A massive Feynman integral and some reduction relations for
Appell functions

M. A. Shpot

Institute for Condensed Matter Physics, 79011 Lviv, Ukraine and
Fachbereich Physik, Universität Duisburg-Essen, D-47048 Duisburg, Germany

Abstract

New explicit expressions are derived for the one-loop two-point Feynman integral with arbitrary external momentum and masses $m_1^2$ and $m_2^2$ in $D$ dimensions. The results are given in terms of Appell functions, \textit{manifestly symmetric} with respect to the masses $m_i^2$. Equating our expressions with previously known results in terms of Gauss hypergeometric functions yields reduction relations for the involved Appell functions that are apparently new mathematical results.

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I. INTRODUCTION

In this paper, we derive simple new analytic expressions for a standard $D$ dimensional one-loop two-point Feynman integral with two different masses $m_1^2 > 0$ and $m_2^2 > 0$ and an arbitrary external momentum $p_x$,

$$I(p_x; m_1^2, m_2^2) = \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2 + m_1^2} \frac{1}{(p + p_x)^2 + m_2^2}. \quad (1)$$

Our results are given terms of a single, manifestly symmetric Appell function$^1 F_4$, or alternatively, the Appell function $F_1$.

The integrals like that in (1) are ubiquitous in quantum field theory and renormalization group theory of phase transitions and critical phenomena. Actually, there exists for long time a standard general result for the integral (1) given by Berends, Davydychev, and Smirnov in Ref. 2 henceforth called BDS. This result was expressed essentially in a form of a linear combination of two Gauss hypergeometric functions of complicated arguments.

Clearly, the Feynman integral (1) is a symmetric function of its parameters $m_1^2$ and $m_2^2$. In the result of BDS, which we shall quote further, this property was implemented through a symmetrization

$$F(m_1^2, m_2^2) + F(m_2^2, m_1^2) \quad (2)$$

of two complicated terms, none of which obeyed the symmetry $m_1^2 \leftrightarrow m_2^2$.

Here we present the alternative general results for the integral (1). These are given in terms of a single Appell function, $F_4$ or $F_1$, manifestly symmetric with respect to the interchange of $m_1^2$ and $m_2^2$. The symmetry $m_1^2 \leftrightarrow m_2^2$ is implemented within the arguments of these functions. That is, our new expressions are given in terms of completely symmetric combinations like $m_1^2 m_2^2$ or $(m_1 \pm m_2)^2$. They do not contain any symmetrizations like that in (2).

We believe that such kind of results are interesting on their own right, and hope that their existence can be useful in the quantum field theory. Moreover, comparing our results with the expression of BDS we arrive at special identities between the Appell and Gauss hypergeometric functions that are apparently new mathematical relations.

Moreover, there exists another reason for practical utility of our results. In fact, our motivation for present calculations was in the theory of the critical behavior of statistical mechanical systems at Lifshitz points$^3$. Reviews and extensive lists of references on this
The field theoretical treatment of $m$-axial Lifshitz points in $d$-dimensional systems frequently requires calculations of momentum integrals

$$J(p_x, q_x) = \int \frac{d^m q}{(2\pi)^m} \int \frac{d^{d-m} p}{(2\pi)^{d-m}} \frac{1}{(p - p_x/2)^2 + (q - q_x/2)^2} \frac{1}{(p + q_x/2)^2 + (q + q_x/2)^4}$$

in contexts of the $\varepsilon$-expansion (see e.g. Ref. 8), or large-$N$ expansion$^9$. In this setting, an $m$-dimensional subspace corresponding to $0 \leq m \leq d$ special anisotropy directions must be split out within the whole $d$-dimensional space and complemented by the remaining $(d-m)$-dimensional subspace with no peculiarities in its physical properties. The momentum integral in (3) is over $\mathbb{R}^d = \mathbb{R}^m \oplus \mathbb{R}^{d-m}$, and the inner integration over $p$ in $\mathbb{R}^{d-m}$ is easily recognized to be the same as in (1) with $D \equiv d - m$ and identifications of masses $m_1^2$ and $m_2^2$ as $(q \pm q_x/2)^4$. In the fully isotropic limit $m = 0$, we recover the usual situation of the Euclidean field theory in $D = 4 - \varepsilon$ dimensions with the standard upper critical dimension $D^* = 4$. The latter follows from $d^*(m) = 4 + m/2$, the upper critical dimension appropriate for the Lifshitz-point theory, as the number of the anisotropy axes $m$ vanishes.

In the generic case of non-isotropic $m$-axial Lifshitz points with $0 < m < d$, when we are interested, for example, in the one-loop energy-energy correlation function, a further integral over $q$ has to be still performed in (3). To be feasible, its integrand must be as simple and symmetric as possible. We could not use the BDS result of the inner $p$ integration to proceed. The reason was the lack of symmetry $m_1^2 \leftrightarrow m_2^2$ in individual terms $F$ of (2). This implied the presence of complicated explicit dependencies on odd powers of the integration variable $q$ in each of these terms. Of course, they cancel out in the symmetric sum (2), which is an even function of $q$ (and $q_x$). But the outer integration of any part of this sum appeared completely hopeless.

In a hope to achieve the desired simple symmetry of the inner integral in (3), we have differently tried to apply the numerous standard transformation formulas of Gauss hypergeometric functions in the BDS result, but in vain. The application of the Feynman-parameter integration also lead us only to results equivalent to that of BDS. Hence, we were urged to attempt a completely different calculation in order to get the inner integral given separately by (1) in a simplest possible and manifestly symmetric form.

Before turning to details of our calculation and its implications, we write down the BDS
result in the following section.

