CHARACTERIZATION OF SECOND TYPE PLANE FOLIATIONS USING NEWTON POLYGONS

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Abstract. In this article we characterize the foliations that have the same Newton polygon that their union of formal separatrices, they are the foliations called of the second type. In the case of cuspidal foliations studied by Loray [Lo], we precise this characterization using the Poincaré-Hopf index. This index also characterizes the cuspidal foliations having the same desingularization that the union of its separatrices. Finally we give necessary and sufficient conditions when these cuspidal foliations are generalized curves, and a characterization when they have only one separatrix.

1. Introduction

Camacho, Lins-Neto and Sad [Cam-Li-Sad] introduced and studied the singularities of foliations of the generalized curve type, these are the foliations without saddle-nodes in their reduction of singularities. These foliations receive this name because they have a behavior similar to the union of their separatrices, where a separatrix is an irreducible analytical curve invariant for the foliation. For these foliations the Poincaré-Hopf index coincides with the Milnor number of the union of their separatrices (Mol-So and Cam-Li-Sad) and the reduction of singularities of these foliations coincides with the desingularization of the union of their separatrices.

The singularities of generalized curved type verify that their Gómez Mont - Seade - Verjovski index [Go-Sea-Ve] is zero and their Camacho - Sad index [Cam-Sad2] and the Baum-Bott index are equal [Br].

The foliations of the second type can be thought of as a generalization of the singularities of the generalized curve type, in which we will allow the existence of formal separatrices. In order to add the formal separatrix we must admit that we have saddle-nodes in their resolution of singularities that generate formal separatrices, they could not be a corner of two divisors, nor could saddle-nodes outside the corners with weak separatrix contained in the divisor. Note that with these restrictions the singularities of a second type foliation with a single separatrix will have to be a generalized curve foliation. The singularities of second type were introduced by Mattei and Salem [Ma-Sal]. They characterized this type of singularities by means of the coincidence of the multiplicity of the foliation with the multiplicity of the

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union of their formal separatrices. For these singularities, the reduction of singularities coincides with the desingularization of its separatrix. It should be noted that the proof made in [Cam-Li-Sad] to prove this property for generalized curve type foliations also proves this property for second type singularities. There are other characterizations of these singularities (see [Can-Co-Mol] and [FP-Mol]).

Merle [Mer] gives a decomposition of the polar curve of an irreducible curve $C$ that determines the topology of $C$. This theorem was generalized for foliations by Rouillé [R], where he gives a decomposition of the polar of a foliation, of the generalized curve type, which determines the topology of its separatrix. The main ingredient for his decomposition is Dulac’s Theorem [Du], this theorem tells us that the Newton polygon of the foliation coincides with the Newton polygon of the separatrix. Merle’s theorem has been generalized for reduced curves by [GB] and [GB-Gw], the first of which was generalized for foliations by [Co] and the second by [Sar]. There are examples of foliations where their Newton polygon coincides with that of their separatrices and is not a foliation of the generalized curve type, however these foliations are of the second type (see Example 3.3). In this paper we give a new characterization of the singularities of the second type in terms of the Newton polygon of their union of separatrices.

**Theorem 1.1.** A non-dicritical foliation is of the second type if and only if its Newton polygon coincides with that of their union of separatrices.

We will prove Theorem 1.1 in Section 4.

According to Loray [Lo], a foliation with a cuspidal singularity is given by

$$F_{\omega_{p,q},\Delta} : \omega_{p,q,\Delta} = d(y^p - x^q) + \Delta(x,y)(pxdy - qydx),$$

where $p$, $q$ are positive natural numbers and $\Delta(x,y) \in \mathbb{C}\{x,y\}$.

Denote by $\text{PH}(F)$ the Poincaré-Hopf index of the foliation $F$. For the foliation $d(y^p - x^q)$ we have $\text{PH}_{(p,q)} := \text{PH}(d(y^p - x^q)) = (p-1)(q-1)$.

**Theorem 1.2.** Let $F_{\omega_{p,q},\Delta}$ be a cuspidal foliation as in (1) and suppose that $F_{\omega_{p,q},\Delta}$ is non-dicritical. The next statements are equivalents:

(a) The cuspidal foliation $F_{\omega_{p,q},\Delta}$ is of the second type.

(b) The intersection number $(\Delta, y^p - x^q)_0 \geq \text{PH}_{(p,q)} - 1$.

(c) The cuspidal foliation $F_{\omega_{p,q},\Delta}$ has the same reduction of singularities that $d(y^p - x^q)$.

In general, if a foliation has the same reduction of singularities as its union of separatrices then the foliation is not of the second type (see Example 2.3). However, after Theorem 1.2, for the cuspidal foliations this property characterizes foliations of the second type.

We also give, in the next theorem, necessary and sufficient conditions when the cuspidal foliations are of the generalized curve type.

**Theorem 1.3.** Let $F_{\omega_{p,q},\Delta}$ be a cuspidal foliation as in (1) and suppose that $F_{\omega_{p,q},\Delta}$ is non-dicritical. We have:
(a) If the intersection number $(\Delta, y^p - x^q)_0 > PH(p,q) - 1$, then $F_{p,q,\Delta}(x,y)$ is of the generalized curve type.

(b) If $p, q$ are coprime then the foliation $F_{n-p,q,\Delta}$ is of generalized curve type if and only if $(\Delta, y^p - x^q)_0 > PH(p,q) - 1$.

We will prove Theorem 1.2 and Theorem 1.3 in Section 5.

2. Basic Definitions and Notations

In order to fix the notation, we will remember basic concepts of local foliation theory and plane curves. Denote by $\mathbb{C}[[x,y]]$ the ring of formal powers series in two variables with coefficients in $\mathbb{C}$ and $\mathbb{C}\{x,y\}$ the sub-ring of $\mathbb{C}[[x,y]]$ formed by formal powers series that converge in a neighborhood of $0 \in \mathbb{C}^2$. Consider the maximal ideals $m$ and $\hat{m}$ of $\mathbb{C}\{x,y\}$ and $\mathbb{C}[[x,y]]$ respectively. The order of a power series $\hat{h}(x,y) = \sum_{i,j} a_{ij} x^i y^j \in \mathbb{C}[[x,y]]$ is $\text{ord}(\hat{h}) := \min\{i + j : a_{ij} \neq 0\}$.

