On the structure of some typical singularities for implicit ordinary differential equations

M V Pomazanov† §
† Department of mathematics, Faculty of Physics, Moscow State University, Moscow, RUSSIA, 119899

Abstract. We study the systems of ordinary differential equations which are implicit with respect to the higher derivatives, appearing in the linear form, and their solutions near the singular points. The invertibility of the higher derivatives reduces to the singular surface where the theorem of uniqueness and existence is violated. A set of the singular solution is obtained for the case requiring the minimal number of additional conditions. The various types of singularities correspond to these cases. These “turning”, “intersection” singularities appear in several physical applications.

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§ Electronic address michael@math356.phys.msu.su
1. Introduction

The implicit differential equations of the form

\[ A(x,t)\dot{x} = b(x,t), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^1 \]  

are considered in this paper where \( A(x,t) \) is a \((n \times n)\)-matrix and \( b(x,t) \) is a \((n \times 1)\)-vector function. The solution of equation (1) is called a singular solution if it includes the points (called singular ones) \( x_0, t_0 \) where the main determinant of matrix \( A \) vanishes. Such singular solutions pass through the regions where the uniqueness and existence theorem is violated. We will study behaviour of the solutions in these regions by the asymptotic methods. The initial condition for equation (1) is set to be

\[ x(t_0) = x_0, \quad \det A(x_0, t_0) = 0. \]  

The surface \( \det A(x, t) = 0 \) could be called the singular surface in the phase space.

In the special cases the system (1) could be a system of Euler-Lagrange equations defined by the Lagrangian \( L(q, \dot{q}, t) \), where \( q, \dot{q} \in \mathbb{R}^n \) are vectors of coordinates and speeds, respectively. One can denote the coupling \( \{q, \dot{q}\}^T \) by \( x \in \mathbb{R}^{2n} \) (a sign \( ^T \)) is a transposition) and present the Euler-Lagrange equations

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \]

in the form (1), where the \((2n \times 2n)\)-matrix \( A(x,t) \) contains the block I (identity \((n \times n)\)-matrix) and the block

\[ \frac{\partial^2 L}{\partial \dot{q}^2} = \left\{ \frac{\partial^2 L}{\partial q_i \partial q_j} \right\}, i, j = 1, 2, ... n. \]

Therefore, the matrix \( A \) has the form

\[ A(x,t) = \begin{pmatrix} I & 0 \\ 0 & \frac{\partial^2 L}{\partial \dot{q}^2} \end{pmatrix}. \]

The vector function of the right side is defined as

\[ b(x,t) = \begin{pmatrix} \frac{\partial L}{\partial q} - \frac{\partial^2 L}{\partial q \partial t} - \frac{\partial^2 L}{\partial q \partial \dot{q}} \dot{q} \\ \dot{\dot{q}} \end{pmatrix}. \]

In such case the singular surface is determined by the equation

\[ H = 0, \]

where Hessian \( H \) is \( H = \det A(x,t) = \det \frac{\partial^2}{\partial \dot{q}^2} L. \)

The singular solutions of the Euler-Lagrange equations, containing the singular points, appear in some physical applications. For example, as it has been obtained in the model of post-Gallelelian approximation of relativistic quasiclassical particle dynamics,
the Lagrangian is singular on some surfaces of the phase space \[1, 2\]. Another example of the singular Lagrangian appears in the model of higher curvature string gravity \[3, 4\].

The structure of the paper is the following. In the section 2 the few typical singularities for the case of degenerated matrix \(A\) by the multiplicity 1 are investigated. Section 3 deals with the simplest case for a degeneration by two. Prime mathematical demonstration examples are presented in section 4. In which it is present also the real physical example of typical singularities, appeared in the black-hole type solutions for the higher curvature string gravity model.

2. Several types of the simplest singularities in the case of the single-valued degeneration

2.1. Change of coordinates

Without loss of generality one can consider the system \((1-2)\) in the following form

\[
\begin{align*}
A(x,t) \dot{x} &= B(x,t), \\
x(0) &= 0, \\
\text{det}A(0,0) &= 0,
\end{align*}
\]

where \(x \in R^n, t \in R^1, A(x,t) = \{a_{ij}(x,t) \in C^2(R^n \times R^1)\}, B(x,t) = \{b_i(x,t) \in C^2(R^n \times R^1)\}, i,j = 1, n.\) Let the matrix \(A(0,0)\) be degenerated by the multiplicity 1, i.e.

\[
\text{rang} \, A = n - 1.
\]

The matrix \(A(x,t)\) can be rewritten as

\[
A(x,t) = \hat{A} + \sum_{i=1}^{n} x_i A_i(x,t) + t A_t(x,t),
\]

where \(A_i, A_t\) are the \((n \times n)\)-matrices. Let some elements of the matrices \(A_i(0,0)\) and \(A_t(0,0)\) be nonzero. We denote by \(e_0\) the eigenvector of matrix \(\hat{A}\)

\[
\hat{A} e_0 = 0, \quad \|e_0\| = 1,
\]

and by \(e_i (i = 1, n - 1)\) the other orthogonal vectors. These vectors form the basis in space, and the following is correct

\[
\hat{A} e_i \neq 0, \quad i = 1, n - 1,
\]

\[
< e_i, e_j > = \delta_{ij}, \quad i,j = 0, n - 1
\]

(sign \(< \cdot, \cdot >\) denotes a scalar product). The coordinate vector \(x\) can be decomposed in \(e_i\) as

\[
x = \tilde{y} e_0 + \sum_{i=1}^{n-1} z_i e_i,
\]