II. THE RESULT OF BERENDS, DAVYDYCHEV, AND SMIRNOV

The Feynman integral of equation (1) obeys the scaling relation

\[ I(p_x; m_1^2, m_2^2) = p_x^{-\varepsilon} I(1; m_1^2/p_x^2, m_2^2/p_x^2) \]  

(4)

where we define, as usual for the critical phenomena theory, \( D = 4 - \varepsilon \) (note that this differs from the space-time dimension \( n = 4 - 2\varepsilon \) of BDS). Hence, without loss of generality we can consider the function (cf. (3))

\[ I(m_1^2, m_2^2) = I(1; m_1^2, m_2^2) = \int \frac{d^Dp}{(2\pi)^D} \frac{1}{(p - 1/2)^2 + m_1^2} \frac{1}{(p + 1/2)^2 + m_2^2} \]  

(5)

where 1 denotes an arbitrary vector of unit length. The dependence of the original integral (1) on external momentum \( p_x \) can be easily reconstructed by scaling at any stage of calculations.

The reference 2 contains the explicit result for this integral in terms of Gauss hypergeometric functions. Before quoting it, let us split from the function \( I(m_1^2, m_2^2) \) an overall numeric factor via

\[ I(m_1^2, m_2^2) = (4\pi)^{-\frac{D}{2}} \Gamma(2 - \frac{D}{2}) \hat{I}(m_1^2, m_2^2). \]  

(6)

While \( \Gamma(2 - D/2) \) is singular as \( D \to 4 \), the non-trivial function \( \hat{I}(m_1^2, m_2^2) \) is finite in this limit. In our conventions, the translation of the BDS result (see Eq. (A.7) of Ref. 2) reads

\[
\hat{I}(m_1^2, m_2^2) = \frac{\Gamma^2(D/2 - 1)}{\Gamma(D - 2)} \sqrt{\Delta}^{D-3} \]

\[
+ \frac{m_1^{D-4}}{D - 2} \left( 1 + m_1^2 - m_2^2 - \sqrt{\Delta} \right) _2F_1 \left( 1, 2 - \frac{D}{2}; \frac{D}{2}; - \frac{(1 + m_2^2 - m_1^2 - \sqrt{\Delta})^2}{4m_1^2} \right) \]

\[
+ (m_1^2 \leftrightarrow m_2^2). \]  

(7)

For brevity, we denoted by \((m_1^2 \leftrightarrow m_2^2)\) the presence of the third term, repeating the function given explicitly in the second line, but with interchange of the masses \( m_1^2 \) and \( m_2^2 \). The symmetric combination

\[ \Delta = (m_1^2 + m_2^2 + 1)^2 - 4m_1^2m_2^2 = [(m_1 + m_2)^2 + 1] [(m_2 - m_1)^2 + 1] \]  

(8)

is a counterpart of the Källen function with unit momentum in the terminology of Ref. 2. On the other hand, this is the discriminant of a quadratic equation that has to be solved in the
course of calculations employing the Feynman parametrization. The apparent singularities at \( D = 2 \) in (7) mutually cancel, and the finite result in this case is given explicitly in section V.

The original BDS result was derived using the technique of Mellin-Barnes contour integral representations\(^\text{10}\). The result (6)-(7) or its immediate generalizations, expressed in terms of generalized hypergeometric functions of two variables, have been reproduced several times by different authors using different means\(^\text{11–14}\). All of them either contained symmetrizations like (2) or comprised some hidden symmetries in apparently non-symmetric expressions.

As discussed in the Introduction, for our purposes we needed a completely different kind of result for the function \( I(m_1^2, m_2^2) \). Thus, we had to calculate the integral (5) by using another procedure, not related to that of BDS or the Feynman parametrization. The way of doing it is described in the next section.

III. THE ALTERNATIVE CALCULATION

Let us return to the integral (5). Denoting the denominators of its integrand by \( A_- \) and \( A_+ \), we write it as

\[
I(m_1^2, m_2^2) = \int \frac{d^Dp}{(2\pi)^D} \frac{1}{A_- A_+} \quad \text{with} \quad A_- = p^2 + m_1^2 + 1/4 - (p \cdot 1), \\
A_+ = p^2 + m_2^2 + 1/4 + (p \cdot 1).
\]

Using the partial fraction expansion we decompose our integral into two terms via

\[
I(m_1^2, m_2^2) = \int \frac{d^Dp}{(2\pi)^D} \frac{1}{A_- + A_+} \left( \frac{1}{A_-} + \frac{1}{A_+} \right).
\]

Now, it is useful to introduce the arithmetic mean \( a = (A_- + A_+)/2 \) and to express the denominators \( A_- \) and \( A_+ \) in terms of their mean value \( a \) and deviation from it \( b \), as \( A_\pm = a \pm b \). Hence,

\[
I(m_1^2, m_2^2) = \int \frac{d^Dp}{(2\pi)^D} \frac{1}{2a} \left( \frac{1}{a - b} + \frac{1}{a + b} \right).
\]

This representation brings into consideration the mass center \( m^2 = (m_1^2 + m_2^2)/2 \) and the deviation \( m_b = (m_2^2 - m_1^2)/2 \geq 0 \) of masses \( m_1^2 \) and \( m_2^2 \) from \( m^2 \). In terms of these values, we have

\[
a = p^2 + m_a^2 \quad \text{with} \quad m_a^2 = m^2 + 1/4 \quad \text{and} \quad m_b = (p \cdot 1) + m_b.
\]
The above simple algebraic manipulations provide us with a basis for further calculations in terms of certain combinations of $m_1^2$ and $m_2^2$, more appropriate for the parametrization of final results than the original masses itself. These allow us to write

$$I(m_1^2, m_2^2) = \frac{1}{2} (I_- + I_+)$$

with

$$I_\pm = \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2 + m_a^2} \frac{1}{p^2 + m_a^2 \pm m_b \pm (p \cdot 1)}.$$  

Here $m_a^2$ is a manifestly symmetric value related to the mass center $m_a^2$. On the other hand, the value $m_b$ is given by the difference of $m_2^2$ and $m_1^2$. It is not symmetric with respect to the interchange of masses and thus still not "good". Besides of dependency on $m^2$, the result must depend on $m_b^2$, not $m_b$. Our aim is now to reach a manifest realization of the symmetry $m_1^2 \leftrightarrow m_2^2$ at an early stage of the calculation, not on the level of final results.

