The infrared $R^*$ operation

S. Larin$^a$ and P. van Nieuwenhuizen $^b$

$^a$ Institute for Nuclear Research of the Russian Academy of Sciences, 60th October Anniversary Prospect 7a, Moscow 117312. E-mail: larin@ms2.inr.ac.ru

$^b$ C.N.Yang Institute for Theoretical Physics, Stony Brook, NY 11790. E-mail: vannieu@insti.physics.sunysb.edu

Abstract

We describe the infrared $R$-operation for subtraction of infrared divergencies in Feynman diagrams.
Massless particles can introduce infrared divergences (IRD) in Feynman graphs. In Minkowski space there are IRD in the $S$ matrix for QED due to the emission of soft real photons, and also in graphs with virtual photons. These IRD cancel in the cross section [1]. Furthermore, in QCD and also in QED in the approximation that quarks are massless, one has the situation that massless particles couple to themselves or to other massless particles, and this leads to further IRD in the $S$ matrix in Minkowski space, the so-called collinear divergences. These cancel also if one averages over the color of the incoming particles and sums over the momenta of the initial states with soft collinear particles [2], or if one uses factorization methods. In this note we discuss IRD in Green’s functions in Euclidean space; they are unrelated to the IRD in Minkowski space which occur in the $S$ matrix.

Four-dimensional quantum field theories with only dimensionless coupling constants contain in Euclidean space for generic external momenta no IRD. Thus in the proof of renormalizability of proper graphs in QED, all divergences which one encounters (in Euclidean space) are ultraviolet divergences (UVD). The $Z$ factors contain thus only information about the small-distance behavior, and for this reason they can be used to construct running coupling constants. The same applies to QCD and massive quarks. Even if one sets the masses of all particles to zero, it remains true that for generic Euclidean external momenta the proper Green function in four dimensions are free from IRD.

Spontaneously broken field theories have in general dimensionful coupling constants. For example, the $O(2)$ Goldstone model and the $SU(2)$ Higgs model contain an interaction term $\lambda v \sigma^3$ with $\lambda v$ a dimensionful coupling constant. Nevertheless these four-dimensional Goldstone and Higgs models do not contain any IRD in the Euclidean proper graphs; this is due to the Goldstone theorem which states that proper selfenergies $\Pi(p)$ for Goldstone bosons vanish at $p^2 = 0$ even when loops with massive scalars $\sigma$ contribute to the Goldstone boson selfenergy.

However for such theories as $\lambda \varphi^4 + h \varphi^3$ theory in four dimensions with a superrenormalizable dimensionful coupling constant $h$, IRD do occur. A

\[ \text{In dimensional regularization one sets the following gluon selfenergy graph } \frac{1}{k^2} \text{ to zero. This graph has an UVD but not an IRD, and its vanishing should be considered as the result of a computation, not as an independent rule. (One may for example replace } \frac{1}{k^2} \text{ by } \frac{n^2}{k^2(k^2+m^2)} + \frac{1}{k^2+m^2}, \text{ and one finds then that the sum of both integrals vanishes). Similarly } \int \frac{d^4k}{k^4} \text{ vanishes, but now the IRD (evaluated at } n > 4) \text{ cancels the UVD (evaluated at } n < 4). \text{ Again this cancellation follows from the rules of dimensional regularization.} \]


simple graph which shows this explicitly is as follows

![Diagram 1](image1)

We have a massless boson in the larger loop with loop momentum $k$ and a selfenergy insertion with loop momentum $q$ due to a massive scalar. The two massless propagators $k^{-2}$ lead to a logarithmic IRD $\int d^4k/k^4$ because now the proper selfenergy of the massive scalar $\Pi(k) = \int \frac{1}{q^2 + m^2} \frac{1}{(k-q)^2 + m^2} d^4q$ does not vanish for small $k$.

Also at exceptional momenta IRD in Euclidean space can occur as the following example shows

![Diagram 2](image2)

The two external lines in the middle carry momenta $p$ and $-p$ and one can also write them as a $\varphi^2$ -insertion. The loop integral yields $\int d^4k \frac{1}{k^4} \frac{1}{(k+Q)^2}$ with $Q = Q_1 + Q_2$, and contains clearly an IRD at $k = 0$.

Another area where IRD create problems is in the calculations of higher-loop $\beta$ functions. Suppose we want to compute the divergences in the following 4-point graph

![Diagram 3](image3)
It simplifies the calculation a great deal if one sets \( p_3 = p_4 = 0 \). When we set some external momenta to zero, we shall call this operation “nullification of momenta”. Nullification of some of the external momenta makes loop calculations a lot easier but it creates spurious IRD. (By spurious we mean that for generic momenta there are no such infrared divergences). To compute the UVD in this graph after having put \( p_3 = p_4 = 0 \), one must first remove the spurious IRD \[3\].

The conclusion is that in four-dimensional Euclidean space Feynman graphs may contain IRD in the following cases

(i) if there are superrenormalizable coupling constants such as \( \lambda \varphi^3 \) interactions in \( D = 4 \).

(ii) at special external momenta. More precisely, when the sum of some external momenta is equal to the sum of some (or none) of the other external momenta; in particular when some external momenta are nullified. (In the example with external momenta \(+p\) and \(−p\), the sum of these two external momenta vanishes).

When there are both massless and massive particles in the theory, the situation is much more complicated, and one must proceed by studying each theory separately. For example the interaction \( g\varphi\chi^2 \) in \( D = 6 \) with a massless field \( \varphi \) and a massive field \( \chi \) leads to IRD in the \( \varphi \) selfenergy with two or more closed \( \chi \) loops, but if one first renormalizes and imposes the renormalization condition that the \( \varphi \) self-energy vanishes like \( k^2 \), the IRD disappears. This example shows that in general one should first renormalize and then study IRD.