(\(\text{sign} < \cdot, \cdot >\) denotes a scalar product). The coordinate vector \(x\) can be decomposed in \(e_i\) as
In this basis the matrix $\hat{A}$, considered as algebraic operator, takes the form

$$
\hat{A} = \begin{pmatrix} 0 & [0]_1^{n-1} \\ [0]_1^{n-1} & \hat{A}_{n-1} \end{pmatrix},
$$

where $[0]_1^{n-1}$ is the $(1 \times n-1)$-matrix of the null elements, $[0]_1^{n-1}$ is the $(n-1 \times 1)$-matrix and $(n-1 \times n-1)$-matrix $\hat{A}_{n-1}$ is the block of $\hat{A}$ where

$$\det \hat{A}_{n-1} \neq 0.$$ 

The vector function $B(x, t)$ reduces to a new form, defined by the basis $e_i$, as

$$B(x, t) \to (C, d)^T,$$

where $C = \langle e_0, B(x, t) \rangle$, $d = \{d_i, i = 1, n-1\}$, $d_i = \langle e_i, B(x, t) \rangle$. Taking into account the doubly continuous differentiability of matrix $A$ elements, one can present

$$A = \hat{A} + \tilde{y} \hat{A} + \sum_{i=1}^{n-1} z_i \tilde{A}_i + t \tilde{A}_t + [O^2(\tilde{y}, z, t)]_n^n,$$

where $\tilde{A}, \tilde{A}_i, \tilde{A}_t$ are the constant $(n \times n)$-matrices and the quantity $O^2$ can be understood with preceding notation as

$$O^2(\xi) = \sum_{i=1}^{m} \phi_i(\xi) \xi_i, \quad \xi \in R^m,$$

$$\phi_i = O^1(\xi), \quad O^1(\xi) = \sum_{j=1}^{m} \psi_j(\xi) \xi_j,$$

$$\|\psi(0)\| \neq 0, \quad \psi_j(\xi) \in C^2[R^m], \quad i, j = 1, m.$$ 

$O^3(\xi), \ O^4, \ldots$ can be defined in the same way.

We denote $D(\tilde{y}, z, t) = \det A(x, t)$ and introduce the new variable $y$ by

$$y = D(\tilde{y}, z, t).$$

2.2. Asymptotic form of equations

For the first considered case it is possible to assume that

$$\frac{\partial D}{\partial \tilde{y}} \bigg|_{\tilde{y}, z, t=0} = r \neq 0.$$ 

Hence

$$y = r \tilde{y} + \langle D^0_z, z \rangle + D^0_t + O^2(\tilde{y}, z, t),$$

(10)
where $D^0 = \partial D / \partial z(0, 0, 0)$ is the constant row vector, $D^0_t = \partial D / \partial t(0, 0, 0)$. In terms of $y$, variable $\tilde{y}$ takes the form
\[
\tilde{y} = \frac{1}{r}(y - \langle D^0_z, z \rangle > - D^0_t t) + O^2(y, z, t) \quad (11)
\]
The matrix $A(x, t)$ can be expressed in terms of $y, z, t$ as
\[
A(x, t) = \begin{pmatrix} \frac{y}{\det A_{n-1}^o} + O^2(y, z, t) & [O^1(y, z, t)]_{1}^{n-1} \\ [O^1(y, z, t)]_{1}^{n-1} & A_{n-1}^{-1}(y, z, t) \end{pmatrix},
\]
where $A_{n-1}(0, 0, 0) = A_{n-1}^o$, $\det A_{n-1}^o \neq 0$. Denoting by $\bar{A}$ the matrix of cofactors to the matrix $A$, one can write the inverse matrix $A^{-1}$ as
\[
A^{-1} = \frac{1}{y} \bar{A}.
\]
Therefore, equations (5) can be written as
\[
y \begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \bar{A}B, \quad (12)
\]
So, $\bar{A}$ have the structure
\[
\bar{A}(y, z, t) = \begin{pmatrix} \det A_{n-1}^{-1} & [O^1(y, z, t)]_{1}^{n-1} \\ [O^1(y, z, t)]_{1}^{n-1} & yA_{n-1}^{-1} + [O^2(y, z, t)]_{1}^{n-1} \end{pmatrix}. \quad (13)
\]
Taking into account equation (8), the derivative of $y$ have the form
\[
\dot{y} = \frac{\partial D}{\partial \tilde{y}} \cdot \dot{y} + \frac{\partial D}{\partial z} \cdot \dot{z} + \frac{\partial D}{\partial t} \cdot \dot{t}. \quad (14)
\]
Substituting the equations (12) to the expressions (14) and taking into account equation (9) and the structure of the matrix $\bar{A}$ (13), we obtain
\[
y \dot{y} = r \det A_{n-1}^{-1} C^0 + \alpha y + < \beta, z > + \gamma t + O^2(y, z, t) \quad (15)
\]
where $C^0 = C(0)$ in (7) and $\alpha \in R^1$, $\beta \in R^{n-1}$, $\gamma \in R^1$ are the constants, obtained from the first derivatives of $\det A_{n-1}(y, z, t), c, d$ at $y, z, t = 0$ and from the vector matrixes $[O^1(y, z, t)]_{1}^{n-1}, [O^1(y, z, t)]_{1}^{n-1}$ disposed in the matrix $\bar{A}$. The other part of (12) we rewrite as
\[
y \dot{z} = [O^1(y, z, t)]_{1}^{n-1} C^0 + yA_{n-1}^{-1} d^0 + [O^2(y, z, t)]_{1}^{n-1},
\]
and, further, in a form like (13)
\[
y \dot{z} = y(\Lambda C^0 + A_{n-1}^{-1} d^0) + (\Delta z + \theta t)C^0 + [O^2(y, z, t)]_{1}^{n-1}, \quad (16)
\]
where $d^0 = d(0)$ in (7) and the coefficients $\Lambda \in R^{n-1}$, $\Delta \in \{R^{n-1} \times R^{n-1}\}$, $\theta \in R^{n-1}$ can be calculated using the elements of matrix $\bar{A}$. 
2.3. Turning and intersection

