DIOPHANTINE APPROXIMATION
BY ORBITS OF EXPANDING MARKOV MAPS

LINGMIN LIAO AND STÉPHANE SEURET

Abstract. In 1995, Hill and Velani introduced the “shrinking targets” theory. Given a dynamical system \([0, 1], T\), they investigated the Hausdorff dimension of sets of points whose orbits are close to some fixed point. In this paper, we study the sets of points well-approximated by orbits \((T^n x)_{n \geq 0}\), where \(T\) is an expanding Markov map with a finite partition supported by \([0, 1]\). The dimensions of these sets are described using the multifractal properties of invariant Gibbs measures.

1. Introduction

Let \((X, d)\) be a compact metric space and \(T : X \to X\) a piecewise continuous transformation. Let \(\mathcal{O}(x) = \{T^n x : n \in \mathbb{N}\}\) be the orbit of \(x \in X\). The distribution of \(\mathcal{O}(x)\), in particular its density over \(X\), is a historical issue, which goes back to Poincaré’s results. In 1995, Hill and Velani [20] introduced the shrinking targets theory, which aims at investigating the Hausdorff dimensions of sets of points whose orbits are close to some fixed point. For a point \(y \in X\), they studied the set

\[ \{ \forall n \in \mathbb{N} \text{ such that } T^n x \in B(y, r_n) \} \]

where \(B(x, r)\) stands for the ball of radius \(r > 0\) centered at \(x \in X\) and \((r_n)_{n \geq 1}\) is a sequence of positive real numbers converging to 0. In this article, we adopt a complementary point of view: we fix a point \(x \in X\) and consider the set of points \(y\) well-approximated by the orbit \(\mathcal{O}(x)\) of \(x\), i.e. we focus on

\[ \{ y \in X : \forall n \in \mathbb{N} \text{ such that } T^n x \in B(y, r_n) \} \]

which can be written as \(\limsup_{n \to \infty} B(T^n x, r_n) = \bigcap_{N \geq 1} \bigcup_{n \geq N} B(T^n x, r_n)\).

In fact, many questions can be asked about the set \(\mathcal{O}(x)\): for which sequence \((r_n)_{n \geq 1}\) does it cover the whole interval \([0, 1]\)? When \([0, 1]\) is not fully covered, what is its Hausdorff dimension? Can the dependence on \(x\) be quantified? Answering these questions provides us with a precise description of the distribution properties of the orbit \(\mathcal{O}(x)\). Such questions have been investigated in several contexts, and can be interpreted as general Diophantine approximation problems. Indeed, the classical Diophantine questions concern the dimension of the set

\[ \mathcal{S}(\delta) = \left\{ y \in [0, 1] : \frac{|y - p/q|}{q^{2\delta}} \leq \frac{1}{q^2} \text{ for infinitely many couples } (p, q) = 1 \right\} \].

arXiv:1111.1081v1 [math.DS] 4 Nov 2011
This set can again be seen as a limsup set \( \limsup_{q \to +\infty} \bigcup_{p \in \mathbb{Z}} B(p/q, 1/q^{2\delta}) \).

The work [20] is precursor on this subject in the dynamical setting, and thereafter, many people studied sets of the form \( (1) \). For instance, see [23] for the case where \( T \) is an irrational rotation on the torus \( T^1 \). In the literature, one often refers to these results as shrinking targets problems or dynamical Borel-Cantelli lemma. The paper [18] by Fan, Schmeling and Troubetzkoy, where the doubling map on \( T^1 \) is studied, is the first one to consider the set \( (2) \). These studies are also related to many other famous works concerned with metric theory of Diophantine approximation, see [13, 15, 21, 24, 25, 6, 16] and references therein.

In this work, we focus on the study of the set \( (2) \) when \( T \) is an expanding Markov map of the interval \([0, 1]\) with a finite partition - Markov map, for short.

**Definition 1.1** (Markov map). A transformation \( T : [0, 1] \to [0, 1] \) is an expanding Markov map with a finite partition if there is a subdivision \( \{a_i\}_{0 \leq i \leq Q} \) of \([0, 1]\) (denoted by \( I(k) = ]a_{k}, a_{k+1}[ \) for \( 0 \leq k \leq Q - 1 \)) such that:

1. there is an integer \( n \) and a real number \( \delta \) such that \( |(T^n)'| \geq \delta > 1 \);
2. \( T \) is strictly monotonic and can be extended to a \( C^2 \) function on each \( T(I(k)) \);
3. if \( I(j) \cap T(I(k)) \neq \emptyset \), then \( I(j) \subset T(I(k)) \);
4. there is an integer \( R \) such that \( I(j) \subset \bigcup_{n=1}^{R} T^n(I(k)) \) for every \( k, j \);
5. for every \( k \in \{0, \cdots, Q - 1\} \), \( \sup_{(x,y,z) \in I(k)^3} \frac{|T'(x)|}{|T'(y)||T'(z)|} < \infty \).

It appears that for Markov maps, the relevant choice for the sequence \( (r_n)_{n \geq 1} \) is \( r_n = 1/n^{\delta} \), for \( \delta > 0 \). We thus introduce the sets

\[
\mathcal{L}^\delta(x) := \limsup_{n \to \infty} B(T^n(x), n^{-\delta}),
\]

\[
\mathcal{F}^\delta(x) := [0, 1] \setminus \mathcal{L}^\delta(x),
\]

which are the set of points covered by infinitely many balls \( B(T^n(x), n^{-\delta}) \), and its complement. We study the Hausdorff dimension of \( \mathcal{L}^\delta(x) \) and \( \mathcal{F}^\delta(x) \).

Before stating our main theorem, we give some recalls on multifractal analysis. Denote by \( \text{Leb} \) the Lebesgue measure in \( \mathbb{R} \), and by \( \dim \) the Hausdorff dimension.

**Definition 1.2.** For any Borel probability measure \( \mu \) on \([0, 1]\), define the lower (resp. upper) local dimension \( d_{\mu}(y) \) (resp. \( d_{\mu}(y) \)) of \( \mu \) at \( y \in [0, 1] \) by

\[
d_{\mu}(y) := \liminf_{r \to 0} \frac{\log \mu(B(y,r))}{\log r} \quad \text{and} \quad \overline{d}_{\mu}(y) := \limsup_{r \to 0} \frac{\log \mu(B(y,r))}{\log r}.
\]

When \( d_{\mu}(y) = \overline{d}_{\mu}(y) \), their common value is denoted by \( d_{\mu}(y) \), and is simply called the local dimension of \( \mu \) at \( y \). The level sets of the local dimension are

\[
E_{\mu}(\alpha) = \{y \in [0, 1] : d_{\mu}(y) = \alpha\}, \quad \alpha \geq 0.
\]
Finally, the multifractal spectrum of $\mu$, is defined as the application
\[ D_\nu : \alpha \geq 0 \mapsto \dim E_\nu(\alpha). \]

Denote by $\mathcal{M}_{\text{inv}}$ (resp. $\mathcal{M}_{\text{erg}}$) the set of $T$-invariant (resp. ergodic) probability measures on $[0, 1]$. The dimension of a Borel probability measure $\mu$ is defined as
\[ \dim \mu = \inf \{ \dim E : E \text{ Borel set } \subseteq [0, 1] \text{ and } \mu(E) > 0 \}. \]

We prove the following theorem (for precise definitions, see Section 2).

**Theorem 1.3.** Let $T : [0, 1] \to [0, 1]$ be an expanding Markov map. Let $\phi$ be a Hölder continuous potential and let $\mu_\phi$ be the corresponding Gibbs measure. Let
\[ \alpha_+ := \max_{\mu \in \mathcal{M}_{\text{inv}}} \frac{\int_{[0, 1]} (-\phi) \, d\mu}{\int_{[0, 1]} \log |T'| \, d\mu} \quad \text{and} \quad \alpha_{\max} := \frac{\int_{[0, 1]} (-\phi) \, d\mu_{\max}}{\int_{[0, 1]} \log |T'| \, d\mu_{\max}}, \]
where $\mu_{\max}$ is the Gibbs measure associated with the potential $\psi = -\log |T'|$.

1. For $\mu_\phi$-almost every $x \in [0, 1]$, the Hausdorff dimension of $\mathcal{L}^\delta(x)$ is
\[ \dim \mathcal{L}^\delta(x) = \begin{cases} 1/\delta & \text{if } 0 < 1/\delta \leq \dim \mu_\phi, \\ D_{\mu_\phi}(1/\delta) & \text{if } \dim \mu_\phi < 1/\delta \leq \alpha_{\max}, \\ 1 & \text{if } 1/\delta > \alpha_{\max}. \end{cases} \]

2. For $\mu_\phi$-almost every $x \in [0, 1]$, the Hausdorff dimension of $\mathcal{F}^\delta(x)$ is
\[ \dim \mathcal{F}^\delta(x) = \begin{cases} 1 & \text{if } 0 < 1/\delta \leq \alpha_{\max}, \\ D_{\mu_\phi}(1/\delta) & \text{if } 1/\delta > \alpha_{\max}. \end{cases} \]

3. Concerning the Lebesgue measure of $\mathcal{L}^\delta(x)$ and $\mathcal{F}^\delta(x)$, we have:
\[ \text{Leb}(\mathcal{L}^\delta(x)) = 1 - \text{Leb}(\mathcal{F}^\delta(x)) = \begin{cases} 0 & \text{if } 0 < 1/\delta < \alpha_{\max}, \\ 1 & \text{if } 1/\delta > \alpha_{\max}. \end{cases} \]

4. If $1/\delta > \alpha_+$, then $\mathcal{F}^\delta(x) = \emptyset$ and hence $\mathcal{L}^\delta(x) = [0, 1]$.

**Remark 1.4.** The dimensions of $\mathcal{L}^{1/\alpha_{\max}}(x)$ and $\mathcal{F}^{1/\alpha_{\max}}(x)$ are 1, but we do not know their Lebesgue measure. We also prove that $\text{Leb}(\mathcal{L}^{1/\alpha_+}(x)) = 1$ and $\dim \mathcal{F}^{1/\alpha_+}(x) \leq \lim_{1/\delta \to \alpha_+} D_{\mu_\phi}(1/\delta)$, but we do not know if $\mathcal{L}^{1/\alpha_+}(x) = [0, 1]$.

The mapping $1/\delta \mapsto \dim \mathcal{L}^\delta(x)$ exhibits clearly four distinct behaviors, respectively denoted by Part I ($1/\delta \leq \dim \mu_\phi$), Part II ($\dim \mu_\phi < 1/\delta \leq \alpha_{\max}$), Part III ($\alpha_{\max} < 1/\delta \leq \alpha_+$) and finally Part IV ($1/\delta > \alpha_+$). See Figure 1 (the definition of $\alpha_-$ is given in Theorem 2.5).

For $\mu_\phi$-typical $x$, the behavior of $\mathcal{L}^\delta(x)$ enjoys two remarkable characteristics compared to classical Diophantine approximation: the map $1/\delta \mapsto \dim \mathcal{L}^\delta(x)$ may have a strictly concave part (Part II), and the smallest $\delta$ for which $\text{Leb}(\mathcal{L}^\delta(x)) = 1$ and the smallest $\delta$ for which $\mathcal{L}^\delta(x) = [0, 1]$ do not coincide (Part III). This contrasts with the classical Diophantine approximation, especially the approximation by rationals. In this historical context, as said above, the analog of the sets $\mathcal{L}^\delta(x)$ are the sets $S(\delta)$ defined by (3). The Dirichlet theorem ensures that $S(1) = [0, 1]$, and it is well-known [22, 1].
that for every \( \delta > 1 \), \( \dim S(\delta) = 1/\delta \). In particular, the dimension of \( S(\delta) \) decreases linearly with respect to \( 1/\delta \), and as soon as the Lebesgue measure of \( S(\delta) \) equals 1, it instantaneously covers the whole interval \([0, 1]\). Comparable results hold for sets of numbers approximated by other families, see for instance [3, 4, 10, 11, 33]. The two characteristics of \( L_\delta(x) \) mentioned above can be interpreted by the fact that, although the orbits \( O(x) \) of \( \mu_\phi \)-typical points are dense, they are not as regularly distributed when \( n \) tends to infinity as the rational numbers are. The exponents \( \dim \mu_\phi \), \( \alpha_{\text{max}} \) and \( \alpha_+ \) characterize this “distortion”.

