Research Article

European Option Pricing under Wishart Processes

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This study deals with a single risky asset pricing model whose volatility is described by Wishart affine processes. This multifactor model with two dependency matrices describing the correlation between the asset dynamic and Wishart processes makes it more flexible enough to fit the market data for short or long maturities. The aim of the study is to derive and solve the call option pricing problem under the double Wishart stochastic volatility model. The Fourier transform techniques combined with perturbation methods are employed in order to price the European call options. The numerical illustrations on pricing predictions show similar behavior of price movements under the double Wishart model with respect to the market price.

1. Introduction

The limitation of the standard Black and Scholes [1] model of not accounting for the fact that implied volatility of derivative products varies by strike and maturity; this makes it less flexible to reproduce certain market conditions observed on derivative prices. This caused the introduction of the Heston [2] model, which has been remarkably famous and heavily applied in financial markets due to its flexibility, financial interpretation of parameters, and analytical tractability property of the model since it belongs to the class of the affine process. This helps it to obtain the call price of a European option by inverting the Fourier transform and forming a closed-form solution of the log-price characteristic function.

However, despite the Heston model popularity, the existing literature documented the limitation of the model. Da Fonseca et al. [3], Christoffersen et al. [4], Ahdida and Alfonsi [5], Kang et al. [6], La Bua and Marazzina [7], and Gourieroux [8] pointed out that the major deficiency of the model does not generate the realistic term structure of the volatility smiles. The Heston model provides too flat implied volatility surface to attain reality. Yet, the implied volatility curve generally has steep slope, and it is convex in short maturity and tends to be linear for long maturity.

The literature recommends that the problem of not capturing such stylized facts can be solved by generalizing the Heston model using two approaches: by adding jump in the stock dynamic or volatility and investigating the multifactor nature of the implied volatility as emphasized in the studies by Benabid et al. [9], Da Fonseca et al. [3], La Bua and Marazzina [7], and Kang and Kang [10]. It is well understood that the multifactor approach is capable of handling the pricing problem of derivative products and volatility skew. This usher in the application of the stochastic matrix defined process, that is, the Wishart multidimensional stochastic volatility model. The Wishart process was introduced in Bru [11] and is defined as a positive semidefinite matrix-valued generalization of the square root process. The square root process is a Bessel process which is a generalization of the multiple chi-squared distribution [12, 13]. The Wishart stochastic volatility model found its application in finance by Gourieroux and Sufana [14], and it has been applied in financial markets because of its nature of matrix specification which makes it flexible.
2. Wishart Process

**Definition 1.** Let $W_t, t \geq 0$ under the probability measure $Q$ be a $n \times n$ matrix-valued Brownian motion. The Wishart matrix process $\Sigma$ satisfies the following equation:

$$\text{d}\Sigma_t = (\beta Q Q^T + \Sigma_t + M \Sigma_t M^T) \text{d}t + \sqrt{\Sigma_t} \text{d}W_t Q + Q^T \text{d}W_t^T \sqrt{\Sigma_t},$$

(1)

where $Q \in GL_n(\mathbb{R})$ is the invertible matrix, $M \in M_n$ is the nonpositive matrix, and $\Sigma_0 \in S_n^+$ is the nonnegative symmetric matrix, while $\beta$ is a real parameter. The condition $\beta > (n - 1)$ is considered to ensure existence and uniqueness of the $\Sigma_t \in S_n^+$ solution for equation (1), and Eigen values of the solution are nonnegative for all $t \geq 0$ a.s. $\Sigma_t \in S_n^+$. Like in the study by Benabid et al. [9], the probability measure $Q$ corresponds to a risk-neutral measure (Appendix: change of the probability measure).

2.1. Wishart Volatility Model in the Stock Market. In Da Fonseca et al. [3], under the risk-neutral probability measure, the risky asset price dynamic and its quadratic variation are follows:

$$\frac{dS(t)}{S(t)} = r dt + \text{Tr}[\sqrt{\Sigma_t} \ dZ(t)],$$

$$S_0 = s,$$

$$d\Sigma_t = (\beta Q Q^T + \Sigma_t + M \Sigma_t M^T) dt + \sqrt{\Sigma_t} dW_t Q + Q^T dW_t^T \sqrt{\Sigma_t},$$

$$\Sigma_0 = \Sigma,$$

(2)

where $r$ denotes risk-less interest rate, $\text{Tr}$ is the trace, $Z \in M_n$ is the Brownian matrix, and $\Sigma_t$ is in a set of symmetric $n \times n$ positive-definite matrices. We observe that asset volatility is a trace of $\Sigma_t$ matrix, which is a multidimensional process with $M, Q \in M_n$, and $W_t \in M_n$ is the Brownian matrix.

In the study by Bru [11], the Wishart process provides a matrix analogue of the square root mean-reverting process and $M$ is considered negative to ensure positivity and mean-reverting property of the volatility with parameter $\beta > n - 1$ for uniqueness and existence of the solution.

2.2. Correlation Structure. The Brownian matrices $W_t$ and $Z_t$ are correlated, in such a way that it gives a constant correlated matrix $R \in M_n$, like in the study by Da Fonseca et al. [3], which describes the correlation structure, in such a way that $Z_t$ can be presented as follows:

$$Z_t = W_t R^T + B_t \sqrt{1 - R R^T},$$

(3)

where $I$ denotes the identity matrix, $T$ is the transpose, and $B_t$ is an independent matrix Brownian motion from $W_t$. The correlation structure is a Brownian motion (Appendix).

2.3. Bivariate Wishart Stochastic Processes in Stock Market. In this section, we present a proposed novel model, the multifactor model with two Wishart variance processes or double Wishart stochastic volatility model with two dependence matrices. The model takes two volatility components which is the trace of Wishart whose diagonal components will be the controlling factors for the dynamics of volatilities.

Under arbitrage-free financial market and probability measure, we consider risky asset dynamic as follows:

$$\frac{dS(t)}{S(t)} = r dt + \text{Tr} \left[ \sqrt{\Sigma_t} dZ_1 + \sqrt{\Sigma_t} dZ_2 \right],$$

$$S_0 = s,$$

$$d\Sigma_t = (\beta Q Q^T + \Sigma_t + \Sigma_t M^T) dt + \sqrt{\Sigma_t} dW_t Q + Q^T dW_t^T \sqrt{\Sigma_t},$$

$$\Sigma_0 = \Sigma,$$

$$d\Sigma_t = (\beta Q Q^T + \Sigma_t + \Sigma_t M^T) dt + \sqrt{\Sigma_t} dW_t Q + Q^T dW_t^T \sqrt{\Sigma_t},$$

$$\Sigma_0 = \Sigma,$$

(4)
where $\beta$ and $\bar{\beta}$ are the real parameters, such that $\beta, \bar{\beta} > n - 1$, $Q, \bar{Q}, M, \bar{M} \in M_n$. $Q$ is the invertible matrix, and $W_t, \bar{W}_t \in M_n$ are the matrices and Brownian motions, also $Z_t, \bar{Z}_t \in M_n$.

**Lemma 1.** The correlations between the Brownian matrices of the stock price dynamic and the Brownian matrices of the Wishart processes in equation (4) are given by

\[
\rho_t = \frac{\text{Tr}(R^T Q \Sigma_t)}{\sqrt{\text{Tr}(\Sigma_t) \text{Tr}(Q^T Q \Sigma_t)}},
\]

\[
\bar{\rho}_t = \frac{\text{Tr}(\bar{R}^T \bar{Q} \bar{\Sigma}_t)}{\sqrt{\text{Tr}(\bar{\Sigma}_t) \text{Tr}(\bar{Q}^T \bar{Q} \bar{\Sigma}_t)}}.
\]

**Proof.** We derive the correlations as follows:

\[
\frac{dS_t}{S_t} = r dt + \text{Tr} \left[ \sqrt{\Sigma_t} dZ_t + \sqrt{\bar{\Sigma}_t} d\bar{Z}_t \right]
= r dt + \sqrt{\text{Tr}(\Sigma_t) \text{Tr}(\bar{\Sigma}_t) \text{Tr}(Q^T Q \Sigma_t)} dt + \sqrt{\text{Tr}(\Sigma_t) \text{Tr}(\bar{\Sigma}_t) \text{Tr}(Q^T Q \Sigma_t)} dX_t,
\]

where $X_t$ and $\bar{X}_t$ are the standard Brownian motions (Appendix), and also taking the trace of the Wishart volatility dynamics (4), we have

\[
d\text{Tr}(\Sigma_t) = \left( (\beta \text{Tr}(Q^T Q)) + 2 \text{Tr}(M \Sigma_t) \right) dt + 2 \sqrt{\text{Tr}(Q^T Q \Sigma_t)} dW_t \sqrt{\Sigma_t},
\]

\[
d\text{Tr}(\bar{\Sigma}_t) = \left( (\bar{\beta} \text{Tr}(\bar{Q}^T \bar{Q})) + 2 \text{Tr}(\bar{M} \bar{\Sigma}_t) \right) dt + 2 \sqrt{\text{Tr}(\bar{Q}^T \bar{Q} \bar{\Sigma}_t)} d\bar{W}_t \sqrt{\bar{\Sigma}_t}.
\]

These processes can still be written in the following form:

\[
d\text{Tr}(\Sigma_t) = \left( (\beta \text{Tr}(Q^T Q)) + 2 \text{Tr}(M \Sigma_t) \right) dt + 2 \sqrt{\text{Tr}(Q^T Q \Sigma_t)} \frac{\text{Tr}(QdW_t \sqrt{\Sigma_t})}{\sqrt{\text{Tr}(Q^T Q \Sigma_t)}}
\]

\[
d\text{Tr}(\bar{\Sigma}_t) = \left( (\bar{\beta} \text{Tr}(\bar{Q}^T \bar{Q})) + 2 \text{Tr}(\bar{M} \bar{\Sigma}_t) \right) dt + 2 \sqrt{\text{Tr}(\bar{Q}^T \bar{Q} \bar{\Sigma}_t)} \frac{\text{Tr}(\bar{Q}d\bar{W}_t \sqrt{\bar{\Sigma}_t})}{\sqrt{\text{Tr}(\bar{Q}^T \bar{Q} \bar{\Sigma}_t)}}
\]

Then,

\[
d\text{Tr}(\Sigma_t) = \left( (\beta \text{Tr}(Q^T Q)) + 2 \text{Tr}(M \Sigma_t) \right) dt + 2 \sqrt{\text{Tr}(Q^T Q \Sigma_t)} \frac{\text{Tr}(QdW_t \sqrt{\Sigma_t})}{\sqrt{\text{Tr}(Q^T Q \Sigma_t)}}
\]

\[
d\text{Tr}(\bar{\Sigma}_t) = \left( (\bar{\beta} \text{Tr}(\bar{Q}^T \bar{Q})) + 2 \text{Tr}(\bar{M} \bar{\Sigma}_t) \right) dt + 2 \sqrt{\text{Tr}(\bar{Q}^T \bar{Q} \bar{\Sigma}_t)} \frac{\text{Tr}(\bar{Q}d\bar{W}_t \sqrt{\bar{\Sigma}_t})}{\sqrt{\text{Tr}(\bar{Q}^T \bar{Q} \bar{\Sigma}_t)}}
\]

where $\xi_t$ and $\eta_t$ are the Brownian motions (Appendix).

