GEOMETRIC QUANTIZATION OF SUPERORBITS: A CASE STUDY

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Abstract. By decomposing the regular representation of a particular (Heisenberg-like) Lie supergroup into irreducible subspaces, we show that not all of them can be obtained by applying geometric quantization to coadjoint orbits with an even symplectic form. However, all of them can be obtained by introducing coadjoint orbits through non-homogeneous points and with non-homogeneous symplectic forms as described in [Tu1]. In this approach it turns out that the choice of a polarization can change (dramatically) the representation associated to an orbit. On the other hand, the procedure is not completely mechanical (meaning that some parts have to be done “by hand”), hence work remains to be done in order to understand all details of what is happening.

1. Introduction

In [Tu1] I introduced the notion of a non-homogeneous symplectic form on a supermanifold and I constructed a possible prequantization of such a symplectic supermanifold. The question remained whether non-homogenous symplectic forms are interesting to study and whether the proposed prequantization is the correct one to use. In this paper I will show by an explicit example that non-homogeneous symplectic forms play a role in representation theory. The example I will study is a Heisenberg-like group, meaning that we have two graded vector spaces $E$ and $C$ and an even graded skew-symmetric bilinear form $\Omega : E \times E \to C$ and that we look at the group $G$ which is as a manifold the even part of $E \times C$: $G = (E \times C)_0$ equipped with the group structure

$$(a,b) \cdot (\hat{a},\hat{b}) = (a + \hat{a}, b + \hat{b} + \frac{1}{2}(a,\hat{a})\Omega).$$

On this group I look at the (left-) regular representation consisting of square-integrable functions on $G$ with the group action

$$(\Phi_g f)(\hat{g}) = f(g^{-1} \cdot \hat{g}).$$

Using ordinary Fourier direct integrals and Berezin-Fourier direct integrals, it is fairly easy to decompose this regular representation into invariant subspaces. Depending upon the Fourier parameters, some of these subspaces can be decomposed further in direct (Berezin-) Fourier integrals and others can be seen to be a direct sum of two invariant subspaces. A summary of the obtained decomposition is given at the end of §3.

Since we use direct integrals, these subspaces are not really subspaces in the strict sense of the word: the functions in question are no longer square-integrable over the
But this is a standard technical detail. More important is the fact that these subspaces for an odd Berezin-Fourier parameter are not graded subspaces in any reasonable sense. We thus provide in §4 a framework for dealing with this kind of subspaces by defining what we mean by an odd family decomposition of a graded vector space and what it should mean when we say that such a decomposition is irreducible. In the next section we apply these definitions to show that the decomposition of our regular representation obtained in §3 is irreducible. We thus have completely decomposed the regular representation into irreducible parts.

The next task is to see how the irreducible representations obtained in the decomposition of the regular representation correspond to the representations obtained by quantizing coadjoint orbits. It turns out that there are four types of coadjoint orbits: 0-dimensional ones, 2|2-dimensional ones with an even symplectic form, 2|2-dimensional ones with an odd symplectic form and 3|3-dimensional ones with a non-homogenous symplectic form. In order to compute the associated representations, it turns out that we have to adjust the prequantization procedure given in [Tu1] slightly by introducing an odd parameter for the odd part of the symplectic form. This parameter is the odd counterpart of the parameter $h$ that is used for the even part. The difference is that the actual value of $h$ can be taken to be 1 by adjusting the physical units, whereas we cannot “rescale” the odd parameter.

In the ungraded case it is (at least morally) true that the representation is independent of the choice of the invariant polarization. In the graded case this is true as long as the graded dimension of the polarization does not change; (invariant) polarizations with different graded dimensions will give rise to different representations. For the orbits of dimension 0 and those of dimension 2|2 with an even symplectic form there is only one graded dimension possible for an invariant polarization, but for 2|2-dimensional orbits with an odd symplectic form there are three different possibilities and for 3|3-dimensional orbits there are two possibilities. Each of these possibilities gives rise to a representation of our supergroup.

When we compare these representations with those obtained in the decomposition of the regular representation, we find the following result: all representations associated to 0-dimensional orbits appear; some but not all representations associated to 2|2-dimensional orbits with an even symplectic form appear; some but not all representations associated to 2|2-dimensional orbits with an odd symplectic form and a 3|3-dimensional polarization appear; and finally, some but not all representations associated to 3|3-dimensional orbits (with a non-homogeneous symplectic form) with a 3|2-dimensional polarization appear. Representations associated to orbits and polarizations of a different graded dimension do not appear in the regular representation. We thus see that representations associated to orbits with a non-homogeneous symplectic form intervene in the regular representation, showing that this kind of symplectic supermanifold is an interesting object to study.

However, several problems still have to be solved. In the first place the question how to interpret the odd parameter introduced in the prequantization for the odd part of the symplectic form. And then how to make the correspondence between the irreducible representations appearing in the regular representation and those obtained by geometric quantization correctly. Because even though there is an obvious correspondence, there remains a problem how to identify real parameters with odd parameters (see the end of this paper). And then a rather long list of technical problems have to be solved, among others how to define densities for the quantization of symplectic supermanifolds with a non-homogeneous form (said
differently, how to define in an intrinsic way the scalar product on the function spaces that appear in quantization) and how to define in a natural way the notion of equivalence between odd families of representations.

2. Preliminaries

I will work with the geometric $H^\infty$ version of DeWitt supermanifolds, which is equivalent to the theory of graded manifolds of Leites and Kostant (see [DW], [Ko], [Le], [Ro], [Tu2]). Any reader using a (slightly) different version of supermanifolds should be able to translate the results to her/his version of supermanifolds.

- The basic graded ring will be denoted as $\mathcal{A}$ and we will think of it as the exterior algebra $\mathcal{A} = \Lambda V$ of an infinite dimensional real vector space $V$.
- Any element $x$ in a graded space splits into an even and an odd part $x = x_0 + x_1$.
- All (graded) objects over the basic ring $\mathcal{A}$ have an underlying real structure, called their body, in which all nilpotent elements in $\mathcal{A}$ are ignored/killed. This forgetful map is called the body map, denoted by the symbol $B$. For the ring $\mathcal{A}$ this is the map/projection $B : \mathcal{A} = \Lambda V \rightarrow \Lambda^0 V = \mathbb{R}$.
- If $\omega$ is a $k$-form and $X$ a vector field, we denote the contraction of the vector field $X$ with the $k$-form $\omega$ by $\iota(X)\omega$, which yields a $k-1$-form. If $X_1, \ldots, X_\ell$ are $\ell \leq k$ vector fields, we denote the repeated contraction of $\omega$ by $\iota(X_1, \cdots, X_\ell)\omega$.

More precisely:

$$\iota(X_1, \cdots, X_\ell)\omega = \left(\iota(X_1) \circ \cdots \circ \iota(X_\ell)\right)\omega.$$ 

In the special case $\ell = k$ this definition differs by a factor $(-1)^{k(k-1)/2}$ from the usual definition of the evaluation of a $k$-form on $k$ vector fields. This difference is due to the fact that in ordinary differential geometry repeated contraction with $k$ vector fields corresponds to the direct evaluation in the reverse order. And indeed, $(-1)^{k(k-1)/2}$ is the signature of the permutation changing $1, 2, \ldots, k$ in $k, k-1, \ldots, 2, 1$. However, in graded differential geometry this permutation not only introduces this signature, but also signs depending upon the parities of the vector fields. These additional signs are avoided by our definition.

- The evaluation of a left linear map $\mu$ on a vector $v$ is denoted as $\langle v | \mu \rangle$. For the contraction of a multi-linear form with a vector we will use the same notation as for the contraction of a differential form with a vector field. In particular, we denote the evaluation of a left bilinear map $\Omega$ on a vector $v$ by $\iota(v)\Omega$, which yields a left linear map $w \mapsto \langle w | \iota(v)\Omega \rangle \equiv \iota(w, v)\Omega$.
- If $E$ is an $\mathcal{A}$-vector space, $E^*$ will denote the left dual of $E$, i.e., the space of all left linear maps from $E$ to $\mathcal{A}$.
- If $G$ is an $\mathcal{A}$-Lie group, then its $\mathcal{A}$-Lie algebra $\mathfrak{g}$ is $\mathfrak{g} = T_e G$, whose Lie algebra structure is given by the commutator of left-invariant vector fields (who are determined by their value at $e \in G$).
- If $\Phi : G \times M \rightarrow M$ denotes the (left) action of an $\mathcal{A}$-Lie group $G$ on an $\mathcal{A}$-manifold $M$, then for all $v \in \mathfrak{g} = T_e G$ the associated fundamental vector field $v^M$ on $M$ is defined as $v^M|_m = -T_{(e, m)}\Phi(v, 0)$. The minus sign is conventional and ensures that the map from $\mathfrak{g}$ to vector fields on $M$ is a homomorphism of $\mathcal{A}$-Lie algebras.

Similarly, if $\Phi : M \times G \rightarrow M$ is a right action of $G$ on $M$, then the fundamental vector field $v^M$ associated to $v \in \mathfrak{g}$ is defined as $v^M|_m = T_{(m, e)}\Phi(0, v)$. And again
the map from \( \mathfrak{g} \) to vector fields on \( M \) is a morphism of \( \mathcal{A} \)-Lie algebras. In the special case when \( M = G \) with the natural right action on itself, the fundamental vector fields are exactly the left-invariant vector fields on \( G \).

3. The Group and its (Left) Regular Representation

Consider the \( \mathcal{A} \)-vector space \( E \) of (graded) dimension 4|4 with basis \( e_1, e_2, e_3, k_0, e_4, e_5, e_6, k_1 \) of which the first four are even (and the last four are odd). The group \( G \) is the even part \( G = E_0 \) with coordinates \( (a^1, a^2, a^3, b, \alpha^4, \alpha^5, \alpha^6, \beta) \) with respect to the given basis. The group law is given as

\[
\begin{pmatrix}
\hat{a}^1 \\
\hat{a}^2 \\
\hat{a}^3 \\
\hat{b} \\
\hat{a}^4 \\
\hat{a}^5 \\
\hat{a}^6 \\
\hat{\beta}
\end{pmatrix}
= 
\begin{pmatrix}
a^1 + a^1 \\
a^2 + a^2 \\
a^3 + a^3 \\
b + \frac{1}{2}(\hat{a}^2 a^1 - \hat{a}^1 a^2 - \hat{a}^5 a^5 + \hat{a}^6 a^6) \\
a^4 + a^4 \\
a^5 + a^5 \\
a^6 + a^6 \\
\hat{\beta} + \frac{1}{2}(\hat{a}^4 a^1 - \hat{a}^1 a^4 + \hat{a}^5 a^3 - \hat{a}^3 a^5)
\end{pmatrix}
\cdot
\begin{pmatrix}
\hat{a}^1 \\
\hat{a}^2 \\
\hat{a}^3 \\
\hat{b} \\
\hat{a}^4 \\
\hat{a}^5 \\
\hat{a}^6 \\
\hat{\beta}
\end{pmatrix}
\]

The neutral element has all coordinates zero and taking the inverse is reversing the sign of all coordinates. The action \( \Phi_g \) of an element \( g \in G \) on a function \( f : G \to \mathcal{A}^\mathbb{C} \) is defined as

\[
(\Phi_g f)(g) = f(\hat{g}^{-1} \cdot g).
\]

Any smooth function \( f : G \to \mathcal{A}^\mathbb{C} \) can be written as

\[
f(a^i, \alpha^j, b, \beta) = f_{(a,i)}(a^i, b) + \beta \cdot f_{(0)}(a^i, b)
+ \sum_{t=1}^{3} \sum_{4 \leq j_1 < \cdots < j_t \leq 6} \alpha^{j_1} \cdots \alpha^{j_t} \cdot f_{(j_1, \ldots, j_t)}(a^i, b)
+ \sum_{t=1}^{3} \sum_{4 \leq j_1 < \cdots < j_t \leq 6} \beta \cdot \alpha^{j_1} \cdots \alpha^{j_t} \cdot f_{(0, j_1, \ldots, j_t)}(a^i, b),
\]

where the 16 functions \( f_{(\ldots)} \) are (equivalent to) ordinary smooth functions of the 4 real (even) coordinates \( a^1, a^2, a^3, b \).

We now wish to study the (left) regular representation \( V \) of \( G \), which consists of functions on \( G \) of the form (3.1) with \( f_P \) an \( L^2 \) function on \( \mathbb{R}^4 \), together with the action \( \Phi \) defined above. Our purpose is to obtain a complete decomposition of this representation into irreducible subspaces. To do so, we start looking at the space \( V_{(t_0, t_2, t_3, \lambda_0, \lambda_4)} \) of functions on \( G \) of the form

\[
f(a^i, \alpha^j, b, \beta) = t_{(t_0, t_2, t_3, \lambda_0, \lambda_4)}(a^1, \alpha^5, \alpha^6, \\
\cdot e^{i t_2 a^2} \cdot e^{i t_3 a^3} \cdot e^{i \lambda_4 a^2} \cdot e^{i \lambda_0(\beta + \frac{1}{2} a^1 a^2 - \frac{1}{2} a^3 a^3)}),
\]

with \( t_{(t_0, t_2, t_3, \lambda_0, \lambda_4)} \) a function of one even and two odd coordinates. This is essentially a Fourier mode with respect to the coordinates \( a^2, a^3, \alpha^4, \beta, \) \( \) More precisely, the functions \( t_{(t_0, t_2, t_3, \lambda_0, \lambda_4)} \) are the Fourier transform with respect to the coordinates \( a^2, a^3, \alpha^4, \) \( c, \gamma \) of a function \( f \) with

\[
c = b + \frac{1}{2} a^1 a^2, \quad \gamma = \beta + \frac{1}{2} a^1 a^4 - \frac{1}{2} a^3 a^5.
\]
given explicitly by
\[
t_{(t_0, t_2, t_3, \lambda_0, \lambda_4)}(a^1, \alpha^5, \alpha^6) = (2\pi)^{-3} \cdot \int f(a^1, \alpha^3, c - \frac{1}{2}a^1 a^2, \gamma - \frac{1}{2}a^1 \alpha^4 + \frac{1}{2}a^3 \alpha^5). \\
e^{-it_2 a^2} \cdot e^{-it_3 a^3} \cdot e^{-it_0 \alpha^4} \cdot e^{-it_0 \gamma} \cdot dc \cdot da^2 \cdot da^3 \cdot da^4,
\]
where we use Berezin integration for the odd variables \(\alpha^4\) and \(\gamma\) (see also [GS,\S7.1]).

With our choice of the normalization constant, the inverse operation is given by
\[
f(a^1, \alpha^3, b, \beta) = \int t_{(t_0, t_2, t_3, \lambda_0, \lambda_4)}(a^1, \alpha^5, \alpha^6) \cdot e^{it_2 a^2} \cdot e^{it_3 a^3} \cdot e^{it_0 \lambda^4 a^4} \\
\cdot e^{it_0 (b + \frac{1}{2}a^1 a^2)} \cdot e^{it_0 (\beta + \frac{1}{2}a^1 \alpha^4 + \frac{1}{4}a^3 \alpha^5)} \cdot d\ell_0 \cdot d\ell_2 \cdot d\lambda_0 \cdot d\lambda_4.
\]

This shows that an arbitrary function on \(G\) can be written as a direct integral of functions of the form (3.2), or more precisely that we have a direct integral decomposition of \(V\) as
\[
V = \int V_{(t_0, t_2, t_3, \lambda_0, \lambda_4)} \cdot d\ell_0 \cdot d\ell_2 \cdot d\lambda_0 \cdot d\lambda_4.
\]

The action of \(\Phi_{\hat{g}}\) on a function \(f\) of the form (3.2) is given by the formula
\[
(\Phi_{\hat{g}} f)(g) = t_{(t_0, t_2, t_3, \lambda_0, \lambda_4)}(a^1 - \hat{a}^1, \alpha^5 - \hat{a}^5, \alpha^6 - \hat{\alpha}^6). \\
e^{-it_2 \alpha^2} \cdot e^{-it_3 \alpha^3} \cdot e^{-it_0 (b + \frac{1}{2}a^1 a^2)} \cdot e^{-it_0 (\beta + \frac{1}{2}a^1 \alpha^4 + \frac{1}{4}a^3 \alpha^5)} \\
e^{it_2 a^2} \cdot e^{it_3 a^3} \cdot e^{it_0 (b + \frac{1}{2}a^1 a^2)} \cdot e^{it_0 (\beta + \frac{1}{2}a^1 \alpha^4 + \frac{1}{4}a^3 \alpha^5)},
\]
which shows that the spaces \(V_{(t_0, t_2, t_3, \lambda_0, \lambda_4)}\) are invariant under the action of \(G\).

The direct integral decomposition (3.3) thus is a decomposition into invariant “subspaces.”

In order to simplify notation, we will identify a function \(f \in V_{(t_0, t_2, t_3, \lambda_0, \lambda_4)}\) i.e. of the form (3.2), with the function \(t_{(t_0, t_2, t_3, \lambda_0, \lambda_4)}\) of the three variables \(a^1, \alpha^5, \alpha^6\).

In terms of such a function \(t\) of these three variables, the action \(\Psi_{\hat{g}}\) of \(\hat{g} \in G\) is given by
\[
(\Psi_{\hat{g}} t)(a^1, \alpha^5, \alpha^6) = t(a^1 - \hat{a}^1, \alpha^5 - \hat{a}^5, \alpha^6 - \hat{\alpha}^6). \\
e^{-it_2 \alpha^2} \cdot e^{-it_3 \alpha^3} \cdot e^{-it_0 (b + \frac{1}{2}a^1 a^2)} \cdot e^{-it_0 (\beta + \frac{1}{2}a^1 \alpha^4 + \frac{1}{4}a^3 \alpha^5)},
\]
\[
(\Psi_{\hat{g}} t)(a^1, \alpha^5, \alpha^6) = t(a^1 - \hat{a}^1, \alpha^5 - \hat{a}^5, \alpha^6 - \hat{\alpha}^6). \\
e^{-it_2 \alpha^2} \cdot e^{-it_3 \alpha^3} \cdot e^{-it_0 (b + \frac{1}{2}a^1 a^2)} \cdot e^{-it_0 (\beta + \frac{1}{2}a^1 \alpha^4 + \frac{1}{4}a^3 \alpha^5)}.
\]

3.5 Proposition. If \(\lambda_0\) is zero but \(t_0\) is non-zero, then the space \(V_{(t_0, t_2, t_3, 0, \lambda_4)}\) splits as a direct sum of two \(G\)-invariant spaces \(V_{(t_0, t_2, t_3, 0, \lambda_4)}^\pm\).

Proof. If \(\lambda_0 = 0\), the \(G\) action in terms of the functions \(t\) (3.4) is given by
\[
(\Psi_{\hat{g}} t)(a^1, \alpha^5, \alpha^6) = t(a^1 - \hat{a}^1, \alpha^5 - \hat{a}^5, \alpha^6 - \hat{\alpha}^6) \cdot e^{-it_2 \alpha^2} \cdot e^{-it_3 \alpha^3} \cdot e^{it_0 (b - \frac{1}{2}a^1 a^2)} \cdot e^{-it_0 (\beta - \frac{1}{2}a^1 \alpha^4 - \frac{1}{4}a^3 \alpha^5)},
\]
It is not hard to see that functions of the form

\[ t(a^1, \alpha^5, \alpha^6) = h_\epsilon(a^1, \alpha^5 + \epsilon \alpha^6) \cdot e^{-\frac{1}{2} \ell_0 \alpha^5 \alpha^6} \]

\[ \text{with } \epsilon = \pm 1 \text{ and } h_\epsilon \text{ a function of one even and one odd variable, transform (in terms of these functions } h_\epsilon \text{) under the action of } \Psi_\tilde{g} \text{ as} \]

\[ \Psi_\tilde{g} h_\epsilon(a^1, \xi) = h_\epsilon(a^1 - \tilde{a}^1, \xi - (\tilde{a}^5 + \epsilon \tilde{a}^6)) \cdot e^{-\ell_0 (\tilde{a}^2 a^1 - \frac{1}{2} (\tilde{a}^5 - \epsilon \tilde{a}^6)) \xi}, \]

\[ e^{-i\ell_2 \tilde{a}^2} \cdot e^{-i\ell_3 \tilde{a}^3} \cdot e^{-i\lambda_4 \tilde{a}^4} \cdot e^{-i\ell_0 (\tilde{a}^2 - \frac{1}{2} \tilde{a}^2 + \frac{1}{2} \tilde{a}^3 \tilde{a}^4\tilde{a}^5)}. \]

This shows that these spaces are invariant under the \( G \)-action. Moreover, any smooth function \( t \) of the variables \( a^1, \alpha^5, \alpha^6 \) can be written as

\[ t(a^1, \alpha^5, \alpha^6) = t_0(a^1) + \alpha^5 \cdot t_5(a^1) + \alpha^6 \cdot t_6(a^1) + \alpha^5 \alpha^6 \cdot t_{56}(a^1) \]

with four smooth functions \( t_0, t_5, t_6, t_{56} \) of the single (even) variable \( a^1 \). In the same way, the functions \( h_\pm(a^1, \xi) \) can be written as

\[ h_\pm(a^1, \xi) = h_{\pm,0}(a^1) + \xi \cdot h_{\pm,1}(a^1). \]

Taking the sum and writing \( e^{-\frac{1}{2} \ell_0 \alpha^5 \alpha^6} = 1 - \frac{i}{\ell_0} \epsilon \alpha^5 \alpha^6 \), we obtain

\[ h_+(a^1, \alpha^5 + \alpha^6) \cdot e^{-\frac{1}{2} \ell_0 \alpha^5 \alpha^6} + h_-(a^1, \alpha^5 - \alpha^6) \cdot e^{-\frac{1}{2} \ell_0 \alpha^5 \alpha^6} = \]

\[ (h_{+,0}(a^1) + h_{-,0}(a^1)) + \alpha^5 \cdot (h_{+,1}(a^1) - h_{-,1}(a^1)) \]

\[ + \alpha^6 \cdot (h_{+,1}(a^1) - h_{-,1}(a^1)) + \frac{i}{\ell_0} \alpha^5 \alpha^6 \cdot (h_{-,0}(a^1) - h_{+,0}(a^1)). \]

Comparing this with (3.8) shows that any (smooth) function of \( (a^1, \alpha^5, \alpha^6) \) can be decomposed in a unique way as the sum of two functions of the form (3.6) (one of each kind).

