Optimal scaling of the ADMM algorithm for distributed quadratic programming

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Abstract—This paper addresses the optimal scaling of the ADMM method for distributed quadratic programming. Scaled ADMM iterations are first derived for generic equality-constrained quadratic problems and then applied to a class of distributed quadratic problems. In this setting, the scaling corresponds to the step-size and the edge-weights of the underlying communication graph. We optimize the convergence factor of the algorithm with respect to the step-size and graph edge-weights. Explicit analytical expressions for the optimal convergence factor and the optimal step-size are derived. Numerical simulations illustrate our results.

I. INTRODUCTION

Recently, a number of applications have triggered a strong interest in distributed algorithms for large-scale quadratic programming. These applications include multi-agent systems [1], [2], distributed model predictive control [3], [4], and state estimation in networks [5], to name a few. As these systems become larger and their complexity increases, more efficient algorithms are required. It has been argued that the alternating direction method of multipliers (ADMM) is a particularly powerful and efficient approach [6]. One attractive feature of ADMM is that it is guaranteed to converge for all (positive) values of its step-size parameter [6]. This contrasts many alternative techniques, such as dual decomposition, where mistuning of the step-size for the gradient updates can render the iterations unstable.

The ADMM method has been observed to converge fast in many applications [6]–[9] and for certain classes of problems it has a guaranteed linear rate of convergence [10]–[12]. However, the solution times are sensitive to the choice of the step-size parameter, and the ADMM iterations can converge (much) slower than the standard gradient algorithm if the parameter is poorly tuned. In practice, the ADMM algorithm is tuned empirically for each specific application. In particular, for distributed quadratic programming, [7]–[9] report various rules of thumb for picking the step-size. However, a thorough analysis and design of optimal step-size and scaling rules for the ADMM algorithm is still missing in the literature. The aim of this paper is to close this gap for a class of distributed quadratic programming problems.

We first consider a particular class of equality-constrained quadratic programming problems and derive the corresponding iterations for the ADMM method. The iterations are shown to be linear and the corresponding eigenvalues are characterized as roots of quadratic polynomials. These results are then used to develop optimally scaled ADMM iterations for a class of distributed quadratic programming problems that appear in power network state-estimation applications [13]. In this class of problems, a number of agents collaborate with neighbors in a graph to minimize a convex objective function with a specific sparsity structure over a mix of shared and private variables. We show that quadratic programming problems with this structure can be reduced to an equality constrained convex quadratic programming problem in terms of private variables only. The ADMM iterations for this quadratic problem are then formulated taking into account the communication network constraints. The network-constrained scaling of the ADMM method includes the step-size and edge weights of the communication graph. Methods to minimize the convergence factor by optimal scaling of the ADMM iterations are proposed for generic connected graphs. In particular, analytical expressions for the optimal step-size and convergence factor are derived in terms of the spectral properties of the communication graph. A tight lower-bound for the convergence factor is also obtained. Finally, given that the optimal step-size is chosen, we propose methods to further minimize the convergence factor by optimizing the edge weights.

The outline of this paper is as follows. Section II gives an elementary background to the ADMM method. The ADMM iterations for a class of equality-constrained quadratic programming problems are formulated and analyzed in Section III. Distributed quadratic programming and optimal networked-constrained scaling of the ADMM algorithm are addressed in Section IV. Numerical examples illustrating our results and comparing them to state-of-the-art techniques are presented in Section V. Finally, a discussion and outlook on future research concludes the paper.

A. Notation

We denote the set of real and complex numbers with \( \mathbb{R} \) and \( \mathbb{C} \), respectively. For a given matrix \( A \in \mathbb{R}^{n \times m} \), denote \( \mathcal{R}(A) \triangleq \{ y \in \mathbb{R}^n \mid y = Ax, x \in \mathbb{R}^m \} \) as its range-space and let \( \mathcal{N}(A) \triangleq \{ x \in \mathbb{R}^m \mid Ax = 0 \} \) be the null-space of \( A \). For \( A \) with full-column rank, define \( A^\dagger \triangleq (A^\top A)^{-1} A^\top \) as the pseudo-inverse of \( A \) and \( \Pi_{\mathcal{N}(A^\top)} \triangleq A A^\dagger \) as the orthogonal projector onto \( \mathcal{R}(A) \). Since \( \mathcal{R}(A) \) and \( \mathcal{N}(A^\top) \) are orthogonal complements, we have \( \Pi_{\mathcal{N}(A^\top)} = I - \Pi_{\mathcal{R}(A)} \).
and \( \Pi_{\mathcal{R}(A)} \Pi_{\mathcal{N}(A^T)} = 0 \). Now consider \( A, D \in \mathbb{R}^{m \times n} \), with \( D \) invertible. The generalized eigenvalues of \( (A, D) \) are defined as the values \( \lambda \in \mathbb{C} \) such that \( (A - \lambda D)v = 0 \) holds for some nonzero vector \( v \in \mathbb{C}^n \). Additionally, \( A \geq 0 \) \((A \geq 0)\) denotes that \( A \) is positive definite (semi-definite).

Consider the sequence \( \{x^k\} \) converging to the fixed-point \( x^* \). The convergence factor of the converging sequence is defined as [14]

\[
\phi^* \triangleq \sup_{x \neq x^*} \frac{\|x^{k+1} - x^*\|}{\|x^k - x^*\|}.
\]

Definitions from graph theory are now presented [15]. Let \( \mathcal{G}(V, E, \mathcal{W}) \) be a connected undirected graph with \( n \) vertices, edge set \( E \) with \( m \) edges, and edge-weights \( \mathcal{W} \). Each vertex \( i \in V \) represents an agent, and an edge \( e_k = (i, j) \in E \) means that agents \( i \) and \( j \) can exchange information. Letting \( w_{e_k} \geq 0 \) be the weight of \( e_k \), the edge-weight matrix is defined as \( W \triangleq \text{diag}(w_{e_1}, \ldots, w_{e_m}) \). Denote \( \mathcal{N}_i \triangleq \{ j \neq i | (i, j) \in E \} \) as the neighbor set of node \( i \). Define \( A \) as the span of real symmetric matrices, \( S^n \), with sparsity pattern induced by \( \mathcal{G} \), \( A \triangleq \{ S \in S^n | \mathcal{S}_{ij} = 0 \text{ if } i \neq j \text{ and } (i, j) \notin E \} \). The adjacency matrix \( A \in \mathbb{A} \) is defined as \( A_{ij} = w_{ij} \) for \( (i, j) \in E \) and \( A_{ii} = 0 \). The diagonal degree matrix \( D \) is given by \( D_{ii} = \sum_{j \in V} A_{ij} \). The incidence matrix \( B \in \mathbb{R}^{m \times n} \) is defined as \( B_{ij} = 1 \) if \( j \in e_i \) and \( B_{ij} = 0 \) otherwise.