We now determine the covariation of the stock price and Wishart processes:

\[
\text{Cov}_t(dX_t, d\xi_t) = \text{Cov}_t\left( \frac{\text{Tr}(\sqrt{\Sigma_t} dZ_t)}{\sqrt{\text{Tr}(\Sigma_t) \text{Tr}(Q^T Q \Sigma_t)}}, \frac{\text{Tr}(QdW_t \sqrt{\Sigma_t})}{\sqrt{\text{Tr}(Q^T Q \Sigma_t)}} \right)
\]

\[
= \mathbb{E}_t\left( \frac{\text{Tr}(\sqrt{\Sigma_t} dW_t \sqrt{\bar{Q}^T \bar{Q} \bar{\Sigma}_t})}{\sqrt{\text{Tr}(\Sigma_t) \text{Tr}(Q^T Q \Sigma_t)}} \right)
\]

\[
= \sum_i \mathbb{E}_t(e_i^T \sqrt{\Sigma_t} dW_t \sqrt{\bar{Q}^T \bar{Q} \bar{\Sigma}_t} e_i) \frac{\text{Tr}(\sqrt{\Sigma_t} \text{Tr}(Q^T Q \Sigma_t) e_i e_j)}{\sqrt{\text{Tr}(\Sigma_t) \text{Tr}(Q^T Q \Sigma_t)}}
\]

\[
= \sum_i \mathbb{E}_t(e_i^T \sqrt{\Sigma_t} dW_t \sqrt{\bar{Q}^T \bar{Q} \bar{\Sigma}_t} e_i) \frac{\text{Tr}(\sqrt{\Sigma_t} \text{Tr}(Q^T Q \Sigma_t) e_i e_j) dt}{\sqrt{\text{Tr}(\Sigma_t) \text{Tr}(Q^T Q \Sigma_t)}}
\]

\[
= \sum_i \mathbb{E}_t(e_i^T \sqrt{\Sigma_t} dW_t \sqrt{\bar{Q}^T \bar{Q} \bar{\Sigma}_t} e_i) \frac{\text{Tr}(\sqrt{\Sigma_t} \text{Tr}(Q^T Q \Sigma_t) e_i e_j) dt}{\sqrt{\text{Tr}(\Sigma_t) \text{Tr}(Q^T Q \Sigma_t)}}
\]

\[
\text{Cov}_t(dX_t, d\xi_t) = \frac{\text{Tr}(R^T Q \Sigma_t)}{\sqrt{\text{Tr}(\Sigma_t) \text{Tr}(Q^T Q \Sigma_t)}}
\]

Similarly, for the second stochastic differential equation of the Wishart processes, the determination of covariation follows the same procedures, that is,

\[
\text{Cov}_t(d\bar{X}_t, d\eta_t) = \frac{\text{Tr}(\bar{R}^T \bar{Q} \bar{\Sigma}_t)}{\sqrt{\text{Tr}(\bar{\Sigma}_t) \text{Tr}(\bar{Q}^T \bar{Q} \bar{\Sigma}_t)}}
\]
2.4. The Correlation Structure of the Model. The correlated Brownian motions \(W_t, Z_t, \) and \(\overline{W}_t, \overline{Z}_t,\) respectively, result in constant correlations \(R, \overline{R} \in \mathbb{M}_{n},\) which describe the two respective correlation structures, such that \(Z_t\) and \(\overline{Z}_t\) can be presented as follows:

\[
\begin{align*}
Z_t &= W_t R^T + \beta_t \sqrt{1 - RR^T}, \\
\overline{Z}_t &= \overline{W}_t \overline{R}^T + \beta_t \sqrt{1 - \overline{R}\overline{R}^T},
\end{align*}
\]

(12)

where \(I\) is the identity matrix, \(T\) is the transposition, and \(\beta_t\) and \(\overline{\beta}_t\) are the Brownian matrices independent of \(W_t\) and \(\overline{W}_t,\) respectively.

2.5. The Dynamic of Log-Price for the Double Wishart Model. The matrices \(R\) and \(\overline{R}\) describe the correlations between the Brownian of the asset price and those of the Wishart processes.

Lemma 2. The log-price dynamic \(Y_t = \log(S_t)\) under the double Wishart model is given as

\[
dY_t = \left( r - \frac{1}{2} \text{Tr}[\Sigma_t + \overline{\Sigma}_t] \right) dt + \text{Tr} \left[ \sqrt{\Sigma_t} dB_t + \sqrt{\overline{\Sigma}_t} \overline{d}B_t \right].
\]

(13)

Proof. Let \(Y_t = \log(S_t),\) and the asset dynamic is given as

\[
\frac{dS_t}{S_t} = r dt + \text{Tr} \left[ \sqrt{\Sigma_t} dZ_t + \sqrt{\overline{\Sigma}_t} \overline{d}Z_t \right].
\]

(14)

By applying Ito’s formula on \(Y_t,\) referring the studies by Björk [18] and Shreve [19], we get

\[
dY_t = d\log(S_t) = \frac{dS_t}{S_t} - \frac{1}{2} \frac{(dS_t)^2}{S_t^2}.
\]

(15)

Substituting the asset process (14) in the derivative expression (15) of \(Y_t,\)

\[
dY_t = \left( r - \frac{1}{2} \text{Tr}[\Sigma_t + \overline{\Sigma}_t] \right) dt + \text{Tr} \left[ \sqrt{\Sigma_t} dZ_t + \sqrt{\overline{\Sigma}_t} \overline{d}Z_t \right].
\]

(16)

which can be written as

\[
dY_t = \left( r - \frac{1}{2} \text{Tr}[\Sigma_t + \overline{\Sigma}_t] \right) dt + \text{Tr} \left[ \sqrt{\Sigma_t} dB_t + \sqrt{\overline{\Sigma}_t} \overline{d}B_t \right].
\]

(17)

3. Option Pricing Problem

This part deals with the European call option pricing problem, given its payoff as

\[
(S_T - K)^+.
\]

(18)

To handle this pricing problem, the infinitesimal generator of the Wishart processes should be obtained, in order to employ the conditional characteristic function on the log asset return. The Riccati ordinary differential equations are linearized to give the closed-form solution to the pricing problem through fast Fourier transforms of Duffie et al. [15]. This is possible because the Wishart processes preserve the property of analytical tractability, since it belongs to the class of affine [20].

3.1. Infinitesimal Generator. The log-price process and Wishart processes, together with corresponding pair of two correlated Brownian motions, \(Z^\ast_t, \overline{Z}^\ast_t\) and \(Z^T_t, \overline{Z}^T_t,\) can be written in the form as in the study by Benabid et al. [9], for easy handling of the complexity of the dynamics:

\[
dY_t = \left( r - \frac{1}{2} \text{Tr}[\Sigma_t + \overline{\Sigma}_t] \right) dt + \text{Tr} \left[ \sqrt{\Sigma_t} dZ^\ast_t + \sqrt{\overline{\Sigma}_t} \overline{d}Z^\ast_t \right],
\]

\[
d\text{Tr}(\Sigma_t) = \left( \beta \text{Tr}(Q^T Q) + 2 \text{Tr}(MQ_i) \right) dt + 2 \sqrt{\text{Tr}(Q^T Q \Sigma_i)} \left( \rho_i dZ^\ast_i + \sqrt{1 - \rho_i^2} dZ^\overline{i}_i \right),
\]

\[
d\text{Tr}(\overline{\Sigma}_t) = \left( \beta \text{Tr}(\overline{Q}^T \overline{Q}) + 2 \text{Tr}(\overline{M} \overline{Q}_i) \right) dt + 2 \sqrt{\text{Tr}(\overline{Q}^T \overline{Q} \overline{\Sigma}_i)} \left( \rho^\overline{i} dZ^\ast_i + \sqrt{1 - \rho^\overline{i}_i^2} dZ^\ast_i \right),
\]

where \(Q, \overline{Q}\) are the Wishart matrices independent of \(W_t, \overline{W}_t,\) respectively.
where
\[
d\langle Z^y, Z^z \rangle_t = p_t = \frac{\text{Tr}(R^T Q \Sigma_t)}{\sqrt{\text{Tr}(\Sigma_t) \sqrt{\text{Tr}(Q^T Q \Sigma_t)}}},
\]
\[
d\langle Z^y, Z^z \rangle_t = p_t = \frac{\text{Tr}(R^T Q \Sigma_t)}{\sqrt{\text{Tr}(\Sigma_t) \sqrt{\text{Tr}(Q^T Q \Sigma_t)}}}.
\]

\(20\)

\[
\mathcal{L}_{Y,Y} = \left( r - \frac{\text{Tr}[\Sigma + \Sigma]}{2} \right) \frac{\partial}{\partial y} + \frac{\text{Tr}[\Sigma + \Sigma]}{2} \frac{\partial^2}{\partial y^2}
\]

\[+ \left( \beta \text{Tr}(Q^T Q) + 2 \text{Tr}(M \Sigma) \right) \frac{\partial}{\partial \Sigma} + 2 \text{Tr} \left( \Sigma \frac{\partial}{\partial \Sigma} Q^T \frac{\partial}{\partial \Sigma} \right)
\]
\[+ \left( \beta \text{Tr}(Q^T Q) + 2 \text{Tr}(M \Sigma) \right) \frac{\partial}{\partial \Sigma} + 2 \text{Tr} \left( \Sigma \frac{\partial}{\partial \Sigma} Q^T \frac{\partial}{\partial \Sigma} \right)
\]
\[+ 2 \text{Tr} \left( \Sigma R Q \frac{\partial}{\partial \Sigma} \right) \frac{\partial}{\partial y} + 2 \text{Tr} \left( \Sigma R Q \frac{\partial}{\partial y} \right) \frac{\partial}{\partial y}
\]

(21)

**Proof.** The infinitesimal generator has a nontrivial term which arises from covariation \(d\langle Y^y, Y \rangle\) corresponding to the coefficients of the term
\[
\frac{\partial^2}{\partial x_{\theta,ij} \partial y} = \frac{\partial}{\partial x_{\theta,ij}} \left( \frac{\partial}{\partial y} \right), \quad i, j = 1, \ldots, n, \quad \theta = 1, 2.
\]

(22)

Proposition 1. The infinitesimal generator of the double Wishart stochastic volatility model for \((Y^y, \Sigma, \Sigma_t)\) is given by

Let \(V_{\theta,t} := \sqrt{\Sigma_{\theta,t}}\) be the square root matrix, such that
\[
\Sigma_{ij} = \sum_{t=1}^{n} V_{\theta,t} Q_{ij} V_{\theta,t}^\top = \sum_{t=1}^{n} V_{\theta,t}^{ij} V_{\theta,t}^{ij}.
\]

(23)

