Functional Determinant on Pseudo-Einstein 3-manifolds

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**Abstract** Given a three dimensional pseudo-Einstein CR manifold \((M, T^{1,0}M, \theta)\), we establish an expression for the difference of determinants of the Paneitz type operators \(A_\theta\), related to the problem of prescribing the \(Q'\)-curvature, under the conformal change \(\theta \mapsto e^w \theta\) with \(w \in \mathcal{P}\) the space of pluriharmonic functions. This generalizes the expression of the functional determinant in four dimensional Riemannian manifolds established in \([6]\).

1 Introduction and statement of the results

There has been extensive work on the study of spectral invariants of differential operators defined on a Riemannian manifold \((M, g)\) and the relations to their conformal invariants, see for instance \([4, 6, 5]\) and the references therein. For instance, consider the two dimensional case with the pair of the Laplace operator \(-\Delta_g\), and the associated invariant which is the scalar curvature \(R_g\). One knows that under conformal change of the metric \(g \mapsto \tilde{g} = e^{2w} g\), one has the relation

\[
R_{\tilde{g}} e^{2w} = -\Delta_g w + R_g.
\]

It is also known that the spectrum of \(-\Delta_g\) is discrete and can be written as \(0 < \lambda_1 \leq \lambda_2 \leq \cdots\) and the corresponding zeta function is then defined by

\[
\zeta_{-\Delta_g}(s) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s}.
\]

This series converges uniformly for \(s > 1\) and can be extended to a meromorphic function in \(\mathbb{C}\) with 0 as a regular value. The determinant of the operator \(-\Delta_g\) can be written as

\[
det(-\Delta_g) = e^{-\zeta'_{-\Delta_g}(0)}.
\]

The celebrated Polyakov formula \([21, 22]\), states that if \(\tilde{g} = e^{2w} g\) then

\[
\ln \left( \frac{\det(-\Delta_{\tilde{g}})}{\det(-\Delta_g)} \right) = \frac{1}{12} \int_M |\nabla w|^2 + R w \, dv_g - \ln \left( \int_M e^{2w} \, dv_g \right).
\]

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Notice that the right hand side is a familiar quantity. It is the Beckner-Onorfi energy. \[ \Pi \] and we know that
\[
\frac{1}{12\pi} \int_{S^2} |\nabla w|^2 + R w \, dv_g - \ln(\int_{S^2} e^{2w} \, dv_g) \geq 0.
\]
This notion of determinant was extended to dimension four for conformally invariant operators. Keeping in mind that the substitute of the pair \((-\Delta_g, R)\) in dimension two is the pair \((P_g, Q_g)\) in dimension four, where \(P_g\) is the Paneitz operator and \(Q_g\) is the Riemannian Q-curvature \((\text{[6, 11]})\). In addition, two new terms appear in the expression of the determinant. Indeed, if \((M, g)\) a 4-dimensional manifold and \(A_g\) a non-negative self-adjoint conformally covariant operator then, there exists \(\beta_1, \beta_2\) and \(\beta_3 \in \mathbb{R}\) such that
\[
\ln \left( \frac{\det(A_g)}{\det(A_g)} \right) = \beta_1 I(w) + \beta_2 II(w) + \beta_3 III(w),
\]
where
\[
I(w) = 4 \int_M w |W_g|^2 \, dv_g - (\int_M |W_g|^2 \, dv_g) \log(\int_M e^{4w} \, dv_g),
\]
\[
II(w) = \int_M w P_g w + 4Q_g w \, dv_g - (\int_M Q_g \, dv_g) \log(\int_M e^{4w} \, dv_g),
\]
\[
III(w) = 12 \int_M (\Delta_g w + |\nabla w|^2)^2 \, dv_g - 4 \int_M w |\nabla w|^2 \, dv_g.
\]
In the case of the sphere \(S^4\), we see that the second term \(II\) corresponds to the Beckner-Onorfi energy.

