$L^p$ COMPACTNESS CRITERIA WITH AN APPLICATION TO VARIATIONAL
CONVERGENCE OF SOME NONLOCAL ENERGY FUNCTIONALS

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Abstract. Motivated by some variational problems from a nonlocal model of mechanics, this
work presents a set of sufficient conditions that guarantee a compact inclusion in the function
space of $L^p$ vector fields defined on a domain $\Omega$ that is either a bounded domain in $\mathbb{R}^d$ or $\mathbb{R}^d$ itself.
The criteria are nonlocal and are given with respect to nonlocal interaction kernels that may not
be necessarily radially symmetric. Moreover, these criteria for vector fields are also different from
those given for scalar fields in that the conditions are based on nonlocal interactions involving
only parts of the components of the vector fields. The $L^p$ compactness criteria are utilized in
demonstrating the convergence of minimizers of parameterized nonlocal energy functionals.

1. Introduction and main results

1.1. Motivation. The present work is motivated by the study of nonlocal peridynamics models
initially proposed by Silling in [32]. In particular, the state-based peridynamics model given in
[31, 32, 33] postulates that the total strain energy for constitutively linear, isotropic solid undergoing
deforrination is given by

\begin{equation}
W_\rho(u) = \beta \int_\Omega (D_\rho(u)(x))^2 \, dx + \alpha \int_\Omega \int_\Omega \rho(x' - x) \left( \mathcal{D}(u)(x, x') - \frac{1}{d} \mathcal{D}_\rho(u)(x) \right)^2 \, dx' \, dx
\end{equation}

where $\Omega \subset \mathbb{R}^d$ is a bounded domain occupied by the solid material, the kernel $\rho(|\xi|)$ is a nonnegative
locally integrable and radial weight function that measures the interaction strength between material
particles at position $x$ and $x'$, $u$ is a displacement field, $\mathcal{D}(u)$ is a rescaled nonlocal operator on $u$
defined by [13]

\begin{equation}
\mathcal{D}(u)(x, x') = \left( \frac{u(x') - u(x)}{|x' - x|} \right) \cdot \frac{|x' - x|}{|x' - x|^2},
\end{equation}

representing a (unit-less) linearized nonlocal strain [33] and the operator $D_\rho$ is a nonlocal linear
operator (a weighted version of $\mathcal{D}$ [13, 14]), called ‘nonlocal divergence’, which is defined as

\begin{equation}
D_\rho(u)(x) := p.v. \int_\Omega \rho(x' - x) \mathcal{D}(u)(x, x') \, dx'
\end{equation}

which is a means of incorporating the effect of the collective deformation of a neighborhood of $x$
into the model. The positive constants $\alpha$ and $\beta$ are proportional to the shear and bulk moduli
of the material, respectively. The quadratic energy in (1.1) is a generalization of the bond-based model
that was introduced in [32] and studied in [1, 15, 17, 24, 38] that takes in to account the linearized
strain due to the dilatation and the deviatoric portions of the deformation. Mathematical analysis
of linearized peridynamics models have been extensively studied in [12, 13, 14, 15, 17, 23, 25, 38]
along with results geared towards nonlinear models in [16, 21, 24, 3, 10].

For $\rho \in L^1_{loc}$, it is not difficult to show (see Lemma 2.3 below) that the energy space associated
with the energy functional $W_\rho$, $\{u \in L^2(\Omega; \mathbb{R}^d) : W_\rho(u) < \infty\}$, is precisely

\begin{equation}
S_{\rho,2}(\Omega) = \left\{ u \in L^2(\Omega; \mathbb{R}^d) : |u|^2_{2, \rho, 2} < \infty \right\},
\end{equation}

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shown in [24, Theorem 1], via an application of Lax-Milgram, that for any applied load the classical Navier-Lamé potential energy
\begin{equation}
E(\rho, u) = \int_\Omega \frac{1}{2} \mu \nabla u : \nabla u + \frac{\lambda}{2} \nabla (\nabla u) : \nabla (\nabla u) - \int_\Omega f \cdot u \, dx
\end{equation}
and
\begin{equation}
\lim_{\delta \to 0} \delta^2 \left( \int_{B_\delta} \rho(\xi) \, d\xi \right)^{-1} = 0.
\end{equation}
and consider a sequence of nonnegative, radial kernels \{\rho_n\}_{n \in \mathbb{N}} each satisfying (1.6) and that
\begin{equation}
\rho_n \leq \rho \quad \text{a.e. and } \rho_n \to \rho \quad \text{a.e. in } \mathbb{R}^d.
\end{equation}
It then follows that \rho_n \to \rho strongly in \text{L}^1_{\text{loc}}(\mathbb{R}^d) as \n \to \infty. We will establish a clear connection between the sequence of energies \{E_{\rho_n}\} and \{E_{\rho}\}. Most importantly, we will show that minimizers of the energies \{E_{\rho_n}\} over an admissible class will converge to a minimizer of \{E_{\rho}\} over the same admissible class. The notion of variational convergence we use is \Gamma-convergence (see [11]) which we define
below. The advantage of $\Gamma$-convergence is that under the additional assumption of equicoercivity of the functionals it implies the convergence of minimizers as well [11 Theorem 7.8 and Corollary 7.202].

**Definition 1.1.** Suppose that $\mathcal{E}_n : L^2(\Omega; \mathbb{R}^d) \to \mathbb{R} \cup \{\infty\}, \forall 1 \leq n \leq \infty$. We say that the sequence $\mathcal{E}_n$ $\Gamma$-converges to $\mathcal{E}_\infty$ in the $L^2$-topology if and only if

a) for every sequence $\{u_n\} \in L^2(\Omega; \mathbb{R}^d)$ with $u_n \to u$ in $L^2(\Omega; \mathbb{R}^d)$, as $n \to \infty$, we have

$$E_\infty(u) \leq \liminf_{n \to \infty} E_n(u_n),$$

b) and for every $u \in L^2(\Omega; \mathbb{R}^d)$ there exists a recovery sequence $u_n \to u$ in $L^2(\Omega; \mathbb{R}^d)$, such that $E_\infty(u) = \lim_{n \to \infty} E_n(u_n)$.

The following is one of the main results of the paper on the variational limit of the nonlocal functionals $\{E_n\}$.

**Theorem 1.2.** Suppose $\rho$ and $\{\rho_n\}$ satisfies (1.6), (1.7), and (1.8). The sequence of functionals $\mathcal{E}_n$ $\Gamma$-converges in the strong $L^2(\Omega; \mathbb{R}^d)$ topology to the functional $E_\rho$, where the extended functionals $\{E_n(u)\}_{n \leq \infty}$ are defined as

$$E_n(u) = \begin{cases} E_{\rho_n}(u), & \text{if } u \in S_{\rho_n,2}(\Omega), \\ \infty, & \text{if } u \in L^2(\Omega; \mathbb{R}^d) \setminus S_{\rho_n,2}(\Omega), \end{cases}$$

where $S_{\rho_n,2}(\Omega)$ and $E_{\rho_n}$ are defined as before in (1.4) and (1.5), respectively, where $\rho$ is replaced by $\rho_n$. The extended functional $E_\rho$ is similarly defined. Moreover, if $V$ is a weakly closed subset of $L^2(\Omega; \mathbb{R}^d)$ such that $V \cap \mathcal{R} = \{0\}$, and for each $n$, $u_n$ minimizes $E_{\rho_n}$ over $V \cap S_{\rho_n,2}(\Omega)$, then the sequence $\{u_n\}$ is precompact in $L^2(\Omega; \mathbb{R}^d)$ with any limit point belonging to $S_{\rho,2}(\Omega)$ and minimizing $E_\rho$ over $V \cap S_{\rho,2}(\Omega)$.

Although the discussion above is focused on the case of quadratic peridynamic energies for ease of explaining the main ideas, the result can naturally be extended to small strain nonlocal nonlinear peridynamic models with $p$-growth, for $p \geq 2$, that have been introduced in [34] and whose variational analysis investigated in [29].

We will prove Theorem 1.2 in the sections that follow. But we would like to highlight that this result has an important implication in the numerical approximation of minimizers of $E_\rho$ over an admissible class. Indeed, compactness results have been quite useful for analyzing numerical approximations of nonlocal problems in various contexts such as [36, 35, 9]. In the context discussed in this work, let us take for an example that $\rho(\xi) = \frac{1}{|\xi|^{d+2(s-1)}}$ for $s \in (0,1)$. This kernel satisfies (1.6) and (1.7). It is also clear that $\rho(\xi)|\xi|^{-2}$ is not integrable on any bounded domain containing $0$. In the event $\Omega$ has a smooth boundary, the energy space $S_{\rho,2}(\Omega)$ coincides with the fractional Sobolev space $H^s(\Omega; \mathbb{R}^d)$ see [31, 28]. In particular, if $1/2 < s < 1$, then all functions in $S_{\rho,2}(\Omega)$ have continuous representative. Now, if $V \subset S_{\rho,2}(\Omega)$ is a weakly closed subset of $L^2(\Omega; \mathbb{R}^d)$ such that $V \cap \mathcal{R} = \{0\}$, a minimizer of $E_\rho$ over $V$ exists (and will be in $H^s(\Omega; \mathbb{R})$). The analysis of the existence and uniqueness of the minimizer $u$ of this quadratic energy can also, equivalently, be found by solving the corresponding Euler-Lagrange equation. The latter gives us a way of numerically solving the equation by writing it first in the weak form and then applying the Galerkin approach of choosing a finite-dimensional subspace $\mathcal{M} \subset V$ to solve for a projected solution of $u$ on $\mathcal{M}$. Notice that for $s \in (0,1/2)$ the finite-dimensional subspace $\mathcal{M}$ can contain discontinuous functions, while for $s \in (1/2, 1)$, all the elements of $\mathcal{M}$ must be continuous in order for $\mathcal{M}$ to be conforming, that is, for $\mathcal{M} \subset S_{\rho,2}(\Omega)$ . In the latter case, if one wants to employ the advantageous discontinuous Galerkin approximation, which is now nonconforming, one needs to find an effective way to implement it to the model problem. The result in Theorem 1.2 will allow us to develop approximation schemes by solving a sequence of Euler-Lagrange equations of modified energies. To demonstrate this, define the sequence of kernels

$$\rho_n(\xi) = \begin{cases} \rho(\xi) & \text{if } \rho(\xi) \leq n |\xi|^2 \\ n |\xi|^2 & \text{if } \rho(\xi) \geq n |\xi|^2. \end{cases}$$
It is not difficult to check that, for each \( n, \rho_n \) satisfies (1.6), (1.8), and that the functions \( \rho_n(\xi)|\xi|^{-2} \in L_{loc}^1(\mathbb{R}^d) \) are just truncations of the fractional kernel \( |\xi|^{-d-2s} \) (at level \( n \)). As discussed before, the energy space associated with \( E_{\rho_n} \) will coincide with \( L^2(\Omega; \mathbb{R}^d) \) and a unique minimizer \( u_n \in V \cap S_n \) of \( E_{\rho_n} \) exists. Since the admissible space is a subspace of \( L^2(\Omega; \mathbb{R}^d) \) that avoids nontrivial infinitesimal rigid displacements, we may use discontinuous finite element spaces, denoted by \( M_{n,h} \), for the standard conforming Galerkin approximation to the solution of the Euler-Lagrange equation associated with the energy \( E_{\rho_n} \). This, in turn, can be viewed as a nonconforming discontinuous Galerkin scheme to numerically solve the original Euler-Lagrange equations when the discretization parameter \( h \) goes to zero and at the same time the truncation level \( n \) goes to infinity. Intuitively, for large \( n \), by Theorem 1.2, \( u \) is approximated by \( u_n \) (in the \( L^2 \) norm), and then \( u_n \) will be numerically approximated by \( u_{n,h} \in M_{n,h} \). The proper convergence analysis of this nonconforming numerical scheme as \( h \to 0 \) and \( n \to \infty \) simultaneously has been carried out in [35] in the special case of scalar nonlinear problems when the subspace \( V \) is the set of scalar-valued functions \( u \in L^2(\Omega; \mathbb{R}^d) \) such that \( u \) vanishes outside of a fixed set \( \Omega' \) which is compactly contained in \( \Omega \). The analysis in [35] makes use of the framework of asymptotically compatible schemes for parameter-dependent problems first developed in [36] and the vanishing of the functions in the admissible class around the volumetric-boundary \( \Omega \setminus \Omega' \) is crucial for employing certain compactness arguments. To extend the convergence analysis in [35] to the case of a system of strongly coupled nonlocal equations, the variations problems associated with (1.5), solved over any admissible set that does not include infinitesimal rigid vector fields, Theorem 1.2 as well as some of the compactness results we prove in this paper will be crucial. Such analysis on nonconforming discontinuous Galerkin numerical schemes to systems of nonlocal equations under discussion will be carried out in a future work.

