LOWER SPECTRAL RADIUS AND SPECTRAL MAPPING THEOREM FOR SUPREMA PRESERVING MAPPINGS

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Abstract. We study Lipschitz, positively homogeneous and finite suprema preserving mappings defined on a max-cone of positive elements in a normed vector lattice. We prove that the lower spectral radius of such a mapping is always a minimum value of its approximate point spectrum. We apply this result to show that the spectral mapping theorem holds for the approximate point spectrum of such a mapping. By applying this spectral mapping theorem we obtain new inequalities for the Bonsall cone spectral radius of max type kernel operators.

1. Introduction. Max-type operators (and corresponding max-plus type operators and their tropical versions known also as Bellman operators) arise in a large field of problems from the theory of differential and difference equations, mathematical physics, optimal control problems, discrete mathematics, turnpike theory, mathematical economics, mathematical biology, games and controlled Markov processes, generalized solutions of the Hamilton-Jacobi-Bellman differential equations, continuously observed and controlled quantum systems, discrete and continuous dynamical systems, ... (see e.g. [28], [20], [27], [26], [4], [32] and the references cited there). The eigenproblem of such operators has so far received substantial attention due to its applicability in the above mentioned problems (see e.g. [28], [20], [4], [3], [32], [22], [29], [13], [31], [30], [2], [41] and the references cited there). However, there seems to be a lack of general treatment of spectral theory for such operators, even though the spectral theory for nonlinear operators on Banach spaces is already quite well developed (see e.g. [11], [10], [12], [16], [40], [32] and the references cited there). One of the reasons for this might lie in the fact that these operators behave

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nicely on a suitable subcone (or subsemimodule), but less nicely on the whole (Banach) space. Therefore it appears, that it is not trivial to directly apply this known non-linear spectral theory to obtain satisfactory information on a restriction to a given cone of a max-type operator. The Bonsall cone spectral radius plays the role of the spectral radius in this theory (see e.g. [28], [29], [32], [4], [21], [17], [31] and the references cited there).

In [32], we studied Lipschitz, positively homogeneous and finite suprema preserving mappings defined on a max-cone of positive elements in a normed vector lattice. We showed that for such a mapping the Bonsall cone spectral radius is the maximum value of its approximate point spectrum (see Theorem 2.1 below). We also proved that an analogue of this result holds also for Lipschitz, positively homogeneous and additive mappings defined on a normal convex cone in a normed space (see Theorem 2.2 below).

The current article may be considered as a continuation of [32]. It is organized as follows. In Section 2 we recall some definitions and results that we will use in the sequel. We show in Section 3 that the lower spectral radius of a mapping from both of the above described settings is a minimum value of its approximate point spectrum (Theorems 3.5 and 3.6). In Section 4 we apply this result to show that the maxpolynomial spectral mapping theorem holds for the approximate point spectrum of Lipschitz, positively homogeneous and finite suprema preserving mappings (Theorem 4.11). In the last section we use this spectral mapping theorem to prove some new inequalities for the Bonsall cone spectral radius of Hadamard products of max type kernel operators (Theorem 5.5).

2. Preliminaries. A subset $C$ of a real vector space $X$ is called a cone (with vertex 0) if $tC \subset C$ for all $t \geq 0$, where $tC = \{tx : x \in C\}$. A map $A : C \to C$ is called positively homogeneous (of degree 1) if $A(tx) = tA(x)$ for all $t \geq 0$ and $x \in C$. We say that the cone $C$ is pointed if $C \cap (-C) = \{0\}$.

A convex pointed cone $C$ of $X$ induces on $X$ a partial ordering $\leq$, which is defined by $x \leq y$ if and only if $y - x \in C$. In this case $C$ is denoted by $X_+$ and $X$ is called an ordered vector space. If, in addition, $X$ is a normed space then it is called an ordered normed space. If, in addition, the norm is complete, then $X$ is called an ordered Banach space.

A convex cone $C$ of $X$ is called a wedge. A wedge induces on $X$ (by the above relation) a vector preordering $\leq$ (which is reflexive, transitive, but not necessary antisymmetric). We say that the cone $C$ is proper if it is closed, convex and pointed. A cone $C$ of a normed space $X$ is called normal if there exists a constant $M$ such that $\|x\| \leq M\|y\|$ whenever $x \leq y$, $x, y \in C$. A convex and pointed cone $C = X_+$ of an ordered normed space $X$ is normal if and only if there exists an equivalent monotone norm $\|\cdot\|$ on $X$, i.e., $\|x\| \leq \|y\|$ whenever $0 \leq x \leq y$ (see e.g. [9, Theorem 2.38]). Every proper cone $C$ in a finite dimensional Banach space is necessarily normal.

If $X$ is a normed linear space, then a cone $C$ in $X$ is said to be complete if it is a complete metric space in the topology induced by $X$. In the case when $X$ is a Banach space this is equivalent to $C$ being closed in $X$.

If $X$ is an ordered vector space, then a cone $C \subset X_+$ is called a max-cone if for every pair $x, y \in C$ there exists a supremum $x \lor y$ (least upper bound) in $C$. We consider here on $C$ an order inherited from $X_+$. A map $A : C \to C$ preserves finite suprema on $C$ if $A(x \lor y) = Ax \lor Ay$ ($x, y \in C$). If $A : C \to C$ preserves finite
suprema, then it is monotone (order preserving) on \( C \), i.e., \( Ax \leq Ay \) whenever \( x \leq y, x, y \in C \).

An ordered vector space \( X \) is called a vector lattice (or a Riesz space) if every two vectors \( x, y \in X \) have a supremum and infimum (greatest lower bound) in \( X \). A positive cone \( X_+ \) of a vector lattice \( X \) is called a lattice cone.

Note that by [9, Corollary 1.18] a pointed convex cone \( C = X_+ \) of an ordered vector space \( X \) is the lattice cone for the vector subspace \( C - C \) generated by \( C \) in \( X \), if and only if \( C \) is a max cone (in this case a supremum of \( x, y \in C \) exists in \( C \) if only if it exists in \( X \); and suprema coincide).

If \( X \) is a vector lattice, then the absolute value of \( x \in X \) is defined by \( |x| = x \vee (-x) \). A vector lattice and normed vector space is called a normed vector lattice (a normed Riesz space) if \( |x| \leq |y| \) implies \( \|x\| \leq \|y\| \). A complete normed vector lattice is called a Banach lattice. A positive cone \( X_+ \) of a normed vector lattice \( X \) is proper and normal.

In a vector lattice \( X \) the following Birkhoff’s inequality for \( x_1, \ldots, x_n, y_1, \ldots, y_n \in X \) holds:

\[
\left| \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} y_j \right| \leq \sum_{j=1}^{n} |x_j - y_j|.
\]

For the theory of vector lattices, Banach lattices, cones, wedges, operators on cones and applications e.g. in financial mathematics we refer the reader to [1], [9], [7], [42], [6], [23], [18], [32], [24], [5] and the references cited there.

