NON-HOMOGENEOUS BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFUSION EQUATIONS IN $L^2$-SETTING

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Abstract. In the present article, we study the diffusion equations with fractional time derivatives. The aim of this paper is to investigate the best possible regularity for the initial value/boundary value problems with non-homogeneous Dirichlet boundary data. The main tool we use here is called the transposition method.

1. Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^d$ with $C^2$ boundary $\Gamma := \partial \Omega$ and set $Q := \Omega \times (0, T)$ and $\Sigma := \Gamma \times (0, T)$. We consider the following initial value/boundary value problem for a partial differential equation with the fractional derivative in time $t$:

\[
\begin{aligned}
\partial_\alpha^t u + Au &= 0 \text{ in } Q, \\
u &= g \quad \text{ on } \Sigma, \\
u(\cdot, 0) &= 0 \text{ in } \Omega
\end{aligned}
\]

with $0 < \alpha < 1$. Here $\partial_\alpha^t$ denotes the Caputo derivative, which is defined by

\[
\partial_\alpha^t u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u}{\partial \tau}(x, \tau) d\tau,
\]

and $\Gamma(\cdot)$ is the Gamma function (see Podlubny [4]). The differential operator $A$ is given by

\[
Au(x) = -\sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j}(x) \right) + c(x)u(x), \quad x \in \Omega,
\]

and the coefficients satisfy the following:

\[
a_{ij} = a_{ji} \in C^1(\overline{\Omega}), \quad 1 \leq i, j \leq d, \quad \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2, \ x \in \overline{\Omega}, \ \xi \in \mathbb{R}^d,
\]

\[
c \in C(\overline{\Omega}), \quad c(x) \geq 0, \ x \in \overline{\Omega},
\]

where $\mu > 0$ is constant. The function $g$ is given on $\Sigma$.

In the present paper, we study the regularity of the solution to (1.1) in the sense of Sobolev spaces. As for this problems, Lions and Magenes [3] showed the result for the parabolic equations.

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2. Main Result

In this section, we prepare the notations and state our main results. We denote by $H^s(\Omega)$, $s \geq 0$, the Sobolev spaces. For $r, s \geq 0$, we abbreviately set

$$H^{r,s}(Q) := L^2(0, T; H^r(\Gamma)) \cap H^s(0, T; L^2(\Omega)).$$

Then $H^{r,s}(Q)$ is a Hilbert space with the norm $\| \cdot \|_{H^{r,s}(Q)}$ given by

$$\|u\|_{H^{r,s}(Q)}^2 := \|u\|_{L^2(0, T; H^r(\Gamma))}^2 + \|u\|_{H^s(0, T; L^2(\Omega))}^2.$$  

In particular,

$$H^{0,0}(Q) = L^2(Q) = L^2(0, T; L^2(\Omega)).$$

Similarly we set

$$H^{r,s}(\Sigma) := L^2(0, T; H^r(\Gamma)) \cap H^s(0, T; L^2(\Gamma)),$$

$$H^{r,0}(\Sigma) := L^2(0, T; H^r(\Gamma)) \cap H^0(0, T; L^2(\Gamma)),$$

with the norm defined by

$$\|u\|_{H^{r,s}(\Sigma)}^2 := \|u\|_{L^2(0, T; H^r(\Gamma))}^2 + \|u\|_{H^s(0, T; L^2(\Gamma))}^2.$$  

We clearly have

$$H^{0,0}(\Sigma) = L^2(\Sigma) = L^2(0, T; L^2(\Gamma)).$$

As for the spaces with negative exponents, we define

$$H^{-r,-s}(\Sigma) := (H^{r,s}(\Sigma))', \quad r, s \geq 0.$$  

The duality pairing between $H^{-r,-s}(\Sigma)$ and $H^{r,s}(\Sigma)$ is denoted by $\langle \psi, u \rangle_{r,s}$ for $\psi \in H^{-r,-s}(\Sigma)$ and $u \in H^{r,s}(\Sigma)$.

