Ω_0 < 1 From Inflation

Martin Bucher†‡
Alfred S. Goldhaber*
Neil Turok†

†Physics Department, Princeton University
Princeton, New Jersey 08544

* Institute for Theoretical Physics, State University of New York
Stony Brook, New York 11794-3840

Abstract

An inflationary scenario that leads to Ω_0 < 1 today is presented. An epoch of ‘old’ inflation during which the smoothness and horizon problems are solved is followed by a shortened epoch of ‘new’ inflation. Old inflation exits through the nucleation of a single bubble, leading to negative spatial curvature on slices of constant cosmic time. We calculate the spectrum of density perturbations in such a scenario.

January 1995

‡ Talk presented at the ASTPART conference in Stockholm, Sweden in September 1994.
1. INTRODUCTION

The ratio of the mean density of the universe to the critical density \( \Omega_0 = \rho_0/\rho_{\text{crit}} \) is a crucial cosmological parameter that determines the geometry of the universe. For \( \Omega = 1 \), slices of constant cosmic time are Euclidean, with vanishing spatial curvature. On the other hand, if \( \Omega < 1 \), slices of constant cosmic time are hyperbolic, with constant negative spatial curvature. Although there is some observational evidence in favor of \( \Omega = 1 \), many observations suggest a universe of subcritical density, with \( \Omega < 1 \).

Inflation is an attractive scenario for the very early universe because of its elegant solution of the horizon and smoothness problems. The exponential expansion during inflation smooths out whatever inhomogeneities may have existed prior to inflation. Moreover, in addition to erasing initial conditions, inflation also provides a predictive mechanism for generating the primordial density perturbations that seed later structure formation.

In this contribution we investigate the compatibility of \( \Omega_0 < 1 \) with inflation. In particular, we investigate the power spectrum of adiabatic density perturbations that results from a low-\( \Omega_0 \) model of inflation. This work is discussed in more detail in ref. 1, which also contains a more extensive set of references.

Although standard inflation predicts \( \Omega_0 \) equal to one to extremely high accuracy, inflation with low \( \Omega_0 \) is also possible. The basic idea is due to Coleman and de Luccia \(^2\) and to Gott \(^3\). When the false vacuum decays through the nucleation of a single bubble, inside the bubble the spatial hypersurfaces on which the scalar field is constant, indicated by the solid lines in Fig. 1, are surfaces of constant negative spatial curvature. In new inflation the surfaces on which the inflaton field is constant are the natural constant time hypersurfaces. The interior of the bubble contains an expanding open FRW universe, described by the metric

\[
d s^2 = -d t^2 + a^2(t) \left[ d \xi^2 + \sinh^2(\xi) d \Omega^2_{(2)} \right]. \tag{1.1}
\]

We call this patch region I. Another coordinate patch (which we call region II) with
the line element

\[
ds^2 = d\sigma^2 + b^2(\sigma) \cdot [-d\tau^2 + \cosh^2[\xi]d\Omega^2_{(2)}]
\]

(i.e., a coordinate system somewhat akin to ‘Rindler’ coordinates) covers the exterior of the bubble and part of the interior as well. For de Sitter space \( a(t) = H^{-1} \sinh[Hz] \) and \( b(\sigma) = H^{-1} \sin[Hz] \).

† Actually to cover maximally extended de Sitter space requires three additional coordinate patches identical in structure to region I, but this point shall not concern us here.

Consider a potential for the inflaton field of the form sketched in Fig. 2. The inflaton field starts in the false vacuum during the initial epoch of old inflation, during which the horizon and smoothness problems are solved. During this time the spacetime geometry is that of pure de Sitter space, with no preferred time direction. Then old inflation is exited through the nucleation of a single bubble. The nucleation event, a classically forbidden process, roughly corresponds to the region below the dashed line in Fig. 1. Then the bubble expands classically, at a velocity approaching the velocity of light. The thin wall picture is inadequate for our purposes here—the bubble wall has no well defined edge. Instead of tunneling directly to the true vacuum, the inflaton field tunnels onto a slowing rolling potential, so that a shortened epoch of new inflation occurs inside the bubble. Formally, \( \Omega = 0 \) on the null surface bounding region I (at \( t = 0 \)). As \( t \) increases, \( \Omega \) flows toward one until reheating, and thereafter flows away from one.

