EPPA FOR TWO-GRAPHS AND ANTIPODAL METRIC SPACES

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Abstract. We prove that the class of two-graphs has the extension property for partial automorphisms (EPPA, or Hrushovski property), thereby answering a question of Macpherson. In other words, we show that the class of graphs has the extension property for switching automorphisms. We present a short self-contained purely combinatorial proof which also proves EPPA for the class of integer valued antipodal metric spaces of diameter 3, answering a question of Aranda et al.

The class of two-graphs is an important new example which behaves differently from all the other known classes with EPPA: Two-graphs do not have the amalgamation property with automorphisms (APA), their Ramsey expansion has to add a graph, it is not known if they have ample generics and coherent EPPA and even EPPA itself cannot be proved using the Herwig–Lascar theorem.

1. Introduction

Two-graphs, introduced by Higman and studied extensively since 1970s [Sei73, Cam99], are 3-uniform hypergraphs with the property that on every four vertices there is an even number of hyperedges. A class \( C \) of finite structures (such as hypergraphs) has the extension property for partial automorphisms (EPPA, sometimes also called Hrushovski property) if for every \( A \in C \) there exists \( B \in C \) containing \( A \) as an (induced) substructure such that every isomorphism of substructures of \( A \) extends to an automorphism of \( B \). We call \( B \) the EPPA-witness for \( A \). We prove:

**Theorem 1.1.** The class \( \mathcal{T} \) of all finite two-graphs has EPPA.

Our result answers a question of Macpherson which is also stated in Siniora’s PhD thesis [Sin17] and can be seen as a contribution to the ongoing effort of identifying new classes of structures with EPPA. This was started in 1992 by Hrushovski’s proof [Hru92] that the class of all finite graphs has EPPA, and followed by a series of papers including [Her95, Her98, HL00, HO03, Sol05, Ver08, Con19, Ott17, ABWH+17, HKN18a, Kon18, HKN18b].

All proofs of EPPA in this paper are purely combinatorial and self-contained. The second part of the paper requires some model-theoretical notions and discusses in more detail the following interesting properties for which there were no known examples before:

(1) The usual procedure for building an EPPA-witness is to construct an incomplete object (where some relations are missing) and later complete it to satisfy axioms of the class without affecting any automorphisms (i.e. one needs to have an automorphism-preserving completion [ABWH+17]). This is not possible for two-graphs and thus makes them related to tournaments which pose a well known open problem in the area, see Remark 8.1.

(2) In fact, \( \mathcal{T} \) does not even have APA (amalgamation property with automorphisms). Hodges, Hodkinson, Lascar, and Shelah [HHL93] introduced...
this notion and showed that APA together with EPPA imply the existence of ample generics (see also [Sin17, Chapter 2]). To the authors’ best knowledge, $\mathcal{T}$ is the only known class with EPPA but not APA besides pathological examples, see Section 6.

(3) In all cases known to the authors except for the class of all finite groups, whenever a class of structures $\mathcal{C}$ has EPPA then expanding a variant of $\mathcal{C}$ by all linear orders gives a Ramsey expansion. This does not seem to be the case for two-graphs, see Section 7.

(4) Solecki and Siniora [Sol09, SS17] introduced the notion of coherent EPPA (see Section 2.2) as a way to prove that the automorphism group of the respective Fraïssé limit contains a dense locally-finite subgroup. Our method does not give coherent EPPA for $\mathcal{T}$ and thus it makes $\mathcal{T}$ the only known example with EPPA for which coherent EPPA is not known. However, our method does give coherent EPPA for the class of all antipodal metric spaces of diameter 3 and using it we are able to obtain a dense locally-finite subgroup of the Fraïssé limit of $\mathcal{T}$. See Section 5.

Another way to interpret our result is to see it as a direct strengthening of the theorem of Hrushovski [Hru92] which states that the class of all finite graphs $\mathcal{G}$ has EPPA. Namely we can consider $\mathcal{G}$ with a richer class of mappings — the switching automorphisms.

Given a graph $G$ with vertex set $G$ and $S \subseteq G$, the (Seidel) switch $G_S$ of $G$ is the graph created from $G$ by complementing the edges between $S$ and $G \setminus S$. (That is, for $s \in S$ and $t \in G \setminus S$ it holds that $\{s, t\}$ is an edge of $G_S$ if and only if $\{s, t\}$ is not an edge of $G$. Edges and non-edges with both endpoints in $S$ or $G \setminus S$ are preserved.)

Given a graph $H$ with vertex set $H$, a function $f: G \to H$ is a switching isomorphism of $G$ and $H$ if there exists $S \subseteq G$ such that $f$ is an isomorphism of $G_S$ and $H$. If $G = H$ we call such a function a switching automorphism.

**Definition 1.1.** We say that a class $\mathcal{C} \subseteq \mathcal{G}$ has the extension property for switching automorphisms if for every $G \in \mathcal{C}$ there exists $S \subseteq G$ such that $f$ is an isomorphism of $G_S$ and $H$. If $G = H$ we call such a function a switching automorphism.

