Free Vibration Analysis Of Horizontal Spinning Beams Using The Differential Quadrature Method

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Abstract. The governing differential equations in free are built vibration by using Hamilton’s principle for a horizontal spinning beam. The differential quadrature (DQ) method is a numerical solution technique for initial and/or boundary problems. By using the DQ method, the dimensionless natural frequencies at different rotational speeds are obtained for the cantilever beam. To ensure the accuracy of the present results by the DQ method, comparisons are made with those available in open literature and very good agreements are achieved.

1. Introduction
Spinning beams is widely used in transmission shaft, turbine blade, propellers and many other mechanical spinning structures, so there has been a growing interest in the investigation of free vibration characteristics of them. Horizontal spinning refers to the rotation of the beam around its axis. Bauer[1] established the horizontal rotation model of Euler beam and used the traditional differential equation method to analyze the free vibration of beams under ten different boundary conditions. Chen and Liao[2] employed an assumed-modes method for solving the free vibration problem of pre-twisted spinning beams. Yu and Cleghorn[3] investigated the free vibration problems of a spinning stepped beam using the finite element method. Song and Librescu[4,5] studied the vibration characteristics of composite rotating beam. Banerjee[6] applied the Wittrick-Williams algorithm to verify and analyze various rotating beams. The DQ method has been projected by its proponents as a potential alternative to the conventional numerical solution techniques such as the finite difference and finite element methods.

The motion governing equations for a horizontal spinning beam are partial differential equations. By using the DQ method, the partial differential governing equations are transformed approximately into a set of linear algebraic governing equations. Imposing the given boundary conditions, the numerical eigenvalue equations for the free vibration of the horizontal spinning beam are derived and then solved.

2. Derivation of the governing differential equations
Figure 1 shows a uniform spinning beam of length L, mass per unit length m and principal axes bending rigidities EIxx and EIyy in the YZ and XZ planes, respectively. The beam is spinning about the Z-axis at a constant angular velocity \( \Omega \). The validity of the Bernoulli–Euler beam theory is assumed so that the effects of shear deformation and rotatory inertia are considered to be small, and hence neglected in the analysis. The theory developed in this paper applies to spinning beams with other cross-sections, but the circular cross-section is shown in the figure only for convenience.
Figure 1. The model of uniform spinning beam.

At a cross-section $z$ from the origin, if $u$ and $v$ are displacements of a point $P$ in the $X$ and $Y$ directions respectively, the position vector $r$ for the total deformation is given by

$$r = ui + vj$$

where $i$ and $j$ are unit vectors in the $X$ and $Y$ directions respectively.

The velocity of the point $P$ is thus given by

$$\mathbf{v} = \mathbf{r}_t + \Phi \mathbf{r} = ((u_t - \Omega v) i + (v_t + \Omega u)) j$$

where $\Phi = \Omega k$, with $k$ as the unit vector in the $Z$-direction.

The kinetic energy $T$ and potential energy $U$ of the Euler beam are respectively

$$T = \frac{1}{2} \int_0^L m |\mathbf{v}|^2 dz = \frac{1}{2} m \int_0^L [u_t^2 + v_t^2 + 2\Omega(uv - u\dot{v}) + \Omega^2(u^2 + v^2)] dz$$

and

$$U = \frac{1}{2} EI \int_0^L v'' dz + \frac{1}{2} EI \int_0^L u'' dz$$

By applying Hamilton's principle, it can be expressed as Lagrangian

$$\delta \int_{t_1}^{t_2} (T - U) dt = 0$$

Where $L = T - U$ is the Lagrangian, and $t_1$ and $t_2$ are the time intervals, we obtain

$$\delta \int_{t_1}^{t_2} (T - U) dt = 0$$

Substituting $T$ and $U$ from equation (3) and (4), and using the $\delta$ operator gives

$$\int_{t_1}^{t_2} \int_0^L m(\ddot{u}\delta u + \ddot{v}\delta v + \Omega^2 \dot{u}\delta u + \Omega^2 \dot{v}\delta v + \Omega \dot{u}\delta v + \Omega \dot{v}\delta u - \Omega \delta u - \Omega \delta v) dz dt -$$

$$\int_{t_1}^{t_2} \left[ EI \int_0^L u'''' \delta u'''' + EI \int_0^L v'''' \delta v'''' \right] dz dt = 0$$

