Comparison of methods to extract an asymmetry parameter from data

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Abstract

Several methods to extract an asymmetry parameter in an event distribution function are discussed and compared in terms of statistical precision and applicability. These methods are: simple counting rate asymmetries, event weighting procedures and the unbinned extended maximum likelihood method. It is known that weighting methods reach the same figure of merit (FOM) as the likelihood method in the limit of vanishing asymmetries. This article presents an improved weighting procedure reaching the FOM of the likelihood method for arbitrary asymmetries. Cases where the maximum likelihood method is not applicable are also discussed.

Key words: event weighting, minimal variance bound, Cramér-Rao inequality, asymmetry extraction, optimal observables, parameter determination, maximum likelihood

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1 Introduction

We consider two differential event distributions $n^\pm(x)$ following the functional form

$$n^\pm(x) = \alpha(x)(1 \pm \beta(x)A).$$

(1)

In a typical experimental situation encountered in particle physics $\alpha(x)$ includes a flux and acceptance factor and $\beta(x)$ is an analyzing power. Both depend on a set of kinematic variables here denoted by $x$. Concrete examples

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are spin cross section asymmetries and muon decay. In the latter case the asymmetry corresponds to the muon polarization. The two data sets (+ and −) are for example obtained by changing the sign of a polarization. The goal is to extract the parameter $A$ by measuring the event distributions $n^\pm(x)$. This would be an easy task if both $\alpha(x)$ and $\beta(x)$ were known. However in many applications the factor $\alpha(x)$ is not known, or not accurately enough known and only $\beta(x)$ is given.

Section 2 presents various methods to extract the parameter $A$: The simplest method, based on counting rate asymmetries, the more efficient extended unbinned maximum likelihood (EML) method and finally methods based on event weighting are discussed. While in Section 2 we assume that the factor $\alpha(x)$ is the same for both data sets, Section 3 extends the discussion to the case where one has two different factors. Up to Section 3 we assume that the number of observed events is large enough so that averages can be replaced by the corresponding expectation values. Effects occurring at low statistics event samples are discussed in Section 4. A summary and conclusions are given in Section 5.

2 Different Methods to extract $A$

2.1 Determining $A$ from counting rate asymmetry

The expectation value of the number of events for the two data sets reads

$$\langle N^\pm \rangle = \int n^\pm(x)dx = (1 \pm \langle \beta \rangle A) \int \alpha(x)dx$$

with $\langle \beta \rangle = \int \alpha \beta dx / \int \alpha dx$. The integrals run over the kinematic range of $x$. The asymmetry $A$ can be extracted without the knowledge of $\int \alpha(x)dx$:

$$A = \frac{1}{\langle \beta \rangle} \frac{\langle N^+ \rangle - \langle N^- \rangle}{\langle N^+ \rangle + \langle N^- \rangle}.$$  

Eq. $A$ leads to following estimator for $A$:

$$\hat{A}_{\text{cnt}} = \frac{N^+ + N^-}{\sum_+ \beta_i + \sum_- \beta_i} \frac{N^+ - N^-}{N^+ + N^-} = \frac{N^+ - N^-}{\sum_+ \beta_i + \sum_- \beta_i}$$

where $\beta_i \equiv \beta(x_i)$, $N^+$ and $N^-$ are the numbers of observed events. The sums $\sum_+$ and $\sum_-$ run over all events in the corresponding data set (+ or −).
As shown in Section 2.3 and App. A the figure of merit (FOM), i.e. the inverse of the variance on $A_{cnt}$, is

$$FOM_{A_{cnt}} = N \frac{\langle \beta \rangle^2}{1 - A^2 \langle \beta^2 \rangle}$$

(4)

where $N = N^+ + N^-$ denotes the total number of events.

This figure of merit may be increased if a cut is set to remove some data with low values of $\beta$. However, it will not reach the FOM of the unbinned extended likelihood method discussed now, unless $\beta(x)$ is constant.

