The investigation of low frequency dilaton generation

V.I. Denisov\textsuperscript{a}, I.P. Denisova\textsuperscript{b}, E.T. Einiev\textsuperscript{c}

\textsuperscript{a} Department of Physics, Moscow State University, 119991, Moscow, Russia
\textsuperscript{b} Moscow Aviation Institute (National Research University), 125993, Moscow, Volokolamskoe Highway 4, Russia

Abstract
The electromagnetic source of dilaton is a first invariant of the electromagnetic field tensor. For electromagnetic waves, this invariant can be non zero only in the near zone. Pulsars and magnetars are natural sources of this type. We calculated the generation of dilatons by coherent electromagnetic field of rotating magnetic dipole moment of pulsars and magnetars. It is shown that the radiation of dilaton waves occurs at the two frequencies: the rotation frequency \( \omega \) of the magnetic dipole moment of the neutron star and the twice frequency.

The generation of dilaton at frequency \( \omega \) is maximal in the case when the angle between the magnetic dipole moment and the axis of its rotation is \( \pi/4 \). If this angle is \( \pi/2 \), then dilaton radiation at frequency \( \omega \) does not occur. The generation of dilaton at frequency \( 2\omega \) is maximal in the case when the angle between the magnetic dipole and the axis of its rotation is \( \pi/2 \).

Angular distribution of radiation of dilatons having a frequency of \( \omega \), has a maximum along conic surfaces \( \theta = \pi/4 \). Angular distribution of radiation of dilatons having a frequency of \( 2\omega \), has a maximum in the plane which is perpendicular to the axis of rotation (\( \theta = \pi/2 \)).

1 Introduction
In theoretical physics, an interest to the theory of dilatons \cite{1,2} has increased again. In these works, the manifestations of the dilaton field in astrophysical and laboratory conditions are theoretically studied.

In particular, the influence of the dilaton scalar field on the properties of strange quark stars and quantum deformed Schwarzschild black holes was studied in \cite{3,4}.

In addition, new ideas about the diversification of the theory in the field of modern physical experiment have appeared. It has led to a necessity of carrying on an extra analytical research of dilatons behavior in a significantly non-liner sector \cite{5,6,7,8} and using fully nonlinear, numerical investigation \cite{9,10} as well as at high energies available for LHC experiments.

Particularly, in paper \cite{11} authors have performed a detailed study of dilatons phenomenology in a composite twin Higgs model; the authors of article \cite{12} explore the possibility that a light dilaton can be the first sign of new physics at the LHC.

However, there are such processes of dilatons generation that are available and at low energies. This paper is devoted to one of these processes.

The density of the Lagrange function in Maxwell-dilaton theory, following the work \cite{9}, we write in the form:

\[
L = a_0 (\partial \Psi)^2 + a_1 e^{-2K\Psi} F_{nm} F_{nm}, \tag{1}
\]

where \( \Psi \) is the scalar field of the dilaton, \( F_{nm} \) is the electromagnetic field tensor, \( a_0, a_1 \) and \( K \) is the coupling constants.

It should be noted that recently the concept of dilaton has been expanded. In effective field theory \cite{11,12,13,14}, a dilaton is a scalar field, whose Lagrangian density, in the particular case of interaction with an electromagnetic field, coincides with the Lagrangian density \eqref{1}, and has no relation to the multidimensional theory of gravity. Therefore, in this paper we will use the Lagrangian density \eqref{1} without discussing the physical nature of the dilaton.

In pseudo-Euclidean space-time, the field equations obtained from the density of the Lagrange function \eqref{1}
\[ \mathbf{\nabla} \psi = \frac{a_2 k}{a_0} e^{-2 \kappa \psi} F_{nn} F^{nm} = \frac{2 a_1 k}{a_0} e^{-2 \kappa \psi} [B^2 - E^2], \]  

where \( B \) and \( E \) are the induction of the magnetic field and the intensity of the electric field, that create the dilaton field.

