Probing Many-Body Localization by Spin Noise Spectroscopy

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We propose to apply spin noise spectroscopy (SNS) to detect many-body localization (MBL) in disordered spin systems. The SNS methods are relatively non-invasive technique to probe spontaneous spin fluctuations. We here show that the spin noise signals obtained by cross-correlation SNS with two probe beams can be used to separate the MBL phase from a noninteracting Anderson localized phase and a delocalized (diffusive) phase in the studied models for which we numerically calculate real time spin noise signals and their power spectra. For the archetypical case of the disordered XXZ spin chain we also develop a simple phenomenological model.

The fate of Anderson localization in the presence of inter-particle interactions in a disordered quantum medium is an exciting frontier in condensed matter physics [1-19]. It is well known that all states of a one-dimensional (1D) disordered chain of noninteracting particles are Anderson localized (AL) for any amount of disorder [20-21]. Thus, the AL state is a perfect insulator as long as the particles are not coupled to other degrees of freedom. In the presence of inter-particle interactions, a dynamical transition from a delocalized (diffusive) phase to a many-body localized (MBL) phase has been predicted in disordered quantum media when the strength of (quenched) randomness is increased [22-24].

The MBL state is also a perfect insulator, but it has different dynamical properties compared to the AL state of noninteracting particles [10-12, 16]. For example, entanglement entropy in the MBL phase shows a slow logarithmic growth following a global quench in an isolated system [10-12]. However, the entanglement entropy is very difficult to measure experimentally, and electrons or spins in conventional solid-state systems are coupled to an environment such as a phonon bath. Recent studies show that signatures of the MBL phase can survive in the presence of weak coupling to a thermalizing environment [22-23]. In particular, the spectral functions of local operators can be used to identify the MBL phase in the presence of weak dissipation [22-24].

A new experimental approach based on a modified non-local spin-echo protocol along with a double electron-electron resonance technique in electron spin resonance has been proposed to distinguish the MBL phase from a noninteracting AL phase and a delocalized phase at infinite temperature [24]. The proposed approach can probe interaction effects, thus is able to separate the MBL phase from the AL phase. The method does involve optical pumping or polarization of local spins by external pulse fields, which can in principle lead to unwanted local heating and excitations. Here we propose a relatively non-invasive method based on spin noise spectroscopy (SNS) to distinguish the MBL phase from the AL phase and the delocalized phase in disordered spin systems.

The optical SNS method has been developed recently as an alternative to conventional perturbation-based (pump-probe) techniques for measuring dynamical spin properties [25-26]. Intrinsic spin fluctuations of electrons and holes are passively detected in SNS by measuring optical Faraday rotation fluctuations of a linearly polarized probe laser beam passing through the sample [27-38]. SNS with a single probe laser has been useful in characterizing various properties (e.g., g-factors, relaxation rates and decoherence times) of different spin ensembles, such as specific alkali atoms [30], itinerant electron spins in semiconductors [33] and localized hole spins in quantum dot ensembles [34]. Last year, an extension of the traditional SNS has been proposed and demonstrated by using two linearly polarized probe lasers for detecting inter-species spin interactions in a heterogeneous two-component spin ensemble interacting via binary exchange coupling in thermal equilibrium [39-40]. In this cross-correlation SNS method, intrinsic spin fluctuations from two different species are independently detected, and interaction effects are determined by the cross-correlation of these two spin noise signals. Interaction effects between spins of a single species in a sample can also be determined by two-beam SNS when the two probe laser beams are spatially separated as shown in Fig. 1(a) [39]. Thus, we propose that the two-beam SNS measurements can be used to separate the MBL phase from an AL phase. In fact, as we show in our example, the responses from single-beam and two-beam SNS can be efficiently employed to distinguish the MBL phase from both the AL and delocalized phases.

