Hierarchical Locally Recoverable Codes

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Abstract

The traditional definition of Integrated Interleaved (II) codes generally assumes that the component nested codes are either Reed-Solomon (RS) or shortened Reed-Solomon codes. By taking general classes of codes, we present a recursive construction of Extended Integrated Interleaved (EII) codes into multiple layers, a problem that brought attention in literature for II codes. The multiple layer approach allows for a hierarchical scheme where each layer of the code provides for a different locality. In particular, we present the erasure-correcting capability of the new codes and we show that they are ideally suited as Locally Recoverable codes (LRC) due to their hierarchical locality and the small finite field required by the construction. Properties of the multiple layer EII codes, like their minimum distance and dimension, as well as their erasure decoding algorithms, parity-check matrices and performance analysis, are provided and illustrated with examples. Finally, we will observe that the parity-check matrices of high layer EII codes have low density.

Keywords: Erasure-correcting codes, product codes, Reed-Solomon (RS) codes, generalized concatenated codes, integrated interleaving, extended integrated interleaving, MDS codes, local and global parities, locally recoverable (LRC) codes.

I. INTRODUCTION

The construction of \( t \)-level Integrated Interleaved (II) \([5]\), \([11]\), \([17]\), \([22]\), \([24]\), \([26]\) and Extended Integrated Interleaved (EII) \([2]\), \([6]\) uses \( t \) nested codes \( C_i \) over a finite field \( GF(q) \) (for simplicity, in this paper we assume that the field \( GF(q) \) has characteristic 2, but the constructions are valid over fields of any characteristic). The idea is to divide \( mn \) symbols into \( m \) distinct codewords, each codeword having a certain correcting capability so they can be corrected locally. In addition, the \( m \) codewords share parity symbols that enhance the correction capability of the individual codewords. There is a vast literature on codes with such characteristics (see for example \([2]\) and references within). In particular, II and EII codes are connected to Generalized Concatenated codes \([7]\), \([27]\) and to Tensor Product codes \([13]\)–\([15]\), \([23]\). In \([5]\), \( t \)-level II codes were proposed as Locally Recoverable (LRC) codes \([10]\), \([19]\), \([21]\) by considering them as erasure-correcting codes. In general II codes are not optimal as LRC codes with respect to the minimum distance as the codes in \([21]\). However, II codes require a much smaller field and they are competitive when metrics different from the minimum distance are considered, like for example, the average number of erasures causing an uncorrectable pattern \([5]\).

Let us point out that LRC codes have important practical applications, for example, in the Windows Azure storage \([12]\) and in HDFS-Xorbas \([20]\).

In this paper, we present a novel definition of EII codes that generalizes previous definitions. The new definition does not require that the minimum distances of the nested codes are decreasing. Although the relaxing of this requirement seems like a small change, the consequences are profound. In effect, we will show how the new definition allows for a natural construction of multiple layer EII codes (a problem already treated in literature for II codes \([17]\), \([26]\)). This multiple layer construction allows for a natural hierarchy of localities in EII codes. The paper is structured as follows: in Section II we give the definition of EII codes, which is similar to the traditional definitions of II \([22]\), \([24]\) and EII \([2]\), \([6]\) codes. As opposed to previous definitions, no assumption is made with respect to the nested codes utilized in the construction. In effect, quite often, it is assumed that the nested codes are MDS and in particular, (shortened) Reed-Solomon codes or at least that the minimum distance of the nested codes is decreasing. By not making this assumption, we will show how to construct EII codes in a hierarchical way. We also present the fundamental properties of EII codes according to the new definition. In particular, we give the erasure-correcting capability of the codes, which we prove constructively by giving an efficient decoding algorithm and we illustrate it with examples. We also give the dimension and the minimum distance of the EII codes defined.

In Section III we define recursively \( \ell \)-layer EII codes by applying the definition of Section II. In particular, we notice that 2-layer EII codes correspond to traditional EII codes and that 3-layer \([26]\) and multiple layer \([17]\) II codes are special cases that were obtained using the so called connection matrices (related to the parity-check matrices) of the codes. We illustrate our recursive construction with several concrete examples.

In Section IV we give parity-check matrices for the EII codes defined in Section II and for the \( \ell \)-layer EII codes defined in Section III. The parity-check matrix of an \( \ell \)-layer EII code is giving recursively based on parity-check matrices of lower layer...
EII codes, and in particular, we give closed formulae for the parity-check matrices of 2 and 3-layer EII codes. We illustrate the construction by revisiting the examples of Section III and obtaining the corresponding parity-check matrices.

In Section V we present a new parameter, the average number of erasures to failure (ANETF). We argue that the ANETF is more important than the minimum distance of the code when the erasures occur one after the other. We run simulations computing the ANETF of codes having the same rate as the examples of 2, 3 and 4-layer codes presented in [26] and in [17]. We tabulate the results and show that in several cases, we obtain codes with better ANETF and minimum distance than those in [26] and in [17]. We also observe that by using the parity-check matrices at the decoding, we can often correct erasure patterns that exceed the erasure-correcting capability of the codes.

We end the paper by drawing some conclusions and ideas for future research in Section VII. In particular, we observe that the parity-check matrix of a high layer EII code has low density.

II. DEFINITION AND PROPERTIES OF EII CODES

We define $t$-level EII codes as follows:

**Definition 1.** Let $\{0\} = \mathcal{C}_t \subset \mathcal{C}_{t-1} \subset \mathcal{C}_{t-2} \subset \cdots \subset \mathcal{C}_0$ be a sequence of $t+1$ nested $[n, n-u, d_i]$ codes over $GF(q)$ for $0 \leq i \leq t-1$, $u_0 \geq 0$, $s_0, s_1, \ldots, s_t$ non-negative integers such that $1 \leq m = s_0 + s_1 + \cdots + s_{t-1} + s_t < q$ and let $\alpha$ be an element of order $O(\alpha) \geq m$ in $GF(q)$.

Define $\mathcal{C}$ as the code of length $(m)(n)$ over $GF(q)$, $m < q$, such that, if

$$s_i = \sum_{j=i}^{t} s_j \quad \text{for} \quad 0 \leq i \leq t, \quad (1)$$

for each $\mathcal{C} \in \mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{m-1})$, $\mathcal{C}_j \in \mathcal{C}_0$ for $0 \leq j \leq m-1$ and

$$\bigoplus_{j=0}^{m-1} \alpha^j \mathcal{C}_j \subset \mathcal{C}_i \quad \text{for} \quad 1 \leq i \leq t \quad \text{and} \quad 0 \leq r \leq s_i - 1. \quad (2)$$

Then we say that $\mathcal{C}$ is an EII code. If $s_i = 0$ we say that $\mathcal{C}$ is an II code. If

$$t' = |\{i : 0 \leq i \leq t-1 \quad \text{and} \quad s_i \neq 0\}|, \quad (3)$$

then we say that $\mathcal{C}$ is a $t'$-level EII code.

Definition 1 is slightly different to the ones traditionally given in literature. Most papers on $t$-level II [17], [24], [26] or EII codes [2] assume that the nested codes $\mathcal{C}_i$ in Definition 1 are Reed-Solomon (RS) or shortened RS codes [13]. In the original definitions of 2-level II [11] and $t$-level II [22] codes, it is not assumed that the nested codes are MDS, although it is required that their minimum distances are decreasing, i.e., $d_i < d_{i+1}$ for $0 \leq i \leq t-1$. There is no such assumption in Definition 1. This subtle difference, though, will be crucial for the construction of $\ell$-layer EII codes to be presented in Section III.

Another difference with traditional definitions is that generally, it is assumed that $s_i \geq 1$ for $0 \leq i \leq t-1$. Taking $s_i \geq 0$ is not fundamentally different, but it is convenient, since it allows that two different EII codes share the same nested codes, and this allows for an easy necessary and sufficient condition, based on the $s_i$s of each code, to determine if one of the codes is contained in the other one. This property will be important for the construction of parity-check matrices of $\ell$-layer EII codes to be given in Section IV.

Let us point out that in the literature on $t$-level II codes, a 2-level II code is often called an II code [11] while a $t$-level II code with $t > 2$ is called a Generalized Integrated Interleaved (GII) code [17], [22], [24], [26]. Since there is no conceptual difference between the cases $t = 2$ and $t > 2$, we prefer calling these codes simply $t$-level II codes.

The next theorem describes the erasure patterns that are guaranteed to be correct by $t$-level EII codes. The result is a generalization of Theorem 6 in [6], where the component codes are MDS codes.

**Theorem 2.** Consider an EII code $\mathcal{C}$ on $t+1$ nested codes $\mathcal{C}_i$ according to Definition 1. Let $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{m-1}) \in \mathcal{C}$ be a codeword with erasures. Assume that the vectors $\mathcal{C}_i$ can be divided into $t+1$ disjoint sets $S_i$, $0 \leq i \leq t$, such that $|S_0| \geq s_0$, $0 \leq |S_i| \leq s_i$ for $1 \leq i \leq t$ and the erasures in each $\mathcal{C}_i \in S_i$, if they occur in a codeword of code $\mathcal{C}_i$, are correctable in $\mathcal{C}_i$. Then, all the erasures in $\mathcal{C}$ can be corrected.

**Proof:** If $\mathcal{C}_i \in S_0$, since, by Definition 1, $\mathcal{C}_i \in \mathcal{C}_0$, the erasures in $\mathcal{C}_i$ can be corrected. So, we may assume that the $\mathcal{C}_i$s with erasures are not correctable in $\mathcal{C}_0$, hence, they are not in $S_0$.

Assume that there are $\ell = |S_1| + |S_2| + \cdots + |S_t|$ $\mathcal{C}_i$s that are uncorrectable in $\mathcal{C}_0$ and the erasures in any $\mathcal{C}_i \in S_i$, $1 \leq i \leq t$, are correctable in $\mathcal{C}_i$ when they occur in a codeword of $\mathcal{C}_i$. We do induction on $\ell$. 

If \( \ell = 0 \), there are no erasures and all the sets \( S_i \) are empty. Assume that \( \ell \geq 1 \) and the erasures in any \( \mathcal{C}_j \in S_i \) are correctable in \( \mathcal{C}_j \) when they occur in a codeword of \( \mathcal{C}_j \). In particular, \( \ell \leq s_1 + s_2 + \ldots + s_t = s_J \).

Let \( i_0, i_1, \ldots, i_{m-1} \) be an ordering of the \( \mathcal{C}_j \)s such that:

1) If the erasures in \( \mathcal{C}_{i_{t-1}} \) are correctable in \( \mathcal{C}_w \) when they occur in a codeword of code \( \mathcal{C}_w \), \( 1 \leq w \leq t \), but not in \( \mathcal{C}_{w-1} \) when they occur in a codeword of code \( \mathcal{C}_{w-1} \), then the erasures in \( \mathcal{C}_j \) for \( 0 \leq j \leq \ell - 2 \) are not correctable either in \( \mathcal{C}_{w-1} \) when they occur in a codeword of code \( \mathcal{C}_{w-1} \).

2) Vectors \( \mathcal{C}_{i_{t+1}}, \ldots, \mathcal{C}_{i_{w-1}} \) have no erasures.

In particular, by 1) and 2) in the ordering of the \( \mathcal{C}_j \)s, \( |S_i| = 0 \) for \( 1 \leq i \leq w - 1 \) and hence

\[
\ell = |S_w| + |S_{w+1}| + \cdots + |S_t| \leq s_w + s_{w+1} + \cdots + s_t = s_J.
\] (4)

Rearranging the order of the elements of the sums in (2), by (4), we have

\[
\bigoplus_{j=0}^{m-1} \alpha^{r_j} \mathcal{C}_j \in \mathcal{C}_w \text{ for } 0 \leq r \leq \ell - 1 \leq s_J - 1.
\] (5)

Since the \( \ell \times m \) matrix corresponding to the coefficients of the \( \mathcal{C}_j \)s in (5) is a Vandermonde type of matrix and \( O(a) \geq m \), this matrix can be triangulated. Taking the last row of the triangulation, we obtain

\[
\mathcal{C}'_{i_{t-1}} = \mathcal{C}_{i_{t-1}} \oplus \left( \bigoplus_{j=\ell}^{m-1} \gamma_j \mathcal{C}_j \right) \in \mathcal{C}_w,
\] (6)

where the coefficients \( \gamma_j \) are obtained from the triangulation. Since \( \mathcal{C}_j \) is erasure free for \( \ell \leq j \leq m-1 \), then, by (6), the erasures of \( \mathcal{C}_{i_{t-1}} \) and of \( \mathcal{C}_{i_{t-1}} \) occur in the same locations. Since \( \mathcal{C}'_{i_{t-1}} \) is in \( \mathcal{C}_w \) by (6), by condition 1) of the ordering of the \( \mathcal{C}_j \)s, the erasures can be corrected. Once \( \mathcal{C}'_{i_{t-1}} \) is corrected, again by (6), \( \mathcal{C}_{i_{t-1}} \) is obtained as

\[
\mathcal{C}_{i_{t-1}} = \mathcal{C}'_{i_{t-1}} \oplus \left( \bigoplus_{j=\ell}^{m-1} \gamma_j \mathcal{C}_j \right).
\]

Now we have \( \ell - 1 \) \( \mathcal{C}_j \)s with erasures. Redefining \( S_0 = S_0 \cup \{i_{t-1}\} \) and \( S_w = S_w - \{i_{t-1}\} \), the \( m \) \( \mathcal{C}_j \)s can be divided into \( t + 1 \) disjoint sets \( S_i \), \( 0 \leq i \leq t \), such that \( |S_0| = s_0 \), \( 0 \leq |S_i| \leq s_i \) for \( 1 \leq i \leq t \) and the erasures in \( \mathcal{C}_j \in S_i \) if they occur in a codeword of code \( \mathcal{C}_i \), are correctable in \( \mathcal{C}_j \). By induction, the \( \ell - 1 \) \( \mathcal{C}_j \)s with erasures can be corrected.

