Lie algebra and conservation laws for the time-fractional heat equation

Amlan K. Halder1*, C. T. Duba2 & P. G. L. Leach3,4

1Department of Mathematics, Pondicherry University, Puducherry, India, 2School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa, 3Institute for Systems Science, Durban University of Technology, Durban, South Africa, 4School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa

*Email: amlanhalder1@gmail.com

The Lie symmetry method is applied to derive the point symmetries for the N-dimensional fractional heat equation. We find that the numbers of symmetries and Lie brackets are reduced significantly as compared to the integer order for all dimensions. In fact for integer order linear heat equation the number of solution symmetries is equal to the product of the order and space dimension, whereas for the fractional case, it is half of the product on the order and space dimension. The Lie algebras for the integer and fractional order equations are mentioned using the subsequent computations of Lie brackets and by inspection. Interestingly, it is observed that for the one-dimensional fractional heat equation, the Lie algebra obtained by inspection of symmetries is similar to the result obtained by computation of Lie brackets, which is \( A_1 \oplus A_2 \oplus \infty A_1 \). The Lie algebra using the symmetries of the two-dimensional heat equation is observed to be \( (2A_1 \oplus \alpha x(2)) \oplus 2A_1 \oplus \infty A_1 \), whereas using the Lie brackets the algebra is deduced to be \( A_{3,6} \oplus \alpha x(2) \oplus A_1 \oplus \infty A_1 \). Hence, it can be concluded that the Lie algebra obtained from the nonzero Lie brackets can be conflated to the algebra which is obtained by inspection. Further, the subsequent Lie algebras are mentioned for the three and four-dimensional integer and fractional equations and the conservation laws are explicitly stated.

Keywords: lie symmetries; solutions; conservation laws; fractional partial differential equations

1. INTRODUCTION

Fractional differential equations (FDEs) are considered generic form of integer type differential equations. One of the significant applications of FDEs is the ingrained nonlocal property. A lot of real-life problems depends on the past state or at various location in a surrounding. Mathematically, it can be termed as a nonlocal property of the derivatives. FDEs play an important role in modelling such problems which could easily comprehend the problem in a locality. The application of fractional partial differential equations (FPDEs) can be detected in various physical and biological systems or phenomena such as viscoelasticity, signal-processing, fluid flow and electrochemistry.

Several methods have been developed to find solutions of FPDEs. Certain standard methods used in the literature involves the exponential function method, the fractional subequation method, the first integral method, the spectral method, the \( G'/G \) expansion method and the Lie symmetry method.

Lie symmetry analysis provides a way to deal with differential equations and systems thereof. The theory of Lie group analysis is used mainly to construct similarity reductions and invariant solutions and for obtaining conservation laws. Lie symmetries are also used to reduce the order of the differential equation as well as the number of independent variables. Lie symmetry analysis has been applied extensively to differential equations of integral order. Recent studies on Lie symmetry analysis now focus on fractional differential equations. Gazizov et al. (2007, 2009, 2015) used Lie symmetry analysis to study FDEs by deriving a prolongation formula for the Riemann–Liouville derivative, which enables one to determine their point symmetries. Others, for example Wang et al. (2015), conducted a study of Lie symmetries and conservation laws of time-fractional nonlinear dispersive equation. Lashkarian and Reza Hejazi (2017) showed that, by using Lie analysis, a FPDE can be reduced to an fractional ordinary differential equation (FODE). Yaşar et al. (2016) and Rui and Zhang (2016) have obtained conservation laws for time-FPDE. Djordjevic and Atanackovic (2008), Sahadevan and Bakkyaraj (2012) and Wang et al. (2013) reduced a FPDE to a nonlinear FODE using the theory of Sophus Lie. Further information on fractional systems is listed in Singla and Gupta (2017).

Certain methods which assists in the study of PDEs can also be used for its corresponding fractional part. We have cited certain papers which was a helping hand in understanding the underlying theories, mainly with respect to conservation laws. Wang et al. (2020) discussed the solutions of the (1 + 2)-dimensional sine-Gordon and sinh-Gordon equations using the (1 + 2)-dimensional zero curvature equation obtained from the AKNS system. In this paper, the authors cited the importance of the computation of solutions of nonlinear PDEs using the methods of Hirota bilinear form and the Lie symmetry analysis. Various solutions such as the kink type solutions, standard travelling-wave solutions were computed and finally the conservation laws were deduced using the multipliers approach. Wang (2021a) discussed a new form of the (1 + 3)-dimensional Schrödinger equation derived from the (1 + 3)-dimensional zero curvature equation by considering the positive cases of the compatibility conditions. Henceforth,
two systems of PDEs were obtained and the authors studied these two systems using the Lie symmetry analysis. Symmetries pertaining to different transformations are listed and certain reductions from the original equation to (1 + 1)- and (1 + 2)-dimensional Schrödinger equation are specified. The paper also presented some soliton and special solutions. The conserved densities are mentioned using the multiplier method.

Wang (2021b) discussed the generalisation of the work done in Wang et al. (2020). In this paper, a newer form of (1 + 3)-dimensional sine-Gordon and sinh-Gordon equation were derived using the (1 + 3)-dimensional zero curvature equation by studying its compatibility conditions for the negative order case. The authors discussed the solutions and the conservation laws using the multiplier’s approach. In Wang (2021c) the symmetry analysis of generalised KdV-Burgers-Kuramoto equation is studied and invariant solutions are explicitly mentioned. A particular series solution of the parent equation is derived. The authors have initiated the derivation of the reciprocal Bäcklund transformations of the subsequent conservation laws of the underlying equation. The Bäcklund transformations using the truncated Painlevé expansion method are also mentioned. The corresponding time fractional version of the generalised KdV-Burgers-Kuramoto equation were analysed using its symmetries and its underlying conservation law.

In Wang and Wazwaz (2022a) the modified Gardner-type equation and its corresponding time fractional form using the Fermi-Pasta-Ulam (FPU) model were derived. The symmetries, reductions and subsequent conservation laws were computed and the relation between the equations and the standard form of nonlinear Schrödinger equation is established using perturbation analysis. The author also computed the Bäcklund transformations of the corresponding conservation laws for the underlying parent equation. In Wang and Wazwaz (2022b) a new form of (1 + 3)-dimensional KdV and modified KdV equation and its corresponding fractional form are discussed. These equations were derived using the extended (1 + 3)-dimensional zero curvature equation. The authors derived the symmetries and conservation laws and showed the existence of one-soliton solution for the time fractional form of the two equations. Koval and Popovych (2022) introduced the concept of pseudo-discrete symmetry of the classical heat equation. This led the authors to strengthen the classification of the subalgebra of the equation.

We present the classification of symmetries of fractional order arising from a study of a time-fractional heat equation

\[ D_\alpha^t u = \nabla^2 u, \]

where \(0 < \alpha \leq 1\) and the Laplacian, \(\nabla\), may take any number of space dimensions. Moreover we investigate the Lie algebras (Patera and Winternitz, 1977) and derive conservation laws arising from the study of heat equation of fractional order. Gazizov et al. (2009) considered a one-dimensional nonlinear time-fractional diffusion equation and derived the Lie point symmetries (Jefferson and Carminati, 2014) and the Lie Brackets using both the Riemann–Liouville and the Caputo derivatives. Lukashchuk (2015) extended their work by constructing conservation laws for the time-fractional diffusion equation. We extend this work by considering the \(N\)-dimensional linear time-fractional diffusion equation. We classify the symmetries, obtain the Lie Brackets and derive conservation laws corresponding to each of the Lie point symmetries of the time-fractional diffusion equations.

A pattern is observed in the subsequent Lie algebras of the time-fractional equation of various dimensions. A similar coherence can also be easily observed in the computed conservation densities. The obtained results are mostly consistent with the work of Gazizov et al. (2009) and the computation of the Lie algebras for various dimension can be considered an enhancement to their work. Computation of the conservation laws also presents a different perception. The work done by Wang et al. (2020), Wang (2021a), Wang (2021b), Wang (2021c), Wang and Wazwaz (2022a), Wang and Wazwaz (2022b) on the computation of conservation laws is mainly based on the multiplier method. In the cited works the impact of the nonlinear terms in the obtained conserved densities can be easily observed. Hence, this becomes a stepping stone to our ongoing work of computing the conservation laws for the nonlinear fractional heat equation. Moreover, in Wang and Wazwaz (2022a) perturbation analysis is used to establish the connection of the fractional parent equation with some well-known equation. In our derived results, one of our hidden objective is to establish the correspondence of the parent equation to some well-known equation using the computation of Lie algebras. This part of the work is ongoing for some other equations as well and the method of perturbation analysis could assist in achieving a great milestone to our desired intuition.

