The model equation of soliton theory

V.E. Adler, A.B. Shabat

Landau Institute for Theoretical Physics, RAS,
1A pr. Ak. Semenova, 142432 Chernogolovka, Russia.
E-mail: adler@itp.ac.ru, shabat@itp.ac.ru

Abstract

We consider an hierarchy of integrable 1 + 2-dimensional equations related to Lie algebra of the vector fields on the line. The solutions in quadratures are constructed depending on $n$ arbitrary functions of one argument. The most interesting result is the simple equation for the generating function of the hierarchy which defines the dynamics for the negative times and also has applications to the second order spectral problems. A rather general theory of integrable 1 + 1-dimensional equations can be developed by study of polynomial solutions of this equation under condition of regularity of the corresponding potentials.

1 Introduction

In this paper we study the equation

$$D_\tau G(\lambda) = \frac{\langle G(\lambda), G(\mu) \rangle}{\lambda - \mu} \quad (1.1)$$

where the unknown function $G(\lambda) = G(x, \tau, \lambda)$ is locally analytic on $\lambda$ and

$$\langle f, g \rangle \overset{\text{def}}{=} fg_x - gf_x.$$

The assumption on the analyticity is essential since otherwise the general solution contains too much arbitrariness (cf Example 1 below).

The solution of the Cauchy problem with initial data $G_0(\lambda) = G_0(x, \lambda)$ can be found as the Taylor expansion

$$G(\lambda) = G_0(x, \lambda) + \tau G_1(x, \lambda) + \tau^2 G_2(x, \lambda) + \ldots \quad (1.2)$$

The substitution into equation yields the recurrence relation

$$(n + 1)G_{n+1}(\lambda) = \frac{1}{\lambda - \mu} \sum_{k=0}^{n} \langle G_k(\lambda), G_{n-k}(\mu) \rangle, \quad n = 0, 1, \ldots$$
which allows to compute all coefficients if \( G_0 \) is a smooth function on \( x \) and \( \lambda \). Notice that if \( G_0 \) is a polynomial in \( \lambda \) of degree \( m \) then this recurrence relation implies that all \( G_k(\lambda) \) are polynomials of degree not greater \( m \) as well. It is not difficult to obtain an estimation for all coefficients of the expansion and to prove that the convergence radius is not zero. Thus, the dynamics on \( \tau \) preserves the polynomial structure which correspond to a class of rather interesting solutions.

The differentiation \( D_\tau \) obviously depends on the choice of \( \mu \) in equation (1.1). It turns out that this arbitrariness leads to the mutually consistent dynamical equations, as the following theorem states.

**Theorem 1.** Let differentiation \( D_i \) is defined by the equation (1.1) at \( \mu = \mu_i \). Then \([D_1, D_2] = 0\).

**Proof.** Denote \( g_i = G(\mu_i) \), then equation (1.1) implies
\[
D_1 g_2 = \frac{\langle g_2, g_1 \rangle}{\mu_2 - \mu_1} = \frac{\langle g_1, g_2 \rangle}{\mu_1 - \mu_2} = D_2 g_1.
\]

It is easy to check that the Jacobi identity
\[
\langle \langle G, g_1 \rangle, g_2 \rangle + \langle \langle g_2, G \rangle, g_1 \rangle + \langle \langle g_1, g_2 \rangle, G \rangle = 0
\]
implies the equality \([D_1, D_2](G) = 0\). \( \blacksquare \)

It is clear that the Theorem 1 remains valid if we replace \( \langle \cdot, \cdot \rangle \) by the bracket in an arbitrary Lie algebra. Our original bracket corresponds to the Lie algebra of the vector fields on the line, another interesting example is the bracket \( \langle f, g \rangle = f_y g_x - f_x g_y \), corresponding to the Hamiltonian vector fields on the plane. However, we restrict ourselves by the simplest case since it is already quite nontrivial and illustrative.

The above equation for the functions \( g_1, g_2 \) can be written as an equation with partial derivatives \( \partial_\tau = D_i \) for the potential \( u \) which is introduced by equations \( g_i = u_\tau_i \):\[
(\mu_i - \mu_j)u_{\tau_i \tau_j} = u_{\tau_i} u_{\tau x_j} - u_{\tau_j} u_{\tau x_i}.
\] (1.3)

This differential equation (cf [1]) can be considered as a simplified version of the original problem, since the analyticity of the solution on the parameter here is not important. The particular solutions of this equation are presented in the Section 4. Obviously, introducing a new independent variable \( \tau_k \) corresponding to \( \mu_k \) leads to equation consistent with (1.3). Some generalizations with similar symmetry properties are discussed in the Section A.2. It would be interesting to compare these examples with the classification results obtained in the papers [3] in the framework of the method of hydrodynamic reductions.

Thus, the important feature of the equation (1.1) is its symmetry properties. In the case of polynomial in \( \lambda \) solutions the sufficient set of symmetries allows to integrate equation (1.1) completely. The general solution contains, in the case of \( n \)-th degree polynomials, \( n \) arbitrary functions of one variable \( \kappa_i(\lambda) \) and the problem is reduced to the interpolation of the polynomial in the given set of points \( G_x|_{\lambda = \gamma_i} = \kappa_i(\gamma_i) \) (see
Section 3. The zeroes $\gamma_i$ of the polynomial $G(\lambda)$ play the role of the Riemannian invariants for the system of hyperbolic equations on the coefficients of $G(\lambda)$ (cf [4]).

In the Section 2 we show that equation (1.1) rewritten in terms of Laurent expansions of the function $G(\lambda)$ generate an infinite sequence of commuting vector fields which can be interpreted as the additional symmetries of equations (1.3).

The Appendix is devoted to the applications of equation (1.1) to the spectral problems of the second order

$$\psi_{xx} = U(x, \lambda)\psi.$$ 

We discuss there the problem of construction of $G(x, \lambda)$ for a wide class of potentials $U(x, \lambda)$. The one-to-one correspondence between the function $G(x, \lambda)$ and the potential $U(x, \lambda)$ can be achieved if we waive the polynomiality in $\lambda$. In the direct problem this is equivalent to solving of Riccati equation, while the solution of the inverse problem defines the potential $U(x, \lambda)$ as the Schwarz derivative of $1/G(x, \lambda)$ with respect to $x$.

The equation (1.1) admits not only polynomial in $\lambda$ solutions, but also the rational ones. The generalization of Dubrovin equations for this case and analysis of possible applications to the KdV-like equations is an interesting open problem.

We finish this introduction by an example which demonstrates the importance of the choice of the suitable analytical structure of $G(\lambda)$.

