Abstract

Design of adaptive algorithms for simultaneous regulation and estimation of MIMO linear dynamical systems is a canonical reinforcement learning problem. Efficient policies whose regret (i.e. increase in the cost due to uncertainty) scales at a square-root rate of time have been studied extensively in the recent literature. Nevertheless, existing strategies are computationally intractable and require a priori knowledge of key system parameters. The only exception is a randomized Greedy regulator, for which asymptotic regret bounds have been recently established. However, randomized Greedy leads to probable fluctuations in the trajectory of the system, which renders its finite time regret suboptimal.

This work addresses the above issues by designing policies that utilize input signals perturbations. We show that perturbed Greedy guarantees non-asymptotic regret bounds of (nearly) square-root magnitude w.r.t. time. More generally, we establish high probability bounds on both the regret and the learning accuracy under arbitrary input perturbations. The settings where Greedy attains the information theoretic lower bound of logarithmic regret are also discussed. To obtain the results, state-of-the-art tools from martingale theory together with the recently introduced method of policy decomposition are leveraged. Beside adaptive regulators, analysis of input perturbations captures key applications including remote sensing and distributed control.

Key words: Finite-time Optimality; Greedy Policies; Adaptive LQRs; Reinforcement Learning; System Identification; Decision-making under Uncertainty; Linear-Quadratic; Exploration-Exploitation; Sequential Decision-Making.

1 Introduction

Multiple Input - Multiple Output (MIMO) systems with linear dynamics represent canonical models in system engineering [1]. Their linear structure renders such systems amenable to rigorous mathematical analysis of their performance, as well as computationally feasible implementation of control policies for their regulation [2,3]. Further, they provide insights on how to deal with nonlinear dynamical models [4,5], and accurately represent the behavior of nonlinear systems around the operating equilibrium [6,7]. Also from the system identification viewpoint, linear models are considered a benchmark for prediction [8,9,10].

The standard formulation is that the system is characterized by autoregressive dynamics (in discrete time), while its operating cost is described by a quadratic function [11]. An extensive and mature literature exists on optimal Linear-Quadratic (LQ) policies when the true dynamical model is known [12]. For unknown dynamics, an adaptive policy is needed to simultaneously learn (identify) the system parameters and plan for regulating the system. This gives rise to designing self-tuning Linear-Quadratic-Regulators (LQR) in order to balance accurate learning (exploration) with efficient regulation (exploitation) [13,14].

There are two major threads in the literature on adaptive LQRs. The first and simpler one, aims to minimize the long run energy (of linear transformations) of output signals only [15,16,17,18,19,20,21], while in the second and more realistic one, the operator aims to minimize the energy (or power) of both input and output signals (or their linear transformations) using an adaptive regulator. A common approach for the latter problem uses an optimistic approximation of the system parameters, assuming that they correspond to the true ones, and subsequently designing control policies accordingly [22]. However, the proposed asymptotic [23,24] and non-asymptotic [25,26,27] algorithms for policy design are not computationally tractable. That is because non-convex matrix optimization problems, requiring high precision accuracy of the optimal solution, need to be solved repeatedly. In addition, the aforementioned adaptive algorithms need to have access to information regarding spectral properties of the closed-loop matrix and the statistical behavior of the noise process.

On the other hand, a Greedy (also referred to as Certainty Equivalence) policy applies the optimal feedback gain of the current estimates of the dynamics parameters. Then, collecting new input-output observations, Greedy learns the unknown parameters through a least squares procedure, and the above learning-planning steps are used in an alternate manner. A recent result
establishes that Greedy self-tuning regulator provides a regret of asymptotically square-root magnitude (with respect to time), if one randomizes the learning step as follows: simply add a statistically independent random matrix to the obtained least squares estimates (LSE) of the unknown dynamics matrix [28]. Randomized Greedy does not require the possibly unavailable a priori knowledge previously mentioned.

However, the non-asymptotic performance of the above randomized Greedy algorithm is not very satisfactory, due to the presence of frequent and large fluctuations in the trajectory of the system. This renders the finite time regret of the randomized Greedy regulator suboptimal, as discussed next. Technically, the probability of “the regret exceeding a factor of the optimal value” decays polynomially (so slowly) with the magnitude of that factor. Hence, in order to ensure that the regret is to remain within a margin of the optimal rate with high probability, the size of the posited margin needs to be large enough (i.e. the magnitude of the regret exceeds the square root of time\(^1\)). Note that the above fluctuations do not compromise the asymptotic optimality of randomized Greedy, since the latter is being compared to the infinitely growing scale of the time.

In this work, we establish that Greedy policy subject to a suitable perturbation of the system’s input achieves finite time optimality. That is, a square root of time upper bound (modulo a logarithmic factor) is established for the regret at all time steps. Further, the non-asymptotic rates at which the unknown true dynamical model is being learned are provided. The analysis leverages martingale concentrations [29,30], together with extending the policy decomposition method [28] to general regulators with perturbed input. The analysis of input perturbation for such systems is also of independent interest both from a theory and applications viewpoint; see for example, distributed control problems where the communication links are unreliable [31,32].

The remainder of the paper is organized as follows. In Section 2, we provide a precise formulation of the problem under consideration. We then address the non-asymptotic regret for general regulation policies with perturbed inputs in Subsection 3.1. Subsequently, the effect of the perturbation signal on the high probability estimation error of the true model is given in Subsection 3.2. In Section 4, we study the growth rate of the regret, as well as the accuracy of learning procedure for the Greedy algorithm with input perturbation. We also discuss the case of restricted uncertainty in Subsection 4.1, where side information such as the support or the rank of the dynamical parameters is available to the system’s operator.

Remark 1 Unless otherwise explicitly stated, all stochastic statements in this work hold almost surely.

1.1 Notation

The following notation will be used throughout this paper. For a matrix \( A \in \mathbb{C}^{p \times q} \), \( A' \) denotes its transpose. When \( p = q \), the smallest (respectively largest) eigenvalue of \( A \) (in magnitude) is denoted by \( \lambda_{\text{min}}(A) \) (respectively \( \lambda_{\text{max}}(A) \)). For \( v \in \mathbb{C}^q \), define the norm \( ||v|| = \left( \sum_{i=1}^{q} |v_i|^2 \right)^{1/2} \). We also use the following notation for the operator norm of matrices. For \( A \in \mathbb{C}^{p \times q} \) let, \( \|A\| = \sup_{\|v\|=1} \|Av\| \). The symbol \( \lor (\land) \) is being used to denote the maximum (minimum) of two or more quantities.

2 Problem Formulation and Preliminaries

The input-output evolution of the system is governed by the following stochastic dynamical model. Starting from an arbitrary initial \( x(0) \in \mathbb{R}^p \), at time \( t = 0, 1, \ldots \), the transition to the next output is determined according to

\[
x(t+1) = Ax(t) + Bu(t) + w(t+1),
\]

namely, by the current output value \( x(t) \in \mathbb{R}^p \), the input signal \( u(t) \in \mathbb{R}^r \), and the noise component \( w(t+1) \in \mathbb{R}^p \), the latter being a mean-zero random vector with \( \mathbb{E}[w(t+1)] = 0 \).

The transition matrix \( A_0 \in \mathbb{R}^{p \times p} \) models the effect of the current output signal, while \( B_0 \in \mathbb{R}^{p \times r} \) is the input matrix governing the influence of the control signal. The noise vectors \( \{w(t)\}_{t=1}^\infty \) are assumed to be independent, but are not required to be identically distributed (strict-sense stationary), and can even be heteroscedastic (wide-sense non-stationary). Note that the independence assumption on the noise vectors is not restrictive and can be replaced by assuming that they form a martingale difference sequence, without impacting the results established in this work.

To proceed, let \( Q \in \mathbb{R}^{p \times p} \), and \( R \in \mathbb{R}^{r \times r} \) be positive definite matrices. Then, the instantaneous cost of regulation policy \( \pi \) at time \( t \) is defined as

\[
c_t(\pi) = x(t)'Qx(t) + u(t)'Ru(t).
\]

Intuitively speaking, a desired control policy aims to minimize the long-run expected average cost. For a precise definition, we first specify the families of causal and adaptive policies. A causal policy, denoted by \( \pi \), determines the input based on the dynamics captured by \( A_0, B_0 \), the matrices \( Q \) and \( R \) assessing the costs of the output and input signals, respectively, and the record of

\[\text{for more details, see Theorem 11 and the ensuing discussion}\]
the signals thus far; i.e.

\[ u(t) = \pi \left(A_0, B_0, Q, R, \{x(i)\}_{i=0}^t, \{u(j)\}_{j=0}^{t-1} \right), \]

for all \( t = 0, 1, \cdots \). Note that the mapping above can be stochastic. In addition, due to the noise process \( \{w(t)\}_{t=1}^\infty \) in (1), causality of the strategies prevents the operator from exactly forecasting the next output signal.

An adaptive policy, denoted by \( \hat{\pi} \), does not have access to the dynamical parameters \( A_0, B_0 \). Therefore, the adaptive operator needs to design, possibly randomly, the input signal according to other available information regarding the system:

\[ u(t) = \hat{\pi} \left(Q, R, \{x(i)\}_{i=0}^t, \{u(j)\}_{j=0}^{t-1} \right). \]

Obviously, an adaptive policy is causal. The control objective is to minimize the expected average cost of the system, subject to the stochastic dynamics equation (1). Naturally, all adaptive and non-adaptive operators are expected to target the same objective being determined by the context of the problem. Therefore, the cost matrices \( Q, R \) are assumed known for design of adaptive policies.

The best performance amongst causal policies belongs to the one that attains the smallest expected average cost:

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} \mathbb{E} \left[ c_t (\pi) \right] = \inf_{\pi} \lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} \mathbb{E} \left[ c_t (\pi) \right],
\]

where the infimum on the right hand side is taken over the set of causal policies. In order to find an optimal regulator \( \pi^* \), one needs to solve a fixed-point (or algebraic) Riccati equation [27]. For this purpose, we assume the necessary condition of stabilizability given below is satisfied.

**Assumption 2** There is a gain matrix \( L \in \mathbb{R}^{r \times p} \) such that \( |\lambda_{\max} (A_0 + B_0 L)| < 1 \).

