CLASSIFICATION OF NEAR-NORMAL SEQUENCES

DRAGOMIR Ž. DOKOVIĆ

Abstract. We introduce a canonical form for near-normal sequences \( N N(n) \), and using it we enumerate the equivalence classes of such sequences for even \( n \leq 30 \).

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1. Introduction

Near-normal sequences were introduced by C.H. Yang in [6]. They can be viewed as quadruples of binary sequences \((A; B; C; D)\) where \( A \) and \( B \) have length \( n + 1 \), while \( C \) and \( D \) have length \( n \), and \( n \) has to be even. By definition, the sequences \( A = a_1, a_2, \ldots, a_{n+1} \) and \( B = b_1, b_2, \ldots, b_{n+1} \) are related by the equalities \( b_i = (-1)^{i-1}a_i \) for \( 1 \leq i \leq n \) and \( b_{n+1} = -a_{n+1} \). Moreover it is required that the sum of the non-periodic autocorrelation functions of the four sequences be a delta function.

Examples of near-normal sequences are known for all even \( n \leq 30 \). Due to the important role that these sequences play in various combinatorial constructions such as that for \( T \)-sequences, orthogonal designs, and Hadamard matrices [1, 5, 4], it is of interest to classify the near-normal sequences of small length. We shall give such classification for all even \( n \leq 30 \). We have recently constructed [3] near-normal sequences for \( n = 32 \) and \( n = 34 \).

In section 2 we recall from [2] the basic properties of base sequences. In section 3 we introduce two equivalence relations for near-normal sequences: \( BS\)-equivalence and \( NN\)-equivalence. The former is finer than the latter. We also introduce the canonical form for the \( BS\)-equivalence classes. By using this canonical form, we are able to compute the representatives of the \( BS\)-equivalence classes and then deduce the set of representatives for the \( NN\)-equivalence classes. In section 4 we tabulate our results giving the list of representatives of the \( NN\)-equivalence classes. The representatives are written in the encoded form used in our previous paper [2].

Key words and phrases. Base sequences, near-normal sequences, canonical form.

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2. Base sequences

We denote finite sequences of integers by capital letters. If, say, $A$ is such a sequence of length $n$ then we denote its elements by the corresponding lower case letters. Thus

$$A = a_1, a_2, \ldots, a_n.$$  

To this sequence we associate the polynomial

$$A(x) = a_1 + a_2x + \cdots + a_n x^{n-1},$$

which we view as an element of the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$. (As usual, $\mathbb{Z}$ denotes the ring of integers.) The non-periodic autocorrelation function $N_A$ of $A$ is defined by:

$$N_A(i) = \sum_{j \in \mathbb{Z}} a_j a_{i+j}, \quad i \in \mathbb{Z},$$

where $a_k = 0$ for $k < 1$ and for $k > n$. Note that $N_A(-i) = N_A(i)$ for all $i \in \mathbb{Z}$ and $N_A(i) = 0$ for $i \geq n$. The norm of $A$ is the Laurent polynomial

$$N(A) = A(x)A(x^{-1}).$$

We have

$$N(A) = \sum_{i \in \mathbb{Z}} N_A(i)x^i.$$  

The negation, $-A$, of $A$ is the sequence

$$-A = -a_1, -a_2, \ldots, -a_n.$$  

The reversed sequence $A'$ and the alternated sequence $A^*$ of the sequence $A$ are defined by

$$A' = a_n, a_{n-1}, \ldots, a_1$$

$$A^* = a_1, -a_2, a_3, -a_4, \ldots, (-1)^{n-1}a_n.$$  

Observe that $N(-A) = N(A') = N(A)$ and $N_{A^*}(i) = (-1)^i N_A(i)$ for all $i \in \mathbb{Z}$. By $A, B$ we denote the concatenation of the sequences $A$ and $B$.

A binary sequence is a sequence whose terms belong to the set $\{\pm 1\}$. When displaying such sequences, we shall often write $+$ for $+1$ and $-$ for $-1$. The base sequences consist of four binary sequences $(A; B; C; D)$, with $A$ and $B$ of length $m$ and $C$ and $D$ of length $n$, such that

$$(2.1) \quad N(A) + N(B) + N(C) + N(D) = 2(m + n).$$

We denote by $BS(m, n)$ the set of such base sequences with $m$ and $n$ fixed.

We recall from [2] that two members of $BS(m, n)$ are said to be equivalent if one can be transformed to the other by applying a finite
sequence of elementary transformations. The elementary transformations of \((A; B; C; D) \in BS(m, n)\) are the following:

(i) Negate one of the four sequences \(A; B; C; D\).
(ii) Reverse one of the sequences \(A; B; C; D\).
(iii) Interchange two of the sequences \(A; B; C; D\) of the same length.
(iv) Alternate all four sequences \(A; B; C; D\).

