EPR-Bohm experiment, interference of probabilities, and imprecision of time

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Abstract

We demonstrate that the EPR-Bohm probabilities can be easily obtained in the classical (but contextual) probabilistic framework by using the formula of interference of probabilities. From this point of view the EPR-Bell experiment is just an experiment on interference of probabilities. We analyze the time structure of contextuality in the EPR-Bohm experiment. The conclusion is that quantum mechanics does not contradict to a local realistic model in which probabilities are calculated as averages over conditionings/measurements for pairs of instances of time $t_1 < t_2$. If we restrict our consideration only to simultaneous measurements at the fixed instance of time $t$ we would get contradiction with Bell’s theorem. One of implications of this fact might be the impossibility to define instances of time with absolute precision on the level of the microscopic realistic model.

1. Introduction. In a series of papers, see, e.g., [1] it was demonstrated that interference of probabilities (which is always considered as one of the main distinguishing features of quantum mechanics, e.g. [2]) can be easily derived in contextual statistical framework (without

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to appeal to the Hilbert space formalism). In this note it is demonstrated that in this framework we also can easily get the EPR-Bohm probabilities. Moreover, the cos-form of the EPR-Bohm probabilities is a consequence of interference of probabilities corresponding to different contexts.

2. Contextual statistical model. Basic structures of the model are physical contexts – complexes of physical conditions.¹ We denote the set of all contexts by the symbol $C$.

Suppose that there is fixed a set of observables $O$ such that any observable $a \in O$ can be measured under a complex of physical conditions $C$ for any $C \in C$. We shall denote observables by Latin letters, $a, b, c, \ldots$, and their values by Greek letters, $\alpha, \beta, \gamma, \ldots$

We do not assume that all these observables can be measured simultaneously; so they need not be compatible. The sets of observables $O$ and contexts $C$ are coupled through

Axiom 1: For any observable $a \in O$, there are well defined contexts $C_\alpha$ corresponding to $\alpha$-filtrations: if we perform a measurement of $a$ under the complex of physical conditions $C_\alpha$, then we obtain the value $a = \alpha$ with probability 1. It is supposed that the set of contexts $C$ contains filtration-contexts $C_\alpha$ for all observables $a \in O$.

Axiom 2: There are defined contextual probabilities $P(a = \alpha/C)$ for any context $C \in C$ and any observable $a \in O$.

3. Interference of probabilities. Consider two dichotomous random variables $a = \pm 1, b = \pm 1$. Let $C$ be a context. There are well defined probabilities:

$$p_C^a(\alpha) = P(a = \alpha/C), p_C^b(\beta) = P(b = \beta/C), p^{b/a}(\beta/\alpha) = P(b = \beta/a = \alpha),$$

We also introduce the matrix of transition probabilities: $P^{b/a} = (p^{b/a}(\beta/\alpha))$. The classical formula of total probability has the form:

$$p^b_C(\beta) = p^a_C(\beta) + p^{b/a}(\beta/+) + p^{b/a}(\beta/-) p^{b/a}(\beta/-).$$

However, for contextual probabilities this formula can be violated, see [1]. We introduce the coefficient of statistical incompatibility of observables $a$ and $b$ in the context $C$ (see [1]):

$$\lambda(\beta/a, C) = \frac{p^b_C(\beta) - [p^a_C(\beta/) + p^{b/a}(\beta/+) + p^{b/a}(\beta/-)]}{2 \sqrt{p^a_C(\beta/) + p^{b/a}(\beta/+) + p^{b/a}(\beta/-)}}$$

¹In principle, the notion of context can be considered as a generalization of a widely used notion of preparation procedure.
and by using this coefficient we can rewrite the probability $p^b_C(\beta)$ in the interference-like form:

$$p^b_C(\beta) = p^a_C(+)p^{b/a}(\beta/+)+ p^a_C(-)p^{b/a}(\beta/-) + 2\lambda(\beta/a, C)\sqrt{p^{b/a}(\beta/+)}p^a_C(-)p^{b/a}(\beta/-).$$

