Solving Cut-Problems in Quadratic Time for Graphs With Bounded Treewidth

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Abstract
In the problem (Unweighted) Max-Cut we are given a graph $G = (V,E)$ and asked for a set $S \subseteq V$ such that the number of edges from $S$ to $V \setminus S$ is maximal. In this paper we consider an even harder problem: (Weighted) Max-Bisection. Here we are given an undirected graph $G = (V,E)$ and a weight function $w: E \to \mathbb{Q}_{>0}$ and the task is to find a set $S \subseteq V$ such that (i) the sum of the weights of edges from $S$ is maximal; and (ii) $S$ contains $\lceil n/2 \rceil$ vertices (where $n = |V|$). We design a framework that allows to solve this problem in time $O(2^{t^2}n^2)$ if a tree decomposition of width $t$ is given as part of the input. This improves the previously best running time for Max-Bisection of Hanaka, Kobayashi, and Sone [8] by a factor $t^2$. Under common hardness assumptions, neither the dependence on $t$ in the exponent nor the dependence on $n$ can be reduced [8, 6, 15]. Our framework can be applied to other cut problems like Min-Edge-Expansion, Sparsest-Cut, Densest-Cut, $\beta$-Balanced-Min-Cut, and Min-Bisection. It also works in the setting with arbitrary weights and directed edges.

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1 Introduction

Unweighted Max-Cut is one of Karp’s 21 NP-complete problems [12]; given a graph $G = (V,E)$ one is asked for a set $S \subseteq V$ such that the number of edges from $S$ to $V \setminus S$ is maximal. Formally, a cut is determined by a set of vertices $S \subseteq V$ of a graph. The size of a cut is given by the number of edges from $S$ to $V \setminus S$. We denote these edges as $\partial S$ and, for the sake of shortness, if $S = \{v\}$ for some $v$, we write $\partial v$ instead of $\partial S = \partial \{v\}$. If the graph is weighted, the size of the cut is given by the sum of the edge weights $w(\partial S) := \sum_{e \in \partial S} w(e)$ instead of their number $|\partial S|$. In this paper we consider different cut problems for directed and weighted graphs, more precisely Max-Cut, $\beta$-Balanced-Min-Cut, Max-Bisection, Min-Bisection, Min-Edge-Expansion, (uniform) Sparsest-Cut, and Densest Cut. See Table 1 for precise formulations of these problems.

Observe that Densest-Cut and Sparsest-Cut can easily be reduced on each other in time $O(n^2)$ (where $n = |V|$) using the complementary graph [3]; however, this reduction might change the treewidth and the corresponding decomposition, hence we have to consider both problems individually. We want to point out that if negative edge weights are allowed, as they are in our algorithm, Max-Bisection and Min-Bisection coincide. This does not hold for Max-Cut and its corresponding minimization variant; Min-Cut is solvable in polynomial time.

The undirected and unweighted versions can easily be modelled as directed and weighted by setting each edge weight to 1 and by replacing each undirected edge between vertices $v_1$ and $v_2$ by two directed edges, $v_1v_2$ and $v_2v_1$. 


We call those more general variants of the problem, where we get rid of the non-negativity restrictions, Max-Bisection’, β-Balanced-Min-Cut’, Min-Edge-Expansion’, Sparsest-Cut’, and Densest-Cut’. As we will see, our framework is able to solve these more general variants of the problems.

Many graph problems are in FPT if parametrized by treewidth. This holds especially for the problems mentioned above [3, 8, 6, 11]. The corresponding algorithms usually assume that a tree decomposition with $O(n^t)$ (or a similar bound like $O(nt)$) nodes of width $t$ is given as part of the input; we will later see why that is a reasonable assumption and what we can do if a tree decomposition is not given. We show that in the above setting, all aforementioned problems can be solved in time $O(2^t n^2)$.

## 2 Related Work

Jansen et al. [10] proposed an algorithm for Max-Bisection and Min-Bisection that runs in time $O(2^t n^3)$ for a graph with $n$ vertices, given a tree decomposition of width $t$ with $O(n)$ nodes[10]. They transform the tree decomposition into a so-called nice tree decomposition [14] and then formulate a dynamic program over the nodes of the tree decomposition. The bottleneck of their analysis are nodes that have more than one child, the so-called join nodes. There might be $Ω(n)$ join nodes and for each the dynamic program might take time $Ω(2^t n^2)$ to compute all the entries. Eiben, Lokshtanov, and Mouawad [6] have been able to

[10] The upper bound on the number of nodes occurs only implicitly in their work within the analysis of their algorithm’s running time.

| Name                  | Weights  | Objective                                                                 |
|-----------------------|----------|---------------------------------------------------------------------------|
| Max-Cut               | arbitrary| $\max_{S \subseteq V} w(\partial S)$                                    |
| β-Balanced-Min-Cut [7]| non-negative| $\min_{S \subseteq V, |V| \leq |S| \leq (1-\beta)|V|} w(\partial S)$          |
| Max-Bisection [10]    | non-negative| $\max_{S \subseteq V, |S| \leq |V|} w(\partial S)$              |
| Min-Bisection [10]    | non-negative| $\min_{S \subseteq V, |S| \leq |V|} w(\partial S)$              |
| Min-Edge-Expansion [16]| non-negative| $\min_{\emptyset \neq S \subseteq V} \frac{w(\partial S)}{|S|}$          |
| Sparsest-Cut [3]      | non-negative| $\min_{\emptyset \neq S \subseteq V} \frac{w(\partial S)}{|S| \cdot |V \setminus S|}$ |
| Densest-Cut [3]       | non-negative| $\max_{\emptyset \neq S \subseteq V} \frac{w(\partial S)}{|S| \cdot |V \setminus S|}$ |

Table 1 Problems that we solve in quadratic time.
improve the running time\textsuperscript{[iii]} in its dependence on \( n \) by balancing the tree decomposition and recognizing that the entries in join nodes can be computed via (max, +)-convolution; this yields a running time of \( \mathcal{O}(8^n t^5 n^2 \log n) \). Hanaka, Kobayashi, and Sone \[8\] proved that the algorithm of Jansen et al. does in fact run in time \( \mathcal{O}(2^n (nt)^2) \), using a clever idea to improve the analysis: while for a single join node computing all entries of the dynamic program might take \( \Omega(2^n n^2) \), the overall time for all join nodes altogether is \( \mathcal{O}(2^n (nt)^2) \).

Lokshtanov, Marx, and Saurabh \[15\] proved that Max-Cut (without a tree decomposition given as part of the input) cannot be solved in time \( \mathcal{O}(2^n (nt)^{3+\epsilon}) \) for any \( \epsilon > 0 \) assuming the Strong Exponential Time Hypothesis (SETH) \[9, 4\]. It is not hard to see that this result can be extended to the case where a tree decomposition is part of the input – for the sake of completeness, we include the corresponding proof in the appendix. By adding isolated vertices, this result can also be applied to Max-Bisection and Min-Bisection \[8\]. Eiben, Lokshtanov, and Mouawad \[6\] proved that Min-Bisection (and hence Max-Bisection\textsuperscript{'} cannot be solved in truly subquadratic time, that is \( \mathcal{O}(n^{2-\epsilon}) \) for some \( \epsilon > 0 \), even if a tree decomposition of width 1 is given as part of the input, unless (\( \min, + \))-convolution can be solved in truly subquadratic time, which is considered unlikely \[5\].

Given a tree decomposition of width \( t \) with \( \mathcal{O}(nt) \) nodes, in time \( \mathcal{O}(2^tn^3) \) the problems Sparsest-Cut \[3\], Densest-Cut \[3\], and Min-Edge-Expansion \[11\] can be solved. To our knowledge, those are the best running times achieved so far.

Our Contribution & Organization Of This Paper

In Section 4 we show how to improve the running time by a factor \( t^2 \) for Max-Bisection. In Section 5 we then generalize this to a framework which can be used to solve different cut problems in time \( \mathcal{O}(2^t n^2) \) (compare Table 1). Some problems (like Sparsest Cut) are improved by a factor \( n \), which is substantial when \( t \) is small. In Section 6 we eventually show how the framework can be used to solve aforementioned problems, including the proofs that the corresponding models are correct.

3 Preliminaries

Notation For tuples we write \((a, b) \oplus (c, d) = (a + c, b + d)\) (and analogously define \(\ominus\)). We write \(\pi\) to denote the projection on tuples, that is: for a tuple \( t \) and an index \( i \), \( \pi_i(t) \) is the \( i\)-th component of \( t \).

For a set \( M \) and a number \( k \) we write \( M_k = \{ M' : |M'| = k \} \). For a graph edge from \( v_1 \) to \( v_2 \) we write \( v_1 v_2 \) for both, directed and undirected graphs. A rooted tree \( T = (V, r, E) \) is a graph \((V, E)\) that is connected, has no circles, and where \( r \in V \).

We use \( n \) as abbreviation for \(|V|\). By \( \mathbb{Q} := \mathbb{Q} \cup \{\infty, -\infty\} \) we denote the rational numbers with positive and negative infinity.

As already mentioned in the previous sections, we make use of tree decompositions, which are defined as follows:

\textbf{Definition 1} (Tree decomposition). Let \( G = (V, E) \) be an undirected graph, \( I \) be a finite set and \( X = (X_i)_{i \in I} \) be a family of sets such that for any \( i \in I \) one has \( X_i \subseteq V \). Moreover, let \( T = (I, r, H) \) be a rooted tree with root \( r \in I \). Then \((I, r, X, H)\) is called a \textit{tree decomposition} of \( G \) if and only if \( T \) has the following properties:

\textsuperscript{[iii]}To be precise, they consider Min-Bisection; however, in their paper as well as in \[10\], all arguments work for both problems.
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(i) **Node coverage**: Every vertex occurs in some $X_i$ for some $i \in I$ (and no further vertices occur): $\bigcup_{i \in I} X_i = V$;

(ii) **Edge coverage**: for every edge $v_1v_2 \in E$ there is a node $i \in I$ such that both, $v_1$ and $v_2$, are contained in $X_i$;

(iii) **Coherence**: for every vertex $v \in V$ the subgraph $T - \{ i \in I : v \notin X_i \}$ is connected. 

By convention the nodes of the graph $G$ are called *vertices* while the nodes of the tree are just called *nodes*. For a node $i \in I$ the set $X_i$ is called a *bag*. The *width* of a decomposition is the largest cardinality of any of its bags minus 1. The minimum width among all decompositions of $G$ is called the *treewidth* of $G$. If the node set of a decomposition is sufficiently small, more precisely if $|I| \leq 4 \cdot (|V| + 1)$, we call the decomposition *small*. A tree decomposition of a directed graph is a tree decomposition of the underlying graph.

