Qubit state guidance without feedback

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New Journal of Physics 7 (2005) 43
Received 8 October 2004
Published 7 February 2005
Online at http://www.njp.org/
doi:10.1088/1367-2630/7/1/043

Abstract. We study a protocol for two-qubit-state guidance that does not rely on feedback mechanisms. In our scheme, entanglement can be concentrated by arranging the interactions of the qubits with a continuous variable ancilla. By properly post-selecting the outcomes of repeated measurements performed on the state of the ancilla, the qubit state is driven to have a desired amount of purity and entanglement. We stress the primary role played by the first iterations of the protocol. Inefficiencies in the detection operations can be fully taken into account. We also discuss the robustness of the guidance protocol to the effects of an experimentally motivated model for mixedness of the ancillary states.

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1. Introduction

Quantum guidance is the ability to drive a quantum system towards desired states having disparate and arbitrary properties in terms of degree of purity and entanglement. The proposal and realization of schemes for the guidance of physical devices is central to the tasks of quantum
information processing (QIP), since many of the protocols proposed so far may be formally interpreted in terms of control of an apparatus operated through classical or quantum potentials [1]. However, it is commonly accepted that any external interference on a physical device opens some channels to the interaction of the system itself with the uncontrollable environment, leading to effects of decoherence [2]. Many different strategies have been suggested previously to correct or avoid these unwanted effects, ranging from quantum error-correcting codes [3] to quantum computing in decoherence-free subspaces [4] or dynamical-decoupling techniques [5]. Another possible strategy is represented by the use of feedback to recursively modify the dynamics of a system.

A common feature of all these proposals is that they operate directly on the system that has to be controlled, thus realizing strategies for local guidance. However, it would be useful and desirable to have a practical procedure allowing for a non-intrusive driving of a quantum device. In this case, the quantum guidance is at a distance in the sense that the active part of a guidance protocol (including unitary and non-unitary operations) is realized on an ancilla connected to the system we are interested in. This strategy is intriguing, for instance, for the purpose of creating a quantum channel to be used for later tasks of a processing protocol [6]. The channel could be generated with minimal interventions on its constituents and its purity and entanglement could be regulated acting on a distant knob. In this paper, we address some recently proposed schemes for the purification of a quantum state and entanglement creation [6], [8]–[11] under the point of view of quantum guidance. We show how these schemes can be suitably modified in order to perform quantum guidance of a two-qubit state.

As we have pointed out, the idea of acting on an ancillary system in order to modify the dynamics of a quantum register has been suggested in [6], where the task is to create an entangled state of two two-level atoms. Plenio et al [6], in particular, have considered the conditioned generation of an entangled state of two atoms through the continuous detection of the leaky field mode of an optical cavity containing the atoms. Despite the elegance of the proposal and its conceptual simplicity, in this scheme the atoms are confined within a resonator, which strongly limits the distances between the nodes of a distributed QIP device. Moreover, the scheme works when no photon from the field mode is observed, a condition which can be prone to indiscernible errors. Some strategies to bypass the problem related to the distance between qubits (and thus increase the length of the distributed channel) were developed in [7].

An intriguing suggestion which bypasses the problem of the distance between the nodes of the distributed device has been put forward in [8], where the entanglement between two qubits is transferred from a continuous variable (CV) system to two initially non-interacting qubits placed far apart. Encoding information in the CV ancilla allows for long haul communication between the qubits. Further formal and practical aspects of these schemes have been analysed, then, in [9]. It is worth stressing that the proposals in [6] and the schemes for CV-mediated discrete entanglement distribution of [8] are intrinsically different as, in the latter, there is no evidence of conditional dynamics. We will comment on these points later in this paper.

More recently, Nakazato et al [10] have proposed a way to probabilistically purify the state of a quantum system (labelled B from now on). The scheme is based on repeated measurements of the state of an ancilla, here referred to as A, which interacts with B through some local Hamiltonian $\hat{H}_{\text{int}}$. System B is prepared in a state $\rho_B$ we want to purify, while system A is initialized in $|\chi\rangle_A$. Let us assume that we are able to implement the projective measurement $\hat{P}_A = |\chi\rangle_A\langle\chi|$. We apply the following procedure: A+B evolves through $\hat{H}_{\text{int}}$ for a time interval
\( \Delta t \), and state A is checked to be in \( |\chi\rangle_A \). Tracing out the ancillary system, B evolves as
\[
\rho_B(\Delta t) = \text{Tr}_A(\hat{P}_A e^{-i\hat{H}_{\Delta t}/\Delta t} |\chi\rangle_A \langle \chi| \otimes \rho_B(0) e^{i\hat{H}_{\Delta t}/\Delta t} \hat{P}_A) = \hat{O}_\chi \rho_B(0) \hat{O}_\chi^\dagger \tag{1}
\]
where \( \hat{O}_\chi = \int \langle \chi| e^{-i\hat{H}_{\Delta t}/\Delta t} |\chi\rangle \). According to Nakazato et al \[10\] if, in the spectral decomposition of \( \hat{O}_\chi \), there is a single term dominating over the rest of the spectrum, then system B is progressively purified by the repeated application of this evolution–detection protocol. The probability of success of the scheme, i.e. the probability that \( \rho_B \) is driven towards a pure state \( |v\rangle_B \), critically depends on \( B \langle v| \rho_B(0) |v\rangle_B \) \[10\]. We notice that, due to the non-unitary nature of the projector \( \hat{P}_A \), it is impossible to describe the effective dynamics of subsystem B alone using the operator-sum representation \[1\]. It has been recognized \[11\] that this same scheme may be generalized to the cases in which the system to purify is intrinsically multipartite (subparties \( B_1, B_2, \ldots, B_m \)). In this case, the repeated measurements operated on the ancilla A project system B onto a pure entangled state of its components \( B_i \) (\( i = 1, \ldots, m \)).