\( \square \)

**Corollary 3.** Consider an EII code \( \mathcal{C} \) on \( t + 1 \) nested codes \( \mathcal{C}_i \) as in Definition 1. Let \( \mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{m-1}) \in \mathcal{C} \) be a codeword with erasures. Assume that the \( m \) \( \mathcal{C}_j \)s can be divided into \( t + 1 \) disjoint sets \( S_i \), \( 0 \leq i \leq t \), such that \( |S_0| = s_0 \), \( 0 \leq |S_i| \leq s_i \) for \( 1 \leq i \leq t \) and each \( \mathcal{C}_j \in S_i \) has up to \( d_i - 1 \) erasures. Then, all the erasures in \( \mathcal{C} \) can be corrected. In particular, if the codes \( \mathcal{C}_i \) are \([n, n - u_i, u_i + 1]\) MDS codes, the code can correct up to \( u_i \) erasures in any \( \mathcal{C}_i \in S_i \), where \( 0 \leq i \leq t \).

**Proof:** Simply observe that any \( d_i - 1 \) erasures can be corrected in any codeword of code \( \mathcal{C}_i \) for \( 0 \leq i \leq t \) and the result follows from Theorem 2. \( \square \)

The MDS case in Corollary 3 corresponds to Theorem 6 in [6].

Given an EII code \( \mathcal{C} \) as in Definition 1, it is often convenient to visualize each codeword \( \mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{m-1}) \in \mathcal{C} \) as an \( m \times n \) array where the \( \mathcal{C}_j \)s are the rows of the array. We will use indistinctly the vector and the array description of a codeword in the next examples.

The proof of Theorem 2 provides a recursive erasure decoding algorithm for \( t \)-level EII codes, which we illustrate in Example 4.

**Example 4.** Assume that \( \{4\} = C_4 \subset C_3 \subset C_2 \subset C_1 \subset C_0 \) are RS codes over GF(8) such that \( C_3 \) is a [7, 2] code, \( C_2 \) is a [7, 3] code, \( C_1 \) is a [7, 5] code and \( C_0 \) is a [7, 6] code. Consider the 4-level EII code \( \mathcal{C} \) with \( m = 7 \) and \( s_0 = 2 \), \( s_1 = s_2 = 1 \), \( s_3 = 2 \) and \( s_4 = 1 \) according to Definition 1.

Assume that the following \( 7 \times 7 \) array is received, where the entries with \( E \) are erased and the blank entries are correct:
Dividing the rows in disjoint sets as in Theorem 2, we have, $S_0 = \{2,5\}$, $S_1 = \{6\}$, $S_2 = \{3\}$, $S_3 = \{0,4\}$ and $S_4 = \{1\}$. Since $|S_0| \geq s_0$, $|S_1| = |S_2| = 1 \leq s_1 = s_2$, $|S_3| = 2 \leq s_3$ and $|S_4| = 1 \leq s_4$, according to Corollary 5 the erasures are correctable. Notice that $\ell = |S_1| + |S_2| + |S_3| + |S_4| = 5$.

The first step of the decoding algorithm is correcting the rows with one erasure, i.e., rows $\mathcal{C}_2$ and $\mathcal{C}_5$, so we may assume that these two rows are erasure-free. Next we reorder the rows according to the order given in the proof of Theorem 2, which in this case would correspond to a non-increasing number of erasures. This gives, $i_0 = 1$, $i_1 = 0$, $i_2 = 4$, $i_3 = 3$, $i_4 = 6$, $i_5 = 2$ and $i_6 = 5$.

Rearranging the order of the elements of the sums in $\mathcal{C}$ as in $\mathcal{S}$, since $w$, as defined in Theorem 2 is equal to 1 and $\alpha$ is primitive in $GF(8)$, thus $\alpha^2 = 1$, we obtain

\[
\begin{align*}
\alpha^4 \mathcal{C}_4 + \alpha^2 \mathcal{C}_0 + \alpha^5 \mathcal{C}_2 + \alpha^3 \mathcal{C}_6 + \alpha^4 \mathcal{C}_5 + \alpha^6 \mathcal{C}_5 & \in C_4 = \{0\} \\
\alpha^3 \mathcal{C}_0 + \alpha^5 \mathcal{C}_4 + \alpha^2 \mathcal{C}_0 + \alpha^4 \mathcal{C}_6 + \alpha^6 \mathcal{C}_2 + \alpha^5 \mathcal{C}_5 & \in C_3 \\
\alpha^2 \mathcal{C}_4 + \alpha^5 \mathcal{C}_0 + \alpha^6 \mathcal{C}_3 + \alpha^4 \mathcal{C}_6 + \alpha^5 \mathcal{C}_6 + \alpha^4 \mathcal{C}_5 + \alpha^3 \mathcal{C}_5 & \in C_3 \\
\alpha^4 \mathcal{C}_0 + \alpha^5 \mathcal{C}_0 + \alpha^4 \mathcal{C}_3 + \alpha^4 \mathcal{C}_6 + \alpha^5 \mathcal{C}_6 + \alpha^5 \mathcal{C}_5 & \in C_2 \\
\alpha^4 \mathcal{C}_1 + \alpha^5 \mathcal{C}_4 + \alpha^5 \mathcal{C}_4 + \alpha^3 \mathcal{C}_6 + \alpha^3 \mathcal{C}_5 + \alpha^5 \mathcal{C}_5 & \in C_1
\end{align*}
\]

Triangulating above and assuming $\alpha^3 = 1 + \alpha$, we obtain

\[
\begin{align*}
\mathcal{C}_1 + \alpha^3 \mathcal{C}_0 + \alpha^5 \mathcal{C}_4 + \alpha^3 \mathcal{C}_6 + \alpha^6 \mathcal{C}_2 + \alpha^4 \mathcal{C}_5 + \alpha^2 \mathcal{C}_5 & = 0 \\
\mathcal{C}_0 + \alpha^4 \mathcal{C}_4 + \alpha^6 \mathcal{C}_4 + \alpha^5 \mathcal{C}_6 + \alpha^2 \mathcal{C}_2 + \alpha^4 \mathcal{C}_5 & \in C_3 \\
\mathcal{C}_4 + \alpha^6 \mathcal{C}_3 + \alpha^4 \mathcal{C}_6 + \alpha^6 \mathcal{C}_2 + \alpha^5 \mathcal{C}_5 & \in C_3 \\
\mathcal{C}_5 + \alpha^6 \mathcal{C}_0 + \alpha^2 \mathcal{C}_2 + \alpha^5 \mathcal{C}_5 & \in C_2 \\
\mathcal{C}_6 + \alpha^2 \mathcal{C}_2 + \alpha^5 \mathcal{C}_5 & \in C_1
\end{align*}
\]

Since $\mathcal{C}_2$ and $\mathcal{C}_5$ are erasure free, the two erasures of $\mathcal{C}_4$ and of $\mathcal{C}_0 = \mathcal{C}_6 + \alpha^2 \mathcal{C}_2 + \alpha^5 \mathcal{C}_5$ occur in the same locations. Since $\mathcal{C}_4 \in C_1$ and $C_1$ is a $[7,5,3]$ code, these two erasures in $\mathcal{C}_4$ can be corrected. Once the erasures are corrected, we obtain $\mathcal{C}_6$ as $\mathcal{C}_6 = \mathcal{C}_4 + \alpha^2 \mathcal{C}_2 + \alpha^5 \mathcal{C}_5$.

Similarly, $\mathcal{C}_0 = \mathcal{C}_3 + \alpha^6 \mathcal{C}_4 + \alpha^3 \mathcal{C}_2 + \alpha^3 \mathcal{C}_5$ and $\mathcal{C}_3$ have both 4 erasures in the same locations. Since $\mathcal{C}_3 \in C_2$ and $C_2$ is a $[7,3,5]$ code, the erasures in $\mathcal{C}_3$ are corrected, and then $\mathcal{C}_3 = \mathcal{C}_3 + \alpha^6 \mathcal{C}_4 + \alpha^3 \mathcal{C}_2 + \alpha^5 \mathcal{C}_5$.

Next, since $\mathcal{C}_4 = \mathcal{C}_4 + \alpha^6 \mathcal{C}_3 + \alpha^4 \mathcal{C}_6 + \alpha^3 \mathcal{C}_5$ and $\mathcal{C}_4$ have both 5 erasures in the same locations and $\mathcal{C}_4 \in C_3$, which is a $[7,2,6]$ code, the 5 erasures in $\mathcal{C}_4$ can be corrected and $\mathcal{C}_4 = \mathcal{C}_4 + \alpha^6 \mathcal{C}_3 + \alpha^4 \mathcal{C}_6 + \alpha^6 \mathcal{C}_2 + \alpha^5 \mathcal{C}_5$.

Similarly, since $\mathcal{C}_0 = \mathcal{C}_0 + \alpha^4 \mathcal{C}_4 + \alpha^6 \mathcal{C}_3 + \alpha^6 \mathcal{C}_2 + \alpha^4 \mathcal{C}_5$, both $\mathcal{C}_0$ and $\mathcal{C}_0$ have 5 erasures in the same locations and $\mathcal{C}_0 \in C_3$. Correcting the 5 erasures in $\mathcal{C}_0$, then $\mathcal{C}_0 = \mathcal{C}_0 + \alpha^4 \mathcal{C}_4 + \alpha^6 \mathcal{C}_3 + \alpha^6 \mathcal{C}_2 + \alpha^4 \mathcal{C}_5$.

Finally, $\mathcal{C}_1$ is obtained as $\mathcal{C}_1 = \mathcal{C}_1 + \alpha^3 \mathcal{C}_0 + \alpha^5 \mathcal{C}_4 + \alpha^3 \mathcal{C}_6 + \alpha^6 \mathcal{C}_6 + \alpha^4 \mathcal{C}_2 + \alpha^2 \mathcal{C}_5$, completing the decoding.

In particular, the encoding is a special case of the decoding. In effect, without loss of generality, we may assume that each of the codes $C_i$, $0 \leq i \leq t$, in Definition 1 admits a systematic encoder such that the first $n - u_i$ symbols in a codeword contain data while the last $u_i$ symbols contain parity. If we view the $u_i$ parity symbols as erasures, then such erasures are correctable by $C_i$. Say, we take an $m \times n$ array such that in the first $s_0$ rows the first $n - u_0$ symbols in each row contain data, in the next $s_1$ rows the first $n - u_1$ symbols in each row contain data, and so on, until the last $s_t$ rows, in which all the symbols contain parity. According to Theorem 2 the erasures can be solved, so the recursive algorithm in the theorem can be used as an encoder. The fact that at the encoding the locations of the erasures are known allows for a simplification of the decoding algorithm. For example, the triangulated matrix in the proof of Theorem 2 may be precomputed. The next example illustrates the encoding algorithm.
Example 5. Let us retake the 4-level EII code of Example 4 and according to the description above, for the encoding we can place the parity and data as follows:

| \( C_0 \) | D | D | D | D | D | P |
|---|---|---|---|---|---|---|
| \( C_1 \) | D | D | D | D | D | D | P |
| \( C_2 \) | D | D | D | D | D | P | P |
| \( C_3 \) | D | D | D | D | P | P | P |
| \( C_4 \) | D | D | P | P | P | P | P |
| \( C_5 \) | D | P | P | P | P | P | P |
| \( C_6 \) | P | P | P | P | P | P | P |

where \( D \) denotes data and \( P \) denotes parity. As stated, we may consider the parities as erasures and apply the decoding algorithm to them.

The disjoint sets as in Theorem 2 are then \( S_0 = \{0, 1\} \), \( S_1 = \{2\} \), \( S_2 = \{3\} \), \( S_3 = \{4, 5\} \) and \( S_4 = \{6\} \).

