Extremal Polynomials and Entire Functions of Exponential Type

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Abstract

In this paper, we discuss asymptotic relations for the approximation of \(|x|^\alpha, \alpha > 0\) in \(L_\infty[-1,1]\) by Lagrange interpolation polynomials based on the zeros of the Chebyshev polynomials of first kind. The limiting process reveals an entire function of exponential type for which we can present an explicit formula. As a consequence, we further deduce an asymptotic relation for the approximation error when \(\alpha \to \infty\). Finally, we present connections of our results together with some recent work of Ganzburg [5] and Lubinsky [10], by presenting numerical results, indicating a possible constructive way towards a representation for the Bernstein constants.

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1 The Bernstein Constant

Let \(\alpha > 0\) be not an even integer. Starting in year 1913 for the case \(\alpha = 1\), and later in 1938 for the general case \(\alpha > 0\), S.N. Bernstein [1], [3] established the limit

\[
\Delta_{\infty,\alpha} = \lim_{n \to \infty} n^\alpha E_n(|x|^\alpha, L_\infty[-1,1]),
\]

where

\[
E_n(f, L_p[a,b]) = \inf \left\{ \|f - p\|_{L_p[a,b]} : \deg(p) \leq n \right\}
\]

denotes the error in best \(L_p\) approximation of a function \(f\) on the interval \([a,b]\) by polynomials of degree less or equal \(n\). The proofs in [1], [3] are highly difficult and long, missing many non-trivial technical details. In his
1938 paper, Bernstein made essential use of the homogeneity property of $|x|^\alpha$, namely that for $c > 0$ one has $|cx|^\alpha = c^\alpha |x|^\alpha$. Using this property, one gets for $a, b > 0$ and all $1 \leq p \leq \infty$ the relation (see [10], Lemma 8.2)

$$E_n (|x|^\alpha, L_p [-b, b]) = \left( \frac{b}{a} \right)^{\alpha + \frac{1}{p}} E_n (|x|^\alpha, L_p [-a, a]) . \quad (1.1)$$

This enabled Bernstein to relate the uniform best approximating error on $[-1, 1]$ to that on $[-n, n]$. A routine argument shows that identity (1.1) sends the best approximating polynomials $P_n^*$ of order $n$ with respect to $[-1, 1]$ into a sequence \( \{ n^\alpha P_n^* \left( \frac{x}{n} \right) : n = 1, 2, \ldots \} \) of scaled polynomials in $[-n, n]$. Bernstein also established a formulation of the limit as the error in approximation on the real line by entire functions of exponential type, namely,

$$\Delta_{\infty, \alpha} = \lim_{n \to \infty} n^\alpha E_n (|x|^\alpha, L_\infty [-1, 1])$$

$$= \lim_{n \to \infty} E_n (|x|^\alpha, L_\infty [-n, n])$$

$$= \lim_{n \to \infty} \left\| |x|^\alpha - n^\alpha P_n^* \left( \frac{x}{n} \right) \right\|_{L_\infty [-n, n]}$$

$$= \inf \left\{ \left\| |x|^\alpha - H \right\|_{L_\infty (\mathbb{R})} : H \text{ is entire of exponential type} \leq 1 \right\} . \quad (1.2)$$

Recall that an entire function $f$ is of exponential type $A \geq 0$ means that for each $\varepsilon > 0$ there is $z_0 = z_0 (\varepsilon)$, such that

$$|f (z)| \leq \exp \left( |z| (A + \varepsilon) \right), \quad \forall z \in \mathbb{C} : |z| \geq |z_0| . \quad (1.3)$$

Moreover, $A$ is taken to be the infimum over all possible numbers for which (1.2) holds. The elegant formulation which introduces now functions of exponential type extends to spaces other than $L_\infty$. Ganzburg [15] and Lubinsky [10] have shown that for all $1 \leq p \leq \infty$ positive constants $\Delta_{p, \alpha}$ exists, where $\Delta_{p, \alpha}$ is defined by

$$\Delta_{p, \alpha} = \lim_{n \to \infty} n^{\alpha + \frac{1}{p}} E_n (|x|^\alpha, L_p [-1, 1])$$

$$= \inf \left\{ \left\| |x|^\alpha - H \right\|_{L_p (\mathbb{R})} : H \text{ is entire of exponential type} \leq 1 \right\} . \quad (1.3)$$

From now on $\Delta_{p, \alpha}$ are called the Bernstein constants. Only for $p = 1, 2$ are the values $\Delta_{p, \alpha}$ known. In 1947, Nikolskii [12] proved that

$$\Delta_{1, \alpha} = \frac{|\sin \frac{\alpha \pi}{2}|}{\pi} 8 \Gamma (\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{(1 + 2n)^{\alpha + 2}}, \quad \alpha > -1,$$
and in 1969, Raitsin [16] established
\[ \Delta_{2,\alpha} = \frac{\left| \sin \frac{2\pi}{2} \right|}{\pi} 2\Gamma(\alpha + 1) \sqrt{\frac{\pi}{2\alpha + 1}}, \quad \alpha > -\frac{1}{2}. \]

In contrast to the case of the \( L_\infty \) norm, no single value of \( \Delta_{\infty,\alpha} \) is known. Bernstein speculated that
\[ \Delta_{\infty,1} = \lim_{n \to \infty} nE_n(|x|, L_\infty [-1, 1]) = \frac{1}{2\sqrt{\pi}} = 0.28209 \ 47917 \ldots \]

Over the years the speculation became known as the Bernstein conjecture in approximation theory. Some 70 years later Varga and Carpenter [20], using sophisticated high precision scientific computational methods, calculated the quantity numerically to
\[ \Delta_{\infty,1} = 0.28016 \ 94990 \ 23869 \ldots \]

Further extensive numerical explorations for the computation of \( \Delta_{\infty,\alpha} \) have been provided later by Varga and Carpenter [21]. Their numerical work gave an enormous impact into the analytical investigation of approximation problems, not only restricted to the Bernstein constants. We would also like to mention the numerical work of Pachón and Trefethen ([13], Figure 4.4) from 2008, when they recomputed \( \{nE_n(|x|, L_\infty [-1, 1]) : n = 1, \ldots, 10^4 \} \) again and provided an graphical illustration indicating a monotonic growth behavior. As the story continued, the approximation of entire functions of exponential type became a much studied topic in function approximation, see [4], [19], but also in connection to problems in number theory, see for instance [22]. As an further application in number theory, we would like to mention a recent paper of Ganzburg [7], where he discusses new asymptotic relations between Zeta-, Dirichlet- and Catalan functions in connection with the asymptotics of Lagrange-Hermite interpolation for \( |x|^\alpha \).

Turning back to the Bernstein constants \( \Delta_{p,\alpha} \), intensive emphasis has been placed on the structure of those entire functions of exponential type which minimize (1.3). For \( p = 1 \), the (unique) minimizing entire function of exponential type 1 may be expressed as an interpolation series at the nodes \( \{(j - \frac{1}{2}) \pi : j = 1, 2, \ldots \} \), see ([5], p. 197) or ([11], Formula 1.8). For \( p = \infty \) an analogous interpolation series at unknown interpolation nodes was derived by Lubinsky in ([11], Theorem 1.1). In ([10], Theorem 1.1) he proved the following result.

Denote by \( P_n^\alpha \) the best approximating polynomial of order \( n \) to \( |x|^\alpha \) in the
\[ \Delta_{p,\alpha} = \lim_{n \to \infty} n^{\alpha + \frac{1}{p}} \| |x|^\alpha - P^*_n \|_{L^p[-1,1]} \]

\[ = \lim_{n \to \infty} n^{\alpha + \frac{1}{p}} E_n (|x|^\alpha, L_p [-1, 1]) \]

\[ = \lim_{n \to \infty} E_n (|x|^\alpha, L_p [-n, n]) \]

\[ = \lim_{n \to \infty} \| |x|^\alpha - n^\alpha P^*_n \left( \frac{\cdot}{n} \right) \|_{L^p[-n,n]} \]

\[ = \| |x|^\alpha - H^*_n \|_{L^p(\mathbb{R})} \]

\[ = \inf \left\{ \| |x|^\alpha - H \|_{L^p(\mathbb{R})} : H \text{ is entire of exponential type } \leq 1 \right\}. \quad (1.4) \]

Moreover, uniformly on compact subsets of \( \mathbb{C} \),

\[ \lim_{n \to \infty} n^\alpha P^*_n \left( \frac{\cdot}{n} \right) = H^*_\alpha (z), \]

and there is exactly one entire function \( H \) of exponential type \( \leq 1 \) which minimizes (1.4). While various versions of this equality and relations (1.4) have been discussed by Bernstein, Raitsin and Ganzburg, the uniqueness of \( H^*_\alpha \) proved in [10] is a highly nontrivial result.

From the Chebyshev alternation theorem it follows that for each integer \( n \) the best approximating polynomial \( P^*_n \) of order \( n \) to \(|x|^\alpha\) in the in \( L^\infty \) norm can be represented as an interpolating polynomial with unknown consecutive nodes in \([-1,1]\). Thus, if one can find something about the nature of those best approximating interpolation nodes in \([-1,1]\), then we would successfully find an approach for a constructive analytical approximation towards some representations for the Bernstein constants \( \Delta_{\infty,\alpha} \). Since \(|x|^\alpha\) is an even function a standard argument allows us to restrict ourselves to interpolation polynomials of even order \( n = 2m \). It is not surprising that Bernstein [2] himself, in 1937, studied the interpolation process to \(|x|^\alpha\) by using the modified Chebyshev system

\[ x_0^{(2n)} = 0, \]

\[ x_j^{(2n)} = \cos \left( \frac{j - 1/2}{2n} \pi \right), \quad j = 1, 2, \ldots 2n, \]

where the \( x_j^{(2n)} \) are the zeros of the Chebyshev polynomial \( T_{2n} \) of first kind, defined by \( T_n(x) = \cos (n \arccos x) \). However, \( x_0^{(2n)} \) is an additional choice, but not a zero of \( T_{2n} \), in order to obtain the corresponding interpolation polynomial \( P^{(1)}_{2n} \) of order \( 2n \) for \(|x|^\alpha\). The final answer for its limit relation
was given not before 2002 by Ganzburg ([5], Formula 2.7). For $\alpha > 0$ one has

$$\lim_{n \to \infty} (2n)^\alpha \| x^\alpha - P_{2n}^{(1)} \|_{L_\infty[-1,1]} = \frac{2}{\pi} \left| \sin \frac{\pi \alpha}{2} \right| \int_0^\infty \frac{t^{\alpha-1}}{\cosh (t)} \, dt.$$  \quad (1.5)

Let us give some remarks on equation (1.5). Firstly, we mention that in [2] Bernstein himself established a slightly weaker solution compared to formula (1.5). Secondly, an extension of limit relation (1.5) to complex values for $\alpha$ was obtained recently in [6].