To analyse the case \( \ell_0 = 0 \), we start looking at the explicit transformation property in this case, which is given in terms of the function \( t(3.4) \) as

\[ (\Psi_\tilde{g} t)(a^1, \alpha^5, \alpha^6) = t(a^1 - \tilde{a}^1, \alpha^5 - \tilde{a}^5, \alpha^6 - \tilde{a}^6) \cdot e^{-i\lambda_0 (\tilde{a}^4 a^1 - \tilde{a}^5 \alpha^6),} \]

\[ e^{-i\ell_2 \tilde{a}^2} \cdot e^{-i\ell_3 \tilde{a}^3} \cdot e^{-i\lambda_4 \tilde{a}^4} \cdot e^{-i\ell_0 (\tilde{a}^2 - \frac{1}{2} \tilde{a}^2 + \frac{1}{2} \tilde{a}^3 \tilde{a}^4\tilde{a}^5)}. \]

Since \( \alpha^6 \) only appears as an argument in \( t \), it is easy to see that we can perform another Berezin-Fourier transform. More precisely, we introduce the spaces \( V_{(0, \ell_2, \ell_3, \lambda_0, \lambda_4), (\lambda_0)} \) of functions \( t \) of the form

\[ t(a^1, \alpha^5, \alpha^6) = h(a^1, \alpha^5) \cdot e^{i\lambda_0 \alpha^6}. \]

In terms of the function \( h \) the action of \( \tilde{g} \in G \) is given by

\[ (\Psi_\tilde{g} h)(a^1, \alpha^5) = h(a^1 - \tilde{a}^1, \alpha^5 - \tilde{a}^5) \cdot e^{-i\lambda_0 (\tilde{a}^4 a^1 - \tilde{a}^5 \alpha^6),} \]

\[ e^{-i(\ell_2 \tilde{a}^2 + \ell_3 \tilde{a}^3)} \cdot e^{-i(\lambda_4 \tilde{a}^4 + \lambda_0 \alpha^6)} \cdot e^{-i\lambda_0 (\tilde{a}^2 - \frac{1}{2} \tilde{a}^2 + \frac{1}{2} \tilde{a}^3 \tilde{a}^4\tilde{a}^5)}. \]
This shows that the spaces \( V_{(0,\ell_2,\ell_3,0,\lambda_4)} \) are invariant under the action of \( G \), meaning that we have an invariant direct integral decomposition

\[
V_{(0,\ell_2,\ell_3,0,\lambda_4)} = \int V_{(0,\ell_2,\ell_3,\lambda_0,\lambda_4)} \, d\lambda_0 .
\]

If \( \lambda_0 = 0 \) the other coordinates \( a^1 \) and \( a^5 \) also appear only in \( t \) and we can perform a triple Berezin-Fourier transform by looking at functions \( t \) of the form

\[
t(a^1, a^5, a^6) = c \cdot e^{i\ell_1 a^1} \cdot e^{i\lambda_5 a^5} \cdot e^{i\lambda_6 a^6}.
\]

These functions transform under the action of \( \hat{g} \in G \) as

\[
(\Psi \hat{g})(a^1, a^5, a^6) = c \cdot e^{-i(\ell_1 a^1 + \ell_2 a^2 + \ell_3 a^3)} \cdot e^{-i(\lambda_4 \delta^1 + \lambda_5 \delta^5 + \lambda_6 \delta^6)}.
\]

This shows that the spaces \( V_{(0,\ell_2,\ell_3,0,\lambda_4)} \) consisting of this kind of functions are also invariant under the action of \( G \) giving us an invariant direct integral decomposition

\[
V_{(0,\ell_2,\ell_3,0,\lambda_4)} = \int V_{(0,\ell_2,\ell_3,0,\lambda_4)} \, d\ell_0 \, d\lambda_5 \, d\lambda_6 .
\]

So far we thus have been able to decompose the regular representation in the following invariant "subspaces":

\[
\begin{array}{ccc}
V & V_{(0,\ell_2,\ell_3,0,\lambda_4)} & V_{(0,\ell_2,\ell_3,0,\lambda_4)} \\
\downarrow & \oplus & \downarrow \\
\ell_0 \neq 0 = \lambda_0 & \ell_0 = 0 \neq \lambda_0 & \ell_0 = 0 = \lambda_0 \\
V_{(\ell_0,\ell_2,\ell_3,0,\lambda_4)} & V_{(0,\ell_2,\ell_3,0,\lambda_4)} & V_{(0,\ell_2,\ell_3,0,\lambda_4)} \\
\downarrow & \oplus & \downarrow \\
V_{(\ell_0,\ell_2,\ell_3,0,\lambda_4)} & V_{(0,\ell_2,\ell_3,\lambda_0,\lambda_4)} & V_{(0,\ell_2,\ell_3,\lambda_5,\lambda_6)}
\end{array}
\]

The main question now is whether this decomposition is irreducible and, for the Berezin direct integrals, in what sense.

### 4. Invariant Odd Families of Subspaces

Throughout this section, \( V \) will denote a graded vector space, \( G \) a graded Lie group and \( \Phi \) a representation of \( G \) on \( V \). This means that \( \Phi \) is a smooth homomorphism from \( G \) to \( \text{Aut}(V) \), the group of automorphisms of \( V \). As is customary, we will note the (even, bijective) linear map \( \Phi(g) \) (for \( g \in G \)) often as \( \Phi_g \). The purpose of this section is to extend the notion of an irreducible subspace and/or irreducible decomposition to incorporate in a convenient way subspaces that are indexed by an odd parameter. Some of our arguments are slightly incomplete and/or vaguely wrong. However, their purpose is to motivate the definitions [4.7], [4.18] and [4.19], not to prove them.
If one starts to think about a family $W_{\lambda} \subset V$ of subspaces indexed by an odd parameter $\lambda \in A_1$, one soon arrives at the following description:

$$W_{\lambda} = \{ w + \lambda I(w) \mid w \in W \}$$

for some subspace $W \subset V$ and a linear map $I : W \to V$. Even though it is not the most general possibility, it also seems reasonable to require that $W$ is a graded subspace and that $I$ is a smooth odd (right-)linear map. This gives us the following definition.

**Definition.** An odd family of subspaces (of $V$) is a couple $(W, I)$ where $W \subset V$ is a graded subspace and $I : W \to V$ a smooth odd right-linear map.

In what follows we will systematically provide two viewpoints for families of subspaces indexed by an odd parameter: the one in which we consider the actual subsets $W_{\lambda}, \lambda \in A_1$, and the one in which the odd parameter $\lambda$ does not appear. The first is the more intuitive approach, the second a more formal one.

**Remark.** Right-linearity of the map $I$ implies that we have $I(v \cdot \lambda) = I(v) \cdot \lambda$ for all $v \in V$ and all $\lambda \in A$. However, the fact that $I$ is odd implies that we don’t have the equality $I(\lambda \cdot v) = \lambda \cdot I(v)$ for all $v$ and $\lambda$. What we do have is the equality $I(\lambda \cdot v) = (-1)^{\alpha} \lambda \cdot I(v)$ for all $\lambda \in A_\alpha$, which means in particular that we obtain a minus sign if $\lambda$ is odd.

**Definition.** Let $(W, I)$ be an odd family of subspaces of $V$. We will say that $(W, I)$ is an odd family decomposition of $V$ if it satisfies the following two conditions:

(i) the map $I : W \to V$ is injective and

(ii) we have a direct sum decomposition $V = W \oplus I(W)$.

The motivation for this definition is the following observation (which should be compared with a Fourier transform). Choosing an element in $W_{\lambda}$ is equivalent to choosing an element in $W$. In particular, choosing for each $\lambda \in A_1$ an element in the corresponding $W_{\lambda}$ corresponds to a map $f : A_1 \to W$, where $f(\lambda) \in W$ determines the element $f(\lambda) + \lambda I(f(\lambda))$ in $W_{\lambda}$. If $f$ is smooth, it has to be of the form $f(\lambda) = w_0 + \lambda w_1$ for two fixed elements $w_0, w_1 \in W$. If we now compute the Berezin integral over $\lambda$ of the corresponding elements in $W_{\lambda} \subset V$, we obtain

$$\int \left( f(\lambda) + \lambda I(f(\lambda)) \right) d\lambda = \int \left( w_0 + \lambda w_1 + \lambda I(w_0 + \lambda w_1) \right) d\lambda$$
$$= \int \left( w_0 + \lambda (w_1 + I(w_0)) \right) d\lambda = w_1 + I(w_0).$$

If $V$ is the direct sum $V = W \oplus I(W)$ and if $I$ is injective, it follows that each element $v \in V$ can be written as a Berezin integral over $\lambda$ of a unique family $f$ of elements in $W_{\lambda}$. In the other direction, if we want that each element of $v$ can be written in a unique way as such an integral, we need that $V$ is the direct sum $V = W \oplus I(W)$ and that $I$ is injective. Our conditions thus are the necessary and sufficient conditions to guarantee that each element of $V$ can be obtained as a Berezin integral of a family of elements in $W_{\lambda}$.
4.2 Lemma. Let \((W,I)\) be an odd family of subspaces of \(V\) and let \(\lambda \neq 0\) be a fixed odd parameter. If the subspace \(W_\lambda\) is invariant under the action \(\Phi\), i.e., for all \(g \in G\) we have \(\Phi_g(W_\lambda) \subset W_\lambda\), then the following two conditions are satisfied.

(i) \(W\) itself is invariant under \(\Phi\), i.e., \(\forall g \in G: \Phi_g(W) \subset W\).

(ii) \(W\) is invariant under all (odd) maps \([\Phi_g,I] = \Phi_g \circ I - I \circ \Phi_g\), i.e., \(\forall g \in G:\)
\[ [\Phi_g,I](W) \subset W. \]

Conversely, if these two conditions are satisfied then all subspaces \(W_\mu, \mu \in A_1\) are invariant under \(\Phi\).

Proof. Let \(g \in G, w \in W\) and \(\mu \in A_1\) be arbitrary. Then, since \(\Phi_g\) is even and thus right- and left-linear, we have
\[ \Phi_g(w + \mu I(w)) = \Phi_g(w) + \mu \Phi_g(I(w)) = \Phi_g(w) + \mu [\Phi_g,I](w) + \mu I(\Phi_g(w)). \]

It follows immediately that if the conditions (i) and (ii) are satisfied, then \(W_\mu\) is invariant under \(\Phi\), simply because \(\Phi_g(w) + \mu [\Phi_g,I](w)\) belongs to \(W\).

For the converse we suppose that \(W_\lambda\) is invariant under \(\Phi\). This means that for all \(w \in W\) and all \(g \in G\) there exists and element \(z \in W\) such that
\[ \Phi_g(w) + \lambda \Phi_g(I(w)) = \Phi_g(w + \lambda I(w)) = z + \lambda I(z). \]

Multiplying this equation on the left with \(\lambda\) gives us the equality \(\lambda \Phi_g(w) = \lambda z\). Since \(z\) and thus \(\lambda z\) belongs to \(W\), it follows that \(\lambda \Phi_g(w)\) belongs to \(W\). By a smoothness argument (to be given below) we deduce that \(\Phi_g(w)\) itself belongs to \(W\). Using the linearity of \(I\) and substituting the equality \(\lambda \Phi_g(w) = \lambda z\) in (4.3), we get
\[ \Phi_g(w) + \lambda \Phi_g(I(w)) = z + \lambda I(\Phi_g(w)). \]

which gives us the solution for \(z\) as
\[ z = \Phi_g(w) + \lambda [\Phi_g,I](w). \]

We know already that \(\Phi_g(w)\) belongs to \(W\), so we deduce that \(\lambda [\Phi_g,I](w)\) belongs to \(W\). As before, it follows by a smoothness argument that \([\Phi_g,I](w)\) itself belongs to \(W\).

The smoothness argument (that \(\lambda \Phi_g(w) \in W\) implies \(\Phi_g(w) \in W\)) goes as follows. We first note that by linearity of \(\Phi_g\) it suffices to know this for \(w \in W\) belonging to a basis of \(W\), which means in particular that such a \(w\) has real coordinates with respect to that basis. We then use the fact that the action \(\Phi\) is smooth, which means in particular that (for a fixed \(w \in W\) with real coordinates) the map \(g \mapsto \Phi_g(w) \in V\) is smooth. If \(G\) is of graded dimension \(p|q\), then in a local coordinate system it is described by \(p\) even coordinates \((g_1,\ldots,g_p)\) and \(q\) odd coordinates \((\chi_1,\ldots,\chi_q)\). Smoothness of the action \(\Phi\) (and the fact that \(w\) has real coordinates) implies that we have ordinary smooth maps of \(p\) real variables \(\Phi_{(i_1,\ldots,i_k)}\) such that
\[ \Phi_g(w) = \sum_{k=0}^{q} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq p} \chi_{i_k} \cdots \chi_{i_2} \cdot \chi_{i_1} \cdot \Phi_{(i_1,\ldots,i_k)}(g_1,\ldots,g_p). \]
We thus know that the element
\begin{equation}
\sum_{k=0}^{q} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq q} \lambda \cdot \chi_{i_1} \cdot \cdots \chi_{i_k} \cdot \Phi_{(i_1, \cdots, i_k)}(g_1, \ldots, g_p)
\end{equation}
lies in \( W \). If we now take the even coordinates \( g_i \) real, then \( \Phi_{(i_1, \cdots, i_k)}(g_1, \ldots, g_p) \) has real coordinates with respect to a basis of \( V \). We will now apply repeatedly the following argument. If \( v \in V \) has real coordinates with respect to a basis and if \( \xi \) is an arbitrary non-zero element of \( \mathcal{A} \), then the condition that \( \xi v \) has a non-zero coordinate with respect to a given basis vector in \( V \) implies that the corresponding (real!) coordinate of \( v \) must be non-zero. Which means in particular that if \( \xi v \) lies in \( W \), then necessarily \( v \) itself lies in \( W \).

We now look at (4.5) and we first take all \( \chi_i = 0 \). Then we get the “equation”
\[ \lambda \cdot \Phi_0(g_1, \ldots, g_p) \in W. \]
But \( \Phi_0(g_1, \ldots, g_p) \) has real coordinates, so by the argument given above we have \( \Phi_0(g_1, \ldots, g_p) \in W \). We then take all \( \chi \) except one (say \( \chi_i \)) to be zero. Knowing that the first term already lies in \( W \) this gives us the “equation”
\[ \lambda \cdot \chi_i \cdot \Phi_{(i)}(g_1, \ldots, g_p) \in W. \]
Choosing \( \chi_i \in \mathcal{A}_i \) such that \( \lambda \cdot \chi_i \neq 0 \) (which is always possible with the restrictions imposed on \( \mathcal{A} \) in [Tu2]) and applying our argument, we deduce that \( \Phi_{(i)}(g_1, \ldots, g_p) \in W \).

The next step is to choose only two non-zero \( \chi \), say \( \chi_i \) and \( \chi_j \) are non-zero. Again knowing that the first terms are in \( W \) we get the “equation”
\[ \lambda \cdot \chi_i \cdot \chi_j \cdot \Phi_{(i,j)}(g_1, \ldots, g_p) \in W. \]
Choosing \( \chi_i \) and \( \chi_j \) such that \( \lambda \cdot \chi_i \cdot \chi_j \neq 0 \) and applying our argument we deduce that \( \Phi_{(i,j)}(g_1, \ldots, g_p) \in W \). Continuing this process, we obtain that all \( \Phi_{(i_1, \cdots, i_k)}(g_1, \ldots, g_p) \) belong to \( W \). But if this is true for \( g_i \) real, it automatically is true for all even \( g_i \). And thus we have shown that \( \Phi_g(w) \) belongs to \( W \) for \( w \in W \) with real coordinates, and thus by linearity for all \( w \in W \).

The same argument applies to the maps \( [\Phi_g, I] \), because \( I \) is a smooth linear map, and thus the maps \( g \mapsto [\Phi_g, I](w) = \Phi_g(I(w)) - I(\Phi_g(w)) \in V \) are smooth when \( w \) has real coordinates.

**Important observation.** In lemma [4.2] we see a phenomenon that we will encounter repeatedly in the sequel: if a property holds for a generic subspace \( W_\lambda \) in the odd family \( (W_\lambda)_{\mu \in \mathcal{A}_1} \) (generic meaning here \( \lambda \neq 0 \)), then it holds for all subspaces in the odd family.

**Definition.** We will say that an odd family of subspaces \((W, I)\) of \( V \) is invariant (under the action \( \Phi \)) if the following two conditions are satisfied:

(i) \( \forall g \in G: \Phi_g(W) \subset W \) and
(ii) \( \forall g \in G: [\Phi_g, I](W) \subset W \).

An invariant odd family decomposition of \( V \) is an odd family decomposition \((W, I)\) of \( V \) such that \((W, I)\) is invariant as an odd family of subspaces.
Remark. It should be noted that if \((W,I)\) is an invariant odd family of subspaces of \(V\) under the action \(\Phi\), then in particular \(W = W_0\) is an invariant subspace. However, it is quite possible that \(W = W_0\) is an invariant subspace without \((W,I)\) being an invariant odd family, i.e., without any \(W_\lambda\) with \(\lambda \neq 0\) being invariant.

Example. Consider the graded vector space \(V\) of all smooth functions on \(A_1\) with values in the complexified ring \(A^C\), \(f : A_1 \to A^C\). Since any such smooth function is of the form \(f(\xi) = c_0 + \xi \cdot c_1\) with \(c_i \in C\), \(V\) has dimension \(1|1\). On \(V\) we let act the group \(G = A_1\) by \((\Phi_\tau(f))(\xi) = f(\xi - \tau)\). In fact, this is the left-regular representation of \(G\).

A basis for \(V\) is given by the functions \(f_0, f_1\) defined by \(f_0(\xi) = 1\) and \(f_1(\xi) = \xi\). In terms of this basis the action of \(G\) is given as

\[
\Phi_\tau f_0 = f_0 \quad \text{and} \quad \Phi_\tau f_1 = \tau \cdot f_0 + f_1.
\]

The graded subspace \(W\) generated by \(f_0\) thus is an invariant subspace, but it does not admit an invariant supplement. However, if we consider the odd right-linear map \(I\) defined on \(W\) with values in \(V\) by \(I(f_0) = i \cdot f_1\), then \((W,I)\) is an invariant odd family decomposition of \(V\). In fact, the subspace \(W_\lambda\) is given as

\[
W_\lambda = \{ c_0 \cdot f_0 + \lambda I(c_0 \cdot f_0) \mid c_0 \in C \} = \{ c_0 \cdot (f_0 + i\lambda \cdot f_1) \mid c_0 \in C \}.
\]

But the function \(f_\lambda = f_0 + i\lambda \cdot f_1\) is defined as

\[
f_\lambda(\xi) = f_0(\xi) + i\lambda \cdot f_1(\xi) = 1 + i\lambda \xi = e^{i\lambda \xi}.
\]

And indeed \(\Phi_\tau f_\lambda = e^{-i\lambda \tau} \cdot f_\lambda\). Moreover, any function \(f \in V\) can be obtained as a Berezin integral over functions of \(\lambda\). This is exactly the odd Fourier transform described (for instance) in [GS, §7.1].

Proposition. Let \((W,I)\) be an odd family of subspaces of \(V\) and let \(\lambda \in A_1\) be a fixed non-zero odd element. If a graded subspace \(X \subset W\) induces (via the isomorphism \(w \mapsto w + \lambda I(w)\)) an invariant subspace \(X_\lambda \subset W_\lambda\) under the action of \(\Phi\) (i.e., \(\Phi_\xi(X_\lambda) \subset X_\lambda\) for all \(\xi \in G\) with \(X_\lambda = \{ x + \lambda I(x) \mid x \in X \}\)), then it induces an invariant subspace of all \(W_\mu, \mu \in A_1\). Moreover, this is the case if and only if the following two conditions are satisfied:

- (i) \(\forall g \in G : \Phi_g(X) \subset X\) and
- (ii) \(\forall g \in G : [\Phi_g, I](X) \subset X\).

Proof. \((X, I|_X)\) is an odd family of subspaces and \(X_\lambda\) is invariant. The result then follows from [4.2]. [Q.E.D]

Definition. Let \((W,I)\) be an odd family of subspaces of \(V\). A graded subspace \(X \subset W\) is called an invariant subspace of the couple \((W,I)\) if \((X, I|_X)\) is an invariant family of subspaces.

Corollary. If \(X\) is an invariant subspace of an odd family \((W,I)\), then not only \(X\), but also \(X \oplus I(X)\) is an invariant subspace of \(V\) under the action \(\Phi\). Moreover, \((X, I|_X)\) is an invariant odd family decomposition of \(X \oplus I(X)\).
Corollary. Suppose \((W, I)\) is an invariant odd family decomposition of \(V\). If \(X\) and \(Y\) are two invariant subspaces of the odd family \((W, I)\) and if \(W = X \oplus Y\), then \((X \oplus I(X)) \oplus (Y \oplus I(Y))\) is a decomposition into two invariant subspaces of \(V\), each of which admits an invariant odd family decomposition.