As in Example 4, we first correct the rows with one erasure, in this case, rows \( C_0 \) and \( C_1 \), so after doing so, these two rows are erasure-free. The order of the rows given in the proof of Theorem 2 is then \( i_j = 6 - j \) for \( 0 \leq j \leq 6 \). Then we have,

\[
\begin{align*}
\alpha^3 c_5 + \alpha^5 c_5 + \alpha^2 c_2 + \alpha^3 c_1 + c_0 &= 0 \\
\alpha^4 c_5 + \alpha^5 c_5 + \alpha^2 c_2 + \alpha^3 c_1 + c_0 &\in C_3 \\
\alpha^5 c_5 + \alpha^3 c_5 + \alpha^2 c_2 + \alpha^3 c_1 + c_0 &\in C_3 \\
\alpha^6 c_5 + \alpha^3 c_5 + \alpha^2 c_2 + \alpha^3 c_1 + c_0 &\in C_2 \\
\alpha_6 + \alpha^5 c_5 + \alpha^2 c_2 + \alpha^3 c_1 + c_0 &\in C_1
\end{align*}
\]

Triangulating above and assuming \( \alpha^3 = 1 + \alpha \), we obtain

\[
\begin{align*}
\alpha^6 c_5 + \alpha^5 c_5 + \alpha^3 c_5 + \alpha^2 c_2 + \alpha^3 c_1 + c_0 &= 0 \\
\alpha^5 c_5 + \alpha^4 c_5 + \alpha^3 c_5 + \alpha^2 c_2 + \alpha^3 c_1 + c_0 &\in C_3 \\
\alpha^4 c_5 + \alpha^3 c_5 + \alpha^2 c_2 + \alpha^3 c_1 + c_0 &\in C_3 \\
\alpha^3 c_5 + \alpha^2 c_2 + \alpha^3 c_1 + c_0 &\in C_2 \\
\alpha^4 c_5 + \alpha^3 c_5 + c_0 &\in C_1
\end{align*}
\]

Since at the encoding we know the location of the erasures, this triangulated matrix can be precomputed. We then obtain successively \( C_4, C_3, C_2, C_1 \) and \( C_0 \) as in Example 4, completing the encoding.

Since the \( mn - \sum_{i=0}^{t} s_i u_i \) data symbols at the encoding are completely arbitrary, we have the following theorem:

**Theorem 6.** Consider an EII code \( C \) as given by Definition 1. Then, \( C \) is an \([ (m)(n), k] \) code, where

\[
k = (m)(n) - \left( \sum_{i=0}^{t} s_i u_i \right). \tag{7}
\]

**Theorem 6** coincides with Theorem 12 in [3], the only difference being that the nested codes in [6] are RS or shortened RS type of codes, while no such limitation is required in Theorem 6.

**Example 7.** Let \( C \) be the 4-level EII code of Example 4. Then, \( C \) is a \([ 49, k ] \) code where, according to (7),

\[
k = 49 - (2)(1) - (1)(2) - (1)(3) - (2)(5) - (1)(7) = 25.
\]

The next theorem gives the minimum distance of an EII code.

**Theorem 8.** Consider an EII code \( C \) as given by Definition 1. Then,
\[ d = \min \{ d_j (\hat{s}_{j+1} + 1) \text{ for } 0 \leq j \leq t - 1 \} \quad (8) \]

**Proof:** Take \( j \) such that \( 0 \leq j \leq t - 1 \). We prove that there is a codeword of weight \( d_j (\hat{s}_{j+1} + 1) \).

Since \( C_j \) is an \([n, n-u_j, d_j]\) code, there is a codeword \( \bar{w} \in C_j \) of weight \( d_j \).

Consider the polynomial \( \bar{v}(x) = (x \oplus 1)(x \oplus \alpha) \cdots (x \oplus \alpha^{\hat{s}_{j+1} - 1}) = \bar{v}_0 + \bar{v}_1 x + \cdots + \bar{v}_{\hat{s}_{j+1}} x^{\hat{s}_{j+1}} \). In particular, \( \bar{v}_s \neq 0 \) for \( 0 \leq s \leq \hat{s}_{j+1} \) and

\[ \bar{v}(\alpha^r) = \sum_{s=0}^{\hat{s}_{j+1}} \alpha^{rs} \bar{v}_s = 0 \text{ for } 0 \leq r \leq \hat{s}_{j+1} - 1. \quad (9) \]

Take a vector \( \bar{c} = (\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_{m-1}) \) such that \( \bar{c}_0 = \bar{v}_s \bar{w} \) for \( 0 \leq s \leq \hat{s}_{j+1} \) and \( \bar{c}_s = 0 \) for \( \hat{s}_{j+1} + 1 \leq s \leq m - 1 \). In particular, \( \bar{c} \) has weight \( d_j (\hat{s}_{j+1} + 1) \) and we will show that \( \bar{c} \in \bar{C} \). Since \( \bar{c}_s \in \bar{C}_j \) by design and \( \bar{C}_j \subseteq \bar{C}_0 \), in particular, \( \bar{c}_s \in \bar{C}_0 \) for \( 0 \leq s \leq m - 1 \). According to (2), we also have to show that

\[ \sum_{s=0}^{m-1} \alpha^{rs} \bar{c}_s = \sum_{s=0}^{\hat{s}_{j+1}} \alpha^{rs} (\bar{v}_s \bar{w}) \subseteq \bar{C}_i \text{ for } 1 \leq i \leq t \text{ and } 0 \leq r \leq \hat{s}_i - 1. \quad (10) \]

Take \( i \) such that \( 1 \leq i \leq t \) and \( r \) such that \( 0 \leq r \leq \hat{s}_i - 1 \). Assume first that \( j < i \), then \( j + 1 \leq i \) and, by (1), \( \hat{s}_{j+1} \geq \hat{s}_i \), so, in particular, \( 0 \leq r \leq \hat{s}_{j+1} - 1 \). Then, by (9),

\[ \sum_{s=0}^{\hat{s}_{j+1}} \alpha^{rs} (\bar{v}_s \bar{w}) = \left( \sum_{s=0}^{\hat{s}_{j+1}} \alpha^{rs} \bar{v}_s \right) \bar{w} = 0 \]

and in particular, (10) holds.

Assume next that \( j \geq i \), then \( C_j \subseteq \bar{C}_i \). Hence, since \( \bar{w} \in \bar{C}_j \), also \( \bar{w} \in \bar{C}_i \) and (10) holds, so \( d \leq d_j (\hat{s}_{j+1} + 1) \). In particular, \( d \leq \min \{ d_j (\hat{s}_{j+1} + 1) \text{ for } 0 \leq j \leq t - 1 \} \).

We prove the other inequality next. Assume that \( d < \min \{ d_j (\hat{s}_{j+1} + 1) \text{ for } 0 \leq j \leq t - 1 \} \) and take a codeword of weight \( d \in C = (\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_{m-1}) \in \bar{C} \). Assume that the non-zero entries of \( \bar{c} \) are erased. Let \( \bar{c}_0, \bar{c}_1, \ldots, \bar{c}_{\ell-1}, \bar{c}_{\ell+1}, \ldots, \bar{c}_{m-1} \) be the vectors of \( \bar{c} \) ordered in non-increasing weighting order and assume that vectors \( \bar{c}_0 \) to \( \bar{c}_{\ell-1} \) have non-zero weight, i.e., if \( \bar{c}_{\ell} \) has weight \( w_{\ell} \), then \( \sum_{s=0}^{\ell-1} w_s = d \) and \( w_0 \geq w_1 \geq \cdots \geq w_{\ell-1} \). If \( w_{\ell-1} < d_0 \), vector \( \bar{c}_{\ell-1} \) would be corrected in \( \bar{C}_0 \) as the zero vector, contradicting that \( \bar{w}_{\ell-1} \neq 0 \). If \( w_{\ell-1} = d_{\ell-1} \), vector \( \bar{c}_{\ell-1} \) would be corrected as the zero array by Theorem 2. Hence, we can define \( i, 1 \leq i \leq t - 1 \), such that \( \hat{s}_i < \ell \leq \hat{s}_i \). Assume that \( w_{\ell-1} \geq d_i \). Then,

\[ d = \sum_{s=0}^{t-1} w_{\ell-1}^s \geq w_{\ell-1} \ell \geq d_i (\hat{s}_{i+1} + 1), \]

contradicting the assumption that \( d < \min \{ d_j (\hat{s}_{j+1} + 1) \text{ for } 0 \leq j \leq t - 1 \} \). Then, \( w_{\ell-1} < d_i \). Since \( \bar{c} \in \bar{C} \), \( \bar{c}_i = 0 \) for \( \ell \leq s \leq m - 1 \) and \( \ell \leq \hat{s}_i \), rearranging the order of the elements of the sums in (2), we obtain

\[ \sum_{s=0}^{m-1} \alpha^{ri_s} \bar{c}_s = \sum_{s=0}^{\ell-1} \alpha^{ri_s} \bar{c}_s \subseteq \bar{C}_i \text{ for } 0 \leq r \leq \ell - 1. \quad (11) \]

Since the \( \ell \times \ell \) matrix corresponding to the coefficients of the \( \bar{c}_i \)s in (11) is a Vandermonde type of matrix and \( O(\alpha) \geq m \), this matrix can be triangulated and \( \bar{c}_{i,\ell-1} \in \bar{C}_i \). Since \( \bar{C}_i \) has minimum distance \( d_i \) and \( \bar{c}_{i,\ell-1} \) has weight \( w_{\ell-1} < d_i \), then \( \bar{c}_{i,\ell-1} = 0 \), a contradiction. \( \square \)

The following corollary corresponds to Theorem 15 in [6].

**Corollary 9.** Consider a \( t \)-level EII code \( \bar{C} \) as given by Definition 1 such that, for \( 0 \leq j \leq t - 1 \), code \( C_j \) is an \([n, n-u_j, u_j + 1]\) MDS code. Then, the minimum distance of \( \bar{C} \) is

\[ d = \min \{ (u_j + 1) (\hat{s}_{j+1} + 1) \text{ for } 0 \leq j \leq t - 1 \}. \quad (12) \]

**Proof:** Simply notice that \( d_j = u_j + 1 \) for \( 0 \leq j \leq t - 1 \) and (3) gives (12). \( \square \)
Example 10. Let \( C \) be again the 4-level EII code of Example 4. Then, according to (12),

\[
d = \min \{ (2)(6), (3)(5), (5)(4), (6)(2) \} = 12.
\]

We end this section with a lemma providing necessary and sufficient conditions to determine whether, given two EII codes sharing the same nested codes, one of them is contained in the other.

Lemma 11. Let \( C \) and \( C' \) be two EII codes with the same nested nested codes \( \{ 0 \} = C_t \subset C_{t-1} \subset C_{t-2} \subset \cdots \subset C_0 \) and non-negative coefficients \( s_i \) and \( s'_i \) respectively for \( 0 \leq i \leq t \) according to Definition 1. Then, \( C' \subseteq C \) if and only if \( \hat{s}_i \leq s'_i \) for \( 0 \leq i \leq t \).

Proof: Let \( \xi = (c_0, c_1, \ldots, c_{m-1}) \in C' \). We have to prove that \( \xi \in C \) if and only if \( \hat{s}_i \leq s'_i \) for \( 0 \leq i \leq t \).

Since \( \xi \in C' \), by Definition 1, \( c_j \in C_0 \) for \( 0 \leq j \leq m-1 \) and, by (2),

\[
\bigoplus_{j=0}^{m-1} \alpha^{r_j} c_j^{(\ell-1)} \in C_i \quad \text{for} \quad 1 \leq i \leq t \quad \text{and} \quad 0 \leq r \leq s'_i - 1.
\]

Then, \( \xi \in C \) if and only if (2) holds, which, by (13), will be the case if and only if \( \hat{s}_i \leq s'_i \) for \( 0 \leq i \leq t \). \( \square \)

Lemma 11 will be useful when constructing the parity-check matrices of \( \ell \)-layer EII codes to be presented in Section IV.

III. RECURSIVE CONSTRUCTION OF \( \ell \)-LAYER EII CODES

The concept of 3-layer II codes is presented in [27] and its generalization to multi-layer II codes in [17]. Next we are going to show that these concepts arise naturally by applying recursively Definition 1 of EII codes, as shown in the next definition. We also automatically obtain the properties of \( \ell \)-layer EII codes discussed in Section II like their dimension and minimum distance.

Definition 12. We say that \( C^{(1)} \) is a 1-layer EII code if it is an \([n, n-u, u+1]\) code over \( GF(q) \). Assuming that \( \ell \)-layer EII codes of length \((m')^n\) over \( GF(q) \) have been defined for \( \ell \geq 1 \), where \( m_0 = 1 \) and \( m^\ell = (m_{\ell-1}) (m_{\ell-2}) \cdots (m_1) (m_0) \), let \( \{ 0 \} = C^{(1)}_t \subset C^{(1)}_{t-1} \subset C^{(1)}_{t-2} \subset \cdots \subset C^{(1)}_0 \) be a sequence of \( t+1 \) nested \( \ell \)-layer EII codes, \( s_0 + s_1 + \cdots + s_t = m_\ell \), \( s_i \geq 0 \) for \( 0 \leq i \leq t \), and \( \alpha \in GF(q) \) such that \( O(\alpha) \geq m_\ell \). Then, we say that \( C^{(\ell+1)} \) is an \((\ell+1)\)-layer EII code of length \((m')^n\) if \( C^{(\ell+1)} \) is an EII code over the nested \( \ell \)-layer codes \( C^{(\ell)}_i \) according to Definition 11.

If \( s_t = 0 \), we say that \( C^{(\ell+1)} \) is an \((\ell+1)\)-layer II code.

Comparing Definitions 1 and 12, we see that an \((\ell+1)\)-layer code is an EII code such that the nested codes are \( \ell \)-layer EII codes. Hence, a 1-layer EII code is simply an MDS code while a 2-layer EII code corresponds to the EII code of Definition 1 such that the nested codes are MDS codes (this assumption is made in most papers on II codes [4, 5, 17, 24, 26]).

Let us point out also that although not required in Definition 12, it is convenient to use a unique element \( \alpha \) in all the layers by requiring \( O(\alpha) \geq \max \{ m_1, m_2, \ldots, m_t \} \).

