Quantum Mechanics on the curved cotangent bundle - a naïve view

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The minimum length paradigm, a cornerstone of quantum gravity phenomenology, and quantum theories on nontrivial momentum space have been brought into explicit correspondence, recently. However, owing to the fact that coordinate transformations introduce additional position-dependence to the otherwise solely momentum-dependent metric, there is no fully consistent formulation of these theories beyond Cartesian coordinates and, in particular, no position representation. This paper is intended to take the first steps in bridging this gap by promoting the measure of the Hilbert space scalar product to an operator and merging it, by analogy with geometric quantization, with the wave functions. Correspondingly, we find a natural action of the position and momentum operators on the resulting wave densities in both of their representations, as well as the free particle-Hamiltonian and the geodesic distance. These reduce to their counterparts in DeWitt’s approach to quantum mechanics on curved spaces in the corresponding circumstances. We apply the formalism perturbatively in a Riemann normal coordinate-like expansion, with curvature in position and momentum space simultaneously, to the isotropic harmonic oscillator and the hydrogen-like atom. Interestingly, under suitable choice of operator ordering, the former retains its symmetry with respect to exchange of positions and momenta, making it impossible to distinguish position-from momentum space-curvature. We thus obtain a true instantiation of Born reciprocity on the curved cotangent bundle.

I. INTRODUCTION

Nontrivial momentum space, i.e., a dependence of the length element on the direction of motion, has been investigated by mathematicians since it was first mentioned by Bernhard Riemann in his famous habilitation dissertation on the workings of geometry [1]. In particular, significant contributions to the field were made by the likes of Paul Finsler [2] and Élie Cartan [3] and in the context of quantum groups by Vladimir Drinfel’d [4] and Shahn Majid [5, 6]. These days, this approach spans the areas of Lagrangian and Hamiltonian geometries (for a detailed overview, consult Refs. [8, 9]).

Physicists, in contrast, only gradually adopted the idea, for example, to cure divergences through the nonlocal field theory-program [10]. Max Born, in particular, took a different road to arrive at the necessity for curvature in the cotangent bundle [11, 12]: He had noticed that Hamiltonian mechanics is flat space and, rather manifestly, quantum mechanics is symmetric under the exchange

$$\hat{x} \rightarrow \hat{p}, \quad \hat{p} \rightarrow -\hat{x}. \quad (1)$$

This duality, nowadays called "Born reciprocity," owing to the term "reciprocal lattice" in condensed matter physics, is badly broken by general relativity, only allowing position space to be curved. Thus, nontrivial, possibly dynamical momentum space was understood as a necessary condition to retain this symmetry, hence, in Born’s view [11], paving the way towards a unification of quantum theory and general relativity.

The inclusion of nontrivial momentum space into physics was further developed mainly by Yuri Gol’dand [13, 15] and Igor Tamm [16, 17]. This, in fact, lead to the canonical quantization of quantum field theories on curved momentum space [18, 20]. Recent endeavours culminated in the Born geometry program [21–25] in search of a construction capturing all mathematical structures behind Hamiltonian mechanics (symplectic), quantum theory (complex) and general relativity (metric) at once. Nowadays, curved momentum space is mostly found in applications to quantum gravity phenomenology, especially in doubly special (or general) relativity [20, 30]. To the knowledge of the author, as of yet, there has been no application of this idea to nonrelativistic quantum mechanics probably because there was just no need for it. This has changed, however, following very recent developments.

Since the earliest days of the development of the theory [31], the minimum length concept is a hallmark of quantum gravity [32–37]. In fact, it suffices to combine arguments from general relativity and quantum mechanics to arrive at fundamental and absolute limitations to measurements [48–63]. This effect can be modelled easily within nonrelativistic quantum mechanics through modifications to the algebra of observables [64–68], resulting in generalized uncertainty principles (GUPs). Similarly, there are analogous deformations leading to extended [69, 74] and, a combination of the two, generalized extended uncertainty principles [71, 74], in short EUPs and GEUPs, respectively. In that vein, for example, a modification of the form

