

On the Relation Between the Index Coding and the Network Coding Problems

Salim El Rouayheb, Alex Sprintson, and Costas Georghiades
Department of Electrical and Computer Engineering
Texas A&M University, College Station, TX 77843
{salim, spalex, c-georghiades}@ece.tamu.edu

Abstract—In this paper we show that the Index Coding problem captures several important properties of the more general Network Coding problem. An instance of the Index Coding problem includes a server that holds a set of information messages \( X = \{x_1, \ldots, x_k\} \) and a set of receivers \( R \). Each receiver has some side information, known to the server, represented by a subset of \( X \) and demands another subset of \( X \). The server uses a noiseless communication channel to broadcast encodings of messages in \( X \) to satisfy the receivers’ demands. The goal of the server is to find an encoding scheme that requires the minimum number of transmissions.

We show that any instance of the Network Coding problem can be efficiently reduced to an instance of the Index Coding problem. Our reduction shows that several important properties of the Network Coding problem carry over to the Index Coding problem. In particular, we prove that both scalar linear and vector linear codes are insufficient for achieving the minimal number of transmissions.

I. INTRODUCTION

Since its introduction by the seminal paper of Ahlswede et al. [1], the network coding paradigm has received a significant interest from the research community (see e.g., [2], [3] and references therein). Network coding extends the functionality of the intermediate network nodes from merely copying and forwarding their received messages to combining the information content of several incoming messages and forwarding the result over the outgoing edges. The network coding approach was shown to produce substantial gain over the traditional approach of routing and tree packing in many scenarios.

The Index Coding problem has been recently introduced in [4] and has been the subject of several studies [5], [6], [7]. An instance of the Index Coding problem includes a server/transmitter that holds a set of information messages \( X \) and a set of receivers \( R \), each one of them has some side information represented by a subset of \( X \), known to the server, and demands another subset of \( X \). The server can broadcast encodings of messages in \( X \) over a noiseless channel. The objective is to identify an encoding scheme that satisfies the demands of all clients with the minimum number of transmissions.

Figure 1 depicts an instance of the Index Coding problem that includes a server with four messages \( x_1, \ldots, x_4 \in \{0, 1\} \) and four clients. For each client, we show the set of messages it has (side information), and the set of messages it demands (demands). Note that the server can always satisfy the demands of the clients by sending all the messages. However, this solution is suboptimal since it is sufficient for the server to broadcast the two messages \( x_1 + x_2 + x_3 \) and \( x_1 + x_4 \) (all operations are over \( GF(2) \)). This shows that by using an efficient encoding scheme, the server can significantly reduce the number of transmissions, and, in turn, reduce the delay and the energy consumption.

In general, each message can be divided into several packets and the encoding scheme can combine packets from different messages to minimize the number of transmissions. With linear index coding, all packets are considered to be elements of a certain finite field \( F \) and each transmitted packet is a linear combination of the packets corresponding to the original messages in \( X \). The linear solutions can be further classified into scalar linear and vector linear. With a scalar linear solution, each message corresponds to exactly one packet, while with a vector linear solution each message can be divided into several packets. Note that the example shown in Figure 1 uses a scalar linear solution over \( F = GF(2) \).

The Index Coding problem was studied from an information theoretical perspective in [5]. The authors of [4] established lower and upper bounds on the minimum number of transmissions based on the properties of a certain related graph. References [6] and [8] present several heuristic solutions for this problem. In addition, the authors of [7] showed the suboptimality of scalar linear encoding schemes, which disproves the conjecture of [4].
Contributions

Index coding can be seen as a special case of the Network Coding problem. In this paper, we show that, nevertheless, several important properties of the more general Network Coding problem carry over to the Index Coding problem. To that end, we present a reduction that maps any instance of the Network Coding problem to a corresponding instance of the Index Coding problem. We use this reduction to establish several fundamental properties of the Index Coding problem.

