Moduli Instability in Warped Compactification

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Abstract

We derive four-dimensional effective theories for warped compactification of the ten-dimensional IIB supergravity. We show that these effective theories allow a much wider class of solutions than the original higher-dimensional theories. This result indicates that the effective four-dimensional theories should be used with caution, if one regards the higher-dimensional theories more fundamental.

1 Introduction

Recently, a new class of dynamical solutions describing a size-modulus instability in the ten-dimensional type IIB supergravity model have been discovered by Gibbons et al. \cite{Gibbons:2005xe} and the authors \cite{Kodama:2005pf}. These solutions can be always obtained by replacing the constant modulus $h_0$ in the warp factor $h = h_0 + h_1(y)$ for supersymmetric solutions by a linear function $h_0(x)$ of the four-dimensional coordinates $x^\mu$. Such extensions exist for many of the well-known solutions compactified with flux on a conifold, resolved conifold, deformed conifold and compact Calabi-Yau manifold \cite{Kodama:2006sz}.

In most of the literature, the dynamics of the internal space, namely the moduli, in a higher-dimensional theory is investigated by utilising a four-dimensional effective theory. In particular, effective four-dimensional theories are used in essential ways in recent important work on the moduli stabilisation problem and the cosmological constant/inflation problem in the IIB sugra framework \cite{Kachru:2003aw}. Hence, it is desirable to find the relation between the above dynamical solutions in the higher-dimensional theories and solutions in the effective four-dimensional theory.

In the conventional approach where the non-trivial warp factor does not exist or is neglected, an effective four-dimensional theory is derived from the original theory assuming the “product-type” ansatz for field variables. This ansatz requires that each basic field of the theory is expressed as the sum of terms of the form $f(x)\omega(y)$, where $f(x)$ is an unknown function of the four-dimensional coordinates $x^\mu$, and $\omega(y)$ is a known harmonic tensor on the internal space. Further, it is assumed that the higher-dimensional metric takes the form $ds^2 = ds^2(X_4) + h_0^2(x)ds^2(Y)$, where $ds^2(X_4) = g_{\mu\nu}(x)dx^\mu dx^\nu$ is an unknown four-dimensional metric, $h_0(x)$ is the size modulus for the internal space depending only on the $x$-coordinates, and $ds^2(Y) = \gamma_{pq}dy^pdq$ is a (Calabi-Yau) metric of the internal space that depends on the $x$-coordinates only through moduli parameters. Under this ansatz, the four-dimensional effective action is obtained by integrating out the known dependence on $y^p$ in the higher-dimensional action.

The dynamical solutions in the warped compactification mentioned at the beginning, however, do not satisfy this ansatz. Hence, in order to incorporate such solutions to the effective theory, we have to modify the ansatz. Taking account of the structure of the supersymmetric solution, the most natural modification of the ansatz is to introduce the non-trivial warp factor $h$ into the metric as $ds^2 = h^\alpha ds^2(X_4) + h^\beta ds^2(Y)$ and assume that $h$ depends on the four-dimensional coordinates $x^\mu$ only through the modulus parameter of the supersymmetric solution as in the case of the internal moduli degrees of freedom. This leads to the form $h = h_0(x) + h_1(y)$ for the IIB models, which is consistent with the structure of the dynamical solutions in the ten-dimensional theory.

In the present work, starting from this modified ansatz, we study the dynamics of the four-dimensional effective theory and its relation to the original higher-dimensional theory for warped compactification of the ten-dimensional type IIB supergravity. For simplicity, we assume that the moduli parameters other than the size parameter are frozen.

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2 Ten-dimensional solutions

In our previous work \[2\], we derived a general dynamical solution for warped compactification with fluxes in the ten-dimensional type IIB supergravity. In that work, we imposed \( d \ast (B_2 \wedge H_3) = 0 \), which led to a slightly strong constraint on the free data for the solution, especially in the case of a compact internal space. Afterward, we have noticed that this condition is not necessary to solve the field equations, and without that condition, we can find a more general class of solutions. Because we take this class as the starting point of our argument, we first briefly explain how to get a general solution without that condition. We omit the details of calculations because they are essentially contained in our previous paper \[2\].

