Regularity for time fractional wave problems

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Abstract
Using the Galerkin method, we obtain the unique existence of the weak solution to a time fractional wave problem, and establish some regularity estimates which reveal the singularity structure of the weak solution in time.

Keywords: time fractional wave equation; weak solution; regularity

1 Introduction
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $C^2$ boundary, $1 < \alpha < 2$, $0 < T < \infty$, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in L^2(\Omega_T)$ with $\Omega_T := \Omega \times (0, T)$. This paper considers the following time fractional wave problem:

$$\partial_t^\alpha (u - u_0 - tu_1) - \Delta u = f \quad \text{in } \Omega_T, \quad (1.1)$$

subject to the boundary value condition that

$$u = 0 \quad \text{on } \partial \Omega \times [0, T].$$

Here $\partial_t^\alpha: L^1(\Omega_T) \rightarrow \mathcal{D}'(\Omega_T)$, a Riemann-Liouville fractional differential operator, is defined by $\partial_t^\alpha := \partial_t^2 I^{2-\alpha}_{0+}$, where $\partial_t: \mathcal{D}'(\Omega_T) \rightarrow \mathcal{D}'(\Omega_T)$ denotes the standard generalized differential operator with respect to the time variable $t$, and $I^{2-\alpha}_{0+}: L^1(\Omega_T) \rightarrow L^1(\Omega_T)$ is given by

$$(I^{2-\alpha}_{1,0+} v)(x, t) := \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} v(x, s) \, ds, \quad (x, t) \in \Omega_T,$$

for all $v \in L^1(\Omega_T)$, with $\Gamma(\cdot)$ denoting the standard Gamma function. It appears that we have not imposed initial value conditions for problem (1.1), but it will be clear later that the initial value conditions are actually contained in the governing equation (1.1), provided $f$, $u_0$ and $u_1$ are regular enough.

The above problem is a special case of a large class of problems, the fractional diffusion-wave problems, that have attracted a considerable amount of research efforts in the field of numerical analysis in the past decade; see [25, 24, 5, 7, 14, 15, 9, 26, 10, 23, 13, 19] and the references therein. Because of the nonlocal property of the fractional differential operator, the cost of memory and computing of

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an accurate approximation to problem (1.1) is much more expensive than that to a corresponding normal wave problem. To reduce the cost, high-accuracy algorithms are often preferred. However, high-accuracy numerical algorithms generally require the solution to be of high regularity; especially, for problem (1.1) the differentiability of the solution with respect to the time variable $t$ is of great importance. This is the primary motivation for us to investigate the regularity for problem (1.1).

Up to now, there have been many works devoted to the mathematical treatments of problem (1.1); see [16, 8, 3, 1, 6, 18, 20] and the references therein. However, these works are not very useful for the numerical analysis. Recently, Li, Xie, and Zhang [12] presented a new smoothness result for Caputo-type fractional ordinary differential equations, which reveals that, subtracting a non-smooth function that can be obtained by the information available, a non-smooth solution belongs to $C^m$ for some positive integer $m$. Later, Li and Xie [11] discussed the regularity for time fractional diffusion problems by the Galerkin method. In this paper, using the same approach as in [11], we obtain the unique existence of the weak solution to problem (1.1), and establish some new regularity estimates. These regularity estimates demonstrate that the weak solution to problem (1.1) generally has singularity in time; however, subtracting some particular forms of singular functions, we can improve the regularity of the weak solution. This is not only of theoretical value, but also can provide insight into developing high-accuracy numerical algorithms.

The rest of this paper is organized as follows. In Section 2 we introduce some properties of the Riemann-Liouville fractional integration/derivative operators. In Section 3 we discuss the regularity for an ordinary equation. Finally, in Section 4 we study the regularity of the weak solution to problem (1.1).

2 Preliminaries

We start by introducing a vector-valued Sobolev space. Let $X$ be a separable Hilbert space with inner product $(\cdot, \cdot)_X$ and an orthonormal basis $\{e_k | k \in \mathbb{N}\}$. For $0 \leq \beta < \infty$, let $H^\beta(0,T)$ denote the standard Sobolev space [22], and define

$$H^\beta(0,T;X) := \left\{ v : (0,T) \to X \mid \sum_{k=0}^{\infty} \|v, e_k\|_{H^\beta(0,T)}^2 < \infty \right\},$$

and equip this space with the following norm: for all $v \in H^\beta(0,T;X)$,

$$\|v\|_{H^\beta(0,T;X)} := \left( \sum_{k=0}^{\infty} \|v, e_k\|^2_{H^\beta(0,T)} \right)^{\frac{1}{2}}.$$

A standard argument in the theory of the $l^2$ space gives that $H^\beta(0,T;X)$ is a Banach space. In particular, we also use $L^2(0,T;X)$ to denote the space $H^0(0,T;X)$. Furthermore, for $v \in H^\beta(0,T;X)$ with $\beta \geq 1$, define

$$v'(t) := \sum_{k=0}^{\infty} d'_k(t)e_k, \quad 0 < t < T,$$

where $d_k(\cdot) := (v(\cdot), e_k)_X$, and $d'_k$ denotes the weak derivative of $d_k$. 

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Remark 2.1. It is evident that the spaces $L^2(0, T; X)$ and $H^1(0, T; X)$ defined above coincide respectively with the corresponding standard $X$-valued Sobolev spaces [4], with the same norms. Using the $K$-method [22], we see that, for $0 < \beta < 1$, the space $H^\beta(0, T; X)$ coincides with the interpolation space

$$(L^2(0, T; X), H^1(0, T; X))_{\beta, 2};$$

with equivalent norms. Thus, the space $H^\beta(0, T; X)$, $0 \leq \beta \leq 1$, is independent of the choice of orthonormal basis $\{e_k| k \in \mathbb{N}\}$ of $X$; the case of $\beta > 1$ is analogous. In addition, the $v'$ defined above coincides with the usual weak derivative of $v$ [4].

