SOME NEW BOUNDS ANALOGOUS TO GENERALIZED PROPORTIONAL FRACTIONAL INTEGRAL OPERATOR WITH RESPECT TO ANOTHER FUNCTION

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Abstract. The present article deals with the new estimates in the view of generalized proportional fractional integral with respect to another function. In the present investigation, we focus on driving certain new classes of integral inequalities utilizing a family of positive functions \(n(n \in \mathbb{N})\) for this newly defined operator. From the computed outcomes, we concluded some new variants for classical generalized proportional fractional and other integrals as remarks. These variants are connected with some existing results in the literature. Certain interesting consequent results of the main theorems are also pointed out.

1. Introduction. Fractional calculus generally referred to as the calculus of non-integer order, which was a trademark outgrowth of traditional definitions of calculus integral and derivative. The concept of fractional calculus has provoked a host of researchers and comprehensively studied in the literature for the last few decades. A continuous effort has been conveyed out on it on an enormous scale and everybody has stimulated its different angles. In the present century, the exceptional idea has been described by several mathematicians with a slightly distinct technique, for instance, in this concern Liouville, Riemann, Grunwald, Letnikov, Hadamard, Weyl, Riesz, Marchaud, Koer and Caputo fractional integral (see, \([2, 3, 7, 11, 17, 26, 39]\)).

Most of these researchers first of all added fractional integrals, on the concept of which the associated fractional derivative and other associated results had been produced. Recently, Khalil et al. \([20]\) and Abdeljawad \([1]\) introduced fractional operators known as fractional conformable derivatives and integrals. In a while in \([9]\), Anderson and Ulness introduced the concept of local derivatives for upgrading the concept of the fractional conformable derivative. The exponential and Mittag-Leffler

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functions are used as kernels by several researchers for developing new fractional techniques. In [15], Jarad et al. presented the concept of generalized proportional-integral operators which have been utilized to characterize certain probability density functions and has intriguing applications in statistics (also see [3, 12, 21, 22, 29, 30]). Recently, Jarad et al. [16] and Rashid et al. [47] proposed another novel fractional approach which comes into existence in the theory of fractional calculus, which is known as generalized proportional fractional operators with respect to another function $\Psi$. The specific features of generalized proportional fractional integral operator with respect to another function is that it can capture diverse number of complex problems on one hand and on the other hand the generalized proportional fractional derivative with respect to another function can also capture different types of complexities, thus putting these two concepts together can help us to understand the complexities of existing nature in a much better way. The generalized proportional fractional integral operators have attracted attention of almost all researchers from different fields of science, see [31, 32, 33, 34]. The idea is quite new and seems to have opened new doors of investigation towards various scientific fields of research including engineering, fluid dynamics, meteorology, analysis, aerodynamics and many more.

Integral inequalities have potential application in several areas of science: technology, mathematics, chemistry, plasma physics, among others; especially we point out the initial value problem, the stability of linear transformation, integral differential equations, and impulse equations [35, 36, 37]. Variants regarding fractional integral operators are the use of noteworthy significant strategies amongst researchers and accumulate fertile functional applications in various areas of science, see [38, 39]. On account of their potential outcomes to be utilized for the presence of nontrivial and positive solutions of distinct kind of fractional differential equations, our findings concerning fractional integrals are appreciably essential.

An enormous heft of present literature comprises generalizations of several variants by fractional integral operators and their applications [4, 5, 6, 8, 13, 18, 23, 27, 40, 41, 42, 43, 44]. We state some of them, that is, the variants of Minkowski, Hardy, Opial, Hermite-Hadamard, Grüss, lyenger, Ostrowski, Chebyshev, and Polya-Szego, amongst others [45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 57]. Such applications of fractional integral operators compelled us to show the generalization by using a family of $n$ positive functions involving generalized proportional fractional integrals operators with respect to another function $\Psi$.

The principal objective of this article is that we demonstrate the notations and primary definitions of our newly introduced operator generalized proportional fractional integral with respect to another function $\Psi$. Also, we present the results concerning for a class of family of $n$ ($n \in \mathbb{N}$) continuous positive decreasing functions on $[\nu_1, \nu_2]$ by employing generalized conformable fractional integral operator. Finally, we advocate essential consequences via the generalized proportional fractional integral operators with respect to another function $\Psi$.

2. Notations. In this segment, we give some significant ideas from fractional calculus utilized in our consequent discourse. The fundamental details are given in the monograph by Kilbas et al. [19].

Now, we present a new fractional operator which is known as the generalized
proportional fractional integral operator of a function with respect to another function \( \Psi \), which is proposed by Jarad et al and Rashid et al. [16, 47].

**Definition 2.1.** ([16, 47]) Let \((v_1, v_2) (-\infty \leq v_1 < v_2 \leq \infty)\) be a finite or infinite real interval and \( \beta > 0 \). Let \( \Psi(\zeta) \) is an increasing and positive monotone function on \((v_1, v_2)\). Then the left-sided and right-sided generalized proportional fractional integral operator of a function \( F \) with respect to another function \( \Psi \) of order \( \beta > 0 \) are defined as:

\[
(\Psi F_{\beta, \mu}^\nu (x) = \frac{1}{\mu^\beta \Gamma(\beta)} \int_{v_1}^x \frac{\exp[\frac{\mu-1}{\mu}(\Psi(x) - \Psi(\xi))]\Psi(\xi)}{(\Psi(x) - \Psi(\xi))^{1-\beta}} F(\xi) d\xi, \quad v_1 < x \quad (1)
\]

and

\[
(\Psi F_{\beta, \mu}^\nu (x) = \frac{1}{\mu^\beta \Gamma(\beta)} \int_x^{v_2} \frac{\exp[\frac{\mu-1}{\mu}(\Psi(\xi) - \Psi(x))]\Psi(\xi)}{(\Psi(\xi) - \Psi(x))^{1-\beta}} F(\xi) d\xi, \quad x < v_2, \quad (2)
\]

where the proportionality index \( \mu \in (0, 1], \beta \in \mathbb{C}, \Re(\beta) > 0 \), and \( \Gamma(x) = \int_0^\infty \xi^{x-1}e^{-\xi}d\xi \) is the Gamma function.