The paper is organized as follows. Section 2 contains some recalls on multifractal analysis and hitting time. Section 3 describes the relations between hitting time, approximation rate and local dimension of \( \mu_\phi \). From these relations we will give a direct proof of item 3. of Theorem 1.3. In Section 4, two key lemmas are proved. They illustrate the fact that intervals which have a small local dimension for \( \mu_\phi \) are hit by the balls \( B(T^n x, 1/n^\delta) \) with big probability, and vice-versa. Then, Sections 5, 6, 7 and 8 contains the proofs of the upper and lower bounds for the Hausdorff dimensions of \( L_\delta(x) \) and \( F_\delta(x) \) for Parts I, IV, III and II, respectively.

Theorem 1.3 is similar to the results in [18] for the doubling map on \( T^1 \), but the proofs require other arguments. First, for \( x \mapsto 2x \), since the Lyapunov exponents are constant, the balls of generation \( n \) have same lengths, while for the Markov maps their lengths may have different order. Second, in [18], the authors focus on the Bowen’s entropy spectrum. Third, the notions of Hölder exponent and hitting time for the doubling map involve only cylinders, while we need centered balls to define similar quantities. Finally,
some arguments (Lemmas 4.2, 4.3 and 4.4) are adapted from [18] to the context of Markov maps, but several others do not. The best example is the difficult lower bound for $\dim L^\delta(x)$ when $0 < 1/\delta \leq \dim \mu_\phi$.

2. First definitions, and recalls on multifractal analysis

2.1. Covering of $[0,1]$ by basic intervals. Let $T$ be a Markov map defined in Definition 1.1. With $T$ are associated generations of basic intervals.

Definition 2.1. Let $\mathcal{A} = \{0,1,\cdots,Q-1\}$. For every integer $n \geq 1$, we denote by $\mathcal{G}_n$ the set of basic intervals of generation $n$ defined by

$$\text{if } (i_1i_2\cdots i_n) \in \mathcal{A}^n, \quad I_{i_1i_2\cdots i_n} := I(i_1) \cap T^{-1}(I(i_2)) \cap \cdots \cap T^{-n+1}(I(i_n)).$$

The following distortion property between intervals will be crucial: there exists a number $L > 1$ such that for every integer $n \geq 2$, for every $(i_1i_2\cdots i_n) \in \mathcal{A}^n$,

$$1 \leq \frac{|I_{i_1i_2\cdots i_{n-1}}|}{|I_{i_1i_2\cdots i_{n-1}i_n}|} \leq L$$

($|I|$ is the length of the interval $I$).

It is obvious that the intervals of a given generation $n$ form a covering of $[0,1]$. This covering of $[0,1]$ is not composed of intervals of the same length. But using (9), for every real number $0 < r < 1$, one easily shows that there is a family of basic intervals $J_1, J_2, \cdots, J_N$ (not belonging to the same $\mathcal{G}_n$’s) such that:

- $\bigcup_{j=1}^N J_j = [0,1]$
- for every $j \neq j'$, the intersection of interiors $\overset{\circ}{J}_j \cap \overset{\circ}{J}_{j'}$ is empty,
- for the same constant $L$ as in (9), for every $j \in \{1,2,\cdots,N\}$,

$$L^{-1}r \leq |J_j| \leq Lr.$$

For every $n \in \mathbb{N}$, we fix $\mathcal{C}_n$ one possible collection of intervals such that (10) holds for $r = 2^{-n}$. From these considerations, one deduces that there exists a number $L' > 1$ such that for all $n \in \mathbb{N}$, the generation $n_J$ of a basic interval $J \in \mathcal{C}_n$ satisfies

$$L'^{-1}n \leq n_J \leq L'n.$$

2.2. Definition of hitting time. Denote $\mathcal{O}^+(x) := \mathcal{O}(x) \setminus \{x\}$.

Definition 2.2. For every $(x,y) \in [0,1]^2$ and $r > 0$, we define the hitting time (first entrance time) of the orbit of $x$ into the ball $B(y,r)$ by

$$\tau_r(x,y) := \inf\{n \geq 1 : T^n x \in B(y,r)\}.$$

Then we set

$$R(x,y) := \liminf_{r \to 0} \frac{\log \tau_r(x,y)}{-\log r}.$$

By convention, when $\mathcal{O}^+(x) \cap B(y,r) = \emptyset$, we set $\tau_r(x,y) = +\infty$ and $R(x,y) = +\infty$.

We define the hitting time $\tau(x,C)$ of a basic interval $C$ by a point $x$. Let $m \geq 1$ and let $C \in \mathcal{G}_m$. If $\mathcal{O}(x) \cap C = \emptyset$, we set $\tau(x,C) = +\infty$. Otherwise,

$$\tau(x,C) := \inf\{n \geq 0 : T^n x \in C\}.$$
Definition 2.3. Let $s \geq 0$ be a real number. We define the sets
\[ R_{\geq s}(x) = \{ y \in [0,1] : R(x,y) \geq s \}, \quad R_{< s}(x) = \{ y \in [0,1] : R(x,y) < s \}, \]
where $R_{\geq s}(x)$ and $R_{< s}(x)$ are the sets of points $y$ in $[0,1]$ such that $R(x,y) \geq s$ and $R(x,y) < s$, respectively.

2.3. Multifractality of Gibbs measures. Here are some facts on Gibbs measures.

Theorem 2.4 ([8, 25]). Let $T : I \to I$ be a Markov map. Then for any Hölder continuous function $\phi : I \to \mathbb{R}$, there exists a unique equilibrium state $\mu_\phi$ which satisfies the following Gibbs property: there exist constants $\gamma > 0$ and $P(\phi)$ (the topological pressure associated with $\phi$), such that
\[ (15) \quad \mu(\phi(I)) \leq e^{\gamma \phi(x)} \mu(I), \quad \forall x \in I, \]
where $\mu(I)$ is the Lebesgue measure $\text{Leb} I$.

A potential $\phi$ is often assumed to be normalized, i.e. $P(\phi) = 0$. If it is not the case, we can replace $\phi$ by $\phi - P(\phi)$.

Now, let us recall some standard facts on multifractal analysis of Borel measures. By an extensive literature [14, 31, 9, 34, 5, 28], the multifractal spectrum $D_{\mu_\phi}$ of $\mu_\phi$ can be computed.

Theorem 2.5. Consider a Markov map $T : [0,1] \to [0,1]$ and a normalized Hölder continuous potential $\phi : [0,1] \to \mathbb{R}$.

1. The multifractal spectrum $D_{\mu_\phi}$ of $\mu_\phi$ is a concave analytic map on the interval $[\alpha_-, \alpha_+]$, where $\alpha_- := \min_{\mu \in \mathcal{M}_{\text{inv}}} \frac{\int_{[0,1]} (-\phi) \, d\mu}{\int_{[0,1]} \log |T| \, d\mu}$ and $\alpha_+$ was defined in [5].

2. The spectrum $D_{\mu_\phi}$ reaches its maximum value 1 at $\alpha_{\text{max}}$ defined in [8].

3. The graph of $D_{\mu_\phi}$ and the first bisector intersect at a unique point which is $(\dim \mu_\phi, \dim \mu_\phi)$. Moreover, $\dim \mu_\phi$ satisfies
\[ \dim \mu_\phi = \frac{\int_{[0,1]} (-\phi) \, d\mu_\phi}{\int_{[0,1]} \log |T| \, d\mu_\phi}. \]

For every $q \in \mathbb{R}$, there is a unique real number $\eta_\phi(q)$ such that the topological pressure $P(-\eta_\phi(q) \log |T| + q\phi)$ associated with the Hölder potential $\phi_q := -\eta_\phi(q) \log |T| + q\phi$ equals 0. Such a number exists since the map $P : t \mapsto (-t \log |T| + q\phi)$ is real-analytic and decreasing in $t$. The resulting function $q \mapsto \eta_\phi(q)$ is real-analytic and concave. We denote by
\[ (16) \quad \mu_q := \mu_{\phi_q}, \quad \text{where } \phi_q := -\eta_\phi(q) \log |T| + q\phi, \]
the Gibbs measure associated with the potential $\phi_q$. Observe that $\eta_\phi(0) = 1$, and $\eta_\phi(1) = 0$. The measures $\mu_0$ and $\mu_1 (= \mu_\phi)$ are associated with the potentials $\phi_0 = -\log |T|$ and $\phi_1 = \phi$ respectively. By a folklore theorem, the Lebesgue measure $\text{Leb}$ is equivalent to $\mu_0$, coinciding with $\mu_{\text{max}}$ used in Theorem 1.3.
For every \( q \in \mathbb{R} \), we introduce the exponent
\[
(17) \quad \alpha(q) = \frac{\int_{[0,1]} (-\phi) \, d\mu_q}{\int_{[0,1]} \log |T'| \, d\mu_q}.
\]

By the Gibbs property of \( \mu_\phi \) and the ergodicity of \( \mu_q \), the measure \( \mu_q \) is supported by the level set \( \mathcal{E}_{\mu_\phi}(\alpha(q)) \) (equation (17)). Hence \( D_{\mu_\phi}(\alpha(q)) = \dim \mu_q = \eta_\phi(q) + q\alpha(q) \). The map \( q \mapsto \alpha(q) \) is decreasing, and
\[
\lim_{q \to +\infty} \alpha(q) = \alpha_-, \quad \lim_{q \to -\infty} \alpha(q) = \alpha_+,
\]
\[
(18) \quad \alpha(1) = \dim \mu_\phi, \quad \alpha(0) = \alpha_{\max}.
\]

For \( \alpha \in [\alpha_-, \alpha_+] \), we write the inverse function of \( q \mapsto \alpha(q) \) as \( \alpha \mapsto q(\alpha) \).

**Remark 2.6.** By a standard result, we have
\[
\sup_{y: \, d_{\mu_\phi}(y) \text{ exists}} d_{\mu_\phi}(y) = \sup_{v \in \mathcal{M}_{\mu_\phi}} \frac{\int_{[0,1]} \phi \, d\nu}{\int_{[0,1]} \log |T'| \, d\nu} = \alpha_+.
\]

**Definition 2.7.** Let \( s \in \mathbb{R}^+ \). We define the sets
\[
(15)_s = \{ y \in [0,1] : d_{\mu_\phi}(y) \geq s \} \quad \text{and} \quad \mathcal{E}_{\leq s} = \{ y \in [0,1] : d_{\mu_\phi}(y) \leq s \},
\]
\[
(16)_s = \{ y \in [0,1] : d_{\mu_\phi}(y) > s \} \quad \text{and} \quad \mathcal{E}_{< s} = \{ y \in [0,1] : d_{\mu_\phi}(y) < s \}.
\]

From large deviations theory \cite{9}, we get the values for the Hausdorff dimensions of the sets \( \mathcal{E} \) in Definition 2.7. These values depend on whether \( s \) is located in the increasing or in the decreasing part of the multifractal spectrum \( D_{\mu_\phi} \).

