EULER CLASS AND FREE GENERATION

ALEXANDER REZNIKOV

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“What can you say in your favour?”
“You see ...”
“Enough. Shoot ’m. Next.” [S].

This paper consists of two parts. In the first auxiliary part, we deal with sets
with cyclic order. For any such set $\mathcal{O}$ we introduce following [BG] a cocycle $\ell : G \times G \to \mathbb{Z}$ valued in $\{0, \pm 1\} \subset \mathbb{Z}$ on the group $G$ of automorphisms of $\mathcal{O}$. The cohomology class of $\ell$ in $H^2(G, \mathbb{Z})$ will be called the Euler class. If $K$ is an ordered
field, then the projective line $\mathbb{P}^1(K)$ has a cyclic order and $PSL_2(K)$ acts order-
 preserving on $\mathbb{P}^1(K)$, so that we get both the cocycle $\ell$ and the Euler class in
$H^2(PSL_2(K), \mathbb{Z})$. If $K = \mathbb{R}$, then our Euler class coincides with the usual Euler
class on $PSL_2(\mathbb{R})$.

In view of our extension of the Euler class to all ordered fields, the following two
problems arise.

**Problem 1.** Let $\rho : \pi_1(S) \to PSL_2(K)$ be a homomorphism of the fundamental
group of a closed oriented surface. Is it true, that $|\rho^*[\ell], [S]| \leq 2g - 2$?

**Problem 2.** Suppose $|\rho^*[\ell], [S]| = 2g - 2$. Is it true that $\rho$ is injective?

For $K = \mathbb{R}$ the theorems of Milnor [M] and Goldman [Go2], answer these pr ob-
lems positively.

In the main second part of this paper, we apply the cocycle $\ell$ and the ideas
from the theory of Hamiltonian systems on the Teichmüller space to the following
classical problem:

**Problem 3.** When $n$ matrices in $SL_2(\mathbb{R})$ generate a free discrete group?

For $n = 2$ this problem has been treated in many papers, see [Gi]. An effective
solution is given in [Gi]. The analogous problem for $SL_2(\mathbb{C})$ also attracted a lot
of attention especially since Jörgensen paper [J]. For $n > 2$ however, the problem
becomes much harder. We will give a simple sufficient condition for $n$ hyperbolic
elements in $SL_2(\mathbb{R})$ to generate a free discrete group. This condition is open, that
is, satisfied on an open domain in $(SL_2(\mathbb{R}))^n$. Here is our main result.

Let $n$ be even and let $a_i, b_i, 1 \leq i \leq n$ be in $SL_2(\mathbb{R})$. Suppose $h = \prod_{i=1}^n [a_i, b_i]$ is hyperbolic. Consider the eigenvectors $x_1, x_2$ of $h$ and a matrix $r$ which takes the

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form \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) in the basis \((x_1, x_2)\). Put \( a_i = rb_{n+1-i}r^{-1}, b_i = ra_{n+1-i}r^{-1} \) for \( n + 1 \leq i \leq 2n \). Let \( I_j = \prod_{i=1}^{j} [a_i, b_i] (j \leq 2n) \).

**Main Theorem.** Let \( f(a, b) = \frac{1}{\pi} \sum_{j=1}^{2n} \ell(I_{j-1}, a_j) + \ell(I_{j-1}a_j, b_j) - \ell(I_{j-1}a_jb_ja_j^{-1}, a_j) - \ell(I_j, b_j) \) Then

(a) \( f(a, b) \) is an integer and \( |f(a, b)| \leq 2n - 1 \)

(b) if \( |f(a, b)| = 2n - 1 \), then \( \{a_i, b_i\} \) generate a free hyperbolic group in \( SL_2(\mathbb{R}) \).

1. Cyclically ordered sets, ordered fields and the Euler class.

1.1. A cyclically ordered set \( O \) is a set with a subset \( \Omega \) in \( O \times O \times O \), satisfying the following conditions:

(i) if \((x, y, z) \in O \) then \( x, y, z \) are all different

(ii) if \( \sigma \) is a permutation of \((x, y, z) \) and \((x, y, z) \in O \) then \( \{\sigma(x), \sigma(y), \sigma(z)\} \in O \) if and only if \( \sigma \) is even

(iii) if \( z \) is fixed then the relation \( x < y \iff (x, y, z) \in O \) is a linear order.