First, we show that a scalar linear solution may require more transmissions than a vector linear one. In particular, we show two instances of the Index Coding problem in which a vector linear solution that divides each message into two packets yields a smaller number of transmissions than a scalar linear solution.

Second, we show that even vector linear solutions for the Index Coding problem are insufficient for achieving the minimal number of transmissions. In particular, we use our reduction and the construction presented in [9] to show an instance of the Index Coding problem for which a non-linear solution requires a lower number of transmissions than the linear one.

II. Model

A. Network Coding

Let $G(V, E)$ be a graph with vertex set $V$ and edge set $E$. For each edge $e(u, v) \in E$, we define the in-degree of $e$ to be the in-degree of its tail node $u$. Similarly, we define the out-degree of $e$ to be the out-degree of its head node $v$. Let $S \subseteq E$ be the subset of edges in $E$ of zero in-degree and let $D \subseteq E$ be the subset of edges in $E$ of zero out-degree. We refer to edges in $S$ and $D$ as input and output edges, respectively.

We denote $m = |E|$, $k = |S|$, $d = |D|$, and assume that the edges in $E$ are indexed such that $S = \{e_1, \ldots, e_k\}$ and $D = \{e_{m-d+1}, \ldots, e_m\}$. Also, for each edge $e = (u, v) \in E$, we define $P(e)$ to be the set of the parent edges of $e$, i.e., $P(e) = \{(w, u); (w, u) \in E\}$.

We represent a communication network by a 3-tuple $\mathbb{N}(G(V, E), X, \delta)$ defined by an acyclic graph $G(V, E)$, a message set $X = \{x_1, \ldots, x_k\}$, and an onto function $\delta : D \rightarrow X$ from the set of output edges to the set of messages. Each message $x_i \in X$ consists of a vector of $n$ packets $x_i = (x_{i1}, \ldots, x_{in})$.

We assume that the message $x_i$ is available at the tail node of the input edge $e_i$. The function $\delta$, referred to as the demand function, represents, for each output edge $e_i \in D$, the message demanded by its head node.

Definition 1 (Network Code): Let $\mathbb{N}(G(V, E), X, \delta)$ be an instance of the Network Coding problem with $k = |X|$ messages, each message is a vector of $n$ packets, $x_i = (x_{i1}, \ldots, x_{in}) \in \Sigma^n$, where $\Sigma = \{0, \ldots, q-1\}$ is a $q$-ary alphabet. Then, an $(n, q)$ network code of block length $n$ is a collection

$$C = \{f_i = (f_1^i, \ldots, f_n^i); f_i : (\Sigma^n)^k \rightarrow \Sigma, e \in E, 1 \leq i \leq n\}$$

of global encoding functions, indexed by the edges of $G$, that satisfy the following conditions:

(N1) $f_{e_i}(X) = x_i$ for $i = 1, \ldots, k$;
(N2) $f_{e_i}(X) = \delta(e_i)$ for $i = m - d + 1, \ldots, m$;
(N3) For each $e = (u, v) \in E \setminus S$ with $P(e) = \{e_1, \ldots, e_p\}$, there exists a function $\phi_e : (\Sigma^n)^{p_e} \rightarrow \Sigma^n$, referred to as the local encoding function of $e$, such that $f_e(X) = \phi_e(f_{e_1}(X), \ldots, f_{e_p}(X))$, where $p_e$ is the in-degree of $e$ and $P(e)$ is the set of parent edges of $e$.

If $n = 1$, the network code is called a scalar network code, otherwise (if $n > 1$) it is called a vector or a block network code. If $\Sigma$ is a certain finite field $\mathbb{F}$, and all the global and local encoding functions are linear functions of the packets, the network code is called linear over $\mathbb{F}$.

B. Index Coding

An instance of the Index Coding problem $\mathcal{I}(X, R)$ includes

1) A set of messages $X = \{x_1, \ldots, x_k\}$;
2) A set of clients $R \subseteq \{(x, H); x \in X, H \subseteq X \setminus \{x\}\}$.