We define the operator $\partial_{\nu_A} : H^s(\Omega) \rightarrow H^{s-3/2}(\Gamma), s > 3/2$, as

$$\frac{\partial u}{\partial \nu_A}(x) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \nu_j(x),$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_d(x))$ is the outward unit normal vector to $\Gamma$ at $x$. Then for $u \in H^{r,s}(Q)$ with $r > 3/2$, the trace theorem (Theorem 2.1 in Chapter 4 of Lions and Magenes [3]) yields

$$\frac{\partial u}{\partial \nu_A} \in H^\mu(\Omega), \quad \frac{\mu}{r} = \frac{s}{r} = \frac{r - 3/2}{r} \quad (\nu = 0 \text{ if } s = 0) \quad (2.1)$$

and

$$\left\| \frac{\partial u}{\partial \nu_A} \right\|_{H^\mu(\Sigma)} \leq C \|u\|_{H^{r,s}(Q)}. \quad (2.2)$$

In order to define the weak solution of (1.1), we introduce the dual system:

$$\begin{cases}
D_\alpha t^\alpha v + Av = f & \text{in } Q, \\
v = 0 & \text{on } \Sigma, \\
I_{T^-}^{1-\alpha}v(\cdot, T) = 0 & \text{in } \Omega,
\end{cases} \quad (2.3)$$
where $D_t^\alpha$ is the backward Riemann-Liouville fractional derivative, which is defined by

$$D_t^\alpha h(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (\tau-t)^{-\alpha} h(\tau) d\tau, \quad 0 < \alpha < 1.$$ 

Moreover $I_T^\nu$ denotes the backward integral of order $\nu$;

$$I_T^\nu h(t) = \frac{1}{\Gamma(\nu)} \int_t^T (\tau-t)^{\nu-1} h(\tau) d\tau, \quad \nu > 0.$$ 

By the same argument as Chapter 4 in Bajlekova [1], for any $f \in L^2(Q)$ there exists a unique solution $v \in H^{2,\alpha}(Q)$ of (2.3) such that

$$\|v\|_{H^{2,\alpha}(Q)} \leq C\|f\|_{L^2(Q)}. \quad (2.4)$$

Henceforth we will denote this solution by $v_f$. Now we apply (2.1) to $v_f \in H^{2,\alpha}(Q)$ and obtain

$$\frac{\partial v_f}{\partial \nu_A} \in H^{1/2,\alpha/4}(\Sigma) \quad \text{and} \quad \left\| \frac{\partial v_f}{\partial \nu_A} \right\|_{H^{1/2,\alpha/4}(\Sigma)} \leq C\|v_f\|_{H^{2,\alpha}(Q)}. \quad (2.5)$$

Since $0 < \alpha < 1$, we have

$$H_0^{\alpha/4}(0, T; L^2(\Gamma)) = H^{\alpha/4}(0, T; L^2(\Gamma))$$

(see (B.1)). That is,

$$\frac{\partial v_f}{\partial \nu_A} \in H^{1/2,\alpha/4}(\Sigma) = H_0^{1/2,\alpha/4}(\Sigma).$$

Now we are ready to define the weak solution of (1.1);

**Definition 2.1.** A function $u$ is a weak solution of (1.1) if

$$(u, f)_{L^2(Q)} + \langle g, \frac{\partial v_f}{\partial \nu_A} \rangle_{1/2,\alpha/4} = 0 \quad (2.6)$$

holds for any $f \in L^2(Q)$.

The main result of this paper is as follows;

**Theorem 2.1.** Let $g \in L^2(\Sigma)$, then (1.1) has a unique weak solution $u \in H^{1/2,\alpha/4}(Q)$ satisfying

$$\|u\|_{H^{1/2,\alpha/4}(Q)} \leq C\|g\|_{L^2(\Sigma)}. \quad (2.7)$$

Now we roughly describe the strategy of the proof. It is not difficult to show the unique existence of the weak solution of (1.1) for $g \in L^2(\Sigma)$, but the regularity of $H^{1/2,\alpha/4}(Q)$ cannot be directly deduced. Therefore we first show the following two results;

(i) Regularity of the solution for $g \in H^{-1/2,-\alpha/4}(\Sigma)$.

