Generalized Slow Roll Conditions and the Possibility of Intermediate Scale Inflation in Scalar-Tensor Theory

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Generalized slow roll conditions and parameters are obtained for a general form of scalar-tensor theory (with no external sources), having arbitrary functions describing a nonminimal gravitational coupling $F(\phi)$, a Kahler-like kinetic function $k(\phi)$, and a scalar potential $V(\phi)$. These results are then used to analyze a simple toy model example of chaotic inflation with a single scalar field $\phi$ and a standard Higgs potential and a simple gravitational coupling function. In this type of model inflation can occur with inflaton field values at an intermediate scale of roughly $10^{11}$ GeV when the particle physics symmetry breaking scale is approximately 1 TeV, provided that the theory is realized within the Jordan frame. If the theory is realized in the Einstein frame, however, the intermediate scale inflation does not occur.

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I. INTRODUCTION

Our experience with gravitation leads us to believe that Einstein gravity provides a good description, at least at low energies. However, we cannot dismiss the possibility that some generalization, such as scalar-tensor (ST) theory, with an appropriate low energy Einstein limit, provides a more accurate description, especially at high energies where the deviation from Einstein gravity can be significant. ST theories include the special cases of Brans-Dicke theory [1] and dilaton gravity, and are physically motivated by ideas such as Mach’s principle [1], string theory [2], [3] and other higher dimensional theories. When investigating the phenomenon of inflation, it is therefore natural to study ST modifications to Einstein gravity and their implications [4], [5].

Here, attention is focused upon a fairly general form of ST theory (equivalent to hyperextended scalar-tensor gravity [6]) where a single scalar field $\phi$ couples nonminimally to gravity through the function $F(\phi)$ and whose dynamics is partially governed by a scalar potential $V(\phi)$. A kinetic function $k(\phi)$, which resembles a Kahler metric in supersymmetric theories, allows for a noncanonical kinetic term and permits the ST theory to be written in a form resembling generalized Brans-Dicke gravity or dilaton gravity, for instance. The Einstein limit is realized when $F = 1$. The usual assumptions and approximations made for slow roll inflation in the Einstein theory [1] can be simply extended to accommodate the functions $F$, $k$, and $V$ in the ST theory with the imposition of a very simple and mild requirement that each function (or its inverse) have a sufficiently rapid convergence of its Taylor series power expansion so that it can be well approximated with a finite number $N$ of terms with $N \ll H/($rate of change of $\phi$). With these conditions, the usual slow roll approximations can be implemented to obtain the slow roll equations of motion (EOM) for the field $\phi$ in a flat Robertson-Walker spacetime. The slow roll parameters can be defined for the ST theory in terms of the functions $F$, $k$, and $V$. (However, an “effective potential” $U(\phi) = V/F^2$ arises in the ST theory and appears in the EOM, and the parameters can be written in a more economical form by using the function $U$.) For a canonical kinetic term ($k = 1$) in the Einstein limit ($F = 1$), the slow roll parameters and EOM reduce to the usual ones.

Applying the generalized results to slow roll chaotic inflation, we explore the possibility that ST modifications can allow chaotic inflation [8] to occur at intermediate energy scales, characterized by inflaton field values $\phi \ll M_P$, that are far below the Planck scale and therefore radically different from those normally assumed. This feature also appears in other models with minimally coupled scalar fields that have recently been suggested, such as “supernatural inflation” [9] or “assisted inflation” [10] in higher dimensional theories [11]. However, these inflationary models involve more than one scalar field, as Randall, Soljacić, and Guth [9] have argued must be the case. But, within the context of ST theory, the nonminimal coupling of a single
inflaton to gravity can have an important effect, allowing inflation to proceed at an intermediate energy scale $E \ll M_P$ for a class of coupling functions $F(\phi)$. This result stems from an interesting possibility of a relative increase in the strength of gravity at an intermediate scale in the ST theory, with the nonminimal coupling $F(\phi)$ giving rise to an effective gravitational coupling $\bar{\kappa}^2(\phi) = \kappa^2 / F(\phi)$ that is an increasing function of the inflaton field $\phi$. This effect can allow enough inflation to occur for a relatively small range of values of $\phi$ at a relatively low energy scale.

