Computational Complexity Aspects of Point Visibility Graphs

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Abstract

A point visibility graph is a graph induced by a set of points in the plane where the vertices of the graph represent the points in the point set and two vertices are adjacent if and only if no other point from the point set lies on the line segment between the two corresponding points. The set of all point visibility graphs form a graph class which is examined from a computational complexity perspective in this paper. We show NP-hardness for several classic graph problems on point visibility graphs such as Feedback Vertex Set, Longest Induced Path, Bisection and F-free Vertex Deletion (for certain sets F). Furthermore, we consider the complexity of the Dominating Set problem on point visibility graphs of points on a grid.

1 Introduction

Visibility graphs are a way to encode the information that certain objects are visible from one another or not. The objects correspond to the vertices and there is an edge between two vertices if and only if the two corresponding objects are visible from each other, for some specified definition of visibility. Different kinds of visibility graphs have been studied, like rectangle visibility graphs [19], hypercube visibility graphs [20], segment visibility graphs [13], and polygon visibility graphs (bearing a slightly different meaning). They find their application in many real world problems, for example, in computing Euclidean shortest paths in the presence of obstacles in the field of robotics [4], decomposition of two-dimensional shapes by graph theoretic clustering in the field of object recognition [23], or even in the diagnosis of Alzheimer’s disease [2]. In point visibility graphs (PVGs) our objects are simply points in the plane and two points are visible if there is a direct line between them, that is, a line that does not contain any other point. PVGs can be thought of the extreme case of other visibility graphs when our viewpoint on the objects is far away. In this case the shapes and diameter of the objects become negligible as the objects shrink to points. The visibility relation between points of a point set in the plane (represented by PVGs) is thus a fundamental structure in computational geometry [4].

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Figure 1: A point visibility graph $G$ and its visibility embedding in the plane.

In this work, we adopt an algorithmic perspective on the class of point visibility graphs. In doing so, we intend to bring this practically relevant graph class to the attention of a broader audience in order to motivate research focused on solving computational graph problems for this graph class. We start by studying several classical graph problems that are NP-complete in general and investigate whether they become polynomial-time solvable on PVGs. It turns out that many of the problems remain NP-complete on point visibility graphs.

1.1 Preliminaries and Properties of Point Visibility Graphs

All graphs in this paper are undirected, without self-loops or parallel edges. We use standard graph notation (see, e.g. Diestel [5]). We start with the definition of point visibility graphs.

**Definition 1.** A graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ is a point visibility graph (PVG), if there exists a set of points $P = \{p_1, \ldots, p_n\}$ in the plane (each point $p_i$ corresponds to vertex $v_i$) such that $\{v_i, v_j\} \in E$ if and only if there exists no other point in $P$ on the line segment between $p_i$ and $p_j$, that is, $p_i$ and $p_j$ are visible to each other. The point set $P$ is also called the visibility embedding of $G$.

In short, a PVG is a graph that has a visibility embedding in the plane. Figure 1 shows a PVG denoted $G$ and its visibility embedding in the plane.

We can divide the graph class of PVGs into two subclasses: Paths and non-path graphs. Every path is a PVG and it holds that the diameter is equal to the length of the path. On the other hand, every non-path PVG has diameter two [16]. For non-path PVGs, this is due to the fact that for each pair of points in the plane that are not visible to each other there exists a third point that can see both: The point closest to the line segment between the two points. Another property of interest of non-path PVGs is that they always have a Hamiltonian cycle. Intuitively, it can be found in a visibility embedding of a non-path PVG in polynomial time by going from the outermost to the innermost convex hull of the points in the embedding (see [14, Theorem 1] for the details).

1.2 Related Work

Structural properties of PVGs have been thoroughly researched. Ghosh and Roy [14] have shown among other results that non-path PVGs always have a Hamiltonian cycle. Kára et al. [16] gave results on the chromatic number of PVGs. They showed that for PVGs with clique size two and three, the chromatic number is two and three, respectively, and characterize those PVGs that are 2- and 3-colorable. It follows that 3-COLORABILITY is polynomial-time solvable
on PVGs when a visibility embedding is given. On the other side, Diwan and Roy \cite{7} showed that it is NP-hard to decide whether a given PVG is $k$-colorable for $k \geq 5$. Recently, the case of $k = 4$ was shown to be polynomial-time solvable (on a given visibility embedding) \cite{1}. Furthermore, Pfender \cite{21} showed that for PVGs with clique size six, the chromatic number can be arbitrary large.

As regards other graph problems, Ghosh and Roy \cite{14} showed that \textsc{Vertex Cover}, \textsc{Independent Set} and \textsc{Maximum Clique} remain NP-hard on PVGs. Notably, the recognition problem for PVGs was shown by Cardinal and Hoffmann \cite{3} to be complete for the existential theory of the reals $\exists \mathbb{R}$ (which implies NP-hardness; note also that $\text{NP} \subseteq \exists \mathbb{R} \subseteq \text{PSpace}$ \cite{3}).