In this work, we assess the protocol for the generation of entangled pure states of a composite system B under the point of view of quantum guidance of a system. We develop the intuition that the effectiveness of any purification scheme through repeated measurements of an ancillary system may be ascribed to some kind of quantum guidance procedure. We suggest a scheme that is nearer to the experimental state of the art. In our analysis, indeed, we get rid of the unrealistic assumption of measurements projecting the ancilla onto the bona fide state \( |\phi\rangle_A \) (a condition which is challenging in many practical cases). Furthermore, no feedback is required, in our procedure, but just the post-selection of favourable events and the proper control of the interaction interval. We show that, by considering the ancillary system A to be embodied by a bipartite CV system, a wide range of possibilities for quantum state guidance is offered. To the best of our knowledge, this point was never addressed in this context. Just one-qubit systems were used to embody the ancilla in conditioned protocols \[10, 11\]. We analyse in detail possible sources of non-ideality such as detection inefficiencies. Moreover, an experimentally motivated model for a mixed state of the CV ancilla is quantitatively taken into account. Our scheme appears to be robust against both these imperfections, a feature which is appealing from a practical viewpoint.

2. The system

We set the scenario in order to show that a two-mode CV mediator, in a proper initial entangled state, is able to drive two qubits, labelled 1 and 2, towards a pure state, distilling entanglement at the same time.

We consider a system of two remote qubits, 1 and 2, each defined in Hilbert spaces spanned by the basis \( \{|g\rangle, |e\rangle\} \). The qubits are initially prepared in a separable state and are mutually independent (there is no direct interaction between the qubits all along the quantum guidance protocol). The qubits represent the subparties of system B whose state we want to drive. Each qubit interacts with a CV system (1 interacts with \( a \) and 2 with \( b \)), which is in an entangled state. We consider local interactions \( \hat{H}_{\text{int}} = \hat{H}_{1a} + \hat{H}_{2b} \) between each qubit and the corresponding CV mode. A sketch of the scheme we consider here is shown in figure 1. After the subsystems mutually interact via \( \hat{H}_{\text{int}} \) for an interval \( \Delta \tau \) (rescaled by the characteristic coupling rate between the qubits and the modes), \( a \) and \( b \) are measured. Excluding the detection step, some
Figure 1. Scheme of the protocol for quantum guidance without feedback. The entangled two-mode ($a$ and $b$) CV system interacts with the (initially separable) qubits 1 and 2, respectively. After the interaction, the state of $a$ and $b$ is detected and the events are post-selected. The conditioned qubit state obtained after $n$ iterations of this procedure is progressively purified and its entanglement is distilled.

An analogy with the entanglement-transfer schemes [8, 9] can be sketched, especially as concerns the particular nature of the ancillary system. However, in [8, 9], the control over the system’s performances (here represented by the post-selection of favourable detection events) was totally absent (no real guidance was performed), the overall procedure being much more passive than here.

A good source of quantum correlated two-mode states of a CV system is a non-degenerate parametric amplifier generating a two-mode squeezed vacuum (TMSV) of its squeezing parameter $s$ [12]. On the other hand, it has been recently shown that the fidelity of teleportation of a coherent state can be improved [13] and the loophole-free non-locality test could be possible [14] by conditioning the CV entangled resource (embodied in a TMSV) through linear optical elements and photodetection [15]. The specific form of the state considered in [13, 14] is the photon-subtracted state (PSS)

$$|\chi\rangle_{ab} = \mathcal{N} \sum_{l=0}^{\infty} \lambda(l) |l, l\rangle_{ab},$$

where $\lambda(l) = (\tanh s)^l(l + 1)$, $|l\rangle$ is a photon-number state and $\mathcal{N}$ is a normalization factor. The probabilistic generation of equation (2) using a TMSV, high-transmittivity beam-splitters and photodetectors is described in [13].

