ON THE BRAUER GROUP OF REAL ALGEBRAIC SURFACES

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Dedicated to Professor Igor R. Shafarevich on the occasion of his seventieth birthday

Abstract. Let $X$ be a real projective algebraic manifold, $s$ numerates connected components of $X(\mathbb{R})$ and $2Br(X)$ the subgroup of elements of order 2 of cohomological Brauer group $Br(X)$.

We study the natural homomorphism $\xi : 2Br(X) \to (\mathbb{Z}/2)^s$ and prove that $\xi$ is epimorphic if $H^3(X(\mathbb{C})/G;\mathbb{Z}/2) \to H^3(X(\mathbb{R});\mathbb{Z}/2)$ is injective. Here $G = Gal(\mathbb{C}/\mathbb{R})$.

For an algebraic surface $X$ with $H^3(X(\mathbb{C})/G;\mathbb{Z}/2) = 0$ and $X(\mathbb{R}) \neq \emptyset$, we give a formula for $\dim 2Br(X)$.

As a corollary, for a real Enriques surfaces $Y$, the $\xi$ is epimorphic and $\dim 2Br(Y) = 2s - 1$ if both liftings of the antiholomorphic involution of $Y$ to the universal covering $K3$- surface $X$ have non-empty sets of real points (this is the general case). For this case, we also give a formula for the number $s_{nor}$ of non-orientable components of $Y$ which is very important for the topological classification of real Enriques surfaces.

§0. Formulation of basic results

In the paper of R. Sujatha and the author [N-S], the Brauer group of a real Enriques surface was studied. Here we continue the study of Brauer group with the remark that most of the results of these paper generally valid for an arbitrary smooth projective real algebraic surface.

Let $X$ be a projective algebraic variety over the field $\mathbb{R}$ of real numbers. Let

$$Br'(X) = H^2_{et}(X; G_m)$$

denote the cohomological Brauer group of $X$. See a definition in the book of J. Milne [Mi], for example. We mention that the cohomological Brauer group is very closely related with the more interesting classical Brauer group $Br(X)$ classifying Azumaya algebras over $X$ (see the papers of A. Grothendieck [Gr2] and the book [Mi]). For example, it is known that $Br(X) \subset Br'(X)$. For curves and smooth surfaces it gives an isomorphism. But we will only consider here the cohomological Brauer group $Br'(X)$.

Let $X(\mathbb{R})$ denote the space of $\mathbb{R}$-rational points of $X$ with the Euclidean topology and $s$ denote the number of real connected components of this space. Let $2Br'(X)$ denote the group of elements of order two in $Br'(X)$. If $P \in X(\mathbb{R})$ is a real point of $X$, we get a natural map $2Br'(X) \to 2Br'(P) \cong \mathbb{Z}/2$. It is shown in [CT-P] that this map does not depend from a choice of the point $P$ in a connected component of $X(\mathbb{R})$. Thus, the canonical map

$$(0-1) \quad 2Br'(X) \to (\mathbb{Z}/2)^s$$
is defined.

We mention that a studying of the map (0–1) and a description of the Brauer group of \(X\) is very important for calculation of a such interesting group connected with \(X\) as the Witt group \(W(X)\). See R. Sujatha [Su].

It is shown in the paper of J.-L. Colliot-Thélène and R. Parimala [CT-P] that the map (0–1) is epimorphic if \(X\) is a smooth projective algebraic surface and \(H^3(X(\mathbb{C}); \mathbb{F}_2) = 0\) (here \(\mathbb{F}_2 = \mathbb{Z}/2\)). This is a generalization of the old result of E. Witt [W] about curves (compare with Remark 1.8 below). There doesn’t seem to be any known example of a surface where the map (0–1) fails to be epimorphic.

This paper is devoted to studying of this map (0–1) and also calculating of \(\dim_2 Br' (X)\). Our idea is to interpret \(\dim_2 Br' (X)\) and the map (0–1) purely topologically, and apply to them topological considerations.

We prove here the following basic result where \(G\) is the group of order two generated by the antiholomorphic involution \(g\) of \(X(\mathbb{C})\) defined by the structure of real algebraic variety on \(X\).

**Theorem 0.1.** Let \(X/\mathbb{R}\) be an algebraic projective manifold (smooth) over the field \(\mathbb{R}\) of real numbers.

Then, the homomorphism (0–1) is epimorphic if \(H^3(X(\mathbb{C}); G; F_2) = 0\). More generally, the homomorphism (0–1) is epimorphic if the kernel of the homomorphism

\[
i^*: H^3(X(\mathbb{C}); G; F_2) \rightarrow H^3(X(\mathbb{R}); F_2)
\]

is equal to zero. Here \(i: X(\mathbb{R}) \subset X(\mathbb{C}); G\) denote the embedding.

**Proof.** See Theorem 1.6 below. In fact, in Theorem 1.6, we give a precise topological obstruction to epimorphicity of the map (0–1). This obstruction is zero if the kernel of the homomorphism \(i^*\) above is zero.

We mention that for smooth curves \(X\) the group \(H^3(X(\mathbb{C}); G; F_2) = 0\), thus the map (0–1) is epimorphic (it is well-known, compare with [W]). For surfaces \(X\) the group \(H^3(X(\mathbb{R}); F_2) = 0\), and \(\text{Ker } i^* = H^3(X(\mathbb{C}); G; F_2)\).

Now, let us show that, from Theorem 0.1, the result of J.-L. Colliot-Thélène and R. Parimala mentioned above follows.

Let \(X\) be a smooth projective algebraic surface and \(H^3(X(\mathbb{C}); F_2) = 0\). Then, by Poincaré duality, we have \(H_1(X(\mathbb{C}); F_2) = 0\). If \(X(\mathbb{R}) \neq \emptyset\), then any loop in \(X(\mathbb{C}); G\) with the beginning on \(X(\mathbb{R})\) has a lifting to a loop on \(X(\mathbb{C})\). It follows that the canonical homomorphism \(H_1(X(\mathbb{C}); F_2) \rightarrow H_1(X(\mathbb{C}); G; F_2)\) is epimorphic. Thus, \(H_1(X(\mathbb{C}); G; F_2) = 0\).

For the dimension 2 the quotient space \(X(\mathbb{C}); G\) is homeomorphic to a smooth compact 4-dimensional manifold. By Poincaré duality, we then get that \(H^3(X(\mathbb{C}); G; F_2) = 0\). This proves the statement.

We apply the Theorem 0.1 to real Enriques surfaces.

By a complex Enriques surface \(Y\) over \(\mathbb{C}\), we mean a non-singular minimal projective algebraic surface \(Y/\mathbb{C}\) such that the invariants \(\kappa(Y) = p_g(Y) = q(Y) = 0\). These are equivalent to irregularity \(q(Y) = 0\) and \(2K_Y = 0\) but \(K_Y \neq 0\) where \(K_Y\) is the canonical class of \(Y\). One may find all information about Enriques surfaces we need in the books [A] and [C-D].

By a real Enriques surface \(Y/\mathbb{R}\), we mean a projective algebraic surface \(Y/\mathbb{R}\) such that \(Y \otimes \mathbb{R} \mathbb{C}\) is a complex Enriques surface. Universal covering complex surface of an Enriques surface \(X(\mathbb{C})\) is a \(K3\) surface, \(X(\mathbb{C})\) (see [A] and [C-D]) which twice gives a complex Enriques surface.
covers the Enriques surface $Y(C)$. We denote by $\tau$ the holomorphic involution on $X(C)$ of this covering. There are precisely two liftings $\sigma$ and $\tau\sigma$ on $X(C)$ of the antiholomorphic involution $\theta$ of $Y(C)$ corresponding to the real structure on $Y$. Besides, one can see very easily that if $Y(R) \neq \emptyset$, then both $\sigma$ and $\tau\sigma$ are antiholomorphic involutions of $X(C)$. Thus, $\sigma$ and $\tau\sigma$ define two real structures $X_\sigma$ and $X_{\tau\sigma}$ on the $K3$-surface $X$. We denote by

$$X_\sigma(R) = X(C)^\sigma, \quad X_{\tau\sigma}(R) = X(C)^{\tau\sigma}$$

the real parts of the real $K3$-surfaces $X_\sigma$ and $X_{\tau\sigma}$ corresponding to these real structures respectively. Since $\tau$ has no fixed points on $X(C)$, it follows that the sets $X_\sigma(R)$ and $X_{\tau\sigma}(R)$ have an empty intersection. From the Theorem 0.1, we get

**Corollary 0.2.** Let $Y$ be a real Enriques surface with the antiholomorphic involution $\theta$, and the real part $Y(R) \neq \emptyset$. Suppose that the real parts $X_\sigma(R)$ and $X_{\tau\sigma}(R)$ of both liftings $\sigma$ and $\tau\sigma$ of $\theta$ to the universal covering $K3$-surface $X(C)$ are non-empty.

Then the canonical map $(0–1)$ corresponding to the real Enriques surface $Y$ is epimorphic.

**Proof.**

$$Y(C)/\{id_{Y(C)}, \theta\} = X(C)/\{id_{X(C)}, \tau, \sigma, \tau\sigma\} = (X(C)/\{id_{X(C)}, \sigma\})/\{id, \tau\sigma \mod \{id_{X(C)}, \sigma\}\}.$$ 

Here involutions $\sigma$ and $\tau\sigma \mod \{id_{X(C)}, \sigma\}$ have non-empty sets of fixed points because real parts of both involutions $\sigma$ and $\tau\sigma$ of $X(C)$ are non-empty and are not coincided. Since for a $K3$-surface $X$, the group $H_1(X(C); F_2) = 0$, it follows like above that $H_1(X(C)/\{id_{X(C)}, \sigma\}; F_2) = 0$ and $H_1(Y(C)/\{id_{Y(C)}, \theta\}; F_2) = 0$. The topological space $Y(C)/\{id_{Y(C)}, \theta\}$ is isomorphic to a smooth compact 4-dimensional manifold. By Poincaré duality, then $H^3(Y(C)/\{id_{Y(C)}, \theta\}; F_2) = 0$. By Theorem 0.1, the map $(0–1)$ is epimorphic for the real Enriques surface $Y$.

