The Fermion Model of Representations of Affine Krichever–Novikov Algebras *

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To a generic holomorphic vector bundle on an algebraic curve and an irreducible finite-dimensional representation of a semisimple Lie algebra, we assign a representation of the corresponding affine Krichever–Novikov algebra in the space of semi-infinite exterior forms. It is shown that equivalent pairs of data give rise to equivalent representations and vice versa.

1. Introduction

Affine Krichever–Novikov algebras appeared in \([5, 6]\) as a generalization of affine Kac–Moody algebras that is related to a compact Riemann surface with two distinguished points. They belong to the new class of quasigraded algebras (see the definition in §2), whose structure and representations are present at the initial stage of investigation. At the same time, the notion of a quasigraded algebra is of importance and enables one to generalize many properties of the well-studied class of graded algebras, in particular, the most important property that one can construct representations generated by the vacuum vector. The first constructions of representations of affine Krichever–Novikov algebras (Verma modules and irreducible modules) were suggested in \([15, 17, 18]\), and, in the case of several distinguished points, in \([13]\). These constructions are of abstract algebraic nature. The problem of finding geometric objects on which these algebras act in a natural way remained open. In this paper we present such an object, namely, function spaces related in a special way to holomorphic vector bundles over the corresponding Riemann surface (these spaces were introduced earlier in \([8, 9]\) in connection with the solution of soliton equations).

Starting from these spaces, we construct representations generated by highest vectors by means of the known construction of an infinite fermion representation (the wedge representation) \([1, 5, 6, 12, 21]\); see also \([2, 3]\). To this end, we use some bases in the function spaces under consideration; these bases were introduced in \([11]\) and generalize the well-known Krichever–Novikov bases in function and tensor spaces on a Riemann surface with two distinguished points \([3, 6, 7]\). In §2 we give a construction of these bases and slightly extend the exposition in \([11]\).

The main theorem of the paper, Theorem 3.2, establishes conditions for the equivalence of fermion representations, and the most substantial condition is the equivalence of bundles generating these representations.

The appearance of bundles in representation theory of Krichever–Novikov algebras is by no means occasional. It was shown in \([14, 16]\) that the orbit space of the coadjoint representation of such an algebra coincides with the space of equivalence classes of finite-dimensional irreducible representations of the fundamental group of a punctured Riemann surface. In fact, these are just the data that arose earlier in the classification of Narasimhan–Seshadri modules of stable bundles on a (compact) Riemann surface. These data are also closely related \([13]\) to the Hitchin construction of Higgs bundles.

Krichever and Novikov \([11]\) arrived at the idea of constructing representations by using algebraic curves for other reasons. The quasigraded structure on the Krichever–Novikov algebras and modules over these algebras enables one to treat the representation theory of these algebras as a part of the theory of difference operators, where the theory of commuting difference operators developed in the papers of these authors \([4, 10, 11]\) plays a substantial role and readily leads to holomorphic bundles over curves. In fact, they had in mind the relationship with the theory of solitons already when writing the papers \([3, 6, 7]\). Note that the deep connection between the representation theory of affine Krichever–Novikov algebras and the theory of commutative rings of difference operators is revealed, in particular, in the proof of Theorem 3.1 in the present paper.

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This paper appeared as the result of my discussions with I. M. Krichever of all specific features of the construction in question, and the author is deeply indebted to I. M. Krichever for these discussions. Moreover, I. M. Krichever told the author about the results of the paper [11] long before the publication, expressed a lot of ideas about the application of these results in the posed problem of representation theory, and patiently discussed versions of their realization. Of these suggestions, the most significant was to pass from the space of sections of a vector bundle (on which, as is known, there is no action of matrix algebras in general) to the corresponding space of vector functions (see §2), where the desired action exists.

2. Krichever–Novikov Bases in Spaces of Sections of Holomorphic Bundles

(a) Krichever–Novikov algebras. Let \( \Sigma \) be a compact algebraic curve over \( \mathbb{C} \) with two distinguished points \( P_\pm \), let \( \mathcal{A}(\Sigma, P_\pm) \) be the algebra of meromorphic functions on \( \Sigma \) regular outside \( P_\pm \), and let \( \mathfrak{g} \) be a complex semisimple Lie algebra. Then

\[
\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}(\Sigma, P_\pm) \oplus \mathbb{C}c
\]

(2.1)

is called a Krichever–Novikov algebra of affine type \([3, 14, 16]\) with one-dimensional center generated by the element \( c \). Elements of \( \hat{\mathfrak{g}} \) will be denoted by \( X = X + ac \), where \( X \in \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}(\Sigma, P_\pm) \) and \( a \in \mathbb{C} \). The bracket on \( \hat{\mathfrak{g}} \) is defined by

\[
[x \otimes A, y \otimes B] = [x, y] \otimes AB + \gamma(x \otimes A, y \otimes B)c, \quad [x \otimes A, c] = 0,
\]

where \( \gamma \) is the cocycle defined by the relation

\[
\gamma(x \otimes A, y \otimes B) = (x, y) \text{ res}_{P_+}(AdB).
\]

(2.2)

As was mentioned in the introduction, the algebra \( \hat{\mathfrak{g}} \) carries a remarkable quasigraded structure.

Definition 2.1. (a) Let \( \mathcal{L} \) be a Lie algebra or an associative algebra that admits a decomposition \( \mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n \) into a direct sum of finite-dimensional subspaces \( \mathcal{L}_n \). The algebra \( \mathcal{L} \) is said to be quasigraded (almost graded, graded in the extended sense) if \( \dim \mathcal{L}_n < \infty \) and there exist constants \( R \) and \( S \) such that

\[
\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h = n + m - R}^{n + m + S} \mathcal{L}_h \quad \forall n, m \in \mathbb{Z}.
\]

(2.3)

The elements of the subspaces \( \mathcal{L}_n \) are called homogeneous elements of degree \( n \).