Next we define recursively the erasure-correcting capability of \( \ell \)-layer EII codes by using a vector \( u \).

Definition 13. If \( C^{(1)} \) is a 1-layer \([n, n-u, u+1]\) EII code, we say that the erasure-correcting capability of \( C^{(1)} \) is the vector of length 1 \( u^{(1)} = (u) \). Let \( \ell \geq 1 \) and consider an \((\ell+1)\)-layer EII code \( C^{(\ell+1)} \) as given by Definition 12. Let the erasure-correcting capability of the \( \ell \)-layer nested EII code \( C^{(\ell)}_i \), \( 0 \leq i \leq t-1 \), be given by a vector \( u^{(\ell)}_i \) of length \((m_{\ell-1})(m_{\ell-2}) \cdots (m_1)(m_0) \), where \( m_0 = 1 \). Then, we denote the erasure-correcting capability of \( C^{(\ell+1)} \) by the vector of length \((m_{\ell})(m_{\ell-1}) \cdots (m_1)(m_0) \)

\[
u^{(\ell+1)} = \left( u^{(\ell)}_0, u^{(\ell)}_1, \ldots, u^{(\ell)}_{t-1}, u^{(\ell)}_{t-1}, u^{(\ell)}_1, \ldots, u^{(\ell)}_1, \ldots, u^{(\ell)}_0, u^{(\ell)}_1, \ldots, u^{(\ell)}_{t-1}, u^{(\ell)}_{t-1} \right).
\]
We illustrate Definitions 12 and 13 in the next examples.

**Example 14.** Let \( n = 7, m_1 = 6, c^{(1)}_i = [7, 7 - i - 1, i + 2] \) MDS code over GF(8) for \( 0 \leq i \leq 6 \) and \( \alpha \) a primitive element in GF(8). Let \( C^{(2)}_0 \) and \( C^{(1)}_1 \) be the two 2-layer 2-level II codes with nested codes \( \{0,7\} = C^{(1)}_2 \subset C^{(1)}_1 \subset C^{(1)}_0 \). Denoting by \( s_{ij} \) the elements of code \( C(2), 0 \leq i \leq 1 \), according to Definition 12 assume that \( s_{0,0} = 5, s_{0,1} = 1, s_{0,2} = 0, s_{1,0} = 4, s_{1,1} = 2 \) and \( s_{0,2} = 0 \). By Lemma 11 \( C^{(2)}_1 \subset C^{(2)}_0 \). By Theorem 2 both codes can correct any of the 6 rows with one erasure, and in addition, \( C^{(2)}_0 \) can correct up to one row with two erasures, while \( C^{(2)}_1 \) can correct any pair of rows with two erasures each.

By (14) in Definition 13 the erasure-correcting capability of \( C^{(2)}_0 \) is \( u^{(2)} = 1, 1, 1, 1, 1, 2 \) and the erasure-correcting capability of \( C^{(1)}_1 \) is \( u^{(1)} = 1, 1, 1, 1, 2, 2 \). By Theorem 5 and Corollary 9 \( C^{(2)}_0 \) is a \([42, 35, 3]\) code and \( C^{(2)}_1 \) is a \([42, 34, 3]\) code.

Since \( \{0, 42\} = C^{(2)}_2 \subset C^{(1)}_2 \subset C^{(2)}_0 \) we can construct a 3-layer 2-level II code \( C^{(3)} \) using Definition 12. We note here that both nested codes \( C^{(2)}_2 \) and \( C^{(1)}_2 \) have the same minimum distance \( d_0 = d_1 = 3 \), while the traditional definition of II codes \([10, 22]\) requires \( d_0 < d_1 \). Dropping this requirement allows us to construct higher layer II codes.

In effect, assume that \( m_2 = 2, s_0 = 1 \) and \( s_2 = 0 \). Then, if \( (3, C^{(3)}, (3, 0) = (c^{(2)}_0, c^{(2)}_1) \), where \( c^{(2)}_0 \) and \( c^{(2)}_1 \) are in \( C^{(2)}_0 \) and, according to (4), \( c^{(2)}_0 \oplus x \) \( x \) \( c^{(2)}_1 \) in \( C^{(2)}_0 \).

We can visualize both \( c^{(2)}_0 \) and \( c^{(2)}_1 \) as \( 6 \times 7 \) arrays. Then, according to Theorem 2, code \( C^{(3)} \) can correct an array with erasures correctable in \( C^{(2)}_0 \) together with an array with erasures correctable in \( C^{(2)}_1 \). For example, consider the two arrays in \( C^{(3)} \)

![Array Visualization](image-url)

where \( E \) denotes an erasure. Each row with only one erasure is in \( C^{(1)}_0 \), so it can be corrected. After correcting the rows with one erasure, since each of the two arrays is in \( C^{(2)}_0 \), the array in the right, which has one row with two erasures while the remaining ones are erasure-free, is corrected without intervention of the first array. Once this array is corrected, the array in the left, which has two rows with two erasures and the remaining ones are erasure-free, is corrected following the method of Theorem 2.

By (14) in Definition 13 the erasure-correcting capability of code \( C^{(3)} \) is \( u^{(3)} = (((1, 1, 1, 1, 2), (1, 1, 1, 2, 2)) \).

By Theorems 5 and 8 code \( C^{(3)} \) is an \([84, 69, 3]\) code.

**Example 15.** This example is similar to the example in Section IV of [17]. Let \( n = 7, C^{(1)}_1 \) 1-layer II codes over GF(8) as in Example 14 and \( \alpha \) a primitive element in GF(8). Let \( m_1 = 3 \) and \( C^{(2)}_2 \) and \( C^{(2)}_3 \) two 2-layer II codes with nested codes \( \{0, 7\} = C^{(3)}_3 \subset C^{(2)}_2 \subset C^{(1)}_1 \subset C^{(1)}_0 \). As in Example 14 we denote by \( s_{ij} \) the elements of code \( C(2), 0 \leq i \leq 1 \), according to Definition 12.

Assume that \( s_{0,0} = 2, s_{0,1} = 1, s_{0,2} = 0, s_{0,3} = 0, s_{1,0} = 1, s_{1,1} = 1, s_{1,2} = 1 \) and \( s_{1,3} = 0 \). Notice that \( C^{(2)} \) is a 2-layer 2-level II code while \( C^{(2)}_3 \) is a 2-layer 3-level II code. By Theorem 5 and Corollary 9 \( C^{(3)}_3 \) is a \([21, 17, 3]\) code and \( C^{(2)}_1 \) is a \([21, 15, 4]\) code. Considering each codeword as a \( 3 \times 7 \) array, by Theorem 2 both codes can correct any of the three rows with one erasure, and in addition, \( C^{(2)}_2 \) can correct up to one row with two erasures, and \( C^{(2)}_0 \) can correct one row with two erasures and one row with three erasures. By (14) in Definition 13 the erasure-correcting capability of \( C^{(2)}_0 \) is \( u^{(2)} = (1, 1, 2) \) and the erasure-correcting capability of \( C^{(2)}_2 \) is \( u^{(2)} = (1, 2) \).

By Lemma 11 \( C^{(2)}_1 \subset C^{(2)}_0 \) and hence we can construct a 3-layer II code \( C^{(3)} \) using Definition 11 with nested codes \( C^{(2)}_0 \) and \( C^{(2)}_1 \). In effect, assume that \( m_2 = 4, s_0 = 1, s_1 = 3, s_2 = 0 \), hence \( C^{(3)} \) is a 3-layer 2-level II code. We may visualize the codewords in \( C^{(3)} \) as four \( 3 \times 7 \) arrays. Then, according to Theorem 2 any erasures correctable in \( C^{(2)}_1 \) can be corrected in any of the arrays, while up to three arrays with erasures correctable in \( C^{(2)}_1 \) are also correctable in \( C^{(3)} \) provided that the fourth array is erasure-free. For example, consider the four arrays in \( C^{(3)} \)

![Array Visualization](image-url)

Each row with only one erasure is in \( C^{(1)}_0 \), so it can be corrected. After correcting the rows with one erasure, since the second array is in \( C^{(2)}_0 \) and it has one row with two erasures while the remaining two are erasure-free, it is corrected. Once this array
is corrected, the three remaining arrays, which have a row with two erasures, a row with three erasures and the remaining one is erasure-free, are corrected following the decoding algorithm of Theorem 2.

By (14) in Definition 13 the erasure-correcting capability of code $C^{(3)}$ is $\text{(3)} = ((1, 1, 2), (1, 2, 3), (1, 2, 3), (1, 2, 3))$.

By Theorems 6 and 8 code $C^{(3)}$ is an $[84, 62, 4]$ code.

As a comparison (and as was done in (17)), consider a 2-layer 2-level code $C^{(0)}$ with $m = 12$, also with nested 1-layer EII codes $C^{(2)}_1 \subset C^{(1)}_1 \subset C^{(0)}_0$ as above, and $s_{0} = 5$, $s_{1} = 4$ and $s_{2} = 3$. However, since $m = 12$ and, according to Definition 12 $m < q$, we cannot use the field $GF(8)$ as in the case of code $C^{(3)}$ above. The next field of characteristic 2 is $GF(16)$, so we assume that the codes $C^{(1)}_1$ are over $GF(16)$. By Theorem 6 and Corollary 9 $C^{(2)}$ is also an $[84, 62, 4]$ code, so $C^{(2)}$ and $C^{(3)}$ have the same rate and minimum distance. However, since by (14) in Definition 13 the erasure-correcting capability of $C^{(2)}$ is $(1,1,1,1,2,2,2,3,3,3,3)$, there are erasures that can be corrected in $C^{(2)}$ but not in $C^{(3)}$. For example, visualizing the codewords in $C^{(2)}$ as $12 \times 7$ arrays, $C^{(2)}$ can correct any three rows with three erasures each while the remaining nine rows are erasure-free by Theorem 2. Certainly this is not true for code $C^{(3)}$. But code $C^{(3)}$ has better locality than code $C^{(2)}$, if a row has two erasures and the remaining 11 are erasure-free, then all such 12 rows are needed to reconstruct the erasures in $C^{(2)}$, while $C^{(3)}$ can do it with only three rows. In addition, since code $C^{(3)}$ is over $GF(8)$ while code $C^{(2)}$ is over $GF(16)$, the implementation of $C^{(3)}$ has less complexity. These tradeoffs that must be taken into account when choosing a code. □

Example 16. This example is similar to Example 15 but we incorporate an EII code (as opposed to an II code) as one of the nested 2-layer EII codes (notice that (17), (26) use only II codes as nested codes).

Let $m_{1} = 3$ and $C^{(2)}_0$ and $C^{(2)}_1$ two 2-layer EII codes with nested 1-layer II codes $\{0_{2}\} = C^{(2)}_2 \subset C^{(1)}_2 \subset C^{(0)}_0$, where $C^{(1)}_0$ and $C^{(1)}_1$ are as in Example 15. We denote by $s_{0}$ the $s$'s of $C^{(2)}_0$ and by $s_{1}$ the $s$'s of $C^{(2)}_1$ according to Definition 12.

Assume that $s_{0} = 0, s_{1} = 1, s_{2} = 0, s_{1} = 1, s_{1} = 1$ and $s_{2} = 1$. Notice that, since $s_{0} = 0$, $C^{(2)}_0$ is a 2-layer 2-level II code while, since $s_{1} \neq 0$, $C^{(2)}_1$ is a 2-layer 2-level EII code. By Theorem 6 and Corollary 9, $C^{(2)}_1$ is a $[21, 17, 3]$ code and $C^{(2)}_0$ is a $[21, 11, 6]$ code. By (14) in Definition 13 the erasure-correcting capability of $C^{(2)}_0$ is $(1, 1, 2)$ while the one of $C^{(2)}_1$ is $(1, 2, 7)$. By Lemma 11, $C^{(2)}_1 \subset C^{(2)}_0$.

Next we construct a 3-layer 2-level II code $C^{(3)}$ using Definition 14 on the nested codes $\{0_{2}\} = C^{(2)}_2 \subset C^{(2)}_1 \subset C^{(2)}_0$ with $m_{1} = 4, s_{0} = 3, s_{1} = 1$ and $s_{2} = 0$. As in Example 15 we visualize a codeword in $C^{(3)}$ as four $3 \times 7$ arrays. According to Theorem 2 any erasures correctable in $C^{(2)}_1$ can be corrected in any of the arrays, while up to one of the arrays with erasures correctable in $C^{(2)}_1$ also correctable in $C^{(3)}$ provided that the remaining three arrays are erasure-free. According to (14), the erasure-correcting capability of $C^{(3)}$ is $\text{(3)} = \text{(1, 1, 2, 1, 1, 2, 1, 2, 7)}$. For example, consider the four arrays in $C^{(3)}$

Each row with only one erasure is in $C^{(3)}$, so it can be corrected. After correcting the rows with one erasure, since each of the four arrays is in $C^{(2)}_0$, and the first, second and fourth arrays, contain one row with two erasures while the remaining ones are erasure-free, they are corrected without intervention of the other arrays. Once these three arrays are corrected, the third array, which has a row that is erasure-free, a row with two erasures and the remaining row erased, is corrected following the decoding algorithm of Theorem 6.

