ON THE BEREZIN TRANSFORMS ON LINE BUNDLES OVER THE COMPLEX HYPERBOLIC SPACES

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Abstract. We define a generalized Berezin transforms on line bundle over the complex hyperbolic space $\mathbb{B}^n = S(n,1)/S(U(n) \times U(n)$, and we give it as a functions of the $G$-invariant laplacian on the line bundles.

1. Introduction

Berezin transform [2] is of particular interest both in quantum theory and operator theory [8], [21]. In a series of papers [3], [1] and references theirs in, F. Berezin introduce a new approach to quantization of Khaier manifolds, based on reproducing kernel function theory. Since then many authors have been interested in the so-called Berezin quantization.

Classically the Berezin transform is defined as follows. Consider a domain $\Omega \subset \mathbb{C}^n$ and a Borel measure $d\mu$ on $\Omega$. Let $\mathcal{H}$ be a closed subspace of $L^2(\Omega, d\mu)$ consisting of continuous function and assume that $\mathcal{H}$ has a reproducing kernel $K(.,.)$. The Berezin symbol $\hat{A}$ of a bounded operator $A$ on $\mathcal{H}$ is the function defined on $\Omega$ by

$$\hat{A}(z) = \frac{\langle AK(.,z), K(.,z) \rangle}{K(z,z)}, \quad z \in \Omega. \tag{1.1}$$

For each $\varphi$ such that $\varphi \mathcal{H} \in L^2(\Omega, d\mu)$-for instance for any $\varphi \in L^\infty(\Omega)$, the Toeplitz operator $T_\varphi$ with symbol $\varphi$ is the operator on $\mathcal{H}$ given by $T_\varphi[f] = P(f \varphi)$; $f \in \mathcal{H}$, where $P$ is the orthogonal projector on $\mathcal{H}$. By definition the Berezin transform $B$ is the integral transform defined by

$$B[\varphi](z) := \hat{T}_\varphi(z) = \int_\Omega \frac{|K(z,\omega)|^2}{K(z,z)} \varphi(\omega) d\mu(\omega). \tag{1.2}$$

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The formula representing the Berezin transform as a function of Laplace Beltrami operator plays an important role in the Berezin quantization theory \[7\].

In the case of a bounded symmetric domain \( \mathcal{D} = G/K \), the Berezin transform intertwines with the group actions. Therefore it is a function, in the sense of the functional calculus for commuting self-adjoint operators, of the \( G \)-invariant differential operators \( \Delta_1, ..., \Delta_r \) generating the algebra of all \( G \)-invariant differential operators on \( \mathcal{D} \). This idea was carried out by Berezin \[3\] in the rank one case (without proof) and proved by Unterberger and Upmeier \[19\] for the general case in the strongest, spectral-theoretic, sense \[20\].

Now, taking into account that the Berezin transform can be defined provided that there is a given closed subspace, which possesses a reproducing kernel, we are here concerned with the domain \( \Omega = B^n \) the unit ball of \( \mathbb{C}^n \) viewed as homogeneous space \( B^n = G/K \) where \( G = SU(n, 1) \) be the group of \( \mathbb{C} \)-linear transforms \( g \) on \( \mathbb{C}^{n+1} \) that preserve the the indefinite Hermitian form \( \sum_{j=1}^n |z_j|^2 - |z_{n+1}|^2 \), with \( \det g = 1 \). Suppose that \( B^n \) is endowed with the measure \( d\mu_\nu = (1 - |z|^2)^{\nu-n-1} dm(z) \), where \( \nu \in \mathbb{R}_+ \setminus \mathbb{Z} \), \( \nu > n \) and \( dm(z) \), be the Lebesgue measure on \( \mathbb{C}^n \). We consider as the the subspace of the space of the \( L^2 \)-integrable functions on \( B^n \) with respect to the measure \( d\mu_\nu \), the range space of the spectral projector \( R^\nu_l \) corresponding the eigenvalue \( \rho_l = -(\nu - n - 2l)^2 \) of the \( G \)-invariant differential operator:

\[
\Delta_\nu = 4(1 - |z|^2) \left\{ \sum_{1 \leq i,j \leq n} (\delta_{i,j} - z_i \overline{z}_j) \frac{\partial^2}{\partial z_i \partial \overline{z}_j} - \nu \sum_{j=1}^n \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \right\}
\]

where \( l \) is a fixed integer such that \( 0 \leq l < \frac{\nu-n}{2} \). In other word, we are concerned with the following eigenspace:

\[
A^{2,\nu}_l = \{ F \in L^2(\mathbb{B}^n, d\mu_\nu), \Delta_\nu F = \rho_l F \}.
\]

\[
K^\nu_l(z, w) = c_l \frac{1}{(1 - \langle z, w \rangle^\nu)} \text{ } _2F_1(-l, \nu + n + 1, 1 - \frac{|1 - \langle z, w \rangle|^2}{(1 - |z|^2)(1 - |w|^2)})
\]

where,

\[
c_l = \frac{2\Gamma(n + l)(\nu - n - 2l)\Gamma(\nu - l)}{\pi^n \Gamma(n)! \Gamma(\nu - n - l + 1)}.
\]

It is known, that the kernel \( K^\nu_0(z, w) \), corresponding to \( l = 0 \), is the Bergman kernel, hence the space \( A^{2,\nu}_0 \) is the classical weighted
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Bergman- space of holomorphic functions that are $(1-|z|^2)^{\nu-n-1}dm(z)$-integrable, while for $l \neq 0$, The space $A^2_{\nu}$ which can be viewed as as a Kernel spaces of the elliptic differential operator $\Delta_{\nu} - \rho_l$, consists of non holomorphic functions. The Berezin transform associated the Hilbert subspace $A^2_{\nu}$ is studied by many Authors [15].

Here, for the case $l \neq 0$, we associate to the sub space $A^2_{\nu}$, the following Berezin transform:

$$L^2_{\nu}(\mathbb{B}^n) \to L^2_{\nu}(\mathbb{B}^n)$$

$$B^l_{\nu}F(z) = \int_{\mathbb{B}^n} F(w)B^\nu_l(z, w)d\mu_{\nu}(w), \quad (1.7)$$

where the kernel $B^\nu_l(z, w)$, is defined by:

$$B^\nu_l(z, w) = (1-|z, w|)^{-\nu} \frac{|K^\nu_l(z, w)|^2}{K^\nu_l(z, z)K^\nu_l(w, w)}, \quad (z, w) \in \mathbb{B}^n \times \mathbb{B}^n. \quad (1.8)$$

In this paper, our aim is to express the above Berezin transform as a function of the $G-$ invariant Laplacian $\Delta_{\nu}$. The method used is based on the $L^2-$ spectral theory of $\Delta_{\nu}$. [4], [23], together with the Fourier-Jacobi transform [13], [11]. Precisely, we establish the following result:

$$B^\nu_l = \frac{\pi^n\Gamma^2(n)}{2\Gamma^2(\nu-l)}|\Gamma\left(i\sqrt{-(\Delta_{\nu} + (n-\nu)^2)} + 3\nu - 4l - n\right)|^2 \times \sum_{q=0}^{2l} (-1)^q A_q \frac{S_q\left(-\frac{\Delta_{\nu} + (n-\nu)^2}{4}, \frac{3\nu-4l-n}{2}, \nu, \frac{n}{2}, n-\nu\right)}{\Gamma(2(\nu-l) + q)\Gamma(\nu-2l+q)}, \quad (1.9)$$

where $S_q$ denotes the continuous dual Hahn polynomial and $A_q$ are the following parameters

$$A_q = 2^{-q} \sum_{p=\max(0,q-l)}^{\min(l,q)} (l-p)_q(l)_p \frac{\Gamma(\nu-l+q-p)\Gamma(\nu-l+p)}{\Gamma(n+p)\Gamma(n+q-p)}. \quad (1.10)$$

The paper is organized as follows. In section 2, we review some well known spectral properties of the operator $\Delta_{\nu}$. In the section 3 we give the spectral density associated with the $G-$invariant Laplacian $\Delta_{\nu}$. As an application we give its heat kernel. The section 4, will be devoted for the $G-$ invariance and the boundedness of the Berezin transform $B^\nu_l$. In the section 5, we give the proof of the main result (1.9).
section 6, as an application we give an expression of the Berezin heat kernel.

2. \textit{L}^2-\textit{Concrete spectral analysis of the invariant Laplacians} $\Delta_\nu$.

In this section we review some results on the $L^2$-Concrete spectral analysis, in the sense of Strichartz [17], of the invariant Laplacians $\Delta_\nu$ in the weighted Hilbert space $L^2_\nu(\mathbb{B}^n)$.