In order to demonstrate our idea we study the spin noise signals of a 1D random-field XXZ spin chain,
perhaps the best-studied “canonical” model system for MBL. In addition, results for the disordered transverse-field Ising model are given in the Suppl. Mat. [43]. The Hamiltonian of disordered XXZ spin chain is given by

$$H = \sum_{i=1}^{N} J_{\perp}(S_{i}^{x}S_{i+1}^{x} + S_{i}^{y}S_{i+1}^{y}) + J_{z}S_{i}^{z}S_{i+1}^{z} + h_{i}S_{i}^{z}(1)$$

where the random fields \( h_{i} \) are chosen from a uniform distribution within the window \([-\eta, \eta]\), and \( S_{N+1}^{x,y,z} = S_{1}^{x,y,z} \) for periodic boundary conditions. For isotropic exchange interactions (\( J_{z} = J_{\perp} \)), this is a disordered Heisenberg chain. The spin Hamiltonian in Eq. 1 can be mapped to a model of interacting spinless Jordan-Wigner fermions on a lattice where \( J_{\perp} \) represents hopping between neighboring sites, \( J_{z} \) is inter-particle interaction strength and \( h_{i} \) denotes a random local chemical potential [42]. For \( J_{z} \neq 0 \), the Hamiltonian in Eq. 1 represents noninteracting fermions subject only to a random local potential, and hence can be used to study the AL phase of noninteracting particles. This model with a non-zero \( J_{z} \) has been extensively studied recently in the context of MBL. A disorder versus energy density phase-diagram separating the delocalized and MBL phases of an isolated chain is shown in Fig. 1(b) following Ref. [44] where \( \epsilon = (\langle H \rangle - E_{\text{max}})/(E_{\text{min}} - E_{\text{max}}) \) with \( E_{\text{max}} \) (\( E_{\text{min}} \)) being maximum (minimum) energy.

For the Hamiltonian in Eq. 1 we now describe the signals measured in the SNS set-ups. The measured response of the spin component transverse to the random magnetic field in the single-beam SNS is

$$C_{\perp}(t) = \langle \langle S_{\perp}^{y}(t), S_{\perp}^{y}(0) \rangle \rangle,$$  

where \( \{ \ldots \} \) is the anti-commutator and \( \langle \langle \ldots \rangle \rangle \) denotes both (canonical) thermal and disorder averaging. Here, the \( x \)-component of total spin polarization at the measurement spot at time \( t \) is \( S_{\perp}^{y}(t) = \sum_{i \in \mathcal{I}} S_{i}^{y}(t) \) where the sum runs over all local spin sites within the sample spot illuminated by the probe laser beam \( I \). The correlation function \( C_{\perp}(t) \) describes the relaxation of spontaneous spin fluctuations at the spot of the sample probed by the single-beam SNS. The single-beam (local) spin-noise power spectrum is obtained by its Fourier transform,

$$P_{\perp}(\omega) = \int_{-\infty}^{\infty} dt \ e^{i\omega t} C_{\perp}(t).$$

Let \( |E_{n}\rangle \) denote a many-body eigenstate with eigenvalue \( E_{n} \), \( H|E_{n}\rangle = E_{n}|E_{n}\rangle \). The single-beam spin-noise power spectrum reads,

$$P_{\perp}(\omega) = \frac{1}{Z} \sum_{n,p} \delta(\omega + E_{n} - E_{p})(e^{-\beta E_{n}} + e^{-\beta E_{p}}) \langle \langle E_{n}|S_{\perp}^{y}|E_{p}\rangle \rangle^{2},$$

with partition function \( Z = \sum_{n} e^{-\beta E_{n}} \), and inverse temperature \( \beta = 1/k_{B}T \). \( \langle \langle \ldots \rangle \rangle \) in Eq. 3 describes averaging over different disorder realizations. The cross-correlation signal of the two-beam SNS is

$$C_{\perp}(t) = \langle \langle S_{\perp}^{y}(t), S_{\perp}^{y}(0) \rangle \rangle + \langle S_{\perp}^{y}(t), S_{\perp}^{y}(0) \rangle,$$

where \( S_{\perp}^{y}(t) = \sum_{m \in \mathcal{I}} S_{m}^{y}(t) \), and the sum over \( m \) runs through all sites which the probe laser \( I \) illuminates. The cross-correlation spin noise power

$$P_{\perp}(\omega) = \int_{-\infty}^{\infty} dt \ e^{i\omega t} C_{\perp}(t)$$

$$= \frac{1}{Z} \sum_{n,p} \delta(\omega + E_{n} - E_{p})(e^{-\beta E_{n}} + e^{-\beta E_{p}}) \langle \langle E_{n}|S_{\perp}^{y}|E_{p}\rangle \rangle \langle \langle E_{p}|S_{\perp}^{y}|E_{n}\rangle \rangle,$$

