Nonlocal strength criteria for the calculation of critical loads in bended structural elements, taking into account the nonlinearity of the material deformation

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Abstract. The destruction of artificial materials (plexiglas, ebonite) in non-uniform stress fields was experimentally investigated. Testing of one batch of samples is carried out using various methods: three-point and clean (four-point) bending of rectangular cross-section beams, Brazilian test (method of splitting a cylinder along a generator). The comparison of the strength characteristics obtained by the Brazilian test with the uniaxial tensile strength, as well as with the tensile strength during bending tests. Experiments on the fracture of beams on a three-point flexural test revealed higher values of ultimate tensile stress than those of a four-point flexural test. The processing of the experimental data obtained by the Brazilian test is carried out using various non-local strength criteria. A sequential analysis of the applicability of the criteria used in the determination of failure loads is carried out. Computer simulation was performed using the finite-element method for the destruction of a cylinder by splitting along a generator, three-point and four-point bending flexural test of a beam in a geometrically nonlinear formulation, taking into account the non-linearity of the material deformation diagram. The problem was solved in the updated Lagrangian formulation taking into account the physical and geometric nonlinearity based on the governing equations of the hypoelastic material. A comparative analysis of the numerical, analytical and experimental results is carried out. A qualitative correspondence was obtained between the experimental data and the results of numerical calculations.

1. Introduction
Most of the nonlocal strength criteria are based on the concept of the formation of a prefracture zone in the material, in which a local redistribution of stresses occurs, while the main material is elastically deformed up to failure. Fracture is considered as a physical process that takes place not at a mathematical point at which the maximum value of equivalent stress is reached, but in some small neighborhood of it (the prefracture zone). A common property of these criteria is the introduction of the internal size of the material that characterizes its structure, which allows one to expand the scope of application compared to traditional criteria.

Criteria (theories) of strength are essentially hypotheses about the mechanisms, causes and conditions for the occurrence of a limiting state. The construction of strength hypotheses is based on the premise that two stress state of any kind are considered equally dangerous and of equal strength if, when the principal stresses increase proportionally in the same number of times, they become limit at the same time. If in two stress states the safety factors are equal, then such stress states are said to be equally dangerous. This makes it possible to compare various stress states according to the degree of
their danger – according to the value of the safety factor. Equally dangerous stress states criteria make it possible to go from a complex stress state to an equally dangerous state of uniaxial tension. But for uniaxial tension, the limiting state is known from the experiment; therefore, it is easy to calculate its safety factor, which, by virtue of equal danger, is equal to the safety factor also for a complex state. Therefore, strength hypotheses should establish an equally dangerous criterion. Therefore, it is usually convenient to formulate them in the form of a criterion for the equally dangerous complex and uniaxial stress states, while the latter is called the equivalent stress state, and the nonzero principal stress of the uniaxial state is called the equivalent stress and is denoted by \( \sigma_{\text{equ}} \) [1].

The reduction of a complex stress state to a linear one that is equally dangerous to it is carried out by replacing the principal stresses \( \sigma_1, \sigma_2, \sigma_3 \) with an equivalent stress that must be created in a deformed specimen in order to obtain a stress state that is equally dangerous to the given one. Strength is assessed by comparing equivalent stresses with limit stresses at tension (compression) or directly with allowable stresses. Replacing a complex stress state with an equivalent stress, we get the opportunity to use the condition of coming of the limiting state (strength criterion) for a complex stress state \( \sigma_{\text{equ}} = \sigma_{\text{crit}} \), where \( \sigma_{\text{crit}} \) is the critical value of some parameter, which can be, for example, tensile strength \( \sigma_t \), compression strength \( \sigma_c \), yield strength \( \sigma_Y \) etc. [2].

In this article, on the basis of nonlocal strength criteria, comparison of the strength characteristics of plexiglass and ebonite, obtained by the Brazilian test, with uniaxial tensile strength, as well as with the tensile strength at bending tests was carried out.

2. Experimental studies
For brittle materials, obtaining tensile/compressive strengths encounters technical difficulties, so there is a need for alternative methods for obtaining the strength characteristics of materials. Such methods include bending tests (Figure 1) and the Brazilian test (Figure 2).