By Theorems 6 and 8 code $C^{(3)}$ is an $[84, 62, 6]$ code, hence, it has the same rate as codes $C^{(2)}$ and $C^{(3)}$ in Example 15. However, both $C^{(2)}$ and $C^{(3)}$ have minimum distance $d = 4$, while $C^{(3)}$ has minimum distance $d = 6$. □

Example 17. This example is similar to the one given in (17). Consider the two nested 2-layer II codes $C^{(2)}_0$ and $C^{(2)}_1$ of Example 15. We construct two 3-layer II codes $C^{(3)}_0$ and $C^{(3)}_1$ with $m_{2} = 2$ sharing the nested codes $\{0_{2}\} = C^{(2)}_2 \subset C^{(2)}_1 \subset C^{(2)}_0$. According to Definition 12 for $C^{(3)}_0$, let $s_{0} = 0, s_{1} = 1$ and $s_{2} = 0$ (hence, $C^{(3)}_0$ is a 3-layer 2-level II code), while for $C^{(3)}_1$, let $s_{1} = 0, s_{1} = 2$ and $s_{2} = 0$ (hence, $C^{(3)}_1$ is a 3-layer 1-level II code). By Lemma 11, $C^{(3)}_1 \subset C^{(3)}_0$. According to Theorems 6 and 8 $C^{(3)}_0$ is a $[42, 32, 4]$ 3-layer II code and $C^{(3)}_1$ is a $[42, 30, 4]$ 3-layer II code. By Definition 13 the erasure-correcting capability of $C^{(3)}_0$ is $(1, 1, 2, (1, 2, 3))$, while the one of $C^{(3)}_1$ is $(1, 2, 3, (1, 2, 3))$.

Next, using the nested codes $\{0_{2}\} = C^{(2)}_2 \subset C^{(2)}_1 \subset C^{(2)}_0$, we construct a 4-layer 2-level II code $C^{(4)}$ with $m_{3} = 2$ and $s_{1} = s_{2} = 1$. By (14), the erasure-correcting capability of code $C^{(4)}$ is $\text{(4)} = \text{((1, 1, 2, (1, 2, 3)), ((1, 2, 3, (1, 2, 3)))}$. Hence, both the 4-layer II code $C^{(4)}$ and the 3-layer II code $C^{(3)}$ of Example 15 can correct the same erasure patterns, but code $C^{(4)}$ has better locality. In effect, if a row has three erasures and the remaining 11 rows are erasure-free, code $C^{(3)}$ needs all 12 rows to recover the erasures, while code $C^{(4)}$ needs only 6 rows.
By Theorems 6 and 8 code $C_0^{(4)}$, like code $C_0^{(3)}$ in Example 15 is an [84, 62, 4] code.

Example 18. We give another example of a 4-layer 2-level II code slightly different to the one of Example 17.

Assume that $n = 7$ and the 1-layer EII codes $C_i^{(1)}$ over $GF(8)$ are as in Example 16. Consider the two nested 2-layer II codes $C_2^{(2)} \subset C_0^{(2)}$ of Example 15. In addition, take two 2-layer 2-level II codes $C_2^{(2)}$ and $C_3^{(2)}$ with nested codes $C_2^{(1)} \subset C_0^{(1)}$ such that, for $C_2^{(2)}$, $s_0 = 1, s_1 = 2$ and $s_2 = 0$, while for $C_3^{(2)}$, $s_0 = 1, s_1 = 0$ and $s_2 = 2$. In particular, notice that $C_3^{(2)} \subset C_2^{(2)}$, and, by Theorems 6 and 8 $C_2^{(2)}$ is an [21, 16, 3] code and $C_3^{(2)}$ is an [21, 14, 4] code. By Definition 13 the erasure-correcting capability of $C_2^{(2)}$ is $(1, 2, 2)$ while the one of $C_3^{(2)}$ is $(1, 3, 3)$.

Next, we construct two 3-layer II codes similar to Example 17 with $m_2 = 2$. The first one, $C_0^{(3)}$, is the same as in Example 17. We have seen that its erasure-correcting capability is $((1, 1, 2), (1, 2, 3))$ and its minimum distance is 4.

The second one, $C_3^{(3)}$, is a 2-level code with nested codes $C_3^{(2)} \subset C_2^{(2)}$, where $C_2^{(2)}$ and $C_3^{(2)}$ were defined above, and $s_0 = s_1 = 1$. Hence $C_3^{(3)} \subset C_3^{(3)}$. We have seen in Example 17 that $C_3^{(3)}$ is a [42, 32, 4] 3-layer II code with erasure-correcting capability $((1, 1, 2), (1, 2, 3))$, while, by Theorems 6 and 8 $C_3^{(3)}$ is a [42, 30, 4] code. By Definition 13 its erasure-correcting capability is $((1, 2, 2), (1, 3, 3))$.

Next, using the nested codes $C_3^{(3)} \subset C_0^{(3)}$, we construct a 4-layer 2-level II code $C_1^{(4)}$ with $m_3 = 2$ and $s_1 = s_2 = 1$.

By Theorems 6 and 8 code $C_1^{(4)}$ is an [84, 62, 4] code. By Definition 13 its erasure-correcting capability is $C_1^{(4)} = (((1, 1, 2), (1, 2, 3)), ((1, 2, 2), (1, 3, 3)))$.

We had seen that the 4-layer II code $C_0^{(4)}$ of Example 17 and the 3-layer II code $C_3^{(3)}$ of Example 15 have the same erasure-correcting capability. This is not true for $C_1^{(4)}$ though. It has the same rate and minimum distance as the previous two, but $C_1^{(4)}$, can correct two of the first three rows (i.e., rows 0, 1 and 2) with 3 erasures each provided that the remaining ten rows are erasure-free, and this is not true for $C_3^{(3)}$ nor for $C_0^{(4)}$ (the same is true for consecutive rows 3, 4 and 5, 6, 7 and 8 and 9, 10 and 11).

For example, one such pattern correctable in $C_1^{(4)}$, but not in $C_3^{(3)}$ nor in $C_0^{(4)}$, consists of the following 4 arrays:

\[
\begin{bmatrix}
E & E & E \\
E & E & E \\
E & E & E \\
E & E & E \\
\end{bmatrix}
\quad
\begin{bmatrix}
E & E & E \\
E & E & E \\
E & E & E \\
E & E & E \\
\end{bmatrix}
\quad
\begin{bmatrix}
E & E & E \\
E & E & E \\
E & E & E \\
E & E & E \\
\end{bmatrix}
\quad
\begin{bmatrix}
E & E & E \\
E & E & E \\
E & E & E \\
E & E & E \\
\end{bmatrix}
\]

On the other hand, the erasure pattern on 4 arrays in Example 15 is guaranteed to be correctable both in $C_3^{(3)}$ and in $C_0^{(4)}$, but not in $C_1^{(4)}$.

In this section, we have seen several examples of [84, 62, 2, 3] and 4-layer EII codes over $GF(16)$ and $GF(8)$. We will retake these examples in the next section in which we give a general method for obtaining a parity-check matrix of an $\ell$-layer EII code. The 4-layer II codes we presented in Examples 17 and 18 have minimum distance $d = 4$. However, it is possible to obtain an [84, 62] 4-layer II code with minimum distance $d = 6$ as well. For example, we can construct, using the same methods as in these two examples, a 4-layer II code with erasure-correcting capability $(((1, 1, 2), (1, 2, 3)), ((1, 1, 2), (1, 2, 5)))$. By Theorem 8 this code has minimum distance 6, and it will be one of the codes whose performance we will analyze in Table 1.

We can increase the minimum distance of an [84, 62] 4-layer II code even further. Take for example the 4-layer II code with erasure-correcting capability $(((0, 0, 1), (1, 1, 3)), ((1, 1, 3), (2, 3, 6)))$. By Theorem 8 this code has minimum distance 7. However, with such a code, the local erasure-correction on rows is lost. Definition 11 allows to have $u_0 = 0$, which would correspond to code $C_0$ being the whole space with minimum distance $d_0 = 1$ (no erasure-correcting capability). In effect, the 2-layer (0,0,1) code can correct one erasure in one of three rows, while the remaining two have to be erasure-free. For the 4-layer code with minimum distance 7, a correctable pattern must have at least three consecutive rows with at most one erasure in them.

IV. Parity-check matrices of $\ell$-layer EII codes

Given integers $m$ and $n$, let $\alpha$ be an element in $GF(q)$ such that $O(\alpha) \geq \max\{m, n\}$. Consider the following Vandermonde matrices of rank $s$ for $s \leq \max\{m, n\}$ and $s \leq w \leq n$:

\[
H_{s,w,v}^{(q)} = \begin{pmatrix}
1 & \alpha^v & \alpha^{2v} & \ldots & \alpha^{(w-1)v} \\
1 & \alpha^{v+1} & \alpha^{2(v+1)} & \ldots & \alpha^{(w-1)(v+1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{v+s-1} & \alpha^{2(v+s-1)} & \ldots & \alpha^{(w-1)(v+s-1)}
\end{pmatrix}.
\]
When the context is clear, we denote $H_{\mu, w, \delta}$ simply as $H_{\delta}$. Also, we denote by $I_n$ the $n \times n$ identity matrix, by $0_{m \times n}$ the $m \times n$ zero matrix, and by $A \otimes B$ the Kronecker product [18] (also called the tensor product in literature) of matrices $A$ and $B$.

The next theorem gives (recursively) a parity-check matrix for an EII code according to Definition [1]

**Theorem 19.** Let $C$ be an EII code with nested codes $C_i$ as given by Definition [1]. Let $H_0$ be the $n_0 \times n$ parity-check matrix of $C_0$ and, assuming that the parity-check matrix of $C_{i-1}$ is $H_{i-1}$ for $i \geq 1$, let

$$
H_i = \left( \frac{H_{i-1}}{B_i} \right)
$$

be the parity-check matrix of $C_i$, where $B_i$ is an $(u_i - u_{i-1}) \times n$ matrix for $1 \leq i \leq t$. Then a parity-check matrix for code $C$ is given by the $(mu_0 + ns + \sum_{i=1}^{t} (u_i - u_{i-1}) \delta_i) \times n$

$$
H = \begin{pmatrix}
I_m & \otimes \ H_0 \\
H_{S_1, m, 0} & \otimes \ B_1 \\
H_{S_2, m, 0} & \otimes \ B_2 \\
& \ddots & \ddots & \ddots \\
H_{S_{t-1}, m, 0} & \otimes \ B_{t-1} \\
H_{S_t, m, 0} & \otimes I_n
\end{pmatrix}
$$

**Proof:** We have to prove that $c = (c_1, \ldots, c_{m-1}) \in C$ if and only if $Hc^T = 0_{m}$, with $w = mu_0 + ns + \sum_{i=1}^{t} (u_i - u_{i-1}) \delta_i$.

Consider Definition [1]. Notice that $c_i \in C_0$ for $0 \leq i \leq m-1$ if and only if $H_0c_i^T = 0_{m_0}$, if and only if $(I_m \otimes H_0)c_i^T = 0_{mu_0}$.

Next we prove that for every $i, 1 \leq i \leq t-1$, $\bigoplus_{j=0}^{m-1} \alpha^j c_i \in C_i$, where $0 \leq r \leq \delta_i - 1$. We have to show that

$$
(\bigoplus_{j=0}^{m-1} \alpha^j B_i) c_i^T = 0_{(u_i - u_{i-1}) \delta_i}.
$$

In effect, assume that for every $i, 1 \leq i \leq t-1$, $\bigoplus_{j=0}^{m-1} \alpha^j c_i \in C_i$, where $0 \leq r \leq \delta_i - 1$. We have to show that

$$
\bigoplus_{j=0}^{m-1} \alpha^j c_i^T = 0_{u_i - u_{i-1}}.
$$

Since $\bigoplus_{j=0}^{m-1} \alpha^j c_i \in C_i$ and, since by (16), $B_i$ is part of the rows of the parity-check matrix $H_i$ of $C_i$, (18) follows.

