SHADOWS OF ORDERED GRAPHS

BÉLA BOLLOBÁS, GRAHAM BRIGHTWELL, AND ROBERT MORRIS

Abstract. We investigate the possible sub-linear speeds of a hereditary property of ordered graphs, \( P \). In [3], Balogh, Bollobás and Morris showed that if \(|P_k| < k\) for some \( k \in \mathbb{N} \), then the speed \(|P_n|\) is eventually constant. Here we substantially strengthen their result by showing, under the same conditions, that in fact \(|P_n|\) is decreasing on \([k, \infty)\).

This follows as an almost immediate corollary of the following theorem about the shadow \( \partial G \) of a collection \( G \) of ordered graphs on \([n]\): if \(|G| < n\), then \(|\partial G| \geq |G|\).

1. Introduction

In this paper we continue the investigation of the possible speeds of a hereditary property of ordered graphs from [3] and [4]. We shall be studying very low (sub-linear) speeds, but we shall prove a very precise result. We begin with some definitions.

An ordered graph \( G \) is a graph together with a linear order on its vertex set. If \(|V(G)| = n\), then we shall identify \( V(G) \) with \([n]\), the set \( \{1, \ldots, n\} \) with the usual order. A hereditary property of ordered graphs, \( P \), is a collection of ordered graphs closed under taking induced ordered subgraphs. For example, the collection of all ordered graphs not containing a given ordered graph \( H \) as an induced ordered subgraph is hereditary. We write \( P_n \) for the collection of ordered graphs in \( P \) with vertex set \([n]\), and call the function \( n \mapsto |P_n| \) the speed of \( P \).

The speed is a very natural measure of the ‘size’ of \( P \), and was introduced (in the context of labelled graphs) by Erdős [11] in 1964. It has been extensively studied for a variety of combinatorial structures (see for example the work of Alekseev [1], Scheinerman and Zito [17], Bollobás and Thomason [9] and others [12, 10, 6, 15, 5, 4]), and a surprising phenomenon has emerged: in many cases, only very ‘few’ speeds are possible. More precisely, there often exists a family of speeds \( F \) and another function \( F \), with each \( f \in F \) much smaller than \( F \), such that either the speed \(|P_n|\) is eventually smaller than \( f(n) \) for some \( f \in F \), or is at least \( F(n) \) for every \( n \in \mathbb{N} \). We refer to this phenomenon as a ‘jump’ in the set of possible speeds.

In [3], Balogh, Bollobás and Morris initiated the study of the possible speeds of a hereditary property of ordered graphs below \( 2^n \). Extending a result of Kaiser and Klazar [13] for permutations (which may be viewed as a special kind of ordered graph), they proved

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the following result: either \(|\mathcal{P}_n|\) is a polynomial, i.e.,

\[
|\mathcal{P}_n| = \sum_{i=0}^{k} a_i \binom{n}{i}
\]

for some \(k \in \mathbb{N}_0\), some integers \(a_0, \ldots, a_k\), and all sufficiently large \(n\), or \(|\mathcal{P}_n| \geq F_n\) for every \(n \in \mathbb{N}\), where \(F_n \sim \left(\frac{1}{2}(1 + \sqrt{5})\right)^n\) is the Fibonacci sequence. They also determined an infinite sequence of further jumps between \(F_n\) and \(2^n\) and, more significantly for the purposes of this article, that if \(|\mathcal{P}_n|\) is unbounded then \(|\mathcal{P}_n| \geq n\) for every \(n \in \mathbb{N}\). In [4] the same authors conjectured that there is a further jump from exponential speed to factorial speed \((\geq n^{cn})\), and proved various special cases. This conjecture substantially generalizes the Stanley-Wilf Conjecture for permutations, which was recently proved by the combined results of Klazar [14] and Marcus and Tardos [15].

In order to aid the reader’s understanding, let us briefly describe some of the extremal properties in the results above. There are exactly two hereditary properties of ordered graphs with speed exactly \(F_n\): the collection of ordered graphs with edge set \(\{i_1(i_1 + 1), \ldots, i_k(i_k + 1)\}\), where \(i_j + 1 < i_{j+1}\) for each \(j \in [k]\), and the collection of complements of such ordered graphs. However, there are several hereditary properties \(\mathcal{P} = \bigcup \mathcal{P}_n\) with speed exactly \(n\): for example, if \(\mathcal{P}_n\) is the collection of ordered graphs on \([n]\) with edge set \(\{ij : 1 \leq i < j \leq k\}\) for some \(1 \leq k \leq n\), then \(\mathcal{P}\) is hereditary and \(|\mathcal{P}_n| = n\). Similarly, we could have used the edge set \(\{ij : i \leq k < j\}, \{1k\}, \{k(k+1)\}, \{1j : 1 < j \leq k\}\), or \(\{1j : k < j\}\). (In each case we also include in \(\mathcal{P}_n\) the empty ordered graph, with edge set \(E(G) = \emptyset\).) Two of these six properties are pictured below. Up to symmetry between left and right, and between edges and non-edges, these are the only extremal families.

![Figure 1: Two ordered graphs from extremal families, and their edge sets](image-url)

In this paper we shall investigate in more detail the range \(|\mathcal{P}_n| < n\), where \(\mathcal{P}\) is a hereditary property of ordered graphs. By the results of [3] stated above, if there exists \(k \in \mathbb{N}\) such that \(|\mathcal{P}_k| < k\), then there exist \(M, N \in \mathbb{N}\) such that \(|\mathcal{P}_n| = M\) if \(n \geq N\). However, [3] does not give bounds on \(M\) and \(N\), and this result does not rule out the possibility that the speed of \(\mathcal{P}\) exhibits large fluctuations before finally settling down at \(M\). We shall show that this is not the case, by showing that in fact \(|\mathcal{P}_n|\) is decreasing for \(n \geq k\) (as long as \(k \geq 136\)). In order to do so, we shall need the concept of a shadow of an ordered graph.

Given an ordered graph \(G\), and a subset \(U \subset V(G)\), let \(G[U]\) denote the ordered graph induced by the set \(U\), with the inherited order. We shall set \(G - v = G[V(G) \setminus \{v\}]\). The
shadow of $G$ is defined to be
\[ \partial G = \{ H : H = G - v \text{ for some } v \in V(G) \} \]
and if $\mathcal{G}$ is a collection of ordered graphs then the shadow of $\mathcal{G}$ is
\[ \partial \mathcal{G} = \bigcup_{G \in \mathcal{G}} \partial G. \]

We shall prove the following result.

**Theorem 1.** Let $n \in \mathbb{N}$, with $n \geq 136$, and let $\mathcal{G}$ be a collection of ordered graphs on $[n]$. If $|\mathcal{G}| < n$, then $|\partial \mathcal{G}| \geq |\mathcal{G}|$.

There is an extensive literature on shadows of set systems (see [7], for example) but, to our knowledge, this is the first result of its kind on shadows of ordered graphs. We remark that although the result seems simple, ordered graphs are complex objects containing a large amount of information. The special case of Theorem 1 in which each ordered graph has at most one edge is equivalent to a special case of an isoperimetric inequality on $\mathbb{N}^3$ first proved by Bollobás and Leader [8] using compressions. The reader is invited to attempt to prove Theorem 1 for himself in the case in which each ordered graph has at most two edges, or indeed in the case $n = 4$.

