ON POLAR FOLLATIONS AND FUNDAMENTAL GROUP

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Abstract. In this work we investigate the relation between the fundamental group of a complete Riemannian manifold $M$ and the quotient between the Weyl group and reflection group of a polar action on $M$, as well as the relation between the fundamental group of $M$ and the quotient between the lifted Weyl group and lifted reflection group. As applications we give alternative proofs of two results. The first one, due to the author and Töben, implies that a polar action does not admit exceptional orbits, if $M$ is simply connected. The second result, due to Lytchak, implies that the orbits of a polar foliation are closed and embedded if $M$ is simply connected. All results are proved in the more general case of polar foliations.

1. Introduction

In this section, we state our main results in Theorem 1.1, Theorem 1.2, Corollary 1.3 and Corollary 1.4.

A singular Riemannian foliation $\mathcal{F}$ on a complete Riemannian manifold $M$ is called a polar foliation if for each regular point $p$, there is an immersed submanifold $\Sigma_p$, called section, that passes through $p$ and that meets all the leaves and always perpendicularly. It follows that $\Sigma_p$ is totally geodesic and that the dimension of $\Sigma_p$ is equal to the codimension of the regular leaf $L_p$; for definitions and properties of singular Riemannian foliations and polar foliations see [10, 4] and [1, 2, 12].

By choosing an appropriate section $\Sigma$, one can define a subgroup of isometries of $\Sigma$ called Weyl group $W(\Sigma)$ of the section $\Sigma$ so that if $w \in W(\Sigma)$ then $w(x) \in L_x$. Therefore $W(\Sigma)$ describes how the leaves of $\mathcal{F}$ intersect the section $\Sigma$; see Section 2. As expected, when $\mathcal{F}$ is the partition of a Riemannian manifold $M$ into the orbits of a polar action $G \times M \rightarrow M$ (i.e., an isometric action with sections), the Weyl group $W(\Sigma)$ is the usual Weyl group $N/Z$, where $N = \{g \in G | g(x) \in \Sigma, \forall x \in \Sigma\}$ and $Z = \{g \in G | g(x) = x, \forall x \in \Sigma\}$. The intersection of the singular leaves of $\mathcal{F}$ with the section $\Sigma$ is a union of totally geodesic hypersurfaces, called walls and the reflections in the walls are elements of the Weyl group $W(\Sigma)$.

By these reflections is called reflection group $\Gamma(\Sigma)$. As explained in Remark 2.1 the definition of $\Gamma(\Sigma)$ can be different from the usual definition of reflection group on a manifold. We are now able to state our first result.

Theorem 1.1. Let $\mathcal{F}$ be a polar foliation on a complete Riemannian manifold $M$, $\Sigma$ a section, $W = W(\Sigma)$ the Weyl group of $\Sigma$ and $\Gamma = \Gamma(\Sigma)$ the reflection group of $\Sigma$. Then there exists a surjective homomorphism $\rho : \pi_1(M) \rightarrow W/\Gamma$. 

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A direct consequence of Theorem 1.1 is that $W(\Sigma) = \Gamma(\Sigma)$ when $M$ is simply connected. The fact that $W(\Sigma) = \Gamma(\Sigma)$ does not imply previous results about polar foliations on simply connected spaces; see details in Example 4.1. In order to solve this problem, we lift the Weyl group $W(\Sigma)$ and reflection group $\Gamma(\Sigma)$ to subgroups of the isometries of the Riemannian universal cover of $\Sigma$, called lifted Weyl group $\tilde{W}$ and lifted reflection group $\tilde{\Gamma}$; see Definition 2.3 and Definition 2.6. This lead us to the second main result of this paper.

**Theorem 1.2.** Let $F$ be a polar foliation on a complete Riemannian manifold $M$. Let $\tilde{W}$ and $\tilde{\Gamma}$ denote the lifted Weyl group and the lifted reflection group respectively. Then there exists a surjective homomorphism $\tilde{\rho} : \pi_1(M) \rightarrow \tilde{W}/\tilde{\Gamma}$.

Theorem 1.2 allow us to give alternative proofs to the next two results, where the first one is due to the author and Töben [3] and the second one is due to Lytchak [7].

**Corollary 1.3.** Let $F$ be a polar foliation on a simply connected complete Riemannian manifold. Then each regular leaf has trivial holonomy.

**Corollary 1.4.** Let $F$ be a polar foliation on a simply connected complete Riemannian manifold. Then the leaves of $F$ are closed and embedded.

