Generation and entanglement study of generalized
$N$-mode single-photon perfect W-states

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Abstract
We propose generation and entanglement detection schemes for generalized $N$-mode single-photon perfect W-states. These states are suitable for perfect teleportation and superdense coding. Based on the evolution of single-photon wavefunction in scalable integrated photonic lattices, we present schemes to generate these states using both planar and ring type waveguide structures. The integrated waveguide structures can be precisely fabricated, offer low photon propagation losses and can be integrated on a chip. In addition, we derive set of generalized entanglement conditions using the sum uncertainty relations of generalized $su(2)$ algebra operators. We show that any given genuinely entangled $N$-mode single-photon state is a squeezed state of a specific $su(2)$ algebra operator and can be expressed as superposition of a pair of orthonormal generalized $N$-mode single-photon perfect W-states which are eigenstates of that specific $su(2)$ algebra operator. Within the single-photon subspace, the generalized entanglement condition reduces to a simplified single-photon separability condition. Detection of entanglement of single-photon states, using this single-photon separability condition, requires finding fidelity with two pairs of orthonormal generalized

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$N$-mode single-photon perfect $W$-states and hence more suitable for our purpose. Finally, we propose an experimental scheme to verify the entanglement using the proposed conditions. This scheme uses a photonic circuit consisting of directional couplers and phase shifters. The same photonic circuit can also be used to generate generalized $N$-mode single-photon perfect $W$-states.

**Keywords** Perfect $W$-state · Quantum entanglement · Integrated waveguides

### 1 Introduction

Multipartite entangled states [1–3] involve complex entanglement structure [4, 5] and have drawn wide interest recently. In the case of three qubits, there exist two inequivalent classes of maximally entangled states, known as the GHZ-class [6] and the $W$-class states [7]. For more than three qubits, we have other interesting entangled states [8–10]. These multipartite entangled states are useful for various applications in quantum information processing [11, 12]. One of such applications is quantum teleportation [13–16], where entangled state is shared as quantum channel between the sending and receiving ends. Teleportation with Bell states [17], GHZ states [18] and other multipartite states [19] were shown to be possible. Among the multipartite entangled states, $W$-class states are known for their robustness against particle loss. A set of states belonging to $W$-class can be used for perfect teleportation [20, 21], and hence, these states are denoted as perfect $W$-states [22–24]. The study of such $W$-class states has been extended to $N$-qubit system [21]. We provide a generalized expression of such $N$-qubit states (up to global phase) as follows.

$$|\tilde{W}\rangle_N = \frac{1}{\sqrt{2}} \left[ \sum_{j=1}^{N-1} \alpha_j |1\rangle_j |0\ldots0\rangle_1\ldots,j-1,j+1..N \right] + \frac{1}{\sqrt{2}} |00\ldots01\rangle_1\ldots N \quad (1)$$

where $\alpha_j$’s are nonzero complex coefficients such that $0 < |\alpha_j| < 1$ and $\sum_{j=1}^{N-1} |\alpha_j|^2 = 1$.

In Eq. 1, each of the first $N-1$ qubits can have either same or different probabilities of finding it in the state $|1\rangle$; however, their sum is equal to $1/2$. Hence, we denote these states as generalized $N$-qubit perfect $W$-states. When $\alpha_j = \frac{1}{\sqrt{N-1}}$, $\forall j$, the state $|\tilde{W}\rangle_N$ reduces to

$$|W\rangle_N = \frac{1}{\sqrt{2(N-1)}} \left[ \sum_{j=1}^{N-1} |1\rangle_j |0\ldots0\rangle_1\ldots,j-1,j+1..N \right] + \frac{1}{\sqrt{2}} |00\ldots01\rangle_1\ldots N, \quad (2)$$

which has been shown to be useful for perfect teleportation and superdense coding [21, 25, 26]. In Eq. 2, the probability amplitudes of first $N-1$ ket vectors are same and it is a real number. We refer this state as $N$-qubit perfect $W$-state. Although the state holds important applications in quantum information processing, generation of such states
with more number of qubits has been a challenging task. A scheme using exchange interaction between electron spins was proposed to generate this state in quantum dots [21]. However, in this system, it is difficult to preserve multiqubit entanglement due to the decoherence of electron spins. A cavity QED scheme which uses two energy levels of atoms as qubits was proposed to prepare perfect W-state [27] and superconducting qubits were used to generate 3-mode perfect W-states [28]. However, these systems are not scalable to generate large \( N \)-qubit perfect W-states. In another scheme, fusion and expansion mechanisms were used to prepare large \( N \)-qubit perfect W-states [26]. This scheme uses polarization degree of freedom of photons as qubits and requires small sized perfect W-states as initial states.

A promising system to realize generalized \( N \)-qubit perfect W-states is a single photon shared between \( N \) spatial modes [29–36]. Single-photon path entangled states have been reported to be useful in quantum random number generation [37–39], quantum repeater [40], optical Bloch oscillation [41], to name a few examples. Optical photonic waveguides can readily be used to generate such single-photon entangled states. Integrated waveguide lattices are gaining interest for their diverse applications in physics. These are scalable, offer low loss and can be precisely fabricated by femtosecond laser direct writing technique [42, 43]. They are compact and are realizable with current technologies and hence can serve as an important tool for optical simulation [44, 45] and generation of entangled states [32, 35, 46, 47].

In this paper, we propose schemes to generate generalized \( N \)-mode single-photon perfect W-states using weakly coupled waveguide structures. Perfect W-states are useful to achieve perfect teleportation and superdense coding. Experimental generation of these states with large number of modes has not been reported and the previously proposed schemes are not scalable because of the use of bulk optical elements. This is the first time the integrated waveguide structures are considered to propose schemes to generate large number of modes generalized single-photon perfect W-states. We also present precisely calculated parameters which will be useful for the experimental realization of the state. In our previous work, we proposed scheme to generate generalized 3-mode single-photon perfect W-states using one-dimensional (1D) integrated waveguide structures [48]. Here we consider both 1D structure (planar structure) and two-dimensional (2D) ring structure of single-mode waveguides. In the 1D structure, the spacing between two successive waveguides is kept different to ensure different coupling strengths between waveguides. In 2D ring structure, \( N \) waveguides are symmetrically arranged to form a ring and another waveguide is kept at the center of ring. The same geometries were considered to propose schemes for single-photon symmetric W-state generation [35]. However, the generation of generalized \( N \)-mode single-photon perfect W-states requires completely different coupling schemes and hence different waveguide structures. Specifically, in our scheme to generate generalized \( N \)-mode single-photon perfect W-states using 2D ring structure, it is possible to have more than 6 waveguides on the ring. In addition, we have studied the effect of dissipation in our scheme, which confirms the robustness of our scheme in the presence of photon loss.

Verification and quantification of entanglement of generated states are very crucial for many quantum information processing applications. Two-qubit entanglement can be quantified using entanglement measures [49, 50]. Extensions of bipartite entan-
lement measures [51, 52] and monogamy relations [53–55] are used to study the multipartite entanglement. Entanglement detection conditions can be used to distinguish entangled states from separable states. For two-qubit systems, the positivity of partial transposition (PPT) is a both necessary and sufficient condition for separability [56, 57]. For multiqubit systems, entanglement detection conditions based on entanglement witness operators [58–60] and spin squeezing inequalities [60] can be used to distinguish genuinely entangled states from separable states. Especially the genuine entanglement of single-photon states can be detected using entanglement detection conditions [61–64] which involves violation of sum and product uncertainty relations constructed using $su(2)$ and $su(1,1)$ algebra operators. These conditions can be experimentally verified [64]. Entanglement witness operators can also be used to detect the entanglement of multimode single-photon states [33, 34, 65].