**Remark 3** Henceforth, we employ the shorthand \( \theta = (A_0, B_0, Q, R) \) to denote the pair \([A, B]\) of dynamics matrices \( A \in \mathbb{R}^{p \times q}, B \in \mathbb{R}^{p \times r} \).

Then, for the dynamics parameter \( \theta \), define the Riccati operator \( \Psi_\theta (\cdot) : \mathbb{R}^{p \times p} \to \mathbb{R}^{p \times p} \):

\[ \Psi_\theta (K) = Q + A'K A - A'K B (B'K B + R)^{-1} B'K A. \]

Suppose that \( K (\theta) \) solves

\[ K (\theta) = \Psi_\theta (K (\theta)). \]

It has been established that whenever \( \theta \) is stabilizable, \( K (\theta) \) exists, is unique and positive definite [33]. In addition, computation based on the recursive Riccati equation of the form \( K_{t+1} = \Psi_\theta (K_t) \), \( K_0 = Q \), is known to provide a sequence of matrices converging exponentially fast to \( K (\theta) \) [33].

Using the matrix \( K (\theta) \) defined above, let

\[ L (\theta) = - (B'K (\theta) B + R)^{-1} B'K (\theta) A. \]

The linear time invariant gain \( L (\theta_0) \) provides an optimal regulator \( \pi^* \) for a system of stabilizable dynamics parameter \( \theta_0 \) [27]. It is also known that \( L (\theta_0) \) is a stabilizer; i.e. \( |\lambda_{\max} (A_0 + B_0 L (\theta_0))| < 1 \). Henceforth, let \( \pi^* \) be the aforementioned optimal gain:

\[ \pi^* : \ u(t) = L (\theta_0) x(t), \quad t = 0, 1, 2, \cdots. \]

The above result on the optimality of linear gains motivates us to consider adaptive regulators of this form. However, since information on the unknown dynamics parameters is acquired over time, such adaptive regulators need to be time-varying so that they can achieve good performance. Further, to address identification of the unknown parameters, the linear feedbacks are subject to exogenous randomness through statistically independent perturbations.

Specifically, we study causal policies of the form

\[ \pi : \ u(t) = L_t x(t) + v(t), \quad t = 0, 1, 2, \cdots, \]

denoted by \( \pi = \{L_t, v(t)\}_{t=0}^\infty \), where \( L_t \) is a \( r \times p \) gain matrix (determined according to \( A_0, B_0, Q, R, \{x(i)\}_{i=0}^t, \{u(j)\}_{j=0}^{t-1} \) to retain causality), and the \( r \) dimensional perturbation vectors are \( \{v(t)\}_{t=0}^\infty \). Similarly, the adaptive regulator \( \hat{\pi} = \{L_t, v(t)\}_{t=0}^\infty \) is defined according to the adaptive gain matrices \( \{L_t\}_{t=0}^\infty \) (that is, \( \hat{L}_t \) is a function of \( Q, R, \{x(i)\}_{i=0}^t, \{u(j)\}_{j=0}^{t-1} \)) in addition to the perturbation signal \( v(t) \).

**Remark 4** Throughout this paper, the perturbations \( \{v(t)\}_{t=0}^\infty \) are assumed to be mean zero independent signals, also independent of the noise process \( \{w(t)\}_{t=1}^\infty \).

The efficiency of an arbitrary causal strategy \( \pi \) is compared to the optimal feedback \( \pi^* \) in (5). In fact, the following natural and commonly used definition of regret will be studied:

\[ R_n (\pi) = \sum_{t=0}^{n-1} \left[ c_t (\pi) - c_t (\pi^*) \right]. \]
On the other hand, the precision of a learning procedure is measured through the error term \( \| \hat{\theta}_n - \theta_0 \| \), which reflects the deviation of estimate \( \hat{\theta}_n \) from the true dynamics parameter \( \theta_0 \), at time \( n \) (i.e. based on a sample of size \( n \)). In the sequel, we provide non-asymptotic analysis of the effect of input perturbation on the regret bounds of general (causal and adaptive) policies, which hold uniformly over time; i.e. we study

\[
\sup_{t \leq n} R_t(\pi).
\]

We also present the high probability upper bounds of the anytime bounds for the identification error by studying

\[
\sup_{t \geq n} \| \hat{\theta}_t - \theta_0 \|.
\]

3 General Input Perturbations

In this section, we study the properties of arbitrary causal control policies with perturbed input signal. The exploitation analysis which considers the effects of both the linear feedback gains, as well as the perturbation signals will be presented in Subsection 3.1. Subsequently, we address the exploration performance in Subsection 3.2, namely how the structure of a causal policy determines the finite sample performance of the learning accuracy for the unknown parameter \( \theta_0 \). The main results, Theorem 6 and Theorem 9, both provide high probability bounds which hold uniformly over time.

3.1 Regret Bound

In this subsection, we analyze the regret when the input signal of a causal policy is perturbed. In absence of a perturbation sequence, i.e. \( v(t) = 0 \), it has been shown that the regret is equal to

\[
\sum_{t=0}^{n-1} \| (L(\theta_0) - L_t) x(t) \|^2,
\]

modulo a constant factor [28]. Theorem 6 generalizes the above result along the following two directions. First, the input signals provided by the linear feedback gains \( \{L_t\}_{t=0}^{\infty} \) are perturbed with exogenous random signals \( \{v(t)\}_{t=0}^{\infty} \). Second, the theorem below provides a high probability upper bound for the maximum of the regret over time. The energy quantities defined below will be used through the forthcoming results of the non-asymptotic analyses.

Definition 5 For a causal policy \( \pi = \{L_t, v(t)\}_{t=0}^{\infty} \), let the deterministic sequences \( \{\pi_t\}_{t=0}^{\infty}, \{\ell_t\}_{t=0}^{\infty} \) be such that for all \( t \geq 0 \), the followings hold:

\[
\| (L(\theta_0) - L_t) x(t) \| \leq \overline{t}_t, \\
\| v(t) \| \leq \overline{v}_t.
\]

Then, define \( E_v = \sum_{t=0}^{n-1} \overline{v}_t^2, E_\ell = \sum_{t=0}^{n-1} \ell_t^2 \). Similarly, define \( \overline{\ell}_t, \overline{x}_t \), as well as \( E_w, E_x \), for \( w(t), x(t) \), respectively. Finally, let

\[
E_\pi = \left( \sum_{t=0}^{n-1} \overline{\ell}_t (\overline{x}_t + \ell_t)^2 \right)^{1/2} + \left( \overline{\ell}_t^2 \sum_{t=0}^{n-1} (\overline{\ell}_t + \overline{x}_t)^2 \right)^{1/2},
\]

where \( \overline{\ell}_t = \max_{1 \leq t \leq n} \overline{\ell}_t \).

The dependence of all terms \( E \) on \( n \) is suppressed for notational convenience. Intuitively, \( E_\pi \) corresponds to the deterministic upper bound for the energy of the signal \( \{y(t)\}_{t=0}^{\infty} \) up to time \( n \). Subsequently, we establish a high probability regret bound in terms of the energy quantities defined above.

Theorem 6 Using Definition 5 for \( \pi = \{L_t, v(t)\}_{t=0}^{\infty} \), with probability at least \( 1 - \delta \) we have

\[
\sup_{n \geq 1} \frac{R_n(\pi)}{\overline{\ell}_t + E_\ell + E_\pi \log^{1/2} (n\delta^{-1}) + \max_{0 \leq t < n} \overline{x}_t} \leq C_1,
\]

where \( C_1 < \infty \) is a fixed constant depending on \( \theta_0 \).

One can explicitly calculate \( C_1 \) using the statements in the proof of Theorem 6 presented in the last section. Also note that the statement of Theorem 6 holds uniformly over time. To provide further intuition and outline the consequences of the above general result, we state the following corollary:

Corollary 7 With the notation of Theorem 6, we have

\[
\sup_{n \geq 1} \frac{E[ R_n(\pi) ]}{\overline{\ell}_t + E_\ell + \max_{0 \leq t < n} \overline{x}_t} \leq C_1,
\]

where the expectation \( E[\cdot]\) is with respect to the probability measures induced by \( \{w(t)\}_{t=1}^{\infty}, \{v(t)\}_{t=0}^{\infty} \).

Considering Theorem 6 and Corollary 7, the only different term is \( E_\pi \log^{1/2} (n\delta^{-1}) \). Thus, the stochastic behavior of the causal policy \( \pi \) is with high probability determined by \( E_\pi \); the other terms reflect the average deviations from the optimal regulator \( \pi^* \). In other words, the cumulative deviation from the desired optimal trajectory consists of the systematic long lasting portion, as well as the spontaneous fluctuations. According to Corollary 7, the former is reflected in the energy of the
perturbation signal \( \mathcal{E}_v \), and the energy of the instantaneous deviations in the feedback gains \( \mathcal{E}_\pi \). The magnitude of the spontaneous fluctuations is with high probability determined by \( \mathcal{E}_\pi \), which essentially reflects the interactions of

\[
x(t), v(t), L_t - L(\theta_0); \quad w(t + 1), v(t), L_t - L(\theta_0).
\]

So, \( \mathcal{E}_\pi \) can be interpreted as the fluctuation energy. We will shortly compare the aforementioned involved quantities for diminishing perturbations.

If there is no perturbation, \( v(t) = 0 \), \( \mathcal{E}_\ell \) is asymptotically shown to be a tight (i.e. both lower and upper) bound for \( R_n(\pi) \) [28]. To compare with the result of Theorem 6, letting \( \gamma_t = 0 \) for all \( t = 0, 1, \cdots \), suppose that the noise is uniformly bounded. Then, since \( \sup_{t \geq 1} \mathcal{E}_t < \infty \), the fluctuation energy \( \mathcal{E}_\pi \) can be replaced with \( \mathcal{E}_\ell^{1/2} \). This leads to the non-asymptotic regret bound \( \mathcal{E}_\ell + \mathcal{E}_\ell^{1/2} \log^{1/2} (n\delta^{-1}) + \max_{0 \leq t < n} \bar{t}_\mathcal{E} \), which provides a finite time version for the asymptotic bound \( \mathcal{E}_\ell \) [28], for bounded noise.

