RELATIVE SINGULAR VALUE DECOMPOSITION AND APPLICATIONS TO LS-CATEGORY

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Abstract. Let \( \text{Sp}(n) \) be the symplectic group of quaternionic \((n \times n)\)-matrices. For any \( 1 \leq k \leq n \), an element \( A \) of \( \text{Sp}(n) \) can be decomposed in \( A = \begin{bmatrix} \alpha & T \\ \beta & P \end{bmatrix} \) with \( P \) a \((k \times k)\)-matrix.

In this work, starting from a singular value decomposition of \( P \), we obtain what we call a relative singular value decomposition of \( A \). This feature is well adapted for the study of the quaternionic Stiefel manifold \( X_{n,k} \), and we apply it to the determination of the Lusternik-Schnirelmann category of \( \text{Sp}(k) \) in \( X_{2k-j,k} \), for \( j = 0, 1, 2 \).

1. Introduction

Let \( \mathbb{H}^n \) be the quaternionic \( n \)-space (with the structure of a right \( \mathbb{H} \)-vector space) endowed with the Hermitian product \( \langle u, v \rangle = u^* v \). For \( 0 < k \leq n \), we denote by \( \text{Sp}(n) \) the Lie group of matrices, \( A \in \mathbb{H}^{n \times n} \), such that \( AA^* = I_n \) and by \( X_{n,k} \) the Stiefel manifold of linear maps \( \phi: \mathbb{H}^k \to \mathbb{H}^n \) which preserve the Hermitian product. Alternatively, the elements of \( X_{n,k} \) are the orthonormal \( k \)-frames of \( \mathbb{H}^n \), represented by a matrix \( x \in \mathbb{H}^{n \times k} \) such that \( x^* x = I_k \). Usually we shall write \( x = \begin{bmatrix} T \\ P \end{bmatrix} \), with \( P \in \mathbb{H}^{k \times k} \). Let \( \phi_0 \in X_{n,k} \) be the inclusion \( \mathfrak{v} \mapsto \begin{bmatrix} 0 \\ \mathfrak{v} \end{bmatrix} \), represented by the matrix \( x_0 = \begin{bmatrix} \mathfrak{v}^T \\ I_k \end{bmatrix} \).

The linear left action of \( \text{Sp}(n) \) on \( X_{n,k} \) is transitive and the isotropy group of \( x_0 \) is isomorphic to \( \text{Sp}(n-k) \). Therefore the Stiefel manifold \( X_{n,k} \) is diffeomorphic to \( \text{Sp}(n)/\text{Sp}(n-k) \) and there is a principal

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\[ \text{Sp}(n-k) \xrightarrow{i} \text{Sp}(n) \xrightarrow{\rho} X_{n,k}. \]

If we write \( A = \begin{bmatrix} \alpha & T \\ \beta & P \end{bmatrix} \in \text{Sp}(n) \), with \( T \in \mathbb{H}^{(n-k)\times k} \) and \( P \in \mathbb{H}^{k\times k} \), the application \( \rho: \text{Sp}(n) \to X_{n,k} \) is defined by \( \rho(A) = \begin{bmatrix} T \\ P \end{bmatrix} \). If \( P \in \text{Sp}(k) \), we may choose \( T = 0 \) and get an element of \( X_{n,k} \). This gives a canonical inclusion,

\[ \iota_{n,k}: \text{Sp}(k) \to X_{n,k}. \]

We shall come back below on some aspects of this inclusion. First, we characterize the matrices \( P \in \mathbb{H}^{k\times k} \) that can be completed with \( T \in \mathbb{H}^{(n-k)\times k} \) for getting an element \( \begin{bmatrix} T \\ P \end{bmatrix} \in X_{n,k} \). In Proposition 2.2, we prove that such \( T \) exists if, and only if, the eigenvalues of \( P^*P \) (that is, the singular values of \( P \)), belong to the interval \([0, 1]\) and the multiplicity of the eigenvalue 1 is greater than or equal to \( 2k - n \).

Next, we use the well-known (8) singular value decomposition (SVD, in short) of \( P \in \mathbb{H}^{k\times k} \) for the determination of the possible completions of it in an element of \( X_{n,k} \). More precisely, in Theorem 3.1 starting from the SVD of \( P \in \mathbb{H}^{k\times k} \), satisfying the previous criterion, we describe the various matrices of \( \text{Sp}(n) \) of the shape \( \begin{bmatrix} \alpha & T \\ \beta & P \end{bmatrix} \). This gives a “relative SVD of a matrix in \( \text{Sp}(n) \)’’.

We apply this decomposition to the study of the Lusternik-Schnirelmann category (in short LS-category). Let us recall first that an open subset \( U \) of a topological space \( X \) is called categorical if \( U \) is contractible in \( X \). The LS-category, \( \text{cat} X \), of \( X \) is defined as the least integer \( m \geq 0 \) such that \( X \) admits a covering by \( m+1 \) categorical open sets (2).

The LS-category is a homotopy invariant that turns out to be useful in areas such as dynamical systems and symplectic geometry. But it is also particularly difficult to compute. A longstanding problem is the determination of the LS-category of Lie groups. In the case of unitary and special unitary Lie groups, Singhof determined \( \text{cat} U(n) = n \) and \( \text{cat} SU(n) = n - 1 \) (17), using eigenvalues. This method cannot be carried out for the symplectic groups \( \text{Sp}(n) \) due to the non-commutativity of quaternions (11). Some progress has been made for small \( n \) with \( \text{cat} \text{Sp}(2) = 3 \) (16), \( \text{cat} \text{Sp}(3) = 5 \) (3), or with bounds as \( \text{cat} \text{Sp}(n) \leq \binom{n+1}{2} \) (12) and \( \text{cat} \text{Sp}(n) \geq n + 2 \) when \( n \geq 3 \) (4).
Proposition 4.6 we show how Theorem 3.1 supplies an explicit minimal categorical open cover of $\text{Sp}(2)$.

Some partial results also exist for the LS-category of symplectic Stiefel manifolds. For instance, in [15], Nishimoto proves $\text{cat}_{X_{n,k}} = k$ when $n \geq 2k$, making use of eigenvalues of associated complex matrices. Different techniques of proof have been given for this result, as the use of the Cayley transform in [14], or Morse-Bott functions in [6]. Let us also mention that Morse-Bott functions are also present in [9], [13] for the study of LS-category. Finally recall the existence of a lower bound for the LS-category of Stiefel manifolds, generally better than the classical cup-length, established by Kishimoto in [7], and recalled in Theorem 4.1.

