Abstract. We give explicit formulas for a pair of linearly independent solutions of \((py')' + qy = (\lambda_1 r_1 + \lambda_2 r_2 + \cdots + \lambda_d r_d)y\), thus generalizing to arbitrary \(d\) previously known formulas for \(d = 1\). These are power series in the spectral parameters \(\lambda_1, \ldots, \lambda_d\) (real or complex), with coefficients which are functions on the interval of definition of the differential equation. The coefficients are obtained recursively using indefinite integrals involving the coefficients of lower degree. Examples are provided in which these formulas are used to solve numerically some boundary value problems for \(d = 2\), as well as an application to transmission and reflectance in optics.

Keywords. Sturm-Liouville problem, spectral parameter power series, SPPS representation, characteristic function, eigencurve, multiparameter spectral problem

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0 Introduction

We will consider the second-order linear differential equation

\[(py')' + qy = (\lambda_1 r_1 + \lambda_2 r_2 + \cdots + \lambda_d r_d)y\]

on a real interval \(x_1 \leq x \leq x_2\), where \(p, q, r_1, \ldots, r_d\) are given functions and \(\lambda_1, \ldots, \lambda_d\) are unknown parameters. Let some appropriate boundary conditions be imposed at \(x_1\) and \(x_2\). Then a spectral problem consists of determining the subset \((\lambda_1, \ldots, \lambda_d) \subset \mathbb{R}^d\) (or \(\mathbb{C}^d\)) for which there exists a solution \(y\) of (1) which satisfies those boundary conditions. While there is a vast literature on spectral theory for general differential equations and on numerical methods specifically developed for (1) for \(d = 1\)—indeed, the expression “spectral problem” commonly implies a single \(\lambda\) there is considerably less available concerning several parameters. One may find qualitative results on this subject in [3, 4, 7, 31, 32, 33]. For \(d = 2\) some properties of the eigencurves are set forth in [4, chapter 6].

An approach for solving spectral problems for \(d = 1\) was presented in [20, 23] which produces two explicit power series in the variable \(\lambda\) with coefficients which are functions on \([x_1, x_2]\). These series represent two functions \(y = u_1(x), y = u_2(x)\) parametrized by \(\lambda\) which are linearly independent solutions of (1). There are similar power series for the derivatives \(u_1'(x), u_2'(x)\). By evaluating these series with \(x\) at the endpoints \(x_1, x_2\) we obtain power series in \(\lambda\) for the boundary values, which upon substitution in the boundary conditions produce a “characteristic function” whose zeroes are the eigenvalues of the spectral problem. (It is not necessary for the boundary conditions to be linear for this procedure to apply.) These series representations have applications beyond spectral problems; for example they provide an effective method for solving initial value problems.
Since its appearance in 2008, consequences of this SPPS (spectral parameter power series) representation have been investigated in many directions. These include completeness properties of the “formal powers” used to define the coefficients of the power series [21, 22]; relationship to transmutation operators, Darboux and other transformations, and Goursat problems [17, 25, 27, 28]; extension to other number systems (quaternions, etc) [8, 9, 27] and equations of higher order [15]; relaxation of regularity conditions on the coefficients of the differential equation [5, 11]. Further, there have appeared numerous applications to problems in physics and engineering [10, 16, 18, 19, 29] as well as in complex analysis [6, 24].

In dealing with physics or engineering models which involve a Sturm-Liouville problem containing several eigenvalues \( \lambda_1 \), it is common practice to fix all but one of them, and then solve the spectral problem for the remaining one. This appears to be due to the difficulties of existing methods of handling more than one spectral parameter. In this paper we work out the SPPS coefficients corresponding to (1) for arbitrary \( d \geq 1 \). This permits treating the spectral parameters in unified way. We give some numerical examples with \( d = 2 \), and then an application to a problem of transmittance of an electromagnetic wave through an inhomogeneous layer, in which the two parameters correspond to physical characteristics of the phenomenon.

1 Formal powers

The Sturm-Liouville linear differential expression on the left-hand side of (1) will be denoted by

\[
Ly = (py')' + qy. \tag{2}
\]

Throughout this paper \( p, q, r_1, \ldots, r_d \) will denote real or complex valued functions on the closed interval \([x_1, x_2]\). In this section we are interested in describing the procedure for constructing the SPPS representation of solutions, while questions of convergence and regularity will be deferred to the next section. A basepoint \( x_0 \) is fixed in \([x_1, x_2]\). For convenience we will use the notation

\[
g = \int f
\]

to mean

\[
g(x) = \int_{x_0}^{x} f(s) \, ds
\]

for any function \( f \) under consideration, inasmuch as we will have no use for other limits of integration. In the following we will set up a notation for describing sums of finitely nested integrals of the form

\[
\cdots \int r_{i_n} u_{n}^2 \int \frac{1}{pu_{n}} \cdots \int r_{i_2} u_{2}^2 \int \frac{1}{pu_{2}} \int r_{i_1} u_{1}^2, \tag{3}
\]

\[
\cdots \int r_{i_n} u_{n}^2 \int \frac{1}{pu_{n}} \cdots \int r_{i_2} u_{2}^2 \int \frac{1}{pu_{2}} \int r_{i_1} u_{1}^2 \int \frac{1}{pu_{1}}. \tag{4}
\]
1.1 Construction of $\tilde{X}(\vec{j})$

For simplicity of handling the indices, we will begin with the form (3) which produces the family of functions called $\tilde{X}(\vec{j})$ (the notation follows that generally used in the SPPS literature). Here $\vec{j} = (j_1, \ldots, j_d)$ is a multiindex with integral entries. It has $d$ predecessors given by

$$\vec{j} - \vec{\delta}_i = (j_1, \ldots, j_{i-1}, j_i - 1, j_{i+1}, \ldots, j_d)$$

where $\vec{\delta}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $i$-th standard basis vector. We will say that $\vec{j}$ is an admissible multiindex when at most one of its entries $j_i$ is odd. An admissible $\vec{j}$ is called even or odd according to the parity of $|\vec{j}| = \sum_{i=1}^{d} j_i$; i.e., it is odd when exactly one $j_i$ is odd. We start from the constant function

$$\tilde{X}(0) = 1 \quad (5)$$

for all $x$, where $\vec{0} = (0, \ldots, 0)$. For definiteness we set $\tilde{X}(\vec{j}) = 0$ whenever $j_i < 0$ for some $i$. Then we define the formal power $\tilde{X}(\vec{j})$ for admissible $\vec{j}$ with nonnegative indices in the following recursive manner:

$$\tilde{X}(\vec{j}) = \begin{cases} |\vec{j}| \int r_i u^2 \tilde{X}(\vec{j} - \vec{\delta}_i), & \vec{j} \text{ odd,} \\ |\vec{j}| \int \frac{1}{p u^2} \sum_{i=1}^{d} \tilde{X}(\vec{j} - \vec{\delta}_i), & \vec{j} \text{ even.} \end{cases} \quad (6)$$

These are all functions on $[x_1, x_2]$. Note that when $\vec{j}$ is odd, the index $i$ referred to in the first clause of (6) is unambiguously defined. The interdependencies of the $\tilde{X}(\vec{j})$ are illustrated for $d = 2$ in Figure 1. We will say that the degree of $\tilde{X}(\vec{j})$ is $|\vec{j}|$.

To motivate to some extent what we have done we give the following relationship.

