Topological Vacuum of the Closed Universe as a Gauging Factor

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Abstract

The only known general base to eliminate the vacuum divergencies of quantized matter fields in quantum geometrodynamics is the fermion-boson supersymmetry. The topological effect of the closed Universe – discretization of the vacuum fluctuations spectra – allows to formulate the conditions of cancellation of the divergencies. In the center of attention of this work is the fact that these conditions result in the considerable restrictions on the gauge and factor-ordering ambiguities peculiar to the equations of the theory and, in the limits of the isotropic model of the Universe, remove these ambiguities completely.

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1 Introduction

Quantization of physical fields in a gravitational field of the Universe encounters the standard problem of vacuum divergencies. As, however, in quantum geometrodynamics (QGD) the standard perturbation theory with its renormalization procedure is inapplicable, the only known base for solution of the problem is supersymmetry [1] (its urgency for QGD at present is a commonplace). The vacuum contributions of fermion and boson fields are opposite on a sign, and the topological effect of a closed universe – discretization of the vacuum fluctuations spectra – allows to formulate conditions of cancellation of the vacuum divergencies. The most remarkable thing is that these conditions sharply limit gauge arbitrariness in QGD, in particular, the homogeneous isotropic model turns out completely deprived of this arbitrariness by them. Moreover, if, taking into account the known ambiguity in ordering and parametrization [2, 3] of the Wheeler – DeWitt (WDW) theory [4], one imposes on it the mentioned conditions, the Hamiltonian of this theory will gain a nontrivial spectrum of eigenvalues, in essence conterminous with an energy spectrum of matter including its vacuum component. This radically changes the structure of the theory, approximating it to the dynamic one proposed in the work [5]. Below, for distinguishing the two versions of geometrodynamics, we understand by QGD the dynamic theory [5], and the WDW structure, in which there is no time as a formal parameter of dynamics, we shall name, as it is accepted now, the quantum cosmology (QC WDW).

In the following section the statement of the problem in the simplest, semiclassical, version of a closed homogeneous isotropic universe is adduced. In Sec. 3. we shall formulate the problem of topological vacuum spectrum in the closed isotropic Universe, give the outcomes of its solution and show that the effect of topological vacuum (TV) can not be coordinated with the semiclassical theory. Sections 4. and 5. are devoted to solution of the problem within the framework of quantum theory: QC WDW (Sec. 4.) and QGD (Sec. 5.).

2 Semiclassical aspect

In the self-consistent semiclassical theory the \{0\}_0-Einstein equation of a closed homogeneous isotropic universe at the presence of matter has the form:

\[
\frac{3}{\kappa} \left( \frac{a^2}{N^2 a^2} + \frac{1}{a^2} \right) = \frac{\mathcal{N}}{2\pi^2 a^5} \langle H_{\text{mat}} \rangle, \tag{1}
\]

where \(\kappa\) is the gravitational constant, \(\mathcal{N} = \sqrt{\det g}\) is the “laps function”, \(a\) is the scale factor, \(\langle H_{\text{mat}} \rangle\) is the quantum average Hamiltonian of material fields, as which we shall consider massless conformal two-component objects: neutral vector, spinor and charged scalar fields:

\[
H^{(s)} = \sum_k H^{(s)}_k,
\]

\(s\) being the spin quantum number,

\[
k = (\lambda, j, m), \quad \lambda = 1 + s, 2 + s, \ldots, \infty, \quad j = s, s + 1, \ldots, \lambda - 1, \quad m = -j, \ldots, +j.
\]

The Hamiltonian of boson fields can be presented as

\[
H^{(0)}_k, H^{(1)}_k = \lambda_k \left( a_k^\dagger a_k + a_k^\dagger a_k + 1 \right),
\]
where the two annihilation operators $a_k$, $\bar{a}_k$ concern two different components of a field (accordingly – the creation operators $a_k^\dagger$, $\bar{a}_k^\dagger$).

