Solution of Simple Prey-Predator Model by Runge Kutta Method

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Abstract. The research on simulating logistic equation and prey predator system of equation using Runge Kutta method has been done. In logistic equation, simulation is agreed to some characteristics of equation. The simple form of prey predator system of equation is taken to be simulated. Simulation shows some similar patterns on graph. The changing on constants produces a certain pattern on graph.

1. Introduction

First, population dynamics has behavior as an exponential function. This model is a very simple, only depends to birth rate or death rate and neglects many factors, like food sources, competition, presence of prey and predator. This model assumes population is always increased or decreased, never changes its trend. In environment, there is a relation between population and food sources, population and predator or prey which are main factors.

If prey and predator are present in environment, type of interaction is added so formulation is not simple and does not have analytical solution. The model of prey - predator has been developed by in coordination with harvesting of prey and predator. Finite difference method [1], stochastic differential equation method [2], finite element method [3] and Runge Kutta scheme [4] are some numerical methods implemented to solve population dynamics.

The output of this paper is numerical solution of logistic equation and numerical solution of prey - predator model by Runge Kutta method.

2. Logistic Equation and Prey - Predator System of Equation
2.1. Logistic Equation

The formula of logistic equation is:

\[
\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right)
\]  
(1)

The exact solution for equation 1 is [5]:

\[
N(t) = \frac{N_0 Ke^{rt}}{K + N_0 \left( e^{rt} - 1\right)}
\]  
(2)

Where r is growth rate and K is carrying capacity. In equation 1, there is a certain value K which makes zero at bracket term. Three key features of the logistic growth are [6]:

- \[\lim_{t \to \infty} N(t) = K\], the population will ultimately reach its carrying capacity,
- The relative growth rate, \[\frac{1}{N} \frac{dN}{dt}\], declines linearly with increasing population size,
- The population at the inflection point (where growth rate is maximum), \[N_{inf}\], is exactly half the carrying capacity, \[N_{inf} = \frac{K}{2}\].

2.2. Prey-Predator System of Equation

An organism can eat or can be eaten by another organism that is drawn in food pyramid. This situation can be formulated mathematically as:

\[
\frac{dx}{dt} = ax - \alpha xy
\]  
(3)

\[
\frac{dy}{dt} = -cy + \gamma xy
\]  
(4)

With a, c, \(\alpha\), \(\gamma\) are positive constants. Parameter a refers to birth rate of prey, c refers to death rate of predator, \(\alpha\) and \(\gamma\) refer to interaction constants. Variable x refers to prey population and variable y refers to predator population. If prey is absent, predator decays exponentially. If predator is absent, prey grows exponentially. Equation 3 and equation 4 has eco-closed system condition which migration of organism in or out ecosystem is not allowed.

Mathematically, equation 3 and equation 4 are coupled system of equation. Both equations should be solved alternately. It is not important which variable first solved.

3. Runge-Kutta Method

Runge Kutta (RK) method is implied from Taylor series using a higher order derivatives. Numerical differential equation starts from:
\[
\frac{dy}{dt} = f(x, y)
\]  

(5)

The formula to calculate the value at next point is:

\[
x_{i+1} = x_i + \phi(x_i, y_i, h)h
\]  

(6)

\[
y_{i+1} = y_i + \phi(x_i, y_i, h)h
\]  

(7)

Where \( \phi(x_i, y_i, h) \) is an increment function. The increment function consists of an expansion \( \phi = a_0k_1 + a_1k_2 + \ldots + a_nk_n \). The detail of increment function is explained in [7].

After some derivations, third order RK is written as:

\[
y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h
\]  

(8)

where \( k_1 = f(x_i, y_i) \), \( k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right) \), \( k_3 = f(x_i + h, y_i - k_ih + 2k_2h) \).

The fourth order RK is written as:

\[
y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h
\]  

(9)

where \( k_1 = f(x_i, y_i) \), \( k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right) \), \( k_3 = f\left(x_i + h, y_i + \frac{1}{2}k_2h\right) \), \( k_4 = f(x_i + h, y_i + k_1h) \).

4. Results and Discussion

4.1 Logistic Equation

We compare some results from RK3 and RK4. The compared components are error, trend of graphic with variation \( r \) and \( K \). Value of initial population is always 2. Figure 1 explains population varies to time with \( K = 1 \) and \( r = 2 \) equals to initial population. That condition produces same population along time.
Figure 1. Variation of population to time with $K = N_0$.

In figure 2, the population disables exponentially at $K$ less than initial population. The convergence reaches faster at smaller $K$ which has meaning that population disables faster if carrying capacity is smaller.

Figure 2. Variation of population to time with $K = 1$ (left), $K = 1.5$ (right).

Figure 3. Variation of population to time with $K = 3$ (left), $K = 4$ (right).

In figure 3, the population grows exponentially at $K$ more than initial population. The convergence reaches faster at smaller $K$ which has meaning that population grows faster if carrying capacity is smaller. At figure 2 and figure 3, logistic equation has value $r = 2$ population has inflection value equals to $\frac{K}{2}$.
The variation of grow rate at $K = 1$ is figured in figure 4. Figure 2 and figure 4 clearly indicates RK 3 has greater error than RK 4 does which is agreed with theorem that RK 4 produces result closer to analytic than RK 3 does. Both method has a greater error due to greater $r$ which has limitation until $r = N_0$.

**Figure 4.** Variation of population to time with $r = 1.25$ (left), $r = 1.5$ (center), $r = 1.75$ (right).

### 4.2 Prey-predator Equation

Prey-predator equation is simulated by RK 4 method because it is a good method as in section 4.1. The simulation has variation of every parameter. Figure 5 shows simulation on condition parameter $a$ is equals to 0.125 (left), 0.15 (center), 0.175 (right) and the other parameters are 0.1.

**Figure 5.** Variation of Prey and Predator Population with varying of $a$.

Figure 6 shows simulation on condition parameter $c$ is equals to 0.125 (left), 0.15 (center), 0.175 (right) and the other parameters are 0.1.

**Figure 6.** Variation of Prey and Predator Population with varying of $c$. 
Figure 5 and figure 6 give some information:

- population of prey is always greater than population of predator if $a > c$,
- population of predator is always greater than population of prey if $c > a$,
- the more $a$ or $c$, the more value of maximum local point predator or prey population.

The results in figure 5 is true because birth rate of prey is more than death rate of predator, as consequence of food is always added. In figure 6, results show like that because death rate of predator is more than birth rate of prey so prey population is always more than predator population is.

The result as variation of alpha and gamma is explained in figure 7 and figure 8. On left value alpha or gamma is 0.125, on center value alpha or gamma is 0.15 and on right value alpha or gamma is 0.175. The other variables set to 0.1.

![Figure 7. Variation of Population with alpha is varied.](image)

![Figure 8. Variation of Population with gamma is varied.](image)

As $a$ or $c$ becomes greater, two characteristics present such as the absolute difference between prey and predator at nearest local maximum point becomes greater but period becomes shorter.

The results show a similarity pattern as figured in figure 5 to figure 8. The most like function that can explain pattern, is exponential harmonic function because the result shows some maximum or minimum local points with its value is greater than previous maximum local points and less than previous minimum local value. Special to harmonic, the input of harmonic function may be a linear function to time.
5. Conclusion

RK method can be implemented to logistic equation and prey predator system of equation. In logistic equation, RK 4 produces result with less error than RK 3 and both produces a convergence and stabilizer result. In prey predator, RK 4 can simulate system of equation with certain characteristic graphic.

6. References

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