Conversely, assume that $(\bigoplus_{j=0}^{m-1} \alpha^j B_i) c_i^T = 0_{(u_i - u_{i-1}) \delta_i}$ for every $i, 1 \leq i \leq t-1$. In particular, for every $r, 0 \leq r \leq \delta_i - 1$, $(\bigoplus_{j=0}^{m-1} \alpha^j B_i) c_i^T = 0_{u_i - u_{i-1}}$. We have to show that

$$
H_i \left( \bigoplus_{j=0}^{m-1} \alpha^j c_i \right)^T = 0_{u_i} \text{ for } 0 \leq r \leq \delta_i - 1,
$$

which will hold if and only if

$$
H_0 \left( \bigoplus_{j=0}^{m-1} \alpha^j c_i \right)^T = 0_{mu_0} \text{ and } B_v \left( \bigoplus_{j=0}^{m-1} \alpha^j c_i \right)^T = 0_{u_v - u_{v-1}} \text{ for } 1 \leq v \leq i \text{ and } 0 \leq r \leq \delta_i - 1.
$$

Notice that $B_v \left( \bigoplus_{j=0}^{m-1} \alpha^j c_i \right)^T = 0_{u_v - u_{v-1}}$ for $0 \leq r \leq \delta_v - 1$, and since $1 \leq v \leq i$, $\delta_v \geq \delta_i$, in particular,

$$
B_v \left( \bigoplus_{j=0}^{m-1} \alpha^j c_i \right)^T = 0_{u_v - u_{v-1}} \text{ for } 0 \leq r \leq \delta_i - 1.
$$

Finally, notice that $\bigoplus_{j=0}^{m-1} \alpha^j c_i = 0_v$ for $0 \leq r \leq \delta_i - 1$ if and only if $(H_{S_1, m, 0} \otimes I_n) c_i^T = 0_{\delta_i n}$, completing the proof. □

Theorem [19] allows us to obtain the parity-check matrix of a 2-layer EII code in the next corollary.
Corollary 20. Let $C^{(2)}$ be a 2-layer EII code of length $(m)(n)$ as given by Definition 12 where $\{C_i\} = C^{(1)}_1 \subset C^{(1)}_t \subset C^{(1)}_{t-2} \subset \cdots \subset C^{(1)}_0$ is the sequence of nested 1-level codes such that $C^{(1)}_i$ is an $[n, n-u_i, u_i+1]$ code with parity-check matrix $H_{u_i, n, 0}$ as given by (15). $0 \leq u_0 < u_1 < \cdots < u_{t-1} < n$ and $m = \sum_{i=0}^{t} s_i$, where $s_i \geq 0$ for $1 \leq i \leq t$. Then, the parity-check matrix of $C^{(2)}$ is given by

$$H^{(2)} = \begin{pmatrix}
I_m & \otimes & H_{u_1, n, 0} \\
H_{s_1, m, 0} & \otimes & H_{u_1-u_0, n, u_0} \\
H_{s_2, m, 0} & \otimes & H_{u_2-u_1, n, u_1} \\
\vdots & \vdots & \vdots \\
H_{s_{t-1}, m, 0} & \otimes & H_{u_{t-1}-u_{t-2}, n, u_{t-2}}
\end{pmatrix} (19)$$

Proof: By (15).

By (20), taking $B_i = H_{u_i-u_{i-1}, n, u_{i-1}}$ in (16) and (17), we obtain (19). \qed

It can be easily proven that the parity-check matrix of a $t$-level EII code as given by (25) in (2) is equivalent to $H^{(2)}$ as given by (19).

The next theorem gives a recursive construction for the parity-check matrix of an $\ell$-layer EII code when $\ell \geq 3$.

Theorem 21. Let $\ell \geq 3$ and consider an $\ell$-layer EII code $C^{(\ell)}$ of length $n_\ell = (m_{\ell-1})(m_{\ell-2}) \cdots (m_1)(n)$ as given by Definition 12 where $\{C_{i_{\ell-1}}\} = C^{(1)}_{t_{\ell-1}} \subset C^{(1)}_{t_{\ell-2}} \subset \cdots \subset C^{(1)}_0$ is the sequence of $t_{\ell-1} + 1$ nested $(\ell - 1)$-layer EII codes of length $n_{\ell-1} = (m_{\ell-2})(m_{\ell-3}) \cdots (m_1)(n)$ and $m_{\ell-1} = \sum_{i=0}^{t_{\ell-1}} s_i$, where $s_i \geq 0$ for $0 \leq i \leq t_{\ell-1}$. According to Definition 12 and Corollary 20 without loss of generality, assume that the $(\ell - 1)$-layer EII codes $C^{(1)}_i$ share the same sequence of $(\ell - 2)$-layer $[n_{\ell-2}, n_{\ell-2} - u_j]$ nested EII codes $\{C_{i_{\ell-2}}\} = C^{(1)}_{t_{\ell-2}} \subset C^{(1)}_{t_{\ell-3}} \subset \cdots \subset C^{(1)}_0$, where $n_{\ell-2} = (m_{\ell-3})(m_{\ell-4}) \cdots (m_1)(n)$ for $\ell \geq 4$ and $n_1 = n$ for $\ell = 3$. Denoting by $s_{ij}$, $0 \leq i \leq t_{\ell-2}$ and $0 \leq j \leq t_{\ell-2}$, the $s_{ij}$s of code $C^{(1)}_i$ according to Definition 12, $\sum_{j=0}^{t_{\ell-2}} s_{ij} = m_{\ell-2}$ and $s_{ij} \geq 0$. Assuming that $B^{(1)}_i = H_{u_i-u_{i-1}, n, u_{i-1}}$ for $1 \leq j \leq t_1 - 1$ when $\ell = 3$, let

$$B^{(\ell-1)}_i = \begin{pmatrix}
H(\hat{s}_{i1}, \hat{s}_{i1-1}), m_{\ell-2}, \hat{s}_{i1-1} & \otimes & B^{(\ell-2)}_1 \\
H(\hat{s}_{i2}, \hat{s}_{i2-1}), m_{\ell-2}, \hat{s}_{i2-1} & \otimes & B^{(\ell-2)}_2 \\
\vdots & \vdots & \vdots \\
H(\hat{s}_{it_{\ell-2}}, \hat{s}_{it_{\ell-2}-1}), m_{\ell-2}, \hat{s}_{it_{\ell-2}-1} & \otimes & B^{(\ell-2)}_{t_{\ell-2}}
\end{pmatrix} (21)$$

for $1 \leq i \leq t_{\ell-1} - 1$. Then, the parity-check matrix of the $\ell$-layer $t_{\ell-1}$-level EII code $C^{(\ell)}$ is given by

$$H^{(\ell)} = \begin{pmatrix}
I_{m_{\ell-1}} & \otimes & H^{(\ell-1)}_1 \\
H\hat{s}_{1, m_{\ell-1}, 0} & \otimes & B^{(\ell-1)}_1 \\
H\hat{s}_{2, m_{\ell-1}, 0} & \otimes & B^{(\ell-1)}_2 \\
\vdots & \vdots & \vdots \\
H\hat{s}_{t_{\ell-1}, m_{\ell-1}, 0} & \otimes & B^{(\ell-1)}_{t_{\ell-1}}
\end{pmatrix} (22)$$

where $B^{(\ell-1)}_i$ is given by (21).

Proof: Let $\ell \geq 3$. By induction on $\ell$ and (22), we may assume that each $(\ell - 1)$-layer EII code $C^{(\ell-1)}_i$ is a code of length $n_{\ell-1} = (m_{\ell-2})(m_{\ell-3}) \cdots (m_1)(n)$ with parity-check matrix
According to Definition 12 and Corollary 20, without loss of generality, we may assume that the 2-layer EII codes \( B_i^{(\ell-2)} \) is given by (21), while, when \( \ell = 3 \), \( B_i^{(1)} = H_{u_i - u_{i-1}, n, u_{i-1}} \) and, by (19),

\[
\mathcal{H}_i^{(\ell-1)} = \begin{pmatrix}
I_{m_{\ell-2}} & \otimes & \mathcal{H}_0^{(\ell-2)} \\
H_{S_{1,1}, m_{\ell-2}, 0} & \otimes & B_1^{(\ell-2)} \\
H_{S_{2,1}, m_{\ell-2}, 0} & \otimes & B_2^{(\ell-2)} \\
\vdots & \vdots & \vdots \\
H_{S_{\ell-1,1}, m_{\ell-2}, 0} & \otimes & B_{\ell-1}^{(\ell-2)} \\
H_{S_{\ell,1}, m_{\ell-2}, 0} & \otimes & I_{n_{\ell-1}}
\end{pmatrix},
\]

(23)

if \( \ell \geq 4 \), where \( B_i^{(\ell-2)} \) is given by taking \( \ell - 1 \) instead of \( \ell \) in (21), so, by (17), we obtain that the parity-check matrix of \( C_i^{(\ell)} \) is given by (22).

\[
\mathcal{H}_i^{(2)} = \begin{pmatrix}
I_{m_1} & \otimes & H_{u_0, n, 0} \\
H_{S_{1,1}, m_1, 0} & \otimes & H_{u_1 - u_0, n, u_0} \\
H_{S_{2,1}, m_1, 0} & \otimes & H_{u_2 - u_1, n, u_1} \\
\vdots & \vdots & \vdots \\
H_{S_{\ell-1,1}, m_1, 0} & \otimes & H_{u_{\ell-1} - u_{\ell-2}, n, u_{\ell-2}} \\
H_{S_{\ell,1}, m_1, 0} & \otimes & I_n
\end{pmatrix}.
\]

(24)

Since the codes \( C_i^{(\ell-1)} \) are nested, by Lemma 11, \( \delta_{i,j} \geq \delta_{i-1,j} \) for \( 1 \leq i \leq t_{\ell-1} - 1 \) and \( 1 \leq j \leq t_{\ell-2} \). From (16) and (23), we can take \( B_i = B_i^{(\ell-1)} \), where \( B_i^{(\ell-1)} \) is given by (21), so, by (17), we obtain that the parity-check matrix of \( C_i^{(\ell)} \) is given by (22).

The next corollary simply applies Theorem 21 to the case \( \ell = 3 \). It is convenient to give it explicitly since it will appear repeatedly in the examples.

**Corollary 22.** Consider a 3-layer EII code \( C_i^{(3)} \) of length \( (m_2)(m_1)(n) \) as given by Definition 12 where \( \{0_{(m_1)}(n)\} = c_i^{(2)} \subset \mathcal{C}_1^{(2)} \subset \mathcal{C}_2^{(2)} \subset \cdots \subset \mathcal{C}_0^{(2)} \) is the sequence of \( t_2 + 1 \) nested 2-layer EII codes and \( m_2 = \sum_{i=0}^{t_2} s_i \), where \( s_i \geq 0 \) for \( 0 \leq i \leq t_2 \). According to Definition 12 and Corollary 20 without loss of generality, we may assume that the 2-layer EII codes \( C_i^{(2)} \) share the same sequence of \( t_1 \) 1-layer \( [n, n - u_i, u_j + 1] \) nested EII codes \( \{0_{(m_2)}\} = c_i^{(1)} \subset \mathcal{C}_1^{(1)} \subset \mathcal{C}_2^{(1)} \subset \cdots \subset \mathcal{C}_0^{(1)} \). Denoting by \( s_{i,j} \), \( 0 \leq i \leq t_2 \) and \( 0 \leq j \leq t_1 \), the \( s_{i,j}s \) of code \( C_i^{(2)} \) according to Definition 12, \( \sum_{j=0}^{t_1} s_{i,j} = m_1 \) and \( s_{i,j} \geq 0 \). Let

\[
B_i^{(2)} = \begin{pmatrix}
H_{(\delta_{1,1} - \delta_{1,1}), m_1, \delta_{1,1,1}} & \otimes & H_{u_1 - u_0, n, u_0} \\
H_{(\delta_{2,1} - \delta_{2,1}), m_1, \delta_{1,1,2}} & \otimes & H_{u_2 - u_1, n, u_1} \\
\vdots & \vdots & \vdots \\
H_{(\delta_{t_2,1} - \delta_{t_2,1}), m_1, \delta_{1,1,t_1}} & \otimes & H_{u_{t_1} - u_{t_1-1}, n, u_{t_1-2}} \\
H_{(\delta_{t_1,1} - \delta_{t_1,1}), m_1, \delta_{1,1,t_1}} & \otimes & I_n
\end{pmatrix}
\]

(25)

for \( 1 \leq i \leq t_2 - 1 \). Then, the parity-check matrix of the 3-layer \( t_2 \)-level EII code \( C_i^{(3)} \) obtained from the \( t_2 \) nested 2-layer EII codes \( C_i^{(2)} \) is given by

\[
\mathcal{H}_i^{(3)} = \begin{pmatrix}
I_{m_2} & \otimes & \mathcal{H}_0^{(2)} \\
H_{S_{1,1}, m_2, 0} & \otimes & B_1^{(2)} \\
H_{S_{2,1}, m_2, 0} & \otimes & B_2^{(2)} \\
\vdots & \vdots & \vdots \\
H_{S_{t_1,1}, m_2, 0} & \otimes & B_{t_2-1}^{(2)} \\
H_{S_{t_2,1}, m_2, 0} & \otimes & I_{(m_1)(n)}
\end{pmatrix},
\]

(26)

where \( B_i^{(2)} \) is given by (25).
Let us revisit next the examples of Section III to illustrate the construction of parity-check matrices of \( \ell \)-layer EII codes.

