CASTELNUOVO-MUMFORD REGULARITY BOUNDS FOR LOW DIMENSIONAL PROJECTIVE VARIETIES

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Abstract. We first give a regularity bound for a dimension zero scheme by using its Loewy length. Then we bound the regularity of a curve allowing embedded or isolated point components in terms of its arithmetic degree. Finally, we verify the Eisenbud-Goto conjecture in the case of a normal surface with rational, Gorenstein elliptic and log canonical singularities.

1. Introduction

The motivation of this paper is to generalize several results on Castelnuovo-Mumford regularity bounds for points, curves and surfaces. Throughout this paper we work over the complex number field \( k := \mathbb{C} \). Let \( X \) be a closed subscheme of \( \mathbb{P}^n \) defined by an ideal sheaf \( \mathcal{I}_X \). \( X \) is said to be \( m \)-regular if \( H^i(\mathbb{P}^n, \mathcal{I}_X(m - i)) = 0 \) for all \( i > 0 \). The minimal such number \( m \) is called the Castelnuovo-Mumford regularity of \( X \) and is denoted by \( \text{reg} X \). In the same way, we can define \( \text{reg} \mathcal{F} \) for any coherent sheaf \( \mathcal{F} \) on \( \mathbb{P}^n \). For further reference on regularity refer to the book [Laz04].

Our first result establishes a regularity bound for a dimension zero subscheme of \( \mathbb{P}^n \). Instead of the length of the scheme, we use the Loewy length (cf. Definition 2.1) to bound the regularity.

Theorem 1.1. Let \( X \subset \mathbb{P}^n \) be a dimension zero subscheme supported at distinct closed points \( \{p_1, ..., p_t\} \). For each \( 1 \leq i \leq t \), set \( \mu_i := \mu_{\mathcal{O}_{X,p_i}} \) to be the Loewy length of the local ring \( \mathcal{O}_{X,p_i} \). Then one has

\[ H^1(\mathbb{P}^n, \mathcal{I}_X(k)) = 0 \text{ for } k \geq \mu_1 + \mu_2 + \cdots + \mu_t + t - 1, \]

i.e., \( X \) is \((\mu_1 + \mu_2 + \cdots + \mu_t + t)\)-regular.

The second result provides a regularity bound for a curve allowing point components(cf. Definition 3.1), which extends the classical result of Gruson-Lazarsfeld-Peskine [GLP83] and Giaino [Gia06].

Theorem 1.2. Let \( X \subset \mathbb{P}^n \) be a curve with point components and \( X_1 \) be its curve part. Then \( X \) is

\[ (\text{adeg}_0 X + \text{adeg}_1 X - \dim(\text{Span} X_1) + 2)\)-regular,

where \( \text{Span} X_1 \) means the minimal linear space containing \( X_1 \).

The last result gives a regularity bound for projective normal surfaces with mild singularities. This extends the result of Lazarsfeld [Laz87] on nonsingular surfaces and verifies a conjecture due to Eisenbud and Goto [EG84] in a special case.

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Theorem 1.3. Let $X$ be a nondegenerate normal surface $X \subset \mathbb{P}^n$ with the following singularities: rational, Gorenstein elliptic and log canonical. Then one has

$$\text{reg } X \leq \text{deg } X - \text{codim } X + 1.$$ 

It should be mentioned that the method to prove the above theorem is a combination of a generic projection used in [Laz87] and Grothendieck duality. The idea of using duality in this problem is due to Lawrence Ein. Some slightly weak regularity bounds for higher dimensional varieties was also obtained by Kwak in his work [Kwa98] and [Kwa00].

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2. Regularity bounds for dimension zero subschemes of $\mathbb{P}^n$

In this section, we give a regularity bound for a dimension zero subscheme of $\mathbb{P}^n$. Classically, the regularity of such a scheme is bounded by its degree, i.e., the length of the structure sheaf of the scheme. This bound works well if the scheme is reduced but becomes worse if the scheme is nonreduced. Thus in order to get a better bound, nilpotent elements must be considered.

Definition 2.1. Let $(A, \mathfrak{m})$ be a local Artinian ring. We define the Loewy length $\mu_A$ of $A$ to be the nonnegative number $\mu_A := \max\{i \mid \mathfrak{m}^i \neq 0\}$. If $\mathfrak{m} = 0$, i.e., $A$ is a field, then we write $\mu_A = 0$. We may also write $\mu$ instead of $\mu_A$ if no confusion arise.

In general, given an Artinian ring, its Loewy length $\mu$ is much smaller than its length. Also, Loewy length can be thought of as an efficient way to measure the size of nilpotent elements of the Artinian ring. Thus instead of using length, we use Loewy length to bound the regularity of a dimension zero scheme.