2.2 Extended Maximum Likelihood (EML) Method

We now turn to the unbinned extended maximum likelihood (EML) method[12] which is known to reach the Cramér-Rao limit of the lowest possible statistical error. The log-likelihood function derived from Eq. (1) reads

$$l = \sum_+ \ln (\alpha_i (1 + \beta_i A)) - \langle N^+ \rangle (A) + \sum_- \ln (\alpha_i (1 - \beta_i A)) - \langle N^- \rangle (A).$$

Using the expression in Eq. (2) for the expectation values $\langle N^\pm \rangle$ results in

$$l = \sum_+ \ln (1 + \beta_i A) + \sum_- \ln (1 - \beta_i A) - 2 \int \alpha(x) dx - \sum_{+,-} \ln \alpha_i. \quad (5)$$

The last two terms do not depend on $A$ and can be ignored in the likelihood maximization. The asymmetry $A$ can thus be determined without knowledge of $\alpha(x)$. For small values of $\beta A$ one can even derive an analytic expression for $\hat{A}_{LH}$ which reads

$$\hat{A}_{LH} = \frac{\sum_+ \beta_i - \sum_- \beta_i}{\sum_+ \beta_i^2 + \sum_- \beta_i^2}.$$ 

(6)

For arbitrary asymmetries the maximization has to be done numerically. Note, that this requires CPU intensive sums over all events in the maximization procedure.

The figure of merit (FOM) is given by

$$FOM_{\hat{A}_{LH}} = -\frac{\partial^2 l}{\partial A^2} = \sum_+ \frac{\beta_i^2}{(1 + \beta_i A)^2} + \sum_- \frac{\beta_i^2}{(1 - \beta_i A)^2}.$$
Replacing the sum over events by integrals one finds

\[
FOM_{\hat{A}_{LH}} = \int \frac{\alpha (1 + \beta A) \beta^2}{(1 + \beta A)^2} \, dx + \int \frac{\alpha (1 - \beta A) \beta^2}{(1 - \beta A)^2} \, dx \\
= \int \frac{2 \alpha \beta^2}{1 - \beta^2 A^2} \, dx.
\]  

(7)

Noting that for an arbitrary function \( f(x) \) the average is defined by

\[
\langle f \rangle = \frac{\int f(x)(n^+(x) + n^-(x)) \, dx}{\int n^+(x) + n^-(x) \, dx} = \frac{\int \alpha(x)f(x) \, dx}{\int \alpha(x) \, dx},
\]

the figure of merit can be written as

\[
FOM_{\hat{A}_{LH}} = N \left( \frac{\beta^2}{1 - \beta^2 A^2} \right).
\]

(8)

2.3 Weighting Method

Next, consider the following estimator

\[
\hat{A}_w = \frac{\sum^+ w_i - \sum^- w_i}{\sum^+ w_i \beta_i + \sum^- w_i \beta_i}
\]

(9)

where \( w_i \equiv w(x_i) \) is a, for the moment arbitrary, weight factor assigned to every event. The expectation value of \( \hat{A}_w \) equals \( A \) independently of the weight function \( w(x) \) used. App. A shows that

\[
FOM_{\hat{A}_w} = N \frac{\langle w \beta \rangle^2}{\langle w^2 (1 - A^2 \beta^2) \rangle}.
\]

(10)

Two cases are of interest:
1.) Setting \( w = 1 \) corresponds to the counting rate asymmetry discussed in Section 2.1 and proves Eq. (4) for the FOM.

2.) In the case \( w = \beta \) the FOM is

\[
FOM_{\hat{A}_{w=\beta}} = N \frac{\langle \beta^2 \rangle}{1 - A^2 \langle \beta^4 \rangle / \langle \beta^2 \rangle}.
\]

A comparison with Eq. (8) indicates that \( FOM_{\hat{A}_{w=\beta}} \) coincides with the FOM of the likelihood method for vanishing \( A \). Actually, in this case, the two es-
timators are identical as can be seen by comparing Eq. (9) with \( w \equiv \beta \) and Eq. (9). Note that the estimator \( \hat{A}_w \) can be applied for arbitrary asymmetries as well, accepting a decrease of the FOM compared the EML estimator as discussed in Section 2.5.

Such a weighting procedure has been used for example in Refs. [3,4] to extract spin asymmetries in the case where \( \langle \beta \rangle A \ll 1 \). In Ref. [5] a weighting method is discussed to simultaneously extract signal and background asymmetries. The fact that a weighting procedure reaches the same FOM as the EML method was first discussed in Ref. [7] in the context of signal and background extractions. The next section shows that one can find a weight factor reaching the FOM of the EML method even in the case of non-vanishing asymmetries.