**Theorem 1.2.** The class of all finite graphs $\mathcal{G}$ has the extension property for switching automorphisms.

Because of the moreover part of Definition 1.1, Theorem 1.2 implies the theorem of Hrushovski. It is also a strengthening of Theorem 1.1 by the following correspondence of two-graphs and switching classes:

Given a graph $G$, its associated two-graph is a two-graph on the same vertex set as $G$ such that $\{a, b, c\}$ is a hyperedge if and only if the three-vertex subgraph induced by $G$ on $\{a, b, c\}$ has an even number of edges. It is a well known fact that there is a switching isomorphism of $G$ and $H$ if and only if their associated two-graphs are isomorphic. In other words, the existence of a switching isomorphism is an equivalence on the class of graphs and two-graphs correspond to its equivalence classes (called switching classes of graphs).

We shall see that the most natural setting for our proof is to work with the class of all finite integer-valued antipodal metric spaces of diameter 3. Following [ACM16] we call a metric space an integer-valued metric space of diameter 3 if the distance of every two distinct points is 1, 2 or 3. It is antipodal if
Figure 1. Two possible antipodal quadruples for choice of $a, b, c, d$, $d_A(a, b) = d_A(c, d) = 3$.

(1) it contains no triangle with distances 2, 2, 3, and
(2) the edges with label 3 form a perfect matching (in other words, for every vertex there is precisely one antipodal vertex at distance 3).

In this language we can state our main theorem as (see Proposition 3.2):

**Theorem 1.3.** The class of all finite integer-valued antipodal metric spaces of diameter 3 has coherent EPPA.

We remark that this theorem generalises to all antipodal metric spaces from Cherlin’s catalogue of metrically homogeneous graphs [Che17]. This is outlined in [ABWH*17] and will appear in full detail elsewhere. This answers affirmatively a question of Aranda, Bradley-Williams, Hubička, Karamanlis, Kompatscher, Konečný and Pawliuk [ABWH*17] and completes the analysis of EPPA for all known amalgamation classes of metrically homogeneous graphs.

2. Notation and preliminaries

It is in the nature of this paper to consider multiple types of structures. We will use bold letter such as $A, B, C, \ldots$ to denote structures ((hyper)graphs or metric spaces defined below) and corresponding normal letters, such as $A, B, C, \ldots$, to denote corresponding vertex sets. Our substructures (sub-(hyper)graphs or subspaces) will always be induced.

Formally, we will consider a metric space to be a complete edge-labelled graph (that is, a complete graph where edges are labelled by the respective distances), or, equivalently, a relational structure with multiple binary relations representing the distances. This justifies that we will speak of pairs of vertices in distance $d$ as of edges of length $d$. We will, however, use both notions (a vertex set with a distance function or a complete edge-labelled graph) interchangeably. We adopt the standard notion of isomorphism, embedding and substructure.

2.1. Correspondence between two-graphs and antipodal metric spaces.

There is a one-to-one correspondence between two-graphs and antipodal metric spaces of diameter 3 (up to isomorphism). This correspondence is not new and was studied as double covers of complete graphs. (By a double cover of a complete graph $K_n$ we mean a graph $G$ with a 2-to-1 covering map from the vertices of $G$ to those of $K_n$, so that each edge of $K_n$ is covered by two edges of $G$.) Considering a double cover as a special metric space makes our constructions more explicit.

In an antipodal metric space $A$, every quadruple of distinct vertices $a, b, c, d$ such that $d_A(a, b) = d_A(c, d) = 3$ (a pair of edges of label 3) induces one of the two (isomorphic) subspaces depicted in Figure 1 — we call these antipodal quadruples. However, three edges with label 3 can induce two non-isomorphic structures; the edges of length 1 either form two triangles, or one 6-cycle (see Figure 2). This motivates the following correspondence:

**Definition 2.1** (Antipodal spaces to two-graphs). Let $A$ be an antipodal metric space. Let $M$ be the set of all edges of $A$ of length 3 (thus, $|M| = \frac{|A|}{2}$). Define $T(A)$ to be the 3-uniform hypergraph on vertex set $M$ where $\{a, b, c\}$ is a hyperedge
if and only if the substructure of $A$ induced on the edges $a, b, c$ is isomorphic to Figure 2 (b).

It is easy to check that $T(A)$ is a two-graph. One only needs to verify that four edges of length 3 always induce a structure with even number of substructures isomorphic Figure 2 (b). It is also straightforward to check that every automorphism of $A$ induces an automorphism of $T(A)$.

In the other direction, it is also possible to assign (uniquely up to isomorphism) an antipodal metric space to a two-graph:

**Proposition 2.1.** Let $T$ be a two-graph. Then there is an antipodal metric space $A$ such that $T = T(A)$. Furthermore every automorphism of $T$ is induced by an automorphism of $A$.

**Proof.** Take arbitrary three vertices of $T$ and put into $A$ three edges of length 3 corresponding to Figure 2 (a) or (b) based on whether the three vertices in $T$ form a hyperedge. Then enumerate all remaining vertices of $T$, for each one add a new edge of length 3 to $A$ and connect it arbitrarily (validly, see Figure 1) to one existing edge of length 3 of $A$. The remaining connections are uniquely determined by the hyperedges of $T$. The statement now follows. \[\square\]

Observe also that given a graph $G$, its associated two-graph and the corresponding antipodal metric-space $A$, the graph $G$ is isomorphic to one of the graphs created from $A$ by considering only edges of distance 1 and removing one vertex from every pair of vertices in distance 3. (Depending on the choice of removed vertices one obtains $G$ or some Seidel switch of $G$.)

