CONSTRANTS ON VELOCITY ANISOTROPY OF SPHERICAL SYSTEMS WITH SEPARABLE AUGMENTED DENSITIES

JIN H. AN
National Astronomical Observatories, Chinese Academy of Sciences, A20 Datun Road, Chaoyang District, Beijing 100012, China; jinan@nao.cas.cn

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ABSTRACT

If the augmented density of a spherical anisotropic system is assumed to be multiplicatively separable into functions of the potential and the radius, the radial function, which can be completely specified by the behavior of the anisotropy parameter alone, also fixes the anisotropic ratios of every higher order velocity moment. It is inferred from this that the non-negativity of the distribution function necessarily limits the allowed behaviors of the radial function. This restriction is translated into the constraints on the behavior of the anisotropy parameter. We find that not all radial variations of the anisotropy parameter satisfy these constraints and thus anisotropy profiles exist that are not consistent with any separable augmented density.

Key words: dark matter – galaxies: kinematics and dynamics – methods: analytical

1. INTRODUCTION

One basic problem in stellar dynamics is to find the distribution function that is consistent with the given local density profile. For the simplest cases, the problem reduces to solving an integral equation. The solution to finding the ergodic distribution function for an isotropic spherical system, that is, the so-called Eddington formula, has been known for as long as its namesake (Eddington 1916). The complete solutions for devising the two-integral even distribution function for an axisymmetric system are also available through works by Fricke (1952), Lynden-Bell (1962), Hunter & Qian (1993), etc.

By contrast, the construction of a two-integral distribution function for an anisotropic spherical system contains an additional difficulty. This is because integrating the two-integral distribution function over the velocity space produces an augmented density, which is a bivariate function of the potential and the radius. While it is an easy exercise to demonstrate that the distribution function from the augmented density is a formally identical problem to the case of the axisymmetric system with a two-integral even distribution function (see, e.g., Qian 1993; An 2011), it is also obvious from the outset that, given the local density and the potential, there is no unique way to specify the bivariate augmented density without any additional prescription regarding the system. In fact, if the potential is known, specifying the augmented density is essentially equivalent to knowing an infinite subset of the velocity moment functions (cf. Dejonghe & Merritt 1992), to the zeroth of which the local density corresponds. The usual line of attack is to restrict either the augmented density or the distribution function to be in the specified functional form (e.g., Osipkov 1979; Merritt 1985; Cuddeford 1991; Cuddeford & Louis 1995; An & Evans 2006a; Wojtak et al. 2008) and match further properties such as the anisotropy parameter to that following the particular ansatz. One advantage of this approach is that these procedures usually simplify the subsequent inversion for the distribution function from the augmented density although these tend to sacrifice the flexibility in the behaviors of the varying anisotropy.

On the other hand, the procedure that uniquely specifies the system and also allows the greatest possible freedom for the radially varying anisotropy has been outlined in Qian & Hunter (1995). If the potential and radial dependencies of the augmented density are assumed to be multiplicatively separable, then the radial portion can be determined by the anisotropy parameter alone. The potential portion then follows immediately once the potential and the local density are specified. In fact, demanding the separable augmented density is the only route that fixes the anisotropy parameter independently of knowing the potential and the density. A practical implementation utilizing a general parametric form of the monotonically varying anisotropy parameter is found in Baes & Van Hese (2007), although adopting any parametric form intrinsically restricts the full flexibility of the technique.

Recently, the separable augmented density has attracted renewed interest in the context of the necessary conditions for the distribution function to be non-negative. For instance, Cioffi & Morganti (2010b) proposed the existence of the so-called global density slope–anisotropy inequality, which conditionally extends the central density slope–anisotropy theorem of An & Evans (2006b) to all radii, for any system with a separable augmented density. This has been subsequently proved with a restriction on the central anisotropy by Van Hese et al. (2011) and An (2011).

This paper further explores the implications of the separability in the augmented density and its limitation. In particular, we examine the relation between the distribution function and the moment functions and derive how the anisotropy parameter together with the potential and density uniquely specifies the separable augmented density. We also find that the radial part of the separable augmented density completely determines the anisotropic behaviors of not only the velocity dispersions but also every higher order even velocity moment. Based on these findings, we also show that the non-negativity of the moment functions, which follows the non-negativity of the distribution function, restricts the physically permitted behaviors of the separable augmented density and the anisotropy parameter described by it. In the following discussion, we consider our constraints in the context of the necessary and sufficient conditions for a separable augmented density corresponding to a physical system, extending prior works. Finally, further discussion concerning the inversion for the anisotropy profile in relation to the separable augmented density and our constraints is also provided.
2. PRELIMINARY

The Jeans theorem (Jeans 1915) indicates that a steady-state spherical dynamical system is described by the phase-space distribution function of the form \( f(E, L^2) \). Here, \( E \), the specific binding energy, and \( L \), the magnitude of the specific angular momentum, are the two classical isotropic isolating integrals admitted by the generic spherical potential. That is, \( E = \Psi(r) - \frac{1}{2}v^2 \) and \( L = rv_i \), where \( \Psi \) is the relative potential with respect to the boundary (hence, \( E > 0 \) for a bound particle).

Finally, \( v = \sqrt{v^2 + v^2 + v^2} \) with \( v_i = \sqrt{v^2 + v^2 + v^2} \) and \( v \) being the tangential and radial velocities. Note that \( (v, v, v) \) constitutes the set of three orthogonal velocity components in a common unit.

Integrating the distribution function over the velocity space,

\[
N(\Psi, r^2) \equiv \iiint_{|v|^2 < 2\Psi} d^3v f(E, L^2) = \frac{2\pi}{r} \int_{E > 0, L^2 > 0} \frac{f dE dL^2}{\sqrt{2\pi(\Psi - E) - L^2}} \tag{1}
\]

results in a bivariate function of \( \Psi \) and \( r^2 \). This is usually referred to as the “augmented density.” Once the potential \( \Psi = \Psi(r) \) (which is not necessarily generated by the following density) is specified, the local density is found to be \( \nu(r) = N(\Psi(\nu), r^2) \).

In a self-consistent system, however, Poisson equation with the augmented density as the source term results in an ordinary differential equation on \( \Psi(\nu) \), which can be solved to determine \( \nu(\nu) \) uniquely.

The local higher order velocity moments are found similarly. Whereas any odd-integral moment must vanish thanks to the spherical symmetry (in particular, the isotropy in the configuration space), all the even-integral moments are found to be

\[
\Psi_{\nu} = \Psi_{\nu} / N \quad \text{where}
\]

\[
m_{p, q}(\Psi, r^2) = \frac{2\pi}{r^{2q+2}} \int_{|v|^2 < 2\Psi} \frac{d^3v v^{2p+2q} f(E, L^2)}{\sqrt{2\pi(\Psi - E) - L^2}} \tag{2a}
\]

Here the transform kernel is given by

\[
K(E, L^2; \Psi, r^2) \equiv 2(\Psi - E) - \frac{L^2}{r^2}, \tag{2b}
\]

which is actually \( v^2 \) expressed as a function of the quadruple \( (E, L^2; \Psi, r^2) \), whereas the integral is over the region in the \((E, L^2)\) space defined to be

\[
T \equiv \{(E, L^2) \mid E \geq 0, L^2 \geq 0, K \geq 0\}, \tag{2c}
\]

that is, the triangular region bound by lines \( E = 0, L^2 = 0 \), and \( K = 0 \). The last line is the same as the diagonal line given by \( E + (2r^2)^{-1}L^2 = \Psi \).

