On Distance-Regular Graphs with Smallest Eigenvalue at Least $-m$

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Abstract

A non-complete geometric distance-regular graph is the point graph of a partial geometry in which the set of lines is a set of Delsarte cliques. In this paper, we prove that for fixed integer $m \geq 2$, there are only finitely many non-geometric distance-regular graphs with smallest eigenvalue at least $-m$, diameter at least three and intersection number $c_2 \geq 2$.

1 Introduction

In this paper, we will show that for fixed integer $m \geq 2$, there are only finitely many distance-regular graphs with smallest eigenvalue at least $-m$, diameter at least three, intersection number $c_2$ at least two which are not the point graph of a partial geometry. This result generalizes earlier results of R. C. Bose \cite{Bose} and A. Neumaier \cite{Neumaier} for strongly regular graphs, and of C. D. Godsil \cite{Godsil} for antipodal distance-regular graphs with diameter 3 (For definitions, see next section.).

Note that any connected graph has smallest eigenvalue at most $-1$ with equality if and only if the graph is complete. For connected regular graphs with smallest eigenvalue at least $-2$, it was shown by P. J. Cameron et al. \cite{Cameron}, cf. \cite[Theorem 3.12.2]{Godsil}, that either it is a line graph, a cocktail party graph or the number of vertices is at most 28.

To introduce the results of R. C. Bose, A. Neumaier and C. D. Godsil mentioned above, we will first introduce the notion of a Delsarte clique in a distance-regular graph. Recall that a clique in a graph is a set of pairwise adjacent vertices. Let $\Gamma$ be a distance-regular graph with valency $k$, diameter
$D \geq 2$ and the smallest eigenvalue $\theta_D$. Then any clique $C$ in $\Gamma$ contains at most $1 + \frac{k}{-\theta_D}$ vertices. This was shown by P. Delsarte [11] for strongly regular graphs and C. D. Godsil generalized it to distance-regular graphs. A clique $C$ in $\Gamma$ is called a Delsarte clique if $C$ contains exactly $1 + \frac{k}{-\theta_D}$ vertices. It is known that for any clique $C$ in $\Gamma$, the clique is Delsarte if and only if it is a completely regular code with covering radius $D - 1$. Moreover, the outer distribution numbers of a Delsarte clique are completely determined by the intersection numbers of $\Gamma$ and hence do not depend on the specific Delsarte clique, cf. [13 Section 13.7]. Note that for a distance-regular graph which contains a Delsarte clique, its smallest eigenvalue must be integral as $\frac{k}{-\theta_D}$ is integral.

C. D. Godsil [14] introduced the following notion of a geometric distance-regular graph. A non-complete distance-regular graph $\Gamma$ is called geometric if there exists a set of Delsarte cliques $C$ such that each edge of $\Gamma$ lies in a unique $C \in C$. We will also say that $\Gamma$ is geometric with respect to $C$ in this case.

Examples of geometric distance-regular graphs include the Hamming graphs, Johnson graphs, Grassmann graphs, dual polar graphs, bilinear forms graphs and so on (See [6, Chapter 9] for more information on these examples).

Note that a set $C$ of Delsarte cliques for a geometric distance-regular graph does not have to be unique. For example, in the Johnson graph $J(2t, t), t \geq 2$, there are (exactly) two different sets of Delsarte cliques such that $J(2t, t)$ is geometric with respect to either one of them.

The definition of geometric distance-regular graphs for diameter two is equivalent to the notion of geometric strongly regular graphs as was introduced by R. C. Bose [5]. In the last section, we will give more details.

Let $\Gamma$ be a geometric distance-regular graph with valency $k$, diameter $D$ and the smallest eigenvalue $\theta_D$ and assume that $\Gamma$ is geometric with respect to $C$. Then the pair $\mathcal{G} := (V(\Gamma), C)$ is a partial geometry \footnote{A partial geometry of order $(s, t)$ is an incidence structure of points and lines such that each line has $s + 1$ points, each point is on $t + 1$ lines and any two distinct lines meet in at most one point.} where a vertex $x$ is incident with a clique $C$ if and only if $x \in C$) of order $(s, t)$ where $s = \frac{k}{-\theta_D}$ and $t = -\theta_D - 1$. This partial geometry is an example of a distance-regular geometry as defined in F. De Clerck, S. De Winter, E. Kuijken and C. Tonesi [10]. The incidence graph of $\mathcal{G}$ is an example of a distance-semiregular graph as introduced by H. Suzuki [20].

A graph $\Gamma$ is called coconnected when its complement graph (i.e., the graph with vertex set $V(\Gamma)$ whose edges are all the non-edges of $\Gamma$) is connected. Note that the only non-complete distance-regular graphs which are not coconnected are the complete multipartite graphs $K_{t \times n}$ with $t, n \geq 2$ (cf. [6 Lemma 1.1.7]).

In [18], A. Neumaier has shown the following result.

\textbf{Theorem 1.1} (cf. [7 Theorem 4.6])
Fix an integer \( m \geq 2 \). Then, there are only finitely many coconnected non-geometric distance-regular graphs with smallest eigenvalue at least \(-m\) and diameter two.

The next result is a generalization of Theorem 1.1 to any diameter at least two.

**Theorem 1.2** Fix integers \( m \geq 2 \) and \( D \geq 2 \). Then there are only finitely many coconnected non-geometric distance-regular graphs with smallest eigenvalue at least \(-m\) and diameter \( D\).

In the next result, we show that we can replace the condition of a fixed diameter by a condition on the intersection number \( c_2 \).

**Theorem 1.3** Fix an integer \( m \geq 2 \). Then there are only finitely many non-complete coconnected non-geometric distance-regular graphs with smallest eigenvalue at least \(-m\), and intersection number \( c_2 \) at least 2.

On [6, p.130], they asked whether any distance-regular graph with valency \( k \geq 3 \) and diameter \( D \geq 3 \) always has an integral eigenvalue \( \theta \neq k\). Theorem 1.3 gives a partial answer for this problem since the smallest eigenvalue of any geometric distance-regular graph is integral.

