OPTIMAL ORDER PRECONDITIONERS FOR LINEAR SYSTEMS ARISING IN THE SEMISMooth NEWTON METHOD SOLUTION PROCESS OF A CLASS OF OPTIMAL CONSTRAINED PROBLEMS

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Abstract. In this article we present a new multigrid preconditioner for the linear systems arising in the semismooth Newton method solution process of certain control-constrained, quadratic distributed optimal control problems. Using a piecewise constant discretization of the control space, each semismooth Newton iteration essentially requires inverting a principal submatrix of the matrix entering the normal equations of the associated unconstrained optimal control problem, the rows (and columns) of the submatrix representing the constraints deemed inactive at the current iteration. Previously developed multigrid preconditioners by Drăganescu [Optim. Methods Softw., 29 (2004), pp. 786–818] for the aforementioned submatrices were based on constructing a sequence of conforming coarser spaces, and proved to be of suboptimal quality for the class of problems considered. Instead, the multigrid preconditioner introduced in this work uses non-conforming coarse spaces, and it is shown that, under reasonable geometric assumptions on the constraints that are deemed inactive, the preconditioner approximates the inverse of the desired submatrix to optimal order. The preconditioner is tested numerically on a classical elliptic-constrained optimal control problem and further on a constrained image-deblurring problem.

Key words. multigrid, semismooth Newton methods, optimization with PDE constraints, large-scale optimization

AMS subject classifications. 65K10, 65M55, 65M32, 90C06

1. Introduction. The goal of this work is to construct optimal order multigrid preconditioners for optimal control problems of the type

\[
\min_{u \in \mathcal{U}} \frac{1}{2} |\mathcal{K}u - y_d|^2 + \frac{\beta}{2} |u|^2, \quad a \leq u \leq b, \quad a.e.,
\]  

(1.1)

where \( \mathcal{U} \) is a bounded domain, \( y_d \in \mathcal{U} \) is given, and \( \mathcal{K} : \mathcal{U} \to \mathcal{V} \) is a linear continuous operator with \( \mathcal{V} \to \mathcal{U} \) being a compactly embedded subspace of \( \mathcal{U} \). Throughout this article \( \| \cdot \| \) denotes the \( \mathcal{U} \)-norm or the operator-norm of a bounded linear operator in \( \mathcal{L}(\mathcal{U}) \). The functions \( a, b \in \mathcal{U} \) defining the inequality constraints in (1.1) satisfy \( a(x) < b(x) \) for all \( x \in \Omega \). These problems arise in the optimal control of partial differential equations (PDEs), case in which \( \mathcal{K} \) represents the solution operator of a PDE. For example, the classical PDE-constrained optimization problem

\[
\begin{align*}
\min \quad & \frac{1}{2} |y - y_d|^2 + \frac{\beta}{2} |u|^2, \quad (y, u) \in \mathcal{V} \times \mathcal{U} \\
\text{subject to:} \quad & -\Delta y = u, \quad a \leq u \leq b, \quad a.e.,
\end{align*}
\]  

(1.2)

reduces to (1.1) when replacing \( y = \mathcal{K}u \) in the cost functional of (1.2), where \( \mathcal{K} = (-\Delta)^{-1} \) and \( \mathcal{V} = H_0^1(\Omega) \). Another example, discussed (without box constraints) in [7], addresses the problem of time-reversal for parabolic equations, a problem that is ill-posed. In this example we set \( \mathcal{K}u = \mathcal{S}(T)u \), where \( t \mapsto \mathcal{S}(t)u \) is the time-t solution.

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operator of a linear parabolic PDE with initial value \( u \), and \( T > 0 \) is a fixed time. If the solution \( u_{\text{min}} \) of the inverse problem needs to satisfy certain inequality constraints, e.g., when \( u_{\text{min}} \), perhaps representing the concentration of a substance, is required to have values in \([0, 1]\), then it is essential to impose these constraints explicitly in the formulation of the optimization problem, as shown in (1.1). For obvious reasons, in the PDE-constrained optimization literature (1.1) is referred to as the reduced problem. For other applications, such as image deblurring, \( \mathcal{K} \) can be an explicitly defined integral operator

\[
\mathcal{K} u \overset{\text{def}}{=} \int_{\Omega} k(\cdot, x) u(x) \, dx ,
\]

with \( k \in \mathcal{U} \otimes \mathcal{U} \); here \( u \) is the original image and \( y = \mathcal{K} u \) is the blurred image. Thus, by solving (1.1), we seek to reconstruct the image \( u \) giving whose blurred version is a given \( y_d \), subject to additional box constraints.

We give a few references to works on multigrid methods for solving (1.1) with no inequality constraints. In this case (1.1) is equivalent to the Tikhonov regularization of the ill-posed problem \( \mathcal{K} u = y_d \), which in turn reduces to the linear system

\[
(\mathcal{K}^* \mathcal{K} + \beta I)u = \mathcal{K}^* y_d , \tag{1.3}
\]

representing the regularized normal equations of \( \mathcal{K} u = y_d \). A significant literature \[15, 17, 10, 14, 11, 9, 8\], to mention just a few references, is devoted to multigrid methods for (1.3) or the unregularized ill-posed problem. Moreover, when \( \mathcal{K} \) is the solution operator of a linear PDE, an alternative strategy is to solve directly the indefinite systems representing the Karush-Kuhn-Tucker (KKT) optimality conditions instead of the reduced system, and many works \[2, 16, 18, 19, 20\] are concerned with multigrid methods for PDE-constrained optimization problems in unreduced form. A comprehensive discussion of the latter strategy is found in [3].

The presence of bound-constraints in (1.1) brings additional challenges to the solution process, since the KKT optimality conditions form a complementarity system as opposed to a linear or a smooth nonlinear system. As shown by Hintermüller et al. [11], the KKT system can be reformulated as a semismooth nonlinear system for which a superlinearly convergent Newton’s method can be devised – the semismooth Newton method (SSNM). Moreover, with controls discretized using piecewise constant finite elements, Hintermüller and Ulbrich [12] have shown that the SSNM converges in a mesh-independent number of iterations for problems like (1.1), so it is a very efficient solution method in terms of number of optimization iterations. A comprehensive discussion of SSNMs can be found in [21]. However, as with Newton’s method, each SSNM iteration requires the solution of a linear system, and the efficiency of the SSNM depends on the availability of high quality preconditioners for the linear systems involved. For the problem (1.1), this requires essentially inverting at each iteration a principal submatrix of the matrix representing a discrete version \( \mathcal{H}_h \) of \( \mathcal{H} = (\mathcal{K}^* \mathcal{K} + \beta I) \), where \( h \) denotes the mesh size. The multigrid preconditioner developed by Drăgănescu and Dupont in [22] for the operator \( \mathcal{H}_h \) arising in the unconstrained problem (1.3) is shown, under reasonable conditions, to be of optimal order with respect to the discretization: namely, if we denote by \( \mathcal{S}_h \) the multigrid preconditioner (thought of as an approximation of \((\mathcal{H}_h)^{-1})\), then

\[
1 - C \frac{h^p}{\beta} \leq \frac{\langle \mathcal{S}_h u, u \rangle}{\langle (\mathcal{H}_h)^{-1} u, u \rangle} \leq 1 + C \frac{h^p}{\beta}, \quad \forall u \in \mathcal{U}_h \setminus \{0\} , \tag{1.4}
\]
where $p > 0$ is the convergence order of the discretization and $U_h \subset U$ is the discrete control space; for continuous piecewise linear discretizations we have $p = 2$. A natural extension of the ideas in [7] led to the suboptimal multigrid preconditioner developed by Drăganescu in [6] for principal submatrices of $H_h$, where $p$ is shown to essentially be $1/2$ for a piecewise linear discretization, in a rather laborious analysis. The key aspect of defining the multigrid preconditioners for principal submatrices of $H_h$ is the definition of the coarse spaces. The natural domain of a principal submatrix of $H_h$, thought as an operator, is a subspace of $U_h$. The multigrid preconditioner developed in [6] is based on constructing coarse spaces that are subspaces of $U_h$, i.e., conforming coarse spaces. A similar strategy was used by Hoppe and Kornhuber [13] in devising multilevel methods for obstacle problems. However, a visual inspection of the eigenvectors corresponding to the extreme joint eigenvalues of $(H_h)_{-1}$ and $S_h$ in [6] suggests that the main source of suboptimality of the multigrid preconditioner is precisely the “conforming” aspect of the coarse spaces. Thus in this work we defined a new multigrid preconditioner based on non-conforming coarse spaces. While the new construction is limited to piecewise constant approximations of the controls, we can show that, under reasonable conditions, the approximation order of the preconditioner in (1.4) is $p = 1$. It also turns out that the analysis of the new preconditioner very different from, and simpler than the analysis in [6].

This article is organized as follows: in Section 2 we give a formal description of the problem and we briefly describe the SSNM to justify the necessity of preconditioning principal submatrices of $H_h$. Section 3 forms the core of the article; here we introduce and analyze the two-grid preconditioner, the main result being Theorem 3.2. In Section 4 we extend the two-grid results to multigrid; this section follows closely the analogue extension in [6] with certain modifications required by the non-conforming coarse spaces. In Section 5 we show numerical experiments conducted on two test problems: the elliptic constrained problem (1.2) and the box-constrained image deblurring problem. We formulate a set of conclusions in Section 6.

2. Problem formulation. To fix ideas we assume the compactly embedded space to be $V = H_0^1(\Omega)$ and $\Omega$ to be polygonal or polyhedral. We denote by $\|u\|_1 = \|u\|_{H_0^1(\Omega)}$ and we use the convention $\|u\|_0 = \|u\|$. We also define the $H^{-1}$-norm by

$$\|u\|_{-1} \overset{\text{def}}{=} \sup_{v \in V \setminus \{0\}} \frac{\langle u, v \rangle}{\|v\|_1},$$

where $\langle u, v \rangle$ denotes the $L^2(\Omega)$-inner product. We focus on a discrete version of (1.1) obtained by discretizing the continuous operator $K$ using piecewise constant finite elements, as considered by Hintermüller and Ulbrich in [12]. Let $(T_j)_{j \in \mathbb{N}}$ be a family of shape-regular, nested triangulations of $\Omega$ with with $h_j = \max_{T \in T_j} \text{diam}(T)$ being the mesh-size of the triangulation $T_j$, and assume that

$$f_\text{low} \leq h_{j+1}/h_j \leq f_\text{high}$$

(2.1)

for some $0 < f_\text{low} \leq f_\text{high} < 1$ independent of $j$; for example, if $n = 2$ a uniform mesh refinement leads to $f_\text{low} = f_\text{high} = 1/2$. Furthermore, let $U_j$ be the space of continuous piecewise constant functions with respect to $T_j$. Since the triangulations are nested, we have $U_j \subset U_{j+1}$ for all $j \in \mathbb{N}$. We assume that a family of discretizations $K_j \in \mathcal{L}(U_j, V_j)$, $j \in \mathbb{N}$, is given, where $V_j \subset V$ is a conforming finite element space with properties specified below, and $K_j$ represents a discrete version of $K$. The discrete
optimization problem under scrutiny is

\[
\min_{u \in \mathcal{U}_j} J_j^\beta(u) \overset{\text{def}}{=} \frac{1}{2} |K_ju - y_d|^2 + \frac{\beta}{2} |u|^2, \quad a^{(j)} \leq u \leq b^{(j)} \ a.e.,
\]  

(2.2)

where \(a^{(j)}, b^{(j)} \in \mathcal{U}_j\) are discrete functions representing \(a, b\). As in \([9]\), we assume that the operators \(K, K_j\) satisfy the smoothed approximation condition (SAC) appended by \(L^2-L^\infty\) stability of \(K_j\):

**Condition 2.1 (SAC).** There exists a constant \(C_1\) depending on \(\Omega, T_0\) and independent of \(j\) so that

[a] **smoothing:**

\[
\max (|K^*u|_m, |Ku|_m) \leq C_1 |u|, \quad \forall u \in L^2(\Omega), \ m = 0, 1 ;
\]  

(2.3)

[b] **smoothed approximation:** for \(j \in \mathbb{N}\)

\[
|Ku - K_ju| \leq C_1 h_j |u|, \quad \forall u \in \mathcal{U}_j ;
\]  

(2.4)

c] **\(L^2-L^\infty\) stability:** for \(j \in \mathbb{N}\)

\[
\|K_j u\|_{L^\infty(\Omega)} \leq C_1 |u|, \quad \forall u \in \mathcal{U}_j .
\]  

(2.5)

**Remark 2.2.** A simple consequence of Condition 2.1 is that

\[
|Ku| \leq C_1 |u|_1 - 1, \quad \forall u \in \mathcal{U} .
\]  

(2.6)

Indeed, we have

\[
|Ku|^2 = \langle u, K^*Ku \rangle \leq |u|_{-1} \cdot \|K^*Ku|_1 \leq C_1 |u|_{-1} \cdot |Ku| ,
\]

and (2.0) follows.