Lemma 1 Let $u_0$ be a nonvanishing function on $[x_1, x_2]$ and suppose that $Lu_0 = 0$. Then for any nonnegative even multiindex $2\vec{n}$,

$$L(u_0 \tilde{X}(2\vec{n})) = 2|\vec{n}|(2|\vec{n}| - 1) u_0 \sum_{i=1}^{d} r_i \tilde{X}(2\vec{n} - 2\vec{\delta}_i).$$

Proof. As a consequence of $Lu_0 = 0$, is easily seen that the operator $L$ admits the Polya factorization [14]

$$L = \frac{1}{u_0} \partial p u^2 \partial \frac{1}{u_0}$$

where $\partial = \partial/\partial x$ and the functions in this expression refer to the corresponding multiplication
operators. Thus by (6),

\[
L(u_0 \tilde{X}^{(2\vec{n})}) = \frac{1}{u_0} \partial_p u_0^2 \partial \left( \frac{1}{pu_0^2} \sum_i \tilde{X}^{(2\vec{n} - \delta_i)} \right) \\
= 2|\vec{n}| \frac{1}{u_0} \partial \left( \sum_i \tilde{X}^{(2\vec{n} - \delta_i)} \right).
\]

A second application of (6) yields that this is equal to

\[
2|\vec{n}| \frac{1}{u_0} \left( (2|\vec{n}| - 1) \sum_i r_i u_0^2 \tilde{X}^{(2\vec{n} - 2\delta_i)} \right)
\]

as required. \[\Box\]

The number \(\tilde{c}_j\) of summands of the form (3) comprising \(\tilde{X}^{(j)}\) is the same as the number of paths which advance (i.e. from predecessors to successors) from \(\tilde{X}^{(0)}\) to \(\tilde{X}^{(j)}\) and can be described as follows. We define \(\tilde{c}_j = 0\) when some \(j_i < 0\). Clearly \(\tilde{c}_0 = 1\). Then by (6) we have recursively

\[
\tilde{c}_j = \begin{cases} 
\tilde{c}_{j-\delta_i}, & \text{if } j \text{ odd}, \\
\sum_{i=1}^d \tilde{c}_{j-\delta_i}, & \text{if } j \text{ even.}
\end{cases} \tag{7}
\]

\[\begin{array}{c}
\tilde{X}^{(0,0)} \xrightarrow{r_2} \tilde{X}^{(0,1)} \xrightarrow{p} \tilde{X}^{(0,2)} \xrightarrow{r_2} \tilde{X}^{(0,3)} \xrightarrow{p} \tilde{X}^{(0,4)} \xrightarrow{r_2} \tilde{X}^{(0,5)} \\
\tilde{X}^{(1,0)} \xrightarrow{p} \tilde{X}^{(1,2)} \xrightarrow{p} \tilde{X}^{(1,4)} \\
\tilde{X}^{(2,0)} \xrightarrow{r_2} \tilde{X}^{(2,1)} \xrightarrow{p} \tilde{X}^{(2,2)} \xrightarrow{r_2} \tilde{X}^{(2,3)} \xrightarrow{p} \tilde{X}^{(2,4)} \xrightarrow{r_2} \tilde{X}^{(2,5)} \\
\tilde{X}^{(3,0)} \xrightarrow{p} \tilde{X}^{(3,2)} \xrightarrow{p} \tilde{X}^{(3,4)} \\
\tilde{X}^{(4,0)} \xrightarrow{r_2} \tilde{X}^{(4,1)} \xrightarrow{p} \tilde{X}^{(4,2)} \xrightarrow{r_2} \tilde{X}^{(4,3)} \xrightarrow{p} \tilde{X}^{(4,4)} \xrightarrow{r_2} \tilde{X}^{(4,5)} \\
\tilde{X}^{(5,0)} \xrightarrow{r_1} \tilde{X}^{(5,2)} \xrightarrow{p} \tilde{X}^{(5,4)}
\end{array}\]

Figure 1: Construction of \(\tilde{X}^{(j)}\).
By induction via predecessors it is readily seen that

\[ \hat{\delta}_j = \frac{\left[\frac{|j|}{d}\right]!}{\left[\frac{|j|}{d}\right]! \cdots \left[\frac{|j|}{d}\right]!}. \] (8)

Consider a single nested integral appearing as a summand in \( \tilde{X}^{(j)} \). The number of integrations following division by \( pu_0^d \) is \( \lceil |j|/2 \rceil \), while for each \( i \) \((1 \leq i \leq d)\), the number of integrations which follow a multiplication by \( r_iu_0^d \) is easily seen to be \( \lceil (j_i + 1)/2 \rceil \). Here and always \([a]\) means the least integer no greater than the real number \(a\). One may verify that these indeed sum to \( |j| \). Define

\[ M_0 = \sup_{[x_1, x_2]} \frac{1}{|pu_0^d|}, \quad M_i = \sup_{[x_1, x_2]} |r_iu_0^d|. \] (9)

**Lemma 2** The formal powers \( \tilde{X}^{(j)} \) satisfy the growth condition

\[ |\tilde{X}^{(j)}(x)| \leq \hat{\delta}_j M_0 \left[\frac{|j|}{d}\right]! M_1 \left[\frac{|j_1|}{d}\right]! \cdots M_i \left[\frac{|j_i|}{d}\right]! \cdots M_d \left[\frac{|j_d|}{d}\right]! |x - x_0|^{\lceil |j| \rceil} \] (10)

for \( x_1 \leq x \leq x_2 \).

**Proof.** Suppose that \( j \) is even. Then by the inductive hypothesis

\[ |\tilde{X}^{(j)}(x)| \leq \hat{\delta}_j \int_{x_0}^x \left( \sup_{[x_1, x_2]} \frac{1}{|pu_0^d|} \right) \sum_{i=1}^d |\tilde{X}^{(j - \delta_i)}(t)| dt 
\leq \int_{x_0}^x M_0 \left( \sum_{i=1}^d \hat{\delta}_{j - \delta_i} M_0 \left[\frac{|j_i|}{d}\right]! M_1 \left[\frac{|j_1|}{d}\right]! \cdots M_i \left[\frac{|j_i|}{d}\right]! \cdots M_d \left[\frac{|j_d|}{d}\right]! \right) |x - x_0|^{\lceil |j| \rceil - 1} dt. \]

We then integrate and note that \( \lceil ji/2 \rceil = \lceil (ji + 1)/2 \rceil \) since all \( j_i \) are even, obtaining the bound

\[ \left( \sum_{i=1}^d \hat{\delta}_{j - \delta_i} M_0 \left[\frac{|j_i|}{d}\right]! M_1 \left[\frac{|j_1|}{d}\right]! \cdots M_i \left[\frac{|j_i|}{d}\right]! \cdots M_d \left[\frac{|j_d|}{d}\right]! \right) |x - x_0|^{\lceil |j| \rceil + k} \]

which by (7) reduces to (10). The verification for \( |j| \) odd is similar and indeed simpler. \( \square \)

### 1.2 Construction of \( X^{(j)} \)

The construction of \( X^{(j)} \) in terms of nested integrals of the form (4) is analogous to that of \( \tilde{X}^{(j)} \). However, there are some notational complications. The indices could be handled in various ways; our choice, perhaps purist, is as follows. Now the \( j \) will have entries with common fractional part \( ji - |ji| = 1/d \). We will call \( j \) admissible when at most one of the the integral parts \( |ji| \) is odd, while the parity of \( j \) is again that of the integer \( |j| \).