Let \( X \) be a normed space and \( C \subset X \) a non-zero cone. Let \( A : C \to C \) be positively homogeneous and bounded, i.e.,

\[
\|A\| := \sup \left\{ \frac{\|Ax\|}{\|x\|} : x \in C, x \neq 0 \right\} < \infty.
\]

It is easy to see that \( \|A\| = \sup \{\|Ax\| : x \in C, \|x\| \leq 1\} \) and \( \|A^{m+n}\| \leq \|A^m\| \cdot \|A^n\| \) for all \( m, n \in \mathbb{N} \). It is well known that this implies that the limit \( \lim_{n \to \infty} \|A^n\|^{1/n} \) exists and is equal to \( \inf_n \|A^n\|^{1/n} \). The limit \( r(A) := \lim_{n \to \infty} \|A^n\|^{1/n} \) is called the Bonsall cone spectral radius of \( A \). The approximate point spectrum \( \sigma_{ap}(A) \) of \( A \) is defined as the set of all \( s \geq 0 \) such that \( \inf \{\|Ax - sx\| : x \in C, \|x\| = 1\} = 0 \). The (distinguished) point spectrum \( \sigma_p(A) \) of \( A \) is defined by

\[
\sigma_p(A) = \left\{ s \geq 0 : \text{there exists } x \in C, x \neq 0 \text{ with } Ax = sx \right\}.
\]

For \( x \in C \) define the local cone spectral radius by \( r_x(A) := \limsup_{n \to \infty} \|A^n x\|^{1/n} \).
Clearly \( r_x(A) \leq r(A) \) for all \( x \in C \). It is known that the equality

\[
\sup \{r_x(A) : x \in C\} = r(A)
\]

is not valid in general. In [28] there is an example of a proper cone \( C \) in a Banach space \( X \) and a positively homogeneous and continuous (hence bounded) map \( A : C \to C \) such that \( \sup r_x(A) : x \in C \) < \( r(A) \). A recent example of such kind, where \( A \) is in addition monotone, is obtained in [17, Example 3.1]. However, if \( C \) is a normal, complete, convex and pointed cone in a normed space \( X \) and \( A : C \to C \) is positively homogeneous, monotone and continuous, then [29, Theorem 3.3], [28, Theorem 2.2] and [17, Theorem 2.1] ensure that (2) is valid.

If \( X \) is a Banach lattice, \( C \subset X_+ \) a max-cone and \( A : C \to C \) a mapping which is bounded, positively homogeneous and preserves finite suprema, then the equality (2) is not necessary valid as shown in [32]. Some additional examples of maps for which (2) is not valid can be found in [17].
Let $C$ be a cone in a normed space $X$ and $A : C \to C$. Then $A$ is called Lipschitz if there exists $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in C$.

The following two results were the main results of [32] (see [32, Theorem 3.6 and Corollaries 1 and 2] and [32, Theorem 4.2 and Corollary 4]).

**Theorem 2.1.** Let $X$ be a normed vector lattice, let $C \subset X_+$ be a non-zero max-cone. Let $A : C \to C$ be a mapping which is bounded, positively homogeneous and preserves finite suprema. Let $C' \subset C$ be a bounded subset satisfying $\|A^n\| = \sup\{|A^n x| : x \in C'\}$ for all $n$. Then

$$\sup\{r_x(A) : x \in C'\}, r(A) \subseteq \sigma_{ap}(A).$$

In particular, $r(A) \in \sigma_{ap}(A)$. Moreover, $r_x(A) \in \sigma_{ap}(A)$ for each $x \in C$.

If, in addition, $A$ is a Lipschitz, then $r(A) = \max\{t : t \in \sigma_{ap}(A)\}$.

**Theorem 2.2.** Let $X$ be a normed space, $C \subset X$ a non-zero normal wedge and let $A : C \to C$ be positively homogeneous, additive and Lipschitz. Let $C' \subset C$ be a bounded subset satisfying $\|A^n\| = \sup\{|A^n x| : x \in C'\}$ for all $n$. Then

$$\sup\{r_x(A) : x \in C'\}, r(A) \subseteq \sigma_{ap}(A).$$

In particular, $r(A) = \max\{t : t \in \sigma_{ap}(A)\}$.

Moreover, $r_x(A) \in \sigma_{ap}(A)$ for each $x \in C, x \neq 0$.

3. **Lower spectral radius and approximate point spectrum.** Let $X$ be a normed space and $C \subset X$ a non-zero cone. Let $A : C \to C$ be positively homogeneous and bounded mapping. Let $m(A) := \inf\{|Ax| : x \in C, \|x\| = 1\}$ be the minimum modulus of $A$ (note that in [11], $m(A)$ is called the inner norm of $A$ in the case when $C = X$). Observe that $m(A) = 0$ if and only if $0 \in \sigma_{ap}(A)$. It is easy to see that $m(A^{n+m}) \geq m(A^n)m(A^m)$ for all $m, n \in \mathbb{N}$. It is well known that this implies that the limit $\lim_{n \to \infty} m(A^n)^{1/n}$ exists and is equal to the supremum $\sup_n m(A^n)^{1/n}$. Let

$$d(A) = \lim_{n \to \infty} m(A^n)^{1/n}$$

be the "lower spectral radius" of $A$.

If $A : C \to C$ is bijective and $A^{-1}$ is bounded, then $m(A) = \|A^{-1}\|^{-1}$ and $d(A) = r(A^{-1})^{-1}$. If $X, C$ and $A^{-1}$ satisfy the assumptions of Theorem 2.1 or of Theorem 2.2, then $r(A^{-1}) \subseteq \sigma_{ap}(A^{-1})$ and $\sigma_{ap}(A) = \{t^{-1} : t \in \sigma_{ap}(A^{-1})\}$ and thus also $d(A) \in \sigma_{ap}(A)$. We show below in Theorems 3.5 and 3.6 that this is true, not only for invertible mappings $A$, but also under similar assumptions as in Theorems 2.1 and 2.2.

First we observe the following result.

**Proposition 3.1.** Let $X$ be a normed space and $C \subset X$ a non-zero cone. If $A : C \to C$ is positively homogeneous and bounded, then

$$d(A) \leq \inf\{r_x(A) : x \in C, x \neq 0\} \leq \sup\{r_x(A) : x \in C, x \neq 0\} \leq r(A). \quad (3)$$

If, in addition, $A$ is Lipschitz, then $\sigma_{ap}(A) \subseteq [d(A), r(A)]$.

**Proof.** If $x \in C$, $\|x\| = 1$ then $m(A^j) \leq \|A^j x\| \leq \|A^j\|$ for all $j \in \mathbb{N}$. So $d(A) \leq r_x(A) \leq r(A)$, which proves (3).

If, in addition, $A$ is Lipschitz and $t \in \sigma_{ap}(A)$ then one can see easily that $m(A) \leq t \leq \|A\|$ and $t^n \in \sigma_{ap}(A^n)$ for all $n \in \mathbb{N}$ (see also the proof of [32, Lemma 3.3]). Since $m(A^n) \leq t^n \leq \|A^n\|$, it follows that $d(A) \leq t \leq r(A)$, which completes the proof. ☐
The following example shows that \( \sigma_{ap}(A) \) may not contain the whole interval \( \left[ d(A), \inf\{r_x(A) : x \in C, x \neq 0\} \right] \).

**Example 3.2.** Let \( X = \ell^\infty \) with the standard basis \( e_{n,k} \) \( (n, k \in \mathbb{N}) \). More precisely, the elements of \( X \) are formal sums \( x = \sum_{n,k \in \mathbb{N}} \alpha_{n,k}e_{n,k} \) with real coefficient \( \alpha_{n,k} \) such that 
\[
\|x\| := \sup\{|\alpha_{n,k}| : n, k \in \mathbb{N}\} < \infty.
\]
Then \( X \) is a Banach lattice with the natural order. Let \( C = X^+ \) and let \( A : C \to C \) be defined by \( Ae_{n,1} = n^{-1}e_{n,2} \), \( Ae_{n,k} = e_{n,k+1} \) \( (k \geq 2) \). More precisely, 
\[
A\left(\sum_{n,k \in \mathbb{N}} \alpha_{n,k}e_{n,k}\right) = \sum_{n \in \mathbb{N}} \left(\alpha_{n,1}n^{-1}e_{n,2} + \sum_{k=2}^{\infty} \alpha_{n,k}e_{n,k+1}\right).
\]
Then \( A \) is positively homogeneous, additive, Lipschitz mapping that preserves finite suprema, such that \( d(A) = 0 \) and \( r_x(A) = 1 \) for all nonzero \( x \in C \). Moreover, \( \sigma_{ap}(A) = \{0,1\} \) and so \( \sigma_{ap}(A) \) does not contain the whole interval \( \left[ d(A), \inf\{r_x(A) : x \in C, x \neq 0\} \right] \).