Here, $X$ represents the set of messages available at the server. A client is represented by a pair $(x, H) \in R$, where $x$ is the message required by the client, and $H \subseteq X$ is the set of messages available to the client as side information. We assume, without loss of generality, that each client needs exactly one message.

As in the Network Coding problem, each message $x_i \in X$ is divided into $n$ packets $x_i = (x_{i1}, \ldots, x_{in})$. We refer to parameter $n$ as the block length of the index code.

Definition 2 (Index Code): Let $\mathcal{I}(X, R)$ be an instance of the Index Coding problem with $k = |X|$ messages, each message $x_i$ is a vector of $n$ packets, $(x_{i1}, \ldots, x_{in}) \in \Sigma^n$, where $\Sigma = \{0, \ldots, q-1\}$ is a $q$-ary alphabet. Then, an optimal $(n, q)$ index code for $\mathcal{I}(X, R)$ is a function $f : (\Sigma^n)^k \rightarrow \Sigma^\ell$, such that

(I1) for each client $r = (x, H) \in R$, there exists a function $\psi_r : \Sigma^{\ell(n+|H|)} \rightarrow \Sigma^n$ such that $\psi_r(f(x_1, \ldots, x_k), x_{i|e \in H}) = x$;

(I2) $\ell = \ell(n, q)$ is the smallest integer such that (I1) holds for the given $q$-ary alphabet and block length $n$.

We refer to $\psi_r$ as the decoding function for client $r$. With a linear index code, the alphabet $\Sigma$ is a field and the functions $f$ and $\psi_r$ are linear in variables $x_{i1}$. Similarly, if $n = 1$ the index code is called a scalar code and for $n > 1$ it is called a vector or block code.

Our formulation of the Index Coding problem here differs from that of [4] and [7] in two aspects. First, the model of [4] and [7] assumes that for each message in $X$ there is exactly one client that requests it. Our model does not make this assumption. Second, and more importantly, [4] and [7] focus on scalar linear codes (vector linear codes are mentioned in the conclusion of [7]), whereas we consider the more general case of vector linear codes.

Let $\mathcal{I}(X, R)$ be an instance of the Index Coding problem. We define by $\lambda(n, q) = \ell(n, q)/n$ the transmission rate of the optimal solution over an alphabet of size $q$. We also denote by $\lambda^*(n, q)$ the minimum rate achieved by a vector linear solution over a finite field $\mathbb{F}_q$. We are interested in the behavior of $\lambda$ and $\lambda^*$ as functions of $n$ and $q$.

Let $\mu(\mathcal{I})$ be the largest set of messages requested by a collection of clients with identical “has” sets, i.e., $\mu(\mathcal{I}) =$
max_{I \subseteq X} |\{x_i; \ (x_i, I) \in R\}|. It is easy to verify that the optimal rate \(\lambda(n, q)\) is lower bounded by \(\mu(I)\), independently of the values of \(n\) and \(q\).

Lemma 3: For any instance \(I(X, R)\) of the Index Coding problem it holds that

\[\lambda(n, q) \geq \mu(I)\].

III. MAIN RESULT

In this section we present a reduction from the Network Coding problem to the Index Coding problem. Specifically, for each instance \(\mathbb{N}(G(V, E), X, \delta)\) of the Network Coding problem, we construct a corresponding instance \(I_n\) of the Index Coding problem such that \(I_n\) has an \((n, q)\) index code of rate \(\lambda'(n, q) = \lambda(n, q) = \mu(I_n)\) if and only if there exists an \((n, q)\) linear network for \(N\).