As in the teleportation protocol, the ancilla A (i.e. modes $a$ and $b$) represents a resource. Its role, in this paper, is the catalysis of the purification process and the distillation of entanglement between the qubits. From this point of view, we have to look for the best choice for system A. Even though the entire analysis and the results we report in this work can be generalized to any form of a two-mode quantum correlated state, we have checked that the choice of a PSS offers a good result in terms of entanglement and purity for the qubit state. This is essentially due to the
3. The ideal protocol

Having identified the form of the state for our ancilla, we can apply the idealized protocol for purification. We have to calculate the structure of the operator $\hat{O}_\chi$ analogous to the one in equation (1). In our case, it is

$$\hat{O}_\chi = O_{11} |gg\rangle\langle gg| + O_{44} |ee\rangle\langle ee| + O_{12} (|eg\rangle\langle eg| + |ge\rangle\langle ge|) - O_{14} (|ee\rangle\langle gg| + |gg\rangle\langle ee|),$$

where $O_{11} = N^2 \sum_{l=0}^{\infty} \lambda^2(l) Q_{i+1}^2$, $O_{44} = N^2 \sum_{l=0}^{\infty} \lambda^2(l) Q_{i+1}^2$, $O_{12} = N^2 \sum_{l=0}^{\infty} \lambda^2(l) Q_i Q_{i+1}$ and $O_{14} = N^2 \sum_{l=0}^{\infty} \lambda(l)(l+1) \sqrt{1 - Q_{i+1}^2}$. Here, $Q_i = \cos(\Delta \tau \sqrt{l})$. To check the effectiveness of the purification procedure, we have to look for the eigen-decomposition of $\hat{O}_\chi$. We find that, together with the double-degenerate eigenvalue $e_0 = O_{12}$, the spectrum of $\hat{O}_\chi$ includes the eigenvalues $e_\pm = [(O_{11} - O_{44}) \pm \sqrt{(O_{11} - O_{44})^2 + 4O_{14}^2}]/2O_{14}$, which correspond to the eigenoperators $|\pm\rangle_{12} (\pm|$, respectively. Here, $|\pm\rangle_{12} = N_{\pm}[(e_\pm - O_{44}) |gg\rangle + O_{14} |ee\rang{12}$. The eigenspectrum is shown in figure 2, where it is evident that $e_+ > e_0$ and $|e_-|$ at any $\Delta \tau > 0$. In particular, at $\Delta \tau \simeq 4.5$, we find $e_+ \simeq 0.998 \gg e_- = |e_0| \simeq 0.028$. This
means that, by repeating the protocol \( n \) times and post-selecting the favourable events in which the subsystems \( a \) and \( b \) are detected in \( |\chi\rangle_{ab} \), the joint state of qubits 1 and 2 is projected onto a state with an increasing degree of purity and entanglement (even if, given that \( |e_s| < 1 \), the state cannot be maximally entangled). In order to quantify the degree of entanglement, we use the entanglement measure based on negativity of partial transposition (NPT) \([17]\). NPT is a necessary and sufficient condition for entanglement of any bipartite qubit state \([17]\). The corresponding degree of entanglement is defined as \( E_{\text{NPT}} = \max \{ 0, -2\epsilon^- \} \) where \( \epsilon^- \) is the negative eigenvalue of the partial transposition of the qubit density matrix. With this tool, we find that for qubits initially prepared in their ground states, after just one iteration the degree of entanglement for the resulting two-qubit state is 0.798, which is improved to 0.858 after \( n = 5 \) iterations.

However, in this protocol, a way to reliably perform the von Neumann measurements onto the prepared state of the ancilla is required. Checking exactly whether or not the ancilla is in its state \( |\chi\rangle_{ab} \) is, in general, a very hard task from a practical point of view. Even if, with a single qubit embodying the ancillary system \( A \) (this is basically the case considered in \([10, 11, 18]\)), one may envisage some strategy to perform the desired projections, this seems not to be the case when a CV system is used. This should strongly affect the performances of the entire procedure. Nevertheless, a way to bypass the problem represented by the measurement step can be found, as is argued in the next section.

4. Guidance without feedback

Here, we look for a way to retain the basic idea of modifying the qubit state through interventions on an ancillary system. However, we abandon the idealized projective measurements. The aim of this section is to show that, with some suitable modifications, the problem represented by the difficult von Neumann projections may be bypassed without affecting the effectiveness of the procedure. We consider a more realistic detection model that can be generalized to include arbitrary detection inefficiencies. Our protocol allows us to track the evolution of the two-qubit state showing that true quantum guidance can be performed with the qubits being actively driven towards a desired state.