The same considerations show that if one of involutions $\sigma$ or $\tau\sigma$ has an empty set of real points, then

$$H_1(Y(C)/\{id_{Y(C)}, \theta\}; F_2) = F_2 \text{ and } H^3(Y(C)/\{id_{Y(C)}, \theta\}; F_2) = F_2.$$ 

If $X_\sigma(R) \neq \emptyset$ but $X_{\tau\sigma}(R) = \emptyset$ then the surface $X(C)/\{id_{X(C)}, \sigma\}$ is a 2-sheeted universal covering of the surface $Y(C)/\{id_{Y(C)}, \theta\}$. Thus, the Corollary 0.2 gives precisely the case when the Theorem 0.1 may be applied to real Enriques surfaces.

We discuss in Remark 1.7 below a chance of constructing a counterexample to epimorphism of the map $(0–1)$ using real Enriques surfaces $Y$ above with $H^3(Y(C)/G; F_2) = F_2$ (equivalently, with $X_\sigma(R) \neq \emptyset$ but $X_{\tau\sigma}(R) = \emptyset$). It is not difficult to show that real Enriques surfaces with the condition $H^3(Y(C)/G; F_2) = F_2$ do exist. For the most part of real Enriques surfaces (from the point of view of the number of connected components of the moduli space) both involutions $\sigma$ and $\tau\sigma$ have a non-empty set of real points. But for some real Enriques surfaces one of these involutions may have an empty set of real points.

The following results are devoted to a calculation of the dimension of étale cohomology groups with coefficients $F_2$ and $Br(X)$.
We begin with the following general remark about a real algebraic variety $X$.
We recall that the Kummer sequence

\[(0–2)\quad 0 \to \mu_2 \to \mathbb{G}_m \to \mathbb{G}_m \to 0\]

yields the exact sequence

\[(0–3)\quad 0 \to \text{Pic} \, X/2\text{Pic} \to H^2_{\text{et}}(X; \mu_2) \to 2 \text{Br}'(X) \to 0.\]

If $X(\mathbb{R})$ is non-empty, $\text{Pic} \, X = (\text{Pic} \, (X \otimes \mathbb{C}))^G$ (it is well-known [Ma] and not difficult to see). Thus, from (0–3), we have

\[(0–4)\quad \dim 2\text{Br}'(X) = \dim H^2_{\text{et}}(X; \mu_2) - \dim (\text{Pic} \, (X \otimes \mathbb{C}))^G/2(\text{Pic} \, (X \otimes \mathbb{C}))^G.\]

The dimension of the étale cohomology group $H^2_{\text{et}}(X; \mu_2) = H^2_{\text{et}}(X; \mathbb{F}_2)$ is estimated using the Serre-Hochschild spectral sequence where $G = \text{Gal}(\mathbb{C}/\mathbb{R})$,

\[(0–5)\quad E_2^{p,q} = H^p(G; H^q_{\text{et}}(X \otimes \mathbb{C}; \mathbb{F}_2)) \Rightarrow H^{p+q}_{\text{et}}(X; \mathbb{F}_2),\]

where for a complex manifold $X \otimes \mathbb{C}$ we have $H^q_{\text{et}}(X \otimes \mathbb{C}; \mathbb{F}_2) = H^q(X(\mathbb{C}); \mathbb{F}_2)$ (see [Mi], for example).

In the §2, we prove the following results which permit to calculate the dimension of the étale cohomology groups with coefficients $\mathbb{F}_2$ and the 2-torsion of the Brauer group for surfaces satisfying to the condition of Theorem 0.1. These results show that the class of real smooth projective surfaces $X$ satisfying to the condition of Theorem 0.1 is very nice (easy to work with).

**Theorem 0.3.** Let $X/\mathbb{R}$ be a real smooth projective algebraic surface such that $X(\mathbb{R}) \neq \emptyset$ and $H^3(X(\mathbb{C})/G; \mathbb{F}_2) = 0$. Then the Serre-Hochschild spectral sequence (0–5) degenerates and

\[
\begin{align*}
\dim H^0_{\text{et}}(X; \mathbb{F}_2) &= 1; \\
\dim H^1_{\text{et}}(X; \mathbb{F}_2) &= \dim H^1(X(\mathbb{C}); \mathbb{F}_2) + 1; \\
\dim H^2_{\text{et}}(X; \mathbb{F}_2) &= \dim H^2(X(\mathbb{C}); \mathbb{F}_2)^G + \dim H^1(X(\mathbb{C}); \mathbb{F}_2) + 1; \\
\dim H^3_{\text{et}}(X; \mathbb{F}_2) &= 2 \dim H^2(X(\mathbb{C}); \mathbb{F}_2)^G - \dim H^2(X(\mathbb{C}); \mathbb{F}_2) \\
&\quad + 2 \dim H^1(X(\mathbb{C}); \mathbb{F}_2) + 1
\end{align*}
\]

\[
\dim H^k_{\text{et}}(X; \mathbb{F}_2) = 2 \dim H^2(X(\mathbb{C}); \mathbb{F}_2)^G - \dim H^2(X(\mathbb{C}); \mathbb{F}_2) \\
&\quad + 2 \dim H^1(X(\mathbb{C}); \mathbb{F}_2) + 2
\]

for $k \geq 4$.

Using Theorem 0.3 and (0–4), (0–5), we get...
Theorem 0.4. Let $X/R$ be a real smooth projective algebraic surface such that $X(R) \neq \emptyset$ and $H^3(X(C)/G; F_2) = 0$.

Then
\[
\dim 2Br'(X) = 2s - 1 + h^{2,0}(X(C)) + h^{1,1}_-(X(C)) - \rho_+(X \otimes C).
\]

Here $h^{1,1}_-(X(C)) = \dim H^{1,1}_-(X(C))$ where
\[
H^{1,1}_-(X(C)) = \{ x \in H^{1,1}(X(C)) \mid g(x) = -x \}
\]
is the set of potentially real algebraic cycles. And $\rho_+(X \otimes C) = \dim (\text{Pic}(X \otimes C) \otimes C)^G$. The characteristic class map gives an injection of $(\text{Pic}(X \otimes C) \otimes C)^G$ to $H^{1,1}_-(X(C))$.

For an Enriques surface, $h^{2,0}(Y(C)) = 0$ and all cycles are algebraic. For a real Enriques surface $Y$, we have seen above that the condition $H^3(Y(C)/G; F_2) = 0$ is equivalent to the condition of Corollary 0.2. Thus, we get

Corollary 0.5. Let $Y$ be a real Enriques surface with the antiholomorphic involution $\theta$ and the real part $Y(R) \neq \emptyset$. Suppose that the real parts $X_\sigma(R)$ and $X_{\tau\sigma}(R)$ of both liftings $\sigma$ and $\tau\sigma$ of $\theta$ to the universal covering $K3$-surface $X(C)$ are not empty.

Then the Serre-Hochschild spectral sequence (0-5) degenerates and
\[
\dim 2Br'(Y) = 2s - 1
\]
where $s$ is the number of real connected components of $Y(R)$.

We mention that for a real rational surface $Z$ with a non-empty set of real points $Z(R)$ the same results were known: The map (0-1) is epimorphic and $\dim 2Br'(Z) = 2s - 1$. It is also known that the Witt group $W(Z) \cong (Z)^s \oplus (Z/2)^{s-1}$. See [Su]. Perhaps, the last result about the Witt group also valid for real Enriques surfaces with non-empty sets $X_\sigma(R)$ and $X_{\tau\sigma}(R)$.

Using results of [N-S], we may prove some additional results about Brauer groups of real Enriques surfaces which also valid if one of the sets $X_\sigma(R), X_{\tau\sigma}(R)$ is empty.

In [N-S], the important invariants $b(Y)$ and $\epsilon(Y)$ of a real Enriques surface $Y$ with an antiholomorphic involution $\theta$ were introduced. Here
\[
b(Y) = \dim H^2(Y(C); F_2)^\theta - \dim (\text{Pic} Y \otimes C)^\theta / 2(\text{Pic} Y \otimes C)^\theta + 1.
\]
The invariant $\epsilon(Y) = 1$ if the differential $d_2^{0,2}$ of the Hochschild-Serre spectral sequence (0-5) is zero, and $\epsilon(Y) = 0$ otherwise. We have the following results from [N-S] about these invariants:

(0-6) $\dim 2Br'(Y) = b(Y) + \epsilon(Y)$ if $Y(R) \neq \emptyset$,

(0-7) $b(Y) \geq 2s - 2$ for any $Y$.

Thus, by (0-6) and (0-7), for any real Enriques surface $Y$,

(0-8) $\dim 2Br'(Y) \geq 2s - 2 + \epsilon(Y)$.

Additionally to Corollary 0.5 and (0-6)-(0-8), we prove...
Theorem 0.6. Let $Y$ be a real Enriques surface. Then:

(i) The inequality $(0-7)$ is an equality, i.e. $b(Y) = 2s-2$, iff the Hochschild–Serre spectral sequence $(0-5)$ degenerates. In particular, by Corollary 0.5, b(Y) = 2s - 2 if $X_\sigma(R) \neq \emptyset$ and $X_{\tau \sigma}(R) \neq \emptyset$.

(ii) $\dim_1 Br'(Y) = 2s - 1$ if the Hochschild–Serre spectral sequence $(0-5)$ degenerates and $Y(R) \neq \emptyset$. In particular, it is true if $X_\sigma(R) \neq \emptyset$ and $X_{\tau \sigma}(R) \neq \emptyset$.

(iii) $\dim_1 Br'(Y) \geq 2s - 1$.

In §3, we give an application of results of [N-S] and Corollary 0.5 and Theorem 0.6 to a topological studying of real Enriques surfaces $Y$. Let $Y$ be a real Enriques surface with the antiholomorphic involution $\theta$ and $Y(R) \neq \emptyset$. Let $s_{nor}$ be the number of non-orientable connected components of $Y(R)$. We denote by

$$\Gamma = \{id, \tau, \sigma, \tau \sigma\} \cong (\mathbb{Z}/2)^2$$

the group acting on the $K3$-surface $X(C)$ (we use notation above). Let us suppose that the both real parts $X_\sigma(R), X_{\tau \sigma}(R)$ are non-empty. Then we give a formula connecting the number $s_{nor}$ with some invariants of the action of the group $\Gamma$ on the lattice $H^2(X(C); \mathbb{Z})$ with the intersection pairing. We mention that it is not clear that one can express the number $s_{nor}$ using the action of $\Gamma$ on the lattice $H^2(X(C); \mathbb{Z})$. This formula is very important for the topological classification of real Enriques surfaces (see [N4]). See §3 for details.

