Darboux transformation: new identities

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Abstract

This letter reports some new identities for multisoliton potentials that are based on the explicit representation provided by the Darboux matrix. These identities can be used to compute the complex gradient of the energy content of the tail of the profile with respect to the discrete eigenvalues and the norming constants. The associated derivatives are well defined in the framework of the so-called Wirtinger calculus which can aid a complex variable based optimization procedure in order to generate multisolitonic signals with desired effective temporal and spectral width.

Keywords: darboux transformation, multisolitons, inverse scattering transforms

(Some figures may appear in colour only in the online journal)

1. Introduction

This letter deals with the Darboux representation of multisoliton solutions of the nonlinear Schrödinger equation. As carriers of information, these multisolitonic signals offer a promising solution to the problem of nonlinear signal distortions in fiber optic channels [1, 2]. In any nonlinear Fourier transform (NFT) based transmission methodology seeking to modulate the discrete spectrum of the multisolitons [3–6], the unbounded support of such signals (as well as its Fourier spectrum) presents some challenges in achieving the best possible spectral efficiency [7, 8]—this forms part of the motivation for this work.

The Darboux transformation has proven to be an extremely powerful tool in handling the discrete part of the nonlinear Fourier spectrum. The rational structure of the associated Darboux matrix was recently exploited to obtain fast inverse NFT algorithms in [9–11]. In the particular case of \textit{K}-soliton solutions, the rational structure of the Darboux matrix facilitates the exact solution of the Zakharov–Shabat problem [12, 13] for a doubly-truncated version of the signal via the solution of an associated Riemann-Hilbert problem [14]. In [10, 14], an exact method for computing the energy content of the ‘tails’ of \textit{K}-soliton solutions was reported which was again based on the Darboux transformation. This method was further used in [15] to establish the sufficient conditions for either one-sided or compact support of the signals resulting from ‘addition’ of boundstates. Given that the complexity of computing the Darboux matrix coefficients is \(O(K^2)\), these methods turn out to be extremely efficient eliminating any need for heuristic approaches based on the asymptotic expansions (with respect to the windowing parameter, say, \(\tau\)). The present work, therefore, tries to further reinforce the idea that the Darboux representation can potentially facilitate a number of design and signal processing aspects of \textit{K}-soliton solutions. For the general case, when the reflection coefficient is bandlimited, the work presented in [16] may allow us to compute the Jost solutions of the seed potential with extremely high accuracy.

The present work is also motivated by the fact that the recent attempts [7, 8, 17] towards optimizing the generated multisolitonic signals are either based on brute-force methods or asymptotic expansions with respect to the windowing parameter. These methods have serious drawbacks either because they do not scale well in complexity when the number of boundstates are only moderately high or because they are not reliable in the absence of a prior knowledge of the goodness of the approximations made. In this letter, we present some new identities that can potentially make the optimization problem amenable to some of the powerful optimization procedures available in the literature (see [18] and the references therein) at the same time completely circumventing the need for any heuristics. Given that the independent variables (i.e. discrete eigenvalues and the norming constants) in the optimization procedure are complex in nature, the framework based on Wirtinger calculus presented in [18] appears to be more convenient.
The main contributions of this work are presented in section 2 which deals with the temporal width which is followed by a brief discussion of estimation of spectral width in section 3. The letter concludes with some examples in section 4 where calculation can be carried out in a simple manner.

2. Temporal width

The temporal width of a $K$-soliton solution can be defined via the $L^2$-norm of the profile which is also related to the energy of the pulse. Let the energy content of the ‘tails’ of the profile, denoted by $E^{(+)}(t)$, be defined by

$$E^{(+)}(t) = \int_{-\infty}^{\infty} |q(s)|^2 ds,$$

so that $E(\tau) = \int_{-\infty}^{\infty} \left[ E^{(+)}(-\tau) + E^{(+)}(\tau) \right]$ characterizes the total energy in the tails ($|\tau| \geq \tau$). The total energy is given by $|q(s)|^2 = 4\sum_{k} \text{Im} \zeta_k$. Now, if the tolerance for the fraction of total energy in the tails is $\epsilon_{\text{tails}}$, then the effective support, $[-\tau, \tau]_\epsilon$, of the profile must be chosen such that $E(\tau) \leq \epsilon_{\text{tails}} |q|^2$.