In this work, we study the subspace LS-category of $\text{Sp}(k)$ in $X_{n,k}$, denoted $\text{cat}_{X_{n,k}} \text{Sp}(k)$. This means that we are looking for families of open sets in $X_{n,k}$ covering $\text{Sp}(k)$ whose elements are contractible in $X_{n,k}$. We prove in Propositions 5.1, 5.2 and 5.3 that $$\text{cat}_{X_{2k-j,k}} \text{Sp}(k) \leq \text{cat} \text{Sp}(j), \quad \text{for } j = 0, 1, 2,$$
and we wonder if this is still true for any $j \geq 0$.

**Notations and Conventions.** For any pair of square matrices (not necessarily of the same size) the relation $A \sim B$ means: “$A$ is invertible if and only if $B$ is so.”

If $(t_1, \ldots, t_q)$ is a sequence of quaternions, we denote by $\text{diag}(t_i)_{q \times q}$ the $(q \times q)$-matrix having the $t_i$'s on the diagonal and 0 otherwise.

**2. Stiefel manifolds**

In this section, we consider a matrix $P \in \mathbb{H}^{k \times k}$ and study the existence of a “companion” $T \in \mathbb{H}^{(n-k) \times k}$ which gives an element $\begin{bmatrix} T \\ P \end{bmatrix}$ of $X_{n,k}$.

An element of $X_{n,k}$ can be represented by a matrix $x = \begin{bmatrix} T \\ P \end{bmatrix}$, with $T \in \mathbb{H}^{(n-k) \times k}$ and $P \in \mathbb{H}^{k \times k}$. The preservation of the Hermitian product corresponds to the equation $x^*x = I_k$, which becomes $$T^*T + P^*P = I_k.$$ 

**Definition 2.1.** A matrix $P \in \mathbb{H}^{k \times k}$ is $n$-admissible if there exists $T \in \mathbb{H}^{(n-k) \times k}$ such that $\begin{bmatrix} T \\ P \end{bmatrix} \in X_{n,k}$. The integer number $e = 2k - n$ is called the excess of $X_{n,k}$.
Admissible matrices can be entirely characterized by eigenvalues.

**Proposition 2.2.** A matrix \( P \in \mathbb{H}^{k \times k} \) is \( n \)-admissible if, and only if, the eigenvalues of \( P^*P \) belong to the interval \([0, 1]\) and the multiplicity of the eigenvalue 1 is greater than or equal to the excess \( e = 2k - n \).

Let us notice that the second condition is automatically verified if \( e \leq 0 \).

**Proof.** Let

\[
P = U \begin{bmatrix} I_{p \times p} & 0 & 0 \\ 0 & \text{diag}(t_i)_{q \times q} & 0 \\ 0 & 0 & 0_{r \times r} \end{bmatrix} V^*
\]

be the SVD of \( P \), with \( p + q + r = k \), \( U, V \in \text{Sp}(k) \), \( p, q, r \geq 0 \) and \( 0 < t_i < 1 \).

- If there exists \( T \in \mathbb{H}^{(n-k) \times k} \) such that \( \begin{bmatrix} T \\ P \end{bmatrix} \in X_{n,k} \), the equality \( T^*T + P^*P = I_k \) implies

  \[
  T^*T = V \begin{bmatrix} 0_{p \times p} & 0 & 0 \\ 0 & \text{diag}(1-t_i^2)_{q \times q} & 0 \\ 0 & 0 & 0_{r \times r} \end{bmatrix} V^* \in \mathbb{H}^{k \times k}.
  \]

As \( T^*T \) is hermitian semi-definite positive, we deduce \( 1-t_i^2 > 0 \) and \( 0 < t_i < 1 \). For any non-square matrix \( T \in \mathbb{H}^{(n-k) \times k} \), it is known that \( \text{rank}(T^*T) = \text{rank}(T) \), see Lemma 2.4. This implies \( q + r \leq \min(n-k, k) \) and

\[
p = k - (q + r) \geq k - \min(n-k, k) = 2k - n = e.
\]

- Suppose now \( t_i \in [0, 1[ \) and \( p \geq e \). We consider the matrix

\[
T = \begin{bmatrix} 0_{p' \times p} & 0 & 0 \\ 0 & \text{diag}(s_i)_{q \times q} & 0 \\ 0 & 0 & 0_{r \times r} \end{bmatrix} V^*,
\]

with \( 0 < s_i = \sqrt{1-t_i^2} < 1 \) and \( p' + q + r = n - k \). Then we have

\[
T^*T + P^*P = I_k \quad \text{and} \quad \begin{bmatrix} T \\ P \end{bmatrix} \in X_{n,k}.
\]

Let us recall the Study determinant (\( \text{[I]} \)) useful for the detection of invertible matrices. As any quaternionic matrix \( M \in \mathbb{H}^{n \times n} \) can be written as \( M = X + jY \) with \( X, Y \in \mathbb{C}^{n \times n} \), we associate to \( M \) a complex matrix, \( \chi(M) \), defined by

\[
\chi(M) = \begin{bmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{bmatrix} \in \mathbb{C}^{2n \times 2n}.
\]
The Study determinant of $M$, defined by $\text{Sdet}(M) = \sqrt{\text{det} \chi(M)}$, verifies the following properties.

1. The matrix $M$ is invertible if, and only if, $\text{Sdet}(M) \neq 0$.
2. If $M, N \in \mathbb{H}^{n \times n}$, then $\text{Sdet}(MN) = \text{Sdet}(M) \text{Sdet}(N)$.
3. If $N$ is obtained from $M$ by adding a left multiple of a row to another row or a right multiple of a column to another column, then we have $\text{Sdet}(M) = \text{Sdet}(N)$.
4. If $M$ is a triangular matrix then $\text{Sdet}(M)$ equals $|m_{11} \cdots m_{nn}|$, the norm of the product of the elements of the diagonal.

We complete these properties by the following one, well adapted to the quaternionic matrices appearing in the last sections.