Specifically, the possibility of an intermediate scale inflation is studied within the context of a toy model with a standard type of quartic Higgs potential $V(\phi)$ and a simple nonminimal scalar coupling function $F(\phi)$ that is a decreasing function of $|\phi|$. The Higgs potential locates the vacuum state at $|\phi| = \eta$, where $\eta$ is assumed to be small in comparison to the Planck mass $M_P$. The function $F$ becomes unity in the vacuum, i.e. $F(\eta) = 1$, so that the nonminimal coupling approaches minimal coupling in the vacuum. From the slow roll conditions, along with the requirement that there be enough $e$-folds of inflation, the final and initial values of $\phi$ during inflation can be determined. Taking $\eta \approx 1 \text{ TeV}$, as an example, we find inflation occurring for $\phi \sim 10^{11} \text{ GeV}$, i.e., at an intermediate scale. The possibility of intermediate scale inflation exists with the proviso that the scalar-tensor theory is physically realized within the Jordan conformal frame, where there is an explicit nonminimal coupling of the scalar field. Of course, this explicit nonminimal coupling can be removed by performing a conformal transformation to the Einstein frame. However, only one conformal frame can be the physical frame, and we find that intermediate scale inflation does not occur if the Einstein frame is physically realized.

The action for the scalar-tensor theory is presented and the field equations and cosmological EOM are obtained in section II. The usual slow roll conditions and approximations are generalized in section III and applied to obtain the slow roll EOM and slow roll parameters. A toy model of intermediate scale chaotic inflation is presented and analyzed in section IV. In section V we mention the debate as to whether it should be the Jordan frame or the Einstein frame that should be regarded as being the physical frame, and show that the intermediate scale inflation of the toy model does not occur in the Einstein frame. A short summary and some concluding remarks are offered in section V.

II. SCALAR-TENSOR THEORY

We consider scalar-tensor (ST) theory formulated in the Jordan conformal frame, where there is an explicit nonminimal coupling, described by the function $F(\phi)$, between the gravitational field and the scalar field $\phi$. The action can be written
in the form

\[ S = \int d^4x \sqrt{-g} \left\{ F(\phi)\frac{R}{2\kappa^2} + \frac{1}{2}k(\phi)(\partial\phi)^2 - V(\phi) \right\}, \]  

(1)

where \(\kappa^2 = 8\pi G\), and the nonminimal coupling function \(F\), the kinetic function \(k\), and the scalar potential \(V\) are arbitrary functions of \(\phi\) only. No metric dependent external source terms have been included in the action, but a dilaton coupled cosmological constant can be accommodated by the function \(V(\phi)\). For a canonical kinetic term \(k(\phi) = 1\), whereas for pure Brans-Dicke theory, we could write \(F(\phi) = \frac{\kappa^2}{8\pi} \phi\), \(k(\phi) = \omega/(8\pi\phi)\), and \(V(\phi) = 0\). With different parametrizations of the functions \(F\), \(k\), and \(V\), we can write the action (1) in the following forms:

- **Generalized Brans-Dicke theory:** Taking \(F(\phi) = \frac{\kappa^2}{8\pi} \phi\) and \(k(\phi) = \frac{\omega(\phi)}{8\pi\phi}\), the action for a generalized Brans-Dicke theory can be written in the form

\[ S_{BD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ \phi R + \frac{\omega(\phi)}{\phi}(\partial\phi)^2 - 16\pi V(\phi) \right\}. \]  

(2)

- **Dilaton gravity:** For dilaton gravity we can write \(F(\phi) = e^{-\phi}\), \(k(\phi) = -\frac{F(\phi)}{\kappa^2} = -\frac{e^{-\phi}}{\kappa^2}\), and \(V(\phi) = \frac{e^{-\phi}}{2\kappa^2}W(\phi)\). In this case the action (1) can be alternatively written in the form

\[ S_{DG} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-ge^{-\phi}} \left\{ R - (\partial\phi)^2 - W(\phi) \right\}. \]  

(3)

- **Hyperextended scalar-tensor gravity:** The case of hyperextended scalar-tensor gravity (HSTG) studied by Torres and Vucetich [1] and Torres [12], can be described with \(F(\phi) = \frac{\kappa^2}{8\pi G(\phi)}\), \(k(\phi) = \frac{\omega(\phi)}{8\pi\phi}\), where \(G(\phi)\) is an arbitrary function of \(\phi\), with the action assuming the form

\[ S_{HSTG} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ \frac{R}{G(\phi)} + \frac{\omega(\phi)}{\phi}(\partial\phi)^2 - 16\pi V(\phi) \right\}. \]  

(4)