$$[\hat{x}, \hat{p}] = i\hbar \left(1 + \frac{\beta_{\min}^{2}}{4\hbar^2} \right) \left(1 + \frac{\beta_{\min}^{2}}{4\hbar^2} \right) \quad (2)$$

leads, according to the Robertson relation [76], of position...
and momentum uncertainties $\Delta x$ and $\Delta p$, respectively, to a minimal length as $\Delta x \geq x_{\text{min}}$ as well as a minimal momentum $\Delta p \geq p_{\text{min}}$, in other words a maximal wave length $\lambda_{\text{max}} = \hbar / p_{\text{min}}$. Therefore, understanding the maximal wavelength as the radius of the cosmological horizon, it combines quantum gravitational as well as horizon-induced effects. Despite a number of open issues such as the inverse soccer ball problem \[28\], the classical limit \[77, 78\], friction with the second law of thermodynamics \[79\], combination with gauge symmetries \[80\] or, depending on the relativistic completion, deformation \[81\] or straight breaking \[82\] of Lorentz-invariance, this endeavour has lead to manifold phenomenological applications \[83-97\].

Recent times have seen a merger of the areas of the GUP and curved momentum space, first through indication \[98-104\] and finally explicitly \[105\]. In particular, in Ref. \[105\] it was shown by direct construction that there exists a duality relating GUP-deformed theories with quantum mechanics on nontrivial momentum space. Thus, in quantum theory a minimum length immediately translates to a momentum-dependent metric. As a result, the curvature in momentum space corresponding to the usually applied quadratic GUP is directly proportional to the noncommutativity of the coordinates in position space, a fact, which continues to hold for the nonanalytic linear quadratic GUP \[106\].

However, these results could only be described in terms of Cartesian coordinates the metric cannot be solely momentum-dependent otherwise. A transition to spherical coordinates, for example, implies that the metric harbours position dependence as well. While Bryce DeWitt’s approach \[107\] proves successful in the regime of curved position space in the position representation, it cannot be applied once the metric is of the form

$$g^{ij} = g^{ij}(\hat{x}, \hat{p}).$$

Furthermore, extrapolating from the results in Ref. \[105\], it is natural to expect that GEUPs, which, we recall, are a combination of both assumptions, in fact constitute an effective description of quantum mechanics on a generally nontrivial cotangent bundle. Thus, there is a strong motivation for the construction of a quantum mechanical formalism which can be applied to backgrounds described by metrics which depend on both positions and momenta

$$\text{d}s^2 = g_{ij}(x, p) \text{d}x^i \text{d}x^j, \quad (4)$$

$$\text{d}a^2 = g^{ij}(x, p) \text{d}p_i \text{d}p_j, \quad (5)$$

where the second equality defines the metric in momentum space as the inverse of the one in position space as usual. This immediately implies that the Hilbert space measure both in momentum and in position space gains a dependence on positions and momenta of some sort. The present paper delineates an avenue towards the quantum description of such a theory.

In particular, we aim at a consistent formulation of the quantum mechanics of a single particle on a background of the form \[9\]. In that vein, we choose to take the most naïve way of generalizing DeWitt’s ideas \[107\]. Correspondingly, we promote the measure of the Hilbert space, \(i.e., the volume form, to a, by assumption, Hermitian operator. Furthermore, by analogy with geometric quantization \[108\], we split it into two pieces to merge them symmetrically with each wave function in the scalar product, thus creating wave densities. Then, assuming the scalar product of their eigenvectors consists of plane waves, the position and momentum operators posses particularly simple position and momentum representations. Furthermore, we find representations for the squared momentum, \(i.e., the Hamiltonian of the free particle, and, equivalently, the geodesic distance from the origin. Correspondingly, we define perturbation theory in this context and finally deal with central potentials, particularly the isotropic harmonic oscillator and the hydrogen-like atom, on a background harbouring curvature in position as well as momentum space, described by an expansion akin to Riemann normal coordinates. In this context, we find that the isotropic harmonic oscillator, given a suitable choice of operator ordering, retains its reciprocal nature, thereby resulting in an exact example of Born reciprocity on the curved cotangent bundle.