**Remark 1.** In Definition 2.1:

1. If we consider \( \Psi(x) = x \), then we will attain the left and right-sided generalized proportional fractional integral operator in [15].
2. If we consider \( \mu = 1 \), then we will attain the left and right-sided generalized Riemann-Liouville fractional integral operator in [19].
3. If we consider \( \mu = 1 \), along \( \Psi(x) = x \), then we will attain the left and right-sided Riemann-Liouville fractional integral operator in [19].
4. If we consider \( \Psi(x) = \ln x \), then we will attain the left and right-sided generalized proportional Hadamard fractional integral operator in [41].
5. If we consider \( \Psi(x) = \ln x \) along with \( \mu = 1 \), then we attain the left and right-sided Hadamard fractional integral operator [19, 55].

3. **Main results.** Now we demonstrate the the left generalized proportional fractional integral operator with respect to another function \( \Psi \) to derive the generalization of some classical inequalities.

**Theorem 3.1.** Suppose \( \Phi \) be a continuous positive decreasing function on \([v_1, v_2]\) with \( v_1 < r \leq v_2, \delta > 0, \) and \( \varepsilon \geq \gamma > 0 \). Then, for generalized proportional fractional integral with respect to another function \( \Psi \) defined in (1), we have

\[
\frac{\Psi F_{\delta, \varepsilon}^\mu [\Phi^\gamma(r)]}{\Psi F_{\delta, \gamma}^\mu [\Phi^\varepsilon(r)]} \geq \frac{\Psi F_{\delta, \varepsilon}^\mu [(r - v_1)^\delta \Phi^\gamma(r)]}{\Psi F_{\delta, \gamma}^\mu [(r - v_1)^\delta \Phi^\varepsilon(r)]}, \quad (3)
\]

where \( \mu \in [0, 1], \varepsilon \in \mathbb{C}, \) and \( \Re(\varepsilon) > 0 \).

**Proof.** Using the hypothesis given in Theorem 3.1, we have

\[
((\sigma - v_1)^\delta - (\eta - v_1)^\delta)(\Phi^\varepsilon(\eta) - \Phi^\varepsilon(\sigma)) \geq 0, \quad (4)
\]

where \( \delta > 0, \varepsilon \geq \gamma > 0, v_1 < r \leq v_2 \) and \( \eta, \sigma \in [v_1, r] \).

By (4), we have

\[
(\sigma - v_1)^\delta \Phi^\varepsilon(\eta) - (\eta - v_1)^\delta \Phi^\varepsilon(\sigma) - (\sigma - v_1)^\delta \Phi^\varepsilon(\sigma) - (\eta - v_1)^\delta \Phi^\varepsilon(\eta) \geq 0. \quad (5)
\]
Let us define a function

$$
\mathcal{J}(\eta, \rho) = \frac{1}{\mu^s \Gamma(s)} \exp\left[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))\right] \Psi'(\eta).
$$

(6)

Accordingly, the function $\mathcal{J}(\eta, \rho)$ is positive for all $\eta \in (v_1, r)$, $v_1 < r \leq v_2$, as every term of the supposed function is positive and explained in Theorem 3.1. Therefore, multiplying (5) by $\mathcal{J}(\eta, \rho) \Phi^\gamma(\eta) = \frac{1}{\mu^s \Gamma(s)} \exp\left[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))\right] \Psi'(\eta)$, $\eta \in (v_1, r)$, $v_1 < r \leq v_2$, we have

$$
\begin{align*}
\mathcal{J}(r, \eta) & \left[ (\vartheta - v_1)^{\delta} \Phi^{\varepsilon-\gamma}(\eta) - (\eta - v_1)^{\delta} \Phi^{\varepsilon-\gamma}(\vartheta) \right] \\
- (\vartheta - v_1)^{\delta} \Phi^{\varepsilon-\gamma}(\vartheta) & - (\eta - v_1)^{\delta} \Phi^{\varepsilon-\gamma}(\eta) \\
= (\vartheta - v_1)^{\delta} & \frac{1}{\mu^s \Gamma(s)} \exp\left[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))\right] \Psi'(\eta) \Phi^{\gamma}(\eta) \Phi^{\varepsilon-\gamma}(\eta) \\
- (\eta - v_1)^{\delta} & \frac{1}{\mu^s \Gamma(s)} \exp\left[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))\right] \Psi'(\eta) \Phi^{\gamma}(\eta) \Phi^{\varepsilon-\gamma}(\eta) \\
- (\vartheta - v_1)^{\delta} & \frac{1}{\mu^s \Gamma(s)} \exp\left[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))\right] \Psi'(\eta) \Phi^{\gamma}(\eta) \Phi^{\varepsilon-\gamma}(\eta) \\
- (\eta - v_1)^{\delta} & \frac{1}{\mu^s \Gamma(s)} \exp\left[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))\right] \Psi'(\eta) \Phi^{\gamma}(\eta) \Phi^{\varepsilon-\gamma}(\eta) \geq 0.
\end{align*}
$$

(7)

Integrating on both sides with respect to $\eta$ from $v_1$ to $r$, we have

$$
\begin{align*}
(\vartheta - v_1)^{\delta} & \frac{1}{\mu^s \Gamma(s)} \int_{v_1}^{r} \exp\left[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))\right] \Psi'(\eta) \Phi^{\gamma}(\eta) \Phi^{\varepsilon-\gamma}(\eta) d\eta \\
- (\eta - v_1)^{\delta} & \frac{1}{\mu^s \Gamma(s)} \int_{v_1}^{r} \exp\left[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))\right] \Psi'(\eta) \Phi^{\gamma}(\eta) \Phi^{\varepsilon-\gamma}(\eta) d\eta \\
- (\vartheta - v_1)^{\delta} & \frac{1}{\mu^s \Gamma(s)} \int_{v_1}^{r} \exp\left[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))\right] \Psi'(\eta) \Phi^{\gamma}(\eta) \Phi^{\varepsilon-\gamma}(\eta) d\eta \\
- (\eta - v_1)^{\delta} & \frac{1}{\mu^s \Gamma(s)} \int_{v_1}^{r} \exp\left[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))\right] \Psi'(\eta) \Phi^{\gamma}(\eta) \Phi^{\varepsilon-\gamma}(\eta) d\eta \geq 0.
\end{align*}
$$

(8)