Although it is beyond the scope of this work, in passing, we would like to note that this way of developing a nonconforming numerical scheme is also applicable to fractional PDEs where singular kernels are more common [5, 37]. The idea is the same where we use less singular kernels with the compact embedding of fractional Sobolev spaces in \( L^p \). It is not difficult to check that, for each \( \rho \), \( \rho > 0 \) for \( r \) close to 0, and \( r^{-p}\rho(r) \) is nonincreasing in \( r \),

\[(1.10) \quad \rho \text{ is radial, } \rho(r) > 0 \text{ for } r \to 0, \text{ and } r^{-p}\rho(r) \text{ is nonincreasing in } r,\]
and

\[ \lim_{\delta \to 0} \delta^p \left( \int_{B_{\delta}(0)} \rho(\xi) d\xi \right)^{-1} = 0. \tag{1.11} \]

We now state the compactness result whose proof is one of the main objectives of the present work.

**Theorem 1.3 (\(L^p\) compactness).** Let \(1 \leq p < \infty\) and let \(\rho \in L^1_{\text{loc}}(\mathbb{R}^d)\) be nonnegative and satisfying \(1.10\) and \(1.11\). Suppose also that \(\Omega \subset \mathbb{R}^d\) is a domain with Lipschitz boundary. Then \(S_{\rho,p}(\Omega)\) is compactly embedded in \(L^p(\Omega; \mathbb{R}^d)\). That is, any bounded sequence \(\{u_n\}\) in \(S_{\rho,p}(\Omega)\) is precompact in \(L^p(\Omega; \mathbb{R}^d)\). Moreover, any limit point is in \(S_{\rho,p}(\Omega)\).

The condition given by \(1.11\) requires \(\rho\) to have an adequate singularity near \(0\). A straightforward calculation shows that the kernels satisfying \(1.11\) include \(\rho(\xi) = |\xi|^{-(d+p(s-1))}\), for any \(p \in [1, \infty)\), and any \(s \in (0, 1)\), and \(\rho(\xi) = -|\xi|^{p-d} \ln(|\xi|)\). It is not surprising that \(1.11\) is violated if \(|\xi|^{-p} \rho(\xi)\) is a locally integrable function (and therefore, \(S_{\rho,p}(\Omega) = L^p(\Omega; \mathbb{R}^d)\)), and in fact, in this case

\[ \liminf_{\delta \to 0} \delta^p \left( \int_{B_{\delta}(0)} \rho(\xi) d\xi \right)^{-1} = \infty, \]

see \([20]\). It is not clear whether condition \(1.11\) is necessary for compact embedding even for the class of kernels that are radial and nonincreasing. There are radial kernels with the property that \(|\xi|^{-p} \rho(\xi)\) is (locally) nonintegrable, and

\[ \lim_{\delta \to 0} \delta^p \left( \int_{B_{\delta}(0)} \rho(\xi) d\xi \right)^{-1} = c_0 > 0 \]

for which we do not know whether there is a compact embedding \(S_{\rho,p}(\Omega)\) into \(L^p(\Omega; \mathbb{R}^d)\). One such kernel is \(\rho(\xi) = |\xi|^{p-1}\). Nevertheless, we can prove that the associated space \(S_{\rho,p}(\Omega)\) is compact in the \(L^p_{\text{loc}}\) topology, a result which we will state and prove in the appendix.

### 1.2.2. Compactness criteria that involve a sequence of kernels.

For scalar fields, compactness results like those stated above are commonplace for spaces corresponding to special kernels such as the standard fractional Sobolev spaces. In \([23\) Lemma 2.2], for more general radial and monotone decreasing kernels \(\rho\), condition \(1.11\) is shown to be sufficient for the compact embedding of the space

\[ \left\{ f \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{\rho(\rho - x) |f(y) - f(x)|^2}{|y - x|^2} < \infty \right\} \]

in \(L^2(\Omega)\). The statement is certainly true for any \(1 \leq p < \infty\). The proof of \([23\) Lemma 2.2] actually relies on and modifies the argument used to prove another type of compactness result by Bourgain, Brezis and Mironescu in \([9\) Theorem 4] that applies criteria involving a sequence of kernels. The argument of \([6]\) uses extensions of functions to \(\mathbb{R}^d\) where the monotonicity of \(\rho\) is used in an essential way to control the semi-norm of the extended functions by the original semi-norm. That is, let us introduce a sequence of radial functions \(\rho_n\) satisfying

\[ \forall n \geq 1, \rho_n \geq 0, \int_{\mathbb{R}^d} \rho_n(\xi) d\xi = 1, \text{ and } \lim_{n \to \infty} \int_{|\xi| > r} \rho_n(\xi) d\xi = 0, \forall r > 0. \tag{1.12} \]

Assuming that for each \(n\), \(\rho_n\) is nonincreasing, and if

\[ \sup_{n \geq 1} \int_\Omega \int_\Omega \rho_n(\rho - x) \frac{|f_n(y) - f_n(x)|^p}{|y - x|^p} < \infty, \tag{1.13} \]

then \(\{f_n\}\) is precompact in \(L^p(\Omega)\), which is the result of \([6\) Theorem 4] obtained by showing that \(1.13\) makes it possible to apply a variant of the Riesz-Frédéch-Kolomogorov theorem \([8]\). In \([23\) Lemma 2.2], for a fixed \(\rho\), the condition \(1.11\) is used to replace the role played by the condition \(1.12\). In \([20\) Theorem 1.2], the same result as in \([6\) Theorem 4] was proved by dropping the monotonicity assumption on \(\rho_n\) for \(d \geq 2\). In addition, the proof in \([20]\) avoids the extension of functions to \(\mathbb{R}^d\) but rather shows that the bulk of the mass of each \(f_n\), that is \(\int_\Omega |f_n|^p\), comes from
The proof relies on a sequence of results on the limiting behavior of functions as well as the action in the interior and quantifies the contribution near the boundary. As a consequence, if (1.13) holds, then as $n \to \infty$ there is no mass concentration or leak at the boundary, two main causes of failure of compactness. The compactness results were applied to establish some variational convergence results in [30]. Clearly if one merely replaces scalar functions in (1.13) by vector fields, both compactness results [6 Theorem 4] and [29 Theorem 1.2] will remain true. It turns out the results will remain valid for vector fields even under a weaker assumption. Indeed, following the argument [6 Theorem 4] and under the monotonicity assumption that for $n$, $\rho_n$ is nonincreasing, it was proved in [22 Theorem 5.1] that if $u_n$ is a bounded sequence of vector fields satisfying

$$\sup_{n \geq 1} \int \int \rho_n(y-x) |\mathcal{D}(u_n)(x,y)|^p \, dy \, dx < \infty$$

then $\{u_n\}$ precompact in the $L^p_{\text{loc}}(\Omega; \mathbb{R}^d)$ topology with any limit point being in $W^{1,p}(\Omega; \mathbb{R}^d)$ when $1 < p < \infty$, and in $\text{BD}(\Omega)$ when $p = 1$. Here, $\text{BD}(\Omega)$ is a space of functions with bounded deformation. Later, again under the monotonicity assumption on $\rho_n$, but using the argument of [29 Theorem 1.2] instead, it was proved in [26 Proposition 4.2] that in fact, (1.14) implies that $\{u_n\}$ is precompact in the $L^p(\Omega; \mathbb{R}^d)$ topology. In this paper, we will prove a similar result relaxing the requirement that $\rho_n$ is a Dirac-Delta sequence.

**Theorem 1.4.** Let $\rho \in L^1_{\text{loc}}$ satisfy (1.10) and (1.11). For each $n$, $\rho_n$ is radial and $\rho_n$ satisfies (1.10) and that

$$\rho_n \geq 0, \quad \rho_n \rightharpoonup \rho, \quad \text{weakly in } L^1_{\text{loc}}(\mathbb{R}^d), \quad \rho_n \leq c \rho$$

for some $c > 0$. Then, if $\{u_n\}$ is a bounded sequence in $L^p(\Omega; \mathbb{R}^d)$ such that (1.14) holds, then $\{u_n\}$ is precompact in $L^p(\Omega; \mathbb{R}^d)$. Moreover, any limit point is in $S_{\rho,p}(\Omega)$.

A natural by-product of Theorem 1.4 is the Poincaré-Korn type inequality stated below.

**Corollary 1.5** (Poincaré-Korn type inequality). Suppose that $1 \leq p < \infty$ and $V$ is a weakly closed subset of $L^p(\Omega; \mathbb{R}^d)$ such that $V \cap R = \{0\}$. Let $\rho \in L^1_{\text{loc}}$ satisfies (1.10) and (1.11). Let $\rho_n$ be a sequence of radial functions, and for each $n$, $\rho_n$ satisfies (1.10) and that

$$\rho_n \geq 0, \quad \rho_n \rightharpoonup \rho, \quad \text{weakly in } L^1_{\text{loc}}(\mathbb{R}^d), \quad \rho_n \leq c \rho$$

for some $c > 0$. Then there exist constants $C > 0$ and $N \geq 1$ such that

$$\int |u|^p \, dx \leq C \int \int \rho_n(y-x) \left| \frac{(u(y) - u(x))}{|y-x|} \right|^p \left| \frac{(y-x)}{|y-x|} \right| \, dy \, dx$$

for all $u \in V \cap L^p(\Omega; \mathbb{R}^d)$ and $n \geq N$. The constant $C$ depends only on $V, d, p, \rho$, and the Lipschitz character of $\Omega$.

We note that the Poincaré-Korn-type inequality has been proved for Dirac-Delta sequence of kernels $\rho_n$ [23, 26]. The corollary extends the result to sequence of kernels that weakly converge to a given function $\rho$ satisfying (1.10) and (1.11).

The rest of the paper is devoted to prove the main results and it is organized as follows. We prove Theorem 1.2 in Section 2, Theorem 1.4 and Proposition 3.5 are proved in Section 3. The proof of Theorems 1.3 and 1.4 and Corollary 1.5 are presented in Section 4. Further discussions are given at the end of the paper.

2. Proof of the variational convergence

In this section we will prove the $\Gamma$-convergence of the sequence of energies $E_{\rho_n}$ defined in (1.5). The proof relies on a sequence of results on the limiting behavior of functions as well as the action of operators. To that end, we assume that $\rho$ and $\{\rho_n\}$ satisfy (1.6), (1.7), and (1.8) throughout this section. We begin with the convergence properties of the nonlocal divergence operator.