Now we study the four-dimensional effective theory that incorporates the dynamical solutions obtained in Ref. \[2\] does not appear. Further, closed ISD 3-forms on \( Y_6 \) are in one-to-one correspondence with real harmonic 3-forms on \( Y_6 \). Hence, this class of dynamical solutions exist even for a generic compact Calabi-Yau internal space, if we allow \( h_1(y) \) to be a singular function. This singular feature of \( h \) in the compact case with flux arises because \( h \) is a solution to the Poisson equation \[3\] and has nothing to do with the dynamical nature of the solution. It is shared by the other flux compactification models.

Next, by using the equations \[4\] and \[3\], the ten-dimensional Einstein equations can be written

\[
R_{\mu\nu}(X_4) - D_\mu D_\nu h + \frac{1}{4} g_{\mu\nu}(X_4) \triangle h = 0, \quad \partial_\mu \partial_\nu h = 0, \quad R_{pq}(Y_6) - \frac{1}{4} g_{pq}(Y_6) \triangle h = 0. \tag{4}
\]

From the second of these equations, we immediately see that the warp factor \( h \) can be expressed as \( h(x, y) = h_0(x) + h_1(y) \). Further, if we require that \( d_y h \neq 0 \), the rest of the equations can be reduced to

\[
R_{\mu\nu}(X_4) = 0, \quad D_\mu D_\nu h_0 = \lambda g_{\mu\nu}(X_4), \quad R_{pq}(Y_6) = \lambda g_{pq}(Y_6). \tag{5}
\]

Thus, we have found that the most general solutions satisfying the conditions \[11\] and \[2\] are specified by a Ricci flat spacetime \( X_4 \), an Einstein space \( Y_6 \), a closed imaginary-self-dual (ISD) 3-form \( G_3 \) on \( Y_6 \), and the function \( h(x, y) \) that is the sum of \( h_0(x) \) satisfying the second of the equations \[11\] and \( h_1(y) \) satisfying the second of the equations \[3\]. The additional constraint on \( G_3 \), \( d_y [h^{-2}(B_2 \cdot dB_2)] = 0 \), in Ref. \[2\] does not appear. Further, closed ISD 3-forms on \( Y_6 \) in one-to-one correspondence with real harmonic 3-forms on \( Y_6 \). Hence, this class of dynamical solutions exist even for a generic compact Calabi-Yau internal space, if we allow \( h_1(y) \) to be a singular function. This singular feature of \( h \) in the compact case with flux arises because \( h \) is a solution to the Poisson equation \[3\] and has nothing to do with the dynamical nature of the solution. It is shared by the other flux compactification models.

Here, note that the Ricci flatness of \( X_4 \) is required from the Einstein equations. This should be contrasted with no-warp case \[3\]. This point is quite important in the effective theory issue, as we see soon. Moreover, one can show that the Ricci flatness of \( X_4 \) and the second in the equations \[5\] are consistent only when \( X_4 \) is locally flat if \( (Dh_0)^2 \neq 0 \).

3 Four-dimensional effective theory

Now we study the four-dimensional effective theory that incorporates the dynamical solutions obtained in the previous section. For simplicity, we do not consider the internal moduli degrees of freedom of the metric of \( Y_6 \) or of the solution \( h_1(y) \) in the present work. Then, in its \( x \)-independent subclass with \( \lambda = 0 \),
we have only one free parameter \( h_0 \). When we rescale \( ds^2(Y_6) \) by a constant \( \ell \) as \( \ell^2 ds^2(Y_6) \rightarrow ds^2(Y_6) \), we have to rescale \( h \) as \( h \rightarrow h/\ell^4 \). We can easily see that the corresponding rescaled \( h_1 \) satisfies \( G_3 \) again with the same \( G_3 \) as that before the rescaling. We can also confirm that the D3 brane charges associated with the 5-form flux do not change by this scaling. In contrast, \( h_0 \) changes its value by this rescaling. Therefore, \( h_0 \) represents the size modulus of the Calabi-Yau space \( Y_6 \).