Let $G = SU(n, 1)$ be the group of all $\mathbb{C}$-linear transforms $g$, on $\mathbb{C}^{n+1}$ that preserve the indefinite hermitian form

\begin{equation}
\sum_{j=1}^n |z_j|^2 - |z_{n+1}|^2,
\end{equation}

with $\det g = 1$.

The group $G$ acts transitively on the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n; |z| < 1\}$ by

\begin{equation}
G \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \rightarrow g.z = (az + b)(cz + d)^{-1},
\end{equation}

where $a, b, c, d$ are $n \times n, n \times 1, 1 \times n$ and $1 \times 1$ matrices respectively.

Recall that this action satisfy the following relation:

\begin{equation}
1 - <gz, gw> = \frac{(1 - <z, w>)}{(cz + d)(cw + d)}
\end{equation}

where $<,>$ is the well known hermitian product on $\mathbb{C}^n$.

As a homogeneous space we have the identification $\mathbb{B}^n = G/K$ where $K$ is the stabilizer of 0. More precisely

\begin{equation}
K = \left\{ k = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, a \in U(n), d \in U(1) ; \det(ad) = 1 \right\}.
\end{equation}

We recall that The $G$- invariant distance associated to the Bergman metric \cite{6} on the unit ball $\mathbb{B}^n = G/K$, is given by:

\begin{equation}
\cosh^2 d(z, w) = \frac{|1 - <z, w>|^2}{(1 - |z|^2)(1 - |w|^2)}, (z, w) \in \mathbb{B}^n \times \mathbb{B}^n.
\end{equation}

Let $\nu \in \mathbb{R} - \mathbb{Z}$, and suppose that $\nu > 0$. By $dm(z)$ we denote the Lebesgue measure on $\mathbb{C}^n$. Denote by $d\mu_\nu$, the weighted measure on $\mathbb{B}^n$ defined by:

\begin{equation}
d\mu_\nu(z) = (1 - |z|^2)^{\nu - n - 1}dm(z),
\end{equation}

and by $L^2_\nu(\mathbb{B}^n)$ its the corresponding $L^2$-space,
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\[ L_\nu^2(\mathbb{B}^n) = \left\{ F : \mathbb{B}^n \mapsto \mathbb{C}, \int_{\mathbb{B}^n} |F(z)|^2 d\mu_\nu(z) < +\infty \right\} \]  

(2.7)

For $g \in G$, we define

\[ T^\nu(g)F(z) = J(g^{-1}, z)\nu^{n+1}F(g^{-1}.z), \]  

(2.8)

where $J(g^{-1}, z)$ is the complex Jacobian of $g^{-1}$ (with a mild ambiguity of its $\nu$ power depending only on $g$).

Then $T^\nu$ gives rise to a continuous projective representation of the group $G$ on $L_\nu^2(\mathbb{B}^n)$.

Notice that the restriction of $J$ to $K$ gives rise to a character $\chi_\nu$ of $K$. Namely,

\[ J(k, z)\nu^{n+1} = d^{-\nu}, \]  

(2.9)

for $k = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$.

The space $L_\nu^2(\mathbb{B}^n)$ is a trivialization of the $L^2$-space of sections of the homogeneous line bundle over $\mathbb{B}^n$ associated to the one dimensional representation $\chi_\nu$ of the compact group $K$.

The invariant Laplacian with respect to the $G$-action (2.8) is given by:

\[ \Delta_\nu = 4(1-|z|^2)\left\{ \sum_{1 \leq i,j \leq n} (\delta_{ij} - z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j} - \nu \sum_{j=1}^n \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right\}. \]  

(2.10)

Remark. Note that, in [4], a more general family of Laplacians $\Delta_{\alpha,\beta}$ has been considered. The above operators $\Delta_\nu$ corresponds to the case $\alpha = 0$ and $\beta = -\nu$.

In [4] we showed that the invariant laplacian $\Delta_\nu$ is a self-adjoint operator in the space $L_\nu^2(\mathbb{B}^n)$.

Besides the continuous spectrum $\{-\lambda^2 + (\nu - n)^2, \lambda \in \mathbb{R}\}$, it might have a discrete spectrum according to the size of $\nu$.

Precisely, if $\nu > n$ then the point spectrum of the $G$-invariant laplacian $\Delta_\nu$ consists of the finite set

\[ \rho_j = -\left(\lambda_j^2 + (\nu - n)^2\right), j = 0, \ldots, \left\lfloor \frac{\nu - n}{2} \right\rfloor \]  

(2.11)

where

\[ \lambda_j = i(2j + n - \nu), \]  

(2.12)
and, \([x] = \text{the greatest integer not exceeding } x\).

Thus, in the case where \(\nu > n\), the spectrum \(\sigma(\Delta_\nu)\) of the operator \(\Delta_\nu\) is given by

\[
\sigma(\Delta_\nu) = \{-(\lambda^2 + (\nu-n)^2), \lambda \in \mathbb{R}\} \cup \{-(\lambda_j^2 + (\nu-n)^2); j = 0, \ldots, \left[\frac{\nu-n}{2}\right]\}
\]

(2.13)

According to [4] and [23] a fundamental family of eigenfunctions of \(\Delta_\nu\) with eigenvalue \(-(\lambda^2 + (\nu-n)^2)\) is given by the following family of Poisson kernels:

\[
P^\nu_{\lambda}(z, \omega) = \left(1 - \frac{|z|^2}{|1 - \langle z, \omega \rangle|^2}\right)^{\frac{i\lambda+n-\nu}{2}}(1 - \langle z, \omega \rangle)^{-\nu},
\]

(2.14)

from which we may obtain an explicit spectral decomposition of the self-adjoint operator \(\Delta_\nu\) in the Hilbert space \(L^2_\nu(\mathbb{B}^n)\). More precisely, let \(F \in L^2_\nu(\mathbb{B}^n)\). Then we have

\[
F = \int_{-\infty}^{+\infty} P^\nu_{\lambda} F d\lambda + \sum_{0 \leq j < \frac{\nu-n}{2}} R^\nu_j F,
\]

(2.15)

where the integral operators \(P^\nu_{\lambda}\) are related to the Fourier-Helgason transform

\[
\tilde{F}(\lambda, \omega) = \int_{\mathbb{B}^n} F(z) P^\nu_{\lambda}(z, \omega) d\mu_\nu(z),
\]

(2.16)

by

\[
P^\nu_{\lambda} F(z) = \frac{\Gamma(n)}{4^2(n-\nu-1)!} |c'_{\nu}(\lambda)|^{-2} \int_{\partial \mathbb{B}^n} P^\nu_{\lambda}(z, \omega) \tilde{F}(\lambda, \omega) d\sigma(\omega).
\]

(2.17)

where

\[
c'_{\nu}(\lambda) = \frac{2^{-\nu+n-i\lambda} \Gamma(n) \Gamma(i\lambda)}{\Gamma(\frac{\lambda+n-\nu}{2}) \Gamma(\frac{\lambda+n+\nu}{2})},
\]

(2.18)

is the analogous of the Harish-Chandra c-function.

In above the orthogonal projector operators \(R^\nu_j\) are given by

\[
R^\nu_j F(z) = c_j \int_{\partial \mathbb{B}^n} \tilde{F}(\lambda_j, \omega) P^\nu_{\lambda_j}(z, \omega) d\omega,
\]

(2.19)

where

\[
c_j = \frac{2\Gamma(n+j)(\nu-n-2j)\Gamma(\nu-j)}{\pi^n \Gamma(n) j! \Gamma(\nu-n-j+1)}.
\]

(2.20)

Note that the poisson kernel defined in (2.14) satisfy the following integral formula:
\[
\int_{\partial B^n} P^\nu_{\lambda}(z, \omega) P^\nu_{-\lambda}(w, \omega) d\sigma(\omega) =
(1 - \langle z, w \rangle)^{-\nu} F\left(\frac{i\lambda + n - \nu}{2}, \frac{i\lambda + n - \nu}{2}, n, -\sinh^2 d(z, w)\right)
\] (2.21)

**Remark 2.1.** Notice that the set \(\{\lambda_j = i(2j + n - \nu), 0 \leq j < \frac{\nu - n}{2}\}\) corresponds to the poles of the Harish-Chandra c-function \(c^\nu(\lambda)-1\) in the region \(\text{Im} \lambda < 0\).

From now on we suppose that \(\nu > n\).