The two conditions for an odd family of subspaces \((W, I)\) to be invariant are not independent in the sense that the second condition \([\Phi_g, I](W) \subset W\) cannot be formulated when we do not know already the first condition. More precisely, the map \(I\) is defined only on \(W\), so for the map \([\Phi_g, I] = \Phi_g \circ I - I \circ \Phi_g\) to make sense on \(W\), we need to know that \(\Phi_g\) maps \(W\) into \(W\).

In the situation of an odd family decomposition \((W, I)\) of \(V\) we can make the two conditions independent by changing/extending the definition of an odd family decomposition slightly. If \(V = W \oplus I(W)\) we can extend the map \(I : W \to V\) to the whole of \(V\) by saying that on \(I(W) \subset V\) it is the inverse of \(I\). We then have an odd bijective smooth right-linear map \(I : V \to V\) satisfying \(I \circ I = \text{id}_V\) such that \(V = W \oplus I(W)\). (Another way to state these conditions is to say that we have an isomorphism of graded vector spaces between \(V\) and \(W \oplus \prod W\), where \(\prod\) is the parity reversal operation.) Once we have defined \(I\) on the whole of \(V\), the two conditions for an odd family decomposition to be invariant are independent. Since this reformulation of an odd family decomposition will be useful in other situations, we thus modify its definition.

Modified definition. An odd family decomposition of \(V\) is a couple \((W, I)\) with \(W \subset V\) a graded subspace and \(I : V \to V\) a smooth odd right-linear map satisfying \(I \circ I = \text{id}_V\) such that \(V\) is the direct sum of \(W\) and \(I(W)\). It thus is in particular an odd family of subspaces of \(V\).

Let us now consider an odd family of subspaces \((W, I)\) of \(V\) and an odd family of subspaces \((X, J)\) of \(W\). This means that \(W \subset V\) is a graded subspace, that \(I : W \to V\) is a smooth odd right-linear map, that \(X \subset W\) is a graded subspace and that \(J : X \to W\) is a smooth odd right-linear map. We can then use the subspaces \(X_\mu \subset W\), \(\mu \in \mathcal{A}_1\) to define subspaces \(X_{\mu, \lambda} \subset W\) of \(V\) by

\[(4.6) \quad X_{\mu, \lambda} = \{ (x + \mu J(x)) + \lambda I(x + \mu J(x)) \mid x \in X \}.\]

It seems reasonable to call the collection \((X_{\mu, \lambda})_{\mu, \lambda \in \mathcal{A}_1}\) a 2-dimensional odd family of subspaces of \(V\). Since we are more interested in odd family decompositions of \(V\), we will not formalize this general definition.

If \((W, I)\) is an odd family decomposition of \(V\) and if \((X, J)\) is an odd family decomposition of \(W\), then we have the direct sum decomposition

\[V = (X \oplus J(X)) \oplus I(X \oplus J(X)) = X \oplus J(X) \oplus I(X) \oplus I(J(X)).\]

We now extend the map \(J\) to a map \(J' : V \to V\) on the whole of \(V\) by

\[J'_{|W} = J \quad \text{and} \quad J'_{|I(W)} = -I \circ J \circ I.\]

The extended (odd right-linear) map \(J'\) satisfies \(J' \circ J' = \text{id}_V\) and it commutes (in the graded sense) with \(I\), i.e., \(I \circ J' = -J' \circ I\) on the whole of \(V\). Moreover, if we define \(Z = X \oplus I(X)\), then \((Z, J')\) is an odd family decomposition of \(V\) and \((X, I_{|Z})\) is an odd family decomposition of \(Z\).
Going one step further, we can consider an odd family decomposition \((Y, K)\) of \(X\). This gives us a direct sum decomposition of \(V\) as
\[
V = Y \oplus K(Y) \oplus J(Y) \oplus J(K(Y)) \oplus I(Y) \oplus I(K(Y)) \oplus I(J(Y)) \oplus I(J(K(Y))) .
\]
The map \(K : X \to X\) can be extended to a smooth odd right-linear map \(K' : V \to V\) by
\[
K'|_X = K , \quad K'|_{I(X)} = -J \circ K \circ J
\]
\[
K'|_{J(X)} = -I \circ K \circ I , \quad K'|_{J(J(X))} = I \circ J \circ K \circ J \circ I .
\]
It is not hard to see that \(K' \circ K' = id_V\) and that \(K'\) commutes (in the graded sense) with the maps \(I\) and \(J'\). Moreover, it is not hard to show (for instance) that \((Y, J)\) is an odd family decomposition of \(Y \oplus J(Y)\), that \((Y \oplus J(Y), K')\) is an odd family decomposition of \(Y \oplus J(Y) \oplus K(Y) \oplus K'(J(Y))\) and that \((Y \oplus J(Y) \oplus K(Y) \oplus K'(J(Y)), I)\) is an odd family decomposition of \(V\), simply because we have the obvious equality of subspaces \(K'(J(Y)) = J(K(Y))\).

In order to prepare the general definition of an \(n\)-dimensional odd family decomposition of \(V\), we introduce some notation. If \(I_1, \ldots, I_n\) are \(n\) linear maps and if \(P \subset \{1, \ldots, n\}\) is given by \(P = \{i_1, \ldots, i_k\}\) with \(i_1 < \cdots < i_k\), then we define the linear map \(I_P\) by
\[
I_P = I_{i_k} \circ \cdots \circ I_{i_1} ,
\]
with the convention that \(I_\emptyset = id_V\). We obviously assume that this composition makes sense, a condition which will always be satisfied in our use of this notation. Similarly, if \(\lambda_1, \ldots, \lambda_n\) are scalars, then the map \((\lambda I)_P\) is defined as the map \(J_P\) for the linear maps \(J_i = \lambda_i \cdot I_i\), i.e.,
\[
(\lambda I)_P = (\lambda_{i_k} \cdot I_{i_k}) \circ \cdots \circ (\lambda_{i_1} \cdot I_{i_1}) .
\]
We also introduce a multi-commutator \([\ [A, (I)]]_P\) for any linear map \(A\) by
\[
[\ [A, (I)] ]_P = [\ [\ [\ [A, I_{i_k}], I_{i_{k-1}}] ], \ldots, I_{i_1} ] ,
\]
where again we suppose that these (graded!) commutators make sense. By convention we define \([\ [A, (I)] ]_\emptyset = A\).

4.7 Definition. An \(n\)-dimensional odd family decomposition of \(V\) is a graded subspace \(X \subset V\) and \(n\) smooth odd right-linear maps \(I_1, \ldots, I_n : V \to V\) satisfying the following three conditions:

(i) \(\forall 1 \leq i \leq n : I_i \circ I_i = id_V\),

(ii) \(\forall 1 \leq i < j \leq n : [I_i, I_j] = 0\) and

(iii) we have a direct sum decomposition
\[
V = \bigoplus_{P \subset \{1, \ldots, n\}} I_P(X) = X \oplus \bigoplus_{k=1}^n \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} I_{i_k} (\cdots (I_{i_2} (I_{i_1} (X))) \cdots)
\]
Remark. Using the fact that for an odd map $I$ we have $[I, I] = I \circ I + I \circ I$, we can write conditions (i) and (ii) in a single formula as $[I_i, I_j] = 2\delta_{ij} \cdot id_V$ for all $1 \leq i, j \leq n$ (where $\delta_{ij}$ is the Kronecker $\delta$).

In terms of subspaces indexed by $n$ odd parameters $\lambda_1, \ldots, \lambda_n \in A_1$ we obtain the subspaces $X_{\lambda_1, \ldots, \lambda_n}$ as

\begin{equation}
X_{\lambda_1, \ldots, \lambda_n} = \left\{ \sum_{P \subseteq \{1, \ldots, n\}} (\lambda I)_P (x) \mid x \in X \right\}
\end{equation}

\begin{equation}
= \left\{ x + \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_k} I_{i_k} (\cdots (\lambda_{i_2} I_{i_2} (\lambda_{i_1} I_{i_1}(x)))) \mid x \in X \right\}.
\end{equation}

Since the maps $\lambda_i, I_i$ are even, commute and are of square zero ($\lambda_i^2 = 0$), it is not hard to show that we can write this in the following shortened form

\begin{equation}
X_{\lambda_1, \ldots, \lambda_n} = \left\{ e^{\lambda_1 I_1 + \cdots + \lambda_n I_n} (x) \mid x \in X \right\},
\end{equation}

where the exponential map $e^A = \sum_{k=0}^{\infty} A^k / k!$ is in this case a finite sum because the map $A = \lambda_1 I_1 + \cdots + \lambda_n I_n$ is nilpotent (of order less than $n + 1$). Written this way, the bijection between smooth functions $f : A_n^\circ \to X$ and elements of $V$ given by the Berezin integral over $A_n^\circ$ really looks like a Fourier transform: any smooth function $f$ is determined uniquely by $2^n$ elements $x_P \in X$ indexed by a subset $P \subseteq \{1, \ldots, n\}$ as

\begin{equation}
f(\lambda_1, \ldots, \lambda_n) = \sum_{P \subseteq \{1, \ldots, n\}} \lambda_P \cdot x_P,
\end{equation}

where $\lambda_P = \lambda_{i_k} \cdots \lambda_{i_2} \cdot \lambda_{i_1}$ when $P = \{i_1, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$ (and $\lambda_{i_0} = 1$). With our convention on maps, we can write this as $\lambda_P \cdot x_P = (\lambda id_V)_P (x_P)$.

And then

\[
\int \cdots \int e^{\lambda_1 I_1 + \cdots + \lambda_n I_n} (f(\lambda_1, \ldots, \lambda_n)) \, d\lambda_1 \cdots d\lambda_n \\
= \sum_{P, Q \subseteq \{1, \ldots, n\}} \int \cdots \int (\lambda I)_Q (\lambda_P \cdot x_P) \, d\lambda_1 \cdots d\lambda_n \\
= \sum_{P \subseteq \{1, \ldots, n\}} \varepsilon_P I_P^c (x_P),
\]

where $P^c = \{1, \ldots, n\} \setminus P$ denotes the complement of $P$ and where the sign $\varepsilon_P = \pm 1$ is determined by

\[
(\lambda I)_P (\lambda_P \cdot x_P) = \varepsilon_P \cdot \lambda_n \cdots \lambda_2 \cdot \lambda_1 \cdot I_{P^c} (x_P).
\]

Since $V = \bigoplus_{P \subseteq \{1, \ldots, n\}} I_P (X)$ is a direct sum decomposition, we thus have a bijection between smooth functions of $n$ odd parameters $\lambda_i$ and elements of $V$ given by the Berezin-Fourier transform.
Lemma. If \((X, I_1, \ldots, I_n)\) is an \(n\)-dimensional odd family decomposition of \(V\) and if \(\sigma\) is a permutation of \(\{1, \ldots, n\}\), then the increasing sequence of graded subspaces \(X = X_0, X_1, \ldots, X_n = V\) defined by \(X_{k+1} = X_k \oplus I_{\sigma(k+1)}(X_k)\) is such that \((X_k, I_{\sigma(k+1)})\) is an odd family decomposition of \(X_{k+1}\).

4.10 Proposition. Let \((W, I)\) be an odd family of subspaces of \(V\), let \((X, J)\) be an odd family of subspaces of \(W\), let \(\lambda, \mu \in A_1\) be two fixed elements and suppose that the set \(X_{\mu,\lambda}\) \((4.6)\) is invariant under the action \(\Phi\).

(a) If \(\lambda\mu \neq 0\), then the following four conditions are satisfied:

(i) \(\forall g \in G: \Phi_g(X) \subset X\),

(ii) \(\forall g \in G: [\Phi_g, I](X) \subset X\),

(iii) \(\forall g \in G: [\Phi_g, J](X) \subset X\) and

(iv) \(\forall g \in G: \Phi_g : I(X) \subset X\) and

Conversely, if these four conditions are satisfied, then all subspaces \(X_{\mu,\nu}\), \(\nu, \rho \in A_1\) \((4.6)\) are invariant under the action \(\Phi\).

(b) If \(\lambda \neq 0\) and \(\mu = 0\), then conditions (i) and (ii) are satisfied and conversely, if (i) and (ii) are satisfied, then all subspaces \(X_{0,\nu}\) are invariant.

(c) If \(\lambda = 0\) and \(\mu \neq 0\), then conditions (i) and (iii) are satisfied and conversely, if (i) and (iii) are satisfied, then all subspaces \(X_{\rho,0}\) are invariant.

(d) If \(\lambda = \mu = 0\), then condition (i) is satisfied.

Proof. The fact that \(X_{\mu,\lambda}\) is invariant under \(\Phi\) means that for all \(x \in X\) there exists \(y \in X\) such that

\(\Phi_g(x + \mu J(x) + \lambda I(x) + \lambda I(\mu J(x))) = y + \mu J(y) + \lambda I(y) + \lambda I(\mu J(y))\).

If we multiply this equation on the left by \(\mu\), we get the equation

\(\mu \lambda \Phi_g(x) = \mu \lambda y\).

Since \(y\) belongs to \(X\), it follows that \(\mu \lambda \Phi_g(x)\) belongs to \(X\). By a smoothness argument as in the proof of [4.2] it follows that \(\Phi_g(x)\) lies in \(X\), which is condition (i).

If we multiply \((4.11)\) on the left by \(\mu\) and substitute the result \((4.12)\) (note that the linear maps \(\lambda I\) and \(\mu J\) are even, so \(\lambda I(\mu J(y)) = \mu \lambda I(J(y)) = I(\mu \lambda y)\)), we get the equation

\(\mu \Phi_g(x) + \mu \lambda \Phi_g(I(x)) = \mu y + \mu \lambda I(\Phi_g(x))\).

Since we already know that \(\mu y\) and \(\mu \Phi_g(x)\) belong to \(X\), it follows that \(\mu \lambda [\Phi_g, I](x)\) belongs to \(X\). But then we can apply our smoothness argument to show that \([\Phi_g, I](x)\) itself belongs to \(X\), which is condition (ii).

We now solve \((4.13)\) for \(\mu y\), which gives

\(\mu y = \mu \Phi_g(x) + \mu \lambda [\Phi_g, I](x)\),

substitute this and \((4.12)\) in \((4.11)\) and multiply the result by \(\lambda\) on the left. This gives us the equation

\(\lambda \Phi_g(x) + \lambda \mu \Phi_g(J(x)) = \lambda y + \lambda \mu J(\Phi_g(x))\).
Again because we already know that $Φ_g(x)$ and $y$ belong to $X$, it follows that $λμ[Φ_g, J](x)$ belongs to $X$. By our smoothness argument it follows that $[Φ_g, J](x)$ belongs to $X$, which is condition (iii).

Solving (4.15) for $λy$, which gives

$$λy = λΦ_g(x) + λμΦ_g(x),$$

and substituting this result as well as (4.14) and (4.12) in (4.11) we get the equation

$$Φ_g(x) + μ[Φ_g, J](x) + λ[Φ_g, I](x) − λμΦ_g(I(J(x))) = y + μλJ([Φ_g, I](x)) + λμI([Φ_g, J](x)) + λμI(Φ_g(x)).$$

As before it follows that

$$λμ(Φ_g(I(J(x))) + J([Φ_g, I](x)) − I([Φ_g, J](x)) − I(J(Φ_g(x))))$$

belongs to $X$, and thus by our smoothness argument

$$Φ_g(I(J(x))) + J([Φ_g, I](x)) − I([Φ_g, J](x)) − I(J(Φ_g(x))) = [Φ_g, I](x)$$

belongs to $X$, which proves condition (iv).

On the other hand, if these four conditions are satisfied, we can define, for each $x ∈ X$ an $y ∈ X$ by

$$y = Φ_g(x) + ρ · [Φ_g, J](x) + ν · [Φ_g, I](x) + ρν · [Φ_g, I](x).$$

An elementary computation then shows that we have the equality

$$Φ_g(x + ρ · J(x) + ν · I(x) + ρν · I(J(x))) = y + ρ · J(y) + ν · I(y) + ρν · I(J(y)).$$

proving that $X_{i, ρ}$ is invariant under the action $Φ$. This finishes the proof of (a).

The proofs of (b), (c) and (d) are similar and are left to the reader.

4.17 Proposition. Suppose we have a sequence of couples $(W_i, I_i)$ such that the couple $(W_i, I_i)$ is an odd family of subspaces of $W_{i+1}$, $0 ≤ i < n$ and $(W_n, I_n)$ an odd family of subspaces of $V$. Associated to $λ_1, . . . , λ_n ∈ A_1$ we define a subspace $W_{λ_1, . . . , λ_n} ⊂ V$ by

$$W_{λ_1, . . . , λ_n} = \left\{ \sum_{P ⊂ \{1, . . . , n\}} (λI)_{P}(w) \mid w ∈ W_0 \right\}.$$

We now fix a subset $Q ⊂ \{1, . . . , n\}$ and $λ_1, . . . , λ_n ∈ A_1$ such that $\prod_{i ∈ Q} λ_i \neq 0$ and $i ∉ Q ⇒ λ_i = 0$, i.e., the product of all non-zero $λ’s$ is non-zero.

If the subspace $W_{λ_1, . . . , λ_n}$ is invariant under the action $Φ$, then the following conditions are satisfied:

$$∀P ⊂ Q \quad ∀g ∈ G : [Φ_g, (I)](P(W_0)) ⊂ W_0.$$

Conversely, if these conditions are satisfied, then all subspaces $W_{μ_1, . . . , μ_n} ⊂ V$ with $μ_i ∈ A_1$ satisfying $[i ∉ Q ⇒ μ_i = 0]$ are invariant under the action $Φ$. 

QED
4.18 Definition. An invariant \(n\)-dimensional odd family decomposition of \(V\) is an \(n\)-dimensional odd family decomposition \((X, I_1, \ldots, I_n)\) of \(V\) satisfying the conditions:

\[
\forall P \subset \{1, \ldots, n\} \quad \forall g \in G : \left[ [\Phi_g, (I)] \right]_P(X) \subset X.
\]

This is equivalent to requiring that all subspaces \(X_{\lambda_1, \ldots, \lambda_n}\) (4.8) are invariant under the action \(\Phi\).

Remark. Instead of looking at the action \(\Phi\) on a subspace \(W_{\lambda_1, \ldots, \lambda_n}\), we can also look at the induced action \(\Psi\) on \(W\) via the map \(w \mapsto e^{\sum_{i=1}^{n} \lambda_i I_i} \Phi_g(w)\) from \(W\) to \(W_{\lambda_1, \ldots, \lambda_n}\). This induced action \(\Psi\) depends (obviously) upon the odd parameters \(\lambda_i\) and satisfies by definition

\[
e^{\sum_{i=1}^{n} \lambda_i I_i}(\Psi_g(w)) = \Phi_g(e^{\sum_{i=1}^{n} \lambda_i I_i}(w)).
\]

According to (4.4), for \(n = 1\) it is given as

\[
w \mapsto \Phi_g(w) + \lambda [\Phi_g, I](w)
\]

and according to (4.16) for \(n = 2\) it is given by

\[
w \mapsto \Phi_g(w) + \lambda [\Phi_g, I](w) + \mu [\Phi_g, J](w) + \lambda \mu [\Phi_g, I], J](w).
\]

If we introduce for \(M \in \text{End}(V)\) the right-adjoint map \(\overrightarrow{\text{Ad}}(M) : \text{End}(V) \to \text{End}(V)\) by

\[
\overrightarrow{\text{Ad}}(M)(X) = [X, M],
\]

then we can write these two cases as

\[
w \mapsto \left(\Phi_g + \overrightarrow{\text{Ad}}(\lambda I)(\Phi_g)\right)(w) \quad \text{and} \quad w \mapsto \left(\Phi_g + \overrightarrow{\text{Ad}}(\lambda I)(\Phi_g) + \overrightarrow{\text{Ad}}(\mu J)(\Phi_g) + \overrightarrow{\text{Ad}}(\mu J)(\overrightarrow{\text{Ad}}(\lambda I)(\Phi_g))\right)(w).
\]

But the maps \(\overrightarrow{\text{Ad}}(\lambda_i I_i)\) are even, of square zero (because \(\lambda_i^2 = 0\)) and commute (because of the (graded) Jacobi identity and the fact that the \(I_i\) commute). It then is not hard to show that the general case can be written as

\[
w \mapsto \Psi_g(w) = \left(e^{\sum_{i=1}^{n} \overrightarrow{\text{Ad}}(\lambda_i I_i)}(\Phi_g)\right)(w),
\]

simply because one can easily show the equality (as maps on \(V\))

\[
\Phi_g \circ e^{\sum_{i=1}^{n} \lambda_i I_i} = e^{\sum_{i=1}^{n} \lambda_i I_i} \circ \overrightarrow{\text{Ad}}(\lambda_i I_i)(\Phi_g).
\]

The separate maps \(\left[ [\Phi_g, (I)] \right]_P\) then can easily be recovered from the action \(\Psi_g\) by taking the derivative with respect to the odd variables in \(P \subset \{1, \ldots, n\}\) (at zero values for the odd variables).

We now want to investigate the notion of invariant subspaces of invariant \(n\)-dimensional odd family decompositions \((X, I_1, \ldots, I_n)\) and more precisely, we want to find criteria for a subspace of a given \(X_{\lambda_1, \ldots, \lambda_n}\) to be invariant. Any subspace of \(X_{\lambda_1, \ldots, \lambda_n}\) can be described in terms of a subspace \(Y \subset X\) via the map ("bijection")
identification. Moreover, a single \( Y_\lambda \) with \( \lambda \neq 0 \) is invariant, then all \( Y_\mu \) are invariant. However, as can be seen in examples, it is quite possible that \( Y_0 \) (which is \( Y \)) is invariant but that no \( Y_\lambda \) with \( \lambda \neq 0 \) is invariant. There thus are just two cases: either only \( Y_0 \) is invariant, or all \( Y_\lambda \) are invariant. This corresponds (in a philosophical sense) with the fact that \( (Y, I_1) \) determines a decomposition of \( V \) as a direct sum of two subspaces: \( Y \) and \( I_1(Y) \).