While the system we consider here is essentially the same as that described in section 3, here we introduce a near to reality detection step. We borrow the language from quantum optics and model the detection step by Geiger-like on/off detectors that discriminate the presence or absence of excitations in the CV states, irrespective of the excitation number. The quantum efficiency of each detector (assumed to be the same) is \( \eta \). An inefficient detector is formally described by the positive operator-valued measure (POVM) \([1]\)

\[
\hat{\Pi}_{\text{inc}}^i(\eta) = \sum_{n=0}^{\infty} (1 - \eta)^n |n\rangle_i \langle n|, \quad \hat{\Pi}_{\text{c}}^i(\eta) = 1 - \hat{\Pi}_{\text{inc}}^i(\eta), \tag{4}
\]

where \( i = a, b \). The effect of detection inefficiencies on the quantum control protocol will be fully taken into account later on. For a while, we consider perfect detectors of \( \eta = 1 \), so that \( \hat{\Pi}_{\text{inc}}^i(1) = \hat{\Pi}_{\text{c}}^i = |0\rangle_i \langle 0| (i = a, b) \). Furthermore, at each step of the control procedure, the qubits will interact with a fresh quantum-correlated state of modes \( a \) and \( b \). That is, there is no direct feedback of the ancilla state (another hard task in the protocols \([10, 11]\)), but the qubits interact repeatedly with the fresh ancilla.
We consider the evolution of the qubit system resulting from the following programme: the quantum correlated state of the CV system is prepared and the qubit register is initialized as \( \rho_{12}(0) \). The subsystems interact through the joint interaction \( \hat{H}_{int} \), which gives rise to the direct product of local unitaries \( \hat{U}_{12ab}(t) = \hat{U}_{1a}(t) \otimes \hat{U}_{2b}(t) \), where \( \hat{U}_{ia} = e^{-i\hat{H}_{ia}t} \) (analogously for \( \hat{U}_{2b} \)). For definiteness, we assume the Hamiltonian models \( \hat{H}_{1a} \) and \( \hat{H}_{2b} \) to be of the resonant Jaynes–Cummings form [19]. This is a natural model that turns out to be valid in many physical situations in which coherent exchange of excitations between spin-like particles and bosonic modes are involved [9, 20] (however, our study can be extended to any other entangling qubit-mode interaction).

After the evolution, the state of the CV system is detected. The event in which both the detectors click is retained and the entire process is repeated \( n \) times. A single step of this programme changes the qubit state as

\[
\rho_{12}(n + 1, \Delta \tau) = \text{Tr}_{ab}[(\hat{\Pi}_c^a \otimes \hat{\Pi}_c^b)\hat{U}_{12ab}(\Delta \tau)\rho_{12}(n, \Delta \tau) \otimes |\chi\rangle_{ab} \langle \chi| \hat{U}_{12ab}^\dagger(\Delta \tau)(\hat{\Pi}_c^a \otimes \hat{\Pi}_c^b)]
\]

with \( n \geq 0 \). Here, we have explicitly indicated that the density matrix depends on the step \( n \) performed and the (dimensionless) interaction interval \( \Delta \tau \), which is supposed to be equal for the two qubit-mode subsystems. Whenever a no-signal event interrupts a sequence (corresponding to either one or both the detectors failing to click), the two-qubit state is discarded and the process is restarted.

It is not difficult to evaluate explicitly the effect of the operator \((\hat{\Pi}_c^a \otimes \hat{\Pi}_c^b)\hat{U}_{12ab}(t)\) after the first step of the procedure. For definiteness, we assume \( \rho_{12}(0) = |gg\rangle_{12} \langle gg| \). Using the resolution of the identity operator \( \mathbf{1}_j = \sum_{l=1}^{\infty} |l\rangle \langle l| \) \((i = a, b)\) and the photon-number basis to compute the trace in equation (5), we get

\[
\rho_{12}(1, \Delta \tau) = A_{gggg} |gg\rangle \langle gg| + B_{gggg} (|ge\rangle \langle ge| + |eg\rangle \langle eg|) + F_{gggg} |ee\rangle \langle ee| + G_{gggg} (|ee\rangle \langle ee| + |ee\rangle \langle gg|),
\]

