}x_0 + \eta \text{d}z \wedge \text{d}x_0}{z^4 - \eta^2 z^2 - \eta^2 r^2}. \quad (3.63)
\end{align*}$$

In [12] it was observed that this background can be also obtained from the $\text{AdS}_5$ $\sigma$-model via a generalized ‘TsT’ transformation in which the shift is replaced with a more complicated non-linear field redefinition[24].

It turns out that this background can be derived in one step by applying the NAD transformation to the $\text{AdS}_5$ $\sigma$-model with respect to the non-semisimple subalgebra $\mathfrak{h} = [D, P_0]$ (with the dual algebra being given by $\mathfrak{h}^\perp = [D, K_0]$). We first use the local $H$ gauge symmetry of (3.10) to fix $f'$ as

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24 The coordinate transformation is required is to make the dilatation symmetry a linear isometry.
\[
\frac{1}{2} \left( x_1 P_1 + x_2 P_2 + x_3 P_3 \right),
\]

(3.64)

where \( x_i \) are again given by (3.62). We also parametrize the Lagrange multiplier \( \nu \) in terms of the coordinates \( x_0 \) and \( z \) as

\[
\nu = \eta^{-1} \left[ -x_0 D + \frac{1}{z} \epsilon K_0 \right].
\]

(3.65)

Substituting this into (3.10) and integrating out the gauge field, we find that the NAD of the \( AdS_5 \) \( \sigma \)-model with respect to \( h = [D, P_0] \) gives the \( \sigma \)-model based on the metric and \( B \)-field (3.63)

25 If we instead consider the centrally extended algebra \( \{ D, P_0 \} = P_0 + Z \), which, as discussed above, amounts to a trivial extension, (with \( A_+ = A_+ D + A_+ P_0 \) and the explicit form of the central extension term as in (3.24)) we again recover the background (3.63) after taking \( f' \) as in (3.64) and \( \nu = \eta^{-1} \left[ -x_0 D + \frac{1}{z} \epsilon K_0 \right] \). This simple shift in the Lagrange multiplier is directly correlated to the shift in \( P_0 \) required to reach the trivially extended algebra. Similar statements should hold for the remaining examples.
These two $r$-matrices (3.66) and (3.67) are built from the generators of the same algebra
\[ h = \{ D, M_{01}, M_{+2}, M_{-3}, P_0, P_1, P_2, P_3 \}, \]
with the dual algebra being \( \bar{h} = \{ D, M_{01}, M_{-2}, M_{-3}, K_0, K_1, K_2, K_3 \} \). Let us then consider the NAD of the \( AdS_5 \) $\sigma$-model with respect to $h$. After partially using the local $H$ gauge symmetry to completely fix $f'$ in (3.10), i.e.
\[ f' = 1, \]
there are still three remaining gauge symmetries, corresponding to the generators $M_{01}, M_{+2}, M_{-3}$. The Lagrange multiplier $\nu \in \bar{h}$ contains a piece $\nu_0 K_0 + \nu_1 K_1 + \ldots$. For $\nu_1^2 > \nu_0^2$ we can use the remaining gauge symmetries to fix
\[ \nu = \eta^{-1}\left[ -x_0 D + \frac{1}{2} z K_0 + x_0 M_{01} + \frac{1}{2} r (\cos \theta M_{-2} + \sin \theta M_{-3}) \right], \]
while for $\nu_1^2 > \nu_0^2$ we fix
\[ \nu = \eta^{-1}\left[ x_0 D + \frac{1}{2} z K_1 - x_0 M_{01} + \frac{1}{2} r (\cos \theta M_{-2} + \sin \theta M_{-3}) \right]. \]
Substituting $\nu$ in (3.73) into (3.10) and integrating out the gauge field we find the $\sigma$-model based on the metric and $B$-field (3.69), while using $\nu$ in (3.74) we recover the $\sigma$-model based on (3.70). Thus the two YB deformations correspond to different 'patches' of the NAD of \( AdS_5 \) model with respect to $h$.

To conclude, let us observe that for the Jordanian examples the parameter $\eta$, which appears in the $r$-matrix as an overall factor, also enters the dual model in a similar way, with the Lagrange multiplier $\nu \sim \eta^{-1}$ (see (3.65), (3.73) and (3.74)). This is consistent as using the automorphism $P_0 \mapsto \lambda P_0, K_0 \mapsto \lambda^{-1} K_0$ of $\mathfrak{so}(2, 4)$ we can set $\eta = 1$ in the $r$-matrix, while the same can be done in the NAD model by rescaling $\nu$. Similar observations also hold for the examples in appendix where in some cases we also use the inner automorphism generated by the $SO(1, 1)$ subgroup of the $SO(1, 3)$ Lorentz algebra.