In a follow-up paper [3], we show that for fixed \( k \) at least three, there are only finitely many distance-regular graphs with valency \( k \). This implies that the condition \( c_2 \geq 2 \) can be replaced by the condition that the valency is at least three in Theorem 1.3. Note that the odd polygons are non-geometric distance-regular graphs with smallest eigenvalue \( > -2 \).

A. Neumaier [18] also showed that except for a finite number of graphs, all geometric strongly regular graphs with a given smallest eigenvalue are either Latin square graphs or Steiner graphs (cf. [7, Theorem 4.6], We will discuss them in more detail in the last section.) In [23], R. M. Wilson showed that there are super-exponentially many Steiner graphs with parameters \( (v, k, \lambda, \mu) = (v, 3s, s+3, 9) \) for \( v = \frac{(s+1)(2s+3)}{3} \) where \( s \equiv 0 \) or 2 (mod 3) and \( s \geq 6 \). There are super-exponentially many Latin square graphs for certain parameter sets, see [7, p. 210]. This shows that the above-mentioned result of A. Neumaier is the best we can hope for the case of distance-regular graphs of diameter two. We will discuss the situation for geometric distance-regular graphs with diameter at least three in the last section.

The paper is organized as follows. In Section 2, we give the definitions. In Section 3, we give some useful results which we will use in the proofs of the above two theorems. In Section 4, we give some properties of geometric distance-regular graphs. In Section 5, we show, using results of K. Metsch, that the distance-regular graphs with fixed smallest eigenvalue and intersection number \( c_2 \) small compared to \( a_1 \), are geometric. This result in combination with the results in Section 3 implies Theorem 1.2. In Section 6, we study the distance-regular Terwilliger graphs with fixed smallest eigenvalue and \( c_2 \geq 2 \), and give a proof of Theorem 1.3. In the last section we will discuss the geometric distance-regular graphs in more detail and we end the paper with three conjectures.
2 Definitions

All the graphs considered in this paper are finite, undirected and simple (for unexplained terminology and more details, see for example [8]).

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph. We write $x \sim \Gamma y$ or simply $x \sim y$ if two vertices $x$ and $y$ are adjacent in $\Gamma$. For a vertex $x$ of $\Gamma$, let $\Gamma(x) := \{y \mid y \sim x\}$, i.e. the set of neighbours of $x$ and the valency of $x$, denoted by $k(x)$, is the number of neighbours of $x$, $|\Gamma(x)|$. The local graph of a vertex $x$ is the subgraph of $\Gamma$ induced by $\Gamma(x)$.

We say that $\Gamma$ is regular with valency $k$ or $k$-regular if $k(x) = k$ for all vertices $x$ of $\Gamma$. A $k$-regular graph $\Gamma$ on $v$-vertices is called a strongly regular graph with parameters $(v,k,\lambda,\mu)$ if there are constants $\lambda$ and $\mu$ such that for any two distinct vertices $x$ and $y$, the number of common neighbours of $x$ and $y$ equals $\lambda$ if $x \sim y$ and $\mu$ otherwise.

The adjacency matrix $A = A(\Gamma)$ of $\Gamma$ is the $(|V(\Gamma)| \times |V(\Gamma)|)$-matrix whose rows and the columns are indexed by $V(\Gamma)$, and the $(x,y)$-entry of $A$ equals 1 whenever $x \sim y$ and 0 otherwise. The eigenvalues of $\Gamma$ are the eigenvalues of $A$ and real as $A$ is a real symmetric matrix.

For the rest of this section, let $\Gamma$ be a connected graph. The distance $d(x, y)$ between any two vertices $x, y$ of $\Gamma$ is the length of a shortest path between $x$ and $y$ in $\Gamma$. The diameter of $\Gamma$ is the maximal distance occurring in $\Gamma$ and we will denote this by $D = D(\Gamma)$. For a vertex $x \in V(\Gamma)$, define $\Gamma_i(x)$ to be the set of vertices which are at distance $i$ from $x$ ($0 \leq i \leq D$). In addition, define $\Gamma_{-1}(x) := \emptyset$ and $\Gamma_{D+1}(x) := \emptyset$. Note that $\Gamma(x)$ defined above is exactly the same as $\Gamma_1(x)$ and $\Gamma_{D+1}(x)$ is the subgraph of $\Gamma$ induced by $\Gamma(x)$.

A connected graph $\Gamma$ with diameter $D$ is called antipodal if for any vertices $x, y, z$ with $d(x, y) = D = d(y, z)$, either $d(x, z) = D$ or $x = z$ hold.

A connected graph $\Gamma$ with diameter $D$ is called distance-regular if there are integers $b_i, c_i$ ($0 \leq i \leq D$) such that for any two vertices $x, y \in V(\Gamma)$ with $d(x, y) = i$, there are precisely $c_i$ neighbours of $y$ in $\Gamma_{i-1}(x)$ and $b_i$ neighbours of $y$ in $\Gamma_{i+1}(x)$ (cf. [8] p.126). In particular, distance-regular graph $\Gamma$ is regular with valency $k := b_0$ and we define $a_i := k - b_i - c_i$ for notational convenience. The numbers $a_i$, $b_i$ and $c_i$ ($0 \leq i \leq D$) are called the intersection numbers of $\Gamma$. Note that $b_D = c_0 = a_0 = 0$ and $c_1 = 1$. The intersection numbers of a distance-regular graph $\Gamma$ with diameter $D$ and valency $k$ satisfy (cf. [8] Proposition 4.1.6)

(i) $k = b_0 > b_1 \geq \cdots \geq b_{D-1}$;
(ii) $1 = c_1 \leq c_2 \leq \cdots \leq c_D$;
(iii) $b_i \geq c_j$ if $i + j \leq D$.

Moreover, if we fix a vertex $x$ of $\Gamma$ and define $k_i := |\Gamma_i(x)|$, then $k_i$ does not depend on the choice of $x$ as $c_{i+1}k_{i+1} = b_kk_i$ hold for $i = 1, 2, \ldots, D - 1$. Note that the non-complete connected strongly regular graphs are exactly the distance-regular graphs with diameter two.