Single-photon perfect W-states, generated by injecting single photon through coupled waveguide structures, are not biseparable and retain bipartite entanglement between each pairs of modes. We derive a set of generalized entanglement detection conditions based on sum uncertainty relation of generalized $su(2)$ algebra operators to detect the entanglement of generalized $N$-mode single-photon perfect W-states. Recently proposed entanglement condition [66] to detect the entanglement of multimode W-type entangled states belongs to this set of conditions. In the case of single-photon states, we show that the genuinely entangled $N$-mode single-photon states are squeezed states of $su(2)$ algebra operators and reduce the generalized entanglement detection condition to a single-photon separability condition which requires finding fidelity with two pairs of orthonormal generalized $N$-mode single-photon perfect W-states. These generalized $N$-mode single-photon perfect W-states are eigenstates of $su(2)$ algebra operators involved in the generalized entanglement detection condition. We propose an integrated photonic circuit [67] consisting of directional couplers and phase shifters for experimental implementation of generalized entanglement detection conditions. It involves detecting the sum and difference of photon numbers at two specific outputs of the photonic circuit. However, entanglement verification of genuinely entangled single-photon states requires detecting only the photon number difference at the two specific outputs. Finally, we show that the same photonic circuit can also be used to generate generalized $N$-mode single-photon perfect W-states.

Our paper is organized as follows. In Sect. 2, we describe the generation of generalized $N$-mode single-photon perfect W-states using 1D-planar waveguide structure. In Sect. 3, the generation scheme using 2D-ring structure is explicated. In Sect. 4, we study the effect of photon loss on our generation scheme. In Sect. 5, we derive the generalized entanglement conditions using sum uncertainty relation of $su(2)$ algebra operators and reduce it to single-photon separability condition. Further, in the same section, we show the application of single-photon separability condition to detect the entanglement of single-photon mixed states. In Sect. 6, the experimental implementation of proposed entanglement conditions using an integrated photonic circuit is explained. Finally, we arrive at conclusion in Sect. 7.
2 Generation of generalized N-mode single-photon perfect W-states using 1D structure

For one-dimensional array of $N$ identical waveguides, as shown in Fig. 1, the Hamiltonian describing the coupling between the waveguides can be written as,

$$
\hat{H} = -\hbar \sum_{j=1}^{N-1} k_{j,j+1} (\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_j \hat{a}_{j+1}^\dagger)
$$

(3)

where $\hat{a}_j^\dagger (\hat{a}_j)$ is the bosonic creation (annihilation) operator. $k_{j,j+1}$ is the coupling strength between waveguides $j$ and $j + 1$ which depends on the separation $(d_{j,j+1})$ between them. The Heisenberg equations of motion are given by,

$$
i \frac{d\hat{a}_j^\dagger}{dz} = k_{1,2} \hat{a}_2^\dagger,$$

$$
i \frac{d\hat{a}_j^\dagger}{dz} = k_{j-1,j} \hat{a}_{j-1}^\dagger + k_{j,j+1} \hat{a}_{j+1}^\dagger, \quad (j = 2, \ldots N - 1)$$

(4)

The solution to the above set of equations can take the form,

$$\hat{\mathcal{A}}^\dagger (z) = e^{-izM} \hat{\mathcal{A}}^\dagger (0)$$

(5)

where $\hat{\mathcal{A}}^\dagger$ is the column of creation operators, $M$ is the coupling matrix and $e^{-izM}$ is the evolution matrix. For given number of waveguides with appropriate coupling strengths between them, Eq. 5 can be used to find the value of $z$ for which the initial state evolves to generalized $N$-mode single-photon perfect W-state. In the following, we consider the generation of 4-mode and 5-mode single-photon perfect W-states.

The 4-mode single-photon perfect W-state is given as,

$$|W\rangle_4 = \frac{1}{\sqrt{6}}(|1000\rangle + |0100\rangle + |0010\rangle + \sqrt{3} |0001\rangle)$$

(6)

To generate 4-mode single-photon perfect W-state, we consider an array of 4 waveguides in which a single photon is injected in the third waveguide. The equations of motion can be written in the matrix form as follows.

$$
i \frac{d}{dz} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \\ a_4^\dagger \end{pmatrix} = \begin{pmatrix} 0 & k_{1,2} & 0 & 0 \\ k_{1,2} & 0 & k_{2,3} & 0 \\ 0 & k_{2,3} & 0 & k_{3,4} \\ 0 & 0 & k_{3,4} & 0 \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \\ a_4^\dagger \end{pmatrix}$$

(7)
One-dimensional array of $N$ identical waveguides. $k_{i,j}$ and $d_{i,j}$ are coupling strength and separation between waveguides $i$ and $j$, respectively.

### Table 1
The parameters required for generation of 4-mode and 5-mode single-photon perfect W-states

| No. of modes | $k_{1,2}$ (cm$^{-1}$) | $k_{2,3}$ (cm$^{-1}$) | $k_{3,4}$ (cm$^{-1}$) | $k_{4,5}$ (cm$^{-1}$) | $z$ (cm) |
|--------------|----------------------|----------------------|----------------------|----------------------|----------|
| 4            | 1.2043               | 0.686372             | 0.781121             | 0.11504              | 1.15042  |
| 5            | 1.08983              | 0.584456             | 0.988893             | 1.23828              | 1.53062  |

The initial state of the system is $|\psi(0)\rangle = \hat{a}_3^\dagger|0000\rangle$. After propagating a distance $z$ the state evolves to $|\psi(z)\rangle = C_1|1000\rangle + C_2|0100\rangle + C_3|0010\rangle + C_4|0001\rangle$. To obtain the desired state, the parameters $z$ and $k_{i,j}$ are to be selected in such a way that $|C_1|^2 = \frac{1}{6}, |C_2|^2 = \frac{1}{6}, |C_3|^2 = \frac{1}{6}$ and $|C_4|^2 = \frac{1}{2}$. For the values of coupling constants and distance $z$ listed in Table 1, the following generalized 4-mode single-photon perfect W-state can be obtained by solving the above matrix equation.

$$|\tilde{W}\rangle_4 = \frac{1}{\sqrt{6}}(-|1000\rangle + i|0100\rangle + |0010\rangle + i\sqrt{3}|0001\rangle)$$

The 4-mode single-photon perfect W-state can be obtained by adjusting the phase terms of individual ket vectors in Eq. 8. This phase adjustment can be performed by adding phase shifters [67] to individual guides. In this particular case, phase shifters have to be added to provide a phase shift of ‘$-1$’ in the 1st waveguide and of ‘$-i$’ in the 2nd and 4th waveguides.

Next we consider 5-mode single-photon perfect W-state which can be written as,

$$|W\rangle_5 = \frac{1}{2\sqrt{2}}(|10000\rangle + |01000\rangle + |00100\rangle + |00010\rangle + 2|00001\rangle)$$
This state can be generated using an array of 5 waveguides. The equations of motion can be written and solved as described above. Assuming that the photon is injected in the central waveguide, for the values of coupling parameters and distance \( z \) given in Table 1, the following generalized 5-mode single-photon perfect state can be obtained.

\[
|\hat{W}_5\rangle = \frac{1}{2\sqrt{2}} (-|10000\rangle + i|01000\rangle + |00100\rangle + i|00010\rangle - 2|00001\rangle)
\]  

In order to get the 5-mode single-photon perfect W-state, the phase shifters have to be added to provide phase shift of ‘\(-1\)’ in the 1st and 5th waveguides and of ‘\(-i\)’ in the 2nd and 4th waveguides.

In similar way, it is possible to have coupling scheme for any \( N \) number of waveguides arranged in planar structure to generate generalized \( N \)-mode single-photon perfect W-states.

### 3 Generation of generalized \( N \)-mode single-photon perfect W-states using 2D structure

In the following, we consider \( N + 1 \) identical waveguides, one at the center and \( N \) waveguides arranged symmetrically around it, as shown in Fig. 2. The central waveguide is coupled with all surrounding waveguides with coupling strength \( \kappa \) and each surrounding waveguide is coupled with nearest neighbors with coupling strength \( C \). This type of two-dimensional ring structure can be fabricated precisely using femtosecond laser direct writing technique [42, 43]. A photon is injected at the central waveguide.

