Type D space-time of Einstein-space with a naked curvature singularity and Closed Time-like Curves

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Abstract

In this paper, we present a type D vacuum solution of the Einstein field equations with a non-zero negative cosmological constant ($\Lambda < 0$), an extension of axially symmetric, asymptotically flat type D vacuum metric which possesses a naked curvature singularity. The study Einstein space metric admits closed time-like curves (CTC) that appear after a certain instant of time; indicating that it is an Einstein space time machine space-time.

Keywords: closed time-like curves, cosmological constant, Misner space

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1 Introduction

Closed time-like curves constitute one of the most intriguing aspects of general relativity. Among the best known metrics admitting closed time-like curves (CTC) is the Gödel Universe [1]. It represents a rotating universe. The metric, which is axially symmetric, is given by

$$ds^2 = dr^2 + dz^2 + (\sinh^2 r - \sinh^4 r) d\theta^2 + 2 \sqrt{2} \sinh^2 r d\theta dt - dt^2$$  (1)

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Clearly, if for some \( r > r_0 \), the metric function, \( g_{\theta \theta} = \sinh^2 r - \sinh^4 r \) is negative, the circle defined by \( r > r_0 \), and \( t = 0 = z \) will be time-like everywhere. This condition is fulfilled when \( r > r_0 = \ln(1 + \sqrt{2}) \) which is therefore the condition for the existence of closed time-like curves in Gödel space-time because one of the coordinate \( \theta \in [0, 2\pi] \) is periodic. Apart from Gödel’s solution, considerable number of space-times admitting CTC have been constructed (see Refs. \[2, 3\] and references therein). Causality violating curves are classified as either "eternal" or as "true" time-machine space-times. Eternal time machines are those which always pre-exist. In this category would be \[1\] or \[4\] (see also, Refs. \[5, 6, 7, 8, 9\]). A true time machine space-time is the one in which CTC evolve at a particular instant of time. In this category would be the Ori time-machine \[10\] (see also, Refs. \[11, 12, 13, 14, 15, 16, 17, 18\]). Most of the causality violating models however suffer from severe drawbacks as workable time machines. In many cases they violate the weak energy condition (WEC) and hence cannot be regarded as plausible time machine space-times. The weak energy condition states that for a time-like observer the matter density cannot be negative, which is true for all types of (classical) matter fields. We note that the “eternal” time machine space-times are unrealistic models for a putative time machine. A workable model of a time machine must be a space-time where CTCs appear at a definite instant of time.

The cosmological constant plays a vital role in explaining the dynamics of the universe. A tiny positive cosmological constant neatly explains the late time accelerated expansion of the universe. Indeed our universe is observed to be undergoing a de Sitter(dS) type expansion in the present epoch. For a negative cosmological constant, the spacetime is labelled anti-de Sitter(AdS). The AdS space has been a subject of intense study in recent times on account of the celebrated AdS/CFT correspondence which provides a link between a quantum theory of gravity on an asymptotically AdS space and a lower dimensional conformal field theory(CFT) on its boundary.
2 Review of type D vacuum space-time with a naked singularity \[12\]

In Ref. \[12\], a type D axially symmetric, asymptotically flat vacuum solution of the field equations with zero cosmological constant, were constructed. This vacuum metric is as follow

\[
 ds^2 = -\cosh t \coth t \sinh^2 r \, dt^2 + \cosh^2 r \sinh r \, dr^2 + \csch r \, dz^2 \\
+ \sinh^2 r (2\sqrt{2} \cosh t \, dt \, d\phi - \sinh t \, d\phi^2). \tag{2}
\]

After doing a number of transformations into the above metric, we arrive at the following

\[
 ds^2 = \cosh^2 r \, \sinh r \, dr^2 + \csch r \, dz^2 - \sinh^2 r (2 \, dt \, d\phi + t \, d\phi^2). \tag{3}
\]

The Kretschmann scalar of the above metric are

\[
 K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{12}{\sinh^6 r}. \tag{4}
\]

For constant \(r, z\), the metric (3) reduces to conformal Misner metric in 2D

\[
 ds^2 = \Omega ( -2 \, dt \, d\phi - t \, d\phi^2), \tag{5}
\]

where \(\Omega = \sinh^2 r\) is the conformal factor.