**Lemma 2.1.** Suppose that $u_n \rightharpoonup u$ strongly in $L^2(\Omega; \mathbb{R}^d)$, $u \in S_{\rho,2}(\Omega)$, and that $\{\mathcal{D}_{\rho_n}(u_n)\}$ is uniformly bounded in $L^2(\Omega)$. Then $\mathcal{D}_{\rho_n}(u_n) \rightharpoonup \mathcal{D}_{\rho}(u)$ weakly in $L^2(\Omega)$. 
Proof. We recall the nonlocal integration by parts formula \((27)\) that for any \(v \in W^{1,2}_0(\Omega)\)

\[
\int_{\Omega} \mathcal{D}_{\rho_n}(u_n)v(x)dx = -\int_{\Omega} \mathcal{G}_n(v)(x) \cdot u_n(x)dx
\]

where \(\mathcal{G}_n(v)(x)\) is the nonlocal gradient operator

\[
\mathcal{G}_{\rho_n}(v)(x) = p.v. \int_{\Omega} \rho_n(y - x) \frac{v(y) + v(x)}{|y - x|} \frac{y - x}{|y - x|} dy.
\]

Now for \(v \in C^1_c(\Omega)\), we may rewrite the nonlocal gradient as

\[
\mathcal{G}_{\rho_n}(v)(x) = \int_{\Omega} \rho_n(y - x) \frac{v(y) - v(x)}{|y - x|} \frac{y - x}{|y - x|} dy + 2 \int_{\Omega} \rho_n(y - x) \frac{v(x)}{|y - x|} \frac{y - x}{|y - x|} dy.
\]

and estimate as \([27]\) Corollary 2.4,

\[
||\mathcal{G}_{\rho_n}(v)||_{L^\infty} \leq 3||\rho_n||_{L^1} ||\nabla v||_{L^\infty} \leq C||\rho||_{L^1} ||\nabla v||_{L^\infty}.
\]

Also, it is not difficult to show that for all \(x \in \Omega\), \(\mathcal{G}_{\rho_n}(v)(x) \to \mathcal{G}_\rho(v)(x)\). This follows from the convergence of \(\rho_n\) to \(\rho\) in \(L^1_{\text{loc}}(\mathbb{R}^d)\). We thus conclude that for \(v \in C^1_c(\Omega)\),

\[
\mathcal{G}_{\rho_n}(v) \to \mathcal{G}_\rho(v) \quad \text{strongly in } L^2.
\]

Thus from the above integration by parts formula we have that for any \(v \in C^1_c(\Omega)\)

\[
\lim_{n \to \infty} \int_{\Omega} \mathcal{D}_{\rho_n}(u_n)v(x)dx = -\lim_{n \to \infty} \int_{\Omega} \mathcal{G}_{\rho_n}(v)(x) \cdot u_n(x)dx
\]

\[
= -\int_{\Omega} \mathcal{G}_\rho(v)(x) \cdot u(x)dx
\]

\[
= \int_{\Omega} \mathcal{D}_\rho(u)v(x)dx.
\]

The last inequality is possible because \(u \in S_{\rho,2}(\Omega)\). Now for any \(v \in L^2(\Omega)\), let us choose \(v_m \in C^1_c(\Omega)\) such that \(v_m \to v\) strongly in \(L^2(\Omega)\). Then we have for each \(n, m\) that

\[
\int_{\Omega} \mathcal{D}_{\rho_n}(u_n)v(x)dx = \int_{\Omega} \mathcal{D}_{\rho_n}(u_n)v_m(x)dx + R_{n,m}
\]

where

\[
|R_{n,m}| = \left| \int_{\Omega} \mathcal{D}_{\rho_n}(u_n)(v(x) - v_m(x))dx \right| \leq ||\mathcal{D}_{\rho_n}(u_n)||_{L^2(\Omega)} ||v_m - v||_{L^2(\Omega)}.
\]

Therefore using the fact that \(||\mathcal{D}_{\rho_n}(u_n)||_{L^2(\Omega)}\) is uniformly bounded in \(n\), we have that \(\lim_{m \to \infty} \sup_{n \in \mathbb{N}} |R_{n,m}| = 0\) and so we have

\[
\lim_{n \to \infty} \int_{\Omega} \mathcal{D}_{\rho_n}(u_n)v(x)dx = \lim_{n \to \infty} \int_{\Omega} \mathcal{D}_{\rho_n}(u_n)v_m(x)dx + \liminf_{n \to \infty} R_{n,m}
\]

\[
= \int_{\Omega} \mathcal{D}_\rho(u)v_m(x)dx + \liminf_{n \to \infty} R_{n,m}.
\]

We now take \(m \to \infty\) and use the fact that \(\mathcal{D}_\rho(u) \in L^2(\Omega)\) to complete the proof of the lemma. \(\square\)

**Lemma 2.2.** Suppose that \(u_n \to u\) strongly in \(L^2(\Omega; \mathbb{R}^d)\), \(u \in S_{\rho,2}(\Omega)\), and that \(\sup_{n \in \mathbb{N}} W_{\rho_n}(u_n) \leq C < \infty\). Then it holds that

\[
\int_{\Omega} \int_{\Omega} \rho(x' - x) \left( \mathcal{D}(u)(x, x') - \frac{1}{d} \mathcal{D}_\rho(u)(x) \right)^2 dx' dx 
\]

\[
\leq \liminf_{n \to \infty} \int_{\Omega} \int_{\Omega} \rho_n(x' - x) \left( \mathcal{D}(u_n)(x, x') - \frac{1}{d} \mathcal{D}_{\rho_n}(u_n)(x) \right)^2 dx' dx.
\]
Proof. Let \( A \subset\subset \Omega \) and \( \varphi \in C_0^\infty(B_1(0)) \). For \( \epsilon < \text{dist}(A, \partial\Omega) \), consider the sequence of functions \( \varphi_\epsilon \star u_n \) and \( \varphi_\epsilon \star \mathcal{D}_{\rho_n}(u_n) \) defined for \( x \in A \), where \( \varphi_\epsilon(z) = \epsilon^{-d} \varphi(z/\epsilon) \) is standard mollifiers. Then since \( u_n \to u \) strongly in \( L^2 \), for a fixed \( \epsilon > 0 \), we have as \( n \to \infty \),

\[
\varphi_\epsilon \star u_n \to \varphi_\epsilon \star u \quad \text{in} \quad C^2(\overline{A};\mathbb{R}^d) \quad \text{and}
\varphi_\epsilon \star \mathcal{D}_{\rho_n}(u_n) \to \varphi_\epsilon \star \mathcal{D}_{\rho}(u) \quad \text{strongly in} \quad L^2(A). \tag{2.2}
\]

The latter follows from Lemma 2.1 and the fact that the convolution is a compact operator. Using Jensen’s inequality, we have that for each \( \epsilon > 0 \) small and \( n \) large,

\[
\int_A \int_A \rho_n(x' - x) \left( \mathcal{D}(\varphi_\epsilon \star u_n)(x, x') - \frac{1}{d} \varphi_\epsilon \star \mathcal{D}_{\rho_n}(u_n)(x) \right)^2 \, dx' \, dx
\]

\[
\leq \int_A \int_A \rho_n(x' - x) \left( \mathcal{D}(u_n)(x, x') - \frac{1}{d} \mathcal{D}_{\rho_n}(u_n)(x) \right)^2 \, dx' \, dx. \tag{2.3}
\]

The left hand side of (2.3) can be rewritten after change of variables as

\[
\int_\mathbb{R}^d \int_A \rho_n(z) \chi_A(x + z) \left( \mathcal{D}(\varphi_\epsilon \star u_n)(x, x + z) - \frac{1}{d} \varphi_\epsilon \star \mathcal{D}_{\rho_n}(u_n)(x) \right)^2 \, dxdz.
\]

Using (2.2), the sequence of functions

\[ z \mapsto \int_A \chi_A(x + z) \left( \mathcal{D}(\varphi_\epsilon \star u_n)(x, x + z) - \frac{1}{d} \varphi_\epsilon \star \mathcal{D}_{\rho_n}(u_n)(x) \right)^2 \, dx \]

converges in \( L^\infty(\Omega) \) as \( n \to \infty \) to

\[ z \mapsto \int_A \chi_A(x + z) \left( \mathcal{D}(\varphi_\epsilon \star u)(x, x + z) - \frac{1}{d} \varphi_\epsilon \star \mathcal{D}_{\rho}(u)(x) \right)^2 \, dx \]

where we use the simple inequality \(|a^2 - b^2| \leq ||a| - |b||a - b| \) and the assumption that \( u \in S_{\rho,2}(\Omega) \). Using the convergence of \( \rho_n \) to \( \rho \) in \( L^1_{\text{loc}}(\mathbb{R}^d) \) and taking the limit in (2.3) we conclude that for each \( \epsilon > 0 \),

\[
\int_A \int_A \rho(x' - x) \left( \mathcal{D}(\varphi_\epsilon \star u)(x, x') - \frac{1}{d} \varphi_\epsilon \star \mathcal{D}_{\rho}(u)(x) \right)^2 \, dx' \, dx
\]

\[
\leq \liminf_{n \to \infty} \int_A \int_A \rho_n(x' - x) \left( \mathcal{D}(u_n)(x, x') - \frac{1}{d} \mathcal{D}_{\rho_n}(u_n)(x) \right)^2 \, dx' \, dx.
\]

Now inequality (2.1) follows after applying first Fatou’s lemma in \( \epsilon \) and noting that \( A \subset\subset \Omega \) was arbitrary.

Let us state some elementary inequalities that relate the energy \( W_\rho(u) \) and its integrand with that of the seminorm \( |u|_{S_{\rho,2}} \). The proof follows from direct calculations and uses a simple application of Hölder’s inequality.

**Lemma 2.3.** For a given \( \rho \in L^1_{\text{loc}}(\mathbb{R}^d) \) and \( \Omega \) bounded such that for \( u \in S_{\rho,2}(\Omega) \) and \( x \in \Omega \) we have

\[
\mathcal{D}_\rho(u)^2(x) \leq \|\rho\|_{L^1(B_R(0))} \int_\Omega \rho(x - y) |\mathcal{D}(u)(x, y)|^2 \, dy
\]

\[
\int_\Omega \rho(y - x) \left( \mathcal{D}(u)(x, y) - \frac{1}{d} \mathcal{D}_\rho(u)(x) \right)^2 \, dy \leq C(d, \|\rho\|_{L^1(B_R(0))}) \int_\Omega \rho(x - y) |\mathcal{D}(u)(x, y)|^2 \, dy.
\]

Moreover, we have positive constants \( C_1 \) and \( C_2 \), depending on \( \rho, \Omega, \) and \( d \), such that for all \( u \in S_{\rho,2}(\Omega) \)

\[
C_1 |u|_{S_{\rho,2}(\Omega)}^2 \leq W_\rho(u) \leq C_2 |u|_{S_{\rho,2}(\Omega)}^2.
\]
Proof Theorem 1.2. The proof has two parts: the demonstration of the Γ-convergence of the energy functionals and the proof of the convergence of minimizers. For the first part, following the definition of Γ-convergence, we prove the two items in Definition 1.1.

**Item a.** Suppose that \( u_n \to u \) strongly in \( L^2 \). We will show that
\[
E(\rho)(u) \leq \liminf_{n \to \infty} E(\rho)(u_n).
\]
Since \( \int f \cdot u_n \, dx \to \int f \cdot u \, dx \) as \( n \to \infty \), we only need to show that
\[
W(\rho)(u) \leq \liminf_{n \to \infty} W(\rho)(u_n).
\]
To that end, we will assume without loss of generality that \( \liminf_{n \to \infty} W(\rho)(u_n) \leq C < \infty \). Then we have that \( \{D(\rho)(u_n)\} \) is uniformly bounded in \( L^2(\Omega) \) and \( \{|u_n|_{S(\rho,2)}\} \) is uniformly bounded as well, by Lemma 2.3. To prove the desired inequality it suffices to show that
\[
\int (\rho(x' - x) \left( \mathcal{D}(\rho)(x, x') - \frac{1}{d} \mathcal{D}(\rho)(x) \right)^2 \, dx' \, dx 
\]
and
\[
\leq \liminf_{n \to \infty} \int \int \rho(x' - x) \left( \mathcal{D}^\prime(u_n)(x, x') - \frac{1}{d} \mathcal{D}^\prime(u_n)(x) \right)^2 \, dx' \, dx.
\]
To show (2.4), using the weak lower semicontinuity of norm, it suffices to show that
\[
\int (\mathcal{D}(\rho)(u))^2 \, dx \leq \liminf_{n \to \infty} \int (\mathcal{D}(\rho)(u_n))^2 \, dx
\]
and
\[
\int \int \rho(x' - x) \left( \mathcal{D}(\rho)(x, x') - \frac{1}{d} \mathcal{D}(\rho)(x) \right)^2 \, dx' \, dx 
\]
and
\[
\leq \liminf_{n \to \infty} \int \int \rho(x' - x) \left( \mathcal{D}^\prime(u_n)(x, x') - \frac{1}{d} \mathcal{D}^\prime(u_n)(x) \right)^2 \, dx' \, dx.
\]

**Item b.** For a given \( u \in L^2(\Omega) \), we take the recovery sequence to be \( u_n = u \). Now if \( u \in L^2(\Omega) \setminus S(\rho,2) \), then by definition \( E(\rho)(u) = \infty \) and necessarily \( \liminf_{n \to \infty} E(\rho)(u) = \infty \). Otherwise, up to a subsequence (nor renamed) \( \sup_{n \to \infty} E(\rho)(u) < \infty \) and
\[
[u]^2_{S(\rho,2)} \leq C(E(\rho)(u) + \|u\|_{L^2}) \leq C + \|u\|_{L^2},
\]
where we used Lemma 2.3. Then by passing to the limit and using Fatou’s lemma, we have \( [u]^2_{S(\rho,2)} < \infty \), that is, \( u \in S(\rho,2)(\Omega) \), which is a contradiction. In the event that \( u \in S(\rho,2)(\Omega) \), we may use (1.8) to get the pointwise convergence and Lemma 2.3 to get appropriate bounds of the integrand of \( W(\rho)(u) \) to apply the Dominated Convergence Theorem and conclude that \( \liminf_{n \to \infty} W(\rho)(u_n) = W(\rho)(u) \), from which Item b follows.