Proof of Theorem 1.1 Without loss of generality we assume that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_t \geq 0$. We proceed by induction on $\mu_i$. If $\mu_1 = \cdots = \mu_t = 0$, i.e., $X$ is reduced, then the result is well-known.

For general case, let $j := \max\{i \mid \mu_i \neq 0\}$, so that we can assume

$$X = \text{Spec } A_1 \oplus \cdots \oplus \text{Spec } A_j \oplus \text{Spec } k \oplus \cdots \oplus \text{Spec } k,$$

where $(A_i, \mathfrak{m}_i)$ is a nonreduced local Artinian ring for $i = 1, \cdots, j$. We defined a subscheme $X_j$ of $X$ as

$$X_j := \text{Spec } A_1 \oplus \cdots \oplus \text{Spec } A_j / \mathfrak{m}_j^{\mu_j} \oplus \text{Spec } k \oplus \cdots \oplus \text{Spec } k.$$

Set $a := \mu_1 + \cdots + (\mu_j - 1) + t - 1$. Then by induction, $H^1(\mathbb{P}^n, \mathcal{I}_{X_j}(a)) = 0$. Consider an exact sequence

$$0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{I}_{X_j} \rightarrow \mathfrak{m}_j^{\mu_j} \rightarrow 0,$$

we then deduce that

$$H^0(\mathbb{P}^n, \mathcal{I}_{X_j}(a + 1)) \overset{\theta_j}{\longrightarrow} \mathfrak{m}_j^{\mu_j} \rightarrow H^1(\mathbb{P}^n, \mathcal{I}_X(a + 1)) \rightarrow 0.$$ 

Thus, all we need is to show that the morphism $\theta_j$ is surjective.

From the exact sequence $0 \rightarrow \mathcal{I}_{X_j} \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{X_j} \rightarrow 0$, we have

$$0 \rightarrow H^0(\mathbb{P}^n, \mathcal{I}_{X_j}(a + 1)) \rightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a + 1)) \overset{\phi}{\longrightarrow} \mathcal{O}_{X_j} \rightarrow 0.$$ 

Assume that $\mathfrak{m}_j$ is generated by the sections $s_1, \cdots, s_e$ of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, where $1 \leq e \leq n$. Then $\mathfrak{m}_j^{\mu_j}$ will be generated by the sections of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mu_j))$ in the form

$$\sigma_{i_1, \cdots, i_{\mu_j}} := s_{i_1} \cdots s_{i_{\mu_j}}, \text{ where } 1 \leq i_1 \leq \cdots \leq i_{\mu_j} \leq e.$$
Also for each $i \neq j$ there is a section $l_i \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ such that $l_i \in \mathfrak{m}_i$ but $l_i \notin \mathfrak{m}_j$ because of the base point freeness of $\mathcal{O}_{\mathbb{P}^n}(1)$. Then we see that the sections in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a + 1))$ of the form

$$s = \sigma_{i_1 \cdots i_p}l_1^{\mu_1 + 1} \cdots l_p^{\mu_p + 1}$$

will satisfy $\phi(s) = 0$ and therefore $s \in H^0(\mathbb{P}^n, \mathcal{I}_{X}(a + 1))$. Thus those sections will give the surjective morphism

$$\theta_j : H^0(\mathbb{P}^n, \mathcal{I}_{X}(a + 1)) \longrightarrow \mathfrak{m}_j^{\mu_j} \longrightarrow 0.$$ 

This proves that $H^1(\mathbb{P}^n, \mathcal{I}_{X}(a + 1)) = 0$.

The following special case of the theorem will be used in the section 3.

**Corollary 2.2.** Let $X \subset \mathbb{P}^n$ be a subscheme supported at one point $x$ with $\mu := \mu_{\mathcal{O}_{\mathbb{P}^n}}$. Then one has $H^1(\mathbb{P}^n, \mathcal{I}_{X}(k)) = 0$ for $k \geq \mu$, i.e., $X$ is $(\mu + 1)$-regular.

