RATE OF CONVERGENCE TOWARDS SEMI-RELATIVISTIC HARTREE DYNAMICS

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Abstract. We consider the semi-relativistic system of \( N \) gravitating Bosons with gravitation constant \( G \). The time evolution of the system is described by the relativistic dispersion law, and we assume the mean-field scaling of the interaction where \( N \to \infty \) and \( G \to 0 \) while \( GN = \lambda \) fixed. In the super-critical regime of large \( \lambda \), we introduce the regularized interaction where the cutoff vanishes as \( N \to \infty \). We show that the difference between the many-body semi-relativistic Schrödinger dynamics and the corresponding semi-relativistic Hartree dynamics is at most of order \( N^{-1} \) for all \( \lambda \), i.e., the result covers the sub-critical regime and the super-critical regime. The \( N \) dependence of the bound is optimal.

1. Introduction

We consider a system of \( N \) gravitating three-dimensional Bosons in \( \mathbb{R}^3 \). When the particles in the system have the relativistic dispersion with Newtonian gravity, the mean-field Hamiltonian of the system is

\[
H_{\text{grav}} = \sum_{j=1}^{N} (1 - \Delta_j)^{1/2} - G \sum_{i<j} \frac{1}{|x_i - x_j|}.
\]  

The Hamiltonian \( H_{\text{grav}} \) acts on the Hilbert space \( L^2(\mathbb{R}^3)^N \), the subspace of \( L^2(\mathbb{R}^3N) \) consisting of all symmetric functions with respect to the permutations of particles. Such a system is known as a Boson star.

We are interested in the mean-field limit, where we let \( G \to 0 \) and \( N \to \infty \) with \( \lambda := GN \) is fixed. The mean-field Hamiltonian is defined by

\[
H_N = \sum_{j=1}^{N} (1 - \Delta_j)^{1/2} - \frac{\lambda}{N} \sum_{i<j} \frac{1}{|x_i - x_j|}.
\]  

In the mean-field Hamiltonian \( H_N \), the kinetic energy and the interaction potential energy scale is of the same order (inverse length), hence the system is critical and its behavior hugely depends on the coupling constant \( \lambda \). It was proved by Lieb and Yau in [21] that there exists a critical coupling constant \( \lambda_{\text{crit}}(N) \), depending on \( N \), such that the minimum energy

\[
E_N^\lambda = \inf_{\psi \in L^2(\mathbb{R}^N)} \frac{\langle \psi, H_N \psi \rangle}{\|\psi\|^2_{L^2}}
\]  

is bounded below if \( \lambda < \lambda_{\text{crit}}(N) \) and \( E_N^\lambda = -\infty \) if \( \lambda > \lambda_{\text{crit}}(N) \). As \( N \to \infty \), \( \lambda_{\text{crit}}(N) \) converges to a number \( \lambda_{\text{crit}}^H \), where

\[
\lambda_{\text{crit}}^H = \sup_{\|\varphi\|_{L^2(\mathbb{R}^3)} = 1} \left( \frac{1}{2} \int dx \int dy \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|} \right) \left( \int dx \left| \nabla |x|^{1/2} \varphi(x) \right|^2 \right).
\]

The exact value of \( \lambda_{\text{crit}}^H \) is not known, but it was shown in [20, 21] that \( 4/\pi \leq \lambda_{\text{crit}}^H \leq 2.7 \).

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In the subcritical case $\lambda < \lambda_{\text{crit}}^H$, the Hamiltonian $H_N$ defines a self-adjoint operator with domain $H^{1/2}(\mathbb{R}^{3N})$ when $N$ is sufficiently large. (Technically, $H_N$ is considered as the Friedrichs extension of $H_{\text{crit}}^N$. This generates the one-parameter group of unitary operators $e^{-itH_N}$ that describes the time evolution of the given system. We focus on the time evolution with respect to $H_N$ of a factorized initial data $\psi_N := \varphi \otimes N$ for some $\varphi \in H^1(\mathbb{R}^3)$. It is expected that $\psi_{N,t} := e^{-itH_N}\psi_N$ satisfies

$$\psi_{N,t} \simeq \varphi \otimes N,$$

where $\varphi_t$ is the solution of the semi-relativistic nonlinear Hartree equation

$$i \partial_t \varphi_t = (1 - \Delta)^{1/2}\varphi_t - \lambda \left( \frac{1}{|\cdot|} * |\varphi_t|^2 \right) \varphi_t$$

with initial data $\varphi_{t=0} = \varphi$.

The factorization (1.5) should be understood in terms of the marginal densities (reduced density matrices) associated with $\psi_{N,t}$. We define the $k$-particle marginal density through its kernel

$$\gamma_{N,t}^{(k)}(x_k, x'_k) := \int dx_{k+1} \cdots dx_N \psi_{N,t}(x_k, x_{k+1}, \ldots, x_N) \overline{\psi_{N,t}(x'_k, x_{k+1}, \ldots, x_N)},$$

where $x_k = (x_1, x_2, \ldots, x_k)$ and $x'_k = (x'_1, x'_2, \ldots, x'_k)$. Since $||\psi_{N,t}||_{L^2} = 1$, we can see that $\text{Tr} \gamma_{N,t}^{(k)} = 1$ for all $1 \leq k \leq N$. Thus, $\gamma_{N,t}^{(k)}$ is a trace class operator. In [5], Elgart and Schlein proved that, in the large $N$ limit, the $k$-particle marginal density associated with $\psi_{N,t}$ converges to $k$-particle marginal density associated with the factorized wavefunction $\varphi \otimes N$, under the condition that $\lambda < \pi/4$ and $\varphi \in H^1(\mathbb{R})$. More precisely, for any fixed $t \in \mathbb{R}$,

$$\text{Tr} \left| \gamma_{N,t}^{(1)} - \langle \varphi_t \rangle \langle \varphi_t \rangle \right| \to 0 \quad \text{as } N \to \infty,$$

where $\langle \varphi_t \rangle \langle \varphi_t \rangle$ denotes the rank one projection onto $\varphi_t$. For $\lambda < \lambda_{\text{crit}}^H$, it is proved by Lenzmann in [19] that the semi-relativistic Hartree equation (1.6) is globally well-posed in $H^s(\mathbb{R}^3)$ for every $s \geq 1/2$. Therefore, (1.8) shows that the solution of the $N$-particle Schrödinger equation $\psi_{N,t}$ can be approximated by products of the solution of the semi-relativistic Hartree equation $\varphi_t$ for all $t \in \mathbb{R}$.

The rate of convergence in (1.8) is attained by Knowles and Pickl [18] for the case $\lambda < \pi/4$ with the initial condition $\varphi \in H^s$ for $s > 1$. More precisely,

$$\text{Tr} \left| \gamma_{N,t}^{(k)} - \langle \varphi_t \rangle \langle \varphi_t \rangle \otimes k \right| \leq \frac{C(k,t)}{\sqrt{N}}$$

for some constant $C(k,t)$ independent of $N$. Here, $C(k,t) = C_t \sqrt{k}$ where $C_t$ grows at most exponentially in $t$.

In the supercritical regime $\lambda > \lambda_{\text{crit}}^H$, on the other hand, solutions of (1.6) may blow up in finite time, which was proved by Fröhlich and Lenzmann [12]. Physically, the blowup of the solution of (1.6) describes the gravitational collapse of a Boson star whose mass is over a critical value, provided that the relativistic dynamics of the system can be approximated by the semi-relativistic Hartree dynamics as in the subcritical case. This assumption was proved by Michelangeli and Schlein [22] with the regularized Hamiltonian

$$H_N^\alpha = \sum_{j=1}^N (1 - \Delta_j)^{1/2} - \frac{\lambda}{N} \sum_{i<j}^N \frac{1}{|x_i - x_j| + \alpha_N}$$

with $\alpha_N > 0$ and $\alpha_N \to 0$ as $N \to \infty$. The regularized Hamiltonian $H_N^\alpha$ defines a quadratic form, which is bounded below, hence we may consider its Friedrichs extension as a self-adjoint operator with domain
If we let \( \gamma_{N,t}^{\alpha} \) be the one-particle marginal density associated with \( \psi_{N,t}^{\alpha} = e^{-itH_N^\alpha} \varphi \otimes N \), then Theorem 1.1 of \[22\] shows that with the initial condition \( \varphi \in H^2(\mathbb{R}^3) \),

\[
\text{Tr} \left| \gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| \right| \leq C(t) \frac{1}{\sqrt{N}} \tag{1.11}
\]

for all \( |t| \leq T \), where \( T \) is the maximal time of the existence of the solution of \( (1.6) \).

The corresponding results for non-relativistic dynamics is relatively well-established. In \[25\], Spohn first proved that \( (1.8) \) holds when the interaction potential is bounded. This result was extended by Erdős and Yau in \[7\] for the Coulomb type interaction. In \[25\], Rodnianski and Schlein obtained an explicit bound on the rate of the convergence in \( (1.8) \) for the Coulomb type interaction. The result in \[25\], which showed that the rate of the convergence in \( (1.8) \) is \( O(N^{-1/2}) \), is extended further by Knowles and Pickl \[18\] for more singular potentials. On the other hand, Erdős and Schlein \[5\] proved that the rate of convergence in \( (1.8) \) is \( O(N^{-1}) \) for bounded potentials, which is considered to be optimal. The same rate of convergence for more singular potentials including Coulomb type potential was obtained in \[2, 3\]. Another important result in this direction is the derivation of the Gross-Pitaevskii equation for describing Bose-Einstein condensates by Erdős, Schlein, and Yau \[8, 9, 10, 11\]. (See also works by Pickl \[23, 24\]).

In this paper, we improve the bound \( (1.9) \) and \( (1.11) \) by applying the method developed in \[25\]. First introduced by Hepp \[16\] and extended by Ginibre and Velo \[13, 14\], this method has been successful in proving various bounds on the rate of convergence as in \[25, 2, 3, 22\]. We show that the left hand sides of \( (1.9) \) and \( (1.11) \), the differences between the one-particle marginal density associated with the solution of the time evolution of the factorized initial data and the orthogonal projection onto the solution of the semi-relativistic Hartree equation \( (1.6) \), are \( O(N^{-1}) \). The first main result of this paper, which considers the supercritical case, is the following theorem:

**Theorem 1.1.** Suppose that \( \lambda < \lambda_{\text{crit}}^H \), \( \varphi \in H^1(\mathbb{R}^3) \) with \( \|\varphi\|_{L^2} = 1 \), and \( \psi_N = \varphi \otimes N \). Let \( \psi_{N,t} = e^{-itH_N} \varphi_N \) be the evolution of the initial wave function \( \psi_N \) with respect to the Hamiltonian \( (1.2) \) and let \( \gamma_{N,t}^{(1)} \) be the one-particle marginal density associated with \( \psi_{N,t} \). Let \( \varphi_t \) be the solution of the \( (1.6) \) with initial data \( \varphi_{t=0} = \varphi \). Let

\[
\nu(t) := \sup_{|s| \leq t} \|\varphi_s\|_{H^1} \tag{1.12}
\]

Then, there exists a constant \( C \), depending only on \( \lambda \) and \( \nu(t) \), such that

\[
\text{Tr} \left| \gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| \right| \leq CN^{-1}. \tag{1.13}
\]

**Remark 1.1.** Since the semi-relativistic Hartree equation \( (1.6) \) is globally well-posed in \( H^1 \) for the subcritical case, \( \nu(t) < \infty \) for all \( t \in \mathbb{R} \). See \[19, 4\] for more detail.