The Hamiltonian describing the coupling between the waveguides is

\[
\hat{H} = -\hbar \kappa \sum_{j=1}^{N} (\hat{a}_{N+1}^{\dagger} \hat{a}_j + \hat{a}_N \hat{a}_{N+1}^{\dagger}) - \hbar C \left[ \hat{a}_1^{\dagger} \hat{a}_N + \hat{a}_1 \hat{a}_N^{\dagger} + \sum_{j=2}^{N} (\hat{a}_j \hat{a}_{j-1}^{\dagger} + \hat{a}_{j-1}^{\dagger} \hat{a}_j) \right].
\]

(11)

Heisenberg’s equations of motion are

\[
i \frac{d\hat{a}_{N+1}^{\dagger}}{dz} = \kappa \sum_{j=1}^{N} \hat{a}_j^{\dagger},
\]

\[
i \frac{d\hat{a}_1^{\dagger}}{dz} = C (\hat{a}_2^{\dagger} + \hat{a}_N^{\dagger}) + \kappa \hat{a}_{N+1}^{\dagger},
\]

\[
i \frac{d\hat{a}_j^{\dagger}}{dz} = C (\hat{a}_{j+1}^{\dagger} + \hat{a}_{j-1}^{\dagger}) + \kappa \hat{a}_{N+1}^{\dagger}, \quad (j = 2, 3, \ldots, N - 1)
\]

and

\[
i \frac{d\hat{a}_N^{\dagger}}{dz} = C (\hat{a}_1^{\dagger} + \hat{a}_{N-1}^{\dagger}) + \kappa \hat{a}_{N+1}^{\dagger}.
\]

(12)
The solution, $\hat{a}^\dagger_{N+1}(z)$, can be written as

$$\hat{a}^\dagger_{N+1}(z) = e^{(-iCz)} \left\{ \cos \left( \sqrt{C^2 + N\kappa^2} z \right) + \frac{iC}{\sqrt{C^2 + N\kappa^2}} \sin \left( \sqrt{C^2 + N\kappa^2} z \right) \right\} \hat{a}^\dagger_{N+1}(0)$$

$$- \frac{i\kappa}{\sqrt{C^2 + N\kappa^2}} \sin \left( \sqrt{C^2 + N\kappa^2} z \right) \sum_{j=1}^{N} \hat{a}^\dagger_{j}(0) \right\} \quad (13)$$

with $N\kappa^2 = C^2$, $\hat{a}^\dagger_{N+1}(z)$ is written as

$$\hat{a}^\dagger_{N+1}(z) = e^{(-iCz)} \left\{ \cos \left( \sqrt{2C} z \right) + \frac{i}{\sqrt{2}} \sin \left( \sqrt{2C} z \right) \right\} \hat{a}^\dagger_{N+1}(0)$$

$$- \frac{i}{\sqrt{2N}} \sin \left( \sqrt{2C} z \right) \sum_{j=1}^{N} \hat{a}^\dagger_{j}(0) \right\}. \quad (14)$$
When $C_z = \frac{n \pi}{2 \sqrt{2}}$, with $n$ being odd integer, the cosine term becomes zero. For $n = 1$, the input state $\hat{a}^\dagger_{N+1}|000...0\rangle_1...N+1$ evolves to

$$|W\rangle_{N+1} = e^{i \pi/2 \sqrt{2}} \left[ i \sqrt{2/N} \left( |10...0\rangle_1...N|0\rangle_{N+1} + \ldots \right. \right. \right.$$  
$$+ \left. \left. |00...1\rangle_1...N|0\rangle_{N+1} \right] - \frac{i}{\sqrt{2}} |00...0\rangle_1...N|1\rangle_{N+1} \right] \right.$$  

$$+ \left. \left. |00...0\rangle_1...N|0\rangle_{N+1} \right] \right.$$  

(15)

In the 2D ring structure, the radius ($r$) of the ring and the nearest neighbor distance ($a$) between surrounding waveguides are related as $a = 2r \sin(\pi/N)$. Due to the condition, $N \kappa^2 = C^2$, the value of $r$ and hence the value of $a$ depend on the number of surrounding waveguides ($N$). This dependence can be found, by taking $\kappa = k e^{-r/d_0}$ and $C = k e^{-a/d_0}$ ($k$ is characteristic coupling strength and $d_0$ describes the rate of exponential decay of coupling strength), as

$$\frac{r}{d_0} = \frac{\log_e(\sqrt{N})}{1 - 2 \sin(\pi/N)} \quad \text{with} \quad N > 6$$  

(16)

For example, when $N = 7$, we have, $r \approx 7.35791 \; d_0$, $a \approx 6.38496 \; d_0$, $C \approx 1.6867 \times 10^{-3} \; k$ and $\kappa \approx 0.6375 \times 10^{-3} \; k$. The probabilities as function of $kz$ are shown in Fig. 3. The blue and green curves represent the probabilities of finding the photon at the central waveguide and surrounding waveguides, respectively. The red curve represents the probability of finding the photon at a specific surrounding waveguide. The values of $kz$ where the blue and green curves intersect, the 8-mode single-photon perfect W-state can be generated by introducing appropriate phase shifts at individual waveguides.

It can be verified that the ratio $a/d_0$ decreases as $N$ is increased. Hence, in this scheme, $N$ cannot have very large values. In addition, for $N > 12$, the second nearest neighbor distance between surrounding waveguides becomes smaller than $r$ which can be verified from the geometry of 2D ring structure (Fig. 2) and hence the higher order coupling between surrounding waveguides are not ignorable. It can be noted that in the coupling scheme, $\kappa = C$, to generate single-photon symmetric W-state [35], one can have only 6 surrounding waveguides.

Further, it can be noted that the Hamiltonian (Eq. 11) with $C = 0$ can be used to describe the intercavity hopping [68] of a single photon in a system consisting of $N + 1$ cavities in which $N$ cavities are coupled to a cavity through optical fibers. The fiber modes can be eliminated by adiabatic elimination process [69–73]. This system can be used to generate $N + 1$-cavity mode single-photon perfect W-state.

Earlier, cavity QED schemes to prepare single-photon cavity mode symmetric W-state were proposed [74–76]. These schemes involve interaction of an atom in the excited state with multiple cavities in the vacuum state. In these schemes, atom-cavity field interaction time has to be precisely controlled to generate the desired entangled states. The lifetime of created entangled states and hence the implementation of any application is limited by the damping time of cavities. Hence, these schemes require high $Q$ cavities. As compared to these cavity QED schemes, generation of single-photon path entangled states using
waveguide systems is more feasible. Generation of 16-mode single-photon symmetric W-state using waveguides has been reported [32]. Hence, the schemes that we have proposed using weakly coupled waveguide structures are more promising to generate single-photon path entangled perfect W-states.

4 Effect of dissipation

In the last two sections, we considered the generation of generalized \(N\)-mode single-photon perfect W-states using 1D and 2D waveguide structures without including dissipation. In this section, we study the effect of photon loss on generating single-photon perfect W-states by solving the following master equation.

\[
\dot{\rho} = -\frac{i}{\hbar} [\hat{H}, \rho] - \frac{\gamma}{2} \sum_{k=1}^{N} (\hat{a}_i^\dagger \hat{a}_i \rho + \rho \hat{a}_i^\dagger \hat{a}_i - 2 \hat{a}_i \rho \hat{a}_i^\dagger) \quad (17)
\]

where \(\gamma\) is the dissipation strength.

Equation 17 is solved, within the subspace spanned by single-photon states and vacuum state, for 1D array of four and five waveguides with coupling constants given in Table 1. Density matrices, at the propagation distance \(z\), mentioned in Table 1, are obtained for both cases. The fidelity \((F)\) of the obtained density matrices is calculated using the formula,

\[
F = Tr(\sigma \rho) \quad (18)
\]

where \(\sigma\) is the density matrix of the target state. For 1D array of four (five) waveguides, \(\sigma\) is the density matrix corresponding to state \(|\tilde{W}\rangle_4\) (\(|\tilde{W}\rangle_5\)) given in Eq. 8 (Eq. 10).

The master equation (Eq. 17) is also solved for 2D ring structure with seven waveguides on the circumference and one at the center. The coupling strengths are chosen as \(C = \frac{\pi}{2\sqrt{2}} \text{ cm}^{-1}\) and \(\kappa = \frac{\pi}{2\sqrt{14}} \text{ cm}^{-1}\). Density matrix is obtained for \(z = 1\) cm and fidelity is calculated by choosing the density matrix corresponding to the state \(|W'\rangle_{N+1}\) (Eq. 15) with \(N = 7\) as target density matrix \((\sigma)\).
Fidelity (F) as function of dissipation strength (γ) is shown in Fig. 4 for all three waveguide structures. For 2D ring structure, 1D array of four and five waveguides, the values of fidelity at γ = 0.01 cm⁻¹ are 0.990382, 0.988562 and 0.987694, respectively. For all three waveguide structures, the fidelity is above 0.8 up to γ ≈ 0.2 cm⁻¹. Hence, the generation scheme is robust in the presence of photon loss.