It is remarkable that, since the beginning with Bernstein, no one has studied in detail the interpolation process by using the node system consisting of the $2n + 1$ zeros of $T_{2n+1}$, since this node system automatically includes $x = 0$ as a node and apparently it seems to be the more natural choice. To go into detail, let

$$x_j^{(2n+1)} = \cos \left( \frac{j - 1/2}{2n + 1} \right) \pi, \quad j = 1, 2, \ldots, 2n + 1,$$

be the zeros of $T_{2n+1}$ and let us denote by $P_{2n}^{(2)}$ the corresponding interpolation polynomial of order $2n$ for $|x|^{\alpha}$. There is one paper [23], dealing with this node system and presenting the result that the approximation order

$$\| |x|^{\alpha} - P_{2n}^{(2)} \|_{L_\infty[-1,1]} = O \left( \frac{1}{\sqrt{n}} \right)$$

when $\alpha \in (0, 1)$. In other words, the interpolation process attains the Jackson order. We also would like to mention a recent monograph by Ganzburg ([8], Theorem 4.2.3, Corollary 4.3.2 and Remark 4.3.3) for a more general approach to pointwise asymptotic relations within this topic.

In 2013, the author [17] established a strong asymptotic formula, valid for all $\alpha > 0$, from which he established an upper estimate for the error term, see ([17], Corollary 2), by showing that

$$\lim_{n \to \infty} (2n)^\alpha \| x^\alpha - P_{2n}^{(2)} \|_{L_\infty[-1,1]} \leq \frac{2}{\pi} \left| \sin \frac{\pi \alpha}{2} \right| \int_0^\infty \frac{t^{\alpha}}{\sinh (t)} \, dt,$$  \quad (1.6)

introducing an integral of similar nature to that in formula (1.5). In this paper we continue the investigation into the precise limiting quantity of

$$(2n)^\alpha \| |x|^{\alpha} - P_{2n}^{(2)} \|_{L_\infty[-1,1]}$$

for all $\alpha > 0$.

The paper is organized as follows.

In section 2 we collect some definitions for several constants and functions together with some standard results for later use.
In section 3 we establish the precise limit relation (Theorem 3.1) and we show that the scaled polynomials \( n^\alpha P_n^{(2)} \left( \frac{x}{n} \right) \) uniformly converge on compact subsets of the real line to an entire function \( H_\alpha \) of exponential type 1 (Theorems 3.2 and 3.3). We may also present an explicit expansion for \( H_\alpha \) as an interpolating series for \( |x|^\alpha \) (Theorem 3.3). As it can be seen later from the representation for the explicit limiting error term, i.e. from
\[
\lim_{n \to \infty} (2n)^\alpha \| |x|^\alpha - P_{2n}^{(3)} \|_{L_\infty[-1,1]} = \| H (\alpha, \cdot) \|_{L_\infty[0,\infty)}, \tag{1.7}
\]
the exact determination of the quantity on the right-hand side in (1.7) for individual values for \( \alpha \) appears to be a rather difficult challenge.

In section 4 we study a certain envelope function \( H_1 (\alpha, \cdot) \) with respect to \( |H (\alpha, \cdot)| \). We then present in Theorem 4.1 an asymptotic formula for \( \| H_1 (\alpha, \cdot) \|_{L_\infty[0,\infty)} \), when \( \alpha \to \infty \), involving again the integral in formula (1.6).

In section 5, by using an higher order asymptotics and investigating into an (itself) interesting integral inequality, see Theorem 5.1 we finally arrive in Theorem 5.3 at an asymptotic connection between \( \| H_1 (\alpha, \cdot) \|_{L_\infty[0,\infty)} \) and \( \| H (\alpha, \cdot) \|_{L_\infty[0,\infty)} \), when \( \alpha \to \infty \).

In Section 6, to emphasize the importance of the interpolation formulas based on the \( P_n^{(1)} \) and \( P_n^{(2)} \) polynomials, we present a compilation of numerical results involving some non-trivial linear combinations of the just mentioned polynomials together with their corresponding Chebyshev polynomials \( T_n \), in order to present explicit formulas for near best approximation polynomials in the \( L_\infty \) norm, see formula (6.3), together with their corresponding entire functions of exponential type, see formula (6.4). Possibly and hopefully these formulas could indicate a feasible direction towards some explicit asymptotic representations of best approximation polynomials for \( |x|^\alpha \) in the \( L_\infty \) norm and thus for the Bernstein constants \( \Delta_{\infty, \alpha} \) themselves.

2 Notation

In this section we record the following constants and functions, together with properties which are used later in the paper. We denote by \( \Gamma (\cdot) \) the usual Gamma function. The Chebyshev polynomials of first kind are denoted by \( T_n \), where \( T_n (x) = \cos (n \arccos x) \). For \( x \in \mathbb{R} \), let \( [x] \) to be the floor function, namely \( [x] = \max \{ m \in \mathbb{Z} : m \leq x \} \). Obviously, then \( x - 1 < [x] \leq x \). We
define the following constants.

\[ C(\alpha) = \int_0^\infty \frac{t^\alpha}{\sinh(t)} dt, \quad \alpha > 0, \]
\[ Z(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha}, \quad \alpha > 1. \]

Next, we define the following functions.

\[ H(\alpha, x) = \int_0^\infty \frac{t^\alpha x \sin(x)}{\sinh(t)^{x^2 + t^2}} dt, \quad \alpha > 0, x > 0, \]
\[ H_1(\alpha, x) = \int_0^\infty \frac{t^\alpha x}{\sinh(t)^{x^2 + t^2}} dt, \quad \alpha > 0, x > 0, \]
\[ H_2(\alpha, x) = \int_0^\infty \frac{t^\alpha x^2}{\sinh(t)^{x^2 + t^2}} dt, \quad \alpha > 0, x > 0. \]

Note that \( H(\alpha, \cdot) \) should not be mixed up with the subsequent following definition of \( H_\alpha \). We proceed further with:

\[ F(\alpha, x) = \int_0^\infty \frac{t^\alpha}{\sinh(xt)^{1 + t^2}} dt, \quad \alpha > 0, x > 0, \]
\[ G(\alpha, x) = \int_0^\infty t^\alpha e^{-xt} \frac{1}{1 + t^2} dt, \quad \alpha > 0, x > 0, \]
\[ R(\alpha, x) = \frac{x}{\alpha} F(\alpha + 1, x) - F(\alpha, x), \quad \alpha > 0, x > 0, \]
\[ S(\alpha, x) = \left(\alpha x^{\alpha - 1}/2\right) (x^2 + \alpha^2) R(\alpha, x), \quad \alpha > 0, x > 0, \]
\[ F_1(\alpha, x) = \left(2 - \frac{1}{2^\alpha}\right) Z(\alpha + 1) G(\alpha, x), \quad \alpha > 0, x > 0, \]
\[ F_2(\alpha, x) = \left(2 - \frac{1}{2^{\alpha-2}}\right) Z(\alpha - 1) G(\alpha, x), \quad \alpha > 2, x > 0. \]

We collect the following easy to establish properties.

\[ (a) \quad H_1(\alpha, x) = x^\alpha F(\alpha, x), \quad \alpha > 0, x > 0, \]
\[ (b) \quad H_2(\alpha, x) = x^{\alpha+1} F(\alpha, x), \quad \alpha > 0, x > 0, \]
\[ (c) \quad 0 \leq H_2(\alpha, x) \leq C(\alpha), \quad \alpha > 0, x > 0, \]
\[ (d) \quad |H(\alpha, x)| \leq H_2(\alpha, x), \quad \alpha > 0, x > 0, \]
\[ (e) \quad C(\alpha) = \alpha^{\alpha+1} \int_0^\infty \frac{t^\alpha}{\sinh(\alpha t)} dt, \quad \alpha > 0, \]
\[ (f) \quad C(\alpha - 1) = \alpha^\alpha \int_0^\infty \frac{t^{\alpha-1}}{\sinh(\alpha t)} dt, \quad \alpha > 1. \]

Note that \((2.1f)\) is not an easy consequence of \((2.1e)\). We also remark, that for \(\alpha \geq 1\) equation \((2.1a)\) remains also valid for \(x = 0\), by interpreting
both sides as their \( \lim_{x \to 0^+} \). The same holds true for (2.1b) and (2.1d) for \( \alpha > 0 \). We then have

\[
H_1(\alpha, 0) = \begin{cases} 
\frac{\pi}{2}, & \alpha = 1, \\
0, & \alpha > 1,
\end{cases}
\]

\[H(\alpha, 0) = H_2(\alpha, 0) = 0, \quad \alpha > 0. \quad (2.2)\]

Then, using (2.2), we define

\[
H_{\alpha}(x) = |x|^{\alpha} - \frac{2}{\pi} \sin \frac{\pi \alpha}{2} H(\alpha, x), \quad \alpha > 0, x \geq 0. \quad (2.3)
\]

Next, we record

(a) \[
\int_0^c x^{\alpha-1} e^{-\alpha x} (1 - x) dx = \int_0^\infty x^{\alpha-1} e^{-\alpha x} (x - 1) dx = \frac{e^\alpha e^{-\alpha c}}{\alpha}, \quad \alpha > 0, c \geq 0,
\]

(b) \[
\int_0^c x^{\alpha-2} e^{-\alpha x} dx = \frac{\Gamma(\alpha - 1)}{\alpha^{\alpha-1}}, \quad \alpha > 1,
\]

(c) \[
\int_0^\infty x^{\alpha-1} e^{-\alpha x} dx = \int_0^\infty x^{\alpha} e^{-\alpha x} dx = \frac{\Gamma(\alpha)}{\alpha^\alpha}, \quad \alpha > 0,
\]

(d) \[
\Gamma(\alpha) > \sqrt{\frac{2\pi}{\alpha}} \left( \frac{\alpha}{e} \right)^\alpha, \quad \alpha \geq 1.
\]

**Proof.** Both equations in (2.4c) as well as (2.4b) are derived directly from [9, 3.381.4]. The equations (2.4a) are then an easy consequence of (2.4c) combined together with ([9], 3.381.3 and 8.356.2). Inequality (2.4d) can be derived from ([9], 8.327). \( \square \)

Finally, we apologize for the repulsive notation \( \|f(x)\| \) instead of \( \|f\| \) that we occasionally use in this paper.

### 3 The limiting error term

Let \( \alpha > 0 \) and \( n \in \mathbb{N} \). We recall the definition of the nodes \( x_j^{(2n+1)} = \cos \left( \frac{(j-1/2)\pi}{2n+1} \right) \) for \( j = 1, 2, \ldots, 2n+1 \) to be the zeros of the Chebyshev polynomial \( T_{2n+1} \). Further denote by \( P_{2n}^{(2)} \) the unique Lagrange interpolation polynomial for \( |x|^\alpha \) in the interval \([-1, 1]\).