We calculate the eigenvalues and eigenstates of the Hamiltonian using exact diagonalization, and evaluate the spin noise signals in Eqs. 2,5 and the corresponding power spectra in Eqs. 4,7. In the following we present results using periodic boundary conditions at high temperature, \( k_{B}T = 50J_{\perp} \), averaging over 3000 disorder realizations. However, the main results of this paper are also valid at moderate temperatures [43]. We quote \( J_{\perp}, h_{i}, \eta \) in the units of \( J_{\perp} \), and fix \( J_{\perp} = 1 \) throughout the paper.

The single-beam spin noise response in real time for the interacting delocalized phase at low disorder (\( \eta = 1 \)) and for the MBL phase at high disorder (\( \eta = 5 \), at which every many-body eigenstate of the isolated chain is localized) of the random XXZ model is shown in Fig. 2(a,b). The single-beam responses are clearly different in the two phases. The transverse spin component relaxes exponentially in the delocalized phase, while in the MBL phase it appears to do so algebraically, and it also oscillates. We show the single-beam spin noise response of the AL phase (\( J_{z} = 0 \)) in Fig. 2(c), which again indicates an algebraic relaxation of the transverse spin component. The relaxation in this phase can be approximated as \( \sin(t/\tau_{A})/(t/\tau_{A}) \) with some characteristic time scale \( \tau_{A} \). We use high disorder (\( \eta = 5 \)) in Fig. 2(c) to ensure that all states are localized even for a finite-length noninteracting chain. We provide log scale plots in the Suppl. Mat. [43] for the real-time spin noise responses in Fig. 2 to highlight the nature of their respective decays.

Within the single-beam SNS measurements, the slow spin relaxation due to interactions in the highly disordered XXZ model is masked by other mechanisms of relaxation (e.g., due to random fields and hopping) also present in the noninteracting case. Thus, it becomes difficult to separate the MBL phase from the AL phase by single-beam SNS. However, interaction effects are nicely diagnosed by two-beam SNS when we cross correlate two different noise signals to exclude their self-correlations.

The two-beam cross-correlation spin noise responses in real time for the three different phases are shown in Fig. 2(d,e,f) when two beams are just next to each other without overlapping (i.e., the separation between the centers of the two beams is two lattice spacings). The responses here are quite different in all three phases.
FIG. 2. Single-beam and two-beam spin noise responses in real time in the three different phases. (a-c) $C^x(t)$ and (d-f) $C^y(t)$ are obtained by exact diagonalization in the delocalized ($J_z = 1$, $\eta = 1$), the MBL ($J_z = 1$, $\eta = 5$) and the Anderson localized ($J_z = 0$, $\eta = 5$) phase. Here $N = 12$ and the beam $I$ and $II$ illuminate respectively spins 5, 6 and 7, 8.

A large cross-correlation between the transverse components of different spins is developed in the diffusive phase, and it relaxes exponentially fast. The cross-correlation in the MBL phase is relatively smaller and it relaxes slowly with many oscillations. In the AL phase, the transverse component hardly shows any cross-correlation between different spins at high temperature. Notice that $C^y(0) = 1$ while $C^y(0) \approx 0$ in Fig. 2, because $\langle S^z(0)S^z_m(0) \rangle = \delta_{lm}/4$ at high temperature [15]. We also perform these calculation for the relaxation of spins along the $z$-direction, and find that it is similar in the MBL and AL phases even within the two-beam SNS measurements [43, 46]. Thus, $z$-direction response is not a good candidate to separate these two phases.