In higher spacetime dimensions the degree of IRD in general decreases because of the measure \( d^Dk \), but in lower spacetime dimensions one encounters more IRD. In particular, in \( D = 2 \) the nonlinear sigma models with action \( g_{ij}\partial_\mu\varphi^i\partial^\nu\varphi^j \) have dimensionless coupling constants (for example \( g_{ij} = \delta_{ij}(1+g\varphi^2) \) has the dimensionless coupling constant \( g \)) but tadpoles and selfenergies are IRD. On the other hand, \( \lambda \varphi^3 \) theory in \( D = 6 \) dimensions is renormalizable but not superrenormalizable (because \( \lambda \) is now dimensionless) and has no IRD \[\S\].

To explain these various results on IRD one can use some simple IR power counting. Consider a proper graph with nonvanishing external momenta \( p_i \) in \( D \) dimensions. We shall assume that the external momenta are nonexcep-

\[2\] As an example, consider in 6 dimensions a selfenergy graph with a massless scalar in the loop, and insert into this loop a string of \( M \) selfenergies with massive scalars in the loops. Then the propagators yield a factor \( (\frac{1}{k^2})^{M+1} \) and the measure yields \( \int d^6k \), but now each massive renormalized selfenergy yields a factor \( \int d^6q (q^2 + m^2)(k-q)^2 + m^2 \sim k^2 \), and there is indeed no infrared divergence.
tional by which we mean that there does not exist a relation
\[
\sum_{j=1}^{M} p_j = \sum_{k=1}^{N} p_k
\]  
for any \( M \geq 0 \) and \( N > 0 \) other than overall energy-momentum conservation. The propagators contain loop momenta and external momenta. Choosing a particular momentum flow through the diagram, there are “soft propagators” with only loop momenta and “hard propagators” with a combination of loop momenta and external momenta. (There are no propagators with only a combination of external momenta since the graph is proper). For vanishing loop momenta, the hard propagators do not become singular if we do not have exceptional external momenta. Hence, for nonexceptional values, the external momenta provide an infrared cut-off for the hard propagators. We can then determine the IRD which occur if one or more loop momenta tend to zero by shrinking all hard propagators to a point. The following example in \( \lambda \phi^4 \) theory illustrates this procedure

\[
\text{The contracted graphs are still proper when the original graph was proper. Because the hard propagators form a connected graph, there is only one contracted vertex for a given proper graph when the external momenta are nonexceptional.}
\]

Just as one can count the degree of UVD of a proper graph by UV counting rules one can also develop IR counting rules. To perform IR counting, consider a contracted proper graph with \( N \) external lines, \( L \) loops, \( I \) internal propagators and vertices \( V_j \) with \( j \) lines, in addition to the contracted vertex. Let \( N_i \) be the number of soft lines at the contracted vertex (the subscript \( i \) stands for internal). Because the contracted graph is still proper the number of soft lines connecting it to the rest of the graph is at least 2, hence \( N_i \geq 2 \). Then the usual counting rule for the number of loops and the relation which
states that any internal (external) line ends at two (one) vertices, lead to

\[ L = I - \left( \sum_j V_j + 1 \right) + 1 \]
\[ N + 2I = \sum_j jV_j + (N + N_i) \quad (6) \]

(In the example \( L = 4, N = 6, V_4 = 2, I = 6 \) and \( N_i = 4 \). Then \( L = 4 = 6 - 2 \) and \( N + 2I = 18 = 8 + 6 + 4 \)). If there are no superrenormalizable couplings, a vertex \( V_j \) in \( D \) dimensions carries \( D - \frac{1}{2} j(D-2) \) momenta attached to it. (For example, in gauge theory in \( D = 4 \), the \( AA\partial A \) vertex carries one momentum and the \( AAAA \) vertex carries no momentum). When all loop momenta tend to zero, minus the overall degree of IRD of a contracted graph is given by

\[ \omega_{IR} = DL - 2I + \sum_j V_j \left( D - \frac{1}{2} j(D-2) \right) \]
\[ = \frac{1}{2} (D-2) N_i \geq D - 2 \quad (7) \]

Therefore in \( D \geq 3 \) there are in general no overall IRD, but in \( D = 2 \) all contracted graphs are logarithmically IRD. In particular the widely used WZWN models contain IRD. On the other hand, gauge theories in 4 dimensions have no IRD, as we already discussed.

What happens if only some of the loop momenta tend to zero, but others do not? If a number \( I_H \leq I \) of the internal momenta are kept hard, one can further contract the original diagram such that only the soft lines \((I - I_H \) in number) remain. For example, one could make the loop momentum of the soft loop at the top in figure (5) hard. If one were to make the two propagators in the loop on top hard, contraction of this new hard lines would yield a second contracted vertex, but contraction of one of the loops on the side would still leave only one contracted vertex.

Let the new hard lines form a subgraph with \( L_H \) loops, \( I_H \) propagators and \( V_{jH} \) vertices. The remaining degree of IRD is in this case

\[ \omega_{IR} \text{ (subcase)} = \omega_{IR} - \Delta \omega_{IR} \]
\[ \Delta \omega_{IR} = DL_H - 2I_H + \sum jV_{jH} \left( D - \frac{1}{2} j(D-2) \right) \quad (8) \]

The subgraph can be worse IR divergent than the original graph because \( \omega_{IR} \text{ (subcase)} \) is equal to \( \omega_{IR} \) of the original graph minus the degree of infrared divergence \( \Delta \omega_{IR} \) of the subgraph. The original set of hard lines together with the new set of hard lines form a new (possibly disconnected) set of hard lines, and we can again apply the IR counting rules to the new contracted
graph. In the example in (3) with $I = 6$ soft lines we can make the two soft lines on the right hard. Then the soft part subdivides as follows

$$\omega_{IR\text{ (subcase)}} = 3D - 8 + (-D + 4) = 2D - 4$$

$$\Delta \omega_{IR} = D - 4 + (D - 2(D - 2)) = 0$$

(9)

The subcase remains indeed IR finite in $D \geq 3$. As another example, one may make the loop on top of figure (3) hard. Then one finds that $\Delta \omega_{IR} = D - 4 + (-D + 4) = -D + 4$, and $\omega_{IR\text{ (subcase)}} = 3D - 8$, which is again IR for $D \geq 3$. However, an example of a subgraph which is divergent while the minimal graph is not is the following self energy in $D=6$:

![Diagram](image)

(10)

The graph is overall IR finite (as we have proven generally) but contracting the two subloops, one finds an IRD proportional to $\int d^6k/k^6$. We must thus learn how to subtract IR subdivergences.