Example 1. The equation (1.1) admits the reduction

$$G = a(\tau, \lambda)x^2 + 2b(\tau, \lambda)x + c(\tau, \lambda)$$

which leads to the system

$$\frac{da}{d\tau} = 2\{a, b\}, \quad \frac{dc}{d\tau} = 2\{b, c\}, \quad \frac{db}{d\tau} = \{a, c\}, \quad \{f, g\} \triangleq \frac{f(\mu)g(\lambda) - g(\mu)f(\lambda)}{\lambda - \mu}.$$ 

It is easy to see that the discriminant $b^2 - ac$ does not depend on $\tau$. The equation (1.3) turns into the matrix equation

$$(\mu_i - \mu_j)S_{\tau_i, \tau_j} = [S_{\tau_i}, S_{\tau_j}], \quad S = \begin{pmatrix} b & a \\ -c & -b \end{pmatrix} \in \mathfrak{sl}_2$$

which possesses the partial integrals $\partial_{\tau_i}(S_{\tau_i}, S_{\tau_j}) = 0$ where $(S_1, S_2) = \text{tr} S_1 S_2$.

In particular, at $a = 0$ the equation becomes linear:

$$\frac{df(\lambda)}{dt} = \frac{f(\lambda) - f(\mu)}{\lambda - \mu}, \quad \frac{d}{d\tau} = 2b(\mu)\frac{d}{dt}, \quad G(\tau, \lambda) = 2b(\lambda)(x + f(\tau, \lambda)).$$

2 The commuting vector fields

We will assume that the locally analytic functions under consideration are regular at $\lambda = 0$ (this can be achieved by a shift) and that the infinity is not the essential singular point. Then the expansions exist

$$G(\lambda) = \begin{cases} A(\lambda) = a_0\lambda^m + a_1\lambda^{m-1} + a_2\lambda^{m-2} + \ldots, & \lambda \to \infty, \\ B(\lambda) = b_0 + b_1\lambda + b_2\lambda^2 + \ldots, & \lambda \to 0 \end{cases}$$

(2.1)
which converge at $|\lambda| > \rho_1$ and $|\lambda| < \rho_2$ respectively.

Let us assume that $\mu$ belongs to the domain of convergence of one of the power series (2.1) and rewrite the equation (1.1) as follows

$$(\lambda - \mu)D_\tau A(\lambda) = \langle A(\lambda), G(\mu) \rangle.$$  

Collecting the coefficients at $\lambda^{n+1}$ one obtains $D_\tau a_0 = 0$. The change of the form $\partial_x \rightarrow \phi(x)\partial_x$, $G \rightarrow G/\phi(x)$ allows to set $a_0 = 1$ without loss of generality. Then

$$D_\tau a_n = \langle A_n(\mu), G(\mu) \rangle, \quad A_n(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n.$$  

(2.2)

Analogously, the equation

$$(\lambda - \mu)D_\tau B(\lambda) = \langle B(\lambda), G(\mu) \rangle$$

in the neighborhood of $\lambda = 0$ implies

$$D_\tau b_n = \langle A_{-n-1}(\mu), G(\mu) \rangle, \quad A_{-1}(\mu) = -\frac{b_0}{\mu}, \quad A_{-2}(\mu) = -\frac{b_0}{\mu^2} - \frac{b_1}{\mu}, \ldots$$

Therefore, the expansions (2.1) and the equation (1.1) bring to an infinite sequence of the polynomials $A_n$:

$$A_n(\lambda) = \lambda A_{n-1}(\lambda) + \left\{ \begin{array}{ll} a_n, & n > 0, \\ b_{-n}, & n < 0 \end{array} \right., \quad A_0 = 1, \quad A_{-1}(\lambda) = -\frac{b_0}{\lambda}.$$  

(2.3)

The polynomials $A_n$, $n \in \mathbb{Z}$ and Theorem 1 allow to rewrite the original equation (1.1) in the form of a system of equations for the coefficients of the series $A(\lambda)$ and $B(\lambda)$. In order to do this we introduce, following the papers [1], the sequence of the vector fields $\mathcal{L}_n$, $n \in \mathbb{Z}$ as the differential operators

$$\mathcal{L}_n = D_n - A_n(\lambda)D_0, \quad D_n = \frac{\partial}{\partial t_n}, \quad D_0 = \frac{\partial}{\partial x}.$$  

(2.4)

The hierarchy of the corresponding times will be denoted as $t = (\ldots, t_{-1}, t_0, t_1, \ldots)$.

One can prove, by use of the above equations for $D_\tau a_n$ and $D_\tau b_n$, that the commutativity of the differentiations from the Theorem 1 leads to the commutation relations

$$[\mathcal{L}_m, \mathcal{L}_n] = 0 \quad \Leftrightarrow \quad D_m A_n - D_n A_m = \langle A_m, A_n \rangle, \quad \forall \ m, n \in \mathbb{Z}.$$  

(2.5)

Finally, these relations between the polynomials (2.3) can be rewritten immediately, as equations for $G(\lambda)$:

$$D_n G(\lambda) = \langle A_n(\lambda), G(\lambda) \rangle, \quad G(\lambda) = G(t, \lambda), \quad n \in \mathbb{Z}.$$  

(2.6)

The equivalence of (2.5) and (2.6) is proved along the standard scheme [2].

Substitution of the series (2.1) into equation (2.6) yields

$$D_n A(\lambda) = \langle A_n(\lambda), A(\lambda) \rangle, \quad D_n B(\lambda) = \langle A_n(\lambda), B(\lambda) \rangle, \quad n \in \mathbb{Z}.$$  

(2.7)
In particular, for the first “negative” time $t_{-1}$ we have

$$D_{-1}A(\lambda) = \frac{b_{0,x}}{\lambda} + \frac{\langle a_1, b_0 \rangle}{\lambda^2} + \frac{\langle a_2, b_0 \rangle}{\lambda^3} + \cdots, \quad D_{-1}B(\lambda) = \langle b_1, b_0 \rangle + \langle b_2, b_0 \rangle \lambda + \cdots.$$  

This leads to an infinite autonomous system of equations for the variables $b_i$:

$$D_{-1}b_0 = \langle b_1, b_0 \rangle, \quad D_{-1}b_1 = \langle b_2, b_0 \rangle, \ldots$$

which correspond to the value of $\mu$ chosen in the convergence domain of the series $B(\lambda)$.

