ON CERTAIN FUNCTIONS AND RELATED PROBLEMS

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Abstract. The present article is devoted to the description of further investigations of the author of this article. These investigations (in terms of various representations of real numbers) include the generalized Salem functions and generalizations of the Gauss-Kuzmin problem.

Let \( Q \equiv (q_k) \) be a fixed sequence of positive integers, \( q_k > 1, \) \( \Theta_k \equiv \{0, 1, \ldots, q_k - 1\} \), and \( \varepsilon_k \in \Theta_k \).

The Cantor series expansion

\[
\varepsilon_1 \frac{1}{q_1} + \varepsilon_2 \frac{1}{q_1q_2} + \cdots + \varepsilon_k \frac{1}{q_1q_2 \cdots q_k} + \cdots
\]

of \( x \in [0, 1] \), first studied by G. Cantor in [2]. It is easy to see that the Cantor series expansion is the \( q \)-ary expansion

\[
\frac{\alpha_1}{q} + \frac{\alpha_2}{q^2} + \cdots + \frac{\alpha_n}{q^n} + \cdots
\]

of numbers from the closed interval \([0, 1]\) whenever the condition \( q_k = q \) holds for all positive integers \( k \). Here \( q \) is a fixed positive integer, \( q > 1 \), and \( \alpha_n \in \{0, 1, \ldots, q - 1\} \).

By \( x = \Delta^{Q}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_k \ldots} \) denote a number \( x \in [0, 1] \) represented by series (1). This notation is called the representation of \( x \) by Cantor series (1). Also, by \( x = \Delta^{Q}_{\alpha_1 \alpha_2 \ldots \alpha_n \ldots} \) denote a number \( x \in [0, 1] \) represented by series (2). This notation is called the \( q \)-ary representation of \( x \).

We note that certain numbers from \([0, 1]\) have two different representations by Cantor series (1), i.e.,

\[
\Delta^{Q}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{m-1} \varepsilon_m 0000 \ldots} = \Delta^{Q}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{m-1} \varepsilon_{m+1} \ldots} = \sum_{i=1}^{m} \varepsilon_i q_1 q_2 \ldots q_i.
\]

Such numbers are called \( Q \)-rational. The other numbers in \([0, 1]\) are called \( Q \)-irrational.

Let \( c_1, c_2, \ldots, c_m \) be an ordered tuple of integers such that \( c_i \in \{0, 1, \ldots, q_i - 1\} \) for \( i = 1, m \).

A cylinder \( \Delta^{Q}_{c_1 c_2 \ldots c_m} \) of rank \( m \) with base \( c_1 c_2 \ldots c_m \) is a set of the form

\[
\Delta^{Q}_{c_1 c_2 \ldots c_m} \equiv \{ x : x = \Delta^{Q}_{c_1 c_2 \ldots c_m \varepsilon_{m+1} \varepsilon_{m+2} \ldots} \}.
\]

That is any cylinder \( \Delta^{Q}_{c_1 c_2 \ldots c_m} \) is a closed interval of the form

\[
[\Delta^{Q}_{c_1 c_2 \ldots c_m 0000 \ldots}, \Delta^{Q}_{c_1 c_2 \ldots c_m 1111 \ldots}].
\]

Define the shift operator \( \sigma \) of expansion (1) by the rule

\[
\sigma(x) = \sigma(\Delta^{Q}_{\varepsilon_1 \varepsilon_2 \ldots}) = \sum_{k=2}^{\infty} \varepsilon_k q_2 q_3 \ldots q_k = q_1 \Delta^{Q}_{\varepsilon_2 \varepsilon_3 \ldots}.
\]

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It is easy to see that

\[ \sigma^n(x) = \sigma^n \left( \Delta^Q_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{k-n}} \right) = \sum_{k=n+1}^{\infty} \frac{\varepsilon_k}{q_{n+1}q_{n+2} \ldots q_k} = q_1 \ldots q_n \Delta^Q_{\sum_{k=n+1}^{\infty} \varepsilon_{n+1} \varepsilon_{n+2} \ldots} \]

Therefore,

\[ x = \sum_{i=1}^{n} \frac{\varepsilon_i}{q_1q_2 \ldots q_i} + \frac{1}{q_1q_2 \ldots q_n} \sigma^n(x). \] (3)

In [8], the notion of the generalized shift operator was introduced (in terms of alternating Cantor series (also, see [12], [11])). One can describe this notion in terms of positive Cantor series.