We next prove the second part of the theorem, the convergence of minimizers. To apply Theorem 1.5 and Corollary 1.202, we need to prove the equicoercivity of the functionals restricted to \( V \cap S(\rho,2) \). That is, for \( u_n \in V \cap S(\rho,2) \) such that \( \sup_{n \geq 1} E(\rho)(u_n) < \infty \), we need to show that the sequence \( \{u_n\} \) is precompact in \( L^2(\Omega; \mathbb{R}^d) \). To that end, first a positive constant \( C > 0 \) and for all \( n \geq 1 \)
\[
[u_n]^2_{S(\rho,2)} \leq C(E(\rho)(u_n) + \|u_n\|_{L^2}) \leq C + \|u_n\|_{L^2}
\]
Using the uniform Poincaré-Korn inequality, Theorem 1.5 for all large \( n \) we have that \( \|u_n\|_{L^2} \leq C[u_n]_{S(\rho,2)} \) and as a consequence
\[
[u_n]^2_{S(\rho,2)} \leq C(1 + [u_n]_{S(\rho,2)}) \quad \text{for all } n \text{ large.}
\]
It then follows that \( [u_n]_{S(\rho,2)} \) is uniformly bounded and therefore, by the uniform Poincaré-Korn inequality, \( \|u_n\|_{L^2(\Omega)} \) is bounded as well. We now use the compactness result, Theorem 1.4 to
conclude that \( \{ u_n \} \) is precompact in \( L^2(\Omega; \mathbb{R}^2) \) with limit point \( u \) in \( S_{\rho,2}(\Omega) \cap V \). We may now apply \cite[Theorem 7.8 and Corollary 7.202]{11} to state that \( u \) is a minimizer of \( E_\rho \) over \( S_{\rho,2}(\Omega) \cap V \). \qed

3. Compactness in \( L^p_{\text{loc}}(\mathbb{R}^d) \)

The proof of the \( L^p \) compactness result, Theorem \ref{thm:compactness}, will be carried out in two steps. We establish first compactness in \( L^p_{\text{loc}} \) topology followed by proving a boundary estimate that controls growth near the boundary of the domain. The \( L^p_{\text{loc}} \) compactness will be proved in this section under a weaker assumption on the kernel. In fact \( L^p_{\text{loc}} \) compactness will be stated and proved for a broader class of kernels that include kernels of the type \( \hat{\rho}(\xi) \chi_{B_1^\Lambda}(\xi) \) where \( \hat{\rho} \) satisfies \((1.10) \) and \((1.11) \), where \( B_1^\Lambda = \{ x \in B_1 : x/|x| \in \Lambda \} \) is a conic region spanned by a given a nontrivial spherical cap \( \Lambda \subset S^{d-1} \).

To make this and the condition of the theorem precise, we begin identifying the kernel \( \rho \) by the representative

\[
\rho(x) = \begin{cases} 
\lim_{h \to 0} \int_{B_h(x)} \rho(\xi) d\xi, & \text{if } x \text{ is a Lebesgue point,} \\
\infty, & \text{otherwise.}
\end{cases}
\]

For \( \theta_0 \in (0, 1) \) and \( v \in S^{d-1} \), let us define

\[
\rho_{\theta_0}(rv) = \inf_{\theta \in [0, 1]} \rho(\theta rv) \theta^{-p}.
\]

It is clear that for a given \( v \in S^{d-1} \), \( \rho_{\theta_0}(rv) \) \( \leq \rho(\theta rv) \theta^{-p} \) for any \( \theta \in [\theta_0, 1] \) and \( r \in (0, \infty) \). In particular, this implies \( \rho_{\theta_0}(\xi) \leq \rho(\xi) \) for any \( \xi \), with the equality holds if \( \rho \) is radial and \( |\xi|^{-p} \rho(\xi) \) is nonincreasing in \( |\xi| \).

We now make a main assumption on \( \rho \) that

\[
\exists \theta_0 \in (0, 1), \Lambda \subset S^{d-1} \) and \( v_0 \in \Lambda \) such that \( H^{d-1}(\Lambda) > 0 \), \( \rho_{\theta_0}(rv) = \rho_{\theta_0}(rv_0), \forall (r, v) \in (0, \infty) \times \Lambda \), and \( \lim_{\delta \to 0} \int_0^\delta \rho_{\theta_0}(rv_0) \delta^{d-1} dr = 0. \)

Assumption \((3.1) \) says that, on a conic region with apex at the origin, the kernel \( \rho \) is above a nonnegative function with appropriate singular growth near the origin. Note that on one hand, it is not difficult to see if \( \rho \in L^1_{\text{loc}}(\mathbb{R}^d) \) is a nonnegative function that satisfies \((1.10) \) and \((1.11) \), then it also satisfies \((3.1) \). On the other hand, if \( \hat{\rho} \) satisfies \((1.10) \) and \((1.11) \), then given a nontrivial spherical cap \( \Lambda \) and conic region \( B_1^\Lambda = \{ x \in B_1 : x/|x| \in \Lambda \} \), the kernel \( \rho(\xi) = \hat{\rho}(\xi) \chi_{B_1^\Lambda}(\xi) \) satisfies \((3.1) \) (with \( \theta_0 \) being any number in \( (0, 1) \) and \( v_0 \) representing any vector in \( \Lambda \)) but not necessarily \((1.10) \) and \((1.11) \). For kernels of this form, we need the formulation in \((3.1) \) to carry out the proof of \( L^p_{\text{loc}} \) compactness. We should also note that one can construct other \( \rho \) that are not of the above form that satisfy \((3.1) \), see \cite[equation (17)\).]

**Theorem 3.1** (\( L^p_{\text{loc}} \) compactness). Suppose that \( 1 \leq p < \infty \). Let \( \rho \in L^1(\mathbb{R}^d) \) be a nonnegative function satisfying \((3.1) \). Suppose also that \( \{ u_n \} \) is a sequence of vector fields that is bounded in \( S_{\rho,p}(\mathbb{R}^d) \). Then for any \( D \subset \mathbb{R}^d \) open and bounded, the sequence \( \{ u_n | D \} \) is precompact in \( L^p(D; \mathbb{R}^d) \).

We should mention that although the focus is different, operators that use non-symmetric kernels like those satisfying the condition \((3.1) \) have been studied in connection with semi-Dirichlet forms and the processes they generate, see \cite[2\) for more discussions. In particular, most of the examples of kernels listed in \cite[Section 6\) satisfy condition \((3.1) \).

3.1. A few technical lemmas. We begin with the following lemma whose proof can be carried out following the argument used in \cite{29\). Let \( u \in L^p(\mathbb{R}^d; \mathbb{R}^d) \) be given, we introduce the function \( F_p[u] : \mathbb{R}^d \to [0, \infty) \) defined by

\[
F_p[u](h) = \int_{\mathbb{R}^d} \left| u(x + h) - u(x) \right|^p \frac{h}{|h|} dx, \quad \text{for } h \in \mathbb{R}^d.
\]
Lemma 3.2. Suppose that \( \theta_0 \) is given as in (3.1). There exists a constant \( C = C(\theta_0, p) > 0 \) such that for any \( \delta > 0 \), and \( v \in \mathbb{S}^{d-1} \)

\[
F_p[u](tv) \leq C \int_0^\infty \rho(hv)^{d-1} F_p[u](hv) \frac{\rho(\theta_0 sv)s^{d-1}ds}{hv} dh,
\]

for any \( 0 < t < \delta \) and any \( u \in L^p(\mathbb{R}^d, \mathbb{R}^d) \).

Proof. For any \( v \in \mathbb{S}^{d-1} \) and \( t \in \mathbb{R} \), we may rewrite the function \( F_p \) as

\[
F_p[u](tv) = \int_{\mathbb{R}^d} |(u(x + tv) - u(x)) \cdot v|^p dx.
\]

It follows from [29] Lemma 3.1 that given \( 0 < s < t \), there exist \( C_p \) and \( \theta = \frac{t}{s} - k \in (0, 1) \) (\( k \) an integer) such that

\[
F_p[u](tv) \leq C_p \left\{ \frac{F_p[u](sv)}{s^p} + \frac{F_p[u](\theta sv)}{t^p} \right\}.
\]

We also have that for a given \( l_0 \in \mathbb{N} \),

\[
F_p[u](\theta sv) \leq l_0 F_p[u] \left( \frac{\theta s}{l_0 v} \right) \leq 2^{(p-1)l_0} \left\{ F_p[u](sv) + F_p[u] \left( s - \frac{s\theta}{l_0} v \right) \right\}.
\]

Combining the above we have that for any \( l_0 \), there exists a constant \( C = C(p, l_0) \) such that

\[
(3.2) \quad \frac{F_p[u](tv)}{t^p} \leq C(p, l_0) \left\{ \frac{F_p[u](sv)}{s^p} + \frac{F_p[u](\tilde{\theta} sv)}{t^p} \right\}, \quad \text{where} \quad \tilde{\theta} = 1 - \frac{\theta}{l_0}.
\]

Now let us take \( \theta_0 \) as given in (3.1) and choose \( l_0 \) large that \( \frac{1}{l_0} < 1 - \theta_0 \). It follows that \( \theta_0 < \tilde{\theta} \leq 1 \).

Then for any \( \delta > 0 \), and any \( 0 < s < \delta \leq \tau \), by multiplying both sides of inequality (3.2) by \( \rho_\theta(v) \) and integrating from 0 to \( \delta \), we obtain

\[
\int_0^\delta \rho_\theta(sv)s^{d-1}ds \frac{F_p[u](\tau v)}{\tau^p} \leq C(p, l_0) \left\{ \int_0^\delta \rho_\theta(sv)s^{d-1} F_p[u](sv) \frac{ds}{s^p} + \int_0^\delta \rho_\theta(sv)s^{d-1} F_p[u](\tilde{\theta} sv) \frac{ds}{t^p} \right\}.
\]

Let us estimate the second integral in the above:

\[
I = \frac{1}{\tau^p} \int_0^\delta \rho_\theta(sv)s^{d-1} F_p[u](\tilde{\theta} sv) ds.
\]

We first note that using the definition of \( \rho_\theta \) and since \( \delta \leq \tau \), we have

\[
I \leq \frac{1}{\theta_0^{d-1}} \int_0^\tau \rho(\tilde{\theta} sv)(\tilde{\theta} s)^{d-1} F_p[u](\tilde{\theta} sv) \frac{ds}{(\theta s)^p}.
\]

The intention is to change variables \( h = \tilde{\theta} s \). However, note that \( \tilde{\theta} \) is a function of \( s \), and by definition

\[
\tilde{\theta} s = \left( \frac{k}{l_0} + 1 \right) s - \frac{\tau}{l_0} \quad \text{for} \quad k \leq \frac{\tau}{s} < k + 1.
\]

It then follows by a change of variables that

\[
I \leq \frac{1}{\theta_0^{d-1}} \sum_{k=1}^{\infty} \int_{\tau(s-1/k)}^{\tau} \rho(\tilde{\theta} sv)(\tilde{\theta} s)^{d-1} F_p[u](\tilde{\theta} sv) \frac{ds}{(\theta s)^p}
\]

\[
= \frac{1}{\theta_0^{d-1}} \sum_{k=1}^{\infty} \int_{\tau(s-1/k)}^{\tau} \rho(hv) h^{d-1} F_p[u](hv) \frac{dh}{h^p} \frac{1}{k + 1}
\]

\[
\leq C \int_0^\infty \rho(hv) h^{d-1} F_p[u](hv) \frac{dh}{h^p} dh,
\]
where in the last estimate integrals over overlapping domains were counted at most a finite number of times. Combining the above estimates we have shown that there exists a constant $C$ such that for any $\mathbf{v} \in S^{d-1}$, $\delta > 0$ and $\tau \geq \delta$

$$
\left( \int_0^\delta \rho_{\theta_0}(s\mathbf{v}) s^{d-1} ds \right) \frac{F_p[\mathbf{u}](\tau \mathbf{v})}{\tau^p} \leq C \int_0^\infty \rho(h\mathbf{v}) h^{d-1} \frac{F_p[\mathbf{u}](h\mathbf{v})}{h^p} dh.
$$