To describe the semismooth Newton method for the discrete optimization problem, we rewrite (2.2) in vector form. Let \(N_j = \dim(\mathcal{U}_j)\) and \(\varphi_1^{(j)}, \ldots, \varphi_N^{(j)}\) be the standard piecewise constant finite element basis in \(T_{h_j}\). First denote by \(\tilde{K}_j\) the matrix representing the operator \(K_j\), and let \(\tilde{M}_j\) be the mass matrix in \(\mathcal{U}_j\), and \(\hat{M}_j\) the mass matrix in \(\mathcal{V}_j\). Note that the mass matrices \(\tilde{M}_j\) are diagonal. Then (2.2) is equivalent to

\[
\min_{u \in \mathbb{R}^N} J_j^\beta(u) \overset{\text{def}}{=} \frac{1}{2} |(K^* + \beta I) u + \lambda_a - \lambda_b|^2_{\tilde{M}_j} + \frac{\beta}{2} |u|^2_{\hat{M}_j}, \quad a^{(j)} \leq u \leq b^{(j)},
\]  

(2.7)

where \(y_d^{(j)}\) is the vector representing \(\text{Proj}_{\mathcal{V}_j} y_d\), and \(|u|_{\tilde{M}} \overset{\text{def}}{=} \sqrt{u^T \tilde{M} u}\). To simplify the exposition we omit the sub- and superscripts \(j\) for the remainder of this section, so that \(K = K_j, N = N_j, a = a^{(j)}\), etc. Similarly to (1.1), the discrete optimization problem (2.7) has a unique solution, which satisfies the KKT system

\[
\begin{aligned}
(K^* + \beta I) u + \lambda_a - \lambda_b &= f \\
(a - u) \cdot \lambda_a &= 0 \\
(b - u) \cdot \lambda_b &= 0,
\end{aligned}
\]  

(2.8)

where \(K^* = \tilde{M}^{-1}K^T \hat{M}\) is the adjoint of \(K\) with respect to the \(L^2\)-inner product, \(f = K^*y_d\), and \(\lambda_a, \lambda_b \in \mathbb{R}^N\) are the Lagrange multipliers. The inequalities \(u \leq v\) and
the vector-valued product $\mathbf{u} \cdot \mathbf{v}$ are to be understood componentwise. The fact that we are able to write the KKT system in the form (2.8) is not completely obvious, and it relies on the mass matrices $\mathbf{M}_j$ being diagonal. It is worth noting that in [9, 6], where the controls were discretized using continuous piecewise linear functions, the mass-matrices were intentionally modified (equivalently to using a quadrature for computing $L^2$-inner products) so that they be diagonal. Following [11] (see also [8]), the complementarity problem (2.8) is equivalent to (2.9) given the solution $(\mathbf{u}, \lambda)$ of (2.8), the following sets play a role in understanding the equivalence of (2.8) and (2.9):

\begin{align*}
I &= \{ i \in \{1, \ldots, N\} : \lambda_i + \beta(a_i - u_i) < 0 \quad \text{and} \quad \lambda_i + \beta(b_i - u_i) > 0 \} \\
A^a &= \{ i \in \{1, \ldots, N\} : \lambda_i + \beta(a_i - u_i) \geq 0 \quad \text{and} \quad \lambda_i + \beta(b_i - u_i) > 0 \} \\
A^b &= \{ i \in \{1, \ldots, N\} : \lambda_i + \beta(a_i - u_i) < 0 \quad \text{and} \quad \lambda_i + \beta(b_i - u_i) \leq 0 \}.
\end{align*}

Assume (2.9) holds. If $i \in I$, the second equality in (2.9) implies that $\lambda_i = 0$; hence $a_i < u_i < b_i$, so the constraint corresponding to the $i$th component of $\mathbf{u}$ is inactive. Instead, if $i \in A^a$, then $u_i = a_i$, so the lower constraints are active; similarly, if $i \in A^b$, then $u_i = b_i$, so the upper constraints are active. So, if $(\mathbf{u}, \lambda)$ is the solution of (2.9), then $I$ is the set of indices where the constraints are inactive, $A^a$ is the set of indices where the lower constraints are active, while $A^b$ is the set of indices where the upper constraints are active.

The system (2.9) is in fact semismooth due to the fact that the function $\mathbf{u} \mapsto \max(\mathbf{u}, \mathbf{0})$ from $\mathbb{R}^N$ to $\mathbb{R}^N$ is slantly differentiable [11]. Consequently, (2.9) can be solved efficiently using the SSNM, which is equivalent to the primal-dual active set method described below, as shown in [11]. The equivalence of the two is used to prove that the convergence is superlinear. The SSNM is an iterative process that attempts to identify the sets $I$, $A^a$, $A^b$ where the inequality constraints are active/inactive. More precisely, at the $k$th iteration, given sets $I_k$, $A^a_k$, $A^b_k$ partitioning $\{1, \ldots, N\}$, we solve the system

\begin{align*}
\begin{cases}
(K^*K + \beta I)\mathbf{u}^{(k+1)} - \lambda^{(k+1)} = \mathbf{f} \\
\mathbf{u}_i^{(k+1)} = a_i, \quad \text{for } i \in A^a_k, \quad \mathbf{u}_i^{(k+1)} = b_i, \quad \text{for } i \in A^b_k, \\
\lambda_i^{(k+1)} = 0, \quad \text{for } i \in I_k.
\end{cases}
\end{align*}

(2.10)

The solution $(\mathbf{u}^{(k+1)}, \lambda^{(k+1)})$ is then used to define the new sets

\begin{align*}
I_{k+1} &= \{ i : \lambda_i^{(k+1)} + \beta(a_i - u_i^{(k+1)}) < 0 \quad \text{and} \quad \lambda_i^{(k+1)} + \beta(b_i - u_i^{(k+1)}) > 0 \}, \\
A^a_{k+1} &= \{ i : \lambda_i^{(k+1)} + \beta(a_i - u_i^{(k+1)}) \geq 0 \quad \text{and} \quad \lambda_i^{(k+1)} + \beta(b_i - u_i^{(k+1)}) > 0 \}, \\
A^b_{k+1} &= \{ i : \lambda_i^{(k+1)} + \beta(a_i - u_i^{(k+1)}) < 0 \quad \text{and} \quad \lambda_i^{(k+1)} + \beta(b_i - u_i^{(k+1)}) \leq 0 \}.
\end{align*}

The key problem in (2.10) is to identify $\mathbf{u}_i^{(k+1)}$ for $i \in I_k$. If we denote the Hessian of $J^\beta$ from (2.7) by $H = K^*K + \beta I$
and partition the matrix $H$ according to the sets $\mathcal{I}_k$ and $\mathcal{A}_k = \mathcal{A}^+_k \cup \mathcal{A}^-_k$

$$H^{(k)}_{II} = H(\mathcal{I}_k, \mathcal{I}_k), \quad H^{(k)}_{IA} = H(\mathcal{I}_k, \mathcal{A}_k), \quad H^{(k)}_{AI} = H(\mathcal{A}_k, \mathcal{I}_k), \quad H^{(k)}_{AA} = H(\mathcal{A}_k, \mathcal{A}_k),$$

(we used MATLAB notation to describe submatrices) and the vectors $u^{(k+1)}_I$ and $f^{(k)}_I$ as

$$u^{(k+1)}_I = u(\mathcal{I}_k), \quad u^{(k+1)}_A = u(\mathcal{A}_k), \quad f^{(k)}_I = u(\mathcal{I}_k), \quad f^{(k)}_A = u(\mathcal{A}_k),$$

then $u^{(k+1)}_A$ is given explicitly in (2.10), $\lambda^{(k+1)}_A = 0$, and $u^{(k+1)}_I$ satisfies

$$H^{(k)}_{II} u^{(k+1)}_I = f^{(k)}_I - H^{(k)}_{IA} u^{(k+1)}_A . \quad (2.11)$$

The remaining components of $\lambda^{(k+1)}_A$ are given by

$$\lambda^{(k+1)}_A = H^{(k)}_{AI} u^{(k+1)}_I + H^{(k)}_{AA} u^{(k+1)}_A - f^{(k)}_A.$$ 

Therefore, the main challenge in solving (2.10) (which is a linear system) is in fact solving (2.11). The goal of this work is to construct and analyze multigrid preconditioners for the matrices $H^{(k)}_{II}$ appearing in (2.11).

3. The two-grid preconditioner. As in [3, 6], we start by designing a two-grid preconditioner for the principal submatrices of the Hessian $H_j = (K_j^2 K_j + \beta I)$ arising in the SSNM solution process of (2.7), then we follow the idea in [7] to extend in Section 3 the two-grid preconditioner to a multigrid preconditioner of similar asymptotic quality. In this section we assume that we are solving (2.10) at a fixed level $j$, and that we reached a certain iterate $k$ in the SSNM process, with a current guess at the inactive set given by $\mathcal{I}^{(j)} = \mathcal{I}_k^{(j)}$. Since we are not changing the SSNM iteration we discard the sub- and superscripts $k$, and we refer to $\mathcal{I}_k^{(j)}$ as the “current inactive set”.

For constructing the preconditioner it is preferable to regard the matrix $H_j$ as a discretization of the operator

$$\mathcal{H}_j = (K_j^2 K_j + \beta I) \in \mathcal{L}(U_j)$$

representing the Hessian of $J^\beta_j$ in (2.2). We first define the (current) inactive space

$$U^I_j \overset{\text{def}}{=} \text{span}(\phi^{(j)}_i : i \in \mathcal{I}^{(j)}) \subseteq U_j .$$

Furthermore, denote by $\pi^I_j$ the $L^2$-projection onto $U^I_j$ and by

$$\Omega^I_j = \bigcup_{i \in \mathcal{I}^{(j)}} \text{supp}(\phi^{(j)}_i) \subset \Omega$$

the inactive domain. The matrix $H^I_j$ represents the operator

$$\mathcal{H}^I_j \overset{\text{def}}{=} \pi^I_j (K_j^2 K_j + \beta I) \mathcal{E}^I_j \in \mathcal{L}(U^I_j), \quad (3.1)$$

called here the inactive Hessian, where $\mathcal{E}^I_j : U^I_j \to U_j$ is the extension operator. Thus, our goal is to construct a two-grid preconditioner for $\mathcal{H}^I_j$.