By Theorem 2.1, \( r(A) = \max\{t : t \in \sigma_{ap}(A)\} \) if \( X \) is a normed vector lattice, \( C \subset X^+ \) a non-zero max-cone and \( T : C \to C \) a mapping, which is Lipschitz, positively homogeneous and preserves finite suprema. We show below in Theorem 3.5 that under these assumptions we also have that \( d(A) = \min\{t : t \in \sigma_{ap}(A)\} \). In the proof we will need the following lemmas ([32, Lemma 3.1, Lemma 3.2]). The first one is based on the inequality (1).

**Lemma 3.3.** Let \( X \) be a normed vector lattice and let \( x_1, \ldots, x_n, y_1, \ldots, y_n \in X \). Then 
\[
\left\| \bigvee_{j=1}^{n} x_j - \bigvee_{j=1}^{n} y_j \right\| \leq \sum_{j=1}^{n} \|x_j - y_j\|.
\]

**Lemma 3.4.** Let \( X \) be a vector lattice and \( x_j, y_j \in X \) for \( j = 1, \ldots, n \). Then 
\[
\bigvee_{j=1}^{n} x_j - \bigvee_{j=1}^{n} y_j \leq \bigvee_{j=1}^{n} (x_j - y_j).
\](4)

If, in addition, \( X \) is a normed vector lattice and \( x_j \geq y_j \geq 0 \) for \( j = 1, \ldots, n \), then 
\[
\left\| \bigvee_{j=1}^{n} x_j - \bigvee_{j=1}^{n} y_j \right\| \leq \bigvee_{j=1}^{n} (x_j - y_j)\|. \quad (5)
\]

The following result is one of the main results of this section.

**Theorem 3.5.** Let \( X \) be a normed vector lattice, let \( C \subset X^+ \) be a non-zero max-cone. Let \( A : C \to C \) be a mapping which is bounded, positively homogeneous and preserves finite suprema. Then \( d(A) \in \sigma_{ap}(A) \).

If, in addition, \( A \) is Lipschitz, then \( d(A) = \min\{t : t \in \sigma_{ap}(A)\} \).

**Proof.** If \( d(A) = 0 \) then \( m(A) = 0 \) and \( 0 \in \sigma_{ap}(A) \).

Let \( d(A) > 0 \). Without loss of generality we may assume that \( d(A) = 1 \).

Let \( \varepsilon > 0 \). We show that there exists \( w \in C, w \neq 0 \) such that \( \frac{\|Aw - w\|}{\|w\|} \leq \varepsilon \).

Let \( n \in \mathbb{N} \) satisfy \( n > \max\{2, 16\varepsilon^{-2}\} \) and \( m(A^n) > 0 \). Find \( \delta > 0 \) such that \((1 - \delta)^n \geq \frac{1}{2}\) and \((1 + \delta)^n \leq 2\). Find \( N_1 \in \mathbb{N}, n|N_1, N_1 \geq 2n\) and 
\[(1 - \delta)^N < m(A^N) < (1 + \delta)^N\]
for all $N \geq N_1$. Find $s \in \mathbb{N}$, $s \geq 16$ with $m(A^s) > 2^{-s/4}$ and let $N = sN_1$.

Find $x \in C$, $\|x\| = 1$ such that $\|A^N x\| < (1 + \delta)^N \leq 2^{N/n}$. We also have $\|A^N x\| > m(A^N) > (1 - \delta)^N \geq 2^{-N/n}$.

Consider the vectors $x, A^nx, A^{2n}x, \ldots, A^N x$. Write for short $a_j = \|A^j x\|$ for $j = 0, \ldots, N/n$.

**Claim.** There exists $m, 1 \leq m < N/n$ with

$$a_m \geq \frac{1}{4} \max\{a_{m-1}, a_{m+1}\}.$$

Suppose on the contrary that $4a_m < \max\{a_{m-1}, a_{m+1}\}$ for all $m = 1, \ldots, N/n - 1$.

If $a_{j+1} \geq a_j$ for some $j \leq \frac{N}{n} - 2$, then this condition for $j+1$ means that $a_{j+2} > 4a_{j+1}$.

By induction, we get

$$a_j \leq a_{j+1} \leq a_{j+2} \leq \cdots \leq a_{N/n}.$$

Similarly, if $a_{j+1} \leq a_j$ for some $j \geq 1$ then $a_{j-1} > 4a_j$.

By induction, we get

$$a_{j+1} \leq a_j \leq a_{j-1} \leq \cdots \leq a_0 = 1.$$

So there exists $k, 0 \leq k \leq N/n$ such that

$$a_0 \geq a_1 \geq \cdots \geq a_{k-1} \geq a_k \leq a_{k+1} \leq \cdots \leq a_{N/n}.$$

Moreover,

$$1 = a_0 \geq 4a_1 \geq 4^2a_2 \geq \cdots \geq 4^{k-1}a_{k-1}\geq 4^{k-1}a_{k+1}.$$

If $k \geq \frac{N}{n} + 1$ then

$$a_{k-1} = \|A^{(k-1)n}x\| \geq m(A^{(k-1)n}) > (1 - \delta)^{(k-1)n} \geq \frac{1}{2^{x-1}},$$

a contradiction with the estimate $a_{k-1} < \frac{1}{2^{x-1}}$.

If $k \leq \frac{N}{n} + 1$ then $k+1 \leq \frac{4N}{n} - 1 = \frac{2N}{s}$. We have

$$a_{N/n} \geq a_{k+1} \cdot 4^{\frac{N}{n} - k-1} \geq (m(A^n))^{k+1} \cdot 4^{\frac{N}{n} - k-1} \geq 2^{-\frac{2N}{s}} \cdot 4^{\frac{N}{n} - \frac{2N}{s}} \geq 2^{-\frac{N}{s}} \cdot 2^{\frac{N}{n} - \frac{2N}{s}} \geq 2^{\frac{N}{s}} (\frac{2}{4} - 1) = 2^{\frac{N}{s}}.$$

However, $a_{N/n} = \|A^N x\| < (1 + \delta)^N \leq 2^{N/n}$, a contradiction.

**Continuation of the Proof of Theorem 3.5.** Let $m, 1 \leq m < N/n$ satisfy $\|A^m x\| \geq \frac{1}{4} \max\{\|A^{(m-1)n} x\|, \|A^{(m+1)n} x\|\}$.

Let $y = \sqrt{(m+1)^n - 1} A^j y$. Then $Ay = \sqrt{(m+1)^n}$ and by Lemma 3.3 we have $\|Ay - y\| \leq \|A^{(m-1)n} x\| + \|A^{(m+1)n} x\| \leq 8\|A^m x\|$. If $\|y\| > 8\|A^m x\|$ then $\|Ay - y\| / \|y\| \leq \epsilon$ and we are done. So assume that $\|y\| < 8\|A^m x\|$.

Let

$$u = \left(\frac{1}{n} A^{(m-1)n} x \lor \frac{2}{n} A^{(m-1)n+1} x \lor \cdots \lor \frac{n-1}{n} A^{mn-2} x \lor A^{mn-1} x\right) \lor \left(\frac{n-1}{n} A^{mn} x \lor \cdots \lor \frac{2}{n} A^{(m+1)n-3} x \lor A^{(m+1)n-2} x\right).$$
We have \( \|u\| \geq \frac{n-1}{n} \|A^{mn}x\| \geq \frac{1}{2} \|A^{mn}x\| \). Furthermore, by Lemma 3.4
\[
Au - u \leq \frac{1}{n} \left( A^{(m-1)n}x \lor A^{(m-1)n+1}x \lor \cdots \lor A^{(m+1)n-1}x \right) = \frac{y}{n},
\]
and similarly, \( u - Au \leq \frac{y}{n}. \) Hence \( \|Au - u\| \leq n^{-1} \|y\| < 8n^{-1} \|A^{mn}x\| \) and
\[
\frac{\|Au - u\|}{\|u\|} \leq \frac{16}{n \varepsilon} < \varepsilon.
\]
Since \( \varepsilon > 0 \) was arbitrary, \( 1 \in \sigma_{ap}(A) \).

If, in addition, \( A \) is Lipschitz, then \( d(A) = \min\{t : t \in \sigma_{ap}(A)\} \) by Proposition 3.1.

