ON THE DYNKIN INDEX OF A PRINCIPAL $\mathfrak{sl}_2$-SUBALGEBRA

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INTRODUCTION

The ground field $k$ is algebraically closed and of characteristic zero. Let $\mathfrak{g}$ be a simple Lie algebra over $k$. The goal of this note is to prove a closed formula for the Dynkin index of a principal $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$, see Theorem 3.2. The key step in the proof uses the “strange formula” of Freudenthal–de Vries. As an application, we (1) compute the Dynkin index any simple $\mathfrak{g}$-module regarded as $\mathfrak{sl}_2$-module and (2) obtain an identity connecting the exponents of $\mathfrak{g}$ and the dual Coxeter numbers of both $\mathfrak{g}$ and $\mathfrak{g}^\vee$, see Section 4.

1. THE DYNKIN INDEX OF REPRESENTATIONS AND SUBALGEBRAS

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra of rank $n$. Let $\mathfrak{t}$ be a Cartan subalgebra, and $\Delta$ the set of roots of $\mathfrak{t}$ in $\mathfrak{g}$. Choose a set of positive roots $\Delta^+$ in $\Delta$. Let $\Pi$ be the set of simple roots and $\theta$ the highest root in $\Delta^+$. As usual, $\rho = \frac{1}{2} \sum_{\gamma > 0} \gamma$. The $\mathbb{Q}$-span of all roots is a ($\mathbb{Q}$-)subspace of $\mathfrak{t}^*$, denoted $E$. Choose a non-degenerate invariant symmetric bilinear form $(\ ,\ )_\mathfrak{g}$ on $\mathfrak{g}$ as follows. The restriction of $(\ ,\ )_\mathfrak{g}$ to $\mathfrak{t}$ is non-degenerate, hence it induces the isomorphism of $\mathfrak{t}$ and $\mathfrak{t}^*$ and a non-degenerate bilinear form on $\mathfrak{t}^*$. We require that $(\theta, \theta)_\mathfrak{g} = 2$, i.e., $(\beta, \beta)_\mathfrak{g} = 2$ of any long root $\beta$ in $\Delta$.

Definition 1 (E.B. Dynkin).

1. Let $s$ be a simple subalgebra of $\mathfrak{g}$. The Dynkin index of $s$ in $\mathfrak{g}$ is defined by

$$\text{ind}(s \hookrightarrow \mathfrak{g}) = \frac{(x,x)_\mathfrak{g}}{(x,x)_s}, \quad x \in s.$$ 

2. If $\nu : \mathfrak{g} \rightarrow \mathfrak{sl}(V)$ is a representation of $\mathfrak{g}$, then the Dynkin index of the representation, denoted $\text{ind}_D(\mathfrak{g}, V)$ or $\text{ind}_D(\mathfrak{g}, \nu)$, is defined by

$$\text{ind}_D(\mathfrak{g}, V) = \text{ind}(\mathfrak{g} \hookrightarrow \mathfrak{sl}(V)).$$

It is not hard to verify that, for the simple Lie algebra $\mathfrak{sl}(V)$, the normalised bilinear form is given by $(x,x)_{\mathfrak{sl}(V)} = \text{tr}(x^2)$, $x \in \mathfrak{sl}(V)$. Therefore, a more explicit expression for the Dynkin index of a representation $\nu : \mathfrak{g} \rightarrow \mathfrak{sl}(V)$ is

$$\text{ind}_D(\mathfrak{g}, V) = \frac{\text{tr}(\nu(x)^2)}{(x,x)_\mathfrak{g}}.$$ 

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Conversely, the index of a simple subalgebra can be expressed via indices of representations. Namely,

\[(1.2) \quad \text{ind}(s \hookrightarrow g) = \frac{\text{ind}_D(s, g)}{\text{ind}_D(g, \text{ad}_g)}.\]

The denominator in the right hand side represents the index of the adjoint representation of \(g\), and the numerator represents the index of the \(s\)-module \(g\).

The following properties easily follow from the definition:

**Multiplicativity:** If \(h \subset s \subset g\) are simple Lie algebras, then
\[
\text{ind}(h \subset s \subset g) = \text{ind}(h \subset s) \cdot \text{ind}(s \subset g).
\]

**Additivity:**
\[
\text{ind}_D(g, V_1 \oplus V_2) = \text{ind}_D(g, V_1) + \text{ind}_D(g, V_2).
\]

It is therefore sufficient to determine the indices for the irreducible representations.