Then, for \( 2n > \alpha > 0 \) and all \( x \in [-1, 1] \), we simply derive from ([17], Theorem 1) the asymptotic formula

\[
(2n)^\alpha \left( |x|^\alpha - P_{2n}^{(2)}(x) \right) = (-1)^n \frac{2}{\pi} \sin \frac{\pi \alpha}{2} \left( 1 - \frac{1}{2n+1} \right)
\]

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\[ T_{2n+1}(x) \int_0^\infty \frac{t^\alpha}{\sinh(t)} \frac{2nx}{(2nx)^2 + t^2} dt + o(1), \quad n \to \infty, \quad (3.1) \]

where \( o(1) \) is independent of \( x \).

The objective now is to find its limiting error term in the \( L_\infty \) norm. Since the error term is symmetric in \([−1,1]\) we prove the following

**Theorem 3.1.** Let \( \alpha > 0 \). Then we have

\[
\lim_{n \to \infty} (2n)^\alpha \left\| |x|^\alpha - P_{2n}^{(2)} \right\|_{L_\infty[0,1]} = \frac{2}{\pi} \left| \sin \frac{\pi \alpha}{2} \right| \|H(\alpha, \cdot)\|_{L_\infty[0,\infty)}
\]

\[
= \frac{2}{\pi} \left| \sin \frac{\pi \alpha}{2} \right| \sup_{x \in [0,\infty)} \int_0^\infty \frac{t^\alpha}{\sinh(t)} \frac{x|\sin x|}{t^2 + t^2} dt.
\]

**Theorem 3.2.** Let \( \alpha > 0 \). Then, uniformly on compact subsets in \([0,\infty)\),

\[
\lim_{n \to \infty} (2n)^\alpha P_{2n}^{(2)} \left( \frac{x}{2n} \right) = H_\alpha(x).
\]

**Theorem 3.3.** Let \( \alpha > 0 \) be not an even integer. Then \( H_\alpha \) (interpreted as its extension into the complex domain) is an entire function of exponential type 1, interpolating \(|x|^\alpha\) at the interpolation points \( \{k\pi : k = 0, 1, 2, \ldots\} \) and \( H_\alpha \) admits a representation as an interpolating series of the following form.

Denote by \( N = [\alpha/2] \). Then, for all \( x \in \mathbb{R} \), we have

\[
H_\alpha(x) = \sin x \left( \frac{2}{\pi} \sum_{n=0}^{N-1} \sin \left( \frac{\pi (\alpha - 2n - 2)}{2} \right) C(\alpha - 2n - 2) x^{2n+1} \right.
\]

\[
+2x^{2N+1} \sum_{k=1}^\infty (-1)^k \frac{(k\pi)^{\alpha-2N}}{x^2 - (k\pi)^2}. \quad (3.2)
\]

For the special case \( 0 < \alpha < 2 \) the expansion is then represented by

\[
H_\alpha(x) = 2x \sin x \sum_{k=1}^\infty (-1)^k \frac{(k\pi)^{\alpha}}{x^2 - (k\pi)^2}. \quad (3.3)
\]

We start with the proof for Theorem 3.1 by splitting it in several Lemmas. First, we present without a proof the following two Lemmas.

**Lemma 3.1.** Let \( x \in [0, \frac{1}{2}] \). Then \(|\arcsin x - x| \leq x^2\).

**Lemma 3.2.** For \( n \in \mathbb{N} \) and \( x \in [-1,1] \) we have

\[
T_{2n+1}(x) = (-1)^n \sin ((2n + 1) \arcsin x).
\]
Lemma 3.3. Let $n \in \mathbb{N}$ and $x \in [-2n, 2n]$. Then we have

$$\left| \frac{T_{2n+1} \left( \frac{x}{2n} \right)}{x} \right| \leq 1 + \frac{1}{2n}.$$ 

Proof. The assertion is an easy consequence from (17, Lemma 10).

Lemma 3.4. Let $C > 0$ be fixed, $\varepsilon > 0$ and $n > \max \left( C, \frac{C}{\varepsilon} \right)$. Then

$$\left\| \frac{T_{2n+1} \left( \frac{x}{2n} \right)}{x} - (-1)^n \sin \left( \frac{(2n+1)}{x} \frac{x}{2n} \right) \right\|_{L_\infty[0,C]} < \varepsilon.$$ 

Proof. For $x \in [0, C]$ we get $0 \leq \frac{x}{2n} \leq \frac{C}{2n} < \frac{C}{2C} = \frac{1}{2}$. Then, using Lemma 3.1 and Lemma 3.2 we estimate

$$\left| \frac{T_{2n+1} \left( \frac{x}{2n} \right)}{x} - (-1)^n \sin \left( (2n+1) \frac{x}{2n} \right) \right|$$

$$= \frac{1}{x} \left| \sin \left( (2n+1) \arcsin \left( \frac{x}{2n} \right) \right) - \sin \left( (2n+1) \frac{x}{2n} \right) \right|$$

$$\leq \frac{2n+1}{x} \left| \arcsin \frac{x}{2n} - \frac{x}{2n} \right| \leq \frac{2n+1}{x} \left( \frac{x}{2n} \right)^2 \leq \frac{C}{n} < \varepsilon.$$ 

Lemma 3.5. Let $C > 0$ be fixed, $\varepsilon > 0$ and $n > \frac{1}{2}$. Then

$$\left\| \sin \left( \frac{(2n+1)}{x} \frac{x}{2n} \right) - \frac{\sin x}{x} \right\|_{L_\infty[0,C]} < \varepsilon.$$
Proof. Let \( x \in [0, C] \). Then by a standard argument we arrive at
\[
\left| \sin \left( (2n + 1) \frac{x}{2n} \right) - \sin x \right| \leq \frac{1}{x} \left| (2n + 1) \frac{x}{2n} - x \right| = \frac{1}{2n} < \varepsilon.
\]

Lemma 3.6. Let \( C > 0 \) be fixed, \( \varepsilon > 0 \) and \( n > \max \left( C, \frac{C}{\varepsilon}, \frac{1}{2\varepsilon} \right) \). Then
\[
\left\| \frac{T_{2n+1} \left( \frac{x}{2n} \right)}{x} - (-1)^n \frac{\sin x}{x} \right\|_{L_{\infty}[0,C]} < 2\varepsilon.
\]
Proof. This follows directly by applying the triangle inequality combined together with Lemma 3.4 and Lemma 3.5.

Lemma 3.7. Let \( C > 0 \) be fixed, \( \varepsilon > 0 \) and \( n > \max \left( C, \frac{C}{\varepsilon}, \frac{1}{2\varepsilon} \right) \). Then, for \( \alpha > 0 \), we have
\[
\left\| T_{2n+1} \left( \frac{x}{2n} \right) H_1 \left( \alpha, x \right) \right\|_{L_{\infty}[0,C]} \leq \left\| H \left( \alpha, x \right) \right\|_{L_{\infty}[0,\infty)} + 2\varepsilon \cdot C \left( \alpha \right).
\]
Proof. First, we remark that for \( \alpha > 0 \) the left-hand side in Lemma 3.7 is well defined by applying (2.2) together with Lemma 3.3. Using again the triangle inequality together with Lemma 3.6 and formula (2.1c), we arrive at
\[
\left\| T_{2n+1} \left( \frac{x}{2n} \right) H_1 \left( \alpha, x \right) \right\|_{L_{\infty}[0,C]} = \left\| \frac{T_{2n+1} \left( \frac{x}{2n} \right)}{x} H_2 \left( \alpha, x \right) \right\|_{L_{\infty}[0,C]}
\leq \left\| \frac{T_{2n+1} \left( \frac{x}{2n} \right)}{x} - (-1)^n \frac{\sin x}{x} \right\|_{L_{\infty}[0,C]} \left\| H_2 \left( \alpha, x \right) \right\|_{L_{\infty}[0,C]}
+ \left\| \sin x \cdot H_1 \left( \alpha, x \right) \right\|_{L_{\infty}[0,C]} \leq 2\varepsilon C \left( \alpha \right) + \left\| H \left( \alpha, x \right) \right\|_{L_{\infty}[0,\infty)}.
\]

Our first substantial result is now the following

Lemma 3.8. Let \( \alpha > 0 \). Then
\[
\lim_{n \to \infty} \left\| T_{2n+1} \left( \frac{x}{2n} \right) H_1 \left( \alpha, x \right) \right\|_{L_{\infty}[0,2n]} \leq \left\| H \left( \alpha, x \right) \right\|_{L_{\infty}[0,\infty)}.
\]
Proof. Let $\varepsilon > 0$, $C > \frac{C(\alpha)}{\varepsilon}$ and $n > \max \left( C, \frac{C}{\varepsilon}, \frac{1}{2\varepsilon} \right)$. Then

$$\left\| T_{2n+1} \left( \frac{x}{2n} \right) H_1(\alpha, x) \right\|_{L_\infty[0,2n]} \leq \left\| T_{2n+1} \left( \frac{x}{2n} \right) H_1(\alpha, x) \right\|_{L_\infty[0,C]} + \left\| T_{2n+1} \left( \frac{x}{2n} \right) H_1(\alpha, x) \right\|_{L_\infty[C,2n]}.$$

Using (2.1c), the latter part can be estimated to

$$\left\| T_{2n+1} \left( \frac{x}{2n} \right) H_1(\alpha, x) \right\|_{L_\infty[C,2n]} \leq \frac{1}{C} \cdot C(\alpha) < \varepsilon.$$

Combined together with the previous estimate and Lemma 3.7 we finally get

$$\left\| T_{2n+1} \left( \frac{x}{2n} \right) H_1(\alpha, x) \right\|_{L_\infty[0,2n]} \leq \left\| H(\alpha, x) \right\|_{L_\infty[0,\infty)} + 2\varepsilon \cdot C(\alpha) + \varepsilon.$$

By taking the $\lim$ the result follows. 

Now, we are turning to the $\lim$ case.