By replacing \( \lor \) with + in the proof above and by suitably adjusting some estimates the following theorem also follows. We include some details of the proof for the sake of clarity.

**Theorem 3.6.** Let \( X \) be a normed space, \( C \subset X \) a non-zero normal wedge and let \( A : C \rightarrow C \) be positively homogeneous, additive and bounded. Then \( d(A) \in \sigma_{ap}(A) \).

If, in addition, \( A \) is Lipschitz, then \( d(A) = \min\{t : t \in \sigma_{ap}(A)\} \).

**Proof.** As in the proof of Theorem 3.5 we may assume that \( d(A) = 1 \). Let \( \varepsilon > 0 \) and let \( n \in \mathbb{N} \) satisfy \( n > \max\{2, 32M\varepsilon^{-2}\} \) and \( m(a^n) > 0 \), where \( M \) is the normality constant of \( C \). If \( N \in \mathbb{N} \) is chosen as in the proof of Theorem 3.5, then it follows in the same way that there exists \( m \in \mathbb{N}, 1 \leq m < N/n \) such that
\[
\|A^{mn}x\| \geq \frac{1}{4} \max\{\|A^{(m-1)n}x\|, \|A^{(m+1)n}x\|\}.
\]
Let \( y = \sum_{j=(m-1)n}^{(m+1)n-1} A^jx \). Then \( Ay - y = A^{(m+1)n}x - A^{(m-1)n}x \) and so \( \|Ay - y\| \leq 8\|A^{mn}x\| \). Without loss of generality we may assume that \( \|y\| < 8\varepsilon^{-1}\|A^{mn}x\| \). Let
\[
u = \left( \frac{1}{n} A^{(m-1)n}x + \frac{2}{n} A^{(m-1)n+1}x + \cdots + \frac{n-1}{n} A^{(m+1)n-2}x + A^{(m+1)n-1}x \right)
\]
\[
+ \left( \frac{n-1}{n} A^{(m+1)n}x + \cdots + \frac{2}{n} A^{(m+1)n-3}x + \frac{1}{n} A^{(m+1)n}x \right).
\]

Then \( 2\|u\| \geq \|A^{mn}x\| \) and
\[
\|Au - u\| = \frac{1}{n} \left\| A^{(m+1)n-1}x + \cdots + A^{mn}x - \left( A^{(m-1)n-1}x + \cdots + A^{(m-1)n}\right) \right\|
\]
\[
\leq \frac{1}{n} \left( \|A^{(m+1)n-1}x + \cdots + A^{mn}x\| + \|A^{(m-1)n-1}x + \cdots + A^{(m-1)n}\| \right) \leq \frac{2M\|y\|}{n}
\]
\[
< \frac{16M}{n\varepsilon} \|A^{mn}x\| \leq \frac{32M}{n\varepsilon} \|u\| < \varepsilon \|u\|.
\]
Since \( \varepsilon > 0 \) was arbitrary, \( 1 \in \sigma_{ap}(A) \).

There are interesting non-trivial examples of operators to which Theorem 3.6 applies. In particular, it applies to the \( C \)-linear Perron-Frobenius operators from [29, Section 5] and [34, Sections 5 and 6].

**Remark 3.7.** In [32, Example 2] there is an example of a Banach lattice \( X \) and a closed max-cone \( C \subset X_+ \), which is normal and convex (a normal wedge), and a positively homogeneous, additive mapping \( A : C \rightarrow C \) that preserves all suprema and satisfies \( \|A\| = 1 \). However, \( A \) is not Lipschitz, \( d(A) = r(A) = 0 \) and \( \{0, 1\} \subset \sigma_{ap}(A) \). So, as pointed out in [32], the Lipschitzity of \( A \) is necessary for the property \( r(A) = \max\{t : t \in \sigma_{ap}(A)\} \) to hold.
The following example shows that the Lipschitz property of $A$ is necessary for the property $d(A) = \min\{t : t \in \sigma_p(A)\}$ to hold in Theorems 3.5 and 3.6.

**Example 3.8.** Let $X$ be as in Example 3.2. Let

$$C = \left\{ \sum_{n,k \in \mathbb{N}} \alpha_{n,k} e_{n,k} : \alpha_{n,k} \geq 0, \alpha_{n,2} \leq n^{-1}\alpha_{n,1} \text{ for all } n, k \in \mathbb{N} \right\}.$$ 

Then $C$ is a max-cone (moreover $C$ is a convex normal cone). Let $A : C \to C$ be defined by

$$A\left( \sum_{n,k \in \mathbb{N}} \alpha_{n,k} e_{n,k} \right) = \sum_{n \in \mathbb{N}} \left( \alpha_{n,1} e_{n,1} + n^{-1}\alpha_{n,1} e_{n,2} + 2n\alpha_{n,2} e_{n,3} + \sum_{k=3}^{\infty} 2\alpha_{n,k} e_{n,k+1} \right).$$

Clearly $A$ is bounded, $\|A\| = 2$ and $A$ is a positively homogeneous, additive mapping on $C$ that preserves finite suprema. We have $\|Ae_{n,1} - e_{n,1}\| = \|n^{-1}e_{n,2}\| = n^{-1}$ and $\|A^2e_{n,1} - Ae_{n,1}\| = \|2e_{n,3}\| = 2$ for all $n$, so $1 \in \sigma_p(A)$ and $A$ is not Lipschitz. On the other hand, it is easy to see that $d(A) = 2$.

4. Spectral mapping theorems for point and approximate point spectrum. In this section we generalize the spectral mapping theorem in max-algebra (see [31, Theorem 3.4] and also [19, Theorem 3.6]) to the infinite dimensional setting. The main results of this section are Corollary 4.3, Theorem 4.6 and Theorem 4.11.

Let $X$ be a Riesz space (i.e., a vector lattice). Let $C \subset X_+$ be a nonzero max-cone and $A : C \to C$ a positively homogeneous mapping that preserves finite suprema. Let $P_+$ denote the set of all polynomials $q(z) = \sum_{j=0}^{n} \alpha_j z^j \in P_+$ with $\alpha_j \geq 0$ for all $j$. For $q \in P_+$ and $t \geq 0$ let $q_v(t) = \max_j \alpha_j t^j$ (i.e., $q_v$ is the maxpolynomial corresponding to $q$). Define $q_v(A) : C \to C$ by

$$q_v(A)x = \bigvee_{j=1}^{n} \alpha_j A^j x \quad (x \in C).$$

**Lemma 4.1.** Let $X$ be a vector lattice and let $C \subset X_+$ be a nonzero max-cone. Let $A : C \to C$ be a positively homogeneous mapping that preserves finite suprema and $q = \sum_{j=0}^{n} \alpha_j z^j \in P_+$. Then

$$q_v(\sigma_p(A)) \subset \sigma_p(q_v(A)).$$

**Proof.** Let $t \geq 0$, $x \in C$ and $Ax = tx$. Then $A^j x = t^j x$ for all $j \geq 0$, and so $q_v(A)x = q_v(t)x$. Hence $q_v(\sigma_p(A)) \subset \sigma_p(q_v(A))$. \[\square\]

**Lemma 4.2.** Let $X$ be a vector lattice and let $C \subset X_+$ be a nonzero max-cone. Let $A : C \to C$ be a positively homogeneous, finite suprema preserving mapping. Assume that $q(z) = \sum_{j=1}^{n} \alpha_j z^j \in P_+$, $q_v(1) = 1$ and $1 \in \sigma_p(q_v(A))$. Then $1 \in \sigma_p(A)$.

**Proof.** Since $q_v(1) = 1$, we have $\alpha_j \leq 1$ for all $j$ and there exists $m, 1 \leq m \leq n$ with $\alpha_m = 1$.

