Parameterized discrete uniformization theorems and curvature flows for polyhedral surfaces, I

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Abstract

In this paper, we introduce a parameterized discrete curvature (α-curvature) on polyhedral surfaces, which is a generalization of the classical discrete curvature. A discrete uniformization theorem is established for the parameterized discrete curvature, which generalizes the discrete uniformization theorem obtained by Gu-Luo-Sun-Wu [38]. We also prove the global rigidity of parameterized discrete curvature with respect to the discrete conformal factors, which confirms a generalized Luo conjecture [45] on rigidity of discrete curvatures. We further introduce a parameterized discrete Yamabe flow for piecewise linear metrics on surfaces. To handle the possible singularities along the flow, we do surgery on the flow by flipping. Then we prove that the flow with surgery converges to a piecewise linear metric with constant discrete α-curvature, which confirms another generalized Luo conjecture [45] on convergence of discrete Yamabe flow with surgery. We also introduce a parameterized discrete Calabi flow and prove the convergence of the flow with surgery, which generalizes the convergence result proved in [70].

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1 Introduction

To study the conformal geometry of piecewise linear metrics on manifolds, Luo [45] and Röcek-Williams [54] independently introduced a discrete conformality for piecewise linear metrics (Euclidean polyhedral metrics), which is now called vertex scaling. Luo [45] further introduced the combinatorial Yamabe flow for piecewise linear metrics on triangulated surfaces and obtained the combinatorial obstruction for the existence of piecewise linear metrics with constant combinatorial curvature. Bobenko-Pinkall-Springborn [3] proved
the global rigidity of vertex scaling and obtained the relationship between the vertex scaling and the geometry of ideal polyhedra in hyperbolic three space. They further introduced vertex scaling for piecewise hyperbolic metrics on triangulated surfaces. Based on Bobenko-Pinkall-Springborn’s work [3] and Penner’s work [51], Gu-Luo-Sun-Wu [38] recently proved a discrete uniformization theorem for piecewise linear metrics on surfaces via a variational principle established by Luo in [45]. Similar discrete uniformization theorem was established by Gu-Guo-Luo-Sun-Wu [37] for piecewise hyperbolic metrics on surfaces. Combinatorial Yamabe flow with surgery for polyhedral metrics were defined in [37, 38], where the long time existence and convergence of the combinatorial Yamabe flow with surgery were proved. Following Luo’s approach, Ge [15, 16] introduced the combinatorial Calabi flow on surfaces. Recently, Zhu and the author [70] proved the long-time existence and convergence of the combinatorial Calabi flow with surgery for vertex scaling of piecewise linear and piecewise hyperbolic metrics on surfaces. Other related work on vertex scaling could be found in [39, 41, 48, 49, 55, 56, 58–61, 66].

In this paper, we introduce a parameterized discrete curvature (combinatorial $\alpha$-curvature) for piecewise linear metrics with respect to the vertex scaling on surfaces, which is a generalization of the classical discrete curvature. We prove the global rigidity and a discrete uniformization theorem for this curvature. We also study the properties of the corresponding discrete curvature flows. The combinatorial $\alpha$-curvature of piecewise hyperbolic metrics on surfaces is studied in [67]. Combinatorial $\alpha$-curvature was introduced by Ge and the author [28, 30] for Thurston’s circle packing metrics as a generalization of the classical combinatorial curvature. There are lots of works on combinatorial curvatures and combinatorial curvature flows on surfaces and 3-dimensional manifolds, see [8, 11–18, 20–30, 32–36, 40, 46, 50, 57, 62–65, 67–69] and others.

Suppose $S$ is a closed connected surface and $V$ is a finite subset of $S$, $(S, V)$ is called a marked surface. A piecewise linear metric (PL metric) on $(S, V)$ is a flat cone metric with cone points contained in $V$. Suppose $T = (V, E, F)$ is a triangulation of $(S, V)$, where $V, E, F$ represent the set of vertices, edges and faces respectively. We use $(S, V, T)$ to denote a triangulated surface. If a map $d : E \rightarrow (0, +\infty)$ satisfies that $d_{rs} < d_{rt} + d_{st}$ for $\{r, s, t\} = \{i, j, k\}$, where $d_{rs} = d(\{rs\})$ and $\{i, j, k\}$ is any triangle in $F$, then $d$ determines a PL metric on $(S, V)$, which is still denoted by $d$. Given $(S, V)$ with a triangulation $T$ and a map $d : E \rightarrow (0, +\infty)$ determined by a PL metric $d$ on $(S, V)$, the vertex scaling [45, 54] of the PL metric $d$ by a function $w : V \rightarrow (0, +\infty)$ is defined to be the PL metric $w \ast d$ on $(S, V)$ determined by the map $w \ast d : E \rightarrow (0, +\infty)$ with

$$(w \ast d)_{ij} := w_i w_j d_{ij}, \quad \forall \{ij\} \in E.$$  

The function $w : V \rightarrow (0, +\infty)$ is called a conformal factor. For a triangulated surface $(S, V, T)$ with a PL metric $d$, we denote the admissible space of conformal factors by
$\Omega^T(d)$, which is the set of discrete conformal factors such that the triangle inequalities are satisfied for every face in $T$. Set $u_i = \ln w_i$, $i = 1, \cdots, N$, and $U^T(d) = \ln \Omega^T(d)$. Here $N = |V|$.

Suppose $(S, V)$ is a marked surface with a PL metric $d$. The combinatorial curvature $K_i$ of $d$ at $v_i \in V$ is $2\pi$ less the cone angle at $v_i$. If $T$ is a geometric triangulation of $(S, V)$ with a PL metric $d$, we denote $\theta^T_i$ as the inner angle at the vertex $v_i$ of the triangle $\triangle ijk$. Then the combinatorial curvature $K_i = 2\pi - \sum_{\triangle ijk \in F} \theta^T_i$. Note that the combinatorial curvature $K$ is independent of the geometric triangulations of $(S, V)$ with a PL metric $d$.

**Definition 1.1.** Suppose $(S, V, T)$ is a triangulated surface with a PL metric $d$ and $w : V \to (0, +\infty)$ is a conformal factor of $d$ on $(S, V, T)$. For any $\alpha \in \mathbb{R}$, the combinatorial $\alpha$-curvature of $w \ast d$ on $(S, V, T)$ is defined to be

$$R_{\alpha,i} = \frac{K_i}{w_i^\alpha}. \quad (1.1)$$

In the case that $\alpha = 0$, the curvature $R_0$ is the classical combinatorial curvature $K$. Furthermore, for any constant $\lambda > 0$, we have $R_{\alpha,i}(\lambda \ast l) = \lambda^{-\alpha} R_{\alpha,i}(l)$. Especially, for $\alpha = 1$, we have $R_{1,i}(\lambda \ast l) = \lambda^{-1} R_{1,i}(l)$, which is parallelling to the transformation of smooth Gaussian curvature $K_{\lambda g} = \lambda^{-1} K_g$.

For combinatorial $\alpha$-curvature of PL metrics on triangulated surfaces, we have the following global rigidity, which confirms a generalized Luo conjecture [45] on rigidity of discrete curvatures.

**Theorem 1.2.** Suppose $(S, V, T)$ is a triangulated closed surface with a PL metric $d$ and $\alpha \in \mathbb{R}$ is a constant such that $\alpha \chi(S) \leq 0$. $R$ is a given function defined on $V$.

1. If $\alpha R \equiv 0$, then there exists at most one conformal factor $\bar{w} \in \Omega^T(d)$ with $\alpha$-curvature $\bar{R}$ up to scaling;
2. If $\alpha R \leq 0$ and $\alpha R \neq 0$, then there exists at most one conformal factor $\bar{w} \in \Omega^T(d)$ with $\alpha$-curvature $\bar{R}$.

If $\alpha = 0$, there is no restriction on $R_0 = K$ and the global rigidity of $R_0 = K$ in Theorem 1.2 is reduced to the rigidity proved in [3, 45].

For $\alpha$-curvature, it is interesting to consider the following Yamabe problem.

**Combinatorial $\alpha$-Yamabe Problem:** Suppose $(S, V, T)$ is a closed triangulated surface with a PL metric $d$, does there exist any conformal factor $w : V \to (0, +\infty)$ in $\Omega^T(d)$ such that $w \ast d$ has constant $\alpha$-curvature?

In this paper, we prove the following result on combinatorial $\alpha$-Yamabe problem, which is also called a parameterized discrete uniformization theorem.
**Theorem 1.3.** Suppose \((S, V)\) is a closed connected marked surface with a PL metric \(d_0\) and \(\alpha \in \mathbb{R}\) is a constant such that \(\alpha \chi(S) \leq 0\). Then there exists a PL metric in the conformal class \(\mathcal{D}(d_0)\) with constant \(\alpha\)-curvature.

Here \(\mathcal{D}(d_0)\) is the discrete conformal class defined in the sense of Gu-Luo-Sun-Wu [38]. Please refer to Definition 3.1 for this. Theorem 1.3 is a parameterized generalization of Gu-Luo-Sun-Wu’s discrete uniformization theorem in [38]. Especially, if \(\alpha = 0\), Theorem 1.3 is the discrete uniformization theorem proved in [38]. To study the combinatorial \(\alpha\)-Yamabe problem, we introduce the combinatorial \(\alpha\)-Yamabe flow and combinatorial \(\alpha\)-Calabi flow for PL metrics on surfaces.