Besides PVGs the structure of other visibility graphs like line segments or polygons has also been investigated \cite{13, 18}.

1.3 Our Contribution and Organization

We study the complexity of several classical computational graph problems on the graph class of PVGs. Due to the fact that PVGs are Hamiltonian, problems like \textsc{Longest Path} and \textsc{Hamilton Path} are trivial. Nevertheless, many graph problems remain NP-hard on PVGs. In Section 2 we prove NP-hardness of \textsc{Feedback Vertex Set}, \textsc{Longest Induced Path}, \textsc{Colorability}, \textsc{Bisection}, \textsc{Cluster Vertex Deletion}, and a restricted version of \textsc{$\mathcal{F}$-free Vertex Deletion} on PVGs. Herein, we build upon a general reduction idea introduced by Ghosh and Roy \cite{14}. In Section 3 we briefly discuss the \textsc{Dominating Set} problem and show that it is unlikely to be NP-hard at least on a subclass of point visibility graphs. We close in Section 4 by pointing to some open questions.

2 NP-hardness Results

In this section we prove that several graph problems remain NP-hard when restricted to PVGs. Our hardness results follow from a transformation (mapping arbitrary graphs to PVGs) that was first introduced by Ghosh and Roy \cite{14} to show NP-hardness of \textsc{Vertex Cover}, \textsc{Independent Set} and \textsc{Maximum Clique} on PVGs and was already used in complexity studies of other point set problems \cite{9}. This transformation, henceforth called $\Phi$, allows us to prove NP-hardness for the following problems on PVGs: \textsc{Feedback Vertex Set}, \textsc{Longest Induced Path}, \textsc{Bisection} and \textsc{$\mathcal{F}$-free Vertex Deletion} for certain sets $\mathcal{F}$ (we are not aware of any other results concerning these problems on Hamiltonian graphs). The formal definition is as follows.

**Definition 2** (Transformation $\Phi$). Given a graph $G = (V, E)$, we add a vertex $b_{uv}$ for every vertex pair $u \neq v \in V$ with $\{u, v\} \notin E$ and we connect $b_{uv}$ to all vertices in $V$. We will call $b_{uv}$ a \textit{blocker} because it blocks the visibility between $u$ and $v$ in a corresponding visibility embedding. In total, $(\binom{|V|}{2} - |E|)$ blockers are added to $G$. Finally, all blockers are connected to each other to obtain $\Phi(G)$.

A small example of the transformation $\Phi$ is shown in Figure 2. It is not hard to see that the resulting graph $\Phi(G)$ can be computed in polynomial time. Keep in mind below that all added blockers in $\Phi(G)$ form a clique and that the graph $G$ is an induced subgraph of $\Phi(G)$. Furthermore, $\Phi(G)$ is always a PVG \cite{9, 14}. A proof sketch is as follows. Let $v_1, \ldots, v_n$ be $n$ distinct points in general position (e.g. on a circle) corresponding to the vertices of $G$. We
can now add blockers inductively as follows: For a non-edge \( \{u, v\} \not\in E \) consider the line segment \( uv \) defined by \( u \) and \( v \). It is clear that we can always choose a blocker \( b_{uv} \) on this segment such that \( b_{uv} \) is not lying on any other line defined by any two other points introduced so far, since there are only finitely many intersection points of these lines with \( uv \). Using a similar argument, we can show that, for each PVG, we can add a universal vertex \( u \), a vertex connected to all other vertices, while maintaining the PVG property. We use this observation below.

In the following, we will use the transformation \( \Phi \) in polynomial-time reductions to prove NP-hardness for the above mentioned problems. It is clearly computable in polynomial time, since it only involves adding a polynomial number of vertices and edges to the input graph in a trivial way.

### 2.1 Feedback Vertex Set

We start with the well-known Feedback Vertex Set problem which is NP-hard on general graphs [11].

**Feedback Vertex Set**

**Input:** A graph \( G = (V, E) \) and an integer \( k \in \mathbb{N} \).

**Question:** Is there a subset \( V' \subseteq V \) with \( |V'| \leq k \) such that all cycles in \( G \) contain at least one vertex of \( V' \)?

We show that it is also NP-hard on PVGs.

**Theorem 1.** Feedback Vertex Set on PVGs is NP-hard.

**Proof.** Given a Feedback Vertex Set instance \((G, k)\), we construct an instance \((G', k')\) of Feedback Vertex Set on PVGs as follows. We set \( G' := \Phi(G) \) and \( k' := k + |B| \), where \( B \) is the set of blockers that where introduced by the transformation \( \Phi \).