II. THE ADMM METHOD

The ADMM algorithm solves problems of the form

\[
\begin{align*}
\text{minimize}_{x, z} & \quad f(x) + g(z) \\
\text{subject to} & \quad Ex + Fz - h = 0
\end{align*}
\]

where \( f \) and \( g \) are convex functions, \( x \in \mathbb{R}^n \), \( z \in \mathbb{R}^m \), \( h \in \mathbb{R}^p \). Moreover, \( E \in \mathbb{R}^{p \times n} \) and \( F \in \mathbb{R}^{p \times m} \) have full-column rank; see [6] for a detailed review. The method is based on the augmented Lagrangian

\[
L_{\rho}(x, z, \mu) = f(x) + g(z) + (\rho/2)||Ex + Fz - h||_2^2 + \mu^\top (Ex + Fz - h)
\]

and performs sequential minimization of the \( x \) and \( z \) variables, followed by a dual variable update:

\[
\begin{align*}
x^{k+1} &= \text{argmin}_x L_{\rho}(x, z^k, \mu^k) \\
z^{k+1} &= \text{argmin}_z L_{\rho}(x^{k+1}, z, \mu^k) \\
\mu^{k+1} &= \mu^k + \rho(Ex^{k+1} + Fz^{k+1} - h).
\end{align*}
\]

These iterations indicate that the method is particularly useful when the \( x \)- and \( z \)-minimizations can be carried out efficiently (e.g. admit closed-form expressions). One advantage of the method is that there is only one single algorithm parameter, \( \rho \), and under rather mild conditions, the method can be shown to converge for all values of the parameter; see, e.g., [6]. However, \( \rho \) has a direct impact on the convergence speed of the algorithm, and inadequate tuning of this parameter may render the method very slow. In the remaining parts of this paper, we will derive explicit expressions for the step-size \( \rho \) that minimizes the convergence factor (1) for some particular classes of problems.

III. ADMM FOR A CLASS OF EQUALITY-CONSTRAINED QUADRATIC PROGRAMMING PROBLEMS

In this section, we develop scaled ADMM iterations for a particular class of equality-constrained convex quadratic programming problems. In terms of the standard formulation [2], these problems have \( f(x) = \frac{1}{2}x^\top Q x + q^\top x \) with \( Q \succeq 0 \) and \( q \in \mathbb{R}^n \), \( g(z) = 0 \), and \( h = 0 \).

An important difference compared to the standard ADMM iterations described in the previous section is the introduction of a matrix \( R \in \mathbb{R}^{p \times p} \) scaling the equality constraints

\[
R(Ex + Fz) = 0.
\]

The underlying assumption on the choice of \( R \) is that all non-zero vectors \( v = Ex + Fz \), \( v \neq 0, z \in \mathbb{R}^m \) do not belong to the null-space of \( R \). In other words, after the transformation (5) the feasible set in (4) remains unchanged. Taking into account the transformation in (5), the penalty term in the augmented Lagrangian becomes

\[
\frac{1}{2}(Ex + Fz)\top \rho R\top R(Ex + Fz).
\]

Definition 1: \( \rho R\top R \) is called the scaling of the augmented Lagrangian (3).

Our aim is to find the optimal scaling that minimizes the convergence factor of the corresponding ADMM iterations. Specifically, introducing \( \bar{E} = RE \) and \( \bar{F} = RF \), the scaled ADMM iterations read

\[
\begin{align*}
x^{k+1} &= (Q + \rho \bar{E}\top \bar{E})^{-1} (-Q - \rho \bar{E}\top \bar{F} z^k + u^k) \\
z^{k+1} &= -(\bar{F}\top \bar{F})^{-1} \bar{F}\top (\bar{E} x^{k+1} + u^k) \\
u^{k+1} &= u^k + \bar{E} x^{k+1} + \bar{F} z^{k+1},
\end{align*}
\]

where \( u^k = \mu^k / \rho \). From the \( z \)- and \( u \)-iterations we observe

\[
\begin{align*}
u^{k+1} &= (u^k + \bar{E} x^{k+1}) - \bar{F}(\bar{F}\top \bar{F})^{-1} \bar{F}\top (\bar{E} x^{k+1} + u^k) \\
&= \Pi_{\mathcal{R}(\bar{F}\top)}(\bar{E} x^{k+1} + u^k).
\end{align*}
\]

Since \( \mathcal{R}(\bar{F}\top) \) and \( \mathcal{R}(\bar{F}) \) are orthogonal complements, then we have \( \Pi_{\mathcal{R}(\bar{F})} u^k = 0 \) for all \( k \), which results in

\[
\bar{F} z^k = -\Pi_{\mathcal{R}(\bar{F})}\bar{E} x^k.
\]

By induction the \( u \)-iterations can be rewritten as

\[
u^{k+1} = \Pi_{\mathcal{N}(\bar{F}\top)} \left( \sum_{i=1}^{k+1} (\bar{E} x^i) + u^0 \right).
\]

Supposing \( u^0 = 0 \), without loss of generality, and given \( \bar{E} \) and \( \bar{F} \), the \( x \)-iterations can be rewritten as

\[
\begin{align*}
x^{k+1} &= (Q + \rho \bar{E}\top \bar{E})^{-1} (-Q - \rho \bar{E}\top \Pi_{\mathcal{R}(\bar{F})}\bar{E} x^k) \\
&- (Q + \rho \bar{E}\top \bar{E})^{-1} \rho \bar{E}\top \Pi_{\mathcal{N}(\bar{F}\top)} \sum_{i=1}^{k+1} (\bar{E} x^i),
\end{align*}
\]
or in matrix form as
\[
\begin{bmatrix}
x^{k+1} \\
x^k
\end{bmatrix} = \begin{bmatrix}
M_{11} & M_{12} \\
I & 0
\end{bmatrix} \begin{bmatrix}
x^k \\
x^{k-1}
\end{bmatrix},
\]
with
\[
M_{11} = \rho(Q + \rho E^T E)^{-1} E^T (\Pi_{\mathcal{R}(\bar{F})} - \Pi_{\mathcal{N}(\bar{F}^T)}) \bar{E} + I,
\]
\[
M_{12} = -\rho(Q + \rho E^T E)^{-1} E^T \Pi_{\mathcal{R}(\bar{F})} \bar{E}.
\]
(11)

The convergence properties of the ADMM iterations are characterized by the spectral properties of the matrix $M$. In particular, denote $\{\phi_i\}$ as the ordered eigenvalues of $M$ so that $|\phi_1| \leq \cdots \leq |\phi_{2n-s}| < |\phi_{2n-s+1}| = \cdots = |\phi_{2n}|$ for $s \geq 1$. The ADMM iterations converge to the optimal solution if $\phi_{2n} = \cdots = \phi_{2n-s+1} = 1$ and the respective convergence factor $\|M\|$ corresponds to $\phi^* = |\phi_{2n-s}|$.

Below we state the main problem to be addressed in the remainder of this paper.

**Problem 1:** What are the optimal scalar $\rho^*$ and matrix $R^*$ in the scaling $\rho R^T R$ that minimize the convergence factor of the ADMM algorithm?