**Example 2.3.** We show that the regularity bound in Theorem 1.1 can be achieved. Consider a line $l$ in the projective space $\mathbb{P}^n$ and $t$ reduced points $P_1, \cdots, P_t$ sitting on the line $l$. Assume that the defining ideal sheaf of $P_1$ in $\mathbb{P}^n$ is $\mathcal{M}_i$ for $i = 1, \cdots, t$. Then we define a subscheme $Z$ defined by the ideal sheaf

$$\mathcal{I}_Z := \mathcal{M}_1 \cap \mathcal{M}_2 \cap \cdots \cap \mathcal{M}_t,$$

where $a_i \geq 1$ are integers for $i = 1, \cdots, t$. Then it is clear that $Z$ is supported at points $P_1, \cdots, P_t$ and at each point $P_i$ the Loewy length of $\mathcal{O}_{Z,P_i}$ is $a_i - 1$. Thus by Theorem 1.1 we have $\text{reg } Z \leq a_1 + a_2 + \cdots + a_t$. On the other hand, since the length of $l \cap Z$ is $a_1 + a_2 + \cdots + a_t$, i.e., $l$ is a $(a_1 + a_2 + \cdots + a_t)$-secant line of $Z$, we see that $\text{reg } Z \geq a_1 + a_2 + \cdots + a_t$. Therefore $\text{reg } Z = a_1 + a_2 + \cdots + a_t$.

3. **Regularity bounds for curves with point components**

In this section, we give a regularity bound for a curve allowing points as its embedded or isolated components. Let $X$ be a closed subscheme of $\mathbb{P}^n$. Denote by $R_1 \mathcal{O}_X$ the subsheaf of $\mathcal{O}_X$ containing sections whose support has dimension $< 1$. Then $R_1 \mathcal{O}_X$ is naturally a $\mathcal{O}_X$-ideal sheaf and it defines a subscheme $X_1$ of $X$ with the structure sheaf $\mathcal{O}_{X_1} = \mathcal{O}_X/R_1 \mathcal{O}_X$. $X_1$ is said to be obtained from $X$ by throwing away dimension zero components of $X$. We denote by $\mathcal{I}_X^*$ the defining ideal sheaf of $X_1$ as a subscheme of $\mathbb{P}^n$. This definition can also be found in [BM93]. For further reference on subsheaves $R_i \mathcal{O}_X$ of any $i > 0$ we refer to [Har66 Chapter 2].

**Definition 3.1.** Let $X$ be a dimension one closed subscheme of $\mathbb{P}^n$ defined by an ideal sheaf $\mathcal{I}_X$. We say $X$ is a curve with point components if by throwing away its dimension zero components we obtain a reduced dimension one subscheme, i.e., the ideal sheaf $\mathcal{I}_X^*$ defines a reduced equidimension one closed subscheme $X_1$ of $\mathbb{P}^n$. $X_1$ is said to be the curve part of $X$.

**Remark 3.2.** If $X$ is a curve with point components then $X$ could have embedded points or/and isolated points, or none of them which means $X$ is already a reduced curve.

**Definition 3.3.** Let $X$ be a curve with point components in $\mathbb{P}^n$. We define its 0-th arithmetic degree to be the length of $\mathcal{I}_X^*/\mathcal{I}_X$, i.e. $\text{adeg}_0 X := l(\mathcal{I}_X^*/\mathcal{I}_X)$. Define its 1-st arithmetic degree $\text{adeg}_1 X$ to be the degree of $X_1$. 

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Lemma 3.4. Let $V$ be a $k$-vector space and $\mathcal{Q}$ be a coherent sheaf on $\mathbb{P}^n$ with $\dim \text{Supp} \mathcal{Q} = 0$ and length $l := l(\mathcal{Q}) \geq 1$. Suppose that there is a surjective morphism $V \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{Q} \rightarrow 0$ as $\mathcal{O}_{\mathbb{P}^n}$-modules. Then by twisting $\mathcal{O}_{\mathbb{P}^n}(l-1)$ one has a surjective morphism on global sections, i.e.

\[(3.4.1) \quad V \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l-1)) \rightarrow H^0(\mathbb{P}^n, \mathcal{Q}(l-1)) \rightarrow 0.\]

Proof. We proceed by induction on the length $l$. Starting with $l = 1$ we may assume that $\mathcal{Q} = k(p)$, a residue field of a closed point $p \in \mathbb{P}^n$. Then the surjective morphism $V \otimes \mathbb{P}^n \rightarrow k(p) \rightarrow 0$ shows that there is one section $s \in V$ generate $k(p)$, which means that $s$ is mapped to $1 \in k(p)$. Then it is clear that the morphism in $(3.4.1)$ is surjective.

