The Tychonoff uniqueness theorem for the $G$-heat equation

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Abstract

In this paper we shall investigate a uniqueness result for solutions of the $G$-heat equation. We obtain the Tychonoff uniqueness theorem for the $G$-heat equation.

Keywords: Tychonoff uniqueness theorem; $G$-heat equation; $G$-expectation; $G$-Brownian motion; Viscosity solution.

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1 Introduction

Motivated by the volatility uncertainty problems, risk measures and superhedging in finance, Peng has introduced recently a new notion of a nonlinear expectation, the so-called $G$-expectation (see [8], [9], [11], [13]), which is generated by the following nonlinear heat equation, called the $G$-heat equation:

$$
\frac{\partial u}{\partial t} = G(D^2u), \quad (t, x) \in (0, T) \times \mathbb{R}^n,
$$

where $D^2u$ is the Hessian matrix of $u$, i.e., $D^2u = (\partial^2_{x_i x_j} u)_{i,j=1}^n$ and

$$
G(A) = \frac{1}{2} \sup_{\alpha \in \Gamma} tr[\alpha \alpha^T A], \quad A = (A_{ij})_{i,j=1}^n \in \mathbb{S}^n, \quad (1.1)
$$

where $\mathbb{S}^n$ denotes the space of $n \times n$ symmetric matrices and $\Gamma$ is a given non-empty, bounded and closed subset of $\mathbb{R}^{n \times n}$, the space of $n \times n$ matrices.

Together with the notion of the $G$-expectation Peng (see [8], [11], [13]) introduced the related $G$-normal distribution and the $G$-Brownian motion, and established an Itô calculus for the $G$-Brownian motion. Peng (see [10], [12] and [14]) also obtained the law of large numbers and central limit theorem under nonlinear expectations, which indicates that the notion of $G$-normal distribution plays an important role in the theory of nonlinear expectations as that of normal distribution in the classical probability theory. The $G$-expectation can be regarded as a coherent risk measure and the conditional $G$-expectation can be regarded as a dynamic risk measure.

Tychonoff (see [16], [17] and Theorem 4.3.3 in [6]) obtained the following uniqueness theorem for the heat equation.

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Theorem 1.1 Let $u_1, u_2 \in C([0, T] \times \mathbb{R})$ be solutions of the heat equation:

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^n,$$

with $u_1(0, x) = u_2(0, x) = \varphi(x)$. If there are two positive constants $c_1, c_2$ such that

$$|u_1(t, x)| \leq c_1 e^{c_2|x|^2}, |u_2(t, x)| \leq c_1 e^{c_2|x|^2}, \text{ uniformly for } t \in [0, T],$$

then $u_1 \equiv u_2$ in $[0, T] \times \mathbb{R}^n$.

The objective of this paper is to investigate the Tychonoff uniqueness theorem for the following generalized $G$-heat equation. We also call it the $G$-heat equation.

$$u_t - G(t, x, D^2 u) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n,$$

where $G : [0, T] \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ and only satisfies the following conditions:

(H) $G$ is continuous, and for all $(t, x) \in [0, T] \times \mathbb{R}^n$, $G(t, x, \cdot)$ is subadditive and uniformly elliptic, $G(t, x, 0) = 0$.

Da Lio and Ley [3] obtained a uniqueness theorem for second-order Bellman-Isaacs equations under quadratic growth assumptions. Peng [14] obtained a uniqueness theorem for a class of second order parabolic equations under the polynomial growth condition. Strömberg [15] considered the Cauchy problem for parabolic Isaacs’s equations:

$$\begin{cases}
u_t + F(t, x, u, Du, D^2 u) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
u(0, x) = \varphi(x), \quad x \in \mathbb{R}^n,
\end{cases}$$

where

$$F(t, x, r, p, X) = \sup_{\gamma, \delta} \inf \{ -tr(A_{\gamma, \delta}(t, x)X) + \langle b_{\gamma, \delta}(t, x), p \rangle + c_{\gamma, \delta}(t, x)r - f_{\gamma, \delta}(t, x) \},$$

$$(t, x, r, p, X) \in (0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n.$$

