TIME-PERIODIC EVANS APPROACH TO WEAK KAM THEORY

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Abstract. We study the time-periodic version of Evans approach to weak KAM theory. Evans minimization problem is equivalent to a first order mean field game system. For the mechanical Hamiltonian we prove the existence of smooth solutions. We introduce the corresponding effective Lagrangian and Hamiltonian and prove that they are smooth. We also consider the limiting behavior of the effective Lagrangian and Hamiltonian, Mather measures and minimizers.

1. Introduction

We consider the extension, to time-periodic Hamiltonians $H: \mathbb{T}^{d+1} \times \mathbb{R}^d \to \mathbb{R}$, of Evans approach to weak KAM theory. For $k \in \mathbb{N}$, we address the problem of minimizing

$$I_k[u] := \int_{\mathbb{T}^{d+1}} e^{k(u_t + H(x,t,\nabla u))}$$

among functions $u: \mathbb{T}^{d+1} \to \mathbb{R}$ with $\int u = 0$. We assume that $H: \mathbb{T}^{d+1} \times \mathbb{R}^d \to \mathbb{R}$ is a smooth function strictly convex and superlinear i.e. we assume that $H_{pp}$ is positive definite and

$$\frac{H(z,p)}{|p|} \to +\infty, \text{ as } |p| \to +\infty.$$

This variational problem is equivalent to the first order mean field game system

$$\begin{cases}
    u_t + H(x,t,\nabla u) = \frac{1}{k} \ln m + \bar{H}_k \\
    m_t + \text{div}(H_p m) = 0 \\
    \int u = 0, \int m = 1
\end{cases}$$

(MFG)

The minimization problem is also related through duality to an entropy penalized extension of Mather problem (section 3). P. Cardaliaguet has also studied first order mean field games using variational principles in duality [C], [CG] (with J. Graber), as well as other approaches [C1]. Although the mean field game we are considering is time dependent, the fact that we are searching for time-periodic solutions $(u,m)$ together with the constant $\bar{H}_k$, makes more appropriate to consider it as an ergodic problem. The main difference with the ergodic problems that have been studied is that the new Hamiltonian $r + H(z,p)$ is neither coercive nor strictly convex in the variable $(p,r)$.

According to [CIS] there exists a unique $\bar{H} \in \mathbb{R}$ such that the Hamilton Jacobi equation

$$u_t + H(z,\nabla u) = \bar{H}, \quad z = (x,t)$$

(2)
has a Lipschitz viscosity solution \( \phi : \mathbb{T}^{d+1} \rightarrow \mathbb{R} \). Moreover \( \bar{H} \) is given by the min-max formula

\[
\bar{H} = \inf_{u \in C^1(\mathbb{T}^{d+1})} \max_{z \in \mathbb{T}^{d+1}} u_t(z) + H(z, \nabla u(z))
\]

which motivates Evans problem of minimizing the functional \( I_k \) \[E1\]. The convexity of the exponential and Hamiltonian functions imply that \( I_k \) is lower semicontinuous on \( W^{1,q}(\mathbb{T}^{d+1}) \) but it is not coercive due to the linear term \( u_t \).

Using Jensen’s inequality and \( \int u_t = 0 \), we have

\[
e^{k \min H} \leq \exp \int_{\mathbb{T}^{d+1}} kH(z, \nabla u) \leq I_k[u],
\]

which can be written as

\[
(4) \quad u_{tt} + 2H_p \nabla u_t + \nabla^2 u(H_p, H_p) + \frac{1}{k} \text{Tr}(H_{ppp} \nabla^2 u) + H_t + H_x \cdot H_p + \frac{1}{k} \text{Tr} H_{px} = 0
\]

where all derivatives of \( H \) are evaluated at \((z, \nabla u(z))\).