One can view the equivalence classes in \(BS(m, n)\) as orbits of an abstract finite group \(G\). We shall assume that \(m \neq n\). In that case \(G\) has order \(|G| = 2^{11}\). To construct \(G\), we start with an elementary abelian group \(E\) of order \(2^8\) with generators \(\varepsilon_i, \varphi_i, i \in \{1, 2, 3, 4\}\), and an elementary abelian group \(V\) of order 4 with generators \(\sigma_1, \sigma_2\). Let \(H\) be the semidirect product of \(E\) and \(V\), with \(V\) acting on \(E\) so that \(\sigma_1\) commutes with \(\varepsilon_3, \varepsilon_4, \varphi_3, \varphi_4\), and \(\sigma_2\) commutes with \(\varepsilon_1, \varepsilon_2, \varphi_1, \varphi_2\), and

\[
\sigma_1 \varepsilon_1 = \varepsilon_2 \sigma_1, \quad \sigma_1 \varphi_1 = \varphi_2 \sigma_1, \quad \sigma_2 \varepsilon_3 = \varepsilon_4 \sigma_2, \quad \sigma_2 \varphi_3 = \varphi_4 \sigma_2.
\]

Finally, we define \(G\) as the semidirect product of \(H\) and the group \(Z_2\) of order 2 with generator \(\psi\). By definition, \(\psi\) commutes with each \(\varepsilon_i\) and we have

\[
\psi \varphi_i = \varepsilon_i^{m-1} \varphi_i \psi, \quad i = 1, 2; \quad \psi \varphi_j = \varepsilon_j^{n-1} \varphi_j \psi, \quad j = 3, 4.
\]

The group \(G\) acts on \(BS(m, n)\) as follows. If \(S = (A; B; C; D) \in BS(m, n)\) then

\[
\varepsilon_1 S = (-A; B; C; D), \quad \varphi_1 S = (A'; B; C; D),
\]

\[
\varepsilon_2 S = (A; -B; C; D), \quad \varphi_2 S = (A; B'; C; D),
\]

\[
\varepsilon_3 S = (A; B; -C; D), \quad \varphi_3 S = (A; B; C'; D),
\]

\[
\varepsilon_4 S = (A; B; C; -D), \quad \varphi_4 S = (A; B; C; D'),
\]

and \(\psi S = (A^*; B^*; C^*; D^*)\). It is easy to verify that the defining relations of \(G\) are satisfied by these transformations and so the action of \(G\) on \(BS(m, n)\) is well defined. Consequently, the following proposition holds.

**Proposition 2.1.** If \(m \neq n\), the orbits of \(G\) in \(BS(m, n)\) are the same as the equivalence classes in \(BS(m, n)\).

We need also the encoding scheme for the base sequences \((A; B; C; D)\) in \(BS(n + 1, n)\) introduced in [2]. We now recall that scheme. We decompose the pair \((A; B)\) into quads

\[
\begin{bmatrix}
  a_i & a_{n+2-i} \\
  b_i & b_{n+2-i}
\end{bmatrix}, \quad i = 1, 2, \ldots, \left\lfloor \frac{n + 1}{2} \right\rfloor,
\]
and, if $n = 2m$ is even, the central column $\begin{bmatrix} a_{m+1} \\ b_{m+1} \end{bmatrix}$. Up to equivalence of base sequences, we can assume that the first quad of $(A; B)$ is $\begin{bmatrix} + & + \\ + & - \end{bmatrix}$. We attach to this particular quad the label 0. The other quads in $(A; B)$ and all the quads of the pair $(B; C)$, shown with their labels, must be one of the following:

1 = $\begin{bmatrix} + & + \\ + & + \end{bmatrix}$,  2 = $\begin{bmatrix} + & + \\ - & + \end{bmatrix}$,  3 = $\begin{bmatrix} - & + \\ - & + \end{bmatrix}$,  4 = $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$,

5 = $\begin{bmatrix} - & + \\ + & - \end{bmatrix}$,  6 = $\begin{bmatrix} + & - \\ + & - \end{bmatrix}$,  7 = $\begin{bmatrix} - & - \\ + & + \end{bmatrix}$,  8 = $\begin{bmatrix} - & - \\ - & - \end{bmatrix}$.

The central column is encoded as

$0 = \begin{bmatrix} + \\ + \end{bmatrix}$,  1 = $\begin{bmatrix} + \\ - \end{bmatrix}$,  2 = $\begin{bmatrix} - \\ + \end{bmatrix}$,  3 = $\begin{bmatrix} - \\ - \end{bmatrix}$.

If $n = 2m$ is even, the pair $(A; B)$ is encoded as the sequence of digits $q_1q_2 \ldots q_mq_{m+1}$, where $q_i$, $1 \leq i \leq m$, is the label of the $i$th quad and $q_{m+1}$ is the label of the central column. If $n = 2m - 1$ is odd, then $(A; B)$ is encoded by $q_1q_2 \ldots q_m$, where $q_i$ is the label of the $i$th quad for each $i$. We use the same recipe to encode the pair $(C; D)$.

As an example, the base sequences

\[ A = +, +, +, +, - , - , +, - , +, +; \]
\[ B = +, +, +, - , +, +, +, - , - , -; \]
\[ C = +, +, - , - , +, - , - , +; \]
\[ D = +, +, +, - , - , - , - , +; \]

are encoded as 06142; 1675.

### 3. Near-normal sequences

Near-normal sequences, originally defined by C.H. Yang [6], can be viewed as a special type of base sequences $(A; B; C; D) \in BS(n+1, n)$ (see [4, 2]) with $n$ even, namely such that $b_i = (-1)^{i+1}a_i$ for $1 \leq i \leq n$. Note that we also must have $b_{n+1} = -a_{n+1}$. Hence, the sequence $B$ is uniquely determined by $A$, and we define $\alpha A = B$. Note that also $\alpha B = A$.

We denote by $NN(n)$ the subset of $BS(n+1, n)$ consisting of near-normal sequences. It has been conjectured (Yang [6]) that $NN(n) \neq \emptyset$.
for all positive even $n$'s. Yang's conjecture has been confirmed for all even $n \leq 34$ [3].

We shall introduce two equivalence relations in $NN(n)$: $BS$-equivalence and $NN$-equivalence. The former is stronger than the latter.