1.2 Example. Let \( K \) be an ordered field and let \( \mathbb{P}^1(K) \) be a projective line over \( K \). We can think of \( \mathbb{P}^1(K) \) as \( K \cup \{\infty\} \). The cyclic order in \( \mathbb{P}^1(K) \) is defined by a condition that the induced order in \( K \) is standard. The group \( PSL_2(K) \) acts on \( \mathbb{P}^1(K) \) preserving the cyclic order.

1.3. Define a function \( \psi : O \times O \times O \to \{0, \pm 1\} \) in a following way:

(i) if any of \((x, y, z) \) are equal, then \( \psi(x, y, z) = 0 \)

(ii) \( \psi \) is odd under permutation of \((x, y, z) \)

(iii) if \((x, y, z) \in \Omega \), then \( \psi(x, y, z) = 1 \).

1.4. Now let \( G \) be a group, acting in order preserving way on \( O \). Fix any element \( p \in O \) and define a function \( \ell : G \times G \to \{0, \pm 1\} \) as \( \ell(g_1, g_2) = \psi(p, g_2p, g_1g_2p) \).

**Lemma (1.4).** \( \ell \) is an integer cocycle on \( G \).

**Proof.** is a direct computation and left to the reader.

**Definition.** The cohomology class of \( \ell \) in \( H^2(G) \) (which does not depend on \( p \)) is called the Euler class. In particular, for an ordered field \( K \) one gets the Euler class in \( H^2(PSL_2(K), \mathbb{Z}) \).

1.5 Comparison theorem ([BG]). For \( K = \mathbb{R} \), the class of \( \ell \) in \( H^2(PSL_2(\mathbb{R})) \) is the usual Euler class of associated \( S^1 \)-bundle over \( BPSL_2^0(\mathbb{R}) \).

**Proof.** Consider the action of \( PSL_2(\mathbb{R}) \) on \( \mathcal{H}^2 \). For any \( p \in \mathcal{H}^2 \) the class \( \ell_p(g_1, g_2) = \text{Area}(p, g_2p, g_1g_2p) \) represents the Euler class \( e \) [Gu]. Here \( \text{Area}(p, q, r) \) is the area of oriented geodesic triangle with vertices in \( p, q, r \). Now \( [\ell_p] \in H^2(PSL_2(\mathbb{R})) \) does not depend on \( p \) and all cocycles \( \ell_p \) are uniformly bounded. For \( p_0 \in \partial \mathcal{H}^2 \) and \( p \to p_0 \), we will have \( \ell_{\infty} \)-convergence of cocycles \( \ell_{p_i} \to \ell_{p_0} \). Moreover the area of ideal triangle \((p, q, r) \) is \( \pi \cdot \psi(p, q, r) \). Any homology class in \( H_2(PSL_2(\mathbb{R})) \) is represented by a map of a surface group \( \pi_1(S) \xrightarrow{\alpha} PSL_2(\mathbb{R}) \). It follows that \( (\ell_{p_0}, \alpha_\ast[S]) = \lim(\ell_{p_i}, \alpha_\ast[S]) = (e, \alpha_\ast[S]) \). This completes the proof.

2. Discrete Goldman twist. We will work with the representation variety \( \mathcal{M} = \text{Hom}(\pi_1(S), SL_2(\mathbb{R}))/SL_2(\mathbb{R}) \), where \( S \) is an oriented closed surface of genus \( g \). It
has $2^{2g+1} + 2g - 3$ connected components, indexed by the value of the Euler class $[H]$. Every such component, say $M_e$, is a symplectic manifold, nonsingular if $e \neq 0$. For any $\gamma$ a conjugacy class in $\pi_1(M)$, there is a natural Hamiltonian $Tr_\gamma : M \rightarrow \mathbb{R}$, and the corresponding Hamiltonian flow has been identified by Goldman. If $\gamma$ can be represented by a simple separating curve, this flow can be described as follows. Write a presentation of $\pi_1(S)$ in the following form:

$$[x_1, x_2] \cdots [x_{2\kappa-1}, x_{2\kappa}] = [\gamma] = [x_{2\kappa+1}, x_{2\kappa+2}] \cdots [x_{2g-1}, x_{2g}]$$

Next, write $[\gamma] = \exp A$ for some $A \in sl_2(\mathbb{R})$. Then put

$$\bar{x}_i^t = \bar{x}_i, i \leq 2\kappa$$
$$\bar{x}_i^t = \exp(-tA)\bar{x}_i \exp tA, i \geq 2\kappa;$$

this is the flow of $Tr_\gamma$. In particular, $f_t : \{\bar{x}_i\} \rightarrow \{\bar{x}_i^t\}$ is a symplectomorphism of $M$. Here $\bar{x}_i$ stands for the representation matrix of $x_i$.