Due to the fact that \(V_{\theta,t}\) is symmetric, we find the covariation terms matching with \(\partial^2/\partial x_{\theta,ij} \partial y\) coefficients:

\[
\langle d\Sigma_{\theta,t} \rangle_Y = \mathbb{E}_t \left[ \left( \sum_{i,k=1}^{n} V_{\theta,t}^{ik} dW_{\theta}^{ik} Q_{kj}^{\theta} + \sum_{i,k=1}^{n} V_{\theta,t}^{ik} dW_{\theta}^{ik} Q_{kj}^{\theta} \right) \left( \sum_{i,k=1}^{n} V_{\theta,t}^{ik} dW_{\theta}^{ik} R_{hk}^{\theta} \right) \right]
\]
\[= \sum_{i,k=1}^{n} \left( \sum_{l=1}^{n} V_{\theta,t}^{il} dW_{\theta}^{il} R_{hk}^{\theta} \right) \frac{\partial^2}{\partial x_{\theta,ij} \partial y} \frac{\partial^2}{\partial x_{\theta,ij} \partial y} dt
\]
\[= \sum_{k,h=1}^{n} \left( \sum_{l=1}^{n} V_{\theta,t}^{il} dW_{\theta}^{il} \right) Q_{kj}^{\theta} + \left( \sum_{l=1}^{n} V_{\theta,t}^{il} dW_{\theta}^{il} \right) Q_{kj}^{\theta} R_{hk}^{\theta} dt
\]
\[= \sum_{k,h=1}^{n} \left( \Sigma_{\theta,t}^{ik} Q_{kj}^{\theta} + \Sigma_{\theta,t}^{ik} Q_{kj}^{\theta} \right) R_{hk}^{\theta} dt.
\]

(24)

This gives the corresponding coefficients of the term since

\[
2 \text{Tr} (\Sigma R Q D) \frac{\partial}{\partial y} = 2 \sum_{i,j,k,h=1}^{n} D_{ij}^{\theta} \Sigma_{\theta,t}^{ik} R_{hk}^{\theta} Q_{kj}^{\theta} \frac{\partial}{\partial y}
\]

(25)

The notation when \(\theta = 1, \Sigma_1 = \Sigma, R^1 = R, Q^1 = Q, D_1 = (\partial/\partial x_{ij})\) for \(\theta = 2, \Sigma_2 = \Sigma, R^2 = R, Q^2 = Q, D_2 = (\partial/\partial x_{ij})\).
3.2. *The Laplace Transform of Asset Returns.* Following Duffie et al. [15] and Da Fonseca et al. [3], in order to solve the European options pricing problem, we consider the Laplace transform of the process (17). The Laplace transform of the Wishart process is exponentially affine as in the study by Bru [11]; hence, the conditional moment generating function of the log-asset returns is the exponential of an affine combination of $Y$ and Wishart components, so we provide for the deterministic functions $\lambda_1(t), \lambda_2(t) \in M_n$, and $\delta(t), \epsilon(t) \in \mathbb{R}$, as parameters of the Laplace transform:

$$
\psi_{\tau,t}(\tau) = E\left(e^{\gamma Y_{\tau,t}}\right) = \exp\left[\text{Tr}\lambda_1(\tau)\Sigma_t + \text{Tr}\lambda_2(\tau)\Sigma_t + \delta(t)Y_t + \epsilon(t)\right],
$$

(26)

where $\gamma \in \mathbb{R}$ and $E_{\tau}$ denote the conditional expected value with respect to the probability measure.

Using Feynman–Kac argument helps us to get matrix Riccati equations:

$$
\begin{align*}
\frac{\partial \psi_{\tau,t}}{\partial \tau} &= \mathcal{L}_{\tau} \psi_{\tau,t}, \\
\psi_{\tau,t}(0) &= \exp\{\gamma Y_t\},
\end{align*}
$$

(27)

**Proposition 2.** *The Laplace transform of the asset returns*

$$
\psi_{\tau,t}(\tau) = \exp[\text{Tr}\{\lambda_1(\tau)\Sigma_t + \lambda_2(\tau)\Sigma_t + \delta(t)Y_t + \epsilon(t)\}],
$$

(28)

where $\lambda_1, \lambda_2,$ and $\epsilon$ are the solutions of the differential equations through linearization methods:

$$
\begin{align*}
\lambda_1'(\tau) &= M\lambda_1(\tau) + (M^T + 2\gamma RQ)\lambda_1(\tau) + 2\lambda_1(\tau)Q^T Q\lambda_1(\tau) + \frac{\gamma(y-1)}{2} I_n, \\
\lambda_2'(\tau) &= M\lambda_2(\tau) + (M^T + 2\gamma RQ)\lambda_2(\tau) + 2\lambda_2(\tau)Q^T Q\lambda_2(\tau) + \frac{\gamma(y-1)}{2} I_n, \\
\epsilon'(\tau) &= r\gamma + \beta \text{Tr}\left[(Q^T Q)\lambda_1(\tau)\right] + \beta \text{Tr}\left[(Q^T Q)\lambda_2(\tau)\right],
\end{align*}
$$

with boundary conditions $\lambda_1(0) = 0, \lambda_2(0) = 0, \epsilon(0) = 0$, and $\delta(t) = \gamma = C_0$.

The solutions $\lambda_1, \lambda_2,$ and $\epsilon$ are obtained as follows:

$$
\begin{align*}
\lambda_1(\tau) &= H_1(\tau)^{-1}H_2(\tau), \\
\lambda_2(\tau) &= I_1^{-1}(\tau)I_2(\tau), \\
\epsilon(\tau) &= -\frac{\beta}{2} \text{Tr}\left[\log H_1(\tau) + (M^T + 2\gamma RQ)\tau\right] - \frac{\beta}{2} \text{Tr}\left[\log I_1(\tau) + (M^T + 2\gamma RQ)\tau\right] + r\gamma \tau,
\end{align*}
$$

(30)

with

$$
\begin{align*}
\begin{pmatrix} H_2(\tau) & H_1(\tau) \end{pmatrix} &= \begin{pmatrix} H_2(0) & H_1(0) \end{pmatrix} \exp\begin{pmatrix} M & -2Q^T Q \\ \frac{\gamma(y-1)}{2} I_n & -(M^T + 2\gamma RQ) \end{pmatrix},
\end{align*}
$$

(31)

$$
\begin{align*}
\begin{pmatrix} I_2(\tau) & I_1(\tau) \end{pmatrix} &= \begin{pmatrix} I_2(0) & I_1(0) \end{pmatrix} \exp\begin{pmatrix} \bar{M} & -2Q^T \bar{Q} \\ \frac{\gamma(y-1)}{2} I_n & -(\bar{M}^T + 2\gamma \bar{R} \bar{Q}) \end{pmatrix},
\end{align*}
$$
Proof

From equation (27) and the Laplace transform (26), we solve the problem by considering Proposition 1 for the double Wishart stochastic volatility model as follows:

\[
\frac{\partial \psi_{Y,t}}{\partial \tau} = \left( r - \frac{\text{Tr}[\Sigma]}{2} \right) \frac{\partial \psi_{Y,t}}{\partial r} + \frac{\text{Tr}[\Sigma]}{2} \frac{\partial^2 \psi_{Y,t}}{\partial r^2} \\
+ \left( \beta \text{Tr}(Q^\top Q) + 2 \text{Tr}(M \Sigma) \right) \frac{\partial^2 \psi_{Y,t}}{\partial \Sigma} + 2 \text{Tr}\left( \Sigma \frac{\partial^2 Q^\top Q}{\partial \Sigma} \right) \psi_{Y,t} \\
+ \left( \beta \text{Tr}(\Sigma^\top Q^\top) + 2 \text{Tr}(M \Sigma) \right) \frac{\partial \psi_{Y,t}}{\partial \Sigma} + 2 \text{Tr}\left( \Sigma \frac{\partial Q^\top Q}{\partial \Sigma} \right) \psi_{Y,t} \\
+ 2 \text{Tr}\left( \Sigma^\top R Q \right) \frac{\partial \psi_{Y,t}}{\partial y} + 2 \text{Tr}\left( \Sigma R Q \right) \frac{\partial^2 \psi_{Y,t}}{\partial y^2}
\]

(32)

From equation (27), we have

\[
\begin{align*}
\lambda_1(0) &= 0, \\
\lambda_2(0) &= 0, \\
\delta(0) &= \gamma, \\
\epsilon(0) &= 0.
\end{align*}
\]

(33)

Then,

\[
\frac{\partial \psi_{Y,t}(r)}{\partial \tau} = \text{Tr}\left[ \left( \frac{\partial \lambda_1(r) \Sigma}{\partial r} + \frac{\partial \lambda_2(r) \Sigma}{\partial r} \right) \frac{d}{dr} \delta(t) Y + \frac{d}{dr} \epsilon(t) \right] \psi_{Y,t}(r)
\]

\[
= \left( r - \frac{[\Sigma + \Sigma]}{2} \right) \frac{\partial \psi_{Y,t}}{\partial r} + \frac{\text{Tr}[\Sigma + \Sigma]}{2} \frac{\partial^2 \psi_{Y,t}}{\partial r^2} \\
+ \left[ \beta \text{Tr}(Q^\top Q) + 2 \text{Tr}(M \Sigma) \right] \lambda_1(r) + \left[ \beta \text{Tr}(\Sigma^\top Q^\top) + 2 \text{Tr}(M \Sigma) \right] \lambda_2(r) \psi_{Y,t} \\
+ 2 \text{Tr}[\Sigma \lambda_1(r)Q^\top \Sigma \lambda_1(r)] \psi_{Y,t} + 2 \text{Tr}[\Sigma \lambda_1(r)Q^\top \Sigma \lambda_1(r)] \psi_{Y,t} \\
+ 2 \text{Tr}[\Sigma R Q \delta(t) \lambda_1(r)] \psi_{Y,t} + 2 \text{Tr}[\Sigma R Q \delta(t) \lambda_1(r)] \psi_{Y,t}
\]

(34)
Now, we need to identify the coefficients for above equations and obtain the matrix Riccati ordinary differential equations:

\[
\begin{align*}
\frac{d}{dr} \lambda_1 (r) &= M \lambda_1 (r) + \left( M^T + 2 \gamma R Q \right) \lambda_1 (r) + 2 \lambda_1 (r) Q^T \lambda_1 (r) + \frac{\gamma (\gamma - 1)}{2} \lambda_1 (r), \\
\lambda_1 (0) &= 0, \\
\frac{d}{dr} \lambda_2 (r) &= M \lambda_2 (r) + \left( M^T + 2 \gamma R Q \right) \lambda_2 (r) + 2 \lambda_2 (r) Q^T \lambda_2 (r) + \frac{\gamma (\gamma - 1)}{2} \lambda_2 (r), \\
\lambda_2 (0) &= 0.
\end{align*}
\]

(35)  

(36)

For the constant \( \epsilon \), the matrix Riccati ordinary differential is obtained as follows:

\[
- \delta' (r) Y_t - \epsilon' (r) + r \delta (r) + \text{Tr} \left( \beta Q^T Q \right) \lambda_1 (r) + \text{Tr} \left( \bar{Q} Q \lambda_2 (r) \right) = 0,
\]

\[
\delta (r) = C_0 = \gamma, \text{ since } \delta' (r) = 0 \Rightarrow \delta (r) = \gamma,
\]

\[
\begin{align*}
\frac{d \epsilon (r)}{dr} &= r \gamma + \beta \text{Tr} \left[ \left( Q^T Q \right) \lambda_1 (r) \right] + \bar{\beta} \text{Tr} \left[ \left( Q^T Q \right) \lambda_2 (r) \right], \\
\epsilon (0) &= 0.
\end{align*}
\]

(37)

Then, \( \epsilon (r) \) is obtained by integrating directly

\[
\epsilon (r) = \int_0^r \gamma y ds + \int_0^r \text{Tr} \left[ \beta Q^T Q \lambda_1 (s) + \bar{\beta} Q^T Q \lambda_2 (s) \right] ds,
\]

(38)

with \( \lambda_1, \lambda_2 \in M_n (\mathbb{R}) \), and \( \delta (r), \epsilon (r) \in \mathbb{R} \).