A singular formal foliation $\hat{F}_\omega$ of codimension one over $\mathbb{C}^2$ is locally given by a 1-form $\omega = \hat{A}(x,y)dx + \hat{B}(x,y)dy$, where $\hat{A}, \hat{B} \in \hat{m}$ are coprime. The power series $\hat{A}$ and $\hat{B}$ are called the coefficients of $\omega$. The multiplicity of the foliation $\hat{F}_\omega$ is defined as $\text{mult}(\omega) := \min\{|\text{ord}(\hat{A}), \text{ord}(\hat{B})|\}$.

Let $T \subseteq \mathbb{N}^2$. Denote by $D(T)$ the convex hull of $(T + \mathbb{R}^2_{\geq 0})$, where $+$ is the Minkowski sum, and by $N(T)$ the polygonal boundary of $D(T)$, which will call Newton polygon determined by $T$.

Let $\hat{h}(x,y) = \sum_{i,j} a_{ij} x^i y^j \in \mathbb{C}[[x,y]]$. The support of $\hat{h}$ is

$$\text{supp}(\hat{h}) := \{ (i,j) \in \mathbb{N}^2 : a_{ij} \neq 0 \}$$

and the Newton polygon of $\hat{h}$ is by definition the Newton polygon $N(\text{supp}(\hat{h}))$.

Let $\omega = \hat{A}(x,y)dx + \hat{B}(x,y)dy$ be a one-form, where $\hat{A}, \hat{B} \in \hat{m}$. The support of $\omega$ is

$$\text{supp}(\omega) = \text{supp}(x\hat{A}) \cup \text{supp}(y\hat{B})$$

If we write $\omega = \sum_{i,j} \hat{\omega}_{ij}$, where $\hat{\omega}_{ij} = \hat{A}_{ij} x^{i-1} y^j dx + \hat{B}_{ij} x^i y^{j-1} dy$, then

$$\text{supp}(\omega) = \{ (i,j) : (\hat{A}_{ij}, \hat{B}_{ij}) \neq (0,0) \}.$$

Let $\hat{F}_\omega : \omega = 0$ be a foliation given by the one-form $\omega$. The Newton polygon of $\hat{F}_\omega$, denoted by $N(\hat{F}_\omega)$ or $N(\omega)$ is the Newton polygon $N(\text{supp}(\omega))$.

Let $\hat{f}(x,y) \in \mathbb{C}[[x,y]]$. We say that the $\hat{S}_f : \hat{f}(x,y) = 0$ is invariant by $\hat{F}_\omega$ if $\omega \wedge d\hat{f} = \tilde{\eta} \hat{f}$, where $\tilde{\eta}$ is a two-form (i.e. $\tilde{\eta} = \hat{g} dx \wedge dy$, for some $\hat{g} \in \mathbb{C}[[x,y]]$). If $\hat{S}_f$ is irreducible then we will say that $\hat{S}_f$ is a formal separatrix of $\hat{F}_\omega : \omega = 0$.

We will consider non-dicritical foliations, that is, foliations having a finite set of separatrices (see [Cam-Li-Sad] page 158 and page 165). Let $(\hat{S}_f)_f^1$ be the set of all formal separatrices of the non-dicritical foliation $\hat{F}_\omega : \omega = 0$. Each separatrix
\( \hat{S}_{f_j} \) corresponds to an irreducible formal power series \( \hat{f}_j(x,y) \). Denote by \( \hat{S}(\hat{F}_\omega) \) the union \( \bigcup \hat{S}_{f_j} \) of all separatrices of the foliation \( \hat{F}_\omega \), which we will call union of formal separatrices of \( \hat{F}_\omega \). In the following we will denote by \( \mathcal{F}_\omega \) a holomorphic foliation and by \( \mathcal{S}(\mathcal{F}_\omega) \) its union of convergent separatrices.

The dual vector field associated to \( \hat{F}_\omega \) is

\[
X = \hat{B}(x,y) \frac{\partial}{\partial x} - \hat{A}(x,y) \frac{\partial}{\partial y}.
\]

We say that the origin \( (x,y) = (0,0) \) is a simple or reduced singularity of \( \hat{F}_\omega \) if the matrix associated with the linear part of the field

\[
\begin{pmatrix}
\frac{\partial \hat{B}(0,0)}{\partial x} & \frac{\partial \hat{B}(0,0)}{\partial y} \\
-\frac{\partial \hat{A}(0,0)}{\partial x} & -\frac{\partial \hat{A}(0,0)}{\partial y}
\end{pmatrix}
\]

has two eigenvalues \( \lambda, \mu \), with \( \frac{\lambda}{\mu} \notin \mathbb{Q}^+ \).

It could happen that

a) \( \lambda \mu \neq 0 \) and \( \frac{\lambda}{\mu} \notin \mathbb{Q}^+ \) in which case we will say that the singularity is not degenerate.

b) \( \lambda \mu = 0 \) and \( (\lambda, \mu) \neq (0,0) \) in which case we will say that the singularity is a saddle-node.

In the b) case, the strong separatrix of a foliation with singularity \( P \) is an analytic invariant curve whose tangent at the singular point \( P \) is the eigenspace associated with the non-zero eigenvalue of the matrix given in (2). The zero eigenvalue is associated with a formal separatrix called weak separatrix.

From now on \( \pi : M \to (\mathbb{C}^2,0) \) represents the process of singularity reduction or desingularization of \( \hat{F}_\omega \), obtained by a finite sequence of point blow-ups, where \( D := \pi^{-1}(0) = \bigcup_{j=1}^n D_j \) is the exceptional divisor, which is a finite union of projective lines with normal crossing (that is, they are locally described by one or two regular and transversal curves). In this process, any separatrix of \( \hat{F}_\omega \) is smooth, disjoint and transverse to \( D_j \subset D \), and it does not pass through a corner (intersection of two components of the divisor \( D \)). Let \( \hat{F}_\omega \) be a non-dicritical formal foliation and consider the minimal reduction of singularities \( \pi : M \to (\mathbb{C}^2,0) \) of \( \hat{F}_\omega \) (this is, a reduction with the minimal number of blows-up that reduces the foliation). The strict transform of the foliation \( \hat{F}_\omega \) is given by \( \hat{F}_\omega' = \pi^* \hat{F}_\omega \) and the exceptional divisor is \( D = \pi^{-1}(0) \).