Rewriting the above and restricting $\mathbf{v} \in \Lambda$ we have that

$$
F_p[\mathbf{u}](\tau \mathbf{v}) \leq C \frac{\tau^p}{\int_0^\delta \rho_{\theta_0}(s\mathbf{v}) s^{d-1} ds} \int_0^\infty \rho(h\mathbf{v}) h^{d-1} \frac{F_p[\mathbf{u}](h\mathbf{v})}{h^p} dh.
$$

Now let $0 < t < \delta$ and applying the above inequality for $\tau = \delta$ and $\tau = t + \delta$, we obtain

$$
F_p[\mathbf{u}](t\mathbf{v}) = F_p[\mathbf{u}]( (t + \delta) \mathbf{v} - \delta \mathbf{v}) \leq 2^{p-1} \{ F_p[\mathbf{u}]( (t + \delta) \mathbf{v}) + F_p[\mathbf{u}](\delta \mathbf{v}) \} \leq C \frac{\delta^p}{\int_0^\delta \rho_{\theta_0}(s\mathbf{v}) s^{d-1} ds} \int_0^\infty \rho(h\mathbf{v}) h^{d-1} \frac{F_p[\mathbf{u}](h\mathbf{v})}{h^p} dh.
$$

This completes the proof. \(\square\)

**Lemma 3.3.** Suppose that $\rho \in L^1_{loc}(\mathbb{R}^d)$ and there exists a spherical cap $\Lambda \subset S^{d-1}$ and a vector $\mathbf{v}_0 \in \Lambda$ such that the function $\rho(r\mathbf{v}) = \rho(r\mathbf{v}_0) = \bar{\rho}(r)$, for all $\mathbf{v} \in \Lambda$ and $r \mapsto r^{-p} \bar{\rho}(r)$ is nonincreasing. Then there exists a constant $C = C(d, p, \Lambda)$ such that for any $\delta > 0$, and $\mathbf{v} \in \Lambda$,

$$
F_p[\mathbf{u}](t\mathbf{v}) \leq C \frac{\delta^p}{\int_0^\delta \rho(h\mathbf{v}) s^{d-1} dh} \int_0^\infty \rho(h\mathbf{v}) h^{d-1} \frac{F_p[\mathbf{u}](h\mathbf{v})}{h^p} dh,
$$

for any $0 < t < \delta$ and any $\mathbf{u} \in L^p(\mathbb{R}^d, \mathbb{R}^d)$.

**Proof.** It suffices to note that for $\rho \in L^1_{loc}(\mathbb{R}^d)$ that satisfies the conditions in the statement of the proposition, we have that for any $\theta_0 \in (0, 1)$, and any $\mathbf{v} \in \Lambda$,

$$
\rho_{\theta_0}(r\mathbf{v}) = r^p \inf_{\theta \in [\theta_0, 1]} \rho(\theta r\mathbf{v})(\theta r)^{-p} = \rho(r\mathbf{v}) = \rho(r\mathbf{v}_0) = \bar{\rho}(r).
$$

We may then repeat the argument in the proof of Lemma 3.2. \(\square\)

Before proving one of the main results, we make an elementary observation.

**Lemma 3.4.** Let $1 \leq p < \infty$. Given a spherical cap $\Lambda$ with aperture $\theta$, there exists a positive constant $c_0$, depending only on $d$, $\theta$ and $p$, such that

$$
\inf_{\mathbf{w} \in S^{d-1}} \int_{\Lambda \cap S^{d-1}} |\mathbf{w} \cdot \mathbf{s}|^p d\sigma(s) \geq c_0 > 0.
$$

The above lemma follows from the fact that the map

$$
\mathbf{w} \mapsto \int_{\Lambda \cap S^{d-1}} |\mathbf{w} \cdot \mathbf{s}|^p d\sigma(s)
$$

is continuous on the compact set $S^{d-1}$, and is positive, for otherwise the portion of the unit sphere $\Lambda$ will be orthogonal to a fixed vector which is not possible since $H^{d-1}(\Lambda) > 0$. 

3.2. Proof of Theorem 3.1

From the assumption we have

\begin{equation}
\sup_{n \geq 1} \|u_n\|_{L^p}^p + \sup_{n \geq 1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x' - x) |\mathcal{D}(u_n)(x, x')|^p \, dx' \, dx < \infty.
\end{equation}

(3.3)

We will use the compactness criterion in [22, Lemma 5.4], which is a variant of the well-known Riesz-Fréchet-Kolmogorov compactness criterion [8, Chapter IV.27]. Let \( \Lambda \) be as given in (3.1). For \( \delta > 0 \), let us introduce the matrix \( Q = (q_{ij}) \), where

\[ q_{ij} = \int_{\Lambda} s_i s_j dH^{d-1}(s) \]

The symmetric matrix \( Q \) is invertible. Indeed, the smallest eigenvalue is given by

\[ \lambda_{\min} = \min_{|x| = 1} (Qx, x) = \min_{|x| = 1} \int_{\Lambda} |x \cdot s|^2 dH^{d-1}(s) \]

which we know is positive by Lemma 3.3. We define the following matrix functions

\[ \mathbb{P}(z) = dQ^{-1} \frac{z \otimes z}{|z|^2} \chi_{B_1^e}(z), \quad \mathbb{P}^\delta(z) = \delta^{-d} \mathbb{P} \left( \frac{z}{\delta} \right) \]

where \( B_1^e = \{ x \in B_1 : x/|x| \in \Lambda \} \), as defined before. Then for any \( \delta > 0 \),

\[ \int_{\mathbb{R}^d} \mathbb{P}^\delta(z) \, dz = 1. \]

To prove the theorem, using [22, Lemma 5.4], it suffices to prove that

\begin{equation}
\lim_{\delta \to 0} \limsup_{n \to \infty} \|u_n - \mathbb{P}^\delta \ast u_n\|_{L^p(\mathbb{R}^d)} = 0.
\end{equation}

(3.4)

We show next that the inequality (3.3) and condition (3.1) imply (3.4). To see this, we begin by introducing the notation \( B_\delta^\Lambda = \{ x \in B_\delta(0) : x/|x| \in \Lambda \} \) and applying Jensen’s inequality to get

\begin{align}
\int_{\mathbb{R}^d} |u_n(x) - \mathbb{P}^\delta \ast u_n(x)|^p \, dx &\leq \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \mathbb{P}^\delta(y - x)(u_n(y) - u_n(x)) \, dy \right|^p \, dx \\
&\leq |\Lambda| |Q|^{-1} \int_{B_\delta^e(x)} \frac{(y - x)}{|y - x|} \cdot (u_n(y) - u_n(x)) \frac{(y - x)}{|y - x|} \, dy \right|^p \, dx \\
&\leq |\Lambda| |Q|^{-1} |\mathbb{P}| \int_{B_\delta^e(x)} \frac{(y - x)}{|y - x|} \cdot (u_n(y) - u_n(x)) \frac{(y - x)}{|y - x|} \, dy \right|^p \, dx \\
&\leq \frac{|\Lambda|^p |Q|^{-1} |\mathbb{P}|}{|B_\delta^\Lambda|} \int_0^\delta \int_{\Lambda} \tau^{d-1} F_p[u_n](\tau v) \, dH^{d-1}(v) \, d\tau \\
&\leq C(d, p) \frac{|\mathbb{P}|}{|B_\delta^\Lambda|} \int_0^\delta \int_{\Lambda} \tau^{d-1} F_p[u_n](\tau v) \, dH^{d-1}(v) \, d\tau \\
\end{align}

(3.5)

where as defined previously

\[ F_p[u_n](\tau v) = \int_{\mathbb{R}^d} \tau v \cdot (u_n(x + \tau v) - u_n(x)) \, dx. \]
Moreover, the fact that $|\Lambda|^p \leq C(d, p, \Lambda)$ for any $\delta > 0$ is also used. We can now apply Lemma 3.2 and use the condition (3.1) to obtain that

$$\frac{C(d, p, \lambda)}{|B_\delta^d|} \int_0^\delta \int_{\Lambda} r^{d-1} F_p[u_n](\tau v) d\mathcal{H}^{d-1}(v) dr \leq C(d, p, \Lambda)$$

Moreover, the fact that $|u_n|_{S_{p, p}^e(\mathbb{R}^d)}$ is precompact in $L^p_{loc}(\mathbb{R}^d; \mathbb{R}^d)$. Therefore from the boundedness assumption (3.3) we have,

$$\int_{\mathbb{R}^d} |u_n(x) - \phi^\delta * u_n(x)|^p dx \leq C(p, d, \Lambda) \frac{\delta^p}{\int_0^\delta p_0(s v_0) s^{d-1} ds}.$$

Equation (3.4) now follows from condition (3.1) after letting $\delta \to 0$. That completes the proof.

3.3. A variant of compactness in $L^p_{loc}(\mathbb{R}^d; \mathbb{R}^d)$. A corollary of the compactness result, Theorem 3.1, is the following result that uses a criterion involving a sequence of kernels. The effort made in the proof above was to show the theorem for kernel $\rho$ satisfying (3.1), but the proposition below limits to those satisfying (1.10) and (1.11).

**Proposition 3.5.** Let $\rho \in L^1_{loc}$ satisfy (1.10) and (1.11). Let $\rho_n$ be a sequence of radial functions satisfying (1.10) and that $\rho_n \to \rho$ weakly in $L^1$ as $n \to \infty$. If

$$\sup_{n \geq 1} \left\{ \|u_n\|_{L^p(\mathbb{R}^d)} + |u_n|_{S_{p, p}^e(\mathbb{R}^d)} \right\} < \infty$$

then $\{u_n\}$ is precompact in $L^p_{loc}(\mathbb{R}^d; \mathbb{R}^d)$. Moreover, if $A \subset \mathbb{R}^d$ is a compact subset, the limit point of the sequence restricting to $A$ is in $S_{p, p}^e(A)$.

**Proof.** Using Lemma 3.3 applied to each $\rho_n$, we can repeat the argument in the proof of Theorem 3.1 to obtain

$$\int_{\mathbb{R}^d} |u_n(x) - \phi^\delta * u_n(x)|^p dx \leq C(p, d) \frac{\delta^p}{\int_0^\delta p_0(r) r^{d-1} dr} \leq C(p, d) \frac{\delta^p}{\int_{B_\delta} \rho_n(\xi) d\xi}.$$

Now since $\rho_n \to \rho$, weakly in $L^1$ as $n \to \infty$, for a fixed $\delta > 0$, it follows that

$$\limsup_{n \to \infty} \int_{\mathbb{R}^d} |u_n(x) - \phi^\delta * u_n(x)|^p dx \leq C(p, d) \frac{\delta^p}{\int_{B_\delta} \rho(\xi) d\xi}.$$

We now let $\delta \to 0$, and use the assumption (1.11) to obtain

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \int_{\mathbb{R}^d} |u_n(x) - \phi^\delta * u_n(x)|^p dx = 0,$$

from which the compactness in the $L^p_{loc}$ topology follows.

We next prove the final conclusion of the proposition. To that end, let $A \subset \mathbb{R}^d$ be a compact subset. For $\phi \in C^\infty_0(B_1)$, we consider the convoluted sequence of function $\phi_* \ast u_n$, where $\phi_*(z) = e^{-d} \phi(z/e)$ is the standard mollifier. Since $u_n \to u$ strongly in $L^p(A; \mathbb{R}^d)$ for a fixed $\varepsilon > 0$, we have as $n \to \infty$,

$$\phi_* \ast u_n \to \phi_* \ast u \in C^2(A; \mathbb{R}^d).$$

(3.6)
Using Jensen’s inequality, we obtain that for any \( \epsilon > 0 \), and \( n \) large,

\[
\int \int_{A} \rho_{n}(y - x) \left( \frac{(\phi_{n} \ast u_{n}(y) - \phi_{n} \ast u_{n}(x)) \cdot (y - x)}{|y - x|^{2}} \right)^{p} dy dx \\
\leq \int \int_{R^{d}} \rho_{n}(y - x) \left( \frac{(u_{n}(y) - u_{n}(x)) \cdot (y - x)}{|y - x|^{2}} \right)^{p} dy dx.
\]

Taking the limit in \( n \) for fixed \( \epsilon \), we obtain for any \( A \) compact that

\[
\int \int_{A} \rho(y - x) \left( \frac{(\phi \ast u(y) - \phi \ast u(x)) \cdot (y - x)}{|y - x|^{2}} \right)^{p} dy dx \leq \sup_{n \geq 1} |u_{n}|_{L, p}^{p} < \infty.
\]

where we have used (3.6) and the fact that \( \rho_{n} \) converges weakly to \( \rho \) in \( L^{1} \). Finally, let \( \epsilon \to 0 \) and use Fatou’s lemma (since \( \phi_{n} \ast u \to u \) almost everywhere) to obtain that for any compact set \( A \),

\[
\int \int_{A} \rho(y - x) \left( \frac{(u(y) - u(x)) \cdot (y - x)}{|y - x|^{2}} \right)^{p} dy dx \leq \sup_{n \geq 1} |u_{n}|_{L, p}^{p} < \infty,
\]

hence completing the proof. \( \square \)

4. Global compactness

In this section we prove Theorem 1.3. We follow the approach presented in [29]. The argument relies on controlling the \( L^{p} \) mass of each \( u_{n} \), \( \int_{\Omega} |u_{n}|^{p} dx \), near the boundary by using the bound on the seminorm to demonstrate that in the limit when \( n \to \infty \) there is no mass concentration or loss of mass at the boundary. This type of control has been done for the sequence of kernels that converge to the Dirac Delta measure in the sense of measures. We will do the same for a fixed locally integrable kernel \( \rho \) satisfying the condition (1.11).