(b) Let \( \mathcal{M} \) be a quasigraded Lie algebra or an associative algebra, and let \( \mathcal{M} \) be an \( \mathcal{L} \)-module that admits a decomposition \( \mathcal{M} = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n \) into a direct sum of subspaces. The module \( \mathcal{M} \) is said to be quasigraded (almost graded, graded in the extended sense) if \( \dim \mathcal{M}_n < \infty \) and there exist constants \( R' \) and \( S' \) such that

\[
\mathcal{M}_n \cdot \mathcal{M}_m \subseteq \bigoplus_{h = n + m - R'}^{n + m + S'} \mathcal{M}_h \quad \forall n, m \in \mathbb{Z}.
\]

(2.4)

The elements of the subspaces \( \mathcal{M}_n \) are called homogeneous elements of degree \( n \).

In the following, we sometimes refer to the constants \( R, S, R', \) and \( S' \) as the coefficients of diversity of the grading.

For Krichever–Novikov algebras, the space of homogeneous elements of degree \( n \) for \( n > 0 \) and \( n < -g \) is \( \mathfrak{g} \otimes \mathcal{A}_n \), where \( \mathcal{A}_n \subset \mathcal{A} \) is the subspace of functions that are of order \( n \) at the point \( P_+ \) and of order \( -n + g \) at the point \( P_- \) (cf. also Example 2.1 below). For the constants \( R \) and \( S \) and other details, see \([3, 14, 13]\).

(b) Holomorphic bundles. Tyurin parameters. Let \( \Sigma \) be a Riemann surface of genus \( g \) and \( F \) a holomorphic bundle of rank \( l \) over \( \Sigma \). Suppose that \( l \) holomorphic sections \( \Psi_1, \ldots, \Psi_l \) of \( F \) are given that form a basis in the fiber over any point except for finitely many points \( \gamma_1, \ldots, \gamma_{gl} \),
referred to as points of degeneration. This set of sections is called a framing, and a bundle that admits a framing is said to be framed.

If one chooses a local trivialization of $F$ and defines sections $\Psi_1, \ldots, \Psi_l$ in this trivialization, say, by column vectors (of functions), then one can take a matrix $\Psi$ formed by these column vectors, which is nondegenerate everywhere except for the points $\gamma_i$, $i = 1, \ldots, gl$, at which $\det \Psi$ has simple zeros. In local coordinates in a neighborhood of the point $s$ in $\Sigma$, relation (2.3) follows the note [11].

Hence, the residues of the functions $\Psi_i$, $i = 1, \ldots, l$, are proportional. Therefore, the functions $\Psi_i$, $i = 1, \ldots, l$, can be extended to the points of the divisor $D$. By assumption, at least one of these determinants is nonzero, and hence $\Psi$ has $l - 1$ linearly independent rows.

In this case, for any $i = 1, \ldots, gl$ there exists a nontrivial solution of the system of linear equations $\Psi(\gamma_i)\alpha_i = 0$; this solution is unique up to proportionality. Let $\alpha_i = (\alpha_{ij})^1_{i=1,\ldots,j} (i = 1, \ldots, gl)$. In [8, 9], the divisor of the bundle and the numbers $(\alpha_{ij})^1_{i=1,\ldots,j}$ are called the Tyurin parameters of the bundle $F$. A framing is defined uniquely up to a right action of elements of the group $GL(l)$ on the matrix $\Psi$ (while the transition functions act on this matrix on the left; we see that the action of $GL(l)$ commutes with the transition functions and hence takes sections to sections). Owing to this fact, the Tyurin parameters are defined uniquely up to proportionality and to multiplication by elements of $GL(l)$ on the left. According to the results in [20], the Tyurin parameters define the corresponding bundle uniquely up to equivalence. We point out that the set of points $\gamma_1, \ldots, \gamma_{gl}$ is the same for equivalent framed bundles.

As was noted in [8, 9], Tyurin’s results in [20] imply the following description of the space of meromorphic sections of $F$ (we consider only meromorphic sections that are holomorphic outside $P_\pm$). The elements $\Psi_j(P)$ ($j = 1, \ldots, l$) form a basis in the fiber over any point $P$ outside the divisor $D$. Hence the value $S(P)$ for any meromorphic section $S$ can be decomposed with respect to this basis. To the section $S$ we assign the vector function $f = (f_1, \ldots, f_l)^T$ on the Riemann surface $\Sigma$ in such a way that outside $D$ one has

$$S(P) = \sum_{j=1}^l \Psi_j(P)f_j(P).$$

(2.5)

It follows from the Kronecker formula that $f_j = \det(\Psi_1, \ldots, \Psi_{j-1}, S, \Psi_{j+1}, \ldots, \Psi_l)(\det \Psi)^{-1}$, and therefore, the functions $f_j$ can be extended to the points of the divisor $D$. The extensions have at most simple poles at these points, since $\Psi_1, \ldots, \Psi_l$ are holomorphic at the points of $D$ and $\det \Psi$ has simple zeros at these points. In local coordinates in a neighborhood of the point $\gamma_i$, it follows from (2.3) that $S(z) = \Psi(\gamma_i)(\text{res}_\gamma f)z^{-1} + O(1)$. The left-hand side of the last relation is holomorphic. Hence, the residues of the functions $f_j$, $j = 1, \ldots, l$, at the point $\gamma_i$ satisfy the system of linear equations $\Psi(\gamma_i)(\text{res}_\gamma f) = 0$, which is just the system satisfied by the Tyurin parameters at this point. By assumption, the matrix $\Psi(\gamma_i)$ is of rank $l - 1$. Hence, the vectors $\text{res}_\gamma f$ and $\alpha_i$ are proportional.

