Dynamical tides excited in rotating stars of different masses and ages and the formation of close in orbits

S. V. Chernov, J. C. B. Papaloizou and P. B. Ivanov

1 Astro Space Centre, P.N. Lebedev Physical Institute, 84/32 Profsoyuznaya Street, Moscow 117997, Russia
2 DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

ABSTRACT
We study the tidal response of rotating solar-mass stars, as well as more massive rotating stars, of different ages in the context of tidal captures leading to either giant exoplanets on close in orbits, or the formation of binary systems in star clusters. To do this, we adopt approaches based on normal mode and associated overlap integral evaluation, developed in a companion paper by Ivanov et al., and direct numerical simulation, to evaluate energy and angular momentum exchanges between the orbit and normal modes. The two approaches are found to be in essential agreement apart from when encounters occur near to pseudo-synchronization, where the stellar angular velocity and the orbital angular velocity at periastron are approximately matched. We find that the strength of tidal interaction being expressed in dimensionless natural units is significantly weaker for the more massive stars, as compared to the solar-mass stars, because of the lack of significant convective envelopes in the former case. On the other hand, the interaction is found to be stronger for retrograde as opposed to prograde orbits in all cases. In addition, for a given pericentre distance, tidal interactions also strengthen for more evolved stars on account of their radial expansion. In agreement with previous work based on simplified polytropic models, we find that energy transferred to their central stars could play a significant role in the early stages of the circularization of potential ‘Hot Jupiters’.

Keywords: hydrodynamics – celestial mechanics – planet–star interactions – binaries: close – stars: oscillations – stars: rotation.

1 INTRODUCTION
Tidal interaction leads to synchronization and orbital circularization of close binary stars (e.g. Zahn 1977; Hut 1981). It may also result in double star or star–planet systems that undergo close encounters in marginally unbound orbits becoming tidally captured into highly eccentric orbits that then begin to circularize (e.g. Press & Teukolsky 1977). A process of this kind is believed to account for giant exoplanets in close orbits with periods of a few days (e.g. Rasio & Ford 1996; Weidenschilling & Marzari 1996).

The determination of the tidal evolution requires the calculation of the response of the tidally perturbed body. This involves energy and angular momentum exchange between its normal modes and the orbit leading to its evolution. In a companion paper, Ivanov, Papaloizou & Chernov (2013), subsequently referred to as Paper 1, we developed general procedures for calculating tidal energy and angular momentum exchange rates, for bodies in periodic orbits, that are associated with an identifiable regular spectrum of low-frequency rotationally modified gravity modes for rotating stars with realistic structure.

These are likely to give rise to the dominant tidal response in bodies with stratification, where the tidal forcing frequencies significantly exceed the inverse of the convective time-scale associated with any convection zone, so that any effective turbulent viscosity is inefficient. This is also expected to be the case for rotating stars, when the dominant tidal forcing frequencies as viewed in the rotating frame exceed twice the rotation frequency with the consequence that inertial modes are not efficiently excited in convective regions.

In Paper 1, we also gave expressions from which the energy and angular momentum transferred to stellar modes of oscillation as a result of parabolic encounters can be calculated. A process that could lead to tidal captures and also governs the initial phase of orbital circularization when the orbit is very eccentric (e.g. Ivanov & Papaloizou 2004). Evaluation of the response arising from normal modes requires calculation of mode eigenfrequencies and corresponding overlap integrals that determine the strength of mode coupling with the tidal potential (e.g. Press & Teukolsky 1977). This procedure was discussed in some detail in Paper 1 for the case when the traditional approximation, appropriate for low-frequency modes in stratified layers, was adopted. We remark that, as discussed in...
more detail in Paper 1, tidal phenomena, such as energy and angular momentum exchange through parabolic encounters, or orbital evolution in the regime of so-called moderately strong viscosity (e.g. Zahn 1977; Goodman & Dickson 1998), where propagating rotationally modified gravity waves attain short wavelengths, and so are dissipated before reaching boundaries from which they can be reflected, are such that results are independent of the precise specification of the dissipation process. In this regime, the wave dissipation should also occur on a time-scale that is significantly longer than the locally excited wave period which will also be characteristic of the time for excitation due to tidal perturbation.

In this paper, we apply the formalism developed in Paper 1, where only Sun-like stars were considered, to calculate the normal modes and their associated overlap integrals for a range of tidal forcing frequencies for two models of a rotating solar-mass star with different ages, as well as several models of more massive rotating stars, with different ages. The dependence of these quantities on the existence and extent of convective regions and the transition between convective and radiative regions is elucidated. We also compare results obtained from the normal mode approach of Paper 1 to those obtained from direct numerical simulations of parabolic encounters (e.g. Papaloizou & Ivanov 2010) delineating when there is good agreement between the two approaches. Our results are then applied to the tidal capture and initial orbital circularization of giant exoplanets for both prograde and retrograde orbits and also the tidal capture of stars to form binary systems in stellar clusters (e.g. Fabian, Pringle & Rees 1975; Press & Teukolsky 1977).

The plan of the paper is as follows. In Section 2, we describe the stellar models for which we calculated the quantities that enable their exchange of energy and angular momentum under tidal gravitational perturbation due to a companion to be calculated. These quantities are the overlap integrals and the low-frequency rotationally modified g-mode spectrum and they are discussed in detail in Paper 1. We consider models in the range of 1–5 M⊙ with a variety of ages. As indicated in Paper 1, the extent of any convective envelope and/or core plays a significant role in determining the strength of tidal interaction as also does the detailed form of the transition between convective and radiative regions.

In Section 3, we discuss the properties of the numerically calculated mode spectra and overlap integrals for the stellar models considered. We also derive the rotational splitting coefficients which give the first-order shifts of mode eigenfrequencies as a result of stellar rotation. In the non-rotating case, the overlap integrals were found to be markedly larger for Sun-like stars as compared to either a polytrope with index 3 or more massive models with much less extensive convective regions. This is because of the convective envelope and is expected from the theory developed in Paper 1.

The overlap integrals are also calculated for rotating models under the neglect of centrifugal distortion and the adoption of the traditional approximation as indicated in Paper 1. Results for angular velocities of rotation in units of the critical rotation rate in the range 0.1–0.4 are presented.

We go on to apply our results to evaluate the energy and angular momentum exchanged as a result of a parabolic encounter with a companion. These enable the possibility of tidal capture from unbound orbits to be assessed. In addition, the time-scale for the initial stages of orbital circularization to occur for low planetary mass companions is estimated.

We compare energy and angular momentum transfers obtained through the normal mode/overlap integral approach to results obtained from solving the encounter problem as an initial value problem numerically (Papaloizou & Ivanov 2010; Ivanov & Papaloizou 2011) for the full range of rotation rates and for pericentre distances that are not too large to make calculation intractable. Both prograde and retrograde encounters are considered. We found that the methods are in good agreement apart from the situation where the system is close to pseudo-synchronization. In this case, the effective tidal forcing frequencies are comparable to the rotation frequency and inertial modes, not taken into account in the normal mode approach can play a significant role. As the characteristic tidal forcing frequencies are expected to be significantly larger than stellar rotation frequencies, inertial modes are unlikely to be excited in the star during the initial stages of the formation of close in giant planet orbits of ‘Hot Jupiters’. Accordingly, we do not pursue the issue of inertial modes further in this paper.

Finally, in Section 5, we summarize and discuss our results. We remark that as in our earlier work (Ivanov & Papaloizou 2011), which considered polytropic models with index, 3, we find that tidal interaction with the central star is significantly stronger for retrograde orbits and that it could play a significant role in the circularization process for giant planets, so potentially reducing the amount of potentially destructive energy dissipation in the planetary interior.

2 FORMULATION OF THE PROBLEM AND DETAILS OF NUMERICAL METHODS

2.1 Coordinate system and notation

The basic definitions and notation adopted in this paper are the same as in Paper 1. We use either a spherical coordinate system (r, φ, θ) or associated cylindrical polar coordinate system (ρr, φ, z) with origin at the centre of mass of the star. When viewed in an inertial frame, the unperturbed star rotates uniformly about the z-axis with angular velocity Ω. For our reference frame, we adopt the rotating frame in which the unperturbed star appears at rest.

2.2 Stellar models considered

Our calculations are for rotating stars of different masses and ages. As in Paper 1, centrifugal distortion is neglected with the consequence that equilibrium structures are not modified by rotation. Therefore, standard spherically symmetric models are used.

We consider models of stars of masses M∗ = 1, 1.5, 2 and 5 M⊙ of different ages. Stellar masses are expressed in solar masses (1 M⊙ = 1.9891 × 1030 g). Radii, R∗, are expressed in solar radii (1 R⊙ = 6.9551 × 1010 cm) and ages, expressed in years, are given in Table 1. Additionally, we have calculated all quantities of interest for a stellar model consisting of a polytrope with index n = 3. The mass and radius are scaled to solar values and the adiabatic index Γ = 5/3. This serves as reference model for our analysis and is referred to as model 1p.

All realistic stellar models apart from model 1b were kindly provided to us by I.W. Roxburgh. The numerical code used to obtain these models is discussed in Roxburgh (2008). Model 1b is for the present-day Sun. It is discussed in Christensen-Dalsgaard et al. (1996). Unlike models described elsewhere in a similar context (see McMillan, McDermott & Taam 1987) our models have metallicity appropriate for Population I stars. The zero-age hydrogen mass fraction X = 0.7 and the mass fraction of heavy elements Z = 0.02 for all models.

Convective heat transport is described by a standard form of mixing length theory (Kippenhahn, Weigert & Weiss 2013). Mixing in convective and semiconvective zones is dealt with by incorporating
diffusion into the equations governing the evolution of the chemical abundances (see Eggleton 1972). The diffusion coefficient is taken to be $D_{\text{conv}} = v_{\text{conv}} l / 6$, where $v_{\text{conv}}$ is the convective velocity and $l$ is the mixing length.