Analogously, for the first “positive” flow $D_1 = \partial_t$, the basic equations (2.6) give

$$D_1A(\lambda) = \langle \lambda + a_1, \frac{a_2}{\lambda^2} + \frac{a_3}{\lambda^3} + \cdots \rangle, \quad D_1B(\lambda) = \langle a_1, b_0 \rangle + (b_{0,x} + \langle a_1, b_1 \rangle) \lambda + \cdots,$$

and this is equivalent to an infinite sequence of equations for the variables $a_i$:

$$D_1a_1 = a_{2,x}, \quad D_1a_2 = a_{3,x} + \langle a_1, a_2 \rangle, \quad D_1a_3 = a_{4,x} + \langle a_1, a_3 \rangle, \ldots$$  \hspace{1cm} (2.8)

corresponding to the value of $\mu$ in the convergence domain of the series $A(\lambda)$.

### 3 Polynomial solutions

In the polynomial case $G = \lambda^n + g_1 \lambda^{n-1} + \cdots + g_n$ the infinite hierarchy of the times $t$ and the corresponding polynomials (2.3) is reduced to the finite basis

$$D_i(G) = \langle G_i, G \rangle, \quad G_i = \lambda^i + g_1 \lambda^{i-1} + \cdots + g_i.$$  \hspace{1cm} (3.1)

Any given differentiation corresponding to an arbitrary $\mu$ can be expanded over this basis. In particular, $\partial_x = D_0$, and the value $\mu = 0$ corresponds to the differentiation $D_{n-1}$. In this latter case the equations for the coefficients are of the form

$$g_{1,t_{n-1}} = g_{n,x}, \quad g_{2,t_{n-1}} = \langle g_1, g_n \rangle, \quad \ldots, \quad g_{n,t_{n-1}} = \langle g_{n-1}, g_n \rangle.$$  \hspace{1cm} (3.2)

This system possesses rather many applications. For example, the case $n = 2$

$$g_{1,t_1} = g_{2,x}, \quad g_{2,t_1} = \langle g_1, g_2 \rangle \quad \Leftrightarrow \quad u_{t_1 t_1} = \langle u_x, u_{t_1} \rangle$$

describes the Chaplygin model in gas dynamics.

In the general case, we choose the zeroes of $G(\lambda)$ as the dynamical variables:

$$G(\lambda) = (\lambda - \gamma_1)(\lambda - \gamma_2) \cdots (\lambda - \gamma_n)$$

then equation (1.1) at $\lambda = \gamma_i$ implies

$$D_\tau G(\lambda) \bigg|_{\lambda = \gamma_i} = \frac{G(\mu)G_x(\lambda)}{\mu - \lambda} \bigg|_{\lambda = \gamma_i} \quad \Leftrightarrow \quad D_\tau \gamma_i = \frac{G(\mu)}{\mu - \gamma_i} \gamma_i x, \quad i = 1, \ldots, n$$

or

$$D_\tau \gamma_1 = (\mu - \gamma_2) \cdots (\mu - \gamma_n) \gamma_{1,x}, \ldots, \quad D_\tau \gamma_n = (\mu - \gamma_1) \cdots (\mu - \gamma_{n-1}) \gamma_{n,x}.$$  \hspace{1cm} (3.3)
Comparing this with (3.1) at \( \lambda = \gamma_i \) we obtain
\[
D \tau = \mu^{n-1} D_0 + \mu^{n-2} D_1 + \cdots + D_{n-1}.
\] (3.4)

The problem of solving (3.3) admits various settings, and we will construct the general solution following the paper [4]. In virtue of the Theorem 1 and the formula (3.4), the differentiations \( D_i \) mutually commute. We identify \( g_1, \ldots, g_n \) with the elementary symmetric polynomials
\[
g_1 = -\sum \gamma_k, \quad g_2 = \sum_{k<l} \gamma_k \gamma_l, \quad g_3 = -\sum_{k<l<m} \gamma_k \gamma_l \gamma_m, \ldots
\]
and introduce the notations
\[
g_{i,j} = g_i \big|_{\gamma_j = 0} \quad \Rightarrow \quad g_{1,1} = -\gamma_2 - \cdots - \gamma_n, \quad \ldots, \quad g_{n-1,n} = (-1)^n \gamma_1 \cdots \gamma_{n-1}.
\]
This bring to the following statement.

**Statement 2.** Integration of the equations (3.3) is equivalent to the integration of consistent system of \( n(n-1) \) equations
\[
D_i(\gamma_j) = g_{i,j}D_0(\gamma_j), \quad i = 1, \ldots, n-1, \quad j = 1, \ldots, n.
\] (3.5)

It is worth to notice that the equation (3.5) with the number \((i,j)\) is obtained from the corresponding equation (3.1) under the substitution \( \lambda = \gamma_j \) and that
\[
g_{i,j} = G_i(\gamma_j) = \gamma_j^i + \gamma_j^{i-1} g_1 + \cdots + g_i = -\gamma_j^{i-n}(\gamma_j^{n-i-1} g_{i+1} + \cdots + g_n), \quad i < n.
\]
Notice that the zeroes \( \gamma_j \) are Riemannian invariants, that is, this change of variables brings each of \( n-1 \) quasilinear system of first order equations (3.1) to the diagonal form (recall that (3.2) corresponds to \( i = n-1 \)).

**Theorem 3.** Let \( D_i = \partial_i, \quad i = 0, \ldots, n-1 \). Then the general solution of the overdetermined system (3.3) is given by the formula
\[
\begin{bmatrix}
\frac{dt_0}{dt_1} \\
\vdots \\
\frac{dt_{n-1}}{dt_1}
\end{bmatrix} =
\begin{bmatrix}
\gamma_1^{n-1} & \gamma_2^{n-1} & \cdots & \gamma_n^{n-1} \\
\gamma_1^{n-2} & \gamma_2^{n-2} & \cdots & \gamma_n^{n-2} \\
1 & 1 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
\kappa_1^{-1}(\gamma_1) d\gamma_1 \\
\kappa_2^{-1}(\gamma_2) d\gamma_2 \\
\kappa_n^{-1}(\gamma_n) d\gamma_n
\end{bmatrix}.
\] (3.6)

**Proof.** Assume, in addition to (3.5), that the dynamics on \( x \) of the zeroes is defined by the system
\[
\gamma_{1,x} = R_1(\gamma_1, \ldots, \gamma_n), \quad \ldots, \quad \gamma_{n,x} = R_n(\gamma_1, \ldots, \gamma_n)
\] (3.7)
and let us find the form of the functions \( R_i \) by use of the conditions of consistency with (3.3) which are obtained by cross-differentiation. For example, the use of (3.3) gives
\[
D_0 \gamma_1 = R_1, \quad D_\tau \gamma_1 = (\mu - \gamma_2) \cdots (\mu - \gamma_n) R_1 \quad \Rightarrow \quad [D_\tau, D_0](\gamma_1) = D_\tau R_1 - D_0(\mu - \gamma_2) \cdots (\mu - \gamma_n) R_1.
\]
Substituting $\mu = \gamma_2, \ldots, \gamma_n$ into the consistency condition $[D_\tau, D_0](\gamma_1) = 0$, we obtain
\[
\frac{\partial \log R_1}{\partial \gamma_j} = \frac{1}{\gamma_1 - \gamma_j}, \quad \frac{\partial}{\partial \gamma_j} \log \left( R_1 \prod_{j=2}^{n} (\gamma_1 - \gamma_j) \right) = 0, \quad j = 2, \ldots, n.
\]
The other consistency conditions give respectively
\[
\frac{\partial \log R_i}{\partial \gamma_j} = \frac{1}{\gamma_i - \gamma_j}, \quad \forall \ i \neq j \implies R_i = \frac{\kappa_i(\gamma_i)}{\prod(\gamma_i - \gamma_j)}
\]
with $n$ arbitrary functions $\kappa_1(\gamma_1), \ldots, \kappa_n(\gamma_n)$.

