ON THE SUM OF THE RECIPROCALS OF THE DIFFERENCES BETWEEN CONSECUTIVE PRIMES

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ABSTRACT. Let \( p_n \) denote the \( n \)-th prime number, and let \( d_n = p_{n+1} - p_n \). Under the Hardy–Littlewood prime-pair conjecture, we prove

\[
\sum_{n \leq X} \log \alpha d_n \sim \begin{cases} 
\frac{X \log \log X}{\log X} & \alpha = -1, \\
\frac{X \log \log X}{1+\alpha} & \alpha > -1,
\end{cases}
\]

and establish asymptotic properties for some series of \( d_n \) without the Hardy–Littlewood prime-pair conjecture.

1. Introduction

Let \( p_n \) denote the \( n \)-th prime number, and let \( d_n = p_{n+1} - p_n \). In [1], Erdős and Nathanson show that for \( c > 2 \),

\[
\sum_{n=3}^{\infty} \frac{1}{d_n n \log \log n} < +\infty.
\]

The authors give a heuristic argument explaining why the series must diverge for \( c = 2 \). We will prove the above (1.1) by some conclusions of the sieve method.

Let \( \mathcal{H} = \{0, h_1, \ldots, h_{k-1}\} \) be a set of \( k(k \geq 2) \) distinct integers satisfying \( 0 < h_1 < h_2 < \cdots < h_{k-1} \) and not covering all residue classes to any prime modulus. Also, denote

\[
\pi(x; \mathcal{H}) = \#\{n \in \mathbb{N} : n + h_{k-1} \leq x, n, n + h_1, \ldots, n + h_{k-1} \text{ are all primes}\}.
\]

The Hardy–Littlewood prime \( k \)-tuple conjecture is that, for \( X \rightarrow +\infty \),

\[
\pi(X; \mathcal{H}) = \mathcal{S}(\mathcal{H}) \frac{X}{\log^k X} (1 + o(1)),
\]

where the singular series

\[
\mathcal{S}(\mathcal{H}) = \prod_p \left(1 - \frac{v_{\mathcal{H}}(p)}{p} \right) \left(1 - \frac{1}{p} \right)^{-k},
\]

with \( p \) running through all the primes and

\[
v_{\mathcal{H}}(p) = \#\{m \pmod{p} : m(m + h_1) \ldots (m + h_{k-1}) \equiv 0 \pmod{p}\}.
\]

We will also need the following well-known sieve bound, for \( X \) sufficiently large,

\[
\pi(X; \{0, h, d\}) \leq 2^3 \times 3! \mathcal{S}(\{0, h, d\}) \frac{X}{\log^2 X} (1 + o(1)),
\]

when \( \mathcal{S}(\{0, h, d\}) \neq 0 \). (See Iwaniec and Kowalski’s excellent monograph [2].)

To prove our main theorem, we will need the following Hardy–Littlewood prime-pair conjecture.

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Conjecture 1.1. Let $X$ be sufficiently large and $d \ll \log X$ be a natural number. Then

\begin{equation}
\pi(X, \{0, d\}) = \mathcal{G}(\{0, d\}) \frac{X}{\log^2 X} (1 + o(1)),
\end{equation}

where

\[
\mathcal{G}(\{0, d\}) = \begin{cases}
2 \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|d, p > 2} \left(\frac{p-1}{p-2}\right) & \text{if } d \text{ is even}, \\
0 & \text{if } d \text{ is odd}.
\end{cases}
\]

with the product extending over all primes $p > 2$.

Our main result can be summarized as follows under the above conjecture.

Theorem 1.1. Assume that the Hardy–Littlewood prime-pair conjecture holds for all sufficiently large $X$. Then we have

\[
\sum_{n \leq X} \frac{\log^\alpha d_n}{d_n} \sim \begin{cases}
\frac{X (\log \log X)^{1+\alpha}}{\log X} & \alpha > -1, \\
\frac{X \log \log \log X}{\log^2 X} & \alpha = -1.
\end{cases}
\]

Letting $\alpha = 0$ in above theorem and using Abel’s summation formula, one can obtain the following corollary.

Corollary 1.2. Let $X$ be sufficiently large. Then

\[
\sum_{3 \leq n \leq X} \frac{\log^\alpha d_n}{d_n} \sim \begin{cases}
\frac{\gamma_c + o(1)}{\log \log X} & c > 2, \\
\frac{O((\log \log X)^2 - c)}{(\log \log X)^2 (1 + o(1))} & c < 2,
\end{cases}
\]

where $\gamma_c$ is a constant.

Without the Hardy–Littlewood prime-pair conjecture, using the same idea one can obtain the following result.

Theorem 1.3. Let $X$ be sufficiently large. Then

\[
\sum_{n \leq X} \frac{\log^\alpha d_n}{d_n} \ll \begin{cases}
\frac{X (\log \log X)^{1+\alpha}}{\log X} & \alpha > -1, \\
\frac{X \log \log \log X}{\log^2 X} & \alpha = -1.
\end{cases}
\]

Similar to Corollary 1.2, one can obtain the following corollary.