Lemma 3.9. Let $\alpha > 0$ and $C > 0$ be fixed. Then

$$\lim_{n \to \infty} \left\| T_{2n+1} \left( \frac{x}{2n} \right) H_1(\alpha, x) \right\|_{L_\infty[0,2n]} \geq \left\| H(\alpha, x) \right\|_{L_\infty[0,C]}.$$

Proof. Let $C > 0$, $\varepsilon > 0$ and $n > \max \left( C, \frac{C}{\varepsilon}, \frac{1}{2\varepsilon} \right)$. Then, by applying again the triangle inequality and combining together with Lemma 3.6 and (2.1c), we estimate

$$\left\| T_{2n+1} \left( \frac{x}{2n} \right) H_1(\alpha, x) \right\|_{L_\infty[0,2n]} \geq \left\| H(\alpha, x) \right\|_{L_\infty[0,C]} - \left\| \left( T_{2n+1} \left( \frac{x}{2n} \right) - \frac{(-1)^n \sin x}{x} \right) H_2(\alpha, x) \right\|_{L_\infty[0,C]} \geq \left\| H(\alpha, x) \right\|_{L_\infty[0,C]} - 2\varepsilon \left\| H_2(\alpha, x) \right\|_{L_\infty[0,C]} \geq \left\| H(\alpha, x) \right\|_{L_\infty[0,C]} - 2\varepsilon \cdot C(\alpha).$$

Now, by taking $\lim$ we establish the result. 

\[\square\]
Our second substantial result is the following

**Lemma 3.10.** Let $\alpha > 0$. Then

$$\lim_{n \to \infty} \left\| T_{2n+1} \left( \frac{x}{2n} \right) H_1 (\alpha, x) \right\|_{L_\infty[0,2n]} \geq \left\| H (\alpha, x) \right\|_{L_\infty[0,\infty)}.$$

**Proof.** Let $\varepsilon > 0$ and $C > \frac{C(\alpha)}{\varepsilon}$. Then, starting with the right-hand side in Lemma 3.10, we estimate

$$\left\| H (\alpha, x) \right\|_{L_\infty[0,\infty)} \leq \left\| H (\alpha, x) \right\|_{L_\infty[0,C]} + \left\| H (\alpha, x) \right\|_{L_\infty[C,\infty)}.$$ 

Using again (2.1), the latter part can be estimated to

$$\left\| H (\alpha, x) \right\|_{L_\infty[C,\infty)} = \left\| \frac{\sin x}{x} H_2 (\alpha, x) \right\|_{L_\infty[C,\infty)} \leq \frac{1}{C} \cdot C (\alpha) < \varepsilon.$$

Combined together with Lemma 3.9 and the previous estimate, we arrive at

$$\left\| H (\alpha, x) \right\|_{L_\infty[0,\infty)} - \varepsilon \leq \left\| H (\alpha, x) \right\|_{L_\infty[0,C]} + \lim_{n \to \infty} \left\| T_{2n+1} \left( \frac{x}{2n} \right) H_1 (\alpha, x) \right\|_{L_\infty[0,2n]}.$$

Since the last expression holds for every $\varepsilon > 0$ we establish the result. \hfill \Box

**Proof of Theorem 3.1.** Let $\alpha > 0$. Then

$$\left\| T_{2n+1} (x) \int_{0}^{\infty} \frac{t^\alpha}{\sinh (t)} \frac{2nx}{(2nx)^2 + t^2} dt \right\|_{L_\infty[0,1]}$$

$$= \left\| T_{2n+1} (\frac{x}{2n}) \int_{0}^{\infty} \frac{t^\alpha}{\sinh (t)} \frac{x}{x^2 + t^2} dt \right\|_{L_\infty[0,2n]}$$

$$= \left\| T_{2n+1} (\frac{x}{2n}) H_1 (\alpha, x) \right\|_{L_\infty[0,2n]}.$$

Combining now Lemma 3.8 and Lemma 3.10 together with (3.1), gives the result and we are finished. \hfill \Box

**Proof of Theorem 3.2.** Let $\alpha > 0$. From (3.1) it follows that for every $\varepsilon > 0$ we can find some $n_0 = n_0 (\varepsilon)$, such that for all $n > n_0$

$$\left\| (2n)^\alpha \left( |x|^\alpha - P_2^{(2)} (x) \right) - (-1)^n \frac{2}{\pi} \sin \frac{\pi \alpha}{2} \left( 1 - \frac{1}{2n + 1} \right) \right\|_{L_\infty[0,1]} < \varepsilon.$$
Let $C > 0$ be fixed, $\varepsilon > 0$ and $n > \max \left( C, \frac{C}{2}, \frac{n}{2}, n_0 \right)$. Then

$$
\left\| (2n)^\alpha P^{(2)}_{2n}\left(\frac{x}{2n}\right) - H_\alpha(x) \right\|_{L^\infty[0,C]} = \left\| \frac{2}{\pi} \sin \frac{\pi \alpha}{2} H(\alpha, 2n x) - (2n)^\alpha \left( |x|^\alpha - P^{(2)}_{2n}(x) \right) \right\|_{L^\infty[0,\frac{C}{2n}]} \leq \frac{2}{\pi} \left| \sin \frac{\pi \alpha}{2} \right| \left\| H(\alpha, x) - (-1)^n \frac{2n}{2n+1} T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L^\infty[0,0,C]} + \varepsilon. \tag{3.4}
$$

We proceed further by use of (2.1c), Lemma 3.3 and Lemma 3.6.

$$
\left\| H(\alpha, x) - (-1)^n \frac{2n}{2n+1} T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L^\infty[0,0,C]} \leq \frac{2}{\pi} \left| \sin \frac{\pi \alpha}{2} \right| \left\| H_1(\alpha, x) \left( \sin x - (-1)^n \frac{2n}{2n+1} T_{2n+1}\left(\frac{x}{2n}\right) \right) \right\|_{L^\infty[0,0,C]} + \frac{1}{2n+1} \left\| T_{2n+1}\left(\frac{x}{2n}\right) \right\|_{L^\infty[0,0,C]} \leq C(\alpha) \left( 2\varepsilon + \frac{1}{2n} \right) \leq C(\alpha) 3\varepsilon.
$$

Combining together with (3.4), we obtain for every $\varepsilon > 0$ and $n$ sufficiently large,

$$
\left\| (2n)^\alpha P^{(2)}_{2n}\left(\frac{x}{2n}\right) - H_\alpha \right\|_{L^\infty[0,C]} \leq \frac{2}{\pi} \left| \sin \frac{\pi \alpha}{2} \right| C(\alpha) 3\varepsilon + \varepsilon.
$$

Since any compact set $K$ in $[0, \infty)$ can be included in some interval $[0, C]$ the result is established.

**Proof of Theorem 3.3** The expansion of $H_\alpha$ into the interpolating series (3.2) follows after some routine arguments from ([5], Formula 4.14). The special case (3.3) can be directly seen from ([5], Formula 4.16). The fact that $H_\alpha$ is an entire function of exponential type 1 can now be deduced from ([18], p. 183, Formula 15). The interpolation property is an easy consequence of (2.3).

**4 The Envelope function**

In this section we consider the envelope error function $H_1(\alpha, \cdot)$ with respect to $|H(\alpha, \cdot)|$. Our next objective is to establish an asymptotics for

$$
\left\| H_1(\alpha, \cdot) \right\|_{L^\infty[0,\infty)} \text{ when } \alpha \to \infty.
$$

We show
Theorem 4.1. Let $\alpha \geq 2$. Then, we have
\[
\frac{C(\alpha)}{1 + 2\alpha} \left(1 - \frac{1}{\sqrt{\alpha}}\right) \leq H_1(\alpha, \alpha) \leq \|H_1(\alpha, \cdot)\|_{L^\infty[0, \infty)} \leq \frac{C(\alpha)}{1 + 2\alpha} \left(1 + \frac{2}{\sqrt{\alpha}}\right).
\]

Figure 2: The error function $|H(\alpha, \cdot)|$, its envelope $H_1(\alpha, \cdot)$ and the point evaluation $H_1(\alpha, \alpha)$.

Figure 2 shows the functions $|H(\alpha, \cdot)|$ and $H_1(\alpha, \cdot)$ as well as their point evaluations for values $\alpha = 1.8$ and $\alpha = 6.4$. The figure suggests that a useful lower estimate for $\|H_1(\alpha, \cdot)\|$ should be derivable when determining its point evaluation, i.e. $H_1(\alpha, \alpha)$, at least for large values for $\alpha$.

We start proving Theorem 4.1 by splitting it in several Lemmas. First, we present the following five Lemmas without proof. They can be derived by some standard analysis arguments.

Lemma 4.1. The function $f(x) = (1 + \frac{1}{x})^x$ is monotonically increasing in $x \in (0, \infty)$ and $f(x) \leq e$.

Lemma 4.2. For $x > 0$ we have
\[
\frac{1}{1 - e^{-x}} - \frac{1}{x} \leq 1.
\]

Lemma 4.3. Let $\alpha > 0$. The function
\[
f(x) = \frac{x}{1 - e^{-2ax}}
\]
is convex for $x \geq 0$. Here $f(0) = \lim_{x \to 0^+} f(x) = \frac{1}{2\alpha}$.

Lemma 4.4. Let $\alpha > 0$. Then, for $x \in [0, 1 + \frac{1}{2\alpha}]$, we have
\[
\frac{x}{1 - e^{-2ax}} \leq \left(\frac{1}{1 - e^{-2a - 1}} - \frac{1}{1 + 2\alpha}\right)x + \frac{1}{2\alpha}.
\]
Lemma 4.5. For \( x \geq 0 \) denote by \( f(x) = x(x+1)/(x^2+1) \). Then, for \( x \geq 0 \), we have
\[
f(x) \leq f\left(1 + \sqrt{2}\right) = \frac{1 + \sqrt{2}}{2}.
\]

Our first substantial result is now the following

Lemma 4.6. Let \( \alpha \geq 1 \). Then
\[
C(\alpha) \leq H_1(\alpha, \alpha).
\]

Proof. By some routine arguments and using Lemma (4.5), (2.4a), (2.4d) and (2.4c), we estimate
\[
\begin{align*}
1 + 2\alpha & \int_0^\infty \frac{x^\alpha}{\sinh \alpha x} \left( \frac{x}{1 + 2\alpha} + \frac{1}{1 + x^2} \right) dx \\
&= \int_0^\infty \frac{x^\alpha}{\sinh \alpha x} \left( \frac{x^2 - 1}{x^2 + 1} \right) dx - \frac{1}{\alpha} \int_0^\infty \frac{x^\alpha}{\sinh \alpha x} \frac{1}{x^2 + 1} dx \\
&\leq \int_1^\infty \frac{x^\alpha}{\sinh \alpha x} \frac{x^2 - 1}{dx} \\
&= 2 \int_1^\infty x^{\alpha-1} e^{-\alpha x} \frac{1}{1 - e^{-2\alpha x}} \left( \frac{x(x-1)(x+1)}{x^2+1} \right) dx \\
&\leq \frac{1 + \sqrt{2}}{1 - e^{-2}} \int_1^\infty x^{\alpha-1} e^{-\alpha x} (x-1) dx \\
&\leq \frac{2}{\sqrt{\alpha}} \int_1^\infty x^{\alpha-1} e^{-\alpha x} (x-1) dx \\
&\leq \frac{2}{\sqrt{\alpha}} \Gamma (\alpha) \\
&= \frac{2}{\sqrt{\alpha}} \Gamma (\alpha) \\
&= \frac{2}{\sqrt{\alpha}} \int_0^\infty x^\alpha e^{-\alpha x} dx \\
&\leq \frac{1}{\sqrt{\alpha}} \int_0^\infty x^\alpha dx.
\end{align*}
\]