The paper is organised in the following manner. In Section 2, we give the expression for the Riemann–Liouville fractional derivative and provide references wherein some definitions and properties of the Lie group method to analyse FPDEs are given. In Sections 3, 4 and 5, we present the group analysis of one- to four-dimensional integral and time-fractional heat equations. In Section 7, we present conservation laws of both fractional and integer-valued heat equations and in Section 8 we summarise the results and conclude.

2. PRELIMINARIES

The fractional derivative is given in several forms in the literature. Some of the significant definitions used in the study of the fractional derivatives are as follows: the Riemann–Liouville, the Grúwald–Letnikov, the Weyl, the Caputo, the Riesz and the Miller and Ross. In this work, we have used the definition of the Riemann–Liouville fractional derivative given by

\[ D_\alpha^t u(t, x) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{u(\tau, x)}{\tau^{m-\alpha-1}} d\tau, \]

\[ \alpha = m \in N; \]

\[ m - 1 < \alpha < m, m \in N. \]

Symmetry analysis of FPDEs has been summarised in Sahadevan and Bakkyaraj (2012), Wang et al. (2013), Bakkyaraj and Sahadevan (2015) and Yaşar et al. (2016) and many others. We explain briefly the method here. The reader can also refer to the articles mentioned above and references therein.

Let a fractional pde be of the form

\[ F(t, x, u, u_1, u_2, u_3, \ldots) = 0, \]

with \(0 < \alpha < 1\), \(t\) and \(x\) represent the spatial independent variables, \(u\) as the dependent variable, \(u_1, u_2, u_3\) are the first, second and third derivatives of \(u\) with respect to spatial variables. The fractional pde (2) is invariant under a one-parameter point transformation

\[ \tilde{t} = t + \epsilon \tilde{t}(t, x, u) + O(\epsilon^2), \]

\[ \tilde{x} = x + \epsilon \tilde{x}(t, x, u) + O(\epsilon^2), \]

\[ \tilde{u} = u + \epsilon \tilde{u}(t, x, u) + O(\epsilon^2), \]

(3)
where \( \varepsilon \) represents an infinitesimal parameter, if and only if
\[
(u, \tilde{t}, \tilde{x})|_{(x, t, \varepsilon)} = (u, t, \tau),
\]
The Lie group \( G \) of transformations consisting of the infinitesimal transformations are generated by
\[
\Gamma = \tilde{\xi}(t, \tau, u) \partial_t + \tilde{\zeta}(t, \tau, u) \partial_u + \eta(t, \tau, u) \partial_u,
\]
where \( \tilde{\xi}, \tilde{\zeta} \) and \( \eta \) are the infinitesimals with respect to \( t \), each \( t, \tau \), \( (x, \tau) \) and \( u \). We seek the group \( G \) admitted by the fractional pde (2).

### 2.1. The conservation laws

We mention the preliminaries for the conservation laws of the time-fractional heat equation. A vector field, \( C = (C^t, C^u, C^x, \ldots, C^n) \), where
\[
C^t = C(t, \tau, u, u_1, \ldots), \quad C^u = C^u(t, \tau, u, u_1, \ldots), \quad \tau = (x, \tau), \quad i = 1, 2, \ldots, n
\]
is called a conserved vector if it satisfies the conservation equation,
\[
D_tC^t + \sum_{i=1}^n D_{n}C^i = 0,
\]
on all solutions of (1). Equation (6) is called a conservation law for (1) (Lukashchuk, 2015). A conserved vector is called a trivial conserved vector for Equation (1) if its components \( C_t \) and \( C^u \) vanish on this solution of the equation.

The conserved vectors for the time-fractional diffusion equation with two independent variables \( (t, x) \) for Equation (1) under the Riemann–Liouville fractional derivative is given by
\[
C^t = D_t^\alpha u, \quad C^x = -u_x, \quad n = 1, 2.
\]

Lukashchuk (2015) gives these components as
\[
C^t = \phi(W) + J(W, \phi), \quad C^x = \phi W - \phi W_u
\]
for \( \alpha \in (0, 1) \), where \( i \) corresponds to the number of appropriate symmetries of (1).

Hence we need to determine the conserved vectors, \((C^t, C^u, C^x, \ldots, C^n)\), in each case.

The formal Lagrangian for the time-fractional one-dimensional heat equation can be introduced as
\[
L = a(t, x)(D_t^\alpha u - \nabla^2 u),
\]
where \( a(t, x) \) is a new dependent variable (Ibragimov, 2011; Cao and Lin, 2014; Lukashchuk, 2015). The Action Integral is
\[
\int_0^T \int_{\Omega} L(t, x, u, a, u_x) \, dt \, dx.
\]

Then the Euler–Lagrange equation (Agrawal, 2002; Atanackovic et al., 2009) for the fractional variational principle can be written as
\[
\frac{\partial}{\partial a} - D_t^\alpha \frac{\partial}{\partial D_t^\alpha u} - D_x \frac{\partial}{\partial u_x} + D_t^\alpha \frac{\partial}{\partial u_{xx}},
\]
where \( D_t^\alpha \) is the adjoint operator of the right Riemann–Liouville derivative of order \( \alpha (\text{Atanackovic et al., 2009}) \). Therefore, the adjoint equation for the linear heat equation is
\[
(D_t^\alpha a) - a_{xx} = 0.
\]

When the adjoint equation for the fractional-time heat equation is satisfied for all solutions, \( u(t, x) \), upon a substitution \( a = \phi(t, x, u) \) such that \( \phi(t, x, u) \neq 0 \), then it is called nonlinearly self-adjoint.

Therefore, the conserved component can be mentioned for \( n = [\alpha] + 1 \) as in (Ibragimov, 2011; Lukashchuk, 2015)
\[
c^t = \xi^t L + \sum_{m=0}^{n-1} (-1)^m D_t^{\alpha+1-m} \{ W D_t^\alpha \frac{\partial L}{\partial u} - (-1)^m J \{ W, D_t^\alpha \frac{\partial L}{\partial u} \} \},
\]
where \( J \) and \( W \) can be defined as
\[
J(f, g) = \frac{1}{(\mu-x)^{\alpha-1}} \int_0^\infty f(\tau, x)(\mu-x)^{\alpha-1} \, d\mu \, d\tau,
\]
\[
W = \tilde{\xi} u - \sum_{i=1}^n \tilde{\zeta} u_x,
\]
with \( \tau \) and \( \mu \) are independent variables for \( f \) and \( g \). They satisfy the property that
\[
D_t^\alpha J(f, g) = f(t, x) \L^\alpha g - \L^\alpha f,
\]
where \( J^\alpha f \) is the right Riemann–Liouville derivative of order \( n - \alpha \). \( \phi(t, x) \neq 0 \) is nonlinear self-adjoint as defined in Lukashchuk (2015).

### 3. Lie point symmetries of the one-dimensional heat equation

#### 3.1. Lie point symmetries of the heat equation when \( \alpha = 1 \)

We perform the Lie symmetry analysis of the one-dimensional heat equation,
\[
u_t = u_{xx},
\]

We obtain the differential operators,
\[
\Gamma_1 = \partial_\tau, \quad \Gamma_2 = 2t \partial_\tau - u_\partial_\tau, \quad \Gamma_3 = \partial_\tau, \quad \Gamma_4 = 2 \partial_\tau + x \partial_x, \quad \Gamma_5 = 4t \partial_\tau + 4 \partial_x - u(2t + x^2) \partial_x, \quad \Gamma_6 = u \partial_u, \quad \Gamma_7 = F(t, \tau) \partial_x,
\]
where \( F(t, \tau) \) satisfies
\[
F_t - F_{xx} = 0.
\]

The symmetry \( \Gamma_3 \) is called a time-translational symmetry, \( \Gamma_1 \) and \( \Gamma_2 \) are solution symmetries, whereas \( \Gamma_6 \) is the homogeneity symmetry and \( \Gamma_7 \) is the infinite-dimensional symmetry.