Then, let us introduce the Riemann-Liouville fractional integration and derivative operators as follows [21, 17].

**Definition 2.1.** For $0 < \beta < \infty$, define $I^\beta_{0+} : L^1(0, T) \to L^1(0, T)$ and $I^\beta_{T-} : L^1(0, T) \to L^1(0, T)$, respectively, by

$$I^\beta_{0+} v(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) \, ds, \quad 0 < t < T,$$

$$I^\beta_{T-} v(t) := \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} v(s) \, ds, \quad 0 < t < T,$$

for all $v \in L^1(0, T)$.

**Definition 2.2.** For $m - 1 < \beta < m$ with $m \in \mathbb{N}_{>0}$, define $D^\beta_{0+} : L^1(0, T) \to \mathcal{D}'(0, T)$ and $D^\beta_{T-} : L^1(0, T) \to \mathcal{D}'(0, T)$, respectively, by

$$D^\beta_{0+} := D^m I^m_{0+} - \beta \quad \text{and} \quad D^\beta_{T-} := (-1)^m D^m I^m_{T-} - \beta,$$

where $D : \mathcal{D}'(0, T) \to \mathcal{D}'(0, T)$ denotes the standard generalized differential operator.

**Lemma 2.1.** ([21]) If $\beta, \gamma > 0$, then

$$I^\beta_{0+} I^\gamma_{0+} = I^\beta_{0+} I^\gamma_{T-}, \quad I^\beta_{T-} I^\gamma_{T-} = I^\beta_{T-} I^\gamma_{T-}.$$

**Lemma 2.2.** ([21]) Let $0 < \beta < \infty$. If $u, v \in L^2(0, T)$, then

$$\left( I^\beta_{0+} u, v \right)_{L^2(0, T)} = \left( u, I^\beta_{T-} v \right)_{L^2(0, T)}.$$

If $v \in L^p(0, T)$ with $1 \leq p \leq \infty$, then

$$\left\| I^\beta_{0+} v \right\|_{L^p(0, T)} \leq C \left\| v \right\|_{L^p(0, T)},$$

$$\left\| I^\beta_{T-} v \right\|_{L^p(0, T)} \leq C \left\| v \right\|_{L^p(0, T)},$$

where $C$ is a positive constant that only depends on $T$, $\beta$ and $p$.

**Lemma 2.3.** Let $1 < \beta < 2$ and $v \in H^1(0, T)$ with $v(0) = 0$. Then

$$D^\beta_{0+} v = D^\beta_{0+} v',$$

where $v'$ denotes the weak derivative of $v$. 

For the proofs of Lemmas 2.1 and 2.2, we refer the reader to [21], and since the proof of Lemma 2.3 is straightforward, we omit it here. In the rest of this paper, we shall use the above three lemmas implicitly since they are frequently used. Also, we will use directly the well-known properties of the standard Sobolev spaces, such as that \( H^{\beta}(0,T) \) is continuously embedded into \( C[0,T] \) for all \( 0.5 < \beta < \infty \), and that

\[
\|v\|_{L^2(0,T)} \leq C \|v'\|_{L^2(0,T)}
\]

for all \( v \in H^1(0,T) \) with \( v(0) = 0 \), where \( C \) is a positive constant that only depends on \( T \).

For convenience we make the following conventions: by \( x \lesssim y \) we mean that there exists a positive constant \( C \) that only depends on \( \alpha \), \( T \) or \( \Omega \), unless otherwise stated, such that \( x \leq Cy \) (the value of \( C \) may differ at its each occurrence); by \( x \sim y \) we mean that \( x \approx y \approx x \).

**Lemma 2.4.** If \( v \in L^2(0,T) \), then

\[
\left\| I_{\alpha}^{0+} v \right\|_{H^\alpha(0,T)} \lesssim \|v\|_{L^2(0,T)}. \tag{2.1}
\]

**Proof.** Since

\[
D I_{\alpha}^{0+} v = D I_{\alpha}^{0} v = I_{\alpha}^{0+} v,
\]

the estimate (2.1) follows directly from the following result:

\[
\left\| I_{\alpha}^{0+} v \right\|_{H^{\alpha-1}(0,T)} \lesssim \|v\|_{L^2(0,T)},
\]

which can be obtained by [11, Lemma 2.4]. This completes the proof. \( \blacksquare \)

**Lemma 2.5.** Let \( v \in L^2(0,T) \). Then \( D_{\alpha}^{\frac{\alpha-1}{T}} v \in L^2(0,T) \) if and only if \( v \in H^{\frac{\alpha-1}{T}}(0,T) \); and \( D_{T}^{\frac{\alpha-1}{T}} v \in L^2(0,T) \) if and only if \( v \in H^{\frac{\alpha-1}{T}}(0,T) \). Moreover, if \( v \in H^{\frac{\alpha-1}{T}}(0,T) \), then

\[
\left\| D_{\alpha}^{\frac{\alpha-1}{T}} v \right\|_{L^2(0,T)}^2 \sim \|v\|^2_{H^{\frac{\alpha-1}{T}}(0,T)} \sim \left\| D_{T}^{\frac{\alpha-1}{T}} v \right\|_{L^2(0,T)}^2 \sim \left( D_{\alpha}^{\frac{\alpha-1}{T}} v, D_{T}^{\frac{\alpha-1}{T}} v \right)_{L^2(0,T)}.
\]

The proof of the above lemma is exactly the same as [11, Lemma 2.5].

