A CHAIN OF NUMERICAL RADIUS INEQUALITIES IN COMPLEX HILBERT SPACE

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Abstract. In this paper, we implement the improvement of numerical radius inequalities that were produced by Alomari MW. [Refinements of some numerical radius inequalities for Hilbert space operators. Linear and Multilinear Algebra. 2019 Jun 4:1-6] and devise a new upper bound for $2 \times 2$ operator matrices on complex Hilbert space with many examples which show that our bound is sharper than the existing bounds proved by Bani-Domi W, Kittaneh F. [Norm and numerical radius inequalities for Hilbert space operators. Linear and Multilinear Algebra. 2020 Jul 28:1-2], Al-Dolat M, Jaradat I, Al-Husban B. A novel numerical radius upper bounds for $2 \times 2$ operator matrices. Linear and Multilinear Algebra. 2020 Apr 23:1-2], Shebrawi K. [Numerical radius inequalities for certain $2 \times 2$ operator matrices II. Linear Algebra and its Applications. 2017 Jun 15; 523:1-2] and Hirzallah O, Kittaneh F, Shebrawi K. [Numerical radius inequalities for $2 \times 2$ operator matrices. Studia Mathematica. 2012; 210:99-115].

1. Introduction

Let $(H, \langle , \rangle)$ be a complex Hilbert space and let $B(H)$ be the Banach algebra of all bounded linear operators from $H$ to $H$ with identity $I$. For $T \in B(H)$, let

$$w(T) = \sup_{||x||=1} |\langle Tx, x \rangle|,$$

$$r(T) = \sup \{ |\lambda| : \lambda \in \sigma(T) \},$$

$$||T|| = \sup_{||x||=1} \langle Tx, Tx \rangle^{\frac{1}{2}},$$

denote the numerical radius, the spectral radius and the usual operator norm respectively.

It is known that the numerical radius and the usual operator norm are equivalent norms on $B(H)$ such that

$$\frac{1}{2} ||T|| \leq w(T) \leq ||T||,$$  \hspace{1cm} (1.1)

for all $T \in B(H)$.

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In [2], Kittaneh provided a refinement to the upper bound of the inequality (1.1) by showing that
\[
\frac{1}{2} \|T\| + \|T^*\| \leq \frac{1}{2} \left( \|T\| + \|T^2\|^{1/2} \right),
\] (1.2)
for all \( T \in B(H) \).

Another improvement for the inequality (1.1) was given by the same author as follows:
\[
\frac{1}{4} \|T^* T + TT^*\| \leq w^2(T) \leq \frac{1}{2} \|T^* T + TT^*\|,
\] (1.3)
for every \( T \in B(H) \).

Precisely, the Numerical radius is not submultiplicative that is \( w(AB) \leq w(A)w(B) \) for all \( A, B \in B(H) \) is not true in general, so many authors are interested to estimate lower and upper bounds for \( w(AB) \) where \( A, B \in B(H) \). For example it is known that \( w(AB) \leq 4w(A)w(B) \); and if \( A, B \) commute, then \( w(AB) \leq 2w(A)w(B) \); also, if \( A, B \) are normal, then \( w(AB) \leq w(A)w(B) \).

Recently, in [3] the author gave a new upper bound for the numerical radius of product of operators, he proved that for \( A, B \in B(H) \) such that \( |A|B = B^*|A| \) and for nonnegative continuous functions \( f \) and \( g \) on \( [0, \infty) \) satisfying \( f(t)g(t) = t \), \( (t \geq 0) \),
\[
w(AB) \leq \frac{1}{2} r(B) \left( |f^2(|A|)| + g^2(|A^*|) \right),
\] (1.4)
where \( |A| = (A^*A)^{1/2} \) denotes the absolute value of \( A \).

Also, he proved if \( p \geq 1, \alpha \geq \beta > 1 \), with \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \) and \( \beta p \geq 2 \), then
\[
w^p(AB) \leq r^p(B) \left| \frac{1}{\alpha} f^{\alpha p}(|A|) + \frac{1}{\beta} g^{\beta p}(|A^*|) \right|,
\] (1.5)
and if \( |A^2|B^2 = (B^2)^*|A^2| \), then
\[
w^{2p}(AB) \leq \frac{1}{2} \left( \|AB\|^{2p} + r^p(B^2) \left| \frac{1}{\alpha} f^{\alpha p}(|A^2|) + \frac{1}{\beta} g^{\beta p}(|(A^2)^*|) \right| \right).
\] (1.6)

On the other hand, many authors are interested to estimate the numerical radius for the operator of matrices. In 2020, Al-Dolat, Jaradat and Al-Husban in [16] showed that if \( A, B, C, D \in B(H) \), then
\[
w \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq \frac{1}{2} \left( w^2(A) + 2w(D) + \sqrt{t^2w(A) + \|B\|^2 + \sqrt{(1-t)^2w^2(A) + \|C\|^2}} \right),
\]
for all \( t \in [0, 1] \).

In 2020, Bani-Domi and Kittaneh proved in [15] if \( A, B, C, D \in B(H) \), then

\[
w^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \max\{w^2(A), w^2(D)\} + w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)
+w\left(\begin{bmatrix} 0 & BD^* \\ CA & 0 \end{bmatrix}\right) + \frac{1}{2} \max\{\lambda, \mu\},
\]

where \( \lambda = ||A||^2 + ||B^*||^2 \) and \( \mu = ||D||^2 + ||C^*||^2 \).

Let \( a, b \geq 0 \). Then we have

- The Power-Mean inequality:

\[
a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq (\alpha a^p + (1-\alpha)b^p)^{\frac{1}{p}}, \tag{1.7}
\]

for all \( \alpha \in [0, 1] \) and \( p \geq 1 \).