\footnote{The conservation law for (1) with respect to the Caputo derivative is \( C^t = D_t^{\alpha+1} \L^\alpha u \) (Lukashchuk, 2015).}
The nonzero Lie brackets are
\[
\begin{align*}
[\Gamma_3, \Gamma_2]_{LB} &= 2\Gamma_1, \\
[\Gamma_3, \Gamma_3]_{LB} &= -2\Gamma_0 + 4\Gamma_4, \\
[\Gamma_1, \Gamma_3]_{LB} &= 2\Gamma_2, \\
[\Gamma_2, \Gamma_3]_{LB} &= -\Gamma_2.
\end{align*}
\]
As it is well known, the Lie algebra is
\[
2A_1 \oplus s(2, R) \oplus A_1 \oplus \infty A_1.
\]

3.2. Lie point symmetries of the one-dimensional time-fractional heat equation

We consider the time-fractional heat equation in one-dimensional space
\[
D_\alpha^\gamma u(x, t) = u_{xx},
\]
where \(0 < \alpha < 1\). The Lie symmetries thereof are given by
\[
\begin{align*}
\Gamma_0 &= \partial_x, \\
\Gamma_2 &= 2\partial_t + \alpha x \partial_x, \\
\Gamma_3 &= u \partial_t, \\
\Gamma_4 &= F(x, t) \partial_x.
\end{align*}
\]
The fourth symmetry, \(\Gamma_4\), is called the infinite-dimensional symmetry, where \(F\) is a solution of Equation (11). It occurs as a result of the linearity of (11) in \(u\). We note that for this fractional case we do not have a time-translation symmetry, \(\partial_t\). Furthermore, we have only one solution symmetry \(\Gamma_0\) which is a reduction by one from the non-fractional case. By inspection, we see that the Lie algebra is \(A_1 \oplus A_2 \oplus \infty A_1\). The Lie bracket relations among the first three symmetries in Equation (11) are represented in the table below:
\[
\begin{align*}
[\Gamma_0, \Gamma_2]_{LB} &= \alpha \Gamma_0, \\
[\Gamma_0, \Gamma_3]_{LB} &= 0, \\
[\Gamma_2, \Gamma_3]_{LB} &= 0.
\end{align*}
\]
By the computation of nonzero Lie brackets we see that the algebra is the same as what we obtained from our inspection.

4. Lie point symmetries of the two-dimensional heat equation

4.1. The case \(\alpha = 1\)

For the two-dimensional heat equation,
\[
u_t = u_{xx} + u_{yy},
\]
we obtain ten Lie point symmetries given by
\[
\begin{align*}
\Gamma_1 &= \partial_x, \\
\Gamma_2 &= \partial_y, \\
\Gamma_3 &= 2\partial_t - u_y \partial_y, \\
\Gamma_4 &= 2\partial_t - u_x \partial_x, \\
\Gamma_5 &= y \partial_x - x \partial_y, \\
\Gamma_6 &= \partial_t, \\
\Gamma_7 &= 2\partial_t + x \partial_x + y \partial_y, \\
\Gamma_8 &= 4\partial_t \partial_t + 4xt \partial_x + 4yt \partial_y - u(4t + x^2 + y^2) \partial_x, \\
\Gamma_9 &= u \partial_x, \\
\Gamma_{10} &= F(x, y, t) \partial_x.
\end{align*}
\]
In this group of symmetries, \(\Gamma_6, \Gamma_7\) and \(\Gamma_8\) constitute an \(s(2, R)\) subalgebra whereas \(\Gamma_9\) constitutes an \(so(2)\) subalgebra. \(\Gamma_2\) is the homogeneity and \(\Gamma_{10}\) are solution symmetries. The nonzero Lie brackets are
\[
\begin{align*}
[\Gamma_{11}, \Gamma_{14}]_{LB} &= 2\alpha \Gamma_{11}, \\
[\Gamma_{11}, \Gamma_{15}]_{LB} &= -\Gamma_{12}, \\
[\Gamma_{11}, \Gamma_{16}]_{LB} &= -\Gamma_{12}, \\
[\Gamma_{14}, \Gamma_{12}]_{LB} &= -2\alpha \Gamma_{12}, \\
[\Gamma_{12}, \Gamma_{13}]_{LB} &= \Gamma_{11}.
\end{align*}
\]
By considering all the subalgebras, the Lie algebra is
\[
(4A_1 \oplus s(2)) \oplus s(2, R) \oplus A_1 \oplus \infty A_1.
\]

4.2. Lie point symmetries of a two-dimensional time-fractional heat equation

We consider the two-dimensional time-fractional heat equation:
\[
D_\alpha^\gamma u(x, y, t) = u_{xx} + u_{yy},
\]
In this case for \(0 < \alpha < 1\) the number of symmetries is reduced by four. These are given by
\[
\begin{align*}
\Gamma_1 &= \partial_x, \\
\Gamma_2 &= \partial_y, \\
\Gamma_3 &= y \partial_x - x \partial_y, \\
\Gamma_4 &= 4\alpha \partial_t \partial_t + 4\alpha x \partial_x + 2\alpha y \partial_y + u(3\alpha - 2) \partial_x, \\
\Gamma_5 &= u \partial_x, \\
\Gamma_6 &= F(x, y, t) \partial_x,
\end{align*}
\]
where \(F(x, y, t)\) satisfies Equation (15). It can be easily deduced by looking at the symmetries that the Lie algebra is \((2A_1 \oplus s(2)) \oplus 2A_1 \oplus \infty A_1\). The nonzero Lie brackets are
\[
\begin{align*}
[\Gamma_{11}, \Gamma_{14}]_{LB} &= 2\alpha \Gamma_{11}, \\
[\Gamma_{11}, \Gamma_{15}]_{LB} &= -\Gamma_{12}, \\
[\Gamma_{14}, \Gamma_{12}]_{LB} &= -2\alpha \Gamma_{12}, \\
[\Gamma_{12}, \Gamma_{13}]_{LB} &= \Gamma_{11}.
\end{align*}
\]

The algebra is \(A_{3,6} \oplus s(2) \oplus A_1 \oplus \infty A_1\). It is to be noted that the Lie algebra obtained from the nonzero Lie brackets can be conflated to the algebra which we have mentioned above. Therefore the Lie algebra obtained by the Lie brackets can be treated as the composition of the algebra obtained by inspection.
5. Lie Point Symmetries of a Three-Dimensional Heat Equation

5.1. The Case $\alpha = 1$

The three-dimensional heat equation is given by

$$u_t = u_{xx} + u_{yy} + u_{zz}. \quad (18)$$

The Lie point symmetries of (18) are given by

$$
\begin{align*}
\Gamma_{11} &= \partial_x, \\
\Gamma_{12} &= \partial_y, \\
\Gamma_{13} &= \partial_z, \\
\Gamma_{14} &= 2\partial_x - y\partial_y, \\
\Gamma_{15} &= 2\partial_y - x\partial_x, \\
\Gamma_{16} &= 2\partial_z - u\partial_u, \\
\Gamma_{17} &= -y\partial_x + x\partial_y, \\
\Gamma_{18} &= x\partial_z - z\partial_y, \\
\Gamma_{19} &= -2\partial_y + y\partial_x, \\
\Gamma_{20} &= \partial_t, \\
\Gamma_{21} &= 2\partial_t + x\partial_y + y\partial_z + z\partial_x, \\
\Gamma_{22} &= 4\partial^2_t + 4\partial_x^2 + 4\partial_y^2 + 4\partial_z^2 - u(6\partial_t + x^2 + y^2 + z^2)\partial_u, \\
\Gamma_{23} &= \partial_u, \\
\Gamma_{24} &= F(x, y, z, t)\partial_u.
\end{align*}
$$

The nonzero Lie brackets are

$$
\begin{align*}
[G_{11}, G_{12}] &= 2G_{13}, \\
[G_{11}, G_{13}] &= -G_{12}, \\
[G_{11}, G_{14}] &= -G_{15}, \\
[G_{11}, G_{16}] &= G_{17}, \\
[G_{11}, G_{18}] &= G_{19}, \\
[G_{11}, G_{19}] &= -G_{18}, \\
[G_{11}, G_{20}] &= -G_{17}, \\
[G_{11}, G_{22}] &= -2G_{13}, \\
[G_{11}, G_{23}] &= -2G_{14}, \\
[G_{11}, G_{24}] &= -G_{15}, \\
[G_{11}, G_{25}] &= -G_{16}, \\
[G_{11}, G_{26}] &= G_{17}, \\
[G_{11}, G_{27}] &= -G_{18}, \\
[G_{11}, G_{28}] &= -G_{19}, \\
[G_{11}, G_{29}] &= G_{18}, \\
[G_{11}, G_{30}] &= -G_{17}, \\
[G_{11}, G_{31}] &= -G_{16}, \\
[G_{11}, G_{32}] &= G_{15}, \\
[G_{11}, G_{33}] &= G_{14}, \\
[G_{11}, G_{34}] &= -G_{13}, \\
[G_{11}, G_{35}] &= -G_{12}, \\
[G_{11}, G_{36}] &= G_{11}, \\
[G_{11}, G_{37}] &= -G_{10}, \\
[G_{11}, G_{38}] &= G_{9}, \\
[G_{11}, G_{39}] &= G_{8}, \\
[G_{11}, G_{40}] &= G_{7}, \\
[G_{11}, G_{41}] &= G_{6}, \\
[G_{11}, G_{42}] &= G_{5}, \\
[G_{11}, G_{43}] &= G_{4}, \\
[G_{11}, G_{44}] &= G_{3}, \\
[G_{11}, G_{45}] &= G_{2}, \\
[G_{11}, G_{46}] &= G_{1}, \\
[G_{11}, G_{47}] &= 0.
\end{align*}
$$