### 3. Spectral density

Recall that our aim in this paper is to express the Berezin transform \(B^\nu_{\lambda}\) as function of the \(G-\)invariant Laplacian \(\Delta^\nu\). For this, we consider the \(G-\)invariant shifted Laplacian defined by:

\[
\tilde{\Delta}^\nu = -\left(\Delta^\nu + (\nu - n)^2\right),
\] (3.1)

with \(C_0^\infty(\mathbb{B}_n)\), as its natural regular domain. Note that the spectrum of the operator \(\tilde{\Delta}^\nu\) can be given easily from (2.13) by:

\[
\sigma(\tilde{\Delta}^\nu) = \{s = \lambda^2, \lambda \in \mathbb{R}\} \cup \{s_j = \lambda_j^2; j = 0, \ldots, \left\lfloor \frac{\nu - n}{2} \right\rfloor\};
\] (3.2)

where \(\lambda_j\) are defined in (2.12). To express The Berezin transform \(B^\nu_{\lambda}\) in term of \(\tilde{\Delta}^\nu\), we will need to compute its spectral density. The extension of \(\tilde{\Delta}^\nu\) will be also denoted by \(\tilde{\Delta}^\nu\). The domain of the extension \(\tilde{\Delta}^\nu\) will be denote by \(\chi\). This extension admits a spectral decomposition [22]:

\[
I = \int_{-\infty}^{+\infty} dE_s,
\] (3.3)

where \(I\) is the identity operator and

\[
\tilde{\Delta}^\nu = \int_{-\infty}^{+\infty} s dE_s,
\] (3.4)

in the weak sense, that is

\[
(\tilde{\Delta}^\nu f, g) = \int_{-\infty}^{+\infty} s d(E_sf, g)
\] (3.5)

for \(f \in \chi\) and \(g \in L^2_\nu(\mathbb{B}_n)\). The spectral density [9]:

\[
e_s = \frac{dE_s}{ds}
\] (3.6)
is understood as an operator-valued distribution, an element of the space $D'(\mathbb{R}, L(\chi, L^2_\nu(\mathbb{B}_n)))$ where $L(\chi, L^2_\nu(\mathbb{B}_n))$ is the space of bounded operators from $\chi$ to $L^2_\nu(\mathbb{B}_n)$. In term of the spectral density $e_s = \frac{dE_s}{ds}$, the equation (3.3) and (3.4) become

$$I = <e_s, 1>,$$  

(3.7)

and

$$\tilde{\Delta}_\nu = <e_s, s>,$$  

(3.8)

where $<f(s), \phi(s)>$ is the evaluation of the distribution $f(s)$ on a test function $\phi(s)$. Since $\tilde{\Delta}_\nu$ is elliptic, then its spectral density $e_s$ admits a distributional kernel (called the spectral function) $e(s,w,z)$, an element of $D'(\mathbb{R}, D'(\mathbb{B}_n \times \mathbb{B}_n))$. Precisely, we have the following proposition.

**Proposition 3.1.** The spectral function $e(s,w,z)$ of the operator $\tilde{\Delta}_\nu$, is given by:

$$e(s,w,z) = \frac{\Gamma(n)}{4\pi^{n+1}2^{(\nu-n)}}(1-<z,w>)^{-\nu}\chi_+(s)|C_\nu(\sqrt{s})|^{-2}(\sqrt{s})^{-1}\phi^{(n-1,-\nu)}_\nu(d(z,w))$$

$$+ \sum_{j=0}^{\nu-n} c_j (1-<z,w>)^{-\nu}\phi^{(n-1,-\nu)}_\nu(d(z,w))\delta(s-s_j),$$

where $\chi_+(s)$ is the characteristic function of the set of real positive numbers, $s$ and $s_j$ are the spectral parameters defined in (3.2) and $\phi^{(\alpha,\beta)}_\lambda(t)$, is the Jacobi function defined by:

$$\phi^{(\alpha,\beta)}_\lambda(t) = {}_2F_1(\frac{\alpha + \beta + 1 - i\lambda}{2}, \frac{\alpha + \beta + 1 + i\lambda}{2}, 1 + \alpha; -\sin^2 t).$$

**Proof**

Let $F$ be a $C^\infty$ - function with compact support in $\mathbb{B}_n$, then from the relation (3.4), we have:

$$F(z) = \int_{-\infty}^{+\infty} P_\nu^\nu[F](z)d\lambda + \sum_{0 \leq j \leq \nu-n} \mathcal{R}_j[F](z).$$  

(3.9)

Now, by inserting (2.17) and (2.19) in the equation (3.9), we obtain:

$$F(z) = \frac{\Gamma(n)}{4\pi^{n+1}2^{(\nu-n)}} \int_{-\infty}^{+\infty} d\lambda |C_\nu(\lambda)|^{-2} \int_{\mathbb{B}_n} \left( \int_{\partial\mathbb{B}_n} P_\nu^\nu(z,\omega)P_\nu^\nu(w,\omega)d\sigma(\omega) \right) F(w)d\mu_\nu(w)$$
\[ + \sum_{0}^{n} c_j \int_{B^n} \left( \int_{\partial B^n} P^\nu_{\lambda_j}(z, \omega) P^\nu_{-\lambda_j}(w, \omega) d\sigma(\omega) \right) F(w) d\mu_\nu(w). \] (3.10)

where \( d\sigma(\omega) \), is the superficial measure on \( \partial B^n \).

Making use of the formula (2.21) in where the hypergeometric function in the right hand side, was replaced by the corresponding Jacobi function to get:

\[ F(z) = \frac{\Gamma(n)}{4\pi^{n+1}2^{(\nu-n)}} \int_{-\infty}^{+\infty} d\lambda |C_\nu(\lambda)|^{-2} \int_{B^n} (1 - <z, w>)^{-\nu}(\phi^{(n-1,-\nu)}_\lambda(d(z, w))) F(w) d\mu_\nu(w) \]

\[ + \sum_{0}^{n} c_j \int_{B^n} (1 - <z, w>)^{-\nu}(\phi^{(n-1,-\nu)}_\lambda(d(z, w))) F(w) d\mu_\nu(w). \] (3.11)

It is not difficult to see that the function involved in the first integral with respect to the variable \( \lambda \) in (3.11) is even. Then the equation (3.11) can be written as:

\[ F(z) = \frac{\Gamma(n)}{42^{2^{(\nu-n)}+1}} \int_{-\infty}^{+\infty} d\lambda |C_\nu(\lambda)|^{-2} \int_{B^n} (1 - <z, w>)^{-\nu}(\phi^{(n-1,-\nu)}_\lambda(d(z, w))) F(w) d\mu_\nu(w) \]

\[ + \sum_{0}^{n} c_j \int_{B^n} (1 - <z, w>)^{-\nu}(\phi^{(n-1,-\nu)}_\lambda(d(z, w))) F(w) d\mu_\nu(w). \] (3.12)

Making use the change of variable \( s = \lambda^2 \) in the first integral of (3.12), and \( s_j = \lambda_j^2 \) in the discreet part then, (3.12) can be rewritten as:

\[ F(z) = \frac{\Gamma(n)}{4\pi^{n+1}2^{(\nu-n)}} \int_{0}^{+\infty} ds |C_\nu(\sqrt{s})|^{-2} s^{\frac{1}{2}} \int_{B^n} (1 - <z, w>)^{-\nu}(\phi^{(n-1,-\nu)}_{\sqrt{s}}(d(z, w))) F(w) d\mu_\nu(w) \]

\[ + \sum_{0}^{n} c_j \int_{B^n} (1 - <z, w>)^{-\nu}(\phi^{(n-1,-\nu)}_{\sqrt{s}}(d(z, w))) F(w) d\mu_\nu(w). \] (3.13)

This last identity can be written in the distributional sense as:

\[ F(z) = \int_{-\infty}^{+\infty} (\int_{B^n} e(s, w, z) F(w) d\mu_\nu(w)) ds, \] (3.14)
where the Schwartz kernel \( e(s, w, z) \) is given by:

\[
e(s, w, z) = \frac{\Gamma(n)}{4\pi^{n+1}2(n - \nu)}(1 - \langle z, w \rangle)^{-\nu}\chi_+ (s)|C_\nu (\sqrt{s})|^{-2}(\sqrt{s})^{-1/2}\phi_{\sqrt{s}}^{(n-1,-\nu)}(d(z, w)) + \sum_{j=0}^{\nu-n} e_j (1 - \langle z, w \rangle)^{-\nu}\phi_{\sqrt{s}}^{(n-1,-\nu)}(d(z, w))\delta(s - s_j).
\]

Now, by returning back to the equation (3.9) and applying the operator \( \tilde{\Delta}_\nu \) to its both sides, we obtain the following equation:

\[
\tilde{\Delta}_\nu[F](z) = \int_{\mathbb{R}} \lambda^2 P^\nu_\lambda[F](z) d\lambda + \sum_{0 \leq j < \nu} \lambda_j^2 R_j^\nu[F](z). (3.15)
\]

As in the above, we make use of the change of variable \( s = \lambda^2 \), in the integral part of (3.15) and put \( s_j = \lambda_j^2 \) in the discreet part. Then by following the same steps, we obtain:

\[
\tilde{\Delta}_\nu[F](z) = \int_{\mathbb{R}} s(\int_{\mathbb{B}} e(s, w, z) F(w) d\mu_\nu(w)) ds. (3.16)
\]

By considering the functional \( T \) which corresponds to a test function \( \varphi \) the operator \( < T, \varphi > \in L(\chi, L^2(B_n)) \) defined by:

\[
<T, \varphi>[F](z) = \int_{\mathbb{R}} \varphi(s)(\int_{\mathbb{B}} e(s, w, z) F(w) d\mu_\nu(w)) ds, (3.17)
\]

we observe that the two equations (3.14) and (3.16) become:

\[
<T, 1 > = I, (3.18)
\]
\[
<T, s > = \tilde{\Delta}_\nu, (3.19)
\]

and by the uniqueness of the spectral density associated with a self-adjoint operator, we conclude that the functional \( T \) is nothing but the spectral density of the operator \( \tilde{\Delta}_\nu \). This end the proof.

