The Zel’dovich effect in harmonically trapped, ultracold quantum gases

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Abstract
We investigate the Zel’dovich effect in the context of ultracold, harmonically trapped quantum gases. We suggest that currently available experimental techniques in cold-atom research offer an exciting opportunity for a direct observation of the Zel’dovich effect without the difficulties imposed by conventional condensed matter and nuclear physics studies. We also demonstrate an interesting scaling symmetry in the level rearrangements which has heretofore gone unnoticed.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The Zel’dovich effect (ZE) [1] occurs in any quantum two-body system for which the constituent particles are under the influence of a long-range attractive potential, supplemented by a short-range attractive two-body interaction, which dominates at short distances. The system first found to exhibit the ZE consists of an electron experiencing an attractive long-range Coulomb potential, which at short distances is modified by a short-range interaction [1].

The characterizing feature of the ZE in this scenario is that as the strength of the attractive two-body interaction reaches a critical value (i.e. when a two-body bound state is supported in the short-range potential alone), the S-wave spectrum of the distorted Coulomb problem evolves such that the ground state 1S level plunges down to large negative energies, while simultaneously, the first radially excited 2S state rapidly falls to fill in the ‘hole’ left by the ground state level. This also occurs for higher levels in which, generally, the \((n+1)S\) level replaces the \(nS\) level. This so-called level rearrangement is the signature of the ZE and continues as the strength of the two-body interaction is further increased to support additional low-energy scattering resonances (see, e.g., figure 2 of [1]).
Recently, Combescure et al have revisited the ZE in the context of ‘exotic atoms’ [2, 3], where a negatively charged hadron replaces the electron, and the short-range interaction is provided by the strong nuclear force. However, tuning the short-range interaction in exotic atoms implies that one must be able to adjust the nuclear force in the laboratory, which is a formidable task. Indeed, while it is theoretically easy to adjust the strength of the short-range interaction between the particles in any of the systems above, the experimental reality is very different. As a result, a direct experimental observation of the ZE has been lacking, despite suggestions for its observation in quantum dots [2], Rydberg atoms [4] and atoms in strong magnetic fields [5].

In this paper, we explore the possibility for a direct observation of the ZE in harmonically trapped, charge neutral, ultracold atomic gases. The charge neutrality of the atoms ensures that the long-range attractive potential is provided solely by the isotropic harmonic oscillator trap (in Zel’dovich’s original work, the long-range attractive interaction was facilitated by a modified Coulomb interaction [1]), while the short-range two-body interaction is naturally present owing to the two-body short-range s-wave scattering, which is known to dominate at ultracold temperatures. Moreover, the short-range interaction between the atoms is completely tuneable in the laboratory via the Feshbach resonance. The multi-channel Feshbach resonance can be treated in a simpler single-channel model by a finite-range, attractive two-body interaction, supporting scattering resonances. Thus, ultracold atoms, at least in principle, provide all of the necessary ingredients for the experimental observation of the ZE.

The plan for the remainder of this paper is as follows. In section 2, we establish a deep connection between the level rearrangements and the two-body energy spectrum as characterized by the s-wave scattering length \( a \). This connection allows us to make contact with recent experimental results on ultracold two-body systems [7], from which we suggest that a direct observation of the ZE is possible. Then, in section 3, we investigate the influence of the range of the two-body interaction on the level rearrangements examined in section 2. In particular, we reveal an interesting scaling symmetry of the two-body energy spectrum which has not been noted before. In section 4, we present our concluding remarks.

### 2. Zel’dovich effect in ultracold atoms

#### 2.1. Universal two-body energy spectrum

The two-body spectrum for a pair of harmonically trapped ultracold atoms is obtained from the following Hamiltonian:

\[
H = \frac{p_1^2}{2M} + \frac{p_2^2}{2M} + \frac{1}{2} M \omega^2 r_1^2 + \frac{1}{2} M \omega^2 r_2^2 + V_{SR}(|r_1 - r_2|),
\]

