Reproducing kernel method for a class of weakly singular Fredholm integral equations

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1. Introduction

Some important applications of weakly singular Fredholm integral equations (WSFIEs) in the fields of fracture mechanics, the theory of porous filtering, elastic contact problems, combined infrared radiation and molecular conduction were provided in [1]. In recent years, several numerical methods have been done in order to find the solution of singular integral and integro-differential equations. For instance, Lifanov et al. [2] introduced hypersingular integral equations with their applications and then introduced some new numerical algorithms for solving them. In [3], the discrete Galerkin method has been proposed and analysed for obtaining the numerical solution of these equations. The trapezoidal method was applied for approximating singular or nonsingular integral equations [4–7]. In [8], a generalization of the Euler–Maclaurin summation formula for solving WSFIEs of the second kind was introduced. One can refer to the methods that were proposed in [9–12].

Recently, the reproducing kernel method to solve a variety of singular or nonsingular integral equations was presented. Alvandi and Paripour solved nonlinear Abel’s integral equations with weakly singular kernel in the reproducing kernel space and removed the singularity of the equation considered [13]. The same authors presented a simple and efficient method to solve linear Volterra integro-differential equations [14], nonlinear Volterra–Fredholm integro-differential equations [15], and see [16–21].

In this paper, a reproducing kernel method is used for the solution of WSFIEs.

Consider the following (WSFIE):

\[ u(t) = f(t) + \int_{0}^{1} k(x, t)u(x) \, dx, \quad 0 \leq t \leq 1, \quad (1) \]

where the singularity of kernel may be stated in the forms \( k(x, t) = 1/(x - t)\alpha \) with the assumption \( 0 < \alpha < 1 \).

This paper is organized into five sections including the introduction. In the next section, two different reproducing kernel spaces are presented in order to construct reproducing kernel functions in the space \( W^m[0, 1] \). The representation of solutions for WSFIEs is obtained in Section 3. The numerical experiments are given in Section 4. Section 5 ends this paper with a brief conclusion.

2. Preliminaries

Let us present the definitions of reproducing kernel Hilbert spaces.

2.1. The reproducing kernel space \( W^m[0, 1] \)

The inner product space \( W^m[0, 1] \) is defined as follows:

Definition 2.1: \( W^m[0, 1] = \{ u(x) | u^{(m-1)}(x) \text{ is an absolutely continuous real value function}, u^{(m)}(x) \in L^2[0, 1] \} \).

The inner product and norm in \( W^m[0, 1] \) are given, respectively, by

\[ (u, v) = \sum_{j=0}^{m-1} u^{(j)}(0)v^{(j)}(0) + \int_{0}^{1} u^{(m)}(x)v^{(m)}(x) \, dx, \quad (2) \]
and
\[ \|u\|_m = \sqrt{\langle u, u \rangle_m}, \quad u, v \in W^m[0, 1]. \tag{3} \]

It is easy to see that \( \langle u, v \rangle_{W^m[0, 1]} \) satisfies all the requirements for the inner product. In \([22]\), Cui and Lin proved that \( W^m[0, 1] \) is a reproducing kernel Hilbert space. That is, for every \( x \in [0, 1], y \in [0, 1] \) and \( u(y) \in W^m[0, 1] \), there exists \( R_y(x) \in W^m[0, 1] \) such that \( \langle u(y), R_y(x) \rangle = u(x) \) and \( R_y(y) \) is called the reproducing kernel function of space \( W^m[0, 1] \).