The value of \( \Omega_0 \) today is determined by the shape of the inflaton field potential. The length of new inflation is proportional to the logarithm of the deviation of \( \Omega \) from unity at reheating. Consequently, no unnatural fine tuning is required to obtain a low value of \( \Omega_0 \) because it is the logarithm of a small number rather than the small number itself that must be adjusted, and the usual naturalness argument against \( \Omega_0 \neq 0 \) is inapplicable. In fact, it seems that inflation is necessary to overcome the naturalness argument.
2. DENSITY PERTURBATIONS

In the scenario for open inflation described above, Gaussian adiabatic density perturbations are generated by the same physical mechanism as in flat inflation. Vacuum quantum fluctuations of the inflaton field, initially well within the horizon, turn into adiabatic density perturbations as they cross the horizon. However, the computation of the power spectrum from open inflation is more complicated because of the effects of curvature on large scales. On small co-moving scales (small compared to the curvature scale) one has an approximately scale invariant spectrum of density perturbations, just as in flat inflation, because those modes cross the horizon during the latter part of the new inflationary epoch, when $\Omega$ was close to one and when there was an approximate time translation symmetry of the physical horizon volume. On the other hand, larger scales (of order the curvature scale) cross the horizon during the earlier part of new inflation, when $\Omega$ was significantly less than one and when the approximate time translation symmetry of the physical horizon volume during inflation is badly broken. Moreover, these large scale modes are more sensitive to the quantum state of the scalar field during the prior old inflationary epoch.

The starting point for a calculation of the power spectrum from open inflation is the quantum state of the perturbations of the inflaton field about the background solution during old inflation. These fluctuations are well described by a free scalar field of mass $m^2 = V''[\phi_f]/V'([\phi_f])$ in a de Sitter background. To simplify the computation, we shall take $m^2 = 2H^2$. It is well known that after a sufficient amount of old inflation, an arbitrary initial state will approach what is known as the Bunch-Davies, or Euclidean vacuum, described by the two-point Wightman function

$$G^+(\omega) = \frac{H^2(1 - \nu^2)}{16\pi} \times _2 F_1 \left( \frac{3}{2} - \nu, \frac{3}{2} + \nu; \frac{I(X, X') + 1}{2} \right)$$

where $\nu^2 = \frac{9}{4} - m^2/H^2$ and $I(X, X') = -\bar{w}\bar{w}' + \bar{u}\bar{u}' + \bar{x}\bar{x}' + \bar{y}\bar{y}' + \bar{z}\bar{z}' - i\epsilon \cdot \epsilon(X, X')$.

The quantum fluctuations of the scalar field coupled to scalar gravity at the end of inflation determine the spectrum of adiabatic density perturbations seen today.
These fluctuations are calculated by propagating the ‘positive frequency’ modes of the Bunch-Davies vacuum into the bubble interior through the bubble wall. As the modes propagate through the bubble wall the effective mass squared of the inflaton field changes, from \( V''[\phi] \) outside the bubble wall, to a negative value inside the bubble wall (near the maximum of the potential), and finally to nearly zero inside the bubble, where the potential is very flat.

In order to calculate the effects of this changing effective mass, it is necessary to work in coordinates that maximally exploit the \( SO(3,1) \) symmetry of the expanding bubble solution. We start in region II, described by the line element in eqn. (1.2), and express the Bunch-Davies two point function as a sum of region II modes. To calculate the power spectrum it is sufficient to consider only the \( s \)-wave. The \( s \)-wave mode functions

\[
f^{(\pm)}(u, \tau; \zeta) = \frac{1}{4\pi \sqrt{|\zeta|}} \frac{e^{\pm i\zeta u}}{\tanh u \cosh \tau} e^{\mp i|\zeta|\tau}
\]

form a basis of normalized modes for the \( s \)-wave sector in region II, where \( \tanh[u] = \cos[H\sigma] \). We use units in which \( H = 1 \) so that \( b(\sigma) = \sin(\sigma) \). Although the mode functions \( f^{(+)}(u, \tau; \zeta) \) formally appear to be ‘positive frequency’ mode functions (when viewed in region II hyperbolic coordinates), the true ‘positive frequency’ mode functions corresponding to modes that annihilate the Bunch-Davies vacuum are the linear combinations

\[
g_{\zeta}^{(+)} = \frac{e^{\zeta|\pi/2} f_{\zeta}^{(+)} - e^{-|\zeta|\pi/2} f_{\zeta}^{(-)}}{(e^{+|\zeta|\pi} - e^{-|\zeta|\pi})^{1/2}}
\]

related by a Bogolubov transformation.

Expanding the \( s \)-wave component of the inflaton field as a sum over region II mode functions, one has

\[
\hat{\phi}_{(s)} = \int \limits_{-\infty}^{+\infty} d\zeta \left[ g_{\zeta}^{(+)} \hat{a}_{\zeta} + g_{\zeta}^{(+)} \hat{a}_{\zeta}^\dagger \right]
\]

where the operators \( \hat{a}_{\zeta} \) annihilate the Bunch-Davies vacuum.
Assuming for simplicity that the size of the bubble is small compared to the Hubble length during old inflation and that the bubble radius and thickness is small compared to the co-moving length scales of interest, we make the approximation that the mass changes discontinuously from \( m^2 = 2H^2 \) in region II to \( m^2 = 0 \) in region I. To calculate expectation values in region I, one propagates the mode functions given in eqn. (2.3) into region I. This involves matching on the null surface separating regions I and II. In region I we take into account the coupling of the inflaton field perturbations to scalar gravity using the gauge invariant formalism, originally developed by Bardeen. Instead of \( \hat{\phi} \) it is convenient to use as the dynamical variable the gauge invariant gravitational potential \( \hat{\Phi} \), regarded as a quantum field. The power spectrum is the hyperbolic Fourier transform of the two point function of \( \hat{\Phi} \) at constant time.