If we start with the simplest case \( n = 1 \), we see that if a single \( Y_\lambda \) with \( \lambda \neq 0 \) is invariant, then all \( Y_\mu \) are invariant. In particular all elements of all families \( Y_\lambda \) (meaning \( \mu, \lambda \neq 0 \)) are invariant. However, as can be seen in examples, it is quite possible that \( Y_0 \) (which is \( Y \)) is invariant but that no \( Y_\lambda \) with \( \lambda \neq 0 \) is invariant.

According to \([4.10]\) we have a similar pattern for the case \( n = 2 \), but with some exceptions. We have four cases which can be described roughly as telling which of the two parameters \( \lambda, \mu \) is zero. In the same sense as before, this corresponds to the fact that \( V \) splits as a direct sum of four subspaces: \( Y, I_1(Y), I_2(Y) \) and \( I_2(I_2(Y)) = I_2(I_1(Y)) \). We could say that the subspace \( Y_{0,0} \) corresponds to \( Y \) itself (it actually is \( Y \)), that the family of subspaces \( Y_{0,0} \) with \( \lambda \neq 0 \) corresponds to \( I_1(Y) \), that all \( Y_{\mu,0} \) corresponds to \( I_2(Y) \) and (thus) that the family \( Y_{\mu,0} \) with both \( \lambda \) and \( \mu \) non-zero corresponds to \( I_1(I_2(Y)) \). And indeed, the subspaces \( Y_{0,0} \) all lie in the subspace \( Y \oplus I_1(Y) \subset V \), just as all \( Y_{\mu,0} \) lie in \( Y \oplus I_2(Y) \), confirming our identification. Moreover, a single \( Y_0 \lambda \) being invariant with \( \lambda \neq 0 \) implies that all are invariant, just as in the case \( n = 1 \) above. However, the fourth case \( \lambda \) and \( \mu \) both non-zero leaves some gaps in the sense that it remains possible that the product \( \lambda \mu \) is zero, something which never happens with real numbers. A single “generic” (meaning \( \mu \lambda \neq 0 \)) \( Y_{\mu,\lambda} \) invariant implies that all \( Y_{\mu,\nu} \) are invariant. However, if for instance \( Y_{\lambda,\lambda} \) is invariant for \( \lambda \neq 0 \), no clear-cut conclusion is available. Looking at the proof of \([4.10]\) one can (in the case under consideration \( \lambda \mu = 0 \) but neither \( \lambda \) nor \( \mu \) zero) still deduce condition \([4.10, 1]\) and we can deduce the condition that

\[
\lambda [\Phi_g, I](y) + \mu [\Phi_g, J](y)
\]

should belong to \( Y \) (when \( y \) does), but without more conditions on \( \lambda \) and \( \mu \) and/or a more detailed knowledge of the structure of \( A \), we cannot deduce that the separate terms \([\Phi_g, I](y)\) and \([\Phi_g, J](y)\) belong to \( Y \).

The general case has exactly the same features. We have a direct sum decomposition of \( V \) into \( 2^n \) subspaces \((I)_P(Y)\) indexed by the \( 2^n \) subsets \( P \subset \{1,\ldots,n\} \). Associated to a subset \( Q \subset \{1,\ldots,n\} \) we can associate a whole family of subspaces \( Y_Q \) consisting of those subspaces \( Y_{\lambda_1,\ldots,\lambda_n} \) with \( \lambda_i = 0 \) when \( i \notin P \). These families correspond exactly to the \( 2^n \) cases described in \([4.17]\) where it is shown that if a generic element of \( Y_Q \) (generic meaning that \( \prod_{i \in Q} \lambda_i \neq 0 \)) is invariant, then all members of the family are invariant. In particular all elements of all families \( Y_R \) with \( R \subset Q \) are also invariant. One could say that this corresponds to the fact that all “elements” of \( Y_Q \) lie in the subspace

\[
\bigoplus_{R \subset Q} (I)_R(Y),
\]

as can easily be seen from the definition \( Y_{\lambda_1,\ldots,\lambda_n} = \{ \exp(\sum_{i=1}^n \lambda_i I_i)(y) \mid y \in Y \} \).

If the \( \lambda_i \) were real numbers, any set \( \lambda_1,\ldots,\lambda_n \) would determine a unique family \( Y_Q \) and a generic element in this family, simply by setting \( Q = \{ i \in \{1,\ldots,n\} \mid \lambda_i \neq 0 \} \), which automatically implies \( \prod_{i \in Q} \lambda_i \neq 0 \). However, for odd elements there remains the grey area of \( \lambda_i \) that are non-zero but whose product is zero.
4.19 Definition. Let $(X, I_1, \ldots, I_n)$ be an invariant $n$-dimensional odd family decomposition of $V$ and let $Q \subset \{1, \ldots, n\}$ be any subset. We now look at the direct summand $I_Q(X)$ in the direct sum decomposition

$$V = \bigoplus_{P \subset \{1, \ldots, n\}} I_P(X).$$

An *invariant subspace* of $I_Q(X) \subset V$ will be a subspace $Y \subset X$ satisfying the conditions

$$\forall P \subset Q \quad \forall g \in G : [[\Phi_g, (I)]]_P(Y) \subset Y.$$

This is equivalent to requiring that a generic element of the family $Y_Q$ is invariant (which is equivalent to all elements of this family being invariant). The subspace $I_Q(X)$ is said to be *irreducible* if it does not admit any proper invariant subspace.

**Nota Bene.** With our definition of an invariant subspace of $I_Q(X)$ we have discarded the grey area in the sense that we ignore the possibility that a non-generic element of the family $Y_Q$ (meaning $i \in Q \Rightarrow \lambda_i \neq 0$ but $\prod_{i \in Q} \lambda_i = 0$) could admit an invariant subspace, while a generic element does not. Note that this grey area does not include the situation with $i \in Q$ and $\lambda_i = 0$, as we then have to change the subset $Q$ by excluding this $i$.

**Corollary.** If the direct summand $I_Q(X)$ is irreducible, then all $I_R(X)$ with $Q \subset R$ are irreducible. If $I_Q(X)$ admits an invariant subspace $Y$, then $Y$ is an invariant subspace for all $I_R(X)$ with $R \subset Q$.

5. The regular representation revisited

In order to cast our decomposition of the (left) regular representation of $G$ into the setting of odd family decompositions, we introduce 4 odd maps $I_0$, $I_4$, $I_5$ and $I_6$ on $V$ (the space of $(L^2)$ functions on $G$) defined as

$$I_0 = i(\beta + \frac{1}{2}a^1\alpha^4 - \frac{1}{4}a^3\alpha^5) - i\partial_6$$

$$I_6 = i\alpha^6 - i\partial_{\alpha^6}$$

$$I_4 = i\alpha^4 - i(\partial_{\alpha^4} - \frac{1}{2}\partial_6)$$

$$I_5 = i\alpha^5 - i(\partial_{\alpha^5} + \frac{1}{2}\partial_6).$$

If we apply a change of coordinates $\beta \rightarrow \gamma = \beta + \frac{1}{2}a^1\alpha^4 - \frac{1}{4}a^3\alpha^5$, then these maps/operators take the simpler form

$$I_0 = i\gamma - i\partial_\gamma,$$

$$I_4 = i\alpha^4 - i\partial_{\alpha^4},$$

$$I_5 = i\alpha^5 - i\partial_{\alpha^5},$$

$$I_6 = i\alpha^6 - i\partial_{\alpha^6}.$$

It is not hard to show that these maps verify the conditions $I_i \circ I_i = id_V$ and $i \neq j \Rightarrow [I_i, I_j] = 0$. We define the graded subspace $E \subset V$ as consisting of functions independent of the odd variables $\alpha^4$, $\alpha^5$, $\alpha^6$ and $\beta$, i.e., $E$ is the space of $(L^2)$ functions of the four even (real) coordinates $a^1$, $a^2$, $a^3$ and $b$. It thus has no odd dimensions. It is not hard to show that $(E, I_0, I_4, I_5, I_6)$ is a 4-dimensional odd family decomposition of $V$. More precisely, $I_i$ with $i = 4, 5, 6$ adds a factor $\alpha^i$ to a function in $E$, whereas $I_0$ adds a factor $\gamma$, which is essentially $\beta$. The various graded subspaces $(I)_P(E)$ thus contain the corresponding factors $\alpha$ (and/or a modified $\beta$).
To make the link with the subspaces \( V_{(\ell_0, \ell_2, \ell_3, \lambda_0, \lambda_4)} \) given in \( \S 3 \) we introduce the space \( W_{(\ell_0, \ell_2, \ell_3)} \) consisting of functions (on \( G \)) of the form
\[
f(a^1, a^3, b, \beta) = h(a^1, a^2, \beta, \gamma) \cdot e^{i \ell_0 a^2} \cdot e^{i \ell_2 a^3} \cdot e^{i \ell_0 (b + \frac{1}{2} a^1 a^2)},
\]
with \( \gamma = \beta + \frac{1}{2} a^1 a^4 - \frac{1}{2} a^3 a^5 \) as before and where \( h \) is a function of one even and 4 odd coordinates. In terms of the functions \( h \) the action of \( \hat{g} \in G \) is given by
\[
(\Psi \hat{g}) (a^1, a^3, \gamma) = h(a^1 - \hat{a}^1, a^3 - \hat{a}^3, \gamma - \hat{a}^5 a^1 + \hat{\alpha}^3 a^5 + \frac{1}{2} \hat{a}^1 a^4 - \frac{1}{2} a^3 a^5 - \hat{\beta}) \cdot e^{-i (\ell_2 \hat{a}^2 + \ell_3 \hat{a}^3)} \cdot e^{i \ell_0 (b + \frac{1}{2} \hat{a}^1 a^2)}.
\]

These spaces are invariant under the action \( \Phi \) and they are preserved by the maps \( \Lambda \) (note that there is an \( \alpha^4 \) and \( \alpha^6 \) dependence via \( \gamma \)). In terms of \( L^2 \) functions this means that we have written \( V \) as the (usual) direct integral of the Fourier modes
\[
V = \int W_{(\ell_0, \ell_2, \ell_3)} \, d\ell_0 \, d\ell_2 \, d\ell_3,
\]
given by the map
\[
f(a^1, a^3, b, \beta) = \int h_{(\ell_0, \ell_2, \ell_3)}(a^1, a^3, \gamma) \cdot e^{i (\ell_2 a^2 + \ell_3 a^3 + \ell_0 (b + \frac{1}{2} a^1 a^2))} \, d\ell_0 \, d\ell_2 \, d\ell_3.
\]

A direct computation shows that the graded subspace \( X_{(\ell_0, \ell_2, \ell_3)} \subset W_{(\ell_0, \ell_2, \ell_3)} \) of functions independent of \( \alpha^4 \) and \( \beta/\gamma \), i.e., of the form
\[
f(a^1, a^3, b, \beta) = t(a^1, a^5, \alpha^6) \cdot e^{i \ell_0 a^2} \cdot e^{i \ell_3 a^3} \cdot e^{i \ell_0 (b + \frac{1}{2} a^1 a^2)},
\]
is invariant and that \( (X_{(\ell_0, \ell_2, \ell_3)}, I_0, I_4) \) is an invariant 2-dimensional odd family decomposition of \( W_{(\ell_0, \ell_2, \ell_3)} \). The link with the subspaces \( V_{(\ell_0, \ell_2, \ell_3, \lambda_0, \lambda_4)} \) described in \( \S 3 \) is given by (4.9):
\[
V_{(\ell_0, \ell_2, \ell_3, \lambda_0, \lambda_4)} = \left\{ e^{\lambda_0 I_0 + \lambda_4 I_4} f \mid f \in X_{(\ell_0, \ell_2, \ell_3)} \right\}.
\]

5.2 Proposition. The subspace \( X_{(\ell_0, \ell_2, \ell_3)} \) can be written as the direct sum of two invariant graded subspaces \( X^\pm_{(\ell_0, \ell_2, \ell_3)} \subset X_{(\ell_0, \ell_2, \ell_3)} \):
\[
X_{(\ell_0, \ell_2, \ell_3)} = X^+_0(\ell_0, \ell_2, \ell_3) \oplus X^-_{(\ell_0, \ell_2, \ell_3)}.
\]

These subspaces are also invariant subspaces of \( I_4(X_{(\ell_0, \ell_2, \ell_3)}) \).

If \( \ell_0 \) is non-zero, these subspaces are irreducible, implying that each of the graded subspaces \( X_{(\ell_0, \ell_2, \ell_3)} \) and \( I_4(X_{(\ell_0, \ell_2, \ell_3)}) \) in the odd family decomposition
\[
W_{(\ell_0, \ell_2, \ell_3)} = X_{(\ell_0, \ell_2, \ell_3)} \oplus I_4(X_{(\ell_0, \ell_2, \ell_3)}) \oplus I_0(X_{(\ell_0, \ell_2, \ell_3)}) \oplus I_0(I_0(X_{(\ell_0, \ell_2, \ell_3)}))
\]
splits into two irreducible subspaces.

Proof. Mimicking the proof of [3.5] we first note that the action of \( G \) on the elements of \( X_{(\ell_0, \ell_2, \ell_3)} \) in terms of the functions \( t \) (5.1) (see also (3.2), (3.4)) is given by the formula
\[
(\Psi \hat{t})(a^1, a^3, \alpha^6) = t(a^1 - \hat{a}^1, \alpha^5 - \hat{\alpha}^5, a^6 - \hat{\alpha}^6) \cdot e^{-i \ell_0 (\hat{\alpha}^2 a^1 - \frac{1}{2} \hat{\alpha}^3 a^5 + \frac{1}{2} \hat{\alpha}^6 a^6)} \cdot e^{-i \ell_2 \hat{a}^2} \cdot e^{-i \ell_3 \hat{a}^3} \cdot e^{-i \ell_0 (b + \frac{1}{2} \hat{a}^1 a^2)}.
\]
As in the proof of [3.5] it is not hard to see that the graded subspaces \( X^\pm_{(t_0,t_2,t_3)} \subset X_{(t_0,t_2,t_3)} \) of functions of the form

\[
(5.4) \quad t(a^1, \alpha^5, \alpha^6) = h_e(a^1, \alpha^5 + \epsilon \alpha^6) \cdot e^{-\frac{i}{2} t_0 \epsilon \alpha^6} \alpha^6
\]

(with \( \epsilon = \pm 1 \) and \( h_e \) a function of one even and one odd variable) transform under the action of \( \Psi_g \) as

\[
(\Psi_g t)(a^1, \alpha^5, \alpha^6) = h_e(a^1 - \hat{a}^1, (\alpha^5 + \epsilon \alpha^6) - (\hat{a}^5 + \epsilon \hat{a}^6)) \cdot e^{-\frac{i}{2} t_0 \epsilon \alpha^6} \alpha^6 \cdot e^{-i t_2 \hat{a}^2} \cdot e^{-i t_3 \hat{a}^3} \cdot e^{-i t_0 (b - \frac{1}{2} \hat{a}^1 \hat{a}^2 + \frac{1}{4} \epsilon \hat{a}^5 \hat{a}^6)} \]

This shows that these spaces are invariant under the \( G \)-action. And again as in the proof of [3.5] one can show that this gives a direct sum decomposition into two invariant subspaces

\[
X_{(t_0,t_2,t_3)} = X^+_{(t_0,t_2,t_3)} \oplus X^-_{(t_0,t_2,t_3)}.
\]

To show that these subspaces are also invariant subspaces of \( I_4(X_{(t_0,t_2,t_3)}) \) [4.19], we must show that we have \((\Psi_g, I_4)|X^\pm_{(t_0,t_2,t_3)} \subset X^\pm_{(t_0,t_2,t_3)}\). An elementary computation shows that we have, in terms of the functions \( t \) of the form (5.4)

\[
([\Psi_g, I_4] t)(a^1, \alpha^5, \alpha^6) = -\alpha^4 \cdot (\Psi_g t)(a^1, \alpha^5, \alpha^6).
\]

Since we already know that we have \( \Psi_g(X_{(t_0,t_2,t_3)}) \subset X^\pm_{(t_0,t_2,t_3)} \), this shows that the \( X^\pm_{(t_0,t_2,t_3)} \) are also invariant subspaces of \( I_4(X_{(t_0,t_2,t_3)}) \).

To show that these two spaces are irreducible, we consider the first kind (the second being similar). If we don’t write the (obligatory) factors \( e^{-\frac{i}{2} t_0 \epsilon \alpha^6} \alpha^6 \cdot e^{i t_2 \hat{a}^2} \cdot e^{i t_3 \hat{a}^3} \cdot e^{-i t_0 (b - \frac{1}{2} \hat{a}^1 \hat{a}^2 + \frac{1}{4} \epsilon \hat{a}^5 \hat{a}^6)} \), we have to deal with functions \( h_e \) of the two variables \( a^1, \xi \) which transform under the \( G \)-action as

\[
(\Psi_g h_e)(a^1, \xi) = h_e(a^1 - \hat{a}^1, \xi - (\hat{a}^5 + \epsilon \hat{a}^6)) \cdot e^{-i t_0 (a^2 a^1 - \frac{1}{2}(\hat{a}^5 - \epsilon \hat{a}^6)\xi)} \cdot e^{-i t_2 \hat{a}^2} \cdot e^{-i t_3 \hat{a}^3} \cdot e^{-i t_0 (b - \frac{1}{2} \hat{a}^1 \hat{a}^2 + \frac{1}{4} \epsilon \hat{a}^5 \hat{a}^6)}.
\]

If we restrict for the moment our attention to the subgroup in which only \( \hat{a}^1, \hat{a}^2, \hat{b} \) are non-zero, the group law is given as

\[
(\hat{a}^1, \hat{a}^2, \hat{b}) \cdot (a^1, a^2, b) = (\hat{a}^1 + a^1, \hat{a}^2 + a^2, \hat{b} + b + \frac{1}{2}(\hat{a}^2 a^1 - \hat{a}^1 a^2)),
\]

which we recognize as the Heisenberg group. On this subgroup we get the transformation property

\[
(\Psi_g h_e)(a^1, \xi) = h_e(a^1 - \hat{a}^1, \xi) \cdot e^{-i t_2 \hat{a}^2} \cdot e^{-i t_0 (b + \hat{a}^2 a^1 - \frac{1}{2} \hat{a}^1 \hat{a}^2)}.
\]

which we recognize as the irreducible representation of the Heisenberg group, provided we disregard the dependence of the odd coordinate \( \xi \). Writing as before
implies that the subspaces $H \otimes V$ of (5.5) on subgroup. But $V$ and each $V_{\epsilon,i}$ consists of the functions $h_{\epsilon,i}$ (the other component being zero). We thus have

$$X_{(\ell_0, \ell_2, \ell_3)}^\epsilon = V_{\epsilon,0} \oplus V_{\epsilon,1}$$

and each $V_{\epsilon,i}$ is invariant and irreducible under the Heisenberg subgroup.

We now look at the generators of the $\hat{\alpha}^5$ and $\hat{\alpha}^6$ action (take the derivative with respect to these variables in the group action), we get

$$h_\epsilon \mapsto -\frac{\partial h_\epsilon}{\partial \xi} \cdot \frac{i \ell_0}{2} + \epsilon \cdot h_\epsilon \quad \text{and} \quad h_\epsilon \mapsto -\frac{\partial h_\epsilon}{\partial \xi} - \epsilon \cdot \frac{i \ell_0}{2} \cdot \xi \cdot h_\epsilon .$$

In terms of the decomposition $h_\epsilon = (h_{\epsilon,0}, h_{\epsilon,1})$ this corresponds to the matrices

$$
\begin{pmatrix}
0 & i \ell_0 \frac{1}{2} \\
\frac{1}{2} & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & -\epsilon \\
-\epsilon & 0
\end{pmatrix} .
$$

If a subspace $H \subset X_{(\ell_0, \ell_2, \ell_3)}$ is $G$-invariant, then it must be invariant under these operations and also under linear combinations of these operations and thus in particular under the matrices ($\ell_0 \neq 0$)

$$
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix} .
$$

Now if $H$ is invariant, it is also invariant under the Heisenberg subgroup. Which implies that the subspaces $H \cap V_{\epsilon,i} \subset V_{\epsilon,i}$ are also invariant under the Heisenberg subgroup. But $V_{\epsilon,i}$ is irreducible under this subgroup, thus $H \cap V_{\epsilon,i}$ is either $V_{\epsilon,i}$ or $\{0\}$. Now suppose that $H \cap V_{\epsilon,0} = V_{\epsilon,0}$, then $V_{\epsilon,0} \subset H$. But if we then act with the second matrix of (5.5) on $V_{\epsilon,0} \subset H$ we get $V_{\epsilon,1}$, which by invariance of $H$ must also lie in $H$. But then $H = V_{\epsilon,0} \oplus V_{\epsilon,1} = X_{(\ell_0, \ell_2, \ell_3)}^{(\ell_0, \ell_2, \ell_3)}$. A similar argument applies when $H \cap V_{\epsilon,1} = V_{\epsilon,1}$. And if both intersections yield $\{0\}$, we must have $H = \{0\}$. This shows that $X_{(\ell_0, \ell_2, \ell_3)}^\epsilon$ is indeed irreducible.