- Kittaneh and Manasrah [1] gave a refinement for (1.7) as follows:

\[
a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b - r_0(\sqrt{a} - \sqrt{b})^2, \tag{1.8}
\]

for all \( \alpha \in [0, 1] \) where \( r_0 = \min\{\alpha, 1-\alpha\} \).

- The authors in [4] presented a generalization for (1.8) as follows:

\[
(a^\alpha b^{1-\alpha})^k \leq (\alpha a + (1-\alpha)b)^k - r_0^k(a^\frac{k}{2} - b^\frac{k}{2})^2, \tag{1.9}
\]

for all \( k \in \mathbb{N} \) and \( \alpha \in [0, 1] \) where \( r_0 = \min\{\alpha, 1-\alpha\} \).

- Recently, Choi [5] improved the Power-Mean inequality as follows:

\[
(a^\alpha b^{1-\alpha})^k \leq (\alpha a + (1-\alpha)b)^k - (2r_0)(\frac{(a+b)}{2})^k - (ab)^\frac{k}{2}, \tag{1.10}
\]

for all \( k \in \mathbb{N} \) and \( \alpha \in [0, 1] \) where \( r_0 = \min\{\alpha, 1-\alpha\} \).

- The Power-Young inequality:

\[
ab \leq \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta} \leq \left(\frac{a^p}{\alpha} + \frac{b^p}{\beta}\right)^{\frac{1}{p}}, \tag{1.11}
\]

for all \( \alpha, \beta > 1 \) with \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \) and \( p \geq 1 \).

In this paper, we present some generalizations and refinements for the numerical radius inequalities. Further, new upper bounds for the numerical radius of 2 \( \times \) 2 operator matrices are given.

## 2. The main results

The aim of this section is to establish a generalizations and refinements for the numerical radius inequalities. To do this, we need the following sequence of lemmas. The first lemma is a result of the spectral Theorem together with Jensen’s inequality (see[7]).
**Lemma 2.1.** Let $T \in B(H)$ be a positive operator and let $x \in H$ be any vector. Then

a. $\langle Tx, x \rangle^s \leq ||x||^{2s-2} \langle T^s x, x \rangle$ for $s \geq 1;$

b. $\langle T^s x, x \rangle \leq ||x||^{2-2s} \langle Tx, x \rangle^s$ for $0 < s \leq 1.$

The second lemma gives an upper bound for the spectral radius which was obtained by Kittaneh [6].

**Lemma 2.2.** Let $A, B \in B(H).$ Then

$$r(AB) \leq \frac{1}{4} \left( ||AB|| + ||BA|| + \sqrt{(||AB|| - ||BA||)^2 + 4 \min\{||A||||BAB||, ||B||||ABA||\}} \right).$$

In particular,

$$r(A) \leq \frac{1}{2}(||A|| + ||A^2||^{\frac{1}{2}}).$$

The next lemma is a consequence of the spectral Theorem [7].

**Lemma 2.3.** Let $A, B \in B(H)$ such that $|A|B = B^*|A|.$ If $f, g$ are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t,$ where $t \geq 0,$ then

$$|\langle ABx, y \rangle| \leq r(B) ||f(|A|)x|| ||g(|A^*|)y||,$$

for every vectors $x, y \in H.$

Our first main result is the following improvement of (1.4).

**Theorem 2.4.** Let $A_i, B_i \in B(H)$ ($i = 1, 2, \ldots, n$) such that $|A_i|B_i = B_i^*|A_i|$ and let $f$ and $g$ be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty).$ Then for every $k \in \mathbb{N}$ and $p, q \geq k,$

$$w^n \left( \sum_{i=1}^{n} A_i B_i \right) \leq \frac{n^{p-k/q}}{2^{k/q}} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \left\| \sum_{i=1}^{n} \left( f^{2p/q}(|A_i|) + g^{2p/q}(|A_i^*|) \right) \right\|^{k/q}$$

$$- \inf_{||x||=1} \phi(x)$$

(2.1)

where $\phi(x) = \frac{n^{p-1}}{2^k} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^{n} \left( \langle f^{2p/k}(|A_i|)x, x \rangle^{k/2} - \langle g^{2p/k}(|A_i^*|)x, x \rangle^{k/2} \right)^2.$
Proof. Let $x \in H$ be any unit vector. Then

$$\left| \left\langle \sum_{i=1}^{n} A_{i}B_{i}x, x \right\rangle \right|^p$$

$$\leq \left( \sum_{i=1}^{n} \left| \left\langle A_{i}B_{i}x, x \right\rangle \right| \right)^p$$

$$\leq n^{p-1} \sum_{i=1}^{n} \left| \left\langle A_{i}B_{i}x, x \right\rangle \right|^p$$

(by convexity of $t^p$)

$$\leq n^{p-1} \sum_{i=1}^{n} (r(B_i) \| f(|A_i|)x \| \| g(|A_i^*|)x \|)^p$$

(by Lemma 2.3)

$$\leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^{n} \left( \left\langle f^2(|A_i|)x, x \right\rangle^{p/2} \left\langle g^2(|A_i^*|)x, x \right\rangle^{p/2} \right)$$

$$\leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^{n} \left( \left\langle f^{2p/k}(|A_i|)x, x \right\rangle^{1/2} \left\langle g^{2p/k}(|A_i^*|)x, x \right\rangle^{1/2} \right)^k$$

(by Lemma 2.1)

$$\leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^{n} \left[ \left( \frac{\left\langle f^{2p/k}(|A_i|)x, x \right\rangle^q + \left\langle g^{2p/k}(|A_i^*|)x, x \right\rangle^q}{2} \right)^{k/q} - \frac{1}{2^k} \left( \left\langle f^{2p/k}(|A_i|)x, x \right\rangle^{k/2} - \left\langle g^{2p/k}(|A_i^*|)x, x \right\rangle^{k/2} \right)^2 \right]$$