where each atom has a mass of \( M \) and \( V_{SR}(|r_1 - r_2|) \) is a short-range potential. Introducing the usual relative, \( r = r_1 - r_2 \), and centre-of-mass, \( R = (r_1 + r_2)/2 \), coordinates, and noting that the centre-of-mass motion may be separated out, the associated Schrödinger equation in the s-wave channel reads [8]

\[
-\frac{\hbar^2}{M} u''(r) + \frac{1}{4} M \omega^2 r^2 u(r) + V_{SR}(r) u(r) + \frac{\hbar^2}{M} (d - 1)(d - 3) 4r^2 u(r) = E u(r),
\]

where \( u(r) = r^{(d-1)/2}\psi(r) \) is the reduced radial two-body wavefunction, the primes denote derivatives, \( d \) is the dimension of the space and \( E \) is the relative energy of the two-body interaction.

\[\text{[1] The reader will find a clear description of the underlying physics of Feshbach resonance in [6].}\]
defining the dimensionless variables $\eta = \frac{2E}{\hbar \omega}$, $\ell_{osc} = \sqrt{\hbar/L\omega}$ and $x = r/\sqrt{2}\ell_{osc}$, equation (2) may be written as

$$-u''(x) + x^2u(x) + \bar{V}_{SR}(x)u(x) + \frac{(d-1)(d-3)}{4x^2}u(x) - \eta u(x) = 0,$$

where $\bar{V}_{SR}(x) = 2V_{SR}/\hbar \omega$.

Exact analytical solutions to (3) exist $\forall d$ if the potential is taken to be an appropriately regularized zero-range contact interaction. For $d = 3$, the spectrum is described by [9, 10]

$$\ell_{osc} = \frac{\Gamma(1/4 - E/(2\hbar \omega))}{\sqrt{2}\Gamma(3/4 - E/(2\hbar \omega))};$$

for $d = 1$, we have [8]

$$\frac{\ell_{osc}}{a} = \frac{\sqrt{2}\Gamma(3/4 - E/(2\hbar \omega))}{\Gamma(1/4 - E/(2\hbar \omega))};$$

and for $d = 2$ [8],

$$\tilde{\psi}(1/2 - E/(2\hbar \omega)) = \ln \frac{\ell_{osc}^2}{2a} + 2 \ln 2 - 2\gamma.$$

In the above, $a$ is the s-wave scattering length in $V_{SR}$ alone, $\Gamma(\cdot)$ is the gamma function, $\tilde{\psi}(\cdot)$ is the digamma function and $\gamma = 0.577215665\ldots$ is the Euler constant [11]. Note that in any dimension, the two-body spectrum is universal in the sense that the relative energy, $E$, is determined entirely by the scattering length. Thus, even for a two-body potential with finite range, it has been shown that provided $b \ll \ell_{osc}$ (practically speaking, $b/\ell_{osc} \lesssim 0.01$), the same two-body energy spectrum as described above will be obtained for an arbitrary two-body interaction evaluated at the same scattering length [8].

2.2. Level rearrangements

In this section, the level rearrangements (i.e. the ZE) exhibited by the two-body energy spectrum are investigated. The results presented here are strictly for three dimensions (3D), although analogous findings are also observed in other dimensions. We will focus on three different interaction potentials, namely a finite square well (FSW), the modified Poshl–Teller potential [12] and an exponential potential [13],

$$V_{SR}(r) =\begin{cases} 
-V_0\Theta(b - r) \\
-V_0\text{sech}^2(r/b) \\
-V_0\exp(-r/b),
\end{cases}$$

respectively. In the above, $\Theta(\cdot)$ is the Heaviside step function and $V_0$ is the depth of the potential. The potentials listed in equation (7) are certainly not meant to be representative of the actual short-range two-body interactions between the atoms, but rather will be used to illustrate the universality in the two-body energy spectrum, which is independent of the details of the potentials in the low-energy limit. The 3D s-wave scattering lengths for the potentials are given by (in the same order as the potentials listed above) [10, 12–16]

$$a = \begin{cases} 
b\left(1 - \frac{\tan \sqrt{g}}{\sqrt{g}}\right) \\
b\left(\gamma + \tilde{\psi}(\lambda) + \frac{\pi}{2} \cot \pi \lambda/2\right) \\
b\left(2\gamma + \ln g - \frac{\pi V_0(2\sqrt{g})}{J_0(2\sqrt{g})}\right),
\end{cases}$$