The matrix Riccati equations above are linearized to provide the closed-form solution, following the techniques of Benabid et al. [9] and Da Fonseca et al. [3]. So from equations (35) and (36), let

\[
\lambda_1 (r) = H_1 (r)^{-1} H_2 (r),
\]

(39)

where \( H_1 (r) \in GL_n (\mathbb{R}) \), \( H_2 (r) \in M_n (\mathbb{R}) \), and therefore,

\[
\frac{d}{dr} \left[ H_1 (r) \lambda_1 (r) \right] = \frac{d H_1 (r)}{dr} \lambda_1 (r) + H_1 (r) \frac{d \lambda_1 (r)}{dr},
\]

\[
H_1 (r) \frac{d \lambda_1 (r)}{dr} = \frac{d}{dr} \left[ H_1 (r) \lambda_1 (r) \right] - \frac{d H_1 (r)}{dr} \lambda_1 (r),
\]

\[
H_1 (r) \lambda_1 (r) M + H_1 (r) \left( M^T + 2 \gamma R Q \right) \lambda_1 (r) + 2 H_1 (r) \lambda_1 (r) Q^T Q \lambda_1 (r)
\]

\[
+ H_1 (r) \frac{\gamma (\gamma - 1)}{2} = H_1 (r) \frac{d \lambda_1 (r)}{dr},
\]

(40)

\[
H_2 (r) M + H_1 (r) \left( M^T + 2 \gamma R Q \right) \lambda_1 (r) + 2 H_2 (r) Q^T Q \lambda_2 (r)
\]

\[
+ H_1 (r) \frac{\gamma (\gamma - 1)}{2} = H_1 (r) \frac{d \lambda_2 (r)}{dr}.
\]
Since $H_2 (\tau) = H_1 (\tau) \lambda_1 (\tau)$, now we have
\[
\frac{dH_2 (\tau)}{d\tau} = \frac{dH_1 (\tau)}{d\tau} \lambda_1 (\tau) + H_1 (\tau) \frac{d\lambda_1 (\tau)}{d\tau}.
\] (41)
\[
H_1 (\tau) \frac{d\lambda_1 (\tau)}{d\tau} = \frac{dH_2 (\tau)}{d\tau} - \frac{dH_1 (\tau)}{d\tau} \lambda_1 (\tau).
\] (42)
Then, we get expressions
\[
\frac{dH_2 (\tau)}{d\tau} - \frac{dH_1 (\tau)}{d\tau} \lambda_1 (\tau) = H_2 (\tau) M + H_1 (\tau) \left( M^T + 2\gamma RQ \right) \lambda_1 (\tau) + 2H_2 (\tau) Q^T Q \lambda_1 (\tau) + H_1 (\tau) \frac{\gamma (y - 1)}{2} \lambda_1 (\tau),
\] (43)
\[
= H_1 (\tau) \frac{\gamma (y - 1)}{2} + H_2 (\tau) M + \left[ H_1 (\tau) \left( M^T + 2\gamma RQ \right) + 2H_2 (\tau) Q^T Q \right] \lambda_1 (\tau),
\] (44)
\[
\frac{dH_2 (\tau)}{d\tau} = H_1 (\tau) \frac{\gamma (y - 1)}{2} + H_2 (\tau) M,
\] (45)
\[
\frac{dH_1 (\tau)}{d\tau} = -H_1 (\tau) \left( M^T + 2\gamma RQ \right) - 2H_2 (\tau) Q^T Q.
\] (46)
\[
\frac{d}{d\tau} \left( H_2 (\tau) \ H_1 (\tau) \right) = \left( H_2 (\tau) \ H_1 (\tau) \right) \begin{pmatrix} M & -2Q^T Q \\ \frac{\gamma (y - 1)}{2} I_n & -\left( M^T + 2\gamma RQ \right) \end{pmatrix}.
\] (47)
\[
\begin{pmatrix} H_2 (\tau) & H_1 (\tau) \end{pmatrix} = \left( H_2 (0) \ H_1 (0) \right) e^{\gamma Q},
\] (48)
where
\[
e^{\gamma Q} = \begin{pmatrix} \lambda_1^{11} (\tau) & \lambda_1^{12} (\tau) \\ \lambda_1^{21} (\tau) & \lambda_1^{22} (\tau) \end{pmatrix}.
\] (49)
\[
\text{This represents the closed-form solution of the Riccati equation (35).}
\]
Now, we look at the solution of the second Riccati equation (36). Let
\[
\lambda_2 (\tau) = I_1^{-1} (\tau) I_2 (\tau),
\] (50)
then,
\[
\frac{dI_2 (\tau)}{d\tau} = I_1 (\tau) \frac{\gamma (y - 1)}{2} + I_2 (\tau) M,
\] (51)
\[
\frac{dI_1 (\tau)}{d\tau} = -I_1 (\tau) \left( M^T + 2\gamma RQ \right) - 2I_2 (\tau) Q^T Q,
\] (52)
\[
\frac{d}{d\tau} \begin{pmatrix} I_2(\tau) \\ I_1(\tau) \end{pmatrix} = \begin{pmatrix} M & -2QQ^T \\ \frac{y(y-1)}{2} & -M^T - 2yRQ \end{pmatrix} \begin{pmatrix} I_2(\tau) \\ I_1(\tau) \end{pmatrix},
\]

such that

\[
\begin{pmatrix} I_2(\tau) \\ I_1(\tau) \end{pmatrix} = \begin{pmatrix} I_2(0) \\ I_1(0) \end{pmatrix} e^{\mathcal{M} \tau}.
\]

From equation (40), we go through the same procedures as above, with conditions

\[
\begin{pmatrix} I_2(\tau) \\ I_1(\tau) \end{pmatrix} = \begin{pmatrix} \lambda_2(0) \lambda_2^1(\tau) \\ \lambda_2^1(\tau) \lambda_2^2(\tau) \end{pmatrix},
\]

since \( \lambda_2(0) = 0 \), then

\[
\begin{pmatrix} I_2(\tau) \\ I_1(\tau) \end{pmatrix} = \begin{pmatrix} \lambda_2^1(\tau) \\ \lambda_2^2(\tau) \end{pmatrix}.
\]

Thus, it provides the expression for a closed-form solution of the Riccati equation:

\[
\lambda_2(\tau) = \lambda_2^2(\tau)^{-1} \lambda_2^1(\tau).
\]

Let compute the last Riccati equation (37) for constant \( \varepsilon \),

\[
\begin{pmatrix} \frac{d}{d\tau}(\varepsilon) \\ \frac{d}{d\tau}(\lambda_1) \end{pmatrix} = \begin{pmatrix} \frac{d}{d\tau}(\varepsilon) \\ \frac{d}{d\tau}(\lambda_1) \end{pmatrix},
\]

From Riccati equations (35) and (36), we have

\[
\begin{pmatrix} \frac{d}{d\tau}(\varepsilon) \\ \frac{d}{d\tau}(\lambda_1) \end{pmatrix} = \begin{pmatrix} \frac{d}{d\tau}(\varepsilon) \\ \frac{d}{d\tau}(\lambda_1) \end{pmatrix},
\]

and as well as from equation (64),

\[
H_2(\tau) = \frac{1}{2} \left[ \frac{dH_1(\tau)}{d\tau} + H_1(\tau) (M^T + 2yRQ) \right] (Q^TQ)^{-1},
\]

\[
I_2(\tau) = \frac{1}{2} \left[ \frac{dI_1(\tau)}{d\tau} + I_1(\tau) (M^T + 2yRQ) \right] (Q^TQ)^{-1}.
\]
and then, substituting equations (66) and (67) in (65) provides

\[
\frac{d\varepsilon(t)}{dt} = \frac{B}{2} \text{Tr} \left[ H^{-1}(t) \frac{dH(t)}{dt} + (M^T + 2\gamma RQ) \right] = \frac{B}{2} \text{Tr} \left[ I^{-1}(t) \frac{dI(t)}{dt} + \left( M^T + 2\gamma RQ \right) \right] + r\gamma, \varepsilon(0) = 0, \tag{68}
\]

\[
\frac{d\varepsilon(t)}{dt} = \frac{B}{2} \text{Tr} \left[ \frac{dH(t)}{dt} + (M^T + 2\gamma RQ) \right] = \frac{B}{2} \text{Tr} \left[ I^{-1}(t) \frac{dI(t)}{dt} + \left( M^T + 2\gamma RQ \right) \right] + r\gamma dt. \tag{69}
\]

By integrating (69), we obtain

\[
\varepsilon(t) = \frac{B}{2} \text{Tr} \left[ \log H(t) + (M^T + 2\gamma RQ) t \right] - \frac{B}{2} \text{Tr} \left[ \log I(t) + (M^T + 2\gamma RQ) t \right] + r\gamma t. \tag{70}
\]

### 3.3. Fast Fourier Transform Method and Characteristic Function

In this section, we consider the fast Fourier transform (FFT) method in the study by Carr and Madan [16] to price European call option with \( \alpha > 0 \), at time \( t \), strike \( k = \log(K) \), and time to maturity \( T \), given as

\[
C_T(T, k) = e^{-r(T-t)} \mathbb{E} \left[ (X_T - K)^+ | \mathcal{F}_t \right],
\]

\[
C_t(T, K) = C_t(T, k) = e^{-r(T-t)} \mathbb{E} \left[ (\exp(Y_t) - \exp(k))^+ | \mathcal{F}_t \right]. \tag{71}
\]

The modified price \( C^\alpha(T, k) \) in the study by Carr and Madan [16] is considered, with \( \alpha = 1.1 \) as a good empirical value for the Heston model. The square integrable function is obtained through the modified price in order to use inverse Fourier transform:

\[
C^\alpha(T, k) = \exp(\alpha k) C_t(T, k). \tag{72}
\]

We introduce the Fourier transform of the modified price together with the application of the Fubini integration theorem:

\[
\psi^\alpha(T, \theta) = \int_{-\infty}^{\infty} C^\alpha(T, k) e^{ik} dk = \int_{-\infty}^{\infty} e^{ik} C_t(T, k) e^{ik} dk = \int_{-\infty}^{\infty} e^{ik} e^{-r(T-t)} \mathbb{E} \left[ (\exp(Y_t) - \exp(k))^+ | \mathcal{F}_t \right] e^{ik} dk = e^{-r(T-t)} \int_{-\infty}^{\infty} \exp[(\alpha + i\theta)k] \mathbb{E} \left[ (\exp(Y_t) - \exp(k))^+ | \mathcal{F}_t \right] dk, \tag{73}
\]

\[
\psi^\alpha(T, \theta) = e^{-r(T-t)} \frac{\phi_1(T, \theta - (1 + \alpha)i)}{(\alpha + i\theta)(\alpha + i\theta + 1)}. \tag{74}
\]