(ii) Regularity of the solution for $g \in H^{3/2,3\alpha/4}(\Sigma)$. 


After showing (i) and (ii), we obtain the regularity for \( g \in L^2(\Sigma) \) by interpolating the above results.

The result for (i) can be easily shown. Indeed, from this definition, we can immediately deduce the following proposition:

**Proposition 2.2.** Let \( g \in \mathcal{H}^{-1/2,-\alpha/4}(\Sigma) \), then (1.1) has a unique weak solution \( u \in L^2(Q) \) satisfying

\[
\|u\|_{L^2(Q)} \leq C\|g\|_{\mathcal{H}^{-1/2,-\alpha/4}(\Sigma)}. \tag{2.8}
\]

**Proof.** Combining (2.4) and (2.5), we obtain

\[
\left\| \frac{\partial v_f}{\partial v_A} \right\|_{H^{1/2,\alpha/4}(\Sigma)} \leq C\|f\|_{L^2(Q)}. \tag{2.9}
\]

Thus the mapping

\[ L^2(Q) \ni f \mapsto \frac{\partial v_f}{\partial v_A} \in H^{1/2,\alpha/4}(\Sigma) \]

is bounded, and so is

\[ L^2(Q) \ni f \mapsto -\langle g, \partial_{v_A} v_f \rangle \in \mathbb{C}. \]

Therefore the Riesz's representation theorem yields the unique existence of \( u \in L^2(Q) \) such that

\[(u,f)_{L^2(Q)} = -\langle g, \partial_{v_A} v_f \rangle \]

holds for any \( f \in L^2(Q) \). Thus we have proven the unique existence of weak solution.

Moreover for any \( f \in L^2(Q) \) we have

\[
|\langle u,f \rangle_{L^2(Q)}| = |\langle g, \partial_{v_A} v_f \rangle| \leq \|g\|_{\mathcal{H}^{-1/2,-\alpha/4}(\Sigma)} \|\partial_{v_A} v_f\|_{H^{1/2,\alpha/4}(\Sigma)} \leq C\|g\|_{\mathcal{H}^{-1/2,-\alpha/4}(\Sigma)}\|f\|_{L^2(Q)},
\]

where we have used (2.9) in the last inequality. Therefore we have (2.8). \( \square \)

Thus we have proved part (i). In the next section, therefore, we will consider case (ii).

### 3. Regular solution

We first formulate the functions in \( \mathcal{H}^s(0,T) \) vanishing at \( t = 0 \). Following (2.10.3.1b) in Triebel [6], for \( \mathbb{R}_+ := (0,\infty) \) we denote by \( \widetilde{\mathcal{H}}^s(\mathbb{R}_+) \) the functions in \( \mathcal{H}^s(\mathbb{R}) \) which are identically zero outside \( \mathbb{R}_+ \):

\[
\widetilde{\mathcal{H}}^s(\mathbb{R}_+) := \left\{ u \in \mathcal{H}^s(\mathbb{R}); \supp u \subset \mathbb{R}_+ \right\}.
\]

Moreover we set

\[
\widetilde{\mathcal{H}}^s_0(0,T) := \{u|_{(0,T)}; \widetilde{\mathcal{H}}^s(\mathbb{R}_+)\}.
\]