Now let us restrict our considerations to the case of relatively small coefficients of incompatibility:

$$|\lambda(\beta/a, C)| \leq 1.$$ 

In this case we can introduce new statistical parameters $\theta(\beta/a, C) \in [0, 2\pi]$ and represent the coefficients of incompatibility in the trigonometric form:

$$\lambda(\beta/a, C) = \cos \theta(\beta/a, C).$$

Parameters $\theta(\beta/a, C)$ are said to be statistical phases. For such contexts we get the standard formula of interference of probabilities:

$$p^b_C(\beta) = p^a_C(+)p^{b/a}(\beta/+)+ p^a_C(-)p^{b/a}(\beta/-) + 2\cos(\beta/a, C)\sqrt{p^{b/a}(\beta/+)}p^a_C(-)p^{b/a}(\beta/-)$$

which is usually derived by using the Hilbert space formalism.\(^2\)

**Important remark** Quantum observables always produce double stochastic matrices of transition probabilities, i.e., $p^{b/a}(+/\alpha) + p^{b/a}(-/\alpha) = 1$ and $p^{b/a}(\beta/+)+ p^{b/a}(\beta/-) = 1$.

4. EPR-Bohm probabilities as interference probabilities.

Let us now consider three dichotomous observables $a, b, c = \pm 1$. There are well defined selection-contexts $C_{\pm}$, selections with respect to values $\gamma = \pm 1$. We choose $C = C_+$ (i.e., the context corresponding to the $c = +1$ selection) and use results of section 3 for this context. There are well defined probabilities

$$p^a_C(\alpha) \equiv p(a = \alpha/C), \alpha = \pm 1, \quad p^b_C(B) \equiv p(b = \beta/C), \beta = \pm 1.$$ 

By taking into account that $C = C_+$ we get:

$$p^a_C(\alpha) = p^{a/c}(\alpha/+), \quad p^b_C(\beta) \equiv p^{b/c}(\beta/+).$$

\(^2\)If the coefficients of incompatibility are larger than 1, we obtain so called hyperbolic interference [1] which is described by hyperbolic quantum mechanics [3]. We do not consider this possibility in the present paper, but we notice that both ordinary (“trigonometric”) quantum mechanics and hyperbolic quantum mechanics can be obtained as deformations of the same classical mechanics [3].
We now can use the formula of total probability with interference term, (1), to the \( b \)-observable. For \( \beta = +1 \) we have:

\[
p^{b/c}(+/+) = p^{a/c}(+/+)p^{b/a}(+/+) + p^{a/c}(-/+p)b^{b/a}(+/-) + 2 \cos \theta_+ \sqrt{p^{a/c}(+/+)p^{b/a}(+/+)p^{a/c}(-/+p)b^{b/a}(+/-)},
\]

where \( \theta_+ = \theta(+/a, C_+) \). We can always represent probabilities in the trigonometric form:

\[
p^{a/c}(+/+) = \cos^2 \xi, \quad p^{a/c}(-/+p) = 1 - \cos^2 \xi = \sin^2 \xi,
\]

where \( \xi = \xi^{a/c}(+/+) \). Suppose now that matrix \( P^{b/a} \) is double stochastic. Then we can represent probabilities:

\[
p^{b/a}(+/+) = \sin^2 \eta, \quad p^{b/a}(+/-) = 1 - \sin^2 \eta = \cos^2 \eta,
\]

where \( \eta = \eta^{a/b}(+/+) \). To simplify considerations, we consider the case when both phases \( \xi, \eta, \in (0, \pi/2) \). Thus we have:

\[
p^{b/c}(+/+) = \cos^2 \xi \sin^2 \eta + \sin^2 \xi \cos^2 \eta + 2 \cos \theta_+ \cos \xi \sin \eta \sin \xi \cos \eta.
\]