Turning to the frequency domain, the disorder averaged spin noise power spectra of single-beam and two-beam SNS are shown in Fig. 3. We find a Lorentzian line-shape for the single-beam noise power spectrum $P^x_{cr}(\omega)$ [Fig. 3(a)] in the diffusive regime reflecting the exponential spin relaxation. In the MBL phase, the shape of $P^x_{cr}(\omega)$ exhibits a plateau as shown in Fig. 3(b). The overall rectangular shape of $P^x_{cr}(\omega)$ in the AL regime [Fig. 3(c)] is somewhat similar to that of the MBL phase but it shows much higher fluctuations.

Crucially, the cross-correlation spin noise power spectra $P^{xy}_{cr}(\omega)$ of the two-beam SNS in Fig. 3(d-f) differ significantly in shape and magnitude in the delocalized (diffusive), MBL and AL phases. The power spectrum in the delocalized phase [Fig. 3(d)] can be perceived as the difference between two equal area Lorentzians of different widths. Thus, the total area under the cross-correlation curve is zero, a signature of high-temperature behaviour. The line-shape of $P^{xy}_{cr}(\omega)$ in the MBL phase is very different from that in the delocalized and AL phases. It is also almost one order of magnitude smaller than in the delocalized phase but is clearly non-zero [Fig. 3(e)]. However, $P^{xy}_{cr}(\omega)$ in the AL phase is very small (one order of magnitude smaller than that in the MBL phase at high temperature) and featureless indicating essentially no correlation between spins. Note that $P^{xy}_{cr}(\omega)$ shows both negative and positive values which signify (anti-)correlations between spins at different frequency/time scales. The anti-correlations in Fig. 3(d,e) at higher frequencies (shorter time) are induced by fast co-flips between different spins in the presence of spin-exchange coupling. This feature is present in the two-beam cross-correlation spectra in the delocalized and MBL phases but it is absent in the AL phase.

$P^{xy}_{cr}(\omega)$ depends on the separation between the two probe beams. With an increasing separation between the beams, $P^{xy}_{cr}(\omega)$ exhibits more oscillations in the delocalized and MBL phases, and its magnitude falls rapidly in the MBL and AL phases as might be expected in the presence of spatial localization. Also the shape of $P^{xy}_{cr}(\omega)$ in the three phases remains unchanged when we explicitly couple the disordered XXZ chain to a weakly thermalizing environment. Both of these features are discussed in the Suppl. Mat. [33].

In order to understand the cross-correlation noise power spectra obtained from numerics, we develop a simple phenomenological model inspired by the experimental set-up of the two-beam SNS. Let us denote the respective total spin polarizations of the spins illuminated by the probe laser beams $I$ and $II$ by $S^I$ and $S^{II}$. We assume that the spins in beam $I$ ($II$) can relax (a) due to spin-exchange interactions with the spins in beam $II$ ($I$),

FIG. 3. Single-beam and two-beam spin noise power spectra in the three different phases. (a-c) $P^x_{cr}(\omega)$, (d-f) $P^{xy}_{cr}(\omega)$ are obtained by exact diagonalization and (g-i) $P^z_{cr}(\omega)$ are calculated from Eq. 10. In the delocalized ($J_z = 1$, $\eta = 1$), the MBL ($J_z = 1$, $\eta = 5$) and the Anderson localized ($J_z = 0$, $\eta = 5$) phase. We use $\gamma = 0.068$, $\gamma_1 = 0.18$ in (g), $\gamma = 0.03$, $\gamma_1 = 0.08$ in (h) and $\gamma = 0$ in (i). The parameters in (a-f) are the same as in Fig. 2.
and (b) due to interactions with other spins in the sample. The spin-exchange interactions between the spins from the two illuminated spots conserve total spin but do not transfer spins between the two spots via the spin co-flips with a rate $\gamma$. This leads to cross-correlations of their fluctuations. For process (b) we define a net spin relaxation rate $\gamma_\alpha$ for spins in beams $\alpha = I, II$. This process does not conserve the total spin in the beams. Combining these with Larmor precession in the random magnetic field, we obtain the following stochastic evolution equations [10].