We summarize
\[
\begin{align*}
\int_0^\infty \frac{x^\alpha}{\sinh \alpha x} \left( \frac{1}{1 + x^2} - \frac{\alpha}{1 + 2\alpha} \right) dx \\
&\geq -\frac{1}{\sqrt{\alpha}} \int_0^\infty \frac{x^\alpha}{\sinh \alpha x} dx.
\end{align*}
\]
Now, using (2.1a), (2.1e) together with (4.1), we obtain the final result
\[ H_1(\alpha, \alpha) = \alpha^\alpha F(\alpha, \alpha) \]
\[ = \frac{C(\alpha)}{1 + 2\alpha} + \alpha^\alpha \int_0^\infty \frac{t^\alpha}{\sinh \alpha t} \left( \frac{1}{1 + t^2} - \frac{\alpha}{1 + 2\alpha} \right) dt \]
\[ \geq \frac{C(\alpha)}{1 + 2\alpha} - \frac{\alpha^\alpha}{\sqrt{\alpha}} \frac{\alpha}{1 + 2\alpha} \int_0^\infty \frac{t^\alpha}{\sinh \alpha t} dt \]
\[ = \frac{C(\alpha)}{1 + 2\alpha} \left( 1 - \frac{1}{\sqrt{\alpha}} \right). \]

Next, we show

**Lemma 4.7.** Let $\alpha > 1$. Then
\[ \|H_1(\alpha, \cdot)\|_{L_\infty[0, \infty)} \leq \frac{1}{2} C(\alpha - 1). \]

**Proof.** From (2.2) it follows that we can restrict ourselves to values $H_1(\alpha, x)$ for $x > 0$. Thus
\[ \|H_1(\alpha, \cdot)\|_{L_\infty[0, \infty)} = \left\| \int_0^\infty \frac{t^\alpha}{\sinh t} \frac{x}{x^2 + t^2} dt \right\|_{L_\infty(0, \infty)} \]
\[ \leq \left\| \int_0^\infty \frac{t^\alpha}{\sinh t} \frac{x}{2xt} dt \right\|_{L_\infty(0, \infty)} \]
\[ = \frac{1}{2} \int_0^\infty \frac{t^{\alpha - 1}}{\sinh t} dt \]
\[ = \frac{1}{2} C(\alpha - 1). \]

**Lemma 4.8.** Let $\alpha \geq 2$. Then
\[ \frac{1}{2} C(\alpha - 1) \leq \frac{C(\alpha)}{1 + 2\alpha} \left( 1 + \frac{2}{\sqrt{\alpha}} \right). \]
Proof. By using Lemma 4.4 and Lemma 4.2 we begin with

\[ \int_0^\infty \frac{x^{\alpha-1}}{\sinh \alpha x} \left( 1 + \frac{1}{2\alpha} - x \right) \, dx \]

\[ \leq 2 \int_0^{1+\frac{1}{2\alpha}} x^{\alpha-2} e^{-\alpha x} \left( \frac{x}{1 - e^{-2\alpha x}} \left( 1 + \frac{1}{2\alpha} - x \right) \right) \, dx \]

\[ \leq 2 \int_0^{1+\frac{1}{2\alpha}} x^{\alpha-2} e^{-\alpha x} \left( \left( \frac{1}{1 - e^{-2\alpha x}} - \frac{1}{1 + 2\alpha} \right) x + \frac{1}{2\alpha} \right) \left( 1 + \frac{1}{2\alpha} - x \right) \, dx \]

\[ \leq 2 \int_0^{1+\frac{1}{2\alpha}} x^{\alpha-2} e^{-\alpha x} \left( x + \frac{1}{2\alpha} \right) \left( 1 + \frac{1}{2\alpha} - x \right) \, dx \]

\[ = 2 \int_0^{1+\frac{1}{2\alpha}} x^{\alpha-2} e^{-\alpha x} \left( x - x^2 + \frac{1}{2\alpha} + \frac{1}{4\alpha^2} \right) \, dx. \]

Note, that for \( \alpha \geq \frac{1}{2} \) we have \( 1/\alpha \geq 1/(2\alpha) + 1/(4\alpha^2) \). From this, by using (2.4a), it follows that

\[ 2 \int_0^{1+\frac{1}{2\alpha}} x^{\alpha-2} e^{-\alpha x} \left( x - x^2 + \frac{1}{2\alpha} + \frac{1}{4\alpha^2} \right) \, dx \]

\[ \leq 2 \int_0^{1+\frac{1}{2\alpha}} x^{\alpha-1} e^{-\alpha x} (1 - x) \, dx + \frac{2}{\alpha} \int_0^{1+\frac{1}{2\alpha}} x^{\alpha-2} e^{-\alpha x} \, dx \]

\[ = \frac{2}{\alpha} \sqrt{\left( 1 + \frac{1}{2\alpha} \right)^{2\alpha}} e^{-\alpha} e^{-\frac{1}{\alpha}} + \frac{2}{\alpha} \int_0^{1+\frac{1}{2\alpha}} x^{\alpha-2} e^{-\alpha x} \, dx. \]

Then, using Lemma 4.1 and (2.4b), we can further estimate to

\[ \frac{2}{\alpha} \sqrt{\left( 1 + \frac{1}{2\alpha} \right)^{2\alpha}} e^{-\alpha} e^{-\frac{1}{\alpha}} \leq \frac{2}{\alpha} e^\frac{1}{2} e^{-\alpha} e^{-\frac{1}{2}} + \frac{2}{\alpha} \int_0^\infty x^{\alpha-2} e^{-\alpha x} \, dx \]

\[ = \frac{2}{\alpha} e^{-\alpha} + \frac{2}{\alpha} \Gamma(\alpha - 1) \]

\[ = \frac{2}{\alpha} e^{-\alpha} + \frac{2}{\alpha - 1} \alpha \Gamma(\alpha). \]

We collect for \( \alpha \geq 2 \) the inequality \( 2/\alpha \geq 1/(\alpha - 1) \). Now, using (2.4a) and
we estimate further
\[
\frac{2}{\alpha} e^{-\alpha} + \frac{2}{\alpha - 1} \frac{1}{\alpha} \Gamma (\alpha) \\
\leq \frac{1}{\sqrt{\alpha}} \frac{\Gamma (\alpha)}{\alpha^\alpha} \left( \frac{2}{\sqrt{2\pi}} + \frac{4}{\sqrt{\alpha}} \right) \\
\leq \frac{4}{\sqrt{\alpha}} \frac{\Gamma (\alpha)}{\alpha^\alpha} = \frac{4}{\sqrt{\alpha}} \int_0^\infty x^\alpha e^{-\alpha x} \, dx \\
\leq \frac{2}{\sqrt{\alpha}} \int_0^\alpha \frac{x^\alpha}{\sinh \alpha x} \, dx.
\]

Combining all together, we obtain for all \( \alpha \geq 2 \),
\[
\int_0^\alpha \frac{t^{\alpha-1}}{\sinh \alpha t} \, dt \leq \frac{2\alpha}{1 + 2\alpha} \left( 1 + \frac{2}{\sqrt{\alpha}} \right) \int_0^\alpha \frac{t^\alpha}{\sinh \alpha t} \, dt.
\]

Finally, using (2.11) and (2.14), we arrive at
\[
\frac{1}{2} C (\alpha - 1) = \frac{\alpha^\alpha}{2} \int_0^\alpha \frac{t^{\alpha-1}}{\sinh \alpha t} \, dt \\
\leq \frac{\alpha^\alpha}{2} \frac{2\alpha}{1 + 2\alpha} \left( 1 + \frac{2}{\sqrt{\alpha}} \right) \int_0^\alpha \frac{t^\alpha}{\sinh \alpha t} \, dt \\
= \left( 1 + \frac{2}{\sqrt{\alpha}} \right) \frac{C (\alpha)}{1 + 2\alpha}.
\]

Proof of Theorem 4.1. The Theorem is now an easy consequence of Lemma 4.6, Lemma 4.7 and Lemma 4.8.

5 Asymptotics of the error function

In this section we establish an asymptotic bound for the norm of the limiting error function, i.e. for \( \| H (\alpha, \cdot) \|_{L_\infty[0, \infty)} \). This section is the most technical part in this paper. Here, we use the generalized Watson Lemma (Laplace method for integrals with large parameter) for deriving an asymptotic expansion used to be later in the context. As it turns out, we need an higher order asymptotics up to order 5 involving the computation of certain rather complicated defined constants. However, the main idea for deriving a lower estimate is quite easy to see. Let us start, once again, with a diagram (Figure 3) involving the functions \( |H (\alpha, \cdot)| \) and \( H_1 (\alpha, \cdot) \).
Figure 3 shows the functions $|H (\alpha, \cdot)|$ and its envelope $H_1(\alpha, \cdot)$ together with the point evaluations $H_1(\alpha, \alpha)$ and $|H (\alpha, \beta)| = H_1 (\alpha, \beta)$, where $\beta = \beta (\alpha) = \pi \left[ \frac{\alpha}{\pi} \right] + \frac{3}{2} \pi$ and $\alpha = 3.9$ and $\alpha = 8.4$. Geometrically, the point $\beta$ is the position of the first or the second relative maximum of $|H (\alpha, \cdot)|$ on the right-hand side of $\alpha$, where $H_1 (\alpha, \cdot)$ appears to be descending. For growing values of $\alpha$, the size of these maxima appear to be of the same magnitude compared to the size $H_1 (\alpha, \alpha)$. We use both observations for the asymptotic analysis. First, we show that $H_1 (\alpha, \cdot)$ is descending at least for values $x \geq \alpha$. Then, we derive the asymptotics for the local maximum in $|H (\alpha, \beta)|$. It turns out that the following integral inequality plays an essential role.

**Theorem 5.1.** There exists a fixed constant $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$,

$$R (\alpha, \alpha) = \int_0^\infty \frac{t^{\alpha+1}}{\sinh \alpha t} \frac{1}{1+t^2} dt - \int_0^\infty \frac{t^\alpha}{\sinh \alpha t} \frac{1}{1+t^2} dt > 0. \quad (5.1)$$

We remark that (5.1) is not true for all $\alpha_0 > 0$. This can be seen out from Figure 4. Also, for growing values of $\alpha$, the positive magnitude becomes rather small. Numerical experiments suggest that the minimal value for $\alpha_0$ such that (5.1) becomes true, is somewhere in the interval $(2.54288, 2.54289)$. However, since we are interested in an asymptotic expansion, the determination of the exact size of the minimal value $\alpha_0$ is not important. From Theorem 5.1 we may derive our first desired property.

**Theorem 5.2.** There exists a fixed constant $\alpha_0 > 0$ such that $H_1 (\alpha, \cdot)$ is decreasing, whenever $x \geq \alpha \geq \alpha_0$.

From Theorem 5.2 we obtain the final asymptotics.

