Gauge anomaly cancellations in 
SU(2)$_L \times$ U(1)$_Y$ Electroweak theory on the lattice

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Abstract

We consider the cohomological classification of the 4+2-dimensional topological field, which is proposed by Lüscher, for SU(2)$_L \times$ U(1)$_Y$ electroweak theory. The dependence on the admissible abelian gauge field of U(1)$_Y$ is determined through topological argument, with SU(2)$_L$ gauge field fixed as background. We then show the exact cancellation of the local gauge anomaly of the mixed type SU(2)$_L^2 \times$ U(1)$_Y$ at finite lattice spacing, as well as U(1)$_Y$-3, using the pseudo reality of SU(2)$_L$ and the anomaly cancellation conditions in the electroweak theory given in terms of the hyper-charges of U(1)$_Y$.

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1 Introduction

The gauge interaction of the Weyl fermions now can be described in the framework of lattice gauge theory. The clue to this development is the construction of gauge covariant and local Dirac operators \[1, 2, 3\] which solve the Ginsparg-Wilson relation \[4\]. The Ginsparg-Wilson relation implies an exact chiral symmetry for the Dirac fermion \[5\] and a gauge-field-dependent chiral projection to the Weyl degrees of freedom \[6, 7\].

The functional measure for the Weyl fermion field is defined based on this chiral projection. It leads to a mathematically reasonable definition of the chiral determinant, which generically has the structure as an overlap of two vacua \[8\]. It has been shown by Lüscher in \[9\] that for anomaly-free abelian chiral gauge theories, the functional measure for the Weyl fermion fields can be constructed so that the gauge invariance is maintained exactly on the lattice. This issue of the gauge-invariant construction of the functional measure in non-abelian chiral theories has been related to the cohomological classification of a certain topological field which is defined on the four-dimensional lattice plus two continuum dimensions \[10, 11\]. It has been shown that in all orders in the lattice spacing \(a\), the topological field has trivial cohomology for anomaly free theories. This problem has also been examined by Suzuki from the point of view of the Wess-Zumino consistency condition and the BRST cohomology in four-dimensions \[12\]. The gauge anomaly cancellation has been proved for general gauge groups in all powers of gauge potential \[1\].

The above result for the abelian chiral gauge theories implies that \(U(1)_Y\) hyper-charge chiral gauge theory now can be constructed on the lattice. In this paper, we consider a first step towards the extension of this work to the case of the \(SU(2)_L \times U(1)_Y\) electroweak theory. We examine the exact cancellation of gauge anomalies in the \(SU(2)_L \times U(1)_Y\) electroweak theory, through the cohomological classification \[14, 21\] of the 4+2-dimensional topological field proposed by Lüscher \[10\]. Here we will discuss the cancellation of the local anomaly in infinite volume lattice only and leave the issue related to possible global obstructions to the non-perturbative construction

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1 The topological aspect of the non-abelian anomaly for Weyl fermions defined based on the overlap formalism / the Ginsparg-Wilson relation has been examined by D.H. Adams in close relation to the argument of L. Alvarez-Gaumé and P. Ginsparg in the continuum theory \[13, 14\]. The global SU(2) anomaly has been examined by H. Neuberger and O. Bär and I. Campos \[15, 16\] in detail. A lattice implementation of the \(n\)-invariant and its relation to the effective action for chiral Ginsparg-Wilson fermions has been examined by T. Aoyama and Y.K. in \[17\]. Non-compact formulation of abelian chiral gauge theories has been considered recently by Neuberger \[18\].
of the theory for future study.

The SU(2)\textsubscript{L} × U(1)\textsubscript{Y} electroweak theory contains the following fermions as the first generation:

\[
\left( \begin{array}{c}
\nu_L(x) \\
\nu_L(x)
\end{array} \right)_{Y=-\frac{1}{2}}, \quad \left( \begin{array}{c}
e_R(x)_{Y=-1} \\
\tau_L(x)
\end{array} \right)_{Y=-\frac{1}{6}}, \quad \left( \begin{array}{c}
u_L(x)_{Y=+\frac{2}{3}} \\
\tau_L(x)
\end{array} \right)_{Y=+\frac{1}{6}},
\]

where \( i \) is the color index (\( i = 1, 2, 3 \)). The left-handed leptons and quarks are SU(2)\textsubscript{L} doublets. The right-handed fermions are SU(2)\textsubscript{L} singlet. Taking into account of the color degrees of freedom, there are four doublets. The hyper-charge \( Y \), which is related to electromagnetic charge \( Q \) by the Gell-Mann-Nishijima relation,

\[ Q = I_3 + Y, \tag{1.2} \]

are assigned as shown above.

There are two types of gauge anomalies in the electroweak theory. The first one is the gauge anomaly associated with the abelian U(1)\textsubscript{Y} gauge group. The second one is the gauge anomaly of the mixed type among SU(2)\textsubscript{L} and U(1)\textsubscript{Y} gauge groups. In the continuum theory, these gauge anomalies are generated from the following diagrams:

Figure 1: Gauge Anomalies in the SU(2)\textsubscript{L} × U(1)\textsubscript{Y} electroweak theory

Then the conditions for the gauge anomaly cancellation in the electroweak theory are given in terms of the hypercharges as

\[ \sum_L Y^3 - \sum_R Y^3 = 0, \tag{1.3} \]

and

\[ \sum_{\text{doublet}(L)} Y = 0. \tag{1.4} \]
We can see that the assignment of the hyper-charges shown above indeed satisfies these conditions and one more condition as

$$\sum_{\text{singlet}(R)} Y = 0. \quad (1.5)$$

In order to show the exact cancellations of the (local) gauge anomalies at finite lattice spacing, we consider the cohomological classification of the 4+2-dimensional topological field for $SU(2)_L \times U(1)_Y$ electroweak theory. Our approach is then to determine the dependence on the admissible abelian gauge field of $U(1)_Y$ through topological argument, with $SU(2)_L$ gauge field fixed as background (cf. [19, 20]). Although it does not determine the explicit dependence on $SU(2)_L$ gauge field, it turns out to be sufficient to show the exact cancellations of the gauge anomalies: we can show the cancellation of the gauge anomaly of the mixed type $SU(2)_L^2 \times U(1)_Y$ at finite lattice spacing, as well as $U(1)_Y^3$, using the pseudo reality of $SU(2)_L$ and the anomaly cancellation conditions in the electroweak theory given in terms of the hyper-charges of $U(1)_Y$.