It follows that

$$
\begin{align*}
(\vartheta - v_1)^{\delta} & \left( \Psi \mathcal{T}_{v_1}^{\mu, \gamma}[\phi^{\varepsilon}(r)] \right) + \Phi^{\varepsilon-\gamma}(\vartheta) \left( \Psi \mathcal{T}_{v_1}^{\mu, \gamma}[\phi^{\gamma}(r)] \right) \\
- (\eta - v_1)^{\delta} & \Phi^{\varepsilon-\gamma}(\eta) \left( \Psi \mathcal{T}_{v_1}^{\mu, \gamma}[\phi^{\gamma}(r)] \right) - (\vartheta - v_1)^{\delta} \Phi^{\varepsilon-\gamma}(\vartheta) \left( \Psi \mathcal{T}_{v_1}^{\mu, \gamma}[\phi^{\gamma}(r)] \right) \geq 0.
\end{align*}
$$

(9)

Multiplying (9) by $\mathcal{J}(r, \vartheta) \Phi^{\gamma}(\vartheta) = \frac{1}{\mu^s \Gamma(s)} \exp\left[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))\right] \Psi'(\eta)$, $\vartheta \in (v_1, r)$, $v_1 < r \leq v_2$, and integrating the subsequent identity w.r.t $\vartheta$ from $v_1$ to $r$ shows

$$
\begin{align*}
\begin{cases}
\Phi^{\gamma}(\vartheta) \\
\Phi^{\varepsilon}(r)
\end{cases}
\end{align*}

(9)

Dividing the above inequality by $\begin{cases}
\Phi^{\gamma}(\vartheta) \\
\Phi^{\varepsilon}(r)
\end{cases}$, we get the desired inequality (3.1). □
A specific case of Theorem 3.1 can be deduced as follows:

**Corollary 1.** If we choose $\Psi(\eta) = \eta$ along with $\mu = 1$, then we have a new result for Riemann-Liouville fractional integral

$$
\frac{T_{\upsilon_1, r}^\varsigma (\Phi^\varsigma(r))}{T_{\upsilon_1, r}^\varsigma (\Phi(r))} \geq \frac{T_{\upsilon_1, r}^\varsigma (\Phi^\varsigma(r))}{T_{\upsilon_1, r}^\varsigma (\Phi(r))}.
$$

**Remark 2.** Letting $\Psi(r) = r$, then Theorem 3.1 reduces to Theorem 2.1 in [40] and if we choose $\Psi(r) = r$ along with $\varsigma = \mu = 1$, then Theorem 3.1 will lead to Theorem 3 in [24]. Moreover, the inequality (3) will reverse if $\Phi$ is an increasing function on $[\upsilon_1, \upsilon_2]$.

**Theorem 3.2.** Suppose $\Phi$ be a continuous positive decreasing function on $[\upsilon_1, \upsilon_2]$ with $\upsilon_1 < r \leq \upsilon_2$, $\delta > 0$, and $\varsigma \geq \gamma > 0$. Then, for generalized proportional fractional integral with respect to another function $\Psi$ defined in (1), we have

$$
\left( \frac{\Phi^\varsigma(\upsilon_1, r) \Phi(r)}{\Phi(\upsilon_1, r)} \right) \left( \frac{\Phi^\varsigma(r)}{\Phi(r)} \right) \geq 1,
$$

where $\varsigma, \lambda \in \mathbb{C}$, $\Re(\varsigma) > 0$, $\Re(\lambda) > 0$, $\mu \in [0, 1]$.

**Proof.** Taking product on both sides of (9) by $\Phi(r)\Phi^\gamma(\vartheta) = \frac{1}{\mu^\gamma(\lambda)}

\times \frac{\exp\left[\frac{\mu - 1}{\mu}(\Phi(r) - \Phi(\vartheta))\right] \Phi^\gamma(\vartheta)}{(\Phi(\upsilon_1, r) - \Phi(\vartheta))^\gamma} \Phi^\gamma(\vartheta) \in (\upsilon_1, r)$, $\upsilon_1 < r \leq \upsilon_2$, and integrating the subsequent identity w.r.t $\vartheta$ from $\upsilon_1$ to $r$ shows

$$
\left( \frac{\Phi^\varsigma(\upsilon_1, r) \Phi(r)}{\Phi(\upsilon_1, r)} \right) \left( \frac{\Phi^\varsigma(r)}{\Phi(r)} \right) \geq 1.
$$

Hence, dividing (11) by

$$
\left( \frac{\Phi^\varsigma(\upsilon_1, r) \Phi(r)}{\Phi(\upsilon_1, r)} \right) \left( \frac{\Phi^\varsigma(r)}{\Phi(r)} \right) - \left( \frac{\Phi^\varsigma(\upsilon_1, r) \Phi(r)}{\Phi(\upsilon_1, r)} \right) \left( \frac{\Phi^\varsigma(r)}{\Phi(r)} \right),
$$

the proof of (10) is complete. \qed

**Remark 3.** Letting $\varsigma = \lambda$ in inequality (10), then we attain the inequality (19).

**Theorem 3.3.** Let $\Phi$ be a continuous positive decreasing function on $[\upsilon_1, \upsilon_2]$ and $h_1$ be a continuous positive increasing function on $[\upsilon_1, \upsilon_2]$ with $\upsilon_1 < r \leq \upsilon_2$, $\delta > 0$, and $\varsigma \geq \gamma > 0$. Then, for generalized proportional fractional integral with respect to another function $\Psi$ defined in (1), we have

$$
\left( \frac{\Phi^\varsigma(r)}{\Phi(r)} \right) \geq 1,
$$

where $\varsigma \in \mathbb{C}$, $\Re(\varsigma) > 0$ and $\mu \in [0, 1]$. 