**Lemma 2.3.** Let $M \in \mathbb{H}^{m \times n}$ and $N \in \mathbb{H}^{n \times m}$. Then we have

$$\text{Sdet}(I_m + MN) = \text{Sdet}(I_n + NM).$$

**Proof.** This is a classical argument,

$$\text{Sdet} \begin{bmatrix} I_m + MN & -M \\ 0 & I_n \end{bmatrix} = \text{Sdet} \begin{bmatrix} I_m & 0 \\ N & I_n + NM \end{bmatrix}. \quad \square$$

We end this section with the following lemma, used in the proof of Theorem 3.1. It is a classical result and we give the proof for the convenience of the reader.

**Lemma 2.4.** Let $M \in \mathbb{H}^{m \times n}$ be a non-necessarily square quaternionic matrix. Then, we have $\ker M^* M = \ker M$ and $\ker MM^* = \ker M^*$.

**Proof.** The inclusion $\ker M \subset \ker M^* M$ is direct. On the other hand, if $u \in \ker M^* M$, we get $|M(u)|^2 = \langle Mu, Mu \rangle = \langle u, M^* Mu \rangle = 0$ and $u \in \ker M$. A similar argument gives the second equality. \quad \square

## 3. Relative singular value decomposition in $\text{Sp}(n)$

In this section, we establish a “relative singular value decomposition” of the elements of $\text{Sp}(n)$. This structure proves to be effective for the study of the injection $\text{Sp}(k) \to X_{n,k}$ as it appears in Section 5.
Theorem 3.1. For any $k \leq n$, an element $A$ of $\text{Sp}(n)$ can be written in blocks as follows,

$$A = \begin{bmatrix}
m \begin{bmatrix} I_{p'} & 0 & 0 \\
0 & \text{diag}(\cos \theta_i)_{q \times q} & 0 \\
0 & 0 & 0_r \end{bmatrix} & \ell^* & m \begin{bmatrix} 0_{p' \times p} & 0 & 0 \\
0 & -\text{diag}(\sin \theta_i)_{q \times q} & 0 \\
0 & 0 & 0 \end{bmatrix} & b^* \\
a \begin{bmatrix} 0_{p \times p'} & 0 & 0 \\
0 & \text{diag}(\sin \theta_i)_{q \times q} & 0 \\
0 & 0 & 0_r \end{bmatrix} & \ell^* & a \begin{bmatrix} I_p & 0 & 0 \\
0 & \text{diag}(\cos \theta_i)_{q \times q} & 0 \\
0 & 0 & 0_r \end{bmatrix} & b^* \end{bmatrix}$$

with $\theta_i \in [0, \pi/2]$, $a, b \in \text{Sp}(k)$, $m, \ell \in \text{Sp}(n-k)$, $p \geq 2k-n$, $p+q+r = k$ and $p'+q+r = n-k$.

Proof. Let $A = \begin{bmatrix} \alpha & T \\
\beta & P \end{bmatrix} \in \text{Sp}(n)$, with $P \in \mathbb{H}^{k \times k}$. The SVD of $P$ gives

$$P = a \begin{bmatrix} I_p & 0 & 0 \\
0 & \text{diag}(c_i)_{q \times q} & 0 \\
0 & 0 & 0_r \end{bmatrix} b^*,$$

with $a, b \in \text{Sp}(k)$, $p+q+r = k$ and $0 < c_i < 1$. From $T^*T + P^*P = I_k$, we deduce

$$T^*T = I_k - b \begin{bmatrix} I_p & 0 & 0 \\
0 & \text{diag}(c_i^2)_{q \times q} & 0 \\
0 & 0 & 0_r \end{bmatrix} b^* \quad (1)$$

with $0 < s_i < 1$ and $s_i^2 = 1-c_i^2$. We proceed in three steps, determining successively $T$, $\beta$ and $\alpha$.

Step 1. Let $p'$ such that $p' + q + r = n - k$. The matrix $T$ can be written as

$$T = m \begin{bmatrix} 0_{p' \times p} & 0 & 0 \\
0 & -\text{diag}(s_i)_{q \times q} & 0 \\
0 & 0 & 0 \end{bmatrix} b^* \quad (2)$$

if, and only if, the columns $(b_i)_{1 \leq i \leq k}$ and $(m_j)_{1 \leq j \leq n-k}$ of the matrices $b$ and $m$, respectively, verify:

$$T(b_i) = 0 \quad \text{for} \quad 1 \leq i \leq p,$$

$$T(b_{p+i}) = -m_{p'+i}s_i \quad \text{for} \quad 1 \leq i \leq q,$$

$$T(b_{p+q+i}) = m_{p'+q+i} \quad \text{for} \quad 1 \leq i \leq r.$$
We therefore have to establish these three properties from (1). The equation (3) is a direct consequence of Lemma 2.4. Next, the equations (4) and (5) define the vectors \((m'_{p+i})_{1 \leq i \leq q+r}\). They constitute an orthogonal system because the same holds for the corresponding \((b_i)\). In fact, from (1) and (4), we deduce for \(1 \leq i \leq q\),

\[
\langle m'_{p+i}, m'_{p+i} \rangle = \frac{1}{s_i^2} \langle T(b_{p+i}), T(b_{p+i}) \rangle = \frac{1}{s_i^2} \langle b_{p+i}, T^*T(b_{p+i}) \rangle = 1,
\]

and analogously \(\langle m'_{p+i}, m'_{p+j} \rangle = 0\) for \(i \neq j\).

We also have \(\langle m'_{p+i}, m'_{p+q+j} \rangle = \delta_{i,j}\) for \(1 \leq i, j \leq r\). Thus it suffices to complete \((m'_{p+i})_{1 \leq i \leq q+r}\) in an orthonormal basis to get the announced expression of \(T\).