5 Generalized entanglement detection conditions

Generalized N-mode single-photon perfect W-states belonging to W-class are not biseparable and retain pairwise entanglement. Hence, the entanglement detection condition proposed in Ref. [66] can be used to detect the entanglement of generalized N-mode single-photon perfect W-states. However, there are certain generalized N-mode single-photon perfect W-states which will not satisfy that condition. For example, the generalized 5-mode single-photon perfect W-state, $|\tilde{W}\rangle_5 = \frac{1}{2\sqrt{2}}(|10000\rangle + |01000\rangle - |00100\rangle - |00010\rangle + 2|00001\rangle)$, will not satisfy the following 5-mode entanglement condition [66].

$$\langle (\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4)^{\dagger} \rangle^2 > 4 \langle \hat{N}_{\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4} \rangle$$

where

$$\hat{N}_{\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4} = \frac{(\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4)^{\dagger}(\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4)}{4}.$$

Hence, we derive a new set of entanglement conditions suitable to detect the entanglement of generalized N-mode single-photon perfect W-states. From the form of generalized N-mode perfect W-states [Eq. 1], we consider the following operators.

$$\hat{L}_1 = \sum_{j=1}^{N-1} [\alpha_j \hat{a}_j^{\dagger} \hat{a}_N + \alpha_j^* \hat{a}_N^{\dagger} \hat{a}_j]$$
\[ \hat{L}_2 = \sum_{j=1}^{N-1} \left[ i\alpha_j \hat{a}_j \hat{a}_N^\dagger - i\alpha_j^* \hat{a}_j^\dagger \hat{a}_N \right] \]

and

\[ \hat{L}_3 = \sum_{j,k=1}^{N-1} \left[ \alpha_j^* \alpha_k \hat{a}_j^\dagger \hat{a}_k - \hat{N} \hat{a}_N \right] \tag{20} \]

where \( \alpha_j \)'s are nonzero complex coefficients as described in the Introduction. For the given set of \( \alpha_j \)'s, these three operators satisfy \( su(2) \) algebra,

\[ \left[ \hat{L}_x, \hat{L}_y \right] = 2i\epsilon_{xyz} \hat{L}_z \]

with \( x,y,z = 1,2,3 \). The sum of variances of \( \hat{L}_1 \) and \( \hat{L}_2 \) can be written as

\[ (\Delta \hat{L}_1)^2 + (\Delta \hat{L}_2)^2 = 2 \left[ \sum_{j,k=1}^{N-1} \langle \alpha_j^* \alpha_k \hat{a}_j^\dagger \hat{a}_k \rangle + \langle \hat{N} \hat{a}_N \rangle \right] \]

\[ + 4 \left[ \sum_{j,k=1}^{N-1} \langle \alpha_j^* \alpha_k \hat{a}_j^\dagger \hat{a}_k \hat{N} \hat{a}_N \rangle - \sum_{j=1}^{N-1} \langle \alpha_j \hat{a}_j \hat{a}_N^\dagger \rangle \right] \tag{21} \]

Following the arguments given in Ref. [66], it can be shown that the following inequality relation is satisfied by fully separable states.

\[ (\Delta \hat{L}_1)^2 + (\Delta \hat{L}_2)^2 \geq 2 \left[ \sum_{j,k=1}^{N-1} \langle \alpha_j^* \alpha_k \hat{a}_j^\dagger \hat{a}_k \rangle + \langle \hat{N} \hat{a}_N \rangle \right] \tag{22} \]

This inequality relation can be shown to be valid for a density matrix, \( \rho = \sum_j p_j \rho_j \), where all \( \rho_j \)'s are fully separable states, as shown in [62]. Hence, violation of this inequality relation implies that the state is entangled and the entanglement condition can be written as

\[ \left| \sum_{j=1}^{N-1} \langle \alpha_j \hat{a}_j \hat{a}_N^\dagger \rangle \right|^2 > \sum_{j,k=1}^{N-1} \langle \alpha_j^* \alpha_k \hat{a}_j^\dagger \hat{a}_k \hat{N} \hat{a}_N \rangle \tag{23} \]

The entanglement condition proposed in Ref. [66] is a special case \( \alpha_j = 1/\sqrt{N-1}, \forall j \) of this condition. Genuine entanglement of multimode states which retain pairwise entanglement between all possible pairs of modes, referred as multimode W-type entangled states in Ref. [66], can be detected using this set of conditions. These states include both single-photon and multiphoton states. Furthermore, it is possible to obtain higher-order entanglement conditions, as discussed in Refs. [62, 66], to detect the genuine entanglement of more W-type multiphoton entangled states.

Since the \( su(2) \) algebra operators (Eq. 20) conserve the total number of photons, we now consider the action of these operators only on the single-photon subspace. Any genuinely
entangled $N$-mode single-photon state of the form

$$|\Psi\rangle = \sum_{j=1}^{N} C_j |1\rangle_j |0\ldots0\rangle_{1..j-1,j+1..N}.$$

(24)

with $C_j = |C_j| e^{i\varphi_j}$ and $\sum_{j=1}^{N} |C_j|^2 = 1$, can be rewritten in two different ways as (global phase term $e^{i\varphi_N}$ is excluded)

$$|\Psi\rangle = \pm \sqrt{\frac{1 + \lambda}{2}} \left[ \sum_{j=1}^{N-1} \alpha_j^* |1\rangle_j |0\ldots0\rangle_{1..j-1,j+1..N} \right] + \sqrt{\frac{1 - \lambda}{2}} |00\ldots01\rangle_{1..N}$$

(25)

where

$$\lambda = 1 - 2|C_N|^2 \quad \text{and} \quad \alpha_j^* = \frac{\pm C_j e^{-i\varphi_N}}{\sqrt{\sum_{j=1}^{N-1} |C_j|^2}}.$$  

(26)

Since $|C_N|$ is neither 1 nor 0, the value of $|\lambda|$ is less than one.

From Eqs. 20 and 25, it can be verified that for the state, $|\Psi\rangle$, we have $(\Delta \hat{L}_1)^2 = \lambda^2$, $(\Delta \hat{L}_2)^2 = 1$, and $\langle \hat{L}_3 \rangle = \lambda$. Hence, the state, $|\Psi\rangle$, is $\hat{L}_1$-squeezed state. That is, it satisfies the following conditions [64, 77].

$$(\Delta \hat{L}_1)(\Delta \hat{L}_2) = |\langle \hat{L}_3 \rangle|$$

$$(\Delta \hat{L}_1)^2 = |\lambda \langle \hat{L}_3 \rangle| < |\langle \hat{L}_3 \rangle|$$

and

$$(\Delta \hat{L}_2)^2 = \frac{\langle \hat{L}_3 \rangle}{\lambda} > |\langle \hat{L}_3 \rangle|$$

(27)

Further, the state $|\Psi\rangle$ can be written as superposition of two orthonormal generalized $N$-mode single-photon perfect W-states. That is,

$$|\Psi\rangle = D_1 |\tilde{W}\rangle_{N+} + D_2 |\tilde{W}\rangle_{N-}$$

(28)

with

$$D_1 = \frac{\sqrt{1 - \lambda} \pm \sqrt{1 + \lambda}}{2},$$

$$D_2 = \frac{\sqrt{1 - \lambda} \mp \sqrt{1 + \lambda}}{2},$$

and

$$|\tilde{W}\rangle_{N\pm} = \pm \frac{1}{\sqrt{2}} \left[ \sum_{j=1}^{N-1} \alpha_j^* |1\rangle_j |0\ldots0\rangle_{1..j-1,j+1..N} \right] + \frac{1}{\sqrt{2}} |00\ldots01\rangle_{1..N}$$

(29)
It can be verified that the states $|\tilde{W}\rangle_{N\pm}$ are eigenstates of $\hat{L}_1$ corresponding to the eigenvalues $\pm 1$.