As the initial step to tackle Problem 1 in what follows we characterize the eigenvalues $\phi_i$ of $M$. Let $[u^T \ v^T]$ be an eigenvector of $M$ associated with the eigenvalue $\phi_i$, from which we conclude $\phi v = u$. Thus the following holds for the eigenvalues and corresponding eigenvectors of $M$
\[
\phi^2 v = \phi M_{11} v + M_{12} v.
\]
(12)

Our analysis will be simplified by picking $R$ such that $E^T \bar{E} = \kappa Q$ for some $\kappa > 0$. The following lemma indicates that such an $R$ can always be found.

**Lemma 1:** For $E \in \mathbb{R}^{p \times n}$ with full-column rank and $\kappa > 0$, there exists an $R$ that does not change the constraint set in (11) and ensures that $E^T \bar{E} = \kappa Q$.

**Proof:** The proof is derived in the appendix.

Now, $E^T \bar{E} = \kappa Q$ in (11) we have
\[
M_{11} = \frac{\rho}{1 + \rho \kappa} (E^T \bar{E})^{-1} E^T (\Pi_{\mathcal{R}(\bar{F})} - \Pi_{\mathcal{N}(\bar{F}^T)}) \bar{E} + I,
\]
\[
M_{12} = -\frac{\rho}{1 + \rho \kappa} (E^T \bar{E})^{-1} E^T \Pi_{\mathcal{R}(\bar{F})} \bar{E},
\]
The next result presents the explicit form of the eigenvalues of $M$ in (11).

**Theorem 1:** Consider the ADMM iterations (10). If $E^T \bar{E} = \kappa Q$, the eigenvalues of $M$ are described by
\[
2\phi = (f(\rho) \bar{\lambda} + 1) + \sqrt{(f(\rho) \bar{\lambda} + 1)^2 - 2f(\rho)(\bar{\lambda} + 1)},
\]
(13)
with
\[
f(\rho) = \frac{\rho \kappa}{1 + \rho \kappa},
\]
\[
\bar{\lambda} = \frac{v^T (E^T (\Pi_{\mathcal{R}(\bar{F})} - \Pi_{\mathcal{N}(\bar{F}^T)}) \bar{E}) v}{v^T (E^T E) v},
\]
\[
\kappa = v^T (E^T \bar{E}) v.
\]
**Proof:** The result follows from (12) and $E^T \bar{E} = \kappa Q$.

From (13) one directly sees how $\rho$ and $R$ affect the eigenvalues of $M$. Specifically, $f(\rho)$ is a function of $\rho$, while $\lambda$ only depends on $R$. In the next section we address and solve Problem 1 for a particular class of problems. The analysis follows by applying Theorem 1 and studying the properties of (13) with respect to $\rho$ and $\lambda$.

**IV. ADMM FOR DISTRIBUTED QUADRATIC PROGRAMMING**

We are now ready to develop optimal scalings for the ADMM iterations for distributed quadratic programming. Specifically, we will consider a scenario where $n$ agents collaborate to minimize an objective function on the form
\[
\min \eta \bar{Q} \eta + \bar{q}^T \eta,
\]
(14)
where $\eta = [\eta_1^T \ \cdots \ \eta_i^T \ \eta_s^T]$ and $\eta_i \in \mathbb{R}^{n_i}$ represents the private decisions of the agent $i$, $\eta_s \in \mathbb{R}$ is a shared decision among all agents, and $\bar{Q}$ has the structure
\[
\bar{Q} = \begin{bmatrix}
Q_{11} & 0 & \cdots & 0 & Q_{1s} \\
0 & Q_{22} & \cdots & Q_{2s} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & Q_{ns} & \cdots & Q_{ss}
\end{bmatrix}
\]
(15)
\[
\bar{q}^T = [q_1^T \ \cdots \ q_s^T]
\]
(16)
Here, $Q_{ss} \in \mathbb{R}$ for simplicity, $Q_{ii} \succ 0$, and $Q_{si} = Q_{is}^T \in \mathbb{R}^{n_i}$. Such structured cost functions are common in optimization for interconnected systems. For instance, the problem of state estimation in electric power networks [13] gives rise to such sparsity structure. Given that $\eta_s$ is scalar, state estimation for an electric power network with the physical structure depicted in Fig. 1(a) results in such structured $\bar{Q}$.

![Fig. 1](image_url)

**Fig. 1.** The cost coupling resulting in $\bar{Q}$ as outlined in (15). In (a) each agent $i \neq s$ represents a large area of the power network, while node $s$ corresponds to the connection point between all the areas. In (b) the agents from different areas need to jointly minimize (14) constrained by the communication network.

The optimization problem is almost decoupled, except for the shared variable $\eta_s$, and can be solved in a distributed fashion by introducing copies of the shared variable $x_{i,s} = \eta_s$ to each agent and solving the optimization problem
\[
\min_{\{\eta_1, \ldots, x_{i,1}, \ldots, x_{i,n}, \ldots, x_{s,1}, \ldots, x_{s,n}, \ldots, x_{s,s}\}} \sum_{i=1}^n f_i(\eta_i, x_{i,s})
\]
subject to
\[
x_{i,s} = x_{j,s}, \quad \forall i, j, i \neq j
\]
with
\[
f_i(\eta_i, x(i,s)) \triangleq \frac{1}{2} \begin{bmatrix} \eta_i \\ x(i,s) \end{bmatrix}^T \begin{bmatrix} Q_{ii} & Q_{is} \\ Q_{si} & \alpha_i Q_{ss} \end{bmatrix} \begin{bmatrix} \eta_i \\ x(i,s) \end{bmatrix} + \frac{1}{2} q_i^T x(i,s),
\]
\[
\hat{q}_i = (Q_{ss} \alpha_i - Q_{si} Q_{ii}^{-1} Q_{si}),
\]
where \(\alpha_i > 0\) indicates how the cost associated with \(\eta_i\) is distributed among the the copies \(x(i,s)\) with \(\sum_{i=1}^{n} \alpha_i = 1\).

Since the private variables \(\eta_i\) are unconstrained, one can solve for them analytically with respect to the shared variables, yielding
\[
f_i(x(i,s)) \triangleq \frac{1}{2} x(i,s)^T \hat{Q}_i x(i,s) + \hat{q}_i^T x(i,s),
\]
\[
\hat{Q}_i = (Q_{ss} \alpha_i - Q_{si} Q_{ii}^{-1} Q_{si}),
\]
When \(\hat{Q}\) is positive definite, there exist a set \(\{\alpha_i\}\) such that each \(f_i(x(i,s))\) is convex, as stated in the following result.

Lemma 2: For \(\hat{Q} > 0\), there exist \(\{\alpha_i\}\) such that \(\sum_{i=1}^{n} \alpha_i = 1\) and \(\hat{Q}_i > 0\) for all \(i = 1, \ldots, n\).

Proof: See the appendix.

