Singular behavior of the multipole electromagnetic field

C Vrejoiu and R Zus

University of Bucharest, Department of Physics, PO Box MG 11, Bucharest-Magurele, RO 077125, Romania

E-mail: vrejoiu@fizica.unibuc.ro and roxana.zus@fizica.unibuc.ro

Received 17 May 2010, in final form 16 August 2010
Published 15 September 2010
Online at stacks.iop.org/JPhysA/43/405208

Abstract

The singularities of the electromagnetic field are derived to include all the pointlike multipoles representing an electric charge and current distribution. In the static case, the \( \delta \)-singularities are expressed for arbitrary multipole orders, while in the dynamic case we restrict ourselves to the lower orders. The algorithm we give can be easily extended to the next orders. In both cases, we show that for higher orders, it is more efficient to have fields represented in terms of symmetric and trace-free moments.

PACS numbers: 03.50.De, 41.20.—q

1. Introduction

In the cases of electrostatic and magnetostatic fields of pointlike dipoles, one has the well-known procedure of introducing Dirac \( \delta \)-function terms for obtaining correct expressions of the electric and magnetic fields defined on the entire space. The corresponding field expressions are written in the following form [1–3]:

\[
E_p(r) = \frac{1}{4\pi \varepsilon_0} \frac{3(\nu \cdot p)\nu - p}{r^3} - \frac{1}{3\varepsilon_0} p\delta(r),
\]

(1)

where \( \nu = r/r \), and

\[
B_m(r) = \mu_0 \frac{3(\nu \cdot m)\nu - m}{r^3} + \frac{2\mu_0}{3} m\delta(r).
\]

(2)

The expressions from equations (1) and (2) are introduced in [1] as conditions of compatibility with the average value of the electric or magnetic field over a spherical domain containing all the charges or currents inside. Another procedure for introducing equations (1) and (2) is based on an extension of the derivative \( \partial_i \partial_j/(1/r) \) to the entire space [2]:

\[
\frac{1}{r} \partial_i \partial_j v_j = \frac{3v_i v_j - \delta_{ij}}{r^3} - \frac{4\pi}{3} \delta_{ij} \delta(r).
\]

(3)
Obviously, equations (1)–(3) must be considered as equations in the space of distributions (generalized functions). The right-hand side of each equation has a structure obtained by a procedure that separates a singular part represented by a distribution with pointlike support. This distribution can only be a linear combination of the Dirac \( \delta \)-functions and the corresponding derivatives. These operations correspond to the extension to the entire space as distributions of the fields \( E, B \) and, implicitly, of the second derivative of \( 1/r \). Therefore, the above distributions are distinguished sometimes by adequate notation from ordinary functions defined only for \( r \neq 0 \). For example, the notation \( \hat{\partial}_i \hat{\partial}_j (1/r) \) and the term ‘distributional derivative’ are employed in [4]. In this paper such notation is not employed, the mathematical meaning of each term in the written equations resulting from the context or being specified when and if it is necessary for a good understanding. The first term on the right-hand side of equation (3), and similar terms in the first two equations, must be considered as a limit of a regular distribution defined by the function inside the bracket. This implies a regularization procedure and we have to specify it. In the next section, we will introduce physical criteria for choosing an adequate regularization for the problem treated here\(^1\).

More detailed discussions on the different regularization procedures in treating the singularities of multipole fields and, more generally, concerning the introduction of distributional derivatives of \( 1/r^m \) can be found in [4–7].

A more pedagogical and suitable approach for understanding the origin of the difference between the electric and magnetic cases is done in [3]. Generalizations of equations (1) and (2) to the dynamic case for oscillating electric and magnetic dipoles are in [8, 9].

The aim of this paper is to derive, in a consistent manner, the singularities of the electromagnetic field such that all the pointlike multipoles representing an electric charge and current distribution are included. We will show that for higher orders, it is more efficient to have fields represented in terms of symmetric and trace-free moments. In the static case, we will determine the \( \delta \)-singularities for arbitrary multipole orders, while in the dynamic case we will restrict ourselves to orders up to \( n = 3 \), the orders important for the current applications. Section 2 is dedicated to the reader less used with the tensorial formalism in handling multipolar expansions. For the informed reader, the section can be seen as an introduction to the notation and formulas used along the paper. In sections 3 and 4, the results for the electric and magnetic fields are presented in the static case. The dynamic case is treated in section 5, and the conclusions are outlined in section 6.

The formalism employed in this paper is a purely algebraic one. With a good understanding of the definitions and notation presented in section 2, we think the reader will be able to verify every detail of the calculation and, possibly, to search and find some more adequate versions.

2. Preliminaries

2.1. Definitions, notation and formulas from the tensorial formalism

Let us consider an \( n \)-th order Cartesian tensor denoted by \( T^{(n)} \) and characterized by the components \( T_{i_1...i_n}, i_q = 1, 2, 3 \). These components are, in fact, the components of a vector in the \( n \)-tensorial product of the Euclidean space \( \mathbb{R}^3 \):

\[
T^{(n)} = T_{i_1...i_n} e_{i_1} \otimes \ldots \otimes e_{i_n} = T_{i_1...i_n} e_{i_1...i_n},
\]

where \( e_i \) are the unit vectors of a Cartesian basis in \( \mathbb{R}^3 \) and \( e_{i_1...i_n} \) are the unit vectors of the tensorial product space. Some vectors from the tensorial product of spaces can be represented

\(^1\) The authors thank an unknown referee for suggesting the elucidation of this point.
by the tensorial products of \( n \) vectors from the different space factors \( \mathbb{R}^3 \). Particularly, in the case of identical factors, we employ the usual notation

\[
\alpha^n = a_i \ldots a_n e_{i_1 \ldots i_n}.
\]

The vector \( \alpha \) can be the differential operator \( \nabla = e_i \partial_i \) and, in this case,

\[
\nabla^n = e_{i_1 \ldots i_n} \partial_{i_1} \ldots \partial_{i_n}.
\]

Because of this simplified notation, for avoiding confusions, the Laplace operator in the space \( \mathbb{R}^3 \) is symbolized by \( \Delta \) and not by the frequent \( \nabla^2 \).

We employ the following notation for the tensorial contractions:

\[
(A^{(n)}|B^{(m)})_{i_1 \ldots i_n j_1 \ldots j_m} = \begin{cases} 
A_{i_1 \ldots i_n j_1 \ldots j_m} B_{j_1 \ldots j_m}, & n > m \\
A_{i_1 \ldots i_n} B_{j_1 \ldots j_m}, & n = m \\
A_{i_1 \ldots i_n j_1 \ldots j_m}, & n < m.
\end{cases}
\]

A fully symmetric tensor \( S^{(n)} \), which here is called simply ‘symmetric’, has a projection on the subspace of symmetric and trace-free (STF) tensors. This projection will be represented by the tensor \( T(S^{(n)}) \equiv S^{(2)} \). For \( n = 2 \), for example, we can write

\[
S_{ij} = S_{ij} + \delta_{ij}\Lambda(S^{(2)}).
\]

The condition \( S_{ii} = 0 \) gives

\[
\Lambda = \frac{1}{3} S_{qq}.
\]

In the case \( n = 3 \):

\[
S_{ijk} = S_{ijk} + \delta_{ij} \Lambda_{kl}(S^{(3)}),
\]

where by the symbol \( A_{i_1 \ldots i_n} \) is denoted the sum over all distinct permutations of the indexes.

The condition \( S_{ijk} = 0 \) for \( k = 1, 2, 3 \) implies

\[
\Lambda_{i}(S^{(3)}) = \frac{1}{5} S_{qqq}.
\]

Though, maybe, only for a theoretical interest, let us consider the general case for the STF projection of the symmetric tensor \( S^{(n)} \) defined by the equation

\[
T_{i_1 \ldots i_n}(S^{(n)}) = S_{i_1 \ldots i_n} - \delta_{i_1 i_2} \Lambda_{i_3 \ldots i_n}(S^{(n)}).
\]

\( \Lambda^{(n-2)} = \Lambda(S^{(n)}) \) is a symmetric tensor and is defined by the condition that \( T^{(n)} \) is a trace-free tensor. For low values of \( n \), the ones of really practical interest, the components \( \Lambda_{i_1 \ldots i_n} \) can be calculated directly from the equation system representing the vanishing relations of all the partial traces of the tensor \( T^{(n)} \). However, we mention here a general formula known from the literature [10, 11] which, with the notation from this paper, is written as

\[
[T[S^{(n)}]]_{i_1 \ldots i_n} = \sum_{m=0}^{[n/2]} \frac{(-1)^m(2n - 1 - 2m)!!}{(2n - 1)!!} \delta_{i_1 i_2} \ldots \delta_{i_{2m+1} i_{2m+2}} S^{(m)}_{i_1 \ldots i_{2m+1}, i_{2m+2} \ldots i_{2m+1}}.
\]

The symbol \( \lfloor \alpha \rfloor \) represents the integer part of \( \alpha \) and \( S^{(m)}_{i_1 \ldots i_n} \) denotes the components of the \( (n - 2m) \)th-order tensor obtained from \( S^{(n)} \) by contracting \( m \) pairs of symbols \( i \). This equation is known as the detracer theorem [11]. As a consequence of this theorem, the components of the tensor \( \Lambda^{(n-2)} \) are written as

\[
\Lambda_{i_1 \ldots i_{n-2}}(S^{(n)}) = \sum_{m=0}^{[n/2-1]} \frac{(-1)^m(2n - 1 - 2(m + 1))!!}{(m + 1)(2n - 1)!!} \delta_{i_1 i_2} \ldots \delta_{i_{2m+1} i_{2m+1}} S^{(m+1)}_{i_1 \ldots i_{2m+2}, i_{2m+3} \ldots i_{2m+2}}.
\]