As the approaches mentioned before, our approach also makes use of a specific kind of tree decompositions, which are of a very simple structure:

**Definition 2** (Nice Tree Decomposition). Let $G = (V,E)$ be an undirected graph and $(I,r,X,H)$ a tree decomposition of $G$. We call $(I,r,X,H)$ a nice tree decomposition if and only if for any $i \in I$ the node $i$ is of one of the following forms:

(i) **Leaf node**: $i$ has no child node in $T$, that is $i$ is a leaf of the tree $T$;

(ii) **Forget node**: $i$ has exactly one child node $j \in I$ in $T$ and $X_i \cup \{v\} = X_j$ for some $v \in X_j$, that is $i$ forgets a vertex from $X_j$;

(iii) **Introduce node**: $i$ has exactly one child $j \in I$ in $T$ and $X_i = X_j \cup \{v\}$ for some $v \in V \setminus X_j$, that is $i$ introduces a new vertex $v \in V \setminus X_j$; or

(iv) **Join node**: $i$ has exactly two child nodes $j \in I, k \in I, j \neq k$, in $T$ such that $X_i = X_j = X_k$, that is joining two branches of the tree $T$.

The conversion of a tree decomposition into a nice tree decomposition can be done in time $O(nt^2)$ as long as the number of nodes of the decomposition is at most linear in the number of vertices.

**Lemma 3** ([14, Lemma 13.1.3, p. 150]). Given a small tree decomposition of a graph $G$ with width $t$ one can find a nice tree decomposition of $G$ with width $t$ and with at most $4n$ nodes in $O(nt^2)$ time, where $n$ is the number of vertices of $G$.

It is reasonable to assume that given tree decompositions are small for the following reason:

No matter how the tree decomposition is constructed, it is always possible to incorporate the following mechanism without asymptotically increasing the running time: If a node $j$ with parent $i$ has $X_j \subseteq X_i$, merge those nodes.

We claim that we now can only have $n$ edges. This is because for every node $j$ with parent $i$, we have $X_j \not\subseteq X_i$; this means, at least one vertex has to disappear when going from $j$ up to $i$. Since by Coherence every vertex can disappear at most once,[iv] this upper bounds the number of edges by $n$ and hence the number of nodes by $n + 1$.

Note that this also allows us to easily extend our approach to the case where a tree decomposition with $O(nt)$ nodes is given as part of the input: simply apply the procedure described above to reduce the number of nodes.

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[iiv] It might disappear on multiple leaf-root-paths; however, the node at which a vertex disappears, is the same on each of those leaf-root-paths.
4 Max-Bisection: From $O(2^t n^3)$ to $O(2^t n^2)$

In this section we focus on our idea on how the running time of the algorithm of Jansen et al. [10] for Max-Bisection can be improved to $O(2^t n^2)$, incorporating the idea of Hanaka, Kobayashi, and Sone [8].

The algorithm of Jansen et al. [10] is a subroutine used in their PTAS for the Max-Bisection problem on planar graphs. Their approach uses Baker’s technique (see [1]) where the idea is to solve the problem for $k$-outerplanar graphs (instead of general planar graphs), for a $k$ depending only on the approximation factor, and then combining the results. Note that $k$-outerplanar graphs have a treewidth in $O(k)$ [13]. For those $k$-outerplanar graphs, the problem is solved exactly using the aforementioned subroutine. Since – as opposed to the general case – tree decompositions for $k$-outerplanar graphs can be computed in time $O(kn)$ [13], this subroutine gets tree decomposition as part of its input; otherwise the running time of the subroutine would be dominated by the computation of the decomposition.

Let us now focus on the subroutine. We will traverse the nice tree decomposition bottom up in the algorithm of Jansen et al., hence the following notations come in handy: For a node $i$ the set $Y_i$ contains all the vertices appearing in bags associated with nodes below $i$. Moreover, we write $F_i := Y_i \setminus X_i$ to describe the set of vertices that “have been forgotten” somewhere below $i$, that is, that they have appeared in bag of some node $j$ below $i$, but are not contained in $X_i$. Due to Definition 1 (iii), those vertices can never reoccur in any bag above $i$.

The algorithm of Jansen et al. uses a dynamic program to compute

$$
\mathcal{B}_i: \{0, \ldots, |Y_i|\} \times 2^{X_i} \rightarrow \bar{Q}_{>0}
$$

$$
\mathcal{B}^i(\ell, S) = \max_{S \subseteq Y_i, |S| = \ell} w(\partial \bar{S} \cap Y_i^e),
$$

(1)

given a small nice tree decomposition of a weighted, undirected graph $G = (V, E)$ with weight function $w: E \rightarrow \bar{Q}_{>0}$. The idea is that for a node $i$ the entry $\mathcal{B}_i(\ell, S)$ is the size of the largest possible cut that consists of $\ell$ vertices from $Y_i$ and includes the set $S \subseteq X_i$. As the table might have preimages $(\ell, S)$ where there does not exist be a cut meeting the requirements above, $\infty$ and $-\infty$ have to be used to deal with those – we omit further details.

For the root $r$ of the tree decomposition we can compute our optimal objective value by iterating over all entries of $\mathcal{B}_r$ and picking the best value where the number of vertices is in the feasible interval for Max-Bisection.

The dynamic program traverses the decomposition bottom up. The bottleneck of the running time comes from the time spent at join nodes – the values for each node of any different type can be computed in time $O(2^t n)$ using a simple DP. For a join node $i$ with left child $j$ and right child $k$, they use the following recurrence to compute $\mathcal{B}_i$:

$$
\mathcal{B}_i(\ell, S) = \max_{|S| \leq \ell_1 \leq |V|} (B_j(\ell_1, S) + B_k(\ell_2, S) - w(\partial S \cap (S \times (X_i \setminus S))))
$$

(2)

We omit the details on how the necessary values of $w$ are computed in their case; it suffices to see that a rough analysis of the equation above, assuming that we are given the value of the $w$ expression, is $O(n^2)$ per entry for a single join node. It is also easy to see that there are indeed instances where the computation of an entry for $(\ell, S)$ takes $\Theta(n^2)$ time. This yields an overall running time of $O(2^t n^3)$ as stated by Jansen et al. [10].
Hanaka, Kobayashi, and Sone [8] provided an improved analysis for the algorithm of Jansen et al. [10]. They defined $v_i$ to be the sum of all $|X_j|$ for nodes $j$ that are below $i$. It is not hard to see that the running time for the computation of a single entry of a join node $i$ with left child $j$ and right child $k$ can be done in time $O(v_i \nu_k)$. Using a labeling argument they then proved that

$$\sum_{i: \text{join node with children } j, k} v_j \nu_k \leq (nt)^2.$$ 

Their idea is that after labeling the vertices in all bags (possibly giving the same vertex different labels for different bags), every pair of those labels can occur at at most one join node. The consequence of the above statement is that the worst case for join nodes cannot occur too often; overall, all entries of all join nodes can be computed in time $O(2^n nt)^2$.

Our approach is now to reformulate the recurrence by something that can be thought of as an index shift; for each node $i$ we define a table

$$\Gamma_i: \{0, \ldots, |F_i|\} \times 2^{X_i} \to \mathbb{Q}$$

$$\Gamma_i(\ell, S) := \max_{S \in (F_i)} w(\partial(S \cup \tilde{S}) \cap Y_i^2).$$

(3)

In comparison to (1), there are two differences.

1. The indices have a different meaning: In $\Gamma_i(\ell, S)$ we store the value of the best cut (with respect to $f$) that consists of the set $S \subseteq X_i$ and $\ell$ further vertices that occur in bags below $i$, but not in $X_i$.

2. The table’s size is now $O(|F_i|^2)$; this is not only smaller but also for every entry $(\ell, S)$ there is a cut consisting of $\ell$ vertices from $F_i$ and the vertices from $S$ (hence we do not have to consider those special cases of undefinedness as it had to be done in [10]).

For the modified recursion, the join nodes are still the bottleneck; their recurrence is\(^{\text{[v]}}\)

$$\Gamma_i(\ell, S) \approx \left( \max_{0 \leq \ell_1 \leq |F_j| \atop 0 \leq \ell_2 \leq |F_k| \atop \ell_1 + \ell_2 = \ell} (\Gamma_j(\ell_1, S) + \Gamma_k(\ell_2, S)) \right) - w(\partial S \cap Y_i^2).$$

The key observation is that the running time is dominated by computing the max expression\(^{\text{[v]}}\), which depends linearly on $|F_j| \cdot |F_k| = |F_j \times F_k|$. We can now show that all of those occurring Cartesian products are disjoint:

\textbf{Proposition 4.} For each pair $(v_1, v_2) \in V^2$ there is at most one join node $i$ with left child $j$ and right child $k$ such that $(v_1, v_2) \in F_j \times F_k$.

\textbf{Proof.} Proof by contradiction. Assume there was a join node $i' \neq i$ with left child $j'$ and right child $k'$ such that $(v_1, v_2) \in F_{j'} \times F_{k'}$. Then, by Coherence, either $i'$ is below $i$ or $i$ is below $i'$. We assume without loss of generality that $i'$ is below $i$. Moreover, we assume without loss of generality that $i'$ is somewhere in the left subtree of $i$. As $(v_1, v_2) \in F_j \times F_k$ by assumption, we have in particular $v_2 \in F_k$, and, additionally taking into account that $F_j \cap F_k = \emptyset$ by Coherence, $v_1 \notin F_k$. Since $i'$ is in the left subtree of $i$, we also have $F_{j'} \subseteq F_{j'} \subseteq F_j$, hence $v_1 \notin F_{j'}$. This is a contradiction to $(v_1, v_2) \in F_{j'} \times F_{k'}$.

\textsuperscript{[v]}Since we only did an index shift, we can reuse the recurrence from [10] by applying the same shift to it.

\textsuperscript{[v]}Note that $\partial S$ can only take on $O(2^n)$ different values at some fixed node $i$. We use a simple DP to precompute the $w(\cdot)$ terms efficiently (for a fixed node in time $O(2^n)$).
Using this statement we can now deduce that

\[
\sum_{i : \text{join node with children } j,k} |F_j \times F_k| = \bigcup_{i : \text{join node with children } j,k} (F_j \times F_k) \leq |V|^2 = n^2. \tag{4}
\]

We can thus deduce the overall running time of computing all entries for all join nodes is \(\mathcal{O}(2^t n^2)\), as we need time \(\mathcal{O}(2^t |F_j \times F_k|)\) for a single join node.

As the running time for the other node types obviously remain unchanged, this yields an algorithm with overall running time \(\mathcal{O}(2^t n^2)\) for Max-Bisection.