### 2.2. Coherent EPPA

Coherence is a natural strengthening of EPPA which is used to construct dense locally finite subgroups [Sol09]. At the moment all previously known EPPA classes are also coherent EPPA classes. Here, we can prove coherent EPPA for the antipodal metric spaces of diameter 3, but not for two-graphs (this is discussed in Section 5). We need to introduce two additional definitions.

**Definition 2.2** (Coherent maps [Sol09, SS17]). Let $X$ be a set and $\mathcal{P}$ be a family of partial bijections between subsets of $X$. A triple $(f, g, h)$ from $\mathcal{P}$ is called a coherent triple if

$$\text{Dom}(f) = \text{Dom}(h), \text{Range}(f) = \text{Dom}(g), \text{Range}(g) = \text{Range}(h)$$

and

$$h = g \circ f.$$

Let $X$ and $Y$ be sets, and $\mathcal{P}$ and $\mathcal{Q}$ be families of partial bijections between subsets of $X$ and between subsets of $Y$, respectively. A function $\varphi : \mathcal{P} \to \mathcal{Q}$ is said to be a coherent map if for each coherent triple $(f, g, h)$ from $\mathcal{P}$, its image $\varphi(f), \varphi(g), \varphi(h)$ in $\mathcal{Q}$ is coherent. In the case of EPPA, we sometimes refer to the image of $\varphi$ as a coherent family of automorphisms extending $\mathcal{P}$. 

![Figure 2. Two non-isomorphic antipodal metric spaces with 6 vertices.](image-url)
Definition 2.3 (Coherent EPPA [Sol09, SS17]). A class $\mathcal{C}$ of finite structures is said to have \textit{coherent EPPA} if $\mathcal{C}$ has EPPA and moreover the extension of partial automorphisms is coherent. That is, for every $A \in \mathcal{C}$, there exists $B \in \mathcal{C}$ such that $A \subseteq B$ (that is, $B$ contains $A$ as an (induced) substructure) and every partial automorphism $f$ of $A$ extends to some $\hat{f} \in \text{Aut}(B)$ with the property that the map $\varphi$ from the partial automorphisms of $A$ to the automorphisms of $B$ given by $\varphi(f) = \hat{f}$ is coherent.

In this case we also call $B$ a \textit{coherent EPPA-witness} of $A$.

3. EPPA for antipodal metric spaces

Given an antipodal metric space $A$ we give a direct construction of its coherent EPPA-witness $B$. Some ideas are based on a construction of Hodkinson and Otto [HO03] and some of the terminology is loosely based on Hodkinson’s exposition of this construction [Hod02]. Our techniques also give a very simple and short proof of EPPA for graphs.

Fix an antipodal metric space $A$. Denote by $M = \{e_1, e_2, \ldots, e_n\}$ the set of all edges of $A$ of length 3. For a function $\chi : M \to \{0, 1\}$ we denote by $1 - \chi$ the function satisfying $(1 - \chi)(x) = 1 - \chi(x)$ for every $x \in M$. We construct $B$ as follows:

1. The vertices of $B$ are all pairs $(x, \chi_x)$ where $x \in M$ and $\chi_x$ is a function from $M$ to $\{0, 1\}$ (called a valuation function).
2. Distances for $(x, \chi_x) \neq (y, \chi_y) \in B$ are given by the following rules:
   i) $d_B((x, \chi_x), (x, 1 - \chi_x)) = 3$,
   ii) $d_B((x, \chi_x), (y, \chi_y)) = 1$ if and only if $\chi_x(y) = \chi_y(x)$,
   iii) $d_B((x, \chi_x), (y, \chi_y)) = 2$ otherwise.

Lemma 3.1. The structure $B$ is an antipodal metric space.

Proof. Given $(x, \chi_x) \in B$, by (i) we know that there is precisely one vertex in distance 3 (namely $(x, 1 - \chi_x)$) and thus the edges of length 3 form a perfect matching.

It remains to check that every quadruple $(x, \chi_x), (x, 1 - \chi_x), (y, \chi_y), (y, 1 - \chi_y)$ of distinct vertices of $B$ is an antipodal metric space. By (ii) we know that precisely one of $(y, \chi_y), (y, 1 - \chi_y)$ is in the distance 1 from $(x, \chi_x)$ and by (iii) that the other in the distance 2, similarly for $(x, 1 - \chi_x)$. It also follows that $d_B((x, \chi_x), (y, \chi_y)) = d_B((x, 1 - \chi_x), (y, 1 - \chi_y))$ and $d_B((x, \chi_x), (y, 1 - \chi_y)) = d_B((x, 1 - \chi_x), (y, \chi_y))$. □

We now define a generic copy $A'$ of $A$ in $B$ by giving a construction of an embedding $\psi : A \to B$.

Fix an arbitrary function $p : A \to \{0, 1\}$ such that whenever $d_A(x, x') = 3$, then $p(x) = 1 - p(x')$. This function partitions vertices of $A$ into two podes such that pairs of vertices in distance 3 are in different podes. For every $1 \leq i \leq n$ we denote by $x_i$ and $y_i$ the endpoints of $e_i$ such that $p(x_i) = 0$ and $p(y_i) = 1$. We construct $\psi$ by induction.

