A FRACTIONAL HELLY THEOREM FOR BOXES

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This paper is dedicated to Javier Bracho on occasion of his sixtieth birthday.

Abstract. Let $F$ be a family of $n$ axis-parallel boxes in $\mathbb{R}^d$ and $\alpha \in (1 - 1/d, 1]$ a real number. There exists a real number $\beta(\alpha) > 0$ such that if there are $\alpha \binom{n}{2}$ intersecting pairs in $F$, then $F$ contains an intersecting subfamily of size $\beta n$. A simple example shows that the above statement is best possible in the sense that if $\alpha \leq 1 - 1/d$, then there may be no point in $\mathbb{R}^d$ that belongs to more than $d$ elements of $F$.

1. Introduction and results

According to the classical theorem of Helly [1], if every $d + 1$-element subfamily of a finite family of convex sets in $\mathbb{R}^d$ has nonempty intersection, then the entire family has nonempty intersection. Although the number $d + 1$ in Helly’s theorem cannot be lowered in general, it can be reduced for some special families of convex sets. For example, if any two elements in a finite family of axis-parallel boxes in $\mathbb{R}^d$ intersect, then all members of the family intersect, cf. [2].

Katchalski and Liu [7] proved the following generalization of Helly’s theorem for the case when not all but only a fraction of $d + 1$-element subfamilies have a nonempty intersection in a family of convex sets.

Fractional Helly Theorem. (Katchalski and Liu [7]) Assume that $\alpha \in (0, 1]$ is a real number and $F$ is a family of $n$ convex sets in $\mathbb{R}^d$. If at least $\alpha \binom{n}{d+1}$ of the $(d+1)$-tuples of $F$ intersect, then $F$ contains an intersecting subfamily of size $\alpha \frac{n}{d+1}$.

The bound on the size of the intersecting subfamily was later improved by Kalai [6] from $\frac{n}{d+1}$ to $(1 - (1 - \alpha)^{1/(d+1)})n$, and this bound is best possible.

In this paper, we study the fractional behaviour of finite families of axis-parallel boxes, or boxes for short. We note that the boxes can be either open or closed, our statements hold for both cases. Our aim is to prove a statement similar to the Fractional Helly Theorem.

The intersection graph $G_F$ of a finite family $F$ of boxes is a graph whose vertex set is the set of elements of $F$, and two vertices are connected by an edge in $G_F$ precisely when the corresponding boxes in $F$ have nonempty intersection.

Recall that for two integers $n \geq m \geq 1$, the Turán-graph $T(n, m)$ is a complete $m$-partite graph on $n$ vertices in which the cardinalities of the $m$ vertex classes are

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as close to each other as possible. Let \( t(n, m) \) denote the number of edges of the Turán graph \( T(n, m) \). It is known that \( t(n, m) \leq (1 - \frac{1}{m}) \frac{n^2}{2} \), and equality holds if \( m \) divides \( n \). Furthermore,

\[
\lim_{n \to \infty} \frac{t(n, m)}{\frac{n^2}{2}} = 1 - \frac{1}{m}, \tag{1}
\]

For more information on the properties of Turán graphs see, for example, the book of Diestel [3].

The following example shows that we cannot hope for a statement for boxes that is completely analogous to the Fractional Helly Theorem.

**Example 1.** Let \( n \geq d + 1 \) and \( m, k \geq 0 \) be integers such that \( n = md + k \) and \( 0 \leq k \leq d - 1 \). Let \( n_1, \ldots, n_d \) be positive integers with \( n = n_1 + \cdots + n_d \) and \( n_i = \left\lceil \frac{n}{d} \right\rceil \) for \( 1 \leq i \leq k \) and \( n_i = \lfloor \frac{n}{d} \rfloor \) for \( k + 1 \leq i \leq d \). For \( 1 \leq i \leq d \), consider \( n_i - 1 \) hyperplanes orthogonal to the \( i \)th coordinate direction. These hyperplanes cut \( \mathbb{R}^d \) into \( n_i \) pairwise disjoint open slabs \( B_{ij}, j = 1, \ldots, n_i \). Let \( C \) be a large open axis-parallel box that intersects each slab and let \( F_i \) consist of the open boxes \( B_{ij} = C \cap B_{ij} \). Define \( F \) as the union of the \( F_i \).