The following assertion represents the results of [20].

Proposition 2.1 [8, 9]. For a generic bundle $F$, the space of meromorphic sections holomorphic outside the distinguished points $P_\pm$ is isomorphic to the space of meromorphic vector functions $f = (f_1, \ldots, f_l)^T$ on the same Riemann surface such that $f$ is holomorphic outside $D$ and $P_\pm$, has at most simple poles at the points of $D$, and satisfies the relations

$$(\text{res}_\gamma f_j)\alpha_{ik} = (\text{res}_\gamma f_k)\alpha_{ij}, \quad i = 1, \ldots, gl, \quad j = 1, \ldots, l.$$

(c) Krichever–Novikov bases. In this subsection, the exposition up to and including Proposition 2.3 follows the note [11].

For any pair of integers $n$, $j$ ($0 \leq j < l$), we shall construct a vector function in the space defined in Proposition 2.1 and denote this function by $\psi_{n,j}$. The number $n$ is called the degree
of \(\psi_{n,j}\). The function \(\psi_{n,j}\) will be specified by its asymptotic behavior at the points \(P_\pm\). Let us treat \(\psi_{n,j}\) as a column vector and form a matrix \(\Psi_n\) of these vectors. At the point \(P_+\), we set

\[
\Psi_n(z_+) = z_+^{n} \sum_{s=0}^{\infty} \xi_{n,s}^+ z_+^s, \quad \text{where} \quad \xi_{n,0}^+ = \begin{pmatrix} 1 & \ast & \ldots & \ast \\ 0 & 1 & \ldots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix},
\]
and at the point \(P_-\), we set

\[
\Psi_n(z_-) = z_-^{-n} \sum_{s=0}^{\infty} \xi_{n,s}^- z_-^s, \quad \text{where} \quad \xi_{n,0}^- = \begin{pmatrix} \ast & 0 & \ldots & 0 \\ \ast & \ast & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ast & \ast & \ldots & \ast \end{pmatrix},
\]

where \(z_\pm\) are the local parameters in neighborhoods of the points \(P_\pm\), respectively, and the asterisks \(\ast\) stand for arbitrary complex numbers.

Thus, the function \(\Psi_n\) has a zero of order \(n\) at the point \(P_+\) and a pole of order \(n\) at the point \(P_-\), and the determinant of \(\Psi_n\) has \(gl\) simple poles at the points of the divisor \(D\) and admits an additional divisor of zeros outside the points \(P_\pm\); the latter divisor is not given \textit{a priori}.

**Example 2.1.** In the Krichever–Novikov basis in the algebra \(\mathfrak{g}(\Sigma, P_\pm)\) with singularities at two points, the asymptotics are of the following form

\[
A_m \cong \alpha_m^{\pm m+\varepsilon_{\pm}} (1 + O(\varepsilon_{\pm})), \quad \alpha_m^+ = 1,
\]

where \(\varepsilon_+ = 0\) and \(\varepsilon_- = -g\), i.e., the sum of orders (the degree of the divisor of distinguished points) is equal to \(-g\) (note that \(l = 1\) in this case). Hence, there are exactly \(g\) zeros outside the points \(P_\pm\). This example differs from the general case studied above in that the additional \(g\) poles are concentrated at the same points \(P_\pm\) and the external divisor of degree \(-g\) is absent. This difference is unessential. The isomorphism is established by the multiplication (division) by a scalar function that has a zero of multiplicity \(g\) at the point \(P_-\) and a given divisor of (simple) poles of degree \(-g\) (such a function exists by the Riemann–Roch theorem).

**Proposition 2.2** \([1]\). 1. There exists a unique matrix function \(\Psi_n\) satisfying conditions \((2.6)\) and \((2.7)\).

2. The dimension of the spaces generated by the vector functions \(\psi_{n,j}\) for a given \(n\) is equal to \(l\).

**Proof.** It is clear that assertions 1 and 2 are equivalent. In the part related to the matrices \(\xi_{n,0}^\pm\), conditions \((2.6)\) and \((2.7)\) are the normalization conditions that uniquely select the vector functions \(\psi_{n,j}\) in the \(l\)-dimensional space.

The dimension of the space of functions with \(gl\) poles at the points of the divisor \(D\) with the orders \(\pm n\) at the points \(P_\pm\), respectively, is equal to \(gl - g + 1\). Since our vector functions are of dimension \(l\), we find that the dimension of the corresponding space is \(l(gl - g + 1)\). However, these functions satisfy the \((l-1)gl\) Tyurin relations. Therefore, the dimension of the function space under consideration is equal to \(l(gl - g + 1) - (l-1)gl = l\).

The algebra \(\mathfrak{g}(\Sigma, P_\pm)\) naturally acts both on the sections of the bundle \(F\) by the multiplication of these sections by functions and on the corresponding vector functions.

**Proposition 2.3.** The action of the elements of the algebra \(\mathfrak{g}(\Sigma, P_\pm)\) on the basis elements \(\psi_{n,j}\) is quasigraded:

\[
A_m \psi_{n,j} = \sum_{k=m+n}^{m+n+g} \sum_{j'=0}^{l-1} C_{m,n,j}^{k,j'} \psi_{k,j'},
\]

where \(\bar{g} = g + 1\) for \(-g \leq m \leq 0\) and \(\bar{g} = g\) otherwise.
Proof. Let us use the asymptotic behavior (2.3), where \( \varepsilon_- = -g \) for \( m > 0 \) and for \( m < -g \) and \( \varepsilon_- = -g - 1 \) for \( -g \leq m \leq 0 \) [3]. Set \( \bar{g} = -\varepsilon_- \). Multiplying the matrix \( \Psi_n \) by \( A_m \), we obtain the order \( z_+^{m+1} \) at the point \( P_+ \) and the order \( z_-^{n-m-\bar{g}} \) at the point \( P_- \). This is sufficient for the validity of relations (2.9) for some values of the constants \( C_{m,n,j}^{k,j'} \). In particular, the diversity of the grading with respect to the index \( n \) in the general case is equal to \( g \) (for \( -g \leq m \leq 0 \) this index is equal to \( g + 1 \)), and hence it is bounded. This proves that the action is quasigraded. In what follows we use the fact that the diversity of the grading with respect to the index \( ln - j \) is equal to \( lg \) in the general case. \( \square \)