Let \((G, k)\) be a yes-instance and let \( v_1, \ldots, v_k \) be vertices in \( G \) such that \( G - \{v_1, \ldots, v_k\} \) does not contain a cycle. Then, removing the vertices \( v_1, \ldots, v_k \) along with all the blockers \( B \) from \( \Phi(G) \) yields the acyclic graph \( G - \{v_1, \ldots, v_k\} \). Hence, \((G', k')\) is a yes-instance.

If \((G, k)\) is a no-instance, then we need to remove at least \( k + 1 \) vertices to delete all cycles in \( G \). Hence, there are more than \( k + 2 \) vertices in \( G \) since, otherwise, deleting \( k \) arbitrary vertices removes all cycles. Moreover, deleting any set of \( k + 1 \) vertices in \( G \) always leaves an edge in the remaining graph since, otherwise, we could only delete \( k \) of these vertices and still remove all cycles.

Now note that we always have to delete at least \( |B| - 2 \) of the blockers in \( \Phi(G) \) since all blockers are connected to all other vertices and any three of them form a cycle. However, we...
know that we cannot delete all blockers because we need at least $k + 1$ vertex deletions in order to delete all cycles in $G$. If at least one blocker remains, then we can delete at most $k + 1$ vertices in $G$. But then, there still exists an edge in $G$ which forms a cycle with the one remaining blocker. If two blockers remain, then we can delete at most $k + 2$ vertices in $G$. Hence, there still remains a vertex in $G$ forming again a cycle with the two remaining blockers. Thus, $(G', k')$ is also a no-instance. 

### 2.2 Longest Induced Path

In the following we define the length of a path as the number of edges it contains.

**Longest Induced Path**

**Input:** A graph $G$ and an integer $k \in \mathbb{N}$.

**Question:** Is there an induced path with at least $k$ edges in $G$?

This problem is NP-hard on general graphs \[11\] and remains NP-hard on PVGs.

**Theorem 2.** Longest Induced Path on PVGs is NP-hard.

**Proof.** Let $(G, k)$ be an instance of Longest Induced Path. If $k \leq 2$, then we solve the instance in polynomial time and return a trivial constant-size yes- or no-instance accordingly. For $k \geq 3$, we construct the instance $(\Phi(G), k)$.

A blocker is connected to all other vertices and forms a cycle of length three with each pair of vertices that are adjacent to each other. Thus, no blocker lies on any induced path of $\Phi(G)$ of length at least three. That is, any induced path of $\Phi(G)$ of length at least three is an induced path in $G$. Consequently, $(G, k)$ is a yes-instance if and only if $(\Phi(G), k)$ is a yes-instance.

### 2.3 Bisection

The Bisection problem is to partition the vertices of a graph into two equally sized disjoint subsets such that the number of edges between these two vertex subsets is minimized. The problem is formally defined as follows.

**Bisection**

**Input:** A graph $G = (V, E)$ and an integer $k \in \mathbb{N}$.

**Question:** Is there a partition $(U, W)$ of $V$, that is, $U, W \subseteq V$, $U \cap W = \emptyset$, and $U \cup W = V$, such that $|U| = |W|$ and $|\{\{u, w\} \in E \mid u \in U \land w \in W\}| \leq k$?

On PVGs the Bisection problem translates to partitioning the points in the plane into equal-size subsets such that the number of pairs of points from both subsets that can see each other is minimized. To show NP-hardness of this problem on PVGs, we use an idea by Garey et al. \[10\] who showed NP-hardness of Bisection by reducing from Max Cut.

**Max Cut**

**Input:** A graph $G = (V, E)$ and an integer $k \in \mathbb{N}$.

**Question:** Is there a partition $(U, W)$ of $V$ such that $|\{\{u, w\} \in E \mid u \in U \land w \in W\}| \geq k$?

**Theorem 3.** Bisection on PVGs is NP-hard.
Proof. Given a MAX CUT instance \((G = (V, E), k)\), we construct an instance \((G', k')\) of BISECTION on PVGs as follows. To obtain \(G'\), we first take the complement graph \(\overline{G}\) of \(G\) and apply the transformation \(\Phi\). We obtain \(\Phi(\overline{G})\) with \(B\) being the set of blockers introduced by \(\Phi\). If \(|B| < |V|\), then we add \(|V| - |B|\) additional vertices to \(\Phi(\overline{G})\) and connect them to all other vertices. If \(|B| \geq |V|\) and \(|B| + |V|\) is odd, then we add another universal vertex to get an even number of vertices. As mentioned before, adding such universal vertices does not destroy the PVG property. We obtain the PVG \(G' = (V', E')\) with \(V' = V \cup B'\), where \(B'\) is the set of all added vertices (original blockers and additional universal vertices). Finally, we set \(k' := (\frac{1}{2}|V'|)^2 - k\).