**Remark.** In terms of the subspaces $V_{(\ell_0, \ell_2, \ell_3, \lambda_0, \lambda_4)}$ and $I_4(X_{(\ell_0, \ell_2, \ell_3)}^{\pm})$ in (5.3) correspond to $\lambda_0 = 0$. And then [5.2] tells us that (for $\ell_0 \neq 0$) the invariant subspaces

$$V_{(\ell_0, \ell_2, \ell_3, 0, \lambda_4)}^{\pm} = \{ e^{i \lambda_4} f \mid f \in X_{(\ell_0, \ell_2, \ell_3)}^{\pm} \}$$

are irreducible.

**Proposition.** If $\ell_0$ is non-zero, then the graded subspaces $I_0(X_{(\ell_0, \ell_2, \ell_3)})$ and $I_4(I_0(X_{(\ell_0, \ell_2, \ell_3)}))$ in the odd family decomposition

$$W_{(\ell_0, \ell_2, \ell_3)} = X_{(\ell_0, \ell_2, \ell_3)} \oplus I_4(X_{(\ell_0, \ell_2, \ell_3)}) \oplus I_0(X_{(\ell_0, \ell_2, \ell_3)}) \oplus I_4(I_0(X_{(\ell_0, \ell_2, \ell_3)}))$$

are irreducible.

**Proof.** An invariant subspace for one of these summands is an invariant subspace of $X_{(\ell_0, \ell_2, \ell_3)}$ compatible with $I_0$ and $I_4$ in the sense that such a subspace should be...
invariant under all maps $\Psi \tilde{g}, [\Psi \tilde{g}, I_0], [\Psi \tilde{g}, I_4]$ and $[[\Psi \tilde{g}, I_0], I_4]$. As we have seen in [5.2], there are only two non-trivial subspaces of $X(\ell_0, \ell_2, \ell_3)$ that are invariant under all maps $\Psi \tilde{g}$ and $[\Psi \tilde{g}, I_4]$, which are $X^\tilde{g}(\ell_0, \ell_2, \ell_3)$. To show that they are not invariant subspaces for the direct summands $I_0(X(\ell_0, \ell_2, \ell_3))$ and $I_4(I_0(X(\ell_0, \ell_2, \ell_3)))$, it suffices to show that $X^\tilde{g}(\ell_0, \ell_2, \ell_3)$ is not invariant under $[\Psi \tilde{g}, I_0]$. An elementary computation shows that we have, in terms of the functions $t$ of the form (5.1)

$$([\Psi \tilde{g}, I_0]t)(a^1,\alpha^5,\alpha^6) = (-\tilde{\beta} + \frac{1}{2}a^1\tilde{a}^4 - \frac{1}{2}a^3\tilde{a}^5 - \tilde{a}^4a^1 + \tilde{a}^3\alpha^5) \cdot (\Psi \tilde{g}t)(a^1,\alpha^5,\alpha^6).$$

This computation is a (partial) confirmation that $(X(\ell_0, \ell_2, \ell_3), I_0, I_4)$ is an invariant 2-dimensional odd family decomposition. But more important is that it shows that the space of functions of the form (5.4) is not invariant under the maps $[\psi \tilde{g}, I_0]$, simply because of the single $\alpha^5$ factor in front of $(\Psi \tilde{g}t)(a^1,\alpha^5,\alpha^6)$, which does not come in the combination $\alpha^5 + \epsilon a^6$. It follows that there cannot be a non-trivial invariant subspace.

Remark. We can rephrase this result as saying that the subspaces $V(\ell_0, \ell_2, \ell_3, \lambda_0, \lambda_4)$

$$V(\ell_0, \ell_2, \ell_3, \lambda_0, \lambda_4) = \{e^{i\lambda_0 I_0 + \lambda_4 I_4} f \mid f \in X(\ell_0, \ell_2, \ell_3)\}$$

are irreducible when $\ell_0$ and $\lambda_0$ are non-zero.

Corollary. If $\ell_0 \neq 0$, then the direct sum decomposition

$$W(\ell_0, \ell_2, \ell_3) = X^+(\ell_0, \ell_2, \ell_3) \oplus X^-_{(\ell_0, \ell_2, \ell_3)} \oplus I_4(X^+_{(\ell_0, \ell_2, \ell_3)}) \oplus I_4(X^-_{(\ell_0, \ell_2, \ell_3)})$$

$$\oplus I_0(X(\ell_0, \ell_2, \ell_3)) \oplus I_4(I_0(X(\ell_0, \ell_2, \ell_3)))$$

is a decomposition into irreducible subspaces.

To analyse the case $\ell_0 = 0$, we start looking at the explicit transformation property of elements in $X(0, \ell_2, \ell_3)$ in terms of the functions $t$ of (5.1) (see also (3.2), (3.4)) when $\ell_0 = 0$. This gives us

$$(\Psi \tilde{g}t)(a^1,\alpha^5,\alpha^6) = t(a^1 - \tilde{a}^1,\alpha^5 - \tilde{a}^5,\alpha^6 - \tilde{a}^6) \cdot e^{-i\ell_0 \tilde{a}^2} \cdot e^{-i\ell_3 \tilde{a}^3}.$$ 

It is obvious that this can be decomposed fully into 1-dimensional invariant subspaces, but not all of them will be compatible with the odd-family decomposition. However, with the results of §3 in mind, we introduce the subspaces $Y(\ell_2, \ell_3) \subset X(0, \ell_2, \ell_3)$ of functions independent of $\alpha^6$. It turns out that $(Y(\ell_2, \ell_3), I_0, I_4, I_6)$ is an invariant 3-dimensional odd family decomposition of $W(0, \ell_2, \ell_3)$ for which the corresponding subspaces indexed by odd parameters are exactly the spaces $V(0, \ell_2, \ell_3, \lambda_0, \lambda_4)$:

$$V(0, \ell_2, \ell_3, \lambda_0, \lambda_4) = \{e^{i\lambda_0 I_0 + \lambda_4 I_4 + \lambda_6 I_6} f \mid f \in Y(\ell_2, \ell_3)\}.$$

Proposition. In the direct sum decomposition associated to the 3-dimensional invariant odd family decomposition $(Y(\ell_2, \ell_3), I_0, I_4, I_6)$,

$$W(0, \ell_2, \ell_3) = Y(\ell_2, \ell_3) \oplus I_4(Y(\ell_2, \ell_3)) \oplus I_6(Y(\ell_2, \ell_3)) \oplus I_0(Y(\ell_2, \ell_3))$$

$$\oplus I_0(Y(\ell_2, \ell_3)) \oplus I_4(I_0(Y(\ell_2, \ell_3))) \oplus I_6(I_0(Y(\ell_2, \ell_3))) \oplus I_6(I_4(Y(\ell_2, \ell_3)))$$

\[\text{QED}\]
the direct summand $I_0(Y_{(t_2, t_3)})$ is irreducible, and thus so are the direct summands $I_4(Y_{(t_2, t_3)})$, $I_6(Y_{(t_2, t_3)})$ and $I_6(Y_{(t_2, t_3)})$.

Proof. According to [4.19], $I_0(Y_{(t_2, t_3)})$ is irreducible if there does not exist a graded subspace of $Y_{(t_2, t_3)}$ which is invariant under the action of $\Phi_g$ and $[\Phi_g, I_0]$. An element $f$ of $Y_{(t_2, t_3)}$ is of the form (compare with (5.1))

$$f(a^i, \alpha^j, b, \beta) = s(a^1, \alpha^5) \cdot e^{i\ell_2 a^2} \cdot e^{i\ell_3 a^3}$$

for some function $s$ of one even and one odd variable. The action of $\Phi_g$ and $[\Phi_g, I_0]$ on such a function is given (in terms of the function $s$) by

$$(\Phi_g s)(a^1, \alpha^5) = s(a^1 - \hat{a}^1, \alpha^5) \cdot e^{-i\ell_2 \hat{a}^2} \cdot e^{-i\ell_3 \hat{a}^3}$$

and

$$([\Phi_g, I_0] s)(a^1, \alpha^5) = -i(\bar{s} + \hat{a}^4 a^1 - \hat{a}^5 \hat{a}^5 - \frac{1}{2} \hat{a}^2 a^4 + \frac{1}{2} \hat{a}^3 \hat{a}^5) s(a^1 - \hat{a}^1, \alpha^5) \cdot e^{-i\ell_2 \hat{a}^2} \cdot e^{-i\ell_3 \hat{a}^3}.$$ 

If we compute the generators of the $\hat{a}^1$ and $\hat{a}^4$ action (taking the derivative with respect to these variables in the group action), we find the operators

$$s \mapsto \frac{\partial s}{\partial a^1}, \quad s \mapsto -ia^1 \cdot s,$$

which we recognize as the action of the Heisenberg algebra. Computing the generators of the $\hat{a}^3$ and $\hat{a}^5$ action, we find the maps

$$(5.6) \quad s \mapsto -i\ell_3 s, \quad s \mapsto -\frac{\partial s}{\partial a^5}, \quad s \mapsto i\alpha^4 s - i\ell_3 s.$$ 

Now recall that a function $s(a^1, \alpha^5)$ can be written as $s(a^1, \alpha^5) = s_0(a^1) + \alpha^5 \cdot s_1(a^1)$ with $s_i$ functions in $L^2(\mathbb{R})$. After taking some linear combinations in (5.6) we find the maps

$$(s_0, s_1) \mapsto (0, is_0), \quad (s_0, s_1) \mapsto (-s_1, 0).$$

We thus are in exactly the same situation as in the proof of [5.2] (except that we have an action of the Heisenberg algebra instead of the Heisenberg group). As in the proof of [5.2] we thus may conclude that there does not exist a non-trivial invariant subspace.

To finish our search for the decomposition into irreducible components, it thus remains to analyse the direct sum

$$R_{(t_2, t_3)} = Y_{(t_2, t_3)} \oplus I_4(Y_{(t_2, t_3)}) \oplus I_6(Y_{(t_2, t_3)}) \oplus I_6(Y_{(t_2, t_3)}).$$

There are two ways to see that these spaces decompose into 1-dimensional invariant spaces. The fast way is to introduce the subspace $S_{(t_2, t_3)} \subset Y_{(t_2, t_3)}$ of functions that are independent of $\alpha^5$, meaning that the elements of $S_{(t_2, t_3)}$ are just ($L^2$) functions of a single even variable $a^1$. It then is easy to show that $S_{(t_2, t_3)}$, $I_4$, $I_5$, $I_6$ is a 3-dimensional invariant odd family decomposition of $R_{(t_2, t_3)}$. If we finally introduce the (1-dimensional) spaces $C_{(t_1, t_2, t_3)}$ consisting of functions (on $G$) of the form

$$f(a^i, \alpha^j, b, \beta) = c \cdot e^{i(\ell_1 a^1 + \ell_2 a^2 + \ell_3 a^3)},$$

then we see that $S_{(t_2, t_3)}$ consists of functions of the form $c \cdot e^{i(\ell_1 a^1 + \ell_2 a^2 + \ell_3 a^3)}$, and $I_4$, $I_5$, $I_6$ are corresponding subspaces.
then $S_{(t_2, t_3)}$ is a direct integral of these spaces:

$$S_{(t_2, t_3)} = \int C_{(t_1, t_2, t_3)} \, d\ell_1 .$$

Moreover, this is an invariant decomposition which is compatible with the odd family decomposition $(S_{(t_2, t_3)}, I_4, I_5, I_6)$, meaning that each of the summands is a direct integral of 1-dimensional invariant subspaces. Being 1-dimensional guarantees that they are irreducible.

If one feels uneasy about distributing direct integrals over the odd-family decomposition, one can start with the subspace $R_{(t_2, t_3)}$, which is the space of functions on $G$ of the form

$$f(a^1, \alpha^4, b, \beta) = s(a^1, \alpha^5, \alpha^6) \cdot e^{it_2 a^2} \cdot e^{it_3 a^3} .$$

The $G$-action on functions of this form, in terms of the function $s$, is given by

$$(\Phi_g s)(a^1, \alpha^4, \alpha^5, \alpha^6) = s(a^1 - \alpha^1, \alpha^4 - \alpha^4, \alpha^5 - \alpha^5, \alpha^6 - \alpha^6) \cdot e^{-it_2 a^2} \cdot e^{-it_3 a^3} .$$

We thus can perform a Fourier transform by introducing the spaces $R_{(t_1, t_2, t_3)}$ of functions of $G$ of the form

$$f(a^1, \alpha^2, b, \beta) = t(\alpha^4, \alpha^5, \alpha^6) \cdot e^{i(t_1 a^1 + t_2 a^2 + t_3 a^3)} ,$$

which gives us the invariant direct integral decomposition

$$R_{(t_2, t_3)} = \int R_{(t_1, t_2, t_3)} \, d\ell_1 .$$

The space $C_{(t_1, t_2, t_3)}$ introduced above is (also) the subspace of $R_{(t_1, t_2, t_3)}$ of functions independent of the odd variables $\alpha^4, \alpha^5, \alpha^6$. And it is not hard to show that $(C_{(t_1, t_2, t_3)}, I_4, I_5, I_6)$ is an invariant 3-dimensional odd-family decomposition of $R_{(t_1, t_2, t_3)}$. The difference between this second approach and the first approach is that here we start with the direct integral decomposition and then introduce the odd-family decomposition, whereas in the first approach we start with the odd-family decomposition and then perform the direct integral decomposition, which we then have to “distribute” over the odd-family decomposition.

**Corollary.** For $\ell_0 = 0$, the space $W_{(0, t_2, t_3)}$ decomposes into the direct sum

$$W_{(0, t_2, t_3)} = R_{(t_2, t_3)} \oplus I_6(Y_{(t_2, t_3)}) \oplus I_6(I_0(Y_{(t_2, t_3)}))$$

of which the last for summands are irreducible. The first can be decomposed as a direct integral

$$R_{(t_2, t_3)} = \int R_{(t_1, t_2, t_3)} \, d\ell_1$$

and inside the spaces $R_{(t_1, t_2, t_3)}$ we have an invariant 3-dimensional odd family decomposition

$$R_{(t_1, t_2, t_3)} = C_{(t_1, t_2, t_3)} \oplus I_4(C_{(t_1, t_2, t_3)}) \oplus I_5(C_{(t_1, t_2, t_3)})$$

$$\oplus I_6(C_{(t_1, t_2, t_3)}),$$

in which all summands are 1-dimensional and irreducible.
Summary. In order to find the irreducible components of $V$ in terms of invariant odd family decompositions, we started by applying a 3-dimensional Fourier transforms $V = \int W(\ell_0, \ell_2, \ell_3) \, d\ell_0 \, d\ell_2 \, d\ell_3$. Elements of $W(\ell_0, \ell_2, \ell_3)$ are functions which depend upon 4 odd variables. Inside the various $W(\ell_0, \ell_2, \ell_3)$ we introduced subspaces of functions depending on less and less odd variables as

$$S(\ell_2, \ell_3) \subset X(\ell_0, \ell_2, \ell_3) \subset W(\ell_0, \ell_2, \ell_3)$$

As a last decomposition we applied a Fourier transform $S(\ell_2, \ell_3) = \int C(\ell_1, \ell_2, \ell_3) \, d\ell_1$. With these spaces, the link with the decomposition given in \S 3 is given by the following table.

- $V(\ell_0, \ell_2, \ell_3, \lambda_0, \lambda_4) \leftrightarrow e^{\lambda_0 \ell_0 + \lambda_4 \ell_2} X(\ell_0, \ell_2, \ell_3)$ irreducible for $\ell_0 \neq 0 \neq \lambda_0$.
- $V(\ell_0, \ell_2, \ell_3, 0, \lambda_4) \leftrightarrow e^{\lambda_4 \ell_2} X(\ell_0, \ell_2, \ell_3)$ irreducible for $\ell_0 \neq 0$.
- $V(\ell_0, \ell_2, \ell_3, \lambda_0, 0) \leftrightarrow e^{\lambda_0 \ell_0 + \lambda_4 \ell_2 + \lambda_0 \ell_3} Y(\ell_2, \ell_3)$ irreducible for $\lambda_0 \neq 0$.
- $V(\ell_0, \ell_2, \ell_3, 0, 0) \leftrightarrow e^{\lambda_4 \ell_2 + \lambda_5 \ell_3 + \lambda_0 \ell_3} S(\ell_1, \ell_2, \ell_3)$ irreducible.

6. Determination of the coadjoint orbits

In this section we determine the coadjoint orbits of our super group. It turns out that there are four families of orbits: a family of $0|0$ dimensional orbits with the trivial symplectic form, a family of $2|2$ dimensional orbits with an even symplectic form, a family of $2|2$ dimensional orbits with an odd symplectic form and a fourth family of $3|3$ dimensional orbits with a mixed symplectic form. The existence of orbits with a mixed symplectic form was already established in [Tu1].

A basis for the left-invariant 1-forms on $G$ is given by $da^1, da^2$ and

$$db - \frac{1}{2} a^2 da^1 + \frac{1}{2} a^1 da^2 - \frac{1}{2} da^5 \cdot \alpha^5 + \frac{1}{2} da^6 \cdot \alpha^6$$

$$d\beta^4 - \frac{1}{2} \alpha^4 da^1 - \frac{1}{2} \alpha^5 da^3 + \frac{1}{2} a^1 da^4 + \frac{1}{2} a^3 da^5.$$}

The corresponding basis of left-invariant vector fields is given by $\partial_\alpha, \partial_\beta$ and

$$\partial_{a^1} + \frac{1}{2} a^2 \partial_\beta + \frac{1}{2} \alpha^4 \partial_\beta$$

$$\partial_{a^2} - \frac{1}{2} a^1 \partial_\beta$$

$$\partial_{a^4} - \frac{1}{2} a^5 \partial_\beta$$

$$\partial_{a^5} + \frac{1}{2} \alpha^5 \partial_\beta$$

$$\partial_{a^6} - \frac{1}{2} \alpha^6 \partial_\beta.$$}

These vector fields can be identified with the given basis of $E$ in the order $k_0, k_1, e_1, \ldots, e_6$. For the Lie algebra this means that the only non-zero commutators (among these basis vectors) are given by

$$[e_2, e_1] = -[e_1, e_2] = [e_5, e_5] = -[e_6, e_6] = k_0$$

$$[e_4, e_1] = -[e_1, e_4] = [e_5, e_3] = -[e_3, e_5] = k_1.$$}

A direct computation shows that the triple product $g \hat{g} g^{-1}$ is given by

$$g \hat{g} g^{-1} = (\hat{a}^1, \hat{a}^2, \hat{a}^3, \hat{b} + a^2 \hat{a}^1 - a^1 \hat{a}^2 + \hat{a}^5 \alpha^5 - \hat{a}^6 \alpha^6, \hat{a}^4, \hat{a}^5, \hat{a}^6, \hat{b} + a^1 \alpha^4 - a^1 \hat{a}^4 + \hat{a}^3 \alpha^5 - \hat{a}^3 \alpha^5).$$
For the adjoint action we get the formulae \( \text{Ad}(g)k_0 = k_0, \text{Ad}(g)k_1 = k_1 \) and
\[
\text{Ad}(g)e_1 = e_1 + a^2 k_0 + \alpha^4 k_1, \quad \text{Ad}(g)e_2 = e_2 - a^1 k_0, \quad \text{Ad}(g)e_3 = e_3 + \alpha^5 k_1 \\
\text{Ad}(g)e_4 = e_4 - a^1 k_1, \quad \text{Ad}(g)e_5 = e_5 + \alpha^5 k_0 - a^3 k_1, \quad \text{Ad}(g)e_6 = e_6 - \alpha^6 k_0.
\]

The (left-)dual Lie algebra \( g^* \) is (isomorphic to) the graded vector space \( E \). Seen as a graded manifold it has dimension 8|8 with even coordinates \( x_1, x_2, x_3, \bar{x}_5, \bar{x}_6, y_0, \bar{y}_1, y_1 \) and odd coordinates \( \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \bar{y}_0, \eta_1 \). These are the even and odd parts of the coordinates of a point \( \mu \in g^* \) with respect to the (left-dual) basis \( e_1^*, e_2^*, e_3^*, e_4^*, e_5^*, e_6^*, k_0^*, k_1^* \) of \( g^* \) determined by the formula
\[
\mu = \sum_{i=1}^{3}(x_i + \bar{\xi}_i) \cdot e_i^* + \sum_{i=4}^{6}(\bar{x}_i + \xi_i) \cdot e_i^* + (y_0 + \bar{\eta}_0) \cdot k_0^* + (\bar{y}_1 + \eta_1) \cdot k_1^*.
\]

The names are chosen such that the unbarred coordinates represent the even part \( \mu_0 \) of \( \mu \in g^* \) and the barred coordinates represent the odd part \( \mu_1 \), while latin characters represent even coordinates and greek characters represent odd coordinates.