As is discussed in Paper 1, when low-frequency gravity modes are considered, their important properties are found to be mainly determined by the functional forms of the density, $\rho$, and the Brunt–Väisälä frequency, $N$. In particular, the locations of stably stratified radiative regions, where $N^2 > 0$, and the behaviour of $N$ in the neighbourhood of transitions from stably stratified to convective regions are particularly significant. For the models described here, the functional forms in these transition regions are strongly affected by the evolutionary history of the chemical composition profiles. Although chemical diffusion is included, the coefficient is zero in radiative regions and large in convective regions so that mixing is efficient there. This results in the possibility of very rapidly varying or even discontinuous abundance profiles according to how the boundary between a radiative and convective zone moves with evolution. Suppose this interface moves with speed $v_1$, we can form the dimensionless quantity $v_1 l / D_{\text{conv}} = 6 v_1 / v_{\text{conv}} \sim t_{\text{conv}} / t_{\text{ev}}$, the latter ratio being the ratio of the convection time-scale to the evolutionary time-scale.

On dimensionless grounds, we might expect the width of the transition region, $w_{\text{tr}}$, between a convective and radiative zone would be given by an expression of the form $w_{\text{tr}} \sim I F (t_{\text{conv}} / t_{\text{ev}})$, where the form of the function $F$ is determined by the way the diffusion coefficient vanishes as the radiative zone is entered and the evolutionary history. Simple modelling suggests that $F$ is a monotonic function of its argument, with the variation being more rapid for diffusion coefficients with more rapid cutoffs. For example, a linear variation is expected when the diffusion coefficient inside the radiative region vanishes as $(r - r_1)^2 / l^2$, with $r_1$ being a distance of one mixing length away from a retreating convection zone boundary. Note that $t_{\text{conv}} / t_{\text{ev}} \sim 10^{-9} - 10^{-10}$ is very small leading to the possibility of very thin transition layers when the evolutionary history is such that the composition in a convective zone differs significantly from its immediate surroundings. At such a transition, the mean molecular weight changes while hydrostatic equilibrium enforces continuity of the pressure. There is then a rapid change in the density with an associated spike in the density gradient and the square of the Brunt–Väisälä frequency. As this results from the density gradient, a large effect can be produced with there being only small jumps in the density or chemical profile.

Plots of the density $\rho$ and the square of the Brunt–Väisälä frequency, $N^2$, against radius, for the models used in our calculations are given in Figs 1–4. In Fig. 1, we illustrate these two quantities, against radius, for the models used in our calculations. Radii, masses and mean densities are expressed in units corresponding to the present-day Sun, stellar ages are expressed in years.

### Table 1. Radii, masses, ages and mean densities for models used in our calculations. Radii, masses and mean densities are expressed in units corresponding to the present-day Sun, stellar ages are expressed in years.

| Model | Mass | Radius | Age | Mean density |
|-------|------|--------|-----|--------------|
| 1p    | 1    | 1      |     | 1            |
| 1a    | 1    | 0.91   | $1.67 \times 10^6$ | 1.33 |
| 1b    | 1    | 1      | $4.41 \times 10^6$ | 1   |
| 1.5a  | 1.5  | 2.08   | $1.27 \times 10^7$ | 0.166 |
| 1.5b  | 1.5  | 1.46   | $5.96 \times 10^7$ | 0.482 |
| 1.5c  | 1.5  | 1.82   | $1.58 \times 10^9$ | 0.249 |
| 2a    | 2    | 2.68   | $6.81 \times 10^6$ | 0.104 |
| 2b    | 2    | 1.63   | $2.93 \times 10^7$ | 0.462 |
| 2c    | 2    | 2.25   | $5.93 \times 10^3$ | 0.175 |
| 2d    | 2    | 2.91   | $8.44 \times 10^6$ | 0.0811 |
| 5a    | 5    | 2.69   | $2.54 \times 10^6$ | 0.257 |

Figure 1. The density $\rho$ [in units of the mean density $\rho = 3 M_\odot / (4\pi R_\odot^3)$] and square of the Brunt–Väisälä frequency, $N^2$ [in units of $GM_\odot / R_\odot^3$], as functions of the radius $r$ expressed in units of $R_\odot$. The solid curves correspond to model 1b while the dashed curves are for the model 1a. The curves monotonically decreasing with $r$ are for the density distributions, while the curves having maxima at some values of $r$ are for the Brunt–Väisälä frequencies.

Figure 2. Same as in Fig. 1 but for models with $M_\odot = 1.5 M_\odot$. The solid, dashed and dotted curves are for models 1.5c, 1.5b and 1.5a, respectively. Note that there are regions very close to the surface where a weak density inversion occurs in these models. However, the values of the density where this occurs are below the minimum level plotted.
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Figure 3. Same as in Fig. 1 but for models with \( M_* = 2 M_\odot \). The solid, dashed, dotted and dot–dashed curves are for models 2a, 2c, 2b and 2a, respectively. Note that the dashed and dot–dashed curves for the density almost coincide.

Sun-like star, model 1a, has a smaller radius than model 1b, which corresponds to the present-day Sun. It accordingly has a larger mean density and is therefore less susceptible to tidal influence from a perturbing companion with a given orbital period.

Models 1.5a–1.5c, illustrated in Fig. 2, are for a star of mass \( M_* = 1.5 M_\odot \) at different ages. Their structure is more complex than that of Sun-like stars. The youngest model 1.5a (plotted with dotted curves) is similar to models 1a and 1b in that it also has convective envelope with a convective core being absent. We remark that a weak density inversion occurs very close to the surface in these models. This is a well-known effect that occurs as hydrogen is ionized in convective envelopes where the energy transport due to convection is inefficient (see e.g. Latour 1970). The ionization zone is very thin in such cases, and such that the reduction in the mean molecular weight, which occurs with weak pressure variation there, causes the density inversion. The density in these layers is very small and so they do not affect any of the tidal response calculations presented in this paper significantly. However, on account of the general similarity of the models, we expect that the overlap integrals, \( Q_x \), characterizing the strength of tidal interactions and discussed in detail in Paper 1, will have similar properties to those obtained for Sun-like stars. But, the density at the base of convection zone, expressed in terms of the mean density \( \bar{\rho} = 3 M_*/(4\pi r R_*^3) \), \( \rho_{cb} \), is smaller than the corresponding quantity for models 1a and 1b. We find \( \rho_{cb} \sim 10^{-5} \bar{\rho} \) and \( \sim 10^{-6} \bar{\rho} \) for model 1.5a, and either of models 1a or 1b, respectively.

The analytic WKBJ theory developed in Paper 1 predicts that the overlap integrals, being determined by the presence of a convective envelope, are proportional to \( \sqrt{\rho_{cb}} \) in the limit of sufficiently small eigenfrequencies (see equations 111, 113 and 115 of Paper 1). Accordingly, we expect that for a given value of the eigenfrequency, \( Q \) values for model 1.5a will be smaller than those for younger models, by a factor corresponding to the square root of the ratio of the respective values of \( \rho_{cb} \). Here, we recall that as the mode spectrum is dense, the overlap integrals are regarded as continuous functions of frequency. Let us note that energy and angular momentum exchanges due to tides associated with a normal mode are, in general, proportional to the square of the appropriate overlap integral (see Paper 1). That means that in a situation where the contribution from the convective envelope and the region of its boundary with the inner radiative zone determines the value of the overlap integral, energy and angular momentum exchanges are approximately proportional to \( \rho_{cb} \), see also equation 13 of Goodman & Dickson (1998).

Our expectation is confirmed by calculation, see Fig. 8. The evolved models 1.5b and 1.5c have both convective envelopes and convective cores. However, \( \rho_{cb} \), at the base of convective envelope is relatively small, being \( \sim 4.6 \times 10^{-5} \) for model 1.5c and \( \sim 6 \times 10^{-6} \) for model 1.5b. This has the consequence that the contribution to the overlap integrals associated with the presence of a convective envelope is strongly suppressed.

Models 1.5b and 1.5c also have convective cores. But, the transition region between the radiative region and the convective core is extremely sharp in these models (see discussion in Section 2.2 above), and must be considered as a discontinuity for eigenfunctions with characteristic wavelength larger than the typical size of the transition zone. The width of the latter, \( \Delta r \), is of the order of or smaller than the grid size with \( \Delta r / r < 2.5 \times 10^{-4}, 2.5 \times 10^{-3} \) for models 1.5b and 1.5c, respectively. We emphasize that the detailed form of this transition region should be determined from a complete treatment of convection, including overshoot, see e.g. Roxburgh (1978), Zahn (1991), and the effects of stellar rotation, etc. This cannot be undertaken at present.\(^1\) However, when eigenfunctions have typical wavelengths larger than the size of the transition zone, one cannot assume that the Brunt–Väisälä frequency increases as a power of distance from the boundary of the convective region and perform a standard WKBJ analysis that assumes that the response wavelength is significantly shorter than the transition width. Accordingly, estimates of quantities characterizing tidal interactions, such as overlap integrals or related quantities, for example the quantity \( E_z \) used by Zahn (1977), based on this assumption are not valid.

Let us estimate typical periods of modes, where such calculations are potentially formally invalid. For definiteness, we consider model 1.5b. We assume that the width of the transition zone is as small as suggested by our numerical model. From Fig. 2, we see that for

\(^1\) Note that future advances in asteroseismology may lead to some observational constraints on details of transitions between radiative and convective regions in the near future, see e.g. Silva Aguirre et al. (2011).
this model, the value of Brunt–Väisälä frequency at the maximum of the ‘spike’ close to the convective core is ≈6.8 and the radius of convective core r_c ≈ 0.087 in our dimensionless units.