5 Our Framework

In this section we discuss how we can generalize the idea from the previous section to other cut problems. More precisely, we present a framework that can solve several cut-problems (for directed, arbitrarily-weighted graphs \(G = (V,E)\) with weight function \(w : E \rightarrow \mathbb{Q}\)) in time \(\mathcal{O}(2^t n^2)\) if a small tree decomposition of width \(t\) is given as part of the input. Without loss of generality we assume that this small tree decomposition is also a nice tree decomposition (if not, we could simply use Lemma 3 to convert it accordingly in sufficiently small time).

The main obstacle is finding an abstraction of the algorithm for Max-Bisection that maintains the running time, but also allows us tackle all the listed problems. Especially extracting the formal arguments hidden implicitly in existing algorithms turned out to be a non-trivial task.

For our framework, we assume that we are given an objective function \(f : \mathbb{N}_0 \times \mathbb{Q} \rightarrow \mathbb{Q}\) that is either monotonic or antitonic\(^{[vii]}\) in its second argument, and a validator function \(\Lambda : \mathbb{N}_0 \rightarrow \{\text{true}, \text{false}\}\). We use a function of arity 2 to be able to not only model Max-Bisection and similar, but also e.g. Sparsest-Cut, where the objective depends on the size of the cut and the number of vertices selected. The validator function is needed e.g. for Max-Bisection, as we somehow have to tell the framework which entries correspond to feasible solutions and which are infeasible; for Max-Bisection we would set \(\Lambda(x) := (|x| - (n - x)| \leq 1)\).

We assume that both, \(f\) and \(\Lambda\), can be evaluated in time \(\mathcal{O}(1)\).

Our task is now to compute an element of all possible preimages (in the sense: there exists a corresponding cut) \(\left\{ \left(|S|, w(\partial S)\right) : S \subseteq V, \Lambda(|S|) \right\}\) of the objective function \(f\) that maximizes \(f\). We can reformulate this task in a more elegant way by introducing the total quasiorder \(\succeq \subseteq \left(\{0, \ldots, n\} \times \mathbb{Q}\right) \cup \{\perp\}\)^2 defined by

\[
a \sqsubseteq b \iff a = \perp \lor (a \neq \perp \neq b \land f(a) \leq f(b)).
\]

The intuition behind that quasiorder is as follows: if we compare two values \(a, b \neq \perp\), then \(a \sqsubseteq b\) iff \(f(a) \leq f(b)\), that is, we compare (non-\(\perp\)) values by their image under \(f\).

The idea of the new symbol \(\perp\) is to represent the case where there is no feasible solution and hence no possible preimage to \(f\) as we have to deal with that case, too.

Our task is now to compute (where \(\bigsqcup\) is the supremum operator regarding \(\sqsubseteq\)).

\[
\Phi = \bigsqcup_{S \subseteq V, \Lambda(|S|)} \left(|S|, w(\partial S)\right).
\tag{5}
\]

\[^{[vii]}\; x \leq y \implies f(a, x) \geq f(a, y)\]
By definition, ⊥ is the smallest element of our order, so if there is a feasible solution, the result cannot be ⊥.

From a strict mathematical perspective, Equation 5 is incorrect as in general there is no such thing as a unique supremum for a total quasiorder (there might be multiple possible preimages of \(f\) taking on the optimal value). Taking this into account would make the description of our approach way more complicated, as we would need to reason about equivalence classes and eventually give a recurrence to compute a representant of the class of element optimizing the objective function. Thus, we identify elements and their corresponding equivalence class (set of possible preimages that have the same objective value) in this paper.

For the sake of shortness, for a subset of edges \(M \subseteq E\) we write \(w_i(M) := w(M \cap Y_i^2)\). We set up the dynamic program similar to the one for Max-Bisection, that is for a node \(i\) we compute

\[
\Gamma_i : \{0, \ldots, |F_i|\} \times 2^{X_i} \to \{0, \ldots, n\} \times Q
\]

\[
\Gamma_i(\ell, S) := \bigcup_{S \in (r'_{\ell})} \big( \ell + |S|, w_i(\partial(S \cup \tilde{S})) \big)
\]

In comparison to Equation 3 there are two differences.
1. An entry is no longer just the size of the corresponding cut, but a 2-tuple consisting of the number of vertices selected and the size of the cut.
2. Instead of storing values for the largest cut, as we did for Max-Bisection, we store the tuple that maximizes the function \(f\).

We can now rewrite Equation 5 in terms of \(\Gamma\) (recall that \(r\) is the root of the given nice tree decomposition):

\[
\Phi = \bigcup_{S \subseteq X_r, \ell \leq |F_r|} \Gamma_r(\ell, S)
\]

Lemma 5.

\[
\Phi = \bigcup_{S \subseteq X_r, \ell \leq |F_r|} \Gamma_r(\ell, S) = \bigcup_{S \subseteq X_r, \ell \leq |F_r|} \bigcup_{S \subseteq \{\ell \in |S|\}} \big( \ell + |S|, w_r(\partial(S \cup \tilde{S})) \big) = \bigcup_{S \subseteq X_r, \ell \leq |F_r|} \big( \ell + |S|, w_r(\partial(S \cup \tilde{S})) \big)
\]

\[
= \bigcup_{S \subseteq X_r} \big( |S| + |\tilde{S}|, w_r(\partial(S \cup \tilde{S})) \big) = \bigcup_{S \subseteq X_r} \big( |S|, w_r(\partial S) \big) = \Phi
\]

It is easy to see that, if we are given the values \(\Gamma_r(\ell, S)\) for all \(0 \leq \ell \leq |F_r|\) and all \(S \subseteq X_r\), we can compute \(\Phi\) in time \(O(2^{|N|})\). We claim that we can compute the table \(\Gamma\) for all nodes together in overall time \(O(2^{|N|^2})\), implying that \(\Phi\) can be computed in time \(O(2^{|N|^2})\). To see this, we now show that we can use a dynamic program to compute the table \(\Gamma\) and eventually \(\Phi\) in the desired time. Therefor, we first set up recurrences for each node type that we can use to efficiently compute the value for a node \(i\) of this type, given that we already know all the values below \(i\).

For our approach there is an important property of \(\subseteq\): Basically, we are able to move the addition with a constant tuple outside of the supremum operator, if the elements all have the same first component.
Lemma 6. For any $a$, any finite set $M \subseteq \{a\} \times \mathbb{Q}$ and any $z$ it holds that

$$z \oplus \bigcup_{x \in M} x = \bigcup_{x \in M} (z \oplus x)$$

Proof. As $M$ is finite, it suffices to show this property for the binary supremum $\sqcup$. Let $z = (b, w)$ and $(a, x) \in M$, $(a, y) \in M$. If $x = y$ or $f(a, x) = f(a, y)$, the property is trivial. Thus we may assume without loss of generality that $x > y$ and $f(a, x) \neq f(a, y)$. We now have two cases, depending on $h \mapsto f(a, h)$. The first case is that $h \mapsto f(a, h)$ monotonic. Then our assumption implies that $f(a, x) > f(a, y)$ and hence:

$$(h, z) \oplus ((a, x) \sqcup (a, y)) = (h, z) \oplus (a, x) = (a + h, x + z) = (a + h, x + z) \sqcup (a + h, y + z)$$

The last step follows from the monotony, as $x + z \geq y + z$. The second case, where $h \mapsto f(a, h)$ is antitonic, can be proven analogously. \hfill ◼️

This property is absolutely crucial as it gives us some freedom for transformations of Equation 6; it is applied multiple times in the proofs in the appendix. Intuitively, this lemma tells us that the optimization process works no different than it does e.g. Max-Cut or Max-Bisection; if we fix the number of vertices we choose, given a set of cuts to pick from the cut maximizing $f$ is either the cut of largest size or the cut of smallest size.

5.1 Recurrences

We now set up a recurrence, depending on the node type, to compute the table $\Gamma$ by traversing the tree decomposition in a bottom-up fashion. In this section we give the intuition behind the recurrences step by step. For correctness proofs of the equations (which are very technical) we refer to the appendix.

Leaf Node

Let $i$ be a leaf node. Then $F_i = \emptyset$. Thus, $\Gamma_i$ is only defined for $\ell = 0$ and $S \subseteq X_i$. If $S = \emptyset$, there are no edges in the cut, hence $\Gamma_i(0, \emptyset) = (0, 0)$. If $S = \{v\}$ consists of a single node, we can simply go through all its edges, that is set

$$\Gamma_i(0, \{v\}) = \left(1, \sum_{v' \in X_i \setminus \{v\}} w(vv')\right).$$

Now let $S = S' \cup \{v\}$ where $S' \neq \emptyset$. We will now argue how we can compute the value $\Gamma_i(0, S)$ given the value $\Gamma_i(0, S')$. In the situation considered for $\Gamma_i(0, S')$ we have $v \notin S'$. If we move $v$ into the selection and are able to track and compute the changes, we can also compute $\Gamma_i(0, S)$. After moving $v$ into the selection, there might be edges from $S'$ to $\{v\}$ (which all have been in the cut before); those edges are no more in the cut for $S$. Also, there might also be some new edges in the cut, more precisely all edges from $v$ to $X_i \setminus S$. There

---

For the reader not familiar with order theory: The binary supremum is an associative and commutative map. If for every pair of elements there is a supremum, that is, a smallest element that is larger than both elements of the pair, then so it does for any finite set $M$. This can be shown by a simple inductive argument using the aforementioned associativity/commutativity.
are no more new edges in the cut and no other edges are removed from the cut. This yields

$$\Gamma_i(0, S' \cup \{v\}) = \Gamma_i(0, S') \oplus \left(1, \sum_{v' \in X_i \setminus S'} w(vv') - \sum_{v' \in S'} w(v'v) \right).$$

(9)

### Forget Node

Let $i$ be a forget node with child $j$. Then there is $v \in V$ such that $X_i \cup \{v\} = X_j$. Now, for the computation of the entries $\Gamma_i(\ell, S)$ we only have to deal with one question: is it better to include $v$ into the selection or not? As $v \in F_i$, this question is only relevant if $\ell \geq 1$; if $\ell = 0$, then $\Gamma_i(\ell, S) = \Gamma_i(\ell, S)$. Now let $\ell \geq 1$. If we included $v$ into our selection, we have to include $\ell - 1$ further vertices from $F_j$; if we do not include $v$, we have to include $\ell$ further from $F_j$. We now simply pick the better result of both options. Overall, this yields the following recurrence:

$$\Gamma_i(\ell, S) = \begin{cases} \Gamma_j(\ell - 1, S \cup \{v\}) \cup \Gamma_j(\ell, S) & \ell \geq 1 \\ \Gamma_j(\ell, S) & \ell = 0. \end{cases}$$

(10)

### Introduce Node

Let $i$ be an introduce node with child $j$. Then there is $v \in V$ such that $X_i = X_j \cup \{v\}$. When computing $\Gamma_i(\ell, S)$ we have to make a case distinction whether $v$ is in $S$ or not.