For the coadjoint action of \( g = (a, b) \) we obtain (from the adjoint action) that the basis vectors \( e_i^* \) are left unchanged and that we have
\[
\text{Coad}(g)k_0^* = k_0^* - e_1^* a^2 - e_2^* \alpha a^3 - e_3^* \alpha^5 + e_6^* \alpha^6 \\
\text{Coad}(g)k_1^* = k_1^* - e_1^* \alpha a^2 - e_2^* \alpha^5 - e_3^* a^3 + e_6^* \alpha^3,
\]

which gives us in terms of coordinates
\[
\begin{align*}
x_1 &\mapsto x_1 - y_0 \cdot a^2 - \eta_1 \cdot \alpha^4, & x_2 &\mapsto x_2 + y_0 \cdot a^1, & x_3 &\mapsto x_3 - \eta_1 \cdot \alpha^5 \\
\xi_4 &\mapsto \xi_4 + \eta_1 \cdot a^1, & \xi_5 &\mapsto \xi_5 + y_0 \cdot \alpha^5 + \eta_1 \cdot a^3, & \xi_6 &\mapsto \xi_6 - y_0 \cdot \alpha^6
\end{align*}
\]

and
\[
\begin{align*}
\bar{\xi}_1 &\mapsto \bar{\xi}_1 - \bar{y}_0 \cdot \alpha^2 - \bar{\eta}_1 \cdot \alpha^4, & \bar{\xi}_2 &\mapsto \bar{\xi}_2 + \bar{y}_0 \cdot a^1, & \bar{\xi}_3 &\mapsto \bar{\xi}_3 - \bar{y}_1 \cdot \alpha^5 \\
\bar{x}_4 &\mapsto \bar{x}_4 + \bar{y}_1 \cdot a^1, & \bar{x}_5 &\mapsto \bar{x}_5 + \bar{y}_0 \cdot \alpha^5 + \bar{y}_1 \cdot a^3, & \bar{x}_6 &\mapsto \bar{x}_6 - \bar{y}_0 \cdot \alpha^6,
\end{align*}
\]

while the coordinates \( y_0, \bar{y}_1, \bar{\eta}_0, \eta_1 \) remain unchanged.

For the fundamental vector fields \((v, z)^\mu\) associated to the element \((v, z) = \sum_{i=1}^{6} v^i \cdot e_i + \sum_{i=0}^{1} z^i k_i \in g\) we obtain
\[
(v, z)^\mu = (v^2 y_0 - v^4 \eta_1) \frac{\partial}{\partial x_1} - v^1 y_0 \frac{\partial}{\partial x_2} - v^5 \eta_1 \frac{\partial}{\partial x_3} - v^1 \eta_1 \frac{\partial}{\partial \xi_4} - (v^2 y_0 + v^4 \eta_1) \frac{\partial}{\partial \xi_5} + v^5 y_0 \frac{\partial}{\partial \xi_6} + (v^2 \bar{y}_0 + v^4 \bar{y}_1) \frac{\partial}{\partial \bar{\xi}_1} - v^5 \bar{y}_0 \frac{\partial}{\partial \bar{\xi}_2} + v^5 \bar{y}_1 \frac{\partial}{\partial \bar{\xi}_3} - v^1 \bar{y}_1 \frac{\partial}{\partial \bar{\xi}_4} + (v^5 \bar{y}_0 - v^3 \bar{y}_1) \frac{\partial}{\partial \bar{\xi}_5} - v^6 \bar{y}_0 \frac{\partial}{\partial \bar{\xi}_6}.
\]

We now look at the orbit \( O_\mu \), through a fixed point \( \mu_0 \in g^* \) with real coordinates (which we need if we want to be sure that the orbit is a bona fide graded manifold embedded in \( g^* \)). Since the coordinates of \( \mu_0 \) are real, we have in particular that
the odd coordinates of \( \mu_o \) are zero: \( \xi_i(\mu_o) = \bar{\xi}_i(\mu_o) = 0 \). Since the coordinates \( y_0 \) and \( \bar{y}_1 \) are constant on the orbit, they remain the same real constant. However, the other even coordinates are in general not constant on \( O_{\mu_o} \), so we will denote the remaining even coordinates of \( \mu_o \) by \( x_1^2, x_2^3, x_3^4, x_5^6, \bar{x}_4^5, \bar{x}_5^6 \in \mathbb{R} \). The odd coordinates \( \bar{y}_0 = \eta_1 = 0 \) are also invariant under the coadjoint action, so they remain constant equal zero on the orbit. Hence the formula for the coadjoint action reduce to:

\[
x_1 \mapsto x_1 - y_0 \cdot \alpha^2, \quad x_2 \mapsto x_2 + y_0 \cdot 1, \quad \xi_5 \mapsto \xi_5 + y_0 \cdot \bar{\alpha}^5, \quad \xi_6 \mapsto \xi_6 - y_0 \cdot \alpha^6
\]

while all other coordinates remain unchanged. For the fundamental vector fields the formula reduces to

\[
(v, z)^\ast = y_0 \cdot \left( v^2 \cdot \frac{\partial}{\partial x_1} - v^1 \cdot \frac{\partial}{\partial x_2} - v^5 \cdot \frac{\partial}{\partial \xi_5} + v^6 \cdot \frac{\partial}{\partial \xi_6} \right) + \bar{y}_1 \cdot \left( v^4 \cdot \frac{\partial}{\partial \xi_1} + v^5 \cdot \frac{\partial}{\partial \xi_3} - v^1 \cdot \frac{\partial}{\partial x_4} - v^3 \cdot \frac{\partial}{\partial x_5} \right).
\]

We now distinguish four cases: (i) \( y_0 = \bar{y}_1 = 0 \), (ii) \( y_0 \neq 0 \) (but real!) and \( \bar{y}_1 = 0 \), (iii) \( \bar{y}_1 \neq 0 \) and \( y_0 = 0 \), and (iv) \( y_0 \cdot \bar{y}_1 \neq 0 \).

**Case (i).** For \( y_0 = \bar{y}_1 = 0 \), the base point \( \mu_o \) of the orbit \( O_{\mu_o} \) is determined by the 6 real values \( x_1^2, x_2^3, x_3^4, x_5^6, \bar{x}_4^5, \bar{x}_5^6 \), while all other coordinates are zero. Since the coadjoint action reduces to the trivial action on such a point, \( O_{\mu_o} = \{ \mu_o \} \) is a point and its symplectic form is the 2-form which is identically zero (which is indeed symplectic as there are no non-zero tangent vectors). This gives us a 6-dimensional family of 0-dimensional orbits labeled by \( x_1^2, x_2^3, x_3^4, x_5^6, \bar{x}_4^5, \bar{x}_5^6 \).

**Case (ii).** For \( \bar{y}_1 = 0 \) and \( y_0 \neq 0 \), the base point \( \mu_o \) is determined by the 7 real values \( x_1^2, x_2^3, x_3^4, x_5^6, \bar{x}_4^5, \bar{x}_5^6, y_0 \), while all other coordinates are zero. The orbit has dimension \( 2|2 \) with even coordinates \( x_1, x_2 \) and odd coordinates \( \xi_5, \xi_6 \) (all other coordinates on \( g^\ast \) remain unchanged for such an orbit). Substituting the fundamental vector fields associated to basis elements \( e_i \) in the formula for the symplectic form gives us the following identities

\[
\iota(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) \omega = \frac{1}{y_0}, \quad \iota(\frac{\partial}{\partial \xi_5}, \frac{\partial}{\partial \xi_6}) \omega = -\frac{1}{y_0}, \quad \iota(\frac{\partial}{\partial \xi_5}, \frac{\partial}{\partial \xi_6}) \omega = \frac{1}{y_0},
\]

all others being either zero or determined by graded skew-symmetry. From these identities we deduce that the (even) symplectic form is given as

\[
\omega = (y_0)^{-1} \cdot \left( dx_2 \wedge dx_1 - \frac{1}{2} d\xi_5 \wedge d\xi_6 + \frac{1}{2} d\xi_5 \wedge d\xi_6 \right).
\]

Since \( x_1 \) and \( x_2 \) are coordinates on such an orbit, the different orbits are labeled by \( x_3^4, \bar{x}_4^5, \bar{x}_5^6, y_0 \), thus giving a 5-dimensional family of \( 2|2 \)-dimensional orbits with an even symplectic form.

**Case (iii).** For \( y_0 = 0 \) and \( \bar{y}_1 \neq 0 \), the base point \( \mu_o \) is again determined by 7 real values, but now \( x_1^2, x_2^3, x_3^4, x_4^5, x_5^6, \bar{x}_1, y_1 \), while all other coordinates are zero. The orbit dimension is still \( 2|2 \), but now with even coordinates \( \bar{x}_4, \bar{x}_5 \) and
odd coordinates $\xi_1, \xi_3$ (as before, the other coordinates on $g^*$ remain unchanged on this orbit). Here we obtain for the symplectic form the identities

$$\iota(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial x_4})\omega = \frac{1}{y_1}, \quad \iota(\frac{\partial}{\partial \xi_3}, \frac{\partial}{\partial x_5})\omega = \frac{1}{y_1},$$

from which we deduce that the (odd) symplectic form is given as

$$\omega = (y_1)^{-1} \cdot (d\bar{x}_4 \wedge d\xi_1 + d\bar{x}_5 \wedge d\xi_3).$$

Here the $\bar{x}_4$ and $\bar{x}_5$ are coordinates on the orbit, so the different orbits are labeled by $x_1^0, x_2^0, x_3^0, \bar{x}_4^0, \bar{x}_5^0, y_0, \bar{y}_1$, giving a 5-dimensional family of $2|2$-dimensional orbits with an odd symplectic form.

**Case (iv).** In the last case $y_0\bar{y}_1 \neq 0$ we have to be slightly more careful. The base point is determined by the 8 real values $x_1^0, x_2^0, x_3^0, \bar{x}_4^0, \bar{x}_5^0, y_0, \bar{y}_1$, while all other coordinates are zero. But the combinations $s = y_0\bar{x}_4 - \bar{y}_1 x_2$ and $\sigma = y_0\bar{\xi}_3 + \bar{y}_1 \xi_5$ are invariant under the group action. Since the base point of the orbit has real coordinates, $\sigma$ is zero at this point, and thus zero on the whole orbit. We introduce the change of coordinates on $g^*$ involving only $x_2, \bar{x}_4, \xi_3, \xi_5$ which change into $\bar{x}_2, s, \sigma, \xi_5$ with $\bar{x}_2 = x_2$ and $\xi_5 = \xi_5$. We then can use the coordinates $x_1, \bar{x}_2, \bar{x}_5, \xi_1, \xi_3, \xi_5$ as independent coordinates on the orbit, which thus has dimension $3|3$. In terms of these new coordinates (on $g^*$) the fundamental vector field is given as

$$(v, z)^9 = y_0 \cdot \left( v^2 \cdot \frac{\partial}{\partial x_1} - v^3 \cdot \frac{\partial}{\partial x_2} - v^5 \cdot \frac{\partial}{\partial \xi_5} + v^6 \cdot \frac{\partial}{\partial \xi_6} \right)
+ \bar{y}_1 \cdot \left( v^4 \cdot \frac{\partial}{\partial \xi_1} - v^3 \cdot \frac{\partial}{\partial \bar{x}_5} \right).$$

As before, substituting suitable basis vectors for $v$ in the formula for the symplectic form gives us the following identities

$$\iota(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})\omega = \frac{1}{y_0}, \quad \iota(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \bar{x}_2})\omega = \frac{1}{y_0}, \quad \iota(\frac{\partial}{\partial \bar{x}_5}, \frac{\partial}{\partial \xi_5})\omega = \frac{1}{y_0},$$

$$\iota(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_5})\omega = \frac{-1}{y_0}, \quad \iota(\frac{\partial}{\partial \xi_6}, \frac{\partial}{\partial \xi_6})\omega = \frac{1}{y_0}.$$

This results in the (non-homogeneous) symplectic form

$$\omega = (y_0)^{-1} \cdot \left( d\bar{x}_2 \wedge dx_1 - \frac{1}{2} d\xi_5 \wedge d\xi_5 + \frac{1}{2} d\xi_6 \wedge d\xi_6 + d\bar{x}_2 \wedge d\xi_1 + d\bar{x}_5 \wedge d\bar{x}_5 \right).$$

In this case the invariants of the orbit are the components $x_3^0, \bar{x}_6^0, y_0, \bar{y}_1$ and the combination $s^0 = y_0 x_2^0 - \bar{y}_1 \bar{x}_4^0$, giving a 5-dimensional family of $3|3$-dimensional orbits with a mixed symplectic form.
7. Geometric quantization of the coadjoint orbits

In this section we discuss the general theory of geometric quantization of a super symplectic manifold in the context of the coadjoint orbits of our supergroup $G$. As explained in [Tu2] a prequantum bundle is a principal fiber bundle $Y \to O_{\mu_o}$ with a connection 1-form $\Gamma$ whose curvature is the (Kostant-Kirillov-Souriau) symplectic form $\omega$. Since the symplectic form can be non-homogeneous, the structure group of $Y$ will be $(A_0/dZ) \times A_1$ of dimension $1$. On $(A_0/dZ) \times A_1$ we will use coordinates $(t, \tau)$ with $t$ the canonical even coordinate on $A_0$ modulo $d$ and $\tau$ the canonical coordinate on $A_1$. The connection 1-form $\Gamma$ on $Y$, an even Lie algebra valued 1-form on $Y$, then takes the form

$$\Gamma = \Gamma_0 \otimes \partial_t + \Gamma_1 \otimes \partial_\tau$$

with $\Gamma_i$ a 1-form on $Y$ of parity $i$. It is useful to identify $\Gamma$ with the mixed 1-form $\Gamma = \Gamma_0 + \Gamma_1$.

In our case the orbits all have a single global chart of the form $(A_0)^p \times (A_1)^q$, hence are simply connected and thus the prequantum bundle is unique and of the form $Y = O_{\mu_o} \times (A_0/dZ) \times A_1$. Moreover, there exists a global potential $\Theta$ for the symplectic form $\omega$, i.e., $d\Theta = \omega$. Splitting $\Theta$ in its homogeneous parts, the connection 1-form $\Gamma$ then is given as

$$\Gamma_0 = dt + \Theta_0, \quad \Gamma_1 = d\tau + \Theta_1.$$

The presence of the (canonical) momentum map on a coadjoint orbit implies that the infinitesimal action of $G$ on the orbit can be lifted to the prequantum bundle (in a way which is compatible with he momentum map). More precisely, if $(v, z) \in g$ is an element of the Lie algebra, the infinitesimal generator $(v, z)^Y$ of the lifted action on $Y$ is determined by the equations

$$\iota((v, z)^Y|_{(\mu, t, \tau)})\Gamma = \langle (v, z)|\mu \rangle \quad \text{and} \quad \mathcal{L}((v, z)^Y)\Gamma = 0.$$ 

In our examples, these infinitesimal actions on the prequantum bundles all integrate to an action of $G$, which we will provide.

Remark. In the non-super orbit method, the lifting of the coadjoint action on an orbit to a prequantum bundle is equivalent to the existence of a character on the isotropy subgroup of the base point whose infinitesimal form is the given base point.

As in the non-super case, an invariant polarization on a coadjoint orbit $O_{\mu_o}$ through $\mu_o \in g^*$ (where $\mu_o$ has real coordinates) is a foliation which is maximal isotropic with respect to the symplectic form and which is invariant under the group action. And just as in the non-super case, such objects are in bijection with (graded) subalgebras $h \subset g$ containing the stabilizer subalgebra $g_{\mu_o}$ and which are maximal with respect to the condition $\langle [h, h]|\mu_o \rangle = 0$.

Since the vectors $k_0, k_1$ generate the center of $g$, they always belong to a stabilizer subalgebra. On the other hand, since the commutator of two elements in $g$ always lies in the graded subspace generated by $k_0, k_1$, it follows easily that any graded subspace containing the vectors $k_0, k_1$ is a graded subalgebra. If we have a
homogeneous basis for \( \mathfrak{h} \), then any commutator between two elements of \( \mathfrak{h} \) is either a multiple of \( k_0 \) or \( k_1 \), but never both at the same time, just because of a parity argument.

There are several equivalent ways to describe geometric quantization (without half-densities or half-forms). In the non-super case one can either look at the associated complex line bundle \( L \to \mathcal{O}_\mu \) and determine the sections that are co-variantly constant in the direction of the (invariant) polarization, but one can also look directly at functions on the principal circle bundle \( Y \to \mathcal{O}_\mu \) that satisfy two conditions: (i) equivariance with respect to the action of the structure group \( S^1 \) and (ii) constant in the direction of the horizontally lifted (invariant) polarization. The lifted action of \( G \) to the principal bundle \( Y \) preserves the space of functions determined by the second method (using the induced action on functions \( (g \cdot f)(y) = f(g^{-1}y) \)). This is the representation of \( G \) obtained by geometric quantization of the orbit.

In the super case we have to be a bit more devious because of the odd dimension of the structure group. We thus introduce two parameters, one even (real) \( \hbar \) and one odd \( \kappa \). The first is usually introduced in physics and can (without loss of generality) be put equal to 1. We will not do so here in order to maintain a minimum of equality between the even and odd coordinates on the structure group. On the other hand, we cannot fix a specific value of \( \kappa \) except \( \kappa = 0 \) because else we obtain results that are no longer within the category of graded manifolds (see [Tu2] for details). We thus are obliged to keep it as a free parameter. With this in mind, the two conditions imposed on functions \( f : Y \to \mathcal{A}^C \) on \( Y \) by geometric quantization become (7.1) and (7.2):

\[
(7.1) \quad f(y \cdot (t, \tau)) = e^{i\tau \kappa + it/\hbar} \cdot f(y),
\]

where \( y \cdot (t, \tau) \) denotes the right-action of the structure group on the principal bundle and where \( e^{i\tau \kappa + it/\hbar} \cdot f(y) \) denotes the usual multiplication in \( \mathcal{A}^C \).

\[
(7.2) \quad (v, z)^\hbar f = 0
\]

for all \( (v, z) \in \mathfrak{g} \), where \( (v, z)^\hbar \) is the horizontal lift to \( Y \) of the fundamental vector field \( (v, z)^\Gamma \). It (thus) satisfies the condition \( \iota((v, z)^\hbar)\Gamma = 0 \). In our situation is is necessarily of the global form

\[
(v, z)^\hbar = (v, z)^\Gamma + p_0 \cdot \partial_t + p_1 \cdot \partial_\tau.
\]

The coefficients \( p_i \) are determined by the equations \( \iota((v, z)^\hbar)\Gamma_i = 0 \).

**Remark.** For the even part \( \mathcal{A}_0/d\mathbb{Z} \) of the structure group we already have a parameter \( d \in \mathbb{R}^\times \) and the choice of \( \hbar \) must be compatible with \( d \) in the sense that \( d/\hbar \) must be a multiple of \( 2\pi \) in order that condition (i) doesn’t reduce to the condition that \( f \) must be identically zero. The parameter \( d \) was introduced for the general case, in which the symplectic form is not exact, but has a non-trivial group of periods. It should satisfy the condition that the group of periods is included in \( d\mathbb{Z} \). In our situation here the group of periods reduces to \( \{0\} \), so no restriction is imposed a priori on \( d \).

**Remark.** The parameter \( \kappa \) is not discussed in [Tu1]. Its presence here will turn out to be crucial if we want to obtain a match between the irreducible representations found in the regular representation and the representations obtained by geometric quantization of coadjoint orbits.
8. Orbits of dimension 0|0

Since the orbit is a point, the prequantum bundle is \( \{ \mu_o \} \times (A_0/dZ) \times A_1 \) together with the connection 1-form \( \Gamma = dt + dr \). Since the canonical momentum map takes the (constant) value \( \mu_o \), the lifted vector fields \( (v, z)^\Gamma \) are given by

\[
(v, z)^\Gamma = \left( v^1 x_1^0 + v^2 x_2^0 + v^3 x_3^0 \right) \cdot \frac{\partial}{\partial t} + \left( v^4 x_4^0 + v^5 x_5^0 + v^6 x_6^0 \right) \cdot \frac{\partial}{\partial \tau},
\]

It follows immediately that the lifted action is given by

\[
g \in G : \begin{pmatrix} t \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} t - \alpha^1 x_1^0 - \alpha^2 x_2^0 - \alpha^3 x_3^0 \\ \tau - \alpha^4 x_4^0 - \alpha^5 x_5^0 - \alpha^6 x_6^0 \end{pmatrix}.
\]

For a point orbit \( \mu_o \), the stabilizer subgroup \( G_{\mu_o} \) is the whole group. An invariant polarization thus corresponds necessarily to \( \mathfrak{h} = \mathfrak{g} \). Since the orbit dimension is 0|0, this corresponds to the trivially zero foliation on the 0-dimensional manifold.