We say that two members of $NN(n)$ are $BS$-equivalent if they are equivalent as base sequences in $BS(n + 1, n)$. One can enumerate the $BS$-equivalence classes by finding suitable representatives of the classes. For that purpose we introduce the concept of canonical form for near-normal sequences.

For convenience we fix the following notation. Let $(A; B = \alpha A; C; D) \in NN(n), n = 2m$, and let

\[ p_1p_2 \cdots p_mp_{m+1} \text{ resp. } q_1q_2 \cdots q_m \]

be the encoding of the pair $(A; B)$ resp. $(C; D)$.

**Definition 3.1.** We say that the near-normal sequences $(A; B; C; D)$ are in the canonical form if the following conditions hold:

(i) $p_1 = 0$, $q_1 = 1$.

(ii) If $q_j = 2$ for some $j$, then $q_i = 7$ for some index $i$ with $1 < i < j$.

(iii) If $q_j \in \{3, 4, 5\}$ for some $j$, then $q_i = 6$ for some index $i$ with $1 < i < j$.

(iv) If $q_k \neq 7$ for all $k$'s and $q_j = 4$ for some $j$, then $q_i = 5$ for some index $i$ with $1 < i < j$.

The following proposition shows how one can enumerate the $BS$-equivalence classes of $NN(n)$.

**Proposition 3.2.** For each $BS$-equivalence class $E \subseteq NN(n), n = 2m$, there is a unique $(A; B; C; D) \in E$ having the canonical form.

**Proof.** Let $(A; B; C; D) \in E$ be arbitrary and let $p_1p_2 \cdots p_mp_{m+1} \text{ resp. } q_1q_2 \cdots q_m$ be the encoding of the pair $(A; B)$ resp. $(C; D)$. By applying the first three types of elementary transformations we can assume that $p_1 = 0$ and $c_1 = d_1 = +1$. Then $q_1$ must be either 1 or 6. In the latter case we apply the elementary transformation (iv). Thus we may assume that $p_1 = 0$ and $q_1 = 1$, i.e., the condition (i) for the canonical form is satisfied.

Now assume that $q_j = 2$ for some $j$ and that $q_i \neq 7$ for all $i < j$. After interchanging the sequences $C$ and $D$, we obtain that $q_j = 7$ and $q_i \neq 2$ for $i < j$. Hence we may also assume that the condition (ii) is satisfied.

Next assume that $q_j \in \{3, 4, 5\}$ for some $j$. We may take $j$ to be minimal with this property. Assume that $q_i \neq 6$ for $i < j$. Consequently, $q_i \in \{1, 2, 7, 8\}$ for all $i < j$. If $q_j = 3$ we replace $(C; D)$ with
(C'; D'). If \(q_j = 4\) we replace \(D\) with \(D'\). If \(q_j = 5\) we replace \(C\) with \(C'\). After this change, we obtain that in all three cases \(q_j = 6\) while the \(q_i\)’s for \(i < j\) remain unchanged. Hence the condition (iii) is also satisfied.

Finally, assume that \(q_k \neq 7\) for all \(k\)’s, \(q_j = 4\) for some \(j\), and \(q_i \neq 5\) for \(i < j\). Since the condition (ii) holds, we have \(q_i \in \{1, 3, 6, 8\}\) for all \(i < j\). After interchanging \(C\) and \(D\), we obtain that \(q_j = 5\) while the \(q_i\) with \(i < j\) remain unchanged. Hence now \((A; B; C; D)\) is in the canonical form.

It remains to prove the uniqueness assertion. Let

\[ S^{(k)} = (A^{(k)}; B^{(k)}; C^{(k)}; D^{(k)}) \in \mathcal{E}, \quad (k = 1, 2) \]

be in the canonical form. By Proposition 2.1 there exists \(g \in G\) such that \(gS^{(1)} = S^{(2)}\). Let \(p_1^{(1)} p_2^{(1)} \cdots p_{m+1}^{(1)}\) resp. \(p_1^{(2)} p_2^{(2)} \cdots p_{m+1}^{(2)}\) be the encoding of the pair \((A^{(1)}; B^{(1)})\) resp. \((A^{(2)}; B^{(2)})\). Let \(q_1^{(1)} q_2^{(1)} \cdots q_m^{(1)}\) resp. \(q_1^{(2)} q_2^{(2)} \cdots q_m^{(2)}\) be the encoding of the pair \((C^{(1)}; D^{(1)})\) resp. \((C^{(2)}; D^{(2)})\). Since \(q_1^{(1)} = q_1^{(2)} = 1\), \(g\) must be in \(H\). Note that \(H = H_1 \times H_2\), where the subgroup \(H_1\) resp. \(H_2\) is generated by \(\{\varepsilon_1, \varepsilon_2, \varphi_1, \varphi_2, \sigma_1\}\) resp. \(\{\varepsilon_3, \varepsilon_4, \varphi_3, \varphi_4, \sigma_2\}\). Thus we have \(g = h_1 h_2\) with \(h_1 \in H_1\) and \(h_2 \in H_2\). Consequently, \(h_1 \cdot (A^{(1)}; B^{(1)}) = (A^{(2)}; B^{(2)})\) and \(h_2 \cdot (C^{(1)}; D^{(1)}) = (C^{(2)}; D^{(2)})\).

We also have the direct decomposition \(E = E_1 \times E_2\), where \(E_1 = E \cap H_1\) and \(E_2 = E \cap H_2\).

Since \(p_1^{(1)} = p_1^{(2)} = 0\), the equality \(h_1 \cdot (A^{(1)}; B^{(1)}) = (A^{(2)}; B^{(2)})\) implies that \(h_1 \in E_1\). Thus \(h_1 = e_1^{e_1} e_2^{e_2} \varphi_1^{f_1} \varphi_2^{f_2}\) with \(e_1, e_2, f_1, f_2 \in \{0, 1\}\).