**Lemma 2.6.** Let \( v \in H^{\frac{\alpha-1}{T}}(0,T) \) such that \( D_{\alpha}^{\alpha} v \in L^2(0,T) \). Then

\[
v = I_{\alpha}^{\alpha} D_{\alpha}^{\alpha} v,
\]

\[
I_{\alpha}^{\alpha} D_{\alpha}^{\alpha} v = D_{\alpha}^{\alpha} I_{\alpha}^{\alpha} v,
\]

and

\[
\|v\|_{H^\alpha(0,T)} \lesssim \|D_{\alpha}^{\alpha} v\|_{L^2(0,T)}. \tag{2.4}
\]

**Proof.** Let us first consider (2.2) and (2.4). Since \( D_{\alpha}^{\alpha} v \in L^2(0,T) \) implies \( I_{\alpha}^{\alpha-\alpha} v \in H^2(0,T) \), a straightforward calculation gives

\[
v(t) = c_0 t^{\alpha-2} + c_1 t^{\alpha-1} + (I_{\alpha}^{\alpha} D_{\alpha}^{\alpha} v)(t), \quad 0 < t < T,
\]
where \( c_0 \) and \( c_1 \) are two real constants. Note that Lemma 2.4 implies \( I_{0+}^\alpha D_{0+}^\alpha v \in H^\alpha(0, T) \), which, together with the fact that \( v \in H^{\frac{\alpha+1}{2}}(0, T) \), shows \( c_0 = c_1 = 0 \). Hence (2.2) holds. Moreover, (2.4) follows directly from Lemma 2.4.

Then, let us prove (2.3). Note that \( v(0) = 0 \) due to (2.2) implies

\[
D_{0+}^{\alpha-1} v' = D_{0+}^\alpha v,
\]

which yields \( I_{0+}^{2-\alpha} v' \in H^1(0, T) \). Moreover, by (2.2) we have

\[
v' = I_{0+}^{\alpha-1} D_{0+}^\alpha v,
\]

and so

\[
(I_{0+}^{2-\alpha} v')(0) = (I_{0+} D_{0+}^\alpha v)(0) = 0.
\]

Consequently, using integration by parts gives

\[
(I_{0+} D_{0+}^\alpha v, \varphi)_{L^2(0,T)} = (D_{0+}^{\alpha-1} v', I_{T-}^\varphi)_{L^2(0,T)} = (I_{0+}^{2-\alpha} v', \varphi)_{L^2(0,T)}
\]

\[
= \langle v', I_{T-}^{2-\alpha} \varphi \rangle_{L^2(0,T)} = \langle v, I_{T-}^{3-\alpha} \varphi' \rangle_{L^2(0,T)} = \langle I_{0+}^{\alpha} v, \varphi'' \rangle_{L^2(0,T)} = \langle D_{0+}^\alpha v + I_{0+} v, \varphi \rangle
\]

for all \( \varphi \in D(0,T) \), where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( D'(0,T) \) and \( D(0,T) \). This proves (2.3) and thus completes the proof of the lemma. \( \blacksquare \)

**Lemma 2.7.** Suppose that \( v \in H^{\frac{\alpha+1}{2}}(0, T) \) with \( v(0) = 0 \). Then (i)-(iii) hold:

(i) We have

\[
\left( D_{0+}^{\frac{\alpha+1}{2}} v, D_{T-}^{\frac{\alpha+1}{2}} v' \right)_{L^2(0,T)} \sim \left\| D_{0+}^{\frac{\alpha+1}{2}} v \right\|_{L^2(0,T)}^2 \sim \| v \|^2_{H^{\frac{\alpha+1}{2}}(0, T)}, \tag{2.5}
\]

(ii) If \( \varphi \in H^{\frac{\alpha+1}{2}}(0, T) \), then

\[
\left| \left( D_{0+}^{\frac{\alpha+1}{2}} v, D_{T-}^{\frac{\alpha+1}{2}} \varphi \right)_{L^2(0,T)} \right| \lesssim \| v \|_{H^{\frac{\alpha+1}{2}}(0, T)} \| \varphi \|_{H^{\frac{\alpha+1}{2}}(0, T)}. \tag{2.6}
\]

(iii) If \( \varphi \in D(0,T) \), then

\[
\langle D_{0+}^\alpha v, \varphi \rangle = \left( D_{0+}^{\frac{\alpha+1}{2}} v, D_{T-}^{\frac{\alpha+1}{2}} \varphi \right)_{L^2(0,T)}. \tag{2.7}
\]

**Proof.** Let us first prove (2.5) and (2.6). By \( v(0) = 0 \) we have

\[
D_{0+}^{\frac{\alpha+1}{2}} v = D_{0+}^{\frac{\alpha+1}{2}} v', \tag{2.8}
\]

so that, by \( v' \in H^{\frac{\alpha+1}{2}}(0, T) \), Lemma 2.5 implies

\[
\left( D_{0+}^{\frac{\alpha+1}{2}} v, D_{T-}^{\frac{\alpha+1}{2}} v' \right)_{L^2(0,T)} \sim \left\| D_{0+}^{\frac{\alpha+1}{2}} v \right\|_{L^2(0,T)}^2 \sim \| v' \|^2_{H^{\frac{\alpha+1}{2}}(0, T)}. \tag{2.9}
\]

Since \( v(0) = 0 \) also gives

\[
\| v' \|^2_{H^{\frac{\alpha+1}{2}}(0, T)} \sim \| v \|_{H^{\frac{\alpha+1}{2}}(0, T)}^2,
\]

and so
the estimate (2.5) follows immediately, and then (2.6) follows from the Cauchy-Schwarz inequality and Lemma 2.5.

Then, let us prove (2.7). Note that $\frac{\alpha+1}{\alpha} v' \in L^2(0,T)$ implies $I_{0+}^{\frac{\alpha+1}{\alpha}} v' \in H^1(0,T)$. Also, by $(3 - \alpha)/2 > 0.5$, a simple computing yields

$$(I_{0+}^{\frac{\alpha+1}{\alpha}} v')(0) = 0.$$  

Therefore, using integration by parts gives

$$\langle D_0^\alpha v, \varphi \rangle = \langle (I_{0+}^{\alpha-\alpha} v, \varphi') \rangle_{L^2(0,T)} = \langle v, (I_{T-}^{\alpha-\alpha} \varphi) \rangle_{L^2(0,T)} = \langle \varphi', (I_{T-}^{\alpha-\alpha} v') \rangle_{L^2(0,T)} = \langle (D_0^{\frac{\alpha+1}{\alpha}} v', D_0^{\frac{\alpha+1}{\alpha}} \varphi') \rangle_{L^2(0,T)},$$

for all $\varphi \in \mathcal{D}(0,T)$, which, together with (2.8), proves (2.7). This completes the proof of the lemma. ■

3 Regularity for an ordinary equation

This section considers the following problem: given $c_0, c_1 \in \mathbb{R}$ and $g \in L^2(0,T)$, seek $y \in H^\alpha(0,T)$ such that

$$D_0^\alpha (y - c_0 - c_1 t) + \lambda y = g,$$  

where $\lambda \geq 1$ is a positive constant.