Next, we discuss a case of diminishing perturbations to compare the contributions of different factors toward the uniform bound of Theorem 6. Assuming \( \sup_{t \geq 1} \mathcal{E}_t < \infty \), let \( \{v(t)\}_{t=0}^{\infty} \) be a diminishing signal such that

\[
\sup_{t \geq 0} t^{\alpha_1} \mathcal{E}_t < \infty,
\]

for some \( \alpha_1 > 0 \). Further, let the deviation from the optimal gain satisfy

\[
\sup_{t \geq 0} t^{\alpha_2} \| L(\theta_0) - L_t \| < \infty,
\]

for some \( \alpha_2 > 0 \). Since the above policy stabilizes the system, the output signal is uniformly bounded as well. Then, Theorem 6 provides the following uniform regret bound:

\[
n^{-1/2} \cdot \log^{1/2} (n\delta^{-1}).
\]

Similarly, it is straightforward to show that uniform boundedness of the noise vectors and the output signal leads to the bound

\[
\mathcal{E}_v + \mathcal{E}_\ell + (\mathcal{E}_v + \mathcal{E}_\ell)^{1/2} \log^{1/2} (n\delta^{-1}).
\]

Therefore, the perturbation signal does not leads to an increase in the high probability regret bound as long as it is diminishes as fast as \( L(\theta_0) - L_t \). For the special case of \( \alpha_1 \land \alpha_2 = 1/2 \), the regret is

\[
\log n + \log^{1/2} n \log^{1/2} \delta^{-1},
\]

which is the information theoretic lower bound for adaptive regulators [28].

Finally, Theorem 6 shows that the non-asymptotic high probability uniform bound of the regret is revocable and memoryless, similar to the asymptotic bound \( \mathcal{E}_\ell \) (when the input is not perturbed [28]). That being said, as soon as the operator stops using perturbed inputs or deviated feedback gains, the regret freezes and does not grow anymore. In other words, the history of the non-optimal actions previously taken, does not significantly influence the future trajectory of the dynamical system.

### 3.2 Identification Bound

Next, we analyze the effect of the input perturbation for learning the unknown dynamics parameter \( \theta_0 = [A_0, B_0] \). Based on the linear structure of the statistical model in (1), a natural estimation procedure for \( \theta_0 \) is linear regression. That is, to regress every output sample \( x(t + 1) \) on the previous input \( w(t) \), as well as the last output sample \( x(t) \). This leads to the following least squares estimator:

\[
\hat{\theta}_n = \arg \min_{\theta \in \mathbb{R}^p \times q} \sum_{t=0}^{n-1} \left[ x(t + 1) - \theta \left[ x(t), w(t) \right] \right]^2.
\]

In the statistical analyses of the accuracy of \( \hat{\theta}_n \), the associated stochastic process \( \{w(t)\}_{t=1}^{\infty} \), as well as the exogenous one \( \{v(t)\}_{t=0}^{\infty} \), play an important role. So, we use the following definition which uses the second moments of the aforementioned stochastic processes.

**Definition 8** Suppose that \( \pi = \{L_t, v(t)\}_{t=0}^{\infty} \) is a causal policy. Then, let

\[
\Sigma_w = \sum_{t=1}^{n-1} \mathbb{E} \left[ w(t)w(t)^\prime \right],
\]

\[
\Sigma_v = \sum_{t=0}^{n-2} \mathbb{E} \left[ v(t)v(t)^\prime \right],
\]

\[
\lambda_n = |\lambda_{\min}(\Sigma_w)| \land |\lambda_{\min}(\Sigma_v)|.
\]

Further, letting \( D_0 = A_0 + B_0L(\theta_0) \), and

\[
\mathcal{E}_t = \left\| B_0 \right\| \bar{t}_\mathcal{E} + \bar{t}_{\ell+1} + 2 \left\| B_0 \right\| \bar{t}_\pi + 2 \left\| D_0 \right\| \bar{t}_v,
\]

define the following energy type of quantities similar to...
\[ \mathcal{E}_1 = \left( \frac{1}{2} \left( \sum_{t=0}^{n-2} \left( \| P_0 \| \| \varpi_t + \varpi_{t+1} \| \right)^2 \right) \right)^{1/2} \]

\[ \mathcal{E}_2 = \left( \frac{1}{2} \left( \sum_{t=0}^{n-2} \left( 2 \left[ 1 + \| L(\theta_0) \| \right] \| \pi_t + 2\varpi_t + \varpi_{t+1} \| \right)^2 \right) \right)^{1/2} \]

Note that \( \Sigma_w, \Sigma_v, \mathcal{E}_1, \mathcal{E}_2 \) all depend on \( n \). In order to interpret \( \mathcal{E}_1, \mathcal{E}_2 \), applying the causal policy \( \pi = \{ L_t, v(t) \}_{t=0}^{\infty} \) to the dynamical model in (1), the closed-loop dynamics becomes \( x(t+1) = D_t x(t) + B_0 v(t) + w(t+1) \), where \( D_t = A_0 + B_0 L_t \). Thus, the noise in the evolution of the system is \( B_0 v(t) + w(t+1) \), which intuitively shows that \( \mathcal{E}_1 \) is reflecting a deterministic upper bound for the interaction energy between the observed signals \( x(t) \), and the unobserved noise \( B_0 v(t) + w(t+1) \). Similarly, \( \mathcal{E}_2 \) is bounding the interaction energy between the output signal \( x(t) \) and the perturbation \( v(t) \).

In order to accurately identify the true parameter \( \theta_0 \), the lower bound \( \lambda_n \) cannot be very small. Broadly speaking, the magnitude of \( \lambda_n \) determines the extend to which both the perturbation and the noise processes excite all coordinates of the matrix \( \theta_0 \). So, the precision of the learning procedure in (8) highly depends on \( \lambda_n \). To establish a uniform upper bound for learning accuracy, we assume the followings for the quantities defined in Definition 8:

\[ |\lambda_{\min}(\Sigma_w)| \geq 4 \mathcal{E}_1 \log^{1/2} (8 \bar{\delta}^{-1}) , \tag{9} \]

\[ |\lambda_{\min}(\Sigma_v)| \geq 4 \mathcal{E}_2 \log^{1/2} (8 \bar{\delta}^{-1}) . \tag{10} \]

We will shortly discuss that the above inequalities are not restrictive. The energy interpretations also confirm them as follows. Simply writing down the closed-loop dynamics as before, \( |\lambda_{\min}(\Sigma_w)| \) denotes the minimum energy in the noise \( B_0 v(t) + w(t+1) \). Going back to the interpretation of \( \mathcal{E}_1 \) after definition 8, (9) is indeed stating that the noise energy is larger than the interaction energy (of the noise and the signal \( x(t) \)), with high probability. Therefore, the interaction is not powerful enough to prevent the noise from exciting all coordinates of the unknown transition matrix \( A_0 \). Similar explanation for (10) states that the excitation of \( B_0 \) by the perturbation signal is not masked by its interaction with the output signal. These are formally presented in the following result.

**Theorem 9** Using the notation of Definitions 5, 8, with probability at least \( 1 - \delta \) we have

\[ \sup_{n \geq 1} \left( \frac{\lambda_n \| \hat{\theta}_n - \theta_0 \|}{\log \left( (\mathcal{E}_1 + \mathcal{E}_x + \mathcal{E}_v) \delta^{-1} \right)} \right) \leq C_2, \]

where the supremum is being taken over all \( n \geq 1 \) satisfying (9), (10).

The constant \( C_2 \) depends on the true dynamics parameter \( \theta_0 \), and can be extracted from the statements of the proof in the last section.

Next, to discuss Theorem 9, suppose that the covariance matrices of the noise vectors are bounded from below:

\[ \inf_{n \geq 1} \frac{1}{n} |\lambda_{\min}(\Sigma_w)| > 0. \]

Further, for the perturbation process assume the conditions:

\[ \inf_{t \geq 1} \left( \frac{1}{n} \right) |\lambda_{\min}(\Sigma_v)| > 0, \]

\[ \sup_{t \geq 1} \left( \frac{t^{\alpha_1} \| \Sigma \|}{\bar{\delta}^{-1}} \right) < \infty, \]

for some \( 0 \leq \alpha_1 < 1 \), and \( 0 \leq \alpha_2 \leq 1 \). Hence, we obtain

\[ \inf_{n \geq 1} \left( \frac{1}{n} \right) \lambda_n > 0. \]

Further, similar to the discussion after Theorem 6, suppose that both the noise and the output signal are uniformly bounded (e.g. \( \bar{\varepsilon}_t \) is diminishing). Assuming \( \alpha_1 - \alpha_2 < 1/2 \), since

\[ \sup_{n \geq 1} n^{-1/2} \mathcal{E}_1 < \infty, \]

\[ \sup_{n \geq 1} n^{\alpha_2 - 1/2} \mathcal{E}_2 < \infty, \]

(9), (10) are satisfied as long as \( n \) is up to a constant factor at least

\[ n^{\alpha_1} \delta^{-1} \]

Then, Theorem 9 implies the following identification bound for uniform learning accuracy of estimating the unknown parameter \( \theta_0 \):

\[ n^{\alpha_1 - 1/2} \log^{1/2} \left( n \delta^{-1} \right). \]

Hence, the identification accuracy is basically determined by the diminishing rate of the perturbation signal. Moreover, if the perturbation is long lasting, i.e. \( \alpha_1 = \alpha_2 = 0 \), the least squares estimator in (8) achieves
the optimal uniform learning rate of
\[ n^{-1/2} \log^{1/2} \left( n \delta^{-1} \right). \]

Note that in comparison to the analysis presented in the previous subsection, a larger perturbation leads to more accurate learning together with larger deviations from the optimal cost, and vice versa. Balancing this trade-off is the main challenge for adaptive policies, and will be addressed in the sequel. Also note that for \( \alpha_1 = 1/2 \), the result corresponds to the finite time extension of the asymptotic identification rates presented in [28].