Hence the optimization problem can be rewritten as
\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{n} f_i(x(i,s)) \\
\text{subject to} & \quad x(i,s) = x(j,s), \quad \forall i, j, i \neq j
\end{align*}
\] (17)
which reduces to an agreement, or consensus, problem on the shared variable \(x_s\), between all the nodes \(i \neq s\) depicted in Fig. [1(b)].

Each agent \(i\) holds a local copy of the shared variable \(x_i \triangleq x(i,s)\) and it only coordinates with its neighbors \(\mathcal{N}_i\) to compute the network-wide optimal solution to the agreement problem (17).

The constraints imposed by the graph can be formulated in different ways, for instance by assigning auxiliary variables to each edge or node [2]. The former is illustrated next.

A. Enforcing agreement with edge variables

Constraints must be imposed on the distributed problem so that consensus is achieved. One such way is to enforce all pairs of nodes connected by an edge to have the same value, i.e., \(x_i = x_j\) for all \((i, j) \in \mathcal{E}\). To include this constraint in the ADMM formulation, the auxiliary variable \(z(i,j)\) is created for each edge \((i, j)\), with \(z(i,j) = z_{j,i}\), and the problem is formulated as
\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \sum_{i \in \mathcal{V}} f_i(x_i) \\
\text{subject to} & \quad x_i = z_{i,j}, \quad \forall i \in \mathcal{V}, \forall (i,j) \in \mathcal{E} \\
& \quad z_{i,j} = z_{j,i}, \quad \forall (i,j) \in \mathcal{E}.
\end{align*}
\]
Consider an arbitrary direction for each edge \(e_i \in \mathcal{E}\). Now decompose the incidence matrix as \(B = B_1 + B_0\), where \(B_1|_{|j} = 1\) if \((B_0)_| = 1\) if, and only if, node \(j\) is the head (tail) of the edge \(e_i = (j, k)\). The optimization problem can be rewritten as
\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^T Q x - q^T x \\
\text{subject to} & \quad \begin{bmatrix} RB_0 \\ RB_1 \end{bmatrix} x - \begin{bmatrix} R \\ R \end{bmatrix} z = 0,
\end{align*}
\] (18)
where \(Q = \text{diag}([\hat{Q}_1 \ldots \hat{Q}_n]), q^T = [\hat{q}_1 \ldots \hat{q}_n]\), and \(W = R^T R\) is the non-negative diagonal matrix corresponding to the edge-weights.

Assumption 1: The graph \(\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})\) is connected.

As derived in the previous section, the ADMM iterations can be written in matrix form as (10). Since \(\Pi_{\mathcal{E}}(\hat{F}) = 2W\),
\[
E^T \Pi_{\mathcal{E}}(\hat{F}) E = \frac{1}{2} (B_1 + B_0)^T W (B_1 + B_0) = \frac{1}{2} (D + A),
\]
and \(E^T \hat{F} E = B_0^T W B_0 + B_1^T W B_1 = D\), we have
\[
\begin{align*}
M_{11} &= \rho(Q + \rho D)^{-1} A + I, \\
M_{12} &= -\frac{\rho}{2} (Q + \rho D)^{-1} (D + A).
\end{align*}
\] (19)

The main result in this paper is stated below and, for given \(W = R^T R\), it explicitly characterizes the optimal \(\rho\) solving Problem (11) and the corresponding convergence factor of (10) with \(M_{11}\) and \(M_{12}\) derived in (19).

Theorem 2: Suppose \(W \succ 0\) is chosen so that \(\mathcal{G}\) is connected and \(D = \kappa Q\) for \(\kappa > 0\). Let \(\{\lambda_i\}\) be the set of ordered generalized eigenvalues of \((A, D)\) for which \(\lambda_1 \leq \cdots \leq \lambda_n = 1\). The optimal step-size \(\rho^*\) that minimizes the convergence factor \(\phi^*\) is
\[
\rho^* = \begin{cases} \frac{1}{\kappa \sqrt{1 - \lambda_n^{-1}}} , & \lambda_n^{-1} \geq 0 \\
\frac{1}{\kappa} , & \lambda_n^{-1} < 0. \end{cases}
\]
Furthermore, the corresponding convergence factor is
\[
\phi^* = |\phi_{2n-1}| = \begin{cases} \frac{1}{2} \left(1 + \frac{\lambda_{n-1}}{1 + \sqrt{1 - \lambda_n^{-1}}} \right) , & \lambda_{n-1} \geq 0 \\
\frac{1}{2} , & \lambda_{n-1} < 0. \end{cases}
\]

Proof: The proof is presented in the appendix.

Note that, for a given \(W\), the optimal \(\rho^*\) and convergence factor \(|\phi_{2n-1}|\) are parameterized by \(\kappa\) and \(\lambda_{n-1}\). Moreover, it is easy to see that \(|\phi_{2n-1}| \geq \frac{1}{2}\) and minimizing \(\lambda_{n-1}\) leads to the minimum convergence factor. Hence, by finding \(W^*\) as the edge-weights minimizing \(\lambda_{n-1}\), the optimal scaling is then given by \(\rho^*(\lambda_{n-1}^*) W^*\). The optimal choice of \(W^*\) is described in the following section.

B. Optimal network-constrained scaling

Here we address the second part of Problem (11) by computing the optimal scaling matrix \(R^*\) that, together with \(\rho^*\), provides the optimal scaling minimizing the ADMM convergence factor. But first we introduce a transformation to relax the assumption that \(D = \kappa Q\). The constraints in the agreement problem (18) enforce \(x = 1_n y\) for some \(y \in \mathbb{R}\), where \(1_n \in \mathbb{R}^n\) is a vector with all entries equal to \(1\). Therefore the optimization problem is equivalent to
\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} y^T Q_1 y - q^T y - 1_n^T y.
\end{align*}
\] (20)

The next result readily follows.

Lemma 3: Consider the optimization problem (18). For given diagonal \(D \succ 0\), the optimal solution to (18) remains unchanged when \(Q\) is replaced by \(\frac{1}{\kappa} D\) if \(\kappa = \frac{1}{\sqrt{\lambda_{n-1}^*}}\).

Proof: The proof follows directly from converting (18) to (20) and having \(1_n^T Q_1 = \frac{1}{\kappa} \frac{1}{n} D 1_n\).

Thus the constraint \(D = \kappa Q\) can be achieved for any \(D \succ 0\) by modifying the original problem (18) by replacing \(Q\) with \(\frac{1}{\kappa} D\) and letting \(\kappa = \frac{1}{\sqrt{\lambda_{n-1}^*}}\). Below we show how the
minimization of $\lambda_{n-1}$ with respect to $W$ can be formulated, where the adjacency matrix $A$ is determined by the edge-weights $W$ and graph-induced sparsity pattern $A$.