3
Figure 1. Left panel: the s-wave two-body energy spectrum versus strength for all three model potentials with fixed $b/\ell_{osc} = 0.01$. The open circles, triangles and squares represent the numerical integration of equation (3) for the FSW, Poshl–Teller and exponential potentials, respectively. The vertical dashed line represents the critical strength, $g_c$, of all three potentials, while the horizontal dashed lines represent the energy values at $a = \pm \infty$. The strength axis is scaled so that the critical strength values lie along the same vertical dashed line. Right panel: energy versus scattering length for all three model potentials. The same symbols as the plots in the left panel are used. The vertical dot-dashed line indicates $a = 0$. In all four plots, the solid black line represents the exact expression obtained from equation (4). Units are scaled as discussed in the text.

where $g \equiv M V b^2 / \hbar^2$ is the dimensionless strength of the potential, $\lambda \equiv (1 - \sqrt{1 + 4 g^2})/2$, and $J_0(\cdot)$ and $Y_0(\cdot)$ are the zeroth-order Bessel functions of the first and second kind, respectively [11].

We proceed by numerically integrating equation (3) for each of the three potentials. For our numerics, we have set $\hbar = \omega = 1$ and $M = 2$ to be consistent with the numerical results in [9, 10]. We plot our numerical results (open symbols) for the relative energy, $E$ (in units of $\hbar \omega$), as characterized by both the strength, $g$, and the s-wave scattering length, $a$, in figure 1. The left panel illustrates the level rearrangements as the strength, $g$, is increased beyond the first scattering resonance, whereas the right panel illustrates the relative energy, $E$, as determined by the s-wave scattering length. The level rearrangements shown in the left panels illustrate the 2S level replacing the 1S level at the first scattering resonance, while the 1S level dives down to large negative values.

A further examination of figure 1 reveals that while $b/\ell_{osc} = 0.01$ for all three potentials, the level rearrangements displayed in the left panels exhibit noticeable differences. In particular, we see that the FSW has a much sharper drop at $g = g_c$ than the Poshl–Teller
or exponential potentials. These level repulsions, or ‘anticrossings’, are known to be as a result of the levels belonging to the same SO(2) symmetry of the Hamiltonian, while the mixing of the levels is dependent on how rapidly the short-range potential ‘shuts off’ [2].

The underlying message here is as follows. While all three plots on the left of figure 1 display the ZE, namely they all undergo level rearrangement at some value of the strength parameter \( g \), all three different potentials map onto the same \( E \) versus \( a \) curve, as illustrated in the right panel of figure 1. This reaffirms that while the details of the ZE are sensitive to the form of the two-body interaction (i.e. whether it is a finite-range, or as is usually taken in the literature, a zero-range pseudopotential [9]), the energy dependence on the scattering length, \( a \), is indeed universal. It is also worth pointing out that the solid curves in the left panels of figure 1 are obtained from substituting the expressions for the scattering length, equation (8), into equation (4), which is exact only for a zero-range interaction. However, it is clear that the numerically obtained open symbols closely follow the solid curve derived from equation (4). Thus, for \( b/\ell_{\text{osc}} \ll 1 \), the level rearrangements in harmonically trapped two-body systems interacting via a finite, short-range potential, are all equivalent to a zero-range interaction. Viewed another way, given a set of data for \( E \) versus \( a \), there must exist some quantum two-body system (i.e. the two-body potential need not be known explicitly) whose \( E \) versus \( g \) dependence exhibits the ZE. This observation has some interesting implications, which we further explore in the following subsection.

2.3. Flow of the spectrum

In order to make the connection between the \( E \) versus \( g \) and \( E \) versus \( a \) curves more apparent, we now study the ‘flow’ of the two-body energy spectrum. Although we focus our attention on the FSW, the same analysis holds for any other potential.