These formulas are useful for defining and processing the multipole expansions of the electrodynamic potentials and fields.
2.2. Multipole expansion of the electromagnetic field in Cartesian coordinates

The multipole expansions of the potentials in the Lorentz gauge are written as [12–15]

\[ \Phi(r, t) = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla^n ||P(n)(\tau)|| r \]

and

\[ A(r, t) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \left[ \nabla \times \left( \nabla^{n-1} ||M(n)(\tau)|| r \right) + \nabla^{n-1} ||\dot{P}(n)(\tau)|| r \right]. \]

The dot symbolizes the time derivative and \( \tau = t - \frac{r}{c} \) is the retarded time with respect to the origin \( O \) of the Cartesian axes in the point corresponding to the vector \( r \). The origin \( O \) is a point from the support of the electric charge and current distribution. The tensors \( P(n)(t) \) and \( M(n)(t) \) are the electric and magnetic moments of the electric charges and currents’ distributions \( \rho(r, t) \) and \( J(r, t) \), the support of these distributions being included in the domain \( D \):

\[ P(n)(t) = \int_D d^3x r^n \rho(r, t) \]

and

\[ M(n)(t) = \frac{n}{n+1} \int_D d^3x r^n \times J(r, t). \]

In the last equation, a tensorial contraction via the Levi-Civita pseudo-tensor \( \varepsilon_{ijk} \) is employed:

\[ \{T^{(n)}, a\} \rightarrow T^{(n)} = \varepsilon_iq_{s} T_{i_{1}...i_{n-1}}a_{l_{1}...l_{n}} \]

which in the particular case of \( T^{(n)} = b^n \) becomes

\[ b^n \times a = b_{i_{1}}...b_{i_{n}} (a \times b)_{l_{1}...l_{n}}. \]

The expansions (6) and (7) are running in the exterior of the minimal radius sphere including the support of \( \rho \) and \( J \).

From equations (6) and (7) one obtains the following expansions of the fields \( E(r, t) \) and \( B(r, t) \):

\[ E(r, t) = -\nabla \Phi(r, t) - \frac{\partial A(r, t)}{\partial t} = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla^{n+1} ||P(n)(\tau)|| r \]

\[-\frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \left[ \nabla \times \left( \nabla^{n-1} ||M(n)(\tau)|| r \right) + \nabla^{n-1} ||\dot{P}(n)(\tau)|| r \right] \]

and

\[ B(r, t) = \nabla \times A(r, t) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \]

\[ \left[ \nabla^{n+1} ||M(n)(\tau)|| r \right] - \nabla^{n-1} ||\Delta M(n)(\tau)|| r + \nabla \times \left( \nabla^{n-1} ||\dot{P}(n)(\tau)|| r \right). \]

For the magnetic moments (8), it is also possible to introduce STF moments \( \mathcal{M}^a = T(M^a) \), but, this time, there are two steps required in order to complete the objective. The tensor \( M^{(n)} \) is symmetric only in the first \( n - 1 \) indexes and satisfies the property

\[ M_{i_{1}...i_{n-2}qq} = 0. \]
In the first step we must obtain the symmetric projection $^{(n)}_{\text{sym}}$ of the tensor $\mathbf{M}^{(n)}$. We begin with the first simple example corresponding to $n = 2$. Let us write the identity

$$M_{ij} = \frac{1}{2} (M_{ij} + M_{ji}) + \frac{1}{2} (M_{ij} - M_{ji}) = ^{\text{sym}}_{\text{sym}} M_{ij} + \frac{1}{2} \epsilon_{ijk} N_k (M^{(2)}),$$

where $\mathbf{M}$ is the symmetric part of $\mathbf{M}^{(2)}$ and

$$N_j (M^{(2)}) = \epsilon_{ijk} M_{jk} = \frac{2}{3} \int_D d^3 x \ [r \times (r \times J)].$$

In this case ($n = 2$), $\mathcal{N}^{(2)} = M^{(2)}$ and, consequently, corresponds to the STF projection. Therefore,

$$M_{ij} = \mathcal{N}_{ij} + \frac{1}{2} \epsilon_{ijk} N_k (M^{(2)}). \quad (11)$$

For $n \geq 3$, we can generalize this result writing the identity

$$M_{i_1 \ldots i_n} = \frac{1}{n} (M_{i_1 \ldots i_n} + M_{i_2 \ldots i_n i_1} + \cdots + M_{i_1 \ldots i_{n-1}})$$

$$+ \frac{1}{n} \left[ (M_{i_1 \ldots i_n} - M_{i_1 \ldots i_{n-1}}) + \cdots + (M_{i_1 \ldots i_n} - M_{i_1 \ldots i_{n-1} i_{n-1}}) \right]$$

$$= M_{i_1 \ldots i_n} + \frac{1}{n} \sum_{k=1}^{n-1} \epsilon_{i_1 \ldots i_{n-1} k} N_{i_1 \ldots i_{n-1} k} (M^{(n)}), \quad (12)$$

where by $N^{(n)}_{i_1 \ldots i_{n-1} i_1}$ we understand the component without the index $i_1$. The tensor $\mathbf{N}^{(n-1)} = \mathbf{N}^{(n)}$ is given by

$$N_{i_1 \ldots i_{n-1}} (M^{(n)}) = \epsilon_{i_1 \ldots i_{n-1} pq} M_{i_1 \ldots i_{n-2} p q} = \frac{n}{n + 1} \int_D d^3 x x_{i_1} \ldots x_{i_{n-2}} [r \times (r \times J)]_{i_{n-1}}.$$

It is a tensor of the same type as $\mathbf{M}^{(n-1)}$, i.e. symmetric in the first $n - 2$ indices and with $n - 1$ vanishing traces ($N_{i_1 \ldots i_{n-2} pq} = 0$). Therefore, the STF moment $\mathcal{N}^{(n)}$ is given by the components

$$\mathcal{N}_{i_1 \ldots i_n} = \delta_{i_1 i_2} M_{i_1 i_2} - \delta_{i_1 i_2} \delta_{i_2 i_1} (\mathbf{N}^{(n)}).$$

As it will be seen in the dynamic case, we have to express even the symmetric projection of the tensor $\mathbf{N}^{(n)}$. For this, it is useful to introduce the operator $\mathcal{N}$ defining the correspondence

$$\mathbf{N}^{(n)} \rightarrow \mathcal{N} (\mathbf{N}^{(n)}) : \mathcal{N} (\mathbf{N}^{(n)})_{i_1 \ldots i_n} = \epsilon_{i_1 \ldots i_2 p q} N_{i_1 \ldots i_n - p q}.$$

Repeating this operation, we obtain

$$\mathcal{N}^{2k} (\mathbf{M}^{(n)}) = \frac{(-1)^k n}{n + 1} \int_D d^3 x r^{2k} r^{n-2k} \times J,$$

$$\mathcal{N}^{2k+1} (\mathbf{M}^{(n)}) = \frac{(-1)^k n}{n + 1} \int_D d^3 x r^{2k} r^{n-2k-1} \times (r \times J).$$

2.3. Delta-function identities and multipole singularities

A pointlike multipole system can be considered as representing a system of electric charges and currents distributed in the domain $\mathcal{D}$ from the perspective of an observer placed in the exterior of $\mathcal{D}$. Formally, for this observer, the domain $\mathcal{D}$ is replaced by the point $O \in \mathcal{D}$ arbitrarily chosen, all the information possessed being only the multipole moments of the charges and currents distributed in this domain and the expansions (9) and (10). The multipole
expansions are valid in the exterior of a minimal radius $r_0$ sphere including the domain $D$ of the given distribution. Consequently, as a model of reducing to zero the dimensions of an electric system such that it becomes a point $O$, one can use a limiting process in which the exterior field is expressed by equations (9), (10) in the exterior of the sphere $\Sigma_\epsilon$ with $\epsilon \to 0$. This is a natural ground for employing a ‘spherical’ regularization in the process of searching the $\delta$-singularities of the multipole fields. Usually, the multipole expansions are considered, term by term, as functions of $r$ defined on $\mathbb{R}^3$ excepting a singular point which is chosen as the origin $O$ of the Cartesian axes. Actually, these multipole terms are mathematical distributions (or generalized functions) considered firstly as regular ones having as support the entire space except the origin point $O$. The observable quantities are expressed as weighted averages on spatial regions or as surface integrals of functions of field variables. No problems appear when in these regions $r \neq 0$, but when we have to calculate, for example, the interaction of a distribution $(\rho, J)$ with an external field $(E, B)$,

$$ W_{\text{int}} = \int d^3x \ (\rho \Phi - J \cdot A) $$

and the system associated with the external field $(E, B)$ is represented by a pointlike multipole system placed in $O$, the singularities of the potentials or fields having as support this point become unavoidable. This is the case of overlapping sources as encountered in atomic physics. As done in [2], these singularities are determined starting from some $\delta$-function identities associated with the extension of multiple spatial derivatives of the functions of the type $1/r$ to the entire space. Having in mind the dynamic case, too, we generalize such identities to the derivatives of the function $f(\tau)/r$. Representing the corresponding derivatives as functions of $r$ and $t$ for $r \neq 0$, we can write their expressions in the form

$$ D_{i_1 \ldots i_n}(f) \equiv \partial_{i_1} \ldots \partial_{i_n} \frac{f(\tau)}{r} = \sum_{j=0}^{n} \frac{1}{(n-j)! j!} c^{(n,j)} \frac{d^{n-j} f(\tau)}{d\rho^{n-j}}, \quad (13) $$

where $c^{(n,j)}_{i_1 \ldots i_n}$ are fully symmetric in the indexes $i_1 \ldots i_n$. The general form of the coefficients $C$ is a simple consequence of the symmetry properties and of the derivative rules:

$$ C^{(n,j)}_{i_1 \ldots i_n} = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} K^{(n,j)}_k \delta_{i_1 i_2} \ldots \delta_{i_{2k-1} i_{2k}} \nu_{2k+1} \ldots \nu_n. \quad (14) $$

For lower $n$, the coefficients $C$ and $K$ can be easily calculated by successive derivative operations. As one can see in the following, for obtaining general results for the $\delta$-type singularities of the multipole terms, it will be necessary to know only the coefficients

$$ K^{(n,n)}_0 = (-1)^n (2n - 1)!!. \quad (15) $$

In appendix A we enumerate the coefficients $c^{(n,j)}_{i_1 \ldots i_n}$ for the first five values of $n$.