Strömberg obtained the following uniqueness of viscosity solution of (1.3):

Theorem 1.2 Let some conditions be satisfied and let $u_1, u_2 \in C(\overline{Q})$ be solutions of (1.3) in the strip $Q = (0, T) \times \mathbb{R}^n$ with $u_1(0, x) = u_2(0, x) = \varphi(x)$. If there are two positive constants $c_1, c_2$ such that

$$|u_1(t, x)| \leq c_1 e^{c_2|x|}, |u_2(t, x)| \leq c_1 e^{c_2|x|}, \text{ uniformly for } t \in [0, T],$$

then $u_1 \equiv u_2$ in $\overline{Q}$.

Strömberg raised a question if solutions of the Cauchy problem for (1.3) under weaker conditions than $|u(t, x)| \leq K e^{k|x|^2}$ are unique. We shall give a positive answer and prove that solutions of the Cauchy problem for the $G$-heat equation (1.2) satisfying $|u(t, x)| \leq K e^{k|x|^2}$ are unique.

The rest of the paper is organized as follows. In Section 2, we give notations and preliminaries which will be needed in what follows. In Section 3, we investigate the Tychonoff uniqueness theorem for the $G$-heat equation.
2 Notations and Preliminaries

The objective of this section is to give some notations and preliminaries, which we will need. We first recall the definition of the parabolic superjet and the parabolic subjet.

Definition 2.1 Let $u : (0,T) \times \mathbb{R}^n \to \mathbb{R}$ and $(t,x) \in (0,T) \times \mathbb{R}^n$. Then we define the parabolic superjet of $u$ at $(t,x)$:

$$\mathcal{P}^{2,+} u(t,x) = \left\{ (p,q,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \mid u(s,y) \leq u(t,x) + p(s-t) + \langle q, y-x \rangle + \frac{1}{2} \langle \nabla_y u(s,y), y-x \rangle + o(|s-t| + \|y-x\|^2), (s,y) \to (t,x) \right\},$$

and its closure:

$$\bar{\mathcal{P}}^{2,+} u(t,x) = \left\{ (p,q,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \mid \exists (t_n,x_n,p_n,q_n,X_n) \text{ such that} \right\}
\begin{align*}
(p_n,q_n,X_n) &\in \mathcal{P}^{2,+} u(t_n,x_n), \\
\lim_{n \to +\infty} (p_n,q_n,X_n) &= (p,q,X), \\
\lim_{n \to +\infty} (t_n,x_n,u(t_n,x_n)) &= (t,x,u(t,x)).
\end{align*}$$

Similarly, we consider the parabolic subjet and its closure:

$$\mathcal{P}^{2,-} u(t,x) = -\mathcal{P}^{2,+}(-u)(t,x), \bar{\mathcal{P}}^{2,-} u(t,x) = -\bar{\mathcal{P}}^{2,+}(-u)(t,x).$$

According to [2], we have

$$\mathcal{P}^{2,\pm} u(t,x) = \left\{ \begin{array}{ll}
\left( \frac{\partial \varphi}{\partial t} (t,x), D \varphi (t,x), D^2 \varphi (t,x) \right), & \varphi \in C^{1,2}([0,T] \times \mathbb{R}^n), \\
& u - \varphi \text{ has a global maximum(minimum) 0 at } (t,x) \right\}. $$

We now recall the definition of viscosity solution of (1.2) from Crandall, Ishii and Lions [2].

Definition 2.2

(i) A viscosity subsolution of (1.2) on $(0,T) \times \mathbb{R}^n$ is a function $u \in \text{USC}((0,T) \times \mathbb{R}^n)$ such that, for all $(t,x) \in (0,T) \times \mathbb{R}^n$,

$$p - G(t,x) \leq 0, \text{ for } (p,q,X) \in \mathcal{P}^{2,+} u(t,x).$$

(ii) A viscosity supersolution of (1.2) on $(0,T) \times \mathbb{R}^n$ is a function $u \in \text{LSC}((0,T) \times \mathbb{R}^n)$ such that, for all $(t,x) \in (0,T) \times \mathbb{R}^n$,

$$p - G(t,x) \geq 0, \text{ for } (p,q,X) \in \mathcal{P}^{2,-} u(t,x).$$

(iii) $u \in C((0,T) \times \mathbb{R}^n)$ is said to be a viscosity solution of (1.2) on $(0,T) \times \mathbb{R}^n$ if it is both a viscosity subsolution and supersolution of (1.2) on $(0,T) \times \mathbb{R}^n$.

