A NOTE ON SUBEXPONENTIAL RATE OF CONVERGENCE TO EQUILIBRIUM FOR PROCESSES ON THE HALF-LINE

ANDREY SARANTSEV

Abstract. A powerful tool for studying long-term convergence of a Markov process to its stationary distribution is a Lyapunov function. In some sense, this is a substitute for eigenfunctions. For a stochastically ordered Markov process on the half-line, Lyapunov functions can be used to easily find explicit rates of convergence. Our earlier research focused on exponential rate of convergence. This note extends these results to slower rates, including power rates, thus improving results of (Douc, Fort, Guillin, 2009).

1. Introduction

Consider a continuous-time Markov processes $X = (X(t), t \geq 0)$ with transition function $P^t(x, A) := \mathbb{P}(X(t) \in A \mid X(0) = x)$, and transition semigroup $P^t f(x) = \mathbb{E} [f(X(t)) \mid X(0) = x]$. Suppose $X$ has a unique stationary distribution $\pi$, a probability measure on its state space $S$ such that if the process starts from $X(0) \sim \pi$, then for every $t \geq 0$ we have: $X(t) \sim \pi$. Does the measure $P^t(x, \cdot)$ converge to $\pi$ for all $x$ as $t \to \infty$, in which distance, and how fast?

For continuous-time Markov chains on a finite state space, the answer depends on the generating matrix $L$ of transition intensities: Loosely speaking, the eigenvalue closest to 0 gives us the exponential convergence rate. For general state spaces $S$, such as domains in $\mathbb{R}^d$, this description of eigenvalues is often unavailable or hard to obtain, since $L$ is no longer a matrix: It is an operator defined by

$$
L f(x) = \lim_{t \downarrow 0} \frac{1}{t} [P^t f(x) - f(x)]
$$
on a certain space of functions $f : S \to \mathbb{R}$. It is often hard to find eigenvalues and eigenfunctions of $L$, but there is a useful substitute: Lyapunov functions. There are various versions of this concept, but the basic property is (loosely speaking) that this is a function $V : S \to [1, \infty)$ such that for some compact subset $K \subseteq S$,

$$
(1) \quad L V(x) \leq -c < 0, \ x \in S \setminus K.
$$

Then under some additional technical conditions the process $X$ is ergodic: That is, $P^t(x, \cdot) \to \pi$ as $t \to \infty$ for every $x \in S$ in the total variation norm, defined later in this article. Let us explain the intuition. The function $V$ measures the “altitude” reaches by the process $X$. When the process is outside of the “valley” which is the compact set $K$, it is compelled by the condition (1) to “decrease altitude”. This random process has negative “drift” until it gets back to the “valley”. This “gravity force” implies (after additional technical work) its ergodicity. See articles [23, 24] for rigorous exposition, and the classic book [26] for similar concepts on discrete-time Markov chains. Under a stronger than (1) condition: For a constant $k > 0$ and a compact $K \subseteq S$,

$$
(2) \quad L V(x) \leq -k V(x), \ x \in S \setminus K,
$$
we get exponential ergodicity: $P^t(x, \cdot) \to \pi$ as $t \to \infty$ for all $x \in S$ as fast as $C e^{-\kappa t}$ for some $C, \kappa > 0$. See rigorous statements in [11]. However, finding or estimating $\kappa$ proved to be very hard (compared to finding or estimating $k$ from (2), which is often easy in practice); see for example [25, 27]. Among other articles on this topic, let us mention [5, 7]; and applications to Markov Chain Monte Carlo techniques, [28].

There is one special case when we simply conclude that $\kappa = k$ from (2). The state space $S = \mathbb{R}_+ := [0, \infty)$, the exceptional compact set is $K = \{0\}$, and the process $X$ is stochastically ordered. The latter means that for any $x_2 \geq x_1 \geq 0$, we can couple two copies $X_1$ and $X_2$ of this process starting from $X_i(0) = x_i$, on the same probability space, so that $X_1(t) \leq X_2(t)$ for all $t \geq 0$. This statement was shown in [19] and improved in [29] using the coupling method. We used this method in risk theory [15] and for more general processes, so-called Walsh diffusions, which include as a particular case diffusions on the real line, [17].

One can also have a condition stronger than (1), but weaker than (2): For some increasing concave function $\varphi : [1, \infty) \to [0, \infty)$ and some compact set $K \subseteq S$,

\begin{equation}
\mathcal{L}V(x) \leq -\varphi(V(x)), \quad x \in S \setminus K.
\end{equation}

It is shown in [9] that this implies (under some additional technical conditions) a subexponential, or subgeometric, convergence rate $P^t(x, \cdot) \to \pi$ to $t \to \infty$. That is, the total variation distance converges to zero, but at a rate slower than exponential, given by $C e^{-\alpha t}$. In that article and earlier ones [14, 31], they were able to find explicit rates, similar to $ct^{-\alpha}$ for some $C, \alpha > 0$, or $C \exp \left[-\lambda t^\beta\right]$ for $C, \lambda > 0$ and $\beta \in (0, 1)$.

In this note, as opposed to [9, 14], we assume that the compact set $K = \{0\}$ on the state space $S = [0, \infty)$, and that the process $X$ is stochastically ordered. We improve upon the results of these articles cited above. In other words, we combine ideas from [9, 19, 29] in the context of subgeometric ergodicity. This leads to simple and elegant proofs. We explore how our new results fit into exponential ergodicity framework from [29].

We give examples of reflected diffusions, jump-diffusions, and Lévy processes on $\mathbb{R}_+$. These processes behave as, correspondingly, diffusions, jump-diffusions, and Lévy processes, as long as they do not hit zero. When this happens, they are reflected back inside the positive half-line. In our examples, they have power rates of convergence: $h(t) := t^{-\beta}$ for $\beta > 0$. These processes have been extensively studied, from the original article [30] on stochastic differential equations with reflection on the half-line, to recent articles [2] on multidimensional reflected jump-diffusions and [3, 4] on reflected Lévy processes. These processes have applications in queueing theory (heavy traffic approximation), see [6, 18] and references therein; and, more recently, in financial mathematics, see [12, 16] and references therein.

We prove convergence not only in total variation norm, but in stronger norms. Such convergence implies the convergence of moments up to a certain order. As in [9], there is a tradeoff between the norm and the rate of convergence.