In the left panel of figure 2, we note that as \( g \) is increased from zero, the energy only slightly varies from the unperturbed energy, until the critical strength, \( g_c \), is reached at which point the ZE occurs. In the right panel of figure 2, the same flow is illustrated, but this time in terms of the scattering length. The lower flow in the left panel (red) illustrates that the trajectory of the ground state \( A \rightarrow B \rightarrow C \rightarrow D \) is continuous through the resonance at

**Figure 2.** Flow of the spectrum for the FSW. The boxed and double arrows (green) represent the flow of the first excited state, while the circles and single arrows (red) follow the flow of the ground state. Identical points in each panel are labelled by the same letter. For example, the point \( B \) on the left is the *exact same* data point as \( B \) on the right but subject to the transformation in equation (8). The horizontal dotted lines in both panels correspond to the asymptotic values for the energy \( E \) at \( |a| = \infty \), as in figure 1. See also the discussion preceding equation (19) in section 3.
Figure 3. The $E$ versus $a$ spectrum being rolled onto a cylinder. Top (from left to right): $a = -10$ to $a = 10$, $a = -30$ to $a = 30$, $a = -35$ to $a = 35$. Bottom (from left to right): $a = -40$ to $a = 40$, $a = -90$ to $a = 90$, $a = -\infty$ to $a = \infty$. The thick vertical line (red) represents $a = 0$. The symmetry axis of the cylinder is the $E$ axis, while the azimuthal angle is connected to the scattering length $a$.

$g = g_c$. However, as we follow the same path in $E$ versus $a$, the point $B$ flows out to $a \rightarrow -\infty$, while $C$ and $D$ flow in from $a \rightarrow +\infty$ and then to $a \rightarrow 0$. Thus, while the flow for the energy spectrum in $g$-space is continuous, the flow in $a$-space appears to be disconnected, similarly for the first excited state (green) where the $B'$ and $C'$ flow is continuous in $g$-space, but rapidly branches off to $a \rightarrow -\infty$ and $a \rightarrow 0$, in $a$-space, respectively. The continuous flow in $g$-space suggests that the $a$-space spectrum is more appropriately viewed on the topology of a cylinder, where $a = \pm \infty$ may be identified.

2.3.1. Cylindrical mapping. The observations made above suggest that we map the $E$ versus $a$ spectrum onto the surface of a cylinder. The details of this mapping are closely related to the mapping of the real line (in our case, the scattering length) onto the unit circle, $S^1$, followed by constructing the Cartesian product, $\mathbb{R} \times S^1$, with $\mathbb{R}$ identified with the energy $E$. The essential point of this mapping is to provide a more natural interpretation for the two-body $E$ versus $a$ spectrum.

To this end, figure 3 illustrates a series of ‘snapshots’ which show how the original $E$ versus $a$ spectrum is mapped onto the surface of a cylinder. Each of the six panels in figure 3 should be viewed as an intermediate step in taking the $E$ versus $a$ spectrum and rolling it onto a cylinder. In figure 4, we present the complete mapping of the $E$ versus $a$ spectrum up to the fourth excited state of the bare harmonic trap. This ‘Zel’dovich spiral’ (ZS) may now be explicitly connected to the level rearrangements discussed in the left panel of figure 1.

Indeed, we observe that the flow of the spectrum shown in the left panels of figure 2 corresponds to clockwise (CW) rotations about the ZS. That is, increasing the strength of the two-body interaction corresponds to moving along the ZS in a CW direction, with the starting
Figure 4. The complete mapping of the $E$ versus $a$ spectrum onto the surface of a cylinder illustrating the ‘ZS’. The solid vertical line (red) identifies the unperturbed system for which $a = 0$. Every $2\pi$ winding along the spiral corresponds to a complete level rearrangement, e.g., $2S \rightarrow 1S$ after $2\pi$ rotations. The lower solid circle indicates the unperturbed $1S$ level, whereas the upper solid circle corresponds to the $2S$ level.

point (i.e. the front of the cylinder) along the thick vertical line (red) in figure 4. A CW rotation of $\pi$ puts us on the back of the cylinder, or $a = -\infty$, whereas a counter-CW rotation of $\pi$ takes us to $a = +\infty$ (i.e. the azimuthal angle $|\phi| = \pi$ is a branch point).