As a next step, we perform the angular integration in $I_\pm$ via

$$I_\pm = K_D \int_0^\infty \frac{p^{D-1} dp}{p^2 + m_a^2} \int_0^\pi \frac{d\theta}{\Omega} \frac{\sin^{D-2} \theta}{p^2 + m_a^2 \pm m_b \pm p \cos \theta}.$$  

Here $K_D$ is a usual geometric factor given by $K_D = 2^{1-D} \pi^{-D/2}/\Gamma(D/2)$. The normalization factor of the angular integration $\Omega = \sqrt{\pi} \Gamma(D/2 - 1/2)/\Gamma(D/2)$ is the value of the $\theta$ integral without the denominator. The required result is

$$I_\pm = K_D \int_0^\infty dp \frac{p^{D-1}}{p^2 + m_a^2} \frac{1}{p^2 + m_a^2 \pm m_b} \, _2F_1 \left( \frac{1}{2}, \frac{D}{2}; \frac{1}{2}; \left( p^2 + m_a^2 \pm m_b \right)^2 \right).$$

Hence we obtain, by definition of the Gauss hypergeometric function, a series representation

$$I(m_1^2, m_2^2) = \frac{K_D}{2} \sum_{n \geq 0} \left( \frac{1}{2} \right)^n \frac{(D/2)_n}{(D/2)_n} \int_0^\infty dp \frac{p^{D-1+2n}}{(p^2 + m_a^2)^{2n+2}}$$  

$$\times \left[ \left( 1 - \frac{m_b}{p^2 + m_a^2} \right)^{2n-1} + \left( 1 + \frac{m_b}{p^2 + m_a^2} \right)^{2n-1} \right]$$

where $(c)_n \equiv \Gamma(c + n)/\Gamma(c)$ is the Pochhammer symbol. Here, inside of the square brackets we get a simple realization of the symmetry $m_1^2 \leftrightarrow m_2^2$ in the form of (2). This symmetry becomes manifest in the representation of the content of square brackets in terms of a Gauss hypergeometric function (Ref. 15, p. 34, Ref. 16, p. 461, Eq. 106)

$$
\left( 1 - \frac{m_b}{p^2 + m_a^2} \right)^{-2n-1} + \left( 1 + \frac{m_b}{p^2 + m_a^2} \right)^{-2n-1} = _2F_1 \left( n + \frac{1}{2}, n + 1; \frac{1}{2}; \left( m_b^2 \right)^2 \right)
$$
the argument of which depends explicitly on \( m_b^2 = (m_2^2 - m_1^2)^2/4 \).

Using equation (10) we can simply perform the remaining integration over the radial distance \( p \) in (9). Its output is again (see (6)) given as \( I(m_1^2, m_2^2) = (4\pi)^{-D/2}\Gamma(2-D/2)\hat{I}(m_1^2, m_2^2) \), while for the function \( \hat{I}(m_1^2, m_2^2) \) we obtain a double series expansion

\[
\hat{I}(m_1^2, m_2^2) = m_a^{D-4} \sum_{k,n \geq 0} \frac{\left(\frac{1}{4}\right)_{k+n}(2-D/2)_{2k+n}}{\left(\frac{1}{2}\right)_{k+n}k!n!}
\left(\frac{m_b^2}{4m_a^2}\right)^k \left(\frac{1}{4m_a^2}\right)^n.
\]

The same result could be reached directly from equation (9) by using binomial expansions for each of two terms in the square brackets.

The double series in (11) does not match directly any item of the Horn list (see Refs. 15,17) of essentially distinct complete hypergeometric functions of two variables. This leaves a possibility that it can be reduced to some standard function through certain algebraic rearrangements.

We see that one of the Pochhammer symbols in the nominator of (11) has a doubling of the summation index \( k \). This is rather unusual in the theory of generalized hypergeometric functions. In order to reduce our double series to a more common form, one has to get rid of this duplication. This can be achieved by applying the exact resummation formula (Ref. 15, p. 100)

\[
\sum_{k,n \geq 0} C(n,k) = \sum_{n \geq 0} \sum_{k=0}^{n} C(n-k,k)
\]

along with the identity (Ref. 15, p. 23)

\[
\frac{1}{(n-k)!} = \frac{(-1)^k(-n)_k}{n!}, \quad 0 \leq k \leq n.
\]

These manipulations lead us to

\[
\hat{I}(m_1^2, m_2^2) = m_a^{D-4} \sum_{n \geq 0} \frac{\left(\frac{1}{2}\right)_n(\frac{\epsilon}{2})_n}{\left(\frac{1}{2}\right)_nk!n!} \left(\frac{1}{4m_a^2}\right)^n \sum_{k=0}^{n} \frac{(-n)_k(\epsilon/2+n)_k}{k!} \left(\frac{m_b^2}{m_a^2}\right)^k
\]

Here we identify the inner finite sum with \( \, _2F_1(-n,\epsilon/2+n;1/2;-m_b^2/m_a^2) \) and convert it to a non-terminating hypergeometric function with the help of the linear transformation formula (see e.g. Ref. 18, p. 559, Eqs. 15.3.4-5)

\[
\, _2F_1(a,b;c;z) = (1-z)^{-a} \, _2F_1\left(a,c-b;c;\frac{z}{z-1}\right).
\]