To start the recursion we use the predecessors of \((1/d)^{\tilde{\delta}_i}\), i.e. \( j = (1/d, 1/d, \ldots, 1/d, -1 + 1/d, 1/d, \ldots, 1/d)\), defining the constant functions

\[ X^{((1/d)^{\tilde{\delta}_i})}(x) = \frac{1}{d} \] (11)
for $i = 1, \ldots, d$. These are functions of degree 0. For definiteness we define $X^{(j)}(x) = 0$ whenever some $j_i < 0$ except as specified by (11). The formal powers $X^{(\vec{j})}$ for the remaining admissible $\vec{j} \geq 0$ are defined by

$$
X^{(\vec{j})} = \begin{cases} 
|\vec{j}| \int r_i u_0^2 X^{(\vec{j} - \delta_i)}, & \vec{j} \text{ even}, \\
|\vec{j}| \int \frac{1}{pu_0^2} \sum_{i=1}^{d} X^{(\vec{j} - \delta_i)}, & \vec{j} \text{ odd},
\end{cases}
$$

(12)
as outlined in Figure 2. This formula differs from (6) not only in the exchange of even and odd, but also in that the indices and coefficients have different interpretations. One justification for this notation is the role of the degree $|\vec{j}|$, cf. Lemma 4.

Analogously to Lemma 1 we find

**Lemma 3**

$$
L(u_0 X^{(2|\vec{n}| + \frac{1}{d})}) = (2(|\vec{n}| + 1)(2|\vec{n}|) u_0 \sum_{i=1}^{d} r_i \tilde{X}^{(2|\vec{n}| - 2\delta_i, +\frac{1}{d})}.
$$

In verifying this it is useful to note that $|\vec{n}| = 1$ and to use the linearity of the degree operator $| \cdot |$.

The number $c_{\vec{j}}$ of terms in $X^{(\vec{j})}$ is determined recursively by setting $c_{(1/d)\vec{l} - \delta} = 1$, while otherwise $c_{\vec{j}} = 0$ if some $j_i < 0$, and then

$$
c_{\vec{j}} = \begin{cases} 
\tilde{c}_{\vec{j} - \delta}, & \vec{j} \text{ even}, \\
\overline{c}_{\vec{j} - \delta}, & \vec{j} \text{ odd},
\end{cases}
$$

(13)

which is analogous but again with a different interpretation of the indices. The number of integrations following division by $pu_0^2$ is now $|\vec{l} + 1)/2|$, and those following multiplication by $r_i u_0^2$ number $|(j_i - 1/d + 1)/2|$. This gives the growth estimate:

**Lemma 4**

$$
|X^{(\vec{j})}(x)| \leq c_{\vec{j}} M_0^{\left[\frac{j_1 + 1}{2}\right]} M_1^{\left[\frac{j_2 - 1/d + 1}{2}\right]} \cdots M_d^{\left[\frac{j_d - 1/d + 1}{2}\right]} |x - x_0|^{|\vec{j}|}.
$$
2 SPPS series and characteristic function

2.1 General solution

We define the SPPS functions \( u_1, u_2 \) as

\[
\begin{align*}
u_1 &= u_0 \sum_{\vec{n} \geq 0} \frac{1}{(2|\vec{n}|)!} X^{(2|\vec{n}|)} \lambda_1^{n_1} \cdots \lambda_d^{n_d}, \\
u_2 &= u_0 \sum_{\vec{n} \geq 0} \frac{1}{(2|\vec{n}| + 1)!} X^{(2|\vec{n}| + 1)} \lambda_1^{n_1} \cdots \lambda_d^{n_d}.
\end{align*}
\]

where the sums are over all nonnegative multiindices \( \vec{n} \). Note that the degree of \( X^{(2|\vec{n}| + 1)} \) is \( 2|\vec{n}| + 1 \). The main result, which mirrors that of [23], is as follows.

**Theorem 5** Let \( p, q, r_1, \ldots, r_d \) be continuous complex-valued functions of the real variable \( x \in [x_0, x_1] \), with \( p \) continuously differentiable and \( p(x) \neq 0 \). Let the differential operator \( L \) be defined by (2). Then for every \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \) the two series in (14) converge uniformly on \( x \in [x_0, x_1] \), and the functions \( u_1, u_2 \) thus defined are linearly independent solutions of (1).
Further, their derivatives are given by uniformly convergent power series,

\[ u'_1 = \frac{u'_0}{u_0} u_1 + \frac{1}{p u_0} \sum_{\vec{n} \geq 0} \frac{1}{(2|\vec{n}| - 1)!} \sum_{i=1}^d \bar{X}^{(2\vec{n} - \vec{\delta})} \lambda_{1i}^{ni} \cdots \lambda_{2d}^{n_d}, \]

\[ u'_2 = \frac{u'_0}{u_0} u_2 + \frac{1}{p u_0} \sum_{\vec{n} \geq 0} \frac{1}{(2|\vec{n}|)!} X^{(2\vec{n} - \vec{\delta} + \vec{\delta})} \lambda_{1i}^{ni} \cdots \lambda_{2d}^{n_d}. \] (15)

For every value of \( \bar{X} \), the initial values of these functions are equal to

\[ u_1(x_0) = u_0(x_0), \quad u'_1(x_0) = u'_0(x_0). \]

\[ u_2(x_0) = 0, \quad u'_2(x_0) = \frac{1}{p(x_0) u_0(x_0)}. \] (16)

**Proof.** This proof is quite analogous to the proof for \( d = 1 \) given in [23], but certain details must be taken into account when there are more spectral parameters. First we verify the convergence.

Let \( \Lambda = \max(|\lambda_1|, \ldots, |\lambda_d|) \). Recalling (9), let \( M = \max(M_0, M_1, \ldots, M_d) \). Now by (10),

\[ |\bar{X}^{(2|\vec{n}|)}(x)| \leq \tilde{c}_{2|\vec{n}|} M^{||\vec{n}||} \cdot M^{[n_1] + \cdots + [n_d]}|x_2 - x_1|^{|\vec{n}|} = \tilde{c}_J M^{||\vec{n}||}|x_2 - x_1|^{|\vec{n}|} \]

so by (8) the summands in the formula for \( u_1 \) in (14) are bounded by \( a^{2|\vec{n}|}/(2|\vec{n}|)! \), where

\[ a = \sqrt{\Lambda|x_2 - x_1|}. \]

Since \( a^{2|\vec{n}|} = a^{2n_1} a^{2n_2} \cdots a^{2n_d} \), we can factor the sum \( \sum_0^\infty a^{2|\vec{n}|}/(2|\vec{n}|)! \) into a product of \( d \) sums, each of which is equal to \( \cosh a \). By comparison with this finite sum it follows that the series for \( u_1 \) converges uniformly to a function bounded by \( \cosh^d a \). By similar arguments the series for \( u'_1, u_2, \) and \( u'_2 \) also converge uniformly, and this justifies the term by term differentiation.