From the WKBJ theory applied to gravity waves, the characteristic wavelength, λ, can be estimated through λ ≈ ro/(√6N), where ω is the eigenfrequency and it is assumed that the star is non-rotating, see e.g. Christensen-Dalsgaard (1998). On the other hand, λ should be larger than the width of the transition region, estimated above as ≈2.5 × 10^{-3}r or smaller. Thus, we obtain ω > 4.2 × 10^{-3} in natural units for this inequality to be valid. From Table 1, it follows that this corresponds to periods <40 d. This means that estimates based on a standard WKBJ analysis may be inapplicable for all periods of interest. Let us stress, however, that the width of the transition region may be much larger than was assumed in order to obtain this estimate (see discussion in Section 2.2).

In Fig. 3, we show the forms of the density and the square of Brunt–Väisälä frequency for models with M_* = 2 M_⊙. The youngest model, model 2a (dot–dashed curve) is fully radiative. We expect that for this model, overlap integrals are suppressed at low frequencies as compared to models with convective regions. The overlap integrals associated with such models are expected to decrease with eigenfrequency, ωo, faster than any power of ωo, as happens for a polytropic star represented as model 1p. Thus, tidal interactions determined by low-frequency gravity modes become rather inefficient in this case. Models 2b–2d are similar to those with mass M_* = 1.5 M_⊙ with the difference that the convective envelope is practically absent. Therefore, contributions to the overlap integrals coming from the envelope region should be very small. There are also almost discontinuous transitions from radiative envelopes to convective cores. Thus, we conclude that previous estimates of the strength of tidal interactions based on the natural behaviour of N2 close to this transition may need revision.

Finally, in Fig. 4, we plot the density and square of the Brunt–Väisälä frequency for a model of a young star with M_* = 5 M_⊙. The structure of this model is rather simple and similar to the cases of evolved stars with M_* = 2 M_⊙. There is no convective envelope in this model and, again, there is a quite sharp transition between the radiative region and convective core.

3 PROPERTIES OF STELLAR EIGENMODES: EIGENSPECTRA, OVERLAP INTEGRALS AND ROTATIONAL SPLITTING COEFFICIENTS

In this section, we consider the quantities that determine the tidal interactions for given orbital parameters and properties of decay of free stellar oscillation either due to viscosity or non-linear effects. These are the eigenfrequencies of free pulsations ωo, normalized overlap integrals Q, and, in the case of small rotational frequency Ω ≪ Ω_0 ≡ √GM_* / R_0^3, the coefficients β which determine the splitting of eigenfrequencies due to rotation in the non-rotating frame. We discuss how these quantities depend on stellar structure.

The overlap integrals are discussed in detail in Paper 1. Here, we briefly recall them for completeness. In general, Q is given by the expression

\[ Q = \sqrt{\Omega} \cdot Q = \left( \xi \int_0^{2\pi} d\phi \rho e^{-i\phi/\Omega} \nabla (r^2 Y^m_0) \right), \]

where it is implied that the inner scalar product of any two complex vectors η_1 and η_2 is determined by integration over the cylindrical coordinates σ and z as

\[ \langle \eta_1 | \eta_2 \rangle = \int \sigma d\sigma d\rho (\eta_1^\ast \cdot \eta_2) \cdot \frac{\partial \xi}{\partial r} \frac{\partial S}{\partial \Omega}, \]

Here, * denotes the complex conjugate, Y^m_0 is the spherical function, ξ is the Lagrangian displacement vector corresponding to a particular eigenmode with eigenfrequency ω. It is assumed that in all expressions the dependence of ξ on the azimuthal angle φ is e^{imφ}, is factored out. In general, the azimuthal mode number, m, is such that |m| < 1/2. However, we shall consider only |m| = 2 below as this is the most important case (see Paper 1).

The norm n is determined by the expression

\[ n = \pi \Omega (\xi \xi^\ast + (\xi \star C \xi) / \omega^2), \]

where C is a integro-differential self-adjoint operator, which when operating on −ξ gives the restoring acceleration due to the action of gravity and pressure forces.

When the star is non-rotating, the overlap integrals reduce to the form given by Press & Teukolsky (1977). A similar expression can be obtained in the so-called traditional approximation discussed below and in Paper 1. In that case, we have

\[ Q = \alpha Q_st, \quad \text{with} \quad Q_st = \int_0^{R_s} r^2 dr \rho \left( 2 \xi + \Lambda \xi^2 \right), \]

and

\[ n = n_{st} + n_v, \]

with \( n_{st} = \int_0^{R_s} r^2 dr \rho \left( \xi^2 + \Lambda (\xi^3)^2 \right) \) and \( n_v = v I_0 I_0 \)

where \( I_0 = \int_0^{R_s} r^2 dr \rho (\xi^2)^2 \) and \( v = 2 \Omega / \omega \). In the expressions (4) and (5), \( \xi(r) \) and \( \xi^2(r) \) give the radial dependences of the radial component of the displacement vector and the angular components, respectively, in the traditional approximation. The quantities \( \Lambda, \alpha \) and \( I_0 \) are functions of \( \nu \), their explicit form and the dependence on \( \nu \) are discussed in Paper 1. Note that \( \Lambda(\nu) \) is an eigenvalue associated with acceptable solutions of the Laplace tidal equation.

When the star does not rotate, the Laplace tidal equation reduces to the Legendre equation, the solution of which is the associated Legendre function. We then have \( \alpha = 1, \Lambda = 6, n = n_{st} \) and the expressions (4) and (5) reduce to their standard form as given by e.g. Press & Teukolsky (1977) and Ivanov & Papaloizou (2004).

Hereafter, we express all quantities of interest in natural units. Thus, eigenfrequencies are expressed in terms of the natural stellar frequency Ω_0, and the overlap integrals in terms of √M_0 R_0.

3.1 Eigenfrequencies and overlap integrals for non-rotating stars

Let us first discuss the eigenfrequencies and overlap integrals for non-rotating models. In this case, we set \( \alpha = 1 \) and \( n_v = 0 \) in (4) and (5), respectively. Eigenfrequencies and eigenfunctions were obtained by a shooting method described in section 5.2 of Paper 1. Here, we note that we integrated the standard full set of four equations describing adiabatic pulsations (e.g. Christensen-Dalsgaard 1998) to find these quantities for relatively large values of eigenfrequencies \( \omega > 0.3–0.5 \) depending on the particular model. These were then used to evaluate the overlap integrals. For smaller eigenfrequencies, the Cowling approximation is used to find the eigenfrequencies and the expression (78) of Paper 1 was employed to find the overlap integrals. This expression is equivalent to the original one presented above provided the Cowling approximation is used to find eigenmodes.

We checked that in the intermediate region for which \( \omega \sim 0.3–0.5 \) both methods give practically the same results. The advantage of using the expression (78) of Paper 1 in the low-frequency limit is the fact that the integrand in this expression is less oscillatory in comparison to the integrand in the original expression, thus allowing
Figure 5. The radial dependence of the radial and tangential components of the displacement vectors, $\xi$ and $\xi^S$, for typical eigenmodes for models 2a and 2d. These are plotted in arbitrary units as functions of the radius $r$, in the inner region $0 < r < 0.15$. The dotted and dot–dashed curves and dashed and solid curves show $\xi$ and $\xi^S$, respectively, for models 2a and 2d. Fig. 2 indicates that model 2a is mostly radiative, while model 2d has a sharp transition from the exterior radiative region to the convective core which is situated at $r_c \sim 0.05$. It is seen that the eigenfunctions corresponding to the mostly radiative model are smooth while those corresponding to the model with a convective core demonstrate a very sharp change of behaviour in the vicinity of $r \sim r_c$.

us to significantly increase the accuracy of determination of $\hat{Q}$ in this limit where the eigenfunctions contain a large number of nodes.

Models of massive stars have very rapid variations of $N^2$, see Figs 2–4 in the transition regions between radiative envelopes and convective cores. Since the characteristic radial extent of these variations can be of the order of the initial stellar model grid size, a treatment of discontinuities may be needed. We prefer, however, to avoid this situation in our approach. To deal with this issue, we worked with computational grids which had a much larger number of grid points than were originally used to represent the structure of the stellar models. State variables were interpolated on to our more refined grid as smooth functions. There were then no discontinuities in the eigenfunctions on the refined grid, see Figs 5 and 6.

The results of calculations of overlap integrals are shown in Figs 7–10. In Fig. 7, we show the results for models 1p, 1a and 1b. Since the polytropic model 1p has been discussed extensively elsewhere (e.g. Press & Teukolsky 1977; Lee & Ostriker 1986; Ivanov & Papaloizou 2004 and references therein) and overlap integrals corresponding to models 1a and 1b are discussed in detail in Paper 1, here we only mention that the polytropic model has much smaller overlap integrals when $\omega < 0.4$. This is due to the fact that $\hat{Q}$ corresponding to Sun-like stars has contributions arising from the presence of a convective envelope. These decay as a power of $\omega$ in the limit $\omega \to 0$, while the overlap integrals of the polytropic model may be shown to decay faster than any power of $\omega$. Note that the overlap integrals for the solar model have also been calculated recently by Weinberg et al. (2012). Our calculations for the model 1b agree quite well with their results. It is of interest to note that the overlap integrals for this model are not monotonic at large frequencies. This effect is even more prominent for the models with $M_\ast = 1.5$ and $2 M_\odot$ discussed below, where several peaks in the values of $\hat{Q}$ at values of frequencies corresponding to pressure modes are observed, see Figs 8 and 9. Since this effect is not important for our purposes, we do not discuss it here. However, we would like to mention that it appears to be rather generic, e.g. it is present in the dependence of $\hat{Q}$ on $\omega$ in a model of red giant star, see Fuller et al. (2012).