The newly introduced formalism allows us to describe systems governed by the metric \[3\] but also, for example, find a momentum representation of curved position space and vice versa. We stress, however, that the ansatz developed in the present paper is based on the sole assumption that the metric is of the kind \(3\) and should be understood as a first step into this direction, neglecting many of the subtleties underlying generalized Hamilton spaces (c.f. \[8, 9\]).

The paper is organized as follows. First, we review the approach to quantum mechanics in curved space, developed by DeWitt in section II. The main formalism is introduced as a generalization of DeWitt’s ideas in section III. Section IV is devoted to a perturbative application of the approach to central potentials. Finally, we summarize the results and conclude in section V.

II. TREATMENT FOLLOWING DEWITT

In general, curved \(d\)-dimensional backgrounds are most intuitively introduced into quantum mechanics by virtue of a modified measure of the Hilbert space scalar product

$$\langle \psi | \phi \rangle \equiv \int \text{d}^d x \sqrt{g(x)} \psi^* \phi, \quad (6)$$

where \(g\) denotes the determinant of the background metric. Then, the wave functions, as well as the scalar product transform as scalars. This can be traced back to the definition of the position operator as

$$\hat{x}^i \equiv \int \text{d}^d x \sqrt{g(x)} |x\rangle \langle x|, \quad (7)$$

with its eigenstates \(|x\rangle\) and eigenvalues \(x^i\). The assumption that \(\hat{x}^i\) is Hermitian implies the orthonormality of
its eigenstates

$$1 = \int d^d x \sqrt{g} |x\rangle \langle x|,$$

$$|x\rangle |x'| = \frac{\delta^d(x-x')}{\sqrt{g}},$$

with Dirac's \(\delta\)-distribution in \(d\) dimensions \(\delta^d(x-x').\) Then, the completeness relation \([8]\) can be used to express every state \(|\psi\rangle \in \mathcal{H}\) in the position representation as

$$|\psi\rangle = \int d^d x \sqrt{g} \langle x\psi |x\rangle \equiv \int d^d x \sqrt{g} \psi(x) |x\rangle.$$  \(10\)

Application of Eq. \([10]\) then immediately leads to the scalar product \([10]\).

Correspondingly, the momentum operator has to be modified in order to be Hermitian, resulting in the position representation \([107]\)

$$\hat{p}_i \psi = -i\hbar \left( \partial_i + \frac{1}{2} \Gamma^j_{ij} \right) \psi,$$  \(11\)

with the Christoffel symbol \(\Gamma^j_{ij}\). This implies that we can express a general amplitude containing the operator as

$$\langle \psi|\hat{p}_i|\phi\rangle = \int d^d x \sqrt{g} \psi^* \left(-i\hbar\right) \left( \partial_i \phi + \frac{1}{2} \Gamma^j_{ij} \phi \right).$$  \(12\)

Due to its self-adjointness, the eigenstates of the momentum operator themselves furnish an orthonormal basis of the given Hilbert space. Yet, as the background curvature should also manifest itself in the momentum space-measure to assure its diffeomorphism invariance, it is not clear how to express the momentum basis within this formalism.

As long as the metric depends solely on momenta, this ansatz can, by Born reciprocity, be equivalently applied to nontrivial momentum space in the momentum representation. But, analogously, it cannot yield a corresponding description in position space. Thus, in order to treat these cases, DeWitt’s approach has to be generalised. In turn, this will also open up the possibility of describing quantum mechanics on curved space in the momentum basis.

