A short note on $q$-analogue of modified Stancu-Beta operators

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Abstract

This paper deals with the modified $q$-Stancu-Beta operators and we have investigated the statistical approximation theorems for these operators with the help of the Korovkin type approximation theorem. We have also established the rates of statistical convergence by means of the modulus of continuity and the Lipschitz type maximal function. Our results show that the rates of convergence of our operators are at least as fast as the classical Stancu-Beta operators.

Keywords: $q$-integers, statistical convergence; $q$-Stancu-Beta operators; rate of statistical convergence; modulus of continuity; positive linear operators; Korovkin type approximation theorem.

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1. Introduction

In 1995, Stancu [16] introduced Beta operators $L_n$ of second kind in order to approximate the Lebesgue integrable functions on the interval $I = (0, \infty)$ as follows:

$$L_n(f; x) = \frac{1}{B(nx, n+1)} \int_0^{\infty} \frac{t^{nx-1}}{(1 + t)^{nx+n+1}} f(t) dt,$$  \hspace{1cm} (1.1)

where, $x \in I, n \in \mathbb{N}$, $f$ is a real-valued functions defined on $I$, and $B$ is the Euler Beta function defined as:

$$B(t, s) = \int_0^{\infty} \frac{x^{t-1}}{(1 + x)^{t+s}} dt, \hspace{0.5cm} t, s > 0.$$ 

In 2012, Aral and Gupta [1] introduced the $q$-analogue of Stancu-Beta operators as follows:

$$L_n^q(f; x) = \frac{K(A, [n]_q x)}{B_q([n]_q x, [n]_q + 1)} \int_0^{\infty/A} \frac{u^{[n]_q x - 1}}{(1 + u)^{[n]_q x + [n]_q + 1}} f(q^{[n]_q x} u) d_q u, \hspace{0.5cm} \forall n \in \mathbb{N}$$  \hspace{1cm} (1.2)

where $q \in (0, \infty)$, and $x \in [0, \infty)$. In this paper they estimated moments, and direct results in terms of modulus of continuity and their asymptotic formulae.

The following theorem on operators (1.2) is given by Aral and Gupta [1]:

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Theorem 1. Let \( q = (q_n) \) satisfy \( 0 < q_n < 1 \) with \( \lim_{n \to \infty} q_n = 1 \). For each \( f \in C_{x^2}^r[0, \infty) \), we have

\[
\lim_{n \to \infty} \| (L_n^q(f); .) - f \|_{x^2} = 0,
\]

where \( C_{x^2}[0, \infty) \) denotes the subspace of all continuous functions on \([0, \infty)\) such that \( |f(x)| \leq M_f \), and \( C_{x^2}^r[0, \infty) \) denotes the spaces of all \( f \in C_{x^2}[0, \infty) \) such that \( \lim_{x \to \infty} \frac{f(x)}{1+x^2} \) is finite. The norm on \( C_{x^2}^r[0, \infty) \) is given by \( \| f \|_{x^2} = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1+x^2} \).

After that Mursaleen and Khan [15] defined modified \( q \)-Stancu-Beta operators as follows:

\[
L_n^*(f; q, x) = q \frac{K(A, [n]_q x)}{B_q([n]_q x, [n]_q + 1)} \int_0^{\infty/A} \frac{u^{[n]_q x - 1}}{(1 + u)^{[n]_q x + [n]_q + 1}} f([n]_q x u) du \quad (1.3)
\]

where \( x \geq 0, 0 < q \leq 1 \). It is easy to verify that if \( q = 1 \), then these operators turns into the classical Stancu-Beta operators.