**Step 2.** We determine \(\beta \in \mathbb{H}^{k \times (n-k)}\) such that \(\beta \beta^* + PP^* = I_k\). This equality gives

\[
\beta \beta^* = I_k - a \begin{bmatrix} I_p & 0 & 0 \\ 0 & \text{diag}(s_i^2)_{q \times q} & 0 \\ 0 & 0 & I_r \end{bmatrix} a^*
\]

\(\text{(6)}\)

\[
= a \begin{bmatrix} 0_p & 0 & 0 \\ 0 & \text{diag}(s_i^2)_{q \times q} & 0 \\ 0 & 0 & I_r \end{bmatrix} a^*.
\]

The argumentation developed in the first step brings a matrix \(\ell \in \text{Sp}(n-k)\) such that

\[
\beta^* = \ell \begin{bmatrix} 0_{p' \times p} & 0 & 0 \\ 0 & \text{diag}(s_i)_{q \times q} & 0 \\ 0 & 0 & I_r \end{bmatrix} a^*.
\]

\(\text{(7)}\)

As in the first step, the columns \((\ell'_{p+i})_{1 \leq i \leq q+r}\) are explicitly determined, but for the family \((\ell_i)_{1 \leq i \leq p'}\) the only requirement is to have an orthonormal basis \((\ell_j)_{1 \leq j \leq p'+q+r}\).

**Step 3.** We are reduced to decompose the matrix \(\alpha \in \mathbb{H}^{(n-k) \times (n-k)}\). From \(AA^* = I_n = A^*A\), we deduce

\[
\alpha \alpha^* + TT^* = I_{n-k},
\]

\(\text{(8)}\)

\[
\alpha^*T + \beta^*P = 0,
\]

\(\text{(9)}\)

\[
\alpha^* \alpha + \beta^* \beta = I_{n-k}.
\]

\(\text{(10)}\)

As before, we denote by \((m_j)_{1 \leq j \leq n-k}\) and \((\ell_j)_{1 \leq j \leq n-k}\) the columns of the matrices \(m\) and \(\ell\) respectively. Replacing \(T\) by its value (2), we deduce from (8) that the family \((m'_{p+q+i})_{1 \leq i \leq r}\) is a basis of \(\ker \alpha \alpha^* = \ker \alpha^*\), see Lemma 2.4.
The replacement of $T, \beta, P$ by their value in (9) gives the equality
\[
\alpha^* m \begin{bmatrix}
0_{p' \times p} & 0 & 0 \\
0 & -\text{diag}(s_i)_{q \times q} & 0 \\
0 & 0 & -I_r
\end{bmatrix} = -\ell \begin{bmatrix}
0_{p' \times p} & 0 & 0 \\
0 & \text{diag}(s_i)_{q \times q} & 0 \\
0 & 0 & 0_r
\end{bmatrix},
\]
which implies the relations
\[
\begin{align*}
\alpha^* m_{p'+i} &= \ell_{p'+i} c_i, \quad \text{for } 1 \leq i \leq q, \\
\alpha^* m_{p'+q+i} &= 0, \quad \text{for } 1 \leq i \leq r.
\end{align*}
\]
Thus, for proving (13)
\[
\alpha = m \begin{bmatrix}
I_{p'} & 0 & 0 \\
0 & \text{diag}(c_i)_{q \times q} & 0 \\
0 & 0 & 0_r
\end{bmatrix} \ell^*,
\]
it remains to establish
\[
\alpha \ell_i = m_i, \quad \text{for } 1 \leq i \leq p'.
\]
For that, starting from an orthonormal basis $(\ell_j)_{1 \leq j \leq n-k}$ built in Step 2, we have to prove that we could have taken $m_i = \alpha \ell_i$ for $1 \leq i \leq p'$ in order to complete an orthonormal basis $(m_j)_{1 \leq j \leq n-k}$ as we built in Step 1.

From (10) and (7), we deduce
\[
\alpha^* \alpha = \ell \begin{bmatrix}
I_{p'} & 0 & 0 \\
0 & \text{diag}(c_i)_{q \times q} & 0 \\
0 & 0 & 0_r
\end{bmatrix} \ell^*,
\]
which implies $\alpha^* \alpha \ell_i = \ell_i$, for all $1 \leq i \leq p'$. We prove now the orthonormality of $(m_j)_{1 \leq j \leq n-k}$.

- Let $1 \leq j, k \leq p'$. We have
  \[
  \langle m_j, m_k \rangle = \langle \alpha \ell_j, \alpha \ell_k \rangle = \langle \ell_j, \alpha^* \alpha \ell_k \rangle = \langle \ell_j, \ell_k \rangle,
  \]
  which gives an orthogonality relation for $j \neq k$ and $\langle m_j, m_j \rangle = 1$.

- Let $1 \leq j \leq p'$ and $1 \leq k \leq q$. We have:
  \[
  \begin{align*}
  \langle m_j, m_{p'+q+k} \rangle &= \langle \alpha \ell_j, m_{p'+q+k} \rangle = \langle \ell_j, \alpha^* m_{p'+q+k} \rangle \\
  &= \langle \ell_j, \ell_{p'+k} \rangle c_k = 0.
  \end{align*}
  \]

- Let $1 \leq j \leq p'$ and $1 \leq k \leq r$. We have:
  \[
  \begin{align*}
  \langle m_j, m_{p'+q+k} \rangle &= \langle \alpha \ell_j, m_{p'+q+k} \rangle = \langle \ell_j, \alpha^* m_{p'+q+k} \rangle \\
  &= \langle \ell_j, 0 \rangle = 0. \quad \square
  \end{align*}
  \]

The following particular case of Theorem 3.1 corresponds to $k = 1$ and $2k - n \leq 0$. 

Corollary 3.2. Let \( n \geq 2 \). Any element of \( \text{Sp}(n) \) can be written as

\[
P = \begin{pmatrix}
  m \begin{bmatrix} I_{n-2} & 0 \\ 0 & \cos \theta \end{bmatrix} \ell^* & m \begin{bmatrix} 0 \\ -\sin \theta \end{bmatrix} E \\
  [0 \sin \theta] \ell^* & (\cos \theta) E
\end{pmatrix},
\]

with \( m, \ell \in \text{Sp}(n-1) \), \( \cos \theta \in [0,1] \), \( E \in \text{Sp}(1) \).

4. Background on LS-category

We recall basic definitions and properties of the Lusternik-Schnirelmann category (LS-category in short). We also state the results on the LS-category of Stiefel manifolds obtained by T. Nishimoto ([15]) and D. Kishimoto ([7]) as well as the technique for the construction of categorical open subsets, introduced by the authors in [14].