Integrating each term by parts, and then collecting terms and noting that $d\dot{u}$ and $d\dot{v}$ are completely arbitrary, the following governing differential equations are obtained

$$\begin{cases} -m\ddot{u} + m\Omega^2 u + 2m\Omega\dot{v} - EI_{yy} u'''' = 0 \\ -m\ddot{v} + m\Omega^2 v - 2m\Omega\dot{u} - EI_{xx} v'''' = 0 \end{cases}$$

3. DQM solution of governing equations

In order to solve the governing differential equations, the differential quadrature technique is used. In DQ, derivative of any order of a function is approximated by a weighted linear sum of the function values at all the discrete points. Taking a function $f(z, t)$ as an example, the mathematical description of the DQ method is given as

$$\left. \frac{\partial^p f(z, t)}{\partial z^p} \right|_{z=z_j} = \sum_{k=1}^{N} C_{jk}^p f(z_k, t)$$

where $N$ is the number of total discrete grid points in $z$ direction. $C_{jk}^p$ is the weighting coefficient related to the $p$th-order derivative and is obtained as follows:

If $p=1$, then

$$C_{jk}^1 = \frac{M^{(1)}(z_j)}{(z_j - z_k)M^{(1)}(z_k)} \text{ for } j \neq k \text{ and } j, k = 1, 2, \ldots, N$$

and
\[ C_{jj} = - \sum_{k=1}^{N} C_{jk} \quad \text{for} \quad j = 1,2,\ldots,N \tag{11} \]

where \( M^{(1)}(z) \) is the first derivative of \( M(z) \) and they can be defined as

\[ M(z) = \prod_{k=1}^{N} (z - z_k), \quad M^{(1)}(z_k) = \prod_{j=1\,(j\neq k)}^{N} (z_k - z_j) \tag{12} \]

If \( m > 1 \), namely for the second and higher order derivatives, the weighting coefficients are obtained by using the following simple recurrence relationship

\[ C_{jk}^p = p \left( C_{jk}^{p-1} - \frac{C_{jk}^{p-2}}{z_j - z_k} \right) \quad \text{for} \quad j \neq k \quad \text{and} \quad j, k = 1,2,\ldots,N, \quad p = 2,3,\ldots,N - 1, \tag{13} \]

and

\[ C_{jj}^p = - \sum_{k=1\,(k\neq j)}^{N} C_{jk}^p \quad \text{for} \quad j = 1,2,\ldots,N \tag{14} \]

In order to get denser population near boundaries, the sampling points are selected based on the Chebyshev–Gauss–Lobatto grid distribution.

\[ z_i = \frac{1}{2} \left[ 1 - \cos \left( \frac{j - 1}{N - 1} \pi \right) \right] \tag{15} \]

4. Boundary conditions

The boundary conditions of the cantilever beam are:

\[
\begin{align*}
\{ & u = 0, \quad u' = 0, \quad v = 0, \quad v' = 0 \quad \text{at} \quad z = 0, \\
& u'' = 0, \quad u''' = 0, \quad v'' = 0, \quad v''' = 0 \quad \text{at} \quad z = L.
\end{align*}
\tag{16}
\]

Application of the DQ method along with boundary conditions to the equations of motion will give a system of equations in matrix form. The eigenvalues of the system would be natural frequencies that are assessed using MATLAB. The results would then be compared to the published literature in the next Section.

5. Numerical results and discussion

It is useful to present results in non-dimensional form so that the following non-dimensional natural frequency and spinning speed parameters are defined enabling a general application of results.

\[ \omega_i^* = \omega_i \omega_0^{-1}, \quad \Omega^* = \Omega \omega_0^{-1} \tag{17} \]

where

\[ \omega_0 = \left[ \left( EI_{xx} EI_{yy} \right)^{1/2} \left( mL^4 \right)^{-1} \right]^{1/2} \tag{18} \]

For comparison purposes, the values of other parameters are the same as those in Ref.[6]:

\[ EI_{xx} = EI_{yy} = 582.996 \text{Nm}^2, \quad m = 2.87 \text{kg/m}, \quad L = 1.29 \text{m} \]

To investigate the convergence of the differential quadrature method, a cantilever spinning beam is studied. The natural frequencies are shown in Table 1 and compared with other available solution. It shows that the first six non-dimensional frequencies converge quickly. When the number of grid points is larger than 14, the six frequencies are virtually invariant. The excellent agreement with those results reported in Ref.[6] indicates that the differential quadrature method is reliable and accurate. In all later computations, 12 uniform grid points are used.