Since the first and the last terms of the sequences \(A^{(1)}\) and \(A^{(2)}\) are +1, we have \(e_1 = 0\). It follows that the middle terms of these two sequences are the same. As \(S^{(1)}\) and \(S^{(2)}\) are near-normal sequences, the sequences \(B^{(1)}\) and \(B^{(2)}\) must also have the same middle terms. Consequently, \(e_2 = 0\). Since the sequences \(B^{(1)}\) and \(B^{(2)}\) have the same first term, +1, and the same last term, −1, we infer that \(f_2 = 0\). Consequently, \(B^{(1)} = B^{(2)}\).

As \(A^{(1)} = \alpha B^{(1)}\) and \(A^{(2)} = \alpha B^{(2)}\), we infer that also \(A^{(1)} = A^{(2)}\).

Since \(q_1^{(1)} = q_1^{(2)} = 1\), the equality \(h_2 \cdot (C^{(1)}; D^{(1)}) = (C^{(2)}; D^{(2)})\) implies that \(h_2\) belongs to the subgroup of \(H_2\) generated by \(\{\varphi_3, \varphi_4, \sigma_2\}\). Thus \(h_2 = \varphi_3^{f_3} \varphi_4^{f_4} \sigma_2^{s}\) with \(f_3, f_4, s \in \{0, 1\}\).

Assume that \(q_j^{(1)} = 7\) for some \(j\). Choose the smallest such \(j\). Then \(q_i^{(1)} \neq 2\) for \(1 < i < j\). The quads 1, 3, 6, 8 are fixed by \(\sigma_2\) and the quads 2, 4, 5, 7 are permuted via the involution \((2, 7)(4, 5)\). On the other hand, the generators \(\varphi_3\) and \(\varphi_4\) fix the quads 1, 2, 7, 8. Since \(S^{(1)}\) and \(S^{(2)}\) are in the canonical form it follows that \(s = 0\) and so \(q_j^{(2)} = 7\) and \(q_i^{(2)} \neq 2\).
for $1 < i < j$. The equality $\varphi_3^f \varphi_4^f \cdot (C^{(1)}; D^{(1)}) = (C^{(2)}; D^{(2)})$ now implies that $C^{(1)} = C^{(2)}$ and $D^{(1)} = D^{(2)}$. Hence $S^{(1)} = S^{(2)}$. The argument is similar if $q_j^{(1)} \neq 7$ for all $j$, which implies that also $q_j^{(1)} \neq 2$ for all $j$. \hfill \Box

We proceed to define the NN-equivalence relation in $NN(n)$. For this we need to introduce the NN-elementary transformations:

(i) Negate both sequences $A; B$ or one of $C; D$.
(ii) Reverse one of the sequences $C; D$.
(iii) Interchange the sequences $A; B$ or $C; D$.
(iv) Replace the sequences $(A; B = \alpha A)$ with $(\hat{A}; \hat{B} = \alpha \hat{A})$ where

$$\hat{A} = a_{n-1}, a_2, a_{n-3}, a_4, \ldots, a_1, a_n, a_{n+1}.$$  

(v) Replace the sequences $(C; D)$ with the pair $(\tilde{C}; \tilde{D})$ which is defined by its encoding $\tilde{q}_1 \tilde{q}_2 \ldots \tilde{q}_m$ with

$$\tilde{q}_i = \begin{cases} 
5, & \text{if } q_i = 4; \\
4, & \text{if } q_i = 5; \\
q_i & \text{otherwise.}
\end{cases}$$

(vi) Alternate all four sequences $A; B; C; D$.

Lemma 3.3. By using the above notation, we have $N(\hat{A}) + N(\hat{B}) = N(A) + N(B)$ and $N(\tilde{C}) + N(\tilde{D}) = N(C) + N(D)$. Consequently, the quadruples $(\hat{A}; \hat{B}; C; D)$ and $(A; B; \tilde{C}; \tilde{D})$ belong to $NN(n)$.

Proof. We sketch the proof only for the first quadruple. It suffices to show that for even $i$ and odd $j < n$ we have

$$\hat{a}_i \hat{a}_j + \hat{b}_i \hat{b}_j = a_ia_j + b_ib_j.$$  

This is indeed true since $\hat{b}_j = \hat{a}_j$ and $\hat{a}_i + \hat{b}_i = 0$. \hfill \Box

We say that two members of $NN(n)$ are $NN$-equivalent if one can be transformed to the other by applying a finite sequence of the $NN$-elementary transformations (i-vi).

Our main objective is to enumerate the $NN$-equivalence classes of $NN(n)$ for even integers $n \leq 30$.

4. Equivalence classes of near-normal sequences

In Table 1 we list the codes for the representatives of the $NN$-equivalence classes of $NN(n)$ for even $n \leq 30$. All representatives are chosen in the canonical form.
Table 1: \(NN\)-equivalence classes of \(NN(n)\)