In the context of CTC, the Misner space metric in 2D is interesting because CTC appear after a certain instant of time from causally well-behaved conditions. The metric for the Misner space in 2D \[19\] is given by

\[
 ds^2_{\text{Mis}} = -2 \, dT \, dX - T \, dX^2 \tag{6}
\]

where \(-\infty < T < \infty\) but the co-ordinate \(X\) is periodic locally. The metric (6) is regular everywhere as \(\det g = -1\) including at \(T = 0\). The curves \(T = T_0\), where \(T_0\) is a constant, are closed since \(X\) is periodic. The curves \(T < 0\) are spacelike, \(T > 0\) are time like, while the null curves \(T = 0\) form
the chronology horizon. The second type of curves, namely, $T = T_0 > 0$ are closed time-like curves. Therefore, the metric (2) or (3) is a four-dimensional generalization of Misner space in curved space-time. Note that the above space-time is vacuum solution of field equations, a Ricci flat, that is, $R_{\mu\nu} = 0$.

In this paper, we extend the above space-time (2) to a Ricci flat space-time with non-zero cosmological constant, an Einstein space satisfying the following criteria:

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \quad R = 4 \Lambda, \quad \Lambda < 0.$$  \hspace{1cm} \text{(7)}

3 Analysis of the space-time

Consider the following line element, modification of the metric (2) still representing a vacuum space-time of non-zero cosmological constant is given by

$$ds^2 = \left( \alpha \csch r + \frac{\Lambda}{3} \sinh^2 r \right) dz^2 - \cosh t \coth t \sinh^2 r \, dt^2 - \sinh^2 r \sinh t \, d\phi^2$$

$$+ 2\sqrt{2} \cosh t \sinh^2 r \, dt \, d\phi + \frac{dr^2}{\alpha \csch r \sech^2 r + \frac{\Lambda}{3} \tanh^2 r}.$$ \hspace{1cm} \text{(8)}

Here $\alpha > 0$ and $\Lambda < 0$ are positive constant and cosmological constant, respectively. The coordinates are labelled $x^0 = t$, $x^1 = r$, $x^2 = \phi$, and $x^3 = z$. The ranges of the coordinates are

$$0 \leq r < \infty, \quad -\infty < z < \infty, \quad -\infty < t < \infty$$  \hspace{1cm} \text{(9)}

and $\phi$ is a periodic coordinate $\phi \sim \phi + \phi_0$, with $\phi_0 > 0$. The metric is Lorentzian with signature $(+,+,+,\) and the determinant of the corresponding metric tensor $g_{\mu\nu}$,

$$\det g = -\cosh^2 r \sinh^4 r \cosh^2 t.$$  \hspace{1cm} \text{(10)}
By doing a number of transformations as done in Ref. [12] into the above metric (8), we arrive at the following

\[ ds^2 = \left( \alpha \cosh r + \frac{\Lambda}{3} \sinh^2 r \right) dz^2 + \frac{\cosh^2 r \, dr^2}{\left( \alpha \cosh r + \frac{\Lambda}{3} \sinh^2 r \right)} - \sinh^2 r \left( 2 \, dt \, d\phi + t \, d\phi^2 \right). \]  

(11)

The space-time (8) satisfies the condition (7) of negative cosmological constant.

For constant \( r = r_0 > 0 \) and \( z = z_0 \), from the metric (11) we obtain

\[ ds^2_{confor} = \sinh^2 r \left( -2 \, dt \, d\phi - t \, d\phi^2 \right) = \Omega \, ds^2_{Mis}, \]  

(12)

a conformal Misner space metric in 2D where, \( \Omega \) is the conformal factor. Therefore, the study space-time admits CTC for \( t = t_0 > 0 \) similar to the Misner space.

We check whether the CTCs evolve from an initially spacelike \( t = constant \) hypersurface (and thus \( t \) is a time coordinate). This is determined by calculating the norm of the vector \( \nabla_\mu t \) (or alternately from the value of \( g^{tt} \) in the inverse metric tensor \( g^{\mu\nu} \)). A hypersurface \( t = constant \) is spacelike when \( g^{tt} < 0 \) at \( t < 0 \), timelike when \( g^{tt} > 0 \) for \( t > 0 \) and null \( g^{tt} = 0 \) for \( t = 0 \).

For our given metric (8),

\[ \nabla_\mu t \nabla^\mu t = g^{tt} = \frac{\sinh t}{\sinh^2 r \cosh^2 t}. \]  

(13)

Thus a hypersurface \( t = constant \) is spacelike for \( t < 0 \), timelike for \( t > 0 \) and null at \( t = 0 \). We restrict our analysis to \( r > 0 \) otherwise no CTC form. Thus the spacelike \( t = constant < 0 \) hypersurface can be choosen as initial hypersurface over which initial data may be specified. There is a Cauchy horizon at \( t = 0 \), called Chronology horizon, which separates the causal past and future in a past directed and a future directed manner.