A foliation \( \hat{F}_\omega \) is a generalized curve if in its reduction of singularities there are no saddle-node points.

If in the desingularization of \( \hat{F}_\omega \), the exceptional divisor \( D \) at point \( P \) contains the weak invariant curve of the saddle-node, then the singularity is called saddle-node tangent. Otherwise we will say that \( \hat{F}_\omega \) is a saddle-node transverse to \( D \) at point \( P \).
Definition 2.1. The foliation $\tilde{F}_\omega$ is of the second type with respect to the divisor $D$ if no singular points of $\tilde{F}_\omega$ is of tangent node type.

Non-dicritical foliations of the second type were studied by Mattei and Salem [Ma-Sal], also by Cano, Corral and Mol [Can-Co-Mol] and in the dicritical case by Genzmer and Mol [Ge-Mol] and Fernández Pérez-Mol [FP-Mol]. Mattei and Salem gave the next characterization of foliations of the second type in terms of the multiplicity of their union of formal separatrices:

Theorem 2.2. [Ma-Sal] Théorème 3.1.9] Let $\tilde{F}_\omega$ be a non-dicritical foliation and let $\tilde{S}(\tilde{F}_\omega): \tilde{f}(x, y) = 0$ be a reduced equation of its union of separatrices. Consider the minimal reduction of singularities $\pi: (M, D) \to (\mathbb{C}^2, 0)$ of $\tilde{F}_\omega$. Then

1. $\pi$ is a reduction of singularities of $\tilde{S}(\tilde{F}_\omega)$. Furthermore, if $\tilde{F}_\omega$ is of the second type then $\pi$ is the minimal reduction of singularities of $\tilde{S}(\tilde{F}_\omega)$.
2. $\text{mult}(\tilde{\omega}) \geq \text{mult}(\tilde{S}(\tilde{F}_\omega))$ and the equality holds if and only if $\tilde{F}_\omega$ is of the second type.

The reciprocal of the first statement of Theorem 2.2 is not true, that is, if the reduction of singularities of the foliation and that of its union of separatrices coincide does not guarantee that the foliation is of the second type, as shown in the following example.

Example 2.3. The union of the separatrices of the foliation $F_\omega = (xy + y^2)dx - x^2dy$ is $S(F_\omega) = xy$. The foliation $F_\omega$ and its union of separatrices are desingularized after a blow-up but the foliation is not of the second type because the strong separatrix that passes through the saddle-node is not contained in the divisor.

3. Foliations and Newton polygons

Non-dicritical generalized curve foliations are those in which no saddle-node points appear in their desingularization [Sei] and they have a finite number of separatrices [Cam-Li-Sad]. These foliations were studied by Camacho, Lins Neto and Sad who proved that

Theorem 3.1. [Cam-Li-Sad] Theorem 2] Let $F_\omega$ be a non-dicritical generalized curve foliation and $S(F_\omega)$ its union of separatrices. Then $F_\omega$ and $S(F_\omega)$ have the same reduction of singularities.

Rouillé obtained the following result on non-dicritical generalized curve foliations. In [R] it is indicated that Mattei reported that this result was known by Dulac [Du].

Proposition 3.2. [R] Proposition 3.8] Let $F_\omega$ be a non-dicritical generalized curve foliation and $S(F_\omega): f(x, y) = 0$ be a reduced equation of its union of separatrices. Then $N(\omega) = N(df)$.

The reciprocal of Theorem 3.1 and Proposition 3.2 are not true, as the following example shows:

Example 3.3. Consider $b \notin \mathbb{Q}$ and the foliation defined by $\omega = ((b - 1)xy - y^3)dx + (xy - bx^2 + xy^2)dy$. A reduced equation of its union of separatrices is $f(x, y) = xy(x - y) = 0$. The foliation $F_\omega$ and the curve $f(x, y) = 0$ are desingularised.
after a blow-up. The Newton polygons $N(\omega)$ and $N(f)$ are equal but $\mathcal{F}_\omega$ is not a curve generalized type foliation since a saddle-node point appears in its reduction of singularities.

In [Br] pag 532 was introduced the Gómez-Mont-Seade-Verjovsky index denoted by $\text{GSV}(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega))$, where $\mathcal{F}_\omega : \omega = 0$ and $\mathcal{S}(\mathcal{F}_\omega) : f(x,y) = 0$ is a reduced equation of union of convergent separatrices of $\mathcal{F}_\omega$. For $f \in \mathbb{C}\{x,y\}$, there are $g, h \in \mathbb{C}\{x,y\}$, with $h$ and $f$ coprime, and an analytic one-form $\eta$ such that $g\omega = hdf + f\eta$. In [Br], Brunella defines

$$\text{GSV}(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = \frac{1}{2\pi i} \int_{\partial \mathcal{S}(\mathcal{F}_\omega)} \frac{g}{h} \frac{d}{d} \left( \frac{h}{y} \right),$$

when $\mathcal{S}(\mathcal{F}_\omega) : f = 0$ is irreducible and $\omega = A(x,y)dx + B(x,y)dy$. We get

$$\text{GSV}(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = \text{ord}_t \left( \frac{B}{f_y}(\gamma(t)) \right),$$

where $\gamma(t)$ is a parametrization of $\mathcal{S}(\mathcal{F}_\omega)$. Now, we remember some results on the index $\text{GSV}(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega))$.

**Theorem 3.4.** [Cav-Le Théorème 3.3][Br] Proposition 7] Let $\mathcal{F}_\omega$ be a non-dicritical foliation and let $\mathcal{S}(\mathcal{F}_\omega) : f(x,y) = 0$ be a reduced equation of its union of separatrices. Then $\mathcal{F}_\omega$ is a generalized curve foliation if and only if $\text{GSV}(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = 0$.

The next result on generalized curve foliations was obtained by Rouillé [R] and it will be very useful in this paper. Denote by $\mathbb{C}[\!(t)\!]$ the ring of formal power series in the variable $t$ and coefficients in $\mathbb{C}$, and $\mathbb{C}\{t\}$ the subring of $\mathbb{C}[\!(t)\!]$ of convergent power series.

