Approximation of tensor fields on surfaces of arbitrary topology based on local Monge parametrizations

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Abstract

We introduce a new method, the Local Monge Parametrizations (LMP) method, to approximate tensor fields on general surfaces given by a collection of local parametrizations, e.g. as in finite element or NURBS surface representations. Our goal is to use this method to solve numerically tensor-valued partial differential equations (PDE) on surfaces. Previous methods use scalar potentials to numerically describe vector fields on surfaces, at the expense of requiring higher-order derivatives of the approximated fields and limited to simply connected surfaces, or represent tangential tensor fields as tensor fields in 3D subjected to constraints, thus increasing the essential number of degrees of freedom. In contrast, the LMP method uses an optimal number of degrees of freedom to represent a tensor, is general with regards to the topology of the surface, and does not increase the order of the PDEs governing the tensor fields. The main idea is to construct maps between the element parametrizations and a local Monge parametrization around each node. We test the LMP method by approximating in a least-squares sense different vector and tensor fields on simply connected and genus-1 surfaces. Furthermore, we apply the LMP method to two physical models on surfaces, involving a tension-driven flow (vector-valued PDE) and nematic ordering (tensor-valued PDE). The LMP method thus solves the long-standing problem of the interpolation of tensors on general surfaces with an optimal number of degrees of freedom.

Keywords: Vector-valued PDE, tensor-valued PDE, finite elements, surface PDE, approximation

1. Introduction

1.1. Motivation for the approximation of tensor fields on surfaces

Tensor fields are fundamental objects in continuum physics. Vector fields, the simplest nontrivial instance of a tensor field, appear as a natural representation of the velocity of material particles [1] or the polarization in dielectrics [2]. Second-order tensor fields are employed to represent fiber distribution or the shape of microscopic inclusions in composite materials [3], to parametrize nematic ordering in liquid crystals [4, 5], or to describe internal variables in plasticity [3]. The steady-state or the time-evolution of these tensorial quantities are governed by tensor-valued partial differential equations (PDEs). A classical way to solve vector- or tensor-valued PDEs numerically in two-dimensional or three-dimensional Euclidean space is through the Finite Element Method (FEM), where the Cartesian components of the tensor field are discretized at the nodes of a mesh and approximated with the help of basis functions. However, as we will show in the next section, this straightforward approximation cannot be applied to the components of a tangential vector or tensor field on a surface without increasing the number of essential degrees of freedom, thus also increasing the computational cost involved in the solution.
Motivated by increasingly quantitative experiments in soft matter and mechanobiology, there is a need to model and simulate the mechanics of thin structures, viewed as surfaces in Euclidean space, including lipid bilayers [6, 7, 8, 9, 10], the cell cortex [11, 12, 10, 13] or epithelial monolayers [14, 15]. In most of these works, vector- and tensor-valued PDEs are solved to describe the time-evolution of the surfaces and the different physical quantities characterizing the material. From a theoretical and computational side, there has also been a growing attention to the formulation and resolution of vector- and tensor-valued PDEs on surfaces including Navier-Stokes equations [16, 17, 18, 19], vector Laplacians [20], models for liquid crystal [21, 22, 23, 24], or viscoelasticity and plasticity formulations [25].

Besides finite element simulations of continuum systems, tensor fields on surfaces also play an important role in computer graphics. Vector and tensor fields are designed [26, 27, 28, 29] and routinely discretized [30, 31, 32, 33, 34, 35, 36, 37, 38] to control the appearance of surfaces, including texture synthesis [39, 40], line illustration, non-photorealistic rendering [41], and anisotropic meshing [42], among many others [43].

A few methods exist for the discretization of vector and tensor fields on surfaces in the context of computational mechanics, such as the Hodge decomposition of vector fields for simply-connected surfaces [16, 44, 17, 10, 18], or methods relying on the Cartesian representation of vector and tensor fields mapped to tangential fields using projections [19, 45, 21, 46]. In computer graphics, vector fields are discretized as piece-wise constant per face [30, 31, 33], per edge [32, 34, 26, 35], per vertex [27, 47], or encoded as linear operators [36, 37]. However, these methods present several drawbacks for the resolution of vector- or tensor-valued PDEs on surfaces: face-based representations lack of a clear notion of covariant derivative, edge-based discretizations are difficult to generalize to higher-order tensors, and existing vertex-based representations depend on the definition of an alternative notion of covariant derivative (connection) which does not match that of the discretization. We refer to [48] for a thorough and illustrative review of these different kind of discretizations in computer graphics and their shortcomings. With the goal of solving tensor-valued PDEs on surfaces, we propose a new method based on local Monge parametrizations for the discretization of tensor field on surfaces. We show the application of our methodology in the context of subdivision surfaces [49, 50] and apply it to numerically approximate different problems relevant to soft matter and biological physics.