To see how the ZS naturally contains the level rearrangements, let us first begin at $E = 3/2$, which in figure 4 is represented by the lower solid circle along the vertical line. As we move in a CW rotation along the spiral, $E = 3/2$ ($a = 0$, $\phi = 0$) goes to $E \rightarrow 1/2^+$ for large negative values of $a$, and finally to $E = 1/2$ at $a = -\infty$ (lower dotted curve in the right panel of figure 2). A further infinitesimal rotation takes us to $E = 1/2^-$ at large positive values of $a$, and finally to $E \rightarrow -\infty$ at $a = 0$ after a full $2\pi$ rotation; we have just followed the flow of the $1S$ level in the left panel of figure 1, namely $A \rightarrow B \rightarrow C \rightarrow D \rightarrow \ldots$. Similarly, the upper solid circle in figure 4 corresponds to $E = 7/2$, which as we rotate CW evolves to $E \rightarrow 5/2^+$ for large negative values of $a$, $E = 5/2$ at $|a| = \infty$ (upper dotted curve in the right panel of figure 2), and subsequently to $E = 3/2$ at $\phi = 2\pi$; this description is precisely the flow of the $2S$ level in the left panel of figure 2. If we were to the continue with our CW rotation (i.e. continue increasing the strength $g$), we would then evolve from $E = 3/2 \rightarrow 1/2$ at $\phi = 3\pi$ followed by $E \rightarrow -\infty$ at $\phi = 4\pi$.

In our opinion, viewing level rearrangements in this way is more natural than the original $E$ versus $a$ spectrum in $\mathbb{R}^2$. We see that critical strengths, $g_\ast$, correspond to CW rotations of odd multiples of $\pi$, whereas a complete level rearrangement occurs for even multiples of $\pi$. In general, the $(n+1)S$ level, with $E_{n+1} = 2n + 3/2$ ($n = 0, 1, 2, \ldots$), will eventually evolve to the $1S$ level after $2n\pi$ CW rotations along the spiral.
The ZS also helps to clarify several misconceptions about the \( E \) versus \( a \) spectrum in the literature. The spectrum is typically understood by taking \( a = \pm \infty \) separately, and assigning different interpretations to \( a \to 0^+ \) and \( a \to 0^- \). An example of this is a recent contribution by Shea et al [10] who describe the spectrum by first ‘starting from the far left’ and making the interaction weaker and weaker as \( a \to 0^- \) and then independently ‘starting from the right’ and making the interaction stronger and stronger as \( a \to 0^+ \). On the ZS, nothing is ambiguous, since one always moves in a CW rotation along the spiral, corresponding to increasing the strength, \( g \), of the interaction; a complete level rearrangement occurs after we undergo an even multiple of \( \pi \) CW rotations. Furthermore, the ‘counter-intuitive’ properties of the \( E \) versus \( a \) spectrum discussed in [9] are now seen to be nothing more than a manifestation of the onset of the ZE. We find it rather surprising that the ZE has been present in the two-body \( E \) versus \( a \) spectrum all along, but until now has gone unnoticed.

2.3.2. Experimental observations. In a recent work, Stöferle et al [7] have experimentally measured the binding energy as a function of the s-wave scattering length between two interacting particles in a harmonic trap. This experiment highlights the versatility of trapped, ultracold atomic systems, in which an analytically solvable model, once only the purview of theoretical physics, has now been realized in the laboratory. Remarkably, the experimental results for the \( E \) versus \( a \) spectrum are in excellent agreement with theory (see figure 2 in [7]), even though the two-body interaction in the experiments is most certainly not a zero-range interaction. Thus, the theoretical prediction that the \( E \) versus \( a \) spectrum is universal has been confirmed experimentally.

What has not been appreciated until now, however, is that the experimental \( E \) versus \( a \) spectrum obtained in [7] is exactly equivalent to obtaining the ground state branch in the left panel of figure 2 (single arrows, red). In other words, the work of Stöferle et al has already been a direct experimental observation of the ground state branch of the two-body system exhibiting the ZE. We therefore suggest that further experiments along the lines of [7] be performed so that data corresponding to the double arrows and primed letters (green branch) in the right panel of figure 2 may be obtained. If such an extension to the experiments in [7] is viable, then according to our analysis, these data would be exactly equivalent to the 2S branch (double arrows, green, in the left panel of figure 2) undergoing the ZE. Therefore, just a few additional data points in the \( E \) versus \( a \) spectrum would provide for a direct experimental confirmation of the ZE for two interacting particles confined in a harmonic trap.