**Definition 1.4.** Suppose \((S, V, \mathcal{T})\) is a triangulated closed surface with a PL metric \(d_0\) and \(\alpha \in \mathbb{R}\) is a constant. The normalized combinatorial \(\alpha\)-Yamabe flow is defined to be

\[
\begin{align*}
\frac{dw_i}{dt} &= (R_{\alpha, av} - R_{\alpha, i}) w_i, \\
\frac{w_i(0)}{} &= 1, \\
\end{align*}
\]

(1.2)

where \(R_{\alpha, av} = \frac{2\pi \chi(M)}{\sum_{i=1}^{N} w_i^\alpha}\).

When \(\alpha = 0\), this is the combinatorial Yamabe flow introduced by Luo [45]. By direct calculations, the combinatorial \(\alpha\)-curvature \(R_{\alpha}\) evolves according to

\[
\frac{dR_{\alpha, i}}{dt} = (\Delta^T_{\alpha} R_{\alpha})_i + \alpha R_{\alpha, i} (R_{\alpha, i} - R_{\alpha, av})
\]

(1.3)

along the combinatorial \(\alpha\)-Yamabe flow (1.2), where the \(\alpha\)-Laplace operator \(\Delta^T_{\alpha}\) on \((S, V, \mathcal{T})\) is defined to be

\[
(\Delta^T_{\alpha} f)_i = \frac{1}{w_i^\alpha} \sum_{j \sim i} \left( \cot \theta_{ij}^k + \cot \theta_{ij}^l \right) (f_j - f_i)
\]

for \(f \in \mathbb{R}^V\). Here \(\theta_{ij}^k\) and \(\theta_{ij}^l\) are two inner angles facing the edge \(\{ij\}\). (1.3) is similar to the evolution of Gaussian curvature along the normalized Ricci flow on surfaces [7,44].

**Definition 1.5.** Suppose \((S, V, \mathcal{T})\) is a closed triangulated surface with a PL metric \(d_0\) and \(\alpha \in \mathbb{R}\) is a constant. The combinatorial \(\alpha\)-Calabi flow is defined to be

\[
\begin{align*}
\frac{dw_i}{dt} &= (\Delta^T_{\alpha} R_{\alpha})_i w_i, \\
\frac{w_i(0)}{} &= 1. \\
\end{align*}
\]

(1.4)

When \(\alpha = 0\), this is the combinatorial Calabi flow introduced by Ge [15]. By direct calculations, the combinatorial \(\alpha\)-curvature \(R_{\alpha}\) evolves according to

\[
\frac{dR_{\alpha, i}}{dt} = -(\Delta^T_{\alpha})^2 R_{\alpha, i} - \alpha R_{\alpha, i} (\Delta^T_{\alpha} R_{\alpha})_i
\]

(1.5)
along the combinatorial $\alpha$-Calabi flow (1.4), which is similar to the evolution of Gaussian curvature along the surface Calabi flow [5][6][9]. If the parameters are chosen properly, the evolution equations of combinatorial $\alpha$-curvature along the combinatorial $\alpha$-Yamabe flow and $\alpha$-Calabi flow are formally the same as the evolution equations of Gaussian curvature along the surface Ricci flow and surface Calabi flow respectively. Please refer to [30] for this.

The combinatorial $\alpha$-flows (combinatorial $\alpha$-Yamabe flow and combinatorial $\alpha$-Calabi flow) may develop singularities. To handle the possible singularities along the combinatorial $\alpha$-flows, we do surgery on the flows by flipping, the idea of which comes from [37,38,45]. Note that the weight in the $\alpha$-Laplace operator is $\omega_{ij} = \cot \theta_{ik} + \cot \theta_{il} w_{ij}^\alpha$, which is not symmetric with the indices $i$ and $j$. To ensure that the discrete $\alpha$-Laplace operator have good properties along the $\alpha$-flows, especially the discrete maximal principle could be applied on the combinatorial $\alpha$-Yamabe flow, we need the weight $\omega_{ij}$ to be nonnegative on every edge, which is equivalent to $\theta_{ijk}^2 + \theta_{ijl}^2 \leq \pi$ for every edge $\{ij\} \in E$. This is exactly the Delaunay condition on the triangulation [4]. This condition is imposed on both the combinatorial $\alpha$-Yamabe flow and the combinatorial $\alpha$-Calabi flow. Note that every PL metric on $(S,V)$ admits at least one Delaunay triangulation [1,4,53], so this additional condition is reasonable. Along the $\alpha$-flows on $(S,V)$ with a triangulation $T$, if $T$ is Delaunay in $w(t) * d_0$ for $t \in [0,T]$ and not Delaunay in $w(t) * d_0$ for $t \in (T,T + \epsilon)$, $\epsilon > 0$, there exists an edge $\{ij\} \in E$ such that $\theta_{ijk}^2(t) + \theta_{ijl}^2(t) \leq \pi$ for $t \in [0,T]$ and $\theta_{ijk}^2(t) + \theta_{ijl}^2(t) > \pi$ for $t \in (T,T + \epsilon)$. Then we replace the triangulation $T$ by a new triangulation $T'$ at time $t = T$ via replacing two triangles $\triangle ijk$ and $\triangle ij l$ adjacent to $\{ij\}$ by two new triangles $\triangle ik l$ and $\triangle jkl$. This is called a surgery by flipping on the triangulation $T$, which is also an isometry of $(S,V)$ with PL metric $w(T) * d_0$. After the surgery at time $t = T$, we run the $\alpha$-flows on $(S,V,T')$ with initial metric coming from the corresponding $\alpha$-flow on $(S,V,T)$ at time $t = T$.

We prove the following result on combinatorial $\alpha$-Yamabe flow and $\alpha$-Calabi flow with surgery.

Theorem 1.6. Suppose $(S,V)$ is a closed connected marked surface with a PL metric $d_0$ and $\alpha \in \mathbb{R}$ is a constant with $\alpha \chi(S) \leq 0$. Then there exists a PL metric in the conformal class $D(d_0)$ with constant combinatorial $\alpha$-curvature if and only if one of the following two conditions is satisfied:

1. The combinatorial $\alpha$-Yamabe flow with surgery exists for all time and converges exponentially fast to a PL metric $d^*$ with constant combinatorial $\alpha$-curvature;

2. The combinatorial $\alpha$-Calabi flow with surgery exists for all time and converges exponentially fast to a PL metric $d^*$ with constant combinatorial $\alpha$-curvature.
Applying the discrete maximal principle to the combinatorial \(\alpha\)-Yamabe flow with surgery suitably, we further have the following result.

**Theorem 1.7.** Suppose \((S, V)\) is a closed connected marked surface with a PL metric \(d_0\) and \(\alpha \in \mathbb{R}\) is a constant with \(\alpha \chi(S) \leq 0\). Then the combinatorial \(\alpha\)-Yamabe flow with surgery and the combinatorial \(\alpha\)-Calabi flow with surgery exist for all time and converge exponentially fast to a PL metric \(d^\ast\) with constant combinatorial \(\alpha\)-curvature.

Theorem 1.7 confirms another generalized Luo conjecture \([45]\) on the convergence of the combinatorial Yamabe flow with surgery. When \(\alpha = 0\), the convergence of combinatorial Yamabe flow with surgery was proved in \([38]\) and the convergence of combinatorial Calabi flow with surgery was proved in \([70]\).

The paper is organized as follows. In Section 2, we prove Theorem 1.2 and study the stability of combinatorial \(\alpha\)-flows on triangulated surfaces. In Section 3, we prove Theorem 1.3, Theorem 1.6 and Theorem 1.7 based on the discrete conformal theory established in \([38]\).

### 2 \(\alpha\)-curvature and \(\alpha\)-flows on triangulated surfaces

#### 2.1 Rigidity of \(\alpha\)-curvature on triangulated surfaces

Suppose \((S, V, T)\) is a triangulated surface with a PL metric \(d\) and \(w : V \to (0, +\infty)\) is a positive function defined on \(V\). Set \(h : \mathbb{R}^n_{>0} \to \mathbb{R}^n\) be the homeomorphism defined by \(u_i = h(w_i) = \ln w_i\). Then \(w\) is a conformal factor of \(d\) on \((S, V, T)\) if and only if \(u : V \to \mathbb{R}\) is in the following space

\[
\mathcal{U}^T_{ijk}(d) \triangleq \{(u_i, u_j, u_k) \in \mathbb{R}^3 \mid \frac{d_{rs}}{e^{u_t}} + \frac{d_{rt}}{e^{u_s}} > \frac{d_{st}}{e^{u_r}}, \{r, s, t\} = \{i, j, k\}\}
\]

for every triangle \(\triangle ijk \in F\). It is observed by Luo \([45]\) that the non-convex simply connected space \(\mathcal{U}^T_{ijk}(d)\) is the image of the convex space \(\{(w_i^{-1}, w_j^{-1}, w_k^{-1}) \in \mathbb{R}^3_{>0} \mid (\frac{d_{rs}}{w_t}, \frac{d_{rt}}{w_s}, \frac{d_{st}}{w_r}) \in \Delta\}\) under the homeomorphism \(-h\), where \(\Delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3_{>0} \mid x_i + x_j > x_k, \text{where } i, j, k \text{ are distinct}\}\). For a nondegenerate triangle \(\triangle ijk\) in \((S, V, T)\) with a PL metric \(d\), we denote the inner angle at \(i\) as \(\theta_i\) for simplicity. Luo \([45]\) proved the following lemma for a triangle.