Hence the spacetime evolves from a partial Cauchy surface (i.e. Cauchy spacelike hypersurface) in a causally well-behaved, up to a moment, i.e., a
null hypersurface $t = 0$ and the formation of CTCs takes place from causally well-behaved initial conditions. The evolution of CTCs is thus identical to the case of the Misner space.

That the space-time represented by (8) satisfy the requirements of axial symmetry is clear from the following. Consider the Killing vector $\eta = \partial_\phi$ having the normal form

$$\eta^\mu = (0, 1, 0, 0)$$  

Its co-vector form

$$\eta_\mu = \sinh^2 r \left(0, -\sinh t, 0, \sqrt{2} \cosh t\right)$$  

The vector (15) satisfies the Killing equation $\eta_{\mu;\nu} + \eta_{\nu;\mu} = 0$. The space-time is axial symmetry if the norm of the Killing vector $\eta^\mu$ vanish on the axis i.e. at $r = 0$ (see [20, 21] and references therein). In our case

$$X = |\eta_\mu \eta^\mu| = |g_{\phi\phi}| = | - \sinh t \sinh^2 r| \to 0,$$

as $r \to 0$.

The metric has a curvature singularity at $r = 0$. We find that the Kretschmann scalar

$$K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{8 \Lambda^2}{3} + \frac{12 \alpha^2}{\sinh^6 r}$$  

We can see that the scalar curvature diverge at $r \to 0$ which indicates the study space-time possess a singularity. Furthermore, the singularity is not covered by an event horizon and therefore, a naked singularity is formed. In addition, the Kretschmann scalar $K \to \frac{8 \Lambda^2}{3}$ for $r \to \infty$ indicating that the metric is asymptotically anti-de Sitter space radially.
3.1 Classification of the study solution and it’s physical Interpretation

To classify the metric (8), we construct the following set of null tetrad $k, l, m, \bar{m}$. Explicitly these co-vectors are

\[
k_\mu = \frac{\sinh r}{\sqrt{2}} \left( \frac{\cosh t}{\sqrt{\sinh t}}, 0, (-\sqrt{2} + 1) \sqrt{\sinh t}, 0 \right),
\]

\[
l_\mu = \frac{\sinh r}{\sqrt{2}} \left( \frac{\cosh t}{\sqrt{\sinh t}}, 0, -(\sqrt{2} + 1) \sqrt{\sinh t}, 0 \right),
\]

\[
m_\mu = \frac{1}{\sqrt{2}} \left( 0, \frac{\cosh r}{\sqrt{\alpha \cosh r + \Lambda}} \sinh^2 r, 0, i \sqrt{\alpha \cosh r + \Lambda} \sinh^2 r \right),
\]

\[
\bar{m}_\mu = \frac{1}{\sqrt{2}} \left( 0, \frac{\cosh r}{\sqrt{\alpha \cosh r + \Lambda}} \sinh^2 r, 0, -i \sqrt{\alpha \cosh r + \Lambda} \sinh^2 r \right).
\]

where $i = \sqrt{-1}$. The set of null tetrad is such that the metric tensor for the line element (8) can be expressed as

\[
g_{\mu\nu} = -k_\mu l_\nu - l_\mu k_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu
\]

(22)

The vectors (18)—(21) are null vector and orthogonal, except for $k_\mu l^\mu = -1$ and $m_\mu \bar{m}^\mu = 1$.

We calculate the five Weyl scalars, of these only

\[
\Psi_2 = C_{\mu\nu\rho\sigma} k^\mu m^\nu \bar{m}^\rho l^\sigma = \frac{\alpha}{2 \sinh^3 r}
\]

(23)

is non-vanishing, while the rest are all vanishes. Thus the study metric is clearly of type D in the Petrov classification scheme.

In order to analyze effect of the local gravitational field of the solution, we used the equations of geodesic deviation frames [2, 6] which in terms of orthonormal frame $e_{(a)}$ can be written as

\[
\ddot{Z}^{(i)} = -R^{(i)}_{(0)(j)(0)} Z^{(j)}, \quad i, j = 1, 2, 3,
\]

(24)
where \( e_{(0)} = u \) is time-like four-velocity vector of the free test particles. We set here \( Z^{(0)} = 0 \) such that all test particles are synchronized by the proper time. From the standard definition of the Weyl tensor and the field equation \( G_{\mu \nu} + \Lambda g_{\mu \nu} = 0 \), we get

\[
R_{(i)(0)(j)(0)} = C_{(i)(0)(j)(0)} - \frac{\Lambda}{3} \delta_{ij},
\]

where \( C_{(i)(0)(j)(0)} \equiv d_{(i)} \varepsilon^{\mu} e_{(j)}^\nu e_{(j)}^\rho e_{(j)}^\sigma C_{\mu \nu \rho \sigma} \) are the components of the Weyl tensor.