In the supercritical case, while we should introduce the regularized Hamiltonian \( (1.10) \) to define a self-adjoint operator, the approximating semi-relativistic Hartree equation does not need to contain the regularized non-linear term, i.e., it suffices to consider the equation \( (1.6) \) for approximating the evolution of the \( N \)-particle factorized initial state. The second main result of this paper is the following theorem:

**Theorem 1.2.** Suppose that \( \lambda \geq \lambda_{\text{crit}}^H \), \( \varphi \in H^1(\mathbb{R}^3) \) with \( \|\varphi\|_{L^2} = 1 \), and \( \psi_N = \varphi \otimes N \). Let \( \psi_{N,t} = e^{-itH_N} \psi_N \) be the evolution of the initial wave function \( \psi_N \) with respect to the Hamiltonian \( (1.10) \) with \( \alpha_N \leq N^{-4} \) and let \( \gamma_{N,t}^{(1)} \) be the one-particle marginal density associated with \( \psi_{N,t} \). Let \( \varphi_t \) be the solution of the \( (1.6) \) with initial data \( \varphi_{t=0} = \varphi \). Fix \( T \) such that

\[
\kappa := \sup_{|t| \leq T} \|\varphi_t\|_{H^{1/2}} < \infty. \tag{1.14}
\]
Then, there exists a constant $C$, depending only on $\lambda$, $\|\varphi\|_{H^1}$, $T$, and $\kappa$, such that
\[
\text{Tr} \left| \gamma_{N,t}^{\alpha,\xi(1)} - |\varphi_t\rangle\langle\varphi_t| \right| \leq CN^{-1}
\] (1.15)
for all $|t| \leq T$.

Remark 1.2. The existence of such $T$ follows from the local well-posedness of the semi-relativistic Hartree equation (1.6). See [19, 4] for more detail.

As in [25, 2, 3, 22], we first consider the case where the initial state is the coherent state in the Fock space. (See (3.21) and (3.23).) For the evolution of the coherent state, we need to control the fluctuation $\mathcal{U}_N(t; s)$, which is defined in (3.20) around the semi-relativistic Hartree dynamics. It was proved in Theorem 4.1 of [22] that for the evolution of the coherent state we can achieve the optimal rate of convergence $O(N^{-1})$ towards the semi-relativistic Hartree dynamics. We then use the information on the evolution of the coherent state to estimate the fluctuations for the dynamics of the factorized state (1.5). See (3.21) and (3.23).

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The paper is organized as follows. In Section 2, we show that the time evolution with original Hamiltonian can be well approximated by the time evolution with regularized Hamiltonian, provided that the cutoff approaches zero sufficiently fast. In Section 3, we define the Fock space and reformulate the problem using the operators defined on the Fock space. In Section 4, we prove Proposition 2.2, which implies the main results of the paper. A series of estimates will be proved in Sections 5-8.

Remark 1.3. Throughout the paper, $C$ and $K$ will denote various constants independent of $N$. The $L^p$-space norm for $1 \leq p \leq \infty$ will be denoted by $\| \cdot \|_p$. The sequence $\alpha_N$ is positive and satisfies $\alpha_N \leq N^{-4}$.

2. Regularization of the Interaction

Recall that the regularized Hamiltonian is defined by
\[
H_N^\alpha = \sum_{j=1}^{N} (1 - \Delta_j)^{1/2} - \frac{\lambda}{N} \sum_{i<j}^{N} \frac{1}{|x_i - x_j| + \alpha_N}.
\] (2.1)

As in [3], we first prove an estimate for the difference between the evolution of the initial $N$-particle wavefunction with respect to the original Hamiltonian $H_N$ and with respect to the regularized Hamiltonian $H_N^\alpha$.

Lemma 2.1. Let $\psi_N = \varphi \otimes^N$ for some $\varphi \in H^1(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$. Let $\psi_{N,t} = e^{-iH_N t} \psi_N$ and $\psi_{N,t}^\alpha = e^{-iH_N^\alpha t} \psi_N$. If $\lambda < \lambda_{\text{crit}}^H$, then there exist constants $C > 0$ and $N_0$ such that, for all $t \in \mathbb{R}$ and positive integer $N > N_0$,
\[
\| \psi_{N,t} - \psi_{N,t}^\alpha \|_2^2 \leq CN^2 \alpha_N |t|.
\] (2.2)
Proof. We first consider the derivative
\[
\frac{d}{dt} \|\psi_{N,t} - \psi_{N,t}^\alpha\|_2^2 = -2 \text{Re} \frac{d}{dt} (\psi_{N,t}, \psi_{N,t}^\alpha) = 2 \text{Im} (\psi_{N,t}, (H_N - H_N^\alpha) \psi_{N,t}^\alpha).
\]
(2.3)

Next, we note that
\[
|\langle \psi_{N,t}, (H_N - H_N^\alpha) \psi_{N,t}^\alpha \rangle| = \frac{\lambda}{N} \left| \left( \sum_{i<j} N \left( \frac{1}{|x_i - x_j|} - \frac{1}{|x_i - x_j| + \alpha_N} \right) \psi_{N,t}^\alpha \right) \right|^2
\leq \frac{\lambda N}{N} \sum_{i<j} (\psi_{N,t}, (1 - \Delta_i)^{1/2} (1 - \Delta_j)^{1/2} \psi_{N,t}^\alpha)^{1/2} (\psi_{N,t}, (1 - \Delta_i)^{1/2} (1 - \Delta_j)^{1/2} \psi_{N,t}^\alpha)^{1/2}
\leq \frac{\lambda N}{N} \sum_{i<j} (\psi_{N,t}, (1 - \Delta_i)^{1/2} (1 - \Delta_j)^{1/2} \psi_{N,t}^\alpha) + (\psi_{N,t}^\alpha, (1 - \Delta_i)^{1/2} (1 - \Delta_j)^{1/2} \psi_{N,t}^\alpha)\right),
\]
where we used the operator inequality
\[
\frac{1}{|x_i - x_j|^2} \leq C (1 - \Delta_i)^{1/2} (1 - \Delta_j)^{1/2}.
\]
(2.5)

(See Lemma 9.1 of \[5\] for the proof.)

Thus, from (2.3), (2.4), and Lemma 6.1, we find that
\[
\frac{d}{dt} \|\psi_{N,t} - \psi_{N,t}^\alpha\|_2^2 \leq CN^2 \alpha_N.
\]
(2.6)

The lemma follows after integrating over $t$.

From Lemma 2.1, we obtain a bound on the difference between the marginal densities associated with the $\psi_{N,t}$ and $\psi_{N,t}^\alpha$.

Corollary 2.1. Let $\phi = \phi^\otimes N$ for some $\phi \in H^1(\mathbb{R}^3)$ with $\|\phi\|_2 = 1$. Let $\psi_{N,t} = e^{-iH_N t} \psi_N$ and $\psi_{N,t}^\alpha = e^{-iH_N^\alpha t} \psi_N$. For any $k \in \mathbb{N}$, let $\gamma_{N,t}^{(k)}$ and $\gamma_{N,t}^{\alpha,(k)}$ be the $k$-particle reduced densities associated with $\psi_{N,t}$ and $\psi_{N,t}^\alpha$, respectively. Suppose $\alpha_N \leq N^{-1}$ in $\mathbb{R}$. If $\lambda < \lambda^{H}_{\text{crit}}$, then there exist a constant $C > 0$ and positive integer $N > N_0$ such that, for all $t \in \mathbb{R}$ and positive integer $N > N_0$,
\[
\text{Tr} \left| \gamma_{N,t}^{(k)} - \gamma_{N,t}^{\alpha,(k)} \right| \leq C|t|^{1/2} N^{-1}.
\]
(2.7)

Proof. See Corollary 2.1 of \[5\].

We next estimate the difference between the solutions of the semi-relativistic Hartree equations with the Coulomb potential and with the regularized potential. The proof of the following proposition will be given in Section 3

Proposition 2.1. Let $\phi \in H^1(\mathbb{R}^3)$ with $\|\phi\|_2 = 1$. Let $\phi_t$ denote the solution of the nonlinear Hartree equation (1.6) with initial condition $\phi_{t=0} = \varphi$ and $\phi_t^\alpha$ the solution of the regularized semi-relativistic Hartree equation
\[
i \partial_t \phi_t^\alpha = (1 - \Delta)^{1/2} \phi_t^\alpha - \lambda \left( \frac{1}{|\cdot| + \alpha_N} * |\phi_t^\alpha|^2 \right) \phi_t^\alpha,
\]
with the same initial condition $\phi_{t=0}^\alpha = \varphi$. Fix $T$ such that
\[
\kappa = \sup_{|t| \leq T} \|\phi_t\|_{H^{1/2}} < \infty.
\]
(2.9)
Then, there exist constants $C$ and $K$, depending only on $\lambda$, $\kappa$, $T$, and $\|\varphi\|_{H^1}$, such that
\[
\|\varphi_t - \varphi_t^0\|_{H^{1/2}} \leq C\alpha_N^{1/2}
\]
for all $|t| < T$. Therefore,
\[
\operatorname{Tr} \left| \|\varphi_t^0\| (\|\varphi_t^0\| - |\varphi_t^0\langle |\varphi_t\rangle |) \right| \leq \|\varphi_t - \varphi_t^0\|_2 \leq C\alpha_N^{1/2}.
\]

As a consequence of Corollary 2.3 and Proposition 2.2, Theorem 1.1 and Theorem 1.2 follow from the next proposition.

**Proposition 2.2.** Let $\varphi \in H^1(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$. Let $\gamma_{N,t}^{\alpha,1}$ be the one-particle marginal density associated with $e^{-itH_N} \varphi \otimes \varphi$ and $\varphi_t^0$ the solution of the regularized semi-relativistic Hartree equation (2.8) with initial data $\varphi_t = 0 = \varphi$. Suppose $\alpha_N \leq N^{-1}$ in (2.1). Then, there exists a constant $C$, depending only on $\lambda$, $\|\varphi\|_{H^1}$, $T$, and $\kappa$, such that
\[
\operatorname{Tr} \left| \gamma_{N,t}^{\alpha,1} - |\varphi_t^0\langle |\varphi_t^0\rangle \right| \leq CN^{-1}
\]
for all $|t| \leq T$.

The proof of Proposition 2.2 will be given in Section 4, where we will use the Fock space representation of the problem.

### 3. Fock Space Representation

Let $\mathcal{F}$ be the Fock space of symmetric functions, i.e.
\[
\mathcal{F} := \bigoplus_{n \geq 0} (L^2(\mathbb{R}^{3n}))_s,
\]
where we let $L^2(\mathbb{R}^{3n})_s = \mathbb{C}$ when $n = 0$ and $s$ denotes the subspace of symmetric functions with respect to the permutation of particles $x_1, x_2, \cdots, x_n$. A vector $\psi$ in $\mathcal{F}$ is a sequence $\psi = \{\psi^{(n)}\}_{n \geq 0}$ of $n$-particle wavefunctions $\psi^{(n)} \in (L^2(\mathbb{R}^{3n}))_s$. The scalar product between $\psi_1, \psi_2 \in \mathcal{F}$ is defined by
\[
\langle \psi_1, \psi_2 \rangle = \sum_{n \geq 0} \langle \psi^{(n)}_1, \psi^{(n)}_2 \rangle_{L^2(\mathbb{R}^{3n})}
\]
and we will omit the subscript $\mathcal{F}$ from now on. We let
\[
\Omega := \{1, 0, 0, \cdots \} \in \mathcal{F},
\]
which is called the vacuum. We will also make use of an operator $P_n$, the projection onto the $n$-particle sector of the Fock space, which is defined by $P_n \psi = \{0, 0, \cdots, \psi^{(n)}, 0, \cdots \}$ for a vector $\psi$ in $\mathcal{F}$.