The equations (15), (16), considered under the initial conditions \( y(0) = 0, z(0) = 0 \), can have two types of singularities. The first type we define as “the turning”. If \( C^0 \neq 0 \) then equation (15) can be rewritten in the following simple form

\[
y \dot{y} = F_0 + O^1(y, z, t),
\]

where \( F_0 = r \det \overset{n-1}{A_n} C^0 \neq 0 \). In such situation equations (16) have the form

\[
y \dot{z} = y g^0 + [O^1(z, t) + O^2(y, z, t)]_1^{n-1},
\]

where \( g^0 = \Lambda C^0 + \overset{n-1}{A_n} d^0 \) is the constant column vector, assumed to be nontrivial. That is the reason why the functions \( y(t), z(t) \) have the unique asymptotical structure

\[
y = \pm \sqrt{2F_0 t} + o(\sqrt{F_0 t}), \quad z = g^0 t + [o(t)]_1^{n-1}
\]
defined in the neighbourhood of the null point (Figure 1(a)). (The value \( o(\cdot) \) may be interpreted in the generally accepted sense.)

The second type of singularity for the case mentioned above appears under the condition \( C^0 = 0 \) and is called “the double intersection”. The equations (15), (16) have the following general form:

\[
\begin{cases}
y \dot{y} = \alpha y + <\beta, z> + \gamma t + O^2(y, z, t), \\
y \dot{z} = y \overset{n-1}{A_n} d^0 + [O^2(y, z, t)]_1^{n-1}.
\end{cases}
\]

Assuming that \( y \sim t^\varepsilon + o(t^\varepsilon), z \sim t^{\nu} + o(t^{\nu}) \) at \( \varepsilon, \nu > 0 \), we evidently obtain \( \varepsilon = \nu = 1 \).

Thus, \( y \) and \( z \) can be represented as

\[
y = \eta t + o(t), \quad z = \xi t + [o(t)]_1^{n-1}
\]

where \( \eta \in R^1, \xi \in R^{n-1} \) are the parameters which must be calculated. It is obvious that

\[
\xi = \overset{n-1}{A_n} d^0,
\]

but for the parameter \( \eta \) the square equation is resulted in the form

\[
\eta^2 = \alpha \eta + G,
\]

where \( G = <\beta, \overset{n-1}{A_n} d^0 > + \gamma \) is the constant. Taking into account the sign of discriminant \( Dis = \alpha^2 - 4G \), we declare the next possible cases: i) \( Dis < 0 \). Solutions are absent, ii)\( Dis = 0 \). One solution is present, and iii) \( Dis > 0 \). Two solutions are present. The case iii) means that there are two solutions passing through a singular point and intersecting under the nonzero angle (Figure 1(b)).
2.4. More complex cases

Further we will study the case of the singularities when the condition (17) is broken. Let us assume that

$$\frac{\partial D}{\partial \hat{y}} \bigg|_0 = 0.$$  (17)

We split the coordinate vector $z \in \mathbb{R}^{n-1}$ to the new coordinates $(\hat{y}, z)$, where the dimension of new vector $z$ is decreased by 1 and becomes equal to dim $z = n - 2$. It is assumed for coordinate $\hat{y}$ that

$$\frac{\partial D}{\partial \hat{y}} \bigg|_0 = \hat{r} \neq 0.$$

The part $d$ of the vector of the right side $B(x, t)$ (17) decomposes to the vector $(\hat{c}, d)$, where dim $d = n - 2$. By the analogue with (10) we can write now

$$y = \hat{r}\hat{y} + < D \hat{z}, z > + D^0_t t + O^2(\hat{y}, \hat{y}, z, t).$$  (18)

The system (12) has the following form:

$$\begin{pmatrix} \hat{y} \\ \hat{\dot{y}} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \det \bar{A}_{n-1} & g_{12} & [O^1]^1_{n-2} \\ g_{21} & y_\bar{l}_{22} + O^2(x, t) & y_\bar{l}_{23} + [O^2]^1_{n-2} \\ [O^1]^n_{n-2} & y_\bar{l}_{32} + [O^2]^n_{n-2} & y_\bar{l}_{33} + [O^2]^n_{n-2} \end{pmatrix} \cdot \begin{pmatrix} c \\ \dot{c} \\ d \end{pmatrix},$$

where $g_{21}, g_{12} = O^1(x, t)$, $l_{ij}$ are the block components of the matrix $\bar{A}$ (13) (for example, $y_\bar{l}_{23}$ is a $(1 \times n-2)$ row matrix which is a part of block $y\bar{A}_{n-1}$ in $\bar{A}$ disposed).

The following relation takes place for $\hat{y}$

$$\hat{\dot{y}} = \frac{\partial D}{\partial \hat{y}} \hat{y} + \frac{\partial D}{\partial \hat{\dot{y}}} \hat{\dot{y}} + \frac{\partial D}{\partial \dot{z}} \dot{z} + \frac{\partial D}{\partial t},$$

where $\partial D/\partial \hat{y} = O^1(\hat{y}, \hat{y}, z, t)$ (see (17)), $\partial D/\partial \hat{\dot{y}} = \hat{r} + O^1(\hat{y}, \hat{y}, z, t)$, $\partial D/\partial \dot{z} = D^0_z + [O^1(\hat{y}, \hat{y}, z, t)]^1_{n-2}$, $\partial D/\partial t = D^0_t + O^1(\hat{y}, \hat{y}, z, t)$.

