Action-angle variables for the particle near extreme Kerr throat

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Abstract

We construct the action-angle variables for the spherical part of conformal mechanics describing the motion of a particle near extreme Kerr throat. We indicate the existence of the critical point |pγ| = mεRsch (with m being the mass of the particle, c denoting the speed of light, Rsch = 2γM/c2 being the Schwarzschild radius of a black hole with mass M, and γ denoting the gravitational constant), where these variables are expressed in terms of elementary functions. Away from this point the action-angle variables are defined by elliptic integrals. The proposed formulation allows one to easily reconstruct the whole dynamics of the particle both in initial coordinates, as well as in the so-called conformal basis, where the Hamiltonian takes the form of conventional non-relativistic conformal mechanics. The related issues, such as semiclassical quantization and supersymmetrization are also discussed.

1 Introduction

The Kerr solution [1] was discovered in 1963 as a solution of the vacuum Einstein equations describing the rotational black holes. Its uniqueness, proven by Carter [2], as well as the separability of variables of the particle moving in the Kerr background [3] gave to the Kerr solution a special role in General Relativity.

A very particular case of the Kerr solution, when the Cauchy’s and event horizons coincide is called extreme Kerr solution. In this special case the angular momentum J of the Kerr black hole is related with the mass of the Kerr black hole M by the expression J = γM2/c (with γ being the gravitational constant and c being the velocity of light). In 1999 Bardeen and Horowitz derived the near-horizon limit of the extreme Kerr solution and found that the isometry group of the limiting metric is SO(1, 2) × U(1) [4]. They conjectured that the extreme Kerr throat solution might admit a dual conformal theory description in the spirit of AdS/CFT duality. The extensive study of AdS/Kerr duality was initiated almost a decade later in [5] which continues today (see e.g. [6] and refs therein). It is clear to the moment, that the near-horizon extreme Kerr solution and its generalizations play a distinguished role in supergravity. Particularly, their thermodynamical properties and connection to the string theory allow one to expect that the quantum gravity should be closely related with these objects.

The simplest way to study of the near-horizon limit of the extreme Kerr black hole is the investigation of a test particle moving in its field. The study of this system may help to reveal some important symmetries or non-trivial constructions related to the field. The direct interpretation of the purely mechanical problem is also motivated, since there are known objects with a set of parameters close to those of the extremal Kerr’s black hole [7]. As a consequence of extended SO(1, 2) × U(1) symmetry of the near-horizon extreme Kerr metric, the dynamics of a test particle is described by an integrable conformal mechanical model. On the other hand, any mechanical system with (dynamical) conformal symmetry can be represented in the “non-relativistic” form

\[ H = \frac{p_R^2}{2} + \frac{2I(u)}{R^2}, \]

where the role of “effective coupling constant” I(u) is played by the Casimir of the conformal algebra. Here u is a set of phase variables such that \{u, R\} = \{u, P_R\} = 0. Together with R and P_R the variables u parametrize the entire phase space of the system. Hence, the whole specific information on the given conformal mechanics could be encoded in the “spherical part” defined by the Hamiltonian I(u) and respective Poisson brackets, including (im)possibility to construct the D(1, 2|α) superconformal extension [8]. So, rather than studying the whole dynamics of the probe particle near the extreme Kerr throat, we can restrict our study to its spherical part, which defines an integrable two-dimensional Hamiltonian system. Let us notice that the study of the probe particle near the extreme Kerr throat and of its \( \mathcal{N} = 2 \) superconformal extension has been performed in [9](see also [11]), and extended in [10] to the case of the Kerr-Newman-AdS-dS black hole. However there were troubles with the construction of its \( \mathcal{N} = 4 \) superconformal extension, which reflected the lack of supersymmetry invariance of the extreme Kerr solution. The investigation of the spherical part should clarify weather these troubles are crucial.
The goal of present paper is the construction of action-angle variables for the integrable two-dimensional system, which plays the role of the spherical part of the dynamic of the particle moving near an extreme Kerr throat. The action-angle variables would provide us with the complete information on (quasi)periodic motions in the system. Because of the uniqueness among all the other canonical variables, these variables allow us to establish a correspondence/discrepancy between different integrable systems of classical mechanics, to perform their Bohr-Sommerfeld quantization, as well as to construct their supersymmetric (at least, formally) extensions. Besides, action-angle variables forms a tool for the developing of classical perturbation theory.