In the first case $v \in S$ we rely on $\Gamma_j(\ell, S \setminus \{v\})$ to compute the value. As we consider a selection of $S$ and $\ell$ further vertices of $F_i$ as opposed to the entry $\Gamma_j(\ell, S \setminus \{v\})$ which only considers a selection of $S \setminus \{v\}$ and $\ell$ further vertices of $F_j = F_i$, we have to add to the first component of $\Gamma_j(\ell, S \setminus \{v\})$ to account for the additional vertex $v$. For the second component, the size of the cut, we only have to add the weight of all edges from $v$ to $X_j \setminus S$. Note that there are no weights that we need to subtract; while there might be edges between $v$ and $S$, those are not considered in the computation of $\Gamma_j(\ell, S \setminus \{v\})$ as the vertex $v$ does not appear in $Y_j$ (due to Coherence and Edge Coverage). Thus, for $v \in S$ we get

we now add the new vertex $v$ to the selection $S \setminus \{v\}$

$$\Gamma_i(\ell, S) = \left(1, w_i(\partial v \cap (\{v\} \times (X_i \setminus S))) \right) \oplus \Gamma_j(\ell, S \setminus \{v\}).$$

(11)

edges from $v$ to $X_i \setminus S$; we will later discuss how to compute this value

Now let us consider the case $v \notin S$. The argumentation is very similar. As $v \notin S$, there is no new vertex we add to the selection. However, there are possibly new edges and that change the value of the cut: the edges from $S$ to $\{v\}$. This gives us the following recurrence for the case $v \notin S$:

$$\Gamma_i(\ell, S) = \left(0, w_i(\partial S \cap (X_i \times \{v\})) \right) \oplus \Gamma_j(\ell, S).$$

(12)

edges from $S$ to $\{v\}$; we will later discuss how to compute this value

In both cases there is a (case-dependent) additive term that has to be added to an entry of $\Gamma_j$. Observe that this additive term is independent of $\ell$. As it turns out, this additive term (depending only on $S$) can be rewritten as

$$\pi_2(\Gamma_i(\ell, S) \oplus \Gamma_j(\ell, S))$$

(13)
We claim that our algorithm has an overall running time of $O(2^n)$ per node for any arbitrary, but fixed $t$; for details we refer to Lemma 22 in the appendix. Thus, we can do the following: for $t = 0$ we need to explicitly compute the additive term for all $S \subseteq X_i$. For all $t \geq 1$ we can simply use $\pi_2(\Gamma_i(\ell - 1, S) \oplus \Gamma_i(\ell - 1, S))$ for all $S \subseteq X_i$.

**Join Node**

Let $i$ be a join node with left child $j$ and right child $k$. Then $X_i = X_j = X_k$. The value $\Gamma_i(\ell, S)$ can then be interpreted as the tuple maximizing $f$ over all selections $S \subseteq Y_i$ where there are $\ell_1$ vertices from $F_j$ and $\ell_2$ from $F_k$ for every $0 \leq \ell_1 \leq |F_j|$, $0 \leq \ell_2 \leq |F_k|$ where $\ell_1 + \ell_2 = \ell$. This is closely related to

$$\Gamma_i(\ell, S) = \bigcup_{\substack{0 \leq \ell_1 \leq |F_j| \\ 0 \leq \ell_2 \leq |F_k| \\ \ell_2 + \ell_1 = \ell}} (\Gamma_j(\ell_1, S) \oplus \Gamma_k(\ell_2, S));$$

it is not hard to see that this equation almost computes the desired tuple, with the exception that each vertex in $S$ and each edge of $E \cap X_i^2$ that is in the cut is counted twice. Thus, by subtracting the doubly counted vertices/edge weights and applying Lemma 6 we get

$$\Gamma_i(\ell, S) = \left( \bigcup_{\substack{0 \leq \ell_1 \leq |F_j| \\ 0 \leq \ell_2 \leq |F_k| \\ \ell_2 + \ell_1 = \ell}} (\Gamma_j(\ell_1, S) \oplus \Gamma_k(\ell_2, S)) \right) \ominus \left( |S|, w_i(\partial S) \right). \quad (14)$$

**5.2 Running Time**

We claim that our algorithm has an overall running time of $O(2^nn^2)$. To prove this we first need to deal with some preprocessing steps. The first step was already mentioned: converting the tree decomposition into a nice tree decomposition. This can be done in $O(nt^2)$ according to Lemma 3. We model sets as bit vectors of length $n$; we omit the (very technical) details on how set operations can be implemented on bit vectors such that they all take only constant time. The second preprocessing step is creating an adjacency matrix of the graph in time $O(n^2)$; this is necessary to be able to compute the explicit sums occurring in our recurrences in time $O(n)$ each. We now claim that our dynamic program takes the running time specified in Figure 1 per node, depending on the node’s type. As we only have $O(n)$ nodes in the decomposition (it is a small decomposition), this directly implies that the computation of $\Gamma_i$ only takes time $O(2^nn^2)$.

**Leaf Node**

Let $i$ be a leaf node. Then $F_i = \emptyset$. Hence $\Gamma_i$ has $O(2^n)$ entries. We make use of Equations 8 and 9 to compute these entries; both equations have a constant number of sums, and each sum can be computed in $O(n)$ using the adjacency matrix and basic set operations.
Forget Node

Let \( i \) be a forget node with child \( j \). Then \( X_i \cup \{v\} = X_j \) for some \( v \in V \). There are \( O(2^t|F_i|) \) entries in \( \Gamma_i \). As we only use Equation 10 to compute the values, which takes constant time per value, the overall running time for a single forget node is \( O(2^t|F_i|) \).

Introduce Node

Let \( i \) be an introduce node with child \( j \). Then \( X_i = X_j \cup \{v\} \) for some \( v \in V \). There are \( O(2^t|F_i|) \subseteq O(2^t n) \) entries in \( \Gamma_i \) we need to compute. For the entries \( \Gamma_i(0,S) \) for \( S \subseteq X_i \) we use Equations 11 and 12 and explicitly compute the \( w_i \) expressions as sums in \( O(n) \) each. There are at most \( O(2^t n) \) entries for \( \ell = 0 \), thus for all these entries we need at most \( O(2^t n) \) time. For all other \( O(2^t n - 2^t) = O(2^t n) \) entries we make use of Equation 13 which allows us now to use Equations 11 and 12 to compute the remaining entries in time \( O(1) \) each. The overall running time for a single introduce node is hence \( O(2^t n) \).

Join Node

Let \( i \) be a join node with left child \( j \) and right child \( k \). Then \( X_i = X_j = X_k \). There are \( O(2^t|F_i|) \subseteq O(2^t|F_j| \cdot |F_k|) \) entries we need to compute. For a fixed \( S \subseteq X_i \), we need to compare \( |F_j| \cdot |F_k| \) different pairs in that equation. This means that the running time – per entry – is in \( O(|F_j| \cdot |F_k|) \) = \( O(|F_j| \cdot |F_k|) \). Using Equation 4 (which is a consequence of Proposition 4) it follows that the overall running time for all entries of all join nodes is bounded by \( O(2^t n^2) \).

6 Applications

We now give applications to several cut problems. Recall that our framework computes \( \Phi \), a 2-tuple that maximizes a given map \( f \). At this point we want to point out that \( \Phi = \bot \) can only occur if there is no partition of the vertices whose cardinality satisfies \( \Lambda \). This does not occur in our applications.[ix] Note that for any instantiation of the framework we have

\[
 f(\Phi) = f(\bigcup_{S \subseteq V} (|S|, w(\partial S))) = \max_{S \subseteq V \Lambda(|S|)} f(\max_{S \subseteq V \Lambda(|S|)} (|S|, w(\partial S))).
\]  

(15)

Max-Cut

Set \( f(v, s) := s \) and \( \Lambda(x) := \text{true} \). It is not hard to see that \( f \) is monotonic in the second argument and hence fulfils the requirements of our framework.

▲ Lemma 7.

\[
 f(\Phi) = \max_{S \subseteq V} w(\partial S)
\]

Proof.

\[
 f(\Phi) = \max_{S \subseteq V \Lambda(|S|)} f(\max_{S \subseteq V \Lambda(|S|)} (|S|, w(\partial S))) = \max_{S \subseteq V} w(\partial S)
\]

[ix]In general, we cannot think of any reason why this case should occur in practice; it seems to us that if it occurs this is likely to indicate an error of the model used.
Max-Bisection’

Set \( f(\nu, s) := s \) and \( \Lambda(x) := (2x - n) \leq 1 \). It is not hard to see that \( f \) is monotonic in the second argument and hence fulfils the requirements of our framework.

\[ \text{Lemma 8.} \]

\[ f(\Phi) = \max_{S \subseteq V} w(\partial S) \]

Proof.

\[ f(\Phi) = \max_{S \subseteq V} f\left( |S|, w(\partial S) \right) = \max_{|S| - |S| \leq 1} w(\partial S) = \max_{|S| - |V \setminus S| \leq 1} w(\partial S) \]

\[ \beta \text{-Balanced-Min-Cut’} \]

Set \( f(\nu, s) := -s \) and \( \Lambda(x) := (\beta n \leq x \leq (1 - \beta)n) \); the negation is required to model a minimization problem in our maximization framework. It is not hard to see that \( f \) is antitonic in the second argument and hence fulfils the requirements of our framework.

\[ \text{Lemma 9.} \]

\[ -f(\Phi) = \min_{\beta |V| \leq |S| \leq (1 - \beta)|V|} w(\partial S) \]

Proof.

\[ -f(\Phi) = -\max_{\Lambda(|S|)} f\left( |S|, w(\partial S) \right) = \min_{\beta n \leq |S| \leq (1 - \beta)n} -f\left( |S|, w(\partial S) \right) = \min_{\beta |V| \leq |S| \leq (1 - \beta)|V|} w(\partial S) \]

Min-Edge-Expansion’

Set \( f(\nu, s) := \begin{cases} -\infty & \nu = 0 \\ -\frac{2}{\beta} \nu & \text{otherwise} \end{cases} \). We claim that \( f \) is antitonic in its second argument. To see this, fix \( \nu \) such that \( 0 \leq \nu \leq n \) and let \( g(s) := f(\nu, s) \). If \( \nu = 0 \), then for all \( s \) we have that \( g(s) = f(\nu, s) = -\infty \) is a constant map and hence antitonic. If \( \nu > 0 \), then then for all \( s \) we have that \( g(s) = f(\nu, s) = -\frac{2}{\beta} \) is a linear map with negative sign and hence also antitonic.