2.4 Improved Weighting Method

Variational calculus shows (s. App. [B]) that the maximum FOM is reached using a weight factor

\[
w = \frac{\beta}{1 - \beta^2 A_0^2}.
\]  

Here, \( A_0 \) is a first estimate of the asymmetry \( A \) obtained for example from the weighting method presented in Section 2.3. The weighting factor defined in Eq. (11) leads to the following estimator (the index \( iw \) stands for improved weight)

\[
\hat{A}_{iw} = \frac{\sum + \frac{\beta}{1 - \beta^2 A_0^2} - \sum - \frac{\beta}{1 - \beta^2 A_0^2}}{\sum + \frac{\beta^2}{1 - \beta^2 A_0^2} + \sum - \frac{\beta^2}{1 - \beta^2 A_0^2}}.
\]  

Eq. (10) gives

\[
\text{FOM}_{\hat{A}_{iw}} = N \frac{\beta^2}{1 - \beta^2 A_0^2} \left( \frac{1 - \beta^2 A_0^2}{(1 - \beta^2 A_0^2)^2} \right)^2.
\]  

Thus given a good estimate \( A_0 \approx A \), we get \( \text{FOM}_{\hat{A}_{iw}} = \text{FOM}_{\hat{A}_{EML}} \), i.e. the improved weighting method reaches the same FOM as the EML method for arbitrary asymmetries as well with the advantage that no CPU consuming maximization procedure is needed.
Before we move to a comparison of the different methods, we note that the estimator

\[ \hat{A} = \frac{\sum_+ \beta_i^+ - \sum_- \beta_i^-}{\sum_+ (\beta_i^+)^2 + \sum_- (\beta_i^-)^2} + A_0 \quad \text{with} \]

\[ \beta^\pm = \frac{\beta}{1 \pm \beta A_0} \quad (14) \]

reaches as well the FOM of the EML for \( A \approx A_0 \). In contrast to \( \hat{A}_{iw} \), its expectation value only equals \( A \) if \( A_0 \approx A \). For \( \tau \) decays the optimal weight factor in Eq. (14) is discussed in Ref. [6].

### 2.5 Comparison of different methods

Tab. 1 summarizes the FOM of the various estimators proposed. Fig. 1 shows the figure of merit of the different estimators vs. \( A \) for the choice

\[ \alpha(x) = \text{const.} = 2500 \quad \text{and} \quad \beta(x) = x, \quad 0 < x < 1. \quad (15) \]

The curves are analytic calculations. The points are results of simulations. For each value of the asymmetry 10000 configurations with \( \alpha = 2500 \), which corresponds on average to 5000 events, were simulated. One configuration consists of a plus and minus data set used to evaluate an asymmetry. For each of the 10000 configurations simulated, the asymmetries were calculated using the estimators discussed above. The FOM was determined from the RMS of the asymmetry distributions.

The results are in perfect agreement with the analytic calculations. The statistical errors of the simulations are of the order of the size of the points. Note, that for all methods no bias was found for the asymmetry. The question of bias and the range of validity of the expressions for the FOM will be discussed in more detail in Section 4. The weighting methods are superior to the
method using simply the counting rates. As expected the improved weighting or the EML method reach a higher FOM than the simple weighting method the larger the asymmetry A. The results depend of course on the shape of \( \alpha(x) \) and \( \beta(x) \). The gain in FOM using weighted events compared to counting rates depends on the spread of \( \beta(x) \). For \( A = 0 \) for example it is \( \langle \beta^2 \rangle / \langle \beta \rangle^2 \) as can be derived from Eq. (10).

Fig. 2 shows the influence of the choice of \( A_0 \) on the FOM in the improved weighting method for the factors \( \alpha \) and \( \beta \) as given in Eq. (15) and an asymmetry \( A = 0.8 \). Choosing \( A_0 \) in a range 0.7–0.86, one reaches at least 99% of FOM_{ALH}. The normal weighting method corresponds to \( A_0 = 0 \).