This way we have obtained a family \( F \) of \( n \) boxes with the property that two elements of \( F \) intersect exactly if they belong to different \( F_i \). The intersection graph of \( F \) is \( T(n, d) \) and thus the number of intersecting pairs in \( F \) is \( t(n, d) \). However, there is no point of \( \mathbb{R}^d \) that belongs to any \( d + 1 \)-element subfamily of \( F \). Thus, (1) shows that in a fractional Helly-type statement for boxes, the percentage \( \alpha \) has to be greater than \( 1 - \frac{1}{d} \).

Let \( n \geq k \geq d \) and let \( T(n, k, d) \) denote the maximal number of intersecting pairs in a family \( F \) of \( n \) boxes in \( \mathbb{R}^d \) with the property that no \( k + 1 \) boxes in \( F \) have a point in common.

**Theorem 1.** With the above notation,

\[
T(n, k, d) < \frac{d - 1}{2d} n^2 + \frac{2k + d}{2d} n.
\]

It is quite easy to precisely determine \( T(n, k, d) \) when \( d = 1 \):

**Proposition 1.** \( T(n, k, 1) = (k - 1)n - \binom{k}{2} \).

Theorem 1 directly implies the following corollary.

**Corollary 1.** Assume that \( \varepsilon > 0 \) is a real number and \( F \) is a family of \( n \) boxes in \( \mathbb{R}^d \). If at least \( \left( \frac{d - 1}{2d} + \varepsilon \right) n^2 \) pairs of \( F \) intersect, then \( F \) contains an intersecting subfamily of size \( dn \varepsilon - \frac{d}{2} + 1 \).

The proof of Corollary 1 is given in Subsection 2.2. Corollary 1 yields the next theorem, which is our main result.

**Fractional Helly Theorem for boxes.** For every \( \alpha \in (1 - \frac{1}{d}, 1] \) there exists a real number \( \beta(\alpha) > 0 \) such that, for every family \( F \) of \( n \) boxes in \( \mathbb{R}^d \), if an \( \alpha \) fraction of pairs are intersecting in \( F \), then \( F \) has an intersecting subfamily of cardinality at least \( \beta n \).

Kalai’s lower bound \( \beta(\alpha) = 1 - (1 - \alpha)^{1/(d+1)} \) for the size of the intersecting subfamily in the fractional Helly theorem yields that if \( \alpha \to 1 \), then \( \beta(\alpha) \to 1 \) as well. The same holds for families of parallel boxes as stated in the following theorem.
**Theorem 2.** Let $\mathcal{F}$ be a family of $n$ boxes in $\mathbb{R}^d$, and let $\alpha \in (1 - \frac{1}{d^2}, 1]$ be a real number. If at least $\alpha \binom{n}{2}$ pairs of boxes in $\mathcal{F}$ intersect, then there exists a point that belongs to at least $(1 - d\sqrt{1 - \alpha})n$ elements of $\mathcal{F}$.

Simple calculations show that Corollary 1 does not imply Theorem 2 so we provide a separate proof for it in Section 2.

2. Proofs

2.1. Proof of Theorem 1. It is enough to prove that if no $k + 1$ elements of $\mathcal{F}$ have a point in common, then there are at least $\frac{n^2 - 2(k + 1)n}{2d}$ non-intersecting pairs. We may assume by standard arguments that the boxes in $\mathcal{F}$ are all open, so $B \in \mathcal{F}$ is of the form $B = (a_1(B), b_1(B)) \times \cdots \times (a_d(B), b_d(B))$. We assume without loss of generality that all numbers $a_i(B), b_i(B) (B \in \mathcal{F})$ are distinct. For $B \in \mathcal{F}$ we define $\deg B$ to be the number of boxes in $\mathcal{F}$ that intersect $B$.