Now assume that the lemma is true for $l$, we show that it is true for $l + 1$, i.e. the case that $\mathcal{Q}$ has length $l + 1$. Since $\dim \text{Supp} \mathcal{Q} = 0$, then $\mathcal{Q}$ has a submodule $k(p)$, a residue field of a closed point $p \in \text{Supp} \mathcal{Q}$. Denote by $\mathcal{Q}' = \mathcal{Q}/k(p)$ then $\mathcal{Q}'$ is a $\mathcal{O}_{\mathbb{P}^n}$-module with dimension zero support and length $l$. We then have the following commutative diagram

\[
\begin{array}{cccccc}
0 & & & & & 0 \\
\downarrow & & & & & \downarrow \\
0 & \rightarrow & M & \rightarrow & V \otimes \mathcal{O}_{\mathbb{P}^n} & \rightarrow & \mathcal{Q} & \rightarrow & 0 \\
\downarrow & & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K & \rightarrow & V \otimes \mathcal{O}_{\mathbb{P}^n} & \rightarrow & \mathcal{Q}' & \rightarrow & 0 \\
\downarrow & & & & & \downarrow & & \downarrow & & \\
k(p) & & & & & 0 & & & & \\
\downarrow & & & & & \downarrow & & & & \\
0 & & & & & & & & \\
\end{array}
\]

(3.4.2)

where $\varphi$ is given by assumption and $\varphi'$ is induced by composing $\varphi$ with the quotient $\mathcal{Q} \rightarrow \mathcal{Q}'$ and $M$ and $K$ are kernels of $\varphi$ and $\varphi'$ respectively. The left hand side vertical short exact sequence is obtained by snake lemma. Now by induction, one has a surjective morphism

\[V \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l-1)) \rightarrow H^0(\mathbb{P}^n, \mathcal{Q}(l-1)) \rightarrow 0.\]

Thus it is easy to check that the coherent sheaf $K$ is $l$-regular. Therefore $K(l)$ is generated by global sections and we then have a surjective morphism $H^0(\mathbb{P}^n, K(l)) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow K(l) \rightarrow 0$. Composing with the morphism $\psi$, we obtain a commutative diagram

\[
\begin{array}{ccc}
H^0(\mathbb{P}^n, K(l)) \otimes \mathcal{O}_{\mathbb{P}^n} & \rightarrow & K(l) \\
\varphi_p \downarrow & & \downarrow \psi \\
k(p)(l) & & \end{array}
\]

The morphism $\varphi_p$ is surjective and $k(p)(l)$ has length 1. Thus by taking global sections we have a surjective morphism $H^0(\mathbb{P}^n, K(l)) \rightarrow H^0(\mathbb{P}^n, k(p)(l))$ by the case $l = 1$ of the induction. Now going back to the diagram $(3.4.2)$ we see $H^1(\mathbb{P}^n, M(l)) = 0$ and therefore the morphism $V \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) \rightarrow H^0(\mathbb{P}^n, \mathcal{Q}(l))$ is surjective as required. \qed
**Theorem 3.5.** Let $X$ be a curve with point components in $\mathbb{P}^n$. Assume that its curve part is $r$-regular. Then $X$ is

$$(r + \deg_0 X)\text{-regular.}$$

**Proof.** Let $\mathcal{I}_X$ be the ideal sheaf defining curve part $X_1$ of $X$. Set $l := \deg_0 X$ which equals the length of $\mathcal{I}_X/\mathcal{I}_X$. Since $\mathcal{I}_X$ is $r$-regular the sheaf $\mathcal{I}_X(r)$ is generated by global sections. We then have a commutative diagram.

$$
\begin{array}{cccc}
H^0(\mathbb{P}^n, \mathcal{I}_X(r)) \otimes \mathcal{O}_{\mathbb{P}^n} & \xrightarrow{\varphi} & \mathcal{I}_X(r) & \xrightarrow{\mathcal{I}_X} \mathcal{I}_X(r) / \mathcal{I}_X(r) \xrightarrow{\mathcal{I}_X} 0.
\end{array}
$$

Now applying Lemma 3.4 to the morphism $\varphi$ and chasing through the diagram we then obtain the assertion. \hfill \Box

**Proof of Theorem 1.2.** By the main theorem of Gia06, one has $X_1 = \deg_1 X_1 - \dim \text{Span} \ X_1 + 2)$-regular. Then the result follows from Theorem 3.5. \hfill \Box

**Remark 3.6.** (1) We can define a variety with point components of arbitrary dimension in a similar way as in Definition 3.1 and then get a similar result as Theorem 3.5 immediately.

(2) Theorem 1.2 says that the regularity of a curve with point components can be bounded by its arithmetic degree. Notice that we require that the curve part must be reduced. It is interesting to ask if arithmetic degrees can be used to bound regularities for arbitrary schemes. Unfortunately, the answer is negative in general. In the following example, we show that even in the curve case if the curve part is nonreduced then we cannot bound the regularity by arithmetic degrees.