If \( p > \frac{1}{2} \), we find that

\[
\frac{\partial(r^{2p+2}m_{p, q})}{\partial X} = (2p - 1)\pi \int_T dE dL^2 K^{p-1} \frac{\partial K}{\partial X} L^{2q} f, \tag{3}
\]

where \( X = \Psi \) or \( r^2 \). Given that \( \partial K/\partial \Psi = 2 \) and \( \partial K/(r^2) = L^2/r^2 \), Equation (3) indicates the existence of differential recursion relations for the moment functions

\[
\frac{\partial m_{p, q}}{\partial \Psi} = (2p - 1)m_{p-1, q}, \tag{4a}
\]

\[
\frac{\partial(r^{2p+2}m_{p, q})}{\partial r^2} = \left( p - \frac{1}{2} \right) r^{2q}m_{p-1, q+1}, \tag{4b}
\]

which is valid for \( p > \frac{1}{2} \). In fact, once the augmented density is specified, every other velocity moment can be recovered without inverting for the distribution function. Specifically, given \( m_{0, 0} = N \) and the initial conditions \( m_{p, 0}(0, r^2) = 0 \), we first find that for \( k \geq 1 \)

\[
m_{k, 0}(\Psi, r^2) = 2^k \left( \frac{1}{2} \right) \frac{\Psi}{2} \int_0^\Psi \Psi \prod_{i=0}^{k-1} \frac{d\Psi_{\nu}}{d\Psi} \left( \frac{\Psi_{\nu}}{\Psi} \right) \tag{5a}
\]

via repeated integrations of Equation (4a) and the Cauchy formula for repeated integration (Equation (A5c)). Here, \( (a_n)_{n=1}^\infty \) is the rising sequential product (the Pochhammer symbol). Next, the repetitions of Equation (4b) yield

\[
\prod_{j=1}^k \left( \frac{1}{2} + j \right) m_{k-q, q} = \frac{1}{r^{2q+2}} \left( r^4 \frac{\partial}{\partial r^2} \right)^q r^2 m_{k, 0}, \tag{5b}
\]

for \( 0 \leq q \leq k \). Combining these and using Equation (A3), we recover the result of Dejonghe & Merritt (1992, Equation (13)),

\[
m_{k-q, q}(\Psi, r^2) = 2^\left( k - \frac{1}{2} \right) \frac{k - q}{k - 1} \prod_{i=0}^{k-1} \frac{d\Psi_{\nu}}{d\Psi} \left( \frac{\Psi_{\nu}}{\Psi} \right) \tag{5c}
\]

for \( k \geq 1 \) and \( 0 \leq q \leq k \). The proper verification that Equation (5c) is the unique solution to Equation (4) given \( m_{p, q}(0, r^2) = 0 \) is provided in Appendix A. An immediate corollary following Equation (5c) is that

\[
\left( \frac{\partial}{\partial r^2} \right)^q \left[ r^{2q} N(\Psi, r^2) \right] \geq 0 \tag{6}
\]

for every non-negative integer \( q \) is a sufficient (but not necessary) condition for every \( m_{p, q} \) to be non-negative.

The behavior of the anisotropic velocity dispersions in a spherical system is usually parameterized by the “velocity anisotropy parameter” (Binney 1980),

\[
\beta(\nu) \equiv 1 - \frac{\sigma_v^2}{2\sigma_v^2} = 1 - \frac{m_{0, 1}(\Psi(\nu), r^2)}{2m_{1, 0}(\Psi(\nu), r^2)}, \tag{7a}
\]

where \( \sigma_v^2 = \frac{\nu_v^2}{v^2} \) and \( \sigma_v^2 = \frac{\nu_v^2}{v^2} \). Meanwhile, Equation (4b) with \( (p, q) = (1, 0) \) reduces to

\[
\frac{\partial(r^{2m_{1, 0}})}{\partial r^2} = \frac{m_{0, 1}}{2}, \tag{7b}
\]

Hence, the anisotropy parameter is directly related to the radial partial derivative of the moment function \( m_{1, 0} \), that is,

\[
\beta(\nu) = 1 - \frac{m_{0, 1}}{2m_{1, 0}} = \frac{\partial \ln m_{1, 0}}{\partial \ln r^2} \bigg|_{\Psi(\nu), r^2}. \tag{7c}
\]
Also notable is that the total radial derivative of \( m_{1,0} \) results in
\[
\frac{dm_{1,0}}{dr} = \frac{\partial m_{1,0}}{\partial r} + \frac{\partial m_{1,0}}{\partial \Psi} \frac{d \Psi}{dr} = \frac{2m_{1,0}}{r} \frac{\partial \ln m_{1,0}}{\partial \ln r^2} + m_{0,0} \frac{d \Psi}{dr}. \tag{8}
\]
With \( \Psi = \Psi(r) \) and Equation (7c), Equation (8) is simply the second-order steady-state spherical Jeans equation. Note in fact that \( \Psi(r) = \Phi_0 - \Phi(r) \) is the relative potential with respect to the reference \( \Phi_0 \) where \( \Phi(r) \) is the true gravitational potential, and thus \(-d\Psi/dr = d\Phi/dr = GM/r^2\) where \( M \) is the enclosed gravitating mass within the sphere of radius \( r \).

More generally, for \( p > \frac{1}{2} \), Equation (4b) indicates that
\[
\frac{\partial m_{p,q}}{\partial r} = -\frac{2}{r} \left[ (q + 1)m_{p,q} - \left( p - \frac{1}{2} \right) m_{p-1,q+1} \right]. \tag{9}
\]
Consequently, for the corresponding total radial derivatives result in
\[
\frac{dm_{p,q}}{dr} = -\frac{2}{r} \left[ (q + 1)m_{p,q} - \left( p - \frac{1}{2} \right) m_{p-1,q+1} \right] + (2p - 1)m_{p-1,q+1} \frac{d \Psi}{dr}, \tag{10}
\]
which in fact constitute the complete set of the Jeans equations (Dejonghe & Merritt 1992). See also Merrifield & Kent (1990) for the fourth-order equations that correspond to \((p, q) = (2, 0)\) and \((1, 1)\) here.