I am grateful to O. Gabber for assuring me that the statement of Proposition 1.1 below should be true. I am grateful to R. Sujatha for very useful discussions, in particular, for pointing out me on the results of E. Witt from [W].

A preliminary variant of this paper [N3] was written during my stay in the University of Notre Dame (USA) at 1991—1992. I am grateful to the University of Notre Dame for hospitality.

§1. The proof of the Theorem 0.1.

We recall that if $X$ is a topological space with an action of a group $G$ and $A$ is a $G$-sheaf of groups on $X$, then the group $H^k(X; G, A)$ of equivariant cohomology (or Galois cohomology) is defined. See A. Grothendieck [Gr1, Ch. 5]. It is the right derived functor $R^k\Gamma^{G}$ to the functor $A \mapsto \Gamma(X; A)^G$ of $G$-invariant sections. This composition of functors $A \mapsto \Gamma(X; A)$ and $M \mapsto M^G$ for a $G$-module $M$ defines two spectral functors which tend to this cohomology:

$$I^p,q_2 = H^p(X/G; \mathcal{H}^q(G; A)) \Rightarrow H^{p+q}(X; G, A)$$

where $\mathcal{H}^q(G; A) = R^qf_*^G A$ is the sheaf corresponding to the presheaf on $X/G$:

$$(1-1) \quad U \mapsto H^q(\pi^{-1}(U); G, A).$$

Here $\pi : X \rightarrow X/G$ is the quotient map. And

$$(1-2) \quad II^p,q_2 = H^p(G; H^q(X; A)) \Rightarrow H^{p+q}(X; G, A).$$

The following statement is fundamental for us. This is analogous to the well-known connection between étale and ordinary cohomology for a complex algebraic manifold $Z$ and a finite abelian group $B$ (see [Mi, Ch. III, §3]): The morphism of the ordinary site (Euclidean topology) to the étale site gives rise an isomorphism

$$H^k_{\text{et}}(Z; B) \cong H^k(Z(C); B).$$
Proposition 1.1. Let $X/R$ be a real algebraic manifold and $G = Gal(C/R)$. Let $A$ be a sheaf in étale topology of $X$ such that this sheaf is a constant $G$-sheaf $A$ on $X \otimes C$, where $A$ is a finite abelian group with an action of the group $G$.

Then there exists the canonical isomorphism $H_{et}^k(X; A) \cong H^k(X(C); G, A)$ together with the canonical isomorphism of the Hochschild-Serre spectral sequence $\II$, which is defined by the canonical isomorphism

$$E_2^{p,q} = H^p(G; H_{et}^q(X \otimes C; A)) \cong II_2^{p,q} = H^p(G; H^q(X(C); A))$$

induced by the canonical isomorphism (1–3).

Proof. We follow to the proof of the isomorphism (1–3) for complex algebraic manifolds. See [Mi,Ch. III, §3], for example.

Let $X(C)_{cx}$ be a small site $X(C)^n_{E}$ of morphisms of complex analytic spaces over $X(C)^n$, which are local isomorphisms. (We use notation of [Mi].) An open subset $U \subset X(C)$ is a local isomorphism. It follows that we have the morphism of sites $X(C)_{cx} \to X(C)$. Every covering of $X(C)$ in the site $X(C)_{cx}$ has a refinement covering of $X(C)$ in the site $X(C)$. It follows that the morphism $X(C)_{cx} \to X(C)$ gives an isomorphism of cohomology

(1–4) $$H^i(X(C); A) \cong H^i(X(C)_{cx}; A).$$

Like above, we can define $G$-equivariant cohomology $H^k(X(C)_{cx}; G, F)$ for the site $X(C)_{cx}$ as the right derived functor to the composition of functors $F \to \Gamma(X(C)_{cx}; F)^G$ for a $G$-sheaf $F$ on the site $X(C)_{cx}$. This equivariant cohomology also has a spectral sequence $\II(X(C)_{cx})$ with the beginning

$$\II_2^{p,q}(X(C)_{cx}) = H^p(G; H^q(X(C)_{cx}; F)) \Rightarrow H^{p+q}(X(C)_{cx}; G, F).$$

One can calculate equivariant cohomology using invariant coverings (see [Gr1, Ch.5]). Any $G$-invariant covering of $X(C)$ in site $X(C)_{cx}$ also contains a refinement $G$-invariant covering of $X(C)$ in the site $X(C)$. It follows that we also have the canonical isomorphism of equivariant cohomology:

(1–5) $$H^i(X(C)_{cx}; G, A) \cong H^i(X(C); G, A),$$

together with the isomorphism of the corresponding spectral sequences $\II$ of these cohomology

(1–6) $$\II_2^{p,q}(X(C)_{cx}) = H^p(G; H^q(X(C)_{cx}; A)) \cong II_2^{p,q}(X(C)) = H^p(G; H^q(X(C); A)).$$

defined by the isomorphism (1–4).

Further, we use standard results about étale cohomology (see [Mi]). Since $X \otimes C \to X$ is an étale covering with Galois group $G$, it follows that a sheaf $F$ on the site $X_{et}$ corresponds to a $G$-sheaf $F$ on the site $(X \otimes C)_{et}$. Étale cohomology is a right derived functor to the composition of functors

$$F \to \{ \text{mod. } \Gamma(X \otimes C; F) \} \quad \text{and} \quad M \to M^G.$$
Here $M$ is a $G$-module. The Hochschild-Serre spectral sequence
\[ E_2^{p,q} = H^p(G; H^q_{et}(X \otimes \mathbb{C}; F)) \Rightarrow H^{p+q}_{et}(X; F) \]
corresponds to the composition of these functors.

Every étale morphism $Y \to X \otimes \mathbb{C}$ gives a morphism $Y(\mathbb{C}) \to X(\mathbb{C})$ in the site $X(\mathbb{C})_{cx}$. This defines the morphism of sites $X(\mathbb{C})_{cx} \to X_{et}$. From the remarks above, this morphism defines the homomorphism of cohomology
\[ H^k_{et}(X; A) \to H^k(X(\mathbb{C})_{cx}; G, A) = H^k(X(\mathbb{C}); G, A) \] (1–7)

together with the homomorphism of spectral sequences
\[ E_r^{p,q} \to II_r^{p,q} \] (1–8)
defined by the homomorphism
\[ E_2^{p,q} = H^p(G; H^q_{et}(X \otimes \mathbb{C}; A)) \Rightarrow II_2^{p,q} = H^p(G; H^q(X(\mathbb{C})_{cx}; A)) = H^p(G; H^q(X(\mathbb{C}); A)). \] (1–9)
The last homomorphism is defined by the homomorphism
\[ H^q_{et}(X \otimes \mathbb{C}; A) \to H^q(X(\mathbb{C})_{cx}; A) = H^q(X(\mathbb{C}); A), \] (1–10)
which is the isomorphism (1–3). It follows that (1–9), (1–8) and (1–7) are isomorphisms too. This finishes the proof.

We mention that D. A. Cox [C] had shown that étale homotopy type of a real algebraic manifold is defined by Euclidean topology.

We recall that for a compact manifold $M$ and a constant sheaf of modules, one can calculate sheaf cohomology using simplicial triangulation of the manifold $M$. One can prove this using the canonical isomorphisms between sheaf, Čech, Alexander–Spanier, singular and simplicial triangulation cohomology for compact manifolds. See [Gr1] and [Sp], for example. Similarly, for a compact manifold $M$ with an action of the group $G$ and an Abelian $G$-group $A$, one can prove that equivariant sheaf, Čech, Alexander–Spanier, singular and simplicial triangulation cohomology are isomorphic. Thus, for this case, we can calculate equivariant cohomology using the following elementary procedure (this definition is used in the book [Bro], for example):

We consider some $G$-equivariant simplicial triangulation $K$ of $M$. Thus, $K$ is a $G$-equivariant simplicial complex. We may suppose that $K/G$ is a simplicial triangulation of $M/G$ and the fixed part $K^G$ is a simplicial triangulation of $M^G$. We consider the corresponding chain complex $C_n = C_n(K; \mathbb{Z})$ and the corresponding cochain complex $C^n(K; A)$, where $C^n(K; A) = \text{Hom}(C_n, A)$. We organize a cochain complex for calculation of group cohomology of $C^n = C^n(K, A)$. For the group $G = \{1, g\}$ of order two this is a complex
\[ 0 \to C^0 = C^0(K; A) \to C^1 = C^1(K; A) \to C^2 = C^2(K; A) \to C^3 = C^3(K; A) \to \cdots \] (1–11)
Thus, we get a double complex

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& & & & & & & & \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

Equivariant cohomology \( H^n(M; G, A) \) are the homology of this double complex. The spectral sequence \( I \) corresponds to the filtration from below on cohomology of this double complex, and the spectral sequence \( II \) corresponds to the filtration from left of this double complex. From this complex, it is clear (see Godement [Go, Chapter 1], for example) that

\[
I\Pi^{p,q}_1(M; G, A) = H^q(M; A) \Longrightarrow H^{p+q}(M; G, A),
\]

and

\[
I\Pi^{p,q}_2(M; G, A) = H^p(G; H^q(M; A)) \Longrightarrow H^{p+q}(M; G, A).
\]

Here, for a \( G \)-module \( R \), we have

\[
H^0(G; R) = R^G,
\]

\[
H^p(G; R) = R^p/(1 + g)R \text{ for } p \text{ even and } p > 0;
\]

and

\[
H^p(G; R) = R^{(-g)}/(1 - g)R \text{ for } p \text{ odd}.
\]

If \( R \) is a vector space over the field \( \mathbf{F}_2 \), the formulae (1–16) and (1–17) give the same.
Further, we calculate the equivariant cohomology for $A = F_2$. For this case, $1 - g = 1 + g$, and the double complex (1–12) is periodic.