where \( A_{gggg} = N^2 \sum_{l=1}^{\infty} \lambda^2(l) \mathcal{Q}_{l}^4 \), \( F_{gggg} = N^2 \sum_{l=2}^{\infty} \lambda^2(l)(1 - \mathcal{Q}_{l}^2)^2 \), \( B_{gggg} = N^2 \sum_{l=1}^{\infty} \lambda(l) \mathcal{Q}_{l}(1 - \mathcal{Q}_{l}^2) \), \( G_{gggg} = -N^2 \sum_{l=1}^{\infty} \lambda(l) \mathcal{Q}_{l}(l + 1) \mathcal{Q}_{l}^2(1 - \mathcal{Q}_{l+1}^2) \). In what follows, \( A \) stands for the coefficient relative to the \( 12 \langle gg| \rho_{12}(n, t) |gg\rangle_{12} \) density matrix elements (the same holds, \textit{mutatis mutandis}, for other coefficients), while the subscripts label the element of the initial density matrix we are considering. It is important to stress the correspondence between the form of \( \rho_{12}(1, \Delta \tau) \) and that of the effective post-measurement operator \( \hat{O}_x \) in equation (3). Indeed, the application of equation (3) to the initial state \( \rho_{12}(0) = |gg\rangle_{12} \langle gg| \) will result in a density matrix having exactly the same form as equation (6). This is an effect due to the particular symmetries in the quantum-correlated state of the ancilla and it assures that our protocol reproduces the right form of the density matrix. Our task is to show that the degree of entanglement and purity for the qubits progressively increases as the protocol is repeated.

It is straightforward to prove that the form of a state as equation (6) is invariant for successive applications of the dynamics described by equation (5). The density operator is trapped in a form that deserves some comments. The absence of the coherences \( 12 \langle eg| \rho_{12}(1, \Delta \tau) |ge\rangle_{12} \) and its Hermitian conjugate and the fact that the populations \( 12 \langle ge| \rho_{12}(1, \Delta \tau) |ge\rangle_{12} \) and

\[1\] This assumption is just a convenient choice that makes the results of the calculations more transparent. Any initial state with any degree of mixedness can be considered. We are simply providing the necessary ingredients to evaluate the evolution of any arbitrary initial state.
Figure 3. Eigenspectrum of the density matrix for the two-qubit state after a single application of the quantum guidance protocol against the rescaled time interval $\Delta \tau$. The eigenvalue $\beta_+$ (---) dominates over the rest of the eigenspectrum. There is a double-degenerate eigenvalue, which is why there are just three curves, in this plot. As in figure 2, here the squeezing parameter in the PSS is $s = 0.3$.

$\langle eg | \rho_{12}(1, \Delta \tau) | eg \rangle_{12}$ are equal suggest that, on the basis of the maximally entangled Bell states $\{|\phi_+\rangle, |\phi_-\rangle, |\psi_+\rangle, |\psi_-\rangle\rangle_{12}$ [1], equation (6) can be written as the mixture

$$\rho_{12}(1, \Delta \tau) = \alpha(|\psi_+\rangle_{12} \langle \psi_+| + |\psi_-\rangle_{12} \langle \psi_-| + \beta |\phi_+\rangle_{12} \langle \phi_+| + \gamma |\phi_-\rangle_{12} \langle \phi_-|.$$

Here, $|\phi_+\rangle_{12} = (1/\sqrt{2})(|gg\rangle - |ee\rangle)_{12}$, $|\phi_-\rangle_{12} = \hat{\sigma}_z |\phi_-\rangle_{12}$, $|\psi_+\rangle_{12} = \hat{\sigma}_x |\phi_+\rangle_{12}$, and $|\psi_-\rangle_{12} = i\hat{\sigma}_y |\phi_-\rangle_{12}$ [1]. The purity and entanglement of the final state depend on the relative weight of the coefficients $\alpha$, $\beta$ and $\gamma$.

The first indications about the state that has to be expected and its corresponding degrees of purity and entanglement are given by the eigenspectrum of $\rho_{12}(1, \Delta \tau)$. As a consequence of the analysis above, it is not surprising that the eigenspectrum presents exactly the same characteristics pointed out about $\hat{O}_{\chi}$ (see the discussion after equation (3), in section 3), i.e. the existence of an eigenvalue that dominates over all the others. We indicate this eigenvalue as $\beta_+ = [(A_{eegg} - F_{eegg}) + \sqrt{(A_{eegg} - F_{eegg})^2 + 4G_{eegg}^2}] / 2G_{eegg}$ and, in figure 3, we plot it (with the rest of the spectrum) against $\Delta \tau$. The eigenvector corresponding to $\beta_+$ turns out to be the (unnormalized) entangled state $(\beta_+ - F_{eegg}) |gg\rangle_{12} + G_{eegg} |ee\rangle_{12}$, which has a large overlap with $|\phi_-\rangle_{12}$, in perfect correspondence with what is obtained using the results described in section 3.

We go on with the procedure summarized by equation (5) and look for the statistical properties of the resulting qubit state. After the first iteration, each non-zero density matrix element becomes more and more involved. However, it is possible to find a recurrence pattern that expresses the (unnormalized) matrix elements after $n + 1$ iterations ($n \geq 1$) as a function of the matrix elements after the first and $n$th iterations as

$$M(n + 1) = N_n[A(n)M_{eegg} + B(n)(M_{eegg} + M_{gege}) + F(n)M_{egee} + G(n)(M_{eegg} + M_{gege})],$$

where $N_n = [A(n) + 2B(n) + F(n)]^{-1}$ is the normalization of the density matrix after $n$ iterations and $M = A, B, F, G$. Obviously, the matrix elements at $n = 1$ depend on the choice of the initial density matrix and coincide with the coefficient given in equation (6) for $\rho_{12}(0) = |gg\rangle_{12} \langle gg|$. 