This paper is organized as follows. In section 2, we introduce the 4+2-dimensional topological field for the electroweak theory and discuss its specific features for $SU(2)_L \times U(1)_Y$ gauge groups. In section 3, we formulate the Poincaré lemma in 4+2 dimensions and determine the dependence on the admissible $U(1)_Y$ field at finite lattice spacing. In section 4, we show the exact cancellations of gauge anomalies of the mixed type $SU(2)_L^2 \times U(1)_Y$ as well as $U(1)_Y^3$, using the pseudo reality of $SU(2)_L$ and the anomaly cancellation conditions in the electroweak theory given in terms of the hyper-charges of $U(1)_Y$. In section 5, we give some discussions.

2 4+2 dimensional topological field for Electroweak theory on the lattice

2.1 Weyl fermions on the lattice

Let us consider lattice Dirac fermion which is described by a gauge-covariant and local lattice Dirac operator which satisfies the Ginsparg-Wilson relation.

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D. \quad (2.1)$$

The action of the Dirac fermion is written as

$$S = a^4 \sum_x \bar{\psi}(x)D\psi(x). \quad (2.2)$$
In the case of Neuberger’s Dirac operator
\[ D = \frac{1}{a} \left( 1 + \gamma_5 \frac{H}{\sqrt{H^2}} \right), \]  
where \( H \) is defined by the hermitian Wilson-Dirac operator
\[ H = \gamma_5 \left( \sum_\mu \left\{ \frac{1}{2} \gamma_\mu \left( \nabla_\mu - \nabla_\mu^\dagger \right) + \frac{a}{2} \nabla_\mu \nabla_\mu^\dagger \right\} - \frac{m_0}{a} \right), \]
locality of the action has been proved rigorously for gauge fields with bounded field strength \[ [3, 21]. \]
\[ \| 1 - U_{\mu\nu}(x) \| \leq \epsilon, \quad 1 - 6(2 + \sqrt{2})\epsilon > |1 - m_0|^2. \]  
This proof has been extended to the case where \( H \) is defined by the transfer matrix of the five-dimensional Wilson fermion \[ [22]. \]

The action is invariant under the transformation which can be regarded as the chiral transformation on the lattice:
\[ \delta \psi(x) = \gamma_5 (1 - aD) \psi(x), \quad \delta \bar{\psi}(x) = \bar{\psi}(x) \gamma_5. \]  
By virtue of this exact chiral symmetry, we can define left-handed Weyl fermion on the lattice by the following projections:
\[ \hat{P}_- \psi_L(x) = \psi_L(x), \quad \bar{\psi}_L(x) P_+ = \bar{\psi}_L(x). \]  
\( \hat{P}_- \) is the chiral projector defined as
\[ \hat{P}_- = \frac{1 - \hat{\gamma}_5}{2}, \quad \hat{\gamma}_5 = \gamma_5 (1 - aD). \]  
\( P_+ \) is the usual chiral projector defined with \( \gamma_5 \). The right-handed Weyl fermions can be defined in the similar manner.

The functional measure for the Weyl fermion can be defined as follows: we first introduce chiral bases \( \{ v_j(x) \} \) and \( \{ \bar{v}_k(x) \} \) as
\[ \hat{P}_- v_j(x) = v_j(x), \quad \bar{v}_k(x) P_+ = \bar{v}_k(x), \]
and expand the Weyl fermion fields in terms of the chiral bases with the coefficients which generate the Grassmann algebra,
\[ \psi(x) = \sum_j v_j(x)c_j, \quad \bar{\psi}(x) = \sum_k \bar{c}_k \bar{v}_k(x). \]
Then the functional measure of the Weyl fermion can be defined as

$$\prod_x d\psi_L(x) d\bar{\psi}_L(x) = \prod_j dc_j \prod_k d\bar{c}_k.$$  \hspace{1cm} (2.11)

Given the definition for the functional measure of the Weyl fermion, the chiral determinant is evaluated as

$$Z_W = \int \prod_x d\psi_L(x) d\bar{\psi}_L(x) \exp \left( -a^4 \sum_x \bar{\psi}_L(x) D\psi_L(x) \right)$$  \hspace{1cm} (2.12)

$$= \det M_{kj},$$  \hspace{1cm} (2.13)

where

$$M_{kj} = a^4 \sum_x \bar{v}_k(x) Dv_j(x) = (\bar{u}_k Dv_j).$$  \hspace{1cm} (2.14)

### 2.2 4+2 dimensional topological field

The question is how to construct the functional measure for the Weyl fermions so that gauge invariance is maintained at finite lattice spacing. As shown by Lüscher [10], this question can be formulated as the cohomological problem of a certain topological field which is defined on the four-dimensional lattice plus two continuum dimensions.