For the rest of this section, let $\Gamma$ be a distance-regular graph with diameter $D$. Let $C, C' \subseteq V(\Gamma)$ be non-empty subsets and $x$ be a vertex of $\Gamma$. We write $d(x, C) := \min\{d(x, y) \mid y \in C\}$ and $d(C', C) := \min\{d(x, y) \mid x \in C', y \in C\}$. The covering radius of $C$, denoted by $\rho(C)$ is defined as $\rho(C) := \max\{d(x, C) \mid x \in V(\Gamma)\}$, and define $C_i := \{x \in V(\Gamma) \mid d(x, C) = i\}$ ($0 \leq i \leq \rho(C)$). For
$x$ a vertex of $\Gamma$ and $C$ a non-empty subset of $V(\Gamma)$, we write $B_{xi}(C) := |C \cap \Gamma_i(x)|$. The numbers $B_{xi}(C), i = 0, 1, \ldots, D$, are called the outer distribution numbers of $C$.

A non-empty subset $C \subseteq V(\Gamma)$ with covering radius $\rho$, is called a completely regular code, if the outer distribution number $B_{xi}(C)$ only depends on $i$ and $d(x, C)$, that is, there exist numbers $e_{\ell i}$ ($\ell = 0, 1, \ldots, \rho, \ i = 0, 1, \ldots, D$) such that for all vertices $x$ of $\Gamma$ and $i \in \{0, 1, \ldots D\}$, we have $B_{xi}(C) = e_{\ell i}$ where $\ell = d(x, C)$. We refer to the numbers $e_{\ell i}$ as the outer distribution numbers of the completely regular code $C$ and write $\psi_i(C) := e_{ii}$ for $i = 0, 1, \ldots, \rho(C)$.

A partition $\Pi = \{P_1, P_2, \ldots, P_\ell\}$ of $V(\Gamma)$ into non-empty parts, is called equitable if there are constants $\beta_{ij}$ such that each vertex $x \in P_i$ has exactly $\beta_{ij}$ neighbours in $P_j$ ($1 \leq i, j \leq \ell$). The quotient matrix of $\Pi$ is the $(\ell \times \ell)$-matrix $Q = Q(\Pi)$ defined by $Q_{ij} := \beta_{ij}$ for $1 \leq i, j \leq \ell$. An equitable partition $\Pi$ of $V(\Gamma)$ is called a uniformly regular partition if there exist numbers $e_{01}$ and $e_{11}$ such for any $C \in \Pi$ and $x \in V(\Gamma)$, the number $B_{x1}(C)$ is equal to $e_{11}$ if $\ell \in \{0, 1\}$ and zero otherwise, where $\ell = d(x, C)$. Moreover, a uniformly regular partition $\Pi$ of $V(\Gamma)$ is called a completely regular partition if each $C \in \Pi$ is a completely regular code and the outer distribution numbers for all $C \in \Pi$ are the same.

Suppose that $\Gamma$ is a distance-regular graph with valency $k \geq 2$ and diameter $D \geq 2$, and $A = A(\Gamma)$ is the adjacency matrix of $\Gamma$. It is well-known that $\Gamma$ has exactly $D + 1$ distinct eigenvalues, $k = \theta_0 > \theta_1 > \cdots > \theta_D$ ([4] p.128]). For an eigenvalue $\theta$ of $\Gamma$, the sequence $u_i = u_i(\theta)$ ($0 \leq i \leq D$) satisfying

$$u_0 = 1, \quad u_1 = \frac{\theta}{k}, \quad c_iu_{i-1} + a_iu_i + b_iu_{i+1} = \theta u_i$$

is called the standard sequence corresponding to the eigenvalue $\theta$ ([6] p.128]). It is known that the standard sequence corresponding to $\theta_i$ has exactly $i$ sign changes ([6] Corollary 4.1.2]).

## 3 Some Useful Results

In this section, we list a number of results which will be used in this paper.

First, we show that for a connected $k$-regular graph of diameter at least three, it has an eigenvalue $\theta \neq k$ satisfying $|\theta| > \sqrt{\frac{k}{2}}$.

**Lemma 3.1** Suppose that $\Gamma$ is a connected regular graph with valency $k \geq 2$ and diameter $D \geq 3$. Then $\Gamma$ has an eigenvalue $\theta$ different from $k$ satisfying

$$|\theta| > \sqrt{\frac{k}{2}}.$$  

**Proof:** Suppose that $\Gamma$ has $n$ vertices and let $k = \eta_1 > \eta_2 \geq \cdots \geq \eta_n$ be the eigenvalues of $\Gamma$. Let $A$ be the adjacency matrix of $\Gamma$. Then it is well-known (cf. [4] Lemma 2.5]) that

$$\sum_{i=1}^{n} \eta_i^2 = \text{Tr}(A^2) = nk,$$

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where \( \text{Tr}(A^2) \) is the trace of \( A^2 \). Let \( x, y \) be two vertices in \( \Gamma \) at distance 3. Then \( n \geq |\Gamma(x) \cup \{x\} \cup \Gamma(y) \cup \{y\}| = 2k+2 \). It follows that \( \sum_{i=2}^{n} \eta_i^2 = (n-k)k > \frac{n}{2}k \). This immediately implies the lemma.

Next we show two results on distance-regular graphs with smallest eigenvalue at least \(-m\).

**Lemma 3.2** Fix an integer \( m \geq 2 \). Suppose that \( \Gamma \) is a distance-regular graph with smallest eigenvalue at least \(-m\), valency \( k \geq 2 \), diameter \( D \geq 2 \) and intersection number \( a_1 \). Then

\[
k < m(a_1 + m).
\]

**Proof:** Let \( \theta_D \) be the smallest eigenvalue of \( \Gamma \). Since the standard sequence of \( \theta_D \) has exactly \( D \) sign changes, it follows that \( u_2(\theta_D) > 0 \). By (1), we have \( u_2(\theta_D) = \frac{\theta_D^2 - a_1 \theta_D - k}{k(k-a_1-1)} \), and by using \( \theta_D \geq -m \), the lemma follows.