**Lemma 3.5.** [R Lemme 3.7] Let $\mathcal{F}_{\omega_1}$ and $\mathcal{F}_{\omega_2}$ two non-dicritical generalized curve foliations with the same union of separatrices. If $\gamma(t) = (x(t), y(t)) \in (\mathbb{C}\{t\})^2$ with $\gamma(0) = 0$, then

$$\text{ord}_t \gamma^* \omega_1 = \text{ord}_t \gamma^* \omega_2.$$

We deduce from Example [3.3] that there are foliations having the same polygon as their union of separatrices but they are not generalized curve foliations. Our objective in this paper is to characterize the foliations having the same Newton polygon that its union of separatrices. They will be the non-dicritical foliations of the second type.

### 4. Characterization of a foliation of the second type in terms of the Newton polygon

In this section, a new characterization of the second-type non-dicritical foliations is given in terms of the Newton polygon of the foliation and that of its union of separatrices.

**Lemma 4.1.** Let $\hat{\mathcal{F}}_{\omega}$ be a non-dicritical foliation and $\hat{f}(x,y) = 0$ be a reduced equation of its union of separatrices. If $N(\hat{\omega}) = N(\hat{f})$ then $\hat{\mathcal{F}}_{\omega}$ is a foliation of the second type.
Proof. Consider the foliation $\hat{F}_\omega$ given by $\hat{\omega} = \sum_{i,j} \hat{A}_{ij}x^{i-1}y^jdx + \sum_{i,j} \hat{B}_{ij}x^iy^{j-1}dy$.

Since $\text{mult}(\hat{\omega}) = \min\{\text{ord}(\hat{A}), \text{ord}(\hat{B})\}$ then
\[
\text{mult}(\hat{\omega}) = \min\{i + j - 1 : (i, j) \in \mathcal{N}(\hat{\omega})\} = \min\{i + j - 1 : (i, j) \in \mathcal{N}(df)\}
\tag{3}
\]

From (3) and the second statement of Theorem 2.2 we conclude that if $N(\hat{\omega}) = N(f)$ then the foliation $\hat{F}_\omega$ and its union of separatrices $\hat{S}(\hat{F}_\omega)$ have the same resolution. In the following proposition we generalize Lemma 3.5 to second type foliations.

**Proposition 4.2.** Let $\hat{F}_\omega$ be a non-dicritical second type foliation and $\hat{S}(\hat{F}_\omega)$ its union of separatrices. If $\gamma(t) = (x(t), y(t)) \in (\mathbb{C},[[t]])^2$, with $\gamma(0) = 0$, then
\[
\text{ord}_t \gamma^* \hat{\omega} = \text{ord}_t \gamma^* df.
\]

Proof. If $\gamma(t) = (x(t), y(t))$ is a parameterization of a separatrix of $\hat{\omega}$ and $df$ then $\hat{\omega}(\gamma(t)), \gamma'(t) = 0 = df(\gamma(t)), \gamma'(t)$ and we conclude the proposition in such a case.

Suppose now that $\gamma(t)$ is not a parameterization of any separatrix of $\hat{\omega}$ and $df$. We proceed by induction on the number of blows-up $n$ needed in the process of the desingularization of the foliation $\hat{F}_\omega$. First we suppose that the number of blows-up is $n = 0$. Then the foliations defined by the one-forms $\hat{\omega}$ and $df$ are reduced. If $\hat{F}_\omega$ is a generalized curve foliation then the proposition follows from Lemma 3.5.

Suppose now that $\hat{F}_\omega$ is a reduced foliation with a saddle-node. We can consider the formal form of the saddle-node given by the equation:
\[
-y^{p+1}dx + (1 + \lambda y^p)xdy \quad \text{with} \quad p \geq 1 \quad \text{and} \quad \lambda \in \mathbb{C},
\]
and the reduced equation of its union of formal separatrices is given by $\hat{f}(x, y) = xy$.

We can write $\gamma(t) = (x(t), y(t)) = (ta^1_n(t), t^bn_2(t))$, where $a, b$ are positive integers and $n_i(t)$ are units of $\mathbb{C}[[t]]$ (that is $n_i(0) \neq 0$ for $i = 1, 2$). We have
\[
\gamma^* \hat{\omega} = [bt^a + b^{a-1}n_1(t)n_2(t) + t^{a+b}n_1(t)n_2'(t)]dt + [t^{a+b}n_1(t)(n_2(t))^{p+1}(\lambda b - a) + \lambda t^{a+b}n_1(t)n_2(t)]^{p+1}(n_2(t))^{p+1}]dt,
\]
so $\text{mult}(\gamma^* \hat{\omega}) = a + b - 1$. On the other hand $df = ydx + xdy$, hence
\[
\gamma^* df = [t^bn_2(t)(at^{a-1}n_1(t) + t^{a+b}n_1'(t))]dt + [t^{a+b}n_1(t)(bt^{b-1}n_2(t) + t^{a+b}n_2'(t))]dt = [t^a + b(n_1(t)n_2(t))(a + b) + t^{a+b}(n_1(t)n_2(t) + n_1(t)n_2')]dt,
\]
so $\text{mult}(\gamma^* df) = a + b - 1$. Therefore, if $\hat{F}_\omega$ is a reduced foliation with a saddle-node, then $\text{ord}_t(\gamma^* \hat{\omega}) = \text{ord}_t(\gamma^* df)$. 

\[\square\]
Now we suppose that the foliations defined by the one-forms $\hat{\omega}$ and $d\hat{f}$ are not reduced and $n > 0$. Let $E$ be the blow-up at the origin $(x, y) = (0, 0)$ given by $E : (x, t) = (x, xt)$, so $E^*\hat{\omega} = x^{m_1}\hat{\omega}$, where $m_1$ is the multiplicity of $\hat{\omega}$ and $\tilde{\omega}$ is the strict transform of $\hat{\omega}$. Denote by $\tilde{\gamma}$ the strict transformation of the curve $\gamma$ by $E$.

By induction hypothesis, we get,

$$\text{ord}_t \tilde{\gamma}^* \tilde{\omega} = \text{ord}_t \tilde{\gamma}^* d\tilde{f}.$$ (4)

Since the foliation $\hat{F}_\omega$ is of the second type, by Theorem 2.2, using the induction hypothesis and replacing in the equation (4) we get $\text{ord}_t \gamma^* \hat{\omega} = \text{ord}_t \gamma^* d\hat{f}$ and we finish the proof of the proposition.

