The odd harmonious labeling of matting graph

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Abstract. Let $G(p, q)$ be a graph that consists of $p$ vertices and $q$ edges, where $V$ is the set of vertices and $E$ is the set of edges of $G$. A graph $G(p, q)$ is odd harmonious if there exists an injective function $f$ that labels the vertices of $G$ by integer from 0 to $2q - 1$ that induced a bijective function $f^*$ defined by $f^*(uv) = f(u) + f(v)$ such that the labels of edges are odd integer from 1 to $2q - 1$. A graph that admits harmonious labeling is called a harmonious graph.

A matting graph is a chain of $C_4$–snake graph. A matting graph can be viewed as a variation of the grid graph. In this paper, we prove that the matting graph is an odd harmonious graph.

1. Introduction
A graph $G(V, E)$ is an ordered pair of sets $V$ and $E$ where the elements of $E$ are unordered pairs of distinct elements of $V$. An element of $V$ and $E$ is called vertex and edge, respectively. A graph $G(p, q)$ is a graph with order $p = |V|$ and size $q = |E|$. A labelling of $G$ is an assignment of integers to element $G$ (vertices and edges).

One of the graph labelling types is odd harmonious labeling. An odd harmonious labelling of graph $G(p, q)$ is an injective function $f: V \to \{0, 1, 2, \ldots, 2q - 1\}$ that induces a bijective function $f^*: E \to \{1, 3, 5, \ldots, 2q - 1\}$ defined by $f^*(uv) = f(u) + f(v)$, for each $uv \in E$ and $u, v \in V$. A graph $G(p, q)$ that admits odd harmonious labelling is called odd harmonious graph.

One type of graph which has many modifications and variations is a cycle $C_n$ graph. Liang and Bai [1] proved that a cycle $C_n$ is odd harmonious if and only if $n \equiv 0 \pmod{4}$. Alyani, Firmansyah, Giyarti, and Sugeng [2] proved that $kC_n$–snake graphs for specific values of $n$, that is, for $n = 4$ and $n = 8$ are odd harmonious. Abdel-Aal [3] proved that all subdivision of $2mΔ_k$ – snake, $k, m \geq 1$ and $kC_4$ – snake $\ominus mk_1$ for each $k, m \geq 1$ are odd harmonious. Jeyanti P, Philo S, and Youssef M [4] proved that path union of $m$ copies of $P_m \times P_n$ is odd harmonious, where $m, n, t \geq 2$. Other researchers who also published their work in odd harmonious labeling are Febriana and Sugeng [5], Tanna [6], Saputri and Sugeng [7], and also Jeyanti and Philo [8, 9]. The compilation of results in graph labeling can be found in Gallian [10].

In this paper, we consider a chain of $C_4$–snake graph that we call matting graph. A matting graph is a graph obtained by arranging the $m$-row $nC_4$–snake graph. We show that the matting graph is odd harmonious.

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2. Result and Discussion

A \( nC_4 \)-snake is a graph obtained from a path \( u_0^n, u_1^n, \ldots, u_l^n \) by joining \( u_i^j \) and \( u_i^{j+1} \) to new vertices \( v_{i-1}^j \) and \( v_i^j \) for \( j = 1, 2, \ldots, n-1 \), where \( n \geq 2 \) is a positive integer. A matting graph \( M_{m,n} \) with \( m, n \geq 2 \) is obtained by arranging \( m \)-row of \( nC_4 \)-snake graph.

The vertex set and the edge set of matting graph \( M_{m,n} \) is as follows:

\[
V(M_{m,n}) = \{v_i^j | 1 \leq j \leq n, 1 \leq i \leq m\} \cup \{u_i^j | 1 \leq j \leq n, 1 \leq i \leq m\} \cup \{v_0^j | 1 \leq j \leq n\}
\]

The order and size of \( M_{m,n} \) is \( 2mn + m + n \) and \( 4mn \), respectively.

A matting graph \( M_{m,n} \) and its vertex notation is shown in Figure 1.

![Figure 1. Matting graph \( M_{m,n} \)](image)

**Theorem 1**

A matting graph is an odd harmonious graph.

**Proof.**

Define the vertex label \( f : V(M_{m,n}) \to \{0, 1, 2, \ldots, 8mn - 1\} \) as follows:

\[
f(v_i^j) = 4ni + 2j - 1, \quad \text{for } 1 \leq j \leq n, 0 \leq i \leq m, \quad (1)
\]

\[
f(u_i^j) = 4ni + 2j - 4n, \quad \text{for } 0 \leq j \leq n, 1 \leq i \leq m, \quad (2)
\]

We show that \( f \) is a one to one function from \( V(M_{m,n}) \) to \( \{0, 1, 2, \ldots, 8mn - 1\} \).