We now perform similar considerations for \( \beta = -1 \) :

\[
p^{b/c}(-/+p) = p^{a/c}(+/+)p^{b/a}(-/+p) + p^{a/c}(-/+p)b^{b/a}(-/-) + 2 \cos \theta_- \sqrt{p^{a/c}(+/+)p^{b/a}(-/+p)p^{a/c}(-/+p)b^{b/a}(-/-)},
\]

where \( \theta_- = \theta(-/a, C_+) \). We have that

\[
p^{b/a}(-/+p) = 1 - p^{b/a}(+/-) = \cos^2 \eta, \quad p^{b/a}(+/-) = 1 - p^{b/a}(+/-) = \sin^2 \eta.
\]

Thus

\[
p^{b/c}(-/+p) = \cos^2 \xi \cos^2 \eta + \sin^2 \xi \sin^2 \eta + 2 \cos \theta_- \cos \xi \cos \eta \sin \xi \sin \eta.
\]

Consider now the case of trigonometric interference the \textit{maximal magnitude}:

\[
|\lambda_\pm| = |\cos \theta_\pm| = 1 \quad (2)
\]

**Lemma 1.** Let \( P^{b/a} \) be double stochastic and let the condition (2) hold. Then

\[
\cos \theta_+ = - \cos \theta_-
\]

(3)
Proof. We have

\[ 1 = p^{b/c}(-/+)+p^{b/c}(+/+) = \]

\[ \cos^2 \xi (\cos^2 \eta + \sin^2 \eta) + \sin^2 \xi (\cos^2 \eta + \sin^2 \eta) + \]

\[ 2 \cos \theta_+ \cos \xi \sin \eta \sin \xi \cos \eta + 2 \cos \theta_- \cos \xi \cos \eta \sin \xi \sin \eta \]

This implies (3) (since \( \xi, \eta \in (0, \pi/2) \)).

Let us say that

\[ \cos \theta_+ = -1, \cos \theta_- = +1 \] (4)

(we can make the opposite choice but this would induce a phase shift). Then we get

\[ p^{b/c}(+/+) = (\cos \xi \sin \eta - \sin \xi \cos \eta)^2 = \sin^2 (\xi - \eta), \]

\[ p^{b/c}(-/+)(\cos \xi \cos \eta + \sin \xi \sin \eta)^2 = \cos^2 (\xi - \eta) \]

We now consider representations of probabilities \( p^{b/c}(\pm/-) \) for the context \( C = C_- \) (selection for the value \( c = -1 \)).

\[ p^{b/c}(+/-) = p^{a/c}(+/+)p^{b/a}(+/+) + p^{a/c}(-/-)p^{b/a}(+/+), \]

\[ 2 \cos \tilde{\theta}_+ \sqrt{p^{a/c}(+/+)p^{b/a}(+/+)p^{a/c}(-/-)p^{b/a}(+/+)} \]

where \( \tilde{\theta}_+ \equiv \theta(+/a, C_-) \). Suppose that the matrix \( P^{a/c} \) is also double stochastic. Then we get

\[ p^{a/c}(+/-) = 1 - \cos^2 \xi = \sin^2 \xi, \quad p^{a/c}(-/-) = \cos^2 \xi. \]

So we have:

\[ p^{b/c}(+/-) = \sin^2 \xi \sin^2 \eta + \cos^2 \xi \cos^2 \eta + 2 \cos \tilde{\theta}_+ \sin \xi \sin \eta \cos \xi \cos \eta. \]

In the same way we get:

\[ p^{b/c}(-/-) = \cos^2 \xi \sin^2 \eta + \sin^2 \xi \cos^2 \eta + 2 \cos \tilde{\theta}_- \cos \xi \sin \eta \sin \xi \cos \eta. \]