Remark 3.1. For given a suitable function \( f : \mathbb{R} \rightarrow \mathbb{C} \), the operator \( f(\tilde{\Delta}_\nu) \), is defined by

\[
f(\tilde{\Delta}_\nu)[\varphi](z) = \int_{\mathbb{B}} \Omega_f(w, z) \varphi(w) d\mu_\nu(w) (3.20)
\]

where the kernel \( \Omega_f(w, z) \) is defined by:

\[
\Omega_f(w, z) = \int_{\sigma(\tilde{\Delta}_\nu)} e(s, w, z) f(s) ds (3.21)
\]

with (3.20) and (3.21) are understand in the distributional sense.
As a direct consequence of the above proposition we can derive the heat kernel of the $G$–invariant operator $\Delta_\nu$. Precisely, we have the following.

**Proposition 3.2.** Let $\psi(t, z)$, be the solution of the heat Cauchy problem associated to the operator $\Delta_\nu$, on $B_n$:

\[
\partial_t \psi(t, z) = \Delta_\nu \psi(t, z), \quad (t, z) \in \mathbb{R}_+ \times B_n
\]

\[
\psi(0, z) = \varphi(z) \in C^\infty_0(B_n).
\]

Then, $\psi(t, z)$, is given by the integral formula:

\[
\psi(t, z) = \int_{B_n} K_\nu(t, z, w) \varphi(w) d\mu_\nu(w),
\]

where $K_\nu(t, z, w)$, is the heat kernel given by:

\[
K_\nu(t, z, w) = (1 - \langle z, w \rangle)^{-\nu} \sum_{j=0}^{\frac{\nu-n}{2}} \tau_j e^{-2j(\nu-n-2j)t} P_j^{(n-1,-\nu)}(\cosh 2d(z, w))
\]

\[
+ (1 - \langle z, w \rangle)^{-\nu} e^{-t(\nu-n)^2} \frac{\Gamma(n)}{2\pi^{n+\frac{1}{2}}(\nu-n)} (3.25)
\]

\[
\times \int_{0}^{+\infty} e^{-t\lambda^2} \left| C_\nu(\lambda) \right|^{-2} F\left( \frac{n-\nu-i\lambda}{2}, \frac{n-\nu+i\lambda}{2}, n; -\sinh^2(d(z, w)) \right) d\lambda.
\]

where $\tau_j = \frac{2(\nu-n-2j)\Gamma(\nu-j)}{\pi^{n+\frac{1}{2}}(\nu-n-j+1)}$ and $C_\nu(\lambda)$ is the Harish-chandra function defined in (2.18).

**Proof.** The solution $\psi(t, z)$, is given by the action of the semigroup $e^{t\Delta_\nu}$ on the initial data $\varphi(z)$:

\[
\psi(t, z) = \int_{B_n} K_\nu(t, z, w) \varphi(w) d\mu_\nu(w),
\]

Note that for the operators $\Delta_\nu$ and $\tilde{\Delta}_\nu = -(\Delta_\nu + (n - \nu)^2)$, we have the following semigroup relation:

\[
e^{t\Delta_\nu} = e^{-t(\nu-n)^2} e^{-t\tilde{\Delta}_\nu}.
\]

Then, the heat kernel $k(t, z, w)$ of $\Delta_\nu$ is given by:

\[
K(t, z, w) = e^{-t(\nu-n)^2} K_\nu(t, z, w),
\]

where $\tilde{K}_\nu(t, z, w)$ is the heat kernel of the operator $\tilde{\Delta}_\nu$.

By using (3.20) and (3.21), we obtain:

\[
\tilde{K}_\nu(t, z, w) = \int_{\sigma(\tilde{\Delta}_\nu)} e(s, w, z) e^{-ts} ds.
\]
Then, we have
\[
\tilde{K}_\nu(t, z, w) = \sum_{j=0}^{\nu+n} c_j (1- \langle z, w \rangle)^{-\nu} \phi_{\sqrt{s}}^{(n-1, -\nu)}(d(z, w)) e^{-ts_j}
\]
\[
+ \frac{\Gamma(n)}{4\pi^{n+1/2}(\nu-n)} (1- \langle z, w \rangle)^{-\nu} \int_0^{+\infty} \left| C_\nu(\sqrt{s}) \right|^{-2} (\sqrt{s})^{-1} \phi_{\sqrt{s}}^{(n-1, -\nu)}(d(z, w)) e^{-ts} ds.
\]
(3.30)
Recall that \(\sqrt{s_i} = \lambda_j = i(2j + n - \nu), j = 0, 1, \ldots, \frac{n-\nu}{2}\). Then the Jacobi function \(\phi_{\sqrt{s}}^{(n-1, -\nu)}(d(z, w))\), involved in the discrete part of the kernel \(\tilde{K}_\nu(t, z, w)\), becomes
\[
\phi_{\sqrt{s}}^{(n-1, -\nu)}(d(z, w)) = 2 F_1(-j, j + n - \nu, n; -\sinh^2(d(z, w))).
\]
(3.31)
Next, by using of the identity ([14], p.39)
\[
P_k^{(\alpha, \beta)}(y) = \left(1 + \frac{\alpha}{k} \right) \binom{k}{n} \frac{1-y}{2} F_1(-k, \alpha + \beta + k + 1, \alpha + 1; \frac{1-y}{2}),
\]
(3.32)
for \(\alpha = n-1, \beta = -\nu, k = j\) and \(y = 1+2\sinh^2(d(z, w)) = \cosh 2d(z, w)\), the equation (3.31) becomes:
\[
\phi_{\sqrt{s}}^{(n-1, -\nu)}(d(z, w)) = \frac{j!}{(n)_j} P_j^{(\alpha, \beta)}(\cosh 2d(z, w)).
\]
(3.33)
By inserting the above expression of Jacobi function in the discrete part of the equation (3.30) and using the change of variable \(s = \lambda^2, \lambda > 0\), in the continuous part, we obtain:
\[
\tilde{K}_\nu(t, z, w) = (1- \langle z, w \rangle)^{-\nu} \sum_{j=0}^{\nu+n} \tau_j e^{-(2j+n-\nu)^2t} P_j^{(n-1, -\nu)}(\cosh 2d(z, w))
\]
(3.34)
\[
+ (1- \langle z, w \rangle)^{-\nu} \frac{\Gamma(n)}{2\pi^{n+1/2}(\nu-n)} \int_0^{+\infty} e^{-t\lambda^2} \left| C_\nu(\lambda) \right|^{-2} F_{\nu-n}^{(n-\nu-i\lambda/2, n-\nu+i\lambda/2)}(n; -\sinh^2(d(z, w))) d\lambda.
\]
(3.35)
where \(\tau_j = \frac{2(\nu-n+2j)\Gamma(n-\nu)}{n! (\nu-n-j+1)^2}\). Then, we get the desired result.
\(\square\)
4. The Berezin Transform

Let $l$ be a fixed integer in the set $j = 0, \ldots, \left \lfloor \frac{\nu-n}{2} \right \rfloor$, and let $A^{2,l}_{\nu}(\mathbb{B}^n) = R^l_{\nu} L^2_{\nu}(\mathbb{B}^n)$ be the subspace appearing in the discrete part of the Plancherel formula (3.4). Then, according to [23], $A^{2,l}_{\nu}(\mathbb{B}^n)$ is a closed invariant subspace of $L^2_{\nu}(\mathbb{B}^n)$, with reproducing kernel given by:

$$K_{l}^{\nu}(z, w) = c_l\frac{1}{(1 - \langle z, w \rangle)^\nu} \frac{2F_1(-l, l - \nu + n, 1; 1 - |z|^2)(1 - |w|^2)}{(1 - |z|^2)(1 - |w|^2)},$$

(4.1)

where $c_l$ is the constant defined by (2.20).