1. Start with empty mapping $\psi_0$.
2. For every $i \in \{1, 2, \ldots, n\}$ assume that $\psi_{i-1}$ is already defined. Extend $\psi_{i-1}$ to $\psi_i$ by mapping $x_i$ to $(e_i, \chi_{x_i})$ and $y_i$ to $(e_i, 1 - \chi_{x_i})$ where $\chi_{x_i}$ is defined as

$$\chi_{x_i}(e_j) = \begin{cases} 0 & \text{if } j \geq i \\ 0 & \text{if } j < i \text{ and } d_A(x_i, x_j) = 1 \\ 1 & \text{otherwise.} \end{cases}$$

We put $\psi = \psi_n$. It follows from the construction that $\psi$ is indeed an embedding.

Now we are ready to show the main result of this section:
Proposition 3.2. The antipodal metric space $\mathbf{B}$ is a coherent EPPA-witness for $\mathbf{A}'$. Moreover, $p \circ \psi^{-1}$ extends to a function $\hat{p}: B \to \{0,1\}$ such that whenever partial automorphism $\varphi$ preserves values of $p \circ \psi^{-1}$, then its coherent extension $\theta$ preserves values of $\hat{p}$.

Proof. Let $\varphi$ be a partial automorphism of $\mathbf{A}'$. Let $\pi: B \to M$ be the projection mapping $(x,\chi_x) \mapsto x$. By this projection $\varphi$ induces a partial permutation of $M$ and we denote by $\hat{\varphi}$ its extension to an permutation of $M$. To obtain coherence we always extend the permutation in an order-preserving way (where the linear order of $M$ can be fixed arbitrarily, for example by its enumeration $\{e_1,e_2,\ldots,e_n\}$).

Let $F$ be the set consisting of unordered pairs $x,y \in M$ (possibly $x = y$) such that there exists $\chi_x$ with the property that $(x,\chi_x) \in \text{Dom}(\varphi)$ and $\chi_x(y) \neq \chi_x(\hat{\varphi}(y))$ for $(z,\chi_z) = \varphi((x,\chi_x))$. We say that these pairs are flipped by $\varphi$. Because of the choice of $\mathbf{A}'$, there are at most two choices for $\chi_x$, and if there are two, then they are $\chi_x$ and $1 - \chi_x$ for some $\chi_x$ and both of them give the same result.

Note that there may be no $\chi_y$ such that $(y,\chi_y) \in \text{Dom}(\varphi)$. However, if such $\chi_y$ exists, the condition above has the same outcome when the role of $x$ and $y$ is exchanged. This follows from the definition of $\mathbf{B}$ and the fact that the distance of $(x,\chi_x)$ and $(y,\chi_y)$ is preserved by $\varphi$.

Define a function $\theta: B \to B$ by putting

$$\theta((x,\chi_x)) = (\hat{\varphi}(x),\chi'_x)$$

where

$$\chi'_x(\hat{\varphi}(y)) = \begin{cases} \chi_x(y) & \text{if } (x,y) \notin F \\ 1 - \chi_x(y) & \text{if } (x,y) \in F. \end{cases}$$

First we verify that $\theta$ is an automorphism extending $\varphi$. It is easy to see that $\theta$ is one-to-one (one can construct its inverse). Because conditions (i), (ii) and (iii) in the definition of $\mathbf{B}$ are not affected by the adjustments (flips) made to the valuation functions by means of the set $F$, we get that $\theta$ is an automorphism. On the other hand, we constructed $F$ precisely to get $\varphi((x,\chi_x)) = \theta((x,\chi_x))$ for every $(x,\chi_x) \in \text{Dom}(\varphi)$ and thus $\theta$ is an extension of $\varphi$. We thus conclude that $\mathbf{B}$ is an EPPA-witness for $\mathbf{A}'$.

Next we verify coherence. Consider partial automorphisms $\varphi_1$, $\varphi_2$ and $\varphi$ of $\mathbf{A}'$ such that $\varphi$ is the composition of $\varphi_1$ and $\varphi_2$. Denote by $\hat{\varphi}_1$, $\hat{\varphi}_2$ and $\hat{\varphi}$ their corresponding permutations of $M$ constructed above, by $F_1$, $F_2$ and $F$ the corresponding sets of flipped pairs and by $\theta_1$, $\theta_2$ and $\theta$ the corresponding extensions. Because permutations $\hat{\varphi}_1$, $\hat{\varphi}_2$ and $\hat{\varphi}$ were constructed by extending projections of $\varphi_1$, $\varphi_2$ and $\varphi$ (which also compose coherently) in an order preserving way, we know that $\hat{\varphi}$ is the composition of $\hat{\varphi}_1$ and $\hat{\varphi}_2$. To see that $\theta$ is the composition of $\theta_1$ and $\theta_2$ it remains to verify that pairs flipped in $\theta$ by the construction of $F$ above are precisely those pairs that are flipped by the composition of $\theta_1$ and $\theta_2$. This follows from the construction of $F$. Only pairs with at least one vertex in the domain of $\pi \circ \varphi_1$ are put into sets $F$ and $F_1$ and again only pairs with at least one vertex in the domain of $\pi \circ \varphi_2$ (which is the same as the value range of $\pi \circ \varphi_1$) are put into $F_2$. Consequently only those pairs with at least one vertex in the domain of $\pi \circ \varphi_1$ (and also of $\pi \circ \varphi$) are possibly flipped by the composition and those adjustments are unique to make $\theta_2 \circ \theta_1$ the extension of $\varphi_2 \circ \varphi_1$ and thus the same flips must be performed by $\theta$.

