Path integrals, SUSY QM and the Atiyah-Singer index theorem for twisted Dirac
Dana Fine
Stephen Sawin

Abstract
Feynman’s time-slicing construction approximates the path integral by a product, determined by a partition of a finite time interval, of approximate propagators. This paper formulates general conditions to impose on a short-time approximation to the propagator in a general class of imaginary-time quantum mechanics on a Riemannian manifold which ensure these products converge. The limit defines a path integral which agrees pointwise with the heat kernel for a generalized Laplacian. The result is a rigorous construction of the propagator for supersymmetric quantum mechanics, with potential, as a path integral. Further, the class of Laplacians includes the square of the twisted Dirac operator, which corresponds to an extension of $N=1/2$ supersymmetric quantum mechanics. General results on the rate of convergence of the approximate path integrals suffice in this case to derive the local version of the Atiyah-Singer index theorem.

Introduction
This paper’s primary goal is to construct imaginary-time path integrals for a class of theories which includes ordinary quantum mechanics and what might be called twisted $N=1/2$ supersymmetric quantum mechanics (whose precise definition appears in Sect. 3) on a Riemannian manifold. The heuristic formulation of such path integrals suggests they should represent the propagator, i.e. the kernel of the time-evolution operator, which in the imaginary-time formulation is the heat operator of the given Laplacian. Further, the steepest-descent approximation should give asymptotics for the heat kernel. Indeed, these two heuristic properties form the basis for path integral “proofs” of index theorems. Therefore this paper constructs the path integral and then goes on to prove it agrees pointwise with the kernel of the heat operator and to give an asymptotic approximation in appropriate circumstances. This ensures the construction yields as a by-product new proofs of index theorems, including the local version of the Atiyah-Singer index theorem for the twisted Dirac operator. The resulting proof of the latter is arguably the closest to the heuristic path-integral argument which Witten [Wit82a, Wit82b] suggests and Alvarez-Gaumé and Friedan and Windey [AG83, FW84] implement.

The present approach, which is a rigorous realization of Feynman’s time-slicing interpretation of the path integral, gives approximate propagators indexed by partitions of a fixed time interval. These approximate propagators can be interpreted as defined by an integral over a finite-dimensional approximation to the space of paths of a discretized version of the action, which in imaginary time is the energy. That is, the approximate propagator is a time-slicing approximation to the path integral. The main work of the first two sections is to prove that these approximations converge as the partitions get finer. More importantly, the convergence must be sufficiently uniform in the parameters defining the theory that steepest descent computes the asymptotics of the propagator. The final section checks the earlier convergence results suffice to obtain the asymptotics of the component of twisted $N=1/2$ supersymmetric quantum mechanics that imply the local index theorem.

This paper is closely related to the authors’ previous work [FS08] constructing the path integral form of the propagator for imaginary-time $N=1$ supersymmetric quantum mechanics and using it to prove the Gauss-Bonnet-Chern theorem. The current work gen-
eralizes the earlier work in two ways. First, it constructs propagators for a much larger class of theories (including the $N = 1$ version as a special case, but also including bosonic quantum mechanics and indeed all theories with elliptic Hamiltonians). Second, to prove Gauss-Bonnet-Chern we needed only the lowest order term in the asymptotics, whereas the current work needs higher order in the asymptotics and therefore requires a delicate interchange of the small-parameter and fine-partition limits. Despite this, the construction below is somewhat simpler and we hope more natural than in the earlier work.

The authors’ above-cited paper discusses the distinctions and relationships between this approach and other work inspired by the path integral heuristics including that of Bismut [Bis84a, Bis84b], Getzler [Get86b, Get86a], Rogers [Rog87, Rog92a, Rog92b] and Andersson and Driver [AD99].

Technical Introduction

Secs. 1 and 2 of this paper use Feynman’s time-slicing approach to construct the path integral representing the propagator for imaginary-time quantum mechanics on a vector bundle $\mathcal{V}$ over a compact (or merely “tame”) manifold $M$ with elliptic quantum Hamiltonian $\Delta$. Time slicing starts with an approximate propagator coming from a discretization of the action and associates a product of these kernels to each partition of a given interval of time $[0, t]$. This leads to a path integral expression for the exact propagator as a fine-partition limit. The reference [FS08] gives a detailed account of the relation between the time-slicing approach to defining the path integral and the refinement limit of a product of approximate kernels, with particular attention to the case of $N = 1$ supersymmetric quantum mechanics on a Riemannian manifold $M$. For a look at how this works in a simple case, consider a (bosonic) Lagrangian $L(\sigma, \dot{\sigma}, s)$ depending on a path $\sigma : [0, t] \to M$, with parameter $s$ and tangent $\dot{\sigma}$. Heuristically, the kernel of the time-evolution operator $e^{-t\Delta/2}$ may be written as the path integral (with imaginary time and $\hbar = 1$ units)

\[
\int e^{-\int_0^t L \, ds} \, d\sigma,
\]

where the integral is over paths with $\sigma(0) = y$ and $\sigma(t) = x$. Note the endpoint conditions and the explicit $t$-dependence mean the path integral is a function on $M \times M \times \mathbb{R}$, as is the propagator. In the imaginary-time formulation, the time-evolution operator is in fact the heat operator associated with the Hamiltonian $\Delta$, which is a generalized Laplacian.

The idea of time-slicing is to partition $[0, t]$ into subintervals of length $t_i$ for $i = 1, 2 \ldots n$, and to write the path integral as a product of $n$ such integrals. Then, in each of these path integrals, replace the integral of $L$ over the subinterval of length $t_i$ with an approximation $\tilde{L}(y_i, y_{i-1}; t_i) t_i$, where the $y$’s are the endpoint values of $\sigma$ on that subinterval. Heuristically, a requirement on this approximation is that $\sum_i \tilde{L}(y_i, y_{i-1}; t_i) t_i$ be a Riemann sum converging under refinement to $\int_0^t L \, ds$. This leads to an approximate heat kernel $K(x, y; t) = (2\pi t)^{-m/2} e^{-\tilde{L}(x, y; t) t}$, and a well-defined approximate path integral which is the kernel product of $n$ copies of $K$. The Riemann sum requirement suggests that if $t$ itself is small enough, the trivial partition should suffice; hence, $K$ must be close to the actual heat kernel when $t$ is small. If $K$ has the semigroup property, then in fact the approximation is independent of the choice of partition, and the convergence of the approximate path integral is trivial.

One obvious choice for $\tilde{L}$ is to ask that $\tilde{L}(x, y; t) = \int_0^t L(\sigma_{cl}, \dot{\sigma}_{cl}; s) \, ds$ where $\sigma_{cl}$ is the path obeying the classical equations of motion subject to $\sigma_{cl}(0) = y$ and $\sigma_{cl}(t) = x$. Surprisingly, this does not lead to a limit with the desired Hamiltonian; correction terms, which may be thought of as resolving operator-ordering ambiguity, must be added. (In physical units, these corrections enter at higher powers of $\hbar$.)
Sec. 1 spells out, in a general setting, an appropriate sense of $K$ being almost the heat kernel for a given choice of $\Delta$, and provides the needed estimates. In particular Def. 4 spells out how close $K$ must be to satisfying the semigroup property to ensure the kernel products defining the path integrals converge, and Def 5 says how close $K$ must be to the true heat kernel of a given $\Delta$ to ensure the limit is the heat kernel. Sec. 2 proves the existence of the fine-partition limit for such $K$, as well as properties of the limiting kernel, and precise results on the convergence. Sec. 3 associates a quantum mechanical system to each generalized Laplacian, by relating a given action to a path integral construction for the corresponding propagator. In particular, it gives the propagator for twisted $N = 1/2$ supersymmetric quantum mechanics. Sec. 4 treats the asymptotic behavior of this propagator, which requires using results from Sec. 2 to interchange the asymptotic and fine-partition limits. The result agrees with the heuristic steepest descent treatment of the path integral.

1 Approximate heat kernels

1.1 Kernels, $*$-products, and local coordinate bounds

The heuristic time-slicing interpretation of the path integral suggests, as above, the approximation to the heat kernel need only get the short-time and near-diagonal (on $M \times M$) behavior right. This suggests formulating the requirements on an approximation locally.

Accordingly, let $O$ be an open contractible subset of $\mathbb{R}^m$, and let $g_{ij}(x)$ be a smooth Riemannian metric on $O$. Require that all derivatives of order $k$ of $g$ and of $g^{-1}$ are bounded in supremum norm for $0 \leq k \leq 5$. This will ultimately ensure given approximations to the short-time behavior have the desired convergence properties.

Let $d(x, y)$ be the distance between $x, y \in O$ in this metric. For $v \in \mathbb{R}^m$, $x \in O$ and $t \in \mathbb{R}$ the geodesic through $x$ with tangent $v$ at $x$ with parameter $t$ proportional to arc length defines the exponential map $\exp_x tv$. If $y \in O$ is close enough to $x$ that there is a unique minimal geodesic connecting them, define $y_x = \exp^{-1}_x y$. Let $(\cdot, \cdot)_x$ denote the inner product with respect to $g$ at $x \in O$, and let $|\cdot|_x$ denote the corresponding norm. If the vectors inside are of the form $y_x$ or the point at which the norm or inner product is computed is otherwise understood from context, drop the subscript. Write $d_{y}g = \det_{y}^{1/2}(g)dy$, where $dy$ is standard Lebesgue measure on $\mathbb{R}^m$ restricted to $O$, and write $dy_x$ for Lebesgue measure on $O$ with respect to the inner product given by $g$ at $x$; that is, the metric measure at $x$ pulled back to $y$ by $\exp_{y}^{-1}$.

Henceforth to say that a quantity, such as $D$ in the following lemma, “depends on the metric bounds” will mean that quantity is a function of the assumed bounds on the supremum norm of $g, g^{-1}$ and their first five derivatives (as well as on the dimension $m$). The concern is that, in later arguments which require rescaling the metric, preserving these bounds should be sufficient to preserve the estimates which follow here.

**Lemma 1.1** There is a $D > 0$ depending on the metric bounds such that, for $x, y, z \in O$ with $d(x, y), d(y, z), d(x, z) < D$ there is a unique minimal geodesic connecting $x$ and $y$, $y_x$ depends smoothly on $x$ and $y$, and $y - x$ depends smoothly on $x$ and on $y_x$. Moreover,

$$y - x = y_x + \mathcal{O}(|y_x|^2)$$  \hspace{1cm} (1.1)

$$|z_x|^2 = |z_y|^2 + |x_y|^2 - 2(x_y, z_y) + \mathcal{O}(|x_y|^2 |z_y|^2)$$  \hspace{1cm} (1.2)

$$\frac{dy_y}{dy_x} = 1 + \mathcal{O}(|y_x|^2)$$  \hspace{1cm} (1.3)
where for example $O(\|x\|^2 \|z\|^2)$ indicates the difference between the left-hand side and the truncated Taylor series is bounded by a constant (depending on the metric bounds) times $\|x\|^2 \|z\|^2$ (as each of these tends towards zero).

**Proof:** As expressed in local coordinates, the components of the Riemann curvature are continuous functions of the first two derivatives of the metric. By assumption, then, the Riemannian and hence sectional curvatures are bounded above, so, by Rauch’s comparison theorem [DC92], the injectivity radius is bounded below (contractibility means the injectivity radius is the minimum distance of a point from its nearest conjugate point). Within the injectivity radius the exponential map $\exp_x$ at each $x$ is defined by the differential equation in local coordinates, writing $\sigma^\mu(t)$ for the $\mu$th component of $\exp_x(tv)$,

$$\frac{d^2\sigma^\mu}{dt^2} + \Gamma^\mu_{\nu\rho} \frac{d\sigma^\nu}{dt} \frac{d\sigma^\rho}{dt} = 0.$$ 

Since the Christoffel symbols $\Gamma^\nu_{\mu\rho}$ are continuous in the first derivatives of the metric, the coefficients of the differential equation have bounded derivatives up to degree four. Standard existence and uniqueness results [Arn98] ensure the solution with the given initial conditions is $C^4$, but a careful reading of the argument shows that the first four derivatives are in fact bounded in terms of the metric bounds. Further, with $t = 1$, $\exp_x v$ has its first four derivatives with respect to both $x$ and $v$ bounded in terms of the metric bounds. (One normally thinks of $\exp$ as a map from the tangent space to the manifold, but in this case each of these is a subset of $\mathbb{R}^n$, so $\exp$ refers to the endomorphism on $\mathbb{R}^n$). Since the injectivity radius is bounded below by the metric bounds, there is a radius $D$ bounded below by the metric bounds such that $\exp_x^{-1}$ has its first four derivatives bounded in terms of the metric bounds on a circle of radius $D$ around $x$.

This means that if $d(x, y) < D$ then $y_x = \exp_x^{-1} y$ as a function of $x$ and $y$ has its first four derivatives bounded in terms of the metric bounds and $y = \exp_x y_x$ and hence $y - x$ as functions of $x$ and $y_x$ have their first four derivatives bounded in terms of the metric bounds.

For Eq. (1.1) the zeroth and first order terms of the Taylor series for $y = \exp_x y_x$ as a function of $x$ are set by the initial conditions of $\exp$, and the second order error term is bounded by the supremum of the second derivative of $\exp$, which is bounded in terms of the metric bounds.

For Eq. (1.2), fixing $y$, notice that $|z_x| = d(x, z) = d(\exp_x x_y, \exp_x z_y)$ and all its first four derivatives in $x_y$ and $z_y$ are bounded in terms of the metric bounds. With Gauss’ Lemma, the Taylor series of $d(x, z)^2$ as a function of $x_y$ is

$$d(x, z)^2 = |z_y|^2 - 2(x_y, z_y) + x_y^2 \frac{\partial^2}{\partial x_y^2} d(x', z)^2$$

the last term on the right-hand side is an abbreviation for a linear combination of quadratic functions of $x_y$ involving second partial derivatives with respect to the components of $x_y$, each evaluated at some point $x'$ on the geodesic from $y$ to $x$. (The point $x'$ will in general be different for each of the derivatives appearing in the linear combination.)

Expanding this last term term as a Taylor series in $z_y$ yields

$$d(x, z)^2 = |x_y|^2 + |z_y|^2 - 2(x_y, z_y) + x_y^2 z_y^2 \frac{\partial^2}{\partial x_y^2} \frac{\partial^2}{\partial z_y^2} d(x', z')^2$$

where $z'$ is on the geodesic between $y$ and $z$. The last term on the right-hand side here, being a fourth derivative of the exponential map, is bounded in terms of the metric bounds.
For Eq. (1.3), note that the Taylor series centered at \( x \) for the components of the metric at \( y \), expressed in the coordinates mapping \( y \) to \( y_x \), will have no term linear in \( y_x \); the quadratic term has coefficients given by second derivatives of the metric at \( x \). \cite{BGV04}. Eq. (1.3) follows by direct calculation, with the implied constants depending on the bounds of the metric. \( \square \)

Define
\[
\chi_{<D}(x, y) = \begin{cases} 
1 & \text{if } d(x, y) < D \\
0 & \text{else,}
\end{cases}
\]
and \( \chi_{>D}(x, y) = 1 - \chi_{<D}(x, y) \).

For \( x, y \in O, \ t > 0, \) and \( D > 0 \) small enough that Lemma 1.1 holds, define
\[
H_D(x, y; t) = \chi_{<D}(x, y)(2\pi t)^{-m/2}e^{-|y_x|^2/(2t)}.
\]

(1.4)

Given \( n \in \mathbb{N} \), let \( f : O \rightarrow \mathbb{R}^n, f^* : O \rightarrow (\mathbb{R}^n)^* \) and \( K : O \times O \rightarrow \text{Matrix}_{n,n}. \) \( K \) represents kernels of left or right operators on the space of functions from \( O \) to \( \mathbb{R}^n \) or \((\mathbb{R}^n)^*\) whose actions are given by
\[
K * f(x) = \int_O K(x, y) \cdot f(y) d_g y
\]
\[
f^* * K(y) = \int_O f^*(x) \cdot K(x, y) d_g x
\]

(1.5)

where \( \cdot \) represents the matrix product. The kernel of the operator product of the operators represented by \( K \) and \( J \) is the \(*\)-product
\[
J * K(x, z) = \int_O J(x, y) \cdot K(y, z) d_g y.
\]

(1.6)

The matrix norm sends \( K \) to a nonnegative function \(|K|\) on \( O \times O \). Use this to define
\[
\|K\|_{\text{op}} = \max\left(\sup_x \int |K(x, y)| d_g y, \sup_y \int |K(x, y)| d_g x\right),
\]
which is the max of the operator norms of \( K \) acting on the left and the right. Define the kernel norm by
\[
\|K\|_{\text{ker}} = \max(\|K\|_{\text{op}}, \|K\|_{\infty}).
\]

Notice \(|J * K|_{\text{ker}} \leq \|J\|_{\text{ker}} \|K\|_{\text{ker}}\) and \(|J \ast K |_{\text{ker}} \leq \|J\|_{\text{op}} \|K\|_{\text{ker}}\).

Notice \( H_D \) of Eq. (1.4) agrees for \( d(x, y) < D \) with the flat-space heat kernel when the metric is flat. The next two lemmas explore classes of kernels whose relation to \( H_D \) are increasingly tenuous, to delineate the extent to which they retain key properties of the heat kernel under kernel products. The purpose of this exploration, which culminates in Prop. 1.1, is to determine the key properties of a time-slicing approximation that ensure the approximate path integrals converge with sufficient rapidity to the heat kernel of a given Laplace-like operator.