The definition of LS-category has been recalled in the introduction, see [2] for more details. If \( X \) is an \( (m-1) \)-connected CW-complex, then there is the upper bound,

\[
\text{cat} \ X \leq \frac{\dim X}{m}.
\]

As \( \dim X_{n,k} = \dim \text{Sp}(n) - \dim \text{Sp}(n-k) = k(4n - 2k + 1) \), we get (see [5, Proposition 2.1] for the connectivity of \( X_{n,k} \))

\[
\text{cat} \ X_{n,k} \leq \frac{k(4n - 2k + 1)}{4(n-k) + 3}.
\]

A lower bound is given by the cup length in the cohomology algebra but, for Stiefel manifolds, there is also a lower bound, due to Kishimoto.

Theorem 4.1 ([7]). We have

\[
\text{cat} \ X_{n,k} \geq \begin{cases}
  k & \text{if } n \geq 2k - 1, \\
  k + 1 & \text{if } n = 2k - 2 \text{ or } n = 2k - 3, \\
  k + 2 & \text{if } n \leq 2k - 4.
\end{cases}
\]

In the particular case \( n \geq 2k \), Nishimoto has computed the LS-category of \( X_{n,k} \), using the number of eigenvalues of an associated complex matrix.

Theorem 4.2 ([15]). If \( n \geq 2k \) then \( \text{cat} \ X_{n,k} = k \).

Remark 4.3. From Theorem 4.1, Theorem 4.2 and (15), we can deduce for instance: \( \text{cat} \ X_{3,2} = 2 \) and \( \text{cat} \ X_{4,3} = 4 \).

Nishimoto’s result can also be proven ([14]) from Cayley open subsets, defined as follows.
Theorem 4.4 ([14, Theorem 1.2]). Let $P \in \mathbb{H}^{k \times k}$ be an $n$-admissible matrix. The Cayley open subset 
\[
\Omega(P) = \left\{ \begin{bmatrix} \tau \\ \pi \end{bmatrix} \in X_{n,k} \mid \pi + P^* \text{ invertible} \right\}
\]
is categorical in $X_{n,k}$.

Remark 4.5. Let $\text{diag}(0_s, -I_t, I_r) = \begin{bmatrix} 0_s & 0 & 0 \\ 0 & -I_t & 0 \\ 0 & 0 & I_r \end{bmatrix} \in \mathbb{H}^{k \times k}$ be the diagonal matrix defined by blocks from the null matrix $0_s \in \mathbb{H}^{s \times s}$ and the identity matrices $I_t \in \mathbb{H}^{t \times t}, I_r \in \mathbb{H}^{r \times r}$, with $s + t + r = k$ and $s, t, r \geq 0$. Then $\text{diag}(0_s, -I_t, I_r)$ is $n$-admissible if and only if $r + t \geq e$. In this case, we have the categorical open subset of $X_{n,k}$ 
\[
\Omega(0_s, -I_t, I_r) = \left\{ \begin{bmatrix} T \\ P \end{bmatrix} \in X_{n,k} \mid P + \text{diag}(0_s, -I_t, I_r) \text{ invertible} \right\}.
\]

From Theorem 3.1, we determine an explicit minimal categorical open cover of $\text{Sp}(2)$.

Proposition 4.6. The four open subsets $\Omega(I_2), \Omega(-I_2), \Omega(I_1, -I_1)$ and $\Omega(-I_1, I_1)$ constitute a categorical open cover of $\text{Sp}(2)$.

Proof. From Theorem 3.1 we know that any element of $\text{Sp}(2)$ can be written as 
\[
P = \begin{bmatrix} m \cos \theta \ell^* & -m \sin \theta b^* \\ a \sin \theta \ell^* & a \cos \theta b^* \end{bmatrix},
\]
where $a, b, m, \ell$ are quaternionic numbers of norm 1 and $\cos \theta \in [0, 1]$. We set $\varepsilon_i = \pm 1$ for $i = 1, 2$ and $\text{diag}(\varepsilon_1, \varepsilon_2) = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}$. We observe $\text{diag}(\varepsilon_1, \varepsilon_2)^2 = I_2$ and we are looking for the property “$P + \text{diag}(\varepsilon_1, \varepsilon_2)$ is invertible”. Lemma 2.3 and easy calculations imply that 
\[
P + \text{diag}(\varepsilon_1, \varepsilon_2) \sim \begin{bmatrix} m & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \ell^* & 0 \\ 0 & b^* \end{bmatrix} + \text{diag}(\varepsilon_1, \varepsilon_2)
\]
\[
\sim \begin{bmatrix} \ell^* & 0 \\ 0 & b^* \end{bmatrix} \text{diag}(\varepsilon_1, \varepsilon_2) \begin{bmatrix} m & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} + I_2
\]
\[
\sim \begin{bmatrix} \varepsilon_1 \ell^* m & 0 \\ 0 & \varepsilon_2 b^* a \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} + I_2
\]
\[
\sim \begin{bmatrix} \varepsilon_1 \ell^* m & 0 \\ 0 & \varepsilon_2 b^* a \end{bmatrix} + \begin{bmatrix} \cos \theta & \varepsilon_1 \ell^* \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.
\]
We set $Q_1 = \varepsilon_1 \ell^* m$ and $Q_2 = \varepsilon_2 b^* a$.

Suppose $\cos \theta \neq 1$. This implies $\sin \theta \neq 0$ and we may use it as a “pivot” in the last matrix. This gives, by adding to the second row a left multiple of the first row,

$$P + \text{diag}(\varepsilon_1, \varepsilon_2) \sim \begin{bmatrix} \cos \theta + Q_1 & \sin \theta \\ -\sin \theta - (\sin \theta)^{-1}(\cos \theta + Q_2)(\cos \theta + Q_1) & 0 \end{bmatrix}.$$  

Thus $P + \text{diag}(\varepsilon_1, \varepsilon_2)$ is not invertible if and only if

$$\sin^2 \theta + (\cos \theta + Q_2)(\cos \theta + Q_1) = 0 \iff 1 + \cos \theta Q_2 + (\cos \theta + Q_2)Q_1 = 0 \iff Q_1 = -(\cos \theta + Q_2)^{-1}(1 + \cos \theta Q_2).$$

The last writing makes sense since $\cos \theta \neq 1$ implies $\cos \theta + Q_2 \neq 0$ because $|Q_2| = 1$. If $(\varepsilon_1, \varepsilon_2)$ is given, the previous equation admits a unique solution $(Q_1, Q_2)$. Therefore, among the matrices of the statement, we can find a matrix diag$(\varepsilon_1, \varepsilon_2)$ for which $P + \text{diag}(\varepsilon_1, \varepsilon_2)$ is invertible. (In fact, two of them suffice in this case.)