There exists a general scheme for subtracting IRD, which is the counter part of the so-called $R$-scheme of BPHZ for subtracting UVD [4]. The scheme which subtracts both UVD and IRD is called the $R^*$ scheme [5], and one can formally write

$$R^* = R_{UV}R_{IR}$$

(11)

According to this prescription, one first removes all IRD of given graph, and only afterwards subtracts all UVD. This is correct for most cases, but there are graphs where $R_{UV}R_{IR}$ is not equal to $R_{IR}R_{UV}$. The question then arises which order is the correct one, and the answer is that the correct order is $R_{IR}R_{UV}$, and not (11). We already argued before that in the case of theories with massive and massless particles one should first renormalize before extracting IRD. In most of our examples we shall follow the prescription in (11) because this is technically easier, but we shall also discuss a 5-loop graph where (11) is not correct.

The UVD can be canceled by the usual UV counter terms, but the IRD are discarded by hand. For the calculation of $\beta$ functions this is no problem because the IR divergences due to nullification of external momenta were anyhow spurious, but for field theories such as $h\varphi^3$ in $D = 4$ discarding genuine
IRD by hand seems a dubious procedure. One would prefer to have also an IR renormalization procedure similar to the UV renormalization procedure, but it does not seem to exist. Speculations have been made that the sum of infrared divergences vanishes in the two-dimensional WZWN model (when properly summed). [4] If this does not happen in this model, or in massless superrenormalizable theories, one would have to exclude such theories.

In the $R$ subtraction scheme of BPHZ, graphs are expanded into a Taylor series in the external momenta. We shall instead use dimensional regularization to compute both the IRD and the UVD. Before going on, we make a comment on the relation $\int \frac{d^4k}{k^4} = 0$ in dimensional regularization. The reason this integral vanishes is that it contains both an UVD and an IRD, whose sum cancels. (One may separately compute the UVD from $\int \frac{d^4k}{(k^2 + m^2)^2}$ and the IRD from $\int \frac{d^4k \frac{1}{k^2+m^2}}{k^4}$ and show that their sum cancels). If one were to use $\int \frac{d^4k}{k^4} = 0$ in higher-loop $\beta$ function calculations, one would drop some UVD, and hence one would make an error. In fact, one never encounters the need for setting $\int \frac{d^4k}{k^4} = 0$ in the computation of the $\beta$ function at the one-loop and two-loop level, but at higher loops care is required not to discard UVD. The claim is that the $R^*$ scheme does not loose UVD even though it sets tadpoles to zero according to the rules of dimensional regularization.

Let us now explain the infrared subtraction procedure by some simple examples. Consider massless $\lambda \varphi^4$ theory in $D = 4$. The proper graph

\begin{equation}
\begin{array}{c}
p \\
\end{array} \rightarrow \begin{array}{c}
p \\
\end{array}
\end{equation}

contains no IRD at generic $p$, hence there is nothing to subtract. This is clear after contracting the graph

\begin{equation}
\begin{array}{c}
\end{array} \rightarrow \begin{array}{c}
\end{array}
\end{equation}

Let us introduce an operator $R_{IR}$ which projects out the infrared finite part from a given graph. Then in this example we obtain

\begin{equation}
R_{IR} = \quad = \quad \text{IR-finite}
\end{equation}
However, consider in $D = 4$ the graph

![Graph Image](image)

where the dot indicates that two external momenta have been nullified. There is then clearly a logarithmic divergence $\int d^4k/k^4$. This IRD is due to the double propagator $\bullet\longrightarrow\bullet$ and we can determine the divergence it produces by inserting it into the simplest graph where it yields an IRD. So we work in two steps: we determine first the IR divergent part associated with $\bullet\longrightarrow\bullet$ and then we use it to subtract the IRD from the original graph. We indicate the IRD induced by the double propagator $\bullet\longrightarrow\bullet$ by $(\int)_{IR}$. Then $(\int)_{IR}$ should make the simplest graph with an $\bullet\longrightarrow\bullet$ insertion finite as far as IR divergences go

$$
\int d^Dk \frac{1}{k^4(Q-k)^2} + \left( c_1 \frac{1}{\epsilon} \right) \frac{1}{Q^2} = \text{IR-finite}
$$

(16)

Note that we obtain the remainder of the graph after extracting the IRD by deleting the subgraph which yields the IRD, not by contracting it. Analytically the meaning of this graphic equation is as follows

$$
\int d^Dk \frac{1}{k^4(Q-k)^2} + \left( c_1 \frac{1}{\epsilon} \right) \frac{1}{Q^2} = \text{IR-finite}
$$

(17)

We can also formulate the subtraction procedure by replacing $\frac{1}{k^4}$ in the original graph by a term with $\delta^4(k)$

$$
\frac{1}{k^4} \rightarrow \frac{1}{k^4} + c_1 \frac{1}{\epsilon} \delta^D(k) \mu^\epsilon , \epsilon = D - 4
$$

(18)

The factor $\mu^\epsilon$ is needed in dimensional regularization to keep the dimension of the last term the same as that of $k^{-4}$. Given the rules of dimensional regularization, one can compute $c_1$.