Considering the distribution representing the derivative (13) extended to the entire space, we understand by $\{\partial_{i_1} \ldots \partial_{i_n} (f(\tau)/r)\}_{(0)}$ the singular part having as support the point $O$. In the following, we are interested in writing explicitly this contribution.

A correct procedure for extracting this singular part is that employed in [2] defining

$$ \left\langle \partial_{i_1} \ldots \partial_{i_n} \frac{f(\tau)}{r} \right\rangle_{(0)} \Phi = \lim_{\epsilon \to 0} \int_{D_\epsilon} d^3x \partial_{i_1} \ldots \partial_{i_n} \frac{f(\tau)}{r} \Phi(r) $$

$$ = \lim_{\epsilon \to 0} \left[ \int_{\Sigma_\epsilon} dS_{\nu_i} \partial_{i_1} \ldots \partial_{i_n} \frac{f(\tau)}{r} \Phi(r) - \int_{D_\epsilon} d^3x \partial_{i_1} \ldots \partial_{i_n} \frac{f(\tau)}{r} \partial_{i_0} \Phi(r) \right]. \quad (16) $$

In this equation, $\Phi(r)$ is supposed to be an element of the domain of distributions, i.e. an infinitely differentiable function. As it can be seen from equation (13), similar restrictive
properties are considered for \( f(\tau) \). Since these results will be employed in searching the singularities of the multipole fields (9), (10), based on the observation from the beginning of this subsection, we can consider the domain \( D_\epsilon \) as the spherical region delimited by the spherical surface \( \Sigma_\epsilon \) with radius \( \epsilon \). Finally, for the calculation process in the present paper, the essential property is the possibility of representing the functions \( f \) and \( \phi \) in the domain \( D_\epsilon \) by their Taylor series upon the origin \( O \):

\[
f(\tau) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{d^k f(\tau)}{d\tau^k} \right)_{\tau=0},
\]

(17)

\[
\phi(r) = \sum_{a=0}^{\infty} \frac{1}{a!} r^a v_{i_1} \ldots v_{i_a} (\partial_{i_1} \ldots \partial_{i_a} \phi(r))_{r=0} = \sum_{a=0}^{\infty} \frac{r^a}{a!} v_{i_1} \ldots v_{i_a} (\partial_{i_1} \ldots \partial_{i_a} \phi(r))_{r=0}.
\]

Equation (16) can be written as

\[
\langle D_{i_1 \ldots i_n}(f) \rangle_{(0)} \phi = \lim_{\epsilon \to 0} \oint_{\Sigma_\epsilon} dS v_i \partial_{i_2} \ldots \partial_{i_n} \frac{f(\tau)}{r} \phi(r) - \langle D_{i_1 \ldots i_n}(f) \rangle_{(0)} v_i \partial_{i_2} \ldots \partial_{i_n} \phi(r),
\]

generating a recursive calculation for a given \( n \).

Let us express the \( \delta \)-singularities for lower order derivatives. Even in these simple cases, the importance of the angular average of a function \( g(\nu) \) defined as

\[
\langle g(\nu) \rangle = \frac{1}{4\pi} \int g(\nu) d\Omega(\nu)
\]

becomes obvious. For this average, we have the well-known formula [10]

\[
\langle v_{i_1} \ldots v_{i_n} \rangle = \begin{cases} 
0, & n = 2k + 1, \\
\frac{1}{(n+1)!} \delta_{i_1 i_2} \ldots \delta_{i_{n+1} i_n}, & n = 2k, \quad k = 0, 1, \ldots 
\end{cases}
\]

(18)

The first \( \delta \)-singularity is obtained for \( n = 2 \), calculating the limit

\[
\langle (D_{ij}(f))_{(0)} \rangle \phi = \lim_{\epsilon \to 0} \oint_{\Sigma_\epsilon} dS v_i \partial_{j} \frac{f(\tau)}{r} \phi(r) - \oint_{D_\epsilon} d^3x \left( \partial_j \frac{f(\tau)}{r} \right) \partial_i \phi(r)
\]

\[
= \lim_{\epsilon \to 0} \left[ \oint_{\Sigma_\epsilon} dS v_i \left( \partial_j \frac{f(\tau)}{r} \right) \phi(r) - \oint_{\Sigma_\epsilon} dS v_i \frac{f(r)}{r} \partial_i \phi(r) + \oint_{D_\epsilon} d^3x \frac{f(\tau)}{r} \partial_i \partial_j \phi(r) \right].
\]

The first surface integral limit can be written as

\[
L_{1\sigma} = \lim_{\epsilon \to 0} \oint_{\Sigma_\epsilon} d\Omega(\nu) \varepsilon^2 v_j \left[ -\frac{v_j}{c\epsilon} f(\tau_\epsilon) - \frac{v_j}{\varepsilon} f(\tau) \right] [\phi(0) + \varepsilon v_k (\partial_k \phi) + \cdots],
\]

where \( \tau_\epsilon = \tau - \varepsilon/c \). Since \( f(\tau_\epsilon) = f(\tau) + \mathcal{O}(\varepsilon) \), and all the terms containing as factors positive powers of \( \varepsilon \) vanish at the limit \( \varepsilon \to 0 \), we can write

\[
L_{1\sigma} = -4\pi \langle v_i v_j \rangle f(\tau) \phi(0) = -\frac{4\pi}{3} \delta_{ij} f(\tau) \phi(0).
\]

The second surface integral limit

\[
L_{2\sigma} = \lim_{\epsilon \to 0} \oint_{\Sigma_\epsilon} d\Omega(\nu) \varepsilon v_j f(\tau_\epsilon) [(\partial_i \phi) + \varepsilon v_k (\partial_i \partial_k \phi) + \cdots] = 0,
\]

since all the terms contain as factors positive powers of \( \varepsilon \). As the volume integral limit also vanishes, we can finally write

\[
\left( \partial_j \frac{f(\tau)}{r} \right)_{(0)} = -\frac{4\pi}{3} f(\tau) \delta_j \delta(r).
\]

(19)

Including the expression of the derivative for \( r \neq 0 \), the above result written for \( f(\tau) = 1 \) verifies equation (3).
Let us consider the next-order derivative and calculate
\[ \langle (D_{ijkl}(f))(0), \phi \rangle = \lim_{\varepsilon \to 0} \left\{ \int d\Omega(\nu)c^2 \left\{ \frac{1}{c^2} v_i v_j v_k f(\tau_\varepsilon) + \frac{1}{c^2} v_i (3v_j v_k - \delta_{jk}) f(\tau_\varepsilon) + \frac{1}{c^2} v_i (3v_j v_k - \delta_{jk}) f(\tau_\varepsilon) \right\} \right\}, \]
where equation (19) is employed in the volume integral. In the surface integral limit, as \( \varepsilon \to 0 \), the terms containing as factor the second-order time derivative \( f \) vanish because of the positive powers of \( \varepsilon \). The terms containing the first-order time derivative \( f \) are proportional to positive powers of \( \varepsilon \) except for the term corresponding to \( \phi(0) \), but in this case the corresponding limit cancels since it contains also as factors the null averages \( \langle v_i v_j v_k \rangle \) and \( \langle v_i \rangle \). The only terms giving a nonvanishing limit are provided by the product \( f(\tau_\varepsilon) (\partial_i \phi)_0 \) such that we can write
\[ \langle (D_{ijkl}(f))(0), \phi \rangle = 4\pi \langle 3v_i v_j v_k v_l - \delta_{jk} v_i v_l \rangle f(\tau)(\partial_i \phi)_0 + \frac{4\pi}{3} f(t) \delta_{jk} (\partial_i \phi)_0. \]
Finally
\[ \left( \partial_i \partial_j \partial_k \frac{f(\tau)}{r} \right)(0) = -\frac{4\pi}{5} f(t) \delta_{ij} \partial_k \delta(r). \]  
For \( f(t) = 1 \), equation (20) becomes the \( \delta \)-singularity corresponding to equation (13) from [2]. Equations (19) and (20) will be applied in the static case for \( f(t) = 1 \).

3. Singularities of the electrostatic field

Let us consider the multipole expansion of the electrostatic field derived from the potential expansion (6) in the static case:
\[ E(r) = \frac{1}{4\pi \varepsilon_0} \sum_{n \geq 0} \frac{(-1)^{n-1}}{n!} V^{n+1} \left| \frac{P^{(n)}}{r} \right| = \frac{1}{4\pi \varepsilon_0} \sum_{n \geq 0} \frac{(-1)^{n-1}}{n!} p^{(n)} \left| \nabla^{n+1} \frac{1}{r} \right|. \]
In the dipolar case \( (n = 1) \), the potential has no \( \delta \)-singularities. For the electric field, we apply equation (19) obtaining the known result (1).