**Theorem 3.1.** Problem (3.1) has a unique solution $y \in H^\alpha(0,T)$, and $y$ satisfies $y(0) = c_0$ and

$$\left(D_0^{\frac{\alpha+1}{\alpha}} (y - c_0 - c_1 t), D_T^{\frac{\alpha+1}{\alpha}} z \right)_{L^2(0,T)} + \lambda (y, z)_{L^2(0,T)} = (f, z)_{L^2(0,T)}$$  

for all $z \in H^{\frac{\alpha+1}{\alpha}}(0,T)$. Moreover,

$$\|y\|_{H^{\frac{\alpha+1}{\alpha}}(0,T)} + \lambda^{\frac{1}{2}} \|y\|_{L^2(0,T)} \lesssim \|g\|_{L^2(0,T)} + \lambda^{\frac{1}{2}} |c_0| + |c_1|.$$  

**Proof.** Let

$$b(z) := (g, z)_{L^2(0,T)} + \left(D_0^{\frac{\alpha+1}{\alpha}} (c_1 t), D_T^{\frac{\alpha+1}{\alpha}} z \right)_{L^2(0,T)} - \lambda (c_0, z)_{L^2(0,T)}$$

for all $z \in H^{\frac{\alpha+1}{\alpha}}(0,T)$. Since Lemma 2.5 implies $b \in H^{\frac{\alpha+1}{\alpha}}(0,T)$ (the dual space of $H^{\frac{\alpha+1}{\alpha}}(0,T)$), Lemma 2.7 and the Babuška-Lax-Milgram Theorem [2] guarantee the unique existence of $w \in H^{\frac{\alpha+1}{\alpha}}(0,T)$ with $w(0) = 0$ such that

$$\left(D_0^{\frac{\alpha+1}{\alpha}} w, D_T^{\frac{\alpha+1}{\alpha}} z \right)_{L^2(0,T)} + \lambda (w, z)_{L^2(0,T)} = b(z)$$  

for all $z \in H^{\frac{\alpha+1}{\alpha}}(0,T)$. Using Lemma 2.7 gives

$$\langle D_0^{\alpha} w, \varphi \rangle = \left(D_0^{\frac{\alpha+1}{\alpha}} w, D_0^{\frac{\alpha+1}{\alpha}} \varphi \right)_{L^2(0,T)},$$

$$\langle D_0^{\alpha} (c_1 t), \varphi \rangle = \left(D_0^{\frac{\alpha+1}{\alpha}} (c_1 t), D_0^{\frac{\alpha+1}{\alpha}} \varphi \right)_{L^2(0,T)},$$
for all $\varphi \in D(0, T)$, so that from (3.4) it follows that
\[ D_{y}^\alpha (w - c_1 t) = g - \lambda (w + c_0). \]

Putting $y := w + c_0$ gives
\[ D_{y}^\alpha (y - c_0 - c_1 t) + \lambda y = g, \]
and then by Lemmas 2.6 and 2.7 it is evident that $y$ is the unique $H^\alpha (0, T)$-solution to problem (3.1). Also, $y(0) = c_0$ is obvious, and (3.2) follows directly from (3.4).

Now let us prove (3.3). Firstly, taking $z := y'$ in (3.4) and using integration by parts yield
\[ \left( D_{0+}^{\alpha+1} (y - c_0 - c_1 t), D_{T-}^{\alpha-1} y' \right)_{L^2(0, T)} + \frac{\lambda}{2} y^2(T) = (g, y')_{L^2(0, T)} + \frac{\lambda}{2} c_0^2, \]
so that
\begin{align*}
\left( D_{0+}^{\alpha+1} (y - c_0 - c_1 t), D_{T-}^{\alpha-1} (y - c_0 - c_1 t)' \right)_{L^2(0, T)} + \frac{\lambda}{2} y^2(T) \\
= (g, y')_{L^2(0, T)} + \frac{\lambda}{2} c_0^2 - \left( D_{0+}^{\alpha+1} (y - c_0 - c_1 t), D_{T-}^{\alpha-1} (c_0 + c_1 t)' \right)_{L^2(0, T)}. 
\end{align*}

Therefore, Lemma 2.7, the Cauchy-Schwarz inequality and the Young’s inequality with $\epsilon$ imply
\[ \|y - c_0 - c_1 t\|^2_{H^{\alpha+1}(0, T)} + \lambda y^2(T) \lesssim \|g\|^2_{L^2(0, T)} + \lambda c_0^2 + c_1^2, \]
and so
\[ \|y\|_{H^{\alpha+1}(0, T)} \lesssim \|g\|_{L^2(0, T)} + \lambda^{\frac{1}{2}} |c_0| + |c_1|. \tag{3.5} \]

Secondly, taking $z := y$ in (3.4) gives
\[ \lambda \|y\|^2_{L^2(0, T)} = (g, y)_{L^2(0, T)} - \left( D_{0+}^{\alpha+1} (y - c_0 - c_1 t), D_{T-}^{\alpha-1} y \right)_{L^2(0, T)}, \]
so that using Lemmas 2.5 and 2.7, the Cauchy-Schwarz inequality and the Young’s inequality with $\epsilon$ gives
\[ \lambda \|y\|^2_{L^2(0, T)} \lesssim \|y - c_0 - c_1 t\|_{H^{\alpha+1}(0, T)} \|y\|_{H^{\alpha-1}(0, T)} + \lambda^{-1} \|g\|^2_{L^2(0, T)}, \]
which, together with (3.5), yields
\[ \lambda^{\frac{1}{2}} \|y\|_{L^2(0, T)} \lesssim \|g\|_{L^2(0, T)} + \lambda^{\frac{1}{2}} |c_0| + |c_1|. \tag{3.6} \]