The reproducing kernel \( R_y(x) \) can be denoted by
\[
R_y(x) = \begin{cases} 
R_1(x, y) = \sum_{i=1}^{2m} c_i(y)x^{i-1}, & y \leq x, \\
R_2(x, y) = \sum_{i=1}^{2m} d_i(y)x^{i-1}, & y > x,
\end{cases} \tag{4}
\]
where coefficients \( c_i(y), d_i(y), \{i = 1, 2, \ldots, 2m\} \), could be obtained by solving the following equations:
\[
\frac{\partial^2 R_y(x)}{\partial x^2} |_{x=y+0} = \frac{\partial^2 R_y(x)}{\partial x^2} |_{x=y-0}, \quad i = 0, 1, 2, \ldots, 2m - 2, \tag{5}
\]
\[
(-1)^m \left( \frac{\partial^{2m-1} R_y(x)}{\partial x^{2m-1}} |_{x=y+0} - \frac{\partial^{2m-1} R_y(x)}{\partial x^{2m-1}} |_{x=y-0} \right) = 1, \tag{6}
\]
\[
\frac{\partial^i R_y(0)}{\partial x^i} - (-1)^{m-i} \frac{\partial^{2m-i} R_y(0)}{\partial x^{2m-i}} = 0, \quad i = 0, 1, \ldots, m - 1, \tag{7}
\]
\[
\frac{\partial^{2m-i} R_y(1)}{\partial x^{2m-i}} = 0, \quad i = 0, 1, \ldots, m - 1.
\]

For computing the approximate solutions of Equation (1), reproducing kernel functions are obtained in \( W^0[0, 1] \) and \( W^2[0, 1] \). Thus, the representation of reproducing kernel functions \( R_1(x, y) \in W^0[0, 1] \) and \( R_2(x, y) \in W^2[0, 1] \) are given by
\[
R_1(x, y) = \frac{1}{5040} (x(-x^6 + 5040y + 7x^2y + 1260xy^2 - 21x^4y^2 + 140x^3y^3 + 35x^3y^2))
\]
and
\[
R_2(x, y) = \frac{1}{362880} (362880 + y^9 + 126x^3y^2(5 + y) - 84x^3y^3(-120 + y^3) + 36x^2y^2(2520 + y^5) - 9xy(-40320 + y^7)).
\]

2.2. Definition of operators

To deal with the system, we consider \( L : W^m[0, 1] \rightarrow L^2[0, 1] \) as
\[
L(u) = u(t) - \int_0^1 \frac{u(x)}{(x-t)^\alpha} \, dx, \quad 0 \leq t \leq 1, \quad 0 < \alpha < 1. \tag{8}
\]

Then Equation (1) can be written as
\[
L(u) = f(t). \tag{9}
\]

We can assume that \( L \) is an invertible bounded linear operator, where \( L^* \) is the adjoint operator of \( L \).

3. The exact and approximate solution of Equation (I)

We choose a countable dense subset \( \{x_i\}_{i=1}^\infty \) in \([0, 1]\) and define \( \psi_i(x) = R(x, x_i) \),
\[
\psi_i(x) = L^* \psi_i(x) = L^* R_y(x_i) = \langle L_y R_y(x_i) \rangle(x_i), \quad i = 1, 2, \ldots; \tag{10}
\]

hence, we have
\[
\psi_i(x) = R(x, x_i) - \int_0^1 \frac{R(x, t)}{\sqrt{1-t}} \, dt, \quad 0 \leq x \leq 1. \tag{11}
\]

The orthonormal system \( \{\tilde{\psi}_i(x)\}_{i=1}^\infty \) of \( W^m[0, 1] \) can be derived from the Gram–Schmidt orthogonalization process of \( \{\psi_i(x)\}_{i=1}^\infty \),
\[
\tilde{\psi}_i(x) = \sum_{k=1}^i \beta_{ik} \psi_k(x), \quad (\beta_{ii} > 0, \ i = 1, 2, \ldots), \tag{12}
\]

where \( \beta_{ik} \) are orthogonal coefficients.

Lemma 3.1: For Equation (1), if \( \{x_i\}_{i=1}^\infty \) is dense on \([0, 1]\), then \( \{\tilde{\psi}_i(x)\}_{i=1}^\infty \) is the complete system of \( W^m[0, 1] \).