By Lemma [1]

\[ Lu_1 = \sum_{|\vec{n}|=0}^\infty \frac{1}{(2|\vec{n}|)!} \lambda_{1i}^{ni} \cdots \lambda_{2d}^{n_d} L(u_0 \bar{X}^{(2|\vec{n}|)}) = \sum_{|\vec{n}|=0}^\infty \frac{\lambda_{1i}^{ni} \cdots \lambda_{2d}^{n_d}}{(2|\vec{n}|)!} (2|\vec{n}| - 1) u_0 \sum_{i=1}^d (r_i \bar{X}^{(2|\vec{n}| - 2\delta_i)}). \]

Rearrange the last double sum as

\[ \sum_{i=1}^d \sum_{|\vec{n}|=0}^{\infty} \frac{\lambda_{1i}^{ni} \cdots \lambda_{2d}^{n_d}}{(2(|\vec{n}| - 1))!} \bar{X}^{(2(|\vec{n}| - \vec{\delta}_i))}. \]
and reindex each \( n_i \) down by 1, using the assumption that \( \tilde{X}(\bar{\vec{n}}) = 0 \) when \( j_i < 0 \) for some \( i \):

\[
\sum_{|\vec{n}|=0}^{\infty} \frac{\lambda_1^{n_1} \cdots \lambda_d^{n_d}}{(2(|\vec{n}| - 1))!} \tilde{X}(2(\vec{n} - \vec{\delta}_i)) = \sum_{|\vec{n}|=0}^{\infty} \frac{\lambda_1^{n_1+1} \cdots \lambda_d^{n_d}}{(2(|\vec{n}|))!} \tilde{X}(2\vec{n})
\]

Thus we have

\[
Lu_1 = \left( \sum_i \lambda_ir_i \right) u_1
\]

and the same argument verifies the corresponding statement for \( u_2 \). The final statement regarding the initial values follows from the fact that \( \tilde{X}(\bar{\vec{n}}) = 0 \) whenever even a single \( j_i \) is positive, so only the constant terms survive in the series for \( u_1, u'_1 \); similarly all but the lowest degree terms involving \( X(\bar{\vec{n}}) \) also vanish.

It is well known that when \( p, q, r_i \) are real-valued, a complex nonvanishing solution \( u_0 \) of \( Ly = 0 \) can be obtained as a complex linear combination of any two linearly independent solutions; in fact, by considerations of dimension one sees it is not necessary for the coefficients be real-valued for such a nonvanishing solution to exist. The hypotheses of Theorem 5 could be weakened, for instance by only requiring \( 1/(pu_0^2) \) and \( r_iu_0^2 \) to be continuous, but we will not enter into such details here.

**Corollary 6** Let \( u_1, u_2 \) be given by (14). Define

\[
v_1 = \frac{1}{u_0(x_0)} u_1 - p(x_0)u_0'(x_0) u_2, \\
v_2 = p(x_0)u_0(x_0) u_2.
\]

Then \( v_1, v_2 \) satisfy the normalizations

\[
v_1(x_0) = 1, \quad v'_1(x_0) = 0, \\
v_2(x_0) = 0, \quad v'_2(x_0) = 1.
\]

Observe that \( v_1, v_2 \) are also represented as power series in \( \lambda_1, \ldots, \lambda_d \).

### 2.2 Generalized Sturm-Liouville equation

We consider now equations of the form

\[
Ly = \sum_{i=1}^{d} \lambda_i R_i[y]
\]

where we define

\[
R_i[y] = r_1y + s_i y'
\]

2
for given functions $r_i, s_i, i = 1, \ldots, d$. Thus (11) is the particular case where all $s_i$ vanish identically. In [26] SPPS formulas were developed for (17) for the case previously but with the following modified recursive definition:

\[
\bar{X}^{(j)} = |j| \int u_0 R_i[u_0 \bar{X}^{(j-\delta_i)}] \quad (j \text{ odd})
\]

\[
\bar{X}^{(j)} = |j| \int \frac{1}{pu_0} \sum_{i=1}^{d} \bar{X}^{(j-\delta_i)} \quad (j \text{ even})
\]

\[
X^{(j)} = |j| \int u_0 R_i[u_0 X^{(j-\delta_i)}] \quad (j \text{ even})
\]

\[
X^{(j)} = |j| \int \frac{1}{pu_0} \sum_{i=1}^{d} X^{(j-\delta_i)} \quad (j \text{ odd});
\]

(19)

again we use integral entries in $j$ for $\bar{X}^{(j)}$ and non-integral entries for $X^{(j)}$.

We will need the common bound

\[
M = \sup_{[x_1, x_2]} \left( \frac{1}{pu_0}, |u_0 R_1[u_0]|, \ldots, |u_0 R_d[u_0]|, \left| \frac{s_1}{p} \right|, \ldots, \left| \frac{s_d}{p} \right| \right)
\]

of the functions appearing in the considerations below. For odd $j$ it follows from (18)–(19) that

\[
R_i[u_0 \bar{X}^{(j-\delta_i)}] = R_i[u_0 \bar{X}^{(j-\bar{\delta}_i)}] + \left( \frac{|j|}{pu_0} - 1 \right) s_i \sum_{i=1}^{d} \bar{X}^{(j-\delta_i-\delta_{i'})}
\]

(20)

which implies that the formal powers $\bar{X}^{(j)}$ may be calculated without recourse to numerical differentiation (other than for $u_0$) and that

\[
\left| u_0 R_i[u_0 \bar{X}^{(j-\bar{\delta}_i)}] \right| \leq M \left( |\bar{X}^{(j-\bar{\delta}_i)}| + \left| |j| - 1 \right| \sum_{i'=1}^{d} \bar{X}^{(j-\delta_i-\delta_{i'})} \right).
\]

(21)

Analogous statements hold for $X^{(j)}$.

**Lemma 7** For all $x \in [x_2, x_2]$, the inequalities $|\bar{X}^{(j)}| \leq \bar{P}_{\bar{j}}(x)$ and $|X^{(j)}| \leq P_{\bar{j}}(x)$ hold, where

\[
\bar{P}_{\bar{j}}(x) = d^{\left( \frac{1}{2} \right)} |\bar{j}|! \sum_{k=\lfloor \frac{j}{2} \rfloor + 1}^{j} \left( \frac{1}{j-k} \right) \frac{M^k}{k!} |x-x_0|^k,
\]

\[
P_{\bar{j}}(x) = d^{\left( \frac{1}{2} \right)} |\bar{j}|! \sum_{k=\lfloor \frac{j}{2} \rfloor - 1}^{j} \left( \frac{1}{j-k} \right) \frac{M^k}{k!} |x-x_0|^k,
\]

for integral $j \geq 0$.

**Proof.** First we consider $\bar{X}^{(j)}$, i.e. $j$ has integer entries. Write $E_k = (M|x-x_0|)^k/k!$ so $|M \int E_{k-1}| = E_k$. The inequalities are clearly valid when $|\bar{j}|$ is 0 or 1. Suppose that it is valid
for $|j|$ up to $n-1$. Now if $|j| = n$ is odd and $j$ has an odd entry in the $i$-th position, we calculate that

\[ \tilde{P}_{n-1}(x) = d^\frac{n-1}{2}(n-1)! \sum_{k=\frac{n-1}{2}}^{n-1} \left( \frac{n-3}{2} \right) \frac{2}{n-1-k} E_k, \]

\[ d(n-1) \tilde{P}_{n-2}(x) = d^\frac{n-1}{2}(n-1)! \sum_{k=\frac{n-1}{2}}^{n-2} \left( \frac{n-3}{2} \right) \frac{2}{n-2-k} E_k. \]

Then by the inductive hypothesis and (19), (21),

\[ \left| \tilde{X}^{(j)}(x) \right| \leq n \int_{x_0}^x M(\tilde{P}_{n-1}(x) + d(n-1) \tilde{P}_{n-2}(x)) \, dx \]

\[ = d^\frac{n-1}{2} n! \left( E_k + \sum_{k=\frac{n-3}{2}}^{n-2} \left( \frac{n-3}{2} \right) \frac{2}{n-1-k} E_k \right) \]

\[ = d^\frac{n-1}{2} n! \sum_{k=\frac{n-3}{2}}^{n-1} \left( \frac{n-1}{2} \right) \frac{2}{n-1-k} E_k+1 \]

\[ = \tilde{P}_n(x) \]

as is seen after reindexing $k+1$ to $k$ and then noting that $|n/2| + 1 = (n+1)/2$. On the other hand, if $j$ is even, then a similar, simpler argument verifies the inequality.