**Theorem 1.1** (Dynkin, [2, Theorem 2.5]). Let \(V_\lambda\) be a simple finite-dimensional \(g\)-module with highest weight \(\lambda\). Then
\[
\text{ind}_D(g, V_\lambda) = \frac{\dim V_\lambda}{\dim g}(\lambda, \lambda + 2\rho)_g.
\]

Although it is not obvious from the definition, the Dynkin index of a representation is an integer. This was proved by E.B. Dynkin [2, Theorem 2.2] using lengthy classification results. Later, he gave a better proof that is based on a topological interpretation of the index. A short algebraic proof is given in [5, Ch. I, §3.10].

**Example 1.2.**

1) Let \(R_d\) be the simple \(\mathfrak{sl}_2\)-module of dimension \(d + 1\). Then \(\text{ind}_D(\mathfrak{sl}_2, R_d) = \binom{d+2}{3}\).

2) Recall that \(\theta\) is the highest root in \(\Delta^+\). By Theorem 1.1,
\[
\text{ind}_D(g, \text{ad}) = (\theta, \theta + 2\rho)_g = (\theta, \theta)_g(1 + (\rho, \theta^\vee)_g) = 2(1 + (\rho, \theta^\vee)_g).
\]

Note that the value \((\rho, \theta^\vee)_g\) does not depend on the normalisation of the bilinear form. The integer \(1 + (\rho, \theta^\vee)_g\) is customarily called the dual Coxeter number of \(g\), and we denote it by \(h^*(g)\). Thus, \(\text{ind}_D(g, \text{ad}) = 2h^*(g)\). In the simply-laced case, \(h^*(g) = h(g)\)—the usual Coxeter number. For the other simple Lie algebras, we have \(h^*(B_n) = 2n-1, h^*(C_n) = n+1, h^*(F_4) = 9, h^*(G_2) = 4\).

Andreev, Vinberg, and Elashvili applied the Dynkin index of representations to some invariant-theoretic problem [1]. To this end, they adjusted the index so that it does not depend on the choice of a bilinear form on \(g\).

**Definition 2** (Andreev–Vinberg–Elashvili, 1967). Let \(\nu : g \to \mathfrak{sl}(V)\) be a finite-dimensional representation of a simple Lie algebra. Then
\[
\text{ind}_{AVE}(g, V) := \frac{\text{ind}_D(g, V)}{\text{ind}_D(g, \text{ad})} = \frac{\text{tr}(\nu(x)^2)}{\text{tr}(\text{ad}_g(x)^2)}, \quad x \in g.
\]
It follows that \( \text{ind}_{AVE}(g, ad_g) = 1 \) and

\[
\text{ind}_{AVE}(g, V_\lambda) = \frac{\dim V_\lambda}{\dim g} \cdot (\lambda, \lambda + 2\rho)_g.
\]

2. THE “STRANGE FORMULA”

Let \( \mathcal{K} \) be the Killing form on \( g \), i.e., \( \mathcal{K}(x, x) = \text{tr}(ad_g(x)^2), x \in g \). The induced bilinear form on \( t^* \) (and \( E \)) is denoted by \( \langle \cdot, \cdot \rangle \). It is the so-called canonical bilinear form on \( E \). The canonical bilinear form is characterised by the following property:

\[
\langle v, v \rangle = \sum_{\gamma \in \Delta} \langle v, \gamma \rangle \langle v, \gamma \rangle = 2 \sum_{\gamma > 0} \langle v, \gamma \rangle \langle v, \gamma \rangle \text{ for any } v \in E.
\]

The “strange formula” of Freudenthal–de Vries (see [3, 47.11]) is

\[
\langle \rho, \rho \rangle = \frac{\dim g}{24}.
\]

Using our normalisation of \( (\cdot, \cdot)_g \), the “strange formula” reads

\[
\langle \rho, \rho \rangle = \frac{\dim g}{12} h^*(g).
\]

Indeed, it is well known that \( \langle \theta, \theta \rangle = 1/h^*(g) \) (see e.g. [6, Lemma 1.1]). Therefore, the transition factor between two forms \( \langle \cdot, \cdot \rangle \) and \( (\cdot, \cdot)_g \) (considered as forms on \( E \)) equals \( 2h^*(g) \). Using the transition factor, we can also rewrite Eq. (2.1) in terms of \( (\cdot, \cdot)_g \):

\[
\langle \rho, \rho \rangle = \sum_{\gamma > 0} \langle v, \gamma \rangle \langle v, \gamma \rangle_g.
\]