On the basis of this observation, we construct the four-dimensional effective theory for the class of ten-dimensional configurations specified as follows. First, we assume that \( X_{10} \) has the metric

\[
ds^2(X_{10}) = h^{-1/2}(x, y) ds^2(X_4) + h^{1/2}(x, y) ds^2(Y_6),
\]

where \( h = h_0(x) + h_1(y) \) and \( ds^2(Y_6) \) is a fixed Einstein metric on \( Y_6 \) satisfying the third of the equations \( \delta \), while \( ds^2(X_4) \) is an arbitrary metric on \( X_4 \). Further, we assume that the dilaton is frozen as in \( \eta \), \( G_3 \) is given by a fixed closed ISD 3-form on \( Y_6 \), \( h_1(y) \) is a fixed solution to the second of the equations \( \delta \), and \( \bar{F}_5 \) is given by the first of the equations \( \delta \). Hence, the metric of \( X_4 \) and the function \( h_0 \) on it are the only dynamical variables in the effective theory.

The four-dimensional effective action for these variables can be obtained by evaluating the ten-dimensional action of the IIB theory

\[
S_{\text{IIB}} = \frac{1}{2\kappa^2} \int_{X_{10}} d\Omega(X_{10}) \left[ R(X_{10}) - \frac{\nabla M \cdot \nabla M}{2(1+M)^2} - \frac{G_3 \cdot \bar{G}_3}{2 \mathrm{Im} \tau} - \frac{1}{4} \bar{F}_5^2 \right] \pm \frac{i}{8\kappa^2} \int_{X_{10}} C_4 \wedge G_3 \wedge \bar{G}_3, \tag{7}
\]

for the class of configurations specified above. In general, there is a subtlety concerning the action of the type IIB supergravity, because the correct field equations can be obtained by imposing the self-duality condition \( *F_5 = \pm \bar{F}_5 \) after taking variation of the action in general. In the present case, however, since we are only considering configurations \( \delta \) satisfying the self-duality condition, this problem does not affect our argument. We can obtain the "correct" effective action by simply inserting \( \delta \) into the above ten-dimensional action.

Inserting the expressions \( \delta \) and the third of the equations \( \delta \) into \( \delta \), we get

\[
S_{\text{IIB}} = \frac{1}{2\kappa^2} \int_{X_4} d\Omega(X_4) [HR(X_4) + 6\lambda], \tag{8}
\]

where we have dropped the surface term coming from \( \triangle_X h_0 \) and neglected the boundary term in the Chern-Simons term, \( \kappa = (V_6)^{-1/2} \bar{k} \), and \( H(x) \) is defined by

\[
H(x) = h_0(x) + c; \quad c := V_6^{-1} \int_{Y_6} d\Omega(Y_6) h_1. \tag{9}
\]

This effective action has the same form as the case of vanishing flux \( \delta \). Hence, it gives the four-dimensional field equations of the same form as in the no-flux case:

\[
R_{\mu\nu}(X_4) = H^{-1} \left[ D_\mu D_\nu H - \lambda g_{\mu\nu}(X_4) \right], \quad \triangle_X H = 4\lambda. \tag{10}
\]

If the four-dimensional spacetime is Ricci flat, these equations reproduce the correct equation for \( h_0(x) = H - c \) obtained from the ten-dimensional theory \( \delta \). However, the Ricci flatness of \( X_4 \) is not required in the effective theory unlike in the ten-dimensional theory. Hence, the class of solutions allowed in the four-dimensional effective theory is much larger than the original ten-dimensional theory.