Let $x \in C$ be a nonzero vector satisfying $q_v(A)x = x$. Set $y = \bigvee_{j=0}^{m-1} A^j x$. We have $A^m x \leq q_v(A)x = x$. Consequently, $A^{m+j} x \leq A^j x$ for all $j \in \mathbb{N}$. Thus...
\[ y = \sqrt[n-1]{A^j x} \]. We have \( Ay = \bigvee_{j=1}^n A^j x \geq q_\nu(A)x = x \). Thus
\[
Ay = x \vee \bigvee_{j=1}^m A^j x \geq \bigvee_{j=0}^{m-1} A^j x = y.
\]
Conversely, \( A^m x \leq x \leq y \) and
\[
Ay = \bigvee_{j=1}^{m-1} A^j x \vee A^m x \leq y.
\]
Hence \( Ay = y \) and \( 1 \in \sigma_p(A) \). \( \Box \)

**Corollary 4.3.** Let \( X \) be a vector lattice and let \( C \subseteq X_+ \) be a nonzero max-cone. Let \( A : C \rightarrow C \) be a positively homogeneous, finite suprema preserving mapping and \( q = \sum_{j=1}^n \alpha_j z^j \in \mathcal{P}_+ \) a non-zero polynomial. Then
\[
\sigma_p(q_\nu(A)) = q_\nu(\sigma_p(A)).
\]

**Proof.** The inclusion \( \supseteq \) was proved above.
\[
\subset: \text{Let } t \geq 0, s = q_\nu(t) \text{ and } s \in \sigma_p(q_\nu(A)). \text{ If } t = 0 = s, x \in C, x \neq 0 \text{ and } q_\nu(A)x = 0, \text{ then there exists } m, 1 \leq m \leq n \text{ with } \alpha_m \neq 0. \text{ So } \alpha_mA^m x = 0 \text{ and } A^mx = 0. \text{ Find } k, 1 \leq k \leq m - 1 \text{ with } A^k x \neq 0 \text{ and } A^{k+1} x = 0. \text{ Then } A^k x \text{ is a nonzero eigenvector and } 0 \in \sigma_p(A).
\]
Let \( s = q_\nu(t) \neq 0. \text{ Then } t \neq 0. \text{ Consider the mapping } A' = t^{-1}A \text{ and polynomial } p(z) = s^{-1}q(tz) = \sum_{j=1}^n \alpha_j t^j z^j. \text{ Then } p_\nu(1) = 1. \text{ Let } x \in C \text{ be a nonzero vector satisfying } q_\nu(A)x = sx. \text{ Then }
\[
p_\nu(A') x = \bigvee_{j=1}^n \frac{\alpha_j t^j}{s} \left( \frac{A}{t} \right)^j x = x.
\]
By Lemma 4.2 there exists \( y \in C, y \neq 0 \text{ and } A'y = y. \text{ Hence } Ay = ty \) and \( t \in \sigma_p(A). \)

**Lemma 4.4.** Let \( X \) be a vector lattice, \( x, y \in X_+, s > 1 \text{ and } x \vee y = sx. \) Then \( y = sx. \)

**Proof.** Let \( k \) be the smallest integer satisfying \( k(s-1) \geq 1. \) We prove by induction that \( y \geq j(s-1)x \) for \( j = 1, \ldots, k. \)
\[
\text{We have } x + y \geq x \vee y = sx, \text{ and so } y \geq (s-1)x. \text{ Let } 1 \leq j \leq k - 1 \text{ and suppose that } y \geq j(s-1)x. \text{ Since } x \vee y = sx \text{, we have }
\]
\[
(x - j(s-1)x) \vee (y - j(s-1)x) = sx - j(s-1)x,
\]
and so
\[
x \vee y - j(s-1)x = x \frac{s - j(s-1)}{1 - j(s-1)}.
\]
By the induction assumption for \( j = 1 \) applied to \( x \vee y' = s'x, \) where \( y' = \frac{x - j(s-1)x}{1 - j(s-1)} \text{ and } s' = \frac{s - j(s-1)}{1 - j(s-1)}, \text{ we have }
\]
\[
y - j(s-1)x \frac{1}{1 - j(s-1)} \geq x \frac{s - j(s-1)}{1 - j(s-1)} - 1).\]
By multiplying both sides with $1 - j(s - 1)$ we obtain
\[ y - j(s - 1)x \geq (s - 1)x \]
and so
\[ y \geq (j + 1)(s - 1)x. \]
By induction, $y \geq j(s - 1)x$ for all $j = 1, \ldots, k$. Hence $y \geq k(s - 1)x \geq x$ and $y = x \vee y = sx$.

**Proposition 4.5.** Let $X$ be a vector lattice and let $C \subset X_+$ be a nonzero max-cone. Let $A : C \to C$ be a positively homogeneous, finite suprema preserving mapping. If $\alpha > 0$ and $q(z) = \alpha + z$, then
\[ \sigma_p(q_v(A)) \cap (\alpha, \infty) = q_v(\sigma_p(A)) \cap (\alpha, \infty). \]

**Proof.** The inclusion $\subset$ follows from Lemma 4.1.
\[ \subset: \text{Let } t > \alpha \text{ and } t \in \sigma_p(q_v(A)). \text{ So there exists a nonzero } x \in C \text{ with } q_v(A)x = \alpha x \vee A x = tx. \text{ So } x \vee \alpha^{-1}Ax = \alpha^{-1}tx, \text{ where } \alpha^{-1}t > 1. \text{ By Lemma 4.4 for } y = \alpha^{-1}Ax \text{ we have } \alpha^{-1}Ax = \alpha^{-1}tx, \text{ and so } Ax = tx. \text{ Hence } t \in \sigma_p(A). \]

Now the following result follows.

**Theorem 4.6.** Let $X$ be a vector lattice and let $C \subset X_+$ be a nonzero max-cone. Let $A : C \to C$ be a positively homogeneous, finite suprema preserving mapping and let $q(z) = \sum_{j=0}^n \alpha_j z^j \in P_+$. Then
\[ q_v(\sigma_p(A)) \subset \sigma_p(q_v(A)) \subset q_v(\sigma_p(A)) \cup \{0\}. \]

**Proof.** The first inclusion follows from Lemma 4.1.
\[ \text{If } \alpha_0 = 0 \text{ then the second inclusion follows from Corollary 4.3. Let } \alpha_0 \neq 0 \text{ and } t > \alpha_0, t \in \sigma_p(q_v(A)). \text{ So there exists a nonzero } x \in C \text{ such that } tx = q_v(A)x = \alpha_0 x \vee A x, \text{ where } y = \sqrt[n]{\sum_{j=1}^n} \alpha_j A^j x. \text{ By Lemma 4.4, we have } y = tx \text{ and } t \in \sigma_p(\sqrt[n]{\sum_{j=1}^n} \alpha_j A^j). \text{ By Corollary 4.3, there exists } s \in \sigma_p(A) \text{ with } t = \max\{\alpha_j s^j : 1 \leq j \leq n\} = \max\{\alpha_j s^j : 0 \leq j \leq n\} = q_v(s). \text{ So } t \in q_v(\sigma_p(A)). \]

**Remark 4.7.** Under the assumptions of Theorem 4.6 it is possible for $q(z) = \sum_{j=0}^n \alpha_j z^j$ that $\sigma_p(q_v(A)) \neq q_v(\sigma_p(A))$. Consider the Banach lattice $\ell^\infty$ with natural order and let $C$ be the positive cone. Let $(e_n)$ be the standard basis in $\ell^\infty$ and define a mapping $A : C \to C$ by $A(\sum_{n} \gamma_n e_n) = \sum_{n} n^{-1} \gamma_n e_{n+1}$. Then $\sigma_p(A) = \emptyset$. Let $q_v(z) = 1 \vee z$. Then for $y = \sum_{n} e_n$ we have $q_v(A)y = y \vee Ay = y$. Hence $\sigma_p(q_v(A)) = \{1\} \neq q_v(\sigma_p(A)) = \emptyset$.

In the following, $X$ will be a normed vector lattice and $C \subset X_+$ a non-zero max cone. Let $A : C \to C$ be positive homogeneous, Lipschitz and finite suprema preserving. The spectral mapping theorem for the approximate point spectrum (see Theorem 4.11 below) can be proved similarly as the above results by repeating similar arguments. However, one can also apply the following standard construction.