Letting \( e^{k\bar{H}_k} = \inf_u I_k[u] \), we have

\[
e^{k \min H} \leq e^{k\bar{H}_k} \leq I_k[0] \leq e^{\max\{H(z,0):z \in \mathbb{T}^{d+1}\}}
\]

and thus

\[
\min H \leq \bar{H}_k \leq \max_{z \in \mathbb{T}^{d+1}} H(z,0).
\]

A minimizer must satisfy the Euler Lagrange equation

\[
(EL) \quad (e^{k(u_t+H(z,\nabla u))})_t + \text{div}(e^{k(u_t+H(z,\nabla u))}H_p(z, \nabla u)) = 0
\]

which can be written as

\[
(5) \quad \int_{\mathbb{T}^{d+1}} e^{k(u_t+H(x,t,\nabla u))} \geq \int_{\mathbb{T}^{d}} e^{\int_0^t kH(x,t,\nabla u)dt}.
\]

If \( H \) does not depend on \( t \), we define \( \bar{u}(x) = \int_0^1 u(x,t)dt \), then \( \nabla \bar{u}(x) = \int_0^1 \nabla u(x,t)dt \).

From the convexity of \( H \)

\[
(5) \quad \int_0^1 kH(x, \nabla u(x,t))dt \geq kH(x, \nabla \bar{u}(x))
\]

\[
(6) \quad \int_{\mathbb{T}^{d}} e^{\int_0^t kH(x,\nabla u)dt} \geq \int_{\mathbb{T}^{d}} e^{kH(x, \nabla \bar{u})} \geq e^{k\bar{H}_k,a}
\]

where \( e^{k\bar{H}_k,a} \) is the minimum for the autonomous problem. If \( v \) is a minimizer for that problem, \( v_t = 0 \) and

\[
\int_{\mathbb{T}^{d+1}} e^{kH(x, \nabla v)} = e^{k\bar{H}_k,a},
\]

so \( v \) is also a minimizer for the time dependent problem and \( \bar{H}_k = \bar{H}_{k,a} \).
2. Existence of classical solutions

Writing \( z = (x, t), q = (p, r), Du(z) = (\nabla u(z), u_t(z)) \), (11) can be written in the form

\[
\operatorname{Tr}(a_k(z, Du)D^2u) + b_k(z, Du) = 0,
\]

where

\[
a_k(z, q) = \left( \frac{1}{k} H_{pp} + H_p \otimes H_p \ H_p \right) = \sigma \sigma^T, \quad \sigma = \left( \frac{1}{k} H_{pp}, H_p \right)
\]

\[
b_k(z, q) = H_t + H_x \cdot H_p + \frac{1}{k} \operatorname{Tr} H_{px}.
\]

The main difficulty to establish the existence of classical solutions of (7) is that the maximal and minimal eigenvalues of \( a_k(z, q) \) go as \( H_P^2 \) and \( 1/H_P^2 \) when \( |p| \) tends to \( \infty \).

One can obtain apriori Lipschitz bounds for solutions in particular cases that include the most typical examples of Hamiltonians. Those are the cases when \( b_k \) is sublinear. More precisely,

**Assumption 1.** There exists a continuous function \( \chi : [0, \infty) \to \mathbb{R} \) such that

\[
\int_0^\infty \frac{ds}{\chi(s) + 1} = \infty, \quad |b_k(z, q)| \leq \chi(|q|), (z, q) \in \mathbb{T}^{d+1} \times \mathbb{R}^{d+1}
\]

We observe that the assumption holds when

\[
H(x, t, p) = \frac{1}{2} |p + \eta(t)|^2 + V(x, t).
\]

In fact,

\[
b_k(z, p) = (p + \eta(x, t)) \cdot \eta'(t) + \nabla V(x, t)(p + \eta(t))
\]

does not depend on \( k \), and we can take \( \chi(s) = cs + d \) for some \( c, d > 0 \).

**Lemma 1.** Under assumption 1 there exists \( K > 0 \) depending only on \( \chi \) such that for a solution \( \phi \in C^2(\mathbb{T}^{d+1}) \) of (7), we have \( \|D\phi\|_{\infty} \leq K \).