The verification for $X^{(j)}$ is analogous.

The following results for the generalized formal powers are now proved in exactly the same way as Lemmas 1 and 3 and Theorem 5.

**Lemma 8**

\[ L[u_0 \tilde{X}^{(2\tilde{r})}] = 2|\tilde{r}|(2|\tilde{r}| - 1) \sum_{i=1}^{d} R_i[u_0 \tilde{X}^{(2\tilde{r}-2\tilde{r}_i)}] \]

and

\[ L(u_0 X^{(2\tilde{r} + \frac{1}{2} 1)}) = (2(|\tilde{r}| + 1)(2|\tilde{r}|)) \sum_{i=1}^{d} R_i[u_0 \tilde{X}^{(2\tilde{r}-2\tilde{r}_i + \frac{1}{2} 1)}]. \]

**Theorem 9** Let $p, q, r_1, \ldots, r_d, s_1, \ldots, s_d$ be continuous on $[x_0, x_1]$ with $p$ continuously differentiable and $p(x) \neq 0$. Define $\tilde{X}^{(j)}$ and $X^{(j)}$ by (12), and then define $u_1, u_2$ by (14). These series converge uniformly on $x \in [x_0, x_1]$ for every fixed $\tilde{X} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$, and are linearly independent solutions of the generalized Sturm-Liouville equation (14). Their derivatives are given by (15) and they satisfy the initial conditions (10).
2.3 Spectral problems

The treatment of multiparameter spectral problems by the SPPS approach is the same as for a single spectral variable. Consider for simplicity linear boundary conditions of the form

\[ \alpha v(x_1) + \alpha' v'(x_1) = 0, \quad \beta v(x_2) + \beta' v'(x_2) = 0. \]  

(22)

For the general solution \( v = c_1 v_1 + c_2 v_2 \) with \( v_1, v_2 \) given by Corollary, this gives rise to a system of two equations in \( c_1, c_2 \) with determinant

\[ \alpha (\beta v_1(x_2) + \beta' v_1'(x_2)) - \alpha' (\beta v_2(x_2) + \beta' v_2'(x_2)). \]

Thus (22) is satisfied when \( \chi(\vec{\lambda}) = 0 \), where

\[ \chi(\vec{\lambda}) = -\alpha' \beta v_1(x_2) + \alpha \beta' v_1'(x_2) - \alpha' \beta' v_1(x_2) + \alpha \beta' v_2'(x_2). \]

(23)

Theorem represents \( \chi(\vec{\lambda}) \) as a power series in \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_d) \). Solutions of the boundary value problem are precisely the zeroes of this analytic function of several complex variables.

In like manner, nonlinear or mixed boundary conditions will also produce a characteristic function. When these conditions are analytic, the result will be expressible as a power series in the \( \lambda_i \), although it may be more convenient to leave it as a function defined as a combination of power series with other types of functions (cf. (28) below).

Similarly, one may impose boundary conditions at more than two points. One way of solving such a problem is by converting it to an integral equation \[1, 2, 13\]. With the approach described here, one simply evaluates the SPPS representation at all boundary points required, in order to obtain the desired set of simultaneous characteristic equations.

2.4 Remarks

2.4.1 Reduction to simple cases

We note that for \( d = 1 \) (i.e. \((\vec{0}) = (0), (\vec{I}) = (1)\)), the starting integral of the \( \vec{X}^{(j)} \) family is \( \vec{X}^{(0)} = 1 \), and by (11) for the family \( X^{(j)} \) the starting integral also reduces to

\[ X^{(1-I-j_1)} = X^{(0)} = \frac{1}{\Gamma} = 1. \]

Further, for general \( d \) the degree-1 power \( X^{((1/d)\vec{I})} \) is simply the integral

\[ \int \frac{1}{\rho u_0^d}. \]

which coincides with \( X^{(1)} \) in the case \( d = 1 \). Thus our notation is consistent with the “classical” definition of \[23\].

Considering \( d > 1 \), let us suppose that \( r_i \) is identically zero for every \( i \neq i_0 \). Then \( \vec{X}^{(j)} \) will vanish whenever \( j \) contains a \( j_i > 0 \) where \( i \neq i_0 \). The surviving powers \( X^{(0,...,0,j_i,0,...,0)} \)}
form the sequence $\tilde{X}(\vec{j})$ of classical 1-spectral-parameter formal powers in the single variable $\lambda_i$. Similarly, the $X^{(j)}$ reduce to the sequence $\tilde{X}(\vec{j})$, and the series $u_1, u_2$ become the classical SPPS solutions.

On the other hand, when all the $r_i$ are equal, the formal power $\tilde{X}(\vec{j})$ is unchanged when the entries $j_1, \ldots, j_d$ are permuted, so the sum only depends on the degree $|\vec{j}|$, giving $\tilde{X}(\vec{j}) = c_{\vec{j}} \tilde{X}(\vec{0})$, and similarly $X^{(j)} = c_{\vec{j}} X^{(\vec{0})}$, which are multiples of the classical formal powers. It follows that $u_1, u_2$ are the classical solutions obtained using $|\vec{\lambda}| = \lambda_1 + \cdots + \lambda_d$ in place of the single spectral parameter.

2.4.2 Computational aspects

We make a few observations to simplify the task of programming the formal powers. One can omit the factors $|\vec{j}|$ in the recursive definitions (6), (12) of $\tilde{X}(\vec{j})$ and $X^{(j)}$, producing “rescaled powers” $\tilde{X}^{*}(\vec{j})$ and $X^{*}(\vec{j})$ defined by

$$
\tilde{X}^{*}(\vec{j}) = \begin{cases}
\int r_i u_0^2 \tilde{X}(\vec{j} - \vec{\delta}_i), & \text{if } \vec{j} \text{ odd}, \\
\frac{1}{pu_0^2} \sum_{i=1}^d \tilde{X}(\vec{j} - \vec{\delta}_i), & \text{if } \vec{j} \text{ even}.
\end{cases}
$$

and similarly for $X^{*}(\vec{j})$. Then by induction

$$
\tilde{X}^{*}(\vec{j}) = \frac{1}{|\vec{j}|!} \tilde{X}(\vec{j}), \quad X^{*}(\vec{j}) = \frac{1}{|\vec{j}|!} X(\vec{j}).
$$

Besides this saving in multiplications when calculating the formal powers (and often avoiding calculating with very large numbers), it is no longer necessary to divide by these factorials to obtain the terms in the sums for $u_1, u_2, u_1', u_2'$; i.e., we have simply

$$u_1 = u_0 \sum_{\vec{n}} \tilde{X}^{*}(\vec{n}) \lambda_1^{n_1} \cdots \lambda_d^{n_d},$$

etc. This is because the coefficient of each formal power in the formulas (14) is precisely the reciprocal of the factorial of its degree.