Thus we get

\[
\hat{I}(m_1^2, m_2^2) = \left(\frac{\Delta}{4}\right)^{-\frac{D}{2}} \sum_{n \geq 0} \frac{\left(\frac{1}{2}\right)_n(\frac{\epsilon}{2})_n}{\left(\frac{1}{2}\right)_nk!n!} \left(\frac{1}{\Delta}\right)^n \, _2F_1\left(\frac{\epsilon}{2} + n,\frac{1}{2} + n;\frac{1}{2};\frac{4m_b^2}{\Delta}\right)
\]
where we again encounter the familiar discriminant $\Delta$ from equation (8). This time it appears in a compact form as

$$\Delta = 4(m_a^2 + m_b^2).$$

(15)

The sum in equation (14) represents the Appell function $F_4$ (Ref. 15, p. 53, Ref. 16, p. 413, Eq. 5) whose original definition is\textsuperscript{1,15,19,20}

$$F_4(a, b; c', c; x, y) = \sum_{k,n \geq 0} \frac{(a)_k (b)_{k+n}}{(c)_k(c')_n} \frac{x^k y^n}{k! n!}, \quad \sqrt{|x|} + \sqrt{|y|} < 1.$$  

(16)

Thus, the basic integral $I(m_1^2, m_2^2) = (4\pi)^{-D/2} \Gamma(\varepsilon/2) \hat{I}(m_1^2, m_2^2)$ is given by

$$\hat{I}(m_1^2, m_2^2) = \left(\frac{\Delta}{4}\right)^{-\frac{\varepsilon}{2}} F_4\left(\frac{\varepsilon}{2}, \frac{1}{2}; \frac{3}{2}, \frac{1}{2}; \frac{1}{\Delta} \frac{4m_2^2}{\Delta}\right).$$  

(17)

For reader’s convenience, we recall the notation $m_b = (m_2^2 - m_1^2)/2$ and note an alternative explicit formula (cf. (8)) for the value $\Delta$, which follows directly from (15):

$$\Delta = 1 + 2(m_1^2 + m_2^2) + (m_2^2 - m_1^2)^2.$$  

(18)

Let us check the convergence region of the Appell function $F_4$ in (17). Following (16), we have to make sure that the combination

$$\frac{1}{\sqrt{\Delta}} + \frac{2|m_b|}{\sqrt{\Delta}} = \frac{1 + |m_2^2 - m_1^2|}{\sqrt{\Delta}} < 1.$$  

(19)

To this end we rewrite the value $\Delta$ as

$$\Delta = (1 + |m_2^2 - m_1^2|)^2 + 2(m_1^2 + m_2^2 - |m_2^2 - m_1^2|)$$

$$= (1 + |m_2^2 - m_1^2|)^2 + 2 \min\{m_1^2, m_2^2\}.$$  

The last equality gives evidence that $\sqrt{\Delta} > 1 + |m_2^2 - m_1^2|$, and thus, the inequality (19) holds for any $m_1^2, m_2^2 > 0$.

The equation (17) is one of our central results. The integral defined by (5) is expressed here in terms of a single Appell function $F_4$. Its variables are given in terms of manifestly symmetric in $m_1^2$ and $m_2^2$ combinations $\Delta$ and $m_2^2$. The result for $\hat{I}(m_1^2, m_2^2)$ is finite and absolutely convergent in the range $0 < D \leq 4$ for arbitrary non-vanishing positive masses $m_1^2$ and $m_2^2$. The limit $D \to 0$ will be considered in section V along with further special cases.
IV. A FURTHER TRANSFORMATION OF THE APPELL FUNCTION

There is a repetition of two parameters in the Appell function $F_4$ appearing in the equation (17). This means that the result (17) can be alternatively expressed in terms of the Horn function $H_3$ (Ref. 15, p. 57). Moreover, it is reducible to the Appell function $F_1^{1,15,19,20}$,

\[
F_1(a; b, b'; c; x, y) = \sum_{k, n \geq 0} \frac{(a)_{k+n}(b)_k(b')_n x^k y^n}{(c)_{k+n} k! n!}, \quad \text{max}\{|x|, |y|\} < 1, \tag{20}
\]

by means of Bailey’s $19$ reduction formula

\[
F_4\left(a, b; c, b'; -\frac{X}{(1 - X)(1 - Y)}, -\frac{Y}{(1 - X)(1 - Y)}\right) = (1 - X)^a(1 - Y)^a F_1(a; c - b, 1 + a - c; c; X, XY). \tag{21}
\]

Denoting for a while the arguments $1/\Delta$ and $4m_1^2/\Delta$ of the Appell function $F_4$ in (17) as $A^2$ and $B^2$, we find (cf. Refs. 22,23)

\[
X = \frac{A^2 + B^2 - 1 \pm \sqrt{\delta}}{2B^2}, \quad Y = \frac{A^2 + B^2 - 1 \pm \sqrt{\delta}}{2A^2} \tag{22}
\]

with the new discriminant

\[
\delta = (A^2 + B^2 - 1)^2 - 4A^2B^2 = [(A + B)^2 - 1][(B - A)^2 - 1], \tag{23}
\]

astonishingly similar to the function $\Delta$ from equation (8).