4 Perturbed Greedy Regulator

Next, we analyze the non-asymptotic regret of perturbed Greedy policy, as well as its identification error. For this purpose, we first state a fairly general condition about the probabilistic properties of the noise process [27]. Then, Greedy adaptive regulator with input perturbation will be formally presented. The subsequent contents consist of establishing that Greedy addresses the main dilemma of reinforcement learning; i.e. balancing the exploration-exploitation trade-off.

In this section, we assume the following about the noise sequence \( \{w(t)\}_{t=1}^{\infty} \) in the dynamical model (1).

**Assumption 10** The noise has a sub-Weibull distribution. That is, for some fixed \( \tilde{\beta}, \beta, \alpha_0 > 0 \), the following holds for all \( t \geq 1 \), and \( \eta > 0 \),
\[ \mathbb{P}(\|w(t)\| > \eta) \leq \tilde{\beta} \exp(-\beta^{-1} \eta^{\alpha_0}). \]

Further, we assume that the covariance matrices of the noise vectors satisfy
\[ \inf_{n \geq 1} \frac{1}{n} \sum_{t=1}^{n} \lambda_{\min}(\mathbb{E}[w(t)w(t)']) \geq \sigma_0 > 0. \]

Note that positive definiteness of the covariance matrices \( \mathbb{E}[w(t)w(t)'] \) is sufficient for the second part of Assumption 10, but is not necessary. Regarding the first part of Assumption 10, sub-Weibull distributions are remarkably general for finite time analyses in learning theory [10]. In fact, they encompass a wide range of distributions being commonly used in the literature, such as uniformly bounded (\( \alpha_0 = \infty \)), sub-Gaussian (\( \alpha_0 = 2 \)), and sub-Exponential (\( \alpha_0 = 1 \)) random vectors, as well as the heavy-tail distributions for which moment generating functions do not exist (\( \alpha_0 < 1 \)). Later on, we will see that the exponent \( \alpha_0 \) which determines the decay rate in the probability distributions of the noise vectors, plays a crucial role in the non-asymptotic analyses for both the regret and the identification error. Note that the noise distributions are not required to have densities (pdf) or continuous cumulative distribution functions (CDF).

Perturbed Greedy is an episodic algorithm; i.e. learning of the unknown parameters is deferred until sufficiently many input-output observations have been collected. More precisely, the algorithm updates the parameter estimates only at the end of epochs of exponentially growing length. This lets the solution of Riccati equation (3) be efficiently used by preventing unnecessary computations. Note that numerical computation of the optimal feedback gain in (4) is practically nontrivial so that the operator needs to spend a decent amount of time for finding \( L(\hat{\theta}_t) \) [34].

Formally, the algorithm starts with the arbitrary initial approximation \( \theta_0 \). At each time \( t = 0, 1, 2, \cdots \), it applies the adaptive policy
\[ u(t) = L(\hat{\theta}_t) x(t) + v(t). \]

The design of the perturbation signals \( v(t) \) will be discussed shortly. Further, the lengths of the epochs are determined by the reinforcement rate \( \gamma > 1 \) as follows. If \( t \) satisfies \( t \in [\gamma^m, \gamma^{m+1}) \) for some \( m = 0, 1, \cdots \), the algorithm tunes the regulator by finding \( \hat{\theta}_t \) according to (8). Otherwise, for \( t \not\in [\gamma^m, \gamma^{m+1}) \), the learning step will be skipped and no update occurs: \( \hat{\theta}_t = \hat{\theta}_{t-1} \).

Then, the perturbations during each epoch \( m \geq 1 \) are wide-sense stationary (homoscedastic) random signals with positive definite covariance matrices. Namely, for all \( \gamma^m \leq t < \gamma^{m+1} \), we have \( \mathbb{E}[v(t)v(t)'] = \Sigma_m \), and \( \|v(t)\| \leq \overline{v}_m \), such that the following holds:
\[ \mathcal{C} < m^{-2}\gamma^{m/2} \|\lambda_{\min}(\Sigma_m)\| \leq m^{-2}\gamma^{m/2} \overline{v}_m^2 < \overline{C}, \]
for some fixed universal constants \( \mathcal{C} > 0, \overline{C} < \infty \) which do not depend on \( m \). Note that we always have \( \mathbb{E}[v(t)] = 0 \). Further, since
\[ r \|\lambda_{\min}(\Sigma_m)\| \leq \text{tr}(\mathbb{E}[v(t)v(t)']) = \mathbb{E}[v(t)'v(t)] \leq \overline{v}_m^2, \]
we have \( r\mathcal{C} < \overline{C} \). The algorithmic procedure of perturbed Greedy regulator is depicted in Algorithm 1.

Although larger values of \( \gamma \) lead to longer epochs and thus less updates of the parameter estimates, the rates of regulation and identification do not depend on the magnitude of \( \gamma \). Intuitively, it is because Greedy regulator utilizes all the random perturbation signals during each epoch when learning the dynamics parameter at the end of that epoch. The following theorem addresses the uniform finite time rates of regulation and identification for the above Greedy algorithm with input perturbation.
we have \( \hat{\theta} \) the true dynamics parameter.

Therefore, the potential stochastic identification error. Hence, the solutions of the Riccati equations (3), (4) need to be repeatedly computed at every time step \( t \) [35].

For the sake of completeness, we briefly discuss the adaptive stabilization of the system for perturbed Greedy regulator. In fact, if the system is not a priori stabilized, the learning procedure in (8) leads to the stabilization in finite time so that \( |\lambda_{\max}(A_0 + B_0 L (\hat{\theta}))| < 1 \). Technically, during the stabilization period, it suffices that the unstable closed-loop transition matrices \( A_0 + B_0 L (\hat{\theta}) \) will not be irregular [10], as defined below. Irregularity occurs if an eigenvalue of \( A_0 + B_0 L (\hat{\theta}) \) outside the unit circle (in the complex plane) has an eigenspace of dimension larger than one [10]. Two methods for an adaptive operator to prevent the closed-loop matrices becoming pathologically irregular are available in the existing literature; random parameter [28], and random feedback [33].

The method of random parameter adds a continuously distributed random matrix to the estimate \( \hat{\theta} \) in (8) [28]. This of course is sufficient to occur in the first few epochs [28], since there is no eigenvalue outside the unit circle once the system is stabilized. The second method relies on employing random feedback gain matrices [33]. Indeed, it is established in the previous reference that the proposed stabilization algorithm terminates in finite time, and is with a high probability guaranteed to stabilize the system [33]. Thus, it suffices to add continuously distributed feedback gains to \( L (\hat{\theta}) \) during the first \([q/p]\) epochs [33]. The details are fully discussed in the aforementioned references, so are omitted here.

4.1 Partial Uncertainty of Parameters

Next, we study the effect of restricted uncertainty about the true dynamical parameter \( \theta_0 \) on the performance of adaptive regulators. In fact, the context of the plant under consideration is capable of providing side information about \( \theta_0 \). Examples of such a priori knowledge include the structural connectivity of networks which determines the support of \( \theta_0 \), and the man-made systems where the construction imposes restriction on possible sets of \( p \times q \) parameter matrices. Further, in the systems with longer memory where the current output signal \( \hat{x}(t) \) depends on multiple previous time steps; \( \hat{x}(t-1), \ldots, \hat{x}(t-k) \), for some \( k > 1 \), the transition matrix \( A_0 \) has a particular structure once the dynamics is written in the form of (1) for the new output signal \( x(t) = [\hat{x}(t), \ldots, \hat{x}(t-k+1)]' \) [13,10]. Therefore, the adaptive operator can restrict the learning of the unknown dynamics parameter to a subset of \( p \times q \) matrices.

An analysis under fairly stronger technical assumptions provides a bound for the regret which is consistent with the bound of Theorem 11, but is not uniform over all time points [35]. Further, the computational cost of the proposed non-episodic self-tuning regulator is exponentially higher than the episodic algorithm of this section; i.e. the solutions of the Riccati equations (3), (4) need to be repeatedly computed at every time step \( t \) [35].

Next, we discuss the relationship between the above statements and the asymptotic results obtained without perturbing the input signal [28]. Technically, if one analyzes the finite time behavior of the adaptive policies presented in the above reference, they lead to polynomials of \( n \delta^{-1} \) appearing in the expressions of both regret and identification error. Therefore, the potential stochastic fluctuations are neither sufficiently scarce (because of \( \delta^{-1} \)), nor small enough (because of \( n \)). So, another important strength of the input perturbations in Greedy policy is the term \( \log(n \delta^{-1}) \) which renders the frequent and large fluctuations exponentially rare and small, respectively.

Algorithm 1: Perturbed Greedy

Inputs: \( \gamma > 1 \), \( 0 < r \mathcal{C} < \mathcal{C} < \infty \)
Let \( \hat{\theta}_0 \in \mathbb{R}^{p \times q} \) be stabilizable
for \( m = 0, 1, 2, \ldots \)
while \( t < \gamma^m \) do
    Draw perturbation \( v(t) \) according to (12)
    Apply input \( u(t) = L(\hat{\theta}) x(t) + v(t) \)
    Update the estimate \( \hat{\theta}_t = \hat{\theta}_t 
end while
Theorem 11 Suppose that the adaptive regulator \( \hat{\pi} \) is Greedy with input perturbation. Let \( \hat{\theta}_n \) be the parameter estimate at time \( n \). Then, with probability at least \( 1 - \delta \) we have

\[
\sup_{n \geq n_0} \frac{\mathcal{R}_n(\hat{\pi})}{n^1/2 \log^{2/\alpha}(n \delta^{-1})} \leq C_3,
\]

\[
\sup_{n \geq n_0} \frac{\left\| \hat{\theta}_n - \theta_0 \right\|^2}{n^{-1/2} \log^{2/\alpha - 1}(n \delta^{-1})} \leq C_4,
\]

where \( n_0 \geq C_0 \left( \log^{2+4/\alpha}(\delta^{-1}) \right) (\log \log \delta^{-1}) \).

The constants \( C_0, C_3, C_4 \) are fixed, finite, and depend on the true dynamics parameter \( \theta_0 \), constants of the noise process \( \beta, \tilde{\beta}, \sigma_0 \), and the constants involved in the design of perturbed Greedy adaptive regulator \( \gamma, \mathcal{C}, \mathcal{C} \).