### 3. Separable Augmented Density

Let us suppose that the \( \Psi \) and \( r^2 \) dependencies of the augmented density are multiplicatively separable as in
\[
N(\Psi, r^2) = P(\Psi) R(r^2), \tag{11}
\]
for some functions \( P(\Psi) \) and \( R(r^2) \). It then follows from Equation (5c) that every moment function \( m_{p,q}(\Psi, r^2) \) is also separable. In particular, \( m_{p,q}(\Psi, r^2) = 2^{p+q}(1/2)_p P_{p+q}(\Psi) R_q(r^2) \) or
\[
m_{k-n,n}(\Psi, r^2) = 2^{k}(1/2)_{k-n} P_k(\Psi) R_n(r^2) \tag{12a}
\]
for \( 0 \leq n \leq k \), where
\[
P_k(\Psi) \equiv \begin{cases} \frac{P(\Psi)}{(k-1)!} & (k = 0) \\ \frac{1}{k} \int_0^\Psi dQ (\Psi - Q)^{k-1} P(Q) & (k \geq 1) \end{cases}, \tag{12b}
\]
\[
R_n(r^2) = \left( \frac{d}{dr^2} \right)^n [r^{2n} R(r^2)] = \frac{1}{r^{2n+2}} \left( r^4 \frac{d}{dr^2} \right)^n [r^2 R(r^2)]. \tag{12c}
\]
We find from Equation (12b) that \( dP_k/d\Psi = P_{k-1} \) for any positive integer \( k \). Similarly, Equations (12c) and (A3) lead to \( d(r^{2n+2} R_n)/dr = r^{2n} R_{n+1} \) for any non-negative integer \( n \). Next, since \( m_{1,0}(\Psi, r^2) = P_1(\Psi) R(r^2) \), Equation (7c) indicates
\[
\beta(r) = -\frac{d \ln R(r^2)}{d \ln r^2}, \tag{13a}
\]
\[
R(r^2) = \exp \left[ 2 \int_r^\infty \frac{\beta(\tilde{r})}{\tilde{r}} d\tilde{r} \right]. \tag{13b}
\]
In other words, the radial function \( R(r^2) \) is completely specified (up to an immaterial scale constant) given the anisotropy parameter \( \beta(r) \). Once \( R(r^2) \) is specified, the potential part \( P(\Psi) \) immediately follows the local density \( \nu(r) \) as \( P(\Psi) = \nu(r)/R(r^2) \) with the inverse function \( r = \Psi^{-1}(\Psi) \) of the potential \( \Psi(r) \) (Qian & Hunter 1995; Baes & Van Hese 2007).

### 3.1. Implications of the Separable Augmented Density

Given the boundary conditions \( \Psi(r_0) = 0 \) and \( m_{1,0}(0, r_0^2) = 0 \) at \( r = r_0 \) (which may be infinite), the radial velocity dispersion is given by
\[
\nu^2 = \frac{m_{1,0}[\Psi(\nu), r^2]}{R(r^2)} = \frac{R(r^2)}{\int_0^{\Psi(r)} dQ P(Q)} \tag{14a}
\]
\[
= R(r^2) \int_0^r v(\nu) \frac{d \Psi}{R(r^2) d\nu} d\nu. \tag{14b}
\]
Here Equation (14b) is actually the solution to the steady-state spherical Jeans equation with \( R^{-1} \) of Equation (13b) being its integrating factor (e.g., van der Marel 1994; An & Evans 2009). In other words, Equation (14b) always provides the velocity dispersions of the system given the potential, the density, and the anisotropy parameter irrespective of the separability assumption.

However, the true implications of the separability assumption lie beyond the behaviors of the velocity dispersions. That is to say, with the separable augmented density assumption, the anisotropy parameter not only specifies the complete augmented density together with the local density and the potential, but also constrains the anisotropic behaviors of every higher order velocity moment (including those of velocity dispersions) by itself.

In particular,
\[
\alpha_n = \frac{R_n}{R_0} = \left( \frac{1}{2} + p \right) \frac{m_{p,n}}{m_{p+n,0}}, \tag{15a}
\]
while \( \alpha_0(r) \) for any non-negative integer \( n \) is determined recursively from \( \beta(r) \) alone (Appendix B) via
\[
\alpha_{n+1} = (n + 1 - \beta) \alpha_n + \alpha'_n \tag{15b}
\]
with \( \alpha_0 = 0 \). Here, \( \alpha'_n = d\alpha_n/dr = d\alpha_n/d ln r^2 \) where \( u = ln r^2 \). For a few small \( n \), this is specifically translated into
\[
\alpha_1 = 1 - \beta = \frac{m_{0,1}}{2m_{1,0}} = \frac{3m_{1,1}}{2m_{2,0}} = \frac{5m_{2,1}}{2m_{3,0}} = \ldots, \tag{15c}
\]
\[
\alpha_2 = (1 - \beta)(2 - \beta) - \beta' = \frac{3m_{0,2}}{4m_{2,0}} = \frac{15m_{1,2}}{4m_{3,0}} = \ldots, \tag{15d}
\]
\[
\alpha_3 = (1 - \beta)(2 - \beta)(3 - \beta) - 3(2 - \beta)^2 \beta'' = \frac{15m_{0,3}}{8m_{3,0}} = \ldots, \tag{15e}
\]
and so on. Here, \( \beta'' = d^2 \beta/du^2 = d^2 \beta/dr^2 \), etc.

Furthermore, we also have
\[
\frac{dm_{p,q}}{dr} = \frac{d}{dr} \left( \frac{m_{p,q}}{m_{p+q,0}} \right) = \frac{C_{p,q} R_q}{C_{p+q,0} R_0} \frac{dm_{p+q,0}}{dr} + C_{p,q} P_{p+q,0} d \alpha_q/dr, \tag{16a}
\]
\[
\frac{d \Psi}{dr} = \frac{d \Psi}{d \ln r^2} = \frac{R(r^2)}{2r} \frac{d R(r^2)}{d \ln r^2} + \frac{\nu(r)}{R(r^2) d \ln r^2}. \tag{16b}
\]
Further explorations of this idea will be given elsewhere.

\[
\frac{\partial m_{p,q}}{\partial r} = C_{p,q} P_{p+q} \frac{d}{dr} \left( \frac{R_a}{R_0} \right)
\]

\[
= C_{p,q} P_{p+q} \frac{R_a}{R_0} \frac{dR_0}{dr} + C_{p,q} P_{p+q} R_0 \frac{da_q}{dr},
\]

(16b)

\[
\frac{\partial m_{p,q}}{\partial \Psi} = C_{p,q} \frac{dP_{p+q}}{d\Psi} R_a = C_{p,q} P_{p+q-1} R_a,
\]

(16c)

where \(C_{p,q} = 2^p q(1/2)_p\). Therefore, expressing the total radial derivative of \(m_{p,q}(\Psi, r^2)\) leads to (note \(R_0 = R\))

\[
\frac{dm_{p,q,0}}{dr} = C_{p,q+1} \left[ P_{p+q} \frac{dR}{dr} + P_{p+q-1} R \frac{d\Psi}{dr} \right].
\]

(17a)

Given Equation (13a), this indicates that if the augmented density is assumed to be separable, the set of the \((2n)\)th order spherical Jeans equations (Equation (10) with \(p + q = n\) and \(p \geq 1\)) reduces to a single equation,

\[
\frac{dm_{n,0}}{dr} + \frac{2\beta}{r} m_{n,0} = (2n-1)m_{n-1,0} \frac{d\Psi}{dr}.
\]

(17b)

This generalizes the fourth-order Jeans equation for “constant anisotropy” introduced by Łokas (2002, see also Łokas & Mamon 2003). We note, however, that Equation (17b) is in fact the result of the separability assumption and not of the constant anisotropy per se. In addition, under the separability assumption, the solution to Equation (17b) is immediately obvious as per \(m_{n,0} = (2n-1)!^1 P_n(\Psi(r)) R(r^2)\) with Equations (12b) and (13b) as well as \(P(\Psi(r)) = \psi(r)/R(r^2)\).