The last three equations yield totally unexpected simple expressions. In particular, we get

\[
\sqrt{\delta} = 4 \frac{\sqrt{m_1^2m_2^2}}{\Delta}
\]

Choosing the upper + sign in the solutions $X$ and $Y$ from (22) and assuming that $m_1 > 0$ and $m_2 > 0$ are the principal square roots of $m_1^2$ and $m_2^2$, we obtain

\[
X = -\frac{1}{(m_1 + m_2)^2} \quad \text{and} \quad Y = -(m_1 - m_2)^2. \tag{24}
\]

An alternative choice of minus signs in (22), as in Ref. 22, gives $X = -1/(m_1 - m_2)^2$ and $Y = -(m_1 + m_2)^2$. This would lead to a much less convenient parametrization of the function $\hat{I}(m_1^2, m_2^2)$ owing to the singular behavior of the variable $X$ by equal parameters $m_1$ and $m_2$. 
Thus, the application of the reduction formula (21) to the function $F_4$ from (17) leads to the Appell function $F_1$ with surprisingly simple symmetric arguments. For $\hat{I}(m_1^2, m_2^2)$ this yields

$$\hat{I}(m_1^2, m_2^2) = \left(\frac{m_1 + m_2}{2}\right)^{-\varepsilon} F_1\left(\frac{\varepsilon}{2}; 1, -\frac{1}{2} + \frac{3}{2}; \frac{-1}{2}, \frac{1 - (m_1 - m_2)^2}{(m_1 + m_2)^2}\right). \quad (25)$$

Reintroducing by scaling (4) the momentum dependence of the original Feynman integral (1) and taking into account the overall numeric factor (6) we obtain a very appealing in its structure result

$$I(p_x; m_1^2, m_2^2) = (4\pi)^{-2+\varepsilon/2} \Gamma\left(\frac{\varepsilon}{2}\right) \times \left(\frac{m_1 + m_2}{2}\right)^{-\varepsilon} F_1\left(\frac{\varepsilon}{2}; 1, -\frac{1}{2} + \frac{3}{2}; \frac{-p_x^2}{2(m_1 + m_2)^2}, \frac{1 - (m_1 - m_2)^2}{2(m_1 + m_2)^2}\right). \quad (26)$$

A note is in order here. The last two equations represent the double series expansions (20) whose domain of absolute convergence is shrunk with respect to the original integral’s validity range, $m_1 > 0$, $m_2 > 0$, and arbitrary $p_x$. Indeed, the definition (20) of the Appell function $F_1$ requires $m_1 + m_2 > 1$ for the first variable of the scaling function $\hat{I}(m_1^2, m_2^2)$ in (25), or $p_x/(m_1 + m_2) < 1$ for the whole integral (26). Nevertheless, this is not an essential drawback. To get rid of it, one can apply the linear transformation (Ref. 1 p. 30, Eq. (53), Ref. 20 p. 218, Eq. (8.3.3))

$$F_1(a; b, b'; c; u, v) = (1 - u)^{-a} F_1\left(a; c - b - b', b'; c; \frac{u}{u - 1}, \frac{u - v}{u - 1}\right) \quad (27)$$

to the Appell function $F_1$ in (25). This provides the necessary analytic continuation for this function, very similar to that given by the linear transformation (13) for the Gauss hypergeometric function: The first argument of $F_1$ behaves in exactly the same way as the variable $z$ in (13). Hence we obtain, for any $m_1, m_2 > 0$,

$$\hat{I}(m_1^2, m_2^2) = 2^\varepsilon \left[1 + (m_1 + m_2)^2\right]^{-\frac{\varepsilon}{2}} \times F_1\left(\frac{\varepsilon}{2}; 1 - \frac{1}{2}, \frac{3}{2}; \frac{1}{1 + (m_1 + m_2)^2}, \frac{1 + (m_1 - m_2)^2}{1 + (m_1 + m_2)^2}\right). \quad (28)$$

The convergence range of this double series expansion now coincides with the initial validity domain of the original integral (1).

Although with a reduced domain of convergence, the compact expressions (25) or (26) can be used in further calculations, where the needed analytic continuation is done at some later
stage or is provided by the extended convergence region of final results. Examples of this kind will be given in the following section. Also, the Appell function $F_1$ is presently a built-in function in the Mathematica$^{24}$, and its analytic continuation is automatically provided by this program.

V. SPECIAL CASES

Our general results (17) and (25)-(26) can be checked by specializing to several integer space dimensions $D$. Here we encounter mathematical simplifications allowing us to get the expressions for $\hat{I}(m_1^2, m_2^2)$ in terms of elementary functions. Some of them have already been calculated by other means before.

At $D = 1$, the integrals (1) or (5) are elementary, and a short calculation using Mathematica$^{24}$ yields

$$\hat{I}(m_1^2, m_2^2)|_{D=1} = 2 \frac{m_1 + m_2}{m_1 m_2} \frac{1}{1 + (m_1 + m_2)^2}. \quad (29)$$

The same result immediately follows from (25): With $\varepsilon = 3$, the Appell function $F_1$ simply decouples there into two geometric progressions.

At $D = 2$, the integral $\hat{I}(m_1^2, m_2^2)$ has been calculated in Ref. 9, in the context of the $1/N$ expansion for uniaxial ($m = 1$) Lifshitz points in three-dimensional ($d = 3$) space. In this special case we have $\varepsilon = 2$, and we obtain from the equation (17)

$$\hat{I}(m_1^2, m_2^2)|_{D=2} = \frac{4}{\Delta} F_4\left(1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\Delta}, \frac{4m_2^2}{\Delta}\right). \quad (30)$$

This simplified Appell function reduces through the relation (Ref. 19, p. 102)

$$F_4\left(a, b; 1 + a - b, b; \frac{-X}{(1 - X)(1 - Y)}, \frac{-Y}{(1 - X)(1 - Y)}\right)$$

$$= (1 - Y)^a \, _2F_1\left(a, b; 1 + a - b; -\frac{X}{1 - X}\right) \quad (31)$$

to a Gaussian hypergeometric function with parameters $1, 1/2, 3/2$ and positive argument $w = (1 + (m_1 - m_2)^2)/(1 + (m_1 + m_2)^2) < 1$. It is known (e.g. Ref. 18, p. 556) to represent a logarithmic function. Thus we obtain

$$\hat{I}(m_1^2, m_2^2)|_{D=2} = \frac{2}{\sqrt{\Delta}} \ln \frac{1 + \sqrt{w}}{1 - \sqrt{w}} = \frac{2}{\sqrt{\Delta}} \ln \frac{1 + m_1^2 + m_2^2 + \sqrt{\Delta}}{2m_1 m_2}. \quad (32)$$
This special result comes as well from (25) through the reduction formula (Ref. 19, p. 79)

\[ F_1(a; b, b'; b + b'; u, v) = (1 - v)^{-a} {}_2F_1(a, b; b'; \frac{u - v}{1 - v}) \] (33)

followed by the linear transformation (13) for the resulting Gauss hypergeometric function. The latter transformation converts the argument \( z \equiv \frac{(u - v)(1 - v)}{1} = -(1 + (m_1 - m_2)^2)/(4m_1m_2) \), which blows up for small values of \( m_1 \) and \( m_2 \), again to the safe combination \( w < 1 \). This is a counterpart of the analytical continuation carried out before through the transformation (27) for the Appell function \( F_1 \).