The dynamics of the Maxwell-dilaton theory (2) in Minkowski spacetime we will only consider in the weak dilaton field approximation: \( |K \psi| \ll 1 \). In this case, the equation for the dilaton field will take the form:

\[ \mathbf{\nabla} \psi = \frac{2 a_1 k}{a_0} [B^2 - E^2]. \]  

Thus, for \( |K \psi| \ll 1 \), the source of dilaton radiation is only the first invariant of the electromagnetic field tensor \( F_{\mu \nu} \). Since this invariant is zero in the wave zone of any electromagnetic waves, the noticeable radiation of dilatons is possible only from the near zone of electromagnetic waves, where this invariant is not zero. Therefore, the effective electromagnetic generators of dilatons are coherent electromagnetic waves, in which in the non-wave zone, the fields \( E \) and \( B \) satisfy the condition: \( B^2 \neq E^2 \). These are the properties of the magnetic dipole radiation of pulsars and magnetars. They have strong magnetic fields, that comparable and even exceed the quantum field \( B_0 = 4.41 \cdot 10^{13} \text{ gauss} \); the induction of a magnetic dipole field on the surface of a pulsar can reach \( 10^{13} \text{ gauss} \) \([15]\), and on the surface of a magnetar – up to \( 2 \cdot 10^{15} \text{ gauss} \) \([16]\).

Therefore, the first invariant of the electromagnetic field \( F_{nn} F^{nm} \) for the magnetic dipole radiation of pulsars and magnetars in the near zone takes on the value, that can hardly be created in other electromagnetic processes.

### 2 Calculation of dilaton generation by magnetic dipole radiation of pulsars and magnetars

Consider a pulsar or magnetar of radius \( R_0 \) with a magnetic dipole moment \( \mathbf{M} \) rotating with a frequency of \( \omega \) around an axis making an angle \( \alpha \) with a vector \( \mathbf{M} \).

Then the pulsar magnetic dipole moment has the components:

\[ \mathbf{M}(t) = |M| (\cos(\omega t) \sin \alpha, \sin(\omega t) \sin \alpha, \cos \alpha), \]

where \( \tau \) is retarded time: \( \tau = t - r/c \).

Due to the rotation of the vector \( \mathbf{M} \), magnetic dipole radiation of electromagnetic waves is being generated. According to work \([17]\), this radiation vectors \( \mathbf{B} \) and \( \mathbf{E} \), can be written in the form:

\[ \mathbf{B}(r, \tau) = \frac{3(\mathbf{M}(\tau) \cdot \mathbf{r}) \mathbf{r} - r^2 \mathbf{M}(\tau)}{r^5} - \frac{\mathbf{M}(\tau)}{cr^2} + \frac{3(\mathbf{M}(\tau) \cdot \mathbf{r}) \mathbf{r} - r^2 \mathbf{M}(\tau)}{cr^2}, \]  

\[ \mathbf{E}(r, \tau) = \frac{(r \times \mathbf{M}(\tau))}{c r^2} + \frac{(r \times \mathbf{M}(\tau))}{c r^2}. \]

where the dot above the vector \( \mathbf{M} \) means the derivative on retarded time \( \tau \).

For pulsars and magnetars condition \( \omega R_0 \ll c \), are met, hence they are in the near zone of their own magnetic dipole radiation.

The total intensity \( I \) – the amount of energy of the electromagnetic waves emitted in all directions by a rotating magnetic dipole per time unit:

\[ I_{EMW} = \frac{2 M^2}{3c^3} = \frac{2 c B_0^2 R_0^6 k^4}{3} \sin^2 \alpha. \]  

where \( k = \omega/c \) and the square of the magnetic dipole moment vector of the neutron star \( M^2 \) is written in terms of the square of the magnetic induction vector on the surface of the star \( B_0^2 \) according to the equation: \( M^2 = B_0^2 R_0^6 \).

Using relations (4), and keeping only the time-dependent part, we calculate the invariant \( B^2 - E^2 \):

\[ B^2 - E^2 = \frac{B_0^2 R_0^6}{2 r^6} \left\{ \sin^2 \alpha \sin^2 \theta \left[ (3 - 2 k^2 r^2) \times \right. \right. \]

\[ \times \left. \cos \left[ 2(\varphi + kr - \omega t) \right] + 6 k r \sin \left[ 2(\varphi + kr - \omega t) \right] \right] + \right. \]

\[ + 2 \sin 2 \alpha \sin \theta \cos \theta \left[ (3 + k^2 r^2) \cos \left[ 2(\varphi + kr - \omega t) \right] + \right. \]

\[ + 3 k r \sin \left[ 2(\varphi + kr - \omega t) \right] \right\}. \]  