3.2. Constraints on the Anisotropy Parameter

3.2.1. General Cases with a Separable Augmented Density

The non-negativity of the distribution function implies that all the even-integral moment functions must also be non-negative. Consequently, if \(N(\Psi, r^2) = P(\Psi) R(r^2)\) is separable, then \(P_n(\Psi) \geq 0\) and \(R_n(r^2) \geq 0\) for any non-negative integer \(n\). While \(P(\Psi) \geq 0\) is a sufficient condition (and also the necessary condition since \(P = P_0\) for \(P_n(\Psi) \geq 0\) for any non-negative integer \(n\)), the condition that

\[
R_n(r^2) = \frac{d^n}{dx^n} \left[ x^n R(x) \right] = 0 \quad (r, x \geq 0)
\]

(18a)

or equivalently (see Equation (A3))

\[
\left( x^n \frac{d}{dx} \right)^n \left[ x^2 R(x) \right] = (-1)^n x^n \frac{d}{dx} \left[ \frac{R(w)}{w} \right] = 0 \quad (r, w \geq 0)
\]

(18b)

for every non-negative integer \(n\) constitutes a set of independent constraints on the behavior of \(R(r^2)\).¹ In other words, Equation (18) is a necessary condition for the radial portion \(R(r^2)\) of any separable augmented density to be generated by a non-negative distribution function.

Combining with Equation (13a), it also forms the set of restrictions on the possible radial variations of the anisotropy parameter allowed for the spherical system with a separable augmented density. In particular, any spherical anisotropic system with a separable augmented density is physical only if

\[
\alpha_n(r) \geq 0 \quad (r \geq 0)
\]

(19a)

for all positive integers \(n\). Here the set of functions \(\alpha_n\) is as defined in Equation (15b) with the anisotropy parameter \(\beta(r)\). For the first few small \(n\)'s, the conditions are equivalent to

\[
\beta \leq 1,
\]

(19b)

\[
r \frac{d\beta}{dr} \leq (1 - \beta)(2 - \beta),
\]

(19c)

\[
r \frac{d^2\beta}{dr^2} + \frac{3}{2}(2 - \beta) r \frac{d\beta}{dr} \leq (1 - \beta)(2 - \beta)(3 - \beta),
\]

(19d)

and so on. Equation (19b) is obvious from the definition of the anisotropy parameter in Equation (7a) and the non-negativity of the velocity dispersions, and thus universal independent of the separability assumption. By contrast, the further constraints involving the radial derivatives of the anisotropy parameter are the consequence of the separability assumption—following Equation (15) and the non-negativity of the higher order velocity moments.

These imply that, even if \(R(r^2)\) could be formally expressed using Equation (13b), not all arbitrarily varying anisotropy parameters are consistent with separable augmented densities because some might produce negative higher order moments. For example, consider the anisotropy parameter behaving as

\[
\beta(r) = \frac{r_{2s}}{r_{2s}^2 + r_{2s}^2}
\]

(20a)

so that the system is isotropic at the center and radially anisotropic in the outskirts. If \(s = 1\), the anisotropy profile of Equation (20a) is that of the Osipkov–Merritt system (Osipkov 1979; Merritt 1985). The corresponding radial function is \(R(r^2) = (1 + r_{2s}^2/x_{2s}^2)^{-1/3}\) within a constant, but this does not satisfy the condition in Equation (18) if \(s > 1\) because

\[
\left. \frac{d^2[R^2(x)]}{dx^2} \right|_{x=x_{2s}/r_{2s}^2} = 2 - (s - 1)x^2 / (1 + x^2)^{1/3}
\]

(20b)

which is negative for \(x > \sqrt{2/(s - 1)}\). Equivalently, we find

\[
(1 - \beta)(2 - \beta) - r \frac{d\beta}{dr} = 2 \left( \frac{r_{2s}^2}{r_{2s}^2 + r_{2s}^2} \right)\frac{2s - 1}{(r_{2s}^2 + r_{2s}^2)^{1/3}},
\]

(20c)

and thus Equation (20a) fails the constraint in Equation (19c) if \(s > 1\) and \(r/r_{2s} > \sqrt{2/(s - 1)}\). Consequently, the anisotropy parameter given by Equation (20a) is consistent with a separable augmented density only if \(s \leq 1\). In fact, the converse is also true, that is, if \(s \leq 1\), then \(R(r^2) = (1 + r_{2s}^2/x_{2s}^2)^{-1/3}\) satisfies Equation (18) for all non-negative integers \(n\)—obviously, if

¹ A function \(\phi(x)\) defined for \(x > 0\) is said to be “completely monotonic” (c.m.) if and only if \((-1)^n \phi^{(n)}(x) > 0\) for all non-negative integers \(n\). Hence, the condition is equivalent to saying \(R(w) = R(w^{-1})/w\) is a c.m. function of \(w\). According to S. Bernstein’s (1880–1968) theorem on c.m. functions (see Widder 1941), an important corollary to this is that \(R(\omega)\) must be the Laplace transformation of a non-negative function—i.e., the inverse Laplace transformation of \(R(\omega)\) exists and is non-negative in the positive real domain. Moreover, expressing the inverse Laplace transformation using E. Post’s (1897–1954) inversion formula (see Hirschman & Widder 1955) indicates that Equation (18) is actually equivalent to \(\lim_{n \to \infty} R_n(t/n)!^1 > 0\) for \(t \geq 0\).

Further explorations of this idea will be given elsewhere.
s = 1, the non-negative Osipkov–Merritt distribution function exists with a properly chosen potential term $P(\Psi)$.

Roughly speaking, Equation (19c) insists that the anisotropy parameter in the system with a separable augmented density cannot radically increase faster than the limiting value determined by the local anisotropy parameter, which tends to get smaller as it becomes more radially anisotropic. Similar interpretations by the local anisotropy parameter, which tends to get smaller as it becomes more radially anisotropic. Similar interpretations for higher order constraints of Equation (19) are less obvious.