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In order to characterize the state $\rho_{12}(n, \Delta \tau)$, we consider the degree of mixedness as measured by the linearized entropy $S_L(n) = (4/3)(1 - \text{Tr}[\rho^2_{12}(n, \Delta \tau)])$ [9]. This quantity is 0 for pure states and 1 for completely mixed ones. Moreover, to get an immediate picture of how near the state we get to the $|\phi_{-}\rangle_{12}$ Bell state, we consider the fidelity $F_{12} = \langle \phi_{-}| \rho_{12}(n, \Delta \tau) |\phi_{-}\rangle_{12}$. These values of merit have been plotted in figures 4(a) and (b), respectively, against the interaction interval $\Delta \tau$. Due to computational limitations, we report in these plots the results obtained for $n = 1, 2, 3$. The results obtained after further iterations can be evaluated numerically, pointwise with respect to the interaction interval.

Figure 4(a) shows that, in agreement with the results presented in figures 2 and 3, the time interval $\Delta \tau \simeq 4.5$ is interesting for purification tasks. When the interactions between subsystems A and B last for such a length, repeated application of our protocol leads to a reduction of the mixedness and an increase of the overlap with $|\phi_{-}\rangle_{12}$. The largest decrease in the mixedness value is obtained in the early iterations, when $S_L$ changes by nearly one order of magnitude from $S_L \simeq 0.09$ to $S_L \simeq 0.01$. The successive reductions are smaller but still considerable, up to the step $n = 9$, which is the last we have considered in our calculations.

It appears that different tasks may be achieved with our quantum guidance protocol by simply choosing a different value of $\Delta \tau$. For example, it is possible to choose $\Delta \tau \simeq 3.3$, corresponding to the situation in which successive applications of the guidance protocol increase the purity of the qubit state, leaving its overlap with $|\phi_{-}\rangle_{12}$ unchanged. This realizes the purification of the qubit state for a given amount of entanglement shared between the qubits. This semi-quantitative analysis should have shown that genuine quantum control without any real feedback (but just post-selection) is performed, in this way. Other possibilities are offered by a dynamic adjustment of the interaction intervals, where the values of $\Delta \tau$ change at each iteration, in order to guide the state of the qubits towards a desired target state. However, it is obvious that such a procedure (representing a true feedback scheme) requires a much finer control on the dynamics of the setup.

The last step required to complete the analysis of the performances of our control protocol is a more direct comparison with the results achieved using the idealized procedure outlined in section 3. This may be performed by contrasting the purity and entanglement of the qubit systems, as shown in figure 5. The evolution of $\rho_{12}(n, \Delta \tau)$, in the purity-entanglement plane, is shown (from right to left) by the star-shaped symbols for up to $n = 4$ iterations. We have taken $\Delta \tau = 4.5$, in this plot. The black square is what is obtained from the idealized protocol. As we have pointed out, the larger reduction of the mixedness is accomplished after the early iterations.
Figure 5. Comparison between the state after the ideal procedure (■) and those obtained for \( n = 1 \rightarrow n = 4 \) in our effective guidance protocol (★). Here, \( \Delta \tau = 4.5 \) and the squeezing parameter of the ancillary state is \( s = 0.3 \). The evolution of the state described by the ★ is from the right to the left of the horizontal axis.

By repeating the protocol, the improvement in the degree of purity (as well as the increases of the degree of entanglement) becomes smaller even if the purification of the state and its entanglement concentration are still effective. The series of points we obtain on the purity-entanglement plane slowly approaches the state produced by the ideal scheme in section 3.

5. Non-idealities in the protocol

In this section, we take into account two classes of non-idealities which can be faced by a realistic implementation of our scheme. We start considering the effect that non-ideal detectors have in the conditional dynamics of the qubits. Tackling this source of error is particularly important as our scheme is based on coincidence detections. However, the effect of the detection inefficiencies can be fully taken into account, in our approach. Indeed, it turns out that the use of non-ideal detectors is formally equivalent to the replacement \( \lambda(l) \rightarrow [1 - (1 - \eta)]^{\lambda(l)} \) in equation (2) followed by the explicit recalculation of equation (5). Following these lines, we find that, for efficiency of detection as small as \( \eta = 0.7 \), the mixedness of the qubit state is about 20% larger when compared with perfect detectors, as shown in figure 6. Even if figure 6 presents these results just for a rescaled time around \( \Delta \tau \approx 4.5 \), the same is true all along the relevant interaction times we have examined and for up to \( n = 9 \) iterations of the control protocol. The same qualitative considerations hold for the overlap of the qubit state with the \( |\phi_-\rangle_{12} \) Bell state (inset to figure 6). Even if it is raised by the non-ideality of the photodetectors, the mixedness of the qubit state is, nevertheless, reduced at each further application of our procedure. It is thus clear that realistic inefficiencies at the detection step do not affect significantly the guidance of the qubit state.