In 2014, Cai [3] introduced a new kind of modification of \( q \)-Stancu-Beta operators which preserve \( x^2 \) based on the concept of \( q \)-integer

\[
\mathcal{L}_{n,q}(f; x) = \frac{K(A, [n]_q v_n(x))}{B_q([n]_q v_n(x), [n]_q + 1)} \int_0^{\infty/A} \frac{u^{[n]_q v_n(x) - 1}}{(1 + u)^{[n]_q v_n(x) + [n]_q + 1}} f([n]_q v_n(x) u) du \quad (1.4)
\]

where

\[
v_n(x) = \sqrt{\frac{q[n]_q - q}{4[n]_q^2} + \frac{1}{2[n]_q^2} - \frac{1}{2[n]_q}} \quad \text{for } f \in C[0, \infty), \ n \in \mathbb{N}, q \in (0, 1).
\]

2. Some preliminary results

In this section we give the following lemmas, which we need to prove our theorems.

Lemma 1. [1] The following equalities hold:

\[
L_n^q(1; x) = 1, \quad L_n^q(t; x) = x \quad \text{and} \quad L_n^q(t^2; x) = \frac{([n]_q x + 1)x}{q([n]_q - 1)}.
\]

Lemma 2. [12] Note that \( L_n^q(f; q, x) = L_n^q(f; x) \) and from the Lemma 1 of Aral and Gupta [3], we have \( L_n^q(1; x) = 1, L_n^q(t; x) = x, L_n^q(t^2; x) = \frac{([n]_q x + 1)x}{q([n]_q - 1)}. \) Hence for \( x \geq 0, \ 0 < q \leq 1 \), we have

\[
L_n^*(1; q, x) = q, \quad L_n^*(t; q, x) = q x \quad \text{and} \quad L_n^*(t^2; q, x) = \frac{([n]_q x + 1)x}{([n]_q - 1)}.
\]

Lemma 3. [3] Let \( q \in (0, 1), \ x \in [0, \infty), \) we have

(i) \( \mathcal{L}_{n,q}(1; x) = 1 \)

(ii) \( \mathcal{L}_{n,q}(t; x) = \sqrt{\frac{q[n]_q - q}{4[n]_q^2} + \frac{1}{4[n]_q^2} - \frac{1}{2[n]_q}} \)

(iii) \( \mathcal{L}_{n,q}(t^2; x) = x^2 \)
Remark 1. Let $q \in (0, 1)$, then for $x \in [0, \infty)$, we have

$$\alpha_n(x) = \mathcal{L}_{n,q}(t-x;x) = 0$$

and

$$\delta_n(x) = \mathcal{L}_{n,q}((t-x)^2;x) = 2x^2 - 2x\sqrt{\frac{q[n]_q - q}{[n]_q}}x^2 + \frac{1}{4[n]_q^2} + \frac{x}{[n]_q}.$$ 

First, we recall certain notations of $q$-calculus and the details on $q$-integers can be found in [6, 11]. The $q$-integer $[k]_q$ for each nonnegative integer is defined as

$$[k]_q := \begin{cases} \frac{(1-q^k)}{(1-q)}, & q \neq 1 \\ k, & q = 1 \end{cases}$$

and

$$[k]_q! := \begin{cases} [k]_q[k-1]_q...[1]_q, & k \geq 1 \\ 1, & k = 0 \end{cases}$$

The $q$-improper integral (Koornwinder [13]) is defined as

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, \quad A > 0.$$ 

Now, we extend the earlier work done by Cai [3]. By considering the operators defined in [3], we shall obtain some approximation properties of Modified $q$ Stancu-Beta operators. We shall also estimate the rate of statistical convergence of these sequence of the operators.

3. Korovkin type statistical approximation properties

Firstly, we recall the concept of statistical convergence for sequences of real numbers which was introduced by Fast [9] and further studied by [7, 8, 10] and many others.

Let $K \subseteq \mathbb{N}$ and $K_n = \{j \leq n : j \in K\}$. Then the natural density of $K$ is defined by $\delta(K) = \lim n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of the set $K_n$.