This paper is organized as follows. In Section 2 we recall the definition of Weyl group and present the definition of lifted Weyl group. In Sections 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2 using the construction of Section 3. In Section 5 we provide alternative proofs of Corollaries 1.3 and 1.4. In particular to read this section it suffices to read Section 2 and accept Theorem 1.2. Finally, in Section 6 we note that the above results admit natural generalizations to the class of infinitesimally polar foliations.

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2. **Weyl and lifted Weyl groups**

Let $F$ be a polar foliation on a complete Riemannian manifold $M$. It follows from [1] that, for each point $x_\alpha$, we can find a neighborhood $U_\alpha$ (an isoparametric neighborhood) such that $F|_{U_\alpha}$ is diffeomorphic to a polar foliation on an Euclidean space, i.e., an isoparametric foliation. Therefore we can find a totally geodesic submanifold $\sigma_\alpha \subset U_\alpha$ transverse to the plaques that we call local section.

Now consider $\{U_\alpha\}$ an open covering of $M$ by isoparametric neighborhoods. If $U_\alpha \cap U_\beta \neq \emptyset$, then we find neighborhoods $V_\alpha^\alpha \subset \sigma_\alpha$, $V_\beta^\beta \subset \sigma_\beta$ and an isometry $\varphi_{\beta,\alpha} : V_\alpha^\alpha \cap C_\alpha \rightarrow V_\beta^\beta \cap C_\beta$ where $C_\alpha$ (respectively $C_\beta$) is a Weyl chamber of $\sigma_\alpha$ (respectively $\sigma_\beta$), i.e., $C_i \subset \sigma_i$ is the closure in $\sigma_i$ of a connected component of the set of nonsingular points in $\sigma_i$, for $i = \alpha, \beta$. Since $C_\alpha, C_\beta$ are Weyl chambers, we extend the map $\varphi_{\alpha,\beta}$ to an isometry $\varphi_{\beta,\alpha} : V_\alpha^\alpha \rightarrow V_\beta^\beta$. If $\beta$ is a curve contained in a regular leaf, we can cover $\beta$ by open neighborhoods $U_i$ and set $\varphi[\beta] := \varphi_{n,n-1} \circ \ldots \circ \varphi_{2,1}$. The germ of $\varphi[\beta]$ is then called holonomy map and depends only on the homotopy class $[\beta]$ of $\beta$. The source (domain) of $\varphi[\beta]$ can contain singular points.

We define the Weyl pseudogroup of a section $\Sigma$ as the pseudogroup generated by the holonomy maps whose source and target are contained in the section $\Sigma$. By
the appropriate choice of section, this pseudogroup turns out to be a group that is called \emph{Weyl group} \( W(\Sigma) \); see \cite{12}. From now on we fix this section where the Weyl group is well defined as the section \( \Sigma \). By definition, if \( w \in W(\Sigma) \) then \( w(x) \in L_x \) and \( W(\Sigma) \) describes how the leaves of \( \mathcal{F} \) intersect the section \( \Sigma \). As expected, when \( \mathcal{F} \) is the partition of a Riemannian manifold \( M \) into the orbits of a polar action \( G \times M \to M \), the Weyl group \( W(\Sigma) \) is the usual Weyl group \( N/Z \), where \( N = \{ g \in G | g(x) \in \Sigma, \forall x \in \Sigma \} \) and \( Z = \{ g \in G | g(x) = x, \forall x \in \Sigma \} \); see \cite{14}.

It follows from \cite{1} that the intersection of the singular leaves of \( \mathcal{F} \) with the section \( \Sigma \) is a union of totally geodesic hypersurfaces, called \emph{walls} and that the reflections in the walls are elements of the Weyl group \( W(\Sigma) \). We define the \emph{reflection group} \( \Gamma(\Sigma) \) as the group generated by these reflections. Again when \( \mathcal{F} \) is the partition of a Riemannian manifold into the orbits of a polar action, \( \Gamma(\Sigma) \) is the usual reflection group in the section.

\begin{remark}
Note that, in the definition of a reflection of a polar foliation, we do not assume that fixed point set \( \Sigma_r \) of a reflection \( r \) separates \( \Sigma \) as usual in the literature. Also note that what we call \emph{wall} is a hypersurface contained in the intersection of the singular leaves of \( \mathcal{F} \) with \( \Sigma \). Therefore a wall is a connected component of the fixed point set of a reflection \( r \), but there can exist a connected component of the fixed point set of \( r \) that is not a wall. This is also different from the usual definition of walls of reflections. These definitions will coincide when we consider the lifted reflection group.
\end{remark}