The 4+2 dimensional topological field is introduced as follows. We consider lattice gauge fields

$$U_\mu(z) \in G, \quad z = (x_\mu, t, s), \quad \mu = 1, 2, 3, 4$$  \hspace{1cm} (2.15)

which depend on two additional real coordinates $t$ and $s$. We also introduce gauge potentials $A_t(z)$ and $A_s(z)$ along these directions and define the associated field tensor by

$$F_{ts}(z) = \partial_t A_s(z) - \partial_s A_t(z) + i [A_t(z), A_s(z)].$$  \hspace{1cm} (2.16)

The covariant derivative in these directions is defined as

$$D^A_r U_\mu(z) = \partial_r U_\mu(z) + i [A_r(z) U_\mu(z) - U_\mu(z) A_r(z + a\hat{\mu})], \quad r = t, s$$  \hspace{1cm} (2.17)

which transforms in the same way as $U_\mu(z)$ under gauge transformations in 4+2 dimensions. Then we consider the following 4+2 dimensional field
which is gauge invariant and local:

\[
q(z) = -i \text{tr} \left\{ \frac{1}{4} \gamma_5 \left[ D_t^A \hat{P}_-, D_s^A \hat{P}_- \right] + \frac{1}{4} \left[ D_t^A \hat{P}_-, D_s^A \hat{P}_- \right] \gamma_5 \right. \\
+ \frac{i}{2} R(F_{ts}) \gamma_5 \left( x, x \right) \right\}.
\]

(2.18)

The trace is taken over the Dirac and flavor indices only.

By noting

\[
a^4 \sum_x q(x) = i \text{Tr} \left\{ \hat{P}_- \left[ \partial_t \hat{P}_-, \partial_s \hat{P}_- \right] - \frac{i}{2} \partial_t \left[ R(A_s) \gamma_5 \right] + \frac{i}{2} \partial_s \left[ R(A_t) \gamma_5 \right] \right\}
\]

(2.19)

and making use of the identity

\[
\text{Tr} \left\{ \delta_1 \hat{P}_- \delta_2 \hat{P}_- \delta_3 \hat{P}_- \right\} = 0,
\]

(2.20)

we can show that this 4+2 dimensional field satisfies

\[
a^4 \sum_x \int dt ds \delta q(z) = 0
\]

(2.21)

for all local variations of the link variables \( U_\mu(z) \) and the potential \( A_r(z) \), i.e. it is a topological field.

It has been shown by Lüscher [10] that if this topological field is in the trivial cohomology class, i.e. it is equal to the divergence of a gauge-invariant local current,

\[
q(z) = \partial_t^\ast k_\mu(z) + \partial_t k_s(z) - \partial_s k_t(z),
\]

(2.22)

then using \( k_r(z) \) it is possible to construct a gauge-covariant local current \( j_{\mu}^a(x) \) which satisfies the integrability condition in differential form and the anomalous conservation law.

### 2.3 4+2 dimensional topological field for SU(2)_L × U(1)_Y electroweak theory

In order to construct the 4+2 dimensional topological field for SU(2)_L × U(1)_Y electroweak theory, we consider lattice SU(2)_L and U(1)_Y gauge fields,

\[
U^{(1)}_\mu(z) \in U(1), \quad U^{(2)}_\mu(z) \in SU(2),
\]

(2.23)
which satisfy the admissibility conditions with sufficiently small constants $\epsilon^{(2)}$ and $\epsilon^{(1)}$:

$$\|1 - U^{(2)}_{\mu\nu}(x)\| < \epsilon^{(2)}, \quad \|1 - U^{(1)}_{\mu\nu}(x)\| < \epsilon^{(1)}.$$  \hfill (2.24)

When $\epsilon^{(1)} < 1/6Y \times \pi/3$, the admissible abelian lattice gauge fields can be expressed in terms of vector potentials

$$U^{(1)}_{\mu}(x) = \exp(iA_\mu(x))$$  \hfill (2.25)

which has the following properties:

$$F_{\mu\nu}(x) \equiv \frac{1}{i} \ln U_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \quad |A_\mu(x)| \leq \pi(1 + 8\|x\|).$$  \hfill (2.26)

This representation of the link variable is unique up to the gauge transformation with the parameter $\omega(x)$ which takes values in integer multiple of $2\pi$.

$$\tilde{A}_\mu(x) = A_\mu(x) + \partial_\mu \omega(x).$$  \hfill (2.27)

We also introduce gauge potentials along the two additional dimensions $A_r(z), r = t, s$ for $U(1)_Y$ and $B_r(z), r = t, s$ for $SU(2)_L$ and denote the 4+2 dimensional gauge fields as follows:

$$A_\mu(z) = (A_k(z), A_t(z), A_s(z)), \quad U_\mu(z) = \left(U^{(2)}_k(z), iB_t(z), iB_s(z)\right),$$  \hfill (2.28)

where $\mu = 1, \cdots, 6$ and we use the Latin index $i = 1, 2, 3, 4$ for four-dimensional lattice here after. Then the 4+2 dimensional topological field $q(z)$ for $SU(2)_L \times U(1)_Y$ electroweak theory can be regarded as a gauge-invariant local functional of the 4+2 dimensional gauge field variables $A_\mu(z)$ and $U_\mu(z)$:

$$q(z) = q(z; A_\mu(z), U_\mu(z))$$  \hfill (2.29)

It follows from the charge conjugation property of the lattice Dirac operator that $q(z)$ changes sign under complex-conjugation of the representations of the gauge fields

$$q\left(z; A_\mu(z), U_\mu(z)\right) = -q\left(z; -A_\mu(z), U^*_\mu(z)\right)$$  \hfill (2.30)
Since SU(2)_L is pseudo real, there exists a unitary transformation S such that

\[ SU_\mu^* S^{-1} = U_\mu. \]  

Then we obtain

\[ q(z; A_\mu(z), U_\mu(z)) = -q(z; -A_\mu(z), U_\mu(z)). \]  

(2.32)

We note also that \( q(z) \) vanishes identically when the U(1)_Y gauge fields are switched off:

\[ q(z; 0, U_\mu(z)) = 0. \]  

(2.33)

3 Cohomological classification of the topological field in 4+2 dimensions

3.1 Analysis of the 4+2 dimensional topological field

In this section, we will formulate the Poincaré lemma in 4+2 dimensions and examine the dependence of \( q(x,t,s) \) on the admissible U(1)_Y field through topological argument. In the course of the analysis, SU(2)_L gauge field is fixed as background.