**Lemma 1.2** If \( B \) is large enough, \( D \) is small enough, and \( t \) is small enough (each depending on the bounds of the metric and the previous quantities); and if
\[
K_{B,D}(x, y; t) = e^{B|y_x|^2/(2m)}H_D(x, y; t).
\]

(1.7)
$$0 < t_1, t_2,$$ and \( t = t_1 + t_2; \) then

$$\chi_{<D}[K_{B,D}(t_1) * K_{B,D}(t_2)] \leq e^{Bt} t_2/t K_{B,D}(t),$$

$$\|\chi_{>D}[K_{B,D}(t_1) * K_{B,D}(t_2)]\|_{op} \leq t^2 e^{-D^2/2t}, \tag{1.8}$$

and

$$|K_{B,D}(t)|_{op} \leq e^{Bt}. \tag{1.9}$$

**Proof:** For the first line of Eq. (1.8), let \( u = \exp \left( \frac{t}{b} z \right) \), so \( u_x = t_1 z_x/t \) and \( u_z = t_2 z_z/t \). These imply \( t_2 x_u + t_1 z_u = 0 \). Let \( b = B/(5m) \) and let \( c > 0 \) be such that Eqs. (1.2) and (1.3) become

$$\| y_x \|^2 - (|y_u|^2 + |x_u|^2 - 2(y_u, x_u)) \leq c |y_u|^2 |x_u|^2,$$

$$\| y_z \|^2 - (|y_v|^2 + |z_v|^2 - 2(y_v, z_v)) \leq c |y_v|^2 |z_v|^2,$$

$$d_{y} y \leq [1 + c |y_u|^2] dy_u.$$

Then

$$\chi_{<D}(x, z)[K_{B,D}(t_1) * K_{B,D}(t_2)](x, z) = \int \chi_{<D}(x, z) K_{B,D}(x, y; t_1) K_{B,D}(y, z; t_2) d_{y} y$$

$$\leq H_D(x, z; t) e^{(t_1^2 + t_2^2)/2} (2\pi t_1 t_2)^{-m/2} \int \chi_{<D}(x, y) \chi_{<D}(y, z)$$

$$\exp \left( \frac{t |y_u|^2}{2 t_1 t_2} (1 - c t_1 t_2 |z_v|^2 / t - 2 c t_1 t_2 / t - 2 c t_1 t_2 |z_v|^2 / t - 2 c t_1 t_2 / t - 2 c t_1 t_2 |z_v|^2 / t - 2 c t_1 t_2 / t) \right) dy_u$$

where the last integral is over all vectors \( y_u \) which are taken by \( \exp_y \) to some \( y \) within a distance \( D \) of \( x \) and \( z \). Because the integrand is positive the inequality still holds if the integral is extended over all of \( \mathbb{R}^n \). Noting that \( e^y \leq x + e^{y^2} \) and that the integral of a Gaussian times a linear function is 0,

$$\chi_{<D}(x, z)[K_{B,D}(t_1) * K_{B,D}(t_2)](x, z) \leq H_D(x, z; t) e^{(t_1^2 + t_2^2)/2} (2\pi t_1 t_2)^{-m/2} e^{-t |y_u|^2 / (2 t_1 t_2)} dy_u$$

where

$$a = c t_1 t_2 |z_v|^2 / t^2 + 2 c t_1 t_2 / t + 4 b c t_1 t_2 / t + 2 b (c + 4 b) t_1 t_2 |z_v|^2 / t$$

$$= (c + 4 b) t_1 t_2 / t + (c + 2 b (c + 4 b) t_1 t_2 |z_v|^2 / t^2 \leq B t_1 t_2 / (m t) + 2 b t_1 t_2 |z_v|^2 / (m t^2)$$

if \( B \) is chosen large enough and \( T \) is chosen small enough. If \( t \) and \( D \) are small enough (depending on \( B \)) then \( a \leq 1/2 \), so \((1 - a)^{-m/2} < e^{m a} \) and hence the Gaussian integral yields

$$\chi_{<D}(x, z)[K_{B,D}(t_1) * K_{B,D}(t_2)](x, z) \leq H_D(x, z) e^{b |x_v|^2 + B t_1 t_2 / t}.$$

Deferring the proof of the second line of Eq. (1.8) for a moment, consider first Eq. (1.9): Defining \( b \) and \( c \) as above

$$|K_{B,D}(t) * f(x)| \leq \int H_D(x, y; t) e^{b |y_v|^2} [f(y)] d_{y} y$$

$$\leq \| f \|_{\infty} \int \chi_{<D}(x, y)(2 \pi t)^{-m/2} e^{-t |y_u|^2 / (2 t_1 t_2)} dy_u.$$
Again extending the integral, choosing $t$ small enough to bound the quantity in braces, and completing the Gaussian integral yields

$$|K_{B,D}(t) \ast f(x)| \leq \|f\| \exp^{2n(t+h)c_t} \leq e^{B t} \|f\|$$

if $B$ is chosen large enough.

Finally, for the second line of Eq. (1.8), if $d(x,z) > D$ then any $y \in O$ satisfies either $d(x,y) > D/2$ or $d(y,z) > D/2$, so

$$\chi_{>D}(x,z) [K_{B,D}(t_1) \ast K_{B,D}(t_2)] (x,z) \leq \int \chi_{>D/2}(x,y) K_{B,D}(x,y,t_1) K_{B,D}(y,z,t_2) d_y y$$

$$+ \int \chi_{>D/2}(y,z) K_{B,D}(x,y,t_1) K_{B,D}(y,z,t_2) d_y y.$$ 

But if $d(x,y) > D/2$, then $K_{B,D}(x,y,t_1) \leq (1/2)t_1^2 e^{-D^2/9t_1}$ if $t$ is small enough, and, by Eq. (1.9), $\|K_{B,D}(t_2)\| \leq 2$ for $T$ small enough. Thus

$$\int \chi_{>D/2}(x,y) K_{B,D}(x,y,t_1) K_{B,D}(y,z,t_2) d_y y \leq t_1^2 e^{-D^2/9t_1},$$

and therefore

$$\|\chi_{>D} [K_{B,D}(t_1) \ast K_{B,D}(t_2)] (x,z)\| \leq t_1^2 e^{-D^2/9t_1} + t_2^2 e^{-D^2/9t_2} \leq t^2 e^{-D^2/9t}$$

by the convexity of $e^{-D^2/(9t)}$, all for small enough $t$ (depending on $D$).

If $D$ is chosen small enough, the volume of the ball of radius $2D$ around any point is less than 1 (based on the bound on the second derivative of $g$) so

$$\|\chi_{>D} [K_{B,D}(t_1) \ast K_{B,D}(t_2)] (x,z)\| \leq t^2 e^{-D^2/9t}.$$

$\square$

### 1.2 Two families of kernels and the $t$-norm

**Definition 1** For $B, D, t > 0$ define $\mathcal{E}_{B,D}(t)$ to be the set of all kernels $K$ for which there exists a probability measure $d\mu$ on the interval $[1, 2]$ such that

$$|K(x,y)| \leq e^{B \sqrt{T}} \int K_{B,D}(x,y; \alpha t) d\mu_\alpha,$$

(1.10)

where $K_{B,D}$ is the particular one-parameter family of kernels defined in Eq. (1.7).

Note that $K_{B,D}(t)$ itself is in $\mathcal{E}_{B,D}(t)$. The following lemma extends the previous one to say that $\mathcal{E}_{B,D}(t)$ is almost closed under the $\ast$ product, and made up of almost contraction maps. The “almost” here refers in both cases to the exponential $\sqrt{T}$ factor, and in the first to an exponentially damped term far from the diagonal. Precisely,

**Lemma 1.3** If $B$ is large enough, $D$ is small enough, and $T$ is small enough (each depending on the bounds of the metric and the previous quantities) and if $K_1$ and $K_2$ are one-parameter families of kernels with $K_1(t), K_2(t) \in \mathcal{E}_{B,D}(t)$ for $t < T$, then, for $0 < t_1, t_2$ and $t = t_1 + t_2 < T$

$$\|K_1(t)\| \leq e^{1.1 \sqrt{B \sqrt{T}}},$$

(1.11)

and

$$\chi_{<D} K_1(t_1) \ast K_2(t_2) \in e^{B \sqrt{T_1 t_2}} \mathcal{E}_{B,D}(t)$$

$$|\chi_{<D} K_1(t_1) \ast K_2(t_2)| \leq t^2 e^{-D^2/(20t)}.$$
Proof: For \( i = 1, 2 \), choose measures \( d\mu_{i,\alpha} \) on \([1, 2]\) such that \( |K_i(t)| \leq e^{B\sqrt{T}} \int_1^2 K_{B,D}(at)d\mu_{i,\alpha} \). For Eq. (1.11),
\[
|K_1(t)|_{op} \leq e^{B\sqrt{T}} \int_1^2 K_{B,D}(at)d\mu_{1,\alpha} \bigg|_{op} \\
\leq e^{B\sqrt{T}} \int_1^2 e^{Bot}d\mu_{1,\alpha} \leq e^{B\sqrt{T}+2Bt} \leq e^{1.1B\sqrt{T}},
\]
using Eq. (1.9), and assuming \( t \) is small enough for the final inequality to hold. For the first line of Eq. (1.12), use the first line of Eq. (1.8) to get
\[
|\chi < D[K_1(t_1) * K_2(t_2)]| \leq e^{B\sqrt{T_1}+B\sqrt{T_2}} \int_1^2 \int_1^2 \chi < D[K_{B,D}(at_1) * K_{B,D}(\beta t_2)]d\mu_{1,\alpha}d\mu_{2,\beta} \\
\leq e^{B\sqrt{T_1}+B\sqrt{T_2}} \int_1^2 \int_1^2 e^{Bot_{12}[(\alpha t_1 + \beta t_2)]}K_{B,D}(at_1 + \beta t_2)d\mu_{1,\alpha}d\mu_{2,\beta} \\
\leq e^{B\sqrt{T_1}+B\sqrt{T_2}+2Bt_{12}/t} \int_1^2 K_{B,D}(\gamma t)d\nu_\gamma \\
\leq e^{B\sqrt{T}e^{B\sqrt{1/t_{12}/t}}} \int_1^2 K_{B,D}(\gamma t)d\nu_\gamma \in e^{B\sqrt{1/t_{12}/t}}E_{B,D}(t),
\]
if \( t \) is small enough. Here \( \gamma t = \alpha t_1 + \beta t_2 \) and \( d\nu \) is the pushforward of the product measure \( d\mu_{1}\mu_{2} \) to this subspace.

For the second line of Eq. (1.12),
\[
|\chi > D[K_1(t_1) * K_2(t_2)]|_{ker} \leq \left\| e^{B\sqrt{T_1}+B\sqrt{T_2}} \int_1^2 \int_1^2 \chi > D[K_{B,D}(at_1) * K_{B,D}(\beta t_2)]d\mu_{1,\alpha}d\mu_{2,\beta} \right\|_{ker} \\
\leq e^{2B\sqrt{T}} \int_1^2 \int_1^2 |\chi > D[K_{B,D}(at_1) * K_{B,D}(\beta t_2)]|_{ker} d\mu_{1,\alpha}d\mu_{2,\beta} \\
\leq e^{2B\sqrt{T}} \int_1^2 \int_1^2 e^{-D^2/(18\alpha)}d\mu_{1,\alpha}d\mu_{2,\beta} \\
\leq t^2 e^{-D^2/(20t)}
\]
if \( t \) is small enough. The third line here follows from Eq. (1.8).

Continue to enlarge the class of kernels which behave well under kernel products to

**Definition 2** For, \( B, D, t > 0 \) define \( E_{B,D}(t) \) to be the set of all kernels which can be written as \( K + J \) where \( K \in E_{B,D}(t) \) and \( |J|_{ker} \leq te^{-D^2/(20t)} \).

This class is also almost closed under kernel products, in a sense which the following proposition makes precise.

**Proposition 1.1** If \( B \) is large enough, \( D \) is small enough and \( T \) is small enough (each depending only on the bounds of the metric and the previous quantities) and if \( K_1 \) and \( K_2 \) are one-parameter families of kernels with \( K_1(t), K_2(t) \in E_{B,D}(t) \) for all \( t < T \), then, for \( 0 < t_1, t_2 \) and \( t = t_1 + t_2 < T \)
\[
|K_i(t)|_{op} \leq e^{2B\sqrt{T}},
\]
(1.13)
\[ |K_i(x, y; t)| \leq 2(2\pi t)^{-m/2}e^{-d(x, y)^2/(4t)} + te^{-D^2/(20t)}, \quad (1.14) \]

and
\[ K_1(t_1) * K_2(t_2) \in e^{B\sqrt{T}}E_{B, D}(t). \quad (1.15) \]

**Remark 1.1** There is a minimum \( B \) and a maximum \( D \) and \( T \) to make Prop. 1.1 hold, and these numbers depend only on the supremum of the first few derivatives of the metric and its inverse (and \( m \)), a fact that will be crucial in Sect. 4. If one chose a larger \( B \), the maximum \( D \) and \( T \) would be smaller but would still exist. If one chose an even smaller \( D \), the maximum \( T \) would be smaller still. In the definition of approximate semigroup and approximate heat kernel below, the choice of constants will also depend on the family of kernels being considered.

**Proof:** For Eq. (1.13), write \( K_i(t) = \tilde{K}_i(t) + J_i(t) \) where \( \tilde{K}_i(t) \in E_{B, D}(t) \) and \( |J_i(t)|_{\text{ker}} \leq te^{-D^2/(20t)} \). Then, using Eq. (1.11),
\[ \|K_i(t)\|_{\text{op}} \leq \|\tilde{K}_i(t)\|_{\text{op}} + |J_i(t)|_{\text{op}} \leq e^{1.1B\sqrt{T}} + e^{-D^2/1} \leq e^{2B\sqrt{T}} \]
for small enough \( t \). Eq. (1.14) follows from the definition of \( E_{B, D} \) for small enough \( D \) and \( t \).

For Eq. (1.15), use Eq. (1.12) to write \( \tilde{K}_3(t_1) \ast \tilde{K}_3(t_2) = \tilde{K}_3(t_1, t_2) + J_3(t_1, t_2) \), where \( \tilde{K}_3(t_1, t_2) \in E_{B, D}(t) \) and \( |J_3(t_1, t_2)|_{\text{ker}} \leq t^2e^{-D^2/(20t)} \). Then
\[ \|K_1(t_1) * K_2(t_2) - \tilde{K}_3(t_1, t_2)\|_{\text{ker}} \]
\[ \leq \|J_3(t_1, t_2)\|_{\text{ker}} + \|\tilde{K}_1(t_1) \ast J_2(t_2)\|_{\text{ker}} + \|J_1(t_1) \ast \tilde{K}_2(t_2)\|_{\text{ker}} + \|J_1(t_1) \ast J_2(t_2)\|_{\text{ker}} \]
\[ \leq t^2e^{-2D/(20t)} + \|\tilde{K}_1(t_1)\|_{\text{op}} + \|J_2(t_2)\|_{\text{op}} + \|J_2(t_2)\|_{\text{op}} + \|J_1(t_1)\|_{\text{op}} + \|J_2(t_2)\|_{\text{op}} \]
\[ \leq t^2e^{-2D/(20t)} + e^{1.1B\sqrt{T}/2}te^{-D^2/(20t)} + t_1e^{-D^2/(20t_1)}e^{1.1B\sqrt{T}/2} + t_1e^{-D^2/(20t_1)}te^{-D^2/(20t_2)} \]
\[ \leq \frac{5}{4}t^2e^{-2D/(20t)} + te^{1.1B\sqrt{T}/2}e^{-D^2/(20t)} \leq e^{1.1B\sqrt{T}/2}te^{-D^2/(20t)} \]
where the third inequality comes from Eq. (1.11). In the fourth, one straightforward estimate gives \( t_1e^{-D^2/(20t_1)}te^{-D^2/(20t_2)} \leq \frac{1}{4}t^2e^{-D^2/(20t)} \). Further, \( t_2e^{1.1B\sqrt{T}/2} + t_1e^{1.1B\sqrt{T}/2} \leq te^{1.1B\sqrt{T}/2} \), for small enough \( t \). The fifth is a straightforward estimate for small enough \( t \).

This proposition provides the basis on which to define a norm:

**Definition 3** For given \( B, D, t > 0 \) define the \( t \)-norm \( |K_i|_{(t)} \) to be the smallest positive real number such that \( K_i|_{(t)} \in E_{B, D}(t) \) if it exists. (Otherwise set \( |K_i|_{(t)} = \infty \).)

**Corollary 1.1** If \( B \) is large enough, \( D \) is small enough and \( t \) is small enough (each depending only on the bounds of the metric and the previous constants), then for the associated \( t \)-norm and for families of kernels \( K_1, \) and \( K_2, \)
\[ \|K_i\|_{\text{op}} \leq e^{2B\sqrt{T}}|K_i|_{(t)}, \quad (1.16) \]
\[ |K_i(x, y; t)| \leq |K_i|_{(t)} \left[ 2(2\pi t)^{-m/2}e^{-d(x, y)^2/(4t)} + te^{-D^2/(20t)} \right]; \quad (1.17) \]
in particular, there is an $A_2 > 0$ such that

$$|K_1(t)|_\infty \leq A_2 t^{-n/2} |K_1|_{(t)}.$$  \hfill (1.18)

Finally,

$$|K_1(t_1) * K_2(t_2)|_{(t)} \leq e^{B \sqrt{T}} |K_1|_{(t_1)} |K_2|_{(t_2)}.$$  \hfill (1.19)

**Proof:** Eqs. (1.16), (1.17) and (1.19) of the corollary are simply restatements of Eqs. (1.13), (1.14) and (1.15) of the proposition. Eq. (1.18) is a separately-useful immediate consequence of Eq. (1.17). \hfill $\blacksquare$

### 1.3 Approximate semigroups and approximate kernels

As noted in the introduction, the definition of approximate semigroup below will ensure that the fine-partition limit of kernel products of approximate semigroups converge. The definition of approximate heat kernel will ensure that it is an approximate semigroup and that the fine-partition limit of its kernel products is in fact the heat kernel of the associated operator.