Consider \( w \in W(\Sigma) \). In what follows we will define a isometry \( \tilde{w} : \tilde{\Sigma} \to \tilde{\Sigma} \) on the Riemannian universal cover of \( \Sigma \) such that \( \pi_\Sigma \circ \tilde{w} = w \circ \pi_\Sigma \) where \( \pi_\Sigma : \tilde{\Sigma} \to \Sigma \) is the Riemannian covering map.

\begin{definition}
Let \( p \) be a regular point of the section \( \Sigma \) and \( \tilde{p} \) a point of \( \tilde{\Sigma} \) such that \( \pi_\Sigma(\tilde{p}) = p \). Consider a curve \( c \in \Sigma \) that joins \( p \) to \( w(p) \) and \( \delta \) be a curve in \( \Sigma \) with \( \delta(0) = p \). Let \( c \ast w \circ \delta \) be the concatenation of \( c \) and the curve \( w \circ \delta \), \( \tilde{\delta} \) the lift of \( \delta \) starting at \( \tilde{p} \) and \( (c \ast w \circ \delta) \) the lift of \( c \ast w \circ \delta \) starting at \( \tilde{p} \). Then we set
\begin{equation}
(2.1) \quad \tilde{w}_{[c]}(\tilde{\delta}(1)) := (c \ast w \circ \delta)(1).
\end{equation}

The isometry \( \tilde{w}_{[c]} \) defined in equation \( (2.1) \) is called a \emph{lift of \( w \) along \( c \)}.
\end{definition}

\begin{definition}
We call \emph{lifted Weyl group} \( \tilde{W} \) the group of isometries on the universal covering \( \tilde{\Sigma} \) generated by all isometries \( \tilde{w}_{[c]} \) constructed above. In other words, \( \tilde{W} = \langle \tilde{w}_{[c]} \rangle \) for all \( w \in W \) and for all curves \( c : [0, 1] \to \Sigma \), such that \( c(0) = p \) and \( c(1) = w(p) \).
\end{definition}

\begin{remark}
Sometimes we will need to lift an isometry \( w \) that is naturally characterized by its action on a neighborhood of a point \( x \in \Sigma \) far from the fixed point \( p \). In this case, it is convenient to consider the following procedure. Let \( c_0 \) be a curve in \( \Sigma \) that joins \( p \) to \( x \) and \( c_1 \) a curve in \( \Sigma \) that joins \( x \) to \( w(x) \). Let \( \delta \) be a curve in \( \Sigma \) with \( \delta(0) = x \), \( c_0 \ast \delta \) the concatenation of \( c_0 \) and \( \delta \) and \( (c_0 \ast \delta) \) the lift of \( c_0 \ast \delta \) starting at \( \tilde{p} \). Then we define
\begin{equation}
(2.2) \quad \tilde{w}((c_0 \ast \delta)(1)) := (c_0 \ast c_1 \ast w \circ \delta)(1).
\end{equation}
\end{remark}
Clearly the construction of $\tilde{w}$ depends on the homotopy class $[c_0]$ and $[c_1]$. It is not difficult to check that the group generated by all these isometries is the lifted Weyl group $\tilde{W}$.

**Remark 2.5.** The lifted Weyl group was used by Töben [12] in his study of desingularization of polar foliations. One can prove that each element $\tilde{w} \in \tilde{W}$ is a lift of some isometry $w \in W$.

We are now able to construct the appropriate reflection group on $\tilde{\Sigma}$.

**Definition 2.6.** Let $H$ be a wall in $\Sigma$. Consider a reflection $r \in \Gamma(\Sigma)$ in this wall $H$ and a point $x$ near to $H$. Let $c_1$ be a curve joining $x$ to $w(x)$ in a simply connected neighborhood. Now define $\tilde{r}$ by equation (2.2). The isometry $\tilde{r}$ is then a reflection in a hypersurface $\tilde{H}$ such that $\tilde{\pi}_\Sigma(\tilde{H}) = H$. The group generated by all reflections $\tilde{r}$ is called *lifted reflection group* $\tilde{\Gamma}$.

**Remark 2.7.** One can prove that $\Gamma$ is a normal subgroup of $\tilde{W}$. From the definition we have that $\pi_1(\Sigma)$ is a subgroup of $\tilde{W}$. If $W(\Sigma)$ is the Weyl group of $\Sigma$ then one can check that $\pi_1(\Sigma)$ is a normal subgroup and $W(\Sigma) = \tilde{W}/\pi_1(\Sigma)$. Nevertheless, in general $\pi_1(\Sigma)$ is not entirely contained in $\tilde{\Gamma}$. A simple example is again the action of $SO(3)$ on itself by conjugation (see Example 4.1). In particular $\tilde{W}/\tilde{\Gamma}$ does not need to be isomorphic to $W/\Gamma$.