**Theorem 3.3** Fix a real number \( \epsilon, 0 < \epsilon < 1 \), and integers \( m \geq 2 \) and \( D \geq 3 \). Then there are only finitely many distance-regular graphs with smallest eigenvalue at least \(-m\), diameter \( D \) and intersection numbers \( a_1, c_2 \) satisfying \( c_2 \geq \epsilon a_1 \).

**Proof:** For a given real number \( \epsilon, 0 < \epsilon < 1 \) and for given integers \( m \geq 2 \) and \( D \geq 3 \), suppose that \( \Gamma \) is a distance-regular graph with valency \( k \geq 2 \) and diameter \( D \) such that its smallest eigenvalue is at least \(-m\) and its intersection numbers \( a_1, c_2 \) satisfy \( c_2 \geq \epsilon a_1 \). We will show that

\[
k < D^2 \left( \frac{2m^2}{\epsilon} \right)^{2D+4}
\]

holds, from which the theorem immediately follows, as the diameter \( D \) is fixed.

By Lemma 3.2, we have \( k < m(a_1 + m) \) and hence, if \( a_1 = 0 \), then \( k < m^2 \) and we are done. Hence, in order to show Inequality (2), we may assume that \( a_1 \neq 0 \) and \( k \geq 2m^2 \left( \frac{2m^2}{\epsilon} \right)^D D^2 \).

As \( k < m(a_1 + m) \) and \( c_2 \geq \epsilon a_1 \), we find

\[
\frac{b_1}{c_2} < \frac{(m-1)(a_1 + m + 1)}{\epsilon a_1} < \frac{2m^2}{\epsilon}.
\]

As \( b_{i-1} \geq b_i \) and \( c_i \leq c_{i+1} \) hold for all \( i = 1, \ldots, D - 1 \) and using Inequality (3), we obtain

\[
k_{i+1} = \frac{b_i}{c_{i+1}} k_i \leq \frac{b_i}{c_2} k_i < \left( \frac{2m^2}{\epsilon} \right)^i k \quad (i = 1, \ldots, D - 1).
\]

This implies

\[
|V(\Gamma)| = \sum_{i=0}^{D} k_i < \left( \frac{2m^2}{\epsilon} \right)^D Dk.
\]
Now Lemma 3.1 implies that $\theta_1 > \sqrt{\frac{k}{2}}$ as $k > 2m^2$, and thus from (4) we obtain

$$0 = \text{Tr}(A) = \sum_{i=0}^{D} m_i \theta_i > m_1 \theta_1 - m|V(\Gamma)| > \sqrt{\frac{k}{2}m_1 - m \left(\frac{2m^2}{\epsilon}\right)^D} Dk.$$  \hspace{1cm} (5)

By [6, Proposition 4.4.8] and by Inequality (5) with $k \geq 2m^2 \left(\frac{2m^2}{\epsilon}\right)^{2D} D^2$, it follows $2 < m_1 < k$. This in turn implies, by [6, Theorem 4.4.4] that any local graph of $\Gamma$ has an eigenvalue $\eta$, where $\eta := -1 - \frac{b_1}{\theta_1 + 1}$. As any eigenvalue of a subgraph of $\Gamma$ is at least the smallest eigenvalue of $\Gamma$, $\eta \geq -m$ holds. As $b_1 \geq c_2$ holds by $D \geq 3$, it follows that $k < m(a_1 + m) \leq 2m^2 a_1 \leq \frac{2m^2}{\epsilon} c_2 \leq \frac{2m^2}{\epsilon} b_1$ which in turn implies

$$\theta_1 \geq \frac{b_1}{m - 1} - 1 > \frac{ek}{2m^2(m - 1)} - 1 \geq \frac{ek}{2m^3}. \hspace{1cm} (6)$$

Moreover, by Inequalities (4), (5) and (6),

$$m_1 \frac{ek}{2m^3} < m_1 \theta_1 < m \left(\frac{2m^2}{\epsilon}\right)^D Dk,$$

and this implies

$$m_1 < Dm^2 \left(\frac{2m^2}{\epsilon}\right)^{D+1} < D \left(\frac{2m^2}{\epsilon}\right)^{D+2}.$$  

As $2k \leq (m_1 - 1)(m_1 + 2)$, by [6, Theorem 5.3.2], we find $2k \leq (m_1 - 1)(m_1 + 2) < 2D^2 \left(\frac{2m^2}{\epsilon}\right)^{2D+4}$ holds, and this completes the proof.

Now, we consider diameter bounds for distance-regular graphs. A special case of the diameter bound given by A. A. Ivanov ([16], cf. [6, Theorem 5.9.8]) gives that the diameter of a distance-regular graph with valency $k$ and $c_2 \geq 2$ satisfies $D \leq 4^k$. As there are only finitely many connected non-isomorphic $k$-regular graphs with diameter at most $4^k$, we obtain:

**Theorem 3.4** Fix an integer $k \geq 3$. Then there are only finitely many distance-regular graphs with valency $k$ and intersection number $c_2 \geq 2$.

The next diameter bound is due to P. Terwilliger ([22], cf. [6, Corollary 5.2.2]).

**Theorem 3.5** Suppose that $\Gamma$ is a distance-regular graph with valency $k \geq 2$, diameter $D \geq 2$ and intersection numbers $a_1$ and $c_D$. If $\Gamma$ contains an induced quadrangle, then

$$D \leq \frac{k + c_D}{a_1 + 2} \leq \frac{2k}{a_1 + 2}.$$
Proposition 3.6  Fix an integer $m \geq 2$. Suppose that $\Gamma$ is a distance-regular graph with smallest eigenvalue at least $-m$, valency $k \geq 2$, diameter $D \geq 2$ and intersection number $a_1$. If $\Gamma$ contains an induced quadrangle, then

$$D < \frac{2m(a_1 + m)}{a_1 + 2} \leq m^2.$$ 

Proof: The result follows immediately from Lemma 3.2 and Theorem 3.5.