Lemma 2. Let matrices \( P^{b/a}, P^{a/c}, P^{b/c} \) be double stochastic and have strictly positive elements. Then

\[ \cos \theta_+ = -\cos \tilde{\theta}_+, \cos \theta_- = -\cos \tilde{\theta}_- \] (5)
Proof. By using double stochasticity of $P^{b/c}$ we get

$$1 = p^{b/c}(+/+) + p^{b/c}(+/-) = \sin^2 \eta (\cos^2 \xi + \sin^2 \xi) +$$

$$\cos^2 \eta (\cos^2 \xi + \sin^2 \xi) + 2(\cos \theta_+ + \cos \bar{\theta}_+) \cos \xi \cos \eta \sin \xi \sin \eta.$$ 

By using that $\xi, \eta \in (0, \frac{\pi}{2})$ (this we can always assume since $p^{b/a}(\beta/\alpha), p^{a/c}(\alpha/\gamma) > 0$), we get $\cos \theta_+ = -\cos \bar{\theta}_-.$

Results of all these considerations and computations can be represented as

**Theorem.** Let all matrices $P^{b/a}, P^{a/c}, P^{b/c}$ be double stochastic and let interference parameter $\lambda_{\pm} = \cos \theta_{\pm}$ have the maximal absolute magnitude, $|\lambda_{\pm}| = 1.$ Then probabilities can be represented in the EPR-Bohm form:

$$p^{b/c}(+/+) = p^{b/c}(−/−) = \sin^2 (\xi − \eta)$$

$$p^{b/c}(+/−) = p^{b/c}(−/+) = \cos^2 (\xi − \eta)$$

(6)

**Conclusion.** In the contextual approach the EPR-Bohm probabilities can be interpreted as interference probabilities.

5. Physical consequences. What are main physical consequences of the contextual probabilistic derivation of the EPR-Bohm probabilities?

The evident consequence is that local realism is compatible with quantum mechanics\(^3\) if contextuality of probabilities is taken into account.

However, to make physicists interested in our contextual approach to the EPR-Bohm experiment, we should be able to find the physical mechanism of contextuality in this experiment. We recall that contextuality is dependence on complexes of physical conditions. To obtain the EPR-Bohm probabilities in our approach, we should first make selection with respect to one fixed context, e.g., the context $C = C_+$ corresponding to the selection of subensemble of pairs such that the measurement of the $c$-projection of spin (or polarization) on the first particle in a pair gives the value $c = +1.$ We emphasize that, despite the fact the measurement is performed only on the first particle in a pair of correlated particles, the selection $C_+$ is selection of pairs. Such a selection creates a new ensemble and on this new ensemble we perform measurements of $a$ or $b$-projections for the second particle in a pair. In this way we obtain the same probabilities which

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\(^3\)We remark that we do not consider Bell’s inequality.
are predicted by quantum mechanics (by using the formalism of complex Hilbert space). The crucial point is that the order structure – contextual selection and then measurement – should always be taken into account. A selection performed at some instant of time $t_1$ gives an ensemble which is used for measurement at some instant of time $t_2 > t_1$. Thus, to have a local realistic picture, we should assume that the contextual selection and measurement are performed in different instances of time. We remark that it is not important on which of particles (on the first or on the second) there is performed the selection measurement and on which the final measurement. Probabilities are symmetric with respect to these procedures. This is a consequence of double stochasticity.

Thus in our model the EPR-Bell probabilities are obtained as the result of the average of contextual $t_2/t_1$-probabilities over all pairs of instances of time $t_1 < t_2$.

The model cannot say anything about simultaneous measurements, $t_1 = t_2$. However, the contribution of simultaneous measurements, $t_1 = t_2$, is negligibly small, since the measure of the diagonal in any two dimensional time-rectangle equals to zero.

If we restrict our consideration only to simultaneous measurements at the fixed instance of time $t$ we would get contradiction with Bell’s theorem. One of implications of this fact might be the impossibility to define instances of time with absolute precision on the level of the microscopic realistic model. Such a conjecture is quite natural if we take into account time-energy uncertainty relation. This relation (derived in quantum formalism), of course, should be also valid for the prequantum local realistic model.

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