It is clear that the value of \( f \) in (1) is odd and the value of \( f \) in (2) is even, for any \( i \) and \( j \), where \( f \) has a different value for different \( i \) and \( j \). Because the values of \( f \) from (1) are odd numbers and the values of \( f \) from (2) are even numbers, then \( \{v_i^j\} \cap \{u_i^j\} = \emptyset \). The values of \( f \) in (1) are increasing from...
1 to $4mn + 2n - 1$, and values of $f$ in (2) are increasing from 0 to $4mn + 2m - 4n$. Since $4mn + 2n - 1 < 8mn - 1$ and $4mn + 2m - 4n < 8mn - 1$, it is proven that $f$ is an injective function from $V(M_{m,n})$ to $\{0, 1, 2, ..., 8mn - 1\}$.

The induced edge labels $f^*: E(M_{m,n}) \rightarrow \{1, 3, 5, ..., 8mn - 1\}$ which defined by $f^*(uv) = f(u) + f(v)$ where $uv \in E(G)$ and $u, v \in V(G)$, are:

\[
\begin{align*}
    f^*(u_i^{-1}v_{i-1}) &= 8ni - 8n + 4j - 3, \quad \text{for } 1 \leq j \leq n, 1 \leq i \leq m, \quad (3) \\
    f^*(u_i^jv_{i-1}) &= 8ni - 8n + 4j - 1, \quad \text{for } 1 \leq j \leq n, 1 \leq i \leq m, \quad (4) \\
    f^*(u_i^{-1}v_i^j) &= 8ni - 4n + 4j - 3, \quad \text{for } 1 \leq j \leq n, 1 \leq i \leq m, \quad (5) \\
    f^*(u_i^jv_i^j) &= 8ni - 4n + 4j - 1, \quad \text{for } 1 \leq j \leq n, 1 \leq i \leq m. \quad (6)
\end{align*}
\]

We show that $f^*$ is a bijective function from $E(M_{m,n})$ to $\{1, 3, 5, ..., 8mn - 1\}$.

It is clear that the values of $f^*$ in (3) and (5) are in the class of $1(mod 4)$ and the values of $f^*$ in (4) and (6) are in the class of $3(mod 4)$, therefore all values of $f^*$ are odd numbers. Since the values of $f^*$ in (3) and (5) are in a different class with the values of $f^*$ in (4) and (6), therefore there is no intersection between them. To prove that $f^*$ is a one-to-one, we only need to show two cases: the set of values of $f^*$ in (3) is disjoint with the set of values in (5), and the set of values of $f^*$ in (4) is disjoint with the set of values in (6).

Case 1. Suppose that $f^*(u_i^{-1}v_{i-1}) = f^*(u_i^{-1}v_{i-1})$. It means $8ni - 8n + 4j - 3 = 8ni - 4n + 4j - 3$ or $n = 0$.

Case 2. Suppose that $f^*(u_i^jv_{i-1}) = f^*(u_i^jv_{i-1})$. It means $8ni - 8n + 4j - 1 = 8ni - 4n + 4j - 1$ or $n = 0$.

In both cases the values of the functions are different.

The values of $f^*$ in (3) are increasing from 1 to $8mn - 4n - 3$. The values of $f^*$ in (4) are increasing from 3 to $8mn - 4n - 1$. The values of $f^*$ in (5) are increasing from $4n + 1$ to $8mn - 3$. And the values of $f^*$ in (6) are increasing from $4n + 3$ to $8mn - 1$. It is clear that $8mn - 4n - 3 < 8mn - 1$, $8mn - 4n - 1 < 8mn - 1$, $8mn - 3 < 8mn - 1$, and $8mn - 1 = 8mn - 1$. The function $f^*$ has 4 sub-functions with each sub-function having $mn$ iterations, therefore $f^*$ has $4mn$ different values, corresponding with the $4mn$ edges the matting graph has. Because the values of $f^*$ are odd numbers with the minimum value is 1 and the maximum value is $8mn - 1$, and $f^*$ has corresponding different values for every edge, it is proven that $f^*$ is a bijective function from $E(M_{m,n})$ to $\{1, 3, 5, ..., 8mn - 1\}$.

The example of odd harmonious labeling of matting $M_{3,5}$ graph is shown in Figure 2. It shows the vertex and the edge labels of a matting graph which consists of 3 rows and 5 columns of $C_4$. 

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3. Conclusion
The matting graph $M_{m,n}$ is proven to be odd harmonious. Further investigation can be conducted to find the variations of the matting graphs and its odd harmonious labeling.

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