As the human assistant, I will now carefully read and translate the contents of this page. The page appears to be from a mathematics journal, discussing Diophantine triples and related concepts. Let's break down the content into coherent segments for easier understanding:

**Title:** STRONG RATIONAL DIOPHANTINE $D(q)$-TRIPLES

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**Abstract:** We show that for infinitely many square-free integers $q$ there exist infinitely many triples of rational numbers $\{a, b, c\}$ such that $a^2 + q$, $b^2 + q$, $c^2 + q$, $ab + q$, $ac + q$ and $bc + q$ are squares of rational numbers.

**1. Introduction**

Classically, a Diophantine $m$-tuple is a set $\{a_1, \ldots, a_m\}$ of $m$ non-zero integers with the property that $a_ia_j + 1$ is a square, whenever $i \neq j$; such an $m$-tuple is called rational if we allow its elements to be non-zero rational numbers.

Fermat found the first Diophantine quadruple in integers $\{1, 3, 8, 120\}$. In 1969, Baker and Davenport [1] proved that Fermat’s set cannot be extended to a Diophantine quintuple. This result motivated the conjecture that there does not exist a Diophantine quintuple in integers. The conjecture has been proved recently by He, Togbé and Ziegler [15].

The first example of a rational Diophantine quadruple, the set $\{11, 192, 35, 155, 512, 1235, 180873\}$ was found by Diophantus, while Euler proved that there exist infinitely many rational Diophantine quintuples (see [16]). In 1999, Gibbs found the first example of rational Diophantine sextuple $\{41, 192, 35, 155, 512, 1235, 180873\}$ (see [13]). In 2017, Dujella, Kazalicki, Mikić and Szikszai [9] proved that there are infinitely many rational Diophantine sextuples, and alternative constructions of families of rational Diophantine sextuples are given in [8], [10] and [11]. It is not known whether there exist any rational Diophantine septuple.

More information on Diophantine $m$-tuples can be found in the survey article [4].

Dujella and Petričević in [12] introduced the notion of strong rational Diophantine $m$-tuple, as a rational Diophantine $m$-tuple with the additional property that $a_i^2 + 1$ is a rational square for every $i = 1, \ldots, m$. They proved that there exist infinitely many strong rational Diophantine triples. One such example is the set $\{1976/5607, 3780/1691, 14596/1197\}$.

Let $q$ be a rational number. A set $\{a_1, \ldots, a_m\}$ of nonzero integers (rationals) is called a (rational) $D(q)$-$m$-tuple, if $a_ia_j + q$ is a square of a rational number for all $1 \leq i < j \leq m$. It is known that for every rational number $q$ there exist infinitely many rational $D(q)$-quadruples,

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and that there are infinitely many square-free integers \( q \) for which there exist infinitely many rational \( D(q) \)-quintuples (see \([3, 5]\)).

In this paper, we will consider the problem which arises if we combine the two above mentioned variants of Diophantine \( m \)-tuples.

**Definition 1.1.** Let \( q \) be a rational number. A **strong rational Diophantine** \( D(q) \)-**\( m \)-tuple** is a set of non-zero rationals \( \{a_1, \ldots, a_m\} \) such that \( a_ia_j+q \) is a square for all \( i, j = 1, \ldots, m \) (including the case \( i = j \)).

As we already mentioned, the case \( q = 1 \) was studied in \([12]\). The case \( q = -1 \) was studied in \([7]\) and it was shown that there exist infinitely many strong rational \( D(-1) \)-triples (in \([7]\) they are called strong Eulerian triples because of the direct connection between \( D(-1) \)-\( m \)-tuples and so called Eulerian \( m \)-tuples, which are sets with property that \( xy + x + y = (x+1)(y+1) - 1 \) is a perfect square for all elements \( x, y \) of the set).

Our main result is the following theorem.

**Theorem 1.2.** There exist infinitely many square-free integers \( q \) with the property that there exist infinitely many strong rational Diophantine \( D(q) \)-**\( m \)-triples.**

**2. Construction of strong rational Diophantine \( D(q) \)-pairs and \( m \)-triples**

One may see easily that if \( \{a_1, \ldots, a_m\} \) is a strong rational Diophantine \( D(q) \)-**\( m \)-tuple, then \( \{za_1, \ldots, za_m\} \) is a strong rational Diophantine \( D(z^2q) \)-**\( m \)-tuple.** Therefore, it is enough to consider the problem of existence of strong rational Diophantine \( D(q) \)-**\( m \)-triples for square-free integers \( q \), and will do so in Section 3. Also, since we may choose \( z = 1/a_1 \) there is no lost of generality if we assume that \( a_1 = 1 \) and consequently \( 1^2 + q = r^2 \), i.e. \( q = r^2 - 1 \).