**Theorem 3:** Consider the weighted graph $G = (\mathcal{V}, \mathcal{E}, W)$ and assume there exist non-negative edge-weights $W = \{w_{ij}\}$ such that $G$ is connected. The non-negative edge-weights $\{w_{ij}\}$ minimizing the second largest generalized eigenvalue of $(A, D)$, $\lambda_{n-1}$, while having $G$ connected are obtained from the optimal solution to the quasi-convex problem

\[
\begin{align*}
\text{minimize}_{\{w_{ij}\}, \lambda} & \quad \lambda \\
\text{subject to} & \quad w_{ij} \geq 0, \quad \forall i, j \in \mathcal{V}, \\
& \quad A_{ij} = w_{ij}, \quad \forall (i, j) \in \mathcal{E}, \\
& \quad A_{ij} = 0, \quad \forall (i, j) \notin \mathcal{E}, \\
& \quad D = \text{diag}(A1_n), \\
& \quad D \succ \epsilon I, \\
& \quad A - D - 1_\mathcal{V}1_n^\top \prec 0, \\
& \quad P^\top (A - \lambda D) P \prec 0,
\end{align*}
\]

where the columns of $P \in \mathbb{R}^{n \times n-1}$ form an orthonormal basis of $\mathcal{N}(1_n^\top)$ and $\epsilon > 0$.

**Proof:** The proof is in the appendix.

Given the results derived in this section, the optimal scaling $\rho^* W^*$ solving Problem 1 can be computed as summarized in Algorithm 1.

**Algorithm 1 Optimal Network-Constrained Scaling**

1) Compute $W^*$ and the corresponding $D^*$ and $\lambda_{n-1}$ according to Theorem 3.
2) Given $D^*$ and $Q$, compute $\kappa^*$ from Lemma 5.
3) Given $\kappa^*$ and $\lambda_{n-1}$, compute the optimal step-size $\rho^*$ as described in Theorem 2.
4) The optimal scaling for the ADMM algorithm with $Q$ replaced by $\frac{1}{\kappa^*} D^*$ is $\rho^* W^*$.

**V. Numerical Examples**

Next we illustrate our results in numerical examples.

**A. Distributed quadratic programming**

Consider a distributed quadratic programming problem with $n = 3$ agents and an objective function defined by

\[
\tilde{Q} = \begin{bmatrix}
4 & 1 & 1 \\
1 & 6 & 1 \\
1 & 1 & 2
\end{bmatrix},
\]

\[
\tilde{q}^\top = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}.
\]

As shown previously, the optimization problem can be reformulated on the form

\[
\begin{align*}
\text{minimize}_{\{x(i,s)\}, \lambda} & \quad \sum_{i=1}^n \frac{1}{2} x_{\{i,s\}}^\top \tilde{Q}_i x_{\{i,s\}} + \hat{q}_i^\top x_{\{i,s\}} \\
\text{subject to} & \quad x_{\{i,s\}} = x_{\{j,s\}}, \quad \forall i \neq j
\end{align*}
\]

with $n = 3$, $\alpha = \frac{1}{4}[0.5 0.9 1.6]$, $\hat{Q}_1 = 0.5507$, $\hat{Q}_2 = 0.0667$, $\hat{Q}_3 = 0.2232$, $\hat{q}_1 = -0.3116$, $\hat{q}_2 = -0.3667$, and $\hat{q}_3 = -0.1623$. As for the communication graph, we consider a line graph with node 2 connected to nodes 1 and 3. Algorithm 1 is applied, resulting in $\lambda_{n-1} = 0$ with the edge weights $w_{e_1} = w_{e_2} = 0.1566$ and degree matrix $D = \text{diag}(0.1566, 0.3132, 0.1566)$. From Theorem 2 we then have $\rho^* = \frac{1}{\kappa^*} = \sum_{i=1}^n \hat{Q}_i$, and $\phi^* = |\phi_{2n-1}| = 0.5$, which is the best achievable convergence factor. The performance of the ADMM algorithm with optimal network-constrained scaling is presented in Fig. 2. The performance the unscaled ADMM algorithm with unitary edge weights and manually optimized step-size $\rho$ is also depicted for comparison. The convergence factor of the manually tuned ADMM algorithm is $|\phi_{2n-1}| = 0.557$, thus exhibiting worse performance than the optimally scaled algorithm as depicted in Fig. 2.

**B. Distributed consensus**

In this section we apply our methodology to derive optimally scaled ADMM iterations for the average consensus problems and compare our convergence factors with the state-of-the-art fast consensus algorithm presented in [2]. The average consensus problem is a particular case of (18) where $x \in \mathbb{R}$, $Q = \alpha I$ for some $\alpha \in \mathbb{R}$, and $q = 0$. As an indicator of the performance, we compute the convergence factors for the two methods on a large number of randomly generated Erdős-Rényi graphs. Fig. 3 presents Monte Carlo simulations of the convergence factors versus the number of nodes $n \in [5, 20]$. Each component $(i, j)$ in the adjacency matrix $A$ is non-zero with probability $p = (1 + \epsilon) \frac{\log(n)}{n}$, where $\epsilon \in (0, 1)$ and $n$ is the number of vertices. In our simulations, we consider two scenarios: sparse graphs with $\epsilon = 0.2$ and dense topologies $\epsilon = 0.8$. For every network size, 50 network instances are generated, the convergence factors are computed and averaged to generate the depicted results. The figure shows two versions of Algorithm 1 with and without weight optimization in Theorem 3. We observe...
a significant improvement compared to the state-of-the-art fast consensus [2] in both sparse and dense topologies.

VI. CONCLUSIONS AND FUTURE WORK

Optimal scaling of the ADMM method for distributed quadratic programming was addressed. In particular, a class of distributed quadratic problems were cast as equality-constrained quadratic problems, to which the scaled ADMM method is applied. For this class of problems, the network-constrained scaling corresponds to the usual step-size constrained quadratic problems, to which the scaled ADMM method was cast as equality-constrained quadratic problems.

In numerical examples and significant performance improvements over state-of-the-art techniques were demonstrated. As a future work, we plan to extend the results to a broader class of distributed quadratic problems.

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APPENDIX

A. Proof of Lemma 2

Let \( Q = R_0^T R_Q \) and choose \( R = \sqrt{\kappa} R_Q (E^T E)^{-1} E^T \). Then we have \( E^T E = E^T R^T R E = \kappa E^T E^T E^{-1} Q (E^T E)^{-1} E^T \). Replacing \( R \) in aforementioned yields \( \sqrt{\kappa} R_Q x + \sqrt{\kappa} R_Q (E^T E)^{-1} E^T F z = 0 \). It indicates \( x = - (\sqrt{\kappa} R_Q)^{-1} \sqrt{\kappa} R_Q Q (E^T E)^{-1} E^T F z = -(E^T E^{-1} E^T F z) = -E^T F z \).