**Example 3.7.** Consider a degree $d$ rational normal curve $X$ in $\mathbb{P}^d$. Its conormal bundle is $\mathcal{N}_X := \mathcal{I}_X/\mathcal{I}_X^2 = \oplus^{d-1} \mathcal{O}_{\mathbb{P}^1}(d-2)$. Let $X_2$ be a subscheme defined by $\mathcal{I}_X^2$. Then from a short exact sequence

$$
\begin{align}
0 & \rightarrow \mathcal{I}_X / \mathcal{I}_X^2 \rightarrow \mathcal{O}_{X_2} \rightarrow \mathcal{O}_X \rightarrow 0,
\end{align}
$$

we have $\chi(\mathcal{O}_{X_2} \otimes L^m) = \chi(\mathcal{O}_X \otimes L^m) + \chi(\mathcal{N}_X^2 \otimes L^m)$ where $L = \mathcal{O}_{\mathbb{P}^d}(1)$ so that we see that $\deg X_2 = d^2$. Now take a sub-line bundle $\mathcal{O}_{\mathbb{P}^1}(\delta)$ of $\mathcal{N}_X^2$ and let $\mathcal{I}$ be its preimage under the quotient morphism

$$
\begin{align}
0 & \rightarrow \mathcal{I}_X^2 \rightarrow \mathcal{I}_X \rightarrow \mathcal{N}_X^* \rightarrow 0
\end{align}
$$

and then we get an exact sequence

$$
\begin{align}
0 & \rightarrow \mathcal{I}_X^2 \rightarrow \mathcal{I}_X \rightarrow \mathcal{O}_{\mathbb{P}^1}(\delta) \rightarrow 0.
\end{align}
$$

Let $Z$ be the subscheme defined by $\mathcal{I}$. Then from an exact sequence

$$
\begin{align}
0 & \rightarrow \mathcal{I} / \mathcal{I}_X^2 \rightarrow \mathcal{O}_{X_2} \rightarrow \mathcal{O}_Z \rightarrow 0,
\end{align}
$$

we can compute that $\deg Z = d^2 - d$. Now let $\delta = dt$ for $t > 0$. Note that $\mathcal{O}_{\mathbb{P}^1}(\delta) = \mathcal{O}_{\mathbb{P}^1}(-t)|_X$. Then we see that $\reg Z = \reg \mathcal{O}_{\mathbb{P}^1}(\delta) = t + 1$, which cannot be bounded by $\deg Z$.

**Example 3.8.** The regularity bound in Theorem 1.2 can be achieved. Consider a nondegenerate rational normal curve $X$ in $\mathbb{P}^n$. Take a secant line $l$ of $X$ intersecting with $X$ in two distinct reduced points. Let $P_1, \cdots, P_d$ be $d$ distinct reduced points sitting on $l$ but not on $X$. Then consider a curve $Z := X \cup \{P_1, \cdots, P_d\}$ which has those $P_i$’s as isolated components. From Theorem 1.2 we have $\reg Z \leq d + 2$. On the other hand, since $l$ is a $(d + 2)$-secant line of $Z$, we see $\reg Z \geq d + 2$. Hence $\reg Z = d + 2$. 

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4. Regularity bounds for surfaces

In this section, we give a regularity bound for a projective normal surfaces allowing the following singularities: rational, Gorenstein elliptic and log canonical. For the definition and classification of those singularities we refer to [Rei97], [KM98] and [Mat02]. Here we only list the results we shall use and give related references for the convenience of the reader.

**Theorem 4.1.** Let \( P \in X \) be a closed point of a normal surface \( X \). Denote by \( \text{mult}_P X \) the multiplicity of \( X \) at \( P \) and by \( \text{embdim}_P X \) the embedding dimension of \( P \) at \( X \).

1. If \( P \) is a rational singular point then \( \text{mult}_P X + 1 = \text{embdim}_P X \).
2. Let \( d \) be the degree of \( P \) (for the definition of degree see [Rei97, Section 4.23]). Suppose that \( P \) is a Gorenstein elliptic singular point, then
   (i) if \( d = 1, 2 \), then \( \text{mult}_P X = 2 \) and \( \text{embdim}_P X = 3 \);
   (ii) if \( d \geq 3 \), then \( \text{mult}_P X = d \) and \( \text{embdim}_P X = d \).
3. If \( P \in X \) is a log terminal singular point then it is rational singular.
4. If \( P \in X \) is a log canonical singular point which is not log terminal and let \( r \) be the index of \( K_X \) at \( P \) then
   (i) If \( r = 1 \), then \( P \) is elliptic singular.
   (ii) If \( r \geq 2 \), then \( P \) is rational singular.