1.1. Organization of the article. In Section 2, we state all notation and definitions, present main results, and discuss their relationship with existing research. In Section 3, we present examples, and Section 4 is devoted to proofs of main results.

1.2. Acknowledgements. We thank Professor Mark M. Meerschaert for invitation to give a talk in November 2016 at the Colloquium at the Department of Statistics & Probability, Michigan State University in East Lansing, and for useful discussion there. We thank the Department of Mathematics & Statistics, University of Nevada in Reno, for a supportive atmosphere for research and professional development.
2. Notation, Definitions, and Main Results

2.1. Notation and definitions. Let \( \mathbb{R}_+ := [0, \infty) \). For two probability measures \( \rho_1 \) and \( \rho_2 \) on \( \mathbb{R}_+ \), we can define their stochastic maximum \( \rho_0 := \rho_1 \lor \rho_2 \) is defined by \( \rho_0((x, \infty)) := \rho_1((x, \infty)) \lor \rho_2((x, \infty)) \), \( x \geq 0 \). For a signed measure \( \nu \) on \( \mathbb{R}_+ \) and a function \( f : \mathbb{R}_+ \to \mathbb{R} \), we define \( \langle \nu, f \rangle := \int_{\mathbb{R}_+} f(x) \nu(dx) \). For a function \( V : \mathbb{R}_+ \to [1, \infty) \), the \( V \)-norm of a signed measure \( \nu \) on \( \mathbb{R}_+ \) is

\[
\| \nu \|_V := \sup_{|f| \leq V} |\langle \nu, f \rangle|,
\]

where the sup is taken over all functions \( f : \mathbb{R}_+ \to \mathbb{R} \) such that \( |f(x)| \leq V(x) \). If \( V \equiv 1 \), then this norm is denoted by \( \|\cdot\|_{TV} \) and is called the total variation norm. A family of Borel measures \( \{Q_x\}_{x \geq 0} \) on \( \mathbb{R}_+ \) is called stochastically ordered if \( Q_x([0, \infty)) \leq Q_y([0, \infty)) \) for \( 0 \leq x \leq y \) and \( z \geq 0 \).

We operate on a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \). Consider a Markov process \( X = (X(t), t \geq 0) \) on \( \mathbb{R}_+ \) with transition kernel \( P^t(x, \cdot) \). Formally, \( P^t(x, A) = \mathbb{P}(X(t) \in A \mid X(0) = x) \). This process \( X \) has positivity property if \( P^t(x, B) > 0 \) for any \( t > 0, x \geq 0, \) and \( B \subseteq \mathbb{R}_+ \) of positive Lebesgue measure. For any initial value \( X(0) = x \), we can construct a copy of this Markov process. If the initial distribution is \( X(0) \sim \rho \), then the distribution of \( X(t) \) is written as

\[
\rho P^t, \quad \text{where} \quad \rho P^t(B) := \int_0^\infty P^t(x, B) \rho(dx) \quad \text{for} \quad B \subseteq \mathbb{R}_+.
\]

The transition semigroup \( (P^t)_{t \geq 0} \) is defined as

\[
P^tf(x) = \mathbb{E}[f(X(t)) \mid X(0) = x] = \int_0^\infty P^t(x, dy)f(y).
\]

This process has generator \( \mathcal{L} \) with domain \( \mathcal{D}(\mathcal{L}) \):

\[
\mathcal{L}f(x) = \lim_{t \downarrow 0} \frac{1}{t} (P^t f(x) - f(x)), \quad f \in \mathcal{D}(\mathcal{L}).
\]

This Markov process is called stochastically ordered if for every \( t \geq 0 \), the family of measures \( (P^t(x, \cdot))_{x \geq 0} \) is stochastically ordered. A probability measure \( \pi \) on \( \mathbb{R}_+ \) is called a stationary distribution for this Markov process if \( X(0) \sim \pi \) implies \( X(t) \sim \pi \) for all \( t > 0 \):

\[
\int_0^\infty P^t(x, A) \pi(dx) = \pi(A) \quad \forall A \subseteq \mathbb{R}_+.
\]

Take a strictly increasing concave \( \varphi : [1, \infty) \to \mathbb{R}_+ \). Define

\[
\Phi(s) := \int_1^s \frac{du}{\varphi(u)}.
\]

Assuming this function is finite for all \( s > 1 \), it is strictly increasing \( \Phi : [1, \infty) \to \mathbb{R}_+ \), thus it has a well-defined inverse \( \Psi := \Phi^{-1} \) on \( [0, \Phi(\infty)) \), with \( \Phi(\Psi(v)) \equiv v \) for \( v \geq 0 \). Note that \( \Psi(\infty) = \infty \): If \( \Psi(\infty) = a < \infty \), then \( \Phi(a) = \infty \), which contradicts our assumptions.

A \( \varphi \)-Lyapunov function for this process \( X \) is a function \( V : \mathbb{R}_+ \to (0, \infty) \) such that the following process is a local \( (\mathcal{F}_t)_{t \geq 0} \)-supermartingale:

\[
\varphi(V(X(t \wedge \tau_0))) + \int_0^{t \wedge \tau_0} \mathcal{L}V(X(s)) \, ds, \quad t \geq 0,
\]

where \( \tau_0 := \inf\{t \geq 0 \mid X(t) = 0\} \) is the hitting time of 0.
Remark 1. If $V \in \mathcal{D}(\mathcal{L})$, this condition \([5]\) is equivalent to $\mathcal{L}V(x) \leq -\varphi(V(x))$ for $x > 0$. But it is more convenient for us to write this condition in the form \([4]\). We consider functions with $V'(0) \neq 0$, but reflected processes have generator domain $\mathcal{D}(\mathcal{L})$ restricted by $V'(0) = 0$.

Example 1. For $\varphi(s) = ks$, $k > 0$, this becomes modified Lyapunov function from \([29]\).