For different $\gamma$, the Hamiltonians $Tr_\gamma$ yield nice commutation relations, discovered by Wolpert [W] and put in a more “representation variety language” by Goldman [Go1]. In fact, the free module on the set of conjugacy classes of $\pi_1(M)$ becomes a Lie ring with Goldman’s bracket. One may wonder what kind of group object correspond to it.

Whatever this eventual “Kac-Moody-Goldman” group may be, we will introduce now some elements from the “other connected components” of it. These are defined for $M_{\pm(g-1)}$, which is naturally symplectomorphic to the Teichmüller space by a theorem of Goldman (see various proofs in [Go2], [H], [Re2]. In this case, all representation matrices are hyperbolic.

For $\gamma$ as above, let $r$ be a unique matrix (up to sign), commuting with $[\gamma]$ with eigenvalues $+1$ and $-1$. Put

$$f(\bar{x}_i) = x_i, i \leq 2\kappa$$
$$f(\bar{x}_i) = r^{-1}\bar{x}_i r, i \geq 2\kappa$$

This is a symplectic diffeomorphism of $M_{\pm(g-1)}$. There is a particularly nice description of the map $f$ if one views $M_{g-1}$ as Teichmüller space. Namely, realise a point in $M_{g-1}$ as a hyperbolic metric on $S$ and find a geodesic, representing $\gamma$. We assume that the marked point lies on $\gamma$. Cut $S$ into two pieces along $\gamma$ and glue again by a reflection, which fixes a marked point. Then the new hyperbolic structure is the image of $f$.

3. **Euler class.** For a representation $x_i \rightarrow \bar{x}_i$ of $\pi_1(S)$ in $SL_2(\mathbb{R})$ the Euler number is an integer between $-(g-1)$ and $(g-1)$ by Milnor [M]. As mentioned above, all representations with the maximal Euler number are discrete faithful hyperbolic by Goldman’s theorem. One can introduce a universal Euler class $e$ in $\tilde{H}^2(SL_2^0(\mathbb{R}), \mathbb{R})$ as the image of a generator in continuous cohomology $\tilde{H}^2_{cont}(SL_2(\mathbb{R}), \mathbb{R})$. A representation $x_i \rightarrow \bar{x}_i$ as above defines a homology class in $H_2(SL_2^0(\mathbb{R}), \mathbb{Z})$, the image of the generator of $H_2(\pi_1(S), \mathbb{Z}) \approx \mathbb{Z}$, and the Euler number is just given by the evaluation of $e$ on this class. Now, the generator of $H_2(\pi_1(S), \mathbb{Z})$ can be realized by an explicit cycle in the standard complex [B], that is, $\sum_{j=1}^{2g}(I_{j-1}|x_j) + \langle I_{j-1}x_j|y_j \rangle - \langle I_{j-1}x_jy_jx_j^{-1}|y_j \rangle - \langle I_j|y_j \rangle$ where
Let $I_j = [x_1, y_1] \ldots [x_j, y_j]$. On the other hand, the universal Euler class may be realized by a cocycle $A, B \mapsto \ell(A, B)$ as defined in Section 1. So the Euler number will be

$$
\sum_{j=1}^y \ell(I_{j-1}, x_j) + \ell(I_{j-1}, x_j, y_i) - \ell(I_{j-1} x_j, x_j^{-1}, y_j) - \ell(I_j, y_j),
$$

where $I_j$ is defined in an obvious way.