4 Properties of Geometric Distance-Regular Graphs 

In this section, we will give some properties of geometric distance-regular graphs, which we will need later in this paper.

First, we recall the notion of a Delsarte pair as was introduced by S. Bang, A. Hiraki and J. Koolen [2]. Suppose that $\Gamma$ is a non-complete distance-regular graph with valency $k$, diameter $D$ and the smallest eigenvalue $\theta_D$, and that $\mathcal{C}$ is a set of Delsarte cliques in $\Gamma$. Then the pair $(\Gamma, \mathcal{C})$ is called a Delsarte pair with parameters $(k, \frac{k}{\theta_D}, n_{\mathcal{C}})$, if every edge lies in exactly $n_{\mathcal{C}}$ cliques of $\mathcal{C}$, where $n_{\mathcal{C}} \geq 1$ (cf. [2, Definition 1.1]).

Suppose that $\Gamma$ is a geometric distance-regular graph with respect to $\mathcal{C}$, that is, $(\Gamma, \mathcal{C})$ is a Delsarte pair with $n_{\mathcal{C}} = 1$. Recall that for a Delsarte clique $C$, the numbers $\psi_i(C)$ for $i = 1, 2, \ldots, D$ do only depend on the intersection numbers of $\Gamma$. Hence, we can write $\psi_i := \psi_i(C)$ for $C \in \mathcal{C}$. In [2, Lemma 4.1, Proposition 4.2 (i)], we showed that for any fixed integer $j$ , $1 \leq j \leq D$ and for any vertices $x, y$ at distance $j$, the number of cliques $C \in \mathcal{C}$ that contain $y$ and satisfy $d(x, C) = j - 1$ is dependent only on $j$ and $n_{\mathcal{C}}$ (and not on the particular pair of $x, y$ at distance $j$) and we denote this number by $\tau_j := \tau_j(C)$. Note that the numbers $\psi_i$ and $\tau_j$ do not depend on the particular set of Delsarte cliques.

Therefore the next lemma is a direct consequence of [2, Proposition 4.2 (i)] with $n_{\mathcal{C}} = 1$.

Lemma 4.1  Suppose that $\Gamma$ is a geometric distance-regular graph with valency $k \geq 2$, diameter $D \geq 2$ and smallest eigenvalue $\theta_D$. Then the following hold :

$$b_i = -(\theta_D + \tau_i) \left(1 + \frac{k}{\theta_D} - \psi_i\right) \quad (1 \leq i \leq D - 1),$$

$$c_i = \tau_i \psi_{i-1} \quad (1 \leq i \leq D).$$

A distance-regular graph does not need to be geometric if its intersection numbers satisfy the equations in Lemma 4.1 for some non-negative integers $\tau_i, \psi_{i-1}$ ($i = 1, 2, \ldots, D$). For example, the Doob graph $D(s, c)$ (the direct product of $s$ Shrikhande graphs and $c$ 4-cliques with $s \geq 1$) is non-geometric and it has the same intersection numbers with the Hamming graph $H(2s + c, 4)$, which is geometric.

Part (ii) of the next lemma is a direct consequence of [2, Lemma 5.1 (iii)], but we include its proof for completeness.
Lemma 4.2 Suppose that \( \Gamma \) is a geometric distance-regular graph with diameter \( D \geq 2 \) and intersection number \( c_2 \) at least two. Then the following holds:

(i) \( \tau_2 \geq \psi_1 \).

(ii) \( \Gamma \) contains an induced quadrangle.

Proof: Let \( C \) be a set of Delsarte cliques with respect of which \( \Gamma \) is geometric. Let \( x, y \) be any two vertices of \( \Gamma \) at distance 2. Let \( z \) be a common neighbour of \( x \) and \( y \), and let \( C \) be the unique clique in \( C \) containing \( x \) and \( z \). Put \( \Gamma(y) \cap C := \{z_1, \ldots, z_{\psi_1}\} \). Then each edge \( \{y, z_i\} \) lies in a unique clique \( C(i) \in C, 1 \leq i \leq \psi_1 \).

(i): As \( C(i) \neq C(j) \) holds for any \( i \neq j \), the result (i) follows.

(ii): If \( \psi_1 = 1 \), then the local graph of any vertex is disjoint union of cliques, and the result holds as \( c_2 \geq 2 \). Now, suppose \( \psi_1 \geq 2 \). Then there exists a vertex \( w \in C(\psi_1) \cap \Gamma(x) \) such that \( w \neq z_1 \) holds, by the definition of \( \psi_1 \) and \( z_1 \sim y \). Hence the subgraph induced on \( x, y, z_1, w \) is a quadrangle. This shows (ii).

By applying Lemma 4.2 (ii) to Proposition 3.6, we obtain the following diameter bound for geometric distance-regular graphs.

Proposition 4.3 Fix an integer \( m \geq 2 \). Suppose that \( \Gamma \) is a geometric distance-regular graph with smallest eigenvalue \(-m\), diameter \( D \geq 2 \) and intersection number \( c_2 \). If \( c_2 \geq 2 \) holds, then \( D < m^2 \).

5 Distance-Regular Graphs with Small \( c_2 \)

In this section, we will show that any distance-regular graphs with intersection number \( c_2 \) much smaller than \( a_1 \) are geometric. Also we will show Theorem 1.2.

To show any distance-regular graphs with intersection number \( c_2 \) much smaller than \( a_1 \) are geometric, we first state the following result of K. Metsch.

Proposition 5.1 ([17, Result 2.1]) Let \( k \geq 2, \mu \geq 1, \lambda \geq 0, s \geq 1 \) be integers. Suppose that \( \Gamma \) is a \( k \)-regular graph such that any two non-adjacent vertices have at most \( \mu \) common neighbours, and any two adjacent vertices have exactly \( \lambda \) common neighbours. Define a line to be a maximal clique \( C \) in \( \Gamma \) such that \( C \) has at least \( \lambda + 2 - (s - 1)(\mu - 1) \) vertices. If

(i) \( \lambda > (2s - 1)(\mu - 1) - 1 \) and

(ii) \( k < (s + 1)(\lambda + 1) - \frac{1}{2} s(s + 1)(\mu - 1) \)

all hold, then every vertex is in at most \( s \) lines, and each edge lies in a unique line.