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Appendix. Further examples corresponding to Jordanian $r$-matrices

In this appendix we present a number of further examples of the relation between YB deformed models based on Jordanian $r$-matrices and NADs of the $AdS_5$ $\sigma$-model. This list (including the cases discussed already in section 3.5), while not a classification, covers the majority of the $r$-matrices considered in [7–12] up to certain automorphisms of $\mathfrak{so}(2, 4)$ including those based on $P_0 \leftrightarrow K_0$ and $+ \leftrightarrow \ldots$.

For each case we will give the $r$-matrix and the parametrization of the group element $f$ for the YB deformed model (3.5). We will then provide the corresponding data for the NAD model (3.10): the algebra $h$ in which we dualize, its dual $\bar{h}$ and the parametrizations of the
group element $f'$ and of the Lagrange multiplier $v \in \mathfrak{h}$. As was mentioned in section 3.5, for the Jordanian r-matrices we do not need to consider central extensions of $\mathfrak{h}$.

For reasons of brevity we will not present the explicit forms of the metrics and $B$-fields, but just state that in all cases one finds the complete agreement between the YB deformed and NAD transformed $AdS_5$ model.

Note that the pair of examples 1 and 2 below follow a similar pattern as the extended Jordanian r-matrices in example 2 in section 3.5: the two YB deformed models both correspond to the same NAD model. The difference on the NAD side appears in the gauge fixing of the Lagrange multiplier $v$. The same is true also for the pair of examples 3 and 4.

1. $r = \eta (D \wedge P_0 + M_{01} \wedge P_1 + M_{+2} \wedge P_2)$, $f = \exp \left[ -x_0 P_0 + x_1 P_1 + x_2 P_2 + x_3 P_3 \right] \times \exp \left[ \log z D \right]$, $\mathfrak{h} = \{ D, P_0, M_{01}, P_1, M_{+2}, P_2 \}$, $f' = \exp \left\{ \frac{x_3}{z} P_3 \right\}$, $\mathfrak{h} = \{ D, M_{01}, M_{01}, M_{+2}, M_{+2} \}$, $v = -\frac{x_0}{\eta} D + \frac{z}{2\eta} K_0 + \frac{x_1}{\eta} M_{01} + \frac{x_2}{2\eta} M_{+2}$.

2. $r = \eta (D \wedge P_1 + M_{01} \wedge P_0 + M_{+2} \wedge P_2)$, $f = \exp \left[ -x_0 P_0 + x_1 P_1 + x_2 P_2 + x_3 P_3 \right] \times \exp \left[ \log z D \right]$, $\mathfrak{h} = \{ D, P_1, M_{01}, P_0, M_{+2}, P_2 \}$, $f' = \exp \left\{ \frac{x_3}{z} P_3 \right\}$, $\mathfrak{h} = \{ D, K_0, M_{01}, K_0, M_{-2}, K_2 \}$, $v = -\frac{x_0}{\eta} D + \frac{z}{2\eta} K_0 - \frac{x_0}{\eta} M_{01} + \frac{x_2}{2\eta} M_{-2}$.

3. $r = \eta (D \wedge P_0 + M_{01} \wedge P_1)$, $f = \exp \left[ -x_0 P_0 + x_1 P_1 + r (\cos \theta P_2 + \sin \theta P_3) \right] \times \exp \left[ \log z D \right]$, $\mathfrak{h} = \{ D, P_0, M_{01}, P_1 \}$, $f' = \exp \left\{ \frac{r}{z} (\cos \theta P_2 + \sin \theta P_3) \right\}$, $\mathfrak{h} = \{ D, K_0, M_{01}, K_1 \}$, $v = -\frac{x_0}{\eta} D + \frac{z}{2\eta} K_0 + \frac{x_1}{\eta} M_{01}$.

4. $r = \eta (D \wedge P_1 + M_{01} \wedge P_0)$, $f = \exp \left[ -x_0 P_0 + x_1 P_1 + r (\cos \theta P_2 + \sin \theta P_3) \right] \times \exp \left[ \log z D \right]$, $\mathfrak{h} = \{ D, P_1, M_{01}, P_0 \}$, $f' = \exp \left\{ \frac{r}{z} (\cos \theta P_2 + \sin \theta P_3) \right\}$, $\mathfrak{h} = \{ D, K_1, M_{01}, K_0 \}$, $v = -\frac{x_0}{\eta} D + \frac{z}{2\eta} K_1 - \frac{x_0}{\eta} M_{01}$.