Next, we discuss the relationship between the above statements and the asymptotic results obtained without perturbing the input signal [28]. Technically, if one analyzes the finite time behavior of the adaptive policies presented in the above reference, they lead to polynomials of \( n \delta^{-1} \) appearing in the expressions of both regret and identification error. Therefore, the potential stochastic fluctuations are neither sufficiently scarce (because of \( \delta^{-1} \)), nor small enough (because of \( n \)). So, another important strength of the input perturbations in Greedy policy is the term \( \log(n \delta^{-1}) \) which renders the frequent and large fluctuations exponentially rare and small, respectively.

An analysis under fairly stronger technical assumptions provides a bound for the regret which is consistent with the bound of Theorem 11, but is not uniform over all time points [35]. Further, the computational cost of the proposed non-episodic self-tuning regulator is exponentially higher than the episodic algorithm of this section; i.e. the solutions of the Riccati equations (3), (4) need to be repeatedly computed at every time step \( t \) [35].
that the underlying MDP has a *distinguishability* property, the regret can be in the order of magnitude of $\log n$ [37]. A similar situation holds for Multi-Arm Bandits (MAB) [38].

To proceed, we define the following identifiability condition as the analogue of the distinguishability condition in Markov Decision Processes. Intuitively, in the identifiable systems, the optimal feedback gain $L(\theta_0)$ can be learned as accurately as the closed-loop transition matrix $A_0 + B_0L(\hat{\theta}_t)$.

**Definition 12** Assuming $\theta_0 \in \Gamma_0$ for some $\Gamma_0 \subset \mathbb{R}^{p \times q}$, $\theta_0$ is called identifiable if

$$\sup_{\theta_1, \theta_2 \in \Gamma_0} \frac{\|L(\theta_2) - L(\theta_0)\|}{\|\theta_2 - \theta_0\| \cdot \|L(\theta_1)\|} < \infty, \quad (13)$$

where $\tilde{L}(\theta) = [I_p, L(\theta)]'$; i.e. $\theta_2 \tilde{L}(\theta_1) = A_2 + B_2L(\theta_1)$.

**Remark 13** It has been established that the gain difference $L(\theta) - L(\theta_0)$ is dominated by the parameter difference $\theta - \theta_0$, modulo a constant factor [27]:

$$\sup_{\theta : \|\theta\| \leq C \kappa} \|L(\theta) - L(\theta_0)\| \leq C_L, \quad (14)$$

where $C_L < \infty$ is a fixed constant determined by $\theta_0$ and $C_\kappa < \infty$. Note that (13) is stronger than (14) since in general, the identifiability condition does not need to hold. Indeed, one can have $L(\theta_2) \neq L(\theta_0)$, but at the same time $\theta_2 \tilde{L}(\theta_1) = \theta_0 \tilde{L}(\theta_1)$ [28].

An extensive discussion of concrete examples of $\Gamma_0$ is provided in [28]. Briefly, $\Gamma_0$ can be a manifold (or a finite union of manifolds) induced by the following conditions.

- **Support**: assume that for some

  $$\mathcal{I} \subset \{(i, j) : 1 \leq i \leq p, 1 \leq j \leq q\},$$

  the matrix $\theta$ belongs to $\Gamma_0$, if and only if the coordinate $(i, j)$ of $\theta$ is zero for all $(i, j) \notin \mathcal{I}$. If the cardinality of $\mathcal{I}$ is not larger than $p^2$, $\Gamma_0$ can satisfy (13) [28].

- **Sparsity**: if $\theta_0$ is known to have at most $p^2$ non-zero entries, $\Gamma_0$ in contained in the finite union of the subsets satisfying the above support condition.

- **Rank**: the set of $p \times q$ matrices of rank at most $d$ is a finite union of manifolds of dimension at most $d (p + q - d)$ [39], which can satisfy (13) if $d (p + q - d) \leq p^2$ [28].

- **Subspace**: suppose that $\theta_0$ is known to belong to a subspace of $\mathbb{R}^{p \times q}$. If the above subspace is of dimension at most $p^2$, it can satisfy the identifiability condition (13) [28].

The quantity $p^2$ appearing in the above examples can be replaced by $p^2 + (p - \text{rank}(A_0)) r$ [28]. The latter dimension is actually determined by the dimensions of invariant manifolds of learning and planning, denoted by $\mathcal{N}$ and $\mathcal{S}$, respectively [28]. Indeed, the planning invariant manifold $\mathcal{S}$ consists of parameters $\theta$ which share the feedback gain with the true parameter $\theta_0$: $L(\theta) = L(\theta_0)$. Further, the learning invariant manifold $\mathcal{N}$ contains the dynamics matrices $\theta$ for which the closed-loop matrix is indistinguishable from the truth $\theta_0$, i.e. $\theta L(\theta_0) = \theta_0 \tilde{L}(\theta_0)$. It has been established that if

$$\Gamma_0 \cap \frac{\mathcal{N}}{\mathcal{N} \cap \mathcal{S}} = \{\theta_0\};$$

$\Gamma_0$ satisfies (13) [28]. Since $\mathcal{N} / (\mathcal{N} \cap \mathcal{S})$ is a linear subspace of dimension rank $(A_0) r$ [28], intuitively the dimension of $\Gamma_0$ needs to be at most $pq - \text{rank}(A_0) r$. For more technical details, we refer the interested reader to the existing results studying the aforementioned manifolds [40,41,28].

In the adaptive regulator for identifiable systems, we replace (12) with $\gamma^m \pi_m < C$; i.e. the perturbation signals are lighter than (12) (diminish faster), and the covariance matrix does not need to be positive definite. Further, the uncertain dynamics parameter $\theta_0$ will be learned through the following least squares procedure on the identifiability set $\Gamma_0$:

$$\hat{\theta}_n = \arg \min_{\theta \in \Gamma_0} \sum_{t=0}^{n-1} \|x(t+1) - \theta \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \|^2, \quad (15)$$

All other parts of the adaptive regulator such as the lengths of the epochs are the same as Algorithm 1. We conclude this section with the next result which compared to Theorem 11 provides a remarkably smaller regret bound.

**Theorem 14** Letting $\theta_0$ be identifiable, denote the adaptive regulator above $\hat{\pi}$. Then, with probability at least $1 - \delta$ we have

$$\sup_{n \geq n_0} \frac{\mathcal{R}_n(\hat{\pi})}{\log n \log^{1+2/\alpha_0} (n\delta^{-1})} \leq C_5,$$

where $n_0$ is defined in Theorem 11, and the fixed constant $C_5 < \infty$ depends on $\theta_0$.

5 Concluding Remarks

We analyzed the finite time performance of adaptive regulation policies with perturbed input for multidimensional LQ systems. Indeed, we established non-asymptotic results for the high probability regret bound.
(Theorem 6), as well as the learning accuracy (Theorem 9), which hold uniformly over time. Leveraging the developed general theory, we showed that Greedy adaptive regulator with suitably designed input perturbation provides (nearly) square root regret with respect to time (Theorem 11), and outperforms previously available reinforcement learning algorithms across different thrusters.

Specifically, perturbed Greedy designs the adaptive regulator independent of the spectral properties of the closed-loop transition matrix, and the noise statistics. Further, the computationally intractable steps being used by optimism-based adaptive policies are not required. Finally, Greedy with input perturbation efficiently regulates the trajectory of the system by preventing fluctuations due to the stochastic nature of the dynamics. We also discussed the situations where the regret is of a logarithmic magnitude, assuming that the adaptive operator has access to some information (such as the support or the rank) of the unknown dynamics matrices $A_0, B_0$ (Theorem 14).

Extending the presented framework to the case of imperfect observations where a linear transformation of the output signal is being observed with some noise is an interesting direction for future work. Moreover, addressing the problem of adaptive regulation for large scale systems -e.g. networks- in a possibly high dimensional setting such as sparse or low rank dynamical models is of interest for further investigation. Finally, settings involving time varying cost functions or time varying dynamics -e.g. switching systems and Markov Jump Processes- are other topics for future studies.

6 Proofs

6.1 Proof of Theorem 6 and Corollary 7

We start by extending the decomposition technique introduced in [28]. For a policy $\pi = \{L_t, v(t)\}_{t=0}^{n-1}$ and a fixed $n \geq 1$, define the sequence of policies $\pi_0, \cdots, \pi_n$ according to $\pi$:

$$
\pi_i : \begin{cases} u(t) = L_t x(t) + v(t), & t < i \\
 u(t) = L(\theta_0)x(t), & t \geq i.
\end{cases}
$$

Indeed, for $i = 0, 1, \cdots, n$, the causal policy $\pi_i$ follows the control strategy of $\pi$ at every time $t < i$, and from $t = i$ on switches to the optimal policy $\pi^*$ defined in (5). Clearly, $\pi_0 = \pi^*$, and $\pi_n = \pi$. So, one only needs to find $c_t(\pi_k) - c_t(\pi_{k-1})$, for $1 \leq k \leq n$, and $0 \leq t \leq n - 1$.