A sequence $x = (x_j)_{j \geq 1}$ of real numbers is said to be statistically convergent to $L$ provided that for every $\epsilon > 0$ the set $\{j \in \mathbb{N} : |x_j - L| \geq \epsilon\}$ has natural density zero, i.e. for each $\epsilon > 0$,

$$\lim_{n} \frac{1}{n}|\{j \leq n : |x_j - L| \geq \epsilon\}| = 0.$$ 

It is denoted by $\text{st} - \lim x_n = L$.

Doru and Kanat [5], defined the Kantorovich-type modification of Lupaş operators as follows:
Theorem 2. Let \( q := (q_n) \), \( 0 < q < 1 \), be a sequence satisfying the following conditions:

\[
st - \lim_{n \to \infty} q_n = 1, \quad st - \lim_{n \to \infty} q_n^n = a \quad (a < 1) \text{ and } st - \lim_{n \to \infty} \frac{1}{n} = 0,
\]

then if \( f \) is any monotone increasing function defined on \([0,1]\), for the positive linear operator \( \tilde{R}_n(f;q;x) \), then

\[
st - \lim_{n \to \infty} \| \tilde{R}_n(f;q) - f \|_{C[0,1]} = 0
\]

holds.

Doğru [4] gave some examples so that \( (q_n) \) is statistically convergent to 1 but it may not convergent to 1 in the ordinary case.

Now, we consider a sequence \( q = (q_n) \), \( q_n \in (0,1) \), such that

\[
\lim_{n \to \infty} q_n = 1.
\]

The condition (3.3) guarantees that \([n]_{q_n} \to \infty\) as \( n \to \infty \).

Theorem 3. Let \( \mathcal{L}_{n,q_n} \) be the sequence of the operators \((1.4)\) and the sequence \( q = (q_n) \) satisfies \((3.2)\). Then for any function \( f \in C[0,\nu] \subset C[0,\infty), \nu > 0 \), we have

\[
st - \lim_{n \to \infty} \| \mathcal{L}_{n,q_n}(f, \cdot) - f \| = 0,
\]

where \( C[0,\nu] \) denotes the space of all real bounded functions \( f \) which are continuous in \([0,\nu]\).

Proof. Let \( f_i = t^i \), where \( i = 0,1,2 \). As \( \mathcal{L}_{n,q_n}(1;x) = 1 \), and \( \mathcal{L}_{n,q_n}(t^2;x) = x^2 \), (see Lemma [3], [3.4]) holds true for \( i=0 \) and \( i=2 \). So, \( \mathcal{L}_{n,q_n}(1;x) = 1 \), it is clear that

\[
st - \lim_{n \to \infty} \| \mathcal{L}_{n,q_n}(1;x) - 1 \| = 0. \quad \text{and} \quad st - \lim_{n \to \infty} \| \mathcal{L}_{n,q_n}(t^2;x) - x^2 \| = 0.
\]

Finally, for \( i=1 \), by Lemma (3)(ii), we have

\[
\| \mathcal{L}_{n,q_n}(t,x) - x \| = \sqrt{\frac{[q]_{n}[q]_{n}^{q} - q}{[n]_{q}^{q}} x^2 + \frac{1}{4[n]_{q}^{q}} - x} \leq \left( 1 - \sqrt{\frac{[q]_{n}[q]_{n}^{q} - q}{[n]_{q}^{q}}} \right) x + \frac{1}{2[n]_{q}^{q}}.
\]

For given \( \epsilon > 0 \), we define the following sets:

\[
U = \{ k : \| \mathcal{L}_{n,q_n}(t,x) - x \| \geq \epsilon \},
\]

and
\[ U' = \left\{ k : \left(1 - \sqrt{\frac{q[k]-q}{[k]}}\right)x + \frac{1}{2[k]} \geq \epsilon \right\}. \] (3.5)

It is obvious that \( U \subset U' \), it can be written as

\[ \delta (\{k \leq n : \| L_{n,q}(t,x) - x \| \geq \epsilon \}) \leq \delta \left( \{k \leq n : \left(1 - \sqrt{\frac{q[k]-q}{[k]}}\right)x + \frac{1}{2[k]} \geq \epsilon \} \right). \]