The construction of the above mentioned variables is not a straightforward task. In [18] the construction was done using action-angle variables. Thus, as a by-product of the present studies we present explicit expressions for these variables.

Performing the action-angle formulation of the spherical part of the “near-horizon extreme Kerr particle” we will find, that the system under consideration has a critical point defined by the value of angular momentum $|p_\phi|$, 

$$
|p_\phi| = \gamma \frac{2mM}{c},
$$

with $m$ being the mass of the particle and $M$ denoting the black hole mass. At this critical point the action-angle variables are given in terms of elementary functions, while the Hamiltonian coincides with that of the one-dimensional Higgs oscillator. Respectively, the system becomes exactly solvable. Away from the critical point the functional dependence of the action-angle variables from the initial ones, as well as the dependence of the Hamiltonian from the action variables are given by elliptic integrals. For the extreme Kerr throat, the existence of this critical point (as the point where the particle motion becomes integrable in terms of elementary functions) has been noticed already in Ref. [9].

The paper is arranged as follows:

In Section 2 we present the general consideration and show that the phase space of the spherical part of the particle moving near the extreme Kerr throat has a critical point (1.2). In Section 3 we construct the action-angle variables for the system. In Section 4 we restore the radial dynamics of the probe particle. Finally, in Section 5 we present some concluding remarks and discuss the issues concerning supersymmetrization.

Throughout the text we will use the gravitational units, where $\gamma = c = 1$(where $\gamma$ is the gravitational constant and $c$ is the velocity of light).

## 2 General consideration

The Kerr solution is the stationary axially symmetric solution of the vacuum Einstein equations, which describes the rotating black hole with mass $M$ and angular momentum $J$. It is defined by the metric

$$
ds^2 = -\left(\frac{\Delta - a^2 \sin^2 \theta}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma}\right) \sin^2 \theta d\phi^2 - \frac{2a(r^2 + a^2 - \Delta) \sin^2 \theta}{\Sigma} dt d\phi, \quad (2.1)
$$

where

$$
\Delta = r^2 + a^2 - 2Mr, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad a = \frac{J}{M}. \quad (2.2)
$$

The extreme solution of the Kerr metric corresponds to the choice $M^2 = J$, so that the event horizon is at $r = M$. The limiting near-horizon metric is given by the expression [4]

$$
ds^2 = \left(1 + \cos^2 \theta \right) \left[ -\frac{r^2}{r_0^2} dt^2 + \frac{r_0^2}{r^2} dr^2 + r_0^2 \sin^2 \theta d\theta^2 \right] + \frac{2r_0^2 \sin^2 \theta}{1 + \cos^2 \theta} \left[ d\phi + \frac{r}{r_0} dt \right]^2, \quad r_0 = \sqrt{2}M. \quad (2.3)
$$

The Kerr metric admits the second rank Killing tensor [12], which allows to integrate the geodesic equation for a massive particle in Kerr space-time by quadratures [3]. The limiting Killing tensor becomes reducible, in the sense that it can be constructed from the Killing vectors corresponding to the $SO(2,1) \times U(1)$ isometry group.

Consequently, the motion of a particle near the horizon of the extreme Kerr black hole is defined by the generators of the conformal algebra $so(1,2)$ [9]

$$
H = \frac{r}{r_0^2} \left( \sqrt{(rp_r)^2 + L(p_\theta, p_\phi, \theta)} - p_\phi \right), \quad K = \frac{r_0^2}{r} \left( \sqrt{(rp_r)^2 + L(p_\theta, p_\phi, \theta)} + p_\phi \right) \quad D = rp_r, \quad (2.4)
$$
\( \{ H, D \} = H, \quad \{ H, K \} = 2D, \quad \{ D, K \} = K, \) \hfill (2.5)

where Poisson brackets are defined by the canonical symplectic structure

\( \omega = dr \wedge dp_r + d\varphi \wedge dp_\varphi + d\theta \wedge dp_\theta . \) \hfill (2.6)

The generator \( H \) plays the role of the Hamiltonian of the system. It has two constants of motion,

\[ L = p_\theta^2 + \frac{(1 + \cos^2 \theta)^2 p_\varphi^2}{4 \sin^2 \theta} + \left( \frac{1 + \cos^2 \theta}{2} \right) (mr_0)^2, \] \hfill (2.7)

defined by the Killing tensor of the second rank, and

\[ p_\varphi : \{ p_\varphi, H \} = 0, \] \hfill (2.8)

corresponding to the \( U(1) \) isometry of the near horizon extreme Kerr metric.