The call price can be provided through inversion of Fourier transform given \( \psi^\alpha(T, \theta) \) function having both odd imaginary and even real parts; so from equation (73), we have

\[
\int_{-\infty}^{\infty} C^\alpha(T, k) e^{ik} dk = \frac{e^{-r(T-t)} \phi_1(T, \theta - (1 + \alpha)i)}{(\alpha + i\theta)(\alpha + i\theta + 1)}, \tag{75}
\]

\[
C^\alpha(T, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi^\alpha(T, \theta) e^{-ik} d\theta. \tag{76}
\]

Then, we have

\[
C_t(T, K) = \frac{e^{-ak}}{2\pi} \int_{-\infty}^{\infty} \psi^\alpha(T, \theta) e^{-ik} d\theta, \tag{77}
\]

\[
C_t(T, K) = \frac{e^{-ak}}{2\pi} \int_{-\infty}^{\infty} e^{-r(T-t)} \phi_1(T, \theta - (1 + \alpha)i) e^{-ik} d\theta. \tag{78}
\]

This is a Fourier transform:
Corollary 1. Let \( D \) be the symmetric matrix, and it is sufficient to find the conditional characteristic function of the double Wishart \( \Sigma_i \) and \( \bar{\Sigma}_i \) given by (Appendix)

\[
\frac{dA_1(t)}{dt} = A_1(t)M + M^T A_1(t) + 2A_1(t)\bar{Q}^T QA_1(t),
\]

\( A_1(0) = iD_1, \)

\[
\frac{dA_2(t)}{dt} = A_2(t)\bar{M} + \bar{M}^T A_2(t) + 2A_2(t)\bar{Q}^T \bar{Q}A_2(t),
\]

\( A_2(0) = iD_2, \)

\[
C(t) = \int_0^t \text{Tr}\left[ \beta\bar{Q}^T QA_1(t)du + \bar{Q}^T \bar{Q}A_2(t)du \right].
\]

Proposition 3. The call price under double Wishart is given as

\[
C_t(T, K) = \frac{e^{-\alpha k}}{2\pi} \text{Re} \left( \int_{-\infty}^{\infty} e^{-r(T-t)}\phi_{\Sigma, \bar{\Sigma}}(t)e^{-ik\theta}d\theta \right),
\]

(80)

where

\[
\frac{dA_1(t)}{dt} = A_1(t)M + M^T A_1(t) + 2A_1(t)\bar{Q}^T QA_1(t),
\]

\( A_1(0) = iD_1, \)

\[
\frac{dA_2(t)}{dt} = A_2(t)\bar{M} + \bar{M}^T A_2(t) + 2A_2(t)\bar{Q}^T \bar{Q}A_2(t),
\]

\( A_2(0) = iD_2, \)

\[
C(t) = \int_0^t \text{Tr}\left[ \beta\bar{Q}^T QA_1(t)du + \bar{Q}^T \bar{Q}A_2(t)du \right].
\]

Proof. Let \( \phi(t, T) \) be the characteristic function of the log-price \( Y_t \). We have,

\[
Y_{t,T} = \ln\left( \frac{X_T}{X_t} \right) = \ln(X_T) - \ln(X_t) = Y_T - Y_t,
\]

\[
\phi_{Y,0}(t, T) = \mathbb{E}\left[ \exp\{i\phi Y_{t,T} \} \right]
\]

\[
= \mathbb{E}\left[ \exp\{-i\phi Y_{t} \} \right] \mathbb{E}\left[ \exp\{i\phi Y_{T} \} \right]
\]

\[
= \mathbb{E}\left[ \exp\{-i\phi Y_{t} \} \exp\left[ \text{Tr}\left[ \lambda_1 (T-t)\Sigma_i + \lambda_2 (T-t)\bar{\Sigma}_i \right] + i\phi Y_{t} + \epsilon(T-t) \right] \right]
\]

\[
= \exp[\epsilon(T-t)] \mathbb{E}\left[ \exp\left[ \text{Tr}\left[ \lambda_1 (T-t)\Sigma_i + \lambda_2 (T-t)\bar{\Sigma}_i \right] \right] \right]
\]

\[
= \exp[\epsilon(T-t)] \mathbb{E}\left[ \exp\left[ \text{Tr}\left[ A_1(t)\Sigma_0 + A_2(t)\bar{\Sigma}_0 \right] + C(t) \right] \right]
\]

\[
= \phi_{Y,0}(t, T) = \exp[\text{Tr}\left[ A_1(t)\Sigma_0 + A_2(t)\bar{\Sigma}_0 \right] + C(t) + \epsilon(T-t)],
\]

(82)
where \( A_j(t) \) is obtained from equation (79) with \( \tau = t \) provided \( A_j(0) = \lambda_j(T - t), \) \( j = 1, 2. \) Hence,

\[
C_t(T, K) = \frac{e^{-\lambda t}}{2\pi} \text{Re} \left( \int_{-\infty}^{\infty} e^{-r(T-t)} e^{i\sum_{n=0}^{\infty} A_n(t) \xi_n} C(t)^2 \frac{d\theta}{(\alpha + i\theta)(\alpha + i\theta + 1)} e^{-i\theta \epsilon} d\theta \right), \quad (83)
\]

where

\[
\epsilon(\tau) = \text{Tr} \left[ \log \left( \sqrt{\frac{1}{(H_1(\tau))^{\beta}} (I_1(\tau))^{\beta}} \right) - \frac{1}{2} (\beta M^T + \beta M^T) + ry \right]. \quad (84)
\]

3.4. Perturbation Techniques of the Riccati Differential Equations. This section deals with implementation of perturbation techniques since the system of the Riccati differential equations does not allow an analytical closed-form solution. This is due to noncommutativity of matrix multiplication, in order to approximate the call option price.

The method procedures retain the affine properties and higher orders comparable to the standard perturbation scheme on partial differential equations that is complex after the first order [9, 17]. Consider the differential Riccati equations of the double Wishart stochastic volatility model and take dimension \( n = 2. \) The two characteristics orders in perturbation \( p \) and \( q \) are considered. The solution of \( A(\tau) \) is given in the following form:

\[
A(\tau) = \sum_{i,j} p^{ij} q^{ij} A_i(\tau). \quad (85)
\]

The developments in the perturbed differential equations are done by comparing the coefficients and identifying the terms in \( p \) and \( q \) to give expected approximations. Let \( p = M_1 \) and \( q = M_2 \) are small while \( \nu_i \) quantities remain constant. Then, we consider approximation at order one \((\sqrt{p}, \sqrt{q})\) and order two \((p, q)\) together with the following notations.

\[
M = \begin{pmatrix} -p & 0 \\ 0 & -q \end{pmatrix} = -p \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - q \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -p M_1 - q M_2,
\]

\[
\overline{M} = \begin{pmatrix} -\overline{p} & 0 \\ 0 & -\overline{q} \end{pmatrix} = -\overline{p} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \overline{q} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -\overline{p} M_1 - \overline{q} M_2,
\]

and also noting that \( Q \) is the volatility,

\[
Q = \sqrt{\nu_1 M_1 + \sqrt{\nu_2 M_2}},
\]

\[
Q^2 = \nu_1^2 M_1 + \nu_2^2 M_2,
\]

which can be written as

\[
Q = \sqrt{PQ_1 + \sqrt{Q_2}}. \quad (88)
\]

We can now rewrite the Riccati equations as in the form

\[
A_1(\tau) = p \left[ -A_1(\tau) M_1 - M_1 A_1(\tau) + 2 A_1(\tau) Q^2 A_1(\tau) \right] + q \left[ -A_1(\tau) M_2 - M_2 A_1(\tau) + 2 A_1(\tau) Q^2 A_1(\tau) \right], \quad (89)
\]

and for

\[
\lambda_1(\tau) = \frac{\nu_1^2}{2} \lambda_1(\tau) + p \left[ -\lambda_1(\tau) M_1 - M_1 \lambda_1(\tau) + 2 \lambda_1(\tau) Q^2 \lambda_1(\tau) \right] + 2 \sqrt{\nu_1} \sqrt{Q_2 \lambda_1(\tau)} \quad (90)
\]

\[
\lambda_2(\tau) = \frac{\nu_2^2}{2} \lambda_2(\tau) + p \left[ -\lambda_2(\tau) M_1 - M_1 \lambda_2(\tau) + 2 \lambda_2(\tau) Q^2 \lambda_2(\tau) \right] + 2 \sqrt{\nu_2} \sqrt{Q_2 \lambda_2(\tau)} \quad (91)
\]
Development of the Riccati functions: by dealing with the development of the Riccati functions \(A_1, A_2, \lambda_1, \lambda_2, C,\) and \(\varepsilon,\) the perturbation of the Riccati equations at order two (order 2) is considered to obtain the European call option pricing formula as follows:

\[
A_k (\tau) = A_k^0 (\tau) + \sqrt{p} A_k^1 (\tau) + \sqrt{q} A_k^{0.1} (\tau) + p A_k^{2.0} (\tau) + q A_k^{0.2} (\tau) + \sqrt{pq} A_k^{1.1} (\tau) + o(\max(p, q)), \quad k = 1, 2. \tag{92}
\]

Let us determine \(A_k^0, \ldots, A_k^{1.1}.\)

By identifying the terms in respective orders, we have

\[
(A_1^0 (\tau))' = 0, \quad A_1^0 (\tau) \in M_2 (C). \tag{93}
\]

Since \(A_1 (0) = iD_1,\)

\[
A_1^{0.0} (0) + \sqrt{p} A_1^{1.0} (0) + \sqrt{q} A_1^{0.1} (0) + p A_1^{2.0} (0) + q A_1^{0.2} (0) + \sqrt{pq} A_1^{1.1} (0) = iD_1. \tag{95}
\]

Then,\[
A_1^{0.0} (0) = iD_1 = i \hat{d}_1 I_2, \quad A_1^{1.0} (0) = A_1^{0.1} (0) = A_1^{2.0} (0) = A_1^{0.2} (0) = A_1^{1.1} (0) = (0), \tag{96}
\]

By proceeding by obtaining

\[
A_1^0 (\tau) = -2d_1 (i + d_1 \gamma_1^0) \tau M_1, \quad A_1^{1.0} (\tau) = -2d_1 (i + d_1 \gamma_1^2) \tau M_2, \quad (A_1^{1.1} (\tau))' = 0, \Rightarrow A_1^{1.1} (\tau) = C, \tag{97}
\]

Then,\[
A_2^0 (\tau) = \mathbb{P}[-A_2 (\tau) M_1 - M_1 A_2 (\tau) + 2A_2 (\tau) C^2 A_2 (\tau)] + \mathbb{Q}[-A_2 (\tau) M_2 - M_2 A_2 (\tau) + 2A_2 (\tau) Q^2 A_2 (\tau)], \tag{98}
\]

does not change the terms in respective orders. Thus, we proceed by obtaining

\[
(A_2^0 (\tau))' = 0, \quad \Rightarrow A_2^0 (\tau) \in M_2 (C). \tag{99}
\]