Substituting expressions (6) in the right part of equation (3), for the convenience of the solution, we will rewrite it in a complex form, assuming that after solving it, we will leave only the real part. Then we have:

\[ \Delta \psi(r, t) - \frac{1}{c^2} \frac{\partial^2 \psi(r, t)}{\partial t^2} = \frac{a_1 k B_0 R_0^6}{a_0 r^6} \left\{ \sin^2 \alpha \sin^2 \theta \left[ 3 - \right. \right. \]

\[ - 2 k^2 r^2 + 6 i k r \right. \left. \cos \left[ (\varphi + kr - \omega t) \right] \right) \]

\[ - 2 \sin 2 \alpha \sin \theta \cos \theta \left[ (3 + k^2 r^2) + 3 \right. \]

\[ \left. \times \left. \left. \right) e^{-i(\varphi + kr - \omega t)} \right\} \right. \}

\[ + 2 \sin 2 \alpha \sin \theta \cos \theta \left[ (3 + k^2 r^2) + 3 i k r \right] \times \]

\[ \right. \times \left. \left. \right) e^{-i(\varphi + kr - \omega t)} \right\}, \]  

\[ \psi(r, t) = - \frac{a_1 B_0 R_0^6}{8 \pi a_0} \int_V \frac{dV'}{r^6 |r - r'|} \left[ \sin^2 \alpha \sin^2 \theta' \left[ 3 - \right. \right. \]

\[ - 2 k^2 r'^2 + 6 i k r' \right. \left. \cos \left[ (\varphi' + kr' - \omega t + k|r - r'|) \right] + \right. \]

\[ + 2 \sin 2 \alpha \sin \theta' \cos \theta' \left[ (3 + k^2 r'^2) + 3 i k r' \right] \times \]
\[
\times e^{-i(\Psi' + kr' - \omega t + k r - r'))},
\]
where \(r' = (r' \sin \theta' \cos \varphi', r' \sin \theta' \sin \varphi', r' \cos \theta')\) and \(r = (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta)\).

Using the Gehenbauer theorem \[19\] and the formulas of the article \[20\], we find the field \(\Psi(r, t)\) in the region \(r > R_s\) (see Appendix A):

\[
\Psi(r, t) = \frac{i \pi^2 B^3_1 R^6_s}{\sqrt{r}} e^{2i\omega t} \sin^2 \alpha \sin^2 \theta e^{-2i\varphi} \times (9)
\]

\[
\times \left\{ \frac{H_{5/2}^{(2)}(2kR_s)}{\alpha_0 \sqrt{r}} e^{i\omega t} \sin 2 \alpha \sin \theta \cos \theta e^{-i\varphi} \times \left( \frac{g_1(2k)}{2a_0} - \frac{g_3(kR_s)}{2a_0} \right) - \frac{J_{5/2}(2kR_s)}{\alpha_0 \sqrt{r}} e^{i\omega t} \sin 2 \alpha \sin \theta \cos \theta e^{-i\varphi} \times \left( \frac{g_1(2k)}{2a_0} - \frac{g_3(kR_s)}{2a_0} \right) \right\},
\]

where the notations are entered:

\[
g_1(z) = \frac{2}{2\pi} \left\{ \left( 2z^2 + 3iz^2 + 12z - 6i \right) \exp(-2iz) + \frac{8z^3 - 2iz^4 + 9iz^2 + 6i}{4z^6} \right\},
\]

\[
g_2(z) = \sqrt{\frac{2}{2\pi}} \frac{2z^2 + 3iz^2 + 12z - 6i}{2z^6} \exp(-2iz),
\]

\[
g_3(z) = \sqrt{\frac{2}{2\pi}} \frac{15iz^2 - 2z^3 + 24z - 12i}{\sqrt{\pi} z^6} \exp(-2iz) + \frac{2z}{\sqrt{\pi} z^6} \frac{2iz^4 + 4z^3 + 9iz^2 + 12i}{4z^6},
\]

\[
g_4(z) = -\frac{4k^9/2[2z^2 - 15iz^2 + 24z + 12i]}{\sqrt{\pi} z^6} \exp(-2iz).
\]

The expression (9) is an exact solution of the equation (7).

3 The angular distribution of the dilaton radiation

Let us study the angular distribution of the arising dilaton radiation. By definition \[18\], the amount of energy \(dI\), emitted by the source per unit time through the solid angle \(d\Omega = \sin \theta d\theta d\varphi\), is given by the formula:

\[
\frac{dI}{d\Omega} = \lim_{r \to \infty} r(W \cdot r),
\]

where vector \(W\) in tensor form has the components \(W'^\alpha = C'^\alpha_{\alpha_0}\), and \(T^{nm}\) is the energy momentum tensor of dilaton radiation.