Let $M > 0, x \in \mathbb{R}^n$. We say that $P(x)$ is a paraboloid of opening $M$ if $P(x) = \pm \frac{M}{2} |x|^2 + l(x) + l_0$, where $l$ is linear and $l_0$ is a constant. $P(x)$ is convex if $+$ appears and concave if $-$ appears. So for $t_0, \rho > 0$, the equation $t = t_0 - \frac{|x|^2}{\rho^2}$ denotes the graph of a concave paraboloid of opening $\frac{2}{\rho^2}$ with vertex at $(t_0,0) \in \mathbb{R}^{n+1}$, which we will henceforth write as $\rho = \frac{|x|}{\sqrt{t_0-t}}$. By concentric concave paraboloids of opening $2\rho_1^{-2}$ and $2\rho_2^{-2}$, we mean these paraboloids have common vertex $(t_0,0)$. 

3
Let \( Q \subset \mathbb{R}^{n+1} \). \( Q \) is bounded below by the line \( t = 0 \) and above by the line \( t = t' \), where \( t' < t_0 \). \( Q \) is bounded laterally by the arcs of the paraboloids \( \rho_1 = \frac{|x|}{\sqrt{t_0-t}} \) and \( \rho_2 = \frac{|x|}{\sqrt{t_0-t}} \) of opening \( 2\rho_1^{-2} \) and \( 2\rho_2^{-2} \) respectively, with \( \rho_1 \leq \rho_2 \). Geometrically, \( Q \) is a concave paraboloid shell, truncated just below the vertex \((t_0,0)\). For \( \rho_1 \leq \rho \leq \rho_2 \), we define the functions as follows:

\[
M_1(\rho) = \max_{|x| = \rho \sqrt{t_0-t}, \ 0 \leq t' \leq t} u(t,x),
M_2 = \max_{\rho_1 \sqrt{t_0} \leq |x| \leq \rho \sqrt{t_0}} u(0,x),
M(\rho) = \max\{M_1(\rho), M_2\}.
\]

For \( f \in C(Q) \) and positive constants \( \lambda \leq \Lambda \), we denote by \( \mathcal{S}(\lambda, \Lambda, f) \) the class of viscosity subsolutions of the equation \( \mathcal{M}(D^2 u, \lambda, \Lambda) - u_t = f(t,x) \), where for any real \( n \times n \) symmetric matrix \( M \)

\[
\mathcal{M}(D^2 u, \lambda, \Lambda) = \lambda \sum_{e_i > 0} e_i(M) + \lambda \sum_{e_i < 0} e_i(M),
\]

where \( e_i(M) \) are the eigenvalue of \( M \).

Finally, the following three curves Lemma was obtained in Kovats [7].

**Lemma 2.3** Let \( u \in \mathcal{S}(\lambda, \Lambda, 0) \) in a domain \( Q \subset \mathbb{R}^{n+1} \) containing two concave concentric paraboloids of opening \( 2\rho_1^{-2} \) and \( 2\rho_2^{-2} \) and the region between them. If \( M(\rho) \) denotes the maximum of \( u \) on any concentric concave paraboloid of opening \( 2\rho^{-2} \), with \( \rho_1 \leq \rho \leq \rho_2 \), then there exists a differential function \( \psi(\rho) \) depending on \( n, \lambda, \Lambda \) and \( \rho \), such that

\[
M(\rho) \leq \frac{M(\rho_1)(\psi(\rho_2) - \psi(\rho)) + M(\rho_2)(\psi(\rho) - \psi(\rho_1))}{\psi(\rho_2) - \psi(\rho_1)}.
\]

Moreover, if \( \psi'(\rho) \geq 0 \), then \( \psi'(\rho) = e^{\frac{\rho^2}{4\lambda}} \rho^{-\frac{\lambda(n-1)}{\lambda}} \); and if \( \psi'(\rho) \leq 0 \), then \( \psi'(\rho) = -e^{\frac{\rho^2}{4\lambda}} \rho^{-\frac{\lambda(n-1)}{\lambda}} \).