We now need the validator function to exclude all sets \( S \) where \( |S| > |V \setminus S| \); we do not have to explicitly exclude the empty set as our measure function evaluates to \( -\infty \) for those cases (but we may do so). Thus, we might define \( \Lambda(x) := (x \leq n - x) \). Then the value of the Min-Edge-Expansion can be recovered from the result of our framework \( \Phi \) by computing \(-f(\Phi)\). That value cannot be \( \infty \) unless the graph is empty.

\[ \text{Lemma 10.} \text{ If } n \geq 1 \text{, then } -f(\Phi) = \min_{\emptyset \neq S \subseteq V} \frac{w(\partial S)}{|S|}. \]

Proof.

\[ -f(\Phi) = -\max_{\Lambda(|S|)} f\left( |S|, w(\partial S) \right) = \min_{|S| \leq n - |S|} -f\left( |S|, w(\partial S) \right) \]

\[ = \min_{S \subseteq V} \begin{cases} \infty & |S| = 0 \\ \frac{w(\partial S)}{|S|} \end{cases} \quad \text{otherwise} \]

\[ = \min_{\emptyset \neq S \subseteq V} \min_{|S| \leq |V \setminus S|} \frac{w(\partial S)}{|S|} \]

\[ \text{Lemma 11.} \text{ If } n \geq 1 \text{, then } -f(\Phi) = \min_{\emptyset \neq S \subseteq V} \frac{w(\partial S)}{|S|}. \]

Proof.

\[ -f(\Phi) = -\max_{\Lambda(|S|)} f\left( |S|, w(\partial S) \right) = \min_{\emptyset \neq S \subseteq V} \min_{|S| \leq |V \setminus S|} \frac{w(\partial S)}{|S|} \]

\[ \text{Lemma 12.} \text{ If } n \geq 1 \text{, then } -f(\Phi) = \min_{\emptyset \neq S \subseteq V} \frac{w(\partial S)}{|S|}. \]

Proof.

\[ -f(\Phi) = -\max_{\Lambda(|S|)} f\left( |S|, w(\partial S) \right) = \min_{\emptyset \neq S \subseteq V} \min_{|S| \leq |V \setminus S|} \frac{w(\partial S)}{|S|} \]
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Sparest-Cut’

Sparest Cut’ is quite similar to Min-Edge-Expansion’. We set \( f(\nu, s) := \begin{cases} -\infty & \text{if } \nu \not\in \{0, n\} \\ \frac{s}{t(n-\nu)} & \text{otherwise} \end{cases} \). This is again an antitonic map, following the same argumentation as for Min-Edge-Expansion’. The validator function becomes trivial for Sparest-Cut’, that is \( \Lambda(x) := \text{true} \). Recovering the Sparest-Cut’ from \( \Phi \) works the same way as before: it can be expressed as \( -f(\Phi) \) which, again, cannot be \( \infty \) unless the graph has at most one vertex.

\textbf{Lemma 11.} If \( n \geq 2 \), then \( -f(\Phi) = \min_{\notin \emptyset \subseteq V} \frac{w(\partial S)}{|S| \cdot |V \setminus S|} \).

\textbf{Proof.}

\[ -f(\Phi) = -\max_{S \subseteq V} \Lambda(|S|) f(|S|, w(\partial S)) = \min_{S \subseteq V} \min_{\emptyset \not\subseteq V} \frac{w(\partial S)}{|S| \cdot |V \setminus S|} = \min \left\{ \infty, \min_{\notin \emptyset \subseteq V} \frac{w(\partial S)}{|S| \cdot |V \setminus S|} \right\} \]

\textbf{Densest-Cut’}

As Densest-Cut’ is the complementary problem of Sparest-Cut’, the instantiation of the framework is very similar: We set \( f(\nu, s) := \begin{cases} -\infty & \text{if } \nu \not\in \{0, n\} \\ \frac{s}{t(n-\nu)} & \text{otherwise} \end{cases} \) and \( \Lambda(x) := \text{true} \). Then, \( f(\Phi) \) is the value of the densest cut (and \( -\infty \) if there is at most one vertex).

\textbf{Lemma 12.} If \( n \geq 2 \), then \( f(\Phi) = \max_{\notin \emptyset \subseteq V} \frac{w(\partial S)}{|S| \cdot |V \setminus S|} \).

\textbf{Proof.}

\[ f(\Phi) = \max_{S \subseteq V} \min_{\emptyset \not\subseteq V} \frac{w(\partial S)}{|S| \cdot |V \setminus S|} = \max_{S \subseteq V} \min_{\emptyset \not\subseteq V} \frac{w(\partial S)}{|S| \cdot |V \setminus S|} = \max \left\{ \infty, \max_{\notin \emptyset \subseteq V} \frac{w(\partial S)}{|S| \cdot |V \setminus S|} \right\} \]

\section{Conclusion}

We showed that given a small tree decomposition of width \( t \) – many cut problems can be solved in time \( O(2^t n^2) \) using our framework. To our knowledge, the running times achieved by our framework are better than the previously known algorithms for the considered problems. Moreover, this running time is unlikely to be improved significantly (improvements by factors \( \text{poly } t \) and/or \( \text{poly log } n \) are not excluded) in general: An algorithm (solving Min-Bisection) that runs in time \( O(n^{2-\epsilon} f(t)) \) for some \( f \) and some \( \epsilon > 0 \) would imply an algorithm of running time \( O(n^2 \delta) \) for some \( \delta > 0 \) for \( (\text{min}, +)\)-convolution \cite{6}, which is considered unlikely \cite{5}. An algorithm (solving Max-Cut) cannot have a running time \( O((2-\epsilon)^t \text{ poly } n) \) for some \( \epsilon > 0 \) unless SETH fails \cite{15, 8, 9, 4}. However, there might be problems that can be solved using our framework that we have not considered yet. Moreover, it might be possible to generalize the framework (with possibly worse running time) to e.g. be able to also cover connectivity problems (see \cite{2}).


8 References

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Appendix

A Correctness of the Recurrences

In order to prove the correctness of our recurrences, we first need a few calculation rules for the interaction of \(\partial A\) and the intersection with specific Cartesian products. This will be done in the following section. Afterwards, there is one section per node type where the correctness of the recurrence for this node type is shown formally.

A.1 Calculation Rules

We start with the simplest and very obvious rule that states that the weight of edges of the disjunct union \(A \cup B\) of two disjoint edge sets \(A, B\) is exactly the sum of the weights of \(A\) plus those of \(B\):

Lemma 13. For two sets disjoint edge sets \(A, B\) we have

\[ w(A \cup B) = w(A) + w(B). \]

Proof. Follows directly from definition.

Now let us consider some vertex set \(A\) and the edges leaving \(A\), that is the edges going from \(A\) to \(\overline{A}\). It is not hard to see that the intersection with \(B \times C\) for \(B \supseteq A\) and \(C \supseteq \overline{A}\) leaves this set unchanged:

Lemma 14. For any \(A, B \supseteq A\), and \(C \supseteq \overline{A}\) we have

\[ \partial A = \partial A \cap (B \times C). \]

Proof.

\[ \partial A = \{ v_1v_2 \in E : v_1 \in A, v_2 \in \overline{A} \} = \{ v_1v_2 \in E : v_1 \in A, v_2 \in \overline{A} \} \cap (B \times C) \]

Now we show rules that are more complex. The following set of rules consider some edge set \(J\) and vertex sets \(K\) and \(L\). More precisely, the expression \(J \cap (K \times L)\) can be decomposed if there are certain relations between the sets involved.

Lemma 15. Let \(A\) and \(B\) be disjoint sets.

(i) \[ \partial (A \cup B) = \partial A \cap (A \times \overline{B}) \cup \partial B \cap (B \times \overline{A}) \]

Now let \(C\) be a set disjoint with both, \(A\) and \(B\).

(ii) \[ \partial A \cap (A \times (B \cup C)) = \partial A \cap (A \times B) \cup \partial A \cap (A \times C) \]

(iii) \[ \partial (A \cup B) \cap ((A \cup B) \times C) = \partial A \cap (A \times C) \cup \partial B \cap (B \times C) \]

Proof. Proof of Statement (i).

\[ \partial (A \cup B) = \{ v_1v_2 \in E : v_1 \in A \cup B, v_2 \in \overline{A \cup B} \} \]

\[ = \{ v_1v_2 \in E : v_1 \in A, v_2 \in \overline{A} \cup B \} \cup \{ v_1v_2 \in E : v_1 \in B, v_2 \in \overline{A \cup B} \} \]

\[ = \{ v_1v_2 \in E : v_1 \in A, v_2 \in \overline{A} \land v_2 \in \overline{B} \} \cup \{ v_1v_2 \in E : v_1 \in B, v_2 \in \overline{A \land v_2 \in B} \} \]

\[ = \{ v_1v_2 \in E : v_1 \in A, v_2 \in \overline{A} \} \cap (A \times \overline{B}) \cup \{ v_1v_2 \in E : v_1 \in B, v_2 \in \overline{B} \} \cap (B \times \overline{A}) \]

\[ = \partial A \cap (A \times \overline{B}) \cup \partial B \cap (B \times \overline{A}) \]
Proof of Statement (ii). Note note that $B \cup C$ and $A$ are disjoint, which implies $B, C \subseteq (B \cup C) \subseteq \overline{A}$.

\[
\partial A \cap (A \times (B \cup C)) = \{ v_1, v_2 \in E : v_1 \in A, v_2 \in \overline{A} \} \cap (A \times (B \cup C))
\]
\[
\subseteq \{ v_1, v_2 \in E : v_1 \in A, v_2 \in B \cup C \}
\]
\[
= \{ v_1, v_2 \in E : v_1 \in A, v_2 \in B \} \cup \{ v_1, v_2 \in E : v_1 \in A, v_2 \in C \}
\]
\[
\implies \partial A \cap (A \times B) \cup \partial A \cap (A \times C)
\]

Proof of Statement (iii). Note note that $A \cup B$ and $C$ are disjoint, which implies $(A \cup B) \subseteq \overline{C}$, which is equivalent to $A \cup \overline{B} \supseteq C$.

\[
\partial (A \cup B) \cap ((A \cup B) \times C) = \{ v_1, v_2 \in E : v_1 \in A \cup B, v_2 \in \overline{A \cup B} \} \cap ((A \cup B) \times C)
\]
\[
\subseteq \{ v_1, v_2 \in E : v_1 \in A \cup B, v_2 \in C \}
\]
\[
= \{ v_1, v_2 \in E : v_1 \in A, v_2 \in C \} \cup \{ v_1, v_2 \in E : v_1 \in B, v_2 \in C \}
\]
\[
= \partial A \cap (A \times C) \cup \partial B \cap (B \times C)
\]

We need those rules for the correctness proof of the join nodes as well as for showing another rule in the following paragraph.

