A note on string interaction on the pp-wave background

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Abstract

We consider type IIB string interaction on the maximally supersymmetric pp-wave background and discuss how the bosonic symmetries of the background are realized. This analysis shows that there are some interesting differences with respect to the flat-space case and suggests modifications to the existing form of the string vertex. We focus on the zero-mode part which is responsible for some puzzling string predictions about the $\mathcal{N} = 4$ SYM side. We show that these puzzles disappear when a symmetry preserving string interaction is used.
1 Introduction

The duality [1] between IIB string theory on $AdS_5 \times S_5$ and $\mathcal{N} = 4$ Super Yang–Mills theory realized, for the first time, the idea that the planar limit of a quantum gauge theory is equivalent to a classical string theory. However, string theory on the $AdS_5 \times S_5$ background remains still largely intractable and even the free spectrum for this case is not known. Thus, most of the checks of the above duality have been done in the supergravity limit ($\alpha' \rightarrow 0$), where the gauge theory is strongly coupled ($\lambda = g^2_{YM} N \rightarrow \infty$). In [2] a concrete rule was given for comparing, in the $\lambda \rightarrow \infty$ limit, dynamical quantities on the two sides of the correspondence and, since then, many checks of the AdS/CFT duality have been performed.

In an apparently unrelated development [3], it has been shown recently that it is possible to extend to supergravity the limiting procedure described by Penrose in the case of pure gravity [4]. The nice feature of this limit is that it deforms a solution of the classical equations of motion into a new universal wave-like solution. The Penrose–limit of the $AdS_5 \times S_5$ background in type IIB supergravity corresponds to the plane–wave solution [5],

$$ds^2 = -4dx^+dx^- - \mu^2 \sum_{I=1}^{8} x_I x^I (dx^+)^2 + \sum_{I=1}^{8} dx_I dx^I, \quad F_{+1234} = F_{+5678} = 2\mu, \quad (1)$$

which was initially obtained in [6] as a new background preserving all IIB supercharges. Since, as showed in [8, 9], IIB string theory in the above plane–wave background is solvable in the Green-Schwarz formalism, this limiting procedure turned out to be very interesting for the AdS/CFT duality. This was first pointed out by Berenstein, Maldacena and Nastase [7] who proposed that, on the Yang–Mills side, this limiting procedure corresponds to focusing on a subset of the full spectrum of composite operators of $\mathcal{N} = 4$ Yang–Mills.

The possibility to work with a tractable string theory showed a glimpse of the full power of the AdS/CFT duality. In fact, the simple computation of the mass spectrum in string theory gave an exact prediction for the conformal dimensions ($\Delta$) of the corresponding gauge theory operators; that is the $\alpha'$ dependence of the string masses translates into a continuous function of the effective coupling interpolating, in the planar limit, between the weak and the strong–coupling behavior of $\Delta$. This prediction has been checked on the gauge theory side first at the perturbative level up to 2–loop order [10], and then at all orders in [11]. The behavior of $\Delta$ has been been studied also at the torus level [12, 13], where the situation is more complicated due to a nontrivial mixing of BMN operators. Of course, it would be very nice to extend the string theory analysis and obtain new exact predictions for the $\mathcal{N} = 4$ Yang–Mills theory. However, there are still two largely unsolved issues preventing a straightforward application of the pp–wave/CFT duality beyond the computation of conformal dimensions.

The first problem is that the background (1) gives rise to a free world–sheet theory only in the light cone gauge. Moreover, a non–trivial R–R field can be handled, so far,
only in the Green-Schwarz formalism\textsuperscript{1}. In this approach the description of the string interaction is quite involved already in the flat-space background and a general form for the tree-level $N$-string amplitude is not known. However, in the 80's a detailed analysis of this formalism was carried out \cite{15, 16} (see also Chap. 11 of \cite{17} and references therein) so that the first interesting amplitudes could be computed. Recently the IIB 3-string vertex in the background (1) has been studied by the authors of Ref. \cite{18} extending the flat space analysis of \cite{16}.

A second problem in the pp-wave/CFT duality is represented by the dictionary between string and gauge theory dynamical computations. The rule given in \cite{2} for comparing Yang–Mills Green functions and supergravity tree–level graphs does not seem to be directly applicable to the gauge theory operators we are interested in, since the corresponding supergravity excitations are confined far away from the AdS conformal boundary. It is not even clear whether it is possible to extract from string theory the complete Yang–Mills Green functions. Actually, contrary to what happens in the supergravity limit, we do not expect that this will be possible in general. A rule for comparing string interactions and Yang-Mills results was proposed in \cite{13}. This was further investigated and generalized in \cite{19}–\cite{23}. The correspondence discussed in these papers involved a particular class of BMN operators – the operators with scalar impurities only.