**Lemma 2.1.** The \(3 \times 3\) matrix \([\frac{\partial w}{\partial u_i}]_{3 \times 3}\) is symmetric, negative semi-definite with null space \(\{(t, t, t) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}\).

Lemma 2.1 and the simply connectness of \(\mathcal{U}^T_{ijk}(d)\) implies that

\[
F_{ijk}(u) = \int_{u_0}^{u} \theta_i du_i + \theta_j du_j + \theta_k du_k
\]
is well-defined on $\mathcal{U}_{ijk}(d)$. Furthermore, $F_{ijk}(u)$ is locally concave on $\mathcal{U}_{ijk}(d)$ and locally strictly concave on $\mathcal{U}_{ijk}(d) \cap \{ u_i + u_j + u_k = c \}$. Bobenko-Pinkall-Springborn [3] obtained the explicit formula of $F_{ijk}$ using Milnor’s Lobachevsky function and extended $F_{ijk}$ to be a globally concave function $\tilde{F}_{ijk}$ defined on $\mathbb{R}^3$. See also [68]. Luo [46] studied Bobenko-Pinkall-Springborn’s extension and obtained a general extension method for similar problems without involving Milnor’s Lobachevsky function, which has lots of applications (see [46, 47, 62, 63] for example). Here we take Luo’s approach.

**Lemma 2.2** ([46]). Let $l_1, l_2, l_3$ and $\theta_1, \theta_2, \theta_3$ be the edge lengths and inner angles of a triangle $\triangle$ in $\mathbb{E}^2$, or $\mathbb{H}^2$, or $\mathbb{S}^2$ so that the $l_i$-th edge is opposite to the angle $\theta_i$. Consider $\theta_i = \theta_i(l)$ as a function of $l = (l_1, l_2, l_3)$.

1. If $\triangle$ is Euclidean or hyperbolic, the angle function $\theta_i$ defined on
   \[ \Omega = \{(l_1, l_2, l_3) \in \mathbb{R}^3 | l_1 + l_2 > l_3, l_1 + l_3 > l_2, l_2 + l_3 > l_1 \} \]
   can be extended continuously by constant functions to a function $\tilde{\theta}_i$ on $\mathbb{R}^3 > 0$.
2. If $\triangle$ is spherical, the angle function $\theta_i$ defined on
   \[ \Omega = \{(l_1, l_2, l_3) \in \mathbb{R}^3 | l_1 + l_2 > l_3, l_1 + l_3 > l_2, l_2 + l_3 > l_1, l_1 + l_2 + l_3 < 2\pi \} \]
   can be extended continuously by constant functions to a function $\tilde{\theta}_i$ on $(0, \pi)^3$.

Before going on, we recall the following result of Luo in [46].

**Definition 2.3.** A differential 1-form $w = \sum_{i=1}^{n} a_i(x)dx^i$ in an open set $U \subset \mathbb{R}^n$ is said to be continuous if each $a_i(x)$ is continuous on $U$. A continuous differential 1-form $w$ is called closed if $\int_{\partial \tau} w = 0$ for each triangle $\tau \subset U$.

**Theorem 2.4** ([46], Corollary 2.6). Suppose $X \subset \mathbb{R}^n$ is an open convex set and $A \subset X$ is an open subset of $X$ bounded by a $C^1$ smooth codimension-1 submanifold in $X$. If $w = \sum_{i=1}^{n} a_i(x)dx_i$ is a continuous closed 1-form on $A$ so that $F(x) = \int_a^x w$ is locally convex on $A$ and each $a_i$ can be extended continuous to $X$ by constant functions to a function $\tilde{a}_i$ on $X$, then $\tilde{F}(x) = \int_a^x \sum_{i=1}^{n} \tilde{a}_i(x)dx_i$ is a $C^1$-smooth convex function on $X$ extending $F$.

Using Lemma 2.2 and Theorem 2.4 we have

**Lemma 2.5** ([3, 46]). The function $F_{ijk}(u)$ in (2.2) could be extended to be a $C^1$-smooth concave function
   \[ \tilde{F}_{ijk}(u) = \int_{u_0}^u \tilde{\theta}_i du_i + \tilde{\theta}_j du_j + \tilde{\theta}_k du_k \quad (2.3) \]
defined for $u \in \mathbb{R}^3$ where the extension $\tilde{\theta}_i$ of $\theta_i$ by constant is defined to be $\tilde{\theta}_i = \pi$ when $l_{jk} \geq l_{ik} + l_{ij}$ and $\tilde{\theta}_i = 0$ when $l_{ik} \geq l_{jk} + l_{ij}$ or $l_{ij} \geq l_{jk} + l_{ik}$.
Proof of Theorem 1.2: The proof is paralleling to that of Theorem 3.3 in [29]. For completeness, we give the proof here. Suppose \( w_0 \in \Omega^T(d) \) is a conformal factor and \( u_0 = \ln w_0 \). Then we can define the following Ricci energy \( F(u) \) by \( \bar{R} \)

\[
F(u) = - \sum_{\Delta_{ijk} \in F} F_{ijk} + \int_{u_0}^{u} \sum_{i=1}^{N} (2\pi - \bar{R}_i u_i^\alpha) du_i. \tag{2.4}
\]

Note that the function \( F_{ijk} \) is smooth on \( \mathcal{U}^T(d) = h(\Omega^T(d)) \). By direct calculations, we have

\[
\text{Hess}_u F = L - \alpha \begin{pmatrix}
\bar{R}_1 w_1^\alpha \\
\vdots \\
\bar{R}_N w_N^\alpha
\end{pmatrix},
\]

where

\[
L = (L_{ij})_{N \times N} = \frac{\partial(K_1, \ldots, K_N)}{\partial(u_1, \ldots, u_N)} = \begin{pmatrix}
\frac{\partial K_1}{\partial u_1} & \cdots & \frac{\partial K_1}{\partial u_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial K_N}{\partial u_1} & \cdots & \frac{\partial K_N}{\partial u_N}
\end{pmatrix}. \tag{2.5}
\]

The matrix \( L \) has the following property [45].

Lemma 2.6 ([45]). For a triangulated surface \((S, V, T)\) with a PL metric \( d \), the matrix \( L \) is symmetric and positive semi-definite on \( \mathcal{U}^T(d) \) with kernel \( \{ t \mid t \in \mathbb{R} \} \), where \( 1 = (1, \ldots, 1) \).

If \( \alpha \bar{R} \equiv 0 \), then \( \text{Hess}_u F \) is positive semi-definite with kernel \( \{ (t, \ldots, t) \mid t \in \mathbb{R} \} \) and \( F \) is locally convex. If \( \alpha \bar{R} \leq 0 \) and \( \alpha \bar{R} \not\equiv 0 \), then \( \text{Hess}_u F \) is positive definite and \( F \) is locally strictly convex.

By Lemma 2.5, \( F_{ijk} \) defined on \( \mathcal{U}^T_{ijk}(d) \) could be extended to be \( \tilde{F}_{ijk} \) defined by (2.3) on \( \mathbb{R}^3 \hookrightarrow \mathbb{R}^N \). And the second term \( \int_{u_0}^{u} \sum_{i=1}^{N} (2\pi - \bar{R}_i u_i^\alpha) du_i \) in (2.4) can be naturally defined on \( \mathbb{R}^N \), then we have the following extension \( \tilde{F}(u) \) defined on \( \mathbb{R}^N \) of the Ricci energy function \( F(u) \)

\[
\tilde{F}(u) = - \sum_{\Delta_{ijk} \in F} \tilde{F}_{ijk} + \int_{u_0}^{u} \sum_{i=1}^{N} (2\pi - \bar{R}_i u_i^\alpha) du_i.
\]

As \( \tilde{F}_{ijk} \) is \( C^1 \)-smooth concave by Lemma 2.5 and \( \int_{u_0}^{u} \sum_{i=1}^{N} (2\pi - \bar{R}_i u_i^\alpha) du_i \) is a well-defined convex function on \( \mathbb{R}^N \) for \( \alpha \bar{R} \leq 0 \), we have \( \tilde{F}(u) \) is a \( C^1 \)-smooth convex function on \( \mathbb{R}^N \). Furthermore,

\[
\nabla_{u_i} \tilde{F} = - \sum_{\Delta_{ijk} \in F} \tilde{\theta}_i + 2\pi - \bar{R}_i u_i^\alpha = \tilde{K}_i - \bar{R}_i u_i^\alpha,
\]
where $\widetilde{K}_i = 2\pi - \sum_{\triangle ijk \in \mathcal{F}} \tilde{\theta}_i$. Then we have $\widetilde{F}(u)$ is convex on $\mathbb{R}^N$ and locally strictly convex on $\mathcal{U}^T(d) \cap \{ \sum_{i=1}^{N} u_i = 0 \}$ for $\alpha \mathcal{R} \equiv 0$. Similarly, $\widetilde{F}(u)$ is convex on $\mathbb{R}^N$ and locally strictly convex on $\mathcal{U}^T(d)$ for $\alpha \mathcal{R} \leq 0$ and $\alpha \mathcal{R} \not\equiv 0$.