2.2. Main results. Take a function $G(t, u) := \Psi(\Phi(u) + t)$, $u \geq 1$, $t \geq 0$.

Theorem 1. Take a concave increasing function $\varphi : [1, \infty) \to [0, \infty)$ with finite $\Phi$ from \([4]\). Assume $X$ is a semimartingale and a stochastically ordered Markov process on $\mathbb{R}_+$. Take a $\varphi$-Lyapunov function $V$ such that for some nondecreasing functions $U, r : \mathbb{R}_+ \to \mathbb{R}_+$,

$$h(t)U(x) \leq G(t, V(x)), \quad t, x \geq 0.$$

Then we have the following results:

(a) Take two copies $X_1$ and $X_2$ of $X$ starting from $x_1$ and $x_2$, with $0 \leq x_1 \leq x_2$. Then

$$\|P^t(x_1, \cdot) - P^t(x_2, \cdot)\|_U \leq 2h^{-1}(t)V(x_2), \quad t \geq 0.$$

(b) Take two copies $X_1$ and $X_2$ of $X$ starting from distributions $\rho_1$ and $\rho_2$ on $\mathbb{R}_+$. Then we can write $X_i(t) \sim \rho_i P^t$, $i = 1, 2$. Then

$$\|\rho_1 P^t - \rho_2 P^t\|_U \leq 2h^{-1}(t)(\rho_1 \lor \rho_2, V), \quad t \geq 0.$$

(c) Take a copy $X$ starting from $X(0) \sim \rho$ with $(\rho, V) < \infty$. Assume $X$ has a unique stationary distribution $\pi$ with $(\pi, V) < \infty$, then

$$\|\rho P^t - \pi\|_U \leq 2h^{-1}(t)(\pi \lor \rho, V).$$

Remark 2. In Theorem\([1]\) we can let $U := 1$ and $h := \Psi$. Then we get convergence in the total variation norm. The rate of this convergence is given by $1/\psi(t)$. This is stronger than in \([9]\), where the rate of convergence for the total variation norm is $1/\varphi(\Psi(t))$. Indeed, in the above example $\varphi(u) := \sqrt{u}$, and so $\varphi(\Psi(t)) = o(\Psi(t))$ as $t \to \infty$.

Remark 3. Let us discuss these results in exponential ergodicity case, when $\varphi(s) = ks$, $k > 0$. Then $\Phi(s) = k \ln(s)$, and $\Psi(v) := e^{kv}$. Thus $G(t, u) = we^{kt}$, and we can take $U := V$ and $r(t) := e^{kt}$. Here, we have perfect decomposition of $G$ into a product form.

Example 2. Even if $\varphi$ is bounded, we can get nontrivial results. For example, $\varphi(x) = k > 0$ implies $\Phi(x) = k^{-1}(x - 1)$ and $\Psi(x) = kx + 1$. Thus $G(t, x) = \Psi(\Phi(x) + t) = x + kt$, and we can let $h(t) := 2kt^{1/2}$, $U(x) := V^{1/2}(x)$, or use more general techniques from the Appendix.

Remark 4. The existence and uniqueness of the stationary distribution $\pi$ and the property $(\pi, \nu) < \infty$ can be obtained from the Lyapunov conditions in their classic form, as in \([23, 24]\). Often we can find (another version of) a Lyapunov function $\hat{V} : \mathbb{R}_+ \to [1, \infty)$ in $\mathcal{D}(\mathcal{L})$ such that for constants $x_0, b > 0$:

$$\mathcal{L}\hat{V}(x) \leq -\varphi(\hat{V}(x)) + b1_{[0,x_0]}(x).$$

We also need the following positivity property: $P^t(x, B) > 0$ for every $t > 0$, $x \geq 0$, and every Borel set $B \subseteq \mathbb{R}_+$ of positive Lebesgue measure. Then the process has a unique stationary distribution $\pi$, and $(\pi, \hat{V}) < \infty$. This follows from \([23, 24]\). In practice, for processes on $\mathbb{R}_+$, this function $\hat{V}$ can sometimes be constructed as $\hat{V}(x) := V(\psi(x))$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function with $\psi(x) = 0$ for $x \leq x_1$ and $\psi(x) = x$ for $x \geq x_2$, where $0 < x_1 < x_2$; and, finally, $\psi(x) \leq x$ for all $x \geq 0$. Then $\hat{V}(x) \leq V(x) \leq C\hat{V}(x)$ for some constant $C > 1$, and $(\pi, \hat{V}) < \infty$ implies $(\pi, V) < \infty$. 

ANDREY SARANTSEV
3. Examples

3.1. Reflected diffusions. Take functions $g, \sigma : \mathbb{R}_+ \to \mathbb{R}$. For a one-dimensional $(\mathcal{F}_t)_{t \geq 0}$-Brownian motion $W = (W(t), t \geq 0)$, consider a stochastic differential equation (SDE):

$$dZ(t) = g(Z(t)) dt + \sigma(Z(t)) dW(t). \tag{8}$$

**Definition 1.** A solution to the SDE (8) with reflection on $\mathbb{R}_+$ starting from $x \geq 0$ is defined as an adapted process $Z = (Z(t), t \geq 0)$ with a.s. continuous trajectories, such that

$$Z(t) = x + \int_0^t g(Z(s)) \, ds + \int_0^t \sigma(Z(s)) \, dW(s) + \ell(t), \quad t \geq 0,$$

where $W$ is an $(\mathcal{F}_t)_{t \geq 0}$-Brownian motion, and $\ell = (\ell(t), t \geq 0)$ is a continuous nondecreasing process with $\ell(0) = 0$, increasing only when $Z(t) = 0$. It is also called a reflected diffusion on $\mathbb{R}_+$ with drift $g$ and diffusion $\sigma^2$.

This process is well-defined for continuous $g, \sigma$, stochastically ordered and has generator

$$\mathcal{L}f(x) = g(x)f'(x) + \frac{1}{2} \sigma^2(x)f''(x), \quad f \in C^2(\mathbb{R}_+), \quad f'(0) = 0.$$ 

Thus for all functions $f \in C^2(\mathbb{R}_+)$, even when $f'(0) \neq 0$ and $f$ is not in the domain of the generator, the following process is an $(\mathcal{F}_t)_{t \geq 0}$-supermartingale:

$$f(Z(t \wedge \tau_0)) - \int_0^{t\wedge \tau_0} \mathcal{L}f(Z(s)) \, ds, \quad t \geq 0.$$ 

We can prove that $\mathcal{L}V(x) \leq -\varphi(V(x))$ for $x > 0$, even when $V'(0) \neq 0$, and this would prove that $V$ is a $\varphi$-Lyapunov function. Assume for some constants $a, c > 0$ and $\alpha \in (0, 1)$,

$$g(x) \leq -a(1 + cx)^{\alpha - 1}, \quad x > 0. \tag{9}$$

**Example 3.** Take a function $V(x) = 1 + cx$. We get: $V'(x) = c$, $V''(x) = 0$, and thus

$$\mathcal{L}V(x) \leq -ac(1 + cx)^{\alpha - 1} = -\varphi(V(x)), \quad \varphi(s) := acs^{1-\alpha}.$$ 