If \((G, k)\) is a yes-instance of MAX CUT, then there exists a partition \((U, W)\) of \(V\) such that \(|\{\{u, w\} \in E \mid u \in U \wedge w \in W\}| \geq k\). In the complement graph \(\overline{G}\) it holds that there are at least \(k\) edges missing between the vertices in \(U\) and \(W\). Now we choose a subset \(U' \subseteq (U \cup B')\) with \(U \subseteq U'\) and a subset \(W' \subseteq (W \cup B')\) with \(W \subseteq W'\) such that \(|U'| = |W'| = \frac{1}{2}|V'|\) and \(U' \cup W' = V'\). Informally speaking, we fill up the vertex sets \(U\) and \(W\) with the vertices in \(B'\) to obtain two disjoint equally sized vertex sets \(U'\) and \(W'\). This is always possible since \(|B'| \geq |V|\) by construction. Clearly, it holds \(|\{\{u, w\} \in E' \mid u \in U' \wedge w \in W'\}| \leq (\frac{1}{2}|V'|)^2 - k\) and hence \((G', k')\) is a yes-instance.

If \((G', k')\) is a yes-instance, then there exist two vertex sets \(U'\) and \(W'\) with \(U' \cup W' = V'\), \(|U'| = |W'|\), and \(|\{\{u, w\} \in E' \mid u \in U' \wedge w \in W'\}| \leq k' = (\frac{1}{2}|V'|)^2 - k\). Since the vertices in \(B'\) are universal vertices, the \(k\) missing edges between \(U'\) and \(W'\) can only be between vertices of \(\overline{G}\). Set \(U := U' \cap V\) and \(W := W' \cap V\). Clearly, we have \(U \cup W = V\) and \(|\{\{u, w\} \in E \mid u \in U \wedge w \in W\}| \geq k\). Hence, \((G, k)\) is a yes-instance.

2.4 \(\mathcal{F}\)-free Vertex Deletion

In this section, we consider a general graph problem called \(\mathcal{F}\)-FREE VERTEX DELETION. In the following, \(\mathcal{F}\) denotes a finite set of graphs, \(K_i, i \in \mathbb{N}\), denotes the complete graph on \(i\) vertices, and \(K_{i,j}, i, j \in \mathbb{N}\) denotes the complete bipartite graph with two partite sets containing \(i\) and \(j\) vertices, respectively. We say that a graph \(G\) is \(\mathcal{F}\)-free if no \(H \in \mathcal{F}\) occurs as an induced subgraph of \(G\). The \(\mathcal{F}\)-FREE VERTEX DELETION problem is then defined as follows:

**\(\mathcal{F}\)-FREE VERTEX DELETION**

**Input:** A graph \(G = (V, E)\) and an integer \(k \in \mathbb{N}\).

**Question:** Is there a subset of vertices \(X \subseteq V\) with \(|X| \leq k\) such that the graph \(G - X := G[V \setminus X]\) obtained by deleting all vertices in \(X\) from \(G\) is \(\mathcal{F}\)-free?

This is a generic graph problem that generalizes (depending on the choice of \(\mathcal{F}\)) various fundamental graph problems such as VERTEX COVER (\(\mathcal{F} = \{K_2\}\)) or CLUSTER VERTEX DELETION (\(\mathcal{F} = \{P_3\}\)), that is, \(\mathcal{F}\) consists of a single path on three vertices. In a vertex-deletion problem, we are given a graph \(G\) and an integer \(k\) and we want to decide whether we can delete at most \(k\) vertices from \(G\) such that the resulting graph has a certain fixed property \(\Pi\). Lewis and Yannakakis \[17\] showed that vertex-deletion problems are NP-hard whenever \(\Pi\) is a nontrivial property, that is, there are infinitely many graphs satisfying \(\Pi\) and infinitely many graphs not satisfying \(\Pi\). Hence, \(\mathcal{F}\)-FREE VERTEX DELETION is NP-hard on general graphs if there are infinitely many \(\mathcal{F}\)-free graphs and infinitely many graphs that are not \(\mathcal{F}\)-free.

For point visibility graphs, we prove the following theorem.
Theorem 4. \(\mathcal{F}\)-free Vertex Deletion is \(\text{NP-hard}\) on PVGs in each of the three individual cases where

(i) \(\mathcal{F}\) contains no complete graphs,

(ii) \(\mathcal{F}\) contains a \(K_t\) with \(t \geq 3\) and no graph in \(\mathcal{F}\) can be made \(K_{t-1}\)-free with less than two vertex deletions,

(iii) \(\mathcal{F}\) contains \(K_t\) and \(K_{t'}\) with \(t \geq 3, t' \geq 2\) and no \(K_2\)-free graphs.

Proof. We prove the three cases separately.