Then similarly to (B.6), we can see that such function spaces have a good property for interpolation;

\[
[\widetilde{\mathcal{H}}^{s_1}_{0+}(0,T), \widetilde{\mathcal{H}}^{s_2}_{0+}(0,T)]_\theta = \widetilde{\mathcal{H}}^{(1-\theta)s_1+\theta s_2}_{0+}(0,T), \quad 0 \leq s_1 < s_2 < \infty, \; 0 \leq \theta \leq 1.
\]
We also note that we have the representation of \( \tilde{H}^s_{0+}(0, T) \) as
\[
\tilde{H}^s_{0+}(0, T) = \left\{ u \in H^s(0, T); \int_0^T |u(t)|^2 \frac{dt}{t^{2s}} < \infty \right\}, \quad 0 \leq s \leq 1. \tag{3.1}
\]
In particular, for \( 0 \leq s \leq 1, s \neq 1/2 \), we also have
\[
\tilde{H}^s_{0+}(0, T) = \begin{cases} \{ u \in H^s(0, T); u(0) = 0 \}, & \frac{1}{2} < s \leq 1 \end{cases} \tag{3.2}
\]
(see (B.1) and (B.5)). For simplicity, we set
\[
H^{r,s}_{0+}(Q) := L^2(0, T; H^r(\Omega)) \cap \tilde{H}^s_{0+}(0, T; L^2(\Omega))
\]
and
\[
H^{r,s}_{0+}(\Sigma) := L^2(0, T; H^r(\Gamma)) \cap \tilde{H}^s_{0+}(0, T; L^2(\Gamma)).
\]
As for these spaces, we have the following properties for trace;

**Proposition 3.1.** Let \( u \in H^{2,\alpha}_{0+}(Q), 0 < \alpha < 1 \), then we have
\[
u|_\Sigma \in H^{3/2,3\alpha/4}_{0+}(\Sigma).
\]
Moreover the mapping
\[
H^{2,\alpha}_{0+}(Q) \ni u \mapsto u|_\Sigma \in H^{3/2,3\alpha/4}_{0+}(\Sigma)
\]
is a continuous surjection.

For the proof of this proposition, we prepare the following lemma;

**Lemma 3.2 (Trace Theorem).** Let \( X \) and \( Y \) be Hilbert spaces such that \( X \) is embedded to \( Y \) densely and continuously. If \( u \in L^2(\mathbb{R}_+; X) \cap H^r(\mathbb{R}_+; Y) \) with \( r > 1/2 \), then
\[
u^{(j)}(0) \in [Y, X]_{1-(j+1/2)/r}, \quad 0 \leq j < r - 1/2.
\]
Moreover, the mapping
\[
L^2(\mathbb{R}_+; X) \cap H^r(\mathbb{R}_+; Y) \ni u \mapsto u^{(j)}(0) \in [Y, X]_{1-(j+1/2)/r}
\]
is a continuous surjection.

For this lemma, see Theorem 4.2 in Chapter 1 of [3].
Proof of Proposition 3.1. Without loss of generality, we may consider the case of
\[ \Omega = \mathbb{R}_+^d = \{(x_1, \ldots, x_{d-1}, x_d); \ x_1, \ldots, x_{d-1} \in \mathbb{R}, \ x_d > 0\}, \]
\[ \Gamma = \mathbb{R}^{d-1} = \{(x_1, \ldots, x_{d-1}); \ x_1, \ldots, x_{d-1} \in \mathbb{R}\}. \]
We abbreviately write \( x' := (x_1, \ldots, x_{d-1}) \). Then we have
\[ u \in H^{2,\alpha}_{0+}(Q) = L^2(0, T; H^2(\mathbb{R}_+^d)) \cap \tilde{H}^{\alpha}_{0+}(0, T; L^2(\mathbb{R}_+^d)) \]
if and only if
\[ u \in L^2(\mathbb{R}_{+,x_d}; H^{2,\alpha}_{0+}(\mathbb{R}_+^{d-1} \times (0, T))) \cap H^2(\mathbb{R}_{+,x_d}; L^2(\mathbb{R}_+^{d-1} \times (0, T))) \]
\[ = L^2(\mathbb{R}_+, H^{2,\alpha}_{0+}(\Sigma)) \cap H^2(\mathbb{R}_+, L^2(\Sigma)). \]
We apply Lemma 3.2 as \( X = H^{2,\alpha}_{0+}(\Sigma) \) and \( Y = L^2(\Sigma) \). Then we have
\[ u|_{\Sigma} = u(0) \in [L^2(\Sigma), H^{2,\alpha}_{0+}(\Sigma)]_{3/4} = H^{3/2,3\alpha/4}_{0+}(\Sigma). \]
The surjectivity also follows from Lemma 3.2. \( \square \)