We now return to the original graph in (15). As before, the infrared finite part of the original graph is projected out by the operator $R_{IR}$, and
pictorially one has the following relation

\[ R_{IR} \left( \begin{array}{c}
\end{array} \right) = \begin{array}{c}
\end{array} + \left( \begin{array}{c}
\end{array} \right)_{IR} = \text{IR-finite} \quad (19) \]

Note that one always begins with the original graph on the right-hand side, and then one adds terms which subtract the IRD of the various subgraphs of the original graph. By convention one always writes plus signs on the right-hand side.

The UV subtraction scheme can be formulated in the same way. Consider for example the graph in \( D = 4 \)

\[ R_{UV} \left( \begin{array}{c}
\end{array} \right) = \begin{array}{c}
\end{array} + \left( \begin{array}{c}
\end{array} \right)_{UV} + 2 \left( \begin{array}{c}
\end{array} \right)_{UV} \quad (21) \]

Again by convention we always use plus signs for the terms to be subtracted.

After the IRD have been subtracted, one may subtract the UVD. One does this for each term on the right-hand side of the \( R_{IR} \) equation separately. Consider for example (19). The subtraction of UVD proceeds as follows

\[ R_{UV} \left( \begin{array}{c}
\end{array} + \left( \begin{array}{c}
\end{array} \right)_{IR} \right) = \]

\[ R_{UV} \left( \begin{array}{c}
\end{array} \right) + \left( \begin{array}{c}
\end{array} \right)_{IR} R_{UV} \left( \begin{array}{c}
\end{array} \right) \quad (22) \]

We never subtract IRD or UVD from counter terms, so there is no “nesting” of the subtraction procedure.

The UVD are easily located. Only the subgraph \( \begin{array}{c}
\end{array} \) is UVD. Hence

\[ R_{UV} = \begin{array}{c}
\end{array} + \left( \begin{array}{c}
\end{array} \right)_{UV} + \left( \begin{array}{c}
\end{array} \right)_{UV} = \text{UV-finite} \quad (23) \]

\[ R_{UV} = \begin{array}{c}
\end{array} + \left( \begin{array}{c}
\end{array} \right)_{UV} = \text{UV-finite} \]
The tadpole graph in the first line vanishes according to the rules of dimensional regularization. (We set it to zero even though it contains an infrared and an ultraviolet divergence. The ultraviolet divergence in this diagram is accounted for by the whole $R^*$ procedure). Note that again we always begin with the original graph on the right-hand side, and then subtract divergences by adding counter terms. (Again by convention we use plus signs for these subtractions).

Combining the IR and UV subtraction procedure, we obtain for the graph in (15)

\[
R^* \left( \begin{array}{c}
\circ \circ \\
\end{array} \right) = \begin{array}{c}
\circ \\
\end{array} + \left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV} + \left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV} \\
+ \left( \begin{array}{c}
\circ \\
\end{array} \right)_{IR} + \left( \begin{array}{c}
\circ \\
\end{array} \right)_{IR} \left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV} = \text{finite}
\]

Combining the IR and UV subtraction procedure, we obtain for the graph in (15)

\[
R^* \left( \begin{array}{c}
\circ \circ \\
\end{array} \right) = \begin{array}{c}
\circ \\
\end{array} + \left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV} + \left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV} \\
+ \left( \begin{array}{c}
\circ \\
\end{array} \right)_{IR} + \left( \begin{array}{c}
\circ \\
\end{array} \right)_{IR} \left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV} = \text{finite}
\]

We can now determine the overall UV counter term which makes the graph finite after all subdivergences have been removed. This is the UV counter term one needs for the $\beta$ function. It is given by $\left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV}$ and can be computed by evaluating the r.h.s. of the following equation

\[
\left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV} = - \left( \begin{array}{c}
\circ \\
\end{array} \right) + \left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV} + \left( \begin{array}{c}
\circ \\
\end{array} \right)_{IR} + \left( \begin{array}{c}
\circ \\
\end{array} \right)_{IR} \left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV} + \text{finite parts}
\]

Taking the pole parts (PP), we can also write

\[
\left[ \begin{array}{c}
\circ \\
\end{array} \right]_{UV, PP} = - \left( \begin{array}{c}
\circ \\
\end{array} \right) + \left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV} + \left( \begin{array}{c}
\circ \\
\end{array} \right)_{IR} + \left( \begin{array}{c}
\circ \\
\end{array} \right)_{IR} \left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV} + \text{finite parts}
\]

Each graph on the r.h.s. is computed with dimensional regularization, including the original graph (which is of course the most difficult to compute). The subtraction terms denoted by $\left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV}$ and $\left( \begin{array}{c}
\circ \\
\end{array} \right)_{IR}$ are polynomials in $\frac{1}{\epsilon}$, so one must compute also some graphs on the right-hand side to order $\epsilon, \epsilon^2$ etc. More precisely when there is a higher order pole $\frac{1}{\epsilon^k}$ due to $\left( \begin{array}{c}
\circ \\
\end{array} \right)_{UV}$ and $\left( \begin{array}{c}
\circ \\
\end{array} \right)_{IR}$, one must compute the corresponding graph to order $\epsilon^{k-1}$.
To illustrate the analogies and differences of $R_{UV}$ and $R_{IR}$ consider the following example

\[ R_{UV} = \begin{array}{c}
\text{hatched part}\end{array} + (\begin{array}{c}
\text{hatched part}\end{array})_{UV} \]

\[ R_{IR} = \begin{array}{c}
\text{hatched part}\end{array} + (\begin{array}{c}
\text{hatched part}\end{array})_{IR} \]

The hatched part of the graph denotes any nonsingular subgraph. As this example shows, to remove UVD one shrinks the divergent proper subgraphs to a point, but one deletes the IR divergent subgraphs. We compute $(\begin{array}{c}
\text{hatched part}\end{array})_{UV}$ as the $\frac{1}{\epsilon}$ pole part of this one-loop graph, and $(\begin{array}{c}
\text{hatched part}\end{array})_{IR}$ is computed from

\[ R_{IR} = \begin{array}{c}
\text{hatched part}\end{array} + (\begin{array}{c}
\text{hatched part}\end{array})_{IR} = \text{finite} \quad (\text{no } \frac{1}{\epsilon}) \quad (28) \]

In both cases one gets $\frac{1}{\epsilon}$ poles (if one uses dimensional regularization).