**Proposition 2.8.** Let \( \mu_\phi \) be a Gibbs measure. Then:
\begin{enumerate}
  \item For every \( s < \alpha_{\max} \), \( \dim (\mathcal{E}_{< s}) = \dim (\mathcal{E}_{\leq s}) = D_{\mu_\phi}(s) \).
  \item For every \( s > \alpha_{\max} \), \( \dim (\mathcal{E}_{> s}) = \dim (\mathcal{E}_{\geq s}) = D_{\mu_\phi}(s) \).
\end{enumerate}

### 3. Results about hitting time

#### 3.1. Orbits and hitting time

The proof of Lemma 3.1 is left to the reader.

**Lemma 3.1.** The following three assertions are equivalent:
\begin{enumerate}
  \item There exists an integer \( n_0 \geq 1 \) such that \( y = T^{n_0}x \) \( \text{i.e.} \ \ y \in \mathcal{O}^+(x) \).
  \item The hitting time \( \tau_r(x, y) \) is bounded for all \( r > 0 \).
  \item There is a sequence \( \tau_i \to 0 \) such that \( \tau_i(x, y) \) is bounded.
\end{enumerate}

The next lemmas investigate the relationship between \( \mathcal{L}^\delta(x) \) and hitting times.

**Lemma 3.2.** For every \( \delta > 0 \), we have the two embedding properties:
\begin{enumerate}
  \item \( \mathcal{R}_{<1/\delta}(x) \setminus \mathcal{O}^+(x) \subset \mathcal{L}^\delta(x) \subset \mathcal{R}_{\leq 1/\delta}(x) \)
  \item \( \mathcal{R}_{>1/\delta}(x) \subset \mathcal{F}^\delta(x) \subset \mathcal{R}_{\geq 1/\delta}(x) \cup \mathcal{O}^+(x) \).
\end{enumerate}
Theorem 3.5. Suppose that

\[ \Theta \]

are the following

\[ \frac{1}{n_i} \]

Consider the sequence of integers \( n_i \). By construction,

\[ r_i < (n_i)^{-1/\delta}(1 - \epsilon) < n_i^{-\delta}. \]

Thus \( T^{n_i}x \in B(y, n_i^{-\delta}). \) Since \( y \notin O^+(x) \), Lemma 3.1 yields that \( (n_i)_{i \geq 1} \) is not bounded. We deduce that \( y \in B(T^{n_i}x, n_i^{-\delta}) \) for infinitely many increasing integers \( n_i \). This proves that \( y \in \mathcal{L}^\delta(x) \).

For the second inclusion of (21), consider \( y \in \mathcal{L}^\delta(x) \). By definition, \( T^{n_i}x \in B(y, n_i^{-\delta}) \) for infinitely many integers \( (n_i)_{i \geq 1} \). Hence, for these \( n_i \), we have \( \tau_{1/n_i}(x, y) \leq n_i \), which implies that

\[ R(x, y) \leq \lim inf_{i \to \infty} \frac{\log \tau_{1/n_i}(x, y)}{-\log(n_i^{\delta})} \leq \lim inf_{n_i \to \infty} \frac{\log n_i}{\delta \log n_i} = \frac{1}{\delta}. \]

This completes the proof. \( \square \)

Lemma 3.3. Suppose that \( x \) is not eventually periodic. If \( y \in O^+(x) \), then:

\[ y \in \mathcal{L}^\delta(x) \quad \text{if} \quad R(y, y) < 1/\delta \quad \text{and} \quad y \in \mathcal{F}^\delta(x) \quad \text{if} \quad R(y, y) > 1/\delta. \]

Observe that the case where \( R(y, y) = 1/\delta \) is not determined yet.

Proof. Suppose that \( y \in O^+(x) \). Since \( x \) is not eventually periodic, there exists a unique positive integer \( n_0 \) such that \( T^{n_0}x = y \) and \( y \notin O^+(y) \). The rest of the proof is the same as that of Lemma 3.2. \( \square \)

Lemma 3.4. Suppose that \( \mu \in \mathcal{M}_{inv} \) has no atom. Then for \( \mu \)-a.e. \( x \), we have

\[ O^+(x) \subset \mathcal{L}^\delta(x) \quad \text{if} \quad 1/\delta > \dim \mu \quad \text{and} \quad O^+(x) \subset \mathcal{F}^\delta(x) \quad \text{if} \quad 1/\delta < \dim \mu. \]

Proof. We remark that the set of eventually periodic points is a countable set, hence it has a \( \mu \)-measure equal to zero. By the Ornstein-Weiss theorem 27.

(23) for \( \mu \)-almost all \( x \), for every \( n \geq 1 \), \( R(T^nx, T^nx) = \dim \mu. \)

Hence, for a \( \mu \)-typical \( x \) (which is not eventually periodic), consider \( y \in O^+(x) \). By the same argument as above, \( y = T^{n_0}x \) for some unique integer \( n_0 \geq 0 \). By (23), \( R(y, y) = \dim \mu. \) Applying Lemma 3.3 we find that if \( 1/\delta > \dim \mu \) (resp. \( 1/\delta < \dim \mu \)) then \( y \in \mathcal{L}^\delta(x) \) (resp. \( y \in \mathcal{F}^\delta(x) \)). \( \square \)

3.2. Local dimension and hitting time. Gibbs measures enjoy exponential decay of correlations 52, 29, 26, 1. More precisely, we have the following theorem.

Theorem 3.5. Suppose that \( f : [0, 1] \to [0, 1] \) has bounded variation and \( g : [0, 1] \to [0, 1] \) is integrable. Then there exist constants \( 0 < \beta < 1 \) and \( \Theta > 0 \), such that for every integer \( n \geq 1 \),

\[ \left| \int f g^\circ T^n \, d\mu_\phi - \int f d\mu_\phi \int g d\mu_\phi \right| \leq \Theta \beta^n \left( \int |f| d\mu_\phi + \text{var}(f) \right) \int |g| d\mu_\phi, \]
where \( \text{var}(f) \) stands for the total variation of \( f \) on \([0,1]\). In particular, if \( f = 1_A \) and \( g = 1_B \) where \( A \) is an interval and \( B \) is a measurable set, then for every \( n \),
\[
\| \mu_\phi(A \cap T^{-n}B) - \mu_\phi(A)\mu_\phi(B) \| \leq \Theta \beta^n (\mu_\phi(A) + 2) \mu_\phi(B).
\]

Theorem 3.5 allows us to use the following theorem which describes the relationship between hitting time and local dimension of invariant measures.

**Theorem 3.6** ([19]). If \((X,T,\mu)\) has superpolynomial decay of correlations and if \(d_\mu(y)\) exists, then for \(\mu\)-almost every \(x\) we have
\[
R(x,y) = d_\mu(y).
\]

We return to the study of the Markov map \(T\) on the interval \([0,1]\).

**Corollary 3.7.** Let \(\mu_\phi,\mu_\psi\) be two \(T\)-invariant Gibbs probability measures on \([0,1]\) associated with normalized Hölder potentials \(\phi\) and \(\psi\). We have
\[
\text{for } \mu_\phi\text{-a.e. } x, \text{ for } \mu_\psi\text{-a.e. } y, \quad R(x,y) = d_{\mu_\phi}(y) = \frac{-\int_{[0,1]} \phi d\mu_\psi}{\int_{[0,1]} \log |T| d\mu_\psi}.
\]

**Proof.** For \(\mu_\psi\)-almost every \(y\), \(\frac{1}{n} S_n \phi(y)\) tends to \(\int_{[0,1]} \phi d\mu_\psi\). Hence, for \(\mu_\psi\)-almost every \(y\), using the Gibbs property of \(\mu_\phi\) and \(\mu_\psi\), the Hölder exponent \(d_{\mu_\phi}(y)\) computed on the basic intervals defined as
\[
\tilde{d}_{\mu_\phi}(y) := \lim_{n \to +\infty} \frac{\log \mu(I_n(y))}{\log |I_n(y)|}
\]
exists and is equal to \(-\frac{\int_{[0,1]} \phi d\mu_\psi}{\int_{[0,1]} \log |T| d\mu_\psi}\). By Theorem 5.1 of [2], for quasi-Bernoulli measures, and in particular for Gibbs measures associated with Markov maps, \(\tilde{d}_{\mu_\phi}(y)\) coincides with \(d_{\mu_\phi}(y)\), for \(\mu_\psi\)-a.e. \(y\). The lemma then follows by Theorem 3.6 and the Fubini theorem. \(\square\)

**Corollary 3.8.** Let \(\mu_\phi,\mu_\psi\) be two \(T\)-invariant Gibbs probability measures on \([0,1]\) associated with normalized Hölder potentials \(\phi\) and \(\psi\). Then
\[
\text{for } \mu_\phi \times \mu_\psi\text{-almost every } (x,y), \quad R(x,y) = d_{\mu_\phi}(y) = \frac{\int_{[0,1]} (-\phi) d\mu_\psi}{\int_{[0,1]} \log |T| d\mu_\psi}.
\]

### 3.3. First results on covering.

We introduce the real number
\[
\delta(\phi,\psi) := \sup \{ \delta \geq 0 : \mu_\psi(\mathcal{L}^\delta(x)) = 1 \text{ for } \mu_\phi\text{-almost every } x \}.
\]

The following proposition, which summarizes the results of the previous sections, will be useful when proving the lower bound for Part I of Theorem 1.3. We can also use it to give a direct proof for the item 3. of Theorem 1.3.

**Proposition 3.9.** For any normalized Hölder potentials \(\phi\) and \(\psi\), we have
\[
\delta(\phi,\psi) = \frac{\int_{[0,1]} \log |T| d\mu_\psi}{\int_{[0,1]} (-\phi) d\mu_\psi}.
\]

In particular, for every \(\alpha \in]0,\infty[\),
\[
\delta(\phi,\phi_{\alpha^\delta})(x) = \sup \{ \delta \geq 0 : \mu_\phi(\mathcal{L}_\delta(x)) = 1 \text{ for } \mu_\phi\text{-a.e. } x \} = 1/\alpha.
\]
Proof. Combine Lemma 3.2, Corollary 3.8 and the definition (17) of \( \alpha(q) \).

We are now able to give a direct proof of the item 3. of Theorem 1.3.

Proof. [Direct proof for Theorem 1.3, 3.] Take the potential \( \psi := \phi_0 = - \log |T'| \). As already observed, the corresponding Gibbs measure \( \mu_0 \) is an invariant measure equivalent to the Lebesgue measure. Thus, “\( \mu_0 \)-almost everywhere” is equivalent to “Lebesgue-almost everywhere”. Hence, \( \delta(\phi, \psi) \) is also equal to

\[
\sup \left\{ \delta \geq 0 : \text{Leb}(\mathcal{L}^\delta(x)) = 1 \text{ for } \mu_0 - a.e. \ x \right\}.
\]

From Proposition 3.9, applying (25) with the measure \( \mu_0 \), the exponent \( \delta(\phi, \psi) \) coincides with \( \frac{1}{\alpha_{\max}} \) (defined by (5)). This concludes the proof.

We investigate other exponents (recall that \( q(\alpha) \) is the inverse function of \( \alpha(q) \)):

- By Proposition 3.9, for any \( \varepsilon > 0 \) and \( \delta \in [1/\alpha_+, 1/\alpha_-] \), for \( \mu_\phi \)-a.e. \( x \), we have

\[
(26) \quad \mu_{q(1/\delta)}(\mathcal{L}^{\delta-\varepsilon}(x)) = 1.
\]

- For \( 1/\delta = \alpha(1) = \dim \mu_\phi \), we have \( q(1/\delta) = 1 \) and \( \mu_q = \mu_1 = \mu_\phi \). Hence, applying Proposition 3.9 and (18), we get

\[
\sup \left\{ \delta \geq 0 : \mu_\phi(\mathcal{L}^\delta(x)) = 1 \text{ for } \mu_\phi - a.e. \ x \right\} = \frac{1}{\dim \mu_\phi}.
\]

Thus for every \( 0 < \delta < \frac{1}{\dim \mu_\phi} \), we have \( \mu_\phi(\mathcal{L}^\delta(x)) = 1 \) for \( \mu_\phi \)-a.e. \( x \).