**Remark 4.1.** Thanks to the formula (2.4) The reproducing kernel $K_{l}^{\nu}(z, w)$, can be written also as:

$$K_{l}^{\nu}(z, w) = c_l(1 - \langle z, w \rangle)^{-\nu} F(-l, l - \nu + n, n, - \sinh^2 d(z, w)), \quad (4.2)$$

Notice that if $l = 0$ the above kernel is the Bergman kernel. Thus the space $A^{2,l}_{\nu}(\mathbb{B}^n)$ is the classical weighted Bergman space of holomorphic functions in $L^2_{\nu}(\mathbb{B}^n)$.

As is well known the classical Berezin transform associated to $A^{2,l}_{\nu}(\mathbb{B}^n)$ is defined by:

$$B_{\nu}F(z) = \frac{2\Gamma(\nu)}{\pi^n \Gamma(\nu - n)} \int_{\mathbb{B}^n} \frac{(1 - |z|^2)^\nu}{|1 - \langle z, w \rangle|^2} F(w) d\mu(w),$$

and has been studied by many authors.

As mentioned in the above, our aim in this section is to define a G-invariant Berezin transform associated to the G-invariant eigenspace $A^{2,l}_{\nu}(\mathbb{B}^n)$, to this end we consider the following kernel function:

$$B_{l}^{\nu}(z, w) = (1 - \langle z, w \rangle)^{-\nu} \frac{|K_{l}^{\nu}(z, w)|^2}{K_{l}^{\nu}(z, z) K_{l}^{\nu}(w, w)} (z, w) \in \mathbb{B}^n \times \mathbb{B}^n.$$

(4.3)

**Definition** Assume that $l \neq 0$. The transformation $B_{l}^{\nu}$ defined by:

$$L^2_{\nu}(\mathbb{B}^n) \to L^2_{\nu}(\mathbb{B}^n)$$

$$B_{l}^{\nu}F(z) = \int_{\mathbb{B}^n} F(w) B_{l}^{\nu}(z, w) d\mu(w), \quad (4.4)$$

is called here the Berezin transform on the the line bundle over the complex hyperbolic space $\mathbb{B}^n = SU(n,1)/SU(U(n) \times U(1))$.  


Explicitly, from the expression of the reproducing kernel $K_\nu^\nu(z, w)$ given in (4.2) combining with the use of the equation (2.5), the expression of the berezin kernel $B_\nu^\nu(z, w)$ is given by:

$$B_\nu^\nu(z, w) = (1 - <z, w >)^{-\nu} \cosh^{-2\nu} d(z, w)|F(-l, l-\nu+n, n, -\sinh^2 d(z, w))|^2.$$  

(4.5)

It is easy to establish the following,

**Proposition 4.1.** The Berezin transform $B_\nu^\nu$ is $G$-invariant with respect to the representation $T^\nu$.

**Proof.** Let

$$g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

$$T_\nu^\nu[B_\nu^\nu F](z) = (cz+d)^{-\nu} \int_{\mathbb{B}^n} (1 - <g^{-1}z, w >)^{-\nu} \frac{|K_\nu^\nu(g^{-1}.z, w)|^2}{K_\nu^\nu(g^{-1}.z, g^{-1}.z)K_\nu^\nu(w, w)} F(w)d\mu_\nu(w).$$

By using the formula:

$$1 - <g^{-1}z, z > = \frac{(1 - <z, \zeta >)}{(cz + d)(c\zeta + d)}$$

(4.7)

for $z \in \mathbb{B}^n$, and $\zeta = g.w$, we obtain:

$$(1 - <g^{-1}z, w >)^{-\nu} = (cz + d)^\nu(cg.w + d)^\nu(1 - <z, g.w >)^{-\nu},$$

(4.8)

$$K_\nu^\nu(g^{-1}.z, w) = (cz + d)^\nu(cg.w + d)^\nu K_\nu^\nu(z, g.w),$$

(4.9)

and

$$K_\nu^\nu(g^{-1}.z, g^{-1}.z) = |cz + d|^{2\nu} K_\nu^\nu(z, z),$$

(4.10)

Then, by inserting (4.8), (4.9) and (4.10) in (4.6) we obtain:

$$T_\nu^\nu[B_\nu^\nu F](z) = \int_{\mathbb{B}^n} (cg.w + d)^\nu|cg.w + d|^{2\nu}(1 - <z, g.w >)^{-\nu} \frac{|K_\nu^\nu(z, g.w)|^2}{K_\nu^\nu(z, z)K_\nu^\nu(w, w)} F(w)d\mu_\nu(w).$$

(4.11)

By using the change of variable $\zeta = g^{-1}w$, The equation (4.11) becomes:

$$T_\nu^\nu[B_\nu^\nu F](z) = \int_{\mathbb{B}^n} (c\zeta + d)^\nu|c\zeta + d|^{2\nu}(1 - <z, \zeta >)^{-\nu} \frac{|K_\nu^\nu(z, \zeta)|^2}{K_\nu^\nu(z, z)K_\nu^\nu(g^{-1}\zeta, g^{-1}\zeta)}$$

$$\times F(g^{-1}\zeta)d\mu_\nu(g^{-1}\zeta).$$

(4.12)

After inserting the equation (4.10) for which $z$ is replaced by $\zeta$ in the equation (4.12) and using the fact that the measure $d\mu_\nu(\zeta) = (1 - |\zeta|^2)^{\nu-n-1}d\mu(\zeta)$, is $G$-invariant, the equation (4.12) becomes
\[
T^\nu_g[B^l_\nu F](z) = \int_{\mathbb{B}^n} (c\zeta + d)^{-\nu} (1 - \langle z, \zeta \rangle)^{-\nu} \frac{|K^\nu(z, \zeta)|^2}{K^\nu(z, z)K^\nu(\zeta, \zeta)} F(g^{-1}\zeta)d\mu_\nu(\zeta).
\]

(4.13)

This ends the proof.

In order to prove that \( B^\nu_l \) is a bounded operator, we have need the following lemma.

**Lemma 4.1.** The generalized berezin kernel \( B^\nu_l(z, w) \) given in (4.13) admits also the following expression:

\[
B^\nu_l(z, w) = (1 - \langle z, w \rangle)^{-\nu} \cosh^{(d-2\nu)}(d(z, w)) |F(-l, -l+\nu, n, \tanh^2(d(z, w)))|^2
\]

(4.14)

**Proof.** From (4.5), we have:

\[
B^\nu_l(z, w) = (1 - \langle z, w \rangle)^{-\nu} \cosh^{-2\nu} d(z, w) |F(-l, -l+\nu+\nu, n, -\sinh^2 d(z, w))|^2.
\]

Making use of the Euler formula \( F(a, b, c) = (1 - z)^{-a} F(a, c - b, \frac{z}{z-1}) \) (4.15)

for \( a = -l, b = l + n - \nu \) and \( c = n \), we obtain:

\[
F(-l, -l+\nu+n, n, -\sinh^2 d(z, w)) = c h^{2l} d(z, w) F(-l, \nu-l, n; \tanh^2 d(z, w)),
\]

(4.16)

then by inserting (4.15) in the above expression of \( B^\nu_l(z, w) \), we get the desired result. This ends the proof.

We have the following proposition:

**Proposition 4.2.** The Berezin transform \( B^l_\nu \) is a bounded operator on \( L^p(\mathbb{B}^n, d\mu_\nu) \) for \( 1 \leq p \leq \infty \)

Proof. We start by proving that \( B^l_\nu \) is bounded operator in \( L^\infty(\mathbb{B}^n, d\mu_\nu) \).

First observe that since \( l \) is a positive integer then

\[
|F(-l, -l+\nu, n, \tanh^2 d(z, w))|^2 \leq M,
\]

for some positive constant \( M = M(\nu, l) \). Thus

\[
|B^l_\nu(z, w)| \leq M |1 - \langle z, w \rangle|^{-3\nu} (1 - |z|^2)^\nu(1 - |w|^2)^\nu |\cosh^2 d(z, w)|^2.
\]

(4.17)
from which we deduce that:
\[
| B^l_\nu(z, w) | \leq M \frac{(1 - | z |^2)^{\nu - 2l}(1 - | w |^2)^{\nu - 2l}}{1 - < z, w >^{3\nu - 4l}}.
\] (4.18)

Now we recall a result in [16] on the asymptotic behavior of certain integrals.

Let \( c \) and \( t \) be real numbers such \( c > 0 \) and \( t > -1 \). Then,
\[
(1 - | z |^2)^{-c} \approx \int_{\mathbb{B}^n} \frac{(1 - | w |^2)^t}{1 - < z, w >^{n+1+t+c}} dm(w).
\] (4.19)

In above the notation \( a(z) \approx b(z) \) means that the ratio \( \frac{a(z)}{b(z)} \) has a positive limit as \( | z | \) goes to 1.