On $\mathcal{F}$, the creation operator $a^*_x$ and the annihilation operator $a_x$ for $x \in \mathbb{R}^3$ are defined by
\[
(a^*_x \psi)^{(n)}(x_1, \cdots, x_n) = \frac{1}{\sqrt{n!}} \sum_{j=1}^n \delta(x - x_j) \psi^{(n-1)}(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)
\]
\[
(a_x \psi)^{(n)}(x_1, \cdots, x_n) = \sqrt{n + 1} \psi^{(n+1)}(x, x_1, \cdots, x_n).
\]
For $f \in L^2(\mathbb{R}^3)$, $a^*(f)$ and $a(f)$ are given by
\[
a^*(f) = \int dx f(x) a^*_x
\]
\[
a(f) = \int dx f(x) a_x.
\]
or equivalently,
\[
(a^*_x \psi)^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{N}} \sum_{j=1}^{n} f(x_j) \psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)
\] (3.8)
\[
(a_x \psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n + 1} \int dx \ f(x) \psi^{(n+1)}(x, x_1, \ldots, x_n).
\] (3.9)

We also use the self-adjoint operator
\[
\phi(f) = a^*(f) + a(f)
\] (3.10)
for \( f \in L^2(\mathbb{R}^3) \). We have the following lemma that will be used to bound the creation operator and the annihilation operator:

**Lemma 3.1.** For any \( f \in L^2(\mathbb{R}^3) \) and \( \psi \in D(N^{1/2}) \), we have
\[
\|a(f)\psi\| \leq \|f\|_2 \|N^{1/2}\psi\|,
\] (3.11)
\[
\|a^*(f)\psi\| \leq \|f\|_2 \|(N+1)^{1/2}\psi\|,
\] (3.12)
\[
\|a(f)\psi\| \leq 2\|f\|_2 \|(N+1)^{1/2}\psi\|.
\] (3.13)

**Proof.** See Lemma 2.1 of [25]. \( \square \)

For an operator \( J \) acting on \( L^2(\mathbb{R}^3) \), we define the second quantization of \( J \), \( d\Gamma(J) \), as the operator on \( \mathcal{F} \) whose action on the \( n \)-particle sector is given by
\[
(d\Gamma(J)\psi)^{(n)} = \sum_{j=1}^{n} J_j \psi^{(n)},
\] (3.14)
where \( J_j = 1 \otimes 1 \otimes \cdots 1 \otimes J \otimes 1 \otimes \cdots 1 \) is the operator acting only on the \( j \)-th particle. If \( J \) has a kernel \( J(x; y) \), then \( d\Gamma(J) \) can be written as
\[
d\Gamma(J) = \int dx dy \ J(x; y) a^*_x a_y.
\] (3.15)
The number operator \( \mathcal{N} \) is defined by
\[
\mathcal{N} := d\Gamma(1) = \int dx \ a^*_x a_x
\] (3.16)
and it also satisfies
\[
(\mathcal{N}\psi)^{(n)} = n\psi^{(n)}.
\] (3.17)

We will use the following lemma to estimate \( d\Gamma(J) \):

**Lemma 3.2.** For any bounded one-particle operator \( J \) on \( L^2(\mathbb{R}^3) \) and for every \( \psi \in D(\psi) \), we have
\[
\|d\Gamma(J)\psi\| \leq \|J\| \|\mathcal{N}\psi\|.
\] (3.18)

**Proof.** See Lemma 3.1 of [23]. \( \square \)

We define the Hamiltonian \( \mathcal{H}_N \) on \( \mathcal{F} \) by
\[
\mathcal{H}_N := \int dx \ a^*_x (1 - \Delta_x)^{1/2} a_x - \frac{\lambda}{2N} \int dx dy \ \frac{1}{|x-y|} a^*_x a_y a_x a_x.
\] (3.19)

Note that for any function \( \psi^{(N)} \in L^2(\mathbb{R}^{3N})_x \), \( \mathcal{H}_N \psi^{(N)} = H_N \psi^{(N)} \). Similarly, we define the regularized Hamiltonian \( \mathcal{H}_N^\alpha \) on \( \mathcal{F} \) by
\[
\mathcal{H}_N^\alpha := \int dx \ a^*_x (1 - \Delta_x)^{1/2} a_x - \frac{\lambda}{2N} \int dx dy \ \frac{1}{|x-y| + \alpha_N} a^*_x a_y a_x a_x.
\] (3.20)
which also satisfies \( \mathcal{H}_N^\alpha \psi^{(N)} = H_N^\alpha \psi^{(N)} \) for any function \( \psi^{(N)} \in L^2(\mathbb{R}^{3N}) \).

For \( f \in L^2(\mathbb{R}^3) \), the Weyl operator \( W(f) \) is defined by
\[
W(f) := \exp(a^*(f) - a(f)),
\]
and it satisfies
\[
W(f) = e^{-\|f\|^2/2} \exp(a^*(f)) \exp(-a(f)).
\]
The coherent state with a one-particle wave function \( f \) is \( W(f) \Omega \), which satisfies
\[
W(f)\Omega = e^{-\|f\|^2/2} \exp(a^*(f))\Omega = e^{-\|f\|^2/2} \sum_{n \geq 0} \frac{(a^*(f))^n}{\sqrt{n!}} \Omega.
\]

Let \( \Gamma^{\alpha,1}_{N,t}(x; y) \) be the kernel of the one-particle marginal density associated with the time evolution of the coherent state \( W(\sqrt{N}\varphi)\Omega \) with respect to the regularized Hamiltonian \( \mathcal{H}_N^\alpha \). By definition,
\[
\Gamma^{\alpha,1}_{N,t}(x; y) = \frac{1}{N} \left( e^{i\mathcal{H}_N^\alpha t} W(\sqrt{N}\varphi)\Omega, a_y^* a_x e^{-i\mathcal{H}_N^\alpha t} W(\sqrt{N}\varphi)\Omega \right).
\]
We expect that the limit of the kernel of one particle marginal density is \( \sqrt{\phi_t^\alpha}(y)\varphi_t^\alpha(x) \), thus we expand \( \Gamma^{\alpha,1}_{N,t}(x; y) \) in terms of \( (a_x - \sqrt{N}\varphi_t^\alpha(x)) \) and \( (a_y^* - \sqrt{N}\varphi_t^\alpha(x)) \). Then, we get
\[
\Gamma^{\alpha,1}_{N,t}(x; y) = \varphi_t^\alpha(x)\overline{\varphi_t^\alpha(y)}
+ \frac{1}{N} \left( \Omega, W^*(\sqrt{N}\varphi) e^{i\mathcal{H}_N^\alpha t} (a_y^* - \sqrt{N}\varphi_t^\alpha(y))(a_x - \sqrt{N}\varphi_t^\alpha(x)) e^{-i\mathcal{H}_N^\alpha t} W(\sqrt{N}\varphi)\Omega \right)
+ \frac{\varphi_t^\alpha(x)}{\sqrt{N}} \left( \Omega, W^*(\sqrt{N}\varphi) e^{i\mathcal{H}_N^\alpha t} (a_y^* - \sqrt{N}\varphi_t^\alpha(y)) e^{-i\mathcal{H}_N^\alpha t} W(\sqrt{N}\varphi)\Omega \right)
+ \frac{\overline{\varphi_t^\alpha(y)}}{\sqrt{N}} \left( \Omega, W^*(\sqrt{N}\varphi) e^{i\mathcal{H}_N^\alpha t} (a_x - \sqrt{N}\varphi_t^\alpha(x)) e^{-i\mathcal{H}_N^\alpha t} W(\sqrt{N}\varphi)\Omega \right).
\]

We define the unitary evolution
\[
U(t; s) := e^{-i\omega(t; s)} W^*(\sqrt{N}\varphi_t^\alpha) e^{-i(t-s)\mathcal{H}_N^\alpha} W(\sqrt{N}\varphi_s^\alpha)
\]
with the phase factor
\[
\omega(t; s) := \frac{N}{2} \int_s^t ds \int dx \left( \frac{\lambda}{\varepsilon + \alpha_N} * |\varphi_t^\alpha|^2 \right)(x) |\varphi_t^\alpha(x)|^2.
\]
It turns out that \( U(t; s) \) is a unitary operator satisfying
\[
i\partial_t U(t; s) = (\mathcal{L}_2(t) + \mathcal{L}_3(t) + \mathcal{L}_4) U(t; s) \quad \text{and} \quad U(s; s) = I,\]
where the generators \( \mathcal{L}_2, \mathcal{L}_3, \) and \( \mathcal{L}_4 \) are defined as follows:
\[
\mathcal{L}_2(t) := \int dx a_x^*(1 - \Delta_x)^{1/2} a_x + \lambda \int dx \left( \frac{1}{\varepsilon + \alpha_N} * |\varphi_t^\alpha|^2 \right)(x) a_x^* a_x
+ \lambda \int dx dy \frac{1}{|x-y| + \alpha_N} \varphi_t^\alpha(x) \varphi_t^\alpha(y) a_x^* a_y
+ \frac{\lambda}{2} \int dx dy \frac{1}{|x-y| + \alpha_N} (\varphi_t^\alpha(x) \varphi_t^\alpha(y) a_x^* a_y^* + \varphi_t^\alpha(x) \overline{\varphi_t^\alpha(y)} a_x a_y),
\]
\[
\mathcal{L}_3(t) := \frac{\lambda}{\sqrt{N}} \int dx dy \frac{1}{|x-y| + \alpha_N} \varphi_t^\alpha(y) a_x^* a_y a_x + \frac{\lambda}{\sqrt{N}} \int dx dy \frac{1}{|x-y| + \alpha_N} \overline{\varphi_t^\alpha(y)} a_x a_y a_x,
\]
\[
\mathcal{L}_4 := \frac{\lambda}{2N} \int dx dy \frac{1}{|x-y| + \alpha_N} a_x^* a_y a_x a_y.
\]
Let
\[ K := \int dx \, a_x^*(1 - \Delta_x)^{1/2} a_x. \] (3.32)

We consider a modified evolution \( \tilde{U}(t; s) \), which is a unitary operator satisfying
\[ i\partial_t \tilde{U}(t; s) = (L_2(t) + L_4(t))\tilde{U}(t; s) \quad \text{and} \quad \tilde{U}(s; s) = I \] (3.33)

We remark that \( \tilde{U}(t; s) \) is bounded in \( Q(K + N^2) \), the form domain of the operator \( (K + N^2) \), and is strongly differentiable from \( Q(K + N^2) \) to \( Q^*(K + N^2) \). See section 8 for more detail.

For simplicity, we will use notations
\[ \gamma_{N,t}^{\alpha(1)}(x; y) = \frac{1}{N} \left\langle \frac{(a_x^*(\varphi))^N}{\sqrt{N!}} \right\rangle \Omega, e^{iH_{N,t}^t a_y^*a_x} \left( a_x^*(\varphi)^N \right)^N \sqrt{N!} \Omega; \] (3.34)

4. Proof of Main Results

In this section, we prove Proposition 2.2, which implies Theorem 1.1 and Theorem 1.2.