The spectral sequence $I$ corresponds to the filtration of this complex from below. Then

\[ I_1^{p,0}(M; G, F_2) = (C^p)^g = C^p(M/G; F_2), \]

and

\[ I_2^{p,0}(M; G, F_2) = (C^p)^g / (1 + g)(C^p) = C^p(M/G; F_2) / C^p(M/G, M^G; F_2) = C^p(M^G; F_2). \]

For $q > 0$,

\[ I_1^{p,q}(M; G, F_2) = (C^p)^g / (1 + g)(C^p) = C^p(M/G; F_2) / C^p(M/G, M^G; F_2) = C^p(M^G; F_2). \]

And we get

\[ I_2^{p,q}(M; G, F_2) = H^p(M^G; F_2) \text{ if } q > 0. \]

In particular, for a real projective algebraic manifold $X$, the set $X(C)$ of complex points with the group $G = \{1, g\}$ generated by antiholomorphic involution $g$ acting on $X(C)$, and the set $X(R) = X(C)^G$ of real points, we have:

\[ II_2^{p,q}(X(C); G, A) = H^p(G; H^q(X(C); A)) \Rightarrow H^{p+q}(X(C); G, A); \]

\[ I_2^{p,0}(X(C); G, F_2) = H^p(X(C)/G; F_2); \]

and

\[ I_2^{p,q}(X(C); G, F_2) = H^p(X(R); F_2) \text{ if } q > 0. \]

Further, we will consider this case of real projective algebraic manifold $X$, but, actually, all results valid for a compact manifold $M$ with an involution.

For $X(R)$, the double complex (1–12) has $1 - g = 1 + g = 0$. Thus, we evidently get (see [Gr1, Corollaire 5.4.1])

**Proposition 1.2.** *We have a canonical isomorphism*

\[ H^k(X(R); G, F_2) = \bigoplus_{i=0}^{k} H^i(X(R); F_2). \]

*Besides, the spectral sequence $I(X(R); G, F_2)$ degenerates from $I_2$, and for all $p, q$ we have $I_2^{p,q}(X(R); G, F_2) = H^p(X(R); F_2)$. Thus, all differentials*

\[ d_r^{p,q}(X(R); G, F_2) \]

*of the spectral sequence $I$ vanish for $r \geq 2$.*

The same is true for a topological space with the trivial action of the group $G$. Now we have (see [Kr])
Proposition 1.3. For $k > 2 \dim X$, 

$$H^k(X(\mathbb{C}); G, \mathbb{F}_2) = H^k(X(\mathbb{R}); G, \mathbb{F}_2) = \bigoplus_{i=0}^{k} H^i(X(\mathbb{R}); \mathbb{F}_2).$$

Proof. Like above, using the double complex, we can define equivariant cohomology of a pair. We evidently have an exact sequence:

$$\ldots \to H^i(X(\mathbb{C}), X(\mathbb{R}); G, \mathbb{F}_2) \to H^i(X(\mathbb{C}); G, \mathbb{F}_2) \to H^i(X(\mathbb{R}); G, \mathbb{F}_2) \to H^{i+1}(X(\mathbb{C}), X(\mathbb{R}); G, \mathbb{F}_2) \to \ldots.$$ 

The group $G$ acts without fixed points on the pair $(X(\mathbb{C}), X(\mathbb{R}))$. – For the corresponding double complex all differentials $1 + g, 1 - g$ give an exact sequence. It follows that for $q > 0$ the spectral sequence $I^{p,q}_1 = 0$ for this double complex and

$$H^i(X(\mathbb{C}), X(\mathbb{R}); G, \mathbb{F}_2) = H^i(X(\mathbb{C})/G, X(\mathbb{R}); G, \mathbb{F}_2).$$

Thus, $H^i(X(\mathbb{C}), X(\mathbb{R}); G, \mathbb{F}_2) = 0$ for $i > 2 \dim X$. It follows the statement.

From the calculation above of the beginning of the spectral sequence $I$ and Proposition 1.2, we get

Proposition 1.4. The embedding $\rho : X(\mathbb{R}) \subset X(\mathbb{C})$ gives the homomorphism of the spectral sequences $\rho^* : I(X(\mathbb{C}); G, \mathbb{F}_2) \to I(X(\mathbb{R}); G, \mathbb{F}_2)$. For $r = 2$ it is defined by

$$I^{p,0}_2(X(\mathbb{C}); G, \mathbb{F}_2) = H^p(X(\mathbb{C})/G; \mathbb{F}_2) \overset{i^*}{\to} H^p(X(\mathbb{R}); \mathbb{F}_2) = I^{p,0}_2(X(\mathbb{R}); G, \mathbb{F}_2),$$

where $i : X(\mathbb{R}) \subset X(\mathbb{C})/G$ is the embedding, and by the identical isomorphism

$$I^{p,q}_2(X(\mathbb{C}); G, \mathbb{F}_2) = H^p(X(\mathbb{R}); \mathbb{F}_2) = I^{p,q}_2(X(\mathbb{R}); G, \mathbb{F}_2) \quad \text{for} \quad q > 0.$$ 

In particular, since the spectral sequence $I(X(\mathbb{R}); G, \mathbb{F}_2)$ degenerates, the differential $d^{r,q}_r$ of $I(X(\mathbb{C}); G, \mathbb{F}_2)$ vanish if $r \geq 2$ and $q \neq r - 1$, and $\rho^* d^{r,q-1}_r = 0$ for $r \geq 2$.

Now we can reformulate the map $(0–1)$

$$\_2Br'(X) \to (\mathbb{Z}/2)^s.$$ 

Let us choose a point $P_i$, $i = 1, 2, \ldots, s$, for every connected component of $X(\mathbb{R})$. Here $i = 1, 2, \cdots, s$ numerates connected components of $X(\mathbb{R})$. From the Kummer exact sequence $(0–3)$, from the isomorphism $Br'([P_i]) = H^2_{et}([P_i]; \mathbb{F}_2) = \mathbb{F}_2$ and Proposition 1.1 (also see Proposition 1.2), the image of the homomorphism $(0–1)$ is the same as the image of the composition of the homomorphisms

$$(1–25) \quad H^2(X(\mathbb{C}); G, \mathbb{F}_2) \to H^2(X(\mathbb{R}); G, \mathbb{F}_2) \to \bigoplus_{i=1}^{s} H^2([P_i]; G, \mathbb{F}_2).$$
defined by the inclusions \( \{ P_i \} \subset X(\mathbb{R}) \subset X(\mathbb{C}) \). By Proposition 1.2, the second homomorphism is epimorphic and has the kernel

\[
H^1(X(\mathbb{R}); F_2) \oplus H^2(X(\mathbb{R}); F_2).
\]

It is clear from the definition of equivariant cohomology using the double complex (1–12). We remark that we then have a canonical identification

\[
H^0(X(\mathbb{R}); F_2) = H^2(X(\mathbb{R}); G, F_2)/(H^1(X(\mathbb{R}); F_2) \oplus H^2(X(\mathbb{R}); F_2) = I^0_{\infty}(X(\mathbb{R}); G, F_2)
\]

for the first spectral sequence \( I \) of the equivariant cohomology of \( X(\mathbb{R}) \). It follows, that the image of the map (1–25) is the image of the canonical homomorphism \( H^2(X(\mathbb{C}); G, F_2) \rightarrow H^2(X(\mathbb{R}); G, F_2) \rightarrow I^0_{\infty}(X(\mathbb{C}); G, F_2) = H^0(X(\mathbb{R}); F_2) \).

This homomorphism preserves the filtration \( I \) on \( H^2(X(\mathbb{C}); G, F_2) \). Thus, this image is the same as the image of the canonical homomorphism

\[
\rho^* : I^0_{\infty}(X(\mathbb{C}); G, F_2) \rightarrow I^0_{\infty}(X(\mathbb{R}); G, F_2) = H^0(X(\mathbb{R}); F_2),
\]

where \( \rho : X(\mathbb{R}) \subset X(\mathbb{C}) \) is the embedding.

Let us look on the part of the spectral sequence \( I^p,q_{\infty}(X(\mathbb{C}); G, F_2) \) which takes part in calculation of the \( I^0_{\infty}(X(\mathbb{C}); G, F_2) \). It is the left-below corner

\[
\begin{array}{cccc}
H^3(X(\mathbb{C}); G, F_2) & H^2(X(\mathbb{C}); G, F_2) & H^2(X(\mathbb{R}); F_2) \\
H^2(X(\mathbb{C}); G, F_2) & H^1(X(\mathbb{C}); G, F_2) & H^1(X(\mathbb{R}); F_2) \\
H^1(X(\mathbb{C}); G, F_2) & H^0(X(\mathbb{C}); G, F_2) & H^0(X(\mathbb{R}); F_2) \\
H^0(X(\mathbb{C}); G, F_2) & H^0(X(\mathbb{R}); F_2) & H^0(X(\mathbb{R}); F_2)
\end{array}
\]

where the differential hits from below to up and from right to left. In particular, we have \( I^0_{\infty}(X(\mathbb{C}); G, F_2) \subset I^0_{\infty}(X(\mathbb{C}); G, F_2) = H^0(X(\mathbb{R}); F_2) \). From the double complex (1–12), the canonical map

\[
\rho^* : I^0_{\infty}(X(\mathbb{C}); G, F_2) = H^0(X(\mathbb{R}); F_2)
\]

\[
\rightarrow I^0_{\infty}(X(\mathbb{R}); G, F_2) \rightarrow I^0_{\infty}(X(\mathbb{R}); G, F_2) = H^0(X(\mathbb{R}); F_2)
\]

is the identical isomorphism of \( H^0(X(\mathbb{R}); F_2) \). Thus, we get

**Proposition 1.5.** The image of the map \( \rho^* \) (equivalently, of the map (0–1)) is equal to the image of the embedding

\[
I^0_{\infty}(X(\mathbb{C}); G, F_2) \subset I^0_{\infty}(X(\mathbb{C}); G, F_2) = H^0(X(\mathbb{R}); F_2) = I^0_{\infty}(X(\mathbb{R}); G, F_2).
\]

From Proposition 1.4, the differential

\[
d^0_{2} : H^0(X(\mathbb{R}); F_2) \rightarrow H^2(X(\mathbb{R}); F_2)
\]

is equal to zero. By Proposition 1.4, we also have \( i^* d^1_{2} = d^3_{3} = 0 \) and \( \rho^* d^0_{2} = 0 \). We then get the basic result.
Theorem 1.6. Let \( i : X(\mathbb{R}) \subset X(\mathbb{C})/G \) be the embedding and \( i^* : H^3(X(\mathbb{C})/G; \mathbb{F}_2) \to H^3(X(\mathbb{R}); \mathbb{F}_2) \) the corresponding canonical homomorphism.