By using (3.2), we get

\[ \text{st- lim}_{n \to \infty} \left(1 - \sqrt{\frac{q[n]-q}{[n]}}\right)x + \frac{1}{2[n]} = 0. \]

So, we have

\[ \delta \left( \{k \leq n : \left(1 - \sqrt{\frac{q[n]-q}{[n]}}\right)x + \frac{1}{2[n]} \geq \epsilon \} \right) = 0, \]

then

\[ \text{st- lim}_{n \to \infty} \| L_{n,q}(t,x) - x \| = 0. \]

This completes the proof of theorem. \( \Box \)

4. Weighted statistical approximation

In this section, we obtain the Korovkin type weighted statistical approximation by the operators defined in (1.4). A real function \( \rho \) is called a weight function if it is continuous on \( \mathbb{R} \) and \( \lim_{|x| \to \infty} \rho(x) = \infty \), \( \rho(x) \geq 1 \) for all \( x \in \mathbb{R} \).

Let by \( B_\rho(\mathbb{R}) \) denote the weighted space of real-valued functions \( f \) defined on \( \mathbb{R} \) with the property \( |f(x)| \leq Mf \rho(x) \) for all \( x \in \mathbb{R} \), where \( Mf \) is a constant depending on the function \( f \). We also consider the weighted subspace \( C_\rho(\mathbb{R}) \) of \( B_\rho(\mathbb{R}) \) given by \( C_\rho(\mathbb{R}) = \{ f \in B_\rho(\mathbb{R}) : f \text{ continuous on } \mathbb{R} \} \). Note that \( B_\rho(\mathbb{R}) \) and \( C_\rho(\mathbb{R}) \) are Banach spaces with \( \| f \|_\rho = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{\rho(x)} \).

In case of weight function \( \rho_0 = 1 + x^2 \), we have \( \| f \|_{\rho_0} = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{1 + x^2} \).

Now we are ready to prove our main result as follows:

**Theorem 4.** Let \( L_{n,q,n}(f;x) \) be the sequence of the operators (1.4) and the sequence \( q = (q_n) \) satisfies (3.2). Then for any function \( f \in C_B[0, \infty) \), we have

\[ \text{st- lim}_{n \to \infty} \| L_{n,q,n}(f;x) - f \|_{\rho_0} = 0. \]

**Proof.** By Lemma (3)(iii), we have \( L_{n,q,n}(t^2,x) \leq Cx^2 \), where \( C \) is a positive constant, \( L_{n,q,n}(f;x) \)
is a sequence of positive linear operator acting from $C_\rho[0, \infty)$ to $B_\rho[0, \infty)$.

Let $f_i = t^i$, where $i = 0, 1, 2$. Since $L_{n,q}(1; x) = 1$, and $L_{n,q}(t^2; x) = x^2$, (see Lemma (3), (3.4)
holds true for $i=0$ and $i=2$. So, $L_{n,q}(1; x) = 1$, it is clear that
$st \lim_n \|L_{n,q}(1; x) - 1\|_{\rho_0} = 0$. and $st \lim_n \|L_{n,q}(t^2; x) - x^2\|_{\rho_0} = 0$.
Finally, for $i=1$, by Lemma (3)(ii), we have

$$
\|L_{n,q}(t, x) - x\|_{\rho_0} = \sup_{x \in [0, \infty)} \frac{|L_{n,q}(t, x) - x|}{1 + x^2}
\leq \left( 1 - \sqrt{q[n]_q - q} \right) \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} + \frac{1}{2[n]_q} \sup_{x \in [0, \infty)} \frac{1}{1 + x^2},
$$

Using (3.2), we get

$$
st \lim_n \left( 1 - \sqrt{q[n]_q - q} \right) + \frac{1}{2[n]_q} = 0,
$$
then

$$
st \lim_n \|L_{n,q}(t, x) - x\|_{\rho_0} = 0.
$$

This completes the proof of the theorem. \hfill \Box

5. Rates of statistical convergence

In this section, by using the modulus of continuity, we will study rates of statistical convergence
of operator (1.4) and Lipschitz type maximal functions are introduced.