Finally put $\hat{p}(x,\chi_x) = \chi_x(x)$. If $\varphi$ preserves values of $p(x)$ we also know that it has the property that $F$ contains only pairs of distinct vertices. By definition of $\theta$ we immediately get $\hat{p}(x,\chi_x) = \hat{p}(\theta((x,\chi_x)))$. \qed
4. Proofs of the main results

Theorem 1.3 is a direct consequence of Proposition 3.2. EPPA for two-graphs follows easily too:

Proof of Theorem 1.1. Let $T$ be a finite two-graph and $A$ be the antipodal metric space associated to $T$ by Proposition 2.1. By Theorem 1.3 we know that there is an EPPA-witness $B$ for $A$. We claim that $T(B)$ is an EPPA-witness for $T$. Indeed, let $\varphi: T \to T$ be a partial automorphism of $T$. This induces a partial automorphism $\psi: A \to A$ (observe that this automorphism is not unique) and by Theorem 1.3 we get an automorphism $\bar{\psi} \supseteq \psi$ of $T(B)$ extending $\varphi$.

Remark 4.1. Observe that the EPPA-witness given in this proof of Theorem 1.1 is not a coherent EPPA-witness. The problem is that for every partial automorphism of $T$ there are two corresponding partial automorphisms of $A$. For example, if the partial automorphism of $T$ is a partial identity, one partial automorphism of $A$ is also a partial identity, while the other flips all the involved edges of length 3. While the first is extended to the identity by the construction in Proposition 3.2, the other is extended to a non-trivial permutation of the edges of length 3. This seems to be a fundamental obstacle for our construction to give coherence for two-graphs.

Proof of Theorem 1.2. Given a graph $G$, we construct an antipodal metric space $A$ on vertex set $G \times \{0,1\}$ with distances defined as follows:

1. $d_A((x,0),(x,1)) = 3$ for every $x \in G$,
2. $d_A((x,i),(y,i)) = 1$ for every $x \neq y$ forming an edge of $G$ and $i \in \{0,1\}$,
3. $d_A((x,i),(y,1-i)) = 1$ for every $x \neq y$ forming a non-edge of $G$ and $i \in \{0,1\}$, and
4. $d_A((x,i),(y,j)) = 2$ otherwise.

Let $p: A \to \{0,1\}$ be a function defined by $p((x,i)) = i$ and apply Proposition 3.2 to construct an antipodal metric space $C$ and a function $\hat{p}$. Construct a graph $H$ with vertex set $\{x \in C : \hat{p}(x) = 0\}$ with $x,y$ forming an edge if and only if $d_C(x,y) = 1$.

Now consider a partial automorphism $\varphi$ of $G$. This automorphism corresponds to a partial automorphism $\varphi'$ of $A$ by putting $\varphi'((x,i)) = (\varphi(x),i)$ for every $x \in \text{Dom}(\varphi)$ and $i \in \{0,1\}$. $\varphi'$ then extends to $\theta$ which preserves values of $\hat{p}$. Consequently, $\theta$ restricted to $H$ is an automorphism of $H$.

Finally consider a partial switching automorphism $\varphi$. Let $S$ be the set of vertices switched by $\varphi$. Now the partial automorphism of $A$ is defined by putting $\varphi'((x,i)) = (\varphi(x),i)$ if $x \notin S$ and $\varphi'((x,i)) = (\varphi(x),1-i)$ otherwise. Again extend $\varphi'$ to $\theta$ and observe that $\theta$ gives a switching automorphism of $H$.

5. Existence of a dense locally-finite subgroup

Solecki and Siniora [Sol09, SS17] introduced the notion of coherent EPPA (see Section 2.2) as a way to prove that the automorphism group of the respective Fraïssé limit contains a dense locally-finite subgroup. While we cannot prove coherent EPPA for $T$, Theorem 1.3 gives coherent EPPA for the class of all antipodal metric spaces of diameter 3 and thus a dense locally-finite subgroup of the automorphism group of their Fraïssé limit. We now show how to pull this subgroup to the automorphism group of the Fraïssé limit of $T$.

Let $\mathbb{T}$ be the Fraïssé limit of $T$ and $\mathbb{M}$ be the Fraïssé limit of the class of all finite antipodal metric spaces of diameter 3.

Theorem 5.1. There is a dense, locally finite subgroup of $\text{Aut}(\mathbb{T})$. 

Proof. It is clear that the two graph $T(M)$ from Proposition 2.1 is a countable homogeneous two-graph which embeds all finite two-graphs, thus it is isomorphic to $T$ and we have a group homomorphism $\alpha: \text{Aut}(M) \to \text{Aut}(T)$ by considering the effect of an automorphism of $M$ on antipodal pairs of vertices. By Proposition 2.1, the image of $\alpha$ is dense in $\text{Aut}(T)$. Thus, if $H$ is a dense locally finite subgroup of $\text{Aut}(M)$ (as guaranteed by coherent EPPA), then $\alpha(H)$ is a locally finite group which is dense in $\text{Aut}(T)$. □

Remark 5.1. As $\alpha$ is induced by a finite-to-one map $T \to M$, it can be shown that its image is closed, and therefore $\alpha$ has image $\text{Aut}(T)$.