Now let us move to the CR setting. We consider a 3-dimensional CR manifold \((M, T^{1,0}, J, \theta)\) and we recall that the substitute for the pair \((P_g, Q_g)\) is \((P_\theta, Q_\theta)\) where \(P_\theta\) is the CR Paneitz operator and \(Q_\theta\) is the CR Q-curve, \([12, 13]\). The problem with this pair is that the total Q-curvature is always zero. In fact in pseudo-Einstein manifolds the Q-curvature vanishes identically. Hence, we do not have a decent substitute for the CR Beckner-Onorfi inequality. Fortunately, if we restrict our study to pseudo-Einstein manifolds and variations in the space of pluri-harmonic functions \(\mathcal{P}\), then we have a better substitute for the pair \((P_g, Q_g)\) namely \((P_\theta', Q_\theta')\). These quantities were first introduced on odd dimensional spheres in \([3]\) and then on pseudo-Einstein manifolds in \([9, 8, 14]\). In particular one has a Beckner-Onorfi type inequality involving the operator \(P_\theta'\) acting on pluri-harmonic functions as proved in \([9]\). We also recall that the total \(Q'\)-curvature corresponds to a geometric invariant, namely the Burns-Epstein invariant \(\mu(M)\) \((\text{[17, 10]}\)).

One is tempted to see what the spectral invariants of the operator \(P'\) are or the restriction of \(P'\) to the space \(\mathcal{P}\) of pluri-harmonic functions and link them to geometric quantities such as the total \(Q'\)-curvature.

We recall that the quantity \(Q_\theta'\) changes as follows: if \(\tilde{\theta} = e^{w} \theta\) with \(w \in \mathcal{P}\), then
\[
P_\theta' w + Q_\theta' = Q_\theta' e^{2w} + \frac{1}{2} P_\theta(\theta^2),
\]
which we can write as
\[
P_\theta' w + Q_\theta' = Q_\theta' e^{2w} \mod \mathcal{P}^\perp.
\]
We let $\tau_\theta : L^2 \to P$ be the orthogonal projection on $P$ with respect to the inner product induced by $\theta$ and set $A_\theta = \tau_\theta P_\theta^\prime \tau_\theta$. Then equation (2) can be rewritten as

$$A_\theta w + \tau_\theta (Q_\theta') = \tau_\theta (Q_\theta'e^{2w}).$$

Prescribing the quantity $\overline{Q}_\theta = \tau_\theta (Q_\theta')$ was thoroughly investigated in [16, 9, 15] mainly because of the property that

$$\int_M Q_\theta' d\nu_\theta = \int_M Q_\theta' d\nu_\theta = -\frac{\mu(M)}{16\pi^2}.$$

We recall that in [17], we proved that the dual of the Beckner-Onofri inequality, namely the logarithmic Hardy-Littlewood-Sobolev inequality can be linked to the regularized zeta function of the operator $A_\theta$ evaluated at one. This was proved in the Riemannian setting in [18, 19, 20].

In this paper, we will generalize the expression (1) by studying the determinant of the operator $A_\theta$. In all that follows we assume that $(M,T^{1,0}M,J,\theta)$ is an embeddable pseudo-Einstein manifold such that $P_\theta'$ is non-negative and has trivial kernel.

First we show that

**Theorem 1.1 (Conformal Index).** Let $\zeta_{A_\theta}$ be the spectral zeta function of the operator $A_\theta$. Then $\zeta_{A_\theta}(0)$ is a conformal invariant in $P$. Moreover,

$$\zeta_{A_\theta}(0) = -\frac{1}{24\pi^2} \int_M Q_\theta d\nu_\theta - 1.$$

In order to compute the determinant of the operator $A_\theta$ we introduce the quantities $\tilde{A}_1(w)$, $\tilde{A}_2(w)$ and $\tilde{A}_3(w)$ defined by

$$\tilde{A}_1(w) := \int_M w A_\theta w + Q_\theta' w - \frac{1}{c_1} \ln \left( \int_M e^{2w} d\nu \right) d\nu$$

$$\tilde{A}_2(w) := 2 \int_M R \left( \Delta_\theta w + \frac{1}{2} |\nabla_\theta w|^2 \right) - \left( \Delta_\theta w + \frac{1}{2} |\nabla_\theta w|^2 \right)^2 d\nu$$

$$\tilde{A}_3(w) := 2 \int_M w_0 R - \frac{1}{3} w_0 |\nabla_\theta w|^2 - w_0 \Delta_\theta w d\nu$$