Proof. Using the hypothesis given in Theorem 3.3, we have
\[
(h_1^2(\vartheta) - h_1^1(\eta)) (\Phi^{\varepsilon-\gamma}(\eta) - \Phi^{\varepsilon-\gamma}(\vartheta)) \geq 0,
\]
where $\delta > 0$, $\varepsilon \geq \gamma > 0$, and $\eta, \vartheta \in [v_1, r]$. From (13), we have
\[
h_1^2(\vartheta)\Phi^{\varepsilon-\gamma}(\eta) + h_1^1(\eta)\Phi^{\varepsilon-\gamma}(\vartheta) + h_1^2(\vartheta)\Phi^{\varepsilon-\gamma}(\vartheta) + h_1^1(\eta)\Phi^{\varepsilon-\gamma}(\eta) \geq 0.
\]
Taking product of (14) by $\mathcal{J}(r, \eta)\Phi^{\gamma}(\eta) = \frac{1}{\mu \Gamma(\varsigma)} \frac{\exp[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))]\psi'(\eta)}{(\Psi(r) - \Psi(\eta))^{1-\varsigma}} \Phi^{\gamma}(\eta)$, $\eta \in (v_1, r)$, $v_1 < r \leq v_2$, where $\mathcal{J}(r, \eta)$ is defined by (6), we have
\[
\mathcal{J}(r, \eta)\Phi^{\gamma}(\eta) [h_1^2(\vartheta)\Phi^{\varepsilon-\gamma}(\eta) + h_1^1(\eta)\Phi^{\varepsilon-\gamma}(\vartheta) + h_1^2(\vartheta)\Phi^{\varepsilon-\gamma}(\vartheta) + h_1^1(\eta)\Phi^{\varepsilon-\gamma}(\eta)]
\]
\[
= h_1^2(\vartheta) \frac{1}{\mu \Gamma(\varsigma)} \frac{\exp[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))]\psi'(\eta)}{(\Psi(r) - \Psi(\eta))^{1-\varsigma}} \Phi^{\gamma}(\eta)
\]
\[
+ h_1^1(\eta) \frac{1}{\mu \Gamma(\varsigma)} \frac{\exp[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))]\psi'(\eta)}{(\Psi(r) - \Psi(\eta))^{1-\varsigma}} \Phi^{\varepsilon-\gamma}(\vartheta) \Phi^{\gamma}(\eta)
\]
\[
+ h_1^2(\vartheta) \frac{1}{\mu \Gamma(\varsigma)} \frac{\exp[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))]\psi'(\eta)}{(\Psi(r) - \Psi(\eta))^{1-\varsigma}} \Phi^{\varepsilon-\gamma}(\vartheta) \Phi^{\gamma}(\eta)
\]
\[
+ h_1^1(\eta) \frac{1}{\mu \Gamma(\varsigma)} \frac{\exp[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))]\psi'(\eta)}{(\Psi(r) - \Psi(\eta))^{1-\varsigma}} \Phi^{\varepsilon-\gamma}(\eta) \geq 0.
\]
Integrating (15) with respect to $\eta$ from $v_1$ to $r$, we have
\[
h_1^2(\vartheta) \frac{1}{\mu \Gamma(\varsigma)} \int_{v_1}^{r} \frac{\exp[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))]\psi'(\eta)}{(\Psi(r) - \Psi(\eta))^{1-\varsigma}} \Phi^{\gamma}(\eta) d\eta
\]
\[
+ h_1^1(\eta) \frac{1}{\mu \Gamma(\varsigma)} \int_{v_1}^{r} \frac{\exp[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))]\psi'(\eta)}{(\Psi(r) - \Psi(\eta))^{1-\varsigma}} \Phi^{\varepsilon-\gamma}(\vartheta) \Phi^{\gamma}(\eta) d\eta
\]
\[
+ h_1^2(\vartheta) \frac{1}{\mu \Gamma(\varsigma)} \int_{v_1}^{r} \frac{\exp[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))]\psi'(\eta)}{(\Psi(r) - \Psi(\eta))^{1-\varsigma}} \Phi^{\varepsilon-\gamma}(\vartheta) \Phi^{\gamma}(\eta) d\eta
\]
\[
+ h_1^1(\eta) \frac{1}{\mu \Gamma(\varsigma)} \int_{v_1}^{r} \frac{\exp[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\eta))]\psi'(\eta)}{(\Psi(r) - \Psi(\eta))^{1-\varsigma}} \Phi^{\varepsilon-\gamma}(\eta) d\eta \geq 0.
\]
From (16), it follows that
\[
h_1^2(\vartheta) \left( \Phi_{v_1, r}^{\varsigma, \mu} [\Phi^{\varepsilon}(r)] \right) + \Phi^{\varepsilon-\gamma}(\eta) \left( \Phi_{v_1, r}^{\varsigma, \mu} [h_1^2(\eta)\Phi^{\gamma}(r)] \right)
\]
\[- h_1^2(\vartheta)\Phi^{\varepsilon-\gamma}(\eta) \left( \Phi_{v_1, r}^{\varsigma, \mu} [\Phi^{\gamma}(r)] \right) - \left( \Phi_{v_1, r}^{\varsigma, \mu} [h_1^1(\eta)\Phi^{\gamma}(r)] \right) \geq 0.
\]
Again, taking product of (9) by $\mathcal{J}(r, \vartheta)\Phi^{\gamma}(\vartheta) = \frac{1}{\mu \Gamma(\varsigma)} \frac{\exp[\frac{\mu-1}{\mu}(\Psi(r) - \Psi(\vartheta))]\psi'(\vartheta)}{(\Psi(r) - \Psi(\vartheta))^{1-\varsigma}} \Phi^{\gamma}(\vartheta)$, $\vartheta \in (v_1, r)$, $v_1 < r \leq v_2$, and integrating the subsequent identity w.r.t $\vartheta$ from $v_1$ to $r$ shows
\[
\left( \Phi_{v_1, r}^{\varsigma, \mu} [\Phi^{\varepsilon}(r)] \right) \left( \Phi_{v_1, r}^{\varsigma, \mu} [\Phi^{\gamma}(r)h_1^2(\eta)] \right)
\]
Corollary 2. If we choose $\Psi(x) = x$ along with $\mu = 1$, then we have a new result for Riemann-Liouville fractional integral
\[
\left( T_{v_1, \tau}^\varsigma \left[ \Phi^\gamma (r) \right] \right) \left( T_{v_1, \tau}^\varsigma \left[ \Phi^\gamma (r) h_1^\delta (r) \right] \right) \geq 1.
\]

which completes the desired inequality (12) of Theorem 3.3.

A specific case of Theorem 3.3 can be deduced as follows:

Remark 4. Letting $\Psi(\eta) = \eta$, then Theorem 3.3 reduces to Theorem 2.3 in [40] and if we choose $\Psi(\eta) = \eta$ along with $\varsigma = \mu = 1$, then Theorem 3.3 will lead to Theorem 4 in [14]. Moreover, the inequality (12) will reverse if $\Phi$ is an increasing function on $[v_1, v_2]$.