1.2. Description of the problem

Our starting point is a surface $\Gamma$ described by piece-wise parametrizations as in finite element approximations, which may be obtained either as an approximation to a given surface or computationally from a physical model. We consider a control mesh consisting on $N_E$ elements, $N_P$ control points and the basis functions $N_{[E,I]}(\xi)$ of control point $I$ for element $E$ taking values in a reference element $\Xi$. The discretization of $\Gamma$ is then given by a collection of finite element parametrizations

$$\psi_{[E]}(\xi) = \sum_{I\in\langle E \rangle} x_{[I]} N_{[E,I]}(\xi),$$

where $\langle E \rangle$ refers to the nodes contributing to the approximation in element $E$ and $x_{[I]}$ denotes the $I$th control point. The collection of parametrized elements $\Gamma_{[E]} = \psi_{[E]}(\Xi)$ conforms the numerical surface $\Gamma$. We also define $\Gamma_{[I]} = \bigcup_{E\in\langle I \rangle} \Gamma_{[E,I]}$, where now $\langle I \rangle$ identifies the elements in the neighborhood of a node (the dual of $\langle E \rangle$). Let us now consider the discretization of a scalar field $\alpha$ on $\Gamma$, given by the nodal coefficients $\alpha_{[I]}$

$$\alpha(x) = \sum_{I\in\langle E \rangle} \alpha_{[I]} N_{[E,I]} \circ \psi_{[E]}^{-1}(x).$$

Note that to evaluate $\alpha$ on $x \in \Gamma$, we need to compose the basis functions, taking values in the reference element $\Xi$, with the inverse of $\psi_{[E]}$. For points $x$ on $\Gamma$ belonging to edges between elements, there will be several elements whose image by $\psi_{[E]}$ contains $x$. Thus, one can compute several different $\psi_{[E]}^{-1}$ and calculate Eq. (2) in different elements. However, the resulting $\alpha(x)$ will be the same in each of these cases provided that the basis functions $N_{[E,I]}$ are continuous across elements, i.e. $N_{[E,I]} \circ \psi_{[E]}^{-1}(x) = N_{[E',I]} \circ \psi_{[E']}^{-1}(x)$ whenever $x \in \Gamma_{[E]} \cap \Gamma_{[E']}$. 




Now, let us consider a vector field $\mathbf{v}$ of the tangent bundle of $\Gamma$, $TT\Gamma = \bigcup_{x \in \Gamma} T_x \Gamma$ with $T_x \Gamma$ the tangent plane to $\Gamma$ at $x$. Given an orthonormal basis of Euclidean space $\{i_1, i_2, i_3\}$ we could discretize the components of $\mathbf{v} = v^a i_a$ in this basis as

$$v^\alpha = \sum_{I \in \langle E \rangle} v^a_I N_{[E,I]} \circ \psi^{-1}_{[E]}(x), \quad \alpha = 1, 2, 3. \quad (3)$$

Note that we use Einstein summation convention for repeated indices, denote with greek letters indices running from 1 to 3, and use super-indices for contravariant components of tensors and sub-indices for covariant components. Because $\mathbf{v}$ is tangent to $\Gamma$, its three components $v^1, v^2$ and $v^3$ are not independent and need to satisfy that $\mathbf{v} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the surface normal. In calculations, one would usually need to introduce this additional constraint through a Lagrange multiplier or a penalty [19, 45, 46]. Alternatively, one could consider the discretization of the components of $\mathbf{v} = v^a e_a$ in a basis $e_a$ of the tangent space of $\Gamma$, i.e.

$$v^a(x) = \sum_{I \in \langle E \rangle} v^a_I N_{[E,I]} \circ \psi^{-1}_{[E]}(x), \quad a = 1, 2. \quad (4)$$

Subsequently, we use latin letters to denote indices running from 1 to 2. Eq. (4) requires that the basis $e_a$ tangent to $\Gamma$ is defined everywhere. Furthermore, if $\mathbf{v}$ is to be continuous, then $e_a$ also needs to be continuous on $\Gamma$ so that $\mathbf{v} = v^a e_a$ is continuous. However, one cannot define such a basis for a general closed surface, e.g. on a sphere, as a consequence of the hairy ball theorem [51]. For instance, polar coordinates in the sphere present singularities at the poles. We also note that the natural basis of the finite element parametrizations, given by the vectors $\partial_1 \psi_{[E]}$ and $\partial_2 \psi_{[E]}$, is discontinuous across elements due to the jump in the definition of local coordinates from one element to another. These are fundamental difficulties to an approximation of the tangential components of a vector field as in Eq. (4).

### 1.3. Hodge decomposition

A classical approach to discretize a vector field on a surface using only tangential calculus rather than relying on its interdependent three-dimensional components is the so-called Hodge decomposition of a vector field in terms of scalar potentials, which are trivial to approximate following Eq. (2). According to this decomposition, a vector field $\mathbf{v}$ tangent to a surface $\Gamma$ can be represented as

$$\mathbf{v} = \nabla \alpha + \nabla \times \beta + \mathbf{h}, \quad (5)$$

where $\alpha$ and $\beta$ are scalar fields on $\Gamma$ and $\mathbf{h}$ is a harmonic vector field, satisfying $\nabla \cdot \mathbf{h} = 0$ and $\nabla \times \mathbf{h} = 0$. Here, $\nabla$ denotes the surface gradient or covariant derivative. Note that the curl operator $\nabla \times$ on a surface (an instance of exterior derivative) applied on a scalar function $\beta$ is a vector with components $(\nabla \times \beta)^a = \epsilon^{ab} \nabla_b \beta$, where $\epsilon$ is the antisymmetric tensor, with components