Definition 4: Let \(\mathbb{N}(G(V, E), X, \delta)\) be an instance of the Network Coding problem. We construct an instance \(I_{2n}(Y, R)\) of the Index Coding problem as follows:

1) The set of messages \(Y\) includes a message for each edge \(e \in E\) and the messages \(x_i \in X\), i.e., \(Y = \{y_1, \ldots, y_m\} \cup X\);
2) The set of clients \(R\) is a union of \(R_1, \ldots, R_5\) defined as follows:
   a) \(R_1 = \{(x_i, \{y_i\}); e_i \in S\}\)
   b) \(R_2 = \{(y_i, \{x_i\}); e_i \in S\}\)
   c) \(R_3 = \{(y_i, \{y_j, \{x_i, e_j \in P(e_i)\}); e_i \in E \setminus S\}\)
   d) \(R_4 = \{(\delta(e_i), \{y_j\}); e_i \in D\}\)
   e) \(R_5 = \{(y_i, X); i = 1, \ldots, m\}\)

It is easy to verify that for instance \(I_{2n}(Y, R)\) it holds that \(\mu(I_{2n}) = m\).

Theorem 5: Let \(\mathbb{N}(G(V, E), X, \delta)\) be an instance of the Network Coding problem and let \(I_{2n}(Y, R)\) be the corresponding instance of the Index Coding problem as defined above. Then, there exists a linear \((n, q)\) index code for \(I_{2n}\) with \(\lambda'(n, q) = \mu(I_{2n})\) iff there exists a linear \((n, q)\) network code for \(N\).

Proof: Suppose there is a linear \((n, q)\) network code \(C = \{f_e(X); f_e : (\mathbb{F}_q)^{n+k} \rightarrow (\mathbb{F}_q)^m, e \in E\}\) for \(N\) over the finite field \(\mathbb{F}_q\) for some integer \(n\).

Define \(g : (\mathbb{F}_q)^{n+k} \rightarrow (\mathbb{F}_q)^m\) such that \(\forall Z = (x_1, \ldots, x_k, y_1, \ldots, y_m) \in (\mathbb{F}_q)^{n+k}, g(Z) = (g_1(Z), \ldots, g_m(Z))\) with

\[g_i(Z) = y_i + x_i, \quad i = 1, \ldots, k\]
\[g_i(Z) = y_i + f_e(X), \quad i = k + 1, \ldots, m\]

Next, we show that \(g(Z)\) is in fact an index code for \(I_{2n}\) by proving the existence of the decoding functions \(\psi_r\). We consider five cases:

1) \(\forall r = (x_i, \{y_i\}) \in R_1, \psi_r = g_i(Z) - y_i, \quad i \in 1, \ldots, k\)
2) \(\forall r = (y_i, \{x_i\}) \in R_2, \psi_r = g_i(Z) - x_i, \quad i \in 1, \ldots, m\)
3) \(\forall r = (y_i, \{y_j, \{x_i, e_j \in P(e_i)\}); e_i \in E \setminus S\}\), since \(C\) is a linear network code for \(N\), there exists a linear function \(\delta(e_i)\) such that \(f_{e_i}(X) = \delta(e_i)(f_{e_i}(X), \ldots, f_{e_i}'(X))\). Thus,
\[
\psi_r = g_i(Z) - \delta(e_i)(g_{i_1}(Z) - y_{i_1}, \ldots, g_{i_p}(Z) - y_{i_p}), \quad \forall r = (\delta(e_i), \{y_j\}) \in R_4, e_i \in D, \psi_r = g_i(Z) - y_i,
\]
4) \(\forall r = (y_i, \{y_j\}) \in R_5, e_i \in D, \psi_r = g_i(Z) - y_i, \quad i \in 1, \ldots, m\)

5) \(\forall r = (y_i, X) \in R_5, \psi_r = g_i(Z) - f_{e_i}(X)\).