Proposition 4.2 was also given in [Can-Co-Mol], Corollary 1], but with other proof.

**Corollary 4.3.** Let $\hat{F}_\omega$ be a non-dicritical second type foliation and let $\hat{S}_f$ be a reduced equation of its union of formal separatrices. Then $N(\hat{\omega}) = N(d\hat{f})$.

**Proof.** Since $\hat{F}_\omega$ is a second type foliation, using Theorem 2.2 we have that $\hat{F}_\omega$ has the same reduction of singularities as its union of formal separatrices and $\text{mult}(\hat{\omega}) = \text{mult}(d\hat{f})$. Reasoning analogously as in the proof given by Rouillé [R] in order to prove the Proposition 3.2 and by Proposition 4.2 we finish the proof.

As a consequence of the Corollary 4.3 we have:

**Corollary 4.4.** Let $\hat{F}_{\omega_1}$ and $\hat{F}_{\omega_2}$ be two non-dicritical second type foliations. If $\hat{F}_{\omega_1}$ and $\hat{F}_{\omega_2}$ have the same union of formal separatrices, then $N(\hat{\omega}_1) = N(\hat{\omega}_2)$.

**Example 4.5.** The foliation $F_\omega$ given by $\omega = (ny + x^n)dx - xdy$, $n \geq 1$ is not a foliation of the second type. The union of separatrices of $F_\omega$ is $S(F_\omega) : x = 0$. We observe that $\text{supp}(\omega) = \{(1, 1), (n + 1, 0)\}$ and $\text{supp}(f) = \{(1, 0)\}$, hence the Newton polygons of $F_\omega$ and $S(F_\omega)$ are different.

**Example 4.6.** Let us go back to Example 3.3. The second type foliation given by $\omega = ((b - 1)xy - y^3)dx + (xy - bx^2 + xy^2)dy$ with $-b, 1 - b \notin \mathbb{Q}^+$ has as union of separatrices to $S(F) = xy(x - y)$. We observe that polygons $N(\omega)$ and $N(f)$ are equal.
Proof of Theorem 1.1. It is an immediate consequence of Corollary 4.3 and Lemma 4.1.

Theorem 1.1 gives a new characterization of the non-dicritical second type foliations using its Newton polygon.

5. Cuspidal Foliations

Cuspidal foliations are inspired by nilpotent foliations. A foliation \( F_\omega \) in \((\mathbb{C}^2, 0)\) is called a nilpotent singularity if it is generated by a vector field \( X \) with a non-zero nilpotent linear part (that is, the matrix associated with the linear part of the field is nilpotent). The nilpotent singularities were generalized to cuspidal singularities by Loray \[Lo\], as we shall see below.

In this section we characterize when foliations with cuspidal singularities are of the second type in terms of weighted order. Furthermore, by means of the weighted order, we give necessary and sufficient conditions for these foliations to be of generalized curve type.

Given \( p, q \in \mathbb{N}^* \), we define the weighted degree of a monomial \( x^iy^j \) as

\[
\operatorname{deg}_{(p,q)}(x^iy^j) = \frac{ip + jq}{\gcd(p, q)},
\]

and the weighted order of a power series \( f(x, y) = \sum_{i,j} a_{i,j}x^iy^j \in \mathbb{C}\{x, y\} \) as

\[
\operatorname{ord}_{(p,q)}(f(x, y)) = \min \left\{ \operatorname{deg}_{(p,q)}(x^iy^j) : a_{i,j} \neq 0 \right\}.
\]

Remember that according to Loray \[Lo\], a foliation with a cuspidal singularity is given as in \((1)\), that is by

\[
F_{\omega_{p,q,\Delta}} : \omega_{p,q,\Delta} = d(y^p - x^q) + \Delta(x, y)(pxdy - qydx),
\]

where \( p, q \) are positive natural numbers and \( \Delta(x, y) \in \mathbb{C}\{x, y\} \).

On the other hand, remember that \( \operatorname{PH}_{(p,q)} := \operatorname{PH}(d(y^p - x^q)) = pq - p - q + 1 \).

Cuspidal foliations are nilpotent foliations when \( p = 2 \).

For Loray, the foliations \( F_{\omega_{p,q,\Delta}} \) and \( d(y^p - x^q) \) have the same resolution of singularities if and only if

\[
\operatorname{ord}_{(p,q)}(\Delta) > \frac{pq - p - q}{\gcd(p, q)} - \frac{\operatorname{PH}_{\omega_{p,q,\Delta}} - 1}{\gcd(p, q)}.
\]

Fernández, Mozo and Neciosup \[F-Moz-N\], find an imprecision in the characterization originally proposed by Loray. These authors mention that the condition is sufficient but not necessary, as can be seen from the following example.
Example 1. The foliation \( \omega = d(y^6 - x^3) + axy(6x^2y - 3y^4dx) \) with \( a \notin \{ -(6r + 1)c/r \in \mathbb{Q}_{>0} \ y \ c^3 = 1 \} \) has the same resolution as the foliation \( d(y^6 - x^3) = 0 \), but the function \( \Delta(x, y) = axy \) satisfies \( \text{ord}_{(6, 3)} \Delta = 3 \), so the inequality \( \text{ord}_{(6, 3)} \Delta > \frac{\text{PH}_{(p, q)} - 1}{\gcd(p, q)} \) does not hold.

For \( d = \gcd(p, q) \), we have

\[
y^p - x^q = \prod_{i=1}^{d}(y^\frac{p}{d} - \zeta^i x^\frac{q}{d}),
\]

and \( \gamma_i(t) = (t^\frac{p}{d}, A_i t^\frac{q}{d}) \) with \( A_i \frac{p}{d} = \zeta^i \) is a parameterization of \( \mathcal{S}_i : f_i(x, y) = (y^\frac{p}{d} - \zeta^i x^\frac{q}{d}) \), with \( \zeta \in \mathbb{C}, \zeta^d = 1 \). We get

\[
(\Delta, y^p - x^q)_0 = \sum_i (\Delta, f_i)_0 = d \cdot \text{ord}_{(p, q)}(\Delta),
\]

where \( (f, g)_0 = \text{dim}_\mathbb{C} \mathbb{C} \langle x, y \rangle / (f, g) \) is the intersection number of \( f \) and \( g \).