The function $\varphi$ is increasing and concave. We can compute

$$\Phi(s) := \int_1^s \frac{dy}{\varphi(y)} = \frac{1}{a\alpha} [s^{\alpha} - 1], \quad \Psi(v) := [a\alpha v + 1]^{1/\alpha} ; \tag{10}$$

$$G(t, x) := \Psi(\Phi(x) + t) = [a\alpha t + x^{\alpha}]^{1/\alpha}.$$ 

Thus we can deduce (10) from (33) from the Appendix:

$$\Psi(t, V(x)) = [a\alpha t + a^\alpha(1 + cx)^{\alpha}]^{1/\alpha} \geq h(t)U(x),$$

$$h(t) := \left[ H^{-1}(a\alpha \cdot t)^{1/\alpha} \right], \quad U(x) := \left[ K^{-1}(a^\alpha(1 + cx)^{\alpha}) \right]^{1/\alpha}.$$ 

For the example in (32), we get: $H^{-1}(x) = p^{1/p}x^{1/p}$ and $K^{-1}(y) := q^{1/q}y^{1/q}$, thus

$$h(t) := (a\alpha p^{1/p} t^{1/\alpha})^{1/\alpha}, \quad U(x) := q^{1/q}x^{1/q}a^{1/\alpha}(1 + cx)^{1/\alpha}.$$ 

If $p$ is larger, then $q$ is smaller. This illustrates the general principle: Taking a stronger $\|\cdot\|_U$, for a larger function $U$, leads to slower convergence rate $h$, and vice versa.
Example 4. Now assume the condition \([11]\) holds, and moreover \(\sigma(x) \equiv \sigma = \text{const.}\) Take a function \(V(x) := (1 + \lambda x)^\beta\) for \(\beta > 1\) and \(\lambda \geq c\) to be determined later. Then we have:

\[
V'(x) = \beta \lambda \cdot (1 + \lambda x)^{\beta - 1}, \quad V''(x) = \lambda^2 \beta (\beta - 1) (1 + \lambda x)^{\beta - 2},
\]

\[
\mathcal{L} V(x) \leq -a \beta (1 + cx)^{\alpha - 1} (1 + \lambda x)^{\beta - 1} + \frac{1}{2} \lambda^2 \sigma^2 \beta (\beta - 1) (1 + \lambda x)^{\beta - 2} \leq -A(\lambda) (1 + \lambda x)^{\beta - \alpha},
\]

\[
A(\lambda) := a \beta \lambda - \frac{1}{2} \sigma^2 \beta (\beta - 1) \lambda^2, \quad x > 0.
\]

Thus we can take \(\varphi(s) := A(\lambda) s^{1 - \alpha/\beta}\). Since \(\alpha < 1 < \beta\), then \(0 < 1 - \alpha/\beta < 1\) and this is a concave increasing function. This is only true if we can make sure that \(A(\lambda) > 0\). We need

\[
(11) \quad \lambda < \frac{2a}{\sigma^2 (\beta - 1)}.
\]

But on the other hand, we need \(\lambda \geq c\). We got a prerequisite inequality:

\[
(12) \quad \frac{2a}{\sigma^2 (\beta - 1)} > c \iff \beta < 1 + \frac{\sigma^2}{2ac}.
\]

If we satisfy conditions \((11)\) and \((12)\), then we can construct this Lyapunov function. It gives us better rates of convergence and stronger \(U\)-norm \(\| \cdot \|_U\). But we did this under \((11)\) combined with constant diffusion coefficient. If we have similar estimates for a non-constant diffusion coefficient, we need to modify our Lyapunov function.

Remark 5. Let us follow Remark 4 to show for both examples that the stationary distribution \(\pi\) exists and is unique, and \((\pi, V) < \infty\). For each case, taking \(\hat{V}(x) := V(\psi(x))\) as at the end of Section 2, we get a function \(\hat{V}\) such that \(\hat{V}'(0) = 0\) and the condition \((7)\) holds. Since the process \(X\) has the positivity property \(P^t(x, B) > 0\), it has a unique stationary distribution \(\pi\) which satisfies \((\pi, \hat{V}) < \infty\). Since \(\hat{V}(x) = V(x)\) for large enough \(x\), we can automatically conclude that \((\pi, V) < \infty\).

3.2. Reflected jump-diffusions. In addition to the above notation, take a family \((\nu_x)_{x \geq 0}\) of finite Borel measures on \(\mathbb{R}_+\) with \(\nu_x(\mathbb{R}_+) = M < \infty\) for all \(x \geq 0\). Then we can augment the above reflected diffusion with these jumps: They occur with intensity \(M\), and the destination of a jump from \(x \geq 0\) is distributed as \(M^{-1} \nu_x(\cdot)\). As long as the family \((\nu_x)_{x \geq 0}\) is weakly continuous: \(\nu_y \to \nu_x\) weakly as \(y \to x\), and \(g, \sigma\) satisfy the same assumptions as in the previous subsection, this process can be constructed by piecing out, see \([29]\), and it exists in the weak sense and is unique in law. This is a Markov process with the following generator, for \(f \in C^2(\mathbb{R}_+)\) with \(f'(0) = 0\):

\[
(13) \quad \mathcal{L} f(x) = g(x) f'(x) + \frac{1}{2} \sigma^2(x) f''(x) + \int_0^\infty [f(y) - f(x)] \nu_x(dy).
\]

It is shown in \([29]\) that if this family of measures is stochastically ordered, then the process \(Z\) is also stochastically ordered. Take a Lyapunov function \(V(x) = 1 + cx\) for \(c > 0\). Applying the generator from \((13)\) even though \(V'(0) \neq 0\), we get:

\[
\mathcal{V}(x) = cm(x), \quad m(x) := g(x) + \int_0^\infty (y - x) \nu_x(dy).
\]

One can think of \(m(x)\) as “average drift” at the point \(x \geq 0\) which is the usual drift \(g(x)\) combined with the “implied drift” created by jumps. This is the “velocity” with which the process “wants to move” to the right while at location \(x\). If \(m(x)\) instead of \(g(x)\) satisfies \((9)\),
then we can make the same conclusions as for the reflected diffusion above. We need to show that the stationary distribution \( \pi \) exists, is unique, and satisfies \((\pi, V) < \infty\), as in Remark 4.