Theorem 3.4. Let $\Phi$ be a continuous positive decreasing function on $[v_1, v_2]$ and $h_1$ be a continuous positive increasing function on $[v_1, v_2]$ with $v_1 < r \leq v_2$, $\delta > 0$, and $\varepsilon \geq \gamma > 0$. Then, for generalized proportional fractional integral with respect to another function $\Psi$ defined in (1), we have
\[
\left( \psi T_{v_1, r}^{\lambda, \mu} \left[ \Phi^\gamma (r) \right] \right) \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ h_1^\delta (r) \Phi^\gamma (r) \right] \right) + \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ \Phi^\gamma (r) \right] \right) \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ h_1^\delta (r) \Phi^\gamma (r) \right] \right) \geq 1,
\]
where $\lambda, \varsigma \in \mathbb{C}$, $\Re(\varsigma) > 0$, $\Re(\lambda) > 0$ and $\mu \in [0, 1]$.

Proof. Taking product on both sides of (17) by $\mathfrak{A}(r, \vartheta) \Phi^\gamma (\vartheta) = \frac{1}{\mu^\varsigma \Gamma(\varsigma)}$
\[
\times \exp \left( \frac{\mu - 1}{\mu} (\Phi(r) - \Phi(\vartheta)) \right) \Phi^\gamma (\vartheta) \vartheta \in (v_1, r), v_1 < r \leq v_2,
\]
and integrating the subsequent identity w.r.t $\vartheta$ from $v_1$ to $r$ shows
\[
\left( \psi T_{v_1, r}^{\lambda, \mu} \left[ \Phi^\gamma (r) \right] \right) \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ h_1^\delta (r) \Phi^\gamma (r) \right] \right) + \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ \Phi^\gamma (r) \right] \right) \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ h_1^\delta (r) \Phi^\gamma (r) \right] \right) - \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ h_1^\delta (r) \Phi^\gamma (r) \right] \right) \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ \Phi^\gamma (r) \right] \right) \geq 0.
\]
It follows that
\[
\left( \psi T_{v_1, r}^{\lambda, \mu} \left[ \Phi^\gamma (r) \right] \right) \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ h_1^\delta (r) \Phi^\gamma (r) \right] \right) + \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ \Phi^\gamma (r) \right] \right) \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ h_1^\delta (r) \Phi^\gamma (r) \right] \right) \geq \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ h_1^\delta (r) \Phi^\gamma (r) \right] \right) \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ \Phi^\gamma (r) \right] \right).
\]
Dividing the above inequality by
\[
\left( \psi T_{v_1, r}^{\lambda, \mu} \left[ h_1^\delta (r) \Phi^\gamma (r) \right] \right) \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ \Phi^\gamma (r) \right] \right) + \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ \Phi^\gamma (r) \right] \right) \left( \psi T_{v_1, r}^{\lambda, \mu} \left[ h_1^\delta (r) \Phi^\gamma (r) \right] \right),
\]
acquires the desired inequality (19).

Remark 5. Letting $\varsigma = \lambda$ in inequality (19), then we attain the inequality (12).

Now, we demonstrate the for generalized proportional fractional integral with respect to another function $\Psi$ to derive some inequalities for a class of $n$-decreasing positive functions.
Theorem 3.5. Let \( \{\Phi_i, i = 1, 2, 3, ..., n\} \) be a sequence of continuous positive decreasing functions on \([v_1, v_2]\). Let \( v_1 < r \leq v_2, \delta > 0, \varepsilon \geq \gamma_p > 0 \) for any fixed \( p \in \{1, 2, 3, ..., n\} \). Then, for generalized proportional fractional integral with respect to another function \( \Psi \) defined in (1), we have

\[
\frac{\Psi T^{a, \mu}_{v_1, r} \left[ \prod_{i \neq p} \Phi_i^\gamma \Phi_p^\varepsilon (v) \right]}{\Psi T^{a, \mu}_{v_1, r} \left[ \prod_{i = 1}^n \Phi_i^\gamma \Phi_p^\varepsilon \right]} \geq \frac{\Psi T^{a, \mu}_{v_1, r} \left[ (r - v_1)^\delta \prod_{i \neq p} \Phi_i^\gamma \Phi_p^\varepsilon (r) \right]}{\Psi T^{a, \mu}_{v_1, r} \left[ (r - v_1)^\delta \prod_{i = 1}^n \Phi_i^\gamma \right]},
\]

(20)

where \( \zeta \in \mathbb{C}, \Re(\zeta) > 0, \) and \( \mu \in (0, 1] \).

Proof. Since \( \{\Phi_i, i = 1, 2, 3, ..., n\} \) is a sequence of continuous positive decreasing functions on \([v_1, r]\), we have

\[
(\vartheta - v_1)^\delta (\eta - v_1)^\delta (\Phi_p^{\varepsilon - \gamma_p} (\eta) - \Phi_p^{\varepsilon - \gamma_p} (\vartheta)) \geq 0
\]

(21)

for any fixed \( p \in \{1, 2, 3, ..., n\}, \delta > 0, \varepsilon \geq \gamma_p > 0, \) and \( \eta, \vartheta \in [v_1, v_2] \).

By (21), we have

\[
(\vartheta - v_1)^\delta \Phi_p^{\varepsilon - \gamma_p} (\eta) + (\eta - v_1)^\delta \Phi_p^{\varepsilon - \gamma_p} (\vartheta) \\
\geq (\vartheta - v_1)^\delta \Phi_p^{\varepsilon - \gamma_p} (\eta) + (\eta - v_1)^\delta \Phi_p^{\varepsilon - \gamma_p} (\vartheta).
\]

(22)

Taking product of (22) by \( \mathcal{J}(r, \eta) \prod_{i = 1}^n \Phi_i^\gamma (\eta) = \frac{1}{\mu^1(\zeta)} \exp\left[ \frac{\mu - 1}{\mu}(\Psi(r) - \Psi(\eta)) \Psi'(\eta) \right] \prod_{i = 1}^n \Phi_i^\gamma (\eta) \Phi_p^{\varepsilon - \gamma_p} (\vartheta) \)

\[
\prod_{i = 1}^n \Phi_i^\gamma (\eta), \eta \in (v_1, r), v_1 < r \leq v_2, \text{ where } \mathcal{J}(r, \eta) \text{ is defined by (6), and integrating the subsequent identity w.r.t } \eta \text{ from } v_1 \text{ to } r \text{ we have}
\]

\[
\mathcal{J}(r, \eta) (\vartheta - v_1)^\delta \Phi_p^{\varepsilon - \gamma_p} (\eta) + (\eta - v_1)^\delta \Phi_p^{\varepsilon - \gamma_p} (\vartheta) - (\vartheta - v_1)^\delta \Phi_p^{\varepsilon - \gamma_p} (\eta)
\]