The Lie algebra is

$$(6A_1 \oplus so(3)) \oplus s(2, R) \oplus A_1 \oplus A_1 \oplus A_1 \oplus \infty A_1.$$
The nonzero Lie brackets are given by

\[ [\Gamma_{11},\Gamma_{11}]_{LB} = 2t_6, \quad [\Gamma_{31},\Gamma_{31}]_{LB} = \Gamma_{31}, \quad [\Gamma_{11},\Gamma_{12}]_{LB} = \Gamma_{31}, \quad [\Gamma_{31},\Gamma_{12}]_{LB} = \Gamma_{31}, \quad [\Gamma_{11},\Gamma_{32}]_{LB} = \Gamma_{31}, \quad [\Gamma_{31},\Gamma_{32}]_{LB} = \Gamma_{31}, \quad [\Gamma_{11},\Gamma_{33}]_{LB} = \Gamma_{31}, \quad [\Gamma_{31},\Gamma_{33}]_{LB} = \Gamma_{31}, \quad [\Gamma_{11},\Gamma_{13}]_{LB} = \Gamma_{31}, \quad [\Gamma_{31},\Gamma_{13}]_{LB} = \Gamma_{31}, \quad [\Gamma_{11},\Gamma_{34}]_{LB} = \Gamma_{31}, \quad [\Gamma_{31},\Gamma_{34}]_{LB} = \Gamma_{31}, \quad [\Gamma_{11},\Gamma_{35}]_{LB} = \Gamma_{31}, \quad [\Gamma_{31},\Gamma_{35}]_{LB} = \Gamma_{31}, \quad [\Gamma_{11},\Gamma_{36}]_{LB} = \Gamma_{31}, \quad [\Gamma_{31},\Gamma_{36}]_{LB} = \Gamma_{31}, \quad [\Gamma_{11},\Gamma_{37}]_{LB} = \Gamma_{31}, \quad [\Gamma_{31},\Gamma_{37}]_{LB} = \Gamma_{31}, \quad [\Gamma_{11},\Gamma_{38}]_{LB} = \Gamma_{31}, \quad [\Gamma_{31},\Gamma_{38}]_{LB} = \Gamma_{31}, \quad [\Gamma_{11},\Gamma_{39}]_{LB} = \Gamma_{31}, \quad [\Gamma_{31},\Gamma_{39}]_{LB} = \Gamma_{31}, \quad [\Gamma_{11},\Gamma_{40}]_{LB} = \Gamma_{31}, \quad [\Gamma_{31},\Gamma_{40}]_{LB} = \Gamma_{31}, \quad [\Gamma_{11},\Gamma_{41}]_{LB} = \Gamma_{31}, \quad [\Gamma_{31},\Gamma_{41}]_{LB} = \Gamma_{31} \]

The Lie algebra is

\[(\mathfrak{so}(4)) \oplus s(2, R) \oplus A_1 \oplus \infty A_1.\]

6.2. The case \(0 < \alpha < 1\)

The heat equation in four-dimensional space with fractional time-derivative,

\[ u_\alpha^t = u_{xx} + u_{yy} + u_{zz} + u_{ww}, \quad (23) \]

has the following Lie point symmetries

\[ \Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = \partial_y, \quad \Gamma_4 = \partial_z, \quad \Gamma_5 = -y \partial_x + x \partial_y, \quad \Gamma_6 = -y \partial_x + z \partial_y, \quad \Gamma_7 = y \partial_x - w \partial_z, \quad \Gamma_8 = \partial_x - x \partial_y, \quad \Gamma_9 = -w \partial_z + z \partial_y, \quad \Gamma_{10} = \partial_x + \alpha \partial_y + \alpha \partial_z, \quad \Gamma_{11} = \partial_x + \alpha \partial_y + \alpha \partial_z + u(\alpha - 1) \partial_z, \quad \Gamma_{12} = \partial_t, \quad \Gamma_{13} = F(x, y, z, w, t) \partial_t. \]

\(\Gamma_{11}\) is the remnant of \(s(2, R), \Gamma_{12}\) is the homogeneity symmetry, \(\Gamma_5 \) to \(\Gamma_{10}\) constitute an \(\mathfrak{so}(4)\) subalgebra. \(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\) and \(\Gamma_6\) are solution symmetries. The nonzero Lie brackets, which are reduced significantly in number compared to the case \(\alpha = 1\), are

\[ [\Gamma_{46},\Gamma_{47}]_{LB} = -\Gamma_{62}, \quad [\Gamma_{46},\Gamma_{47}]_{LB} = -\Gamma_{61}, \quad [\Gamma_{46},\Gamma_{57}]_{LB} = \Gamma_{49}, \quad [\Gamma_{46},\Gamma_{57}]_{LB} = \Gamma_{49}, \quad [\Gamma_{46},\Gamma_{67}]_{LB} = \Gamma_{48}, \quad [\Gamma_{46},\Gamma_{67}]_{LB} = \Gamma_{48}, \quad [\Gamma_{46},\Gamma_{68}]_{LB} = \Gamma_{48}, \quad [\Gamma_{46},\Gamma_{68}]_{LB} = \Gamma_{48}, \quad [\Gamma_{61},\Gamma_{62}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{62}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{63}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{63}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{64}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{64}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{65}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{65}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{66}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{66}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{67}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{67}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{68}]_{LB} = \Gamma_{61}, \quad [\Gamma_{61},\Gamma_{68}]_{LB} = \Gamma_{61}. \]

The algebra obtained by inspection of the symmetries is

\[(4A_1 \oplus s(4)) \oplus 2A_1 \oplus \infty A_1,\]

7. LIE POINT SYMMETRIES OF THE N-DIMENSIONAL TIME-FRACTIONAL HEAT EQUATION

The heat equation in \(N\)-dimensional space with fractional time-derivative can be defined as follows,

\[ u_\alpha^t = u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_N x_N}, \quad (24) \]

where \(x_1, x_2, x_3, \ldots, x_n\) are independent variables. It has the following Lie point symmetries

\[ \Gamma_{71} = \partial_{x_i}, \quad 1 \leq i \leq n \]

\[ \Gamma_{72} = -x_j \partial_{x_i} + x_i \partial_{x_j}, \quad i < j \]

\[ \Gamma_{73} = \partial_t + \alpha \sum_{i=1}^{n} x_i \partial_{x_i} + \alpha(\alpha - 1) \partial_t, \]

\[ \Gamma_{74} = u \partial_{x_i}, \]

\[ \Gamma_{75} = F(t, x_1, x_2, \ldots, x_n) \partial_{x_i}. \]

8. THE CONSERVATION LAWS FOR THE NONFRACTIONAL HEAT EQUATION

8.1. Introduction

We now construct the conservation laws of the heat equation (1). The time-fractional diffusion equation (1), with the Riemann–Liouville fractional derivative can be rewritten in the form of conservation laws.

8.2. Conservation laws for the one-dimensional heat equation

The Lie point symmetries for the one-dimensional heat equation are given in (10). The formal Lagrangian can be introduced as

\[ L = \phi(t, x)(u_t - u_{xx}), \]

where \(\phi(t, x)\) is a new dependent variable. The components of the conserved vector for \(\Gamma_1\) are

\[ C^i = -u_x \phi, \]

\[ C^e = \phi u_t - u_x \phi. \]

For \(\Gamma_2\) the components of the conserved vector are

\[ C^i = W \phi, \]

\[ C^e = 2x \phi u_t + W \phi_x + \phi(ux + xu), \]

\[ W = -ux - 2u_x. \]

The components of the conserved vectors for \(\Gamma_3\) are

\[ C^i = -u_x \phi, \]

\[ C^e = W \phi_x + \phi u_t, \]

\[ W = -u_t. \]

The conserved vectors for \(\Gamma_4\) are

\[ C^i = -2x^2 u_t x - \phi u_{xx}, \]

\[ C^e = x \phi(u_t - u_{xx}) + W \phi_x + \phi(u_t + xu + 2tu), \]

\[ W = -2u_t - xu. \]

The components of the conserved vectors for \(\Gamma_5\) are

\[ C^i = -4t^2 \phi u_{xx} - \phi(2u + x^2) + 4txu, \]

\[ C^e = 4tx \phi(u_t - u_{xx}) + W \phi_x + \phi(2tu + 2ux + x^2 u + 4tu + 4tu + 4tu + 4tu + 4tu + 4tu), \]

\[ W = -u_t - x^2 - 4t^2 u - 4tu. \]
The components of the conserved vectors for $\Gamma_6$ are
\begin{align}
C^t &= W \phi, \\
C^x &= W \phi_x - \phi u_t, \\
W &= u.
\end{align}
\tag{31}

