A lattice Poisson algebra for the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring

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**Abstract.** The Poisson algebra of the Lax matrix associated with the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring is computed from first principles. The resulting non-ultralocality is mild, which enables to write down a corresponding lattice Poisson algebra.

1 Introduction

We recently showed in [1] that the Poisson algebra of the Lax matrix associated with symmetric space sine-Gordon models, defined through a gauged Wess-Zumino-Witten action with an integrable potential [2], admits an integrable lattice discretization. In the present letter we compute the $r/s$-matrix structure [3] associated with the Pohlmeyer reduction of $AdS_5 \times S^5$ superstring theory [4] directly from its representation in terms of a fermionic extension of a gauged WZW action with an integrable potential. We similarly find that it is precisely of the type which, after regularization as in [6], admits an integrable lattice discretization of the general form identified in [7, 8].

2 Canonical analysis and Hamiltonian

To begin with we briefly recall some usual notations. We refer the reader to [4] for more details concerning this setup. The superalgebra $\mathfrak{f} = \mathfrak{psu}(2,2|4)$ admits a $\mathbb{Z}_4$-grading, $\mathfrak{f} = \mathfrak{f}^{(0)} \oplus \mathfrak{f}^{(1)} \oplus \mathfrak{f}^{(2)} \oplus \mathfrak{f}^{(3)}$ where $\mathfrak{g} = \mathfrak{f}^{(0)} = \mathfrak{so}(4,1) \oplus \mathfrak{so}(5)$. Let $G$ denote the corresponding Lie group. The supertrace is compatible with the $\mathbb{Z}_4$-grading, in the sense that $\text{Str}(A^{(m)} B^{(n)}) = 0$ for $m + n \neq 0 \mod 4$. The
reduced theory relies on the element \( T = \frac{1}{2} \text{diag}(1, 1, -1, -1, 1, 1, -1, -1) \in \mathfrak{f}^{(2)} \). It defines a \( \mathbb{Z}_2 \)-grading of \( \mathfrak{f} \) with \( \mathfrak{f}^{[0]} = \text{Ker}(Ad_T) \) and \( \mathfrak{f}^{[1]} = \text{Im}(Ad_T) \). Elements of \( \mathfrak{f}^{[0]} \) commute with \( T \) while those of \( \mathfrak{f}^{[1]} \) anti-commute with \( T \) and we have \( \text{Str}(A^{[0]}B^{[1]}) = 0 \). Finally, projectors on \( \mathfrak{f}^{[0]} \) and \( \mathfrak{f}^{[1]} \) are given respectively by \( P^{[0]} = -[T, [T, \cdot]]_+ \) and \( P^{[1]} = -[T, [T, \cdot]] \). Let \( \mathfrak{h} = \mathfrak{g}^{[0]} \) be the subalgebra in \( \mathfrak{g} \) of elements commuting with \( T \). The corresponding Lie group \( H \) is \([SU(2)]^4\).

Our starting point is the field theory introduced in \([4]\). It corresponds to a fermionic extension of a \( G/H \) gauged WZW with a potential term. The action we start with is, taking \( \epsilon^{\tau \sigma \xi} = 1 \),

\[
S = \frac{1}{2} \int \! d\tau d\sigma \text{Str}(g^{-1} \partial_+ gg^{-1} \partial_- g) + \frac{1}{3} \int \! d\tau d\sigma \! d\xi \epsilon^{\alpha \beta \gamma} \text{Str}(g^{-1} \partial_\alpha gg^{-1} \partial_\beta gg^{-1} \partial_\gamma g)
- \int \! d\tau d\sigma \text{Str}(A_+ \partial_- gg^{-1} - A_- \partial_+ g + g^{-1} A_+ g A_- - A_+ A_-)
+ \frac{1}{4} \int \! d\tau d\sigma \left( \psi_L[T, D_+ \psi_L] + \psi_R[T, D_- \psi_R] \right)
+ \int \! d\tau d\sigma \left( \mu^2 \text{Str}(g^{-1} T g T) + \mu \text{Str}(g^{-1} \psi_L g \psi_R) \right).
\]

The fields \( g, \psi_R, \psi_L \) and the gauge fields \( A_\pm \) respectively take values in \( G, \mathfrak{f}^{(1)[1]}, \mathfrak{f}^{(3)[1]} \) and in \( \mathfrak{h} \). The covariant derivatives are \( D_\pm = \partial_\pm - [A_\pm, \cdot] \) with \( \partial_\pm = \partial_\tau \pm \partial_\sigma \).