| N | \( \omega_1^* \) | \( \omega_2^* \) | \( \omega_3^* \) | \( \omega_4^* \) | \( \omega_5^* \) | \( \omega_6^* \) |
|---|---|---|---|---|---|---|
| 9 | 1.516 | 5.516 | 20.059 | 24.059 | 59.733 | 63.733 |
| 10 | 1.516 | 5.516 | 20.042 | 24.042 | 59.835 | 63.835 |
| 11 | 1.516 | 5.516 | 20.034 | 24.034 | 59.780 | 63.780 |
Table 2. Nature frequencies of cantilever beams ($E_{II} = E_{IY}$).

| \( \Omega^* \) | \( \omega_1 \) | \( \omega_2 \) | \( \omega_3 \) | \( \omega_4 \) | \( \omega_5 \) | \( \omega_6 \) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0              | 3.5160         | 3.5160         | 22.0345        | 22.0345        | 61.6975        | 61.6975        |
| Ref.[6]        | 3.5160         | 3.5160         | 22.0340        | 22.0340        | 61.6970        | 61.6970        |
| 2              | 1.5160         | 5.5160         | 20.0345        | 24.0345        | 59.6975        | 63.6975        |
| Ref.[6]        | 1.5160         | 5.5160         | 20.0340        | 24.0340        | 59.6970        | 63.6970        |
| 3.5            | 0.0160         | 7.0160         | 18.5345        | 25.5345        | 58.1975        | 65.1975        |
| Ref.[6]        | -              | 7.0160         | 18.5340        | 25.5340        | 58.1970        | 65.1970        |
| 4              | 0.4840         | 7.5160         | 18.0345        | 26.0345        | 57.6975        | 65.6975        |
| Ref.[6]        | -              | 7.5160         | 18.0340        | 26.0340        | 57.6970        | 65.6970        |
| 8              | 4.4839         | 11.5160        | 14.0345        | 30.0345        | 53.6975        | 69.6975        |
| Ref.[6]        | -              | 11.5160        | 14.0340        | 30.0340        | 53.6970        | 69.6970        |

The natural frequencies with the spinning speed using the present method together with those obtained in Ref.[6] are shown in Table 2 for different spinning speeds. A perfect agreement of numerical results with those in Ref.[6] can be seen. The first natural frequency becomes zero when the \( \Omega^* \) is 3.5 in Ref.[6], but the result is 0.016 using the present method. The vibration frequency is close to zero when the \( \Omega^* \) is around 3.5, and hence the beam becomes unstable.

Similar results are obtained for beams with the ratio of $E_{II} = 0.01 E_{IY}$. For illustrative purpose, the first four natural frequencies are shown in Table 3. The values of the first two critical spinning speeds are 1.1 and 7.1 respectively in Ref.[6], but the frequencies become zero when the \( \Omega^* \) are 1.11 and 6.97 using the present method.

Table 3. Nature frequencies of cantilever beams ($E_{II} = 0.01 E_{IY}$).

| \( \Omega^* \) | \( \omega_1 \) | \( \omega_2 \) | \( \omega_3 \) | \( \omega_4 \) |
|----------------|----------------|----------------|----------------|----------------|
| 0              | 1.1119         | 6.9679         | 11.1186        | 19.5105        |
| Ref.[6]        | 1.1119         | 6.9680         | 11.1190        | 19.5110        |
| 1.1            | 0.1589         | 6.8771         | 11.2807        | 19.4782        |
| Ref.[6]        | 0              | 6.8623         | 11.3070        | 19.4730        |
| 2              | -              | 6.6636         | 11.6320        | 19.4036        |
| Ref.[6]        | -              | 6.6637         | 11.6320        | 19.4040        |
| 4              | -              | 5.6678         | 12.9000        | 19.0800        |
| Ref.[6]        | -              | 5.6679         | 12.9000        | 19.0800        |
| 6              | -              | 3.4910         | 14.5354        | 18.5300        |
| Ref.[6]        | -              | 3.4910         | 14.5360        | 18.5300        |
| 7.1            | -              | -              | 15.5126        | 18.1243        |
| Ref.[6]        | -              | 0              | 15.526         | 18.1180        |
| 8              | -              | -              | 16.3365        | 17.7348        |
| Ref.[6]        | -              | -              | 16.3367        | 17.7348        |

6. Conclusion

This paper deals with free vibration analysis of horizontal spinning beams with various boundary conditions using the differential quadrature method. The numerical results are compared with other available solution. The excellent agreement with those results reported in literature indicates that the differential quadrature method is reliable and accurate. It provides a new method to study the free vibration of rotating beam.
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