**Example 23.** Consider the conditions of Example 14. Both \( C_0^{(2)} \) and \( C_1^{(2)} \) in Example 14 have as nested codes the 1-layer codes \( \{ \emptyset \} = C_2^{(1)} \subset C_1^{(1)} \subset C_0^{(1)} \) and we had \( s_{0,0} = 5, s_{0,1} = 1, s_{0,2} = 0, s_{1,0} = 4, s_{1,1} = 2 \) and \( s_{0,2} = 0 \). Since \( m_1 = 6, u_0 = 1 \) and \( u_1 = 2 \), according to (24), the parity-check matrix of \( C_0^{(2)} \) is the \( 7 \times 42 \) matrix

\[
H_0^{(2)} = \begin{pmatrix}
I_6 & \otimes & H_{1,7,0} \\
1,6,0 & \otimes & H_{1,7,1}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & H_{1,7,0} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & H_{1,7,0} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & H_{1,7,0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & H_{1,7,0} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & H_{1,7,0} & 0 \\
H_{1,7,1} & H_{1,7,1} & H_{1,7,1} & H_{1,7,1} & H_{1,7,1} & H_{1,7,1} & H_{1,7,1}
\end{pmatrix}
\]

Similarly, according to (24), the parity-check matrix of \( C_1^{(2)} \) is the \( 8 \times 42 \) matrix

\[
H_1^{(2)} = \begin{pmatrix}
I_6 & \otimes & H_{1,7,0} \\
1,6,0 & \otimes & H_{1,7,1}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & H_{1,7,0} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & H_{1,7,0} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & H_{1,7,0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & H_{1,7,0} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & H_{1,7,0} & 0 \\
H_{1,7,1} & H_{1,7,1} & H_{1,7,1} & H_{1,7,1} & H_{1,7,1} & H_{1,7,1} & H_{1,7,1}
\end{pmatrix}
\]

The 3-layer 2-level II code \( C^{(3)} \) of Example 14 has as nested codes \( \{ \emptyset \} = C_2^{(2)} \subset C_1^{(2)} \subset C_0^{(2)} \), \( m_2 = 2 \) and \( s_0 = s_1 = 1 \) and \( s_2 = 0 \). Since \( \hat{s}_{1,1} = 2 \geq \hat{s}_{0,1} = 1 \), according to (25),

\[
B_1^{(2)} = H_{1,7,1} \otimes H_{1,7,1}
\]

so, according to (26), the parity-check matrix of \( C^{(3)} \) is the \( 15 \times 84 \) matrix

\[
H^{(3)} = \begin{pmatrix}
I_2 & \otimes & H_0^{(2)} \\
H_{1,2,0} & \otimes & B_1^{(2)}
\end{pmatrix}
\begin{pmatrix}
0 & \otimes & \emptyset_{2 \times 42} \\
\emptyset_{2 \times 42} & \otimes & H_0^{(2)} \\
\emptyset_{2 \times 42} & \otimes & B_1^{(2)}
\end{pmatrix}
\begin{pmatrix}
I_6 & \otimes & H_{1,7,0} \\
H_{1,6,0} & \otimes & H_{1,7,1}
\end{pmatrix}
\begin{pmatrix}
0 & \otimes & \emptyset_{2 \times 42} \\
\emptyset_{2 \times 42} & \otimes & \emptyset_{2 \times 42}
\end{pmatrix}
\begin{pmatrix}
H_{1,7,1} & H_{1,7,1} \\
\emptyset_{2 \times 42}
\end{pmatrix}
\]

\[\square\]

**Example 24.** Let us take now the conditions of Example 15. Both \( C_0^{(2)} \) and \( C_1^{(2)} \) in Example 15 have as nested codes the 1-layer codes \( \{ \emptyset \} = C_3^{(1)} \subset C_2^{(1)} \subset C_1^{(1)} \subset C_0^{(1)} \) and \( u_0 = 1, u_1 = 2, u_2 = 2, m_1 = 3, s_{0,0} = 2, s_{0,1} = 1, s_{0,2} = 0, s_{0,3} = 0, s_{1,0} = s_{1,1} = s_{1,2} = 1 \) and \( s_{1,3} = 0 \). According to (24), the parity-check matrix of the 2-layer 2-level code \( C_0^{(2)} \) is the \( 4 \times 21 \) matrix
Similarly, by (24), the parity-check matrix of the 2-layer 3-level code $C_{1}^{(2)}$ is the $6 \times 21$ matrix

$$
\mathcal{H}_{1}^{(2)} = \begin{pmatrix}
\frac{I_{3} \otimes H_{1,7,0}}{H_{1,7,1}} \\
\frac{H_{1,7,0} \otimes \varnothing}{H_{1,7,1}} \\
\frac{\varnothing \otimes H_{1,7,0}}{H_{1,7,1}} \\
\frac{H_{1,7,1} \otimes \varnothing}{H_{1,7,1}} \\
\frac{\varnothing \otimes H_{1,7,1}}{H_{1,7,1}} \\
\frac{H_{1,7,1} \otimes \varnothing}{H_{1,7,1}} \\
\end{pmatrix}. 
$$

The 3-layer 2-level EII code $C^{(3)}$ of Example 15 has as nested codes $\{\mathcal{C}_{0}^{(2)}\} = C_{2}^{(2)} \subset C_{1}^{(2)} \subset C_{0}^{(2)}$, $t_2 = 2$, $m_2 = 4$, $s_0 = 1$, $s_1 = 3$ and $s_2 = 0$. Since $s_{0,1} = 1$, $s_{0,2} = 0$, $s_{0,3} = 0$, $s_{1,1} = 2$, $s_{1,2} = 1$ and $s_{1,3} = 0$, according to (25),

$$
B_{1}^{(2)} = \begin{pmatrix}
H_{1,7,1} & \alpha H_{1,7,1} & \alpha^{2} H_{1,7,1} \\
\end{pmatrix},
$$

(29)

so, according to (26), the parity-check matrix of $C^{(3)}$ is the $22 \times 84$ matrix

$$
\mathcal{H}^{(3)} = \begin{pmatrix}
I_{4} \otimes \mathcal{H}_{1}^{(2)} \\
\frac{H_{1,7,0} \otimes \varnothing}{H_{1,7,1}} \\
\frac{\varnothing \otimes H_{1,7,0}}{H_{1,7,1}} \\
\frac{H_{1,7,1} \otimes \varnothing}{H_{1,7,1}} \\
\frac{\varnothing \otimes H_{1,7,1}}{H_{1,7,1}} \\
\frac{H_{1,7,1} \otimes \varnothing}{H_{1,7,1}} \\
\end{pmatrix},
$$

(30)

where $\mathcal{H}^{(2)}_{1}$ is given by (27) and $B_{1}^{(2)}$ by (29).

Consider next the 2-layer 2-level II code $C^{(2)}$ over $GF(16)$ in Example 15 then, according to (17), its parity-check matrix is given by the $22 \times 84$ matrix

$$
\mathcal{H}^{(2)} = \begin{pmatrix}
I_{12} \otimes H_{1,7,0} \\
H_{1,7,0} \otimes \varnothing \\
\varnothing \otimes H_{1,7,1} \\
H_{1,7,1} \otimes \varnothing \\
\varnothing \otimes H_{1,7,1} \\
\varnothing \otimes H_{1,7,1} \\
\end{pmatrix}.
$$

Example 25. Consider the conditions of Example 16. Since $m_1 = 3$, $s_{0,0} = 2$, $s_{0,1} = 1$, $s_{0,2} = 0$ and $s_{1,0} = s_{1,1} = s_{1,2} = 1$. The parity-check matrix $\mathcal{H}^{(2)}_{0}$ of the 2-layer 2-level II code $C^{(2)}_{0}$ is given by (27), while, by (24), the parity-check matrix of $C^{(2)}_{1}$ is the $12 \times 21$ matrix

\[\square\]
\[ \mathcal{H}^{(2)}_1 = \begin{pmatrix} I_3 \otimes H_{1,7,0} \\ H_{2,3,0} \otimes H_{1,7,1} \\ H_{1,3,0} \otimes I_7 \end{pmatrix} = \begin{pmatrix} H_{1,7,0} & 0 \alpha & 0 \\ 0 & H_{1,7,0} & 0 \alpha \\ 0 & 0 & H_{1,7,0} \\ I_7 & I_7 & I_7 \end{pmatrix}. \] (30)

Since \( C_1^{(2)} \) is a [21,11] code, the number of parities is 10, so two of the rows of \( \mathcal{H}^{(2)}_1 \) as given by (30) are dependent since the matrix has rank 10. We can delete the last two rows of \( \mathcal{H}^{(2)}_1 \) to obtain a parity-check matrix of rank 10.

The 3-layer 2-level EII code \( C^{(3)} \) of Example 18 has as nested codes \( \{0,1\} = C^{(2)}_2 \subset C^{(2)}_1 \subset C^{(2)}_0 \), \( m_2 = 4 \), \( s_0 = 3 \) and \( s_1 = 1 \). Since \( s_{1,1} = 2 \geq s_{0,1} = 1 \) and \( s_{1,2} = 1 \geq s_{0,2} = 0 \), according to (25).

\[ B^{(2)}_1 = \begin{pmatrix} H_{1,3,1} \\ H_{1,3,0} \end{pmatrix} = \begin{pmatrix} H_{1,7,1} \\ I_7 \end{pmatrix}, \] (31)

so, according to (26), the parity-check matrix of the 3-layer 2-level II code \( C^{(3)} \) of Example 16 is the 24 \( \times \) 84 matrix

\[ \mathcal{H}^{(3)} = \begin{pmatrix} I_4 \otimes \mathcal{H}^{(2)}_1 \\ H^{(2)}_1 \otimes B^{(2)}_1 \end{pmatrix} = \begin{pmatrix} \mathcal{H}^{(2)}_1 & 0 & 0 & 0 & 4 \times 7 \\ 0 & \mathcal{H}^{(2)}_1 & 0 & 0 & 4 \times 7 \\ 0 & 0 & \mathcal{H}^{(2)}_1 & 0 & 4 \times 7 \\ 0 & 0 & 0 & \mathcal{H}^{(2)}_1 & 4 \times 7 \\ B^{(2)}_1 & B^{(2)}_1 & B^{(2)}_1 & B^{(2)}_1 & B^{(2)}_1 \end{pmatrix}. \]

where \( B^{(2)}_1 \) is given by (31).

Matrix \( \mathcal{H}^{(3)} \) has rank 22 since \( C^{(3)} \) has dimension 22. We can delete the last two rows of \( \mathcal{H}^{(3)} \) to obtain a 22 \( \times \) 84 parity-check matrix of \( C^{(3)} \).

**Example 26.** We revisit now Example 17. We had the 3-layer nested codes \( \{0,1\} = C^{(2)}_2 \subset C^{(2)}_1 \subset C^{(2)}_0 \) sharing the 2-layer nested codes \( \{0,1\} = C^{(2)}_2 \subset C^{(2)}_1 \subset C^{(2)}_0 \), where \( s_{0,0} = s_{0,1} = 1 \) and \( s_{0,2} = 0 \) correspond to \( C^{(3)}_0 \) and \( s_{1,0} = 0 \), \( s_{1,1} = 2 \) and \( s_{1,2} = 0 \) correspond to \( C^{(3)}_1 \).

Proceeding as in previous examples, applying Corollary 22 we can verify that the parity-check matrix of \( C^{(3)}_0 \) is given by the 10 \( \times \) 42 matrix

\[ \mathcal{H}^{(3)}_0 = \begin{pmatrix} I_2 \otimes \mathcal{H}^{(2)}_1 \\ H_{1,2,0} \otimes B^{(2)}_1 \end{pmatrix} = \begin{pmatrix} \mathcal{H}^{(2)}_1 & 0 & 0 & 4 \times 21 \\ 0 & \mathcal{H}^{(2)}_1 & 0 & 4 \times 21 \\ 0 & 0 & \mathcal{H}^{(2)}_1 & 4 \times 21 \\ B^{(2)}_1 & B^{(2)}_1 & B^{(2)}_1 & B^{(2)}_1 \end{pmatrix}. \] (32)

where \( B^{(2)}_1 \) is given by (29).

Similarly, the parity-check matrix of \( C^{(3)}_1 \) is given by the 12 \( \times \) 42 matrix
Consider the 4-layer 2-level II code $C^{(4)}_0$ of Example [7] with nested codes $\{0_{12}\} = C^{(3)}_2 \subset C^{(3)}_1 \subset C^{(3)}_0$. We had, $s_0 = s_1 = 1$, $s_2 = 0$, $m_3 = m_2 = 2$, $s_{0,0} = s_{0,1} = 1$, $s_{0,2} = 0$, $s_{1,0} = 0$, $s_{1,1} = 2$ and $s_{1,2} = 0$. Since $s_{1,1} = 2$ and $s_{0,1} = 1$, according to (21),

$$B^{(3)}_1 = \left( B^{(2)}_1 \right), \quad (33)$$

where $B^{(2)}_1$ is given by (29), while according to (17), (32) and (33), the parity-check matrix of $C^{(4)}_0$ is given by the $22 \times 84$ matrix

$$H^{(4)}_0 = \left( \begin{array}{c} I_2 \otimes H^{(3)}_0 \\ H^{(3)}_0 \end{array} \right)$$

(34)

Example 27. Assume that we take the four nested codes 1-layer II codes $\{0_{12}\} = C^{(3)}_3 \subset C^{(3)}_2 \subset C^{(3)}_1 \subset C^{(3)}_0$, where $C^{(3)}_1$ is a $[7, 7-i-1, i+1]$ code over $GF(8)$. With these three codes, we construct, using Definition [12] the three nested 2-layer II codes $C^{(2)}_2 \subset C^{(2)}_1 \subset C^{(2)}_0$ with $m_1 = 5$ and $n = 7$, where $s_{0,0} = 4$, $s_{0,1} = 1$, $s_{0,2} = 0$, $s_{0,3} = 0$, $s_{1,0} = 3$, $s_{1,1} = 2$, $s_{1,2} = 0$, $s_{1,3} = 0$, $s_{2,0} = 3$, $s_{2,1} = 1$, $s_{2,2} = 1$, $s_{2,3} = 0$, (hence, $s_{i,0} + s_{i,1} + s_{i,2} = 5$ for $0 \leq i \leq 2$), let $H^{(2)}_i$ be the parity-check matrix of code $C^{(2)}_i$ according to (24) and assume that we want to construct the parity-check matrix $H^{(3)}_0$ of a 3-layer 2-level II code $C^{(3)}_0$ with $m_2 = 4$, $s_0 = 2$, $s_1 = 2$ and $s_1 = 1$ (hence, $s_0 + s_1 = 4$). Notice that the erasure-correcting capability of $C^{(2)}_0$ is $(1, 1, 1, 1, 2)$, the one of $C^{(2)}_1$ is $(1, 1, 1, 2)$ and the one of $C^{(2)}_2$ is $(1, 1, 1, 2, 3)$. Hence, the erasure-correcting capability of $C^{(3)}_0$ is $(1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 3)$ and its minimum distance, according to Theorem 8 is $d^{(3)}_0 = 3$.