Denote by $\ell^\infty(X)$ the set of all bounded sequences $(x_j)_{j=1}^\infty$ of elements of $X$. With the norm $\|x_j\| = \sup_j \|x_j\|$ and order $(x_j) \leq (y_j) \iff x_j \leq y_j$ for all $j$, $\ell^\infty(X)$ is again a normed vector lattice. Let $C^\infty \subset \ell^\infty(X)$ be the set of all bounded sequences of elements of $C$ and let $A^\infty : C^\infty \to C^\infty$ be defined by $A^\infty((e_j)) = (Ae_j)$. 
Let $c_0(X)$ be the set of all null sequences $(x_j)$ of elements of $X$, $\|x_j\| \to 0$. Clearly $c_0(X)$ is an ideal in $\ell^\infty(X)$. Let $\bar{X} = \ell^\infty(X)/c_0(X)$. Then $\bar{X}$ is again a normed lattice. Let $\bar{C} = (C^\infty + c_0(X))/c_0(X)$ and $\bar{A} : \bar{C} \to \bar{C}$ be the natural quotient mapping, which is well defined since $A$ is Lipschitz. Then $\bar{C}$ is a max cone and $\bar{A}$ is positive homogeneous and finite suprema preserving. Moreover, it is easy to show that $\sigma_{ap}(A) = \sigma_p(\bar{A})$.

Thus the following result follows.

**Theorem 4.8.** Let $X$ be a normed vector lattice and let $C \subseteq X_+$ be a nonzero max-cone. Let $A : C \to C$ be a Lipschitz, positively homogeneous, finite suprema preserving mapping. Let $q(z) = \sum_{j=0}^n \alpha_j z^j \in \mathcal{P}_+$. Then

$$q_\nu(\sigma_{ap}(A)) \subseteq \sigma_{ap}(q_\nu(A)) \subseteq q_\nu(\sigma_{ap}(A)) \cup \{0\}.$$ 

If $\alpha_0 = 0$ then $q_\nu(\sigma_{ap}(A)) = \sigma_{ap}(q_\nu(A))$.

**Corollary 4.9.** Let $X$ be a normed vector lattice and let $C \subseteq X_+$ be a nonzero max-cone. Let $A : C \to C$ be a Lipschitz, positively homogeneous, finite suprema preserving mapping. If $q \in \mathcal{P}_+$, $q = \sum_{j=0}^{\deg q} \alpha_j z^j$, then

$$r(q_\nu(A)) = q_\nu(r(A)).$$

**Proof.** By Theorems 4.8 and 2.1 it follows that

$$q_\nu(r(A)) \leq r(q_\nu(A)) \leq \max\{q_\nu(r(A)), \alpha_0\} = q_\nu(r(A)),$$

which completes the proof. \qed

The equality (6) suggests that the situation for the approximate point spectrum is even better. Indeed, the equality $\sigma_{ap}(q_\nu(A)) = q_\nu(\sigma_{ap}(A))$ is true for all polynomials from $\mathcal{P}_+$ as we prove below in Theorem 4.11.

**Proposition 4.10.** Let $X$ be a normed vector lattice and let $C \subseteq X_+$ be a nonzero max-cone. Let $A : C \to C$ be a Lipschitz, positively homogeneous, finite suprema preserving mapping. Let $q(z) = \alpha + z \in \mathcal{P}_+$. Then $\sigma_{ap}(q_\nu(A)) = q_\nu(\sigma_{ap}(A))$.

**Proof.** If $\alpha = 0$ then the statement is clear. Let $\alpha \neq 0$. Without loss of generality we may assume that $\alpha = 1$. Let $B = I \vee A$.

We know that $\sigma_{ap}(B) \supset q_\nu(\sigma_{ap}(A))$ and

$$\sigma_{ap}(B) \cap (1, \infty) = q_\nu(\sigma_{ap}(A)) \cap (1, \infty).$$

Moreover, $m(B) \geq 1$ and $\sigma_{ap}(B) \subset [1, \infty)$.

Let $1 \in \sigma_{ap}(B)$. For each $j \in \mathbb{N}$ we have $1 \in \sigma_{ap}(B^j)$. Let $(x_k)$ be a sequence of unit vectors in $C$ satisfying $\|B^j x_k - x_k\| \to 0$. Thus $m(B^j) \leq 1$ and $m(A^j) \leq m(B^j) \leq 1$. Hence $d(A) = \lim m(A^j)^{1/j} \leq 1$ and so $q_\nu(d(A)) = 1$. Since $d(A) \in \sigma_{ap}(A)$ by Theorem 3.5 it follows that $1 \in q_\nu(\sigma_{ap}(A))$ and so $\sigma_{ap}(q_\nu(A)) = q_\nu(\sigma_{ap}(A))$ \qed

Now the following result follows from Theorem 4.8, Proposition 4.10 and Theorem 3.5.
Theorem 4.11. Let $X$ be a normed vector lattice and let $C \subset X_+$ be a nonzero max-cone. Let $A : C \to C$ be a Lipschitz, positively homogeneous, finite suprema preserving mapping. Let $q(z) = \sum_{j=0}^n \alpha_j z^j \in P_+$. Then

$$\sigma_{ap}(q(A)) = q(\sigma_{ap}(A))$$

and so

$$d(q(A)) = q(d(A)).$$

In particular, the results above apply to the following two classes of examples from [28] and [32].

Example 4.12. Given $a > 0$, consider the following max-type kernel operators $A : C[0, a] \to C[0, a]$ of the form

$$(A(x))(s) = \max_{t \in [\alpha(s), \beta(s)]} k(s, t)x(t),$$

where $x \in C[0, a]$ and $\alpha, \beta : [0, a] \to [0, a]$ are given continuous functions satisfying $\alpha \leq \beta$. The kernel $k : S \to [0, \infty)$ is a given non-negative continuous function, where $S$ denotes the compact set

$$S = \{(s, t) \in [0, a] \times [0, a] : t \in [\alpha(s), \beta(s)]\}.$$

It is clear that for $C = C_+[0, a]$ it holds $AC \subset C$. We will denote the restriction $A|C$ again by $A$. The eigenproblem of these operators arises in the study of periodic solutions of a class of differential-delay equations

$$\varepsilon y'(t) = g(y(t), y(t - \tau)), \quad \tau = \tau(y(t)),$$

with state-dependent delay (see e.g. [28]).

By [28, Proposition 4.8] and its proof, the operator $A : C \to C$ is a positively homogeneous, Lipschitz mapping that preserves finite suprema. Hence $r(A) = \max\{t : t \in \sigma_{ap}(A)\}$ and $d(A) = \min\{t : t \in \sigma_{ap}(A)\}$. By [28, Theorem 4.3] it also holds that $r(A) = \lim_{n \to \infty} b_n^{1/n} = \inf_{n \geq 1} b_n^{1/n}$, where $b_n = \|A^n\| = \max_{s \in S_n} k_n(\sigma)$,

$$k_n(\sigma) = k(s_0, s_1)k(s_1, s_2) \cdots k(s_{n-1}, s_n)$$

and

$$S_n = \{(s_0, s_1, s_2, \ldots, s_n) : s_0 \in [0, a], s_i \in [\alpha(s_{i-1}), \beta(s_{i-1})], i = 1, 2, \ldots, n\}.$$  

On the other hand,

$$d(A) = \lim_{n \to \infty} d_n^{1/n} = \sup_{n \in N} d_n^{1/n},$$

where

$$d_n = m(A^n) = \inf\{\max_{\sigma \in S_n} k_n(\sigma)x(s_n) : x \in C, \|x\| = 1\}.$$

We also point out the following related example from [32].