For the special case that $B$ contains exactly one element in the first rule of the previous statement, we get the following rule by incorporating Lemma 14, too:

\[\textbf{Corollary 16.} \text{ For all } v \text{ and all } S \text{ where } v \not\in S \text{ it holds that }
\]
\[
\partial(S \cup \{v\}) = \partial S \cap \overrightarrow{v}^2 \cup \partial v \cap \overrightarrow{S}^2.
\]

Proof.

\[\text{Lemma 15i}
\]
\[
\partial(S \cup \{v\}) \subseteq \partial S \cap (S \times \{v\}) \cup \partial v \cap (\{v\} \times \overrightarrow{S})
\]
\[
= \partial S \cap \overrightarrow{v}^2 \cup \partial v \cap \overrightarrow{S}^2
\]
\[\text{Lemma 14, using } S \subseteq \overrightarrow{v}^1 \text{ and } \{v\} \subseteq \overrightarrow{S}
\]

This rule is very important as our recurrences are usually (except join nodes) built by inductively adding vertices to the selection set $S$.

A.2 Leaf Node

\[\textbf{Lemma 17 (Correctness of Equation 8).} \text{ Let } i \text{ be a leaf node and } v \in X_i. \text{ Then for all } S \subseteq X_i, \text{ we have }
\]
\[
\Gamma_i(0, \{v\}) = \left(1, \sum_{v' \in X_i, vv' \in E} w(vv')\right).
\]

Proof. We can simply plug in the definition of $\Gamma$ and make use of $Y_i = X_i$ as follows:
We can use this and Equation 16 to deduce

\[
\Gamma_i(0, \{v\}) = \bigcup_{\hat{S} \in \binom{X_i}{S}} \left( |\{v\}|, w_i(\partial(\{v\} \cup \hat{S})) \right) = \left( 1, w_i(\partial v) \right)
\]

\[
= \left( 1, w(\partial v \cap Y_i^2) \right) = \left( 1, w(\partial v \cap X_i^2) \right) = \sum_{v' \in X_i \setminus S} w(vv') \uparrow_{Y_i = X_i} \subseteq S
\]

Lemma 18 (Correctness of Equation 9). Let \( i \) be a leaf node and \( v \in X_i \). Then for all \( S \subseteq X_i \) and all \( \emptyset \neq S' \subseteq X_i \setminus \{v\} \) we have

\[
\Gamma_i(0, S' \cup \{v\}) = \Gamma_i(0, S') \oplus \left( 1, \sum_{v' \in X_i \setminus S'} w(vv') - \sum_{v' \in S'} w(v'v) \right)
\]  (9)

Proof. We start by plugging in the definition of \( \Gamma_i \):

\[
\Gamma_i(0, S' \cup \{v\}) = \bigcup_{\hat{S} \in \binom{X_i}{S'}} \left( |S' \cup \{v\}|, w_i(\partial(S' \cup \{v\})) \right) = \left( 1 + |S'|, w_i(\partial(S' \cup \{v\})) \right)
\]  (16)

The question is now: how can we show that the size of the cut changes (in comparison to the size of the cut when only \( S' \) is selected instead of \( S' \cup \{v\} \)) as claimed in the statement, that is

\[
w_i(\partial(S' \cup \{v\})) = \sum_{v' \in X_i \setminus S'} w(vv') - \sum_{v' \in S'} w(v'v)?
\]

To answer this, we explicitly consider the change of the cut, that is

Corollary 16

\[
w_i(\partial(S' \cup \{v\})) = w_i(\partial(S' \cup \{v\})) - w_i(\partial S') - \sum_{v' \in X_i \setminus S'} w(vv') + \sum_{v' \in S'} w(v'v).
\]

Lemma 13

\[
w(\partial S' \cap (X_i \setminus \{v\})) = w(\partial S' \cap (X_i \setminus \{v\})) - w_i(\partial S')
\]

Corollary 16

\[
w(\partial S' \cap (X_i \setminus \{v\})) = w(\partial S' \cap (X_i \setminus \{v\})) - w_i(\partial S')
\]

Lemma 13

\[
w(\partial S' \cap (X_i \setminus \{v\})) = w(\partial S' \cap (X_i \setminus \{v\})) - w_i(\partial S' \cap (X_i \setminus \{v\})) - \sum_{v' \in X_i \setminus S'} w(vv') + \sum_{v' \in S'} w(v'v).
\]

We can use this and Equation 16 to deduce

Equation 16

\[
\Gamma_i(0, S' \cup \{v\}) \equiv \left( 1 + |S'|, w_i(\partial(S' \cup \{v\})) \right)
\]
Lemma 20

\[
\Gamma_i(\ell, S) = \begin{cases}
\Gamma_j(\ell - 1, S \cup \{v\}) \cup \Gamma_j(\ell, S) & \ell \geq 1 \\
\Gamma_j(\ell, S) & \ell = 0
\end{cases}.
\]

Proof. Fix \( S \subseteq X_i \) and \( 0 \leq \ell \leq |F_i| \).

\[
\Gamma_i(\ell, S) = \bigcup_{S \in \ell'} \left( \ell + |S|, w_i(\partial(S \cup \tilde{S})) \right)
\]

Now we make a case distinction whether \( v \in \tilde{S} \); if \( v \in \tilde{S} \) we may select \( \ell - 1 \) further vertices from \( F_j \setminus \{v\} \), otherwise we may select \( \ell \) vertices from \( F_j \setminus \{v\} \). This allows us to split the supremum into the supremum of two suprema:

\[
= \left( \bigcup_{\mathcal{S} \in \ell''} \left( \ell + |S|, w_i(\partial(S \cup \tilde{S} \cup \{v\})) \right) \right) \cup \left( \bigcup_{\mathcal{S} \in \ell'} \left( \ell + |S|, w_i(\partial(S \cup \tilde{S})) \right) \right)
\]

Note that the left supremum is applied on an empty set if \( \ell = 0 \) (and thus vanishes in that case, as \( \bigcup \emptyset = \bot \)); incorporating this and using the definition of \( \Gamma_j \) we eventually get

\[
\Gamma_i(\ell, S) = \begin{cases}
\Gamma_j(\ell - 1, S \cup \{v\}) \cup \Gamma_j(\ell, S) & \ell \geq 1 \\
\Gamma_j(\ell, S) & \ell = 0
\end{cases}.
\]

A.3 Forget Node

Lemma 19 (Correctness of Equation 10). Let \( i \) be a forget node with child \( j \). Let \( v \in V \) such that \( X_i \cup \{v\} = X_j \). Then for all \( S \subseteq X_i \) and all \( 0 \leq \ell \leq |F_i| \) it holds that

\[
\Gamma_i(\ell, S) = \begin{cases}
\Gamma_j(\ell - 1, S \cup \{v\}) \cup \Gamma_j(\ell, S) & \ell \geq 1 \\
\Gamma_j(\ell, S) & \ell = 0
\end{cases}.
\]

A.4 Introduce Node

Lemma 20 (Correctness of Equation 11). Let \( i \) be an introduce node with child \( j \). Let \( v \in V \) such that \( X_i = X_j \cup \{v\} \). Then for all \( S \subseteq X_i \) where \( v \in S \) and all \( 0 \leq \ell \leq |F_i| \) it holds that

\[
\Gamma_i(\ell, S) = \begin{cases}
\Gamma_j(\ell - 1, S \cup \{v\}) \cup \Gamma_j(\ell, S \setminus \{v\}) & \ell \geq 1 \\
\Gamma_j(\ell, S \setminus \{v\}) & \ell = 0
\end{cases}.
\]
We want to focus on the $w_i(\ldots)$ expression in the equation above. Therefore, let $S' := S \setminus \{v\}$. Consider the term $w_i(\vartheta(S \cup \tilde{S}))$. For the sake of shortness, let $\tilde{S} := S' \cup \tilde{S}$. We then have

Corollary 16

$w_i(\vartheta(S \cup \{v\})) \leq w_i(\vartheta(S \cup \{v\}^2 \cup \vartheta v \cap S^2)$

Lemma 13

$= w_i(\vartheta(S \cup \{v\}^2) + w_i(\vartheta v \cap S^2)$

$\subseteq w(\vartheta(S \cup \{v\}^2 \cap S^2) + w(\vartheta v \cap S^2 \cap S^2)$

$\subseteq w(\vartheta(S \cap (Y_i \setminus \{v\})) + w(\vartheta v \cap (Y_i \setminus \tilde{S}^2))$

Now note that $v$ has just been introduced; due to Definition 1 (ii) and Definition 1 (iii) there cannot be an edge from $v$ to any vertex in $F_i$, thus, using $Y_i = X_i \cup F_i$, we have

$= w(\vartheta S \cap (X_i \setminus \{v\})) + w(\vartheta v \cap (X_i \setminus \tilde{S}^2))$

$\subseteq X_i = X_i \cup \{v\}$

$\subseteq X_i \subseteq X_i \cup \{v\}$

$\subseteq w_i(\vartheta S \cap X_i^2) + w(\vartheta v \cap (X_i \setminus \tilde{S}^2))$

$\subseteq w_i(\vartheta S \cap X_i^2) + w(\vartheta v \cap \tilde{S}^2)$

$\subseteq w_i(\vartheta S \cap X_i^2) + w(\vartheta v \cap (S_i \setminus \tilde{S}^2))$

$\subseteq w_i(\vartheta S \cap \tilde{S}^2)$

$\subseteq w_i(\vartheta S \cap \tilde{S}^2)$

$\subseteq w_i(\vartheta S \cap \tilde{S}^2)$

Hence, we can rewrite $\Gamma_i(\ell, S)$ as follows:

$\Gamma_i(\ell, S) = \bigcup_{S \in (F_i \cap X_i \cup \{v\})} \left( \ell + |S|, w_i(\vartheta(S \cup \tilde{S})) \right)$

$\subseteq \bigcup_{S \in (F_i \cap X_i \cup \{v\})} \left( 1 + \ell + |S|, w_i(\vartheta(S' \cup \tilde{S})) \right)$

Statement above

$\subseteq \bigcup_{S \in (F_i \cap X_i \cup \{v\})} \left( 1 + \ell + |S|, w_j(\vartheta(S' \cup \tilde{S})) \right) + w(\vartheta v \cap (\{v\} \times (X_i \setminus S'))$

$\subseteq \bigcup_{S \in (F_i \cap X_i \cup \{v\})} \left( 1, w(\vartheta v \cap (\{v\} \times (X_i \setminus S'))) \right) \odot \left( \ell + |S'|, w_j(\vartheta(S' \cup \tilde{S})) \right)$
Lemma 6
\[
\begin{align*}
&= \left(1, \, w(\partial v \cap \{v\} \times (X_i \setminus S'))\right) \oplus \bigcup_{S' \in (\ell, \ell + |S'|]} (\ell + |S'|, \, w_j(S' \cup \tilde{S})) \\
&= \left(1, \, w(\partial v \cap \{v\} \times (X_i \setminus S'))\right) \oplus \Gamma_j(\ell, S') \\
\end{align*}
\]

Let \( \tilde{S} = S \cup \tilde{S} \), thus we will have a closer look at the term \( \tilde{S} \cap Y_2 \).