The last expression in (32) agrees with the integrand of \( I(1, q) \) from equation (70) of Ref. 9 after the shift of the integration variable via \( q' \to q' - q/2 \) and identifications \( m_1 = (q' - q/2)^2 \) and \( m_2 = (q' + q/2)^2 \). After some work, it can also be seen to be equivalent with the formula (4.3) of Ref. 11.

At \( D = 3 \), the dimensional parameter \( \varepsilon = 1 \), and in the Appell function \( F_1 \) from (25) one of the nominator parameters vanishes. Thus, the function \( F_1 \) reduces to an ordinary series expansion, and we get

\[ \hat{I}(m_1^2, m_2^2) \big|_{D=3} = 2 \arctan(m_1 + m_2)^{-1} \] (34)

in agreement with Ref. 25, Eq. (A.2) and Ref. 11, Eq. (4.6).

When \( D = 4 \), the value of \( \varepsilon \) vanishes, and both the equations (17) and (25) yield the trivial result \( \hat{I}(m_1^2, m_2^2) \big|_{D=4} = 1 \). Now, of interest is the first-order term of a small-\( \varepsilon \) expansion of \( \hat{I}(m_1^2, m_2^2) \) taken at \( D = 4 - \varepsilon \). Its calculation will yield the essential non-trivial contribution to the finite part of the whole integral \( I(m_1^2, m_2^2) \) given by equations (5)-(6). This term has been calculated previously by BDS using the Feynman parametrization.

Expanding the result (17) in small \( \varepsilon \) we obtain (see Appendix A)

\[
\hat{I}(m_1^2, m_2^2) = 1 + \varepsilon + \frac{\varepsilon}{4} \sqrt{\Delta} \ln \frac{1 + m_1^2 + m_2^2 - \sqrt{\Delta}}{1 + m_1^2 + m_2^2 + \sqrt{\Delta}} + \frac{\varepsilon}{4} (m_2^2 - m_1^2) \ln \frac{m_2^2}{m_1^2} - \frac{\varepsilon}{4} \ln(m_1^2m_2^2) + O(\varepsilon^2) 
\] (35)

in agreement with Refs. 2 and 11.

The limiting case \( D = 0 \) is somewhat special. It can be well illustrated by turning to the "double" integral (3) stemming from the Lifshitz-point theory. Let us accept again that \( D = d - m \) while the \( d \)-dimensional space is split into mutually complementing \( m \)- and \( D \)-dimensional subspaces. Then, by inspecting (3), it becomes evident that when the dimension
$D$ shrinks to zero, we must remain with an $m$ dimensional integral over $q$ where no trace of $D$ dimensional integration remains. This implies that in the limit $D \to 0$, the result of the integration over $p$ must yield its integrand at $p = p_x = 0$. Physically relevant limiting regimes of this kind have been considered in Refs. 8,9. For the integral $I(p_x; m_1^2, m_2^2)$ from (1) this means that

$$I(p_x; m_1^2, m_2^2)|_{D=0} = \frac{1}{m_1^2 m_2^2}.$$  

(36)

This is easily reproduced from equation (26). At $p_x = 0$, the first argument of $F_1$ vanishes, and it reduces to a Gauss hypergeometric function yielding

$$I(p_x = 0; m_1^2, m_2^2) = (4\pi)^{-2+\varepsilon/2} \Gamma \left( \frac{\varepsilon}{2} \right) \frac{\Delta}{2 - \varepsilon} \frac{m_1^{2-\varepsilon} - m_2^{2-\varepsilon}}{m_1^2 - m_2^2}$$

in agreement, up to the normalization of the integral, with Ref. 10. By setting $\varepsilon = 4$ here, we obtain (36). Note that this correct limit could not be obtained by setting $D = 0$ directly in the scaling function (25). There, no vanishing of the first argument in $F_1$ occurs, which was achieved by restoring the external momentum dependence in (26).

VI. REDUCTION RELATIONS FOR THE APPELL FUNCTIONS

By equating the results (17), (25), or (28) for the integral $\hat{I}(m_1^2, m_2^2)$ with its BDS expression (7) we obtain apparently new reduction relations for the involved Appell functions. Thus, for the function $F_4$ we get

$$F_4(\alpha, \frac{1}{2}; \frac{3}{2}, \frac{1}{2}; \Delta, \frac{(m_2^2 - m_1^2)^2}{\Delta}) = 4^{-\alpha} \Gamma \left( \frac{1}{2} - \alpha \right) \Gamma \left( 2 - \frac{2}{\alpha} \right) \sqrt{\Delta}$$

(37)

$$+ \left( \frac{\Delta}{4m_1^2} \right)^{\alpha} \frac{1 + m_2^2 - m_1^2 - \sqrt{\Delta}}{2(1 - \alpha)} \text{$_2F_1$} \left( 1, \alpha; 2 - \alpha; -\frac{1 + m_2^2 - m_1^2 - \sqrt{\Delta}}{4m_1^2} \right)$$

$$+ \left( \frac{\Delta}{4m_2^2} \right)^{\alpha} \frac{1 + m_1^2 - m_2^2 - \sqrt{\Delta}}{2(1 - \alpha)} \text{$_2F_1$} \left( 1, \alpha; 2 - \alpha; -\frac{1 + m_1^2 - m_2^2 - \sqrt{\Delta}}{4m_2^2} \right)$$

where a parameter $\alpha$ stands in place of $2 - D/2 = \varepsilon/2$. The relation (37) was derived for arbitrary real positive masses $m_1^2$ and $m_2^2$. This is in conformity with the standard definition of the convergence region of the Appell function $F_4$, $(1 + |m_2^2 - m_1^2|)/\sqrt{\Delta} < 1$, as discussed at the end of the section III. Once again we recall that $\Delta = 1 + 2(m_1^2 + m_2^2) + (m_2^2 - m_1^2)^2$.