The energy momentum tensor of the dilaton field has the form:

\[
T^{ak} = 2a_0 \eta^{ak} g^{km} \left\{ \frac{\partial \Psi}{\partial x^n} \frac{\partial \Psi}{\partial x^m} - \frac{1}{2} g^{mn} \frac{\partial \Psi}{\partial x^k} \frac{\partial \Psi}{\partial x^p} g^{kp} \right\}.
\]

Then the angular distribution of the dilaton radiation will be determined by the expression:

\[
\frac{dI}{d\Omega} = -2a_0 \lim_{r \to \infty} r(r \nabla \Psi) \frac{\partial \Psi}{\partial t}.
\]

For further investigation of the generation of dilaton radiation by the electromagnetic field of a rotating magnetic dipole, we only need the wave part of expression (9), which decreases at \(kr >> 1\) as \(1/r\). In addition, we take into account that for most pulsars and magnetars \(kR_s << 1\), and keeping in the resulting expression only the asymptotically main term in the expansions with respect to this small parameter. Then, keeping in expression (9) the real part and discarding the non-wave terms, we get (see Appendix B):

\[
\Psi(r, t) = \frac{2a_0 K R^3_s}{5a_0 r} \left\{ 2 \sin^2 \alpha \sin^2 \theta \cos(2(\omega t - kr - \varphi)) - \sin 2 \alpha \sin \theta \cos(\omega t - kr - \varphi) \right\}. \tag{11}
\]

Substituting the expression (11) into the ratio (10) and averaging the resulting formula for the wave period \(T = 2\pi/\omega\), we come to the expression:

\[
\frac{dI}{d\Omega} = \frac{dI}{d\Omega}(\omega) + \frac{dI}{d\Omega}(2\omega),
\]

where

\[
\frac{dI}{d\Omega}(\omega) = \frac{4ca_0^2 K^2 B^4_{10} R^6_{10}}{25a_0} \sin^2 2\alpha \sin^2 \theta \cos^2 \theta \tag{12}
\]

is the angular distribution of the radiation of dilatons having the frequency \(\omega\), and

\[
\frac{dI}{d\Omega}(2\omega) = \frac{64ca_0^2 K^2 B^4_{10} R^6_{10}}{25a_0} \sin^4 \alpha \sin^4 \theta \tag{13}
\]

is the angular distribution of the radiation of dilatons having the frequency \(2\omega\).

Integrating these expressions over the angles \(\theta\) and \(\varphi\), we obtain the total intensity \(I\) – the amount of energy of the dilatonic waves emitted in all directions by a rotating magnetic dipole per time unit:

\[
I(\omega) = \frac{32ca_0^2 K^2 B^4_{10} R^6_{10}}{375a_0} \sin^2 2\alpha, \tag{14}
\]

\[
I(2\omega) = \frac{2048ca_0^2 K^2 B^4_{10} R^6_{10}}{375a_0} \sin^4 \alpha.
\]

It follows from the expressions (12)-(14) that both the angular distributions and the total intensities of dilaton radiation at the frequencies \(2\omega\) and \(\omega\) differ significantly, although they are generated by the same electromagnetic fields (4) of the rotating magnetic dipole of a neutron star.
4 Discussion

The calculation showed that dilaton radiation in general occurs at the two frequencies: the rotation frequency \( \omega \) of the magnetic dipole moment of the neutron star and the twice frequency \( 2\omega \).

Angular distribution of radiation of dilatons having a frequency of \( \omega \), has a maximum along conic surfaces \( \theta = \pi/4 \).

Angular distribution of radiation of dilatons having a frequency of \( 2\omega \), has a maximum in the plane which is perpendicular to the axis of rotation \( (\theta = \pi/2) \).

As it follows from the expression (12), the generation of dilatons at frequency \( \omega \) is maximal in the case when the angle between the magnetic dipole moment and the axis of its rotation is \( \pi/4 \). If this angle is \( \pi/2 \), then dilaton radiation at frequency \( \omega \) does not occur.

The generation of dilatons at frequency \( 2\omega \) is maximal in the case when the angle between the magnetic dipole moment and the axis of its rotation is \( \pi/2 \).