Lemma 14
\[
\begin{align*}
\partial S \cap Y_2 \cap Y_i &= \partial S \cap (\tilde{S} \times (Y_i \setminus \tilde{S})) \\
&= \partial \tilde{S} \cap ((Y_j \cup \{v\}) \setminus \tilde{S}) \\
&= \partial \tilde{S} \cap ((Y_j \setminus \tilde{S}) \cup \{v\}) \\
&= \partial \tilde{S} \cap (\tilde{S} \times (Y_j \setminus \tilde{S})) \cup \partial \tilde{S} \cap (\tilde{S} \times \{v\}) \\
&= \partial \tilde{S} \cap (\tilde{S} \times (Y_j \setminus \tilde{S})) \cup \partial \tilde{S} \cap (X_i \setminus \{v\}) \\
\end{align*}
\]

Now focus on the second term of the union above:

Lemma 15i
\[
\begin{align*}
\partial S \cap (X_i \times \{v\}) &= \partial(S \cup \tilde{S}) \cap (X_i \times \{v\}) \\
&= \partial S \cap (S \times \tilde{S}) \cup \partial \tilde{S} \cap (\tilde{S} \times S) \cap (X_i \times \{v\}) \\
&= \partial S \cap (S \times \tilde{S}) \cup \partial \tilde{S} \cap (\tilde{S} \times S) \cap (X_i \times \{v\}) \\
&= \partial S \cap (S \times \tilde{S}) \cup \partial \tilde{S} \cap (\tilde{S} \times \{v\}) \cup \partial \tilde{S} \cap (S \cap \{v\}) \\
&= \partial S \cap (X_i \times \{v\}) \\
\end{align*}
\]
Combining both just shown equalities we get that
\[ \partial \tilde{S} \cap Y_i^2 = \partial \tilde{S} \cap Y_j^2 \cup \partial S \cap (X_i \times \{v\}). \]

We can thus deduce, using Lemma 13, that
\[ w_1(\partial \tilde{S}) = w_j(\partial \tilde{S}) + w(\partial S \cap (X_i \times \{v\})). \]

This allows us to rewrite \( \Gamma_i(\ell, S) \) as follows:
\[
\Gamma_i(\ell, S) = \bigcup_{\tilde{S} \in (\ell\ell)} \left( \ell + |S|, w_i(\partial(S \cup \tilde{S})) \right)
\]

\[ \text{Statement above:} \]
\[
\downarrow = \bigcup_{\tilde{S} \in (\ell\ell)} \left( \ell + |S|, w_j(\partial(S \cup \tilde{S})) + w(\partial S \cap (X_i \times \{v\})) \right)
\]
\[
= \bigcup_{\tilde{S} \in (\ell\ell)} \left( \left( \ell + |S|, w_j(\partial(S \cup \tilde{S})) \right) \oplus \left( 0, w(\partial S \cap (X_i \times \{v\})) \right) \right)
\]

\[ \text{Lemma 6} \]
\[ F_i \leq F_j \]
\[
\downarrow = \left( 0, w(\partial S \cap (X_i \times \{v\})) \right) \oplus \bigcup_{\tilde{S} \in (\ell\ell)} \left( \ell + |S|, w_j(\partial(S \cup \tilde{S})) \right)
\]
\[
= \left( 0, w(\partial S \cap (X_i \times \{v\})) \right) \oplus \Gamma_j(\ell, S) \]

\[ \text{Lemma 22 (Correctness of Equation 13). Let } i \text{ be an introduce node with child } j. \text{ Let } v \in V \text{ such that } X_i = X_j \cup \{v\}. \text{ Then for all } S \subseteq X_i \text{ and all } 0 \leq \ell \leq |F_i| \text{ it holds that} \]
\[ \pi_2(\Gamma_i(\ell, S) \oplus \Gamma_j(\ell, S \cap X_j)) = \begin{cases} 
  w_i(\partial v \cap \{v\} \times (X_i \setminus S)) & v \in S \\
  w_i(\partial S \cap (X_i \times \{v\})) & v \notin S.
\end{cases} \quad (13)
\]

**Proof.** Fix \( S \subseteq X_i \) and \( 0 \leq \ell \leq |F_i| \). If \( v \in S \), then by Lemma 20
\[ \Gamma_i(\ell, S) = \left( 1, w_i(\partial v \cap \{v\} \times (X_i \setminus S)) \right) \oplus \Gamma_j(\ell, S \setminus \{v\}) \]
and hence
\[
\pi_2(\Gamma_i(\ell, S) \oplus \Gamma_j(\ell, S \cap X_j)) \]
\[ = \pi_2 \left( \left( 1, w_i(\partial v \cap \{v\} \times (X_i \setminus S)) \right) \oplus \Gamma_j(\ell, S \setminus \{v\}) \oplus \Gamma_j(\ell, S \cap X_j) \right) \]
\[ = \pi_2 \left( \left( 1, w_i(\partial v \cap \{v\} \times (X_i \setminus S)) \right) \oplus \pi_2(\Gamma_j(\ell, S \setminus \{v\})) \oplus \pi_2(\Gamma_j(\ell, S \cap X_j)) \right) \]
\[ = \pi_2 \left( 1, w_i(\partial v \cap \{v\} \times (X_i \setminus S)) \right) \oplus \pi_2(\Gamma_j(\ell, S \setminus \{v\})) \]
\[ = w_i(\partial v \cap \{v\} \times (X_i \setminus S)). \]
When considering the special case of Max-Bisection, this simplifies to

\[ \pi_2(\Gamma_i(\ell, S) \cup \Gamma_j(\ell, S \cap X_j)) \]

\[ \Rightarrow \pi_2\left(\left(0, w_i(\partial S \cap (X_i \times \{v\}))\right) \cup \Gamma_j(\ell, S) \cup \pi_2(\Gamma_j(\ell, S)) \right) \]

or only in building the maximum. Recall that for a join node we claimed that

\[ \Gamma_j(\ell, S) = \left( \bigcup_{0 \leq \ell_1 \leq |F_i|, 0 \leq \ell_2 \leq |F_k|, \ell_1 + \ell_2 = \ell} (\Gamma_j(\ell_1, S) \cup \Gamma_k(\ell_2, S)) \right) \cup \left( |S|, w_i(\partial S) \right). \] (14)

When considering the special case of Max-Bisection, this simplifies to

\[ \Gamma_i(\ell, S) = \max_{0 \leq \ell_1 \leq |F_i|, 0 \leq \ell_2 \leq |F_k|, \ell_1 + \ell_2 = \ell} \left( \Gamma_j(\ell_1, S) + \Gamma_k(\ell_2, S) \right) - w_i(\partial S). \]

The intuition behind the subtraction is somewhat clear: If \( \Gamma_j(\ell_1, S) \) is the largest possible cut in \( G[Y_j] \) using \( S \) and further \( \ell_1 \) vertices, and \( \Gamma_k(\ell_2, S) \) is the largest possible cut in \( G[Y_k] \) using \( S \) and further \( \ell_2 \) vertices, their sum includes some summands twice: The term \( w_i(S) \). This is because all other edges in the largest cut have at least on endpoint that is only in \( Y_j \) or only in \( Y_k \) (outside \( X_i \)). The following proposition now shows this formally; we need it for the proof of the correctness of the equation for join nodes in the general case as well as for the specific case for Max-Bisection.

\[ \text{Proposition 23.} \quad \text{Let} \quad \ell \quad \text{be a join node with left child} \quad j \quad \text{and right child} \quad j. \quad \text{Then for all} \quad X_i \subseteq F_j, \quad \tilde{S}_1 \subseteq F_j, \quad \text{and} \quad \tilde{S}_2 \subseteq F_k \quad \text{we have} \]

\[ w_i(\partial(S \cup \tilde{S}_1 \cup \tilde{S}_2)) = w_j(\partial(S \cup \tilde{S}_1)) + w_k(\partial(S \cup \tilde{S}_2)) - w_i(\partial S \cap X_i^2). \] \[ \text{Proof.} \quad \text{Fix} \quad S \subseteq X_i, \quad \tilde{S}_1 \subseteq F_j, \quad \text{and} \quad \tilde{S}_2 \subseteq F_k. \quad \text{We claim that it suffices to show that} \]

(i) Show that \( \partial(S \cup \tilde{S}_1 \cup \tilde{S}_2) \cap Y_i^2 = \partial(S \cup \tilde{S}_1) \cap Y_i^2 \cup \partial(S \cup \tilde{S}_2) \cap Y_i^2 \); and

(ii) \( \partial(S \cup \tilde{S}_1) \cap Y_j^2 \cap \partial(S \cup \tilde{S}_2) \cap Y_k^2 = \partial S \cap X_i^2 \).