### III. PROMOTING THE VOLUME ELEMENT TO AN OPERATOR

Clearly, once it depends on positions and momenta, the root of the determinant of the metric appearing in the volume form in the scalar product has to be promoted to an operator \(\sqrt{g}\), which is assumed to be Hermitian and whose eigenvalues are, by definition, positive. In order to do this symmetrically, denote its square root as \(\sqrt{g}\). Then, the position operator can be defined as (c.f. Eq. \([4]\))

$$\hat{x}^i = \int d^d x x^i |\hat{\mu}x\rangle \langle \hat{\mu}x|.$$  \(13\)

Thus, we denote its eigenstates as \(|\hat{\mu}x\rangle = \hat{\mu} |x\rangle\), with the flat-space position eigenstate \(|x\rangle\), such that \(\hat{x}^i |\hat{\mu}x\rangle = x^i |\hat{\mu}x\rangle\). As for DeWitt’s approach, these eigenstates furnish an orthonormal basis, hence satisfying

$$\int d^d x |\hat{\mu}x\rangle \langle \hat{\mu}x| = 1,$$  \(14\)

$$\langle \hat{\mu}x|\hat{\mu}x'\rangle = \delta^d(x-x'),$$  \(15\)

which generalize Eqs. \([5]\) and \([9]\). Then, every state in the Hilbert space can be expanded in the given basis, yielding

$$|\psi\rangle = \int d^d x \langle \hat{\mu}x|\psi\rangle |\hat{\mu}x\rangle.$$  \(16\)

Defining the position space wave function as \(\psi \equiv \langle x|\psi\rangle\), we can express amplitudes in terms of the wave density \(\Phi\), such that \(\Phi = \langle x|\phi\rangle\), leading to the description à la DeWitt (c.f. Eqs. \([7]\), \([5]\) and \([9]\). As can be inferred from Eq. \([18]\), this procedure provides a map from quantum mechanics on a curved cotangent bundle into ordinary flat space-quantum mechanics, given the actions of operators on wave densities like \(\Phi\) and \(\Psi\).

Analogously, we can define the momentum operator as

$$\hat{p}_i = \int d^d p p_i |\hat{\mu}^{-1}p\rangle \langle \hat{\mu}^{-1}p|,$$  \(19\)

whose eigenstates \(|\hat{\mu}^{-1}p\rangle\) obey the relations

$$\int d^d p |\hat{\mu}^{-1}p\rangle \langle \hat{\mu}^{-1}p| = 1,$$  \(20\)

$$\langle \hat{\mu}^{-1}p|\hat{\mu}^{-1}p'\rangle = \delta^d(p-p').$$  \(21\)

Accordingly, this provides us with the momentum space representation

$$|\psi\rangle = \int d^d p \langle \hat{\mu}^{-1}p|\psi\rangle |\hat{\mu}^{-1}p\rangle \equiv \int d^d p \mu^{-1}(\tilde{\psi})(p) |\mu p\rangle$$  \(22\)

We use the term “wave density” because the quantities \(\Phi, \Psi\ldots\) and their correspondents in momentum space \(\Phi, \Psi\ldots\) change under coordinate transformations not as scalar functions but as scalar densities of weight \(1/2\) and \(-1/2\) respectively.
and, defining \( \tilde{\Gamma}(p) \equiv \langle p|\psi \rangle \) and \( \tilde{\Psi}(p) \equiv \langle \tilde{\mu}^{-1} p|\psi \rangle = \tilde{\mu}^{-1}(\tilde{\psi}) \), with the scalar product
\[
\langle \psi|\phi \rangle = \int d^d p \langle \psi|\tilde{\mu}^{-1} p \rangle \langle \tilde{\mu}^{-1} p|\phi \rangle = \int d^d p \tilde{\psi}^* \tilde{\Phi}.
\]
Before this formalism can be applied to examples, though, it remains to be shown that this construction is well-defined.

### A. Plane wave basis

From flat-space quantum mechanics we know that the eigenfunctions of the momentum operator in the position representation are plane waves. We equally assume in the general case that
\[
\langle \tilde{\mu} x|\tilde{\mu}^{-1} p \rangle = \langle x|p \rangle = e^{\frac{ipx}{\hbar}} / \sqrt{2\pi\hbar}.
\]
This assumption is vital for the approach introduced here which would be inconsistent otherwise.