In contrast with the algebra \( \mathcal{A}(\Sigma, P_x) \), the Lie algebra \( \mathfrak{g} \) admits no natural action on sections of the bundle. However, suppose that a representation \( \tau \) of the Lie algebra \( \mathfrak{g} \) in an \( l \)-dimensional space is given. Then this representation induces an action of this algebra on the vector functions \( \psi_{n,j} \). Namely, let \( x \in \mathfrak{g} \) and let \( \tau(x) = (x_i^i) \), where \( (x_i^i) \) is a matrix and the indices \( i \) and \( i' \) range from 1 to \( l \). We represent the vector function \( \psi_{n,j} \) in the form of the set of its coordinates, \( \psi_{n,j} = (\psi_{n,j}^i) \). In the coordinate representation, the action of \( x \) on \( \psi_{n,j} \) can be written in the form \( (x \psi_{n,j})^i = \sum_{i'} x_i^i \psi_{n,j}^i \). Having the action of the algebras \( \mathfrak{g} \) and \( \mathcal{A}(\Sigma, P_x) \) on vector functions, we define the corresponding action of the Lie algebra \( \mathfrak{g} \otimes \mathcal{A}(\Sigma, P_x) \) as follows:

\[
((x A_m) \psi_{n,j})^i = \sum_{k=m+n-j}^{m+n+g} \sum_{l=0}^{l-1} \sum_{i'} c_{m,n,j}^{k,j'} x_i^i \psi_{n,j}'^i.
\]

(2.10)

Formula (2.10) is an expression for the tensor product of the representation \( \tau \) of \( \mathfrak{g} \) and the action (2.9) of \( \mathcal{A}(\Sigma, P_x) \). Thus, this formula defines a representation of the tensor product of these algebras. It follows from Proposition 2.3 that the action (2.10) is quasigraded and the diversity of the grading with respect to the index \( n \) (respectively, \( ln - j \)) is equal to \( g \) (respectively, \( gl \)).

In the sequel, we interpret the \( \psi_{n,j}^i \) as formal symbols rather than coordinates of vector functions and treat relations (2.10) as formal linear transformations of these symbols. Under this approach, the action of \( \mathfrak{g} \) does not change the indices \( n \) and \( j \), and the role of a highest vector is played by the symbol \( \psi_{n,j}^i \) for given \( n \) and \( j \).

3. Fermion Representations

Let us consider a framed holomorphic bundle \( F \) of rank \( l \) and the corresponding space \( F_{KN} \) of Krichever–Novikov vector functions. In this space, we consider the basis \( \{ \psi_{n,j} \mid n \in \mathbb{Z}, j = 0, \ldots, l - 1 \} \), the corresponding set of symbols \( \psi_{n,j}^i (i = 1, \ldots, l) \), and the action (2.10) on this set; these objects were defined in the preceding section. Recall that this action depends on some representation \( \tau : \mathfrak{g} \mapsto gl(l) \). Starting from these data, we will construct a space \( V_{F} \) of semi-infinite forms and a representation \( \pi_{F,\tau} \) of \( \mathfrak{g} \) in this space.

Let us number the symbols \( \psi_{n,j}^i \) in lexicographically ascending order of the triples \( (n, -j, i) \), setting the index of the element \( \psi_{0,0}^i \) to be equal to \(-1\). Let \( N = N(n,j,i) \) be the index of a triple \( (n,j,i) \); then \( N(n,j,i) = l^2 n - l j + i - l - 1 \). Set \( \psi_N = \psi_{n,j}^i \). Formal finite linear combinations of the symbols \( \{ \psi_N \mid N \in \mathbb{Z} \} \) form a vector space, which we denote by \( F_{KN}^l \).

The space \( V_F \) is constructed in the following way. It is spanned over \( \mathbb{C} \) by formal expressions \( \psi_{N_0} \wedge \psi_{N_1} \wedge \ldots \), referred to as semi-infinite monomials, where \( N_0 < N_1 < \ldots \). It is required that, under a transposition of \( \psi_N \) and \( \psi_{N'} \) (\( N \neq N' \)), a semi-infinite monomial changes its sign and one has \( N_k = k + m \) for sufficiently large \( k \) (\( k \gg 1 \)), where \( m \) is an integer; following [2], \( m \) is called the charge of the monomial (the reader should not confuse this with the central charge: it follows from Remark 3.1 that the charge \( m \) is rather a component of the weight).

By the degree of a monomial \( \psi \) of charge \( m \) one means

\[
\deg \psi = \sum_{k=0}^{\infty} (N_k - k - m).
\]

(3.1)
The degree thus defined equips the space $V_F$ with a quasigraded structure, i.e., a decomposition into a direct sum of finite-dimensional subspaces. Note that the definition of the numbers $N_k$ is ambiguous; namely, it depends on the numbering of the elements $\psi_{n,j}^i$ for a given $n$. It is of importance that the definition of degree is independent of this ambiguity since the modification of the above numbering amounts to a permutation of some summands in formula (3.1).

