Preprint

A NEW EXTENSION OF THE SUN-ZAGIER RESULT INVOLVING BELL NUMBERS AND DERANGEMENT NUMBERS

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Abstract. Let \( p \) be any prime and let \( a \) and \( n \) be positive integers with \( p \nmid n \). We show that

\[
\sum_{k=1}^{p^n-1} \frac{B_k}{(-n)^k} \equiv a(-1)^{n-1}D_{n-1} \pmod{p},
\]

where \( B_0, B_1, \ldots \) are the Bell numbers and \( D_0, D_1, \ldots \) are the derangement numbers. This extends a result of Sun and Zagier published in 2011. Furthermore, we prove that

\[
(-x)^n \sum_{k=1}^{p^n-1} \frac{B_k(x)}{(-n)^k} \equiv -a\sum_{r=1}^{a} x^{p^r} \sum_{k=0}^{n-1} \frac{(n-1)!}{k!}(-x)^k \pmod{p\mathbb{Z}_p[x]},
\]

where \( B_k(x) = \sum_{l=0}^{k} S(k, l)x^l \) is the Bell polynomial of degree \( k \) with \( S(k, l) (0 \leq l \leq k) \) the Stirling numbers of the second kind, and \( \mathbb{Z}_p \) is the ring of all \( p \)-adic integers.

1. Introduction

Let \( B_0 = 1 \). For each \( n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \) let \( B_n \) denote the number of partitions of a set of cardinality \( n \). For example, \( B_3 = 5 \) since there are totally 5 partitions of \( \{1, 2, 3\} \):

\[
\{\{1\}\}, \{\{2\}\}, \{\{3\}\}, \{\{1, 2\}\}, \{\{1, 3\}\}, \{\{2, 3\}\}, \{\{1\}\}, \{\{1, 2, 3\}\}.
\]

The Bell numbers \( B_0, B_1, \ldots \), named after E. T. Bell who studied them in the 1930s, play important roles in combinatorics. Here are values of \( B_1, \ldots, B_7 \):

\[
B_1 = 1, \ B_2 = 2, \ B_3 = 5, \ B_4 = 15, \ B_5 = 52, \ B_6 = 203, \ B_7 = 877.
\]

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It is known that
\[
\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = e^{e^x - 1} \quad \text{and} \quad B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \quad (n = 0, 1, 2, \ldots).
\]

The author’s conjecture (cf. [5, Conjecture 3.2]) that the sequence \((\sqrt[n]{B_{n+1}}/\sqrt[n]{B_n})_{n \geq 1}\) is strictly decreasing (with limit 1), is still open.

For any prime \(p\) and \(m, n \in \mathbb{N} = \{0, 1, 2, \ldots\}\), we have the classical Touchard congruence (cf. [7])
\[
B_{p^m n + 1} \equiv mB_n + B_{n+1} \pmod{p}.
\]

Let \(D_0 = 1\), and define \(D_n \ (n \in \mathbb{Z}^+)\) by
\[
D_n = |\{\pi \in S_n : \pi(k) \neq k \ \text{for all} \ k = 1, \ldots, n\}|.
\]
Those \(D_0, D_1, D_2, \ldots\) are called the derangement numbers, and they were first introduced by Euler. It is well known that
\[
D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \quad \text{for all} \ n \in \mathbb{N}.
\]

In 2011, the author and D. Zagier [6] showed that for any prime \(p\) and \(n \in \mathbb{Z}^+\) with \(p \nmid n\) we have
\[
\sum_{k=1}^{p-1} \frac{B_k}{(-n)^k} \equiv (-1)^{n-1}D_{n-1} \pmod{p}, \quad (1.1)
\]
which relates the Bell numbers to the derangement numbers. The surprising congruence (1.1) was called the Sun-Zagier congruence by Y. Sun, X. Wu and J. Zhuang [4] who used the umbral calculus to give a generalization, by I. Mező and T. L. Ramirez [3] in 2017 who extended it to the so-called \(r\)-Bell numbers, and by Q. Mu [2] in 2018 who re-proved via an identity of R.J. Clarke and M. Sved [1] relating the Bell numbers to the derangement numbers.

In this paper we extend the fundamental Sun-Zagier result in a new way.