We can continue the analysis that non-ideality has in the performance of this protocol addressing the problem of no signalling at the detection step. As pointed out earlier, the scheme in [10] discards negative events in which the ancilla is found in a state different from the bona fide one. In our protocol, a positive event is a coincidence at the detection stage. Even if we have
demonstrated that a chain of detection coincidences is the desirable concatenation of events, the probability of such a result is certainly quite small. Our question is about the effect of a missing coincidence with respect to the purity of the qubit state. To answer this question, we have simulated a sequence of two applications of the procedure. The first application is positive (in the sense that a coincidence of detection is found) while the second one is negative (no signal at the detectors, which is formally described by the action of the operator \( \bigotimes_{i=1}^{2} \hat{1}_{\eta_{ie}}(\eta) \)).

It is possible to derive an explicit expression for the resulting density matrix and to evaluate the relative eigenspectrum. We find that the effectiveness of the purification process is not critically affected, even if the overlap with \( |\phi_{-}\rangle_{12} \) is smaller than in the case of an ideal sequence of events. We have checked what happens to the qubit density matrix when the sequence of positive and negative events is inverted. In this case, the entanglement distillation process is no more effective, up to the second iteration. There is a very small increase in the overlap between the resulting qubit state and the target state \( |\phi_{-}\rangle_{12} \).

On the other hand, so far we have considered a pure state of the two-mode CV ancilla. However, we know that a PSS is generated, from a TMSV, only with a finite probability [13]. It turns out, for example, that, in very recent experiments performed to generate a single-mode PSS, the conditional protocol which simulates the effective single-photon subtraction produces a state that is an incoherent mixture of a (successful) PSS and a single-mode squeezed state [16]. Motivated by these experimental limitations, in the remainder of this section, we consider the model for a mixed state of the two-mode CV ancilla reading

\[
\sigma_{ab} = p |\chi\rangle_{ab} \langle \chi | + (1-p) |S\rangle_{ab} \langle S |. \tag{9}
\]

Here, \( |S\rangle_{ab} = (\cosh s)^{-2} \sum_{l,l'}^{\infty} (\tanh s)^l |l,l\rangle_{ab} \) is the explicit form of a TMSV and \( p \) is the probability of getting the PSS \( |\chi\rangle_{ab} \) through the conditional subtraction of a photon in each mode. In [16], the corresponding probability for the single-mode case is estimated to be \( \simeq 0.7 \). The two-mode nature of the PSS considered in our scheme and the independence of the effective photon subtractions in each mode lead us to consider \( p \simeq 0.49 \). In order to give a flavour of the effect that equation (9) has in our protocol, we have simulated a single step of the quantum
Figure 7. Differences between the corresponding eigenvalues of the spectra of \( \rho'_{12}(1, \Delta \tau) \) and \( \rho_{12}(1, \Delta \tau) \) plotted against the rescaled interaction interval \( \Delta \tau \), for \( s = 0.3 \) and \( p = 0.49 \). The solid line represents \( \beta'_+ - \beta_+ \).

guidance scheme evaluating, this time, the conditional qubit state

\[
\rho'_{12}(1, \Delta \tau) = \text{Tr}_{ab}[(\hat{\Pi}_c^a \otimes \hat{\Pi}_c^b) \hat{U}_{12ab}(\Delta \tau) \rho_{12}(0) \otimes \sigma_{ab} \hat{U}^\dagger_{12ab}(\Delta \tau)(\hat{\Pi}_c^a \otimes \hat{\Pi}_c^b)].
\]

(10)

A useful value of merit in evaluating possible discrepancies introduced by the model equation (9) is given by the differences between homonymous eigenvalues of \( \rho_{12}(1, \Delta \tau) \) and \( \rho'_{12}(1, \Delta \tau) \) (which means that, for instance, we are interested in \( \beta'_+ - \beta_+ \), where \( \beta'_+ \) is the largest of the eigenvalues of the qubit density matrix in equation (10) and \( \beta_+ \) is the analogous quantity as defined in section 3). In figure 7, we plot these differences against the interaction interval \( \Delta \tau \). This plot gives us information about the magnitude of \( \beta'_+ \) compared to \( \beta_+ \) (solid line) and the order relation between these two quantities. It is apparent that the deviations between the largest eigenvalues in the spectrum of the two different density matrices are, most of the time, below 10%, being mainly negative. On the other hand, the analogous deviations calculated for the rest of the spectrum of the density matrices (dashed and dot-dashed lines) indicate that the mixedness of the ancillary state slightly decreases the contrast between the dominant eigenvalue and the rest of the eigenspectrum of \( \rho'_{12}(1, \Delta \tau) \). The scheme appears to be robust against the spoiling effect of a mixed CV ancilla described by equation (9). Taking apart the mere quantitative details, this can be understood considering that \( \sigma_{ab} \) has the effective structure