Recall that the graphical identity

\[ R_{IR} = \begin{array}{c}
\text{hatched part}\end{array} + (\begin{array}{c}
\text{hatched part}\end{array})_{IR} \]

corresponds to the following analytical expression

\[ R_{IR} \int d^4k \frac{1}{k^4 (k - Q)^2} = \int d^4k \left( \frac{1}{k^4 + \frac{c}{\epsilon} \delta^4(k)} \right) \frac{1}{(k - Q)^2} \quad (30) \]

The IRD occurs at $k = 0$ and is taken care of by the $\frac{1}{\epsilon} \delta^4(k)$ insertion. One may find products of such IR factors $\delta^4(k)/\epsilon$ with different loop momenta, but never with the same loop momentum. As an example consider the following Feynman graph in $\varphi^4 + \varphi^3$ theory

\[ F = \int \frac{d^4kd^4qd^4p}{(k-p)^4p^2k^2 (Q-k)^2 (Q-q)^2(p-q)^2} \quad (31) \]
In this example there is an IRD at $k = p$, and an overall IRD at $k = p = 0$. To subtract these IRD we replace some propagators by delta function in $D$ dimensions

$$R_{IR}(F) = F + \int \left( \frac{1}{(k-p)^4} \rightarrow \mu^4 C_1 \delta^D(k-p) \right) \frac{1}{p^2} \frac{1}{k^2} \frac{1}{(Q-k)^2} \frac{1}{(Q-q)^2} \frac{1}{(p-q)^2} + \int \left( \frac{1}{(p-k)^4} \frac{1}{p^2} \frac{1}{k^2} \rightarrow (\mu^2)^2 C_2 \delta^D(p) \delta^D(k) \right) \frac{1}{(Q-k)^2} \frac{1}{(Q-q)^2} \frac{1}{(p-q)^2}$$

(32)

Because there are now two $D$-dimensional Dirac functions one needs the factor $\mu^2 \epsilon$ with $\epsilon = D - 4$, as usual in dimensional regularization, to make the dimensions come out correctly.

We can write this pictorially as follows

$$R_{IR} \left( \begin{array}{c} \text{propagator} \\ \text{propagator} \end{array} \right) = \begin{array}{c} \text{propagator} \\ \text{propagator} \end{array} + \begin{array}{c} \text{propagator} \\ \text{propagator} \end{array} + \begin{array}{c} \text{propagator} \\ \text{propagator} \end{array} = \text{IR-finite}$$

(33)

We recall that we must first compute $C_1$ and $C_2$, and then by substitution we should find that the r.h.s. is IR-finite.

The computation of $C_1$ was discussed before, it follows from $\begin{array}{c} \text{propagator} \\ \text{propagator} \end{array}$.

The computation of $C_2$ follows from requiring that the following expression be IR-finite

$$R_{IR} \left( \begin{array}{c} \text{propagator} \\ \text{propagator} \end{array} \right) = \begin{array}{c} \text{propagator} \\ \text{propagator} \end{array} + \begin{array}{c} \text{propagator} \\ \text{propagator} \end{array} + 2 \begin{array}{c} \text{propagator} \\ \text{propagator} \end{array} = \text{IR-finite}$$

(34)

The factor 2 is due to the two overall IRD, one upstairs and one downstairs. Note that $\begin{array}{c} \text{propagator} \\ \text{propagator} \end{array} = \frac{2}{\epsilon}$ but $\begin{array}{c} \text{propagator} \\ \text{propagator} \end{array} = \frac{b}{\epsilon} + \frac{c}{\epsilon^2}$. (If the two lower propagators would have been massive, we would have needed a factor one). Note also that $\begin{array}{c} \text{propagator} \\ \text{propagator} \end{array} = 1/Q^4$. Having determined $C_1$ and $C_2$ we can determine $R_{IR} \begin{array}{c} \text{propagator} \end{array}$.

The IR counter terms are always polynomials in $\frac{1}{\epsilon}$ and $\delta^4(k_j)$. However, as our next example shows, one sometimes needs derivatives of $\delta^4(k)$. Consider the following massless graph

13
There is now a quadratic IRD proportional to \( \int d^4p/p^6 \). We want to extract it from the original graph by adding a term involving \( \delta^D(p) \) but in order that dimensions match, we need a Dalembertian acting on \( \delta^D(p) \)

\[
R_{IR} \int \frac{d^Dp}{p^6(p - Q)^2} = \int d^Dp \left[ \frac{1}{p^6} + \mu^\epsilon \bar{C}_1 \square_p \delta^D(p) \right] \frac{1}{(p - Q)^2} \quad (36)
\]

Partially integrating the operator \( \square_p = \frac{\partial}{\partial p^\mu} \frac{\partial}{\partial p_\mu} \) we obtain for the last term in the integrand

\[
\int d^Dp \delta^D(p) \delta^D(p) \frac{1}{(p - Q)^2} = \left[ \square_p \frac{1}{(p - Q)^2} \right]_{p=0} \quad (37)
\]