Fig. 8 shows results for models 1.5a–1.5c. For the range of frequencies plotted, only model 1.5a has overlap integrals larger than those of the polytrope at small values of $\omega$. As discussed above this...
is because of the fact that this model has a rather extended convective envelope. From our analytical theory developed in Paper 1, it follows that in the case of model 1.5a, the contribution determined by the presence of this region should be roughly three times smaller than that arising for Sun-like stars. As seen from Fig. 8, this is confirmed by our numerical results. More evolved models 1.5b and 1.5c have rather small overlap integrals in the range of frequencies plotted. They are even smaller than those of the polytropic star. This indicates that tidal interactions determined by the excitation of eigenmodes in the shown range of frequencies are relatively weak for these models.  

Results for stars of mass $M_*=2 M_\odot$ are shown in Fig. 9. This case is rather similar to the previous one. However, the more massive stars do not have well-pronounced convective envelopes and, therefore, their overlap integrals are rather small for the range of frequencies shown, being several times smaller than those for a polytropic star. Finally, in Fig. 10 the overlap integrals of a young star with $M_*=5 M_\odot$ are shown. They are rather similar to, though slightly smaller than those of a polytropic star for the range of frequencies shown. This means that in this frequency range the contribution determined by the presence of a convective core is probably not seen.

### 3.2 Rotational splitting coefficients

When the rotation frequency of a star, $\Omega$, is small in comparison to its natural frequency: $\Omega \ll \Omega_*$, one may treat effects due rotation in a simplified manner (e.g. Lai 1997; Ivanov & Papaloizou 2011). From first-order perturbation theory, the eigenfrequencies, $\omega_j$, are shifted with respect to their values for a non-rotating star, $\omega_{0,j}$, by an amount proportional to $\Omega$. We take this shift into account but assume that the overlap integrals are unchanged. In the inertial frame, we have (see e.g. Christensen-Dalsgaard 1998)

$$\omega_j = \omega_{0,j} + m \beta r \Omega, \quad (6)$$

where $m = 0, \pm 1, \pm 2$ is the azimuthal number and $\beta r$ are dimensionless coefficients determining the magnitude of the rotational splitting. Note that when the rotation axis is perpendicular to the orbital plane, terms with $m = \pm 1$ do not contribute to the energy and angular momentum transfer as a result of a periastron flyby. The rotational splitting coefficients $\beta r$ can be expressed as a ratio of two integrals involving the components of the mode Lagrangian displacement. When $\omega_{0,j} \gg 1$ they are close to unity and when

\[ \beta r = \frac{\int \rho \chi^2 \, dV}{\int \rho \chi \chi_\ast \, dV}, \]

\[ \chi = \frac{\partial \chi}{\partial \xi}, \]

\[ \chi_\ast = \frac{\partial \chi_\ast}{\partial \xi}. \]

\[ \rho = \frac{\partial \rho}{\partial \xi}. \]
Figure 11. The rotational splitting coefficients $\beta_r$ as functions of $\omega$ for models with $M_*=M_\odot$ and $M_*=5M_\odot$. The dotted, solid, dashed and dot–dashed curves are for models 1p, 1b, 1a and 5a, respectively. The open circles show positions of numerically calculated eigenfrequencies.

Figure 12. Same as Fig. 11 but for $M_*=1.5M_\odot$. The solid, dashed and dotted curves are for models 1.5c, 1.5b and 1.5a, respectively. One can see from these figures that the dependence of $\beta_r$ on $\omega$ is not necessarily monotonic, as was also found for the overlap integrals.

$\omega_0,l \ll 1$ they tend to $1-1/(ll+1) = 5/6$ for spherical harmonic index, $l=2$ (e.g. Christensen-Dalsgaard 1998). In the intermediate range of eigenfrequencies, these integrals have to be evaluated numerically. We calculate them for our stellar models and illustrate them for models 1p, 1a, 1b and 5a in Fig. 11. Results for models 1.5a–1.5c are illustrated in Fig. 12, and for models 1b–2d in Fig. 13. One can see from these figures that the dependence of $\beta_r$ on $\omega_0,l$ is not necessarily monotonic, as was also found for the overlap integrals.

From the results presented below, we find that use of the perturbative description of the influence of rotation on tides described above yields results that agree quite well with those obtained from a more accurate approach based on the traditional approximation (Unno et al. 1989), even for quite large stellar angular velocities $\Omega \sim 0.4$. This is the case when either the stellar rotation is retrograde with respect to that of the orbital motion, or prograde with respect to the orbital motion, but with the angular frequency being smaller in magnitude than approximately the value corresponding to pseudo-synchronization, for which there is zero angular momentum transfer.

3.3 The overlap integrals in the traditional approximation

As explained in Paper 1 and above, when the traditional approximation is used the overlap integrals $\hat{Q}$ can be represented as products of ‘angular’ and ‘radial’ contributions. Thus, $\hat{Q} = \alpha \hat{Q}_r$, where the angular contribution $\alpha$ is given in fig. 1 of Paper 1. The radial contribution, $\hat{Q}_r$, is illustrated in Figs 14 and 15, for models 1b and 1.5a, respectively. Note that the curves corresponding to the largest retrograde rotation are not monotonic, with the overlap integrals having a pronounced minimum at some value of the eigenfrequency. This effect is explained in Paper 1. Namely, as follows from the discussion above, the values of the radial contributions to overlap integrals for stars with extended convective envelopes are mainly determined by contributions coming from the convective envelope and from the vicinity of the base of the convective zone. It can be shown (see Paper 1, equation 111) that these contributions are proportional to the factor $(1 - 30/\Lambda(v))$, where we recall that $\Lambda$ is the eigenvalue associated with acceptable solutions of the Laplace tidal equation and $v = 2\Omega/\omega$. It turns out that in the case of retrograde rotation, the value of $\Lambda$ corresponding to the frequencies, where this minimum is observed, is approximately equal to 30, and, therefore, the main contributions to the overlap integrals are strongly suppressed.

We use below the traditional approximation described in detail in Paper 1 to calculate the energy and angular momentum transfer for model 1b as a result of a flyby of a perturber on a parabolic orbit for $\Omega = \pm 0.11, \pm 0.21$ and $\pm 0.42$ as well as for model 1.5a for $\Omega = \pm 0.25$ and $\pm 0.5$. These are compared with results obtained by numerically solving the flyby problem directly as an initial value problem.
as a point mass. We consider realistic stellar models. In general, we assume that the stellar rotation axis is perpendicular to the orbital plane. For a discussion of the case of a general inclination between the stellar rotation axis and the orbital angular momentum vector, see Ivanov & Papaloizou (2011).

The magnitude of the energy transferred, $\Delta E$, depends on whether it is calculated in the inertial or rotating frame. In general, we have

$$\Delta E = \Delta E_1 + \Omega \Delta L,$$

where $\Delta E_1$ and $\Delta E$ are the energy transferred as calculated in the inertial and rotating frame, respectively, $\Omega$ is the stellar angular velocity, and $\Delta L$ is the amount of angular momentum transferred.

It is convenient to introduce natural units for the energy and angular momentum transferred. Thus, we express $\Delta E_1$ and $\Delta E$ in units of $E_\star = Gm_\star^2/(1+q)^2 R_\star$, where $G$ is the gravitational constant, $m_\star$ is the mass of a perturbing body and $q = m_\nu/M_\star$. We express $\Delta L$ in units of $L_\star = q^3(1+q)^{-2}M_\star \sqrt{GM_\star R_\star}$. We remark that all quantities in equation (54) of Paper 1 can also be represented in natural units. Once the ratio $\Omega/\Omega_1$ is specified, the energy and angular momentum transferred expressed in natural units are functions of only one parameter (see e.g. Press & Teukolsky 1977; Ivanov & Papaloizou 2004, 2007)

$$\eta = \sqrt{\frac{1}{1+q} \left( \frac{R_\nu}{R_\star} \right)^3} = 3.05 \sqrt{\tilde{\rho}} P_{\text{orb}},$$

where we recall that $R_\nu$ is the periastron distance. The quantity $\tilde{\rho}$ is the ratio of the mean stellar density to the solar value, $\tilde{\rho} = R_\odot^3 M_\odot/(R_\star^3 M_\star)$ and $P_{\text{orb}}$ is orbital period of a circular orbit which has the same value of the orbital angular momentum as the parabolic orbit under consideration, expressed in units of one day.

Assuming that tidal evolution approximately conserves angular momentum (see e.g. Ivanov & Papaloizou 2011) for a discussion of this approximation) $P_{\text{orb}}$ characterizes the orbital period of the binary system after the process of tidal circularisation is complete. Note that when $q \ll 1$ as for dynamic tides induced in a central star in exoplanetary systems, the condition $\eta = 1$ corresponds to a grazing encounter with periastron distance equal to the stellar radius. Thus, for these systems only $P_{\text{orb}} > P_{\text{crit}} = 0.325/\sqrt{\tilde{\rho}}$ are possible.

We remark that a number of authors (e.g. Press & Teukolsky 1977; Giersz 1986; Lee & Ostriker 1986; McMillan et al. 1987) express results in terms of another dimensionless quantity, $T_3(\eta)$, which is related to the dimensionless energy transfer, $\Delta E$, through

$$T_3(\eta) = \eta^2 \Delta E.$$

We use equation (9) to compare our results with those obtained by previous authors.

