Exact bit error rate expressions for interference-limited Poisson networks

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Closed-form expressions for the bit error rate (BER) in an interference-limited wireless network are derived. Users are distributed according to a Poisson point process and communicate over a Nakagami fading channel using the widely employed Gray-coded quadrature amplitude modulation. The way how interference influences the BER provides insight into the interference control mechanism to be implemented and helps to analyse the achievable data rates.

Introduction and contributions: Stochastic geometry has established as a mathematical discipline for analysing the properties of large wireless networks [1–3]. By modelling transmitter and receiver locations through a spatial stochastic process, for example, approximations and general tradeoffs of the network interference behaviour can be derived analytically. Insights on the design of communication protocols and resource allocation schemes have been reported [4–6]. The link outage rate has been the main performance metric considered so far. The bit error rate (BER), however, has not been investigated yet in a comprehensive manner, despite the fact that it is an ultimate performance metric.

In parallel to the advances on stochastic geometry analysis, Hamdi made contributions to the characterisation of wireless interference, but without accounting for spatial characteristics [7–9]. Specifically, he characterises the signal-to-interference-ratio (SIR) assuming knowledge of the involved interferer statistics. He reports closed-form expressions for usual factors featured in classical BER expressions. However, these expressions only approximate the interference case, and the lack of spatial features limits their range of application.

This Letter fills this research gap by deriving closed-form expressions for the BER in large, interference-limited networks with users randomly distributed according to a homogeneous Poisson point process (PPP) and communicating with quadrature amplitude modulation (QAM) over a Nakagami multipath fading channel. We denote this type of network as Poisson network. We also show how these expressions can be applied in different contexts. As a side result, we derive an expression for the moments of the SIR in a Poisson network.

Network interference model: Consider a wireless network with distributed users in space according to a two-dimensional homogeneous PPP Φ of intensity λ. Users access the channel in time slots without carrier sensing. We are interested in a typical receiver in the network, denoted by r0, communicating with a certain transmitter. Without loss of generality, we assume that r0 is located at the origin. The transmitter is not included in Φ but has a known distance d0 to r0; it transmits with power ps over a channel with path loss exponent α and fading coefficient h0. Gray-coded QAM is employed as modulation scheme.

All users are potential interferers to r0. The probability for a user to transmit in a given time slot is \( \text{Pr} \) (i.e. elements of a Bernoulli distribution). This determines a thinning on Φ, that is a subset of Φ contains all transmitting users at time symbol t. This subset Φs is a PPP with intensity \( \lambda s \). The interference caused by user i at location r0 depends on the transmit power ps, the distance di from i to r0, the path loss exponent α, and a fading coefficient hi of this link.

The signal arriving at symbol time t at r0 can be written as

\[
\gamma_0(t) = \sqrt{p_s d_0^{-\alpha} h_0} x(t) + \sum_{i \in \Phi} \sqrt{p_s d_i^{-\alpha} h_i} x(t) + n(t),
\]

(1)

where \( x(t), x_i(t) \) are random symbols drawn from a Gray-coded QAM constellation of size \( \mathcal{M} \) and unitary mean power \( \mathbb{E}[x_i(t)] = 1 \) is the mean noise at r0, and \( h_0 \) is assumed known. From (1), the power received from the desired transmitter is \( S = p_s d_0^{-\alpha} h_0 \| x \|^2 \) and the total interference power at the origin is \( I = \sum_{i \in \Phi} p_s d_i^{-\alpha} h_i \| x_i \|^2 \). The SIR at the origin is thus

\[
\text{SIR} = \frac{\gamma_0}{\gamma} = \frac{S}{I} = \frac{p_s d_0^{-\alpha} h_0 \| x \|^2}{\sum_{i \in \Phi} p_s d_i^{-\alpha} h_i \| x_i \|^2}.
\]

Considering an interference-limited network, we disregard thermal noise \( n(t) \) in (1) and focus on the SIR in what follows.

BER in AWGN channels: The BER of Gray-coded QAM is [10]

\[
P_{BER} = \xi \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \text{erfc}(\sqrt{2} \sqrt{\frac{k}{K}}),
\]

where \( \gamma_0 \) emphasises that AWGN is considered. The term \( M \) is the constellation size and \( 2D \) is the scaling factor for the Euclidean distance between adjacent signal points, with \( D = \sqrt{\log_2 M / 2(M-1)} \). The factor \( (2i+1) \) affecting \( D \) stand for the multiples of \( D \) that result in a bit error, and \( \xi = 1/\sqrt{\log_2 M} \). The weighting factors \( x(k,i) \in \mathbb{Z} \) account for the number of times a certain multiple of \( D \) results in an error on bit \( k \):

\[
X(k,i) = (-1)^{2i+1} \frac{1}{\sqrt{2M+2}}
\]

(4)

BER in Poisson networks: Let us consider Nakagami-m fading with shape factor m. Let interferers transmit with probability \( \lambda \) and non-unitary power \( p_s \). This setup applies to a variety of practical settings of interest. Note that the SIR in (2) depends on the network model. Therefore, the characterisation of its moments is needed to analyse how the BER is influenced by the interference.