(i) Let \(\mathcal{F}'\) be the set resulting from \(\mathcal{F}\) by removing all universal vertices (that is, vertices that are adjacent to all other vertices) from every graph in \(\mathcal{F}\). Clearly, there are infinitely many graphs that are not \(\mathcal{F}'\)-free (e.g. by adding vertices to any graph in \(\mathcal{F}'\)). All complete graphs, however, are \(\mathcal{F}'\)-free. Therefore, \(\mathcal{F}'\)-free Vertex Deletion is \(\text{NP-hard}\) on general graphs \([17]\). We give a reduction from \(\mathcal{F}'\)-free Vertex Deletion on general graphs to \(\mathcal{F}\)-free Vertex Deletion on PVGs.

Given an instance \((G, k)\), we construct the instance \((G', k)\), where \(G'\) is the PVG obtained by adding \(k + \ell\) additional universal vertices to \(\Phi(G)\), where \(\ell\) is the maximum number of universal vertices of any graph in \(\mathcal{F}\). Note that \(\ell\) is a fixed constant (for every \(\mathcal{F}\)) and, hence, we can add these vertices in polynomial time. As mentioned before, adding universal vertices to a PVG yields another PVG.

Assume that there is a size-\(k\) vertex subset \(X \subseteq V\) such that \(G - X\) is \(\mathcal{F}'\)-free. We claim that \(G' - X\) is \(\mathcal{F}\)-free. If \(G' - X\) contains \(H \in \mathcal{F}\) as an induced subgraph, then all vertices of that subgraph that are not universal are also contained in \(G - X\). This implies that \(G - X\) contains an induced subgraph \(H' \in \mathcal{F}'\) obtained from \(H\) by removing all universal vertices, which is a contradiction. Therefore, \(G' - X\) is indeed \(\mathcal{F}\)-free.

Now assume that \(G'\) contains a size-\(k\) vertex set \(X\) such that \(G' - X\) is \(\mathcal{F}\)-free. Since \(G'\) contains at least \(k + \ell\) universal vertices, it follows that \(G' - X\) contains at least \(\ell\) universal vertices. Therefore, \(G' - X\) contains all the required universal vertices for every graph \(H \in \mathcal{F}\). This implies that for all \(H \in \mathcal{F}\), the graph \(G' - X\) does not contain a copy of \(H' \in \mathcal{F}'\), where \(H'\) is again obtained from \(H\) by removing all universal vertices. Hence, also \(G - X\) is \(\mathcal{F}'\)-free.

(ii) Note that \(\mathcal{F}\) contains no \(K_2\)-free graphs. Hence, all edgeless graphs are \(\mathcal{F}\)-free. Moreover, every complete graph with at least \(t\) vertices is not \(\mathcal{F}\)-free, implying that \(\mathcal{F}\)-free Vertex Deletion is \(\text{NP-hard}\) on general graphs \([17]\). We reduce the problem on general graphs to its restriction on PVGs. Given an instance \((G, k)\), we construct the instance \((\Phi(G), k' := k + |B|)\), where \(B\) is the set of blockers introduced in \(\Phi(G)\).

Assume that \(G\) contains a vertex subset \(X \subseteq V\) of size \(k\) such that \(G - X\) is \(\mathcal{F}\)-free. Removing \(X\) along with all blockers \(B\) from \(\Phi(G)\) clearly yields an \(\mathcal{F}\)-free graph.

Now, let \(\Phi(G)\) contain a vertex subset \(X\) of size \(k + |B|\) such that \(\Phi(G) - X\) is \(\mathcal{F}\)-free. If \(B \subseteq X\), then \(X \setminus B\) is a set of \(k\) vertices such that \(G - (X \setminus B)\) is \(\mathcal{F}\)-free. If \(B \not\subseteq X\), then it holds that \(1 \leq t' := |B \setminus X| < t\) since otherwise the blockers contain a \(K_t\).

Also, it follows that \(G - (X \setminus B)\) is \(K_{t'-1}\)-free since all blockers are universal vertices. Let \(X' \subseteq (X \setminus B)\) contain arbitrary \(k\) vertices from \(X \setminus B\). Note that \(G - X'\) is \(K_t\)-free and can be made \(K_{t'-1}\)-free by deleting at most \(|(X \setminus B) \setminus X'| = k + t' - k = t'\) vertices. Hence, \(G - X'\) can be made \(K_{t-1}\)-free by at most one vertex deletion. It follows that \(G - X'\) cannot contain any graph in \(\mathcal{F}\).
(iii) Note that all edgeless graphs are $\mathcal{F}$-free and all complete graphs with at least $t$ vertices are not. Hence, $\mathcal{F}$-free Vertex Deletion is NP-hard on general graphs \cite{17}. We reduce the general case to the restriction on PVGs.