The Hamiltonian of fermions

$$H^{(1/2)}_k = \lambda_k \left( b_k^\dagger b_k + \bar{b}_k^\dagger \bar{b}_k - 1 \right).$$

In a state with definite energy

$$\langle H^{(s)} \rangle = \mathcal{E}^{(s)} = \sum_k \mathcal{E}_k^{(s)}, \quad \mathcal{E}_k^{(s)} = \lambda_k^{(s)} \left( n_{k1}^{(s)} + n_{k2}^{(s)} \pm 1 \right),$$

(2)

where “±”, correspondingly, for bosons and fermions, $n_k$ are the occupation numbers.

According to (2), the quantity $\mathcal{E}^{(s)}$ consists of two physically different parts,

$$\mathcal{E}^{(s)} = \mathcal{E}_\text{part}^{(s)} + \mathcal{E}_\text{vac}^{(s)},$$

$$\mathcal{E}_\text{part}^{(s)} = \sum_k \lambda_k^{(s)} \left( n_{k1}^{(s)} + n_{k2}^{(s)} \right)$$

is the particle energy,

$$\mathcal{E}_\text{vac}^{(s)} = \pm \sum_k \lambda_k^{(s)} = \pm \sum_{\lambda=1+s}^{\infty} \sum_{j=s}^{\lambda-1} \sum_{m=-j}^{j} \lambda = \pm \sum_{\lambda=1+s}^{\infty} \lambda \left( \lambda^2 - s^2 \right)$$

(3)

is the vacuum component. The equation (4) takes the form:

$$\frac{3}{\kappa} \left( \dot{a}^2 + \frac{1}{a^2} \right) = \frac{\mathcal{N}}{2\pi^2 a^5} \mathcal{E},$$

$$\mathcal{E} = \sum_s \left( \mathcal{E}_\text{part}^{(s)} + \mathcal{E}_\text{vac}^{(s)} \right).$$

In particular, in the gauge of “conformal time”

$$\mathcal{N} = a$$

(4)

we have

$$\frac{6\pi^2}{\kappa} \left( \dot{a}^2 + a^2 \right) = \mathcal{E}.$$  

(5)

As was mentioned in Introduction, the diverging vacuum contributions (3) of various fields can not be renormalized by the standard quantum field methods. However opposition of signs of these contributions from bosons and fermions specifies that the cancellation of divergencies is possible in the framework of the boson-fermion supersymmetry [1].
3 Topological vacuum. Inconsistency of semiclassical approximation

Let us consider an elementary supermultiplet of the two-component boson field and the two-component fermion one. According to (3), the vacuum energy of $N$ scalar-spinor supermultiplets is determined by the

$$E_{\text{vac}} = \sum_{a=1}^{2N} \sum_{\lambda=1}^{\infty} \lambda^3 - \sum_{a=1}^{2N} \sum_{\lambda=3/2}^{\infty} \lambda \left( \lambda^2 - \frac{1}{4} \right).$$

(6)

To reduce the sums to common limits let us make in the spinor sum the replacement $\lambda \to \lambda + 1/2$, and then in both sums $\lambda^3$ put shifts $\lambda = k + \alpha_a$ with $\alpha_a$ integers:

$$\sum_{a} \sum_{\lambda=1}^{\infty} \lambda^3 = \sum_{a} \sum_{k=1}^{\infty} (k + \alpha_a)^3.$$