For the specific case of $K$-soliton solutions, an exact recipe for computing $E(\tau)$ was reported in [10, 14] which we summarize briefly as follows: let $v(t; \zeta) = (\phi, \psi)$ be the matrix form of the Jost solutions. Then, from the standard theory of scattering transforms [13], it is known that, for $\zeta \in \mathbb{C}_+$,

$$v e^{i\zeta_0} = \begin{pmatrix} 1 + \frac{1}{\zeta_0} & -\frac{i}{2\zeta_0} E^{(+)}(t) \\ -\frac{1}{2i\zeta_0} E^{(+)}(t) & 1 + \frac{1}{\zeta_0} E^{(+)}(t) \end{pmatrix} + O \left( \frac{1}{\zeta_0^2} \right).$$

The $K$-soliton potentials/signals along with their Jost solutions can be computed using the Darboux transformation (DT) which can be implemented as a recursive scheme [10, 14]. Let $\mathcal{S}_K$ be the discrete spectrum of a $K$-soliton potential. The seed solution here corresponds to the null potential; therefore, $v_0(t; \zeta) = e^{-i\zeta_0 \zeta}$. The augmented matrix Jost solution $v_k(t; \zeta)$ can be obtained from the seed solution $v_0(t; \zeta)$ using the Darboux matrix as $v_k(t; \zeta) = \mu_k(\zeta) D_K(t; \zeta, \mathcal{S}_K) v_0(t; \zeta)$ for $\zeta \in \mathbb{C}_+$. Let us define the successive discrete spectra $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots \subset \mathcal{S}_K$ such that $\mathcal{S}_j = \{ \{ \zeta_j, b_j \} \} \cup \mathcal{S}_{j-1}$ for $j = 1, 2, \ldots, K$ where $(\zeta_j, b_j)$ are distinct elements of $\mathcal{S}_j$. The Darboux matrix of degree $K > 1$ can be factorized into Darboux matrices of degree one as

$$D_K(t; \zeta, \mathcal{S}_K | \mathcal{S}_0) = D_1(t; \zeta, \mathcal{S}_K | \mathcal{S}_{K-1}) \times D_1(t; \zeta, \mathcal{S}_{K-1} | \mathcal{S}_{K-2}) \times \ldots \times D_1(t; \zeta, \mathcal{S}_1 | \mathcal{S}_0),$$

where

$$D_1(t; \zeta, \mathcal{S}_j | \mathcal{S}_{j-1}) = \begin{pmatrix} \zeta - \frac{1}{b_j} \beta_{j-1}^2 \zeta_j + \zeta_j^2 & \zeta_0 \zeta_j \beta_{j-1}^2 \zeta_j \beta_{j-1}^2 - \zeta_0 \zeta_j \beta_{j-1}^2 \zeta_j \beta_{j-1}^2 \\ \zeta_0 \zeta_j \beta_{j-1}^2 \zeta_j \beta_{j-1}^2 - \zeta_0 \zeta_j \beta_{j-1}^2 \zeta_j \beta_{j-1}^2 & \zeta - \frac{1}{b_j} \beta_{j-1}^2 \zeta_j + \zeta_j^2 \end{pmatrix}$$

for $j = 1, \ldots, K$ be the successive Darboux matrices of degree one and

$$\beta_{j-1}(t; \zeta_j, b_j) = \frac{\phi_1^{(j-1)}(t; \zeta_j) - b_j \psi_1^{(j-1)}(t; \zeta_j)}{\phi_2^{(j-1)}(t; \zeta_j) - b_j \psi_2^{(j-1)}(t; \zeta_j)}.$$  \hspace{1cm} (5)