Example 4.13. Let $M$ be a nonempty set and let $X$ be the set of all bounded real functions on $M$. With the norm $\|f\|_\infty = \sup\{|f(t)| : t \in M\}$ and natural operations, $X$ is a normed vector lattice. Let $C = X_+$ and let $k : M \times M \to [0, \infty)$ satisfy $\sup\{k(t, s) : t, s \in M\} < \infty$. Let $A : C \to C$ be defined by $(Af)(s) = \sup\{k(s, t)f(t) : t \in M\}$ and so $\|A\| = \sup\{k(t, s) : t, s \in M\}$. Clearly $C$ is a max-cone, $A$ is bounded, positive homogeneous and preserves finite suprema. Moreover, $A$ is Lipschitz. So we have that $r(A) = \max\{t : t \in \sigma_{ap}(A)\}$ and $d(A) = \min\{t : t \in \sigma_{ap}(A)\}$. 

In particular, if \( M \) is the set of all natural numbers \( \mathbb{N} \), our results apply to infinite bounded non-negative matrices \( k = [k(i,j)] \) (i.e., \( k(i,j) \geq 0 \) for all \( i, j \in \mathbb{N} \) and \( \|k\|_{\infty} = \sup_{i,j \in \mathbb{N}} k(i,j) < \infty \)). In this case, \( X = \ell^\infty \) and \( C = \mathbb{R}_+^n \) and \( \|A\| = \|k\|_{\infty} \).

5. **Application to inequalities involving Hadamard products.** Throughout this section let \( X, C \) and all the mappings \( A, B, A_1, \ldots, A_m, A_{11}, \ldots, A_{mn} \) that map \( C \) to \( C \) be as in Example 4.12 (where the functions \( \alpha \) and \( \beta \) are fixed - the same for all operators \( A, B, A_1, \ldots, A_m, A_{11}, \ldots, A_{mn} \)) or let \( X, C \) and all the mappings \( A, B, A_1, \ldots, A_m, A_{11}, \ldots, A_{mn} \) that map \( C \) to \( C \) be as in Example 4.13. We denote the set of such mappings by \( \mathcal{C} \).

In this section we apply \((6)\) to prove some new inequalities on Hadamard products (Theorem 5.5) by applying an idea from \([15]\). Let \( A \circ B \) denote the Hadamard (or Schur) product of mappings \( A \) and \( B \) from \( C \), i.e., \( A \circ B \in \mathcal{C} \) is a mapping with a kernel \( k(s,t)h(s,t) \), where \( k \) and \( h \) are the kernels of \( A \) and \( B \), respectively. Similarly, for \( \gamma > 0 \) let \( A^{(\gamma)} \) denote the Hadamard (or Schur) power of \( A \), i.e., a mapping with a kernel \( k^\gamma(s,t) \).

The following result was stated in \([37\], Theorem 4.1\) in the special case of \( n \times n \) non-negative matrices and was essentially proved in \([36\], Theorem 5.1 and Remark 5.2\] and the fact that for \( A_1, \ldots, A_m, A \in \mathcal{C} \) and \( \gamma > 0 \) we have

\[
A_1^{(\gamma)} \cdots A_m^{(\gamma)} = (A_1 \cdots A_m)^{(\gamma)} \quad \text{and} \quad \|A^{(\gamma)}\| = \|A\|^\gamma
\]

and consequently \( r(A^{(\gamma)}) = r(A)^\gamma \). Observe also that \( A \leq B \) implies \( r(A) \leq r(B) \).

**Theorem 5.1.** Let \( A_{ij} \in \mathcal{C} \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \) and let \( \alpha_1, \alpha_2, \ldots, \alpha_m \) be positive numbers. Then we have

\[
( A_{11}^{(\alpha_1)} \circ A_{12}^{(\alpha_2)} \circ \cdots \circ A_{1m}^{(\alpha_m)} ) \cdots ( A_{n1}^{(\alpha_1)} \circ A_{n2}^{(\alpha_2)} \circ \cdots \circ A_{nm}^{(\alpha_m)} ) \\
\preceq (A_{11} \cdots A_{1n})^{(\alpha_1)} \circ (A_{12} \cdots A_{n2})^{(\alpha_2)} \circ \cdots \circ (A_{1m} \cdots A_{nm})^{(\alpha_m)}
\]

and

\[
r \left( ( A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)} ) \cdots ( A_{n1}^{(\alpha_1)} \circ \cdots \circ A_{nm}^{(\alpha_m)} ) \right) \\
\preceq r (A_{11} \cdots A_{1n})^{\alpha_1} \cdots r (A_{1m} \cdots A_{nm})^{\alpha_m}.
\]

(7)

**Remark 5.2.** An analogue to \((7)\) for \( \| \cdot \| \) also holds. As pointed out in \([36\], [37] \) and \([30]\), Theorem 5.1 is in fact a result on the generalized and joint spectral radius in max-algebra (see also \([38]\)). The logarithm of the latter is also known as the maximal Lyapunov exponent in max algebra and is important in the study of certain discrete event systems (see e.g. the references cited in \([30]\)).

Inequalities \((10)\) and \((11)\) below were established in \([37\], Corollary 4.8\] in the special case of \( n \times n \) matrices, while the inequality \((9)\) is a max-algebra version of \([39\], Corollary 3.3\) and \([15\], Theorem 3.2\]. The proofs of these inequalities are similar to the proofs of the results from \([37\], [15] \) and \([39]\) and are included for the convenience of readers. In the proof we use the fact that

\[
r(AB) = r(BA).
\]

(8)
Corollary 5.3. Let $A_1, \ldots, A_m, A, B \in C$ and let $P_i = A_iA_{i+1} \cdots A_mA_1 \cdots A_{i-1}$ for $i = 1, \ldots, m$. Then the following inequalities hold:

\begin{align}
q_{1} &= (A_1 \cdots \circ A_m) \leq r((A_1 \circ \cdots \circ A_m)^m) \leq r(A_1 \cdots A_m),
q_{2} &= (A \circ B) \leq r(AB \circ BA)^{1/2} \leq r(AB),
q_{3} &= (AB \circ BA) \leq r(A^2B^2),
\end{align}

Proof. By Theorem 5.1 and (8) we have

\begin{align}
q_{1} &= (A_1 \cdots \circ A_m)^m = r((A_1 \circ \cdots \circ A_m)^m) \\
q_{2} &= r((A_1 \circ \cdots \circ A_m)(A_2 \circ \cdots \circ A_m \cdots A_1 \cdots A_{m-1})) \leq r(P_1 \cdots \circ P_m) \leq r(P_1) \cdots r(P_m) = r(A_1 \cdots A_m)^m,
\end{align}

which proves (9).

Inequality (10) is a special case of (9), while (11) follows from (10) and (8). □

Remark 5.4. As pointed out in [37, Example 4.10] the inequalities in (9) are sharp and may be strict, and in some cases the inequality (11) may be better than (10).

If $m \in \mathbb{N}$ and $q \in P_+$, $q = \sum_{j=0}^{\deg q} \alpha_j z^j$, let us define the polynomial $q^{[m]}$ by $q^{[m]} = \sum_{j=0}^{\deg q} \alpha_j^{m} z^j$.

By applying (6) and an idea from [15], we extend Corollary 5.3 in the following way.

Theorem 5.5. Let $q \in P_+$, $q = \sum_{j=0}^{\deg q} \alpha_j z^j$ and $A_1, \ldots, A_m, A, B \in C$. If $P_i$ for $i = 1, \ldots, m$ are as in Corollary 5.3, then the following inequalities hold:

\begin{align}
q_{1}(A_1 \cdots \circ A_m) &\leq r(q^{[m]}_v(P_1 \cdots \circ P_m))^{1/m} \leq r(q_v(A_1 \cdots A_m)),
q_{2}(A \circ B) &\leq r(q^{[2]}_v(AB \circ BA))^{1/2} \leq r(q_v(AB)),
q_{3}(AB \circ BA) &\leq r(q_v(A^2B^2)).
\end{align}

Proof. Since $q_v(t) = \sqrt[m]{q^{[m]}_v(t)}$ for $t \geq 0$, it follows from (6) and (9) that

\begin{align}
q_v(A_1 \cdots \circ A_m) &= q_v(r(A_1 \circ \cdots \circ A_m)) \\
&\leq q_v(r(P_1 \cdots \circ P_m)^{1/m}) = q^{[m]}_v(r(P_1 \circ \cdots \circ P_m)^{1/m}) \\
&= r(q^{[m]}_v(P_1 \cdots \circ P_m)^{1/m}) \leq q^{[m]}_v(r(A_1 \cdots A_m)^{1/m}) \\
&= q_v(r(A_1 \cdots A_m)) = r(q_v(A_1 \cdots A_m)),
\end{align}

which proves (12).