Finally, collecting (3.5) and (3.6) leads to (3.3), and thus proves this theorem. \[ \blacksquare \]

Denote, for $0 < t < T$,
\begin{align*}
\tilde{S}_1(t) &:= \frac{g(0) - \lambda c_0}{\Gamma(\alpha + 1)} t^\alpha, &\tilde{S}_2(t) &:= \frac{g'(0) - \lambda c_1}{\Gamma(\alpha + 2)} t^{\alpha+1}.
\end{align*}
**Theorem 3.2.** Suppose that \( g \in H^1(0, T) \) and \( y \) is the solution to problem (3.1). Then \( y \in C^1[0, T] \) with \( y'(0) = c_1 \), and

\[
\|y - \tilde{S}_1\|_{H^{\frac{3}{2}}(0, T)} + \lambda^{\frac{1}{2}} \|y\|_{H^1(0, T)} + \lambda \|y\|_{L^2(0, T)} \lesssim \|g\|_{H^1(0, T)} + \lambda |c_0| + \lambda |c_1| + \lambda |g(0) - \lambda c_0|. \tag{3.7}
\]

Furthermore, if \( 1.5 < \alpha < 2 \) and \( g \in H^2(0, T) \), then

\[
\|y - \tilde{S}_1 - \tilde{S}_2\|_{H^{\frac{3}{2}}(0, T)} + \lambda^{\frac{1}{2}} \|y\|_{H^2(0, T)} + \lambda \|y\|_{H^1(0, T)} \lesssim \|g\|_{H^2(0, T)} + \lambda |c_0| + \lambda |c_1| + \lambda |g(0) - \lambda c_0| + \lambda |g'(0) - \lambda c_1|. \tag{3.8}
\]

**Proof.** Let us first prove that \( y \in C^1[0, T] \) with \( y'(0) = c_1 \). By Theorem 3.1, there exists a unique \( w \in H^\alpha(0, T) \) with \( w(0) = 0 \) such that

\[
D_0^\alpha w + \lambda w = g' - \lambda (c_1 + \tilde{S}_1'), \tag{3.9}
\]

and

\[
\|w\|_{H^{\frac{3}{2}}(0, T)} + \lambda^{\frac{1}{2}} \|w\|_{L^2(0, T)} \lesssim \|g'\|_{L^2(0, T)} + \lambda |c_1| + \lambda |g(0) - \lambda c_0|. \tag{3.10}
\]

Integrating both sides of (3.9) in \((0, T)\), by Lemma 2.6 we obtain

\[
D_0^\alpha I_{0+} w + \lambda I_{0+} w = g - g(0) - \lambda c_1 t - \lambda \tilde{S}_1,
\]

so that, setting

\[
y := c_0 + c_1 t + \tilde{S}_1 + I_{0+} w, \tag{3.11}
\]

it follows

\[
D_0^\alpha (y - c_0 - c_1 t) + \lambda y - D_0^\alpha \tilde{S}_1 - \lambda c_0 + g(0) = g.
\]

Since a straightforward calculation gives

\[
D_0^\alpha \tilde{S}_1 = g(0) - \lambda c_0,
\]

we see that \( y \) is the solution to problem (3.1). Finally, by (3.11) and the fact that \( w \in H^\alpha(0, T) \) with \( w(0) = 0 \), it is evident that \( y \in C^1[0, T] \) with \( y'(0) = c_1 \).

Next, let us prove (3.7). Note that

\[
\|\lambda y\|_{L^2(0, T)} = \|g - D_0^\alpha (y - c_0 - c_1 t)\|_{L^2(0, T)} = \|g - D_0^\alpha \tilde{S}_1 - D_0^\alpha (y - c_0 - c_1 t - \tilde{S}_1)\|_{L^2(0, T)} \lesssim \|g\|_{L^2(0, T)} + \|D_0^\alpha \tilde{S}_1\|_{L^2(0, T)} + \|D_0^\alpha (y - c_0 - c_1 t - \tilde{S}_1)\|_{L^2(0, T)} \lesssim \|g\|_{L^2(0, T)} + |g(0) - \lambda c_0| + \|D_0^\alpha (y - c_0 - c_1 t - \tilde{S}_1)\|_{L^2(0, T)}.
\]

Also, (3.11) implies

\[
(y - c_0 - c_1 t - \tilde{S}_1)(0) = (y - c_0 - c_1 t - \tilde{S}_1)'(0) = 0,
\]

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and hence
\[ \|D_0^+ (y - c_0 - c_1 t - \tilde{S}_1)\|_{L^2(0,T)} = \|I_0^{\alpha} (y - c_0 - c_1 t - \tilde{S}_1)\|_{L^2(0,T)} \]
\[ \lesssim \|y - \tilde{S}_1\|_{H^2(0,T)} \lesssim \|y - \tilde{S}_1\|_{H^{\frac{n+3}{2}}(0,T)} . \]
Consequently,
\[ \lambda \|y\|_{L^2(0,T)} \lesssim \|g\|_{L^2(0,T)} + |g(0) - \lambda c_0| + \|y - \tilde{S}_1\|_{H^{\frac{n+3}{2}}(0,T)} , \]
and then (3.7) follows from the following estimate:
\[ \|y - \tilde{S}_1\|_{H^{\frac{n+3}{2}}(0,T)} + \lambda \|y\|_{H^1(0,T)} \lesssim \|g'\|_{L^2(0,T)} + \lambda^{\frac{1}{2}} |c_0| + \lambda |c_1| + \lambda |g(0) - \lambda c_0| , \]
which is a direct consequence of (3.10) and (3.11).