If there are two different conformal factors $\overline{w}_A, \overline{w}_B$ with the same combinatorial $\alpha$-curvature $\overline{\mathcal{R}}$, then $\overline{w}_A = \ln \overline{w}_A \in \mathcal{U}^T(d), \overline{w}_B = \ln \overline{w}_B \in \mathcal{U}^T(d)$ are both critical points of the extended Ricci potential $\overline{F}(u)$. It follows that

$$\nabla \overline{F}(\overline{w}_A) = \nabla \overline{F}(\overline{w}_B) = 0.$$ 

Set

$$f(t) = \overline{F}((1-t)\overline{w}_A + t\overline{w}_B)$$

$$= \sum_{\triangle ijk \in \mathcal{F}} f_{ijk}(t) + \int_{u_0}^{(1-t)\overline{w}_A + t\overline{w}_B} \sum_{i=1}^{N} (2\pi - \mathcal{R}_i w_i^\alpha) du_i,$$

where

$$f_{ijk}(t) = -\overline{F}_{ijk}((1-t)\overline{w}_A + t\overline{w}_B).$$

Then $f(t)$ is a $C^1$-smooth convex function on $[0, 1]$ and $f'(0) = f'(1) = 0$, which implies that $f'(t) \equiv 0$ for $t \in [0, 1]$. Note that $\overline{w}_A$ is in the open set $\mathcal{U}^T(d)$, there exists $\epsilon > 0$ such that $(1-t)\overline{w}_A + t\overline{w}_B \in \mathcal{U}^T(d)$ for $t \in [0, \epsilon]$. Then $f(t)$ is smooth on $[0, \epsilon]$.

In the case of $\alpha \mathcal{R} \leq 0$ and $\alpha \mathcal{R} \not\equiv 0$, the strict convexity of $\overline{F}(u)$ on $\mathcal{U}^T(d)$ implies that $f(t)$ is strictly convex on $[0, \epsilon]$ and $f'(t)$ is a strictly increasing function on $[0, \epsilon]$. Then $f'(0) = 0$ implies $f'(\epsilon) > 0$, which contradicts $f'(t) \equiv 0$ on $[0, 1]$. So there exists at most one conformal factor with combinatorial $\alpha$-curvature $\overline{\mathcal{R}}$.

For the case of $\alpha \mathcal{R} \equiv 0$, we have $f(t)$ is $C^1$ convex on $[0, 1]$ and smooth on $[0, \epsilon]$. $f'(t) \equiv 0$ on $[0, 1]$ implies that $f''(t) \equiv 0$ on $[0, \epsilon]$. Note that, for $t \in [0, \epsilon]$,

$$f''(t) = (\overline{w}_A - \overline{w}_B)L(\overline{w}_A - \overline{w}_B)^T.$$ 

By Lemma 2.6 we have $\overline{w}_A - \overline{w}_B = c1$ for some constant $c \in \mathbb{R}$, which implies that $\overline{w}_A = e^c \overline{w}_B$. So there exists at most one conformal factor with combinatorial $\alpha$-curvature $\overline{\mathcal{R}}$ up to scaling.

Theorem 1.2 has a direct corollary.

**Corollary 2.7.** Suppose $(S, V, \mathcal{T})$ is a triangulated closed surface with a PL metric $d$ and $\alpha \in \mathbb{R}$ is a constant such that $\alpha \chi(S) \leq 0$. Then there exists at most one $u^* \in \mathcal{U}^T(d)$ such that the PL metric $e^{u^*} \ast d$ has constant combinatorial $\alpha$-curvature (up to scaling for $\alpha \chi(S) = 0$).
2.2 Combinatorial Yamabe flow of $\alpha$-curvature on triangulated surfaces

By direct calculations, we have the following properties of combinatorial $\alpha$-Yamabe flow.

**Lemma 2.8.** If $\alpha = 0$, $\sum_{i=1}^{N} u_i$ is invariant along the normalized combinatorial $\alpha$-Yamabe flow (1.2). If $\alpha \neq 0$, $||w||_{\alpha}^\alpha = \sum_{i=1}^{N} w_i^\alpha$ is invariant along the normalized combinatorial $\alpha$-Yamabe flow (1.2).

For simplicity, we denote the hypersurface invariant along the normalized combinatorial $\alpha$-Yamabe flow (1.2) in Lemma 2.8 as $P$ in the following. Note that the hypersurface $P$ is determined by the initial value $u(0)$.

**Theorem 2.9.** Suppose $d_0$ is a PL metric on a triangulated surface $(S,V,T)$ and $\alpha \in \mathbb{R}$. If the solution of normlized combinatorial $\alpha$-Yamabe flow (1.2) on $(S,V,T)$ converges, then the limit metric is a constant combinatorial $\alpha$-curvature PL metric. Furthermore, suppose there exists a constant combinatorial $\alpha$-curvature PL metric $d^* = e^{u^*} \ast d_0$ on a triangulated surface $(S,V,T)$ with $\alpha \chi(S) \leq 0$, there exists a constant $\delta > 0$ such that if $||R_\alpha(u(0)) - R_\alpha(u^*)|| < \delta$ and $u^* \in P$, then the combinatorial $\alpha$-Yamabe flow (1.2) on $(S,V,T)$ exists for all time and converges exponentially fast to $u^*$.

**Proof.** Suppose $u(t)$ is a solution of the normalized combinatorial $\alpha$-Yamabe flow (1.2). If $u(\infty) = \lim_{t \to +\infty} u(t)$ exists in $U^T(d)$, then we have $R_\alpha(u(\infty)) = \lim_{t \to +\infty} R_\alpha(u(t))$ exists. Furthermore, there exists $\xi_n \in (n, n+1)$ such that $u_i(n+1) - u_i(n) = u_i'(\xi_n) = R_{\alpha,av} - \alpha R_{\alpha,i}(\xi_n) \to 0$,

which implies that $R_{\alpha}(u(\infty)) = R_{av}$ and $u(\infty) \ast d_0$ is a constant $\alpha$-curvature PL metric.

Suppose $u^*$ corresponds to a constant $\alpha$-curvature metric. Set $\Gamma_i(u) = R_{\alpha,av} - \alpha R_{\alpha,i}$. By direct calculations, we have

$$\frac{\partial \Gamma_i}{\partial u_j}_{|u=u^*} = -\frac{1}{w_i^\alpha} \frac{\partial K_i}{\partial u_j} + \alpha R_{\alpha,av}(\delta_{ij} - \frac{w_j^\alpha}{||w||_{\alpha}^\alpha})$$

$$= \alpha R_{\alpha,av} \delta_{ij} - \frac{1}{w_i^\alpha} \frac{\partial K_i}{\partial u_j} + \alpha R_{\alpha,av} \frac{w_i^\alpha w_j^\alpha}{||w||_{\alpha}^\alpha}.$$ 

Set $w^\alpha = (w_1^\alpha, \cdots, w_N^\alpha)^T$ and $\Sigma = diag\{w_1^\alpha, \cdots, w_N^\alpha\}$, then

$$\left(\frac{\partial \Gamma}{\partial u}\right)_{|u=u^*} = \alpha R_{\alpha,av} I - \Sigma^{-\alpha}(L + \alpha R_{\alpha,av} \frac{w^\alpha \cdot (w^\alpha)^T}{||w||_{\alpha}^\alpha})$$

$$= -\Sigma^{-\alpha/2} \left(\Lambda_{\alpha} - \alpha R_{\alpha,av}[I - \frac{w^{\alpha/2} \cdot (w^{\alpha/2})^T}{||w||_{\alpha}^\alpha}]\right) \Sigma^{\alpha/2}, \quad (2.6)$$
where $\Lambda_\alpha = \Sigma^{-\alpha/2}L\Sigma^{-\alpha/2}$. Note that the matrix $I - \frac{\mu_{\alpha/2}}{||\mu||_2^2} \Gamma$ has eigenvalues 1 ($N - 1$ times) and 0 (1 time) and kernel $\{cw^\frac{1}{2}\alpha| c \in \mathbb{R}\}$ and $\Lambda_\alpha$ is positive semi-definite with 1-dimensional kernel $\{cw^\frac{1}{2}\alpha| c \in \mathbb{R}\}$. Then if the first nonzero eigenvalue $\lambda_1(\Lambda_\alpha)$ of $\Lambda_\alpha$ satisfies $\lambda_1(\Lambda_\alpha) > \alpha R_{\alpha,av}$, especially if $\alpha R_{\alpha,av} \leq 0$, we have $(\frac{\partial \Gamma}{\partial u})|_{u=u^*}$ has $N - 1$ negative eigenvalues and a zero eigenvalue with eigenspace $\{(c, c, \cdots, c) \in \mathbb{R}^N | c \in \mathbb{R}\}$.