**Theorem 1.1.** Let \(p\) be any prime and let \(a\) be a positive integer. For any \(n \in \mathbb{Z}^+\) with \(p \nmid n\), we have
\[
\sum_{k=1}^{p^n-1} \frac{B_k}{(-n)^k} \equiv a(-1)^{n-1}D_{n-1} \pmod{p}. \quad (1.2)
\]

**Remark 1.1.** Note that (1.2) in the case \(a = 1\) gives (1.1).
For $n \in \mathbb{Z}^+$ and $k \in \{0, \ldots, n\}$, the Stirling number $S(n, k)$ of the second kind denotes the number of ways to partition the set $\{1, \ldots, n\}$ into $k$ disjoint nonempty parts. In addition, we adopt the usual convention $S(0, 0) = 1$. For $n \geq k \geq 0$, it is well known that

$$k!S(n, k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n. \quad (1.3)$$

For any $n \in \mathbb{N}$, the Bell polynomial (or the Touchard polynomial) of degree $n$ is given by

$$B_n(x) = \sum_{k=0}^{n} S(n, k)x^k. \quad (1.4)$$

Clearly, $B_n(1) = B_n$ for all $n \in \mathbb{N}$, and $B_n(x) = x \sum_{k=1}^{n} S(n, k)x^{k-1}$ for all $n \in \mathbb{Z}^+$. Theorem 1.1 actually follows from our following theorem concerning the Bell polynomials.

**Theorem 1.2.** Let $a$ be any positive integer. For any $n \in \mathbb{Z}^+$ and prime $p \nmid n$, we have

$$(-x)^n \sum_{k=1}^{p^a-1} \frac{B_k(x)}{(-n)^k} \equiv -\sum_{r=1}^{a} x^p \sum_{k=0}^{n-1} \frac{(n-1)!}{k!} (-x)^k \pmod{p\mathbb{Z}_p[x]}, \quad (1.5)$$

where $\mathbb{Z}_p$ denotes the ring of all $p$-adic integers.

**Remark 1.2.** The congruence (1.5) in the case $a = 1$ was deduced by Sun and Zagier [6] via the usual explicit formula (1.3) for Stirling numbers of the second kind. Our Theorem 1.2 can be further extended in the spirit of [4, 3], we omit the details.

We will show Theorem 1.2 in the next section.

## 2. Proof of Theorem 1.2

**Lemma 2.1.** Let $p$ be a prime and let $a \in \mathbb{Z}^+$.

(i) For any $j, k \in \mathbb{N}$ with $j + k \leq p^a - 1$, we have

$$\binom{p^a - 1 - k}{j} \equiv \binom{-1 - k}{j} \equiv 1 \pmod{p}.$$ 

In particular,

$$\binom{p^a - 1}{j} \equiv (-1)^j \pmod{p} \quad \text{for all } j = 0, \ldots, p^a - 1.$$
(ii) We have

$$B_{p^n}(x) \equiv \sum_{r=0}^{a} x^{p^r} \pmod{p\mathbb{Z}_p[x]}.$$  

Proof. (i) Since $j + k < p^a$ we have

$$\left(\frac{p^a - 1 - k}{j}\right) = \prod_{0<i\leq j} \left(\frac{p^a - i - k}{i}\right) = \prod_{0<i\leq j} \left(1 - \frac{p^a}{i + k}\right) \equiv 1 \pmod{p}.$$  

When $k = 0$, this yields

$$\left(\frac{p^a - 1}{j}\right) \equiv (-1)^j \pmod{p}.$$  

(ii) By A. Gertsch and A. M. Robert’s extension [2] of Touchard’s congruence, for any $n \in \mathbb{N}$ we have

$$B_{p^n+n}(x) \equiv B_{n+1}(x) + B_n(x) \sum_{r=1}^{a} x^{p^r} \pmod{p\mathbb{Z}_p[x]}.$$  

In particular,

$$B_{p^n}(x) \equiv B_1(x) + B_0(x) \sum_{r=1}^{a} x^{p^r} = x + \sum_{r=1}^{a} x^{p^r} = \sum_{r=0}^{a} x^{p^r} \pmod{p\mathbb{Z}_p[x]}.$$  