The first step, and perhaps the most notable achievement in this work, is the construction of an appropriate coarse space: we define the coarse inactive space
as the span of all coarse basis functions whose support intersect $\Omega^{I}_j$ non-trivially, i.e.,

$$U^{I}_{j-1} \overset{\text{def}}{=} \text{span}\{\varphi^{(j-1)}_i : i \in I^{(j-1)}\} \subseteq U_{j-1},$$

with

$$I^{(j-1)} \overset{\text{def}}{=} \left\{i \in \{1, \ldots, N_{j-1}\} : \mu(\text{supp}(\varphi^{(j-1)}_i) \cap \Omega^{I}_j) > 0\right\}, \quad (3.2)$$

where $\mu$ is the Lebesgue measure in $\mathbb{R}^n$. Similarly, we define the coarse inactive domain by

$$\Omega^{I}_{j-1} = \bigcup_{i \in I^{(j-1)}} \text{supp}(\varphi^{(j-1)}_i) \subseteq \Omega, \quad (3.3)$$

A few remarks are in order. First, since $\mathcal{T}_j$ is a refinement of $\mathcal{T}_{j-1}$ the set $\text{supp}(\varphi^{(j-1)}_i) \cap \Omega^{I}_j$ is a (possibly empty) union of $\mathcal{T}_j$-elements. Since each element making up $\Omega^{I}_j$ lies inside one element making up $\Omega^{I}_{j-1}$ we have the inclusion

$$\Omega^{I}_j \subseteq \Omega^{I}_{j-1}. \quad (3.4)$$

Second, we do not expect in general that $U^{I}_{j-1} \subseteq U^{I}_j$. However, we have

$$U^{I}_{j-1} \subseteq U^{I}_j \quad \text{if and only if} \quad \Omega^{I}_{j-1} = \Omega^{I}_j. \quad (3.5)$$

We also denote

$$\partial_n \Omega^{I}_j \overset{\text{def}}{=} \Omega^{I}_{j-1} \setminus \Omega^{I}_j,$$

a set we call the numerical boundary of $\Omega^{I}_j$ with respect to the coarse mesh.

We would like to contrast the definition (3.2) of the coarse indices with that of Drăgănescu [6], where a coarse basis function enters the span of the coarse inactive space if $\text{supp}(\varphi^{(j-1)}_i) \subseteq \Omega^{I}_j$, this would define a coarse inactive space that lies inside the fine inactive space $U^{I}_j$, and the inclusion (3.4) would be reversed, that is, the coarse inactive domain would be included in the fine inactive domain.

We now define the two-grid preconditioner $M_j \in \mathfrak{L}(U^{I}_j)$ by

$$M_j \overset{\text{def}}{=} \pi^{I}_j \left((\mathcal{H}^{I}_{j-1})^{-1} \pi^{I}_{j-1} + \beta^{-1}(I - \pi^{I}_{j-1})\right). \quad (3.6)$$

The definition (3.6) is rooted in the two-grid preconditioner definition from [7, 6]; the difference lies the presence of the action of the projection $\pi^{I}_j$ as the last step in (3.6) (left-most term), which is necessary precisely because $U^{I}_{j-1}$ is not expected to be a subspace of $U^{I}_j$. An operator related to $M_j$, necessary for the analysis, is

$$S_j \overset{\text{def}}{=} \pi^{I}_j \left(\mathcal{H}^{I}_{j-1} \pi^{I}_{j-1} + \beta(I - \pi^{I}_{j-1})\right). \quad (3.7)$$

**Remark 3.1.** Both $M_j$ and $S_j$ are symmetric with respect to the $L^2$-inner product, that is,

$$\langle M_j u, v \rangle = \langle u, M_j v \rangle, \quad \langle S_j u, v \rangle = \langle u, S_j v \rangle, \quad \forall u, v \in U^{I}_j.$$


In addition, if \( U_{j-1}^0 \subseteq U_j^1 \), then \( \mathcal{M}_j = (\mathcal{S}_j)^{-1} \).

The key to the last assertions in Remark 3.1 is that \( \pi^1_j \) has no effect (hence can be discarded) when \( U_{j-1}^0 \subseteq U_j^1 \). Our ultimate goal is to estimate the spectral distance between \( (\mathcal{H}^1_j)^{-1} \) and \( \mathcal{M}_j \), as a measure of their spectral equivalence (see definition below). As an intermediate step we will estimate the spectral distance between \( \mathcal{H}^1_j \) and \( \mathcal{S}_j \).

Given a Hilbert space \((\mathcal{X}, (\cdot, \cdot))\), we denote by \( \Sigma_+ (\mathcal{X}) \) the set of symmetric positive definite operators in \( \Sigma (\mathcal{X}) \). The spectral distance between \( A, B \in \Sigma_+ (\mathcal{X}) \), introduced in [7] to analyze multigrid preconditioners for inverse problems like (1.3), is given by

\[
d_{\mathcal{X}}(A, B) = \sup_{u \in \mathcal{X} \setminus \{0\}} \left| \min \frac{\langle Au, u \rangle}{\langle Bu, u \rangle} \right|.
\]

If \( \delta \) is the smallest number for which the following inequalities hold

\[
1 - \delta \leq \frac{\langle Au, u \rangle}{\langle Bu, u \rangle} \leq 1 + \delta, \quad \forall u \neq 0,
\]

and \( \delta \ll 1 \), then \( d_{\mathcal{X}}(A, B) \approx \delta \). The spectral distance not only allows to write the above inequalities in a more compact form, but some of its properties (including the fact that it is a distance function) are also used in the analysis. The main result in this article is the following theorem.

**Theorem 3.2.** Assuming Condition 2.1 holds, there exists constants \( \delta > 0 \) and \( C_{1g} > 0 \) independent of \( j \) and the inactive set \( \mathcal{I}^{(j)} \) so that if \( \beta^{-1}(h_j + \mu(\partial_n \Omega^1_j)) < \delta \) the following holds:

\[
d_{\mathcal{X}}(\mathcal{M}_j, (\mathcal{H}^1_j)^{-1}) \leq C_{1g} \beta^{-1}(h_j + \mu(\partial_n \Omega^1_j)). \quad (3.8)
\]

We postpone the proof of Theorem 3.2 after a few preliminary results.

**Remark 3.3.** Without further formalizing the argument, we would like to comment on the optimality of the result in Theorem 3.2. First, it should be recognized that \( \mu(\partial_n \Omega^1_j) \) can be \( O(1) \) for certain choices of \( \Omega^1_j \). For example, if \( T_j^1 \) is a uniform refinement of \( T_{j-1}^1 \) in two dimensions, and \( \Omega_{j-1}^1 \) contains exactly one level-\( j \) subdivision of each of the level-\( (j-1) \) triangles that make up \( \Omega \), then \( U_{j-1}^0 = U_{j-1} \) (the entire coarse space) and \( \Omega_{j-1}^1 = \Omega \); thus \( \mu(\partial_n \Omega_{j-1}^1) = \frac{2}{3} \mu(\Omega) \). In this case the two-grid preconditioner is not efficient. However, if \( \Omega^1_j \) is a good approximation of the correct inactive domain \( \Omega^1 = \{ x \in \Omega : a < u_{\min}(x) < b \} \), and \( \Omega^1 \) is sufficiently regular, e.g., has a Lipschitz boundary, then we expect \( \mu(\partial_n \Omega^1_j) \approx Ch_j \). It is in this sense that we regard Theorem 3.2 as proof of the fact that the two-grid preconditioner \( \mathcal{M}_j \) approximates the operator \( (\mathcal{H}^1_j)^{-1} \) with optimal order.

The optimality result in the following lemma is a critical component for the proof of Theorem 3.2.

**Lemma 3.4.** There exists a constant \( C_2 \) independent of the mesh-size \( h_j \) and on the inactive set \( \mathcal{I}^{(j)} \) so that

\[
\| (I - \pi_{j-1}^1) u \|_{-1} \leq C_2 h_j \| u \|, \quad \forall u \in U_j^1, \quad (3.9)
\]

where \( (I - \pi_{j-1}^1) u \) is extended with zero outside its support.

**Proof.** For \( u \in U_j^1 \) we have

\[
\| (I - \pi_{j-1}^1) u \|_{-1} = \sup_{v \in H^1_0(\Omega)} \frac{\langle (I - \pi_{j-1}^1) u, v \rangle}{\| v \|_1}.
\]
Since supp\((I - \pi_{j-1}^I)u\) ⊆ \(\Omega^1_j \cup \Omega^1_{j-1} = \Omega^1_{j-1}\),

\[
\langle (I - \pi_{j-1}^I)u, v \rangle = \int_{\Omega^1_{j-1}} (I - \pi_{j-1}^I)u(v - \pi_{j-1}^Iv) = \int_{\Omega^1_{j-1}} (I - \pi_{j-1}^I)u(v - \pi_{j-1}^Iv) \\
\leq \| (I - \pi_{j-1}^I)u \|_{L^2(\Omega^1_{j-1})} \cdot \| (I - \pi_{j-1}^I)v \|_{L^2(\Omega^1_{j-1})} \\
\leq 2\|u\| \cdot \| (I - \pi_{j-1}^I)v \|_{L^2(\Omega^1_{j-1})},
\]

Now

\[
\| (I - \pi_{j-1}^I)v \|^2_{L^2(\Omega^1_j)} = \sum_{i \notin \mathcal{T}^{(j-1)}} \| (I - \pi_{j-1}^I)v \|^2_{L^2(\mathcal{T}^{(j-1)})} \\
\leq \tilde{C}^2 h_{j-1}^{-2} \sum_{i \notin \mathcal{T}^{(j-1)}} \|v\|^2_{H^1(\mathcal{T}^{(j-1)})} \leq f_{low}^{-2} \tilde{C}^2 h_{j-1} \|v\|^2_{H^1(\Omega)},
\]

where \(\tilde{C}\) is the constant (uniform with respect to \(i\) and \(j\) due to shape-regularity) appearing in the Bramble-Hilbert Lemma on each element \(T_i^{(j-1)}\) in \(\mathcal{T}_{j-1}\) with \(i \in \mathcal{T}^{(j-1)}\); we also used the fact that the \(L^2\)-projection is local for the finite element space under consideration, that is, \(\pi_{j-1}^Iv\mid_{T_i^{(j-1)}}\) is the average of \(v\) on the element \(T_i^{(j-1)}\). It follows that

\[
\langle (I - \pi_{j-1}^I)u, v \rangle \leq f_{low}^{-1} \tilde{C} h_j \|u\| \cdot \|v\|_1, \quad \forall v \in H^1_0(\Omega),
\]

which implies the desired result with \(C_2 = f_{low}^{-1} \tilde{C}\).

**Remark 3.5.** It is remarkable that the constant \(C_2\) is independent of the inactive set, and depends only on the constant appearing in the Bramble-Hilbert lemma and the refinement ratio. Also, what makes the optimal estimate (3.9) possible, is the inclusion \(\Omega_j^1 \subseteq \Omega^1_{j-1}\). If our choice of spaces had led to \(\Omega^1_{j-1} \subsetneq \Omega^1_j\), then the term to estimate would be \(\| (I - \pi_{j-1}^I)v \|_{L^2(\Omega_j^1)}\), which is expected to be of size \(\sqrt{\mu(\Omega_j^1 \setminus \Omega^1_{j-1})}\); as shown in [4], the latter term is of often size \(\sqrt{h_j}\).