### 3 Main result

The objective of this section is to investigate the Tychonoff uniqueness theorem for the \( G \)-heat equation. In order to get this theorem, we first give the following lemma.

**Lemma 3.1** Let \( u \) be a viscosity subsolution and \( v \) be a viscosity supersolution of \((I.2)\). Then \( u - v \) is a viscosity subsolution of \((I.2)\).

**Proof:** Let \( \varphi \in C^2((0,T] \times \mathbb{R}^n) \) and \((t_0, x_0) \in [0,T] \times \mathbb{R}^n \) be a strict local maximum point of \( u - v - \varphi \), and more precisely a strict maximum point in \([t_0-r, t_0+r] \times \overline{B}(x_0,r)\), where \( \overline{B}(x_0,r) = \{ x \in \mathbb{R}^n : |x-x_0| \leq r \} \) is a ball with a radius \( r > 0 \). Then we consider the function:

\[
\Phi_\varepsilon(t,x,y) = u(t,x) - v(t,y) - \frac{|x-y|^2}{\varepsilon^2} - \varphi(t,x),
\]

where \( \varepsilon \) is a positive parameter.

Since \((t_0, x_0)\) is a strict maximum point of \( u - v - \varphi \) in \([t_0-r, t_0+r] \times \overline{B}(x_0,r)\), then by virtue of the classical argument of viscosity solutions, there exists \((t_\varepsilon, x_\varepsilon, y_\varepsilon)\) such that

1. \((t_\varepsilon, x_\varepsilon, y_\varepsilon)\) is a strict maximum point of \( \Phi_\varepsilon(t, x, y) \) in \([t_0-r, t_0+r] \times \overline{B}(x_0,r) \times \overline{B}(x_0,r)\);
(ii) \((t_\varepsilon, x_\varepsilon, y_\varepsilon) \to (t_0, x_0, x_0)\), as \(\varepsilon \to 0\);

(iii) \(\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2}\) is bounded and \(\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0\), as \(\varepsilon \to 0\).

Thanks to Theorem 8.3 in [2], for every \(\alpha > 0\), there exist \(p, q \in \mathbb{R}\) and \(X, Y \in \mathbb{S}^n\) such that

\[
\begin{align*}
\left(p, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + D\varphi(t_\varepsilon, x_\varepsilon), X\right) &\in \mathcal{P}_{2+}^0 u(t_\varepsilon, x_\varepsilon), \\
\left(q, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}, Y\right) &\in \mathcal{P}_{2-}^0 v(t_\varepsilon, y_\varepsilon),
\end{align*}
\]

\[
p - q = \frac{\partial \varphi(t_\varepsilon, x_\varepsilon)}{\partial t},
\]

and

\[
\begin{pmatrix}
X & 0 \\
0 & -Y
\end{pmatrix} \leq A + \alpha A^2,
\]

where

\[
A = \begin{pmatrix}
D^2\varphi(t_\varepsilon, x_\varepsilon) + \frac{2I}{\varepsilon^2} & -\frac{2I}{\varepsilon^2} \\
-\frac{2I}{\varepsilon^2} & \frac{2I}{\varepsilon^2}
\end{pmatrix}.
\]

Taking \(\alpha = \frac{\varepsilon^2}{2}\), we get

\[
\begin{pmatrix}
X & 0 \\
0 & -Y
\end{pmatrix} \leq \frac{6}{\varepsilon^2} \begin{pmatrix}
I & -I \\
-1 & I
\end{pmatrix} + \frac{\varepsilon^2}{2} \begin{pmatrix}
(D^2\varphi(t_\varepsilon, x_\varepsilon))^2 & 0 \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
3D^2\varphi(t_\varepsilon, x_\varepsilon) & -D^2\varphi(t_\varepsilon, x_\varepsilon) \\
-D^2\varphi(t_\varepsilon, x_\varepsilon) & 0
\end{pmatrix}.
\]