The following theorem about hereditary properties of ordered graphs follows easily from Theorem 1.

**Theorem 2.** Let $\mathcal{P}$ be a hereditary property of ordered graphs. If $|\mathcal{P}_k| < k$ for some $k \geq 136$, then $|\mathcal{P}_n| \leq |\mathcal{P}_k|$ for every $k \leq n \in \mathbb{N}$.

We remark that the first result of this type was proved in 1938 by Morse and Hedlund [16], for properties of words. They showed that an infinite word is either periodic, or contains at least $n + 1$ different words of length $n$ for each $n \in \mathbb{N}$. Recently Balogh and Bollobás [2] extended this result to general hereditary properties of words.

Finally, let us note that Theorem 2 is sharp, in the sense that there are hereditary properties with speed exactly $n$, and in fact, as described earlier, there are (up to symmetry) exactly six such properties. Similarly, Theorem 1 is sharp, in the sense that there exists a collection $\mathcal{G}_1$ with $|\mathcal{G}_1| = n$, but $|\partial \mathcal{G}_1| < |\mathcal{G}_1|$, and a collection $\mathcal{G}_2$ with $|\mathcal{G}_2| < n$, but $|\partial \mathcal{G}_2| = |\mathcal{G}_2|$. For example, the collection $\mathcal{G}_1 = \{ G_k : k \in [n] \}$ of ordered graphs on $[n]$ with edge set $E(G_k) = \{ ij : i < j \leq k \}$ has $|\mathcal{G}_1| = n$ and $|\partial \mathcal{G}_1| = n - 1$, and the collection $\mathcal{G}_2 = \{ G_k : k \in [n - 1] \}$ of ordered graphs on $[n]$ with the single edge $k(k + 1)$ has $|\mathcal{G}_2| = |\partial \mathcal{G}_2| = n - 1$. The condition $n \geq 136$, on the other hand, is an artifact of the proof, and is almost certainly unnecessary.

The rest of the paper is organised as follows. In Section 2 we list some of the notation we shall use in the paper, and in Section 3 we prove a lemma on shadows of set systems. In Section 4 we describe a partition of the ordered graphs on $[n]$ into ‘types’, and use the result of Section 3 to prove a special case of Theorem 1. The most substantial part of the paper is Section 5, in which we deal with ordered graphs with many small ‘homogenous blocks’. After all this preparation, the pieces are put together in Section 6 to prove
Theorem 1 and Theorem 2. Finally, in Section 7, we discuss possible avenues for further research, including an extension to general linear speeds.

2. Notation

In this section we collect, for easy reference, some of the notation which we shall use throughout the paper.

As usual, we write $\mathbb{Z}$ for the integers, $\mathbb{N}$ for the natural numbers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for the non-negative integers. Given an ordered graph $G$, we shall use the relation $<$ to denote both the ordering on the vertices of $G$ and the usual order on the integers, and trust that this will cause no confusion. Let $|G| = |V(G)|$ denote the number of vertices of $G$, and $e(G) = |E(G)|$ denote the number of edges of $G$. If $u, v \in V(G)$ then $[u, v] = \{w \in V(G) : u \leq w \leq v\}$.

If $x \in \mathbb{Z}^d$ for some $d \in \mathbb{N}$, then $x_i \in \mathbb{Z}$ will denote the $i$th coordinate of $x$. Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d$ denote the vector with a single 1 in position $j \in [d]$. If $x, y \in \mathbb{Z}^d$, then $x \leq y$ if $x_i \leq y_i$ for each $i \in [d]$.

If $A \subset \mathbb{N}_0$, then we write $K[A] = \{(i, j) : i, j \in A, i \neq j\}$ for the set of pairs (i.e., potential edges) on $A$. We shall write $\mathbb{1}[\cdot]$ to denote the usual indicator function.

Finally, by ‘symmetry’ we mean both the symmetry between edges and non-edges, and that between left and right.

3. A lemma on shadows of sets

In this section we shall prove a lemma on the shadows of sets which, although simple, will be an important tool in the proof of Theorem 1. We begin with some definitions. For each $n, d \in \mathbb{N}_0$, we write

$$\mathbb{Z}^d(n) = \left\{ x \in \mathbb{Z}^d : x_i \geq 0 \text{ and } \sum_i x_i = n \right\}.$$

The shadow of a set $A \subset \mathbb{Z}^d(n)$ is the set

$$\partial A = \{ x \in \mathbb{Z}^d(n-1) : x \leq y \text{ for some } y \in A \}.$$

A line in $\mathbb{Z}^d(n)$ is a set

$$\left\{ x \in \mathbb{Z}^d(n) : x_i = 0 \text{ if } i \notin \{j, k\} \right\}$$

for some $j, k \in [d]$. Observe that if $A$ is a line in $\mathbb{Z}^d(n)$, then $|A| = n + 1$ and $|\partial A| = n$. Moreover, if $A$ is a proper subset of a line, then $|\partial A| > |A|$.

It follows from Corollary 11 of [8] that $|\partial A| > |A| - 1$ for any set $A \subset \mathbb{Z}^d(n)$ with $|A| < 2n$. The following lemma tells us about the case of equality.

**Lemma 3.** Let $n, d \in \mathbb{N}$, and let $A \subset \mathbb{Z}^d(n)$ with $|A| < 2n$. Then either $|\partial A| \geq |A|$, or $A$ contains a line.
Proof. The proof is by induction on \( n + d \). When \( n = 1 \) or \( d = 1 \) the result is trivial since \(|A| \leq 1\), and if \( d = 2 \) then \( A \) is a subset of the line \( \mathbb{Z}^2(n) \). If \( n = 2 \) but \( A \) is not a subset of a line, then \(|\{i \in [d] : x_i \geq 1 \text{ for some } x \in A\}| \geq 3\), and so \(|\partial A| \geq 3\).

So let \( n, d \geq 3\), and consider the sets \( F_j = \{x \in A : x_j = 0\} \) for each \( j \in [d] \). We shall refer to the set \( \{x \in \partial A : x + e_j \in A\} \) as the set obtained by ‘compressing \( A \) in direction \( j \)’.

There are three cases to consider.

Case 1: \( |F_d| = 0 \).

This case is easy: we simply compress in direction \( d \). The set obtained has size \(|A|\) and is contained in the shadow.

Case 2: \( |F_d| = 1 \).

Let \( u = (a_1, \ldots, a_{d-1}, 0) \in A \) be the element of \( F_d \). Compressing in direction \( d \) gives us a set of size \(|A| - 1\). We claim that either \(|\partial A| \geq |A|\) or

\[
S(u) = \{x \in \mathbb{Z}^d(n) : x_i \leq a_i \text{ for each } 1 \leq i \leq d - 1\} \subset A.
\]

Indeed, consider an element \( b \in S(u) \setminus A \) with \((b_1, \ldots, b_{d-1})\) maximal in the usual partial order. Noting that \( b \neq u \), by maximality we have \( b + e_j - e_d \in A \) for some \( j \in [d - 1]\), and so \( b - e_d \in \partial A \). But \( b \notin A \), so \( b - e_d \) was not obtained by compressing in direction \( d \). Thus \(|\partial A| \geq |A|\) as claimed.