In this note we consider the extension of the pp–wave/CFT duality to BMN operators of a different kind and, correspondingly, to different 3-string interactions. We mainly focus on the string side with the aim of giving an explanation for the puzzle posed by the Yang–Mills computation of \cite{24}, where a BMN operator containing a derivative impurity ($D_{\mu}Z$) has been considered explicitly for the first time. A detailed computation of its conformal dimension is presented both at the planar and at the torus level, generalizing the ideas and the results of \cite{12, 13} to this new kind of BMN operators. It turns out that the result coincides exactly with the one found for a purely scalar BMN operator. If this result is interpreted via the unitarity argument presented in \cite{13}, one gets information about the 3–string tree–level interaction that should be directly compared with the string cubic vertex of \cite{18}. However, as noticed in various points in the literature \cite{13, 22, 23, 24}, the string interactions seems to be vanishing and would imply a zero torus-level contribution to the anomalous dimension of the operator under study.

This is in conflict with the field theory result. A possible explanation of this mismatch (beyond those suggested in \cite{24}) is that the unitarity argument of \cite{13} is incorrect and thus cannot be reliably used to derive the 3–point function from the value of the anomalous dimension. In fact, this possibility has been confirmed in the very recent literature based on field theory calculations.

In this note, we examine the string theory side of the correspondence. Our main point is that the light–cone quantization of string theory on the pp–wave background so far considered does not realize in an explicit way all the symmetries of the background

\textsuperscript{1}In \cite{14} an alternative formalism has been applied to the case of pp–wave background. Even if this looks like a promising step toward a covariant quantization, the presence of a non–trivial background gives rise to a complicate world-sheet action, and explicit computations of string interactions in this framework have not been done so far.
Thus we propose that some modifications have to be made and that they affect the structure of the zero–modes in the interaction Hamiltonian, which is the part responsible for the puzzle described above. We argue that it is also possible that the non–zero mode part of the vertex has to be modified.

The plan of the paper is the following. We start with a discussion of a discrete $Z_2$ symmetry of the background (1) and its realization in the string quantization procedure. We show that implementing this $Z_2$ symmetry requires a modification in the choice of vacuum and in the zero–mode structure of the interaction Hamiltonian. We then focus on the part of the vertex relevant for computing amplitudes among string states without any fermionic oscillator. On one hand we show that these modifications do not spoil the exact matching between gauge and string theory found so far in the case of scalar BMN operators. On the other hand we are able to reconcile the 3–point string computation with the field theory result of [24]. In fact it turns out that amplitudes among string states with only bosonic excitations display the $SO(8)$ symmetry already present in the bosonic part of the background (1).

2 The string interaction and its symmetries

In this section we will analyze the symmetries of the interaction Hamiltonian proposed in Ref. [18]. As it is well known, symmetries play a crucial role in determining the string interaction in the Green–Schwarz formalism. Contrary to what happens in the covariant formalism, one does not have at his disposal a world-sheet BRST charge that can be used to define vertex operators. Thus the strategy used in [15, 16] is to first look for a string interaction realizing locally on the world–sheet all the kinematical symmetries of the light–cone algebra. In the case of flat space, this implies that the string coordinates are continuous at the interaction point and that the conjugate momenta are conserved. By applying the same idea to the pp–wave background, in [18] it was shown that $|H_3\rangle$ enjoys exactly the same features as in flat space–time, even if this is not the case for the quadratic part of the Hamiltonian. Then one has to look at the dynamical part of the supersymmetry algebra. It turns out that, in order to have a string interaction respecting the dynamical supersymmetries, one has to add a particular prefactor term. Also in this part, the analysis of [18] follows closely the flat space–time case [16]. However, the bosonic symmetries of the background (1) are not those of flat–space and this suggests to introduce some different choice in the treatment of the zero–modes, thus yielding a different form of $|H_3\rangle$.

2.1 The $Z_2$ symmetry and the choice of vacuum

The point we want to stress is that the presence of a non–trivial R–R field in the plane–wave background breaks the light–cone Lorentz symmetry $SO(8)$ down to $SO(4) \times SO(4) \times Z_2$. The two $SO(4)$ rotate the first and the last four directions among themselves respectively, while the discrete $Z_2$ symmetry swaps simultaneously the two groups of four
Of course one is free to define the action of $Z_2$ in a different way, for instance by assigning a minus sign to all the components of the r.h.s. of (2). These different choices, however, are perfectly equivalent, since they differ just by a rotation in one of the two $SO(4)$ and $SO(4) \times Z_2$ group. Notice that the $Z_2$ transformation above is just a particular rotation in the full $SO(8)$ group. Indeed, a generic $SO(8)$ rotation by an angle $\omega_{IJ}$ is \[ M_{IJ} = \exp \left( i \omega_{IJ} M^{IJ} \right), \] where $M_{IJ}$ are the standard $SO(8)$ Lorentz generators.