(23)

Integrating (23) with respect to \( \eta \) from \( v_1 \) to \( r \), we have

\[
(\vartheta - v_1)^\delta \frac{1}{\mu^1(\zeta)} \int_{v_1}^r \exp\left[ \frac{\mu - 1}{\mu}(\Psi(r) - \Psi(\eta)) \Psi'(\eta) \right] \prod_{i = 1}^n \Phi_i^\gamma (\eta) \Phi_p^{\varepsilon - \gamma_p} (\eta) \ d\eta
\]

(24)
From (24), it follows that

\[(\theta - v_1)^\delta \Psi_{\nu_1}^{\mu} \left[ \prod_{i \neq p} \Phi_i^{\gamma_i} \Phi_p^{\varepsilon_i} \right] + \Psi_{\nu_1}^{\mu} \left[ (r-v_1)^\delta \prod_{i=1}^n \Phi_i^{\gamma_i} \right]
\geq (\theta - v_1)^\delta \Phi_p^{-\gamma_p} \Psi_{\nu_1}^{\mu} \left[ \prod_{i \neq p} \Phi_i^{\gamma_i} \right] + \Psi_{\nu_1}^{\mu} \left[ (r-v_1)^\delta \prod_{i=1}^n \Phi_i^{\gamma_i} \Phi_p^{\varepsilon_i} \right]. \tag{25}\]

Again, taking product of (25) by \(\mathcal{J}(r, \theta) \prod_{i=1}^n \Phi_i^{\gamma_i} \), \(\theta \in (v_1, r), v_1 < r \leq v_2\), where \(\mathcal{J}(r, \theta)\) is defined by (6), and integrating the subsequent identity w.r.t \(\theta\) from \(v_1\) to \(r\), we have

\[
\Psi_{\nu_1}^{\mu} \left[ \prod_{i \neq p} \Phi_i^{\gamma_i} \Phi_p^{\varepsilon_i} \right] \Psi_{\nu_1}^{\mu} \left[ (r-v_1)^\delta \prod_{i=1}^n \Phi_i^{\gamma_i} \right] 
\geq \Psi_{\nu_1}^{\mu} \left[ (r-v_1)^\delta \prod_{i \neq p} \Phi_i^{\gamma_i} \Phi_p^{\varepsilon_i} \right] \Psi_{\nu_1}^{\mu} \left[ \prod_{i=1}^n \Phi_i^{\gamma_i} \right]. \tag{26}\]

which gives the desired inequality (20). \(\square\)

\textbf{Remark 6.} Letting \(\Psi(r) = r\), then Theorem 3.5 reduces to Theorem 2.5 in [40] and if we choose \(\Psi(r) = r \) along with \(\mu = 1\), then Theorem 3.5 will lead to Theorem 3.1 in [14]. Moreover, the inequality (20) will reverse if \(\Phi\) is an increasing function on \([v_1, v_2]\).

\textbf{Theorem 3.6.} Let \(\{\Phi_i, i = 1, 2, 3, \ldots, n\}\) be a sequence of continuous positive decreasing functions on \([v_1, v_2]\). Let \(v_1 < r \leq v_2\), \(\delta > 0\), \(\varepsilon \geq \gamma_p > 0\) for any fixed \(p \in \{1, 2, 3, \ldots, n\}\). Then, for generalized proportional fractional integral with respect to another function \(\Psi\) defined in (1), we have

\[
\left( \Psi_{\nu_1}^{\mu} \left[ \prod_{i \neq p} \Phi_i^{\gamma_i} \Phi_p^{\varepsilon_i} \right] \Psi_{\nu_1}^{\mu} \left[ (r-v_1)^\delta \prod_{i=1}^n \Phi_i^{\gamma_i} \right] \right) 
+ \Psi_{\nu_1}^{\mu} \left[ \prod_{i \neq p} \Phi_i^{\gamma_i} \Phi_p^{\varepsilon_i} \right] \Psi_{\nu_1}^{\mu} \left[ (r-v_1)^\delta \prod_{i=1}^n \Phi_i^{\gamma_i} \right] 
\geq 1, \tag{27}\]

where \(\varsigma, \lambda \in \mathbb{C}\), \(\Re(\varsigma) > 0\), \(\Re(\lambda) > 0\), and \(\mu \in [0, 1]\).

\textbf{Proof.} Taking product on both sides of (25) by \(\mathcal{J}(r, \theta) \prod_{i=1}^n \Phi_i^{\gamma_i} \), \(\theta \in (v_1, r), v_1 < r \leq v_2\), where \(\mathcal{J}(r, \theta)\) is defined by (6), and integrating the subsequent identity w.r.t \(\theta\) from \(v_1\) to \(r\), we have

\[
\Psi_{\nu_1}^{\mu} \left[ \prod_{i \neq p} \Phi_i^{\gamma_i} \Phi_p^{\varepsilon_i} \right] \Psi_{\nu_1}^{\mu} \left[ (r-v_1)^\delta \prod_{i=1}^n \Phi_i^{\gamma_i} \right] 
\geq 1. \]
Let \( \eta \) be defined by (6), and integrating the subsequent identity w.r.t \( r \) gives the desired inequality (27).

Let \( \upsilon \) for any fixed \( p \). 

Proof. Under the given hypothesis, we have

\[
\sum_{i=1}^{n} \Phi_{i}^{-\gamma_{i}}(r) \geq \sum_{i=1}^{n} \Phi_{i}^{-\gamma_{i}}(r)
\]

Taking product on both sides of (34) by \( \Theta(r, \eta) \prod_{i=1}^{n} \Phi_{i}^{-\gamma_{i}}(r) \) we have

\[
\prod_{i=1}^{n} \Phi_{i}^{-\gamma_{i}}(r) \geq 1,
\]

which gives the desired inequality (27). 

\[ \square \]

Remark 7. Letting \( \zeta = \lambda \) in inequality (27), then we attain the inequality (20).

