THE MODULAR CURVE AS THE SPACE OF STABILITY CONDITIONS OF A CY$_3$ ALGEBRA

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Abstract. We prove that a connected component of the space of stability conditions of a CY$_3$ triangulated category generated by an $A_2$-collection of 3-spherical objects is isomorphic to the universal cover of the $\mathbb{C}^*$-bundle of non-zero holomorphic differentials on the moduli space of elliptic curves.

1. Introduction

The space of stability conditions $\text{Stab}(\mathcal{D})$ of a triangulated category $\mathcal{D}$ was introduced in [1]. As a set it has a description as the pairs $(\mathcal{A}, Z)$ where $\mathcal{A}$ is the heart of a t-structure on $\mathcal{D}$, and $Z : K(\mathcal{A}) \cong K(\mathcal{D}) \to \mathbb{C}$ is a stability function on $\mathcal{A}$ known as the central charge. As the forgetful map $\text{Stab}(\mathcal{D}) \to \text{Hom}(K(\mathcal{D}), \mathbb{C})$ remembering just the central charge is a local homeomorphism [1, Prop 6.3], $\text{Stab}(\mathcal{D})$ has the structure of a complex manifold. It carries an action of the group of autoequivalences $\text{Aut}(\mathcal{D})$ and a free action of $\mathbb{C}$ for which $\mathbb{Z} \subset \mathbb{C}$ acts as the autoequivalence $[1]$, the shift functor of $\mathcal{D}$.

In this paper we compute a connected component $\text{Stab}^0(\mathcal{D})$ of the space of stability conditions of $\mathcal{D} = \mathcal{D}_{fd}(GA_2)$, the derived category of finite dimensional modules over the Ginzburg dg algebra of the $A_2$ quiver. This is a CY$_3$ triangulated category generated (cf [6, Sect 2]) by two objects $S$ and $T$ with

$$\text{Hom}(S, S) \cong \mathbb{C} \cong \text{Hom}(T, T) \quad \text{Ext}^1(S, T) \cong \mathbb{C}$$

We will call the heart $\mathcal{A}^0$ consisting of all modules supported in degree zero the standard heart. It is equivalent to the abelian category of finitely generated modules over the path algebra of the $A_2$ quiver, and its two simple objects are $S$ and $T$. We study the connected component $\text{Stab}^0(\mathcal{D})$ which contains stability conditions supported on the standard heart $\mathcal{A}^0$.

In section two we study the subquotient $\text{Aut}^0(\mathcal{D})$ of $\text{Aut}(\mathcal{D})$ of those autoequivalences preserving the connected component $\text{Stab}^0(\mathcal{D})$ modulo those which act trivially on it. We show that the set of hearts supporting a stability condition in $\text{Stab}^0(\mathcal{D})$ is an $\text{Aut}^0(\mathcal{D})$-torsor and deduce that

**Theorem 1.1.** $\text{Aut}^0(\mathcal{D})$ is isomorphic to the braid group $\text{Br}_3$ on three strings.

In section three we show how to define central charges using periods of a meromorphic differential $\lambda$ on the universal family of framed elliptic curves $E \to \mathcal{M}_{1,1}$. Restricted to a fibre $E$, $\lambda$ has a single pole of order 6 at the marked point $p$ and double zeroes at each of the half-periods. Using the framing $\{\alpha, \beta\}$ and the basis $\{[S], [T]\}$ of $K(\mathcal{D})$ to identify the lattices $H_1(E \setminus p, \mathbb{Z}) \cong K(\mathcal{D})$, we prove...
Theorem 1.2. There is a biholomorphic map

\[ \widetilde{M}_{1,1} \xrightarrow{f} \text{Stab}^0(\mathcal{D})/\mathbb{C} \]

lifting the period map of \( \lambda \). It is equivariant with respect to the actions of \( \text{PSL}(2,\mathbb{Z}) \) on the left by deck transformations and on the right by \( \text{Aut}^0(\mathcal{D})/\mathbb{Z} \) which are both determined by their induced actions on \( K(\mathcal{D}) \).

As a corollary we obtain a \( \text{Br}_3 \)-equivariant biholomorphism from the universal cover of the \( \mathbb{C}^* \)-bundle \( L^* \) of non-zero holomorphic differentials on \( M_{1,1} \) to \( \text{Stab}^0(\mathcal{D}) \).