The $\hat{L}_2$-squeezed states corresponding to the $\hat{L}_1$-squeezed states given in Eq. 24 can be written as

$$|\Psi'\rangle = -i \sum_{j=1}^{N-1} C_j |1\rangle_j |0\ldots 0\rangle_{1..j-1,j+1..N} + C_N |00\ldots 01\rangle_{1..N}$$  \hspace{1cm} (30)

The relation between $\alpha_j$'s and $C_j$'s, and the squeezing parameter $\lambda$ and $C_N$ are the same as given in Eq. 26. Similar to $\hat{L}_1$-squeezed states, the $\hat{L}_2$-squeezed states can be expressed as linear combination of two orthonormal generalized $N$-mode single-photon perfect W-states as follows.

$$|\Psi'\rangle = D_1 |W'\rangle_+ + D_2 |W'\rangle_-$$ \hspace{1cm} (31)

where

$$|W'\rangle_{N\pm} = \mp i \frac{1}{\sqrt{2}} \left[ \sum_{j=1}^{N-1} \alpha_j^* |1\rangle_j |0\ldots 0\rangle_{1..j-1,j+1..N} + \frac{1}{\sqrt{2}} |00\ldots 01\rangle_{1..N} \right]$$  \hspace{1cm} (32)

are eigenstates of $\hat{L}_2$ corresponding to the eigenvalues $\pm 1$.

Further the action of operators involved in the separability condition [Eq. 22], within the single-photon subspace, can be represented by the action of the projection operators as shown below.

$$\hat{L}_1 |\Phi\rangle = \left[ |\tilde{W}\rangle_{N++N} \langle \tilde{W}| - |\tilde{W}\rangle_{N--N} \langle \tilde{W}| \right] |\Phi\rangle$$ \hspace{1cm} (33)

$$\hat{L}_2 |\Phi\rangle = \left[ |W'\rangle_{N++N} \langle W'| - |W'\rangle_{N--N} \langle W'| \right] |\Phi\rangle$$ \hspace{1cm} (34)

$$\left[ \sum_{j,k=1}^{N-1} \alpha_j^* \alpha_k \hat{a}_j^\dagger \hat{a}_k + \hat{N}_\alpha \right] |\Phi\rangle = \left[ |\tilde{W}\rangle_{N++N} \langle \tilde{W}| + |\tilde{W}\rangle_{N--N} \langle \tilde{W}| \right] |\Phi\rangle$$

$$= \left[ |W'\rangle_{N++N} \langle W'| + |W'\rangle_{N--N} \langle W'| \right] |\Phi\rangle$$ \hspace{1cm} (35)

where $|\Phi\rangle$ is generated $N$-mode single-photon state.

In terms of these projection operators, the separability condition given in Eq. 22 can be rewritten as

$$\left[ |\langle \Phi | \tilde{W}\rangle_{N+}|^2 - |\langle \Phi | \tilde{W}\rangle_{N-}|^2 \right]^2 + \left[ |\langle \Phi | W'\rangle_{N+}|^2 - |\langle \Phi | W'\rangle_{N-}|^2 \right]^2 \leq 0 \hspace{1cm} (36)$$

We refer this condition as single-photon separability condition. The two terms on the left hand side (LHS) of single-photon separability condition are always positive, and hence, the terms on the LHS being not equal to zero indicate that the single-photon state, $|\Phi\rangle$, is
entangled. It can be verified that for $\hat{L}_1(\hat{L}_2)$-squeezed state, $|\Psi\rangle$ ($|\Psi'\rangle$), the second (first) term on the LHS of single-photon separability condition becomes zero and that condition becomes

$$1 - \lambda^2 \leq 0.$$  \hfill (37)

With $\lambda \in (-1, 1)$, the single-photon separability condition is violated by $\hat{L}_1$ and $\hat{L}_2$-squeezed states. Hence, in order to verify the entanglement of any given genuinely entangled $N$-mode single-photon state, one has to find the difference between the fidelities of the given $N$-mode single-photon state with properly chosen set of two orthonormal generalized $N$-mode single-photon perfect $W$-states.

The single-photon separability condition (Eq. 36) can also be used to detect the entanglement of single-photon mixed states and it requires finding the expectation value of the projection operators $|\tilde{W}\rangle_{N_{+}\pm N} \langle \tilde{W}|$ and $|\tilde{W}'\rangle_{N_{+}\pm N} \langle \tilde{W}'|$. To illustrate this, we consider four different single-photon mixed states. The mixed states and their separability conditions are given in Table 2.

The mixed state $\rho_1$ is the mixture of eigenstates of $\hat{L}_1$ corresponding to the eigenvalues $\pm 1$. It can be noted that the separability condition is violated for all values of $p$, except at $p = 1/2$. When $p = 1/2$, we have

$$\rho_1 = \frac{1}{2} |\psi_1\rangle \langle \psi_1| + \frac{1}{2} |\psi_2\rangle \langle \psi_2|,$$  \hfill (38)

with $|\psi_1\rangle = \sum_{j=1}^{-1} \alpha_j^+ |1\rangle |0\ldots0\rangle_{1\ldots j-1,j+1,N}$ and $|\psi_2\rangle = |00\ldots01\rangle_{1\ldots N}$. In the state, $|\psi_1\rangle$, the $N$th mode is separable and the state $|\psi_2\rangle$ is fully separable.

The mixed state $\rho_2$ is the mixture of eigenstates of $\hat{L}_1$ and $\hat{L}_2$ corresponding to the eigenvalue $+1$. For $\rho_2$, the separability condition is violated for all values of $p$. Hence the state, $\rho_2$, is entangled for values of $p$.

The mixed state $\rho_3$ is the mixture of $\hat{L}_1$-squeezed and $\hat{L}_2$-squeezed states. Since, $[1 - 2p(p - 1)] > 0$, $\forall p \in [0, 1]$, and $1 - \lambda^2 > 0$, $\forall \lambda \in (-1, 1)$, the separability condition is violated and the state $\rho_3$ is entangled for all values of $p$ and $\lambda$.

In the mixed state, $\rho_4$, $P_0$ and $P_k$ ($\forall k = 1, 2, \ldots, N$) denote the projection operators onto the states, $|00\ldots00\rangle_{1\ldots N}$ and $|1\rangle_k |0\ldots0\rangle_{1\ldots k-1,k+1\ldots N}$, respectively. The state $|\Psi\rangle$ is $\hat{L}_1$-squeezed state (Eq. 28). For this mixed state, the separability condition is violated for all values of $p$ except at $p = 0$.

### Table 2  Separability condition for four different single-photon mixed states. For all four mixed states, we have $0 \leq p \leq 1$

| Mixed state | Separability condition |
|-------------|------------------------|
| $\rho_1 = p|\tilde{W}\rangle_{N_{+}N_{-}} \langle \tilde{W}| + (1 - p)|\tilde{W}\rangle_{N_{-}N_{+}} \langle \tilde{W}|$ | $(2p - 1)^2 \leq 0$ |
| $\rho_2 = p|\tilde{W}\rangle_{N_{+}N_{+}} \langle \tilde{W}| + (1 - p)|\tilde{W}\rangle_{N_{-}N_{-}} \langle \tilde{W}|$ | $2p(1 - p) \geq 1$ |
| $\rho_3 = |\Psi\rangle \langle \Psi| - (1 - \lambda^2) \geq 0$ | $1 - 2p(1 - p)(1 - \lambda^2) \leq 0$ |
| $\rho_4 = p|\Psi\rangle \langle \Psi| + (1 - p)|\Psi\rangle \langle \Psi'|$ | $p^2(1 - \lambda^2) \leq 0$ |
Fig. 5 Integrated photonic circuit to verify the violation of separability conditions (Eqs. 22, 36). It has \((N-1)\) directional couplers (DC \(j\)’s) and \(N\) phase shifters (PS \(j\)’s) as described in the main text. The dashed line represents the repetition of the pattern of circuit from DC\(_3\) to DC\(_{N-2}\). D\(_1\) and D\(_2\) are photodetectors.

6 Experimental implementation of generalized entanglement detection conditions

We now consider the experimental verification of violation of separability condition proposed in the last section. In order to do that the operators \(\hat{L}_1, \hat{L}_2\) and the one on the right hand side of Eq. 22 have to be measured. An integrated photonic circuit to measure these operators is shown in Fig. 5. The circuit consists of \(N-1\) directional couplers and \(N\) phase shifters. This circuit can be fabricated on a chip.