For this purpose, let

$$
D_0 = A_0 + B_0 L(\theta_0),
$$

$$
P_0 = Q + L(\theta_0)' RL(\theta_0),
$$

$$
M = B_0^* K(\theta_0) B_0 + R,
$$

and fixing $k$, define the matrices

$$
\Delta_k = B_0 (L_k - L(\theta_0)),
$$

$$
K_k = \sum_{j=n-k}^{\infty} D_0^j P_0 D_0^j.
$$

Let \(\{x(t)\}_{t=0}^{n-1}, \{y(t)\}_{t=0}^{n-1}\) be the output signals under the policies $\pi_k, \pi_{k-1}$, respectively. So, $\begin{aligned} x(t) &= y(t), \quad 0 \leq t \leq k - 1, \\
x(k) &= D_0 x(k - 1) + w(k) + z_{k-1}, \\
y(k) &= D_0 x(k - 1) + w(k), \\
x(t) &= y(t) = D_0^{t-k} z_{k-1}, \quad \text{for } t \geq k,
\end{aligned}$

where $z_{k-1} = \Delta_{k-1} x(k - 1) + B_0 v(k - 1)$. Therefore, $c_t(\pi_k) - c_t(\pi_{k-1}) = x(t)' P_0 x(t) - y(t)' P_0 y(t)$, for $t \geq k$, we have

$$
c_t(\pi_{k-1}) = x(t)' P_0 x(t),
$$

$$
c_t(\pi_k) = x(t)' P_{k-1} x(t) + v(t)' R v(t) + 2x(t)' L_{k-1}' R v(t),
$$

where $P_{k-1} = Q + L_{k-1}' R L_{k-1}$. So, according to $c_t(\pi_k) - c_t(\pi_{k-1}) = x(t)' P_0 x(t) - y(t)' P_0 y(t)$, for $t \geq k$, we have

$$
\begin{aligned}
\sum_{t=0}^{n-1} [c_t(\pi_k) - c_t(\pi_{k-1})] &= c_{k-1}(\pi_k) - c_{k-1}(\pi_{k-1}) \\
+ &\sum_{t=k}^{n-1} \left[ 2 y(t)' P_0 D_0^{t-k} z_{k-1} + z_{k-1}' D_0^{t-k} P_0 D_0^{t-k} z_{k-1} \right].
\end{aligned}
$$

Substituting for $y(t)$, and rearranging the terms, the above expression leads to

$$
\begin{aligned}
\sum_{t=0}^{n-1} [c_t(\pi_k) - c_t(\pi_{k-1})] &= x(k - 1)' E_{k-1} x(k - 1) \\
+ &v(k - 1)' F_{k-1} v(k - 1) + 2x(k - 1)' G_{k-1} v(k - 1) \\
+ &2 \sum_{j=k}^{n-1} w(j)' (K_{n-j} - K_j) D_0^{t-k} z_{k-1},
\end{aligned}
$$

where using the Lyapunov equation

$$
K(\theta_0) - D_0 K(\theta_0) D_0 = P_0,
$$

the matrices $E_{k-1}, F_{k-1}, G_{k-1}$ can be calculated as follows:

$$
\begin{aligned}
E_{k-1} &= L_{k-1}' R L_{k-1} - L(\theta_0)' RL(\theta_0) + D_0' (K_{n-k} - K_{k-1}) \Delta_{k-1} \\
&+ \Delta_{k-1}' (K_{n-k} - K_{k-1}) D_0 + \Delta_{k-1}' (K_{n-k} - K_{k-1}) \Delta_{k-1}, \\
F_{k-1} &= R + B_0' (K_{n-k} - K_k) B_0, \\
G_{k-1} &= L_{k-1}' R + (D_0' + \Delta_{k-1}') (K_{n-k} - K_k) B_0.
\end{aligned}
$$
Since $K_n = K(\theta_0)$, the definition of $L(\theta_0)$ in (4) yields
\[ E_{k-1} = \left(L(\theta_0) - L_{k-1}\right)' M \left(L(\theta_0) - L_{k-1}\right) - H_{k-1}, \]
where
\[ H_{k-1} = D_{0}' K_k \Delta_{k-1} + \Delta_{k-1}' K_k D_0 + \Delta_{k-1}' K_k \Delta_{k-1}. \]

Because $R_n(\pi) = \sum \sum [c(\pi_k) - c(\pi_{k-1})]$, the summation of the above expressions implies that
\[ R_n(\pi) = -\phi_n + \zeta_n + \psi_n + \sum_{j=1}^{n-1} s(j)' w(j), \quad (16) \]
where
\[ \phi_n = \sum_{k=0}^{n-1} x(k)' H_k x(k), \quad \zeta_n = \sum_{k=0}^{n-1} v(k)' F_k v(k), \]
\[ \xi_n = \sum_{k=0}^{n-1} x(k)' G_k v(k), \quad \psi_n = \sum_{k=0}^{n-1} \left(\left\| M^{1/2} (L_k - L(\theta_0)) x(k) \right\|^2 \right), \]
and $s(j) = 2 \sum_{k=1}^{j} (K_n - K_j) D_0^{j-k} z_{k-1}$.

Then, by $K_n = K(\theta_0)$, positive semi-definiteness of $K_k$ implies
\[ \zeta_n \leq \lambda_{\text{max}}(M) \mathcal{E}_v, \quad (17) \]
as well as the following:
\[
\frac{-\phi_n}{\|K(\theta_0)\|} \leq 2 \left\| B_0 \right\| \left\| D_0 \right\| \sum_{k=0}^{n-1} \left\| D_0^{n-k-1} \right\| \left(\ell_k x_k \right)^2 \\
\leq 2 \left\| B_0 \right\| \left\| D_0 \right\| \left(\sum_{k=0}^{n-1} \left\| D_0^k \right\|^2 \right) \max_{0 \leq k < n} \ell_k x_k. \quad (18)
\]

Further, one can bound $\xi_n$:
\[ |x(k)' G_k v(k)| \leq \mathfrak{g}_k \left(\ell_k + \mathfrak{x}_k\right) \mathfrak{v}_k, \]
where
\[ \mathfrak{g}_k = (1 + \|L(\theta_0)\|) \|R\| + \left\| D_0 \right\| \left\| B_0 \right\| \|K_n - K_{k+1}\| \left\| B_0 \right\|. \]
So, applying Azuma's Inequality [30] we obtain
\[
\Pr \left(\frac{\zeta_n^2}{8 \log \left( \frac{20n^2}{\delta^2} \right)} > 2 \sum_{k=0}^{n-1} \mathfrak{g}_k^2 \left(\ell_k + \mathfrak{x}_k\right)^2 \mathfrak{v}_k^2 \right) \leq \frac{3\delta}{10n^2}. \quad (19)
\]
To proceed toward bounding the last term in (16), note that $|s(j)' w(j)| \leq \mathfrak{s}_j \mathfrak{w}_j$, where
\[ \mathfrak{s}_j = 2 \left\| K_n - K_j \right\| B_0 \left\| \sum_{k=1}^{j} \left\| D_0^{k-1} \right\| \left(\ell_{k-1} + \mathfrak{w}_{k-1}\right) \right\|. \]

Therefore, Azuma's Inequality [30] implies that with probability at least $1 - 0.3\delta n^{-2}$, we have
\[
\left(\sum_{j=1}^{n-1} s(j)' w(j)\right)^2 \leq 8 \log \left( \frac{20n^2}{\delta^2} \right) \sum_{j=1}^{n-1} \mathfrak{s}_j^2 \mathfrak{w}_j^2. \quad (20)
\]
Then, stability of $D_0$ clearly implies that all the following quantities being used in (17), (18), (19), and (20) are finite, with upper bounds depending only on $\theta_0, Q, R$:
\[
\sup_{k \geq 0} \left\| D_0^k \right\|, \quad \sup_{0 \leq k < n} \left\| K_n - K_k \right\|, \quad \sup_{0 \leq k < n} \frac{\sum_{j=1}^{n-1} \mathfrak{s}_j^2 \mathfrak{w}_j^2}{w^2 \sum_{t=0}^{n-1} (\ell_t + \mathfrak{v}_t)^2}. \]

Since $\psi_n \leq \lambda_{\text{max}}(M) \mathcal{E}_v$, plugging (17), (18), (19), and (20) in (16), with probability at least $1 - 0.6\delta n^{-2}$ it holds that
\[ \frac{R_n(\pi)}{C_1} \leq \mathcal{E}_v + \mathcal{E}_\ell + \mathcal{E}_\pi \log^{1/2} \left( nd^{-1}\right) + \max_{0 \leq t < n} \ell_t x_t, \]
where $C_1 < \infty$ is fixed. Taking a union bound, we get the desired result of Theorem 6 since $\sum_{n=1}^{\infty} 0.6n^{-2} < 1$. Moreover, since $E \left[ \xi_n \right] = 0, E \left[ s(j)' w(j) \right] = 0$, clearly Corollary 7 is concluded from (17), (18), and $\psi_n \leq \lambda_{\text{max}}(M) \mathcal{E}_v$.

### 6.2 Proof of Theorem 9

First, solving for the least squares estimate in (8), we get
\[ \hat{\theta}_n \Sigma = \sum_{t=0}^{n-1} x(t + 1) \left[ x(t)' u(t) \right], \]
where
\[ \Sigma = \sum_{t=0}^{n-1} \left[ x(t)' u(t) \right]. \quad (21) \]
is the (unnormalized) empirical covariance matrix of the
covariates \( x(t), u(t) \). Since \( x(t+1) = \theta_0 \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + w(t+1) \),
we obtain

\[
\left( \hat{\theta}_n - \theta_0 \right) \hat{\Sigma} = \sum_{t=0}^{n-1} w(t+1) [x(t)', u(t)'].
\]

(22)

In the sequel we investigate \( \hat{\Sigma} \). The design of the perturbed input according to \( u(t) = L_t x(t) + v(t) \) leads to the closed-loop dynamics

\[ x(t+1) = D_t x(t) + z(t), \]

where

\[
D_t = A_0 + B_0 L_t,
\]

\[ z(t) = B_0 v(t) + w(t+1). \]

So, in order to study the \( p \times p \) left upper block of \( \hat{\Sigma} \) which is the Gram matrix of the output signal, we write

\[
\hat{\Sigma}_x = \sum_{t=0}^{n-1} x(t)x(t)' = x(0)x(0)' + \sum_{t=0}^{n-2} x(t+1)x(t+1)'
\]

\[ = \Sigma_x + x(0)x(0)' + \sum_{t=0}^{n-2} [M_t + D_t x(t)x(t)'D_t'], \]

where \( \Sigma_x = \sum_{t=0}^{n-2} \mathbb{E} [z(t)z(t)'] \), and

\[
M_t = z(t)z(t)' - \mathbb{E} [z(t)z(t)'] + D_t x(t)z(t)' + z(t)x(t)'D_t'.
\]

Then, \( M_t \) is a martingale difference sequence that according to Definition 8 is bounded:

\[ |\lambda_{\max} (M_t)| \leq \sqrt{n} \left( \| B_0 \| \sqrt{T_l} + \sqrt{T_{l+1}} \right). \]