By (28),

$$B^{(2)}_1 = \left( H_{1,5,1} \otimes H_{1,7,1} \right) \quad (34)$$

Applying (24), (26) and (34), we obtain that the parity-check matrix of $C^{(3)}_0$ is the $26 \times 140$ matrix

$$H^{(3)}_0 = \left( \begin{array}{c} I_4 \otimes H^{(2)}_0 \\ H^{(2)}_0 \end{array} \right)$$

(35)

Consider next the parity-check matrix $H^{(3)}_1$ of a 3-layer 3-level II code $C^{(3)}_1$ with $m_2 = 4$, $s_0 = 2$, $s_1 = 1$ and $s_2 = 1$. Notice that the erasure-correcting capability of $C^{(3)}_1$ is $(1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 3)$ and its minimum distance, according to Theorem 8 is $d^{(3)}_1 = 4$.

By (28),

$$B^{(2)}_2 = \left( H_{1,5,0} \otimes H_{1,7,2} \right) \quad (36)$$

Applying (24), (34), (36) and (26), we obtain the $27 \times 140$ matrix
Finally, consider the parity-check matrix $\mathcal{H}_2^{(3)}$ of a 3-layer 3-level code $C_2^{(3)}$ with $m_2 = 4$, $s_0 = 1$, $s_1 = 2$ and $s_2 = 1$. The erasure-correcting capability of $C_2^{(3)}$ is $((1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 2), (1, 1, 2, 3))$ and its minimum distance, according to Theorem 8, is $d_2^{(3)} = 4$.

Applying (24), (34), (36) and (26), we obtain the $28 \times 140$ matrix

\[
\mathcal{H}_2^{(3)} = \begin{pmatrix}
I_4 & \otimes & \mathcal{H}_0^{(2)} \\
H_3,4,0 & \otimes & B_2^{(2)} \\
H_2,4,0 & \otimes & B_2^{(2)} \\
H_1,4,0 & \otimes & B_2^{(2)} \\
\end{pmatrix}.
\]

(37)

We will use the three nested codes $C_2^{(3)} \subset C_1^{(3)} \subset C_0^{(3)}$ to construct the parity-check matrix of a 4-layer II code in the next example. □

**Example 28.** Consider the four nested 3-layer II codes $\{0_{140}\} = C_2^{(3)} \subset C_2^{(3)} \subset C_1^{(3)} \subset C_0^{(3)}$ of Example 27. Assume that we want to construct the parity-check matrix $\mathcal{H}^{(4)}$ of a 4-layer 3-level II code $C^{(4)}$ with $m_3 = 3$ and $s_0 = s_1 = s_2 = 1$ using (22) and (21) in Theorem 21.

Explicitly, since $s_{0,0} = 2$, $s_{0,1} = 2$, $s_{0,2} = 0$, $s_{0,3} = 0$, $s_{1,0} = 2$, $s_{1,1} = 1$, $s_{1,2} = 1$, $s_{1,3} = 0$, $s_{2,0} = 1$, $s_{2,1} = 2$, $s_{2,2} = 1$ and $s_{2,3} = 0$, according to (21), (34) and (36),

\[
B_1^{(3)} = B_1^{(2)} = H_{1,4,0} \otimes B_2^{(2)} = H_{1,4,0} \otimes (H_{1,5,0} \otimes H_{1,7,2})
\]

(39)

and

\[
B_2^{(3)} = B_1^{(2)} = H_{1,4,2} \otimes B_1^{(2)} = H_{1,4,2} \otimes (H_{1,5,1} \otimes H_{1,7,1})
\]

(40)

According to (17), (26), (35), (39) and (40), the parity-check matrix of $C^{(4)}$ is given by the $81 \times 420$ matrix

\[
\mathcal{H}^{(4)} = \begin{pmatrix}
I_3 & \otimes & \mathcal{H}_0^{(3)} \\
H_3,3,0 & \otimes & B_2^{(3)} \\
H_2,3,0 & \otimes & B_2^{(3)} \\
H_1,3,0 & \otimes & B_2^{(3)} \\
\end{pmatrix}.
\]

(41)

The parity-check of a code allows for decoding erasures in a traditional way, that is, by inverting the submatrix with columns corresponding to the erasures. The decoding algorithm for erasures as illustrated in Theorem 2 is certainly more efficient than the straightforward inverting method, but the advantage of using the parity-check matrix is that some extra erasures may
be corrected. The decoding algorithm in Theorem 2 can only correct those erasures that can be guaranteed to be corrected according to Theorem 2 but there may be more possible correctable erasures. This point has also been made in other papers [2], [6]. We will elaborate further in Section VI in particular, in Table II.

V. AVERAGE NUMBER OF UNCORRECTABLE ERASES

In this section, as done in [5] for 2-layer II codes, we examine the average number of erasures causing an uncorrectable pattern in $\ell$-layer EII codes. Let us call this parameter the Average Number of Erasures to Failure (ANETF). The ANETF is more relevant than the minimum distance of the code when failures do not occur all at the same time, but one after the other, for example, by arriving following a Poisson distribution [3], [8], [9].

So, assume that failures (erasures) occur consecutively, one after the other. The question is, given an $\ell$-layer EII code, what is its ANETF? In [5], it was found that in some cases, 2-layer II codes having lower minimum distance than others, nevertheless had higher ANETF.

As an example, we take a number of multiple layer EII codes with rate 62/84, as illustrated in Table I. We choose this rate since [26] gives the example of a 3-layer 2-level II code with erasure-correcting capability $\{(1, 1, 2), (1, 2, 3), (1, 2, 3)\}$, while [17] gives a 4-layer 2-level II code with erasure-correcting capability $\{(1, 1, 2), (1, 2, 3), (1, 2, 3)\}$, both cases with rate 62/84. We retake these two examples in Table I as well as several others with the same rate.

The first column in Table I gives the $\ell$ corresponding to the layer of the EII code described, in this case, $1 \leq \ell \leq 4$. The second column gives the erasure-correcting capability of the code. The third column gives the minimum distance of the code according to Theorem 8. The fourth column gives the result of simulations where the ANETF was computed in two ways. One was by using the algorithm of Theorem 2 where, each time the erasure-correcting capability in the third column is exceeded, an uncorrectable pattern is declared. The second computation of the ANETF in the fourth column is in parenthesis and with an asterisk. It is obtained by doing erasure-decoding using the parity-check of the code as given in Section IV (certainly, both decoding methods can be combined: if an erasure pattern exceeds the erasure-correcting capability of the code, decoding may be attempted using the parity-check matrix). We can see that the ANETF improves considerably when using the parity-check matrix to correct erasure patterns that exceed the erasure-correcting capability of the codes. In some cases, the improvement is dramatic. For example, for the 4-layer code in the next to last row, the ANETF improves from 11.8 to 22.3 erasures. We can see in Table I that, taking codes with the same value $\ell$ of their layers, their ANET and their minimum distance are roughly correlated when using the parity-check matrix for erasure decoding. In general, the largest the minimum distance, the largest the ANETF. An exception is the 2-layer II code with erasure-correcting capability $\{(0, 0, 1, 1, 1, 1, 1, 1, 1, 2, 3, 3, 3, 6)\}$, which has minimum distance 7 and ANETF 22.7, and the 2-layer EII code with erasure-correcting capability $\{(0, 0, 1, 1, 1, 1, 1, 2, 3, 4, 7)\}$, which has minimum distance 10 and ANETF 22.6. In spite of the large difference in minimum distance, the code with minimum distance 7 has slightly better ANETF than the code with minimum distance 10.

When using the decoding algorithm of Theorem 2 there is less correlation between minimum distance and ANETF. For example, the 4-layer II code $\{(1, 1, 2, (1, 2, 3)), ((1, 2, 3), (1, 2, 3))\}$ in Table I has minimum distance 4 and ANETF 15 when using the decoding algorithm of Theorem 2 while the 4-layer II code $\{(0, 0, 1, 1, 1, 1, 2, 3, 3, 3, 6), (1, 1, 3, (2, 3, 6))\}$ has minimum distance 7 and ANETF 11.8 when using the decoding algorithm. However, we can see that the ANETF is 17 in the first case and 22.3 in the second one when decoding using the parity-check matrices.

The first row in Table I incorporates the (unique) 1-layer code of rate 62/84 and minimum field size of characteristic 2. According to Definition 12 it is an [84, 62, 23] MDS code over $GF(128)$. If we take the 4-layer code with erasure-correcting capability $\{(0, 0, 1, 1, 1, 1, 1, 2, 3, 3, 3, 6)\}$ in Table I its minimum distance is 7, very far from the MDS bound of 23 (the MDS bound of course coincides with the ANETF upper bound). However, the MDS code has no locality (all 84 symbols need to be accessed in the event of a single erasure) and it requires a field of size at least the length of the code, which in this case is $GF(128)$. The ANETF of the 4-layer code, though, is 22.3, which is at 97% of the upper bound. In addition, the 4-layer code has multiple localities in the event of erasures and it is defined over the field $GF(8)$, much smaller than the field $GF(128)$ required by the MDS code.

Let us point out that computing the ANETF is related to birthday surprise types of problems [3], [8], [9], [10] and obtaining exact formulae is possible, but in our case they would be too complicated. Simulations provide good approximations though.

VI. CONCLUSIONS AND FUTURE WORK

We have presented a new definition of Extended Integrated Interleaved (EII) codes that introduces a slight difference with respect to traditional definitions in literature. Mainly, we do not require that the nested codes in the definition have decreasing minimum distances. This slight difference, though, allows for the construction of $\ell$-layer EII codes, a new family of codes that establishes a hierarchy of localities. We showed the properties of the new codes, in particular, their erasure-correcting capability, dimension, minimum distance and parity-check matrices. We introduced a new parameter, the Average Number of Erasures to Failure (ANETF). An upper bound to the ANETF is the MDS bound. We provided some examples of constructions approaching the ANETF upper bound, although the codes are defined over fields much smaller than MDS codes with the same parameters, have different layers of locality and sparse parity-check matrices.
Future research will include adapting the constructions of $\ell$-layer EII codes to codes over any field, as done in \cite{15} for 2-layer EII codes, and for decoding of errors as well.

An intriguing topic of research would be to check the performance of $\ell$-layer EII codes as LDPC codes. In effect, given several $\ell$-layer codes with the same parameters, the larger $\ell$ is, the sparser the parity-check matrix of the code is with respect to the other EII codes with lower layer. For example, take the three next to last rows in Table I corresponding to a 2, 3 and 4-layer code respectively. For the 2-layer code, all the entries in the parity-check matrix are non-zero, for the 3-layer code, 86% of the entries in the parity-check matrix are non-zero, while for the 4-layer code, 56% of the entries are non-zero. We can see that as the layer goes up, the density of non-zero entries goes down significantly. Of course, more than half of the entries of the parity-check being non-zero does not qualify for a code being low density. However, these are toy examples. Normally LDPC codes involve very long codes. For example, if we take the 4-layer code whose parity-check matrix is the $81 \times 420$ matrix given by \cite{11}, we can verify that the density of non-zero entries is only 8.6%.

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| Code Layer | Erasure-Correcting Capability | Minimum Distance | ANETF | Finite Field |
|------------|-------------------------------|-----------------|-------|-------------|
| 1          | (22)                          | 23              | 16.6  | GF(16)      |
| 2          | (1,1,1,1,1,2,2,2,3,3,3,3)     | 4               | 16.0  | GF(16)      |
| 3          | (1,1,2,1,1,2,3,1,2,3,1,2,3)   | 4               | 15.0  | GF(16)      |
| 4          | (((1,1,2),(1,2,3)),((1,2,3),(1,1,2))) | 4  | 15.0  | GF(16)      |
| 5          | (1,1,1,1,1,2,2,3,3,3,4)       | 5               | 18.8  | GF(16)      |
| 6          | (((1,1,2),(1,1,2)),((1,1,2),(1,1,2))) | 5  | 16.4  | GF(16)      |
| 7          | (((1,1,2),(1,2,3)),((1,2,3),(1,2,3))) | 5  | 15.4  | GF(16)      |
| 8          | (0,0,1,1,1,1,2,3,3,3,6)       | 7               | 17.5  | GF(16)      |
| 9          | ((0,0,1),(1,1,3)),((1,1,3),(2,3,6)) | 7  | 12.4  | GF(16)      |
| 10         | ((0,0,1),(1,1,3)),((1,1,3),(2,3,6)) | 7  | 11.8  | GF(16)      |
| 11         | (0,0,1,1,1,1,1,2,3,4,7)       | 10              | 15.9  | GF(16)      |

**TABLE I**

PARAMETERS OF SOME $\ell$-LAYER II AND EII CODES OF RATE 62/84
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