\[
\sigma_{ab} = \sum_{n=0}^{\infty} \mu(n, m, \lambda, p) |nn\rangle_{ab} \langle mm|,
\]

with the amplitudes \( \mu(n, m, \lambda, p) \), which can be easily derived from equation (9). This state still projects the conditioned qubit density matrix onto the trapped form of equation (7). Then, the magnitudes of the eigenvalues of \( \rho'_{12}(1, \Delta \tau) \) guarantee that the coefficient \( \gamma \), in equation (7), is still the dominant one. It is, thus, the symmetric structure of the ancilla state, with respect to the modes \( a \) and \( b \), which forces the qubits into a state near \( |\phi_-\rangle_{12} \). We thus conjecture that, qualitatively, this same effect has to be expected whenever a two-mode CV state as \( \sum_n \nu(n, m) |nn\rangle_{ab} \langle mm| \) is considered. The efficiency of the purification scheme (in terms of overlap with a single Bell state) depends on the statistics of the ancilla.

Another evidence of the importance of the first iteration can be found looking for the asymptotic structure of the density matrix elements. A numerical analysis shows that, within a reasonable degree of accuracy, the following approximations hold

\[
\mathcal{F}(n+1) \simeq N_n \mathcal{F}(n) \mathcal{F}_{eece} + A(n) \mathcal{A}_{eeeg}, \quad A(n+1) \simeq N_n \mathcal{F}(n) \mathcal{A}_{eece} + A(n) \mathcal{A}_{eeeg}, \quad B(n+1) \simeq 0 \quad \text{and} \quad \mathcal{C}(n+1) \simeq N_n \mathcal{F}(n) \mathcal{C}_{eece} + A(n) \mathcal{A}_{eeeg}.
\]

The final state of the qubit system at which the purification procedure ends is obtained when the density matrix elements no longer change after repeated
application of the control protocol. More specifically, we are looking for $\rho_{12}(n, \Delta \tau) \simeq \rho_{12}(n + 1, \Delta \tau) = \rho_f$, which is found to be

$$
\begin{pmatrix}
A_{eee}F_{eee} & 0 & 0 & A_{eee}G_{eee} + (F_{eee} - A_{ggg})G_{eee} \\
0 & 0 & 0 & 0 \\
A_{eee}G_{eee} + (F_{eee} - A_{ggg})G_{eee} & 0 & 0 & F_{eee}(F_{eee} - A_{ggg}) \\
F_{eee} + A_{ggg} & 0 & 0 & F_{eee} + A_{ggg}
\end{pmatrix}.
$$

This shows how, in our case, the purity and entanglement of the final state entirely depend on the form of the density matrix after a single iteration of the purification protocol. The first step is the most important one and the control procedure has to be designed in such a way that the mixedness of $\rho_f$ is the smallest possible. It is worth noticing that, for our choice of the quantum-correlated CV state, this approximated asymptotic state leads to the values of the characterizing benchmarks equal to $S_L(f, \Delta \tau = 4.5) \simeq 10^{-2}$, $E_{\text{NPT}}(f, \Delta \tau = 4.5) \simeq 0.94$ and a fidelity with $|\phi_-\rangle_{12} \simeq 0.91$.

6. Remarks

We have studied the purification and entanglement distillation of a two-qubit system through its interaction with a CV ancilla on which repeated measurements are operated. This attempt is motivated by the somehow natural advantages in using photonic systems as ancillae. A longer distance between the qubits may be achieved and more reliable detectors are available for these systems. Taking advantage of our previous studies about entanglement-transfer processes [8, 9], we have explicitly considered the case of a bipartite entangled state of the ancilla to show that an efficient and fast generation of entanglement is possible in this scenario. Our investigation has gone further in addressing the possibility of true quantum guidance of the state of two qubits without real feedback. This is certainly an advantage as feedback procedures are in general difficult to implement.

Our protocol is able to contemplate inefficiencies in the detection steps. Moreover, the effect of an imperfect ancilla has been considered, using an experimentally motivated model for the mixedness of the two-mode CV state. An analysis of these non-idealities is important as they are faced in realistic implementations of quantum optics setups. We have quantitatively shown that the protocol is reasonably robust against negative detection events, realistic detector inefficiencies and mixedness of the entanglement distributor.

Acknowledgments

This work has been supported by the EPSRC, UK, and the KRF (2003-070-C00024). M.P. acknowledges IRCEP for financial support.

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