Let $(G, k)$ be the input instance. By Ramsey’s theorem, there exists a number $R(t, t')$ such that every graph with at least $R(t, t')$ vertices contains a $K_t$ or an edgeless induced subgraph with $t'$ vertices \cite{22}. First, assume that $n < k + R(t, t')$. Then, there are at most $O(n R(t, t'))$ possible ways to choose $k$ vertices. Since $R(t, t')$ is a constant, we can solve the input instance in polynomial time by brute force and output a trivial yes-or no-instance. If $n \geq k + R(t, t')$, then we construct the instance $(\Phi(G), k' := k + |B|)$, where $B$ is the set of blockers introduced in $\Phi(G)$.

First, assume that $G$ contains a size-$k$ vertex subset $X$ such that $G - X$ is $\mathcal{F}$-free. Then, again, removing $X$ along with all blockers from $\Phi(G)$ yields an $\mathcal{F}$-free graph. Now assume that $X$ is a vertex set of size $k + |B|$ such that $\Phi(G) - X$ is $\mathcal{F}$-free. First, we claim that $X$ contains all blockers in $\Phi(G)$. The graph $\Phi(G) - X$ contains at least $n + |B| - (k + |B|) = n - k \geq R(t, t')$ vertices. Since it is $K_t$-free, it follows that it contains $t'$ pairwise non-adjacent vertices. These $t'$ vertices have to be from $G$, because blockers are universal. If $\Phi(G) - X$ additionally contains a blocker, then there exists a $K_{1,t'}$, which is not possible. Therefore, $X$ contains all blockers and only $k$ vertices from $G$ and $G - (X \setminus B)$ is $\mathcal{F}$-free.

We remark that case (ii) of Theorem 4 subsumes the case that $\mathcal{F}$ contains only complete graphs (excluding $K_1$).

### 3 Dominating Set on Point Visibility Graphs

In this section, we focus on the following NP-hard problem \cite{11}:

**DOMINATING SET**

**Input:** A graph $G = (V, E)$ and a parameter $k \in \mathbb{N}$

**Question:** Is there a set $D \subseteq V$ with $|D| \leq k$ such that every vertex is contained or has at least one neighbor in $D$?

DOMINATING SET in PVGs can be interpreted as a guarding problem in which the vertices represent places to be observed and we want to select a small number of observation posts among them that see all other places (also known as the art gallery problem \cite{12}).

Notably, the transformation $\Phi$ which has been used in Section 2 to show NP-hardness for several graph problems does not work for DOMINATING SET: If the input graph $G$ is not complete, then at least one blocker will be added to $G$. Then, regardless of the input graph $G$, the PVG $\Phi(G)$ always has a dominating set of size one containing a single blocker. In fact, the complexity of DOMINATING SET on PVGs remains unresolved. However, for a restricted subclass of PVGs, there exists some indication that it is unlikely to be NP-hard, which we will briefly discuss. To this end, we define the subclass of grid point visibility graphs.

**Definition 3.** An $n \times m$ grid point visibility graph (GPVG) is a PVG that is induced by the point set $P = \{(x, y) \mid 1 \leq x \leq n \land 1 \leq y \leq m\}$. For $n = m$, we call the graph a square GPVG.
Interestingly, for the point set that induces an $n \times m$ GPVG, it is not hard to see that two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ are visible to each other if and only if the numbers $|x_1 - x_2|$ and $|y_1 - y_2|$ are relatively prime. Two integers $a, b \in \mathbb{N}$ are relatively prime if there is no prime number that divides both of them. Therefore, the structure of GPVGs can be analyzed using number theory. For example, the following bounds on the size of a minimum dominating set are already known:

**Theorem 5** (Abbot [1]). Let $f(n)$ be the size of a minimum dominating set of the $n \times n$ GPVG. Then, for sufficiently large $n$, it holds

$$\frac{\log(n)}{2\log\log(n)} < f(n) < 4\log(n).$$

Due to the upper bound in **Theorem 5**, we can conclude that DOMINATING SET is unlikely to be NP-hard on square GPVGs since this implies that all problems in NP are solvable in quasi-polynomial time, that is, in time $2^{(\log n)^c}$ for some constant $c$, where $n$ is the input size, which contradicts the Exponential Time Hypothesis [15].

**Corollary 1.** If DOMINATING SET on square GPVGs is NP-hard, then every problem in NP can be solved in quasi-polynomial time.

**Proof.** Given an $n \times n$ GPVG, we know by **Theorem 5** that the size of an optimal dominating set is upper-bounded by $4\log n$. Hence, an optimal solution can be found by brute-force in quasi-polynomial time $O(n^{4\log n} \cdot \text{poly}(n))$. If DOMINATING SET is indeed NP-hard on GPVGs, then every problem in NP is quasi-polynomial-time solvable by reducing it first in polynomial time to DOMINATING SET on a square GPVG which can then be solved by the above brute-force algorithm.

Note that the encoding of the graph is crucial above: We could encode a square GPVG in a single integer but then the running time of the brute-force algorithm would not be quasi-polynomial in the input length anymore.