Apparently, the found additional dynamical system
\[
\gamma_{1,x} = \frac{\kappa_1(\gamma_1)}{(\gamma_1 - \gamma_2) \cdots (\gamma_1 - \gamma_n)}, \quad \gamma_{2,x} = \frac{\kappa_2(\gamma_2)}{(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3) \cdots (\gamma_2 - \gamma_n)}, \ldots
\]
explicitly defines, together with (3.5), the Jacobi matrix $n \times n$
\[
J = \frac{\partial(\gamma_1, \ldots, \gamma_n)}{\partial(t_0, \ldots, t_{n-1})}
\]
of the partial derivatives of $n$ zeroes with respect to $n$ independent variables ($t_0 = x$). One can straightforwardly check that $J$ coincide with the inverse of the Vandermonde matrix $W$ from (3.6), up to the left multiplication by the diagonal matrix:
\[
J = \text{diag}(\kappa_1(\gamma_1), \kappa_2(\gamma_2), \ldots, \kappa_n(\gamma_n)) W^{-1}.
\]

In order to finish the proof, we note that the formula (3.6) rewritten as
\[
dt_{n-1} = \frac{d\gamma_1}{\kappa_1(\gamma_1)} + \cdots + \frac{d\gamma_n}{\kappa_n(\gamma_n)}, \quad \ldots, \quad dx = \frac{\gamma_1^{n-1} d\gamma_1}{\kappa_1(\gamma_1)} + \cdots + \frac{\gamma_n^{n-1} d\gamma_n}{\kappa_n(\gamma_n)}
\]
allows to solve in quadratures the problem of finding $t_0, \ldots, t_{n-1}$ as functions on $\gamma_1, \ldots, \gamma_n$.

Obviously, the degree of the polynomial $G_x(\lambda)$ is $(n - 1)$ and equation (3.8) and relations
\[
G_x(\lambda) = -\gamma_{1,x}(\lambda - \gamma_2) \cdots (\lambda - \gamma_n) - \cdots - \gamma_{n,x}(\lambda - \gamma_1) \cdots (\lambda - \gamma_{n-1})
\]
imply that in order to calculate the Jacobian (3.9) and to integrate the differential equations (3.1) it is sufficient to choose, arbitrarily, the functions
\[
G_x(\lambda)\big|_{\lambda = \gamma_j} = -\kappa_j(\gamma_j), \quad j = 1, \ldots, n.
\]

**Corollary 4.** Theorem 3 allows to find all derivatives $D_i(G)$ directly in terms of $\gamma_1, \ldots, \gamma_n$, without integrating differential equations (3.7). In particular, $D_0(G) = G_x$ is defined by Lagrange interpolation formula:
\[
G_x(\lambda) = -\kappa_1(\gamma_1) \frac{(\lambda - \gamma_2) \cdots (\lambda - \gamma_n)}{(\gamma_1 - \gamma_2) \cdots (\gamma_1 - \gamma_n)} - \cdots - \kappa_n(\gamma_n) \frac{(\lambda - \gamma_1) \cdots (\lambda - \gamma_{n-1})}{(\gamma_n - \gamma_1) \cdots (\gamma_n - \gamma_{n-1})}.
\]
The independent variable $t_{n-1}$ corresponding to $\mu = 0$ can be replaced with $\tau$. The transition from the variables $t_0 = x, t_1, \ldots, t_{n-1}$ to the variables $t_0, t_1, \ldots, t_{n-2}, \tau$ is defined by the Jacobian

$$\frac{\partial(t_0, \ldots, \tau)}{\partial(\gamma_1, \ldots, \gamma_n)} = \frac{\partial(t_0, \ldots, \tau)}{\partial(t_0, \ldots, t_{n-1})} \frac{\partial(t_0, \ldots, t_{n-1})}{\partial(\gamma_1, \ldots, \gamma_n)}.$$ 

Since $d\tau = \mu^{n-1} dt_0 + \mu^{n-2} dt_1 + \cdots + dt_{n-1}$ we obtain

$$d\tau = \sum \frac{\kappa(\gamma_i) d\gamma_i}{\kappa_i(\gamma_i)}, \quad \kappa(\gamma) = (\mu \gamma)^{n-1} + (\mu \gamma)^{n-2} + \cdots + 1.$$ 

In order to apply Theorem 3 to equation (1.3) it is sufficient to express the potential through the series $\gamma_j(x, \alpha, \beta), j = 1, 2 \ldots$:

$$du = \frac{\gamma_1^2 d\gamma_1}{\kappa_1(\gamma_1)} + \cdots + \frac{\gamma_n^2 d\gamma_n}{\kappa_n(\gamma_n)} \quad \Rightarrow \quad D_j u = g_j.$$ 

4 The map $G \rightarrow U$

Let us consider the classification of the polynomial solutions $G(\lambda)$ of equation (1.1) based on the map

$$G \rightarrow U : 4U = \frac{K(\lambda)}{G^2} - \frac{G_x^2}{G^2} + 2 \frac{G_{xx}}{G}. \quad (4.1)$$

Notice, that the kernel of the map $G \rightarrow U$ is not trivial: it consists of the quadratic polynomials in $x$ (1.4) (see Example 1.) Indeed, the equation (4.1) after multiplying by $G^2$ and differentiating yields the third order linear equation

$$2UG_x + U_x G = \frac{1}{2} G_{xxx}.$$ 

Therefore, if $U = 0$ then $G_{xxx} = 0$. Also, notice that in virtue of the homogeneity of the equation (4.1)), the change

$$\tilde{G} = \kappa(\lambda) G, \quad \tilde{K} = \kappa^2 K \quad \Rightarrow \quad \tilde{U} = U$$

does not change $U$. This allows to consider as equivalent the polynomial solutions $\tilde{G} = \kappa(\lambda) G$ and $G$ which differ by a factor $\kappa(\lambda)$ depending on $\lambda$ only.