The construction of the tables for $\tilde{X}$ and $X$ is seen to be identical when we disregard the initial terms $X^{((1/d)\vec{1} - \vec{\delta}_i)}$ from the second table. That is, according to whether we insert the function $1 = \tilde{X}(\vec{0})$ or $1/(pu_0^2) = X^{((1/d)\vec{1})}$ in the upper left hand corner, the same procedure of multiplying and then integrating will produce the entire table for $\tilde{X}(\vec{j})$ or $X^{(j)}$ respectively. Both tables and the corresponding power series can thus be computed via a single program, except that in the formula (13) for $u_2'$, the first term corresponding to $\vec{n} = \vec{0}$ contains negative exponents and is not found in the truncated table. Its value is

$$
\frac{1}{pu_0} \left( \frac{1}{0!} \sum_{i=1}^d X^{((1/d)\vec{1} - \vec{\delta}_i)} \right) \lambda_1^0 \cdots \lambda_d^0 = \frac{1}{pu_0}.
$$
This term must be added in separately to obtain \( u'_2 \).

When programming, one may likely prefer to drop the fractional parts of the indices, using effectively
\[
\hat{X}^{(j)} = \hat{X}^{(j + (1/d)\mathbf{i})}.
\]

In the development of the theory given above, this amounts to replacing the coefficient \(|\mathbf{j}|\) with \(|\mathbf{j}| + 1\), which is the true degree of \( \hat{X}^{(j)} \).

It is easily seen that if \( u_0 \) is a solution of (1) for a fixed multiparameter \((\lambda_1,0,\ldots,\lambda_d,0) \in \mathbb{C}^d\), then our construction of \( \hat{X}^{(j)} \), \( X^{(j)} \) will produce series in powers of \( \lambda_1 - \lambda_{1,0}, \ldots, \lambda_d - \lambda_{d,0} \) analogous to (14)–(15). This can be used to recenter the series for obtaining increased accuracy as in [23].

When calculating one must truncate the problem, say by using a finite number \( M \) of points of \([x_1, x_2]\) when integrating, and by approximating the series (14)–(15) with polynomials formed of the terms for \(|\mathbf{n}| \leq N\). The total number of formal powers in \( \{\hat{X}^{(j)}, X^{(j)}\}_{|\mathbf{n}| \leq N} \) grows as \( O(N^d) \), so the memory requirement is of the order \( O(MN^d) \). For boundary value problems this can be reduced by saving only the last value \( \hat{X}^{(j)}(x_2), X^{(j)}(x_2) \) once the values interior to the interval are no longer needed for further integrations. The resulting memory cost \( O(MN) + O(N^d) \) is in fact a great savings since often \( M \) is much larger than \( N \).

3 Numerical examples

We give some examples for \( d = 2 \). The operational parameters \( M, N \) are as described at the end of the last section; calculations were carried out in Mathematica.

3.1 Boundary value problems

Example 1. This simple example uses constant coefficients \( p = 1, q = 0, r_1 = r_2 = -1 \). The equation \( u'' = -(\lambda_1 + \lambda_2)u \) has normalized solutions \( v_1(x) = \cos(\sqrt{\lambda_1 + \lambda_2}x), v_2(x) = \sin(\sqrt{\lambda_1 + \lambda_2}x)/\sqrt{\lambda_1 + \lambda_2} \). On the interval \([x_1, x_2] = [0, \pi]\), the SPPS solutions of Corollary 6 with \( M = 800, N = 20 \) are found to agree with these formulas to within \( 10^{-9} \) for \( |\lambda_i| \leq 1 \). As is common with polynomial approximations, the accuracy drops rapidly for larger values of \( |\lambda_i| \) when the truncation limit \( N \) is fixed. We impose the boundary conditions \( u(0) = 0, u(\pi) = 0 \). The graph of the characteristic function \( \chi(\lambda_1, \lambda_2) \) (eigensurface) is shown in Figure 3. The eigencurves \( \chi = 0 \), calculated numerically from \( \chi \) via the function ContourPlot in the figure, coincide with the solutions of
\[
\lambda_1 + \lambda_2 = \frac{k^2 \pi^2}{b^2}
\]
for \( k = 1, 2, 3, 4 \). Indeed, the values of \( |\chi(\lambda_1, \lambda_2)| \) for \( |\lambda_i| \leq 5 \) for \( k = 1, 2, 3, 4 \) are less than \( 10^{-12}, 10^{-12}, 10^{-10}, 10^{-5} \) respectively. When the maximal degree of the powers is reduced to \( N = 16 \), the level curve for \( k = 4 \) is visibly far off the mark.
Example 2. This example, with \( p(x) = 1 \), \( q(x) = \cos x \), \( r_1(x) = \cos(x^2) \), \( r_2(x) = \cos x \), which is not amenable to a solution in closed form, is chosen to illustrate level sets which are not connected and which contain closed curves. Using the same interval \([0, \pi]\) and boundary conditions \( u(0) = 0 \), \( u(\pi) = 0 \), we find the characteristic function and its zero set as depicted in Figure 4. For illustration we take an arbitrary section \( \lambda_2 = 1.0 \), and restrict \( \chi \) to this value (Figure 5). The corresponding numerical pairs \((\lambda_1, \lambda_2)\) determine an ordinary differential equation which can be solved numerically by \texttt{NDSolve} using the boundary condition at \( x = 0 \) to define an initial condition. The resulting values at \( x = \pi \) were found to differ from \( \chi(\lambda_1, \lambda_2) \) by less than \( 10^{-6} \) when the experiment was carried out with \( M = 100 \), \( N = 12 \). The calculation of the characteristic function took about 0.3 seconds, and then each value of \( \lambda_1 \) less than a thousandth of a second on a portable computer (this does not include the time for checking by solving the initial value problem). The three eigenvalues \( \lambda_1 \approx -9.5644, -4.3944, 3.9177 \) in the range considered are easily located by techniques of numerical approximation of zeroes of polynomials.

Example 3. The following example involves consideration of complex eigenvalues. The boundary value problem

\[
y''(t) + (E + z \text{sgn } t)y(t) = 0, \quad y(-1) = y(1) = 0,
\]

where \( \text{sgn } x \) is the sign of \( x \), was studied in detail in [30]. A spectral surface is formed of pairs
(E, z) ∈ C². Let λ₁ = E, λ₂ = z, r₁(x) = -1, r₂(x) = sgn(x) for -1 ≤ x ≤ 1.

For the SPPS calculation, due to the jump singularity in r₂, it would be appropriate to integrate separately on [-1, 0] and [0, 1]. For this example, however, we simply calculate the formal powers with M = 10,000 mesh points, and settle for about five significant figures in the integrations. Since |χ| does not change sign near its zeros, we take the logarithm; then the plotting routine (Plot3D) easily reveals the set where χ(λ₁, λ₂) = 0 as shown in Figure 6, where we have taken λ₁ real and λ₂ purely imaginary.

In [30] certain curves in the spectral Riemann surface were explicitly parametrized as

λ₁(s) = s² - h(s)²,  \quad λ₂(s) = 2ish(s)  \quad (25)

where s ∈ \bigcup_{n=0}^{∞} [(n + 1/2)π, (n + 1)π] and where h is defined implicitly by the relation

s\sin(2s) + h(s)\sinh(2h(s)) = 0.