**Proof.** For a suitable increasing concave function \( \psi : [0, \infty) \to \mathbb{R} \) with \( \psi(0) = 0 \), consider \( h : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R} \)

\[
h(z, w) = \phi(z) - \phi(w) - \psi(|z - w|).
\]

Since \( \phi \) is periodic and \( \psi \) is increasing, \( h \) achieves its global maximum in the the cube \([0, 1]^{d+1} \times [0, 1]^{d+1}\). So we build \( \psi \) on \([0, 2]\) and extend it by keeping it increasing and concave. Using Assumption 1 one can choose \( \psi \) to be a solution of

\[
2\psi'' = -\chi(\psi') - 1 \text{ on } (0, 2), \quad \psi(0) = 0
\]

with \( \psi' > 0 \) on \([0, 2]\). In fact, for \( K \) sufficiently large the function

\[
g(t) = \int_t^K \frac{2du}{\chi(u) + 1}
\]

is strictly decreasing and for some \( a > 0, [0, 2] \subset g([a, K]) \). Letting \( \psi : [0, 2] \to \mathbb{R} \) be the primitive of \( g^{-1} \) with \( \psi(0) = 0 \), we have that \( \psi \) satisfies the requeriments.

Observe that \( \max h(z, w) \geq 0 \) and we want to prove that \( \max h = 0 \) because in that case

\[
\phi(z) - \phi(w) \leq \psi(|z - w|) \leq K|z - w|,
\]
the last inequality being a consequence of the concavity of \( \psi \). Assume by contradiction that \( \max h \) is positive and it is achieved at \((\bar{z}, \bar{w})\). Then
\[
(0, 0) = Dh(\bar{z}, \bar{w}) = (D\phi(\bar{z}) - \psi'(|\bar{z} - \bar{w}|))q \quad \text{and} \quad 0 \geq D^2h(\bar{z}, \bar{w})
\]
\[
= \begin{pmatrix}
    D^2\phi(\bar{z}) & 0 \\
    0 & -D^2\phi(\bar{w})
\end{pmatrix} - \psi'(|\bar{z} - \bar{w}|) \begin{pmatrix}
    B & -B \\
    -B & B
\end{pmatrix} - \psi''(|\bar{z} - \bar{w}|) \begin{pmatrix}
    q \odot q & -q \odot q \\
    -q \odot q & q \odot q
\end{pmatrix},
\]
where \( B = \frac{I - q \otimes q}{|\bar{z} - \bar{w}|} \). Thus, \( Bq = 0 \) and then
\[
q^T (D^2\phi(\bar{z}) - D^2\phi(\bar{w}))q = (q^T - q^T) \begin{pmatrix}
    D^2\phi(\bar{z}) & 0 \\
    0 & -D^2\phi(\bar{w})
\end{pmatrix} (q \odot q) \leq 4\psi''(|\bar{z} - \bar{w}|).
\]
Taking \( v = \sigma^{-1}q \) we have
\[
(\sigma v)^T D^2\phi(\bar{z}) \sigma v - (\sigma v)^T D^2\phi(\bar{w}) \sigma v \leq 4\psi''(|\bar{z} - \bar{w}|).
\]
Similarly, for any \( r \in \mathbb{R}^{d+1} \) we have
\[
(\sigma r)^T D^2\phi(\bar{z}) \sigma r - (\sigma r)^T D^2\phi(\bar{w}) \sigma r = ((\sigma r)^T \sigma r)^T \begin{pmatrix}
    D^2\phi(\bar{z}) & 0 \\
    0 & -D^2\phi(\bar{w})
\end{pmatrix} (\sigma r) \leq 0.
\]
Inequalities (9) and (10) imply that
\[
\text{Tr} \sigma^T D^2\phi(\bar{z}) \sigma - \text{Tr} \sigma^T D^2\phi(\bar{w}) \sigma \leq 4\psi''(|\bar{z} - \bar{w}|).
\]
Since \( \phi \) is a solution of (11), we have
\[
\text{Tr} \sigma^T D^2\phi(\bar{z}) \sigma + b_k(\bar{z}, \psi'(|\bar{z} - \bar{w}|))q = 0
\quad \text{and} \quad \text{Tr} \sigma^T D^2\phi(\bar{w}) \sigma + b_k(\bar{w}, \psi'(|\bar{z} - \bar{w}|))q = 0
\]
Thus
\[
4\psi''(|\bar{z} - \bar{w}|) \geq b_k(\bar{w}, \psi'(|\bar{z} - \bar{w}|))q - b_k(\bar{z}, \psi'(|\bar{z} - \bar{w}|))q \geq -2\chi(\psi'(|\bar{z} - \bar{w}|)),
\]
giving \(-2 \geq 0\).
\( \square \)