The components of the conserved vectors for $\Gamma_7$ are
\begin{align}
C^t &= W \phi, \\
C^x &= W \phi_x - \phi F_y, \\
W &= F.
\end{align}
\tag{32}

### 8.3. Conservation laws for the two-dimensional nonfractional heat equation

The symmetries for the two-dimensional nonfractional heat equation are given in (14). The formal Lagrangian can be introduced as
\begin{equation}
L = \phi(t, x, y)(u_t - u_{xx} - u_{yy}),
\end{equation}
where $\phi(t, x, y)$ is a new dependent variable. The components of the conserved vectors for $\Gamma_{21}$ are
\begin{align}
C^t &= -u_x \phi, \\
C^x &= \phi(u_t - u_{xx} - u_{yy}) + W \phi_x + u_{xx}, \\
C^y &= W \phi_y, \\
W &= -u_y.
\end{align}
\tag{33}

The components of the conserved vectors for $\Gamma_{22}$ are
\begin{align}
C^t &= -u_y \phi, \\
C^y &= W \phi_y, \\
C^x &= \phi(u_t - u_{xx} - u_{yy}) + W \phi_x + u_{yy}, \\
W &= -u_y.
\end{align}
\tag{34}

The components of the conserved vectors for $\Gamma_{23}$ are
\begin{align}
C^t &= W \phi, \\
C^x &= W \phi_x + \phi u_t, \\
C^y &= 2t \phi(u_t - u_{xx}) + W \phi_y + \phi(u_{yy} + u), \\
W &= -u_y - 2u_t.
\end{align}
\tag{35}

The components of the conserved vectors for $\Gamma_{24}$ are
\begin{align}
C^t &= W \phi, \\
C^x &= 2t \phi(u_t - u_{xx} - u_{yy}) + W \phi_y + \phi(u + xu_t), \\
C^y &= W \phi_y + \phi u_t, \\
W &= -ux - 2u_t.
\end{align}
\tag{36}

The components of the conserved vectors for $\Gamma_{25}$ are
\begin{align}
C^t &= W \phi, \\
C^x &= y \phi(u_t - u_{yy}) + W \phi_x - \phi u_t, \\
C^y &= -x \phi(u_t - u_{xx}) + W \phi_y + \phi u_t, \\
W &= -yu_x + xu_y.
\end{align}
\tag{37}

The components of the conserved vectors for $\Gamma_{26}$ are
\begin{align}
C^t &= -\phi(u_{xx} + u_{yy}), \\
C^x &= W \phi_x + \phi u_{tx}, \\
C^y &= W \phi_y + \phi u_{ty}, \\
W &= -u_t.
\end{align}
\tag{38}

The components of the conserved vectors for $\Gamma_{27}$ are
\begin{align}
C^t &= 2t \phi(u_{xx} + u_{yy}) - \phi(xu_t + yu_y), \\
C^x &= x \phi(t, x, y)(u_t - u_{xx} - u_{yy}) + W \phi_x + \phi(2tu_{xx} + u_x), \\
C^y &= y \phi(t, x, y)(u_t - u_{xx} - u_{yy}) + W \phi_y + \phi(2tu_{yy} + u_y), \\
W &= -2tu_t - xu_t - yu_y.
\end{align}
\tag{39}

The components of the conserved vectors for $\Gamma_{28}$ are
\begin{align}
C^t &= -4t \phi(t, x, y)(u_t + u_x) - \phi(u(4u + x^2 + y^2) + 4u_x + 4uy), \\
C^x &= 4xt \phi(u_t - u_y) + W \phi_x + \phi(8u_{xx} + x^2u_t + 2xu + y^2u_t + 2yu + 4t^2u_{xx}), \\
C^y &= 4yt \phi(u_t - u_x) + W \phi_y + \phi(8u_{yy} + x^2u_t + 2yu + y^2u_t + 4t^2u_{yy}), \\
W &= -u(4t + x^2 + y^2) - 4t^2u_t - 4xu_x - 4yu_y.
\end{align}
\tag{40}

The components of the conserved vectors for $\Gamma_{29}$ are
\begin{align}
C^t &= W \phi, \\
C^x &= W \phi_x - \phi F_x, \\
W &= u.
\end{align}
\tag{41}

The components of the conserved vectors for $\Gamma_{210}$ are
\begin{align}
C^t &= W \phi, \\
C^x &= W \phi_x - \phi F_x, \\
C^y &= W \phi_y - \phi F_y, \\
W &= F.
\end{align}
\tag{42}

### 8.4. Conservation laws of the three-dimensional nonfractional heat equation

The symmetries for the nonfractional three-dimensional heat equation are given in (19). The formal Lagrangian can be introduced as
\begin{equation}
L = \phi(t, x, y, z)(u_t - u_{xx} - u_{yy} - u_{zz}),
\end{equation}
where $\phi(t, x, y, z)$ is a new dependent variable. The components of the conserved vectors for $\Gamma_{31}$ are
\begin{align}
C^t &= W \phi, \\
C^x &= \phi(u_t - u_{xx} - u_{yy} - u_{zz}) + W \phi_x + u_{xx}, \\
C^y &= W \phi_y, \\
W &= -u_z.
\end{align}
\tag{43}
The components of the conserved vectors for \( \Gamma_{32} \) are

\[
\begin{align*}
C' &= W \phi, \\
C'' &= W \phi_x, \\
C''' &= \phi(u_t - u_{xx} - u_{yy} - u_{zz}) + W \phi_y + u_{yy}, \\
C'''' &= W \phi_z, \\
W &= -u_z.
\end{align*}
\]

The components of the conserved vectors for \( \Gamma_{33} \) are

\[
\begin{align*}
C' &= W \phi, \\
C'' &= W \phi_x, \\
C''' &= W \phi_y, \\
C''''' &= \phi(u_t - u_{xx} - u_{yy} - u_{zz}) + W \phi_y + u_{yy}, \\
W &= -u_z.
\end{align*}
\]

The components of the conserved vectors for \( \Gamma_{34} \) are

\[
\begin{align*}
C' &= W \phi, \\
C'' &= W \phi_x + \phi u_y, \\
C''' &= 2t \phi(u_t - u_{xx} - u_{yy}) + W \phi_y + \phi(u + u_t), \\
C'''' &= W \phi_z + \phi u_t, \\
W &= -ix - 2u_x.
\end{align*}
\]

The components of the conserved vectors for \( \Gamma_{35} \) are

\[
\begin{align*}
C' &= W \phi, \\
C'' &= 2t \phi(u_t - u_{xx} - u_{yy}) + W \phi_y + \phi(u + u_t), \\
C''' &= W \phi_y + \phi u_t, \\
C''''' &= W \phi_z + \phi u_t, \\
W &= -2u_t - 2u_{xx}.
\end{align*}
\]

The components of the conserved vectors for \( \Gamma_{36} \) are

\[
\begin{align*}
C' &= W \phi, \\
C'' &= W \phi_x + \phi u_t, \\
C''' &= W \phi_y + \phi u_t, \\
C''''' &= 2t \phi(u_t - u_{xx} - u_{yy}) + W \phi_y + \phi(u + u_t), \\
W &= -ux - 2u_{xx}.
\end{align*}
\]

The components of the conserved vectors for \( \Gamma_{37} \) are

\[
\begin{align*}
C' &= W \phi, \\
C'' &= -y \phi(u_t - u_{yy} - u_{zz}) + W \phi_y + \phi u_t, \\
C''' &= x \phi(t, x, y, z)(u_t - u_{xx} - u_{yy}) + W \phi_y + \phi u_t, \\
C''''' &= W \phi_z, \\
W &= yu_t - xu_y.
\end{align*}
\]

The components of the conserved vectors for \( \Gamma_{38} \) are

\[
\begin{align*}
C' &= W \phi, \\
C'' &= -z \phi(u_t - u_{xx} - u_{yy}) + W \phi_y + \phi u_t, \\
C''' &= W \phi_z, \\
C''''' &= x \phi(t, x, y, z)(u_t - u_{xx} - u_{yy}) + W \phi_y + \phi u_t, \\
W &= zu_t - xu_z.
\end{align*}
\]

The components of the conserved vectors for \( \Gamma_{39} \) are

\[
\begin{align*}
C' &= W \phi, \\
C'' &= W \phi_y, \\
C''' &= -z \phi(u_t - u_{xx} - u_{yy}) + W \phi_y + \phi u_t, \\
C''''' &= y \phi(u_t - u_{xx} - u_{yy}) + W \phi_z + \phi u_t, \\
W &= zu_t - yu_z.
\end{align*}
\]

The components of the conserved vectors for \( \Gamma_{310} \) are

\[
\begin{align*}
C' &= -\phi(u_t + u_{xx} + u_{yy}) + W \phi, \\
C'' &= W \phi_x + \phi u_t, \\
C''' &= W \phi_y + \phi u_t, \\
C''''' &= W \phi_z + \phi u_t, \\
W &= -u_t.
\end{align*}
\]