For the vector representation $\mathcal{M}$ one can easily derive the explicit action of $Z_2$ on the spinors. Once a particular realization of the $SO(8)$ $\gamma$-matrices is chosen, it is sufficient to use the appropriate generators $M^{IJ} = \frac{1}{4} [\gamma^I, \gamma^J]$ in the above formula to find the rotation matrix. As noticed in various points in the literature, computations get simplified if one works with a specific representation where $\Pi = \gamma^1 \gamma^2 \gamma^3 \gamma^4 = 1_{2 \times 2} \otimes \sigma_3 \otimes 1_{4 \times 4}$. This can be realized as follows

$$
\begin{align*}
\gamma^1 &= (i \sigma_2) \otimes (i \sigma_2) \otimes (i \sigma_2) \otimes (i \sigma_2), \\
\gamma^2 &= (i \sigma_2) \otimes \sigma_1 \otimes (i \sigma_2) \otimes 1_{2 \times 2}, \\
\gamma^3 &= (i \sigma_2) \otimes (i \sigma_2) \otimes 1_{2 \times 2} \otimes \sigma_1, \\
\gamma^4 &= (i \sigma_2) \otimes (i \sigma_2) \otimes 1_{2 \times 2} \otimes \sigma_3, \\
\gamma^5 &= (i \sigma_2) \otimes 1_{2 \times 2} \otimes \sigma_1 \otimes (i \sigma_2), \\
\gamma^6 &= (i \sigma_2) \otimes \sigma_3 \otimes (i \sigma_2) \otimes 1_{2 \times 2}, \\
\gamma^7 &= \sigma_1 \otimes 1_{2 \times 2} \otimes 1_{2 \times 2} \otimes 1_{2 \times 2}, \\
\gamma^8 &= (i \sigma_2) \otimes 1_{2 \times 2} \otimes \sigma_3 \otimes (i \sigma_2).
\end{align*}
$$

From the above $\gamma$'s one can, as usual, construct the $SO(1,9)$ real and chiral $\Gamma$'s: $\Gamma^0 = (i \sigma_2) \otimes 1_{16 \times 16}$ and $\Gamma^\mu = \sigma_1 \otimes \gamma^\mu$ for $\mu = 1, \ldots, 9$, with $\gamma^9 = \prod_{I=1}^8 \gamma^I$. However, we will not need the 10D $\Gamma$'s since all our spinors are both Majorana-Weyl and satisfy the light-cone constraint $(\Gamma^0 + \Gamma^9) \theta = 0$. This constraint means that, with the chosen $\gamma$-representation, only the first eight components of a spinor are non-vanishing. With the definitions (3), one can easily verify that the $Z_2$ reflection (2) simultaneously exchanges some components of the 8-dimensional spinor:

$$
\theta^3 \leftrightarrow \theta^4, \quad \text{and} \quad \theta^7 \leftrightarrow -\theta^8.
$$

Notice that the 2D string action in the light-cone gauge [8] is $Z_2$ invariant, even if the combination $\Pi$ appears explicitly. In fact $\Pi$ and $\Pi' = \gamma^5 \gamma^6 \gamma^7 \gamma^8 = \sigma_3 \otimes \sigma_3 \otimes 1_{4 \times 4}$ have exactly the same action on the first eight components (those relevant for us). The $Z_2$ invariance of the string action is reflected at the level of the energy spectrum.

It is then natural to require that the $Z_2$ symmetry (2) is preserved by the interaction terms of the Hamiltonian. At first sight this seems obvious. In fact the construction in [18] parallels the one of [16] and thus one may think to have a 3-string vertex which

\footnote{See the detailed analysis of [9] and in particular tables I and II, apart from the typo in $b_{ij}^6(6)$, whose $SO(4) \times SO(4)$ labels should be $(1, -1) \times (0, 0)$.}
is invariant under the full $SO(8)$. However this is not the case. In the notations of [18] and [23], the 3-string vertex reads:

$$|H_3\rangle = \left[K_i\tilde{K}_j\psi^{IJ}(\Lambda)\ E_a\ E'_b\ E_0^a\big|0\rangle_1 \otimes \big|0\rangle_2 \otimes \big|0\rangle_3\right].$$  \hspace{1cm} (5)

For our purposes, we have separated the contributions of the fermion non-zero modes $E'_b$ and of the fermion zero modes $E_0^b$. We will argue below that (5) does not respect the $Z_2$ symmetry of the background and that both $|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$ and $E_0^b$ have to be modified.