4. Multiple-quasi-Bernoulli inequalities and hitting lemmas

4.1. Multiple-quasi-Bernoulli inequalities. Let \( \phi \) be a normalized potential, i.e. \( P(\phi) = 0 \). The Gibbs property (15) of \( \mu_\phi \) can be written as

\[
(27) \quad \forall x \in [0, 1], \forall n \geq 1, \quad \frac{1}{\gamma} e^{S_n \phi(x)} \leq \mu_\phi(I_n(x)) \leq \gamma e^{S_n \phi(x)}.
\]

It is classical that (27) implies the quasi-Bernoulli property of \( \mu_\phi \).

Lemma 4.1. For any basic intervals \( A \) and \( B \) of generation \( n_A \) and \( n_B \) respectively,

\[
(28) \quad \frac{1}{\gamma^3} \mu_\phi(A) \mu_\phi(B) \leq \mu_\phi(A \cap T^{-n_A} B) \leq \gamma^3 \mu_\phi(A) \mu_\phi(B).
\]

Proof. Consider any \( x \in A \cap T^{-n_A} B \). Applying (27) three times, we get

\[
\mu_\phi(A \cap T^{-n_A} B) \geq \gamma^{-1} e^{S_{n_A} \phi(x) + S_{n_B}(T^{n_A} x)} \geq \gamma^{-3} \mu_\phi(A) \mu_\phi(B),
\]

\[
\mu_\phi(A \cap T^{-n_A} B) \leq \gamma e^{S_{n_A} \phi(x) + S_{n_B}(T^{n_A} x)} \leq \gamma^3 \mu_\phi(A) \mu_\phi(B).
\]

Moreover, by (24), the following multiple-quasi-Bernoulli inequalities hold. The same inequalities are referred to as a multi-relation for the doubling map in [18].
Lemma 4.2. Let $\mu_\phi$ be the Gibbs measure associated with a normalized potential $\phi$. Let $n \in \mathbb{N}$, $n \geq 1$ and let $C_0, C_1, \ldots, C_k$ be $(k+1)$ basic intervals in $C_n$. Then there exist a constant $M > 0$ independent of the choice of $n$ such that for all integer $\omega$ large enough ($\beta$ is the constant appearing in $(24)$),

$$
\frac{1}{\gamma^3} (1 - M \beta^{\omega n})^{k-1} \leq \frac{\mu_\phi \left( C_0 \cap \bigcap_{j=1}^{k} T^{-2j\omega n} C_j \right)}{\prod_{j=0}^{k} \mu_\phi (C_j)} \leq \gamma^3 (1 + M \beta^{\omega n})^{k-1}.
$$

Proof. Let $n_j$ be the generation of $C_j$, and let $\omega$ be an integer large enough that $n_j - 2\omega n \leq -1$ for all $0 \leq j \leq k$. Observe that

$$
C_0 \cap \bigcap_{j=1}^{k} T^{-2j\omega n} C_j = C_0 \cap T^{-n_0} B, \text{ where } B = \bigcap_{j=1}^{k} B_j \text{ and } B_j := T^{n_0 - 2j\omega n} C_j.
$$

Applying $(28)$ to $A = C_0$ and to each $B = B_j$, we obtain

$$
\gamma^{-3} \mu_\phi (C_0) \mu_\phi (B) \leq \mu_\phi (C_0 \cap T^{-n_0} B_j) \leq \gamma^3 \mu_\phi (C_0) \mu_\phi (B_j).
$$

Then, summing over all the $B_j$'s, we get

$$
\gamma^{-3} \mu_\phi (C_0) \mu_\phi (B) \leq \mu_\phi (C_0 \cap T^{-n_0} B) \leq \gamma^3 \mu_\phi (C_0) \mu_\phi (B).
$$

The invariance of $\mu_\phi$ implies that

$$
\mu_\phi (B) = \mu_\phi \left( \bigcap_{j=1}^{k} T^{-2j\omega n} C_j \right).
$$

Thus, in order to get $(29)$, we need only to prove that for some constant $M$,

$$
(1 - M \beta^{\omega n})^{k-1} \leq \frac{\mu_\phi \left( \bigcap_{j=1}^{k} T^{-2j\omega n} C_j \right)}{\prod_{j=1}^{k} \mu_\phi (C_j)} \leq (1 + M \beta^{\omega n})^{k-1}.
$$

Recalling the exponential decay of correlation $(24)$, for every choice of two basic intervals $A, B$, and for every integer $m$, we have

$$
|\mu_\phi (A \cap T^{-m} B) - \mu_\phi (A) \mu_\phi (B)| \leq \Theta \beta^m \left( \frac{\mu_\phi (A) + 2}{\mu_\phi (A)} \right).
$$

which can be rewritten as

$$
\left(1 - \Theta \beta^m \frac{\mu_\phi (A) + 2}{\mu_\phi (A)} \right) \leq \frac{\mu_\phi (A \cap T^{-m} B)}{\mu_\phi (A) \mu_\phi (B)} \leq \left(1 + \Theta \beta^m \frac{\mu_\phi (A) + 2}{\mu_\phi (A)} \right).
$$

Consider the intervals $C_1, \cdots, C_k$ and observe that

$$
\bigcap_{j=1}^{k} T^{-2j\omega n} C_j = \left( T^{-2\omega n} C_1 \right) \bigcap \left( T^{-2\omega n} \bigcap_{j=2}^{k} T^{-2(j-1)\omega n} C_j \right).
$$

Iterating $(33)$, we apply the double-sided inequality $(32)$ inductively to obtain

$$
\prod_{j=1}^{k-1} \left(1 - \Theta \beta^{2\omega n} \frac{\mu_\phi (C_j) + 2}{\mu_\phi (C_j)} \right) \leq \frac{\mu_\phi \left( \bigcap_{j=1}^{k} T^{-2j\omega n} C_j \right)}{\prod_{j=1}^{k} \mu_\phi (C_j)} \leq \prod_{j=1}^{k-1} \left(1 + \Theta \beta^{2\omega n} \frac{\mu_\phi (C_j) + 2}{\mu_\phi (C_j)} \right).
$$
By the Gibbs property \(^{27}\), we have for \(1 \leq j \leq k\),
\[
\frac{\mu_\phi(C_j)}{\mu_\phi(C_j)} + 2 \leq \frac{3}{\mu_\phi(C_j)} \leq 3\gamma \cdot e^{-S_{n_j} \phi(x)} \leq 3\gamma \cdot e^{-n_j (\min_{x \in [0,1]} (\phi(x)))}.
\]
Since \(\phi\) is normalized, \(\min_{x \in [0,1]} (\phi(x))\) is negative. Recalling \(^{11}\), we find that
\[
\frac{\mu_\phi(C_j)}{\mu_\phi(C_j)} + 2 \leq 3\gamma \cdot e^{-L'(\min_{x \in [0,1]} (\phi(x)))}.
\]

4.2. **Big hitting probability lemma.** Lemma \(^{4.3}\) illustrates that intervals with small local dimension for \(\mu_\phi\) are hit by the balls \(B(T^n,1/\delta^\mu)\) with big probability.

**Lemma 4.3.** Let \(h\) and \(\varepsilon\) be two positive real numbers. Consider \(N\) distinct basic intervals \(C_1, \ldots, C_N\) in \(C_n\) satisfying \(\mu_\phi(C_i) \geq |C_i|^{h-\varepsilon}\). Set
\[
C_{n,N,h} := \left\{ x \in [0,1] : \exists C \in \{C_i\}_{i=1}^N \text{ such that } \tau(x,C) > |C|^{-h} \right\}.
\]
Then there exists an integer \(n_h \in \mathbb{N}\) independent of \(N\) such that
\[
\text{for every } n \geq n_h, \quad \mu_\phi(C_{n,N,h}) \leq 2^{-n}.
\]

**Proof.** Fix one interval \(\tilde{C}\) among the \(N\) basic intervals \(C_1, \ldots, C_N\). Let
\[
X_{\tilde{C}} := \left\{ x \in [0,1] : \forall n \leq |\tilde{C}|^{-h}, \quad T^n x \notin \tilde{C} \right\}.
\]

Obviously we have the embedding property \(C_{n,N,h} \subset \bigcup_{i=1}^N X_{C_i}\), so we are going to bound from above each \(\mu_\phi(X_{\tilde{C}})\). Pick up an integer \(\omega\) such that \(2\omega > L\) (\(L\) appears in \(^{10}\)), and set \(m_{\omega} = \lfloor |\tilde{C}|^{-h} / (2\omega n) \rfloor\). Then by definition of \(m_{\omega}\), we have
\[
X_{\tilde{C}} \subset \bigcap_{j=0}^{m_{\omega}} \left\{ x \in [0,1] : T^{2j+1} x \notin \tilde{C} \right\} = \bigcap_{j=0}^{m_{\omega}} \left( [0,1] \setminus T^{-2j \omega n}(\tilde{C}) \right).
\]

We know that the union of the intervals belonging to \(C_n\) is the whole interval \([0,1]\). Observe also that the cardinality of \(C_n\) is of order \(2^n\). Let us denote by \(C_n(\tilde{C})\) the subset of \(C_n\) constituted of the basic intervals disjoint from \(\tilde{C}\).

Since \(2\omega > L\), the definition of \(X_{\tilde{C}}\) implies that for any point \(x \in X_{\tilde{C}}\), there is a choice of \(m_{\omega} + 1\) basic intervals \((D_0, \ldots, D_{m_{\omega}})\) all belonging to \(C_n(\tilde{C})\), such that \(x \in D_0 \cap T^{-2\omega n} D_1 \cap \cdots \cap T^{-2m_{\omega} \omega n} D_{m_{\omega}}\). From this, we deduce that
\[
\mu_\phi(X_{\tilde{C}}) \leq \sum_{(D_0, \ldots, D_{m_{\omega}}) \in (C_n(\tilde{C}))^n} \mu_\phi(D_0 \cap T^{-2\omega n} D_1 \cap \cdots \cap T^{-2m_{\omega} \omega n} D_{m_{\omega}}).
\]

We choose \(\omega\) large enough that Lemma \(^{4.2}\) can be applied. Inequality \(^{29}\)
Thus, for all \( \mu (36) \)

\[
\mu_{\phi}(X_{\tilde{C}}) \leq \gamma^3 (1 + M \beta^{\omega n})^m \bar{c} \sum_{(D_0, \ldots, D_m) \in C_n(\tilde{C})^m} \prod_{j=0}^{m-1} \mu_{\phi}(T^{-2j\omega n} D_j)
\]

= \[
\gamma^3 (1 + M \beta^{\omega n})^m \bar{c} \left( \sum_{D \in C_n(\tilde{C})} \mu_{\phi}(D) \right)^{m \bar{c} + 1}
\]

Since the intervals of \( C_n \) have disjoint interiors, \( \sum_{D \in C_n(\tilde{C})} \mu_{\phi}(D) \leq 1 - \mu_{\phi}(\tilde{C}) \). Hence

\[
\mu_{\phi}(X_{\tilde{C}}) \leq \gamma^3 (1 + M \beta^{\omega n})^m \bar{c} (1 - \mu_{\phi}(\tilde{C}))^{m \bar{c} + 1} \leq \frac{\gamma^3}{1 + M \beta^{\omega n}} \left( (1 + M \beta^{\omega n})(1 - \mu_{\phi}(\tilde{C})) \right)^{m \bar{c} + 1}.
\]

If \( \omega \) is chosen large enough,

\[
(35) \quad (1 + M \beta^{\omega n})(1 - \mu_{\phi}(\tilde{C})) \leq 1 - \mu_{\phi}(\tilde{C})/2.
\]

Thus we finally obtain

\[
(36) \quad \mu_{\phi}(X_{\tilde{C}}) \leq \frac{\gamma^3}{1 + M \beta^{\omega n}} (1 - \mu_{\phi}(\tilde{C})/2)^{m \bar{c} + 1}.
\]

Here, we emphasize that \( \omega \) can be chosen large enough that \( (35) \), and thus \( (36) \), can be realized simultaneously for all \( \tilde{C} \) and for all \( n \). In fact, from the Gibbs property \( (15) \) of \( \mu_{\phi} \), there exists a maximal exponent \( H > 0 \) such that for every basic interval \( \tilde{C} \) of any generation \( n \), \( \mu_{\phi}(\tilde{C}) \geq |\tilde{C}|^H \geq L^{-H} 2^{-nH} \).