B. Proof of Lemma 2

Let \( T_i = Q_{ss}^{-1} Q_{si} \) and note that if \( \bar{Q} > 0 \) then \( Q_{ss} = \sum_{i=1}^n Q_{ss}^{-1} Q_{si} = \sum_{i=1}^n T_i \). Then, by taking the Schur complement of \( Q \) with respect to \( Q_{ss} \). For \( i = 2, \ldots, n \) and \( \epsilon > 0 \), let \( Q_{ss}^{-1} Q_{si} - T_i = \epsilon \) and define \( \alpha_l = (T_i + \epsilon) Q_{ss}^{-1} \). All that remains is to compute \( \alpha_l \) such that...
\[ \sum_{i=1}^{n} \alpha_i = 1 \quad \text{and} \quad A0_{n} - T_i > 0. \] Using the former equation we have \( \alpha_1 + \sum_{i=2}^{n}(T_i + \epsilon)Q_{ss}^{-1} = 1, \) and by subtracting \( T_iQ_{ss}^{-1} \) to both sides it can be rewritten as \( Q_{ss}\alpha_1 - T_i = Q_{ss} - \sum_{i=1}^{n}T_i - (N-1)\epsilon. \) Since \( \epsilon \) is arbitrary, we let \( Q_{ss}\alpha_1 - T_i = \epsilon \) and so we have \( \epsilon = \frac{1}{n}(Q_{ss} - \sum_{i=1}^{n}T_i) > 0, \) which concludes the proof.

C. Proof of Theorem 2

We are interested in minimizing the convergence factor of (7), which is expressed in terms of the magnitude of eigenvalues of \( M, \{\phi_i\}. \) From Theorem 1 the eigenvalues of \( M \) are given by

\[ 2\phi(\rho, \lambda) = 1 + \bar{\lambda} f(\rho) \pm \sqrt{1 + \lambda^2 f(\rho)^2 - 2f(\rho)} = 2\phi_v(\rho, \lambda) + 2\phi_c(\rho, \lambda), \] (22)

with \( f(\rho) = \frac{\rho \kappa}{1 + \rho \kappa}, \kappa > 0, \) and \( \bar{\lambda} = \frac{v^\top A v}{v^\top D v}. \) Recalling \( D = \kappa Q, \) in the sequel we select \( v \in \mathbb{R}^n \) such that \( v^\top D v = \kappa. \) Taking into account this scaling and considering (19) we conclude \( \lambda = \kappa^{-1}v^\top A v. \) Some properties of \( \bar{\lambda} \) follow.

**Lemma 4:** Let \( v^\top D v = \kappa \) for \( \kappa > 0 \) and define \( \kappa \lambda = v^\top A v. \) Then the following holds

\[ -1 \leq \lambda_1 \triangleq \min \frac{v^\top A v}{v^\top D v} \leq \lambda \leq \max \frac{v^\top A v}{v^\top D v} \triangleq \lambda_n = 1, \]

where \( \lambda_i \) is the \( i \)-th general eigenvalue of the matrix pencil \((A, D), \) ordered as \( \lambda_n \geq \cdots \geq \lambda_i \geq \cdots \geq \lambda_1. \)

**Proof:** The proof comes from Assumption 1. Specifically, if \( G \) is connected with \( W \geq 0 \) it is well-known from graph theory [15] that \( D > 0, D - A \geq 0, \) and \( D + A \geq 0. \) Therefore we have \( -v^\top D v \leq v^\top A v \leq v^\top D v. \) One can divide former inequalities by \( v^\top D v \) and conclude the claimed bounds, since \( D > 0. \) Moreover, from \( (D - A)1_n = 0 \) we conclude that \( D - A = \) singular and thus \( \lambda_n = 1 \) is the largest general eigenvalue of \((A, D), \) i.e., the largest value for which we have \([A - \lambda_n D] = 0. \) Similarly, \( \lambda_1 = -1 \) if \( D + A = \) singular, which is not necessarily implied by Assumption 1.

We have seen that the largest eigenvalue of \( M \) in magnitude is equal to 1 and it corresponds to the fixed-point of the ADMM algorithm [13]. So we discard it and focus on its second largest eigenvalue in magnitude. The following result characterizes the second largest eigenvalue of \( M. \)

**Lemma 5:** The magnitude of the second largest eigenvalue of \( M, |\phi_{2n-1}|, \) is given by the following equation

\[ |\phi_{2n-1}| = \max \left\{ \max_{\lambda \in \{\lambda_1, \lambda_{n-1}\}} |\phi(\rho, \lambda)|, \frac{\rho \kappa}{1 + \rho \kappa} \right\}. \] (23)

**Proof:** Recall that all the eigenvalues of \( M \) satisfy (22). Consider \( v = a1_n \) with \( a \in \mathbb{R}^n \) then \( \kappa = v^\top D v = a^21_n^\top D 1_n. \) Hence (22) becomes

\[ \phi(\rho, a^21_n^\top A 1_n) = \{1, \frac{\rho \kappa}{1 + \rho \kappa} \}. \] (24)

But \( \phi = 1 \) is the simple maximum eigenvalue of \( M \) and we discard it. Still the second term of (24) might be the magnitude of the second largest eigenvalue of \( M. \) Another possibility for the maximum magnitude of \( \phi_{2n-1} \) is when we have \( v \) orthogonal to \( 1_n \) in the definition of \( \phi \); i.e.,

\[ \max_{v, 1_n^\top v = 0} |\phi(\rho, \frac{1}{\kappa} v^\top A v)|, \]

Since \( 1_n \) is also the eigenvector associated with the largest general eigenvalue of \((A, D), \lambda_n, \) the optimization can be performed over \( \lambda = v^\top A v / \kappa. \) As a result we have

\[ \max_{\lambda \in \{\lambda_1, \lambda_{n-1}\}} |\phi(\rho, \lambda)|, \]

where \( \lambda \) is bounded above by \( \lambda_{n-1} \) as requiring \( 1_n^\top v = 0 \) excludes the largest general eigenvalue \( \lambda_n = 1. \)

The optimal step-size \( \rho^* \) that minimizes \( |\phi_{2n-1}| \) is

\[ \rho^* \triangleq \arg \max_{\rho} \max_{\lambda \in \{\lambda_1, \lambda_{n-1}\}} |\phi(\rho, \lambda)|, \frac{\rho \kappa}{1 + \rho \kappa}. \] (25)

In the following we focus on the first term of (25) and characterize \((\rho^*, \bar{\lambda}^*)\) as the solution to

\[ (\rho^*, \bar{\lambda}^*) = \arg \max_{\rho} \arg \max_{\lambda \in \{\lambda_1, \lambda_{n-1}\}} |\phi(\rho, \lambda)|. \] (26)

The next result characterizes the behavior of \( |\phi(\rho, \lambda)| \) with respect to \( \rho \) for the case where the value of (22) is complex, i.e., \( \phi_v(\rho, \lambda) \neq 0. \)

**Lemma 6:** Let \( \lambda \in \{\lambda_1, \lambda_{n-1}\} \) and \( \phi_v(\rho, \lambda) \neq 0. \) Then \( |\phi(\rho, \lambda)| \) is monotonically increasing with respect to \( \rho. \)