In view of the above, we have completed the proof of Lemma 2.1. □

Proof of Theorem 1.2. It is known that

$$B_{m+1}(x) = x \sum_{k=0}^{m} \binom{m}{k} B_k(x) \quad \text{for all } m \in \mathbb{N}. \quad (2.1)$$

In light of this and Lemma 2.1, for any prime $p$ we have

$$\sum_{k=1}^{p^a-1} (-1)^k B_k(x) \equiv \sum_{k=1}^{p^a-1} \binom{p^a - 1}{k} B_k(x) = \frac{B_{p^n}(x)}{x} - B_0(x)$$

$$\equiv \sum_{r=1}^{a} x^{p^r-1} \pmod{p\mathbb{Z}_p[x]}.$$  

So the desired result holds when $n = 1$.

Now we fix $n \in \mathbb{Z}^+$ and assume that (1.5) holds for every prime $p \mid n$.  

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Let $p$ be any prime not dividing $n+1$. If $p \mid n$, then $n!/k! \equiv 0 \pmod{p}$ for all $k = 0, \ldots, n-1$, and hence

$$( -x )^{n+1} \sum_{k=1}^{p^a-1} \frac{B_k(x)}{(-n-1)^k} \equiv ( -x )^{n+1} \sum_{k=1}^{p^a-1} \frac{B_k(x)}{(-1)^k} \equiv ( -x )^{n+1} \sum_{r=1}^{a} x^{p^r-1}$$

$$\equiv - \sum_{r=1}^{a} x^{p^r} \sum_{k=0}^{n} \frac{n!}{k!} ( -x )^k \pmod{p\mathbb{Z}_p[x]}.$$ 

Now we suppose that $p \nmid n$. In view of (2.1) and Lemma 2.1, we have

$$\sum_{k=1}^{p^a-1} \frac{B_k(x)}{(-n)^k} = \sum_{l=0}^{p^a-2} \frac{B_l(x)}{(-n)^l} \sum_{r=1}^{p^a-l-1} \frac{(-1)^{r-1}}{(r-1)!}$$

$$= \sum_{l=0}^{p^a-2} \frac{B_l(x)}{(-n)^l+1} \sum_{r=1}^{p^a-l} \frac{(-1)^{r-1}}{n^{r-1}}$$

$$\equiv \sum_{l=0}^{p^a-2} \frac{B_l(x)}{(-n)^l+1} \sum_{r=1}^{p^a-l-1} \left( \frac{1}{1/n} \right)^{p^{a-1}-l} - \frac{1}{n^{p^a-1-l}}$$

$$\equiv \sum_{l=0}^{p^a-1} \frac{n^l B_l(x)}{(-n)^{l+1}} \left( 1 - \frac{1}{(n+1)^l} \right) \pmod{p}$$

with the aid of Fermat’s little theorem. Therefore

$$-n \sum_{k=1}^{p^a-1} \frac{B_k(x)}{(-n)^k} \equiv \sum_{k=1}^{p^a-1} \frac{B_k(x)}{(-n-1)^k} - \sum_{l=1}^{p^a-1} \frac{B_l(x)}{(-1)^l} \pmod{p\mathbb{Z}_p[x]}$$

and hence

$$\sum_{k=1}^{p^a-1} \frac{B_k(x)}{(-n-1)^k} \equiv -n \sum_{k=1}^{p^a-1} \frac{B_k(x)}{(-n)^k} + \sum_{r=1}^{a} x^{p^r-1} \pmod{p\mathbb{Z}_p[x]}.$$
Combining this with (1.5), we obtain
\[ (-x)^{n+1} \sum_{k=1}^{p^a-1} \frac{B_k(x)}{(-n-1)^k} \equiv -n \sum_{r=1}^{a} x^{p^r} \sum_{k=0}^{n-1} \frac{(n-1)!}{k!} (-x)^{k} + (-x)^{n+1} \sum_{r=1}^{a} x^{p^r-1} \]
\[ = - \sum_{r=1}^{a} x^{p^r} \sum_{k=0}^{n-1} \frac{n!}{k!} (-x)^{k} \pmod{p\mathbb{Z}_p[x]}. \]
This concludes the induction step.
By the above, the proof of Theorem 1.1 is now complete. \( \square \)

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