Generalizing the analysis of \([9]\) to the case considered here, one finds that the phase space is spanned by the fields \( (g, J_L, A_\pm, P_\pm, \psi_L, \psi_R) \). The field \( J_L \) corresponds to the left-invariant WZW current. Alternatively, one can use instead the right-invariant current \( J_R \), related to \( J_L \) by

\[
J_R = -2 \partial_\sigma gg^{-1} + g J_L g^{-1}.
\]

The fields \( P_\pm \) are the canonical momenta of \( A_\pm \). The non-vanishing Poisson brackets are

\[
\{ J_{L1}(\sigma), J_{L2}(\sigma') \} = [C^{(00)}_{12}, J_{L2}(\sigma')] \delta_{\sigma \sigma'} + 2C^{(00)}_{12} \partial_\sigma \delta_{\sigma \sigma'},
\{ J_{R1}(\sigma), J_{R2}(\sigma') \} = -[C^{(00)}_{12}, J_{R2}(\sigma')] \delta_{\sigma \sigma'} - 2C^{(00)}_{12} \partial_\sigma \delta_{\sigma \sigma'},
\{ J_{L1}(\sigma), g_{2}(\sigma') \} = -g_2 C^{(00)}_{12} \delta_{\sigma \sigma'},
\{ J_{R1}(\sigma), g_{2}(\sigma') \} = -C^{(00)}_{12} g_2 \delta_{\sigma \sigma'},
\{ A_{\pm 1}(\sigma), P_{\pm 2}(\sigma') \} = C^{(00)[00]}_{12} \delta_{\sigma \sigma'},
\{ \psi_{R1}(\sigma), \psi_{R2}(\sigma') \} = [T_{2}, C^{(13)}_{12}] \delta_{\sigma \sigma'},
\{ \psi_{L1}(\sigma), \psi_{L2}(\sigma') \} = [T_{2}, C^{(31)}_{12}] \delta_{\sigma \sigma'}.
\]

In these expressions \( C^{(ij)}_{kl} \in \mathfrak{f}^{(i)} \otimes \mathfrak{f}^{(j)} \) are the components of the tensor Casimir (see \([10]\) for its properties) in the decomposition \( C^{(00)}_{12} = C^{(00)}_{12} + C^{(13)}_{12} + C^{(22)}_{12} + C^{(31)}_{12} \) with respect to the \( \mathbb{Z}_2 \)-grading. The component \( C^{(00)[00]}_{12} \) is defined in a similar way relative to the \( \mathbb{Z}_2 \)-grading.
The standard analysis shows that there is a total of four constraints,

\[
\begin{align*}
\chi_1 &= P_+, \\
\chi_2 &= P_- \\
\chi_3 &= \mathcal{J}^0_R + A_+ - A_ - \frac{1}{2} [\psi_L, [T, \psi_L]], \\
\chi_4 &= \mathcal{J}^0_L + A_+ - A_ - \frac{1}{2} [\psi_R, [T, \psi_R]].
\end{align*}
\] (2.2a, 2.2b)

The extended Hamiltonian, which has weakly vanishing Poisson brackets with the constraints (2.2), is

\[
H = \int d\sigma \left( \frac{1}{2} \text{Str}(\mathcal{J}_L^2 + \mathcal{J}_R^2) + \text{Str}(\mathcal{J}_R^0 A_+ - \mathcal{J}_L^0 A_-) + \frac{1}{2} \text{Str}((A_+ - A_-)^2) \right.
\]
\[
- \frac{1}{2} \text{Str}(\psi_L [T, \partial_\sigma \psi_L] - [A_+, \psi_L]) - \frac{1}{2} \text{Str}(\psi_R [T, -\partial_\sigma \psi_R - [A_-, \psi_R]])
\]
\[
- \mu^2 \text{Str}(g^{-1} T g T) - \mu \text{Str}(g^{-1} \psi_L g \psi_R) + v_+ P_+ + v_- P_- + \lambda (\chi_3 - \chi_4) \bigg) \]

with \(v_+ - v_- = \partial_\sigma (A_+ + A_-) - [A_+, A_-]\). The combination \(\chi_3 - \chi_4\) of the constraints generates a gauge invariance.