**Proof:** Recall from Lemma 5 that \([\lambda_1, \lambda_{n-1}] \subseteq [-1, 1]. \) For \( \phi_v(\rho, \lambda) \neq 0, \) \( |\phi(\rho, \lambda)| = \sqrt{\frac{\rho \kappa + \rho \lambda}{2(1 + \rho \kappa)}}. \) The derivative of \( |\phi(\rho, \lambda)| \) with respect to \( \rho \) is

\[ \nabla_\rho |\phi(\rho, \lambda)| = \frac{1}{2} \sqrt{\frac{2(1 + \rho \kappa)}{\rho \lambda + \rho \kappa}} \frac{\kappa \rho + \rho \kappa}{2(1 + \rho \kappa)} \geq 0 \]

since \(|\lambda| \leq 1, \) which proves that \( |\phi(\rho, \lambda)| \) is monotonically increasing for \( \phi_v(\rho, \lambda) \neq 0. \)

We now analyze the monotonicity of \( |\phi(\rho, v)| \) with respect to \( \lambda. \)

**Lemma 7:** Let \( \lambda \in \{\lambda_1, \lambda_{n-1}\} \) and \( \phi_v(\rho, \lambda) \neq 0. \) Then \( |\phi(\rho, \lambda)| \) is monotonically increasing with respect to \( \lambda. \)

**Proof:** For \( \phi_v(\rho, \lambda) \neq 0, \) \( |\phi(\rho, \lambda)| = \sqrt{\frac{\rho \kappa + \rho \lambda}{2(1 + \rho \kappa)}}. \) The derivative of \( |\phi(\rho, \lambda)| \) with respect to \( \lambda \) is

\[ \nabla_\lambda |\phi(\rho, \lambda)| = \frac{1}{2} \sqrt{\frac{2(1 + \rho \kappa)}{\rho \kappa + \rho \lambda}} \frac{\kappa \rho}{2(1 + \rho \kappa)} \frac{1}{1 + \lambda} \geq 0, \]

since \(|\lambda| \leq 1. \)

For \( \phi_v \neq 0, \) Lemma 6 indicates that \( \rho \) should be as small as possible, while Lemma 7 indicates that the optimal \( \lambda \) corresponds to \( \lambda_{n-1} \) \((A, D)\). Therefore, since for \( \lambda = \lambda_{n-1} \) we have that \( |\phi_v| \) is monotonically increasing with \( \rho, \) the second largest eigenvalue of \( M \) under the optimal \( \rho^* \) is real and \( \phi_v(\rho^*, \bar{\lambda}^*) = 0. \) The following lemmas address this case and characterize the derivatives of \( |\phi(\rho, \lambda)| \) with respect to \( \rho \) and \( \lambda \) when \( \phi(\rho, \lambda) \) is real, i.e., \( \phi_v(\rho, \lambda) = 0. \)
Lemma 8: Let $\hat{\lambda} \in [\lambda_1, \lambda_{n-1}]$ and $\phi_\nu(\rho, \hat{\lambda}) = 0$. Then $|\phi(\rho, \hat{\lambda})|$ is monotonically decreasing with respect to $\rho$.

Proof: When $\phi_\nu(\rho, \lambda) = 0$, we have $|\phi(\rho, \lambda)| = |\phi_\nu(\rho, \lambda)|$

and

$$2|\phi(\rho, \hat{\lambda})| = 1 + \lambda f(\rho) \pm \sqrt{1 + \lambda^2 f(\rho)^2 - 2f(\rho)},$$

where $\kappa \hat{\lambda} = v^T Av, \text{ and } v \perp 1_n$. Moreover, since $|\hat{\lambda}| \leq 1$ we have

$$2|\phi(\rho, \hat{\lambda})| = 1 + \lambda f(\rho) + \sqrt{1 + \lambda^2 f(\rho)^2 - 2f(\rho)}.$$

Let $g(\rho, \hat{\lambda}) \equiv 2|\phi(\rho, \hat{\lambda})|$. Recall that we are interested in the values of $\lambda_1 \leq \hat{\lambda} \leq \lambda_{n-1} < 1$. Taking the derivative of $g(\rho, \lambda)$ with respect to $\rho$ yields

$$\nabla g = f'(\rho) \left( \lambda + (1 + \lambda^2 f(\rho)^2 - 2f(\rho))^{-\frac{1}{2}} (\lambda^2 f(\rho) - 1) \right).$$

Since $f'(\rho) = \frac{\kappa}{(1 + \rho \rho)^2} > 0$, we can further simplify the above derivative and check its negativity:

$$\nabla g < 0 \iff \lambda + (1 + \lambda^2 f(\rho)^2 - 2f(\rho))^{-\frac{1}{2}} (\lambda^2 f(\rho) - 1) < 0.$$

By replacing $f(\rho)$ in the second term of the right hand side inequality we have

$$\lambda - (1 + \lambda^2 f(\rho)^2 - 2f(\rho))^\frac{1}{2} \left( \frac{1 + \rho(1 - \lambda^2)}{1 + \rho \kappa} \right)$$

$$\leq (1 - \lambda + \lambda_n \kappa)^{-1} \left( \frac{1 + \rho(1 - \lambda^2)}{1 + \rho \kappa} \right)$$

$$= -(1 - \lambda + \rho(1 - \lambda^2)) < 0,$$

where in (a) we have replaced $\hat{\lambda} < 1$ with its upper bound in the inverse square root term. \hfill \blacksquare

Lemma 9: Let $\hat{\lambda} \in [\lambda_1, \lambda_{n-1}]$ and $\phi_\nu(\rho, \hat{\lambda}) = 0$. Then $|\phi(\rho, \hat{\lambda})|$ is monotonically increasing with respect to $\hat{\lambda}$ if and only if either of the following holds:

1) $\hat{\lambda} \geq 0$;
2) $\hat{\lambda} < 0$ and $\rho \kappa \leq 1$.

Proof: Recall that for $\phi_\nu(\rho, \hat{\lambda}) = 0$ we have

$$2|\phi(\rho, \hat{\lambda})| = 1 + \lambda f(\rho) + \sqrt{1 + \lambda^2 f(\rho)^2 - 2f(\rho)}.$$

Defining $g(\rho, \hat{\lambda}) \equiv 2|\phi(\rho, \hat{\lambda})|$ we have

$$\nabla g = f(\rho) + \lambda f(\rho) \pm \sqrt{1 + \lambda^2 f(\rho)^2 - 2f(\rho)}^{-1} (\lambda f(\rho)^2).$$

For $\lambda > 0$ it follows that $\nabla g \geq f(\rho) > 0$.

Supposing that $\lambda < 0$, we see that $\nabla g \geq 0$ is equivalent to having $\sqrt{1 + \lambda^2 f(\rho)^2 - 2f(\rho)} \geq |\hat{\lambda}| f(\rho)$. Taking the square of both sides of the latter inequality we conclude that $\nabla g \geq 0$ for $\hat{\lambda} < 0$ holds if and only if $\rho \kappa \leq 1$. \hfill \blacksquare

The former results are now used to characterize the optimal $\hat{\lambda}$ that maximizes $|\phi(\rho, \hat{\lambda})|$.