By using the trace theorem stated above, problem (1.1) with \( g \in H^{3/2,3\alpha/4}_{0+}(\Sigma) \) is directly reduced to the following problem with homogeneous boundary condition;
\[
\begin{cases}
\partial_t^\alpha u + Au = F & \text{in } Q, \\
u = 0 & \text{on } \Sigma, \\
u(\cdot, 0) = 0 & \text{in } \Omega.
\end{cases}
\]
(3.3)

For (3.3), Gorenflo, Luchko and Yamamoto [2] showed the \( L^2 \)-maximal regularity. In their setting, the Caputo derivative \( \partial_t^\alpha \) equipped with the initial value \( u(0) = 0 \) is formulated as an operator in \( L^2(0, T) \) with its domain given by
\[ \mathcal{D}(\partial_t^\alpha) = \left\{ \begin{array}{l}
\{ u \in H^{1/2}(0, T); \ \int_0^t |u(t)|^2 \frac{dt}{t} < \infty \}, \quad \alpha = 1/2, \\
\{ u \in H^{\alpha}(0, T); \ u(0) = 0 \}, \quad 1/2 < \alpha \leq 1.
\end{array} \right. \]
(3.4)

By (3.1) and (3.2), this can be rewritten as
\[ \mathcal{D}(\partial_t^\alpha) = \tilde{H}^{\alpha}_{0+}(0, T). \]
(3.5)

Thus we can see that if (3.3) has a “solution” in \( \mathcal{D}(\partial_t^\alpha) \), then the initial condition \( u(\cdot, 0) = 0 \) is satisfied in a weaker sense. They also revealed that the above operator \( \partial_t^\alpha \) is essentially equivalent to the Riemann-Liouville derivatives, which were already discussed in [1]. Anyway we obtain the following result;

**Lemma 3.3.** Let \( 0 < \alpha < 1 \) and \( F \in L^2(Q) \), then (3.3) has a unique solution \( u \in H^{2,\alpha}_{0+}(Q) \) satisfying
\[ \|u\|_{H^{2,\alpha}(Q)} \leq C\|F\|_{L^2(Q)}. \]
For the proof of this lemma, see Theorem 4.3 in [2].

**Proposition 3.4.** Let $0 < \alpha < 1$ and $g \in H^{3/2,3\alpha/4}_{0+}(\Sigma)$. Then problem (1.1) has a unique solution $u \in H^{2,\alpha}_{0+}(Q)$ satisfying

\[ \|u\|_{H^{2,\alpha}(Q)} \leq C\|g\|_{H^{3/2,3\alpha/4}(\Sigma)}. \]

**Proof.** Since $g \in H^{3/2,3\alpha/4}_{0+}(\Sigma)$, Proposition 3.1 yields that there exists $\tilde{g} \in H^{2,\alpha}_{0+}(Q)$ such that

\[ \tilde{g}|\Sigma = g \quad \text{and} \quad \|\tilde{g}\|_{H^{2,\alpha}(Q)} \leq C\|g\|_{H^{3/2,3\alpha/4}(\Sigma)}. \]

By setting $F := -A\tilde{g} - \partial_t^\alpha \tilde{g} \in L^2(Q)$ and applying Lemma 3.3, there exists a solution $w \in H^{2,\alpha}_{0+}(Q)$ of

\[
\begin{aligned}
\partial_t^\alpha w + Aw &= F \quad \text{in } Q, \\
w &= 0 \quad \text{on } \Sigma, \\
w(\cdot, 0) &= 0 \quad \text{in } \Omega,
\end{aligned}
\]

which satisfies

\[ \|w\|_{H^{2,\alpha}(Q)} \leq C\|F\|_{L^2(Q)} \leq C\|\tilde{g}\|_{H^{2,\alpha}(Q)} \leq C\|g\|_{H^{3/2,3\alpha/4}(\Sigma)}. \]

