Stress and Strain of Cracked Orthotropic Sheet

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Abstract. The analytic function of complex variable including a material parameter is analyzed fully. The new stress function has been formed by the complex function method. Typical shear mode II crack states are considered to orthotropic materials. The boundary shear loading problem of cracked orthotropic sheet is studied and the formulae for stress fields are derived. The singular strain fields are also discussed. And the expressions of the singular stress and strain in the crack-tip are determined with inclusion of the material elasticity coefficients.

1. Introduction
Linear elastic fracture mechanics (LEFM) has been found to be a very useful tool for design purposes and investigated in great detail for many engineering materials, whether isotropic or anisotropic [1]. Nowadays the fracture mechanics of anisotropic materials may be relative to many structural aspects in engineering evaluation. The prediction of crack initiation and propagation must be based on the fracture mechanics method [2]. Fiber-reinforced polymer matrix materials are as the most typical composites and usually modeled as anisotropic materials at the macroscopic level. And the orthotropic plates have been for the base of composites in common use [3, 4]. The plane fracture mechanics of composite materials may be more important, particularly the study for the shear mode II crack problem is very necessary.

The investigation of singular stress or strain fields near the crack tip holds an essential part of LEFM. Generally the method of elastic mechanics can be used to obtain stresses and displacements in cracked bodies [5]. The valid method to solve crack tip field problems in anisotropic composites may be in using complex analytic function theory, and the results have been reported [6]. But the general solution may have some weakness. So it is necessary to make up a new solution for the singular stresses and strains of cracked composite materials.

2. General Complex Function
Complex variable theory has been a very powerful tool for the solution of many engineering mechanical problems. The technique is also useful for anisotropic materials. It is well known that the basic complex variable (z) and its conjugate (z̄) are defined as:

\[ z = x + iy, \quad \bar{z} = x - iy \quad (i = \sqrt{-1}) \]

For the convenience of general investigation, it is truly necessary to introduce another complex variable (w) and its conjugate (w̄), which are defined as follows:

\[ w = x + ihy, \quad \bar{w} = x - ihy \]

(1)
Where, the parameter $h$ is real. Now we suppose the constant $h$ to be positive ($h > 0$), and it can be called tensile or compressive ratio for the coordinate system. The derivative relation of $w(x, y)$ must be given as follows:

$$\frac{\partial w}{\partial x} = \frac{\partial \bar{w}}{\partial x} = \frac{\partial \bar{z}}{\partial x} = \frac{\partial \bar{x}}{\partial x} = 1, \quad \frac{\partial w}{\partial y} = -\frac{\partial \bar{w}}{\partial y} = ih$$  \hspace{1cm} (2)

For the mode II crack problem, rectangular and polar coordinates are shown in Figure 1. The polar coordinate system centered at the crack tip may be adequate for local stress analysis. In terms of the polar coordinates, the complex variables can be written as:

$$z = x + iy = a + r \cos \theta + ir \sin \theta$$

$$w = x + ihy = a + r \cos \theta + ihr \sin \theta$$  \hspace{1cm} (3)

![Figure 1. Basic model of the composite with mode II crack](image)

Consider the analytic functions of complex variable. $\Psi = \Psi(w)$, $\Phi = \Phi(w)$. The analytic functions can be expressed as follows:

$$\Psi(w) = \text{Re} \, \Psi + i \text{Im} \, \Psi, \quad \Psi(w) = \text{Re} \, \Psi - i \text{Im} \, \Psi$$

$$\frac{d\Psi}{dw} = \Phi(w) = \text{Re} \, \Phi + i \text{Im} \, \Phi$$

$$\frac{d\bar{\Psi}}{d\bar{w}} = \bar{\Phi}(\bar{w}) = \text{Re} \, \Phi - i \text{Im} \, \Phi$$  \hspace{1cm} (4)