Inequality (13) is a special case of (12) and inequality (14) is proved in a similar way as (12). □

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REFERENCES

[1] Y.A. Abramovich and C.D. Aliprantis, *An invitation to operator theory*, American Mathematical Society, Providence, 2002.

[2] M. Akian, S. Gaubert and A. Hochart, Ergodicity conditions for zero-sum games, *Discrete and Continuous Dynamical Systems - A*, 35 (9) (2015), 3901–3931.

[3] M. Akian, S. Gaubert and C. Walsh, Discrete max-plus spectral theory, in *Idempotent Mathematics and Mathematical Physics*, G.L. Litvinov and V.P. Maslov, Eds, Contemporary Mathematics 377, 53–77, AMS, 2005. arXiv:math.SP/0405225

[4] M. Akian, S. Gaubert and R.D. Nussbaum, A Collatz-Wielandt characterization of the spectral radius of order-preserving homogeneous maps on cones, preprint, arXiv:1112.5968

[5] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis, A Hitchhiker’s Guide*, Third Edition, Springer, 2006.

[6] C.D. Aliprantis, D.J. Brown and O. Burkinshaw, *Existence and optimality of competitive equilibria*, Springer-Verlag, Berlin, 1990.

[7] C. Aliprantis and O. Burkinshaw, *Positive operators*, Reprint of the 1985 original, Springer, Dordrecht, 2006.

[8] C.D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces with applications to economics*, Second edition, Mathematical Surveys and Monographs 105, American Mathematical Society, Providence, RI, 2003.

[9] C.D. Aliprantis and R. Tourky, *Cones and duality*, American Mathematical Society, Providence, 2007.

[10] J. Appell, E. De Pascale and A. Vignoli, A comparison of different spectra for nonlinear operators, *Nonlinear Anal. Anoal. 40* (2000), 73–90.

[11] J. Appell, E. De Pascale and A. Vignoli, *Nonlinear Spectral Theory*, Walter de Gruyter GmbH and Co. KG, Berlin, 2004.

[12] J. Appell, E. Giorgieri and M. Väth, Nonlinear spectral theory for homogeneous operators, *Nonlinear Funct. Anal. Appl. 7* (2002), 589–618.

[13] R.B. Bapat, A max version of the Perron-Frobenius theorem, *Linear Algebra Appl. 275-276*, (1998), 3–18.

[14] P. Butkovič, *Max-linear systems: theory and algorithms*, Springer-Verlag, London, 2010.

[15] R. Drnovšek and A. Peperko, Inequalities on the spectral radius and the operator norm of Hadamard products of positive operators on sequence spaces, *Banach J. Math. Anal.* (2016), Vol. 10, Num. 4 (2016), 800–814.

[16] W. Feng, A new spectral theory for nonlinear operators and its applications, *Abstr. Appl. Anal. Anal. 2* (1997), 163–183.

[17] G. Gripenberg, On the definition of the cone spectral radius, *Proc. Amer. Math. Soc. 143* (2015), 1617–1625.

[18] M. de Jeu and M. Messerschmidt, A strong open mapping theorem for surjections from cones onto Banach spaces, *Advances in Math.* 259 (2014), 43–66.

[19] R.D. Katz, H. Schneider and S. Sergeev, On commuting matrices in max algebra and in nonnegative matrix algebra, *Linear Algebra Appl.* 436(2), (2012), 276–292.

[20] V.N. Kolokoltsov and V.P. Maslov, *Idempotent analysis and its applications*, Kluwer Acad. Publ., 1997.

[21] B. Lins and R.D. Nussbaum, Continuity of the cone spectral radius, *Proc. Amer. Math. Soc.* 141 (2013), 2741–2754. arXiv:1107.4532

[22] B. Lins and R.D. Nussbaum, *Nonlinear Perron-Frobenius Theory*, Cambridge University Press, 2012.

[23] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I and II*, A reprint of the 1977 and 1979 editions, Springer, 1996.

[24] B. Lins and R.D. Nussbaum, Denjoy-Wolff theorems, Hilbert metric nonexpansive maps on reproduction-decimation operators, *J. Funct. Anal.* 254 (2008), 203–245.

[25] G.L. Litvinov, The Maslov dequantization, idempotent and tropical mathematics: A brief introduction, *J. Math. Sci.(N. Y.) 140*, no.3 (2007), 426–444. arXiv:math/0507014

[26] G.L. Litvinov, V.P. Maslov and G.B. Shpiz, Idempotent functional analysis: An algebraic approach, *Math Notes 69*, no. 5-6 (2001) 696–729. arXiv:math.FA/0009128

[27] G.L. Litvinov and V.P. Maslov (eds.), Idempotent mathematics and mathematical physics, *Contemp. Math.* 377, Amer.Math. Soc., Providence, RI, 2005.
[28] J. Mallet-Paret and R.D. Nussbaum, Eigenvalues for a class of homogeneous cone maps arising from max-plus operators, *Discrete and Continuous Dynamical Systems*, 8, num 3 (2002), 519–562.

[29] J. Mallet-Paret and R. D. Nussbaum, Generalizing the Krein-Rutman theorem, measures of noncompactness and the fixed point index, *J. Fixed Point Theory and Applications* 7 (2010), 103–143.

[30] V. Müller and A. Peperko, Generalized spectral radius and its max algebra version, *Linear Algebra Appl.* 439 (2013), 1006–1016.

[31] V. Müller and A. Peperko, On the spectrum in max-algebra, *Linear Algebra Appl.* 485 (2015), 250–266.

[32] V. Müller and A. Peperko, On the Bonsall cone spectral radius and the approximate point spectrum, *Discrete and Continuous Dynamical Systems - Series A*, vol. 37, no 10. (2017), 5337–5364.

[33] R. D. Nussbaum, Eigenvalues of nonlinear operators and the linear Krein-Rutman, in: Fixed Point Theory (Sherbrooke, Quebec, 1980), E. Fadell and G. Fournier, editors, Lecture notes in Mathematics 886, Springer-Verlag, Berlin (1981), 309–331.

[34] R. D. Nussbaum, Periodic points of positive linear operators and Perron-Frobenius operators, *Integral Equations Operator Theory* 39 (2001), 41–97.

[35] L. Pachter and B. Sturmfels (eds.), *Algebraic statistics for computational biology*, Cambridge Univ. Press, New York, 2005.

[36] A. Peperko, Inequalities for the spectral radius of non-negative functions, *Positivity* 13 (2009), 255–272.

[37] A. Peperko, Bounds on the generalized and the joint spectral radius of Hadamard products of bounded sets of positive operators on sequence spaces, *Linear Algebra Appl.* 437 (2012), 189–201.

[38] A. Peperko, Bounds on the joint and generalized spectral radius of the Hadamard geometric mean of bounded sets of positive kernel operators, *Linear Algebra Appl.* 533 (2017), 418–427.

[39] A. Peperko, Inequalities on the spectral radius, operator norm and numerical radius of the Hadamard weighted geometric mean of positive kernel operators, *submitted*, arXiv:1612.01767.

[40] P. Santucci and M. Väth, On the definition of eigenvalues of nonlinear operators, *Nonlinear Anal.* 40 (2000), 565-576.

[41] G. B. Shpiz, An eigenvector existence theorem in idempotent analysis, *Mathematical Notes* 82, 3-4 (2007), 410–417.

[42] W. Wnuk, *Banach lattices with order continuous norms*, Polish Scientific Publ., PWN, Warszawa, 1999.

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