Therefore, we have

\[
X - Y \leq \frac{\varepsilon^2}{2}(D^2\varphi(t_\varepsilon, x_\varepsilon))^2 + D^2\varphi(t_\varepsilon, x_\varepsilon). \tag{3.1}
\]

Since \(u\) is a viscosity subsolution and \(v\) is a viscosity supersolution of (1.2), then we have

\[
p - G(t_\varepsilon, x_\varepsilon, X) \leq 0, \quad q - G(t_\varepsilon, x_\varepsilon, Y) \geq 0,
\]

and the above inequality and the subadditivity of \(G(t_\varepsilon, x_\varepsilon, \cdot)\) yield

\[
\frac{\partial \varphi(t_\varepsilon, x_\varepsilon)}{\partial t} = p - q \leq G(t_\varepsilon, x_\varepsilon, X) - G(t_\varepsilon, x_\varepsilon, Y) \leq G(t_\varepsilon, x_\varepsilon, X - Y).
\]

By the above inequality and (3.1) we have

\[
\frac{\partial \varphi(t_\varepsilon, x_\varepsilon)}{\partial t} \leq G(t_\varepsilon, x_\varepsilon, X - Y) \leq G \left( t_\varepsilon, x_\varepsilon, \frac{\varepsilon^2}{2}(D^2\varphi(t_\varepsilon, x_\varepsilon))^2 + D^2\varphi(t_\varepsilon, x_\varepsilon) \right).
\]
Letting $\varepsilon \to 0$, since $(t_\varepsilon, x_\varepsilon, y_\varepsilon) \to (t_0, x_0, x_0)$, as $\varepsilon \to 0$, and $G$ is continuous, we get
\[
\frac{\partial \varphi(t_0, x_0)}{\partial t} - G(t_0, x_0, D^2 \varphi(t_0, x_0)) \leq 0.
\]
Therefore, $u - v$ is a viscosity subsolution of (1.2). The proof is complete. \hfill \Box

We now state and prove the main result in this paper.

**Theorem 3.2** Let (H) be satisfied and let $u_1, u_2 \in C(\overline{Q})$ be solutions of (1.2) in the strip $Q = (0, T) \times \mathbb{R}^n$ with $u_1(0, x) = u_2(0, x) = \varphi(x)$. If there are two positive constants $c_1, c_2$ such that
\[
|u_1(t, x)| \leq c_1 e^{c_2|x|^2}, |u_2(t, x)| \leq c_1 e^{c_2|x|^2}, \text{ uniformly for } t \in [0, T],
\]
then $u_1 \equiv u_2$ in $\overline{Q}$.

**Proof:** Since $u_1$ is a viscosity subsolution and $u_2$ is a viscosity supersolution of (1.2), then by Lemma 3.1 we have $v = u_1 - u_2$ is a viscosity subsolution of (1.2) with $v(0, x) = 0$. Due to [1], we have $v \in S(\frac{n}{4}, \Lambda, 0)$, where $\lambda \leq \Lambda$ are two positive constants.

Putting $t_0 \leq \frac{1}{4\Lambda c_2}$, we first consider $v$ in a domain $Q_1 = [0, \frac{t_0}{4}] \times \mathbb{R}^n$. For $\rho_1 \leq \rho \leq \rho_2$, from Lemma 2.3 there exists a differential function $\psi(\rho)$ depending on $n, \lambda, \Lambda$ and $\rho$, such that
\[
M(\rho) \leq \frac{M(\rho_1)(\psi(\rho_2) - \psi(\rho_1)) + M(\rho_2)(\psi(\rho) - \psi(\rho_1))}{\psi(\rho_2) - \psi(\rho_1)}.
\]
Moreover, if $\psi'(\rho) \geq 0$, then $\psi'(\rho) = \frac{e^{\frac{\rho^2}{4}}}{e^{\frac{\rho^2}{4}} - \frac{n\Lambda(n-1)}{4}}$; and if $\psi'(\rho) \leq 0$, then $\psi'(\rho) = -\frac{e^{\frac{\rho^2}{4}}}{e^{\frac{\rho^2}{4}} - \frac{n\Lambda(n-1)}{4}}$.