When $\Delta E_1$ and $\Delta E$ are expressed in natural units, their dependence on $\eta$ may be used to compare the strength of tidal interactions of stars with different masses and radii. However, for systems with given orbital parameters and $m_\nu$, it also depends on the average density being larger for stars with smaller $\tilde{\rho}$, mainly through the dependence of $\eta$ on this quantity. Therefore, in a similar way to

\[ \text{Figure 14.} \text{ We show the radial contribution to the overlap integrals } \hat{Q}_i \text{ for model 1b of the present-day Sun as functions of eigenfrequency } \omega \text{ for different values of } \Omega. \text{ The cases of } |\Omega| = 0.42, 0.21, 0.11 \text{ and the non-rotating case are shown using solid, dashed, dot-dashed and black dotted curves, respectively. The curves of the same type with smaller (larger) values of } \hat{Q}_i \text{ for a given value of } \omega \text{ are calculated for retrograde (prograde) directions of rotation with respect to the pattern rotation associated with the forcing potential. The curves corresponding to the three cases with prograde rotation almost coincide. Symbols show the positions of eigenfrequencies for } |\Omega| = 0.42, 0.21, 0.11 \text{ and 0, respectively.} \]

\[ \text{Figure 15.} \text{ As in Fig. 14 but for model 1.5a. The cases } |\Omega| = 0, 0.25 \text{ and 0.5 are plotted using black dotted, dot-dashed and solid curves, respectively. Squares, triangles and circles show positions of eigenfrequencies for } |\Omega| = 0.5, 0.25 \text{ and 0, respectively.} \]

\[ \text{4 TRANSFERS OF ENERGY AND ANGULAR MOMENTUM ARISING FROM PARABOLIC ENCOUNTERS AND THE TIDAL CAPTURE PROBLEM} \]

In this section, we discuss the well-known problem of calculating the energy and angular momentum transferred to the normal modes of a star as a result of a parabolic encounter with a perturber treated

\[ \text{as a point mass. We consider realistic stellar models. In general, we assume that the stellar rotation axis is perpendicular to the orbital plane. For a discussion of the case of a general inclination between the stellar rotation axis and the orbital angular momentum vector, see Ivanov & Papaloizou (2011).} \]
our previous studies (Ivanov & Papaloizou 2004, 2007, 2011), we introduce the tidal circularization time

\[ T_{cv} = 15 \left( \frac{M_i}{M_\odot} \right) \left( \frac{R_i}{R_\odot} \right) \left( \frac{M_1}{m_p} \right) \frac{1}{\Delta E_1} \sqrt{a_\infty}, \]  

(10)

where \( M_i \) is the mass of Jupiter (see equation 104 of Ivanov & Papaloizou 2007). From here on, it is implied that energy and angular momentum transfers are expressed in natural units, and \( T_{cv} \) is expressed in years. Under the assumption that the energy transfers arising from consecutive periastron passages can be simply added, \( T_{cv} \) gives a characteristic time-scale for the tidal evolution of the semimajor axis of a highly eccentric orbit with initial semimajor axis, \( a_\infty \), in units of 10 au. We stress that this time-scale applies only to the initial stages of circularization and will be characteristic of the whole process, only if it can proceed efficiently enough at small eccentricities (see Ivanov & Papaloizou 2011 for a discussion). We set \( m_p = M_1 \) and \( a_\infty = 1 \) hereafter, generalization to other values of \( m_p \) simply follows from the form of equation (10).

It is convenient to use equation (10) when considering tidal interactions in systems containing exoplanets. However, dynamic tides may also be important for other problems, such as e.g. the tidal capture of stars to form binary systems in stellar clusters (e.g. Fabian et al. 1975; Press & Teukolsky 1977). To characterize the strength of tidal interactions in such a setting it is convenient to introduce ‘the capture radius’ \( R_{cap} \), defined by the condition that when the periastron distance for a tidal encounter between two initially unbound stars is equal to \( R_{cap} \), the initial relative kinetic energy of these stars, when they are very far apart, is equal to the amount of energy transferred due to tidal interactions, i.e.

\[ \Delta E_i^{(1)} + \Delta E_i^{(2)} = \frac{1}{2} \frac{M_i^2 \dot{v}_e^2}{M_\odot^2 + M_e^2}, \]  

(11)

where the upper index, \( i \), denotes quantities associated with star \( i \), and \( \dot{v}_e \) is the initial relative velocity of the stars with respect to each other. Assuming that the binary consists of two identical stars, which rotate in the same sense with respect to their orbital motion, we get

\[ \Delta E_i(\eta_{cap}) = 3.125 \times 10^{-4} v_s^2, \quad v_s = \sqrt{\frac{M_\odot}{M_e}} \frac{R_e}{10 \text{ km s}^{-1}}, \]  

(12)

and \( \eta_{cap} \) is related to \( R_{cap} \) through equation (8) with \( R_p = R_{cap} \).

### 4.1 Energy and angular momentum transfer as a result of a parabolic encounter from direct numerical solution of the linear initial value problem

We have calculated the energy and angular momentum transferred to a star as a result of an encounter with a perturber on a parabolic orbit by solving the linear initial value problem directly. We refer to this procedure as the direct numerical approach. The method adopted follows from that described in Papaloizou & Ivanov (2010) and Ivanov & Papaloizou (2011). The equations solved are (36–41) of Ivanov & Papaloizou (2011) with the following modifications. In that work, a polytrope of index \( n = 3 \) and a constant adiabatic index \( \gamma = 5/3 \) was considered. However, here we consider realistic stellar models for which this varies. Accordingly, the quantity \( P^{(i)/\gamma}/P \nabla(P/P^{(i)/\gamma}) \) in equation (36) of Ivanov & Papaloizou (2011) was replaced by \( F_{ad}/\rho \nabla(P/F_{ad}) \), where

\[ F_{ad} = \int_{P}^{P_0} \frac{1}{\Gamma_1 P} dP, \]  

(13)

where \( \Gamma_1 = (d\ln P/d\ln \rho)_{\text{adiabatic}} \) and \( P_s \) is the surface boundary or photospheric pressure. This quantity is readily obtainable for the models provided by numerical integration. Elsewhere in equations (36–41) of Ivanov & Papaloizou (2011), \( \gamma \) was replaced by \( \Gamma_1 \).

The presence of regions with negative \( N^2 \equiv \omega^2 \eta \) gave rise to linearly convectively unstable eigenmodes which would ultimately dominate the solution. In order to remove such modes, as long as the density gradient was negative, we redefined \( \Gamma_1 \) in such regions such that \( N^2 \to 0 \) there. This procedure amounts to stating that during linear perturbation, the relationship between \( P \) and \( \rho \) in these regions is maintained. Here, we are adopting the common approximation that the layers are effectively adiabatically stratified (e.g. Ogilvie & Lin 2007). This is equivalent to the condition that the convective, or frictional time-scale (cf. Zahn 1977) be significantly longer than the rotation period, which is expected to be satisfied for the cases we consider.

In some models, there were small low-density regions where the density gradient became positive (see discussion in Section 2.2 above). For these regions, again \( N^2 \) was set to zero, but \( \Gamma_1 \) was not allowed to become negative, instead being set to be the largest positive value attained on the grid from the first procedure. In this way, an incompressibility condition is approached. However, as the values of the density and pressure were very small, this did not lead to numerical difficulties or, as shown by numerical tests, affect results significantly. As in our previous work, most simulations were carried out on a 200 × 200 numerical grid with \( m = 2 \) which gives the dominant contribution. Resolution tests were carried out by doubling the resolution to 400 × 400 and as in our previous work showed good convergence in these cases with variable \( \Gamma_1 \).

### 4.2 Non-rotating stars

In this section, we consider non rotating stars, accordingly \( \Omega = 0 \). In this case \( \Delta E_i = \Delta E \). The results of numerical calculations of \( \Delta E \) and \( T_{cv} \) for the stellar models presented in Table 1 are shown in Figs 16–23.

In Fig. 16, we show the dependence of \( \Delta E \) on \( \eta \) calculated for our models of Sun-like stars, 1a and 1b, together with the same quantities calculated for a polytropic star with solar mass and radius. Circles show results obtained from the direct numerical approach. One can see that the approach based on normal mode calculation gives practically the same energy transfer as the direct numerical approach. A small deviation for \( \eta = 8 \) is probably due to the effect of numerical diffusion and finite integration time. As seen from Fig. 16, the energy transfers for realistic models are significantly larger than those for the polytropic model for large enough values of \( \eta \). Clearly, this is due to much larger values of the overlap integrals for the realistic models at small eigenfrequencies, see Fig. 7. Interestingly, the energy transfers for models 1a and 1b are quite close to each other regardless of the fact that the overlap integrals corresponding to model 1a are larger than those of model 1b. This is explained by observation that the number of eigenmodes contributing to the tidal interaction is larger in the case of model 1b. This compensates for smaller values of the overlap integrals.

The dot–dashed curve shown in Fig. 16 is obtained using the formalism described in Paper 1 for model 1b. This curve is calculated by purely analytic means. We see that there is quite good agreement between the analytic and numerical results. In the range \( 2 < \eta < 10 \), the deviation is at most about 40 per cent, when
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Figure 16. Energy transferred to the normal modes of a non-rotating star with $M_\ast = M_\odot$ as the result of a parabolic flyby of a perturber of mass $m_p = M_J$, expressed in units of $E_\ast$, as a function of the parameter $\eta$. The solid, dashed and dotted curves are for models 1b, 1a and 1p, respectively. The dot–dashed curve shows results calculated using the purely analytic expressions for the mode eigenfrequencies and overlap integrals obtained in Paper 1 for model 1b. This almost coincides with the solid line. The dot–dot–dashed curve shows the result of Giersz (1986). The circles show the energy transfer calculated for model 1b using the direct numerical approach.

Figure 17. The evolution time $T_{ev}$ as a function of the orbital period after circularization, $P_{orb}$, for the same models as illustrated in Fig. 16. Again, the solid, dashed and dotted curves are for models 1b, 1a and 1p, respectively. The dot–dot–dashed curve in the same figure shows the result of Giersz (1986) for a solar model. The energy transfer found Giersz (1986) is significantly smaller than we obtain in this paper. Since our results are obtained using three independent methods, we believe that the Giersz (1986) result underestimates the energy transfer, though the origin of the discrepancy is unclear. One possible explanation is that the number of eigenmodes used in his calculation was too small.

$\eta > 10$, corresponding curves practically coincide. The dot–dot–dashed curve in the same figure shows the result of Giersz (1986) for a solar model. The energy transfer found Giersz (1986) is significantly smaller than we obtain in this paper. Since our results are obtained using three independent methods, we believe that the Giersz (1986) result underestimates the energy transfer, though the origin of the discrepancy is unclear. One possible explanation is that the number of eigenmodes used in his calculation was too small.