Notice that the quantity $\tilde{A}_1$ is related to the CR Beckner-Onofri inequality studied in [3]. One can also write $\tilde{A}_2(w)$ as

$$\tilde{A}_2(w) = 2 \int_M R \left( \frac{\Delta_\theta e^{\frac{1}{2}w}}{2e^{\frac{1}{2}w}} \right) - \left( \frac{\Delta_\theta e^{\frac{1}{2}w}}{2e^{\frac{1}{2}w}} \right)^2 d\nu.$$

Then we have the following

**Theorem 1.2.** There exists $c_2$ and $c_3 \in \mathbb{R}$ such that for all $w \in P$, we have

$$\ln \left( \frac{\det(A_\theta)}{\det(A_{e^w\theta})} \right) = -\frac{1}{24\pi^2} \tilde{A}_1(w) + c_2 \tilde{A}_2(w) - c_3 \tilde{A}_3(w).$$

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2 Heat Coefficients and Conformal Invariance

Let \((M, T^{1,0}M, \theta)\) be a pseudo-Einstein 3-manifold and \(P'_\theta\) its \(P'\)-operator defined by
\[
P'_\theta f = 4\Delta^2 f - 8\text{Im} \left( \nabla^1 (A_{11} \nabla^1 f) \right) - 4\text{Re} \left( \nabla^1 (R \nabla^1 f) \right).
\]
Denote by \(\tau: L^2(M) \to \mathcal{P}\) the orthogonal projection on the space of pluriharmonic functions with respect to the \(L^2\)-inner product induced by \(\theta\). We consider the operator \(A_\theta = \tau P'_\theta \tau\) and for the conformal change \(\tilde{\theta} = e^{w}\theta\), with \(w \in \mathcal{P}\), we let
\[
A_{\tilde{\theta}} = \tau_{\tilde{\theta}}^* e^{-2w} A_\theta,
\]
where \(\tau_{\tilde{\theta}}\) is the orthogonal projection with respect to the \(L^2\)-inner product induced by \(\tilde{\theta}\).

In order to evaluate and manipulate the spectral invariants, we need to study the expression of the heat kernel of the operator \(A_\theta\). Unfortunately, this operator is not elliptic or sub-elliptic (as an operator on \(C^\infty(M)\)), and does not have an invertible principal symbol in the sense of \(\Psi^4_H(M)\)-calculus (see [23]). In fact \(A_\theta\) can be seen as a Toeplitz operator, and one might adopt the approach introduced in [2] in order to study it. But instead, we will modify the operator in order to be able to use the classical computations done for the heat kernel.

Consider the operator \(L = A_\theta + \tau L \tau = A_\theta\), where \(L\) is chosen so that \(L\) has an invertible principal symbol in \(\Psi^4_H(M)\). Notice that \(\tau L = L \tau = A_\theta\). Based on [23], if \(K\) is the heat kernel of \(L\) one has the following expansion near zero:
\[
K(t, x, x) \sim \sum_{j=0}^{\infty} \tilde{a}_j(x)t^{j-4} + \ln(t) \sum_{j=1}^{\infty} t^j \tilde{b}_j(x).
\]
Since \(e^{-tL} = e^{-tA_\theta} + e^{-tL_{\perp}}\), we have that the kernel \(K\) of \(e^{-tA_\theta}\) which is the restriction to \(\mathcal{P}\) of \(K\), reads as
\[
K(t, x, x) \sim \sum_{j=0}^{\infty} t^{j+4} \tilde{a}_j(x) + \ln(t) \sum_{j=1}^{\infty} t^j \tilde{b}_j(x),
\]
and this will be the main expansion that we will be using for the rest of the paper.

Now we want to define the infinitesimal variation of a quantity under a conformal change. Fix \(w \in \mathcal{P}\) and for a given quantity \(F_\theta\) depending on \(\theta\) denote \(\delta F_\theta := \frac{d}{dr}|_{r=0} F_{e^{r\theta}}\). Next, we define the zeta function of \(A_\theta\) by
\[
\zeta_{A_\theta}(s) := \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s},
\]
where \(0 < \lambda_1 \leq \lambda_2 \leq \cdots\), is the spectrum of the operator \(A_\theta : W^{2,2}(M) \cap \mathcal{P} \to \mathcal{P}\). In what follows \(\text{TR}[A]\) is to be understood as the trace of the operator \(A\) in \(\mathcal{P}\). Then we have the following proposition.