The angular part of the system, given by the Casimir of \( so(1, 2) \) defines the following integrable Hamiltonian system:

\[ \mathcal{I} = KH - D^2 = L - p_\varphi^2 = p_\theta^2 + \left( \frac{\cos^4 \theta + 6 \cos^2 \theta - 3}{4 \sin^2 \theta} \right) p_\varphi^2 + \left( \frac{1 + \cos^2 \theta}{2} \right) (mr_0)^2, \quad \omega_0 = dp_\theta \wedge d\theta + dp_\varphi \wedge d\varphi. \] \hfill (2.9)

Let us notice, that effective metric of the configuration space of the system has the singularities at latitudes defined by angles \( \theta_1 = \arccos \sqrt{3} - 3 \) and \( \pi - \theta_0. \)

Besides, the above system has some distinguished points in the phase space as well, which could be visualized after formulating the system in terms of action-angle variables. In accordance with Liouville’s theorem, the existence of action-angle variables requires the level surface to be a compact and connective manifold [13]. To show that it is a case, let us re-write the Hamiltonian of the system in the following form:

\[ \mathcal{I} = \frac{(p_\varphi^2 - 2(mr_0)^2) \cos^4 \theta + 6p_\varphi^2 \cos^2 \theta + 2(mr_0)^2 - 3p_\varphi^2}{4 \sin^2 \theta} + p_\theta^2. \] \hfill (2.10)

From this expression it follows that \( \mathcal{I} \sim 4p_\varphi^2/\sin^2 \theta > 0 \) when \( \theta \sim 0 \), i.e. the requirement of compactness is satisfied.

In order to get the action-angle formulation of the system we have to consider the generating function (“effective action”)

\[ S(p_\varphi, \mathcal{I}, \theta, \varphi) = p_\varphi \varphi + S_0(p_\varphi, \mathcal{I}, \theta), \] \hfill (2.11)

where

\[ S_0 = \int_{p_\varphi = const} \int_{\mathcal{I} = const} \frac{d\theta}{\sqrt{\mathcal{I} - \left( \frac{\cos^4 \theta + 6 \cos^2 \theta - 3}{4 \sin^2 \theta} \right) p_\varphi^2 - \left( \frac{1 + \cos^2 \theta}{2} \right) (mr_0)^2}}. \] \hfill (2.12)

By its use we define the action variables,

\[ I_1(\mathcal{I}, p_\varphi) = \frac{1}{2\pi} \int d\theta \sqrt{\mathcal{I} - \left( \frac{\cos^4 \theta + 6 \cos^2 \theta - 3}{4 \sin^2 \theta} \right) p_\varphi^2 - \left( \frac{1 + \cos^2 \theta}{2} \right) (mr_0)^2}, \quad I_2 = p_\varphi, \] \hfill (2.13)

and the angle ones

\[ \Phi_1 = \frac{\partial S_0(p_\varphi, \mathcal{I}, \theta)}{\partial I_1}, \quad \Phi_2 = \varphi + \frac{\partial S_0}{\partial I_2}. \] \hfill (2.14)

From (2.10) one can notice that the condition \((p_\varphi^2 - 2(mr_0)^2) = 0\) simplifies the expression for \( \mathcal{I} \) and makes it exactly coincide with the Hamiltonian of the Higgs oscillator. We will discuss this particular case in more detail later in this article.

\(^1\)We can also extract the spherical part for the system of a particle moving on the near-extreme Kerr-Newman background and perform similar analyses, however, because of the complexity of the resulting system, we avoid including explicit formulae in this work.
3 Action-angle variables

In this Section first we assume that \( |p_\varphi| \neq \sqrt{2} mr_0 \). The case \( |p_\varphi| = \sqrt{2} mr_0 \) will be analyzed in a separate subsection.