Remark 1.3. In [11] the authors list 9 families of rank two connections on \( \mathbb{P}^1 \) having at least one irregular singularity which have precisely a one-parameter family of isomonodromic deformations described by one of the Painlevé equations. To each such family we associate a quiver \( Q \) as in [3], where \( Q = A_2 \) corresponds to the family whose isomonodromic deformations are given by solutions to the first Painlevé equation. It is anticipated that similar considerations to those of this paper will give a description of the space of numerical stability conditions of \( \mathcal{D}_{fd}(GQ) \) as the universal cover of a \( \mathbb{C}^* \)-bundle of meromorphic differentials over a moduli space of elliptic curves. We intend to return to this in future work.

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2. Autoequivalences

In this section we prove Theorem [11]. We show that every heart supporting a stability condition in \( \text{Stab}^0(\mathcal{D}) \) is a translate of the standard heart \( \mathcal{A}^0 = \text{mod}(\mathbb{C}A_2) \) by a composite of a spherical twist and the shift functor [1]. We deduce that every element of \( \text{Aut}^0(\mathcal{D}) \) is expressible in this way. The group of spherical twists \( \text{Sph}(\mathcal{D}) \) is a subgroup of \( \text{Aut}^0(\mathcal{D}) \) of index five, and we use a result of Seidel-Thomas that \( \text{Sph}(\mathcal{D}) \cong \text{Br}_3 \) to deduce that \( \text{Aut}^0(\mathcal{D}) \cong \text{Br}_3 \), the braid group on three strings.

Definition 2.1. An object \( X \in \mathcal{D} \) is spherical if \( \text{Hom}(\mathcal{D}(X, X) \cong \mathbb{C} \oplus \mathbb{C}[-3]. \) For \( X \) spherical there is a twist functor \( \Phi_X \) such that

\[ \Phi_X(Y) = \text{Cone}(X \otimes \text{Hom}(X, Y) \to Y) \]

There are two spherical objects \( S \) and \( T \) in \( \mathcal{D} \) which are the simple objects in the standard heart \( \mathcal{A}^0 \). They form an \( A_2 \)-collection [10, Def 1.1] as \( \text{Ext}^1(S, T) \cong \mathbb{C}. \)

Theorem 2.2. [10] Thms 1.2, 1.3 The spherical twists \( \Phi_S, \Phi_T \) satisfy the braid relations

\[ \Phi_S \Phi_T \Phi_S = \Phi_T \Phi_S \Phi_T \]

and generate a subgroup \( \text{Sph}(\mathcal{D}) \) of the group of autoequivalences \( \text{Aut}(\mathcal{D}) \) isomorphic to the braid group on three strings \( \text{Br}_3 \).

The braid group \( \text{Br}_3 \) has the following presentation by generators and relations [11, Sect 1.14]

\[ \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle \]
Its centre is the infinite cyclic subgroup generated by the element \( u = (\sigma_1 \sigma_2)^3 \) [1] Thm 1.24 giving us the short exact sequence
\[
1 \to \mathbb{Z} \to Br_3 \to PSL(2, \mathbb{Z}) \to 1
\]
where the quotient map sends the generators \( \sigma_1, \sigma_2 \) to
\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix}
\]
We note that the action of a spherical twist \( \Phi_X \) on \( K(D) \) is given by the formula
\[
\Phi_X([Y]) = [Y] - \chi(X, Y)[X]
\]
and so the map \( \text{Sph}(D) \to PSL(2, \mathbb{Z}) \) sends a spherical twist to the matrix given by its action on the lattice \( K(D) \) with respect to the basis \([S, [T]]\).

We now study the combinatorial backbone of the space of stability conditions, namely (a connected component of) the exchange graph of hearts of \( D \).

**Definition 2.3.** We say \( \mathcal{A}' \) is a simple tilt of \( \mathcal{A} \) at \( S \) if either
- \( \mathcal{A}' \) is the left tilt of \( \mathcal{A} \) with respect to the torsion pair \( \mathcal{T} = \langle S \rangle = \{ S^\oplus n \mid n \in \mathbb{N}_0 \} \) \( \mathcal{F} = \{ X \mid \text{Hom}_\mathcal{A}(S, X) = 0 \} \)
- \( \mathcal{A}' \) is the right tilt of \( \mathcal{A} \) with respect to the torsion pair \( \mathcal{T} = \{ X \mid \text{Hom}_\mathcal{A}(X, S) = 0 \} \) \( \mathcal{F} = \langle S \rangle = \{ S^\oplus n \mid n \in \mathbb{N}_0 \} \)

The relevance of simple tilts is that they occur precisely at the codimension 1 components of the boundary of the space of stability conditions \( U(\mathcal{A}) \) supported on a given heart \( \mathcal{A} \) by [2] Lemma 5.5. Thus \( \text{Stab}(D) \) is glued together from the \( U(\mathcal{A}) \) according to the exchange graph.