**Proposition 3.6.** Under the assumptions of Theorem 3.2 there exists \(C_3,\delta > 0\) independent of \(j,\beta\) and the fine inactive set \(\mathcal{I}^{(j)}\) so that, if \(h_j/\beta < \delta\), then

\[
d_{U_j} \langle S_j, H^1_j \rangle \leq C_3 h_j \frac{h_j}{\beta}.
\]

**Proof.** As in Lemma [4], functions are extended with zero outside their support when necessary. We have for \(u \in U_j^1\)

\[
\langle (H_j^1 - S_j)u, u \rangle = \langle \pi_j^1 (\mathcal{K}_j - \pi_{j-1}^I \mathcal{K}_{j-1} - \pi_{j-1}^I)u, u \rangle = \langle \mathcal{K}_j u, \mathcal{K}_j u \rangle - \langle \mathcal{K}_{j-1} \pi_{j-1}^I u, \mathcal{K}_{j-1} \pi_{j-1}^I u \rangle \\
= \| \mathcal{K}_j u \|^2 - \| \mathcal{K}_j u \|^2 + \| \mathcal{K}_j u \|^2 - \| \mathcal{K}_{j-1} \pi_{j-1}^I u \|^2 + \| \mathcal{K}_{j-1} \pi_{j-1}^I u \|^2 - \| \mathcal{K}_{j-1} \pi_{j-1}^I u \|^2.
\]

Condition [2.1] implies that

\[
|A_1| = \| (\mathcal{K}_{j-1} - \mathcal{K})u \|^2 \leq \| (\mathcal{K}_{j-1} - \mathcal{K})u \| \cdot (\| \mathcal{K}_j u \| + \| \mathcal{K}_j u \|) \leq 2C^2 h_j \|u\|,
\]

\[
|A_2| = \| (\mathcal{K}_{j-1} - \mathcal{K})u \|^2 \leq \| (\mathcal{K}_{j-1} - \mathcal{K})u \|^2 \leq 2C^2 h_j \|u\|,
\]

\[
|A_3| = \| (\mathcal{K}_{j-1} - \mathcal{K})u \|^2 \leq \| (\mathcal{K}_{j-1} - \mathcal{K})u \|^2 \leq 2C^2 h_j \|u\|,
\]

\[
|A_4| = \| (\mathcal{K}_{j-1} - \mathcal{K})u \|^2 \leq \| (\mathcal{K}_{j-1} - \mathcal{K})u \|^2 \leq 2C^2 h_j \|u\|,
\]

\[
|A_5| = \| (\mathcal{K}_{j-1} - \mathcal{K})u \|^2 \leq \| (\mathcal{K}_{j-1} - \mathcal{K})u \|^2 \leq 2C^2 h_j \|u\|,
\]

\[
|A_6| = \| (\mathcal{K}_{j-1} - \mathcal{K})u \|^2 \leq \| (\mathcal{K}_{j-1} - \mathcal{K})u \|^2 \leq 2C^2 h_j \|u\|.
\]
and a similar estimate holds for the term $A_3$ with a constant depending on $C_1$ and $f_{low}$. For the second term $A_2$ we have

$$
|A_2| = \left| |Ku|^2 - |\mathcal{K}1_{j-1}u|^2 \right| \leq \|K(I - \pi_{j-1})u\| \cdot (|Ku| + |\mathcal{K}1_{j-1}u|)
$$

$$
\leq 2C_i^2C_2|u|.
$$

The symmetry of $(\mathcal{H}_{j}^1 - S_j)$ implies that

$$
\|\mathcal{H}_{j}^1 - S_j\| \leq C'h_j,
$$

with $C'$ depending on $C_1, C_2, f_{low}$, but not on $h_j$ or the inactive set $\mathcal{I}^{(j)}$. The rest of the argument follows closely the proof of Theorem 4.9 in [9], and we provide it for completeness. Since $\mathcal{H}_{j}^1$ is symmetric and $\langle \mathcal{H}_{j}^1 u, u \rangle \geq \beta \|u\|^2$, $\forall u \in \mathcal{U}_j$, it follows that

$$
\sigma \left( (\mathcal{H}_{j}^1)^{-\frac{1}{2}} \right) \subseteq (0, \beta^{-\frac{1}{2}}], \quad \text{therefore } \| (\mathcal{H}_{j}^1)^{-\frac{1}{2}} \| \leq \beta^{-\frac{1}{2}}.
$$

Hence

$$
|I - (\mathcal{H}_{j}^1)^{-\frac{1}{2}} S_j (\mathcal{H}_{j}^1)^{-\frac{1}{2}}| \leq \beta^{-1} |\mathcal{H}_{j}^1 - S_j| \leq C'h_j/\beta.
$$

If $C'h_j/\beta < 1/2$, then

$$
W(S_j, \mathcal{H}_{j}^1) = \left\{ \frac{\langle S_j u, u \rangle}{\langle \mathcal{H}_{j}^1 u, u \rangle} : u \in \mathcal{U}_j \right\} \subseteq \left[ \frac{1}{2}, \frac{3}{2} \right].
$$

By Lemma 3.2 in [7]

$$
\sup_{u \in \mathcal{U}_j} \left| \ln \left( \frac{\langle S_j u, u \rangle}{\langle \mathcal{H}_{j}^1 u, u \rangle} \right) \right| \leq \frac{3}{2} |I - (\mathcal{H}_{j}^1)^{-\frac{1}{2}} S_j (\mathcal{H}_{j}^1)^{-\frac{1}{2}}| \leq \frac{3C'h_j}{2\beta},
$$

which proves (3.10) with $C_3 = 3C'/2$. □

Another essential element in the proof of Theorem 3.10 is the following additional enriched level-$j$ inactive set and associated space:

$$
\hat{\mathcal{I}}^{(j)} \overset{\text{def}}{=} \{ i \in \{1, \ldots, N_j \} : \text{supp}(\varphi^{(j)}) \subseteq \Omega_{j-1} \}, \quad (3.12)
$$

$$
\hat{\mathcal{U}}_{j} \overset{\text{def}}{=} \text{span}\{\varphi^{(j)} : i \in \hat{\mathcal{I}}^{(j)}\}. \quad (3.13)
$$

It is obvious that $\hat{\mathcal{U}}_{j}$ includes both $\mathcal{U}_{j-1}$ and $\mathcal{U}_j$; it could also be regarded as the level-$j$ inactive space whose inactive domain is identical to $\Omega_{j-1}$. We should also point out that the coarse inactive index set generated by $\hat{\mathcal{I}}^{(j)}$ is still $\mathcal{I}^{(j-1)}$, therefore the coarse inactive space associated with $\hat{\mathcal{U}}_{j}$ is identical to that associated with $\mathcal{U}_{j}$, namely $\mathcal{U}_{j-1}$.

Let $\hat{\pi}_{j}^1$ be the $L^2$-projection onto $\hat{\mathcal{U}}_{j}$ and $\hat{\mathcal{E}}_{j} : \hat{\mathcal{U}}_{j} \rightarrow \mathcal{U}_j$ be the extension operator. We now define the inactive Hessian and two-grid preconditioners associated with $\hat{\mathcal{I}}^{(j)}$, all of which are to be regarded as operators in $\mathfrak{L}(\hat{\mathcal{U}}_{j})$:

$$
\hat{H}_{j} \overset{\text{def}}{=} \hat{\pi}_{j}^1(\mathcal{K}_j^2 + \beta I)\hat{\mathcal{E}}_{j}, \quad (3.14)
$$

$$
\hat{S}_j \overset{\text{def}}{=} \mathcal{H}_{j-1}^{-1} \hat{\pi}^1_{j-1} + \beta(I - \hat{\pi}_{j-1}), \quad (3.15)
$$

$$
\hat{\mathcal{M}}_j \overset{\text{def}}{=} (\mathcal{H}_{j-1})^{-1} \pi_{j-1} + \beta^{-1}(I - \pi_{j-1}). \quad (3.16)
$$
Also, let \( \eta_j^1 : U^1_j \to \widehat{U}_j^1 \) be the extension operator.

**Lemma 3.7.** There exists constants \( \delta > 0 \) and \( C_4 > 0 \) independent of \( j \) and the inactive set \( \widehat{T}^{(j)} \) so that if \( \beta^{-1} \mu(\partial_n \Omega_j^1) < \delta \) then

\[
d_{\mathcal{U}_j^1} \left( (\mathcal{H}^1_j)^{-1}, \pi_j^1(\widehat{\mathcal{H}}_j^1)^{-1} \eta_j^1 \right) \leq C_4 \frac{\mu(\partial_n \Omega_j^1)}{\beta}. \tag{3.17}
\]

**Proof.** The first task is to find a practical expression for the operator \( \pi_j^1(\widehat{\mathcal{H}}_j^1)^{-1} \eta_j^1 \).

Let \( \mathcal{U}^1_{j,c} \) be the orthogonal complement of \( \mathcal{U}_j^1 \) in \( \widehat{U}_j^1 \), so that \( \widehat{U}^1_j = \mathcal{U}_j^1 \oplus \mathcal{U}^1_{j,c} \); note that functions in \( \mathcal{U}^1_{j,c} \) are supported in \( \partial_n \Omega_j^1 \). Furthermore, let \( \pi_{j,c}^1 \) be the orthogonal projector on \( \mathcal{U}^1_{j,c} \) be the projector and \( \eta_{j,c}^1 : \mathcal{U}^1_{j,c} \to \widehat{U}_j^1 \) be the extension operator.

Following the block-splitting of the matrix representing \( \widehat{\mathcal{H}}_j^1 \), we define the operators

\[
\mathcal{H}^1_{j,oc} = \pi_{j,c}^1 \widehat{\mathcal{H}}_j^1 \eta_{j,c}^1 \in \mathcal{L}(\mathcal{U}^1_{j,c}, \mathcal{U}_j^1), \quad \mathcal{H}^1_{j,co} = \pi_{j,c}^1 \widehat{\mathcal{H}}_j^1 \eta_{j,c}^1 \in \mathcal{L}(\mathcal{U}_j^1, \mathcal{U}^1_{j,c})
\]

\[
\mathcal{H}^1_{j,cc} = \pi_{j,c}^1 \widehat{\mathcal{H}}_j^1 \eta_{j,c}^1 \in \mathcal{L}(\mathcal{U}^1_{j,c}, \mathcal{U}^1_{j,c}).
\]

Naturally, \( \mathcal{H}^1_{j,oc} = \pi_{j,c}^1 \widehat{\mathcal{H}}_j^1 \eta_{j,c}^1 \), and \( \mathcal{H}^1_{j,co} = (\mathcal{H}^1_{j,co})^* \). To ease notation, in the first part of this analysis we eliminate the sub- or super-scripts “\( j \)”, since the level does not vary, so \( \widehat{T} = \widehat{T}^{(j)} \), \( \widehat{U}^1 = \widehat{U}_j^1 \), \( \mathcal{H}^1_{co} = \mathcal{H}^1_{j,co} \), etc. Accordingly, if \( \widehat{u} = u + u_c \) with \( u \in \mathcal{U}^1 \), \( u_c \in \mathcal{U}^1_{c} \), we have

\[
\widehat{\mathcal{H}}^1 \widehat{u} = \underbrace{(\mathcal{H}^1 u + \mathcal{H}^1_{oc} u_c)}_{\in \mathcal{U}^1} + \underbrace{(\mathcal{H}^1_{co} u + \mathcal{H}^1_{cc} u_c)}_{\in \mathcal{U}^1_{c}}. \tag{3.18}
\]

We also define the Schur-complement of \( \mathcal{H}^1 \) in \( \widehat{\mathcal{H}}^1 \)

\[
\mathcal{G} = \mathcal{H}^1_{cc} - \mathcal{H}^1_{co}(\mathcal{H}^1_{co})^{-1} \mathcal{H}^1_{oc} \in \mathcal{L}(\mathcal{U}^1_{c}, \mathcal{U}^1_{c}).
\]

Note that \( \mathcal{G} \) is symmetric. For \( u_c \in \mathcal{U}^1_{c} \) define \( u = -(\mathcal{H}^1_{oc})^{-1} \mathcal{H}^1_{co} u_c \in \mathcal{U}^1 \) and \( \widehat{u} = u + u_c \). A simple calculation shows that

\[
\langle \mathcal{G} u_c, u_c \rangle = \langle \widehat{\mathcal{H}}^1 \widehat{u}, \widehat{u} \rangle \geq \beta \langle \widehat{u}, \widehat{u} \rangle = \beta (|u|^2 + |u_c|^2) > \beta |u_c|^2. \tag{3.19}
\]

(This is simply saying that the smallest eigenvalue of the Schur-complement is greater than the smallest eigenvalue of the original operator). Hence, it follows that

\[
\| \mathcal{G}^{-1} \| \leq \beta^{-1}. \tag{3.20}
\]