Proof of Proposition 2.2. From (3.20), we find that
\[ W^*(\sqrt{N} \varphi) e^{iH_{N,t}^t} (a_x - \sqrt{N} \varphi_\alpha^t(x))e^{-iH_{N,t}^t} W(\sqrt{N} \varphi) = U^t(t) a_x U(t). \] (4.1)

By definition, we have that
\[ P_N W(\sqrt{N} \varphi) \Omega = e^{-N/2} \left( \frac{(a_x^*(\varphi))^N}{N!} \right) \Omega. \] (4.2)

where \( P_N \) is the projection onto the \( N \)-particle sector of the Fock space. Here, \( d_N \) denotes the constant
\[ d_N := \frac{\sqrt{N!}}{N^{N/2} e^{-N/2}} \approx N^{1/4}. \] (4.3)

For factorized initial data, it follows from (4.1) and (4.2) that
\[ \gamma_{N,t}^{(1)}(x; y) = \frac{1}{N} \left\langle \frac{(a_x^*(\varphi))^N}{\sqrt{N!}} \right\rangle \Omega, e^{iH_{N,t}^t a_y^*a_x} \left( a_x^*(\varphi)^N \right)^N \sqrt{N!} \Omega; \]
\[ = \frac{d_N}{N} \left\langle \frac{(a_x^*(\varphi))^N}{\sqrt{N!}} \right\rangle \Omega, e^{iH_{N,t}^t a_y^*a_x} \left( a_x^*(\varphi)^N \right)^N \sqrt{N!} \Omega; \]
\[ = \frac{d_N}{N} \left\langle \frac{(a_x^*(\varphi))^N}{\sqrt{N!}} \right\rangle \Omega, P_N W(\sqrt{N} \varphi) U^t(t) \left( a_y^* + \sqrt{N} \varphi_\alpha^t(y) \right) U(t) \Omega \]
\[ = \frac{d_N}{N} \left\langle \frac{(a_x^*(\varphi))^N}{\sqrt{N!}} \right\rangle \Omega, W(\sqrt{N} \varphi) U^t(t) \left( a_y^* + \sqrt{N} \varphi_\alpha^t(y) \right) (a_x + \sqrt{N} \varphi_\alpha^t(x)) U(t) \Omega \]. (4.4)

Thus, we obtain the following equation for the one-particle marginal.
\[ \gamma_{N,t}^{(1)}(x; y) - \varphi_t^\alpha(y) \varphi_t^\alpha(x) = \frac{d_N}{N} \left\langle \frac{(a_x^*(\varphi))^N}{\sqrt{N!}} \right\rangle \Omega, W(\sqrt{N} \varphi) U^t(t) a_x^* a_x U(t) \Omega \]
\[ + \varphi_t^\alpha(y) \frac{d_N}{N} \left\langle \frac{(a_x^*(\varphi))^N}{\sqrt{N!}} \right\rangle \Omega, W(\sqrt{N} \varphi) U^t(t) a_x U(t) \Omega \]
\[ + \varphi_t^\alpha(x) \frac{d_N}{N} \left\langle \frac{(a_x^*(\varphi))^N}{\sqrt{N!}} \right\rangle \Omega, W(\sqrt{N} \varphi) U^t(t) a_x U(t) \Omega \]. (4.5)
For any compact one-particle Hermitian operator $J$ on $L^2(\mathbb{R}^3)$, we find
\[
\text{Tr} \left[ J \left( \gamma_{N,t}^{(1)} - |\varphi_t^\alpha\rangle \langle \varphi_t^\alpha| \right) \right] 
= \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega, W(\sqrt{N}\varphi)U^*(t)d\Gamma(J)U(t)\Omega \right\rangle + \frac{d_N}{\sqrt{N}} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega, W(\sqrt{N}\varphi)U^*(t)\phi(J\varphi_t^\alpha)U(t)\Omega \right\rangle 
=: E_t^1(J) + E_t^2(J). 
\] (4.6)

Lemma 5.1 and Lemma 5.2 show that
\[
\left| \text{Tr} \left[ J \left( \gamma_{N,t}^{(1)} - |\varphi_t^\alpha\rangle \langle \varphi_t^\alpha| \right) \right] \right| \leq |E_t^1(J)| + |E_t^2(J)| \leq \frac{C}{N} ||J||e^{Kt} 
\] (4.7)
for all compact Hermitian operators $J$ on $L^2(\mathbb{R}^3)$. Since the space of compact operators is the dual to the space of trace class operators, and since $\gamma_{N,t}^{(1)}$ and $|\varphi_t^\alpha\rangle \langle \varphi_t^\alpha|$ are Hermitian, we obtain that
\[
\text{Tr} \left[ \gamma_{N,t}^{(1)} - |\varphi_t^\alpha\rangle \langle \varphi_t^\alpha| \right] \leq \frac{Ce^{Kt}}{N}, 
\] (4.8)
which was to be proved.

5. COMPARISON OF DYNAMICS

In this section, we prove important lemmas that were used in the proof of Theorem 1 by estimating the difference between $U(t; s)$ and $\tilde{U}(t; s)$.

Lemma 5.1. For a Hermitian Operator $J$ on $L^2(\mathbb{R}^3)$, let
\[
E_t^1(J) = \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega, W(\sqrt{N}\varphi)U^*(t)d\Gamma(J)U(t)\Omega \right\rangle. 
\] (5.1)
Then, there exist constants $C$ and $K$, depending only on $\lambda$ and $\sup_{|s| \leq t} ||\varphi_s||_{H^1}$, such that
\[
|E_t^1(J)| \leq \frac{C||J||e^{Kt}}{N}. 
\] (5.2)

Proof. We first observe that
\[
|E_t^1(J)| = \left| \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega, W(\sqrt{N}\varphi)U^*(t)d\Gamma(J)U(t)\Omega \right\rangle \right| 
\leq \frac{d_N}{N} \left\| (N+1)^{-1/2} W^*(\sqrt{N}\varphi) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \right\| \left\| (N+1)^{1/2} U^*(t)d\Gamma(J)U(t)\Omega \right\|. 
\] (5.3)

Lemma 5.3 shows that
\[
\left\| (N+1)^{-1/2} W^*(\sqrt{N}\varphi) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \right\| \leq \frac{C}{d_N}. 
\] (5.4)

Lemma 5.5 shows that
\[
\left\| (N+1)^{1/2} U^*(t)d\Gamma(J)U(t)\Omega \right\| \leq C||J||e^{Kt}(N+1)^{3} \Omega \leq C||J||e^{Kt} \leq C\lambda^N \|\varphi\|^2 \Omega 
\] (5.5)
Thus, we obtain
\[
|E_t^1(J)| \leq \frac{C||J||e^{Kt}}{N}, 
\] (5.6)
which proves the desired lemma.
Lemma 5.2. For a Hermitian Operator $J$ on $L^2(\mathbb{R}^3)$, let

$$E^2_J = \frac{dN}{\sqrt{N}} \left\langle \frac{(a^*(\phi))^N}{\sqrt{N!}} \Omega, W(\sqrt{N}\phi)\mathcal{U}(t)\phi(J\varphi_t)\mathcal{U}(t)\Omega \right\rangle. \quad (5.7)$$

Then, there exist constants $C$ and $k$, depending only on $\lambda$ and $\sup_{|s|\leq 1} \|\varphi_s\|_{H^3}$, such that

$$|E^2_J| \leq \frac{C\|J\|e^{Kt}}{N}. \quad (5.8)$$

Proof. Let

$$\mathcal{R}(J\varphi_t) := \mathcal{U}(t)\phi(J\varphi_t)\mathcal{U}(t) - \tilde{\mathcal{U}}(t)\phi(J\varphi_t)\tilde{\mathcal{U}}(t). \quad (5.9)$$

We know from Lemma 5.2 that $P_k\tilde{\mathcal{U}}(t)\phi(J\varphi_t)\tilde{\mathcal{U}}(t)\Omega = 0$ for any $k = 0, 1, 2, \cdots$. Thus, we have

$$|E^2_J| = \frac{dN}{\sqrt{N}} \left\langle \frac{(a^*(\phi))^N}{\sqrt{N!}} \Omega, W(\sqrt{N}\phi)\mathcal{U}(t)\phi(J\varphi_t)\mathcal{U}(t)\Omega \right\rangle + \frac{dN}{\sqrt{N}} \left\langle \frac{(a^*(\phi))^N}{\sqrt{N!}} \Omega, W(\sqrt{N}\phi)\mathcal{R}(J\varphi_t)\Omega \right\rangle$$

$$\leq \frac{dN}{\sqrt{N}} \left( \sum_{k=1}^{\infty} (N+1)^{-5/2} P_{k-1} W^*(\sqrt{N}\varphi) \frac{a^*(\phi)^N}{\sqrt{N!}} \Omega \right) \left\| (N+1)^{5/2} \tilde{\mathcal{U}}(t)\phi(J\varphi_t)\tilde{\mathcal{U}}(t)\Omega \right\|$$

$$+ \frac{dN}{\sqrt{N}} \left( \sum_{k=1}^{\infty} (N+1)^{-1/2} W^*(\sqrt{N}\varphi) \frac{a^*(\phi)^N}{\sqrt{N!}} \Omega \right) \left\| (N+1)^{1/2} \mathcal{R}(J\varphi_t)\Omega \right\| \quad (5.10)$$

Let $M := (1/2)N^{1/3}$. Lemma 7.2 shows that

$$\left\| \left( \sum_{k=1}^{\infty} (N+1)^{-5/2} P_{k-1} W^*(\sqrt{N}\varphi) \frac{a^*(\phi)^N}{\sqrt{N!}} \Omega \right) \right\|^2 \leq \sum_{k=1}^{M} \left( (N+1)^{-5/2} P_{k-1} W^*(\sqrt{N}\varphi) \frac{a^*(\phi)^N}{\sqrt{N!}} \Omega \right)^2 \leq \frac{C}{M^5} \left( \sum_{k=1}^{M} \frac{C}{d^2N^3} \right) + \frac{C}{N^{5/3}} \left\| W^*(\sqrt{N}\varphi) \frac{a^*(\phi)^N}{\sqrt{N!}} \Omega \right\|^2 \leq \frac{C}{d^2N^3}. \quad (5.11)$$

Lemma 5.3 shows that

$$\| (N+1)^{5/2} \tilde{\mathcal{U}}(t)\phi(J\varphi_t)\tilde{\mathcal{U}}(t)\Omega \| \leq C e^{Kt} \| (N+1)^{5/2} \phi(J\varphi_t)\tilde{\mathcal{U}}(t)\Omega \| \leq C \| J\varphi_t \|_2 e^{Kt} \| (N+1)^{3/2} \tilde{\mathcal{U}}(t)\Omega \|^2 \leq C \| J\| e^{Kt} \| (N+1)^{3/2} \tilde{\mathcal{U}}(t)\Omega \|^2 \leq C \| J\| e^{Kt}. \quad (5.12)$$

Lemma 7.3 shows that

$$\left\| (N+1)^{-1/2} W^*(\sqrt{N}\varphi) \frac{a^*(\phi)^N}{\sqrt{N!}} \Omega \right\| \leq \frac{C}{d^2N}. \quad (5.13)$$

Finally, Lemma 5.6 shows that

$$\| (N+1)^{1/2} \mathcal{R}(J\varphi_t)\Omega \| \leq \frac{C \| J\varphi_t \|_2 e^{Kt}}{N} \leq \frac{C \| J\| e^{Kt}}{N}. \quad (5.14)$$

Therefore,

$$|E^2_J| \leq \frac{C \| J\| e^{Kt}}{N}, \quad (5.15)$$

which was to be proved. \qed
Lemma 5.3. For any $\psi \in \mathcal{F}$ and $j \in \mathbb{N}$, there exist constants $C$ and $K$, depending on $\lambda$, $j$, and $\sup_{|\tau| \leq |t|, |s|} \|\varphi_{\tau}\|_{H^{1/2}}$, such that

$$\langle \tilde{U}(t; s)\psi, N^j \tilde{U}(t; s)\psi \rangle \leq Ce^{K(t-s)}\langle \psi, (N+1)^j \psi \rangle. \quad (5.16)$$

Proof. Let $\tilde{\psi} = \tilde{U}(t; s)\psi$. We have

$$\frac{d}{dt} \langle \tilde{\psi}, (N+1)^j \tilde{\psi} \rangle = \langle \tilde{\psi}, [i\mathcal{L}_2, (N+1)^j] \tilde{\psi} \rangle$$

Using (5.21) and (5.22) the conclusion follows directly from the Gronwall’s lemma.