By (3.2), we know that $|v(t, x)| \leq 2c_1 e^{c_2|x|^2}$. Then $M(\rho_2) \leq 2c_1 e^{c_2\rho_2^2 t_0}$. If $\psi' \geq 0$, then we have
\[
\lim_{\rho_2 \to \infty} \frac{M(\rho_2)}{\psi(\rho_2)} \leq \lim_{\rho_2 \to \infty} \frac{2c_1 e^{c_2\rho_2^2 t_0}}{\psi(\rho_2)} = \lim_{\rho_2 \to \infty} \frac{4c_1c_2\rho_2 t_0 e^{c_2\rho_2^2 t_0}}{e^{\frac{\rho^2}{4}} - \frac{n\Lambda(n-1)}{4}} = \lim_{\rho_2 \to \infty} \frac{4c_1c_2 t_0 \rho_2^{1+n\Lambda(n-1)}}{e^{\frac{\rho^2}{4}} - c_2 e^{c_2\rho_2^2 t_0}}.
\]
Since $t_0 \leq \frac{1}{4\Lambda c_2}$, we have
\[
\lim_{\rho_2 \to \infty} \frac{M(\rho_2)}{\psi(\rho_2)} \leq 0.
\]
Therefore,
\[
M(\rho) \leq \lim_{\rho_2 \to \infty} \frac{M(\rho_1)(\psi(\rho_2) - \psi(\rho_1)) + M(\rho_2)(\psi(\rho) - \psi(\rho_1))}{\psi(\rho_2) - \psi(\rho_1)} \leq M(\rho_1) + \frac{M(\rho_1)(\psi(\rho_1) - \psi(\rho_1)) + M(\rho_2)(\psi(\rho) - \psi(\rho_1))}{\psi(\rho_2) - \psi(\rho_1)} \leq M(\rho_1).
\]

Letting $\rho_1 \to 0$, we know that the maximum value of $v$ in $Q_1$ occurs on the hyperplane $x = 0$. If the maximum value of $v$ in $Q_1$ occurs at $(x = 0, t = 0)$, then $v \leq v(0, 0) = 0$ in $Q_1$. We consider $-v$ in $Q_1$. By the similar argument, we have $-v \geq -v(0, 0) = 0$ in $Q_1$. Thus, $v = 0$ in $Q_1$. If the maximum value of $v$ in $Q_1$ occurs at $(x = 0, t = t_1)$, where $t_1 \in [0, t_0]$, then by the strong maximum principle in [1] we have $v = v(0, 0) = 0$ in $[0, t_1] \times \mathbb{R}^n$. 


Repeating the above process, using $t = t_1$ as the initial line and there exists $t_2$ such that $t_1 < t_2 \leq t_0$, we obtain that $v \equiv 0$ in $Q_2 = [t_1, t_2] \times \mathbb{R}^n$. After a finite number of steps, we get $v \equiv 0$ in $Q$. Therefore, $u_1 \equiv u_2$ in $Q$.

If $\psi' \leq 0$, by the similar argument we can get the desired result. The proof is complete. □

We give a counterexample (see [5]) to show that if (3.2) is not satisfied, then solution of the heat equation are not unique.

**Example 3.3** We consider the following heat equation:

\[
\begin{aligned}
&u_t - u_{xx} = 0, \quad (t, x) \in (0, T] \times \mathbb{R}^n, \\
u(0, x) = 0, \quad x \in \mathbb{R}^n.
\end{aligned}
\]

The above equation has many solutions. In fact, for any $\alpha > 1$, putting

\[
g(t) = \begin{cases} 
e^{-t^\alpha}, & t > 0; \\
0, & \text{otherwise},
\end{cases}
\]

we can check that

\[
u(t, x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)x^{2k}}{2k!}.
\]

are solutions of the above heat equation.

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