When solving \( \widehat{\mathcal{H}}^1 \widehat{u} = y \) for \( y \in \mathcal{U}^1 \), standard block-elimination yields \( \widehat{u} = u + u_c \) with

\[
u = (\mathcal{H}^1)^{-1} \left( I + \mathcal{H}^1_{oc} \mathcal{G}^{-1} \mathcal{H}^1_{co}(\mathcal{H}^1)^{-1} \right) y, \quad u_c = -\mathcal{G}^{-1} \mathcal{H}^1_{co}(\mathcal{H}^1)^{-1} y.
\]

The first equation above shows that

\[
\pi^1(\widehat{\mathcal{H}}^1)^{-1} \eta^1 = (\mathcal{H}^1)^{-1} (I + \mathcal{H}^1_{oc} \mathcal{G}^{-1} \mathcal{H}^1_{co}(\mathcal{H}^1)^{-1}). \tag{3.21}
\]
To estimate the spectral distance between \( \pi^1(\mathcal{H}^l)^{-1} \eta^l \) and \( (\mathcal{H}^l)^{-1} \) we bound

\[
\sup_{u \in U^l} \left| 1 - \frac{\langle \pi^1(\mathcal{H}^l)^{-1} \eta^l u, u \rangle}{\langle (\mathcal{H}^l)^{-1} u, u \rangle} \right| \leq \sup_{v \in U^l \setminus \{0\}} \left| \frac{\langle (\mathcal{H}^l)^{-1} \eta^l G^{-1} \mathcal{H}^l_{oc} (\mathcal{H}^l)^{-1} u, u \rangle}{\langle (\mathcal{H}^l)^{-1} u, u \rangle} \right| \tag{3.21}
\]

\[
= \sup_{v \in U^l \setminus \{0\}} \left| \frac{\langle (\mathcal{H}^l)^{-1} \eta^l G^{-1} \mathcal{H}^l_{oc} (\mathcal{H}^l)^{-1} v, v \rangle}{\langle v, v \rangle} \right| = \langle (\mathcal{H}^l)^{-1} \eta^l G^{-1} \mathcal{H}^l_{oc} (\mathcal{H}^l)^{-1} \rangle \tag{3.22}
\]

since \( (\mathcal{H}^l)^{-1} \eta^l \mathcal{H}^l_{oc} = \mathcal{H}^l_{oc} (\mathcal{H}^l)^{-1} \) as they are dual to each other.

We resume using the index \( j \) as we estimate \( \langle (\mathcal{H}^l)^{-1} \mathcal{H}^l_{j,oc} \rangle \). Following (3.19), for all \( u_c \in U^l_{j,c} \)

\[
|\langle (\mathcal{H}^l)^{-1} \mathcal{H}^l_{j,oc} u_c \rangle|^2 = \langle (\mathcal{H}^l_{j,co} (\mathcal{H}^l_{j})^{-1} \mathcal{H}^l_{j,oc} u_c, u_c \rangle \leq \langle (\mathcal{H}^l_{j,cc} - \beta I) u_c, u_c \rangle \leq \langle \pi^1_{j,c} (\mathcal{H}^l_{j} - \beta I) u_c, u_c \rangle.
\]

If we define by \( \mathcal{E}^l_{j,c} \in L(U^l_{j,c}, U_j) \) the extension-with-zero operator, then

\[
\pi^1_{j,c} (\mathcal{H}^l_{j} - \beta I) u_c = \pi^1_{j,c} (\mathcal{K}^*_j \mathcal{K}_j) \mathcal{E}^l_{j,c}.
\]

Therefore,

\[
|\langle (\mathcal{H}^l)^{-1} \mathcal{H}^l_{j,oc} u_c \rangle|^2 \leq \langle \pi^1_{j,c} (\mathcal{K}^*_j \mathcal{K}_j) \mathcal{E}^l_{j,c} u_c, u_c \rangle = \| \mathcal{K}_j \mathcal{E}^l_{j,c} u_c \|^2 \leq \| \mathcal{K}_j \mathcal{E}^l_{j,c} u_c \|^2 \leq C_i^2 \| \mathcal{E}^l_{j,c} u_c \|^2 \mu(\partial_n \Omega_j^l) = C_i^2 \| u_c \|^2 \mu(\partial_n \Omega_j^l).
\]

It follows that

\[
\| (\mathcal{H}^l)^{-1} \mathcal{H}^l_{j,oc} \| \leq C_1 \sqrt{\mu(\partial_n \Omega_j^l)} \tag{3.23}
\]

So, by (3.22) and (3.23) we get

\[
\sup_{u \in U^l} \left| 1 - \frac{\langle \pi^1(\mathcal{H}^l)^{-1} \eta^l u, u \rangle}{\langle (\mathcal{H}^l)^{-1} u, u \rangle} \right| \leq C_i^2 \beta^{-1} \mu(\partial_n \Omega_j^l).
\]

If \( C_i^2 \beta^{-1} \mu(\partial_n \Omega_j^l) < \frac{1}{2} \), by Lemma 3.2 in [7]

\[
\left| \ln \frac{\langle \pi^1(\mathcal{H}^l)^{-1} \eta^l u, u \rangle}{\langle (\mathcal{H}^l)^{-1} u, u \rangle} \right| \leq \frac{3}{2} C_i^2 \beta^{-1} \mu(\partial_n \Omega_j^l),
\]

which concludes the proof. \( \square \)
We now return to the proof of Theorem 3.2.

Proof. We refer to the space \( \tilde{U}_j \) defined in (3.13) and the associated operators \( \tilde{H}_j, \tilde{S}_j, \tilde{M}_j \) defined in (3.14)-(3.16). Cf. Remark 3.1 because \( U_{j-1} \subseteq \tilde{U}_j \), we have \((\tilde{S}_j)^{-1} = \tilde{M}_j \). By Lemma 3.10 in [7] we have
\[
d_{\tilde{U}_j} \left( \tilde{M}_j, (\tilde{H}_j)^{-1} \right) = d_{\tilde{U}_j} \left( \tilde{S}_j, (\tilde{H}_j)^{-1} \right).
\]
Hence,
\[
d_{U_j} \left( M_j, (H_j)^{-1} \right) = \sup_{u \in U_j \setminus \{0\}} \left| \frac{\langle \pi_j \left( (H_j^{-1})^{-1} \pi_j + \beta^{-1}(I - \pi_j) \right) u, u \rangle}{\langle (H_j^{-1}) u, u \rangle} \right|
\]
\[
\leq \sup_{u \in U_j \setminus \{0\}} \left| \frac{\langle \tilde{M}_j u, u \rangle}{\langle (H_j)^{-1} u, u \rangle} \right| + \sup_{u \in U_j \setminus \{0\}} \left| \frac{\langle \hat{H}_j^{-1} u, u \rangle}{\langle (H_j^{-1})^{-1} u, u \rangle} \right|
\]
\[
\leq d_{\tilde{U}_j} \left( \tilde{M}_j, (\tilde{H}_j)^{-1} \right) + \sup_{u \in U_j \setminus \{0\}} \left| \frac{\langle \pi_j (\hat{H}_j)^{-1} u, u \rangle}{\langle (H_j)^{-1} u, u \rangle} \right|.
\]

By Proposition 3.6, the first term above is bounded by \( C_3 \beta^{-1} h_j \), assuming \( \beta^{-1} h_j \) is sufficiently small. The second term expresses the spectral distance between \( \pi_j (\hat{H}_j)^{-1} \eta_j \) and \( (\hat{H}_j)^{-1} \), and is bounded by \( C_4 \beta^{-1} \mu(\partial_\Omega_j) \) provided \( \beta^{-1} \mu(\partial_\Omega_j) \) is sufficiently small, cf. Lemma 3.7 which concludes the proof. \( \square \)

4. The multigrid preconditioner. The extension of the two-grid preconditioner introduced in Section 3 to a multigrid preconditioner follows closely [6]. However, since the use of non-conforming spaces requires a few changes both in the construction and the analysis, we give here a full description of the extension process.

As in [6], we adopt the following point of view: the level for which we construct a multigrid preconditioner is given to be \( j \) and is considered fixed, and we also fix an inactive set \( I^{(j)} \), which corresponds to one of the SSNM iterations. This leads to the definition of \( H_j \) as in (3.11). As with other multigrid methods for integral equations of the second kind, the base level, denoted by \( j_0 \), may not necessarily be the coarsest case available, i.e., \( j_0 = 0 \), but has to sufficiently fine for the conditions in Theorem 4.6 below to be satisfied. The goal is to construct the operator \( Z_j \) representing the multigrid preconditioner for \( H_j \), i.e., an approximation of \( (H_j)^{-1} \).

4.1. Construction and complexity. The first step in building the multigrid preconditioner is to construct the coarse inactive spaces and operators for the levels \( k = j - 1, j - 2, \ldots, j_0 \), in accordance with (3.2). More precisely, after defining
\[
\Omega_j^i = \bigcup_{i \in I^{(j)}} \text{supp}(\varphi_{i}^{(j)}),
\]

...
we construct recursively the coarser inactive index-sets, domains, and spaces.

**Algorithm 4.1** (Inactive set, inactive domain definition).
1. for $k = (j - 1) : -1 : j_0$
2. \[ \mathcal{I}^{(k)} \text{ def } \left\{ i \in \{1, \ldots, N_k \} : \mu(\text{supp}(\varphi_i^{(k)}) \cap \Omega_k^{1}) > 0 \right\} \]
3. \[ \Omega_k^{1} \text{ def } \bigcup_{i \in \mathcal{I}^{(k)}} \text{supp}(\varphi_i^{(k)}) \]
4. end

With inactive index-sets constructed, we now define, as before, the inactive spaces and operators for $k = j_0, \ldots, j$:

\[ \mathcal{U}_k^1 \text{ def } \text{span}\{\varphi_i^{(k)} : i \in \mathcal{I}^{(k)}\}, \]
\[ \mathcal{H}_k^1 \text{ def } \pi_k^{1}(K_k^{*}K_k + \beta I)\mathcal{E}_k^1 \in \mathcal{L}(\mathcal{U}_k^1). \]

Recall that $\mathcal{U}_k^1 \subseteq \mathcal{U}_k$, but we do not expect in general that $\mathcal{U}_k^1 \subseteq \mathcal{U}_{k+1}^1$. However, the inclusion $\Omega^1_{k+1} \subseteq \Omega^1_k$ holds for $k = j_0, \ldots, j - 1$. We also define for $k = 1, 2, \ldots$ the operators

\[ \mathcal{J}^{k-1}_k : \mathcal{L}(\mathcal{U}^1_{k-1}) \to \mathcal{L}(\mathcal{U}^1_{k}), \quad \mathcal{J}^{k-1}_k(\mathcal{X}) = \pi_k^{1}(\mathcal{X} \cdot \pi^{1}_{k-1} + \beta^{-1}(I - \pi^{1}_{k-1})). \quad (4.1) \]

Note that the two-grid preconditioner $\mathcal{M}_j$ can be written as

\[ \mathcal{M}_j = \mathcal{J}^{j-1}_j ((\mathcal{H}^{j-1}_j)^{-1}) . \quad (4.2) \]

Another essential element in defining the multigrid preconditioner is the family of operators $\mathcal{M}_k, k = j_0, \ldots, j$, given by

\[ \mathcal{R}_k : \mathcal{L}(\mathcal{U}_k^1) \to \mathcal{L}(\mathcal{U}_k^1), \quad \mathcal{R}_k(\mathcal{X}) \text{ def } 2\mathcal{X} - \mathcal{X} \cdot \mathcal{H}_k^1 \cdot \mathcal{X}. \]

It is known that $\mathcal{X}_{l+1} = \mathcal{R}_k(\mathcal{X}_l), l = 1, 2, \ldots$, represents the Newton iteration for solving the nonlinear operator-equation $\mathcal{X}^{-1} - \mathcal{H}_k^1 = 0$ (e.g., see [7]).