Then $u := w + \tilde{g}$ satisfies (1.1) and

\[ \|u\|_{H^{2,\alpha}(Q)} \leq \|w\|_{H^{2,\alpha}(Q)} + \|\tilde{g}\|_{H^{2,\alpha}(Q)} \leq C\|g\|_{H^{3/2,3\alpha/4}(\Sigma)}. \]

Thus we have completed the proof. \qed

4. **Proof of the main result**

In this section, we complete the proof of Theorem 2.1 by interpolation.

**Proof of Theorem 2.1.** Let $\pi$ be the operator which operates the boundary data $g$ to the weak solution $u$ of (1.1). Then, by Propositions 2.2 and 3.4, we have

\[ \pi \in \mathcal{L} \left( H^{-1/2,-\alpha/4}(\Sigma); L^2(Q) \right) \cap \mathcal{L} \left( H^{3/2,3\alpha/4}_{0+}(\Sigma); H^{2,\alpha}(Q) \right), \]

where $\mathcal{L}(X,Y)$ denotes the set of linear and bounded operators from $X$ to $Y$. By Proposition A.1, the operator $\pi$ also belongs to

\[ \mathcal{L} \left( [H^{-1/2,-\alpha/4}(\Sigma), H^{3/2,3\alpha/4}_{0+}(\Sigma)]_\theta; [L^2(Q), H^{2,\alpha}(Q)]_\theta \right) \]

for any $0 \leq \theta \leq 1$. In particular, if we set $\theta = 1/4$, then

\[ [H^{-1/2,-\alpha/4}(\Sigma), H^{3/2,3\alpha/4}_{0+}(\Sigma)]_{1/4} = L^2(\Sigma) \quad \text{and} \quad [L^2(Q), H^{2,\alpha}(Q)]_{1/4} = H^{1/2,\alpha/4}(Q), \]

and therefore we have

\[ \pi \in \mathcal{L}(L^2(\Sigma); H^{1/2,\alpha/4}(Q)). \]

Thus we have completed the proof. \qed
Appendix A. Interpolation

Throughout this article, we often use the word “interpolation” as a complex interpolation defined below. As for the detailed argument on this topic, we can refer to Triebel [6], Yagi [7] and the references therein. On the other hand, in some classical works such as Lions and Magenes [3], the “interpolation” of two Hilbert spaces is defined as the domain of fractional powers of positive and self-adjoint operator. We will see that these two kinds of definitions coincide with each other (see Proposition A.2). Therefore, we can refer to [3] and use some of their results (e.g., Theorem 4.2 in Chapter 1 of [3]) without any confusion. In this section, we recall the definition of complex interpolation of Banach spaces and summarize their fundamental properties.

Let $X_i$ be a Banach space equipped with the norm $\| \cdot \|_{X_i}$ ($i = 0, 1$) and suppose that $X_1$ is embedded in $X_0$ continuously and densely. Let $S$ be defined by

$$S := \{ z \in \mathbb{C}; \ 0 < \text{Re} \ z < 1 \}.$$

We say that a function $F : S \to X_0$ belongs to $\mathcal{H}(X_0, X_1)$ if and only if the following conditions (H1)-(H3) are satisfied;

(H1) $F$ is analytic in $S$.
(H2) $F$ is bounded and continuous in $S$.
(H3) $\mathbb{R} \ni y \mapsto F(1 + iy) \in X_1$ is bounded and continuous.