The representation $\pi_{F,r}$ is defined as follows. Consider the Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{A}(\Sigma, P_{\pm})$ embedded in the Lie algebra $\hat{\mathfrak{g}}$ as a linear space. Every element $x \in \hat{\mathfrak{g}}$ acts on the symbols $\psi_N$ by linear changes in a quasigraded way by formula (2.10). This precisely means that if the symbols $\psi_N$ are regarded as a formal basis of an infinite-dimensional linear space, then the action of an element $x \in \hat{\mathfrak{g}}$ in this basis is defined by an infinite matrix with finitely many nonzero diagonals, or, equivalently, by a difference operator. Following [3], we denote the algebra of these matrices by $\mathfrak{a}_\infty$. Thus, to the basis $\{\psi_N\}$ there corresponds an embedding of $\hat{\mathfrak{g}}$ in the algebra $\mathfrak{a}_\infty$. It suffices to define the action of the algebra $\mathfrak{a}_\infty$ on $V_F$, and then, by virtue of this embedding, we obtain a representation of $\hat{\mathfrak{g}}$ in the same space.

Let us take a basis element $E_{IJ} \in \mathfrak{a}_\infty$. We define its action on a semi-infinite monomial $\psi = \psi_{I_0} \wedge \psi_{I_1} \wedge \ldots$ as follows:

$$r(E_{IJ})\psi = (E_{IJ}\psi_{I_0}) \wedge \psi_{I_1} \wedge \ldots + \psi_{I_0} \wedge (E_{IJ}\psi_{I_1}) \wedge \ldots + \ldots.$$  \hfill (3.2)

By virtue of the condition $I_k = k + m$ ($k \gg 1$), the action (3.2) is well defined for $I > J$ and for $I < J$. The operator $r(E_{II})$ acts on a semi-infinite monomial by multiplication by 1 or by 0, depending on whether the monomial contains $\psi_I$. Therefore, for a diagonal matrix $D = \sum_{I=-\infty}^{\infty} \lambda_I E_{II}$ with infinitely many coefficients $\lambda_I \neq 0$ such that $I > 0$, the sum $D\psi$ (which is well defined by (3.2)) can contain infinitely many terms. In this case, we apply the following regularization [2]. Let

$$\hat{r}(E_{IJ}) = r(E_{IJ}) \quad \text{for } I \neq J \text{ and for } I = J < 0,$$  \hfill (3.3)

$$\hat{r}(E_{II}) = r(E_{II}) - \text{Id} \quad \text{for } I \geq 0.$$  \hfill (3.4)

Then the following commutation relations hold [2]:

$$[\hat{r}(E_{IJ}), \hat{r}(E_{MN})] = \delta_{JM} \hat{r}(E_{IN}) - \delta_{NI} \hat{r}(E_{MJ}) \quad \text{for } (I, J) \neq (N, M),$$  \hfill (3.5)

$$[\hat{r}(E_{IJ}), \hat{r}(E_{II})] = \hat{r}(E_{II}) - \hat{r}(E_{IJ}) + \text{Id}.$$  \hfill (3.6)

Thus, the regularization gives the following cocycle $\alpha$:

$$\alpha(E_{IJ}, E_{II}) = -\alpha(E_{II}, E_{IJ}) = 1 \quad \text{for } I \geq 0, J < 0,$$  \hfill (3.7)

$$\alpha(E_{IJ}, E_{MN}) = 0 \quad \text{otherwise}.$$  \hfill (3.8)

It is quite clear that the action of the algebra $\mathfrak{a}_\infty$ preserves the charge of a monomial, since the infinite “tail” of each of the monomials on the right-hand side in (3.2) coincides with the “tail” of the monomial on the left-hand side in (3.2). Moreover, $\pi_{F,r}$ is a quasigraded representation. This follows from the fact that the action on the space $F_{KN}$ is quasigraded (Proposition 2.3 and relations (2.9) and (2.10)).

By highest (or vacuum) monomials we mean semi-infinite monomials of the form $\tilde{\psi}_M = \psi_M \wedge \psi_{M+1} \wedge \psi_{M+2} \wedge \ldots$ (the indices are successive, starting from some number).

The algebra $\mathcal{A}(\Sigma, P_{\pm})$ can also be embedded in the algebra $\mathfrak{a}_\infty$ of matrices with finitely many nonzero diagonals. To prove this fact, it suffices to rewrite relation (2.9) by replacing every $\psi_{n,j}^i$ by $\psi_{n,j}^i$ and every triple of indices $n, j, i$ by the corresponding index $N = N(n, j, i)$:

$$A_{m}\psi_N = \sum_{N' = l^2m + N} C_{m,N}'\psi_{N'},$$

where $g_l = l^2\tilde{g}$ (see Proposition 2.3). The following lemma clarifies the character of the action of the algebra $\mathcal{A}(\Sigma, P_{\pm})$ on the highest monomials. Let us take a highest monomial $\psi_M = \wedge_{N=M}^{\infty} \psi_N.$
Lemma 3.1. The element $A_m \tilde{\psi}_M$ has a nonzero projection on $\tilde{\psi}_M$ only for $m = -g, \ldots, -1, 0$, and this projection is equal to $\sum_{K=M}^{-1} C_{MK}^K$.

Proof. It follows from relation (2.9) that, for arbitrary integer $N$, the element $A_j \psi_N$ has a nonzero projection on $\psi_N$ if and only if $\psi_N$ occurs on the right-hand side in (2.9), i.e., only for $n + m \leq n \leq n + m + g$. This implies that $-g \leq m \leq 0$.