The following algorithm produces for $k = j_0 + 1, \ldots, j$ a sequence of operators $\mathcal{Z}_k \in \mathcal{L}(\mathcal{U}_k^1)$, of which $\mathcal{Z}_j$ is the desired multigrid preconditioner.

**Algorithm 4.2** (Operator-form definition of $\mathcal{Z}_k$; input arguments: $j \geq j_0 + 1$).
1. $\mathcal{Z}_{j_0} \text{ def } (\mathcal{H}^{j_0}_j)^{-1}$ % base level
2. for $k = j_0 + 1 : j - 1$ % intermediate levels (if any)
3. $\mathcal{Z}_k \text{ def } \mathcal{R}_k(\mathcal{J}^{k}_{k-1}(\mathcal{Z}_{k-1}))$
4. end
5. $\mathcal{Z}_j \text{ def } \mathcal{J}^{j}_{j-1}(\mathcal{Z}_{j-1})$ % finest level

Algorithm 4.2 shows that $\mathcal{Z}_j$ has a W-cycle structure. Moreover, for $k < j$, applying $\mathcal{Z}_k$ involves one application of $\mathcal{H}^{j}_{j_0}$. To estimate the cost of applying $\mathcal{Z}_j$ we make some assumptions with respect to the cost of applying $\mathcal{H}^{1}_{j_0}$ and the cost of inverting $\mathcal{H}^{1}_{j_0}$ at step 1 using unpreconditioned conjugate gradient (CG). Recall that $N_k = \dim(\mathcal{U}_k)$, and assume that there exists $\alpha \in (0,1)$ so that $N_{k-1} \leq \alpha N_k$, $k = 1, 2, \ldots$; we expect $\alpha \approx 2^{-n}$, where $n$ is the dimension of the ambient space. We also assume that the cost of applying the Hessian $\mathcal{H}_k$, and hence $\mathcal{H}^{1}_{j_0}$, is

\[ t(k) \approx C_\text{op} N_k^p, \quad p \geq 1. \]
For the elliptic-constrained problem (1.2) we take \( p = 1 \) if we use classical multigrid for solving the elliptic problems, while for the image deblurring example we have \( p = 2 \). We assume that the cost of applying \( \mathcal{H}_k \) dominates the added \( O(N_k) \)-costs of projecting vectors onto the coarse space and other usual vector additions in the preconditioner, hence we discard the latter from the cost computation. The last hypothesis is that for any level \( k \), CG converges to the desired tolerance in at most \( F_{cg} \) iterations at a cost of \( cF_{cg}N_k \) Flops. In practice we have seen \( F_{cg} \) to range between 10 – 100 on a variety of problems. It follows from Algorithm 4.2 that the cost \( f(k) \) of applying \( \mathcal{Z}_k \) satisfies the recursion:

\[
\begin{align*}
    f(j) & \leq f(j - 1) + O(N_j) \approx f(j - 1) \\
    f(k) & \leq 2f(k - 1) + t(k), \quad k = j_0 + 1, \ldots, j - 1 \\
    f(j_0) & \leq cF_{cg}N_{j_0}.
\end{align*}
\]

(4.3)

Assuming that \( 2\alpha^p < 1 \), a standard argument shows that

\[
    f(j - 1) \leq 2^{j - j_0 - 1}cF_{cg}N_{j_0} + C_{op}\frac{1 - (2\alpha^p)^{j - j_0 - 1}}{1 - 2\alpha^p}N_{j - 1}^p.
\]

(4.4)

If we denote by \( l = j - j_0 + 1 \) the number of levels used (i.e., \( j = j_0 + 2 \) meaning three levels) and discard the \( O(N_j) \) term in (4.3), then

\[
    f(j) \leq \left( 2\alpha \right)^{l - 1}F_{cg}\frac{c}{2C_{op}} + \frac{\alpha^p}{1 - 2\alpha^p}(1 - (2\alpha^p)^{l - 2}) \frac{N_{j - 1}^p}{C_{op}N_j^p}.
\]

The expression above is not expected to be consistent with the cases \( l = 1, 2 \) due to the neglect of the costs of projections. Formula (4.3) shows that it is certainly advantageous to use as many levels as possible to keep the cost \( f(j) \) of applying the preconditioner \( \mathcal{Z}_j \) low relative to the cost \( t(j) \) of applying the inactive Hessian \( \mathcal{H}_j \). Asymptotically, if \( l \) is large, then \( f(j) \approx \frac{2^l\alpha^p}{1 - 2\alpha^p}t(j) \). However, if \( \alpha \) is truly small due to high-dimensionality and/or the cost of applying the Hessian is high (either \( C_{op} >> c \) or \( p > 1 \)), then the relative cost \( f(j)/t(j) \) can be small even with a low number of levels. We expect the wall-clock timings we show in Section 5 to give a better picture of the computational savings of using the multigrid preconditioned conjugate gradient (MGCG) versus CG.

### 4.2. Analysis

Estimating the spectral distance between the multigrid preconditioner \( \mathcal{Z}_j \) and \( (\mathcal{H}_j)^{-1} \) follows the same path as the analysis in [6]. The only significant difference lies in the presence of the projection \( \pi^1_k \) in the operator \( \mathcal{Y}^j_{k-1} \) defined in (4.1). We now verify that \( \mathcal{Y}^j_{k-1} \) is non-expansive in the spectral distance, a result similar to Lemma 4.2 in [6].

**Lemma 4.3.** For \( k = 1, 2, \ldots, \) and \( \mathcal{X} \in \mathcal{L}_+(U^j_{k-1}) \), we have \( \mathcal{Y}^j_{k-1}(\mathcal{X}) \in \mathcal{L}_+(U^j_{k-1}) \). Moreover, if \( \mathcal{X}, \mathcal{Y} \in \mathcal{L}_+(U^j_{k-1}) \), then

\[
    d_{U^j_{k-1}}(\mathcal{Y}^j_{k-1}(\mathcal{X}), \mathcal{Y}^j_{k-1}(\mathcal{Y})) \leq d_{U^j_{k-1}}(\mathcal{X}, \mathcal{Y})
\]

(4.5)

**Proof.** If \( \mathcal{X} \in \mathcal{L}_+(U^j_{k-1}) \), then for \( u, v \in U^j_k \) we have

\[
\begin{align*}
    \langle \pi^1_k \mathcal{X} \pi^1_{k-1} u, v \rangle &= \langle \mathcal{X} \pi^1_{k-1} u, \pi^1_{k-1} v \rangle = \langle \mathcal{X} \pi^1_{k-1} u, \pi^1_{k-1} v \rangle \\
    &= \langle u, \pi^1_k \mathcal{X} \pi^1_{k-1} v \rangle.
\end{align*}
\]

\( \footnote{Erratum: On p. 800 of [6] the correct definition is \( \mathcal{Y}^j_{j-1}(\mathcal{X}) = \mathcal{X} \cdot \pi^1_{j-1} + \beta^{-1}(I - \pi^1_{j-1}) \).} \)
which shows that \( \pi_k^I \mathcal{A} \pi_I^{k-1} \) is symmetric (recall that neither of \( \mathcal{U}_k^I \) and \( \mathcal{U}_k^I \) are assumed to be subspaces of each other). Given the symmetry of the orthogonal projection \( \pi_k^I(I - \pi_k^I) \) onto \((\mathcal{U}_k^{I-1})^\perp \cap \mathcal{U}_k^I\), the symmetry of \( \mathcal{L}_k^{k-1}(\mathcal{A}) \) follows. We leave the positive definiteness of \( \mathcal{L}_k^{k-1}(\mathcal{A}) \) as an exercise to the reader.

Let \( \mathcal{X}, \mathcal{Y} \in \mathfrak{L}_I(\mathcal{U}_k^{I-1}) \). By Lemma 4.1 in [6] we have

\[
\ln \frac{w_1 + x}{w_2 + x} \leq \ln \frac{w_1}{w_2}, \forall x > 0,
\]

for any \( w_1, w_2 \) complex numbers in the right half-plane. So

\[
d_{\mathcal{L}_k^{k-1}}(\mathcal{X}, \mathcal{Y}) = \sup_{u \in \mathcal{U}_k^{I-1}\setminus\{0\}} \left| \ln \frac{\langle \mathcal{X} \pi_k^I + \beta^{-1}(I - \pi_k^I)u, u \rangle}{\langle \mathcal{Y} \pi_k^I + \beta^{-1}(I - \pi_k^I)u, u \rangle} \right|
\]

\[
= \sup_{u \in \mathcal{U}_k^{I-1}\setminus\{0\}} \left| \ln \left( \frac{\langle \mathcal{X} \pi_k^I, \pi_k^I u \rangle - \langle \mathcal{Y} \pi_k^I, \pi_k^I u \rangle}{\langle \mathcal{Y} \pi_k^I, \pi_k^I u \rangle} \right) \right| \leq \sup_{u \in \mathcal{U}_k^{I-1}\setminus\{0\}} \left| \ln \frac{\langle \mathcal{X} u, v \rangle}{\langle \mathcal{Y} u, v \rangle} \right| = d_{\mathcal{U}_k^{I-1}}(\mathcal{X}, \mathcal{Y}).
\]

We also recall two technical results from [7]. The next result appears as Lemma 5.3 in [7].

**Lemma 4.4.** Let \((e_k)_{k \geq 0}\) and \((a_k)_{k \geq 0}\) be positive numbers satisfying

\[
e_k \leq C(e_{k-1} + a_k)^2, \quad a_k \leq a_{k-1} \leq f^{-1} a_k, \quad k = 1, 2, \ldots,
\]

for some \(0 < f < 1\). If \(a_0 \leq \frac{f}{10}\) and if \(e_0 \leq 4Ca_0^2\), then

\[
e_k \leq 4Ca_k^2, \quad \forall k > 0.
\]

Second, from Theorem 3.12 in [7] we extract the following result signifying the quadratic convergence of Newton’s method for the operator equation \( X^{-1} - A = 0 \) measured in the spectral distance.

**Lemma 4.5.** Given a Hilbert space \( \mathcal{X} \) and \( A, H \in \mathfrak{L}_+(\mathcal{X}) \) so that \( d_{\mathcal{X}}(A, H^{-1}) < 0.4 \), we have

\[
d_{\mathcal{X}}(2A - AHA, H^{-1}) \leq 2 \left( d_{\mathcal{X}}(A, H^{-1}) \right)^2.
\]

We are now in the position to prove the main result of this section.