Lemma 4. Let $0 < q < 1$ and $a \in [0, bq]$, $b > 0$. The inequality

$$
\int_a^b |t - x| d_q t \leq \left( \int_a^b |t - x|^2 d_q t \right)^{1/2} \left( \int_a^b d_q t \right)^{1/2}
$$

(5.1)
is satisfied.

Let $C_B[0, \infty)$, the space of all bounded and continuous functions on $[0, \infty)$ and $x \geq 0$. Then, for $\delta > 0$, the modulus of continuity of $f$ denoted by $\omega(f; \delta)$ is defined to be

$$
\omega(f; \delta) = \sup_{|t - x| \leq \delta} |f(t) - f(x)|, \ t \in [0, \infty).
$$

It is known that $\lim_{\delta \to 0} \omega(f; \delta) = 0$ for $f \in C_B[0, \infty)$ and also, for any $\delta > 0$ and each $t, x \geq 0$, we have
\[|f(t) - f(x)| \leq \omega(f; \delta) \left(1 + \frac{|t - x|}{\delta}\right). \quad (5.2)\]

**Theorem 5.** Let \((q_n)\) be a sequence satisfying (3.2). For every non-decreasing \(f \in C_B[0, \infty), \ x \geq 0 \) and \(n \in \mathbb{N}\), we have
\[
|\mathcal{L}_{n, q_n}(f, x) - f(x)| \leq 2\omega(f; \sqrt{\delta_n(x)}),
\]
where
\[
\delta_n(x) = 2x^2 - 2x \sqrt{\frac{q[n]_q - q[n]_q}{[n]_q^2}} + \frac{1}{4[n]_q} + \frac{x}{[n]_q}.
\]

**Proof.** Let \(f \in C_B[0, \infty)\) be a non-decreasing function and \(x \geq 0\). Using linearity and positivity of the operators \(\mathcal{L}_{n, q_n}\) and then applying (5.2), we get for \(\delta > 0\)
\[
|\mathcal{L}_{n, q_n}(f, x) - f(x)| \leq \mathcal{L}_{n, q_n}(|f(t) - f(x)|, x) \\
\leq \omega(f, \delta) \{\mathcal{L}_{n, q_n}(1, x) + \frac{1}{\delta}\mathcal{L}_{n, q_n}(|t - x|, x)\}.
\]
Taking \(\mathcal{L}_{n, q_n}(1, x) = 1\) and using Cauchy-Schwartz inequality, we have
\[
|\mathcal{L}_{n, q_n}(f, x) - f(x)| \leq \omega(f; \delta) \left\{1 + \frac{1}{\delta} \left(\mathcal{L}_{n, q_n}((t - x)^2, x)^{1/2} \mathcal{L}_{n, q_n}(1, x)^{1/2}\right)\right\} \\
\leq \omega(f; \delta) \left[1 + \frac{1}{\delta} \left\{2x^2 - 2x \sqrt{\frac{q[n]_q - q[n]_q}{[n]_q^2}} + \frac{1}{4[n]_q} + \frac{x}{[n]_q}\right\}^{1/2}\right].
\]
Taking \(q = (q_n)\), a sequence satisfying (3.2) and choosing \(\delta = \delta_n(x)\) as in (5.3), the theorem is proved. \(\square\)

Now we will give an estimate concerning the rate of approximation by means of Lipschitz type maximal functions.