The derivatives of $\Psi$ are given by

$$\frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial w} = \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial w} = \frac{\partial \Phi}{\partial x} = \Psi' = \Phi(w) = \text{Re} \, \Phi + i \text{Im} \, \Phi = \frac{\partial \text{Re} \, \Psi}{\partial x} + i \frac{\partial \text{Im} \, \Psi}{\partial x}$$

$$\frac{1}{h} \frac{\partial \Psi}{\partial y} = \frac{1}{h} \frac{\partial \Psi}{\partial w} = \frac{\partial \Phi}{\partial x} = j \Phi' = \frac{\partial j \Phi}{\partial x} = i \text{Re} \, \Phi - \text{Im} \, \Phi = \frac{1}{h} \left( \frac{\partial \text{Re} \, \Psi}{\partial y} + i \frac{\partial \text{Im} \, \Psi}{\partial y} \right)$$  \hspace{1cm} (5)

Obviously, it is easy to derive the following differential equations:

$$\text{Re} \, \Psi' = \text{Re} \, \Phi = \frac{\partial \text{Re} \, \Psi}{\partial x} = \frac{1}{h} \frac{\partial \text{Im} \, \Psi}{\partial y}$$

$$\text{Im} \, \Psi' = \text{Im} \, \Phi = \frac{\partial \text{Im} \, \Psi}{\partial x} = -\frac{1}{h} \frac{\partial \text{Re} \, \Psi}{\partial y}$$  \hspace{1cm} (6)
3. Basic Equations and Stress Function

The plane stress state of composite sheets is common and very important for the application. It is the key point to solve the stress-field problems in orthotropic materials. In plane elasticity, the equilibrium equations are (body forces are absent):

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0
\]  

(7)

Suppose the principal elastic directions of the plate coincide with the coordinate directions, and let the directions 1, 2 parallel to the axes x, y, respectively. Now consider the linear elastic strain-stress relations, the constitutive equations for the orthotropic materials are given as (plane stress state):

\[
\varepsilon_x = \frac{\sigma_x}{E_1} - \nu_{12}\gamma_{xy}, \quad \varepsilon_y = \frac{\sigma_y}{E_2} - \nu_{12}\gamma_{xy}, \quad \gamma_{xy} = \frac{\tau_{xy}}{G_{12}}
\]  

(8)

The compatibility condition of strains must be satisfied as follows:

\[
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}
\]  

(9)

It is well known that the Airy stress function \( U = U(x, y) \) is defined by:

\[
\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}
\]  

(10)

The equilibrium equations in Eqs (7) can be satisfied. Next by using above relations, the governing equation of the compatibility condition can be expressed by the Airy stress function, that is:

\[
\frac{\partial^4 U}{\partial y^4} + \left(\frac{E_1}{G_{12}} - 2\nu_{12}\right) \frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{E_1}{E_2} \frac{\partial^4 U}{\partial x^4} = 0
\]  

(11)

The Airy stress function \( U \) can be expressed by the complex variable function. Consider an infinite plane with the crack along the x-axis shown in Figure 1, and also consider the problem of Mode II shear loading. The Airy stress function can be determined by the form:

\[
U = y \frac{\Psi + \overline{\Psi}}{2} = y \text{Re} \Psi
\]  

(12)

The derivatives of \( U \) with respect to x or y are given by:

\[
\frac{\partial U}{\partial x} = y \frac{\partial \text{Re} \Psi}{\partial x} = y \text{Re} \Psi' = y \text{Re} \Phi', \quad \frac{\partial U}{\partial y} = \text{Re} \Psi + y \frac{\partial \text{Re} \Psi}{\partial y} = \text{Re} \Psi - hy \text{Im} \Phi
\]

\[
\frac{\partial^2 U}{\partial x^2} = y \text{Re} \Psi'' = y \text{Re} \Phi'', \quad \frac{\partial^2 U}{\partial y^2} = -2h \text{Im} \Phi - h^2 y \text{Re} \Phi', \quad \frac{\partial^2 U}{\partial x \partial y} = \text{Re} \Phi - hy \text{Im} \Phi'
\]