**Definition 4** A family of kernels $K(t)$ is an approximate semigroup with constants $(B, C, D, T)$ if for $0 < t_1, t_2$ and $t = t_1 + t_2 < T$ with the $t$-norm of Def. 3

$$|K(t)|_{(t)} \leq 1$$  \hfill (1.20)

and

$$|K(t_1) * K(t_2) - K(t)|_{(t)} \leq C t^{3/2}.$$  \hfill (1.21)

**Remark 1.2** Note that Eq. (1.20) implies an approximate semigroup $K(t)$ must be in $E'(t)$ for all $t < T$. Moreover, accordingly writing $K(t) = \tilde{K}(t) + J(t)$ for $\tilde{K}(t) \in E(t)$, the following lemma says it suffices to check Eq. (1.21) only on $\tilde{K}(t)$.

**Lemma 1.4** If $K(t) = \tilde{K}(t) + J(t) \in E'(t)$ with $\tilde{K}(t) \in E(t)$ satisfying Eq. (1.21), then $K(t)$ satisfies Eq. (1.21), albeit with potentially smaller $D$, larger $C$, and smaller $T$.

**Proof:** Consider

$$(\tilde{K} + J)(t_1) * (\tilde{K} + J)(t_2) - (\tilde{K} + J)(t) =$$

$$\tilde{K}(t_1) * \tilde{K}(t_2) - \tilde{K}(t) + \tilde{K}(t_1) * J(t_2) + J(t_1) * \tilde{K}(t_2) + J(t_1) * J(t_2) - J(t).$$

By hypothesis, the first two terms on the right-hand side combine to give $C t^{3/2}$ times an element of $E'(t)$. Applying Eq (1.11) bounds each of the next two terms by $e^{1.1 B \sqrt{T} t e^{-D^2/(20t)}}$.

Replacing $D$ with $D/2$, these terms are thus each bounded by $t^{3/2} e^{-D^2/(20t)}$, for small $T$. Easy estimates give the same bound for the remaining two terms, so the sum on the right-hand side, after division by $(C + 3)t^{3/2}$ lies in $E'$. \hfill $\blacksquare$
Definition 5 Let $\Delta$ denote a second order elliptic differential operator defined on $O \in \mathbb{R}^m$ acting on functions with values in $\mathbb{R}^n$. Suppose the second-order coefficients of $\Delta$ are the inverse of the metric $g$ (i.e., $\Delta$ is a generalized Laplacian) and the lower-order coefficients are bounded in sup norm. A family of kernels $K(t)$ is an approximate heat kernel for $\Delta$ with constants $(B, C, D, T)$, all positive, if it is differentiable to first order in $t \in (0, T)$ and to second order in the spatial variables, and if, for $t < T$ and using the $t$-norm with constants $(B, D)$,

$$|K(t)|_{(t)} \leq 1,$$  \hspace{1cm} (1.22)

for all $f: O \rightarrow \mathbb{R}^n$

$$\lim_{t \to 0} K(t) * f = f,$$  \hspace{1cm} (1.23)

$$\lim_{t \to 0} \frac{K(t) * f - f}{t} = \frac{\Delta}{2} f$$  \hspace{1cm} (1.24)

(both pointwise),

$$\left\| \frac{\partial}{\partial x} K(x, y; t) \right\|_{(t)} , \left\| \frac{\partial}{\partial y} K(x, y; t) \right\|_{(t)} \leq B/t,$$  \hspace{1cm} (1.25)

and

$$\left\| \left( \frac{1}{2} \Delta_x - \frac{\partial}{\partial t} \right) K(x, y; t) \right\|_{(t)} \leq C t^{1/2}$$

$$\left\| \left( \frac{1}{2} \Delta_y^* - \frac{\partial}{\partial t} \right) K(x, y; t) \right\|_{(t)} \leq C t^{1/2},$$  \hspace{1cm} (1.26)

where $\Delta_x$ acts from the left on $\text{End}(\mathbb{R}^n)$ and $\Delta_y^*$ acts from the right via $\int_O \Delta_y^*[h^*(y)] \cdot f(y) d_y y = \int_O h^*(y) \cdot \Delta_y f(y) d_y y$.

Proposition 1.2 Suppose $K(t)$ is an approximate heat kernel for the elliptic operator $\Delta$ and metric $g$ with constants $(B, C, D, T)$. Then there exist positive constants $B_1, C_1, D_1, T_1$ (each depending on the bounds of the metric and $\Delta$, on $B, C, D, T$ and on the previous constants) such that $K$ is an approximate semigroup with constants $(B_1, C_1, D_1, T_1)$.

Proof: Make $B$ large enough, and $D$ and $T$ small enough that Prop. 1.1, Cor. 1.1 and Lemma 1.4 hold. According to Lemma 1.4, it suffices to prove Eq. (1.21) for $K(x, y; t) \in \mathcal{E}_{B, D}(t)$. For $d(x, z) \geq D/2$ Eqs. (1.19) and (1.17) imply

$$\l|K(t_1) * K(t_2)(x, z)\r| \leq 2e^{B_1 t_1/2} (2\pi t_2)^{-m/2} e^{-D^2/(16t_2)} + t^2 e^{-D^2/(20t)} \leq C_1 t^{5/2} e^{-D^2/(20t)}$$

for large enough $C_1$, and small enough $D_1$ and $t$, giving Eq. (1.21).
For \(d(x, z) \leq D/2\), the left hand side of Eq. (1.21) is

\[
\left\|K(t_1) * K(t_2) - K(t)\right\|_{(t)} \leq \int_0^{t_1} \left\|\frac{\partial}{\partial \tau} [K(\tau) * K(t - \tau)]\right\|_{(t)} d\tau
\]

\[
= \int_0^{t_1} \left\|K(\tau) * K(t - \tau) - K(\tau) * K(t - \tau)\right\|_{(t)} d\tau
\]

\[
\leq \int_0^{t_1} \frac{1}{2} \{\Delta_x[K(\tau)] * K(t - \tau) - K(\tau) * \Delta_y[K(t - \tau)]\} \right\|_{(t)} d\tau
\]

\[
+ \int_0^{t_1} C \tau \frac{1}{2} e^{Bt^{1/2}} + C(t - \tau)^{1/2} e^{Bt^{1/2}} d\tau
\]

\[
\leq \int_0^{t_1} \frac{1}{2} \int_{\partial \tau} \delta_y[K(\tau)] \cdot_y K(t - \tau) d_y d\tau + \frac{4}{3} C \tau \frac{1}{2} t^{3/2}
\]

where the third inequality uses Eqs. (1.26) and (1.22) of the definition of an approximate heat kernel and Eq. (1.19) of Prop. 1.1, and the subscript \(y\) indicates the operators act on the fiber over the middle copy of \(\mathbb{R}^n\) (the one that \(*\) contracts over). The first term of the last equation, call it \(\|J(x, z; t)\|_{(t)}\), is the boundary term obtained using the formal adjoint of \(\Delta\). That is, \(\delta\) is the first order operator for which

\[
\int_R (f \cdot \Delta h - \Delta^* f \cdot h) d_y y = \int_{\partial R} \delta f \cdot h d_y y,
\]

and the subscript means the integral is over \(y\) such that one of \(d(x, y)\) and \(d(y, z)\) is equal to \(D\) and the other less. So, since \(d(x, z) \leq D/2\), both \(d(x, y) \geq D/2\) and \(d(y, z) \geq D/2\).

The boundary integral (notice it has finite volume with a bound depending on \(D\) and bounds on the first two derivatives of the metric) can thus be bounded by a multiple of \(e^{-D^2/(20t)} e^{-D^2/(20t)} |P_1(t_1) P_2(t_1)|\) where \(P_i\) are polynomials (uses Eq. (1.25)). Therefore \(\|J(t)\|_{\infty} \leq c t^{5/2} e^{-D^2/t}\) for some \(c, d\).

For fixed \(x\) the set of \(z\) for which \(J(x, z; t)\) is nonzero is a ball of radius \(D/2\) (and likewise for \(x\) and \(z\) reversed) which has bounded volume (depending on \(D\) and the bounds of the first two derivatives of the metric), so also \(\|J(t)\|_{op} = c' t^{5/2} e^{-D^2/t}\) and therefore

\[
\|J(t)\|_{(t)} \leq C_1 t^{3/2}
\]

with the appropriate constants in the definition of the \(t\)-norm. Eq. (1.21) follows.

\[\square\]

1.4 Manifolds

**Definition 6** Suppose \(\mathcal{V}\) is an \(n\)-dimensional vector bundle over an \(m\)-dimensional manifold \(M\) with Riemannian metric \(g\). An atlas of charts for \(\mathcal{V}\) over \(M\) is tame if

- All derivatives of \(g\) and \(g^{-1}\) expressed in coordinates of order \(0 \leq k \leq 6\) are uniformly bounded in sup norm on all charts.
- There is a \(D_0 > 0\) such that the ball of radius \(D_0\) around any point is contained in a single chart.

The tuple \((M, g, \mathcal{V})\) is tame if it admits a tame atlas. If \(\Delta\) is a generalized Laplacian, i.e. a second-order elliptic operator on sections of \(\mathcal{V}\) which in local coordinates is of the form

\[
\Delta = g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + A^i \frac{\partial}{\partial x_i} + B
\]
(with $A^i$ and $B$ valued in $\text{Matrix}_{m,n}$), and if there is a tame atlas so that the derivatives of order $0 \leq k \leq 2$ of $A^i$ and $B$ in all charts are uniformly bounded in sup norm, then $(M, g, \mathcal{V}, \Delta)$ is tame.

Of course any such data is tame if $M$ is compact and everything is smooth.

Let $\pi_i : M \times M \to M$ be the projection onto the $i$th copy of $M$, and consider the bundle $\text{Hom}_{\pi^2}$ of homomorphisms from $\pi_2^*\mathcal{V}$ to $\pi_1^*\mathcal{V}$. Its fiber over $(x, y)$ is $\text{Hom}(\mathcal{V}_y, \mathcal{V}_x)$. Call a section $K(x, y)$ of $\text{Hom}_{\pi^2}$ a kernel on $\mathcal{V}$. $K$ is then a kernel in the sense of the Subsection 1.1 on any chart for $\mathcal{V}$ (where $m$ is the dimension of $M$ and $n$ the dimension of $\mathcal{V}$). On any tame atlas, for sufficiently large $B$ and sufficiently small $D$, there is a sufficiently small $t$ such that the $t$-norm with constants $(B, D)$ can be defined on each chart, and thus it makes sense to define $\|K\|_t$ to be the supremum of the $t$-norms of its image in each chart.

**Corollary 1.2** If $(M, g, \mathcal{V})$ is tame the $t$-norm defined in terms of any tame atlas will satisfy Eqs. (1.16)-(1.19) for sufficiently large $B$ and sufficiently small $D$.

**Definition 7** A family of kernels $K(t)$ on $\mathcal{V}$ for $t > 0$ is an approximate semigroup with constants $(B, C, D, T)$ if $(M, g, \mathcal{V})$ admits a tame atlas on each chart of which $K$ is represented as an approximate semigroup with constants $(B, C, D, T)$, with $D \leq D_0$ above. A family of kernels $K(t)$ on $\mathcal{V}$ is an approximate heat kernel with constants $(B, C, D, T)$ if $(M, g, \mathcal{V})$ admits a tame atlas on each chart of which $K$ is represented as an approximate heat kernel with constants $(B, C, D, T)$ with $D \leq D_0$.

**Corollary 1.3** An approximate semigroup on a vector bundle satisfies Eqs. (1.19)-(1.21). An approximate heat kernel for some $\Delta$ on $\mathcal{V}$ is an approximate semigroup, with constants $(B, C, D, T)$ for the approximate semigroup whose constants can be made to depend only on the corresponding constants for the approximate heat kernel and the bounds on the defining atlas.

**Remark 1.3** While it suffices for the rest of the work, the dependence of the structures defined on the choice of tame atlas is mathematically distressing. In fact there is a natural notion of the comparability of tame structures, which simply involves requiring that the diffeomorphisms between charts induced by the identity on $\mathcal{V}$ have all derivatives up to the appropriate order uniformly bounded. It is then straightforward if laborious to check that the $t$-norms associated to compatible tame atlases are comparable (each bounded by a multiple of the other), that families of kernels that are approximate semigroups or heat kernels with respect to one atlas are the same with respect to the other, and therefore that the limit results of the following section depend only on the “tame equivalence class” of the vector bundle, Riemannian manifold and operator.

## 2 The fine-partition limit

If $P = (t_1, t_2, \ldots, t_k)$ is a partition of a positive real number $t$ (that is, $t_i > 0$ and $\sum_i t_i = t$) define $|P| = \max_i t_i$, $\#P = k$, and for any kernel $K$

$$K^{*P}(t) = K(t_1) * K(t_2) * \cdots * K(t_k).$$

(2.1)

If $P$ is a partition of $t$ and $P'$ is a partition of $t'$, then the concatenation $PP'$ is a partition of $t + t'$; if $P_i$ is a partition of $t_i$ for $1 \leq i \leq k$, then the partition $P_1 P_2 \cdots P_k$ is a refinement of $P = (t_1, \ldots, t_k)$.
In the language of the introduction, $K^{tP}$ is the approximate path integral corresponding to the approximate heat kernel $K$ and a choice of partition $P$. Thm. 2.1 below asserts the convergence of these approximations and provides a key estimate on the rate of convergence in terms of $t$ and $|P|$, valid provided $K$ is an approximate semigroup in the precise sense of Defs. 7 and 4.

2.1 PARTITIONS AND THE REFINEMENT LIMIT

**Lemma 2.1** Suppose $K(t)$ is a family of kernels and $\| \cdot \|_{(t)}$ is a family of norms for which Eqs. (1.19), (1.20), and (1.21) hold for some constants $B, C$, and $T$. Then there is an $A > 0$ depending on $B, C$ such that, if $T$ is chosen small enough,

$$\left\| K^{*Q}(t) - K^{tP}(t) \right\|_{(t)} < A t^{5/4} |P|^{1/4}$$

(2.2)

for all refinements $Q$ of all partitions $P$ of $t < T$.

**Proof:** First observe that by Eqs. (1.19), (1.20), and (1.21) there is a $c_2 > 0$ so that for all sufficiently small $t = t_1 + t_2 + t_3$

$$\left\| K(t_1) * K(t_2) * K(t_3) - K(t) \right\|_{(t)} \leq c_2 t^{3/2}.$$

Next, argue by induction on the number of entries in $Q$ that there are positive reals $b_2, c_3 > 0$ such that

$$\left\| K^{*Q}(t) - K(t) \right\|_{(t)} \leq c_3 e^{b_2 t^{1/2}} t^{3/2}.$$  

(2.3)

For that note one can always write $Q = Q_1(t_2)Q_3$, where $Q_1$ is a partition of $t_1$ and $Q_3$ is a partition of $t_3$, $t_2$ is a component of $Q$, $t_1 \leq t/2$ and $t_3 \leq t/2$ (one or both of $t_1, t_3$ may be 0). Then

$$\left\| K^{*Q}(t) - K(t) \right\|_{(t)}$$

$$\leq \left\| K^{*Q_1}(t_1) - K(t_1) \right\| * K(t_2) * K(t_3) \right\|_{(t)}$$

$$+ \left\| K(t_1) * K(t_2) * \left[ K^{*Q_3}(t_3) - K(t_3) \right] \right\|_{(t)}$$

$$+ \left\| K^{*Q_1}(t_3) - K(t_3) \right\| * K(t_2) * \left[ K^{*Q_2}(t_3) - K(t_3) \right] \right\|_{(t)}$$

$$+ \left\| K(t_1) * K(t_2) * \left[ K^{*Q_3}(t_3) - K(t_3) \right] \right\|_{(t)}$$

$$\leq e^{2B t^{1/2}} \left\| K^{*Q_1}(t_1) - K(t_1) \right\|_{(t_1)} + e^{2B t^{1/2}} \left\| K^{*Q_2}(t_3) - K(t_3) \right\|_{(t_3)}$$

$$+ e^{2B t^{1/2}} \left\| K^{*Q_1}(t_3) - K(t_3) \right\|_{(t_3)}$$

$$\leq e^{2B t^{1/2}} \left[ c_3 e^{b_2 t_1^{1/2}} t_1^{3/2} + c_3 e^{b_2 t_3^{1/2}} t_3^{3/2} + c_3 e^{b_2 (t_1^{1/2} + t_3^{1/2}) t_1^{3/2} t_3^{3/2} + c_2 t^{3/2}} \right]$$

$$\leq c_3 t^{1/2} e^{2B t^{1/2} + b_2 (t_2^{3/2})} \left[ t^{2-1/2} + c_3 e^{b_2 t^{1/2}} t^{5/2} (c_2/c_3) t \right]$$

$$\leq c_3 t^{1/2} e^{b_2 t^{1/2}} \left( 2^{1/4} t + c_3 e^{b_2 t^{1/2}} t^{5/2} \right) \leq c_3 e^{b_2 t^{1/2}} t^{3/2}$$

where the second inequality follows from Eqs. (1.19) and (1.20), the third from the inductive hypothesis, the fourth from $t_1 < t/2$, $t_3 < t/2$, the fifth from choosing $b_2 > 2B/(1 - 2^{-1/2})$ and $c_3 > c_2/(2^{-1/4} - 2^{-1/2})$ (this condition also covers the base case) and the last by setting $T$ small enough that $c_3 e^{b_2 t^{1/2}} t^{3/2}$ is less than $(1 - 2^{-1/4})$. 