(by inequalities (1.9) and (1.7))

$$\leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^{n} \left[ \left( \frac{\left\langle f^{2p/k}(|A_i|)x, x \right\rangle + \left\langle g^{2p/k}(|A_i^*|)x, x \right\rangle}{2} \right)^{k/q} - \frac{1}{2^k} \left( \left\langle f^{2p/k}(|A_i|)x, x \right\rangle^{k/2} - \left\langle g^{2p/k}(|A_i^*|)x, x \right\rangle^{k/2} \right)^2 \right]$$

(by Lemma 2.1)

$$\leq n^{p-k/q} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \left( \sum_{i=1}^{n} \left\langle f^{2p/k}(|A_i|)x, x \right\rangle + \left\langle g^{2p/k}(|A_i^*|)x, x \right\rangle \right)^{k/q} x, x$$

$$\leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^{n} \left( \left\langle f^{2p/k}(|A_i|)x, x \right\rangle^{k/2} - \left\langle g^{2p/k}(|A_i^*|)x, x \right\rangle^{k/2} \right)^2$$

(by concavity of $t^{k/q}$).
Thus,

\[
w^p \left( \sum_{i=1}^{n} A_i B_i \right) = \sup \left\{ \left\| \sum_{i=1}^{n} A_i B_i x, x \right\|^p : x \in H, \|x\| = 1 \right\}
\]

\[
\leq \frac{n^{p-k/q}}{2^{k/q}} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \left\| \sum_{i=1}^{n} \left( f^{2pq/k}(|A_i|) + g^{2pq/k}(|A_i^*|) \right) \right\|^{k/q} - \inf_{\|x\|=1} \phi(x).
\]

Choosing \( n = 1 \) in Theorem 2.4 then using Lemma 2.2 we obtain the following corollary.

**Corollary 2.5.** Let \( A, B \in B(H) \) such that \(|A|B = B^*|A|\) and let \( f \) and \( g \) be nonnegative continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t \) for \( t \geq 0 \). Then for \( k \in \mathbb{N} \) and \( p, q \geq k \),

\[
w^p(AB) \leq \frac{r^p(B)}{2^k} \left\| f^{2pq/k}(|A|) + g^{2pq/k}(|A^*|) \right\| - \inf_{\|x\|=1} \phi(x)
\]

\[
\leq \frac{1}{2^{p+k/q}} (\|B\| + \|B^2\|^{1/2}) \left\| f^{2pq/k}(|A|) + g^{2pq/k}(|A^*|) \right\| - \inf_{\|x\|=1} \phi(x),
\]

where \( \phi(x) = \frac{r^p(B)}{2^k} \left( \left\| f^{2pq/k}(|A|)|x, x\rangle \right\|^2 - \left\| g^{2pq/k}(|A^*|)|x, x\rangle \right\|^2 \right)^{1/2} \).

The next result follows from Corollary 2.5 by setting \( p = q = k = 1 \) and \( f(t) = t^\alpha \), \( g(t) = t^{1-\alpha} \) for \( \alpha \in [0, 1] \).

**Corollary 2.6.** Let \( A, B \in B(H) \) such that \(|A|B = B^*|A|\). Then for \( \alpha \in [0, 1] \),

\[
w(AB) \leq \frac{r(B)}{2} \left\| |A|^{2\alpha} + |A^*|^{2(1-\alpha)} \right\| - \inf_{\|x\|=1} \phi(x)
\]

\[
\leq \frac{1}{4} (\|B\| + \|B^2\|^{1/2}) \left\| |A|^{2\alpha} + |A^*|^{2(1-\alpha)} \right\| - \inf_{\|x\|=1} \phi(x),
\]

where \( \phi(x) = \frac{r(B)}{2} \left( \left\| |A|^{\alpha} x, x\rangle \right\|^2 - \left\| |A^*|^{2(1-\alpha)} x, x\rangle \right\|^2 \right)^{1/2} \).

The next result is a simple form follows from Corollary 2.6 by letting \( \alpha = 1/2 \).

**Corollary 2.7.** Let \( A, B \in B(H) \) such that \(|A|B = B^*|A|\). Then

\[
w(AB) \leq \frac{r(B)}{2} \| |A| + |A^*| \| - \inf_{\|x\|=1} \phi(x)
\]

\[
\leq \frac{1}{4} (\|B\| + \|B^2\|^{1/2}) \| |A| + |A^*| \| - \inf_{\|x\|=1} \phi(x),
\]

where \( \phi(x) = \frac{r(B)}{2} \left( \left\| |A| x, x\rangle \right\|^2 - \left\| |A^*| x, x\rangle \right\|^2 \right)^{1/2} \).
The next lemma is a result of Shebrawi [8] that will be used in the proof of Corollary 2.9.

**Lemma 2.8.** Let \( A, B \in B(H) \) and let \( t \in [0, 1] \). Then

\[
\|A + B\| \leq \max(\|A\|, \|B\|) + \frac{1}{2} \left( \|\|A\|B\|^{1-t}\| + \|A^*B\|^{1-t}\| \right).
\]

Using Corollary 2.7, Lemma 2.8 with \( t = \frac{1}{2} \) and the fact \( \|\|A\|B\|^{1/2}\| \leq \|AB\|^{1/2} \) we obtain the following corollary.

**Corollary 2.9.** Let \( A, B \in B(H) \) such that \( \|A\|B\| = B^*\| \). Then

\[
w(AB) \leq \frac{1}{4} \left( \|B\| + \|B^2\|^{1/2} \right) \left( \|A\| + \|A^2\|^{1/2} \right) - \inf_{\|x\|=1} \phi(x),
\]

where \( \phi(x) = \frac{r(B)}{2} \left( \langle |A|x, x \rangle^{1/2} - \langle A^*|x, x \rangle^{1/2} \right)^2 \).