Assume that there exist \( 0 < c_1 < c_2 \) such that for \( x \geq c_2 \), \( \nu_x([0, c_1]) = 0 \). Take \( \psi(x) = 0 \) for \( x \leq c_1/2 \) and \( \psi(x) = x \) for \( x \geq c_1 \). Then for \( x \geq c_2 \) we have:

\[
\int_0^\infty (\dot{V}(y) - \dot{V}(x)) \nu_x(dy) = \int_0^\infty (V(\psi(y)) - V(\psi(x))) \nu_x(dy) = \int_{c_1}^\infty (V(y) - V(x)) \nu_x(dy)
\]

\[
= \int_{c_1}^\infty (V(y) - V(x)) \nu_x(dy) = \int_{c_1}^\infty V(y)(d\nu_x) - V(x)(d\nu_x)
\]

\[
g(x)\dot{V}'(x) + \frac{1}{2}\sigma^2(x)\dot{V}''(x) = g(x)V'(x) + \frac{1}{2}\sigma^2(x)V''(x);
\]

and thus the generator formally applied to \( V \) will be the same as if applied to \( \hat{V} \in D(\mathcal{L}) \):

\[
\mathcal{L}\hat{V}(x) = \mathcal{L}V(x), \quad x \geq c_2.
\]

If \( \mathcal{L}\hat{V}(x) \) is bounded on \([0, c_2]\) (it is easy to check in practice) then we have the condition \( (7) \). Together with positivity property, proved in [29], this implies existence and uniqueness of the stationary distribution \( \pi \) and \((\pi, V) < \infty\).

**Example 5.** Try the following process: \( g(x) = -3(x + 1)^{-0.5} \), \( M = 2 \) and \( \nu_x \) is defined as

\[
\nu_x(dy) = \lambda(x) \exp[\lambda(x)(x - y)] 1_{\{y > x\}} dy, \quad \lambda(x) := (x + 1)^{0.5}.
\]

In other words, the process jumps with constant intensity 2 to the right, and the displacement is distributed as an exponential random variable with rate \( \lambda(x) \) (and therefore mean \( \lambda^{-1}(x) \)). Then the average rate of displacement from jumps is \( 2\lambda^{-1}(x) \). Thus

\[
m(x) = -3(x + 1)^{-0.5} + 2(x + 1)^{-0.5} = -(x + 1)^{-0.5}, \quad x \geq 0,
\]

This gives us subexponential convergence, as in Example 3.

### 3.3. Reflected Lévy processes.

Any Lévy process on the real line can be decomposed into the sum of two components: a Brownian motion with constant drift and diffusion (the continuous component), and a pure jump Lévy process \( \mathcal{J} \). In this article, we assume this pure jump process is nondecreasing. Its jumps are governed by a spectral measure \( \nu \). If the spectral measure is finite, then there are a.s. finitely many jumps (a Poisson number) during any time interval \([0, t]\), and \( \mathcal{J} \) is, in fact, a compound Poisson process. However, if \( \nu(\mathbb{R}_+) = \infty \), then this process \( \mathcal{J} \) makes infinitely many jumps during any time interval \([0, t]\). This \( \nu \) is a \( \sigma \)-finite Borel measure on \( \mathbb{R}_+ \). This measure must satisfy

\[
(14) \quad \int_0^\infty (1 \wedge x) \nu(dx) < \infty.
\]

We take a reflected version \( X \) of \( L \) on the half-line using the Skorohod reflection mapping

\[
X(t) := L(t) + \sup_{0 \leq s \leq t} [\max(-L(s), 0)].
\]

The resulting process will have values in \( \mathbb{R}_+ \) and behave like a Lévy process \( L \) as long as it is strictly inside \( \mathbb{R}_+ \). When it hits 0, it is reflected back. If \( \nu \) is finite, we are back in the case of reflected jump-diffusions, discussed in the previous subsection. In this subsection, we are
mostly interested in the case when \( \nu(\mathbb{R}_+) = \infty \). This process is defined by a triple \((g, \sigma, \nu)\) where \( g \in \mathbb{R}_+ \) is a drift, \( \sigma > 0 \) is a diffusion coefficient, and it can be decomposed as
\[
L(t) = L(0) + gt + \sigma W(t) + \mathcal{J}(t), \quad t \geq 0,
\]
where \( W \) is a Brownian motion. The generator of \( L \) is given by
\[
\mathcal{L} f(x) = g f'(x) + \frac{1}{2} \sigma^2 f''(x) + \int_{0}^{\infty} [f(x + z) - f(x)] \nu(dz).
\]
Assume that \( \nu \) is supported on \([0,1]\); from (14), we get:
\[
(17) \quad \mathcal{P} := \int_{0}^{1} z \nu(dz) < \infty.
\]

The reflected version has the same generator (16), but for functions \( f \in C^2(\mathbb{R}_+) \) with \( f'(0) = 0 \). Formally apply the generator to the function \( V(x) := 1 + x \), although \( V'(0) \neq 0 \). Then \( \mathcal{L} V(x) = g + \mathcal{P} \). If \( g < -\mathcal{P} \), then this process is ergodic. We prove this by taking \( \hat{V}(x) := V(\varphi(x)) \) as in Remark 1. Moreover, take \( V(x) := e^{\lambda x} \). Then
\[
\mathcal{L} V(x) = V(x)k(\lambda), \quad k(\lambda) := \lambda g + \frac{\sigma^2 \lambda^2}{2} + \int_{0}^{1} [e^{\lambda z} - 1] \nu(dz).
\]

This follows from differentiability under the integral with respect to \( \lambda \) at \( \lambda = 0 \), which in turn follows from (17). From (17) and bounded support of \( \nu \), we get: for all \( \lambda > 0 \),
\[
\int_{0}^{1} [ze^{\lambda z}] \nu(dz) < \infty.
\]

From here, we can deduce by Lebesgue dominated convergence theorem that
\[
\frac{d}{d\lambda} \int_{0}^{1} [e^{\lambda z} - 1] \nu(dz) = \int_{0}^{1} \frac{\partial}{\partial \lambda} [e^{\lambda z} - 1] \nu(dz) = \int_{0}^{1} [ze^{\lambda z}] \nu(dz).
\]

Thus, taking derivative of \( k(\lambda) \) and letting \( \lambda = 0 \), we get: \( k'(0) = g + \int_{0}^{1} z \nu(dz) = g + \mathcal{P} < 0 \).