3. Proof of the Main Theorem. Consider a closed surface $S$ of genus $2n$. A map $x_i \to a_i, y_i \to b_i, 1 \leq i \leq 2n$ defines a homomorphism from $\pi_1(S)$ to $SL_2(\mathbb{R})$. Indeed, $\prod_{i=1}^n [a_i, b_i] \cdot \prod_{i=n+1}^{2n} [a_i, b_i] = h \cdot r^{-1} h^{-1} r = 1$. Next, the Euler number of this representation is computed as above, so $f(a, b)$ is always an integer. Moreover, if $f(a, b) = 4n - 2$, then the representation above is discrete and faithful. Q.E.D.

**SU(1, n)-case, I.** Consider a standard action of $SU(1, n)$ on the unit ball $B \subset \mathbb{C}^n$ with the complex hyperbolic metric. Let $\omega$ be the Kähler form of $B$. Fix a point $\infty$ in the sphere at infinity and consider a function $\varphi(A, B) = \psi(\infty, A(\infty), AB(\infty))$, where $\psi(x, y, z)$ is an integral of $\omega$ over any surface, spanning the geodesic triangle with vertices $x, y, z$. Let $a_i, b_i, 1 \leq i \leq g$ be matrices in $SU(1, n)$. Let $h = \prod_{i=1}^g [a_i, b_i]$ and suppose $h$ has a nonisotropic eigenvector. Then there exists a reflection $r$, commuting with $h$. Define $a_i, b_i, i \geq g + 1$ as in Introduction.

3.1 Theorem. Define $f(a, b)$ by the formula in the Main Theorem. Then

(a) $f(a, b)$ is an integer and $|f(a, b)| \leq 2g - 1$

(b) if $f(a, b) = 2g - 1$, then $\{a_i, b_i\}$ generate a discrete group in $SU(1, n)$.

Proof. Same as above with Toledo’s theorem [To], [Re1] instead of Goldman’s.

4. Bounded cohomology. The bounded cohomology theory is an invention of Mikhail Gromov. The idea is as follows: in the standard complex, computing the real group cohomology we look only at bounded cochains, that is, bounded functions $f : \prod_i G \to \mathbb{R}$. The resulted cohomology spaces are called $H^i_b(G, \mathbb{R})$.

There is a canonical homomorphism $H^i_b(G, \mathbb{R}) \to H^i(G, \mathbb{R})$.

4.1 Example. Let $M$ be a symmetric space of negative curvature with isometry group $G$. Let $\omega$ be any $G$-invariant $i$-form on $M$. Then one gets a (Borel) class $Bor(\omega) \in H^i_{cont}(G, \mathbb{R})$ (see [Re 1] for example), which may be represented by bounded cocycle [Gr 1]. The Euler class in $SU(1, n), n \geq 1$ is a further specialization.

4.2 Second bounded cohomology and combinatorial group theory. Consider a kernel of the map $H^2_b(G, \mathbb{R}) \to H^2(G, \mathbb{R})$. It gives rise to a function $f : G \to \mathbb{R}$ satisfying $|f(xy) - f(x) - f(y)| \leq C$. Moreover, this function may be chosen a class function, that is, $f(xyx^{-1}) = f(y)$ and such that $f(x^n) = nf(x)$ [BG].

Next, for an element $z \in G' = [G, G]$ a genus norm is the smallest integer $g$ such that $z$ is a product of $g$ generators. A theorem of Culler [C] states:

**Theorem 4.2 (Culler).** If $G$ is a f.g. free group. There is a positive constant such that for any $z \in G'$, $\|z^n\|_{genus} \geq \text{const} \cdot n$.

The following result has been proved by Gromov [Gr2] and author [Re3].
Theorem 4.3. Let $G$ be geometrically hyperbolic, that is a fundamental group of a manifold of pinched negative curvature with $\lim_{x \to \infty} i(x) = \infty$ (e.g. compact). Then the conclusion of the Theorem 4.2 holds. Here $i(x)$ is the injectivity radius at the point $x$.

Now, we have

**Proposition 4.4.** Let $f : G \to \mathbb{R}$ be as above. If $f(z) \neq 0$, then $\| z^n \|_{\text{genus}} \geq \text{const} \cdot n$.

**Proof.** Let $z^n = \prod_{i=1}^{g}[x_i, y_i]$. Then

$$|n \cdot f(z)| = |f(z^n)| = |f(\prod_{i=1}^{g}[x_i, y_i])| = \sum_{i=1}^{g} f([x_i, y_i]) + C \cdot g \leq 3c \cdot g + C \cdot g = 4C \cdot g,$$

so $g \geq \frac{n|f(z)|}{4C}$,

Q.E.D.