**Definition 2.4.** The exchange graph \( \text{EG}(D) \) of \( D \) has vertices the set of hearts \( \mathcal{A} \subset D \) and an edge between any two hearts related by a simple tilt. Define \( \text{EG}^0(D) \) to be the connected component containing the standard heart \( \mathcal{A}^0 \).

We compute the four simple tilts of the standard heart \( \mathcal{A}^0 \).

**Proposition 2.5.** Denote by \( E \) and \( X \) the unique non-trivial extensions up to isomorphism of \( S \) by \( T \) and \( T \) by \( S[1] \) respectively. Let \( (\mathcal{A}, \mathcal{B})_C \) denote the abelian category supported by two simple objects \( \mathcal{A} \) and \( \mathcal{B} \) having a unique up to isomorphism non-trivial extension \( C \) of \( \mathcal{B} \) by \( \mathcal{A} \), so that the standard heart \( \mathcal{A}^0 = (T, S)_E \).

Then
\[
\begin{align*}
R_S(\mathcal{A}^0) &= (S[1], T)_{X[1]} \\
R_T(\mathcal{A}^0) &= (E, T[1])_{S} \\
L_T(\mathcal{A}^0) &= (T[-1], S)_X \\
L_S(\mathcal{A}^0) &= (S[-1], E)_T
\end{align*}
\]
Moreover the tilted hearts are obtained from \( \mathcal{A}^0 \) by applying the following autoequivalences.
\[
\begin{align*}
R_S(\mathcal{A}^0) &= (\Phi_T \Phi_S \Phi_T)[3] (\mathcal{A}^0) \\
R_T(\mathcal{A}^0) &= (\Phi_S \Phi_T)[2] (\mathcal{A}^0) \\
L_T(\mathcal{A}^0) &= ((\Phi_T \Phi_S \Phi_T)[3])^{-1} (\mathcal{A}^0) \\
L_S(\mathcal{A}^0) &= ((\Phi_S \Phi_T)[2])^{-1} (\mathcal{A}^0)
\end{align*}
\]
We will prove the statement about the left tilt at \( T \), the rest being similar. The torsion pair in this case is
\[
\mathcal{T} = \langle T \rangle \quad \mathcal{F} = \{ X \in \mathcal{A}^0 \mid \text{Hom}_{\mathcal{A}^0}(T, X) = 0 \} = \langle S \rangle
\]
We will use the long exact sequence in cohomology with respect to the original t-structure \( \mathcal{A}^0 \), the groups being non-zero only in degrees 0 and 1.

**Lemma 2.6.** \( T[1] \) is simple in \( L_T(\mathcal{A}^0) \)
Thus $\Phi = \left[\begin{array}{c}
S
\end{array}\right]_{\text{generators } \Phi}$ component of the exchange graph. If an element of $\text{Sph}(D)$ is simple, e.g. $\Phi$.

Proof. Consider a short exact sequence in $L_T(A^0)$

$$0 \to X \to T[-1] \to Y \to 0$$

giving a long exact sequence in $A^0$.

$$0 \to H^0(X) \to H^0(T[-1]) \to H^0(Y) \to H^1(X) \to H^1(T[-1]) \to H^1(Y) \to 0$$

We have $H^0(T[-1]) = 0$ so $H^0(X) = 0$. Splitting the remaining 4-term exact sequence into two short exact sequences

$$0 \to H^0(Y) \to H^1(X) \to Z \to 0$$

$$0 \to Z \to T \to H^1(Y) \to 0$$

$Z$ is either 0 or $T$ as $T$ is simple in $A^0$. But there are no non-zero maps from $H^0(Y) \in F$ to $H^1(X) \in T$ so $H^1(X) \cong Z$. So $X$ is either 0 or $T[-1]$ and so $T[-1]$ is simple.

Lemma 2.7. $S$ is simple in $L_T(A^0)$.