In particular, the effective theory has a modular invariance similar to that found in the no-flux Calabi-Yau case with \( \lambda = 0 \). In fact, by the conformal transformation \( ds^2(X_4) = H^{-1} ds^2(\bar{X}_4) \), \( \delta \) is expressed in terms of the variables in the Einstein frame as

\[
S_{\text{IIB}} = \frac{1}{2\kappa^2} \int_{\bar{X}_4} d\Omega(\bar{X}_4) \left[ R(\bar{X}_4) - \frac{3}{2}(\bar{D} \ln H)^2 + 6\lambda H^{-2} \right], \tag{11}
\]

where \( R(\bar{X}_4) \) and \( \bar{D}_\mu \) are the scalar curvature and the covariant derivative with respect to the metric \( ds^2(\bar{X}_4) \). The corresponding four-dimensional Einstein equations in the Einstein frame and the field equation for \( H \) are given by

\[
R_{\mu\nu}(\bar{X}_4) = \frac{3}{2} \bar{D}_\mu \ln H \bar{D}_\nu \ln H - 3\lambda H^{-2} g_{\mu\nu}(\bar{X}_4), \quad \triangle_{\bar{X}} \ln H = 4\lambda H^{-2}, \quad \nabla^2 \ln H = 4\lambda H^{-2}, \tag{12}
\]
where $\Delta_{\bar{X}}$ is the Laplacian with respect to the metric $ds^2(X_4)$. It is clear that for $\lambda = 0$, this action and the equations of motion are invariant under the transformation $H \rightarrow k/H$, where $k$ is an arbitrary positive constant.

This transformation corresponds to the following transformation in the original ten-dimensional metric. Let us denote the new metric of $X_4$ and the function $h$ obtained by this transformation by $ds'^2(X_4)$ and $h'$, respectively. Then, since the transformation preserves the four-dimensional metric in the Einstein frame, $ds'^2(X_4)$ is related to $ds^2(X_4)$ as $ds'^2(X_4) = (H^2/k)ds^2(X_4)$. In the meanwhile, from $H' = k/H = h'_0 + c$, $h'$ is expressed in terms of the original $h_0$ as

$$h' = k \frac{1}{h_0(x) + c} - c + h_1(y). \quad (13)$$

The corresponding ten-dimensional metric is written

$$ds^2 = k^{-1}H^2(h')^{-1/2}ds^2(X_4) + (h')^{1/2}ds^2(Y_6). \quad (14)$$

It is clear that this metric and $h'$ do not satisfy the original ten-dimensional field equations. Hence, the modular-type invariance of the four-dimensional effective theory is not the invariance of the original ten-dimensional theory.

4 Summary

In the present work, we have derived four-dimensional effective theories for the spacetime metric and the size modulus of the internal space for warped compactification with flux in the ten-dimensional type IIB supergravity. The basic idea was to consider field configurations in higher dimensions that are obtained by replacing the constant size modulus in supersymmetric solutions for warped compactifications, by a field on the four-dimensional spacetime. The effective action for this moduli field and the four-dimensional metric has been determined by evaluating the higher-dimensional action for such configurations. In all cases, the dynamical solutions in the ten-dimensional theories found by Gibbons et al. [1], Kodama and Uzawa [2] were reproduced in the four-dimensional effective theories [4].

In addition to this, we have found that these four-dimensional effective theories have some unexpected features. First, the effective actions of both theories are exactly identical to the four-dimensional effective action for direct-product type compactifications with no flux in ten-dimensional supergravities. In particular, the corresponding effective theory has a kind of modular invariance with respect to the size modulus field in the Einstein frame. This implies that if there is a solution in which the internal space expands with the cosmic expansion, there is always a conjugate solution in which the internal space shrinks with the cosmic expansion.

Second, the four-dimensional effective theory for warped compactification allows solutions that cannot be obtained from solutions in the original higher-dimensional theories. The modular invariance in the four-dimensional theory mentioned above is not respected in the original higher-dimensional theory either. The same results hold for the heterotic M-theory [4]. This situation should be contrasted with the no-warp case in which the four-dimensional effective theory and the original higher-dimensional theory are equivalent under the product-type ansatz for the metric structure. This result implies that we have to be careful when we use a four-dimensional effective theory to analyse the moduli stabilisation problem and the cosmological problems in the framework of warped compactification of supergravity or M-theory.

References

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