If $\alpha \neq 0$, set $u_i = e^{-\frac{\alpha}{2}u^*_i}\tilde{u}_i$ and $\Sigma^* = \Sigma|_{u=u^*}$, then $u = (\Sigma^*)^{-\frac{1}{2}}\tilde{u}$ in matrix form. The combinatorial $\alpha$-Yamabe flow (1.2) could be written as

$$\frac{d\tilde{u}}{dt} = (\Sigma^*)^{-\frac{1}{2}}\Gamma((\Sigma^*)^{-\frac{1}{2}}\tilde{u})$$

(2.7) in the variable $\tilde{u}$. Set $\bar{\Gamma}(\tilde{u}) = (\Sigma^*)^{-\frac{1}{2}}\Gamma((\Sigma^*)^{-\frac{1}{2}}\tilde{u})$ and $\bar{u}^* = (\Sigma^*)^{-\frac{1}{2}}u^*$. Then

$$(\frac{\partial \bar{\Gamma}}{\partial \bar{u}})|_{\bar{u}=\bar{u}^*} = (\Sigma^*)^{-\frac{1}{2}} \cdot (\frac{\partial \Gamma}{\partial u})|_{u=u^*} \cdot (\Sigma^*)^{-\frac{1}{2}},$$

which is symmetric by (2.6) and negative semi-definite with kernel $\{c(e^{\frac{\alpha}{2}u_1^*}, \cdots, e^{\frac{\alpha}{2}u_N^*})| c \in \mathbb{R}\}$ under the condition $\alpha R_{\alpha,av} \leq 0$ by the property of $\frac{\partial \Gamma}{\partial u}|_{u=u^*}$. Note that the hypersurface $P$ invariant along the combinatorial $\alpha$-Yamabe flow (1.2) could be written as

$$\sum_{i=1}^N e^{\alpha e^{-\frac{\alpha}{2}u^*_i} \tilde{u}_i} = \sum_{i=1}^N e^{\alpha e^{-\frac{\alpha}{2}u^*_i} \tilde{u}_i(0)}$$

(2.8) in $\tilde{u}$. The normal space of the hypersurface (2.8) in $\tilde{u}$ at $\tilde{u}^*$ is $\{c(e^{\frac{\alpha}{2}u_1^*}, \cdots, e^{\frac{\alpha}{2}u_N^*})| c \in \mathbb{R}\}$, which is exactly the kernel space of $\frac{\partial \bar{\Gamma}}{\partial \bar{u}}|_{\bar{u}=\bar{u}^*}$. Therefore, $\frac{\partial \bar{\Gamma}}{\partial \bar{u}}|_{\bar{u}=\bar{u}^*}$ is negative definite on the tangential space of $P$ at $\bar{u}^*$ and $\bar{u}^*$ is a local attractor of the combinatorial $\alpha$-Yamabe flow (2.7) in $\tilde{u}$. As a result, the local convergence of the solution $\tilde{u}(t)$ of the combinatorial $\alpha$-Yamabe flow (2.7) follows from the Lyapunov stability theorem (52, Chapter 5), from which the local convergence of the solution of the combinatorial $\alpha$-Yamabe flow (1.2) follows. The proof for the case of $\alpha = 0$ is similar.

\[ \square \]

### 2.3 Combinatorial Calabi flow of $\alpha$-curvature on triangulated surfaces

Similar to the combinatorial $\alpha$-Yamabe flow, we have the following properties of combinatorial $\alpha$-Calabi flow.

**Lemma 2.10.** If $\alpha = 0$, $\sum_{i=1}^N u_i$ is invariant along the combinatorial $\alpha$-Calabi flow (1.4). If $\alpha \neq 0$, $||u||_\alpha^\alpha = \sum_{i=1}^N w_i^\alpha$ is invariant along the combinatorial $\alpha$-Calabi flow (1.4).

**Theorem 2.11.** Suppose $d_0$ is a PL metric on a triangulated surface $(S, V, T)$ and $\alpha \in \mathbb{R}$. If the solution of combinatorial $\alpha$-Calabi flow on $(S, V, T)$ converges, then the limit metric...
is a constant combinatorial $\alpha$-curvature PL metric. Furthermore, suppose there exists a constant combinatorial $\alpha$-curvature PL metric $d^* = e^{u^*} * d_0$ on $(S,V,T)$ with $\alpha\chi(S) \leq 0$, there exists a constant $\delta > 0$ such that if $||R_\alpha(u(0)) - R_\alpha(u^*)|| < \delta$ and $u^* \in P$, then the combinatorial $\alpha$-Calabi flow (1.4) on $(S,V,T)$ exists for all time and converges exponentially fast to $u^*$.

**Proof.** The proof of Theorem 2.11 is similar to that of Theorem 2.9, we just give some key calculations. Set $\Gamma_i(u) = (\Delta T R_\alpha)^i$, then

$$\frac{\partial \Gamma_i}{\partial u_j}|_{u=u^*} = -\frac{1}{w_i^2} \sum_{k=1}^N L_{ik} \frac{1}{w_k^2} L_{kj} + \alpha R_{\alpha,av} \frac{1}{w_i^2} L_{ij}.$$ 

In matrix form, we have

$$\left(\frac{\partial \Gamma}{\partial u}\right)|_{u=u^*} = -\Sigma^{-\alpha} L \Sigma^{-\alpha} L + \alpha R_{\alpha,av} \Sigma^{-\alpha} L$$

$$= -\Sigma^{-\alpha/2} \left(\Sigma^{-\alpha/2} L \Sigma^{-\alpha} L \Sigma^{-\alpha/2} - \alpha R_{\alpha,av} \Sigma^{-\alpha/2} L \Sigma^{-\alpha/2}\right) \Sigma^{\alpha/2}.$$ 

If $\alpha\chi(S) \leq 0$, then $\left(\frac{\partial \Gamma}{\partial u}\right)|_{u=u^*}$ has $N - 1$ negative eigenvalue and a zero eigenvalue with 1-dimensional kernel $\{ (c, c, \cdots, c) \in \mathbb{R}^N | c \in \mathbb{R} \}$. The following of the proof is the same as that for Theorem 2.9, we omit the details here. \[\square\]

### 3 $\alpha$-curvature and $\alpha$-flows on discrete Riemann surfaces

Theorem 2.9 and Theorem 2.11 gives the long time existence and convergence of the combinatorial $\alpha$-Yamabe flow (1.2) and combinatorial $\alpha$-Calabi flow (1.4) for initial PL metrics with small initial energy. However, for general initial PL metrics, the combinatorial $\alpha$-Yamabe flow and combinatorial $\alpha$-Calabi flow may develop singularities, including the conformal factor tends to infinity and some triangle degenerates along the combinatorial $\alpha$-Yamabe flow and combinatorial $\alpha$-Calabi flow. To handle the possible singularities along the $\alpha$-flows, we do surgery on the flows by flipping as described in Section 1.

To analyze the behavior of the $\alpha$-flows with surgery, we need to use the discrete conformal theory established by Gu-Luo-Sun-Wu [38] for PL metrics. In the following, we briefly recall some results in [38]. For details of the theory, please refer to Gu-Luo-Sun-Wu’s important work [38].

#### 3.1 Gu-Luo-Sun-Wu’s work on discrete uniformization theorem

**Definition 3.1** ([38] Definition 1.1). Two PL metrics $d, d'$ on $(S,V)$ are discrete conformal if there exist sequences of PL metrics $d_1 = d, \cdots, d_m = d'$ on $(S,V)$ and triangulations $T_1, \cdots, T_m$ of $(S,V)$ satisfying
(a) (Delaunay condition) each $\mathcal{T}_i$ is Delaunay in $d_i$.

(b) (Vertex scaling condition) if $\mathcal{T}_i = \mathcal{T}_{i+1}$, there exists a function $u : V \to \mathbb{R}$ so that if $e$ is an edge in $\mathcal{T}_i$ with end points $v$ and $v'$, then the lengths $l_{d_{i+1}}(e)$ and $l_{d_i}(e)$ of $e$ in $d_i$ and $d_{i+1}$ are related by

$$l_{d_{i+1}}(e) = l_{d_i}(e)e^{u(v)+u(v')}.$$ 

(c) if $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then $(S, d_i)$ is isometric to $(S, d_{i+1})$ by an isometry homotopic to identity in $(S, V)$.

The discrete conformal class of a PL metric is called a discrete Riemann surface.

The space of PL metrics on $(S, V)$ discrete conformal to $d$ is called the conformal class of $d$ and denoted by $D(d)$.

The following discrete uniformization theorem was established in [38].

**Theorem 3.2** ([38] Theorem 1.2). Suppose $(S, V)$ is a closed connected marked surface and $d$ is a PL metric on $(S, V)$. Then for any $K^* : V \to (-\infty, 2\pi)$ with $\sum_{v \in V} K^*(v) = 2\pi \chi(S)$, there exists a PL metric $d'$, unique up to scaling and isometry homotopic to the identity on $(S, V)$, such that $d'$ is discrete conformal to $d$ and the discrete curvature of $d'$ is $K^*$.