A first interesting feature of the \( \delta \)-singularities problem appears beginning with the electric quadrupole potential and field. The singularity of the potential is obtained employing equation (19):
\[ \Phi^{(2)}(r) = \left( \frac{1}{8\pi \varepsilon_0} \nabla^2 \left| \frac{P^{(2)}}{r} \right| \right)_{r \neq 0} = \frac{1}{8\pi \varepsilon_0} \mathbf{P}_{qq} \delta(\mathbf{r}). \]  
Beginning with this equation, we employ the notation \( (f(r))_{r \neq 0} \) for the regular distribution defined by the function \( f(r) \) such that, with the spherical regularization,
\[ \langle (f(r))_{r \neq 0}, \phi \rangle = \lim_{\varepsilon \to 0} \int d^3 x f(r) \theta(r - \varepsilon) \phi(r), \]  
where \( \theta(x) \) is the Heaviside step function. The first term from the right-hand side of equation (21) is invariant to the substitution \( \mathbf{P}^{(2)} \to \mathbf{P}^{(2)} \) and, employing equations (4) and (5) with \( S = \mathbf{P}^{(2)} \), we can write
\[ \Phi^{(2)}(r) = \left( \frac{1}{8\pi \varepsilon_0} \nabla^2 \left| \frac{P^{(2)}}{r} \right| \right)_{r \neq 0} - \frac{1}{2\varepsilon_0} \Lambda \delta(\mathbf{r}) \].
Usually, since in the exterior of the charge distribution the potential $\Phi(r)$ is the solution of the Laplace equation, it is represented by a series of spherical functions:

$$
\Phi(r) = \sum_{l=0}^{\infty} \frac{1}{r^l} \sum_{m=-l}^{l} Q_{lm} Y_{lm}(\theta, \varphi).
$$

(23)

The $2l + 1$ spherical moments corresponding to a given $l$ are linear combinations of the $2l + 1$ components of the $l$th-order STF tensor $\mathcal{P}^{(l)}$. For the above example of the 4-polar potential, it is obvious that using the expansion representing the particular case of equation (23), the $\delta$-singularity of the potential is lost. The conclusion can be extended to the higher-order multipolar potentials. For the electrostatic field $E(r)$, the employment of the corresponding expansion in spherical functions leads to an incomplete description of the $\delta$-form singularities.

Based on this observation, in our opinion, some care is necessary when passing to the representations by spherical functions since it is equivalent to working only in a subspace of the mathematical objects used correctly in the theory. The direct substitution $\mathcal{P} \rightarrow \mathcal{P}$ or passing to the spherical tensorial representations is not always justified. This observation is pointed out in another context also in [16].

Searching the $\delta$-singularities of $E^{(2)}(r)$, instead of a straightforward processing of the corresponding expression in terms of the primitive moment $\mathcal{P}^{(2)}$, we can introduce from the beginning the STF moment $\mathcal{P}^{(2)}$ writing

$$
(E^{(2)}(r))_{(0)} = -\frac{1}{8\pi \varepsilon_0} \left( \nabla^2 \right) \mathcal{P}^{(2)} (\frac{r}{r})_{(0)} = -\frac{1}{8\pi \varepsilon_0} \left( \nabla^2 \right) \mathcal{P}^{(2)} (\frac{r}{r} + \delta_i \delta \Delta \frac{\Lambda}{r})_{(0)} = -\frac{1}{8\pi \varepsilon_0} \left( \nabla^2 \right) \mathcal{P}^{(2)} (\frac{r}{r})_{(0)} + \frac{1}{2\varepsilon_0} \Lambda \delta(r)
$$

since $\Delta(1/r) = -4\pi \delta(r)$. Equation (20) finally gives

$$
E^{(2)}(r) = (E^{(2)}(r))_{r \neq 0} + \frac{1}{2\varepsilon_0} \mathcal{P}^{(2)} || \nabla \delta(r) + \frac{1}{2\varepsilon_0} \Lambda \nabla \delta(r), \quad \Lambda = \Lambda(\mathcal{P}^{(2)}),
$$

(24)

where

$$
(E^{(2)}(r))_{r \neq 0} = -\frac{1}{8\pi \varepsilon_0} \left( \nabla^2 \right) \mathcal{P}^{(2)} (\frac{r}{r})_{r \neq 0} = -\frac{1}{8\pi \varepsilon_0} \left( \nabla^2 \right) \mathcal{P}^{(2)} (\frac{r}{r})_{r \neq 0}.
$$

Even from this simple example it becomes obvious that this calculation version is convenient for higher $n$ since some contractions of $\mathcal{P}^{(n)}$ with the Kronecker symbols vanish.

In case $n = 3$, the advantage of inserting from the very beginning the STF moments is more evident. Instead of searching the singularities of the fourth-order derivative $\delta_i \delta_j \delta_k \delta_l (1/r)$, we search the singularities of the contraction of this derivative with the STF moment $\mathcal{P}^{(3)}$ writing

$$
\left( \nabla^4 || \mathcal{P}^{(3)} (\frac{r}{r}) \right)_{(0)} \cdot \phi = e_i \lim_{\varepsilon \rightarrow 0} \left[ \int \frac{d\Sigma}{\Sigma} \delta_i \delta_j \delta_k \delta_l \mathcal{P}_{ijkl} (\frac{r}{r}) \phi(r) - \int _{P_i} d^3x \delta_i \delta_j \delta_k \delta_l \mathcal{P}_{ijkl} (\frac{r}{r}) \delta_i \phi(r) \right].
$$

For the surface integral limit, we apply equation (13) particularized to the static case. Denoting by $L_{\alpha}$ this limit,

$$
L_{\alpha} = e_i \lim_{\varepsilon \rightarrow 0} \int \frac{d\Sigma}{\Sigma} \frac{d\Omega(\nu)}{g^2} v_i c^{(3)}_{ijkl} \mathcal{P}_{ijkl} (\frac{r}{r})
$$

and, inserting equation (A.4), one obtains

$$
L_{\alpha} = e_i \lim_{\varepsilon \rightarrow 0} \int \frac{d\Sigma}{\Sigma} \frac{d\Omega(\nu)}{g^2} (-15v_i v_j v_k v_l + 3v_i \delta_{ijkl} v_j) \mathcal{P}_{ijkl} (\frac{r}{r}).
$$
Since $\delta_{ijkl} v_i v_j v_k v_l = 0$, we get
\[ L_\sigma = -15 e_i \sum_{\nu} \frac{d \Omega(\nu)}{\nu^2} v_i v_j v_k v_l \mathcal{P}_{ijkl} \left( \phi(0) + \varepsilon v_j \left( \partial_i \partial_j \phi_0 \right) + \frac{1}{2} \varepsilon^2 v_k v_l \left( \partial_i \partial_j \phi_0 \right) \right) \]
\[ = -4\pi \times 15 e_i \lim_{\varepsilon \to 0} \left\{ \frac{1}{\varepsilon^3} v_i v_j v_k v_l \mathcal{P}_{ijkl} \left( \phi(0) + \varepsilon v_j \left( \partial_i \partial_j \phi_0 \right) + \frac{1}{2} \varepsilon^2 v_k v_l \left( \partial_i \partial_j \phi_0 \right) \right) \right\}. \]

Denoting by $\alpha$ the power of $\varepsilon$ in the Taylor series of $\phi(r)$, we easily conclude that for $\alpha \geq 3$ all the terms from the last equation vanish for $\varepsilon \to 0$. For $\alpha = 0$, the corresponding term vanishes since $\langle v_i v_j v_k v_l \mathcal{P}_{ijkl} \rangle = 0$, as it is seen by inserting equation (18). For $\alpha = 1$, the related term contains the average of an odd number of factors $v$ yielding also a null result. The only nonvanishing result corresponds to $\alpha = 2$, such that
\[ L_\sigma = -4\pi \frac{15 \times 3!}{2 \times 7!} e_i \mathcal{P}_{ijkl} \left( \partial_i \partial_j \phi_0 \right) = -\frac{24\pi}{14} \mathcal{P}^{(3)} || \nabla^2 \phi_0. \] (25)

The volume integral limit is zero since
\[ \left( \nabla^4 || \mathcal{P}^{(3)} \right)_{(0)} = 0, \]
and it can be easily verified. Equation (25) implies
\[ \left( \nabla^4 || \mathcal{P}^{(3)} \right)_{(0)} = -\frac{24\pi}{14} \mathcal{P}^{(3)} || \nabla^2 \delta(r). \] (26)

In appendix B, this equality is proven for the general case. Let us write
\[ (E^{(3)}(r))_{(0)} = \frac{1}{24\pi \varepsilon_0} \left( \nabla^4 || \mathcal{P}^{(3)} \right)_{(0)} = -\frac{1}{14\varepsilon_0} \mathcal{P}^{(3)} || \nabla^2 \delta(r) - \frac{1}{2\varepsilon_0} \mathcal{A} || \nabla^2 \delta(r) \]
\[ \text{or} \]
\[ E^{(3)}(r) = (E^{(3)}(r))_{r \neq 0} - \frac{1}{14\varepsilon_0} \mathcal{P}^{(3)} || \nabla^2 \delta(r) - \frac{1}{2\varepsilon_0} \mathcal{A} || \nabla^2 \delta(r), \]
where
\[ (E^{(3)}(r))_{r \neq 0} = \frac{1}{24\pi \varepsilon_0} \nabla^4 || \mathcal{P}^{(3)} = \frac{1}{24\pi \varepsilon_0} \mathcal{P}^{(3)} || \nabla^2 \delta(r). \]