3.2.2. A Family of Monotonic Anisotropy Parameters

Consider the anisotropy profile,

$$\beta(r) = \frac{\beta_\infty r^{2s} + \beta_0 r^{2s}}{r^{2s} + s} \quad (s > 0). \quad (21a)$$

This parameterization has also been introduced by Baes & Van Hese (2007) for their construction of dynamical models with a flexible anisotropy parameter. Equation (20a) corresponds to Equation (21a) with $(\beta_0, \beta_\infty) = (0, 1)$. Note that the transform of $s \rightarrow -s$ is actually equivalent to switching $\beta_0 \leftrightarrow \beta_\infty$, and thus the restriction $s > 0$ is actually not necessary. Nevertheless, to assign definite physical meanings to the parameters, we retain the restriction. Then $\beta$ varies monotonically from $\beta_0$ at the center to $\beta_\infty$ as $r \rightarrow \infty$, with the constant anisotropy case represented by $\beta_0 = \beta_\infty$. A choice of $r_s = 0$ or $r_s = \infty$ also produces a constant anisotropy model although these cases will not be explicitly considered here. The case $s = \frac{1}{2}$ reduces to the generalized Mamon–Łokas anisotropy model (cf. Mamon & Boué 2010) with the original Mamon & Łokas (2005) model given by $(\beta_0, \beta_\infty) = (0, \frac{1}{2})$. The separable augmented densities with $s = 1$ include the Cuddeford (1991) system (see also Ciotti & Morganti 2010a) for which $\beta_\infty = 1$ and the Osipkov–Merritt system with $(\beta_0, \beta_\infty) = (0, 1)$. As Baes & Van Hese (2007) have noted, this parameterization is notable as it yields a simple analytic integrating factor for the second-order Jeans equation (Equation (13b)),

$$[R(r^2)]^{-1} = r^2 \beta_0 \left( r^{2s} + s \right)^{(\beta_\infty - \beta_0) / s}. \quad (21b)$$

With $\nu(r)$ and $\Psi(r)$ specified, the radial velocity dispersion can be found in quadrature by Equation (14b), regardless of the separability of the augmented density.

Under the separable augmented density assumption, however, this completely specifies the resulting system; $N(\Psi, r^2) = P(\Psi)R(r^2)$ where $P[\Psi(r)] = \nu(r) / R(r^2)$ and $R(r^2)$ is deduced from Equation (21b). The distribution function can then be found by inverting the integral Equation (1)—see, e.g., Dejonghe (1986) and Baes & Van Hese (2007) for the technique based on the Laplace–Mellin transform or Qian (1993) and Hunter & Qian (1993; see also An 2011) for the complex contour integral method.

However, the preceding arguments indicate that the resulting model is not necessarily physical for an arbitrary parameter set. Obviously, the condition that $\beta(r) \leq 1$ for $\forall r \geq 0$ restricts the parameters to $\beta_0 \leq 1$ and $\beta_\infty \leq 1$. As for the system with a separable augmented density, more constraints on the parameters also follow Equations (18) and (19). The first of these corresponding to Equation (18) with $n = 2$ reduces to

$$[2 - \beta_0 + (2 - \beta_\infty)x^1][1 - \beta_0 + (1 - \beta_\infty)x^1] \geq s(\beta_\infty - \beta_0) x^1$$

for $x \geq 0$,

$$\geq s(\beta_\infty - \beta_0) x^1 \quad (22a)$$

which is also equivalent to Equation (19c);

$$\left( r^{2s} + 1 \right)^2 \left( (1 - \beta)(2 - \beta) - \frac{r d\beta}{2 dr} \right) = \left[ (2 - \beta_\infty)r^{2s} + 2 - \beta_0 \right]$$

$$\times \left[ (1 - \beta_\infty)r^{2s} + 1 - \beta_0 \right] - s(\beta_\infty - \beta_0)(1 - \beta_\infty) \geq 0$$

for $r \geq 0$.

Here, we have set $r_s = 1$ for brevity, but this does not affect the following results. Since the necessary and sufficient condition for the real-coefficient monic quadratic equation $x^2 + bx + c = 0$ to possess no non-degenerate positive real root is $c \geq 0$ and $b^2 \leq 4c$, Equation (22) for $\beta_0, \beta_\infty \leq 1$ is also equivalent to

$$\frac{(2 - \beta_\infty)(1 - \beta_0) + (2 - \beta_0)(1 - \beta_\infty)}{\beta_\infty - \beta_0}$$

$$\geq s(\beta_\infty - \beta_0). \quad (22c)$$

With $(2 - \beta_\infty)(1 - \beta_0) + (2 - \beta_0)(1 - \beta_\infty) = 2(2 - \beta_0)(1 - \beta_\infty) + \beta_\infty - \beta_0$, we therefore find for fixed $\beta_0, \beta_\infty \leq 1$ that Equation (22) fails if $\beta_0 < \beta_\infty$ and

$$s > 1 + \frac{2\sqrt{(2 - \beta_\infty)(1 - \beta_\infty)}}{\beta_\infty - \beta_0}$$

$$\times \left[ \sqrt{(2 - \beta_\infty)(1 - \beta_\infty) + \sqrt{(1 - \beta_\infty)(2 - \beta_\infty)}} \right]. \quad (23)$$

In other words, certain parameter combinations exist for Equation (21) that cannot be consistent with any separable physical augmented density.

More higher order constraints may be derived similarly, but direct calculations for general cases become rather complicated as the order increases. Instead, here we just note that if $0 < s \leq 1$ or $(\beta_0 - \beta_\infty)/s$ is a non-negative integer, then $R(r^2)$ in Equation (21b) satisfies the condition of Equation (18) and so $\beta(r)$ in Equation (21a) can be consistent with a separable physical augmented density. (An elementary proof is provided in Appendix C.) The sufficiency of the condition that $0 < s \leq 1$ for the parameterization given in Equation (21) can also be deduced by the existence of the corresponding non-negative distribution function with a separable augmented density as demonstrated by Baes & Van Hese (2007), who explicitly constructed the particular distribution function in terms of the convergent Fox H-function. We also suspect that if $\beta_0 < \beta_\infty$, the condition that $0 < s \leq 1$ is the necessary condition for Equation (21b) to satisfy Equation (18) but have no definite proof at this time.

In addition, we also note that the condition in Equation (18) is linear on $R(r^2)$. Hence, if both $A(r^2)$ and $B(r^2)$ meet the necessary condition in Equation (18), the radial function given by the linear combination, $R(r^2) = aA(r^2) + bB(r^2)$ where $a$ and $b$ are positive constants, also satisfies the same necessary condition (and therefore the anisotropy parameter resulting from it is consistent with Equation (19) and a separable physical augmented density). For example, this indicates that the multicomponent generalized Cuddeford systems studied by Ciotti & Morganti (2010a) do satisfy Equation (18) as their radial functions are given by the sums of the functions in the form of Equation (21b) with $s = 1$ and $\beta_\infty = 1$ and different $r_s$.