| \(A \times B\) | \(C \times D\) | \(a, b, c, d\) | \(a^*, b^*, c^*, d^*\) |
|---------------|---------------|----------------|-------------------|
| \(n = 2\)     |               |                |                   |
| 1             | 02            | 1              | 1, 1, 2, 2        | 3, -1, 0, 0       |
| \(n = 4\)     |               |                |                   |
| 1             | 050           | 16             | 3, 1, 2, 2        | 3, 1, -2, -2     |
| 2             | 073           | 17             | -1, 1, 0, 4       | 3, -3, 0, 0      |
| \(n = 6\)     |               |                |                   |
| 1             | 0711          | 188            | 3, 3, -2, -2      | 5, 1, 0, 0       |
| 2             | 0512          | 172            | 3, 3, 2, 2        | 5, 1, 0, 0       |
| \(n = 8\)     |               |                |                   |
| 1             | 07643         | 1651           | -1, 1, 4, 4       | 3, -3, -4, 0     |
| 2             | 05850         | 1163           | 1, -1, 4, 4       | 1, -1, 4, 4      |
| 3             | 05323         | 1637           | 3, -3, 0, 4       | -1, 1, -4, -4    |
| \(n = 10\)    |               |                |                   |
| 1             | 076462        | 16712          | -1, 3, 4, 4       | 5, -3, -2, -2    |
| 2             | 078211        | 16561          | 3, -1, 4, 4       | 1, 1, -6, -2     |
| 3             | 078412        | 16787          | -1, 3, -4, 4      | 5, -3, -2, -2    |
| 4             | 076761        | 17621          | -1, 3, 4, 4       | 5, -3, 2, 2      |
| 5             | 056732        | 11726          | -1, 3, 4, 4       | 5, -3, 2, 2      |
| 6             | 058511        | 11635          | 3, -1, 4, 4       | 1, 1, 2, 6       |
| 7             | 053281        | 16355          | 3, -5, 2, 2       | -3, 1, -4, -4    |
| 8             | 053781        | 17616          | -1, -1, 2, 6      | 1, -3, 4, 4      |
| \(n = 12\)    |               |                |                   |
| 1             | 0765373       | 161762         | -3, 3, 4, 4       | 5, -5, 0, 0      |
| 2             | 0764373       | 165513         | -3, 3, 4, 4       | 5, -5, 0, 0      |
| 3             | 0764320       | 165776         | 3, 1, -2, 6       | 3, 1, -6, -2     |
| 4             | 0764870       | 167162         | -3, 3, 4, 4       | 5, -5, 0, 0      |
| 5             | 0784370       | 167867         | -3, 3, -4, 4      | 5, -5, 0, 0      |
| 6             | 0765373       | 176738         | -3, 3, -4, 4      | 5, -5, 0, 0      |
| 7             | 0715873       | 187766         | -3, 3, -4, 4      | 5, -5, 0, 0      |
| 8             | 0737653       | 187222         | -3, 3, -4, 4      | 5, -5, 0, 0      |
| 9             | 0585140       | 116754         | 3, 1, 2, 6        | 3, 1, -2, 6      |
| 10            | 0517820       | 161675         | 3, 1, 2, 6        | 3, 1, -2, -6     |
| 11            | 0512673       | 165714         | 3, 1, 2, 6        | 3, 1, -6, 2      |
| 12            | 0512870       | 167575         | 3, 1, -2, 6       | 3, 1, -2, -6     |
| 13            | 0515643       | 176153         | 3, 1, 2, 6        | 3, 1, 2, 6       |
| 14            | 0515343       | 176547         | 3, 1, -2, 6       | 3, 1, 6, -2      |
## Classification of Near-Normal Sequences

Table 1 (continued)

|   | $A \& B$ | $C \& D$ | $a, b, c, d$ | $a^*, b^*, c^*, d^*$ |
|---|---------|---------|-------------|---------------------|
| $n = 14$ |         |         |             |                     |
| 1  | 07623211 | 1637668 | $7, -1, -2, 2$ | $1, 5, -4, -4$     |
| 2  | 07621231 | 1651468 | $7, -1, 2, 2$ | $1, 5, -4, -4$     |
| 3  | 07643511 | 1675657 | $3, 3, -2, 6$ | $5, 1, 4, -4$      |
| 4  | 07676212 | 1763321 | $1, 5, 4, 4$ | $7, -1, 2, 2$      |
| 5  | 07176262 | 1868866 | $1, 5, -4, -4$ | $7, -1, 2, 2$     |
| 6  | 07378282 | 1865311 | $-5, -1, 4, 4$ | $1, -7, 2, -2$    |
| 7  | 05673512 | 1172663 | $1, 5, 4, 4$ | $7, -1, -2, -2$   |
| 8  | 05821712 | 1187763 | $3, 3, -2, 6$ | $5, 1, -4, -4$    |
| 9  | 05128712 | 1638177 | $3, 3, -2, 6$ | $5, 1, -4, -4$    |
| 10 | 05121562 | 1678524 | $7, 3, 0, 0$ | $5, 5, -2, -2$    |
| 11 | 05146762 | 1678376 | $1, 5, -4, 4$ | $7, -1, -2, -2$   |