![Figure 1](https://example.com/figure1.png)

**Figure 1.** The three-point bending flexural test (a) and four-point bending flexural test (b).

The performance of direct tests for the uniaxial tension of non-metallic fragile materials is associated with the technical problem of applying tensile axial forces to a sample. This problem is solved by embedding the ends of the sample in special tips by means of epoxy or other adhesive materials, Wood alloy, etc. Equipping of samples with this kind of devices significantly increases the cost of experimentation and virtually eliminates their massive and somewhat operational execution. For this reason, in materials science and production (construction, mining, etc.), an indirect method for determining the tensile strength of a material, proposed in 1947 by Brazilian engineer F. Carneiro, has been widely used. In this method the tensile strength of a material is determined from the compression...
test of a cylindrical specimen in the diametrical plane of a uniformly distributed along a generator load transmitted by the edge of a triangular prism. Such tests in scientific publications received the name of the Brazilian test or the Brazilian method. In regulatory documents (national standards, departmental instructions, etc.), the Brazilian test is often called the “splitting method” [3].

![Figure 2. Brazilian test.](image)

Experiments were conducted on the fracture of samples of plexiglas and ebonite. Each series contained 3 samples, made from one plate in one direction, which failed along a selected plane. Beams for determining bending strength had a length of 110 mm, the distance between the supports in all experiments was 100 mm. To determine the effect of beam thickness on flexural strength, beams with a thickness of 11 to 25 mm and a width of 11 to 19 mm were tested. The side with which the destruction began was carefully planished and polished. Discs for the Brazilian test were cut from plates with a thickness of 11 to 25 mm, with a diameter of 24 and 48 mm. The beams were supported on cylinders with a diameter of 4 mm, the load was transmitted through a wedge with a radius of 2.5 mm. The scatter of critical load values \( P_c \) did not exceed 5-10% of the average value of the three tests for each batch of samples. The scatter of values \( P_c \) is mainly due to the inaccuracy of the installation of samples on the supports and places of application of the loading wedges. A photo of the destroyed specimens is shown in Figure 3.

The strength of the beams was determined by the theory of bending in the assumption of linear elastic deformation: for three-point bending flexural test

\[
\sigma_{3f} = \frac{3P_cL}{2hb^2}\,
\]

for four-point bending flexural test

\[
\sigma_{4f} = \frac{3P_cL}{4hb^2}\,
\]

tensile strength according to the method of the Brazilian test was calculated by the formula

\[
\sigma_{bt} = \frac{2P_c}{\pi Dt}.
\]

Here \( h \) is the beam thickness, \( b \) is the beam width, \( D \) is the disc diameter, \( t \) is the disc thickness.
In addition, experiments were carried out on uniaxial tension of flat samples in the form of spatulas and a dumbbell-shaped form with a gage length of \( \approx 50 \) mm and a cross section of \( 13 \times 10 \) mm in order to determine the deformation diagram of plexiglas and ebonite. Typical \( (\sigma - \varepsilon) \) diagrams of uniaxial deformation for plexiglas (curve 1) and ebonite (curve 2) are shown in Figure 4. Diagrams similar to those shown in Figure 4, have a pronounced nonlinearity, but without the presence of a plastic segment. The idea that plexiglass and ebonite are usually brittle is not always justified. Uniaxial tension experiments show that these materials are nonlinearly elastic.

![Figure 3. Samples after destruction.](image)

Figure 3. Samples after destruction.

![Figure 4. Diagrams of uniaxial deformation for plexiglas (curve 1) and ebonite (curve 2).](image)

Figure 4. Diagrams of uniaxial deformation for plexiglas (curve 1) and ebonite (curve 2).