Now assume that \(g : (\mathbb{F}_q)^{n+k} \rightarrow (\mathbb{F}_q)^m\) is a linear \((n, q)\) index code for \(I_n\) over the field \(\mathbb{F}_q\), such that \(\forall Z = (x_1, \ldots, x_k, y_1, \ldots, y_m) \in (\mathbb{F}_q)^{n+k}, g(Z) = (g_1(Z), \ldots, g_m(Z))\), \(x_i, y_i, g_i(Z) \in \mathbb{F}_q^n\). We write

\[g_i(Z) = \sum_{j=1}^k x_j A_{ij} + \sum_{j=1}^m y_j B_{ij},\]

where \(i = 1, \ldots, m\) and \(A_{ij}, B_{ij} \in M_{\mathbb{F}_q}(n,n)\) are sets of \(n \times n\) matrices with elements in \(\mathbb{F}_q\).

The functions \(\psi_r\) exist for all \(r \in R_5\) iff the matrix \(M = [B_{ij}] \in M_{\mathbb{F}_q}(nm, nm)\), which has the matrix \(B_{ij}\) as a block submatrix in the \((i,j)\)th position, is invertible. Define \(h : (\mathbb{F}_q)^{n+k} \rightarrow (\mathbb{F}_q)^m\), such that \(h(Z) = g(Z)M^{-1}, \forall Z \in (\mathbb{F}_q)^{n+k}\). So, we obtain

\[h_i(Z) = y_i + \sum_{j=1}^k x_j C_{ij}, i = 1, \ldots, m,\]

where \([C_{ij}] \in M_{\mathbb{F}_q}(n,n)\). We note this \(h(Z)\) is a valid index code for \(I_n\). In fact, \(\forall r = (x, H) \in R\) with \(\psi_r(g_i(x, x \in H) = x, \psi'_r = (h, x, x \in H) = h(M, x \in H)\) is a valid decoding function corresponding to the client \(r\) and the index code \(h(Z)\).

For all \(r \in R_1 \cup R_4\), \(\psi_i\) exists iff \(i \neq j\) and \(j \neq i\) holds that \(C_{ij} = [0] \in M_{\mathbb{F}_q}(n, n)\) and \(C_{ii}\) is invertible, where \([0]\) denotes the all zeros matrix. This implies that

\[h_i(Z) = y_i + \sum_{j=1}^k x_j C_{ij}, i = k + 1, \ldots, m-d \quad (1)\]

Next, we define the functions \(f_{e_i} : (\mathbb{F}_q)^{n+k} \rightarrow (\mathbb{F}_q)^m, e_i \in E\) as follows:

1) \(f_{e_i}(X) = x_i, i = 1, \ldots, k\)
2) \(f_{e_i}(X) = \sum_{j=1}^k x_j C_{ij}, i = k + 1, \ldots, m-d\)
3) \(f_{e_i}(X) = \delta(e_i), i = m-d + 1, \ldots, m\)

Then \(C = \{f_{e_i}; e_i \in E\}\) is a linear \((n, q)\) network code for \(N\). To show that it suffices to prove that condition N3 holds.

Let \(e_i\) be an edge in \(E \setminus S\) with the set \(P(e_i) = \{e_{i_1}, \ldots, e_{i_p}\}\) of parent edges. We denote by \(I_i = \{i_1, \ldots, i_p\}\) and \(r_i = (y_i, \{y_{i_1}, \ldots, y_{i_p}\}) \in R_5\). Then, there is a linear function \(\psi_{r_i}\) such that \(y_i = \psi_{r_i}(h_{i_1}, \ldots, h_{i_p}, y_{i_1}, \ldots, y_{i_p})\). Hence, there exist \(T_{i_1}, T_{i_2}' \in M_{\mathbb{F}_q}(n, n)\) such that

\[y_i = \sum_{j=1}^m h_j T_{i_j} + \sum_{\alpha \in I_i} y_{\alpha} T'_{i_\alpha}\]