The shifts $\alpha_a$ should be chosen so that the divergencies were mutually compensated, and then the residual $E_{\text{vac}}$ will develop out of the sums $\sum_{k=1}^{0} - \alpha_a$ if $\alpha_a > 0$, and $- \sum_{k=1}^{\infty} - \alpha_a$ if $\alpha_a < 0$. The degrees of the divergencies are determined by the formulas:

$$\sum_{k=1}^{K} k^3 = \frac{1}{4}K^2(K+1)^2;$$

$$\sum_{k=1}^{K} k^2 = \frac{1}{6}K(K+1)(2K+1);$$

$$\sum_{k=1}^{K} k = \frac{1}{2}K(K+1).$$

So,

$$\frac{1}{6}AK(K+1)(2K+1) + \frac{1}{2}BK(K+1) + CK = 0;$$

$$A = 3 \sum_{a=1}^{N} (\alpha_a - \beta_a) - 3\frac{N}{2},$$

$$B = 3 \sum_{a=1}^{N} (\alpha_a^2 - \beta_a^2) - \frac{N}{2},$$

$$C = \sum_{a=1}^{N} (\alpha_a^3 - \beta_a^3),$$

$\alpha_a$ are the shifts in the scalar sums, $\beta_a$ are those in the spinor sums.
The condition of divergency cancellation is $A = B = C = 0$, that means the set of equations:

$$
\sum_{a=1}^{N} (\alpha_a - \beta_a) = \frac{N}{2},
$$

$$
\sum_{a=1}^{N} (\alpha_a^2 - \beta_a^2) = \frac{N}{6},
$$

$$
\sum_{a=1}^{N} (\alpha_a^3 - \beta_a^3) = 0.
$$

The second version of divergency cancellation appears after the replacement $\lambda \rightarrow \lambda - 1/2$ in the spinor sum (6). In this case the set of equations for $\alpha_a, \beta_a$ takes the form

$$
\sum_{a=1}^{N} (\alpha_a - \beta_a) = -\frac{N}{2},
$$

$$
\sum_{a=1}^{N} (\alpha_a^2 - \beta_a^2) = -\frac{N}{2},
$$

$$
\sum_{a=1}^{N} (\alpha_a^3 - \beta_a^3) = 0.
$$

For $N$ vector-spinor supermultiplets there are 4 different versions in correspondence with the two possible appropriate replacements of $\lambda$ in each of the two sums – boson and fermion:

1.

$$
\sum_{a=1}^{N} (\alpha_a - \beta_a) = -\frac{N}{2},
$$

$$
\sum_{a=1}^{N} (\alpha_a^2 - \beta_a^2) = -\frac{N}{2},
$$

$$
\sum_{a=1}^{N} (\alpha_a^3 - \beta_a^3) = 0.
$$

2.

$$
\sum_{a=1}^{N} (\alpha_a - \beta_a) = -3\frac{N}{2},
$$

5
\[
\sum_{a=1}^{N} \left( \alpha_a^2 - \beta_a^2 \right) = -\frac{N}{2},
\]
\[
\sum_{a=1}^{N} \left( \alpha_a^3 - \beta_a^3 \right) = 0.
\]

3.

\[
\sum_{a=1}^{N} (\alpha_a - \beta_a) = -\frac{N}{2},
\]
\[
\sum_{a=1}^{N} \left( \alpha_a^2 - \beta_a^2 \right) = \frac{N}{2},
\]
\[
\sum_{a=1}^{N} \left( \alpha_a^3 - \beta_a^3 \right) = 0.
\]

4.

\[
\sum_{a=1}^{N} (\alpha_a - \beta_a) = \frac{N}{2},
\]
\[
\sum_{a=1}^{N} \left( \alpha_a^2 - \beta_a^2 \right) = \frac{N}{2},
\]
\[
\sum_{a=1}^{N} \left( \alpha_a^3 - \beta_a^3 \right) = 0.
\]

These sets of equations are solvable at \( N \) multiple 12, and the vector-spinor sets 1. and 2. are solvable at \( N \) multiple 8 as well. The vacuum spectrum \( \mathcal{E}_{\text{vac}} \) obtained as a result of solution of these sets of equations for various \( N \) has proved to be equidistant, extending from \( -\infty \) up to \( +\infty \) with the same pitch equal 6:

\[
\mathcal{E}_{\text{vac}} = 6n + v, \ n = 0, \pm 1, \pm 2, \ldots, \tag{7}
\]

and \( v \) can take values 0, \( \pm 2 \) depending on a set of supermultiples.