The following theorem gives a generalization for (1.5) which can be stated as follows.

**Theorem 2.10.** Let \( A_i, B_i \in B(H) \) \((i = 1, 2, \ldots, n)\) such that \( |A_i|B_i = B_i^*|A_i| \) and let \( f, g \) be nonnegative continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t, \) \((t \geq 0). \) Then for \( \alpha \geq \beta > 1 \) with \( \frac{1}{\alpha} + \frac{1}{\beta} = 1, \) \( s \geq 1 \) and \( p \geq \max\{1, 2\beta\}, \)

\[
\begin{align*}
w^p \left( \sum_{i=1}^n A_i B_i \right) & \leq n^{p-1/s} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \left( \sum_{i=1}^n \left( \frac{1}{\alpha} f^{ps\alpha}(\|A_i\|) + \frac{1}{\beta} g^{ps\beta}(\|A_i^*\|) \right) \right)^1/s \\
& \leq \frac{n^{p-1/s}}{2^p} \sqrt{\max \left\{ \frac{1}{\alpha}, \frac{1}{\beta} \right\} \left( \max_{1 \leq i \leq n} (\|B_i\| + \|B_i^2\|)^p \right) } \\
& \times \left| \left\| \sum_{i=1}^n \left( f^{ps\alpha}(\|A_i\|) + g^{ps\beta}(\|A_i^*\|) \right) \right\|^{1/s} \right|
\end{align*}
\]

**Proof.** Let \( x \in H \) be any unit vector. Then, we have

\[
\left| \left\langle \sum_{i=1}^n A_i B_i x, x \right\rangle \right|^p \leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^n \left( \langle f^2(|A_i|)x, x \rangle^{ps/2} \langle g^2(|A_i^*|)x, x \rangle^{ps/2} \right)^{1/s} \\
\leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^n \left( \frac{1}{\alpha} f^2(|A_i|)x, x \rangle^{ps\alpha/2} + \frac{1}{\beta} \langle g^2(|A_i^*|)x, x \rangle^{ps\beta/2} \right)^{1/s}
\]

(by inequality 1.11)
\[ \leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^n \left( \frac{1}{\alpha} \langle f^{p\alpha}(A_i) | x, x \rangle + \frac{1}{\beta} \langle g^{p\beta}(A^*_i) | x, x \rangle \right)^{1/s} \]

(by Lemma 2.1)

\[ = n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^n \left( \frac{1}{\alpha} f^{p\alpha}(A_i) + \frac{1}{\beta} g^{p\beta}(A^*_i) \right) x, x \right)^{1/s} \]

\[ \leq n^{p-1/s} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \left( \sum_{i=1}^n \left( \frac{1}{\alpha} f^{p\alpha}(A_i) + \frac{1}{\beta} g^{p\beta}(A^*_i) \right) x, x \right)^{1/s} \]

(by concavity of \( t^{1/s} \)).

Now, the first bound of Theorem 2.10 is obtained by taking the supremum over all unit vectors \( x \in H \). We obtain the second bound by applying Lemma 2.2 on the first inequality. \( \square \)

As a direct consequence of Theorem 2.10 we get the following result which can be considered as a generalization for the first bound of the inequality (1.2)

**Corollary 2.11.** Let \( A \in B(H) \). Then for all \( p, s \geq 1 \),

\[ w^p(A) \leq \left| \frac{|A| |A^*|}{2} \right|^{1/s}. \]

Setting \( p = s = 1 \) in Corollary 2.11 we get the first bound in the inequality (1.2).

On the other hand the next result is obtained by letting \( n = 1 \) in Theorem 2.10.

**Corollary 2.12.** Let \( A, B \in B(H) \) such that \( |A|B = B^*|A| \). If \( f, g \) are nonnegative continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t \ (t \geq 0) \). Then

\[ w^p(AB) \leq r^p(B) \left| \frac{1}{\alpha} f^{p\alpha}(|A|) + \frac{1}{\beta} g^{p\beta}(|A^*|) \right|^{1/2}, \]

where \( \alpha \geq \beta > 1 \) with \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \), \( s \geq 1 \) and \( p \geq \max\{1, \frac{2}{s\beta} \} \).

A general refinement for the second bound of (1.3) will be given in the following theorem.

**Theorem 2.13.** Let \( A_i, B_i \in B(H) \ (i = 1, 2, \ldots, n) \) such that \( |A_i|B_i = B_i^*|A_i| \) and let \( f, g \) be nonnegative continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t \ (t \geq 0) \). Then for \( \alpha \in [0, 1], \ q \geq 1, \ k \in \mathbb{N} \) and \( p \geq k \),

\[ w^p \left( \sum_{i=1}^n A_iB_i \right) \leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^n \left( \left| \| f^{2pq/\alpha k}(|A_i|) + (1 - \alpha) g^{2pq/(1-\alpha)k}(|A^*_i|) \right| \right)^{1/2} \]

\[ - \inf_{|x|=1} \phi(x)^{1/2}, \]
where

\[
\phi(x) = (2 \min \{\alpha, 1 - \alpha\})^k \times \left( \left\langle \frac{f^{2p/\alpha k}(|A_i|)}{2} + g^{2p/(1-\alpha)k}(|A_i^+|) \right\rangle_x, x \right)^k
- \left( \left\langle f^{2p/\alpha k}(|A_i|)x, x \right\rangle \left\langle g^{2p/(1-\alpha)k}(|A_i^+|)x, x \right\rangle \right)^{k/2}.
\]