Corollary 1.4. Let $X$ be sufficiently large. Then

\[
\sum_{3 \leq n \leq X} \frac{1}{d_n (\log \log n)^c} = \begin{cases}
\frac{\gamma_c + o(1)}{\log \log X} & c > 2, \\
O((\log \log \log X)^2 - c) & c < 2,
\end{cases}
\]

where $\gamma_c$ is a constant.

2. Basic Lemma

To prove Theorem 1.1, we need the following lemmas.

Lemma 2.1. (See [3, Proposition 1]). Let $X$ be sufficiently large. Then

\[
\sum_{d \leq X} \mathcal{G}(\{0, d\}) - X + \frac{\log X}{2} \ll \log^2 X.
\]

As a special case of [4, Lemma 2], we have
Lemma 2.2. Let $d$ be an even integer. Then

$$\sum_{h=1}^{d-1} \mathcal{S}(\{0, h, d\}) = \mathcal{S}(\{0, d\})d(1 + o_d(1)).$$

The following lemma is important in this paper.

Lemma 2.3. Let $f(x) \in \mathcal{C}^1[2, +\infty)$ be strictly monotonically decreasing to 0, and $\int_2^\infty f(t) \, dt$ divergence. Also, let $X$ sufficiently large, $y = o(\log X)$ and $y \gg \log \log X$.

(a) Using Conjecture 1.1, we have

$$\sum_{d_n \leq y \atop p_n+1 \leq X} f(d_n) \sim \frac{X}{\log X} \int_2^y f(t) \, dt.$$

(b). Without using Conjecture 1.1, we have

$$\sum_{d_n \leq y \atop p_n+1 \leq X} f(d_n) \ll \frac{X}{\log X} \int_2^y f(t) \, dt.$$

Proof. The proof of parts (a) and (b) are essentially the same. Therefore, we prove part (a) only. Firstly, we have

$$\sum_{d_n \leq y \atop p_n+1 \leq X} f(d_n) = \sum_{d \leq y} f(d) \sum_{\substack{d_n = d \atop p_n+1 \leq X}} 1$$

$$= \sum_{d \leq y} f(d) \pi(X; \{0, d\}) + \sum_{d \leq y} f(d) \left( \sum_{\substack{d_n = d \atop p_n+1 \leq X}} 1 - \pi(X; \{0, d\}) \right).$$

By the inclusion-exclusion principle, it is easy to see that

$$\pi(X; \{0, d\}) - \sum_{h=1}^{d-1} \pi(X; \{0, h, d\}) \leq \sum_{\substack{d_n = d \atop p_n+1 \leq X}} 1 \leq \pi(X; \{0, d\}).$$

Hence

$$(2.1) \quad \sum_{d_n \leq y \atop p_n+1 \leq X} f(d_n) = \sum_{d \leq y} f(d) \pi(X; \{0, d\}) + O \left( \sum_{d \leq y} f(d) \sum_{h=1}^{d-1} \pi(X; \{0, h, d\}) \right).$$

Combining (1.2) with Lemma 2.2, we see that the error term in (2.1) is

$$\ll \frac{X}{\log^3 X} \sum_{d \leq y} f(d) \sum_{h=1}^{d-1} \mathcal{S}(\{0, h, d\}) \ll \frac{X}{\log^3 X} \sum_{d \leq y} f(d) d \mathcal{S}(\{0, d\}).$$

Using Abel’s summation formula, noting that $f(x) \in \mathcal{C}^1[2, +\infty)$ is strictly monotonically decreasing to 0 and $y \gg \log \log X$, we have