We prove Theorem 1 by induction on $n$. The starting case $n = k$ is simple since then $\frac{n^2 - 2(k + d)n}{2d} < 0$. In the induction step $n - 1 \rightarrow n$ we consider two cases.

**Case 1.** When there is a box $B$ with $\deg B \leq (1 - \frac{1}{d})n + \frac{2k + 1}{2d}$.

By induction, we have at least $\frac{(n - 1)^2 - 2(k + d)(n - 1)}{2d}$ non-intersecting pairs after removing $B$ from $\mathcal{F}$. Since $B$ is involved in at least $(n - 1) - (1 - \frac{1}{d})n - \frac{2k + 1}{2d}$ non-intersecting pairs, there are at least

$$\frac{(n - 1)^2 - 2(k + d)(n - 1)}{2d} - 1 + \frac{n}{d} - \frac{2k + 1}{2d} = \frac{n^2 - 2(k + d)n}{2d}$$

non-intersecting pairs in $\mathcal{F}$, indeed.

**Case 2.** For every $B \in \mathcal{F} \deg B \geq (1 - \frac{1}{d})n + \frac{2k + 1}{2d}$.

We show by contradiction that this cannot happen which finishes the proof.

We define $d$ distinct boxes $B_1, \ldots, B_d \in \mathcal{F}$ the following way. Set

$$c_1 = \min\{b_1(B) : B \in \mathcal{F}\}$$

and define $B_1$ via $c_1 = b_1(B_1)$. The box $B_1$ is uniquely determined as all $b_1(B)$ are distinct numbers. Assume now that $i < d$ and that the numbers $c_1, \ldots, c_{i - 1}$, and boxes $B_1, \ldots, B_{i - 1}$ have been defined. Set

$$c_i = \min\{b_i(B) : B \in \mathcal{F} \setminus \{B_1, \ldots, B_{i - 1}\}\}$$

and define $B_i$ via $c_i = b_i(B_i)$ which is unique, again.

Let $\mathcal{F}' = \mathcal{F} \setminus \{B_1, \ldots, B_d\}$. We partition $\mathcal{F}'$ into $d + 2$ parts. Let $\mathcal{F}_0$ be the set of all boxes of $\mathcal{F}'$ that intersect every $B_i$. For $i = 1, \ldots, d$ let $\mathcal{F}_i$ be the set of all boxes in $\mathcal{F}'$ that intersect every $B_j$ for $j \neq i$ but do not intersect $B_i$. Let $\mathcal{F}^*$ be the set of all boxes of $\mathcal{F}'$ that intersect at most $d - 2$ of the $B_i$ boxes. As this is a partition of $\mathcal{F}'$ we have

$$|\mathcal{F}_0| + \sum_{i=1}^{d} |\mathcal{F}_i| + |\mathcal{F}^*| = |\mathcal{F}'| = n - d.$$

Note that $|\mathcal{F}_0| \leq k$ since every box in $\mathcal{F}_0$ contains the point $(c_1, \ldots, c_d)$.

Let $N$ be the number of intersecting pairs between $\{B_1, \ldots, B_d\}$ and $\mathcal{F}'$. Each $B_i$ intersects at least $\deg B_i - (d - 2)$ boxes from $\mathcal{F}'$ as $B_i$ may intersect $B_j$ for all
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$j \in [d], j \neq i$. Since every \( \deg B_i \geq (1 - \frac{1}{d})n + \frac{2k+1}{2d} \) we have

\[
d \left( (1 - \frac{1}{d})n + \frac{2k+1}{2d} - (d - 1) \right) \leq N
\]

Every box in \( F_0 \) intersects every \( B_i, i \in [d] \), every box in \( F_i \) intersects every \( B_j \)
except for \( B_i \) and every box in \( F^* \) intersects at most \( (d-2) \) of the \( B_i \). Consequently

\[
N \leq d|F_0| + (d - 1) \sum_{i=1}^{d} |F_i| + (d - 2)|F^*|.
\]