We will now explain a construction of strong rational Diophantine \( D(q) \)-**pairs which use properties of related elliptic curves.**

**Proposition 2.1.** For all rational numbers \( r, r \neq 0, \pm 1, \pm \frac{1}{2} \), there exist infinitely many rational numbers \( b \) such that \( \{1, b\} \) is a strong rational Diophantine \( D(r^2 - 1) \)-**pair.**

**Proof:** For convenience, we set \( q = r^2 - 1 \). We consider the elliptic curve \( E^q \) defined by the equation

\[
E^q : \quad y^2 = (x + q)(x^2 + q) = x^3 + qx^2 + qx + q^2.
\]

The curve \( E^q \) is non-singular for \( q \neq 0, -1 \), i.e. for \( r \neq 0, \pm 1 \), so in what follows we will always assume that \( r \neq 0, \pm 1 \). Some obvious rational points on \( E^q \) are

\[
T^q = (-q, 0), \quad P^q = (0, q), \quad S^q = (1, 1 + q).
\]

It is easily checked that \( T^q + P^q + S^q = O \).

Any rational number \( b \) such that \( \{1, b\} \) is a strong rational Diophantine \( D(q) \)-**pair, is the** \( x \)-coordinate of a point on \( E^q \).
Standard 2-descent (see e.g. [17, 4.2, p.85]) yields that the \( x \)-coordinate \( b \) of any point in \( 2E_q(\mathbb{Q}) \) satisfies that \( \{1, b\} \) is a strong rational Diophantine \( D(q) \)-pair. Hence, we will finish the proof if we show that \( E_q \) has rank at least 1 over \( \mathbb{Q} \).

We notice that

\[
2S^q = \left( \frac{1}{4} + q, \frac{q}{2} + \frac{1}{8} \right) = \left( \frac{5}{4} - r^2, \frac{r^2}{2} - \frac{3}{8} \right).
\]

Assume for the moment that \( r \) is an integer. Since the \( y \)-coordinate of \( 2S^q \) cannot be an integer, by the Lutz-Nagell theorem \( S^q \) has infinite order and \( \text{rank}(E_q) \) is at least 1. Let us consider now the general case when \( r \) is a rational number. We want to show that again the point \( S^q \) has infinite order. By Mazur’s classification of torsion points of elliptic curves over \( \mathbb{Q} \), it is enough to check that \( kS^q \) is not the point at infinity for \( k \leq 12 \) by considering rational roots of the denominators of the coordinates. We obtain that the only rational roots of denominators are \( r = \pm \frac{1}{2} \), in which cases the point \( S^q \) is of order 3. For all other rational numbers \( r \), the point \( S^q \) is of infinite order.

By the proof of Proposition 2.1, we may use the \( x \)-coordinate of \( 2kS^{r^2-1} \), \( k \) is an integer, to construct families of strong rational Diophantine \( D(r^2 - 1) \)-pairs. However, since the \( x \)-coordinate of \( S^{r^2-1} \) (which is equal to 1) satisfies that conditions that both \( x + r^2 - 1 \) and \( x^2 + r^2 - 1 \) are rational squares, by 2-descent, we conclude that we may also use the \( x \)-coordinate of \( (2k + 1)S^{r^2-1} \).

For example, the \( x \)-coordinates of \( 2S^{r^2-1} \), \( 3S^{r^2-1} \) and \( 4S^{r^2-1} \) yield that the pairs

\[
\left\{ \frac{1}{4} - r^2, 1 \right\}, \quad \left\{ \frac{-16r^4 + 16r^2 + 1}{16r^4 - 8r^2 + 1}, 1 \right\}, \quad \left\{ \frac{256r^8 - 768r^6 + 864r^4 - 496r^2 + 145}{256r^4 - 384r^2 + 144}, 1 \right\}
\]

are \( D(r^2 - 1) \)-pairs.

By extending the first of these three families of pairs, we will construct infinitely many strong rational Diophantine \( D(r^2 - 1) \)-triples for rational numbers \( r \) of certain form. More precisely, we prove the following proposition.