On the basis of above relations, the governing equation (11) becomes:

\[
4\left(\frac{E_1}{2G_{12}} - \nu_{12} - h^2\right) \frac{\partial^3 \text{Re} \Psi}{\partial x^3 \partial y} + \left[\frac{E_1}{E_2} - h^2 \left(\frac{E_1}{G_{12}} - 2\nu_{12}\right) + h^4\right] y \frac{\partial^4 \text{Re} \Psi}{\partial x^4} = 0
\]  

(13)

Evidently, the solution must be followed by the characteristic equations, and also reduced to:
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\[
\frac{E_1}{2G_{12}} - \nu_{12} - h^2 = 0
\]

\[
\frac{E_1}{E_2} - 2h^2\left(\frac{E_1}{2G_{12}} - \nu_{12}\right) + h^4 = 0
\]  \hspace{1cm} (14)

The solution of the characteristic equation is given as:

\[
h^2 = \frac{E_1}{2G_{12}} - \nu_{12}, \quad h^4 = \frac{E_1}{E_2}
\]

Because the real constant \( h \) is positive, so the positive real root must be as:

\[
h = \sqrt{\frac{E_1}{2G_{12}} - \nu_{12}} = \frac{\sqrt[4]{E_1}}{\sqrt{E_2}}
\]  \hspace{1cm} (15)

The stresses are considered again. By substituting the above derivative relations of \( U \) into stress expressions (10), the stresses can be expressed as:

\[
\sigma_x = -2h \text{Im} \Phi - h^3y \text{Re} \Phi' \\
\sigma_y = y \text{Re} \Phi' \\
\tau_{xy} = -\text{Re} \Phi + hy \text{Im} \Phi'
\]  \hspace{1cm} (16)

The stress expressions of stresses are similar to the traditional stress formulations that have been given in the books of fracture mechanics for the isotropic materials.

4. Solution of Stress and Strain

For Mode II crack problem, the sheet with a line crack is subjected to uniform shear loading \( \tau \) at remote edges. Consider the following stress boundary conditions:

\[
\sigma_y = \tau_{xy} = 0 \quad \text{at} \quad |x| < a \quad \text{and} \quad y = 0 \quad (\text{free crack surfaces}) \\
\sigma_x = \sigma_y = 0, \quad \tau_{xy} = \tau \quad \text{at} \quad x^2 + y^2 \rightarrow \infty
\]

In order for the stress function to meet the preceding boundary value problem, the complex function can be selected as:

\[
\Psi = Cw\sqrt{1 - \frac{a^2}{w^2}}
\]  \hspace{1cm} (17)

Then the derivatives of the complex functions \( \Psi(w), \Phi(w) \) are given as:

\[
\Psi' = \Phi = Cw\sqrt{\frac{w^2}{w^2 - a^2}} \\
\Phi' = -C\frac{a^2}{w}\sqrt{\frac{w^2}{(w^2 - a^2)^3}}
\]  \hspace{1cm} (18)

By substituting the stress functions \( \Phi \) and \( \Phi' \) into Eq. (16), and also using the remote boundary conditions, the coefficient \( C \) can be derived. It is given out: \( C = -\tau \). So the stress expressions can be determined as follows:
\[
\frac{\sigma_r}{r} = 2h \text{Im} \left( \frac{w^2}{w^2 - a^2} - h^2 \text{Re} \left[ \frac{a^2y}{w} \left( \frac{w^2}{(w^2 - a^2)^2} \right) \right] \right) \\
\frac{\sigma_\theta}{r} = \text{Re} \left[ \frac{a^2y}{w} \left( \frac{w^2}{(w^2 - a^2)^2} \right) \right] \\
\frac{\tau_{\theta r}}{r} = \text{Re} \left( \frac{w^2}{w^2 - a^2} + h \text{Im} \left[ \frac{a^2y}{w} \left( \frac{w^2}{(w^2 - a^2)^2} \right) \right] \right) \tag{19}
\]