The action of directional coupler DC\(_j\) on the input modes \(\hat{b}_{j-1}\) and \(\hat{a}_{j+1}\) is shown below

\[
\begin{pmatrix}
\hat{b}_j \\
\hat{c}_j
\end{pmatrix} = \begin{pmatrix} T_j & R_j \\ R_j & T_j \end{pmatrix} \begin{pmatrix}
\hat{b}_{j-1} \\
\hat{a}_{j+1}
\end{pmatrix} \tag{39}
\]

where \(T_j\) and \(R_j\) satisfy the conditions, \(|T_j|^2 + |R_j|^2 = 1\) and \(R_j^* T_j + T_j^* R_j = 0\).

The input modes of directional coupler DC\(_1\) are \(\hat{a}_1 = (\hat{b}_0)\) and \(\hat{a}_2\). The action of phase shifter PS\(_j\) on the mode \(\hat{a}_j\) is given by

\[
\hat{a}_j \rightarrow \hat{a}_j e^{-i\phi_j} \tag{40}
\]

From Eqs. 39 and 40, the mode \(\hat{b}_{N-2}\) can be written as follows

\[
\hat{b}_{N-2} = \sum_{j=1}^{N-1} \alpha_j \hat{a}_j \tag{41}
\]

where

\[
\alpha_1 = \prod_{k=1}^{N-2} T_{N-1-k} e^{-i\phi_1},
\]

\[
\alpha_j = \prod_{k=1}^{N-1-j} T_{N-1-k} R_{j-1} e^{-i\phi_j}, \quad j = 2, \ldots, N - 2
\]
and

\[ \alpha_{N-1} = R_{N-2}e^{-i\phi_{N-1}}. \]  

(42)

It can be verified that \( \sum_{j=1}^{N} |\alpha_j|^2 = 1. \)

For directional coupler, DC\(_{N-1}\), the values of \( R_{N-1} \) and \( T_{N-1} \) are chosen as \( i/\sqrt{2} \) and \( 1/\sqrt{2} \), respectively. The phase shift introduced by the phase shifter \( \text{PS}_N \) is denoted as \( \phi_N \). The output modes \( \hat{b}_{N-1} \) and \( \hat{c}_{N-1} \) of DC\(_{N-1}\) can be written as

\[ \hat{b}_{N-1} = \frac{\hat{b}_{N-2} + i\hat{a}_Ne^{-i\phi_N}}{\sqrt{2}} \]  

(43)

\[ \hat{c}_{N-1} = \frac{i\hat{b}_{N-2} + \hat{a}_Ne^{-i\phi_N}}{\sqrt{2}} \]  

(44)

For \( \phi_N = \pi/2 \) and \( \pi \), the photon number difference \( \hat{b}_{N-1}^\dagger\hat{b}_{N-1} - \hat{c}_{N-1}^\dagger\hat{c}_{N-1} \) gives the operators \( \hat{L}_1 \) and \( \hat{L}_2 \), respectively. Similarly, \( \hat{b}_{N-1}^\dagger\hat{b}_{N-1} + \hat{c}_{N-1}^\dagger\hat{c}_{N-1} \) yields the operator on the right hand side of Eq. 22 (excluding the factor 2) for any value of \( \phi_N \). Thus by measuring the sum and difference of photon numbers at photodetectors D\(_1\) and D\(_2\), the violation of separability condition can be verified.

Measurement of the two terms involved in the single-photon separability condition [Eq. 36] can be explained as follows. When the input state is \( |\tilde{W}\rangle_{N+} \) (\( |\tilde{W}\rangle_{N-} \)), the photonic circuit with \( \phi_N = \pi/2 \) does a unitary transformation such that the photon will be found in the mode \( \hat{b}_{N-1} \) (\( \hat{c}_{N-1} \)) at the output. This can be verified from Eq. 43 (Eq. 44). When the input state is one of the remaining \( N - 2 \) degenerate eigenstates of \( \hat{L}_1 \) corresponding to the eigenvalue 0, the photon will be found in one of the remaining \( N - 2 \) output modes which can be verified from the input–output relations of this circuit. The transformation of generated \( N \)-mode single-photon state, \( |\Phi\rangle \), can be found by expressing the state in the eigenbasis of \( \hat{L}_1 \). Thus, measuring the photon number difference at the photodetectors D\(_1\) and D\(_2\) gives the square root of the first term of Eq. 36. Similarly, the measurement of second term can be explained by choosing \( \phi_N = \pi \) and expressing the state, \( |\Phi\rangle \), in the eigenbasis of \( \hat{L}_2 \).

The values of \( R_j \)'s, \( T_j \)'s (\( j = 1, 2, ..., N - 2 \)) and \( \phi_k \)'s (\( k = 1, 2, ..., N - 1 \)) can be chosen suitably to detect the entanglement of a given generalized \( N \)-mode single-photon perfect W-state. For example, in the case of \( N \)-mode single-photon perfect W-state [Eq. 2], the values of \( R_j \) and \( T_j \) (\( j = 1, 2, 3, ..., N - 2 \)) have to be chosen as \( i/\sqrt{j+1} \) and \( \sqrt{j/(j+1)} \), respectively. The phase shift \( (\phi_j) \) introduced by the phase shifter \( \text{PS}_j \) for \( j = 2, 3, ..., N - 1 \) have to be \( \pi/2 \) and 0 for \( j = 1 \).

As compared to the single-photon entanglement detection scheme presented in Ref. [78], our scheme involves very much reduced measurement setup. In our scheme, in order to detect the entanglement of genuinely entangled single-photon states, the photon has to be detected only at two output modes irrespective of the number of input modes. It is also possible to experimentally characterize the unitary transformation performed by the photonic circuit (Fig. 5) as described in Ref. [46]. In the earlier experimental work [32], the events due to imperfections, discussed in Ref. [78], were shown to have negligible influence on the generation and verification of single-photon entangled states using waveguide structures, single-photon source based on spontaneous parametric down-conversion and
avalanche photodiodes. Hence, with reduced measurement setup, our entanglement detection scheme can easily be implemented and is more suitable for detecting the entanglement of genuinely entangled single-photon states with large number of modes. It can be noted that the application of entanglement condition given in Eq. 23, goes beyond single-photon entangled states as mentioned in the last section.

Finally, it can be verified from Eqs. 43 and 44 that the generalized \( N \)-mode single-photon perfect W-states can be generated, using the photonic circuit given in Fig. 5, by injecting a photon in either the mode \( \hat{b}_{N-1} \) or \( \hat{c}_{N-1} \).

7 Conclusion

We have proposed generation and entanglement verification schemes for generalized \( N \)-mode single-photon perfect W-states. The single-photon realization of generalized \( N \)-qubit perfect W-states are more advantageous than the other realizations [21, 26–28] as the single-photon states can be generated easily using waveguide structures. These structures are stable, scalable, and offer low photon propagation losses [79, 80]. The state generation schemes involve injecting a photon into one of \( N \) weakly coupled waveguides arranged in two different geometrical structures: 1D planar and 2D ring structures. 1D planar structure is scalable to generate large \( N \)-mode single-photon perfect W-states, whereas the 2D ring structure has restrictions on the number of surrounding waveguides.

The generalized entanglement detection conditions are obtained from sum uncertainty relations of generalized set of \( su(2) \) algebra operators. They are suitable to detect the entanglement of multimode W-type entangled states [66]. The entanglement verification of generalized \( N \)-mode single-photon perfect W-states can be performed using the single-photon separability conditions. As the generalized \( N \)-mode single-photon perfect W-states are single-photon eigenstates of \( su(2) \) algebra operators corresponding to nonzero eigenvalues, verification of entanglement of single-photon states requires finding fidelity with generalized \( N \)-mode perfect W-states. The genuinely entangled single-photon states are squeezed states of suitably chosen set of \( su(2) \) algebra operators and they violate the corresponding single-photon separability condition. Experimental implementation of entanglement conditions require doing measurement at two specific outputs of a photonic circuit consisting of directional couplers and phase shifters, irrespective of number of inputs. Thus, the proposed entanglement conditions are more suitable not only for entanglement verification of generalized \( N \)-mode single-photon perfect W-states but also to detect the entanglement of any genuinely entangled \( N \)-mode single-photon states with large \( N \).