It is known that $\mathcal{H}(X_0, X_1)$ is a Banach space with the norm $\| \cdot \|_{\mathcal{H}}$ given by

$$\| F \|_{\mathcal{H}} := \max \left( \sup_{y \in \mathbb{R}} \| F(iy) \|_{X_0}, \sup_{y \in \mathbb{R}} \| F(1 + iy) \|_{X_1} \right), \quad F \in \mathcal{H}(X_0, X_1).$$

For each $0 \leq \theta \leq 1$, we define the space $[X_0, X_1]_\theta$ by

$$[X_0, X_1]_\theta := \{ u \in X_0; \ u = F(\theta) \text{ for some } F \in \mathcal{H}(X_0, X_1) \}.$$

Moreover $[X_0, X_1]_\theta$ is a Banach space with the norm $\| \cdot \|_\theta$ defined by

$$\| u \|_\theta := \inf_{F \in \mathcal{H}(X_0, X_1), \ F(\theta) = u} \| F \|_{\mathcal{H}}, \quad u \in [X_0, X_1]_\theta.$$

By the interpolation, we can show various kinds of “intermediate properties”. For example, if a linear operator $T$ is bounded from $X_0$ into $Y_0$ and from $X_1$ into $Y_1$ at the same time, then we can deduce that $T$ is also a bounded operator from $[X_0, X_1]_\theta$ into $[Y_0, Y_1]_\theta$ for any $0 < \theta < 1$.

Proposition A.1. Let $X_1$ (resp. $Y_1$) be embedded to $X_0$ (resp. $Y_0$) densely and continuously. Then for any $0 < \theta < 1$,

$$\mathcal{L}(X_0, Y_0) \cap \mathcal{L}(X_1, Y_1) \subset \mathcal{L}([X_0, X_1]_\theta, [Y_0, Y_1]_\theta)$$

and we have

$$\| T \|_{\mathcal{L}([X_0, X_1]_\theta, [Y_0, Y_1]_\theta)} \leq \| T \|_{\mathcal{L}(X_0, Y_0)}^{1-\theta} \| T \|_{\mathcal{L}(X_1, Y_1)}^\theta, \quad T \in \mathcal{L}(X_0, Y_0) \cap \mathcal{L}(X_1, Y_1).$$
Moreover we can also characterize the domain of fractional power of operators;

**Proposition A.2.** Let $X$ be a Hilbert space and $A : X \to X$ be a positive and self-adjoint operator. Then we have

$$D(A^\theta) = [X, D(A)]_\vartheta, \quad 0 \leq \theta \leq 1$$

with isometry.

Here we note that $[X, D(A)]_\vartheta$ stated above coincides with $[D(A), X]_{1-\vartheta}$ in the notation by Lions and Magenes [3].

**Appendix B. Sobolev spaces**

Let $\Omega$ be a domain of $\mathbb{R}^d$ with smooth boundary $\Gamma = \partial \Omega$. We denote by $H^s(\Omega)$, $s \geq 0$, the space of Bessel potentials (see e.g., [6]). We can see that $H^s(\Omega)$, $s \geq 0$, has a good property with respect to the interpolation;

$$[H^{s_1}(\Omega), H^{s_2}(\Omega)]_\theta = H^{(1-\theta)s_1 + \theta s_2}(\Omega), \quad 0 \leq s_1 < s_2 < \infty.$$  

As a characterization of subspaces of $H^s(\Omega)$ consisting of the functions vanishing on $\Gamma$, we often use $H^s_0(\Omega)$ (also denoted by $\overset{\circ}{H}^s(\Omega)$), the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$. We note that $H^s_0(\Omega)$ for $0 \leq s \leq 1$ has the representation as

$$H^s_0(\Omega) = \begin{cases} H^s(\Omega), & 0 \leq s \leq \frac{1}{2}; \\ \{ u \in H^s(\Omega); u|_\Gamma = 0 \}, & \frac{1}{2} < s \leq 1. \end{cases} \quad (B.1)$$