$$\epsilon^{ab} = J^{-1} \epsilon^{ab}, \quad (6)$$

with $\epsilon^{ab}$ the Levi-Civita symbols, defined by $\epsilon^{11} = \epsilon^{22} = 0, \epsilon^{12} = -\epsilon^{21} = 1$, $J = \sqrt{\det g}$ and $g$ is the metric tensor on the surface, whose components in a basis $\{e_1, e_2\}$ of $TT$ are given by $g_{ab} = e_a \cdot e_b$. When acting on a vector, $\nabla \times$ leads to a scalar $\nabla \times \mathbf{h} = \epsilon^{ab} \nabla_b h^a$. It is not clear how Eq. (5) could help us discretize a vector field since, while the scalar fields $\alpha$ and $\beta$ are easily dealt with, one still needs to approximate the harmonic vector field $\mathbf{h}$. For simply connected surfaces, i.e. closed surfaces with genus equal to 0 topologically equivalent to a sphere, there is only a trivial harmonic vector field, $\mathbf{h} = \mathbf{0}$, and thus $\mathbf{v}$ can be described in terms of the two scalar fields $\alpha$ and $\beta$. In this case, one can approximate $\alpha$ and $\beta$

$$\alpha(x) = \sum_{I \in \langle E \rangle} \alpha_I N_{[E,I]} \circ \psi^{-1}_{[E]}(x), \quad \beta(x) = \sum_{I \in \langle E \rangle} \beta_I N_{[E,I]} \circ \psi^{-1}_{[E]}(x), \quad (7)$$

and consider the discretization of $\mathbf{v}$ as

$$\mathbf{v}(x) = \sum_{I \in \langle E \rangle} \left( \left( \alpha_I g^{ab} \nabla_b N_{[E,I]} + \beta_I \epsilon^{ab} \nabla_b N_{[E,I]} \right) \partial_a \psi_{[E]} \right) \circ \psi^{-1}_{[E]}(x). \quad (8)$$
Local Monge parametrization. In the neighborhood of node $I$, defined as the support of the basis function associated with it (light red), a Monge parametrization relative to a plane $\Theta_{[I]}$ passing through $x_{[I]}$ can be defined. The two coordinates $\theta^1$ and $\theta^2$ define the position of a point in $\Theta_{[I]}$ and parametrize the surface with the help of a height function $h_{[I]}$. Given a triangle of the finite element discretization $\Gamma_{[E]}$ and its parametrization from the reference element $\Xi, \psi_{[E]}$, one can construct in each point of this element a linear map between the bases of the tangent plane at this point given by $B_{\psi_{[E]}} = \{ \partial_a \psi_{[E]} \}$ to $B_{\phi_{[I]}} = \{ \partial_a \phi_{[I]} \}$.

This method, which we refer to as the Hodge Decomposition (HD) method, was introduced in the context of fluid interfaces by Secomb and Skalak [52] and subsequently used by many others, mostly for inextensible flows on stationary surfaces where $v = \nabla \times \beta$ (since $\nabla \cdot v = \Delta \alpha = 0$) [16, 53, 17, 54, 55, 56]. More recently, its full representation involving both $\alpha$ and $\beta$ has been used to solve flows on surfaces with imposed evolution [17] or in self-consistent models including equations for shape evolution [10]. In computer graphics, the HD is routinely used for vector field design [44, 48]. However, this method is restricted to simply connected surfaces and cannot be applied to higher-order tensor fields, although an extension for second-order tensor fields based on discrete exterior calculus has been recently proposed [38]. Furthermore, as the representation of the vector field already involves derivatives, it leads to higher-order PDEs. In the next section, we introduce a new method for the approximation of general tensor fields on surfaces of arbitrary topology.

2. Description of the LMP method

Discretizing tensor fields for the resolution of tensor-valued PDEs on surfaces of arbitrary topology based only on tangential calculus remains an open problem. The main goal of this work is to provide a general method to address this issue, which we call the Local Monge Parametrizations (LMP) method. The LMP method can be synthesized in three steps. First, one needs to construct a set of local Monge parametrizations of $\Gamma$ around each node of the mesh. Then, one needs to find the changes of coordinates between local Monge parametrizations and the finite element bases. We describe how to find a suitable set of local Monge parametrizations and how to compute the changes of basis in section 2.1. In the LMP method, the components of a tensor are discretized at the nodes of the mesh in terms of their respective local Monge parametrizations. As we describe in section 2.2, the LMP method then uses the changes of basis from local Monge parametrizations to the finite element parametrizations to reconstruct the components of the tensor in the finite element parametrization used for calculations.

2.1. Local Monge parametrizations

We introduce next an atlas of local Monge parametrizations, one per node of the control mesh. Given node $I$, we recall that its neighborhood $\Gamma_{[I]}$ is the image of all element parametrizations of the form in Eq. (1)
to which node \( I \) contributes. We further define a plane \( \Theta_{[I]} \) passing through \( x_{[I]} \), which does not need to be tangent to \( \Gamma \) but must satisfy that in \( \Gamma_{[I]} \) the projection of \( \Gamma \) onto \( \Theta_{[I]} \) is one-to-one. We define a basis of \( \Theta_{[I]} \) given by the orthonormal vectors \( i_{[I]1} \) and \( i_{[I]2} \) and we denote by \( n_{[I]} \) the unit normal to \( \Theta_{[I]} \) (see Fig. 1). Thus, \( \{i_{[I]1}, i_{[I]2}, n_{[I]}\} \) forms an orthonormal basis of Euclidean space.

