SOME PROPERTIES OF A HILBERTIAN NORM FOR PERIMETER

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Abstract. We investigate a relationship first described in [FJ14] between the perimeter of a set and a related fractional Sobolev norm. In particular, we derive a new characterization of sets of finite perimeter, and demonstrate that the fractional Sobolev norm does not recover the $BV$ norm but rather a certain quadratic integral.

In a recent paper of Jerison and Figalli [FJ14], a relationship is developed between the perimeter of a set and a fractional Sobolev norm of its indicator function. More precisely, letting $\gamma(x)$ denote the standard Gaussian in $\mathbb{R}^n$, and defining the scaled Gaussian $\gamma_\varepsilon(x) = \varepsilon^{-n} \gamma(x/\varepsilon)$, Jerison and Figalli showed that

$$\limsup_{\varepsilon \to 0^+} \frac{1}{|\log \varepsilon|} \| \gamma_\varepsilon * 1_E \|_{H^{1/2}}^2 \approx \liminf_{\varepsilon \to 0^+} \frac{1}{|\log \varepsilon|} \| \gamma_\varepsilon * 1_E \|_{H^{1/2}}^2 \approx P(E).$$

Here $E$ is a set of finite perimeter and $1_E$ is its indicator function. The definition of the $H^{1/2}$ norm that we shall use for the paper is given in Section 1. The formula is remarkable for its quadratic scaling and for its apparent Fourier-analytic nature. The motivation for writing down this expression for the perimeter actually came from a purely geometric question about characterizing convex sets in terms of there marginals.

A very similar quantity has appeared in the literature before, in the foundational work of Bourgain, Brezis, and Mironescu [BBM01]. There the one-dimensional expression

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int \int_{|x-y| > \varepsilon} \frac{|f(x) - f(y)|^2}{|x-y|^2} \, dx \, dy$$

appears as a remark referencing an earlier unpublished work of Mironescu and Shafrir. The motivation for studying this functional came from studying $\Gamma$-limits of Ginzburg-Landau type functionals, as for example in [Kur06a, Kur06b]. More recently Poliakovsky introduced a similar functional in connection to the $\Gamma$-limit of the Aviles-Gila problem [Pol17]. Poliakovsky introduced the notion of $BV^q$ spaces and

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showed that a certain nonlocal functional very similar to that appearing in Equation (1) captures the $L^q$ norm of the jump set of a function.  

We mention also several other works which relate perimeter and total variation to nonlocal functionals. A paper of Leoni and Spector [LS11] studies a fractional Sobolev expression that recovers the total variation of a function. The main difference is that the integrand in their expression scales linearly in the function $u$, whereas the formula (1) scales quadratically. Similarly, a recent paper of Ambrosio, Bourgain, Brezis, and Figalli [ABBF16] introduced a very interesting $BMO$-type norm which recovers the perimeter of a set. In [FMS16] this norm was shown to also give the total variation of functions in $SBV$.

0.1. Summary of Results. In this paper we prove several results with the purpose of trying to better understand (1). The first question we investigate is what happens to sets that do not have finite perimeter. The answer comes in two parts. First, we show in Theorem 2.1 that there is a set $E \subset [0,1]^n$ for which the limit inferior in (1) vanishes. We do this by showing it is possible to construct a set of infinite perimeter such that, for an sequence $\varepsilon_k \to 0$ the functions $\gamma_{\varepsilon_k} * 1_E$ are much smoother than $\gamma_{\varepsilon_k}$. The construction is presented in Section 2.

The second part of the answer is that the limit superior does characterize sets of finite perimeter. This is proven in Section 3 using in a strong way the $L^2$ structure of the norm. The proof relies on the characterization of sets of finite perimeter provided in [ABBF16].

The next question this paper addresses is whether the limit in the expression (1) exists at all. From the previous results it is clear that the limit cannot always exist, as it is possible for

$$0 = \liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\gamma_{\varepsilon} * 1_E\|_{H^{1/2}}^2 < \limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\gamma_{\varepsilon} * 1_E\|_{H^{1/2}}^2 = \infty$$

when $E$ is a set of infinite perimeter. We show however that this cannot be the case when $E$ is a set of finite perimeter. Indeed, we prove in Theorem 4.1 that

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\gamma_{\varepsilon} * 1_E\|_{H^{1/2}}^2 = c_n P(E).$$

The argument consists of a few local calculations to cover the smooth case and is finished by classical structural results about sets of finite perimeter and a typical covering argument.

The reader is invited to compare the results of this paper with Theorem 1.1 of [Pol17]. It is not clear to the author if there are any direct implications in either direction, but the works certainly seem related.
The question of what happens to other functions in $BV$ remains open, and we are only able to give partial results. Our first result is that the leading order in the divergence of the $H^{1/2}$ norm only recovers information about the jump set. That is, under the condition that the total variation of a function in $u \in BV \cap L^{\infty}(\mathbb{R}^n)$ vanishes on all sets of Hausdorff dimension $n - 1$,

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\gamma_\varepsilon * u\|_{H^{1/2}}^2 = 0.$$  

This can be interpreted as a trichotomy that it interesting in its own right: if a function $u$ has sufficient energy at high frequencies (as measured by the $H^{1/2}$ norm of $\gamma_\varepsilon * u$), then it either fails to have bounded variation, it is unbounded, or it has a jump discontinuity. This result is stated in a more quantitative form as Theorem 5.1 in Section 5.

We are able to use this result to completely resolve the situation in one dimension. In fact, Theorem 5.2 implies that for $u \in BV(\mathbb{R})$

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\gamma_\varepsilon * u\|_{H^{1/2}}^2 = c_1 \sum_{x \in J_u} |u_+ (x) - u_- (x)|^2,$$

where $J_u$ is the jump part of $u$, as described for example in [EG15, Section 5.9] or [AFP00, Section 3.8].

This one-dimensional formula is suggestive of what should occur in higher dimensions, but we are not able to prove anything definitive here.

0.2. A note on organization. Preliminary results and basic definitions appear in Section 1. Each of the other sections can be read independently of each other, so the reader should feel free to skip to the result that is most interesting to them.

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1. Setup and Basic Estimates

First we define more clearly the Sobolev norm that we shall use. Given a smooth function $u \in \mathcal{S}(\mathbb{R}^n)$ with rapid decay, we define

$$\|u\|_{H^{1/2}}^2 = \int |\xi||\hat{u}(\xi)|^2 d\xi.$$
We would like to study the expression
\[ \frac{1}{|\log \varepsilon|} \| \gamma_\varepsilon * u \|_{H^{1/2}}^2 \]
for functions \( u \in BV \cap L^{\infty}(\mathbb{R}^n) \) by decomposing the contributions from different scales. To do this, we write
\[ \| \gamma_\varepsilon * u \|_{H^{1/2}}^2 = \sum_{k=1}^{\lfloor \log \varepsilon \rfloor} \varepsilon^{-1-2^{-k}} \| \varphi_{\varepsilon 2^k} * u \|_{L^2}^2 + O(\| u \|_{L^2}) \]
with the convolution kernel \( \varphi \) chosen such that \( \hat{\varphi}(\xi) \) is nonnegative and
\[ |\hat{\varphi}(\xi)|^2 = |\xi|(|\hat{\gamma}(\xi)|^2 - |\hat{\gamma}(2\xi)|^2). \]
Observe that \( |\xi|^{-3/2} \hat{\varphi}(\xi) \) is continuous and bounded, and that \( \hat{\varphi}(\xi) \) is smooth outside the origin. From this we deduce that \( f \) has the decay
\[ |\varphi(x)| \leq C(1 + |x|)^{-3-n}. \]
In particular, \( |x|\varphi \in L^1(\mathbb{R}^n) \). In addition, \( \varphi \) is smooth and satisfies the cancellation condition \( \int f = 0 \). These three conditions are sufficient for us to prove the following elementary estimates that we will use throughout the paper.

**Lemma 1.1.** Let \( f \) be any smooth function such that \( \int f = 0 \) and \( |x|f \in L^1(\mathbb{R}^n) \). Then we have the global \( L^{\infty} \) bound
\[ \| f_r * u \|_{L^\infty} \leq C \| u \|_{L^\infty}, \]
and the \( BV \) bound
\[ \frac{1}{r} \| f_r * u \|_{L^1} \leq C \| u \|_{BV}. \]
Moreover, for every \( \varepsilon > 0 \), there exists \( C(\varepsilon) \) such that the pointwise bound
\[ |f_r * u(x)| \leq C r^{1-n} |Du|(B_{C(\varepsilon)r}(x)) + \varepsilon \| u \|_{L^\infty}. \]
holds for any \( x \in \mathbb{R}^n \) and \( r > 0 \) and the local bounds
\[ \frac{1}{r} \| f_r * u \|_{L^1(A)} \leq C |Du|(A + B_{C(\varepsilon)r}) + \varepsilon \| u \|_{BV} \]
\[ \| f_r * u \|_{L^\infty(A)} \leq C \| u \|_{L^\infty(A + B_{C(\varepsilon)r})} + \varepsilon \| u \|_{L^\infty} \]
are satisfied for any measurable set \( A \subseteq \mathbb{R}^n \).