4.1. Some technical estimates. In order to control the behavior of functions near the boundary by the semi-norm \( | \cdot |_{S, p} \), we first present a few technical lemmas.

**Lemma 4.1.** [29] Suppose that \( 1 \leq p < \infty \) and that \( g \in L^{p}(0, \infty) \). Then there exists a constant \( C = C(p) \) such that for any \( \delta > 0 \) and \( t \in (0, \delta) \)

\[
\int_{0}^{\delta} |g(x)|^{p} dx \leq C \delta^{p} \int_{0}^{2\delta} \frac{|g(x + t) - g(x)|^{p}}{t^{p}} dx + 2^{p-1} \int_{\delta}^{3\delta} |g(x)|^{p} dx
\]

**Proof.** For a given \( t \in (0, \delta) \), choose \( k \) to be the first positive integer such that \( kt > \delta \). Observe that \( (k - 1)t \leq \delta \), and so \( k \leq 2\delta \). Now let us write

\[
|g(x)|^{p} \leq 2^{p-1}(|g(x + kt) - g(x)|^{p} + |g(x + kt)|^{p}) \\
\leq 2^{p-1}k^{p-1} \sum_{j=0}^{k-1} |g(x + j\delta + t) - g(x + j\delta)|^{p} + 2^{p-1}|g(x + kt)|^{p}.
\]

We now integrate in \( x \) on both side over \((0, \delta)\) to obtain that

\[
\int_{0}^{\delta} |g(x)|^{p} dx \leq 2^{p-1}k^{p-1} \sum_{j=0}^{k-1} \int_{0}^{\delta} |g(x + j\delta + t) - g(x + j\delta)|^{p} dx + 2^{p-1} \int_{0}^{\delta} |g(x + kt)|^{p} dx \\
\leq 2^{p-1}k^{p-1} \sum_{j=0}^{k-1} \int_{j\delta}^{(j+1)\delta} |g(x + j\delta) - g(x)|^{p} dx + 2^{p-1} \int_{\delta}^{3\delta} |g(x)|^{p} dx \\
\leq 2^{p-1}k^{p} \int_{0}^{2\delta} |g(x + t) - g(x)|^{p} dx + 2^{p-1} \int_{\delta}^{3\delta} |g(x)|^{p} dx.
\]

Recalling that \( k \leq 2\delta \), we have that \( k^{p} \leq 2^{p} \delta^{p}/t^{p} \) and we finally obtain the conclusion of the lemma with \( C = 2^{2p-1} \). \( \square \)
Indeed, let us pick \( x \cap \) showing that \( \zeta \) is larger than or equal to \( C \).

The constant \( \theta \) uniform aperture \( \in \) function \( C \) the uniform aperture \( \theta \) and does not depend on \( \xi \). We also assume that the Lipschitz constant \( \kappa = 1/2 \) and the uniform aperture \( \theta = \pi/4 \). As a consequence, \( \zeta(x') \leq |x'|/2 \) for all \( x' \in B_{4r_{0}}(0') \). Given any \( 0 < r < r_{0} \), we consider the graph of \( \zeta : \Gamma_{r} := \{ x = (x', \zeta(x')) \in \mathbb{R}^{d} : x' \in B_{r}(0') \} \).

We now begin to work on local boundary estimates. To do that without loss of generality, after translation and rotation (if necessary) we may assume that \( \xi = 0 \) and

\[
\Omega \cap B(0, 4r_{0}) = \{ (x', x_{d}) : x_{d} > \zeta(x') \} \cap B(0, 4r_{0}),
\]

where \( \zeta(0') = 0 \) and \( |\zeta(x') - \zeta(y')| \leq \kappa|x' - y'| \). We also assume that the Lipschitz constant \( \kappa = 1/2 \) and the uniform aperture \( \theta = \pi/4 \). As a consequence, \( \zeta(x') \leq |x'|/2 \) for all \( x' \in B_{4r_{0}}(0') \).

We denote the upper cone with aperture \( \pi/4 \) by \( \Sigma \) and is given by

\[
\Sigma = \{ x = (x', x_{d}) \in \mathbb{R}^{d} : |x'| \leq x_{d} \}.
\]

Finally we define \( \Omega_{r} = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \tau \} \) to be the set of points in \( \Omega \) at least \( r \) units away from the boundary. Based on the above discussion we have that for any \( r \in (0, r_{0}), \)

\[
\Omega \cap B_{r/2} \subset \Gamma_{r} + (\Sigma \cap B_{r}) \subset \Omega \cap B_{3r}.
\]

Indeed, let us pick \( x = (x', x_{d}) \in \Omega \cap B_{r/2} \). The point \( \xi = x - (x', \zeta(x')) = (0', x_{d} - \zeta(x')) \in \Sigma \), since \( 0 < x_{d} - \zeta(x') \). Moreover, by the bound on the Lipschitz constant \( \zeta \) that \( |x_{d} - \zeta(x')| < r/2 + r/4 < r \).

On the other hand, for any \( x = (x'_{1}, \zeta(x'_{1})) + (x'_{2}, (x_{2})_{d}) \in \Gamma_{r} + (\Sigma \cap B_{r}), \)

we have

\[
\zeta(x'_{1} + x'_{2}) - \zeta(x'_{1}) \leq |x'_{2}|/2 \leq (x_{2})_{d}/2,
\]

showing that \( \zeta(x'_{1} + x'_{2}) < \zeta(x'_{1}) + (x_{2})_{d} \) and therefore \( x \in \Omega \). It easily follows that \( x \in B_{3r}, \) as well.

For any \( r \in (0, r_{0}), \) and \( x' \in B_{r}(0') \), and \( v \in \Sigma \cap S^{d-1} \), dist \( \{ x' = \zeta(x') \} + rv, \partial \Omega \) \( \geq r/\sqrt{10}. \)

Indeed, dist \( \{ x' = \zeta(x') \} + rv, \partial \Omega \) is larger than or equal to the length of the black dashed line, which is larger than or equal to \( r \sin(\pi/4 - \arctan(1/2)) = r/\sqrt{10}. \)

**4.2. Near boundary estimate.** In this subsection we establish the near boundary estimate in the following lemma.

**Lemma 4.2.** Suppose that \( \Omega \subset \mathbb{R}^{d} \) is a domain with Lipschitz boundary. Let \( 1 \leq p < \infty \). Then there exist positive constants \( C_{1}, C_{2}, r_{0} \) and \( e_{0} \in (0, 1) \) with the property that for any \( r \in (0, r_{0}), \)

\[
\int_{\Omega} |u|^{p} dx \leq C_{1}(r) \left( \int_{\Omega_{r}} |u|^{p} dx + C_{2} \int_{B_{r}(0)} \right) \int_{\Omega_{r}} \rho(x - y) |\mathcal{D}(u)(x, y)|^{p} dx dy.
\]

The constant \( C_{1} \) may depend on \( r \) but the other constants \( C_{2} \) and \( r_{0} \) depend only on \( d, p \) and the Lipschitz constant of \( \Omega \). Here for any \( \tau > 0 \), we define \( \Omega_{r} = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \tau \} \).
The next step involves a change of variable $y$. A constant $\pi/2$ and the aperture is $\pi/4$. Therefore, $\hat{\zeta}$ is not bigger than 1 and $\partial \Omega$ is the function $\zeta$. We then apply Lemma 4.1 to get a constant $C_p > 0$ such that for any $t \in (0, r)$,

$$
\int_0^r |u(\xi + sv) \cdot v|^p ds \leq C r^p \int_0^r \frac{|(u(\xi + sv + t \cdot v) - u(\xi + sv)) \cdot v|^p}{t^p} ds,
$$

where we used the fact that $u$ vanishes on $\Omega / \sqrt{B}$. Noting that $\xi = (\xi', \zeta(\xi'))$ for some $\xi' \in B_r \subset \mathbb{R}^{d-1}$, we integrate first in the above estimate with respect to $\xi' \in B'_r$ to obtain that

$$
\int_{B'_r} \int_0^r |u(\xi + sv) \cdot v|^p ds d\xi' 
\leq C r^p \int_{B'_r} \int_0^r \frac{|(u(\xi + sv + t \cdot v) - u(\xi + sv)) \cdot v|^p}{t^p} ds d\xi'.
$$

The next step involves a change of variable $y = (\xi', \zeta(\xi')) + sv$. Define a mapping $G : (\xi', s) \mapsto (\xi', \zeta(\xi')) + sv$. Then the Jacobian of the mapping is defined almost everywhere and is given by

$$
J_G = v \cdot (-\nabla \zeta'(\xi'), 1).
$$

Notice that $|J_G|$ is bounded from above and below by two constants since $v \in \Sigma \cap \mathbb{S}^{d-1}$ and the Lipschitz constant of $\zeta$ is not bigger than 1/2. Also notice that $G(B'_r \times (0, 1)) \subset \Gamma_r + \Sigma \cap B_r$. Therefore,

$$
\int_{B'_r} \int_0^r \frac{|(u(\xi + sv + t \cdot v) - u(\xi + sv)) \cdot v|^p}{t^p} ds d\xi' 
\leq C \int_{\Gamma_r + \Sigma \cap B_r} \frac{|(u(y + t \cdot v) - u(y)) \cdot v|^p}{t^p} dy,
$$

$$
\leq C \int_{\Omega \cap B_r} \frac{|(u(y + t \cdot v) - u(y)) \cdot v|^p}{t^p} dy,
$$

\textbf{Proof.} Following the above discussion, let us pick $\eta \in \partial \Omega$ and assume without loss of generality that $\eta = 0$, the function $\zeta$ that defines the boundary $\partial \Omega$ has a Lipschitz constant not bigger than 1/2 and the aperture is $\pi/4$.