By using (4.19) with \( t = 2\nu - 2l - n - 1 \) and \( c = \nu - 2l \) we deduce easily that,
\[
\int_{\mathbb{B}^n} | B^l_\nu(z, w) | d\mu_\nu(w) \leq C,
\] (4.20)
for some positive constant \( C = C_{\nu,l} \).

It follows from above that for every \( F \in L^\infty(\mathbb{B}^n, d\mu_\nu) \) we have
\[
\| B^l_\nu F \|_\infty \leq C \| F \|_\infty,
\]
therefore \( B^l_\nu \) is bounded on \( L^\infty(\mathbb{B}^n, d\mu_\nu) \).

Next let \( F \in L^1(\mathbb{B}^n, d\mu_\nu) \). Then,
\[
\| B^l_\nu F \|_1 \leq \int_{\mathbb{B}^n} F(w) (\int_{\mathbb{B}^n} | B^l_\nu(z, w) | d\mu_\nu(z)) d\mu_\nu(w),
\]

it follows from (4.20) that
\[
\| B^l_\nu F \|_1 \leq C \| F \|_1,
\]
therefore the Berezin transform is bounded on \( L^1(\mathbb{B}^n, d\mu_\nu) \) and the result follows from The Riesz-Thorin Theorem.

5. The Berezin Transform as Function of the Invariant Laplacian.

In this section, we shall express the transform \( B^l_\nu \) as a function of the invariant Laplacian \( \Delta_\nu \). To this end we need the following lemma.
Lemma 5.1. Let \( j = 0, 1, \ldots, \left\lfloor \frac{\nu - n}{2} \right\rfloor \). Then, the complex numbers
\[
\xi_j = (\nu - n - 2j)i, \quad \text{Im}(\xi_j) > 0,
\]  
(5.1)
are poles of the function \( \eta(\lambda) = (C_{\nu}(\lambda)C_{\nu}(-\lambda))^{-1} \) and we have:
\[
c_j = 2^{2(n-\nu)} \frac{\Gamma(n)}{\pi^n} (-i \text{Res}(\eta; \lambda = \xi_j)),
\]  
(5.2)
where \( \text{Res}(\eta; \lambda = \xi_j) \) means the residue of the function \( \eta(\lambda) \) at \( \lambda = \xi_j \), and the constant \( c_j \) is the constant defined by (2.20).

Proof. From the expression of \( C_{\nu} \), the function \( \eta(\lambda) \) can be written as
\[
\eta(\lambda) = \frac{2^{2(n-\nu)}}{(\Gamma(n))^2} \frac{\Gamma(\frac{n+\nu+i\lambda}{2})\Gamma(\frac{n-\nu-i\lambda}{2})\Gamma(\frac{n-\nu+i\lambda}{2})}{\Gamma(\nu - n - 2j)\Gamma(2j + n - \nu)} \Gamma(\frac{n-\nu+i\lambda}{2}),
\]  
(5.3)
is not difficult to see that,
\[
\text{Res}(\eta; \lambda = \xi_j) = \frac{2^{2(n-\nu)}}{(\Gamma(n))^2} \frac{\Gamma(\nu - j)\Gamma(j + n)\Gamma(j + n - \nu)}{\Gamma(\nu - n - 2j)\Gamma(2j + n - \nu)} \text{Res}(\Gamma(\frac{n-\nu+i\lambda}{2}); \lambda = \xi_j).
\]  
(5.4)
By a direct calculus we get,
\[
\text{Res}(\eta; \lambda = \xi_j) = \frac{2^{2(n-\nu)}}{(\Gamma(n))^2} \frac{\Gamma(\nu - j)\Gamma(j + n)\Gamma(j + n - \nu)}{\Gamma(\nu - n - 2j)\Gamma(2j + n - \nu)} \frac{2(\nu - n)}{\pi^2} \frac{\Gamma(\nu - j)\Gamma(j + n)\Gamma(j + n - \nu)}{\Gamma(\nu - n - j + 1)} \frac{\pi i}{j!}.
\]  
(5.5)
Recall that the constant \( c_j \), is given by
\[
c_j = \frac{2\Gamma(n + j)(\nu - n - 2j)\Gamma(\nu - j)}{\pi^n \Gamma(n)j! \Gamma(\nu - n - j + 1)},
\]
From this expression of \( c_j \) and the equation (5.3) we obtain:
\[
\frac{c_j}{\text{Res}(f; \lambda = \xi_j)} = \frac{(-1)^j 2^{2(n-\nu)} \Gamma(n) \Gamma(2j + n - \nu) \Gamma(1 - (2j + n - \nu))}{\pi^n \Gamma(j + n - \nu) \Gamma(1 - (j + n - \nu))} (-i),
\]  
(5.6)
By using the formula (14), p.2,
\[
\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}, \quad z \neq 0, -1, 1, -2, 2, \ldots,
\]  
the equation (5.6) becomes,
\[
\frac{c_j}{\text{Res}(f; \lambda = \xi_j)} = \frac{(-1)^j 2^{2(n-\nu)} \Gamma(n) \sin \pi (j + n - \nu)}{\pi^n \sin \pi (2j + n - \nu)} (-i),
\]  
(5.7)
Thus we get the desired formula (5.2). This ends the proof.

Since the Berezin transform $B_{l}^{\nu}$ is a bounded operator on $L^{2}(\mathbb{B}^n, d\mu_{\nu})$ commuting with the representation $T_{\nu}$, it follows that it is a function of the $G$-invariant Laplacian $\Delta_{\nu}$.

Namely there exists a $\mathbb{C}$-valued Borelian function $h_{l}$ such that

$$B_{l}^{\nu} = h_{l}(\Delta_{\nu}).$$

In other hand, since we dispose of the spectral function (3.1) of the shifted $G$-invariant Laplacian $\tilde{\Delta}_{\nu} = -(\Delta_{\nu} + (n - \nu)^2)$, it will be natural to give $B_{l}^{\nu}$ in terms of $\tilde{\Delta}_{\nu}$, instead of $\Delta_{\nu}$. That is

$$B_{l}^{\nu} = f(\tilde{\Delta}_{\nu}),$$

for some complex valued Borelian function $f$ on $\mathbb{R}$.

The main result of this paper is:

**Theorem 5.1.** Let $l = 0, 1, 2, \ldots; [\frac{\nu - n}{2}]$, Then, the Berezin transform $B_{l}^{\nu}$ defined by (4.3) and (4.4) can be expressed in terms of the $G$-invariant operator $\Delta_{\nu}$ as:

$$B_{l}^{\nu} = \frac{\pi^{n}\Gamma(2(n))}{\Gamma(\nu - l)}(\frac{i}{2} \sqrt{-(\Delta_{\nu} + (n - \nu)^2)} + 3\nu - 4l - n)^2$$

$$\times \sum_{q=0}^{2l} (-1)^q A_q \frac{S_q(-((\Delta_{\nu} + (n - \nu)^2), 3\nu - 4l - n, \nu + n, n - \nu))}{\Gamma(2(\nu - l) + q)\Gamma(\nu - 2l + q)}$$

Where $S_q$ denotes the continuous dual Hahn polynomial.

**Proof.** Recall from (3.20) and (3.21) that the operator $f(\tilde{\Delta}_{\nu})$ is the Distributional operator with the Schwartz kernel:

$$\Psi_{g}(z, w) = \int_{\mathbb{R}} e(s, z, w)f(s)ds,$$

where $e(s, z, w)$ is the spectral function (3.1) associated to the operator $\tilde{\Delta}_{\nu}$.

Then, the equation $B_{l}^{\nu} = f(\tilde{\Delta}_{\nu})$, implies that:

$$B_{l}^{\nu}(z, w) = \int_{\mathbb{R}} e(s, z, w)f(s)ds.$$
Hence,

\[ B^\nu_t(z, w) = \frac{\Gamma(n)}{4\pi^{n+1}2^{(\nu-n)}}(1 - <z, w>^{\nu})^\nu \int_0^{+\infty} s^{\nu} |C_\nu(S^2)|^{-2}\phi^{n-1,-\nu}_{\frac{1}{2}}(t)f(s)ds \]

\[ + \sum_{j=0}^{\nu+n} c_j \phi^{(n-1,-\nu)}_{\lambda_j}(t)f(s_j) \]  

(5.12)

where \( s_j^2 = \lambda_j \) and \( \lambda_j \) are the parameters defined in (2.12).