6. Amalgamation property with automorphisms

Let $A$, $B_1$ and $B_2$ be structures, $\alpha_1$ an embedding of $A$ into $B_1$ and $\alpha_2$ an embedding of $A$ into $B_2$. Then every structure $C$ with embeddings $\beta_1: B_1 \to C$ and $\beta_2: B_2 \to C$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ is called an amalgamation of $B_1$ and $B_2$ over $A$ with respect to $\alpha_1$ and $\alpha_2$. Amalgamation is strong if $\beta_1(x_1) = \beta_2(x_2)$ if and only if $x_1 \in \alpha_1(A)$ and $x_2 \in \alpha_2(A)$.

For simplicity, in the following definition we will assume that all the embeddings in the definition of amalgamation are in fact inclusions.

Definition 6.1 (APA). Let $C$ be a class of finite structures. We say that $C$ has the amalgamation property with automorphisms (APA) if for every $A$, $B_1$, $B_2 \in C$ such that $A \subseteq B_1$, $B_2$ there exists $C \in C$ which is the amalgamation of $B_1$ and $B_2$ over $A$, has $B_1$, $B_2 \subseteq C$ and furthermore the following holds:

For every pair of automorphisms $f_1 \in \text{Aut}(B_1)$, $f_2 \in \text{Aut}(B_2)$ such that $f_i(A) = A$ for $i \in \{1, 2\}$ and $f_1|_A = f_2|_A$, there is an automorphism $g \in \text{Aut}(C)$ such that $g|_{B_i} = f_i$ for $i \in \{1, 2\}$.

In other words, APA is a strengthening of the amalgamation property which requires that every pair of automorphisms of $B_1$ and $B_2$ which agree on $A$ extends to an automorphism of $C$.

As was mentioned in the introduction, EPPA + APA imply the existence of ample generics [HHLS93] and it turns out that most of the known EPPA classes also have APA. We now show that this is not the case for two-graphs.

Proposition 6.1. Let $A$ be the two-graph on two vertices, $B_1$ be the two-graph on three vertices with no hyper-edge and $B_2$ be the two-graph on three vertices which form a hyper-edge. Let $C_0$ be the free amalgam of $B_1$ and $B_2$ over $A$ (see Figure 3). Then $C_0$ has a completion in $T$, but has no automorphism-preserving completion in $T$.

Figure 3. A failure of APA for two-graphs
Proof. For convenience, we name the vertices as in Figure 3: $A = \{u, v\}$, $B_1 = \{u, v, x_1\}$ and $B_2 = \{u, v, x_2\}$. For $i \in \{1, 2\}$ let $f_i$ be the automorphism of $B_i$ such that $f_i(x_i) = x_i$, $f_i(u) = v$ and $f_i(v) = u$. Clearly $f_1$ and $f_2$ agree on $A$.

Consider the amalgamation problem for $B_1$ and $B_2$ over $A$ (with inclusions and with automorphisms $f_1$, $f_2$) and assume for a contradiction that there is $C \in \mathcal{T}$ and an automorphism $g$ of $C$ as in Definition 6.1. By the definition of $\mathcal{T}$, we get that there has to be an even number of triples in $C$ on $\{u, v, x_1, x_2\}$ and since we know that $ux_1$ is not a triple and $ux_2$ is a triple, there actually have to be precisely two triples on $\{u, v, x_1, x_2\}$. Therefore, exactly one of $ux_1x_2$ and $vx_1x_2$ has to form a triple in $C$. But this means that $g$ is not an automorphism (as $g$ fixes $x_1$ and $x_2$ and swaps $u$ and $v$), a contradiction.

On the other hand, if we only want to amalgamate $B_1$ and $B_2$ over $A$ (not with automorphisms), then this is clearly possible. □

Remark 6.1. In the introduction we mentioned that there are also some pathological examples with EPPA but not APA:

1. Let $M$ be the two-vertex set with no structure. Its age consists of the empty set, the one-element set and $M$. Consider the amalgamation problem for $A$ the empty set and $B_1 = B_2 = M$. The amalgam has to be $M$ again. But then it is impossible to preserve all four possible pairs of automorphisms of $B_1$ and $B_2$.

   This phenomenon clearly happens because this is not a strong amalgamation class (and is exhibited by other non-strong amalgamation classes) and, indeed, disappears when we only consider closed structures.

2. Let now $M$ be the disjoint union of two infinite cliques, that is, an equivalence relation with two equivalence classes and consider its age. Let $A$ be the empty structure and $B_1 = B_2 = M$. This amalgamation problem, again, does not have an APA-solution, because one needs to decide which pairs of vertices will be equivalent and this cannot preserve all four pairs of automorphisms.

   This generalises to situations where the homogeneous structure has a definable equivalence relation with finitely many equivalence classes (or, more generally, of finite index).

   However, the reason here is that the equivalence classes are algebraic in a quotient, i.e. a similar reason as in the previous point. One can either require the amalgamation problem to specify which classes go to which ones or, equivalently, consider an expansion which weakly eliminates imaginaries and the problem disappears.¹

¹For example, let $C$ be a class of all finite structures in language consisting of three equivalence relations, $E_1$, $E_2$ and $E_3$ such that $E_2$ is a refinement of $E_1$, $E_3$ is a refinement of $E_2$ and furthermore $E_1$ has only two equivalence classes and inside each equivalence class of $E_2$ there are only two equivalence classes of $E_3$.

