The Perturbative Method Fails in Non-Abelian Models

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It is shown that perturbation theory in 2\(D\) nonlinear \(\sigma\)-models as well gauge theories in dimension \(D \geq 2\) produces answers that depend on boundary conditions even after the infinite volume limit has been taken. This unphysical phenomenon occurs only in the non-Abelian versions of those models, starting at \(O(1/\beta^2)\). It is not present in the true (nonperturbatively defined) models and represents a failure of the perturbative method. It is related to a hitherto unnoticed type of low-lying excitation, dubbed super-instanton, that dominates the low-temperature (= weak coupling) regime of these models.

1. Introduction

At the conference \textit{Lattice 92} in Amsterdam we argued \cite{1} on the basis of percolation theoretic considerations that all 2\(D\) \(O(N)\) models must have at low temperature a soft spin wave phase characterized by power-like decay of correlations. This argument, while not fully rigorous, is intuitively compelling and was not challenged by anybody. Maybe the most striking conclusion to be drawn from it is that asymptotic scaling has to fail and that the perturbatively computed \(\beta\)-function does not describe correctly the variation of the correlation length with the bare coupling or its inverse \(\beta\).

How can that be? Only if, as we have been arguing for many years,\textsuperscript{3,4} perturbation theory (PT) does not produce the correct asymptotic expansion for expectation values in the infinite volume limit in those models. Several colleagues questioned this possibility, arguing that such a failure of PT should show up in PT itself. While this argument is not logically compelling, it induced us nevertheless to search for possible pathologies in PT. Here I want to report that we indeed found such a pathology, and in fact not only in 2\(D\) non-Abelian nonlinear \(\sigma\)-models but also in non-Abelian lattice Yang-Mills (LYM) models:

\textit{PT produces results that remain sensitive to boundary conditions (b.c.) even in the infinite volume limit, whereas the true (nonperturbatively defined) models do not show such a dependence.}

2. The Low-Lying Excitations

In order to motivate the b.c. we are studying, we first have to look at the low-lying excitations of these models, which will be relevant for the behavior of the Gibbs state at low temperature (= weak coupling).

Let us first look at the 2\(D\) \(O(N)\) models. Since the work of Kosterlitz and Thouless\textsuperscript{5} it has been assumed that for \(N = 2\) the crucial excitations are the vortices, which in isolation have an energy \(O(\ln L)\) (\(L\) is the linear size of the system), and which therefore will be bound into pairs at low temperature. Kosterlitz and Thouless also argued that for \(N = 3\) the relevant excitations are the instantons (even though that name was coined later), which have an energy \(O(L^2)\). But there is a different type of excitation which has arbitrarily low energy and which we therefore dubbed ‘super-instanton’. It will truly dominate at low temperature.

The super-instantons can be loosely described

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as follows: Fix a spin at the origin to a direction $e_o \in S^{N-1}$ and fix the spins at the boundary to a (generally different) direction $e_1$, forming an angle $\phi_o$ with $e_o$. The super-instanton is then the configuration of lowest energy satisfying these b.c.. It describes spins $s_i$ that turn gradually from $e_o$ to $e_1$ in such a way that the angle between $e_o$ and $s_i$ satisfies the lattice Laplace equation (as does the KT vortex). It is not hard to convince oneself that the energy $E$ of such a super-instanton will be $O(\phi_o^2/\ln L)$. Analogous configurations can also be constructed in other dimensions. In $1D$ the energy will be $O(1/L)$, whereas for $D \geq 3$ the minimal energy will satisfy $E \geq E_o > 0$.

The fact that in $D \leq 2$ the energy can be made arbitrarily small is responsible for the Mermin-Wagner theorem [4], which forbids spontaneous symmetry breaking (SSB): the abundance of super-instantons will disorder the system. In $D \geq 3$, on the other hand, super-instantons will be strongly suppressed at low temperature, and SSB will occur.

How about LYM models? It is well known that in the axial gauge ($U_{i,i+e_o} = 1$ for $\mu = 1$), LYM theory looks like a collection of $1D$ spin chains, where for $D \geq 3$ the chains are coupled via the plaquettes orthogonal to the 1-direction. The super-instantons in this case are simply the super-instantons of the corresponding spin chains. Of course they can also be described in a gauge invariant way as configurations which have a $L \times 1$ Wilson loop fixed to a (generally nontrivial) value. These super-instantons have an energy $O(1/L)$ in all $D \geq 2$ and enforce the analogue of the Mermin-Wagner theorem in gauge theories, which can be stated as the absence of SSB in the complete axial gauge and has been proven long ago [5].

The central point is that both in $2D$ $O(N)$ and in LYM models there are large fluctuations present at all values of the bare coupling and that these fluctuations make PT untrustworthy.