To fix \( \bar{C}_1 \) we consider the simplest graph with this divergence; this is unfortunately the graph itself but we nevertheless proceed (we could make the original graph more complicated to lift this degeneracy). Hence we fix \( \bar{C}_1 \) from

\[
\int d^Dp \delta^D(p) \frac{1}{(p - Q)^2} = \left[ \square_p \frac{1}{(p - Q)^2} \right]_{p=0} \quad (37)
\]

The two slashes in the last part of the equation indicate the action of \( \square_p \) on \( 1/(p - Q)^2 \). However, \( \square_p (p - Q)^{-2} \bigg|_{p=0} = 2\epsilon Q^{-4} \), so we find the equation

\[
\square_p \frac{1}{(p - Q)^2} = -2\epsilon \quad (39)
\]

One would expect that \( \bar{C}_1 \) is proportional to \( 1/\epsilon \), so the original graph happens to be IR-finite (due to the peculiar properties of dimensional regularization).

To evaluate \( \bar{C}_1 \) we therefore need another graph. One could take a massive propagator with \( (p - Q)^2 \) in which case

\[
\square_p [(p - Q)^2 + m^2]^{-1} = \frac{(8 - 2D)(p - Q)^2 - 2Dm^2}{[(p - Q)^2 + m^2]^3} \quad (40)
\]

\[
\int d^Dp \frac{1}{p^6(p - Q)^2} + \bar{C}_1 2\epsilon \frac{1}{Q^4} = \text{IR-finite} \quad (39)
\]
is no longer proportional to $\epsilon$. One can then determine $\bar{C}_1$.

We now discuss a subtlety having to do with the order in which one applies $R_{UV}$ and $R_{IR}$. The combined operation is denoted by $R^\star$. A priori one might expect that $R_{IR}R_{UV}$ is equal to $R_{UV}R_{IR}$, but there exist counterexamples at the five-loop level \[7\]. Consider the following 5-loop graph

Operating with $R_{UV}R_{IR}$ gives the incorrect result

$$\text{(41)}$$
The IRD arise when the following loop momenta vanish: $p$, $pql$, and $pqls$. When $pt$ vanish (or $pl$), one finds an IRD of the term $\int dpdt t^{-2}$, but this was already considered in the IRD with $\int dpp^{-6}$ so we do not count it separately. We repeatedly used that tadpoles vanish, no matter how many loops they contain. We also used (36)–(39), but because there are two propagators on which $\Box_p$ can act, we get also cross terms where each propagator carries...
one derivative. Operating with $R_{IR} R_{UV}$ gives the correct result

$$\begin{align*}
R_{IR} R_{UV} \left( \begin{array}{c}
\text{Diagram 1}
\end{array} \right) &= \left[ \begin{array}{c}
\text{Diagram 2}
\end{array} \right] \quad + \left( \begin{array}{c}
\text{Diagram 3}
\end{array} \right)_{IR} \quad \{ -2 \epsilon \left( \begin{array}{c}
\text{Diagram 4}
\end{array} \right) \}
\end{align*}$$

As another example, we consider a 5-loop graph which is needed to compute the $\beta$ function at 5-loops. The original diagram gives a vertex correction for the $\lambda \varphi^4$ coupling, namely a graph with 4 external lines, but we nullify all 4 lines. Since vacuum graphs vanish in dimensional regularization we add two
new external lines carrying a new external momentum $Q$.

The reason we let momenta $Q$ flow in and out the graph at these particular points is that this allows to compute the original graph easily. Indeed, to compute the graph itself (which is always needed), one may first compute the subgraphs

$$
\begin{align*}
\sim & \frac{1}{(Q^2)^{2\epsilon}} \quad ; \\
\sim & \frac{1}{(Q^2)^{2\epsilon}}
\end{align*}
$$

Then one evaluates $\int d^D q (Q - q)^2)^{-2\epsilon} (q^2)^{-2\epsilon}$. We now evaluate $R^*$ on this diagram. Since there are no $\frac{1}{p^0}$ terms, there is no term with $\Box_p \delta^D(p)$ and thus no ambiguity whether one should choose $R_{UV} R_{IR}$ or $R_{IR} R_{UV}$. We choose the former. We first record the result for acting with $R_{IR}$ on the graph, and then record the results due to acting with $R_{UV}$ on each of the terms in the result for $R_{IR}$. We write the results such that each column in the result for $R_{UV}$ corresponds to one term in the result for $R_{IR}$.

$$
R_{IR} \quad = \quad + \quad (\text{diagram})
$$

The subgraph $\left( \begin{array}{c} \text{diagram} \end{array} \right)_{IR}$ (which corresponds to the IR divergent integral $\int d^4k d^4p (k - p)^{-2\epsilon} k^2 p^{-4}$ for small $k$ and $p$) is written as $c\delta^D(k)\delta^D(p)$ whereas the subgraph $\left( \begin{array}{c} \text{diagram} \end{array} \right)_{IR}$ contains only one delta function $\delta^D(k + q)$ (since there is only one external momentum, one can only use one $\delta^D(p)$).
At the end no IRD are left (\( \Delta \) is IR finite).