3. Level rearrangements in the zero-range limit

We conclude this work with a discussion of an interesting scaling symmetry present in the level rearrangements. Specifically, we show that in the \( b \to 0 \) limit, the entire two-body \( E \) versus \( g \) spectrum is determined by only the first-level rearrangement. Figure 5 illustrates the spectrum through the first three low-energy scattering resonances, namely \( g = g_0, g_1, g_2 \) for the FSW. From this figure, we immediately note the similarities between the level rearrangements as we move from one region to the next, along a fixed value for the energy \( E \). Each region begins at \( a = 0 \), undergoes a level rearrangement and then returns to \( a = 0 \). In the language of the ZS, each region corresponds to one complete \( 2\pi \) CW rotation along the spiral. Figure 5 suggests that it may be possible to map every \( g_n \) (\( n \neq 0 \)) region onto \( g_0 \) by some appropriate scaling of the \( g \)-axis. This mapping should ensure that \( a = 0 \) in, say, \( g_1 \), matches with \( a = 0 \) in \( g_0 \), and that the critical \( g_1 \) value in the first region overlaps with the critical \( g_0 \) value in the zeroth region, and so on.
Figure 5. Several level rearrangements for a FSW interaction with $b/\ell_{\text{osc}} = 0.01$. The dotted horizontal lines (green) represent unperturbed energy values, while the dotted vertical lines (red) represent values at which $a = 0$. Each subsequent region corresponds to a CW rotation of $2\pi$ along the ZS. The solid black circle represents a representative data point which we wish to map back onto the $g_0$ region, as schematically illustrated by the open circle in $g_0$. The single arrows on the $g$-axis indicate the various critical values, $g_{c,n}$, in the $n$th region, while the double arrows indicate $g_n^{(n)}$ for which the scattering length vanishes.

Let us first define some useful nomenclature. We define $g_n^{(n)}$ as the $n$th value of $g$ for which $a = 0$ (double arrows in figure 5). Next, $g_{c,n}$ is defined as the $n$th critical $g$ value; that is, the $g$ value at which the $n$th level rearrangement occurs (single arrows in figure 5). Last, $\tilde{g}$ is the value of $g$ outside the region $g_0$ which we intend to map back onto $g_0$. For example, consider the point labelled by the solid dot in the $g_2$ region of figure 5. Considering this point, which we wish to map back onto $g_0$ (represented by the open circle in figure 5), we have $\tilde{g} = 75$, $g_{c,2} = 25\pi^2/4$ and $g_0^{(2)} = 59.67951594410\ldots$. The mapping that takes this $\tilde{g}$ back onto $g_0$ is

$$g = \frac{(\tilde{g} - g_0^{(2)}) g_{c,0}}{g_{c,2} - g_0^{(2)}} \approx 18.84894603\ldots,$$

where $g_{c,0} = \pi^2/4$ is the zeroth critical $g$ value. We may generalize this example to any region by employing the following prescription:

$$g = \frac{g_{c,n} - g_n^{(n)}}{g_{c,0} - g_0^{(0)}} (n \neq 0).$$

With this remapped value of $g$, we also have the associated energy $E$. If the mapping is indeed exact, the energy $E$ of the remapped point should be identical to the energy for the same $g$ value in $g_0$.

In figure 6, we study this mapping for the FSW with $b/\ell_{\text{osc}} = 0.01$. At first glance, the left panel figure 6 appears to be showing that the mapping is exact, but a closer examination of the spectrum for values of $g$ near the resonance (right panel in figure 6) reveals noticeable discrepancies between data in the $g_0$ and $g_n > g_0$ regions. Remarkably, even for $b/\ell_{\text{osc}} = 0.01$, the mapping of the data from $g_1$ and $g_2$ onto $g_0$ agrees almost perfectly (i.e. the dashed (red)
Figure 6. Left: equation (10) as applied to data from the FSW to different regions with \( b/\ell_{\text{osc}} = 0.01 \). In both panels, the solid (blue), dashed (red) and dot-dashed (green) curves correspond to the \( g_0, g_1 \) and \( g_2 \) regions, respectively. In these figures, the remapped data from \( g_1 \) and \( g_2 \) are indistinguishable on the scale of the plots.

and the dot-dashed (green) curves, respectively). Regardless, the mapping given by equation (10) is not exact for any finite range, \( b \).