The successive Jost solutions, $v_j = (\phi_j, \psi_j)$, needed in this ratio are computed as

$$v_j(t; \zeta) = (\zeta - \zeta_j)^{-1} D_{1j}(t; \zeta, \mathcal{S}_j) v_{j-1}(t; \zeta).$$

The signal and the energy in the tails are given by

$$q_j = q_{j-1} - 2i \frac{(\zeta_j - \zeta_j^2) \beta_{j-1}}{1 + \mid \beta_{j-1} \mid^2},$$

$$E_j^{(+)} = E_{j-1}^{(+)} + \frac{4 \text{Im}(\zeta_j)}{1 + \mid \beta_{j-1} \mid^2},$$

so that $E^{(+)}(t) = E_0^{(+)}(t) + E_k^{(+)}(t)$ with $E_0^{(+)}(t) = 0$. Next, our objective is to compute the derivatives of $E(\tau)$ with respect to the discrete spectra of $K$-soliton in order to facilitate optimization procedures that are based on gradients. In the following, we use the notation $\partial \zeta_j = \partial / \partial \zeta_j$ and $\partial \beta_j = \partial / \partial \beta_j$. For real-valued function $f$, it is known that $(\partial \zeta_j f)(\zeta_j) = \partial \beta_j f(\beta_j) \partial \zeta_j f(\zeta_j)$. In Wirtinger calculus, the complex gradient is defined as $\partial \beta_j f(\beta_j) \partial \zeta_j f(\zeta_j)$.

Remark 2.1. Here, we have described the Darboux transformation only for the case of $K$-soliton potentials; however, the recipe can be easily adapted to the case of arbitrary seed potentials. Note that this would require explicit knowledge of $v_0(t; \zeta) = E_0^{(+)}(t)$.

2.1. Derivatives with respect to norming constants

Note that $E_{K-1}^{(+)}(-\tau)$ and $E_{K-1}^{(+)}(\tau)$ are independent of $b_K$; therefore,

$$\partial_{b_K} E_{K-1}^{(+)}(\tau) = \pm 4 \text{Im}(\zeta_j) \partial_{b_K} [1 + \mid \beta_{K-1}(\tau) \mid^2]^{-1}.$$ \hspace{1cm} (9)

By direct calculation, we have

$$\partial_{b_K} \beta_{K-1}(t; \zeta, b_K) = \frac{a_{K-1}(\zeta_K)}{[\phi_2^{(K)}(t; \zeta) - \beta_{K-1}(\tau) \phi_1^{(K)}(t; \zeta)]^2},$$

where we have used the Wronskian relation

$$a_{K-1}(\zeta) = \partial(x_K - \psi_{K-1}) = \prod_{k=1}^{K-1} \frac{\zeta - \zeta_k}{\zeta - \zeta_k}.$$ \hspace{1cm} (11)

Using the identity (10), it is straightforward to work out

$$\partial_{b_K} E_{K-1}^{(+)}(\tau) = \pm 4 \text{Im}(\zeta_j) \left[ \frac{a_{K-1}(\zeta_K)}{[\phi_2^{(K)}(t; \zeta) - \beta_{K-1}(\tau) \phi_1^{(K)}(t; \zeta)]^2} \right] \partial_{b_K} [1 + \mid \beta_{K-1} \mid^2]^{-1}.$$ \hspace{1cm} (12)

Note that $\zeta_K$ is the last eigenvalue to be added using the DT iterations. Given that there is no restriction on the order in which the eigenvalues can be added, we can always choose $\zeta_K$ to be added last. This would determine $\partial_{b_K} E_{K}^{(+)}$ using DT iterations for arbitrary $K$. Thus, the complexity of computing $K$ derivatives works out to be $O(K^3)$. 