Since \(A_2 (0) = iD_2,\)

\[
(A_2^0 (0))' + \sqrt{p} (A_2^{2.0} (0))' + \sqrt{q} (A_2^{0.2} (0))' + \sqrt{pq} (A_2^{1.1} (0))' = iD_2, \tag{100}
\]

and \( (A_2^{1.1} (\tau))' = 0, A_2^{1.1} (\tau) = 0; \) since \(A_2^{1.1} (0) = 0,\) for all \(\tau \in \mathbb{R}_+, C = 0\) is constant. So now, we can get the functions of \(\lambda_1\) and \(\lambda_2\) from the ordinary differential equations (90) and (91) as follows:
\[ \lambda_1^l (r) = \frac{y(y-1)}{2} r^2 + p\left(-\lambda_1 (r) M_1 - M_1 \lambda_1^l (r) + 2 \lambda_1 (r) Q_1^2 \lambda_1 (r) \right) + 2 \sqrt{p} y R Q \lambda_1 (r) \]
\[ + 2 \sqrt{q} y R Q_2 \lambda_1 (r) + q\left(-\lambda_1 (r) M_2 - M_2 \lambda_1^l (r) + 2 \lambda_1 (r) Q_2^2 \lambda_1 (r) \right), \]
\[ \lambda_1 (r) = \lambda_1^{0,0} (r) + \sqrt{p} \lambda_1^{1,0} (r) + \sqrt{q} \lambda_1^{0,1} (r) + p \lambda_1^{1,0} (r) + q \lambda_1^{0,2} (r) + \sqrt{pq} \lambda_1^{1,1} (r) + o(\max(p, q)), \]
such that

\[ \begin{align*}
\lambda_1^{0,0} (r) &= \frac{y(y-1)}{2} r^2, \\
\lambda_1^{1,0} (r) &= \frac{y^2 (y-1)}{2} \eta_1 r^2 (R M_1), \\
\lambda_1^{0,1} (r) &= \frac{y^2 (y-1)}{2} \eta_2 r^2 (R M_2), \\
\lambda_1^{2,0} (r) &= \frac{y^2 (y-1)}{2} r^2 M_1 + \frac{y^3 (y-1)}{6} \eta_1^2 r^3 M_1 + \frac{y^3 (y-1)}{3} \eta_1^2 r^3 (R M_1)^2, \\
\lambda_1^{0,2} (r) &= \frac{y^2 (y-1)}{2} r^2 M_2 + \frac{y^3 (y-1)}{6} \eta_2^2 r^3 M_2 + \frac{y^3 (y-1)}{3} \eta_2^2 r^3 (R M_2)^2, \\
\lambda_1^{1,1} (r) &= \frac{y^3 (y-1)}{3} \eta_1 \eta_2 r^3 \left[(R M_1) (R M_2) + (R M_2) (R M_1) \right].
\end{align*} \]

Then, similar for \( \lambda_2 \), we have the following approximations:

\[ \begin{align*}
\lambda_2^{0,0} (r) &= \frac{y(y-1)}{2} r^2, \\
\lambda_2^{1,0} (r) &= \frac{y^2 (y-1)}{2} \bar{\eta}_1 r^2 (\bar{R} M_1), \\
\lambda_2^{0,1} (r) &= \frac{y^2 (y-1)}{2} \bar{\eta}_2 r^2 (\bar{R} M_2), \\
\lambda_2^{2,0} (r) &= \frac{y^2 (y-1)}{2} r^2 M_1 + \frac{y^3 (y-1)}{6} \bar{\eta}_1^2 r^3 M_1 + \frac{y^3 (y-1)}{3} \bar{\eta}_1^2 r^3 (\bar{R} M_1)^2, \\
\lambda_2^{0,2} (r) &= \frac{y^2 (y-1)}{2} r^2 M_2 + \frac{y^3 (y-1)}{6} \bar{\eta}_2^2 r^3 M_2 + \frac{y^3 (y-1)}{3} \bar{\eta}_2^2 r^3 (\bar{R} M_2)^2, \\
\lambda_2^{1,1} (r) &= \frac{y^3 (y-1)}{3} \bar{\eta}_1 \bar{\eta}_2 r^3 \left[(\bar{R} M_1) (\bar{R} M_2) + (\bar{R} M_2) (\bar{R} M_1) \right].
\end{align*} \]
Then, we look at the Riccati differential equation for $C(r)$:

\[
C'(r) = \beta Tr[pv_1^2M_1A_1(r) + qv_2^2M_2A_1(r)] + \overline{\beta} Tr[p\overline{v}_1^2M_1A_2(r) + q\overline{v}_2^2M_2A_2(r)],
\]
\[
C^{0,0}(r) = 0.
\]  

Since $C(0) = 0$,

\[
C^{1,0}(r) = C^{0,1}(r) = 0,
\]
\[
C^{2,0}(r) = i\beta v_1^2 d_1 r + i\overline{\beta} \overline{v}_1^2 d_2 r,
\]
\[
C^{0,2}(r) = i\beta v_2^2 d_1 r + i\overline{\beta} \overline{v}_2^2 d_2 r,
\]
\[
C^{1,1}(r) = 0.
\]  

For the $\epsilon$ differential equation,

\[
\epsilon' (r) = r \gamma + \beta Tr[pv_1^2M_1\lambda_1(r) + qv_2^2M_2\lambda_1(r)] + \overline{\beta} Tr[p\overline{v}_1^2M_1\lambda_2(r) + q\overline{v}_2^2M_2\lambda_2(r)],
\]

we have,

\[
\epsilon^{0,0}(r) = r \gamma r,
\]
\[
\epsilon^{0,1}(r) = 0,
\]
\[
\epsilon^{2,0}(r) = \beta v_1^2 \frac{\gamma^{(y - 1)}}{4} r^2 + \overline{\beta} \overline{v}_1^2 \frac{\gamma^{(y - 1)}}{4} r^2,
\]
\[
\epsilon^{0,2}(r) = \beta v_2^2 \frac{\gamma^{(y - 1)}}{4} r^2 + \overline{\beta} \overline{v}_2^2 \frac{\gamma^{(y - 1)}}{4} r^2,
\]
\[
\epsilon^{1,1}(r) = 0.
\]  

Substituting back in the perturbation function, we obtain

\[
A_1(r) = A_1^{0,0}(r) + \sqrt{p} A_1^{1,0}(r) + \sqrt{q} A_1^{0,1}(r) + p A_1^{2,0}(r) + q A_1^{0,2}(r) + \sqrt{pq} A_1^{1,1}(r) + o(\max(p, q)),
\]
\[
A_1(r) = \left( \begin{array}{cc} id_1 & 0 \\ 0 & id_1 \end{array} \right) + \left( \begin{array}{cc} -2pd_1(i + d_1 v_2^2) r & 0 \\ 0 & 0 \end{array} \right) + o(\max(p, q)),
\]
\[
A_1(r) = \left( \begin{array}{cc} -2pd_1^2 v_1^2 r + id_1 (1 - 2pr) & 0 \\ 0 & -2qd_2^2 v_2^2 r + id_2 (1 - 2qr) \end{array} \right),
\]
\[
A_2(r) = \left( \begin{array}{cc} -2pd_2^2 v_1^2 r + id_2 (1 - 2pr) & 0 \\ 0 & -2qd_1^2 v_2^2 r + id_1 (1 - 2qr) \end{array} \right).
\]
The variance processes $\Sigma_0$ and $\Sigma_\beta$ are given as

\[
\Sigma_0 = \begin{pmatrix} u & v \\ v & w \end{pmatrix},
\]

\[
\Sigma_\beta = \begin{pmatrix} \beta & v \\ v & w \end{pmatrix},
\]

\[
A_1(\tau)\Sigma_0 = \begin{pmatrix} -2pd_1^2v_1^2\tau + id_1(1-2pr) & 0 \\ 0 & -2qd_1^2v_1^2\tau + id_1(1-2qr) \end{pmatrix} \begin{pmatrix} u & v \\ v & w \end{pmatrix},
\]

\[
A_1(\tau)\Sigma_\beta = \begin{pmatrix} -2upd_1^2v_1^2\tau + id_1(1-2pr)u & -2vpd_1^2v_1^2\tau + id_1(1-2pr)v \\ -2qv^2d_1^2v_1^2\tau + id_1(1-2qr)v & -2qvd_1^2v_1^2\tau + id_1(1-2qr)w \end{pmatrix},
\]

\[
A_2(\tau)\Sigma_0 = \begin{pmatrix} -2\pi pd_2^2v_2^2\tau + id_2(1-2pr)\pi & -2\pi rd_2^2v_2^2\tau + id_2(1-2pr)\pi \\ -2qvd_2^2v_2^2\tau + id_2(1-2qr)\pi & -2qvd_2^2v_2^2\tau + id_2(1-2qr)\pi \end{pmatrix}.
\]

Taking the trace,

\[
Tr(A_1(\tau)\Sigma_0) = -2d_1^2[u\pi v_1^2 + wq\pi v_2^2]\tau + i[(1-2pr)u + (1-2qr)\pi]d_1,
\]

\[
Tr(A_2(\tau)\Sigma_0) = -2d_2^2[p\pi v_1^2 + wq\pi v_2^2]\tau + i[(1-2pr)\pi + (1-2qr)\pi]d_2.
\]

For functions $C$ and $\varepsilon$,

\[
C(\tau) = pC^{2,0}(\tau) + qC^{0,2}(\tau) + o(\max(p,q))
\]

\[
= p(i\beta v_1^2d_1\tau + i\beta v_1^2d_2\tau) + q(i\beta v_1^2d_1\tau + i\beta v_1^2d_2\tau) + o(\max(p,q))
\]

\[
= i\beta d_1(pv_1^2 + qv_2^2)\tau + i\beta d_2(pv_1^2 + qv_2^2)\tau + o(\max(p,q)).
\]

Then,

\[
\varepsilon(\tau) = \varepsilon^{0,0}(\tau) + p\varepsilon^{2,0}(\tau) + q\varepsilon^{0,2}(\tau) + o(\max(p,q))
\]

\[
= ry\tau + \frac{y(y-1)}{4}p(pv_1^2 + qv_2^2)\tau^2 + \frac{y(y-1)}{4}q(pv_1^2 + qv_2^2)\tau^2 + o(\max(p,q)).
\]

Substituting back in equation (80), we have
\[ \bar{\phi}_\Sigma(t) = \exp \left\{ ry(T-t) + p \left[ -2d_1^2\nu^2_t - 2d_2^2\nu^2_t + \frac{y(y-1)}{4} \beta \nu_1^2(T-t)^2 + \frac{y(y-1)}{4} \beta \nu_2^2(T-t)^2 \right] ight\} + q \left[ -2d_1^2\nu^2_t - 2d_2^2\nu^2_t + \frac{y(y-1)}{4} \beta \nu_1^2(T-t)^2 + \frac{y(y-1)}{4} \beta \nu_2^2(T-t)^2 \right] \times \exp \left\{ \frac{i}{2}(1 - 2pt)ud_1 + (1 - 2qt)wd_1 + (1 - 2pt)\bar{u}d_2 + (1 - 2qt)\bar{w}d_2 \right\} \]