Even if only for the mathematical interest, we mention the possibility of expressing the multipole electrostatic field for arbitrary $n$. In appendix B, the general relation corresponding to equation (26) is introduced:
\[ \left( \nabla^{n+1} || \mathcal{P}^{(n)} \right)_{(r)} = -\frac{4\pi}{2n+1} \mathcal{P}^{(n)} || \nabla^{n-1} \delta(r), \]
such that
\[ (E^{(n)}(r))_{(0)} = \frac{(-1)^{n-1}}{4\pi \varepsilon_0 n!} \left( \nabla^{n+1} || \mathcal{P}^{(n)} + \frac{n(n-1)}{2} \nabla^{n-1} || \Delta \mathcal{A}^{(n-2)} \right)_{(0)} \]
\[ = \frac{(-1)^n}{\varepsilon_0} \left( \frac{1}{n!(2n+1)} \mathcal{P}^{(n)} + \frac{n-1}{2n!} \mathcal{A}^{(n-2)} \right) || \nabla^{n-1} \delta(r). \] (27)
We point out that for $n \geq 4$, for obtaining the complete singular structure of the field, we have to introduce the irreducible tensors in the expression from equation (27) corresponding to the $\delta$-singularities. This implies a recursive calculation introducing in equation (27) the STF projections of $\Lambda^{(n-3)}$, $\Lambda^{(n-4)}$, $\ldots$. For example, in the case $n = 4$, we have to reduce the tensor $\Lambda^{(4)} = \Lambda(\mathcal{P}^{(4)})$ writing
\[ \Lambda^{(4)}|\nabla^3 \delta(r) = T(\Lambda^{(2)})|\nabla^3 \delta(r) + \Lambda(\Lambda^{(2)}) \nabla \Delta \delta(r), \]
where
\[ \Lambda(\Lambda^{(2)}) = \frac{1}{160} \mathcal{P}_{qss}, \]
\[ T_{ij}(\Lambda^{(2)}) = \frac{1}{3}(\mathcal{P}_{qss} - \frac{1}{8} \delta_{ij}). \]

4. Singularities of the magnetostatic field

Let us write the multipole expansions of the vector potential $A(r)$ and of the magnetic field $B(r)$ in the static case:
\[ A(r) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla \times (\nabla^{n-1}|M^{(n)}) \]
and
\[ B(r) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \left[ \nabla \cdot (\nabla^{n-1}|M^{(n)}) - \Delta \left( \frac{\nabla^{n-1}|M^{(n)}}{r} \right) \right], \tag{28} \]
respectively. For $r \neq 0$, the multipole magnetic field $B(r)$ is expressed, as in the electric case, only by a scalar potential since the second term from the right-hand side of equation (28) vanishes. However, the $\delta$-singularity of the magnetic field is not similar to the corresponding singularity of the electric field, this second term giving the difference. The difference becomes easily obvious if one considers the fictitious magnetic shells (or sheets) employed in the Ampère formalism. It suffices to consider the case of the pointlike magnetic dipole which can be taken as the limit of a current loop of infinitesimal size. For finite dimensions, the field of the loop is derived from a scalar potential $\Phi_m$ which is defined by an integral on the corresponding sheet and having a 'jump' in all their points. Just this jump generates the $\delta$-singularity corresponding to the second term from equation (28). An explicit calculation of this limit when the loop concentrates in a point is given in [17].

In the dipolar case ($n = 1$), retaining the singular term, we have
\[ (B^{(1)}(r))_{(0)} = \frac{\mu_0}{4\pi} \left( e_i \partial_i \partial_j m_{j} \right)_{(0)} + 4\pi m \delta(r). \]
Processing the first term as in the electrostatic case, we obtain the well-known expression (2). No $\delta$-singularity is present in the dipolar vector potential.

Let us consider the quadrupole term from equation (28) and the corresponding singularities:
\[ (B^{(2)}(r))_{(0)} = -\frac{\mu_0}{8\pi} \left( \nabla^3 |M^{(2)}| - \nabla |M^{(2)}| \right)_{(0)} \]
By introducing the STF moment $\mathcal{M}^{(2)} = M^{(2)}$ defined by equation (11),
\[ (B^{(2)}(r))_{(0)} = -\frac{\mu_0}{8\pi} \left( \nabla^3 |M^{(2)}| - \frac{1}{2} e_i \delta_{j} \delta_{k} \partial_{j} \partial_{k} \frac{N_{j}}{r} r - e_i \partial_{j} \Delta \frac{M^{(2)}_{j}}{r} + \frac{1}{2} \delta_{ij} \partial_{k} \Delta \frac{N_{j}}{r} \right)_{(0)} \]
\[ = -\frac{3\mu_0}{10} \mathcal{M}^{(2)} \nabla \delta(r) - \frac{\mu_0}{4} N \times \nabla \delta(r), \]
where \( N = e_i N_i \). It follows that

\[
B^{(2)}(r) = (B^{(2)}(r))_{r \neq 0} - \frac{3\mu_0}{10} \mathcal{M}^{(2)} || \nabla \delta(r) - \frac{\mu_0}{4} N \times \nabla \delta(r) ,
\]

with

\[
(B^{(2)}(r))_{r \neq 0} = \left( -\frac{\mu_0}{8\pi} \nabla^3 \right) r_{\neq 0} = \left( -\frac{\mu_0}{8\pi} \nabla^3 \right) \mathcal{M}^{(2)} \]

(29)

Let us consider the quadrupolar vector potential \( A^{(2)}(r) \) with equation (11) already inserted:

\[
A^{(2)}(r) = -\frac{\mu_0}{8\pi} e_i \varepsilon_{ijk} \partial_j \partial_k \frac{M_k}{r} = -\frac{\mu_0}{8\pi} \nabla \left( \nabla || \mathcal{M}^{(2)} \right)
\]

\[
+ \frac{\mu_0}{16\pi} \nabla \left( \nabla \cdot \frac{N}{r} \right) - \frac{\mu_0}{16\pi} N \Delta \frac{1}{r} .
\]

(30)

The term proportional to \( \Delta (1/r) \) vanishes for \( r \neq 0 \). Whereas \( B^{(2)} \) is invariant to the substitution \( M \rightarrow \mathcal{M}^{(2)} \), this property is not verified by the vector potential \( A^{(2)} \). This substitution has as a result an additional gradient term, i.e. a gauge transformation of the potential. Though, by the extension to the entire space, the gradient term from expression (30) generates a \( \delta \)-singularity, this term gives no contribution to the singularities of the magnetic field. The singularities of \( B \) are, by definition, only the singularities corresponding to the curl of the vector potential. It remains still the problem of independence of the physical results on the \( A \) \( \delta \)-singularities. Maybe, the presence of such singularities in the expression of an interaction Hamiltonian corresponding to the density term \( J \cdot A \) can generate the problem of proving the gauge independence of physical results as in the case of the Aharonov–Bohm effect.

We can give, as in the electrostatic case, the general formula for the singularities of the magnetic field writing

\[
(B^{(n)}(r))_{(0)} = \left( \frac{(-1)^{n-1}\mu_0}{4\pi n!} \left( \nabla^{n+1} || \mathcal{M}^{(n)} \right) \right)_{(0)} ,
\]

(31)

The introduction of the symmetric moment \( \mathcal{M}^{(n)} \), equation (12), gives

\[
\nabla^{n+1} || \mathcal{M}^{(n)} \right) = \nabla^{n+1} || M^{(n)} \right) + \frac{1}{n} \varepsilon_{ilq} \partial_l \partial_q \sum_{k=1}^{n-1} \varepsilon_{ikl} N_{(l_1...l_{n-1}-q)}^{(n)} = \nabla^{n+1} || M^{(n)} \right) ,
\]

since in all the terms of the sum there are the contractions \( \varepsilon_{ilq} \partial_l \partial_q \) for \( l = 1, \ldots n - 1 \) which vanish. From equation (27), we can write a corresponding result for the magnetic field with the substitutions \( P^{(n)} \rightarrow \mathcal{M}^{(n)} \), \( P^{(n)} \rightarrow \mathcal{M}^{(n)} \), \( \varepsilon_0 \rightarrow 1/\mu_0 \):

\[
\left( \frac{(-1)^{n-1}\mu_0}{4\pi n!} \left( \nabla^{n+1} || \mathcal{M}^{(n)} \right) \right)_{(0)} = \left( \frac{(-1)^{n-1}\mu_0}{4\pi n!} \left( \nabla^{n+1} || \mathcal{M}^{(n)} \right) \right)_{(0)} + \frac{n(n-1)}{2n!} \mathcal{A}^{(n-2)} || \nabla^{n+1} \delta(r) \right) ,
\]

(32)

where \( \mathcal{A}^{(n-2)} = \mathcal{A}(\mathcal{M}^{(n)}) \). The insertion of equation (32) in equation (31) gives

\[
(B^{(n)}(r))_{(0)} = \left( \frac{(-1)^{n-1}\mu_0}{n!} \left( \frac{n}{2n+1} \mathcal{M}^{(n)} || \nabla^{n+1} \delta(r) \right) \right) + \frac{n(n-1)}{2n!} \mathcal{A}^{(n-2)} || \nabla^{n+1} \delta(r) - \nabla^{n+1} \delta(r) || \mathcal{M}^{(n)} \right) ,
\]

(33)
With this result, the separation of the $\delta$-singularities for the magnetic field can be considered finished. Obviously, the analysis of the higher-order singular terms is more complicated for the magnetic field than for the electric one, but it is a purely technical problem.

5. The dynamic case

Considering the $n$th-order multipole electric and magnetic fields, writing separately the contributions of the electric and magnetic multipoles,

$$E^{(n,p)}(r,t) = \frac{(-1)^n}{4\pi \varepsilon_0 n!} \left( \nabla^{n+1} || \mathbf{P}^{(n)}(\tau) \right) + \frac{1}{c^2} \nabla^{n-1} \mathbf{P}^{(n)}(\tau),$$  \hspace{1cm} (34)

and

$$B^{(n,p)}(r,t) = \frac{(-1)^n}{4\pi \varepsilon_0 c^2 n!} \nabla \times \left( \nabla^{n+1} || \mathbf{P}^{(n)}(\tau) \right),$$  \hspace{1cm} (35)

we search the corresponding $\delta$-singularities. Processing the different terms from equations (34) and (35) for reducing the moment tensors to the STF ones, we deal with the Laplace operator $\Delta$ applied to functions of the type $f(\tau)/r$. Since the equation verified by these functions is

$$\Delta f(\tau) - \frac{1}{c^2} \frac{\partial^2 f(\tau)}{\partial t^2} = -4\pi f(t) \delta(r),$$  \hspace{1cm} (36)

the processing is more complicated in the dynamic case compared with the static one.