Expressing the variable $\hat{y}$ in terms of $y$ by using (18) and substituting the result to the system (14), we can write equations (19) in the general form

$$\begin{cases} \hat{\dot{y}} = C^0 \det \bar{A}_{n-1} + \bar{a} \hat{y} + a \hat{y} + < b, z > + b_t t + O^2(\hat{y}, y, z, t), \\ \hat{\dot{y}} = \hat{\alpha} \hat{y} + \alpha \hat{y} + < \beta, z > + \gamma t + O^2(\hat{y}, y, z, t), \\ \hat{\dot{z}} = C^0(\bar{g} \hat{y} + [O^1(y, z, t)]^n_{n-2}) + G \hat{y} + [O^2(\hat{y}, y, z, t)]^n_{n-2}, \end{cases}$$  (20)

where $\bar{a} \in \mathbb{R}^1$, $a \in \mathbb{R}^1$, $b \in \mathbb{R}^{n-2}$, $b_t \in \mathbb{R}^1$, $\hat{\alpha} \in \mathbb{R}^1$, $\alpha \in \mathbb{R}^1$, $\beta \in \mathbb{R}^{n-2}$, $\gamma \in \mathbb{R}^1$, $g \in \mathbb{R}^{n-2}$, $G \in \mathbb{R}^{n-2}$ are the coefficients calculated from the matrix $\bar{A}$. The initial conditions for the system (20) are $\hat{y}(0) = y(0) = 0$, $z(0) = 0$. Similarly to the situation considered above, the equations (20) have a few types of the singular solutions. Here we investigate some of the most realizable of them.
2.5. Reflecting

The first type appears under the condition

\[ C^0 \neq 0, \quad \tilde{\alpha} \neq 0, \quad \|g\| \neq 0. \]

We would like to call this case as “the reflecting”. We can find the coordinates \( \tilde{y}(t), y(t), z(t) \) in the following asymptotical expression in the same manner as in the above cases

\[ \tilde{y} \sim t^\varepsilon + o(t^\varepsilon), \quad y \sim t^{\nu} + o(t^{\nu}), \quad z \sim t^{\omega} + o(t^{\omega}) \tag{21} \]

where \( \varepsilon > 0, \nu > 0, \omega > 0 \) are the unknown coefficients to be determined. After the substitution of Eq. (21) to (20) and the equalization of the minimal extents, one obtains the following system of the algebraic equations

\[
\begin{align*}
\varepsilon + \nu - 1 &= 0, \\
2\nu - 1 &= \min(\varepsilon, \nu, \omega, 1), \\
\varepsilon + \omega - 1 &= \min(\varepsilon, \nu, \omega, 1). 
\end{align*} \tag{22}
\]

The unique solution for this system is \( \varepsilon = 1/3, \nu = \omega = 2/3 \). Let us introduce the new parameters \( \delta \in R^1, \chi \in R^1 \) and \( \Delta \in R^{n-2} \) characterizing the singular solutions by

\[ \tilde{y} = \delta \sqrt[3]{t} + o(\sqrt[3]{t}), \quad y = \chi \sqrt[3]{t^2} + o(\sqrt[3]{t^2}), \quad z = \Delta \sqrt[3]{t^2} + [o(\sqrt[3]{t^2})]^{n-2}. \tag{23} \]

By substituting (23) to (20) and setting equal the coefficients in the terms with the minimal extents of \( t \), we obtain the equations for this coefficients

\[
\begin{align*}
\delta \chi &= 3C^0 \det \tilde{A}_{n-1}, \\
2\chi^2 &= 3\tilde{\alpha} \delta, \\
2\chi \Delta &= 3C^0 g \delta. 
\end{align*} \tag{24}
\]

System (24) has obviously the unique solution

\[
\chi = \frac{3}{2} \sqrt[3]{\frac{9}{2} \tilde{\alpha} C^0 \det \tilde{A}_{n-1}}, \quad \delta = \frac{2\chi^2}{3\tilde{\alpha}}, \quad \Delta = \frac{3C^0 \delta}{\chi} g. 
\]

So, this type means that there is only a single solution passing through a singular point. The phase point moves at this solution to the singular surface \( y = 0 \) and after the contact with it goes away at the same side (Figure 1(c)).

2.6. Complicated branching

The second case defined by (17) appears under the condition

\[ C^0 \neq 0, \quad \tilde{\alpha} = 0, \quad \|g\| \neq 0. \tag{25} \]
It is called “the complicated branching”. For the coefficients \( \varepsilon, \nu, \omega \) we must write the system like (22)
\[
\begin{align*}
\varepsilon + \nu - 1 &= 0, \\
2\nu - 1 &= \min(2\varepsilon, \nu, \omega, 1), \\
\varepsilon + \omega - 1 &= \min(\varepsilon, \nu, \omega, 1).
\end{align*}
\] (26)

We assume here that the second equation of system (20) has the form
\[
y \dot{y} = \alpha y + \langle \beta, z \rangle + \delta \tilde{y}^2 + \tilde{y} \cdot O^1(y, z, t) + O^3(\tilde{y}, y, z, t),
\]
where the coefficient \( \delta \), which is assumed nonzero, can be found from \( \tilde{A} \) in (13) like \( \alpha, \beta, \gamma \). This assumption must be added to (17). By solving the system (26), we obtain the unique solution \( \varepsilon = 1/4, \nu = 3/4, \omega = 1/2 \). Therefore, the functions \( \tilde{y}(t), y(t) \) and \( z(t) \) have the asymptotic form
\[
\tilde{y} = \delta \sqrt{s}t + o(\sqrt{s}t), \quad y = \chi \sqrt{s}t^3 + o(\sqrt{s}t^3), \quad z = \Delta \sqrt{s}t + [o(\sqrt{s}t)]^{n-2}
\] (27)
where \( \delta, \chi, \Delta \) are the unknown coefficients and sign \( s = +1 \) or \( -1 \). This sign indicates the domain of definition of the solutions. After substituting the equations (27) to the system (20), like in the previous case (24), we obtain the system
\[
\begin{align*}
\delta \chi &= 4sC^0 \det \tilde{A}_{n-1}, \\
3\chi^2 &= 4s(\hat{\alpha} \delta^2 + \langle \beta, \Delta \rangle), \\
\chi \Delta &= 2sC^0 g.
\end{align*}
\] (28)