The components of the conserved vectors for \( \Gamma_{311} \) are

\[
\begin{align*}
C' &= -2t \phi(u_{xx} + u_{yy} + u_{zz}) - \phi(u_t + yu_y + zu_z), \\
C'' &= x \phi(u_t - u_{xx} - u_{yy}) + W \phi_x + \phi(u + 2u_t), \\
C''' &= y \phi(u_t - u_{xx} - u_{yy}) + W \phi_y + \phi(u + 2u_t), \\
C''''' &= z \phi(u_t - u_{xx} - u_{yy}) + W \phi_z + \phi(u + 2u_t), \\
W &= -yu_t - yu_x - zu_z.
\end{align*}
\]

The components of the conserved vectors for \( \Gamma_{312} \) are

\[
\begin{align*}
C' &= -4t^2 \phi(u_{xx} + u_{yy} + u_{zz}) - \phi(u_t + t^2 + t^2 + t^2 + 4u_x + 4u_y + 4u_z), \\
C'' &= 4x \phi(u_t - u_{xx} - u_{yy}) + W \phi_x + \phi(10u_t + z^2 u_x + 2u_x + z^2 u_y + z^2 u_z), \\
C''' &= 4y \phi(u_t - u_{xx} - u_{yy}) + W \phi_y + \phi(10u_t + z^2 u_x + 2u_x + z^2 u_y + z^2 u_z), \\
C''''' &= 4z \phi(u_t - u_{xx} - u_{yy}) + W \phi_z + \phi(10u_t + z^2 u_x + 2u_x + z^2 u_y + z^2 u_z), \\
W &= -u(6u + x^2 + y^2 + z^2) - 4u_t - 4u_x - 4u_y - 4u_z.
\end{align*}
\]

The components of the conserved vectors for \( \Gamma_{313} \) are

\[
\begin{align*}
C' &= W \phi, \\
C'' &= W \phi_x - \phi u_t, \\
C''' &= W \phi_y - \phi u_t, \\
C''''' &= W \phi_z - \phi u_t, \\
W &= u.
\end{align*}
\]

The components of the conserved vectors for \( \Gamma_{314} \) are

\[
\begin{align*}
C' &= W \phi, \\
C'' &= W \phi_x - \phi F_z, \\
C''' &= W \phi_y - \phi F_y, \\
C''''' &= W \phi_z - \phi F_z, \\
W &= F.
\end{align*}
\]

8.5. Conservation laws of the four-dimensional nonfractional heat equation

The formal Lagrangian can be introduced as

\[ L = \phi(t, x, y, z, w)(u_t - u_{xx} - u_{yy} - u_{zz} - u_{ww}), \]

where \( \phi(t, x, y, z, w) \) is a new dependent variable.
The components of the conserved vectors for $\Gamma_{51}$ are

\[
\begin{align*}
C^t &= W\phi_t, \\
C^x &= \phi(u_z - u_{zz} - u_{ww}) + W\phi_x, \\
C^y &= W\phi_y, \\
C^z &= W\phi_z, \\
C^w &= W\phi_w, \\
W &= -u_t.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{52}$ are

\[
\begin{align*}
C^t &= W\phi_t, \\
C^x &= W\phi_x, \\
C^y &= \phi(u_x - u_{xx} - u_{ww}) + W\phi_y, \\
C^z &= W\phi_z, \\
C^w &= W\phi_w, \\
W &= -u_y.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{53}$ are

\[
\begin{align*}
C^t &= W\phi_t, \\
C^x &= W\phi_x, \\
C^y &= W\phi_y, \\
C^z &= \phi(u_t - u_{tt} - u_{yy} - u_{ww}) + W\phi_z, \\
C^w &= W\phi_w, \\
W &= -u_z.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{54}$ are

\[
\begin{align*}
C^t &= W\phi_t, \\
C^x &= W\phi_x, \\
C^y &= W\phi_y, \\
C^z &= W\phi_z, \\
C^w &= \phi(u_t - u_{xx} - u_{yy} - u_{zz}) + W\phi_w, \\
W &= -u_u.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{55}$ are

\[
\begin{align*}
C^t &= W\phi_t, \\
C^x &= W\phi_x + \phi u_x, \\
C^y &= 2t\phi(u_t - u_{xx} - u_{zz} - u_{ww}) + W\phi_y + \phi(u + uy), \\
C^z &= W\phi_z + \phi u_z, \\
C^w &= W\phi_w + \phi u_w, \\
W &= -uy - 2u_y.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{56}$ are

\[
\begin{align*}
C^t &= W\phi_t, \\
C^x &= 2t\phi(u_t - u_{xx} - u_{zz} - u_{ww}) + W\phi_x + \phi(u + xu), \\
C^y &= W\phi_y + \phi u_y, \\
C^z &= W\phi_z + \phi u_z, \\
C^w &= W\phi_w + \phi u_w, \\
W &= -ux - 2u_x.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{57}$ are

\[
\begin{align*}
C^t &= W\phi_t, \\
C^x &= W\phi_x + \phi u_x, \\
C^y &= 2t\phi(u_t - u_{xx} - u_{yy} - u_{uu}) + W\phi_y + \phi(u + zu), \\
C^z &= W\phi_z + \phi u_z, \\
C^w &= W\phi_w + \phi u_w, \\
W &= -uz - 2u_z.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{58}$ are

\[
\begin{align*}
C^t &= W\phi_t, \\
C^x &= W\phi_x + \phi u_x, \\
C^y &= 2t\phi(u_t - u_{xx} - u_{yy} - u_{uu}) + W\phi_y + \phi(u + wu), \\
C^z &= W\phi_z + \phi u_z, \\
C^w &= W\phi_w + \phi u_w, \\
W &= -uw - 2u_u.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{59}$ are

\[
\begin{align*}
C^t &= W\phi_t, \\
C^x &= -y\phi(u_t - u_{xy} - u_{ww}) + W\phi_x + \phi u_y, \\
C^y &= x\phi(u_t - u_{xx} - u_{yz} - u_{ww}) + W\phi_y - \phi u_x, \\
C^z &= W\phi_z, \\
C^w &= W\phi_w, \\
W &= u_t - uy.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{60}$ are

\[
\begin{align*}
C^t &= W\phi_t, \\
C^x &= W\phi_x + \phi u_x, \\
C^y &= -w\phi(u_t - u_{xx} - u_{yz} - u_{ww}) + W\phi_y + \phi u_y, \\
C^z &= W\phi_z, \\
C^w &= y\phi(u_t - u_{xx} - u_{yy} - u_{zz}) + W\phi_w - \phi u_y, \\
W &= wu_y - yu_y.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{61}$ are

\[
\begin{align*}
C^t &= W\phi_t, \\
C^x &= -z\phi(u_t - u_{xy} - u_{zz} - u_{ww}) + W\phi_x + \phi u_z, \\
C^y &= x\phi(u_t - u_{xx} - u_{yz} - u_{ww}) + W\phi_y - \phi u_x, \\
C^z &= W\phi_z, \\
C^w &= W\phi_w, \\
W &= zu_t - xu_t.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{62}$ are

\[
\begin{align*}
C^t &= W\phi_t, \\
C^x &= W\phi_x + \phi u_x, \\
C^y &= -z\phi(u_t - u_{xx} - u_{yy} - u_{ww}) + W\phi_y + \phi u_z, \\
C^z &= W\phi_z, \\
C^w &= W\phi_w, \\
W &= zu_t - yu_y.
\end{align*}
\]
The components of the conserved vector for $\Gamma_{512}$ are

\[
\begin{align*}
C_x &= W \phi, \\
C_y &= -w \phi (u_t - u_{3y}) + W \phi_x + \phi u_z, \\
C_z &= W \phi_z, \\
C^e_w &= x \phi (u_t - u_{3y}) + W \phi_w - \phi u_z, \\
W &= w u_t - x u_z.
\end{align*}
\]

The components of the conserved vector for $\Gamma_{514}$ are

\[
\begin{align*}
C^e_x &= W \phi_x, \\
C^e_y &= W \phi_y, \\
C^e_z &= -w \phi (u_t - u_{3y}) + W \phi_z + \phi u_x, \\
C^e_w &= z \phi (x, y, z, w)(u_t - u_{3y}) + W \phi_w - \phi u_z, \\
W &= w u_t - x u_z.
\end{align*}
\]

The components of the conserved vector for $\Gamma_{515}$ are

\[
\begin{align*}
C^e_x &= \phi (u_t - u_{3x} - u_{3y} - u_{3w}) + W \phi_x, \\
C^e_y &= W \phi_y + \phi u_z, \\
C^e_z &= W \phi_z + \phi u_x, \\
C^e_w &= W \phi_w + \phi u_y, \\
W &= -u_t.
\end{align*}
\]