Lemma 5.4. For any $\psi \in \mathcal{F}$ and $j \in \mathbb{N}$, there exist constants $C$ and $K$, depending on $\lambda$, $j$, and $\sup_{|\tau| \leq |t|, |s|} \|\varphi_{\tau}\|_{H^1}$ such that

$$\langle \tilde{U}(t; s)\psi, N^j \tilde{U}(t; s)\psi \rangle \leq Ce^{K(t-s)}\langle \psi, (N+1)^{2j+2} \psi \rangle. \quad (5.23)$$

Proof. See Proposition 3.3 of [25].
Lemma 5.5. For any \( \psi \in \mathcal{F} \) and \( j \in \mathbb{N} \), there exist a constant \( C \), depending on \( \lambda \), \( j \), and \( \| \varphi_t \|_{H^1} \), such that

\[
\|(N+1)^{j/2} L_3(t)\psi\| \leq C \frac{\sqrt[4]{N}}{N^{(j+3)/2}} \|\varphi_t\|_{H^1}.
\] (5.24)

Proof. While this lemma is proved in Lemma 6.3, we give a shorter proof here. Let

\[
A_3(t) = \iint dx dy \frac{1}{|x-y| + \alpha N} \nabla_3(y) a_x a_y a_z.
\] (5.25)

Then,

\[
(N+1)^{j/2} L_3(t) = \frac{\lambda}{\sqrt[4]{N}} \left((N+1)^{j/2} A_3(t) + (N+1)^{j/2} A_3^*(t)\right).
\] (5.26)

We estimate \((N+1)^j A_3(t)\) and \((N+1)^j A_3^*(t)\) separately. The first term \((N+1)^j A_3(t)\) satisfies for any \( \xi \in \mathcal{F} \) that

\[
|\langle \xi, (N+1)^{j/2} A_3(t)\psi \rangle| = \left| \iint dx dy \frac{\nabla_3(y)}{|x-y| + \alpha N} \langle \xi, (N+1)^{j/2} a_x a_y a_z \psi \rangle \right|
\]

\[
= \left| \iint dx dy \frac{\nabla_3(y)}{|x-y| + \alpha N} \langle (N+1)^{-1/2} \xi, (N+1)^{j+1/2} a_x a_y a_z \psi \rangle \right|
\]

\[
\leq \left( \iint dx dy \frac{|\nabla_3(y)|^2}{|x-y|^{2}} \|a_x (N+1)^{-1/2} \xi\|^2 \right)^{1/2} \left( \iint dx dy \|a_y a_z (N+1)^{j+1/2} \psi\|^2 \right)^{1/2}
\]

\[
\leq C \|\varphi_t\|_{H^1} \|\xi\| \|N^{(j+3)/2} \psi\|,
\] (5.27)

where we used Hardy inequality in the last inequality. Since \( \xi \) was arbitrary, we obtain that

\[
\|(N+1)^{j/2} A_3(t)\psi\| \leq C \|\varphi_t\|_{H^1} \|N^{(j+3)/2} \psi\|.
\] (5.28)

Similarly, we can find that

\[
\|(N+1)^{j/2} A_3^*(t)\psi\| \leq C \|\varphi_t\|_{H^1} \|(N+2)^{(j+3)/2} \psi\|.
\] (5.29)

Hence, from (5.26), (5.28), and (5.29) we get

\[
\|(N+1)^{j/2} L_3(t)\psi\| \leq C \frac{\sqrt[4]{N}}{N^{(j+3)/2}} \|(N+1)^{(j+3)/2} \psi\|,
\] (5.30)

which was to be proved. \(\square\)

Lemma 5.6. For all \( j \in \mathbb{N} \), there exist constants \( C \) and \( K \) depending only on \( \lambda \), \( j \), and \( \sup_{|s| \leq t} \|\varphi_s\|_{H^1} \) such that, for any \( f \in L^2(\mathbb{R}^3) \),

\[
\|(N+1)^{j/2} (\tilde{U}^*(t) \phi(f) \tilde{U}(t) - \tilde{U}^*(t) \phi(f) \tilde{U}(t)) \Omega \| \leq C \frac{\|f\|_{L^2(\mathbb{R}^3)}}{N}.
\] (5.31)

Proof. Let

\[
\mathcal{R}_1(f) := (\tilde{U}^*(t) - \tilde{U}^*(t)) \phi(f) \tilde{U}(t)
\] (5.32)

and

\[
\mathcal{R}_2(f) := \tilde{U}^*(t) \phi(f) (\tilde{U}(t) - \tilde{U}(t))
\] (5.33)

so that

\[
\tilde{U}^*(t) \phi(f) \tilde{U}(t) - \tilde{U}^*(t) \phi(f) \tilde{U}(t) = \mathcal{R}_1(f) + \mathcal{R}_2(f).
\] (5.34)
Then, from Lemma 5.3, Lemma 5.4, and Lemma 5.5, we find that
\[
\| (N + 1)^{1/2} R_1(f) \Omega \| = \left\| \int_0^t ds \, (N + 1)^{1/2} U^*(s; 0) L_3(s) \tilde{U}^*(t; s) \phi(f) \tilde{U}(t) \Omega \right\|
\leq \int_0^t ds \| (N + 1)^{1/2} U^*(s; 0) L_3(s) \tilde{U}^*(t; s) \phi(f) \tilde{U}(t) \Omega \|
\leq C e^{Kt} \int_0^t ds \| (N + 1)^{1+1/2} L_3(s) \tilde{U}^*(t; s) \phi(f) \tilde{U}(t) \Omega \|
\leq \frac{C e^{Kt}}{\sqrt{N}} \| (N + 1)^{j+(5/2)} \phi(f) \tilde{U}(t) \Omega \|
\] (5.35)

Thus, we can get the following bound for \( R_1(f) \).
\[
\| (N + 1)^{1/2} R_1(f) \Omega \|
\leq \frac{C e^{Kt}}{\sqrt{N}} \left( \| \alpha(f) (N + 1)^{j+(5/2)} \tilde{U}(t) \Omega \| + \| \alpha^*(f) (N + 1)^{j+(5/2)} \tilde{U}(t) \Omega \| \right)
\leq \frac{C \| f \|_{2 e^{Kt}}}{\sqrt{N}} \| (N + 1)^{j+(5/2)} \Omega \| \leq \frac{C \| f \|_{2 e^{Kt}}}{\sqrt{N}}.
\] (5.36)

The study of \( R_2(f) \) is similar and gives
\[
\| (N + 1)^{1/2} R_2(f) \Omega \| \leq \frac{C \| f \|_{2 e^{Kt}}}{\sqrt{N}}.
\] (5.37)

Therefore,
\[
\| (N + 1)^{1/2} \left( U^*(t) \phi(f) U(t) - \tilde{U}^*(t) \phi(f) \tilde{U}(t) \right) \Omega \| \leq \| (N + 1)^{1/2} R_1(f) \Omega \| + \| (N + 1)^{1/2} R_2(f) \Omega \|
\leq \frac{C \| f \|_{2 e^{Kt}}}{\sqrt{N}},
\] (5.38)

which was to be proved. \( \square \)

6. Properties of Regularized Dynamics

In this section, we prove various lemmas, which allowed us to use the regularized dynamics instead of the full dynamics.

**Lemma 6.1.** Let \( \psi_N = \varphi \otimes N \) for some \( \varphi \in H^1(\mathbb{R}^3) \) with \( \| \varphi \| = 1 \). Let \( \psi_{N,t} = e^{-i H_N t} \psi_N \) and \( \psi_{N,t}^{\alpha} = e^{-i H_N^\alpha t} \psi_N \). If \( \lambda < \lambda_{\text{crit}}^H \), then there exists a constant \( C > 0 \) and \( N_0 \) such that, for all \( t \in \mathbb{R} \) and for any positive integer \( N > N_0 \),
\[
\sum_{i < j}^{N} \langle \psi_{N,t} \rangle (1 - \Delta_i)^{1/2} (1 - \Delta_j)^{1/2} \psi_{N,t} \leq C N^3 \quad (6.1)
\]
and
\[
\sum_{i < j}^{N} \langle \psi_{N,t}^{\alpha} \rangle (1 - \Delta_i)^{1/2} (1 - \Delta_j)^{1/2} \psi_{N,t}^{\alpha} \leq C N^3. \quad (6.2)
\]

**Proof.** Let
\[
S_j = (1 - \Delta_j)^{1/2}, \quad V_{ij} = \frac{\lambda}{|x_i - x_j|},
\] (6.3)
so that

\[ H_N = \sum_{j=1}^{N} S_j - \frac{1}{N} \sum_{i<j}^{N} V_{ij}. \]  

(6.4)

We first consider the operator

\[ H_{N-1} = \sum_{j=1}^{N-1} S_j - \frac{1}{N-1} \sum_{i<j}^{N-1} V_{ij}. \]  

(6.5)

Let \( \eta = (\lambda_{\text{crit}}^H/\lambda)^{1/2} \) so that \( \eta > 1 \) and \( \lambda \eta < \lambda_{\text{crit}}^H \). Then,

\[ H_{N-1} = \eta^{-1} \left( (\eta - 1) \sum_{j=1}^{N-1} S_j + \sum_{j=1}^{N-1} S_j - \frac{\lambda \eta}{N-1} \sum_{i<j}^{N-1} V_{ij} \right). \]  

(6.6)

When \( N \) is sufficiently large, we have the following operator inequality

\[ \sum_{j=1}^{N-1} S_j - \frac{\lambda \eta}{N-1} \sum_{i<j}^{N-1} V_{ij} \geq -M(N-1) \]  

(6.7)

for some \( M \geq 0 \). (See Theorem 1 of [21].) Thus,

\[ H_{N-1} \geq -\eta^{-1} MN + \left(1 - \eta^{-1}\right) \sum_{j=1}^{N-1} S_j. \]  

(6.8)

Let

\[ H_{N}^{(N-1)} = \sum_{j=1}^{N-1} S_j - \frac{1}{N} \sum_{i<j}^{N-1} V_{ij}. \]  

(6.9)

We consider the operator

\[ H_N^2 = \left( H_N^{(N-1)} + S_N - \frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right)^2 \]

\[ = \left( H_N^{(N-1)} - \frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right)^2 + S_N^2 + 2H_N^{(N-1)} S_N - S_N \left( \frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right) - \left( \frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right) S_N, \]  

(6.10)

where we used that \([H_N^{(N-1)}, S_N] = 0\). Now, we find that

\[ H_N^2 \geq S_N^2 + 2H_N^{(N-1)} S_N - S_N \left( \frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right) - \left( \frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right) S_N. \]  

(6.11)

Since

\[ H_N^{(N-1)} \geq H_{N-1} \geq -\eta^{-1} MN + \left(1 - \eta^{-1}\right) \sum_{j=1}^{N-1} S_j, \]  

(6.12)
we have that
\[
H_N^{(N-1)} S_N = S_N^{1/2} H_N^{(N-1)} S_N^{1/2} \geq S_N^{1/2} \left( -\eta^{-1} M N + (1 - \eta^{-1}) \sum_{j=1}^{N-1} S_j \right) S_N^{1/2}
\]
\[
\geq -\eta^{-1} M N S_N + (1 - \eta^{-1}) \sum_{j=1}^{N-1} S_j S_N.
\]

Let \( C_0 \) be a constant satisfying the operator inequality
\[
C_0 S_j S_N \geq V_j^2 N, \tag{6.14}
\]
and choose \( N_1 \) large so that \((1 - \eta^{-1}) \geq C_0 N_1^{-1}\). Then, for all \( N > N_1 \),
\[
2 H_N^{(N-1)} S_N + 2 M N S_N \geq 2 (1 - \eta^{-1}) \sum_{j=1}^{N-1} S_j S_N \geq (1 - \eta^{-1}) \sum_{j=1}^{N-1} S_j S_N + \frac{1}{N} \sum_{j=1}^{N-1} V_j^2 N
\]
\[
\geq (1 - \eta^{-1}) \sum_{j=1}^{N-1} S_j S_N + \left( \frac{1}{N} \sum_{j=1}^{N-1} V_j N \right)^2,
\]
where the last inequality comes from the Schwarz inequality. Hence, we obtain from (6.11) and (6.15) that
\[
H_N^2 + 2 M N S_N \geq (1 - \eta^{-1})(\sum_{i:i\neq j}^{N} S_i S_j).
\]
Similarly, for any \( 1 \leq j \leq N \),
\[
H_N^2 + 2 M N S_j \geq (1 - \eta^{-1}) \sum_{i:i\neq j}^{N} S_i S_j.
\]