We can now define a local Monge parametrization of the surface around \( I \), \( \phi_{[I]} : \Theta_{[I]} \to \Gamma_{[I]} \) given by

\[
\phi_{[I]}(\theta^1, \theta^2) = x_{[I]} + \theta^1 i_{[I]1} + \theta^2 i_{[I]2} + h_{[I]}(\theta^1, \theta^2)n_{[I]},
\]

where \( h_{[I]} \) is the height function from \( \Theta_{[I]} \) to \( \Gamma_{[I]} \). The basis vectors of the tangent plane, that induced by the parametrization of the corresponding element \( B_{\psi_{[E]}} = \{\partial_A \psi_{[E]}\} \), and of the local Monge parametrization of node \( I \) \( B_{\phi_{[I]}} = \{\partial_A \phi_I\} \). We can then express a tangent vector \( v \) locally in either of these bases

\[
v = v^a \partial_a \psi_{[E]} = V^A \partial_A \phi_{[I]}.
\]

The change of basis between \( B_{\psi_{[E]}} \) and \( B_{\phi_{[I]}} \) is just a linear map \( T_{[E,I]} \) from the tangent plane to itself given by four coefficients. A simple calculation shows (see Appendix A) that

\[
T_{[E,I]}^A_a (\theta_1, \theta_2) = i_{[I]A} \cdot \partial_a \psi_{[E]}(\theta_1, \theta_2),
\]

from which we can relate the tangent vectors in \( B_{\psi_{[E]}} \) and \( B_{\phi_{[I]}} \) as

\[
\partial_a \psi_{[E]} = T_{[E,I]}^A_a \partial_A \phi_{[I]} \quad \text{and} \quad \partial_A \phi_{[I]} = T_{[E,I]}^{-1 A}_a \partial_a \psi_{[E]},
\]

where \( T_{[E,I]}^{-1} = T_{[E,I]}^{-1} \). Remarkably, \( T_{[E,I]} \) is independent of \( h_{[I]} \).

2.2. Approximation of tensor fields

Let us first describe the interpolation of a vector field \( v \) by the LMP method and leave the interpolation of higher-order tensors for later in this section. We will approximate a vector field over the surface in element \( E \) as a linear combination of the form

\[
v(x) = \sum_{I \in (E)} \left( v_{[I]} N_{[E,I]} \right) \circ \psi^{-1}_{[E]}(x).
\]

In a usual FEM discretization, \( v_{[I]} \) would be a constant vector. For general surfaces with Gaussian curvature, however, there is a fundamental difficulty to define a constant vector field \([57, 48]\). For this reason, around each node \( I \) we define a vector field \( v_{[I]}(x) \) with support in \( \Gamma_{[I]} \) and given by

\[
v_{[I]}(x) = V_{[I]}^A \left( \partial_A \phi_{[I]} \circ \phi_{[I]}^{-1}(x) \right),
\]

which plays the role of the coefficient of node \( I \). In fact, this local vector field is such that its components are constant in the basis \( B_{\phi_{[I]}} \) of the \( T_{x} \Gamma \). This local vector field is continuous provided that \( \phi_{[I]} \), the local Monge parametrization of node \( I \), is continuously differentiable, which is the case if \( \Gamma_{[I]} \) is continuously differentiable. Since all the local Monge parametrizations form an atlas of the surface, this requires \( \Gamma \) to be continuously differentiable. If this is the case, since the basis functions \( N_{[E,I]} \) are continuous across elements, then the element-wise approximations in Eq. (13) define a continuous vector field over \( \Gamma \). Following this argument, a surface defined by \( C^0 \) finite elements would not lead to a continuous vector field using this method. Instead, Loop subdivision surfaces are \( C^2 \) everywhere except at irregular points \([49, 50]\), where they present jumps in second-order derivatives, and hence \( v \) will be continuously differentiable everywhere except at irregular points.
method is able to approximate a vector field $v = 2$. The rate of convergence is unaffected by geometric non-axisymmetric perturbations of the sphere, $r = 3$, see Fig. 2 top. A similar analysis with the HD method, shows that the latter converges with a rate $w$ mesh size. We compute $\Gamma$ by a least-squares fit to the sphere. We find that the field $v$ solving a linear system of a given vector field $w$ the approximation of the vector field to $\Gamma$, we compute the projection onto $\Gamma$ of a three-dimensional vector field $W$. Here, $g$ are the components of the metric tensor in the basis $B_{\psi[E]}$. Although the basis vectors $\partial_a \psi[E]$ are discontinuous, the also discontinuous linear transformations $\hat{T}_{[E,I]}$ make the resulting approximation continuous as discussed above.

This method can be generalized to any kind of tensor field over $\Gamma$, noting that the dual bases of $B_{\psi[E]}$ and $B_{\psi[E]}$ transform analogously to Eq. (12) but with the inverse matrix. For a 1-form $\alpha$, one can thus write its components in the dual of $B_{\psi[E]}$ as

$$\alpha_a(x) = \sum_{l \in (E)} \alpha_{[l],a} \left( T_{[E,I]}^{A} N_{[E,I]} \right) \circ \psi_{[E]}^{-1}(x).$$

For a general tensor, this can be trivially generalized to

$$q_{a_1 \ldots a_m b_1 \ldots b_n}(x) = \sum_{l \in (E)} q_{[l]C_1 \ldots C_m D_1 \ldots D_n} \left( T_{[E,I]}^{C_1} \cdots T_{[E,I]}^{C_n} \hat{T}_{[E,I]}^{a} \cdots \hat{T}_{[E,I]}^{b} \right) \circ \psi_{[E]}^{-1}(x).$$