$$\sum_{d \leq y} f(d) d \mathcal{S}(\{0, d\}) = \int_2^y f(x) d \left( \sum_{d \leq x} \mathcal{S}(\{0, d\}) \right) \ll \int_2^y f(x) x \, dx.$$
Together with (1.1), we have
\[
(2.2) \quad \sum_{d_n \leq y \atop p_{n+1} \leq X} f(d_n) = \frac{X}{\log^2 X} (1 + o(1)) \sum_{d \leq y} f(d) \mathcal{G}(\{0, d\}) + O \left( \frac{X}{\log^2 X} \int_{2}^{y} f(x) x \, dx \right).
\]
Combining Lemma 2.1 and using Abel’s summation formula again, we obtain
\[
\sum_{d \leq y} f(d) \mathcal{G}(\{0, d\}) = \int_{2}^{y} f(x) d \left( \sum_{d \leq x} \mathcal{G}(\{0, d\}) \right)
= \int_{2}^{y} f(x) \, dx + O(1) + O(f(y) \log y) + O \left( \int_{2}^{y} \frac{f(x)}{x} \, dx \right).
\]
Since
\[
\int_{2}^{y} \frac{f(x)}{x} \, dx = f(y) \log y - \int_{2}^{y} f'(x) \log x \, dx + O(1)
\]
and \(-\int_{2}^{y} f'(x) \log x \, dx > 0\) by the assumption on \(f\), hence by (2.2) and (2.3) we have
\[
\sum_{d_n \leq y \atop p_{n+1} \leq X} f(d_n) = \frac{X}{\log^2 X} \left( 1 + o(1) \right) \int_{2}^{y} f(x) \, dx + O \left( \int_{2}^{y} \frac{f(x)}{x} \, dx + \int_{2}^{y} \frac{f(x) x \, dx}{\log X} \right).
\]
By using L’Hospital’s rule, we get
\[
\lim_{y \to +\infty} \frac{\int_{2}^{y} f(x) x^{-1} \, dx}{\int_{2}^{y} f(x) \, dx} = 0 \quad \text{and} \quad \lim_{y \to +\infty} \left| \frac{\int_{2}^{y} f(x) x \, dx}{y \int_{2}^{y} f(x) \, dx} \right| \leq 1.
\]
Hence
\[
\sum_{d_n \leq y \atop p_{n+1} \leq X} f(d_n) = \frac{X}{\log^2 X} \left( 1 + o(1) + O \left( \frac{y}{\log X} \right) \right) \int_{2}^{y} f(x) \, dx.
\]
On noting that \(y = o(\log X)\), we obtain the proof of part (a).

\[\square\]

Lemma 2.4. Let \(f(x) \in C^1[2, +\infty)\) be strictly monotonically decreasing to 0, and \(\int_{2}^{\infty} f(t) \, dt\) divergence. Also, let \(X\) sufficiently large and \(\log^2 X \leq x < y \leq \log^2 X\). Then we have
\[
\sum_{x < d_n \leq y \atop p_{n+1} \leq X} f(d_n) \ll \frac{X}{\log^2 X} \left( \int_{x}^{y} f(t) \, dt + f(x) \log \log X \right).
\]

Proof. Since \(f(x)\) is strictly monotonically decreasing and \(\log^2 X \leq x < y \leq \log^2 X\), we have
\[
\sum_{x < d_n \leq y \atop p_{n+1} \leq X} f(d_n) = \sum_{x < d \leq y} f(d) \sum_{d_n = d \atop p_{n+1} \leq X} \sum_{x < d \leq y} f(d) \pi(X; \{0, d\})
\ll \frac{X}{\log^2 X} \sum_{x < d \leq y} f(d) \mathcal{G}(\{0, d\}) \ll \frac{X}{\log^2 X} \left( \int_{x}^{y} f(t) \, dt + O(\log t) \right)
\ll \frac{X}{\log^2 X} \left( \int_{x}^{y} f(t) \, dt + f(x) \log y + \int_{x}^{y} |f'(t)| \log t \, dt \right)
\ll \frac{X}{\log^2 X} \left( \int_{x}^{y} f(t) \, dt + f(x) \log \log X \right).
\]
This completes the proof of the lemma. \[\square\]
3. The proof of main theorem

Let \( y = \log X (\log \log X)^{-1} \) and \( f(t) = t^{-1} \log \alpha \), \( \alpha \geq -1 \). Using Lemma 2.3 and Lemma 2.4, we have

\[
\sum_{p_{n+1} \leq X} f(d_n) = \sum_{d_n \leq y} f(d_n) + \sum_{\substack{d_n \leq \log X \leq y \leq d_n \leq \log X}} f(d_n) + \sum_{\substack{d_n > \log X \leq p_{n+1} \leq X}} f(d_n)
\]

\[
= \frac{X}{\log^2 X} \left( 1 + o(1) \right) \int_2^y f(x) \, dx + O \left( \frac{X \int_y^{\log X} f(t) \, dt}{\log^2 X} + \frac{X \log X}{\log^2 X} f(y) \right) + O \left( \frac{X f(\log X)}{\log X} \right).
\]

Substituting the values of \( f \) and \( y \) into the above equation, we obtain

\[
\sum_{p_{n+1} \leq X} \frac{\log^\alpha d_n}{d_n} = \frac{X}{\log^2 X} (1 + o(1)) \int_{\log 2}^{\log(\log X (\log \log X)^{-1})} u^\alpha \, du + \frac{X}{\log^2 X} \left( o(1) + O \left( (\log \log X)^\alpha \log \log \log X \right) \right).
\]

Hence we get

\[
(3.1) \quad \sum_{p_{n+1} \leq X} \frac{\log^\alpha d_n}{d_n} \sim \begin{cases} \frac{X}{\log^2 X} \frac{(\log \log X)^{1+\alpha}}{\log^{1+\alpha} X} & \alpha > -1, \\ \frac{X}{\log^2 X} \frac{\log \log \log X}{\log^2 X} & \alpha = -1. \end{cases}
\]

By prime number theorem, the maximum integer \( n \) satisfying \( p_{n+1} \leq X \) is \( X \log X (1 + o(1)) \) and substituting these values into (3.1) above completes the proof of the theorem.

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