Denote the Teichmüller space of all PL metrics on $(S, V)$ by $T_{PL}(S, V)$ and decorated Teichmüller space of all equivalence class of decorated hyperbolic metrics on $S - V$ by $T_{D}(S - V)$. In the proof of Theorem 3.2, Gu-Luo-Sun-Wu proved the following result.

**Theorem 3.3** ([38]). There is a $C^1$-diffeomorphism $A : T_{PL}(S, V) \to T_{D}(S - V)$ between $T_{PL}(S, V)$ and $T_{D}(S - V)$. Furthermore, the space $D(d) \subset T_{PL}(S, V)$ of all equivalence classes of PL metrics discrete conformal to $d$ is $C^1$-diffeomorphic to $\{p\} \times \mathbb{R}_{>0}$ under the diffeomorphism $A$, where $p$ is the unique hyperbolic metric on $S - V$ determined by the PL metric $d$ on $(S, V)$.

Set $u_i = \ln w_i$ for $w = (w_1, w_2, \cdots, w_n) \in \mathbb{R}_{>0}^n$. Using the map $A$, Gu-Luo-Sun-Wu defined the curvature map

$$F : \mathbb{R}^n \to (-\infty, 2\pi)^n$$

$$u \mapsto K_{A^{-1}(p, w(u))}$$

and proved the following property of $F$.

**Proposition 3.4** ([38]). 1. For any $k \in \mathbb{R}$, $F(v + k(1, 1, \cdots, 1)) = F(v)$.

2. There exists a $C^2$-smooth convex function $W : \mathbb{R}^n \to \mathbb{R}$ so that its gradient $\nabla W$ is $F$ and the restriction $W : \{u \in \mathbb{R}^n | \sum_{i=1}^{n} u_i = 0\} \to \mathbb{R}$ is strictly convex.
Theorem 3.3 implies that the union of the admissible spaces $\Omega^T_D(d')$ of conformal factors such that $T$ is Delaunay for $d' \in \mathcal{D}(d)$ is $\mathbb{R}^n_{>0}$. Furthermore, $F$, which is defined on $\mathbb{R}^n_{>0}$, is a $C^1$-extension of the curvature $K$ defined on the space of conformal factors $\Omega^T_D(d')$ for $d' \in \mathcal{D}(d)$. Then we can extend the Euclidean discrete $\alpha$-Laplace operator to be defined on $\mathbb{R}^n_{>0}$, which is the space of the conformal factors for the discrete conformal class $\mathcal{D}(d)$.

**Definition 3.5.** Suppose $(S, V)$ is a marked surface with a PL metric $d_0$. For a function $f : V \rightarrow \mathbb{R}$ on the vertices, the discrete conformal $\alpha$-Laplace operator of $d \in \mathcal{D}(d_0)$ on $(S, V)$ is defined to be the map

$$\Delta_{\alpha} : \mathbb{R}^V \rightarrow \mathbb{R}^V$$

$$f \mapsto \Delta_{\alpha} f,$$

where the value of $\Delta_{\alpha} f$ at $v_i$ is

$$\Delta_{\alpha} f_i = \frac{1}{w_i^\alpha} \sum_{j \sim i} \left( - \frac{\partial F_i}{\partial u_j} (f_j - f_i) = -\frac{1}{w_i^\alpha} (\tilde{L} f)_i, \right. \tag{3.2}$$

where $(p, w) = A(d)$ and $\tilde{L}_{ij} = \frac{\partial F_i}{\partial u_j}$ is an extension of $L_{ij} = \frac{\partial K_i}{\partial u_j}$ for $u = \ln w \in \mathcal{U}^T_D(d') = \ln \Omega^T_D(d')$, $d' \in \mathcal{D}(d)$.

**Remark 1.** Note that $F$ is $C^1$-smooth in $u \in \mathbb{R}^n$ and $\Delta^T$ is independent of the Delaunay triangulations of a PL metric, so the operator $\Delta_{\alpha}$ is well-defined on $\mathbb{R}^n$. Furthermore, $\Delta_{\alpha}$ is continuous and piecewise smooth on $\mathbb{R}^n$ as a matrix-valued function of $u$ (38, Lemma 5.1).

### 3.2 Rigidity of $\alpha$-curvature on discrete Riemannian surfaces

Following Gu-Luo-Sun-Wu’s approach, we can define the $\alpha$-curvature on discrete Riemannian surfaces as follows.

**Definition 3.6.** Suppose $(S, V)$ is a marked closed surface with a PL metric $d$, $\alpha \in \mathbb{R}$ is a constant and $F$ is the curvature map in (3.1). The $\alpha$-curvature on the discrete Riemann surface $\mathcal{D}(d)$ is defined to be

$$F_{\alpha,i} = \frac{F_i}{w_i^\alpha}. \tag{3.3}$$

**Remark 2.** Note that $\alpha$-curvature on a discrete Riemann surface is well-defined and an extension of the combinatorial $\alpha$-curvature on a triangulated surface. If $T$ is a Delaunay triangulation of the marked surface $(S, V)$, $\Omega^T_D(d')$ is the space of conformal factors such that $T$ is Delaunay for $d' \in \mathcal{D}(d)$, then $F_{\alpha,|\mathcal{U}^T_D(d')} = R_\alpha$. 

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Denote the space of conformal factors by $\Omega(d)$ and set $U(d) = \ln \Omega(d)$. Similar to Theorem 1.2 for $\alpha$-curvature on triangulated surfaces, we have the following global rigidity for $\alpha$-curvature on discrete Riemann surfaces.

**Theorem 3.7.** Suppose $(S, V)$ is a marked surface with a PL metric $d$ and $\alpha \in \mathbb{R}$ is a constant with $\alpha \chi(S) \leq 0$. $F$ is a function defined on the vertices.

1. If $\alpha F \equiv 0$, then there exists at most one conformal factor $u^* \in U(d)$ up to scaling such that $A^{-1}(p, w(u^*)) \in D(d)$ has combinatorial $\alpha$-curvature $F$.

2. If $\alpha F \leq 0$ and $\alpha F \neq 0$, then there exists at most one conformal factor $u^* \in U(d)$ such that $A^{-1}(p, w(u^*)) \in D(d)$ has combinatorial $\alpha$-curvature $F$.

**Proof.** Define the energy function

$$W_\alpha(u) = W(u) - \int_0^u \sum_{i=1}^N F_i u_i^\alpha du_i.$$  \hfill (3.4)

By Proposition 3.4, $W_\alpha$ is a well-defined $C^2$-smooth function defined on $\mathbb{R}^n$. Furthermore, we have

$$\nabla_{u_i} W_\alpha = F_i - F_i w_i^\alpha.$$

$A^{-1}(p, w(u^*)) \in D(d)$ for $u^* \in U(d)$ has combinatorial $\alpha$-curvature $F$ if and only if $\nabla W_\alpha(u^*) = 0$. By direct calculations, we have

$$\text{Hess} W_\alpha = L - \alpha \begin{pmatrix} F_1 w_1^\alpha \\ \vdots \\ F_N w_N^\alpha \end{pmatrix}.$$

If $\alpha F \equiv 0$, Hess $W_\alpha$ is positive semi-definite with kernel $t(1, \ldots, t)$ and $W_\alpha|_{\Sigma_0}$ is a strictly convex function on $\Sigma_0 = \{u_1 + \cdots + u_N = 0\}$. If $\alpha F \leq 0$ and $\alpha F \neq 0$, then Hess $W_\alpha$ is positive definite and $W_\alpha$ is strictly convex on $\mathbb{R}^n$.

Recall the following well known fact from analysis.

**Lemma 3.8.** If $W : \Omega \to \mathbb{R}$ is a $C^1$-smooth strictly convex function on an open convex set $\Omega \subset \mathbb{R}^m$, then its gradient $\nabla W : \Omega \to \mathbb{R}^m$ is an embedding.

Then the rigidity follows from Lemma 3.8. \hfill $\square$

**Corollary 3.9.** Suppose $(S, V)$ is a marked surface with a PL metric $d$ and $\alpha \in \mathbb{R}$ is a constant with $\alpha \chi(S) \leq 0$. Then the constant combinatorial $\alpha$-curvature PL metric in $D(d)$ is unique (up to scaling if $\alpha \chi(S) = 0$).
3.3 Combinatorial $\alpha$-Yamabe flow with surgery

By Gu-Luo-Sun-Wu’s discrete conformal theory [38], the normalized combinatorial $\alpha$-Yamabe flow with surgery takes the following form.

**Definition 3.10.** Suppose $(S, V)$ is a marked surface with a PL metric $d_0$. The combinatorial $\alpha$-Yamabe flow with surgery is defined to be

$$\begin{cases} \frac{du_i}{dt} = F_{\alpha, av} - F_{\alpha, i}, \\ u_i(0) = 0, \end{cases}$$

where $F_{\alpha, av} = \frac{2\pi \chi(S)}{\sum_{i=1}^{N} w_i^\alpha}$.