The components of the conserved vector for $\Gamma_{516}$ are

\[
\begin{align*}
C^e_x &= 2 \phi (u_t - u_{3x} - u_{3y} - u_{3w}) + W \phi_x, \\
C^e_y &= x \phi (u_t - u_{3y} - u_{3z} - u_{3w}) + W \phi_y + \phi u_z, \\
C^e_z &= y \phi (u_t - u_{3x} - u_{3w}) + W \phi_z + \phi u_x, \\
C^e_w &= z \phi (u_t - u_{3x} - u_{3y}) + W \phi_z + \phi u_x, \\
C^e_w &= w \phi (u_t - u_{3x} - u_{3y} - u_{3z} - u_{3w}) + W \phi_w + \phi u_z, \\
W &= -2u_t - x u_x - y u_y - z u_z - w u_w.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{517}$ are

\[
\begin{align*}
C^e_x &= 4 x^2 \phi (u_t - u_{3x} - u_{3y} - u_{3w}) + W \phi_x, \\
C^e_y &= 4 x^4 \phi (u_t - u_{3y} - u_{3z} - u_{3w}) + W \phi_y + \phi (12 u_t + x^2 u_x + 2 u_t^3 + x^2 u_{x^2} + 2 u_t x u_{x^2} + 4 u_{3x} u_{x^2}), \\
C^e_z &= 4 y^2 \phi (u_t - u_{3x} - u_{3w}) + W \phi_z + \phi (12 u_t + x^2 u_x + y^2 u_{y^2} + x^2 u_{x^2} u_{y^2} + 4 u_{3x} u_{x^2} + 4 u_{3z} u_{y^2}), \\
C^e_w &= 4 z^2 \phi (u_t - u_{3x} - u_{3y}) + W \phi_z + \phi (12 u_t + x^2 u_x + y^2 u_{y^2} + z^2 u_{z^2} + 2 u_t x u_{x^2} u_{z^2}), \\
W &= -u_t (8 + x^2 + y^2 + z^2 + u_{3x} - 4 u_{3z} u_{x^2} - 4 u_{3x} u_{y^2} - 4 u_{3z} u_{y^2} - 4 u_{3y} u_{y^2} + 4 u_{3w} u_{z^2}).
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{518}$ are

\[
\begin{align*}
C^e_x &= W \phi, \\
C^e_y &= W \phi_x + \phi u_z, \\
C^e_z &= W \phi_z, \\
C^e_w &= W \phi_w, \\
W &= u.
\end{align*}
\]

The components of the conserved vectors are $\Gamma_{519}$ are

\[
\begin{align*}
C^e_x &= W \phi_x, \\
C^e_y &= W \phi_y - \phi F_z, \\
C^e_z &= W \phi_z - \phi F_z, \\
C^e_w &= W \phi_w - \phi F_w, \\
W &= F.
\end{align*}
\]

9. CONSERVATION LAWS WITH THE RIEMANN–LIOUVILLE FRACTIONAL DERIVATIVE FOR ONE-DIMENSIONAL HEAT EQUATION

The conservation laws with respect to each $\Gamma_{i0}, i = 1, 2, 3, 4$, are as follows: For $\Gamma_{i0}$, the conserved vectors are

\[
\begin{align*}
C^e_x &= \phi (t, x) D^\alpha_{1-x} (W) + J(W, \phi), \\
C^e_y &= \phi \alpha \phi D^\alpha_{1-x} (W) + \phi f_x, \\
W &= 2u_t - \alpha u_x.
\end{align*}
\]

For $\Gamma_{i2}$ the conserved vectors are

\[
\begin{align*}
C^e_x &= 2 \phi (D^\alpha_{1-x} u - u_x) + \phi D^\alpha_{1-x} (W) + J(W, \phi), \\
C^e_y &= \alpha \phi D^\alpha_{1-x} u + W \phi_y - \phi 2 u_{tx}, \\
W &= 2u_t - \alpha u_x.
\end{align*}
\]

For $\Gamma_{i3}$ we have the following components of the conserved vector

\[
\begin{align*}
C^e_x &= \phi (t, x) D^\alpha_{1-x} (W) + J(W, \phi), \\
C^e_y &= W \phi_x - 2 \phi u_z, \\
W &= u.
\end{align*}
\]

For $\Gamma_{i4}$ the conserved vectors are

\[
\begin{align*}
C^e_x &= \phi D^\alpha_{1-x} (F) + J(F, \phi), \\
C^e_y &= F \phi_y - \phi (t, x) F_x.
\end{align*}
\]

9.1. The conservation laws for the two-dimensional time-fractional heat equation

The symmetries are given in (16). For $\Gamma_{11}$ the components of the conserved vectors are

\[
\begin{align*}
C^e_x &= \phi (D^\alpha_{1-x} (W) + J(W, \phi), \\
C^e_y &= \phi (D^\alpha_{1-x} u - u_x) + W \phi_x, \\
C^e_z &= W \phi_z + \phi u_{xy}, \\
W &= -u_t.
\end{align*}
\]

For $\Gamma_{12}$ the components of the conserved vectors are

\[
\begin{align*}
C^e_x &= \phi (D^\alpha_{1-x} (W) + J(-u_x, \phi_y), \\
C^e_y &= W \phi_y + \phi u_{xy}, \\
C^e_z &= \phi (D^\alpha_{1-x} u - u_{xy}) + W \phi_y, \\
W &= -u_x.
\end{align*}
\]
For $\Gamma_{13}$ the components of the conserved vectors are
\[
C = \phi D_1^{-\alpha}(W) + J(W, \phi), \\
C^u = y\phi(D_y^\alpha u - u_y) + W\phi_1 - \phi(u_y + xu_y), \\
C^z = -x\phi(D_z^\alpha u - u_z) + W\phi_y + \phi(yu_y + u_z), \\
W = -yu_y + xu_y. \\
\]
(82)

For $\Gamma_{14}$ the components are
\[
C = 4t(D_y^\alpha u - u_y) + \phi D_1^{-\alpha}(W) + J(W, \phi), \\
C^u = 2\alpha\phi(D_y^\alpha u - u_y) + W\phi_1 - \phi(au_y - 2u_y - 4tu_y - 2axu_y), \\
C^z = 2\alpha\phi(D_z^\alpha u - u_z) + W\phi_y + \phi(au_y - 2u_y - 4tu_y - 2axu_y), \\
W = u(3\alpha - 2) - 4tu_y - 2\alpha xu_y - 2\alpha yu_y. \\
\]
(83)

For $\Gamma_{15}$ the components are
\[
C = \phi D_1^{-\alpha}(W) + J(u, \phi), \\
C^u = u\phi_1 - u_y\phi, \\
C^z = u\phi_y - u_{yy}\phi, \\
W = u. \\
\]
(84)

For $\Gamma_{16}$ the conserved vectors are
\[
C = \phi D_1^{-\alpha}(F) + J(F, \phi), \\
C^u = F\phi_1 - F_y\phi, \\
C^z = F\phi_y - F_{yy}\phi, \\
W = u. \\
\]
(85)

9.2. The conservation laws for the three-dimensional time-fractional heat equation

The symmetries are given in (21). The conserved vectors with respect to each symmetry are as follows: For $\Gamma_{41}$ the components of the conserved vectors are
\[
C = \phi D_1^{-\alpha}(W) + J(W, \phi), \\
C^u = \phi(D_1^{-\alpha}u - u_y - u_z) + W\phi_1, \\
C^z = W\phi_y + \phi(u_{yy}), \\
C^t = W\phi_z + \phi(u_{zz}), \\
W = -u_z. \\
\]
(86)

For $\Gamma_{42}$ the components of the conserved vectors are
\[
C = \phi D_2^{-\alpha}(W) + J(W, \phi), \\
C^u = W\phi_1 + \phi(u_{xx}), \\
C^z = \phi(D_1^{-\alpha}u - u_{xx} - u_y) + W\phi_y, \\
W = -u_x. \\
\]
(87)

For $\Gamma_{43}$ the components of the conserved vectors are
\[
C = \phi D_3^{-\alpha}(W) + J(W, \phi), \\
C^u = W\phi_1 + \phi(u_{xx}), \\
C^z = W\phi_y + \phi(u_{zy}), \\
C^t = \phi(D_1^{-\alpha}u - u_{xx} - u_{yy}) + W\phi_z, \\
W = -u_z. \\
\]
(88)

For $\Gamma_{44}$ the components of the conserved vectors are
\[
C = \phi D_1^{-\alpha}(W) + J(W, \phi), \\
C^u = -y\phi(D_1^{-\alpha}u - u_y - u_x) + W\phi_1 + \phi(xu_y + u_x), \\
C^z = x\phi(D_1^{-\alpha}u - u_x - u_z) + W\phi_y - \phi(u_z + yu_z), \\
C^t = W\phi_z + \phi(yu_z - xu_z), \\
W = yu_z - xu_z. \\
\]
(89)