The point is that the Lorentz structure of the operator inside the square brackets is very similar to the flat–space vertex, the only novelty being the presence of the $Z_2$-invariant combination $\Pi$. Thus the expression in the brackets commutes with the $Z_2$ generator. However one has still to define the action of (2) on the “vacuum” state $|0\rangle$ defined by

$$a_n|0\rangle = 0, \forall n, \quad b_n|0\rangle = 0, n \neq 0, \quad \theta_0|0\rangle = 0.$$  \hspace{1cm} (6)

In order to have a $Z_2$ preserving interaction it is natural to define

$$Z_2|0\rangle = |0\rangle.$$  \hspace{1cm} (7)

This choice also guarantees that in the limit $\mu \to 0$, one smoothly recovers the flat space theory, where the vacuum is a $SO(8)$ scalar. However, the innocent looking choice (7) is quite strange. In fact, as shown in [9], $|0\rangle$ preserves the full $SO(8)$ symmetry, since it is defined as $\theta_0^a|0\rangle = 0$, but it is not the state of minimal light–cone energy $|v\rangle$. Its energy is given by $4\mu$. The vacuum state $|v\rangle$ with zero energy is defined by

$$a_n|v\rangle = b_n|v\rangle = 0 \quad \forall n.$$  \hspace{1cm} (8)

The definition of the “zero–mode” oscillators $b_0^a$ in terms of the $\theta_0^a$ is given in Eq. (19). This implies that $|v\rangle$ is related to $|0\rangle$ as follows (for example, for positive $p^+$):

$$|0\rangle = \theta_0^5 \theta_0^6 \theta_0^7 \theta_0^8|v\rangle.$$  \hspace{1cm} (9)

The relations (4), (7) and (9) imply that $|v\rangle$ is odd under $Z_2$, since we found that under this transformation $\theta^7$ and $\theta^8$ are exchanged. We note that the minus sign arising from the action of $Z_2$ on the product of $\theta$’s above does not depend on the particular realization of the $\gamma$-matrices chosen in (3). The specific form of the action may change, but the minus sign is always present. This is consistent with the supergravity analysis of [9] where the polarizations of the string “massless” states are mapped into the various supergravity states. One can check that the lowest energy field $h$ is odd under $Z_2$. This flip of sign in the field matches with the fact that the product of the four $\theta$’s in (9) represents the polarization of the field.

Thus $|v\rangle$ and $|0\rangle$ cannot have the same $Z_2$–parity. With the choice (7) $|0\rangle$ is $Z_2$-invariant. However this assignment is puzzling both on the string and the gauge theory side. On the string side, defining $|0\rangle$ to be a scalar is natural only in flat–space where all the supergravity modes are degenerate in energy and $|0\rangle$ plays a special role, being
In the pp–wave background, however, (7) is not natural because the zero–modes are not degenerate in energy; it is the state \( |v\rangle \) that plays a special role because it represents the real vacuum of the theory (i.e. the state of minimal energy and supersymmetry preserving). The conventional choice in string quantization is to define the vacuum to be invariant under all the symmetries of the background (including the discrete symmetries, present, for instance, in orbifold compactifications). This choice is also supported by the analysis of different, but closely related, setups\(^3\).

The assignment (7) is also puzzling from the point of view of the string/gauge–theory correspondence. In fact, \( |v\rangle \) corresponds to the operator \( O_{\text{vac}}^J \sim \text{tr} \, Z^J \) which is naturally defined to be \( Z_2 \) invariant, since it does not have any Lorentz index along the directions where the \( Z_2 \) action is non–trivial.

Thus, there are two reasons why one should change the quantum number assignment of \( |v\rangle \) and declare it to be a \( SO(4) \times SO(4) \times Z_2 \) scalar. The first reason is to construct a string interaction explicitly realizing the \( SO(4) \times SO(4) \times Z_2 \). The second reason is to keep a close relation with the gauge–theory correspondence. In particular,

\[
Z_2 |v\rangle = |v\rangle \quad \Leftrightarrow \quad Z_2 |0\rangle = -|0\rangle .
\]

This change however is not without consequences since it implies that the 3–string interaction (5) does not preserve the \( Z_2 \) invariance of the background (1). Notice that, since \( Z_2^2 = 1 \), (7) and (10) are the only 2 possible assignments, if one insists that \( |v\rangle \) is an eigenstate of \( Z_2 \).

In summary, the more desirable choice (10) is possible only if the form of the interacting Hamiltonian (5) is modified too.