| $n = 16$ |         |         |             |                     |
| 1  | 076567350 | 16117872 | $-1, 5, 2, 6$ | $7, -3, -2, -2$   |
| 2  | 076215320 | 16333817 | $7, 1, 0, 4$ | $3, 5, -4, -4$    |
| 3  | 076212650 | 16373355 | $7, 1, 0, 4$ | $3, 5, -4, -4$    |
| 4  | 076214670 | 16377568 | $3, 5, -4, 4$ | $7, 1, 0, -4$     |
| 5  | 076487150 | 16716223 | $-1, 5, 6, 2$ | $7, -3, 2, 2$     |
| 6  | 076417643 | 16752321 | $-1, 5, 6, 2$ | $7, -3, 2, -2$    |
| 7  | 076417343 | 16756467 | $-1, 5, -2, 6$ | $7, -3, 2, 2$   |
| 8  | 076715643 | 17265377 | $-1, 5, -2, 6$ | $7, -3, -2, -2$  |
| 9  | 076517353 | 17661518 | $-1, 5, 2, 6$ | $7, -3, 2, -2$   |
| 10 | 076534120 | 17665214 | $5, 3, 4, 4$ | $5, 3, -4, 4$    |
| 11 | 076587150 | 17766813 | $-1, 5, -2, 6$ | $7, -3, 2, 2$   |
| 12 | 076487150 | 17816788 | $-1, 5, -6, 2$ | $7, -3, 2, 2$ |
| 13 | 071564320 | 18676557 | $5, 3, -4, 4$ | $5, 3, 4, 4$    |
| 14 | 071265620 | 18863557 | $7, 1, -4, 0$ | $3, 5, -4, -4$   |
| 15 | 051284823 | 16546732 | $3, -7, 2, 2$ | $-5, 1, -6, 2$   |
| 16 | 051235623 | 16637385 | $7, -3, -2, 2$ | $-1, 5, 6, 2$    |
| 17 | 051462323 | 16654553 | $7, -3, 2, 2$ | $-1, 5, 6, -2$   |
| 18 | 051267640 | 16753874 | $5, 3, -4, 4$ | $5, 3, -4, -4$   |
| 19 | 053467670 | 16537515 | $-1, 5, 2, 6$ | $7, -3, 2, -2$   |
| 20 | 053467873 | 16754414 | $-5, 1, 2, 6$ | $3, -7, -2, -2$  |
| 21 | 053462823 | 16758534 | $3, -7, -2, 2$ | $-5, 1, -2, -6$ |
| 22 | 051712820 | 17268876 | $7, 1, -4, 0$ | $3, 5, -4, -4$   |
| 23 | 051564173 | 17726215 | $3, 5, 4, 4$ | $7, 1, 4, 0$    |
| 24 | 051467373 | 17886836 | $-1, 5, -6, -2$ | $7, -3, -2, -2$ |
Table 1 (continued)

|      | $A \& B$       | $C \& D$        | $a, b, c, d$ | $a^*, b^*, c^*, d^*$ |
|------|----------------|-----------------|-------------|---------------------|
| $n = 18$ |                |                 |             |                     |
| 1    | 0767653462     | 161544647       | $-3, 5, 2, 6$ | $7, -5, 0, 0$       |
| 2    | 0762328231     | 163544668       | $5, -7, 0, 0$ | $-5, 3, -2, -6$     |
| 3    | 0762328211     | 163554338       | $7, -5, 0, 0$ | $-3, 5, -6, -2$     |
| 4    | 0762328211     | 165138835       | $7, -5, 0, 0$ | $-3, 5, -6, -2$     |
| 5    | 0782156561     | 165413577       | $3, -1, 0, 8$ | $1, 1, -6, 6$       |
| 6    | 0782143782     | 165726177       | $-3, 1, 0, 8$ | $3, -5, -6, -2$     |
| 7    | 0767178262     | 172782221       | $-3, 5, 6, -2$ | $7, -5, 0, 0$       |
| 8    | 0767643462     | 176672155       | $-3, 5, 2, 6$ | $7, -5, 0, 0$       |
| 9    | 0765153782     | 177821181       | $-3, 5, 2, 6$ | $7, -5, 0, 0$       |
| 10   | 0785153762     | 172212188       | $-3, 5, 6, -2$ | $7, -5, 0, 0$       |
| 11   | 0737846761     | 186557167       | $-5, 3, -2, 6$ | $5, -7, 0, 0$       |
| 12   | 0567156482     | 117763815       | $-1, 3, 0, 8$ | $5, -3, 2, 6$       |
| 13   | 0512876462     | 165351136       | $1, 1, 6, 6$ | $3, -1, 0, 8$       |
| 14   | 0514846732     | 166751736       | $-1, 3, 0, 8$ | $5, -3, 2, 6$       |
| 15   | 0512846282     | 167654744       | $3, -5, -2, 6$ | $-3, 1, -8, 0$      |
| 16   | 0512846262     | 167655745       | $5, -3, -2, 6$ | $-1, 3, -8, 0$      |
| 17   | 0512848232     | 167655745       | $3, -5, -2, 6$ | $-3, 1, -8, 0$      |
| 18   | 0532376482     | 165351136       | $-1, -1, 6, 6$ | $1, -3, 0, 8$       |
| 19   | 0517846761     | 176567164       | $-1, 3, 0, 8$ | $5, -3, 6, -2$      |
| 20   | 0537346781     | 176567164       | $-3, 1, 0, 8$ | $3, -5, 6, -2$      |