Coming back to the semiclassical equation (5), we must note the following. The \( \mathcal{E}_{\text{part}} \) being a function of the occupation numbers, the \( \mathcal{E}_{\text{vac}} \) is an independent variable, and the negative half of its rigid discrete spectrum cannot be coordinated with the positive definite left part of the equation. So, the semiclassical approach to the physics of the early Universe is incomplete: the presence of the TV demands appealing to the full quantum theory.
4 The Wheeler – DeWitt quantum cosmology.

The WDW equation

\[ H \Psi = 0 \]  \hspace{1cm} (8)

can be arranged in such a way that the operator \( H \) divides into tree parts,

\[ H = H_{\text{grav}} + H_{\text{part}} + \mathcal{E}_{\text{vac}}, \]

where \( H_{\text{grav}} \) is its gravitational component, \( H_{\text{part}} = \sum_i \sum_k H_{i}^{k} \), \( H_{i}^{k} = \lambda_{k} a_{i}^{k} a_{i}^{k} \), the index \( i \) numbers components of material fields, \( a_{i}^{k} \) are the annihilation operators of the appropriate particles. The solution of the equation (8) can be factored: \( \Psi = \Psi_{\text{part}} \cdot \Psi_{\text{grav}} \), so that

\[ H_{\text{part}} \Psi_{\text{part}} = \mathcal{E}_{\text{part}} \Psi_{\text{part}}, \]  \hspace{1cm} (9)

\[ H_{\text{grav}} \Psi_{\text{grav}} = (\mathcal{E}_{\text{grav}} - \mathcal{E}_{\text{part}} - \mathcal{E}_{\text{vac}}) \Psi_{\text{grav}} . \]  \hspace{1cm} (10)

Thus, an eigenvalue \( \mathcal{E}_{\text{part}} \) of the operator \( H_{\text{part}} \) has to be considered as a parameter of the equation (10). As to the spectrum \( \mathcal{E}_{\text{vac}} \), it should be obtained as a result of solution of the WDW equation responsible for a state of the Universe as a whole. Fortunately, parametrization/gauge and permutation ambiguities of this equation allow to come near the solution of the problem.

At first, the oscillator aspect of the left part of the equation (5) and the spectrum nature of the \( \mathcal{E}_{\text{vac}} \) specify the definite advantage of the gauge (4). If, in addition, one puts \( \mathcal{E}_{\text{part}} = 0 \) and chooses a system of units in which \( 6\pi^{2}/\kappa = 1/2 \), the appropriate WDW equation can be presented as:

\[ -\frac{1}{2} \left( \frac{\partial^{2}}{\partial a^{2}} - a^{2} \right) \Psi_{\text{vac}} = E_{\text{vac}} \Psi_{\text{vac}}, \hspace{0.5cm} a \geq 0, \]

\[ E_{\text{vac}} = 2m + 3/2, \hspace{0.5cm} m = 0, 1, 2, . . . . \]

Comparing the spectrum of eigenvalues of this equation with (7), we should, first of all, remove all the values of the \( m \) that are not multiple 3. Existing \( \mathcal{E}_{\text{part}} \neq 0 \) requires presence at the left part of the equation of an element that could absorb this component. And here comes to the aid the second ambiguity of the quantum theory in general and of the WDW equation in particular, known as the ordering problem. Here it manifests itself in that in the general case the kinetic part of the WDW Hamiltonian can be presented as

\[ \frac{1}{2} f^{-1}(a) \frac{\partial}{\partial a} f^{2}(a) \frac{\partial}{\partial a} f^{-1}(a), \]

where \( f(a) \) is an arbitrary differentiable function of the scale factor. Let us put \( f = a^{l+1} \). Then the WDW equation takes the form of the equation for a “centrifugal oscillator”

\[ \left[ \frac{1}{2} \left( \frac{\partial^{2}}{\partial a^{2}} - a^{2} - \frac{l(l+1)}{a^{2}} \right) + \mathcal{E} \right] \Psi_{\mathcal{E}} = 0 \]
with the spectrum (comp. [3])

\[ E = 2m + l + \frac{3}{2}, \quad l = \varepsilon_{\text{part}}. \]  \hspace{1cm} (11)

What do we have in the result? Presence of the TV with its distinctive spectrum [4]

1. fixes the gauge and solves the ordering problem;

2. transforms QC WDW to the theory with a nontrivial spectrum.