Recall from (4.14), that the Berezin kernel \( B^\nu_t(z, w) \) has the following expression:

\[ B^\nu_t(z, w) = (1 - <z, w>)^{-\nu}\cosh^{(d-2\nu)}(d(z, w))|F(-l, -l+\nu, n, \tanh^2(d(z, w)))|^2. \]

Then, after replacing the Berezin kernel \( B^\nu_t(z, w) \) by its above expression in where we have set \( t = d(z, w) \), and using the change of variable \( s = \lambda^2 \), in the first integral of (5.12) and \( s_j = \lambda_j^2 \), in the discreet part, we obtain:

\[ \cosh^{(d-2\nu)}(t)|F(-l, -l+\nu, n, \tanh^2(t))|^2 = \]

\[ \frac{\Gamma(n)}{4\pi^{n+1}2^{(\nu-n)}} \int_0^{+\infty} 2|C_\nu(\lambda)|^{-2}\phi^{n-1,-\nu}_{\lambda}(t)f(\lambda^2)d\lambda \]

\[ + \sum_{j=0}^{\nu+n} c_j \phi^{(n-1,-\nu)}_{\lambda_j}(t)f(\lambda_j^2). \]  

(5.13)

Putting \( g(\lambda) = f(\lambda^2) \). Then, the above equation becomes:

\[ \frac{\Gamma(n)}{2\pi^{n+1}2^{(\nu-n)}} \int_0^{+\infty} |C_\nu(\lambda)|^{-2}\phi^{(n-1,-\nu)}_{\lambda}(t)g(\lambda)d\lambda + \sum_{j=0}^{\nu+n} c_j \phi^{(n-1,-\nu)}_{\lambda_j}(t)g(\lambda_j). \]

\[ = \cosh^{(d-2\nu)}(t)|F(-l, -l+\nu, n, \tanh^2(t))|^2. \]  

(5.14)

Now, by using the fact :\( \phi^{(n-1,-\nu)}_{\lambda}(t) = \phi^{(n-1,-\nu)}_{-\lambda}(t) \) and \( g(\lambda) = g(-\lambda) \)

and replacing the constants \( c_j \) by theirs expressions given in the equation (5.2) with taking into a count that \( \zeta_j = -\lambda_j \) the equation (5.14) becomes:

\[ \frac{1}{2\pi} \int_0^{+\infty} |C_\nu(\lambda)|^{-2}\phi^{(n-1,-\nu)}_{\lambda}(t)g(\lambda)d\lambda \]

\[ + \sum_{\zeta_j \in D_\nu} (-i\text{Res} ((C_\nu(\lambda)C_\nu(-\lambda))^{-1}), \lambda = \xi_j) \phi^{(n-1,-\nu)}_{\zeta_j}(t)g(\xi_j), \]
\[
\frac{2^{2(\nu-n)}\pi^n}{\Gamma(n)}(\cosh t)^{-\mu}(F(-l, \nu - l, n, \tanh^2 t))^2,
\]  \hspace{1cm} (5.15)

where \( \mu = 2\nu - 4l \), and the set \( D_\nu \) is given by,
\[
D_\nu = \{ \zeta_j = i(\nu - n - 2j), j = 0, 1, 2, ..., \nu - n - 2j > 0 \}.
\]

Now, we recall from [13], [11], some properties of the Fourier-Jacobi transform. Assume that \( \alpha > 1 \) and \( |\beta| > \alpha + 1 \). Then, the Fourier-Jacobi transform of a \( C^\infty \) - compactly supported function \( f \) on \( \mathbb{R} \) is defined by
\[
\hat{\varphi}(\lambda) = \int_0^{+\infty} \phi(t)\phi^{(\alpha,\beta)}_\lambda(t)\Delta_{\alpha,\beta}(t)dt
\]  \hspace{1cm} (5.16)

its inverse is given by:
\[
\varphi(t) = \frac{1}{2\pi} \int_0^{+\infty} \hat{\varphi}(\lambda)|C_{\alpha,\beta}(\lambda)|^{-2}d\lambda + \sum_{\lambda \in D_{\alpha,\beta}} d_{\alpha,\beta}(\lambda)\hat{\varphi}(\lambda)
\]  \hspace{1cm} (5.17)

Where,
\[
d_{\alpha,\beta}(\lambda) = -i\text{Res}(C_{\alpha,\beta}(z)C_{\alpha,\beta}(z)^{-1}, z = \lambda)\phi^{(\alpha,\beta)}_\lambda(\lambda),
\]  \hspace{1cm} (5.18)

with
\[
D_{\alpha,\beta} = \{ i(|\beta| - \alpha - 1 - 2j) : j = 0, ..., |\beta| - \alpha - 1 - 2j > 0 \} \hspace{1cm} (5.19)
\]
\[
\Delta_{\alpha,\beta}(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1},
\]  \hspace{1cm} (5.20)
\[
C_{\alpha,\beta}(\lambda) = 2^{\rho-\i\lambda}\frac{\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{\alpha+\beta+1+i\lambda}{2})\Gamma(\frac{\alpha-\beta+1+i\lambda}{2})}, \rho = \alpha + \beta + 1
\]  \hspace{1cm} (5.21)

and \( \phi^{(\alpha,\beta)}_\lambda(t) \), is the Jacobi function defined in the proposition [3,1].

Here, from (5.17), is not difficult to see that the the left hand side of the integral equation, (5.15), is nothing other than the inverse of Fourier-Jacobi transform of the function \( g \), with \( \alpha = n - 1; \beta = -\nu. \)

In other hand, the function:
\[
h(t) = \frac{2^{2(\nu-n)}\pi^n}{\Gamma(n)}(\cosh t)^{-\mu}(F(-l, \nu - l, n, \tanh^2 t))^2
\]  \hspace{1cm} (5.22)

given in the right hand side of the equation (5.15) can be written as:
\[
h(t) = (\cosh t)^{-\mu}\psi(\cosh^{-2} t)
\]  \hspace{1cm} (5.23)

where \( \mu = 2\nu - 4l \) and \( \psi(t) \), is the \( C^\infty \) function on \([0, 1]\) given by:
\[
\psi(t) = \frac{2^{2(\nu-n)}\pi^n}{\Gamma(n)}(F(-l, \nu - l, n, 1 - t))^2. \tag{5.24}
\]

More, in our case \(\alpha = n - 1, \beta = -\nu\), it is easy to see that we have the inequality \(\mu > \rho\), where \(\rho\), is the parameter defined in (5.21). Then, thanks to (11), \(\hat{h}(\lambda)\) is well defined on \(\mathbb{R}\). Then by (5.16) the function \(g\) involved in the integral equation (5.15) is given by:

\[
g(\lambda) = \int_{0}^{+\infty} h(t) \phi^{(n-1,-\nu)}_\lambda(t) \Delta_{n-1,-\nu}(t) dt
\]

\[
= \frac{\pi^n}{\Gamma(n)} \int_{0}^{+\infty} \cosh^{-\mu} t \left| F(-l, \nu - l, n, \tanh^2 t) \right|^2 \phi^{(n-1,-\nu)}_\lambda(t)
\times (\sinh t)^{2n-1} (\cosh t)^{1-2\nu} dt. \tag{5.25}
\]

Using the expression of Jacobi-polynomials \([14], p.39\)

\[
P_k^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)_k}{k!} F(-k, \alpha + \beta + 1 + k, \alpha + 1; \frac{1 - x}{2}), \tag{5.26}
\]

for \(k = l, \alpha = n - 1, \beta = \nu - n - 2l\) and \(x = 1 - 2 \tanh t\), we obtain:

\[
F(-l, \nu - l, n, \tanh^2 t) = \frac{l!}{(n)_l} P_l^{(n-1,\nu-n-2l)}(1 - 2 \tanh^2 t). \tag{5.27}
\]

Then, by using (5.27), the Equation (5.25) becomes:

\[
g(\lambda) = C_{n,l,\nu} \int_{0}^{+\infty} (\cosh t)^{(4l-2\nu)} (P_l^{(n-1,\nu-n-2l)}(1 - 2 \tanh^2 t))^2
\times \phi^{(n-1,-\nu)}_\lambda(t) (\sinh t)^{2n-1} (\cosh t)^{1-2\nu} dt. \tag{5.28}
\]

where,

\[
C_{n,l,\nu} = \frac{\pi^n (l!)^2}{(n)_l^2 \Gamma(n)} \tag{5.29}
\]

Use again (4.15) to rewrite the Jacobi function \(\phi^{(n-1,-\nu)}_\lambda\) as follows

\[
\phi^{(n-1,-\nu)}_\lambda(t) = (1 - \tanh^2 t)^{-\frac{i\lambda + n - \nu}{2}} F\left(\frac{i\lambda + n - \nu}{2}, \frac{i\lambda + n + \nu}{2}, n, \tanh^2 t\right).
\]

Henceforth
\[ g(\lambda) = C_{n,l,\nu} \int_0^{+\infty} (1 - \tanh^2 t) \frac{i\lambda - n - 4l + 3\nu}{2} \left\{ P_t^{(n-1,\nu-n-2l)}(1 - 2\tanh^2 t) \right\}^2 \]
\[ \times F\left(\frac{i\lambda + n - \nu}{2}, \frac{i\lambda + n + \nu}{2}, n, \tan^2 t\right) \tan^2 n^2 \, t \, dt. \] (5.30)