For the electric field of the electric dipole, we write the singular part as

$$(E^{(1,p)}(r,t))_0 = \frac{1}{4\pi \varepsilon_0} \left[ \nabla \left( \nabla \cdot \mathbf{p}(\tau) \right) - \frac{1}{c^2} \frac{\dot{\mathbf{p}}(\tau)}{r} \right]_0 = \frac{1}{4\pi \varepsilon_0} e_i \left( \frac{\partial_i \partial_j p_j(\tau)}{r} \right)_0.$$

Note that the second term proportional to $\dot{\mathbf{p}}$ has no $\delta$-singularity. Equation (19) gives a result similar to that from the static case:

$$(E^{(1,p)}(r,t))_0 = -\frac{1}{3\varepsilon_0} \mathbf{p}(t) \delta(r).$$

The magnetic field $B^{(1,p)}(r,t)$ of the electric dipole has no $\delta$-singularity.

Obviously, the electric field $E^{(1,m)}(r,t)$ has no $\delta$-singularity, either. Writing the singular part of $B^{(1,m)}(r,t)$ as

$$(B^{(1,m)}(r,t))_0 = \frac{\mu_0}{4\pi} \left[ \nabla \left( \nabla \cdot \mathbf{m}(\tau) \right) - \frac{1}{c^2} \frac{\dot{\mathbf{m}}(\tau)}{r} + 4\pi \mathbf{m}(t) \delta(r) \right]_0,$$

one obtains the result similar to that from the static case:

$$(B^{(1,m)}(r,t))_0 = \frac{2\mu_0}{3} \mathbf{m}(t) \delta(r).$$
Let us consider the electric field $E^{(2, p)}(r, t)$ and search the corresponding $\delta$-singularities by introducing firstly the STF moment $\mathbf{P}^{(2)}$:

$$(E^{(2, p)}(r, t))_{(0)} = -\frac{1}{8\pi \varepsilon_0} \left[ \nabla^3 || \frac{\mathbf{P}^{(2)}(r)}{r} \right]_{(0)} - \frac{1}{c^2} \nabla || \frac{\mathbf{P}^{(2)}(r)}{r} + \nabla \left( \frac{\Delta \Lambda(t)}{r} \right) - \frac{1}{c^2} \nabla \frac{\dot{\Lambda}(t)}{r} .$$

The insertion of equation (36) gives

$$E^{(2, p)}(0) = -\frac{1}{8\pi \varepsilon_0} \left[ \nabla^3 || \frac{\mathbf{P}^{(2)}(r)}{r} - \frac{1}{c^2} \nabla || \frac{\mathbf{P}^{(2)}(r)}{r} - 4\pi \Lambda(t) \nabla \delta(r) \right]_{(0)} .$$

Expressing the singularity of the first term from the parenthesis by the insertion of equation (20), we can finally write

$$E^{(2, p)}(r, t) = (E^{(2, p)}(r, t))_{r \neq 0} + \frac{1}{8\pi \varepsilon_0} \mathbf{P}^{(2)}(t)[\nabla \delta(r) + \frac{1}{2\varepsilon_0} \Lambda(t) \delta(r)] ,$$

where

$$(E^{(2, p)}(r, t))_{r \neq 0} = -\frac{1}{8\pi \varepsilon_0} \left[ \nabla^3 || \frac{\mathbf{P}^{(2)}(r)}{r} - \frac{1}{c^2} \nabla || \frac{\mathbf{P}^{(2)}(r)}{r} \right]_{r \neq 0} .$$

The result (37) is similar to the result (24) from the static case. For the electric 4-polar term, the electric field expression for $r \neq 0$ is invariant to the substitution $\mathbf{P}^{(2)} \to \mathbf{M}^{(2)}$ as in the static case but, as we will see in the following, such property is not yet verified for higher-order multipoles in the dynamical case.

Searching the singularities corresponding to the electric field $E^{(2, m)}$ of the magnetic quadrupole, the difference from the static case is obvious. Let us express this field with the help of the STF magnetic moment $\mathbf{M}^{(2)} = M^{(2)}$:

$$E^{(2, m)}(r, t) = \frac{1}{8\pi \varepsilon_0 c^2} \nabla \times \left( \nabla || \frac{\mathbf{M}^{(2)}}{r} \right)$$

$$= \frac{1}{8\pi \varepsilon_0 c^2} \left[ \nabla \times \left( \nabla || \frac{\mathbf{M}^{(2)}}{r} \right) + \frac{1}{2} \epsilon_i \epsilon_{ijk} \epsilon_{qk} \partial_j \partial_q \frac{N_q(t)}{r} \right]$$

$$= \frac{1}{8\pi \varepsilon_0 c^2} \left[ \nabla \times \left( \nabla || \frac{\mathbf{M}^{(2)}}{r} \right) - \frac{1}{2} \nabla^2 || \frac{N(t)}{r} + \frac{1}{2} \frac{\Delta N(t)}{r} \right].$$

The same notation as in equation (29) is used. The singularities added by the extension of this expression to the entire space are the following:

$$\left[ \nabla \times \left( \nabla || \frac{\mathbf{M}^{(2)}}{r} \right) \right]_{(0)} = \epsilon_i \epsilon_{ijk} \partial_j \partial_k \frac{M_{jk}}{r} = -\frac{4\pi}{3} \epsilon_i \epsilon_{ijk} M_{jk}(t) \delta(r) = 0,$$

$$\left[ \nabla^2 || \frac{N(t)}{r} \right]_{(0)} = -\frac{4\pi}{3} \dot{N}(t) \delta(r) .$$

Related to the last term from the expression of $E^{(2, m)}$, we have to consider equation (36), i.e.

$$\Delta \frac{N(t)}{r} = \frac{1}{c^2} \frac{\dot{N}(t)}{r} - 4\pi \dot{N}(t) \delta(r) ,$$

$\Delta \frac{N(t)}{r}$
In equation (39), we have written separately the term which represents the delta-singularity 
\( \delta \) resulting in the expression of the electric field corresponding to an electric dipole with the moment 
\( \delta p' = -\frac{1}{4\varepsilon_0 c^2} \tilde{N} \).

In equation (39), we have written separately the term which represents the \( \delta \)-singularity corresponding to this dipole.

Let us express the field \( \mathbf{B}^{(2)}(r, t) \). For the part \( \mathbf{B}^{(2, p)} \), we insert the STF moment \( \mathcal{P}^{(2)} \) which gives

\[
\mathbf{B}^{(2, p)}(r, t) = \frac{\mu_0}{8\pi} \nabla \times \left( \mathbf{v} \left( \mathcal{M}^{(2)}(\tau) \right) \left( \frac{\mathcal{P}^{(2)}(\tau)}{r} \right) \right) = \frac{\mu_0}{8\pi} \nabla \times \left( \mathbf{v} \left( \mathcal{M}^{(2)}(\tau) \right) \left( \frac{\mathcal{P}^{(2)}(\tau)}{r} \right) + e_i \epsilon_{ijk} \partial_j \partial_k \Lambda(\tau) \delta_{ij} \right)
\]

(41)

since \( \epsilon_{ijk} \partial_j \partial_k \delta_{ij} = 0 \). It follows that \( \mathbf{B}^{(2, p)} \) is invariant to the substitution \( \mathcal{P}^{(2)} \rightarrow \mathbf{P}^{(2)} \) and has no \( \delta \)-singularities, due to equation (38) written for \( \mathbf{M}^{(2)} \rightarrow \mathbf{M}^{(2)} \).

The introduction of the STF magnetic moment \( \mathcal{M}^{(2)} = \mathbf{M}^{(2)} \) in the expression of \( \mathbf{B}^{(2, m)} \) results in

\[
\mathbf{B}^{(2, m)}(r, t) = -\frac{\mu_0}{8\pi} \left( \mathbf{v} \left( \mathcal{M}^{(2)}(\tau) \right) \left( \frac{\mathcal{M}^{(2)}(\tau)}{r} \right) \right) = -\frac{\mu_0}{8\pi} \left( \mathbf{v} \left( \mathcal{M}^{(2)}(\tau) \right) \left( \frac{\mathcal{M}^{(2)}(\tau)}{r} \right) + \frac{1}{2} \mathbf{v} \times \Delta \left( \frac{\mathcal{M}^{(2)}(\tau)}{r} \right) \right).
\]

Separating the \( \delta \)-singularities with the help of equations (20) and (36), we obtain

\[
\mathbf{B}^{(2, m)}(r, t) = -\frac{\mu_0}{8\pi} \left( \mathbf{v} \left( \mathcal{M}^{(2)}(\tau) \right) \left( \frac{\mathcal{M}^{(2)}(\tau)}{r} \right) - \frac{1}{c^2} \mathbf{v} \times \Delta \left( \frac{\mathcal{M}^{(2)}(\tau)}{r} \right) + \frac{1}{2c^2} \mathbf{v} \times \Delta \left( \frac{\mathcal{M}^{(2)}(\tau)}{r} \right) \right)_{r \neq 0}
\]

(42)

Since for \( r \neq 0 \) the magnetic field of the magnetic quadrupole can be written as

\[
\left( \mathbf{B}^{(2, m)}(r, t) \right)_{r \neq 0} = -\frac{\mu_0}{8\pi} \left( \mathbf{v} \left( \mathcal{M}^{(2)}(\tau) \right) \left( \frac{\mathcal{M}^{(2)}(\tau)}{r} \right) - \frac{1}{c^2} \mathbf{v} \times \Delta \left( \frac{\mathcal{M}^{(2)}(\tau)}{r} \right) \right)_{r \neq 0},
\]

we can conclude that for \( r \neq 0 \) this field is not invariant to the substitution \( \mathbf{M}^{(2)} \rightarrow \mathbf{M}^{(2)} \) and the additional term introduced by this substitution is equivalent to that of an electric dipole.
having the moment $\delta p'$ given by equation (40). Obviously, the corresponding term has no $\delta$-singularity such that it is not represented by a singular term in equation (42).