**Proposition 2.2.** For any rational number \( t \) different from \( 0, \pm \frac{1}{5}, \pm \frac{3}{5}, \pm \frac{7}{5} \) or \( \pm \frac{7}{15} \), the triple

\[
\left\{ \frac{625t^4 - 930t^2 + 49}{1024t^2}, 1, -\frac{(5t + 1)(5t - 1)(5t + 7)(5t - 7)}{1600} \right\}
\]

is a strong rational Diophantine \( D(q) \)-triple, with

\[
q = \frac{(t - 1)(t + 1)(25t + 7)(25t - 7)}{1024t^2}.
\]

**Proof:** In what follows we will use the symbol \( \Box \) to mean a square of a rational number. A strong rational Diophantine \( D(q) \)-triple \( \{a, b, c\} \) amounts to the following conditions being
simultaneously verified:

\[ a^2 + q = b^2, \quad ac + q = ab, \quad bc + q = bc. \]

We set \( q = r^2 - 1, \ a = 1, \) and \( b = \frac{5}{4} - r^2, \) for a rational number \( r \neq 0, \pm 1, \pm \frac{1}{2}. \)

We want to find \( c, \) different from 1 and \( b, \) such that \( \{1, b, c\} \) is a strong Diophantine \( D(q) \)-triple. From the condition \( c + q = s^2, \) we shall write \( c = s^2 - r^2 + 1, \) for some rational number \( s. \) We search for such values of the form \( s = kr. \) The condition \( bc + q = \Box_{bc} \) then becomes

\[ p(k, r) = \frac{5}{4} r^2 k^2 - r^4 k^2 - \frac{5}{4} r^2 + r^4 + \frac{1}{4} = \Box_{bc}. \]

This is possible for the values of \( k \) that make the discriminant of \( p(k, r) \) vanish. The discriminant of \( p(k, r), \) with respect to \( r, \) is equal to

\[ -\frac{1}{64} (5k - 3)^2 (5k + 3)^2 (k - 1)^3 (k + 1)^3, \]

so to have \( c \neq 1 \) we can choose \( k = 3/5. \) Then \( p(3/5, r) = \left( \frac{5r^2 - 5}{16} \right)^2. \) Thus, the only condition left is \( c^2 + q = \Box_{cc}, \) with \( c = -\frac{16}{25} r^2 + 1, \) that translates into

\[ \frac{1}{625} r^2 (256r^2 - 175) = \Box_{cc}. \]

This implies that we need to find \( t \in \mathbb{Q} \) such that

\[ (256t^2 - 175) = (16r + 25t)^2, \]

that results into the equality \( r = -\frac{1}{32} \frac{25t^2 + 7}{t}. \) Substituting this value in the formulas for \( b, c, \) and \( q, \) we finally obtain that the triple

\[ \left\{ 1, \frac{625t^4 - 930t^2 + 49}{1024 t^2}, \frac{(5t + 1)(5t - 1)(5t + 7)(5t - 7)}{1600 t^2} \right\} \]

is a strong rational Diophantine \( D \left( \frac{(t - 1)(t + 1)(25t + 7)(25t - 7)}{1024 t^2} \right) \)-tuple. Finally, the two elements different from 1 are distinct if and only if \( t \) is different from \( \pm \frac{3}{5} \) or \( \pm \frac{7}{15}. \) \( \square \)
3. Proof of Theorem 1.2

From Proposition 2.2, we only need to prove that, for infinitely many square-free integers \( q \), there are infinitely many rational numbers \( t \) such that

\[
\frac{(t-1)(t+1)(25t+7)(25t-7)}{1024t^2} = qw^2,
\]

for some rational number \( w \). Then by dividing all elements of the triple from Proposition 2.2 by \( w \), we get a strong rational \( D(q) \)-triple.

In other words, we need to study the quartic curve \( Q_q : qv^2 = (t-1)(t+1)(25t+7)(25t-7) \). The latter curve is the \( n \)-quadratic twist of the curve \( Q : v^2 = (t-1)(t+1)(25t+7)(25t-7) \).

The quartic curve \( Q \) is birationally equivalent, by the substitutions \( t = \frac{144+x}{144-x}, v = \frac{576y}{(x-144)^2} \), to the elliptic curve described by the Weierstrass equation:

\[
E_1 : y^2 = x(x+81)(x+256);
\]

similarly, \( Q_q \) is birationally equivalent, by the same substitutions, to

\[
E_q : qy^2 = x(x+81)(x+256).
\]