The use of the map (4.1) for the classification of the polynomial solutions $G(\lambda)$ is related, above all, with the theory, developed by S.P. Novikov school, of the finite-gap potentials $U$ for the second order spectral problems

$$\psi_{xx} = U(x, \lambda) \psi. \quad (4.3)$$

In connection with these applications to the spectral theory the following formulae are useful:

$$f^\pm \equiv \frac{G_x}{2G} \pm \frac{\sqrt{K(\lambda)}}{2G} \quad \Rightarrow \quad f^\pm_x + (f^\pm)^2 = \frac{K(\lambda)}{4G^2} + \frac{G_{xx}}{2G^2} - \frac{G_x^2}{4G^2} = U, \quad (4.4)$$
relating the equation (4.1) with Riccati equation $f_x + f^2 = U$. In Appendix we discuss the direct problem of construction, for the given potential $U$, of two special solutions $f = f^\pm$ of Riccati equation such that

$$f^+ - f^- = \frac{\sqrt{K(\lambda)}}{G}.$$ 

This allows, in particular, to interpret the function $K(\lambda)$ in equation (4.1) as the Wronskian of two solutions $\psi^\pm$ of the spectral problem (4.3) related to $f = f^\pm$ by the standard rule $(\log \psi^\pm)_x = f^\pm$.

The zeroes $\lambda = \gamma_i$ of the polynomial solutions $G(\lambda)$ of the equation (1.1) lead, in general, to the poles of the potential defined by the formula (4.1). This restricts its use in the applications where the analytic structure of $U(\lambda)$ is fixed a priori. The theorem below explains how to get rid of these unwanted poles by the special choice of the functions $\kappa_j(\lambda)$ in equation (3.11).

**Theorem 5.** The potential $U(\lambda, x)$ as function on $\lambda$ does not possess the moving poles $\lambda = \gamma_j(x)$, $\gamma_j, x \neq 0$ if and only if the following regularity conditions are fulfilled:

$$\kappa_i(\lambda) = \pm \sqrt{K(\lambda)}, \quad i = 1, \ldots, n. \quad (4.5)$$

**Proof.** Let us deduce the equations analogous to (3.8) directly from the basic equation (4.1), using only the regularity condition and without use of the Statement 2. To do this substitute

$$\frac{1}{G} = \frac{\varepsilon_1}{\gamma_1 - \lambda} + \cdots + \frac{\varepsilon_n}{\gamma_n - \lambda}, \quad \Gamma = \frac{G_x}{G} = \frac{\gamma_1, x}{\gamma_1 - \lambda} + \cdots + \frac{\gamma_n, x}{\gamma_n - \lambda}, \quad \varepsilon_i \overset{\text{def}}{=} \prod_{j \neq i} \frac{1}{\gamma_i - \gamma_j}$$

into the equation

$$4U = \frac{K(\lambda)}{G^2} + 2\Gamma_x + \Gamma^2. \quad (4.6)$$

Vanishing of the coefficients at the poles

$$(\lambda - \gamma_1)^{-2}, \quad (\lambda - \gamma_2)^{-2}, \ldots \quad (\lambda - \gamma_n)^{-2}$$

yields $n$ Dubrovin equations for $\gamma_1, \ldots, \gamma_n$:

$$\gamma_1, x = \frac{\sqrt{K(\gamma_1)}}{(\gamma_1 - \gamma_2) \cdots (\gamma_1 - \gamma_n)}, \quad \cdots \quad \gamma_n, x = \frac{\sqrt{K(\gamma_n)}}{(\gamma_n - \gamma_1) \cdots (\gamma_n - \gamma_{n-1})}. \quad (4.7)$$

Comparing these equations with (3.8) we see that the conditions (4.5) are fulfilled.

It turns out that these conditions are not only necessary but also sufficient for the regularity of the potential $U$ defined accordingly to (4.1) by solution $G$ of equations (3.8). Indeed, differentiation of equations (4.7) with respect to $x$ yields

$$\gamma_i, xx = \gamma_i^2 \left(\frac{1}{2} \frac{K'}{K} (\gamma_i) - \sum_{j \neq i} \frac{1}{\gamma_i - \gamma_j}\right) + \sum_{j \neq i} \frac{\gamma_i, x \gamma_j, x}{\gamma_i - \gamma_j}, \quad i = 1, \ldots, n. \quad (4.8)$$
On the other hand

$$\frac{K(\lambda)}{G^2(\lambda)} = K_2 + K_1 + K_0, \quad K_2 = \sum_{i=1}^{n} \frac{\varepsilon_i^2 K(\gamma_i)}{(\lambda - \gamma_i)^2}, \quad K_1 = \sum_{i=1}^{n} \frac{\varepsilon_i^2 K(\gamma_i) K_{1,i}}{\lambda - \gamma_i}$$

(4.9)

where

$$K_{1,i} = \frac{d}{d\lambda} \log K(\lambda) \bigg|_{\lambda = \gamma_i} - 2 \sum_{j \neq i} \frac{1}{\gamma_i - \gamma_j},$$

and $K_0$ is a regular function. Since, accordingly to (4.6),

$$4U = \frac{K(\lambda)}{G^2} - \sum_{i=1}^{n} \frac{\gamma_i^2 x}{(\gamma_i - \lambda)^2} + 2 \sum_{i<j} \frac{\gamma_i x \gamma_j x}{(\gamma_i - \lambda)(\gamma_j - \lambda)} + 2 \sum_{i=1}^{n} \frac{\gamma_i x x}{(\gamma_i - \lambda)},$$

hence we prove that the first and second order poles are cancelled in virtue of Dubrovin equations (4.7).