**Proof.** The first estimate, Equation (3), follows from the fact that \( f \in L^1 \) and the scale invariance \( \| f_r \|_{L^1} = \| f \|_{L^1} \).
We first prove the bound (4) for smooth functions. Using the fundamental theorem of calculus,

$$|u(x - y) - u(x)| \leq \int_0^1 |\nabla u(x - ty)||y|\,dt.$$  

Combining this with the cancellation condition \( \int f = 0 \), and then applying the change of variables \( w = ty \),

$$\frac{1}{r} |f_r \ast u(x)| \leq \frac{1}{r} \int_{\mathbb{R}^n} |u(x - y) - u(x)||f_r(y)|\,dy$$  

(8)  

$$\leq \frac{1}{r} \int_{\mathbb{R}^n} \int_0^1 |y||f_r(y)||\nabla u(x - ty)|\,dt\,dy$$  

$$= \frac{1}{r} \int_{\mathbb{R}^n} \int_0^1 |f_r(w/t)|t^{-n-1}|w||\nabla u(x - w)|\,dt\,dw.$$  

This is a convolution of \(|\nabla u|\) against the function \( g(r, w) \) defined by

$$g(r, w) = r^{-1}|w|\int_0^1 t^{-n-1}|f_r(w/t)|\,dt.$$  

The condition \(|x|f \in L^1\) implies that \( g \in L^1 \). Moreover we have the scaling relationship that \( g(r, w) = r^{-n}g(1, w/r) \). Now Young\'s inequality yields (4). To complete the proof for a general function \( u \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), form a sequence of smooth approximations \( u_k \) such that \( u_k \to u \) in \( L^1 \) and \( \|u_k\|_{BV} \to \|u\|_{BV} \).

We now prove the pointwise bound (5). Let \( \varepsilon > 0 \), and choose \( C(\varepsilon) \) so large that

$$\int_{|x|>Cr} |f_r(x)|\,dx = \int_{|x|>C} |f(x)|\,dx < \varepsilon/2.$$  

Using the cancellation condition again, and splitting the integral into parts,

$$|f_r \ast u(x)| \leq \int |u(x - y) - u(x)||f_r(y)|\,dy$$  

$$\leq \int_{|y|<Cr} |u(x - y) - u(x)||f_r(y)|\,dy + \varepsilon\|u\|_{L^\infty}.$$  

Define \( \tilde{f}_r(y) = f_r(y)1_{\{|y|<Cr\}} \), and notice that the first term can be written as

$$\int_{\mathbb{R}^n} |u(x - y) - u(x)||\tilde{f}_r(y)|\,dy.$$  

Now applying again the fundamental theorem of calculus and a change of variables as in (8), we obtain
\begin{equation}
\frac{1}{r} \int_{\mathbb{R}^n} |u(x-y) - u(x)| |\tilde{f}_r(y)| \, dy \leq \int_{\mathbb{R}^n} \tilde{g}(r, w) |\nabla u|(x-w) \, dw
\end{equation}
where
\[ \tilde{g}(r, w) = r^{-1} |w| \int_0^1 t^{-n-1} |\tilde{f}_r(w/t)| \, dt. \]
By scaling, \( \tilde{g}(r, w) = r^{1-n} g(1, w/r) \), so \( \|\tilde{g}(r, w)\|_{L^\infty} \leq Cr^{-n} \). Moreover \( \tilde{g}(r, w) \) has support in \( |w| < Cr \).

Multiplying both sides of (9) by \( r \), we therefore obtain
\[ \int_{\mathbb{R}^n} \tilde{g}(r, w) |\nabla u|(x-w) \, dw = \int_{|x-w| < Cr} \tilde{g}(r, w) |\nabla u|(x-w) \, dw \]
\[ \leq Cr^{1-n} |Du|(B_{Cr}(x)). \]

For the local \( L^1 \) bound (6) we write
\[ \frac{1}{r} |f_r * u(x)| \leq \int_{\mathbb{R}^n} \tilde{g}(r, w) |\nabla u|(x-w) \, dw + \int_{\mathbb{R}^n} (g - \tilde{g})(r, w) |\nabla u|(x-w) \, dw. \]
Choosing \( C(\varepsilon) \) larger if necessary, we may assume that \( \|g - \tilde{g}\|_{L^1} < \varepsilon \).

Now applying Young’s inequality and to both pieces and keeping track of the support yields (6). Finally the local \( L^\infty \) bound follows simply from the same type of argument. Writing
\[ |f_r * u(x)| \leq |\tilde{f}_r * u| + |(f - \tilde{f}) * u|. \]
and applying Young’s inequality to both sides completes the proof. \( \square \)

We conclude the section with a few basic properties of the limit of
\[ \frac{1}{r} \|f_r * u\|_{L^2(K)}^2 \] that will let us perform various decompositions. The first property is a kind of locality.

**Proposition 1.1.** For any closed set \( K \subset \mathbb{R}^n \),
\[ \limsup_{r \to 0} \frac{1}{r} \|f_r * u\|_{L^2(K)}^2 \leq C \|u\|_{BV(K)} \|u\|_{L^\infty(K)} \]

**Proof.** Applying (6) and (7) we certainly have for each \( \varepsilon > 0 \) that
\[ \limsup_{r \to 0} \frac{1}{r} \|f_r * u\|_{L^2(K)}^2 \leq \|u\|_{BV(K + B_\varepsilon)} \|u\|_{L^\infty(K + B_\varepsilon)} + \varepsilon \|u\|_{BV} \|u\|_{L^\infty}. \]
By monotone convergence and the fact that \( K = \bigcap_{\varepsilon>0}(K + B_\varepsilon) \), the result follows from taking \( \varepsilon \to 0 \). \( \square \)

The second result is that the functional is a continuity to perturbations.
Proposition 1.2. Let $u \in BV \cap L^\infty(\mathbb{R}^n)$. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $v \in BV \cap L^\infty(\mathbb{R}^n)$ satisfies
\[
\limsup_{r \to 0} \frac{1}{r} \| f_r * v \|_{L^2}^2 < \delta,
\]
then
\[
\limsup_{r \to 0} \frac{1}{r} \left| \| f_r * (u + v) \|_{L^2}^2 - \| f_r * u \|_{L^2}^2 \right| < \varepsilon.
\]

Proof. Using linearity of convolution and expanding the square, one has
\[
\frac{1}{r} \left| \| f_r * (u + v) \|_{L^2}^2 - \| f_r * u \|_{L^2}^2 \right| \leq \frac{1}{r} \| f_r * v \|_{L^2}^2 + \frac{1}{r} | \langle f_r * u, f_r * v \rangle |.
\]
The first term is bounded by $2\delta$ for sufficiently small $r$. We may apply Cauchy-Schwartz to the second term to obtain the bound
\[
\frac{1}{r} | \langle f_r * u, f_r * v \rangle | \leq \left( \frac{1}{r} \| f_r * u \|_{L^2} \right) \left( \frac{1}{r} \| f_r * v \|_{L^2} \right).
\]
By the global $L^1$ and $BV$ estimates of Lemma 1.1 the first term remains bounded, while the second term is eventually bounded by $2\sqrt{\delta}$ by hypothesis. Thus one can choose $\delta$ sufficiently small to conclude. \(\square\)

2. An example with infinite perimeter.

In this section we show that $\liminf_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \| \gamma_{\varepsilon} \ast 1_E \|_{H^{1/2}}^2$ does not characterize sets of finite perimeter. Indeed we show that one can find a set with infinite perimeter for which the $\liminf$ is zero.

Theorem 2.1. For any $n > 0$ there exists $E \subset \mathbb{R}^n$ with $P(E) = \infty$ and such that
\[
\liminf_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \| \gamma_{\varepsilon} \ast 1_E \|_{H^{1/2}}^2 = 0.
\]

This will follow from the result in one dimension.

Lemma 2.1. There exists a set $E \subset [0, 1]$ with $0 < |E| < 1$ and
\[
\liminf_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \| \gamma_{\varepsilon} \ast 1_E \|_{H^{1/2}}^2 = 0.
\]

We show now that Theorem 2.1 follows from the one-dimensional case.
Proof of Theorem 2.1 using Lemma 2.1. Choose $E \subset [0,1]$ according to Lemma 2.1. For $n > 1$, consider the Cartesian product $E^n \subset \mathbb{R}^n$. That is, the indicator function can be written

$$1_{E^n}(x_1, \ldots, x_n) = 1_E(x_1)1_E(x_2) \cdots 1_E(x_n).$$

We can use the fact that the Gaussian separates to estimate $\gamma_\varepsilon * 1_{E^n}$:

$$\|\gamma_\varepsilon * 1_{E^n}\|_{H^{1/2}}^2 = \int \cdots \int \left(\sum_j \xi_j^2\right)^{1/2} \prod_{i=1}^n \left|\widehat{\gamma_\varepsilon}(\xi_i)1_E(\xi_i)\right|^2 d\xi_i.
\leq \sum_{j=1}^n \int \cdots \int |\xi_j| \prod_{i=1}^n \left|\widehat{\gamma_\varepsilon}(\xi_i)1_E(\xi_i)\right|^2 d\xi_i.
= n\|\gamma_\varepsilon * 1_E\|_{H^{1/2}}^2 \|\gamma_\varepsilon * 1_E\|_{L^2}^{2(n-1)}
\leq n\|\gamma_\varepsilon * 1_E\|_{H^{1/2}}^2.
$$

In the last step we used the fact that $|E| < 1$. □

The rest of the section will be devoted to proving Lemma 2.1.