Finally, let us prove (3.8). Since \( g \in H^2(0,T) \) and \( 1.5 < \alpha < 2 \) imply
\[ g' - \lambda (c_1 + \tilde{S}_1' ) \in H^1(0,T) , \]
applying (3.7) to problem (3.9) gives
\[ \|w - \tilde{S}_1\|_{H^{\frac{n+3}{2}}(0,T)} + \lambda \|w\|_{H^1(0,T)} + \lambda \|w\|_{L^2(0,T)} \lesssim \|g\|_{H^2(0,T)} + \lambda |c_1| + \lambda |g(0) - \lambda c_0| + \lambda |g'(0) - \lambda c_1| , \]
which, together with (3.11), yields (3.8). This completes the proof of the theorem. ■

Remark 3.1. Theorem 3.2 shows that the solution \( y \) to problem (3.1) generally has singularity despite how smooth \( g \) is; however, it also shows that we can improve the regularity of \( y \) by subtracting some particular singular functions, provided \( g \) is sufficiently regular. Although Theorem 3.2 only considers the cases of \( g \in H^1(0,T) \), and \( g \in H^2(0,T) \) with restriction \( 1.5 < \alpha < 2 \), using the same technique used in the proof of Theorem 3.2, we can also obtain the singularity information of the solution to problem (3.1) when \( g \) is of higher regularity than \( H^1(0,T) \). For example, if \( g \in H^2(0,T) \) then we can obtain the following regularity estimate for all \( 1 < \alpha < 2 \):
\[ \|y - \tilde{S}_1 - \tilde{S}_2 - \tilde{S}_3\|_{H^{\frac{n+3}{2}}(0,T)} + \lambda^{\frac{1}{2}} \|y - \tilde{S}_1\|_{H^2(0,T)} + \lambda \|y\|_{H^1(0,T)} \lesssim \|g\|_{H^2(0,T)} + \lambda |c_0| + \lambda |c_1| + \lambda |g'(0) - \lambda c_1| + \lambda^2 |g(0) - \lambda c_0| , \]
where \( S_1 \) and \( S_2 \) are defined as in Theorem 3.2, and
\[ \tilde{S}_3(t) := -\lambda \frac{g(0) - \lambda c_0}{\Gamma(2\alpha + 1)} t^{2\alpha}, \quad 0 < t < T . \]
4 Main results

This section is to study the regularity of the weak solution to problem (1.1). Let us first introduce some notations and conventions. We use $C([0, T]; L^2(\Omega))$ to denote the set of continuous $L^2(\Omega)$-valued functions with domain $[0, T]$. Given $v \in L^2(\Omega_T)$, we regard it as an $L^2(\Omega)$-valued function with domain $(0, T)$ as usual, and, for convenience, we also use $v$ to denote this $L^2(\Omega)$-valued function.

We introduce the following two fractional differential operators:

$$\partial^{\frac{\alpha}{2}}_{t, 0^+} := \partial^2_t I^{\frac{\alpha}{2}}_{t, 0^+} \quad \text{and} \quad \partial^{\frac{\alpha}{2}}_{t, T^-} := -\partial_t I^{\frac{\alpha}{2}}_{t, T^-},$$

where $I^{\frac{\alpha}{2}}_{t, 0^+}, I^{\frac{\alpha}{2}}_{t, T^-} : L^1(\Omega_T) \to L^1(\Omega_T)$ are defined, respectively, by

$$I^{\frac{\alpha}{2}}_{t, 0^+} v(x, t) := \frac{1}{\Gamma(\frac{\alpha}{2} + 1)} \int_0^t (t - s)^{\frac{\alpha}{2} - 1} v(x, s) \, ds, \quad (x, t) \in \Omega_T,$$

$$I^{\frac{\alpha}{2}}_{t, T^-} v(x, t) := \frac{1}{\Gamma(\frac{\alpha}{2} + 1)} \int_t^T (s - t)^{\frac{\alpha}{2} - 1} v(x, s) \, ds, \quad (x, t) \in \Omega_T,$$

for all $v \in L^1(\Omega_T)$. Moreover, from Lemmas 2.5 and 2.7 it is easy to know that the above two operators have the following fundamental properties.

**Lemma 4.1.** If $v \in H^{\frac{\alpha}{2}}(0, T; L^2(\Omega))$, then

$$\partial^{\frac{\alpha}{2}}_{t, T^-} v = \sum_{k=0}^{\infty} \phi_k D^{\frac{\alpha}{2}}_{T^-} (v, \phi_k)_{L^2(\Omega)} \quad \text{and}$$

$$\left\| \partial^{\frac{\alpha}{2}}_{t, T^-} v \right\|_{L^2(\Omega_T)} \sim \| v \|_{H^{\frac{\alpha}{2}}(0, T; L^2(\Omega))}.$$

If $v \in H^{\frac{\alpha}{2}}(0, T; L^2(\Omega))$ with $v(0) = 0$, then

$$\partial^{\frac{\alpha}{2}}_{t, 0^+} v = \sum_{k=0}^{\infty} \phi_k D^{\frac{\alpha}{2}}_{0^+} (v, \phi_k)_{L^2(\Omega)} \quad \text{and}$$

$$\left\| \partial^{\frac{\alpha}{2}}_{t, 0^+} v \right\|_{L^2(\Omega_T)} \sim \| v \|_{H^{\frac{\alpha}{2}}(0, T; L^2(\Omega))}.$$

Next let us introduce the definition of a weak solution to problem (1.1).