Now, if \( S(u) \subset A \) then \(|A| \geq \prod_i (a_i + 1)\). But \( \sum_i a_i = n \), and therefore either \(|A| \geq 2n\), or \( u = (n, 0, \ldots, 0) \), say. But in the latter case \( S(u) \) is a line, so \( A \) contains a line, and we are done.

Case 3: \( |F_d| \geq 2 \).

Assume, without loss of generality, that \(|F_d| \geq |F_i|\) for each \( i \leq d \). We apply the induction hypothesis (for \( n \) and \( d - 1 \)) to the set \( F_d \) and (for \( n - 1 \) and \( d \)) to the set \((A \setminus F_d) - e_d\). Note that in the latter case we use the fact that \(|F_d| \geq 2\).

Suppose first that neither contains a line. Then \(|\partial F_d| \geq |F_d|\) and \(|\partial((A \setminus F_d) - e_d)| \geq |A| - |F_d|\), and the sets \( \partial F_d \) and \( \partial((A \setminus F_d) - e_d) + e_d \) are disjoint, so \(|\partial A| \geq |A|\). Note also that a line in \( F_d \) is a line in \( A \), so we are done in this case as well.

Hence we may assume that there is an \((n - 1)\)-line \( L - e_d \) in \((A \setminus F_d) - e_d\). Note that \(|L| = n\), and that \( L \) lies in at least \( d - 3 \) faces of \( \mathbb{Z}^d(n) \) (it lies in \( d - 2 \) faces if it uses the direction \( d \)). Thus \( L \subset F_j \) for some \( j \neq d \), unless \( d = 3 \) and \( L = (\ast \ast 1) \). In the former case \(|F_j \setminus F_d| \geq n\), and so \(|A| \geq 2n\) since we chose \(|F_d| \geq |F_j|\). In the latter case however, the set \((\ast \ast 0) \cup (\ast \ast 1)\) of size \(2n - 1\) is in \( \partial A\), and so we are done.

We conclude this section by remarking that the result above is sharp. To see this, consider the sets \( A_1 = (\ast \ast 0) \cup (\ast \ast 1) \setminus (0 0 0) \) and \( A_2 = (\ast \ast 0) \cup (0 \ast \ast) \setminus (0 0 0) \), which have size \(2n\), have shadows of size \(2n - 1\), and do not contain a line.
4. Homogeneous blocks and types of ordered graph

In this section we shall define the type and the excess of an ordered graph, notions which will be crucial in what follows. We shall then deduce a special case of Theorem 1 from Lemma 3.

We begin by recalling the definition of a homogeneous block in an ordered graph from [3] (see also [6]). Let $G$ be an ordered graph, and for each $x, y \in V(G)$, say that $x \sim y$ if $\Gamma(x) \setminus \{y\} = \Gamma(y) \setminus \{x\}$. A homogeneous block is a maximal collection $B$ of consecutive vertices in $G$ such that $x \sim y$ for every $x, y \in B$. It is easy to see that $\sim$ is an equivalence relation, and that the homogeneous blocks are subsets of equivalence classes, and thus uniquely determined by $G$.

Now, let $G$ be an ordered graph with homogeneous blocks $B_1, \ldots, B_k$, where $B_1 < \cdots < B_k$. Define $H(G)$ to be the ordered graph with loops, with vertex set $[k]$, in which $ij \in E(H)$ if and only if $u_iu_j \in E(G)$ for every $u_i \in B_i$ and $u_j \in B_j$. Furthermore, let $b_G = (b_1, \ldots, b_k) \in \{1, 2\}^k$ satisfy $b_i = 1$ if $|B_i| = 1$, and $b_i = 2$ otherwise.

**Definition.** The type $T(G)$ of $G$ is defined to be the pair $(H(G), b_G)$.

Let $H(G)$ denote the set of homogeneous blocks of an ordered graph $G$.

**Definition.** For any ordered graph $G$, let

$$m(G) = \sum_{B \in H(G), |B| \geq 3} \left(|B| - 2\right),$$

and call $m(G)$ the excess of $G$.

Note that if two ordered graphs have the same number of vertices and are of the same type, then they have the same excess. We may therefore write $m_n(T)$ for the excess of an ordered graph on $[n]$ of type $T$.

In order to prove the next lemma, we shall also need the concept of shadows and lines within types. First, given a collection of ordered graphs $\mathcal{G}$ on $[n]$, and a type $T$, let

$$\mathcal{G}_T = \{G \in \mathcal{G} : T(G) = T\},$$

and let $\phi$ denote the obvious map which takes $\mathcal{G}_T$ to a subset of $\mathbb{Z}^{d_T}(m_n(T))$, where $d_T$ is the number of homogeneous blocks of size at least two in an ordered graph of type $T$. To spell it out, if the homogeneous blocks in $G$ of size at least two are $B_1, \ldots, B_k$, and $B_1 < \cdots < B_k$, then $\phi(G) = ([B_1] - 2, \ldots, [B_k] - 2)$. A line in $\mathcal{G}_T$ is a set of ordered graphs $S \subset \mathcal{G}_T$ such that $\phi(S) \subset \mathbb{Z}^{d_T}(m_n(T))$ is a line in $\mathbb{Z}^{d_T}(m_n(T))$, as defined in Section 3.

Finally, we define the operation $\partial_T$, which we call taking the shadow within types, as follows. Given an ordered graph $G$, let

$$\partial_T G = \{H \in \partial G : T(H) = T(G)\},$$

and for any collection $\mathcal{G}$ of ordered graphs,

$$\partial_T \mathcal{G} = \bigcup_{G \in \mathcal{G}} \partial_T G.$$
We can now deduce the following lemma from Lemma 3.

**Lemma 4.** Let \( n \in \mathbb{N} \), and let \( \mathcal{G} \) be a collection of ordered graphs on \([n]\). Either \(|\partial \mathcal{G}| \geq |\mathcal{G}|\), or there exists a type \( T \) such that \( |\partial_T \mathcal{G}_T| < |\mathcal{G}_T|\), and hence, such that either \( |\mathcal{G}_T| \geq 2m_n(T)\) or \( \mathcal{G}_T \) contains a line.

**Proof.** Since \( \partial_T \mathcal{G} \subset \partial \mathcal{G} \), either \(|\partial \mathcal{G}| \geq |\mathcal{G}|\) or there exists a type \( T \) such that \( |\partial_T \mathcal{G}_T| < |\mathcal{G}_T|\). Applying Lemma 3 to the set \( \phi(\mathcal{G}_T) \subset Z^{dr}(m_n(T)) \), we deduce that either \( |\mathcal{G}_T| = |\phi(\mathcal{G}_T)| \geq 2m_n(T)\), or \( \phi(\mathcal{G}_T) \) contains a line, as required.