### 2.2 The string interaction \( H_3 \)

The existing form (5) for \( |H_3\rangle \) and its behavior under \( Z_2 \) are responsible for a few puzzling features noticed in the literature. The authors of [13] noticed that 3-string amplitudes involving only states dual to scalar BMN operators have a relative minus sign with respect to other amplitudes involving only operators with derivative impurities. Even more puzzling, it seems that string theory predicts a vanishing 3-point interaction when the incoming state is dual to an operator with one scalar and one derivative impurity. These properties of the string amplitudes have been checked by explicit computations [19, 22, 23], and follow from the vacuum choice (5) and the \( \gamma \)-matrix relations

\[
\sum_{K=1}^{8} \gamma^{iK}_{[ab} \gamma^{jK}_{cd]} = \delta^{ij} \epsilon_{abcd} , \quad \sum_{K=1}^{8} \gamma^{i'K}_{[ab} \gamma^{j'K}_{cd]} = -\delta^{i'j'} \epsilon_{abcd} , \quad \sum_{K=1}^{8} \gamma^{iK}_{[ab} \gamma^{j'K}_{cd]} = 0 , \quad (11)
\]

\(^3\)For instance, the pp–waves background supported by a NS–NS flux were reconsidered in [26] with the goal of studying the holographic properties of these backgrounds. These pp–waves can be analyzed by means of CFT techniques in the RNS quantization, and it turns out that the ground state is always invariant under all discrete symmetries of the background. Moreover, it is interesting to note that the Matrix String Theory analysis of certain pp–wave backgrounds also yields a symmetry–preserving interaction [27].
where the spinor indices $a, b, \ldots$ are restricted to be in the positive $\Pi$–chirality (i.e. $a = 1, \ldots, 4$) or in the negative $\Pi$–chirality (i.e. $a = 5, \ldots, 8$) subspace, while the vector indices $i, j$ run from 1 to 4, $i', j'$ from 5 to 8 and $K$ runs from 1 to 8. Eq. (11) is derived by a direct computation from the $\gamma$–matrix realization (3) and the relative minus sign between the first two relations is responsible for the puzzling features described above. However, in order to see the effects of the minus sign in Eq. (11), one does not have to go through the $\gamma$–matrix algebra. It is sufficient to use the transformation rules under $Z_2$ of the 3–string amplitudes. Let us define the string amplitude

$$A^{ij} := \left( \langle v, p_3^+ | \alpha^{I}_{0(n(3))} \alpha^{-}_{0(n(3))} \otimes \langle v, p_1^+ | \alpha^{I}_{m(1)} \alpha^{-}_{m(1)} \otimes \langle v, p_2^+ | \right) |H_3\rangle,$$

(12)

and consider the string amplitudes $A^{ij}, A^{i'j'}$. The oscillators$^4$ inserted in $A^{i'j'}$ are just the $Z_2$ images of those inserted in $A^{ij}$. Moreover the operatorial content of the interaction (5) is $Z_2$ invariant. Thus one can insert $Z_2^2 = 1$ in the amplitude $A^{ij}$ and relate it to $A^{i'j'}$ with a coefficient of proportionality of $(-1)^3$. In fact, the states $\ket{0}$ and $\ket{v}$ cannot be even at the same time and the factor of $(-1)$ comes either from the action of $Z_2$ on the external states or from the action on the kets in $|H_3\rangle$, according to the $Z_2$–parity chosen for $|v\rangle$. Thus one is led to the conclusion that $A^{ij} = -A^{i'j'}$. A similar argument, where one uses in addition the $SO(4) \times SO(4)$ invariance and the fact that the exchange $(n, m) \rightarrow (-n, -m)$ leaves invariant the oscillator contribution, implies that the mixed amplitudes $A^{ij} = 0$.$^5$ Through more direct computations, these features of the vertex (5) were also noted in [22, 23, 24].

It is clear that, in order to obtain a different result for the amplitudes, it is not sufficient to switch from choice (7) to (10) keeping the operator part of $|H_3\rangle$ unchanged. So we will modify the operator part of $|H_3\rangle$ in order to construct a string interaction where both the 3-point vertex and the vacuum are $Z_2$–invariant (as it happens in flat–space with the $SO(8)$ invariance).

For this purpose, we just need to focus on the fermionic zero–modes. This part is constrained by the requirements that the string coordinates $\theta$ are continuous and that the conjugate momentum $\lambda$ is conserved. This means that we should find a state $|\delta\rangle$ in the product of the three Hilbert spaces of the external strings satisfying simultaneously

$$\sum_{r=1}^{3} \lambda_{0(r)}^{a} |\delta\rangle = 0 , \quad \sum_{r=1}^{3} p_{r}^{+} \theta_{0(r)}^{a} |\delta\rangle = 0 ,$$

(13)

where $\lambda_{(r)}$ are the conjugate momenta of the fermions $\theta_{(r)}$. The solution of [18] is basically unique if one wants to keep the $SO(8)$ invariance. However, in the pp–wave background

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$^4$As already said we follow the conventions of [18] or [23] and denote with $\alpha_{n}$ the BMN string oscillators and with $a_{n}$ the oscillators usually employed in the string vertex. The relations are $\alpha_{n} = \frac{1}{\sqrt{2}}(a_{|n|} - i \text{sign}(n) a_{-|n|})$, for $n \neq 0$ and $\alpha_{0} = a_{0}$.