Using Eq. (1), we get that \(T_i\) is the identity matrix, \(T'_{i_\alpha} = -T'_{i_\alpha} \forall \alpha \in I_i, i_j = \{0\} \forall j \notin I_i \cup \{i\}\). Therefore,

\[f_{e_i} = -\sum_{\alpha \in I_i} f_{e_\alpha} T_{i_\alpha}, \forall e_i \in E \setminus S,\]

and \(C\) is a feasible network code for \(N\).
Lemma 6: Let \( \mathbb{N}(G(V,E),X,\delta) \) be an instance of the Network Coding problem and let \( \mathcal{I}_n(Y,R) \) be the corresponding index problem. If there is an \((n,q)\) network code (not necessarily linear) with \( \lambda(n,q) = \mu(\mathcal{I}_n) = m \), then there is a \((n,q)\) index code for \( \mathcal{I}_n \) with \( \lambda(n,q) = \mu(\mathcal{I}_n) = m \), where \( m = |E| \).

Proof: The proof can be obtained by slightly modifying the first part of the proof of Theorem 5.

IV. APPLICATIONS

A. Block Encoding

Index coding, as noted in [4], [7], is reminiscent of the zero-error source coding with side information problem discussed by Witsenhausen in [10]. Two cases were studied there depending on whether the transmitter knows the side information available to the receiver or not. It was shown that in the former case repeated scalar encoding is optimal, i.e. block encoding does not provide any benefit. We will demonstrate in this section that this result does not always hold for the Index Coding problem which can be regarded as an extension of the point to point problem discussed in [10].

Let \( \mathcal{N}_1 \) be the M-network introduced in [11] and depicted in Figure 2(a). It was shown in [12] that this network does not have a scalar linear solution, but has a vector linear one of block length 2. Interestingly, such a vector linear solution does not require encoding. In fact, reference [12] proves a more general theorem:

Theorem 7: The M-network has a linear network code of block length \( n \) iff \( n \) is even.

Next, we present another network \( \mathcal{N}_2 \), that we refer to as the non-Pappus network, and that has the same property as the M-network. Both of these networks will be used to construct two instances of the Index Coding problem where vector linear outperforms scalar linear coding.

Definition 8 (non-Pappus Network): Let \( S_0 = \{\{1,2,3\},\{1,5,7\},\{3,5,9\},\{2,4,7\},\{4,5,6\},\{2,6,9\},\{1,6,8\},\{3,4,8\}\} \), and \( S_1 = \{I \subseteq \{1,2,\ldots,9\}; |I| = 3 \} \setminus S_0 \). The non-Pappus network \( \mathcal{N}_2 \) is obtained by adding to the network depicted in Figure 2(b) a node \( n_I \) for each \( I = \{i,j,k\} \in S_1 \), the edges \((n_i,n_j),(n_i,n_k),(n_k,n_j)\) and three output edges outgoing from \( n_I \), each one of them demands a different \( x_i \).

Theorem 9: There is no scalar linear network code for the non-Pappus network over any field, but there is a \((2,3)\) linear one.

Proof: Let \( C = \{f_e; e \in \mathcal{N}_2\} \) be a scalar linear network code for \( \mathcal{N}_2 \) over a certain field \( \mathbb{F} \). Without loss of generality, we assume that for each node \( n_I \) of \( \mathcal{N}_2 \), the functions associated with its output edges are identical. We define then \( f_i = f_e \) where \( e \) is an outgoing edge to \( n_I \), \( i = 1,\ldots,9 \), and write \( f_i = a_1 x_1 + a_2 x_2 + a_3 x_3 = a_i \cdot X^T \), where \( X = (x_1,x_2,x_3) \) and \( a_i = (a_{i1},a_{i2},a_{i3}) \).

Since \( \forall I = \{i,j,k\} \in S_1 \), the outgoing edges to node \( n_I \) demand \( x_1, x_2 \) and \( x_3 \), we have \( \text{rank}\{a_1,a_2,a_3\} = 3 \). Furthermore, from the connectivity of \( \mathcal{N}_2 \), we deduce that \( a_2 \) should be a linear combination of \( a_1 \) and \( a_3 \), giving \( \text{rank}\{a_1,a_2,a_3\} = 3 \). But \( \text{rank}\{a_1,a_2,a_4\} = 3 \), which implies that \( \text{rank}\{a_1,a_2,a_3\} > 1 \), hence \( \text{rank}\{a_1,a_2,a_3\} = 2 \). Similarly, \( \forall \{i,j,k\} \in S_0, \text{rank}\{a_1,a_j,a_k\} = 2 \).