As a consequence of this result, we obtain the following result on distance-regular graphs.
Proposition 5.2 Fix an integer \( m \geq 2 \). Suppose that \( \Gamma \) is a distance-regular graph with smallest eigenvalue at least \(-m\), valency \( k \geq 3 \), diameter \( D \geq 2 \) and intersection numbers \( a_1 \) and \( c_2 \). We define a line to be a maximal clique \( C \) in \( \Gamma \) such that \( C \) has at least \( a_1 + 2 - (m - 1)(c_2 - 1) \) vertices. If \( a_1 > m^2 c_2 \), then every vertex is in at most \( m \) lines, and each edge lies in a unique line.

Proof: Suppose that \( \Gamma \) is a distance-regular graph with smallest eigenvalue at least \(-m\), valency \( k \geq 3 \), diameter \( D \geq 2 \) satisfying \( a_1 > m^2 c_2 \). By Proposition 5.1, it is enough to show that the following inequalities all hold:

\[
a_1 > (2m - 1)(c_2 - 1) - 1, \tag{7}
\]
\[
k < (m + 1)(a_1 + 1) - \frac{1}{2} m(m + 1)(c_2 - 1). \tag{8}
\]

Inequality (7) follows immediately from \( a_1 > m^2 c_2 \). To see that Inequality (8) holds, note that

\[
m(a_1 + m) \leq (m + 1)(a_1 + 1) - \frac{1}{2} m(m + 1)(c_2 - 1)
\]

holds, from which Inequality (8) follows by Lemma 3.2.

The following theorem is the main result of this section.

Theorem 5.3 Fix an integer \( m \geq 2 \). Suppose that \( \Gamma \) is a distance-regular graph with diameter \( D \geq 2 \) such that its smallest eigenvalue \( \theta_D \) satisfies \(-m \leq \theta_D < 1 - m \). If intersection numbers \( a_1 \) and \( c_2 \) satisfy \( a_1 > m^2 c_2 \), then \( \Gamma \) is geometric (and \( \theta_D = -m \)).

Proof: Suppose that \( \Gamma \) is a distance-regular graph with diameter \( D \geq 2 \) such that its smallest eigenvalue \( \theta_D \) satisfies \(-m \leq \theta_D < 1 - m \), satisfying \( a_1 > m^2 c_2 \). Define lines as in Proposition 5.2 and for each vertex \( x \in V(\Gamma) \), let \( M_x \) be the number of lines containing \( x \). By Proposition 5.2, we have \( M_x \leq m \). On the other hand, as any maximal clique \( C \) satisfies \( |C| \leq 1 + \frac{k}{\theta_D} < 1 + \frac{k}{m-1} \),

\[
M_x > \frac{k}{1 + \frac{k}{m-1}} = m - 1
\]

follows as every edge lies in a unique line, by Proposition 5.2. Hence, \( M_x = m \) for all \( x \).

Let \( B \) be the vertex-line incidence matrix (i.e. the \((0,1)\)-matrix whose rows and columns are indexed by the vertex set and the set of lines of \( \Gamma \), respectively where \((x,C)\)-entry of \( B \) is 1 if the vertex \( x \) is contained in the line \( C \) and 0 otherwise). Then \( BB^T = A(\Gamma) + mI \) holds, where \( B^T \) is the transpose of \( B \) and \( I \) is the identity matrix. Since each line contains more than \( m \) vertices, by \( a_1 + 2 - (m - 1)(c_2 - 1) > mc_2 \geq m \), the matrix \( BB^T \) is singular. This implies that 0 is an eigenvalue of \( BB^T \) and thus \(-m\) is an eigenvalue of \( A \). As \( \theta_D \geq -m \), \( \theta_D = -m \). But then, every line has exactly \( 1 + \frac{k}{m} \) vertices as any maximal clique has cardinality at most \( 1 + \frac{k}{\theta_D} = 1 + \frac{k}{m} \), \( M_x = m \) for all \( x \) and each vertex lies in a unique line. This proves that \( \Gamma \) is geometric with \( \theta_D = -m \).
Proof of Theorem 1.2: For given integers \( m \geq 2 \) and \( D \geq 2 \), let \( \Gamma \) be a non-geometric distance-regular graph with diameter \( D \) and smallest eigenvalue at least \(-m\). By Theorem 5.3, the intersection numbers \( a_1 \) and \( c_2 \) of \( \Gamma \) satisfy \( c_2 \geq \frac{1}{m^2}a_1 \). The result now immediately follows from Theorem 3.3 and Theorem 1.1.

6 Terwilliger Graphs and Proof of Theorem 1.3

In this section, we will show Theorem 1.3. Before we do this, we first need to consider Terwilliger graphs which are distance-regular (i.e., distance-regular graphs without induced quadrangles). In [6, p.36, Problems (ii)], it was asked whether it is possible to classify the distance-regular Terwilliger graphs with intersection number \( c_2 \geq 2 \). We will show in Theorem 6.2 that for a fixed integer \( m \geq 2 \), there are only finitely many distance-regular Terwilliger graphs with \( c_2 \geq 2 \) and smallest eigenvalue at least \(-m\).

For an integer \( \alpha \geq 1 \), a \( \alpha \)-clique extension of a graph \( \Gamma \) is the graph \( \Sigma \) obtained from \( \Gamma \) by replacing each vertex \( x \in V(\Gamma) \) by a clique \( C(x) \) of \( \alpha \) vertices, where for any \( x, y \in V(\Gamma) \), \( x' \in C(x) \) and \( y' \in C(y) \), \( x \sim_\Gamma y \) if and only if \( x' \sim_\Sigma y' \).

Proposition 6.1 Suppose that \( \Gamma \) is a graph and that \( \Sigma \) is the \( \alpha \)-clique extension of \( \Gamma \) for an integer \( \alpha \geq 1 \). Then for each eigenvalue \( \theta \) of \( \Gamma \), \((\alpha(\theta + 1) - 1)\) is an eigenvalue of \( \Sigma \).