We can, however, show that this mapping becomes exact in the \( b \to 0 \) limit by considering equation (4). We write this expression in the notationally convenient form

\[
\tilde{a}(g) = R(E) = \frac{\sqrt{2/\Gamma(1/4-E/2\hbar \omega)}}{\Gamma(3/4-E/2\hbar \omega)}.
\]

Our goal is now to map \( g \) values in two different regions, \( g_k \) and \( g_k' \), onto a value in the region \( g_0 \) and investigate the difference in their energy values. We denote \( \delta E = E_k - E_k' \) and note that we have two expressions for the spectrum \( \bar{a}(g_k) = R(E_k) \) and \( \bar{a}(g_k') = R(E_k') \). The difference between these two expressions is

\[
\bar{a}(g_k') - \bar{a}(g_k) = R(E_k') - R(E_k) = R(\delta E + E_k) - R(E_k).
\]

Taylor expanding up to first order in \( \delta E \) gives

\[
\delta E = \frac{\delta a}{R'(E_k)}.
\]

where \( \delta a = \bar{a}(g_k') - \bar{a}(g_k) \) and \( R'(E_k) = \frac{\partial R(E_k)}{\partial E_k} \). From equation (4), we may re-express equation (11) as

\[
\delta E = \frac{\delta a}{\bar{a}(g_k)} \frac{dE_k}{dg_k},
\]

which upon noting that \( E_k = R^{-1}(a(g_k)) \) becomes the implicit expression

\[
R(a(g_k)) = R\left( \frac{\delta a}{\delta E} \right).
\]

Assuming \( \delta E \) to be small compared to \( \delta a \), we seek an asymptotic expression for

\[
R\left( \frac{\delta a}{\delta E} \right) = \frac{\sqrt{2/\Gamma(1/4 - \frac{\delta a}{2\pi})}}{\Gamma(3/4 - \frac{\delta a}{2\pi})}.
\]

An application of Euler’s reflection formula [1] and Stirling’s approximation to the above gives the approximate expression

\[
R\left( \frac{\delta a}{\delta E} \right) \simeq 2 \sqrt{\frac{\delta E}{\delta a}} \tan \left( \frac{\pi \delta a}{2\delta E} - \frac{3\pi}{4} \right).
\]
Equation (15) along with equation (13) gives

\[
\frac{\delta a}{\delta E} \simeq \frac{2}{\pi} \cot^{-1} \left( 2 \sqrt{\frac{\delta E}{\delta a \ R'(a(g_k))}} \right) + \frac{3}{2}.
\]

(16)

With the approximation \(\cot^{-1}(x) \simeq \pi/2, x \ll 1\), the difference in the energies becomes

\[
\delta E = \frac{2}{5} \delta a,
\]

(17)

or, defining \(a(g) = bA(g)\),

\[
\delta E = \frac{4b}{5 \ell_{osc}} (A(g_{k'}) - A(g_k)).
\]

(18)

The above equation analytically shows that \(\delta E \to 0\) as \(b \to 0\), and the \(g\) values in the two different regions get mapped back onto the \(g_0\) region at the exact same energy. Therefore, the mapping of all subsequent regions, \(g_n (n \neq 0)\) back onto \(g_0\), is exact in the zero-range limit. It is important to note that our analysis has not relied upon specifying the details of interaction, and so is equally valid for any short-range two-body interaction supporting bound states.