**Remark 2.8.** It follows from Davis [6] Lemma 1.1] that $\tilde{\Gamma}$ is a *reflection group in the classical sense*, i.e., is a discrete group generated by reflections acting properly, effectively on $\Sigma$. In particular a Weyl chamber $C$ of $\tilde{\Gamma}$ is a fundamental domain of the action of $\tilde{\Gamma}$; see [6] Theorem 4.1.

### 3. Proof of Theorem 1.1

#### 3.1. The construction of the map $\rho$

In this subsection we construct a map $\rho$ that associate each loop $\alpha$ with an element of $W/\Gamma$. Let $p$ be a regular point and $\alpha : [0, 1] \to M$ be a loop with $\alpha(0) = p = \alpha(1)$.

We start by considering the following objects:

1. A partition $0 = t_0 < t_1 < \ldots < t_n = 1$ and an open covering $\{U_i\}$ of $\alpha$ by isoparametric neighborhood (i.e., $\mathcal{F}|_{U_i}$ is diffeomorphic to an isoparametric foliation) such that $\alpha_i := \alpha|_{t_i = [a_i, b_i]} \subset U_i$, where $a_i = t_{i-1}$ and $b_i = t_i$;
2. $\sigma_i \subset U_i$ a local section, i.e., a normal neighborhood of a section $\Sigma_i$, and $C_i \subset \sigma_i$ a Weyl chamber, i.e., the closure in $\sigma_i$ of a connected component of the set of nonsingular points in $\sigma_i$;
3. $\pi_i : U_i \to C_i$ the chamber map, i.e., the continuous map such that $L_y \cap C_i = \{\pi_i(y)\}$.

Now we consider an holonomy map $\varphi_i := \varphi_{[\beta_i]} : V_i \subset \sigma_i \to V_{i+1} \subset \sigma_{i+1}$ with the following properties:

1. $\beta_i$ is a curve in an isoparametric neighborhood $U_{i,i+1} \supset U_i \cup U_{i+1}$,
2. $\pi_i \circ \alpha_i(b_i) \in V_i$ and $\pi_{i+1} \circ \alpha_{i+1}(a_{i+1}) \in V_{i+1}$,
3. $\varphi_i := \pi_i \circ \alpha_i(b_i) = \pi_{i+1} \circ \alpha_{i+1}(a_{i+1})$.

We also consider holonomies maps $\psi_i : B_i \subset \Sigma \to \sigma_i \subset \Sigma_i$ with $\psi_1 = Id = \psi_{n+1}$.

Finally we define, for $1 \leq i \leq n$, the following objects:

1. $\gamma_i := \psi_1^{-1} \circ \pi_i \circ \alpha_i$,
We have then constructed a $W$-loop $(\gamma_i, w_i)$ with base point $p \in \Sigma$, i.e.,

\begin{enumerate}
  \item a sequence $0 = t_0 < \cdots < t_n = 1$,
  \item \text{continuous paths } $\gamma_i : [t_{i-1}, t_i] \to \Sigma$, $1 \leq i \leq n$,
  \item elements $w_i \in W$ defined in a neighborhood of $\gamma_i(t_i)$ for $1 \leq i \leq n$ such that $\gamma_1(0) = w_n \gamma_n(1) = p$ and $w_i \gamma_i(t_i) = \gamma_{i+1}(t_i)$, where $1 \leq i \leq n - 1$.
\end{enumerate}

Note that we can use the $W$-loop $(\gamma_i, w_i)$ to construct a connected curve $\gamma$ with endpoint $w_1^{-1} \cdots w_n^{-1} p$. For example, if we have a $W$-loop $(\gamma_1, w_1)$, $(\gamma_2, w_2)$, $(\gamma_3, w_3)$, we have the connected curve $\gamma_1 \circ (w_1^{-1} \gamma_2) \circ (w_1^{-1} w_2^{-1} \gamma_3)$ with endpoint $w_1^{-1} w_2^{-1} w_3^{-1} p$. We finally define the map $\rho$ as follows:

\begin{equation}
\rho(\alpha) := (w_1)^{-1} \cdot (w_2)^{-1} \cdots (w_n)^{-1} \Gamma.
\end{equation}

### 3.2. The map $\rho : \pi_1(M, p) \to W/\Gamma$ is well defined and surjective.

We start by checking that the definition of the map $\rho$ in equation (5.1) does not depend on the choice of the Weyl chamber $\{C_i\}$, the local sections $\{\psi_i\}$ and the maps $\{\psi_i\}$.