Another corollary is related to fixed-parameter tractability. A problem is fixed-parameter tractable with respect to some parameter $k$, an integer-valued function of the input, if it admits an algorithm with running time $f(k) \cdot n^c$, where $c$ is a constant and $n$ the input size. We obtain fixed-parameter tractability on GPVGs with respect to the size of the dominating set. Note that DOMINATING SET is W[2]-hard on general graphs and thus presumably not fixed-parameter tractable, see Downey and Fellows [8].

**Corollary 2.** DOMINATING SET on square GPVGs is fixed-parameter tractable with respect to the sought solution size.

**Proof.** Given a DOMINATING SET instance $(G, k)$, where $G$ is an $n \times n$ GPVG, we know that if $k < \frac{\log(n)}{2\log\log(n)}$, then we can answer “no” due to the lower bound in **Theorem 5**. Otherwise, we have $k \geq \frac{\log(n)}{2\log\log(n)}$ and consequently $n \leq g(k)$ for some function $g$. This implies that DOMINATING SET on square GPVGs is fixed-parameter tractable with respect to $k$ since we can check every possible solution in a running time only depending on $k$.

We show that there also is an upper bound for the DOMINATING SET problem on general PVGs depending on the minimum vertex degree in the PVG. In the following, we say a PVG is a non-path PVG if it is not a path. Our result is based on the observation that the
neighborhood of any vertex in a non-path PVG is a dominating set since every non-path PVG has diameter two. We can even tighten up this observation by taking a closer look at the visibility embedding of a non-path PVG.

Theorem 6. For every non-path PVG with minimum vertex degree $\delta$, there exists a dominating set of size $\left\lfloor \frac{\delta}{2} \right\rfloor + 1$.

Proof. Let $G$ be a non-path PVG with minimum vertex degree $\delta \geq 2$ and let $v$ be a vertex of degree $\delta$. Let $u_1, \ldots, u_\delta$ be the neighbors of $v$. Consider a visibility embedding of $G$ with the point set $P$, where $v$ corresponds to the point $p \in P$ and $u_1, \ldots, u_\delta$ correspond to the points $p_1, \ldots, p_\delta$. Then, all points in $P$ lie on one of the at most $\delta$ lines defined by the pair $(p, p_i)$ for each $i = 1, \ldots, \delta$. See Figure 3 for examples. Let $L_i$ denote the line defined by $p$ and $p_i$ and assume that $L_1, \ldots, L_\delta$ are in clockwise order. Then, each point on $L_i$ is visible to all the points on $L_{i-1}$ and $L_{i+1}$ since otherwise there would be another neighbor of $v$ corresponding to a point (and defining a line) in between $L_{i-1}$ and $L_i$ or $L_i$ and $L_{i+1}$. Hence, picking the vertices $u_i$ and $u_{i+1}$ into the dominating set dominates all vertices corresponding to points on the four lines $L_{i-1}, \ldots, L_{i+2}$.

Now, if $\delta \mod 4 = 0$, then we can select the vertices $u_{4i-1}$ and $u_{4i-2}$ for each $i = 1, \ldots, \left\lfloor \frac{\delta}{4} \right\rfloor$. This yields a dominating set of size $\frac{\delta}{2}$. If $\delta \mod 4 = 1$, then we additionally select the vertex $u_\delta$ to obtain a dominating set of size $\frac{\delta-1}{2} + 1 = \left\lfloor \frac{\delta}{4} \right\rfloor + 1$. Finally, for the case $\delta \mod 4 \in \{2, 3\}$, we additionally select the vertices $u_{\delta-1}$ and $u_\delta$ and obtain a dominating set of size $\left\lfloor \frac{\delta}{2} \right\rfloor + 1$. \qed

Note that Theorem 6 implies that DOMINATING SET is polynomial-time solvable on PVGs with constant minimum vertex degree.

We close this section by mentioning a conjecture by O’Rourke [18] stating that a logarithmic upper bound for the size of an optimal dominating set also holds for arbitrary PVGs.

Conjecture 1 ([18]). Every PVG $G = (V, E)$ has a dominating set of size $O(\log |V|)$.

4 Outlook

In this paper we made some effort towards examining point visibility graphs from the viewpoint of algorithmic complexity. We surveyed some structural properties of point visibility graphs and started to investigate the complexity of graph problems when restricted to such graphs. Even though some problems turn out to be efficiently (sometimes even trivially) solvable
in special cases, we showed that also several classical graph problems still remain NP-hard on point visibility graphs. Our main goal was to initiate productive research on solving computational problems for this natural graph class. Thus, we conclude with open questions and further directions for research.