For $\Gamma_{45}$ the components of the conserved vectors are
\[
C = \phi D_1^{-\alpha}(W) + J(W, \phi), \\
C^u = W\phi_1 + \phi(yu_z - zy_z), \\
C^z = z\phi(D_1^{-\alpha}u - u_x - u_z) + W\phi_y - \phi(u_z + yu_z), \\
C^t = -y\phi(D_1^{-\alpha}u - u_y - u_z) + W\phi_z + \phi(u_x + zy_z), \\
W = yu_z - zy_z. \\
\]
(90)

For $\Gamma_{46}$ the components of the conserved vectors are
\[
C = \phi D_1^{-\alpha}(W) + J(W, \phi), \\
C^u = z\phi(D_1^{-\alpha}u - u_x - u_z) + W\phi_y - \phi(u_z + yu_z), \\
C^z = W\phi_z + \phi(xu_z - zy_z), \\
W = xu_z - zy_z. \\
\]
(91)

For $\Gamma_{47}$ the components of the conserved vectors are
\[
C = 2\alpha\phi(D_y^\alpha u - u_y) + \phi D_1^{-\alpha}(W) + J(W, \phi), \\
C^u = 2\alpha\phi(D_y^\alpha u - u_y) + W\phi_1 - \phi(au_y + 2tu_y + axu_y + \alpha yu_y), \\
C^z = 2\alpha\phi(D_z^\alpha u - u_z) + W\phi_y + \phi(au_y + 2tu_y + axu_y + \alpha yu_y), \\
C^t = \phi(u_x + 2tu_x + axu_x + \alpha yu_y), \\
W = u(\alpha - 1) - 2tu_y - axu_y - \alpha yu_y - \alpha zu_z. \\
\]
(92)

For $\Gamma_{48}$ the components of the conserved vectors are
\[
C = \phi D_1^{-\alpha}(W) + J(W, \phi), \\
C^u = W\phi_1 - \phi(u_z), \\
C^z = W\phi_y - \phi(u_y), \\
W = u. \\
\]
(93)

For $\Gamma_{49}$ the components of the conserved vectors are
\[
C = \phi D_1^{-\alpha}(W) + J(W, \phi), \\
C^u = W\phi_1 - \phi(F_z), \\
C^z = W\phi_y - \phi(F_y), \\
W = F. \\
\]
(94)
9.3. The conservation laws for the four-dimensional time-fractional heat equation

The components of the conserved vectors for $\Gamma_{61}$ are

\[
\begin{align*}
C^r &= \phi D_t^{1-a}(W) + J(W, \phi_t), \\
C^s &= \phi (D_t^{1-a}u - u_{xy} - u_{zz} - u_{ww}) + W \phi_x, \\
C^t &= W \phi_x + \phi u_{z}, \\
C^u &= W \phi_x + \phi u_{w}, \\
W &= -u_x.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{62}$ are

\[
\begin{align*}
C^r &= \phi D_t^{1-a}(W) + J(W, \phi), \\
C^s &= W \phi_x + \phi u_{xy}, \\
C^t &= \phi (D_t^{1-a}u - u_{xl} - u_{zz} - u_{ww}) + W \phi_y, \\
C^u &= W \phi_y + \phi u_{wz}, \\
W &= -u_y.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{63}$ are

\[
\begin{align*}
C^r &= \phi D_t^{1-a}(W) + J(W, \phi), \\
C^s &= W \phi_x + \phi u_{tx}, \\
C^t &= \phi (D_t^{1-a}u - u_{xl} - u_{yy} - u_{ww}) + W \phi_z, \\
C^u &= W \phi_z + \phi u_{ww}, \\
W &= -u_z.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{64}$ are

\[
\begin{align*}
C^r &= \phi D_t^{1-a}(W) + J(W, \phi), \\
C^s &= W \phi_x + \phi u_{tx}, \\
C^t &= \phi (D_t^{1-a}u - u_{xy} - u_{zz} - u_{ww}) + W \phi_y, \\
C^u &= W \phi_y + \phi u_{wz}, \\
W &= -u_w.
\end{align*}
\]

The components of the conserved vectors for $\Gamma_{65}$ are

\[
\begin{align*}
C^r &= \phi D_t^{1-a}(W) + J(W, \phi), \\
C^s &= -y \phi (D_t^{1-a}u - u_{xy} - u_{zz} - u_{ww}) + W \phi_y + \phi (uy + u_y), \\
C^t &= x \phi (D_t^{1-a}u - u_{xy} - u_{zz} - u_{ww}) + W \phi_x + \phi (ux + u_x), \\
C^u &= W \phi_x + \phi (uy + u_y), \\
W &= uy - xu_y.
\end{align*}
\]
The components of the conserved vectors for $\Gamma_{611}$ are

\[
C' = 2\phi[D_{1}^{-\alpha}u - u_{\partial z} - u_{\partial z} - u_{\partial w}] + \phi D_{1}^{-\alpha}(W) + J(W, \phi),
\]

\[
C^\omega = \alpha\phi[D_{1}^{-\alpha}u - u_{\partial z} - u_{\partial w}] + W \phi_{z} + \phi(u_{x} + \alpha u_{x} + \alpha u_{x} + \alpha u_{x} + 2tu_{x}),
\]

\[
C = \alpha\phi[D_{1}^{-\alpha}u - u_{\partial z} - u_{\partial w}] + W \phi_{z} + \phi(u_{x} + \alpha u_{x} + \alpha u_{x} + \alpha u_{x} + 2tu_{x}),
\]

\[
C^\omega = \alpha\phi[D_{1}^{-\alpha}u - u_{\partial z} - u_{\partial w}] + W \phi_{z} + \phi(u_{x} + \alpha u_{x} + \alpha u_{x} + \alpha u_{x} + 2tu_{x}),
\]

\[
W = u(\alpha - 1) - \alpha u_{x} - \alpha u_{x} - \alpha u_{x} - \alpha u_{x} - 2u_{x}.
\]

For $\Gamma_{612}$ the components of the conserved vectors are

\[
C' = \phi D_{1}^{-\alpha}(W) + J(W, \phi),
\]

\[
C' = W \phi_{x} - \phi u_{x},
\]

\[
C = W \phi_{x} - \phi u_{x},
\]

\[
C' = W \phi_{x} - \phi u_{x},
\]

\[
W = u.
\]

For $\Gamma_{613}$ the components of the conserved vectors are

\[
C' = \phi D_{1}^{-\alpha}(W) + J(W, \phi),
\]

\[
C' = W \phi_{x} - \phi u_{x},
\]

\[
C = W \phi_{x} - \phi u_{x},
\]

\[
C = W \phi_{x} - \phi u_{x},
\]

\[
W = F.
\]

9.4. The conservation laws for the $N$-dimensional time-fractional heat equation

The conservation laws for each of the symmetry $\Gamma_{ij}$ $j = 1, 2, \ldots, 5$ can be summarised as:

\[
C' = \xi L + D_{1}^{-\alpha}(W)\phi(t, x, y, z, w) + J(W, \phi),
\]

\[
C^\omega = \xi L + W \left(\frac{\partial L}{\partial u_{x}} - D_{1} \frac{\partial L}{\partial u_{x}}\right) + D_{1}(W) \frac{\partial L}{\partial u_{x}}, \quad i = 1, 2, \ldots, n.
\]

The terms $L$, $W$, $\xi$ and $\xi$ for various integer values of $i$, are already defined in the preliminaries.

10. CONCLUDING REMARKS AND DISCUSSION

We have investigated time-fractional heat equation using Lie symmetries and obtaining a classification of these symmetries.

We note that for all dimensions, when we consider the case $0 < \alpha < 1$, we lose the translational symmetry, $\phi_{x}$, as explained in Gazzizov et al. (2009).

The number of symmetries is reduced significantly and the differences in Lie Algebras for the fractional- and integral-order PDEs can be attributed to this fact. The significance of the reduction in conservation laws for the fractional form of the nonfractional case can be a matter of further research.

According to Myeni and Leach (2009) in the case of linear ODEs the number of solution symmetries is equal to the order of the equation. From this paper we see that for integer-order linear PDEs the number of solution symmetries is equal to the product of the order and space dimension, whereas for the fractional PDEs it is half of the product of the order and space dimension.

We have generalised the number of symmetries we can find for an $n$-dimensional time-fractional heat equation. For the case of $\alpha = 1$, the number of symmetries for the $n$-dimensional case is,

\[
\frac{1}{2}(n^{2} + 3n + 10).
\]

In the case of $0 < \alpha < 1$, the number of symmetries is,

\[
\frac{1}{2}(n^{2} + n + 6).
\]

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