Thus, for all \( \tilde{C} \in C_n \), \( 1 - \frac{1}{2} \mu_{\phi}(\tilde{C}) \geq 1 - \frac{1}{2} L^{-H} 2^{-nH} \). So, we can choose \( \omega \) so that

\[
1 + M \beta^{\omega n} \leq 1 - \frac{1}{2} L^{-H} 2^{-nH} \leq 1 - \frac{1}{2} \mu_{\phi}(\tilde{C}) \leq 1 - \mu_{\phi}(\tilde{C}),
\]

which implies \( (35) \) for all \( \tilde{C} \). Summing over all \( \tilde{C} \in \{C_1, \ldots, C_N\} \), by \( (36) \) and the definition of \( m_{\tilde{C}} \), we have

\[
\mu_{\phi}\left(C_{n,N,h}\right) \leq \frac{\gamma^3}{1 + M \beta^{\omega n}} \sum_{\tilde{C}} \left(1 - \mu_{\phi}(\tilde{C})/2\right)^{|\tilde{C}|^{-\varepsilon}/(2\omega n)}
\]

\[
= \frac{\gamma^3}{1 + M \beta^{\omega n}} \sum_{\tilde{C}} \exp \left( \frac{|\tilde{C}|^{-\varepsilon}}{2\omega n \mu_{\phi}(\tilde{C})} \log \left(1 - \mu_{\phi}(\tilde{C})/2\right) \right)
\]

\[
\leq \frac{\gamma^3}{1 + M \beta^{\omega n}} \sum_{\tilde{C}} \exp \left( \frac{-|\tilde{C}|^{-\varepsilon}}{4\omega n} \right).
\]
Now, recalling (10), we have $|\tilde{C}|^{-\varepsilon} \geq L^{-\varepsilon}2^m$. Since the number of possible choices for $\tilde{C}$ is less than $L \cdot 2^n$, we have
\[
\mu_\phi(C_{n,N,h}) \leq \frac{\gamma^3}{1 + M \beta^{2m}} \sum_{\tilde{C}} \exp \left(-\frac{2^m \varepsilon}{4 \omega_n L^2}\right) \leq \frac{L \gamma^3}{1 + M \beta^{2m}} 2^{\frac{1}{\omega n} 2(\varepsilon n \log 2 - \frac{2^m \varepsilon}{4 \omega_n L^2})}.
\]
This last term is independent of $N$ and less than $2^{-n}$ for sufficiently large $n$. \hfill \Box

4.3. Small hitting probability lemma. We now study the probability of hitting points with high local dimension for $\mu_\phi$. The arguments are close to those of [18].

**Lemma 4.4.** Let $0 < a < 1$, $0 < c < b < 1$ and $\eta > b - c$. Consider $2^m$ different basic intervals $C_1, \ldots, C_{2^m}$ in $C_n$. Assume that for every $j \in \{1, \ldots, 2^m\}$,
\[
\mu_\phi(C_j) \leq 2^{-(a+\eta)n}.
\]
Set
\[
X_{a,b,c} := \{x : \tau(x, C_i) \leq 2^m \text{ for } 2^m \text{ distinct intervals among the } \{C_i\}_{i=1}^{2^m}\}.
\]
Then there exists an integer $n_{a,b,c} \in \mathbb{N}$ such that as soon as $n \geq n_{a,b,c}$,
\[
\mu_\phi(X_{a,b,c}) \leq 2^{-n}.
\]

**Proof.** Let us denote $K := 2^m$, $P := 2^m$, $N := 2^m$. When $x \in X_{a,b,c}$, there exist $N$ integers $0 < \ell_1 < \ell_2 < \cdots < \ell_N \leq K$ and $N$ different basic intervals $C_{i_1}, C_{i_2}, \ldots, C_{i_N}$ such that
\[
T^{\ell_1} x \in C_{i_1}, \quad T^{\ell_2} x \in C_{i_2}, \quad \cdots, \quad T^{\ell_N} x \in C_{i_N}.
\]
Let $N' := \lfloor N/(2\omega n) \rfloor$ and let $(\ell_j)_{j=1}^{N'}$ be a subset of $(\ell_j)_{j=1}^{N}$ defined by $t_p = \ell_{2^mp}$. Let $j_p$ be the unique index $i$ such that $T^{\ell_j} x \in C_i$ in (37). If $x \in X_{a,b,c}$,
\[
T^{\ell_1} x \in C_{j_1}, \quad T^{\ell_2} x \in C_{j_2}, \quad \cdots, \quad T^{\ell_{N'}} x \in C_{j_{N'}},
\]
where $C_{j_1}, \ldots, C_{j_{N'}}$ are $N'$ different basic intervals among the intervals $C_{i_1}, \ldots, C_{i_P}$.

Fix now $N'$ basic intervals $C_{j_1}, \ldots, C_{j_{N'}}$ among the intervals $C_1, \ldots, C_P$ and fix also the integers $t_1 < \ldots < t_{N'} \leq K$. Consider the set $\tilde{X}$ of points $x$ such that (38) is satisfied. This set $\tilde{X}$ depends on $a$, $b$, $c$, and on the intervals and the integers we have chosen. As said above, $X_{a,b,c} \subset \bigcup \tilde{X}$, where the union is taken over all possible choices of parameters $C_{j_1}, \ldots, C_{j_{N'}}$ and $t_1 < \ldots < t_{N'}$. In order to bound from above the $\mu_\phi$-measure of $X_{a,b,c}$, we will first study the $\mu_\phi$-measure of one set $\tilde{X}$. Applying (31) again and using the same arguments as in Lemma 4.3, we see that the $\mu_\phi$-measure of $\tilde{X}$ is bounded from above by
\[
\mu_\phi(\tilde{X}) \leq \max_{1 \leq i \leq L} \mu_\phi(C_i)^{N'} (1 + M \beta^{2\omega_n})^{N'}.
\]
It remains us to estimate the maximal number of choices for the associated intervals $C_{j_1}, \ldots, C_{j_{N'}}$ and integers $(t_1, \ldots, t_{N'})$. We have $\binom{P}{N'}$ possible choices for the $N'$ different basic intervals among the list of $P$
integers $C_1, \ldots, C_P$, and there are at most $\binom{K}{N'}$ choices for the integers $t_1 < t_2 < \cdots < t_{N'} < K$. Finally there are $N'$ ways to arrange the $N'$ intervals. Combining this and (39), we find
\[
\mu_{\phi}(\mathcal{X}_{a,b,c}) \leq \sum_{\lambda} \mu_{\phi}(\tilde{\lambda}) \leq \binom{P}{N'} \binom{K}{N'} \cdot N'! \cdot \max_{C_i} \mu_{\phi}(C_i) N' \cdot (1 + M \beta^{\omega_n})^{N'}.
\]

Since
\[
\binom{P}{N'} \binom{K}{N'} = \frac{P!}{(P-N')!} \cdot \frac{K!}{(K-N')!} \cdot \frac{1}{N'!},
\]
using the estimates $\frac{P!}{(P-N')!} \leq P^{N'}$, $\frac{K!}{(K-N')!} \leq K^{N'}$, $\frac{1}{N'!} \leq \xi \cdot \frac{\epsilon^{N'}}{N'}$ for some universal constant $\xi$, we conclude that
\[
\mu_{\phi}(\mathcal{X}_{a,b,c}) \leq \xi \cdot P^{N'} \cdot K^{N'} \cdot e^{N'} \cdot N'! \cdot (\max_{C_i} \mu_{\phi}(C_i)) N' \cdot (1 + M \beta^{\omega_n})^{N'}.
\]

Replacing all constants $K$, $P$, $N$ by their values, we get
\[
\mu_{\phi}(\mathcal{X}_{a,b,c}) \leq \xi \cdot \left(2^{\ln 2} \cdot 2^{an} \cdot e \cdot (N')^{-1} \cdot 2^{-(a+n)\eta} \cdot (1 + M \beta^{\omega_n})\right)^{N'}.
\]

By definition of $N'$, we have $(N')^{-1} \leq \frac{2\omega n}{\omega n} = 2\omega n 2^{-\eta}$ when $\omega$ is large enough. Consequently, the last inequality yields
\[
\mu_{\phi}(\mathcal{X}_{a,b,c}) \leq \xi \cdot \left(e \cdot 2\omega n \cdot (1 + M \beta^{\omega_n}) \cdot 2^{(b-c-n)\eta}\right)^{N'}.
\]

By assumption, $\eta > b - c$, so the quantity between brackets tends to 0 exponentially fast. In particular, it is less than $1/2$. Using the fact that $N' \geq 2^{cn}/2^{\omega n}$, we get
\[
\mu_{\phi}(\mathcal{X}_{a,b,c}) \leq \xi \cdot 2^{-\frac{c\omega n}{2^\omega n}}.
\]

The right term in the above inequality is less than $2^{-n}$ when $n$ becomes large. \qed

5. Part I of the spectrum: $1/\delta < \alpha(1) = \dim \mu_{\phi}$

5.1. Upper bound for $\dim \mathcal{L}^\delta(x)$. By (21), we need only to show the following.

**Proposition 5.1.** For every $0 < s \leq \dim \mu_{\phi}$, for every $x \in [0, 1]$, we have
\[
(40) \quad \dim (\mathcal{R}_{\leq s}(x)) \leq s.
\]

**Proof.** Notice that in the definition of $R(x, y)$, one can replace the limit process of $r \to 0$ by the sequence $2^{-n}$ with $n \to \infty$. Then for $x \in [0, 1]$ and any $a > s$,
\[
\mathcal{R}_{\leq s}(x) \subset \limsup_{n \to \infty} \{y : \tau_{2^{-n}}(x, y) \leq 2^{an}\}.
\]

In other words, given $y \in \mathcal{R}_{\leq s}(x)$, there exists an integer $1 \leq k_n \leq 2^{an}$ such that $y \in B(T^{k_n} x, 2^{-n})$, for infinitely many $n$. Assume that the sequence of integers $(k_n)$ tends to infinity. Using that $2^{-n} \leq (k_n)^{-1/a}$ for such a couple of integers $(k_n, n)$, we have $y \in B(T^{k_n} x, (k_n)^{-1/a})$ for infinitely many integers $k_n$. Hence
\[
\mathcal{R}_{\leq s}(x) \subset \limsup_{n \to \infty} B(T^{k_n} x, k_n^{-1/a}).
\]
For each integer $n$, we deduce that the set of balls $\{B(T^k x, k^{-1/n})\}_{k \geq n}$ forms a covering of $\mathcal{C}^s(x)$ by intervals of length smaller than $n^{-1/a}$. Let $\mathcal{H}_{\alpha}^n$ stand for the $\alpha$-Hausdorff measure obtained by using coverings by balls of size less than $\varepsilon$. Using $\{B(T^k x, k^{-1/n})\}_{k \geq n}$ as covering, we see that for any $a' > a$,

$$\mathcal{H}_{n^{-1/a}}(R_{\leq s}(x)) \leq \sum_{k \geq n} |B(T^k x, k^{-1/a})|^a' \leq 2^{a'/a} \sum_{k \geq n} k^{-a'/a} \leq \xi' n^{1-a'/a},$$

which tends to 0 when $n$ tends to infinity. Here $\xi'$ is a universal constant. We deduce that the $a'$-Hausdorff measure of $\mathcal{C}_{\leq s}(x)$ is necessarily 0. Thus $\dim \mathcal{C}_{\leq s}(x) \leq a'$. Since this holds for any $a' > a$, and then for any $a > s$, (40) follows.