Next make the change of variable \( y = \tanh^2 t \) to rewrite the above integral as,
\[ g(\lambda) = C_{n,l,\nu} \int_0^1 (1 - y) \frac{i\lambda - n - 4l + 3\nu}{2} y^{n-1} \left\{ (P_t^{(n-1,\nu-n-2l)}(1 - 2y)) \right\}^2 \]
\[ \times F\left(\frac{i\lambda + n - \nu}{2}, \frac{i\lambda + n + \nu}{2}, n, y\right) dy. \] (5.31)

By using the following formula power expansion for the product of two Jacobi polynomials \[18\] we get:
\[ \{(P_t^{(n-1,\nu-n-2l)}(1 - 2y))\}^2 = \frac{\Gamma^2(l + n)}{l!2^2\Gamma^2(\nu - l)} \sum_{q=0}^{2l} (-1)^q A_q(2y)^q, \] (5.32)

where
\[ A_q = 2^{-q} \sum_{p=\max(0, q-l)}^{\min(l, q)} \binom{l}{q-p} \binom{l}{p} \frac{\Gamma(\nu - l + q - p)\Gamma(\nu - l + p)}{\Gamma(n + p)\Gamma(n + q - p)}. \] (5.33)

Hence, we are lead to compute the following integral:
\[ I_q(\lambda) = \int_0^1 (1 - y) \frac{i\lambda - n - 4l + 3\nu}{2} y^{n+q-1} F\left(\frac{i\lambda + n - \nu}{2}, \frac{i\lambda + n + \nu}{2}, n, y\right) dy. \] (5.34)

For this, use the following identity \[10\] p:813
\[ \int_0^1 x^{\rho-1}(1 - x)^{\sigma-1} F(\alpha, \beta, \gamma; x) \, dx = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho + \sigma)} b_{2}(\alpha, \beta, \rho; \gamma, \rho + \sigma; 1), \] (5.35)

for all \( \alpha, \beta, \gamma, \rho \) and \( \sigma \) such that \( \Re \rho > 0, \Re \sigma > 0 \) and \( \Re(\gamma + \sigma - \alpha - \beta) > 0 \), to get:
\[ I_q(\lambda) = \frac{\Gamma(n + q)\Gamma\left(\frac{i\lambda + 3\nu - 4l - n}{2}\right)}{\Gamma\left(\frac{i\lambda + 3\nu - 4l + n}{2} + q\right)} \]
Thus, we have established that:

\[ g(\lambda) = \frac{\Gamma(n)\pi^n}{2\Gamma^2(\nu-l)} \sum_{q=0}^{2l} (-1)^q A_q I_q(\lambda). \] (5.37)

We will use the following identity on hypergeometric functions (2, p.593)

\[ \begin{align*}
3F_2(a, b, c; d, e; 1) &= \frac{\Gamma(d(e)\Gamma(d+e-c-a-b)}{\Gamma(c)\Gamma(d+e-a-c)\Gamma(d+e-b-c)} \\
3F_2(d-e-c, d+e-c-a-b; d+e-a-c, d+e-b-c; 1); \quad (5.38)
\end{align*} \]

Finally, we get,

\[ I_q(\lambda) = \frac{\Gamma(n)\Gamma(-\frac{i\lambda+3\nu-4l-n}{2})\Gamma(i\lambda+3\nu-4l-n)}{\Gamma(2(\nu-l))\Gamma(\nu-2l)} \\
\times 3F_2(-q, \frac{i\lambda+3\nu-4l-n}{2}, -\frac{i\lambda+3\nu-4l-n}{2} ; 2(\nu-l), \nu-2l; 1). \] (5.39)

By using the formula (12, p.29)

\[ 3F_2(-j, a+ix, a-ix; a+b, a+c; 1) = \frac{S_j(x^2, a, b, c)}{(a+b)_j(a+c)_j} \] (5.40)

for \( x = \frac{\lambda}{2}, a = \frac{3\nu-4l-n}{2}, b = \frac{\nu+n}{2}, c = \frac{n-\nu}{2} \) and \( j = q \).

The above hypergeometric involved in (5.39) becomes:

\[ 3F_2(-q, \frac{i\lambda+3\nu-4l-n}{2}, -\frac{i\lambda+3\nu-4l-n}{2} ; 2(\nu-l), \nu-2l; 1) = \frac{S_q(\frac{\lambda^2}{4}, \frac{3\nu-4l-n}{2}, \frac{n-\nu}{2})}{(2\nu-l)_q((\nu-2l)_q \] (5.41)

where \( S_j(x^2, a, b, c) \), is the continuous dual Hahn polynomial.

Henceforth, we obtain:

\[ I_q(\lambda) = \frac{\Gamma(n)\Gamma(i\lambda+3\nu-4l-n)\Gamma(-i\lambda+3\nu-4l-n)}{\Gamma(2(\nu-l)+q)\Gamma(\nu-2l+q)} \frac{S_q(\frac{\lambda^2}{4}, \frac{3\nu-4l-n}{2}, \frac{n-\nu}{2})}{(2\nu-l)_q((\nu-2l)_q} \] (5.42)

finally, we get,

\[ g(\lambda) = \frac{\pi^n\Gamma^2(n)}{2\Gamma^2(\nu-l)} |\Gamma(\frac{i\lambda+3\nu-4l-n}{2})|^2 \sum_{q=0}^{2l} (-1)^q A_q \frac{S_q(\frac{\lambda^2}{4}, \frac{3\nu-4l-n}{2}, \frac{n-\nu}{2})}{\Gamma(2(\nu-l)+q)\Gamma(\nu-2l+q)} \] (5.43)
Using the the fact \( f(s) = g(s^2) \), with \( s = \lambda^2 \), Hence, if we replace the spectral parameter \( s \) by \(-(\Delta_\nu + (n - \nu)^2)\), we obtain the desired result. This ends the proof.

6. The Berezin heat kernel.

As a direct consequence of the above theorem, we can derive easily the Heat kernel of the Berezin operator \( B_\nu^l \). Precisely we have the following.

**Proposition 6.1.** Let \( u(t, z) \), be the solution of the heat Cauchy problem associated to the \( B_\nu^l \), on \( \mathbb{B}^n \).

\[
(\partial_t + B_\nu^l)u(t, z) = 0, \quad (t, z) \in \mathbb{R} \times \mathbb{B}^n \tag{6.1}
\]

\[
u u(0, z) = \varphi(z), \quad \varphi \in C_0^\infty(\mathbb{B}^n). \tag{6.2}
\]

Then, \( u(t, z) \), is given by the integral formula,

\[
u u(t, z) = \int_{\mathbb{B}^n} H_\nu^l(t, z, w)\varphi(w)\,d\mu_\nu(w), \tag{6.3}
\]

where \( H_\nu^l(t, z, w) \), is the heat kernel given by:

\[
H_\nu^l(t, z, w) = (1 - \langle z, w \rangle)^{-\nu}\{ \frac{\Gamma(n)}{2\pi^{n/2}} \int_0^{+\infty} |C_\nu(\lambda)|^{-2} \phi_{\lambda-\nu}^{n-1,\nu}(d(z, w)) \exp(tg(\lambda)d\lambda)

+ (1 - \langle z, w \rangle)^{-\nu}\sum_{j=0}^{\nu-n} c_j \phi_{\lambda_j}^{(n-1,\nu)}(d(z, w)) \exp(tg(\lambda_j)) \}. \tag{6.4}
\]

where

\[
g(\lambda) = \frac{\pi^n\Gamma^2(n)}{2\Gamma^2(\nu-l)} |\Gamma(\frac{i\lambda + 3\nu - 4l - n}{2})|^2 \sum_{q=0}^{2l} (-1)^q A_q S_q(\frac{\lambda^2}{4}, \frac{3\nu-4l-n}{2}, \frac{\nu+n}{2}, \frac{n-\nu}{2}, \frac{\nu-l}{2}, \frac{\nu+2l}{2}; \frac{\nu}{2}) \tag{6.5}
\]

Where \( S_j \) is the continuous dual Hahn polynomial defined by (5.40), \( c_j \) the constant defined by (2.20) and \( \lambda_j = i(2j + n - \nu) \).

**Proof.**

It is not difficult to see that the solution \( u(t, z) \), is given by the action of the semigroup \( e^{-tB_\nu^l} \) on the initial data \( \varphi(z) \),

\[
u u(t, z) = e^{-tB_\nu^l} \varphi(z) = h_t(\Delta_\nu), \quad h_t(s) = \exp(-tf(s)), \tag{6.6}
\]

where \( f(s) \) is the function given by \( f(s) = g(s^2) \) and \( g \) the function given in (5.43). Then, by using (3.20) and (3.21), a direct calculus gives the above expression of the heat kernel.
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