14
Note this implies for $K$ as above there is a $b_3 > 0$ so that for any kernel $J$, for $t = t_1 + t_2 < T$ for small enough $T$, and any partition $Q$ of $t_1$
\[
\left\|K^{*Q}(t_1) * J(t_2)\right\|_{\mathcal{L}}(t) \leq e^{b_3 t^{1/2}} |J(t_2)|, \quad \text{and} \quad \left\|K^{*Q}(t)\right\|_{\mathcal{L}}(t) \leq e^{b_3 t^{1/2}}. \quad (2.4)
\]
This follows from simply writing $K^{*Q}$ as $K^{*Q} - K + K$.

Now argue by induction on the number of entries in $P$ that there is an $A > 0$ such that for all partitions $P$ and all refinements $Q$ of $P$
\[
\left\|K^{*Q}(t) - K^{*P}(t)\right\|_{\mathcal{L}}(t) < A t^{5/4} |P|^{1/4}.
\]
Suppose $t = t_1 + t_2 + t_3$, $P = P_1(t_2) P_2$ with $P_i$ a partition of $t_i$, and $Q = Q_1 Q_2 Q_3$ with $Q_i$ a refinement of $P_i$, $Q_2$ a partition of $t_2$, chosen so that $t_1 < t/2$ and $t_3 < t/2$ (one or both of $t_1, t_3$ may be 0). Then
\[
\begin{align*}
\left\|K^{*Q}(t) - K^{*P}(t)\right\|_{\mathcal{L}}(t) &= \left\|K^{*Q_1}(t_1) * K^{*Q_2}(t_2) * K^{*Q_3}(t_3) - K^{*P_1}(t_1) * K^{*P_2}(t_2) * K^{*P_3}(t_3)\right\|_{\mathcal{L}}(t) \\
&\leq \left\|K^{*Q_1}(t_1) - K^{*P_1}(t_1)\right\|_{\mathcal{L}}(t) * K^{*Q_2}(t_2) * K^{*Q_3}(t_3) + \left\|K^{*P_1}(t_1) * K^{*Q_2}(t_2) - K(t_2)\right\|_{\mathcal{L}}(t) * K^{*Q_3}(t_3) + \left\|K^{*P_2}(t_2) + K^{*Q_3}(t_3) - K^{*P_3}(t_3)\right\|_{\mathcal{L}}(t) \\
&\leq e^{2 b_1 t^{1/2}} \left[ A t_1^{5/4} |P_1|^{1/4} + c_3 e^{b_2 t_2^{1/2}} t_2^{3/2} + A t_3^{5/4} |P_3|^{1/4} \right] \\
&\leq A e^{2 b_1 t^{1/2}} t_2^{5/4} |P|^{1/4} \left[ 2^{-1/4} + c_3 A^{-1} e^{b_2 t_2^{1/2}} \right] \\
&\leq A e^{2 b_1 t^{1/2}} t_2^{5/4} |P|^{1/4} 2^{-1/8} \leq A t^{5/4} |P|^{1/4}
\end{align*}
\]
where the second inequality follows from Eqs. 2.3 and 2.4 with the inductive hypothesis, and the third from $t_1 < t/2$, $t_3 < t/2$ and $t_2^{3/2} \leq t^{5/4} |P|^{1/4}$. The fourth inequality follows by choosing $A$ large enough and $T$ small enough that $A > c_3 b_2 t_2^{1/2} / (2^{-1/8} - 2^{-1/4})$ (a similar choice covers the base case), and the last follows by choosing $T$ small enough that $e^{2 b_1 t^{1/2}} < 2^{1/8}$.

**Theorem 2.1** Suppose $K(t)$ is an approximate semigroup with constants $(B,C,D,T)$ on a bundle $V$ over $M$ with Riemannian $g$, all tame, and $\|\cdot\|_{\mathcal{L}}(t)$ is the norm on kernels on $V$ guaranteed by Prop. 1.1. Then there is a family of kernels $K^\infty(t)$ and a constant $A > 0$ depending on $B,C$ such that if $T$ is chosen small enough
\[
\left\|K^\infty(t) - K^{*P}(t)\right\|_{\mathcal{L}}(t) \leq A t^{5/4} |P|^{1/4} \quad (2.5)
\]
for any partition $P$ of any $t < T$. In particular $K^\infty$ can be extended to all $t > 0$ and there are $A_1, T_1, D_1$ depending only on these constants (and dimensions) such that
\[
\left\|K^\infty(t) - K^{*P}(t)\right\|_\infty \leq A_1 t e^{B_1 t} |P|^{D_1} \quad (2.6)
\]
for all $P$ with $|P| < T_1$, so that in fact
\[
K^\infty(t) = \lim_{|P| \to 0} K^{*P}(t) \quad (2.7)
\]
in supremum norm for each fixed $t > 0$. 

Proof: For the short-time construction of $K^\infty$ and Eq. (2.5), consider a sequence $P_1 = (t), P_2, \ldots$ for sufficiently small $t$, each a refinement of the previous and with $|P_1| \to 0$. By Eq. (2.2), $K^{*P_1}(x, y, t)$ is a Cauchy sequence in the $t$-norm, and therefore by Eq. (1.18) is Cauchy in supremum norm and by completeness converges to some $K^\infty(x, y; t)$. If $P$ is any partition of $t$, let $P'_i$ be a refinement of $P_i$ and $P$ for each $i > 1$. Then

$$
\|K^{*P}(t) - K^\infty(t)\|_t \leq \|K^{*P(t)} - K^{*P'}(t)\|_t + \|K^{*P'}(t) - K^{*P'\infty}(t)\|_t + \|K^{*P'\infty}(t) - K^\infty(t)\|_t
$$

$$
< At^{5/4} |P|^{1/4} + 2At^{5/4} |P|^{1/4}
$$

$$
\leq At^{5/4} |P|^{1/4},
$$

taking $i$ to infinity. This proves Eq. (2.5).

Eq. (2.6) and hence Eq. (2.7) will follow for an approximate semigroup $K$ from the observation that, for $P$ a sufficiently fine partition of a given arbitrary $t > 0$ and $Q$ any refinement of $P$, there are constants $A_1, b_1 > 0$ such that

$$
\|K^{*Q}(t) - K^{*P}(t)\|_\infty \leq A_1te^{b_1t} |P|^{1/8(m)}.
$$

(2.8)

To see Eq. (2.8) suffices, consider a sequence $P_i$ of partitions with $|P_i| \to 0$. Consider any two $P_{i_1}, P_{i_2}$ with $i_1 < i_2$ far enough out in the sequence for Eq. (2.8) to apply, and let $Q$ be a common refinement. Then the bounds on $\|K^{*Q}(t) - K^{*P_i}(t)\|_\infty$ imply $\|K^{*P_{i_1}}(t) - K^{*P_{i_2}}(t)\|_\infty \in O\left(|P_{i_1}|^{1/8(m)}\right)$ for fixed $t$. Thus the sequence is Cauchy in the supremum norm, so a limit $K^\infty(t)$ exists. For $P$ in the given sequence, the obvious estimate shows the limit satisfies Eq. (2.6), which is the crux of the theorem. If $t$ is small, this limit clearly agrees with the short-time construction above, and the argument above extends to show Eq. (2.6) and therefore Eq. (2.7) in fact follow from Eq. (2.8) for all partitions.

To see Eq. (2.8) holds, let $P$ be a partition of $t > 0$, where $t$ need not be particularly small, and let $T$ be small enough that Cor. 1.1 holds. Assume $|P|^{1/2(2m)} < T$ and $|P| < 1$, so $|P| < |P|^{1/2(2m)}$. Let $P_0$ be another partition of $t$ such that $P$ is a refinement of $P_0$ and such that each component $t_j$ of $P_0$ satisfies $|P|^{1/2(2m)} \leq t_j \leq 2|P|^{1/2(2m)}$. (To define $P_0$, proceed inductively, using $|P| < |P|^{1/2(2m)}$.) If the partition $P$ is sufficiently fine, then the upper bound on $t_j$ will ensure Eqs. (2.2) and (1.13) hold with $t_j$ replacing the generic $t$ in these equations. For each $t_j$ in $P_0$, the partitions $P$ and $Q$ restrict to partitions $P_j$ and $Q_j$ respectively of $t_j$. In terms of these,

$$
K^{*Q}(t) - K^{*P}(t) = \sum_j K^{*P_1}(t_1) \cdots K^{*P_{j-1}}(t_{j-1}) \left[ K^{*Q_j}(t_j) - K^{*P_j}(t_j) \right]
$$

$$
\times K^{*Q_{j+1}}(t_{j+1}) \cdots K^{*Q_k}(t_k).
$$

(2.9)

Eq. (2.2) together with Eq. (1.18) and the bounds on $t_j$ give

$$
\|K^{*Q_j}(t_j) - K^{*P_j}(t_j)\|_t \leq A_2t_j^{-m/2} \|K^{*Q_j}(t_j) - K^{*P_j}(t_j)\|_{t_j}
$$

$$
\leq A_3t_j^{5/4} \leq A_4t_j |P|^{1/8(m)},
$$

where the second inequality follows from $|P| \leq |P|$ and the lower bound on $t_j$, while the final inequality follows from the upper bound on $t_j$. 

16
 defined approximate semigroups locally. Even local values of the fine partition
such that for some \( r \) boundary of \( O \) approximate semigroups. We can assume
of \( a \) tame Riemannian manifold with an approximate semigroup. Suppose
is a neighborhood of that point, and \( \tau _i \) will change the fine partition limit at the given point only by an exponentially damped term.

Then there are constants \( c, d \) depending on these \( j \) and on the constants \( B, C, D \) associated to the two approximate semigroups
\( \sum _{j} \sum _{k} b_j t^j \) ensure
\( \| K^{*Q}(t) - K^*(t) \|_{op} \leq c^2 e^{b t} \| K^{*Q_j}(t_j) - K^*(t_j) \|_{op} \)
\( \leq c^2 e^{b t} \sum _{j} A_t t_j |P|^{1/(8m)} \leq A_t t e^{b t} |P|^{1/(8m)} . \)

2.2 Relating different kernels

Sect. 1 defined approximate semigroups locally. Even local values of the fine partition limit depend globally on the value of the approximate semigroup, but the next proposition shows this to be true rather weakly. In fact, changing the kernel or even the underlying manifold outside a neighborhood of a point, as long as the bounds \( B, C \) and \( D \) remain fixed,
will change the fine partition limit at the given point only by an exponentially damped term.

Proposition 2.1 Suppose \( (V_i, M_i, g_i, K_i(t)) \) for \( i = 1, 2 \) each represent a tame Riemannian manifold with an approximate semigroup. Suppose \( x \) is a point in \( M_1 \), \( O \) is a neighborhood of that point, and \( \Phi \) is an isomorphism of all this structure to an open set \( \Phi[O] \subset M_2 \). That is to say in a neighborhood of \( x \) the bundle \( V_1 \) can be identified isometrically with the bundle \( V_2 \) over a neighborhood of \( \Phi(x) \) so that \( K_1(t) \) is the pullback of \( K_2(t) \). Then there are constants \( c, d > 0 \) depending only on the distance \( r \) to the boundary of \( O \) and on the constants \( (B, C, D) \) associated to the two approximate semigroups such that for some \( T \) depending on these \( (B, C, D) \) and all \( 0 < t \leq T \),
\( |K_1(x, x; t) - K_2(\Phi(x), \Phi(x); t)| \leq cc^{-d/t} , \) (2.10)

which is to say their difference is exponentially damped.

Proof: Choose constants \( (B, C, D) \) and the associated \( T \) for which both kernels are approximate semigroups. We can assume \( r < D \) and \( O \) (resp. \( \Phi(O) \)) contains a ball of radius \( r \) around \( x \) (resp. \( \Phi(x) \)) in \( g_1 \) (resp. \( g_2 \)). For simplicity of notation, write \( K \) for \( \Phi^* K \) and \( O \) for \( \Phi[O] \). Since Eq. (2.10) is obvious for large \( t \) by Eq. (1.18), assume \( t < T \). Let \( \chi(y) \) be

17
a real valued function on both $M$, which is 1 if $y \in O$ and 0 otherwise. For simplicity write $\chi K$ for the kernel whose value on $y, z$ is $\chi(y)K(y, z)$ and $K\chi$ for the one whose value at $y, z$ is $K(y, z)\chi(z)$. All this notational slight of hand and the fact that $K_1 = K_2$ on $O \times O$ allows both sides of the following expression to make unambiguous sense and to be equal: 

$$(1 - \chi)K_1\chi + (1 - \chi)K_2(1 - \chi) = K_1\chi + (1 - \chi)K_2.$$ 

This implies, for any partition $P$ of $t < T$, the sum

$$\sum_j K_1^{\star P_j} * (1 - \chi)K_1(t_j)\chi * K_2^{\star P_j} - K_1^{\star P_j} * \chi K_2(t_j) * (1 - \chi) * K_2^{\star P_j},$$

where $P = P_j(t_j)P_j$, telescopes to $K_1^{\star P} - K_2^{\star P}$, which reduces to $K_1^{\star P} - K_2^{\star P}$ on $O \times O$. Turning this around,

$$\left| K_1^{\star P} - K_2^{\star P} \right| \leq \sum_j \left| K_1^{\star P_j}(1 - \chi)K_1(t_j)\chi * K_2^{\star P_j} \right| + \left| K_1^{\star P_j} * \chi K_2(t_j)(1 - \chi) * K_2^{\star P_j} \right|$$

on $O \times O$, and in particular at $(x, x; t)$.

In the above, $P_j$ is a partition of some $\tau_j$, and $P_j'$ a partition of some $\tau_j'$, with $\tau_j + t_j + \tau_j' = t$. Eq. (2.4) gives $\left\| K_1^{\star P_j} \right\|_{(\tau_j)} \leq e^{b_3 t^{1/2}}$ for some $b_3 > 0$. Thus, provided $y$ is not in $O$, Eq. (1.17) ensures

$$\left| K_1^{\star P_j}(x, y; \tau_j) \right| \leq e^{b_3 t^{1/2}} \left[ 2e^{-D^2/20t} \right] \leq e^{-c_2 / t}$$

for some $c_2$. The bounds in operator norm on $K_1$, multiplication by $\chi$, and $K_2^{\star P_j}$ readily give

$$\left| K_1^{\star P_j} * (1 - \chi)K_1(t_j)\chi * K_2^{\star P_j}(x, x; t) \right| \leq e^{-c_3 / t}$$

for some $c_3$. The same bound applies to $\left| K_1^{\star P_j} + \chi K_2(t_j)(1 - \chi) * K_2^{\star P_j} \right|$, so

$$\left| K_1^{\star P}(x, x; t) - K_2^{\star P}(x, x; t) \right| \leq \# P \cdot 2e^{-c_3 / t}.$$

Let $P$ consist of equal intervals with $\# P$ the least integer greater than $e^{c_3 / (2t)}$. Thus

$$\left| K_1^{\star P}(x, x; t) - K_2^{\star P}(x, x; t) \right| \leq 3e^{-c_3 / (2t)}.$$ 

On the other hand by Eq. (2.5)

$$\left\| K_1^{\star P} - K_2^{\star P} \right\|_{(t)} \leq A^{5/4} |P|^{1/4} \leq A^{3/2} e^{-c_3 / (8t)}$$

and therefore by Eq. (1.18)

$$\left| K_1^{\star P}(x, x; t) - K_2^{\star P}(x, x; t) \right| \leq e^{-d / t}$$

for some $c, d$ and small enough $t$. Eq. (2.10) then follows. □

### 2.3 The Heat Kernel

Notice that an approximate heat kernel is an approximate semigroup by Prop. 1.2, and thus by Thm. 2.1 has a fine partition limit. Thm. 2.2 below equates this limit with the heat kernel of the same Laplacian as appears in the definition of the approximate heat kernel. Thus the fine partition limit offers an alternate construction of the heat kernel of a generalized Laplacian on a manifold. As noted in the introduction, the time-slicing
interpretation of the path integral depends on a choice of kernel (reflecting a discretization of the action) which in some sense approximates the heat kernel for a generalized Laplacian quantizing the Hamiltonian. If this choice is in fact an approximate heat kernel in the precise sense of Def. 5, then Thm. 2.2 provides a rigorous construction of the heat kernel as a time-sliced path integral with the appropriate Lagrangian. Section 3 spells this out in some detail.

**Lemma 2.2** Suppose $K(t)$ is an approximate heat kernel for the operator $\Delta$ as in Defs. 6 and 7 and $K^\infty(t)$ is the limit guaranteed by Thm. 2.1. If $f$ is a smooth section of $V$ bounded in each coordinate patch and $t < T$,

$$f(t) = K^\infty(t) * f$$  \hfill (2.11)

agrees with the unique solution $f(t)$ of the heat equation $\partial f(t)/\partial t = \frac{1}{2} \Delta f(t)$ subject to $\lim_{t \to 0} f(t) = f$.