Theorem 3.7. Let \( h_{1} \) be a continuous positive increasing functions and \( \{ \Phi_{i}, i = 1, 2, 3, ..., n \} \) be a sequence of continuous positive decreasing functions on \([v_{1}, v_{2}]\). Let \( v_{1} < r < v_{2}, \delta > 0, \varepsilon \geq \gamma_{p} > 0 \) for any fixed \( p \in \{1, 2, 3, ..., n\} \). Then, for generalized fractional integral with respect to another function \( \Psi \) defined in (1), we have

\[
\prod_{i=1}^{n} \Phi_{i}^{-\gamma_{i}}(r) \geq 1,
\]

where \( \zeta \in \mathbb{C}, \Re(\zeta) > 0 \), and \( \mu \in [0, 1] \).

Proof. Under the given hypothesis, we have

\[
(h_{1}^{\delta}(\vartheta) - h_{1}^{\eta}(\vartheta))(\Phi_{p}^{-\gamma_{p}}(\eta) - \Phi_{p}^{-\gamma_{p}}(\vartheta)) \geq 0,
\]

for any fixed \( p \in \{1, 2, 3, ..., n\}, \delta > 0, \varepsilon \geq \gamma_{p} > 0 \), and \( \eta, \vartheta \in [v_{1}, r] \).

From (36), we have

\[
h_{1}^{\delta}(\vartheta)\Phi_{p}^{-\gamma_{p}}(\eta) + h_{1}^{\eta}(\eta)\Phi_{p}^{-\gamma_{p}}(\vartheta) - h_{1}^{\delta}(\vartheta)\Phi_{p}^{-\gamma_{p}}(\vartheta) - h_{1}^{\eta}(\eta)\Phi_{p}^{-\gamma_{p}}(\eta) \geq 0.
\]

Taking product on both sides of (34) by \( \Theta(r, \eta) \prod_{i=1}^{n} \Phi_{i}^{-\gamma_{i}}(r) \) we have

\[
\prod_{i=1}^{n} \Phi_{i}^{-\gamma_{i}}(r) \geq 1,
\]

and integrating the subsequent identity w.r.t \( \eta \) from \( v_{1} \) to \( r \), we have

\[
\prod_{i=1}^{n} \Phi_{i}^{-\gamma_{i}}(r) \geq 1,
\]
\[-h_1^{\gamma}(\vartheta) \frac{1}{\mu \Gamma(\varsigma)} \int_{v_1}^{r} \frac{\exp[-\frac{\mu - 1}{\mu} (\Psi(r) - \Psi(\eta))]\Psi'(\eta)}{(\Psi(r) - \Psi(\eta))^{1-\varsigma}} \prod_{i=1}^{n} \Phi_i^{\gamma_i}(\eta) \Phi_p^{\gamma_p}(\vartheta) \, d\eta \]

\[-h_1^{\gamma}(\eta) \frac{1}{\mu \Gamma(\varsigma)} \int_{v_1}^{r} \frac{\exp[-\frac{\mu - 1}{\mu} (\Psi(r) - \Psi(\eta))]\Psi'(\eta)}{(\Psi(r) - \Psi(\eta))^{1-\varsigma}} \prod_{i=1}^{n} \Phi_i^{\gamma_i}(\eta) \Phi_p^{\gamma_p}(\vartheta) \, d\eta \]

Integrating (36) with respect to \( \eta \) from \( v_1 \) to \( r \), we have

\[h_1^{\gamma}(\vartheta) \frac{1}{\mu \Gamma(\varsigma)} \int_{v_1}^{r} \frac{\exp[-\frac{\mu - 1}{\mu} (\Psi(r) - \Psi(\eta))]\Psi'(\eta)}{(\Psi(r) - \Psi(\eta))^{1-\varsigma}} \prod_{i=1}^{n} \Phi_i^{\gamma_i}(\eta) \Phi_p^{\gamma_p}(\vartheta) \, d\eta\]

From (34), it follows that

\[h_1^{\gamma}(\vartheta) \Psi_{T_{v_1, r}}^{\gamma, \mu} \left[ \prod_{i \neq p}^{n} \Phi_i^{\gamma_i}(r) + \Phi_p^{\gamma_p}(r) \right] - h_1^{\gamma}(\eta) \Psi_{T_{v_1, r}}^{\gamma, \mu} \left[ \prod_{i \neq p}^{n} \Phi_i^{\gamma_i}(r) \right] \geq 0. \] (35)

Again, taking product on both sides of (35) by \( \tilde{\Psi}(r, \vartheta) \prod_{i=1}^{n} \Phi_i^{\gamma_i}(\vartheta) = \frac{1}{\mu \Gamma(\varsigma)} \prod_{i=1}^{n} \Phi_i^{\gamma_i}(\vartheta), \quad \vartheta \in (v_1, r), v_1 < r \leq v_2 \), where \( \tilde{\Psi}(r, \vartheta) \) is defined by (6), and integrating the subsequent identity w.r.t \( \vartheta \) from \( v_1 \) to \( r \), we have

\[\Psi_{T_{v_1, r}}^{\gamma, \mu} \left[ \prod_{i \neq p}^{n} \Phi_i^{\gamma_i}(r) \right] \Psi_{T_{v_1, r}}^{\gamma, \mu} \left[ h_1^{\gamma}(r) \prod_{i \neq p}^{n} \Phi_i^{\gamma_i}(r) \right] \Psi_{T_{v_1, r}}^{\gamma, \mu} \left[ h_1^{\gamma}(r) \prod_{i \neq p}^{n} \Phi_i^{\gamma_i}(r) \right] \geq 0, \] (36)

which establishes the desired inequality (30). \( \square \)

**Remark 8.** Letting \( \Psi(r) = r \), then Theorem 3.5 reduces to Theorem 2.7 in [40] and if we choose \( \Psi(r) = r \) along with \( \mu = 1 \), then Theorem 3.7 will lead to Theorem 3.6 in [14]. Moreover, the inequality (30) will reverse if \( \Phi \) is an increasing function on \([v_1, v_2] \).