Proof: Let \( \Pi \) be the set of all \( \alpha \)-cliques of \( \Sigma \) which correspond to the vertices of \( \Gamma \), respectively. Then \( \Pi \) is a uniformly regular partition of \( V(\Sigma) \) and the quotient matrix \( Q(\Pi) \) of \( \Pi \) satisfies \( Q(\Pi) = \alpha A(\Gamma) + (\alpha - 1)I \). Hence if \( \theta \) is an eigenvalue of \( \Gamma \), then \((\alpha(\theta + 1) - 1)\) is an eigenvalue of \( Q(\Pi) \). By [13, Lemma 5.2.2], \((\alpha(\theta + 1) - 1)\) is also an eigenvalue of \( \Sigma \).

A graph \( \Gamma \) with diameter at least 2 is called a Terwilliger graph if for any two vertices \( x, y \) with \( d(x, y) = 2 \), the set \( \Gamma(x) \cap \Gamma(y) \) induces a clique. In particular, any distance-regular graphs with intersection number \( c_2 = 1 \) are distance-regular Terwilliger graphs.

Theorem 6.2 Fix an integer \( m \geq 2 \). Then there are only finitely many distance-regular Terwilliger graphs with smallest eigenvalue at least \(-m\) and intersection number \( c_2 \geq 2 \).

Proof: For given integer \( m \geq 2 \), suppose that \( \Gamma \) is a distance-regular Terwilliger graph with \( c_2 \geq 2 \), valency \( k \) and smallest eigenvalue \( \theta \geq -m \). By [6, Theorem 1.16.3], for any vertex \( x \in V(\Gamma) \), the local graph \( \Sigma \) of \( x \) is the \( \alpha \)-clique extension of a non-complete connected strongly regular Terwilliger graph \( \Delta \) for an integer \( \alpha \geq 1 \). Let \( \eta \) be the smallest eigenvalue of \( \Delta \). Then by Proposition 6.1, \( \alpha(\eta + 1) - 1 \) is an eigenvalue of \( \Sigma \) and hence \( \alpha(\eta + 1) - 1 \geq -m \) as \( \Sigma \) is a subgraph of \( \Gamma \) and \( \Gamma \).
has smallest eigenvalue at least \(-m\). As every connected non-complete strongly regular graph has smallest eigenvalue at most \(-\frac{1-\sqrt{5}}{2}\), we obtain

\[
\frac{1 - m - \alpha}{\alpha} \leq \eta \leq \frac{-1 - \sqrt{5}}{2},
\]

which immediately gives us

\[
\alpha \leq \frac{2(m-1)}{\sqrt{5}-1}.
\] (9)

By Lemma 4.1 any geometric strongly regular graph satisfies \(c_2 \geq 2\) and thus it is not a Terwilliger graph by Lemma 4.2 (ii). Hence, it follows by Theorem 1.1 that there are only finitely many non-complete strongly regular Terwilliger graphs with smallest eigenvalue at least \(-m\). Hence, the valency \(k\) is bounded above by a function of \(m\) by

\[
k \leq |\Sigma| = |\Delta| = \alpha |V(\Delta)| = \alpha \alpha \leq 2(m-1)/\sqrt{5}-1.
\]

Theorem now immediately follows from Theorem 3.4.

Proof of Theorem 1.3: For given integer \(m \geq 2\), suppose that \(\Gamma\) is a coconnected non-geometric distance-regular graph with diameter \(D \geq 2\), smallest eigenvalue at least \(-m\) and intersection number \(c_2\) satisfying \(c_2 \geq 2\). If \(\Gamma\) is a Terwilliger graph, then we are done by Theorem 6.2. Now we may assume that \(\Gamma\) contains an induced quadrangle. Proposition 3.6 implies that \(D < m^2\), from which the theorem follows by Theorem 1.2.

\[\square\]

7 Geometric Distance-Regular Graphs

In this section, we will discuss geometric distance-regular graphs in more detail.

Let us first return to the strongly regular graph case.

Recall that a partial geometry of order \((s, t, \alpha)\) is a partial geometry \(\mathcal{G} = (P, L)\) of order \((s, t)\) such that for each \(p \in P\) and \(L \in L\) satisfying \(p \notin L\), there are exactly \(\alpha\) lines through \(p\) that meet \(L\). For any partial geometry \(\mathcal{G} = (P, L)\) of order \((s, t, \alpha)\) with \(t \geq 1\), the point graph of \(\mathcal{G}\) (i.e., its vertex set is \(P\), and two vertices are adjacent if they lie on a common line) is a strongly regular graph with intersection numbers \(k = s(t+1), b_1 = (s-\alpha+1)t, c_2 = \alpha(t+1)\) and distinct eigenvalues \(k, s-1, -t-1\), see for example [4, Additional result 20f on p.162]. As Delsarte cliques in these graphs have size \(s+1\), it follows that each line (each line is considered as the set of points which lie on it) forms a Delsarte clique and hence the point graph of a partial geometry of order \((s, t, \alpha)\) with \(t \geq 1\) is geometric. On the other hand, it is easy to see that a geometric distance-regular graph of diameter two, is the point graph of a partial geometry of order \((s, t, \alpha)\) for some \(s, t, \alpha\). So these two notions are equivalent for distance-regular graphs of diameter two.

As we already mentioned in Section 1, A. Neumaier showed that for fixed integer \(m \geq 2\), except for a finite number of graphs, all geometric strongly regular graphs with a given smallest eigenvalue are either Latin square graphs or Steiner graphs. Now the Latin square graphs (the Steiner graphs, respectively) arrive as the point graph of a partial geometry of order \((s, t, \alpha)\) with \(\alpha\) equals to \(t+1\) (\(t\), respectively), see for example [7, Section 4].

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As it follows from the above discussion that the point graph of a partial geometry of order \((s, t, \alpha)\) is geometric with \(\psi_1 = \alpha, \tau_2 = t + 1\) and smallest eigenvalue \(-t - 1\), A. Neumaier’s result implies that geometric strongly regular graphs with fixed smallest eigenvalue \(-t - 1 \leq -2\) satisfy \(\psi_1 \in \{\tau_2, \tau_2 - 1\}\), except for a finite number of cases.