**Theorem 5.3.** We have

$$\|H (\alpha, \cdot)\|_{L^\infty[0, \infty)} = \frac{C (\alpha)}{1 + 2\alpha} (1 + o (1)), \quad \alpha \to \infty.$$
We first establish Theorem 5.2 by assuming that Theorem 5.1 holds true. Then, we present the proof for Theorem 5.1 which is completely independent of the forthcoming Lemmas related to Theorem 5.2. Finally, we present the proof for Theorem 5.3. Without proof, we first present the following

**Lemma 5.1.** Let $\alpha > 0$ be fixed and $x > 0$. Then $S(\alpha, x)$ has the representation

$$S(\alpha, x) = \int_{0}^{\infty} \frac{t^{\alpha} (t - \alpha)}{2 \sinh t} \frac{x^{2} + \alpha^{2}}{x^{2} + t^{2}} dt.$$ 

**Lemma 5.2.** Let $\alpha > 0$ be fixed and $x > 0$. Then

$$\frac{d}{dx} H_{1}(\alpha, x) \leq -\frac{2}{x^{2} + \alpha^{2}} S(\alpha, x).$$

**Proof.** Using (2.1a) and by differentiating under the integral, we get

$$\frac{d}{dx} H_{1}(\alpha, x) = \frac{d}{dx} (x^{\alpha} F(\alpha, x))$$

$$= \alpha x^{\alpha-1} \left( \int_{0}^{\infty} \frac{t^{\alpha}}{\sinh xt} \frac{dt}{1 + t^{2}} - \frac{x}{\alpha} \int_{0}^{\infty} \frac{t^{\alpha+1} \cosh xt}{\sinh xt \sinh xt} \frac{dt}{1 + t^{2}} \right)$$

$$\leq \alpha x^{\alpha-1} \left( \int_{0}^{\infty} \frac{t^{\alpha}}{\sinh xt} \frac{dt}{1 + t^{2}} - \frac{x}{\alpha} \int_{0}^{\infty} \frac{t^{\alpha+1}}{\sinh xt} \frac{dt}{1 + t^{2}} \right)$$

$$= -\alpha x^{\alpha-1} \left( \frac{x}{\alpha} F(\alpha + 1, x) - F(\alpha, x) \right) = -\alpha x^{\alpha-1} R(\alpha, x)$$

$$= -\alpha x^{\alpha-1} \frac{1}{x^{2} + \alpha^{2}} S(\alpha, x) = -\frac{2}{x^{2} + \alpha^{2}} S(\alpha, x).$$
Lemma 5.3. Let \( \alpha > 0 \) be fixed and \( x > 0 \). Then
\[
S(\alpha, x) = \frac{\alpha}{2} x^{\alpha-1} \left( x^2 + \alpha^2 \right) R(\alpha, x)
\]
is an increasing function in \( x \).

Proof. Using Lemma (5.1) and by differentiating under the integral again, we get
\[
\frac{d}{dx} S(\alpha, x) = \int_0^\infty \frac{t^\alpha (t - \alpha)}{2 \sinh t} \frac{\partial}{\partial x} \left( \frac{x^2 + \alpha^2}{x^2 + t^2} \right) dt
\]
\[
= \int_0^\infty \frac{xt^\alpha (t - \alpha)^2 t + \alpha}{\sinh t (x^2 + t^2)^2} dt > 0.
\]

The rescaling of \( R(\alpha, \cdot) \) in Lemma 5.3 is now extremely useful in proving Theorem 5.2. Considering formula (5.2) contributes to my colleague, Dr. Maximilian Thaler, for which I thank him.

Proof of Theorem 5.2. By assuming the validity of Theorem 5.1 there exists some \( \alpha_0 > 0 \), such that \( R(\alpha, \alpha) > 0 \), \( \forall \alpha \geq \alpha_0 \). From this fact and (5.2) we deduce \( S(\alpha, \alpha) = \alpha^{\alpha+2} R(\alpha, \alpha) > 0 \), \( \forall \alpha \geq \alpha_0 \). Now, combining Lemma 5.2 together with Lemma 5.3 we establish for all \( x \geq \alpha \geq \alpha_0 \),
\[
\frac{d}{dx} H_1(\alpha, x) \leq -2 \frac{x^2 + \alpha^2}{S(\alpha, x)} \leq -2 \frac{x^2 + \alpha^2}{S(\alpha, \alpha)} < 0.
\]

We turn now to the proof for Theorem 5.1. As before, we derive several Lemmas.

Lemma 5.4. Let \( \alpha > 2 \) be fixed and \( x > 0 \). Then
\[
F_1(\alpha, x) \leq F(\alpha, x) \leq F_2(\alpha, x).
\]

Proof. By some routine calculations we obtain the representation
\[
F(\alpha, x) = 2 \sum_{n=0}^{\infty} \frac{1}{(1 + 2n)^{\alpha-1}} \int_0^\infty \frac{t^\alpha e^{-xt}}{(1 + 2n)^2 + t^2} dt.
\]
Since \( Z(\alpha) \) is the well known zeta function, from \([\text{II}], 9.522.2\) we derive for \( \alpha > 1 \),
\[
Z(\alpha) \left( 2 - 2^{1-\alpha} \right) = 2 \sum_{n=0}^{\infty} \frac{1}{(1 + 2n)^\alpha}.
\]
Combining (5.3) together with (5.4), we obtain for $\alpha > 2$ the right-hand side in Lemma 5.4 by

$$F(\alpha, x) \leq 2 \sum_{n=0}^{\infty} \frac{1}{(1 + 2n)^{\alpha-1}} \int_{0}^{\infty} t^{\alpha-1} e^{-xt} \frac{dt}{1 + t^2} \leq \left(2 - \frac{1}{2^{\alpha-2}}\right) Z(\alpha - 1) G(\alpha, x) = F_2(\alpha, x).$$

Similarly, for $\alpha > 0$, the left-hand side in Lemma 5.4 can be derived by

$$F(\alpha, x) \geq 2 \sum_{n=0}^{\infty} \frac{1}{(1 + 2n)^{\alpha-1}} \int_{0}^{\infty} \frac{t^{\alpha-1} e^{-xt}}{(1 + 2n)^2 + (1 + 2n)^2 t^2} dt = \left(2 - \frac{1}{2^\alpha}\right) Z(\alpha + 1) G(\alpha, x) = F_1(\alpha, x).$$

**Lemma 5.5.** Let $\alpha > 2$. Then

$$R(\alpha, \alpha) \geq \left(2 - \frac{1}{2^{\alpha+1}}\right) G(\alpha + 1, \alpha) - \left(2 - \frac{1}{2^{\alpha-2}}\right) \left(1 + \frac{1}{2^{\alpha-1}} + \frac{1}{\alpha - 2^{\alpha-2}}\right) G(\alpha, \alpha).$$

**Proof.** By using a routine estimate for the zeta function, namely

$$1 < Z(\alpha) < 1 + \frac{1}{2^\alpha} + \frac{1}{\alpha - 1} \frac{1}{2^{\alpha-1}}, \quad \alpha > 1,$$

we combine this together with Lemma 5.4. For $\alpha > 2$ it then follows

$$R(\alpha, \alpha) = F(\alpha + 1, \alpha) - F(\alpha, \alpha) \geq F_1(\alpha + 1, \alpha) - F_2(\alpha, \alpha) = \left(2 - \frac{1}{2^{\alpha+1}}\right) Z(\alpha + 2) G(\alpha + 1, \alpha) - \left(2 - \frac{1}{2^{\alpha-2}}\right) Z(\alpha - 1) G(\alpha, \alpha) \geq \left(2 - \frac{1}{2^{\alpha+1}}\right) \cdot 1 \cdot G(\alpha + 1, \alpha) - \left(2 - \frac{1}{2^{\alpha-2}}\right) \left(1 + \frac{1}{2^{\alpha-1}} + \frac{1}{\alpha - 2^{\alpha-2}}\right) G(\alpha, \alpha).$$

$\square$
Lemma 5.6. Let \( \alpha > 0 \) and \( c \geq 0 \). Then, as \( \alpha \to \infty \), we have the following asymptotics.

\[
G(\alpha, \alpha) = \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha} \left( \frac{1}{2} - \frac{5}{24 \alpha} + \frac{61}{576 \alpha^2} + O\left(\alpha^{-3}\right) \right),
\]

\[
G(\alpha + 1, \alpha) = \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha} \left( \frac{1}{2} - \frac{5}{24 \alpha} + \frac{205}{576 \alpha^2} + O\left(\alpha^{-3}\right) \right),
\]

\[
G(\alpha, \alpha + c) = \sqrt{\frac{2\pi}{\alpha}} e^{-a} \left( \frac{e^{-c}}{2} + O\left(\alpha^{-1}\right) \right).
\]

Proof. We prove the relations with the generalized Watson Lemma. Let \( \alpha > 0 \), \( k = 0, 1 \) and \( c \geq 0 \).

Then

\[
G(\alpha + k, \alpha + c) = \int_0^\infty \frac{t^k}{e^{ct} (1 + t^2)} e^{-\alpha(t - \log t)} dt
\]

\[
= \int_0^\infty f_{k,c}(t) e^{-\alpha g(t)} dt,
\]

with \( f_{k,c}(t) = t^k / (e^{ct} (1 + t^2)) \) and \( g(t) = t - \log t \). Before applying the Watson Lemma, we have to split the integral in two parts \( G(\alpha + k, \alpha + c) = \int_0^\infty + \int_1^1 \), because \( g \) has exactly one single minimum at \( a = 1 \). After verifying the conditions for the Watson Lemma ([14], Theorem 8.1) it allows us to expand the integral \( \int_0^\infty \) into an asymptotic series of the form

\[
\int_0^\infty f_{k,c}(t) e^{-\alpha g(t)} dt \simeq e^{-\alpha g(a)} \sum_{n=0}^\infty \Gamma\left(\frac{n + \lambda}{\mu}\right) a_n^{(k,c)} \frac{a_n^{(k,c)}}{\alpha^{(n+\lambda)/\mu}}, \quad \alpha \to \infty,
\]

with certain coefficients \( \lambda, \mu \) and \( a_n^{(k,c)} \). For the second integral \( \int_1^1 \) we have to apply a suitable transformation before expanding it. It is worth mentioning, that in the classical textbooks on asymptotic analysis (compare [14], p. 86) there is no general formula for the coefficients \( a_n \) available. Only the first one or two coefficients are derived and as it can be easily checked, they are of rather complicated nature. Surprisingly, in the newer literature ([15], Formula 2.3.18) one can find a remarkable easy representation for these coefficients in terms of some residues as well as a reference for its derivation, namely (in our context)

\[
a_n^{(k,c)} = \frac{1}{\mu} \text{Res}_{t=a} \left( \frac{f_{k,c}(t)}{(g(t) - g(a))^{(n+\lambda)/\mu}} \right), \quad n = 0, 1, 2, \ldots \quad (5.5)
\]