The action in (1) is equivalent to the action in (4), where there are three free functions. The variation of the action in (1) gives the field equations

\[ \textnormal{We use a metric with signature \((+,-,-,-)\) and a Ricci tensor } R_{\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\mu\nu} \Gamma^\sigma_{\rho\sigma} + \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\rho\mu}. \]
\[ G_{\mu\nu} = -\frac{F_{,\phi} \nabla_\mu \phi - \frac{1}{F} [F_{,\phi\phi} + \kappa^2 k] \partial_\mu \phi \partial_\nu \phi}{\nabla_\mu \phi - \frac{1}{F} F_{,\phi\phi} + \kappa^2 k} \left( \frac{\partial \phi}{\partial \phi} \right)^2 + F_{,\phi} \Box \phi - \kappa^2 V \right) \] \] (5)

\[ \kappa^2 k \Box \phi + \frac{1}{2} \kappa^2 k_{,\phi} (\partial \phi)^2 + \kappa^2 V_{,\phi} - \frac{1}{2} F_{,\phi} R = 0. \] \] (6)

where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor. Taking the trace of (5) to obtain an expression for the Ricci scalar \( R \) and inserting into (6) gives

\[ \left[ \frac{3(F_{,\phi})^2}{2F} + \kappa^2 k \right] \Box \phi + \left\{ \frac{F_{,\phi}}{2F} [3F_{,\phi\phi} + \kappa^2 k] + \frac{1}{2} \kappa^2 k_{,\phi} \right\} (\partial \phi)^2 + \kappa^2 F^2 U_{,\phi} = 0, \] \] (7)

where \( F_{,\phi} = \partial F / \partial \phi \), \( F_{,\phi\phi} = \partial^2 F / \partial \phi^2 \), etc. and

\[ U = \frac{V}{F^2} \] (8)

is an effective potential induced by the scalar curvature \( R \). We shall take the field equations to be given by (6) and (7).

To obtain the cosmological equations of motion (EOM), we adopt the metric for a flat Robertson-Walker spacetime,

\[ ds^2 = dt^2 - a^2(t) d\vec{x} \cdot d\vec{x}. \] (9)

Eqs. (5), (7), and (9) then give the EOM

\[ H^2 = \frac{\kappa^2}{3F} \left[ \frac{1}{2} \kappa \dot{\phi}^2 + V \right] - \frac{F_{,\phi}}{F} H \dot{\phi}, \] \] (10)

\[ \left[ \frac{3(F_{,\phi})^2}{2F} + \kappa^2 k \right] (\ddot{\phi} + 3H \dot{\phi}) + \left\{ \frac{F_{,\phi}}{2F} [3F_{,\phi\phi} + \kappa^2 k] + \frac{1}{2} \kappa^2 k_{,\phi} \right\} \dot{\phi}^2 + \kappa^2 F^2 U_{,\phi} = 0, \] \] (11)

where \( H = \dot{a} / a \) is the Hubble parameter. Notice that there is an effective gravitational coupling \( \bar{\kappa} \) defined by \( \bar{\kappa}^2 = \kappa^2 / F \).

III. GENERALIZED SLOW ROLL CONDITIONS AND PARAMETERS

In most inflationary models, inflation takes place under the slow roll conditions \[ \] that (i) the inflaton field \( \phi \) evolves slowly in comparison to the expansion rate of the universe, and (ii) the kinetic energy density of the inflaton is negligible in comparison
to the potential energy density, i.e., the inflation is driven by the potential. These conditions, which can be stated more quantitatively as

\[ |\ddot{\phi}| \ll H|\dot{\phi}| \ll H^2|\phi|, \]  

(12)

\[ \frac{1}{2}|k(\phi)|\dot{\phi}^2 \ll |V(\phi)|, \]  

(13)

will be assumed to hold during an inflationary epoch. However, as pointed out by Torres [12], the condition given by (12) can be generalized to read

\[ |\ddot{f}| \ll H|\dot{f}| \ll H^2|f|, \]  

(14)

whenever the function \( f(\phi) \) has a sufficiently rapidly convergent power series Taylor expansion. Actually, this convergence assumption can be relaxed somewhat. For instance, even if a power series expansion for \( f \) does not converge within some domain, but \( g = 1/f \) does converge with sufficient rapidity, then \( |\ddot{g}| \ll H|\dot{g}| \ll H^2|g| \) implies that (14) is satisfied for the function \( f \). (As an example, \( f = (1 + c\phi^2)^{-1} \), where \( c \) is a constant, does not have a convergent Maclaurin series expansion for \( |c|\phi^2 > 1 \), but \( g = f^{-1} \) does, and therefore both \( g \) and \( f \) satisfy the condition (14) whenever the condition (12) is satisfied.) To obtain our generalized slow roll EOM, it will be assumed that we are dealing with functions \( F, F_{,\phi}, k, \) and \( U \) (or \( V \)) that satisfy (14).

