Derivation of the Raychaudhuri Equation

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Abstract

As a homage to A K Raychaudhuri, I derive in a straightforward way his famous equation and also indicate the problems he was last engaged in.

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Let us consider a collection of bodies falling freely under their own gravity. They would all attract each other and would tend to converge. If they are expanding or contracting and their gravitational potential energy is greater in magnitude than the kinetic energy, they would all meet in future while if they are expanding with velocity greater than a critical (escape) value, it could be inferred by extrapolation back in time that they would have all been together at a point in the past. In general we can have a distribution of matter in any form, yet the same result will be expected. At the point of convergence, there will be divergence of density. That is what will characterize singularity signaling breakdown of the gravitational theory there.

Following this Newtonian argument, we could say that if matter distribution is homogeneous and isotropic (all points and all directions in space are equivalent and there is no way to distinguish one from the other), gravitational force at any point will vanish. Isotropy of space demands that in any direction there will be equal and opposite force and hence its sum will vanish because force is a vector quantity. Homogeneity will ask for this to happen at all points. This means in a perfectly homogeneous and isotropic Universe, there is no gravitational force on any body. That is gravity is completely annulled
out. This should not be true because matter density is non-zero and we have the Poisson equation to solve. This strange result critically hinges on gravity being a vector force. We are thus led to a very important inference that gravity can not be a vector force, it should rather be a tensor force. It is indeed a very novel argument again due to Raychaudhuri [1] that homogeneity and isotropy of matter distribution at large scale in the Universe can not be consistent with the Newtonian gravity. A new theory of gravity is required in which gravitational force is not a vector but a tensor.

We come to the Einsteinian theory of gravitation, General Relativity, simply by appealing to the universal property of gravity that it links/interacts with everything that physically exists. Its interaction with light which always propagates with a constant universal velocity can be realized if and only if gravity curves space (rather spacetime, because space and time are bound together into one whole, spacetime by universal speed of light in vacuum) [2, 3]. That is gravity can only be described by curvature of spacetime. It becomes a property of spacetime and hence no longer remains an external force. Motion under gravity is now simply free motion (straight line, geodetic) relative to curved geometry of spacetime. The difficulty that arose earlier for homogeneous and isotropic matter distribution evaporates simply because matter curves spacetime and particles follow its curvature. Gravity has now become a tensor force and hence unlike a vector force it can not be canceled out.

The law of gravity, the analogue of the Newtonian inverse square law, should now follow from the geometry of spacetime. It really does. The Riemann curvature satisfies the differential Bianchi identity which on contraction yields a divergence free second rank symmetric tensor, i.e

\[ G^{ab} = 0, \quad G^{ab} = R^{ab} - \frac{1}{2} R g^{ab} \]  

where \( R_{ab} = R^c_{acb} \). Here a semicolon denotes covariant derivative relative to the metric \( g_{ab} \) which is generalization of the ordinary derivative in curved spacetime. We can then write

\[ G_{ab} = -\kappa T_{ab} - \Lambda g_{ab} \]  

with \( T^{ab} = 0 \). On the left is a second order differential expression.
in the spacetime metric, $g_{ab}$, potential for gravity. On the right there must be its matter source given by $T_{ab}$, the stress energy tensor, with vanishing divergence which automatically takes care of conservation of energy and momentum. $\Lambda$ is a constant which naturally arises as a new constant of integration. Ignoring $\Lambda$ and demanding agreement with the Newtonian gravity in the limit determines $\kappa = -8\pi G/c^2$. This has simply followed from spacetime curvature without any prescription. Why there is an extra constant? Perhaps because here we did not have the comfort of a fixed spacetime background as is the case for all other forces, spacetime itself determined the dynamics of gravity. The new constant is a measure of this fact \[2, 3\].

The Einsteinian gravity could intuitively be viewed as the Newtonian inverse square law in curved space which takes care of light’s interaction with gravity. In the weak field limit, curvature of space will appear as a correction to the Newtonian effect as planetary orbits not being exactly closed but precessing ellipses. This is all fine but the theory does admit solutions which possess singularity characterized by divergence of spacetime curvature and physical (like density) as well as kinematic (like expansion) parameters. Is the occurrence of singularity an artifact of symmetry of spacetime and matter distribution like homogeneity and isotropy for the FRW cosmological model or is it generic feature of the theory? Is the big-bang singularity predicted by the FRW model generic or special to homogeneity and isotropy of matter distribution? This was the precise and profound question which Raychaudhuri addressed in mid 50s and obtained his celebrated equation \[4\]. We shall now consider the derivation of this beautiful equation.