We used a symbolic computation software for the computation of the residues in (5.5), but we do not present the general outcome of these formulas. This
would fill several pages. However, since the calculations are of crucial importance in the proof for Theorem 5.1 we present all relevant outputs. For $k = 0, 1$ and $c = 0$ we calculate

$$
a_0^{(k,0)} = \frac{1}{2\sqrt{2}}, \quad a_3^{(k,0)} = \frac{45k^3 - 90k^2 - 90k + 86}{270}, $$

$$a_1^{(k,0)} = \frac{3k - 1}{6}, \quad a_4^{(k,0)} = \frac{36k^4 - 120k^3 - 96k^2 + 324k + 61}{432\sqrt{2}}, $$

$$a_2^{(k,0)} = \frac{6k^2 - 6k - 5}{12\sqrt{2}}, \quad a_5^{(k,0)} = \frac{189k^5 - 945k^4 - 315k^3 + 4683k^2 + 168k - 3730}{11340}. $$

For $c \geq 0$, we compute $a_0^{(0,c)} = e^{-c} \frac{1}{2\sqrt{2}}$ and $a_1^{(0,c)} = -e^{-c+3c} \frac{1}{6}$. With $\lambda = 1$ and $\mu = 2$ we obtain for $\alpha \to \infty$,

$$
\int_1^\infty f_{0,0}(t) e^{-\alpha g(t)} \, dt = \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha} \left( \frac{1}{4} - \frac{1}{6\sqrt{2\pi}} \frac{1}{\sqrt{\alpha}} - \frac{5}{48\alpha} + \frac{43}{135\sqrt{2\pi}} \frac{1}{\alpha^{3/2}} \right) + 61 \frac{1}{1152\alpha^2} - \frac{746}{1143\sqrt{2\pi}} \frac{1}{\alpha^2} + O(\alpha^{-3}), $$

$$
\int_1^\infty f_{1,0}(t) e^{-\alpha g(t)} \, dt = \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha} \left( \frac{1}{4} + \frac{1}{3\sqrt{2\pi}} \frac{1}{\sqrt{\alpha}} - \frac{5}{48\alpha} - \frac{49}{270\sqrt{2\pi}} \frac{1}{\alpha^2} \right) + 205 \frac{1}{1152\alpha^2} + \frac{5}{567\sqrt{2\pi}} \frac{1}{\alpha^2} + O(\alpha^{-3}), $$

$$
\int_1^\infty f_{0,c}(t) e^{-\alpha g(t)} \, dt = \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha} \left( \frac{e^{-c}}{4} - e^{-c} \frac{1 + 3c}{6\sqrt{2\pi}} \frac{1}{\sqrt{\alpha}} + O(\alpha^{-1}) \right). $$

Proceeding in the same way for the second integral $\int_0^1$, we compute

$$
a_0^{(k,0)} = \frac{1}{2\sqrt{2}}, \quad a_3^{(k,0)} = \frac{45k^3 - 90k^2 - 90k + 86}{270}, $$

$$a_1^{(k,0)} = \frac{3k - 1}{6}, \quad a_4^{(k,0)} = \frac{36k^4 - 120k^3 - 96k^2 + 324k + 61}{432\sqrt{2}}, $$

$$a_2^{(k,0)} = \frac{6k^2 - 6k - 5}{12\sqrt{2}}, \quad a_5^{(k,0)} = \frac{189k^5 - 945k^4 - 315k^3 + 4683k^2 + 168k - 3730}{11340}. $$

For $c \geq 0$, we compute $a_0^{(0,c)} = \frac{e^{-c}}{2\sqrt{2}}$ and $a_1^{(0,c)} = -e^{-c+3c} \frac{1}{6}$. Again with $\lambda = 1$
and \( \mu = 2 \) we obtain for \( \alpha \to \infty \),

\[
\int_0^1 f_{0,0}(t) e^{-\alpha g(t)} dt = \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha \left( \frac{1}{4} + \frac{1}{6\sqrt{2\pi}} \frac{1}{\sqrt{\alpha}} - \frac{5}{48\alpha} - \frac{43}{135\sqrt{2\pi}} \frac{1}{\alpha\sqrt{\alpha}} \right)}
+ \frac{61}{1152\alpha^2} + \frac{746}{1143\sqrt{2\pi} \alpha^2 \sqrt{\alpha}} + O(\alpha^{-3}) ,
\]

\[
\int_0^1 f_{1,0}(t) e^{-\alpha g(t)} dt = \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha \left( \frac{1}{4} - \frac{1}{3\sqrt{2\pi}} \frac{1}{\sqrt{\alpha}} - \frac{5}{48\alpha} + \frac{49}{270\sqrt{2\pi}} \frac{1}{\alpha\sqrt{\alpha}} \right)}
+ \frac{205}{1152\alpha^2} - \frac{5}{567\sqrt{2\pi} \alpha^2 \sqrt{\alpha}} + O(\alpha^{-3}) ,
\]

\[
\int_0^1 f_{0,e}(t) e^{-\alpha g(t)} dt = \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha \left( \frac{e^{-c}}{4} + e^{-c} \frac{1}{3\sqrt{2\pi}} \frac{1}{\sqrt{\alpha}} + O(\alpha^{-1}) \right)} .
\]

Collecting the results we finally arrive at the expansions in Lemma 5.6.

**Lemma 5.7.** There exists some \( \alpha_1 > 0 \), such that

\[
G(\alpha + 1, \alpha) - \left( 1 + \frac{1}{\alpha^3} \right) G(\alpha, \alpha) > 0, \quad \forall \alpha \geq \alpha_1 .
\]

**Proof.** From Lemma 5.6 we calculate

\[
G(\alpha + 1, \alpha) - G(\alpha, \alpha) = \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha \left( \frac{1}{4\alpha^2} + O(\alpha^{-3}) \right)} , \quad \alpha \to \infty .
\]

Now, combining the last expression together with Lemma 5.6 we obtain for \( \alpha \to \infty \) the asymptotics

\[
G(\alpha + 1, \alpha) - G(\alpha, \alpha) - \frac{1}{\alpha^3} G(\alpha, \alpha)
= \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha \left( \frac{1}{4\alpha^2} + O(\alpha^{-3}) \right)} - \frac{1}{\alpha^3} \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha \left( \frac{1}{2} + O(\alpha^{-1}) \right)}
= \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha \frac{1}{4\alpha^2} \left( 1 + O(\alpha^{-1}) \right)} .
\]

The assertion now follows.

**Proof of Theorem 5.1.** Let \( \alpha > \max(2, \alpha_1) \). Combining Lemma 5.5 together with Lemma 5.7, we deduce

\[
R(\alpha, \alpha) \geq \left( -\frac{1}{2^{\alpha+1}} + \frac{2}{2\alpha+1} \frac{1}{\alpha^3} + \frac{1}{2^{2\alpha-3}} - \frac{1}{(\alpha-2)2^{\alpha-3}} \right) G(\alpha, \alpha) .
\]

Since \( G(\alpha, \alpha) > 0, \forall \alpha > 0 \), an easy calculation reveals that the remaining term in the last expression becomes positive, at least for all \( \alpha \geq \alpha_0 = \max(14, \alpha_1) .
\]
We turn now to the proof for Theorem 5.3, again by establishing some Lemmas. Without proof, we first present the following

**Lemma 5.8.** Let $\alpha > 0$ and $\beta = \beta (\alpha) = \frac{\alpha}{\pi} + \frac{3}{2}\pi$. Then

(a) $\alpha + \frac{\pi}{2} < \beta \leq \alpha + \frac{3}{2}\pi$,

(b) $|H (\alpha, \beta)| = H_1 (\alpha, \beta)$.

**Lemma 5.9.** Let $\alpha > 0$ and $c \geq 0$. Then

$$G (\alpha, \alpha + c) = e^{-c} (1 + O (\alpha^{-1})),$$  \quad $\alpha \to \infty$.

**Proof.** From Lemma 5.6 we simply derive

$$\frac{G (\alpha, \alpha + c)}{G (\alpha, \alpha)} = e^{-c} (1 + O (\alpha^{-1})).$$

**Lemma 5.10.** Let $\alpha > 2$. Then

$$H_1 (\alpha, \alpha + \frac{3}{2}\pi) = H_1 (\alpha, \alpha) (1 + o (1)), \quad \alpha \to \infty.$$

**Proof.** Using (2.1a), we obtain

$$\frac{H_1 (\alpha, \alpha + \frac{3}{2}\pi)}{H_1 (\alpha, \alpha)} = \left( 1 + \frac{\frac{3}{2}\pi}{\alpha} \right)^\alpha \frac{F (\alpha, \alpha + \frac{3}{2}\pi)}{F (\alpha, \alpha)}. \quad (5.6)$$

Next, using Lemma 5.4 together with a standard estimate for the zeta function, we establish

$$\frac{F (\alpha, \alpha + \frac{3}{2}\pi)}{F (\alpha, \alpha)} \leq \frac{F_2 (\alpha, \alpha + \frac{3}{2}\pi)}{F_1 (\alpha, \alpha)} \leq \frac{2 - \frac{4}{\pi^{\alpha}}}{2 - \frac{4}{\pi}} \left( 1 + \frac{1}{\alpha - 2} \right) \frac{G (\alpha, \alpha + \frac{3}{2}\pi)}{G (\alpha, \alpha)}, \quad (5.7)$$

and

$$\frac{F (\alpha, \alpha + \frac{3}{2}\pi)}{F (\alpha, \alpha)} \geq \frac{F_1 (\alpha, \alpha + \frac{3}{2}\pi)}{F_2 (\alpha, \alpha)} \geq \frac{2 - \frac{4}{\pi^{\alpha}}}{2 - \frac{4}{\pi}} \left( 1 - \frac{1}{\alpha - 1} \right) \frac{G (\alpha, \alpha + \frac{3}{2}\pi)}{G (\alpha, \alpha)}. \quad (5.8)$$

Now, combining (5.6), (5.7), (5.8) together with Lemma 5.9 we establish the result.
Proof of Theorem 5.3. For \( \alpha \geq 2 \), it follows from Theorem 4.1 that
\[
\| H(\alpha, \cdot) \|_{L_\infty[0, \infty)} \leq \| H_1(\alpha, \cdot) \|_{L_\infty[0, \infty)} \leq \frac{C(\alpha)}{1 + \frac{2}{\sqrt{\alpha}}} \left( 1 + \frac{2}{\sqrt{\alpha}} \right). \tag{5.9}
\]
For the reverse side, let \( \varepsilon > 0 \) be arbitrary small. From Lemma 5.10 we can find some \( \alpha_2 > 0 \), such that for \( \alpha \geq \alpha_2 \),
\[
H_1\left( \alpha, \alpha + \frac{3}{2} \pi \right) \geq H_1(\alpha, \alpha)(1 - \varepsilon).
\]
Using Lemma 5.8, Theorem 5.2 and Theorem 4.1 we further obtain for \( \alpha \geq \max(2, \alpha_0, \alpha_2) \) the estimate
\[
\| H(\alpha, \cdot) \|_{L_\infty[0, \infty)} \geq | H(\alpha, \beta) | = H_1(\alpha, \beta) \geq H_1\left( \alpha, \alpha + \frac{3}{2} \pi \right) \geq H_1(\alpha, \alpha)(1 - \varepsilon) \geq \frac{C(\alpha)}{1 + \frac{2}{\sqrt{\alpha}}} \left( 1 - \frac{1}{\sqrt{\alpha}} \right)(1 - \varepsilon). \tag{5.10}
\]
Finally, combining (5.9) together with (5.10), establishes the result and we are finished.