Let us represent the action of an element $A_m$, $-g \leq m \leq 0$, by an infinite matrix with finitely many nonzero diagonals. Then one has $A_m = \cdots + \sum_K C_{MK}^K E_{KK} + \cdots$, where the dots stand for terms with $E_{IJ}$ for $I \neq J$. For $I < J$, these terms send $\tilde{\psi}_M$ to 0 and, for $I > J$, to either 0 or a monomial of some degree less than that of $\tilde{\psi}_M$. For $E_{KK}$, where $K$ is arbitrary, one has $E_{KK} \psi_N = \delta_{KN} \psi_N$, and for $K \geq 0$ it follows from the above regularization that $E_{KK} \tilde{\psi}_M = 0$. Hence,

$$A_m \tilde{\psi}_M = \left( \sum_{K=M}^{-1} C_{MK}^K \right) \tilde{\psi}_M + \cdots,$$

(3.7)

where the dots stand for the sum of all terms whose degrees are less than that of $\tilde{\psi}_M$. \hfill \Box

Remark 3.1. It follows from the proof of Lemma 3.1 that the eigenvalue of the operator $I_d = \sum N E_{NN}$ (the identity operator of the space $F^s_{KN}$) on $\tilde{\psi}_M$ coincides with $-M$ for $M < 0$ and is zero for $M \geq 0$.

Definition 3.1. Two fermion representations are said to be strongly equivalent (isomorphic) if there exists an isomorphism of the underlying linear spaces that commutes with the action of the algebra $\hat{g}$, is quasi-homogeneous (i.e., takes every homogeneous component of the first module to a homogeneous component of the other), and takes any highest monomial to a highest monomial of the same charge.

Theorem 3.1. Let a fermion representation $\pi$ be defined by a framed holomorphic bundle $F$ and by an irreducible representation $\tau$ of the algebra $g$, and let a fermion representation $\pi'$ be defined by a framed holomorphic bundle $F'$ and by an irreducible representation $\tau'$. The representations $\pi$ and $\pi'$ are equivalent if and only if the corresponding framed bundles $F$ and $F'$ are equivalent and the representations $\tau$ and $\tau'$ of the algebra $g$ are equal.

Proof. Let the bundles $F$ and $F'$ be equivalent, and let the equivalence be defined by a mapping $C: F \to F'$. Obviously, $C$ induces the following isomorphisms:

(1) of the spaces of global holomorphic sections, $H^0(\Sigma, F) \cong H^0(\Sigma, F')$; under this isomorphism, the matrices $\Psi$ and $\Psi'$ composed of the basis holomorphic sections correspond to each other, $\Psi' = C \Psi$;

(2) of the spaces of meromorphic Krichever–Novikov vector functions regarded as $\mathcal{A}(\Sigma, P_{\pm})$-modules: $F_{KN} \cong F'_{KN}$, and the same holds for the related spaces $F^s_{KN}$ and $F'^s_{KN}$.

Since the representations $\tau$ and $\tau'$ are equal, it follows that the spaces $F^s_{KN}$ and $F'^s_{KN}$ are isomorphic as $g$-modules as well. Hence, the corresponding representations of the algebra $\hat{g}$ in the spaces of semi-infinite forms over $F^s_{KN}$ and $F'^s_{KN}$ are equivalent.

Let us pass to the proof of the second part of the theorem. Consider two equivalent fermion representations $\tau$ and $\tau'$ of $g$ generated by framed bundles $F$ and $F'$ and by irreducible representations $\tau$ and $\tau'$, respectively, in the sense of the above construction. Let $C$ be a strong isomorphism, $\pi' = C \pi C^{-1}$.

Let $v$ be a highest monomial in the representation space of $\pi$, and let $v' = C v$. Then it follows from Definition 3.1 that $v'$ is also a highest monomial. Our immediate aim is to prove that the representations $\tau$ and $\tau'$ are equivalent. To this end, we consider the structure of the highest monomials $v$ and $v'$.

Let $v = \tilde{\psi}_M$. A triple of indices $(n, j, i)$ such that $M = N(n, j, i)$ is determined uniquely (see above). We write $n = n(M)$, $j = j(M)$, and $i = i(M)$. Let us choose $v$ in such a way that $i = l$. As was noted at the end of the previous section, this is equivalent to the condition that $\tilde{\psi}_M$ is a highest vector of a subrepresentation equivalent to $\tau$ and acting on the space $F^s_{KN}$. Let $v' = \tilde{\psi}'_{M'}$, 

\[ v = \tilde{\psi}_M, \quad v' = \tilde{\psi}'_{M'} \]
and accordingly, let \( n' = n(M'), j' = j(M'), \) and \( i' = i(M') \). By definition, a quasi-homogeneous isomorphism preserves the charge. Therefore, \( M = M' \), and hence \( i = i' = i \) and \( \psi_{M}' \) is a highest vector of a subrepresentation equivalent to \( \tau' \).

Thus, \( \psi_{M} \) and \( \psi_{M}' \) are highest vectors of the representations \( \tau \) and \( \tau' \), respectively. Hence, the structure of the monomial \( v = \hat{\psi}_{M} \) is as follows: the first place is occupied by the highest vector \( \hat{\psi}_{M} \) of the representation \( \tau \), and to the right of this place one can find an infinite exterior product of the symbols \( \psi_{N} \), which can be partitioned into consecutive groups (we call them bags) each of which is the exterior product of (all) basis elements of the representation \( \tau \). This readily implies that

\[
\pi(x)v = (\tau(x)\psi_{M}) \wedge \psi_{M+1} \wedge \ldots
\]

(3.8)

for an arbitrary \( x \in \mathfrak{g} \). Indeed, if \( \tau(x) \) is a nilpotent element, then the other terms (that occur under the action of \( \pi(x) \) on \( v \) by the Leibniz formula) are equal to 0. If \( \tau(x) \) is a diagonal element, then their sum is annihilated for any bag, since the sum of the weights (counted according to their multiplicity) of a finite-dimensional representation of a semisimple Lie algebra is equal to 0. (This is obvious for any irreducible representation of \( sl(2) \), and the other cases can be reduced to this one by decomposing into irreducible representations of simple three-dimensional subalgebras corresponding to simple roots. In the case of a reductive algebra, this argument would fail for a central element.) A similar consideration shows that

\[
\tau'(x)v' = (\tau'(x)\psi_{M}') \wedge \psi_{M'+1} \wedge \ldots
\]