Smooth functions on \( Y = (A_0/dZ) \times A_1 \) satisfying condition (7.1) are determined by a complex number \( c \in \mathbb{C} \) as

\[
f(t, \tau) = c \cdot e^{t \xi + it/\hbar},
\]

whereas the second condition (7.2) is void. These functions constitute a 1-dimensional graded vector space of dimension 1|0. The induced action of \( G \) on such a function is given (using (8.1)) as

\[
(g \cdot f)(t, \tau) = f(g^{-1}(t, \tau)) = f(t + \alpha^1 x_1^0 + \alpha^2 x_2^0 + \alpha^3 x_3^0, \tau + \alpha^4 x_4^0 + \alpha^5 x_5^0 + \alpha^6 x_6^0) = e^{i(\alpha^1 x_1^0 + \alpha^2 x_2^0 + \alpha^3 x_3^0) \lambda + (\alpha^1 x_1^0 + \alpha^2 x_2^0 + \alpha^3 x_3^0) \lambda} f(t, \tau).
\]

9. Orbits of dimension 2|2 with an even symplectic form

As said in §7, the prequantum bundle is just the direct product \( Y = O_{\mu_o} \times (A_0/dZ) \times A_1 \). Using the explicit expression of the symplectic form, it is not hard to see that a connection form \( \Gamma \) is given by

\[
\Gamma_0 = dt + (y_0)^{-1}(-x_1 dx_2 - \frac{1}{2} \xi_5 d\xi_5 + \frac{1}{2} \xi_6 d\xi_6) \quad \text{and} \quad \Gamma_1 = d\tau.
\]

With this choice, the lifted fundamental vector fields \( (v, z)^\Gamma \) are given by

\[
(v, z)^\Gamma = y_0 \cdot \left( \frac{v^2}{\partial x_1} \cdot v^1 \cdot \frac{\partial}{\partial x_2} - v^5 \cdot \frac{\partial}{\partial \xi_5} + v^6 \cdot \frac{\partial}{\partial \xi_6} \right)
\]

\[
+ \left( v^2 x_2 + v^3 x_3 - \frac{1}{2} v^5 \xi_5 - \frac{1}{2} v^6 \xi_6 + z^0 y_0 \right) \cdot \frac{\partial}{\partial \tau},
\]

which integrate to an action of \( G \) on \( Y \) by

\[
g \in G : \begin{pmatrix} x_1 \\ x_2 \\ \xi_5 \\ \xi_6 \\ t \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} x_1 - y_0 a^2 \\ x_2 + y_0 a^1 \\ \xi_5 + y_0 a^5 \\ \xi_6 - y_0 a^6 \\ t - \alpha^2 x_2 - \alpha^3 x_3 + \frac{1}{2} a^5 \xi_5 + \frac{1}{2} a^6 \xi_6 - b y_0 - \frac{1}{2} y_0 a^1 a^2 \\ \tau - \alpha^4 x_4 - \alpha^5 x_5 - \alpha^6 x_6 \end{pmatrix}.
\]
For all these orbits the stabilizer subgroup $G_{\mu_0}$ is given by

$$G_{\mu_0} = \{ (a', \alpha', b, \beta) \mid a^3 - a^2 = a^5 = a^6 = 0 \}.$$  

The stabilizer subalgebra $\mathfrak{g}_{\mu_0}$ of dimension 2|2 thus is generated by the vectors $e_3, e_4, k_0, k_1$. If a subalgebra $\mathfrak{h} \supset \mathfrak{g}_{\mu_0}$ has even dimension 4, then necessarily it contains all 4 even vectors $e_1, e_2, e_3, k_0$. But then

$$\langle [e_1, e_2] | \mu_0 \rangle = \langle -k_0 | \mu_0 \rangle = -y_0 \neq 0.$$  

Hence for a polarization the even dimension of $\mathfrak{h}$ is at most 3. Similarly, if $\mathfrak{h}$ has odd dimension 4, then it necessarily contains all odd vectors $e_4, e_5, e_6, k_1$ and in particular $e_5$. And then we have

$$\langle [e_5, e_5] | \mu_0 \rangle = y_0 \neq 0.$$  

Since the same argument applies to $e_6$, it follows that the odd dimension of $\mathfrak{h}$ for a polarization is at most 3 and that $\mathfrak{h}$ cannot contain the (isolated) vectors $e_5$ or $e_6$. On the other hand, it is easy to check that the subalgebra $\mathfrak{h}$ of dimension 3|3 generated by the vectors $ae_1 + be_2, e_3, e_4, e_5 - ce_6, k_0, k_1$ with $|a|^2 + |b|^2 = 1$ and $\epsilon = \pm 1$ satisfies the condition $\langle [\mathfrak{h}, \mathfrak{h}] | \mu_0 \rangle = 0$. It thus represents an invariant polarization; with the given degrees of freedom in $a, b, \epsilon$ these are the only ones.

Functions $f$ on $Y$ satisfying condition (7.1) are determined by functions $g$ on the orbit according to the formula

$$f(x_2, x_3, \xi_5, \xi_6; t, \tau) = e^{i\epsilon\tau + it/\hbar} \cdot g(x_1, x_2, \xi_5, \xi_6).$$

We now choose the polarization generated by $e_2, e_3, e_4, e_5 - ce_6, k_0, k_1$ (the others will give equivalent representations), which is equivalent to the invariant polarization (foliation) on the orbit generated by the vector fields

$$\partial_{x_1} \quad \text{and} \quad \partial_{\xi_5} + \epsilon \partial_{\xi_6}.$$  

The horizontal lifts of these vector fields are

$$\frac{\partial}{\partial x_1} \quad \text{and} \quad \frac{\partial}{\partial \xi_5} + \epsilon \frac{\partial}{\partial \xi_6} = \frac{\xi_5 - \xi_6}{2y_0} \frac{\partial}{\partial t}.$$  

Condition (7.2) applied to (9.2) then gives us the equations

$$\frac{\partial g}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial g}{\partial \xi_5} + \epsilon \frac{\partial g}{\partial \xi_6} - i \left( \frac{\xi_5 - \xi_6}{2y_0} \right) \cdot g = 0.$$  

The solutions to these equations are determined by functions $h$ of one even and one odd variable according to

$$g(x_1, x_2, \xi_5, \xi_6) = h(x_2, \xi_5 - \epsilon \xi_6) \cdot e^{i\epsilon\tau - \epsilon\xi_6}.$$  

The induced action of $G$ on such a function is given (using (9.1)) as

$$(\Phi_{\mu_0}f)(x_1, x_2, \xi_5, \xi_6; t, \tau) = h(x_2 - y_0a^1, (\xi_5 - \epsilon \xi_6) - y_0(\alpha^3 + \alpha^6)) \cdot e^{i\epsilon\tau \cdot \bar{\xi}_5 \xi_6} \cdot e^{i\epsilon\tau \cdot x_2} \cdot e^{i\epsilon\tau \cdot x_3} \cdot e^{i\epsilon\tau \cdot x_4} \cdot e^{i\epsilon\tau \cdot x_5} \cdot e^{i\epsilon\tau \cdot x_6},$$

$$e^{i\epsilon\tau \cdot \bar{\xi}_5 \xi_6} \cdot e^{i\epsilon\tau \cdot (x_2^2 + a^2) / 2} \cdot e^{i\epsilon\tau \cdot (x_3^2 + a^2) / 2} \cdot e^{i\epsilon\tau \cdot (x_4^2 + a^2) / 2} \cdot e^{i\epsilon\tau \cdot (x_5^2 + a^2) / 2} \cdot e^{i\epsilon\tau \cdot (x_6^2 + a^2) / 2}.$$
If we introduce the variables \( z_1 = (x_2 - x_3^2)/y_0 \) and \( \zeta = (\xi_5 - \epsilon \xi_6)/y_0 \), then the action of \( \hat{g} \in G \) on the functions \( h \) in terms of these new variables becomes

\[
(\Phi_{\hat{g}} h)(z_1, \zeta) = h(z_1 - \hat{a}_1, \zeta - (\hat{a}_5 + c \hat{a}_6)) \cdot e^{i \frac{\hbar}{\pi} (\hat{a}_2 z_1 - \frac{1}{2} (\hat{a}_5 - c \hat{a}_6) \zeta)} \cdot e^{i \frac{\hbar}{\pi} (b - \frac{1}{2} \hat{a}_1 \hat{a}_2 + \frac{1}{2} \epsilon \hat{a}_5 \hat{a}_6)} \cdot e^{i (\hat{a}_3 \hat{x}_4^2 + \hat{a}_5 \hat{x}_6^2 + \hat{a}_6 \hat{x}_6^2) \kappa}.
\]

This action depends upon the coordinates \( x_2, x_3, \bar{x}_4, \bar{x}_5, \bar{x}_6, y_0 \) of the base point \( \mu_0 \).

The last five are the parameters of the orbit, but the first is not an invariant of the orbit. To see that \( x_2^2 \) is a spurious parameter in this action, it suffices to change the variable \( z_1 \) to \( x_2/y_0 \), in which case the \( \hat{a}_2 x_2^2 \) term in the exponential disappears.

The “introduction” of the parameter \( x_2^2 \) thus just gives an equivalent description of the same representation. We introduced it to facilitate future comparisons.

10. Orbits of dimension \( 2|2 \) with an odd symplectic form

As before, the prequantum bundle is the direct product \( Y = G_{\mu_0} \times (A_0/oZ) \times A_1 \).

Here a connection form \( \Gamma \) is given by

\[
\Gamma_0 = dt \quad \text{and} \quad \Gamma_1 = d\tau + (\bar{y}_1)^{-1}(\bar{x}_5 \, d\bar{x}_3 - \bar{\xi}_1 \, d\bar{x}_4).
\]

The lifted fundamental vector fields \( (v, z)^Y \) are given by

\[
(v, z)^Y = (\bar{y}_1) \cdot \left( v^4 \frac{\partial}{\partial \bar{\xi}_1} + v^5 \frac{\partial}{\partial \bar{\xi}_3} - v^1 \frac{\partial}{\partial \bar{x}_4} - v^3 \frac{\partial}{\partial \bar{x}_5} \right)
+ (v^1 x_2^2 + v^2 \bar{x}_4^2 + v^3 \bar{x}_6^2) \frac{\partial}{\partial t}
+ (v^3 \bar{\xi}_3 + v^4 \bar{x}_4 + v^6 \bar{x}_6^2 + \bar{z}^1 \bar{y}_1) \frac{\partial}{\partial \bar{\tau}},
\]

which integrate to an action of \( G \) on \( Y \) by

(10.1)

\[
g \in G : 
\begin{pmatrix} \bar{\xi}_1 \\ \bar{\xi}_3 \\ \bar{x}_4 \\ \bar{x}_5 \\ t \end{pmatrix} \mapsto 
\begin{pmatrix} \bar{\xi}_1 - \bar{y}_1 a^4 \\ \bar{\xi}_3 - \bar{y}_1 a^5 \\ \bar{x}_4 + \bar{y}_1 a^3 \\ \bar{x}_5 + \bar{y}_1 a^3 \\ \bar{t} - a^1 x_2^2 - a^2 \bar{x}_4^2 - a^3 \bar{x}_6^2 - \beta \bar{y}_1 - \frac{1}{2} \bar{y}_1 a^1 a^4 + \frac{1}{2} \bar{y}_1 a^3 a^5 \end{pmatrix}
\]

For all orbits of this type the stabilizer subgroup \( G_{\mu_0} \) is given by

\[
G_{\mu_0} = \{(a^i, \alpha^j, b, \beta) \mid a^1 = a^3 = \alpha^4 = a^5 = 0\}.
\]

The stabilizer subalgebra \( g_{\mu_0} \) of dimension \( 2|2 \) is generated by the vectors \( e_2, e_6, k_0, k_1 \). Since \( y_0 = 0 \), the subalgebra \( h \) representing an invariant polarization can have an even dimension \( 4 \), i.e., containing all 4 even vectors \( e_1, e_2, e_3, k_0 \). But then it cannot contain any combination of \( e_4 \) and \( e_5 \) as we have

\[
\langle [ae_4 + be_5, e_1] \mid \mu_0 \rangle = a\bar{y}_1 \quad \text{and} \quad \langle [ae_4 + be_5, e_3] \mid \mu_0 \rangle = b\bar{y}_1.
\]

Since \( \bar{y}_1 \neq 0 \), the condition that these values are zero implies that we have \( a = b = 0 \).

The subalgebra \( h \) of dimension \( 4|2 \) is generated by the vectors \( e_1, e_2, e_3, e_6, k_0, k_1 \) thus represents an invariant polarization.
Similarly, the subalgebra $\mathfrak{h}$ representing an invariant polarization can have an odd dimension 4, i.e., containing all 4 odd vectors $e_4, e_5, e_6, k_1$. But then it cannot contain any combination of $e_1$ and $e_3$ as we have
\[
\langle [ae_1 + be_3, e_4] | \mu_o \rangle = -a\bar{y}_1 \quad \text{and} \quad \langle [ae_1 + be_3, e_5] | \mu_o \rangle = -b\bar{y}_1 .
\]
The subalgebra $\mathfrak{h}$ of dimension 2|4 generated by the vectors $e_2, e_4, e_5, e_6, k_0, k_1$ thus represents an invariant polarization.

If we now look at a subalgebra $\mathfrak{g} \supset \mathfrak{g}_{\mu_0}$ of dimension 3|3, it must be generated by $ae_1 + be_3, e_2, ce_4 + de_5, e_6, k_0, k_1$ for some constants $a, b, c, d$. The condition $\langle [\mathfrak{h}, \mathfrak{g}] | \mu_o \rangle = 0$ then implies in particular that we must have
\[
\langle [ae_1 + be_3, ce_4 + de_5] | \mu_o \rangle = -ac - bd = 0 .
\]
Since the couples $(a, b)$ and $(c, d)$ can be multiplied by a constant without changing $\mathfrak{h}$, it follows easily that the “only” solution is given by $c = b$ and $d = -a$ and that we can require $|a|^2 + |b|^2 = 1$. Since $\langle [\mathfrak{h}, \mathfrak{h}] | \mu_o \rangle = 0$ does not give any other conditions, it follows that the subalgebra $\mathfrak{h}$ of dimension 3|3 generated by the vectors $ae_1 + be_3, e_2, be_4 - ac_5, e_6, k_0, k_1$ represents an invariant polarization.

As before, functions $f$ on the prequantum bundle $Y$ satisfying condition (7.1) are determined by functions $g$ on the orbit according to
\[
(10.2) \quad f(\bar{x}_4, \bar{x}_5, \bar{\xi}_1, \bar{\xi}_3, t, \tau) = e^{i\tau \kappa + i\kappa /\hbar} \cdot g(\bar{x}_4, \bar{x}_5, \bar{\xi}_1, \bar{\xi}_3) .
\]
However, in order to proceed with the quantization, we have to choose an invariant polarization.

- We start with the invariant polarization represented by the subalgebra of dimension 4|2, which corresponds to the foliation on the orbit generated by the vector fields
  \[
  \partial_{\bar{x}_4} \quad \text{and} \quad \partial_{\bar{x}_5} .
  \]
The corresponding horizontal lifts are given by
  \[
  \frac{\partial}{\partial \bar{x}_4} + \frac{i\bar{\xi}_1 \kappa}{\bar{y}_1} \frac{\partial}{\partial \tau} \quad \text{and} \quad \frac{\partial}{\partial \bar{x}_5} ,
  \]
Condition (7.2) applied to functions of the form (10.2) then gives the equations
\[
\frac{\partial g}{\partial \bar{x}_4} + \frac{i\bar{\xi}_1 \kappa}{\bar{y}_1} g = 0 \quad \text{and} \quad \frac{\partial g}{\partial \bar{x}_5} = 0 .
\]
The solutions are determined by functions $h$ of two odd variables according to
\[
g(\bar{x}_4, \bar{x}_5, \bar{\xi}_1, \bar{\xi}_3) = e^{-i\bar{x}_4 \bar{\xi}_1 \kappa /\bar{y}_1} \cdot h(\bar{\xi}_1, \bar{\xi}_3) .
\]
The induced action of $G$ on such a function is given (using (10.1)) as
\[
(\Phi_g f)(\bar{x}_4, \bar{x}_5, \bar{\xi}_1, \bar{\xi}_3, t, \tau) = h(\bar{\xi}_1 + \bar{y}_1 \alpha^4, \bar{\xi}_3 + \bar{y}_1 \alpha^5) \cdot e^{i\tau \kappa + i\kappa /\hbar} \cdot e^{-i\bar{x}_4 \bar{\xi}_1 \kappa /\bar{y}_1} \cdot \cdot e^{i(\alpha^1 \bar{x}_4 + \alpha^2 \bar{x}_5 + \alpha^5 \bar{x}_5) /\hbar} \\
\cdot e^{i(\alpha^1 \bar{\xi}_1 + \alpha^2 \bar{\xi}_3 + \alpha^5 \bar{\xi}_3) + i\bar{y}_1 \alpha^4 + i\bar{y}_1 \alpha^5} .
\]
If we introduce the variables \( \zeta_4 = -\bar{\xi}_1/\bar{y}_1 \) and \( \zeta_5 = -\bar{\xi}_3/\bar{y}_1 \), then the action of \( \hat{g} \in G \) on the functions \( h \) in terms of the new variables becomes
\[
(\Phi_{g} h)(\zeta_4, \zeta_5) = h(\zeta_4 - \hat{a}^\xi, \zeta_5 - \hat{a}^\zeta) \cdot e^{i(\hat{a}^\xi \zeta_4 + \hat{a}^\zeta \zeta_5)/\bar{y}_1} \cdot e^{i(\hat{a}^\xi x_1^o + \hat{a}^\zeta x_2^o)/h} \cdot e^{i(\hat{a}^\xi x_3^o)/\bar{y}_1} .
\]

- Next we choose the invariant polarization represented by the subalgebra of dimension 2|4, which corresponds to the foliation on the orbit generated by the vector fields
\[
\partial_{\bar{\xi}_1} \quad \text{and} \quad \partial_{\bar{\xi}_3} .
\]

The corresponding horizontal lifts are given by
\[
\frac{\partial}{\partial \bar{\xi}_1} \quad \text{and} \quad \frac{\partial}{\partial \bar{\xi}_3} - \frac{\bar{x}_5}{\bar{y}_1} \cdot \frac{\partial}{\partial \tau} ,
\]
which gives us the equations
\[
\frac{\partial g}{\partial \bar{\xi}_1} = 0 \quad \text{and} \quad \frac{\partial g}{\partial \bar{\xi}_3} - \frac{i\bar{x}_5}{\bar{y}_1} \cdot g = 0 ,
\]
whose solutions are determined by functions \( h \) of two even variables according to
\[
g(\bar{x}_4, \bar{x}_5, \bar{\xi}_1, \bar{\xi}_3) = e^{i\bar{\xi}_3 \bar{x}_5 \kappa/\bar{y}_1} \cdot h(\bar{x}_4, \bar{x}_5) .
\]

The induced action of \( G \) on such a function is given (using again (10.1)) as
\[
(\Phi_{g_f})(\bar{x}_4, \bar{x}_5, \bar{\xi}_1, \bar{\xi}_3, t, \tau) = h(\bar{x}_4 - \bar{y}_1 \bar{a}^1, \bar{x}_5 - \bar{y}_1 \bar{a}^3) \cdot e^{i\tau \kappa + it/h} \cdot e^{i\bar{\xi}_3 \bar{x}_5 \kappa/\bar{y}_1} \cdot e^{i(\bar{a}^1 \bar{x}_1^o + \bar{a}^2 \bar{x}_2^o + \bar{a}^3 \bar{x}_3^o)/h} \cdot e^{i(\bar{a}^1 \bar{x}_4 + \bar{a}^2 \bar{x}_5 + \bar{a}^3 \bar{x}_3^o + \bar{y}_1 \alpha + \varpi \bar{y}_1 \alpha^4 - \varpi \bar{y}_1 \alpha^5 \kappa)} .
\]

If we introduce the variables \( \bar{z}_1 = (\bar{x}_4 - \bar{x}_4^o)/\bar{y}_1 \) and \( \bar{z}_3 = (\bar{x}_5 - \bar{x}_5^o)/\bar{y}_1 \), then the action of \( \hat{g} \in G \) on the functions \( h \) in terms of the new variables becomes
\[
(\Phi_{g_h})(\bar{z}_1, \bar{z}_3) = h(\bar{z}_1 - \hat{a}^1, \bar{z}_3 - \hat{a}^3) \cdot e^{i(\hat{a}^1 \bar{z}_1^o + \hat{a}^2 \bar{z}_2^o + \hat{a}^3 \bar{z}_3^o)/h} \cdot e^{i(\hat{a}^1 \bar{z}_4 + \hat{a}^2 \bar{z}_5 + \hat{a}^3 \bar{z}_3^o)/\bar{y}_1} ,
\]

As for the case of the representation associated to an orbit of dimension 2|2 with an even symplectic form, the appearance of the parameters \( \bar{x}_4^o \) and \( \bar{x}_5^o \) is artificial; they disappear when we use the coordinates \( \bar{z}_1 = \bar{x}_4/\bar{y}_1 \) and \( \bar{z}_3 = \bar{x}_5/\bar{y}_1 \).

- Our last choice is the invariant polarization represented by the subalgebra of dimension 3|3 generated by \( e_2, e_3, e_4, e_6, k_0, k_1 \) (the other choices for \( a, b \) will give equivalent representations), which corresponds to the foliation on the orbit generated by the vector fields
\[
\partial_{\bar{x}_5} \quad \text{and} \quad \partial_{\bar{\xi}_1} .
\]
The corresponding horizontal lifts are given by the same vectors, so we have to look for functions $g$ satisfying $\partial x_{\xi} g = \partial \xi g = 0$. These are determined by functions $h$ of even one and one odd variable according to

$$g(\bar{x}, \bar{x}, \bar{\xi}, \bar{\xi}) = h(\bar{x}, \bar{\xi}) .$$

The induced action of $G$ on such a function is given (using once again (10.1)) as

$$(\Phi_{\bar{g}} f)(\bar{x}, \bar{x}, \bar{\xi}, t, \tau) = h(\bar{x} - \bar{y} a^1, \bar{\xi} + \bar{y} \alpha^5) \cdot e^{i(\bar{a}^1 \bar{x} + \bar{a}^6 \bar{x})/h} .$$

If we introduce the variables $\bar{z}_1 = (\bar{x} - \bar{x}_3)/\bar{y}$ and $\bar{z}_5 = -\bar{z}_3/\bar{y}$, then the action of $\bar{g} \in G$ on the functions $h$ in terms of the new variables becomes

$$h(\bar{z}_1, \bar{z}_5) = h(\bar{z}_1 - \bar{a}^1, \bar{z}_5 - \bar{a}^5) .$$

(10.3)

And like previous cases, the appearance of the parameter $\bar{x}_3$ is artificial and disappears when we use the coordinate $\bar{z}_1 = \bar{x}/\bar{y}$ instead of the shifted version $\bar{z}_1 = (\bar{x} - \bar{x}_3)/\bar{y}$.