Thus, summing (6.17) over \( j \), we get
\[
N H_N^2 + 2 M N \sum_{j=1}^{N} S_j \geq (1 - \eta^{-1}) \sum_{i\neq j}^{N} S_i S_j.
\]

For the operator \( H_N \), similarly to (6.8), we have
\[
H_N \geq -\eta^{-1} M N + (1 - \eta^{-1}) \sum_{j=1}^{N} S_j,
\]
thus,
\[
(1 - \eta^{-1})^{-1} H_N + (\eta - 1)^{-1} M N \geq \sum_{j=1}^{N} S_j.
\]

Together with (6.18), we have shown that
\[
\eta N \frac{H_N^2}{\eta - 1} + 2(\frac{\eta}{\eta - 1})^2 M N H_N + 2 \frac{\eta M^2 N^2}{(\eta - 1)^2} \geq \sum_{i\neq j}^{N} S_i S_j,
\]

Since \( H_N \) and \( H_N^2 \) have the upper bounds
\[
H_N \leq \sum_{j=1}^{N} S_j
\]
and
\[
H_N^2 = \left( \sum_{j=1}^{N} S_j - \frac{1}{N} \sum_{i<j} V_{ij} \right)^2 \leq 2 \left( \sum_{j=1}^{N} S_j \right)^2 + \frac{2}{N^2} \left( \sum_{i<j} V_{ij} \right)^2 \]
\[
\leq 2N \sum_{j=1}^{N} S_j^2 + \frac{N - 1}{N} \sum_{i<j} V_{ij}^2 \leq CN \sum_{j=1}^{N} S_j^2, \tag{6.23}
\]
respectively, we have that
\[
\langle \psi_{N,t}, H_N \psi_{N,t} \rangle = \langle \varphi^{\otimes N}, H_N \varphi^{\otimes N} \rangle \leq \langle \varphi^{\otimes N}, \sum_{j=1}^{N} S_j \varphi^{\otimes N} \rangle \leq CN \| \varphi \|^2_{H^{1/2}} \tag{6.24}
\]
and
\[
\langle \psi_{N,t}, H_N^2 \psi_{N,t} \rangle = \langle \varphi^{\otimes N}, H_N^2 \varphi^{\otimes N} \rangle \leq CN \langle \varphi^{\otimes N}, \sum_{j=1}^{N} S_j^2 \varphi^{\otimes N} \rangle \leq CN^2 \| \varphi \|^2_{H^1}. \tag{6.25}
\]
Therefore, from (6.21), (6.24), and (6.25), we find
\[
\sum_{i<j} \langle \psi_{N,t}, S_i S_j \psi_{N,t} \rangle \leq CN^3, \tag{6.26}
\]
which proves the first part of the lemma. The second part of the lemma can be proved analogously.

We now consider the regularized semi-relativistic Hartree equation (2.8) given by
\[
i \partial_t \varphi_t^{\alpha} = (1 - \Delta)^{1/2} \varphi_t^{\alpha} - \lambda \left( \frac{1}{|\cdot| + \alpha_N} * |\varphi_t^{\alpha}|^2 \right) \varphi_t^{\alpha}, \tag{6.27}
\]
and study properties of the solution of (2.8).

The following results will be used in the proof of Proposition 2.1.

**Lemma 6.2** (Generalized Leibniz Rule). Suppose that $1 < p < \infty$, $s \geq 0$, $\alpha \geq 0$, $\beta \geq 0$, and $1/p_i + 1/q_i = 1/p$ with $i = 1, 2$, $1 < q_1 \leq \infty$, $1 < p_2 \leq \infty$. Then
\[
\|(\Delta)^{s/2}(fg)\|_p \leq C \left( \|(\Delta)^{(s+\alpha)/2}f\|_{p_1}\|(\Delta)^{\alpha/2}g\|_{q_1} + \|(\Delta)^{\beta/2}f\|_{p_2}\|(\Delta)^{(s+\beta)/2}g\|_{q_2} \right), \tag{6.28}
\]
where the positive constant $C$ depends on all of the parameters above but not on $f$ and $g$.

**Proof.** See Theorem 1.4 of [15].

**Lemma 6.3** (Propagation of Regularity). Fix $s > 1/2$. Let $\varphi \in H^s(\mathbb{R}^3)$ with $\| \varphi \|_2 = 1$. Let $\varphi_t$ and $\varphi_t^{\alpha}$ denote the solutions of the semi-relativistic Hartree equations (1.0) and Hartree equation with cutoff, respectively, with the initial condition $\varphi_{t=0} = \varphi$. Fix $T > 0$ such that
\[
\kappa = \sup_{|t| \leq T} \| \varphi_t^{\alpha} \|_{H^{1/2}} < \infty. \tag{6.29}
\]
Then, there exists a constant $\nu = \nu(\kappa, T, s, \| \varphi \|_{H^s}) < \infty$ (but independent of $\alpha_N$) such that
\[
\sup_{|t| \leq T} \| \varphi_t \|_{H^s}, \sup_{|t| \leq T} \| \varphi_t^{\alpha} \|_{H^s} \leq \nu. \tag{6.30}
\]

**Proof.** See Proposition 2.1 of [22].

To prove Proposition 2.1, we first consider the following a priori bound on the difference in $L^2$-norm:
Lemma 6.4. Suppose that the assumptions of Proposition 2.1 are satisfied. Then, there exist constants $C$ and $K$, depending only on $\lambda$, $\kappa$, $T$, and $\|\varphi\|_{H^1}$, such that

$$\|\varphi_t - \varphi_t^o\|_2 \leq C\alpha_N$$

for all $|t| < T$.

Proof. See Proposition 2.2 of [22]

Using Lemma 6.4, we prove Proposition 2.1. In the following proof, we generally follow the proof of Proposition 2.2 of [22] except in a few estimates.

Proof of Proposition 2.1. First, note that, for any $|t| \leq T$, $\|\varphi_t\|_{H^1} \leq \nu$ for some constant $\nu$ depending only on $T$, $\kappa$, and $\|\varphi\|_{H^1}$, which follows from Lemma 6.2. To prove the proposition, it suffices to show that

$$\|(-\Delta)^{1/4}(\varphi_t - \varphi_t^o)\|_2 \leq C\alpha_N^{1/2}. \quad (6.32)$$

From Schwarz inequality, we obtain that

$$\left| \frac{d}{dt} (-\Delta)^{1/4}(\varphi_t - \varphi_t^o) \right|_2^2 = \left| -2\lambda \text{Im} \left( (-\Delta)^{1/4}(\varphi_t - \varphi_t^o), (-\Delta)^{1/4} \left[ \left( \frac{1}{|\cdot|} * |\varphi_t|^2 \right) \varphi_t - \left( \frac{1}{|\cdot| + \alpha_N} * |\varphi_t^o|^2 \right) \varphi_t^o \right] \right) \right| \leq 2\lambda \|(-\Delta)^{1/4}(\varphi_t - \varphi_t^o)\|_2 \|(-\Delta)^{1/4} \left[ \left( \frac{1}{|\cdot|} * |\varphi_t|^2 \right) \varphi_t - \left( \frac{1}{|\cdot| + \alpha_N} * |\varphi_t^o|^2 \right) \varphi_t^o \right]\|_2. \quad (6.33)$$

To estimate the right hand side, we use the following decomposition:

$$\|(-\Delta)^{1/4} \left[ \left( \frac{1}{|\cdot|} * |\varphi_t|^2 \right) \varphi_t - \left( \frac{1}{|\cdot| + \alpha_N} * |\varphi_t^o|^2 \right) \varphi_t^o \right]\|_2 \leq \|(-\Delta)^{1/4} \left[ \left( \frac{1}{|\cdot|} * |\varphi_t|^2 \right) \varphi_t - \left( \frac{1}{|\cdot|} - \frac{1}{|\cdot| + \alpha_N} \right) |\varphi_t|^2 \right) \varphi_t^o \|_2 + \|(-\Delta)^{1/4} \left[ \left( \frac{1}{|\cdot|} - \frac{1}{|\cdot| + \alpha_N} \right) |\varphi_t|^2 \right) \varphi_t^o \|_2 \| + \|(-\Delta)^{1/4} \left( \left( \frac{1}{|\cdot| + \alpha_N} \right) |\varphi_t|^2 - |\varphi_t^o|^2 \right) \varphi_t - \varphi_t^o \|_2 + \|(-\Delta)^{1/4} \left( \left| \frac{1}{|\cdot| + \alpha_N} \right| \left( |\varphi_t|^2 - |\varphi_t^o|^2 \right) \varphi_t \|_2. \quad (6.34)$$

The first term in the right hand side of (6.34) is bounded by

$$\|(-\Delta)^{1/4} \left[ \left( \frac{1}{|\cdot|} * |\varphi_t|^2 \right) \varphi_t - \varphi_t^o \right]\|_2 \leq C \|(-\Delta)^{1/4} \left( \frac{1}{|\cdot|} * |\varphi_t|^2 \right) \|_6 \|\varphi_t - \varphi_t^o\|_3 + C \frac{1}{|\cdot|} * |\varphi_t|^2 \|_\infty \|(-\Delta)^{1/4}(\varphi_t - \varphi_t^o)\|_2 \quad (6.35)$$

where we used the generalized Leibniz rule, Lemma 6.2. By Sobolev inequality,

$$\|\varphi_t - \varphi_t^o\|_3 \leq C\|(-\Delta)^{1/4}(\varphi_t - \varphi_t^o)\|_2, \quad (6.36)$$

and by Kato’s inequality,

$$\left\| \frac{1}{|\cdot|} * |\varphi_t|^2 \right\|_\infty \leq C\|\varphi_t\|_{H^{1/2}}. \quad (6.37)$$
From (6.35), (6.36), (6.37), and (6.41), we get

\[
\frac{1}{|\cdot|} * |\varphi_t|^2 = -4\pi(-\Delta)^{-1}|\varphi_t|^2,
\]  

(6.38)

we find that

\[
(-\Delta)^{1/4} \left( \frac{1}{|\cdot|} * |\varphi_t|^2 \right) = -4\pi(-\Delta)^{-3/4}|\varphi_t|^2 = -4\pi G_{3/2} * |\varphi_t|^2,
\]  

(6.39)

where \( G_{3/2} \) is the kernel of the operator \((-\Delta)^{-3/4}\) that is given by

\[
G_{3/2}(x) = \frac{\pi^2 \sqrt{2}}{\Gamma(3/4)} |x|^{-3/2}.
\]  

(6.40)

Thus, from Hardy-Littlewood-Sobolev inequality and Sobolev inequality,

\[
\left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot|} * |\varphi_t|^2 \right) \right\|_6 = C \left\| |\cdot|^{-3/2} * |\varphi_t|^2 \right\|_6 \leq C \|\varphi_t\|_3^2 \leq C \|\varphi_t\|_{H^{1/2}}.
\]  

(6.41)

From (6.33), (6.36), (6.37), and (6.41), we get

\[
\left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot|} * |\varphi_t|^2 \right) (\varphi_t - \varphi_t^0) \right\|_2 \leq C \|(-\Delta)^{1/4}(\varphi_t - \varphi_t^0)\|_2.
\]  

(6.42)