The only non-vanishing Weyl scalars are given by (23) so that

\[
C_{(1)(0)(1)(0)} = -\Psi_2 = C_{(3)(0)(3)(0)} \quad \Rightarrow \quad C_{(2)(0)(2)(0)} = 2 \Psi_2. \quad (26)
\]

Therefore, the equations of geodesic deviation (24) takes the following form

\[
\ddot{Z}^{(1)} = -R_{(0)(j)(0)}^{(1)(0)} Z^{(j)} = -(C_{(1)(0)(1)(0)} - \frac{\Lambda}{3}) Z^{(1)} = (\Psi_2 + \frac{\Lambda}{3}) Z^{(1)},
\]

\[
\ddot{Z}^{(2)} = -R_{(0)(j)(0)}^{(2)(0)} Z^{(j)} = -(C_{(2)(0)(2)(0)} - \frac{\Lambda}{3}) Z^{(2)} = (-2 \Psi_2 + \frac{\Lambda}{3}) Z^{(2)},
\]

\[
\ddot{Z}^{(3)} = -R_{(0)(j)(0)}^{(3)(0)} Z^{(j)} = -(C_{(3)(0)(3)(0)} - \frac{\Lambda}{3}) Z^{(3)} = (\Psi_2 + \frac{\Lambda}{3}) Z^{(3)}. \quad (27)
\]

Note that in the limit \( \alpha \to 0 \), all the Weyl scalars including \( \Psi_2 \) vanishes. In that case, the study space-time (8) becomes conformally flat, maximally symmetric spaces satisfying the following criteria

\[
R_{\mu \nu \rho \sigma} = \frac{\Lambda}{3} (g_{\mu \rho} g_{\nu \sigma} - g_{\mu \sigma} g_{\nu \rho}), \quad \Lambda < 0. \quad (28)
\]

Thus the space-time (8) in the limit \( \alpha \to 0 \) becomes anti-de Sitter (AdS) space. So the equations of deviation (27) in this limit becomes

\[
\ddot{Z}^{(i)} = \frac{\Lambda}{3} Z^{(i)} \quad (29)
\]

with the solution

\[
Z^{(i)} = a_i \tau + b_i, \quad (30)
\]

where \( a_i, b_i, i = 1, 2, 3 \) are the arbitrary constants.
Again in the limit \( \Lambda \to 0 \), the only non-vanishing Weyl scalars is \( \Psi_2 \) given by (23). The study space-time (8) reduces to a type D vacuum space-time of zero cosmological constant with a naked curvature singularity which we discussed, in details in Ref. [12]. In this limit, the equations of geodesic deviation (27) becomes

\[
\begin{align*}
\ddot{Z}^{(1)} &= \Psi_2 Z^{(1)}, \\
\ddot{Z}^{(2)} &= -2 \Psi_2 Z^{(2)}, \\
\ddot{Z}^{(3)} &= \Psi_2 Z^{(3)}
\end{align*}
\]

(31)

with the solution

\[
\begin{align*}
Z^{(1)} &= c_1 \cosh(\sqrt{\Psi_2} \tau) + d_1 \sinh(\sqrt{\Psi_2} \tau), \\
Z^{(2)} &= c_2 \cos(2 \sqrt{\Psi_2} \tau) + d_2 \sin(2 \sqrt{\Psi_2} \tau), \\
Z^{(3)} &= c_3 \cosh(\sqrt{\Psi_2} \tau) + d_3 \sinh(\sqrt{\Psi_2} \tau),
\end{align*}
\]

(32)

where \( c_i, d_i, i = 1, 2, 3 \) are the arbitrary constants.

### 4 Conclusions

In this paper, we generalize a Ricci flat space-time of zero cosmological constant [12] to the case of non-zero cosmological constant in four-dimensional curved space-time. Our motivation in this paper is to write down a metric that highlights causality violation within the theory of general relativity. We can see that the generalize metric for constant \( r, z \) reduces to 2D conformal Misner space metric. Although causality violating space-times have been studied extensively in the literature, a very few of them have been identified to be true time-machine space-times (e.g., [10, 11, 12, 13, 14, 15, 16, 17, 18]).

Our study space-time is a vacuum solution of the field equations of non-zero negative cosmological constant with a naked curvature singularity satisfying all the energy conditions. Also the metric admits CTC which appear at an instant of time from an initial spacelike hypersurface in a causally well-behaved manner and may represent a true time-machine space-time.
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