Applying the above slow roll conditions to (10) and (11) (see the appendix for details) gives the slow roll EOM

\[ H^2 = \frac{\kappa^2}{3F} V, \]  

(15)

\[ 3Hk\dot{\phi} + F^2U_{,\phi} = 0. \]  

(16)
We now define the slow roll parameters

\[ \varepsilon_{SR} = \frac{1}{2\kappa^2} \left[ \frac{F}{k} \left( \frac{U_\phi}{U} \right)^2 \right], \tag{17} \]

\[ \eta_{SR} = \frac{1}{\kappa^2} \left[ \frac{1}{FU} \frac{\partial}{\partial \phi} \left( \frac{F^2 U_\phi}{k} \right) \right]. \tag{18} \]

For the case of minimal coupling \((F = 1)\) and a canonical kinetic term \((k = 1)\) these parameters collapse to the usual expressions. The condition given by (13), when used in conjunction with (15) and (16), then translates into the condition

\[ \left| \frac{1}{2k} \dot{\phi}^2 \right| = \frac{|\varepsilon_{SR}|}{3} \ll 1. \]

(The condition \(|\varepsilon_{SR}| \ll 1\) also follows from the condition \(|\dot{U}/U| \ll H\).)

Next, we use the fact that an application of (14) to the functions \(F(\phi)\) and \(U(\phi)\) implies that \(|\dot{H}/H| \ll 1\). Then taking the time derivative of \(\dot{\phi}\), using (10), and applying (15), the condition that \(|\dot{\phi}/(H\dot{\phi})| \ll 1\) translates into the condition \(|\eta_{SR}| \ll 1\). The slow roll conditions can then be stated in terms of the parameters \(\varepsilon_{SR}\) and \(\eta_{SR}\):

\[ |\varepsilon_{SR}| \ll 1, \quad |\eta_{SR}| \ll 1. \tag{19} \]

The violation of these conditions signals the end of the inflationary period.

The onset of inflation occurs at a time \(t_i\) when the inflaton has a value \(\phi = \phi_i\) and inflation ends at a time \(t_f\) when \(\phi = \phi_f\). The amount of inflation is given by the number \(N \approx \int_{t_i}^{t_f} H dt\) of e-folds of the scale factor, which from the slow roll EOM is

\[ N \approx \int_{t_i}^{t_f} \frac{H^2 \dot{\phi}}{H \dot{\phi}} dt = \int_{\phi_i}^{\phi_f} \frac{H^2}{H \dot{\phi}} d\phi \approx -\kappa^2 \int_{\phi_i}^{\phi_f} \left[ \frac{k}{F} \left( \frac{U}{U_\phi} \right) \right] d\phi. \tag{20} \]

### IV. INTERMEDIATE SCALE CHAOTIC INFLATION

We have seen that the slow roll inflationary conditions and parameters for a scalar-tensor theory, where the inflaton is nonminimally coupled to gravity through the function \(F(\phi)\), are modified from the usual conditions for a minimally coupled inflaton field. In other words, the conditions under which inflation begins and ends are controlled not just by the potential \(V\), but also by the coupling function \(F\). (Here, we will consider the case of a canonical kinetic term, \(k = 1\).) It therefore seems plausible that the ST theory can accommodate a class of coupling functions \(F(\phi)\) that would allow the onset of chaotic inflation to appear when the inflaton acquires a value \(\phi \ll M_P = G^{-1/2}\), well below the Planck scale.
In the usual case of a minimally coupled inflaton (and no other scalar fields), we typically find that inflation begins and ends for \( \phi \gtrsim M_P \), as we can easily see from the following example. Consider a minimally coupled theory with \( F = 1, k = 1 \), and a quartic Higgs potential \( V = \lambda \phi^2 - \eta^2 = \lambda \eta^4 (\bar{\phi}^2 - 1)^2 \), where \( \bar{\phi} = \phi/\eta \), and the mass scale, determined by the parameter \( \eta \) in the potential, comes purely from the particle physics. At large values of \( \phi \), i.e. \( \bar{\phi} \gg 1 \), we have a simple power behavior for the potential, \( V \approx \lambda \eta^4 \bar{\phi}^4 \). The end of the inflationary period is signalled by \( \varepsilon_{SR} \sim 1 \), which implies that \( \dot{\phi}_f \sim 4/(\kappa \eta) \), or \( \phi_f \sim 4M \sim M_P \), where we define the mass \( M = \kappa^{-1} = M_P/\sqrt{8\pi} \). The onset of inflation, say about 70 e-folds earlier, occurs at a value \( \phi_i \), determined by \( N \approx \kappa^2 \int_{\phi_i}^{\phi_f} \frac{V}{\sqrt{\bar{V}}} d\bar{\phi} \), which gives \( \phi_i \sim \sqrt{8N}(\kappa \eta)^{-1/2} \approx 5M_P \) for \( N \approx 70 \). Actually, a condition on the vacuum density can be obtained from the slow roll EOM and approximations. From the motion equations for \( H \) and \( \dot{\phi} \), we have \( \left| F^2 U,_{\phi} \right| = \left| 3Hk \dot{\phi} \right| \ll 3H^2 |k\phi| = \kappa^2 |k\phi FU| \), which implies that