The fourth order equation for the coefficient \( \chi \) follows from the above system. It looks like
\[
\frac{3}{4} \chi^4 - 2sC^0 < \beta, g > + \chi - s\hat{\alpha}(4C^0 \det \tilde{A}_{n-1})^2 = 0.
\] (29)

The other coefficients \( \delta \) and \( \Delta \) can be found consequently by the formulas
\[
\delta = \frac{4sC^0 \det \tilde{A}_{n-1}}{\chi}, \quad \Delta = \frac{2sC^0}{\chi} g.
\]

Now one has the question about the number of real roots of the above equation.

For answering this question one can use the well known Shturm’s method [5]. The algebraic form (29) is equivalent to
\[
P(\chi) = \chi^4 + a_1 \chi + a_0,
\] (30)
where the constants \( a_1, a_0 \) are
\[
a_1 = \frac{8}{3} sC^0 < \beta, g >, \quad a_0 = \frac{1}{3} s\hat{\alpha}(8C^0 \det \tilde{A}_{n-1})^2.
\]
In order to find the number of real roots, we must construct the Shturm’s sequence and then count up the difference between the number of the alternations of the signs in this
sequence at $x = \pm \infty$. Using the Shturm’s algorithm, we obtain that the number of
roots of the equation (30) depends on sign of the expression

$$R = a_4^4 - \frac{256}{27}a_0^3.$$ 

When $R < 0$, roots do not exist. When $R = 0$ only one root (31) can exist and when
$R > 0$ two roots can be found. In our case $a_1 = s\hat{a}_1$, $a_0 = s\hat{a}_0$ and sign $s$ can be positive
or negative.

Therefore, we can declare the next possibilities:

$$a) \ a_4^4 < \frac{256}{27} | a_0 |^3, \quad b) \ a_4^4 > \frac{256}{27} | a_0 |^3, \quad c) \ a_4^4 = \frac{256}{27} | a_0 |^3, \quad (31)$$

for which the various types of the singular solutions of the differential equations (21)
can result. The existence of the real singular solutions for the item (31a) determines
the sign $s$ as

$$s = \text{sign}(\hat{\alpha}).$$

In this case the singular solution of equations (20) has the “turning”-like type and can
exist only at $t > 0$ or $t < 0$ (Figure 1(d1)).

In the item (31b) the sign $s$ could be chosen positive or negative. The solution like
the “turning” exists for each of the possible signs. Each solution contains two branches
defined by two roots of the equation (29). This singular solution, placed on both sides
of the point $t = 0$ (Figure 1 (d2)).

The item (31c) differs from (31b) with R is equal to zero at sign($t$) = $s = -\text{sign}(\hat{\alpha})$.
There is only one branch defined by a single root of equation (24) for this sign $t$. There
are two roots (29) in the other side of $t = 0$ at sign($t$) = $+\text{sign}(\hat{\alpha})$, so, there are two
branches of the singular solution. So, in this case three branches pass through the
singular point.

2.7. Triple intersection

The third case defined by (17) appears under additional condition

$$C^0 = 0.$$ 

It can be called “the triple intersection”. The differential equations (20) are now
rewritten in the form

$$\begin{cases}
y\ddot{y} = \tilde{a}\ddot{y} + a\dot{y} + < b, z > + b_1 t + O^2(\tilde{y}, y, z, t), \\
y\ddot{y} = \tilde{a}\ddot{y} + a\dot{y} + < \beta, z > + \gamma t + O^2(\tilde{y}, y, z, t), \\
y\dot{z} = G y + [O^2(\tilde{y}, y, z, t)]^{n-2}. \quad (32)
\end{cases}$$

It is easy to establish that $\varepsilon = \nu = \omega = 1$ using the analysis of extents. Therefore, we
have the right to suggest the following behaviour

$$\ddot{y} = \delta t + o(t), \ y = \chi t + o(t), \ z = \Delta t + [o(t)]^{n-1}, \quad (33)$$
where obviously
\[ \Delta = G. \]

The unknown coefficients \( \delta, \chi \) can be obtained from the algebraic equations
\[
\begin{align*}
\chi \delta &= \tilde{a} \delta + a \chi + F, \\
\chi^2 &= \tilde{a} \delta + a \chi + f,
\end{align*}
\] (34)
where \( F = <b, G> + bt \), \( f = <\beta, G> + \gamma \). We do not pay attention to any degenerate case of solution (34). In general, the system (34) is equivalent the algebraic third order equation. Hence, it can have three (Figure 1(e)), two or one root.