If $\cos \theta = 1$, then we have

$$P + \text{diag}(\varepsilon_1, \varepsilon_2) \sim \begin{bmatrix} 1 + Q_1' & 0 \\ 0 & 1 + Q_2' \end{bmatrix}. \quad \square$$

Let us notice that we need the four matrices of the statement to ensure the existence of one case such that $P + \text{diag}(\varepsilon_1, \varepsilon_2)$ is invertible. In fact, we already know from [16] that there is no categorical open cover of $\text{Sp}(2)$ with strictly less than 4 elements.

5. Subspace LS-category of $\text{Sp}(k)$ in the Stiefel manifold $X_{n,k}$

We give an upper bound for the subspace LS-category, $\text{cat}_{X_{2k-1,k}} \text{Sp}(k)$, of $\text{Sp}(k)$ in $X_{n,k}$, for $n \geq 2k$, $n = 2k - 1$ and $n = 2k - 2$. A question for the general case is also proposed.

If $n \geq 2k$, we first notice that the zero matrix $0_k \in \mathbb{H}^{k \times k}$ is $n$-admissible. Therefore $\text{Sp}(k)$ is included in the categorical open subset $\Omega(0_k)$ and the next result follows.

**Proposition 5.1.** If $0 < 2k \leq n$, we have $\text{cat}_{X_{n,k}} \text{Sp}(k) = 0$. 

Consider now the second case.

**Proposition 5.2.** If \( n = 2k - 1 \), \( 0 < k \), we have \( \text{cat}_{X_{2k-1,k}} \text{Sp}(k) \leq 1 \).

**Proof.** Observe that the matrices \( \text{diag}(0_{k-1}, I_1) \) and \( \text{diag}(0_{k-1}, -I_1) \) are \((2k - 1)\)-admissible. We decompose an element of \( P \in \text{Sp}(k) \) as

\[
P = \begin{bmatrix}
m & \begin{bmatrix} I_{k-2} & 0 \\ 0 & \cos \theta \end{bmatrix} \ell^* & m & \begin{bmatrix} 0 \\ -\sin \theta \end{bmatrix} E \\
[0 & \sin \theta] \ell^* & (\cos \theta) E & & 
\end{bmatrix},
\]

with \( m, \ell \in \text{Sp}(k-1) \), \( \cos \theta \in [0, 1] \), \( E \in \text{Sp}(1) \).

- Suppose \( 1 + E \cos \theta \neq 0 \). Then we have

\[
P + \text{diag}(0_{k-1}, I_1) \\
\sim m \begin{bmatrix} I_{k-2} & 0 \\ 0 & \cos \theta \end{bmatrix} \ell^* + m \begin{bmatrix} 0 \\ \sin \theta \end{bmatrix} EE^*(E^* + \cos \theta)^{-1}[0 \sin \theta] \ell^*
\sim \begin{bmatrix} I_{k-2} & 0 \\ 0 & \cos \theta + \sin^2 \theta (E^* + \cos \theta)^{-1} \end{bmatrix}.
\]

Let us notice that

\[
\cos \theta + \sin^2 \theta (E^* + \cos \theta)^{-1} = E^*(E + \cos \theta)(E^* + \cos \theta)^{-1}
\]

is a quaternion of norm 1. Thus the matrix \( P + \text{diag}(0_{k-1}, I_1) \) is invertible.

- If \( 1 + E \cos \theta = 0 \), then we have \( \cos \theta = 1 \) and \( E = -1 \). This implies \( P = \begin{bmatrix} m \ell^* & 0 \\ 0 & -1 \end{bmatrix} \) and \( P \in \Omega(0_{k-1}, -I_1) \).

In conclusion, we cover \( \text{Sp}(k) \) by the two open subsets \( \Omega(0_{k-1}, I_1) \) and \( \Omega(0_{k-1}, -I_1) \), which are contractible in \( X_{2k-1,k} \).

□

Finally, we state our last result in this direction.

**Proposition 5.3.** If \( n = 2k-2 \), with \( 1 < k \), we have \( \text{cat}_{X_{2k-2,k}} \text{Sp}(k) \leq 3 \).

**Proof.** We prove that the four open subsets \( \Omega(0_{k-2}, 1, 1), \Omega(0_{k-2}, 1, -1), \Omega(0_{k-2}, -1, 1), \Omega(0_{k-2}, -1, -1) \) form a categorical open cover of \( \text{Sp}(k) \) in \( X_{2k-2,k} \). Let \( P \in \text{Sp}(k) \) that we write, by taking, in Theorem 3.1 a
block of size $2 \times 2$ at the bottom right corner, as

$$P = \begin{bmatrix} 0_{k-4,1} & 0_{k-4,1} \\ -\sin \theta_1 & 0 \\ 0 & -\sin \theta_2 \end{bmatrix},$$

(16)

$$P = \begin{bmatrix} I_{k-4} & 0 & 0 \\ 0 & \cos \theta_1 & 0 \\ 0 & 0 & \cos \theta_2 \end{bmatrix} \ell^* \begin{bmatrix} 0_{k-4,1} & 0_{k-4,1} \\ -\sin \theta_1 & 0 \\ 0 & -\sin \theta_2 \end{bmatrix},$$

where $\cos \theta_1, \cos \theta_2 \in [0, 1]$, $a, b \in \text{Sp}(2)$ and $m, \ell \in \text{Sp}(k-2)$.

**First step.** Claim: if $a^*b \in \Omega \begin{bmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{bmatrix}$ then $P \in \Omega(0_{k-2}, I_2)$.