5.2. **Lower bound for $\dim L^\delta(x)$**. Let $(x_n)_{n \geq 1}$ be a sequence in $[0,1]$, and let $(l_n)_{n \geq 1}$ be a positive decreasing sequence. Consider the limsup sets of the form

$$L_\zeta := \bigcap_{N \geq 1} \bigcup_{n \geq N} B(x_n, (l_n)^\zeta).$$

Provided that $\mu_\phi(L_\zeta) = 1$ for some $\zeta_0 > 0$, the dimensions of $L_\zeta$ can be bounded from below using the heterogeneous ubiquity theorems developed in [3]. To apply such theorems, some assumptions need to be checked for $\mu_\phi$. We refer to Definition 2 of [3] for the precise description of these assumptions. We explain now why these assumptions are fulfilled in our framework. From Theorem 1.11(2) of Baladi [1], Theorem 7.1 of Philipp and Stout [30], we deduce the following properties for $\mu_\phi$.

**Theorem 5.2.** Assume that the potential $\phi$ associated with $\mu_\phi$ is Hölderian. There exists a non-decreasing continuous function $\chi$ defined on $\mathbb{R}_+$ with the properties:

- $\chi(0) = 0$, $r \mapsto r^{-\chi(r)}$ is non-increasing near $0^+$,
- $\lim_{r \to 0^+} r^{-\chi(r)} = +\infty$, and $\forall \varepsilon > 0$, $r \mapsto r^{\varepsilon - \chi(r)}$ is non-decreasing near 0,

such that for $\mu_\phi$-almost every $y \in [0,1]^d$, there exists $r(y) > 0$, such that

(41) for all $0 < r \leq r(y)$, $r^{\dim \mu_\phi + \chi(r)} \leq \mu_\phi(B(y, r)) \leq r^{\dim \mu_\phi - \chi(r)}$.

Property (41) shall be viewed as an illustration of the iterated logarithm law for invariant measures. By the theorems of [1] and [30], one can take $\chi$ equal to

(42) $\chi(0) = 0$ and $\chi : r \mapsto \left(\frac{\log \log |\log(r)|}{|\log r|}\right)^{1/2}$ if $r > 0$.

In the previous section, we proved the following: for any $\delta$ such that $1/\delta > \dim \mu_\phi = \alpha(1)$, for $\mu_\phi$-almost every $x \in [0,1]$, $\mu_\phi(L^\delta(x)) = 1$. Theorem 5.2 and the quasi-Bernoulli property of $\mu_\phi$ and $\mu_\eta$ imply that the conditions of Definition 2 of [3] are fulfilled for $\mu_\phi$-a.e. $x \in [0,1]$. We can then apply the heterogeneous ubiquity Theorem 4 of [3] to get the following lower bound.
Theorem 5.3. For any $\delta$ such that $1/\delta > \dim \mu_\phi$, for $\mu_\phi$-a.e. $x \in [0,1]$, we have

$$\dim(L^C(x)) \geq (\dim \mu_\phi)/\zeta.$$  

By considering an increasing countable sequence $(\delta_n)$ tending to $\delta_0 = 1/\dim \mu_\phi$ and applying Theorem 5.3 to each $\delta_n$, we get immediately:

Corollary 5.4. For $\mu_\phi$-almost every $x$, for every $\zeta > 1$,

$$\dim(L^C(x)) \geq (\dim \mu_\phi)/\zeta = 1/(\zeta \cdot \delta_0).$$

In other words, for every $\delta$ such that $1/\delta < \dim \mu_\phi$, we have the lower bound

$$\dim(L^C(x)) \geq 1/\delta.$$

6. Part IV of the spectrum: $1/\delta > \alpha_+$

Proposition 6.1. Let $s \geq 0$. For $\mu_\phi$-almost every $x$,

$$\mathcal{R}_{>s}(x) \subset \mathcal{E}_{>s}.$$  

Moreover, for any Gibbs measure $\mu_\psi$ on $[0,1]$,

$$\text{for } \mu_\phi\text{-almost every } x \in [0,1], \quad \mathcal{R}_{>s}(x) = \mathcal{E}_{>s},$$

where the equality means that the two sets differ from a set of $\mu_\psi$-measure zero.

Remark 6.2. The full $\mu_\phi$-measure set satisfying the first assertion of Proposition 6.1 depends on $s$.

Proof. The case $s = 0$ is obvious, we assume that $s > 0$. For any integer $n \geq 1$, let $I_n(y)$ be the basic interval in $\mathcal{C}_n$ containing $y$. Observe that a priori the generation of $I_n(y)$ is not $n$. For any real number $\varepsilon > 0$, we introduce the sets

$$\mathcal{R}_{n,s,\varepsilon}(x) = \{y : \tau(x,I_n(y)) \geq |I_n(y)|^{s-\varepsilon}\}, \quad \mathcal{E}_{n,s,\varepsilon} = \{y : \mu_\phi(I_n(y)) \leq |I_n(y)|^{s-2\varepsilon}\}.$$  

By definition of $R(x,y)$ and $d_{\mu_\phi}(y)$, we have

$$\mathcal{R}_{>s}(x) = \bigcap_{\varepsilon > 0} \liminf_{n \to \infty} \mathcal{R}_{n,s,\varepsilon}(x) \quad \text{and} \quad \mathcal{E}_{>s} = \bigcap_{\varepsilon > 0} \liminf_{n \to \infty} \mathcal{E}_{n,s,\varepsilon}.$$  

In order to prove (44), it is sufficient to prove that for $\mu_\phi$-almost every $x$, there exists some integer $n(x)$ such that

$$\forall n \geq n(x), \quad \mathcal{R}_{n,s,\varepsilon}(x) \subset \mathcal{E}_{n,s,\varepsilon}.$$  

Notice that $\mathcal{E}_{n,s,\varepsilon}$ is the union of basic intervals $C$ in $\mathcal{C}_n$ such that $\mu_\phi(C) > |C|^{s-2\varepsilon}$. Let $\mathcal{D}_{n,s,\varepsilon} := \{C_1, \cdots, C_N\}$ be the set of these basic intervals. Using Lemma 4.3 to the basic intervals $\mathcal{D}_{n,s,\varepsilon}$ and to $h = s - \varepsilon$, we see that for $n$ larger than some $n_{s,\varepsilon}$,

$$P_n := \mu_\phi\left( \left\{ x : \exists C \in \mathcal{D}_{n,s,\varepsilon} \text{ such that } \tau(x,C) \geq |C|^{s-\varepsilon} \right\} \right) \leq 2^{-n}.$$  

The sum over $n \geq n_{s,\varepsilon}$ of the $P_n$’s is finite. Applying the Borel-Cantelli lemma, the following holds for $\mu_\phi$-a.e. $x$: there exists an integer $n(x)$ such that

$$\forall n \geq n(x), \quad \forall C \in \mathcal{D}_{n,s,\varepsilon}, \quad \tau(x,C) < |C|^{s-\varepsilon}.$$
Remark 6.2, the full $\mu_n$-measure on every $L$ that $1$ to get formula (22) and Lemma 3.4, we deduce that when $1$, i.e. there is no point with hitting times larger than $s > \alpha$.

This last statement is directly deduced from Corollary 3.7. □

We are now ready to prove some of the statements of Theorem 1.3. 

Proof. [Part IV of the spectrum: Theorem 1.3, 4.] By Remark 2.6 and Proposition 6.1, for each $s > \alpha$, for $\mu_\phi$-almost every $x \in [0, 1]$, we have

$$R_{\geq s}(x) = \left\{ y \in [0, 1] : R(x, y) \geq s \right\} = \emptyset,$$

i.e. there is no point with hitting times larger than $s > \alpha$. Then, applying formula (22) and Lemma 3.4, we deduce that when $1/\delta > \alpha$, for $\mu_\phi$-almost every $x \in [0, 1]$, $\mathcal{F}^\delta(x) = \emptyset$ and thus $\mathcal{L}^\delta(x) = [0, 1]$. But as mentioned in Remark 6.2, the full $\mu_\phi$-measure set depends on $\delta$. To solve this problem, i.e. to get $\mathcal{F}^\delta(x) = \emptyset$ for every $\delta$ satisfying $1/\delta > \alpha$, we take a sequence $(\delta_n)_{n \geq 1}$ such that $(1/\delta_n)$ is dense in $[\alpha, \infty]$. By taking intersection of countable full $\mu_\phi$-measure sets, we obtain that for $\mu_\phi$-almost every $x \in [0, 1]$, for all $n$, $\mathcal{F}^{\delta_n}(x) = \emptyset$ and $\mathcal{L}^{\delta_n}(x) = [0, 1]$. Finally, the case of an arbitrary $\delta$ such that $1/\delta > \alpha$, is obtained by using the monotonicity of the sets $\mathcal{F}^\delta(x)$ and $\mathcal{L}^\delta(x)$ with respect to $\delta$. □

7. Part III of the spectrum: $\alpha_{\max} < 1/\delta \leq \alpha_+$

In this short section, we gather the previous results to obtain Part III of the spectrum and item 3. of Theorem 1.3. We adopt the notations of Section 3. Let $\delta$ be such that $\alpha_{\max} < 1/\delta \leq \alpha_+$, and consider the unique real number $q(1/\delta)$. Then the associated Gibbs measure $\mu_{q(1/\delta)}$ is supported on the level set $\mathcal{E}_{\mu_{\phi}}(1/\delta)$, which has Hausdorff dimension $D_{\mu_{\phi}}(1/\delta)$.

Further, we apply the second part of Proposition 6.1 to the measure $\mu_{\psi} = \mu_{q(1/\delta)}$. Then for $\mu_{\phi}$-almost every $x$, the measure $\mu_{q(1/\delta)}$ is also supported on the set $R_{\geq 1/\delta}(x)$. In particular, we conclude that $\dim R_{\geq 1/\delta}(x) \geq \dim \mu_{q(1/\delta)}$.

Now, consider a countable sequence $(\delta_n)_{n \geq 1}$ such that $1/\delta_n$ is dense in the interval $[\alpha_{\max}, \alpha_+]$. The above argument applies to each $\delta_n$. Taking a countable intersection of full $\mu_\phi$-measure sets, we find a set of full $\mu_\phi$-measure of points $x$ such that for all $n \geq 1$, $\mu_{q(1/\delta_n)}$ is also supported on the set $R_{\geq 1/\delta_n}(x)$.