3. The double-valued degeneration

3.1. Equations in neighborhood of a singular point

The double-valued degeneration means that
\[ \text{rang } A = n - 2. \] (35)

Most generally in this case, there are two independent eigenvectors \( e_1, e_2 \) corresponding to the zero eigenvalue of the matrix \( A \). Let us denote \( e_i (i = \overline{3, n}) \) other basis vectors orthogonal to \( e_1, e_2 \). As in (3)
\[
x = e_1 y_1 + e_2 y_2 + \sum_{i=3}^{n} e_i z_i. \]
The \((n \times n)\) matrix \( \overset{o}{A} \) has the representation
\[
\overset{o}{A} = \begin{pmatrix}
0 & [0]^2 \\
[0]^2 & \overset{o}{A}_{n-2}
\end{pmatrix},
\] (36)
where \( \text{det } \overset{o}{A}_{n-2} \neq 0 \). So, the matrix \( A(x, t) \) has the block form
\[
A(x, t) = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & [O^1(y_1, y_2, z, t)]_{n-2}^2 \\
\lambda_{21} & \lambda_{22} & [O^1(y_1, y_2, z, t)]_{n-2}^2 \\
[O^1(y_1, y_2, z, t)]_{n-2}^2 & \overset{o}{A}_{n-2} + [O^1(y_1, y_2, z, t)]_{n-2}^2
\end{pmatrix},
\] (37)
where \( \lambda_{ij} = O^1(y_1, y_2, z, t) \). The vector of right part \( B(x, t) \) reduces to the form \( (C_1(x, t), C_2(x, t), d(x, t))^T \), where \( C_i, d_i = <e_i, B>, \ i = \overline{1, n} \). The main determinant \( \text{det} A(x, t) \) is
\[
\text{det} A(x, t) = \lambda \cdot \text{det } \overset{o}{A}_{n-2} + O^3(x, t),
\]
where \( \lambda = \lambda_{11} \lambda_{22} - \lambda_{21} \lambda_{12} \). Let us choose the orthogonal eigenvectors \( e_1, e_2 \) in order to present the following form for the value \( \lambda \)
\[
\lambda = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \sum_{i=1}^{2} y_i O^1(z, t) + O^2(z, t) + O^3(y_1, y_2, z, t) \] (38)
By defining the system (40) more concretely, we write it as

\[ \bar{A}(x, t) = \begin{pmatrix} \lambda_{22} & -\lambda_{12} \\ -\lambda_{21} & \lambda_{11} \end{pmatrix} \cdot \det \bar{A}_{n-2} + [O^2]_2^{n-2} \left[ O^2(y_1, y_2, z, t) \right]_{n-2}^{n-2} + \lambda \bar{A}_{n-2}^{-1} + [O^3]_{n-2}^{n-2} \right), \] (39)

and the differential equations (12) are rewritten as

\[ \lambda \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{pmatrix} = \bar{A} \begin{pmatrix} C_1(y_1, y_2, z, t) \\ C_2(y_1, y_2, z, t) \\ d(y_1, y_2, z, t) \end{pmatrix}. \]

So,

\[
\begin{align*}
\lambda \dot{y}_1 &= (\lambda_{22}C_1^0 - \lambda_{12}C_2^0) \det \bar{A}_{n-2} + O^2(y_1, y_2, z, t), \\
\lambda \dot{y}_2 &= (-\lambda_{21}C_1^0 + \lambda_{11}C_2^0) \det \bar{A}_{n-1} + O^2(y_1, y_2, z, t), \\
\lambda \dot{z} &= [O^2(y_1, y_2, z, t)]_{n-2}^{n-2} \cdot \begin{pmatrix} C_1^0 \\ C_2^0 \end{pmatrix} + \lambda \bar{A}_{n-2}^{-1} d^0 \\
&\quad + [O^3(y_1, y_2, z, t)]_{n-2}^{n-2} ,
\end{align*}
\]

where \( C_1^0 = C_1(0, ...), C_2^0 = C_2(0, ...), d^0 = d(0, ...). \)

3.2. The simplest case

We'll discuss here only one case appearing at

\[ | C_1^0 | + | C_2^0 | \neq 0, \] (41)

and call it “the linking of turnings”.

It is easy to see that the variables \( y_1, y_2, z \) have the following unique structure

\[ y_i \sim t^{1/2}, \quad z \sim t. \]

By defining the system (40) more concretely, we write it as

\[
\begin{align*}
\lambda \dot{y}_1 &= a_1 y_1 + b_1 y_2 + O^1(z, t) + O^2(y_1, y_2, z, t), \\
\lambda \dot{y}_2 &= a_2 y_1 + b_2 y_2 + O^1(z, t) + O^2(y_1, y_2, z, t), \\
\lambda \dot{z} &= g_{11} y_1^2 + g_{12} y_1 y_2 + g_{22} y_2^2 + \lambda \bar{A}_{n-2}^{-1} d^0 \\
&\quad + \sum_{i=1}^2 y_i [O^1(z, t)]_{1}^{n-2} + [O^2(z, t)]_{1}^{n-2} + [O^3(y_1, y_2, z, t)]_{1}^{n-2} ,
\end{align*}
\]

where \( a_i, b_i \in R^1, g_{ij} \in R^{n-2} \) are the coefficients which can be obtained from actual form of \( \lambda_{ij} \), from the \( O^2 \) — blocks of the matrix \( \bar{A} \) (44) and from \( C_i^0, \quad i, j = 1, 2 \). Thus the following expressions take place:

\[ y_1 = \eta_1 \sqrt{st} + o(\sqrt{st}), \quad y_2 = \eta_2 \sqrt{st} + o(\sqrt{st}), \quad z = \Delta t + [o(t)]_{1}^{n-2} \] (43)
where the coefficients \( \eta_i \in R^1, \Delta \in R^{n-2} \) must be calculated. The coefficient \( s \) is the sign and can be equal only +1 or -1 like the case (17). By substituting (43) to (42) and setting equal the coefficients at the terms with minimal extents of variable \( t \), we obtain the algebraic system

\[
\begin{cases}
(\lambda_1 \eta_1^2 + \lambda_2 \eta_2^2) \eta_1 = 2s(a_1 \eta_1 + b_1 \eta_2), \\
(\lambda_1 \eta_1^2 + \lambda_2 \eta_2^2) \eta_2 = 2s(a_2 \eta_1 + b_2 \eta_2).
\end{cases}
\] (44)