$$\frac{dS_\alpha}{dt} = S_\alpha \times h_\alpha - \gamma_\alpha S_\alpha - \gamma(S_\alpha - S_\beta) + \xi_\alpha,$$

where $\bar{I} = II, \bar{T} = I$. Here we have included stochastic fluctuations $\xi_\alpha$. For simplicity, we consider these as Gaussian white noise with zero mean,

$$\langle \xi_\alpha(t)\xi_{\beta}(t') \rangle = \delta(t-t')\frac{\delta_{\beta\beta}}{2}\left(\delta_{\alpha\beta}\gamma_\alpha + \gamma(\delta_{\alpha\beta} - \delta_{\beta\beta})\right).$$

with $i, j = x, y, z$. The relation in Eq. 9 ensures stationary equal-time correlation $\langle S_\alpha(t)S_\beta(t) \rangle \propto \delta_{\alpha\beta}/4$ in the high temperature limit [10]. We coarse-grain the random magnetic fields within the spot of each laser beams, and represent them by a single random field. Here we choose the random field along $z$-direction following the Hamiltonian in Eq. 1 i.e., $h_\alpha = h_\alpha \hat{z}$ where $h_\alpha$ are two i.i.d. random numbers chosen from a uniform distribution over $[-\eta, \eta]$. Averaging over noise, we find the cross-correlator of spin polarizations in Eq. 7 in compact form,

$$P^*_c(\omega) = \gamma \sum_{q=\pm} \frac{\langle \chi_q \rangle}{\chi^2 + \kappa^2_q},$$

where $\chi_\pm = \gamma_1 \gamma_2 - (\omega \pm h_I)(\omega \pm h_{II})$, $\kappa_\pm = \gamma_3(2\omega \pm (h_I + h_{II})$. Here $\gamma_1 = \gamma_{II}, \gamma_2 = \gamma_1 + 2\gamma$ and $\gamma_3 = \gamma_1 + \gamma$ are characteristic rates that influence broadening of the peaks. $\langle . . \rangle$ in Eq. 10 denotes averaging over the random fields $h_\alpha$. Due to the presence of the random field throughout the sample, the phenomenological relaxation rates $\gamma, \gamma_1$ and $\gamma_2$ vary with disorder strength $\eta$, and we use them as fitting parameters. We plot the cross-correlator $P^*_c(\omega)$ after averaging over the random fields for $\eta = 1.5$ in Fig. 3(h). We find remarkably good agreement between the numerical results obtained at high temperature and the results from our phenomenological model. From Eq. 10 $P^*_c(\omega) = 0$ for $\gamma = 0$, in the AL phase ($J_z = 0$). We note, however, that our model assumes instantaneous transfer of polarization density through direct co-flips of spins, appropriate for directly adjacent spots. If this condition is not fulfilled, a delay for the propagation of polarization density over the distance between the illuminated spots will have to be added to our simplified model. Secondly, our model works only at relatively high temperatures and the results obtained from the phenomenological model would deviate from the numerical results at low temperature [33, 34].

To summarize, we have calculated spin noise signals in the diffusive, AL and MBL regimes of the disordered XXZ and transverse-field Ising spin chains, for different temperatures as well as beam separations. We have shown how these signals can be used to identify the presence of MBL. The current proposal addresses disordered spin systems beyond previously addressed regimes of infinite temperature and strictly isolated systems. For an actual experiment, it is necessary to have a disordered spin system whose interaction energies and relaxation time scales are within the bandwidth of the recent optical SNS measurements (nearly 100 GHz) [8]. In this regard, solid state systems, such as localized spin defects in solids (e.g., nitrogen-vacancy centers in diamond) [21, 22, 23], are suitable candidates for SNS measurements to detect MBL. Apart from SNS, spatially resolved spin noise from quasi-1D or 2D layers of interacting spins can also be measured using recently demonstrated sensitive nanoscale magnetic resonance imaging using nitrogen-vacancy centers in diamond [16, 17].