2.1. Plan for the construction. The idea of the construction is to design a sequence of smooth functions $\phi_k$ that act as the smoothed versions of the set $E$ at varying scales. That is, we would like

$$\gamma_\delta * 1_E \approx \gamma_\delta * \phi_k$$

for any $\delta \geq \delta_k$, where $\delta_k$ is a sequence of scales converging to zero. If the scales $\delta_k$ are sufficiently small, then because $\phi_k$ are smooth we should have

$$\frac{1}{|\log \varepsilon|} \|\gamma_{\delta_k} * 1_E\|_{H^{1/2}}^2 \approx \frac{1}{|\log \varepsilon|} \|\gamma_\delta * \phi_k\|_{H^{1/2}}^2 \approx 0.$$

To ensure that the smooth functions $\phi_k$ converge to a measurable set, we enforce that $0 \leq \phi_k \leq 1$ and that the sets $\{\phi_k = 0\}$ and $\{\phi_k = 1\}$ are strictly increasing. Moreover the functions $\phi_k$ face a compatibility condition whereby local averages of $\phi_{k+1}$ must match local averages of $\phi_k$. The compatibility condition is of the form

$$\gamma_{\delta_k} \phi_{k+1} \approx \gamma_{\delta_k} \phi_k.$$

To reconcile this with our need for the set $\{\phi_{k+1} \in \{0,1\}\}$ to increase, we construct $\phi_{k+1}$ to be highly oscillatory compared to the scale $\delta_k$, so that the $\delta_k$ smoothing recovers only the smoother function $\phi_k$. A cartoon of the first step of the construction is given in Figure 1.

The following definition quantifies the conditions outlined above.
Figure 1. The beginning of the compatible sequence \( \phi_k \). On the left, a smooth function \( \phi_1 \) is chosen which takes values in \([0, 1]\). On the right, the function \( \phi_2 \) is depicted on a magnified portion of the interval (colored in gray on the left). In this interval, \( \phi_2 \) oscillates between 0 and 1 in such a way to preserve the local averages of \( \phi_1 \).

**Definition 2.1.** Let \( \phi_k \in C^\infty_c((0, 1)) \) be a sequence of smooth functions and let \( \varepsilon_k > 0 \) be a decreasing sequence of scales with \( \varepsilon_k \to 0 \). We say that \( \phi_k \) is a compatible sequence if the following properties hold:

- **Nontriviality:**
  \[ 0 < |\{ \phi_1 = 1 \}|. \]

- **Convergence to a set:**
  \[ |\{ \phi_k \not\in \{0, 1\} \}| < (0.99)^k, \]
  \[ 0 \leq \phi_k \leq 1. \]

- **Smoothness to scale \( \varepsilon_k \):**
  \[ \frac{1}{|\log \varepsilon_k|} \| \gamma_{\varepsilon_k} \ast \phi_k \|_{H^{1/2}}^2 < 2^{-n}. \]

- **Compatibility across scales:** One has
  \[ \| \gamma_r \ast (\phi_k - \phi_{k+1}) \|_{L^2} < \varepsilon_k^2 2^{-k} \]
  for all \( r \geq \varepsilon_k \).

**Lemma 2.2.** Let \( \phi_k \) be a compatible sequence with scales \( \varepsilon_k \). Then there exists a measurable set \( E \subset [0, 1] \) with \( 0 < |E| < 1 \), \( \phi_k \to E \) in \( L^p \) for all \( p < \infty \), and

\[ \lim_{k \to \infty} \frac{1}{|\log \varepsilon_k|} \| \gamma_{\varepsilon_k} \ast 1_E \|_{H^{1/2}}^2 = 0. \]
Proof. The existence of the limiting set $E$ follows straightforwardly from the first two hypotheses on $\phi_k$.

To show that $\|\gamma_{\varepsilon_k} * 1_E\|_{H^{1/2}}^2$ grows manageable, we write $1_E$ as a telescoping sum and apply the triangle inequality:

$$\frac{1}{|\log \varepsilon_k|} \|\gamma_{\varepsilon_k} * 1_E\|_{H^{1/2}}^2 \leq \frac{2}{|\log \varepsilon_k|} \|\gamma_{\varepsilon_k} * \phi_k\|_{H^{1/2}}^2 + \frac{2}{|\log \varepsilon_k|} \left( \sum_{m=k}^{\infty} \|\gamma_{\varepsilon_k} * (\phi_{m+1} - \phi_m)\|_{H^{1/2}} \right)^2$$

The first term we bound using (11). For the terms in the sum we interpolate between $L^2$ and $H^1$,

$$\|\gamma_{\varepsilon_k} * (\phi_{m+1} - \phi_m)\|_{H^{1/2}} \leq \|\gamma_{\varepsilon_k} * (\phi_{m+1} - \phi_m)\|_{L^2}^{1/2} \|\gamma_{\varepsilon_k} * (\phi_{m+1} - \phi_m)\|_{H^1}^{1/2}$$

$$\leq (\varepsilon_m^2 2^{-m})^{1/2} (2\|\gamma_{\varepsilon_k}\|_{H^1})^{1/2}$$

$$\leq C(\varepsilon_m^2 2^{-m})^{1/2} (\varepsilon_k)^{-3/2}$$

The bound on the $H^1$ norm follows from the fact that $\|\phi_{m+1} - \phi_m\|_{L^1} < 2$. Since $\varepsilon_m < \varepsilon_k$ for $m > k$, this is bounded by $2^{-m/2}$, and is thus clearly summable over all $m \geq k$.

Thus our task is reduced to showing the existence of a compatible set. This will be done inductively in the next subsection.

2.2. Technical constructions. In this section we prove three short facts that allow us to construct $\phi_{n+1}$ from $\phi_n$. The first says that, in order to get the local approximation $\gamma_r * \phi_k \approx \gamma_r * \phi_k+1$, it suffices to demonstrate that the averages of $\phi_k$ and $\phi_k+1$ are equal on many short intervals. Then we show how to actually construct a function $\phi_{n+1}$ from $\phi_n$ such that the averages on short intervals are correct, with the constraint that $\phi_{n+1}$ takes values in $\{0, 1\}$ more often. This is done with the help of a short proposition that takes care of the case of one single interval.

**Proposition 2.1.** Let $\phi \in C^\infty_c((0, 1))$ with $0 \leq \phi \leq 1$, $\delta > 0$ be a scale, $\varepsilon > 0$ be some tolerance, and $k > 0$ be an integer. Then for sufficiently large $N$ we have the following: For every $\psi \in C^\infty_c((0, 1))$ with $0 \leq \psi \leq 1$, if

$$\phi\left(\frac{i}{N}\right) = N \cdot \int_{i/N}^{(i+1)/N} \psi(t) \, dt$$

for all but at most $k$ values of $i \in [N]$, then

$$\|\gamma_r * (\phi - \psi)\|_{L^\infty} < \varepsilon.$$

for all $r > \delta$. 
Proof. Let \( \chi \) be the indicator function for the interval \([0, 1]\). We will consider the function \( g = \chi_{M/N} * \psi \). We show that by choosing \( M \) large enough, and then \( N \) to be a sufficiently large multiple of \( M \), we can enforce \( \| g - \phi \|_\infty < \varepsilon/2 \).

Indeed, since (13) holds on all but at most \( k \) intervals,

\[
\left| g \left( \frac{i}{N} \right) - \frac{1}{M} \sum_{j=i}^{i+M-1} \phi \left( \frac{j}{N} \right) \right| \leq k/M.
\]

Assuming we take \( M \) to be sufficiently large, and then \( N \) to be a sufficiently large multiple of \( M \), we have \( |g(i/N) - \phi(i/N)| < \varepsilon/10 \).

The definition of \( g \) also yields the Lipschitz bound \( |g'(x)| \leq 2NM^{-1} \), so that we can conclude \( \| g - \phi \|_\infty \leq \varepsilon/2 \) as desired, provided again \( M \) is large enough.

Finally, we take \( N \) large enough that \( \| \gamma * (\phi - \psi) \|_{L^\infty} < \varepsilon/100 \) and \( \| \phi - \chi_{M/N} * \phi \|_{L^\infty} < \varepsilon/100 \).

Then we conclude since

\[
\| \gamma * (\phi - \psi) \|_{L^\infty} \leq \| (\gamma - \delta) * (\phi - \psi) \|_{L^\infty} + \| \delta * (\chi_{M/N} * (\phi - \psi)) \|_{L^\infty} < \varepsilon.
\]

The next proposition lets us satisfy the local average condition on an interval with a function that looks more like an indicator function.

**Proposition 2.2.** Let \( a \in [0, 1) \) be a target average. Then there exists a function \( \psi \in C_c^\infty((0, 1)) \) satisfying \( |\{ \psi \in \{0, 1\} \}| > 0.1 \) and \( \int_0^1 \psi = a \), and such that the sets \( \{ \psi = 0 \} \) and \( \{ \psi = 1 \} \) are unions of finitely many closed intervals.

**Proof.** We will split into the cases \( a > 1/2 \) and \( a < 1/2 \). We begin with the case \( a > 1/2 \). Let \( \sigma \) be a smooth increasing function satisfying \( \sigma(x) = 0 \) for \( x \leq 1/2 \) and \( \sigma(x) = 1 \) for \( x \geq 1 \).

Consider the following function \( \psi_t \in C_c^\infty((0, 1)) \) defined for \( 0 < t \leq 1/2 \):

\[
\psi_t(x) = \begin{cases} 
\sigma(x/t), & x < t \\
1, & x \in [t, 1-t] \\
\sigma((1-x)/t), & x > 1-t
\end{cases}.
\]

Each of \( \psi_t \) satisfy the condition \( |\{ \psi \in \{0, 1\} \}| > 0.1 \). Moreover \( I(t) = \int_0^1 \psi_t \) is a continuous function in \( t \) with \( I(1/2) < 1/2 \) and \( \lim_{t \to 0} I(t) = \).
1. Thus by the intermediate value theorem we have that, for any $a \in (1/2, 1)$, there exists $t$ such that $\int \psi_t = a$.