3. Perturbation Theory

In a fixed volume $L^D$, for large $\beta$, PT produces an asymptotic expansion in $1/\beta$ for quantities such as the energy:

$$E(L) = 1 - \frac{c_1(L)}{\beta} - \frac{c_2(L)}{\beta^2} + O(\frac{1}{\beta^3})$$

The conventional procedure is to take the limits $\lim_{L \to \infty} c_j(L)$ and hope that they will be the coefficients of the (unique if it exists) asymptotic expansion in the infinite volume limit. The point made here is that this procedure is ambiguous, because the limits depend on the b.c. used. For $1D$ $O(N)$ models this is a well known fact [6], but the belief has been [7] that for $2D$ spin models or LYM theory such an effect does not occur.

We will see this effect when we compare the conventional answers, obtained with periodic b.c., with those obtained with the so-called super-instanton b.c. (s.i.b.c.): the latter are motivated by the fact that it should be as legitimate to do PT around a super-instanton (which is a local minimum of the energy) as to do it around the trivial, completely ordered ground state.

For computational reasons we only use trivial s.i.b.c., where the turning angle $\phi_o$ in the spin models is 0 and the long thin Wilson loop in LYM theory is fixed to the identity. It is important to note that these are legitimate b.c. even though we fix a variable in the center of the lattice, because of the Mermin-Wagner theorem [8] (or its analogue in LYM [9]): Fixing one spin (or fixing one link variable) does not change any expectation values of invariant quantities. Fixing then in addition the boundary variables is clearly a b.c. and will leave no effect in the thermodynamic limit.

We computed with s.i.b.c. the PT coefficients $c_2(L)$ of order $1/\beta^2$ for the energy $E(L)$ in the $2D$ $O(N)$ models and in the $3D$ LYM model with gauge groups $U(1)$ and $SU(2)$, and compared them with the results obtained using periodic boundary conditions (p.b.c.). In the $O(N)$ model $E = \langle s_o \cdot s_1 \rangle$, whereas in the LYM model $E = 1/N \langle \text{tr} U p_1 \rangle$, where the plaquette $P_1$ is in the center of lattice, at the end of the Wilson loop that was fixed by the s.i.b.c.. We find the following:

1. Abelian models:

$$\lim_{L \to \infty} c_2^{s.i.b.c.}(L) = \lim_{L \to \infty} c_2^{p.b.c.}(L)$$

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2. Non-Abelian models:

$$\lim_{L \to \infty} c_2^{s.i.b.c.}(L) = \lim_{L \to \infty} c_2^{p.b.c.}(L)$$
The second order PT coefficients of the energy for 2D $O(N)$ models with super-instanton b.c.

For $O(2)$ we give $2c_2(L)$, for $O(3)$ $16c_2(L)$. The last column (labeled ‘p.b.c.’) gives the infinite volume limit obtained using periodic b.c., [1].

| $L$  | 8   | 10   | 12   | 16   | 20   | 30   | $\infty$ | p.b.c. |
|------|-----|------|------|------|------|------|----------|--------|
| $O(2)$ | .7973 | .8161 | .8265 | .8384 | .8455 | .8544 | .8599 | .8670 | 1.0 | 1.0 |
| $O(3)$ | 1.1283 | 1.1852 | 1.2171 | 1.2539 | 1.2759 | 1.3027 | 1.3193 | 1.3401 | 1.6663 | 1.0 |

The PT coefficients $c_2(L)$ for the energy of the plaquette $P_1$ computed with super-instanton b.c.
in LYM models with gauge group $U(1)$ and $SU(2)$. In the last column we give the infinite volume limits of the energy obtained with periodic b.c. [4].

| $L$  | 8   | 10   | 12   | 16   | 20   | 30   | $\infty$ | p.b.c. |
|------|-----|------|------|------|------|------|----------|--------|
| $U(1)$ | .03834 | .04152 | .04369 | .04648 | .04820 | .05056 | 1/18 | 1/18 |
| $SU(2)$ | .2536 | .2852 | .3061 | .3320 | .3472 | .3669 | .4063 | .2325 |

(2) Non-Abelian models:

$$\lim_{L \to \infty} c_2^{s.i.b.c.}(L) \neq \lim_{L \to \infty} c_2^{p.b.c.}(L)$$

In other words, PT fails for non-Abelian models, at least with some b.c.!

The computation was done, following a suggestion by A.Sokal, [9], with the use of an eigenfunction expansion. The numbers are given in Tab.1 for the 2D $O(2)$ and $O(3)$ models (for general $O(N)$ models see [1]) and in Tab.2 for 3D LYM with gauge groups $U(1)$ and $SU(2)$.

The $L \to \infty$ limits for the 2D models are obtained by fitting the data to a 3rd order polynomial in $1/\ln L$, which represents the numbers to all the digits given. For the LYM models the infinite volume limits are obtained by fitting to a 3rd order polynomial in $1/L$ which again represents the data to all digits given. For the spin models Niedermayer and Weisz, [10] confirmed Fact (2) above, going to even larger lattices.

A final remark concerns the $\beta$-function. Of course if we find such an effect for a short distance quantity such as the energy, it should not come as a surprise that also long range quantities are affected, in particular the $\beta$-function. We verified this for the 2D $O(N)$ model.

Our findings are described in more detail in two papers, [1] and [2].

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