**Proof.** (1) is Theorem in [Rei97, section 4.17]. (2.i) is Corollary of [loc. cit., Section 4.25] and (2.ii) is Main Theorem of [loc. cit., Section 4.23]. (3) is Theorem 4-6-18 of [Mat02]. (4) is Theorem 4-6-28 of [loc. cit.]. \( \square \)

One of the classical approach to bounding regularity is using generic projection, which was first used, as far as we known, in the work of Lazarsfeld [Laz87] for nonsingular surface case. Then Kwak use this method to study regularity bounds for higher dimensional varieties. We refer to [Kwa98] and [Kwa00] for the details about the construction of generic projection.

Here let us focus on the case of surfaces. Let \( X \) be a nondegenerate projective surface in \( \mathbb{P}^n \) \((n \geq 4)\). Take a linear space \( \Lambda \) in \( \mathbb{P}^n \) of codimension 4 and disjoint with \( X \). By blowing up \( \mathbb{P}^n \) along the center \( \Lambda \) and then projecting to \( \mathbb{P}^3 \), we obtain the diagram

\[
\begin{array}{ccc}
\text{Bl}_\Lambda \mathbb{P}^n & \xrightarrow{q} & \mathbb{P}^3 \\
\downarrow p & & \downarrow \\
\mathbb{P}^n & & \\
\end{array}
\]

Denote by \( f : X \rightarrow \mathbb{P}^3 \) the linear projection of \( X \) to \( \mathbb{P}^3 \) determined by the center \( \Lambda \). Consider the morphism \( q_* (p^* \mathcal{O}_{\mathbb{P}^n} (2)) \rightarrow q_* (p^* \mathcal{O}_X (2)) \) induced by the restriction morphism \( \mathcal{O}_{\mathbb{P}^n} (2) \rightarrow \mathcal{O}_X (2) \). Notice that \( q_* (p^* \mathcal{O}_X (2)) = f_* \mathcal{O}_X (2) \). Then we get a morphism

\[ w_2 : q_* (p^* \mathcal{O}_{\mathbb{P}^n} (2)) \rightarrow f_* \mathcal{O}_X (2) \]

If we choose the coordinates of \( \mathbb{P}^n \) as \( T_0, \cdots, T_n \) such that \( \Lambda \) is defined by the linear forms \( T_0 = T_1 = T_2 = T_3 = 0 \) and denote by \( V = < T_4, \cdots, T_n > \) the vector subspace of \( H^0 (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n} (1)) \), then we can identify

\[ q_* (p^* \mathcal{O}_{\mathbb{P}^n} (2)) = S^2 V \otimes \mathcal{O}_{\mathbb{P}^3} (-2) \oplus V \otimes \mathcal{O}_{\mathbb{P}^3} (-1) \oplus \mathcal{O}_{\mathbb{P}^3} , \]

where \( S^2 V \) is the second symmetric power of \( V \).

When \( X \) has isolated singularities we can further choose the center \( \Lambda \) to be general so that each singular point of \( X \) is the only point in the fiber of the projection \( f \). We state this fact in the following lemma.
Lemma 4.2. Suppose that $X$ is a normal surface with $\{P_1, \cdots, P_r\}$ as all its singular points. Then for a general center $\Lambda$ the projection $f$ satisfies the condition that $\text{Supp } f^{-1}(f(P)) = \{P_i\}$, for $i = 1, \cdots, r$.

Proof. Let $P$ be a singular point of $X$. We define $\Sigma_P$ to be the algebraic set swapped by the line connecting $P$ and any other point $Q$ of $X$. It is clear that the dimension of $\Sigma_P$ is at most 3. Thus the general $\Lambda$ would not touch $\Sigma_P$ for any singular point $P$ since $\Lambda$ has codimension 4. Then the projection $f$ determined by $\Lambda$ will have the desired property. \hfill $\square$

In the sequel, we always assume that the center $\Lambda$ is general so that the projection $f$ has the property in Lemma 4.2. The key point to obtain a regularity bound for $X$ is to prove that the morphism $w_2$ is surjective. If $X$ is nonsingular then the classical result on generic projection says that the each fiber of $f$ has length no more than three and then by base change the morphism $w_2$ is surjective. This is how generic projection was used in the work [Laz87]. However, if $X$ has singular points, the fiber of $f$ would become complicated and its length could be every large. Thus we need to give a reasonable condition on the singular points of $X$ to make $w_2$ surjective.

Condition 4.3. Let $X$ be a normal surface in $\mathbb{P}^n$. For each singular point $P$ of $X$ the Loewy length of the local ring $\mathcal{O}_{X,P}/(l_1,l_2,l_3)$ is no more than 2, where $l_1,l_2,l_3$ are three general linear forms of $\mathbb{P}^n$ passing through the point $P$.

Proposition 4.4. Let $X$ be a normal surface satisfying Condition 4.3. Then for a general center $\Lambda$ the morphism $w_2$ is surjective.