5. $r = \eta D \wedge P_1$, $f = \exp \left[ -t \cosh \chi P_0 + x_1 P_1 + t \sinh \chi (\cos \theta P_2 + \sin \theta P_3) \right] \times \exp \left[ \log z D \right]$, $\mathfrak{h} = \{ D, P_1 \}$, $f' = \exp \left\{ \frac{t}{z} (\cosh \chi P_0 + \sinh \chi (\cos \theta P_2 + \sin \theta P_3)) \right\}$, $\mathfrak{h} = \{ D, K_1 \}$, $v = -\frac{x_1}{\eta} D + \frac{z}{2\eta} K_1$. 


6. \( r = \eta (D \wedge P_2 + M_{23} \wedge P_3), \quad f = \exp \{ t(-\cosh \chi P_0 + \sinh \chi P_1) + x_2 P_2 + x_3 P_3 \} \times \exp \{ \log z D \}, \)

\( \mathfrak{h} = \{ D, P_2, M_{23}, P_3 \}, \quad f' = \exp \left[ \frac{1}{\varepsilon} (-\cosh \chi P_0 + \sinh \chi P_1) \right], \)

\( \mathfrak{f} = \{ D, K_2, M_{23}, K_3 \}, \quad \nu = \frac{x_2}{\eta} D + \frac{z}{2\eta} K_2 + \frac{x_3}{\eta} M_{23}. \)

7. \( r = \eta (M_{01} \wedge M_{+2} + M_{23} \wedge M_{+3}), \quad f = \exp \left[ -\frac{1}{2} (x_+ + \frac{1}{2} x_+ (\psi_2^2 + \psi_3^2)) P_0 - \frac{1}{2} x_+ \right] \times \exp \{ \log z D \}, \)

\( \mathfrak{h} = \{ M_{01}, M_{+2}, M_{23}, M_{+3} \}, \quad f' = \exp \left[ \sqrt{x_+} P_0 \right] \exp \{ \log z D \}, \)

\( \mathfrak{f} = \{ M_{01}, M_{-2}, M_{23}, M_{-3} \}, \quad \nu = \frac{\psi_2}{2\eta} M_{01} + \frac{1}{2\eta} \sqrt{x_+} M_{-2} + \frac{\psi_3}{2\eta} M_{23}. \)

8. \( r = \eta ((D + M_{01}) \wedge P_+ + 2M_{+2} \wedge P_2 + 2M_{+3} \wedge P_3), \quad f = \exp \left[ -\frac{1}{2} (x_+ P_+ + x_0 P_0) \right] + r (\cos \theta P_2 + \sin \theta P_3) \times \exp \{ \log z D \}, \)

\( \mathfrak{h} = \{ D + M_{01}, P_+, M_{+2}, P_2, M_{+3}, P_3 \}, \quad f' = \exp \left[ -\frac{x_+}{2} P_+ \right], \)

\( \mathfrak{f} = \{ D + M_{01}, K_-, M_{-2}, K_2, M_{-3}, K_3 \}, \quad \nu = -\frac{x_+}{4\eta} (D + M_{01}) + \frac{z^2}{8\eta} K_- \)

\( + \frac{z^r}{4\eta} (\cos \theta M_{-2} + \sin \theta M_{-3}). \)

9. \( r = \eta ((D + M_{01}) \wedge P_+ + 2M_{+2} \wedge P_2), \quad f = \exp \left[ -\frac{1}{2} (x_+ P_+ + x_0 P_0) \right] + r (\cos \theta P_2 + \sin \theta P_3) \exp \{ \log z D \}, \)

\( \mathfrak{h} = \{ D + M_{01}, P_+, M_{+2}, P_2 \}, \quad f' = \exp \left[ -\frac{x_+}{2} P_+ + \frac{r}{z} \sin \theta P_3 \right], \)

\( \mathfrak{f} = \{ D + M_{01}, K_-, M_{-2}, K_2 \}, \quad \nu = -\frac{x_+}{4\eta} (D + M_{01}) + \frac{z^2}{8\eta} K_- \)

\( + \frac{z^r}{4\eta} \cos \theta M_{-2}. \)