(3.9)

Let \( x = h \) be a Cartan element, and let \( \alpha_{\tau} \) and \( \alpha_{\tau'} \) be the highest weights of the representations \( \tau \) and \( \tau' \), respectively. Then it follows from formulas (3.8) and (3.9) that \( \pi(h)v = \alpha_{\tau}(h)v \) and \( \pi'(h)v' = \alpha_{\tau'}(h)v' \). Since the representations \( \pi \) and \( \pi' \) are equivalent, it follows that \( \alpha_{\tau} = \alpha_{\tau'} \).

Hence, \( \tau \) and \( \tau' \) are equivalent.

The Lie algebra \( \mathfrak{g} \) acts on the spaces \( F_{KN}^s \) and \( V_{F} \), and hence the same holds for the universal enveloping algebra \( U(\mathfrak{g}) \). The associative algebra \( \mathcal{A}(\Sigma, P_{\pm}) \) and the tensor product \( U(\mathfrak{g}) \otimes \mathcal{A}(\Sigma, P_{\pm}) \) also act on these spaces. Moreover, by construction, the representation of the algebra \( U(\mathfrak{g}) \otimes \mathcal{A}(\Sigma, P_{\pm}) \) is decomposed into a product of representations of the algebras \( U(\mathfrak{g}) \) and \( \mathcal{A}(\Sigma, P_{\pm}) \). Namely, if \( u \otimes A \in U(\mathfrak{g}) \otimes \mathcal{A}(\Sigma, P_{\pm}) \), then \( \pi(u \otimes A) = \tau(u)\pi(A) \). We claim that the equivalence of representations of the algebra \( \mathfrak{g} \) implies the equivalence of the corresponding representations of the algebra \( \mathcal{A}(\Sigma, P_{\pm}) \). Let \( u = u^{i_1 \ldots i_n}e_{i_1} \ldots e_{i_n} \in U(\mathfrak{g}) \) (where summation with respect to repeated sub- and superscript is assumed). Consider the element \( u' = u^{i_1 \ldots i_n}e_{i_1} \ldots e_{i_n}A \) in which the function \( A \) in any monomial is placed in the last factor. Then \( \pi(u') = \tau(u)\pi(A) \).

For the equivalent representation \( \pi' \) of the algebra \( \mathfrak{g} \), one has \( \pi' = C\pi C^{-1} \), where \( C \) is an intertwining operator, and hence

\[
\tau'(u)\pi'(A) = C(\tau(u)\pi(A))C^{-1}.
\]

(3.10)

Let \( Z(\mathfrak{g}) \) be the center of the ring \( U(\mathfrak{g}) \). Since the representations \( \tau \) and \( \tau' \) are irreducible, it follows that the ring \( Z(\mathfrak{g}) \) acts on \( F_{KN}^s \) and \( F_{KN}^s \) by scalar operators. For \( u \in Z(\mathfrak{g}) \), we set \( \tau(u) = \lambda_u \circ \Id \) and \( \tau'(u) = \lambda'_u \circ \Id \). Since \( \tau \cong \tau' \), it follows that \( \lambda_u = \lambda'_u \). Now it follows from relation (3.10) that, cancelling \( \lambda_u \) (which can be done for at least one element \( u \)), we obtain \( \pi'(A) = C\pi(A)C^{-1} \), as desired.

Let us show that the representations of the ring \( \mathcal{A}(\Sigma, P_{\pm}) \) in the spaces \( F_{KN} \) and \( F_{KN}^s \) are equivalent. To this end, consider the action of an element \( A_n \in \mathcal{A}(\Sigma, P_{\pm}) \) on monomials of the form \( \psi_{N,M} = \psi_{N} \wedge \psi_{M+1,\psi_{M+2}} \ldots \). If \( M - N \) is large compared with \( n \), then the structure constants of the action of the element \( A_n \) on \( \psi_{N} \) are contained in the set of the structure constants of the action of \( A_n \) on \( \psi_{N,M} \). Thus, one can recover the first action from the other one. As usual, let \( N = N(n, j, i) \). The structure constants of the action of the element \( A_m \) on \( \psi_{N,j} \) are independent of \( i \) and coincide with the structure constants of the action of \( A_m \) on \( \psi_{n,j} \), which can be observed by comparing formulas (2.3) and (2.10). Thus, we have obtained the desired equivalence of \( F_{KN} \) and \( F_{KN}^s \) treated as \( \mathcal{A}(\Sigma, P_{\pm}) \)-modules. As was shown in [11], in this case the framed bundles \( F \)
and \( F' \) are equivalent. Let us return to the representations \( \tau \) and \( \tau' \). As was shown above, they are equivalent, i.e., there exists an element \( \gamma \in GL(l) \) such that \( \tau' = \gamma \tau \gamma^{-1} \). If the matrix \( \gamma \) were not scalar, then it would “mix” homogeneous components \( \psi^i_n \) with respect to the index \( i \). Thus, the isomorphism \( C \) would be not quasi-homogeneous, which contradicts our assumption. Hence, \( \gamma \) belongs to the center of the group \( GL(l) \), and therefore, \( \tau = \tau' \). This completes the proof.

There is a more general notion of equivalence for fermion representations than that in Definition 3.1.