Proof: The subscript \( y \) by the operator \( L \) indicates that the operator \( L \) applies to the function of \( y \). So, we have
\[
\psi_i(x) = L^* \psi_i(x) = \langle L^* \psi_i(y), R_y(x_i) \rangle = \langle \psi_i(y), L^* R_y(x_i) \rangle = \langle \psi_i(y), \psi_i(x_i) \rangle = \psi_i(x_i),
\]

It is clear that \( \psi_i(x) \in W^m[0, 1] \). For each fixed \( u(x) \in W^m[0, 1] \), \( \langle u(x), \psi_i(x) \rangle = 0 \) \( (i = 1, 2, \ldots) \), which means that
\[
\langle u(x), L^* \psi_i(x) \rangle = \langle L(u(x)), \psi_i(x) \rangle = L(u(x)) = 0.
\]

Note that \( \{x_i\}_{i=1}^\infty \) is dense on \([0, 1]\); therefore, \( L(u(x)) = 0 \). It follows that \( u(x) = 0 \) from the existence of \( L^{-1} \). So, the proof of the theorem is complete.
Theorem 3.1: If \( u(x) \) is the exact solution of Equation (1), then
\[
u(x) = \sum_{j=1}^{\infty} \sum_{k=1}^{i} \beta_{jk} f(x_k) \bar{\psi}_i(x).
\]
where \([x_i]\) is dense set in \([0, 1]\).

Proof: Note that \((v(x), \phi_k(x)) = v(x_k)\) for each \( v(x) \in W^m[0, 1]\), and \(u(x)\) can be expanded to Fourier series in terms of normal orthogonal basis \(\bar{\psi}_i(x)\) in \(W^m[0, 1]\),
\[
u(x) = \sum_{j=1}^{\infty} (u(x), \bar{\psi}_i(x)) \bar{\psi}_i(x)
\]
\[
= \sum_{j=1}^{\infty} \sum_{k=1}^{i} \beta_{jk} (u(x), \psi_k(x)) \bar{\psi}_i(x)
\]
\[
= \sum_{j=1}^{\infty} \sum_{k=1}^{i} \beta_{jk} (u(x), \psi_k(x)) \bar{\psi}_i(x)
\]
\[
= \sum_{j=1}^{\infty} \sum_{k=1}^{i} \beta_{jk} (f(x), \phi_k(x)) \bar{\psi}_i(x)
\]
\[
= \sum_{j=1}^{\infty} \sum_{k=1}^{i} \beta_{jk} f(x_k) \bar{\psi}_i(x).
\]
(14)

So, the proof is complete.

Now, the approximate solution of Equation (1) can be obtained by the \(n\)-term intercept of Equation (13) and
\[
u_n(x) = \sum_{j=1}^{\infty} \sum_{k=1}^{i} \beta_{jk} f(x_k) \bar{\psi}_i(x).
\]
(15)

Lemma 3.2: If \( u(x) \in W^m[0, 1]\), then there exists a constant \( c \) such that \( |u(x)| \leq c \| u(x) \|_m \). 

Proof: 
\[ |u(x)| = |(u(y), R_x(y))| \leq \| u(y) \| \| R_x(y) \|_m, \]
there exists a constant \( c > 0 \) such that
\[ |u(x)| \leq c \| u \|_m. \]

The proof of the lemma is complete.

Theorem 3.2: Assume that \( \| u_n(x) \| \) is bounded and Equation (9) has a unique solution. If \([x_i]\) is dense in the interval \([0, 1]\), then \(n\)-term approximate solution \(u_n(x)\) converges to the exact solution \(u(x)\) of Equation (9) and the exact solution is expressed as
\[
u(x) = \sum_{j=1}^{\infty} B_j \bar{\psi}_j(x),
\]
(16)

where \(B_j = \sum_{k=1}^{i} \beta_{jk} f(x_k)\).

Proof: First, we will prove \( ||u_n(x)|| \) convergent. we infer that
\[
u_n(x) = \nu_{n-1}(x) + B_n \bar{\psi}_n(x).
\]
(17)

From the orthogonality of \(\{\bar{\psi}_i(x)\}\), it follows that
\[
u_n(x) = \nu_{n-1}(x) + ||\beta_n||^2.
\]
(18)

By eliminating \(||\beta_n||^2\), it is clear that \(||u_n||_{W^m} \geq ||u_{n-1}||_{W^m} \).
Due to the condition that \(||u_n||_{W^m} \) is bounded and descending, \(||u_n||_{W^m} \) converges as soon as \(n \rightarrow \infty\). Then there exists a constant \( c \) such that
\[
\sum_{j=1}^{\infty} B_j^2 = c.
\]
(19)