Figure 18. Same as in Fig. 16 but for the models with $M_\ast = 1.5 M_\odot$. The two dotted curves are for our ‘reference’ models 1p and 1b. The polytropic star, model 1p, has a smaller value of $\Delta E$ for a given value of $\eta$. The solid, dashed and dot–dashed curves are for models 1.5c, 1.5b and 1.5a, respectively. As in Fig. 16, the symbols indicate the amount of energy transferred that was obtained adopting the direct numerical approach. The circle and square are for models 1.5a and 1.5c, respectively. The dot–dot–dashed curve shows the results of McMillan et al. (1987) for a Population II model.

Figure 19. The evolution time $T_{ev}$ for models with $M_\ast = 1.5 M_\odot$ as a function of $P_{orb}$. Curves of a given type plotted in Figs 18 and 19 correspond to the same models.

From Fig. 17, it follows that the evolution time-scale $T_{ev}$ is smaller for model 1b than for model 1a, for a given value of $P_{orb}$. The fact that the model of present-day Sun, model 1b, can be tidally excited more efficiently than the corresponding polytropic model leads to

$^5$ Note that our calculations for model 1a are not realistic when $T_{ev}$ exceeds its age $\approx 1.7 \times 10^8$ yr. More realistic calculations of $T_{ev}$ should employ a set of overlap integrals and eigenfrequencies calculated for a grid of stellar models of different ages.
the conclusion that taking into account realistic stellar models can significantly increase estimates of the contribution of tides exerted on the star for the orbital evolution of Jupiter mass exoplanets on highly eccentric orbits. The enhanced tidal interaction can produce a significant change of the orbital semimajor axis in a time of less than $4 \times 10^9$ yr when $P_{\text{orb}} < 4$. Note that this contribution can be further amplified for stars rotating in the opposite sense to that of the orbital motion, as discussed in Lai (1997), Ivanov & Papaloizou (2011) and below. Furthermore, tides exerted on the star become even more efficient for more massive planets, see e.g. Ivanov & Papaloizou (2004, 2007). We also remark that a consequence of the above is that a significant component of the energy liberated by orbital circularization may be dissipated in the star rather than the planet, thus alleviating the possibility of the potential destruction of the planet (see e.g. the discussion in Ivanov & Papaloizou 2004).
normal mode approach is excellent. The dot–dashed curve shows the results of McMillan et al. (1987) for a Population II 1.5 M⊙ star. At sufficiently large values of $\eta$, the energy transfer for their model is significantly smaller than that for model 1.5a, but larger than that for models 1.5b and 1.5c. Unfortunately, McMillan et al. (1987) did not provide details of their model. Thus, it is unclear as to whether this behaviour is a consequence of the form of the Brunt–Väisälä frequency, as is indicated from consideration of our models.

However, it is important to point out that, for a given $P_{\text{orb}}$, the circularization times, $T_{\text{ev}}$, corresponding to the more massive models are significantly smaller than those for the Sun-like models. This is due to the fact that the mean density of the more massive models is significantly smaller than the mean density of the Sun-like models, see Table 1, which leads to smaller values of $\eta$ for the more massive models for a given value of $P_{\text{orb}}$, see equation (8). In particular, model 1.5c, having the largest age $\sim 1.6 \times 10^6$ yr, has $T_{\text{ev}}$ of the order of or smaller than its age, for planets having $P_{\text{orb}} < 4$. The increase of the efficiency of the tidal interaction for model 1.5c, as compared to models 1.5b and 1b, can also be viewed as a consequence of the expansion of the star that takes place as a result of evolution.

Models with $M = 2$ and 5 M⊙ are qualitatively similar to models 1.5b and 1.5c. In all cases, the energy transfer is even smaller that for the polytropic model, while the evolution times $T_{\text{ev}}$ are smaller than those for the reference models 1p and 1b, due to the smaller mean densities of the massive stars. Note that the energy transfers calculated for models 2c and 2d are very close to each other. It is also interesting to note that the value of the mean density evolves non-monotonically with time. For $M = 2$ M⊙, model 2d with the greatest age $\sim 10^9$ yr, is smallest. This leads to the possibility of significant evolution of the semimajor axis, with $T_{\text{ev}} < 10^9$ yr, for exoplanets orbiting stars with $M = 2$ M⊙ and final orbital periods, assuming the circularization process can be completed of $P_{\text{orb}} < 7$ d, solely due to tides exerted on the star.

4.3 Rotating stars in the traditional approximation

In this section, we present results for rotating models 1b and 1.5a within the framework of the traditional approximation described in detail in Paper 1. We compare these results to those obtained from the perturbative approach, where it is assumed that the overlap integrals are not changed by rotation, and the mode eigenfrequencies as viewed in the inertial frame are shifted by a factor $m\beta/\Omega$, as given by equation (6). The perturbative approach is described in more detail in Ivanov & Papaloizou (2011) and references therein. Results are also compared to those obtained following the numerical approach, that is, by solving the encounter problem directly as an initial value problem (see Section 4.1). In all of this, it is assumed that the rotation axis is perpendicular to the orbital plane, with both prograde and retrograde encounters with respect to the direction of the stellar rotation being considered.\textsuperscript{6}

4.4 A comparison of eigenspectra obtained from normal mode calculations with those obtained from the direct numerical approach

In order to check whether our direct numerical method and the normal modes approach agree with each other, we consider $\theta$-component of the perturbed velocity at a characteristic position in the star as a function of time and make a Fourier transform of the signal. The results are shown in Figs 24 and 25 for a tidal encounter of a non-rotating star and a retrograde encounter with $\Omega = 0.58$, respectively. In the latter case, positive and negative values of the frequency $\omega$ correspond to perturbations propagating in the direction of stellar rotation and opposite to this direction, respectively. The value of the amplitude scaling of the Fourier transform is arbitrary. Peaks in these figures indicate the approximate positions of free normal mode pulsations. Symbols show the positions of eigenfrequencies calculated using the normal mode approach.

\textsuperscript{6}For discussion of the general case of an arbitrary inclination of the stellar rotational axis, see Ivanov & Papaloizou (2011).
As seen from Fig. 24, in the case of the non-rotating star, mainly positive eigenfrequencies are excited in the course of tidal encounter. Positions of peaks are in rather good agreement with the normal mode calculations for modes having $|\omega| > 1$. Smaller values of $\omega$ are not well resolved on account of the finite time duration of the run.

Contrary to the non-rotating case, the retrograde tidal encounter mainly excites eigenmodes with negative eigenfrequencies, see Fig. 25. Again, the most prominent peaks approximately in the range $-3 < \omega < -1$ are in a rather good agreement with the normal mode method. The eigenfrequencies found from the normal mode method are, however, shifted towards $\omega = 0$ with respect to the positions of the corresponding peaks, the shift being larger for eigenmodes with larger absolute values of $\omega$. The situation is similar for modes with positive values of $\omega$ in the range $1 < \omega < 3.5$, but in this case mode eigenvalues are shifted towards larger values of $\omega$ with respect to corresponding peak positions. This disagreement is possibly determined by the fact that we use the Cowling and traditional approximations to calculate the eigenspectrum in the normal mode approach. Since smaller absolute values of the eigenfrequencies result in larger values of the associated transfers of energy and angular momentum, we expect that the direct numerical approach gives smaller values of these quantities in the case of retrograde encounters and larger values for prograde encounters. This is indeed obtained in our calculations, see the discussion below. Note too the absence of significant peaks in the inertial range for which $-1.16 < \omega < 1.16$. This is where inertial modes associated with the convective envelope would be expected to show up. However, when confined to a spherical annulus inertial waves may focus on to wave attractors becoming singular in the inviscid limit. Then, corresponding discrete inviscid modes may not exist (e.g. Ogilvie & Lin 2004; Ogilvie 2013), but instead a continuous spectrum, with the consequence that very prominent resonant spikes are not seen in the response. Nonetheless, the relatively small response in the inertial range indicates that inertial waves are not important in this particular case.

4.5 Energy and angular momentum transfer for rotating stars

Figs 26–30 are for model 1b. In these figures, we show results obtained from the full numerical approach applied to the Sun-like stars having $|\Omega| = 0.42, 0.21, 0.11$ and 0. These are represented by solid, dashed, dot–dashed and dotted curves, respectively. In Figs 26–29, the retrograde encounters have a larger value of the transferred quantity at a given value of $\eta$ than the prograde encounters. The squares, triangles, diamonds and circles in Figs 26–28 show the corresponding results obtained using the direct numerical approach, respectively for $|\Omega| = 0.42, 0.21, 0.11$ and 0.

In Fig. 26, we show the transfer of energy in the inertial frame, $\Delta E$, versus $\eta$. We see that retrograde encounters always have a larger value of $\Delta E$ than prograde encounters, for a given $\eta$, with this effect being more extreme for larger absolute values of the angular velocity. The physical reason for this behaviour is discussed in detail in Lai (1997) and Ivanov & Papaloizou (2011). It is interesting to note that prograde encounters with non-zero angular velocity produce larger values of $\Delta E$ than the case with $\Omega = 0$, when $\eta$ is large. Within the framework of rotationally modified gravity modes under traditional approximation, this is explained by noting that in the limit of very large values of $\eta$, the eigenmodes mainly determining the value of $\Delta E$ propagate in retrograde direction with respect to the stellar rotation in the rotating frame, but at the same time, they propagate in the prograde direction in the inertial frame, since the pattern speed of

![Figure 26](https://example.com/figure26.png)

**Figure 26.** The energy transferred in the rotating frame expressed in natural units as a function of $\eta$. Curves of different style correspond to different absolute values of the angular velocity $\Omega$ and symbols indicate results obtained using the direct numerical approach. See the text for the allocation of the curves with different styles.