Proof. As $H^1(S) = 0$ we have as before

$$0 \to H^0(X) \to S \to Z \to 0$$

$$0 \to Z \to T \to H^1(X) \to 0$$

Thus as $S$ is simple in $A^0$, $H^0(X)$ is either 0 or $S$, and so $Z$ is either $S$ or 0. Then as there are no non-zero maps from $H^0(Y) \in F$ to $H^1(X) \in T$, $H^1(X) = 0$ and so $S$ is simple in $L_T(A^0)$.

We remark that all four simple tilts of $A^0$ are isomorphic to $A^0$ so the above is the local structure of the exchange graph at any vertex of the connected component $EG^0(D)$.

Definition 2.8. Let $\text{Aut}^0(D)$ be the subquotient of $\text{Aut}(D)$ consisting of all autoequivalences preserving the connected component $EG^0(D)$ of the exchange graph modulo those acting trivially on it.

We will see later that in fact $\text{Aut}^0(D)$ is the subquotient preserving the connected component $\text{Stab}^0(D)/\mathbb{C}$ modulo those acting trivially on it.

Proposition 2.9. The vertices of the connected component $EG^0(D)$ of the exchange graph are a torsor for $\text{Aut}^0(D)$.

Proof. From the above computation every heart in $EG^0(D)$ can be obtained by applying an autoequivalence in $\langle \Phi_S, \Phi_T, [1] \rangle$ to the standard heart $A^0$. Thus $\text{Aut}^0(D)$ acts transitively on $EG^0(D)$ and acts freely by definition.

Lemma 2.10. The centre of $\text{Sph}(D)$ is generated by $[-5]$.

Proof. As $\text{Sph}(D) \cong \text{Br}_3$ the centre is generated by $\Phi = (\Phi_S \Phi_T)^3$. We compute $\Phi$ on $S$ and $T$

$$S \mapsto X \mapsto T[-1] \mapsto T[-3] \mapsto E[-3] \mapsto S[-3] \mapsto S[-5]$$

$$T \mapsto T[-2] \mapsto E[-2] \mapsto S[-2] \mapsto S[-4] \mapsto X[-4] \mapsto T[-5]$$

Thus $\Phi = [-5]$ in $\text{Aut}^0(D)$.

We note that $\text{Sph}(D)$ defines a subgroup of $\text{Aut}^0(D)$ isomorphic to $\text{Br}_3$. The generators $\Phi_S$ and $\Phi_T$ are composites of two autoequivalences corresponding to simple tilts, e.g. $\Phi_S^{-1} = (\Phi_T \Phi_S \Phi_T[3])/(\Phi_S \Phi_T[2])$ and so preserve the connected component of the exchange graph. If an element of $\text{Sph}(D)$ acts trivially on $K(D)$
then it belongs to the centre which we have just seen is generated by a non-trivial element of \( \text{Aut}^0(\mathcal{D}) \) so the only element of \( \text{Sph}(\mathcal{D}) \) acting trivially is the identity.

**Theorem 2.11.** The map \( \text{Br}_3 \to \text{Aut}^0(\mathcal{D}) \) given by \( (\sigma_1, \sigma_2) \mapsto (\Phi_S[1], \Phi_T[1]) \) is an isomorphism.

**Proof.** As the exchange graph is an \( \text{Aut}^0(\mathcal{D}) \)-torsor we know that \( \text{Aut}^0(\mathcal{D}) = \langle \Phi_S, \Phi_T, [1] \rangle \). As the shift functor commutes with the spherical twists, we find that \( ((\Phi_S[1])(\Phi_T[1]))^3 = [-5][6] = [1] \) so \( \text{Aut}^0(\mathcal{D}) = \langle \Phi_S[1], \Phi_T[1] \rangle \). These two generators satisfy the braid relation as \( \Phi_S, \Phi_T \) do.

Now consider a word \( w \) in the generators \( \Phi_S[1], \Phi_T[1] \) and their inverses which is equal to the identity of \( \text{Aut}^0(\mathcal{D}) \). As \( [1] \) is in the centre of \( \text{Aut}^0(\mathcal{D}) \), we have \( \Phi_S^{n_1} \cdots \Phi_T^{n_k} = [-1]^{\sum n_i} \) in \( \text{Sph}(\mathcal{D}) \). By the above lemma the centre of \( \text{Sph}(\mathcal{D}) \) is generated by \( (\Phi_S \Phi_T)^3 = [-5] \), so the right hand side is equal to \( [-5]^{\lfloor \sum n_i \rfloor / 5} \). As the braid relation is homogeneous, every element of \( \text{Sph}(\mathcal{D}) \) has a well-defined word length in the generators \( \Phi_S \) and \( \Phi_T \). But applying the word length homomorphism gives \( \sum n_i = \frac{5}{2} \sum n_i \) so \( \sum n_i = 0 \). Thus the relations satisfied by the generators \( \Phi_S[1], \Phi_T[1] \) of \( \text{Aut}^0(\mathcal{D}) \) are precisely those satisfied by the generators \( \Phi_S, \Phi_T \) of \( \text{Sph}(\mathcal{D}) \). \( \square \)