The parameter $\alpha$ was originally constrained to the interval $0 \leq \alpha < 2$ by physical applicability range of the integral (5). In (37), this limitation can be removed. The relation (37) holds for negative values of $\alpha$, where the original integrals are divergent in the ultraviolet, or
for $\alpha > 2$ where the integrals would be negative dimensional. We have numerical evidence that (37) is valid for integer numbers $\alpha \geq 2$ where the hypergeometric functions become singular and one might beware of possible exceptions. This is in agreement with the smooth dependence of $F_4$ on $\alpha$ that indicates no peculiarities for positive integers. Moreover, in the relation (37), the parameter $\alpha$ can be considered to extend to complex values, along with the variables $m_1^2$ and $m_2^2$.

Eliminating the parameters $m_1^2$ and $m_2^2$ in favor of variables that appear in the function $F_4$ we express the last relation in a standard fashion,

$$F_4\left(\alpha, \frac{1}{2}, \frac{3}{2}; x, y \right) = 4^{\alpha} \frac{\Gamma^2(1 - \alpha)}{\Gamma(2 - 2\alpha)} \frac{1}{\sqrt{x}}$$

$$+ \left[1 - (\sqrt{x} + \sqrt{y})^2\right]^{-\alpha} \frac{\sqrt{x} + \sqrt{y} - 1}{2(1 - \alpha)\sqrt{x}} \, _2F_1\left(1, \alpha; 2 - \alpha; \frac{\sqrt{x} + \sqrt{y} - 1}{\sqrt{x} + \sqrt{y} + 1}\right)$$

$$+ \left[1 - (\sqrt{x} - \sqrt{y})^2\right]^{-\alpha} \frac{\sqrt{x} - \sqrt{y} - 1}{2(1 - \alpha)\sqrt{x}} \, _2F_1\left(1, \alpha; 2 - \alpha; \frac{\sqrt{x} - \sqrt{y} - 1}{\sqrt{x} - \sqrt{y} + 1}\right).$$

Comparing the equations (25) and (7) we write down a similar reduction relation for the Appell function $F_1$,

$$F_1\left(\alpha; 1, \alpha - \frac{1}{2}; \frac{3}{2}, -p^2, q^2 \right) = 4^{-\alpha} \frac{\Gamma^2(1 - \alpha)}{\Gamma(2 - 2\alpha)} (p^2)^{-1+\alpha} s^{\frac{1}{2} - \alpha}$$

$$+ \frac{(1 + q)^{-2\alpha}}{2(1 - \alpha)p^2} (p^2 - q - \sqrt{s}) \, _2F_1\left(1, \alpha; 2 - \alpha; -\frac{(p^2 - q - \sqrt{s})^2}{p^2(1 + q)^2}\right)$$

$$+ \frac{(1 - q)^{-2\alpha}}{2(1 - \alpha)p^2} (p^2 + q - \sqrt{s}) \, _2F_1\left(1, \alpha; 2 - \alpha; -\frac{(p^2 + q - \sqrt{s})^2}{p^2(1 - q)^2}\right),$$

where

$$s = (1 + p^2)(p^2 + q^2).$$

By writing in (39) the variables of $F_1$ as $-p^2$ and $q^2$ we stress that this function is even in $p$ and $q$. So is the combination on the right, too. But here, each of terms with a Gauss hypergeometric function contains an explicit dependence on $q$ and is not even in this variable. All the odd powers of $q$ must cancel in the whole combination through the symmetrization $f(q) + f(-q)$ present here. Its role is analogous to that of (2), implemented in (7) and (37).

The Appell function $F_1$ from the relation (39) exists for any real or complex values of the parameter $\alpha$. It is absolutely convergent for $p^2$ and $q^2 < 1$. These variables can also be both real and complex. The convergence region of the function $F_1$ can be extended by using the transformation formulas like that in (27). A further reduction relation for the Appell function $F_1$ can be written in a similar way by using the formula (28).
VII. CONCLUDING REMARKS

In this paper we have discussed the functional form of a standard but non-trivial one-loop Feynman integral (1) with non-vanishing masses \( m_1^2 \) and \( m_2^2 \) and external momentum in \( D \) spatial dimensions. Previously the results for this Feynman integral have been obtained by Berends, Davydychev, and Smirnov\(^2\) and reproduced later\(^{11-14}\) in several equivalent forms. The BDS result contained a linear combination of two Gauss hypergeometric functions neither of which was symmetric with respect to the interchange of the masses \( m_1^2 \) and \( m_2^2 \). The correct symmetric part of this combination had to remain after very complicated and obscured cancelations of non-symmetric terms.

Here we represent the new explicit results for the Feynman integral (1) in terms of Appell functions, manifestly symmetric with respect to the masses \( m_1^2 \) and \( m_2^2 \). The advantage of our results over other representations is that they are expressed by a single function with completely symmetric and rather simple arguments. In equation (17) the result is given in terms of \( F_4 \), "perhaps the most intractable Appell function"\(^{28}\). A further transformation of this function led us to elegant results (25)-(26) and (28) expressed in terms of the Appell function \( F_1 \) with extremely simple and symmetric arguments.