$n = 20$

|      | $A \& B$       | $C \& D$        | $a, b, c, d$ | $a^*, b^*, c^*, d^*$ |
|------|----------------|-----------------|-------------|---------------------|
| 1    | 07621517870    | 1633771868      | $1, 7, -4, 4$ | $9, -1, 0, 0$       |
| 2    | 07643282143    | 1657513537      | $3, -3, 0, 8$ | $-1, 1, -8, -4$     |
| 3    | 07643215823    | 1657761672      | $3, -3, 0, 8$ | $-1, 1, -8, -4$     |
| 4    | 07643285123    | 1676715655      | $3, -3, 0, 8$ | $-1, 1, -8, -4$     |
| 5    | 07821417670    | 1655337213      | $1, 7, 4, 4$ | $9, -1, 0, 0$       |
| 6    | 07821464623    | 1657367551      | $3, -3, 0, 8$ | $-1, 1, -8, -4$     |
| 7    | 07156514620    | 1876332551      | $7, 5, 2, 2$ | $7, 5, -2, -2$      |
| 8    | 07356484873    | 1871628611      | $-7, -1, 4, 4$ | $1, -9, 0, 0$       |
| 9    | 05673282320    | 1166536724      | $5, -5, 4, 4$ | $-3, 3, 0, 8$       |
| 10   | 05153467820    | 1616571625      | $3, 1, 6, 6$ | $3, 1, -6, -6$      |
| 11   | 05178262840    | 1616372252      | $3, -3, 8, 0$ | $-1, 1, -8, -4$     |
| 12   | 05146784840    | 1663611547      | $-1, 1, 4, 8$ | $3, -3, 8, 0$       |
| 13   | 05146214173    | 1665572814      | $7, 5, 2, 2$ | $7, 5, -2, -2$      |
| 14   | 05146515153    | 1678325325      | $7, 5, 2, -2$ | $7, 5, -2, -2$      |
| 15   | 05146265620    | 1678813524      | $9, -1, 0, 0$ | $1, 7, -4, -4$      |
| 16   | 05346265873    | 1661754125      | $-1, -3, 6, 6$ | $-1, -3, 6, -6$     |
| 17   | 05171564620    | 1726655445      | $7, 5, 2, 2$ | $7, 5, -2, -2$      |
| 18   | 05126532340    | 1786556323      | $9, -1, 0, 0$ | $1, 7, 4, 4$        |
Table 1 (continued)

| n = 22 | A & B | C & D | a, b, c, d | a*, b*, c*, d* |
|--------|-------|-------|------------|----------------|
| 1      | 076537321212 | 16156871224 | 5, 5, 6, 2 | 7, 3, 4, −4 |
| 2      | 076212641431 | 1635377225  | 9, 1, 2, 2 | 3, 7, −4, −4 |
| 3      | 076487121512 | 16337381132 | 3, 7, 4, 4 | 9, 1, 2, 2 |
| 4      | 076414343562 | 1655178513  | 1, 5, 0, 8 | 7, −1, −6, −2 |
| 5      | 076414343562 | 16561764357 | 1, 5, 0, 8 | 7, −1, −6, −2 |
| 6      | 076414378212 | 16617767256 | 1, 5, 0, 8 | 7, −1, 6, 2 |
| 7      | 076435641411 | 16761544847 | 5, 5, −2, 6 | 7, 3, −4, −4 |
| 8      | 078212153261 | 16778255254 | 9, −3, 0, 0 | −1, 7, 2, −6 |
| 9      | 078435173511 | 1678586863  | 1, 5, −8, 0 | 7, −1, −2, −6 |
| 10     | 076512676432 | 17633151578 | 1, 5, 0, 8 | 7, −1, −2, 6 |
| 11     | 076515671481 | 17675144618 | 1, 5, 0, 8 | 7, −1, 2, 6 |
| 12     | 076782121711 | 17652175378 | 5, 5, −2, 6 | 7, 3, 4, −4 |
| 13     | 071567356562 | 18767883255 | −1, 7, −6, −2 | 9, −3, 0, 0 |
| 14     | 071584328782 | 18768533758 | −5, −1, −8, 0 | 1, −7, 2, −6 |
| 15     | 056414173761 | 11868766736 | 3, 7, −4, 4 | 9, 1, 2, 2 |
| 16     | 058512141532 | 11635676523 | 7, 3, 4, 4 | 5, 5, −6, 2 |
| 17     | 058512328781 | 11637254662 | 1, −7, 6, 2 | −5, −1, 0, 8 |
| 18     | 051715853212 | 16187155327 | 5, 5, 2, 6 | 7, 3, −4, −4 |
| 19     | 051265126462 | 16534782626 | 9, 1, 2, −2 | 3, 7, 4, 4 |
| 20     | 051265128432 | 16535712626 | 7, −1, 6, 2 | 1, 5, 0, 8 |
| 21     | 051284651712 | 16576127148 | 5, 5, 2, 6 | 7, 3, −4, 4 |
| 22     | 051234648212 | 16654867176 | 7, −1, −2, 6 | 1, 5, 8, 0 |
| 23     | 051464641232 | 16675487723 | 7, 3, −4, 4 | 5, 5, −6, 2 |
| 24     | 051265673412 | 16723155718 | 5, 5, 2, 6 | 7, 3, −4, −4 |
| 25     | 053482826781 | 16383573582 | −1, −9, −2, −2 | −7, −3, −4, −4 |
| 26     | 053465151281 | 16537353721 | 7, −1, 2, 6 | 1, 5, 0, 8 |
| 27     | 053265626512 | 16712758341 | 7, −1, 2, 6 | 1, 5, −8, 0 |
| 28     | 053464621711 | 16728657537 | 7, 3, −4, 4 | 5, 5, −6, 2 |
| 29     | 051515841782 | 17631554474 | 1, 5, 0, 8 | 7, −1, 6, 2 |
| 30     | 051512658432 | 17653363147 | 5, 1, 0, 8 | 3, 3, 6, 6 |
| 31     | 051762482211 | 17657861418 | 7, −1, −2, 6 | 1, 5, 8, 0 |
| 32     | 051567326261 | 17763546681 | 7, −1, −2, 6 | 1, 5, 0, −8 |
Table 1 (continued)