Now suppose $a \leq 1/2$. Observe that $I(0.1) > 1/2$, so the function $\psi = \frac{a}{\int(0.1)} \psi_{1/10}$ satisfies the constraints. \hfill $\square$

Finally we use Proposition 2.2 to construct a function satisfying the local average constraints of Proposition 2.1.

**Proposition 2.3.** Let $\phi \in C^\infty_c((0,1))$ satisfy $0 \leq \phi \leq 1$, and let $N > 0$. Suppose that the sets $\{ \phi = 1 \}$ and $\{ \phi = 0 \}$ can be written as unions of at most $k$ intervals. Then there exists $\psi \in C^\infty_c((0,1))$ satisfying the following constraints:

- $\{ \phi = 1 \} \subset \{ \psi = 1 \}$ and $\{ \phi = 0 \} \subset \{ \psi = 0 \}$.
- $|\{ \psi \notin \{0,1\}\}| \leq 0.99 \cdot |\{ \phi \notin \{0,1\}\}|$.
- The level sets $\{ \psi = 0 \}$ and $\{ \psi = 1 \}$ can each be written as a union of finitely many intervals.
- The function $\psi$ satisfies the following local average constraints

$$\phi \left( \frac{i}{N'} \right) = N' \int_{i/N'}^{(i+1)/N'} \psi(t) \, dt$$

for some $N' > N$, and on all but at most $2k$ intervals.

**Proof.** Choose $N' > N$ such that each interval $[i/N', (i + 1)/N']$ contains at most point in $\partial \{ \phi = 0 \} \cup \partial \{ \phi = 1 \}$. Let $I_i$ be the interval $[i/N', (i + 1)/N']$. Let $A$ be the set of indices such that their intervals contain such an endpoint, that is

$$A := \{ i; I_i \cap (\partial \{ \phi = 0 \} \cup \partial \{ \phi = 1 \}) \neq \emptyset \}.$$

We will define functions $F_i \in C^\infty([0,1])$ for $0 \leq i \leq N$ and set

$$\psi(x) = F_{\lfloor N \cdot x \rfloor}(\text{frac}(N \cdot x))$$

where frac($x$) denotes the fractional part of $x$. We split the choice of $F_i$ into three cases.

**Case I:** $i \notin A$ and $I_i \subset \{ \phi = 1 \}$. In this case we simply set $F_i = 1$.

**Case II:** $i \notin A$ and $\phi(i/N') < 1$. Simply use Proposition 2.2 to choose $F_i$ such that $\int F_i = \phi(i/N')$.

**Case III:** $i \in A$. Choose any $F_i$ subject to the constraints $0 \leq F_i \leq 1$, $\phi \in C^\infty_c$, $\{ \phi = 1 \} \subset \{ \psi = 1 \}$ and $\{ \phi = 0 \} \subset \{ \psi = 0 \}$.

Our choice of $N'$ guarantees that the above three cases are exhaustive. The resulting function $\psi$ satisfies all the conditions of the lemma. \hfill $\square$
2.3. The iterative algorithm. In this section we combine the main lemmas above to inductively define a compatible sequence $\phi_n$.

Proof of Lemma 2.1. Using Lemma 2.2, it suffices to construct a compatible sequence. We begin with any valid function $\phi_1 \in C^\infty_c((0,1))$ satisfying the nontriviality constraint $|\{\phi = 1\}| > 0$ and such that the sets $\{\phi_1 = 1\}$ and $\{\phi_1 = 0\}$ are finite unions of closed intervals. Since $\phi_1$ is smooth, and thus in $H^{1/2}$ we have

$$\lim_{\varepsilon \to 0^+} \frac{1}{|\log \varepsilon|} \|\gamma_\varepsilon * \phi_1\|_{H^{1/2}}^2 \to 0,$$

so we may choose $\varepsilon_1$ small enough to satisfy the smoothness constraint (11).

We now induct on $k$. Suppose that the sets $\{\phi_k = 1\}$ and $\{\phi_k = 0\}$ are unions of at most $K$ intervals. Applying Proposition 2.1 with $\phi = \phi_k$, $\delta = \varepsilon_k$, $\varepsilon = \varepsilon_k 2^{-k}$, and $k = K$, we obtain a value $N_k$ for which the interval average constraints (13) imply the compatibility bound (12). We can then use Proposition 2.3 with $\phi = \phi_k$ and $N_k$ to construct $\phi_{k+1}$. The function $\phi_{k+1}$ is smooth, so we can find $\varepsilon_{k+1}$ to satisfy (11). Moreover, we have that the sets $\{\phi_{k+1} = 1\}$ and $\{\phi_{k+1} = 0\}$ are finite unions of closed intervals, so the induction is closed. \qed

3. Characterizing Sets of Finite Perimeter

3.1. The lower bound. In this section we prove the following characterization of sets of finite perimeter.

Theorem 3.1. Let $E \subset \mathbb{R}^n$ be a set with $P(E) = \infty$. Then

$$\limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\gamma_\varepsilon * 1_E\|_{H^{1/2}}^2 = \infty.$$ 

The proof of this theorem goes through an analysis of the smoothed functions $\gamma_\varepsilon * 1_E$. The difficulty is that these functions may be so smooth that $\|\gamma_\varepsilon * 1_E\|_{H^{1/2}}$ could be very small. However, using a characterization of sets of finite perimeter due to Ambrosio, Bourgain, Brezis, and Figalli, we will be able to see that

$$\varepsilon^{-1}\|1_E - \gamma_\varepsilon * 1_E\|^2_{L^2},$$

grows to be large if $E$ is a set of infinite perimeter [ABB16]. Decomposing the difference $1_E - \gamma_\varepsilon * 1_E$ over many scales in the Fourier domain, we will see that there must be at least some wavelength $\varepsilon' < \varepsilon$ that contributes significantly to the difference. It is at this wavelength that $\|\gamma_{\varepsilon'} * 1_E\|_{H^{1/2}}^2$ is large.
To make this analysis convenient, we will make our smoothing kernels compactly supported in Fourier space. That is, let $\psi \in C^\infty_c(\mathbb{R})$ have support in $[-1, 1]$ with $\psi(\xi) = 1$ for $|\xi| < 1/2$. Then by construction, the differences $(\psi_r - \psi_{r/2}) * 1_E$ and $(\psi_h - \psi_{h/2}) * 1_E$ are orthogonal so long as $r \notin (h/4, 4f)$. From this we deduce the following approximate orthogonality property:

$$
\|\psi_r * 1_E - 1_E\|_{L^2}^2 \leq C \sum_{k=0}^{\infty} \|\psi_{r/2^k} * 1_E - \psi_{r/2^k+1} * 1_E\|_{L^2}^2.
$$

Next we demonstrate the connection between these differences and the quantity $\|\gamma_\varepsilon * 1_E\|_{H^{1/2}}$ via the kernel described in (2).

**Proposition 3.1.** With $\varphi$ as defined by Equation (2), we have

$$
\|(\psi_r - \psi_{r/2}) * 1_E\|_{L^2}^2 \leq C \|\varphi_r * 1_E\|_{L^2}^2
$$

for any measurable set $E \subset \mathbb{R}^n$.

**Proof.** By Plancherel’s theorem and homogeneity it suffices to check that there exists some constant $C$ such that

$$
\left| \hat{\psi}(\xi) - \hat{\psi}(\xi) \right|^2 \leq C \left| \hat{\varphi}(\xi) \right|^2
$$

for all $\xi \in \mathbb{R}^n$. By construction of $\varphi$, the left hand side has support in the annulus $\frac{1}{4} \leq r|\xi| \leq 1$. The result follows from the compactness of the annulus and the positivity of $\hat{\varphi}$ on $\mathbb{R}^n \setminus \{0\}$. $\square$

**Lemma 3.1.** Suppose that $E \subset \mathbb{R}^n$ satisfies

$$
\limsup_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \|\gamma_\varepsilon * 1_E\|_{H^{1/2}}^2 < \infty.
$$

Then

$$
\liminf_{n \to \infty} 2^n \sum_{k=n}^{\infty} \|\varphi_{2^{-k}} * 1_E\|_{L^2}^2 < \infty.
$$

**Proof.** Using the definition of $\varphi$, the condition on $E$ implies that there exists $C$ such that for every integer $n > 0$,

$$
\sum_{k=1}^{n} 2^k \|\varphi_{2^{-k}} * 1_E\|_{L^2}^2 < Cn.
$$

We first use this inequality to bound the infinite sum in (15) in terms of a finite one. Indeed, by grouping the infinite sum into dyadic pieces
and applying the bound above in each piece, we have

$$2^n \sum_{k=2n}^{\infty} \|\varphi_{2-k} \ast 1_E\|_{L^2}^2 \leq 2^n \sum_{N=1}^{\infty} \sum_{k=n2^N}^{n2^{N+1}} \|\varphi_{2-k} \ast 1_E\|_{L^2}^2$$

$$\leq 2^n \sum_{N=1}^{\infty} 2^{-n2^N} \sum_{k=n2^N}^{n2^{N+1}} 2^k \|\varphi_{2-k} \ast 1_E\|_{L^2}^2$$

$$\leq C2^n \sum_{N=1}^{\infty} 2^{-n2^N} n2^{N+1}$$

which is clearly bounded independently of $n$. Thus, it suffices to show that

$$\liminf_{n \to \infty} 2^n \sum_{k=n}^{2n} \|\varphi_{2-k} \ast 1_E\|_{L^2}^2 < \infty.$$  

We do this by finding, for each $n > 0$, a suitable scale $n \leq m \leq 2n$. To see that at least one $m$ suffices we average over all such scales:

$$\frac{1}{n} \sum_{m=n}^{2n} 2^m \sum_{k=m}^{2m} \|\varphi_{2-k} \ast 1_E\|_{L^2}^2 \leq \frac{1}{n} \sum_{k=n}^{4n} \|\varphi_{2-k} \ast 1_E\|_{L^2}^2 \sum_{m=0}^{2n} 2^m$$

$$\leq \frac{2}{n} \sum_{k=n}^{4n} 2^k \|\varphi_{2-k} \ast 1_E\|_{L^2}^2.$$  

The last sum is bounded by using again (16). Thus it is possible to find $m > n$ such that

$$2^m \sum_{k=m}^{2m} \|\varphi_{2-k} \ast 1_E\|_{L^2}^2$$

is bounded independent of $n$. \hfill \Box

The following lemma is a characterization of sets of finite perimeter that appears in [ABBFI6] that we will rely on. A $\delta$-cube is any cube in $\mathbb{R}^n$ with side length $\delta$.