In [14], Lenze introduced a Lipschitz type maximal function as
\[
f_\alpha(x, y) = \sup_{t > 0, t \neq x} \frac{|f(t) - f(x)|}{|t - x|^\alpha}.
\]

In [2], the Lipschitz type maximal function space on \(E \subset [0, \infty)\) is defined as follows
\[
\tilde{V}_\alpha,E = \{f = \sup(1 + x)^\alpha f_\alpha(x, y) \leq M\frac{1}{(1+y)^\alpha}; \ x \geq 0 \ \text{and} \ y \in E\},
\]
where \(f\) is bounded and continuous function on \([0, \infty)\), \(M\) is a positive constant and \(0 < \alpha \leq 1\).

Also, let \(d(x, E)\) be the distance between \(x\) and \(E\), that is,
\[
d(x, E) = \inf\{|x - y|; y \in E\}.
\]

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Theorem 6. If \( \mathcal{L}_{n,q_n} \) be defined by (1.4), then for all \( f \in \tilde{V}_{\alpha,E} \)

\[
| \mathcal{L}_{n,q_n}(f, x) - f(x) | \leq M (\delta_n^\alpha + d(x, E)),
\]

where

\[
\delta_n(x) = 2x^2 - 2x \sqrt{\frac{q[n]_q - q}{n_q} x^2 + \frac{1}{4[n]_q^2} + \frac{x}{n_q}}
\]

Proof. Let \( x_0 \in \bar{E} \), where \( \bar{E} \) denote the closure of the set \( E \). Then we have

\[
| f(t) - f(x) | \leq | f(t) - f(x_0) | + | f(x_0) - f(x) |.
\]

Since \( \mathcal{L}_{n,q_n} \) is a positive and linear operator, \( f \in \tilde{V}_{\alpha,E} \) and using the above inequality

\[
| \mathcal{L}_{n,q_n}(f, x) - f(x) | \leq \mathcal{L}_{n,q_n}(| f(t) - f(x_0) |; q_n; x) + (| f(x_0) - f(x) |) \mathcal{L}_{n,q_n}(1, x)
\]

\[
\leq M (\mathcal{L}_{n,q_n}(| t - x_0 |^\alpha; q_n; x) + | x - x_0 |^\alpha \mathcal{L}_{n,q_n}(1; q_n; x)).
\]

Therefore, we have

\[
\mathcal{L}_{n,q_n}(| t - x_0 |^\alpha; q_n; x) \leq \mathcal{L}_{n,q_n}(| t - x |^\alpha; q_n; x) + | x - x_0 |^\alpha \mathcal{L}_{n,q_n}(1, x).
\]

Now, we take \( p = \frac{2}{\alpha} \) and \( q = \frac{2}{2-\alpha} \) and by using the Hölder’s inequality, one can write

\[
\mathcal{L}_{n,q_n}((t - x)^\alpha; q_n; x) \leq \mathcal{L}_{n,q_n}((t - x)^2, x)^{\alpha/2} (\mathcal{L}_{n,q_n}(1, x))^{(2-\alpha)/2}
\]

\[
+ | x - x_0 |^\alpha \mathcal{L}_{n,q_n}(1, x)
\]

\[
= \delta_n^\alpha + | x - x_0 |^\alpha.
\]

Substituting this in (5.3), we get (5.3). This completes the proof of the theorem.

\[
\square
\]

6. Concluding remarks

Remark 2. Observe that by the conditions in (3.2),

\[
\text{st} - \lim_{n} \delta_n = 0.
\]

By (5.4), we have

\[
\text{st} - \lim_{n} \omega(f; \delta_n) = 0.
\]

This gives us the pointwise rate of statistical convergence of the operators \( \mathcal{L}_{n,q_n}(f, x) \) to \( f(x) \).

Remark 3. If we take \( E = [0, \infty) \) in Theorem 7 since \( d(x, E) = 0 \), then we get for every \( f \in \tilde{V}_{\alpha,[0,\infty)} \)

\[
| \mathcal{L}_{n,q_n}(f, x) - f(x) | \leq M \delta_n^\frac{2}{\alpha}
\]

where \( \delta_n \) is defined as in (5.4).
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