Proof. Let \( x \in H \) be a unit vector. Then, we have

\[
\left\langle \sum_{i=1}^n A_i B_i x, x \right\rangle^p
\leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^n \left\langle f^2(|A_i|)x, x \right\rangle^{p/2} \left\langle g^2(|A_i^+|)x, x \right\rangle^{p/2}
\leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^n \left[ \left( \left\langle f^{2p/\alpha k}(|A_i|)x, x \right\rangle^{p\alpha/k} \left\langle g^{2/(1-\alpha)}(|A_i^+|)x, x \right\rangle^{p(1-\alpha)/k} \right)^{1/2} \right]
\leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^n \left[ \left( \left\langle f^{2p/\alpha k}(|A_i|)x, x \right\rangle^{\alpha} \left\langle g^{2p/(1-\alpha)k}(|A_i^+|)x, x \right\rangle^{1-\alpha} \right)^{1/2} \right]
\leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^n \left[ \left( \alpha \left\langle f^{2p/\alpha k}(|A_i|)x, x \right\rangle^q + (1 - \alpha) \left\langle g^{2p/(1-\alpha)k}(|A_i^+|)x, x \right\rangle^q \right)^{k/q}
- (2 \min \{\alpha, 1 - \alpha\})^k \left( \left\langle f^{2p/\alpha k}(|A_i|) + g^{2p/(1-\alpha)k}(|A_i^+|) \right\rangle_{x, x} \right)^k
- \left( \left\langle f^{2p/\alpha k}(|A_i|)x, x \right\rangle \left\langle g^{2p/(1-\alpha)k}(|A_i^+|)x, x \right\rangle \right)^{1/2} \right)^{1/2} \right]
\leq n^{p-1} \left( \max_{1 \leq i \leq n} r^p(B_i) \right) \sum_{i=1}^n \left[ \left( \alpha f^{2pq/\alpha k}(|A_i|) + (1 - \alpha) g^{2pq/(1-\alpha)k}(|A_i^+|) \right) x, x \right]^q
- (2 \min \{\alpha, 1 - \alpha\})^k \left( \left\langle f^{2p/\alpha k}(|A_i|) + g^{2p/(1-\alpha)k}(|A_i^+|) \right\rangle_{x, x} \right)^k
- \left( \left\langle f^{2p/\alpha k}(|A_i|)x, x \right\rangle \left\langle g^{2p/(1-\alpha)k}(|A_i^+|)x, x \right\rangle \right)^{1/2} \right]^{1/2} \right]
\]

The desired bound is obtained by taking the supremum over all unit vectors \( x \in H \). \( \square \)

Choosing \( \alpha = \frac{1}{2}, n = p = q = k = 1, B = I \) and \( f(t) = g(t) = t^{1/2} \) in Theorem 2.13 we get the following corollary.
COROLLARY 2.14. Let $A \in B(H)$. Then

$$w^2(A) \leq \frac{1}{2}||A||^2 + ||A^*||^2 - \inf_{||x||=1} \phi(x),$$

where

$$\phi(x) = \left\langle \frac{||A||^2 + ||A^*||^2}{2} x, x \right\rangle - \left( \left\langle |A|^2 x, x \right\rangle \left\langle |A^*|^2 x, x \right\rangle \right)^{1/2}.$$

The next lemma is a result of Dragomir [9] that will be used in the proof of Theorem 2.16.

LEMMA 2.15. Let $x, y, e \in H$ such that $||x|| = 1$. Then

$$|\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} \left( |\langle x, y \rangle| + ||x|| ||y|| \right).$$

Let $A, B \in B(H)$ and let $u \in H$ be unit vector. Then for $e = u$, $x = ABu$ and $y = B^*A^*u$ in Lemma 2.15 we have

$$|\langle ABu, u \rangle|^2 \leq \frac{1}{2} \left( |\langle AB \rangle^2 u, u \rangle| + ||ABu|| ||B^*A^*u|| \right). \quad (2.2)$$

The next result provides a generalization for the inequality (1.6).

THEOREM 2.16. Let $A_i, B_i \in B(H) \ (i = 1, 2, \ldots, n)$ such that $A_iB_i = B_iA_i$ and $|A_i^2| |B_i^2| = (B_i^2)^*|A_i^2|$ and let $f, g$ be nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, $(t \geq 0)$. Then for $\alpha \geq \beta > 1$ with $1/\alpha + 1/\beta = 1$, $s \geq 1$ and $p \geq \max\{1, \frac{2}{s^p} \}$,

$$w^2p \left( \sum_{i=1}^{n} A_iB_i \right) \leq \frac{n^{2p-1}}{2} \sum_{i=1}^{n} ||A_iB_i||^{2p} + \frac{n^{2p-1/s}}{2} \left( \max_{1 \leq i \leq n} r^p(B_i^2) \right)$$

$$\times \left\| \sum_{i=1}^{n} \left( \frac{1}{\alpha} f^{p\alpha\beta}(|A_i^2|) + \frac{1}{\beta} g^{p\beta\alpha}(|A_i^2|^*) \right) \right\|^{1/s}$$

$$\leq \frac{n^{2p-1}}{2} \left( \sum_{i=1}^{n} ||A_iB_i|| + \frac{(||B_i^2|| + ||B_i^2||^{1/2})}{n^{-1+1/s}} \right)$$

$$\times \left\| \sum_{i=1}^{n} \left( \frac{1}{\alpha} f^{p\alpha\beta}(|A_i^2|) + \frac{1}{\beta} g^{p\beta\alpha}(|A_i^2|^*) \right) \right\|^{1/s}.$$ 