Then for each $A \in C$ one can construct a structure $A^+$ by the following procedure:

(a) Add a new vertex (ball vertex) for each equivalence class of $E_1$, $i \in \{1, 2, 3\}$ (for $E_1$ we in fact add two ball vertices even if $A$ contains only one equivalence class of $E_1$, for $E_2$ we add two ball vertices for each existing equivalence class of $E_2$, even if it only refines to one equivalence class of $E_3$).

(b) Add unary functions $F_1$, $F_2$, $F_3$ and let each original vertex map to its corresponding ball vertex of $E_1$ by $F_i$.

(c) Add unary functions $F_2^1$ and $F_2^2$ and let each ball vertex of $E_3$ map via $F_2^i$ to the ball vertex of $E_2$ representing a superclass, the same for $F_3^2$ and $E_2$ and $E_1$.

(d) Add a unary function (a constant) pointing to the unordered pair of ball vertices of $E_1$.

(e) And add a unary function pointing from each ball vertex of $E_2$ to the unordered pair of the ball vertices of $E_3$ refining the given equivalence class.
While these two examples point out that, at least from the combinatorial point of view, one needs a more robust definition for APA, two-graphs seem to innately not have APA.

7. Ramsey expansion of two-graphs

As it was pointed out recently, the methods for proving EPPA and the Ramsey property have converged, see e.g. [ABWH + 17]. The standard strategy is to study the completion problem for given classes and their expansions, see [ABWH + 17, EHN17, Kon18, HKN18b]. EPPA is usually a corollary of one of the steps towards finding a Ramsey expansion.

The situation is very different with two-graphs. As we shall observe in this section, the Ramsey question can be easily solved using known techniques, while it is provably not the case for EPPA (see Remark 8.1). This makes two-graphs an important example of the limits of the current methods and shows that the novel approach of this paper is in fact necessary.

We now give the very basic definitions of the structural Ramsey theory.

Class $\mathcal{K}$ of structures is Ramsey if for every $A, B \in \mathcal{K}$ there exists $C \in \mathcal{K}$ such that for every colouring of copies of $A$ in $C$ there exists a copy of $B$ in $C$ that is monochromatic. (By a copy of $A$ in $C$ we mean any substructure of $C$ isomorphic to $A$.)

If class $\mathcal{K}$ has joint embedding property (which means that for every $A_1, A_2 \in \mathcal{K}$ there is $C \in \mathcal{K}$ which contains a copy of both $A_1$ and $A_2$) then EPPA of $\mathcal{K}$ implies amalgamation. It is well known that Ramsey property also implies amalgamation property [Neˇ s89, Neˇ s05] under the assumption of joint embedding.

Every Ramsey class consist only of rigid structures, i.e. structures with no non-trivial automorphism. The usual way to establish rigidity is to extend the language (in a model-theoretical way) by an additional binary relation $\leq$ which fixes the ordering of vertices. It is thus natural question whether class $\rightarrow T$ of all two-graphs with linear ordering of vertices is Ramsey. We now show that the answer is negative:

**Proposition 7.1** (Graphs have expansion property with respect to two-graphs).

For every two-graph $A$ there exists a two-graph $B$ such that every graph in the switching class of $B$ contains a copy of every graph in the switching class of $A$.

**Proof.** Denote by $G$ the disjoint union of all graphs in the switching class of $A$. Now let $H$ be a graph such that every colouring of vertices of $H$ by 2 colours contains a monochromatic copy of $G$ (that is, $H$ is vertex-Ramsey for $G$) — it exists by a theorem of Folkman [Fol70, NR77a]).

Every graph $H'$ in the switching class of $H$ induces a colouring of vertices of $H$ by two colours: $H'$ being in the switching class of $H$ means that there is a set $S \subseteq H$ such that $H' = H_S$, the colour classes are then $S$ and $H \setminus S$ respectively.

By the construction of $H$ we find a copy $\tilde{G} \subseteq H$ of $G$ which is monochromatic with respect to this colouring. This however implies that the graphs induced by $H$ and $H'$ on the set $\tilde{G}$ are isomorphic and thus $H'$ indeed contains every graph in the switching class of $A$. Therefore, we can put $B$ to be the two-graph associated to $H$.

**Corollary 7.2.** The class $\rightarrow T$ is not Ramsey for colouring pairs of vertices.

The class $\mathcal{C}^+$ of all structures $A^+$ which one can get in this way from some $A \in \mathcal{C}$ has APA. $\mathcal{C}$ and $\mathcal{C}^+$ are not isomorphic as categories precisely because we added two ball vertices for $E_1$ even if $A$ had only one equivalence class and similarly with $E_3$. 

Proof. Let $\mathbf{A}$ be the two-graph associated to an arbitrary graph containing both an edge and a non-edge, and let $\mathbf{B}$ be the two-graph given by Proposition 7.1 for $\mathbf{A}$. Let $\overrightarrow{\mathbf{B}}$ be an arbitrary linear ordering of $\mathbf{B}$.

Assume that there exists a ordered two-graph $\overrightarrow{\mathbf{C}}$ such that

$$\overrightarrow{\mathbf{C}} \rightarrow (\overrightarrow{\mathbf{B}})^{\overrightarrow{\mathbf{E}}}$$

where $\overrightarrow{\mathbf{E}}$ is the unique ordered two-graph on 2 vertices.