We can now deduce Theorem 1 when there are not too many homogeneous blocks. First we make the following observation.

**Observation 5.** Let \( G \) be an ordered graph, let \( B \subset V(G) \) be a homogeneous block of size at most 2, and let \( v \in B \). Then \( T(G-v) \neq T(G)\).

**Proof.** Removing a vertex cannot increase the number of homogeneous blocks; it can only remove them, or cause them to merge. Thus, if \( T(G-v) = T(G) \), then removing \( v \) did neither of these. So \( |B| = 2 \), but now \( G-v \) has a singleton where \( G \) had a pair, so they have different types, as claimed.

The following lemma, together with Lemma 4, proves Theorem 1 in the case that all ordered graphs in \( \mathcal{G} \) have excess at least \( n/2 \).

**Lemma 6.** Let \( n \in \mathbb{N} \), let \( \mathcal{G} \) be a collection of ordered graphs on \([n]\), and let \( T \) be a type. If \( \mathcal{G}_T \) contains a line \( L \) then \( |\partial \mathcal{G}_T| \geq \min \{2m_n(T) + 1, |\mathcal{G}_T|\}\).

**Proof.** Let \( G \) be an arbitrary ordered graph in the line \( L \), and observe that, since \( G \) is in a line, \( G \) has at most two homogeneous blocks with three or more elements. We shall refer to these ‘large’ blocks as \( A \) and \( B \), and we shall let \( |A| = a+2 \) and \( |B| = b+2 \), where \( a, b \geq 0 \), and \( a + b = m = m_n(T) \).

Suppose first that there exists a vertex \( v \in V(G) \setminus (A \cup B) \) such that \( A \) and \( B \) do not merge in \( G-v \). We claim that \( |\partial \mathcal{G}_T| \geq 2m \). Indeed, \( \partial \mathcal{G}_T \) contains \( m \) ordered graphs of type \( T(G) \) (remove a vertex from \( A \cup B \)), and it also contains \( m + 1 \) ordered graphs of type \( T(G-v) \) (remove \( v \) from some ordered graph in the line). By Observation 5, these are all distinct.

So assume from now on that no such vertex \( v \) exists. The rest of the proof is a rather tedious case analysis to show that \( |\partial \mathcal{G}_T| \geq |\mathcal{G}_T|\). First observe that if \( G[A \cup B] \) is neither complete nor empty then \( V(G) = A \cup B \), since \( A \) and \( B \) cannot merge. Using symmetry, there are only two cases: \( E(G) = K[A] \cup K[B] \) and \( E(G) = K[A] \). It is easy to check that in either case \( |\partial \mathcal{G}| \geq |\mathcal{G}_T|\).

Next suppose that \( G[A \cup B] \) is either complete or empty. Thus, either there exists a homogeneous block \( C \) with \( A < C < B \), or there exists a vertex \( u \in V(G) \) such that \( \Gamma(u) \cap (A \cup B) \in \{A, B\} \). In the latter case, note that \( A \) and \( B \) do not merge in \( G-v \) for any vertex \( v \in V(G) \setminus (A \cup B \cup \{u\}) \), so \( V(G) = A \cup B \cup \{u\} \). Using symmetry, we may assume that \( e(G) = |A| \in \{0, \ldots, n-1\} \), and hence deduce that \( |\partial \mathcal{G}| \geq |\mathcal{G}_T|\).
Thus we may assume, using symmetry and without loss of generality, that \( G[A \cup B] \) is empty, that there exists a homogeneous block \( C \) with \( A < C < B \), and that \( \Gamma(u) \cap (A \cup B) \) for every \( u \in V(G) \). If \( V(G) = A \cup B \cup C \), then either \( |C| = 1 \) and \( \Gamma(u) = A \cup B \), where \( u \in C \), or \( |C| = 2 \) and \( E(G) = \{w(w + 1)\} \), where \( w = |A| + 1 \). In either case it is easy to check that \( |\partial G| \geq |G_T| \).

So, finally, assume that \( v \in V(G) \setminus (A \cup B \cup C) \). Since \( A \) and \( B \) merge when \( v \) is removed, it follows that \( G[A \cup B \cup C] \) is empty. But \( C \) is a homogeneous block, distinct from \( A \) and \( B \), so there must be a vertex \( w \in V(G) \setminus (A \cup B \cup C) \) which distinguishes \( B \) and \( C \). Now, if \( |C| = 2 \) then removing one of the vertices of \( C \) does not cause \( A \) and \( B \) to merge, so \( C = \{u\} \), say. Moreover, \( V(G) = A \cup B \cup C \cup \{w\} \), since any other vertex could be removed without causing \( A \) and \( B \) to merge. Thus either \( E(G) = \{uw\} \) or \( E(G) = \{wz : z \in A \cup B\} \), and again we have \( |\partial G| \geq |G_T| \), as required. \( \Box \)

5. Ordered graphs with small excess

In this section we shall prove the following pair of lemmas, which deal with the case in which \( G \) contains graphs of small excess. The first tells us that several graphs of small excess have a large joint shadow.

**Lemma 7.** Let \( m, n, t \in \mathbb{N} \), and let \( \mathcal{G} = \{G_1, \ldots, G_t\} \), where the \( G_i \) are distinct ordered graphs on \([n]\). Suppose that \( m(G_i) \leq m \) for each \( i \in [t] \). Then

\[
|\partial \mathcal{G}| \geq \frac{t(n-m)^2}{2(n-m)+32t}.
\]

We remark that, for small values of \( t \), there exist families \( \mathcal{G} \) with \( |\partial \mathcal{G}| \leq \frac{t(n-m)}{2} \), so the lemma is close to being best possible. The next lemma tells us that if \( m(G) \) is small, then the shadow of \( G \) also contains a large number of ordered graphs with small excess.

**Lemma 8.** Let \( G \) be an ordered graph and let \( r \in \mathbb{N} \). Then

\[
|\{H \in \partial G : m(H) \leq m(G) + 2r + 1\}| \geq \frac{1}{2} \left(1 - \frac{1}{r}\right)n - \frac{m(G)}{2}.
\]

We begin with a lemma about homogeneous blocks. This simple statement drives the entire proof.

**Lemma 9.** Let \( G \) be an ordered graph and let \( a, b, c \in V(G) \), with \( a < b < c \). If \( G - a = G - b = G - c \) then \([a, c]\) is a homogeneous block.

We prove Lemma 9 using the following simpler statement. A *semi-homogeneous block* in an ordered graph is a collection \( B \) of consecutive vertices such that, for some set \( L \subseteq \mathbb{N} \) and every \( x, y \in B \), \( \Gamma(x) \setminus B = \Gamma(y) \setminus B \), and \( xy \in E(G) \) if and only if \( |x - y| \in L \).

**Lemma 10.** Let \( G \) be an ordered graph and let \( a, b \in V(G) \), with \( a < b \). If \( G - a = G - b \), then \([a, b]\) is a semi-homogeneous block in \( G \).
Proof. Let \( V(G) = [n] \), and define an equivalence relation \( \sim \) on the edges of \( K_n \) as follows: \( e \sim f \) if \( G - a = G - b \) implies that either both or neither of \( e \) and \( f \) are in \( G \). We must show that various collections of edges are in the same equivalence class.