Therefore, letting \( A = \{a_1,a_2,\ldots,a_9\} \), the matroid \( \mathcal{M}(A,\text{rank}) \) is the non-Pappus matroid shown in Figure 3[13, p.43]. Therefore, the vectors \( a_i \), form a linear representation of \( \mathcal{M} \) over \( \mathbb{F} \). But, by Pappus theorem [13, p.173], the non-Pappus matroid is not linearly representable over any field, which leads to a contradiction. So, \( \mathcal{N}_2 \) does not have a scalar linear solution.

Let \( x_1 = (x,y), x_2 = (w,z), x_3 = (u,v) \in \mathbb{F}_2^3 \). Define \( f_1(X) = x_1, f_2(X) = (x+w,y+z), f_3(X) = x_2, f_4(X) = (x+u+2z,y+2v+w+z), f_5(X) = x_3, f_6(X) = (x+2u+2v+2z,y+u+w+z), f_7(X) = (x+y+u+2v), f_8(X) = (x+u+w+z,y+2v+w), f_9(X) = (u+w,v+z) \). These functions correspond to the multilinear (or partition) representation of the non-Pappus matroid discussed in [14], [15]. For each edge \( e \in G \) outgoing from node \( n_i, i = 1,\ldots,9, \) define \( f_e = f_i \). And for each edge \( e \in D, \) let \( f_e = \delta(e) \). Then, \( \{f_e; e \in \mathcal{N}_2\} \) is a \((2,3)\) network code for the non-Pappus network.

Now, consider \( I_{\mathcal{N}_1} \) and \( I_{\mathcal{N}_2} \) the two Index Coding problems corresponding respectively to the M-network and the non-Pappus network obtained by the construction of the previous section. Both do not admit scalar linear index codes that achieve the bound of Lemma 3 but have linear index codes
of length 2, $I_{N_1}$ over $\mathbb{F}_2$ and $I_{N_2}$ over $\mathbb{F}_3$, that meet this bound. Thus, $I_{N_1}$ and $I_{N_2}$ are two instances of the Index Coding problem where vector linear coding outperforms scalar linear coding. This result can be summarized by the following corollary:

**Corollary 10:** For $I_{N_1}, \lambda^*(2,2) < \lambda^*(1,2)$. And for $I_{N_2}, \lambda^*(2,3) < \lambda^*(1,3)$.

**Proof:** Follows directly from Theorems 5, 7 and 9.

---

**B. Linearity vs. Non-Linearity**

Linearity is a desired property for any code, including index codes. It was conjectured in [4] that binary scalar linear index codes are optimal, meaning that $\lambda^*(1,2) = \lambda(1,2)$ for any instance of the Index Coding problem. The authors of [7] disproved this conjecture for scalar linear codes by providing for any given number of messages $k$ and field $\mathbb{F}_q$, an instance of the Index Coding problem with a large gap between $\lambda^*(1,q)$ and $\lambda(1,q)$.

In this section we show that vector linear codes are insufficient for minimizing the number of transmissions. In particular, we prove that non-linear index codes outperform vector linear codes for any choice of field and block length $n$. Our proof is based on the insufficiency of linear network codes shown in [9].

First, we present the network $N_3$ depicted in Figure 4 which was introduced and studied in [9]. The following theorem was proved in [9].