It follows from above that the solution $G$ is defined by the functions $\kappa_j(\lambda)$ in (3.10), and if these functions are subjected to the regularity condition (4.5) with an analytic function $K(\lambda)$ then the potential $U$ is analytic in $\lambda$. The Liouville theorem says that a function on $\lambda$ analytic in the whole extended complex plane is constant. In particular (see (4.1)) the polynomial functions $K(\lambda)$ correspond to the polynomial in $\lambda$ potentials $U$. This simple structure of the potential $U$ is defined in this case by expansion in the neighborhood of $\lambda = \infty$ of the function

$$H(\lambda) = \frac{\lambda^n}{G} = 1 + \frac{h_1}{\lambda} + \frac{h_2}{\lambda^2} + \ldots, \quad h_1 = -g_1, \quad h_2 = g_2^2 - g_1, \ldots$$

(4.10)

and by the coefficients $c_i$ of the expansion

$$\lambda^{-m} K(\lambda) = 4 \left(1 + \frac{c_1}{\lambda} + \frac{c_2}{\lambda^2} + \ldots\right)$$

where $m$ denotes the order of the pole $K(\lambda)$ at $\lambda = \infty$. Notice, that the coefficients of the expansion (4.10) define the conservation laws of the equations (3.1), since

$$D_i(H) = D_i \left(\frac{\lambda^n}{G}\right) = \frac{\lambda^n \frac{GG_{i,x} - G_x G_i}{G^2}}{G^2} = HG_{i,x} + H_x G_i = D_0(H G_i).$$

(4.11)

Thus, in the polynomial case the form of the potential $U$ is defined by the first term in the formula (4.1) and if the degree $m$ of the polynomial $K(\lambda)$ exceeds the degree $2n$ of the polynomial $G^2(\lambda)$ by 1 or 2 then, correspondingly,

$$\frac{K(\lambda)}{4G^2} = \begin{cases} 
\lambda + 2h_1 + c_1 + \ldots, \\
\lambda^2 + \lambda(2h_1 + c_1) + h_1^2 + 2h_2 + 2c_1 h_1 + c_2 + \ldots.
\end{cases}$$

Since the rest terms in the formula (4.1) vanish at $\lambda = \infty$, hence we obtain in the latter case, at $m = 2n + 2$,

$$U = \lambda^2 + u_1 \lambda + u_2, \quad u_1 = 2h_1 + c_1, \quad u_2 = h_1^2 + 2h_2 + 2c_1 h_1 + c_2.$$
Analogously, in the case $m = 2n + 1$

$$U = \lambda + u(x), \quad u(x) = c_1 + 2h_1 = 2 \sum_{j=1}^{n} \gamma_j(x) - \sum_{i=1}^{2n+1} e_i \tag{4.12}$$

where $e_i, i = 1, 2, \ldots$ denote $2n + 1$ zeroes of the polynomial $K(\lambda)$. It is well known \[7\] that in the case of the real potential the necessary and sufficient condition of its regularity is that the initial values of $\gamma_i$ lie in the restricted gaps of the spectrum:

$$e_1 < e_2 < g_1 < e_3 < \cdots < e_{2n} < \gamma_n < e_{2n+1}.$$  

If the function $K(\lambda)$ possesses, for example, the first order pole at $\lambda = 0$ in addition to the pole of order $m = 2n + 1$ at infinity then the equation (4.1) defines the potential in the form

$$U = \lambda + u + v/\lambda = \lambda + 2h_1 + c_1 + \frac{1}{\lambda}(h_1^2 + 2h_2 - \frac{1}{2}h_{1,xx}). \tag{4.13}$$

In this case the scheme of construction of particular solutions similar to the presented above can be found in the paper \[5\], see also \[2\].

The observation that the map (4.1) is factorizable gives additional possibilities of the classification of the polynomial solutions $G(\lambda)$. Consider the intermediate map $G \rightarrow V$ defined as follows

$$V = \frac{G_x + \kappa(\lambda)}{2G}. \tag{4.14}$$

Then, in virtue of (4.4),

$$U = V_x + V^2, \quad K(\lambda) = \kappa^2(\lambda).$$

Moreover, the regularity condition $V$ is equivalent, due to (3.11), to the relations $\kappa_i = -\kappa, i = 1, \ldots, n$ (cf (4.5)). Like in the case (4.1), the regularity condition guarantees that if $\kappa(\lambda)$ is polynomial then $V$ is polynomial as well. In particular, the choice of the polynomial $\kappa(\lambda)$ of $m = n + 1$ degree we find, analogous to (4.12) that

$$V = \lambda + v(x), \quad v = -g_1, \quad U = \lambda^2 + 2v\lambda + v^2 + v_x. \tag{4.15}$$

In order to illustrate the variety of the applications of the regularity conditions of the form (4.3) and Theorem 3 we present a list of scalar partial differential equations which correspond to equations (3.1) in the cases (4.12) and (4.15). Since the analytic structure of $U$ and $V$ in our case is essentially simpler than the one of $G$, hence it is natural to rewrite the equations (3.1) in terms of $U$ and $V$. To do this, it is sufficient to notice that

$$G_t = RG_x - GR_x \quad \Rightarrow \quad V_t = \left( RV - \frac{1}{2}R_x \right)_x \tag{4.16}$$

and analogously

$$G_t = RG_x - GR_x \quad \Leftrightarrow \quad f_t = \left( Rf - \frac{1}{2}R_x \right)_x \quad \Rightarrow \quad U_t = 2UR_x + U_x R - \frac{1}{2}R_{xxx}. \tag{4.17}$$
Example 2. The choice $R = G_1 = \lambda + g_1$ leads, in the case (4.15), (4.16) to the Burgers equation

$$v_t = v_{xx} + 2vv_x = [(\lambda + g_1)(\lambda + v) - g_{1,x}]_x,$$

and in the case (4.12), (4.17) to the Korteweg-de Vries equation

$$4u_t = u_{xxx} - 6uu_x + \varepsilon u_x. \quad (4.19)$$

On the other hand, the use in (4.17) of the equation for $f$, allows easily to obtain the modified KdV equation. Indeed, the second equation (4.17) gives at $R = G_1 = \lambda + g_1$

$$D_1(f) = D_x\left(\lambda f + fg_1 - \frac{1}{2}g_{1,x}\right), \quad f_x + f^2 = U = \lambda - 2g_1 + c_1.$$

We find, by substitution of the expression of $g_1$ through $f$ into the first equation under consideration:

$$4f_t = D_x(f_{xx} - 2f^3 + 6\lambda f) + \varepsilon f_x. \quad (4.20)$$

The relation between the solutions of equations (4.19), (4.20) defines the Miura map $u = f_x + f^2 + \lambda$ depending on $\lambda$.

Finally, we demonstrate that the original system of equations (3.2) leads, in the case $U = \lambda + u$, directly to sinh-Gordon equation:

$$\varphi_{xt} = \frac{1}{2}k_1e^{-\varphi} - 2e^{\varphi}, \quad e^\varphi = G(0), \quad k_1 = \frac{dK}{d\lambda}\big|_{\lambda=0}, \quad (K(0) = 0). \quad (4.21)$$

Indeed, we obtain from (3.2), (4.1), correspondingly,

$$g_t = \langle \tilde{g}, g \rangle = \tilde{g}g_x - g\tilde{g}_x, \quad 4(\lambda + u)G^2 = K(\lambda) - G_x^2 + 2G_{xx}G \quad (4.22)$$

where

$$g = G(0), \quad \tilde{g} = \frac{dG}{d\lambda}\big|_{\lambda=0}.$$

We find by differentiating with respect to $\lambda$ of the second equation (4.22) and then eliminating $u$:

$$2g^2 - \frac{1}{2}k_1 + k_0 \frac{\tilde{g}}{g} + \langle \tilde{g}, g \rangle_x - (\log g)_x \langle \tilde{g}, g \rangle = 0, \quad k_0 \overset{\text{def}}{=} K(0).$$

Next, setting $k_0 = 0$ and replacing $\langle \tilde{g}, g \rangle$ with $g_t$ bring the latter equation to (4.21), after the obvious transformations.