The knowledge of new functional forms of the integral (1) gave us the possibility to derive the equalities relating the involved Appell functions to previously known combinations of Gauss hypergeometric functions. The way of producing the new mathematical identities by evaluating some integrals in different ways is not new. It was successfully employed by Inayat-Hussain\(^{29,30}\) (see also Ref. 31) while considering certain Feynman integrals arising in phase transition theory, or by Srivastava, Glasser, and Adamchik\(^{32}\) while studying the Riemann Zeta function.

Apart from equations (37)-(39) for \( F_4 \) and \( F_1 \), similar reduction relations can be derived also for the Appell functions \( F_3 \) and \( F_2 \) since \( F_1 \) can always be expressed in terms of one of these latter functions\(^1,19\). Of special interest would be the relations involving \( F_2 \) in view of attention attracted by this function in the recent mathematical literature\(^{33-35}\).
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APPENDIX A: THE EPSILON EXPANSION OF $\hat{I}(m_1^2, m_2^2)$

Let us consider the equation (17) at small $\varepsilon$. For a while, we denote the arguments $1/\Delta$ and $4m_b^2/\Delta$ of the Appell function $F_4$ as $x$ and $y$, respectively. We use the definition (16) of $F_4$ and split there the $k = 0$ term. Denoting it by $\sigma_0$ we have

$$\sigma_0 = \sum_{n \geq 0} \frac{(\varepsilon/2)_n}{n!} y^n = 1 + \frac{\varepsilon}{2} \sum_{n \geq 1} \frac{y^n}{n} + O(\varepsilon^2) = 1 - \frac{\varepsilon}{2} \ln(1 - y) + O(\varepsilon^2) \quad (A1)$$

since

$$(a)_n = a(n - 1)! + O(a^2) \quad \text{if} \quad n \geq 1. \quad (A2)$$

After a rearrangement of Pochhammer symbols via $(a)_{n+k} = (a)_k(a+k)_n$, the rest of the double series expansion can be written as

$$\sigma_1 = \sum_{k \geq 1} \frac{(\varepsilon/2)(1/2)_k}{(3/2)_k} \frac{x^k}{k!} 2F_1(\varepsilon/2 + k, 1/2 + k; 1/2; y).$$

Again we use the property (A2) of $(\varepsilon/2)_k$ to obtain

$$\sigma_1 = \frac{\varepsilon}{2} \sum_{k \geq 1} \frac{(1/2)_k}{(3/2)_k} \frac{x^k}{k} 2F_1(k, 1/2 + k; 1/2; y) + O(\varepsilon^2).$$

Here, the hypergeometric function reduces to an algebraic function (Ref. 15, p. 34, Ref. 16, p. 461, Eq. 106) through the same relation that was used in (10). Thus we get

$$\sigma_1 = \frac{\varepsilon}{4} \sum_{k \geq 1} \frac{(1/2)_k}{(3/2)_k} \frac{x^k}{k} [(1 + \sqrt{y})^{-2k} + (1 - \sqrt{y})^{-2k}] + O(\varepsilon^2).$$

This formula implies that, if we introduce the series expansion

$$\sigma(z) \equiv \sum_{k \geq 1} \frac{(1/2)_k}{(3/2)_k} \frac{z^k}{k}, \quad (A3)$$

the function $\sigma_1$ is given by

$$\sigma_1 = \frac{\varepsilon}{4} [\sigma(X_+) + \sigma(X_-)] + O(\varepsilon^2) \quad \text{where} \quad X_\pm \equiv \frac{x}{(1 \pm \sqrt{y})^2}, \quad (A4)$$
By shifting the summation index in (A3), we can write \( \sigma(z) \) as a a generalized hypergeometric function \( \, _3F_2 \):

\[
\sigma(z) = \frac{z}{3} \, _3F_2\left(\frac{3}{2}, 1, 1; \frac{5}{2}, 2; z\right).
\]

This can be expressed via (Ref. 16, p. 519, Eq. 366) in terms of logarithmic functions:

\[
\sigma(z) = 2 - \frac{1}{\sqrt{z}} \ln \frac{1 + \sqrt{z}}{1 - \sqrt{z}} - \ln(1 - z). \tag{A5}
\]

The same result could be reached by noticing that the derivative of \( \sigma(z) \) is given by

\[
\sigma'(z) = \sum_{k \geq 1} \frac{(1/2)_k}{(3/2)_k} z^{k-1} = \frac{1}{3} \, _2F_1\left(\frac{3}{2}, 1; \frac{5}{2}; z\right) = \frac{1}{z} \left[ \frac{1}{2\sqrt{z}} \ln \frac{1 + \sqrt{z}}{1 - \sqrt{z}} - 1 \right],
\]

where the last equality follows from Ref. 16, p. 477, Eq. 157, and integrating back \( \sigma'(z) \) with respect to \( z \).

The summary of the above calculation is that the first-order \( \varepsilon \) expansion of the Appell function \( F_4 \) from (17) reads

\[
F_4\left(\frac{\varepsilon}{2}, \frac{1}{2}; \frac{3}{2}, \frac{1}{2}; x, y\right) = 1 - \frac{\varepsilon}{2} \ln(1 - y) + \frac{\varepsilon}{4} \left[ \sigma\left(\frac{x}{(1 + \sqrt{y})^2}\right) + \sigma\left(\frac{x}{(1 - \sqrt{y})^2}\right) \right] + O(\varepsilon^2)
\]

where the function \( \sigma(z) \) is given explicitly by (A5).

Finally, the small-\( \varepsilon \) expansion of the complete formula (17) appears in the equation (35) of the main text.

\[\text{Electronic mail: shpot@ph.icmp.lviv.ua}\]

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