**Definition 3.2.** Two fermion representations are said to be *equivalent* (isomorphic) if there is an isomorphism of the form \( C \gamma = C \cdot \tilde{\gamma} \) between the representation spaces that commutes with the action of the algebra \( \hat{g} \), where \( C \) is a strong isomorphism in the sense of Definition 3.1, \( \gamma \in GL(l) \), and the mapping \( \tilde{\gamma} \) is defined as follows: \( \tilde{\gamma}(\psi^1 \wedge \psi^2 \wedge \ldots) = \gamma \psi^1 \wedge \gamma \psi^2 \wedge \ldots \).

Each of the mappings \( C \) and \( \tilde{\gamma} \) in this definition appears as a result of some operation over bundles. As was shown in the proof of Theorem 3.1, to a strong isomorphism \( C \) there corresponds a bundle isomorphism that can be defined by an equivalent change of the gluing functions. In Theorem 3.2 we will prove that the mapping \( \tilde{\gamma} \) arises from the change of framing in a bundle. Any two operations of these two types commute (see \( \S 2 \), Sec. (b)). Hence, the mappings \( C \) and \( \tilde{\gamma} \) commute, and thus an isomorphism of the form \( C \gamma \) is an equivalence relation indeed. The following lemma will be used in the proof of Theorem 3.2.

**Lemma 3.2.** For any irreducible \( l \)-dimensional representation \( \tau \) of the Lie algebra \( g \) and any generic holomorphic bundle \( F \) of rank \( l \) and of degree \( gl \), the equivalence class of the representation \( \tau \) is in a one-to-one correspondence with the set of framings of the bundle \( F \).

**Proof.** The group \( GL(l) \) acts on the irreducible representations of the algebra \( g \) in \( \mathbb{C}^l \) as follows: an element \( \gamma \in GL(l) \) takes the representation \( \tau \) to the equivalent representation \( \gamma^{-1} \tau \gamma \), and then the representation \( \tau \) remains stable if and only if the element \( \gamma \) is scalar, \( \gamma \in \mathbb{C} \cdot \mathrm{Id} \).

At the same time, the group \( GL(l) \) acts on the framings of the bundle \( F \) as follows: an element \( \gamma \) takes a framing defined by the matrix \( \Psi \) of sections to the framing defined by the matrix \( \gamma \Psi \) of sections (see \( \S 2 \), Sec. (b)). Moreover, the scalar operators (and only these operators) preserve the Tyurin parameters as points of the projective space, since they multiply all parameters by the same number.

Thus, both sets under consideration are equal to \( GL(l)/\mathbb{C} \cdot \mathrm{Id} \) as manifolds.

**Theorem 3.2.** The isomorphism relation introduced by Definition 3.2 is an equivalence relation on the set of fermion representations. The corresponding set of equivalence classes of fermion representations is in a one-to-one correspondence with the set of pairs \( ([F],[\tau]) \), where \( F \) is an equivalence class of (nonframed) holomorphic bundles of rank \( l \) and of degree \( gl \) (in general position) and \( [\tau] \) is an equivalence class of \( l \)-dimensional irreducible representations of the algebra \( g \).

**Proof.** Let a generic holomorphic (nonframed) bundle \( F \) of rank \( l \) of degree \( gl \) and an irreducible \( l \)-dimensional representation \( \tau \) of the algebra \( g \) be given. Choose any two framings in \( F \) that are defined by matrices of holomorphic sections \( \Psi \) and \( \Psi' \). Then there exists a matrix \( \gamma \in GL(l) \) such that \( \Psi' = \gamma \Psi^{-1} \). Suppose that to the data \( \{F,\tau,\Psi\} \) \((\{F,\tau',\Psi'\})\) there corresponds a fermion representation \( \pi = \pi(F,\tau,\Psi) \) \((\pi' = \pi(F,\tau',\Psi'))\), respectively. In this case, we have an isomorphism \( \tilde{\gamma} \) between the representation spaces \( \pi \) and \( \pi' \) by the construction of the fermion representation, for each monomial \( \psi^1 \wedge \psi^2 \wedge \ldots \) (if it belongs to the representation space of \( \pi \)) we have the corresponding monomial \( \gamma \psi^1 \wedge \gamma \psi^2 \wedge \ldots \). This can be seen from relation \( (2.5) \). For the framing \( \Psi \), this relation (in matrix form) becomes \( S = \Psi f \). Respectively, for the framing \( \Psi' \) this relation is of the form \( S = \Psi^{-1} \cdot \gamma f \), i.e., the elements \( f \in F_K \) are subjected to the automorphism \( \gamma \).

Thus, we have proved that the mapping \( \tilde{\gamma} \) appears as the result of the change of framing \( \Psi \mapsto \Psi \gamma^{-1} \) in the holomorphic bundle. As was noted above, this implies that an isomorphism in the sense of Definition 3.2 is an equivalence relation on the set of fermion representations.
For the mapping $\tilde{\gamma}$ to commute with the action of the algebra $g$ (this is also one of the conditions in Definition 3.2), it is necessary and sufficient that $\tau' = \gamma \tau \gamma^{-1}$. Thus, we obtain the following picture. By Theorem 3.1, the classes of strong equivalence of fermion representations are parametrized by the data sets $\{F, \tau, \Psi\}$. For a given structure of a holomorphic bundle $F$, to an isomorphism $\tilde{\gamma}$ there corresponds a transformation of the related data set, $\Psi \mapsto \Psi \gamma^{-1}$ and $\tau \mapsto \gamma \gamma^{-1}$. Hence, to the equivalence classes of fermion representations (in the sense of Definition 3.2) we can assign the orbits in the space of the data sets $\{F, \tau, \Psi\}$ under the action of the transformations $F \mapsto F$, $\Psi \mapsto \Psi \gamma^{-1}$, and $\tau \mapsto \gamma \gamma^{-1}$. It follows from Lemma 3.2 that these orbits are in a one-to-one correspondence with the pairs $([F], [\tau])$. 

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