**Proposition 2.1.** With the notations above, we have
\[
\zeta_{A_\theta}(0) = \int_M a_4(x)dx - 1.
\]
Moreover,
\[
\delta \zeta_{A_\theta}(0) = 0.
\]
and
\[ \delta \zeta_{A_0}'(0) = 2 \int_M w(a_4(x) - \frac{1}{V}) \, d\nu, \]

where \( V = \int_M d\nu_0 \) is the volume of \( M \).

Proof: Most of the computations in this part are relatively standard and they can be found in [4, 6, 5] in the Riemannian setting. First we use the Mellin transform and (4) to write
\[
\zeta_{A_0}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( TR[e^{-tA_0}] - 1 \right) \, dt
\]
\[
= \frac{1}{\Gamma(s)} \left( -\frac{1}{s} + \int_0^1 t^{s-1} \sum_{j=0}^N t^{j+1} \int_M a_j(x) \, d\nu \, dt + \int_M t^{s-1} O(t^{N+1/4}) \, dt \right)
\]
\[
+ \sum_{j=1}^N \int_0^1 t^{j+s-1} \ln(t) \int_M b_j \, d\nu \, dt + \int_0^1 t^{s-1} O(t^{N+1} \ln(t)) \, dt + \int_1^\infty t^{s-1} \sum_{j=1}^\infty e^{-\lambda_j t} \, dt \right)
\]
\[
= \frac{1}{\Gamma(s)} \left( -\frac{1}{s} + \sum_{j=0}^N \frac{1}{s + \frac{j+1}{4}} \int_M a_j(x) \, d\nu + \int_0^1 t^{s-1} O(t^{N+1/4}) \, dt \right)
\]
\[
+ \sum_{j=1}^N \frac{1}{(s+2)^j} \int_M b_j \, d\nu + \int_0^1 t^{s-1} O(t^{N+1} \ln(t)) \, dt \int_1^\infty t^{s-1} \sum_{j=1}^\infty e^{-\lambda_j t} \, dt \right). \]

Since, \( \Gamma \) has a simple pole at \( s = 0 \) with residue 1, we see that by taking \( s \to 0 \), there are only two terms that survive, leading to
\[
\zeta_{A_0}(0) = \int_M a_4(x) \, d\nu - 1.
\]

Next we move to the study of the variation of \( \zeta_{A_0} \). Let \( f \in C^\infty(M) \) and \( v \in \mathcal{P} \). Then
\[
\int_M \tau_{rw}(f) \, d\nu_{rw} = \int_M f \, we^{2rw} \, d\nu.
\]

Differentiating with respect to \( r \) and evaluating at 0 yields
\[
\int_M (\delta \tau(f) + 2w\tau(f) - 2wF) \, d\nu = 0.
\]

Hence,
\[
\delta \tau(f) = 2\tau(wf - w\tau(f)).
\]

If we let \( M_w \) be the multiplication by \( w \) then
\[
\delta \tau = 2(\tau M_w - \tau M_w \tau).
\]

In particular, if \( f \in \mathcal{P} \) then \( \delta \tau(f) = 0 \).

Next we want to evaluate \( \delta A_0 \). Recall that \( A_{e^rA_0} = \tau_{e^rA_0} e^{-2rw} A_0 \). Therefore,
\[
\delta A_0 = \delta \tau A_0 - 2\tau M_w A_0 = -2\tau M_w A_0.
\]
Thus,

\[
\delta T R[e^{-tA\theta}] = -t T R[\delta A \theta e^{-tA\theta}]
\]

\[
= 2t T R[\tau M_w A \theta e^{-tA\theta}] = 2t T R[M_w A \theta e^{-tA\theta}].
\]

The last equality follows from \( T R[AB] = T R[BA] \) and \( e^{-tA\theta} \tau = e^{-tA\theta} \). But

\[
-t M_w A \theta e^{-tA\theta} = \frac{d}{d\varepsilon}|_{\varepsilon=0} M_w e^{-t(1+\varepsilon)A\theta}
\]

Using the expansion (4), we have

\[
K(t(1 + \varepsilon), x, x) \sim \sum_{j=0}^{\infty} (1 + \varepsilon)^\frac{j-4}{4} t^\frac{j-4}{4} a_j(x) + H,
\]

where \( H \) is the logarithmic part. Hence, comparing the terms in the expansion after integration, we get:

\[
\delta \int_M a_j \, d\nu = \frac{4 - j}{2} \int_M w a_j \, d\nu.
\]

In particular, we have \( \delta \int_M a_4 \, d\nu = 0 \).