Since \( q(x,t,s) \) smoothly depends on 4+2 dimensional U(1) gauge potential \( A_\mu(x,t,s) \) and its differentials, the variation of the topological field can be expressed as

\[ \delta q(x,t,s) = \sum_{m,n=0,1} \sum_y \frac{\partial q(x,t,s)}{\partial [\partial_s^m \partial_t^n A_\mu(y,t,s)]} \partial_s^m \partial_t^n \delta A_\mu(y,t,s). \]  

(3.1)

By definition, \( q(x,t,s) \) contains at most the first-order differentials of the vector potential \( A_\mu(x,t,s) \) in the continuous coordinates. Therefore, we may restrict the sum over \( m, n \) to 0, 1.

If we define the differential operator in 4+2 dimensions in the above expression as

\[ L_\mu(x,y,t,s) = \sum_{m,n=0,1} \frac{\partial q(x,t,s)}{\partial [\partial_s^m \partial_t^n A_\mu(y,t,s)]} \partial_s^m \partial_t^n, \]  

(3.2)

then the topological property and the gauge invariance of the 4+2 dimensional field lead to the following conditions for the differential operator \( L_\mu \).

\[ \int dt ds a^4 \sum_x \sum_y L_\mu(x,y,t,s) \delta A_\mu(y,t,s) = 0. \]  

(3.3)
The topological field \( q(x, t, s) \) itself can be expressed with this operator as

\[
q(x, t, s) = \alpha(x, t, s) + \sum_y \left. \int_0^1 du L_\mu(x, y, t, s) \right|_{A \rightarrow uA} A_\mu(y, t, s),
\]

where \( \alpha(x, t, s) \) is the part which does not depend on the abelian gauge field \( A_\mu(x, t, s) \). Then the problem reduces to examine the cohomological properties of the differential operator \( L_\mu \). In order to examine such an operator and determine the form of the topological field \( q(x, t, s) \), we will next formulate the Poincaré lemma which is applicable to the differential operators in 4+2 dimension. This is the extension of the Poincaré lemma on the lattice given in [19] along the line of the analysis in the continuum theory of [23].

### 3.2 Poincaré lemma in 4+2 dimensions

We first introduce a Grassmann algebra with basis element

\[
dx_1, dx_2, dx_3, dx_4, dx_5 = dt, dx_6 = ds
\]

and denote them by \( dx_\mu(\mu = 1, \cdots, 6) \) collectively. A \( k \)-form \( (0 \leq k \leq 6) \) in 4+2 dimensions is then defined as:

\[
f(z) = \frac{1}{k!} f_{\mu_1 \cdots \mu_k}(z) dx_{\mu_1} \cdots dx_{\mu_k} \in \Omega_k, \quad z = (x_i, t, s).
\]

We assume that \( f_{\mu_1 \cdots \mu_k}(z) \) is a smooth function in the continuous coordinates \( t, s \) and is locally supported in the lattice coordinate \( x_i \).

The exterior differential operator, which is a map \( \Omega_k \rightarrow \Omega_{k+1} \), is defined by \( d = \sum_{i=1}^4 dx_i \partial_i + \sum_{r=t,s} dr \partial_r \) where \( \partial_i \) is forward difference operator on the four-dimensional lattice. It acts on the \( k \)-forms according to

\[
df(z) = \frac{1}{k!} \partial_\mu f_{\mu_1 \cdots \mu_k}(z) dx_\mu dx_{\mu_1} \cdots dx_{\mu_k}.
\]

The associated divergence operator \( d^* : \Omega_k \rightarrow \Omega_{k-1} \) is defined as

\[
d^* f(z) = \frac{1}{(k-1)!} \partial^*_\mu f_{\mu_2 \cdots \mu_k}(z) dx_{\mu_2} \cdots dx_{\mu_k},
\]
where $\partial_r^*$ is backward difference operator.

Now we consider a class of differential operators $L$ which is a map $L : \Omega_l \rightarrow \Omega_k$ such that

$$L f(x, t, s) = \frac{1}{k!l!} dx_{\mu_1} \cdots dx_{\mu_k} \sum_{y, n, m} L_{\mu_1, \ldots, \mu_k; \nu_1, \ldots, \nu_l}^n m \partial_s^{\nu_1, \ldots, \nu_l} f_{\nu_1, \ldots, \nu_l}(y, t, s),$$

(3.10)

where $L_{\mu_1, \ldots, \mu_k; \nu_1, \ldots, \nu_l}(x; y, t, s)$ is a local function in $x(y), t, s$ which is exponentially decaying in $x(y)$ with respect to the reference point $y(x)$. We assume $m, n = 0, 1$ in the following discussions. When $L$ is not a differential operator, i.e. $n = m = 0$ and is proportional to the Kronecker delta $\delta_{x,y}$, we refer such an operator as zero degree.

$$L^0 f(x, t, s) = \frac{1}{k!l!} dx_{\mu_1} \cdots dx_{\mu_k} L^0_{\mu_1, \ldots, \mu_k; \nu_1, \ldots, \nu_l}(x, t, s) f_{\nu_1, \ldots, \nu_l}(x, t, s).$$

(3.11)

We can show the following Poincaré lemma for these differential operators in 4+2 dimensions.

**Lemma** (Poincaré lemma) If $L : \Omega_l \rightarrow \Omega_k$ is any differential operator satisfying

$$dL(x, y, t, s) = 0,$$

(3.12)

then there exists a differential operator $M(x, y, t, s)$ and an operator of zero degree $L^0$ such that

$$L(x, y, t, s) = \delta_{k,6} \delta_{x,y} L^0(x, t, s) + dM(x, y, t, s),$$

(3.13)

Note that the product of $d$ and $M$ is a product of operators.