$^5$We emphasize that these relations for $A^{ij}, A^{i'j'}$ and $A^{ij}$ follow immediately from the fact that $|0\rangle$ is used in the construction of the vertex (5), while the perturbative string states are built on the true vacuum $|v\rangle$. Thus they are independent of the choice of $Z_2$ parity assignment given to $|0\rangle$ and $|v\rangle$. 
the symmetry is reduced and it is the smaller symmetry $SO(4) \times SO(4) \times Z_2$ that has to be preserved. This can be achieved by choosing
\[
|\delta\rangle = \prod_{a=1}^{8} \left( \sum_{r=1}^{3} \lambda^a_0(r) \right) \prod_{a=1}^{8} \left( \sum_{r=1}^{3} p^+_r \theta^a_0(r) \right) |v,p^+_1\rangle \otimes |v,p^+_2\rangle \otimes |v,p^+_3\rangle.
\]
Here the fermionic delta function $|\delta\rangle$ satisfies the constraints (13). In fact, because of momentum conservation $\sum_{r=1}^{3} p^+_r = 0$, the operators $\sum_{r=1}^{3} \lambda^a_0(r)$, $\sum_{r=1}^{3} p^+_r \theta^a_0(r)$ anti-commute for all $a,b$. Therefore in checking (13), one always encounters the square of a fermion oscillator and (13) are satisfied. Notice that in Eq. (14) it is crucial to use the $SO(8)$ breaking vacuum $|v\rangle$. If the ground state $|0\rangle$ had been used in the r.h.s. of (14), one would have found a trivially vanishing result. Thus our proposal is to use this new solution to the fermionic constraints (13) and replace the combination $E^b_0|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$ in the vertex (5) with the fermionic delta-function defined in (14):
\[
|H^3\rangle_{\text{new}} = \left[ K_I \tilde{K}_J v^{IJ} E^a_a \right] |\delta\rangle.
\]
However the explicit form of $K_I$, $\tilde{K}_J$ and $v^{IJ}$ is likely to be different from the one of flat-space. For the purposes of this letter we will only need to assume that $v^{IJ}$ contains a constant term proportional to $\delta^{IJ}$.

### 2.3 Consequences on the bosonic amplitudes

An immediate advantage of the above modification is that now the zero-mode structure of the interaction Hamiltonian is $Z_2$–even with the natural choice (10). Thus the argument yielding $A_{ij} = -A_{i'j'}$ cannot be applied if (14) is used and one can hope to avoid the puzzling features deriving from the interaction (5). Now we will prove that this is indeed the case.

Let us focus on string amplitudes involving external states built with only bosonic oscillators acting on the true vacuum $|v\rangle$. In this case we will not need the precise form of the prefactor $v^{IJ}$. In fact, as noticed in [16] the prefactor can only contain creation oscillators. The lowering modes are irrelevant since they can read the vacuum structure of the vertex. The creation modes are defined with respect to the vacuum chosen in the interaction Hamiltonian. Since we write the string vertex in terms of the new zero–mode structure (14), the prefactor can only contain $a^\dagger$ and $b^\dagger$ oscillators. If we use external states with no fermionic oscillators, all the terms in the prefactor containing $b^\dagger$’s will not contribute to the amplitude, since they can act directly on the external states. This means that the only term of the prefactor matrix $v^{IJ}$ we will need is the constant part $v^{IJ} = \delta^{IJ}$. This is conflict with the form $v^{IJ} = \pm \delta^{IJ}$, $+$ for $I,J = 1,2,3,4$, $-$ for $I,J = 5,6,7,8$, as obtained in [22, 23], which would give the following string amplitudes
\[
A_{ij} = -A_{i'j'}, \quad A_{ij} = 0, \quad \text{if one uses } |H^3\rangle \text{ in Eq. (5)}.
\]
On the other hand, it is clear that, due to the presence of the $SO(8)$ invariant $v^{IJ} = \delta^{IJ}$, the string vertex (15) leads to
\[
A_{ij} = A_{i'j'} = A_{ij}, \quad \text{if one uses } |H^3_{\text{new}}\rangle \text{ in Eq. (15)}.
\]
Notice that these amplitudes are nonzero. To see this, let us denote the fermionic part common to all of them by

\[ F := 1 \langle v \rangle \otimes 2 \langle v \rangle \otimes 3 \langle v \rangle \prod_{a=1}^{8} \left( \sum_{r=1}^{3} \lambda_{0(r)}^{a} \right) \prod_{a=1}^{8} \left( \sum_{r=1}^{3} p_{i}^{a} \theta_{0(r)}^{a} \right) |v\rangle_{1} \otimes |v\rangle_{2} \otimes |v\rangle_{3}. \]  