**Claim 1:** If \( e = ij \) and \( f = i(j + 1) \), with \( i < a \leq j < b \), then \( e \sim f \).

To see this, simply observe that
\[
ij \in G \iff ij \in G - b \iff ij \in G - a \iff (i + 1)j \in G.
\]

**Claim 2:** If \( e = ij \) and \( f = (i + 1)j \), with \( a \leq i < b < j \), then \( e \sim f \).

This follows from Case 1 by symmetry, or since
\[
ij \in G \iff i(j - 1) \in G - b \iff i(j - 1) \in G - a \iff (i + 1)j \in G.
\]

It follows from Cases 1 and 2 that \( \Gamma(x) \setminus [a, b] = \Gamma(y) \setminus [a, b] \) for every \( x, y \in [a, b] \).

**Claim 3:** If \( e = ij \) and \( f = (i + 1)(j + 1) \), with \( a \leq i < j < b \), then \( e \sim f \).

This again follows easily, since
\[
ij \in G \iff ij \in G - b \iff ij \in G - a \iff (i + 1)(j + 1) \in G.
\]

Cases 1, 2 and 3 together imply that \([a, b]\) is a semi-homogeneous block, as required. \( \square \)

Lemma 9 now follows almost immediately.

**Proof of Lemma 9** By Lemma 10 \([a, c]\) and \([b, c]\) are semi-homogenous blocks. We shall deduce that \([a, c]\) is a homogeneous block. Indeed, since \([a, c]\) is semi-homogeneous there is a set \( L \subseteq [c - a] \) such that, if \( x, y \in [a, c] \), then \( xy \in E(G) \) if and only if \(|x - y| \in L\). Note that \([a, c]\) is homogeneous if and only if \( L \in \{\emptyset, [c - a]\}\).

Suppose without loss of generality that \( 1 \in L \), and let \( 1 \leq x \leq c - a \) satisfy \( 1, x \in L \) but \( x + 1 \notin L \). Assume that \( x \neq c - a \), and observe that there must exist a vertex \( a \leq j < b \leq j + x < c \) such that \( j(j + x) \in E(G) \). Since \([b, c]\) is a semi-homogenous block it follows that \( j(j + x + 1) \in E(G) \). But \( j \) and \( j + x + 1 \) are in \([a, c]\), so this contradicts the assumption that \( x + 1 \notin L \), and so we are done. \( \square \)

Using Lemma 9 we can now easily prove Lemma 8. The proof is via the following simple lemma. Given an ordered graph \( G \) and \( A \subset V(G) \), define
\[
\partial_{|A|}G = \{ H \in \partial G : H = G - a \text{ for some } a \in A \}.
\]

**Lemma 11.** Let \( G \) be an ordered graph and \( A \subset V(G) \). Then \( |\partial_{|A|}G| \geq \frac{|A| - m(G)}{2} \).

**Proof.** Let \( A' \subset A \) contain at most two elements of each homogeneous block. By the definition of \( m(G) \), we may choose \( A' \) so that \( |A'| = |A| - m(G) \). By Lemma 9 no three elements of \( A' \) give the same ordered graph when they are removed from \( G \). Hence \( |\partial_{|A|}G| \geq |A'|/2 \), as claimed. \( \square \)
Note that this implies that a single ordered graph $G$ has a shadow of size at least $(n - m(G))/2$. Lemma \[\text{13}\] says that most of these ordered graphs have excess not much larger than $m(G)$.

**Proof of Lemma \[\text{13}\]** First suppose that $v \in V(G)$ lies in a homogeneous block $B$ of size at least two. We claim that $m(G - v) \leq m(G) + 3$. Indeed, the only homogeneous blocks which may merge are $B$ and its neighbours, since any other two which were distinguished by $v$ are still distinguished by the elements of $B \setminus \{v\}$. Furthermore, these blocks only merge if $|B| = 2$, and so $|B \setminus \{v\}| = 1$. When three adjacent blocks merge and one of them is a singleton (a homogeneous block of size one), the excess increases by at most three. Hence $m(G - v) \leq m(G) + 3$, as claimed.

So suppose that $v \in V(G)$ is a singleton, and suppose that $m(G - v) \geq m(G) + 2r + 2$. Then the removal of $v$ must cause at least $r$ pairs of adjacent blocks to merge (it could also cause the pair either side of it to merge), since when two blocks merge the excess increases by at most two. But each pair of adjacent blocks can be caused to merge by at most one singleton. There are fewer than $n$ adjacent pairs of blocks; thus, there can be at most $n/r$ singletons such that $m(G - v) \geq m(G) + 2r + 2$.

Letting $A = \{v \in V(G) : m(G - v) \leq m(G) + 2r + 1\}$, it now follows, by Lemma \[\text{11}\] that

$$|\{H \in \partial G : m(H) \leq m(G) + 2r + 1\}| \geq \frac{|A| - m(G)}{2} \geq \frac{1}{2} \left(1 - \frac{1}{r}\right)n - \frac{m(G)}{2},$$

as claimed. \[\square\]

We now turn to the proof of Lemma \[\text{7}\] which is somewhat more complicated. We begin with an easy observation.

**Observation \[\text{12}\].** Let $G$ and $H$ be ordered graphs on $[n]$, and let $a, b \in [n]$, with $a \leq b$. If $G - a = H - b$, then every edge in $G \triangle H$ has an endpoint in $[a, b]$.

**Proof.** Let $i, j \in [n] \setminus [a, b]$, and suppose the edge $ij$ in $G$ corresponds to the edge $f$ in $G - a$. Then the edge $ij$ in $H$ also corresponds to the edge $f$ in $H - b$. Since $G - a = H - b$, it follows that $ij \notin G \triangle H$. \[\square\]

Observation \[\text{12}\] has the following simple, but important, consequence.

**Lemma \[\text{13}\].** Let $G$ and $H$ be ordered graphs on $[n]$, and let $1 \leq a_1 \leq b_1 < a_2 \leq b_2 < a_3 \leq b_3 \leq n$. If $G - a_i = H - b_i$ for each $i \in \{1, 2, 3\}$, then $G = H$.

**Proof.** Suppose $a_1 \leq b_1 < a_2 \leq b_2 < a_3 \leq b_3$. Since $G - a_i = H - b_i$, each edge of $G \triangle H$ has an endpoint in $[a_i, b_i]$, by Observation \[\text{12}\]. But each edge has only two endpoints, so this is impossible, unless $E(G \triangle H) = \emptyset$, in which case $G = H$. \[\square\]

Lemma \[\text{13}\] deals with the case in which the intervals $(a_i, b_i)$ are disjoint; we need a similar result when we have overlapping intervals. The following lemma provides such a result, but turns out to be somewhat harder to prove.
Lemma 14. Let \( G \) and \( H \) be ordered graphs, and let \( a_1, \ldots, a_4 \in V(G) \) and \( b_1, \ldots, b_4 \in V(H) \) be distinct vertices, with \( a_1 < a_2 < a_3 < a_4 \leq b_i \) for each \( i \in [4] \). Suppose that \( G - a_i = H - b_i \) for each \( i \in [4] \). Then \( [a_2, a_3] \) is a homogeneous block in \( G \).