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[37] See Supplemental Material for spin noise signals in disordered transverse-field Ising model, a log scale plot of Fig. 2 real-time spin noise response of longitudinal spin component, dependence of $F_{\omega}^{c}(\omega)$ on temperature and separation between the probe beams, and $F_{\omega}^{c}(\omega)$ in the presence of a weakly thermalizing environment.
Supplemental Material for Probing Many-Body Localization by Spin Noise Spectroscopy
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We show plots in log-linear and log-log scales of real-time spin noise responses of the transverse spin component in real time in three different phases of disordered XXZ spin chain. (a,d) in log-linear scale in the delocalized phase \((J_z = 1, \eta = 1)\) and (b,c,e,f) in log-log scale in the MBL \((J_z = 1, \eta = 5)\) and the Anderson localized \((J_z = 0, \eta = 5)\) phase. Here \(N = 12\) and the beam \(I\) and \(II\) illuminate respectively spins 5,6 and 7,8.

Next, we discuss how cross-correlation spin noise spectra \(P_{\text{cr}}(\omega)\) of transverse spin component of the disordered XXZ spin chain depend on temperature and separation between the probe beams in the three different phases. For this we change temperature and separation between the probe beams from those in Fig. 3 of the main text. For comparison in Fig. S3(d-f) we use the same temperature and separation between the probe beams as in the main text. We reduce temperature to \(10J_z\) in Fig. S3[a-c], and we find that the line-shapes of \(P_{\text{cr}}(\omega)\) in all three phases remain similar to that in Fig. S3(d-f) at high temperature \(50J_z\). Thus, the two-beam SNS is still a good probe to separate the three different phases. However, fluctuations in \(P_{\text{cr}}(\omega)\) in the AL phase at low temperature are increased by one order of magnitude from that at high temperature. In fact, \(P_{\text{cr}}(\omega)\) is similar in magnitude both in the MBL and AL phases but still has different shapes. In Fig. S3[g-i] we move the beam \(II\) by one lattice spacing, and we find extra oscillations in \(P_{\text{cr}}(\omega)\) compared to Fig. S3[d,e] in the delocalized and MBL phases. We also notice that the magnitude of \(P_{\text{cr}}(\omega)\) is reduced by one order of magnitude in the MBL phase and by two orders in the AL phase compared to Fig. S3[e-f]. We use periodic boundary conditions and averaging over 3000 disorder realizations in Figs. S1,S2,S3 except in Fig. S3(h) where 10000 disorder realizations are used.

In Fig. S2 we show single-beam and two-beam spin noise responses of the longitudinal spin component \((z\text{-direction})\) of disordered XXZ spin chain in real time in the three different phases. Both spin noise responses show different relaxation behaviour in the delocalized and the localized phases. However, the relaxation of the longitudinal spin component is quite similar in the MBL and the AL phases within both single-beam and two-beam cross-correlation measurements.

FIG. S1. Log scale plots of single-beam and two-beam spin noise responses of transverse spin component in real time in three different phases of disordered XXZ spin chain. (a,d) in log-linear scale in the delocalized phase \((J_z = 1, \eta = 1)\) and (b,c,e,f) in log-log scale in the MBL \((J_z = 1, \eta = 5)\) and the Anderson localized \((J_z = 0, \eta = 5)\) phase. Here \(N = 12\) and the beam \(I\) and \(II\) illuminate respectively spins 5,6 and 7,8.

FIG. S2. Single-beam and two-beam spin noise responses of longitudinal spin component in real time in the delocalized, MBL and Anderson localized phases of disordered XXZ spin chain. The parameters are the same as in Fig. S1.
We check the nature of cross-correlation spin noise power spectra of disordered XXZ spin chain by explicitly coupling the chain (system) to a thermalizing bath. We model the bath here by a weakly disordered XXZ spin chain in the diffusive phase (nonintegrable). The Hamiltonians of the bath and system-bath coupling are respectively,

\[ H_{\text{bath}} = \sum_{i=1}^{N} J_i \hat{\sigma}_i \hat{\sigma}_{i+1} + h_i^B \hat{\sigma}_i^z, \]  
\[ H_c = \sum_{i=1}^{N} J_c \hat{S}_i^z \hat{\sigma}_i, \]  