|   | A & B       | C & D         | a, b, c, d | a*, b*, c*, d* |
|---|-------------|---------------|------------|---------------|
| n = 24 |            |               |            |               |
| 1  | 0765321785873 | 161653745512 | -5, 1, 6, 6 | 3, -7, -6, 2 |
| 2  | 0764156484370 | 165371478678 | -1, 5, -6, 6 | 7, -3, -6, 2 |
| 3  | 0767621532650 | 176577612445 | 5, 3, 0, 8 | 5, 3, 8, 0 |
| 4  | 0767328512140 | 178833835874 | 5, 3, -8, 0 | 5, 3, 0, 8 |
| 5  | 0715653785350 | 187677653446 | -1, 5, -6, 6 | 7, -3, -2, -6 |
| 6  | 0737626512340 | 186537753131 | 5, 3, 0, 8 | 5, 3, 0, 8 |
| 7  | 0734876235823 | 188663535787 | -3, -5, -8, 0 | -3, -5, 0, 8 |
| 8  | 0512653237623 | 16535436747 | 7, -3, -2, 6 | -1, 5, 6, 6 |
| 9  | 0532651262673 | 167167854247 | 7, -3, -2, 6 | -1, 5, 6, 6 |
| 10 | 0515178265340 | 176765741452 | 5, 3, 0, 8 | 5, 3, 0, 8 |
| 11 | 0517646732123 | 176654163667 | 5, 3, 0, 8 | 5, 3, -8, 0 |
| 12 | 0512158564370 | 178868726547 | 5, 3, -8, 0 | 5, 3, 8, 0 |

n = 26

|   | A & B       | C & D         | a, b, c, d | a*, b*, c*, d* |
|---|-------------|---------------|------------|---------------|
| 1  | 0764148784342 | 1654611475266 | -3, 5, 6, 6 | 7, -5, -4, 4 |
| 2  | 0512826456262 | 165477652733 | 7, -5, -4, 4 | -3, 5, -6, 6 |
| 3  | 0512625841481 | 1782657541321 | 7, -5, 4, 4 | -3, 5, 6, -6 |

n = 28

|   | A & B       | C & D         | a, b, c, d | a*, b*, c*, d* |
|---|-------------|---------------|------------|---------------|
| 1  | 076534321432170 | 16178852836758 | 7, 5, -6, -2 | 7, 5, -2, 6 |
| 2  | 07623264878780 | 16354457772331 | -7, -1, 0, 8 | 1, -9, -4, -4 |
| 3  | 078232123565140 | 1653873577542 | 9, -1, -4, 4 | 1, 7, 0, -8 |
| 4  | 078215348487673 | 16754388724478 | -7, -1, -8, 0 | 1, -9, 4, -4 |
| 5  | 078214148264370 | 16767651613356 | 3, 1, 2, 10 | 3, 1, -10, -2 |
| 6  | 076512326587853 | 17635447785113 | -1, -3, -2, 10 | -1, -3, 2, 10 |
| 7  | 076514414635673 | 17655216547871 | 1, 7, 0, 8 | 9, 1, -4, -4 |
| 8  | 076537321737843 | 17661856774521 | -5, 5, 0, 8 | 7, -7, 0, -4 |
| 9  | 076582151735173 | 17727186654441 | 1, 7, 0, 8 | 9, -1, 4, -4 |
| 10 | 078517356737323 | 17262157855212 | -5, 5, 8, 0 | 7, -7, -4, 0 |
| 11 | 078235123464120 | 18683835255348 | 9, -1, -4, -4 | 1, 7, 0, -8 |
| 12 | 071564621714873 | 1863725548877 | 1, 7, -8, 0 | 9, -1, 4, 4 |
| 13 | 071564148764650 | 18667117672553 | 1, 7, 0, 8 | 9, -1, 4, 4 |
| 14 | 071287651232320 | 18876654763441 | 9, -1, -4, 4 | 1, 7, -8, 0 |
| 15 | 053465153484843 | 16754378876583 | -1, -3, -10, 2 | -1, -3, 10, -2 |
| 16 | 051765414647353 | 17631822512665 | 1, 7, 8, 0 | 9, -1, 4, 4 |
| 17 | 051762846767140 | 1765478116581 | 1, 7, 0, 8 | 9, -1, 4, -4 |
| 18 | 051784828462343 | 17656316516487 | -1, -7, 0, 8 | -5, -3, 4, 8 |
| 19 | 051782353215153 | 17678365277211 | 7, 1, 0, 8 | 3, 5, 8, 4 |
| 20 | 051567121285343 | 17765468271156 | 7, 1, 0, 8 | 3, 5, -8, 4 |
Table 1 (continued)

|   | \( A \) & \( B \) | \( C \) & \( D \) | \( a, b, c, d \) | \( a^*, b^*, c^*, d^* \) |
|---|-----------------|-----------------|-----------------|-----------------|
| 1 | 0782321435141431 | 167587656743842 | 9, 1, −6, 2     | 3, 7, 8, 0      |
| 2 | 0784151482828782 | 167835857653471 | −5, −5, −6, 6   | −3, −7, 0, −8   |
| 3 | 0784351765121731 | 167838232233854 | 3, 7, 0, −8     | 9, 1, 2, −6     |
| 4 | 0767641512178561 | 176536611456768 | 3, 7, 0, 8      | 9, 1, −6, −2    |
| 5 | 0564376515151581 | 118772615545132 | 5, 5, 6, 6      | 7, 3, 8, 0      |
| 6 | 0512656235371531 | 165711846213678 | 9, 1, 2, 6      | 3, 7, 0, 8      |
| 7 | 0534678534348481 | 165344387727573 | −5, −5, −6, 6   | −3, −7, −8, 0   |
| 8 | 0532678482348461 | 167165812256464 | −1, −9, 6, 2    | −7, −3, 0, −8   |
| 9 | 0515153564821232 | 177863718512664 | 9, 1, −2, 6     | 3, 7, 8, 0      |

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, N2L 3G1, CANADA

E-mail address: djokovic@uwaterloo.ca