**Theorem 1.** For the Hamiltonian (8) equation (EL) has a smooth solution.

**Proof.** We use the continuation method. Consider the family of Hamiltonians
\[
(11) \quad H_\lambda(x, t, p) = \frac{1}{2}|p|^2 + \lambda p + \lambda \eta(t) + \lambda V(x, t)
\]
and the PDE
\[
(\text{EL}_\lambda) \quad (e^{k(u_t + H_\lambda(x, t, \nabla u))} )_t + \text{div}(e^{k(u_t + H_\lambda(x, t, \nabla u))} D_p H_\lambda(x, t, \nabla u)) = 0
\]
for \( u : \mathbb{T}^{d+1} \to \mathbb{R} \) with \( \int_{\mathbb{T}^{d+1}} u = 0 \).

Define \( \Lambda := \{ \lambda \in [0, 1] : (\text{EL}_\lambda) \text{ has a smooth solution} \} \).

It is clear that \( 0 \in \Lambda \), with \( u \equiv 0 \).

We claim that \( \Lambda \) is closed. In fact, it is clear that Assumption (1) holds for \( H_\lambda \) with the same \( \chi(s) = cs + d \) for all \( \lambda \in [0, 1] \). Therefore Lemma (1) implies that there is \( K > 0 \) such that for any \( \lambda \in \Lambda \) the corresponding solution satisfies \( |Du_\lambda|_\infty \leq K \).

Elliptic regularity theory implies that we can bound uniformly in \( \lambda \in \Lambda \) derivatives...
of \( u_\lambda \) of any order. Thus, any convergent sequence in \( \Lambda \) has a subsequence whose corresponding sequence of solutions converge uniformly, along with all derivatives.

We claim that \( \Lambda \) is open. Indeed, for \( \lambda \in \Lambda \) the linearization of (EL\(_\lambda\)) about the solution \( u \) is given by

\[
L_v := -(e^{k(u_t + H_\lambda(z, \nabla u))}(v_t + D_pH_\lambda(z, \nabla u) \cdot \nabla v)) + \frac{1}{k} \nabla v D_{pp}H_\lambda(z, \nabla u)
\]

so that

\[
\int_{T^{d+1}} vLv = \int_{T^{d+1}} e^{k(u_t + H_\lambda(z, \nabla u))} DvD_T a_k(z, Du) Dv
\]

and then \( L \) is a symmetric, uniformly elliptic operator, whose null space consists of the constants. The Implicit Function Theorem yields a unique solution for any value in a neighborhood of \( \lambda \).

Since \( \Lambda \) is nonempty, closed and open it coincides with \([0, 1]\). Thus equation (EL) has a smooth solution. \( \square \)

3. Entropy penalized Mather theory

Given a Borel probability \( \mu \in \mathcal{P}(T^{d+1} \times \mathbb{R}^d) \) we consider its push forward \( m_\mu \in \mathcal{P}(T^{d+1}) \) given by

\[
\int_{T^{d+1}} \varphi(z)dm_\mu(z) = \int_{T^{d+1} \times \mathbb{R}^d} \varphi(z)d\mu(z, v).
\]