Thus, the TV has properties intrinsic to the gravitational vacuum condensate (GVC) which existence the consecutive construction of QGD on the basis of the usual quantum theory [5] inevitably leads to.

On the other hand, the WDW equation cuts the vacuum spectrum removing, in particular, the negative values \( n \), but does not contain any absorption mechanism for this part of the energy spectrum of the subsystem “topological vacuum”. This is a superfluous testimony of the stated in [5] incompleteness of the theory.

The other situation is in QGD.

5 Quantum geometrodynamics

In the dynamic Schrödinger equation for the physical part of a wave function in the gauge [4] the Hamiltonian has the form:

\[ H_{\text{ph}} = \frac{1}{8} \frac{1}{M(a)} \frac{\partial}{\partial a} M(a) \frac{\partial}{\partial a} - 2a^2, \]  \hspace{1cm} (12)

\( M \) being a probability measure. Let us put

\[ M = a^{2(l+1)}, \quad l \geq 0. \]  \hspace{1cm} (13)

The probability measure, being as well a measure of a path integral (PI), is generated by the certain dependence on \( a \) of a pitch \( \epsilon \) of splitting up the time interval at a final PI definition through a multiple integral, as the measure is connected with \( \epsilon \) by the relation [5], which in this case has the form

\[ \gamma^{-\frac{1}{2}} M = \text{const}, \quad \gamma = a^{4(l+1)}, \]

where \( \gamma \) is defined by

\[ \epsilon = \chi \gamma, \quad \chi \to 0. \]

By substituting the measure (13) into the Hamiltonian (12), we come to the equation for stationary states

\[ \left[ \frac{1}{8} \frac{\partial^2}{\partial a^2} + \frac{1}{4a} \frac{\partial}{\partial a} - 2a^2 - \frac{l(l+1)}{8a^2} + E \right] \Psi = E \Psi, \]
$E$ being a parameter of GVC. The eigenvalue spectrum of the equation is defined by the same expression
\begin{equation}
E = \mathcal{E} - E = \mathcal{E}_{\text{part}} + \mathcal{E}_{\text{vac}} - E = 2n + l + \frac{3}{2}, \quad n = 0, 1, 2, \ldots; \quad l = \mathcal{E}_{\text{part}},
\end{equation}
but this time there is the GVC parameter $E$ allowing to put it in agreement with the TV: presenting (10) in the form $\mathcal{E}_{\text{vac}} = 6(n - m + v), \quad n, m = 1, 2, \ldots$ we have
\begin{equation}
E = 4n - 6m + 6v - \frac{3}{2}.
\end{equation}
Thus, GVC and TV represent a united subsystem, breaking gauge symmetry concerning the diffeomorphism group.

6 Conclusion.

The above-stated analysis shows that presence of material fields with physical topological vacuum breaks the basic equation of the WDW theory – the quantum version of the classical Hamilton constraint, – revealing GVC. This confirms our conclusion made in the work that the quantum principle in cosmology is incompatible to the principle of gauge invariance concerning the diffeomorphism group and that QC WDW has not any general-theoretical foundation.

Here we have considered the elementary cosmological model, which, however, most likely can serve a good “zero approximation” to the exact model of the early Universe. It definitely allows to expect that the obtained results, though will not be so hard for the real model, nevertheless, in essential will be applicable to it. Or else, it is possible to state, that TV considerably limits the gauge/parametrization arbitrariness and the ordering ambiguity (choice of a probability measure) in QGD.

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