3. THE INDEX OF A PRINCIPAL \( sl_2 \)-SUBALGEBRA

If \( e \in g \) is nilpotent, then there exists a subalgebra \( a \subset g \) such that \( a \cong sl_2 \) and \( e \in a \) (Morozov, Jacobson). If \( e \) is a principal nilpotent element, then the corresponding \( sl_2 \)-subalgebra is also called principal. (See [2, § 9] and [4, Sect. 5] for properties of principal \( sl_2 \)-subalgebras.) Let \( (sl_2)^{pr} \) be a principal \( sl_2 \)-subalgebra of \( g \). In this section, we obtain a uniform expression for \( \text{ind}((sl_2)^{pr} \hookrightarrow g) \).

Recall that \( \Delta \) has at most two root lengths. Let \( \theta_s \) denote the short dominant root in \( \Delta^+ \). (Hence \( \theta = \theta_s \) if and only if \( \Delta \) is simply-laced.) Set \( r = \|\theta\|^2/\|\theta_s\|^2 \in \{1, 2, 3\} \). Along with \( g \), we also consider the Langlands dual algebra \( g^\vee \), which is determined by the dual root system \( \Delta^\vee \). Since the Weyl groups of \( g \) and \( g^\vee \) are isomorphic, we have \( h(g) = h(g^\vee) \). However, the dual Coxeter numbers can be different (cf. \( B_n \) and \( C_n \)).

The half-sum of positive roots for \( g^\vee \) is

\[
\rho^\vee := \frac{1}{2} \sum_{\gamma > 0} \gamma^\vee = \sum_{\gamma > 0} \frac{\gamma}{(\gamma, \gamma)_g}.
\]
It is well-known (and easily verified) that \((\rho^\vee, \gamma)_\theta = \text{ht} (\gamma)\) for any \(\gamma \in \Delta^+\). (This equality does not depend on the normalisation of a bilinear form.) It follows that \(h^*(g^\vee) = (\rho^\vee, \theta_\ast) = \text{ht}(\theta_\ast)\).

**Proposition 3.1.** For a simple Lie algebra \(g\) with the corresponding root system \(\Delta\), we have

\[
\sum_{\gamma > 0} \text{ht}^2(\gamma) = \frac{\dim g}{12} h^*(g) h^*(g^\vee) r.
\]

**Proof.** The equality in (3.1) is essentially equivalent to the "strange formula".

Applying Eq. (2.3) to \(v = \rho^\vee\), we obtain

\[
h^*(g)(\rho^\vee, \rho^\vee)_\theta = \sum_{\gamma > 0} (\rho^\vee, \gamma)_\theta (\rho^\vee, \gamma)_\theta = \sum_{\gamma > 0} \text{ht}^2(\gamma).
\]

For \(g^\vee\), the strange formula says that \((\rho^\vee, \rho^\vee)_\theta = \frac{\dim g}{12} h^*(g^\vee)\). Although the normalised bilinear forms \((, )_\theta\) and \((, )_{\theta^\vee}\) are proportional upon restriction to \(E\), they are not equal in general. Indeed, the square of the length of a long root in \(\Delta^\vee\) with respect to \((, )_\theta\) equals \(2r\). Hence the transition factor is \(r\) and

\[
(\rho^\vee, \rho^\vee)_\theta = r (\rho^\vee, \rho^\vee)_\theta = \frac{\dim g}{12} h^*(g^\vee) r.
\]

Then the assertion follows from (3.2) and (3.3). \(\square\)

**Theorem 3.2.** \(\text{ind}((\mathfrak{sl}_2)^{pr} \hookrightarrow g) = \frac{\dim g}{6} h^*(g^\vee) r\).

**Proof.** Combining Eq. (1.2), Example 1.2(2), and Definition 2 yields the following formula for the index of a simple subalgebra \(s\) in \(g\):

\[
\text{ind}(s \hookrightarrow g) = \frac{h^*(s)}{h^*(g)} \cdot \text{ind}_{AVE}(s, g).
\]