This can still be written as

\[ \bar{\phi}_\Sigma(t) = e^{\Delta_1(t)}e^{i\Delta_2(t)}, \]

where

\[ \Delta_1(t) = \left\{ ry(T-t) + p \left[ -2d_1^2\nu^2_t - 2d_2^2\nu^2_t + \frac{y(y-1)}{4} \beta \nu_1^2(T-t)^2 + \frac{y(y-1)}{4} \beta \nu_2^2(T-t)^2 \right] ight\} + q \left[ -2d_1^2\nu^2_t - 2d_2^2\nu^2_t + \frac{y(y-1)}{4} \beta \nu_1^2(T-t)^2 + \frac{y(y-1)}{4} \beta \nu_2^2(T-t)^2 \right] \]

\[ \Delta_2(t) = \left\{ (1 - 2pt)ud_1 + (1 - 2qt)wd_1 + (1 - 2pt)\bar{u}d_2 + (1 - 2qt)\bar{w}d_2 \right\} \]

such that the call price is as follows:

\[ C_t(T, K) = \frac{e^{-ak}}{2\pi} \text{Re} \left( \int_{-\infty}^{\infty} e^{-r(T-t)}\bar{\phi}_\Sigma(t)\phi(\theta)e^{-i\theta k} d\theta \right) \]

\[ \approx \frac{e^{-ak}}{2\pi} \text{Re} \left( \int_{-\infty}^{\infty} \frac{e^{-r(T-t)}\Delta_1(t)e^{i\Delta_2(t)}}{(\alpha + i\theta)(\alpha + 1 + i\theta)} d\theta \right) \]

since

\[ \int_{-\infty}^{\infty} \frac{e^{-r(T-t)}\Delta_1(t)e^{i\Delta_2(t)}}{(\alpha + i\theta)(\alpha + 1 + i\theta)} d\theta = e^{-r(T-t)}\Delta_1(t)e^{i\Delta_2(t)} \int_{-\infty}^{\infty} \frac{e^{-i\theta k}}{(\alpha + i\theta)(\alpha + 1 + i\theta)} d\theta \]

\[ = e^{-r(T-t)}\Delta_1(t)e^{i\Delta_2(t)} \left[ 2\pi \left( e^{ak} - e^{k(\alpha + 1)} \right) \right]. \]

Finally, we obtain the approximated European call option price as follows:

\[ C_t(T, K) = e^{\left( -r(T-t) + \Delta_1(t) \right)} \left( 1 - e^k \right) \cos \Delta_2(t). \]

4. Numerical Illustration

In this section, we deal with illustrative numerical examples to examine the implications of volatility in option pricing under the double Wishart stochastic volatility model.
Comparison between the Double Wishart Model Price and the Market Price. The volatility specification of the double Wishart model remarkably makes it flexible to produce price predictions which exhibit a similar price behavior with the market price. We quote market data drawn from QQQ (a fund by Invesco that tracks the performance of the stocks listed under the NASDAQ Index) options, April 2020.

Let us consider the estimated parameter values from the market data; the variance matrices $\Sigma$ and $\bar{\Sigma}$, with volatility of volatility matrices $Q$ and $\bar{Q}$, are considered since they are very important parameters in a stochastic volatility model.

$$
\Sigma_0 = \begin{pmatrix}
0.00008 & 0 \\
0 & 0.000095
\end{pmatrix},
$$

$$
\bar{\Sigma}_0 = \begin{pmatrix}
0.00012 & 0 \\
0 & 0.0001
\end{pmatrix},
$$

$$
Q = \begin{pmatrix}
0.10 & 0 \\
0 & 0.23
\end{pmatrix},
$$

$$
\bar{Q} = \begin{pmatrix}
0.20 & 0 \\
0 & 0.25
\end{pmatrix}.
$$

While, the strike price is $K = 189$ and interest rate is $r = 0.05$.

Figure 1 illustrates the effect of implied volatilities on the call price under the double Wishart model. Given the parameter values, we observe that the price predictions under the double Wishart model exhibits the same behavior with respect to the market price in short maturity time. This depends also on the choice of the model parameters, such as $\gamma = 0.5, \beta = 4, \bar{\beta} = 3, d_1 = 0.45$, and $d_2 = 0.479$. This example proves that the double Wishart volatility model has greater flexibility.

Figure 2 shows that a slight change in model parameters leads to changes in the model call option price predictions. We observe that as time tends to maturity time of six months, $(d_2 = 0.478)$, the model price movement behaves almost the same with the market price.

The parameters $\gamma = 0.5, \beta = 4, \bar{\beta} = 3$, and $d_1 = 0.45$ are maintained.

Figure 3 demonstrates the effect of parameters on the call European option price under the double Wishart model. The parameters $\gamma = 1, \beta = 4, \bar{\beta} = 3$, $d_1 = 0.45$, and $d_2 = 0.4968$. The model shows price predictions exhibiting the same market price behavior, in long maturity time of 1 year. We note that the model parameters influence the prices greatly and help long position holders not to experience arbitrage profits.

5. Conclusion

The generalization of the Heston model into a multifactor form with two dependence matrices efficiently solves the problem of pricing European call option and financial market data fitting for short or long maturities. The new Wishart affine model retains the remarkable property of analytical tractability, and this allows the model to form a closed-form solution of the conditional characteristic
function for the call option price expression, which is obtained through Fourier transforms together with perturbation methods.

The numerical results show that the double Wishart stochastic volatility model produces price predictions similar to the market prices. Notably the effect of parameters on our model makes it flexible enough for short or long maturities. We recommend future work on discretization schemes and also to investigate the behavior of nondiagonal matrix components in the model.

Appendix

A. Change of Probability Measure

From mathematics perspective, it is important to discuss the change of the probability measure to allow a change of the drift in the process dynamics. In the financial application, more so in the practical aspects, the Wishart process has to be simulated in its general form with \( \beta \geq n + 1 \), such that \( \beta \in \mathbb{R} \). The function allows to write \( K = \beta + 2\lambda \) with \( \beta = K \geq n + 1 \) and \( \lambda \) real number, such that \( 0 \leq \lambda \leq 1/2 \).

The objective of this idea is to find a change of the probability measure in order to change the generalized Wishart diffusion into the simple one, where \( \beta \) is an integer. Therefore, the new probability measure \( \mathbb{P} \), following Benabid et al. [9] and Shreve [19], can be expressed as follows.

**Theorem 1.** Let \( (\Omega, \mathcal{F}, \mathbb{Q}) \) be the probability space equipped with filtration \( \mathcal{F}_t \), satisfying usual conditions.

Let \( q = \beta + \lambda - n - 1 \). If \( h_T(\mathbb{Q}, \mathbb{P}) = (d\mathbb{Q}/d\mathbb{P}) \) defines the Radon–Nikodym derivative of \( \mathbb{Q} \) with respect to \( \mathbb{P} \), then

\[
   h_T(\mathbb{Q}, \mathbb{P}) = \left( \frac{\det(\Sigma_T)}{\det(\Sigma_0)} \right)^{\lambda/2} \exp\left( -\lambda T \text{Tr}(M) \right) \exp\left[ -\frac{\lambda}{2} q \int_0^T \text{Tr}(\Sigma_s^{-1} Q^T Q) ds \right].
\]

(\ref{eq:1})

\[\text{Proof.}\] The probability measure \( \mathbb{P} \) can be specified through an exponential martingale [9]:

\[
   \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left[ \lambda \int_0^T \text{Tr}\left( \sqrt{\Sigma_s^{-1}} dW_s Q \right) - \frac{\lambda^2}{2} \int_0^T \text{Tr}(\Sigma_s^{-1} Q^T Q) ds \right].
\]

(\ref{eq:2})

We can define a new process as

\[
   W_t = \tilde{W}_t + \lambda \int_0^t \sqrt{\Sigma_s^{-1}} Q^T ds.
\]

(\ref{eq:3})

It is simple to check by Girsanov theorem that \( W \) is a matrix-valued Brownian motion under the probability measure \( \mathbb{P} \). Consequently, the dynamics of the Wishart process under the probability measure \( \mathbb{P} \) can be written as follows:

\[
   d\Sigma_t = \left( \beta QQ^T + M \Sigma_t + \Sigma_t M^T \right) dt + \sqrt{\Sigma_t} \left( d\tilde{W}_t + \lambda \sqrt{\Sigma_t^{-1}} Q^T ds \right) Q + Q^T \left( d\tilde{W}_t + \lambda \sqrt{\Sigma_t^{-1}} Q^T ds \right)^T \sqrt{\Sigma_t},
\]

(\ref{eq:4})

\[
   d\Sigma_t = \left( \beta QQ^T + M \Sigma_t + \Sigma_t M^T \right) dt + \sqrt{\Sigma_t} d\tilde{W}_t Q + Q^T d\tilde{W}_t^T \sqrt{\Sigma_t} + \lambda Q^T Q dt + \lambda Q^T Q d\tilde{W}_t.
\]

(\ref{eq:5})

Then,

\[
   d\Sigma_t = \left( (\beta + 2\lambda) QQ^T + M \Sigma_t + \Sigma_t M^T \right) dt + \sqrt{\Sigma_t} d\tilde{W}_t Q + Q^T d\tilde{W}_t^T \sqrt{\Sigma_t}.
\]

(\ref{eq:6})

The Radon–Nikodym derivative can be handled using the determinant dynamics.
\[
\int_0^T \log(\det \Sigma_t) = \int_0^T (\beta - n - 1) \text{Tr}(\Sigma_t^{-1} Q^T \Sigma_t Q) + 2 \text{Tr}(M)\] ds + 2 \int_0^T \text{Tr} \left[ \sqrt{\Sigma_t^{-1}} dW_s Q \right],
\]

\[
\log \left( \frac{\det \Sigma_T}{\det \Sigma_0} \right) = 2 T \text{Tr}(M) + \int_0^T (\beta - n - 1) \text{Tr}(\Sigma_t^{-1} Q^T \Sigma_t Q) \] ds + \int_0^T \text{Tr} \left[ \sqrt{\Sigma_t^{-1}} dW_s Q \right],
\]

\[
\log \left( \frac{\det \Sigma_T}{\det \Sigma_0} \right)^{\lambda/2} = \lambda T \text{Tr}(M) + \frac{\lambda}{2} \int_0^T (\beta - n - 1) \text{Tr}(\Sigma_t^{-1} Q^T \Sigma_t Q) \] ds + \lambda \int_0^T \text{Tr} \left[ \sqrt{\Sigma_t^{-1}} dW_s Q \right], \quad (A.8)
\]

\[
\left( \frac{\det \Sigma_T}{\det \Sigma_0} \right)^{-\lambda/2} = \exp \left\{ -\lambda T \text{Tr}(M) - \frac{\lambda}{2} \int_0^T (\beta - n - 1) \text{Tr}(\Sigma_t^{-1} Q^T \Sigma_t Q) \] ds - \lambda \int_0^T \text{Tr} \left[ \sqrt{\Sigma_t^{-1}} dW_s Q \right] \right\},
\]

\[
e^{\lambda \int_0^T \text{Tr}[\Sigma_t^{-1} dW_s Q]} = \left( \frac{\det \Sigma_T}{\det \Sigma_0} \right)^{\lambda/2} \exp \left\{ -\lambda T \text{Tr}(M) \right\} \exp \left\{ -\frac{\lambda}{2} \int_0^T (\beta - n - 1) \text{Tr}(\Sigma_t^{-1} Q^T \Sigma_t Q) \] ds \right\}. \quad (A.9)
\]