Assume first that $u \in L^p(\Omega; \mathbb{R}^d)$, and vanishes on $\Omega / \sqrt{B}$. Let us pick $\xi = (\xi', \zeta(\xi'))$ such that $|\xi'| < r$ and $v \in \Sigma \cap \mathbb{S}^{d-1}$. Let us introduce the function

$$
g_\xi(t) = u(\xi + tv) \cdot v, \quad t \in (0, r_0).
$$

Then for all $\xi \in \Gamma_r$ and $v \in \Sigma \cap \mathbb{S}^{d-1}$, $\xi + rv \in \Omega / \sqrt{B}$. It follows that, by assumption on the vector field $u$, the function $g_\xi(t) \in L^p(0, 2r)$ and $g_\xi(t) = 0$ for $t \in (r, 2r)$. We then apply Lemma 4.1 to get a constant $C_p > 0$ such that for any $t \in (0, r),

$$
\int_0^r |u(\xi + sv) \cdot v|^p ds \leq C r^p \int_0^r \frac{|(u(\xi + sv + t \cdot v) - u(\xi + sv)) \cdot v|^p}{t^p} ds,
$$

Figure 1. $\Sigma$ is the cone with aperture $\pi/4$ (depicted by the blue lines). The Lipschitz graph $\zeta$ remains outside the double cone with aperture $\pi/2 - \arctan(1/2)$ (depicted by the red lines). The red dashed line has length $r$. 

\[ \text{Figure 1.} \Sigma \text{ is the cone with aperture } \pi/4 \text{ (depicted by the blue lines). The Lipschitz graph } \zeta \text{ remains outside the double cone with aperture } \pi/2 - \arctan(1/2) \text{ (depicted by the red lines). The red dashed line has length } r. \]
where we have used (4.1) in the last step. By some straightforward calculations, one can also find that $\Omega \cap B_{t/4} \subseteq G(B_r \times (0, 1))$. Then

$$
\int_{\Omega \cap B_r} |u(y) \cdot v|^p \, dy \leq \int_{G(B_r \times (0, 1))} |u(y) \cdot v|^p \, dy \leq C \int_{B_r} \int_0^r |u(\xi + sv) \cdot v|^p \, ds \, dx'.
$$

It then follows from the above calculations that for all $v \in \Sigma \cap \mathbb{S}^{d-1}$ and all $t \in (0, r)$,

$$
\int_{\Omega \cap B_r} |u(y) \cdot v|^p \, dy \leq C r^p \int_{\Omega \cap B_r} \frac{|(u(y + tv) - u(y)) \cdot v|^p}{t^p} \, dy.
$$

Multiplying the left hand side of (4.2) by $\rho(tv)t^{d-1}$ and integrating in $t \in (0, r)$ and in $v \in \Sigma \cap \mathbb{S}^{d-1}$, we get

$$
\int_0^r \int_{\Sigma \cap \mathbb{S}^{d-1}} \int_{\Omega \cap B_r} \rho(t^d) |u(y) \cdot v|^p \, dy \, ds(v) \, dt
$$

$$
= \int_{\Omega \cap B_r} \int_{\Sigma \cap \mathbb{S}^{d-1}} \frac{|u(y) \cdot \rho(z)|}{|z|^d} \, dz \, dy.
$$

Using Lemma 3.3, we observe that

$$
\int_{\Omega \cap B_r} \int_{\Sigma \cap \mathbb{S}^{d-1}} |u(y) \cdot \frac{z}{|z|}|^p \rho(z) \, dz \, dy
$$

$$
= \int_{\Omega \cap B_r} \frac{|u(y)|| \rho(z)|}{|z|^d} \, dz \, dy
$$

$$
\geq \left( \int_0^r \rho(t^d) \, dt \right) \int_{\Omega \cap B_r} \frac{|u(y)|| \rho(z)|}{|z|^d} \, dz \, dy
$$

$$
\geq c_0 \left( \int_{B_r} \rho(\xi) \, d\xi \right) \int_{\Omega \cap B_r} |u(y)|^p \, dy.
$$

Similarly, we have

$$
\int_0^r \int_{\Sigma \cap \mathbb{S}^{d-1}} \int_{\Omega \cap B_r} \frac{|(u(y + tv) - u(y)) \cdot v|^p}{t^p} \rho(tv)t^{d-1} \, dy \, ds(v) \, dt
$$

$$
= \int_{\Omega \cap B_r} \int_{\Sigma \cap \mathbb{S}^{d-1}} \frac{|(u(y + z) - u(y)) \cdot \frac{z}{|z|}|^p}{|z|^d} \rho(|z|) \, dz \, dy
$$

$$
\leq \int_{\Omega \cap B_r} \int_{\Sigma \cap \mathbb{S}^{d-1}} \frac{|\mathcal{D}(u(x, y))|^p}{|z|^d} \rho(x - y) \, dy \, dx.
$$

Combining inequalities (4.3), (4.4) and (4.2) we obtain that

$$
c_0 \int_{\Omega \cap B_r} |u(y)|^p \, dy \leq \frac{r^p}{\rho(\xi)} \int_{\Omega \cap B_r} \int_{\Omega \cap B_r} |\mathcal{D}(u(x, y))|^p \rho(x - y) \, dy \, dx
$$

for some positive constant $c_0$ which only depends on $d, p$ and the Lipschitz constant of the domain. In particular, the estimate (4.5) holds true at all boundary points $\partial \Omega$. The next argument is used in the proof of [29, Lemma 5.1]. By applying standard covering argument, it follows from the inequality (4.5) that there exist positive constants $c_0 \in (0, 1/(2\sqrt{10}))$ and $C$ with the property that for all $r \in (0, r_0)$, such that for all $u \in L^p(\Omega; \mathbb{R}^d)$ that vanishes in $\Omega \cap \sqrt{\Omega}$

$$
\int_{\Omega \cap B_{2r}} |u|^p \, dx \leq C \int_{B_r} \int_{\Omega \cap B_r} |\mathcal{D}(u(x, y))|^p \rho(x - y) \, dy \, dx.
$$

The positive constants $c_0$ and $C$ depend only on $p$ and the Lipschitz character of the boundary of $\Omega$. For ease of calculation, set $\tilde{r} = 2r/\sqrt{10}$. Then $\tilde{r}/2 = r/\sqrt{10}$. 

Now let $u \in L^p(\Omega; \mathbb{R}^d)$, and let $\phi \in C^\infty(\Omega)$ be such that: $\phi(x) = 0$, if $x \in \Omega_{r/2}$; $0 \leq \phi(x) \leq 1$, if $x \in \Omega_{r/4} \setminus \Omega_{r/2}$; $\phi(x) = 1$, if $x \in \Omega \setminus \Omega_{r/4}$ and $|\nabla \phi| \leq C/r$ on $\Omega$. Applying (4.6) to the vector field $\phi(x)u(x)$, we obtain that

$$
\int_{\Omega \setminus \Omega_{r/2}} |u|^p \rho^p \leq C \int_{B_r} \frac{\rho^p}{\rho(\xi) d\xi} \int_{\Omega} |\nabla (\phi u)(x,y)|^p \rho(x-y) dy \, dx
$$

We may rewrite $\nabla (\phi u)$ as follows

$$
\nabla (\phi u)(x,y) = (\phi(x) + \phi(y)) \nabla (u)(x,y) - \frac{(\phi(x)u(y) - \phi(y)u(x))}{|y-x|} \cdot (y-x).
$$

It then follows that

$$
\int_{\Omega} \int_{\Omega} |\nabla (\phi u)(x,y)|^p \rho(x-y) dy \, dx
$$

$$
\leq 2^{p-1} \int_{\Omega} \int_{\Omega} ||(\phi(x) + \phi(y)) \nabla (u)(x,y)||^p \rho(x-y) dy \, dx
$$

$$
+ 2^{p-1} \int_{\Omega} \int_{\Omega} \left| \frac{\phi(x)u(y) - \phi(y)u(x)}{|y-x|} \right|^p \rho(x-y) dy \, dx
$$

$$
= 2^{p-1} (I_1 + I_2).
$$

The first term $I_1$ can be easily estimated as

$$
I_1 = \int_{\Omega} \int_{\Omega} ||(\phi(x) + \phi(y)) \nabla (u)(x,y)||^p \rho(x-y) dy \, dx
$$

$$
\leq 2 \int_{\Omega} \int_{\Omega} |\nabla (u)(x,y)|^p \rho(x-y) dy \, dx.
$$

Let us estimate the second term, $I_2$. We first break it into three integrals.

$$
I_2 = \int_{\Omega} \int_{\Omega} \left| \frac{\phi(x)u(y) - \phi(y)u(x)}{|y-x|} \right|^p \rho(x-y) dy \, dx
$$

$$
= \int_A + \int_B + \int_C
$$

where $A = \Omega \setminus \Omega_{r/4} \times \Omega \setminus \Omega_{r/4}$, $B = (\Omega \setminus \Omega_{r/8}) \times \Omega_{r/4} \cup (\Omega_{r/4} \times (\Omega \setminus \Omega_{r/8}))$ and $C = \Omega \times \Omega \setminus (A \cup B)$.

We estimate each of these integrals. Let us begin with the simple one: $\int_A$. After observing that $\phi(x) = \phi(y) = 1$ for all $x, y \in \Omega \setminus \Omega_{r/4}$, we have that

$$
\int_A = \int_{\Omega \setminus \Omega_{r/4}} \int_{\Omega \setminus \Omega_{r/4}} |\nabla (u)(x,y)|^p \rho(x-y) dy \, dx,
$$

and the latter is bounded by the semi norm. Next, we note that set $B$ is symmetric with respect to the diagonal, and as a result,

$$
\int_B = 2 \int_{\Omega \setminus \Omega_{r/8}} \int_{\Omega_{r/4}}
$$

and when \((x, y) \in (\Omega \setminus \Omega_{\tilde{r}/8}) \times \Omega_{\tilde{r}/4}\), we have \(\phi(x) = 1\), and so we have

\[
\iint_B = 2 \int_{\Omega \setminus \Omega_{\tilde{r}/8}} \int_{\Omega_{\tilde{r}/4}} \frac{|u(y) - \phi(y)u(x)|}{|y - x|} \frac{|y - x|}{|y - x|} \rho(x - y) dy \, dx
\]

\[
\leq 2^p \int_{\Omega \setminus \Omega_{\tilde{r}/8}} \int_{\Omega_{\tilde{r}/4}} |\phi(y)\mathcal{D}(u)(x, y)|^p \rho(x - y) dy \, dx
\]

\[
+ 2^p \int_{\Omega \setminus \Omega_{\tilde{r}/8}} \int_{\Omega_{\tilde{r}/4}} \left| \frac{u(y)}{|y - x|} \right|^p \rho(x - y) dy \, dx
\]

\[
\leq 2^p \int_{\Omega \setminus \Omega_{\tilde{r}/8}} \int_{\Omega_{\tilde{r}/4}} |\mathcal{D}(u)(x, y)|^p \rho(x - y) dy \, dx
\]

\[
+ \frac{24p}{r^p} \int_{\Omega \setminus \Omega_{\tilde{r}/8}} \int_{\Omega_{\tilde{r}/4}} |u(y)|^p \rho(x - y) dy \, dx
\]

where we have used the fact that \(\text{dist}(\Omega \setminus \Omega_{\tilde{r}/8}, \Omega_{\tilde{r}/4}) = \tilde{r}/8\). As a consequence we have that

\[
\iint_B \leq 2^p \int_{\Omega \setminus \Omega_{\tilde{r}/8}} \int_{\Omega_{\tilde{r}/4}} |\mathcal{D}(u)(x, y)|^p \rho(y - x) dy \, dx + \frac{24p}{r^p} \left( \int_{|h| > \tilde{r}/8} \rho(h) dh \right) \int_{\Omega_{\tilde{r}/4}} |u(y)|^p dy.
\]

To estimate the integral on \(C\), we first observe that for any \((x, y) \in C\), then \(\text{dist}(x, \partial \Omega) \geq \tilde{r}/8\) and \(\text{dist}(y, \partial \Omega) \geq \tilde{r}/8\). Using this information, adding and subtracting \(\phi(x)u(x)\) we can then estimate as follows:

\[
\iint_C \leq 2^{p-1} \int_C |\mathcal{D}(u)(x, y)|^p \rho(y - x) dy \, dx
\]

\[
+ 2^{p-1} \int_C |u(x)|^p \frac{|\phi(x) - \phi(y)|^p}{|x - y|^p} \rho(|x - y|) dy \, dx
\]

\[
\leq 2^{p-1} \int_{\Omega} \int_{\Omega_{\tilde{r}/4}} |\mathcal{D}(u)(x, y)|^p \rho(y - x) dy \, dx
\]

\[
+ \frac{C}{r^p} \int_{B_R} \rho(h) dh \int_{\Omega_{\tilde{r}/8}} |u(x)|^p dx
\]

where we used the estimate \(\|\nabla \phi\| \leq \frac{C}{r}\), and denoted \(R = \text{diam}(\Omega)\).

We then conclude that there exists a universal constant \(C > 0\) such that for any \(r\) small

\[
\int_{\Omega \setminus \Omega_{\tilde{r}/8}} |u|^p dx \leq C \left( \int_{B_r} \rho(|y|) dy \right) \int_{\Omega} \int_{\Omega_{\tilde{r}/4}} |\mathcal{D}(u)(x, y)|^p \rho(y - x) dy \, dx
\]

\[
+ \frac{1}{r^p} \int_{B_R} \rho(h) dh \int_{\Omega_{\tilde{r}/8}} |u|^p dx.
\]

It then follows that

\[
\int_{\Omega} |u|^p dx = \int_{\Omega_{\tilde{r}/8}} |u|^p dx + \int_{\Omega \setminus \Omega_{\tilde{r}/8}} |u|^p dx
\]

\[
\leq \int_{\Omega_{\tilde{r}/8}} |u|^p dx + C \frac{r^p}{\int_{B_r} \rho(|y|) dy} \int_{\Omega} \int_{\Omega_{\tilde{r}/4}} |\mathcal{D}(u)(x, y)|^p \rho(y - x) dy \, dx
\]

\[
+ C \frac{\|\rho\| L^1(B_R)}{r^p} \int_{\Omega_{\tilde{r}/8}} |u|^p dx.
\]
We hence complete the proof of Lemma 4.2 after choosing \( \epsilon_0 \) sufficiently small, say for example \( \epsilon_0 < 1/4\sqrt{10} \), that

\[
\int_{\Omega} |u|^p dx \leq C(r) \int_{\Omega_{\rho}} |u|^p dx + C \frac{r^p}{\rho(|y|)} \int_{\Omega} \int_{B_r(0)} |\mathcal{D}(u)(x,y)|^p \rho(y - x) dy dx,
\]

as desired.