Let us fix $\delta_0$ such that $\alpha_{\max} < 1/\delta_0 \leq \alpha_+$, and consider a subsequence $(\delta_{\varphi(n)})_{n \geq 1}$ decreasing to $\delta_0$. By (22), for every integer $n$,

$$\dim(\mathcal{F}^{\delta_0}) \geq \dim R_{\geq 1/\delta_{\varphi(n)}}(x) \geq \dim(\mu_{q(1/\delta_{\varphi(n)})}) = D_{\mu_{\phi}}(1/\delta_{\varphi(n)}).$$

Using the continuity of $D_{\mu_{\phi}}$ on its support, we see that for $\mu_{\phi}$-almost every $x \in [0, 1]$,

$$\dim(\mathcal{F}^{\delta_0}) \geq \dim(\mu_{q(1/\delta_0)}) = D_{\mu_{\phi}}(1/\delta_0).$$
Conversely, by choosing an increasing subsequence \((\delta_{x(n)})_{n \geq 1}\) converging to \(\delta_0\), by (22) and Proposition 6.1, we have for \(\mu_\phi\)-almost every \(x\),

\[
\dim(F^{(n)}) \leq \inf_n \dim (E_{1/\delta_{x(n)}}) = \inf_n D_{\mu_\phi}(1/\delta_{x(n)}) = D_{\mu_\phi}(1/\delta_0).
\]

This completes the proofs for the Part III and for item 3. of Theorem 1.3.

8. Part II of the spectrum: \(\dim \mu_\phi < 1/\delta \leq \alpha_{\text{max}}\)

**Proposition 8.1.** If \(\dim \mu_\phi < s < \alpha_{\text{max}}\) then for \(\mu_\phi\)-almost every \(x\) we have

\[
(46) \quad \dim R_{\leq s}(x) \geq D_{\mu_\phi}(s).
\]

**Proof.** For \(\dim \mu_\phi < s < \alpha_{\text{max}}\), there exists a real number \(q_s > 0\) such that

\[
\frac{\int (-\phi) \, d\mu_{q_s}}{\int \log |T'| \, d\mu_{q_s}} = s.
\]

By the Gibbs property of \(\mu_\phi\) and the ergodicity of \(\mu_{q_s}\), the measure \(\mu_{q_s}\) is supported on \(E_{\mu_\phi}(s)\). Then by Corollary 3.8 applied to \(\phi_0\) and \(\mu_{q_s}\), for \(\mu_\phi\)-a.e. \(x\) we have

\[
\dim R_{\leq s}(x) \geq \dim \mu_{q_s} = D_{\mu_\phi}(s).
\]

\[\square\]

We finish by bounding from above the spectrum \(\dim L^\delta(x)\).

**Proposition 8.2.** If \(\dim \mu_\phi < s < \alpha_{\text{max}}\) then for \(\mu_\phi\)-almost every \(x\) we have

\[
(47) \quad \dim R_{\leq s}(x) \leq D_{\mu_\phi}(s).
\]

**Proof.** Fix \(s \in (\dim \mu_\phi, \alpha_{\text{max}})\), and let us decompose \(R_{\leq s}(x)\) into

\[
R_{\leq s}(x) = (R_{\leq s}(x) \cap E_{\leq s}) \cup (R_{\leq s}(x) \cap E_{> s}).
\]

Since \(s\) lies in the increasing part of the spectrum, by Proposition 2.8, we have the upper bound \(\dim E_{\leq s} \leq D_{\mu_\phi}(s)\). Thus, to obtain (47), it suffices to prove that

\[
\dim (R_{\leq s}(x) \cap E_{> s}) \leq D_{\mu_\phi}(s).
\]

Recall that \(C_n\) forms a covering of \([0, 1]\) by basic intervals of size \(\sim 2^{-n}\), these intervals having disjoint interiors.

Let \(0 < h' < h''\). We define the subsets \(C_n(h', h'')\) of \(C_n\) and \(Y_n(h', h'')\) of \([0, 1]\)

\[
C_n(h', h'') = \{C \in C_n : |C|^{h''} \leq \mu_\phi(C) \leq |C|^{h'}\} \subset C_n,
\]

\[
Y_n(h', h'') = \{y \in [0, 1] : \exists C \in C_n(h', h'') \text{ such that } y \in C\} \subset [0, 1].
\]

**Lemma 8.3.** For every \(\varepsilon > 0\), there exists an integer \(n_{h', h'', \varepsilon}\) large enough so that as soon as \(n \geq n_{h', h'', \varepsilon}\),

\[
(48) \quad \text{Card } C_n(h', h'') \leq 2^{nD_{\mu_\phi}(h'') + \varepsilon} \quad \text{if } h'' < \alpha_{\text{max}},
\]

\[
(49) \quad \text{Card } C_n(h', h'') \leq 2^{nD_{\mu_\phi}(h') + \varepsilon} \quad \text{if } h' > \alpha_{\text{max}}.
\]
These properties follow again from standard large deviations properties (see \cite{9}).

Let \( \zeta > 0 \) be a positive real number that we will soon choose in a suitable manner. Set \( h'_1 = s \) and \( h''_1 = s + \frac{1}{2} \zeta \). It is possible to cover the interval \([s + \frac{1}{2} \zeta, \alpha_+]\) by a finite number of open intervals \( \{(h'_i, h''_i)\}_{2 \leq i \leq \ell} \) with length less than \( \zeta \). We have the inclusion

\[
\mathcal{E}_{> s} \subset \bigcup_{N=1}^{+\infty} \bigcap_{n \geq N}^{\ell} \bigcup_{i=1}^{\ell} \mathcal{Y}_n(h'_i, h''_i).
\]

Recall that \( I_n(y) \) is the unique basic interval contained in \( C_n \) containing \( y \). The embedding property \((50)\) emphasizes that when \( y \in [0, 1] \), then necessarily \( \mu_\phi(I_n(y)) < |I_n(y)|^s \) for every integer \( n \) large enough, not only for an infinite number of integers.

We introduce the subset \( C_{n,a}(x) \) of \( C_n \)

\[
C_{n,a}(x) := \{ C \in C_n : \tau(x, C) < 2^{an} \}.
\]

By the definition of \( R_{\lesssim s}(x) \), for any real number \( a > s \), \( R_{\lesssim s}(x) \subset \{ y \in [0, 1] : R(x, y) < a \} \). The distortion property \((9)\) guarantees that \( I_n(y) \) tends to zero very regularly when \( n \) tends to infinity. Hence, if \( y \in R_{\lesssim s}(x) \), there is an infinite number of integers \( n \) such that \( \tau_{2^{-n}}(x, y) \leq 2^{an} \), which means that \( T^p x \in B(y, 2^{-n}) \) for some \( p \leq 2^an \).

Denote by \( d(y, C) \) the distance from \( y \) to the set \( C \). We introduce the sets \( \tilde{\mathcal{Y}}_{n,a}(x) = \{ y \in [0, 1] : \exists C \in C_{n,a}(x) \text{ such that } d(y, C) \leq 2^{-n} \} \). Since \( T^p x \in B(y, 2^{-n}) \) implies that \( d(y, I_n(T^p x)) \leq 2^{-n} \), we have

\[
R_{\lesssim s}(x) \subset \bigcap_{N=1}^{+\infty} \bigcup_{n \geq N}^{\ell} \tilde{\mathcal{Y}}_{n,a}(x).
\]

Thus combining \((50)\) and \((51)\), we get that \( R_{\lesssim s}(x) \cap \mathcal{E}_{> s} \) is included in

\[
\bigcap_{N=1}^{+\infty} \bigcup_{n \geq N}^{\ell} \bigcup_{i=1}^{\ell} \mathcal{Y}_n(h'_i, h''_i) \subset \bigcup_{N=1}^{+\infty} \bigcap_{n \geq N}^{\ell} \bigcup_{i=1}^{\ell} \tilde{\mathcal{Y}}_{n,a}(x) \cap \mathcal{Y}_n(h'_i, h''_i).
\]

The above inversion of \( \cap \) and \( \cup \) follows from the fact that there is a finite number of intervals \([h'_i, h''_i]\). Thus, we need only to show that for all \( 1 \leq i \leq \ell \),

\[
\forall \varepsilon > 0, \quad \dim \left( \limsup_{n \to \infty} (\tilde{\mathcal{Y}}_{n,a}(x) \cap \mathcal{Y}_n(h'_i, h''_i)) \right) \leq D_{\mu_\phi}(s) + \varepsilon.
\]

Let \( C_{n,a,h'_i,h''_i}(x) \) be the subset of the basic intervals of \( C_n \) belonging to both \( C_{n,a}(x) \) and \( C_n(h'_i, h''_i) \).

**Lemma 8.4.** For every \( a \in (s, \alpha_{\max}) \), for every \( \varepsilon > 0 \), for each \( 2 \leq i \leq \ell \),

\[
\sum_n \mu_\phi \left( \left\{ x : \text{Card} C_{n,a,h'_i,h''_i}(x) > 2^{n(D_{\mu_\phi}(a)+\varepsilon)} \right\} \right) < \infty.
\]
Observe that in Lemma 8.4, we do not consider the first interval \([h'_1, h''_1]\). For this interval, \(\sum_{n \in n(a,h')} D_{\mu_{\phi}}(a) \leq 2^{n(D_{\mu_{\phi}}(a)+\varepsilon)}\).

In order to obtain a covering of the set \(\limsup_{n \to \infty} \left( \tilde{Y}_{n,a}(x) \cap Y_n(h'_n, h''_n) \right)\), by construction one considers, for any \(N \geq 1\), the union
\[
\bigcup_{n \geq N} \bigcup_{C \in \mathcal{C}_{n,a,h'_n,h''_n}(x)} \{ y \in [0,1] : d(y, C) \leq 2^{-n} \}.
\]

Using this family of coverings, if \(N \geq n(x)\), then for any \(\varepsilon > 0\), the \((D_{\mu_{\phi}}(a)+2\varepsilon)\)-Hausdorff measure of the above limsup set is bounded by
\[
\sum_{n \geq N} \sum_{C \in \mathcal{C}_{n,a,h'_n,h''_n}(x)} (L + 2) \cdot 2^{-n} D_{\mu_{\phi}}(a) + 2\varepsilon \cdot 2^{n(D_{\mu_{\phi}}(a)+\varepsilon)} \\
\leq \Theta' \sum_{n \geq N} 2^{-n(D_{\mu_{\phi}}(a)+2\varepsilon)} \cdot 2^{n(D_{\mu_{\phi}}(a)+\varepsilon)} \\
\leq \Theta' \sum_{n \geq N} 2^{-n\varepsilon} < \infty,
\]

where \(\Theta'\) is some constant depending on \(L, a, \mu_{\phi}\) and \(\varepsilon\). Hence, letting \(N \) tend to infinity, we see that the \((D_{\mu_{\phi}}(a)+2\varepsilon)\)-Hausdorff measure of the limsup set \(\limsup_{n \to \infty} \left( \tilde{Y}_{n,a}(x) \cap Y_n(h'_n, h''_n) \right)\) is necessarily 0. This implies that
\[
\dim \left( \limsup_{n \to \infty} \left( \tilde{Y}_{n,a}(x) \cap Y_n(h'_n, h''_n) \right) \right) \leq D_{\mu_{\phi}}(a) + 2\varepsilon.
\]

We finish the proof of Proposition 8.2 by letting first \(\varepsilon \downarrow 0\) and then \(a \downarrow s\).

It remains us to prove Lemma 8.4. Let \(a \in (s, \alpha_{\max})\). It is enough to prove Lemma 8.4 for \(a\) close to \(s\). Hence we suppose that \(a < h'_2\).