Let us consider in all generality what happens to a congruence (collection) of particles having timelike 4-velocity $u^a$ as they fall under their own gravity. We know from fluid motion that it could suffer the following effects:

(a) Expansion/Contraction of volume which is given by the divergence of $u^a$, defined as $\theta = u^a_{\; ;a}$.

(b) Shear, distortion in shape without change in volume, is given by a symmetric tensor which is trace free (for no change in volume)
and orthogonal to \( u^a \) and hence it is defined by
\[
\sigma_{ab} = u_{(a;b)} - \frac{1}{3} \theta h_{ab} - \dot{u}_{(a} u_{b)}
\] (3)

where \( u_{(a;b)} = \frac{1}{2} (u_{a;b} + u_{b;a}) \), \( h_{ab} = g_{ab} - u_a u_b \).

(c) Rotation/Vorticity, rotation without change in shape and it is given by an antisymmetric tensor orthogonal to \( u^a \),
\[
\omega_{ab} = u_{[a;b]} - \dot{u}_{[a} u_{b]} \] (4)

where \( u_{[a;b]} = \frac{1}{2} (u_{a;b} - u_{b;a}) \).

(d) Acceleration due to non gravitational force like pressure gradient is a vector defined by \( \dot{u}_a = u_{a;b} u^b \) which is orthogonal to \( u^a \).

Now we can resolve in general
\[
u_{a;b} = \sigma_{ab} + \omega_{ab} + \frac{1}{3} \theta h_{ab} + \dot{u}_a u_b. \] (5)

Note that since any second rank tensor could be written in terms of its symmetric and anti-symmetric parts and hence the terms involving \( h_{ab} \) and \( \dot{u}_a \) cancel out in the above expression. They are however required for the definitions of shear and rotation in view of them being orthogonal to \( u^a \) and trace free.

For the definition of the Riemann curvature, we have
\[
u^a_{\ ;bc} - \nu^a_{\ ;cb} = R^a_{\ ;dcb} u^d. \] (6)

Contracting by putting \( a = b \) and multiplying with \( u^c \), we get
\[
\theta^a c u^c - \nu^a_{\ ;ca} u^c = R_{\ ;dca} u^d u^c. \] (7)

The second term on the left could be written as
\[
(u^a_{\ ;c} u^c)_{;a} - u^a_{\ ;c} u^c_{\ ;a} = \ddot{u}^a_{\ ;a} - u_{a;b} u^b_{\ ;a}. \] (8)

Also note that
\[
u_{a;b} u^{b;a} = 2(\sigma^2 - \omega^2) + \frac{1}{3} \theta^2 \] (9)
where \( \sigma_{ab}\sigma^{ab} = 2\sigma^2, \omega_{ab}\omega^{ab} = 2\omega^2 \). Substituting all this in equation (7), we obtain,

\[
\dot{\theta} - \dot{u}^a_{;a} + 2(\sigma^2 - \omega^2) + \frac{1}{3}\theta^2 = R_{ab}u^a u^b 
\] (10)

where \( \dot{\theta} = \theta_{;a} u^a \).

The Einstein equation reads as

\[
R_{ab} = -8\pi(T_{ab} - \frac{1}{2}T g_{ab}) 
\] (11)

where we have set \( G = c = 1 \), and \( T_{ab} \) is the stress energy tensor of matter distribution, which for the perfect fluid distribution reads as

\[
T_{ab} = (\rho + p)u_a u_b - pg_{ab}. 
\] (12)

Then for perfect fluid, \( R_{ab}u^a u^b = -4\pi(\rho + 3p) \), and we thus derive the Raychaudhuri equation

\[
\dot{\theta} = -2(\sigma^2 - \omega^2) - \frac{1}{3}\theta^2 - 4\pi(\rho + 3p) + \dot{u}^a_{;a}. 
\] (13)