The second term in the right hand side of (6.34) can be bounded analogously, hence it satisfies

\[
\left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot|} - \frac{1}{|\cdot| + \alpha_N} \right) * |\varphi_t|^2 \right\|_2 \leq C \|(-\Delta)^{1/4}(\varphi_t - \varphi_t^0)\|_2.
\]  

(6.43)

The third term in the right hand side of (6.34) is again bounded using Lemma 6.2 by

\[
\left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot|} - \frac{1}{|\cdot| + \alpha_N} \right) * |\varphi_t|^2 \right\|_2 \leq C \left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot|} - \frac{1}{|\cdot| + \alpha_N} \right) * |\varphi_t|^2 \right\|_3 \|\varphi_t\|_6
\]  

(6.44)

\[
+ C \left\| \left( \frac{1}{|\cdot|} - \frac{1}{|\cdot| + \alpha_N} \right) * |\varphi_t|^2 \right\|_\infty \|(-\Delta)^{1/4}\varphi_t\|_2.
\]

We have from Hardy-Littlewood-Sobolev inequality, generalized Leibniz rule, and Sobolev inequality that

\[
\left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot|} - \frac{1}{|\cdot| + \alpha_N} \right) * |\varphi_t|^2 \right\|_3 \leq C \|\alpha_N\| \left\| (-\Delta)^{1/4}(\varphi_t - \varphi_t^0) \right\|_3 \leq C \|\alpha_N\| \left\| (-\Delta)^{1/4}\varphi_t \right\|_2 \|\varphi_t\|_6 \leq C \|\alpha_N\| \kappa \nu.
\]  

(6.45)

From Hardy inequality, we get that

\[
\left\| \left( \frac{1}{|\cdot|} - \frac{1}{|\cdot| + \alpha_N} \right) * |\varphi_t|^2 \right\|_\infty \leq \alpha_N \left\| \frac{1}{|\cdot|^2} * (-\Delta)^{1/4}\varphi_t \right\|_3 \leq C \|\alpha_N\| \nu^2.
\]  

(6.46)

Thus, from (6.44), (6.45), and (6.44), we obtain that

\[
\left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot|} - \frac{1}{|\cdot| + \alpha_N} \right) * |\varphi_t|^2 \right\|_2 \leq C \|\alpha_N\|.
\]  

(6.47)
The fourth term in the right hand side of (6.34) is bounded by
\[
\left\| (-\Delta)^{1/4} \left[ \frac{1}{|\cdot| + \alpha N} \ast (|\varphi_1|^2 - |\varphi_1^\alpha|^2) \right] (\varphi_1 - \varphi_1^\alpha) \right\|_2
\leq C \left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot| + \alpha N} \ast (|\varphi_1|^2 - |\varphi_1^\alpha|^2) \right) \right\|_\infty \|\varphi_1 - \varphi_1^\alpha\|_2
+ C \left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot| + \alpha N} \ast (|\varphi_1|^2 - |\varphi_1^\alpha|^2) \right) \right\|_\infty \left\| (-\Delta)^{1/4} (\varphi_1 - \varphi_1^\alpha) \right\|_2
\] (6.48)

We notice that
\[
\left( (-\Delta)^{1/4} \left( \frac{1}{|\cdot| + \alpha N} \right) (x) \right) \leq \frac{C}{(|x| + \alpha)^{3/2}},
\] (6.49)

which is proved in Proposition 2.2 of [22]. Thus,
\[
\left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot| + \alpha N} \ast (|\varphi_1|^2 - |\varphi_1^\alpha|^2) \right) \right\|_\infty \|\varphi_1 - \varphi_1^\alpha\|_2 \leq C \alpha_N^{-3/2} \|\varphi_1|^2 - |\varphi_1^\alpha|^2\|_1 \|\varphi_1 - \varphi_1^\alpha\|_2
\leq C \alpha_N^{-3/2} \|\varphi_1| + |\varphi_1^\alpha|\|_2 \|\varphi_1 - \varphi_1^\alpha\|_2 \leq C \alpha_N^{1/2},
\] (6.50)

where we used Lemma 6.4 in the last inequality. We also have that
\[
\left\| \frac{1}{|\cdot| + \alpha N} \ast (|\varphi_1|^2 - |\varphi_1^\alpha|^2) \right\|_\infty \leq \alpha_N^{-1} \|\varphi_1|^2 - |\varphi_1^\alpha|^2\|_1 \leq C,
\] (6.51)

where we used the same argument as in (6.50). Thus, from (6.48), (6.50), and (6.51), we obtain that
\[
\left\| (-\Delta)^{1/4} \left[ \frac{1}{|\cdot| + \alpha N} \ast (|\varphi_1|^2 - |\varphi_1^\alpha|^2) \right] (\varphi_1 - \varphi_1^\alpha) \right\|_2 \leq C \alpha_N^{1/2} + C \left\| (-\Delta)^{1/4} (\varphi_1 - \varphi_1^\alpha) \right\|_2.
\] (6.52)

The last term of the right hand side (6.34) is bounded by
\[
\left\| (-\Delta)^{1/4} \left[ \frac{1}{|\cdot| + \alpha N} \ast (|\varphi_1|^2 - |\varphi_1^\alpha|^2) \right] \varphi_1 \right\|_2
\leq C \left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot| + \alpha N} \ast (|\varphi_1|^2 - |\varphi_1^\alpha|^2) \right) \right\|_3 \|\varphi_1\|_6
+ C \left\| \frac{1}{|\cdot| + \alpha N} \ast (|\varphi_1|^2 - |\varphi_1^\alpha|^2) \right\|_6 \left\| (-\Delta)^{1/4} \varphi_1 \right\|_3.
\] (6.53)

The first term in the right hand side of (6.53) is bounded by
\[
\left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot| + \alpha N} \ast (|\varphi_1|^2 - |\varphi_1^\alpha|^2) \right) \right\|_3 \leq \left\| (-\Delta)^{1/4} \frac{1}{|\cdot| + \alpha N} \right\|_3 \|\varphi_1|^2 - |\varphi_1^\alpha|^2\|_1
\leq C \alpha_N \left\| \frac{1}{(|\cdot| + \alpha N)^{3/2}} \right\|_3,
\] (6.54)

where we used the bound (6.49). An explicit computation shows that
\[
\left\| \frac{1}{(|\cdot| + \alpha N)^{3/2}} \right\|_3^3 = 4\pi \int_0^\infty \frac{r^2}{(r + \alpha N)^{9/2}} dr = \frac{64\pi}{105} \alpha_N^{-3/2}.
\] (6.55)

Hence,
\[
\left\| (-\Delta)^{1/4} \left( \frac{1}{|\cdot| + \alpha N} \ast (|\varphi_1|^2 - |\varphi_1^\alpha|^2) \right) \right\|_3 \leq C \alpha_N^{1/2}.
\] (6.56)
The second term in the right hand side of (6.33) is estimated as
\[
\left\| \frac{1}{|\cdot| + \alpha_N} \ast (|\varphi_t|^2 - |\varphi_t^0|^2) \right\|_6 \|(\Delta)^{1/4} \varphi_t\|_3 \leq \left\| \frac{1}{|\cdot|} \ast (|\varphi_t|^2 - |\varphi_t^0|^2) \right\|_6 \|\varphi_t\|_{H^1}
\]
\[
\leq C \left\| (|\varphi_t|^2 - |\varphi_t^0|^2) \right\|_{6/5} \||\varphi_t\|_{H^1} \leq C \|\varphi_t\| + \|\varphi_t^0\|_2 \|\varphi_t - \varphi_t^0\|_3 \|\varphi_t\|_{H^1}
\]
(6.57)
\[
\leq C \nu \| (\Delta)^{1/4} (\varphi_t - \varphi_t^0) \|_2,
\]
where we used Sobolev inequality and Hardy-Littlewood-Sobolev inequality. Thus, from (6.33), (6.34), (6.42), (6.43), (6.47), (6.52), and (6.57), we obtain that
\[
\left\| (\Delta)^{1/4} \left( \frac{1}{|\cdot| + \alpha_N} \ast (|\varphi_t|^2 - |\varphi_t^0|^2) \right) \varphi_t \right\|_2 \leq C \alpha_N^{1/2} + C \| (\Delta)^{1/4} (\varphi_t - \varphi_t^0) \|_2.
\]
(6.58)
Therefore, from (6.33), (6.34), (6.42), (6.43), (6.47), (6.52), and (6.58), we find that
\[
\left\| \frac{d}{dt} (\Delta)^{1/4} (\varphi_t - \varphi_t^0) \right\|_2^2 \leq C \| (\Delta)^{1/4} (\varphi_t - \varphi_t^0) \|_2 \left( \alpha_N^{1/2} + \| (\Delta)^{1/4} (\varphi_t - \varphi_t^0) \|_2 \right)
\]
(6.59)
\[
\leq C \alpha_N + C \| (\Delta)^{1/4} (\varphi_t - \varphi_t^0) \|_2.
\]
Now, (6.32) follows from Gronwall’s lemma. This concludes the proof of the Proposition 2.1.

7. Properties of Weyl Operator

In this section, we prove various estimates on the following state:
\[
W^* \left( \sqrt{N} \varphi \right) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega.
\]
(7.1)

Lemma 7.1. There exists a constant \( C > 0 \) such that, for any \( \varphi \in L^2(\mathbb{R}^3) \) with \( \| \varphi \|_2 = 1 \), we have
\[
\left\| (N + 1)^{-1/2} W^* \left( \sqrt{N} \varphi \right) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \right\| \leq \frac{C}{d_N}.
\]
(7.2)

Proof. See Lemma 6.3 of [2].

In the next lemma, we prove an estimate on the state (7.1), which primarily shows that the state has a very small probability of having an odd number of particles.

Lemma 7.2. For all non-negative integers \( k \leq (1/2)N^{1/3} \),
\[
\left\| P_{2k} W^* \left( \sqrt{N} \varphi \right) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \right\| \leq \frac{2}{d_N}
\]
(7.3)
and
\[
\left\| P_{2k+1} W^* \left( \sqrt{N} \varphi \right) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \right\| \leq \frac{2(k + 1)^{3/2}}{d_N \sqrt{N}}.
\]
(7.4)

Proof. Since
\[
W^* \left( \sqrt{N} \varphi \right) = W(-\sqrt{N} \varphi) = e^{-N/2} \exp \left( a^*(-\sqrt{N} \varphi) \right) \exp \left( a(\sqrt{N} \varphi) \right),
\]
(7.5)
we find for any $\ell \leq N$ that

$$
P^{\ast}(\sqrt{N}Q)(a^{\ast}(\varphi))^{N}
= e^{-N/2} \sum_{m=0}^{\ell} \frac{(-1)^{m} N^{m}}{m!(N-\ell+m)!} \left( \frac{a^{\ast}(\varphi)}{\sqrt{N}} \right)^{m} \left( a^{\ast}(\varphi) \right)^{N-\ell+m} \left( a^{\ast}(\varphi) \right)^{N}
= \frac{e^{-N/2}}{\sqrt{N}!} N^{\ell} \sum_{m=0}^{\ell} \frac{(-1)^{m} N^{m}}{m!(N-\ell+m)!} \left( \frac{a^{\ast}(\varphi)}{\sqrt{N}} \right)^{m} \left( a^{\ast}(\varphi) \right)^{N-\ell+m} \left( a^{\ast}(\varphi) \right)^{N} \quad (7.6)
= \frac{1}{d_{N}} N^{\ell/2} L_{\ell}^{(N-\ell)}(N) \left( a^{\ast}(\varphi) \right)^{\ell},
$$

where $L^{(\alpha)}_{n}(x)$ denotes the generalized Laguerre polynomial.