To conclude the section we consider briefly the applications of equation (1.1) to the question of integrability of the analogs of equations (4.17), (4.16) discussed in the Example 2 obtained by the change of polynomials by solutions $G(\lambda)$ of the form (2.1). Notice that if we increase the degree of the polynomials $G(\lambda)$ then the solutions corresponding to the degree $n$ remain automatically the solutions of the system (3.1) after the change $n$ by $n + 1$. Therefore it is natural to suggest that many formulae of the previous Section,
including (4.4)–(4.17), remain valid in the general case (2.1). The proof follows from the comparison of the equations (3.1) and (2.6) under the change $G(\lambda) \rightarrow A(\lambda)$ where

$$A = 1 + \sum_{j=1}^{\infty} \lambda^{-j}a_j = 1 + \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \ldots .$$

(4.23)

The main difference between the equations for the coefficients of the series (4.23) under consideration

$$D_nA(\lambda) = \langle A_n(\lambda), A(\lambda) \rangle, \quad n = 1, 2, 3, \ldots$$

(4.24)

and the corresponding equations (3.1) is the problem of closing of the infinite chain of equations of the form (2.8). The papers [1] are devoted to the problems which arise after the change of regularity conditions by the conditions of finiteness of the series

$$U = \lambda^m + \sum_{j \geq 1} \lambda^{m-j}u_j, \quad V = \lambda^{m'} + \sum_{j \geq 1} \lambda^{m'-j}v_j$$

(4.25)

obtained from (4.23) via the maps (4.1) and (4.14). These papers contain also some interesting examples of the nonstandard closing of equations (4.24).

A Appendix

A.1 Riccati equation

We prove the following elementary statement in order to establish the relation of the map (4.1) and equation (4.4) with the Schwarz derivative and Riccati equation.

Statement 6. Let $\psi_1$, $\psi_2$ be two linearly independent solutions of the second order equation $\psi_{xx} = U\psi$, then the functions

$$A_1 = \psi_1^2, \quad A_2 = \psi_2^2, \quad A_3 = \psi_1\psi_2$$

form the basis of the solution space of the third order equation (cf (4.2)):

$$A_{xxx} = 4UA_x + 2U_xA.$$

Moreover the function $\varphi = \psi_1/\psi_2$ satisfies Schwarz equation

$$\frac{3\varphi_{xx}^2}{4\varphi_x^2} - \frac{\varphi_{xxx}}{2\varphi_x} = U(x),$$

(A.1)

and the function $A = \psi_1\psi_2$ satisfies equation

$$4U(x)A^2 + A_x^2 - 2AA_{xx} = w^2$$

(A.2)

where $w$ is the Wronskian $w = \langle \psi_1, \psi_2 \rangle = \psi_1\psi_{2,x} - \psi_{1,x}\psi_2$.

1From the algebraical point of view this problem is, probably, equivalent to the choice of the corresponding subalgebras of the algebra (2.5).
Proof. The Wronskian of $A_1$, $A_2$, $A_3$ is

$$W = \langle A_1, A_2, A_3 \rangle = (\psi_1 \psi_{2,x} - \psi_2 \psi_{1,x})^3 = (\psi_1, \psi_2)^3.$$ 

therefore, $W = \text{const} \neq 0$ and the functions $A_i$ are linearly independent. Let us denote

$$\varphi = \frac{\psi_1}{\psi_2}, \quad f_j = \frac{\psi_{j,x}}{\psi_j}.$$

It is not difficult to prove that

$$\varphi_x = \frac{\langle \psi_2, \psi_1 \rangle}{\psi_2^2} = \frac{w}{\psi_2^2}, \quad \varphi_{xx} = -2 \frac{\psi_{2x}}{\psi_2} = -2f_2$$

and therefore equation (A.1) follows from the Riccati equation $f_{2,x} + f_2^2 = U$.

Next, it is easy to check that

$$\frac{w}{A_3} = f_2 - f_1, \quad \frac{A_{3,x}}{A_3} = f_2 + f_1,$$

and therefore

$$f_1 = \frac{A_{3,x} - w}{2A_3}, \quad f_2 = \frac{A_{3,x} + w}{2A_3}.$$

The substitution of these expressions for $f_j$ into Riccati equation brings to equation (A.2) with $A = A_3$ and then the differentiation with respect to $x$ yields (A.2). The proof of the fact that $A_1$, $A_2$ also satisfy (A.2) is analogous, in these cases $w = 0$ in (A.2).

Let us rewrite equation (A.2) in the form

$$4U + \frac{A_x^2}{A^2} - \frac{2A_{xx}}{A} = \frac{w^2}{A^2}$$

and denote $H = 1/A$. Then we find that

$$U = \frac{3H_x^2}{4H^2} - \frac{H_{xx}}{2H} + w^2H^2. \quad (A.3)$$

It follows immediately from the comparison of this equation with Schwarz equation (A.1) that it correspond to the special case (A.3) with $w = 0$ and $H = \varphi_x$.

Therefore we prove that the change

$$F \overset{\text{def}}{=} f_1 - f_2 = \frac{w}{G}, \quad \frac{3F_x^2}{4F^2} - \frac{F_{xx}}{2F} + \frac{F^2}{4} = U$$

brings to equation (A.1) used for the definition of the map $G \rightarrow U$ (in Section 4). Taking into account the obvious relation between the formula (A.3) with the Schwarz derivative one may say that this transition from $G$ to $U$ is equivalent in some sense to the computation of the Schwarz derivative of $F = G^{-1}$. 

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Now consider the direct problem of construction of $G$ on the given a priori potential $U$. We assume that the potential is defined as the formal series (4.24):

$$U = \lambda^m + \sum_{j \geq 1} \lambda^{m-j} u_j, \quad m > 0.$$ 

It turns out that the difficulties which appear if we try to solve the Riccati equation explicitly disappear on the level of the formal series (4.23). For the sake of simplicity we restrict ourselves by the particular case $m = 2$.