**Theorem 11:** The network $N_3$ does not have a linear solution, but has a $(2,4)$ non-linear solution.

Let $\mathcal{I}_{N_3}$ be an instance of the Index Coding problem that corresponds to $N_3$ constructed according to Definition 4. Theorem 11 implies that $\mathcal{I}_{N_3}$ does not have a linear solution that achieves $\mu(\mathcal{I}_{N_3})$, the lower bound of Lemma 3. However, by Lemma 6 the $(2,4)$ non-linear code of $N_3$ can be used to construct a $(2,4)$ non-linear index code for $\mathcal{I}_{N_3}$ that achieves the lower bound of Lemma 6. Hence, we obtain the following result:

**Corollary 12:** For the instance $\mathcal{I}_{N_3}$ of the Index Coding problem it holds that $\lambda(2,4) = \mu(\mathcal{I}_{N_3})$. Furthermore, for any positive integers $n$ and $q$, it holds that $\lambda^*(n,q) < \lambda(n,q)$.

---

**V. Conclusion**

In this paper we studied the connection between the Index Coding and Network Coding problems. We showed a reduction that maps each communication network $N$ to an instance of the Index Coding problem $\mathcal{I}_N$ such that $N$ has a linear network code if and only if $\mathcal{I}_N$ has a linear index code over the same field that satisfies a certain condition on the number of transmissions.

This reduction allowed us to apply many important results for network coding to index coding. For instance, we introduced the non-Pappus network and showed that it does not have a scalar linear network code, but has a vector linear one. The non-Pappus network in addition to the M-network of [11] were used to construct index coding instances where vector linear solutions outperform scalar linear solutions. Another application of this reduction concerns the comparison of linear and non-linear index codes. Using the results of Dougherty et al. in [9] we proved the insufficiency of vector linear solutions for the Index Coding problem.

---

**REFERENCES**

[1] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung. Network Information Flow. IEEE Transactions on Information Theory, 46(4):1204–1216, 2000.
[2] C. Fragouli and E. Soljanin. Network Coding Fundamentals (Foundations and Trends in Networking). Now Publishers Inc, 2007.
[3] R. Yeung, S-Y. Li, and N. Cai. Network Coding Theory (Foundations and Trends in Communications and Information Theory). Now Publishers Inc, 2006.
[4] Z. Bar-Yossef, Y. Birn, T. S. Jayram, and T. Kol. Index coding with side information. In Proc. of the 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS), 2006.
[5] Y. Wu, J. Padhye, R. Chandra, V. Padmanabhan, and P. A. Chou. The local mixing problem. In Proc. Information Theory and Applications Workshop, San Diego, Feb. 2006.
[6] S. El Rouayheb, M.A.R. Chaudhry, and A. Sprintson. On the minimum number of transmissions in single-hop wireless coding networks. In IEEE Information Theory Workshop (Lake Tahoe), 2007.
[7] E. Lubetzky and U. Stav. Non-linear index coding outperforming the linear optimum. In Proc. of the 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS), pages 161–167, 2007.
[8] M. A. R. Chaudhry and A. Sprintson. Efficient algorithms for index coding. 2008.
[9] R. Dougherty, C. Freiling, and K. Zeger. Insufficiency of linear coding in network information flow. IEEE Transactions on Information Theory, 51(8):2745–2759, 2005.
[10] D.J.A. Welsh. Matroid Theory. Academic Press, London, London, 1976.
[11] M. Effros, M. Medard, T. Ho, and D. Karger. On coding for non-multicast networks. In Proceedings of the IEEE International Symposium on Information Theory, 2003.
[12] R. Dougherty, C. Freiling, and K. Zeger. Networks, matroids, and non-shannon information inequalities. IEEE Transactions on Information Theory, 53(6), June 2007.
[13] J. G. Oxley. Matroid Theory. Oxford University Press, USA, New York, NY, USA, January 1993.
[14] J. Simonis and A. Ashikhmin. Almost affine codes. Designs, Codes and Cryptography, 14:179–797, 1998.
[15] F. Matus. Matroid representations by partitions. Discrete Mathematics, 203:169–194, 1999.

---

**Fig. 4.** The network $N_3$ of [9]. $N_3$ does not have a linear network code over any field, but has a non-linear one over a quaternary alphabet.