4. DISCUSSION

4.1. Sufficient Conditions for Separable Physical Augmented Densities

For multicomponent Cuddeford systems, Ciotti & Morganti (2010a) have proved that the condition $d^{\mu+1}P/d\Psi^{\mu+1} \geq 0$...
is sufficient to guarantee the non-negativity of the posited distribution function. Here $\mu$ is the greatest integer not larger than $\frac{3}{2} - \beta_0$. Subsequently, Van Hese et al. (2011) asked whether the same condition should be the sufficient condition for any separable augmented density to be generated by the non-negative distribution function.

The present paper clarifies the answer to their question in the simplest form to be negative. The existence of the necessary condition involving only the radial function $R(r^2)$, that is, Equation (18), implies that any sufficient condition must also contain some restrictions on the same. The hypothesis as stated entirely with the potential portion $P(\Psi)$ thus cannot be a sufficient condition by itself given the independent nature of the potential and radial parts of the separable augmented density. A simple counterexample may be constructed with Equation (21a) and the choice of parameters such that $\frac{1}{2} < \beta_0 < \beta_\infty = 1$ and $s > 1$. With the radial function $R(r^2)$ that follows (Equation (21b)), no function $P(\Psi)$, regardless of whether or not $dP/d\Psi \geq 0$, can lead to a non-negative distribution function—here the choice of $P(\Psi)$ is equivalent to specifying the local density as $P(\Psi(r)) = \nu(r)/R(r^2)$ and insisting $dP/d\Psi \geq 0$ imposes the so-called global density slope anisotropy inequality (Ciottì & Morganti 2010b; Van Hese et al. 2011). Nonetheless, the findings in this paper do not preclude the possibility that the constraint $d^{n+1}P/d\Psi^{n+1} \geq 0$ combined with additional conditions on $R(r^2)$ may constitute a sufficient condition for the system with a separable augmented density. In fact, after the original version of this paper was submitted, E. Van Hese (2011, private communication) discovered the existence of such conditions, e.g., together $dP/d\Psi > 0$ and the set of conditions on $R(r^2)$ that includes Equation (18) are sufficient for the existence of a non-negative distribution function.

4.2. Universal Constraints on the Anisotropy Parameter?

Since the constraints in Equation (19) are actually put on the anisotropy parameter without any explicit reference to the separable augmented density, it seems fair to ponder how general these constraints actually are—that is, whether the constraints exist for any physical augmented density. Although we find, for the time being, no reason to argue that these constraints are universal (for they are derived based on the particular assumption of separable augmented density) beyond the obvious restriction that $\beta \leq 1$, we will not attempt to settle the answer in this paper. However, we do note that, if the constraints are entirely the consequence of the separability of the augmented density, one must be able to construct a pair of the non-negative distribution function and the inseparable augmented density for a spherical anisotropic system with its anisotropy parameter violating the conditions in Equation (19). Unfortunately, the task is complicated by the fact that, with inseparable augmented densities, the anisotropy parameter cannot be specified independently without imposing the fixed behavior of the potential; the pair of $f(\mathcal{E}, L^2)$ and $N(\Psi, r^2)$ typically prescribes only $\beta(\Psi, r^2)$ and thus varying $\Psi = \Psi(r)$ results in a different $\beta(r) = \beta[\Psi(r), r^2]$ unless $\Psi(r)$ and $\nu(r) = N[\Psi(r), r^2]$ are related to each other through the Poisson equation and so the freedom to choose $\Psi(r)$ is subsequently removed.

4.3. Separable Augmented Densities and the Jeans Degeneracy

With real data, our observations are typically limited by projection, and thus the usual kinematical observables available to us are restricted to the surface brightness profile (or the column density profile for the discrete number count data) and the line-of-sight velocity dispersion. It is a well-known fact that, while the three-dimensional density profile can be uniquely inverted from the surface density (the Abel transformation) under the spherical symmetry assumption, the radial and tangential velocity dispersions, $\sigma_r^2(r)$ and $\sigma_\theta^2(r)$, which are needed to find the potential through the Jeans equation, cannot be determined from the line-of-sight velocity dispersion alone. This is because many different sets of $\sigma_r^2$ and $\sigma_\theta^2$ can reproduce the same observations of the line-of-sight velocity dispersion (Dejonghe 1987; Merritt 1987) unless the system is known to have isotropic velocity distributions. One may lift this so-called Jeans degeneracy by imposing additional constraints on the system coming from observations or a priori assumptions.

For example, Mamon & Boué (2010) and Wolf et al. (2010) have shown that it is in general possible to find a value of $\sigma_r^2$ that is consistent with the observed line-of-sight velocity dispersion profile and any arbitrarily specified anisotropy parameter $\beta(r)$. Evans et al. (2009), by contrast, demonstrated that one can also proceed the opposite way, finding a value of $\beta$ that is consistent with the observed line-of-sight velocity dispersion and the arbitrary assumed form of $\sigma_r^2$. Alternatively, under the assumption of a constant mass-to-light ratio, one can also find the unique solution to the coupled Jeans–Poisson equations from the line-of-sight velocity dispersion profile (Binney & Mamon 1982; Toney 1983; Bicknell et al. 1989). This is equivalent to specifying the potential first and inverting the line-of-sight velocity dispersion to determine $\beta$ given the Jeans equation (Solanes & Salvador-Solé 1990; Dejonghe & Merritt 1992). However, for the purpose of constraining the potential, these anisotropy inversion algorithms can only, at best, reject some choices for the gravitational potential as unphysical where $\beta$ reaches values above unity (formally this indicates negative velocity dispersion).

The best observational constraint for lifting the Jeans degeneracy, however, would be some handle on the proper motions of tracers (e.g., Leonard & Merritt 1989; van der Marel & Anderson 2010) because they are the velocity projections that are orthogonal to the line-of-sight velocity. Alternatively, with a data set consisting of discrete tracers, the precisely measured differential distances (which ultimately yield the distances to the center of the system) can also break the Jeans degeneracy (cf. Watkins et al. 2010). Unfortunately, with our current and near-future observational capabilities, their uses are mostly limited to very nearby objects.