Let $\overrightarrow{\mathbf{T}}$ be an arbitrary graph from the switching class of $\overrightarrow{\mathbf{C}}$ (with the inherited order) and colour copies of $\overrightarrow{\mathbf{E}}$ red if they correspond to an edge of $\overrightarrow{\mathbf{T}}$ and blue otherwise. By the construction of $\mathbf{B}$ it follows that there is no monochromatic copy of $\overrightarrow{\mathbf{B}}$, a contradiction. $\square$

Proposition 7.1 says that adding a particular graph from the switching class has the so-called expansion property. As a consequence of the Kechris–Pestov–Todorčević correspondence [KPT05] one then gets that every Ramsey expansion of $\mathcal{T}$ has to fix a particular representative of the switching class (see i.e. [NVT15] for details). On the other hand, expanding any two-graph by a linear order and a particular graph from the given switching class is a Ramsey expansion by the Nešetřil-Rödl theorem [NR77b]. Thus, as opposed to EPPA, Ramsey expansion of $\mathcal{T}$ is an easy problem.

8. Remarks

Remark 8.1. The mentioned strategy for proving EPPA for class $\mathcal{C}$ in, say, relational language $L$ is the following:

1. Assume that, every pair of vertices of every structure in $\mathcal{C}$ is in some relation. If it is not, we can always add a new binary relation to $L$ and put all pairs into the relation.
2. Study the class $\mathcal{C}^-$ which consists of all $L$-structures $\mathbf{A}^-$ such that there is $\mathbf{A} \in \mathcal{C}$ which is a completion of $\mathbf{A}^-$, that is, $\mathbf{A}^-$ and $\mathbf{A}$ have the same vertex set $A$ and there is $X \subseteq P(A)$, a subset of the powerset of $A$, such that if $Y \subseteq X$ and $Z \subseteq Y$, then $Z \subseteq X$, and for each relation $R \in L$ it holds that $R^{\mathbf{A}^-} = R^{\mathbf{A}} \cap X$.
3. Find a finite family of $L$-structures $\mathcal{F}$ such that $\mathcal{C}^-$ is precisely $\text{Forb}(\mathcal{F})$, that is, the class of all finite $L$-structures $\mathbf{B}$ such that there is no $\mathbf{F} \in \mathcal{F}$ with a homomorphism to $\mathbf{B}$.
4. Prove that in fact for every $\mathbf{A}^- \in \mathcal{C}^-$ there is $\mathbf{A} \in \mathcal{C}$ which is its automorphism-preserving completion, that is, $\mathbf{A}^-$ can be obtained from $\mathbf{A}$ as in point 2 and furthermore $\mathbf{A}^-$ and $\mathbf{A}$ have the same automorphisms.
5. Use the Herwig–Lascar theorem [HL00] omitting homomorphisms from $\mathcal{F}$ to get EPPA-witnesses in $\text{Forb}(\mathcal{F})$.
6. Take the automorphism-preserving completion of the witnesses to get EPPA-witnesses in $\mathcal{C}$ and thus prove EPPA for $\mathcal{C}$.

This strategy was applied, for example, in [Sol05, Con19, ABWH +17, Kon18, HKN18b]. See also [HN16] where the notion of completions was introduced.

As we have seen in Section 6, $\mathcal{T}$ does not admit automorphism-preserving completions (because APA is a weaker property). One can also prove (and it will appear elsewhere), using the negative result of Proposition 7.1 and Theorem 2.1 of [HN16], that $\mathcal{T}$ cannot be described by finitely many forbidden homomorphisms (hence in particular there is no finite family $\mathcal{F}$ satisfying point 3 above). This we believe is the first time the correspondence of EPPA and Ramsey techniques has been used to prove a negative result.
Remark 8.2. $\mathcal{T}$ is one of the five reducts of the random graph [Tho91]. Besides $\mathcal{T}$, the random graph itself and the countable set with no structure, the remaining two corresponding automorphism groups can be obtained by adding an isomorphism between the random graph and its complement and an isomorphism between the generic two-graph and its complement respectively.

By a similar argument, based on the Kechris–Pestov–Todorčević correspondence, one can prove that the “best” Ramsey expansion (that is, with the expansion property) of these structures is still the ordered random graph. On the other hand, EPPA for these two classes is an open problem and we would very much like to see it resolved.

Remark 8.3. Theorem 1.2 implies the following. For every graph $G$ there exists an EPPA-witness $H$ with the property that the two-graph associated to $H$ is an EPPA-witness for the two-graph associated for $G$, in other words, it implies that the class of all graphs and two-graphs respectively are a non-trivial positive example for the following question.

Question 8.1. For which pairs of classes $C, C^-$ such that $C^-$ is a reduct of $C$ it holds that for every $A \in C$ there is $B \in C$ such that $B$ is an EPPA-witness for $A$ (in $C$) and furthermore if $A^-$ and $B^-$ are the corresponding reducts in $C^-$ then $B^-$ is an EPPA-witness for $A^-$ (in $C^-$)?

Remark 8.4. As was already mentioned, ample generics are usually proved by showing the combination of EPPA and APA. Siniora in his thesis [Sin17] asked if two-graphs have ample generics. This question still remains open, although we conjecture that it is not the case (ample generics are equivalent to the so-called weak amalgamation property for partial automorphisms and it seems that the reasons for two-graphs not having APA are fundamental enough to also hold in the weak amalgamation context).

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