Lemma 1: The average BER of Gray-coded QAM for the signal in (1) considering Nakagami-m fading, transmit power \( p_s = d_0^2 \), and average transmit power \( p_t \) and transmit probability \( \lambda \), for all interferers is

\[
P_{BER} = \xi \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} X(k, i) \left( 1 - \sum_{j=0}^{\infty} \frac{1}{\eta_{k,\lambda, p}} \right),
\]

where we introduced \( \lambda = 2\alpha \), where \( N \in [0, 1] \) is a matrix with rows having non-zero sum and with columns summing to \( j \) (i.e. elements \( n_{ij} \) are non-zero and sum up to \( j \)). \( \mathcal{N} \) is the family of all matrices \( N \) with that property and \( \mathcal{C}_N = \{ j! \prod_{i=1}^{j} n_{ij} \} \). Finally

\[
\eta_{k,\lambda, p} = \exp(-\lambda ||k||^2),
\]

Proof: See Appendix 1.

Lemma 2: The moments of the SIR defined in (2) considering Nakagami-m fading, transmit power \( p_s = d_0^2 \), and unitary transmit power \( p_t = 1 \) for all interferers i is

\[
E(\gamma^\alpha) = \frac{\pi \lambda (m+1) \Gamma(m+1) \Gamma(m+1)}{\Gamma(m+1) \Gamma(m+1)},
\]

(5)

Proof: See Appendix 2.

Fig. 1 shows (5) together with simulation results for 16-QAM for the case of \( m = 1 \) (Rayleigh fading). We consider \( \lambda \)-values from 0.001 to 0.5 and path loss exponents \( \alpha = 3, 4, 5 \), which result from (6) (with \( n = 1 \)) in the SIR range shown. Simulation results are obtained by averaging over 100,000 realisations of a PPP to double check the analytical results; they show good agreement. The BER obtained with the approximated method reported in [8, 9] is also shown for reference. The underestimation of the interference impact using the approximation is because the probability density function (pdf) of the interference has much heavier tails than the Gaussian distribution assumed in [8, 9].

Fig. 1 Average BER on Poisson interference network with Rayleigh fading

How the network model affects the expected SIR is depicted in Fig. 2. The relation between \( \lambda \) and \( \gamma \) in (6) is shown for \( n = 1 \) and different \( \alpha \)
Applications: Expression (5) can be applied in various types of wireless systems, such as wireless sensor networks using ALOHA and the up- and downlinks of cellular systems using orthogonal frequency division multiple access. For the latter, interference arises from frequency reuse in different cells [4]. The value of $\kappa$ reflects the traffic regime of the network with $\kappa = 1$ for a fully loaded network.

For a cellular network with device-to-device (D2D) capability, (5) needs to be adapted to account for the different interference sources involved: intercell interference $I_c$ and D2D interference $I_d$.

$$ I = I_c + I_d = \sum_{i \in \Phi_c} p_d i \|hi\|^2 + \sum_{i \in \Phi_d} p_d i \|hi\|^2. \quad (7) $$

With Rayleigh fading, the total BER is the sum of the BER from each source; this is not the case for Nakagami fading, giving the exponentiation of $I$ in (10). Nevertheless, applying the binomial expansion to $I$ (assuming $I_c$ and $I_d$ to be independent) we have [6]

$$ \Pr \{ y \geq \theta \} = \sum_{m=0}^{n-1} \sum_{i=0}^{m} \frac{1}{i!} E_k \left[ \left( \frac{2}{\lambda} m d_i \right)^i \exp \left( -m \theta d_i \right) \right] \times E_{k, \Phi_c} \left[ \left( \frac{2}{\lambda} d_{i, c} \right)^i \exp \left( -m \theta d_{i, c} \right) \right]. \quad (8) $$

The BER can be computed using (8) with $\theta$ replaced by $p_c/(2i + 1)D^2$ for $I_c$ and by $p_d/(2i + 1)D^2$ for $I_d$, where $p_c$ and $p_d$ are the average transmit powers of cellular and D2D mode interferers, respectively.

Our BER expressions hold for flat fading links between the interferers and $r_0$. In a multiuser system, they can be applied for each subcarrier and then averaged over the entire subcarrier set. Channel coding can also be accounted for by adjusting the SIR scale with the coding gain.

These results can be applied to find a good interference management policy. Given a certain network setup (density of transmitters, medium access etc.) and performance goal (e.g. network throughput), we can evaluate the rates that can be achieved by the users under a BER constraint. This helps us to assess whether an interference management policy is required and gives insights as to which policy is preferable.

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