Then we have: The differential \( d_{1,1} \) is the composition

\[
d_{1,1}^2 : H^1(X(\mathbb{R}); \mathbb{F}_2) \to \text{Ker} \ i^* \subset H^3(X(\mathbb{C})/G; \mathbb{F}_2),
\]

and the image of the homomorphism \( \rho^* \) from Proposition 1.5 (equivalently, of the homomorphism \( (0–1) \) from Introduction) is equal to the kernel of the differential

\[
d_{0,2}^3 : H^0(X(\mathbb{R}); \mathbb{F}_2) \to \text{Ker} \ i^*/d_{1,1}^2(H^1(X(\mathbb{R}); \mathbb{F}_2)) \subset H^3(X(\mathbb{C})/G; \mathbb{F}_2)/d_{1,1}^2(H^1(X(\mathbb{R}); \mathbb{F}_2)).
\]

In particular, \( \rho^* \) is epimorphic if \( \text{Ker} \ i^* = 0 \) or \( H^3(X(\mathbb{C})/G; \mathbb{F}_2) = 0 \). It is epimorphic too if the differential

\[
d_{1,1}^2 : H^1(X(\mathbb{R}); \mathbb{F}_2) \to \text{Ker} \ i^*
\]

is epimorphic.

As a corollary, we get the Theorem 0.1 from Introduction.

We remark that for surfaces \( X \) the dimension \( \text{dim} \ X(\mathbb{R}) = 2 \), and \( H^3(X(\mathbb{R}); \mathbb{F}_2) = 0 \). Thus, \( \text{Ker} \ i^* = H^3(X(\mathbb{C})/G; \mathbb{F}_2) \).

Remark 1.7. In Introduction (after Corollary 0.2), we showed real Enriques surfaces \( Y \) such that

\[
H^3(Y(\mathbb{C})/G; \mathbb{F}_2) = \mathbb{F}_2.
\]

Theorem 1.6 shows that such a \( Y \) gives an example when the map \((0–1)\) is not epimorphic, exactly if simultaneously the homomorphism

\[
d_{2,1}^{1,1} : H^1(Y(\mathbb{R}); \mathbb{F}_2) \to H^3(Y(\mathbb{C})/G; \mathbb{F}_2) = \mathbb{F}_2
\]

is zero, and the homomorphism

\[
d_{0,2}^{0,2} : H^0(Y(\mathbb{R}); \mathbb{F}_2) \to H^3(Y(\mathbb{C})/G; \mathbb{F}_2) = \mathbb{F}_2
\]

is not zero.

Of course, these homomorphisms are "the same" for Enriques surfaces which belong to one connected component of the moduli space of real Enriques surfaces. Using Global Torelli Theorem [PŠ-S], epimorphism of Torelli map [Ku] for K3-surfaces and methods developed in [N1,N2], in principle, it is possible to enumerate all this connected components using some invariants. Thus, the problem is to rewrite, using these invariants, the invariants of differentials \( d_{1,1}^{1,1}, d_{0,2}^{0,2} \) above. We hope to do it later. Some important results in this direction were obtained in [N-S].

Remark 1.8. We can consider the homomorphism

\[
(1–26) \quad H^1_{et}(X; \mathbb{F}_2) \to (\mathbb{Z}/2)^s
\]

which is defined like the map \((0–1)\) using the isomorphism \( H^1_{et}(\{\text{real point}\}; \mathbb{F}_2) = \mathbb{F}_2).
Like above, we can interpret the right side of (1–26) as the group 
\( H^0(X; F_2) = I^0_{\infty}(X(G); G; F_2) \), and the image of the homomorphism (1–26) as the

\[
\text{Ker} \{ d_{1}^{0,1} : H^0(X; F_2) \to H^2(X(G)/G; F_2) \}
\]

for the differential \( d_{1}^{0,1} \) of the spectral sequence \( I(X); G, F_2 \).

Suppose that \( X \) is a smooth projective real curve and \( X(R) \neq \emptyset \). Then the quotient space \( X(C)/G \) is a connected 2-dimensional manifold with a not-trivial boundary \( X(R) \). By Poincaré duality, 
\( H^2(X(C)/G; F_2) = H_0(X(C)/G, X(R); F_2) = 0 \), and the map (1–26) is epimorphic. This gives a geometrical interpretation of the result of E.Witt from [W].

We mention, that similarly to the homomorphisms (0–1) and (1–26), one may define and study the general homomorphism (1–27) below

\[
(1–27) \quad \quad H^1_{et}(X; F_2) \to (\mathbb{Z}/2)^s
\]

using the isomorphism \( H^1_{et}(\{\text{real point}\}; F_2) = F_2 \) for \( n \geq 0 \).

§2. The Proof of the Theorems 0.3 — 0.6.

We prove the following

**Theorem 2.1.** Let \( X/R \) be a smooth real projective algebraic surface such that \( X(R) \neq \emptyset \) and \( H^3(X(C)/G; F_2) = 0 \).

Then the Serre-Hochschild spectral sequence for the \( H^1_{et}(X; F_2) \) and the spectral sequence \( II(X(C); G, F_2) \) for the equivariant cohomology \( H^*(X(C); G, F_2) \) degenerate. Besides (by Proposition 1.1), \( H^k_{et}(X; F_2) \cong H^k(X(C); G, F_2) \), and we have the formulae for their dimensions:

\[
\dim H^0(X(C); G, F_2) = 1;
\]

\[
\dim H^1(X(C); G, F_2) = \dim H^1(X(C); F_2) + 1;
\]

\[
\dim H^2(X(C); G, F_2) = \dim H^2(X(C); F_2)^G + \dim H^1(X(C); F_2) + 1;
\]

\[
\dim H^3(X(C); G, F_2) = 2 \dim H^2(X(C); F_2)^G - \dim H^2(X(C); F_2) + 2 \dim H^1(X(C); F_2) + 1;
\]

\[
\dim H^k(X(C); G, F_2) = 2 \dim H^2(X(C); F_2)^G - \dim H^2(X(C); F_2) + 2 \dim H^1(X(C); F_2) + 2
\]

for \( k \geq 4 \).

**Proof.** By Proposition 1.1, we should prove that the spectral sequence \( II(X(C); G, F_2) \) degenerates. To prove this, we use the following important result of V. A. Krasnov [Kr].
Proposition 2.2. Let $X/\mathbb{R}$ be a real projective algebraic manifold. Then the spectral sequence $\mathcal{H}(X(\mathbb{C}); G, \mathbb{F}_2)$ with the beginning

\[ \mathcal{II}^{p,q} = H^p(G; H^q(X(\mathbb{C}), \mathbb{F}_2)) \implies H^{p+q}(X(\mathbb{C}); G, \mathbb{F}_2) \]

degenerates iff

\[ \dim H^*(X(\mathbb{R}); \mathbb{F}_2) = \dim H^1(G; H^*(X(\mathbb{C}); \mathbb{F}_2)). \]

Proof. Let $k > 2 \dim X$. By Propositions 1.2 and 1.3,

\[ \dim H^*(X(\mathbb{R}); \mathbb{F}_2) = \dim H^k(X(\mathbb{C}); G, \mathbb{F}_2). \]

On the other hand,

\[ \bigoplus_{p+q=k} \mathcal{II}^{p,q} = \bigoplus_{p+q=k} H^p(G; H^q(X(\mathbb{C}); \mathbb{F}_2)) = H^1(G; H^*(X(\mathbb{C}); \mathbb{F}_2)). \]

Thus, we have the inequality

\[ 2 \dim H^*(X(\mathbb{R}); \mathbb{F}_2) = \dim H^k(X(\mathbb{C}); G, \mathbb{F}_2) \leq \dim H^1(G; H^*(X(\mathbb{C}); \mathbb{F}_2)). \]

Besides, \( 2 \dim \) gives an equality iff \( \mathcal{II}^{p,q} = \mathcal{II}_G^{p,q} \) for \( p + q = k > 2 \dim X \). Thus, if the spectral sequence \( \mathcal{H}(X(\mathbb{C}); G, \mathbb{F}_2) \) degenerates, we have the equality of the Proposition.

Now we assume that the equality of Proposition holds. We have then proven that all differentials \( d^{p,q}_r \) vanish for \( r \geq 2 \) and \( p + q > 2 \dim X \). From periodicity of the double complex \( (1 \rightarrow 12) \), it follows that all differentials \( d^{p,q}_r \) vanish for \( r \geq 2 \).

We recall the Smith exact sequence for an action of a group \( G = \{1, g\} \) of order two (see [Bre, Ch. III, §3], for example).

We have the exact sequence for the chain complex \( C_n(K; \mathbb{F}_2) \) above with an action of \( G \):

\[ 0 \rightarrow C_n(K; \mathbb{F}_2)^G \rightarrow C_n(K; \mathbb{F}_2) \rightarrow (1 + g)(C_n(K; \mathbb{F}_2)) \rightarrow 0. \]

Here we have the canonical identifications

\[ (1 + g)(C_n(K; \mathbb{F}_2)) = C_n(K/G; \mathbb{F}_2)/C_n(K^G; \mathbb{F}_2) = C_n(K/G, K^G; \mathbb{F}_2) \]

and

\[ C_n(K; \mathbb{F}_2)^G = (1 + g)C_n(K; \mathbb{F}_2) \oplus C_n(K^G; \mathbb{F}_2) = C_n(K/G, K^G; \mathbb{F}_2) \oplus C_n(K^G; \mathbb{F}_2). \]

Thus, we get the exact sequence

\[ 0 \rightarrow C_n(K/G, K^G; \mathbb{F}_2) \oplus C_n(K^G; \mathbb{F}_2) \rightarrow C_n(K; \mathbb{F}_2) \rightarrow C_n(K/G, K^G; \mathbb{F}_2) \rightarrow 0. \]