Proof. For any unit vector $x \in H$ we have

$$\left\| \sum_{i=1}^{n} A_iB_i x, x \right\|^{2p} \leq n^{2p-1} \sum_{i=1}^{n} |\langle A_iB_i x, x \rangle|^{2p} \quad \text{(by convexity of } t^{2p})$$

$$\leq n^{2p-1} \sum_{i=1}^{n} \left( \left\langle (A_iB_i)^2 x, x \right\rangle + ||A_iB_i|| ||B_i^*A_i^*|| \right)^{p} \quad \text{(by (2.2))}$$
\[ \leq n^{2p-1} \sum_{i=1}^{n} \left( \left( \langle A_i B_i \rangle^2 x, x \right) \right)^{p} \leq \frac{1}{2} \left( \sum_{i=1}^{n} \left| A_i B_i \right|^2 + \left( \max_{1 \leq i \leq n} \frac{r}{p}(B_i^2) \right) \sum_{i=1}^{n} \left( \langle f^2 \left( |A_i|^2 \right), x, x \rangle \right)^{p/2} \right) \]

(by Lemma 2.3)

\[ \leq n^{2p-1} \left( \sum_{i=1}^{n} \left| A_i B_i \right|^2 + \frac{\left( \max_{1 \leq i \leq n} \frac{r}{p}(B_i^2) \right)}{n^{-1} + 1/s} \right) \left( \sum_{i=1}^{n} \frac{1}{n} \left( \frac{1}{\alpha} f^{p\alpha}(|A_i|^2) + \frac{1}{\beta} g^{p\beta}(|(A_i^2)^*|) \right) \right) \]

(by Theorem 2.16)

The last inequality above is obtained by following the same steps of Theorem 2.10. The proof is finish by taking the supremum over all unit vectors \( x \in H \).

**Corollary 2.17.** Let \( A \in B(H) \), and let \( f, g \) be nonnegative continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t, \ (t \geq 0) \). Then for \( \alpha \geq \beta > 1 \) with \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \) and \( p > \max\{\frac{2}{\alpha}, 1\} \),

\[ w^{2p}(A) \leq \frac{1}{2} \left( \left| A \right|^2 + \left( \frac{1}{\alpha} f^{p\alpha}(|A|^2) + \frac{1}{\beta} g^{p\beta}(|(A|^2)^*) \right) \right)^{1/s} \]

The final result in this section is the following refinement of [[10], Theorem 3.3].

**Theorem 2.18.** Let \( A_i, B_i, X_i \in B(H), \ (i = 1, 2, \ldots, n) \) such that \( A_i, B_i \) positive for each \( i = 1, 2, \ldots, n \). Then for \( \alpha \in [0, 1], \ k \in \mathbb{N}, \ q \geq k \) and \( p \geq 2k \),

\[ w^p \left( \sum_{i=1}^{n} A_i^\alpha X_i B_i^{1-\alpha} \right) \leq \left( \max_{1 \leq i \leq n} \left| X_i \right|^p \right) \min\{\lambda, \mu\}, \]

where

\[ \lambda = n^{p-q/k} \left[ \sum_{i=1}^{n} \left( \alpha A_i^{pq/k} + (1 - \alpha) B_i^{pq/k} \right) \right]^{k/q} - \inf_{\left| x \right| = 1} \phi(x), \]

and

\[ \mu = n^{p-q/k} \left[ \sum_{i=1}^{n} \left( \alpha A_i^{pq/k} + (1 - \alpha) B_i^{pq/k} \right) \right]^{k/q} - \inf_{\left| x \right| = 1} \phi(x), \]

where

\[ \phi(x) = (2r_0)^k n^{p-1} \sum_{i=1}^{n} \left( \frac{A_i^{p/k} + B_i^{p/k}}{2} x, x \right)^k - \left( \frac{A_i^{p/k} x, x}{B_i^{p/k} x, x} \right)^{k/2} \]
and
\[ \varphi(x) = n^{p-1} r_0^k \sum_{i=1}^{n} \left( \left| \left\langle A_i^{p/k} x, x \right\rangle \right|^{k/2} - \left| \left\langle B_i^{p/k} x, x \right\rangle \right|^{k/2} \right)^2, \]
where \( r_0 = \min \{ \alpha, 1 - \alpha \} \).

**Proof.** Let \( x \in H \) be any unit vector. Then by the Cauchy-Schwartz inequality, Lemma 2.3, Lemma 2.1 and the inequality (1.10) we have
\[
\left| \sum_{i=1}^{n} A_i^\alpha X_i B_i^{1-\alpha} x, x \right|^p
\leq n^{p-1} \left( \max_{1 \leq i \leq n} \left| A_i^\alpha \right| \right)^p \sum_{i=1}^{n} \left( \left| \left\langle A_i^{p/k} x, x \right\rangle \right|^{p/2k} \left| \left\langle B_i^{p/k} x, x \right\rangle \right|^{p/2k} \right)^k
\leq n^{p-1} \left( \max_{1 \leq i \leq n} \left| X_i \right| \right)^p \sum_{i=1}^{n} \left( \left| \left\langle A_i^{p/k} x, x \right\rangle \right|^{\alpha} \left| \left\langle B_i^{p/k} x, x \right\rangle \right|^{1-\alpha} \right)^k
\leq \left( \max_{1 \leq i \leq n} \left| X_i \right| \right)^p \left[ \left( \sum_{i=1}^{n} \left( \alpha A_i^{p/k} + (1 - \alpha) B_i^{p/k} \right) \right)^{k/q} - (2r_0)^k n^{p-1} \sum_{i=1}^{n} \left( \left| A_i^{p/k} + B_i^{p/k} \right|_{x,x}^k - \left( \left| A_i^{p/k} x, x \right\rangle \right|^{k/2} \right) \right]^{k/q}
\right]
\]
The last inequality above is obtained by follows the same technique of Theorem 2.4 together with the inequalities (1.10) and (1.7). Taking the supremum over all unit vectors \( x \in H \), we get the first bound. Finally, by using (1.9), (1.7) and the same method that gave the first bound we get the second bound. □