The active gravitational density is really \( \rho + 3p \) which also has pressure contribution and further shear contributes in favour of it while rotation opposes it. This density is certainly positive for all known forms of ponderable matter and its positivity is known as the strong energy condition. Then in the absence of rotation and acceleration, we have

\[
\dot{\theta} \leq -\frac{1}{3}\theta^2 
\] (14)

which integrates out to

\[
\theta(\tau)^{-1} \geq \theta_0^{-1} + \frac{1}{3}\tau 
\] (15)

where \( \theta_0 \) is the initial value of \( \theta \) and \( \tau \) is the proper time. If the congruence is initially contracting with \( \theta_0 < 0 \), \( \theta \) will diverge (\( \theta \to -\infty \)) in finite proper time \( \tau \leq 3/|\theta_0| \). This simply follows for all matter satisfying the strong energy condition (\( \rho + 3p \geq 0 \)) from the Raychaudhuri equation without reference to any symmetry. Divergence of
the expansion parameter by itself does not however imply singularity of spacetime. But this aided with some global arguments does lead to spacetime singularity in certain cases. This important consequence of the Raychaudhuri equation ultimately played a key role in the proof of the celebrated singularity theorems due to Penrose, Hawking and Geroch [5]. In here, singularity was defined in terms of termination or incompleteness of a timelike or null geodesic. This definition facilitated proof of certain detailed theorems. A spacetime singularity marks not only the breakdown of the Einsteinian gravity but also of whole of physics - “End of Everything”.

The Raychaudhuri equation has also found applications in the recent developments in holography principle and quantum computation of black hole entropy. In particular a covariant bound on the black hole entropy can be obtained by studying the evolution of light sheets [6] as well as the flow of the Renormalization group in the theory space could be evolved [7]. In fact it would be applicable in any situation which is universal like gravity and its evolution could be modeled geometrically in an appropriate space. It thus has universal validity and applicability.

The singularity theorems reigned supreme. Particularly because the observation of CMBR [8] also pointed to a singular birth of the Universe in a Big-Bang. Nothing could be happier and more persuasive than observation verifying theory’s prediction. This gave rise to a general belief that singularity was inevitable in GR so long as positive energy condition and causality are respected. This belief was however shaken by Senovilla’s discovery in 1990 of a singularity free cosmological solution [9]. It did not violate the energy and causality conditions. How did it happen then? It brought forth the main suspect in the theorems. Among all the self evident assumptions, the theorems also required existence of closed trapped surface which is certainly not so obvious and self evident. That is gravity should become so strong in some bound region of space that even light can’t escape from it. This is a very limiting assumption for where gravity should become how strong should be determined by the field equation rather than being prescribed. This assumption is quite justifiable for the case of gravitational collapse of an isolated body. From the study of stellar structure we know that a sufficiently massive body could, as its nu-
clear fuel exhausts, ultimately undergo indefinite collapse and thereby reaching the trapped surface limit. But it is certainly not so obvious for cosmology. This limitation was though known to the experts in the field but not much talked of, perhaps in the belief that a singularity free solution will never be found.

This prompted Raychaudhuri once again to the question of singularity theorems. He argued that existence of singularity free cosmological solutions should be recognized and proposed vanishing of space and time averages of all the scalars appearing in the Raychaudhuri equation as a necessary condition for their existence [10]. He later proved a new singularity theorem. It states that some scalar built of the Ricci curvature will blow up if (a) the strong energy condition \( \rho + 3p \geq 0 \) is satisfied, (b) the timelike eigenvector of the Ricci tensor is hypersurface orthogonal (this excludes rotation as well as the vacuum with vanishing Ricci), and (c) the space average of any of the scalars occurring in the Raychaudhuri equation does not vanish [11]. He replaces occurrence of closed trapped surface by non vanishing of space average of scalars (vanishing of which is the necessary condition for occurrence of non-singular solutions).

The last paper he wrote was in 2004 and in which he attempted to deduce the Ruiz-Senovilla family [13] of non-singular solutions for non rotating perfect fluid from very general considerations in a novel manner [12]. It is however known that for imperfect fluid it is easy to construct a non-singular and even oscillating models [14, 15]. Even for stiff fluid, it has been shown that there exists a very large family of singularity free cosmological models [16]. The real challenge is to obtain rotating perfect fluid solutions. Apart from mathematical complexity, rotation brings in question occurrence of closed timelike lines and thereby causality. We have the well-known rotating Gödel universe which has closed timelike lines [17]. It would be interesting to find a rotating perfect fluid solution without closed timelike lines, and that was precisely what he was last working on before he died on 18th June 2005.

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