Now, we return to geometric distance-regular graphs with diameter at least three. In Lemma 4.2 (i), we have shown that \(c_2 \geq 2\) implies \(\psi_1 \leq \tau_2\). For \(c_2 = 1\), we have \(\psi_1 = \tau_2 = 1\).

In the following result, we characterize geometric distance-regular graphs with diameter at least three satisfying \(\psi_1 = \tau_2 \geq 2\). Note that the situation here is completely different from the case of geometric strongly regular where such a characterization is impossible.

**Theorem 7.1** Fix an integer \(m \geq 2\). Suppose that \(\Gamma\) is a geometric distance-regular graph with smallest eigenvalue \(-m\) and diameter at least three. If \(\psi_1 = \tau_2 \geq 2\), then one of the following holds.

(i) \(\psi_1 = 2\) and \(\Gamma\) is a Johnson graph.

(ii) \(\psi_1 = 2\) and \(\Gamma\) is the folded Johnson graph \(\overline{J}(4D, 2D)\) where \(D \geq 3\).

(iii) \(\psi_1 \geq 3\) and \(\Gamma\) is a Grassmann graph defined over the field \(\mathbb{F}_{\psi_1 - 1}\).

(iv) \(\psi_1 \geq 3\) and \(k < \psi_1(\psi_1 - 1)m < m^3\).

**Proof:** (i)-(ii): Assume that \(\tau_2 = \psi_1 = 2\). Then \(c_2 = 4\) and the subgraph induced by the common neighbours of two vertices at distance 2 forms a quadrangle. Now, by [6, Theorem 9.1.3], the graph \(\Gamma\) is either a Johnson graph or a quotient \(J(2s, s)/\Pi\) for some integer \(s \geq 1\), where \(\Pi\) is a uniformly regular partition of \(J(2s, s)\) and each part of \(\Pi\) has size 2. By [6, Theorem 11.1.6], the partition \(\Pi\) must be a completely regular partition of \(J(2s, s)\), and hence each part is a completely regular code of size 2 in \(J(2s, s)\). If \(C = \{x, y\}\) is a completely regular code of size 2, then either \(d(x, y) = 1\) or \(d(x, y) = s\) holds. The case \(d(x, y) = 1\) only occurs if either \(s = 1\) or \(a_1 = 0\). Hence the only completely regular codes of size 2 in the Johnson graph \(J(2s, s)\) are the antipodal pairs (i.e., \(d(x, y) = s\)), and this shows that if \(\Gamma\) is not a Johnson graph then it has to be a folded Johnson graph. For a folded Johnson graph \(\overline{J}(2s, s)\), it is geometric with \(c_2 = 4\) if and only if \(s\) is even and \(s \geq 6\) (cf. [2, Section 3]). This proves (i)-(ii).

(iii)-(iv): Suppose that \(\tau_2 = \psi_1 \geq 3\) holds. Then by applying [6, Theorem 9.3.9] (a result of D. K. Ray-Chaudhuri and A. P. Sprague [19]) with \(q := m - 1\), we find that \(\Gamma\) is a Grassmann graph or \(|C| \leq (m - 1)^2 + m - 1\) for any Delsarte clique \(C\). This shows that one of (iii) and (iv) holds.

The following corollary follows from Proposition 4.3 and Theorem 7.1.

**Corollary 7.2** Fix an integer \(m \geq 2\). Suppose that \(\Gamma\) is a geometric distance-regular graph with smallest eigenvalue \(-m\) and diameter \(D \geq 3\). If \(\psi_1 = \tau_2 \geq 2\) holds, then the graph \(\Gamma\) is either a Johnson graph, a folded Johnson graph, a Grassmann graph or the number of vertices is bounded above by a function of \(m\).

Note that the Hamming graph \(H(e, D)\) and the bilinear forms graph \(H_q(n, D)\) are geometric distance-regular graphs with \(\psi_1 = 1 = \tau_2 - 1\) and \(\psi_1 = q = \tau_2 - 1\), respectively. In [12], Y.
Egawa showed that the Hamming graphs are characterized by its intersection numbers, and K. Metsch [17] showed that the bilinear forms graph $H_q(n, D)$ is characterized by its intersection numbers if $n \geq D + 4 \geq 7$, by generalizing results of T. Y. Huang [15] and H. Cuypers [9]. As far as the authors know, there is no local geometric characterization known (as in the case of the Johnson and Grassmann graphs) of neither the bilinear forms graphs nor the Hamming graphs.

We close this section with three conjectures concerning geometric distance-regular graphs.

**Conjecture 7.3** For fixed integer $m \geq 2$, there are only finitely many coconnected geometric distance-regular graphs with smallest eigenvalue $-m$ and $\psi_1 \leq \tau_2 - 2$.

**Conjecture 7.4** For a fixed integer $m \geq 2$, any geometric distance-regular graph with smallest eigenvalue $-m$, diameter $D \geq 3$ and $c_2 \geq 2$ is either a Johnson graph, a Grassmann graph, a Hamming graph, a bilinear forms graph or the number of vertices is bounded above by a function of $m$.

**Conjecture 7.5** For a fixed integer $m \geq 2$, the diameter of a geometric distance-regular graph with smallest eigenvalue $-m$ and valency at least three is bounded above by a function of $m$.

**Remark 7.6** (i) Conjecture 7.4 is shown for $\psi_1 = \tau_2 \geq 2$ in Corollary 7.2. It is known that it is true for $m = 2$, but in the case $c_2 = 1$ we also have the even polygons. This follows from the result of Cameron et al. for regular graphs with the smallest eigenvalue $-2$, as mentioned in the introduction, and the classification of distance-regular line-graphs, see [6, Theorem 4.2.16]. The second author [1] has shown it for $m = -3$.

(ii) Conjecture 7.5 is shown for $c_2 \geq 2$ in Proposition 4.3. In [24, Problem 3.1.1], H. Suzuki asks a similar question.

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