6 Approximation polynomials in \( L_\infty \)

This section is devoted to an explicit construction for near best approximation polynomials to \( |x|^\alpha, \alpha > 0 \) in the \( L_\infty \) norm. The construction involves the polynomials \( P_n^{(1)} \) and \( P_n^{(2)} \) together with the Chebyshev polynomials \( T_n \). The construction method is based on numerical results. The resulting formulas could indicate a general possible approach and structure for the Bernstein constants \( \Delta_{\alpha, \infty} \).

Let \( \alpha > 0 \) be not an even integer.

First, let us collect some details on the interpolating polynomials \( P_{2n}^{(1)} \). Recall, that the interpolation points are given by \( x_{j}^{(2n)} = \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi}{2n} \right) \) for \( j = 1, 2, \ldots, 2n \) and \( x_0^{(2n)} = 0 \). From Ganzburg [5], Formulas 2.1, 2.7 and

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4.14) it follows

\[
\lim_{n \to \infty} (2n)^\alpha \| |x|^\alpha - P^{(1)}_{2n}(x) \|_{L_\infty[-1,1]} = \left\| |x|^\alpha - G_\alpha \right\|_{L_\infty[0,\infty)} = 2 \pi \left| \sin \frac{\pi \alpha}{2} \right| \int_0^\infty \frac{t^{\alpha-1}}{\cosh(t)} \frac{x^2 \cos x}{x^2 + t^2} dt,
\]

where

\[
G_\alpha(x) = |x|^\alpha - \frac{2}{\pi} \sin \frac{\pi \alpha}{2} \int_0^\infty \frac{t^{\alpha-1}}{\cosh(t)} \frac{x^2 \cos x}{x^2 + t^2} dt,
\]

is an entire function of exponential type 1 that interpolates $|x|^\alpha$ at the nodes \( \{ (k + \frac{1}{2}) \pi : k \in \mathbb{Z} \} \cup \{0\} \). There also exists ([5], Formula 4.15) a representation for $G_\alpha$ as an interpolating series, similar to formula (3.2) in Theorem 3.3.

By an analogue method as that was used in the proof for Theorem 3.2 one can show that uniformly on compact subsets of \([0, \infty)\) we have the scaled limit

\[
\lim_{n \to \infty} (2n)^\alpha P^{(1)}_{2n}\left(\frac{x}{2n}\right) = G_\alpha(x).
\]

Now, based on numerical computations, we made the following observations. For all $\alpha > 0$ not an even integer we find that, beginning with the second positive note, all interpolation points of the best approximation polynomials $P_{2n}$ are located somewhere between two consecutive interpolation points for the $P^{(1)}_{2n}$ and $P^{(2)}_{2n}$ polynomials. See Figure 5.

It is well known that \([1, x, \ldots, x^n; x^{n/2}]\) is an hypernormal Haar space of dimension $n + 2$ on the interval \([0, 1]\), see ([21], p. 199). Consequently it follows that we have always an alteration point at $x = 0$. Thus we cannot expect to perform in the quality of best approximation solely by using the polynomials $P^{(1)}_{2n}$ and $P^{(2)}_{2n}$, since both of them interpolate at $x = 0$. Thus we consider the following polynomials

\[
P^{(3)}_{2n}(x) = c_{1,\alpha} P^{(1)}_{2n}(x) + (1 - c_{1,\alpha}) P^{(2)}_{2n}(x) + \frac{2}{\pi} \sin \frac{\pi \alpha}{2} c_{2,\alpha} \frac{(-1)^n}{(2n)!} T_{2n+1}(x),
\]

where $c_{1,\alpha}$ and $c_{2,\alpha}$ are numerical constants, depending only on $\alpha$. As we see later, for good choices of $c_{1,\alpha}$ and $c_{2,\alpha}$ the linear combination of $P^{(1)}_{2n}$ and
$P_{2n}^{(2)}$ results in a polynomial with almost all the same interpolation points as its best approximation $P_{2n}^*$, while at the same time the last term in (6.3) establishes the alternation property at $x = 0$ and leaves the new interpolation points largely unchanged.

Since we are interested into the asymptotic behavior of the polynomials $P_{2n}^{(3)}$ we directly pass to the resulting scaled limit. From Theorem 3.2 formulas (2.3), (6.1), (6.2) and Lemma 3.6 it follows that uniformly on compact subsets of $[0, \infty)$ we have

$$
\lim_{n \to \infty} (2n)^\alpha P_{2n}^{(3)} \left( \frac{x}{2n} \right) = |x|^\alpha - \frac{2}{\pi} \sin \frac{\pi \alpha}{2} \left( c_{1, \alpha} \int_0^\infty t^{\alpha-1} \frac{x^2 \cos x}{\cosh t} \frac{dt}{x^2 + t^2} + (1 - c_{1, \alpha}) \int_0^\infty \frac{t^\alpha x \sin x}{\sinh t} \frac{dt}{x^2 + t^2} - c_{2, \alpha} \frac{\sin x}{x} \right). \quad (6.4)
$$

Thus, we try to numerically minimize the quantity

$$
\left\| c_{1, \alpha} \int_0^\infty t^{\alpha-1} \frac{x^2 \cos x}{\cosh t} \frac{dt}{x^2 + t^2} + (1 - c_{1, \alpha}) \int_0^\infty \frac{t^\alpha x \sin x}{\sinh t} \frac{dt}{x^2 + t^2} - c_{2, \alpha} \frac{\sin x}{x} \right\|_{L_\infty[0, \infty)}.
$$

For the moment, we cannot present an explicit formula for the constants $c_{1, \alpha}$ and $c_{2, \alpha}$, but based on numerical calculations, we present the following

---

Figure 5: Interpolation points for the best approximation to $|x|^\alpha$. 
Using these numerical values, we present some illustrations for the $P^{(3)}_n$ polynomials from (6.3). In Figure 6 we present the polynomials $P^{(3)}_4$, $P^{(3)}_8$ together with the best approximations $P^*_4$, $P^*_8$ and $\alpha = 0.5$. The same is done in Figure 7 for $\alpha = 1.0$.

![Figure 6: $\alpha = 0.5$. Polynomials $P^{(3)}_4$, $P^*_4$ and $P^{(3)}_8$, $P^*_8$.](image)

We also tried to find some approximations for the minimizing best entire functions $H^*_\alpha$ defined by

$$
\Delta_{\infty,\alpha} = \| |x|^{\alpha} - H^*_\alpha \|_{L_{\infty}[0,\infty)} = \inf \left\{ \| |x|^{\alpha} - H \|_{L_{\infty}(\mathbb{R})} : H \text{ is entire of exponential type } \leq 1 \right\}.
$$

Especially we are interested into the locations of its corresponding interpolation points. Recall, that from ([10], [11]) it follows, that uniformly on compact subsets of $\mathbb{C}$ we have

$$
\lim_{n \to \infty} (2n)^\alpha P^*_2 \left( \frac{z}{2n} \right) = H^*_\alpha (z), \quad (6.5)
$$
There is also a representation for $H_\alpha^*$ as an interpolation series with (unknown) interpolation points $0 < x_1^* < x_2^* < x_3^* < \cdots$. However, it is known ([11], Theorem 1.1) that

$$x_j^* \in \left[ \left( j - \frac{3}{2} \right) \pi, \left( j - \frac{1}{2} \right) \pi \right], \quad \forall j \geq 2.$$

Moreover, from ([11], Formulas 1.6 and 1.7) it follows that there exists alternation points $0 = y_0^* < y_1^* < y_2^* < \cdots$ with

$$|y_j^*|^{\alpha} - H_\alpha^*(\pm y_j^*) = (-1)^{j+\alpha/2} \|x|^\alpha - H_\alpha^*\|_{L_\infty(\mathbb{R})},$$

where $\overline{\alpha/2}$ is the least integer exceeding $\alpha/2$. For the alternation points it is also known that

$$y_j^* \in [(j - 1)\pi, j\pi], \quad \forall j \geq 1.$$

We use formula (6.4) as an approximation for $H_\alpha^*$. In Figure 8 we present some illustrations from (6.4) for $\alpha = 0.5$ and $\alpha = 1.0$. In Figure 9 we illustrate the near equioscillating behavior of the error term in (6.4), again for $\alpha = 0.5$ and $\alpha = 1.0$, and we compare the maximal error magnitude with the corresponding numerical values for the Bernstein constants

$$\Delta_{\infty,0.5} = 0.348648 \ldots,$$
$$\Delta_{\infty,1} = 0.280169 \ldots$$

The values for the Bernstein constants are taken from ([21], Table 1.1).

In the following table we present the approximations for the best interpolation points $x_j^*$ for $j = 1, \ldots, 10$ from (6.4), respectively from Figure 8.

| $\alpha$ | $x_1^*$ | $x_2^*$ | $x_3^*$ | $x_4^*$ | $x_5^*$ | $x_6^*$ | $x_7^*$ | $x_8^*$ | $x_9^*$ | $x_{10}^*$ |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.5     | 0.13    | 2.10    | 4.99    | 8.04    | 11.13   | 14.25   | 17.37   | 20.50   | 23.63   | 26.76   |
| 0.8     | 0.25    | 2.30    | 5.15    | 8.16    | 11.22   | 14.32   | 17.43   | 20.55   | 23.67   | 26.80   |
| 1.0     | 0.34    | 2.38    | 5.24    | 8.23    | 11.28   | 14.36   | 17.47   | 20.58   | 23.70   | 26.83   |
The last table suggests that, for small positive values $\alpha$, all interpolation points are slightly shifted to the left. Apparently this effect becomes greater for those interpolation points which are located closer to the origin. On the other hand, the values suggest that

$$x_{n+1}^* - x_n^* \to \pi, \quad n \to \infty,$$

from below.

Finally, we remark that the overall quality of the $P_n^{(3)}$ polynomials appears to be very encouraging in search for some representations of the Bernstein constants. Their approximation properties with respect to the corresponding best approximation polynomials $P_n^*$ are of high quality, even for small values of $n$. Thus, formula 6.4 though it is at the present time not in its full explicit form, appears to be an important step towards a possible representation for the Bernstein constants $\Delta_{\infty, \alpha}$.
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