The integrated photonic circuit used to implement the entanglement detection conditions can also be used to generate generalized \( N \)-mode single-photon perfect W-states. To the best of our knowledge, the generalized \( N \)-qubit perfect W-states with \( N > 3 \) are not generated experimentally. Hence, the waveguide structures are more promising candidates to generate generalized \( N \)-mode single-photon perfect states with large \( N \) than bulk optical elements [26] and superconducting qubits [24, 28] as the waveguide structures are already proved to be suitable to generate maximally entangled single-photon states [32].

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**Author Contributions** MS derived the results given in Sect. 2. MKS derived the results of Sects. 5 and 6. MKS and MS derived the results presented in Sects. 3 and 4 and prepared the manuscript. A.R and P.K.P verified the results and reviewed the manuscript.

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**References**

1. Borras, A., Plastino, A., Batle, J., Zander, C., Casas, M., Plastino, A.: Multiqubit systems: highly entangled states and entanglement distribution. J. Phys. A Math. Theor. **40**(44), 13407 (2007)
2. Facchi, P., Florio, G., Parisi, G., Pascazio, S.: Maximally multipartite entangled states. Phys. Rev. A **77**(6), 060304 (2008)
3. Enríquez, M., Wintrowicz, I., Życzkowski, K.: Maximally entangled multipartite states: a brief survey. J. Phys. Conf. Ser. **698**, 012003 (2016)
4. Miyake, A.: Classification of multipartite entangled states by multidimensional determinants. Phys. Rev. A **67**(1), 012108 (2003)
5. Koashi, M., Bužek, V., Imoto, N.: Entangled webs: tight bound for symmetric sharing of entanglement. Phys. Rev. A **62**(5), 050302 (2000)
6. Kafatos, M.: Bell’s Theorem, Quantum Theory and Conceptions of the Universe Bell’s theorem, Quantum Theory and Conceptions of the Universe, vol. 37. Springer, New York (2013)
7. Dür, W., Vidal, G., Cirac, J.I.: Three qubits can be entangled in two inequivalent ways. Phys. Rev. A **62**(6), 062314 (2000)
8. Dür, W., Aschauer, H., Briegel, H.J.: Multiparticle entanglement purification for graph states. Phys. Rev. Lett. **91**(10), 107903 (2003)
9. Hein, M., Eisert, J., Briegel, H.J.: Multiparty entanglement in graph states. Phys. Rev. A **69**(6), 062311 (2004)
10. Nielsen, M.A.: Cluster-state quantum computation. Rep. Math. Phys. **57**(1), 147 (2006)
11. Kempe, J.: Multiparticle entanglement and its applications to cryptography. Phys. Rev. A **60**(2), 910 (1999)
12. Yeo, Y., Chua, W.K.: Teleportation and dense coding with genuine multipartite entanglement. Phys. Rev. Lett. **96**(6), 060502 (2006)
13. Pirandola, S., Eisert, J., Weedbrook, C., Furusawa, A., Braunstein, S.L.: Advances in quantum teleportation. Nat. Photon. **9**(10), 641 (2015)
14. Muralidharan, S., Panigrahi, P.K.: Perfect teleportation, quantum-state sharing, and superdense coding through a genuinely entangled five-qubit state. Phys. Rev. A **77**(3), 032321 (2008)
15. Choudhury, S., Muralidharan, S., Panigrahi, P.K.: Quantum teleportation and state sharing using a genuinely entangled six-qubit state. J. Phys. A. Math. Theor. **42**(11), 115303 (2009)
16. Saha, D., Panigrahi, P.K.: N-qubit quantum teleportation, information splitting and superdense coding through the composite GHZ-Bell channel. Quant. Inf. Process. **11**(2), 615 (2012)
17. Bennett, C.H., Brassard, G., Crépeau, C., Jozsa, R., Peres, A., Wootters, W.K.: Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels. Phys. Rev. Lett. **70**(13), 1895 (1993)
18. Karlsson, A., Bourennane, M.: Quantum teleportation using three-particle entanglement. Phys. Rev. A **58**(6), 4394 (1998)
19. Kumar, A., Haddadi, S., Pourkarim, M.R., Behera, B.K., Panigrahi, P.K.: Experimental realization of controlled quantum teleportation of arbitrary qubit states via cluster states. Sci. Rep. **10**(1), 1 (2020)
20. Agrawal, P., Pati, A.: Perfect teleportation and superdense coding with W states. Phys. Rev. A **74**(6), 062320 (2006)
21. Rao, D.B., Ghosh, S., Panigrahi, P.K.: Generation of entangled channels for perfect teleportation using multielectron quantum dots. Phys. Rev. A **78**(4), 042328 (2008)
22. Dong, L., Wang, J.X., Li, Q.Y., Shen, H.Z., Dong, H.K., Xiu, X.M., Gao, Y.J., Oh, C.H.: Nearly deterministic preparation of the perfect W state with weak cross-Kerr nonlinearities. Phys. Rev. A **93**(1), 012308 (2016)
23. Xiu, X.M., Cui, C., Lin, Y.F., Dong, L., Dong, H.K., Gao, Y.J.: Splitting and acquiring quantum information with perfect states based on weak cross-Kerr nonlinearities. Progress of Theoretical and Experimental Physics 1, 013A03 (2018)
24. Li, X.K., Zhou, Y., Wang, G.H., Lv, D.Y., Badshah, F., Huang, H.M.: Generation of microwave photon perfect W states of three coupled superconducting resonators. Chin. Phys. B 32(4), 040306 (2022)
25. Li, L., Qiu, D.: The states of W-class as shared resources for perfect teleportation and superdense coding. J. Phys. A Math. Theor. 40(35), 10871 (2007)
26. Li, K., Kong, F.Z., Yang, M., Ozaydin, F., Yang, Q., Cao, Z.L.: Generating multi-photon W-like states for perfect quantum teleportation and superdense coding. Quantum Inf. Process. 15(8), 3137 (2016)
27. Zhao-Hui, P., Chun-Xia, J., Jun-Gang, L.: Scheme for implementing perfect quantum teleportation with non-maximally entangled W-class state in cavity QED. Commun. Theor. Phys. 50(2), 375 (2008)
28. Swain, M., Devrari, V., Rai, A., Behera, B.K., Panigrahi, P.K.: Generation of perfect W-state and demonstration of its application to quantum information splitting. arXiv preprint arXiv:2006.01742 (2020)
29. Van Enk, S.: Single-particle entanglement. Phys. Rev. A 72(6), 064306 (2005)
30. Morin, O., Bancal, J.D., Ho, M., Sekatski, P., D’Auria, V., Gisin, N., Laurat, J., Sangouard, N.: Witnessing trustworthy single-photon entanglement with local homodyne measurements. Phys. Rev. Lett. 110(13), 130401 (2013)
31. Shi, J., Xu, P., Zhong, M., Gong, Y., Bai, Y., Yu, W., Li, Q., Jin, H., Zhu, S.: Heralded generation of multipartite entanglement for one photon by using a single two-dimensional nonlinear photonic crystal. Opt. Express 21(7), 7875 (2013)
32. Gräfe, M., Heilmann, R., Perez-Leija, A., Keil, R., Dreisow, F., Heinrich, M., Moya-Cessa, H., Nolte, S., Christodoulides, D.N., Szameit, A.: On-chip generation of high-order single-photon W-states. Nat. Photon. 8(10), 791 (2014)
33. Monteiro, F., Vivoli, V.C., Guerreiro, T., Martin, A., Bancal, J.D., Zbinden, H., Thew, R.T., Sangouard, N.: Revealing genuine optical-path entanglement. Phys. Rev. Lett. 114(17), 170504 (2015)
34. Capar, P., Verbanis, E., Oudot, E., Maring, N., Samara, F., Caloz, M., Perrenoud, M., Sekatski, P., Martin, A., Sangouard, N., et al.: Heralded distribution of single-photon path entanglement. Phys. Rev. Lett. 125(11), 110506 (2020)