\[
\left| \frac{k\phi U}{FU,_{\phi}} \right| \gg M^2
\]

which, for the present example, gives \( \phi_i \gg 4 \left( \frac{M}{\eta} \right) M \gg M_P \) for \( \eta/M \ll 1 \), implying that there are many more e-folds of inflation than the minimal number required. If \( \eta \ll M \), i.e. if the symmetry breaking takes place a little below the Planck scale (or if there is no symmetry breaking, as in \( m^2 \phi^2 \) or \( \lambda \phi^4 \) chaotic inflation models), then we have the condition \( \phi \gg M_P \), but if the symmetry breaking takes place at a low energy scale, the condition on the vacuum density implies that \( \phi_i \) is absurdly large.

### A. Toy Model Example

Here we consider a very simple toy model using the same potential \( V = \lambda \eta^4 (\bar{\phi}^2 - 1)^2 \), \( k = 1 \), but for the coupling function we choose the simple function \( F = \frac{\bar{\phi}}{(\bar{\phi}^2 + 1)^{1/2}} \), with \( F \to 1 \) as \( \phi \to \eta \). The basic idea is to investigate how the function \( F \) can alter the inflationary conditions to give an inflationary period at a scale \( \phi \ll M \). To get plenty of e-folds over a sufficiently small range of \( \phi \) values, we want \( H \) to be sufficiently large, or from (20), we want \( |U/FU,_{\phi}| \) to be sufficiently large. Now, if within the context of a scalar-tensor theory we have an effective gravitational coupling \( \bar{\kappa} = F^{-1/2} \kappa \) which is an increasing function of \( \phi \), i.e. \( F \) is a decreasing function of \( \phi \), then for a given value of \( \phi \), \( H^2 = \bar{\kappa}^2 V/3 > \kappa^2 V/3 \), giving a larger value of \( H \) than in the minimally coupled case. This may allow inflation to proceed at a smaller value of \( \phi \) than in the usual scenario. Naively, if in the usual case we have inflation proceeding at \( \phi \sim M \), we expect that in the ST model we would have \( \phi \sim \bar{\kappa}^{-1} = M = F^{1/2} M \). Therefore we expect \( \phi \ll M \) if \( F(\phi) \ll 1 \). For the simple function \( F = 2/(\bar{\phi}^2 + 1) \approx 2/\bar{\phi}^2 \) for \( \bar{\phi} \gg 1 \), the condition \( \phi \sim F^{1/2} M \) gives \( \bar{\phi} \sim (\kappa \eta)^{-1/2} \), or \( \phi \sim (\eta M)^{1/2} \ll M \) for \( \eta/M \ll 1 \).
For example, if we take $\eta \approx 1$ TeV and $M \approx 10^{18}$ GeV, we find $\phi \sim 3 \times 10^{10}$ GeV, implying inflation at an intermediate scale.

To check this qualitative argument, we can use the slow roll conditions obtained above. Because we have simple functions, which for $\bar{\phi} \gg 1$ behave as simple powers, the estimates are easy to get. Taking the end of inflation to coincide with $\varepsilon_{SR} \sim 1$, we find $\phi_f \sim \frac{3(\kappa \eta)}{\sqrt{8\pi}}$, or $\phi_i \sim \frac{3}{\sqrt{8\pi}} \approx 10^{11}$ GeV. These results agree with our qualitative estimate. The condition on the vacuum density $|\phi U/(FU,\phi)| \gg M^2$ in this case gives $\phi_i \gtrsim 3(\eta M)^{1/2} \approx 10^{11}$ GeV. This can be compared to the condition in the previous (minimally coupled) case where $\phi_i \gg (M/\eta)M_p \approx 10^{15}M_p$ (for $\eta \approx 1$ TeV), an enormous value! In a minimally coupled model of inflation (without additional source terms), it appears difficult to have inflation associated with low scale symmetry breaking.

We conclude that the scale at which inflation occurs can be substantially modified by a nonminimal coupling of the scalar field. For a symmetry breaking scale $\eta \approx 1$ TeV, the scale for the inflaton field $\phi$ and the effective Planck scale $M_p = \sqrt{8\pi \bar{M}} = (\sqrt{8\pi / \bar{k}}) \sim (8\pi \eta M)^{1/2}$ are both brought down to an intermediate scale. Another way to state this is that the effective dimensionless gravitational “coupling constant” $\alpha_g = (\bar{k}^2 / 8\pi) E^2$ approaches unity at an energy scale of $E \sim \sqrt{8\pi \bar{k}}^{-1} \sim (8\pi \eta M)^{1/2}$. Therefore, if reality is described by a scalar-tensor theory with a nonminimal coupling of the scalar field which is a decreasing function of $\phi$, and the scalar field is associated with a low energy symmetry breaking, then it is possible for nontrivial gravitational effects to begin showing up at an intermediate scale, well below the Planck scale, which is an intriguing prospect.