Let us go further to the field of the third-order electric multipole. Beginning with the electric field, we have to extend to the entire space the expression

$$E^{(3,p)}(r, t) = \frac{1}{24\pi \varepsilon_0} \left( \nabla^4 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| - \frac{1}{c^2} \nabla^2 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| \right).$$

We consider separately each term from the last equation introducing the STF moment $\mathcal{P}^{(3)}$:

$$\nabla^4 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| = \nabla^4 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| + e_i \partial_i \partial_{j} \partial_{k} \delta_{i} \mathcal{A}_{j}^{i} \left( \frac{r}{r} \right) = \nabla^4 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| + 3e_i \partial_i \partial_{j} \mathcal{A}_{j} \left( \frac{r}{r} \right).$$

Inserting equation (B.8) for the $\delta$-singularity of the first term from the right-hand side of the last equation and employing equation (36) for the second one, we can write

$$\nabla^2 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| = \left( \nabla^2 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| + 2 \nabla^2 \left| \frac{\mathcal{A}(r)}{r} \right| + \frac{\mathcal{A}(r)}{r} \right) - \frac{20\pi}{3} \mathcal{A}(r) \delta(r),$$

where $\Lambda = \Lambda e_i$. Regarding the second term from equation (43), we similarly obtain

$$\nabla^2 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| = \left( \nabla^2 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| + 2 \nabla^2 \left| \frac{\mathcal{A}(r)}{r} \right| + \frac{\mathcal{A}(r)}{r} \right) - \frac{20\pi}{3} \mathcal{A}(r) \delta(r),$$

since the first term containing $\nabla^2 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right|$ has no $\delta$-singularity. The insertion of equations (44) and (45) in equation (43) gives

$$E^{(3,p)}(r, t) = \frac{1}{24\pi \varepsilon_0} \left( \nabla^4 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| - \frac{1}{c^2} \nabla^2 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| \right)_{r \neq 0} + \frac{1}{4\pi \varepsilon_0} \left( \nabla^2 \left| \frac{\mathcal{A}(r)}{r} \right| - \frac{1}{6c^2} \mathcal{A}(r) \right)_{r \neq 0} - \frac{1}{14\varepsilon_0} \mathcal{P}^{(3)}(r) \nabla^2 \delta(r) - \frac{1}{2\varepsilon_0} \mathcal{A}(r) \nabla^2 \delta(r) + \frac{1}{9\varepsilon_0} \mathcal{A}(r) \delta(r).$$

From the above equation we can conclude that $E^{(3,p)}(r, t)$ is not invariant to the substitution $\mathcal{P}^{(3)} \rightarrow \mathcal{P}^{(3)}$. This substitution introduces an additional term which has precisely the expression of the electric field of an electric dipole of moment

$$\delta p'' = \frac{1}{6c^2} \mathcal{A}(r).$$

Let us express the magnetic field $B^{(3,p)}$:

$$B^{(3,p)}(r, t) = \frac{\mu_0}{24\pi} \nabla \times \left( \nabla^2 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| \right) = \frac{\mu_0}{24\pi} e_i e_{j k} \partial_j \partial_{k} \mathcal{P}_{i j k}(r).$$

The introduction of the STF moment $\mathcal{P}^{(3)}$ has as result

$$B^{(3,p)}(r, t) = \frac{\mu_0}{24\pi} \nabla \times \left( \nabla^2 \left| \frac{\mathcal{P}^{(3)}(r)}{r} \right| \right) + \frac{\mu_0}{24\pi} e_i e_{j k} \partial_j \partial_{k} \delta_{i j k} \mathcal{A}_k(r).$$
and since \( \nabla \times (\nabla^2||\mathcal{P}^{(3)}(\tau)/r) \) has no \( \delta \)-singularities, we can write after a simple algebraic calculation and after using equation (36) that

\[
B^{(3,p)}(r, t) = \left( \frac{\mu_0}{24\pi} \nabla \times (\nabla^2||\mathcal{P}^{(3)}(\tau)/r + \frac{\mu_0}{4\pi} \frac{1}{6c^2} \nabla \times \mathcal{A}(\tau) \right)_{r \neq 0} + \frac{\mu_0}{6c^2} \mathbf{\Lambda}(t) \times \nabla \delta(r).
\]

(48)

For \( r \neq 0 \), the additional term introduced by the substitution \( \mathcal{P}(3) \rightarrow \mathcal{P}^{(3)} \) in equation (47) represents the magnetic field corresponding to the electric dipole with the moment \( \delta p^\prime \) defined by equation (46).

Let us consider the electric moment up to \( n = 3 \), the magnetic one up to \( n = 2 \) and the corresponding sums of fields representing one of the first approximations of a complex system assimilated with a pointlike multipolar system. From the above results, as it was expected, we see that the substitutions of the primitive moments by the corresponding STF ones introduce some additional terms. The sum of these terms is equivalent to the substitution \( p \rightarrow \tilde{p} = p + \delta p \),

where

\[
\delta p = \delta p^\prime + \delta p^\prime' = -\frac{1}{c^2} \mathbf{t},
\]

and the vector \( \mathbf{t} \) is defined as

\[
\mathbf{t} = \frac{1}{4} \mathbf{N} - \frac{1}{6} \mathbf{\Lambda} = \frac{1}{10} \int_{\mathcal{D}} d^3x ((r . J)r - 2r^2 J).
\]

This vector is the so-called electric dipolar toroidal moment and represents a first term from a series of toroidal moments introduced by Dubovik et al [18–20] by generalizing Zeldovich’s original idea that a closed toroidal current represents a certain new kind of dipole [21]. The corresponding \( \delta \)-singularities can be easily identified in the expressions established above.

The procedure can be continued for the next orders, the technique being the same as in the case of the first orders treated above. Unlike in the static case, in the dynamic one it is more complicated to establish expressions of the singularities for an arbitrary \( n \). The recursive character involving a general number of \( n \) steps makes such relations difficult to derive and apply [15]. Furthermore, at the current level of applications of such results, we believe that it is mandatory to have expressions up to \( n = 3 \).

6. Conclusion

In this paper, we have expressed the \( \delta \)-singularities of the electric and magnetic multipoles. The results were given for arbitrary multipole orders \( n \) in the static case. In the dynamic case, the results correspond to the lower orders, but the calculation algorithm is presented in a manner that facilitates the continuation step by step of the processing to the next orders. The central idea of the paper is that, instead of employing the \( \delta \)-singularities of the functions \( f(\tau)/r \), it is more efficient, for the higher orders, to search directly such singularities for the multipole fields represented in terms of the STF moments. Working in Cartesian coordinates and employing a particular system of parameters and notation adapted to the technique of reducing the primitive Cartesian moments to the corresponding STF ones, we have stressed the significance of the first type of moments in the process of searching the field singularities. Thus, an important improvement is introducing terms with \( \delta \)-singularities when reducing the primitive tensors to STF tensors. This approach can have some significant consequences. For
example, one obtains new singular terms, not mentioned before in the literature. It may be possible that results from some applications, e.g. the study of the hyperfine interaction where higher-order multipoles are involved and where one uses only the symmetric and trace-free electric or magnetic moments, to be incomplete. A classical argumentation of the Fermi contact term firstly for the dipolar hyperfine interaction and secondly with generalization to arbitrary multipole–multipole hyperfine interactions, is encountered in the literature [1, 5, 22–24]. With some exceptions, the singularities are treated only in the cases of the electric and magnetic dipoles. In [5, 23, 24], the $\delta$-singularity of an electric point quadrupole is discussed, but characterized only by the traceless electric moment tensor $Q_{ij} = 3 P_{ij}$. The approach can be correct if the multipole is associated with an elementary particle case when the description only by irreducible tensors is, maybe, reasonable. In our opinion, in the case when at least a subsystem in the interaction considered is characterized by a finite spatial distribution of charges and currents, the reducible moments are unavoidable.

Acknowledgments

The authors would like to thank Dr R Paiu for her help with the documentation. The work of RZ was supported by grant ID946 (no 44/2007) of the Romanian National Authority for Scientific Research.

Appendix A. Some derivatives of $f(\tau)/r$

\[ \partial_i \ldots \partial_n \frac{f(\tau)}{r} = \sum_{l=0}^{n} \frac{1}{c^{n-l} l!} C_{i_1 \ldots i_n}^{(n,l)} \frac{d^{n-l} f(\tau)}{dr^{n-l}}, \quad \tau = t - \frac{r}{c}. \]

(A.1)

\[ C_{i}^{(0,0)} = 1. \]

(A.2)

\[ C_{i}^{(1,0)} = -v_i, \quad C_{i}^{(1,1)} = -v_i. \]

\[ C_{ij}^{(2,0)} = v_i v_j, \quad C_{ij}^{(2,1)} = 3 v_i v_j - \delta_{ij}, \quad C_{ij}^{(2,2)} = 3 v_i v_j - \delta_{ij}. \]

(A.3)

\[ C_{ijk}^{(3,0)} = -v_i v_j v_k, \quad C_{ijk}^{(3,1)} = -6v_i v_j v_k + \delta_{ij} v_k, \]

\[ C_{ijkl}^{(4,0)} = -15v_i v_j v_k v_l, \quad C_{ijkl}^{(4,1)} = -15v_i v_j v_k v_l + 3\delta_{i[j} v_k v_l]. \]

(A.4)

\[ C_{ijkl}^{(4,2)} = 45v_i v_j v_k v_l - 6\delta_{i[j} v_k v_l], \]

\[ C_{ijkl}^{(4,3)} = 105v_i v_j v_k v_l - 15\delta_{i[j} v_k v_l] + 3\delta_{i[j} \delta_{kl]}, \]

\[ C_{ijkl}^{(4,4)} = 105v_i v_j v_k v_l - 15\delta_{i[j} v_k v_l] + 3\delta_{i[j} \delta_{kl]}. \]

(A.5)
Appendix B. Some general formulas for STF tensors singularities

When searching the \( \delta \)-singularities of the electromagnetic fields, we have to calculate singularities of some derivatives of STF electric and magnetic moments as

\[
\nabla^{n+1} \left( \frac{T^{(n)}(r)}{r} \right), \quad \nabla \times \left( \nabla^{n-1} \left( \frac{T^{(n)}(r)}{r} \right) \right),
\]

where \( T^{(n)} \) is one of the STF electric or magnetic moments.