35. Perez-Leija, A., Hernandez-Herrejon, J., Moya-Cessa, H., Szameit, A., Christodoulides, D.N.: Generating photon-encoded W states in multiport waveguide-array systems. Phys. Rev. A 87(1), 013842 (2013)
36. Hessmo, B., Usachev, P., Heydari, H., Björk, G.: Experimental demonstration of single photon nonlocality. Phys. Rev. Lett. 92(18), 180401 (2004)
37. White, S.J., Klauck, F., Tran, T.T., Schmitt, N., Kianinia, M., Steinfurth, A., Heinrich, M., Toth, M., Szameit, A., Aharonovich, I., et al.: Quantum random number generation using a hexagonal boron nitride single photon emitter. J. Opt. 23(1), 01LT01 (2020)
38. Chen, X., Greiner, J.N., Wrachtrup, J., Gerhardt, I.: Single photon randomness based on a defect center in diamond. Sci. Rep. 9(1), 1 (2019)
39. Luo, Q., Cheng, Z., Fan, J., Tan, L., Song, H., Deng, G., Wang, Y., Zhou, Q.: Quantum random number generator based on single-photon emitter in gallium nitride. Opt. Lett. 45(15), 4224 (2020)
40. Gottesman, D., Jennewein, T., Croke, S.: Longer-baseine telescopes using quantum repeaters. Phys. Rev. Lett. 109(7), 070503 (2012)
41. Rai, A., Agarwal, G.: Possibility of coherent phenomena such as Bloch oscillations with single photons via W states. Phys. Rev. A 79(5), 053849 (2009)
42. Szameit, A., Nolte, S.: Discrete optics in femtosecond-laser-written photonic structures Journal of Physics B: Atomic. Mol. Opt. Phys. 43(16), 163001 (2010)
43. Meany, T., Gräfe, M., Heilmann, R., Perez-Leija, A., Gross, S., Steel, M.J., Withford, M.J., Szameit, A.: Laser written circuits for quantumphotons. Laser Photon. Rev. 9(4), 363 (2015)
44. Keil, R., Noh, C., Rai, A., Stützer, S., Nolte, S., Angelakis, D.G., Szameit, A.: Optical simulation of charge conservation violation and Majorana dynamics. Optica 2(5), 454 (2015)
45. Rai, A., Lee, C., Noh, C., Angelakis, D.G.: Photonic lattice simulation of dissipation-induced correlations in bosonic systems. Sci. Rep. 5(1), 1 (2015)
46. Heilmann, R., Gräfe, M., Nolte, S., Szameit, A.: A novel integrated quantum circuit for high-order W-state generation and its highly precise characterization. Sci. Bull. 60(1), 96 (2015)
47. Rai, A., Das, S., Agarwal, G.: Quantum entanglement in coupled lossy waveguides. Opt. Express 18(6), 6241 (2010)
48. Swain, M., Rai, A., Selvan, M.K., Panigrahi, P.K.: Single photon generation and non-locality of perfect W-state. J. Opt. 22(7), 075202 (2020)
49. Wootters, W.K.: Entanglement of formation of an arbitrary state of two qubits. Phys. Rev. Lett. 80(10), 2245 (1998)
50. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. 81(2), 865 (2009)
51. Ma, Z.H., Chen, Z.H., Chen, J.L., Spengler, C., Gabriel, A., Huber, M.: Measure of genuine multipartite entanglement with computable lower bounds. Phys. Rev. A 83(6), 062325 (2011)
52. Rafa-sanjani, S.H., Huber, M., Broadbent, C.J., Eberly, J.H.: Genuinely multipartite concurrence of N-qubit X matrices. Phys. Rev. A 86(6), 062303 (2012)
53. Coffman, V., Kundu, J., Wootters, W.K.: Distributed entanglement. Phys. Rev. A 61(5), 052306 (2000)
54. Koashi, M., Winter, A.: Monogamy of quantum entanglement and other correlations. Phys. Rev. A 69(2), 022309 (2004)
55. Zhu, X.N., Fei, S.M.: Generalized monogamy relations of concurrence for N-qubit systems. Phys. Rev. A 92(6), 062345 (2015)
56. Peres, A.: Separability criterion for density matrices. Phys. Rev. Lett. 77(8), 1413 (1996)
57. Horodecki, M., Horodecki, P., Horodecki, R.: On the necessary and sufficient conditions for separability of mixed quantum states. Phys. Lett. A 223(1) (1996)
58. Terhal, B.M.: Detecting quantum entanglement. Theor. Comput. Sci. 287(1), 313 (2002)
59. Bourennane, M., Eibl, M., Kurtsiefer, C., Gaertner, S., Weinfurter, H., Gühne, O., Hyllus, P., Bruß, D., Lewenstein, M., Sanpera, A.: Experimental detection of multipartite entanglement using witness operators. Phys. Rev. Lett. 92(8), 087902 (2004)
60. Gühne, O., Tóth, G.: Entanglement detection. Phys. Rep. 474(1–6), 1 (2009)
61. Agarwal, G.S., Biswas, A.: Inseparability inequalities for higher order moments for bipartite systems. New J. Phys. 7(1), 211 (2005)
62. Hillery, M., Zubairy, M.S.: Entanglement conditions for two-mode states. Phys. Rev. Lett. 96(5), 050503 (2006)
63. Hillery, M., Zubairy, M.S.: Entanglement conditions for two-mode states: applications. Phys. Rev. A 74(3), 032333 (2006)
64. Nha, H., Kim, J.: Entanglement criteria via the uncertainty relations in su(2) and su(1, 1) algebras: detection of non-Gaussian entangled states. Phys. Rev. A 74(1), 012317 (2006)
65. Nha, H.: Linear optical scheme to demonstrate genuine multipartite entanglement for single-particle W states. Phys. Rev. A 77(6), 062328 (2008)
66. Selvan, K., Panigrahi, P.K.: Entanglement condition for W type multimode states and schemes for experimental realization. Eur. Phys. J. D 73(6), 1 (2019)
67. Matthews, J.C., Politi, A., Stefano, A., O'brien, J.L.: Manipulation of multiphoton entanglement in waveguide quantum circuits. Nat. Photon. 3(6), 346 (2009)
68. Meher, N., Sivakumar, S., Panigrahi, P.K.: Duality and quantum state engineering in cavity arrays. Sci. Rep. 7(1), 1 (2017)
69. Pellizzari, T.: Quantum networking with optical fibres. Phys. Rev. Lett. 79(26), 5242 (1997)
70. Van Enk, S., Kimble, H., Cirac, J., Zoller, P.: Quantum communication with dark photons. Phys. Rev. A 59(4), 2659 (1999)
71. Lu, D.M., Chen, L.H.: Geometrical quantum discord in the coupled cavities system with tetrahedral structure. Int. J. Theor. Phys. 58(2), 605 (2019)
72. Kyoseva, E., Beige, A., Kwek, L.C.: Coherent cavity networks with complete connectivity. New J. Phys. 14(2), 023023 (2012)
73. Cho, J., Angelakis, D.G., Bose, S.: Heralded generation of entanglement with coupled cavities. Phys. Rev. A 78(2), 022323 (2008)
74. Bergou, J.A.: Entangled fields in multiple cavities as a testing ground for quantum mechanics. Found. Phys. 29(4), 503 (1999)
75. Guo, G.C., Zhang, Y.S.: Scheme for preparation of the W state via cavity quantum electrodynamics. Phys. Rev. A 65(5), 054302 (2002)
76. Yang, M., Yi, Y.M., Cao, Z.L.: Scheme for preparation of W state via cavity QED. Int. J. Quant. Inf. 2(02), 231 (2004)
77. Hillery, M., Mlodinow, L.: Interferometers and minimum-uncertainty states. Phys. Rev. A 48(2), 1548 (1993)
78. Lougovski, P., van Enk, S.J., Choi, K.S., Papp, S.B., Deng, H., Kimble, H.: Verifying multipartite mode entanglement of W states. New J. Phys. 11(6), 063029 (2009)
79. Chen, Z., Zhou, Y., Shen, J.T.: Exact dissipation model for arbitrary photonic Fock state transport in waveguide QED systems. Opt. Lett. 42(4), 887 (2017)
80. Chen, Z., Zhou, Y., Shen, J.T.: Entanglement-preserving approach for reservoir-induced photonic dissipation in waveguide QED systems. Phys. Rev. A 98(5), 053830 (2018)

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