This gives the corresponding homological Smith exact sequence:

\[ \ldots \rightarrow H_n(K/G, K^G; \mathbb{F}_2) \oplus H_n(K^G; \mathbb{F}_2) \rightarrow H_n(K; \mathbb{F}_2) \rightarrow H_n(K/G, K^G; \mathbb{F}_2) \rightarrow \ldots \]
Thus, for a real algebraic variety, we have the homological Smith exact sequence

\[ \cdots \xrightarrow{\delta_{n+1}} H_n(X(C)/G, X(R); F_2) \oplus H_n(X(R); F_2) \xrightarrow{i_n} H_n(X(C); F_2) \xrightarrow{\rho_n} \]

\[ H_n(X(C)/G, X(R); F_2) \xrightarrow{\delta_n} H_{n-1}(X(C)/G, X(R); F_2) \oplus H_{n-1}(X(R); F_2) \to \cdots \]

We repeat some well-known standard facts connected with the Smith exact sequence (see V.A.Rokhlin [R], for example). From this exact sequence, we get

\[ \dim H_{n-1}(X(C)/G, X(R); F_2) + \dim H_{n-1}(X(R); F_2) \]

\[ = \dim \text{Im} \delta_n + \dim \text{Im} i_{n-1}. \]

Moreover,

\[ \dim \text{Im} \delta_n = \dim H_n(X(C)/G, X(R); F_2) - \dim \text{Im} \rho_n \]

\[ = \dim H_n(X(C)/G, X(R); F_2) - \dim H_n(X(C); F_2) + \dim \text{Im} i_n. \]

Thus, we get

\[ \dim H_{n-1}(X(C)/G, X(R); F_2) + \dim H_{n-1}(X(R); F_2) \]

\[ = \dim H_n(X(C)/G, X(R); F_2) - \dim H_n(X(C); F_2) \]

\[ + \dim \text{Im} i_n + \dim \text{Im} i_{n-1}. \]

Considering the sum by \( n \), we then get

\[ (2-2) \quad \dim H_*(X(R); F_2) = 2 \sum_n \dim \text{Im} i_n - \dim H_*(X(C); F_2). \]

From the exact sequence,

\[ 0 \to H_*(X(C); F_2)^G \to H_*(X(C); F_2) \to (1+g)H_*(X(C); F_2) \to 0, \]

we have

\[ (2-3) \quad \dim H^1(G; H_*(X(C); F_2)) = \dim H_*(X(C); F_2)^G - \dim (1+g)H_*(X(C); F_2) \]

\[ = 2 \dim H_*(X(C); F_2)^G - \dim H_*(X(C); F_2). \]

It is clear that

\[ (2-4) \quad \text{Im} i_n \subset H_n(X(C); F_2)^G. \]

Thus, from (2–2), (2–3) and (2–4), we have an inequality

\[ (2-5) \quad \dim H_*(X(R); F_2) \leq \dim H^1(G; H_*(X(C); F_2)). \]

Of course, the inequalities (2–1) and (2–5) are equivalent. Moreover, we see that the inequality (2–5) is an equality iff for any \( n \) we have for the Smith exact sequence

\[ (2-6) \quad \text{Im} i_n = H_n(X(C); F_2)^G. \]

Thus, to prove that the spectral sequence \( II \) degenerates, we have to prove the equalities (2–6) for \( 0 \leq n \leq 4 \) for a surface \( X \) with the condition \( H^3(X(C)/G; F_2) = 0 \).

It will be convenient for us using the following general statement which follows from Smith exact sequence (compare with the proof of [H, Lemme 3.7]).
Proposition 2.3. For the Smith exact sequence,

\[ \rho_n(H_n(X(C); F_2)^G) = \text{Im} \{ H_{n+1}(X(C)/G; F_2) \to H_{n+1}(X(C)/G, X(R); F_2) \} \]

In particular, \( \text{Im} i_n = H_n(X(C); F_2)^G \) iff the image on the right is zero.

Proof. We use the following properties of the Smith exact sequence which follow from the definition above of this sequence:

(2–7) \[ i_n(\rho_n \oplus 0) = \text{id} + g; \]

and the homomorphism

(2–8) \[ H_n(X(C)/G; X(R); F_2) \overset{\delta_n}{\to} H_{n-1}(X(C)/G, X(R); F_2) \oplus H_{n-1}(X(R); F_2) \]

\[ \overset{\pi_{X(R)}}{\to} H_{n-1}(X(R); F_2) \]

is equal to the homomorphism \( \partial_n : H_n(X(C)/G, X(R); F_2) \to H_{n-1}(X(R); F_2) \) in the homological exact sequence of the pair \((X(C)/G, X(R))\).

Now, let \( x_n \in H_n(X(C); F_2)^G \). Then, from (2–7), it is equivalent to \( i_n(\rho_n(x_n) \oplus 0) = 0 \). By Smith exact sequence, it is equivalent to existence of an element \( y_{n+1} \in H_{n+1}(X(C)/G, X(R); F_2) \) such that \( \delta_{n+1}(y_{n+1}) = \rho_n(x_n) \oplus 0 \). By (2–8), it is equivalent to

\[ \rho_n(x_n) \in \text{Im} \{ H_{n+1}(X(C)/G; F_2) \to H_{n+1}(X(C)/G, X(R); F_2) \} \]

\[ \overset{\delta_{n+1}}{\to} H_n(X(C)/G, X(R); F_2) \} \]

Now, we should only remark that, by Smith exact sequence, \( \rho_n(x_n) \in \text{Ker} \delta_n \).

From the Proposition 2.3, we have:

- \( \text{Im} i_4 = H_4(X(C); F_2)^G \) for any surface since \( H_5(X(C)/G; F_2) = 0 \).
- \( \text{Im} i_3 = H_3(X(C); F_2)^G \) since for our case \( \text{Ker} \delta_3 = 0 \), because \( \text{Ker} \delta_3 \subset \text{Ker} \partial_3 = 0 \). Here \( \text{Ker} \partial_3 = 0 \), because \( H_3(X(C)/G; F_2) = 0 \) for our case.
- \( \text{Im} i_2 = H_2(X(C); F_2)^G \) since \( H_3(X(C)/G; F_2) = 0 \) in our case.
- \( \text{Im} i_1 = H_1(X(C); F_2)^G \) since \( \text{Ker} \delta_1 \subset \text{Ker} \partial_1 = 0 \) in our case. Here \( \text{Ker} \partial_1 = 0 \) since \( H_1(X(C)/G; F_2) = 0 \) in our case.
- \( \text{Im} i_0 = H_0(X(C); F_2)^G \) since \( H_0(X(C), X(R); F_2) = 0 \) because \( X(R) \neq \emptyset \). Thus, we proved that the spectral sequence II degenerates.

Now let us prove the formulae of Theorems 1.2 and 0.3. Since the spectral sequence II degenerates, we have

\[ \dim H^k(X(C); G, F_2) = \bigoplus_{p+q=k} H^p(G; H^q(X(C); F_2)). \]

To get formulae, by Poincaré duality, we should only prove that \( G \) is trivial on \( H^0(X(C); F_2) \) and \( \partial H_{-1}X(C); F_2 \). It is true for \( H^0(X(C); F_2) \) since...
\(H^0(\mathcal{T}(\mathfrak{C}); \mathbb{F}_2) \cong \mathbb{F}_2\). Since \(H_1(\mathcal{T}(\mathfrak{C}); \mathbb{F}_2) = 0\), the homomorphism \(\partial_1\) is injective. By (2–8), then the homomorphism \(\delta_1\) for the Smith exact sequence is injective too. Thus, the homomorphism \(\rho_1\) is zero and \(H_1(\mathcal{T}(\mathfrak{C}); \mathbb{F}_2) = \text{Im} \ i_1 \subset H_1(\mathcal{T}(\mathfrak{C}); \mathbb{F}_2)^G\). It follows the statement. It finishes the proof of Theorems 2.1 and 0.3.

**Proof of Theorem 0.4.** For \(n \in \mathbb{N}\), the exact sequence of sheafs

\[
0 \to \mathbb{Z} \xrightarrow{x^n} \mathbb{Z} \to \mathbb{Z}/n \to 0
\]

gives the exact sequence of cohomology (universal coefficient sequence)

\[
\cdots \to H^{k-1}(\mathcal{M}; \mathbb{Z}/n) \to H^k(\mathcal{M}; \mathbb{Z}) \xrightarrow{x^n} H^k(\mathcal{M}; \mathbb{Z}) \to H^k(\mathcal{M}; \mathbb{Z}/n) \to H^{k+1}(\mathcal{M}; \mathbb{Z}) \xrightarrow{x^n} H^{k+1}(\mathcal{M}; \mathbb{Z}) \to \cdots
\]

(2–9)

For a compact manifold \(\mathcal{M}\), the beginning of this sequence gives the exact sequences

\[
0 \to H^0(\mathcal{M}; \mathbb{Z}) \xrightarrow{x^n} H^0(\mathcal{M}; \mathbb{Z}) \to H^0(\mathcal{M}; \mathbb{Z}/n) \to 0,
\]

and

\[
0 \to H^1(\mathcal{M}; \mathbb{Z}) \xrightarrow{x^n} H^1(\mathcal{M}; \mathbb{Z}) \to H^1(\mathcal{M}; \mathbb{Z}/n) \to \cdots.
\]

(2–10)

(2–11)

In particular, \(H^1(\mathcal{M}; \mathbb{Z})\) has no torsion. As we had mentioned in Introduction, for a smooth surface \(\mathcal{X}\), the quotient \(\mathcal{T}(\mathfrak{C})/G\) is a smooth manifold. The group \(G\) preserves the canonical orientation of \(\mathcal{T}(\mathfrak{C})\). It follows that \(\mathcal{T}(\mathfrak{C})/G\) is a smooth oriented manifold. Since \(H^3(\mathcal{T}(\mathfrak{C})/G; \mathbb{F}_2) = 0\), by Poincaré duality \(H^1(\mathcal{T}(\mathfrak{C})/G; \mathbb{F}_2) = 0\). Thus, by (2–11), we then get that \(H^1(\mathcal{T}(\mathfrak{C})/G; \mathbb{Z}) = 0\). It follows \(\mathcal{T}(\mathfrak{C})/G\) is a smooth manifold.