These curves were used to show that the surface is connected by joining various 1-complex-dimensional parts. The first interval s ∈ [π/2, π] corresponds approximately to 2.467 ≤ λ₁ ≤ π², 0 ≤ λ₂/i ≤ 4.475, and is the smallest eigencurve revealed in the plot. For the values given by (25) with values of s in this interval we find numerically that |χ(λ₁(s), λ₂(s))| < 10⁻⁴ for s ∈ [π/2, π] when N = 20.

Figure 6: log |χ(λ₁, λ₂)| for (24) with N = 40 (left); detail of region around smallest eigencurve with N = 16 (right).
3.2 Application to electromagnetic transmission

Example 4. This example is based on [10] from which we restate the minimum possible of background material. The plane \( \mathbb{R}^2 = \{(x,y)\} \) is partitioned into the regions

\[
\Omega_1 = \{x < 0\}, \quad \Omega_0 = \{0 < x < b\}, \quad \Omega_2 = \{x > b\},
\]

which are assumed to be composed of materials such that the index of refraction in \( \Omega_1 \) and \( \Omega_2 \) takes constant values denoted \( n_1, n_2 \) respectively, while in the inhomogeneous region \( \Omega_0 \) it is a function \( n = n(x) \) independently of \( y \). These values are bounded below by \( 1 \). An electromagnetic wave of the form \( e^{-ik_1x} \) travelling in \( \Omega_1 \) strikes the boundary line \( x = 0 \) with \( \Omega_0 \) at an angle \( \theta \) from the perpendicular, and is partially reflected back into \( \Omega_1 \) as \( u(x) = e^{-ik_1x} + R e^{ik_1x} \) and partially transmitted into \( \Omega_2 \) at \( x = d \) as \( u(x) = e^{-ik_2x} \). The parameter

\[
\beta = k \sin \theta \tag{26}
\]

is introduced, where \( k = 2\pi/\lambda \) is the wave number in terms of the wavelength \( \lambda \) (here \( \lambda \) will not denote an eigenvalue). In \( \Omega_0 \) the wave is governed by the differential equation

\[
u''(x) + (k^2n(x)^2 - \beta^2)u(x) = 0 \tag{27}
\]

(for the “s-polarization”, and a similar equation for the “p-polarization”). The problem is the determination of the complex constants \( R \) and \( T \), known as the reflection and transmission coefficients. In [10] the formulas

\[
R = \frac{-k_1k_2v_2(b) - v_1'(b) - ik_2v_1(b) + ik_1v_2'(b)}{(v_1'(b) - k_1k_2v_2(b)) + i(k_2v_1(b) + k_1v_2'(b))}, \quad T = \frac{2ik_1(v_1(b)v_2'(b) - v_1'(b)v_2(b))e^{-ik_2b}}{(v_1'(b) - k_1k_2v_2(b)) + i(k_2v_1(b) + k_1v_2'(b))} \tag{28}
\]

were given, where \( k_1 = \sqrt{k^2n_1^2 - \beta^2}, \quad k_2 = \sqrt{k^2n_2^2 - \beta^2} \). It was shown how by fixing \( k \) in (26) and then using \( \beta^2 \) as the spectral parameter, the SPPS formulas for dimension \( d = 1 \) can be used to calculate \( R \) and \( T \) for varying angles of incidence \( \theta \). Examples were given for three sample functions \( n(x) \). All were for normal incidence \( \beta = 0 \), for which it is not difficult to calculate the solution of the differential equation analytically in terms of special functions for the examples considered (see for example [34]), and thus compare the accuracy. Similar calculations using SPPS were carried out in [12], again for normal incidence, with many graphs comparing the results to other numerical methods used in optics.

Equation (11) for \( d = 2 \) with \( \lambda_1 = \beta^2, \quad \lambda_2 = -k^2, \quad v_1(x) = 1, \quad v_2(x) = n^2 \) takes the form (27). We apply Corollary 9 to obtain normalized solutions \( v_1, v_2 \), and then substitute these together with

\[
k_1 = \sqrt{-\lambda_1 - \lambda_2 n_1^2}, \quad k_2 = \sqrt{-\lambda_1 - \lambda_2 n_2^2} \tag{29}
\]

in (28). This produces analytic expressions \( R(\lambda_1, \lambda_2), T(\lambda_1, \lambda_2) \) which, while they are not simple power series, serve conveniently for calculations.

We will take one example, the “hyperbolic” refractive profile

\[
n(x) = n(0)e^{(x/b)\log(n(b)/n(0))} \tag{30}
\]
with $d = 1$, $n_1 = 1.0$, $n(0) = 1.4$, $n(b) = 2.1$, $n_2 = 1.5$. Further, we set $b = 1$. In Figure 7 all graphs were plotted after a single calculation of the series for $\chi(\lambda_1, \lambda_2)$ and its substitution in the expressions (28) for the parameters given above. It follows from (26) that $b/\lambda \geq b/\beta (2\pi)$, which determines our starting point for plotting the curves. For normal incidence $\beta = 0$, conservation laws require the expression

$$|R|^2 + \frac{n_2}{n_1}|T|^2$$

(31)

to be equal to 1; this is seen in the first graph, which agrees with Figure 6 of [10]. For other values of $\beta$ we have spot-checked numerically by selecting various values of the dimensionless quantities $\beta$ and $b/\lambda$, then solving the corresponding (28) numerically with NDSolve, as in the previous example. The final values $v_1(b), \ldots, v_2(b)$ produce values of $R, T$ via (28) for checking against the $\chi$-values plotted here. The results are given in Table 1. All of the data here is affected by the fact, observed in [10], that arithmetic operations in (28) reduce the accuracy produced by the differential equations by several significant figures.

4 Closing remarks

We have shown how the representation of the solutions of the Sturm-Liouville differential equation in terms of power series in a single spectral parameter may be generalized to several parameters $\lambda_1, \ldots, \lambda_d$. We hope that this will make possible a deeper analysis and simplified computation for many problems in physics and engineering, which have been approached up to now by fixing the values of all parameters but one, and solving by uniparameter methods.

Regarding the many aspects of uniparameter SPPS theory which have been developed up
to now, we point out as illustrative examples only two possible areas for using several spectral parameters.

The so-called \textit{Sturm-Liouville pencils}

$$(pu')' + qu = \left(\sum r_i \lambda^i\right)u$$

have been investigated from the SPPS perspective in \texttt{arXiv:1404.1520}. This equation is a particular case of (1) with $\lambda_1 = 1$, $\lambda_2 = \lambda, \ldots, \lambda_d = \lambda^{d-1}$. Thus our formulas provide the SPPS series for this equation directly.

In another direction, coefficient functions with singularities at one of the endpoints $[x_1, x_2]$, such as occur in Bessel’s equation, have led to modified versions of the SPPS formulas \cite{11}. Similar results can be expected to hold also for several spectral parameters.