The density perturbations are most elegantly described in terms of the variable

\[
\chi = \frac{2H^{-1}\phi' + \Phi}{3} + \frac{\Phi}{1 + w} \tag{2.5}
\]

which in flat models is conserved on superhorizon scales. In open models \( \chi \) is conserved on superhorizon scales while \( \Omega \) is close to one—that is during the latter part of new inflation until the latter part of matter domination, when the universe starts to become curvature dominated.

In terms of \( \chi \) the power spectrum is

\[
P_\chi(\zeta) \sim \left( \frac{H^3}{V,\phi} \right)^2 \frac{\coth[\pi \zeta]}{\zeta(\zeta^2 + 1)} \tag{2.6}
\]

where

\[
\langle \chi(\zeta) \chi(\zeta') \rangle = P_\chi(\zeta)\delta(\zeta - \zeta') \tag{2.7}
\]

and

\[
\langle \chi(\xi) \chi(0) \rangle = \int_0^\infty d\zeta \zeta^2 \frac{\sin[\zeta \xi]}{\sinh[\xi]} P_\chi(\zeta) \tag{2.8}
\]

With these conventions \( P \sim \zeta^{-3} \) corresponds to scale invariance. This is seen by
computing $\langle \chi^2(0) \rangle$ using the small $\xi$ limit of eqn. (2.6) and noting the logarithmic divergence for large $\zeta$.

The spectrum of density perturbations from open inflation has also been considered by Lyth and Stewart\cite{4} and by Ratra and Peebles\cite{5}. Their treatments differ from ours in the choice of initial conditions. Instead of using the Bunch-Davies vacuum as an initial condition in the epoch of old inflation, they use a conformal vacuum for region I considered in isolation. Consequently, the power spectrum given in eqn. (2.6) differs from that given in refs. 4 and 5 by the factor of $\coth[\pi \zeta]$, precisely as one would expect from the Bogolubov coefficients in eqn. (2.3).

In related work Allen and Caldwell\cite{6} and Yamamoto, Tanaka, and Sasaki\cite{7} independently obtained the same Bogolubov coefficients for the Bunch-Davies vacuum in terms of hyperbolic modes.

We have also investigated the more general case where $m^2/H^2 \neq 2$. It turns out that for reasonable values of $\Omega$, large enough to be consistent with lower bounds on the observed mass density of the universe, varying $m^2/H^2$ only slightly changes the power spectrum for observationally accessible wave numbers. These results will be presented elsewhere.\cite{8}

We acknowledge useful conversations with Bruce Allen, Robert Caldwell, David Lyth, Jim Peebles, Bharat Ratra, Misao Sasaki, and Frank Wilczek. This work was partially supported by NSF contract PHY90-21984 and by the David and Lucile Packard foundation.

**Figure Captions.**

*Fig.1— Bubble Nucleation.* An open universe emerges through the bubble nucleation, described semiclassically by the Coleman–de Luccia instanton and its continuation to Lorentzian spacetime.

*Fig.2— Potential for Open Inflation.*
REFERENCES

1. M. Bucher, A.S. Goldhaber and N. Turok, “An Open Universe From Inflation,” Princeton Preprint (10-94) hep-ph 94-11206

2. S. Coleman and F. De Luccia, “Gravitational Effects on and of Vacuum Decay,” Phys. Rev. D21, 3305 (1980).

3. J.R. Gott, III, “Creation of Open Universes from de Sitter Space,” Nature 295, 304 (1982); J.R. Gott, III, and T.S. Statler, Phys. Lett. 136B, 157 (1984); J.R. Gott, III, “Conditions for the Formation of Bubble Universes,” in E.W. Kolb et al., Eds. Inner Space/Outer Space, (Chicago, U. of Chicago Press, 1986).

4. D. Lyth and E. Stewart, “Inflationary Density Perturbations with $\Omega < 1$,” Phys. Lett. B252, 336 (1990).

5. B. Ratra and P.J.E. Peebles, “Inflation in an Open Universe,” PUPT-1444 (Feb. 1994); B. Ratra and P.J.E. Peebles, “CDM Cosmogony in an Open Universe,” PUPT-1445 (Feb. 1994).

6. B. Allen and R. Caldwell, Preprint in preparation.

7. K. Yamamoto, T. Tanaka, and M. Sasaki, Preprint in preparation.

8. M. Bucher and N. Turok, “Power Spectrum from Open Inflation for Arbitrary $m_{fv}^2/H_{f}^2$,” In preparation.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9501396v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9501396v1