Therefore, the newly discovered "Magnificent Seven" magnetars \([21,22]\) should emit dilatons only at the frequency \( 2\omega \), since dilatonic radiation at frequency \( \omega \) is either absent or strongly suppressed, since they have the angle between the magnetic dipole moment and the rotation axis close to \( \pi/2 \).

It should be noted that the dipole radiation generated by the rotation of the magnetic dipole moment of pulsars and magnetars is also a source of generation of two types of axion-like particles: massive axions and strictly massless axions. Unlike dilatons, the electromagnetic source for which is the invariant \( \mathcal{F}_{nk}\mathcal{F}^{nk} = 2(B^2 - E^2) \) of the electromagnetic field tensor, the source of axion-like particles is the pseudo-invariant \((\mathbf{B E})\).

Therefore the radiation of axions, as shown in \([23]\), occurs only at the rotation frequency \( \omega \).

Currently, the values of the constants \( a_0, a_1 \) and \( K \) are unknown.

Let’s roughly estimate the values of combination \( \eta = a_0^2 K^2/a_0 \) these constants. To do this, it is reasonable to require that the maximum value of the intensity of the dilaton radiation (14) was significantly less than the maximum value of the intensity of electromagnetic radiation (5):

\[
\max I_{\text{Dilaton}} \ll \max I_{\text{EMW}}.
\]

We suppose, that the dilaton radiation, like any physical radiation carries non-negative energy, then the constant \( a_0 > 0 \). From expressions (5) and (14) in order of magnitude, we get the inequality:

\[
\eta \ll k^{-2} R_s^{-4} B_s^{-2}.
\]  \hspace{1cm} (15)
where
\[
\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi' - \varphi).
\]

Substituting the expressions (A1-A2) in the retarded integral (8), we bring it to the form:
\[
\Psi(r, t) = \frac{i a_1 K B_0^2 R^6}{8 a_0 \sqrt{r}} e^{i 2 \omega t} \sin^2 \alpha \sum_{n=0}^{\infty} (2n+1) \int_0^\pi \sin^3 \theta \, d\theta' \times
\]
\[
\times \int_0^{2\pi} e^{-i 2\varphi'} P_n(\cos \gamma) d\varphi' \left\{ H_{n+1/2}^{(2)}(2kr) \int_0^{r/\rho_{0/2}} \left[ 3 - 2 k^2 r'^2 + 6ikr' \right] e^{-i 2kr' J_{n+1/2}(2kr')} dr' + + J_{n+1/2}(2kr) \int_0^{\infty} \frac{1}{r/\rho_{0/2}} \left[ 3 - 2 k^2 r'^2 + 6ikr' \right] e^{i 2kr' H_{n+1/2}^{(2)}(2kr')} dr' \right\} + + \frac{i \pi a_1 K B_0^2 R^6}{4 a_0 \sqrt{r}} e^{i 2 \omega t} \sin 2\alpha \sin \theta \cos \theta e^{-i \varphi} \times
\]
\[
\times \left\{ H_{5/2}^{(2)}(2kr) \int_0^{r/\rho_{0/2}} \left[ 3 - 2 k^2 r'^2 + 6ikr' \right] e^{-i 2kr' J_{5/2}(2kr')} dr' + J_{5/2}(2kr) \int_0^{\infty} \frac{1}{r/\rho_{0/2}} \left[ 3 - 2 k^2 r'^2 + 6ikr' \right] e^{i 2kr' H_{5/2}^{(2)}(2kr')} dr' \right\}
\]
\[
\times \left\{ J_{5/2}(2kr) \int_0^{r/\rho_{0/2}} \left[ 3 + k^2 r'^2 + 3ikr' \right] e^{-i kr' J_{5/2}(2kr')} dr' + + J_{5/2}(2kr) \int_0^{\infty} \frac{1}{r/\rho_{0/2}} \left[ 3 + k^2 r'^2 + 3ikr' \right] e^{i kr' H_{5/2}^{(2)}(2kr')} dr' \right\}.
\]

Let note that
\[
J_{5/2}(w) = \sqrt{\frac{2}{\pi w}} \left\{ \frac{3}{w^2} - 1 \right\} \sin w - \frac{3 \cos w}{w} \}
\]
\[
H_{5/2}^{(2)}(w) = \sqrt{\frac{2}{\pi w}} \left\{ \frac{3}{w^2} - 1 \right\} e^{-iw}.
\]