This implies the conclusion of the proof. \( \square \)

**Lemma 1.** If \( g < -\mathcal{P} \), then there exists a \( \lambda > 0 \) such that \( k(\lambda) < 0 \).

**Proof.** This follows from differentiability under the integral with respect to \( \lambda \) at \( \lambda = 0 \), which in turn follows from (17). From (17) and bounded support of \( \nu \), we get: for all \( \lambda > 0 \),
\[
\int_{0}^{1} [ze^{\lambda z}] \nu(dz) < \infty.
\]

From here, we can deduce by Lebesgue dominated convergence theorem that
\[
\frac{d}{d\lambda} \int_{0}^{1} [e^{\lambda z} - 1] \nu(dz) = \int_{0}^{1} \frac{\partial}{\partial \lambda} [e^{\lambda z} - 1] \nu(dz) = \int_{0}^{1} [ze^{\lambda z}] \nu(dz).
\]

Thus, taking derivative of \( k(\lambda) \) and letting \( \lambda = 0 \), we get: \( k'(0) = g + \int_{0}^{1} z \nu(dz) = g + \mathcal{P} < 0 \).

This implies the conclusion of the proof. \( \square \)

**Lemma 2.** For any \( t > 0 \), \( x \geq 0 \), and a set \( B \subseteq \mathbb{R}_+ \) of positive Lebesgue measure, the transition kernel of the reflected process satisfies \( P^t(x, B) > 0 \).

**Proof.** Assume \( x > 0 \). Without loss of generality, we can assume \( B \subseteq [\delta, \infty) \) for some \( \delta > 0 \). Then with probability \( p > 0 \), this process never hits 0 until time \( t \). In this case, it behaves as a non-reflected Lévy process, which has positivity property \( Q^t(x, B) > 0 \). Thus \( P^t(x, B) \geq pQ^t(x, B) > 0 \). Next, if \( x = 0 \), then \( P^{t/2}(0, (0, \infty)) > 0 \), and thus \( P^t(x, B) \geq \int_{0}^{\infty} P^{t/2}(y, B)P^{t/2}(0, dy) > 0 \). Thus we reduced the case \( x = 0 \) to the case \( x > 0 \). \( \square \)
4. Proof of Theorem 1

4.1. Overview of the proof. The main idea is similar to [29, Theorem 4.1, Theorem 5.2]. It is known from [21, Theorem 5] that if \( X \) has trajectories which are a.s. right-continuous with left limits, then being stochastically ordered is equivalent to the following statement: For every \( 0 \leq x_1 \leq x_2 \), there exists a coupling of two versions \( X_1 \) and \( X_2 \) of \( X \), starting from \( X_1(0) = x_1 \) and \( X_2(0) = x_2 \). A probability space and copies \( X_1 \) and \( X_2 \) defined on this space such that \( X_1(t) \leq X_2(t) \) for all \( t \geq 0 \) a.s. Let \( \tau := \inf \{ t \geq 0 \mid X_2(t) = 0 \} \). Then \( X_1(\tau) \leq X_2(\tau) = 0 \) and thus \( X_1(\tau) = 0 \). Therefore, \( \tau \) is a coupling time for \( X_1 \) and \( X_2 \): We can assume \( X_1(t) = X_2(t) \) for \( t > \tau \), so after this coupling time, the processes coincide. For any function \( g : \mathbb{R}_+ \to \mathbb{R} \) with \( |g| \leq U \), we get (the last inequality from nondecreasing \( U \)):

\[
\begin{align*}
(18) & \quad h(t) \left| \mathbb{E}[g(X_1(t))] - \mathbb{E}[g(X_2(t))] \right| = h(t) \left| \mathbb{E}[g(X_1(t))]1_{\{t > \tau\}} - \mathbb{E}[g(X_2(t))]1_{\{t > \tau\}} \right| \\
& \quad \leq h(t) \left( \mathbb{E} \left[ U(X_1(t))1_{\{t > \tau\}} \right] + \mathbb{E} \left[ U(X_2(t))1_{\{t > \tau\}} \right] \right) \leq 2\mathbb{E} \left[ h(t)U(X_2(t))1_{\{t > \tau\}} \right].
\end{align*}
\]

**Lemma 3.** The function \( G \) satisfies the boundary conditions

\[
G(t, 1) = \Psi(t), \quad t \geq 0; \quad G(0, u) = u, \quad u \geq 1,
\]

and for \( t \geq 0, u \geq 1 \) the following equation and inequalities hold:

\[
(19) \quad \frac{\partial G}{\partial t}(t, u) = \varphi(u) \frac{\partial G}{\partial u}(t, u), \quad \frac{\partial G}{\partial u}(t, u) \geq 0, \quad \frac{\partial^2 G}{\partial u^2}(t, u) \leq 0.
\]

This technical lemma was partially proved in [9], but we want to collect these results and give (a straightforward) proof for the sake of completeness. The key part of the proof of Theorem 1 is the following lemma. Take a copy of \( X \) starting from \( X(0) = x_0 \geq 0 \).

**Lemma 4.** The following process \( K = (K(t), t \geq 0) \) is a local \((\mathcal{F}_t)_{t \geq 0}\)-supermartingale:

\[
K(t) := G(t \land \tau, V(X(t \land \tau))), \quad t \geq 0.
\]

Assume Lemma 4 is already proved. Let us show that \( \tau < \infty \) a.s. Since \( G \) is nondecreasing with respect to \( u \), we get: \( G(t, u) \geq G(t, 1) = \Psi(t) \). Letting \( n = 1, 2, \ldots \), we get:

\[
(20) \quad G(0, V(X(0))) = \mathbb{E}K(0) \geq \mathbb{E}K(n) \geq \mathbb{E} \left[ G(n \land \tau, V(X(n \land \tau))) \right] \geq \mathbb{E}\Psi(n \land \tau).
\]

Assume the converse: \( \tau = \infty \) with positive probability. Then

\[
(21) \quad \mathbb{E}\Psi(n \land \tau) \geq \mathbb{E}\Psi(n)\mathbb{P}(\tau = \infty).
\]