**Lemma 3.2 ([ABBFI16, Lemma 3.2]).** Let $K > 0$ and $E \subset \mathbb{R}^n$ be a measurable set with $P(E) = \infty$. Then there exists $\delta_0 = \delta_0(K, A)$ such that for every $\delta < \delta_0$ it is possible to find a disjoint collection $\mathcal{U}_\delta$ of $\delta$-cubes $Q'$ with $\# \mathcal{U}_\delta > K \delta^{-n+1}$ and

$$2^{-n-1} \leq \frac{|Q' \cap E|}{|E|} \leq 1 - 2^{-n-1}$$

for every $Q' \in \mathcal{U}_\delta$. 

Proposition 3.2. Let $Q = (-\frac{1}{2}, \frac{1}{2})^n \subset \mathbb{R}^n$ be the unit cube. Suppose that $E \subset \mathbb{R}^n$ is a measurable set with $2^{-n-1} \leq |E \cap Q| \leq 1 - 2^{-n-1}$. Then there exists constants $c_n, r_n > 0$ such that
\[ \| \psi_{r_n} * 1_E - 1_E \|_{L^2(Q)} > c_n. \]

Proof. We choose $r_n$ so small that
\[ \| \psi_{r_n} * 1_Q - 1_Q \|_{L^1} < 2^{-n-2}. \]
With this choice for $r_n$,
\[ \left| \int_Q \psi_{r_n}(x) dQ - |E \cap Q| \right| = \left| \int_E (\psi_{r_n} - 1_Q) dQ \right| < 2^{-n-2}. \]

It follows from the continuity of $\psi_{r_n} * 1_E$ that for some point $x_0 \in Q$, $\psi_{r_n} * 1_E(x_0) \in (2^{-n-2}, 1 - 2^{-n-2})$. Since $\| \nabla \psi_{r_n} \|_{L^\infty} < r_n^{-1}$, one has
\[ 2^{-n-2} \leq \psi_{r_n} * 1_E(y) < 1 - 2^{-n-2} \]
for any $|y - x_0| < 2^{-n-2}r_n$. In particular,
\[ |\psi_{r_n} * 1_E(y) - 1_E(y)| > 2^{-n-2} \]
for $y \in B_{2^{-n-2}r_n}$. The claim follows upon integrating the above bound over $B_{2^{-n-2}r_n} \cap Q$. \qed

The above lemmas combine in a straightforward manner to prove our main result for this section.

Proof of Theorem 3.1. Suppose that $E \subset \mathbb{R}^n$ is a set with
\[ \limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \| \gamma_{\varepsilon} * 1_E \|_{H^{1/2}}^2 < \infty. \]
We will then show that
\[ \liminf_{\delta > 0} \delta^{-1} \| \psi_{\delta} * 1_E - 1_E \|_{L^2} < \infty. \]
To show that this implies that $E$ is a set of finite perimeter, let $\delta > 0$ be very small and such that
\[ \delta^{-1} \| \psi_{\delta} * 1_E - 1_E \|_{L^2} < C. \]
Consider a collection $\mathcal{U}_{\delta/r_n}$ of $\delta/r_n$-cubes such that $|Q' \cap E| / |Q'| \in (2^{-n-1}, 1 - 2^{-n-1})$ for all $Q' \in \mathcal{U}_\delta$. Appropriately scaling the conclusion of Proposition 3.2,
\[ \| \psi_{\delta} * 1_E - 1_E \|_{L^2(Q')}^2 > \delta^n c_n \]
for all $Q' \in \mathcal{U}_\delta$. In particular,
\[ \delta^n c_n \# \mathcal{U}_\delta < \| \psi_\delta \ast 1_E - 1_E \|_{L^2}^2 < \delta C. \]
Thus $\# \mathcal{U}_\delta < K\delta^{1-n}$ for some $K$. Since this holds for arbitrarily small $\delta$, it follows from Lemma 3.2 that $P(E) < \infty$.

Now we prove (17). Indeed, according to Lemma 3.1, we may find $C > 0$ and arbitrarily large $n$ such that
\[ 2^n \sum_{k=n}^{\infty} \| \varphi_{2^{-k}} \ast 1_E \|_{L^2}^2 < C. \]
Setting $\delta = 2^{-n}$ and applying Proposition 3.1 and the orthogonality property (14) we obtain
\[ \| \psi_{2^{-n}} \ast 1_E - 1_E \|_{L^2}^2 \leq C \sum_{k=n}^{\infty} \| \varphi_{2^{-k}} \ast 1_E \|_{L^2}^2 \leq 2^{-n} C \]
as desired. \[ \square \]

4. APPROXIMATING THE PERIMETER OF A SET

In this section we consider the case where $E \subset \mathbb{R}^n$ is a set of finite perimeter and calculate the limit $\lim_{r \to 0^+} \frac{1}{r} \| f_r \ast 1_E \|_{L^2}^2$ for kernels $f$ which satisfy the conditions of Lemma 1.1.

**Theorem 4.1.** Let $f \in C^\infty(\mathbb{R}^n)$ be a smooth function with $\int f = 0$ and $\int |x||f| \, dx < \infty$. Then for any set $E \subset \mathbb{R}^n$ of finite perimeter,

\[ \lim_{r \to 0^+} \frac{1}{r} \| f_r \ast 1_E \|_{L^2}^2 = \int_{\partial^* E} F(\nu) \, d\mathcal{H}^{n-1}. \]

where the function $F \in C^{0,1}(S^{n-1})$ is given by the expression in (22).

**Remark 4.1.** Choosing $f = \varphi$ as defined in (2), we can conclude that for a set of finite perimeter $E$,

\[ \lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \| \gamma_\varepsilon \ast 1_E \|_{H^{1/2}}^2 = c_n P(E). \]

The proof of Theorem 4.1 consists of a covering argument using De Giorgi’s structure theorem, which reduces the problem to the following local computation. The local computation takes place over small balls.
around points in the reduced boundary $\partial^* E$, where $E$ looks like a half-plane. To set up the notation, for each $\nu \in S^{n-1}$ define the half plane

$$H_{\nu} := \{ x \in \mathbb{R}^n \mid x \cdot \nu \leq 0 \},$$

and let $B$ be the unit ball centered at the origin.

**Lemma 4.1.** Let $f$ be as in Theorem 4.1 and let $\varepsilon > 0$. Then there exists $\delta > 0$ and $\alpha > 0$ such that the following holds: If $E \subset \mathbb{R}^n$ is a set of finite perimeter which is close to a half-space $H_{\nu_0}$ in the sense that $|(E \Delta H_{\nu_0}) \cap B| < \delta$ and $P(E; B) - \omega_{n-1} < \delta$, then

$$\limsup_{r \to 0^+} \left| \frac{1}{r} \| f_r \ast 1_E \|_{L^2(B)}^2 - \int_{\partial^* E} F(\nu) \, d\mathcal{H}^{n-1} \right| < \varepsilon^{\alpha},$$

where $\nu$ denotes the unit normal to $\partial^* E$.

Before proving this lemma, we first show how it implies Theorem 4.1.

**Proof of Theorem 4.1 using Lemma 4.1.** Let $\varepsilon > 0$, and choose $\delta > 0$ according to Lemma 4.1. For each $x \in \partial^* E$ and $\rho > 0$, we say that the ball $B_\rho(x)$ is $\delta$-acceptable if

$$|(E \Delta H_{\nu}(x)) \cap B_\rho(x)| < \rho^n \delta$$

and $P(E; B_\rho(x)) - \omega_{n-1} \rho^{n-1} < \rho^{n-1} \delta$. (19)

De Giorgi’s structure theorem ensures that the set $B_\delta$ of $\delta$-acceptable balls is a Vitali covering of $\partial^* E$. Thus we may choose a finite set $\{B_\rho_j(x_j)\}_{j=1}^N$ of disjoint balls such that

$$\mathcal{H}^{n-1}(\partial^* E \setminus \bigcup_j B_\rho_j(x_j)) < \varepsilon.$$

Let $R$ denote the remainder set $\mathbb{R}^n \setminus \bigcup_{j=1}^N B_\rho_j(x_j)$. The set $R$ is closed and $\|1_E\|_{BV(R)} = P(E; R) < \varepsilon$, so Proposition 1.1 guarantees that

$$\limsup_{r \to 0} \frac{1}{r} \| f_r \ast 1_E \|_{L^2(R)}^2 < C\varepsilon,$$

and so trivially

$$\limsup_{r \to 0} \left| \frac{1}{r} \| f_r \ast 1_E \|_{L^2(R)}^2 - \int_{\partial^* E \cap R} F(\hat{n}) \, d\mathcal{H}^{n-1} \right| < C\varepsilon^{\alpha}.$$

On each $B_\rho_j(x_j)$, scaling and applying Lemma 4.1 yields

$$\limsup_{r \to 0} \left| \frac{1}{r} \| f_r \ast 1_E \|_{L^2(B_\rho_j(x_j))}^2 - \int_{\partial^* E \cap B_\rho_j(x_j)} F(\hat{n}) \, d\mathcal{H}^{n-1} \right| < \rho_j^{n-1} \varepsilon^{\alpha}.$$
Summing everything together we obtain
\[ \lim_{r \to 0} \frac{1}{r} \| f_r \ast 1_E \|_{L^2(B^n)}^2 - \int_{\partial^* E} F(\hat{n}) \, d\mathcal{H}^{n-1} \|_{\mathcal{L}^2(\mathbb{R}^n)} \leq C \varepsilon^\alpha (1 + \sum_j \rho_j^{n-1}). \]

Using again that $B_{\rho_j}(x_j)$ was delta acceptable, $\rho_j^{n-1} \leq 2P(E; B_{\rho_j}(x_j))$. The sum on the right is therefore bounded by $2P(E)$. The conclusion follows in the limit $\varepsilon \to 0$.