So we have

\[
d \left( (1 - \frac{1}{d})n + \frac{2k+1}{2d} - (d - 1) \right) \leq d|F_0| + (d - 1) \sum_{i=1}^{d} |F_i| + (d - 2)|F^*|
\]

\[
= |F_0| + (d - 1) \left( |F_0| + \sum_{i=1}^{d} |F_i| + |F^*| \right) - |F^*|
\]

\[
= |F_0| + (d - 1)(n - d) - |F^*|.
\]

Simplifying the inequality and using \( |F_0| \leq k \) give

\[
k + \frac{1}{2} \leq |F_0| - |F^*| \leq k - |F^*|
\]

implying \( |F^*| \leq -\frac{1}{2} \), which is a contradiction.

2.2. **Proof of Corollary 1.** If no point of \( \mathbb{R}^d \) belongs to \( d n \varepsilon - \frac{d}{2} + 1 \) elements of \( F \), then by Theorem 1 the number of intersecting pairs of \( F \) is smaller than

\[
d - 1 \frac{n^2}{2d} + \frac{2d(n \varepsilon - \frac{d}{2}) + d}{n^2} = \left( \frac{d-1}{2d} + \varepsilon \right) n^2,
\]

which yields a contradiction.

2.3. **Proof of Theorem 2.** Let \( \pi_i \) denote the orthogonal projection to the \( i \)th dimension in \( \mathbb{R}^d \), that is, \( \pi_i(B) = (a_i(B), b_i(B)) \) for \( B \in F \). Set \( \varepsilon = 1 - \alpha \). Define \( T_i = \{ \pi_i(B) : B \in F \} \); this is a family of \( n \) intervals, and all but at most \( \varepsilon^n/2 \) of the pairs in \( T_i \) intersect. According to the sharp version of the fractional Helly theorem (cf. [6]), \( T_i \) contains an intersecting subfamily \( T_i' \) of size \( (1 - \sqrt{\varepsilon})n \), let \( c_i \) be a common point of all the intervals in \( T_i' \). Define \( D_i = \{ B \in F : c_i \notin \pi_i(B) \} \).

Then \( F \setminus \bigcup_1^d D_i \) consists of at least \( (1 - d\sqrt{\varepsilon})n = (1 - d\sqrt{1-\alpha})n \) boxes and all of them contain the point \( (c_1, \ldots, c_d) \).

2.4. **Proof of Proposition 1.** Let \( k \in \{1, \ldots, n\} \) be an integer, and let \( F \) be the family of open intervals \( (i, i+k) \) for \( i = 1, 2, \ldots, n \). Thus \( F \) consists of \( n \) intervals, no \( k + 1 \) of them have a point in common, and there are \( (k - 1)n - \binom{k}{2} \) intersecting pairs in \( F \). Consequently \( T(n, k, 1) \geq (k - 1)n - \binom{k}{2} \).

Next we show, by induction on \( n \) that \( T(n, k, 1) \leq (k - 1)n - \binom{k}{2} \). Let \( F \) be a family of \( n \) intervals such that no \( k + 1 \) of them have a common point. We assume that these intervals are closed which is no loss of generality. The statement is clearly true when \( n = k \). Let \( [a, b] \in F \) be the interval where \( b \) is minimal. Since any interval intersecting \( [a, b] \) contains \( b \), there are at most \( k - 1 \) intervals intersecting \( [a, b] \). Removing \( [a, b] \) from \( F \) and applying induction, we find there
are at most \((k - 1)(n - 1) - \binom{k}{2}\) intersecting pairs in \(\mathcal{F} \setminus \{[a, b]\}\). That is, there are
at most \(k - 1 + (k - 1)(n - 1) - \binom{k}{2} = (k - 1)n - \binom{k}{2}\) intersecting pairs in \(\mathcal{F}\).

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