It is straightforward to check that $\sum_{i=1}^{N} w_i^\alpha \left( \sum_{i=1}^{N} u_i \right)$ for $\alpha = 0$ is invariant along the combinatorial $\alpha$-Yamabe flow with surgery (3.5).

Similar to the results in [8, 16, 30], we have the following result for combinatorial $\alpha$-Yamabe flow with surgery.

**Theorem 3.11.** Suppose $(S, V)$ is a closed connected marked surface with a PL metric $d_0$. $\alpha \in \mathbb{R}$ is a constant such that $\alpha \chi(S) \leq 0$. Then there exists a constant $\alpha$-curvature PL metric in $D(d_0)$ if and only if the combinatorial $\alpha$-Yamabe flow with surgery (3.5) exists for all time and converges to some $u^* \in U(d_0)$.

**Proof.** When $\alpha = 0$, the combinatorial $\alpha$-Yamabe flow with surgery (3.5) is the Yamabe flow with surgery studied in [38][15], where the conclusion has been proved. We only prove the case $\alpha \neq 0$ here.

If the solution $u(t)$ of combinatorial $\alpha$-Yamabe flow with surgery (3.5) converges to $u^* \in U(d_0)$, then we have $F_{\alpha}(u^*) = \lim_{t \to +\infty} F_{\alpha}(u(t))$ by the $C^1$ smoothness of $F$. For any $n \in \mathbb{N}$, there exists $\xi_n \in (n, n+1)$ such that

$$u_i(n+1) - u_i(n) = u'_i(\xi_n) = F_{\alpha, av} - F_{\alpha, i}(u(\xi_n)).$$

Set $n \to +\infty$, then we have

$$F_{\alpha, i}(u^*) = \lim_{n \to +\infty} F_{\alpha, i}(u(\xi_n)) = F_{\alpha, av},$$

which implies that $u^*$ is a conformal factor in $U(d_0)$ with constant combinatorial $\alpha$-curvature.

Conversely, suppose $u^*$ is a conformal factor in $U(d_0)$ with constant combinatorial $\alpha$-curvature. Then the constant curvature must be the constant $F_{\alpha, av} = \frac{2\pi \chi(S)}{\sum_{i=1}^{N} w_i^\alpha}$. Set

$$W_\alpha(u) = W(u) - F_{\alpha, av} \int_{u^*}^{u} \sum_{i=1}^{N} w_i^\alpha du_i.$$
Then $W_\alpha$ is a well-defined $C^2$-smooth convex function defined on $\mathbb{R}^n$ and $W_\alpha(u + k1) = W_\alpha(u)$. Note that $\nabla W_\alpha(u^*) = 0$, we have $\lim_{u \to \infty} W_\alpha(u)|_P = +\infty$, where $P = \{u \in \mathbb{R}^n| \sum_{i=1}^N w_i^\alpha = N\}$. This implies that $W_\alpha(u)|_P$ is a proper function on $P$.

Note that $\frac{d(W_\alpha(u(t)))}{dt} = \sum_{i=1}^N \frac{\partial W_\alpha}{\partial u_i} \cdot \frac{du_i}{dt} = \sum_{i=1}^N (F_i - F_{\alpha,av} w_i^\alpha)(F_{\alpha,av} - F_{\alpha,i})$

$$= - \sum_{i=1}^N (F_{\alpha,av} - F_{\alpha,i})^2 w_i^\alpha \leq 0.$$ 

So we have $0 \leq W_\alpha(u(t)) \leq W_\alpha(u(0))$. Note that $\sum_{i=1}^N w_i^\alpha$ is invariant along the combinatorial $\alpha$-Yamabe flow with surgery \[\text{[3.3]},\] we have the solution $u(t)$ of the combinatorial $\alpha$-Yamabe flow with surgery lies in a compact subset of $P$ by the properness of $W_\alpha$ on $P$. Then the solution of the combinatorial $\alpha$-Yamabe flow with surgery \[\text{[3.3]},\] exists for all time and $\lim_{t \to +\infty} W_\alpha(u(t))$ exists. Furthermore,

$$0 = \lim_{n \to +\infty} (W_\alpha(u(n+1)) - W_\alpha(u(n))) = \lim_{n \to +\infty} \frac{dW_\alpha(u(t))}{dt}|_{t=\xi_n}$$

$$= - \lim_{n \to +\infty} \sum_{i=1}^N (F_{\alpha,av} - F_{\alpha,i})^2 w_i^\alpha|_{t=\xi_n}. $$

Then we have $\lim_{n \to +\infty} F_\alpha(u(\xi_n)) = F_{\alpha,av} = F_\alpha(u^*)$, which implies that $\lim_{t \to +\infty} u(\xi_n) = u^*$ by Theorem 3.7.

Set $\Gamma_i(u) = F_{\alpha,av} - F_{\alpha,i}$. Similar to the proof of Theorem 2.9, we can check that $(\frac{\partial \Gamma_i}{\partial u_i})|_{u^*}$ has $N-1$ negative eigenvalue and a zero eigenvalue. The local convergence of the solution of combinatorial $\alpha$-Yamabe flow with surgery to $u^*$ follows by the same argument in the proof of Theorem 2.9. The conclusion then follows by the local convergence and $\lim_{n \to +\infty} u(\xi_n) = u^*$. □

**Remark 3.** The proof of Theorem 3.11 suggests the following generalization of the combinatorial $\alpha$-Yamabe flow with surgery. Suppose $\overline{\mathcal{F}}$ is a function defined on the vertices, then $\overline{\mathcal{F}}$ is the combinatorial $\alpha$-curvature of a PL metric in $\mathcal{D}(d_0)$ if and only if the solution of combinatorial $\alpha$-Yamabe flow with surgery for $\overline{\mathcal{F}}$ (defined similarly) exists for all time and converges to a conformal factor $u^* \in \mathcal{U}(d_0)$.

In the case of $\alpha\chi(S) \leq 0$, we can further prove the existence of constant combinatorial $\alpha$-curvature metric and then obtain a generalization of the discrete uniformization theorem obtained in [38].

By direct calculations, the curvature $F_{\alpha,i}$ evolves according to the following equation

$$\frac{dF_{\alpha,i}}{dt} = (\Delta_\alpha F_{\alpha})_i + \alpha F_{\alpha,i}(F_{\alpha,i} - F_{\alpha,av})$$

(3.6)

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along the combinatorial $\alpha$-Ricci flow.

Note that the surgery ensures that the weight

$$\omega_{ij} = \frac{1}{w_i^\alpha} \frac{\partial F_i}{\partial u_j} = \frac{\cot \theta_k^{ij} + \cot \theta_l^{ij}}{w_i^\alpha} \geq 0$$

along the combinatorial $\alpha$-Yamabe flow with surgery (3.5). This motives us to use the following discrete maximal principle. The readers can refer to [30] for a proof.

**Theorem 3.12.** (Maximum Principle) Let $f : V \times [0, T) \to \mathbb{R}$ be a $C^1$ function such that

$$\frac{\partial f_i}{\partial t} \geq \Delta f_i + \Phi_i(f_i), \ \forall (i,t) \in V \times [0,T),$$

where the Laplacian operator is defined as

$$\Delta f_i = \sum_{j \sim i} a_{ij}(t)(f_j - f_i)$$

with $a_{ij} \geq 0$ and $\Phi_i : \mathbb{R} \to \mathbb{R}$ is a local Lipschitz function. Suppose there exists $C_1 \in \mathbb{R}$ such that $f_i(0) \geq C_1$ for all $i \in V$. Let $\varphi$ be the solution to the associated ODE

$$\begin{cases}
\frac{d\varphi}{dt} = \Phi_i(\varphi), \\
\varphi(0) = C_1,
\end{cases}$$

then

$$f_i(t) \geq \varphi(t)$$

for all $(i,t) \in V \times [0,T)$ such that $\varphi(t)$ exists.

Similarly, suppose $f : V \times [0, T) \to \mathbb{R}$ be a $C^1$ function such that

$$\frac{\partial f_i}{\partial t} \leq \Delta f_i + \Phi_i(f_i), \ \forall (i,t) \in V \times [0,T).$$

Suppose there exists $C_2 \in \mathbb{R}$ such that $f_i(0) \leq C_2$ for all $i \in V$. Let $\psi$ be the solution to the associated ODE

$$\begin{cases}
\frac{d\psi}{dt} = \Phi_i(\psi), \\
\psi(0) = C_2,
\end{cases}$$

then

$$f_i(t) \leq \psi(t)$$

for all $(i,t) \in V \times [0,T)$ such that $\psi(t)$ exists.

Applying the discrete maximal principle, we have the following result.
Theorem 3.13. Suppose \((S, V)\) is a marked surface with a PL metric \(d_0\). \(\alpha \in \mathbb{R}\) is a constant such that \(\alpha F_{\alpha, i}(u(0)) < 0\) for all \(i \in V\), then the normalized \(\alpha\)-Yamabe flow with surgery (3.5) exists for all time and converges exponentially fast to a constant \(\alpha\)-curvature PL metric.