![Figure 27](https://example.com/figure27.png)

**Figure 27.** Illustration of inertial waves in the convective envelope for the prograde encounter of model 1b with $\eta = 8$ and $\Omega = 0.21$. Contours of the product of the Lagrangian displacement in the $\phi$ direction and $\sqrt{\rho}$ are shown in an upper quadrant at a time after the encounter is over and the energy transfer has been completed. The presence of rotationally modified $g$ modes of the order of up to 15 can be seen in the radiative core while the convective envelope shows the presence of inertial waves that show reflections as well as graze the boundary with the radiative core. The radial coordinate is expressed in $cm$. 
these modes is smaller than the angular velocity of rotation. From the results of e.g. Ivanov & Papaloizou (2011), it follows that the energy transfer due to these modes is proportional to $\eta^{-2}\sum_i Q^2 I_{i,2}^2(y)$, where $y = \eta(2\Omega - |\omega_i|)$, the quantities $I_{i,2}(y)$ are discussed in Press & Teukolsky (1977), see also Ivanov & Papaloizou (2007). The function $I_{2,2}(y)$ has a maximum at $y \approx 2$ and decreases towards smaller and larger values of $y$. Thus, when $y = 1, 3$, it has values approximately one half of its maximum value. Therefore, in order to crudely estimate the energy transfer one may assume that only the contribution of eigenmodes having eigenfrequencies such that $1 < y < 3$ need to be considered and that all these modes have $I_{2,2}(y) = I_{2,2}(y = 2)$. The absolute values of the eigenfrequencies are close to $\omega_{\text{max}} = 2\Omega - 2/\eta$, being defined by the condition $\Delta E_i < \Delta E$ through equation (7). Additionally, we show the energy transfer calculated with the framework of the perturbative approach by dot–dot–dashed lines. Different symbols on these lines correspond to different rotation rates. 

In this range $\Delta E \approx \eta^{-2}$ and both $\Delta E_i$ and $\Delta L$ are negative. Note that this regime persists as long as the number of terms in the sum is larger than 1, which corresponds to $\eta < \eta_{\text{max}} \approx 2/\Delta\omega$, where $\Delta\omega$ is the distance between two neighbouring eigenfrequencies having $\omega_i \approx \omega_{\text{max}}$. When $\eta > \eta_{\text{max}}$, $\Delta E$ decreases faster than $\eta^{-2}$. When $\Omega = 0.42$, we find $\eta_{\text{max}} \approx 70$ for the Sun-like models and we have checked that indeed this behaviour is observed in our results.

As discussed above, the values of $\Delta E$ obtained from the direct numerical approach for the non-rotating star are in excellent agreement with the normal mode method, with only the values corresponding to $\eta = 8$ deviating by about 20 per cent. This deviation may be explained by the influence of numerical viscosity, which leads to relatively more dissipation over the long run times necessary when $\eta$ is large. Such runs become prohibitive for $\eta > 8$. The case of $\Omega = 0.11$ is quite similar to the non-rotating case, with only one sizeable deviation of the order of 40 per cent associated with the retrograde encounter with $\eta = 8$. When $\Omega = 0.21$, the deviations are less than 25 per cent for retrograde encounters and less than 30 per cent for prograde encounters with $\eta \leq 4\sqrt{2}$. There is, however, large disagreement for the prograde encounter with $\eta = 8$ for which the direct numerical approach gives $\Delta E$ approximately 2.5 times larger than the normal mode method. Convergence checks showed that this discrepancy was not due to lack of numerical convergence of the direct numerical approach.

As seen from Fig. 28, for $\Omega = 0.21$ this $\eta$ is close to the value where $\Delta L = 0$, where the star is in a state of pseudo-synchronization (see e.g. Papaloizou & Ivanov 2005; Ivanov & Papaloizou 2004, 2007, 2011 and references therein). In a similar problem of a tidal
encounter of a polytropic rotating star discussed in Papaloizou & Ivanov (2011), an analogous disagreement between the direct numerical and normal mode approaches was observed.

Close to the state of pseudo-synchronization, the normal mode approach indicates that when \( \eta \) is fixed and \( \Delta E \) and the absolute value of \( \Delta \) are considered to be functions of \( \Omega \), \( \Delta E \) has a deep minimum at \( \Omega = \Omega_{\text{ps}} \), being the value for which pseudo-synchronization occurs. At that point \( \Delta \ell = 0 \). The differences between the two numerical approaches may result in a shift in the location of this minimum, which because of its depth causes a large discrepancy when the methods are compared.

In addition, our normal mode approach does not take into account the contribution of inertial waves, which can be excited in the convective regions and increase the amount of transferred energy as viewed in the rotating frame. This effect would be most marked at pseudo-synchronization. One would expect that the inclusion of inertial modes would increase \( \Delta E \). To estimate the possible magnitude of the effect, we note that Papaloizou & Ivanov (2005) found that for a polytrope with \( n = 1.5 \),

\[
\Delta E = 6.5 \times 10^{-3} \frac{E}{\eta^6}.
\]  

(14)

Although only the convective envelope resembles such a polytrope, we use this estimate. In fact there are two corrections, the first arising from the truncation of the envelope at \( r \sim 0.7R_\ast \), is expected to increase \( \Delta E \) by about an order of magnitude (Ogilvie 2013). The second, due to the fact that the envelope is on top of a more centrally condensed model than the polytrope and so has a lower base density, is expected to decrease \( \Delta E \) by a similar factor; thus, in order to make rough estimates, we simply assume these effects approximately cancel out. Use of equation (14) for \( \eta = 8 \), gives \( \Delta E \sim 2.5 \times 10^{-3} \) which is about five times the value indicated in Fig. 26. This indicates that inertial modes are likely to be significant under conditions of pseudo-synchronization for \( \eta = 8 \). Similar estimates indicate that is also the case for \( \eta > 8 \).

In support of the above discussion, we comment that the excitation of inertial waves is seen in our simulation of the prograde encounter of model 1b with \( \eta = 8 \) and \( \Omega = 0.21 \). To illustrate this, contours of the product of the Lagrangian displacement in the \( \phi \) direction and \( \sqrt{\rho} \) are shown in an upper quadrant at a time after the encounter is over and the energy transfer has been completed in Fig. 27. The square of this quantity is proportional to the kinetic energy density of the disturbance associated with motion in the \( \phi \) direction. The presence of rotationally modified \( g \) modes is evident in the radiative core. Inertial waves are seen in the convective envelope. These show some reflections and graze the boundary with the radiative core as would be expected for critical latitude phenomena (see Papaloizou & Ivanov 2010).

When \( \Omega = 0.42 \), the agreement between two approaches is less good. The difference between \( \Delta E \) is typically a factor of 2 or for the prograde encounters and a factor of 3 for retrograde encounters at larger \( \eta \). The fact that such retrograde encounters give larger disagreement can be explained by the shift of the eigenfrequencies of the dominant excited modes as viewed in the rotating frame. These propagate against the sense of rotation of the star towards larger absolute values as \( \eta \) increases. In this case, both the traditional and Cowling approximations used in our normal mode approach become less appropriate.

In Fig. 28, we plot the amount of angular momentum transferred, \( \Delta L \), as a function of \( \eta \). For prograde encounters, in contrast to \( \Delta \), \( \Delta L \) changes sign at a value of \( \eta \), where the star rotates at the pseudo-synchronization rate. Accordingly, we plot the absolute value of \( \Delta L \) in this figure. The form of \( |\Delta L| \) for prograde encounters is non-monotonic, having a deep minimum for \( \eta \equiv \eta_1 \), where \( \Omega = \Omega_{\text{ps}} \). When \( \eta < \eta_1 \), \( \Delta \) is positive (i.e. directed in the sense of stellar rotation), on the other hand when \( \eta > \eta_1 \), \( \Delta \) is negative. In the case of retrograde encounters, \( \Delta \) is always positive (i.e. directed in the sense of the orbital motion). The behaviour of the deviation between the direct numerical and normal mode approaches is similar to that found for \( \Delta E \). We see again better agreement for prograde encounters, with the exception of the encounter having \( \Omega = 0.21 \) and \( \eta = 8 \), where the deviation is rather large. This may be explained as before.

Overall, our results indicate quantitative agreement between the two approaches when the angular frequency is relatively small, say \( \Omega \leq 0.2 \) except for rotation rates close to \( \Omega_{\text{ps}} \). For faster rotators, the agreement is not so good with the direct numerical approach giving values of \( \Delta E \) that are a factor of 2–3 smaller for retrograde encounters and a factor of 2–3 larger for prograde encounters as long as \( \eta < 8 \). These discrepancies probably arise from the neglect of inertial waves in the normal mode treatment as well as use of the traditional approximation and the neglect of self-gravity. Excitation of inertial waves would cause \( \Delta E \) to increase near to pseudo-synchronization while the neglect of self-gravity and the use of the traditional approximation become less appropriate for modes excited at the high relative forcing frequencies that occurs for large retrograde stellar rotation.

In Fig. 29, we plot the energy transfer in the inertial frame, \( \Delta E_i \) related to \( \Delta E \) and \( \Delta L \) through equation (7). As for the angular momentum transfer, it is negative for prograde encounters with \( \eta > \eta_2 \), where \( \eta_2 \sim \eta_1 \), and, therefore, absolute values are plotted. The energy exchanged for prograde encounters has a sharp minimum at \( \eta = \eta_2 \), which moves towards larger values of \( \eta \) as the magnitude of \( \Omega \) decreases. Note that \( \Delta E_i \) is always positive for retrograde encounters. Let us recall that solid, dashed, dot-dashed and dotted curves apply to \( |\Omega| = 0.42, 0.21, 0.11 \) and 0, respectively. Together with these results we also show the energy transfer calculated adopting the ‘perturbative’ approach where it is assumed that the overlap integrals are not modified by rotation and the eigenfrequencies can be calculated using equation (6). The respective curves are represented by dot–dot–dashed lines. Symbols on these lines show different values of \( |\Omega| \) with circles, squares and diamonds corresponding to \( |\Omega| = 0.42, 0.21, 0.11 \), respectively. Remarkably, the perturbative approach agrees quite well with the one based on the traditional approximation, especially for retrograde encounters, even for the largest value of \( |\Omega| = 0.42 \) adopted. In the case of prograde encounters, there is quantitative agreement when \( \eta < \eta_2 \), and, accordingly, \( \Delta E_i > 0 \). Since the perturbative approach does not require the calculation of the overlap integrals for every given value of \( \Omega \), the evaluation of \( \Delta E_i \) is simplified to a great extent. It suffices to use the overlap integrals obtained for non-rotating stars together with the frequency splitting coefficients, \( \beta_\nu \), given above for a number of stellar models (see e.g. Ivanov & Papaloizou 2011).