Before we conclude this discussion, let us examine the case of multisoliton solutions when \( \tau \) is large. In this limit, we have

\[
\beta_{K-1}(-\tau) \sim \frac{b_k e^{2i\zeta_k \tau}}{ak-1(\zeta_k)}, \quad \beta_{K-1}(+\tau) \sim \frac{b_k e^{2i\zeta_k \tau}}{ak-1(\zeta_k)},
\]

so that

\[
\partial_{b_k} \mathcal{E}(\pm \tau) \sim \mp \frac{4 \text{Im}(\zeta_k)}{b_k} |\beta_{K-1}(\pm \tau)|^2.
\]

Thus, the stationary condition \( \partial_{b_k} \mathcal{E}(\tau) = 0 \) translates into \(|b_k| = 1\). Therefore, asymptotically, \(|b_k| = 1\) minimizes the energy in the tails. This result can be easily verified from (8) which in the limit of large \( \tau \) gives

\[
\mathcal{E}(\tau) \sim \sum_{j=1}^{K} \frac{4 \text{Im}(\zeta_j)}{|a_{j-1}\zeta_j|} \left( |b_j|^2 + \frac{1}{|b_j|^2} \right) e^{-4\eta_j \tau},
\]

\[
\leq \sum_{j=1}^{K} \frac{8 \text{Im}(\zeta_j)}{|a_{j-1}\zeta_j|} e^{-4\eta_j \tau}.
\]

\[
(13)
\]

2.2. Derivatives with respect to discrete eigenvalues

Let \( V_j(t; \zeta) = (\zeta - \zeta_j) V_j(t; \zeta) \) and define \( V_j = (\Phi_j, \Psi_j) \) so that \( V_j(t; \zeta) = D_j(t; \zeta, \zeta_j) V_j(t; \zeta) \). The ratio \( \beta_j \) can also be computed in terms of the modified Jost solutions on account of the fact that \( (\zeta - \zeta_j) \) falls out of the equation while taking the ratio:

\[
\beta_{j-1}(t; \zeta_j, b_j) = \frac{\Phi_{j-1}(t; \zeta_j) - b_j \Psi_{j-1}(t; \zeta_j)}{\Phi_j(t; \zeta_j) - b_j \Psi_j(t; \zeta_j)},
\]

\[
(14)
\]

This gives us the opportunity to compute the derivatives with respect to \( \zeta \) recursively:

\[
\partial_{\zeta} V_j = V_{j+1} + D(t; \zeta, \zeta_j) \partial_{\zeta} V_{j-1}.
\]

\[
(15)
\]

Using the notation \( \mathcal{W}(u, v) = (u \partial_v - v \partial_u) \) for the Wronskian of scalar functions, let us introduce

\[
W_1(K-1)(t; \zeta_k) = \mathcal{W}(\Phi_{K-1}(t; \zeta_k), \Phi_{1}(t; \zeta_k)),
\]

\[
W_2(K-1)(t; \zeta_k) = \mathcal{W}(\Psi_{K-1}(t; \zeta_k), \Psi_{1}(t; \zeta_k)).
\]

\[
(16)
\]

By direct calculation, we have

\[
\partial_{\zeta} \beta_{K-1}(t; \zeta_k, b_k) = \frac{W_{K-1}}{\Phi_{K-1}^2 - b_k \Phi_{1}^2}(t; \zeta_k) - \frac{b_k \partial_{b_k} \mathcal{W}(\zeta_k)}{\Phi_{K-1}^2 - b_k \Phi_{1}^2}(t; \zeta_k).
\]

\[
(17)
\]

Note that \( \mathcal{E}_{K-1}^{(\pm)}(-\tau) \) and \( \mathcal{E}_{K-1}^{(\pm)}(\tau) \) are independent of \( \zeta_k \); therefore, using the above identity, it is straightforward to obtain

\[
\partial_{\zeta} \mathcal{E}(\pm \tau) = \frac{2}{b_k} \left[ \frac{W_{K-1}^2}{\Phi_{K-1}^2 - b_k \Phi_{1}^2} + 4 \text{Im}(\zeta_k) \right] \left( \frac{|b_k|^2}{|b_k|^2 + 1} \right)^2.
\]

\[
(18)
\]

Following as in the case of norming constants, we can always choose \( \zeta_k \) to be added last so that \( \partial_{\zeta} \mathcal{E}(\pm) \) can be determined using DT iterations for arbitrary \( k \). Thus, the complexity of computing \( K \) derivatives again works out to be \( \mathcal{O}(K^3) \).