Let $H = a \begin{bmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{bmatrix} b^* + I_2$. The hypothesis on $a^*b$ implies the invertibility of $H$. Thus, we can use $H$ as a “pivot” to add to the first block of columns the second block multiplied on the right by

$$X = H^{-1}a \begin{bmatrix} 0_{1,k-4} & \sin \theta_1 & 0 \\ 0_{1,k-4} & 0 & \sin \theta_2 \end{bmatrix} \ell^*$$

and we get

$$P + \text{diag}(0_{k-2}, I_2)$$

$$\sim m \begin{bmatrix} I_{k-4} & 0 & 0 \\ 0 & \cos \theta_1 & 0 \\ 0 & 0 & \cos \theta_2 \end{bmatrix} \ell^*$$

$$- m \begin{bmatrix} 0_{k-4,1} & 0_{k-4,1} \\ -\sin \theta_1 & 0 \\ 0 & -\sin \theta_2 \end{bmatrix} b^*H^{-1}a \begin{bmatrix} 0_{1,k-4} & \sin \theta_1 & 0 \\ 0_{1,k-4} & 0 & \sin \theta_2 \end{bmatrix} \ell^*$$

$$\sim \begin{bmatrix} I_{k-4} \\ 0_{2,k-4} \end{bmatrix} \begin{bmatrix} \cos \theta_1 & 0 & 0 \\ 0 & \cos \theta_2 \end{bmatrix} + \begin{bmatrix} \sin \theta_1 & 0 \\ 0 & \sin \theta_2 \end{bmatrix} b^*H^{-1}a \begin{bmatrix} \sin \theta_1 & 0 \\ 0 & \sin \theta_2 \end{bmatrix} \ell^*$$

(17)

We observe that

$$b^*H^{-1}a = \left( \begin{bmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{bmatrix} + a^*b \right)^{-1}.$$

We examine the different values of $\cos \theta_1$ and $\cos \theta_2$. 
First, suppose “\(\cos \theta_1 \neq 1\) and \(\cos \theta_2 \neq 1\)””. With usual arguments, we deduce:

\[
P + \text{diag}(0_{k-2}, I_2) \sim \begin{bmatrix}
\frac{\cos \theta_1}{\sin^2 \theta_1} & 0 \\
0 & \frac{\cos \theta_2}{\sin^2 \theta_2}
\end{bmatrix} + b^* H^{-1} a \\
\sim a^* b \begin{bmatrix}
\cos \theta_1 & 0 \\
0 & \cos \theta_2
\end{bmatrix} + I_2 \\
\sim \begin{bmatrix}
\cos \theta_1 & 0 \\
0 & \cos \theta_2
\end{bmatrix} + b^* a.
\]

Thus the hypothesis on \(a^* b\) implies \(P \in \Omega(0_{k-2}, I_2)\) in this case.

- If \(\cos \theta_1 = \cos \theta_2 = 1\), then the hypothesis implies immediately that \(P \in \Omega(0_{k-2}, I_2)\).

- It only remains to consider \(\cos \theta_1 = 1\) and \(\cos \theta_2 \neq 1\). (Notice that the case \(\cos \theta_1 \neq 1\) and \(\cos \theta_2 = 1\) is similar.) We denote \(\theta = \theta_2\).

  — Suppose \(a^* b\) is diagonal, i.e., \(a^* b = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}\). The equality (17) becomes

\[
P + \text{diag}(0_{k-2}, I_2) \sim \begin{bmatrix} 1 & 0 \\ 0 & \cos \theta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & (v + \cos \theta)^{-1} \sin^2 \theta \end{bmatrix} \\
\sim \begin{bmatrix} 1 & 0 \\ 0 & \cos \theta + (v + \cos \theta)^{-1} \sin^2 \theta \end{bmatrix} \\
\sim (v + \cos \theta)^{-1} ((v + \cos \theta) \cos \theta + \sin^2 \theta) \\
\sim 1 + v \cos \theta.
\]

As \(\cos \theta \neq 1\), the quaternionic number \(1 + v \cos \theta\) is different from 0 and \(P + \text{diag}(0_{k-2}, I_2)\) is invertible.

  — If the matrix \(a^* b\) is not diagonal, we know from [10, Proposition 5.1] that it has the form \(a^* b = \begin{bmatrix} u & -\gamma v \\ v & \gamma u \end{bmatrix}\) with \(|\gamma| = 1\), \(v \neq 0\) and \(|v|^2 + |u|^2 = 1\). The equality (17) becomes

\[
P + \text{diag}(0_{k-2}, I_2) \sim \begin{bmatrix} 1 & 0 \\ 0 & \cos \theta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sin \theta \end{bmatrix} b^* H^{-1} a \begin{bmatrix} 0 & 0 \\ 0 & \sin \theta \end{bmatrix}.
\]

We compute

\[
b^* H^{-1} a = \left(\begin{bmatrix} 1 & 0 \\ 0 & \cos \theta \end{bmatrix} + a^* b\right)^{-1}.
\]
Denote
\[ K = \begin{bmatrix} 1 & 0 \\ 0 & \cos \theta \end{bmatrix} + a^*b = \begin{bmatrix} u + 1 \\ v \end{bmatrix} \begin{bmatrix} -\bar{\nu}\gamma \\ v\bar{\nu}^{-1}\gamma + \cos \theta \end{bmatrix}. \]

If \( X \) is such that
\[ v\bar{\nu}^{-1}\gamma + \cos \theta - vX = 0, \]
then we have
\[ K^{-1} = \begin{bmatrix} 1 & -X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u + 1 & -\bar{\nu}\gamma - (u + 1)X \\ v & 0 \end{bmatrix}^{-1} \]
\[ = \begin{bmatrix} 1 & -X \\ 0 & 1 \end{bmatrix} \left( (-\bar{\nu}\gamma - (u + 1)X)^{-1} (\bar{\nu}\gamma + (u + 1)X)^{-1}(u + 1)v^{-1} \right). \]

This implies
\[ P + \text{diag}(0_{k-2}, I_2) \]
\[ \sim \cos \theta(\bar{\nu}\gamma + (u + 1)X) + \sin^2 \theta(u + 1)v^{-1} \]
\[ \sim \cos \theta(\bar{\nu}\gamma + (u + 1)\bar{\nu}v^{-1}\gamma) + \cos \theta(u + 1)v^{-1} \cos \theta + \sin^2 \theta(u + 1)v^{-1} \]
\[ \sim \cos \theta(\bar{\nu}v + u\bar{\nu} + \bar{\nu})v^{-1}\gamma + (u + 1)v^{-1} \]
\[ \sim \cos \theta(1 + \bar{\nu})v^{-1}\gamma + (u + 1)v^{-1}. \]

If this last quaternionic number is equal to zero, we have an equality of modules:
\[ (\cos \theta) |1 + \bar{\nu}| |v|^{-1} = |1 + u| |v|^{-1} \]
which is impossible since \( \cos \theta \neq 1 \) and \( |1 + u| \neq 0 \). Therefore, in this last case, we have also the invertibility of \( P + \text{diag}(0_{k-2}, I_2) \) and the claim is proven.