Similarly

\[
\Gamma(s) \zeta A \theta(s) = \Gamma(s) (\zeta A \theta(0) + s \zeta' A \theta(0) + O(s^2)).
\]

Hence, since \( \delta \zeta A \theta(0) = 0 \), and \( s \Gamma(s) \sim 0 \) when \( s \to 0 \), we have

\[
\delta \zeta' A \theta(0) = [\Gamma(s) \delta \zeta A \theta(s)]_{s=0}.
\]

But,

\[
\Gamma(s) \delta \zeta A \theta(s) = \int_0^\infty 2 t^s T R[w A \theta e^{-tA\theta}] \, dt
\]

\[
= - \int_0^\infty 2 t^s \frac{d}{dt} T R[w e^{-tA\theta}] \, dt
\]

\[
= \int_0^\infty 2s t^{s-1} T R[w(e^{-tA\theta} - \frac{1}{V})] \, dt. \tag{5}
\]

Using again the expansion (4) and a similar computation as in the previous case, yields

\[
\delta \zeta' A \theta(0) = 2 \int_M w(a_4 - \frac{1}{V}) \, d\nu.
\]

\[\square\]

**Proposition 2.2.** There exists \( c \neq 0 \) such that

\[
\zeta A \theta(0) = c \int_M Q' \theta \, d\nu - 1.
\]

Moreover \( c = -\frac{1}{24\pi^2} \).
Proof: First we notice that $a_4$ is a pseudo-Hermitian invariant of order $-2$, that is

$$a_4 e^{r\theta} = e^{-2r} a_4 \theta,$$

for all $r \in \mathbb{R}$. So from [14], we have the existence of $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$ such that

$$a_4 = c_1 Q'_\theta + c_2 \Delta_b R + c_3 R_{\theta,0} + c_4 R^2_\theta + c_5 Q_\theta,$$

where $Q'_\theta = 2\Delta_b R - 4|A|^2 + R^2$ and $Q_\theta = -2\Delta_b R + 2Im(A_{11,11})$. Since we are in a pseudo-Einstein manifold and $w \in \mathcal{P}$ we can assume that $Q_\theta = 0$. So after integration, we have

$$\int_M a_4 \, d\nu = c_1 \int_M Q'_\theta \, d\nu + c_4 \int_M R^2 \, d\nu.$$ 

Since $\int_M a_4 \, d\nu$ is invariant under the conformal change $e^w \theta$, it is easy to see that $c_4 = 0$. Hence,

$$\int_M a_4 \, d\nu = c_1 \int_M Q'_\theta \, d\nu.$$ 

Next we want to calculate $c_1$ (compare to [24], where the invariant $k_2$ is always 0). We take the case of the sphere $S^3$. Based on the computations in [3], we have

$$\zeta_{A_\theta}(s) = 2 \sum_{j=1}^{\infty} \frac{j + 1}{(j(j + 1))^s} = 2 \sum_{j=2}^{\infty} \frac{1}{j^{2s-1}} \left( \frac{1}{1 - \frac{1}{j}} \right)^s.$$ 

Using the expansion of $(1 - \frac{1}{j})^{-s} = 1 + \frac{s}{j} + \frac{s(s+1)}{2j^2} + O(\frac{1}{j^3})$, we see that

$$\zeta_{A_\theta}(s) = 2(\zeta_R(2s - 1) - 1 + s(\zeta_R(2s) - 1) - \frac{s(s+1)}{2} (\zeta_R(2s + 1) - 1)) + sH(s).$$ 

with $H(s)$ holomorphic near $s = 0$ and $\zeta_R$ the classical Riemann Zeta function. Now we recall that $\zeta_R$ is regular at $s = 0$ and $s = -1$ but has a simple pole at $s = 1$ with residue equal to 1. Hence

$$\zeta_{A_\theta}(0) = 2(-\frac{1}{12} - 1 + \frac{1}{4}) = -\frac{5}{3} \neq 0.$$ 