3. Numerical approximation of vector and tensor fields with LMP and subdivision finite elements

Here, we consider a FEM setup based on subdivision surfaces [49, 58] similar to that in [10] to examine numerically the approximation power of the LMP method by computing the $L_2$ projection $v(x)$ as in Eq. (16) of a given vector field $w(x)$ on a given surface. For a vector field, finding the least-squares fit amounts to solving a linear system

$$Kv = b,$$

where

$$K_{2I+A,2J+B} = \sum_{E \in (I) \cap (J)} \int_{V_{[E]}} g_{ab}(x) \left( N_{[E,I]} N_{[E,J]} \hat{T}_{[E,I]}^{a} \hat{T}_{[E,J]}^{b} \right) \circ \psi_{[E]}^{-1}(x) dS,$$

$$b_{2I+A} = \sum_{E \in (I) \cap (J)} \int_{V_{[E]}} w^b(x) g_{ab}(x) \left( N_{[E,I]} \hat{T}_{[E,I]}^{a} \right) \circ \psi_{[E]}^{-1}(x) dS.$$

Here, $g_{ab}$ are the components of the metric tensor in the basis $B_{\psi[E]}$. To define a vector field $w(x)$ tangent to $\Gamma$, we compute the projection onto $\Gamma$ of a three-dimensional vector field $W(x)$. We start by examining the approximation of the vector field $W(x) = \cos(6\pi z)(-y,x,0)$ on a sphere of radius 1 as we refine the mesh size. We compute $\Gamma$ by a least-squares fit to the sphere. We find that the field $v$ represented using the LMP method converges to $w$ with an error scaling as $h^{-r}$, with $h$ being the average mesh size and with rate $r = 3$, see Fig. 2 top. A similar analysis with the HD method, shows that the latter converges with a rate $r = 2$. The rate of convergence is unaffected by geometric non-axisymmetric perturbations of the sphere, see Fig. 2 bottom. The faster convergence rate of the LMP method can be expected since HD involves one additional derivative of the basis functions.

Considering now a non-simply connected surface with genus 1, a torus, Fig. 3 shows that while the LMP method is able to approximate a vector field $v$ with non-zero harmonic component, the HD method fails
Figure 2: Comparison between the approximation power of the LMP and HD methods on simply connected surfaces with a discretization based on subdivision surfaces. We observe a convergence rate of the $L_2$ error as a function of element size of 3 for the LMP method and of 2 for the HD method for two different surface geometries. In the right, we show the solution found with the LMP method for the finest mesh and the coarser mesh used for geometry.

Figure 3: Comparison between the LMP method and the HD method for the approximation of a vector field $\mathbf{v} = (-y, 0, 0)$ on a torus. While the LMP method is able to represent $\mathbf{v}$ on the torus, the HD method fails because the harmonic part of $\mathbf{v}$ cannot be represented using vector potentials.
LMP method applied to the interpolation of general second order tensor fields on a surface. (left) $L_2$ error convergence as a function of mesh size. We observe a cubic convergence for LMP method on a sphere (with irregular points) and a quartic convergence on a torus (no irregular points). (center) Example of tensor field on a sphere reproduced with the LMP method. The antisymmetric component $\epsilon_{ab} \sigma_{ab}$ is plotted as a colormap, while the symmetric part $(\sigma_{ab} + \sigma_{ba})/2$ is represented with two pairs of arrows whose direction indicate the two principal directions of the tensor and whose size is proportional to the (signed) eigenvalues (inward and outward arrows indicate negative and positive eigenvalues respectively). (right) Same plot on a torus.

due to its inability to represent the harmonic component, see [18] for a detailed discussion on this issue. We also note that, for the torus in Fig. 3 with no irregular points in the subdivision triangulation, the rate of converge of the LMP method increases to fourth order ($r = 4$).

Finally, we consider the discretization of a general second-order tensor field $\sigma$. In this case, the matrix and right-hand sides of the system have the form

$$K^h_{I+2A+B, I+2C+D} = \sum_{E \in \{I\} \cap \{J\}} \int_{\Gamma_{[E]}} g_{ac}(x) g_{bd}(x) \left( N_{[E,I]} N_{[E,J]} \hat{T}_{[E,I]}^a \hat{T}_{[E,J]}^b C \hat{T}_{[E,J]}^c D \right) \circ \psi_{[E]}^{-1}(x) dS,$$

$$b^h_{I+2A+B} = \sum_{E \in \{I\} \cap \{J\}} \int_{\Gamma_{[E]}} \sigma^{cd}(x) g_{ca}(x) g_{db}(x) \left( N_{[E,I]} \hat{T}_{[E,I]}^a A \hat{T}_{[E,J]}^b C \hat{T}_{[E,J]}^d \right) \circ \psi_{[E]}^{-1}(x) dS.$$  

To test the convergence of the method, we consider the three dimensional tensor $\Sigma = ((0, -1, -1), (-1, 0, 1), (-1, 0, -1))$, and compute its projection onto $\Gamma$, $\sigma^{ab} = (e^a \cdot i_\alpha)(e^b \cdot i_\beta)\Sigma^{\alpha\beta}$. Again, we find an approximation power of $r = 3$ for a sphere and $r = 4$ for a torus (no irregular points), see Fig. 4. To plot the tensor field, we first decompose its antisymmetric part $\epsilon_{ab} \sigma_{ab}$, a scalar on $\Gamma$, and its symmetric part $(\sigma_{ab} + \sigma_{ba})/2$.