Lemma 5.1. If the cuspidal foliation \( \mathcal{F}_{\omega_{p, q}, \Delta} : \omega_{p, q}, \Delta = 0 \) is a non-dicritical foliation, then \( \mathcal{S}(\mathcal{F}_{\omega_{p, q}, \Delta}) = y^p - x^q = 0 \) is its union of separatrices.

Proof. The curve \( \mathcal{S}_f : y^p - x^q = 0 \) is an invariant curve of the foliation \( \mathcal{F}_{\omega_{p, q}, \Delta} \). Put \( \alpha = \text{ord}(\Delta) \). Then

\[
\text{mult}(\omega_{p, q}, \Delta) = \min\{q - 1, p - 1, \alpha + 1\}. \tag{5}
\]

Suppose that \( p < q \). The multiplicity of the curve \( \mathcal{S}_f \) is \( p \). If we assume that the curve \( \mathcal{S}_f \) is not the only separatrix of the foliation \( \mathcal{F}_{\omega_{p, q}, \Delta} \), then \( \text{mult}(\mathcal{S}(\mathcal{F}_{\omega_{p, q}, \Delta})) > p \). Using (5), we have \( \text{mult}(\omega_{p, q}, \Delta) = \min\{p - 1, \alpha + 1\} \). We will study both possibilities:

(i) If \( \text{mult}(\omega_{p, q}, \Delta) = p - 1 \) then \( p - 1 = \text{mult}(\omega_{p, q}, \Delta) \geq \text{mult}(\mathcal{F}_{\omega_{p, q}, \Delta})) - 1 > p - 1 \), which is a contradiction.

(ii) If \( \text{mult}(\omega_{p, q}, \Delta) = \alpha + 1 \) then \( \alpha + 1 = \text{mult}(\omega_{p, q}, \Delta) \geq \text{mult}(\mathcal{F}_{\omega_{p, q}, \Delta)}) - 1 > p - 1 \), which is a contradiction since \( \alpha + 1 \leq p - 1 \).

Therefore the union of separatrices of the foliation \( \mathcal{F}_{\omega_{p, q}, \Delta} \) is \( \mathcal{S}(\mathcal{F}_{\omega_{p, q}, \Delta}) = y^p - x^q \). The same reasoning happens when \( p \geq q \) and we conclude that \( \mathcal{S}(\mathcal{F}_{\omega_{p, q}, \Delta}) = y^p - x^q \).

\[
\tag*{\square}
\]

Proposition 5.2. Suppose that the cuspidal foliation \( \mathcal{F}_{\omega_{p, q}, \Delta} : \omega_{p, q}, \Delta = 0 \) is non-dicritical. If \( (\Delta, y^p - x^q)_0 \geq \text{PH}_{(p, q)} - 1 \), with \( d = \gcd(p, q) \), then the foliation \( \mathcal{F}_{\omega_{p, q}, \Delta} \) is of the second type.

Proof. Suppose without lost of generality that \( p < q \) and \( \text{ord}(\Delta) = i_0 + j_0 \). Since \( (\Delta, y^p - x^q)_0 \geq \text{PH}_{(p, q)} - 1 \) we have \( i_0p + j_0q \geq \text{PH}_{(p, q)} - 1 \). After \( p < q \) we get

\[
i_0q + j_0q > i_0p + j_0q \geq \text{PH}_{(p, q)} - 1,
\]

so \( i_0 + j_0 > p - 1 - \frac{q}{d} > p - 2 \) and \( \alpha = \text{ord}(\Delta) \geq p - 1 \). Since \( \text{mult}(df) = p - 1 \) for \( \mathcal{S}(\mathcal{F}_{\omega_{p, q}, \Delta}) : f(x, y) = y^p - x^q = 0 \), using (5) we have \( \text{mult}(\omega_{p, q}, \Delta) = p - 1 \). Hence \( \text{mult}(\omega_{p, q}, \Delta) = \text{mult}(df) \) and we conclude that the foliation \( \mathcal{F}_{\omega_{p, q}, \Delta} \) is of the second type.

\[
\tag*{\square}
\]

Proposition 5.3. Suppose that the cuspidal foliation \( \mathcal{F}_{\omega_{p, q}, \Delta} : \omega_{p, q}, \Delta = 0 \) is non-dicritical. If \( \mathcal{F}_{\omega_{p, q}, \Delta} \) is the second type, then \( (\Delta, y^p - x^q)_0 \geq \text{PH}_{(p, q)} - 1 \).
Proof. From Lemma 5.1 we get $\mathcal{S}(\mathcal{F}_{\omega_{p,q,\Delta}}) = y^p - x^q$. Put $d := \gcd(p, q)$. The line containing the only compact side of Newton polygon $\mathcal{N}(df)$ is $L : \frac{2}{5}i + \frac{2}{5}j = \frac{pq}{d}$. Since $\mathcal{F}_{\omega_{p,q,\Delta}}$ is of the second type, using Theorem 1.1 we have $\mathcal{N}(\omega_{p,q,\Delta}) = \mathcal{N}(f)$. Therefore, the line $L$ contains the only compact side of the Newton polygon of $\mathcal{N}(\omega_{p,q,\Delta})$, that is any $(a, b) \in \text{supp}(\omega_{p,q,\Delta})$ verifies $a\frac{d}{q} + b\frac{d}{p} \geq \frac{pq}{d}$. Suppose that $\Delta(x, y) = \sum_{i,j} a_{ij} x^i y^j \in \mathbb{C}\{x, y\}$, then

$$\omega_{p,q,\Delta} = \left( -qx^{q-1} - q \sum_{i,j} a_{ij} x^i y^{i+1} \right) dx + \left( py^{p-1} + p \sum_{i,j} a_{ij} x^{i+1} y^j \right) dy,$$

and $\text{supp}(\omega_{p,q,\Delta}) = \{(q, 0), (i+1, j+1)\}$ for $(i, j) \in \text{supp}(\Delta)$. If $(i+1, j+1) \in \text{supp}(\omega_{p,q,\Delta})$ then $(i+1)\frac{d}{q} + (j+1)\frac{d}{p} \geq \frac{pq}{d}$, so we conclude that $(\Delta, y^p - x^q)_0 = ip + jq \geq \text{PH}(p, q) - 1$. \hfill $\square$

Proposition 5.4. Suppose that the cuspidal foliation $\mathcal{F}_{\omega_{p,q,\Delta}} : \omega_{p,q,\Delta} = 0$ is non-dicritical. The foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ has the same reduction of singularities that $d(y^p - x^q)$, if and only if, $(\Delta, y^p - x^q)_0 \geq \text{PH}(p, q) - 1$.