The proof of the lemma is given as follows. We first note the fact that for any differential operator $L$ in concern and any fixed continuous dimension $r$, there is a unique decomposition of the operator into two operators so that

$$L = S + \partial_r R,$$

(3.14)

where $S$ is zero degree with respect to $\partial_r$. This is because $L$ is at most a first-order differential operator in terms of $\partial_r$ assuming the form

$$L = L_0 + L_1 \partial_r$$

(3.15)

with $L_0$ and $L_1$ zero degree with respect to $\partial_r$ and it can be rewritten uniquely into

$$L = (L_0 - [\partial_r L_1]) + \partial_r L_1,$$

(3.16)
where the bracket $[\partial_r, L_1]$ stands for the fact that $\partial_r$ in it acts only on $L_1$. When $L$ is gauge invariant, by acting it on a constant, we infer that $L_0$ is also gauge invariant. Then it also follows that $L_1$ is gauge invariant. Therefore, in the decomposition of Eq. (3.14), both $S$ and $R$ are gauge invariant.

Then we can decompose any differential operator $L : \Omega_l \to \Omega_k$ into the sequence:

$$L = (L_5 + \partial_s R_5) ds + Z_6,$$

and

$$L_5 = (L_4 + \partial_t R_4) dt + Z_5,$$

where $L_5$ is degree zero with respect to $\partial_s$ and $L_4$ is degree zero with respect to both $\partial_s$ and $\partial_t$.

From the condition Eq. (3.12),

$$dL = \tilde{d} (L_5 + \partial_s R_5) ds + \left( \tilde{d} + ds\partial_s \right) Z_6 = 0,$$

where $\tilde{d} = \sum_{i=1}^{4} dx_i \partial_i + dt \partial_t$. Then the coefficient of $ds$ must vanish

$$\tilde{d}L_5 + \partial_s \left( \tilde{d}R_5 + (-)^k Z_6 \right) = 0.$$  

Since this can be regarded as the unique decomposition of zero operator with respect to $\partial_s$, we have

$$\tilde{d}L_5 = 0,$$

$$\tilde{d}R_5 + (-)^k Z_6 = 0.$$  

In a similar manner, from the condition Eq. (3.21), we obtain

$$\tilde{d}L_4 = 0,$$

$$\tilde{d}R_4 + (-)^{k-1} Z_5 = 0,$$

where $\tilde{d} = \sum_{i=1}^{4} dx_i \partial_i$.

Combing these results, we have

$$L = L_4 dtds + dK,$$

where

$$K = (-)^{k-1} R_5 + (-)^{k-2} R_4 ds.$$  

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$L_4$ is zero degree with respect to both $\partial_s$ and $\partial_t$ and satisfies $\bar{d}L_4 = 0$.

Now we recall the Poincaré lemma on four-dimensional lattice given in [19]. From the condition $\bar{d}L_4 = 0$, we have

$$
L_4(x; y, t, s) = \delta_{k-2,4}\delta_{x,y}L^0_4(x, t, s) + \bar{d}T(x; y, t, s) \tag{3.27}
$$

where

$$
L^0_4(x, t, s) = \sum_y L_4(y; x, t, s). \tag{3.28}
$$

Then we can write $L$

$$
L(x, y, t, s) = \delta_{k,6}\delta_{x,y}L^0_4(x, t, s) + dM(x, y, t, s), \tag{3.29}
$$

where

$$
M = Tdtds + K, \quad L^0 = L^0_4dtds. \tag{3.30}
$$

It is clear from the above construction that $L^0$ and $M$ possesses the same locality and gauge-transformation properties as $L$. When $L$ is gauge invariant under SU(2)$_L$ and U(1)$_Y$ gauge transformations, $L_0$ and $M$ are also gauge invariant.

For the operators $L$ which satisfy $d^*L = 0$ and $Ld^* = 0$, the lemma can be derived in similar manners.

### 3.3 Structure of the 4+2 dimensional topological field

By using the Poincaré lemma in 4+2 dimensions, we next determine the dependence of $q(z)$ on the admissible abelian gauge field of U(1)$_L$. We will show the following lemma:
Lemma The 4+2 dimensional topological field \( q(z) \) for the SU(2)_L \times U(1)_Y electroweak theory is written in the following form.

\[
q(z) = \alpha(z) + \beta_{\mu\nu}(z - \hat{\mu} - \hat{\nu}) F_{\mu\nu}(z - \hat{\mu} - \hat{\nu}) \\
+ \gamma_{\mu\nu\rho\sigma}(z - \hat{\mu} - \hat{\nu} - \hat{\rho} - \hat{\sigma}) F_{\rho\sigma}(z - \hat{\mu} - \hat{\nu} - \hat{\rho} - \hat{\sigma}) F_{\mu\nu}(z - \hat{\mu} - \hat{\nu}) \\
+ \delta \epsilon_{\mu\nu\rho\sigma\lambda\tau} F_{\lambda\tau}(z - \hat{\mu} - \hat{\nu} - \hat{\rho} - \hat{\sigma} - \hat{\lambda} - \hat{\tau}) \times \\
F_{\rho\sigma}(z - \hat{\mu} - \hat{\nu} - \hat{\rho} - \hat{\sigma}) F_{\mu\nu}(z - \hat{\mu} - \hat{\nu}) \\
+ \partial^*_\mu k_\mu(z) \\
\tag{3.33}
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the 4+2 dimensional field strength of U(1)_Y gauge potential. \( k_\mu(z) \) is a local current which is gauge invariant under SU(2)_L and U(1)_Y gauge transformations. \( \delta \) is a constant. \( \alpha(z), \beta_{\mu\nu}(z) \) and \( \gamma_{\mu\nu\rho\sigma}(z) \) are certain gauge invariant local functions which may depend on the SU(2)_L gauge field and satisfy

\[
\partial^*_\mu \beta_{\mu\nu}(z) = 0, \quad \partial^*_\mu \gamma_{\mu\nu\rho\sigma}(z) = 0. \tag{3.34}
\]

Note also that \( \hat{5} = \hat{6} = 0 \) and \( z = (x_i, t, s) \).