The rest of this section is dedicated to proving Lemma 4.1. This is done in a series of steps. First, we consider the case that $E$ is a half-space, and so derive a formula for $F(\nu)$. Second, we allow $E$ to be given by a $C^1$ surface whose normal is everywhere close to $\nu$, and perform a computation which involves changing coordinates. Finally, we prove an approximation lemma which allows us to modify a set of finite perimeter on a small set so that its surface is $C^1$.

4.1. The half-space. In this brief subsection, we derive a formula for $F(\nu)$ so that Lemma 4.1 holds.

We first compute
\[ f \ast 1_{H\nu}(x) = \int_{y \cdot \nu \leq 0} f(x - y) dy, \]
which only depends on $x \cdot \nu$. Now, using scale and translation symmetries
\[ \lim_{r \to 0} \frac{1}{r} \| f_r \ast 1_{H\nu} \|_{L^2(B^n)}^2 = \omega_{n-1} \int_R |(f \ast 1_{H\nu})(t\nu)|^2 \, dt. \]

We now work on simplifying the expression $\int_R |(f \ast H\nu)(t\nu)|^2 \, dt$. We make a first simplification, which is to define the function $f_\nu \in C^\infty(\mathbb{R})$ which takes the integral of $f$ along planes perpendicular to $\nu$:
\[ f_\nu(t) = \int_{y \cdot \nu = t} f(y) \, d\mathcal{H}^{n-1}(y). \]

We may now write $(f \ast H\nu)(t\nu) = f_\nu \ast H(t)$, where $H$ is the usual Heaviside function on $\mathbb{R}$. Moreover, we observe the identity
\[ \int_R |1_H \ast \phi(t)|^2 \, dt = -\frac{1}{2} \int_R \int_R f(t)f(s) |s - t| \, ds \, dt. \]

This can be checked by expanding the left hand side and changing the order of integration, or else by observing that the left hand side is equivalent to $\int_R \phi \Delta^{-1} \phi$, and then using the fact that $|x|$ is the fundamental solution to the Laplacian in one dimension.
Applying this identity to the right hand side of Equation (20), we obtain
\[ \int_R \left| (f * 1_{H_\nu})(t\nu) \right|^2 dt = -\frac{1}{2} \int_R \int_R f_\nu(t)f_\nu(s)|s-t| ds dt. \]
This allows us to give a reasonably explicit formula for \( F(\nu) \):
\[ F(\nu) = -\frac{1}{2} \int_{R^n} \int_{R^n} f(x)f(y)|(x-y) \cdot \nu| ds dt. \]
We may use this expression to verify that \( F(\nu) \) is Lipschitz in \( \nu \), as
\[ |F(\nu) - F(\nu')| \leq |\nu - \nu'| \int_{R^n} \int_{R^n} |f(x)||f(y)||x-y| dxdy \]
\[ \leq |\nu - \nu'| \int_{R^n} \int_{R^n} |f(x)||f(y)|(|x|+|y|) dxdy \]
\[ \leq 2|\nu - \nu'| \|f\|_{L_1} \|\|x\|_{L_1}. \]

4.2. Differentiable graphs. We now show that Lemma 4.1 holds in the case that \( E \) is the graph of a \( C^1 \) function with small gradient. To state the result we need a little notation. Given a unit vector \( \nu_0 \in S^{n-1} \), we use \( \pi_{\nu_0} \) to mean the orthogonal projection onto the hyperplane orthogonal to \( \nu_0 \).

Lemma 4.2. Let \( B \subset R^n \) denote the unit ball, and let \( G \subset R^n \) be a subgraph for a function \( g \in C^1(R^{n-1}) \):
\[ G = \{ x \in R^n; x \cdot \nu_0 \leq g(\pi_{\nu_0}(x)) \}. \]
Suppose that \( \|g\|_{C^0} < \varepsilon \) and \( \|g\|_{C^1} < \varepsilon \). Then
\begin{equation}
\limsup_{r \to 0} \frac{1}{r} \|f_r * 1_G\|_{L_2(B)}^2 - \omega_{n-1} F(\nu_0) \leq C\varepsilon \alpha
\end{equation}
for constants \( C > 0 \) and \( \alpha > 0 \) depending only on the kernel \( f \) and the dimension \( n \).

Remark 4.2. According to Theorem 4.1 and the Lipschitz continuity of \( F \), the correct exponent is \( \alpha = 1 \). However it is more elementary to prove the lemma with smaller \( \alpha \).

Proof. From the calculation in the previous section, it is sufficient to bound
\[ \frac{1}{r} \left| \|f_r * 1_G\|_{L_2(B)}^2 - \|f_r * 1_{H_{\nu_0}}\|_{L_2(B)}^2 \right|. \]
As \( \nu_0 \) is fixed we will write \( H = H_{\nu_0} \) for short. First we will control the contribution to the integral from points far away from \( \partial G \) and \( \partial H \).
Fix a point $x$ away from the boundary $\partial G$, and let $\ell = |x \cdot \nu_0 - g(\pi_{\nu_0}(x))|$. Because the function $g$ is Lipschitz, there exists a constant $c$ such that the ball $B_c(x) \cap \partial G = 0$. In particular, by the cancellation property $\int f = 0$, we have
\[
 f_r * 1_G(x) \leq \int_{|x| > \ell} |f_r(x)| \, dx \\
\leq \int_{|y| > c \ell - 1} |f(y)| \, dy \\
\leq \frac{Cr}{\ell} \int |y||f(y)| \, dy \leq \frac{Cr}{\ell}. 
\]
Define the set $G_\beta$ of points sufficiently far from $\partial G$ by
\[
 G_\beta := \{x \in B \mid |x \cdot \nu_0 - g(\pi_{\nu_0}(x))| > \varepsilon^{-\beta} r\},
\]
where the exponent $\beta$ will be chosen later. Then from the estimate above we conclude
\[
 \frac{1}{r} \int_{G_\beta} |f_r * 1_G(x)|^2 \, dx \leq C\varepsilon^\beta.
\]
Analogously, defining
\[
 H_\beta := \{x \in B \mid |x \cdot \nu_0| > \varepsilon^{-\beta} r\}
\]
we have
\[
 \frac{1}{r} \int_{H_\beta} |f_r * 1_H(x)|^2 \, dx \leq C\varepsilon^\beta.
\]
It remains to control the contribution of the integrand near $\partial G$ and $\partial H$. When $r \ll \varepsilon$, it does not suffice to use the triangle inequality on the difference $|f_r * (1_G - 1_G)(x)|$ because there is not much cancellation. To achieve cancellation we must change coordinates to match the regions on which $f_r * 1_G$ and $f_r * 1_H$ agree. To define this change of variables we write $z = (z', t) \in \mathbb{R}^n$ where $t = z \cdot \nu_0$ and $z' = \pi_{\nu_0}(z)$. Then the coordinate transformation we need is the map $\Phi(z', t) = (z', t + g(z'))$.