**Proof:** For $f(t)$ given by Eq. (2.11), Eq. (2.5) of Thm. 2.1 and Eq. (1.16) combine to give

$$\|f(t) - K(t) * f\|_\infty \leq A t^{3/2} e^{2B/\tau} \|f\|_\infty$$

for $t < T$. In particular

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} K(t) * f = f.$$

To see that the heat equation holds, note $K^\infty$ is a semigroup: $K^\infty(t) = K^\infty(t_1) * K^\infty(t_2)$, for $t = t_1 + t_2$ and $t_1, t_2 > 0$; this follows from considering the limit of $K(t_1) * K(t_2)$ under refinements of the partition $(t_1, t_2)$ of $t$. Thus

$$\left| \frac{\partial f(t)}{\partial t} - \frac{1}{2} \Delta f(t) \right| = \left| \lim_{\tau \to 0} \frac{f(t + \tau) - f(t)}{\tau} - \frac{1}{2} \Delta f(t) \right| = \left| \lim_{\tau \to 0} \frac{K^\infty(\tau) * f(t) - f(t)}{\tau} - \frac{1}{2} \Delta f(t) \right| \leq \left| \lim_{\tau \to 0} \frac{K^\infty(\tau) * f(t) - f(t)}{\tau} - \frac{1}{2} \Delta f(t) \right| + \left| \lim_{\tau \to 0} \frac{A t^{3/2} e^{2B/\tau} |f(t)|_\infty}{\tau} \right| = 0,$$

where the first line follows from the semigroup property of $K^\infty$, the second from the preceding estimate, and the last from Eq. (1.24) of Def. 5.

**Theorem 2.2** Suppose $K$ is an approximate heat kernel for elliptic $\Delta$ on a bundle $V$ over $M$ with Riemannian $g$. Then the fine partition limit $K^\infty(t)$ defined by Thm. 2.1 is the heat kernel of $\Delta$.

**Proof:** Thm. 2.1 implies that the refinement limit $K^\infty(t)$ exists for all $t > 0$ and satisfies Eq. (2.6) for sufficiently small $t$. Lemma 2.2 implies as a distribution $K^\infty$ is a solution to the heat equation for $t < T$. If $t$ is too large to apply this lemma directly, note that $K^\infty(t) = (K^\infty)^* Q_t$ for some partition $Q_t$, with each $t_i < T$. Thus, $K^\infty$ is a distributional heat kernel for all $t > 0$. Since $\Delta$ is elliptic, elliptic regularity [Eva98] says $K^\infty(x, y; t)$ is smooth in $x$, $y$, and $t$ and thus is the heat kernel of $\Delta$. 

\[ \Box \]
3 Kernels for Generalized Laplacians and $N = 1/2$ SUSY

The results of Sect. 2.3 ensure the products of a kernel which satisfies the conditions defining an approximate heat kernel will in fact converge to the heat kernel. This begs the question of how to find such a kernel for a given Laplacian. Eq. (3.2) answers this by defining a specific kernel for each generalized Laplacian. Thm. 3.1 applies the results of Sect. 2.3 to show the fine-partition limit of products of this kernel is the heat kernel for that Laplacian.

Sect. 3.2 interprets the approximate heat kernels for these generalized Laplacians as exponentiated, discrete actions for an associated supersymmetric theory. As such these approximate heat kernels provide the basis for a time-slicing approximation to the path integral for this theory. Thus Thm. 3.1 provides a rigorous realization of the time-slicing construction of the path integral and confirms it represents the heat kernel.

Sect. 3.3 specializes this to twisted $N = 1/2$ supersymmetric quantum mechanics in the imaginary-time formulation. Note that an appropriate choice of twisting (the Levi-Civita connection on the dual spinor bundle) yields $N = 1$ supersymmetric quantum mechanics.

3.1 Approximate heat kernel for elliptic operators

Let $V$ be a vector bundle over a manifold $M$ with Riemannian metric $g$. Berline, Getzler and Vergne [BGV04] observe that every generalized Laplacian can be written locally as

$$\Delta V = g^{ij} \left[ \nabla^V_{\partial_i} \nabla^V_{\partial_j} - \Gamma^V_{ij} \nabla^V_{\partial_k} \right] - V, \quad (3.1)$$

where $\nabla^V$ is a connection on $V$, $\nabla^V_{\partial_i}$ defines the Christoffel symbols for the Levi-Civita connection on the tangent bundle, and $V$ is a section of $\text{End}(V)$. If $(V, M, g, \Delta)$ is tame, then, for $d(x, y) < D$, let $\Psi^y_x \in \text{Hom}(V_y, V_x)$ denote the parallel transport map from $V_y$ to $V_x$ along the unique minimal geodesic. Define the section of $\text{Hom}_{g|_x}$

$$K_{\Delta}(x, y; t) = H_D(x, y; t)e^{-\text{Ricci}(x, x)/12-tV(x)/2} \Psi^y_x, \quad (3.2)$$

where the Ricci and scalar curvatures are evaluated at $y$.

The following lemma provides some basic estimates on the effect of having modified $H_D$ of Eq. (1.4) by a known factor.

Lemma 3.1 For any $k \in \mathbb{N}$

$$d(x, y)^k H_D(x, y; t) \leq 2^{(m+k)/2} (k/e)^{k/2} H_D(x, y; 2t). \quad (3.3)$$

Moreover, if $F(x, y; t) = O((|x| + t)^{-b})$, there is a $B > 0$ so that, in the $t$-norm based on this choice, $\|F(x, y; t)H_D(x, y; t)\|_{(t)} = O((t^{b-2})$. If in particular $F(x, y; t)$ is differentiable as a function of $y$, $x_y$ and $t$, and $F(y, y; 0) = 1$, then there are $B, D$ such that $F(x, y; t)H_D(x, y; t) \in E_{B, D}$.

Proof:

Eq. (3.3) follows immediately from the definition of $H_D$ in Eq. (1.4) and the fact that $x^k e^{-x^2/2}$ is bounded by $(k/e)^{k/2}$.

For $|F| = O((|x| + t)^{-b})$,

$$\|F(t)H_D(t)\|_{(t)} = O\left(t^{b/2-b}\right)H_D(2t)\|_{(t)} = O\left(t^{b/2-b}\right),$$

where the first equality follows immediately from Eq. (3.3).
Similarly, for \( F(x, y; t) = 1 + \mathcal{O}(|x_0|) + \mathcal{O}(t) \),

\[
F(x, y; t)H_D(x, y; t) = H_D(x, y; t) + \mathcal{O}\left(t^{1/2}\right) H_D(x, y; 2t)
\]

and therefore

\[
|F(t)H_D(t)| \leq H_D(t) + \left(e^{Bt^{1/2}} - 1\right) H_D(2t)
\]

\[
\leq e^{Bt^{1/2}} \left[ e^{-Bt^{1/2}} H_D(t) + \left(1 - e^{-Bt^{1/2}}\right) H_D(2t)\right] = e^{Bt^{1/2}} \int_1^t H_D(\alpha t)d\mu_\alpha \in \mathcal{E}_{B,D}(t)
\]

for small enough \( t \) and an appropriate \( \mu \) by Def. 1.

\[ \square \]

**Theorem 3.1** Suppose \( \nabla^V \) is a connection on \( V \) a vector bundle over a manifold \( M \), \( V \) is a section of \( \text{End}(V) \), \( \Delta \) is the generalized Laplacian associated to this data, and \( K_\Delta \) is the kernel given by Eq. (3.2). If \( (V, M, g, \Delta) \) is tame, then \( K_\Delta \) is an approximate heat kernel with constants \( (B, C, D) \) depending only on the bounds on \( g \) and on \( \Delta \), and therefore its large partition limit \( K_\Delta^\infty \) is the heat kernel for \( \Delta \).

**Proof:** By Thm. 2.2, it suffices to verify \( K_\Delta \) satisfies the conditions defining an approximate heat kernel as spelled out in Eqs. (1.22) through (1.26). These are all local conditions, so let \( y \in M \) and work in Riemann normal coordinates around \( y \). That is, pick an orthonormal basis for \( T_yM \). Each point \( x \in M \) near \( y \) is the value of the exponential map at a unique vector \( x \in T_yM \) near 0. (The \( x \) was \( x_0 \) earlier; the subscript is implicit here where there is no danger of confusion.) The components of \( x \) with respect to the chosen basis define the Riemann normal coordinates of the point \( x \). As in the proof of Lemma 1.1, tameness implies that in Riemann normal coordinates \( g_{ij} \) has bounded \( k \)th derivatives for \( 0 \leq k \leq 4 \).

If \( X \) and \( Y \) are tangent vectors at \( x \in M \), let \( R_{[X,Y]} \) be the Riemannian curvature (endomorphism on \( T_xM \)), \( R_{ij}(X,Y) \) be the Ricci curvature, and \( \tau_x \) be the scalar curvature. The coordinate derivatives \( \partial_i \) for \( i = 1, \ldots, m \) at each \( x \in M \) near \( y \in M \) form a basis of \( T_xM \) and define vector fields in a neighborhood of \( y \) (commuting but not in general orthonormal). At \( y \) these agree with the original choice of orthonormal basis. Define a second basis \( e_i \in T_xM \) (orthonormal but not commuting as vector fields) by parallel transporting the same orthonormal basis of \( T_yM \) along a minimal geodesic from \( y \) to (nearby) \( x \). The two bases are related by [BGV04] (Prop. 1.28)

\[
e_i = \left[ \delta_i^j + \frac{1}{6} R_{iklj} x^k x^l \right] \partial_j + \mathcal{O}(|x|^3) \tag{3.4}
\]

where \( R_{iklj} \partial_j = R_{ij}(\partial_i, \partial_k) \partial_l \) defines the coordinates of the curvature at \( y \). If \( g_{ij}(x) = (\partial_i, \partial_j) \), with inverse \( g^{ij}(x) \), and \( \Gamma_{ij}^k(x) \partial_k = \nabla_{\partial_i} \partial_j \), Eq. (3.4) implies

\[
g_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{iklj} x^k x^l + \mathcal{O}(|x|^3) \tag{3.5}
\]

\[
g^{ij}(x) = \delta_{ij} - \frac{1}{3} R_{iklj} x^k x^l + \mathcal{O}(|x|^3) \tag{3.6}
\]

\[
\Gamma_{ij}^k(x) = -\frac{1}{3} \left[ R_{ij}^k + R_{ik}^l \right] x^l + \mathcal{O}(|x|^3) \tag{3.7}
\]

\[
det^{1/2} g(x) = 1 + \frac{1}{6} R_{ij}^k x^i x^j + \mathcal{O}(|x|^3) \tag{3.8}
\]

21
freely raising and lowering indices using \( g_{ij}(0) = \delta_{ij} \). At \( y \), abbreviate \( \text{Ricci}_{ij} = R_{ik}^j = \text{Ricci}_{ij}(\partial_i, \partial_j) \) and \( \tau = \text{Ricci}_{ij}^i y_j \).

The bounds implicit in \( \mathcal{O}(|x|^p) \) above depend only the bounds on \( g_{ij} \) and its derivatives up to order three. Trivialize the bundle \( \mathcal{V} \) in a ball of radius \( D \) around \( y \) by identifying \( \mathcal{V}_s \) with \( \mathcal{V}_y \) via parallel transport along the unique minimal geodesic connecting \( y \) and \( x \), so that

\[
\nabla^V_i = \partial_i + \frac{1}{2} x^j F^V_{ij} + \mathcal{O}(|x|^2) \tag{3.9}
\]

where \( F^V_{ij} \) is the curvature of \( \nabla^V \) evaluated at \( y \) in the \( \partial_i \wedge \partial_j \) direction [BGG04](Prop. 1.18), the bound depending on the bound on the coefficients of \( \nabla \) to order 2.

Eq. (3.2) and Lemma 3.1 give \( K_\Delta(t) \in \mathcal{E}_{B,D} \) (Def. 1) for some \( B > 0 \). Thus, \( |K_\Delta(t)|_{(t)} \leq 1 \), verifying Eq. (1.22) of the definition of an approximate heat kernel.

Using Eqs. (3.5)-(3.9) and the antisymmetry of \( F^V \),

\[
\Delta = g^{ij} \left[ \nabla^V_i \nabla^V_j - \Gamma^k_{ij} \nabla^V_k \right] - V
\]

Compute

\[
\frac{\partial}{\partial t} K_\Delta(x, y; t) = \left[ -\frac{m}{2t^2} + \frac{|x|^2}{2t^2} - \frac{r}{12} - \frac{V}{2} \right] K_\Delta(x, y; t),
\]

\[
\partial_{x_i} K_\Delta(x, y; t) = \left[ -\frac{x_i}{t} - \frac{\text{Ricci}_{ij} x_j}{6} + \mathcal{O}(t) \right] K_\Delta(x, y; t),
\]

so

\[
\left[ \frac{\partial}{\partial t} - \frac{1}{2} \Delta \right] K_\Delta = \left[ -\frac{m}{2t^2} + \frac{|x|^2}{2t^2} - \frac{r}{12} - \frac{V}{2} \right] K_\Delta(x, y; t) + \frac{m}{2t} + \frac{r}{12} - \frac{|x|^2}{2t^2} - \frac{x^i \text{Ricci}_{ij} x^j}{6t} - \frac{x^i \text{Ricci}_{ij} x^j}{6t} + \frac{1}{6t^2} R_{iklj} x^i x^j x^k x^l + \frac{1}{2t} x^k F^V_{kx^i} x^j + \frac{V}{2} + \frac{x^i \text{Ricci}_{ij} x^j}{3t} + \mathcal{O}(|x| + |x|^3/t + |x|^5/t^2 + t) \right] K_\Delta(x, y; t)
\]

after taking into account the antisymmetry of \( F^V \) and the fourfold symmetry of \( R \). By Lemma 3.1, the right-hand side has \( t \)-norm bounded by a multiple of \( t^{1/2} \), so the first line of Eq. (1.26) holds. Since the Laplace-Beltrami operator is self-adjoint, \( \Delta^* \) is the operator associated to \( g, \nabla^* \) and \( V^* \), where \( \dagger \) represents the canonical map sending \( \text{End}(\mathbb{R}^n) \) to \( \text{End}((\mathbb{R}^n)^*) \). So for the second line of Eq. (1.26) it suffices to observe that \( K_{\Delta^*}(x, y; t) = K_\Delta^*(y, x; t) + \mathcal{O}(|x|^3 + |x|^5) \). This estimate follows from the tameness assumption which more directly implies \( \text{Ricci}_x(x, y) = \text{Ricci}_y(x, y) = \mathcal{O}(|x|^3) \), \( r_x - r_y = \mathcal{O}(|x|^3) \), and \( V(y) - (\mathcal{P} y)^{-1} V(x) \mathcal{P} y = \mathcal{O}(|x|^5) \) with the bounds depending on the bounds on the metric. Eq. (1.26) now follows.

For Eq. (1.23), let \( f \) be a smooth function on \( O \) valued in \( \mathbb{R}^n \). Then working in Riemann normal coordinates around \( x \) with the the bundle trivialized by parallel transport in radial
directions,
\[
\lim_{t \to 0} \int K_\Delta(x, y; t) \cdot f(y) \, dy = \int H_D(x, y; t) f(y) \left[ 1 + \mathcal{O}(|y_x|^2) + \mathcal{O}(t) \right] \, dy_x
\]
\[
= f(x) + \mathcal{O}(t).
\]
Similarly, for Eq. (1.24) it suffices by the Mean Value Theorem to show \( \lim_{t \to 0} \frac{\partial}{\partial t} K_\Delta \ast f = \frac{i}{\hbar} \Delta f \). In Riemann normal coordinates
\[
\lim_{t \to 0} \frac{\partial}{\partial t} K_\Delta \ast f(x) = \lim_{t \to 0} \int \left[ -\frac{m}{2t} + \frac{|x_y|^2}{2t^2} - \frac{r}{12} - \frac{V}{2} \right] K_\Delta(x, y; t) f(y) \, dy
\]
\[
= \lim_{t \to 0} \frac{1}{2} \partial_t \partial_y - V \right) f(x) + \mathcal{O}(t^{1/2}) = \frac{1}{2} \Delta f
\]
by straightforward Gaussian integrals. Finally Eq. (1.25) follows for appropriate \( B \) from the above calculation for \( \partial_t K_\Delta \).

\[\square\]

**Remark 3.1** The calculations verifying Eq. (1.26) shed some light on the role of the Ricci and scalar curvature terms in the definition of \( K_\Delta \). Adding \( \alpha t + b(\text{Ricci}(x_\gamma, x_\gamma) - \nabla t) \) to the exponent in \( K_\Delta \), changes Eq. (1.26) in two ways: it would add \( \alpha t \) to the operator \( \Delta \), and \( b(\text{Ricci}(x_\gamma, x_\gamma) - \nabla t) \in \mathcal{O}(1) \) to the bound \( Ct^{1/2} \) on the right-hand side. In units where \( \hbar \) is not 1, the addition to \( \Delta \) is \( \alpha \hbar^2 \) and thus is a quantum correction to the Hamiltonian which presumably corresponds to a different resolution of the operator ordering ambiguity in \( g_{ij} p_i^* p_j \). Although the Ricci\( (x_\gamma, x_\gamma) / t - \tau \) term is of too large an order in \( t \) for Eq. (1.26) to hold, it surprisingly does not change the fine-partition limit. However, the convergence argument in this paper would not suffice in that case.