The proof consists of three steps.

3.3.1 Step one

We first consider the differential operator

\[
L_\mu(x, y, t, s) = \sum_{m,n=0,1} \frac{\partial q(x, s, t)}{\partial [\partial^m_s \partial^n_t A_\mu(y, s, t)]} \partial^m_s \partial^n_t. \\
\tag{3.35}
\]

We may regard this operator as a map \( L_\mu : \Omega_1 \rightarrow \Omega_0 \). Then using the Poincaré lemma, we can write as

\[
L_\mu(x, y, t, s) = \delta_{x,y} L^0_\mu(x, t, s) + \partial^*_\nu K_{\nu;\mu}, \\
\tag{3.36}
\]

where \( L^0 \) is zero degree and \( K_{\nu;\lambda} \) is a map \( K_{\nu;\lambda} : \Omega_1 \rightarrow \Omega_1 \). The topological property of the 4+2 dimensional field implies

\[
0 = \int dt ds a^4 \sum_x \delta q(x, t, s) = \int dt ds a^4 \sum_x L^0_\mu(x, t, s) \delta A_\mu(x, t, s). \\
\tag{3.37}
\]

Since \( L^0_\mu(x, t, s) \) is zero degree, it must vanish identically,

\[
L^0_\mu(x, t, s) = 0. \tag{3.38}
\]
On the other hand, the gauge invariance of the 4+2 dimensional field implies

\[ L_\mu(x, y, t, s) \partial_\mu = 0 \quad (3.39) \]

or

\[ \partial_\nu^* K_{\nu;\mu}(x, y, t, s) \partial_\mu = 0. \quad (3.40) \]

Then, using the Poincaré lemma, we obtain

\[ K_{\nu;\mu}(x, y, t, s) \partial_\mu = \partial_\nu^* H_{\lambda\nu}(x, y, t, s), \quad (3.41) \]

where \( H_{\lambda\nu} : \Omega_0 \rightarrow \Omega_2 \). Using the Poincaré lemma again, it can be cast into the form

\[ H_{\lambda\nu}(x, y, t, s) = \delta_{x,y} H_{\lambda\nu}^0(x, t, s) + R_{\lambda\nu;\rho}(x, y, t, s) \partial_\rho, \quad (3.42) \]

where \( R_{\lambda\nu;\rho} : \Omega_1 \rightarrow \Omega_2 \). We can eliminate \( R_{\lambda\nu;\rho} \) term in the above expression by redefining \( K_{\nu;\mu} \) so that

\[ K_{\nu;\mu} \rightarrow K_{\nu;\mu} + \partial_\nu^* R_{\lambda\nu;\mu}(x, y, t, s), \quad (3.43) \]

which does not affect the relation \( L_\mu = \partial_\nu^* K_{\nu;\mu} \). Thus we obtain

\[ H_{\lambda\nu}(x, y, t, s) = \delta_{x,y} H_{\lambda\nu}^0(x, t, s). \quad (3.44) \]

We substitute this result into Eq. (3.41) and make it act on a constant, we obtain

\[ [\partial_\nu^* H_{\lambda\nu}^0(x, t, s)] = 0, \quad (3.45) \]

where the square bracket means that \( \partial_\nu^* \) acts on the function \( H_{\lambda\nu}^0 \) rather than a product of the differential operators. Then it follows that

\[ \partial_\nu^* \delta_{x,y} H_{\lambda\nu}^0(x, t, s) = \delta_{x-\hat{\lambda},y} H_{\lambda\nu}^0(x-\hat{\lambda}, t, s) \partial_\lambda. \quad (3.46) \]
Then using the Poincaré lemma we obtain

\[ K_{\nu;\mu}(x, y, t, s) = \delta_{x-\hat{\mu}, y} H_{\mu\nu}^0(x - \hat{\nu} - \hat{\mu}, t, s) + \omega_{\nu;\mu}(x, y, t, s) \partial_\rho. \] (3.47)

We now substitute this result into Eq. (3.36) and obtain

\[ L_\mu(x, y, t, s) = \delta_{x-\hat{\nu} - \hat{\mu}} H_{\mu\nu}^0(x - \hat{\nu} - \hat{\mu}, t, s) \partial_\nu + \partial_\nu^* \omega_{\nu;\mu}(x, y, t, s) \partial_\rho. \] (3.48)

Then the 4+2 dimensional topological field is given as follows.

\[ q(x, t, s) = \alpha(x, t, s) + \int_0^1 du \sum_y L_\mu(x, y, t, s) \left| A_\mu(y, t, s) \right| \rightarrow uA \]

\[ = \alpha(x, t, s) + \int_0^1 du \left[ H_{\mu\nu}^0(x - \hat{\nu} - \hat{\mu}, t, s) \right] \partial_\nu A_\mu(x - \hat{\nu} - \hat{\mu}, t, s) \]

\[ + \partial_\nu^* \left( \sum_y \int_0^1 du \omega_{\nu;\mu}(x, y, t, s) \left| \partial_\rho A_\mu(y, t, s) \right| \rightarrow uA \right). \] (3.49)