Proof. Let $y \in P^n$ and let $L_y = q^{-1}(y)$ be the fiber of $q$ over $y$, which is a linear space of $\mathbb{P}^n$ of codimension 3. Suppose that the point $y$ is cut out by linear forms $l_1,l_2,l_3$ in $\mathbb{P}^3$, then the linear space $L_y$ is cut out by the forms $l_1,l_2,l_3$ in $\mathbb{P}^n$. It is clear that the fiber $X_y = f^{-1}(y)$ is the scheme-theoretical intersection $X \cap L_y$. In order to show $w_2$ is surjective, by the base change, it is enough to show the surjectivity of the morphism

$$w_{2,y} : H^0(L_y, \mathcal{O}_{L_y}(2)) \rightarrow H^0(L_y, \mathcal{O}_{X_y}(2)).$$

Since $\Lambda$ is general so the projection $f$ satisfies the results of Lemma 4.2. We also assume that $y = f(P)$ for some point $p \in X$. If $P$ is a nonsingular point then by the choice of $\Lambda$, $L_y$ will cut $X$ only in its nonsingular locus. From the classical generic projection theory of nonsingular surface, we see that the fiber $X_y$ has the length at most 3 and thus the morphism $w_{2,y}$ is surjective at $y$.

On the other hand, if $P$ is a singular point of $X$, then by the choice of $\Lambda$ we have $\text{Supp } X_y = \{P\}$. Denote by $A = \mathcal{O}_{X,P}$ the local ring of $P$ on $X$. Locally at the point $P$, $l_1,l_2,l_3$ are three general element in the maximal ideal $m$ of $A$. It is clear that the local ring $\mathcal{O}_{X_y,P} = A/(l_1,l_2,l_3)$. Since we assume that the local ring $A$ satisfies Condition 4.3 we see that the Lowey length of the local ring $\mathcal{O}_{X_y,P}$ is no more than 2. By Corollary 2.2, $X_y$ is 3-regular in the space $L_y$ and then the morphism $w_{2,y}$ is surjective at $y$, which finishes the proof. \hfill $\square$

Lemma 4.5. Suppose that $P \in X$ is a singular point such that $\text{mult}_P X \leq \text{embdim}_P X$, then the local ring $\mathcal{O}_{X,P}$ satisfies Condition 4.3.

Proof. Denote by $A = \mathcal{O}_{X,P}$ the local ring of $P$ on $X$. Let $l_1,l_2,l_3$ be three general elements in the maximal ideal $m$ of $A$. Then $l_1,l_2$ is a regular sequence of $A$ and is in the cotangent space. Thus if denote by $B = A/(l_1,l_2)$, we have $\text{mult}_P X = l(B)$ and $\text{embdim}(B) = \text{embdim} B + 2$. By assumption that $\text{mult}_P X \leq \text{embdim}_P X$, we see that $l(B) \leq \text{embdim} B + 2$ which implies that the Loewy length of the artinian ring $B$ satisfies the inequality $\mu_B \leq 2$. Then the Loewy length of the local ring $A/(l_1,l_2,l_3)$ is also $\leq 2$ since $\mathcal{O}_{X_y,P} = B/(l_3)$. \hfill $\square$
The following Kodaira type vanishing theorem for a projective normal surface is known to experts but we include its proof here since it is very short.

**Lemma 4.6.** Let $X$ be a projective normal surface and $L$ be a ample line bundle on $X$. Then $H^1(X, \omega_X \otimes L) = H^2(X, \omega_X \otimes L) = 0$, where $\omega_X$ is a dualizing sheaf of $X$.

**Proof.** Let $f : X' \to X$ be a resolution of singularities. Then we have a short exact sequence

$$0 \to f_*\omega_{X'} \to \omega_X \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ has support of dimension zero since $X$ is normal. Notice that $R^if_*\omega_{X'} = 0$ for $i > 0$. Then by applying Kawamata-Viehweg vanishing theorem on $X'$ the result then follows. □

Now we come to our main theorem of this section. As we mentioned in Introduction, the main idea is to combine generic projection used by Lazarsfeld [Laz87] with Grothendieck duality.