The map $\Phi$ is differentiable with differentiable inverse, and is designed so that $\Phi(H) = G$. Moreover we compute the Jacobian $\nabla \Phi = Id + \nu_0 \otimes (\nabla (g \circ \pi_{\nu_0}))$, so in particular
\[
 |\nabla \Phi - Id| \leq \varepsilon. 
\]
Now we compute, using the change of coordinates $y = \Phi(z)$,

$$f_r \ast 1_G(\Phi(x_0)) = \int_G f_r(\Phi(x_0) - y) \, dy$$

$$= \int_G f_r(\Phi(x_0) - \Phi(z)) |\det \nabla \Phi| \, dz.$$

We can now compare this pointwise against $f_r \ast 1_H(x_0)$,

$$|f_r \ast 1_G(\Phi(x_0)) - f_r \ast 1_H(x_0)| \leq \int_H |f_r(\Phi(x_0) - \Phi(z))| |\det \nabla \Phi| - f_r(x_0 - z) | \, dz$$

$$\leq \int_H |\det \nabla \Phi| |f_r(\Phi(x_0) - \Phi(z)) - f_r(x_0 - z)| \, dz$$

$$+ \int_H ||\det \nabla \Phi| - 1||f_r(x_0 - z)| \, dz$$

$$= e_1 + e_2.$$ 

To control the first error term $e_1$ we split the integral into a near and a far part. For the near part, we use the fact that $|\nabla f_r| < C r^{-1-n}$ and the bound (24) to deduce that

$$|f_r(\Phi(x_0) - \Phi(z)) - f_r(x_0 - z)| \leq |x_0 - z| \varepsilon r^{-1-n}.$$ 

We now estimate

$$e_1 \leq \varepsilon \int_{|x_0 - z| < \varepsilon^{-a} r} |x_0 - z|r^{-1-n} \, dz + C \int_{|x_0 - z| \geq \varepsilon^{-a} r} |f_r(x_0 - z)| \, dz$$

$$\leq C \varepsilon^{1-(n+1)a} + C \varepsilon^a.$$ 

Choosing $a = (n+2)^{-1}$ we arrive at $e_1 \leq C \varepsilon^{1/(n+2)}$. Using again the Jacobian bound (24) and the fact that $f_r$ is bounded in $L^1$, we can easily estimate $e_2 < C \varepsilon$. In summary

$$|f_r \ast 1_G(\Phi(x_0)) - f_r \ast 1_H(x_0)| < C \varepsilon^{1/(n+2)}.$$ 

We can finally estimate

$$\frac{1}{r} \left| \int_{B \setminus H_\beta} |f_r \ast 1_G(x)|^2 \, dx - \int_{B \setminus G_\beta} |f_r \ast 1_H(x)|^2 \, dx \right|$$

by changing variables. The main contribution comes from the integral over $B \setminus H_\beta \cap \Phi^{-1}(B \setminus G_\beta)$ where the integrand is bounded by $C \varepsilon^{1/(n+2)}$ and the volume of integration is $\varepsilon^{-\beta} r$. Choosing now $\beta = (2n + 4)^{-1}$, we have shown the lemma with exponent $\alpha = (2n + 4)^{-1}$. 

Because $F(\nu)$ is Lipschitz, the normal to $\partial G$ is $\varepsilon$-close to $\nu$, and the map from $B'$ to $\partial G$ nearly preserves area, we are led to the following corollary.
Corollary 4.1. Let $B$ and $G$ as in Lemma 4.2, and let $\nu$ denote the unit normal to $\partial^* G$. Then for the same constant $\alpha$ appearing in Lemma 4.2,

$$\limsup_{r \to 0} \left| \frac{1}{r} \| f_r * G \|_{L^2(B)}^2 - \int_{\partial^* G} F(\nu) \, d\mathcal{H}^{n-1} \right| \leq C \varepsilon^\alpha.$$ 

4.3. Sets of finite perimeter. Finally, the case of sets of finite perimeter can be resolved using an approximation result that allows us to locally replace the reduced boundary $\partial^* E$ by the graph of a Lipschitz function. This can be done for sets which are close to half-spaces in the sense of Lemma 4.1.

Definition 4.1. If $E \subset B_1$, and $\delta > 0$, we say that $E$ is $\delta$-close to the half space $H_\nu$ if the following bounds hold:

$$|E \Delta H_\nu| < \delta$$

$$|P(E; B) - \omega_{n-1}| < \delta.$$ 

The following structural result from the theory of minimal surfaces allows us to approximate the boundaries of sets $E$ that are $\delta$-close to the half space $H_\nu$ by the graph of a $C^1$ function. The theorem is analogous to the Lipschitz truncation of Sobolev functions introduced by Acerbi and Fusco to study lower semicontinuity of functionals appearing in calculus of variations [AF88] and can be proven using similar methods. The theorem in [Mag12] is more general, and shows that if $E$ is close to having minimal area, then one can even ask that $g$ is close to a harmonic function.

Theorem 4.2 (Consequence of [Mag12, Theorem 23.7]). For any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all sets of finite perimeter $E \subset B$ which are $\delta$-close to $H_\nu$, there exists a function $g \in C^1(\mathbb{R}^{n-1})$ whose subgraph

$$G = \{ x \in B; x \cdot \nu \leq g(\pi_\nu(x)) \}$$

satisfies $|G \Delta E| < \varepsilon$, $P(G \Delta E) < \varepsilon$, and $\|g\|_{C^1} < \varepsilon$.

Proof of Lemma 4.1 using Theorem 4.2. Let $\varepsilon > 0$, and take $\delta > 0$ according to Theorem 4.2. If $E$ is $\delta$-close to $H_\nu$, then we can apply Theorem 4.2 to obtain a $G \subset B$ with the property that $P(G \Delta E) < \varepsilon$. From Propositions 1.1 and 1.2 we see that

$$\limsup_{r \to 0} \left| \frac{1}{r} \| f_r * 1_E \|_{L^2(B)}^2 - \frac{1}{r} \| f_r * 1_G \|_{L^2(B)}^2 \right| < C \varepsilon.$$
Moreover, since the normals of $G$ and $E$ agree except on a small set,
\[
\left| \int_{\partial^* E} F(\nu) \, d\mathcal{H}^{n-1} - \int_{\partial G} F(\nu) \, d\mathcal{H}^{n-1} \right| < C\varepsilon,
\]
we can conclude upon using Corollary 4.1. \hfill \Box

5. Results on bounded functions of bounded variation

5.1. The $H^{1/2}$ norm finds the jump set. In this section we prove that the decay of the Fourier modes of a function in $BV \cap L^\infty$ is tied to a quadratic integral over the jump set.

**Theorem 5.1.** Let $\eta > 0$ be a small parameter and $D \subset \mathbb{R}^n$ be a closed set. Then there exists a constant $C(\eta)$ such that for any $u \in BV \cap L^\infty(\mathbb{R}^n)$,
\[
(26) \quad \limsup_{r \to 0^+} \frac{1}{r} \left\| f_r * u \right\|_{L^2(D)}^2 \leq \eta \left\| u \right\|_{BV(\mathbb{R}^n)} \left\| u \right\|_{L^\infty(\mathbb{R}^n)} + C(\eta) \int_{J_u \cap D} |u^+ - u^-|^2 \, d\mathcal{H}^{n-1}.
\]

**Remark 5.1.** When $f$ is a function with compact support it is possible to refine the argument below to prove that
\[
\limsup_{r \to 0^+} \frac{1}{r} \left\| f_r * u \right\|_{L^2(D)}^2 \leq C \int_{J_u \cap D} |u^+ - u^-|^2 \, d\mathcal{H}^{n-1}.
\]
However, the constant $C$ depends on the radius of the support of $f$. I would guess that this bound remains true when $|x|f \in L^1$, but the methods here do not seem to suffice to prove it.

**Proof.** We can assume that
\[
(27) \quad \limsup_{r \to 0^+} \frac{1}{r} \left\| f_r * u \right\|_{L^2(D)}^2 > \eta \left\| u \right\|_{BV(\mathbb{R}^n)} \left\| u \right\|_{L^\infty(\mathbb{R}^n)},
\]
for otherwise the bound is trivial. In this case we will find a set $A \subset D$ such that $\mathcal{H}^{n-1}(A) \lesssim \left\| u \right\|_{BV} \left\| u \right\|_{L^{1\infty}}$ and $|Du|(A) \gtrsim \left\| u \right\|_{BV}$. In particular we will use this to show that
\[
\left\| u \right\|_{BV} \left\| u \right\|_{L^\infty} \lesssim \int_A |u^+ - u^-|^2 \, d\mathcal{H}^{n-1},
\]
from which the claim will follow.

The proof is divided into three steps. In the first step, we find approximations $A_k$ to the set $A$ and analyze them. Next, the sets $A_k$ are used to show that the limit $A$ has the desired properties. Finally, the existence of $A$ is used to prove (26). These sets are depicted in Figure 2.
Step 1: Finding $A_k$. With the assumption (27) we can find a sequence $r_k \to 0$ such that
\[
\limsup_{r \to 0^+} \frac{1}{r} \| f_r \ast u \|^2_{L^2(D)} > \frac{\eta}{2} \| u \|_{BV(R^n)} \| u \|_{L^\infty(R^n)}.
\]
On the other hand we have the bound
\[
\frac{1}{r} \| f_r \ast u \|^2_{L^2(D)} \leq \left( \frac{1}{r} \| f_r \ast u \|_{L^1(D)} \right) \left( \| f_r \ast u \|_{L^\infty(D)} \right).
\]
Thus, applying (3) and (4) from Lemma 1.1 and rearranging we obtain the bounds
\begin{align*}
(28) & \quad \frac{1}{r_k} \| f_{r_k} \ast u \|_{L^1(D)} \geq c_\eta \| u \|_{BV(R^n)} \\
(29) & \quad \| f_{r_k} \ast u \|_{L^1(D)} \geq c_\eta \| u \|_{L^\infty(R^n)} \\
(30) & \quad \| f_r \ast u \|^2_{L^2(D)} \geq c_\eta \| f_{r_k} \ast u \|_{L^1} \| f_{r_k} \ast u \|_{L^\infty}
\end{align*}
Given $r_k$, define
\[
A'_k := \{ x \in D \mid | f_{r_k} \ast u(x) | \geq \frac{1}{2} c_\eta \| u \|_{L^\infty} \}.
\]
The idea behind setting $A'_k$ this way is this: one should expect that $f_r \ast u(x)$ can only be large near large gradients of $u$. Thus $A'_k$ should help us locate some of the mass of $|Du|$. First we will show that in fact
\( A'_k \) is a set of an appreciable size. Indeed, using the definition of \( A'_k \) we can write
\[
\| f^* u \|_{L^2(D)}^2 \leq \frac{1}{2} c_\eta \| u \|_{L^\infty} \| f^* u \|_{L^1} + \int_{A'_k} | f^* u (x) |^2 \, dx.
\]
Applying the interpolation lower bound (30) on the left, rearranging, and bounding the integrand using the \( L^\infty \) norm we arrive at
\[
\frac{1}{2} c_\eta \| f^* u \|_{L^1} / \| f^* u \|_{L^\infty} \leq | A'_k |
\]
Now using the BV lower bound (28) and the upper bound (3) this becomes
\[
| A'_k | \geq \frac{r_k c_n^2}{2C} \| u \|_{BV} / \| u \|_{L^\infty}.
\]
This is the size one would expect if \( u \) had a jump of size \( \| u \|_{L^\infty} \) along a surface of Hausdorff measure \( \| u \|_{BV} / \| u \|_{L^\infty} \). In this case \( A'_k \) would be contained in a strip of width \( r_k \) near the jump. To corroborate this story we would like to show that in fact the total variation \( |Du| \) has appreciable mass near \( A'_k \). To do this we recall the pointwise bound (5)
\[
| f^* u (x) | \leq C r_k^1 | Du | (B_{C^r}(x)) + \varepsilon \| u \|_{L^\infty}.
\]
Setting \( \varepsilon < \frac{1}{4} \) and rearranging we have, for any \( x \in A'_k \),
\[
\frac{c_n}{4} \| u \|_{L^\infty} \leq C r_k^{1-N} | Du | (B_{C^r_k}(x)).
\]
Integrating this over all \( x \in A'_k \), using (31)
\[
\frac{r_k c_n^3}{8C} \| u \|_{BV} \leq C \int_{A'_k} r_k^{1-N} | Du | (B_{C^r_k}(x)) \, dx
\]
\[
\leq C r_k | Du | (A_k),
\]
where we have defined
\[
A_k = \bigcup_{x \in A'_k} B_{C^r_k}(x).
\]
Now relabeling constants (for we have no longer have need to be careful) we can summarize our main result from this step as
\[
| Du | (A_k) \geq c'_\eta \| u \|_{BV}.
\]