Setting

\[ \phi_{\mu\nu}(x, t, s) = -\frac{1}{2} \int_0^1 du \left[ H_{\mu\nu}^0(x, t, s) \right] \left| A_\mu \rightarrow uA \right], \] (3.50)

\[ \theta_\nu(x, t, s) = \sum_y \int_0^1 du \omega_{\nu;\mu}(x, y, t, s) \left| \partial_\rho A_\mu(y, t, s) \right| \rightarrow uA, \] (3.51)

we obtain

\[ q(z) = \alpha(z) + \phi_{\mu\nu}(z - \hat{\nu} - \hat{\mu}) F_{\mu\nu}(z - \hat{\nu} - \hat{\mu}) + \partial_\nu^* \theta_\nu(z). \] (3.52)

From the topological property of the 4+2 topological field,

\[ 0 = \int dt ds a^4 \sum_x \delta q(x, t, s) \]

\[ = \int ds dt a^4 \sum_x \frac{1}{2} \phi_{\mu\nu}(z - \hat{\nu} - \hat{\mu}) \partial_\nu \delta A_\mu(z - \hat{\nu} - \hat{\mu}). \] (3.53)

By the integration by parts, we obtain

\[ \partial_\nu^* \phi_{\mu\nu}(z) = 0. \] (3.54)
3.3.2 Step two

Next we examine the differential operator which is obtained from the variation of $\phi_{\mu\nu}(x,t,s)$.

$$
\delta \phi_{\mu\nu}(z) = \sum_y \sum_{m,n=0,1} \frac{\partial \phi_{\mu\nu}(z)}{\partial (\partial_s^m \partial_t^n A_\lambda(y,t,s))} \partial_s^m \partial_t^n \delta A_\lambda(y,t,s).
$$

(3.55)

We denote the differential operator in the above expression as $L_{\mu\nu,\lambda}(x,y,t,s)$.

$$
L_{\mu\nu,\lambda}(x,y,t,s) = \sum_{m,n=0,1} \frac{\partial \phi_{\mu\nu}(z)}{\partial (\partial_s^m \partial_t^n A_\lambda(y,t,s))} \partial_s^m \partial_t^n.
$$

(3.56)

$\phi_{\mu\nu}(x,t,s)$ itself can be expressed with $L_{\mu\nu,\lambda}(x,y,t,s)$ as

$$
\phi_{\mu\nu}(z) = \beta_{\mu\nu}(z) + \int_0^1 dt \sum_y L_{\mu\nu,\lambda}(x,y,t,s) \bigg|_{A \to \mu A} A_\lambda(y,t,s).
$$

(3.57)

It follows from the property Eq. (3.54) that

$$
\partial^*_\mu \delta \phi_{\mu\nu}(z) = \delta \left[ \partial^*_\mu \phi_{\mu\nu}(z) \right] = 0,
$$

(3.58)

which implies

$$
\partial^*_\mu L_{\mu\nu,\lambda}(x,y,t,s) = 0.
$$

(3.59)

and in turn implies

$$
\partial^*_\mu \beta_{\mu\nu}(z) = 0.
$$

(3.60)

From the gauge invariance of $\phi_{\mu\nu}(z)$, we have

$$
L_{\mu\nu,\lambda}(x,y,t,s) \partial_\lambda = 0.
$$

(3.61)

Then following the similar argument as the first step, we obtain

$$
\phi_{\mu\nu}(z) = \beta_{\mu\nu}(z) + \eta_{\mu\nu\lambda\rho}(z - \hat{\lambda} - \hat{\rho}) F_{\lambda\rho}(z - \hat{\lambda} - \hat{\rho}) + \partial^*_\rho \theta_{\mu\nu\rho}(z),
$$

(3.62)

and

$$
\partial^*_\rho \eta_{\mu\nu\lambda\rho}(z) = 0, \quad \partial^*_\mu \beta_{\mu\nu}(z) = 0.
$$

(3.63)
3.3.3 Step three

Finally, we examine the differential operator which is obtained from the variation of $\eta_{\mu\nu\lambda\rho}$. In the course, we encounter the operator of zero degree $H_{\mu\nu\lambda\rho\sigma\tau}^0: \Omega_0 \to \Omega_6$, which satisfies the condition

$$\partial^*_\tau H_{\mu\nu\lambda\rho\sigma\tau}^0(x, t, s) = 0.$$  (3.64)

$H^0$ is the six form and it may be written with the totally antisymmetric tensor $\epsilon_{\mu\nu\lambda\rho\sigma\tau}$ as

$$H_{\mu\nu\lambda\rho\sigma\tau}^0(x, t, s) = \delta(x, t, s) \epsilon_{\mu\nu\lambda\rho\sigma\tau}.$$  (3.65)

Then the above condition implies that $\delta$ does not depend on $(x, t, s)$ and is a constant.

Following the similar argument as the first and second steps, we obtain

$$\eta_{\mu\nu\lambda\rho}(z) = \gamma_{\mu\nu\lambda\rho}(z)$$

$$+ \delta \epsilon_{\mu\nu\lambda\rho\sigma\tau} F_{\sigma\tau}(z - \hat{\sigma} - \hat{\tau})$$

$$+ \partial^*_\sigma \theta_{\mu\nu\lambda\rho}(z),$$  (3.66)

and

$$\partial^*_\mu \gamma_{\mu\nu\lambda\rho}(z) = 0,$$  (3.67)

where $\epsilon_{\mu\nu\lambda\rho\sigma\tau}$ is the totally antisymmetric tensor and $\delta$ is a constant.

Combining these three results and using the Bianchi identity,

$$\partial^\rho_{[\mu} F_{\nu\rho]}(x - \hat{\mu} - \hat{\nu}) = \partial_{[\rho} F_{\mu\nu]}(x - \hat{\mu} - \hat{\nu}) = 0,$$  (3.68)

we finally obtain Eq. (3.33) and complete the proof of the lemma.

In Eq. (3.33), there is a difference in the shifts of the lattice indices from the result obtained in [19, 20]. This difference, however, can be shown to be a total divergence.