**Theorem 3.8.** Let \( h_1 \) be a continuous positive increasing functions and \( \{\Phi_i, i = 1, 2, 3, ..., n\} \) be a sequence of continuous positive decreasing functions on \([v_1, v_2] \). Let \( v_1 < r \leq v_2, \delta > 0, \varepsilon \geq \gamma_p > 0 \) for any fixed \( p \in \{1, 2, 3, ..., n\} \). Then, for generalized proportional fractional integral with respect to another function \( \Psi \) defined in (1), we have
\[
\left( \Psi T_{v_1,r}^{\lambda,\mu} \left[ \prod_{i \neq p}^{n} \Phi_i^\gamma \Phi_p^\epsilon (r) \right] \Psi T_{v_1,r}^{\lambda,\mu} \left[ h_1^\delta (r) \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \\
+ \Psi T_{v_1,r}^{\lambda,\mu} \left[ h_1^\delta (r) \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \Psi T_{v_1,r}^{\lambda,\mu} \left[ \prod_{i \neq p}^{n} \Phi_i^\gamma \Phi_p^\epsilon (r) \right] \right) \left( \Psi T_{v_1,r}^{\lambda,\mu} \left[ \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \Psi T_{v_1,r}^{\lambda,\mu} \left[ \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \right) \\
+ \Psi T_{v_1,r}^{\lambda,\mu} \left[ h_1^\delta (r) \prod_{i = 1}^{n} \Phi_i^\gamma \Phi_p^\epsilon (r) \right] \Psi T_{v_1,r}^{\lambda,\mu} \left[ \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \right) \right) \geq 1, 
\]

where \( \varsigma, \lambda \in \mathbb{C} \), \( \Re(\varsigma) > 0, \Re(\lambda) > 0 \), and \( \mu \in [0, 1] \).

**Proof.** Taking product on both sides of (35) by \( \delta(r, \vartheta) \prod_{i = 1}^{n} \Phi_i^\gamma (r) = \frac{1}{\mu + \Gamma(\lambda)} \times \frac{\exp[\mu \Gamma(\lambda) - \Psi(r) - \Psi(\vartheta)]}{{\Psi(r) - \Psi(\vartheta)}} \prod_{i = 1}^{n} \Phi_i^\gamma (r), \vartheta \in (v_1, r), v_1 < r \leq v_2 \), where \( \delta(r, \vartheta) \) is defined by (6), and integrating the subsequent identity w.r.t \( \vartheta \) from \( v_1 \) to \( r \), we have

\[
\left( \Psi T_{v_1,r}^{\varsigma,\mu} \left[ \prod_{i \neq p}^{n} \Phi_i^\gamma \Phi_p^\epsilon (r) \right] \Psi T_{v_1,r}^{\lambda,\mu} \left[ h_1^\delta (r) \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \\
+ \Psi T_{v_1,r}^{\varsigma,\mu} \left[ h_1^\delta (r) \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \Psi T_{v_1,r}^{\lambda,\mu} \left[ \prod_{i \neq p}^{n} \Phi_i^\gamma \Phi_p^\epsilon (r) \right] \right) \left( \Psi T_{v_1,r}^{\varsigma,\mu} \left[ \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \right)
\]

It follows that

\[
\Psi T_{v_1,r}^{\varsigma,\mu} \left[ \prod_{i \neq p}^{n} \Phi_i^\gamma \Phi_p^\epsilon (r) \right] \Psi T_{v_1,r}^{\lambda,\mu} \left[ h_1^\delta (r) \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \\
+ \Psi T_{v_1,r}^{\varsigma,\mu} \left[ h_1^\delta (r) \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \Psi T_{v_1,r}^{\lambda,\mu} \left[ \prod_{i \neq p}^{n} \Phi_i^\gamma \Phi_p^\epsilon (r) \right] \geq \Psi T_{v_1,r}^{\varsigma,\mu} \left[ \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \Psi T_{v_1,r}^{\lambda,\mu} \left[ \prod_{i \neq p}^{n} \Phi_i^\gamma \Phi_p^\epsilon (r) \right] \Psi T_{v_1,r}^{\varsigma,\mu} \left[ \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right]
\]

Dividing both sides by

\[
\left( \Psi T_{v_1,r}^{\lambda,\mu} \left[ \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \Psi T_{v_1,r}^{\varsigma,\mu} \left[ \prod_{i = 1}^{n} \Phi_i^\gamma (r) \right] \right)
\]
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+ \Psi T^{\lambda,\mu}_{\upsilon_1,\upsilon_2} \left[ h_1^\delta(r) \prod_{i=p}^{n} \Phi_i^{\gamma_i}(r) \right] \Psi T^{\varsigma,\mu}_{\upsilon_1,\upsilon_2} \left[ \prod_{i=1}^{n} \Phi_i^{\gamma_i}(r) \right],

acquire the desired inequality (37).

Remark 9. Letting \varsigma = \lambda in inequality (37), then we attain the inequality (30).

4. Conclusion. In this paper, we derived certain variants by employing the newly defined operator generalized proportional fractional integral with respect operator related to a class of \( n \) positive continuous and decreasing functions on the interval \([\upsilon_1, \upsilon_2]\) are presented. In [24], Liu et al. investigated thought-provoking integral inequalities for continuous functions on \([\upsilon_1, \upsilon_2]\). Recently, Dahmani [14] has presented the more generalizations of the work of [24] by utilizing the Riemann-Liouville fractional integral operators. Therefore our findings in the present paper are the generalization of integral inequalities involving the Riemann-Liouville fractional integral operators. If we take into account \( \mu = 1 \) and \( \Psi(x) = x \), then our findings derived in this the paper will reduce the integral inequalities involving the Riemann-Liouville fractional integral operators introduced by Dahmani [14]. Some of the particular cases of our consequences are observed in [24]. The consequences acquired in this paper deliver some contributions to the direction of the idea of integral inequalities and fractional calculus and are anticipated to result in some applications for organizing the uniqueness of solutions in integrodifferential equations. Additionally, as an application part, we are capable of an attempt to find out the analytical solutions of some space-time fractional differential equations by using the unique method in our future work. These numerical schemes probably can provide new insights regarding application to fractional partial differential equations and further new conclusions in the future, also we can investigate the existence and global exponential stability of a class of quaternion-valued cellular neural networks (QVNNs) with time-varying delays by applying a continuation theorem of coincidence degree theory and by constructing an appropriate Lyapunov functional via direct methods. Our results are new and our proposed methods can be used to study the anti-periodic problem for other types of QVNNs.

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