Noncommutative spacetime and the fractional quantum Hall effect

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Abstract

We propose the two formalisms for obtaining the noncommutative spacetime in a magnetic field. One is the first-order formalism and the other is the second-order formalism. Although the noncommutative spacetime is realized manifestly in the first-order formalism, the second-order formalism would be more useful for calculating the physical quantities in the noncommutative geometry than the first-order one. Several interesting points for string theory and fractional quantum Hall effect are discussed. In particular, we point out that the noncommutative geometry is closely related to the fractional quantum Hall effect (FQHE).
Recently noncommutative geometry has attracted much interest in string and M-theory in the $B$-field \[1-6\]. Gauge theory on noncommutative boundary space are relevant to the quantization of D-branes in $B_{\mu\nu}$ fields. It is very important to notice that the noncommutativity can be easily realized by turning on a magnetic field in the plane. This situation is also applicable to studying the FQHE \[7\].

In this letter we study the two models which give us the same noncommutative effect. It is well known that the quantum mechanics of a nonrelativistic particle in a constant magnetic field ($B = B\hat{k}$) produces the noncommutative momentum $\pi$. Hence we start with the second-order Lagrangian \[8,9\]

$$L_m = \frac{1}{2} m \dot{r}^2 + \frac{q}{c} \dot{r} \cdot \mathbf{A} - V(r)$$

(1)

where $r = (x, y)$ is the coordinates and the vector potential ($\nabla \times \mathbf{A} = B$) is given by $A^i = -\epsilon^{ij} r^j B/2$. The corresponding Hamiltonian is given by

$$H_m = \frac{1}{2m} \left( p - \frac{q}{c} \mathbf{A} \right)^2 + V(r).$$

(2)

Here we have two momenta ($p, \pi = m \dot{r} = p - \frac{q}{c} \mathbf{A}$). $p$ is the canonical momentum which is a gauge variant quantity and thus is not a physical observable. On the other hand $\pi$ is the mechanical momentum which is a gauge invariant quantity and thus is a physical observable. Now let us calculate the commutator of $\pi$

$$[\pi^i, \pi^j] = iBe^{ij}$$

(3)

with $q/c = -1$. This shows that $\pi$ becomes a noncommutative momentum when a magnetic field is turned on. However, one finds the commutative spacetime such as $[r^i, r^j] = 0$.

In order to obtain the noncommutative geometry, we take the $m \to 0$ limit \[8\]

$$L_{m \to 0} = -\frac{B}{2} r^i \epsilon_{ij} r^j - V(r).$$

(4)

This is the first-order Lagrangian and thus, has a sympletic structure which enforces the noncommutative relation \[10\].
\[ [r^i, r^j] = \frac{i}{B} \epsilon^{ij}. \]  

(5)

The corresponding Hamiltonian \( \mathcal{H}_{m \to 0} \) is solely given by the potential

\[ \mathcal{H}_{m \to 0} = \frac{\partial \mathcal{L}_{m \to 0}}{\partial \dot{r}} \cdot \dot{r} - \mathcal{L}_{m \to 0} = V(r). \]  

(6)

This can be also derived from (2) by imposing the constraint

\[ \pi = m \dot{r} = p + A \simeq 0. \]  

(7)

Actually (5) emerges from obtaining the same equation of motion \( \dot{r}^i = -\epsilon^{ij} \frac{\partial V}{\partial r^j} \) by using both the Lagrangian formalism of (4) and the Hamiltonian framework of (6).

We wish to comment the following points:

1. We note that in the limit of \( m \to 0 \), the phase space of four \((p_x, p_y, x, y)\) is reduced to two \((x, y)\). This is so because of a consequence of the constraint (7). This implies that there is a reduction of degrees of freedom in the noncommutative spacetime, compared to the ordinary case. Hence we expect that this leads to a negative entropy correction in noncommutative super Yang-Mills theory [11].