---

In order to show this, we can use the following identity in Eqs. (3.52), (3.62), (3.64).

$$f_{\mu\nu}(z) g_{\mu\nu}(z) - f_{\mu\nu}(z - \hat{\mu} - \hat{\nu}) g_{\mu\nu}(z - \hat{\mu} - \hat{\nu})$$

$$= f_{\mu\nu}(z) g_{\mu\nu}(z) - f_{\mu\nu}(z - \hat{\mu}) g_{\mu\nu}(z - \hat{\mu})$$

$$+ f_{\mu\nu}(z - \hat{\mu}) g_{\mu\nu}(z - \hat{\mu}) - f_{\mu\nu}(z - \hat{\mu} - \hat{\nu}) g_{\mu\nu}(z - \hat{\mu} - \hat{\nu})$$

$$= \partial^*_\rho (f_{\omega}(z) g_{\omega}(z) + f_{\mu}(z - \hat{\mu}) g_{\mu}(z - \hat{\mu})).$$

18
4 Exact cancellations of gauge anomalies in
SU(2)$_L \times$ U(1)$_Y$ Electroweak theory

The result Eq. (3.33) concerning the dependence on the admissible U(1)$_Y$ gauge field of the 4+2 dimensional topological field may be written symbolically in following form.

\[ q(z; A_\mu, U_\mu) = \alpha(z; U_\mu) + \beta(z; U_\mu)F + \gamma(z; U_\mu)F^2 + \delta F^3 + d^* k(z; U_\mu, A_\mu). \]  (4.1)

As we discussed in section 3, the 4+2 dimensional topological field for SU(2)$_L \times$ U(1)$_Y$ electroweak theory has the properties

\[ q(z; A_\mu(z), U_\mu(z)) = -q(z; -A_\mu(z), U_\mu(z)) \]  (4.2)

and

\[ q(z; 0, U_\mu(z)) = 0. \]  (4.3)

From these two conditions, we infer

\[ \alpha(z; U_\mu) = 0. \]  (4.4)

and

\[ \gamma(z; U_\mu)F^2 = -d^* \frac{1}{2} (k(z; U_\mu, -A_\mu) + k(z; U_\mu, A_\mu)). \]  (4.5)

Therefore, the 4+2 topological field turns out to have the following structure

\[ q(z; A_\mu, U_\mu) = \beta(z; U_\mu)F + \delta F^3 + d^* \frac{1}{2} (k(z; U_\mu, A_\mu) - k(z; U_\mu, -A_\mu)). \]  (4.6)

We note that \( \beta(z; U_\mu) \) is a gauge-invariant local functional of SU(2)$_L$ gauge field, while \( \delta \) is a constant.

Now we recall the anomaly cancellation conditions for the electroweak theory which is given in terms of U(1)$_Y$ hyper-charges. Because of the cubic condition

\[ \sum_L Y^3 - \sum_R Y^3 = 0, \]  (4.7)
the term $\delta F^3$ vanishes identically, if all the contributions from the fermions are summed up. On the other hand, because of the linear conditions

$$\sum_{\text{doublet}(L)} Y = 0. \quad (4.8)$$

and

$$\sum_{\text{singlet}(R)} Y = 0. \quad (4.9)$$

the term $\beta(z; U_\mu) F$, which represents the gauge anomaly of the mixed type, also vanishes identically, in each sectors of doublets and singlets. Thus we can see that the 4+2 dimensional topological field for the electroweak theory is indeed in the trivial cohomology class.

$$q(z; A_\mu, U_\mu) = \partial_\mu^* \frac{1}{2} (k_\mu(z; U_\mu, A_\mu) - k_\mu(z; U_\mu, -A_\mu)). \quad (4.10)$$

## 5 Summary and discussion

We have shown the exact cancellations of gauge anomalies of the $SU(2)_L \times U(1)_Y$ electroweak theory on the lattice, which is formulated based on the lattice Dirac operator satisfying the Ginsparg-Wilson relation. Our approach is to consider the cohomological classification of the 4+2-dimensional topological field proposed by Lüscher for $SU(2)_L \times U(1)_Y$ electroweak theory. Using the Poincaré lemma in 4+2 dimensions, we have determined the dependence on the admissible abelian gauge field of $U(1)_Y$ through topological argument, with $SU(2)_L$ gauge field fixed as background (cf. [19, 20]). This turned out to be sufficient to show the exact cancellations of the gauge anomalies: using the pseudo reality of $SU(2)_L$ and the anomaly cancellation conditions in the electroweak theory given in terms of the hyper-charges of $U(1)_Y$, we have shown the exact cancellation of the gauge anomaly of the mixed type $SU(2)_L^2 \times U(1)_Y$ at finite lattice spacing, as well as $U(1)_Y^3$.

As to the question of the cohomological classification of the 4+2 dimensional topological field for the electroweak theory, we may also invoke the elegant method based on the non-commutative differential calculus and BRST cohomology in order to explore the structure of the 4+2 dimensional topological field [20].

Towards the lattice construction of the $SU(2)_L \times U(1)_Y$ electroweak theory, the next step would be to show the integrability condition given in
which assures the existence of the functional measure of fermions with desired properties. For this purpose, we need to examine the possible global anomalies in SU(2)$_L \times$ U(1)$_Y$ electroweak theory. Global SU(2) anomaly has been examined by Neuberger and by Bär and Campos in detail [15, 16]. It is also desirable to establish the existence of the model in a finite volume, as in the case of the abelian chiral gauge theories [3].

In order to extend our result to the whole SU(3)$_C \times$ SU(2)$_L \times$ U(1)$_Y$ standard model, we need to attack directly the non-abelian nature of the gauge anomalies of the mixed type of SU(3)$_C \times$ SU(2)$_L$, although there is no corresponding anomaly of this type in the continuum theory. This is also true, of course, when we consider more general non-abelian chiral gauge theories. The recent work by Suzuki [12] based on the BRST cohomology in four-dimensions could shed lights on this issue.

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