Proof. Say that \( e \sim f \) if \( G - a_i = H - b_i \) for each \( i \in \{1, 2, 3, 4\} \) implies that either both or neither of the edges \( e \) and \( f \) are in \( G \). Clearly \( \sim \) is an equivalence relation. We must show that various collections of edges are in the same equivalence class.

Let \((c_1, \ldots, c_4)\) be the reordering of \((b_1, \ldots, b_4)\) so that \( c_1 < \cdots < c_4 \), and let \( \sigma \) be the permutation of \( \{1, 2, 3, 4\} \) such that \( c_j = b_{\sigma(j)} \) for each \( j \in [4] \). We shall write \( \hat{\sigma} = \sigma^{-1} \).

Claim 1: Let \( e = ij \) and \( f = (i+1)j \), with \( i \in [a_2, a_3) \) and \( j \in [n \setminus ([a_2, a_3) \cup (c_2, c_3)] \). Then \( e \sim f \).

The claim follows from the following, slightly more complicated statement.

Subclaim: Let \( 1 \leq u < v \leq 4 \), \( i \in [a_u, a_v) \) and \( j \in [n \setminus ([a_u, a_v) \cup (\min\{b_u, b_v\}, \max\{b_u, b_v\})] \). Then \( ij \sim (i+1)j \).

Proof of subclaim. Let
\[
\varepsilon_1 = \mathbb{1}_{\{j > a_v\}} = \mathbb{1}_{\{a_v < j \leq \min\{b_u, b_v\}\}} + \mathbb{1}_{\{j > \max\{b_u, b_v\}\}}
\]
and
\[
\varepsilon_2 = \mathbb{1}_{\{a_v < j \leq b_v\}} = \mathbb{1}_{\{a_v < j \leq \min\{b_u, b_v\}\}} = \varepsilon_1 - \mathbb{1}_{\{j > \max\{b_u, b_v\}\}}.
\]

Then
\[
ij \in G \iff i(j-\varepsilon_1) \in G - a_v \iff i(j-\varepsilon_1) \in H - b_v \iff i(j-\varepsilon_2) \in H,
\]
\[
\iff i(j-\varepsilon_1) \in H - b_u \iff i(j-\varepsilon_1) \in G - a_u \iff (i+1)j \in G.
\]

Now, let \( S = \{(1,3), (1,4), (2,3), (2,4)\} \), and observe that the intersection of the sets \( (\min\{b_u, b_v\}, \max\{b_u, b_v\}) \) over all pairs \( (u, v) \in S \) is either \( (c_2, c_3) \) (in the case that either \( b_1, b_2 \leq b_3, b_4 \) or \( b_3, b_4 \leq b_1, b_2 \) ), or empty (otherwise). Hence, if we apply the subclaim to each pair \( (u, v) \in S \) then Claim 1 follows.

Claim 2: If \( e = ij \) and \( f = (i+1)j \), with \( i \in [a_2, a_3) \) and \( j \in (c_2, c_3) \), then \( e \sim f \).

First observe that either \( b_1, b_2 \leq c_2 < c_3 \leq b_3, b_4 \) or \( b_3, b_4 \leq c_2 < c_3 \leq b_1, b_2 \) or we are done by the subclaim. We consider the former two cases separately.

Case 1: \( b_1 < b_4 \).

By the comments above, we may assume that \( \sigma(1), \sigma(2) \in \{1, 2\} \) and \( \sigma(3), \sigma(4) \in \{3, 4\} \).

Thus, for any \( p \in (a_1, a_{\sigma(4)}) \) and \( q \in (c_2, c_4) \), we have
\[
pq \in G \iff (p-1)(q-1) \in G - a_1 \iff (p-1)(q-1) \in H - c_{\sigma(1)} \iff (p-1)q \in H
\]
\[
\iff (p-1)q \in H - c_4 \iff (p-1)q \in G - a_{\sigma(4)} \iff (p-1)(q+1) \in G.
\]
Applying this fact to the edge $ij$, we deduce that $ij \in G \iff i'j' \in G$, where $i' + j' = i + j$, and either $i' = a_2 - 1$ and $j' \in (c_2, c_3 + 1)$, or $j' = c_3 + 1$ and $i' \in [a_2, a_3)$. In the same way, we moreover deduce that $(i + 1)j \in G \iff (i' + 1)j' \in G$.

We claim that $i'j' \in G \iff (i' + 1)j' \in G$. If $i' = a_2 - 1$ and $j' \in (c_2, c_3 + 1)$ then this follows by the subclaim with $(u, v) = (1, 2)$, since $b_1, b_2 \leq c_2$. If $j' = c_3 + 1$ and $i' \in [a_2, a_3)$, then it follows by Claim 1. Hence

$$ij \in G \iff i'j' \in G \iff (i' + 1)j' \in G \iff (i + 1)j \in G,$$

and so $ij \in G \iff (i + 1)j \in G$, as claimed.

Case 2: $b_1 < b_2$.

This time we may assume that $\sigma(1), \sigma(2) \in \{3, 4\}$ and $\sigma(3), \sigma(4) \in \{1, 2\}$. Thus, for any $p \in [a_1, a_\sigma(1)]$ and $q \in (c_1, c_3)$, we have

$$pq \in G \iff p(q - 1) \in G - a_\sigma(1) \iff p(q - 1) \in H - c_1 \iff pq \in H$$

$$\iff pq \in H - c_{\sigma(1)} \iff pq \in G - a_1 \iff (p + 1)(q + 1) \in G.$$ Applying this to the edge $ij$, we deduce that $ij \in G \iff i'j' \in G$, where $j' - i' = j - i$, and either $i' = a_2 - 1$ and $j' \in [c_2, c_3)$, or $j' = c_2$ and $i' \in [a_2, a_3)$. In the same way, we deduce that $(i + 1)j \in G \iff (i' + 1)j' \in G$.

We claim that $i'j' \in G \iff (i' + 1)j' \in G$. If $i' = a_2 - 1$ and $j' \in [c_2, c_3)$, this follows by the subclaim with $(u, v) = (1, 2)$, since $b_1, b_2 \geq c_3$. If $j' = c_2$ and $i' \in [a_2, a_3)$ then it follows by Claim 1. Hence

$$ij \in G \iff i'j' \in G \iff (i' + 1)j' \in G \iff (i + 1)j \in G,$$

as claimed.