**Corollary 2.19.** Let \( A, B, X \in B(H) \) such that \( A \) and \( B \) are positive. Then for \( \alpha \in [0, 1] \), \( k \in \mathbb{N} \) and \( p \geq 2k \),
\[ w^p \left( A^\alpha X B^{1-\alpha} \right) \leq || X ||^p \min \{ \lambda, \mu \}, \]
where
\[ \lambda = \| \alpha A^p + (1 - \alpha) B^p \| - \inf_{\| x \| = 1} \varphi(x) \]
and

\[ \mu = \| \alpha A^p + (1 - \alpha)B^p \| - \inf_{\|x\|=1} \varphi(x), \]

where

\[ \varphi(x) = (2r_0)^k \left( \left\langle \frac{A^{p/k} + B^{p/k}}{2}, x, x \right\rangle - \left( \left\langle \frac{A^{p/k}x}{k}, x \right\rangle \left\langle \frac{B^{p/k}x}{k}, x \right\rangle \right)^{k/2} \right), \]

and

\[ \varphi(x) = r_0^k \left( \left\langle \frac{A^{p/k}x}{k}, x \right\rangle^{k/2} - \left\langle \frac{B^{p/k}x}{k}, x \right\rangle^{k/2} \right)^2, \]

where \( r_0 = \min\{\alpha, 1 - \alpha\} \).

### 3. New upper bounds for \( 2 \times 2 \) operator matrices

In this section we will give new upper bounds for \( 2 \times 2 \) operator matrices. To do this we need some facts about the spectral radius and the numerical radius. The first fact gives some basic properties for the spectral radius.

**Lemma 3.1.** Let \( A, B, C, D \in B(H) \). The following statements hold

a. If \( AB = BA \), then \( r(A + B) \leq r(A) + r(B) \) and \( r(AB) \leq r(A)r(B) \).

b. \( r(A^n) = r^n(A) \) for all \( n \in \mathbb{N} \).

c. \( r\left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq r\left( \begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix} \right) \).

d. \( r\left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) = \sqrt{r(BC)} \).

e. If \( A \) is normal, then \( r(A) = w(A) = \|A\| \).

The second fact gives a useful form for the numerical radius (see [11]).

**Lemma 3.2.** Let \( T \in B(H) \). Then \( w(T) = \max_{\theta \in \mathbb{R}} \| \text{Re}(e^{i\theta}T) \| \).

Also, we need the following fact see [12] and [13] respectively.

**Lemma 3.3.** Let \( A, B, C, D \in B(H) \). Then

a. \( w\left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \geq \max\left\{ w(A), w(B), w\left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \right\} \).

b. \( w\left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) = \frac{1}{2} \max_{\theta \in \mathbb{R}} \| e^{i\theta}B + e^{-i\theta}C^* \| \).

Our first estimate can be stated as follows
**Theorem 3.4.** Let $A, B, C, D \in B(H)$. Then

$$w^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \frac{1}{4} \left( w(A) + \sqrt{w^2(A) + 4w^2(E)} \right)^2$$

$$+ \frac{1}{2} w^2(D) + \frac{1}{2} w(D) \sqrt{w^2(A) + 4w^2(E)},$$

where $E = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$.

**Proof.** Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and let $X = e^{i\theta}A + e^{-i\theta}A^*$, $Z = e^{i\theta}D + e^{-i\theta}D^*$ and $Y = e^{i\theta}B + e^{-i\theta}C^*$ where $\theta \in \mathbb{R}$. Then

$$(2w(T))^2 = \max_{\theta \in \mathbb{R}} \left| 2\text{Re}(e^{i\theta}T) \right|^2$$

$$= \max_{\theta \in \mathbb{R}} \left| e^{i\theta}T + e^{-i\theta}T^* \right|^2$$

$$= \max_{\theta \in \mathbb{R}} \left| (e^{i\theta}T + e^{-i\theta}T^*)^2 \right|$$

$$= \max_{\theta \in \mathbb{R}} \left| TT^* + T^*T + 2\text{Re}(e^{2i\theta}T^2) \right|$$

$$= \max_{\theta \in \mathbb{R}} \left| \begin{bmatrix} X^2 + YY^* & XY + YZ \\ Y^*X + ZY^* & Z^2 + Y^*Y \end{bmatrix} \right|$$

$$\leq \max_{\theta \in \mathbb{R}} \left( r\left( \begin{bmatrix} X & Y \\ Y & 0 \end{bmatrix}^2 \right) + r\left( \begin{bmatrix} 0 & YZ \\ ZY^* & Z^2 \end{bmatrix} \right) \right) \quad \text{(by Lemma 3.1(e))}$$

$$= \max_{\theta \in \mathbb{R}} \left( r^2\left( \begin{bmatrix} X & Y \\ Y & 0 \end{bmatrix}^2 \right) + r\left( \begin{bmatrix} 0 & YZ \\ ZY^* & Z^2 \end{bmatrix} \right) \right) \quad \text{(by Lemma 3.1(b))}$$

$$= \max_{\theta \in \mathbb{R}} \left( r^2\left( \begin{bmatrix} ||X|| & ||Y|| \\ ||Y|| & 0 \end{bmatrix} \right) + r\left( \begin{bmatrix} 0 & ||YZ|| \\ ||YZ|| & ||Z^2|| \end{bmatrix} \right) \right) \quad \text{(by Lemma 3.1(c))}$$

$$= \max_{\theta \in \mathbb{R}} \left( \frac{1}{2} \left( ||X|| + \sqrt{||X||^2 + 4||Y||^2} \right)^2 + \frac{1}{2} \left( ||Z|| + \sqrt{||Z||^2 + 4||YZ||^2} \right) \right)$$

$$= \left( w(A) + \sqrt{w^2(A) + 4w^2(E)} \right)^2 + 2w^2(D) + 2w(D) \sqrt{w^2(D) + 4w^2(E)}.$$