If \(m > n\), in view of \((u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{n+1} - u_n)\), it follows that
\[
||u_m - u_n||_{W^m}^2 = \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots + u_{n+1} - u_n\|^2_{W^m}
\]
\[
= \|u_m - u_{m-1}\|^2_{W^m} + \|u_{m-1} - u_{m-2}\|^2_{W^m} + \cdots + \|u_{n+1} - u_n\|^2_{W^m}
\]
\[
= \sum_{i=m+1}^{n} (B_i)^2 \rightarrow 0, (n \rightarrow \infty).
\]
(20)

In [22], it was proved that the space \(W^m[0, 1]\) is complete. Considering the completeness of \(W^m[0, 1]\), it has
\[
u_n(x) \xrightarrow{||\cdot||_{W^m}} u(x), \quad n \rightarrow \infty.
\]

It was proved that \(u_n(x)\) is convergent to \(u(x)\). Second, we will prove that \(u(x)\) is the solution of Equation (9).

From \((\bar{\psi}_i(x), \psi_j(x)) = \bar{\psi}_j(x)\) and Equation (16), it follows
\[
(\bar{\psi}_i(x)) = \sum_{j=1}^{\infty} B_j (\bar{\psi}_i(x), \psi_j(x)).
\]

Therefore,
\[
\sum_{j=1}^{n} B_j (\bar{\psi}_i(x), \psi_j(x)) = \sum_{j=1}^{\infty} B_j (\bar{\psi}_i(x), \psi_j(x))_{W^m} = \sum_{j=1}^{\infty} B_j (\bar{\psi}_i(x), \psi_j(x))_{W^m} = B_n.
\]

If \(n = 1\), then \((\bar{\psi}_1(x)) = \bar{\psi}_1(x)\). If \(n = 2\) then \(B_2 \bar{\psi}_2(x) = \beta_{21} f(x_1) + \beta_{22} f(x_2)\). It is clear that
The proof is complete.

4. Numerical experiments

In this section, some examples will be presented by using the method discussed above. All experiments were performed in MATHEMATICA 8.0. The numerical results are compared with their exact solution and approximate solution. In this regard, we have reported in tables and figures the values of the absolute error function $|u_n(x) - u(x)|$, then the maximum absolute error function of this method is compared with two methods presented in [4, 8].

Example 4.1: In this example, we solve the WSFIE of the second kind with the exact solution $u(t) = t^2$

$$u(t) - \int_0^1 \frac{u(x)}{\sqrt{1-x}} \, dx = t^2 - \frac{16}{15}, \quad 0 \leq t \leq 1.$$

Using the present method (RKHS), taking $n = 32, x_i = 0.03, i = 1, 2, \ldots, n$. Table 1 shows the absolute errors in spaces $W^4[0, 1], W^5[0, 1]$ for this example. The approximate solution, the absolute errors $|u_n(x) - u(x)|$ in $W^4[0, 1]$ and $W^5[0, 1]$ are graphically shown in Figure 1, respectively. However, by increasing $m$, the behaviour improves. The maximum value of the absolute errors in space $W^5[0, 1]$ and $n = 32$ in Figure 1 is $4.3 \times 10^{-11}$.

### Table 2. Numerical results of Example 4.2.

| Node   | $|u_n(x) - u(x)|_{W^4}, n = 32$ | $|u_n(x) - u(x)|_{W^5}, n = 32$ |
|--------|--------------------------------|--------------------------------|
| 0.0    | 0.0                            | 0.0                            |
| 0.1    | 2.04839E−9                     | 1.84376E−11                   |
| 0.2    | 1.58562E−9                     | 2.53714E−11                   |
| 0.3    | 1.76140E−9                     | 3.69912E−11                   |
| 0.4    | 1.68460E−9                     | 4.04320E−11                   |
| 0.5    | 1.71967E−9                     | 4.29935E−11                   |
| 0.6    | 1.70243E−9                     | 4.08601E−11                   |
| 0.7    | 1.71266E−9                     | 3.59675E−11                   |
| 0.8    | 1.70477E−9                     | 2.72871E−11                   |
| 0.9    | 1.70844E−9                     | 1.53777E−11                   |
| 1.0    | 3.63735E−9                     | 3.63735E−11                   |