From equation (A.2), the new measure with \( \tilde{W} \) is given by

\[
\frac{dQ}{dp} = \exp \left\{ \lambda \int_0^T \text{Tr} \left[ \Sigma_t^{-1} d\tilde{W}_s Q \right] \right\} e^{\{a,\Sigma_t^{-1} d\tilde{W}_s Q \}},
\]

\[
\frac{dQ}{dp} = \left( \frac{\det \Sigma_T}{\det \Sigma_0} \right)^{\lambda/2} \exp \left\{ -\lambda T \text{Tr}(M) \right\} \exp \left\{ -\frac{\lambda}{2} \int_0^T (\beta + 2\lambda - n - 1) \text{Tr}(\Sigma_t^{-1} Q^T \Sigma_t Q) \] ds \right\}
\cdot \exp \left\{ -\frac{\lambda^2}{2} \int_0^T \text{Tr}(\Sigma_t^{-1} Q^T \Sigma_t Q) \] ds \right\}
\]

\[
= \left( \frac{\det \Sigma_T}{\det \Sigma_0} \right)^{\lambda/2} \exp \left\{ -\lambda T \text{Tr}(M) \right\} \exp \left\{ -\frac{\lambda}{2} \int_0^T (\beta + 2\lambda - n - 1) \text{Tr}(\Sigma_t^{-1} Q^T \Sigma_t Q) \] ds \right\}
\]

\[
= \left( \frac{\det \Sigma_T}{\det \Sigma_0} \right)^{\lambda/2} \exp \left\{ -\lambda T \text{Tr}(M) \right\} \exp \left\{ -\frac{\lambda}{2} \int_0^T \text{Tr}(\Sigma_t^{-1} Q^T \Sigma_t Q) \] ds \right\},
\]

\[
\frac{dQ}{dp} = \left( \frac{\det \Sigma_T}{\det \Sigma_0} \right)^{\lambda/2} \exp \left\{ -\lambda T \text{Tr}(M) \right\} \exp \left\{ -\frac{\lambda}{2} (\beta + \lambda - n - 1) \int_0^T \text{Tr}(\Sigma_t^{-1} Q^T \Sigma_t Q) \] ds \right\}.
\]

The change of the probability measure is obtained. \( \square \)

**B. The Correlation Structure**

The Brownian matrices \( W_t \) and \( Z_t \) are correlated to give a constant correlated matrix \( R \in \mathbb{M}_{n,n} \) in the study by Da Fonseca et al. [3], which describes the correlation structure for \( Z_t \):

\[
Z_t := W_t R^T + B_t \sqrt{1 - RR^T}, \quad (B.1)
\]

where \( I \) is the identity matrix, \( T \) is the transpose, and \( B_t \) is an independent Brownian motion matrix from \( W_t \). The correlation structure is a Brownian motion.

**Proof.** \( Z_t \) is matrix Brownian motion if and only if \( a, b \in \mathbb{R}^n \):

\[
\text{Cov}_t(dZ_t, a, dZ_t, b) = E_t \left[ (dZ_t, a) (dZ_t, b)^T \right] = a^T b d\text{t}. \quad (B.2)
\]

Since,
\[ \text{Cov}_t (dZ_t, dZ_t) = \mathbb{E}_t \left[ \left( dW_t R^T a + dB_t \sqrt{\| - RR^T a} \right) \left( dW_t R^T b + dB_t \sqrt{\| - RR^T b} \right) \right] \]
\[ = \text{Cov}_t (dW_t R^T a, dW_t R^T b) + \text{Cov}_t (dB_t \sqrt{\| - RR^T a}, dB_t \sqrt{\| - RR^T b}) \]
\[ = a^T RR^T b dt + a^T (1 - RR^T) b dt \]
\[ = a^T b dt. \]

(B.3)

C. Brownian Motions

In Lemma 1, we show that the following processes \( X_t, X_t, \xi_t, \) and \( \eta_t \) are the Brownian motions.

Proof

\[ \text{Cov}_t dX_t, dX_t = \mathbb{E}_t \left[ (dX_t) (dX_t)^T \right] \]
\[ = \mathbb{E}_t \left[ \frac{\text{Tr}(\sqrt{\Sigma_i} dZ_i) \text{Tr}(\sqrt{\Sigma_i} dZ_i)}{\text{Tr}(\Sigma_i)} \right] \]
\[ = \sum_{i,j} \text{Cov}_t(\sqrt{\Sigma_i} dZ_i, \sqrt{\Sigma_j} dZ_j) \frac{\text{Tr}(\Sigma_i)}{\text{Tr}(\Sigma_i)} \]
\[ = \sum_{i,j} \text{Tr}(e_i^T \sqrt{\Sigma_i} dZ_i, e_j^T \sqrt{\Sigma_j} dZ_j) \frac{\text{Tr}(\Sigma_i)}{\text{Tr}(\Sigma_i)} \]
\[ = \sum_{i,j} \frac{\text{Tr}(e_i^T \sqrt{\Sigma_i} e_j) \sqrt{\Sigma_j} e_j dt}{\text{Tr}(\Sigma_i)} \]
\[ = \sum_{i,j} \frac{\text{Tr}(e_i^T \sqrt{\Sigma_i} e_j) \sqrt{\Sigma_j} e_j dt}{\text{Tr}(\Sigma_i)} \]
\[ = \sum_{i,j} \frac{\text{Tr}(e_i^T \sqrt{\Sigma_i} e_j) \sqrt{\Sigma_j} e_j dt}{\text{Tr}(\Sigma_i)} \]
\[ = \frac{\text{Tr}(\Sigma_i) dt}{\text{Tr}(\Sigma_i)} \]
\[ = dt. \]

(C.1)

Similarly, for Brownian motion \( X_t, \) its proof can be obtained through the same procedures as above.

Then, we can also show that \( \xi_t, \) is a Brownian motion as follows.

\[ \text{Proof} \]

\[ \text{Cov}_t d\xi_t, d\xi_t = \mathbb{E}_t \left[ \frac{\text{Tr}(Q dW_t \sqrt{\Sigma_i}) \text{Tr}(Q dW_t \sqrt{\Sigma_i})}{\text{Tr}(Q^T Q \Sigma_i)} \right] \]
\[ = \sum_{i,j} \text{Cov}_t(\sqrt{\Sigma_i} dZ_i, \sqrt{\Sigma_j} dZ_j) \frac{\text{Tr}(\Sigma_i)}{\text{Tr}(\Sigma_i)} \]
\[ = \sum_{i,j} \frac{\text{Tr}(e_i^T \sqrt{\Sigma_i} e_j) \sqrt{\Sigma_j} e_j dt}{\text{Tr}(\Sigma_i)} \]
\[ = \sum_{i,j} \frac{\text{Tr}(e_i^T \sqrt{\Sigma_i} e_j) \sqrt{\Sigma_j} e_j dt}{\text{Tr}(\Sigma_i)} \]
\[ = \frac{\text{Tr}(\Sigma_i) dt}{\text{Tr}(\Sigma_i)} \]
\[ = dt. \]

(C.2)

The same procedures can be followed to show that \( \eta_t, \) is a Brownian motion.

D. Characteristic Function of the Double Wishart Model

Let \( D \) be symmetric matrix, and the conditional characteristic function of the double Wishart \( \Sigma_t \) and \( \Sigma_t \) is given by
\[ \phi_{\Sigma, \tau}^{- D, D} = \mathbb{E}[\exp \{ i \text{Tr}[D_1 \Sigma_\tau + D_2 \Sigma_\tau] \}] \]
\[ = \exp \{ \text{Tr}[A_1(\tau)\Sigma_\tau + A_2(\tau)\Sigma_\tau + C(\tau)] \}, \]

where \( A_1(\tau), A_2(\tau) \in M_n \), and \( C \in \mathbb{C} \) verify the following dynamics.

\[
\begin{align*}
\frac{dA_1(\tau)}{d\tau} &= A_1(\tau)M + M^T A_1(\tau) + 2A_1(\tau)Q A_1(\tau), \\
A_1(0) &= iD_1, \\
\frac{dA_2(\tau)}{d\tau} &= A_2(\tau)\overline{M} + \overline{M}^T A_2(\tau) + 2A_2(\tau)\overline{Q} A_2(\tau), \\
A_2(0) &= iD_2, \\
\frac{dC(\tau)}{d\tau} &= \beta \text{Tr} \left[ Q^T Q A_1(\tau) \right] + \overline{\beta} \text{Tr} \left[ \overline{Q}^T Q A_2(\tau) \right], \\
C(0) &= 0.
\end{align*}
\]

Proof. Obtaining expressions of \( A_1(\tau), A_2(\tau), \) and \( C(\tau) \), through the Riccati equations,

\[
A_1(\tau) = h_1^{-1}(\tau)h_2(\tau), \\
h_1(\tau) = iD_1 A_{11}^{12}(\tau) + A_{11}^{22}(\tau), \\
h_2(\tau) = iD_1 A_{11}^{11}(\tau) + A_{11}^{21}(\tau), \\
A_1(\tau) = \left( iD_1 A_{11}^{12}(\tau) + A_{11}^{22}(\tau) \right)^{-1} \left( iD_1 A_{11}^{11}(\tau) + A_{11}^{21}(\tau) \right), \\
A_2(\tau) = i_1^{-1}(\tau)i_2(\tau), \\
i_1(\tau) = iD_2 A_{21}^{12}(\tau) + A_{21}^{22}(\tau), \\
i_2(\tau) = iD_2 A_{21}^{11}(\tau) + A_{21}^{21}(\tau), \\
A_2(\tau) = \left( iD_2 A_{21}^{12}(\tau) + A_{21}^{22}(\tau) \right)^{-1} \left( iD_2 A_{21}^{11}(\tau) + A_{21}^{21}(\tau) \right).
\]

\[
A_1(\tau) = h_1^{-1}(\tau)h_2(\tau), \\
h_1(\tau) = iD_1 A_{11}^{12}(\tau) + A_{11}^{22}(\tau), \\
h_2(\tau) = iD_1 A_{11}^{11}(\tau) + A_{11}^{21}(\tau), \\
A_1(\tau) = \left( iD_1 A_{11}^{12}(\tau) + A_{11}^{22}(\tau) \right)^{-1} \left( iD_1 A_{11}^{11}(\tau) + A_{11}^{21}(\tau) \right), \\
A_2(\tau) = i_1^{-1}(\tau)i_2(\tau), \\
i_1(\tau) = iD_2 A_{21}^{12}(\tau) + A_{21}^{22}(\tau), \\
i_2(\tau) = iD_2 A_{21}^{11}(\tau) + A_{21}^{21}(\tau), \\
A_2(\tau) = \left( iD_2 A_{21}^{12}(\tau) + A_{21}^{22}(\tau) \right)^{-1} \left( iD_2 A_{21}^{11}(\tau) + A_{21}^{21}(\tau) \right).
\]

\[ \square \]

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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