Knowing that $\int_{S^3} Q'_\theta \, d\nu = 16\pi^2$ we have

$$16\pi^2 c - 1 = -\frac{5}{3}.$$ 

Thus,

$$c = -\frac{1}{24\pi^2}.$$ 

\[\square\]
3 The expression for the Determinant

Recall that in the previous section, we found that \( a_4 = c_1 Q' + c_2 \Delta_b R + c_3 R_0 \). In particular

\[
\delta'_{\lambda_b}(0) = \int_M 2w \left( a_4(x) - \frac{1}{V} \right) \, d\nu
\]

\[
= c_1 \int_M 2w \left( Q'_\theta - \frac{1}{c_1 V} \right) \, d\nu + c_2 \int_M 2R \Delta_b w \, d\nu - c_3 \int_M 2w_0 R \, d\nu
\]

\[
= c_1 A_1 + c_2 A_2 + c_3 A_3.
\]

We will calculate the change of each term under conformal change of \( \theta \). The easiest term to handle is the first one. Indeed, recall that if \( \tilde{\theta} = e^w \theta \) then

\[
\tilde{Q}'e^{2w} = P'_{\tilde{\theta}} w + Q'_{\tilde{\theta}} \mod \mathcal{P}^1,
\]

\[
\tilde{R} = \left[ R - |\nabla_b w|^2 - 2\Delta_b w \right] e^{-w},
\]

and

\[
\tilde{\Delta}_b f = e^{-w} \left[ \Delta_b f + \nabla_b f \cdot \nabla_b w \right].
\]

So if \( \tilde{\theta} = e^{uw} \theta \), we have

\[
\int_M 2w \left[ \tilde{Q}'_\theta - \frac{1}{c_1} \frac{1}{V} \right] \, d\tilde{\nu} = \int_M 2uw P'_{\tilde{\theta}} w + 2uQ'_{\tilde{\theta}} w - \frac{1}{c_1} \int_M \frac{2we^{2uw}}{e^{2uw}} \, d\nu.
\]

Integrating \( u \) in \([0,1]\) yields

\[
\tilde{A}_1(w) = \int_M w A_\theta w + Q'_{\tilde{\theta}} w - \frac{1}{c_1} \ln(\int_M e^{2w} \, d\nu) \, d\nu.
\]

For the second term, we have

\[
\int_M \tilde{R} \Delta_b w \, d\tilde{\nu} = \int_M \left[ R - u^2 |\nabla_b w|^2 - 2u \Delta_b w \right] \left[ \Delta_b w + u |\nabla_b w|^2 \right] \, d\nu
\]

\[
= \int_M R \Delta_b w - u^2 |\nabla_b w|^2 \Delta_b w - 2u (\Delta_b w)^2 + Ru |\nabla_b w|^2 - u^3 |\nabla_b w|^4 - 2u^2 |\nabla_b w|^2 \Delta_b w \, d\nu.
\]

In particular after integrating over \( u \) between 0 and 1, we get

\[
\tilde{A}_2(w) = 2 \int_M R \Delta_b w - |\nabla_b w|^2 \Delta_b w - (\Delta_b w)^2 + \frac{1}{2} R |\nabla_b w|^2 - \frac{1}{4} |\nabla_b w|^4 \, d\nu
\]

\[
= 2 \int_M R \Delta_b w - \left( \Delta_b w + \frac{1}{2} |\nabla_b w|^2 \right)^2 + \frac{R}{2} |\nabla_b w|^2 \, d\nu.
\]

Next compute

\[
\int_M \tilde{T} w \tilde{R} \, d\tilde{\nu} = \int_M \left[ w_0 R - u^2 w_0 |\nabla_b w|^2 - 2uw_0 \Delta_b w \right] \, d\nu,
\]

where \( T \) is the characteristic vectorfield of \( \theta \) and we are adopting the notation \( Tf = f_0 \).

Integrating as above yields:

\[
\tilde{A}_3(w) = 2 \int_M w_0 R - \frac{1}{3} w_0 |\nabla_b w|^2 - w_0 \Delta_b w \, d\nu.
\]
Therefore, one has
\[ \zeta'_{\bar{g}}(0) - \zeta'_{g}(0) = c_1 \tilde{A}_1(w) + c_2 \tilde{A}_2(w) - c_3 \tilde{A}_3(w) \]
or equivalently
\[ \ln \left( \frac{\det(A_{\theta})}{\det(A_{\bar{\theta}})} \right) = c_1 \tilde{A}_1(w) + c_2 \tilde{A}_2(w) - c_3 \tilde{A}_3(w). \]

\[ \square \]

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