### Table 3. Numerical results of Example 4.3.

| Node   | $|u_n(x) - u(x)|_{W^4}, n = 32$ | $|u_n(x) - u(x)|_{W^5}, n = 32$ |
|--------|--------------------------------|--------------------------------|
| 0.0    | 6.44363E−8                     | 6.44363E−10                    |
| 0.1    | 7.94112E−7                     | 7.93319E−8                    |
| 0.2    | 1.11779E−6                     | 5.38755E−8                    |
| 0.3    | 1.67794E−6                     | 5.59250E−9                    |
| 0.4    | 1.45614E−6                     | 3.64013E−8                    |
| 0.5    | 1.23001E−6                     | 2.45993E−8                    |
| 0.6    | 1.74552E−6                     | 2.90911E−8                    |
| 0.7    | 1.63501E−6                     | 2.33633E−8                    |
| 0.8    | 1.48007E−6                     | 1.85083E−8                    |
| 0.9    | 9.24550E−7                     | 1.02277E−8                    |
| 1.0    | 1.37886E−9                     | 1.37886E−11                   |
The maximum value of the absolute errors of two other methods, the classic Trapezoidal rule and the Euler–Maclaurin summation formula, is introduced in [8], $1.273 \times 10^{-1}$ and $5.075 \times 10^{-5}$, respectively. It is obviously seen that the reproducing kernel method is more accurate than the classic Trapezoidal rule and the Euler–Maclaurin summation formula.

Example 4.2: In this example, we solve the WSFIE of the second kind with the exact solution $u(t) = e^t$

$$u(t) - \int_0^1 \frac{u(x)}{\sqrt{1-x}} \, dx = e^t - 4.0602, \quad 0 \leq t \leq 1.$$  

Using the present method (RKHS), taking $n = 32$, $x_i = 0.3i$, $i = 1, 2, \ldots, n$, Table 2 shows the absolute errors in spaces $W^4[0, 1], W^5[0, 1]$ for this example. The approximate solution, the absolute errors $|u_n(x) - u(x)|$ in $W^4[0, 1]$ and $W^5[0, 1]$ are graphically shown in Figure 2, respectively. However, by increasing $m$, the behaviour improves. The maximum value of the absolute errors in space $W^5[0, 1]$ is $5.075 \times 10^{-5}$, respectively. It is obviously seen that the reproducing kernel method is more accurate than the classic Trapezoidal rule and the Euler–Maclaurin summation formula.

Example 4.3: In this example, we solve the WSFIE of the second kind with the exact solution $u(t) = \sqrt{t}$

$$u(t) - \int_0^1 \frac{u(x)}{\sqrt{1-x}} \, dx = \sqrt{t} - \frac{\pi}{2}, \quad 0 \leq t \leq 1.$$  

Using the present method (RKHS), taking $n = 32$, $x_i = 0.3i$, $i = 1, 2, \ldots, n$. Table 3 shows the absolute errors in spaces $W^4[0, 1], W^5[0, 1]$ for this example. The approximate solution, the absolute errors $|u_n(x) - u(x)|$ in $W^4[0, 1]$ and $W^5[0, 1]$ are graphically shown in Figure 3, respectively. However, by increasing $m$, the behaviour improves. The maximum value of the absolute errors in space $W^5[0, 1]$ and $n = 32$ in Figure 3 is $5.3 \times 10^{-7}$. The maximum value of the absolute errors of two other methods, the classic Trapezoidal rule and the Euler–Maclaurin summation formula, is introduced in [8], $1.273 \times 10^{-1}$ and $5.075 \times 10^{-5}$, respectively. It is obviously seen that the reproducing kernel method is more accurate than the classic Trapezoidal rule and the Euler–Maclaurin summation formula.

5. Concluding remarks

In this paper, we presented the new implementation of the reproducing kernel Hilbert space method to obtain the approximate solution of a class of WSFIEs. We obtain the sequence which is proved to converge to the exact solution uniformly. The comparison of numerical results in two spaces shows that our method is an accurate, efficient and reliable analytical technique for these equations. However, to obtain better results, use of the larger parameter $m$ is recommended.

Disclosure statement

No potential conflict of interest was reported by the authors.

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