But \( \mathbb{P}(\infty) = \infty \), as stated in subsection 2.1. Letting \( n \to \infty \), we compare (20) and (21) and arrive at a contradiction. This proves that \( \tau < \infty \) a.s. Since \( K(t) \geq 0 \) for all \( t \geq 0 \), then by Fatou’s lemma we can remove the word “local” from Lemma 4. From (6) we get:

\[
(22) \quad h(t)\mathbb{E} \left[ U(X_2(t))1_{\{t > \tau\}} \right] = \mathbb{E} \left[ h(t \land \tau)U(X_2(t \land \tau))1_{\{t > \tau\}} \right] \\
\leq \mathbb{E} \left[ G(t \land \tau, V(X_2(t \land \tau)))1_{\{t > \tau\}} \right] \leq \mathbb{E} \left[ G(t \land \tau, V(X_2(t \land \tau))) \right] \\
= \mathbb{E}K(t) \leq \mathbb{E}K(0) = G(0, V(x_2)) = V(x_2).
\]

Combining (18) with (22), we get:

\[
(23) \quad h(t) \left| \mathbb{E}[g(X_1(t))] - \mathbb{E}[g(X_2(t))] \right| \leq 2V(x_2).
\]
Dividing by \( h(t) \) and taking the supremum over all \( g : \mathbb{R}_+ \rightarrow \mathbb{R} \) with \( |g| \leq U \), we complete the proof of (a). The proof of (b) is done similarly, and (c) follows from (b) and the observation that \( \pi P^t = \pi \) for a stationary distribution \( \pi \) and any \( t \geq 0 \).

4.2. **Proof of Lemma** \([3]\). The boundary conditions are easy to show: \( G(0, u) = \Psi(\Phi(u)) = u \) for \( u \geq 1 \), and \( G(t, 1) = \Psi(\Phi(1) + t) = \Psi(t) \) for \( t \geq 0 \). Next, the function \( \varphi : [1, \infty) \rightarrow \mathbb{R}_+ \) is increasing. Thus the function \( \Phi : [1, \infty) \rightarrow \mathbb{R}_+ \) satisfies \( \Phi(1) = 0 \), increasing, and concave. Next, \( \Psi : \mathbb{R}_+ \rightarrow [1, \infty) \) satisfies \( \Psi(0) = 1 \), increasing, and convex. By the chain rule, \( \Psi'(v) = [\Psi'(\Psi(v))]^{-1} = \varphi(\Psi(v)) \). Let us do computation of derivatives of \( G \):

\[
\frac{\partial G}{\partial t}(t, u) = \Psi'(\Phi(u) + t) = \varphi(\Psi(\Phi(u) + t)) = \varphi(G(t, u));
\]

\[
\frac{\partial G}{\partial u}(t, u) = \Psi'(\Phi(u) + t) \cdot \Phi'(u) = \frac{\varphi(\Psi(\Phi(u) + t))}{\varphi(u)} \geq 0.
\]

These two equations from (23) prove the equality and the first inequality in (19). Finally, we need to prove the second inequality in (19), that is, concavity of \( G \) with respect to \( u \):

\[
\frac{\partial G^2}{\partial u^2}(t, u) = \frac{\partial}{\partial u} \left( \frac{\varphi(G(t, u))}{\varphi(u)} \right) = \frac{1}{\varphi^2(u)} \left[ \varphi'(G(t, u)) \frac{\partial G}{\partial u}(t, u) \varphi(u) - \varphi(G(t, u)) \varphi'(u) \right].
\]

Next, using the second equation in (23), we get:

\[
\varphi'(G(t, u)) \frac{\partial G}{\partial u}(t, u) \varphi(u) = \varphi'(G(t, u)) \varphi(G(t, u)).
\]

Finally, \( \varphi' \) is nonincreasing. Since \( \Psi \) is nondecreasing, \( G(t, u) = \Psi(\Phi(u) + t) \geq \Psi(\Phi(u)) = u \) for all \( u \geq 1 \) and \( t \geq 0 \). Thus \( \varphi'(G(t, u)) \leq \varphi'(u) \). Multiplying this inequality by \( \varphi(G(t, u)) \geq 0 \) and using (24) and (25), we complete the proof.

4.3. **Proof of Lemma** \([4]\). We know that the following process is an \((\mathcal{F}_t)_{t \geq 0}\)-supermartingale:

\[
N(t) = V(X(t \wedge \tau)) + \int_0^{t \wedge \tau} \varphi(V(X(s))) \, ds, \quad t \geq 0.
\]

When we write in differential notation, we let \( df(t) \leq 0 \) if \( f \) is a nonincreasing function, and \( df(t) \leq dg(t) \) if \( g - f \) is a nondecreasing function. For \( t < \tau \), since \( V \) is smooth and \( X \) is a semimartingale, then \( V(X(t)) \) is also a semimartingale. Thus we can apply Itô’s formula:

\[
dK(t) = dG(t, V(X(t)))
\]

\[
= \frac{\partial G}{\partial t}(t, V(X(t))) \, dt + \frac{\partial G}{\partial u}(t, V(X(t))) \, dV(X(t)) + \frac{1}{2} \frac{\partial^2 G}{\partial u^2}(t, V(X(t))) \, d\langle V(X) \rangle_t.
\]

From concavity of \( G \) with respect to \( u \) found in Lemma \([3]\) and from (27), we get:

\[
dK(t) \leq \frac{\partial G}{\partial t}(t, V(X(t))) \, dt + \frac{\partial G}{\partial u}(t, V(X(t))) \, dV(X(t)).
\]

On the other hand, from (26) we get:

\[
dN(t) \leq dV(X(t)) + \varphi(V(X(t))) \, dt + dM(t)
\]

for some local \((\mathcal{F}_t)_{t \geq 0}\)-martingale \( M = (M(t), \ t \geq 0) \). Thus for \( t < \tau \) we have:

\[
dV(X(t)) = dN(t) - dM(t) - \varphi(V(X(t))) \, dt.
\]
Next, from above calculations in (28) and (26), we get:

\[ dK(t) \leq \frac{\partial G}{\partial t}(t, V(X(t))) dt + \frac{\partial G}{\partial u}(t, V(X(t))) V_t(X(t)) dN(t) \]

\[ - \frac{\partial G}{\partial u}(t, V(X(t))) M_t(X(t)) dt - \frac{\partial G}{\partial u}(t, V(X(t))) \varphi(V(X(t))) dt. \]

(29)

From computations of derivatives of \( G \) we get:

\[ \frac{\partial G}{\partial t}(t, u) = \varphi(u) \frac{\partial G}{\partial u}(t, u). \]

(30)

Plugging (30) into (29), we get:

\[ dK(t) \leq \frac{\partial G}{\partial u}(t, V(X(t))) (dN(t) - dM(t)). \]

But \( N \) is an \( (\mathcal{F}_t)_{t \geq 0} \)-supermartingale, and \( M \) is a local \( (\mathcal{F}_t)_{t \geq 0} \)-martingale. Thus \( N - M \) is also an \( (\mathcal{F}_t)_{t \geq 0} \)-supermartingale, and the same can be said about \( K \), since \( \frac{\partial G}{\partial u} \geq 0 \). This completes the proof.