Lemma 10: Let $\hat{\lambda} \in [\lambda_1, \lambda_{n-1}]$. For a given $\rho$ we have

$$\bar{\lambda}^* = \underset{\hat{\lambda} \in [\lambda_1, \lambda_{n-1}]}{\text{argmax}} |\phi(\rho, \hat{\lambda})|$$

$$= \begin{cases} 
\lambda_{n-1}, & \text{if } \lambda_{n-1} \geq 0 \text{ and } \rho \leq 1/\kappa \\
\lambda_{n-1}, & \text{if } \lambda_{n-1} < 0 \text{ and } \rho \leq 1/\kappa \\
\lambda_1, & \text{if } \lambda_{n-1} < 0 \text{ and } \rho > 1/\kappa 
\end{cases}$$

Similarly, the following result describes the optimal $\rho$ for a given $\lambda$.

Lemma 11: For given $\hat{\lambda} \in [\lambda_1, \lambda_{n-1}]$ we have

$$\rho^* = \underset{\rho}{\text{argmin}} |\phi(\rho, \hat{\lambda})| = \frac{1}{\kappa \sqrt{1 - \lambda^2}}$$

Proof: The proof follows from the monotonic properties of $|\phi(\rho, \hat{\lambda})|$ in Lemma 8 and Lemma 9 which indicate that the optimal $\rho$ yields $\sqrt{1 + \lambda^2 f(\rho)^2 - 2f(\rho)} = 0$ with $f(\rho) = \frac{\rho}{\rho \kappa}$.

The proof of Theorem 2 follows from the previous results and is now presented.

Proof: [Proof of Theorem 2] The proof comes from (25). Lemma 11 and Lemma 10 First suppose that $\lambda_{n-1} \geq 0$ and define $\rho_i = \frac{1}{\kappa \sqrt{1 - \lambda_i^2}}$ Given the monotonic properties of $|\phi(\rho, \hat{\lambda})|$ for $\hat{\lambda} \in [\lambda_1, \lambda_{n-1}]$ and the fact that $|\phi(\rho_{n-1}, \lambda_{n-1})| \geq |\phi(\rho_{n-1}, \lambda_1)|$ for any $\lambda_1 \leq \lambda_{n-1}$, we have that $(\rho_{n-1}, \lambda_{n-1})$ is a saddle-point of $|\phi(\rho, \hat{\lambda})|$ when $\lambda_{n-1} \geq 0$. Moreover we have

$$|\phi(\rho_{n-1}, \lambda_{n-1})| = \frac{1}{2} \left( 1 + \frac{\lambda_{n-1}}{1 + \sqrt{1 - \lambda_{n-1}^2}} \right)$$

$$f(\rho_{n-1}) = \frac{\rho_{n-1} \kappa}{1 + \rho_{n-1} \kappa} = \frac{1}{1 + \sqrt{1 - \lambda_{n-1}^2}},$$

and so the second largest eigenvalue of $M$ given by (25) is max $\{|\phi(\rho_{n-1}, \lambda_{n-1})|, f(\rho_{n-1})\}$. Observing that $|\phi(\rho_{n-1}, \lambda_{n-1})| - f(\rho_{n-1}) \geq 0$ holds for $1 > \lambda_{n-1} \geq 0$ concludes the first part of the proof.

Now suppose that $\lambda_1 \leq \lambda_{n-1} < 0$. From Lemma 11 the optimal $\rho$ for a given $\lambda$ is greater than $1/\kappa$. Therefore the pair $(\rho_1, \lambda_1)$ with $\rho_1 = \frac{1}{\kappa \sqrt{1 - \lambda_1^2}}$ is a saddle-point of $|\phi(\rho, \hat{\lambda})|$, since $\rho_1 \kappa > 1$ for $\lambda_1 \neq 0$ and given the monotonicity properties in Lemma 10. However, note that $|\phi(\rho_1, \lambda_1)| - f(\rho_1) < 0$ for $\lambda_1 < 0$, and hence the second largest eigenvalue of $M$ is governed by $f(\rho_1)$ in (25). Furthermore $f(\rho)$ is monotone increasing with respect to $\rho$; i.e., we have $f(\rho_1) > f(\rho)$ for all $\rho < \rho_1$. From this and Lemma 11 we conclude that $\rho^* = 1/\kappa$ as the intersection point of braces belonging to the negative $\lambda_1$ and $\lambda_{n-1}$ is the minimizer of $|\phi_{2n-1}|$. In fact, $\rho^* = 1/\kappa$ yields $|\phi(\rho^*, \lambda_1)| = \cdots = |\phi(\rho^*, \lambda_{n-1})| = f(\rho^*) = 1/2$ and it is the optimal step-size when $\lambda_1 \leq \lambda_{n-1} < 0$. \hfill \blacksquare

D. Proof of Theorem 2

The first inequality constraint ensures the non-negativity of the edge-weights, while the equality constraints merely construct the adjacency and degree matrices $A$ and $D$, respectively, and $D > \epsilon I$ ensures the problem is not numerically ill-conditioned. Therefore the main elements of the optimization problem are the last two inequality constraints.

First we show that the second-last inequality constraint ensures the graph is connected. From graph theory we have that $A - D$ is negative semi-definite for non-negative edge-weights. Moreover, $A - D$ has a single zero eigenvalue if,
and only if, the respective graph $G$ is connected, and the corresponding eigenvector is $1_n$. Hence $G$ is connected if and only if $A - D - 1_n1_n^\top$ is negative definite.

The final part of the proof shows that, for connected graphs and $P \in \mathbb{R}^{n \times n-1}$ being an orthonormal basis of $\mathcal{N}(1_n^\top)$, $P^\top (A - \lambda D) P < 0$ holds if and only if $\lambda > \lambda_{n-1}$. Consider the matrix pencil $(A, D) = A - \lambda D$ for $\lambda \in \mathbb{R}$ and let $\{\lambda_i\}$ be the ordered set of generalized eigenvalues of $(A, D)$ so that $\lambda_1 \leq \cdots \leq \lambda_n$. Recall that $\lambda_n = 1$ has $1_n$ as the corresponding eigenvector and that for $\lambda > \lambda_n$ we have $A - \lambda D < 0$. Additionally, for $\lambda_n \leq \lambda > \lambda_{n-1}$ the matrix $A - \lambda D$ has one non-negative eigenvalue and $v^\top (A - \lambda D) v < 0$ if and only if $v \not\in \mathcal{R}(1_n)$. Defining $P$ as an orthonormal basis for $\mathcal{N}(1_n^\top)$, we have $y^\top P^\top (A - \lambda D) P y < 0$, $\forall y \in \mathbb{R}^{n-1}$, since $1_n^\top P y = 0$. Hence we conclude that $P^\top (A - \lambda D) P < 0$ if and only if $\lambda > \lambda_{n-1}$. 