Let us search the \( \delta \)-singularities of the first expression for \( n \geq 2 \), starting from the definition

\[
\left( \nabla^{n+1} \frac{T^{(n)}(r)}{r} \right)_{(0)}, \quad \phi = \lim_{\varepsilon \to 0} e^{i \int_{\Sigma_1} dS_{\nu_i} \partial_i \cdots \partial_n T^{(n)}(\tau)_{(r)} \phi \big|_{(r)}}.
\]

Denoting \( L_\sigma \) the limit corresponding to the surface integral from the last equation, the insertion of equations (13), (17) gives

\[
L_\sigma = \lim_{\varepsilon \to 0} e^{i \sum_{l=0}^{n} \sum_{\alpha \geq 0} \sum_{\lambda \geq 0} (-1)^{\lambda} \varepsilon^{n-l+1+\lambda} \frac{e^{n-l+\lambda}}{\alpha!} x_\alpha \nu_1 \cdots \nu_{n-l+1} \frac{d^{n-l+\lambda}}{d^{n-l+\lambda} T_1 \cdots T_{n-l+1}(t) \left( \partial_{j_1} \cdots \partial_{j_{n-l+1}} \phi \right)_{(r)}}_{(0)}.
\]

Considering the expressions (14) of the coefficients \( C^{(n, l)} \), in equation (B.3), we can perform the substitutions

\[
C^{(n, l)}_{i_1 \cdots i_{n-l}} \to K^{(n, l)}_{0} v_{i_1} \cdots v_{i_{n}},
\]

without changing the result since the contractions of \( T_{i_1 \cdots i_{n-l}} \) with all the terms from the coefficients \( C^{(n, l)}_{i_1 \cdots i_{n-l}} \) containing at least a symbol Kronecker vanish. Expressing the integral in this equation as angular average, we can write

\[
L_\sigma = 4\pi e^{i \sum_{l=0}^{n} \sum_{\alpha \geq 0} \sum_{\lambda \geq 0} (-1)^{\lambda} \varepsilon^{n-l+1+\lambda} \frac{e^{n-l+\lambda}}{\alpha!} x_\alpha \nu_1 \cdots \nu_{n-l+1} \frac{d^{n-l+\lambda}}{d^{n-l+\lambda} T_1 \cdots T_{n-l+1}(t) \left( \partial_{j_1} \cdots \partial_{j_{n-l+1}} \phi \right)_{(r)}}_{(0)}.
\]

As we can see from equation (18), the contractions

\[
\{ v_{i_1} \cdots v_{i_{n-l}} v_{j_{1}} \cdots v_{j_{n-l}} \} T_{i_1 \cdots i_{n-l}}(t)
\]

are different from zero only if \( \alpha + 1 + n \geq 2n \), i.e.

\[
\alpha \geq n - 1.
\]

Let \( e = \alpha - l + 1 + \lambda \) be the power of \( \varepsilon \) in equation (B.4). Since \( l \leq n \) and \( \lambda \geq 0 \), the inequality (B.5) implies \( e \geq 0 \), the equality holding only for \( \alpha = n - 1, \ l = n, \ \lambda = 0 \). \( L_\sigma \neq 0 \) only in this case since for \( e > 0 \) the limit vanishes because of the positive values of the powers of \( \varepsilon \). Therefore, equation (B.4) becomes

\[
L_\sigma = 4\pi e^{i \sum_{l=0}^{n} \sum_{\alpha \geq 0} \sum_{\lambda \geq 0} (-1)^{\lambda} \varepsilon^{n-l+1+\lambda} \frac{e^{n-l+\lambda}}{\alpha!} x_\alpha \nu_1 \cdots \nu_{n-l+1} \frac{d^{n-l+\lambda}}{d^{n-l+\lambda} T_1 \cdots T_{n-l+1}(t) \left( \partial_{j_1} \cdots \partial_{j_{n-l+1}} \phi \right)_{(r)}}_{(0)}.
\]
As one can easily see from equation (18), the remaining contractions are different from zero only for the \( n! \) terms of the form \( \delta_{i_1 j_1} \cdots \delta_{i_{n-1} j_{n-1}} \delta_{i n} \) and inserting the value (15) of \( K_{0(n,n)} \) we can finally write

\[
L_\sigma = e_i \frac{(-1)^n 4\pi n}{2n+1} T_{i_1 \ldots i_{n-1}}(t) \left( \partial_{i_1} \cdots \partial_{i_{n-1}} \phi \right)_0 = \frac{(-1)^n 4\pi n}{2n+1} T^{(\sigma)}(t) \left( \nabla^{n-1} \phi \right)_0 .
\] (B.7)

The volume integral limit from equation (B.2) can be processed applying repeatedly the Gauss theorem. The new surface integral limit can be treated as in the previous case being represented by an equation similar to equation (B.4), with \( n \to n - 1 \). The contractions to be considered this time are

\[
\left[ v_{i_1} \cdots v_{i_k} v_{j_1} \cdots v_{j_k} \right] T_{i_1 \ldots i_k} .
\]

They are different from zero only if \( \sigma \geq n \) and, since \( l \leq n - 1 \) the exponents of \( \epsilon \) are \( e = \sigma - l + 1 + \sigma \geq 2 \) such that the limit of this integral for \( \epsilon \to 0 \) cancels. Therefore, equation (B.7) inserted in equation (B.2) has as the final result

\[
\left( \nabla^{n+1} \left[ \frac{T^{(\sigma)}(\tau)}{r} \right] \right)_0 = \frac{4\pi n}{2n+1} T^{(\sigma)}(t) \left| \nabla^{n-1} \delta(r) \right . .
\] (B.8)

The curl expression from equation (B.1) has no \( \delta \)-singularities and this result is obtained applying the same procedure as in the case of the first expression from this equation. Indeed, processing the corresponding surface integral limit, we can reduce it to terms having as factors the following contractions:

\[
\epsilon_{ijk} \left[ v_{i_1} \cdots v_{i_3} v_{j_1} \cdots v_{j_{n-2}} \right] T_{i_1 \ldots i_{n-1}}(t) .
\]

But

\[
\epsilon_{ijk} \left[ v_{i_1} \cdots v_{i_3} v_{j_1} \cdots v_{j_{n-2}} \right] T_{i_1 \ldots i_{n-1}}(t) \sim \epsilon_{ijk} T_{i_1 \ldots i_{n-1} j k} = 0 ;
\]

therefore,

\[
\left[ \nabla \times \left( \nabla^{n-1} \left[ \frac{T^{(\sigma)}(\tau)}{r} \right] \right) \right)_0 = 0 .
\] (B.9)

References

[1] Jackson J D 1975 *Classical Electrodynamics* 2nd edn (New York: Wiley)
[2] Frahm C P 1983 *Am. J. Phys.* 51 826
[3] Leung P T and Ni G J 2006 *Eur. J. Phys.* 27 N1
[4] Estrada R and Kanwal R P 1985 *Proc. R. Soc. A* 401 281
[5] Namias V 1987 *Int. J. Math. Sci. Technol.* 18 767
[6] Bowen J M 1994 *Am. J. Phys.* 62 511
[7] Hnizdo V 2006 arXiv:physics/0409072
[8] Weiglhofer W 1989 *Am. J. Phys.* 57 455
[9] Leung P T 2008 *Eur. J. Phys.* 29 137
[10] Thorne K S 1980 *Rev. Mod. Phys.* 52 299
[11] Applequist J 1989 *J. Phys. A: Math. Gen.* 22 4303
[12] Castellanos A, Panizo M and Rivas J 1978 *Am. J. Phys.* 46 1116
[13] Vrejoiu C 1984 *St. Cerc. Fiz.* 36 863 (in Romanian)
[14] Gonzales H, Juarez S R, Kielanowski P and Loewe M 1998 *Am. J. Phys.* 66 228
[15] Vrejoiu C 2002 *J. Phys. A: Math. Gen.* 35 9911
[16] Raub R E and De Lange O L 2005 *Multipole Theory in Electromagnetism* (Oxford: Clarendon)
[17] Corbò G and Massimo T 2009 *Am. J. Phys.* 77 818
[18] Dubovik V M and Tscheschkov A A 1974 Sov. J. Part. Nucl. 5 318
[19] Dubovik V M and Tugushev V V 1990 Phys. Rep. 187 145
[20] Rădescu E E and Vaman G 2002 Phys. Rev. E 65 046609
[21] Zeldovich Ya B 1958 Sov. Phys. JETP 6 1184
[22] Bucher M 2000 Eur. J. Phys. 21 19
[23] Karl G and Novikov V 2005 Fizika B 14 75
[24] Karl G and Novikov V 2006 arXiv:hep-ph/0602105