Now we perform \( R_{UV} \) on each of these terms. We write the result of each \( R_{UV} \) operation as a column.

\[
\begin{align*}
\begin{pmatrix}
\text{} \\
\begin{pmatrix}
\text{} \\
\end{pmatrix}
\end{pmatrix}
\end{align*}
\]

\[+ \quad \frac{1}{\epsilon} \quad \begin{pmatrix}
\begin{pmatrix}
\text{} \\
\begin{pmatrix}
\text{} \\
\end{pmatrix}
\end{pmatrix}
\end{pmatrix}
\]

\[+ \quad \frac{2}{\epsilon^2} \quad \begin{pmatrix}
\begin{pmatrix}
\text{} \\
\begin{pmatrix}
\text{} \\
\end{pmatrix}
\end{pmatrix}
\end{pmatrix}
\]

\[+ \quad \frac{1}{\epsilon^3} \quad \begin{pmatrix}
\begin{pmatrix}
\text{} \\
\begin{pmatrix}
\text{} \\
\end{pmatrix}
\end{pmatrix}
\end{pmatrix}
\]

Only the term at the bottom of the first column is needed for the \( \beta \) function. Since we need all \( \frac{1}{\epsilon} \) poles, we must calculate each graph to some power in \( \epsilon \). In particular, since the IRD in the second column is proportional to \( \frac{1}{\epsilon} \) we need the graphs in this column to order \( \epsilon^0 \); but since the IRD in front of the third column is \( \frac{1}{\epsilon} + \frac{2}{\epsilon^2} \), we need the graphs in this column to order \( \epsilon \), and we need the terms proportional to \( 1, \epsilon \) and \( \epsilon^2 \) in the graphs of the last column since the IRD in front of the last column contains a leading term \( 1/\epsilon^3 \). The three 1-loop, 2-loop and 3-loop tadpoles graphs in the first three columns in the third row all vanish according to the rules of dimensional regularization.

All these calculations were done using dimensional regularization, not dimensional reduction (the latter is inconsistent at higher loops \[8\]). For theories with \( \gamma_5 \) and \( \epsilon_{\mu\nu\rho\sigma} \) the approach followed is to first compute graphs without \( \epsilon \) tensors (substituting for \( \gamma_5 \) the product \( \epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \)), and only after the calculation is finished one contracts with four-dimensional \( \epsilon \) tensors. In fact, to compute \( R_{UV}F \) of a divergent graph with several \( \epsilon \) tensors, one can use the fact that \( R_{UV}F \) is \( UV \) finite, and write the product of two \( \epsilon \) tensors in terms of \( D \)-dimensional \( \delta \) functions. The error one makes is of order \( \epsilon \), so vanishes as \( \epsilon \to 0 \). These \( D \)-dimensional Kronecker delta function one can then insert inside the expression \( R_{UV}F \) to obtain a scalar. If one has a single \( \epsilon_{\mu\nu\rho\sigma} \), one can multiply by \( k_1^\alpha k_2^{\beta\gamma} k_3^{\delta} \epsilon_{\alpha\beta\gamma\delta} \) and work out the product \( \epsilon_{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma} \) in terms of \( D \)-dimensional \( \delta \) functions. Effectively this means contracting the open indices in \( R_{UV}F \) with momenta \( k^\mu \) to obtain a Lorentz scalar. So, in the end one never computes with open indices. One obtains the correct answer for the 3-loop chiral anomaly \[9\]. This approach works only for multiplicatively renormalizable quantities, and not diagram-by-diagram.
The reason is that for multiplicatively renormalizable models

$$ R_{UV}F = ZF $$

Then the error in using $D$-dimensional contractions is of order $D - 4$.

Another example where the correct subtraction of IRD is crucial for determining the UVD is the massless WZWN model in $D = 2$ dimensions. The simplest IR subtraction corresponds to

$$ R_{IR} \int \frac{d^2k}{k^2} = \int d^2k \left( \frac{1}{k^2} + \frac{\pi \delta^2(k)}{\epsilon} \right), \quad \epsilon = n - 2 $$

$$ R_{IR} \quad = \quad + 2 \left( \right)_IR = \text{IR-finite} $$

A more complicated example is

$$ \int \frac{q_\rho(k - 2q)_{\sigma}}{(k - q)^2q^2(k - p)^2} d^2q d^2p $$

We write the numerator in terms of the momenta which appear in the propagators, $q_\rho(k - 2q)_{\sigma} = q_\rho(k - q)_{\sigma} - q_\rho q_{\sigma}$, and obtain then graphically

$$ \quad = \quad + \quad \quad $$

The slashes denote momenta, and reduce the IRD. We obtain then

$$ R_{IR} \quad = \quad + \left( \right)_IR \quad = \quad + 2 \left( \right)_IR \quad + \left( \right)_IR $$

Having subtracted the IRD, one may then proceed to compute the UVD. These are supposed to cancel in the $D = 2$ WZWN model at $L \geq 2$ loops, but one clearly needs to be careful with first subtracting the correct amount of IRD.
If there are no momenta in the numerator, one obtains

$$R_{IR} = \text{---} + 3 \left( \begin{array}{c} \text{---} \\ \text{---} \end{array} \right)_{IR} + 3 \left( \text{---} \right)_{IR} = IR \text{ finite}$$

(52)

If one of the propagators is massive, we obtain (denoting the massive propagator by a solid line)

$$R_{IR} = \text{---} + 2 \left( \begin{array}{c} \text{---} \\ \text{---} \end{array} \right)_{IR} \text{---} + \left( \text{---} \right)_{IR} \text{---}$$

(53)

We close with two further examples. First consider in $D = 4$ the following 2-loop graph in massless $\lambda \varphi^4$

$$\begin{array}{c}
\end{array}$$

(54)

Suppose we want to compute the 2-loop contribution to the $Z$ factor of the $\varphi^4$ vertex. We first nullify two external momenta because this simplifies the calculation

$$\begin{array}{c}
\end{array} \rightarrow \begin{array}{c}
\end{array}$$

(55)

Next we subtract the IRD

$$R_{IR} = \text{---} + \left( \begin{array}{c} \text{---} \\ \text{---} \end{array} \right)_{IR} \text{---} = IR \text{ finite}$$