To complete the picture we show that \( \text{Sph}(\mathcal{D}) \) is a normal subgroup of index 5.

**Proposition 2.12.** There is a short exact sequence

\[
1 \to \text{Sph}(\mathcal{D}) \to \text{Aut}^0(\mathcal{D}) \to \mathbb{Z}/5\mathbb{Z} \to 1
\]

where the quotient map \( l \) is the modulo 5 word length map in the generators \( \Phi_S[1] \) and \( \Phi_T[1] \)

**Proof.** \( \text{Sph}(\mathcal{D}) \) is in the kernel of \( l \) as

\[
l(\Phi_S) = l(\Phi_S[1]) - l([1]) = 1 - 6 = 0
\]

Conversely the smallest power of \( [1] \) in the kernel is \( [5] = (\Phi_S \Phi_T)^{-3} \in \text{Sph}(\mathcal{D}) \) and so \( \text{Aut}^0(\mathcal{D}) = \langle \Phi_S, \Phi_T, [1] \rangle \) the kernel is contained in \( \text{Sph}(\mathcal{D}) \). \( \square \)

**Remark 2.13.** By Sabidussi’s Theorem [3] Thm 4], \( \text{EG}^0(\mathcal{D}) \) is isomorphic to the Cayley graph of the braid group \( \text{Br}_3 \) with respect to the generators \( \Delta = (\Phi_T \Phi_S \Phi_T)[3] \) and \( \Sigma = (\Phi_S \Phi_T)[2] \) which give the simple tilted hearts. Indeed this gives an alternative presentation of \( \text{Br}_3 \) [2] Sect 1.14]

\[
(\Sigma, \Delta | \Sigma^3 = \Delta^2)
\]

The quotient of \( \text{EG}^0(\mathcal{D}) \) by \( \text{Sph}(\mathcal{D}) \) is the \( A_2 \) cluster exchange graph which is isomorphic to the Cayley graph of \( \mathbb{Z}/5\mathbb{Z} \). This recovers a special case of a result of Keller and Nicolas [5] Thm 5.6.

3. Stability Conditions

In this section we prove Theorem 1.2. We derive the Picard-Fuchs equations satisfied by the periods of the family of meromorphic differentials \( \lambda \) on the fibres \( E \) of the universal family of framed elliptic curves \( E \to \mathcal{M}_{1,1} \). Identifying the lattices \( H_1(E, \mathbb{Z}) \cong K(\mathcal{D}) \), the image in \( \mathbb{P} \text{Hom}(K(\mathcal{D}), \mathbb{C}) \) of a certain branch of the period map is a double of the Schwarz triangle with angles \( (\pi, \pi/3, \pi/2) \). We show that this coincides with the image under the local homeomorphism \( \hat{\text{Stab}}^0(\mathcal{D})/\mathbb{C} \to \mathbb{P} \text{Hom}(K(\mathcal{D}), \mathbb{C}) \) of a fundamental domain for the action of \( \text{Aut}^0(\mathcal{D})/\mathbb{Z} \cong \text{PSL}(2, \mathbb{Z}) \) on \( \text{Stab}^0(\mathcal{D})/\mathbb{C} \). We use our understanding of the exchange graph of \( \mathcal{D} \) to lift the period map to our desired biholomorphism \( f : \mathcal{M}_{1,1} \to \text{Stab}^0(\mathcal{D})/\mathbb{C} \).
Definition 3.1. On an elliptic curve \( y^2 = z^3 + az + b \) define the meromorphic differential \( \lambda = y \, dz \)

This has a pole of order 6 at the point at infinity and double zeroes at each of the three other branch points of \( y \). This is the divisor of the function \( y^2 \). It is the unique differential up to scale with this property as the above divisor has degree zero.