3. Computer simulation
We use the finite element method to find the critical stresses in bending the beam and in splitting the disk. Loading schemes are shown in Figures 1 and 2, the contact problem was solved: the beam and the
disk are deformable bodies, supporting cylinders and the load wedge are rigid. The computational domain was discretized with a uniform mesh of quadrilateral elements 0.25 mm in size with a quadratic approximation of the displacements. Two material models were used: 1) isotropic linearly elastic with Young's modulus $E = 3263$ MPa and 3631 MPa, Poisson’s ratio $\nu = 0.33$ and 0.35 for plexiglas and ebonite, respectively; 2) physically nonlinear hypoelastic material with strain diagrams shown in Figure 4. The constitutive relations of the hypoelastic material are

$$\sigma_{ij} = C^{ijkl} \varepsilon_{kl},$$

where $\sigma_{ij}$ are the components of the true Cauchy stress tensor, $\varepsilon_{kl}$ are the components of the Henky’s logarithmic strain tensor, fourth rank tensor components $C^{ijkl}$ are functions of $\varepsilon_{kl}$.

The load was taken equal to the average critical load found in the experiment. The calculation of the stress-strain state of the beam was performed in the finite element analysis package MSC.Marc 2017. For a three-point bending flexural test, the largest value of the maximum principal stress $\sigma_1$ is achieved in the lower fiber of the beam in the middle of the span. For a four-point bending flexural test in engineering theory, the moment between the lines of application of loads is constant (the so-called pure bend), but the calculation shows that this is not so. Figure 5 shows a plot of $\sigma_1$ on which two symmetric about the middle maxima are observed.

4. Results and discussion

The calculation results are presented in the following tables. The critical stresses in the Brazilian test were calculated using the following nonlocal criteria: Neuber-Novozhilov [4, 5]

$$\sigma_{equ} = \frac{1}{d} \int_0^d \sigma_\rho(r) dr = \sigma_1,$$

where $d$ is size of structural components of the material; Lajtai [6]

$$\sigma_{equ} = \sigma_{max} \left( 1 - \frac{d}{L_\sigma} \right) = \sigma_1,$$

where $L_\sigma = \sigma_{max} / (d \sigma_\rho / dr)_{max}$; Novopashin-Suknev [7, 8]

$$\sigma_{equ} = \frac{\sigma_\rho}{1 + \sqrt{d / L_\sigma}} = \sigma_1.$$
Legan [9, 10]

\[ \sigma_{eq} = \frac{\sigma_u}{1 - \beta + \sqrt{\beta^2 + d/L_u}} = \sigma, \]

where \( \beta \geq 0 \) is a dimensionless parameter that can be thought of as an approximation parameter.

All criteria give approximately the same values, therefore one value is given in the tables. The following notation is accepted: \( \sigma_{anl} \) is the critical stresses found by formulas (1)-(3); \( \sigma_{num}^{LE} \) is the numerical calculation assuming linear elastic material; \( \sigma_{num}^{NLE} \) is the numerical calculation assuming nonlinear elastic material. Specimen sizes are given in mm, critical load \( P_c \) – in N, critical stresses – in MPa.

Table 1. Three-point flexural test, plexiglas.

| \( h \) | \( b \) | \( P_c \) | \( \sigma_{anl} \) | \( \sigma_{num}^{LE} \) | \( \sigma_{num}^{NLE} \) |
|---|---|---|---|---|---|
| 11 | 19 | 1576 | 102.827 | 97.371 | 79.067 |
| 19 | 11 | 3028 | 114.379 | 106.962 | 86.356 |
| 25 | 11 | 5356 | 116.858 | 108.547 | 86.730 |

Table 2. Four-point flexural test, plexiglas.

| \( h \) | \( b \) | \( P_c \) | \( \sigma_{anl} \) | \( \sigma_{num}^{LE} \) | \( \sigma_{num}^{NLE} \) |
|---|---|---|---|---|---|
| 11 | 19 | 2496 | 81.427 | 71.461 | 60.796 |
| 19 | 11 | 4876 | 92.093 | 85.161 | 65.172 |

Table 3. Brazilian test, plexiglas.

| \( D \) | \( t \) | \( P_c \) | \( \sigma_{anl} \) | \( \sigma_{num}^{LE} \) | \( \sigma_{num}^{NLE} \) |
|---|---|---|---|---|---|
| 24 | 25 | 32373 | 34.35 | 33.42 | 41.37 |
| 48 | 11 | 34561 | 41.07 | 40.35 | 48.31 |

Table 4. Three-point flexural test, ebonite.