Without loss of generality, let us take \( p_{1}^{+}, p_{2}^{+} > 0, p_{3}^{+} < 0 \). In terms of string oscillators \( b \)'s, we have (apart from numerical factors)

\[ \theta_{0(r)} \sim \frac{1}{\sqrt{p_{i}^{+}}} \left( b_{0(r)}^{\dagger} b_{0(r)} \right), \quad r=1,2; \quad \theta_{0(3)} \sim \frac{1}{\sqrt{|p_{3}^{+}|}} \left( b_{0(3)}^{\dagger} b_{0(3)} \right), \]  

thus

\[ \prod_{a=1}^{8} \sum_{r=1}^{3} p_{i}^{a} \theta_{0(r)}^{a} |v\rangle_{1} \otimes |v\rangle_{2} \otimes |v\rangle_{3} \]

\[ = (p_{3}^{+})^{2} \cdot b_{0(3)}^{\dagger} \cdots b_{0(3)}^{\dagger} \left( \sum_{r=1}^{2} \sqrt{p_{i}^{+} b_{0(r)}^{\dagger}} \right) \cdots \left( \sum_{r=1}^{2} \sqrt{p_{i}^{+} b_{0(r)}^{\dagger}} \right) |v\rangle_{1} \otimes |v\rangle_{2} \otimes |v\rangle_{3}. \]  

The eight fermion creators have to be annihilated by selecting the corresponding eight annihilators from the factor \( \prod_{a=1}^{8} \sum_{r=1}^{3} \lambda_{0(r)}^{a} \). From the expression in terms of oscillators

\[ \lambda_{0(r)} \sim \sqrt{p_{i}^{+}} \left( b_{0(r)}^{\dagger} b_{0(r)} \right), \quad r=1,2; \quad \lambda_{0(3)} \sim \sqrt{|p_{3}^{+}|} \left( b_{0(3)}^{\dagger} b_{0(3)} \right), \]  

and \( \{ \sum_{r=1}^{2} \sqrt{p_{i}^{+} b_{0(r)}^{\dagger}}, \sum_{r=1}^{2} \sqrt{p_{i}^{+} b_{0(r)}^{\dagger}} \} = -p_{3}^{+} \delta^{ab} \), it is easy to obtain

\[ F = (p_{3}^{+})^{8}. \]  

### 3 Discussion

In this note we proposed a modification in the treatment of the fermionic zero–modes which explicitly satisfies all the symmetries of the plane–wave background. We also found that the new form of the string vertex gives results in direct agreement with the expected \( SO(8) \) symmetry of the bosonic excitations\(^6\). Moreover, the interaction vertex discussed here matches in a simple way, from the string point of view, the explicit Yang–Mills computation of [24]. However, one may think that even a small modification in the string vertex could ruin all the subtle cancellations necessary to have a consistent realization of the supersymmetry algebra on the interaction Hamiltonian. One may also worry about the assignment (10), because it is not what is done in flat–space. Let us first make some comments on this second issue.

\(^6\)Notice, in fact, that the metric in (1) is \( SO(8) \) symmetric and the breaking of this group is just due to a term in the fermionic Lagrangian coming from the R-R form.
The change of the definition of the fermionic vacuum in the pp-wave background is easy to understand. In fact an analogous change in the definition of the bosonic vacuum has already been performed [18]. For strings in flat-space, one considers the eigenstates of the position or momentum operator as possible vacua. In particular, it has been known for a long time that the 3-string interaction contains the following vacuum structure [28]

\[ |0\rangle_{\text{F.s.}} = \prod_{r=1}^{3} |\tilde{x}_{0(i)}^r = 0; 0_a\rangle = \prod_{r=1}^{3} \int dp_{(i)}^I |p_{(i)}^I; 0_a\rangle. \tag{23} \]

However, when \( \mu \) is switched on in the world-sheet Lagrangian both the fermionic and the bosonic coordinates acquire a potential term of the harmonic oscillator type. Thus one cannot use the eigenstates of the position or momentum operator any more and has to change the definition of the vacuum. As usual the combination \( a^i = \frac{1}{\sqrt{2m}} (p^i + imx_0^i) \) is introduced and the vacuum is defined as \( a^i |0_a\rangle = 0 \) (here \( m \) is, of course, proportional to \( \mu \)). This vacuum also appears [18] in the 3-string interaction, instead of the one (23) typical of flat space. This is precisely what is usually done in quantum mechanics when passing from a free particle to the case of a harmonic oscillator. What we are claiming is that also on the fermionic side, a different vacuum \( |v\rangle \) has to be chosen once a nonzero \( \mu \) is turned on. We also claim that the vacuum state \( |v\rangle \) has to be a scalar in order to realize explicitly the \( SO(4) \times SO(4) \times Z_2 \) symmetry of the background.