3 Different acceptance/flux factor in the two data sets

We now turn to the case where the acceptance and flux factor \( \alpha \) is not the same in the two data sets. We assume that they differ by a known factor \( c \) which is independent of \( x \). In this case the differential event distributions are given by

\[
\begin{align*}
n^+(x) &= \frac{2c}{1+c} \alpha(x)(1 + \beta(x)A) \quad \text{and} \\
n^-(x) &= \frac{2}{1+c} \alpha(x)(1 - \beta(x)A).
\end{align*}
\]
The factor $2/(1+c)$ has been introduced in order to normalize the distributions to the same number of events for all values of $c$ at $A = 0$. The log likelihood function reads

\[
l = \sum_+ \ln \left( \frac{2c}{1+c} \alpha_i (1 + \beta_i A) \right) - \langle N^+ \rangle (A) + \sum_- \ln \left( \frac{2}{1+c} \alpha_i (1 - \beta_i A) \right) - \langle N^- \rangle (A)
\]

\[
= \sum_+ \ln (1 + \beta_i A) + \sum_- \ln (1 - \beta_i A) + \sum_+ \ln \frac{2c}{1+c} \alpha_i + \sum_- \ln \frac{2}{1+c} \alpha_i
\]

\[
-2 \int \alpha dx - 2A \frac{c-1}{1+c} \int \alpha \beta dx . \quad (18)
\]

Here the last term cannot be ignored because it contains the parameter $A$ and thus the likelihood method cannot be applied without knowledge of the factor $\int \alpha \beta dx$. The weighting method on the other hand can be applied with a small modification:

\[
\hat{A}_{w,c} = \frac{\sum_+ w_i - c \sum_- w_i}{\sum_+ w_i \beta_i + c \sum_- w_i \beta_i} . \quad (19)
\]

The expectation value of $\hat{A}_{w,c}$ equals again $A$. The figure of merit reads (deriva-
The FOM is shown in Fig. 3 for different values of $c$ for the improved weighting method. As in Fig. 1 the lines correspond to an analytic calculation using Eq. (20), the points are results of simulations. Note, that for arbitrary $c$ the weight reaching the highest FOM is

$$w = \frac{\beta}{(1 - \beta^2 A^2) \left(1 - \beta A \frac{1-c}{1+c}\right)}.$$  

The EML method could be used for $c \neq 1$, if one uses an estimate for $\int \alpha \beta dx \approx \sum_+ \beta(x_i) + c \sum_- \beta(x_i)$ from data. Simulations showed that the FOM of this modified EML method equals the one of the improved weighting method.

4 Validity at low number of events

In this section we discuss the validity of the equations derived for the various FOMs and possible biases of the estimators. The formulas were derived us-
Fig. 4. Combining data in different configurations. The boxes denote the plus and minus data sets.

The results usually error propagation and can thus only be approximations which are the better the higher \( N \). In the simulations presented in Section 2.5, for every value of the asymmetry 10000 configurations were simulated with on average 5000 events (corresponding to \( \alpha = 2500 \) in Eq. (15)). In each of these configurations the asymmetry was determined using the various estimators. In this section we discuss effects occurring if the asymmetries are extracted in smaller configurations, as indicated in Fig. 4.

For lower number of events in one configuration one reaches a point where due to statistical fluctuations the estimated asymmetry in a given configuration can be larger than 1 or smaller than \(-1\). In this case the EML method is no more applicable since the term \((1 \pm \beta A)\) can get negative. For \( \alpha = 250 \) this happens in about 1% of the configurations for an asymmetry of \( A = 0.8 \).

Dividing the sample further in many smaller configurations one reaches a point where one has only 0 or 1 event in one configuration. This limit can also be obtained by dividing the sample in many narrow bins of \( \beta \) having at most one event in a bin. In this case the remaining estimators are identical: \( \hat{A}_{\text{cnt}} \equiv \hat{A}_{\text{w} = \beta} \equiv \hat{A}_{\text{w}_i} = \pm 1/\beta_i \). The sign depends whether the event occurred in the plus or the minus data set. One finds \( \langle \hat{A}_{\text{w}_i} \rangle = A \) and \( \langle \hat{A}_{\text{w}_i}^2 \rangle = 1/\beta_i^2 \), thus the FOM for this event reads

\[
FOM_i = \frac{1}{\langle \hat{A}_{\text{w}_i} \rangle^2 - \langle \hat{A}_{\text{w}_i}^2 \rangle} = \frac{\beta_i^2}{1 - \beta_i^2 A^2} .
\]

Note that \( A \) is not the estimated asymmetry from a single event but rather taken from a larger event sample. Combining all the asymmetries determined on single events leads to

\[
\sum_i \frac{\pm \beta_i}{\beta_i} \cdot \text{FOM}_i = \frac{\sum_i \pm \beta_i}{\sum_i \beta_i} \frac{\beta_i^2}{1 - \beta_i^2 A^2} = \frac{\sum_i \pm \beta_i}{\sum_i \beta_i} \frac{\beta_i^2}{1 - \beta_i^2 A^2} .
\]
Assuming $A = A_0$, Eq. (21) is nothing but the estimator of the improved weighting method, $\hat{A}_{iw}$, defined in Eq. (12). In other words, for the improved weighting method it makes no differences whether the data are analyzed in one large configuration or in many small ones. The improved weighting is also equivalent to using an infinite number of bins in $\beta$ and evaluating the asymmetries in every bin and then combining the results. The advantage of the improved weighting method is that this binning has not to be performed.