2. The limit of \( m \to 0 \) with finite \( B \) is actually a projection onto the Lowest Landau Level(LLL). In other words, the LLL can be singled out by setting \( m \to 0 \) [8]. The same projection is also performed by the limit of \( B \to \infty \) with finite \( m \) [9,12].

3. It is difficult to compute some physical quantities with the constraint system \((\mathcal{L}_{m \to 0}, \mathcal{H}_{m \to 0})\). However, a similar computation with \((\mathcal{L}_m, \mathcal{H}_m)\) may be straightforward because of the simpler, expanded sympletic structure. It was argued that if an operator \( \mathcal{O} \) commutes with the constraint (7) when the latter vanishes, the results in two approaches will be the same [10]. Hence it would be correct that starting with the second-order formalism \( \mathcal{H}_m \), and then one requires either the limit of \( m \to 0 \) or the limit of \( B \to \infty \) to obtain the effect of noncommutative geometry. Our previous calculation was performed along this line [8,13]. The limiting procedure was carried
out through the action of $\omega_c = B/2m \to \infty$ at the final stage of calculation. For a dipole configuration with the harmonic interaction $[4]$, it would be better to use the second-order formalism than the first-order formalism.

4. For the string(gauge) theory calculations $[6,14]$, we propose a new procedure with the second-order formalism to obtain the noncommutative effect, since the calculation on noncommutative spacetime is not easy.

5. The noncommutative spacetime (5) leads to the FQHE. Using the second-quantized formalism, one finds the Hamiltonian $H_0$

$$H_0 = \frac{1}{2m} \psi^\dagger (p + A)^2 \psi. \quad (8)$$

This Hamiltonian is a cornerstone to study the FQHE $[15]$. The limit of $m \to 0$ leads to $(p + A) \psi_{LLL} = 0$, which means that the Fermi particles reside in the LLL. A general solution to this operator constraint is given by

$$\psi_{LLL}(z, \bar{z}) = \sum_{n=0}^{\infty} c_n \varphi_n(z, \bar{z}), \quad (9)$$

where $z = \sqrt{B/2}(x + iy)$, $\bar{z} = \sqrt{B/2}(x - iy)$, $\{c_n, c_m^\dagger\} = \delta_{nm}$. $\varphi_n$ is the $n$-th single particle state in the LLL,

$$\varphi_n(z, \bar{z}) = \frac{1}{\sqrt{\pi n!}} z^n \exp \left( -\frac{1}{2} |z|^2 \right) \quad (10)$$

with $B = 2$ for convenience. Since the LLL wave functions $\{\varphi_n\}$ are incomplete in the view of the total Hilbert space (full Landau levels), the fields have the unconventional commutator $[16]$:

$$\{\psi_{LLL}^\dagger(z_1, \bar{z}_1), \psi_{LLL}(z_2, \bar{z}_2)\} = \frac{1}{\pi} \exp \left( -\frac{1}{2} |z_1 - z_2|^2 + \frac{1}{2} (\bar{z}_1 z_2 - \bar{z}_2 z_1) \right) \equiv \{z_1 | z_2\} \quad (11)$$

instead of a conventional form of $\{\psi^\dagger(z_1, \bar{z}_1), \psi(z_2, \bar{z}_2)\} = \delta^2(z_1 - z_2)$. That is, a bilocal kernel $\{z_1 | z_2\}$ is introduced as a LLL analogue of the delta function and retains its reproducing property.
\[
\int d^2 z_1 F(z_1) \{ z_1 | z_2 \} = F(z_2)
\]  \hspace{1cm} (12)

where \(F(z)\) is any function of the form \(F(z) = f(z)e^{-|z|^2/2}\). The field \(\psi_{\text{LLL}}\) takes this form and thus one finds

\[
\psi_{\text{LLL}}(z, \bar{z}) = \int d^2 z \psi(z', \bar{z}') \{ z' | z \},
\]  \hspace{1cm} (13)

which means the overcompleteness of the LLL. In other words, we cannot vary \(\psi_{\text{LLL}}(z, \bar{z})\) independently at different point \((z, z')\) in the LLL, since neighboring fields are linked by (13). Actually this overcompleteness comes from \([z, \bar{z}] = 1\) (another form of (5)) and thus it is related to the noncommutative space.