In Fig. 30, we plot the characteristic time-scale of evolution of the semimajor axis given by equation (10) as a function of \( P_{\text{orb}} \). The line styles are as for Fig. 26. It is seen that for a given value of \( P_{\text{orb}} \), retrograde encounters have smaller values of \( T_{\text{e}} \) than the corresponding prograde encounter. One can see, that for fast rotators, rotation has a significant influence on the strength of tidal encounters (see also Lai 1997 and Ivanov & Papaloizou 2011). For example, when \( |\Omega| = 0.11 \), the binary system may significantly change its semimajor axis in less than \( 10^6 \) yr for \( P_{\text{orb}} < 2.7 \)

\[^{7}\text{This corresponds to a rotation period of the star of approximately one day.}\]
Figure 31. The capture radius $R_{\text{cap}}$ calculated according to equation (12) for model 1b as a function of a ‘typical’ relative velocity $v_\ast$. Curves of a given style apply to the same rotation rates as in Fig. 26, with larger values of $R_{\text{cap}}$ for a given value of $v_\ast$ corresponding to retrograde encounters.

and $<4$ d for prograde and retrograde encounters, respectively. Although stellar rotation significantly slows down in time this effect may contribute to explaining observed exoplanetary systems containing Hot Jupiters with a significant mismatch between directions of their orbital angular momentum and the rotation axis of their central stars.

In Fig. 31, we show the tidal capture radius $R_{\text{cap}}$ for the rotating 1b model. As seen from this figure, the value of $R_{\text{cap}}$ is larger for retrograde encounters as compared to prograde encounters as was first noted by Lai (1997). However, this effect is prominent only when either, the rotation rate is quite large, or the characteristic relative velocity, $v_\ast$, is small. It is instructive to compare the dependence of $R_{\text{cap}}$ on the rotation rate found here with results of Lai (1997), bearing in mind, however, that in that work, a tidal encounter of an $n=1.5$ polytrope having $M_\ast=0.4\,M_\odot$ and $R_\ast=0.5\,R_\odot$ with a point mass with $m_\odot=1.5\,M_\odot$ was considered. When $v_\text{rel}=2.5$ km s$^{-1}$, we find $R_{\text{cap}} \approx 2.7R_\ast$ and $4.2R_\ast$ for prograde and retrograde encounters with the rather large value, $\Omega=0.42$. On the other hand, Lai (1997) gives values of $R_{\text{cap}} \approx 4.2R_\ast$ and $6.4R_\ast$ for the same encounter, but with $\Omega=0.6$. The ratio of the radii is approximately 0.65 in both cases. Since the rotation rate of our model is smaller, the relative variation of the capture radius produced by changing from retrograde to prograde rotation is somewhat larger in models of stars with a realistic Sun-like structure as expected.

Finally, in Figs 32–35, we plot the same quantities as in Figs 26, 28, 30 and 31, but for the rotating model 1.5a. The absolute values of $\Omega$ are $|\Omega|=0.5, 0.25$ and 0. The results behave in a similar way to those found for the rotating Sun-like star, with the difference that the transfers of energy and angular momentum for a given value of $\eta$ are smaller for the more massive star, as is found in the non-rotating case. The evolution time-scales $T_{\text{ev}}$ are, however, larger for the Sun-like star due to its larger average density. For example, model 1.5a with $\Omega=0.25$ gives an evolution time-scale of less than $10^9$ yr for retrograde encounters, when the orbital period after circularisation $P_{\text{orb}}<12$ d. Of course, as in the previous case, in order to make realistic calculations, one must take into account the evolution of the stellar structure as it affects the mean density of the star, as well as the braking of the stellar rotation with age.

### 5 Conclusions and Discussion

In this paper, we have calculated the energy and angular momentum transferred to a number of Population I stellar models with different masses, ages and states of rotation through dynamical tides, as a result of an encounter with a companion on a parabolic orbit. The results were used to estimate the initial evolution time-scale of the semimajor axis of a highly eccentric orbit. Complementary methods based on the calculation of the normal mode response and a direct numerical approach involving the solution of the encounter problem as an initial value problem were used. These showed quantitative agreement for small and moderate rotation rates $|\Omega|<0.2$ as long as...
The effect of rotation was found to play an important role, with dynamic tides being significantly amplified for retrograde encounters, and weakened for prograde ones, see also Lai (1997) and Ivanov & Papaloizou (2011).

We studied the effect of rotation using the direct numerical approach, the normal mode approach adopting the traditional approximation and also simply treating the effects of rotation by perturbation theory. In the latter treatment, it was assumed that the overlap integrals are not modified by rotation but that eigenfrequencies are shifted by an amount proportional to the product of the splitting coefficient \( \beta \) and rotation frequency \( \Omega \), as expected from first-order perturbation theory.

It was shown that the perturbative approach gives results in quantitative agreement with the treatment based on the normal mode approach with the traditional approximation, even for fast rotators, as long as the energy transferred in the inertial frame \( \Delta E_i \) was positive, being approximately equivalent to the condition that the star rotated at less than the pseudo-synchronization rate.

As implied by our discussion in Section 4.5, this is also the condition for the forcing frequencies to be large enough that the excitation of inertial modes is not expected to play an important role. A condition for this to apply for prograde encounters can be approximately found by requiring that the characteristic forcing frequency, \( \Omega_\ast/\eta \), exceed 2\( \Omega \). Making use of equation (8), we obtain the condition as

\[
P_{\text{orb}} < \frac{\Omega_\ast}{6.1 \sqrt{\rho \Omega^2}}.
\]

(15)

Noting that for a typical rotation period of a T Tauri star of 6 d and solar parameters, equation (15) gives \( P_{\text{orb}} \sim 8.4 \) d, the implication is that the excitation of stellar inertial modes do not play a significant role in the tidal capture of hot Jupiters into final prograde orbits with periods of a few days.

This simplifies applications of the theory to particular systems since use of the perturbative approach requires only the calculation of the overlap integrals, and the splitting coefficients for a given non-rotating stellar model. These are provided for all models considered in this paper. In addition to the methods described above, we also applied the purely analytic approach developed in Paper 1 to the solar model 1b, and showed that it gave results differing from those obtained numerically by at most 40 per cent in the range 2 < \( \eta \) < 30. It is important to stress again that the energy and angular momentum transfers are significantly larger for this model as compared to models of more massive stars and a ‘reference’ model of \( n = 3 \) polytrope.

The stellar mean density plays an important role in applications to particular astrophysical systems. This is because tides become relatively more efficient for radially extended low-density objects. In particular, time-scale \( T_{\text{ev}} \) for the evolution of the semimajor axis, considered as a function of the orbital period after the period of circularization \( P_{\text{orb}} \), becomes smaller for more rarefied evolved massive models regardless of the fact that other effects, such as possessing a smaller convective envelope, act in the direction of
making tidal interaction less efficient as discussed above. Thus, $T_{eq}$ is less than $10^9$ yr for non-rotating models 1b and 1.5c when $P_{orb} < 3.3$ and $< 4.3$ d, respectively. For fast rotators, this time can be significantly reduced for retrograde encounters. Thus, in the case of a 1.5a model rotating with the angular frequency $\Omega = 0.25 T_{eq} < 10^9$ yr when $P_{orb} < 12$ d.

When the theory of dynamic tides is applied to particular astrophysical processes, such as the tidal circularization of exoplanet orbits starting with a high eccentricity induced by gravitational scattering, or the process of tidal capture of stars in stellar clusters, it is important to calculate the overlap integrals and the coefficients $\beta$, for a grid of stellar models of different ages. It is also important to understand the evolution of the stellar structure, and rotation rate as a function of time. The outcome of tidal evolution may differ significantly for stars with different masses and different rotational history.

As discussed in Paper 1, the overlap integrals are also important for discussing the tidal evolution of binaries with small orbital eccentricity. In particular, for forcing frequencies large enough that inertial modes are not expected to be excited, they fully determine the effect of tidal interactions in the regime of ‘moderately large viscosity’, see e.g. Goodman & Dickson 1998 and Paper 1. The results of Zahn (1977) can only be recovered only when the overlap integrals are $\propto \Omega^{17/6}$. This dependence approximately holds for the Sun-like stars in the limit of $\omega \to 0$. It is determined by the functional form of the square of the Brunt–Väisälä frequency in the neighbourhood of the transition from radiative to convective regions. In particular, the $\Omega^{17/6}$ dependence of the overlap integrals requires that, in the neighbourhood of the transition, the square of the Brunt–Väisälä frequency is a linear function of the difference between a given radius and the radius of the transition. This assumption may not be valid for a massive star, where the transition from the radiative envelope to the convective core can be extremely sharp. In particular, the Zahn (1977) theory does not apply to binaries with ultrashort periods $P_{orb} \sim 10 \Omega^{-1}$, where the overlap integrals decrease much faster with $\omega$, see the discussion of Figs 7–10 in the text. A theory appropriate for large orbital periods must consider the origin of, and take into account, possible rapid variations of the Brunt–Väisälä frequency in the neighbourhood of the convective to radiative transition. This is left for future work.

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