The observations discussed above are confirmed by simulations. In total $10^9$ configurations with $\alpha = 0.025$ were simulated. The simulated data were analyzed as follows. First the asymmetries were calculated in the approximately 5% of the configurations actually containing at least one event. Then the weighted average of these asymmetries is calculated. The same data were analyzed in a different way by combining 10 configurations and calculating the asymmetries in these larger configurations corresponding to $\alpha = 0.25$. This procedure was repeated until reaching $10^4$ configurations with $\alpha = 2500$. Fig. 4 illustrates the procedure. These simulations were performed for an asymmetry of 0.8 generating events in the range $0.01 < \beta(x) < 0.99$.

Fig. 4 shows the mean value of the asymmetries and the statistical error for the various estimators for the different values of $\alpha$. As expected the estimator $\hat{A}_{iw}$ gives the same result independent of $\alpha$. No bias is observed within the statistical error which is of the order of $10^{-4}$. The asymmetry for the EML method is only shown for $\alpha = 2500$ since at lower values, as explained above, the EML method is no more applicable.

Fig. 5 shows the distribution of the asymmetry $\hat{A}_{iw}$ for different values of $\alpha$. The entries in the histograms in Fig. 5 are weighted by their corresponding FOM. In the case $\alpha = 2500$ this would not be necessary because all entries have essentially the same FOM for a given method since the relative variation of the number of events $N$ and the averages like $\sum_+ \beta_i^2/N$ entering the FOM vary only very little from configuration to configuration. At lower values of $\alpha$, however, this is no more the case. Taking again the extreme case where the asymmetries is calculated from single events, the FOM depends on the value of $\beta$ for this event. This explains why the number of entries is smaller than 1 in some bins of the histograms. At $\alpha = 0.025$ the number of configurations is $10^9$. The corresponding histogram has only approximately $4.9 \cdot 10^7$ entries reflecting the fact that in most of the configuration there is no event.

Finally, Fig. 7 shows the FOM/$N$ calculated from the RMS of the asymmetry distributions presented in Fig. 5 for different values of $\alpha$. The three lines correspond to $\text{FOM}_{\hat{A}_{iw}}/N$, $\text{FOM}_{\hat{A}_{iw},\beta}/N$ and $\text{FOM}_{\hat{A}_{cnt}}/N$ calculated using the expressions given in Tab. 1. For $\alpha \geq 25$ there is good agreement with the FOM derived in Section 2 since the points coincide with the corresponding lines. At lower values of $\alpha$ the FOM of the weighting and the counting rate
Fig. 5. Results for the asymmetry of the simulations as a function of the average number of events in one configuration. The points are at α = 0.025, 0.25, 2.5, 25, 250, 2500, respectively. For a given value of α they are slightly displaced on the horizontal axis for better readability. Note, that values at different values of α are correlated since the same data were used.

method start to increase and finally reach as expected $\text{FOM}_{A_{iw}}$ at $\alpha = 0.025$, where the three estimators are practically identical.

5 Summary & Conclusions

We presented several estimators to extract an asymmetry parameter $A$ in a number density function. These estimators were based on counting rates, event weighting and the unbinned extended maximum likelihood method. A weighting procedure was derived that reaches the same figure of merit as the unbinned maximum likelihood method, known to reach the minimal variance bound. This weighting estimator is given as an analytic expression, whereas in the EML method the maximization of the likelihood function has to be done numerically. Moreover this estimator can be used (with a small modification)
Fig. 6. Distributions of the estimated asymmetries $\hat{A}_{iw}$ for different values of $\alpha$.

Fig. 7. Figure of merit per event for different methods as a function of $\alpha$ for an asymmetry $A = 0.8$. The lines show the expectation calculated from the expressions given in Tab. 1.
in cases where the EML method cannot be applied because of an incomplete knowledge of event distribution function.