**Theorem 4.7.** Let $X \subset \mathbb{P}^n$ be a nondegenerate normal surface and suppose that $w_2$ is surjective for a general center $\Lambda$, then

$$\text{reg } X \leq \deg X - \text{codim } X + 1.$$

**Proof.** We choose coordinates of $\mathbb{P}^n$ such that $\Lambda$ is defined by $T_0 = T_1 = T_2 = T_3 = 0$ and denote by $V = \langle T_4, \ldots, T_n \rangle$ the vector subspace of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, then

$$q_*(p^*\mathcal{O}_{\mathbb{P}^n}(2)) = S^2V \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2),$$

where $S^2V$ is the second symmetric power of $V$. Twisting $w_2$ by $\mathcal{O}_{\mathbb{P}^3}(-2)$ and writing $E$ to be the kernel, then we have an exact sequence

$$0 \to E \to S^2V \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3} \to f_*\omega_X \to 0.$$

Since $X$ is Cohen-Macaulay and $f$ is finite, $f_*\omega_X$ is a sheaf of codimension one Cohen-Macaulay $\mathcal{O}_{\mathbb{P}^3}$-module and therefore $E$ is a locally free sheaf of rank

$$r = \text{rank } S^2V \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3} = \frac{(n-2)(n-1)}{2}.$$

We claim that $E^\vee$ is $(-2)$-regular. To see this, let $\omega_X$ be a dualizing sheaf of $X$. Applying $\mathcal{H}\text{om}(\mathcal{O}_X, \omega_{\mathbb{P}^3})$ to the exact sequence (4.7.1), we get an exact sequence

$$0 \to S^2V \otimes \omega_{\mathbb{P}^3}(2) \otimes V \otimes \omega_{\mathbb{P}^3}(1) \oplus \omega_{\mathbb{P}^3} \to E^\vee(-4) \to f_*\omega_X \to 0.$$

Twisting it by $\mathcal{O}_{\mathbb{P}^3}(1)$ and taking $H^1$ cohomology, we see that $H^1(\mathbb{P}^3, E^\vee(-3)) = 0$. Then taking $H^2$ cohomology of the exact sequence (4.7.2), we have

$$0 \to H^2(\mathbb{P}^3, E^\vee(-4)) \to H^2(\mathbb{P}^3, f_*\omega_X) \to H^3(\mathbb{P}^3, \omega_{\mathbb{P}^3}) \to \cdots .$$

Since $H^2(\mathbb{P}^3, f_*\omega_X) = H^2(X, \omega_X) = H^3(\mathbb{P}^3, \omega_{\mathbb{P}^3}) = k$, we obtain that $H^2(\mathbb{P}^3, E^\vee(-4)) = 0$. For cohomology $H^3$ of $E^\vee$, twist the exact sequence (4.7.2) by $\mathcal{O}_{\mathbb{P}^3}(-1)$ and then take $H^3$ cohomology to get an exact sequence

$$H^2(f_*\omega_X(-1)) \to H^3(\mathbb{P}^3, V \otimes \omega_{\mathbb{P}^3}(1)) \to H^3(\mathbb{P}^3, E^\vee(-5)) \to 0.$$

We shall show that the morphism $\theta$ is surjective. By duality, it is the same as

$$H^0(X, \omega_X(1))^\vee \to H^0(\mathbb{P}^3, V \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1))^\vee$$

which is the dual of the morphism

$$H^0(\mathbb{P}^3, V \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(X, \omega_X(1)).$$
Note that $H^0(\mathbb{P}^3, V \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Since $X$ is nondegenerate in $\mathbb{P}^n$ the morphism

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \rightarrow H^0(X, \mathcal{O}_X(1))$$

is injective and therefore $\theta$ is surjective. Thus we obtain $H^3(\mathbb{P}^3, E^\vee(-5)) = 0$ and conclude that $E^\vee$ is $(-2)$-regular.

Back to the exact sequence (4.7.1) and let $d := \deg X$. Since $\text{Supp} f_*\mathcal{O}_X$ is a degree $d$ hypersurface of $\mathbb{P}^3$ we obtain

$$c_1(E) = -d + c_1(S^2V \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3})$$

$$= -d - (n-1)(n-3),$$

and therefore $\det E = \mathcal{O}_{\mathbb{P}^3}((-d - (n-1)(n-3))$. Now from the canonical identity

$$E = (\wedge^{r-1}E)^\vee \otimes \det E,$$

we have that $E$ is $(-2)(r-1) + d + (n-1)(n-3)$-regular, i.e. $(d - n + 3)$-regular. From the exact sequence (4.7.1), we get that $f_*\mathcal{O}_X$ and hence $\mathcal{O}_X$ is $(d - n + 3)$-regular. Finally, by using [Laz87] Lemma 1.5, we conclude that reg $X = (d - n + 3)$. □

Now we can write down the proof of Theorem 1.3 easily as a corollary of the above theorem.

**Proof of Theorem 1.3** This is from Theorem 4.1, Lemma 4.5 and Theorem 4.7. □

**Remark 4.8.** The argument in the proof of Theorem 4.5 can be used directly to prove a regularity bound for a nonsingular curve, which was obtained in [GLP83].

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