Hence

$$w^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \frac{1}{4} \left( w(A) + \sqrt{w^2(A) + 4w^2(E)} \right)^2$$

$$+ \frac{1}{2} w^2(D) + \frac{1}{2} w(D) \sqrt{w^2(D) + 4w^2(E)}. \quad \square$$
REMARK 3.5. 1. The inequality in Theorem 3.4 is sharper than the upper bound provided in [[14], Theorem 2.6], to see this take \( B = C = D = 0 \) which implies that the inequality in our theorem becomes equality while in [[14], Theorem 2.6] we obtain \( w(A) \leq ||A|| \).

2. In [[15], Theorem 2.2] if we choose \( A = D = 0 \) and \( B = C = I \), we obtain \( w^2 \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq 1.5 \) whereas in Theorem 3.4 we have \( w^2 \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq 1 \).

3. In [[16], Theorem 2.8] take \( A = B = C = D = I \) with \( t = 1 \) to obtain \( w \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq \frac{4+\sqrt{2}}{2} \approx 2.7 \), while in Theorem 3.4 we get \( w \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq \sqrt{2+\sqrt{5}} \approx 2.06. \)

4. In [8, Corollary 3.4] if we choose \( A = D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \) and \( B = C = 0 \), we obtain \( w^2 \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq 16 \) while in Theorem 3.4 we obtain \( w^2 \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq 8. \)

The second estimate which concerns with certain \( 2 \times 2 \) operator matrix will be given in the following theorem.

**Theorem 3.6.** Let \( X, Y \in B(H) \) and suppose \( f, g \) are nonnegative continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t \) \((t \geq 0)\). Then

\[
w \left( \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right) \leq \frac{1}{2} \left( 1 + \sqrt{r(||X||Y||)} \right) \max \{ ||f^2(||X||) + g^2(||Y^*||)||, ||f^2(||Y||) + g^2(||X^*||)|| \}.
\]

Also,

\[
w(X||Y||) \leq \frac{1}{2} \left( 1 + \sqrt{r(||X||Y||)} \right) \max \{ ||f^2(||X||) + g^2(||Y^*||)||, ||f^2(||Y||) + g^2(||X^*||)|| \}.
\]

\[
w(Y||X||) \leq \frac{1}{2} \left( 1 + \sqrt{r(||X||Y||)} \right) \max \{ ||f^2(||X||) + g^2(||Y^*||)||, ||f^2(||Y||) + g^2(||X^*||)|| \}.
\]

**Proof.** Let \( A = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} I & |X| \\ |Y| & I \end{bmatrix} \). Then it is easy to see that \( |A|B = B^*|A| \) and so by Lemma 2.3 we have

\[
|\langle ABx,x \rangle| \leq r(B)||f(||A||)|x||g(||A^*||)|x|| \quad \text{(where } x \in H \oplus H) \]
\[
\leq r(B)\langle f^2(||A||)|x,x\rangle^{1/2}g^2(||A^*||)|x,x\rangle^{1/2} \]
\[
\leq \frac{1}{2}r(B)\langle (f^2(||A||) + g^2(||A^*||))|x,x\rangle.
\]
Thus,
\[
\begin{align*}
  w\left(\begin{bmatrix} X|Y| & X \\ Y & Y|X| \end{bmatrix}\right) &= w(AB) = \sup \left\{ |\langle ABx, x \rangle| : x \in H \oplus H, ||x|| = 1 \right\} \\
  &\leq \frac{1}{2} \left\| f^2(|A|) + g^2(|A^*|) \right\| r\left(\begin{bmatrix} I & |X| \\ |Y| & I \end{bmatrix}\right) \\
  &= \frac{1}{2} \left\| \begin{bmatrix} f^2(|Y|) + g^2(|X^*|) & 0 \\ 0 & f^2(|X|) + g^2(|Y^*|) \end{bmatrix} \right\| r\left(\begin{bmatrix} I & |X| \\ |Y| & I \end{bmatrix}\right) \\
  &= \frac{1}{2} \max \left\{ \left\| f^2(|X|) + g^2(|Y^*|) \right\|, \left\| f^2(|Y|) + g^2(|X^*|) \right\| \right\} \times \left( r\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\right) + r\left(\begin{bmatrix} 0 & |X| \\ |Y| & 0 \end{bmatrix}\right) \right) \\
  &\leq \frac{1}{2} \max \left\{ \left\| f^2(|X|) + g^2(|Y^*|) \right\|, \left\| f^2(|Y|) + g^2(|X^*|) \right\| \right\} \times \left( 1 + \sqrt{r(|X||Y|)} \right) \quad \text{(by Lemma 3.1(a))} \\
  &= \frac{1}{2} \max \left\{ \left\| f^2(|X|) + g^2(|Y^*|) \right\|, \left\| f^2(|Y|) + g^2(|X^*|) \right\| \right\} \times \left( 1 + \sqrt{r(|X||Y|)} \right) \quad \text{(by Lemma 3.1(d))}.
\end{align*}
\]

Using the above inequality and Lemma 3.3(a) we get our bounds. \(\Box\)

By Theorem 3.6 and Lemma 2.8 we get the following result.

**Corollary 3.7.** Let \( X \in B(H) \). Then
\[
  w(X|X|) \leq \frac{1}{2} (1 + ||X|| ||X| + |X^*||) \leq \frac{1}{2} (1 + ||X|| (||X|| + ||X^2||)^{1/2}).
\]

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