3 Continuum and lattice Poisson algebras

Up to a gauge transformation, the equations of motion for the fields \((\mathcal{J}_L, g, \psi_L, \psi_R)\) under the Hamiltonian (2.3) are equivalent to the zero curvature equation \(\{\mathcal{L}, H\} = \partial_\sigma \mathcal{M} + [\mathcal{M}, \mathcal{L}]\) for the following Lax connection [4]

\[
\mathcal{L}(z) = -\frac{1}{2} \mathcal{J}_L - \frac{1}{2} z \sqrt{\mu} \psi_R - \frac{1}{2} z^2 \mu T + \frac{1}{2} z^{-1} \sqrt{\mu} g^{-1} \psi_L g + \frac{1}{2} z^{-2} \mu g^{-1} T g, \quad (3.1a)
\]
\[
\mathcal{M}(z) = -\frac{1}{2} \mathcal{J}_L + A_- - \frac{1}{2} z \sqrt{\mu} \psi_R - \frac{1}{2} z^2 \mu T - \frac{1}{2} z^{-1} \sqrt{\mu} g^{-1} \psi_L g - \frac{1}{2} z^{-2} \mu g^{-1} T g. \quad (3.1b)
\]

The field \(A_+\) entering the equations appears as an arbitrary element of \(\mathfrak{h}\). We now have all the ingredients needed to compute the Poisson bracket of the Lax matrix (3.1a). The result reads

\[
4 \{\mathcal{L}_1(z_1), \mathcal{L}_2(z_2)\} = [r_{12}(z_1, z_2), \mathcal{L}_1(z_1) + \mathcal{L}_2(z_2)] \delta_{\sigma\sigma'}
\]
\[
+ [s_{12}(z_1, z_2), \mathcal{L}_1(z_1) - \mathcal{L}_2(z_2)] \delta_{\sigma\sigma'} + 2 s_{12}(z_1, z_2) \partial_\sigma \delta_{\sigma\sigma'}, \quad (3.2)
\]

where the kernels of the \(r/s\)-matrices are given by

\[
r_{12}(z_1, z_2) = \frac{z_1^4 + z_2^4}{z_2^4 - z_1^4} C_{12}^{(00)} + \frac{2 z_1 z_2^2}{z_2^4 - z_1^4} C_{12}^{(13)} + \frac{2 z_1^2 z_2}{z_2^4 - z_1^4} C_{12}^{(22)} + \frac{2 z_1^3 z_2}{z_2^4 - z_1^4} C_{12}^{(31)}, \quad (3.3a)
\]
\[
s_{12}(z_1, z_2) = C_{12}^{(00)}. \quad (3.3b)
\]

One can check explicitly that the kernels (3.3) coincide exactly with the ones that would be obtained from the generalization of the alleviation procedure proposed in [4] to semi-symmetric space \(\sigma\)-models. This is simply a matter of replacing the twisted inner product on the twisted loop
algebra considered in [11] by the trigonometric one and to compute the corresponding kernels as explained in [1].

An important property of the above $r/s$-matrix structure is that $s$ is simply the projection onto the subalgebra $\mathfrak{g}$. In this case, the corresponding Poisson algebra (3.2) can be discretized following [6] by introducing a skew-symmetric solution $\alpha \in \text{End} \mathfrak{g}$ of the modified classical Yang-Baxter equation on $\mathfrak{g}$. Then the matrices

\[ a_{12} = (r + \alpha)_{12}, \quad b_{12} = (-s - \alpha)_{12}, \quad c_{12} = (-s + \alpha)_{12}, \quad d_{12} = (r - \alpha)_{12}, \]

satisfy all the requirements of [7, 8] in order to define the following consistent lattice algebra,

\[ 4\{L_1^n, L_2^m\} = a_{12}L_1^n L_2^m \delta_{mn} - L_1^n L_1^m d_{12} \delta_{mn} + L_1^n b_{12} L_2^m \delta_{m+1,n} - L_2^n c_{12} L_1^m \delta_{m,n+1}. \]

This algebra reduces to (3.2) in the continuum limit (see [1]). The corresponding algebra for the monodromy may be found in [1].

4 Conclusion

We have constructed a quadratic lattice Poisson algebra associated with the fermionic extension of the $\left( \text{SO}(4,1) \times \text{SO}(5) \right)/\text{[SU}(2)^4$ gauged WZW model with an integrable potential. The fact that one is able to write down such a lattice algebra is quite appealing and in sharp contrast with what happens for the canonical Poisson structure of the $\text{AdS}_5 \times \text{S}^5$ superstring [10]. Indeed, it brings hope of being able to construct a lattice quantum algebra related to the Pohlmeyer reduction of the $\text{AdS}_5 \times \text{S}^5$ superstring. The precise link of this Pohlmeyer reduction with the alleviation procedure presented in [1] is under study.

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