We assume that the intervals \([h'_i, h''_i]\) are chosen so that, except for at most one of them, either \(h'_i > \alpha_{\max}\) or \(h''_i < \alpha_{\max}\). In other words, we suppose that there is only one integer \(i \in \{2, \ldots, l\}\) such that \(\alpha_{\max} \in (h'_i, h''_i)\). We use two key properties:

- The multifractal spectrum \(D_{\mu_{\phi}}\) is real-analytic and concave on \([\alpha_{-}, \alpha_{+}[^{\cdot}\).
- For every \(h \geq a > s > \dim \mu_{\phi}\), the derivative of \(D_{\mu_{\phi}}\) at \(s\) is strictly less than 1, and the derivative \((D_{\mu_{\phi}})'(s)\) is decreasing. Hence, there is a real number \(0 < \xi_a = (D_{\mu_{\phi}})'(a) < 1\) such that for every \(h\) in every interval \([h'_i, h''_i]\) \((i \geq 2)\),

  \[
  \text{for every } h \geq a, \quad (D_{\mu_{\phi}})'(h) \leq \xi_a.
  \]

We distinguish three cases.

- **If \(h''_i < \alpha_{\max}\):** Take \(b = D_{\mu_{\phi}}(h''_i) + \varepsilon, \ c = D_{\mu_{\phi}}(a) + \varepsilon, \) and \(\eta = h'_i - a\).
  Then on the one hand, by \(\sum_{n \in n(a,h')}, \) for \(n\) large enough there are at most \(2^{2n}\) basic intervals \(C\) in \(\mathcal{C}_{n}(h'_i, h''_i) = \mathcal{C}_{n}(a + \eta, h''_i)\). On the other hand, by the mean value theorem and the fact that \(D_{\mu_{\phi}}(\cdot)\) is increasing on \((\alpha_{-}, \alpha_{\max})\),

  \[
  b - c = D_{\mu_{\phi}}(h''_i) - D_{\mu_{\phi}}(a) < \xi_a(h''_i - a) = \xi_a(h'_i - a) + \xi_a(h''_i - h'_i).
  \]
Since $\xi_a < 1$ and $h_i^\prime\prime - h_i^\prime < \zeta$, we can choose $\zeta$ small enough such that
\[ b - c < h_i^\prime < a = \eta. \]

This choice of $\zeta$ can be uniform, i.e. valid for every index $i$ such that $h_i^\prime < \alpha_{\max}$.

By Lemma 4.4, for sufficiently large $n$,
\[
\mu_\phi \left( \left\{ x : \left\{ \begin{array}{l}
\tau(x, C) \leq 2^{\alpha_2 n} \text{ for } 2^{\alpha_1 n} \text{ distinct } \\
\text{intervals } C \text{ among the } 2^{\alpha_2 n} \text{ intervals } \end{array} \right\} \right\} \right) \leq 2^{-n}.
\]

This is equivalent to say that
\[
\mu_\phi \left( \left\{ x : \text{Card } C_{n,a,h_i^\prime,h_i^\prime\prime}(x) > 2^n(D_{\mu_\phi}(a) + \varepsilon) \right\} \right) \leq 2^{-n}.
\]

Then (52) follows.

- **If $i$ is the unique integer such that $h_i^\prime < \alpha_{\max} < h_i^\prime\prime$: This occurs for one and only one interval $[h_i', h_i'']$. Recall that the cardinality of $C_n$ is less than $L \cdot 2^n$. Take $b = 1$, $c = D_{\mu_\phi}(a)$, and $\eta = h_i - a$. Then
  \[ b - c = 1 - D_{\mu_\phi}(a) < \xi_a(\alpha_{\max} - a) = \xi_a(h_i - a) - \xi_a(\alpha_{\max} - h_i'). \]

Since $D'_{\mu_\phi}(a) < 1$ and $\alpha_{\max} - h_i'' < h_i' - \zeta$, we can choose $\zeta$ small enough that
\[ b - c < h_i' - a = \eta. \]

By Lemma 4.4 and applying the same arguments as above, for large $n$, we have
\[
\mu_\phi \left( \left\{ x : \left\{ \begin{array}{l}
\tau(x, C) \leq 2^{\alpha_2 n} \text{ for } 2^{\alpha_1 n} \text{ distinct } \\
\text{intervals } C \text{ among } 2^{\alpha_2 n} \text{ intervals of } C_n \end{array} \right\} \right\} \right) \leq 2^{-n}.
\]

It is not difficult to prove that we can replace $2^{\alpha_2 n}$ by $L \cdot 2^{\alpha_1 n}$, since constants do not infer in the proofs of Lemma 4.4. In other words,
\[
\mu_\phi \left( \left\{ x : \text{Card } C_{n,a,h_i',h_i''}(x) > 2^n(D_{\mu_\phi}(a)) \right\} \right) \leq 2^{-n},
\]

and (52) is proved.

- **If $\alpha_{\max} < h_i'$:** Take $b = D_{\mu_\phi}(h_i') + \varepsilon$, $c = D_{\mu_\phi}(a) + \varepsilon$, and $\eta = h_i - a$. On the one hand, by (49), for $n$ large enough there are at most $2^{\alpha_2 n}$ basic intervals in $C_n(h_i', h_i'') = C_n(a + \eta, h_i'')$. On the other hand,
\[ b - c = D_{\mu_\phi}(h_i') - D_{\mu_\phi}(a) < \xi_a(h_i' - a) < (h_i' - a). \]

Thus by Lemma 4.4, for sufficiently large $n$, (53) follows, and (52) is proved.
we can obtain that for $\mu_\delta$-almost every $x \in [0, 1]$ for every $\delta$, \[\mu_\delta\] holds. This concludes the proof.

References

[1] V. Baladi. Positive transfer operators and decay of correlations. Advanced Series in Nonlinear Dynamics, 16. World Scientific Publishing Co., Inc., River Edge, NJ, 2000.
[2] J. Barra, F. Ben Nasr and J. Peyrière. Comparing multifractal formalisms: The neighboring boxes conditions. Asian J. Math. 7, (2003),149–165.
[3] J. Barra and S. Seuret. Heterogeneous ubiquitous systems in $\mathbb{R}^d$ and Hausdorff dimension. Bull. Braz. Math. Soc. (N.S.) 38(3) (2007), 467–515.
[4] J. Barra and S. Seuret. Ubiquity and large intersections properties under digit frequencies constraints. Math. Proc. Cambridge Philos. Soc. 145 (3) (2008), 527–548.
[5] L. Barreira, Y. Pesin and J. Schmeling. On a general concept of multifractality: multifractal spectra for dimensions, entropies, and Lyapunov exponents. Multifractal rigidity. Chaos 7 (1997), 27–38.
[6] V. Beresnevich and S. Velani. A Mass Transference Principle and the Duffin-Schaeffer conjecture for Hausdorff measures. Ann. Math. (2) 164(3) (2006), 971–992.
[7] A. S. Besicovitch. Sets of fractional dimension (IV): on rational approximation to real numbers. J. London Math. Soc. 9 (1934), 126–131.
[8] R. Bowen. Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Springer-Verlag, Berlin, 1975.
[9] G. Brown, G. Michon and J. Peyrière. On the multifractal analysis of measures. J. Stat. Phys. 66 (1992), 775–790.
[10] Y. Bugeaud. Approximation by algebraic integers and Hausdorff dimension. J. London Math. Soc. 65 (2002), 547–559.
[11] Y. Bugeaud. A note on inhomogeneous diophantine approximation. Glasg. Math. J. 45 (2003), 105–110.
[12] Y. Bugeaud, S. Harrap, S. Kristensen and S. Velani. On shrinking targets for $\mathbb{Z}^m$-actions on the torii. Mathematika 56 (2010), 193–202.
[13] J.W.S. Cassels. An Introduction to Diophantine Approximation. Cambridge Tracts in Mathematics and Mathematical Physics. Cambridge University Press, New York, 1957.
[14] P. Collet, J. Lebowitz and A. Porzio. The dimension spectrum of some dynamical systems. J. Stat. Phys. 47 (1987), 609–644.
[15] M. M. Dodson, M. V. Melián and D. Pestana, S. L. Velani. Patterson measure and Ubiquity. Ann. Acad. Sci. Fenn. Ser. A I Math. 20 (1995), 37–60.
[16] M. Einsiedler, A. Katok and E. Lindenstrauss. Invariant measures and the set of exceptions to Littlewood’s conjecture. Ann. of Math. (2) 164 (2) (2006), 513–560.
[17] K. J. Falconer. Fractal geometry. Mathematical foundations and applications. Second edition. John Wiley & Sons, Inc., Hoboken, NJ, 2003.
[18] A.-H. Fan, J. Schmeling and S. Troubetzkoy, Dynamical Diophantine approximation. Preprint. 2009.
[19] S. Galatolo. Dimension and hitting time in rapidly mixing systems. Math. Res. Lett. 14(5) (2007), 797–805.
[20] R. Hill and S. L. Velani. Ergodic theory of shrinking targets. Invent. math. 119, (1995), 175–198.
[21] R. Hill and S. L. Velani. The shrinking target problem for matrix transformations of tori. J. London Math. Soc. (2) 60(2) (1999), 381–398.
[22] V. Jarnik. Diophantischen Approximationen und Hausdorffsches Mass. Mat. Sbornik 36 (1929), 371–381.
[23] D. H. Kim. The shrinking target property of irrational rotations. Nonlinearity 20(7) (2007), 1637–1643.
[24] D. Kleinbock and G. A. Margulis. Flows on homogeneous spaces and Diophantine approximation on manifolds. Ann. Math. (2) 148 (1998), 339–360.
[25] D. Kleinbock, E. Lindenstrauss and B. Weiss. On fractal measures and Diophantine approximation. Selecta Math. (N.S.), 10 (2004), 479–523.
[26] C. Liverani, B. Saussol and S. Vaienti. Conformal measure and decay of correlation for covering weighted systems. *Ergod. Th. & Dynam. Sys.* 18(6) (1998), 1399–1420.

[27] D. Ornstein and B. Weiss. Entropy and data compression schemes. *IEEE Trans. Inform. Theory* 39(1) (1993), 78–83.

[28] Y. Pesin and H. Weiss. The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples. *Chaos* 7(1) (1997), 89–106.

[29] W. Parry and M. Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque* No. 187-188 (1990).

[30] W. Philipp and W. Stout. Almost Sure Invariance Principles for Partial Sums of Weakly Dependent Random Variables. *Mem. Amer. Math. Soc.* 2 161, 1975.

[31] D. A. Rand. The singularity spectrum $f(\alpha)$ for cookie-cutters. *Ergod. Th. Dyn. Syst.* 9(3) (1989), 527–541.

[32] D. Ruelle. Thermodynamic formalism. The mathematical structures of equilibrium statistical mechanics. Second edition. *Cambridge Mathematical Library*. Cambridge University Press, Cambridge, 2004.

[33] J. Schmeling and S. Troubetzkoy. Inhomogeneous Diophantine approximation and angular recurrence properties of the billiard flow in certain polygons. *Math. Sbornik* 194 (2003), 295–309.

[34] D. Simpelaere. Dimension spectrum of Axiom A diffeomorphisms. II. Gibbs measures. *J. Statist. Phys.* 76(5-6) (1994), 1359–1375.

[35] P. Walters. Invariant Measures and Equilibrium States for Some Mappings which Expand Distances. *Trans. Amer. Math. Soc.* 236 (1978), 121–153.

LAMA, CNRS UMR 8050, UNIVERSITÉ PARIS-EST CRÉTEIL, 61 AVENUE DU GÉNÉRAL DE GAULLE, 94010 CRÉTEIL CEDEX, FRANCE

E-mail address: lingmin.liao@u-pec.fr

LAMA, CNRS UMR 8050, UNIVERSITÉ PARIS-EST CRÉTEIL, 61 AVENUE DU GÉNÉRAL DE GAULLE, 94010 CRÉTEIL CEDEX, FRANCE

E-mail address: seuret@u-pec.fr