3. Spectral width

Consider the Fourier spectrum of the multisoliton potential denoted by \( Q(\xi) = \int \mathcal{G}(t)e^{-i\xi dt} \), \( \xi \in \mathbb{R} \). Let us observe that the following quantities can be expressed entirely in terms of the discrete eigenvalues:

\[
C_1 = - \int q(\theta) d\theta = 4i \sum_k \text{Im} \zeta_k^2,
\]

\[
C_2 = \int (|q|^4 - |\partial_\theta q|^2) d\theta = -\frac{16}{k} \sum_k \text{Im} \zeta_k^3.
\]

\[
(19)
\]

with \( C_0 = ||q||^2 \). These quantities do not evolve as the pulse propagates along the fiber. From [14], the variance \( \langle \Delta \xi^2 \rangle \) is given by

\[
\langle \Delta \xi^2 \rangle = \frac{\int |q|^4 dt}{C_0} + \frac{\sum_k C_k^2}{C_0} - \frac{C_2}{C_0} \leq ||q||^2 + \frac{\sum_k C_k^2}{C_0} - \frac{C_2}{C_0}.
\]

\[
(20)
\]

This quantity characterizes the width of the Fourier spectrum. The biquadratic integral above cannot be computed exactly in general, however, \( ||q||^2 \) can be computed in a straightforward manner: from (7), we have \( ||q||^2 \leq ||q_k||^2 + 2 \text{Im}(\xi_k) \) so that \( ||q||^2 \leq 2 \sum_k \text{Im}(\xi_k) \) which yields \( \langle \Delta \xi^2 \rangle \leq (C_0^2/4 + C_k^2/C_0^2 - C_2/C_0) \) (correcting a typographical error in [14]). Note that this inequality holds irrespectively of how the pulse evolves as it propagates along the fiber.

4. Examples

4.1. One-sided effective support

Let us consider the case where we want to introduce a boundstate with eigenvalue \( \zeta_k \) to any arbitrary seed profile assumed to be different from a null-potential. It is known that the energy content of the tail \( [\tau, \infty)(\tau > 0) \) of the profile is \( \mathcal{E}_k^{(\pm)}(\tau) \). The problem is to determine the norming constant \( b_k \) which minimizes \( \mathcal{E}^{(\pm)} = \mathcal{E}_k^{(\pm)}(\tau) \). To this end, setting \( \partial_{b_k} \mathcal{E}^{(\pm)}(\tau) = 0 \), we have

\[
[\phi^{(0)}_1 - b_1 \phi^{(0)}_2](\tau) = \partial_{b_k} \mathcal{E}^{(\pm)}(\tau) = 0.
\]

\[
(21)
\]

which yields \( b_1 = [\phi^{(0)}_1/\phi^{(0)}_2](\tau; \zeta_k) \) or \( b_1 = [\phi^{(0)}_2/\phi^{(0)}_1](\tau; \zeta_k) \). It is easy to verify that the first choice corresponds to maximum \( \mathcal{E}^{(\pm)}(\tau) \) which leaves us with the second choice for which \( \mathcal{E}^{(\pm)}(\tau) = \mathcal{E}_k^{(\pm)}(\tau) \), i.e. no part of the soliton’s energy goes into the tail \( [\tau, \infty) \). By a recursive argument, the conclusion holds for any number of boundstates provided \( b_j = [\phi^{(0)}_j/\phi^{(0)}_j](\tau; \zeta_k) \).