Second step. Now we assume that \( a^*b \notin \Omega \begin{bmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{bmatrix} \). We observe:
- if \( a^*b = \begin{bmatrix} u \\ 0 \end{bmatrix} \) then the hypothesis implies \( \cos \theta_1 = 1 \) and \( u = -1 \) or \( \cos \theta_2 = 1 \) and \( v = -1 \).

We develop the different cases.
- Let \( a^*b = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \) with \( \cos \theta_1 = 1 \) and \( \cos \theta_2 \neq 1 \). We denote \( \theta = \theta_2 \). We replace \( b^* = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \) by its value in the expression of \( P \)
and get

\[
P = \begin{bmatrix}
m & \begin{bmatrix} I_{k-4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos \theta \end{bmatrix} \ell^* & m & \begin{bmatrix} 0_{k-4,1} & 0_{k-4,1} \\ 0 & 0 \\ 0 & -\overline{v} \sin \theta \end{bmatrix} a^* \\
a & \begin{bmatrix} 0_{1,k-4} & 0 \\ 0 & \sin \theta \end{bmatrix} \ell^* & a & \begin{bmatrix} -1 & 0 \\ 0 & \overline{v} \cos \theta \end{bmatrix} a^* 
\end{bmatrix}.
\]

Using the bottom right-hand term as pivot of \(P + \text{diag}(0, -I_2)\) gives, with computations similar to those in the first step, that

\[
P \in \Omega(0_{k-2}, -1, -1).
\]

- The second case with \(\cos \theta_2 = 1\) and \(\cos \theta_1 \neq 1\), \(v = -1\) gives the same result,

\[
P \in \Omega(0_{k-2}, -1, -1).
\]

- The last case, \(\cos \theta_1 = \cos \theta_2 = 1\) corresponds to

\[
P = \begin{bmatrix} m \ell^* & 0 \\ 0 & ab^* \end{bmatrix}
\]

and \(P \in \Omega(0_{k-2}, -1, -1)\).

- If \(a^*b\) is not diagonal, we shall prove that \(\cos \theta_1 = \cos \theta_2 = 1\).

In fact, \(a^*b\) has the form \(a^*b = \begin{bmatrix} u & -\overline{v} \gamma \\ v & \overline{v} u v^{-1} \gamma \end{bmatrix}\) with \(|\gamma| = 1\), \(v \neq 0\) and \(|v|^2 + |u|^2 = 1\). Then, \(a^*b \notin \Omega \begin{bmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{bmatrix}\) if and only if

\[
a^*b + \begin{bmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{bmatrix} = \begin{bmatrix} u + \cos \theta_1 & -\overline{v} \gamma \\ v & \overline{v} u v^{-1} \gamma + \cos \theta_2 \end{bmatrix}
\]

is not invertible.

From now on we shall denote \(c_1 = \cos \theta_1\) and \(c_2 = \cos \theta_2\).

As \(v \neq 0\), we can take the matrix \(X = v^{-1}(\overline{v} u v^{-1} \gamma + c_2),\) so

\[
\begin{bmatrix} u + c_1 & -\overline{v} \gamma \\ v & \overline{v} u v^{-1} \gamma + c_2 \end{bmatrix} \sim \begin{bmatrix} u + c_1 & -\overline{v} \gamma - (u + c_1)X \\ v & 0 \end{bmatrix},
\]

which is not invertible if and only if

\[-\overline{v} \gamma - (u + c_1)X = 0.\]

It follows

\[-\overline{v} \gamma = (u + c_1)v^{-1}(\overline{v} u v^{-1} \gamma + c_2) = \frac{1}{|v|^2}(|u|^2 \overline{v} \gamma + u \overline{v} c_2 + c_1 u \overline{v} \gamma + \overline{v} c_1 c_2),\]

hence

\[-|v|^2 \overline{v} \gamma = |u|^2 \overline{v} \gamma + u \overline{v} c_2 + c_1 u \overline{v} \gamma + \overline{v} c_1 c_2\]
and
\[-\bar{v}\gamma = u\bar{v}c_2 + c_1 \bar{u}\bar{v}\gamma + \bar{v}c_1 c_2,\]
because $|u|^2 + |v|^2 = 1$. Finally,
\[-(1 + c_1 \bar{u})\bar{v}\gamma = (u + c_1)c_2 \bar{v}\]
and, taking modules,
\[(19) \quad |1 + c_1 u| = |u + c_1|c_2.\]

We have:
(1) $c_2 \neq 0$: if $c_2 = 0$, $c_1 u = -1$ then $c_1|u| = 1$ so $|u| \geq 1$, which is impossible because $v \neq 0$;
(2) let us suppose $c_2 < 1$: from equation (19) we have $|1 + c_1 u|^2 < |u + c_1|^2$ and we deduce $1 - c_2^2 < (1 - c_2^2)|u|^2$, but $1 - c_2^2 \neq 0$, so $|u|^2 > 1$, a contradiction;
(3) now, as $c_2 = 1$, equation (19) is $|1 + c_1 u| = |u + c_1|$, which is equivalent to $1 - c_2^2 = (1 - c_2^2)|u|^2$, but $|u|^2 < 1$, so $c_2^2 = 1$.

Hence, $\cos \theta_1 = \cos \theta_2 = 1$ as stated.

From Proposition 4.6 we deduce the result. \qed

The previous results lead naturally to the following intriguing question.

**Problem 5.4.** Let $k$ and $j$ with $k \geq j$, do we have

\[\text{cat}_{X_{2k-j,k}} \text{Sp}(k) \leq \text{cat} \text{Sp}(j)?\]

Propositions 5.1, 5.2 and Proposition 5.3 give an affirmative answer for $j = 0, 1, 2$.

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