| \( h \) | \( b \) | \( P_c \) | \( \sigma_{anl} \) | \( \sigma_{num}^{LE} \) | \( \sigma_{num}^{NLE} \) |
|---|---|---|---|---|---|
| 11 | 15 | 893 | 73.80 | 70.713 | 56.368 |
| 15 | 11 | 1324 | 80.24 | 76.137 | 57.832 |
| 15 | 13 | 1472 | 75.49 | 71.847 | 56.702 |

Table 5. Four-point flexural test, ebonite.

| \( h \) | \( b \) | \( P_c \) | \( \sigma_{anl} \) | \( \sigma_{num}^{LE} \) | \( \sigma_{num}^{NLE} \) |
|---|---|---|---|---|---|
| 11 | 15 | 1442 | 59.59 | 54.675 | 47.121 |
| 15 | 11 | 1816 | 55.03 | 51.093 | 44.462 |
| 15 | 16 | 2560 | 64.00 | 59.754 | 50.575 |
Table 6. Brazilian test, ebonite.

| D  | t  | \(P_t\) | \(\sigma_{anl}\) | \(\sigma_{num}^{LE}\) | \(\sigma_{num}^{NLE}\) |
|----|----|---------|------------------|-----------------|-----------------|
| 24 | 25 | 20925   | 22.20            | 21.87           | 42.353          |
| 48 | 11 | 20925   | 24.80            | 25.23           | 45.962          |

As is known, the three-point bending strength is always higher than the uniaxial tensile strength. For plexiglas 1.5÷1.8 times, for ebonite about 1.5 times. This is explained by the heterogeneity of the stress state at the point of maximum deflection of the beam. As can be seen from Tables 1, 2, 4, 5 tests on a three-point bend give higher values of limit tensile stress than similar ones with four-point bending. This fact was noted by many researchers [11]. The calculation by the finite element method and experiments on the loading of the beams show that the tensile strain at four-point bending is approximately 15% higher than at three-point bending, which leads to an overestimation of the strength when testing for three-point bending.

According to Tables 3, 6, the tensile strength in bending essentially exceeds the tensile strength according to the Brazilian test. This discrepancy may be caused by several factors. The deformation diagrams (Figure 4) are inelastic in nature, and the calculation by formula (1) gives an overestimated result, especially tangible, if the material has plastic properties. If the material is really brittle, the effect of the sample volume and stress gradient becomes relevant [12]. The use of the constitutive relations of the hypoeelastic material (4) in the calculations made it possible to reduce the value of the critical stresses. The size effect is manifested in a decrease of flexural strength with a decrease of the thickness of the beam.

The biaxiality of the stress field in the case of the Brazilian test should reduce the strength of the material compared to uniaxial tension, since the compressive stress perpendicular to the tension makes an additional contribution to the tensile strain. The use of a non-linear elastic material model in combination with non-local strength criteria allows an increase in the tensile strength in the Brazilian test for plexiglas by 24%, for ebonite by 82%.

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References
[1] Osetsyk V M 1977 Applied Mechanics (Moscow: Mashinostroenie) (in Russian)
[2] Gorshkov A G, Troshin V N and Shalashilin V I 2002 Strength of Materials (Moscow: Fizmatlit) (in Russian)
[3] Molotnikov V Ya and Molotnikova A A 2014 DSTU Bulletin 14 30 (in Russian)
[4] Neuber G 1937 Kerbspannunglehre: Grunlagen fur Genaua Spannungsrechnung (Berlin: Springer-Verlag)
[5] Novozhilov V V 1969 Prikl. Mat. Mekh 33 212 (in Russian)
[6] Lajtai E Z 1972 Int. J. Rock Mech. Min. Sci. 9 569
[7] Suknev S V and Novopashin M D 2000 Dokl. Phys. 45 339
[8] Novopashin M D and Suknev S V 2007 Vest. Samara State Univ. Natur. Sci. Series 54 316 (in Russian)
[9] Legan M A 1993 J. Appl. Mech. Tech. Phys. 34 146
[10] Legan M A 1994 J. Appl. Mech. Tech. Phys. 35 117
[11] Obert L 1972 Fracture vol 7 (New York and London: Academic Press)
[12] Efimov V P 2009 Phys. Tech. Problems of Mining 6 61