**Theorem 4.6.** Assume that the operators \( \mathcal{K}_j, (\mathcal{K}_j)_{j \geq 0} \) satisfy Condition [2.21] and let \( 0 \leq j_0 < j \) be fixed indices. Consider the inactive index-sets and inactive domains defined by Algorithm [4.1] and the sequence of operators \( \mathcal{Z}_k, j_0 \leq k \leq j \) defined by Algorithm [4.2]. Denote by \( \mu_k = \beta^{-1}(\mathcal{O}_{k-1}^I \setminus \mathcal{O}_k^I) \), and assume there exists \(0 < f \leq f_{\text{low}}\) so that \( \mu_k \leq \mu_{k-1} \leq f^{-1} \mu_k \) for \( k = j_0 + 1, j_0 + 2, \ldots, j \), with \( f_{\text{low}} \) given in [2.1]. If

\[
C_{\text{tg}} \beta^{-1}(h_{j_0} + \mu_{j_0}) < \min(0.1, f/8),
\]

then there exists \( C_{\text{mg}} > 0 \) independent of \( j \) and the inactive set \( \widehat{I}^{(j)} \) so that

\[
d_{\mathcal{U}_j^I}(Z_j, (\mathcal{H}_j^I)^{-1}) \leq C_{\text{mg}} \beta^{-1}(h_j + \mu_j).
\]
Proof. For \( j_0 \leq k \leq j \) denote \( e_k = d_{\Omega_k^d} (Z_k, (H_k^d)^{-1}) \), and \( a_k = C_{tg} \beta^{-1} (h_k + \mu_k) \). The assumptions on \( \mu_k \) and \( f \) imply that

\[
a_k \leq a_{k-1} \leq f^{-1} a_k, \quad \forall \; j_0 + 1 \leq k \leq j,
\]

and that \( a_k \leq 0.1 \) for \( j_0 \leq k \leq j \). Since for \( k < j \) the operator \( Z_k \) is defined as \( \Omega_k (\Omega_{k-1}^d (Z_{k-1})) \), our first goal is to ensure that (4.9) holds for all \( k \geq j_0 \) with \( A = \Omega_{k-1}^d (Z_{k-1}) \) and \( H = H_k^d \). Thus we prove by induction that \( e_k < 0.2 \) for \( j_0 \leq k \leq j-1 \), and that the sequences \( e_k \) and \( a_k \) satisfy (4.11) with \( C = 2 \) for \( j_0 \leq k < j \). Note that \( e_{j_0} = 0 \). For \( k \geq j_0 + 1 \), after recalling that \( \mathcal{M}_k = \Omega_{k-1}^d ((H_{k-1}^d)^{-1}) \), we have

\[
d_{\Omega_k^d} (\Omega_{k-1}^d (Z_{k-1}), (H_k^d)^{-1}) \leq d_{\Omega_k} (\Omega_{k-1}^d (Z_{k-1}), \mathcal{M}_k) + d_{\Omega_k^d} (\mathcal{M}_k, (H_k^d)^{-1})
\]

\[
\leq e_{k-1} + a_k \leq 0.2 + 0.1 = 0.3. \quad (4.12)
\]

So Lemma 4.5 together with (4.12) implies that

\[
e_k = d_{\Omega_k^d} (\Omega_{k-1}^d (Z_{k-1}), (H_k^d)^{-1}) < 2(e_{k-1} + a_k)^2 < 2(0.3)^2 < 0.2, \quad (4.13)
\]

and the inductive statement is proved. Since \( a_{j_0} < f/8 \) by assumption, Lemma 4.4 now implies that

\[
e_{j-1} \leq 8a_{j-1}^2.
\]

Since \( Z_j = \Omega_{j-1}^d (Z_{j-1}) \), it follows, as above, that

\[
e_j \leq e_{j-1} + a_j \leq 8a_{j-1}^2 + a_j \leq (0.8 \cdot f^{-1} + 1)a_j.
\]

Therefore (4.11) holds with \( C_{mg} = (0.8 \cdot f^{-1} + 1)C_{tg} \).

Remark 4.7. We should note that the hypotheses of Theorem 4.6 are consistent with the scenario discussed in Remark 3.3 under which the correct inactive domain \( \Omega_j^d \) is sufficiently regular and the sets \( \Omega_j^d \), \( j_0 \leq k \leq j \), approximate \( \Omega_j^d \) sufficiently well so that \( \mu_k \approx C h_k \). Under these conditions, Theorem 4.6 also shows that the multigrid preconditioner is of optimal order, assuming that the coarsest grid \( j_0 \) is sufficiently fine for (4.10) to hold.

5. Numerical experiments. We test our multigrid preconditioner on two problems. In Section 5.1 we consider a classical elliptic-constrained optimization problem, while in Section 5.2 we showcase the behavior of our algorithm on a constrained optimization method related to image deblurring. Essentially, in these numerical experiments we are looking, first, for a validation of our theoretical results and, second, we would like to estimate the practical value of our preconditioning technique. With respect to the first aim we would like to see that the two-grid preconditioner gives rise to a number of linear iterations per SSNM step that decreases (in average) with respect to increasing resolution. A similar behavior is expected to hold for three-grid preconditioners, four-grid preconditioners, etc; we call this the weak test, and we expect all computations to pass this. We are also interested to see if the experiments pass the following strong test: for a fixed, acceptable (cf. Theorem 4.6) base level \( j_0 \), we should observe the number of linear iterations per SSNM to be decreasing with an increasing number of levels. The strong test is expected to hold only asymptotically in general, since \( C_{mg} \) from Theorem 4.6 is larger than \( C_{tg} \) from Theorem 3.2 this normally results in an increase in number of iterations from two-grid to three-grid
preconditioning, only to begin decreasing when the number of levels is sufficiently large. If the multigrid preconditioner passes the strong test for a given set of parameters, then we expect to see an increase wall-clock efficiency as well. We also expect that the multigrid preconditioner is inefficient or even fails if the base level resolution $h_{j_0}$ is too large relative to $\beta$. With respect to the second aim we simply want to observe the wall-clock efficiency of the multigrid preconditioner. All computations were performed using MATLAB on a system with two eight-core 2.9 GHz Intel Xeon E5-2690 CPUs and 256 GB memory.

5.1. An elliptic-constrained optimal control problem. For the first numerical experiment we consider the classical elliptic-constrained optimization problem

$$
\begin{aligned}
\min_{u \in L^2(\Omega)} & \frac{1}{2} \| y - y_d \|^2 + \frac{\beta}{2} \| u \|^2 \\
\text{subject to} & \quad -\Delta y = u, \quad y \in H^1_0(\Omega), \quad 0 \leq u \leq 1 \text{ a.e. in } \Omega,
\end{aligned}
$$

(5.1)

where $\Delta$ is the Laplace operator acting on $H^1_0(\Omega)$ with $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$. Therefore, $K = (-\Delta)^{-1}$. We define the data by $y_d = Ku_d$, where the so-called target control $u_d$ is the step function shown in the left-side of Figure 5.1. Note that $u_d$ is bounded between 0 and 1, and is supported inside the domain $\Omega$. Naturally, for $\beta \ll 1$ we expect $u_{\min} \approx u_d$. In absence of any box-constraints, or when the constraints turn out to be everywhere inactive, the solution $u_{\min}$ of (5.1) also solves the Tikhonov-regularized inverse problem $Ku = y_d$. It is well known that in this case $u_{\min}$ may not be localized and can exhibit an oscillatory behavior near the support of $u_d$. In order to showcase the behavior of our algorithm we selected a range of values for $\beta$ that render the constraints to be active on a significant portion of $\Omega$ (which requires a sufficiently small $\beta$), while allowing at the same time for a relatively fast convergence, e.g., less than ten SSNM iterations. We thus present results for $\beta = 10^{-4}, 10^{-5}$, and $10^{-6}$. For the $\beta$-values listed we show the solution $u_{\min}$ in Figures 5.1(right image) and 5.2. For $\beta = 10^{-6}$ both constraints are active at the solution, while for $\beta = 10^{-4}$ and $\beta = 10^{-5}$ only the lower constraints are active.

Given $n \in \mathbb{N}$, we divide $\Omega$ uniformly in $n^2$ squares and we discretize the control space using piecewise constant functions; the departure from the theoretical framework in the earlier sections is minimal, we just replaced triangular elements with
rectangular ones. We then use a standard Galerkin formulation to produce a discrete version of $\mathcal{K}$ on each grid using continuous bilinear finite elements. Standard finite element analysis (e.g., see [5]) shows that the SAC Condition 2.1 is satisfied; in particular, part [c] of Condition 2.1 follows from the $H^2$-regularity of the elliptic equation coupled with $L^\infty$-convergence (see also [4]). For each

$$n_j = 64 \times 2^j, \quad j = 0, \ldots, 6,$$

we initialize the SSNM using the solution obtained from a coarser level; we solve the linear systems in the SSNM solution process using MGCG, and we compare the results against CG. For each run, we report in Tables 5.1, 5.2, and 5.3 the average number of MGCG/CG iterations per SSNM step as well as the added wall-clock times used by the MGCG/CG solves during the entire solution process. The relative tolerance for the linear solves is set at $10^{-8}$. The elliptic problem, i.e., the application of $\mathcal{K}_j$ and $\mathcal{K}_j^*$ needed for applying the inactive Hessian $\mathcal{H}_j^{Ij}$, is solved numerically using either direct methods (for $n \leq 256$) or classical multigrid (the full approximation scheme FAS) using a relative tolerance of $10^{-8}$; the base case for FAS was taken to be $n = 256$, a choice that effectively minimized wall-clock times for solving the elliptic problem on our system. For solving the base case (Step 1 in Algorithm 4.2) in the multigrid preconditioner application we use (unpreconditioned) CG with a matrix-free application of $\mathcal{H}_j^{Ij}$, and a tolerance of $10^{-10}$. We should emphasize that the multigrid FAS for solving the elliptic problem is used only for applying $\mathcal{K}_j$ and $\mathcal{K}_j^*$, and is completely independent from the multigrid preconditioner from Algorithm 4.2 although in the implementation they share part of the infrastructure.

First we remark that, for each $\beta$, the SSNM converged in a relatively mesh-independent number of iterations; that number is also independent of the way we solve the linear systems, assuming they are solved to the given tolerance. In the interest of the exposition we do not report the number of SSNM iterations, since the focus is on the linear solves. We also point out that all cases pass the weak test. This is best seen when $\beta = 10^{-6}$ where we see the average number of two-grid iterations decreasing from 9.75 at $n = 512$ to 8.75 at $n = 1024$, and down to 6.25 at $n = 2048$; we did not run the two-grid preconditioned problem for $n = 4096$. For the same value of $\beta$ we see the three-grid average number of iterations decreasing from 11.75
Table 5.1  
Comparison of iteration counts and runtimes for MGCG vs. CG; $\beta = 10^{-4}$.

| $n$ | 128 | 256 | 512 | 1024 | 2048 | 4096 |
|-----|-----|-----|-----|------|------|------|
| \# cg / iteration | 11.25 | 11.67 | 12 | 12 | 12 | 12 |
| $t_{cg}$ (s) | 3.65 | 14 | 84 | 427 | 1915 | 10681 |
| \# mg / iteration, $j_0 = 0$ | 5 | 5 | 4 | 4 | 3 | 3 |
| $t_{mg}$ (s) | 10.5 | 13.6 | 100 | 373 | 1242 | 5090 |
| eff = $t_{mg}/t_{cg}$ | 2.87 | 1.16 | 1.19 | 0.87 | 0.65 | 0.48 |

Table 5.2  
Comparison of iteration counts and runtimes for MGCG vs. CG; $\beta = 10^{-5}$.

| $n$ | 128 | 256 | 512 | 1024 | 2048 | 4096 |
|-----|-----|-----|-----|------|------|------|
| \# cg / iteration | 20 | 19.5 | 19.25 | 19.75 | 20 | 20 |
| $t_{cg}$ (s) | 6.24 | 42 | 197 | 793 | 3081 | 17579 |
| \# mg / iteration, $j_0 = 0$ | 7.75 | 9 | 8.25 | 7.75 | 6 | 6.67 |
| $t_{mg}$ (s) | 15.23 | 32 | 257 | 896 | 1949 | 8227 |
| eff = $t_{mg}/t_{cg}$ | 2.44 | 0.76 | 1.3 | 1.12 | 0.63 | 0.47 |
| \# mg / iteration, $j_0 = 1$ | - | 6.5 | 7 | 6 | 5 | 4 |
| $t_{mg}$ (s) | - | 63 | 254 | 796 | 1865 | 6742 |
| eff = $t_{mg}/t_{cg}$ | - | 1.5 | 1.29 | 1.004 | 0.605 | 0.38 |

Table 5.3  
Comparison of iteration counts and runtimes for MGCG vs. CG; $\beta = 10^{-6}$.

| $n$ | 128 | 256 | 512 | 1024 | 2048 | 4096 |
|-----|-----|-----|-----|------|------|------|
| \# cg / iteration | 32.5 | 31.75 | 31 | 31.75 | 33.25 | 33 |
| $t_{cg}$ (s) | 9.9 | 48 | 241 | 1312 | 6030 | 42007 |
| \# mg / iteration, $j_0 = 2$ | - | - | 9.75 | 11.75 | 16 | 11.25 |
| $t_{mg}$ (s) | - | - | 1135 | 1986 | 7417 | 25134 |
| $t_{mg}/t_{cg}$ | - | - | 4.71 | 1.51 | 1.07 | 0.59 |
| \# mg / iteration, $j_0 = 3$ | - | - | - | 8.75 | 10 | 12 |
| $t_{mg}$ (s) | - | - | - | 4155 | 6403 | 18970 |
| $t_{mg}/t_{cg}$ | - | - | - | 3.16 | 0.92 | 0.45 |
| \# mg / iteration, $j_0 = 4$ | - | - | - | - | 6.25 | 7.75 |
| $t_{mg}$ (s) | - | - | - | - | 16615 | 27677 |
| $t_{mg}/t_{cg}$ | - | - | - | - | 2.39 | 0.65 |

at $n = 1024$ to 10 at $n = 2048$, down to 7.75 at $n = 4096$.