**Definition 4.1.** We call $u \in H^{\frac{\alpha+1}{2}}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ with $u(0) = u_0$ a weak solution to problem (1.1) if

$$\left( \partial_{t, 0^+} (u - u_0 - t u_1), \partial^{\frac{\alpha}{2}}_{t, T^-} \varphi \right)_{L^2(\Omega_T)} + (\nabla u, \nabla \varphi)_{L^2(\Omega_T)} = (f, \varphi)_{L^2(\Omega_T)} \quad (4.1)$$

for all $\varphi \in H^{\frac{\alpha+1}{2}}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)).$

**Remark 4.1.** By Lemma 4.1 it is easy to see that the above weak solution is well-defined. Also, it is easy to verify that, if $u$ is a weak solution to problem (1.1), then

$$\left( \partial_t^\alpha (u - u_0 - t u_1) - \Delta u, \varphi \right)_{L^2(\Omega_T)} = (f, \varphi)_{L^2(\Omega_T)}$$

for all $\varphi \in \mathcal{D}(\Omega_T)$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{D}'(\Omega_T)$ and $\mathcal{D}(\Omega_T)$, namely, $u$ satisfies equation (1.1) in the distribution sense.
Moreover, there exists, in $H^1_0(\Omega) \cap H^2(\Omega)$, an orthonormal basis $\{\phi_k| k \in \mathbb{N}\}$ of $L^2(\Omega)$, and a nondecreasing sequence $\{\lambda_k > 0| k \in \mathbb{N}\}$ such that

$$-\Delta \phi_k = \lambda_k \phi_k \text{ in } \Omega, \text{ for all } k \in \mathbb{N}.$$  

Also, $\{\lambda_k^{-1/2} \phi_k| k \in \mathbb{N}\}$ is an orthonormal basis of $H^1_0(\Omega)$ equipped with the inner product $(\nabla \cdot, \nabla \cdot)_{L^2(\Omega)}$. For each $k \in \mathbb{N}$, define $c_k \in H^\alpha(0, T)$ by

$$D^\alpha_0(c_k - c_{k,0} - c_{k,1} t) + \lambda_k c_k = f_k, \quad (4.2)$$

where

$$c_{k,0} := (u_0, \phi_k)_{L^2(\Omega)}, \quad c_{k,1} := (u_1, \phi_k)_{L^2(\Omega)}, \quad f_k := (f, \phi_k)_{L^2(\Omega)},$$

and we recall that $f \in L^2(\Omega_T)$, $u_0 \in H^1_0(\Omega)$ and $u_1 \in L^2(\Omega)$. Finally, define

$$u(t) := \sum_{k=0}^{\infty} c_k(t) \phi_k, \quad 0 < t < T. \quad (4.3)$$

**Theorem 4.1.** Problem (1.1) has a unique weak solution $u$ given by (4.3). Moreover,

$$\|u\|_{H^{\frac{\alpha+2}{2\alpha+1}}(0, T; L^2(\Omega))} + \|u\|_{L^2(0,T; H^1_0(\Omega))} \leq C \left( \|f\|_{L^2(0,T; L^2(\Omega))} + \|u_0\|_{H^1_0(\Omega)} + \|u_1\|_{L^2(\Omega)} \right). \quad (4.4)$$

Denote, for $0 < t < T$,

$$S_1(t) := \sum_{k=0}^{\infty} \frac{f_k(0) - \lambda_k c_{k,0} t^\alpha}{\Gamma(\alpha + 1)} \phi_k, \quad S_2(t) := \sum_{k=0}^{\infty} \frac{f_k'(0) - \lambda_k c_{k,1} t^{\alpha+1}}{\Gamma(\alpha + 2)} \phi_k.$$

**Theorem 4.2.** Suppose that $u$ is the weak solution to problem (1.1). Then (i)- (ii) hold:

(i) If $f \in H^1(0,T; L^2(\Omega))$, and

$$u_0, u_1, f(0) + \Delta u_0 \in H^1_0(\Omega) \cap H^2(\Omega),$$

then

$$\|u - S_1\|_{H^{\frac{\alpha+2}{2\alpha+1}}(0, T; L^2(\Omega))} + \|u\|_{H^1(0,T; H^1_0(\Omega))} + \|u\|_{L^2(0,T; H^2(\Omega))} \leq C \left( \|f\|_{H^1(0,T; L^2(\Omega))} + \|u_0\|_{H^1_0(\Omega)} + \|u_1\|_{H^2(\Omega)} + \|f(0) + \Delta u_0\|_{H^2(\Omega)} \right). \quad (4.5)$$

(ii) If $1.5 < \alpha < 2$, $f \in H^2(0,T; L^2(\Omega))$, and

$$u_0, u_1, f(0) + \Delta u_0, f'(0) + \Delta u_1 \in H^1_0(\Omega) \cap H^2(\Omega),$$

then

$$\|u - S_1 - S_2\|_{H^{\frac{\alpha+5}{2\alpha+1}}(0, T; L^2(\Omega))} + \|u\|_{H^2(0,T; H^1_0(\Omega))} + \|u\|_{L^2(0,T; H^2(\Omega))} \leq C \left( \|f\|_{H^2(0,T; L^2(\Omega))} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^2(\Omega)} + \|f(0) + \Delta u_0\|_{H^2(\Omega)} + \|f'(0) + \Delta u_1\|_{H^2(\Omega)} \right). \quad (4.6)$$
Remark 4.2. Theorem 4.2 reveals that the solution to problem (1.1) generally has singularity in time. As mentioned in Remark 3.1, we can obtain more precise singularity information of the solution to problem (3.1) when $g$ is of higher regularity than stated in Theorem 3.2. Correspondingly, we can also prove Theorems 4.1 and 4.3 in the remainder of this section. (Suppose that $f \in H^2(0,T;L^2(\Omega))$, $f(0) + \Delta u_0 \in H^3_0(\Omega) \cap H^4(\Omega)$, and $u_0, u_1, f'(0) + \Delta u_1 \in H^3_0(\Omega) \cap H^4(\Omega)$, then $\|u - S_1 - S_2 - S_3\|_{H^{\frac{\alpha+1}{2}}(0,T;L^2(\Omega))} + \|u - S_1\|_{H^2(0,T;H^3(\Omega))} + \|u\|_{H^1(0,T;H^2(\Omega))}$
\[
\leq \|f\|_{H^2(0,T;L^2(\Omega))} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^2(\Omega)} + \|f'(0) + \Delta u_1\|_{H^2(\Omega)}
+ \|f(0) + \Delta u_0\|_{H^4(\Omega)},
\]
where $S_1$ and $S_2$ are defined as in Theorem 4.2, and

$S_3(t) := \sum_{k=0}^{\infty} -\frac{\lambda_k}{\Gamma(2\alpha+1)} f_k(0) - \frac{\lambda_k c_k}{2}\alpha t^{2\alpha}, \quad 0 < t < T.$

Theorem 4.3. Suppose that $u$ is the weak solution to problem (1.1). If $f \in H^1(0,T;L^2(\Omega))$, and $u_0, u_1, f(0) + \Delta u_0 \in H^3_0(\Omega) \cap H^4(\Omega)$, then $u' \in C([0,T];L^2(\Omega))$ with $u'(0) = u_1$.