We close with the observation that an alternative construction to the one described in this paper may be developed by first setting all but one of the spectral parameters to zero, for example considering

$$(py')' + qy = \lambda_1 r_1 y,$$

and writing down the classical formulas for solutions $w^{[\lambda_1]}_1, w^{[\lambda_1]}_2$ depending on this parameter. These can be regarded as solutions of

$$(py')' + (q - \lambda_1 r_1)y = 0,$$

and after choosing a suitable nonvanishing linear combination, this can be used as the seed for solving

$$(py')' + (q - \lambda_1 r_1)y = \lambda_2 r_2 u$$

to obtain $w^{[\lambda_1, \lambda_2]}_1, w^{[\lambda_1, \lambda_2]}_2$, and so forth. Even for the case $d = 2$ the resulting calculations to recover the coefficients of the SPPS series turn out to be surprisingly complicated, and involve many products of the nested integrals which cancel out at the end. The author is grateful to S. Torba for suggesting the simpler approach followed in the present work.

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References

[1] Arscott, F. M.: Integral-equation formulation of two-parameter eigenvalue problems. Spectral theory and asymptotics of differential equations (Proc. Conf., Scheveningen, 1973). North-Holland Math. Studies 13, 95–102 (1974)

[2] Arscott, F. M.: Two-parameter eigenvalue problems in differential equations. Proc. London Math. Soc. 14, 459–470 (1964)

[3] Atkinson, F. V.: Multiparameter spectral theory. Bull. Amer. Math. Soc. 74, 1–27 (1968)

[4] Atkinson, F., Mingarelli, A. B.: Multiparameter Eigenvalue Problems: Sturm-Liouville Theory. CRC Press, Boca Raton (2011)

[5] Blancarte, H., Campos, H. M., Khmelnytskaya, K. V.: Spectral parameter power series method for discontinuous coefficients. Math. Methods Appl. Sci., DOI:10.1002/mma.3282

[6] Brown, P. R., Porter, R. M.: Conformal mapping of circular quadrilaterals and Weierstrass elliptic functions. Comp. Methods Funct. Theory 11:2, 463–486 (2011)

[7] Browne, P. J., Sleeman, B. D.: Nonlinear multiparameter Sturm-Liouville problems. J. Differential Equations 34, 139–146 (1979)

[8] Campos, H., Kravchenko, V. V.: Fundamentals of Bicomplex Pseudoanalytic Function Theory: Cauchy Integral Formulas, Negative Formal Powers and Schrödinger Equations with Complex Coefficients. Complex Anal. Oper. Theory 7, 485–518 (2013)

[9] Campos, H., Kravchenko, V. V., Méndez L. M.: Fundamentals of bicomplex pseudoanalytic function theory: Cauchy integral formulas, negative formal powers and Schrödinger equations with complex coefficients. Complex Anal. Oper. Theory 7:2 485–518

[10] Castillo-Pérez, R., Khmelnytskaya, K. V., Kravchenko, V. V., Oviedo-Galdeano, H.: Efficient calculation of the reflectance and transmittance of finite inhomogeneous layers. J. Opt. A: Pure Appl. Opt. 11 (2009), doi:10.1088/1464-4258/11/6/065707

[11] Castillo-Pérez, R., Kravchenko, V. V., Torba, S. M.: Spectral parameter power series for perturbed Bessel equations. Appl. Math. Comput. 220, 676–694 (2013)

[12] Cedillo Diaz, A.: Análisis para el cálculo de la reflectancia y transmittancia en un medio estratificado no homogéneo con punto de retorno, Master's thesis, Instituto Politécnico Nacional, Mexico (2012)

[13] Chanane, B., Boucherif, A.: Computation of the Eigenpairs of Two-Parameter Sturm-Liouville Problems Using the Regularized Sampling Method, Abstract and Applied Analysis, Volume 2014, Article ID 695303, doi:10.1155/2014/695303

[14] Kelley, W. G., Peterson, A. C.: The Theory of Differential Equations: Classical and Qualitative, Springer Science & Business Media, New York (2010)
[15] Khmelnytskaya, K. V., Kravchenko, V. V., Baldenebro-Obeso, J. A.: Spectral parameter power series for fourth-order Sturm-Liouville problems. Appl. Math. Comput. 219:9, 3610–3624 (2012)

[16] Khmelnytskaya, K. V., Kravchenko, Rosu, H. C.: Eigenvalue problems, spectral parameter power series, and modern applications. Math. Methods Appl. Sci., DOI:10.1002/mma.3213

[17] Khmelnytskaya, K. V., Kravchenko, V. V., Torba, S. M., Tremblay, S.: Wave polynomials, transmutations and Cauchy’s problem for the Klein-Gordon equation. J. Math. Anal. Appl. 399:1, 191–212 (2013)

[18] Khmelnytskaya, K. V., Serroukh, I.: The heat transfer problem for inhomogeneous materials in photoacoustic applications and spectral parameter power series. Math. Methods Appl. Sci. 36:14, 1878–1891 (2013)

[19] Khmelnytskaya, K. V., Torchynska, T. V.: Reconstruction of potentials in quantum dots and other small symmetric structures. Math. Methods Appl. Sci. 33:4, 469–472 (2010)

[20] Kravchenko, V. V.: A representation for solutions of the Sturm-Liouville equation. Complex Var. Elliptic Eq. 53, 775–789 (2008)

[21] Kravchenko, V. V.: On the completeness of systems of recursive integrals. Commun. Math. Anal. Conf. 3, 172–176 (2011)

[22] Kravchenko, V. V., Morelos, S., Tremblay S.: Complete systems of recursive integrals and Taylor series for solutions of Sturm-Liouville equations. Math. Methods ApplSci., published online, doi:10.1002/mma.1596.

[23] Kravchenko V. V., Porter, R. M.: Spectral parameter power series for Sturm-Liouville problems. Math. Meth. Appl. Sci. 33, 459–468 (2010)

[24] Kravchenko V. V., Porter, R. M.: Conformal mapping of right circular quadrilaterals. Complex Var. Elliptic Eq. 56:5, 1747–6941 (2011)

[25] Kravchenko V. V., Torba, S. M.: Transmutations for Darboux transformed operators with applications. J. Phys. A: Math. Theor. 45, (21 pp.) #075201 (2012)

[26] Kravchenko V. V., Torba, S. M.: Modified spectral parameter power series representations for solutions of Sturm–Liouville equations and their applications. Applied Mathematics and Computation 238 82–105 (2014)

[27] Kravchenko V. V., Torba, S. M.: Construction of transmutation operators and hyperbolic pseudoanalytic functions. Complex Anal. Oper. Theory 9:2, 379–429 (2015)

[28] Kravchenko V. V., Torba, S. M.: Analytic approximation of transmutation operators and applications to highly accurate solution of spectral problems. J. Comput. Appl. Math. 275, 1–26 (2015)

[29] Kravchenko V. V., Velasco-García, U.: Dispersion equation and eigenvalues for the Zakharov-Shabat system using spectral parameter power series. J. Math. Phys. 52, #063517 (2011)
[30] Michel, J., Volkmer, H.: On the spectral surface of a model two-parameter Sturm-Liouville problem. Complex Var. Elliptic Eq. 58:3, 333–350 (2013)

[31] Sleeman, B. D.: Multiparameter spectral theory in Hilbert space. J. Math. Anal. Appl. 65, 511-530 (1978)

[32] Turyn, L.: Sturm-Liouville problems with several parameters. J. Differential Equations 38:2, 239–259 (1980)

[33] Volkmer, H.: Multiparameter Eigenvalue Problems and Expansion Theorems. Springer-Verlag, Lecture Notes in Math. 1356 (1988)

[34] Yeh, P.: Optical Waves in Layered Media. Wiley, New York (2005)