A popular idea for possible observational constraints is the use of the higher order moments (Merrifield & Kent 1990) or the distribution of the line-of-sight velocities. Note that specifying the distribution is equivalent to knowing the infinite set of the entire moments. Similar to the velocity dispersions (which are the second moments), Dejonghe & Merritt (1992) have shown that, with the potential specified, the complete set of independent velocity moments can be solved from the observed line-of-sight velocity moments up to the same order. In infinite order, this implies that the distribution function is uniquely specified by the observed distribution of the line-of-sight velocities provided that the potential is known a priori. However, it is easy to argue that this will not solve the degeneracy problem (in particular for observationally tracing the potential) because, under the spherical symmetry, introducing each new $(2n+1)$th moment adds $(n+1)$ new variables and $n$ constraining Jeans equations (Equation (10)) with one further observational constraint into the mix and therefore there is no net increase in the constraints.
Łokas (2002) and Łokas & Mamon (2003) introduced a hybrid of theoretical and observational constraints, “constant anisotropy,” and the fourth moment (kurtosis) to produce a unique solution to the degeneracy problem. However, their “constant anisotropy” is actually in the form of a strictly stronger assumption that the distribution function and the augmented density are given by the ansatz2 $f(E, L^2) = L^{-2\beta}g(E)$ and $N(\Psi, \tau^2) = \tau^{-2\beta}P(\Psi)$, while Section 3.1 of the present paper (in particular, Equation (17b)) indicates that this is a rather unnecessarily restrictive assumption for their method to work. That is to say, under the separability assumption of the augmented density, to bring higher-than-the-second-order moments into the problem only adds one new independent variable (cf. Equation (15a)) and the single Jeans equation (Equation (17b)). Therefore, the observations of the line-of-sight velocity moment at the same order actually act as an additional net constraint on the system given the separable augmented density. Specifically, the introduction of the fourth moment is enough to uniquely solve the Jeans degeneracy if the augmented density is assumed to be separable whereas adding further higher order moments actually overconstrains the problem.

However, the assumption of a separable augmented density is purely formal and its physical interpretation is unclear, although it is a weaker hypothesis than the power-law ansatz for the $L$ part of distribution function and the augmented density which produces the constant anisotropy. The most conservative statement that can be drawn regarding the Jeans degeneracy and the separable augmented density is thus that given the observations of the second and fourth moments (the dispersion and the kurtosis) of the line-of-sight velocities, a unique spherical model exists with a separable augmented density that is consistent with these moments. The resulting model is complete in that it essentially specifies the distribution function as well as the underlying potential. Although the non-negativity of the resulting distribution function is not guaranteed, the condition in Equation (18) is both necessary and sufficient to prove that the model will produce non-negative velocity moments of every order—of course, the model is not necessarily “real” and the predicted higher-than-fourth-order moments should be compared to the observations (if available) in order for it to be acceptable.

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APPENDIX A

PROOF OF EQUATION (5c)

Lemma A1. For a non-negative integer $n$ and arbitrary real $a$,

\[ \frac{d^n(x^a)}{dx^n} = \left[ \prod_{j=0}^{n-1} (a-j) \right] x^{a-n}, \quad (A1) \]

which is easily proven by the induction on $n$. For a positive integer-power monomial, this simplifies to (i.e., $k$ is also a non-negative integer)

\[ \frac{d^n(x^k)}{dx^n} = \begin{cases} \frac{k! x^{k-n}}{(k-n)!} & (0 \leq n \leq k) \\ 0 & (n \geq k+1) \end{cases}. \quad (A1a) \]

Next, using Lemma A1 and the extended Leibniz rule, we find Lemma A2.

Lemma A2. For a non-negative integer $n$ and any function $f$,

\[
\frac{d^n(x^n f)}{dx^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}f}{dx^{n-k}} (n!)(n-k)! x^k f^{(k)}. \quad (A2)
\]

Here $(\binom{n}{k})$ is the binomial coefficient and $f^{(k)}(x) = d^k f/dx^k$. Now, we are able to prove Lemma A3.

Theorem A3. For a non-negative integer $n$ and any function $f$,

\[ \left( x^2 \frac{d}{dx} \right)^n (x f) = x^{n+1} \frac{d^n(x^n f)}{dx^n}. \quad (A3) \]

Proof. We prove this by the induction on $n$. First, Equation (A3) is trivial for $n = 0, 1$. The induction step is proven as

\[
\left( x^2 \frac{d}{dx} \right)^n (x f) = x^2 \frac{d}{dx} \left[ \left( x^2 \frac{d}{dx} \right)^n (x f) \right]
= x^2 \frac{d}{dx} \left[ x^{n+1} \frac{d^n(x^n f)}{dx^n} \right]
= x^2 \frac{d}{dx} \left[ \sum_{k=0}^{n} \binom{n}{k} (n!)^2 x^{n+1+k} f^{(k)} \right]
= x^2 \sum_{k=0}^{n} (n!)^2 ((n+1+k)(n+1+k) + x^{n+k+1} f^{(k+1)})
= x^2 \sum_{k=0}^{n} (n!)^2 ((n+1+k)(n+1+k) + x^{n+k} f^{(k)})
= x^{n+2} \sum_{k=0}^{n} (n!)^2 ((n+1+k)(n+1+k) + x^{n+k} f^{(k)})
= x^{n+2} \frac{d^{n+1}(x^{n+1} f)}{dx^{n+1}}. \quad (A3a)
\]

Immediately following this is Corollary A4.
Corollary A4. For any non-negative integer \(k\),
\[
\frac{d}{dx} \left[ x^{k+1} \frac{d^k (x^f)}{dx^k} \right] = \frac{d}{dx} \left[ \left( x^2 \frac{d}{dx} \right)^k (xf) \right] = \left( 1 + n + x \frac{d}{dx} \right)^{\frac{1}{2}} (x^f) \] (A4)
\]

With \( f = N(\Psi, r^2), x = r^2, \) and \( k = q \), this results in
\[
\frac{\partial}{\partial r^2} \left[ 2q+1 \left( 1 + n + x \frac{d}{dx} \right)^{q-1/2} \right] = r^{2q} \left( 1 + n + x \frac{d}{dx} \right)^{q-1} \left[ 2q+1 \left( \Psi, r^2 \right) \right] \] (A4a)

It is now easy to show that \( m_{p,q}(\Psi, r^2) \) in Equation (5c) satisfies Equation (4b) by direct calculations using Equation (A4a) and the Pochhammer symbol \( \left( \frac{1}{2} \right)_p = \left( \frac{1}{2} \right)_{p-1}(p - \frac{1}{2}) \) for any positive integer \( p \).

One can show that Equation (5c) satisfies Equation (4a) using
\[
\frac{d}{dx} \int_{x_0}^x dy(x - y)^k g(y) = \delta_{k,0} g(x) + k \int_{x_0}^x dy(x - y)^k g(y) \] (A5a)

where \( \delta_{m,n} \) is the Kronecker delta and
\[
\int_{x_0}^x \int_{x_0}^{x_1} dy(x_1 - y)^k g(y) = \int_{x_0}^x dy g(y) \int_y^x dx_1 (x_1 - y)^k
= \frac{1}{k+1} \int_{x_0}^x dy(x - y)^k g(y) \] (A5b)

for any \( k \geq 0 \). By repeatedly applying this, one finds that a simple iterated integral in general reduces to an integral transform,
\[
g_k(x) = \int_0^x dx_k \cdots \int_0^{x_1} dx_1 g(x_1)
= \frac{1}{(k-1)!} \int_0^x dy(x - y)^{k-1} g(y), \] (A5c)
where \( k \) is now a positive integer. This is sometimes known as the “Cauchy formula for repeated integration,” and can be strictly proven through the induction on \( k \). In fact, the function \( g_k(x) \) defined as such is the particular solution to the differential equation \( d^k g_k(x)/dx^k = g(x) \) with the set of initial conditions \( g_j^{(m)}(0) = 0 \) where \( j \in \{0, \ldots, k-1\} \). Formally, Equation (A5c) extends to the case \( k = 0 \) by noting that \( \lim_{x \to 0^+} e^{-x} x^{-1} = 2\delta(x) \) where \( \delta(x) \) is the Dirac delta. The extra factor of two is due to the fact that the integral interval in Equation (A5c) is one-sided extending from \( x \).