Define the coordinates \( j \) and \( u \) on \( \widetilde{M}_{1,1} \) by
\[
(1) \quad J = 1728/j \quad j = 4u(1-u)
\]

where \( J \) denotes the usual \( J \)-invariant. We note that the family of differentials \( \lambda = \sqrt{z^3 - 3z + (4u - 2)} \, dz \) satisfy \( 2\partial_u \lambda = \omega \), where \( \omega = dz/y \) is the family of holomorphic differentials on \( \widetilde{M}_{1,1} \). Using this we show that the periods of \( \lambda \) satisfy hypergeometric equations in \( u \) and \( j \).

Definition 3.2. A hypergeometric differential equation is a second order ordinary differential equation on \( \mathbb{P}^1 \) of the form
\[
w(1-w)f'' + (\gamma - (\alpha + \beta - 1)w)f' - \alpha\beta w = 0
\]
with \( \alpha, \beta, \gamma \in \mathbb{R} \).

It has regular singularities at 0, \( \infty \) and 1 with exponents
\[
\lambda = 1 - \gamma \quad \mu = \alpha - \beta \quad \nu = \gamma - \alpha - \beta
\]

Lemma 3.3. The periods of \( \lambda \) satisfy the hypergeometric equation in \( j \) with exponents \( (1,\frac{1}{2},\frac{1}{2}) \)

Proof. Suppose the periods of \( \lambda \) satisfy the hypergeometric equation in \( u \)
\[
u(1-u)\partial_u^2 f + (\gamma - (\alpha + \beta - 1)u)\partial_u f - (\alpha + \beta - 1) f - \alpha\beta f = 0
\]
Taking the derivative with respect to the dependent variable \( u \), we find that the periods of \( \omega \) must satisfy
\[
u(1-u)\partial_u^2 f + (1 - 2u)\partial_u f + (\gamma - (\alpha + \beta - 1)u)\partial_u f - (\alpha + \beta - 1) f - \alpha\beta f = 0
\]
which is hypergeometric of the form
\[
u(1-u)\partial_u^2 f + ((\gamma + 1) - ((\alpha + 1) + (\beta + 1) - 1)u)\partial_u f - (\alpha + 1)(\beta + 1) f = 0
\]

It is well-known the periods of \( \omega \) satisfy the hypergeometric equation in \( j \) with exponents \( (0,\frac{1}{2},\frac{1}{2}) \). By the quadratic transformation law for the change of variable given above \[12\] Eq 2, they satisfy the hypergeometric equation in \( u \) with exponents \( (0,\frac{1}{2},0) \). By the above computation, the periods of \( \lambda \) satisfy the hypergeometric equation with exponents \( (1,\frac{1}{2},1) \) and so reversing the change of variable gives the result. \( \square \)

Remark 3.4. The coordinate transformation \[17\] defines a double cover \( B \to M_{1,1} \) of the coarse moduli space of elliptic curves. There is a family of elliptic curves on \( B \) whose total space is the complement of the three singular fibres of types \( (I_1, I_1, II^*) \) over \( u = 0, 1 \) and \( \infty \) respectively of a rational elliptic surface \( \Sigma_u \to \mathbb{P}^1_u \). This is the smooth part of Hitchin’s fibration of the moduli space of meromorphic SU(2)-Higgs bundles on \( \mathbb{P}^1_u \) with a single pole of order 4 at \( z = \infty \) whose leading term is nilpotent. The meromorphic differential \( \lambda \) is the Seiberg-Witten differential of this integrable system, that is the exterior derivative of \( \lambda \) defines a holomorphic symplectic form on \( \Sigma \).

In fact \( \Sigma \) is a hyperkähler manifold \[13\], which was studied in \[3\] Sect 9.3.3. In another complex structure \( \Sigma \) is isomorphic to the moduli space of flat \( \text{SL}(2,\mathbb{C}) \)-connections on \( \mathbb{P}^1_u \) with a single pole at \( z = \infty \) of Katz invariant 5/2. This complex
manifold was studied in [11,13] as the moduli space of initial conditions of the first Painlevé equation (cf Remark [12,3]). Its image under the Riemann-Hilbert map is an affine cubic surface which is isomorphic as a complex variety to the cluster algebra of $A_2$.

Now consider the moduli space of elliptic curves $M_{1,1} \cong \mathbb{P}(2,3) \setminus \{\circ\}$ where $\circ$ is the point corresponding to $j = 0$. We make branch cuts on $M_{1,1}$ along the line $3(j) = 0$ between $\circ$ and each of the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifold points $\times, \ast$ at $j = 1, \infty$. We deduce the image of this branch of the period map $p$ of $\lambda$ from the Schwarz triangle theorem.