Since Theorem 4.2 follows from Theorems 3.1 and 3.2 easily, we shall only prove Theorems 4.1 and 4.3 in the remainder of this section.

Proof of Theorem 4.1. If $u$ is given by (4.3), then (4.4) is straightforward by Theorem 3.1; therefore, we only need to prove that $u$ given by (4.3) is the unique weak solution to problem (1.1).

Let us first show that $u$ in (4.3) is a weak solution to problem (1.1). Using the definitions of the $c_k$'s and Theorem 3.1 gives

$\left( D_0^{\frac{\alpha-1}{2}} (c_k - c_k,0 - c_k,1), D_0^{\frac{\alpha-1}{2}} \varphi \right)_{L^2(\Omega,T)} + \lambda_k (c_k, \varphi)_{L^2(0,T)} = (f_k, \varphi)_{L^2(0,T)}$

for all $\varphi \in H^{\frac{\alpha-1}{2}}(0,T)$ and $k \in \mathbb{N}$. From Theorem 3.1 it follows $u \in H^{\frac{\alpha-1}{2}}(0,T;L^2(\Omega))$ with $u(0) = u_0$, then Lemma 4.1 implies

$\left( \partial_{1,T}^{\frac{\alpha-1}{2}} (u - u_0 - tu_1), \partial_{1,T}^{\frac{\alpha-1}{2}}(\varphi \phi_j) \right)_{L^2(\Omega,T)} + (\nabla u, \nabla (\varphi \phi_j))_{L^2(\Omega,T)} = (f, \varphi \phi_j)_{L^2(\Omega,T)}$

for all $\varphi \in H^{\frac{\alpha-1}{2}}(0,T)$ and $j \in \mathbb{N}$. As

$\text{span} \left\{ \varphi \phi_j \mid \varphi \in H^{\frac{\alpha-1}{2}}(0,T), \ j \in \mathbb{N} \right\}$

is dense in $H^{\frac{\alpha-1}{2}}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$, by Lemma 4.1 a standard density argument yields

$\left( \partial_{1,T}^{\frac{\alpha-1}{2}} (u - u_0 - tu_1), \partial_{1,T}^{\frac{\alpha-1}{2}} \varphi \right)_{L^2(\Omega,T)} + (\nabla u, \nabla \varphi)_{L^2(\Omega,T)} = (f, \varphi)_{L^2(\Omega,T)}$
for all $\varphi \in H^{\frac{\alpha+1}{2}}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$, which proves that $u$ is indeed a weak solution to problem (1.1).

Now let us prove that $u$ in (4.3) is the unique weak solution to problem (1.1). To this end, assume that $e \in H^{\frac{\alpha+1}{2}}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ with $e(0) = 0$ satisfies
\[
\left( \partial_t^{\frac{\alpha+1}{2}} e, \partial_t^{\frac{\alpha+1}{2}} \varphi \right)_{L^2(\Omega_T)} + (\nabla e, \nabla \varphi)_{L^2(\Omega_T)} = 0 \quad (4.7)
\]
for all $\varphi \in H^{\frac{\alpha+1}{2}}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$. Then it suffices to show that $e = 0$ in $\Omega_T$. To do so, let $k \in \mathbb{N}$ and define
\[
d_k(t) := (e, \phi_k)_{L^2(\Omega)}, \quad 0 < t < T.
\]
It is obvious that $d_k \in H^{\frac{\alpha+1}{2}}(0, T)$ with $d_k(0) = 0$. By Lemma 4.1, taking $\varphi := d'_k \phi_k$ in (4.7) gives
\[
\left( D_0^{\frac{\alpha+1}{2}} d_k, D_{T-}^{\frac{\alpha+1}{2}} d'_k \right)_{L^2(0, T)} + \lambda_k (d_k, d'_k)_{L^2(0, T)} = 0.
\]
Using integration by parts, by Lemma 2.7 we obtain
\[
||d_k||^2_{H^{\frac{\alpha+1}{2}}(0, T)} + \lambda_k |d_k(T)|^2 = 0,
\]
which yields $d_k = 0$ in $(0, T)$. Since $k \in \mathbb{N}$ is arbitrary, we deduce that $e = 0$ in $\Omega_T$, and hence finish the proof. 

\textbf{Proof of Theorem 4.3.} Note that Theorem 4.2 implies $u - S_1 \in H^{\frac{\alpha+3}{2}}(0, T; L^2(\Omega))$.

Also, using
\[
l^\alpha \in H^{1 + \frac{\alpha}{2}}(0, T) \text{ and } f(0) + \Delta u_0 \in H^1_0(\Omega) \cap H^2(\Omega)
\]
gives $S_1 \in H^{1 + \frac{\alpha}{2}}(0, T; L^2(\Omega))$. As a result, we obtain $u \in H^{1 + \frac{\alpha}{2}}(0, T; L^2(\Omega))$ and so $u' \in H^\frac{\alpha}{2}(0, T; L^2(\Omega))$. As $\alpha/2 > 0.5$ implies $u' \in C([0, T]; L^2(\Omega))$, it remains to show that
\[
c_k'(0) = c_{k,1} \text{ for all } k \in \mathbb{N}.
\]
This assertion holds indeed by the definition of $c_k$ and Theorem 3.2. This proves the theorem. 

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