Proof. Note that the combinatorial \(\alpha\)-curvature \(F_{\alpha, i}\) evolves according to (3.6) along the normalized \(\alpha\)-Yamabe flow with surgery (3.5). The maximum principle, i.e. Theorem 3.12, is valid for this equation. By the maximum principle, if \(\alpha > 0\) and \(F_{\alpha, i}(u(0)) < 0\) for all \(i \in V\), we have

\[
(F_{\alpha, \min}(0) - F_{\alpha, av}) e^{\alpha F_{\alpha, av} t} \leq F_{\alpha, i} - F_{\alpha, av} \leq (F_{\alpha, \max}(0) - F_{\alpha, av}) e^{\alpha F_{\alpha, av} t}.
\]

If \(\alpha < 0\) and \(F_{\alpha, i}(u(0)) > 0\) for all \(i \in V\), we have

\[
\frac{F_{\alpha, av}}{F_{\alpha, \min}(0)} (F_{\alpha, \min}(0) - F_{\alpha, av}) e^{\alpha F_{\alpha, av} t} \leq F_{\alpha, i} - F_{\alpha, av} \leq (F_{\alpha, \max}(0) - F_{\alpha, av}) e^{\alpha F_{\alpha, av} t}.
\]

In summary, if \(\alpha F_{\alpha, i}(u(0)) < 0\) for all \(i \in V\), there exists constants \(C_1\) and \(C_2\) such that

\[
C_1 e^{\alpha F_{\alpha, av}} \leq F_{\alpha, i}(u(t)) - F_{\alpha, av} \leq C_2 e^{\alpha F_{\alpha, av}},
\]

which implies the long-time existence and exponential convergence of the normalized \(\alpha\)-Yamabe flow with surgery (3.5).

Proof of Theorem 1.3: In the case \(\alpha \chi(S) = 0\), we have \(\alpha = 0\) or \(\chi(S) = 0\). For \(\alpha = 0\), \(\alpha\)-curvature \(F_{\alpha}\) is the classical discrete curvature \(F\). The existence of constant curvature PL metric is ensured by Theorem 3.2. If \(\chi(S) = 0\), the constant \(\alpha\)-curvature metric is a zero \(\alpha\)-curvature metric for all \(\alpha \in \mathbb{R}\). Especially, it is a PL metric with zero \(F\) curvature, the existence of which is ensured by Theorem 3.2.

In the case of \(\alpha \chi(S) < 0\), by Theorem 3.2 there is a PL metric \(d' \in \mathcal{D}(d_0)\) with constant \(F\) curvature \(\frac{2\pi \chi(S)}{N}\), which implies that the combinatorial \(\alpha\)-curvature of \(d'\) satisfies \(\alpha F_{\alpha, i} < 0\) for all \(i \in V\). Applying Theorem 3.13 with initial metric \(d'\) gives the conclusion.

Remark 4. Theorem 1.3 and Theorem 3.11 together implies the long time existence and convergence of the combinatorial \(\alpha\)-Yamabe flow. i.e. the \(\alpha\)-Yamabe flow part of Theorem 1.7. This confirms a generalized Luo conjecture [45] on convergence of combinatorial Yamabe flow with surgery.

Remark 5. There is another way to extend the combinatorial Yamabe flow initiated by Ge-Jiang [19]. Ge-Jiang’s extension comes from [3, 46] and is designed for surfaces with fixed triangulations. For a fixed triangulated surface, Ge-Jiang’s extension of \(\alpha\)-Yamabe flow ensures the long-time existence of the extended flow, while the extended \(\alpha\)-Yamabe flow may converge to a virtual constant \(\alpha\)-curvature PL metric. Gu-Luo-Sun-Wu’s
extension we use here ensures the combinatorial \( \alpha \)-Yamabe flow with surgery converges to a real constant \( \alpha \)-curvature metric. Furthermore, Gu-Luo-Sun-Wu’s extension could be applied to extend the combinatorial \( \alpha \)-Calabi flow, while Ge-Jiang’s extension is not valid for this case. The readers can refer to Subsection 3.4 for the \( \alpha \)-Calabi flow with surgery.

### 3.4 Combinatorial \( \alpha \)-Calabi flow with surgery

We can also define the combinatorial \( \alpha \)-Calabi flow with surgery.

**Definition 3.14.** Suppose \( d_0 \) is a PL metric on a marked surface \((S,V)\) and \( \alpha \in \mathbb{R} \). The combinatorial \( \alpha \)-Calabi flow with surgery on \((S,V)\) is defined as

\[
\begin{cases}
\frac{du_i}{dt} = (\Delta_\alpha F_\alpha)_i, \\
u_i(0) = 0,
\end{cases}
\tag{3.7}
\]

where \( \Delta_\alpha \) is the discrete \( \alpha \)-Laplace operator of \( A^{-1}(p, w(u(t))) \) on \((S,V)\) defined by (3.2).

Similar to the combinatorial \( \alpha \)-Yamabe flow on discrete Riemann surface, \( \sum_{i=1}^{N} w_\alpha^i \) (\( \sum_{i=1}^{N} u_i \) for \( \alpha = 0 \)) is invariant along the combinatorial \( \alpha \)-Calabi flow with surgery (3.7). It is straightway to check that if the combinatorial \( \alpha \)-Calabi flow with surgery (3.7) converges, the limit metric is a constant \( \alpha \)-curvature PL metric.

We have the following result for combinatorial \( \alpha \)-Calabi flow with surgery (3.7), which proves the combinatorial \( \alpha \)-Calabi flow part of Theorem 1.7.

**Theorem 3.15.** Suppose \((S,V)\) is a closed connected marked surface with a PL metric \( d_0 \) and \( \alpha \in \mathbb{R} \) is a constant such that \( \alpha \chi(S) \leq 0 \). Then the combinatorial \( \alpha \)-Calabi flow with surgery (3.7) exists for all time and converges to a constant \( \alpha \)-curvature metric in \( \mathcal{D}(d_0) \).

**Proof.** By Theorem 1.3 there exists a unique PL metric \( d = A^{-1}(p, e^{u^*}) \in \mathcal{D}(d_0) \) with \( \sum_{i=1}^{N} e^{u_i^*} = N \) such that \( d \) has constant combinatorial \( \alpha \)-curvature \( F_\alpha \).

Similar to the proof of Theorem 3.11 we can define

\[
W_\alpha(u) = W(u) - F_{\alpha,av} \int_{u} \sum_{i=1}^{N} w_i^\alpha du_i.
\]

Then \( W_\alpha \) is a well-defined \( C^2 \)-smooth convex function defined on \( \mathbb{R}^n \) under the condition \( \alpha \chi(S) \leq 0 \). Furthermore, \( W_\alpha(u) = W_\alpha(u + k1), k \in \mathbb{R} \). Note that \( \nabla W_\alpha(u^*) = 0 \), we have \( \lim_{u \to \infty} W_\alpha(u)|_P = +\infty \), where \( P = \{ u \in \mathbb{R}^n | \sum_{i=1}^{N} w_i^\alpha = N \} \). This implies that \( W_\alpha(u)|_P \) is a proper function on \( P \).
By direct calculations, we have
\[
\frac{dW_\alpha(u(t))}{dt} = \sum_{i=1}^{N} \frac{\partial W_\alpha}{\partial u_i} \frac{du_i}{dt} = \sum_{i=1}^{N} (F_i - F_{\alpha,av}u_i^{\alpha})(\Delta_\alpha F_\alpha)_i \\
= - (F_\alpha - F_{\alpha,av})^T \cdot L \cdot (F_\alpha - F_{\alpha,av}) \leq 0.
\]
Then \(W_\alpha(u(t))\) is bounded along the combinatorial \(\alpha\)-Calabi flow with surgery (3.7). By the properness of \(W_\alpha\), \(u(t)\) is bounded along the combinatorial \(\alpha\)-Calabi flow with surgery (3.7), which implies the long-time existence of combinatorial \(\alpha\)-Calabi flow with surgery.

As \(W_\alpha(u(t))\) is bounded along the combinatorial \(\alpha\)-Calabi flow with surgery and \(\frac{dW_\alpha(u(t))}{dt} \leq 0\), we have \(\lim_{t \to +\infty} W_\alpha(u(t))\) exists.

Note that
\[
0 = \lim_{n \to +\infty} (W_\alpha(u(n+1)) - W_\alpha(u(n))) \\
= - \lim_{n \to +\infty} (F_\alpha - F_{\alpha,av})^T \cdot L \cdot (F_\alpha - F_{\alpha,av})|_{t=\xi_n},
\]
there is a subsequence \(\xi_{n_k}\) of \(\xi_n \in (n, n+1)\) such that \(F_\alpha(u(\xi_{n_k})) \to F_{\alpha,av}\), which implies that \(u(\xi_{n_k}) \to u^* \in P\). So we have \(\lim_{t \to +\infty} W_\alpha(u(t)) = W_\alpha(u^*)\). By the strictly convexity of \(W_\alpha\) on \(P\), we have \(\lim_{t \to +\infty} u(t) = u^*\).

**Remark 6.** In the case of \(\alpha = 0\), the combinatorial Calabi flow with surgery was studied in [70], where the long time existence and convergence of the combinatorial Calabi flow with surgery were proved. Theorem 3.15 generalizes the results obtained in [70].

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