**Genus norm in lattices in $SU(1,n)$.**

**Theorem (4.5).** Let $\Gamma \subset SU(1,n)$ be a lattice with $H_2(\Gamma, \mathbb{R}) = 0$. There exists a nonzero function $f : \Gamma \to \mathbb{R}$ as above such that if $f(z) \neq 0$, then $\| z^n \|_{\text{genus}} \geq \text{const} \cdot n$.

**Remarks.** 1. Observe that if $f(z) \neq 0$, then for any $y \in G$ and $\kappa$ big enough, $f(z^\kappa y) \neq 0$, so there are “many” $z$ for which the Theorem 6.1 applies.

2. If $\Gamma$ is cocompact or if $\lim_{\gamma \to \infty} |\text{Tr} \gamma| \to \infty$ in $\Gamma$ then the conclusion of the Theorem follows from 4.3.

The proof of theorem 4.5 will be completed in 5.2.

**5. Ergodic cocycle and measurable transfer.** Let $G$ be a locally compact group, $H$ a closed subgroup and $X = G/H$. Suppose $G$ has an invariant finite Borel measure $\mu$ on $X$. Suppose we have a measurable section $\delta : X \to G$. For any $g \in G$ and $x \in X$ we define $\lambda(g, x) \in H$ as unique element such that

$$s(gx) = gs(x)\lambda(g, x)$$

This defines a map of groups

$$G \xrightarrow{\lambda} H^X$$

Now, $G$ acts on $H^X$ by changing the argument. The map $\lambda$ is well-known to be a *cocycle* for first non-abelian cohomology. Suppose $f(h_1, \ldots, h_n)$ is a measurable $n$-cocycle on $H$. Then the composition $f \circ \lambda : G^{(n)} \to \mathbb{R}$ is a measurable cocycle valued in the $G$-module of measurable functions on $X$. If $f$ is bounded, so is $f \circ \lambda$.

In this case, $\int_X f \cdot \lambda$ is a real bounded $n$-cocycle on $G$ and we have a well-defined map

$$H^n_b(H, \mathbb{R}) \xrightarrow{t} H^n_b(G, \mathbb{R})$$
Moreover, the composition $H^n_b(G, \mathbb{R}) \xrightarrow{res} H^n_b(H, \mathbb{R}) \xrightarrow{f} H^n_a(G, \mathbb{R})$ is a multiplication by Vol $X$.

The proof of all this facts is easily adopted from [Gr1]. See [Re4].

5.2. As an immediate corollary, we state: 

**Theorem (5.2).** Let $\Gamma$ be a lattice in either 1) $SO(n,1)$ or 2) $SU(n,1)$ or 3) $SU_H(n,1)$. Then in case 1) $H^0_b(\Gamma) \neq 0$; in case 2) $H^2_b(\Gamma, \mathbb{R}) \neq 0$ for all $1 \leq \kappa \leq n$; in case 3) $H^4_b(\Gamma, \mathbb{R})$ for all $1 \leq \kappa \leq n$.

**Proof.** Let us prove 2), since the rest is similar. $SU(n,1)$ acts isometrically on the complex ball $B^n$. The Kähler form $\omega$ defined, as in 4.1, a class $\omega$ in $H^2_b(\Gamma, \mathbb{R})$. It is nontrivial since for any cocompact $\Gamma$ the restriction on $H^2(\Gamma)$ gives the Kähler class of $B/\Gamma$. Now for any $\Gamma$, the restriction on $H^2_b(\Gamma, \mathbb{R})$ must be nontrivial, otherwise $\text{Vol}(SU(n,1)/\Gamma) \cdot \omega = 0$ even as a class in $H^2(SU\delta(n,1), \mathbb{R})$.

**Proof of the theorem 4.5.** We need only to handle the case $H^2(\Gamma, \mathbb{R}) = 0$. Then the restriction of the just defined class in $H^2_b(\Gamma, \mathbb{R})$ on $H^2(\Gamma, \mathbb{R})$ is zero, so by 4.2 we have a function $f$ with desired properties.

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Institute of Mathematics, Hebrew University, Giv’at Ram 91904, Jerusalem, Israel.