In the set \( A \subset \mathcal{P}(T^{d+1}) \) of measures absolutely continuous with respect to Lebesgue measure, the mapping

\[
m \mapsto S^*[m] := \int_{T^{d+1}} \log m(z) dm(z)
\]

is convex and lower semicontinuous. This mapping can be extended in a unique way as a convex lower semicontinuous functional to \( \mathcal{P}(T^{d+1}) \) by

\[
\bar{S}[m] = \liminf_{m_n \in A, m_n \rightharpoonup m} S^*[m_n].
\]

Note that the map \( \bar{S} \) is allowed to take the value \(+\infty\). Furthermore, since \( y \log y \geq -1/e \) we have \( \bar{S} \geq -1/e \). Finally define \( S[\mu] = \bar{S}[m_\mu] \).

Let \( L : T^{d+1} \times \mathbb{R}^d \to \mathbb{R} \) be a \( C^2 \) function, strictly convex and superlinear in \( v \)

\[
\frac{L(z, v)}{|v|} \to +\infty, \quad \text{as } |v| \to +\infty.
\]

We consider the convex lower semicontinuous functional

\[
A_{L,k}(\mu) = \int_{T^{d} \times \mathbb{R}^d} L(z, v)d\mu + \frac{1}{k} \bar{S}[\mu],
\]

defined on the space \( \mathcal{C} \) of measures \( \mu \in \mathcal{P}(T^{d+1} \times \mathbb{R}^d) \) such that

(a) \( \int_{T^{d+1} \times \mathbb{R}^d} |v|d\mu(z, v) < +\infty \)

(b) For all \( \varphi \in C^1(T^{d+1}) \), \( \int_{T^{d+1} \times \mathbb{R}^d} (\varphi_t + \nabla \varphi \cdot v)d\mu(x, t, v) = 0. \)
From Mather theory we know that for any $Q \in \mathbb{R}^d$ there is $\mu \in C$ such that $Q = \int_{\mathbb{T}^{d+1} \times \mathbb{R}^d} v \, d\mu(z,v)$. We define the effective Lagrangian $\bar{L}_k : \mathbb{R}^d \to \mathbb{R}$ by

$$
\bar{L}_k(Q) := \inf \{ A_{L,k}(\mu) : \mu \in C, Q = \int_{\mathbb{T}^{d+1} \times \mathbb{R}^d} v \, d\mu(z,v) \}.
$$

Let $H : \mathbb{T}^{d+1} \times \mathbb{R}^d \to \mathbb{R}$ be Legendre transform of $L$.

**Proposition 1.**

$$
\inf_{\mu \in C} A_{L,k}(\mu) \geq -\frac{1}{k} \inf_{\varphi \in C^1} \log I_k[\varphi].
$$

**Proof.** Let $\mu \in C$ and $\varphi \in C^1(\mathbb{T}^{d+1})$. Let $m_\varphi$ be given by (13). We have

$$
\int L(z,v) \, d\mu = \int (L(z,v) - \varphi_t - \nabla \varphi : v) \, d\mu \geq - \int (\varphi_t + H(z, \nabla \varphi)) \, dm_\varphi,
$$

since, for all $(z,v) \in \mathbb{T}^{d+1} \times \mathbb{R}^d$, $L(z,v) - \nabla \varphi(z)v \geq -H(z, \nabla \varphi(z))$, with equality only when $v = H_p(z, \nabla \varphi(z))$. Define $h_\varphi : \mathcal{P}(\mathbb{T}^{d+1}) \to \mathbb{R}$ by

$$
h_\varphi(m) = \int_{\mathbb{T}^{d+1}} (\varphi_t + H(z, \nabla \varphi)) \, dm - \frac{1}{k} S[m]
$$

Then

$$
A_{L,k}(\mu) \geq -h_\varphi(m_\varphi).
$$

Letting $m_\varphi(z) = e^{k(\varphi_t + H(z, \nabla \varphi))} / I_k[\varphi]$, we have $h_\varphi(m_\varphi) = \frac{1}{k} \log I_k[\varphi]$.