where the random fields \( h_i^B \) are chosen from a uniform distribution within the window \([-\eta_b, \eta_b]\), and \( \hat{\sigma}_{N+1} = \hat{\sigma}_1 \) for periodic boundary conditions. Here \( J_c \) determines strength of the system-bath coupling, and the bath should be effective in thermalizing the system when the coupling matrix element is of the order of the many-body level spacing in the bath. The full Hamiltonian of the system plus bath is \( H = H_c + H_{\text{bath}} + H_b \), where \( H_b \) is given in Eq. 1 of the main text. We calculate spin noise signals in an eigenstate \( |n\rangle \) of \( H_f \) at an energy \( E_n \) corresponding to that of the thermal ensemble at inverse temperature \( \beta \), \( E_n = \langle n | H_1 | n \rangle = \text{Tr}[\{ e^{-\beta H} H \}] / \text{Tr}[e^{-\beta H}] \). In Fig. S4 we show two-beam cross-correlation spin noise spectra of the transverse spin component in the three different phases for two different values of \( J_c \). Noise spectra are shown after averaging over 30000 (except 10000 in Fig. S4(a,d)) disorder realizations. We find that the shapes of two-beam spin noise spectra in Fig. S4 are similar to those obtained earlier without explicitly coupling the system to a thermalizing bath. This shows the robustness of two-beam spin noise signals to distinguish the three different phases. We note from Fig. S3 that the fluctuation in the noise spectra falls with increasing bath coupling as long as the coupling is smaller than the characteristic energy scales in the system. It signals better thermalization with increasing bath coupling without destroying the signatures of the three different phases in the isolated disordered XXZ chain. We mention that the large fluctuations in the noise spectra shown in Fig. S4 are also due to relatively shorter chain length which we were able to simulate for this case.

Finally, we present cross-correlation spin noise signals measured by two-beam SNS for a disordered transverse-field Ising model with next-nearest neighbor coupling. The Hamiltonian of this model is

\[ H_1 = -\sum_{i=1}^{N-1} J_i S_i^z S_{i+1}^z + J_2 \sum_{i=1}^{N-2} S_i^z S_{i+2}^z + h \sum_{i=1}^{N} S_i^z, \]  

where the nearest neighbor couplings \( J_i = J + \delta J_i \), with all random \( \delta J_i \) chosen from a uniform distribution within the window \([-\eta, \eta]\); \( h \) is a constant magnetic field. For finite \( J_2 \), the above model has a delocalized phase at low disorder and an MBL phase at high disorder [1]. The system is in the AL phase when \( J_2 = 0 \). We numerically calculate cross-correlation spin noise signals of the spin component along \( y \)-direction. We quote \( J_2, h, \eta \) in the units of \( J \), and fix \( J = 1 \). The cross-correlation spin noise signals in real time in the delocalized, MBL and AL phases are shown in Fig. S5(a-c) at high temperature. A large cross-correlation between the transverse spin components of separate spins is developed in the delocalized phase.
and it relaxes relatively fast, while the cross-correlation is much weaker in the MBL phase and it relaxes very slowly. The transverse spin component hardly shows any cross-correlation between different spins at high temperature in the AL phase. We plot cross-correlation spin noise power spectra in Fig. S5 (d-f) which are different in shape and magnitude in the three phases. Especially, the cross-correlation power spectra in the MBL and AL phases are very different in shape. We use periodic boundary conditions, averaging over 3000 (in Fig. S5 (a-c)), 6000 (in Fig. S5 (d)) and 20000 (in Fig. S5 (e,f)) disorder realizations.

FIG. S5. Two-beam spin noise responses of transverse spin component in the three different phases of disordered transverse-field Ising chain. (a-c) $C_{y}^{cr}(t)$ and (d-f) $P_{y}^{cr}(\omega)$ are obtained numerically in the delocalized ($J_{z} = 0.3, \eta = 1$), the MBL ($J_{z} = 0.3, \eta = 5$) and the Anderson localized ($J_{z} = 0, \eta = 5$) phase. Here $N = 12$, $J = 1$, $h = 0.6$, temperature is $50 J$ and beams I and II illuminate respectively spins 5,6 and 7,8.

[1] J. A. Kjäll, J. H. Bardarson, and F. Pollmann, Phys. Rev. Lett. 113, 107204 (2014).