We will conclude our proof if we can find infinitely many square-free \( q \) for which \( \text{rank}(E_q) \geq 1 \). We will follow the reasoning from [5]. It is well-known (see e.g. [6]) that for the elliptic curve given by the equation \( y^2 = f(x) \), the point \((u, 1)\) is a rational point of infinite order in \( E_{f(u)}(\mathbb{Q}) \). By writing \( u = u_1/u_2 \), we get that for all integers \( q \) of the form

\[
q = u_1u_2(u_1+81u_2)(u_1+256u_2)
\]

the curve \( E_q \) has positive rank. This gives us infinitely many square-free values of \( q \) for which the rank is positive, and thus for which there exist infinitely many strong rational \( D(q) \)-quintuples. Indeed, for fixed \( \epsilon > 0 \) and sufficiently large \( N \), there are at least \( N^{1/2-\epsilon} \) square-free integers \( q, |q| \leq N \), of the form (9) (see e.g. [GM]).

\[\square\]

4. Examples and remarks

We computed the rank of \( E_q \) for small values \( q \) by \texttt{mwrang} [2], and obtained that rank is positive for the following square-free integers in the range \(|q| < 100:\)

\[-5, -6, -7, -11, -17, -19, -21, -22, -23, -29, -30, -34, -35, -37, -38, -39, -43, -46, -51, -55, -57, -58, -61, -62, -66, -67, -69, -74, -77, -78, -79, -83, -85, -86, -87, -91, -93, -94, -95, 2, 6, 10, 13, 15, 17, 23, 26, 29, 30, 31, 33, 35, 37, 42, 46, 47, 53, 55, 58, 59, 66, 69, 77, 78, 79, 82, 91, 93, 95.\]
In next table we give some examples of strong rational $D(q)$-triples \( \{a, b, c\} \), for small values of \( q \), obtained by the construction from Theorem 1.2. We provide also the corresponding parameter \( t \).

| \( t \) | \( q \)  | \( a \)  | \( b \)  | \( c \)  |
|------|------|------|------|------|
| \( \frac{37}{125} \) | \(-11\) | \( \frac{370}{27} \) | \( \frac{21122}{4995} \) | \( 75578 \) |
| \( \frac{11}{25} \) | \(-7\)  | \( \frac{44}{9} \)  | \( \frac{1051}{396} \)  | \( 736 \)  |
| \( \frac{101}{155} \) | \(-6\)  | \( \frac{3131}{684} \) | \( \frac{21031705}{8566416} \) | \( 591745 \) |
| \( \frac{23}{25} \) | \(-5\)  | \( \frac{23}{3} \)  | \( \frac{709}{276} \)  | \( 1827 \)  |
| \( \frac{119}{457} \) | \( 2\)   | \( \frac{7769}{1638} \) | \( \frac{38893009}{50902488} \) | \( 50817649 \) |
| \( \frac{23}{265} \) | \( 6\)   | \( \frac{1219}{1188} \) | \( \frac{32386295}{5792688} \) | \( 542735 \) |
| \( \frac{1}{31} \)  | \( 10\)  | \( \frac{31}{66} \)  | \( \frac{173279}{8184} \)  | \( -229437 \) |
| \( \frac{1}{25} \)  | \( 13\)  | \( \frac{2}{3} \)  | \( \frac{58}{3} \)  | \( -306 \) |

Just for fun, we also give a triple for \( q = 2019 \):

\[
\begin{align*}
a &= \frac{10842564894099462722723028577175690286281358594075905}{19799560082731784603837091066497564538879492519592} \\
b &= \frac{85871209066745518172033052694741882273167216912703276356128183994549391723676884264069999284696990523040}{231487576147616062211320060590257154156501721172189311604105086986008327988715912282514899652958055759} \\
c &= \frac{59632781988051772952800731039850157585883450782661641362001277739927065919199780738891666169990715641006}{59632781988051772952800731039850157585883450782661641362001277739927065919199780738891666169990715641006}
\end{align*}
\]

**Remark 4.1.** In Theorem 1.2, the existence of infinitely many square-free integers \( n \) for which there are infinitely many \( D(n) \)-triples mounts down to investigating the Mordell-Weil rank of the quadratic twists \( E_q : y^2 = x(x + 81)(x + 256) \). Goldfeld's minimalist conjecture asserts that for 50% of square-free integers \( q \), one would expect that \( \text{rank}(E_q) \) is positive, hence there are infinitely many strong rational \( D(q) \)-triples for at least 50% of square-free integers \( q \). See [5] for reasoning how the Parity Conjecture implies that for \( q \)'s in certain arithmetic progressions the rank of \( E_q \) is odd, and hence positive.
Remark 4.2. Note that the elliptic curve $E_1$, given by the equation $y^2 = x(x+81)(x+256)$ has rank 0 and torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$. For curves with such torsion group it is known that there are infinitely many quadratic twists with rank $\geq 4$ (see [18, 19]).

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