11. Orbits of dimension $3|\bar{3}$

As in the other cases, the prequantum bundle is the direct product $Y = O_{\mu} \times (A_0/d\mathbf{Z}) \times A_1$. Here a connection form $\Gamma$ is given by

$$\Gamma_0 = dt + (y_0)^{-1}(-x_1 dx_2 - \frac{1}{2} \xi_5 d\xi_1 + \frac{1}{2} \xi_6 d\xi_6)$$

$$\Gamma_1 = d\tau + (y_0)^{-1}(-x_5 d\xi_5 - \xi_1 d\xi_2) .$$

The lifted fundamental vector fields $(v, z)^Y$ are given by

$$(v, z)^Y = y_0 \cdot \left( v^2 \cdot \frac{\partial}{\partial x_1} - v^1 \cdot \frac{\partial}{\partial x_2} - v^5 \cdot \frac{\partial}{\partial \xi_5} + v^6 \cdot \frac{\partial}{\partial \xi_6} \right)$$

$$+ y_1 \cdot \left( v^4 \cdot \frac{\partial}{\partial \xi_1} - v^3 \cdot \frac{\partial}{\partial \xi_5} \right)$$

$$+ (v^2 x_2 + v^3 x_3 - \frac{1}{2} v^5 \xi_5 - \frac{1}{2} v^6 \xi_6 + z^0 y_0) \cdot \frac{\partial}{\partial t}$$

$$+ (v^3 \xi_3 + v^4 x_4 + v^6 x_6 + z^1 y_1) \cdot \frac{\partial}{\partial \tau} ,$$

where the functions $\bar{x}_4$ and $\bar{\xi}_3$ must be seen as the functions $y_0^{-1}(x_4 - \bar{y}_1 \bar{x}_2) = y_0^{-1}(y_0 \bar{x}_4 - \bar{y}_1 x_2)$ and $-y_0^{-1} \bar{y}_1 \bar{\xi}_3$ respectively. These vector fields integrate to an action of $G$ on $Y$ by

$$(11.1)$$

$$g \in G : \begin{pmatrix} x_1 \\ \dot{x}_2 \\ x_5 \\ \xi_1 \\ \xi_5 \\ \xi_6 \\ \tau \end{pmatrix} \rightarrow \begin{pmatrix} x_1 - y_0 a^2 \\ \dot{x}_2 + y_0 a^1 \\ \dot{x}_5 + y_0 a^3 \\ \xi_1 - y_1 \alpha^4 \\ \xi_5 + y_0 a^5 \\ \xi_6 - y_0 a^6 \\ \tau - a^3 \xi_3 - a^4 x_4 - a^6 x_6 - \beta \bar{y}_1 - \frac{1}{2} \bar{y}_1 a^1 \alpha^4 + \frac{1}{2} \bar{y}_1 a^3 \alpha^5 \end{pmatrix} \cdot$$
For all these orbits the stabilizer subgroup is given by

$$G_{\mu_0} = \{(a^i, \alpha^i, b, \beta) \mid a^i = \alpha^i = 0\}.$$  

The stabilizer subalgebra $g_{\mu_0}$ of dimension 1|1 thus is generated by $k_0, k_1$. If a subalgebra $h$ representing an invariant polarization has even dimension 4, it contains all 4 even basis vectors, but then

$$\langle [e_1, e_2] | \mu_0 \rangle = -y_0 \neq 0.$$  

The conclusion is that the even dimension of such $h$ is at most 3. Similarly, if the odd dimension is 4, it contains in particular the vector $e_5$, but then

$$\langle [e_5, e_5] | \mu_0 \rangle = y_0 \neq 0.$$  

Hence the odd dimension of such $h$ is at most 3. A detailed analysis when the even dimension is 3 shows that this is possible if and only if the even basis vectors are $ae_1 + be_2, e_3, k_0$ (with $|a|^2 + |b|^2 = 1$). Similarly, the only way to have three odd basis vectors in $h$ is when they are $e_4, e_5 + ce_6, k_1$ with $c = \pm 1$. If $h$ has dimension 3|3, it thus contains the vectors $ae_1 + be_2, e_3, e_4, e_5 + ce_6, k_0, k_1$. But then

$$\langle [e_3, e_5 + ce_6] | \mu_0 \rangle = -\bar{y}_1 \neq 0,$$

which shows that dimension 3|3 is excluded too. A detailed analysis of dimensions 3|2 and 2|3 shows that the only possibilities are the following: for dimension 3|2 there is a single $h$ possible, generated by $e_2, e_3, e_4, k_0, k_1$ and for dimension 2|3 there are exactly two possibilities, generated by $e_2, e_4, e_5 + ce_6, k_0, k_1, \epsilon = \pm 1$.

As in the previous cases, functions $f$ on $Y$ satisfying condition (7.1) are determined by functions $g$ on the orbit according to

$$f(x_1, \hat{x}_2, \bar{x}_5, \xi_1, \xi_5, t, \tau) = e^{i \tau \kappa + i t/\hbar} \cdot g(x_1, \hat{x}_2, \bar{x}_5, \xi_1, \xi_5, \xi_6).$$

**The invariant polarization corresponding to the subalgebra of dimension 3|2** corresponds to the foliation on the orbit generated by the vector fields

$$\partial_{x_1}, \partial_{\hat{x}_5} \text{ and } \partial_{\xi_1}.$$  

The horizontal lifts are given by “the same” vector fields, hence we have to look for functions $g$ determined by functions $h$ of one even and two odd variables according to

$$g(x_1, \hat{x}_2, \bar{x}_5, \xi_1, \xi_5, \xi_6) = h(\hat{x}_2, \xi_5, \xi_6).$$  

The induced action of $G$ on such a function is given (using (11.1)) as

$$(\Phi g f)(x_1, \hat{x}_2, \bar{x}_5, \xi_1, \xi_5, \xi_6, t, \tau) = h(\hat{x}_2 - y_0 a^1 \cdot \xi_5 - y_0 a^5 \cdot \xi_6 + y_0 a^6) \cdot e^{i \tau \kappa + i t/\hbar}$$

$$\cdot e^{(a^2 \hat{x}_2 + a^3 \bar{x}_5 + b y_0 - \frac{1}{2} a^2 \xi_5 + \frac{1}{2} a^5 \xi_6 - \frac{1}{2} y_0 a^1 a^3)/\hbar}$$

$$\cdot e^{(a^4 \xi_3 + a^3 x_4 + a^5 x_6 + b^1 \cdot \bar{y}_1 - \frac{1}{2} \bar{y}_1 a^1 a^4 + \frac{1}{2} \bar{y}_1 a^3 a^5) \kappa},$$

where as before the dependent functions $\bar{\xi}_4$ and $\bar{x}_4$ represent the functions $-\bar{y}_0^{-1} \bar{y}_1 \bar{\xi}_5$ and $y_0^{-1} (s^a + \bar{y}_1 \hat{x}_2) = y_0^{-1} (y_0 \bar{x}_4 - \bar{y}_1 x_2^0 + \bar{y}_1 \hat{x}_2)$ respectively. If we introduce the
variables $z_1 = (\hat{x}_2 - x_2^0)/y_0$, $\zeta_5 = \hat{\xi}_5/y_0$ and $\zeta_6 = -\xi_6/y_0$, then the action of $\hat{g} \in G$ on the functions $h$ in terms of the new variables becomes

$$
(\Phi_\hat{g} h)(z_1, \zeta_5, \zeta_6) = h(z_1 - \hat{a}^1, \zeta_5 - \hat{a}^5, \zeta_6 - \hat{a}^6) \cdot e^{i(\hat{a}^2 x_2^0 + \hat{a}^3 x_5^0)/\hbar} \cdot e^{i(\hat{a}^4 x_6^0 + \hat{a}^5 x_6^0)/\hbar} \cdot e^{i(-\hat{a}^3 z_1 + \hat{b}^1 a^4 + \hat{b}^3 a^5)/\hbar} \cdot e^{i(-\hat{a}^3 \zeta_5 + \hat{a}^4 z_1 + \hat{b}^1 a^4 + \hat{b}^3 a^5)/\hbar}.
$$

If we introduce the variables $z_1 = (\hat{x}_2 - x_2^0)/y_0$, $\zeta_5 = \hat{\xi}_5/y_0$ and $\zeta_6 = -\xi_6/y_0$, then the action of $\hat{g} \in G$ on the functions $h$ in terms of the new variables becomes

$$
(\Phi_\hat{g} h)(z_1, \zeta_5, \zeta_6) = h(z_1 - \hat{a}^1, \zeta_5 - \hat{a}^5, \zeta_6 - \hat{a}^6) \cdot e^{i(\hat{a}^2 x_2^0 + \hat{a}^3 x_5^0)/\hbar} \cdot e^{i(\hat{a}^4 x_6^0 + \hat{a}^5 x_6^0)/\hbar} \cdot e^{i(-\hat{a}^3 z_1 + \hat{b}^1 a^4 + \hat{b}^3 a^5)/\hbar} \cdot e^{i(-\hat{a}^3 \zeta_5 + \hat{a}^4 z_1 + \hat{b}^1 a^4 + \hat{b}^3 a^5)/\hbar}.
$$

(11.3)

This action depends upon the basepoint parameters $x_2^0$, $x_2^0$, $\bar{x}_4^0$, $\bar{x}_6^0$, $y_0$, $\bar{y}_1$. As in previous cases, the appearance of the parameters $x_2^0$ and $\bar{x}_4^0$ is artificial. Using the coordinate $z_1 = \hat{x}_2/y_0$ instead of the shifted version, the term $\hat{a}^2 x_2^0$ in the exponent would have disappeared and the term $\hat{a}^4 x_4^0$ would have become $\hat{a}^2 s^0/y_0$. And then it is obvious that the action only depends upon the orbit parameters $x_2^0$, $s^0$, $\bar{x}_4^0$, $y_0$, $\bar{y}_1$. Another shift would have made the term $\hat{a}^4 x_4^0$ disappear and changed the term $\hat{a}^2 x_2^0$ to $-s^0 \hat{a}^2/\bar{y}_1$.

- The invariant polarization corresponding to the subalgebra of dimension 2|3 corresponds to the foliation on the orbit generated by the vector fields

$$
\partial_{x_1}, \quad \partial_{\zeta_5} \quad \text{and} \quad \partial_{\xi_5} - \epsilon \partial_{\xi_6}.
$$

The horizontal lifts are given by the vector fields

$$
\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial \zeta_1} \quad \text{and} \quad \frac{\partial}{\partial \xi_5} - \epsilon \frac{\partial}{\partial \xi_6} - \frac{\xi_5 + \xi_6}{2y_0} \cdot \frac{\partial}{\partial \tau} + \frac{\bar{x}_5}{y_0} \cdot \frac{\partial}{\partial \tau},
$$

which gives us the equations for the functions $g$

$$
\frac{\partial g}{\partial x_1} = 0 \quad \frac{\partial g}{\partial \zeta_1} = 0 \quad \text{and} \quad \frac{\partial g}{\partial \xi_5} - \epsilon \frac{\partial g}{\partial \xi_6} - i \cdot \frac{\xi_5 + \xi_6}{2y_0} \cdot g + i \cdot \frac{\bar{x}_5}{y_0} \cdot g = 0,
$$

whose solutions are determined by functions $h$ of two even and one odd variable according to

$$
g(x_1, \hat{x}_2, \bar{x}_4, \hat{\xi}_5, \hat{\xi}_6) = h(\hat{x}_2, \bar{x}_5 + \xi_6) \cdot e^{i\epsilon \xi_5 \xi_6} \cdot e^{-\frac{\bar{x}_5}{y_0} \xi_5 \xi_6}.
$$

The induced action of $G$ on such a function is given (using again (11.1)) as

$$
(\Phi_\hat{g} h)(x_1, \hat{x}_2, \bar{x}_4, \hat{\xi}_5, \hat{\xi}_6, \epsilon, \tau) = h(x_2 - y_0 a^1, \bar{x}_5 - \bar{y}_1 a^3, (\xi_5 + \xi_6) - y_0 (a^5 - \epsilon a^6)) \cdot e^{i\epsilon \xi_5 \xi_6 \xi_6} \cdot e^{-\frac{\bar{x}_5}{y_0} \xi_5 \xi_6} \cdot e^{i(\epsilon^2 \hat{x}_2^0 + \epsilon^3 \bar{x}_4^0 + \bar{y}_0 - \frac{1}{2} (\epsilon^2 + \epsilon a^6) (\xi_5 + \xi_6) - y_0 a^2 - \frac{1}{2} \bar{y}_1 a^3) \xi_5 \xi_6} \cdot e^{i(\epsilon a \hat{x}_4^0 + \epsilon \bar{x}_5 a^3 \bar{x}_5 + \bar{y}_1 a^4 \bar{y}_1 a^4 - \frac{1}{2} \bar{y}_1 a^3 a^5) \xi_5 \xi_6}.
$$
where as before the dependent functions \( \hat{\xi}_3 \) and \( \bar{x}_4 \) represent the functions \(-y_0^{-1}\hat{y}_1\hat{\xi}_5 \) and \( y_0^{-1}(s^0 + \hat{y}_1\bar{x}_2) = y_0^{-1}(\hat{y}_0\bar{x}_4 - \hat{y}_1\bar{x}_2 + \hat{y}_1\bar{x}_2) \) respectively. If we introduce the variables \( z_1 = (\bar{x}_2 - x_2^0)/y_0 \), \( \bar{z}_3 = (\bar{x}_5 - x_5^0)/\hat{y}_1 \) and \( \zeta = \xi/y_0 \), then the action of \( \hat{g} \in G \) on the functions \( h \) in terms of the new variables becomes

\[
(\Phi_{\hat{g}}h)(z_1, \bar{z}_3, \zeta) = h(z_1 - \hat{a}^1, \bar{z}_3 - \hat{a}^3, \zeta - (\hat{a}^5 - \epsilon\hat{a}^0)) \cdot e^{i\hat{\rho}_0(\hat{a}^2 z_1 - \frac{1}{2}(\hat{a}^5 + \epsilon\hat{a}^0)\zeta)} \cdot e^{i(\hat{a}^4 z_1 + \hat{a}^5 \bar{z}_3)\hat{y}_1\kappa} \cdot e^{i(\hat{a}^2 x_2^0 + \hat{a}^3 x_5^0)/y_0} \cdot e^{i(\hat{a}^4 x_2^0 + \hat{a}^5 x_5^0 + \epsilon\hat{a}^0 \bar{x}_2^0)\kappa} \cdot e^{i(\hat{a}^4 \bar{x}_5^0 + \hat{a}^5 \bar{x}_2^0 + \epsilon\hat{a}^0 \bar{x}_2^0)\bar{y}_1\kappa}.
\]

And just as in previous cases, the dependence on the parameters \( x_2^0, \bar{x}_3^0 \) and \( \bar{x}_5^0 \) is artificial. If we use the coordinates \( z_1 = \bar{x}_2/y_0 \) and \( \bar{z}_3 = \bar{x}_5/\hat{y}_1 \) instead of the shifted ones given above, the terms \( \hat{a}^2 x_2^0 \) and \( \hat{a}^5 x_5^0 \) in the exponents disappear and the term \( \hat{a}^4 \bar{x}_5^0 \) becomes \( \hat{a}^4 s^0/y_0 \). And in that form this representation only depends upon the parameters \( x_2^0, \bar{x}_5^0, y_0, \hat{y}_1, s^0 \) describing the orbit. Shifting \( z_1 \) differently would have made the term \( \hat{a}^4 \bar{x}_5^0 \) disappear and changed the term \( \hat{a}^2 x_2^0 \) to \(-\hat{a}^2 s^0/\hat{y}_1 \).

12. Recapitulation and comparison

If we gather the results for all orbits, we get the following list:

(i) For each 0/0-dimensional orbit we have a (1-dimensional) representation.
(ii) For each 2/2-dimensional orbit with an even symplectic form we have a representation by functions on a 1/1-dimensional manifold.
(iii) For each 2/2-dimensional orbit with an odd symplectic form we have three different representations by functions on a manifold of dimension 2/0, 0/2 and 1/1 respectively.
(iv) For each 3/3-dimensional orbit (with a mixed symplectic form) we have two different representations by functions on a manifold of dimension 2/1 and 1/2 respectively.

In order to compare this list with the representations we found in the regular representation in §3, we start with the spaces \( V_{(0,t_2,t_3,0,\lambda_5),(t_1,1,\lambda_6)} \). Comparison of the \( G \) action given in (3.10) with the action given in the representation associated to a 0/0-dimensional orbit \( (8.2) \) tells us that they are the same, provided we make the following identifications

\[
\ell_1 = -x_1^0/h, \quad \ell_2 = -x_2^0/h, \quad \ell_3 = -x_3^0/h, \quad \lambda_4 = -\bar{x}_2^0\kappa, \quad \lambda_5 = -\bar{x}_5^0\kappa, \quad \lambda_6 = -\bar{x}_6^0\kappa.
\]

Instead of commenting directly on these identifications, let us first complete our list, starting with \( V_{(0,t_2,t_3,0,\lambda_5),(t_1,1,\lambda_6)} \) with \( \lambda_0 \neq 0 \). Comparing the action (3.9) with the action given in the representation associated to a 2/2-dimensional orbit with an odd symplectic form and a 3/3-dimensional polarization \( (10.3) \) tells us that they are the same, provided we make the identifications

\[
0 = -x_1^0/h, \quad \ell_2 = -x_2^0/h, \quad \ell_3 = -x_3^0/h, \quad \lambda_0 = -\hat{y}_1\kappa, \quad \lambda_4 = -\bar{x}_2^0\kappa, \quad \lambda_6 = -\bar{x}_6^0\kappa.
\]

We then turn our attention to \( V_{(t_0,t_2,t_3,0,\lambda_5)} \) with \( \ell_0 \neq 0 \). Comparing the action (3.7) with the action (9.3) given in the representation associated to a 2/2-dimensional orbit with an even symplectic form tells us that they are the same,
provided we make the identifications
\[ \ell_0 = -y_0/\hbar, \quad \ell_2 = -x_0^2/\hbar, \quad \ell_3 = -x_3^0/\hbar \]
\[ \lambda_4 = -\bar{x}_2^\kappa, \quad 0 = -\bar{x}_3^\kappa, \quad 0 = -\bar{x}_3^\kappa. \]

We finally look at \( V(\ell_0, \ell_2, \ell_3, \lambda_0, \lambda_4) \) with \( \ell_0 \neq 0 \neq \lambda_0 \). Comparing the action (3.4) with the action (11.3) given in the representation associated to a 3|3-dimensional orbit with a 3|2-dimensional polarization tells us that they are the same, provided we make the identifications
\[ \ell_0 = -y_0/\hbar, \quad \ell_2 = -x_0^2/\hbar, \quad \ell_3 = -x_3^0/\hbar \]
\[ \lambda_0 = -\bar{y}_0\kappa, \quad \lambda_4 = -\bar{x}_2^\kappa, \quad 0 = -\bar{x}_3^\kappa. \]

If we look at the required identifications, we see that they all have the same form. The even parameters \( \ell_0, \ell_1, \ell_2, \ell_3 \) have to be the same as the coordinates of the basepoint \( \mu_o \) of the orbit \( y_0, x_0^0, x_2^0, x_3^0 \) up to a factor \( -\hbar \). And the odd parameters \( \lambda_0, \lambda_4, \lambda_5, \lambda_6 \) have to be the same as the coordinates (of \( \mu_o \)) \( \bar{y}_1, \bar{x}_1^0, \bar{x}_5^0, \bar{x}_6^0 \) up to a factor \( -\kappa \). But here the situation is not as simple as for the even parameters. For the even parameters we performed a honest Fourier transform of (smooth) functions of even coordinates. As such the even parameters \( \ell_0, \ell_1, \ell_2, \ell_3 \) are real. And identifying them up to a factor \( \pm \hbar \) with the real coordinates \( y_0, x_1^0, x_2^0, x_3^0 \) poses no problem. On the other hand, the parameters \( \lambda_0, \lambda_4, \lambda_5, \lambda_6 \) are odd, so identifying them with the real coordinates \( \bar{y}_1, \bar{x}_1^0, \bar{x}_5^0, \bar{x}_6^0 \) is not obvious. But as said, it should be up to the odd factor \( \kappa \). But then all four odd parameters \( \lambda_0, \lambda_4, \lambda_5, \lambda_6 \) should be real multiples of the same odd factor \( \kappa \). Which would suggest that we cannot choose these four parameters freely.

We thus reach the unsatisfactory situation that we have a nice identification between the irreducible representations appearing in the regular representation and representations associated to coadjoint orbits, but there is nevertheless something strange about this identification. If we take the odd parameters seriously, then we are embarrassed by the fact that the identification makes them all real multiples of the same odd factor. But we also can turn our attention to the odd family decomposition, where the odd parameter just is a way to indicate a direct sum decomposition into two factors. Seen that way, we just should distinguish the cases “odd parameter zero” and “odd parameter non-zero.” But if we do that, then we have a problem at the orbit level where the orbits are parametrized by the real coordinates \( \bar{x}_i^0 \), and there is no reason to assume that we only should distinguish between zero and non-zero values.

Another point which we have not addressed here is how to define a scalar product on the representation obtained by geometric quantization. It is fairly easy to see that the natural scalar product consisting of integrating over the coordinates on which the functions depend will yield an invariant non-degenerate sesquilinear form. But when the odd dimension is odd (no pun intended; it is the case for the 2|2-dimensional orbit with an even form, for the 2|2-dimensional orbit with an odd form and the 3|3-dimensional polarization and for the 3|3-dimensional orbit with the 2|3-dimensional polarization), this form will be an odd sesquilinear form. Which means that if we take this approach seriously, then the notion of unitary representations in the super setting should not restrict attention to even non-degenerate sesquilinear forms.

A third problem to which I have not found a satisfactory answer is how to define the notion of equivalent representations for odd family decompositions. As the
reader can see, there remain a lot of questions to be answered. However, I hope to have convinced her/him that super symplectic manifolds with a non-homogeneous symplectic form are interesting objects to study.

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