The computational complexity of several graph problems when restricted to PVGs is yet to be determined. Among them are for example DOMINATING SET (see Section 3), MAX CUT or $\mathcal{F}$-FREE VERTEX DELETION (see Section 2.4). An interesting open problem is the case where $\mathcal{F}$ contains a $K_3$ and the graph consisting of a single edge and a single isolated vertex (that is, $K_2 + K_1$). A $\{K_3, K_2 + K_1\}$-free graph is either $K_2$-free or a complete bipartite graph. It is open whether this problem is polynomial-time solvable on PVGs. Also, there are open cases when $\mathcal{F}$ contains infinitely many graphs. Note that FEEDBACK VERTEX SET corresponds to the infinite set $\mathcal{F}$ containing all cycles. For infinitely many complete graphs, NP-hardness still holds by Theorem 4).

Furthermore, for those problems that are NP-hard on point visibility graphs, the existence of efficient approximation algorithms certainly is an interesting question. Another interesting line of research would be the parameterized complexity of graph problems restricted to PVGs. More specifically, for those problems that are $W[1]$- or $W[2]$-hard on general graphs, are they fixed-parameter tractable on PVGs? Recall that we have seen that DOMINATING SET becomes fixed-parameter tractable with respect to the solution size on quadratic grid point visibility graphs. As a final remark, we mention that the NP-hardness reduction by [14] for INDEPENDENT SET on PVGs also proves that it is $W[1]$-hard with respect to the solution size. However, for the CLIQUE problem, the parameterized complexity is still open.

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Bibliography

References

[1] H. L. Abbot. Some results in combinatorial geometry. Discrete Mathematics, 9(3):199–204, 1974.  
[2] M. Ahmadlou, H. Adeli, and A. Adeli. New diagnostic EEG markers of the Alzheimer’s disease using visibility graph. Journal of Neural Transmission, 117(9):1099–1109, 2010.  
[3] J. Cardinal and U. Hoffmann. Recognition and Complexity of Point Visibility Graphs. Discrete & Computational Geometry, 57(1):164–178, 2017.  
[4] M. de Berg, O. Cheong, M. van Kreveld, and M. Overmars. Computational Geometry: Algorithms and Applications. Springer, 2008.  
[5] R. Diestel. Graph Theory, volume 173 of Graduate Texts in Mathematics. Springer, 5th edition, 2016.  
[6] A. A. Diwan and B. Roy. On colouring point visibility graphs. CoRR/abs/1610.00952, 2016.
[7] A. A. Diwan and B. Roy. On colouring point visibility graphs. In Proceedings of the Third International Conference on Algorithms and Discrete Applied Mathematics (CALDAM 2017), volume 10156 of LNCS, pages 156–165. Springer, 2017.

[8] R. G. Downey and M. R. Fellows. Fundamentals of Parameterized Complexity. Texts in Computer Science. Springer, 2013.

[9] V. Froese, I. Kanj, A. Nichterlein, and R. Niedermeier. Finding points in general position. In Proceedings of the 28th Canadian Conference on Computational Geometry (CCCG ’16), pages 7–14, 2016.

[10] M. Garey, D. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. Theoretical Computer Science, 1(3):237–267, 1976.

[11] M. R. Garey and D. S. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, 1979.

[12] S. K. Ghosh. Visibility Algorithms in the Plane. Cambridge University Press, 2007.

[13] S. K. Ghosh and P. P. Goswami. Unsolved Problems in Visibility Graphs of Points, Segments, and Polygons. ACM Computing Surveys, 46(2):22:1–22:29, 2013.

[14] S. K. Ghosh and B. Roy. Some results on point visibility graphs. Theoretical Computer Science, 575:17–32, 2015.

[15] R. Impagliazzo and R. Paturi. On the complexity of k-SAT. Journal of Computer and System Sciences, 62(2):367–375, 2001.

[16] J. Kára, A. Pór, and D. R. Wood. On the chromatic number of the visibility graph of a set of points in the plane. Discrete & Computational Geometry, 34(3):497–506, 2005.

[17] J. M. Lewis and M. Yannakakis. The node-deletion problem for hereditary properties is NP-complete. Journal of Computer and System Sciences, 20(2):219–230, 1980.

[18] J. O’Rourke. Art Gallery Theorems and Algorithms. Oxford University Press, 1987.

[19] E. Peterson. Rectangle Visibility Numbers of Graphs. Master’s thesis, Rochester Institute of Technology, 2016.

[20] E. Peterson and P. S. Wenger. Unit Hypercube Visibility Numbers of Trees. CoRR, abs/1609.00983, 2016.

[21] F. Pfender. Visibility graphs of point sets in the plane. Discrete & Computational Geometry, 39(1):455–459, 2008.

[22] F. P. Ramsey. On a problem of formal logic. Proceedings of the London Mathematical Society (series 2), 30(1):264–286, 1930.

[23] L. G. Shapiro and R. M. Haralick. Decomposition of two-dimensional shapes by graph-theoretic clustering. IEEE Transactions on Pattern Analysis and Machine Intelligence, PAMI-1(1):10–20, 1979.