**Lemma 7.** The Riccati equation with the potential

$$U(x, k) = \lambda^2 + \lambda u_1(x) + u_2(x) + \lambda^{-1} u_3 + \ldots$$  \hspace{1cm} (A.4)

has exactly two solutions $f = f^\pm$ in the form of the formal power series in $\lambda$:

$$f^+ = \lambda + \frac{1}{2} u_1 + f_1^+ \lambda^{-1} + \ldots, \quad f^- = -\lambda - \frac{1}{2} u_1 + f_1^- \lambda^{-1} + \ldots.$$  \hspace{1cm} (A.5)

**Proof.** The formulae for the coefficients of $f_{\pm 0}$ are obtained by substitution of the series $f = \sum_{j=0}^{\infty} \lambda^{-j} f_j$ into the equation $f_x + f^2 = U(x, \lambda)$. Collecting the coefficients at the powers of $\lambda$ we find $(f^\pm_1)^2 = 1, \ j_0 = -1$. The next coefficients of $f_j^+$ and $f_j^-$ are calculated recursively along the relations

$$2 f_{j+1}^+ + f_{j,x}^+ + \sum_{j'+j''=j} f_{j'}^+ f_{j''}^+ = u_j, \quad 2 f_{j+1}^- = f_{j,x}^- + \sum_{j'+j''=j} f_{j'}^- f_{j''}^- - u_j \quad (j > 1).$$

Therefore the coefficients of the series $f_j^\pm$ are uniquely defined differential polynomials on $u_j$. $\blacksquare$

### A.2 The consistent triples of equations

The equation (1.3) gives an example of consistent triple of the form

$$u_{\xi\eta} = f(u, u_\xi, u_\eta, u_x, u_\xi, u_\eta),$$

$$u_{\xi\zeta} = g(u, u_\xi, u_\zeta, u_x, u_\xi, u_\zeta),$$

$$u_{\eta\zeta} = h(u, u_\eta, u_\zeta, u_x, u_\eta, u_\zeta)$$  \hspace{1cm} (A.6)

(here and further on $(\tau_1, \tau_2, \tau_3) = (\xi, \eta, \zeta)$). A more symmetric example is given by the triple

$$(\mu_j - \mu_i) u_x u_{\tau_i \tau_j} - \mu_j u_{\tau_i} u_{x \tau_j} + \mu_i u_{\tau_j} u_{x \tau_i} = 0,$$  \hspace{1cm} (A.7)

where the variable $x$ is actually on the equal footing with the other ones. Notice that this equation is invariant under the changes $u = a(\tilde{u}), \ x = b(\tilde{x}), \ \tau_i = c_i(\tilde{\tau}_i)$ with arbitrary functions, so that a particular solution is $u = a(b(x)c(\tau_i)d(\tau_j))$.

Equations (1.3) and (A.7) are in close relation. Indeed, (A.7) is obtained from (1.3) after eliminating the derivatives $u_{x \tau_i}$ and the choice of one of $\tau_i$ as the new $x$. Moreover, the equation (A.7) can be rewritten in the form

$$\left( \frac{u_{\tau_i}}{\mu_i u_x} \right)_{\tau_j} = \left( \frac{u_{\tau_j}}{\mu_j u_x} \right)_{\tau_i}$$  \hspace{1cm} (15)
and the substitution $v_{ri} = u_{ri}/(\mu_i u_x)$ brings to the equation (1.3) again:

$$(\mu_i^{-1} - \mu_j^{-1})v_{ri\tau_j} = v_{ri} v_{x\tau_j} - v_{rj} v_{xri}.$$  

The classification problem of the consistent systems of the form (A.6) seems not actual until the integration methods are not well understood. Nevertheless, the simple preliminary analysis shows that the class of such systems can be rich enough. Let us assume that the right hand sides are quasilinear and do not depend on $u$ explicitly. This subclass of equations contains the following family of consistent triples.

**Statement 8.** Let $(X, \tilde{X}), (A, \tilde{A}), (B, \tilde{B}), (C, \tilde{C})$ be solutions of the ODE system

$$X' = k_1X^2 + k_2X\tilde{X} + k_3\tilde{X}^2, \quad \tilde{X}' = k_4X^2 + k_5X\tilde{X} + k_6\tilde{X}^2,$$  

(A.8)  

and the following functions are not identically zero:

$$a(u_\eta, u_\zeta) = B(u_\eta)\tilde{C}(u_\zeta) - \tilde{B}(u_\eta)C(u_\zeta), \quad p(u_\zeta, u_x) = A(u_\zeta)\tilde{X}(u_x) - \tilde{A}(u_\zeta)X(u_x),$$  

$$b(u_\zeta, u_\xi) = C(u_\zeta)\tilde{A}(u_\xi) - \tilde{C}(u_\zeta)A(u_\xi), \quad q(u_\eta, u_x) = B(u_\eta)\tilde{X}(u_x) - \tilde{B}(u_\eta)X(u_x),$$  

$$c(u_\xi, u_\eta) = A(u_\xi)\tilde{B}(u_\eta) - \tilde{A}(u_\xi)B(u_\eta), \quad r(u_\zeta, u_x) = C(u_\zeta)\tilde{X}(u_x) - \tilde{C}(u_\zeta)X(u_x).$$  

Then the following equations are consistent:

$$c(u_\zeta, u_\eta)u_{\eta\zeta} = p(u_\zeta, u_x)u_{x\zeta} - q(u_\eta, u_x)u_{x\eta},$$  

$$a(u_\eta, u_\zeta)u_{\eta\zeta} = q(u_\eta, u_x)u_{x\eta} - r(u_\zeta, u_x)u_{x\zeta},$$  

$$b(u_\zeta, u_\xi)u_{\xi\zeta} = r(u_\zeta, u_x)u_{x\xi} - p(u_\xi, u_x)u_{x\zeta}.$$  

The linear changes bring the second equation of the system (A.8) to the form

$$\tilde{X}' = kX\tilde{X} \quad \text{or} \quad \tilde{X}' = k\tilde{X}^2,$$  

and the general solution can be found in quadratures. In particular, the system (A.7) correspond to the functions $A_i = \mu_i/u_{ri}, \tilde{A}_i = 1/u_{ri}, X = \delta/u_x, \tilde{X} = 1/u_x$, which solve the system $X' = -X\tilde{X}, \tilde{X}' = -\tilde{X}^2$. In this case one of the integration constants can be neglected without loss of generality while the second one plays the role of parameter in the resulting system. In the examples

$$(u_{ri} - u_{rj})u_{ri\tau_j} + (u_{rj} - u_{x})u_{x\tau_j} + (u_{x} - u_{ri})u_{xri} = 0,$$  

$$u_{ri\tau_j} = \frac{u_{xri} - u_{x\tau_j}}{e^{u_{ri}} - e^{u_{rj}}}$$  

both integration constants are not essential.

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