We use this formula with \(s = (\mathfrak{sl}_2)^{pr}\). Let \(h\) be the semisimple element of a principal \(\mathfrak{sl}_2\)-triple. Without loss of generality, we may assume that \(h\) is dominant. Then \(\alpha(h) = 2\) for any \(\alpha \in \Pi\). Put \(\tilde{h} = h/2\). Then \(\gamma(\tilde{h}) = \text{ht}(\gamma)\) for any \(\gamma \in \Delta\) and \(\text{ad} \tilde{h}\) has the eigenvalues \(-1, 0, 1\) in \((\mathfrak{sl}_2)^{pr}\). Hence

\[
\text{ind}_{AVE}((\mathfrak{sl}_2)^{pr}, g) = \frac{\text{tr} (\text{ad}_g \tilde{h})^2}{\text{tr} (\text{ad}_g h)^2} = \frac{\sum_{\gamma \in \Delta} \text{ht}^2(\gamma)}{2} = \sum_{\gamma > 0} \text{ht}^2(\gamma).
\]

Since \(h^*(\mathfrak{sl}_2) = 2\), the theorem follows from Proposition 3.1 and Eq. (3.4). \(\square\)

Below, we tabulate the values of index for all simple Lie algebras.

| \(g\) | \(A_n\) | \(B_n\) | \(C_n\) | \(D_n\) | \(E_6\) | \(E_7\) | \(E_8\) | \(F_4\) | \(G_2\) |
|---|---|---|---|---|---|---|---|---|---|
| \(\text{ind}((\mathfrak{sl}_2)^{pr} \hookrightarrow g)\) | \(\frac{n+2}{3}\) | \(\frac{n(n+1)(2n+1)}{3}\) | \(\frac{(2n+1)}{3}\) | \(\frac{(n-1)n(2n-1)}{3}\) | 156 | 399 | 1240 | 156 | 28 |
Remark 3.3. For the exceptional Lie algebras, Dynkin computed the indices of all $\mathfrak{sl}_2$-subalgebras, see [2, Tables 16–20].

Note that the index of a principal $\mathfrak{sl}_2$ is preserved under the unfolding procedure $\mathfrak{g} \sim \tilde{\mathfrak{g}}$ applied to multiply laced Dynkin diagram. Namely, $\text{ind}((\mathfrak{sl}_2)^{pr} \hookrightarrow \mathfrak{g}) = \text{ind}((\mathfrak{sl}_2)^{pr} \hookrightarrow \tilde{\mathfrak{g}})$, where the four pairs $(\mathfrak{g}, \tilde{\mathfrak{g}})$ are: $(\mathfrak{C}_n, \mathfrak{A}_{2n-1}), (\mathfrak{B}_n, \mathfrak{D}_{n+1}), (\mathfrak{F}_4, \mathfrak{E}_6), (\mathfrak{G}_2, \mathfrak{D}_4)$. This is, of course, explained by the multiplicativity of the index of subalgebras and the fact that $\text{ind}(\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}) = 1$.

Remark 3.4. Proposition 3.1 provides a uniform expression for $\sum_{\gamma > 0} \text{ht}^2(\gamma)$. One might ask for a similar formula for $\sum_{\gamma > 0} \text{ht}(\gamma)$. However, such a formula seems to only exist in the simply-laced case. Indeed, for any $\mathfrak{g}$ we have $2(\rho, \rho)_{\mathfrak{g}} = \sum_{\gamma > 0} (\gamma, \rho^\vee)_{\mathfrak{g}} = \sum_{\gamma > 0} \text{ht}(\gamma)$. If $\Delta$ is simply-laced, then $\rho^\vee = 2\rho/(\theta, \theta)_{\mathfrak{g}} = \rho$, and using the “strange formula” one obtains

$$\sum_{\gamma > 0} \text{ht}(\gamma) = 2(\rho, \rho)_{\mathfrak{g}} = \frac{\dim \mathfrak{g}}{6} h(\mathfrak{g}).$$

Question. Consider the function $s \mapsto f(s) = \sum_{\gamma > 0} \text{ht}^s(\gamma)$. Are there some other values of $s$ such that $f(s)$ has a nice closed expression?