The new realization of the fermionic constraints (14) may have consequences also on the general structure of the vertex and, in particular, of the prefactor. The kinematical part of the fermionic vertex can in principle be constructed exploiting the same idea used for the zero-modes. In fact, one can just multiply \( |\delta\rangle \) by all the other modes of the two fermionic constraints (momentum conservation and coordinate continuity) and obtain an expression for \( E_b^i \) in Eq. (15) satisfying all the kinematical constraints. On the other hand, the explicit form of the prefactor is related to the realization of the dynamical generators and is more subtle. However, there is also a technical reason to suspect that the functional form of the fermionic vertex has to be changed with respect to the flat-space case. In fact, following the computation of [16], one sees that the realization of the supersymmetry algebra on \( |H_3\rangle \) requires the identity \( (\theta_{0(1)}^a - \theta_{0(2)}^a)|V\rangle = 0 \) (here with \( |V\rangle \) we indicate the ket state in the interaction Hamiltonian enforcing the string coordinate continuity and momentum conservation). In flat space this identity simply follows from the fact that the zero-modes \( \theta_{0(i)}^a \) are all destruction operators, since the vacuum \( |0\rangle \) was used. Notice that this choice is basically forced by the requirement of \( SO(8) \) invariance.

On the pp-wave background, we have to use the real vacuum \( |v\rangle \) to construct the vertex \( |V\rangle \), so one may have to change this step of the derivation. Thus the presence of \( |v\rangle \) may affect the realization of the supersymmetry algebra on \( |H_3\rangle \), and the form of the prefactor in the complete vertex may be different from that appearing in the flat-space expression.

Let us conclude by noticing that the vertex in Eq. (5) yields a third puzzling prediction on the Yang-Mills side. In fact, beyond \( A_{ij} = -A_{i'j'} \) and \( A_{ij'} = 0 \) already discussed, it turns out that all the amplitudes with external states dual to operators with just fermionic impurities should vanish. This again looks strange from the field theory point of view. This zero is not related to fermion zero-mode counting as the other two problems,
but is instead a consequence of the full form of the interaction (5) where the vertex is at least quadratic in the bosonic oscillators. We have presented evidence in the above that the actual form of the functional prefactor may be different from the one of the flat–space vertex. Thus in order to see what are the predictions of the parity conserving interaction (15), we need to work out the exact form of the new $K^I$, $\tilde{K}^I$, $v^{IJ}$ and $\Lambda^a$.

**Note added:** The zero–mode structure (14) proposed in this paper was used to build a full kinematical vertex in [29]. Two supersymmetric completions are possible for this vertex and have indeed been discussed in subsequent literature. One was obtained in [30] by requiring the continuity of the flat space limit $\mu \to 0$, which implies assigning an odd $Z_2$ parity both to the string vacuum $|v\rangle$ and the prefactor. In [31] the resulting vertex was shown to be equivalent to that proposed in [18, 23, 32, 33]. In [34] an alternative solution is put forward, where the choice proposed in this paper of an even $Z_2$ parity for the string vacuum is maintained, and this symmetry is therefore realized explicitly (i.e. both the interaction and the vacuum state are $Z_2$–invariant at the same time). In this approach, one gives up the smoothness of the flat space $\mu \to 0$ limit. In fact, this second solution follows the behaviour of supergravity in $AdS_5 \times S^5$ more closely. The idea is that, since the PP–wave background can be seen as an approximate description of $AdS_5 \times S^5$, even for small curvatures, the 3–state interaction has to be compared with the results in $AdS_5 \times S^5$ rather than with the results of flat–space.

Concerning the comparison with the field theory description, two different approaches have been proposed, [13] and [35] (see [36] for an updated discussion). This paper is placed in the framework of [13], where string theory amplitudes are compared with field theory correlators. This proposal was explicitly checked in [19, 20] for scalar BMN operators and these computations were subsequently extended also to BMN operators containing vector and fermion impurities [34]. Finally, we would like to remark that a more general motivation for the proposal of [13] was provided in [37, 38] by considering the Penrose limit of the AdS/CFT bulk–to–boundary formula of [2]. This approach was extended in [39], where non–planar corrections to the gauge theory correlators are reproduced from the string side.

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