If the values of \( \eta_1, \eta_2 \) was known, the coefficients \( \Delta \) can be determined by the formula

\[
\Delta = A_{n-2}^{-1} d^0 + \left( g_{11} \eta_2^2 + g_{12} \eta_1 \eta_2 + g_{22} \eta_2^2 \right) \left( \lambda_1 \eta_1^2 + \lambda_2 \eta_2^2 \right). 
\] (45)

followed from the third equation (42) and (38). Let assume that \( b_1 \) is not equal to zero (or \( a_2 \neq 0 \)). The relation between \( \eta_1 \) and \( \eta_2 \) has a form of square equation, as follows from equations (44):

\[
b_1 \eta_2^2 + (a_1 - b_2) \eta_1 \eta_2 - a_2 \eta_1^2 = 0.
\] (46)

It means that real roots of equation (46) can exist under the condition \( Dis \geq 0 \), where \( Dis = (a_1 - b_2)^2 + 4b_1a_2 \) is the discriminant of equation (46). Thus the relation is

\[
\eta_2 = g^\pm \eta_1,
\]

where \( g^\pm = (-a_1 + b_2 \pm \sqrt{Dis})/(2b_1) \) and the coefficient \( \eta_1 \) can be determined by

\[
\eta_1^2 \cdot \left( \lambda_1 + \lambda_2 (g^\pm)^2 \right) = s(a_1 + b_2 \pm \sqrt{Dis}).
\]

Let us satisfy the condition

\[
\lambda_1 + \lambda_2 (g^\pm)^2 \neq 0
\]

which must be added to the condition (44) with the restriction \( |b_1| + |a_2| \neq 0 \).

In this way, sign \( s \) must be chosen as

\[
s = \text{sign}(R^\pm),
\]

where \( R^\pm = (a_1 + b_2 \pm \sqrt{Dis})/(\lambda_1 + \lambda_2 (g^\pm)^2) \) and by joining of them, the singular solution has the next kind

\[
\begin{pmatrix}
  y_1(t) \\
  y_2(t)
\end{pmatrix} = \pm \left( \frac{\sqrt{R^\pm t}}{g^\pm \sqrt{R^\pm t}} \right) + [o(\sqrt{R^\pm t})]_1^n,
\]

\[
z = \left( A_{n-2}^{-1} d^0 + (\lambda_1 + \lambda_2 (g^\pm)^2)^{-1} \cdot (g_{11} + g^\pm g_{12} + (g^\pm)^2 g_{22}) \right) t + [o(t)]_1^{n-2}.
\]

The singular solution passing through the singular point is a linking of two turnings at \( Dis > 0 \). Moreover, if the signs of the value \( R^\pm \) are identical, the both turnings
are placed on the same side of point $t = 0$ (Figure 1($f_1$)). Else they are placed on the various sides (Figure 1($f_2$)). Only one turning exists at $Dis = 0$.

We have considered here only the typical case taking place under the condition (35), but there are a few degenerate cases, for example, one of them can appear at $R^+ = 0$ or $R^- = 0$. We have restricted oneself only to the general situation.

4. Examples and discussion

4.1. A few prime examples

By using the prime model examples we would like to demonstrate several types of the singularities discussed above. For singular differential equation

$$x\dot{x} = t$$

(47)

considered at the initial condition placed on the singular surface $x = 0$

$$x(t_0) = 0,$$

we easily determine that under the condition $t_0 \neq 0$ “the turning” takes place:

$$x(t) = \pm \sqrt{2(t - t_0)t_0} + o(\sqrt{(t - t_0)t_0}).$$

At $t_0 = 0$ the “intersection” takes place $x = \eta t + o(t)$, where $\eta = \pm 1$. It is easy to solve understand Eq. (47) directly. This solution is $x^2 = t^2 + t_0^2$.

For the equation

$$x\dot{x} = -t,$$

under the initial condition mentioned above, we have “the turning” at $t_0 \neq 0$. However, at $t_0 = 0$ we do not obtain “the intersections” case, because the real roots of square equation for the parameter $\eta$ do not exist. It is easy to see from the direct solution $x^2 + t^2 = t_0^2$, that it does not pass through the singular point $x(0) = 0$.

To demonstrate so-called “the reflection” case, we’ll turn into the next primary example of system of two differential equations

$$\begin{cases} y\dot{x} = -1, \\ (y - x)\dot{x} + \dot{y} = 0, \end{cases}$$

(48)

with the initial conditions $x(0) = y(0) = 0$ placed on the singular surface $D = \det A = y = 0$. The matrix $\hat{A}$ of system (48) is

$$\hat{A} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and egenvector is $e_0 = (1, 0)^T$, $\hat{A} e_0 = 0$. The vector function of the right part is $B = (-1, 0)^T$. The vector $\nabla D$ is $\nabla D = (\partial D/\partial x, \partial D/\partial y) = (0, 1)$, so, $r = (e_0, \nabla D) = 0,$
but $<e_0, b> \neq 0$. Thus, the next structure of the singular solution may take place in the most common case

$$x = a \sqrt[3]{t} + o(\sqrt[3]{t}), \quad y = b \sqrt[3]{t^2} + o(\sqrt[3]{t^2}).$$

(49)

By substituting this to the equations (48), we can obtain the following algebraic equations for the coefficients $a, b$

$$\frac{1}{3}ab = -1, \quad a^2 = 2b,$$

which have the unique roots $a = -\sqrt[3]{6}$, $b = \sqrt[3]{9/2}$. The direct solution of the system (48) gives the functions $x(t), y(t)$ in the implicit form

$$t = e^{-x} - (1 - x + \frac{x^2}{2}), \quad y = t + \frac{x^2}{2}.$$  

(50)

The first equation (50) is rewritten as

$$t = \frac{x^3}{3!}(-1 + \frac{3!x}{4!} - \frac{3!x^2}{5!} + ...),$$

from which it is easy to see the unique structure of function $x(t)$, and further $y(t)$, near the singular point $x(0) = y(0) = 0$. It coincides entirely to (49).