To see this, assume that both statements hold. We then have

\[ w_i(\partial S \cup \tilde{S}_1 \cup \tilde{S}_2) \]

\[ = \sum_{e \in \partial(S \cup \tilde{S}_1 \cup \tilde{S}_2) \cap Y_i^2} w(e). \]
Lemma 15.iii

In order to prove (i) and (ii), we approach as follows: We simplify \(\partial(S \cup \hat{S}_1) \cap Y^2\) and \(\partial(S \cup \hat{S}_2) \cap Y^2\) each to a disjunct union of simple terms. We compare those “summands” to prove our goals. We start with the left hand side of the equation in (i), that is, the term \(\partial(S \cup \hat{S}_1 \cup \hat{S}_2) \cap Y^2\):

\[
\partial(S \cup \hat{S}_1 \cup \hat{S}_2) \cap Y^2
\]

Lemma 14

1. \(\partial(S \cup \hat{S}_1 \cup \hat{S}_2) \cap ((S \cup \hat{S}_1 \cup \hat{S}_2) \cap (Y_1 \cap (S \cup \hat{S}_1 \cup \hat{S}_2)))\)

Def \(Y_i\):

1. \(\partial(S \cup \hat{S}_1 \cup \hat{S}_2) \cap ((S \cup \hat{S}_1 \cup \hat{S}_2) \cap ((X_i \cup F_i) \cap (S \cup \hat{S}_1 \cup \hat{S}_2)))\)

\(i\) is join node:

1. \(\partial(S \cup \hat{S}_1 \cup \hat{S}_2) \cap ((S \cup \hat{S}_1 \cup \hat{S}_2) \cap ((X_i \cup F_j \cup F_k) \cap (S \cup \hat{S}_1 \cup \hat{S}_2)))\)

Lemma 15.iii, Lemma 15ii

\[
\begin{align*}
\partial S \cap (S \times (X_i \setminus S)) \cup \partial S \cap (S \times (F_j \setminus \hat{S}_1)) \cup \partial S \cap (S \times (F_k \setminus \hat{S}_2)) \\
\cup \partial \hat{S}_1 \cap (S \times (X_i \setminus S)) \cup \partial \hat{S}_1 \cap (S \times (F_j \setminus \hat{S}_1)) \cup \partial \hat{S}_1 \cap (S \times (F_k \setminus \hat{S}_2)) \\
\cup \partial \hat{S}_2 \cap (S \times (X_i \setminus S)) \cup \partial \hat{S}_2 \cap (S \times (F_j \setminus \hat{S}_1)) \cup \partial \hat{S}_2 \cap (S \times (F_k \setminus \hat{S}_2))
\end{align*}
\]

Let us now consider the first expression on the right side of the equation of (i):

\[
\partial(S \cup \hat{S}_1) \cap Y^2
\]

Lemma 14

1. \(\partial(S \cup \hat{S}_1) \cap ((S \cup \hat{S}_1) \times (Y_1 \setminus (S \cup \hat{S}_1)))\)

\(X_i = X_j\):

1. \(\partial(S \cup \hat{S}_1) \cap ((S \cup \hat{S}_1) \times ((F_j \cup X_j) \setminus (S \cup \hat{S}_1)))\)

\(S \subseteq X_i, \hat{S}_i \subseteq F_j\):

1. \(\partial(S \cup \hat{S}_1) \cap ((S \cup \hat{S}_1) \times ((X_i \setminus S) \cup (F_j \setminus \hat{S}_1)))\)
Lemma 15iii, Lemma 15i

\[ \partial S \cap (S \times (X_i \setminus S)) \cup \partial S \cap (S \times (F_j \setminus \hat{S}_1)) \]
\[ \cup \partial \hat{S}_1 \cap (\hat{S}_1 \times (X_i \setminus S)) \cup \partial \hat{S}_1 \cap (\hat{S}_1 \times (F_j \setminus \hat{S}_1)) \]

(H2)

Analogously, we can also show

\[ \partial (S \cup \hat{S}_1) \cap Y_k^2 \]
\[ = \partial S \cap (S \times (X_i \setminus S)) \cup \partial S \cap (S \times (F_k \setminus \hat{S}_2)) \]
\[ \cup \partial \hat{S}_2 \cap (\hat{S}_2 \times (X_i \setminus S)) \cup \partial \hat{S}_2 \cap (\hat{S}_2 \times (F_k \setminus \hat{S}_2)) \]

(H3)

Now observe that each “summand” (that is not the empty set) in (H1) appears in (H2) and (H3), and vice versa. This proves (i).

For (ii), let us compare the “summands” of (H2) and (H3). First observe, that each pair of those has to be identical or disjunct. This is due to the fact that all “summands” of (H2) and (H3) also appear in (H1), as we observed above, and (H1) is a disjunct union. It is now not hard to see which pair of summands are identical and which are disjunct: Any “summand” starting with \( \partial \hat{S}_1 \) (recall that \( \hat{S}_1 \subseteq F_j \)) in (H2) cannot occur in (H3) due to coherence (unless the “summand” is \( \emptyset \)), and analogously, no “summand” starting with \( \partial \hat{S}_2 \) (recall that \( \hat{S}_2 \subseteq F_k \)) in (H3) can occur in (H2). Moreover, the same applies to “summands” in (H2) that are intersected with a subset of \( V \times F_j \); and analogously, to “summands” in (H3) that are intersected with a subset of \( V \times F_k \). The only “summand” left is \( \partial S \cap (S \times (X_i \setminus S)) \), and this summand occurs (in this very syntactical form) in both, (H2) and (H3). This shows (ii) and hence eventually the claim.

\[ \textbf{Lemma 24 (Correctness of Equation 14).} \text{ Let } i \text{ be a join node with left child } j \text{ and right child } j. \text{ Then for all } 0 \leq \ell \leq |F_i| \text{ and all } S \subseteq X_i \text{ we have} \]
\[ \Gamma_i(\ell, S) = \left( \bigcup_{0 \leq \ell_1 \leq |F_j| \atop 0 \leq \ell_2 \leq |F_k| \atop \ell_2 + \ell_1 = \ell} (\Gamma_j(\ell_1, S) \oplus \Gamma_k(\ell_2, S)) \right) \odot (|S|, w_i(\partial S)). \] (14)

\[ \text{Proof.} \text{ Fix } S \subseteq X_i \text{ and } 0 \leq \ell \leq |F_i|. \text{ By the definition of } \Gamma \text{ we have:} \]
\[ \Gamma_i(\ell, S) \]
\[ = \bigcup_{S \in \ell_1(i)} \left( \ell + |S|, w_i(\partial (S \cup \hat{S})) \right) \]
\[ = \bigcup_{0 \leq \ell_1 \leq \ell \atop \ell_2 = \ell - \ell_1 \atop S_1 \in \ell_1(i)} \left( \ell + |S|, w_i(\partial (S \cup \hat{S}_1 \cup \hat{S}_2)) \right) \]

\[ \text{Proposition 23} \]
\[ = \bigcup_{0 \leq \ell_1 \leq \ell \atop \ell_2 = \ell - \ell_1 \atop S_1 \in \ell_1(i), S_2 \in \ell_2(i)} \left( \ell + |S|, w_j(\partial (S \cup \hat{S}_1)) + w_k(\partial (S \cup \hat{S}_2)) - w_i(\partial S) \right) \]
\[ = \bigcup_{0 \leq \ell_1 \leq \ell \atop \ell_2 = \ell - \ell_1 \atop S_1 \in \ell_1(i), S_2 \in \ell_2(i)} \left( \ell + |S|, w_j(\partial (S \cup \hat{S}_1)) + w_k(\partial (S \cup \hat{S}_2)) - w_i(\partial S) \right) \]
Lemma 6
\[
\bigoplus_{0 \leq t_1 \leq \ell} \bigoplus_{t_2 = \ell - t_1} \bigoplus_{\tilde{S} \in (t_2)} \left( \ell + 2|S|, w_j(\partial(S \cup \tilde{S})) + w_k(\partial(S \cup \tilde{S})) \right)
\]
\[
\bigoplus \left( |S|, w_i(\partial S) \right)
\]

Lemma 6
\[
\bigoplus_{0 \leq t_1 \leq \ell} \bigoplus_{t_2 = \ell - t_1} \bigoplus_{\tilde{S} \in (t_2)} \left( \ell + |S|, w_j(\partial(S \cup \tilde{S})) \right) \oplus \left( \ell + |S|, w_k(\partial(S \cup \tilde{S})) \right)
\]
\[
\bigoplus \left( |S|, w_i(\partial S) \right)
\]
\[
= \bigoplus_{\tilde{S} \in (t_2)} \left( \ell + |S|, w_j(\partial(S \cup \tilde{S})) \right) \oplus \bigoplus_{\tilde{S} \in (t_2)} \left( \ell + |S|, w_k(\partial(S \cup \tilde{S})) \right)
\]
\[
\bigoplus \left( |S|, w_i(\partial S) \right)
\]
\[
= \bigoplus_{0 \leq t_1 \leq \ell} (\Gamma_j(\ell_1, S) \oplus \Gamma_k(\ell_2, S)) \bigoplus \left( |S|, w_i(\partial S) \right).
\]

B Running Time Of Our Framework

C Hardness

We need the following Proposition in order to be able to show that the hardness result of Max-Cut by Lokshantov, Marx, and Saurabh [15] can be extended to the case where a tree decomposition is given as part of the input. The idea is to compute such a decomposition for the result of their reduction from SAT to Max-Cut in polynomial time.

Proposition 25. Let \( G = (V, E) \) and \( S \subseteq V \). If \( G - S \) is a forest, then (given \( S \)) one can compute a small tree decomposition of \( G \) that has width \( |S| + 1 \) in time \( O(\text{poly}(|E| + |V|)) \).

Proof. Fix \( G = (V, E) \) and \( S \). Note that it is trivially possible to compute a small tree decomposition of a tree in time linear in its nodes and edges. Compute the tree decompositions for all connected components in \( G' := G - S \) (which are all trees by assumption). Rename the node sets (and all the occurrences accordingly) of the decompositions such that the union of all node sets is of the form \( [k] \setminus \{1\} \) for some \( k \in \mathbb{N} \) and the node sets are disjunct. Add a new node 1 to the forest of tree decompositions and set the associated bag to be \( X_1 := \emptyset \). Connect the new node 1 to an arbitrary node of each tree decomposition. It is easy to check that we just constructed a small tree decomposition for \( G' \) of width 1. Now modify this decomposition to a small tree decomposition of \( G \) as follows: Add the set \( S \) to all bags. Again, it is easy to check that we obtain a small tree decomposition of \( G \). The same holds for the running time. The width of the constructed decomposition is bounded from above by the term \( 1 + |S| \) as every bag of the decomposition for \( G' \) had at most 2 vertices before we added the vertices within \( S \).
Proof. Consider the graph created in the reduction by Lokshtanov, Marx, and Saurabh [15] and observe that following statements:

1. The number of vertices after the reduction is $O(|\phi| + n) = O(|\phi|)$;
2. For every clause there is exactly one cycle created;
3. Any pair of such cycles does share exactly one vertex: $x_0$;
4. Apart from those partially overlapping cycles, there are only $n$ additional vertices;
5. Every vertex on those cycles that is not $x_0$ has degree at most 3 as it is connected to at most one $\hat{v}_i$;

From (4) and (5) we can now deduce that the number of edges after the reduction is $O(|\phi|)$. Thus the encoding of the graph has asymptotically the same size as the encoding of $\phi$. The correctness of the reduction was shown in [15].

Now observe that (3) and (4) imply that removing all variable vertices $\hat{v}_i$ and $x_0$ does result in a set of paths. This is exactly the statement we need in order to be able to apply Proposition 25 which completes the proof. ▷