For the Hodge decomposition \(H^1(\mathcal{X}(\mathfrak{C}); \mathbb{C}) = H^{1,0}(\mathcal{X}(\mathfrak{C})) + H^{0,1}(\mathcal{X}(\mathfrak{C}))\), the antiholomorphic involution \(g\) on \(\mathcal{X}(\mathfrak{C})\) evidently maps \(H^{1,0}(\mathcal{X}(\mathfrak{C}))\) to \(H^{0,1}(\mathcal{X}(\mathfrak{C}))\). It follows that

\[
0 = \dim H^1(\mathcal{X}(\mathfrak{C}); \mathbb{C})^G = \frac{1}{2} \dim H^1(\mathcal{X}(\mathfrak{C}); \mathbb{C}).
\]

Thus, applying (2–11) to \(\mathcal{X}(\mathfrak{C})\), we get

\[
H^1(\mathcal{X}(\mathfrak{C}); \mathbb{Z}) = 0.
\]

(2–12)

Thus, \(\mathcal{X}\) is a regular surface: the irregularity \(q(\mathcal{X}) = \dim H^{1,0}(\mathcal{X}(\mathfrak{C})) = 0\). It follows that the characteristic class map gives an embedding

\[
\text{Pic} (\mathcal{X} \otimes \mathfrak{C}) \subset H^2(\mathcal{X}(\mathfrak{C}); \mathbb{Z}),
\]

and the image of this map is defined by the condition

\[
\{x \in \text{Pic} (\mathcal{X} \otimes \mathfrak{C}) \mid x \cdot H^{2,0}(\mathcal{X}(\mathfrak{C})) = 0\} = \gamma^{-1}(H^{1,1}(\mathcal{X}(\mathfrak{C}))),
\]

(2–13)

where

\[
\gamma : H^2(\mathcal{X}(\mathfrak{C}); \mathbb{Z}) \to H^2(\mathcal{X}(\mathfrak{C}); \mathbb{C})
\]
is a coefficient map. See [G-H], for example. From the definition of the characteristic class map, we have

\[ g(\gamma(x)) = -\gamma(g(x)) \text{ for } x \in \text{Pic}(X \otimes \mathbb{C}). \]

We remark that Pic\((X \otimes \mathbb{C})\) contains the all torsion of \(H^2(X(\mathbb{C}); \mathbb{Z})\) by (2.14).

Since \(H^1(X(\mathbb{C}); \mathbb{Z}) = 0\) and the sequence

\[ 0 \to H^4(X(\mathbb{C}); \mathbb{Z}) \xrightarrow{\times 2} H^4(X(\mathbb{C}); \mathbb{Z}) \to H^4(X(\mathbb{C}); \mathbb{F}_2) \to 0 \]

is exact, from (2.9), we get the exact sequence

\[ (2.15) \]

\[ 0 \to H^1(X(\mathbb{C}); \mathbb{F}_2) \to H^2(X(\mathbb{C}); \mathbb{Z}) \xrightarrow{\times 2} H^2(X(\mathbb{C}); \mathbb{Z}) \to H^2(X(\mathbb{C}); \mathbb{F}_2) \to 0. \]

Besides, by Poincaré duality, \(\dim H^1(X(\mathbb{C}); \mathbb{F}_2) = \dim H^3(X(\mathbb{C}); \mathbb{F}_2)\). It follows that

\[ (2.16) \]

\[ \dim H^2(X(\mathbb{C}); \mathbb{Z})/2H^2(X(\mathbb{C}); \mathbb{Z}) = b_2 + \dim H^1(X(\mathbb{C}); \mathbb{F}_2), \]

and

\[ (2.17) \]

\[ \dim H^2(X(\mathbb{C}); \mathbb{F}_2) = b_2 + 2 \dim H^1(X(\mathbb{C}); \mathbb{F}_2) \]

where the Betti number \(b_2 = \dim H^2(X(\mathbb{C}); \mathbb{C}) = \dim H^2(X(\mathbb{C}); \mathbb{Z}) \otimes \mathbb{C}.

Let Pic\((X \otimes \mathbb{C})\) = \(T \oplus \mathbb{Z}^{\rho(X \otimes \mathbb{C})}\) where \(T\) is the torsion of Pic\((X \otimes \mathbb{C})\) and \(\mathbb{Z}^{\rho(X \otimes \mathbb{C})} = \text{Pic}(X \otimes \mathbb{C})/T\). Since \(X(\mathbb{R}) \neq \emptyset\), we have

\[ \text{Pic } X = \text{Pic } (X \otimes \mathbb{C})^G \]

(this is well-known, see [Ma]). Let

\[ \text{Pic } X = \text{Pic } (X \otimes \mathbb{C})^G = T' \oplus \mathbb{Z}^{\rho(X)} \]

where \(T'\) is the torsion of Pic\(X\) and \(\mathbb{Z}^{\rho(X)} = \text{Pic } X/T'\). If for \(a \in \text{Pic } (X \otimes \mathbb{C})\) we have \(g(a) = a \mod T\), then \(g(ma) = ma\) for some \(m \in \mathbb{N}\) such that \(mT = 0\). It follows that

\[ (\text{Pic } (X \otimes \mathbb{C})/T)^G \cong (\text{Pic } X)/T' = \mathbb{Z}^{\rho(X)}. \]

Thus,

\[ (2.18) \]

\[ \rho(X) = \rho_+(X \otimes \mathbb{C}), \]

where \((\text{Pic } (X \otimes \mathbb{C})/T)^G \cong \mathbb{Z}^{\rho_+(X \otimes \mathbb{C})}\). We had proven above that \(G\) is trivial on \(H_1(X(\mathbb{C}); \mathbb{F}_2)\). Then, it is trivial on \(H^1(X(\mathbb{C}); \mathbb{F}_2) = H_1(X(\mathbb{C}); \mathbb{F}_2)^*\). From (2.15) and the remarks above, we then get that the group \(G\) is trivial on

\[ \ker \{ \text{Pic } (X \otimes \mathbb{C})/\times 2, \text{Pic } (X \otimes \mathbb{C})\} = H^1(X(\mathbb{C}); \mathbb{F}_2) \]
Thus,
\[ \ker \{ \text{Pic } X \xrightarrow{\times 2} \text{Pic } X \} = H^1(X(\mathbb{C}); \mathbb{F}_2). \]

As a result, we get that
\[ (2-19) \quad \dim \text{Pic } X/2\text{Pic } X = \rho_+(X \otimes \mathbb{C}) + \dim H^1(X(\mathbb{C}); \mathbb{F}_2). \]

Here, by the remarks above about the map (2–13),
\[ \rho_+(X \otimes \mathbb{C}) \leq h^{1,1}_-(X(\mathbb{C})). \]

From Proposition 1.3 and formulae of Theorem 2.1, we have
\[ \dim H^*(X(\mathbb{R}); \mathbb{F}_2) = \dim H^5(X(\mathbb{C}); G, \mathbb{F}_2) = 2 \dim H^2(X(\mathbb{C}); G, \mathbb{F}_2) - \dim H^2(X(\mathbb{C}); \mathbb{F}_2). \]

Thus,
\[ (2-20) \quad \dim H^2(X(\mathbb{C}); G, \mathbb{F}_2) = (1/2) \dim H^*(X(\mathbb{R}); \mathbb{F}_2) + (1/2) \dim H^2(X(\mathbb{C}); \mathbb{F}_2). \]

By Proposition 1.1, \( H^2(X(\mathbb{C}); G, \mathbb{F}_2) = H^2_{et}(X; \mathbb{F}_2). \) By the exact sequence (0–3), we get
\[ \dim 2Br'(X) = \dim H^2_{et}(X; \mathbb{F}_2) - \dim \text{Pic } X/2\text{Pic } X. \]

Thus, from (2–17), (2–19) and (2–20), we get
\[ (2-21) \quad \dim 2Br'(X) = (1/2) \dim H^*(X(\mathbb{R}); \mathbb{F}_2) + b_2/2 - \rho_+(X \otimes \mathbb{C}). \]

Let \( (b_2)_+ = \dim H^2(X(\mathbb{C}); \mathbb{C})^G \) and \( (b_2)_- = b_2 - (b_2)_+ \). From the Lefschetz fixed point formula for the involution \( g \) (see [Sp], for example), we get
\[ (2-22) \quad \chi(X(\mathbb{R})) = 2 + 2(b_2)_+ - b_2 = 2 + b_2 - 2(b_2)_-. \]

Thus, from (2–21) and (2–22), we get
\[ (2-23) \quad \dim 2Br'(X) = (1/2) \dim H^*(X(\mathbb{R}); \mathbb{F}_2) + (1/2)\chi(X(\mathbb{R})) - 1 + (b_2)_- - \rho_+(X \otimes \mathbb{C}). \]

For the Hodge decomposition
\[ H^2(X(\mathbb{C}); \mathbb{C}) = H^{2,0}(X(\mathbb{C})) + H^{0,2}(X(\mathbb{C})) + H^{1,1}(X(\mathbb{C})), \]

the antiholomorphic involution \( g \) sends \( H^{2,0}(X(\mathbb{C})) \to H^{0,2}(X(\mathbb{C})) \) and \( H^{1,1}(X(\mathbb{C})) \to H^{1,1}(X(\mathbb{C})). \) It follows that
\[ (b_2)_- = \dim H^{2,0}(X(\mathbb{C})) + \dim H^{1,1}(X(\mathbb{C})). \]

For a connected compact surface \( F \) we have \( \dim H^*(F; \mathbb{F}_2) + \chi(F) = 4. \) Thus, from (2–23), we get the formula of Theorem 0.4.

**Proof of Theorem 0.6.** Let \( Y \) be a real Enriques surface. In [N-S], the inequality (0–7), i.e. \( b(Y) \geq 2s - 2 \), was proved. The proof was similar to the proof above of the Theorem 0–4 and used Lefschetz fixed-point formula and the inequality (2–1).

From the proof, it follows that the equality \( b(Y) = 2s - 2 \) holds iff the inequality (2–1) is an equality. By Proposition 2.2 (of V.A.Krasnov), it then follows that the spectral sequence \( II \) degenerates iff \( b(Y) = 2s - 2 \). By Proposition 1.1, the Hochschild–Serre spectral sequence degenerates iff \( b(Y) = 2s - 2 \). It follows the statement (i) of Theorem 0.6. From the statement (i), the definition of the invariant \( \epsilon(Y) \), and from (0–6), (0–7), the statements (ii) and (iii) of Theorem 0.6 follow.
§3. Applications to topology of real Enriques surfaces

We use notation on real Enriques surfaces of Introduction. Thus, for a real
Enriques surface $Y$, we denote by $\theta$ the antiholomorphic involution of $Y$, by $\tau$ the holomorphic involution of the 2-sheeted
universal covering $\pi : X(C) \rightarrow Y(C)$, and by $\sigma, \tau \sigma$ two liftings of $\theta$ on $X(C)$. We suppose that the automorphism group

$$\Gamma = \{ \text{id}, \tau, \sigma, \tau \sigma \}$$

on $X(C)$ is isomorphic to $(\mathbb{Z}/2)^2$. In particular, it is true if $Y(R) \neq \emptyset$ (see [N-S]).