Then, the overlap of position and momentum eigenstates satisfies
\[
\langle \tilde{\mu} x|\tilde{\mu}^{-1} p \rangle = \langle x|p \rangle = -i\hbar \partial_x \langle \tilde{\mu} x|\tilde{\mu}^{-1} p \rangle.
\]
This provides us with the position representation of the momentum operator
\[
\langle \tilde{\mu} x|\tilde{\mu}^{-1} p \rangle = \hat{p}_i \Psi = -i\hbar \partial_x \Psi,
\]
leading to the general amplitude
\[
\langle \psi|\tilde{\mu}^{-1} p \rangle = \int d^d x \Psi^* (-i\hbar \partial_x \Psi),
\]
which in the case of a purely position-dependent measure just recovers Eq. (12). Analogously, we obtain for the momentum representation of the position operator
\[
\langle \tilde{\mu}^{-1} p|\tilde{x}^i \psi \rangle = \hat{x}^i \tilde{\Psi} = i\hbar \delta^i_0 \tilde{\Psi},
\]
with \( \delta^i_0 \equiv \partial / \partial p_i \).

Using these relations, it is easy to show that the position and momentum operators defined this way are indeed Hermitian (due to the symmetric choices in Eqs. (13) and (19)) and satisfy the canonical commutation relations
\[
[\hat{x}^i, \hat{p}_j] = i\hbar \delta^i_j, \quad [\hat{x}^i, \hat{x}^j] = [\hat{p}_i, \hat{p}_j] = 0.
\]
Thus, \( \tilde{\mu} \), itself a real function of \( \hat{x}^i \) and \( \hat{p}_i \), can be made Hermitian by suitable symmetrization and is well-defined modulo operator ordering ambiguities.

Furthermore, the Fourier transform is generalized such that it reads
\[
\tilde{\Psi} = \int d^d x \langle \tilde{\mu}^{-1} p|\tilde{x} \rangle \langle \tilde{\mu} x|\psi \rangle = \int d^d x \tilde{\Psi} e^{-ip\hat{x}^i / \hbar},
\]
and inversely
\[
\Psi = \frac{1}{\sqrt{2\pi\hbar}} \int d^d p \tilde{\Psi} e^{ip\hat{x}^i / \hbar}.
\]
We conclude that the procedure outlined above is consistent.

### B. Free particle

The Hamiltonian of a free particle of mass \( M \) subject to a general metric \( g^{ij}(x,p) \) reads \( H_{fp} = \frac{1}{2} g^{ij} p_i p_j / 2M \). If the metric is just a function of the coordinates, i.e. the particle is living in curved space only, the quantum operator reflecting the Hamiltonian in the position representation is aptly chosen to be the Laplace-Beltrami operator
\[
\tilde{H}_{fp} \Psi = -\frac{\hbar^2}{2m} \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} g^{ij} \partial_j \Psi \right)
\]
leading to the general amplitude
\[
\langle \psi|\tilde{H}_{fp} \phi \rangle = -\frac{\hbar^2}{2m} \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} g^{ij} \partial_j \phi \right).
\]
In the more general case treated here this can be naturally generalized to the form
\[
\langle \tilde{\mu} x|\tilde{H}_{fp} \psi \rangle = -\frac{\hbar^2}{4m} \tilde{\mu}^{-1} \partial_i \left( \{ \tilde{\mu}^2, \tilde{g}^{ij} \} \partial_j \tilde{\Psi} \right),
\]
where the metric in its dependence on positions and momenta has been promoted to a Hermitian operator, which does not necessarily commute with the square root of its determinant \( \tilde{\mu}^2 \), and we introduced the anti-commutator \( \{ A, B \} = AB + BA \). In other words, we obtain
\[
\tilde{H}_{fp} \Psi = \frac{1}{4m} \tilde{\mu}^{-1} \tilde{p}_i \{ \tilde{\mu}^2, \tilde{g}^{ij} \} \tilde{p}_j \tilde{\mu}^{-1} \Psi,
\]
which as a symmetric product of Hermitian operators is Hermitian. Furthermore, the expression \( \tilde{H}_{fp} \Psi \) transforms as a scalar density \( \tilde{\mu} \) multiplying a scalar \( \tilde{\mu}^{-2} \partial_i \left( \{ \tilde{\mu}^2, \tilde{g}^{ij} \} \partial_j \psi \right) \), i.e. a scalar density of weight 1/2, as it should.