The convex function $t \mapsto t \log t$ has Legendre transform $s \mapsto e^{s-1}$. In particular this implies that $t \log t + 1 \geq t$, and so, for any $m \in A$ we obtain

$$
h_\varphi(m) \leq h(m_\varphi, \varphi) + \int (\varphi_t + H(z, \nabla \varphi) - \frac{1}{k} \log m_\varphi - \frac{1}{k}) (m(x) - m_\varphi(x)) \, dx.
$$

The convexity and an approximation argument show that in fact the previous inequality holds for any $m \in \mathcal{P}(\mathbb{T}^{d+1})$. From the definition of $m_\varphi$, and since $m$ and $m_\varphi$ are probability measures, the second term on the rhs vanishes and then

$$
\frac{1}{k} \log I_k[\varphi] = \max_m h_\varphi(m).
$$

Therefore,

$$
A_{L,k}(\mu) \geq -\frac{1}{k} \log I_k[\varphi],
$$

and (16) follows. \hfill \Box

**Proposition 2.** If (EL) has a smooth solution $u_k$, $m_k = m_{u_k}$ then

$$
\mu_k(z,v) = \delta(v - H_p(z, \nabla u_k(z)))m_k(z),
$$

is a minimizer of (13) and $u_k$ is a minimizer of (1).

**Proof.** Definition (17) means that for all continuous function $F : \mathbb{T}^{d+1} \times \mathbb{R}^d \to \mathbb{R}$ we have

$$
\int_{\mathbb{T}^{d+1} \times \mathbb{R}^d} F(z,v) \, d\mu_k = \int_{\mathbb{T}^{d+1}} F(z, H_p(z, \nabla u)) \, dm_k
$$

\hfill \Box
and since \((u_k, m_k)\) is a solution of (MFG), we get that \(\mu_k \in C\). Moreover,

\[
A_{L,k}(\mu_k) = \int_{\mathbb{T}^{d+1}} L(z, H_p(z, \nabla u_k)) + \frac{1}{\epsilon} \log m_k dm_k = -h(m_k, u_k) = -\frac{1}{\epsilon} \log I_k[u_k],
\]

and then

\[
A_{L,k}(\mu_k) = \inf_{\mu \in C} A_{L,k}(\mu) = -\frac{1}{\epsilon} \inf_{\varphi \in C^1} \log I_k[\varphi] = -\frac{1}{\epsilon} \log I_k[u_k].
\]

(18) \[A_{L,k}(\mu_k) = \inf_{\mu \in C} A_{L,k}(\mu) = -\frac{1}{\epsilon} \inf_{\varphi \in C^1} \log I_k[\varphi] = -\frac{1}{\epsilon} \log I_k[u_k]. \]

\[ \square \]

**Lemma 2.** The effective Lagrangian \(\bar{L}_k\) is convex.

**Proof.** Given \(\varepsilon > 0\), let \(\nu_1, \nu_2\) be such that \(A_{L,k}(\nu_i) < \bar{L}_k(Q_i) + \varepsilon\) and \(\int v d\nu_i = Q_i\). Note that, for \(0 \leq \lambda \leq 1\),

\[
\int v d(\lambda \nu_1 + (1 - \lambda)\nu_2) = \lambda \int v d\nu_1 + (1 - \lambda) \int v d\nu_2 = \lambda Q_1 + (1 - \lambda)Q_2,
\]

\[
m_{\lambda \nu_1 + (1 - \lambda)\nu_2} = \lambda m_{\nu_1} + (1 - \lambda)m_{\nu_2}.
\]

Since \(\bar{S}\) is convex, we have that

\[
\bar{L}_k(\lambda \nu_1 + (1 - \lambda)\nu_2) \leq A_{L,k}(\lambda \nu_1 + (1 - \lambda)\nu_2) \leq \lambda A_{L,k}(\nu_1) + (1 - \lambda)A_{L,k}(\nu_2)
\]

\[
= \lambda \bar{L}_k(Q_1) + (1 - \lambda)\bar{L}_k(Q_2) + \varepsilon.
\]

\[ \square \]