So, using \( \mathcal{E}_1 \) in Definition 8, Azuma’s Matrix Inequality [30] implies that

\[
\mathbb{P} \left( \lambda_{\max} \left( \sum_{t=0}^{n-2} M_t \right) > 2^{-1/2} |\lambda_{\min} (\Sigma_x)| \right) \leq 2p \exp \left( 16^{-1} \mathcal{E}_1^{-2} |\lambda_{\min} (\Sigma_x)|^2 \right) \leq 4^{-1} \delta,
\]

where in the last inequality above we used (9) and \( |\lambda_{\min} (\Sigma_x)| \geq |\lambda_{\min} (\Sigma_w)| \). Therefore, positive semidefiniteness of \( x(0)x(0)' + \sum_{t=0}^{n-2} D_t x(t)x(t)'D_t' \) leads to

\[
\mathbb{P} \left( \lambda_{\min} (\hat{\Sigma}_x) \right) < 0.29 |\lambda_{\min} (\Sigma_w)| \leq 4^{-1} \delta. \tag{23}
\]

To proceed, note that letting

\[
\tilde{L}_t = \begin{bmatrix} I_p \\ L_t \end{bmatrix}, \quad \tilde{v}(t) = \begin{bmatrix} 0_p \\ v(t) \end{bmatrix},
\]

we have

\[
\hat{\Sigma} = \Sigma_\tilde{v} + \sum_{t=0}^{n-1} \left( \tilde{L}_t x(t)x(t)' \tilde{L}_t' + N_t \right),
\]

where \( \Sigma_\tilde{v} = \sum_{t=0}^{n-1} \mathbb{E} [\tilde{v}(t) \tilde{v}(t)'], \) and

\[
N_t = \tilde{L}_t x(t)\tilde{v}(t)' + \tilde{v}(t)x(t)' \tilde{L}_t' + \tilde{v}(t)\tilde{v}(t)' - \mathbb{E} [\tilde{v}(t) \tilde{v}(t)'].
\]

We show that

\[
\mathbb{P} \left( \lambda_{\max} \left( \sum_{t=0}^{n-1} N_t \right) \right) \geq 2^{-1/2} |\lambda_{\min} (\Sigma_\tilde{v})| \leq 4^{-1} \delta. \tag{24}
\]

For this purpose, we leverage Azuma’s Matrix Inequality [30] as follows. The martingale difference sequence \( N_t \) is bounded:

\[
|\lambda_{\max} (N_t)| \leq \sqrt{n} (2 |L (\theta_0)| \sqrt{T_l} + 2T_l + T_{l+1}).
\]

According to Definition 8, (10) leads to (24). Next, note that the upper \( p \times p \) block of \( \tilde{L}_t \) is the identity matrix \( I_p \), and the upper \( p \times 1 \) block of \( \tilde{v}(t) \) is the zero vector \( 0_p \). Hence, putting (23), (24) together, we get the following for the quantity \( \lambda_n \) defined in Definition 8:

\[
\mathbb{P} \left( \lambda_{\min} (\hat{\Sigma}) \right) < 0.29 \lambda_n < 2^{-1} \delta. \tag{25}
\]

Going back to (22), we use the following non-asymptotic result for matrix valued martingales [29]. The asymptotic version can be found in [42].

**Lemma 15** [29] With probability at least \( 1 - 2^{-1} \delta \), the following holds for all \( n \geq 1 \):

\[
\left\| \left( \lambda_n I_q + \hat{\Sigma} \right)^{-1/2} \sum_{t=0}^{n-1} \frac{x(t)u(t)}{w(t+1)} \right\|^2 \leq pq \log |\lambda_{\max} (\lambda_n I_q + \hat{\Sigma})| - pq \log \lambda_n + 2p \log \left( 2p \delta^{-1} \right).
\]
Then, we clearly have \( \max_{1 \leq t \leq n} \overline{w_t} \leq \overline{w} \), as well as
\[
\left| \lambda_{\max}(\hat{\Sigma}) \right| \leq \sum_{t=0}^{n-1} \left( \|x(t)\|^2 + \|u(t)\|^2 \right)
\leq \left( 1 + 2\|L(\theta_0)\|^2 \right) \mathcal{E}_x + 2\mathcal{E}_v + 2\mathcal{E}_t.
\]
Moreover, for an arbitrary \( \theta \in \mathbb{R}^{p \times q} \), we have
\[
\|\theta \hat{\Sigma}^{-1} \theta'\| \leq 1.29 \|\theta (\lambda_n I_q + \hat{\Sigma})^{-1} \theta'\|,
\]
as long as \( |\lambda_{\min}(\hat{\Sigma})| \geq 0.29\lambda_n \). Therefore, plugging (25) and the result of Lemma 15 in equation (22), we have
\[
0.29\lambda_n \|\hat{n} - \theta_0\|^2 \\
\leq \left\| (\hat{n} - \theta_0) \hat{\Sigma} (\hat{n} - \theta_0)' \right\| \\
\leq 1.29 \left( \|\lambda_n I_q + \hat{\Sigma}\|^{-1/2} \right) \left( \|\hat{n} - \theta_0\|'' \right)^2 \\
\leq 2pq\overline{w^2} \log \left( 4p \left( 1 + \|L(\theta_0)\|^2 \right) \mathcal{E}_x + \mathcal{E}_t + \mathcal{E}_v \right) \delta^{-1},
\]
with probability at least \( 1 - \delta \), which is the desired result.

### 6.3 Proof of Theorem 11

First, suppose that \( \max_{1 \leq t \leq n} \|w(t)\| \leq \overline{w} \). To find the growth rate of the regret, according to Theorem 6 it suffices to determine \( \mathcal{E}_t, \mathcal{E}_v, \mathcal{E}_x \). Further, by (14), Theorem 9 implies that in order to determine the above quantities, one needs to address the behavior of \( \mathcal{E}_1, \mathcal{E}_2, \lambda_n, \mathcal{E}_x \).

Let \( \lambda_n \) be as defined in Definition 8. At the end of epoch \( n \) that corresponds to the time step \( n = \lfloor \gamma n \rfloor \), the adaptive policy \( \hat{\pi} \) updates the parameter estimates according to (8). Then, the second part of Assumption 10 and the design of perturbation in (12), lead to the following lower bound:
\[
\lambda_n \geq \sigma_0 \bigwedge \sum (\gamma - 1) (\log^{-2} \gamma) n^{1/2} \log^2 n. \tag{26}
\]
Since the adaptive policy \( \hat{\pi} \) is stabilizing the system, we have \( \overline{w_t} \leq \overline{w^2 \hat{\pi}} \), \( \overline{\ell_t} \leq \overline{\eta \overline{w^2 \hat{\pi}}} \), for some constants \( \eta, \eta < \infty \). Thus, since the perturbation signal \( v(t) \) is diminishing with the rate specified in (12), we have
\[
\mathcal{E}_1 \leq \overline{\eta \overline{w^2 \hat{\pi}}} n^{1/2},
\mathcal{E}_2 \leq \overline{\eta_2 w^2 \overline{n^{1/4}} \log n},
\]
for some \( \eta_1, \eta_2 < \infty \). Therefore, according to (26), both (9), (10) will be satisfied if
\[
-\eta_0 \log \delta \leq \overline{w^{2-2} n^{1/2} \log n} \vee \overline{w^{2-4} n}, \tag{27}
\]
where \( \eta_0 < \infty \) is a fixed constant determined by \( \overline{C_7, C_1, \sigma_0, \eta_1, \eta_2, \log q} \). Let \( n_0 \) be large enough such that (27) holds for all \( n \geq n_0 \).

Next, plugging (26) in Theorem 9, we obtain
\[
\sup_{n \geq n_0} \frac{n^{1/2} \log n \|\hat{n} - \theta_0\|^2}{\overline{w^2} \log \left( n \frac{\overline{w^2}}{\delta^{-1}} \right)} < \overline{C_4}, \tag{28}
\]
with probability at least \( 1 - \delta \). In addition, using (14), we get the following high probability result:
\[
\sup_{n \geq n_0} \frac{\left( n \log n \right)^{1/4} \|L(\theta_0) - L_n\|}{\overline{w^2} \log^{1/2} \left( n \frac{\overline{w^2}}{\delta^{-1}} \right)} < \infty. \tag{29}
\]
By Definition 5, (29) yields
\[
\sup_{n \geq n_0} \frac{\|E_t\|}{\left( n \log^{-2} n \right)^{1/2} \overline{w^2} \log \left( n \frac{\overline{w^2}}{\delta^{-1}} \right)} \leq C_t, \tag{30}
\]
\[
\sup_{n \geq n_0} \frac{\overline{E_v}}{\overline{w^2} \left( \mathcal{E}_t + \mathcal{E}_v \right)^{1/2}} \leq \overline{C_v}, \tag{31}
\]
where \( C_t, \overline{C_v} \) are fixed and finite.

Now, according to Assumption 10, with probability at least \( 1 - \delta \) we have [27]:
\[
\max_{1 \leq t \leq n} \|w(t)\| \leq \beta^{1/\alpha_0} \log^{1/\alpha_0} \left( \overline{3n} \delta^{-1} \right). \tag{32}
\]
Thus, substituting the above value for \( \overline{w^2} \) in (28), we get the desired result for the identification error. Moreover, since
\[
\mathcal{E}_v \leq \gamma \log^{-2} \gamma \overline{C_n^{1/2} \log^2 n}, \tag{30}, (31), \text{ and Theorem 6 imply the desired result for the regret bound. Finally, for a fixed constant } \overline{C_0}, \text{ letting }
\]
\[
n_0 \geq \overline{C_0} \left( \log^{4/\alpha_0} \delta^{-1} \right) \left( \log \log \delta^{-1} \right) \tag{33}
\]
is sufficient to satisfy (27).