(56)
Finally we subtract the UVD

\[ R_{UV} \left( \begin{array}{c} \circlearrowleft \\ \end{array} + (I)_{IR} \right) = \begin{array}{c} \circlearrowleft \\ \end{array} + (\begin{array}{c} \circlearrowleft \\ \end{array})_{UV} \]

\[ + \begin{array}{c} \circlearrowleft \\ \end{array}_{UV} + (I)_{IR} \quad + (I)_{IR} (\begin{array}{c} \circlearrowleft \\ \end{array})_{UV} \]

The tadpole graph vanishes, and the contribution to the Z factor follows then from

\[ \begin{array}{c} \circlearrowleft \\ \end{array}_{UV} = \left[ \begin{array}{c} \circlearrowleft \\ \end{array} + (I)_{IR} \quad + (I)_{IR} (\begin{array}{c} \circlearrowleft \\ \end{array})_{UV} \right]_{\text{pole part}} \]

\[ (57) \]

For fun we give one last example \[10\]. We consider the following 3-loop graph in massless $\lambda \phi^4$ in $D = 4$

\[ \begin{array}{c} \circlearrowleft \\ \end{array} \]

\[ \begin{array}{c} \circlearrowleft \\ \end{array} \quad \equiv \gamma_1; \quad \begin{array}{c} \circlearrowleft \\ \end{array} \quad \equiv \gamma_2 \]

\[ \begin{array}{c} \circlearrowleft \\ \end{array} \quad \equiv \gamma_3; \quad \begin{array}{c} \circlearrowleft \\ \end{array} \quad \equiv \gamma_4 \]

\[ (59) \]

The infrared subtraction yields

\[ R_{IR} \begin{array}{c} \circlearrowleft \\ \end{array} = \begin{array}{c} \circlearrowleft \\ \end{array} + (I)_{IR} \quad + (\begin{array}{c} \circlearrowleft \\ \end{array})_{IR} \]

\[ (60) \]

Note the appearance of a disconnected graph. Next ultraviolet subtraction
yields

\[
R^* \equiv \left\{ \begin{array}{c}
+ (\ )_{\text{UV}} \\
+ (\ )_{\text{UV}} \\
+ (\ )_{\text{UV}} (\ )_{\text{UV}} \\
+ (\ )_{\text{UV}} \\
+ (\ )_{\text{UV}} (\ )_{\text{UV}} \\
+ (\ )_{\text{UV}} (\ )_{\text{UV}} \\
\end{array} \right\}
\]

\[
\begin{pmatrix}
+ (\ )_{\text{UV}} \\
+ (\ )_{\text{UV}} \\
+ (\ )_{\text{UV}} (\ )_{\text{UV}} \\
+ (\ )_{\text{UV}} (\ )_{\text{UV}} \\
\end{pmatrix}
\begin{pmatrix}
(\ )_{\text{UV}} \\
(\ )_{\text{UV}} \\
(\ )_{\text{UV}} \\
(\ )_{\text{UV}} \\
\end{pmatrix} + (\ )_{\text{IR}}
\]

\[
(61)
\]

The UVD are \( \text{UVD}(\gamma_1) = \text{UVD}(\gamma_2) = \frac{1}{(4\pi)^{\epsilon}} \) with \( \epsilon = n - 4 \), while the IRD are \( \text{IRD}(\gamma_3) = \frac{2}{(4\pi)^{2\epsilon}} \) and \( \text{IRD}(\gamma_4) \) is of the order \( \frac{1}{\epsilon^2} \). The overall UVD of this graph is \( P_G = \frac{1}{3\epsilon^3 (4\pi)^6} \) (where \( G \) denotes the original graph), and \( R^*G \) is UV and IR finite.

We end with final conclusions and comments

(i) Graphs in which all masses have been set to zero allow one to compute \( \beta \) functions in a much simpler way than keeping masses, but one introduces spurious IRD which one should subtract. We have explained the rules for subtracting IRD from Feynman graphs and for determining the final UVD which are relevant for \( \beta \) functions.

(ii) All UV counter terms and IR counter terms are polynomials in \( \frac{1}{\epsilon} \). Tadpoles can be set to zero; even though they may contain UVD, in the total result for the UVD as given by the \( R^* \) scheme no UVD are lost.

(iii) One can write the counter terms as Feynman graphs with some propagators replaced by \( \delta^D(k) \) or \( \Box_k \delta^D(k) \). For example, in \( D = 2 \) one has \( R_{IR} \int \frac{dk}{k^2} = \int d^2k (\frac{1}{2k^2} + \frac{z}{2} \delta^2(k)) \). In this sense the IR counter terms are local in \( p \)-space. In \( x \)-space they would be nonlocal, because infrared divergences deal with the large \( x \) behaviour.

(iv) To remove UVD one shrinks subgraphs, but to remove IRD one deletes
subgraphs. The final UVD is the one one needs for $\beta$ functions. For lower loops $R_{UV}R_{IR}$ is equal to $R_{IR}R_{UV}$, but for higher loops the order matters, and the correct order is $R^* = R_{IR}R_{UV}$. The order only matters if a graph contains a factor $\Box_p \delta(p)$.

(v) in the original BPHZ approach [4, 11], one starts with $R_{UV}F = (1 - t^F)\Pi_H \phi(1 - t^H)F$ where $H$ are all proper subgraphs of the Feynman diagram $F$ which are superficially divergent (power-counting divergent), while $t^F$ yields the overall divergence after all subdivergences have been subtracted. Furthermore, if $H \supset H'$ one should write $1 - t^H$ to the left of $(1 - t^{H'})$, but if $H$ and $H'$ are disjoint or overlapping, it does not matter in which order they appear. There exist refinements which show that one only needs subsets of subgraphs which are nonoverlapping (“forests”). One can then prove the forest formula $R_{UV}F = (1 - t^F)\sum_i \Pi_{H_i \phi_i}(-t^H)F$ where the $\phi_i$ are a forest which includes the empty set. In all examples above we have evaluated this forest formula both for UVD and IRD.

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