For another choice of local sections $\{\psi_i\}$, Weyl chambers $\{C_i\}$ and maps $\{\psi_i\}$ we have, as explained in the last subsection, a $W$-loop $(\gamma_i, \tilde{w}_i)$.

By straightforward calculations on can check the next equation

\begin{equation}
\tilde{w}_i = (g_i)^{-1} (r_{i+1})^{-1} \tilde{w}_i r_{i+1} g_i,
\end{equation}

where $g_i = e = g_{n+1}$, $g_i \in W$, $r_{i+1} \in \Gamma$. Equation (3.2) then implies

\[(w_1)^{-1} \cdot (w_2)^{-1} \cdots (w_n)^{-1} \Gamma = (\tilde{w}_1)^{-1} \cdot (\tilde{w}_2)^{-1} \cdots (\tilde{w}_n)^{-1} \Gamma.\]

Note that the construction does not depend on the open covering $\{U_i\}$. In fact, for another open covering $\{\tilde{U}_j\}$ we assume, without lost of generality, that for each $j$ there exists $i$ such that $\tilde{U}_j \subset U_i$. For this new open covering we have a $W$-loop $(\hat{\gamma}_j, \hat{w}_j)$. Now one can infer equation (3.2) by considering the appropriate subdivision of the $W$-loop $(\gamma_i, w_i)$, i.e., by adding new points to the interval $[0, 1]$, by taking the restriction of the $\gamma_i$ to these new intervals and $w = id$ at the new points. Finally we have to verify that the construction does not depend on the homotopy class of $\alpha$. Consider a fixed choice of $\{U_i\}$, $\{\sigma_i\}$, $\{C_i\}$ and $\{\psi_i\}$. Then one can check that for each curve $\alpha^*$ near to $\alpha$ we produce a $W$-loop $(\gamma_i^*, w_i^*)$ where $w_i^* = w_i$. Therefore $\rho(\alpha^*) = \rho(\alpha)$.

We have proved that the map $\rho : \pi_1(M, p) \to W/\Gamma$ is well defined. By choosing the appropriate open covering, one can see that $\rho$ is also a homomorphism of groups. We conclude this section proving that it is surjective.

Consider $w \in W$ and let $\gamma$ be a curve that joins $p$ to $w(p)$. Now consider $\beta \subset L_p$ a curve such that joins $p$ to $w(p)$ and such that $\varphi[\beta] = \tilde{w}$. Then consider the concatenation $\alpha = \gamma \ast \beta^{-1}$. With the appropriate choice of $\{U_i\}$, $\{\sigma_i\}$, $\{C_i\}$ and $\{\psi_i\}$, we produce the $W$-loop $(\gamma, \tilde{w}^{-1})$ and hence $\rho(\alpha) = (\tilde{w}^{-1})^{-1} \Gamma = \tilde{w} \Gamma$.

#### 4. Proof of Theorem 1.2

In this section we will use the notations and definitions of the previous sections.
4.1. The construction of the map $\hat{\rho}$ and an example. Let $p$ be a regular point and $\alpha : [0, 1] \to M$ be a loop with $\alpha(0) = p = \alpha(1)$. In this subsection we associate $\alpha$ with an element $\hat{\rho}(\alpha) \in \tilde{W}/\tilde{\Gamma}$, briefly sketch the rest of the proof of Theorem 1.2 and illustrate the lifted groups and maps $\rho$ and $\hat{\rho}$ in an example.

Consider $\{U_i\}, \{\sigma_i\}, \{c_i\}$ and $\{\psi_i\}$ defined in Section 3. We will also assume that, for $1 \leq i < n$, the open set $B_i \cup B_{i+1}$ is contained in a simply connected neighborhood $B_{i,i+1}$ such that $B_i = \psi_i^{-1}(\sigma_i)$. As explained in Section 3 we produce a $W$-loop $(\gamma_i, w_i)$ and with this $W$-loop we construct a connected curve $\gamma$ such that $\gamma(0) = \gamma(1)$, where $w = (w_1)^{-1} \cdot (w_2)^{-1} \cdots (w_n)^{-1}$. Finally we set

$$\hat{\rho}(\alpha) := \tilde{w}[\gamma] \tilde{\Gamma},$$

where $\tilde{w}[\gamma]$ is the lift of the isometry $w$ along the curve $\gamma$; recall equation (2.1). Different choices of Weyl chambers will produce different curves $\gamma$ and isometries $w$, but the fact that we are considering the quotient space $\tilde{W}/\tilde{\Gamma}$ will imply that $\hat{\rho}$ is well defined; see Subsection 1.2.