### 3.2 Path integrals

The previous subsection argued that the heat kernel for any generalized Laplacian \( \Delta \) can be expressed as a fine-partition limit of products of an approximate heat kernel constructed directly from \( \Delta \). As noted in the introduction, the product associated to a partition can be viewed as an integral over all elements of a discretized space of paths. Formally, the limit can be interpreted as an integral over all paths of a function on the space of paths. However, it is not obvious that this function is necessarily the exponential of the integral of a classical Lagrangian. The generalized Laplacian relevant to the path integral proof of the index theorem for the twisted Dirac operator cannot be the quantization of some classical Hamiltonian, because there is not even any symplectic space on which such a classical Hamiltonian could be defined. Thus, there is no classical Lagrangian with which to begin formulating a path integral, even heuristically. However, for this case, Friedan and Windey [FW84] suggest a natural extension of the generalized Laplacian to a larger space where the heat kernel can be written as an integral over all paths of the exponentiated integral of a Lagrangian, and such that a natural restriction of the quantum state space recovers the heat kernel for the original Laplacian. Sect. 3.3 shows this trick is unnecessary in the case of untwisted \( N = 1/2 \) supersymmetric quantum mechanics and, for the twisted \( N = 1/2 \) theory, is only necessary to deal with the twisted portion. Interestingly, the perturbative approximation for the restricted operator is the same as the restriction of the perturbative approximation for the unrestricted operator, so either provides an interpretation of the path integral proof of the index theorem.

If \( f(v_1, \ldots, v_n) \) is a multilinear function of \( V^* \) for some vector space \( V \), then the antisymmetrization of \( f \) represents an element of \( \Lambda V \). To say \( \psi \) is a Grassman variable valued
in $V^*$, means that the expression $f(\psi, \ldots, \psi)$ represents that element. If $V$ has an inner product the Berezin integral $\int f(\psi)d\psi$ is the coefficient of the canonical top-degree element of $\Lambda V$ in $f(\psi)$. The inner product induces a nondegenerate pairing on elements of $\Lambda V$ which in this language becomes

$$ (f(\psi), g(\psi)) = \int f(\psi)g(\psi)d\psi. $$

See [MQ86] for a standard reference on Grassman variables; [Rog92a] and [FS14] give examples relevant to SUSYQM.

Suppose $(M, g, V, \nabla, V)$ are as in Eq. (3.2). Let $X = \Lambda V$, and let $\Delta^X$ be the generalized Laplacian associated to $\nabla$ and $V$ promoted to a connection and operator on $X$ (using $\nabla(a \wedge b) = \nabla(a) \wedge b + a \wedge \nabla(b)$ and $V(a \wedge b) = V(a) \wedge b + a \wedge V(b)$). For each point $x \in M$ let $\psi_x$ be a Grassman variable valued in $V^*_x$ so as to write kernels on $X$ as superkernels $K(x, y, \psi_x, \psi_y)$, with the understanding $K$ acts on a section of $X$, which is represented by a superfunction $f(x, \psi_x)$, as

$$ (K \ast f)(x, \psi_x) = \int \int K(x, y, \psi_x, \psi_y)f(y, \psi_y)d\psi_y dy. $$

As outlined in the introduction, given a Lagrangian, a Riemann sum approximation to the action defines a kernel which, after some corrections (higher-order in $\hbar$) defines an approximate kernel. Let $x(t)$ be a path in $M$, let $\Psi, \Psi^\dagger$ be Grassman variables valued in lifts of $\sigma$ to $V^*$ and $V$ respectively, and consider the action

$$ \int \frac{1}{2}(\dot{\sigma}(t), \dot{\sigma}(t)) + i \langle \Psi^\dagger, \nabla_x^\dagger \Psi \rangle - \frac{i}{2} \langle V\Psi^\dagger, \Psi \rangle ds. $$

On a small interval of parameter length $t$, approximating the path connecting $x$ and $y$ by a geodesic gives $\int \frac{1}{2}(\dot{\sigma}(t), \dot{\sigma}(t)) dt \sim (x_{2t}/t, x_{2t}/t) t/2 \sim |x_{2t}|^2/(2t)$, which agrees with the exponent in $H_D$. Assuming $\Psi^\dagger$ and $\nabla_x \Psi$ are covariantly slowly varying, $\int i \langle \Psi^\dagger, \nabla_x \Psi \rangle ds \sim i \langle \Psi^\dagger(t_y), \nabla_x^\dagger \Psi(t_x) - \Psi(t_y) \rangle \sim i \langle \psi_{y_1}^\dagger, \nabla_x^\dagger \Psi(t_x) - \Psi(t_y) \rangle + i \langle \psi_{y_2}^\dagger, \nabla_x^\dagger \Psi(t_x) - \Psi(t_y) \rangle \sim i \langle \psi_{y_1}^\dagger, \nabla_x^\dagger \Psi(t_x) \rangle. \hspace{1cm}$

This suggests an approximate heat kernel

$$ K_{\Delta^X}(x, y, \psi_x, \psi_y; t) = H_D(x, y; t)e^{-\text{Ric}(x_{y, x_{2t}})/12 - t/12 + i \langle \psi_y^\dagger, \psi_{y_2}^\dagger(1-tV^*(x)/2)\psi_x - \psi_y \rangle d\psi_y. $$

The Ricci and scalar curvature terms do not follow directly from the approximation to the action. Rather, referring to Rem. 3.1, they correspond to the resolution of the operator-ordering ambiguity that gives $\Delta^X$ as the operator whose kernel is the path integral with this Lagrangian, and, among such choices, they are of the particular form to make $K_{\Delta^X}$ an approximate heat kernel for $\Delta^X$. Indeed,

**Proposition 3.1** $K_{\Delta^X}$ is an approximate heat kernel for $\Delta^X$ and therefore its fine-partition limit $K_{\Delta^X}^\infty$ is the heat kernel. Furthermore the component of $K_{\Delta^X}^\infty$ of degree 1 in $\psi_x$ and degree dim $V - 1$ in $\psi_y$ is the heat kernel for $\Delta^V$.

**Proof:** If $A: W_x \to W_y$ is a linear map between vector spaces of the same even dimension, $\psi_x, \psi_y$ are Grassman variables taking values in $W_x$ and $W_y$ respectively, and $\psi_y^\dagger$ takes values in $W_y^*$, then for $f$ a function defined on $W_y^*$,

$$ \int \int e^{i \langle \psi_y^\dagger, A\psi_x - \psi_y \rangle} f(\psi_y)d\psi_y d\psi_y = f(A\psi_x). $$

24
This is an immediate consequence of the definitions; the authors’ earlier paper spells out the details for the special case $A = \mathcal{P}_y$ [FS08]. Thus, the quantity

$$\int e^{\left(\psi_y^\dagger \mathcal{P}_y \right)\left(1-\mathcal{P}_y^* \mathcal{P}_y \right)/2} \psi_y \, d\psi_y$$

is, up to terms in $O(t^2)$, the superkernel for the operator

$$e^{-t\mathcal{P}_y}: \mathcal{X}_y \rightarrow \mathcal{X}_x,$$

which is the extension of $e^{-t\mathcal{P}_y}/2 \mathcal{P}_y: \mathcal{V}_y \rightarrow \mathcal{V}_x$. Therefore Eq. (3.10) differs from Eq. (3.2) by $O(t^2)K_{\Delta^\psi}$, which means that it also defines an approximate heat kernel for $\Delta^\psi$. \hfill \Box

### 3.3 Twisted N=1/2 SUSYQM

#### 3.3.1 The generalized Laplacian

The heuristic path integral for twisted $N = 1/2$ SUSYQM in imaginary time is supposed to be related to the kernel of the heat operator for a Laplacian which is the square of the twisted Dirac operator [BGV04]. To define this operator, recall some Clifford algebra facts and terminology detailed in Ch. 3 of [BGV04]. If $M$ is a Riemannian manifold define $\mathcal{C} = C(T^*M)$ to be the bundle which at each point $x \in M$ is the complexified $\mathbb{Z}/2\mathbb{Z}$-graded (and $\mathbb{Z}$-filtrated) algebra generated by $T^*_x M$, subject to the relation

$$v^* \cdot w^* + w^* \cdot v^* = -2(v^*, w^*). \quad (3.11)$$

A Clifford module is a graded vector bundle $\mathcal{V}$ over $M$ with a graded homomorphism $c_\mathcal{V}: \mathcal{C} \rightarrow \text{End}(\mathcal{V})$. $\Lambda(T^*M)$ is a Clifford module with the action $c_\Lambda(v^*)\alpha = v^* \wedge \alpha - i_v(\alpha)$ where $v$ is dual to $v^*$ in the inner product.

If $M$ is even-dimensional and spin, the spinor bundle $\mathcal{S} = \Lambda\mathcal{P}$, where $\mathcal{P}$ is a polarization of the complexified cotangent bundle of $M$ is a Clifford module. Indeed, with this action, $\mathcal{C} \cong \text{End}(\mathcal{S})$, and any Clifford module can be written as $\mathcal{V} = \mathcal{S} \otimes \mathcal{T}$, where $\mathcal{T}$ is a vector bundle on which $\mathcal{C}$ acts trivially.

If $\mathcal{V}$ is a Clifford module, a connection $\nabla^\mathcal{V}$ is a Clifford connection if, for any vector field $X$ and section $Y$ of $T^*M$,

$$\left[\nabla^\mathcal{V}_X, c_\mathcal{V}(Y)\right] = c_\mathcal{V}\left(\nabla^\mathcal{V}_Xc_\mathcal{V}Y\right). \quad (3.12)$$

(The bracket on the left-hand side is graded.) In the case where $M$ is even-dimensional and spin, any Clifford connection $\nabla^\mathcal{V}$ can be written as

$$\nabla^\mathcal{V} = \nabla^\mathcal{S} \otimes 1 + 1 \otimes \nabla^\mathcal{T} \quad (3.13)$$

for some connection $\nabla^\mathcal{T}$ on $\mathcal{T}$ and the Levi-Civita connection $\nabla^\mathcal{S}$ on $\mathcal{S}$. If $M$ is even-dimensional but not spin, the Clifford action is still faithful and the curvature of a Clifford connection still decomposes as $R + F^T$, where $R$ is Riemannian curvature and $F^T$ is the component of the curvature in $\text{End}_{\mathcal{C}(M)}(\mathcal{V})$ [BGV04] (Props. 3.35, 3.40 & 3.43).

If $\mathcal{V}$ is a Clifford module and $\nabla^\mathcal{V}$ a Clifford connection, the twisted Dirac operator is

$$\mathcal{D}^\mathcal{V} = c_\mathcal{V}(dx^i)\nabla^\mathcal{V}_i. \quad (3.14)$$

This provides a square root of the generalized Laplacian $\Delta^\mathcal{V}$ with the choice of section $V = c_\mathcal{V}(F^T) - t/4$, where $c_\mathcal{V}$ acts on two-forms by $c_\mathcal{V}(v^* \wedge w^*) = \frac{1}{2}[c_\mathcal{V}(v^*)c_\mathcal{V}(w^*) - c_\mathcal{V}(w^*)c_\mathcal{V}(v^*)]$. That is, with this $V$,

$$\Delta^\mathcal{V} = (\mathcal{D}^\mathcal{V})^2. \quad (3.15)$$
(In the special case $\mathcal{V} = \mathcal{S}$, the operator $D^\mathcal{V}$ is the ordinary Dirac operator.) If $(\mathcal{V}, M, g, \Delta)$ is tame then the kernel $K_{\Delta^\mathcal{V}}$ associated by Eq. (3.2) to this data is an approximate heat kernel and converges to the heat kernel of the square of the Dirac operator by Thm. 2.2.

### 3.3.2 The action

If $M$ is even-dimensional and spin and $\mathcal{T}$ is a bundle over $M$ with a connection whose curvature is $F$, define twisted $N = 1/2$ SUSYQM via the action

$$
\int \frac{1}{2} (\sigma, \dot{\sigma}) + i \left( \Psi^\dagger, \nabla^S_x \Psi \right) + i \left( \Pi^\dagger, \nabla^T \Pi \right) - \frac{i}{2} \left( F(\Psi, \Psi^\dagger, \Pi, \Pi^\dagger) \right) ds,
$$

for $\Psi$ and $\Psi^\dagger$ Grassman-valued lifts of $\sigma$ to $\mathcal{P}^*$ and $\mathcal{P}$ respectively, and $\Pi$ and $\Pi^\dagger$ Grassman-valued lifts to $\mathcal{T}^*$ and $\mathcal{T}$ respectively. This action was first written down by Friedan and Windey [FW84] (with slightly different normalization conventions). If $\mathcal{T}$ is the trivial bundle it reduces to the action for $N = 1/2$ SUSYQM [AG83].

Discretize as above to get a kernel on $V = \mathcal{S} \otimes X T$

$$
K_{\text{SUSY}} = \int H_D(x, y; t) e^{-R_{\text{cc}}(x_y, x_p)/12} \int \frac{1}{2} e^i (\psi_x, \nabla^T \psi_y - \psi_y) + i(\eta_x, \Psi^\dagger x - \eta_y) + i(\eta^\dagger_x, \Psi x \psi_y^\dagger - \psi_y) ds, ds,
$$

where the parallel transports are with respect to the connections $\nabla^S_x$ and $\nabla^T$. As in the general case, the terms with parallel transport represent, under Berezin integration, the kernel of $e^{-\frac{i}{4}c(F) - t/4} P_{\eta^\dagger}$, with this parallel transport being with respect to the connection on $V$. Thus the discretization is exactly the approximate heat kernel $K_{\Delta^\mathcal{V}}$ for $V = c(F) - t/4$.

The results of Sect. 2 apply to rigorously construct the twisted $N = 1/2$ SUSYQM path integral as $K_{\text{SUSY}}^\infty$ which will agree with the heat kernel for $\Delta^\mathcal{V}$. Restricting $K_{\text{SUSY}}^\infty$ to the appropriate degrees in $\eta, \eta^\dagger$ gives the heat kernel for $\Delta^\mathcal{V} = (D^\mathcal{V})^2$ where $\mathcal{V} = \mathcal{S} \otimes \mathcal{T}$.

Of course the heat kernel for the ordinary Dirac operator is the fine-partition limit of products of the above without the terms referring to $\mathcal{T}$. The corresponding action is just

$$
\int \frac{1}{2} (\sigma, \dot{\sigma}) + i \left( \Psi^\dagger, \nabla^S_x \Psi \right) ds,
$$

which is the usual action for $N = 1/2$ SUSYQM.

### 4 The Small $t$ Asymptotics

McKean & Singer [MS67] recognized that the Gauss-Bonnet-Chern theorem would follow from a sufficiently detailed knowledge of the short-time diagonal behavior of the heat kernel of the Laplace-de Rham operator on differential forms. They used Duhamel’s formula to derive the behavior in degree zero. Gilkey [Gil84] summarizes an approach of Seeley [Sec67], Patodi [Pat71] and Atiyah, Patodi & Singer [APS75, FPS75, APS76] which extends this to cover the square of the Dirac operator of Sect. 3.3, writes the corresponding heat operator as a contour integral, and ultimately approximates the heat kernel by approximating the operator in the integrand. This approximation leads to a heat kernel proof of the Atiyah-Singer index theorem.

Witten [Wit82a, Wit82b] observed that McKean and Singer’s argument fit naturally into the language of supersymmetry, that the heat kernel for the Dirac operator was the
(imaginary-time) propagator for an appropriate supersymmetric quantum mechanical theory, and that standard physics calculations of stationary phase/steepest descent should give the small-time behavior of this propagator. Álvarez-Gaumé [AG83] and, independently, Friedan and Windey [FW84] implemented this program to give path integral “proofs” of the index theorem. Their arguments differ from earlier heat kernel proofs in that the these small-time asymptotics are computed not from the heat equation directly but from steepest descent based on the Lagrangian appearing in the path integral representation. Friedan and Windey in particular cover the general case of a Dirac operator associated to an arbitrary Clifford bundle and Clifford connection, leading to what Berline, Getzler and Vergne [BGV04] refer to as the local index theorem. The argument below follows Friedan and Windey closely, although the more mathematician-friendly notation and terminology are those of Berline et al.

Prop. 2.1 implies the asymptotics of the heat kernel at the diagonal are local, so it suffices to work over \( \mathbb{R}^m \) with a nonstandard metric. Eq. (4.2) rescales the corresponding approximate kernel on \( \mathbb{R}^m \) in a way familiar from standard uses of steepest descent, with the extra wrinkle that the Clifford bundle is also rescaled. The idea is that the rescaling does not affect the small-\( t \) behavior on the diagonal. In fact, Prop. 4.1 shows the rescaling operation commutes with taking the fine-partition limit. On the other hand, Prop. 4.2 shows the rescaled kernel on a given partition approaches, in a certain limit of the rescaling parameter, that of a flat theory with a magnetic term, which is exactly solved in Prop. 4.3. Thm. 4.1 uses the strong results of Eq. (2.5) to interchange the rescaling parameter and the fine partition limits, from which the local version of the the Atiyah-Singer index theorem follows directly.

Suppose \( V \) is a Clifford module over an even-dimensional manifold \( M \) with Riemannian \( g \), \( \nabla^V \) is a Clifford connection, and \( V = c_V (F^T) - t/4 \), so the associated elliptic operator \( \Delta^V = (D^V)^2 \) as in Subsection 3.3. If all of that data is tame (for example, compact and smooth), then \( K_{\Delta^V} = K_{\Delta^2} \) as defined by Thm. 3.1 has a large partition limit \( K^\infty_{\Delta^2} \) which is the heat kernel for \( D^2 \).