### 6.4 Proof of Theorem 14

First, similar to the proof of Theorem 11, assume \( \max_{1 \leq t \leq n} \|w(t)\| \leq \overline{w} \). Let \( \hat{\theta}_{\gamma j} \) be the least squares esti-
mate over all parameter space \( \mathbb{R}^{p \times q} \) given by (8) for the input-output observations being collected until the end of epoch \( i \). Note that \( \theta_{[\gamma^i]} \) is not necessarily equal to the solution of (15) denoted by \( \hat{\theta}_{[\gamma^i]} \). Letting \( n = |\gamma^i| \), define the Gram matrix \( \Sigma \) according to (21). Then, \( \theta_0 \in \Gamma_0 \) implies that

\[
\text{tr} \left( \left( \hat{\theta}_{[\gamma^i]} - \theta_0 \right) \Sigma \left( \hat{\theta}_{[\gamma^i]} - \theta_0 \right)' \right) \leq 4\text{tr} \left( \left( \hat{\theta}_{[\gamma^i]} - \theta_0 \right) \Sigma \left( \hat{\theta}_{[\gamma^i]} - \theta_0 \right)' \right).
\]

(34)

Then, using (22), Lemma 15 leads to

\[
\left\| \Sigma^{1/2} \left( \hat{\theta}_{[\gamma^i]} - \theta_0 \right)' \right\|^2 \leq 4pqw^2 \log \frac{2p}{\delta} \left\| \lambda_{\max}(\Sigma) \right\|_2,$n

(35)

with probability at least \( 1 - \delta/2 \). Next, note that during each epoch the parameter estimates and so the feedback gains are fixed. So, let \( L_i \) be the feedback gain during the \( i \)-th epoch: \( L_i = L \left( \hat{\theta}_{[\gamma^i]} \right) \), and define \( \tilde{L}_i = [I_p, L_i]' \). Hence, the stable closed-loop dynamics takes the form

\[
x(t + 1) = D_i x(t) + B_0 v(t) + w(t + 1),
\]

where \( D_i = \theta_0 \tilde{L}_i \). Further, define

\[
V_i = \sum_{t=1}^{\gamma^i - 1} x(t)x(t)', \quad U_i = \tilde{L}_i V_i \tilde{L}_i' + W_i,
\]

where \( \tilde{v}(t) = [0_p, v(t)'] \), and

\[
W_i = \sum_{t=1}^{\gamma^i - 1} \left[ \tilde{L}_i x(t) \tilde{v}(t)' + \tilde{v}(t)x(t)' \tilde{L}_i' + \tilde{v}(t)\tilde{v}(t)' \right].
\]

Thus, (34), (35) yield

\[
\left\| U_i^{1/2} \left( \hat{\theta}_{[\gamma^i]} - \theta_0 \right)' \right\|^2 \leq 4p^2 qw^2 \log \left( 2p \left\| \lambda_{\max}(\Sigma) \right\|_2 \delta^{-1} \right).
\]

Hence, applying Cauchy-Schwarz inequality, and using the design of perturbation; \( \gamma \delta < C \), we obtain

\[
\left\| U_i^{1/2} \tilde{L}_i \left( \hat{\theta}_{[\gamma^i]} - \theta_0 \right)' \right\|^2 \leq \sigma_i^2 \left\| V_i^{1/2} \tilde{L}_i \left( \hat{\theta}_{[\gamma^i]} - \theta_0 \right)' \right\| \leq 4p^2 qw^2 \log \left( 2p \left\| \lambda_{\max}(\Sigma) \right\|_2 \delta^{-1} \right).}

Then, we use the following result.

**Lemma 16** [27] With probability at least \( 1 - \delta/2 \) we have:

\[
\frac{\sigma_0}{2} \leq \inf_{i \geq m_0} \left\| \lambda_{\min}(V_i) \right\| \leq \sup_{i \geq m_0} \left\| \lambda_{\max}(V_i) \right\| \leq \eta_0 w^2,
\]

for a constant \( \eta_0 < \infty \), and \( m_0 = \log (\eta_0 \sigma_0^{-1} w^2 \log \delta^{-1}) \).

By Lemma 16, and (13) we get

\[
\sup_{n \geq m_0} \left\| L_i (\hat{\theta}_n) - L(\theta_0) \right\|^2 \leq C < \infty,
\]

with probability at least \( 1 - \delta \). Finally, according to (32), Theorem 6 implies the desired result.

**References**

[1] T. Kailath, *Linear systems*. Prentice-Hall Englewood Cliffs, NJ, 1980, vol. 156.

[2] S. Meyn, *Control techniques for complex networks*. Cambridge University Press, 2008.

[3] M. K. S. Faradonbeh, A. Tewari, and G. Michaelidis, “Optimality of fast matching algorithms for random networks with applications to structural controllability,” *IEEE Transactions on Control of Network Systems*, vol. 4, no. 4, pp. 770–780, 2017.

[4] R. Marino and P. Tomei, *Nonlinear control design: geometric, adaptive and robust*. Prentice Hall London, 1995, vol. 1.

[5] C. Li and J. Lam, “Stabilization of discrete-time nonlinear uncertain systems by feedback based on ls algorithm,” *SIAM Journal on Control and Optimization*, vol. 51, no. 2, pp. 1128–1151, 2013.

[6] Y. Chen, M. U. Hashmi, J. Mathias, A. Busi, and S. Meyn, “Distributed control design for balancing the grid using flexible loads,” in *Energy Markets and Responsive Grids*. Springer, 2018, pp. 383–411.

[7] N. Lazic, T. Lu, C. Boutilier, M. Ryu, E. Wong, B. Roy, and G. Imwalle, “Data center cooling using model-predictive control.” *NIPS*, 2018.

[8] M. H. Pesaran and A. Timmermann, “Small sample properties of forecasts from autoregressive models under structural breaks,” *Journal of Econometrics*, vol. 129, no. 1-2, pp. 183–217, 2005.

[9] T. Söderström, *Discrete-time stochastic systems: estimation and control*. Springer Science & Business Media, 2012.

[10] M. K. S. Faradonbeh, A. Tewari, and G. Michaelidis, “Finite time identification in unstable linear systems,” *Automatica*, vol. 96, pp. 342–353, 2018.

[11] D. F. Bertsekas, *Dynamic programming and optimal control*. Athena Scientific Belmont, MA, 1995, vol. 1, no. 2.

[12] P. Dorato, C. T. Abdallah, V. Cerone, and D. H. Jacobson, *Linear-quadratic control: an introduction*. Prentice Hall Englewood Cliffs, NJ, 1995.

[13] L. Guo and H.-F. Chen, “The astrom-wittenmark self-tuning regulator revisited and els-based adaptive trackers,” *IEEE Transactions on Automatic Control*, vol. 36, no. 7, pp. 802–812, 1991.
[14] L. Guo, “Convergence and logarithm laws of self-tuning regulators,” *Automatica*, vol. 31, no. 3, pp. 435–450, 1995.

[15] T. Lai and C.-Z. Wei, “Extended least squares and their applications to adaptive control and prediction in linear systems,” *IEEE Transactions on Automatic Control*, vol. 31, no. 10, pp. 989–906, 1986.

[16] T. Lai, “Asymptotically efficient adaptive control in stochastic regression models,” *Advances in Applied Mathematics*, vol. 7, no. 1, pp. 23–45, 1986.

[17] L. Guo and H. Chen, “Convergence rate of els based adaptive tracker,” *Syst. Sci & Math. Sci.*, vol. 1, pp. 131–138, 1988.

[18] H.-F. Chen and J.-F. Zhang, “Convergence rates in stochastic adaptive tracking,” *International Journal of Control*, vol. 49, no. 6, pp. 1915–1935, 1989.

[19] P. Kumar, “Convergence of adaptive control schemes using least-squares parameter estimates,” *IEEE Transactions on Automatic Control*, vol. 35, no. 4, pp. 416–424, 1990.

[20] T. L. Lai and Z. Ying, “Parallel recursive algorithms in asymptotically efficient adaptive control of linear stochastic systems,” *SIAM Journal on control and optimization*, vol. 29, no. 5, pp. 1091–1127, 1991.

[21] B. Bercu, “Weighted estimation and tracking for armax models,” *SIAM Journal on Control and Optimization*, vol. 33, no. 1, pp. 89–106, 1995.

[22] T. L. Lai and H. Robbins, “Asymptotically efficient adaptive allocation rules,” *Advances in applied mathematics*, vol. 6, no. 1, pp. 4–22, 1985.

[23] M. C. Campi and P. Kumar, “Adaptive linear quadratic gaussian control: the cost-biased approach revisited,” *SIAM Journal on Control and Optimization*, vol. 36, no. 6, pp. 1890–1907, 1998.

[24] S. Bittanti and M. C. Campi, “Adaptive control of linear time invariant systems: the bet on the best principle,” *Communications in Information & Systems*, vol. 6, no. 4, pp. 299–320, 2006.

[25] Y. Abbasi-Yadkori and C. Szepesvári, “Regret bounds for the adaptive control of linear quadratic systems.” in *COLT*, 2011, pp. 1–26.

[26] M. Ibrahimi, A. Javanmard, and B. V. Roy, “Efficient reinforcement learning for high dimensional linear quadratic systems,” in *Advances in Neural Information Processing Systems*, 2012, pp. 2636–2644.

[27] M. K. S. Faradonbeh, A. Tewari, and G. Michailidis, “Regret analysis for adaptive linear-quadratic policies,” *arXiv preprint arXiv:1711.07230* 2017.

[28] ——, “On optimality of adaptive linear-quadratic regulators,” *arXiv preprint arXiv:1806.10749* 2018.

[29] Y. Abbasi-Yadkori, D. Pál, and C. Szepesvári, “Improved algorithms for linear stochastic bandits,” *Advances in Neural Information Processing Systems*, pp. 2312–2320, 2011.

[30] J. A. Tropp, “User-friendly tail bounds for sums of random matrices,” *Foundations of computational mathematics*, vol. 12, no. 4, pp. 389–434, 2012.

[31] O. C. Imer, S. Yüksel, and T. Başar, “Optimal control of Ilt systems over unreliable communication links,” *Automatica*, vol. 42, no. 9, pp. 1429–1439, 2006.

[32] Y. Ouyang, S. M. Ashgari, and A. Nayyar, “Optimal infinite horizon decentralized networked controllers with unreliable communication,” *arXiv preprint arXiv:1806.06497* 2018.

[33] M. K. S. Faradonbeh, A. Tewari, and G. Michailidis, “Finite time adaptive stabilization of LQ systems,” *arXiv preprint arXiv:1807.09120*. 2018.