**Claim 3:** Let $e = ij$, $f = i(j + 1)$ and $g = (i + 1)(j + 1)$, with $a_2 \leq i < j < a_3$. Then $e \sim f \sim g$.

Since $G - a_1 = H - b_1$ and $G - a_4 = H - b_4$, for any $a_1 \leq p < q < a_4$ we have

$$pq \in G \iff pq \in G - a_4 \iff pq \in H - b_4 \iff pq \in H$$

$$\iff pq \in H - b_1 \iff pq \in G - a_1 \iff (p + 1)(q + 1) \in G.$$ Thus $ij \in G$ if and only if $i'j' \in G$, where $i' = a_2 - 1$, and $j' - i' = j - i$, and also $i(j + 1) \in G$ if and only if $i'(j' + 1) \in G$. But $j' \in [a_2, a_3)$, so by Claim 1, $i'j' \in G$ if and only if $i'(j' + 1) \in G$. Hence

$$ij \in G \iff i'j' \in G \iff i'(j' + 1) \in G \iff i(j + 1) \in G$$

as claimed.

Cases 1, 2 and 3 together imply that $[a_2, a_3]$ is a homogeneous block, as required. $\square$

We need one more simple observation.
Lemma 15. Let \( G \) be a graph on \( n \) vertices whose components are all cliques. Then \( G \) has at least \( \frac{n^2}{n + 2e(G)} \) components.

Proof. Let us fix the number of components. The expression \( \frac{n^2}{n + 2e(G)} \) is maximized when the number of edges is minimized, i.e., when the cliques all have (roughly) the same size. Thus, if the average size of a component is \( k \), then

\[
e(G) \geq \left( \frac{k}{2} \right) \frac{n}{k} = \frac{n(k-1)}{2}.
\]

Rearranging the above expression gives the required result. \( \square \)

We now can reap our reward: the proof of Lemma 7.

Proof of Lemma 7. Let \( m, n, t \in \mathbb{N} \), and let \( \mathcal{G} = \{G_1, \ldots, G_t\} \) be a collection of distinct ordered graphs on \([n]\), with \( m(G_i) \leq m \) for each \( i \in [t] \). We are required to show that

\[
|\partial \mathcal{G}| \geq t(n-m)^2 \geq 2(n-m) + 32t.
\]

First, for each ordered graph \( G_i \), choose a set \( X_i \subset V(G_i) \), with \( |X_i| = m(G_i) \), such that \( G_i - X_i \) only has homogeneous blocks of size at most two. This is possible because if \( B \) is a homogeneous block in \( G \) with \( |B| \geq 3 \) and \( v \in B \), then \( B - v \) is a homogeneous block in \( G - v \). The set \( X_i \) represents the excess of \( G_i \).

Now, for each pair \( \{i, j\} \subset [t] \), let

\[
P(i, j) = \{(u, v) \in (V(G_i) \setminus X_i, V(G_j) \setminus X_j) : G_i - u = G_j - v\}.
\]

The result follows from the following claim.

Claim: \(|P(i, j)| \leq 32\) for each \( i, j \in [t] \) with \( i \neq j \).

Proof of claim. Let \( i, j \in [t] \), and let \( H \) be the bipartite graph with vertex set \( V(G_i) \cup V(G_j) \) and edge set \( P(i, j) \). We begin with a simple but crucial observation.

Subclaim: \( d_H(u) \leq 2 \) for every \( u \in V(H) \).

Suppose \((u, v_1), (u, v_2), (u, v_3) \in P(i, j)\). Then \( G_j - v_1 = G_j - v_2 = G_j - v_3 \). Therefore, by Lemma 9 the set \( \{v_1, v_2, v_3\} \) is contained in some homogeneous block of size at most two. By the definition of \( X_i, G_j - X_j \) only has homogeneous blocks of size at most two. Thus \( d_H(u) \leq 2 \) for every \( u \in V(H) \), as claimed.

Now, suppose \(|P(i, j)| \geq 33\). By the subclaim the components of \( H \) are paths and cycles, so there exists a matching in \( H \) consisting of at least half of the edges of \( H \), i.e., on at least 17 edges. Without loss of generality, at least nine of these edges \((a_k, b_k)\) (where \( a_k \in G_i \) and \( b_k \in G_j \)) have \( a_k \leq b_k \).

Consider the poset formed by these nine intervals \([a_k, b_k]\) in the interval order. A chain of height three in the poset corresponds to three disjoint intervals, and by Lemma 13 if
three such intervals exist then \( G_i = G_j \), which is a contradiction. Thus the poset has height at most two, and so it has width at least five.

Let \([a_1, b_1], \ldots, [a_5, b_5]\) be five non-comparable intervals, with \( a_1 < \cdots < a_5 \) say, such that \( G_i - a_k = G_j - b_k \) for each \( k \in [5] \). Since the intervals are incomparable, they have a common point, so \( a_k \leq b_k \) for each \( k, \ell \in [5] \). Now, by Lemma 14 the set \( \{a_2, a_3, a_4\} \) lies in a homogeneous block. But all homogeneous blocks have size at most two, so this is a contradiction, and so \(|P(i, j)| \leq 32\), as claimed.

It is now easy to deduce that \(|\partial G| \geq (n - m)t/2 - O(t^2)\). However, we shall work a little to improve the error term. Consider the graph \( J \) with vertex set \( \bigcup_i V(G_i) \setminus X_i \), and edge set

\[
E(J) = \bigcup_{i \neq j} P(i, j) \cup \bigcup_i \{uv : u, v \in V(G_i) \setminus X_i, G_i - u = G_i - v\}.
\]

Note that the components of \( J \) are all cliques, and that \(|J| \geq t(n - m)\), by the definition of \( X_i \). Moreover, we have \( e(J) \leq 32\binom{t}{2} + |J|/2\), since by the claim there are at most \( 32\binom{t}{2} \) ‘cross-edges’, and by Lemma 9 and the definition of \( X_i \), the set \( V(G_i) \setminus X_i \) induces a matching for each \( i \in [t] \).

Thus, applying Lemma 15 to the graph \( J \), we deduce that \( J \) has at least

\[
\frac{|J|^2}{2|J| + 32t^2} = \frac{t(n - m)^2}{2(n - m) + 32t}
\]

components. Since each component corresponds to a distinct ordered graph in the shadow, this is a lower bound for \(|\partial G|\).

**Question 1.** *What is the best possible lower bound in Lemma 7? In particular, is it true that when \( t = 2 \) and \( m = 0 \), then \(|\partial G| \geq n - 1\)?*

### 6. Proof of Theorem 1

In this section we shall put together the pieces and prove Theorem 1 and Theorem 2. The proof involves some simple calculations, which we collect in the following observation.

**Observation 16.** If \( 2 \leq m \leq n/2 \) and \( n \geq 136 \), then

\[
\frac{(m + 1)(n - m)^2}{2n + 30m + 32} \geq n - 1.
\]

If \( t \geq 3 \) and \( n \geq 4m + 94 \) then

\[
\frac{t(n - m)^2}{2(n - m) + 32t} \geq n - 1.
\]

**Proof.** For the first part, simple calculus shows that, for fixed \( n \), the minimum of the left hand side occurs at one of the extreme points, \( m = 2 \) and \( m = n/2 \). When \( m = 2 \) the inequality reduces to \( n^2 - 102n + 104 \geq 0 \), and for \( m = n/2 \) it is implied by \( n^2 - 134n - 256 \geq 0 \).
For the second part, simple manipulation gives a quadratic in $n$ of the form $an^2 - bn + c > 0$, with $a, b, c \geq 0$. The result now follows from the sufficient condition $n \geq b/a$, and the assumption $t \geq 3$. \hfill \Box

We can now prove Theorem 1.