To prove that $\hat{\rho}$ is surjective, for each $\tilde{w} \in \tilde{W}$ we have to find a loop $\alpha$ such that $\hat{\rho}(\alpha) = \tilde{w} \tilde{\Gamma}$. Here we have to use the fact that each $\tilde{w}$ is a lift $\tilde{w}_c$ of an isometry $w$. Then we can choose $\alpha = c \cdot \beta^{-1}$ where $\varphi[\beta] = w$.

Example 4.1. Assuming that $\hat{\rho}$ is well defined, we now illustrate the lifted groups and maps $\rho$ and $\hat{\rho}$ in the simple example of the action of $G = SO(3)$ on itself by conjugation. In this example the section $\Sigma$ is isomorphic to $S^1$. The identity $\text{Id} \in SO(3)$ is the only singular point in $\Sigma$ and the antipodal point $p \in \Sigma$ is the only point such that the orbit $G(p)$ is exceptional. Note that $W = \Gamma = \{e, w\}$. Therefore the fact that $W(\Sigma) = \Gamma(\Sigma)$ does not imply that all regular orbits have trivial holonomy. Clearly $\Sigma = \mathbb{R}$ and we can identify the elements of $\tilde{\Gamma}$ with the reflection in integer numbers. Note that the group $\pi_1(\Sigma)$ of deck transformations is not entirely contained in $\tilde{\Gamma}$, because the translation of length 1 is a deck transformation and is not contained in $\tilde{\Gamma}$. We can identify $\tilde{\rho}$, the lift of the exceptional point $p$, with the point $1/2 \in \mathbb{R}$. Note that, by the definition of lifted isometries, $R_{1/2}$, the reflection in $1/2$, is an element of $\tilde{W}$. In fact all elements of $\tilde{W}$ can be generated by $R_{1/2}$ and $R_1$, the reflection in the point 1.

Now we exam the maps $\rho$ and $\hat{\rho}$. Consider $\beta$ a curve in $G(p)$ such that $\varphi[\beta] = w$. Set $\alpha_1 = \beta^{-1}$. Then by the process explained in Section 3 we produce a $W$-loop $(p, w^{-1})$. Therefore $\rho(\alpha_1) = w \Gamma = \Gamma$. One can check that $\hat{\rho}(\alpha_1) = R_{1/2} \tilde{\Gamma}$. Finally set $\alpha_2 = \gamma \cdot \beta^{-1}$ where $\gamma$ is a parametrization of $\Sigma = S^1$. Then by the process explained in Section 3 we produce a $W$-loop $(\gamma, w^{-1})$. Therefore $\rho(\alpha_2) = w \Gamma = \Gamma$. One can check that $\hat{\rho}(\alpha_2) = R_1 \tilde{\Gamma} = \tilde{\Gamma}$.

4.2. The map $\hat{\rho} : \pi_1(M, p) \to \tilde{W}/\tilde{\Gamma}$ is well defined. Let $(\gamma_i, w_i)$ be the $W$-loop defined in Subsection 4.1. In order to prove that $\hat{\rho}$ is well defined, we will need an equivalent definition of $\hat{\rho}$ written in terms of lifts of the isometries $w_i$.

Let $\tilde{\gamma}_1$ be the lift of $\gamma_1$ starting at $\tilde{p}$. We join $\gamma_1(b_1)$ to $\gamma_2(a_2)$ by a curve $c_1 \subset B_{1,2}$ and define $\tilde{\gamma}_2$ the lift of $\gamma_2$ starting at $(\gamma_1 \cdot c_1)(1)$. We define the lift $\tilde{w}_1$ replacing the curve $c_0$ with the curve $\gamma_1$ in equation (2.2). Since $B_{1,2}$ is simply connected, the definition of $\tilde{\gamma}_2$ and $\tilde{w}_1$ does not depend on the choice of $c_1$. We join $\gamma_1(b_1)$ with $\gamma_{i+1}(a_{i+1})$ by a curve $c_i \subset B_{i,i+1}$ and, following the same procedure, define by induction $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ and $\tilde{w}_1, \ldots, \tilde{w}_{n-1}$. The fact that $B_{i,i+1}$ is simply connected
imply that this construction does not depend on the choice of \( c_i \). In order to define \( \tilde{w}_n \), set\( c_0 := \gamma_1 * c_1 * \cdots * c_{n-1} * \gamma_n \), set \( c_n := c_0^{-1} \) and replace \( c_1 \) with \( c_n \) in equation (2.2). Note that, for another choice of curves \( c_1, \ldots, c_{n-1} \), we would have an homotopic curve joining \( p \) to \( \gamma_n(b_n) \). Therefore the definition of \( \tilde{w}_n \) does not depend on the choice of the curves \( c_1, \ldots, c_{n-1} \). The next lemma is not difficult to check.