A general amplitude featuring the Hamiltonian of a free particle can thus be expressed as
\[
\langle \psi|\tilde{H}_{fp} \phi \rangle = -\frac{\hbar^2}{4m} \int d^d x \Psi^* \tilde{\mu}^{-1} \partial_i \left( \{ \tilde{\mu}^2, \tilde{g}^{ij} \} \partial_j \tilde{\mu}^{-1} \Psi \right),
\]
which in the case of a purely position-dependent metric exactly recovers (35).

Having derived the action of the basic operators in the given framework, we can now compute perturbative corrections to classical textbook problems.
IV. PERTURBATIVE APPLICATION

Classical and quantum gravity-induced corrections to quantum mechanical experiments are usually negligibly small. Thus, it suffices to treat the corresponding effects perturbatively, i.e. we can assume the background to be flat at leading order. If solely position space is curved, this expansion around a point \( x_0 \) may be given in terms of Riemann normal coordinates \( x^I \) [100], yielding corrections to second order, depending on the Riemann tensor \( R_{kjl} \) evaluated at \( x_0 \). When both, position and momentum space, are curved, the corresponding curvature tensors (denoted \( R_{kjl} \) in position space and \( S^{kjl} \) in momentum space) can still be defined even though they can both be position and momentum dependent. Assuming that there is no mixing of positions and momenta to second order, we then propose an analogous expansion at

\[
\Psi = \Psi^{(0)} + \Psi^{(2)} + \cdots
\]

in terms of Riemann normal coordinates. Moreover, linearising Eq. (47)

\[
V(\sigma) = V(r) + \frac{V'(r)}{r} x^i x^j S_{ij}^k |\gamma_k p_k p_l|
\]

which, depending on the curvature tensor in momentum space, may actually cease to be isotropic.

As this potential is both position and momentum-dependent, promoting it to a Hermitian operator is an ambiguous task. The resulting operator ordering ambiguities have to be treated with care on a case-by-case basis, which will be done in the following examples.
1. Isotropic harmonic oscillator

According to Eq. [48], the potential describing an harmonic oscillator reads

\[
V(\sigma) \simeq \frac{1}{2} M \omega^2 \left( r^2 + \frac{1}{3} x^i x^j S_{i,j}^{k,l} |y_0 \hat{p}_k \hat{p}_l \right),
\]

with the oscillation frequency \( \omega \). Taking into account all possible operator orderings combining the four non-commuting contributions \( \hat{x}^i, \hat{x}^j, \hat{p}_k, \hat{p}_l \) and after application of simple algebra, we obtain the correction to the corresponding operator

\[
\hat{V}_{(2)} = \frac{1}{6} M \omega^2 \left( \hat{x}^i \hat{x}^j S_{i,j}^{k,l} |y_0 \hat{p}_k \hat{p}_l + i h S_{i,j}^l |y_0 \hat{x}^i \hat{p}_j + A h^2 S |y_0 \right),
\]

with \( n \)-dimensionless \( A \) parametrizes the operator ordering ambiguity. A close comparison of Eqs. (54) and (50) makes apparent that it can take a somewhat distinguished value: if \( A = -1/2 \), the implications of position and momentum space-curvature are analogous and Born reciprocity, a crucial feature of the harmonic oscillator mentioned in the introduction, is restored in adapted "natural units" \( (M = \omega = 1) \).

The eigenstates and -values of the unperturbed Hamiltonian describing the harmonic oscillator \( \hat{H}_{(0)} = \hat{H}^{(0)} + \hat{V}_{(0)} \) read in the position representation and in spherical coordinates [110]

\[
\psi_{nlm}^{(0)} = N_{nl} r^l e^{-\frac{M}{2h} r^2} L^{l+1/2}_{\frac{n-l}{2}} \left( \frac{M \omega^2}{h} r^2 \right) Y_{nlm}^{(0)}(\theta, \phi),
\]

\[
N_{nl} = \sqrt{\frac{M^3 \omega^{5/2} 2^{n-l+3} (n-l+1)!}{\pi h^6 (2n + l + 1)!}},
\]