10. \( r = \eta (D + M_{01}) \wedge P_+, \quad f = \exp \left[ -\frac{1}{2} (x_+ P_+ + x_0 P_0) + r (\cos \theta P_2 + \sin \theta P_3) \right] \times \exp \{ \log z D \}, \)

\( \mathfrak{h} = \{ D + M_{01}, P_+ \}, \quad f' = \exp \left[ -\frac{x_+}{2} P_+ + \frac{r}{z} (\cos \theta P_2 + \sin \theta P_3) \right], \)

\( \mathfrak{f} = \{ D + M_{01}, K_- \}, \quad \nu = -\frac{x_+}{4\eta} (D + M_{01}) + \frac{z^2}{8\eta} K_- \)
11. \( r = \eta (D \wedge P_+ + M_{+2} \wedge P_2) \), \( f = \exp \left[ -\frac{1}{2}(x_\bot P_+ + x_\perp P_2) \right. \)
\( + M_{+3} \wedge P_3), \)
\( \hat{h} = [D, P_+, M_{+2}, P_2, M_{+3}, P_3], \)
\( f' = \exp \left[ -\frac{x_\bot P_+}{2\zeta} + \frac{r}{z} \right] \)
\( \hat{b} = [D, K_-, M_{-2}, K_2, M_{-3}, K_3], \)
\( v = -\frac{x_\bot}{2\eta} D + \frac{z}{4\eta} K_- \)
\( + \frac{r}{2\eta} (\cos \theta M_{-2} + \sin \theta M_{-3}). \)

12. \( r = \eta (D \wedge P_+ + M_{+2} \wedge P_2), \)
\( f = \exp \left[ -\frac{1}{2}(x_\bot P_+ + x_\perp P_2) \right. \)
\( + r(\cos \theta P_2 + \sin \theta P_3) \exp [\log z D], \)
\( \hat{h} = [D, P_+, M_{+2}, P_2], \)
\( f' = \exp \left[ -\frac{x_\bot P_+}{2\zeta} + \frac{r}{z} \sin \theta P_3 \right] \)
\( \hat{b} = [D, K_-, M_{-2}, K_2], \)
\( v = -\frac{x_\bot}{2\eta} D + \frac{z}{4\eta} K_- + \frac{r}{2\eta} \cos \theta M_{-2}. \)

13. \( r = \eta D \wedge P_+, \)
\( f = \exp \left[ -\frac{1}{2}(x_\bot P_+ + x_\perp P_2) + r(\cos \theta P_2 + \sin \theta P_3) \right] \exp [\log z D], \)
\( \hat{h} = [D, P_+], \)
\( f' = \exp \left[ -\frac{x_\bot P_+}{2\zeta} + \frac{r}{z} \cos \theta P_2 + \sin \theta P_3 \right] \)
\( \hat{b} = [D, K_-, \]
\( v = -\frac{x_\bot}{2\eta} D + \frac{z}{4\eta} K_. \)

14. \( r = \eta (M_{01} \wedge P_+ + M_{+2} \wedge P_2 + M_{+3} \wedge P_3), \)
\( f = \exp \left[ -\frac{1}{2}(x_\bot P_+ + x_\perp P_2) \right. \)
\( + r(\cos \theta P_2 + \sin \theta P_3)] \)
\( \times \exp [\log z D], \)
\( \hat{h} = [M_{01}, P_+, M_{+2}, P_2, M_{+3}, P_3], \)
\( f' = \exp \left[ -\frac{x_\bot P_+}{2\zeta} \right] \exp [\log z D], \)
\( \hat{b} = [M_{01}, K_-, M_{-2}, K_2, M_{-3}, K_3], \)
\( v = -\frac{x_\bot}{2\eta} M_{01} + \frac{1}{4\eta z} K_- + \frac{r}{2\eta} (\cos \theta M_{-2} + \sin \theta M_{-3}). \)