**Step 2: The limit \( A \).** In this step we analyze the limsup \( A \) of the sets \( A_k \). That is, we set
\[
A = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k.
\]
By the definition of $A_k$, one has that $d(x, D) < C r_k$ for each $x \in A_k$. Thus, $A \subset D$. Moreover by Fatou’s lemma we have that

$$|D u|(A) \geq c'_\eta \|u\|_{BV}.$$  

(33)

It remains to show that $\mathcal{H}^{n-1}(A)$ is small. We will do this by using the sets $A'_k$ to construct efficient coverings of $A_k$.

Indeed, let $\delta > 0$ be a radius for our covering. For each $x \in A$, one has that $x \in A_k$ for infinitely many $k$. In particular, for each point $x \in A$ one has that $x \in B_{C r_m}(y)$ for some $y \in A'_m$ and $C r_m < \delta$. Thus the set

$$B_\delta = \bigcup_{C r_m < \delta} \{B_{C r_m}(y) \mid y \in A'_m\}$$

is a collection of balls of radius less than $\delta$ covering $A$. Using the 5r-covering lemma, can choose a subset $\{B_{5C r_i}(y_i)\}_{i=1}^N$ such that the balls are pairwise disjoint and such that $\{B_{5C r_i}(y_i)\}_{i=1}^N$ covers $A$. Applying the pointwise bound (5) to the balls in this cover we estimate

$$\mathcal{H}^{n-1}_\delta(A) \leq \sum_{i=1}^N c_n(5C r_i)^{n-1} \leq \frac{C_n}{\|u\|_{L^\infty}} \sum_{i=1}^N |D u|(B_{C r_i}(y_i)) \leq C_n \|u\|_{BV} / \|u\|_{L^\infty}.$$  

Since this holds for any $\delta > 0$, we conclude that

$$\mathcal{H}^{n-1}(A) \leq C_\eta \|u\|_{BV} / \|u\|_{L^\infty},$$

(34)

which we observe is what one would expect if $A$ were a portion of the jump set with nearly maximal jump magnitude.

Step 3: Conclusion. We now use the set $A$ to obtain the lower bound

$$\|u\|_{BV} \|u\|_{L^\infty} \leq C(\eta) \int_A |u^+ - u^-|^2 d\mathcal{H}^{n-1}.$$  

First observe that $\mathcal{H}^{n-1}(A) < \infty$ implies that only the jump part $|u^+ - u^-| \mathcal{H}^{n-1} \subset J_u$ of the measure $|D u|$ contributes to $|D u|(A)$. Since $\mathcal{H}^{n-1}(A)$ is small we should expect that at most points in $A$, the magnitude of the jump is nearly optimal. Thus we define

$$A' = \{x \in A \mid |u^+ - u^-| \geq \frac{1}{2C(\eta)^2} \|u\|_{L^\infty}\}.$$
Now we can bound
\[ \|u\|_{BV} \leq C(\eta) \int_A |u^+ - u^-| \, d\mathcal{H}^{n-1} \]
\[ \leq \frac{1}{2} \|u\|_{L^{\infty}} \mathcal{H}^{n-1}(A) + 2 \|u\|_{L^{\infty}} \mathcal{H}^{n-1}(A') \]
\[ \leq \frac{1}{2} \|u\|_{BV} + 2C(\eta) \|u\|_{L^{\infty}} \mathcal{H}^{n-1}(A'). \]

Rearranging we conclude that
\[ \mathcal{H}^{n-1}(A') > c_\eta \|u\|_{BV}/\|u\|_{L^{\infty}}. \]

Finally, we may bound
\[ \int_{A'} |u^+ - u^-|^2 \, d\mathcal{H}^{n-1} \gtrsim c_\eta \|u\|^2_{L^{\infty}} \mathcal{H}^{n-1}(A') \gtrsim c_\eta \|u\|_{L^{\infty}} \|u\|_{BV}. \]

\[ \square \]

5.2. The one-dimensional case. In this section we use Theorem 26 to prove an exact result in one dimension. We first perform a calculation in the case that \( u \in BV(\mathbb{R}) \) has finitely many jumps.

**Proposition 5.1.** Let \( u \in BV(\mathbb{R}) \) be piecewise constant with jumps at finitely many points \( x_j, 1 \leq j \leq N \). Then
\[ \lim_{r \to 0} \frac{1}{r} \|f_r * u\|^2_{L^2} = c_f \sum_{j=1}^{N} |u^+(x_j) - u^-(x_j)|^2. \]

**Proof.** Choose \( \varepsilon < \frac{1}{2} \min_{i,j} |x_i - x_j| \) and partition \( \mathbb{R} \) into a collection of closed intervals \( \{I_k\}_{k=1}^{2N+1} \) such that
\[ I_m = [x_m - \varepsilon, x_m + \varepsilon] \]
for \( 1 \leq m \leq N \). The remaining intervals do not contain any of the jump points \( x_j \), so \( u \) is constant on such intervals. Thus by Proposition 1.1
\[ \lim_{r \to 0} \frac{1}{r} \|f_r * u\|^2_{L^2(I_n)} = 0 \]
for \( n > N \). On the other hand by scaling it must be that
\[ \lim_{r \to 0} \frac{1}{r} \|f_r * u\|^2_{L^2(I_m)} = |u^+(x_m) - u^-(x_m)|^2 \lim_{r \to 0} \frac{1}{r} \|f_r * H\|^2_{L^2((-1,1))} \]
where \( H \) is the heaviside function on \( \mathbb{R} \). Taking
\[ c_f = \lim_{r \to 0} \frac{1}{r} \|f_r * H\|^2_{L^2((-1,1))} \]
concludes the proof. \[ \square \]
Theorem 5.2. Let $u \in BV(R)$ and $f$ be a kernel as in Lemma 1.1. Then for some constant $c_f$,
\[
\lim_{r \to 0^+} \frac{1}{r} \| f_r * u \|_{L^2}^2 = c_f \sum_{x \in J_u} |u^+(x) - u^-(x)|^2
\]

Proof. Let $u \in BV(R)$. We will decompose $u$ into a main part that is a piecewise continuous function with finitely many jumps, a continuous part that contributes nothing according to Theorem 5.1 and an error term that is small in $BV$.

For each jump height $\delta > 0$, define the set $J_\delta$ of jumps of height at least $\delta$:
\[
J_\delta = \{ x \in J_u \mid |u^+(x) - u^-(x)| > \delta \}
\]
Then one can decompose $Du$ as
\[
Du = Du \upharpoonright J_\delta + Du^c + Du \upharpoonright (J_u \setminus J_\delta).
\]
Integrating, we can define the decomposition
\[
u = u_b + u_c + u_s
\]
where the subscripts refer to the parts that are ‘big’, ‘continuous’, and ‘small’. We have, using Proposition 5.1, Theorem 5.1, and Lemma 1.1 respectively:
\[
\lim_{r \to 0^+} \frac{1}{r} \| f_r * u_b \|_{L^2}^2 = \sum_{J_\delta} |u^+(x) - u^-(x)|^2
\]
\[
\lim_{r \to 0^+} \frac{1}{r} \| f_r * u_c \|_{L^2}^2 = 0
\]
\[
\limsup_{r \to 0^+} \frac{1}{r} \| f_r * u_s \|_{L^2}^2 \leq C |Du|(J_u \setminus J_\delta)^2.
\]
In the last bound the square comes from the fact that $\|u_s\|_{L^\infty} \leq \|u_s\|_{BV}$. The theorem follows from applying the continuity from Proposition 1.2 and sending $\delta \to 0$.

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