Proof. Suppose that $(\Delta, y^p - x^q)_0 \geq \text{PH}(p, q) - 1$. By Proposition 5.2 the foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ is of the second type and by Theorem 2.2 we conclude that $\mathcal{F}_{\omega_{p,q,\Delta}}$ and $d(y^p - x^q)$ have the same reduction of singularities. Suppose now that $\mathcal{F}_{\omega_{p,q,\Delta}}$ and $d(y^p - x^q)$ have the same reduction of singularities. The curve $y^p = x^q = 0$ with $p > q$ and $d = \gcd(p, q)$ is desingularized by

$$E : (x, y) = (u^\frac{d}{q}, v^\frac{d}{p}),$$

such that $mp - nq = d$ and $m, n \in \mathbb{N}^*$. The transformation of

$$\omega_{p,q,\Delta} = (-qx^{q-1} - qy\Delta(x, y))dx + (py^{p-1} + px\Delta(x, y))dy,$$

by $E$ defined as (6) is

$$E^*\omega_{p,q,\Delta} = \left[ u^{nq-1}v^{\frac{dp}{q}} \right. \left. (-nq + mpv^{mp-nq}) + dv^{\frac{dp}{q}}u^{nq-1}(um^{n-nq}v^{\frac{mp}{p}+\frac{nq}{q}} - E^*(\Delta(x, y))) \right] du + \left[ \frac{pq}{d}u^{nq-1}(-1 + up^{mp-nq}) \right] dv$$

$$= \left( u^{nq-1}v^{\frac{dp}{q}-1} \right) v(-qn + mpud + \tilde{\Delta}(u, v))du + \frac{pq}{d}u(u^d - 1)dv, \tag{7}$$

where

$$\tilde{\Delta}(u, v) = \frac{dE^*(\Delta(x, y))u^{m+n-nq}v^{\frac{mp}{p}+\frac{nq}{q}}}{dE^*(\Delta(x, y))um^{m-nq}v^{\frac{mp}{p}+\frac{nq}{q}}} = \sum_{i,j} da_{ij} u^{n+m+j+m-nq}v^{\frac{mp}{p}+\frac{nq}{q}+j+nq}.$$

Hence

$$\frac{E^*\omega_{p,q,\Delta}}{u^{nq-1}v^{\frac{dp}{q}-1}} = v(-qn + mpud + \tilde{\Delta}(u, v))du + \frac{pq}{d}u(u^d - 1)dv, \tag{8}$$

which singularities are $(0, 0)$ and $(\zeta^j, 0)$, where $\zeta$ is a $d$th-primitive root of the unity. The dual vector field associated with the foliation defined by (8) is

$$X = \frac{pq}{d}(u^d - 1)u \frac{\partial}{\partial u} - v(-qn + mpud + \tilde{\Delta}(u, v)) \frac{\partial}{\partial v}.$$
and the matrix associated with this field is
\[
 DX = \begin{pmatrix}
 -\frac{pq}{d} + \frac{(d+1)pq}{d} u^d & 0 \\
 * & nq - mpv^d - \Delta(u,v) - v\frac{\partial \Delta(u,v)}{\partial v} 
\end{pmatrix}.
\]

(1) In (0, 0) we have \( DX = \begin{pmatrix}
 -\frac{pq}{d} & 0 \\
 * & nq 
\end{pmatrix} \). Therefore, the singularity (0, 0) is not degenerate.

(2) If \( u^d = 1 \) and \( v = 0 \) then we get \( DX = \begin{pmatrix}
 -\frac{pq}{d} & 0 \\
 * & -d - \Delta(u,v) 
\end{pmatrix} \). Since the foliation is reduced, it could happen that
- \(-d - \Delta(\zeta', 0) = 0\), from where \( \Delta(\zeta', 0) = -d \), in which case the singularity is of saddle-node type.
- \(-d - \Delta(\zeta', 0) = -a\), so that \( \lambda = \frac{pq}{d} \notin \mathbb{Q}^+ \), in this case, the singularity is of a no degenerate type.

We conclude that \( \text{ord}_a \Delta \geq 0 \), so \( p_i + q_j + p + q \geq 0 \) for some \((i, j)\). Therefore \( (\Delta, y^p - x^q)_0 \geq \text{PH}(p,q) - 1 \) for some \((i, j)\) \( \in \text{supp}(\Delta) \).

\[\square\]

**Proof of Theorem** \[\text{[1,2]}\] The equivalence of statements (a) and (b) is a direct consequence of Propositions \[\text{[5,2]}\] and \[\text{[5,3]}\]. The equivalence of statements (b) and (c) is Proposition \[\text{[5,4]}\].

**Corollary 5.5.** Suppose that the cuspidal foliation \( F_{\omega_{p,q},\Delta} : \omega_{p,q,\Delta} = 0 \) is non-dicritical. If the foliation \( F_{\omega_{p,q},\Delta} : \omega_{p,q,\Delta} = 0 \) is of the generalized curve type then \( (\Delta, y^p - x^q)_0 \geq \text{PH}(p,q) - 1 \).

The fact that the foliation \( F_{\omega_{p,q},\Delta} \) is of generalized curve type does not imply that \( (\Delta, y^p - x^q)_0 > \text{PH}(p,q) - 1 \), as the next example shows:

**Example 2.** The foliation
\[\omega = d(y^6 - x^3) + axy(6xy - 3y^2),\]
with \( a \in \{-(6r+1)\zeta' / r \in \mathbb{Q} \} \subseteq \mathbb{C}^* \), and \( a^3 \neq -1 \) is of the generalized curve type, but \( (\Delta, y^p - x^q)_0 = 3 = \text{PH}(p,q) - 1 \), where \( p = 6, q = 3 \) and \( d = 3 \).