4.2. Adding a boundstate to a symmetric profile

Let us consider the case where we want to introduce a boundstate with eigenvalue \( \zeta_k \) to any arbitrary seed profile. The energy content of the tail \( \mathbb{R} \setminus (-\tau, \tau) \) of the seed profile is
\( \mathcal{E}_0(\tau) \). The problem is to determine the norming constant \( b_1 \) which minimizes \( \mathcal{E}(\tau) \). To this end, setting \( \partial_t \mathcal{E}(\tau) = 0 \), we have

\[
\frac{[\varphi(0)^* - b_1 \varphi(0)]^2}{\| \varphi(0)^* - b_1 \varphi(0) \|^2 + | \varphi(0)^* - b_1 \varphi(0) \|^2} \bigg|_{\tau = -\tau} = \frac{[\varphi(0)^* - b_1 \varphi(0)]^2}{\| \varphi(0)^* - b_1 \varphi(0) \|^2 + | \varphi(0)^* - b_1 \varphi(0) \|^2} \bigg|_{\tau = \tau}.
\]

(21)

For the sake of simplicity, we assume that \( \zeta = i \eta \) so that

\[
\begin{cases}
\phi_0(t; i \eta) = \psi_{10}^{(t)}(t; i \eta), \\
\phi_2(t; i \eta) = \psi_{10}^{(t)}(t; i \eta),
\end{cases}
\]

(22)

with \( a(i \eta) = a^*(i \eta) \). In the following, we set \( \tau = t \). Then, using the symmetry relations, we obtain

\[
\frac{[\varphi(0) - \lambda \varphi(0)]^2}{\| \varphi(0) - \lambda \varphi(0) \|^2 + | \varphi(0) - \lambda \varphi(0) \|^2} b_1^2 = \frac{[\varphi(0) - \lambda \varphi(0)]^2}{\| \varphi(0) - \lambda \varphi(0) \|^2 + | \varphi(0) - \lambda \varphi(0) \|^2} b_1^2.
\]

(23)

Physically, \( \log |b_1| \) is related to the translation of the profile; therefore, it is easy to conclude, for a symmetrical profile, that the extremum is obtained for \( |b_1| = 1 \). Putting

\[
A = i(\psi_{10}^{(0)} \psi_{10}^{(0)} - \psi_{10}^{(0)} \psi_{10}^{(0)}) \quad B = i(\psi_{10}^{(0)} \psi_{10}^{(0)} - \psi_{10}^{(0)} \psi_{10}^{(0)})
\]

(24)

in (23) and using \( |b_1| = 1 \), we have \( Ab_1^2 - 2Bb_1 + A^* = 0 \). The solution of this equation works out to be

\[
b_1 = \frac{B \pm i \sqrt{|A|^2 - B^2}}{A} = B \pm i \sqrt{\Delta}.
\]

(25)

From the relations

\[
B^2 = 2[\psi_{10}^{(0)} \psi_{10}^{(0)} - \psi_{10}^{(0)} \psi_{10}^{(0)}] - 2 \text{Re}[\psi_{10}^{(0)} \psi_{10}^{(0)}] \quad |A|^2 = |\psi_{10}^{(0)} \psi_{10}^{(0)}|^2 + |\psi_{10}^{(0)} \psi_{10}^{(0)}|^2 - 2 \text{Re}[\psi_{10}^{(0)} \psi_{10}^{(0)} \psi_{10}^{(0)} \psi_{10}^{(0)}],
\]

we have

\[
\Delta = |a(i \eta)|^2 + |\psi_{10}^{(0)}|^2 |\psi_{10}^{(0)}|^2 + |\psi_{10}^{(0)}|^2 |\psi_{10}^{(0)}|^2 - |\psi_{10}^{(0)}|^2 |\psi_{10}^{(0)}|^2.
\]

(26)