Substituting these relations for the integrals in the expression (A4) and integrating them in parts, we have:
\[
\Psi(r, t) = \frac{i \pi^2 a_1 K B_0^2 R^6}{\sqrt{r}} e^{i 2 \omega t} \sin^2 \alpha \sin^2 \theta e^{-i \varphi} \times
\]
\[
\times \left\{ H_{5/2}^{(2)}(2kg_1(2kr) - g_1(kR_a)) - J_{5/2}(2kr) g_2(2kr) \right\} + + \frac{i \pi a_1 K B_0^2 R^6}{4 a_0 \sqrt{r}} e^{i 2 \omega t} \sin 2\alpha \sin \theta \cos \theta e^{-i \varphi} \times
\]
\[
\times \left\{ H_{5/2}^{(2)}(kr) g_3(kR_a) - g_3(kR_a) - J_{5/2}(kr) g_4(kR_a) \right\},
\]
where the notation is used:
\[
g_1(z) = \sqrt{\frac{3}{2\pi}} \left\{ \frac{2z^2 + 3iz^2 + 12z - 6i}{z^2} \right\} \exp(-2iz) + + \frac{8z^3 - 2z^4 + 9iz^2 + 6i}{z^2},
\]
\[
g_2(z) = \sqrt{\frac{3}{2\pi}} \left\{ \frac{2z^2 + 3iz^2 + 12z - 6i}{z^2} \right\} \exp(-2iz),
\]
\[
g_3(z) = \frac{2}{\sqrt{z^2}} \left[ 15iz^2 - 2z^3 + 24z - 12i \right] \exp(-2iz) + + 2 \sqrt{\pi} \left[ 2iz^2 + 4z^3 + 9iz^2 + 12i \right],
\]
\[
g_4(z) = \frac{4k^{3/2} \left[ 2z^3 - 15iz^2 - 24z + 12i \right] \exp(-2iz) - 2iz^2 + 4z^3 + 9iz^2 + 12i}{\sqrt{\pi} \left[ 2z^3 - 15iz^2 - 24z + 12i \right]}.
\]
Appendix B. Dilaton field in the wave zone

Let us now construct the asymptotically main part of the dilaton field in the wave zone, i.e., at $kr >> 1$. In this zone, we have the asymptotics:

$$H^{(2)}_{5/2}(2kr) = \frac{i}{r\sqrt{\pi k}} e^{-2ikr}, \quad (B1)$$

$$H^{(2)}_{5/2}(x) = -i \left( \frac{2}{\pi x} \right)^{1/2} e^{-ix}, \quad \frac{H^{(2)}_{5/2}(kr)}{\sqrt{r}} = \frac{i\sqrt{2}}{r\sqrt{\pi k}} e^{-ikr}.$$ 

Since $kr << 1$, the function $g_1(2kr)$ and $g_3(2kr)$ we leave only the asymptotically principal parts.

From the relations (A5), it follows that for $z << 1$, the estimates are valid:

$$\lim_{z \to 0} z g_1(z) = -\frac{16k^7/2}{5\sqrt{\pi}}, \quad \lim_{z \to 0} z g_3(z) = -\frac{2k^7/2}{5\sqrt{2}\pi} \quad (B2)$$

Therefore

$$g_1(2kr) = -\frac{8k^{5/2}}{5\sqrt{\pi} r_s}, \quad g_3(2kr) = -\frac{2k^{5/2}}{5\sqrt{2}\pi r_s}.$$ 

The expression (9), taking into account the relations (B1)-(B2), takes the form:

$$\Psi(r, t) = \frac{2a_1KB_0^2k^2R_5^5}{5\omega_0 r} \left\{ 2\sin^2 \alpha \sin^2 \theta e^{2i(\omega t - kr - \varphi)} - \sin 2\alpha \sin \theta \cos \theta e^{i(\omega t - kr - \varphi)} \right\}.$$ 

The real part of this expression has the form:

$$\Psi(r, t) = \frac{2a_1KB_0^2k^2R_5^5}{5\omega_0 r} \left\{ 2\sin^2 \alpha \sin^2 \theta \cos[2(\omega t - kr - \varphi)] - \sin 2\alpha \sin \theta \cos \theta \cos(\omega t - kr - \varphi) \right\}.$$ 

This expression is dilaton field in the wave zone.