First, we discuss the following problem. We have $\dim H^2(Y(C); F_2) = 12$. A
subgroup $H^2(Y(C); Z) \otimes F_2 \subset H^2(Y(C); F_2)$ has $\dim H^2(Y(C); Z) \otimes F_2 = 11$. Thus, we can introduce the invariant

$$\beta(Y) = \dim H^2(Y(C); F_2)^\theta - \dim (H^2(Y(C); Z) \otimes F_2)^\theta.$$

This invariant is very important for real Enriques surfaces, and first, we want to calculate $\beta(Y)$ in some cases. Evidently, $\beta(Y) = 0$ or 1. For the invariant

$$b(Y) = \dim H^2(Y(C); F_2)^\theta - \dim (Pic Y \otimes C)^\theta / 2(Pic Y \otimes C)^\theta + 1,$$

(see Introduction), we have

$$(3-1) \quad b(Y) = b'(Y) + \beta(Y),$$

where we denote

$$b'(Y) = \dim (H^2(Y(C); Z) \otimes F_2)^\theta - \dim (Pic Y \otimes C)^\theta / 2(Pic Y \otimes C)^\theta + 1.$$

In [N-S, Theorem 3.4.7], there was obtained a formula for $b'(Y)$:

$$(3-2) \quad b'(Y) = r(\theta) - a(\theta) + \max\{1 - a(\sigma), (\delta_{\sigma L^r, \sigma} + \delta_{\sigma L^s}) / 2\}.$$

Here $r(\theta), a(\theta), \alpha(\sigma), \delta_{\sigma L^r, \sigma}, \delta_{\sigma L^s}$ are some invariants of the action of $\Gamma$ on the lattice $L$ which is the lattice $H^2(X(C); Z)$ with the intersection pairing. The invariants $r(\sigma), a(\sigma)$ are some non-negative integers, and

$$(3-3) \quad r(\sigma) \equiv a(\sigma) \pmod{2}.$$

The invariants $\alpha(\sigma), \delta_{\sigma L^r, \sigma}, \delta_{\sigma L^s}$ are equal to 0 or 1, and

$$(3-4) \quad \delta_{\sigma L^r, \sigma} = \delta_{\sigma L^s}.$$

The precise definition of these invariants is very long, and we refer to [N-S] for their
definition. Actually, these invariants are some specialization to Enriques surfaces
of general invariants from [N1, N2a].

Besides, in [N-S], it was proved that the invariant $\beta(Y) = 0$ if

$$\max\{1 - \alpha(\sigma), (\delta_{\sigma L^r, \sigma} + \delta_{\sigma L^s}) / 2\} = 0$$

or, equivalently, $\alpha(\sigma) = 1$ and $\delta_{\sigma L^r, \sigma} = \delta_{\sigma L^s} = 0$.

We want to prove here the following result which also gives another prove of the
statement about $\beta(Y)$ above, but only in the case if both real parts $X_{\sigma}(R)$ and $X_{\tau}(R)$ are non-empty.
Theorem 3.1. Let $Y$ be a real Enriques surface and both $X_\sigma(\mathbb{R})$ and $X_{\tau\sigma}(\mathbb{R})$ are non-empty.

Then:

$$\beta(Y) = \max\{1 - \alpha(\sigma), (\delta_{\sigma L^r,\sigma} + \delta_{\sigma L^z})/2\},$$

and

$$b(Y) = r(\theta) - a(\theta) + 2 \max\{1 - \alpha(\sigma), (\delta_{\sigma L^r,\sigma} + \delta_{\sigma L^z})/2\},$$

where $r(\theta), a(\theta), \alpha(\sigma), \delta_{\sigma L^r,\sigma}, \delta_{\sigma L^z}$ are some invariants of the action of the group $\Gamma$ on the lattice $H^2(X(\mathbb{C});\mathbb{Z})$ with the intersection pairing.

Proof. By Theorem 0.6, $b(Y) = 2s - 2 \equiv 0 \mod 2$. By (3–1) — (3–3), we then get

$$\beta(Y) \equiv \max\{1 - \alpha(\sigma), (\delta_{\sigma L^r,\sigma} + \delta_{\sigma L^z})/2\} \mod 2.$$

The right side of this congruence is equal to 0 or 1 since $\alpha(\sigma) = 0$ or 1, and $\delta_{\sigma L^r,\sigma} = \delta_{\sigma L^z}$ is 0 or 1. It follows the first formula since $\beta(Y)$ is equal to 0 or 1 too. From the first formula and (3–1), (3–2), the second one follows.

We don’t know if this statement valid when $\max\{1 - \alpha(\sigma), (\delta_{\sigma L^r,\sigma} + \delta_{\sigma L^z})/2\} = 1$ and one of $X_\sigma(\mathbb{R})$ or $X_{\tau\sigma}(\mathbb{R})$ is empty.

In [N-S, Theorems 3.5.1—3.5.3, formula (3-5-1)], there was obtained a formula for $b(Y)$ using the numbers $s_{or}$ and $s_{nor}$ of orientable and non-orientable connected components of $Y(\mathbb{R})$ respectively. The numbers $s_{or}$ and $s_{nor}$ are connected with the numbers $s(\sigma)$ and $s(\tau\sigma)$ of connected components of $X_\sigma(\mathbb{R})$ and $X_{\tau\sigma}(\mathbb{R})$ respectively by the formula $s(\sigma) + s(\tau\sigma) = 2s_{or} + s_{nor}$ (see [N-S, Lemma 3.2.1]). For the case $s(\sigma) > 0$ and $s(\tau\sigma) > 0$ (or when both sets $X_\sigma(\mathbb{R})$ and $X_{\tau\sigma}(\mathbb{R})$ are non-empty) which is necessary for us, this formula for $b(Y)$ claims that

$$b(Y) = 2s_{or} + s_{nor} - 2 + \min\{\alpha(\sigma), (\delta_{\sigma L^r,\sigma} + \delta_{\sigma L^z})/2\}$$

$$+ \dim H(\sigma)_- - \dim H(\sigma)_+ \cap H(\sigma)_- + \beta(Y).$$

Here $\dim H(\sigma)_-$ and $\dim H(\sigma)_+ \cap H(\sigma)_-$ are some other invariants of the action of the group $\Gamma$ on the lattice $\mathbb{L}$ (see [N-S]). From the formula for $\beta(Y)$ of Theorem 3.1, we get the formula

$$(3\cdot5) \quad b(Y) = 2s_{or} + s_{nor} - 2 + \min\{\alpha(\sigma), (\delta_{\sigma L^r,\sigma} + \delta_{\sigma L^z})/2\}$$

$$+ \max\{1 - \alpha(\sigma), (\delta_{\sigma L^r,\sigma} + \delta_{\sigma L^z})/2\} + \dim H(\sigma)_- - \dim H(\sigma)_+ \cap H(\sigma)_-$$

$$= 2s_{or} + s_{nor} - 1 + \alpha(\sigma)(\delta_{\sigma L^r,\sigma} + \delta_{\sigma L^z} - 1)$$

$$+ \dim H(\sigma)_- - \dim H(\sigma)_+ \cap H(\sigma)_-,$$

if both $X_\sigma(\mathbb{R})$ and $X_{\tau\sigma}(\mathbb{R})$ are non-empty.

By the formula $b(Y) = 2s - 2 = 2s_{or} + 2s_{nor} - 2$ of Theorem 0.6, we get

Theorem 3.2. Let $Y$ be a real Enriques surface and both $X_\sigma(\mathbb{R})$ and $X_{\tau\sigma}(\mathbb{R})$ are non-empty.

Then for the number $s_{nor}$ of non-orientable connected components of $Y(\mathbb{R})$ we have the formula:

$$(3\cdot6) \quad s_{nor} = 1 + \alpha(\sigma)(\delta_{\sigma L^r,\sigma} + \delta_{\sigma L^z} - 1) + \dim H(\sigma)_- - \dim H(\sigma)_+ \cap H(\sigma)_-.$$
where $\alpha(\sigma)$, $\delta_{\sigma^{L^\sigma}}$, $\delta_{\sigma L^\sigma}$, $\dim H(\sigma)_-$, $\dim H(\sigma)^+_\perp \cap H(\sigma)_-$ are some invariants of the action of the group $\Gamma$ on the lattice $H^2(X(C);\mathbb{Z})$ with the intersection pairing.

We mention that by Theorem 0.6, $b(Y) = 2s - 2$ if both $X_\sigma(R)$ and $X_{\tau\sigma}(R)$ are non-empty. Thus, by the formula for $b(Y)$ of Theorem 3.1, we also have the formula for the number $s = s_{or} + s_{nor}$ of all connected components of $Y(R)$ if both $X_\sigma(R)$ and $X_{\tau\sigma}(R)$ are non-empty.

**Theorem 3.3.** Let $Y$ be a real Enriques surface and both $X_\sigma(R)$ and $X_{\tau\sigma}(R)$ are non-empty.

Then for the number $s$ of all connected components of $Y(R)$ we have the formula

$$s = 1 + (r(\theta) - a(\theta))/2 + \max\{1 - \alpha(\sigma), (\delta_{\sigma^{L^\sigma}} + \delta_{\sigma L^\sigma})/2\},$$

where $r(\theta)$, $a(\theta)$, $\alpha(\sigma)$, $\delta_{\sigma^{L^\sigma}}$, $\delta_{\sigma L^\sigma}$ are some invariants of the action of the group $\Gamma$ on the lattice $H^2(X(C);\mathbb{Z})$ with the intersection pairing.

Of course, from the formulae for $s$ and $s_{nor}$ or Theorems 3.2, 3.3, we get a formula for $s_{or} = s - s_{nor}$. These Theorems 3.1—3.3 are very important for the topological classification of real Enriques surfaces—describing of all possible topological types of $Y(R)$ for real Enriques surfaces $Y$. See [N4] for these applications.
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