It is also instructive to consider how the shape of the level rearrangement curves evolves as \(b \to 0\). The shape dependence of the curves can be established by expanding the right-hand side of equation (4) about \(g = g_c\) (some resonant strength value) and the left-hand side about \(E = E_c\) where, in 3D, \(E_c \equiv 1/2, 5/2, 9/2, \ldots\) (i.e. the energies at the back of the ZS). The result is

\[
\frac{c_L b}{g/g_c - 1} = \frac{c_k \ell_{osc}}{1 - E/E_c}.
\]

(19)

where \(c_L\) and \(c_k\) are constants unimportant to our overall discussion. Choosing two points equally spaced away from \(g_c\), call these \(g_1 = g_c - \Delta g, g_2 = g_c + \Delta g\), and their corresponding energies \(E_1 = E_c + \Delta E, E_2 = E_c - \Delta E\), we may use two versions of the approximation in equation (19) to write

\[
\frac{\Delta g}{g_c} = \frac{b}{\ell_{osc}} c_k \frac{\Delta E}{E_c}.
\]

(20)

The important point to take away from this analysis is that the width of the rearrangement region, i.e. the range in \(g\) over which the rearrangement occurs, is \(\Delta g \sim b/\ell_{osc}\). An analogous result to equation (20) is briefly discussed in [4] in the context of exotic atoms. There, the width is stated to be \(\sim b/a_B\), where \(a_B\) is the Bohr radius. We see that our equation (20) is consistent with the result for exotic atoms, in that the width of the rearrangement region is of the order of the range of the potential over the characteristic length of the problem.

In figure 7, we numerically verify our analytical expression, namely equation (20), by plotting the lowest two branches of the FSW for decreasing values of the range, \(b\), of the potential. It is evident that as \(b \to 0\), the level rearrangement curves evolve to a series of staircase functions, which is entirely expected given the collective results of equations (18) and (20). This staircase property of the spectrum has also been discussed in [4, 17] in the context of the quantum defect of atomic physics, but using an entirely different approach to the one presented here. Note that in the inset to figure 7, all four curves intersect at a common point, namely at \(g = g_c\), which corresponds to the back (i.e. \(|\phi| = \pi\)) of the ZS.

There are two noteworthy points to be taken from this staircase-like behaviour. The first is that any other panel of the spectrum, e.g. \(g_1, g_2\) etc in figure 5 (provided \(b/\ell_{osc} \ll 1\), can be obtained by simply applying the scaling transformation, equation (10), to the data in the \(g_0\) region. In addition, the staircase property of the level rearrangements as \(b \to 0\) is not specific to the FSW, which implies that for any short-range two-body potential, the \(E\) versus \(g\) curves...
Figure 7. Level rearrangement spectrum for several different values of the range, $b$, for the FSW. The solid line (orange) represents $b/\ell_{osc} = 0.01$. The dashed line (black) represents $b/\ell_{osc} = 0.001$, the dot–dot–dashed line (red) $b/\ell_{osc} = 0.0001$ and the dot-dashed line (green) is $b/\ell_{osc} = 0.00001$. Inset: a magnification of the data in the main figure near $g_{c,0}$ further illustrating how the level rearrangements evolve to the staircase profile as $b \to 0$.

will exhibit the same scaling symmetry provided the critical values, $g_{c,n}$ are properly scaled as $b \to 0$. It is also important to realize that even the staircase level rearrangements are mapped onto the universal $E$ versus $a$ spectrum, just as with the other potentials listed in equation (7) with $b/\ell_{osc} \ll 1$ in the right panel of figure 1.

4. Conclusions

In this paper, we have examined the two-body problem of ultracold harmonically trapped interacting atoms and its relation to the Zel’dovich effect. We have shown, through our construction of the ‘Zel’dovich spiral’, that the universal spectrum in terms of the scattering length is exactly equivalent to the Zel’dovich effect. This non-trivial observation has been used to motivate further experimental studies in order to provide additional data for the $E$ versus $a$ spectrum, which may then be used to establish the first direct experimental observation of the Zel’dovich effect. Finally, we have shown that in the $b \to 0$ limit, the level rearrangement spectrum exhibits an exact scaling symmetry, which has until now, gone unnoticed. The exact mapping means that the entire $E$ versus $g$ spectrum (and therefore the $E$ versus $a$ spectrum) may be obtained solely from knowledge of the $g_0$ region as $b \to 0$.

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