4. Numerical solution of vector- and tensor-valued PDEs on surfaces with the LMP method

4.1. Tension-driven flow

In this section, we apply our methodology to a tension-driven flow on a surface, described by the PDE

$$2\mu \nabla \cdot \nabla^S v + \nabla \gamma = \eta v.$$  

(24)
The first term in the left-hand side of the previous equation represents the divergence of a viscous stress $2\mu\nabla^S v$, where $(\nabla^S v)_{ab} = (\nabla_a v_b + \nabla_b v_a)/2$ is the symmetric gradient of $v$ and $\mu$ the shear viscosity, whereas the second term represents the gradient of a surface tension $\gamma$. The right-hand side represents friction with the surrounding medium characterized by the friction coefficient $\eta$. A similar equation, but allowing for shape deformations, can be shown to describe actin flows on the cell cortex in the limit of very fast turnover [10, 11, 13]. Here, we restrict ourselves to non-evolving surfaces for simplicity, but combining the LMP method with the procedure described in [10], where we used the HD method instead, is straightforward. The unknown in this problem is the field $v$. Here, we discretize $v$ in terms of its covariant components using the LMP representation

$$v_a(x) = \sum_{I \in \langle E \rangle} \sum_{A} v^I A N_{[E,I]} T_{[E,I]} A a \circ \psi_{[E]}^{-1}(x).$$

To solve this problem numerically, we consider a Galerkin method and multiply Eq. (24) by the “tensorial” test functions

$$u^I A a(x) = N_{[E,I]} T_{[E,I]} a \circ \psi_{[E]}^{-1}(x), \quad u^I b(x) = N_{[E,I]} T_{[E,I]} b \circ \psi_{[E]}^{-1}(x).$$

After integration by parts, we obtain the linear system of equations

$$Kv = b,$$

where

$$K_{2I+a,2J+b} = \sum_{E} \int_{\Gamma_{[E]}} \left[ 2\mu\nabla_b(N_{[E,I]} T_{[E,I]} A a)^g_{ac} g^{bd}\nabla_a(N_{[E,J]} T_{[E,J]} B b) + \frac{\eta}{2} N_{[E,I]} N_{[E,J]} T_{[E,I]} A a g_{ab} T_{[E,J]} B b \right] dS,$$

and

$$b_{2I+a} = \sum_{E} \int_{\Gamma_{[E]}} \gamma\nabla_b(N_{[E,I]} T_{[E,I]} A a)^g_{ab} dS.$$

To simplify notation, we have defined the symbol

$$\nabla_b(N_{[E,I]} T_{[E,I]} A a) = \partial_b N_{[E,I]} T_{[E,I]} A a + N_{[E,I]} \partial_b T_{[E,I]} A a - \Gamma^e_{ba} N_{[E,I]} T_{[E,I]} A e,$$

where $\Gamma^e_{ba} = \partial_b \partial_a \psi_{[E]} \cdot \partial_e \psi_{[E]}$ are the Christoffel symbols on $\Gamma$ in the basis $B_{\psi_{[E]}}$. The derivative $\partial_b T_{[E,I]} A a$ can be computed as

$$\partial_b T_{[E,I]} A a = i_{[I],A} \cdot \partial_b \partial_a \psi_{[E]}.$$
Figure 6: Dynamics of a nematic system with $S_0 = 1$ on a sphere. Starting with a random configuration both for $S$ and $p$ (A), we let the system evolve towards the configuration of minimal energy, formed by four +1/2 defects on the vertices of a tetrahedron (B). We observe that the free energy decreases along the dynamics, as expected from Onsager’s formalism and the variational time-integrator.

On a sphere, for $\gamma = z$, the analytical solution to this problem is $\mathbf{v}(x) = \left(x\sqrt{1 - \rho^2}, y\sqrt{1 - \rho^2}, -\rho^2\right)$, where $\rho = \sqrt{x^2 + y^2}$. In Fig. 5 we plot the convergence rate to the analytical solution of the approximated vector with the LMP and the HD methods respectively, finding equivalent results to those in the previous section.

4.2. Nematic gel on a surface

Finally, we describe the application of the LMP method to the discretization of a nematic tensor on a surface. A nematic tensor $Q$ is a traceless symmetric tensor, often represented by

$$Q_{ab} = S \left(p_ap_b - \frac{1}{2} \delta_{ab}\right),$$

(32)

where $S = \sqrt{2Q : Q} = \sqrt{2Q_{abcd}g^{ac}g^{bd}Q_{cd}}$ is the so-called nematic order parameter and $p$ identifies the unit eigenvector of eigenvalue $S$ of $Q$ [4]. Physically, $Q$ is used to characterize an ensemble of rods, e.g. filaments in the actin cortex [59] or cells in epithelial tissues [60, 61], with average alignment given by $p$ and with ordering measured by $S$. For $S = 0$ the rods are randomly distributed whereas they are completely aligned in the direction of $p$ for $S = 1$. Note that for $S = 0$, $p$ is ill-defined. A simplified evolution law for $Q$ favoring a nematic phase is given by

$$\partial_t Q = \frac{1}{\mu} \left(\frac{S_0}{S} - 1\right) Q + \frac{L}{\mu} \Delta Q,$$

(33)

where $\chi$ is the inverse susceptibility (positive), $S_0$ is a preferred nematic order, $L$ is an elastic Frank constant, $\mu$ measures the rotational viscosity and $\Delta = \nabla \cdot \nabla$ [21, 4]. This evolution law can be shown to derive from a variational principle, which we state next. First, we define the free energy of the system as

$$F[Q] = \frac{1}{2} \int_T \left\{\chi \left(\frac{S}{S_0} - 1\right)^2 + L||\nabla Q||^2\right\} dS,$$