4. Some applications

(A) Let $\nu : \mathfrak{g} \to \mathfrak{sl}(V_\lambda)$ be an irreducible representation. Our first observation is that using Theorems 1.1 and 3.2 we can immediately compute the Dynkin index of $V_\lambda$ as $(\mathfrak{sl}_2)^{pr}$-module:

$$\text{ind}_D((\mathfrak{sl}_2)^{pr}, V_\lambda) = \text{ind}((\mathfrak{sl}_2)^{pr} \hookrightarrow \mathfrak{sl}(V_\lambda)) = \text{ind}((\mathfrak{sl}_2)^{pr} \hookrightarrow \mathfrak{g}) \cdot \text{ind}(\mathfrak{g} \hookrightarrow \mathfrak{sl}(V_\lambda)) =$$

$$\text{ind}((\mathfrak{sl}_2)^{pr} \hookrightarrow \mathfrak{g}) \cdot \text{ind}_D(\mathfrak{g}, V_\lambda) = \frac{\dim \mathfrak{g}}{6} h^*(\mathfrak{g}^\vee) r \cdot \frac{\dim V_\lambda}{\dim \mathfrak{g}} (\lambda, \lambda+2\rho)_{\mathfrak{g}} = \frac{\dim V_\lambda}{6} h^*(\mathfrak{g}^\vee) r \cdot (\lambda, \lambda+2\rho)_{\mathfrak{g}}.$$ 

Furthermore, we have

$$(4.1) \quad \text{ind}_D((\mathfrak{sl}_2)^{pr}, V_\lambda) = \text{ind}_D(\mathfrak{sl}_2, \mathfrak{ad}) \cdot \text{ind}_{AVE}((\mathfrak{sl}_2)^{pr}, V_\lambda) = 4 \cdot \text{ind}_{AVE}((\mathfrak{sl}_2)^{pr}, V_\lambda)$$

and

$$\text{ind}_{AVE}((\mathfrak{sl}_2)^{pr}, V_\lambda) = \frac{\text{tr} (\nu(\tilde{h})^2)}{\text{tr} ((\mathfrak{ad} \tilde{h})^2)} = \frac{\sum_{\mu \vdash V_\lambda} \mu(\tilde{h})^2}{2}.$$ 

where notation $\mu \vdash V_\lambda$ means that $\mu$ is a weight of $V_\lambda$, and the sum runs over all weights according to their multiplicities. Since $\mu(\tilde{h}) = (\mu, \rho^\vee)_{\mathfrak{g}}$, we finally obtain

$$(4.2) \quad \sum_{\mu \vdash V_\lambda} (\mu, \rho^\vee)_{\mathfrak{g}}^2 = \frac{\dim V_\lambda}{12} h^*(\mathfrak{g}^\vee) r \cdot (\lambda, \lambda+2\rho)_{\mathfrak{g}}.$$
This can be compared with the formula of Freudenthal–de Vries (see [3, 47.10.2]):

\[(4·2)\]

\[
\sum_{\mu \vdash V_\lambda} (\mu, \rho)^2 = \frac{\dim V_\lambda}{24} (\langle \lambda, \lambda + 2\rho \rangle).
\]

One can verify that Eq. (4·2) and (4·3) agree in the simply-laced case, where \(\rho\) is proportional to \(\rho^\vee\).

(B) Let \(m_1, \ldots, m_n\) be the exponents of \(\mathfrak{g}\). Regarding \(\mathfrak{g}\) as \((\mathfrak{sl}_2)^{pr}\)-module, one has \(\mathfrak{g} = \bigoplus_{i=1}^{n} \mathbb{R}^2 m_i\) [4, Cor. 8.7]. Then using Example 1.2(1), Eq. (3·4), (4·1), and the additivity of the index of representations, we obtain the identity

\[
\frac{\dim \mathfrak{g}}{6} h^*(\mathfrak{g}^{\vee}) = \text{ind}((\mathfrak{sl}_2)^{pr} \hookrightarrow \mathfrak{g}) = \frac{h^*(\mathfrak{sl}_2)}{h^*(\mathfrak{g})} \sum_{i=1}^{n} \text{ind}_{AVE}(\mathfrak{sl}_2, \mathbb{R}^2 m_i) = \frac{1}{2h^*(\mathfrak{g})} \sum_{i=1}^{n} \text{ind}_{D}(\mathfrak{sl}_2, \mathbb{R}^2 m_i) = \frac{1}{2h^*(\mathfrak{g})} \sum_{i=1}^{n} \left(2m_i + 2\right).
\]

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