Let \( x_0 \in M \). Endow a ball of radius \( D_1 > 0 \) around \( x_0 \) with Riemann normal coordinates, and identify the bundle over it with \( V_{x_0} \) via parallel transport along minimal geodesics. This defines a metric \( g_1 \), a trivial bundle \( V_1 \), and a connection \( \nabla^1 \) over a neighborhood of the origin in \( \mathbb{R}^m \), all with bounded derivatives up to order four. Extend all of these to all of \( \mathbb{R}^m \) so that the derivatives remain bounded and so that both \( \nabla^1 \) and the Levi-Civita connection \( \nabla^{g_1} \) continue to be 0 on radial directions. Let \( C \) denote the Clifford algebra \( C(T_{x_0} M) \) at \( x_0 = 0 \), whose action on \( V_{x_0} \) splits it into \( S \otimes T \), where \( S \) is the spinor representation of \( C \) and \( C \) acts trivially on \( T \). \( V_1 \) can be identified with the trivial bundle \( S \otimes T \) over \( \mathbb{R}^m \). Identifying the Clifford algebra at any point in \( \mathbb{R}^m \) with \( C \) by radial translation gives it an action on \( S \otimes T \) that makes \( \nabla^1 \) a Clifford connection agreeing with \( \nabla^V \) in the ball of radius \( D_1 \). In fact then \( \nabla^1 = \nabla^{g_1} \otimes 1 + 1 \otimes \nabla^T \), where \( \nabla^{g_1} \) is the Levi-Civita connection on \( S \) and \( \nabla^T \) is some connection on \( T \) with curvature \( F^T \). The choice \( V_1 = c(F^T) - t_1/4 \) defines a Dirac operator \( D_1 \) on \( (g_1, S \otimes T \times \mathbb{R}^m, \nabla^1) \) whose associated approximate heat kernel \( K_1 = K_{D_1} \) can be identified with \( K_{D_2} \) in that ball by the obvious isomorphism, and therefore by Prop. 2.1

\[
K^\infty_{D_2}(x_0, x_0; t) - K^\infty_1(0, 0; t) = O\left(e^{-(d_1)^2/t}\right)
\]  

(4.1)

for some \( d_1 > 0 \).

To investigate the small-time asymptotics of \( K_{D_2} \) at \( x_0 \) it thus suffices to consider only \( K_1(0, 0; t) \). Because the bundle is trivial, \( K_1 \) can be taken not as a section but as a function with values in \( \text{End}(S) \otimes \text{End}(T) \sim C \otimes \text{End}(T) \). The Clifford algebra action \( c_\Lambda \) on \( \Lambda T^*_{x_0} M \)
means $K_1$ also picks out a function with values in $\text{End}(\Lambda T_{x_0}^* M) \otimes \text{End}(T)$. Mildly abuse notation to let $K_1$ also refer to this function. Thus, $K_1$ is a kernel on the trivial bundle $\Lambda T_{x_0}^* M \times T$ over $\mathbb{R}^m$, and $K_1$ is still an approximate semigroup with the same constants as before, call them $(B, C, D, T)$.

To rescale $K_1$, define a family of metrics $g_r$ on $\mathbb{R}^m$ for $0 \leq r \leq 1$ as follows: Define $\phi_r: \mathbb{R}^m \to \mathbb{R}^m$ by $\phi_r(x) = r x$, define $\psi_r: \Lambda T_{x_0}^* M \to \Lambda T_{x_0}^* M$ by $\psi_r(\alpha) = r^{\deg(\alpha)} \alpha$ for $\alpha$ homogeneous. Finally, define $g_r = r^{-2} \phi_r^*[g_1]$, and extend by continuity to $g_0 = g_{1,0}$. By construction, $g_{1,0}(v, w) = (v, w)$, the standard inner product on $\mathbb{R}^m$. This family has the following properties (extending each formula by continuity to $r = 0$):

\[
g_{r, x}(v, w) = g_{1, x}(v, w)
\]
\[
d_{g_r}(x, y) = r^{-1} d_{g_1}(r x, r y)
\]
\[
(y)_{g_r} = r^{-1} ((r y)_x)_{g_1}
\]
\[
\text{Ricci}_r(y, x) = \text{Ricci}((r y)_x, (r y)_x)
\]
\[
\tau_r = r^{2} \tau_1
\]
\[
d_{g_r, y} = r^{-m} d_{g_1}(r y).
\]

If $K(x, y; t)$ is a kernel on the bundle $\Lambda T_{x_0}^* M \times T$ over $\mathbb{R}^m$, let $\Phi_r$ rescale $K$ according to

\[
\Phi_r[K](x, y; t) = r^{m} \psi_r^{-1} K(r x, r y; r^2 t) \psi_r.
\]

(4.2)

This rescaling extends $K_1$ to a family of kernels $K_r$ on the same bundle via

\[
K_r = \Phi_r(K_1).
\]

**Proposition 4.1** $\Phi_r$ is a homomorphism from the kernel product $\ast$ using the metric $g_1$ to the kernel product $\ast$ using the metric $g_r$ for $r > 0$. A $t$-norm can be chosen for each metric $g_r$ and constants $(B, C, D, T)$ independent of $0 < r < 1$ such that $\Phi_r$ is a map of norm at most 1 between the respective $t$-norms and such that $K_r$ with the metric $g_r$ is an approximate semigroup with constants $(B, C, D, T)$ independent of $r$. Finally $K_r^\infty = \Phi_r[K_1^\infty]$.

**Proof:** For the first claim

\[
\Phi_r[K](t_1) \ast \Phi_r[J](t_2)(x, z) = \int \Phi_r[K](x, y; t_1)\Phi_r[J](y, z; t_2)d_{g_r, y}
\]
\[
= \int \psi_r^{-1} r^{-m} K(r x, r y, r^2 t_1)J(r y, r z; r^2 t_2)\psi_r d_{g_r, y}
\]
\[
= r^{m} \psi_r^{-1} \int K(r x, u, r^2 t_1)J(u, r z; r^2 t_2) d_{g_r, u} \psi_r
\]
\[
= r^{m} \psi_r^{-1} K \ast J(r x, r z; r^2 t) \psi_r = \Phi_r[K \ast J](x, z; t).
\]

For the second, write $B_1, D_1$ for the corresponding constants in Prop. 1.1 and Cor. 1.1 as determined by the bounds for $g_1$. Since the supremum norm on $g_r$ and all its derivatives are bounded by the corresponding quantities for $g_1$, these constants work for any $g_r$. In particular, there is a $t$-norm satisfying Cor. 1.1 independent of $r$. Write $H_{D, g}$ for the kernel $H_D$ defined by Eq. (1.4) and $K_{B, D, g}$ for the kernel $K_{B, D}$ defined by Eq. (1.7) to emphasize their dependence on a given metric. Defining $J$ by

\[
K_{B_1, D_1, g_1} = \chi_{< r D_1} K_{B_1, D_1, g_1} + J,
\]

28
conclude that

$$\Phi_r[K_{B_1,D_1,g_1}] = K_{B_1,D_1,g_r} + \Phi_r[J]$$

with \( |\Phi_r[J]|_{\ker} \leq t e^{-D^2/(2ot)} \) for small enough \( t \). Suppose \( |K|_{(t)} = 1 \), so \( K = \tilde{K} + \tilde{J} \) where \( \tilde{K} \leq e^{B_1\sqrt{t}} \int K_{B_1,D_1,g_1}(at) \, d\mu_\alpha \) and \( \tilde{J} \leq e^{-D^2/(2ot)} \). Then

$$|\Phi_r(\tilde{K})| \leq e^{B_1\sqrt{t}} \int \Phi_r(K_{B_1,D_1,g_1}(at)) \, d\mu_\alpha$$

$$\leq e^{B_1\sqrt{t}} e^{-D^2/(2ot)} \int K_{B_1,D_1,g_r}(at) + \Phi_r[J(at)] \, d\mu_\alpha,$$

so \( \Phi_r(\tilde{K}) \) is an element of \( E_{B_1,D_1,g_r}(t) \) plus a kernel \( J' \) with \( |J'|_{\ker} \leq 2te^{B_1\sqrt{t}} e^{-D^2/(4ot)} \). Meanwhile \( \|\Phi_r(\tilde{J})\|_{\ker} \leq \|\tilde{J}\|_{\ker} \leq t e^{-D^2/(2ot)} \). So replacing \( D_1 \) with a smaller \( D \) makes \( \Phi_r(K) \in E_{B_1,D_1,g_r}(t) \). With these choices of constants, which still depend only the bounds on \( g_r \), \( \Phi_r \) is norm at most 1 as a map between the corresponding \( t \)-norms.

Since \( |K_t|_{(t)} \leq 1 \), the preceding argument implies \( |K_r|_{(t)} \leq 1 \). Notice also that

$$|K_r(t_1) * K_r(t_2) - K_r(t)|_{(t)} = |\Phi_r(K_1(t_1) * K_1(t_2) - K_1(t))|_{(t)}$$

$$\leq |K_1(t_1) * K_1(t_2) - K_1(t)|_{(t)} \leq C r^{3/2}$$

so \( K_r \) is an approximate semigroup with constants independent of \( r \).

\[ \square \]

As \( r \to 0 \), the rescaled kernel \( K_r \) will approach a kernel \( K_0 \) defined as follows: First, define \( R \in \Lambda T_{x_0}^* M \otimes \text{End}(T_{x_0} M) \) by

$$R^i_k \equiv \frac{1}{2} R^i_{jk} dx^j \wedge dx^k,$$

where \( R \) is evaluated at \( x_0 \), and then define \( F \in \Lambda T_{x_0}^* M \otimes \text{End}(T) \) by

$$F = \frac{1}{2} R^i_{jk} dx^j \wedge dx^k,$$

where \( F \) is likewise evaluated at \( x_0 \). Finally, define

$$H_0(x, y; t) = (2\pi t)^{-m/2} e^{-|y-x|^2/(2t)}$$

(4.3)

and the kernel on \( \Lambda T_{x_0}^* M \)

$$K_0(x, y; t) = H_0(x, y; t) e^{i[R(x,y) - tF]/2}$$

(4.4)

where the elements of \( \Lambda T_{x_0}^* M \) on the right-hand side act by multiplication.

**Remark 4.1** These analytic functions of \( R \) and \( F \) are defined via power series, and are well-defined for all \( t \) because \( R \) and \( F \) are nilpotent.

**Lemma 4.1**

$$\lim_{r \to 0} K_r = K_0$$

(4.5)

pointwise.
Proof: Using $K_r = \Phi_r(K_1)$, the definition of $K_1$ as $K_1(D_t)^2$ of Eq. (3.2) for an elliptic operator $\Delta_1 = (D_t)^2$ over $V_1$ and the definition of $K_0$ above, the statement expands to
\[
\lim_{r \to 0} r^m (2\pi t)^{-m/2} e^{-[d_{g_0}(x,y)]^2/(2t)} \times e^{-\text{Ricci}_r(y, y)} / 12 + \text{tr}_r / 24 - \frac{1}{2} F^T_{ij}(y) \psi_r^{-1} c(dx_i)c(dx_j) \psi_r^{-1} \frac{\partial}{\partial x_i} \psi_r \psi_r^{-1} \frac{\partial}{\partial x_j} \psi_r = H_0 e^{[\text{Ricci}_r(y, y) - 1] / 4}.
\]

Thus the lemma reduces to the the following assertions:
\[
\lim_{r \to 0} d_{g_0}(x, y) = |x - y|,
\]
\[
\lim_{r \to 0} \text{Ricci}_r(y, y) = 0,
\]
\[
\lim_{r \to 0} \tau_r = 0,
\]
\[
\lim_{r \to 0} F^T_{ij}(y) \psi_r^{-1} r^2 c(dx_i)c(dx_j) \psi_r / 2 = F \quad \text{and}
\]
\[
\lim_{r \to 0} \psi_r^{-1} \frac{\partial}{\partial x} \psi_r = e^{[\text{Ricci}_r(y, y) - 1] / 4}.
\]

The first three are immediate from the fact that $g_1 = g_0 + O(|x|^2)$. The fourth follows from the fact that $\lim_{r \to 0} \psi_r^{-1} c(dx) \psi_r = dx$.

For the fifth limit, having trivialized the bundle radially at the origin, the parallel transport from $rx$ to $ry$ is the holonomy of the geodesic triangle from 0 to $rx$ to $ry$ to 0. Treat this separately on $\mathcal{T}$ and on $\Lambda(T^*_0M)$. On $\mathcal{T}$ the holonomy differs from 1 by a quantity proportional to the area enclosed, which is $O(r^2)$. For the $\Lambda(T^*_0M)$ piece, the holonomy is an element of the spin group and therefore an exponential of a degree-two element of $C$. This exponential in turn is the image under $c$ of the two-form generating the holonomy about the same geodesic triangle with respect to the Levi-Civita connection. It is standard [AS53] that this is $(R \cdot rx, ry - rx) / 4 + O(|r| y - rx)| r y + r x|$. Thus, this piece is the exponential of the image under $c$ of $(R \cdot rx, ry) / 4 + O(r^3)$. Conjugation by $\psi_r$ will reduce the power of $r$ by two, giving the result. \qed

**Proposition 4.2** Given any partition $P$ of any $t > 0$
\[
\lim_{r \to 0} K^*_r P (0, 0; t) = K^* P (0, 0; t).
\]

**Proof:** This follows from Lemma 4.1, Lebesgue Dominated Convergence and the fact that $K_0$ and $K_r$ are bounded by $C_1 H(x, y; C_2 t)$ for some $C_1, C_2$ where $H(x, y; t) = (2\pi t)^{-m/2} e^{-d^2_{g_0}(x, y)/(2t)}$, which in turn follows from the same bound on $K_1$. \qed

**Proposition 4.3**
\[
\lim_{|P| \to 0} K^* P = K^* \quad \text{(4.6)}
\]
converges pointwise, and is the heat kernel for the operator
\[
\Delta = \frac{\partial^2}{\partial x_i \partial x_i} + \frac{1}{2} R^y_{ij} x_j \frac{\partial}{\partial x_i} - F + |R |^2 / 16. \quad \text{(4.7)}
\]

30
Therefore,  
\[
K_0^\infty(0, 0; t) = (2\pi t)^{-m/2} \det^{1/2} \left( \frac{tR/4}{\sinh(tR/4)} \right) e^{-2tR/2}. \tag{4.8}
\]

**Proof:**

A slight modification of Roger’s proof of Theorem 8.2 in [Rog03] would give this result. However, that argument refers to expectations in a variant of Wiener measure. The following proof uses the language of products of approximate kernels. First check that for small enough \( t = t_1 + t_2 > 0 \),

\[
K_0(t_1) * H_0(t_2) = \left[ 1 + \frac{t_1}{2}(\Delta - \Delta_i) \right] H_0(t) + \mathcal{O}\left( \frac{t_1^2}{t} \right) (1 + |x|^m) H_0(2t).
\]

To see this, begin by directly computing the Gaussian integral, and use the skew-symmetry of \( R \) to obtain

\[
[K_0(t_1) * H_0(t_2)](x, z; t) = (4\pi^2 t_1 t_2)^{-m/2} \int e^{(x, R(y-x))/4 - t_1 t_2/2 - |y-x|^2/(2t_1) - |x-y|^2/(2t_2)} dy
\]

\[
= H_0(x, z; t)e^{\frac{1}{4t}(x, R(z-x))}. \tag{12a}
\]

Notice that since \( F \) and \( R \) take values in the algebra \( \Lambda T^*_{x_0} M \otimes \text{End}(T) \) and are therefore nilpotent, the exponential truncates to multinomials. Expanding the exponential, and comparing the definition of \( \Delta \) with the Laplacian \( \Delta_i \), corresponding to the Euclidean metric on \( R^m \), gives

\[
[K_0(t_1) * H_0(t_2)](x, z; t) = \left[ 1 + \frac{t_1}{2}(\Delta - \Delta_i) \right] H_0(x, z; t) + \frac{t_1}{32} |Ry|^2 \left( \frac{t_2}{t} - 1 \right) H_0(x, z; t)
\]

\[
+ \mathcal{O} \left( \frac{t_1}{4t} (x, R(z-x)) - \frac{t_1}{2} F + \frac{t_1 t_2}{32t} |Ry|^2 \right) H_0(x, z; t).
\]

Noting \( 1 - \frac{t_1}{4t} = \frac{t_1}{4t} \leq t_1 \) and, as in the proof of Lemma 3.1, \( |z - x|^k H_0(x, z; t) \) is bounded by a multiple of \( H_0(x, z; 2t) \), it is easy to bound the error term by \( \mathcal{O}\left( \frac{t_1^2}{t} \right) (1 + |x|^m) \) \( H_0(x, z; 2t) \) as claimed.

If \( P_n \) is the partition \( (t/n, t/n, \ldots, t/n) \) then

\[
K_0^{*P_n} = [H_0 + (K_0 - H_0)]^{*P_n}
\]

\[
= \sum_{k=0}^m \sum_{i_0 + i_1 + \cdots + i_n = n-k} H_0(i_0 t/n) \ast [K_0(t/n) - H_0(t/n)] \ast H_0(i_1 t/n) \ast \cdots \ast [K_0(t/n) - H_0(t/n)] \ast H_0(i_k t/n) \ast [K_0(t/n) - H_0(t/n)] \ast H_0(i_k t/n),
\]

where the first sum is only to \( m \) because \( K_0 - H_0 \) is of degree at least one in \( \Lambda T^*_{x_0} M \).