(34)

where $||\nabla Q|| = \sqrt{\nabla Q : \nabla Q}$ and $\nabla Q : \nabla R = \nabla_a Q_{ab}g^{ad}g^{be}g^{cf} \nabla_d R_{ce}$. The first term in Eq. (34) penalizes deviations of $S$ from $S_0$ whereas the second term penalizes gradients in $Q$. We also define a dissipation potential, depending on the rate of change of $Q$, as

$$D[\partial_t Q] = \frac{\mu}{2} \int_T ||\partial_t Q||^2 dS.$$

(35)

Then, by defining the Rayleighian

$$R[Q, \partial_t Q] = \partial_t F[Q, \partial_t Q] + D[\partial_t Q],$$

(36)
and one can show that the Euler-Lagrange equations associated to minimizing this functional,

$$\partial_t Q = \arg \min_R R[Q, R],$$  

are precisely Eq. (33). To solve this problem numerically, we resort to the variational time-integrator developed in [10], in which we minimize the discrete Rayleighian

$$\mathcal{R}^n[Q^n, Q^{n+1}] = \frac{\mathcal{F}[Q^{n+1}] - \mathcal{F}[Q^n]}{\Delta t^n} + \mathcal{D}\left[\frac{Q^{n+1} - Q^n}{\Delta t^n}\right],$$  

where $\Delta t^n$ is the (possibly non-uniform) time-step. As shown in [10], this variational time-integrator is unconditionally stable since the free energy is a Lyapunov function of the discrete dynamics, as in the continuous case. In space, we discretize $Q$ with a variation of the LMP method, which imposes by construction that $Q$ is symmetric and traceless. We can express this equation as

$$Q_I = 2 \sum_{E \in \langle \Gamma \rangle} \left( \sum_{a \in \{1, 2\}} \frac{Q_{E,I}^{ab}(x)}{g^{ab}(x)} \right) q_{E,I}^{ab}(x) + \frac{1}{2},$$

where obviously the components of $g$ are those in $B_{\Phi[I]}$ and $Q_{I,C}, C = 1, 2$, are the nodal coefficients. It is straightforward to see that this tensor is symmetric and traceless. We can express this equation as

$$Q_{I,AB}(x) = L_{I,AB}^C(x) Q_{I,C},$$  

where $L_{I,AB}^C$ is the third order symbol defined in matrix notation by

$$\left[ L_{I,AB}^C(x) \right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -g^{11}(x) \\ g^{11}(x) & -g^{12}(x) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -g^{22}(x) & g^{12}(x) \end{pmatrix}.$$

Transforming the components of $Q_{I}(x)$ from the basis $B_{\Phi[I]}$ to the natural finite element basis $B_{\psi[E]}$ as previously described and then using the general LMP representation of tensors, we obtain

$$Q_{ab}(x) = \sum_{I \in \langle E \rangle} Q_{I,C} L_{I,AB}^C(x) \left( N_{E,I}^T A_a T_{E,I}^B b \right) \circ \psi^{-1}_E(x).$$

To compute the covariant derivative of $Q$ in the free energy, we apply the definition

$$\nabla_c Q_{ab} = \partial_c Q_{ab} - \Gamma^d_{ca} Q_{db} - \Gamma^d_{cb} Q_{da},$$

and

$$\partial_c Q_{ab} = \sum_{I \in \langle E \rangle} \left\{ \partial_c N_{E,I}^T A_a T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T \partial_c T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T A_a T_{E,I}^B b \partial_c L_{I,AB}^C Q_{I,C} \right\},$$

where

$$Q_{I,AB}(x) = L_{I,AB}^C(x) Q_{I,C},$$  

and

$$\nabla_c Q_{ab} = \sum_{I \in \langle E \rangle} \left\{ \partial_c N_{E,I}^T A_a T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T \partial_c T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T A_a T_{E,I}^B b \partial_c L_{I,AB}^C Q_{I,C} \right\},$$

where

$$Q_{I,AB}(x) = L_{I,AB}^C(x) Q_{I,C},$$  

and

$$\nabla_c Q_{ab} = \sum_{I \in \langle E \rangle} \left\{ \partial_c N_{E,I}^T A_a T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T \partial_c T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T A_a T_{E,I}^B b \partial_c L_{I,AB}^C Q_{I,C} \right\},$$

where

$$Q_{I,AB}(x) = L_{I,AB}^C(x) Q_{I,C},$$  

and

$$\nabla_c Q_{ab} = \sum_{I \in \langle E \rangle} \left\{ \partial_c N_{E,I}^T A_a T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T \partial_c T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T A_a T_{E,I}^B b \partial_c L_{I,AB}^C Q_{I,C} \right\},$$

where

$$Q_{I,AB}(x) = L_{I,AB}^C(x) Q_{I,C},$$  

and

$$\nabla_c Q_{ab} = \sum_{I \in \langle E \rangle} \left\{ \partial_c N_{E,I}^T A_a T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T \partial_c T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T A_a T_{E,I}^B b \partial_c L_{I,AB}^C Q_{I,C} \right\},$$

where

$$Q_{I,AB}(x) = L_{I,AB}^C(x) Q_{I,C},$$  

and

$$\nabla_c Q_{ab} = \sum_{I \in \langle E \rangle} \left\{ \partial_c N_{E,I}^T A_a T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T \partial_c T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T A_a T_{E,I}^B b \partial_c L_{I,AB}^C Q_{I,C} \right\},$$