\[
E_{nlm}^{(0)} = \langle \psi_{nlm}^{(0)} | \hat{H}_{(0)} | \psi_{nlm}^{(0)} \rangle = h \omega \left( n + \frac{3}{2} \right),
\]

with the generalized Laguerre polynomials \( L^n_\alpha(x) \), the spherical harmonics \( Y_{nlm}(\theta, \phi) \) and where we introduced the quantum numbers \( n \geq 0 \) (radial), \( l = n \) (mod 2), \( n \) (mod 2) + 2, \ldots \( n \) and \( m = -l, -l + 1, \ldots, l \) (angular). Correspondingly, the position and momentum space curvature induces corrections, which can be calculated, according to Eq. (54), as

\[
E_{nlm}^{(2)} = \int d^3x \psi_{nlm}^{(0)} \hat{H}^{(2)} | \psi_{nlm}^{(0)} \rangle.
\]

These integrals can be solved numerically. The resulting contribution exactly follows the pattern

\[
E_{nlm}^{(2)} = \frac{h^2}{6} \left[ \left( l(l+1) - 3m^2 \right) \left( \frac{M \omega^2}{2} S_{zz} + \frac{R_{zz}}{2M} \right) + (m^2 - 1) \frac{R}{2M} + (m^2 + 2A) \frac{M \omega^2}{2} S \right] |y_0\rangle,
\]

This has been checked for all quantum numbers up to \( n = 10 \). In the case \( A = -1/2 \), the correction reduces to

\[
E_{nlm}^{(2)} = \frac{h^2}{6} \left[ \left( l(l+1) - 3m^2 \right) \left( \frac{M \omega^2}{2} S_{zz} + \frac{R_{zz}}{2M} \right) + (m^2 - 1) \left( \frac{M \omega^2}{2} S + \frac{R}{2M} \right) \right] |y_0\rangle,
\]

highlighting the distinction of this value. As a corollary, for this choice it is impossible to discriminate between a position space curvature with a single harmonic oscillator of mass \( M \) and frequency \( \omega \). Evidently, this could have been concluded immediately from the reciprocal property of the quantum Hamiltonian.

Thus, under the assumption of a reciprocal Hamiltonian like the isotropic harmonic oscillator, we exactly obtain the behaviour, Born was striving for, when trying to merge quantum theory with general relativity through curving momentum space.

2. Coulomb potential

The coulomb potential, describing a hydrogen-like atom, is corrected in a way similar to the isotropic harmonic oscillator. Again, using Eq. (48), it reads to second order

\[
V(\sigma) \simeq \frac{Z \alpha h}{r} \left( 1 - \frac{1}{2} \frac{S_{i,j}^{k,l}}{r^2} x^i x^j \hat{p}_k \hat{p}_l \right),
\]

with the fine-structure constant \( \alpha \approx 1/137 \) and the number of elementary charges in the origin \( Z \). In this case, the corresponding quantum operator can be found unambiguously under the assumption that a perturbative treatment is indeed possible: The corrections induced by terms which are dependent on the ordering scale as \( r^{-3} \). Thus, the corresponding expectation values with respect to the unperturbed eigenstates of the Hamiltonian diverge. These contributions to the potential are clearly too singular. Hence, omitting them is equivalent to renormalizing the problem. In short, the quantum operator representing the corrections to the potential may be unambiguously expressed as

\[
\hat{V}_{(2)} = -\frac{\alpha h}{6r^3} \left( \frac{S_{i,j}^{k,l}}{a_0^2} |y_0 \hat{x}^i \hat{x}^j \hat{p}_k \hat{p}_l + i h S_{i,j}^l |y_0 \hat{x}^i \hat{p}_j \rangle \right).
\]

The unperturbed version of this problem leads to the eigenstates and -values [110]

\[
\psi_{nlm}^{(0)} = N_{nl} \left( \frac{2n!}{a_0^2} \right)^{l/2} e^{-\frac{2\pi}{a_0} L_{n-l-1}^{2l+1} \left( \frac{2n r}{a_0} \right) Y_{nlm}^{(0)}(\theta, \phi),
\]

\[
N_{nl} = \sqrt{\frac{2}{na_0^3} \left( n-l-1 \right)! \left( 2n(l+l+1) \right)! \left( 2n(l+l+1) \right)^{l+1}},
\]

\[
E_{nlm}^{(0)} = -\frac{Z^2 \alpha^4 M}{2n^2},
\]