In order to show that \( \Delta \geq 0 \), consider

\[
\Delta = |(\psi_{10}^{(0)} \psi_{10}^{(0)} - \psi_{10}^{(0)} \psi_{10}^{(0)})|^2 + |(\psi_{10}^{(0)} \psi_{10}^{(0)} - \psi_{10}^{(0)} \psi_{10}^{(0)})|^2 - |(\psi_{10}^{(0)} \psi_{10}^{(0)})|^2
\]

\[
= |(\psi_{10}^{(0)} \psi_{10}^{(0)} - \psi_{10}^{(0)} \psi_{10}^{(0)})|^2 + |(\psi_{10}^{(0)} \psi_{10}^{(0)} - \psi_{10}^{(0)} \psi_{10}^{(0)})|^2 - |(\psi_{10}^{(0)} \psi_{10}^{(0)})|^2
\]

\[
\times |(\psi_{10}^{(0)} \psi_{10}^{(0)} - \psi_{10}^{(0)} \psi_{10}^{(0)})|^2 - |(\psi_{10}^{(0)} \psi_{10}^{(0)})|^2 - |(\psi_{10}^{(0)} \psi_{10}^{(0)})|^2|,
\]

which shows that \( \Delta \geq 0 \). Therefore, the extremal points for \( b_1 \) are given by

\[
\arg b_1 = \pm \text{arg} \left[ \frac{B + i \sqrt{\Delta}}{|A|} \right] - \text{arg} A.
\]

(27)

4.2.1. Symmetric 2-soliton. The general result derived above can be applied to a symmetric 2-soliton potential. Let us assume that the seed potential is a symmetric 1-soliton potential with the discrete spectrum given by \( \{ (i \eta_0, e^{i \theta_0}) \} \). The boundstate being introduced is characterized by \( (i \eta_0, e^{i \theta_0}) \). Expression for the Jost solutions can be obtained from (4) which leads to \( B = 0 \) so that \( b_1 = \pm e^{i \theta_0} \). It can be directly verified that \( b_1 = e^{i \theta_0} \) corresponds to the minima of \( \mathcal{E}_1(\tau) \) for all \( \tau > 0 \) as follows: given the symmetric nature of the profile, it suffices to find the minima of \( \mathcal{E}_1(\tau) \) which reads as

\[
\mathcal{E}_1(\tau) = \mathcal{E}_0(\tau) + \frac{4 \eta_1 Y^{-1}}{X^{-1} + Y^{-1}} \left( 1 - \frac{G \cos \theta}{1 - H \cos \theta} \right),
\]

where \( \theta = \theta_1 - \theta_0 \) and

\[
X^{-1} = e^{-2 \eta_0 \tau} + a(0) e^{2 \eta_0 \tau} \quad Y^{-1} = e^{-2 \eta_0 \tau} + e^{2 \eta_0 \tau} \quad G = \frac{2}{\eta_1 + \eta_1} \quad H = \frac{2}{\eta_1 + \eta_1}.
\]

It is straightforward to show that \( 0 < G, H \leq 1, Y^{-1} > 0 \) and \( X^{-1} + Y^{-1} > 0 \). Now, from

\[
\left( 1 - \frac{G \cos \theta}{1 - H \cos \theta} \right) \quad \frac{1 - \frac{G}{H}}{1 - \frac{G}{H} + \frac{G}{H} + \frac{G}{H}},
\]

and

\[
1 - \frac{G}{H} = \frac{[\eta_1 - \eta_0]^2}{2 \eta_1 + \eta_1} [1 + 2 \cosh(2 \eta_0 \tau)] \quad \cosh(2 \eta_0 \tau) \quad \sinh(2 \eta_1 + \eta_1 \tau) \quad \eta_1 + \eta_1 \quad \eta_1 - \eta_1.
\]

it follows that \( (1 - G/H) \leq 0 \); therefore, the minima of \( \mathcal{E}_1(\tau) \) occurs at \( \theta = 2 \pi n \) or \( b_1 = e^{i \theta_0} \).

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