Finally, it should be pointed out that the particular toy model used here as an example, while computationally convenient, is oversimplified and not quite realistic, in that it does not satisfy observational constraints. In the near-vacuum sector, where $\bar{\phi} \approx 1$, we have $V(\phi) \approx 0$ and the Lagrangian takes an approximate Brans-Dicke form

$$\frac{L}{\sqrt{-g}} \approx \frac{\Phi}{16\pi} R + \frac{\kappa^2}{16\pi} \left[ \frac{F}{(F,\phi)^2} \right] \frac{(\partial \Phi)^2}{\Phi},$$

(22)

where $\Phi = (8\pi / \kappa^2) F(\phi)$ is a Brans-Dicke field, and we can identify the Brans-Dicke parameter

$$\omega \approx \kappa^2 \frac{F}{(F,\phi)^2} \bigg|_{\phi=\eta}.$$

(23)

Note that, although inflation may occur for $\phi \sim \bar{M} = \bar{k}^{-1}$, the energy density of the inflaton field $\sim \lambda \phi^4 \sim \lambda M^4$ may still be small compared to $M^4$ provided that $\lambda$ is small, so that the energy scale for inflation can still be below the effective Planck scale.
For our toy model with a simple power function for $F$, $\omega \approx (\kappa \eta)^2 \ll 1$, which violates the observational constraint $\omega \gtrsim 500$. For a realistic model we should therefore have

$$\left| \frac{F_\phi}{F^{1/2}} \right|_{\phi=\eta} \approx \frac{\kappa}{\omega^{1/2}} \lesssim \frac{\kappa}{\sqrt{500}}.$$  \hspace{1cm} (24)

implying that the function $F(\phi)$ should be quite flat near $\phi \approx \eta$ and then decrease for $\phi \gg \eta$ where inflation occurs.

For the case of a minimally coupled inflaton, it has been shown how the inflaton potential can be reconstructed from slow roll parameters (see, e.g., [13,14]). This type of approach of reconstructing the form of the theory has been extended to scalar-tensor theory by Boisseau, Esposito-Farese, Polarski, and Starobinsky [15], who have shown how both of the functions $F(\phi)$ and $V(\phi)$ can be determined (with a rescaling of $\phi$ to set $k = 1$) from future observations.

V. JORDAN AND EINSTEIN CONFORMAL FRAMES

Slow roll conditions and the possibility of intermediate scale inflation in scalar-tensor theory have been presented here in the Jordan frame, where the scalar field $\phi$ is nonminimally coupled to the gravitational field. However, a conformal transformation allows the model to be recast in the Einstein frame, where the scalar field $\phi$ is minimally coupled to gravity. Of course, only one of these frames is the physical frame, but there has existed quite a lot of debate and variance in the literature as to exactly which frame should be considered the physical frame. (For a review of this situation, see, for example, [16] and references therein.) The Jordan frame is taken as the physical frame in many studies of scalar-tensor cosmology (e.g., ref. [17]), including studies of string cosmology (e.g., ref. [3]). The low energy scalar-tensor field theoretic action for string theory, in the string frame, follows from the original action for the strings, and this frame is often considered to provide the physical interpretation, with the strings “seeing” the string (Jordan) frame, not the conformally related Einstein frame. Also, in many forms of scalar-tensor theory, the weak equivalence principle is violated in the Einstein frame, due to the anomalous coupling of the dilaton to matter [16]. On the other hand, some objections have been raised against the consideration of the Jordan frame as the physical frame in ST models [16], especially those having a negative kinetic energy term for the dilaton field at the tree level. In addition, Torres, Schunck, and Liddle [17] have pointed out that whereas several mass definitions for boson stars in a scalar-tensor theory lead to different results in the Jordan frame, they all coincide in the Einstein frame, leading to the suspicion that the Einstein frame must be regarded as the physical frame. (For the case of a massless scalar field, Damour and Nordtvedt [18] have shown that general relativity acts as a cosmological attractor of scalar-tensor theories, so that they may become indistinguishable after early times.)
If the Jordan frame is the physical frame, i.e. the frame in which the physical parameters of the theory coincide with those that are measured in experiments (see, e.g., ref. [15]), we see the possibility arising for intermediate scale inflation for a certain class of coupling functions $F(\phi)$ and a relatively low energy ($\eta \ll M_P$) symmetry breaking scale from the particle physics sector. However, it should be pointed out that our toy model considered previously does not lead to intermediate scale inflation in the Einstein conformal frame. A conformal transformation to the Einstein frame can be accomplished with the rescaling $g_{\mu \nu} \to \hat{g}_{\mu \nu}$, where

$$\hat{g}_{\mu \nu} = F(\phi) g_{\mu \nu}.$$  \hspace{1cm} (25)