For the strong test the key issue is the choice of the base case $j_0$ for the multigrid preconditioner. The hypotheses of Theorem 4.6 show that the base level has to be sufficiently fine (relative to $\beta$) in order for MGCG to run efficiently, as shown in (4.10). For $\beta = 10^{-4}$, the choice $j_0 = 0$ ($n_0 = 64$) seems to be sufficiently fine, as the MGCG requires fewer and fewer iterations as $n$ increases, as predicted by theory. The effective efficiency factor $eff = time(MGCG)/time(CG)$ is also presented; it is shown to decrease with increasing resolution, but it decreases below the value one (e.g., MGCG becomes more efficient than CG) only at higher resolution, as expected. For example, at $n = 2048$, while CG required an average number of 12 iterations per SSNM iteration...
(actually exactly 12 at each iteration), the 5-grid MGCG required an average of 3 iterations per SSNM iterations. In terms of wall-clock time, the linear solves for MGCG required 0.65 of the wall-clock time of CG. The situation is somewhat similar for $\beta = 10^{-5}$, except for the fact that $j_0 = 0$ turns out to be borderline acceptable, in that the average number of MGCG iterations does not decrease with increasing resolution right from the beginning, so $j_0 = 0$ does not pass the strong test. Instead, the case $j_0 = 1$ ($n_1 = 128$) clearly passes the strong test with the exception of the mild increase in number of iterations from two-grid to three-grid. Also, the efficiency factor decreases to 0.38 at $n = 4096$ (with a 5-grid preconditioner). Finally, for $\beta = 10^{-6}$ we see that neither of the values $j_0 = 2, 3, 4$ give rise to the expected decrease in the number of iterations for the MGCG, at least not for small number of levels, thus failing the strong test. However, for high-resolution computations MGCG is still more efficient than CG: for example, a 3-grid MGCG based solve at $n = 4096$ ($j_0 = 3$ and $n_3 = 512$, see Table 5.3) requires an average of 12 inner iteration per SSNM iteration and 0.45 of the time needed for the 33 inner CG iterations per SSNM iteration.

5.2. Image deblurring with box constraints. For the second application we consider again $\Omega = (0, 1) \times (0, 1)$ and we define the Gaussian-blur operator by

$$K u(x) = \int_{\mathbb{R}^2} G(x - y) u(y) dy, \quad \text{where} \quad G(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{|x|^2}{2\sigma^2}}, \quad \sigma > 0. \quad (5.2)$$

Here $u : \Omega \to [0, 1]$ is an $L^2$-function representing a grey-scale image, and is usually extended with zero outside of $\Omega$. For practical considerations we limit the support of $G$ to the square $D_w = (-w, w) \times (w, w)$ for some small $w > 0$ (typically $w = 3\sigma$) by setting $G$ to be zero outside of $D_w$. This allows us to regard $K$ as defined from $L^2(\Omega)$ to $L^2(\Omega_w)$, where $\Omega_w = (w, 1 - w) \times (w, 1 - w)$, thus no longer requiring an extension of $u$ with zero outside of $\Omega$. Here we take $w = 0.1$ and $\sigma = w/3$.

In our numerical solution of the optimization problem (1.1) we discretize $u$, as before, using piecewise constant functions on a uniform $n \times n$ grid. We compute the discrete version $K_h (h = 1/n)$ of $K u$ (representing the blurred image) using a cell-centered cubature rule to integrate (5.2) numerically. Essentially, the value of $K_h u$ on an element is a weighted average of the values of $u$ in disk of radius $w$, with the weights being computed using the function $G$, and rescaled to add up to 1. As customary in Gaussian filtering, the separability of the kernel $G$ allows for a more efficient implementation: namely we have

$$K_h = K_h^x K_h^y = K_h^{xy} K_h^x,$$

where $K_h^x u$ (resp., $K_h^y u$) defines the application of a Gaussian filter to the image $u$ in the $x$-direction only (resp., $y$-direction).

The setup and result presentation is similar to the experiments presented in Section 5.1. Again, we define the data by $y_d = K u_d$, where the target control $u_d$, i.e., the original image (shown as a surface), is the same step function as in the previous experiment. In Figure 5.3 we show both $u_d$ (left) and the blurred image $y_d = K u_d$ (right) as surfaces. For the constrained optimization problem we use the constant constraints $a(x) = 0$ and $b(x) = 1$. In Figures 5.4 and 5.5 we show the solutions of the constrained problem for $\beta = 0.04, 0.02, 0.01, 0.005$.

For the multigrid solves we consider the cases

$$n_j = 128 \times 2^j, \quad j = 0, \ldots, 4.$$
We report the results for $\beta = 0.04$, 0.02 in Table 5.4 and for $\beta = 0.01$, 0.005 in Table 5.5, but we no longer report the effective efficiency factor.

The results are essentially similar with the elliptic-constrained experiments. All cases clearly pass the weak test. However, the only case where there is a hint of the strong test being passed is for $\beta = 0.04$; for $j_0 = 0$ we see the average number of iterations first increasing with resolution from 12.2 ($n_1 = 256$) to 14.5 ($n_2 = 512$) up to 21 ($n_3 = 1024$), only to decrease to 12.5 for ($n_4 = 2048$), all compared to an average number of 40 CG iterations. This certainly reflected in the wall-clock efficiency: the five-level MGCG linear solves required 1442 seconds compared to the 4023 seconds for CG.

As in the elliptic-constrained experiments, by lowering $\beta$ (here $\beta = 0.02$) we also have to raise the base case level in order for MGCG to run efficiently; here $j_0 = 2$ seems to be sufficiently fine, but even $j_0 = 1$ seems to be acceptable, i.e., lead to reasonably efficient linear solves. By contrast we see how even lower values for $\beta$ lead,
Fig. 5.5. Left: optimal control $u_{\text{min}}$ for $\beta = 0.01$. Right: optimal control $u_{\text{min}}$ for $\beta = 0.005$.

| $n_j$ | 256 | 512 | 1024 | 2048 |
|-------|-----|-----|------|------|
| $\beta = 0.04$ |       |     |      |      |
| # cg / iteration | 40  | 40  | 40   | 40   |
| $t_{\text{cg}}$ (s) | 3.6 | 25  | 215  | 4023 |
| # mg / iteration, $j_0 = 0$ | 12.2| 14.5| 21   | 12.5 |
| $t_{\text{mg}}$ (s) | 7.2 | 18  | 149  | 1442 |
| # mg / iteration, $j_0 = 1$ | -   | 9.25| 11.5 | 10.25|
| $t_{\text{mg}}$ (s) | -   | 39  | 101  | 1240 |
| # mg / iteration, $j_0 = 2$ | -   | -   | 7.5  | 8.75 |
| $t_{\text{mg}}$ (s) | -   | -   | 290  | 1347 |
| # mg / iteration, $j_0 = 3$ | -   | -   | -    | 6.5  |
| $t_{\text{mg}}$ (s) | -   | -   | -    | 2665 |
| $\beta = 0.02$ |       |     |      |      |
| # cg / iteration | 51.2| 51  | 51   | 51   |
| $t_{\text{cg}}$ (s) | 4.6 | 32  | 270  | 5354 |
| # mg / iteration, $j_0 = 0$ | 15.8| 18.75| 61.75| 57   |
| $t_{\text{mg}}$ (s) | 12  | 23  | 450  | 6256 |
| # mg / iteration, $j_0 = 1$ | -   | 11  | 14   | 23.75|
| $t_{\text{mg}}$ (s) | -   | 53  | 142  | 2797 |
| # mg / iteration, $j_0 = 2$ | -   | -   | 9.5  | 11.75|
| $t_{\text{mg}}$ (s) | -   | -   | 418  | 1682 |
| # mg / iteration, $j_0 = 3$ | -   | -   | -    | 7.25 |
| $t_{\text{mg}}$ (s) | -   | -   | -    | 3072 |

first to very slowly convergent linear solves (see the cases $\beta = 0.01$ and $j_0 = 0, 1$) and even non-convergent linear solves in the cases (see the cases $\beta = 0.005$ and $j_0 = 0, 1$).
Table 5.5
Comparison of iteration counts and runtimes for multigrid vs. unpreconditioned CG for image deblurring; \( w = 0.1 \), \( \beta = 0.01, 0.005 \).

| \( n_j \) | 256 | 512 | 1024 | 2048 |
|-----------|------|------|------|------|
| \# \( \text{cg} \) / iteration | 65.8 | 66  | 66  | 66  |
| \( t_{\text{cg}} \) (s) | 6   | 60  | 446 | 6576|
| \# \( \text{mg} \) / iteration, \( j_0 = 0 \) | 20.8 | 25.2| > 100| > 100|
| \( t_{\text{mg}} \) (s) | 18  | 46  |     |     |
| \# \( \text{mg} \) / iteration, \( j_0 = 1 \) | -   | 15.4| 19.8| > 100|
| \( t_{\text{mg}} \) (s) | -   | 119 | 279 |     |
| \# \( \text{mg} \) / iteration, \( j_0 = 2 \) | -   | -   | 11.8| 15.5|
| \( t_{\text{mg}} \) (s) | -   | -   | 846 | 2316|
| \# \( \text{mg} \) / iteration, \( j_0 = 3 \) | -   | -   | -   | 9   |
| \( t_{\text{mg}} \) (s) | -   | -   | -   | 4390|

\( \beta = 0.005 \)

| \# \( \text{cg} \) / iteration | 84.67| 84.25| 84.6| 84.74|
| \( t_{\text{cg}} \) (s) | 9   | 52  | 560 | 8926|
| \# \( \text{mg} \) / iteration, \( j_0 = 0 \) | 31.5| 37.5| failed| failed|
| \( t_{\text{mg}} \) (s) | 42  | 64  |     |     |
| \# \( \text{mg} \) / iteration, \( j_0 = 1 \) | -   | 19.75| 26.2| failed|
| \( t_{\text{mg}} \) (s) | -   | 152 | 402 |     |
| \# \( \text{mg} \) / iteration, \( j_0 = 2 \) | -   | -   | 15.2| 20.25|
| \( t_{\text{mg}} \) (s) | -   | -   | 1218| 3260|
| \# \( \text{mg} \) / iteration, \( j_0 = 3 \) | -   | -   | -   | 11.5|
| \( t_{\text{mg}} \) (s) | -   | -   | -   | 6779|

6. Conclusions. We have developed a multigrid preconditioning technique to be used in connection to SSNMIs for certain control-constrained distributed optimal control problems. The multigrid preconditioners exhibit a provably optimal order behavior with respect to the mesh-size, in that the quality of the preconditioners increases at the optimal rate with increasing mesh-size, assuming a piecewise constant representation of the control and a sufficiently fine base level. A natural question is whether the method can be extended to higher order discontinuous piecewise polynomial discretizations such that the optimality of the preconditioner is preserved. Following the analysis of the piecewise constant case, it is apparent that the answer is negative. However, this does not preclude the existence of alternate optimal order preconditioners for higher order discretizations of the controls. The search for such preconditioners is subject of ongoing research.

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