**Lemma 4.2.** Let \( \tilde{w}_i \) defined above. Then \( \tilde{\rho}(\alpha) = (\tilde{w}_1)^{-1} \cdot (\tilde{w}_2)^{-1} \cdots (\tilde{w}_n)^{-1} \tilde{\Gamma} \).

Consider another choice of local sections \( \{\tilde{\sigma}_i\} \), Weyl chambers \( \{\tilde{C}_i\} \) and holonomies \( \{\tilde{\psi}_i\} \). As before, we also assume that, for \( 1 \leq i < n \), the open set \( \tilde{B}_i \cup \tilde{B}_{i+1} \) is contained in a simply connected neighborhood \( \tilde{B}_{i+1} \) where \( \tilde{B}_i = \tilde{\psi}_i^{-1}(\tilde{\sigma}_i) \). As explained in Section 3 we can produce another \( \tilde{W} \)-loop \( (\tilde{\gamma}_i, \tilde{w}_i) \) and then we have a \( \tilde{W} \)-loop \( (\tilde{\gamma}_i, \tilde{w}_i) \). Using equation (2.2) one can prove that

\[
(\tilde{\sigma}_{i+1})^{-1} \tilde{w}_i \tilde{\psi}_i \tilde{\sigma}_i = \tilde{\gamma}_{i+1}^{-1} \tilde{w}_i \tilde{\gamma}_i.
\]

where \( \tilde{\gamma}_i = \tilde{\gamma}_i(1) \in \tilde{W} \) and \( \tilde{\gamma}_i = Id = \tilde{\gamma}_n+1 \). This equation and the fact that \( \tilde{\Gamma} \) is a normal subgroup of \( \tilde{W} \) then imply

\[
(\tilde{w}_1)^{-1} \cdot (\tilde{w}_2)^{-1} \cdots (\tilde{w}_n)^{-1} \tilde{\Gamma} = (\tilde{w}_1)^{-1} \cdot (\tilde{w}_2)^{-1} \cdots (\tilde{w}_n)^{-1} \tilde{\Gamma}.
\]

From Lemma 4.2 we conclude that the construction does not depend on the choice of \( \{\tilde{\sigma}_i\}, \{\tilde{C}_i\} \) and \( \{\tilde{\psi}_i\} \).

By subdivision of \( \tilde{W} \)-loop we see that the definition also does not depend on the covering \( \{\tilde{U}_i\} \). We have concluded that the map \( \tilde{\rho} \) is well defined. The independence of the homotopy class of \( \alpha \) is again proved considering a fixed choice of \( \{\tilde{U}_i\}, \{\tilde{\sigma}_i\} \), \( \{\tilde{C}_i\} \) and \( \{\tilde{\psi}_i\} \) and noting that, for each curve \( \alpha^* \) near to \( \alpha \), we produce an \( \tilde{W} \)-loop \( (\tilde{\gamma}_i^*, \tilde{w}_i^*) \) where \( \tilde{w}_i^* = w_i \). Therefore \( \tilde{\rho}(\alpha^*) = \tilde{\rho}(\alpha) \).

5. Proof of the corollaries

5.1. **Proof of Corollary 1.3.** Assume that there exists a regular leaf with a non-trivial holonomy, i.e., assume that there exists a regular point \( p \in \Sigma \) and \( w \in \tilde{W} \) such that \( w(p) = p \) and \( w \neq e \).

On one hand we define a lift of \( w \) as \( \tilde{w}(\tilde{\delta}(1)) := \tilde{w} \circ \tilde{\delta}(1) \). Note that \( \tilde{w}(\tilde{\rho}) = \tilde{\rho} \) and \( \tilde{w} \) is not the identity.

On the other hand, since \( M \) is simply connected, Theorem 1.2 implies that \( \tilde{W} = \tilde{\Gamma} \). It follows from Remark 2.8 that the only points that can be fixed by elements of \( \tilde{\Gamma} \) are singular points, i.e., points whose projection by \( \pi_{\Sigma} \) are singular points of \( \Sigma \). Therefore \( \tilde{w}(\tilde{\rho}) \neq \tilde{\rho} \) and we have arrived at a contradiction.

5.2. **Proof of Corollary 1.4.** Due to equifocality [3], in order to prove Corollary 1.4 it suffices to prove that the leaves are embedded.