With this conformal rescaling the action (1) now takes the form

$$\hat{S} = \int d^4x \sqrt{-\hat{g}} \left\{ \frac{\hat{R}}{2\kappa^2} + \frac{1}{2} \hat{k}(\phi)(\hat{\partial} \phi)^2 - U(\phi) \right\},$$  \hspace{1cm} (26)

where $\hat{R}$ is built from the Einstein metric $\hat{g}_{\mu \nu}$, $(\hat{\partial} \phi)^2 = \hat{g}^{\mu \nu} \partial_\mu \phi \partial_\nu \phi$, and

$$\hat{k} = \frac{1}{\kappa^2} \left[ \frac{3(F(\phi))^2}{2F^2} + \frac{\kappa^2 k}{F} \right].$$  \hspace{1cm} (27)

The field equations and cosmological EOM can be simply obtained from those presented for the Jordan frame by making the replacements $F \to \hat{F} = 1$, $k \to \hat{k}$, $V \to U$, along with a rescaling of the time coordinate $t \to \hat{t}$ defined by $d\hat{t} = F^{1/2} dt$ and a rescaling of the scale factor $a \to \hat{a} = F^{1/2} a$, so that the metric takes the RW form in the Einstein frame:

$$d\hat{s}^2 = \hat{g}_{\mu \nu} dx^\mu dx^\nu = F[dt^2 - a^2(t) d\vec{x}^2] = d\hat{t}^2 - \hat{a}^2(\hat{t}) d\vec{x}^2.$$  \hspace{1cm} (28)

When the toy model of the previous section is analyzed within the Einstein frame, intermediate scale inflation disappears, since a field redefinition $\phi \to \varphi$ of the scalar field is possible to obtain a canonical kinetic term. This resets the scale of inflation $\phi_i \approx 10^{11}$GeV back to the usual Planck scale; roughly, we have $\varphi_i \sim \hat{k}^{1/2} \phi \sim M_P$, resulting in a conventional type of scenario. Therefore, intermediate scale inflation is a possible consequence of a physical Jordan frame, due to the fact that the effective Planck mass is also brought down to this scale. However, in a physical Einstein frame, the toy model intermediate scale inflation is not realized.

**VI. SUMMARY AND CONCLUDING REMARKS**

Physically motivated scalar-tensor theories of gravity can arise in various contexts. We have considered a general form of scalar-tensor theory for a single scalar field $\phi$, with no external sources. The theory is characterized by three arbitrary functions
describing the nonminimal gravitational coupling of the scalar $F(\phi)$, the scalar kinetic function $k(\phi)$, and the scalar potential $V(\phi)$. Upon adopting a flat Robertson-Walker metric, the cosmological equations of motion were obtained. The usual slow roll conditions for inflation and the slow roll parameters have been generalized to accommodate the arbitrariness of the functions $F$, $k$, and $V$. The generality of these results allows them to be applicable to various models, including generalized Brans-Dicke theory and dilaton gravity. The dependence of the generalized slow roll parameters upon the functions $F$, $k$, and $V$ (or the “effective potential” $U = V/F^2$) shows that several aspects of inflation, such as the onset and the end of inflation, the amount of inflation, and the inflationary solutions themselves, are controlled by more than just the scalar potential, and that scalar-tensor theory can therefore introduce nontrivial modifications to inflationary scenarios based upon minimally coupled models.

Next, the possibility of inflation occurring at an intermediate scale, for which $\phi$ takes values well below the usual Planck scale, was investigated by using a simple toy model where the coupling function $F$ is a decreasing function of $\phi$, leading to an increase in the strength of gravity at intermediate scales. The model has a standard type of quartic Higgs potential with a low energy symmetry breaking scale $\eta$, with $\eta \approx 1$ TeV taken as an example. In this type of model, intermediate scale inflation occurs for values of $\phi$ on the order of $(\eta M)^{1/2}$ (where $M = M_P/\sqrt{8\pi} = \kappa^{-1}$), corresponding to a scale of $10^{10} - 10^{11}$ GeV. The “effective” Planck scale itself is $\bar{M}_P = (\eta M_P)^{1/2}$, which leads to an intriguing possibility of strong gravitational effects showing up at this scale. Although the scale of inflation is not distanced from the effective Planck scale any more than in the usual minimally coupled case, the model demonstrates how a low energy symmetry breaking scale can give rise to an effective Planck scale $\bar{M}_P \ll M_P$. After a period of inflation in the toy model, the scalar field evolves and eventually enters its vacuum state where $F(\eta) = 1$, and the nonminimally coupled model is approximated by a minimally coupled one (with small corrections) for $\phi \approx \eta$. (The evolution of density perturbations within a scalar-tensor theory is a separate problem which has not been addressed here, and so it is not clear what constraints may be imposed by these considerations.)