4.3. Compactness in \( L^p(\Omega) \): proof of Theorem 1.3. Let \( u_n \) be a bounded sequence in \( S_{r,p}(\Omega) \). Let \( \phi_j \in C_c^\infty(\Omega) \) such that \( \phi_j \equiv 1 \) in \( \Omega_{1/j} \). Then the sequence \( \{\phi_j u_n\}_n \) is bounded in \( S_{r,p}(\mathbb{R}^d) \), and so by Theorem 3.1 \( \phi_j u_n \) is precompact in \( \Omega \). In particular, \( \{u_n\} \) is relatively compact in \( L^p(\Omega) \). From this one can extract a subsequence \( u_{n_j} \) such that \( u_{n_j} \to u \) in \( L^p_{loc}(\Omega) \). It is easy to see that \( u \in L^p(\Omega) \). In fact, using the pointwise convergence and Fatou’s lemma, we can see that \( u \in S_{r,p}(\Omega) \). What remains is to show that \( u_{n_j} \to u \) in \( L^p(\Omega) \). To that end, we apply Lemma 4.2 for the function \( u_{n_j} - u \), to obtain that

\[
\int_{\Omega} |u_{n_j} - u|^p dx \leq C_1(r) \int_{\Omega_{\rho}} |u_{n_j} - u|^p dx + C_2 \frac{r^p}{\rho(|y|)} \int_{B_r} |u_{n_j} - u|_{S_{r,p}(\Omega)}^p dy dx.
\]

for all \( r \). We now fix \( r \) and let \( j \to \infty \) to obtain that

\[
\limsup_{j \to \infty} \int_{\Omega} |u_{n_j} - u|^p dx \leq C \frac{r^p}{\rho(|y|)} (1 + |u|^p_{S_{r,p}}).
\]

We then let \( r \to 0 \), to obtain that \( \limsup_{j \to \infty} \int_{\Omega} |u_{n_j} - u|^p dx = 0 \). \( \square \)

4.4. Compactness for a sequence of kernels: proof of Theorem 1.4. Arguing as above and by Proposition 3.5 we have that there is a subsequence \( u_{n_j} \to u \) in \( L^p_{loc}(\Omega) \), and that \( u \in S_{r,p}(\Omega) \). To conclude, we apply Lemma 4.2 for the function \( u_{n_j} - u \) corresponding to \( \rho_{n_j} \) to obtain

\[
\int_{\Omega} |u_{n_j} - u|^p dx \leq C_1(r) \int_{\Omega_{\rho}} |u_{n_j} - u|^p dx + C_2 \frac{r^p}{\rho_{n_j}(|h|)} \int_{B_r} |u_{n_j} - u|_{S_{r,p}(\Omega)}^p dy dx.
\]

By assumption \( \rho_{n_j} \leq C \rho \) and so \( |u_{n_j} - u|_{S_{r,p}(\Omega)}^p \leq C |u_{n_j} - u|_{S_{r,p}(\Omega)}^p \). We then let \( j \to \infty \) and apply the weak convergence of \( \rho_n \) to obtain that

\[
\limsup_{j \to \infty} \int_{\Omega} |u_{n_j} - u|^p dx \leq C \frac{r^p}{\rho(|h|)} (1 + |u|^p_{S_{r,p}}).
\]

Finally, we let \( r \to 0 \) to conclude the proof.

4.5. Poincaré-Korn type inequality: proof of Corollary 1.5. We recall that given \( V \subset L^p(\Omega; \mathbb{R}^d) \) satisfying the hypothesis of the corollary, there exists a constant \( P_0 \) such that for any \( u \in V \),

\[
(4.7) \quad \int_{\Omega} |u|^p dx \leq P_0 \int_{\Omega} \int_{\Omega} \rho(y - x) \left( \frac{|u(y) - u(x)|}{|y - x|} \cdot \frac{y - x}{|y - x|} \right)^p dy dx.
\]

This result is proved in [14] or [24]. We take \( P_0 \) to be the best constant. We claim that given any \( \epsilon > 0 \), there exists \( N = N(\epsilon) \in \mathbb{N} \) such that for all \( n \geq N, (1.15) \) holds for \( C = P_0 + \epsilon \). We prove this by contradiction. Assume otherwise and that there exists \( C > P_0 \) such that for every \( n \), there exists \( u_n \in V \cap L^p(\Omega; \mathbb{R}^d), \|u_n\|_{L^p} = 1 \), and

\[
\int_{\Omega} \int_{\Omega} \rho_n(y - x) \left( \frac{|u_n(y) - u_n(x)|}{|y - x|} \cdot \frac{y - x}{|y - x|} \right)^p dy dx < \frac{1}{C}.
\]
By Theorem 4.4, \( u_n \) is precompact in \( L^p(\Omega; \mathbb{R}^d) \) and therefore any limit point \( u \) will have \( \|u\|_{L^p} = 1 \), and will be in \( \hat{V} \cap L^p(\Omega; \mathbb{R}^d) \). Moreover, following the same procedure as in the proof of Proposition 3.5 we obtain that
\[
\int_\Omega \int_\Omega \rho(y - x) \left| \frac{(u(y) - u(x))}{y - x} \right|^p \frac{(y - x)}{|y - x|} \, dy \, dx \\
\leq \liminf_{n \to \infty} \int_\Omega \int_\Omega \rho_n(y - x) \left| \frac{(u_n(y) - u_n(x))}{y - x} \right|^p \frac{(y - x)}{|y - x|} \, dy \, dx \leq \frac{1}{C} \frac{1}{F_0}
\]
which gives the desired contradiction since \( F_0 \) is the best constant in (4.7).

5. DISCUSSIONS

In this work, we have presented a set of sufficient conditions that guarantee a compact inclusion of a set of \( L^p \)-vector fields in the Banach space of \( L^p \) vector fields. The criteria are nonlocal and given with respect to nonlocal interaction kernels that may not be necessarily radially symmetric. We note that, in addition to the mathematical generality, relaxing the radial symmetry assumption on nonlocal interactions can be useful when modeling anisotropic behavior and directional transport. The \( L^p \)-compactness is established for a sequence of vector fields where the nonlocal interactions involve only part of their components, so that the results and discussions represent a significant departure from those known for scalar fields. It is not clear yet to what extent the conditions assumed here can be weakened to reach the same conclusions. In this regard, there are still some outstanding questions in relation to the set of minimal conditions on the interaction kernel as well as on the set of vector fields that imply \( L^p \)-compactness. An application of the compactness result that will be explored elsewhere includes designing approximation schemes for nonlocal system of equations of peridynamic-type similar to the one done in [35] for nonlocal equations.

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APPENDIX A. COMPACTNESS IN \( L^p_{\text{loc}} \) TOPOLOGY

The following theorem as well as the proof we present here is inspired by the compactness result proved in [20] (see also [19]) for scalar functions which uses the more flexible nonintegrability condition of the \( \frac{\rho(z)}{|z|^p} \) than the one stated in (1.11).

**Theorem A.1** (\( L^p_{\text{loc}} \) compactness). Suppose that \( 1 \leq p < \infty \). Let \( \rho \in L^1(\mathbb{R}^d) \) be a nonnegative radial function satisfying
\[
(A.1) \quad \lim_{\delta \to 0} \int_{|z| > \delta} \frac{\rho(z)}{|z|^p} \, dz = \infty.
\]
Suppose also that \( \{u_n\} \) is a sequence of vector fields that is bounded in \( S_{\rho,p}(\mathbb{R}^d) \). Then for any \( D \subset \mathbb{R}^d \) open and bounded, the sequence \( \{u_n\}_{|D} \) is precompact in \( L^p(D; \mathbb{R}^d) \).

As stated earlier, for radial functions with compact support, condition (A.1) is weaker than (1.11). Indeed, (1.11) implies that \( \rho(z)|z|^{-p} \) is not integrable near 0 which implies (A.1). One the other hand, the kernel \( \rho(z) = |z|^{-d-p} \chi_{B}(z) \) satisfies (A.1) but not (1.11).

Similar to the argument we gave in section 2, the proof of the theorem will make use of the following variant of the Riesz-Fréchet-Kolmogorov theorem [6][22].
Lemma A.2. ([22] Lemma 5.4) Let the sequence \( \{G^\delta\}_{\delta > 0} \subset L^1(\mathbb{R}^d; \mathbb{R}^{d \times d}) \) be an approximation to the identity. That is
\[
\forall \delta > 0, \int_{\mathbb{R}^d} G^\delta(x) dx = 1, \quad \text{for any } r > 0, \lim_{\delta \to 0} \int_{|x| > r} G^\delta(x) dx = 0
\]
If \( \{f_n\}_n \) is a bounded sequence in \( L^p(\mathbb{R}^d; \mathbb{R}^d) \) and
\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \|f_n - G^\delta \ast f_n\|_{L^p} = 0,
\]
then for any open and bounded subset \( \Omega \) of \( \mathbb{R}^d \) the sequence \( \{f_n\} \) is relatively compact in \( L^p(\Omega; \mathbb{R}^d) \).

Proof of Theorem A.1. From the assumption we have
\[
\sup_{n \geq 1} \|u_n\|_{L^p}^p + \sup_{n \geq 1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x') |\mathcal{D}(u_n)(x,x')|^p dxdx < \infty.
\]
Let \( \Gamma^\delta(z) = \frac{\rho(z)}{|z|^p} \chi_{B_1}(z) \). Then for each \( \delta \), \( \Gamma^\delta \in L^1(\mathbb{R}^d) \) and is radial, since \( \rho \) is radial. Moreover, by assumption on \( \rho \) [(A.1)], \( \|\Gamma^\delta\|_{L^1} \to \infty \) as \( \delta \to 0 \). We next introduce the following sequence of integrable matrix functions
\[
G^\delta(z) = \frac{d \Gamma^\delta(z) z \otimes z}{\|\Gamma^\delta\|_{L^1} |z|^2}.
\]
Notice that since \( \Gamma^\delta \) is radial, we have
\[
\int_{\mathbb{R}^d} \frac{\Gamma^\delta(z) z_i^2}{|z|^2} dz = \frac{\|\Gamma^\delta\|_{L^1}}{d}, \quad i = 1, \ldots, d.
\]
As a consequence \( \{G^\delta\} \) is an approximation to the identity. Now for each \( n \) we have
\[
\|u_n - G^\delta \ast u_n\|_{L^p}^p = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} G^\delta(y-x)(u_n(x) - u_n(y)) dy \right)^p dx
\]
\[
\leq d^p \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{\Gamma^\delta(z)}{\|\Gamma^\delta\|_{L^1}} \frac{z}{|z|} \cdot (u_n(x) - u_n(z + x)) dy \right)^p dx
\]
\[
\leq d^p \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{\Gamma^\delta(z)}{\|\Gamma^\delta\|_{L^1}} \frac{z}{|z|} \cdot (u_n(x) - u_n(z + x)) dy \right)^p \rho(z) |z|^p dz dx
\]
\[
= \frac{d^p}{\|\Gamma^\delta\|_{L^1}} \|u_n\|_{S_{p,p}(\mathbb{R}^d)}^p,
\]
By assumption on the sequence \( \{u_n\} \) [(A.2)], we have that for all \( n \),
\[
\|u_n - G^\delta \ast u_n\|_{L^p}^p \leq C \frac{d^p}{\|\Gamma^\delta\|_{L^1}}.
\]
We take the limit as \( \delta \to 0 \) (uniformly in \( n \)) and use Lemma A.2 to conclude that \( u_n \) is compact in \( L^p(\Omega; \mathbb{R}^d) \). \( \square \)

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