15. \( r = \eta (M_{01} \wedge P_+ + M_{+2} \wedge P_2), \)
\( f = \exp \left[ -\frac{1}{2}(x_\bot P_+ + x_\perp P_2) \right. \)
\( + r(\cos \theta P_2 + \sin \theta P_3) \exp [\log z D], \)
\( \hat{h} = [M_{01}, P_+, M_{+2}, P_2], \)
\( f' = \exp \left[ -\frac{x_\bot P_+}{2\zeta} + \frac{r}{z} \sin \theta P_3 \right] \exp [\log z D], \)
\( \hat{b} = [M_{01}, K_-, M_{-2}, K_2], \)
\( v = -\frac{x_\bot}{2\eta} M_{01} + \frac{1}{4\eta z} K_- + \frac{r}{2\eta} \cos \theta M_{-2}. \)
16. \[ r = \eta \, M_{01} \land P_+ , \quad f = \exp \left[ \frac{-1}{2} \left( x_- P_+ + x_+ P_- \right) + r \left( \cos \theta \, P_2 + \sin \theta \, P_3 \right) \right] \exp \left[ \log z \, D \right] , \]
\[ \mathfrak{h} = \left( M_{01}, \, P_+ \right) , \quad f' = \exp \left[ -\frac{x_+}{2} P_+ + r \left( \cos \theta \, P_2 + \sin \theta \, P_3 \right) \right] \exp \left[ \log z \, D \right] , \]
\[ \mathfrak{f} = \left( M_{01}, \, K_- \right) , \quad v = -\frac{x_-}{2\eta} M_{01} + \frac{1}{4\eta} K_- . \]

17. \[ r = \eta \, M_{01} \land (P_+ + K_+), \quad f = \exp \left[ -\frac{1}{2} \left( x_- P_+ + x_+ P_- \right) \right] + r \left( \cos \theta \, P_2 + \sin \theta \, P_3 \right) \exp \left[ \log z \, D \right] , \]
\[ \mathfrak{h} = \left( M_{01}, \, P_+ + K_+ \right) , \quad f' = \exp \left[ \sqrt{x_- x_+} P_0 + r \left( \cos \theta \, P_2 + \sin \theta \, P_3 \right) \right] \exp \left[ \log z \, D \right] , \]
\[ \mathfrak{f} = \left( M_{01}, \, P_+ + K_- \right) , \quad v = \frac{1}{8\eta} \sqrt{x_- \left( P_+ + K_- \right)} . \]

18. \[ r = \eta \, M_{01} \land (P_+ - K_+), \quad f = \exp \left[ -\frac{1}{2} \left( x_- P_+ + x_+ P_- \right) \right] + r \left( \cos \theta \, P_2 + \sin \theta \, P_3 \right) \exp \left[ \log z \, D \right] , \]
\[ \mathfrak{h} = \left( M_{01}, \, P_+ - K_+ \right) , \quad f' = \exp \left[ \sqrt{x_- x_+} P_0 + r \left( \cos \theta \, P_2 + \sin \theta \, P_3 \right) \right] \exp \left[ \log z \, D \right] , \]
\[ \mathfrak{f} = \left( M_{01}, \, P_+ - K_- \right) , \quad v = -\frac{1}{8\eta} \sqrt{x_- \left( P_+ - K_- \right)} . \]

19. \[ r = \eta \, (D - \zeta P_1) \land P_0, \quad f = \exp \left[ -x_0 P_0 + x_1 P_1 + r \left( \cos \theta \, P_2 + \sin \theta \, P_3 \right) \right] \exp \left[ \log z \, D \right] , \]
\[ \mathfrak{h} = \left( D - \zeta P_1, \, P_0 \right) , \quad f' = \exp \left[ \frac{x_1 + \zeta}{z} P_1 + \frac{\zeta}{z} \left( \cos \theta \, P_2 + \sin \theta \, P_3 \right) \right] , \]
\[ \mathfrak{f} = \left( D - \zeta K_1, \, K_0 \right) , \quad v = -\frac{x_0}{\eta (1 + 2\zeta^2)} \left( D - \zeta K_1 \right) + \frac{z}{2\eta} K_0 . \]

20. \[ r = \eta \, (D + M_{01} - \zeta M_{23}) \land P_+, \quad f = \exp \left[ -\frac{1}{2} \left( x_- P_+ + x_+ P_- \right) \right] + r \left( \cos \theta \, P_2 + \sin \theta \, P_3 \right) \exp \left[ \log z \, D \right] , \]
\[ \mathfrak{h} = \left( D + M_{01} - \zeta M_{23}, \, P_+ \right) , \quad f' = \exp \left[ -\frac{x_+}{2} P_+ + r \left( \cos \theta - \zeta \log z \right) P_2 \right] + \sin (\theta - \zeta \log z) P_3 , \]
\[ \mathfrak{f} = \left( D + M_{01} + \zeta M_{23}, \, K_- \right) , \quad v = -\frac{x_-}{2\eta (2 + \zeta^2)} \left( D + M_{01} + \zeta M_{23} \right) + \frac{z^2}{8\eta} K_- . \]

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