APPENDIX B

GENERAL EXPRESSION FOR EQUATIONS (15) AND (19)

For any function \( f(x) \) of \( x \), we find using Equation (A4) that
\[
\frac{d^{n+1}[x^{n+1} f(x)]}{dx^{n+1}} = \frac{1}{x^n} \frac{d}{dx} \left[ x^{n+1} \frac{d^n (x^f)}{dx^n} \right] = \left( 1 + n + x \frac{d}{dx} \right) \frac{d^n [x^n f(x)]}{dx^n}. \] (B1)
Next, with the definitions of \( \alpha_n \) in Equation (15a) and \( R_n \) in Equation (12c)
\[
\alpha_n(r) = \frac{R_n(r^2)}{R_0(r^2)} = \frac{1}{R} \left. \frac{d^n [x^n R(x)]}{dx^n} \right|_{x=r^2} \] (B2)
from the given radial function \( R(r^2) \) of a separable augmented density, the recursion formula for \( \alpha_n \) in Equation (15b) follows as
\[
\alpha_{n+1} = \frac{1}{R} \left( 1 + n + x \frac{d}{dx} \right) \left( R \alpha_n \right) \] (B3)
\[
\left. \frac{d}{d \ln r} \left( \ln \right) \alpha_n + d_{\alpha_n} \right|_{x=r^2}. \]
Replacing \( R \) with \( \beta^2 \) utilizing Equation (13a) yields Equation (15b). Equation (18) then indicates Equation (19) whereas Equation (15b) may be considered the recursive definition of \( \alpha_n(r) \) from the anisotropy parameter \( \beta(r) \) given the initial term \( \alpha_0 = 1 \). Note also that \( \alpha_n \) in Equation (15b) is defined as such without referring to the radial function or the separable augmented density at all.

APPENDIX C

PROOF OF CONSISTENCY OF EQUATION (21)

For \( \beta(r) \) and \( R(x) \) given by Equations (21), if we define
\[
\tau_n \equiv (1 + x^\lambda) \alpha_n = x^{\beta}(1 + x^\lambda) \sum_{n=0}^\infty \left[ 1 + x \frac{d}{dx} \right]^n \left( \frac{x^{n+1} R(x)}{x^{n+1} \left( 1 + x^\lambda \right)^2} \right), \] (C1)
where \( \lambda = (\beta_\infty - \beta_0)/s \), Equation (15b) results in
\[
\tau_{n+1} = \left[ n + 1 - \beta_0 + (n + 1 - \beta_\infty - sn) y \right] \tau_n + s yn(1 + y) \frac{d \tau_n}{dy}, \] (C2)
where \( y \equiv x^s \) for \( k \geq 0 \). Since \( \tau_0 = \alpha_0 = 1 \), this indicates that \( \tau_n \) is an (at most) \( n \)th-order polynomial of \( y \). If \( k = 0 \), Equation (C4a) reduces to
\[
\tau_n = \sum_{k=0}^n \tilde{t}_{n,k} y^k, \] (C3)
we can derive the recursion relation for the coefficients,
\[
\tilde{t}_{n+1,k} = \left[ n + 1 - \beta_0 + s k \right] \tilde{t}_{n,k} + \left[ (1 - s)n + 1 - \beta_\infty + s(k - 1) \right] \tilde{t}_{n+1,k-1}, \] (C4a)
by substituting Equation (C3) into Equation (22), and also using \( \tilde{t}_{n,k} = 0 \) for \( k < 0 \) or \( k > n \). If \( k = 0 \), Equation (C4a) reduces to
\[
\tilde{t}_{n+1,0} = (n + 1 - \beta_0) \tilde{t}_{n,0} \Rightarrow \tilde{t}_{n,0} = (1 - \beta_0) n \geq 0 \] (C4b)
provided that \( \beta_0 \leq 1 \) because \( \tilde{t}_{n-1,0} = 0 \) and \( \tilde{t}_{0,0} = 1 \). Next, for a positive integer pair \( n \geq 1 \), Equation (C4a) indicates that, if \( 0 < s \leq 1 \), the non-negativity of \( \tilde{t}_{n-1,k} \) and \( \tilde{t}_{n-1,1-k} \) can guarantee the non-negativity of \( \tilde{t}_{n,k} \) provided that \( \beta_0 \leq \beta_\infty \leq 1 \). Since we have already found that \( \tilde{t}_{n,k} = 0 \) for \( k < 0 \) and \( \tilde{t}_{n,0} = (1 - \beta_0) n \geq 0 \), we can conclude that if \( 0 < s \leq 1 \) and \( \beta_0, \beta_\infty \leq 1 \), then \( \tau_n(y) \) is a polynomial with all non-negative coefficients and therefore \( \tau_n(y) \geq 0 \) for \( y \geq 0 \) and any non-negative integer \( n \). Since \( \tau_n = (1 + x^s)^n \alpha_n = (1 + x^s) R_n/R \), Equation (18) also follows immediately.

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As for the cases where \( \xi = (\beta_0 - \beta_\infty)/s = -\lambda \) is a non-negative integer, we first consider the constant-\( \beta \) case, that is, \( \xi = 0 \) and \( \beta_0 = \beta_\infty = \beta \). Then, using Equation (A1),

\[
R_n(x) = \frac{d^n x^{n-\beta}}{dx^n} = \left[ \prod_{j=0}^{n-1}(n-\beta-j) \right] x^{-\beta} = \frac{(1-\beta)_n}{x^\beta} \geq 0, \tag{C5a}
\]

for \( x > 0 \), provided that \( \beta \leq 1 \). In general, if \( \xi = (\beta_0 - \beta_\infty)/s \) is a non-negative integer, we can simply extend this result to

\[
R = \frac{(1+x)^\xi}{x^{\beta_0}} = \sum_{k=0}^{\xi} \left( \frac{\xi}{k} \right) x^{sk-\beta_0} \tag{C5b}
\]

\[
R_n = \sum_{k=0}^{\xi} \left( \frac{\xi}{k} \right) \frac{d^n(x^{sk+\beta_0})}{dx^n} = \sum_{k=0}^{\xi} \left( \frac{\xi}{k} \right) (sk+1-\beta_0)_n x^{sk-\beta_0}. \tag{C5c}
\]

Again, \( R_n(x) \geq 0 \) for \( x > 0 \), provided that \( \beta_\infty = \beta_0 - s\xi \leq \beta_0 \leq 1 \).

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