(58)
with the reduced Bohr radius \( a_0^* = \hbar/Z\alpha M \) and the quantum numbers \( n > 0, \ell = 0, 1, \ldots, n \) and \( m = -\ell, -\ell + 1, \ldots, \ell \).

By analogy with the treatment of the harmonic oscillator, the following pattern found for the corrections to the energy eigenvalues has been verified for all quantum numbers until \( n = 10 \). On the one hand, we obtain

\[
E^{(2)}_{n00} = -\frac{\hbar^2}{12M} R|_{Y_0}. \tag{62}
\]

On the other hand, if \( \ell \neq 0 \), the resulting corrections read

\[
E^{(2)}_{nlm} = \frac{\hbar^2}{12M} \left[ (m^2 - 1)R + (l(l+1) - 3m^2)R_{zz} \right]|_{Y_0} + \frac{M^3\alpha^4}{6n^2(l+1)(2l+1)} \times \left[ m^2S + (l(l+1) - 3m^2)S_{zz} \right]|_{Y_0}. \tag{63}
\]

Thus, a hydrogen-like atom could clearly be used as a means of distinction between curvature in position and momentum space. Although the resulting splitting receives quite similar contributions from both sides, the terms resulting changes induced by momentum space-curvature strongly decay towards higher values of \( n \) and \( \ell \) and vanish for \( l = 0 \). Moreover, they are much stronger for particles of higher mass. Indeed, the relative contribution of both types of curvature scales as \( \sim M^4 \), given that \( R\hbar^2 \sim 1/S \), where \( R \) and \( S \) stand for respective curvature related scales.

V. CONCLUSION

Recent developments at the intersection of the minimum length-paradigm, a cornerstone of quantum gravity phenomenology, and nontrivial momentum space necessitate the creation of a formalism, capable of describing quantum mechanics on background metrics, which are simultaneously position- and momentum dependent. In particular, this approach is required to investigate the position representation of quantum mechanics on a nontrivial momentum space, which was recently found to be dual to GUP-deformed quantum theory. Furthermore, this formalism sets the stage to find position and momentum representations of the GEUP, a task which had proven to be illusive thus far.

The present paper marks a first step towards such a description, taking a rather intuitive route. In particular, the Hilbert space measure, derived from the metric á la DeWitt, is promoted to an operator. It is subsequently split into two pieces and merged symmetrically with the wave functions entering the scalar product, thus yielding wave densities by analogy with the geometric quantization program. Resultingly, under the assumption that the scalar product of their eigenstates is given by plane waves, the position and momentum operators can be defined by analogy with their Euclidean counterparts for both of their representations. Furthermore, we find a representation of the Hamiltonian of a free particle and, analogously, the geodesic distance to the origin.

This makes it possible to investigate a metric, which is expanded in position and momentum space by analogy with Riemann normal coordinates. After a short discussion of perturbation theory in this context, we apply the formalism to two central potentials, the harmonic oscillator and the hydrogen-like atom. As a result, we analytically obtain corrections to the eigenvalues of the Hamiltonian in terms of the Ricci scalars and tensors in position and momentum space. Interestingly, given the right choice of ordering, the isotropic harmonic oscillator retains its symmetry between positions and momenta, thus, in principle, making it impossible to distinguish between curvature in position and momentum space. Resultingly, we achieve an instantiation of Born reciprocity in quantum mechanics on the curved cotangent bundle - exactly as intended by Born.

These encouraging results will make it possible to investigate the position representation of theories of curved momentum space dual to noncommutative geometries and GUPs, i.e., instances of quantum spaces. Furthermore, it would be interesting to see, whether it is possible to describe theories of GEUP-deformed quantum mechanics in this language. These ideas will be the subject of future research.

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