However, we see that for these spaces, good properties concerning the interpolation fails for some “singular” cases—when $s = \text{integer}+1/2$. Indeed, according to Theorem 11.6 in Chapter 1 of [3], the identity

$$[H^{s_1}_0(\Omega), H^{s_2}_0(\Omega)]_\theta = H^{(1-\theta)s_1 + \theta s_2}_0(\Omega)$$

is valid for the case in which $s_1, s_2, (1 - \theta)s_1 + \theta s_2 \neq \text{integer}+1/2$. On the other hand, by Theorem 11.7 in Chapter 1 of [3], if $(1 - \theta)s_1 + \theta s_2 = \mu + 1/2$ for some integer $\mu$ ($s_1, s_2$ are still assumed not to be integer+1/2), then

$$H^{\mu+1/2}_0(\Omega) := [H^{s_1}_0(\Omega), H^{s_2}_0(\Omega)]_\theta \subsetneq H^\mu_{00}(\Omega). \quad (B.2)$$

In particular, for $\mu = 0$ we have

$$[L^2(\Omega), H^0_0(\Omega)]_{1/2} = H^{1/2}_0(\Omega) \subsetneq H^0_{00}(\Omega) = H^{1/2}(\Omega).$$

We note that the space $H^{\mu+1/2}_0(\Omega)$ also can be characterized without interpolation. In fact, by (11.52) in Chapter 1 of [3], we have the representation as

$$H^{\mu+1/2}_0(\Omega) = \left\{ u \in H^{\mu+1/2}_0(\Omega); \rho^{-1/2}\partial_\rho^2 u \in L^2(\Omega) \text{ for } |\alpha| = \mu \right\}, \quad (B.3)$$
where \( \rho : \Omega \to [0, \infty) \) is a smooth function such that
\[
\lim_{x \to x_0} \frac{\rho(x)}{d(x, \Gamma)} = d \neq 0, \quad x_0 \in \Gamma.
\]
Moreover by substituting \( \mu = 0 \) in (B.3), we can rewrite (B.3) as
\[
H^{1/2}_{00}(\Omega) = \left\{ u \in H^{1/2}(\Omega); \int_{\Omega} \frac{|u(x)|^2}{\rho(x)} dx < \infty \right\}.
\]
Thus we see that each element in \( H^{1/2}_{00}(\Omega) \) vanishes on \( \Gamma \) in a certain sense.

In the following, therefore, we introduce another formulation of functions in \( H^s(\Omega) \) vanishing on \( \Gamma \), which includes \( H^{\mu+1/2}_{00}(\Omega) \) as a particular case. We denote by \( \tilde{H}^s(\Omega) \) the subspace of \( H^s(\mathbb{R}^d) \) consisting of the functions which are identically zero outside \( \Omega \) (see (4.3.2.1b) in [6]). That is,
\[
\tilde{H}^s(\Omega) := \left\{ u \in H^s(\mathbb{R}^d); \text{ supp } u \subset \Omega \right\}.
\]
By regarding elements of \( \tilde{H}^s(\Omega) \) as functions defined on \( \Omega \), we have
\[
\tilde{H}^s(\Omega) = H^s_0(\Omega), \quad s \geq 0, \quad s \neq \text{integer} + \frac{1}{2}.
\]
Moreover, we have the following good property for interpolation (see Corollary 1.6 in Chapter 3 of Strichartz [5]);
\[
[\tilde{H}^{s_1}(\Omega), \tilde{H}^{s_2}(\Omega)]_\theta = \tilde{H}^{(1-\theta)s_1 + \theta s_2}(\Omega), \quad 0 \leq s_1 < s_2 < \infty, \quad 0 \leq \theta \leq 1.
\]
By comparing (B.2) with (B.6) and noting (B.5), we have
\[
H^{\mu+1/2}_{00}(\Omega) = \tilde{H}^{\mu+1/2}(\Omega), \quad \mu \in \mathbb{N}_0.
\]
This identity also can be verified by comparing the representations (B.3) and (2.4.2.7) in [6] with \( p = 2 \). Thus the subspace \( \tilde{H}^s(\Omega) \) introduced in (B.4) is more appropriate than \( H^s_0(\Omega) \) when we deal with interpolation.

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