Proof of Theorem 1. Let $136 \leq n \in \mathbb{N}$, let $G$ be a collection of ordered graphs on $[n]$, and suppose that $|G| \leq n - 1$. We are required to show that $|\partial G| \geq |G|$.

First, by Lemma 4, either $|\partial G| \geq |G|$, or there exists a type $T$ such that $|\partial_c G_T| < |G_T|$, and hence either $|G_T| \geq 2m_n(T)$ or $G_T$ contains a line. Among types such that $|\partial_c G_T| < |G_T|$, choose $T$ with $m_n(T)$ maximal.

Suppose first that $m_n(T) \geq n/2$, and so $|G_T| \leq |G| \leq n - 1 < 2m_n(T)$. Then $G_T$ must contain a line, and so

$$|\partial G_T| \geq \min \{2m(T) + 1, |G_T|\} = |G_T|,$$

by Lemma 6 which contradicts our choice of $T$. Thus we may assume that $m := m_n(T) < n/2$.

Recall that a line in $G_T$ contains $m_n(T) + 1$ ordered graphs, so, in either case, $G_T$ contains at least this many distinct ordered graphs. By Lemma 7 it follows that

$$|\partial G| \geq \frac{(m+1)(n-m)^2}{2n+30m+32}.$$

Thus, if $2 \leq m \leq n/2$ and $n \geq 136$, then $|\partial G| \geq n - 1 \geq |G|$, by Observation 16.

Therefore we may assume that $m_n(T) \leq 1$, and hence that $|\partial_c G_T| \geq |G_T|$ for every type $T'$ with $m_n(T') \geq 2$. But, by Lemma 8 applied with $r = 2$, if there exists $G \in G$ with $m(G) \leq 1$, then $\partial G$ contains at least $(n-2)/4$ ordered graphs with excess at most 6. But if $G$ contains at least four ordered graphs with excess at most 6, then

$$|\partial G| \geq \frac{4(n-6)^2}{2(n-6) + 128} \geq n - 1,$$

since $n \geq 118$, by Lemma 7 and Observation 16. Thus, we may assume that all but at most 3 ordered graphs in $G$ have excess at least 7, so

$$|\partial G| \geq \left( \sum_{T : m_n(T) \geq 7} |\partial_c G_T| \right) + \frac{n-2}{4} \geq |G| - 3 + \frac{n-2}{4} \geq |G|,$$

and we are done. \hfill \Box

Finally, we deduce Theorem 2 from Theorem 1.

Proof of Theorem 2. Let $P$ be a hereditary property of ordered graphs, let $136 \leq k \in \mathbb{N}$, and suppose that $|P_k| < k$. We are required to prove that $|P_{k+1}| \leq |P_k|$.

Indeed, suppose that $|P_{k+1}| > |P_k|$. Then $P_{k+1}$ contains a collection of ordered graphs $G_{k+1}$ with $|G_{k+1}| = |P_k| + 1 < k + 1$. Applying Theorem 1 to $G_{k+1}$, we obtain

$$|P_k| \geq |\partial P_{k+1}| \geq |\partial G_{k+1}| \geq |G_{k+1}| = |P_k| + 1,$$

Thus, $|P_{k+1}| \leq |P_k|$. \hfill \Box
which is a contradiction. □

7. Extensions to higher speeds

Theorem 1 is only one step towards understanding shadows of collections of ordered graphs, and we expect that corresponding results should hold for larger families / for hereditary properties with higher speeds. The following questions and conjectures make this explicit.

By the results of [3], there exists a function \( f : \mathbb{N} \to \mathbb{Z} \) such that the following holds for every \( k \in \mathbb{N} \). If \( P \) is a hereditary property of ordered graphs, and

\[
\limsup_{n \to \infty} \left( |P_n| - (k - 1)n \right) = \infty,
\]

then \( |P_n| \geq kn - f(k) \) for every sufficiently large \( n \in \mathbb{N} \). Let \( f(k) \) be chosen to be minimal, subject to this condition. We remark that \( f(1) = 0 \), and that \( f(k) \geq (k - 1)(k + 4)/2 \). To see the latter, consider the smallest hereditary property containing all the ordered graphs with edge set \( E(G) = \{(i(i + 1), j(j + 1)) \mid i \leq k - 1 \text{ and } i + 1 < j \} \). It is likely, in fact, that this bound is optimal.

**Conjecture 1.** Let \( n, k \in \mathbb{N} \), and let \( f(k) \in \mathbb{Z} \) be as described above. Let \( G \) be a collection of ordered graphs on \([n]\), and suppose that \(|G| < kn - f(k)|. Then| \( |\partial G| \geq |G| - k + 1 \).

Note that Theorem 1 is exactly Conjecture 1 in the case \( k = 1 \) and \( n \geq 136 \), and so Conjecture 1 includes the extension of Theorem 1 to all \( n \in \mathbb{N} \). It is conceivable that the techniques of this paper could be extended in order to prove the conjecture for all \( k \in \mathbb{N} \) (and sufficiently large \( n \)), although one would require a more general version of Lemma 3. The following problem, on the other hand, is likely to require further new ideas.

For each \( k \in \mathbb{N} \), let \( h(k) \) denote the smallest possible quadratic speed of a hereditary property of ordered graphs, \( \mathcal{P} \), i.e., the largest integer such that \(|\mathcal{P}_n| = \Theta(n^2)| implies \(|\mathcal{P}_k| \geq h(k)\).

**Question 2.** Let \( 3 \leq n \in \mathbb{N} \), and let \( h(n) \) be as described above. Let \( G \) be a collection of ordered graphs on \([n]\), and suppose \(|G| < h(n)|. Is it true that \(|\partial G| \geq |G| - n + 3 |?\)

**Conjecture 2.** Let \( \mathcal{P} \) be a hereditary property of ordered graphs, and let \( k \in \mathbb{N} \). If \(|\mathcal{P}_n| = \Theta(n^k)|, then

\[
|\mathcal{P}_n| \geq \sum_{i=0}^{k} \binom{n-i}{i}
\]

for every \( n \in \mathbb{N} \).
We remark that Conjecture 2 holds in the case $k = 1$ (by the results of [3]), and that if Conjecture 2 is true, then the bound in Question 2 (if true) is best possible. To see this, consider the collection $Q_2$ minus the empty ordered graph: it has $\binom{n-2}{2} + n - 1$ elements, and there are $\binom{n-3}{2} + n - 1$ ordered graphs in its shadow. We suspect that Question 2 is much harder than Conjecture 2.

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Trinity College, Cambridge CB2 1TQ, England, and Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152
E-mail address: B.Bollobas@dpmms.cam.ac.uk

Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, England
E-mail address: graham@tutte.lse.ac.uk

Murray Edwards College, The University of Cambridge, Cambridge CB3 0DF, England
(Work partly done whilst at the Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, Brazil)
E-mail address: rdm30@cam.ac.uk