In the vicinity region of the crack tip \(( r << a )\), as shown in Figure 1, the functions can be simplified by the polar coordinate system:

\[
x = a + r \cos \theta , \quad y = r \sin \theta , \quad w = a + r \cos \theta + i hr \sin \theta = a + r \eta \\
w + a \approx 2a , \quad w^2 - a^2 \approx 2ar(\cos \theta + ih \sin \theta) = 2ar \eta , \quad \eta = \cos \theta + ih \sin \theta
\]

Then the stresses at near crack-tip can be expressed as:

\[
\frac{\sigma_r}{r} = \frac{a}{2r} \left( 2h \text{Im} \left( \frac{1}{\sqrt{\eta}} \right) - h^2 \frac{\sin \theta}{2} \text{Re} \left( \frac{1}{\sqrt{\eta}} \right) \right) \]
\[
\frac{\sigma_\theta}{r} = \frac{a}{2r} \sin \theta \text{Re} \left( \frac{1}{\sqrt{\eta^3}} \right) \]
\[
\frac{\tau_{\theta r}}{r} = \frac{a}{2r} \left( \text{Re} \left( \frac{1}{\sqrt{\eta}} \right) + h \frac{\sin \theta}{2} \text{Im} \left( \frac{1}{\sqrt{\eta^3}} \right) \right)
\]

These expressions show that the stress components tend to infinity at the crack tip, this is so-called stress singularity. By substituting the stress expressions into the constitutive equations (8), the strain equations can be determined as follows:

\[
\varepsilon_x = \frac{r}{E_1} \left( \frac{a}{2r} \left[ 2h \text{Im} \left( \frac{1}{\sqrt{\eta}} \right) - (h^2 + \nu_{12}) \frac{\sin \theta}{2} \text{Re} \left( \frac{1}{\sqrt{\eta}} \right) \right] \right) \\
\varepsilon_y = \frac{r}{E_1} \left( \frac{a}{2r} \left[ -2\nu_{12}h \text{Im} \left( \frac{1}{\sqrt{\eta}} \right) + (\nu_{12}h^2 + \frac{E_1}{E_2}) \frac{\sin \theta}{2} \text{Re} \left( \frac{1}{\sqrt{\eta}} \right) \right] \right) \tag{21}
\]
\[
\gamma_{xy} = \frac{r}{G_{12}} \left( \frac{a}{2r} \left( \text{Re} \left( \frac{1}{\sqrt{\eta}} \right) + h \frac{\sin \theta}{2} \text{Im} \left( \frac{1}{\sqrt{\eta^3}} \right) \right) \right)
\]

It is evidently shown that the strain components tend to infinity at the crack tip, and also the material elasticity must be closely related with the strain fields.

For the isotropic materials \(( h = 1 )\), the complex variable \( \eta \) is as: \( \eta = e^{i\theta} = \cos \theta + i \sin \theta \). So the expressions of stresses and strains at the crack-tip region can be reduced to:

\[
\frac{\sigma_x}{r} = \sqrt{\frac{a}{2r} \frac{\sin \theta}{2} \left( 2 + \cos \frac{3\theta}{2} \right)} \\
\frac{\sigma_\theta}{r} = \sqrt{\frac{a}{2r} \sin \frac{\theta}{2} \cos \frac{3\theta}{2}} \\
\frac{\tau_{\theta r}}{r} = \sqrt{\frac{a}{2r} \cos \frac{\theta}{2} \left( 1 - \sin \frac{3\theta}{2} \right)} \tag{22}
\]

And
\[
\varepsilon_x = -\frac{\tau}{E \sqrt{2r}} \sin \frac{\theta}{2} \left[ 2 + (1 + \nu) \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right]
\]
\[
\varepsilon_y = \frac{\tau}{E \sqrt{2r}} \sin \frac{\theta}{2} \left[ 2\nu + (1 + \nu) \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right]
\]
\[
\gamma_{xy} = \frac{\tau}{G \sqrt{2r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)
\]

(23)

Obviously, the expressions are conventional stress and strain fields of isotropic materials.

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6. References
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