where

$$Q_{I,AB}(x) = L_{I,AB}^C(x) Q_{I,C},$$  

and

$$\nabla_c Q_{ab} = \sum_{I \in \langle E \rangle} \left\{ \partial_c N_{E,I}^T A_a T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T \partial_c T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T A_a T_{E,I}^B b \partial_c L_{I,AB}^C Q_{I,C} \right\},$$

where

$$Q_{I,AB}(x) = L_{I,AB}^C(x) Q_{I,C},$$  

and

$$\nabla_c Q_{ab} = \sum_{I \in \langle E \rangle} \left\{ \partial_c N_{E,I}^T A_a T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T \partial_c T_{E,I}^B b L_{I,AB}^C Q_{I,C} + N_{E,I}^T A_a T_{E,I}^B b \partial_c L_{I,AB}^C Q_{I,C} \right\},$$

where
where
\[ \partial_c L_{[IJ]11} = \frac{\partial_c g^{22} g^{11} - \partial_c g^{11} g^{22}}{(g^{22})^2}, \quad \partial_c L_{[IJ]12} = \frac{2 \partial_c g^{22} g^{12} - \partial_c g^{12} g^{22}}{(g^{22})^2}, \quad \partial_c L_{[IJ]1C} = 0 \text{ otherwise.} \] (45)

To calculate \( \partial_c g^{AB} \) we note that
\[ g^{AB} = T_{[E,I]}^A \cdot T_{[E,I]}^B g^{ab}, \] (46)
and thus
\[ \partial_c g^{AB} = \partial_c T_{[E,I]}^A T_{[E,I]}^B g^{ab} + T_{[E,I]}^A \partial_c T_{[E,I]}^B g^{ab} + T_{[E,I]}^A T_{[E,I]}^B \partial_c g^{ab}. \] (47)

Finally, we have that
\[ \partial_c g^{ab} = -g^{ad} g^{bc} \partial_c g_{dc}, \] (48)
and
\[ \partial_c \psi_{[E]} = \partial_a \psi_{[E]} \cdot \partial_b \psi_{[E]} + \partial_a \psi_{[E]} \cdot \partial_c \partial_b \psi_{[E]} \cdot \partial_c \psi_{[E]}. \] (49)

To solve the non-linear problem resulting from this discretization and Eq. (38), we use a Newton-Raphson method.

Assuming \( S_0 = 1 \), we now show the dynamics of the model simulated with the LMP method. On a sphere, starting with a random configuration for \( p \) and \( S \), we let the system relax its free energy. On the one hand, the system tries to reach a state where \( S = S_0 = 1 \) everywhere to minimize the first part of the free energy. On the other hand, it also tries to reach a state where \( \nabla Q \) is small to minimize the second part. On a sphere, however, due to the hairy ball theorem, it is impossible to get a homogeneous \( Q \) with \( S = 1 \) that leads to a vanishing free energy. Instead, the system relaxes towards a state in which four defects (of +1/2 charge and with a length-scale given by \( \sqrt{\chi/L} \)) are created forming a tetrahedron, as expected from classical theories [62] and recent experiments [63].

5. Discussion

We have presented a novel method, the Local Monge Parametrizations (LMP) method, for the discretization of differentiable tensor fields on surfaces of arbitrary topology based only on tangential calculus. With respect to previous methods, the LMP method is general with regards to the topology of the surface and does not increase the order of vector- and tensor-valued PDEs. We have shown the applicability of the method to approximate vector and second-order tensor fields on surfaces of different topology in combination with subdivision surface finite elements. We have also used the LMP method to solve two different vector- and tensor-valued PDEs, representing a surface-driven flow and a nematic system, on surfaces. This method can be used to discretize the models in [8, 9, 10, 12] involving vector- and tensor-valued PDEs coupled to surface evolution laws. Our method could also find application in the discretization of smooth tensor fields in computer graphics as an alternative to existing methods [48].

6. Acknowledgments

We acknowledge the support of the European Research Council (CoG-681434), the European Commission (project H2020-FETPROACT-01-2016-731957), the Spanish Ministry of Economy and Competitiveness/FEDER (DPI2015-71789-R to MA), and the Generalitat de Catalunya (SGR-1471, ICREA Academia award to MA). We also thank Sohan Kale and Guillermo Vilanova for useful discussions.
Appendix A. Change of basis

To compute the matrix for the change of basis, we note that it should satisfy

\[ T_{[E,I]} A_a \partial_A \phi_{[I]} = \partial_a \psi_{[E]} \].

We can use the definition of \( \phi_{[I]} \) Eq. (9) to compute

\[ \partial_A \phi_{[I]} = i_{[I,A]} + \partial_A h_{[I]} n_{[I]} \].

It is easy to note that the dual basis of \( \{ \partial_A \phi_{[I]} \} \) is simply given by \( \{ i_{[I,A]} \} \), since

\[ i_{[I,B]} \cdot \partial_A \phi_{[I]} = \delta_{AB} \].

Thus, multiplying Eq. (A.1) by \( i_{[I,B]} \), we get

\[ T_{[E,I]} A_a i_{[I,B]} \cdot \partial_A \phi_{[I]} = T_{[E,I]} A_a \delta_{AB} = i_{[I,B]} \cdot \partial_a \psi_{[E]} \],

which leads to Eq. (11).

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