The issues considered here indicate that if reality is described by some type of effective scalar-tensor theory with a Jordan frame realization, then there could be interesting modifications to some of our conventional ideas concerning relevant energy scales for high energy physics and cosmology.

APPENDIX A: SLOW ROLL EQUATIONS OF MOTION

Here we apply the slow roll conditions (12)–(14) to reduce the cosmological equations (10) and (11) to simpler forms. The equations are
\[ H^2 = \frac{\kappa^2}{3F} \left[ \frac{1}{2} k \dot{\phi}^2 + V \right] - \frac{F_{,\phi}}{F} H \dot{\phi}, \]  
(A1)

\[ \left[ \frac{3(F_{,\phi})^2}{2F} + \kappa^2 k \right] (\ddot{\phi} + 3H \dot{\phi}) + \left\{ \frac{F_{,\phi}}{2F} [3F_{,\phi\phi} + \kappa^2 k] + \frac{1}{2} \kappa^2 k_{,\phi} \right\} \dot{\phi}^2 \]  
(A2)

and we can drop the first term in brackets in (A1). The last term on the right hand side (RHS) of this equation has a magnitude proportional to \( H|\dot{F}/F| \ll H^2 \), and so it can also be dropped. Equation (A1) therefore reduces to

\[ H^2 = \frac{\kappa^2}{3F} V. \]  
(A3)

For the second equation, we can first drop the acceleration term \( \ddot{\phi} \). Now let us first focus upon the first two terms on the left hand side (LHS). Although we can not directly compare the relative magnitudes of the terms \( \frac{3(F_{,\phi})^2}{2F} \) and \( \kappa^2 k \), we can see that the first term, proportional to \( \left[ \frac{3(F_{,\phi})^2}{2F} \right] (3H \dot{\phi}) = \frac{3}{2} F_{,\phi} (3H \dot{F}/F) \ll H^2 F_{,\phi} \) is negligible in comparison to the last term on the RHS, \( \kappa^2 F^2 U_{,\phi} \). To see this, we write out this term,

\[ \kappa^2 F^2 U_{,\phi} = \kappa^2 V_{,\phi} - \kappa^2 F_{,\phi} \frac{V}{F} = \kappa^2 V_{,\phi} - F_{,\phi}(3H^2), \]  
(A4)

which contains a part on the order of \( H^2 F_{,\phi} \). Equation (A2) now has reduced to

\[ \frac{F_{,\phi}}{2F} \left[ 3F_{,\phi\phi} + \kappa^2 k \right] + \frac{1}{2} \kappa^2 k_{,\phi} \dot{\phi}^2 + \kappa^2 k(3H \dot{\phi}) + \kappa^2 F^2 U_{,\phi} = 0. \]  
(A5)

The entire first term on the LHS of this equation can now be dropped. This can be seen by looking at the magnitudes of each piece:

\[ \frac{3}{2} |F_{,\phi}(F_{,\phi}/F)| \dot{\phi}^2 = \frac{3}{2} |(F_{,\phi}/F)(\partial F_{,\phi}/\partial t)\dot{\phi}| = \frac{3}{2} \left| \frac{\dot{F}(\partial F_{,\phi}/\partial t)}{F} \right| \ll \frac{3}{2} H^2 |F_{,\phi}| \]

\[ \frac{1}{2} \kappa^2 F_{,\phi} k \dot{\phi}^2 = \frac{1}{2} \kappa^2 \left| \frac{\dot{F}}{F} k \dot{\phi} \right| \ll \frac{1}{2} \kappa^2 H \left| k \dot{\phi} \right| \]

\[ \frac{1}{2} \kappa^2 |k_{,\phi}| \dot{\phi}^2 = \frac{1}{2} \kappa^2 \left| k_{,\phi} \right| \ll \frac{1}{2} \kappa^2 H \left| k_{,\phi} \right| \]

Each of these pieces is dominated by remaining terms in (A5) and are therefore discarded. Equation (A2) therefore reduces to the slow roll EOM

\[ 3H k \dot{\phi} + F^2 U_{,\phi} = 0. \]  
(A6)
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