We wish to describe the many-electron state out of any antisymmetric holomorphic function

\[
|f\rangle = \int \left[ \prod_{k=1}^{N} d^2 z_k \right] f(z_1, z_2, \ldots, z_N) e^{-\frac{1}{2} \sum_{k=1}^{N} |z_k|^2} \psi_{\text{LLL}}^\dagger(z_1) \psi_{\text{LLL}}^\dagger(z_2) \cdots \psi_{\text{LLL}}^\dagger(z_N) |0\rangle.
\]  \hspace{1cm} (14)

Thanks to the anticommutation relation (11) and the reproducing kernel (12), one enables to to express the inner product of two such states as

\[
\langle f | g \rangle = N! \int \left[ \prod_{k=1}^{N} d^2 z_k \right] \overline{f(z_k)g(z_k)} e^{-\frac{1}{2} \sum_{k=1}^{N} |z_k|^2}.
\]  \hspace{1cm} (15)

In particular, defining the state

\[
|z_1, z_2, \cdots, z_N\rangle = \psi_{\text{LLL}}^\dagger(z_1) \psi_{\text{LLL}}^\dagger(z_2) \cdots \psi_{\text{LLL}}^\dagger(z_N) |0\rangle,
\]  \hspace{1cm} (16)

we recover the many-electron wavefunction

\[
\langle z_1, z_2, \cdots, z_N | f \rangle = f(z_1, z_2, \ldots, z_N) e^{-\frac{1}{2} \sum_{k=1}^{N} |z_k|^2}.
\]  \hspace{1cm} (17)

Finally we obtain the Laughlin wavefunction which is essential to the FQHE by taking \(f = \prod_{k<l} (z_k - z_l)^{2n+1}\). The key physics of the FQHE results from the rearrangement of the nearly degenerate states \(\{\varphi_n\}\) in the LLL. Here the noncommutative feature of
$[z, \bar{z}] = 1$ plays an important role in constructing the holomorphic eigenfunction $\{\varphi_n\}$ of $\mathcal{H}_{m \to 0}$ in [3] [8]. Hence we insist that although the FQHE is an effect of many-body interaction, the noncommutative spacetime leads to the FQHE.

6. In order to get the Moyal bracket, we introduce the charge density operator $\hat{\rho}(z) = \psi_{\text{LLL}}^\dagger(z)\psi_{\text{LLL}}(z)$. And the second-quantized Hamiltonian $\mathcal{H} = \mathcal{V}$ is given by

$$\mathcal{V} = \int d^2z \psi_{\text{LLL}}^\dagger V(z, \bar{z}) \psi_{\text{LLL}}(z). \quad (18)$$

We commute $\mathcal{H}$ through $\psi_{\text{LLL}}$ to find

$$i\partial_t \psi_{\text{LLL}}(z) = \int d^2z' V(z') \psi_{\text{LLL}}(z') \{z'|z\}. \quad (19)$$

Introducing an apodized potential

$$\tilde{V}(z_1) = \int \frac{d^2z}{\pi} e^{-|z-z_1|^2} V(z), \quad (20)$$

we derive an analogue of the Moyal bracket

$$\partial_t \hat{\rho}(z) = \frac{1}{i} \sum_{n=1}^\infty \frac{1}{n!} \left\{ \partial^n_z \tilde{V}(z, \bar{z}) \partial^n_{\bar{z}} \hat{\rho}(z) - \partial^n_{\bar{z}} \tilde{V}(z, \bar{z}) \partial^n_z \hat{\rho}(z) \right\}. \quad (21)$$

In conclusion, we propose two formalisms which give us the same noncommutative effect. In particular, we show that the noncommutative geometry is closely related to the FQHE.

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