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A Figure of merit of $\hat{A}_w$

To calculate the FOM of $\hat{A}_w$ defined in Eq. (9), one needs $\sigma^2(\sum w_i)$, $\sigma^2(\sum w_i\beta_i)$ and $\text{cov}(\sum w_i, \sum w_i\beta_i)$. For two arbitrary quantities $f$ and $g$ the covariance between $\sum f_i$ and $\sum g_i$ is

\begin{align}
\text{cov}(\sum_i f_i, \sum_j g_j) &= \langle \sum_{i=j} f_i g_i + \sum_{i\neq j} f_i g_i \rangle - \langle \sum_i f_i \rangle \langle \sum_j g_j \rangle \\
&= \langle N \rangle \langle fg \rangle + \langle N(N-1) \rangle \langle f \rangle \langle g \rangle - \langle N \rangle \langle f \rangle \langle g \rangle \\
&= \langle N \rangle \langle fg \rangle + \left( \langle N^2 \rangle - \langle N \rangle - \langle N \rangle^2 \right) \langle f \rangle \langle g \rangle.
\end{align}

(A.1)

If the number of events $N$ is Poisson distributed, i.e. $\langle N^2 \rangle - \langle N \rangle - \langle N \rangle^2 = 0$, one finds

\begin{equation}
\text{cov}(\sum_i f_i, \sum_j g_j) = \langle N \rangle \langle fg \rangle \approx \sum_i f_i g_i.
\end{equation}

(A.5)

Setting $f = g = w$, $f = g = w\beta$ and $f = w, g = w\beta$ results in

\begin{align}
\sigma^2(\sum w_i) &= \langle N \rangle \langle w^2 \rangle, \\
\sigma^2(\sum w_i\beta_i) &= \langle N \rangle \langle (w\beta)^2 \rangle, \\
\text{cov}(\sum w_i, \sum w_i\beta_i) &= \langle N \rangle \langle w^2 \beta \rangle.
\end{align}

(A.6) (A.7) (A.8)

Simple error propagation in Eq. (9) finally leads to Eq. (10) for the figure of merit.

B Optimal weight

Denoting the weight factor which maximizes the FOM by $w_0$, we consider small deviation from this optimum by

\begin{equation}
w(x) = w_0(x) + \epsilon \eta(x)
\end{equation}

(B.1)

where $\eta(x)$ is arbitrary and $\epsilon \ll 1$. 
Inserting Eq. \((B.1)\) in Eq. \((10)\) keeping terms of 1st order in \(\epsilon\) one finds
\[
FOM = \frac{\left(\langle w_0 \beta \rangle + \epsilon \langle \eta \beta \rangle\right)^2}{\langle w_0^2 + 2\epsilon w_0 \eta \rangle \left(1 - \beta^2 A^2\right)}.
\]
(B.2)

The condition \(\partial FOM / \partial \epsilon = 0\) gives
\[
w_0 = \frac{\beta}{1 - \beta^2 A^2}.
\]

C  FOM for the case \(c \neq 1\)

The error for the estimator defined in Eq. \((19)\) is obtained by simple error propagation taking into account the correlations between \(\sum w_\beta\) and \(\sum w\).

\[
\left(\text{FOM}_{A_w,c}\right)^{-1} = \tilde{v}^T C \tilde{v}
\]

with
\[
\tilde{v}^T = \left(\frac{\partial \hat{A}_{w,c}}{\partial \left(\sum w_i\right)}, \frac{\partial \hat{A}_{w,c}}{\partial \left(\sum w_i\right)}, \frac{\partial \hat{A}_{w,c}}{\partial \left(\sum w_i \beta_i\right)}, \frac{\partial \hat{A}_{w,c}}{\partial \left(\sum w_i \beta_i\right)}\right)
\]
\[
= \frac{1}{\sum w_i \beta_i + c \sum w_i \beta_i} \left(1, -c, -A, -cA\right)
\]
(C.1)

and
\[
C = \begin{pmatrix}
\sum w_i^2 & 0 & \sum w_i^2 \beta_i & 0 \\
0 & \sum w_i^2 & 0 & \sum w_i^2 \beta_i \\
\sum w_i^2 \beta_i & 0 & \sum (w_i \beta_i)^2 & 0 \\
0 & \sum w_i^2 \beta_i & 0 & \sum (w_i \beta_i)^2
\end{pmatrix}.
\]
(C.2)

This leads to Eq. \((20)\).

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