**Corollary 1.** The Legendre transform of \(\bar{L}_k\) is \(\bar{H}_k\).

**Proof.** Since the Legendre transform of \(L(z, v) - P v\) is \(H(z, P + p)\) and (10) is an equality, we have that

\[
\sup_Q PQ - \bar{L}_\varepsilon(Q) = -\inf_{\mu \in C} A_{L_\varepsilon - (P, \cdot), k}(\mu) = \frac{1}{\epsilon} \log \inf_{\varphi} \int_{\mathbb{T}^{d+1}} e^{k(\varphi + H(\cdot, P + \nabla \varphi))} dx.
\]

\[ \square \]

**Lemma 3.** For Hamiltonian \(\mathbb{H}\), \(\bar{H}_k(P)\) is strictly convex. Furthermore for each \(P\), (19) admits at most one minimizer, up to the addition of constants.

**Proof.** For \(P \in \mathbb{R}^d\) the Hamiltonian \(H(x, t, P + P) = \frac{1}{2} |P + \eta(x, t)|^2 + V(x, t)\) is of the same type. Suppose there are \(P_0, P_1 \in \mathbb{R}^d\) and \(0 < \lambda < 1\) such that

\[
\bar{H}_k(\lambda P_0 + (1 - \lambda)P_1) = \lambda \bar{H}_k(P_0) + (1 - \lambda) \bar{H}_k(P_1).
\]

Let \(f_i \in C^2(\mathbb{T}^{d+1})\) be a solution of (EL) with \(u = P_i x + f_i, i = 0, 1\) so that

\[
\bar{H}_k(P_i) = \frac{1}{\epsilon} \log \int_{\mathbb{T}^{d+1}} e^{k(f_i + H(\cdot, P_i + \nabla f_i))} dx.
\]

For \(\varphi = \lambda f_0 + (1 - \lambda)f_1\) we have

\[
\varphi_t = \lambda f_{1t} + (1 - \lambda)f_{2t},
\]

\[
\nabla \varphi = \lambda P_0 + (1 - \lambda)P_1 = \lambda(\nabla f_0 + P_0) + (1 - \lambda)(\nabla f_1 + P_1),
\]

and, by convexity of \(H\),

\[
H(z, \lambda P_0 + (1 - \lambda)P_1 + \nabla \varphi) \leq \lambda H(z, \nabla f_0 + P_0) + (1 - \lambda)(H(z, \nabla f_1 + P_1)).
\]

(20) \[H(z, \lambda P_0 + (1 - \lambda)P_1 + \nabla \varphi) \leq \lambda H(z, \nabla f_0 + P_0) + (1 - \lambda)(H(z, \nabla f_1 + P_1)). \]
Convexity of the exponential function and Hölder inequality yield
\[
e^{k\tilde{H}_k(\lambda P_0+(1-\lambda)P_1)} \leq \int_{\mathbb{T}^{d+1}} e^{k(\varphi_z+H(z,\lambda P_0+(1-\lambda)P_1+\nabla \varphi))}
\]
\[
\leq \int_{\mathbb{T}^{d+1}} e^{k(f_0z+H(z,\nabla f_0+P_1))+(1-\lambda)(f_1z+H(z,\nabla f_1+P_1))}
\]
\[
\leq e^{\lambda k\tilde{H}_k(P_0)(1-\lambda)k\tilde{H}_k(P_1)} = e^{k\tilde{H}_k(\lambda P_0+(1-\lambda)P_1)}.
\]
(21)

Therefore all inequalities in (21) are equalities and so is (20). Since \( H \) is strictly convex \( \nabla f \) such that \( \nabla \) is given by (8) so that \( \tilde{H}_k \) is a Mather measure.