**Theorem 3.5.** [9, p 206] Suppose $f_1, f_2$ are linearly independent solutions to the hypergeometric equation with exponents $(\lambda, \mu, \nu)$. Suppose further that their ratio $s = f_1/f_2$ restricted to the upper-half plane $\mathbb{H} \subset \mathbb{C} \setminus \{0, 1, \infty\}$ is an injection. Then $s$ maps $\mathbb{H}$ biholomorphically onto the interior of a curvilinear triangle $\Delta_{\lambda, \mu, \nu}$ of angles $(\lambda \pi, \mu \pi, \nu \pi)$.

The image is determined up to a Möbius map and so specified uniquely by the positions of the three vertices of the triangle $\Delta$. By the Schwarz reflection principle we have

**Corollary 3.6.** The image $\hat{\Delta} = p(M_{1,1})$ is the double of the curvilinear triangle $\Delta_{1, \frac{1}{2}, \frac{1}{2}}$ along the edge connecting the image of the two orbifold points $\times$ and $\ast$.

We now define a fundamental domain $V = V(\mathcal{A}^0)$ for the action of $\text{Aut}^0(D) / \mathbb{Z}$ on $\text{Stab}^0(D) / \mathbb{C}$ which maps bijectively under the local homeomorphism $\mathbb{Z}$ to $\hat{\Delta}$. Although the vertices of the quotient of the exchange graph $\text{EG}^0(D) = \text{EG}^0(D) / \mathbb{Z}[1]$ are indeed an $\text{Aut}^0(D) / \mathbb{Z}$-torsor, the notion of a projective stability condition $\mathfrak{c} \in \text{Stab}^0(D) / \mathbb{C}$ being supported at a given vertex $\mathcal{A}$ of $\text{EG}^0(D)$ is not a priori well-defined. This is because points of $\text{Stab}^0(D)$ in the same $\mathbb{C}$-orbit can be supported on different hearts, even modulo the shift functor. We define $\mathfrak{c}$ to be supported on $\mathcal{A}$ using the following width function.

**Definition 3.7.** Define the width $\varphi$ of a stability condition $\sigma = (Z, \mathcal{A}) \in \text{Stab}(D)$

$$\varphi(\sigma) = \phi^+(\sigma) - \phi^-(\sigma)$$

where $\phi^+(\sigma)$ and $\phi^-(\sigma)$ denote the maximal and minimal phases respectively of an object in $\mathcal{A}$.

The width is the angle of the image under $Z$ of the cone $C(\mathcal{A}) \subset K(\mathcal{A})$ generated by classes of objects in $\mathcal{A}$.

**Definition 3.8.** We say that $\mathfrak{c} \in \text{Stab}^0(D) / \mathbb{C}$ is supported on $\mathcal{A}$ if the width function is minimised on a lift $\mathcal{A}$ of $\mathcal{A}$.

Note that $\mathfrak{c}$ is supported on more than one $\mathcal{A}$ where the width function is minimised on more than one such $\mathcal{A}$. We will write $V(\mathcal{A}) \subset \text{Stab}^0(D) / \mathbb{C}$ for the subset supported uniquely on $\mathcal{A}$, whose closure $\check{V}(\mathcal{A})$ is the subset supported on $\mathcal{A}$.

**Proposition 3.9.** $V = V(\mathcal{A}^0)$ is the interior of a fundamental domain for the action on $\text{Aut}^0(D) / \mathbb{Z}$ on $\text{Stab}^0(D) / \mathbb{C}$

**Proof.** As the vertices of $\text{EG}^0(D)$ are an $\text{Aut}^0(D) / \mathbb{Z}$-torsor, every point in the set $T = \bigsqcup_{\mathcal{A}} V(\mathcal{A})$ belongs to a unique $V(\mathcal{A})$. The points $\mathfrak{c}$ in $\check{V} \setminus V$ lie on the three codimension 1 walls pictured below where $\mathfrak{c}$ is also supported on some other $\mathcal{A}$ for some simple tilt $\mathcal{A}$ of $\mathcal{A}^0$. These walls of the $V(\mathcal{A})$ are locally finite as there is only one other wall intersecting $V$, namely $\check{V}(L_\mathcal{S}(\mathcal{A}^0)) \cap \check{V}(R_T(\mathcal{A}^0))$. Thus the closure $\check{T} = \bigsqcup_{\mathcal{A}} \check{V}(\mathcal{A})$. But $\check{T}$ is clearly open and so is the entire connected component $\text{Stab}^0(D) / \mathbb{C}$. \qed
Remark 3.10. The above proof shows that an autoequivalence $\Phi$ which preserves $\text{EG}^0(D)$ preserves the connected component $\text{Stab}^0(D)/\mathbb{C}$. Also if $\Phi$ acts trivially on $\text{EG}^0(D)$ then $\Phi$ fixes the central charge $\bar{Z}$ and so $\Phi$ acts trivially on $\text{Stab}^0(D)/\mathbb{C}$.