References

1. A. Salam and J.A. Strathdee, *Nonlinear realizations. 2. Conformal symmetry*, Phys. Rev. **184**, (1969) 1760.
2. T. E. Clark, C. N. Leung, and S. T. Love, *Properties of the dilaton*, Phys. Rev. D **35**, (1987) 997.
3. Manisha Kumari and Arvind Kumar, *Properties of strange quark matter and strange quark stars*, Eur. Phys. J. C **81**, (2021) 791. https://doi.org/10.1140/epjc/s10052-021-09576-w
4. Xu Lu and Yi Xie, *Gravitational lensing by a quantum deformed Schwarzschild black hole*, Eur. Phys. J. C **81** (2021) 627. https://doi.org/10.1140/epjc/s10052-021-09440-x
5. O.V.Kechkin and P.A.Mosharev, *A General Harmonic Solution in Dilaton Electrodynamics: An Exact Expression for the Fields and the Generalized Lorentz Force*, Moscow University Physics Bulletin, **75**, (2020) 427.
6. Bardia H. Fahima and Masoud Ghezelbash, *New class of exact solutions to Einstein–Maxwell-dilaton theory on four-dimensional Bianchi type IX geometry*, Eur. Phys. J. C **81** (2021) 587. https://doi.org/10.1140/epjc/s10052-021-09395-z
7. Justin L. Ripley and Frans Pretorius, *Scalarized black hole dynamics in Einstein-dilaton-Gauss-Bonnet gravity*, Phys. Rev. D **101**, (2020) 044015.
8. A. N. Malybayev, K. A. Boshkayev, V. D. Ivashchuk, *Quasinormal modes in the field of a dyon-like dilaton black hole*, Eur. Phys. J. C **81** (2021) 475. https://doi.org/10.1140/epjc/s10052-021-09252-z
9. Steven L. Liebling, *Maxwell-dilaton dynamics*, Phys. Rev. D, **100**, (2019) 104040.
10. L.R. Colaco, R.F.L. Holanda and R. Silva, *Probing variation of the fine-structure constant in runaway dilaton models using Strong Gravitational Lensing and Type Ia Supernovae*, Eur. Phys. J. C **81** (2021) 822. https://doi.org/10.1140/epjc/s10052-021-09625-4
11. A. Ahmed, B.M. Dillon and S. Najjari, *Dilaton portal in strongly interacting twin Higgs models*, JHEP, **02**, (2020) 124.
12. Aqeel Ahmed, Alberto Mariotti and Saereh Najjari, *A light dilaton at the LHC*, JHEP, **05**, (2020) 093.
13. Thomas Appelquist, James Ingoldby and Maurizio Piai, *Analysis of a dilaton EFT for lattice data*, JHEP, **03**, (2018) 039.
14. Thomas Appelquist, James Ingoldby, and Maurizio Piai, *Dilaton potential and lattice data*, Phys. Rev. D, **101**, (2020) 075025.
15. R.N.Manchester, G.B.Hobbs, A.Teoh, M.Hobbs, *The Australia telescope national facility pulsar catalogue*, The Astronomical Journal, **129**, (2005) 1993.
16. S.A.Olausen and V.M.Kaspi, *The McGill magnetar catalog*, Astrophys. J. Suppl. **212**, (2014) 6.
17. V. I. Denisov, B. N. Shvilkin, and V. A. Sokolov, *Pulsar radiation in post-Maxwellian vacuum nonlinear electrodynamics*, Physical Review D **94**, 045021 (2016) 045021.
18. L.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields*, (Butterworth - Heinemann, 1975).
19. G.N.Watson. *A Treatise on the theory of Bessel functions*, (Cambridge University Press, Cambridge, second edition, 1944).
20. P.A.Vshivtseva, V.I.Denisov and I.P.Denisova, An integral relation for tensor polynomials, Theor Math Phys 166, (2011) 186.

21. R.Turolla, Isolated neutron stars: the challenge of simplicity, Astrophysics and Space Science Library, 357, (2009) 141.

22. R.P.Mignani, V.Testa, D.Gonz’alez Caniulef, et al, Evidence for vacuum birefringence from the first optical polarimetry measurement of the isolated neutron star RXJ1856.5-3754, Mon. Not. R. Astron. Soc., 465 (2017) 492.

23. V.I. Denisov, B.D. Garmaev, and I.P. Denisova, Radiation of arions by electromagnetic field of rotating magnetic dipole, Phys. Rev. D, 104, (2021) 055018 https://doi.org/10.1103/PhysRevD.104.055018