Nevertheless

**Proposition 5.6.** Suppose that the cuspidal foliation \( F_{\omega_{p,q},\Delta} : \omega_{p,q,\Delta} = 0 \) is non-dicritical and \( p \) and \( q \) are coprimes. The foliation \( F_{\omega_{p,q},\Delta} \) is of generalized curve type, if and only if \( (\Delta, y^p - x^q)_0 > \text{PH}(p,q) - 1 \).

**Proof.** Let us consider \( \omega_{p,q,\Delta} = (-g_{x^p-1} - qy\Delta)dx + (py^{p-1} + px\Delta)dy, f(x,y) = y^p - x^q \) and \( \gamma(t) = (t^p, t^q) \) a parameterization of \( f(x,y) = 0 \). Thus \[
\text{GSV}(F_{\omega_{p,q},\Delta}, S(F_{\omega_{p,q},\Delta})) = \text{ord}_t \left( \frac{py^{p-1} + px\Delta}{py^{p-1}} (t^p, t^q) \right) = \text{ord}_t \left( 1 + \frac{t^p \Delta(t^p, t^q)}{t^q (t^p-1)} \right),
\]
where \( \Delta(t^p, t^q) = \sum_{ij} a_{ij} t^{p_i + q_j} \). Note that \[
\text{GSV}(F_{\omega_{p,q},\Delta}, S(F_{\omega_{p,q},\Delta})) = 0, \text{ if and only if } \text{ord}_t \left( 1 + \frac{t^p \Delta(t^p, t^q)}{t^q (t^p-1)} \right) = 0,
\]
what is equivalent to $(\Delta, y^p - x^q)_0 > \text{PH}(p, q) - 1$. From Theorem 3.4, we conclude that $F_{\omega_{p,q,\Delta}}$ is of the generalized curve type, if and only if $(\Delta, y^p - x^q)_0 > \text{PH}(p, q) - 1$. □

Suppose now that $p$ and $q$ are not coprime. We will analyze if the strict inequality $(\Delta, y^p - x^q)_0 > \text{PH}(p, q) - 1$ is a sufficient condition for $F_{\omega_{p,q,\Delta}}$ to be a foliation of generalized curve type. We begin studying what happens when $d = \gcd(p, q) = 2$.

Let us consider $S : f = f_1 f_2$ and $g \omega = h d (f_1 f_2) + f_1 f_2 \eta$. For $S_i : f_i(x, y) = 0$, we have $\text{GSV}(F, S_1) = \frac{1}{2\pi i} \int_{\partial S_1} \frac{d(\frac{\omega}{g})}{\bar{g}} + (f_2, f_1)_0$. Analogously, $\text{GSV}(F, S_2) = \frac{1}{2\pi i} \int_{\partial S_2} \frac{d(\frac{\omega}{g})}{\bar{g}} + (f_1, f_2)_0$. We have

$$\frac{1}{2\pi i} \int_{\partial S_1 \cup \partial S_2} \frac{d(\frac{\omega}{g})}{\bar{g}} = \text{GSV}(F, S_1) + \text{GSV}(F, S_2) - 2(f_1, f_2)_0.$$  

Therefore (see [Br page 532]),

$$\text{GSV}(F, S) = \text{GSV}(F, S_1) + \text{GSV}(F, S_2) - 2(f_1, f_2)_0. \quad (10)$$

For $d = 2 = \gcd(p, q)$, we have

$$y^p - x^q = \prod_{i=1}^{2}(y^\frac{p}{2} - \zeta^i x^\frac{q}{2}), \text{ with } \zeta^2 = 1.$$  

Let $S_i : f_i(x, y) = (y^\frac{p}{2} - \zeta^i x^\frac{q}{2})$ and $\gamma_i(t) = (t^\frac{p}{2}, A_i t^\frac{q}{2})$ with $A_i^\frac{q}{2} = \zeta^i$ a parameterization of $S_i$. Then

$$(f_1, f_2)_0 = \text{ord}_i(f_1(\gamma_2(t))) = \text{ord}_i(t^\frac{p}{2}(1 - \zeta)) = \frac{pq}{4}. \quad (11)$$

Remember that $\omega_{p,q,\Delta} = (-qx^{q-1} - qy\Delta)dx + (py^{p-1} + px\Delta)dy$, thus

$$\text{GSV}(F, S_1) = \text{ord}_i \left( t^\frac{p}{2}(1 - \zeta) + \frac{\frac{q}{2} + \frac{p}{2} - \frac{pq}{4}}{A_i^\frac{q}{2}} t^\frac{q}{2} + \Delta(t^\frac{p}{2}, A_1^\frac{q}{2}) \right). \quad (12)$$

If we consider $(\Delta, y^p - x^q)_0 > \text{PH}(p, q) - 1$, from (12) we have that $\text{GSV}(F, S_1) = \frac{pq}{4}$. Similarly, it turns out that $\text{GSV}(F, S_2) = \frac{pq}{4}$.

After (11) and (10) we have $\text{GSV}(F, S) = 0$, which is equivalent to $\omega_{p,q,\Delta}$, so the foliation $F$ generalized curve type when $d = 2$.

In general [Br], when $S : f = f_1 \cdots f_d$, we have to

$$\text{GSV}(F, S) = \sum_{i=1}^{d} \text{GSV}(F, S_1) - 2 \sum_{i \neq j}^{N} (f_i, f_j)_0, \quad (13)$$

where $N = \binom{d}{2}$, $\text{GSV}(F, S_i) = \frac{(d-1)pq}{d^2}$, and $(f_i, f_j)_0 = \frac{pq}{d^2}$. Therefore, from (13) we get

$$\text{GSV}(F, S) = 0. \quad (14)$$

Hence the following proposition holds.

**Proposition 5.7.** Let $F_{\omega_{p,q,\Delta}}$ be a non dicritical foliation and suppose that $(\Delta, y^p - x^q)_0 > \text{PH}(p,q) - 1$. Then $F_{\omega_{p,q,\Delta}}$ is of the generalized curve type.
Proof of Theorem 1.3. It is an immediate consequence of Propositions 5.6 and 5.7.

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CHARACTERIZATION OF SECOND TYPE PLANE FOLIATIONS USING NEWTON POLYGONS

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