This means that $\text{Stab}^0(D)/\mathbb{C}$ is glued together from the $V(\mathcal{A})$ according to the quotient of the exchange graph $\text{EG}^0(D)$ just as $\text{Stab}^0(D)$ is glued from the $U(\mathcal{A})$ according to $\text{EG}^0(D)$.

Proposition 3.11. The image of $V$ under the map $\bar{Z}$ is $\Diamond$.

Proof. The boundary of $V$ consists of stability conditions supported on one of the three walls which we picture below, whose image under $\bar{Z}$ is the boundary of $\Diamond$. 

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Proof of Theorem 1.2. Using the identification $V(A) \cong \diamondsuit = p(M_{1,1})$, we can extend the branch of the period map to a map $f: \tilde{M}_{1,1} \to \text{Stab}(\mathcal{D})/\mathbb{C}$ by equivariance. We only have to check continuity on the boundary of $\mathcal{M}_{1,1}$, i.e. the action of the monodromy on $H_1(E,\mathbb{Z})$ on crossing one of the two branch cuts in either direction is identical to the action of the four simple tilts on $K(\mathcal{D})$. But these both act by
\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\quad \begin{pmatrix}
0 & 1 \\
-1 & 1
\end{pmatrix}
\]
and their inverses. \hfill \Box

Figure 3. The image of the $\mathbb{Z}/2$- and $\mathbb{Z}/3$-orbifold points $\times$ and $\ast$

We denote by $L^\times$ the total space of the $\mathbb{C}^*$-bundle of non-zero holomorphic differentials on $\tilde{M}$. It is isomorphic to the complement of the discriminant locus in the space $\mathbb{C}^2_{a,b}$ of cubic polynomials $z^3 + az + b$. The fundamental group of $L^\times$ is isomorphic to the braid group $\text{Br}_3$ as the discriminant locus describes the trefoil knot.

Corollary 3.12. There is a biholomorphic map
\[
\begin{array}{ccc}
\tilde{L}^\times & \xrightarrow{F} & \text{Stab}^0(\mathcal{D}) \\
\downarrow \scriptstyle (f_\alpha \lambda, f_\beta \lambda) \quad & & \quad \downarrow \scriptstyle (Z(S), Z(T)) \\
\text{Hom}(K(\mathcal{D}), \mathbb{C})
\end{array}
\]
lifting the periods of the differential $\lambda$. It is equivariant with respect to the actions of $\text{Br}_3$ on the left by deck transformations and on the right by $\text{Aut}(\mathcal{D})$.

Proof. We can lift the map $f: \tilde{M}_{1,1} \to \text{Stab}^0(\mathcal{D})/\mathbb{C}$ to the desired $F$ by equivariance with respect to the $\mathbb{C}$-actions on both sides. It is a bijection as both $\mathbb{C}$-actions are free, and holomorphic as it is locally given by the periods of $\lambda$. We know that the two braid groups act identically on $K(\mathcal{D})$ via their maps to $\text{PSL}(2,\mathbb{Z})$ and so define identical actions on the $\mathbb{C}^*$-bundle $L^\times$. Also the actions of the central subgroup $\mathbb{Z} \subset \text{Br}_3$ are identical by construction as it acts as $\mathbb{Z} \subset \mathbb{C}$. But given these data the actions are determined by a group homomorphism $\text{PSL}(2,\mathbb{Z}) \to \mathbb{Z}$ giving a lifting of the $\text{Br}_3$-action on the $\mathbb{C}^*$ bundle $L^\times$ factoring through $\text{PSL}(2,\mathbb{Z})$ to the universal cover. As the only such homomorphism is the trivial one the two $\text{Br}_3$ actions are identical. \hfill \Box

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