5. Appendix

Below is a method to find \( h \) and \( U \) such that (6) holds. Take any Young pair \((H, K)\) of strictly increasing functions \( \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[ H(0) = K(0) = 0, \quad H(\infty) = K(\infty) = \infty, \quad xy \leq H(x) + K(y), \quad x, y \geq 0. \]

The following example follows from Young’s inequality:

\[ H(x) = \frac{x^p}{p}, \quad K(y) := \frac{y^q}{q}, \quad p, q > 0, \quad \frac{1}{p} + \frac{1}{q} = 1. \]

(32)

Assume (31) holds. Since \( H \) and \( K \) are strictly increasing, define their inverses \( H^{-1}, K^{-1} : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( H(H^{-1}(x)) = x \) and \( K(K^{-1}(y)) = y \) for all \( x, y \geq 0 \). Then

\[ H^{-1}(x)K^{-1}(y) \leq x + y, \quad x, y \geq 0. \]

(33)

For example, inverses of (32) are given by \( H^{-1}(x) := (px)^{1/p}, \quad K^{-1}(y) := (qy)^{1/q}. \)

REFERENCES

[1] Soren Asmussen (1998). Subexponential Asymptotics for Stochastic Processes: Extremal Behavior, Stationary Distributions and First Passage Probabilities. Annals of Applied Probability 8, 354–374.

[2] Rami Atar, Amarjit Budhiraja (2002). Stability Properties of Constrained Jump-Diffusion Processes. Electronic Journal of Probability 7, no. 22.

[3] Boris Baeumer, Mihály Kovács, Mark M. Meerschaert, René L. Schilling, Peter Straka (2016). Reflected Spectrally Negative Stable Processes and Their Governing Equations. Transactions of the American Mathematical Society 368, 227–248.

[4] Boris Baeumer, Mihály Kovács, Mark M. Meerschaert, Harish Sankaranarayanan Boundary Conditions for Fractional Diffusion, Journal of Computational and Applied Mathematics, 336, 408–424.

[5] Dominique Bakry, Patrick Cattiaux, Arnaud Guillin (2008). Rate of Convergence of Ergodic Continuous Markov Chains: Lyapunov vs Poincaré. Journal of Functional Analysis 254, 727–754.

[6] Anton Braverman, Jim G. Dai, Masakiyo Miyazawa (2017). Heavy Traffic Approximation for the Stationary Distribution of a Generalized Jackson Network: The BAR Approach. Stochastic Systems 7, 143–196.

[7] P. Laurie Davies (1986). Rates of Convergence to the Stationary Distribution for \( k \)-Dimensional Diffusion Processes. Journal of Applied Probability 23, 370–384.
[8] Krzysztof Debicki, Kamil Marcin Kosiński, Michel Mandjes (2012). On the Infimum Attained by a Reflected Lévy Process. *Queueing Systems* **70**, 23–35.

[9] Randal Douc, Gersende Fort, Arnaud Guillin (2009). Subgeometric Rates of Convergence of $f$-Ergodic Strong Markov Processes. *Stochastic Processes and Their Applications* **119**, 897–923.

[10] Randal Douc, Gersende Fort, Eric Moulines, Philippe Soulier (2004). Practical Drift Conditions for Subgeometric Rates of Convergence. *Annals of Applied Probability* **23**, 1671–1691.

[11] Douglas Down, Sean P. Meyn, Richard L. Tweedie (1995). Exponential and Uniform Ergodicity of Markov Processes. *Annals of Probability* **23**, 1671–1691.

[12] E. Robert Fernholz, Ioannis Karatzas (2009). Stochastic Portfolio Theory: an Overview. *Handbook of Numerical Analysis* **15**, 89–167.

[13] Gersende Fort, Eric Moulines V-Subgeometric Ergodicity for a Hastings-Metropolis Algorithm. *Statistics & Probability Letters* **49**, 401–410.

[14] Pierre-Olivier Goffard, Andrey Sarantsev (2019). Exponential Convergence Rate of Ruin Probabilities for Level-Dependent Lévy-Driven Risk Processes. *Journal of Applied Probability* **56**, 1244–1268.

[15] Tomoyuki Ichiba, Soumik Pal, Mykhaylo Shkolnikov (2013). Convergence Rates for Rank-Based Models with Applications to Portfolio Theory. *Probability Theory and Related Fields* **156**, 415–448.

[16] Offer Kella, Ward Whitt (1990). Diffusion Approximations for Queues with Server Vacations. *Advances in Applied Probability* **22**, 706–729.

[17] Robert B. Lund, Richard L. Tweedie (1996). Computable Exponential Convergence Rates for Stochastically Ordered Markov Processes. *Annals of Applied Probability* **6**, 218–237.

[18] Robert B. Lund, Richard L. Tweedie (1996). Geometric Convergence Rates for Stochastically Ordered Markov Chains. *Mathematics of Operations Research* **21**, 182–194.

[19] Tetsuro Kamae, Ulrich Krengel, George L. O’Brien (1977). Stochastic Inequalities on Partially Ordered Spaces. *Annals of Probability* **5**, 899–912.

[20] Anatolii V. Skorohod (1961). Stochastic Equations for Diffusion Processes in a Bounded Region. *Theory of Probability and Its Applications* **6**, 264–274.

[21] Pekka Tuominen, Richard L. Tweedie (1994). Subgeometric Rates of Convergence of $f$-Ergodic Markov Chains. *Advances in Applied Probability* **26**, 775–798.

[22] Alexander I. Zeifman (1991). Some Estimates of the Rate of Convergence for Birth and Death Processes. *Journal of Applied Probability* **28**, 268–277.

Department of Mathematics and Statistics, University of Nevada in Reno
E-mail address: asarantsev@unr.edu