Generalized Laguerre polynomials $L^{(\alpha)}_{n}(x)$ satisfy the following recurrence relations:

$$
L^{(\alpha-1)}_{n}(x) = L^{(\alpha)}_{n}(x) - L^{(\alpha)}_{n-1}(x), \quad (7.7)
$$

$$
xL^{(\alpha+1)}_{n}(x) = (n + \alpha + 1)L^{(\alpha)}_{n}(x) - (n + 1)L^{(\alpha)}_{n+1}(x). \quad (7.8)
$$

(See [1] for more detail.) From the recurrence relations, we find that

$$
xL^{(\alpha+2)}_{n-1}(x) = xL^{(\alpha+2)}_{n-1}(x) - xL^{(\alpha+1)}_{n-1}(x) = [(n + \alpha + 1)L^{(\alpha+1)}_{n-1}(x) - nL^{(\alpha+1)}_{n}(x)] - xL^{(\alpha+1)}_{n-1}(x)
= (\alpha + 1 - x)L^{(\alpha+1)}_{n-1}(x) - nL^{(\alpha+1)}_{n-1}(x) + nL^{(\alpha+1)}_{n}(x)
= (\alpha + 1 - x)L^{(\alpha+1)}_{n-1}(x) - nL^{(\alpha)}_{n}(x).
$$

Hence we get,

$$
L^{(\alpha)}_{n}(x) = \frac{\alpha + 1 - x}{n} L^{(\alpha+1)}_{n-1}(x) - \frac{x}{n} L^{(\alpha+2)}_{n-2}(x). \quad (7.10)
$$

Define

$$
A_{\ell} := \begin{cases} 
N^{(1-\ell)/2} L_{\ell}^{(N-\ell)}(N) & \text{if } \ell \text{ odd} \\
N^{-\ell/2} L_{\ell}^{(N-\ell)}(N) & \text{if } \ell \text{ even}
\end{cases}. \quad (7.11)
$$

Then, from (7.10), we can find the following recurrence relations for $A_{\ell}$:

$$
A_{2k+1} = -\frac{2k}{2k+1} A_{2k} - \frac{A_{2k-1}}{2k+1}, \quad A_{2k} = -\frac{2k - 1}{2k} \cdot \frac{A_{2k-1}}{N} - \frac{A_{2k-2}}{2k}, \quad (7.12)
$$

where $k$ is a non-negative integer. It can be easily computed that $A_{0} = 1$ and $A_{1} = 0$. Now, we consider the following claim:

**Claim.** For any $1 \leq k \leq (1/2)N^{1/3}$,

$$
|A_{2k-2}| \leq \frac{1}{\sqrt{2k - 2)!}}, \quad |A_{2k-1}| \leq \frac{k\sqrt{k}}{\sqrt{(2k - 1)!}}. \quad (7.13)
$$
Lemma 8.1. Assume that

\[ |A_{2k}| \leq \frac{k \sqrt{k}}{N \sqrt{(2k-1)!}} + \frac{1}{k \sqrt{(2k-2)!}} = \frac{1}{\sqrt{(2k)!}} \left( \frac{\sqrt{2}k^2}{N} + \frac{2k-1}{2k} \right) \leq \frac{1}{\sqrt{(2k)!}} \left( \frac{\sqrt{2}k^2}{N} + 1 - \frac{1}{4k} \right) \leq \frac{1}{\sqrt{(2k)!}}, \]

since \( k \leq (1/2)N^{1/3} \). We also have that

\[ |A_{2k+1}| \leq \frac{1}{\sqrt{(2k+1)!}} + \frac{k \sqrt{k}}{(2k+1) \sqrt{(2k-1)!}} = \frac{1}{\sqrt{(2k+1)!}} \left( \frac{\sqrt{2}k^2}{N} + \frac{2k}{2k+1} \right)^{1/2} \leq \frac{1}{\sqrt{(2k+1)!}} (k^3 + 3k^2 + 2k + 1)^{1/2} \]

Thus, the claim (7.13) is proved.

Now, we observe that

\[ \left\| P_k W^* (\sqrt{N} \varphi) \left( \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \right) \right\| = \frac{A_{2k}}{d_N} \left\| (a^*(\varphi))^{2k} \Omega \right\| \leq \frac{1}{d_N} \left\| (a^*(\varphi))^{2k} \Omega \right\| \leq \frac{1}{d_N} \]  (7.16)

and

\[ \left\| P_{k+1} W^* (\sqrt{N} \varphi) \left( \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \right) \right\| \leq \frac{A_{2k+1}}{d_N \sqrt{N}} \left\| (a^*(\varphi))^{2k+1} \Omega \right\| \leq \frac{(k+1)^{3/2}}{d_N \sqrt{N}} \left\| (a^*(\varphi))^{2k+1} \Omega \right\| \]

This proves the desired lemma. \qed

8. Properties of the evolution operator \( \tilde{U}(t;s) \)

In this section, we prove some basic properties of the operator \( \tilde{U}(t;s) \).

Following Proposition 2.2 of [14], we can prove that \( \tilde{U}(t;s) \) is bounded in \( Q(K+N^2) \), provided that \( (L_2(t)+L_4) \) is stable. (See Proposition 3.4 of [17] and Lemma 7.1 of [2] for more detail.) The following lemma shows that \( (L_2(t)+L_4) \) is stable:

Lemma 8.1. Assume that \( \nu(t) = \sup_{|s| \leq 2} \| \varphi_s^* \|_{H^1} < \infty \). Then, there exist constants \( C', K' > 0 \), depending on \( N \), \( \alpha_N \), \( \lambda \), \( t \), and \( \nu(t) \), such that, for the operator \( A_2(t) = L_2(t) + L_4 + C(N^2 + 1) \), we have the operator inequality

\[ \frac{d}{dt} A_2(t) \leq K' A_2(t). \]  (8.1)
Proof. Note that
\[
\frac{d}{dt} A_2(t) = \frac{d}{dt} \mathcal{L}_2(t) = \int \lambda \left[ \frac{\phi^2_t(x) \phi^2_t(y) a^*_x a_y - \int \int \lambda \left[ \frac{\varphi^2_t(x) \phi^2_t(y) a^*_y a_x}{|x - y| + \alpha_N} - \int \int \lambda \left[ \frac{\varphi^2_t(x) \phi^2_t(y) a^*_y a_x}{|x - y| + \alpha_N} + \text{h.c.} \right] \right] \right] dx dy \tag{8.2}
\]
where h.c. denotes the Hermitian conjugate and \( \varphi^2_t = \partial_t \phi^2_t \). Recall that \( \phi^2_t \) is the solution of \( \Box \).

Since \( L \) Lemma 6.1 of \[3\] shows that
\[
\frac{1}{\alpha_N^2} \frac{d}{dt} \int \int |\phi_t||^2 dx \leq C N^2 \| \phi_t \|_{H^1}^2 < \infty, \tag{8.3}
\]
we find that \( \phi^2_t \in L^2(\mathbb{R}^\alpha) \). Thus, for any \( \psi \in \mathcal{F} \),
\[
\left| \left\langle \psi, \int \int \lambda \left[ \frac{\phi^2_t(x) \phi^2_t(y) a^*_x a_y}{|x - y| + \alpha_N} - \int \int \lambda \left[ \frac{\varphi^2_t(x) \phi^2_t(y) a^*_y a_x}{|x - y| + \alpha_N} + \text{h.c.} \right] \right] dx dy \right\rangle \right| \leq \int \int \int \int |\phi_t||^2 dx \leq C N^2 \| \phi_t \|_{H^1}^2 \| \psi \| \tag{8.4}
\]
Other terms in the right hand side of \( \Box \) can be bounded similarly. Thus, we find that
\[
\frac{d}{dt} A_2(t) \leq C (N^2 + 1). \tag{8.6}
\]

Lemma 6.1 of \[3\] shows that \( -C (N^2 + 1) \leq \mathcal{L}_2(t) - \mathcal{K} \leq C (N^2 + 1) \) for some constant \( C > 0 \). Moreover, for any \( \psi \in \mathcal{F} \),
\[
|\langle \psi, \mathcal{L}_4 \psi \rangle| = \left| \left\langle \psi, \frac{\lambda}{2 \alpha_N^2} \int \int \lambda \left[ \frac{\phi_t^2(x) \phi_t^2(y) a^*_x a^*_y a_y a_x}{|x - y| + \alpha_N} \right] dx dy \right\rangle \right| \leq C N^{-1} \alpha_N^{-1} \langle \psi, N^2 \psi \rangle \tag{8.7}
\]
hence \( \mathcal{L}_4 \leq C N^{-1} \alpha_N^{-1} N^2 \). In summary, we showed that there exist constants \( C', K' \geq 0 \) such that
\[
\frac{d}{dt} A_2(t) \leq K'(N^2 + 1) \leq K'(\mathcal{L}_2(t) + \mathcal{L}_4 + C'(N^2 + 1)) = K' A_2(t), \tag{8.8}
\]
which was to be proved. \( \square \)

The following lemma that shows the number of the particles in the state \( \tilde{\mathcal{U}}^*(t) \phi(f) \tilde{\mathcal{U}}(t) \Omega \) cannot be even.

**Lemma 8.2.** Let \( f \in L^2(\mathbb{R}^\alpha) \). Then, for any \( k = 0, 1, 2, \cdots \),
\[
P_{2k} \tilde{\mathcal{U}}^*(t) \phi(f) \tilde{\mathcal{U}}(t) \Omega = 0. \tag{8.9}
\]

Proof. We first show that the parity \((-1)^N\) and the operator \( \tilde{\mathcal{U}}(t) \) commute. We note that
\[
\frac{d}{dt} \left( \tilde{\mathcal{U}}^*(t)(-1)^N \tilde{\mathcal{U}}(t) \right) = \tilde{\mathcal{U}}^*(t) \left[ (-1)^N, (\mathcal{L}_2(t) + \mathcal{L}_4) \right] \tilde{\mathcal{U}}(t). \tag{8.10}
\]
Since \( (\mathcal{L}_2(t) + \mathcal{L}_4) \) and \((-1)^N\) commute, we have that
\[
\frac{d}{dt} \left( \tilde{\mathcal{U}}^*(t)(-1)^N \tilde{\mathcal{U}}(t) \right) = 0. \tag{8.11}
\]
We also know that $\tilde{U}(0) = I$, hence,
\[
\tilde{U}^*(t)(-1)^N \tilde{U}(t) = \tilde{U}^*(0)(-1)^N \tilde{U}(0) = (-1)^N.
\tag{8.12}
\]
Thus, $(-1)^N \tilde{U}(t) = \tilde{U}(t)(-1)^N$ for all $t$. Similarly, $\tilde{U}^*(t)$ and $(-1)^N$ also commute.

Since $\tilde{U}(t)$ and $\tilde{U}^*(t)$ commute with the parity $(-1)^N$, we have that for any non-negative integer $k$ and any $\eta \in F$,
\[
\langle \eta, P_{2k} \tilde{U}^*(t) a(f) \tilde{U}(t) \Omega \rangle = \langle \eta, P_{2k}(-1)^N \tilde{U}^*(t) a(f) \tilde{U}(t) \Omega \rangle = \langle \eta, P_{2k} \tilde{U}^*(t)(-1)^N a(f) \tilde{U}(t) \Omega \rangle
\]
\[
= \langle \eta, P_{2k} \tilde{U}^*(t) a(f)(-1)^N a(f) \tilde{U}(t) \Omega \rangle = \langle \eta, P_{2k} \tilde{U}^*(t)(-1)^N a(f) \tilde{U}(t)(-1)^N \Omega \rangle
\]
\[
= -\langle \eta, P_{2k} \tilde{U}^*(t) a(f) \tilde{U}(t) \Omega \rangle,
\tag{8.13}
\]
which shows that $P_{2k} \tilde{U}^*(t) a(f) \tilde{U}(t) \Omega = 0$. The proof for that $P_{2k} \tilde{U}^*(t) a^*(f) \tilde{U}(t) \Omega = 0$ is similar. Therefore, we get the desired lemma.

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