Define the generalized shift operator \( \sigma_m \) of expansion (1) by the rule

\[ \sigma_m(x) = \sigma_m \left( \Delta^Q_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{k-n}} \right) = \sum_{i=1}^{m-1} \frac{\varepsilon_i}{q_1q_2 \ldots q_i} + \sum_{j=m+1}^{m-1} \frac{\varepsilon_j}{q_1q_2 \ldots q_{m-1}q_{m+1} \ldots q_j} = \Delta^Q_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{m-n} \varepsilon_{m+1} \ldots} \]

where \( Q'_m \) is the following sequence: \( (q_1, q_2, \ldots, q_{m-1}, q_{m+1}, q_{m+2}, \ldots) \).

It is easy to see that

\[ \sigma_m(x) = \sigma_m \left( \Delta^Q_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{k-n}} \right) = xq_m - \frac{\varepsilon_m}{q_1q_2 \ldots q_{m-1}} - (q_m - 1) \sum_{i=1}^{m-1} \frac{\varepsilon_i}{q_1q_2 \ldots q_i}. \]

Note that \( \sigma_m = \sigma \) whenever \( m = 1 \).

**Lemma 1.** The mapping \( \sigma_m \) has the following properties:

1. The mapping \( \sigma_m \) is continuous at each point of the interval \( \left( \inf \Delta^Q_{c_1c_2 \ldots c_m}, \sup \Delta^Q_{c_1c_2 \ldots c_m} \right) \). The endpoints of \( \Delta^Q_{c_1c_2 \ldots c_m} \) are points of discontinuity of the mapping.
2. The mapping \( \sigma_m \) has a derivative almost everywhere (with respect to the Lebesgue measure).
   If the mapping has a derivative at the point \( x = \Delta^Q_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{k-n}} \), then \( (\sigma_m)'(x) = q_m \).
3. The derivative of \( \sigma_m \) does not exist at a \( Q \)-rational point
   \[ x = \Delta^Q_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n 000 \ldots} = \Delta^Q_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{n-1} \varepsilon_n \left[ q_{n+1} \right] \left[ q_{n+2} \right] \ldots} \]
   whenever \( n = m \).
4. \( \sigma \circ \sigma_2 \circ \ldots \circ \sigma_2(x) = \sigma^{m+1}(x) \).
5. Suppose \( \left( k_n \right) \) is an arbitrary increasing sequence of positive integers. Then
   \[ \sigma^{k_n-1} \circ \sigma_{k_n} \circ \sigma_{k_{n-1}} \circ \ldots \circ \sigma_{k_1}(x) = \sigma^{k_n+n-1}(x). \]

In the next articles of the author of this paper, the notion of the generalized shift operator will be investigated in more detail and applied by the author of the present article in terms of various representation of real numbers (e.g., positive and alternating Cantor series and their generalizations, as well as Luroth, Engel series, etc., various continued fractions).

Let us consider certain applications of the generalized shift operator. One can model generalizations of the Gauss-Kuzmin problem and generalizations of the Salem function.
1. Generalizations of the Gauss-Kuzmin problem

The Gauss-Kuzmin problem is to calculate the limit
\[
\lim_{n \to \infty} \lambda(E_n(x)),
\]
where \(\lambda(\cdot)\) is the Lebesgue measure of a set and the set \(E_n(x)\) is a set of the form
\[
E_n = \{ z : \sigma^n(z) < x \}.
\]
Here \(z = \Delta_{i_1i_2...i_k}...\) \(i_1i_2...i_k...\) is a certain representation of real numbers, \(\sigma\) is the shift operator.

Generalizations of the Gauss-Kuzmin problem are to calculate the limit
\[
\lim_{k \to \infty} \lambda(\tilde{E}_{n_k}(x)),
\]
for sets of the following forms:

- \(\tilde{E}_{n_k}(x) = \{ z : \sigma_{n_k} \circ \sigma_{n_{k-1}} \circ ... \circ \sigma_{n_1}(z) < x \}\) including (here \((n_k)\) is a certain fixed sequence of positive integers) the cases when \((n_k)\) is a constant sequence.
- the set \(\tilde{E}_{n_k}(x)\) under the condition that \(n_k = \psi(k)\), where \(\psi\) is a certain function of the positive integer argument.
- \(\tilde{E}_{n_k}(x) = \left\{ z : \sigma_{\psi(\varphi(m,k,c))} \circ ... \circ \sigma_{\psi(1)}(z) < x \right\}\),

where \(\varphi\) is a certain function and \(m, c\) are some parameters (if applicable). That is, for example,
\[
\tilde{E}_{n_k}(x) = \left\{ z : \sigma_m \circ \sigma_m \circ ... \circ \sigma_m(z) < x \right\},
\]
where \(k > c\) and \(c\) is a fixed positive integer, or
\[
\tilde{E}_{n_k}(x) = \left\{ z : \sigma_m \circ \sigma_m \circ ... \circ \sigma_m(z) < x \right\},
\]
where \(k \equiv 1(\mod c)\) and \(c > 1\) is a fixed positive integer.
- In the general case,
\[
\tilde{E}_{n_k}(x) = \left\{ z : \sigma_{\psi(\varphi(m,k,c))} \circ ... \circ \sigma_{\psi(1)}(z) < x \right\},
\]
In addition, one can formulate such problems in terms of the shift operator. For example, one can formulate the Gauss-Kuzmin problem for the following sets:
\[
\hat{E}_{n_k}(z) = \{ z : \sigma^{n_k}(z) < \sigma^{k_0}(z) \},
\]
where \(k_0\), \((n_k)\) are a fixed number and a fixed sequence.
\[
\hat{E}_{n_k}(x) = \{ z : \sigma^{n_k}(z) < \sigma^{k_0}(x) \},
\]
\[
\tilde{E}_n(x) = \{ z : \sigma^{\psi(n)}(z) < x \},
\]
where \(\psi(n)\) is a certain function of the positive integer argument.
In addition,
\[
\hat{E}_n(z) = \{ z : \sigma^{\psi(n)}(z) < \sigma^{\varphi(n)}(z) \},
\]
where \( \psi, \varphi \) are certain functions of the positive integer arguments. It is easy to see that similar problems can be formulated for the case of the generalized shift operator.

In next articles of the author of the present article, such problems will be considered by the author of this article in terms of various numeral systems (with a finite or infinite alphabet, with a constant or variable alphabet, positive, alternating, and sign-variable expansions, etc.).

2. Generalizations of the Salem function

In \([3]\), Salem modeled the function

\[
\psi(x) = \psi(\Delta^2_{\alpha_2\alpha_3...\alpha_k} x) = \beta\alpha_2 + \sum_{n=2}^{\infty} \left( \beta\alpha_n \prod_{i=1}^{n-1} q_i \right) = y = \Delta^2_{\alpha_2\alpha_3...\alpha_k},
\]

where \( q_0 > 0, q_1 > 0, \) and \( q_0 + q_1 = 1. \) This function is a singular function. However, generalizations of the Salem function can be non-differentiable functions or do not have the derivative on a certain set.

Note that certain Salem function generalizations are considered in \([10, 4, 5, 13]\) as well.

Suppose \( (n_k) \) is a fixed sequence of positive integers such that \( n_i \neq n_j \) for \( i \neq j \) and such that for any \( n \in \mathbb{N} \) there exists a number \( k_0 \) for which the condition \( n_{k_0} = n \) holds.

Let us consider the following infinite system of functional equations

\[
f(\sigma_{n_{k-1}} \circ \sigma_{n_{k-2}} \circ ... \circ \sigma_{n_1}(x)) = \beta\alpha_{n_k} + \sum_{i=0}^{m_n} p_{i,n} f(\sigma_{n_k} \circ \sigma_{n_{k-1}} \circ ... \circ \sigma_{n_1}(x)),
\]

where \( k = 1, 2, ..., \) \( \sigma_0(x) = x, \) and \( x \) represented in terms of a certain given numeral system, i.e., \( x = \Delta_{\alpha_2\alpha_3...\alpha_k}, \) and \( \alpha_n \in \{0, 1, ..., m_n\} \) for all positive integers \( n. \) Also, here \( P = ||p_{i,n}|| \) is a fixed matrix, where \( i = 0, m_n, m_n \in \mathbb{N} \cup \{0\}, n = 1, 2, ..., \) and for elements \( p_{i,n} \) of \( P \) the following system of conditions is true:

\[
\begin{align*}
1^o . & \quad p_{i,n} \in (-1, 1) \\
2^o . & \quad \forall n \in \mathbb{N} : \sum_{i=0}^{m_n} p_{i,n} = 1 \\
3^o . & \quad \forall (i_n), i_n \in \mathbb{N} \cup \{0\} : \prod_{n=1}^{\infty} |p_{i_n,n}| = 0 \\
4^o . & \quad \forall i_n \in \mathbb{N} : 0 = \beta^0_{i,n} < \beta^n_{i_n,n} = \sum_{i=0}^{i_n-1} p_{i,n} < 1.
\end{align*}
\]