4.2. Black Hole example and discussion

The singular solutions of system (1) can be very diverse. We considered just the cases which have required the least number of additional conditions. In our opinion, the smaller is the number of additional conditions, the more probably the case may occurs in the nature.

In the model of higher curvature string gravity the four-dimensional black hole solutions are generated by the low energy string effective action:

$$S = \int d^4x L = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ m_{Pl}^2 (-R + 2 \partial_{\mu} \phi \partial^\mu \phi) + \lambda e^{-2\phi} S_{GB} \right].$$

(51)

where $R$ is the scalar curvature; $\phi$ is the dilaton field; $m_{Pl}$ is the Plank mass; $\lambda$ is the string coupling parameter. The latter describes Gauss-Bonnet (GB) contribution to the action (51). For consideration of the static, asymptotically flat, spherically symmetrical black-hole-like solutions, the most convenient choice of metric is

$$ds^2 = \Delta dt^2 - \frac{\sigma^2}{\Delta} dr^2 - f^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

(52)

where functions $\Delta$, $\sigma$, $f$ and the dilaton function $\phi$ depend only on the radial coordinate $r$. In this metric the Lagrangian, transformed to the most convenient for analysis form, is

$$L(\Delta', f', \phi', \Delta, \phi, \sigma, f) = L_0 + \lambda L_{GB},$$

(53)
where $L_0 = m_{Pl}^2[\Delta f'f + \Delta (f')^2 + \sigma^2 - \Delta f^2(\phi')^2]/\sigma$, $L_{GB} = 4e^{-2\phi}\phi'[\Delta\Delta'(f')^2 - \Delta'\sigma^2]/\sigma^3$ and the stroke denotes $d/dr$.

The main determinant of Euler-Lagrange equations followed from (53) in the curvature gauge $f(r) = r$ has the structure

$$D_{main} = \Delta(A\lambda^2\Delta^2 + B\lambda\Delta + C)$$

where

$$A = (-32)e^{-4\phi}\sigma^2(4\phi'^2r^2 - 1)\sigma^2m_{Pl}^2 + \lambda\cdot 12e^{-2\phi}\Delta'\phi'$$

$$B = (-32)e^{2\phi}\sigma^4[\sigma^2\phi'm_{Pl}^4r^3 + \lambda2e^{-2\phi}\sigma^2m_{Pl}^2 - \lambda^2\cdot 8e^{-6\phi}\Delta'\phi'$$

$$C = 2\sigma^6[-\sigma^2m_{Pl}^6r^6 + \lambda^2\cdot 32e^{-4\phi}m_{Pl}^2(\sigma^2 + 2\Delta'r) + \lambda^364e^{-6\phi}\Delta'\phi']$$

At $\lambda = 0$ (GB term is absent) the asymptotically flat solution followed from the Euler-Lagrange (Einstein) equations is well known Schwarzschild’s one. This solution is

$$\Delta = 1 - \frac{2M}{r}, \quad \sigma = 1, \quad \phi = \text{const.}$$

The domain of its definition is $r = (0, +\infty)$. This solution has “the intersection” type singular point $r_h$ at the event horizon $\Delta = 0$. Here the Schwarzschild’s solution intersect with symmetrical solution which usually has been rejected because it hasn’t physical sense (Figure 2 curve (c)). But in a case of nonzero, positive $\lambda$, under the regular event horizon ($\Delta(r_h) = 0$), the Lagrangian (53) has the further singular point at $r = r_s < r_h$. This singular “turning” point forms by passing trajectory of solution from the singular surface $A\lambda^2\Delta^2 + b\lambda\Delta + C = 0$, which is nontrivial only at $\lambda \neq 0$. The domain of definition of the new solution becomes $(r_s, +\infty)$ and the solution is subjected to a turning showed in Figure 2. This singularity is absent in the first order curvature gravity and it is found as a surprise for the classical gravity.

The investigation, performed in the present work, may be efficient for analysis of the compound system of the ordinary differential equation with the implicit linear higher derivatives when it is hard to obtain the direct solution. The investigation of the equations (1) by the numerical methods can be made without the significant difficulties only on the range of invertibility of matrix $A$. In order to extend the solution through the singular point, it is necessary to know the structure of the singularity. In the Appendix to Ref. [3] the numerical method of integration the equations (1) by additional parameter is discussed. It allows passing through a singular “turning” point automatically.

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Figure captions

**Figure 1.** The typical singularities in implicit differential equations.

**Figure 2.** Metric functions $\Delta$ and $\sigma$ versus the radial coordinate $r$ when the event horizon value $r_h$ is equal to 20.0 Plank unit values (P.u.v.). The curve (a) is calculated with $\lambda = 1$, the curve (b) is calculated with $\lambda = 0$ (Schwarzschild’s solution). The curve (c) is the nonphysical branch passing from point $r_h$. The curve (d) shows $\sigma(r)$ function at $\lambda = 1$. The arrows pointed $r_s$ shows the positions of $r_s$-singularity.
(a) turning

(b) double intersection

(c) reflecting

(d) complicated branching

(e) triple intersection

(f) linking of turnings