Assume that \( L_p \) is not embedded. Then we can find a sequence \( \{w_n\} \subset W \) such that \( w_n(p) \rightarrow p \). Define \( \tilde{w}_n(\tilde{\delta}(1)) := (c_n \circ w_n \circ \tilde{\delta})(1) \), where \( c_n \) are curves with small lengths contained in a simply connected neighborhood of \( p \). Then one can check that \( \tilde{w}_n(\tilde{\rho}) \rightarrow \tilde{\rho} \).

On the other hand, since \( M \) is simply connected, Theorem 1.2 implies that \( \tilde{W} = \tilde{\Gamma} \). By Remark 2.8 the sequence \( \{\tilde{w}_n(\tilde{\rho})\} \) can not converge to \( \tilde{\rho} \). Hence we have arrived at a contradiction.
6. Infinitesimally polar foliations

A singular Riemannian foliation $\mathcal{F}$ is called *infinitesimally polar foliation* if the restriction of $\mathcal{F}$ to each slice is diffeomorphic (by the composition of the exponential map with a linear map) to an isoparametric foliation. Lytchak and Thorbergsson [8] proved that $\mathcal{F}$ is an infinitesimally polar foliation if and only if for each point $x \in M$ we can find a neighborhood $U$ of $x$ such the leave space of the restricted foliation $\mathcal{F}|_U$ is an orbifold. Since the leave space of an isoparametric foliation is a Co xeter orbifold, they concluded that the leave space $U/\mathcal{F}|_U$ is an orbifold if and only if it is a Coxeter orbifold. Typical examples of infinitesimally polar foliations are polar foliations (see [1]), singular Riemannian foliations without horizontal conjugate points (see [8, 9]) and singular Riemannian foliations with codimension lower then three (see [9]).

**Remark 6.1.** Let $\mathcal{F}$ be an infinitesimally polar foliation on a complete Riemannian manifold $M$ and $\{U_\alpha\}$ a covering of $M$ by isoparametric neighborhoods. We can find submanifolds $\sigma_\alpha$ transverse to the plaques that we still call *local sections*. Using the same argument as in [9, Subsection 3.2] we can change the metric of local sections $\sigma_\alpha$ so that the holonomy maps turn out to be isometries. Like in the classical theory of foliations, the structure of $M/\mathcal{F}$ is described by the *pseudogroup of holonomies* of $\mathcal{F}$, i.e., the pseudogroup generated by holonomy maps acting on the disjoint union of local sections.

Now assume that $M/\mathcal{F}$ is equivalent to $\Sigma/W$ where $\Sigma$ is a connected Riemannian manifold and $W$ is a subgroup of isometries of $\Sigma$; for details about pseudogroup see Salem [10, Appendix D]. Then $W$ is called *Weyl group* of $\mathcal{F}$. Let $\{\psi_\alpha\}$ be a maximal collection of isometries of open subsets of $\Sigma$ onto local sections of $\mathcal{F}$ that gives the equivalence between $\Sigma/W$ and $M/\mathcal{F}$. Let $r$ be a reflection in a wall of a local section $\sigma_\alpha$. Then $r := \psi_\alpha^{-1} \circ r \circ \psi_\alpha \in W$ is called a *reflection* on $\Sigma$ and $\Gamma$ is the group generated by these reflections. One can also define the *lifted Weyl group* $\tilde{W}$ and the *lifted reflection group* $\tilde{\Gamma}$ as in Definitions 2.3 and 2.6. The same proofs of Theorem 1.1 and Theorem 1.2 allow us to conclude that there exist surjective homomorphisms $\rho : \pi_1(M) \to W/\Gamma$ and $\tilde{\rho} : \pi_1(M) \to \tilde{W}/\tilde{\Gamma}$. Also the same proof of Corollary 1.4 implies that the leaves of $\mathcal{F}$ are closed and embedded and $M/\mathcal{F}$ is a good orbifold, if $M$ is simply connected.

Taking the above remark into account, one can adapt the proof of Corollary 1.3 and get the next result due to Lytchak [7].

**Corollary 6.2.** Let $\mathcal{F}$ be a singular Riemannian foliation on a simply connected complete Riemannian manifold $M$. Assume that $M/\mathcal{F}$ is a good orbifold $\Sigma/W$. Then the regular leaves have trivial holonomy.

**Remark 6.3.** It is not difficult to see that, if the regular leaves have trivial holonomies, then $M/\mathcal{F}$ is a Coxeter orbifold; see [3]. Therefore, if $M$ is simply connected, $M/\mathcal{F}$ is a Coxeter good orbifold. This result is used by the author and Javaloyes [5] to prove that $M/\mathcal{F}$ admits nontrivial closed geodesic if $M/\mathcal{F}$ is a compact orbifold and $\mathcal{F}$ is a singular Riemannian foliation on a simply connected complete manifold $M$.

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