Replace each of the \( [K_0(t/n) - H_0(t/n)] \ast H_0(i_j t/n) \) with \( \frac{t}{n(t_j + 1)} (\Delta - \Delta_i) H_0(i_j t/n) + \mathcal{O}\left( \frac{t}{n(t_j + 1)} (1 + |x|^m) \right) H_0(2i_j t/n) \). The second term introduces into the sum a finite
number, independent of \( n \), of “error terms”. Each contributes a summand

\[
\mathcal{O}\left[ \frac{t}{n(tj + 1)} (1 + |x|^m) \right] \sum_{i_0 + i_1 + \cdots + i_k = n - k} H_\infty(i_0 t/n) \ast \frac{t}{2n} (\Delta - \Delta_\infty) H_\infty([i_1 + 1|t/n])
\]

\[
\cdots \ast H_\infty(2ij t/n) \ast \cdots \ast \frac{t}{2n} (\Delta - \Delta_\infty) H_\infty([i_k + 1|t/n])
\]

\[
= \mathcal{O}\left[ \frac{t}{n(tj + 1)} (|x||)_H H_\infty(2t) \right]
\]

where \( P \) is some polynomial. As \( n \) goes to infinity, the contribution of each of the finitely-many error terms goes to zero, leaving

\[
K_{0,k}^\infty = \sum_{k=0}^m \frac{t^k}{n^k} \sum_{i_0 + i_1 + \cdots + i_k = n - k} H_\infty(i_0 t/n) \ast \frac{1}{2} (\Delta - \Delta_\infty) H_\infty([i_1 + 1|t/n])
\]

\[
\cdots \ast \frac{1}{2} (\Delta - \Delta_\infty) H_\infty([i_k + 1|t/n])
\]

\[
= \sum_{k=0}^m \frac{t^k}{n^k} \sum_{i_0 + i_1 + \cdots + i_k = n} H_\infty(i_0 t/n) \ast \frac{1}{2} (\Delta - \Delta_\infty) H_\infty([i_1 t/n]) \ast \cdots \ast \frac{1}{2} (\Delta - \Delta_\infty) H_\infty([i_k t/n])
\]

\[
\xrightarrow{n \to \infty} H_\infty(t) + \sum_{k=1}^m \int \cdots \int_{tj \geq 0} \cdots H_\infty(t_k) \ast \frac{1}{2} (\Delta - \Delta_\infty) H_\infty(t_1)
\]

\[
\cdots \ast \frac{1}{2} (\Delta - \Delta_\infty) H_\infty(t_k) dt_k \cdots dt_1.
\]

This sum agrees with McKean and Singer’s expression [MS67] for the heat kernel for \( \Delta \), as a sum of \( k\)-fold \# products, which they derive from Duhamel’s formula. Their expression has the sum taken over all nonnegative integers \( k \), but, again, terms with more than \( m \) factors of \( \Delta - \Delta_\infty \) vanish due to their form degrees.

\[\square\]

**Lemma 4.2** If \( A \in \mathcal{C}_{2k} \), the degree-2k subset in the Clifford filtration, then the map taking \( A \) to \( \lim_{n \to \infty} \psi_r^{-1} c_{2k}(A) \psi_r r^{2k} \) is multiplication by \( \rho_k(A) \in \mathcal{L}^{2k} T_{x_0}^* M \), where \( \rho_k(A) \) denotes the degree 2k component of \( c_{2k}(A) \). In particular \( \rho_k(A) \) and hence the limit is zero on \( \mathcal{C}_{2k-1} \), and gives the standard identification of \( \mathcal{C}_{2k}/\mathcal{C}_{2k-1} \) with forms of degree 2k.

**Proof:** Conjugation by \( \psi_r \) on \( \text{End}(\Lambda T_{x_0}^* M) \) multiplies homogeneous operators of degree \( k \) on \( \Lambda T_{x_0}^* M \) by \( r^{-2k} \). Since \( c_{2k}(A) \) is a sum of maps on forms of homogeneous degrees ranging from 0 to 2k, the small \( r \) limit will project onto the degree 2k component. In fact, this projection acts as multiplication by \( \rho_k(A) \). In particular it is zero on \( \mathcal{C}_{2k-1} \) and sends \( v_1^* v_2^* \cdots v_{2k}^* \) to \( v_1^* \wedge v_2^* \wedge \cdots \wedge v_{2k}^* \), proving the second sentence. \[\square\]

The following is what Berline, Getzler and Vergne call the local version of the Atiyah-Singer index theorem for a general Dirac operator. The index theorem follows directly from this as in [BGV04].
Theorem 4.1 If \( D \) is a Dirac operator on a Clifford bundle \( \mathcal{V} \) over a smooth, compact, oriented Riemannian manifold \((M, g)\) of dimension \( m \), and \( x_0 \in M \), then the diagonal of the heat kernel \( K^{\infty}_{D}(x_0, x_0; t) \) of \( D^2 \) is asymptotic to a Laurent series in \( t \) of the form

\[
P(t) = \sum_{k=0}^{\infty} A_k t^{k-m/2},
\]

with \( A_k \in C_2 \otimes \text{End}(\mathcal{T}) \). Writing \( \rho(P(t)) = \sum_{k=0}^{m/2} \rho_k(A_k) t^{k-m/2} \) for \( \rho_k \) as in Lemma 4.2,

\[
\rho(P(t)) = (2\pi t)^{-m/2} \det^{1/2}(\frac{tR/4}{\sinh(tR/4)}) e^{-tF/2}
\]

where \( R \) and \( F \) are the curvature forms at \( x_0 \) as in the definition of \( K_0 \) in Eq. 4.4.

Proof: By Eq. (4.1), it suffices to prove the result for \( K_1 = K_{(D^1)^2} \). Using [BGV04][Thm. 2.30], \( K^{\infty}(0, 0; t) = \sum_{i=0}^{\infty} \psi_i c_\lambda(A_i) \psi_i r^{2i} t^{i-m/2} + O(t) \) for some \( A_i \in C \otimes \text{End}(\mathcal{T}) \). Eqs. (3.14) and (3.15) imply each \( A_i \) is of degree \( 2i \) in the Clifford filtration, so \( c_\lambda(A_i) \) is of degree at most \( 2i \) as an element of \( \text{End}(\mathcal{T}) \).

By Lemma 4.2, continuing to write \( K_1 \) for both the kernel and its image under \( c_\lambda \),

\[
\lim_{r \to 0} \Phi_{r}[K^{\infty}_1](0, 0; t) = \lim_{r \to 0} \sum_{k=0}^{m/2} \psi_k^{-1} c_\lambda(A_k) \psi_k r^{2i} t^{i-m/2} = \rho(P(t)),
\]

since the last term in the above sum for \( K^{\infty}_1(0, 0; t) \) and the error term get taken to \( \lim_{r \to 0} \psi_k^{-1} c_\lambda(A_k) \psi_k r^{m+2} \), which is 0 for \( A \in C \otimes \text{End}(\mathcal{T}) \). Thus the theorem is equivalent in light of Prop. 4.3 to

\[
\lim_{r \to 0} \Phi_{r}[K^{\infty}](0, 0; t) = \lim_{|P| \to 0} K^{\infty}_P(0, 0; t).
\]

To prove this statement, fix \( t > 0 \). Given \( \epsilon \), choose \( P \) so that both

\[
\left| K^{\infty}_P(0, 0; t) - \lim_{|P| \to 0} K^{\infty}_P(0, 0; t) \right| < \epsilon/3,
\]

\[
\left| K^{\infty}_P(0, 0; t) - \Phi_{r}[K^{\infty}](0, 0; t) \right| < \epsilon/3
\]

for all \( 0 < r \leq 1 \), where the second estimate follows from Thm. 2.1 and the uniformity of the constants for the \( K_r \). For that \( P \) choose \( r \) so small by Prop. 4.2 that

\[
\left| K^{\infty}_P(0, 0; t) - K^{\infty}_0(0, 0; t) \right| < \epsilon/3
\]

by the pointwise convergence of \( K_r \).

Remark 4.2 Roughly speaking, this proof implements the steepest descent approximation (which is the imaginary-time version of the stationary phase) of the rigorous path integral, for the leading terms. Recall that steepest descent approximates \( \int e^{\phi(x)/i} dx \) by expanding \( \phi \) in a Taylor series about a critical point, and rescaling \( x \) by \( \sqrt{t} \). Choosing to throw away all terms of positive power in \( \epsilon \) replaces \( \phi \) by a quadratic approximation \( \phi_\hbar \), and the approximation to the integral is \( \int e^{\phi_\hbar(x)/i} dx \). Applying this reasoning heuristically to the path integral, using \( \hbar \) as the parameter, results in a Lagrangian with, in general, harmonic oscillator and linear
magnetic terms. Various standard approaches, including Wiener measure, apply to evaluate
this path integral with a purely quadratic exponent, giving what is termed the “semiclassical
approximation”.

Lemma 4.1, and Prop. 4.2 give a rigorous version of this argument, except with \( r \) as
the small parameter instead of \( h \). Moreover, the rescaling here involves both space and the
Clifford bundle, and the expansion is about the constant path. Prop. 4.3 rigorously defines
the path integral with the quadratic action by time-slicing and the fine partition limit. The
interchange of the small-\( r \) and fine-partition limits concluding the proof of the theorem above
thus provides, in this sense, a rigorous proof of the leading terms of the steepest descent
approximation for this nontrivial path integral.

5 Conclusion

The argument culminating in Thm. 4.1 is a direct translation of the heuristic path inte-
gral proof of the Atiyah-Singer index theorem for the twisted Dirac operator into rigorous
mathematics. Prop. 3.1 and Thm. 2.2 provide a rigorous version of the relevant time-sliced
path integral for each of a set of theories including twisted SUSYQM; as expected, the
path integral agrees with the heat kernel. In fact, this gives a new construction of the heat
kernel. That the steepest descent approximation \( K_\infty^0 \) to the path integral indeed gives its
asymptotic behavior on the diagonal is the crux of Thm. 4.1. The explicit calculation in
Prop. 4.3 of \( K_\infty^0 \) thus gives the asymptotic behavior of the heat kernel and with it the index
theorem.

With the appropriate choice of twisting bundle, namely \( \mathcal{T} \) being the dual spinor bundle,
Thm. 4.1 also gives the Gauss-Bonnet-Chern theorem, which was the subject of the authors’
recent work [FS14]. Friedan and Windey [FW84] prove this by explicitly reducing \( K_\infty^0 \) for
this case to the Pfaffian expression of the Chern form.

Of course the Laplace-Beltrami operator on functions on \( M \) is a generalized Laplacian
with trivial vector bundle \( V \). In this special case, the limit of products of approximate kernels
constructs the path integral for ordinary (bosonic) quantum mechanics, and Thm. 3.1 shows
this path integral computes the heat kernel for the Laplace-Beltrami operator. This implies
results similar to those of Andersson and Driver [AD99], though the convergence here is
uniform rather than weak.

The time-slicing approach to the path integral can incorporate functions of paths. In
fact, the given construction of the path integral readily extends to rigorously define the
path integrals for \( n \)-point functions. The form of the resulting expression suggests the path
integral of Prop. 3.1 agrees with a generalization of Wiener measure based on the heat kernel
for generalized Laplacians.

It seems entirely plausible that Thm. 3.1 and Prop. 3.1 carry over to other sufficiently
simple quantum theories. Indeed, this was the authors’ original motivation for constructing
path integrals in SUSYQM. For a first instance, the approach of one of the authors to
Yang-Mills on a Riemann surface [Fln91] should easily combine with the construction of
the (bosonic) path integral given here to provide a rigorous construction of the functional
integral for the expectation of certain classes of Wilson lines in that theory. In this and
other cases, the rigorous stationary phase argument of Thm. 4.1 may apply, but with \( h \) as
the parameter. This would reproduce the semiclassical approximation or perhaps even the
full Feynman diagram expansion. This would be of particular interest in cohomological field
theories, where the stationary phase approximation is exact. A rigorous interpretation of the
path integral, in which the semiclassical approximation proves valid, would be the obvious
starting point to make rigorous several powerful path integral arguments in cohomological
field theories that yield interesting mathematical results.

References

[AD99] Lars Andersson and Bruce Driver. Finite-dimensional approximations to Wiener measure and path integral formulas on manifolds. *J. Funct. Anal.*, 165(2):430–498, 1999.

[AG83] Luis Alvarez-Gaumé. Supersymmetry and the Atiyah-Singer index theorem. *Commun. Math. Phys.*, 90:161, 1983.

[AJ90] Michael F. Atiyah and Lisa Jeffrey. Topological Lagrangians and cohomology. *J. Geom. Phys.*, 7(1):119–136, 1990.

[APS75] Michael F. Atiyah, Vijay K. Patodi, and Isadore M. Singer. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.

[APS76] Michael F. Atiyah, Vijay K. Patodi, and Isadore M. Singer. Spectral asymmetry and Riemannian geometry. III. *Math. Proc. Cambridge Philos. Soc.*, 79(1):71–99, 1976.

[Arn98] V. I. Arnold. *Ordinary Differential Equations*. MIT Press, 1998.

[AS53] W. Ambrose and I. M. Singer. A theorem on holonomy. *Trans. Amer. Math. Soc.*, 75:428–443, 1953.

[Ati85] Michael F. Atiyah. Circular symmetry and stationary-phase approximation. Colloquium in honor of Laurent Schwartz, vol. 1, (Palaiseau, 1983). *Astérisque*, 1(131):43–59, 1985.

[BGV04] Nicole Berline, Ezra Getzler, and Michèile Vergne. *Heat Kernels and Dirac Operators*. Springer, 2004.

[Bis84a] Jean-Michel Bismut. The Atiyah-Singer theorems: a probabilistic approach. I. The index theorem. *J. Funct. Anal.*, 57(1):56–99, 1984.

[Bis84b] Jean-Michel Bismut. The Atiyah-Singer theorems: a probabilistic approach. II. The Lefschetz fixed point formulas. *J. Funct. Anal.*, 57(3):329–348, 1984.

[Bla93] Matthias Blau. The Mathai-Quillen formalism and topological field theory. *J. Geom. Phys.*, 11(1-4):95–127, 1993. Infinite-dimensional geometry in physics (Karpacz, 1992).

[BP] Christian Bär and Frank Pfäffle. Path integrals on manifolds by finite dimensional approximations. AP/07032731v1.

[BT93] Matthias Blau and George Thompson. $N = 2$ topological gauge theory, the Euler characteristic of moduli spaces, and the Casson invariant. *Comm. Math. Phys.*, 152(1):41–71, 1993.

[dC92] Manfredo Perdigão do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[DH82] J. J. Duistermaat and G. J. Heckman. On the variation in the cohomology of the symplectic form of the reduced phase space. *Invent. Math.*, 69(2):259–268, 1982.

[Eva98] Lawrence C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, 1998.
[Fin91] Dana Fine. Quantum Yang-Mills on a Riemann surface. Comm. Math. Phys., 140:321–338, 1991.

[FPS75] Atiyah Michael F., Vijay K. Patodi, and Isadore M. Singer. Spectral asymmetry and Riemannian geometry. II. Math. Proc. Cambridge Philos. Soc., 78(3):405–432, 1975.

[FS08] Dana Fine and Stephen Sawin. A rigorous path integral for supersymmetric quantum mechanics and the heat kernel. Comm. Math. Phys., 284(1):79–91, 2008. arXiv:0705.0638.

[FS14] Dana Fine and Stephen Sawin. Short-time asymptotics of a rigorous path integral for n = 1 supersymmetric quantum mechanics on a riemannian manifold. J. Math. Phys., 55(6), 2014. arXiv:1207.2751.

[FW84] Dan Friedan and Paul Windey. Supersymmetric derivation of the Atiyah-Singer index and the chiral anomaly. Nuclear Phys. B, 235(3):395–416, 1984.

[Get86a] Ezra Getzler. The local Atiyah-Singer index theorem. In Phénomènes critiques, systèmes aléatoires, théories de jauge, Part I, II (Les Houches, 1984), pages 967–974. North-Holland, Amsterdam, 1986.

[Get86b] Ezra Getzler. A short proof of the local Atiyah-Singer index theorem. Topology, 25(1):111–117, 1986.

[Get91] Ezra Getzler. The Thom class of Mathai and Quillen and probability theory. In Stochastic analysis and applications (Lisbon, 1989), volume 26 of Progr. Probab., pages 111–122. Birkhäuser Boston, Boston, MA, 1991.

[Gil84] Peter B. Gilkey. Invariance theory, the heat equation, and the Atiyah-Singer index theorem, volume 11 of Mathematics Lecture Series. Publish or Perish, Inc., Wilmington, DE, 1984.

[MQ86] Varghese Mathai and Daniel Quillen. Superconnections, Thom classes, and equivariant characteristic classes. Topology, 1986.

[MS67] Henry P. McKean, Jr. and Isadore M. Singer. Curvature and the eigenvalues of the Laplacian. J. Differential Geometry, 1(1):43–69, 1967.

[Pat71] V. K. Patodi. Curvature and the eigenforms of the Laplace operator. J. Differential Geometry, 5:233–249, 1971.

[Rog87] Alice Rogers. A superspace path integral proof of the Gauss-Bonnet-Chern theorem. J. Geom. Phys., 4(4):417–437, 1987.

[Rog92a] Alice Rogers. Superspace calculus in superspace. I. Supersymmetric Hamiltonians. J. Phys. A, 25(2):447–468, 1992.

[Rog92b] Alice Rogers. Superspace calculus in superspace. II. Differential forms, supermanifolds and the Atiyah-Singer index theorem. J. Phys. A, 25(22):6043–6062, 1992.

[Rog03] Alice Rogers. Supersymmetry and Brownian motion on supermanifolds. Infin. Dimens. Anal. Quantum Probab. Relat. Top., 6(suppl.):83–102, 2003.

[See67] R. T. Seeley. Complex powers of an elliptic operator. In Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966), pages 288–307. Amer. Math. Soc., Providence, R.I., 1967.

[Wit82a] Edward Witten. Constraints on supersymmetry breaking. Nuclear Phys. B, 202(2):253–316, 1982.

[Wit82b] Edward Witten. Supersymmetry and morse theory. J. Differential Geom., 17(4):661–692 (1983), 1982.