**Theorem 2.** For the Hamiltonian given by (8) the effective functions \( \tilde{L}_k, \tilde{H}_k \) are smooth.

**Proof.** For \( P \in \mathbb{R}^d \) consider equation (EL) with \( u = P \times \phi \) and define \( F(P, \phi) \) as the l.h.s. of that equation. We have seen that for a solution \( \phi = \phi(\cdot, P) \) of \( F(P, \phi) = 0 \), \( \mathcal{L} = D_2 F(P, \phi) \) is given by (12). The Implicit Function Theorem implies that \( \phi(\cdot, P) \) is smooth in \( P \) and so \( \tilde{H}_k \) are smooth. Moreover \( \tilde{H}_k \) is strictly convex so \( D\tilde{H}_k \) has a smooth inverse \( G_k \) and therefore \( \tilde{L}_k(Q) = QG_k(Q) - \tilde{H}_k(G_k(Q)) \) is smooth.

\[\square\]

4. **Approximating weak KAM theory**

In this section we assume the Hamiltonian is given by (8) so that \( b_k \) satisfies assumption (1) with \( \lambda \) independent of \( k \). Let \( u_k \) be the minimizer of (1) with \( \int u_k = 0 \). From Lemma [4] there is \( K \) such that \( \|Du_k\|_\infty \leq K \) for any \( k \), so passing to a subsequence, \( u_k \) converges uniformly to a Lipschitz \( u : \mathbb{T}^{d+1} \to \mathbb{R} \) and \( Du_k \to Du \) weakly in \( L^q(\mathbb{T}^{d+1}, \mathbb{R}^{d+1}) \), for any \( 1 \leq q < \infty \).

As in the autonomous case we have the following Theorem which has a similar proof

**Theorem 3.**

(i) We have \( \lim_{k \to \infty} \tilde{H}_k = \tilde{H} \).

(ii) Function \( u \) is a viscosity solution of

\[u_t + 2H_p \nabla u + \nabla^2 u(H_p, H_p) + H_t + H_x \cdot H_p = 0\]

(22)

(iii) Moreover, \( u_t + H(z, \nabla u) \leq \tilde{H} \) Lebesgue a.e. in \( \mathbb{T}^{d+1} \).

**Proposition 3.** Let \( \mu_k \) be the measure defined by (17). Passing to a subsequence such that \( \mu_k \rightharpoonup \mu \), we have

(a) \( \mu \) is a Mather measure

(b) \( \lim_{k \to \infty} \frac{1}{k} S[\mu_k] = 0 \)
Proof. The function \( \frac{1}{k} \log m_k = u_k t + H(z, \nabla u_k) \) is uniformly bounded. For \( \lambda > 0 \),
\[
\int_{T_{t+1}^k} \frac{1}{k} \log m_k dm_k = \int_{\{ \log m_k \geq -k\lambda \}} \frac{1}{k} \log m_k dm_k + \int_{\{ \log m_k < -k\lambda \}} \frac{1}{k} \log m_k dm_k \\
\geq -\lambda \int_{\{ \log m_k \geq -k\lambda \}} dm_k - \int_{\{ \log m_k < -k\lambda \}} C e^{-k\lambda} dz \\
\geq -\lambda + C e^{-k\lambda}.
\]
Thus,
\[
\liminf_{k \to \infty} \frac{1}{k} S[\mu_k] \geq -\lambda
\]
Since \( \lambda > 0 \) is arbitrary

(23) \[
\liminf_{k \to \infty} \frac{1}{k} S[\mu_k] \geq 0.
\]
We recall that
\[-\bar{H} = \min_{\mu \in \mathcal{C}} \int L d\mu.
\]
From (23)
\[
\int L d\mu = \lim_{k \to \infty} \int L d\mu_k = \lim_{k \to \infty} A_{L,k}(\mu_k) - \frac{1}{k} S[\mu_k] \\
= -\lim_{k \to \infty} \bar{H}_k - \frac{1}{k} S[\mu_k] \leq -\bar{H}.
\]
Therefore, \( \mu \) is a minimizing measure, the inequality is an equality and (b) holds.

\[\Box\]

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