In the next articles of the author of this paper, properties of solutions of the last system of functional equations will be investigated for the cases of various numeral systems (with a finite or infinite alphabet, with a constant or variable alphabet, positive, alternating, and sign-variable expansions, etc.).

Now one can begin this investigation with the case of the \( q \)-ary representation of real numbers.

**Theorem 1.** Let \( P_q = \{p_0, p_1, ..., p_{q-1}\} \) be a fixed tuple of real numbers such that \( p_i \in (-1, 1), \) where \( i = 0, q - 1, \sum_i p_i = 1, \) and \( 0 = \beta_0 < \beta_i = \sum_{j=0}^{i-1} p_j < 1 \) for all \( i \neq 0. \) Then the system of functional equations

\[
f(\sigma_{n_{k-1}} \circ \sigma_{n_{k-2}} \circ ... \circ \sigma_{n_1}(x)) = \beta\alpha_{n_k} + \sum_{i=0}^{m_n} p_{i,n} f(\sigma_{n_k} \circ \sigma_{n_{k-1}} \circ ... \circ \sigma_{n_1}(x)),
\]

(4)
where \( x = \Delta_{a_1a_2...a_k...} \), has the unique solution

\[
g(x) = \beta_{a_{n_1}} + \sum_{k=2}^{\infty} \left( \frac{1}{\beta_{a_{n_k}}} \prod_{j=1}^{k-1} p_{a_{n_j}} \right)
\]

in the class of determined and bounded on \([0,1]\) functions.

**Proof.** Since the function \( g \) is a determined on \([0,1]\) function, using system (1), we get

\[
g(x) = \beta_{a_{n_1}} + p_{a_{n_1}} g(\sigma_{n_1}(x)) = \beta_{a_{n_1}} + p_{a_{n_1}}(\beta_{a_{n_2}} + p_{a_{n_2}} g(\sigma_{n_2} \circ \sigma_{n_1}(x))) = \ldots
\]

\[
\cdots = \beta_{a_{n_1}} + p_{a_{n_1}} \sum_{k=2}^{\infty} \left( \frac{1}{\beta_{a_{n_k}}} \prod_{j=1}^{k-1} p_{a_{n_j}} \right) g(\sigma_{n_k} \circ \cdots \circ \sigma_{n_2} \circ \sigma_{n_1}(x)).
\]

So,

\[
g(x) = \beta_{a_{n_1}} + \sum_{k=2}^{\infty} \left( \frac{1}{\beta_{a_{n_k}}} \prod_{j=1}^{k-1} p_{a_{n_j}} \right)
\]

since \( g \) is a determined and bounded on \([0,1]\) function and

\[
\lim_{k \to \infty} g(\sigma_{n_k} \circ \cdots \circ \sigma_{n_2} \circ \sigma_{n_1}(x)) \prod_{t=1}^{k} p_{a_{n_t}} = 0,
\]

where

\[
\prod_{t=1}^{k} p_{a_{n_t}} \leq \left( \max_{0 \leq i \leq q-1} p_i \right)^k \to 0, \quad k \to \infty.
\]

□

**Theorem 2.** The function \( g \) is continuous at \( q \)-irrational points of \([0,1]\). The set of all points of discontinuities of the function \( g \) is a countable, finite, or empty set. It depends on a sequence \((n_k)\).

**Theorem 3.** Lebesgue integral of the function \( g \) can be calculated by the formula

\[
\int_0^1 g(x) dx = \frac{\beta_1 + \beta_2 + \cdots + \beta_{q-1}}{q - 1}.
\]

**Remark 1.** It can be interesting to consider the case when \((n_k)\) is an arbitrary fixed sequence (finite or infinite) of positive integers. Then the function \( g \) can be a constant function, a linear function, or a function having pathological (complicated) structure, etc. It depends on \((n_k)\). Such problems will be investigated in the next papers of the author of this article.

To be continue...

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