Introducing the variable \( x = \cos^2 \theta \), we transform (2.13) to the following form:

\[
I_1(I, p_\varphi) = \frac{1}{2\pi} \int_0^{2\sqrt{x(1-x)}} \frac{dx}{2\sqrt{1-x}} \sqrt{\frac{(2(mr_0)^2 - p_\varphi^2)(x-a_1)(x-a_2)}{4(1-x)}}.
\]

where

\[
a_{1,2} = \frac{2I + 3p_\varphi^2 \pm 2\sqrt{I^2 - 2I(mr_0)^2 + (mr_0)^4 + 4I^2 p_\varphi^2 - 2(mr_0)^2 p_\varphi^2 + 3p_\varphi^4}}{2(mr_0)^2 - p_\varphi^2}.
\]

In order to calculate the action variable \( I_1 \), we have to determine the integration range in the first integral in (3.1). For this purpose, we notice that for any values of the parameters in the considered range we have:

\[
(2(mr_0)^2 - p_\varphi^2)a_2 > 0, \quad 1 > a_1 > 0
\]

and, from the requirement that the subroot expression in (3.1) is positive (and, therefore, the integrand is a real function), we find that the integration range is always \([0, a_1]\).

Since the substitution of variable \( x = \cos^2 \theta \) identifies the points \( \pm \cos \theta \), this corresponds to only one quarter of a cycle. Therefore, the final expression should be multiplied by 4. As a result, we get

\[
I_1(I, p_\varphi) = \frac{\sqrt{a_2(2(mr_0)^2 - p_\varphi^2)}}{4a_1} F_1(1/2, 1, -1/2, 2, a_1, a_1/a_2).
\]

where \( F_1(1/2, 1, -1/2, 2, a_1, a_1/a_2) \) is the Appel’s hypergeometric function (see [14] and Appendix).

The angle variables are expressed via elliptic integrals of the first, second and third kind (see e.g. [14, 15] and Appendix):

\[
\Phi_1(I, p_\varphi, \theta, \varphi) = \frac{1}{\sqrt{a_2(2(mr_0)^2 - p_\varphi^2)}} \frac{\partial I}{\partial I_1} F\left(\arcsin\frac{\cos \theta}{\sqrt{a_1}}; \frac{a_1}{a_2}\right)
\]

and

\[
\Phi_2(I, p_\varphi, \theta, \varphi) = \frac{\partial I}{\partial I_2} \frac{1}{\sqrt{a_2(2(mr_0)^2 - p_\varphi^2)}} F\left(\arcsin\frac{\cos \theta}{\sqrt{a_1}}; \frac{a_1}{a_2}\right) - \frac{2I_2}{\sqrt{a_2(2(mr_0)^2 - p_\varphi^2)}} \Pi\left(a_1, \arcsin\frac{\cos \theta}{\sqrt{a_1}}; \frac{a_1}{a_2}\right)
\]

The effective frequencies \( \partial I/\partial I_1, 2 \) can be found from (3.4):

\[
\frac{\partial I}{\partial I_1} = \left(\frac{\partial I_1}{\partial I}\right)^{-1}, \quad \frac{\partial I}{\partial I_2} = -\frac{\partial I_1}{\partial I_1}/\partial I_2
\]

Namely, we have:

\[
\Omega_1 = \frac{\partial I}{\partial I_1} = \pi \sqrt{a_2(2(mr_0)^2 - p_\varphi^2)}
\]

and

\[
\Omega_2 = \frac{\partial I}{\partial I_2} = -\frac{p_\varphi}{2K\left(\frac{a_1}{a_2}\right)} \left(7 + a_2\right) K\left(\frac{a_1}{a_2}\right) - a_2 E\left(\frac{a_1}{a_2}\right) - 4\Pi\left(\frac{a_1}{a_2}\right)
\]

where \( K(\phi), E(\pi) \) and \( \Pi(n, \phi) \) are complete elliptic integrals of the first, second and the third kind respectively.
Critical point

At the critical point \(|p_\varphi| = \sqrt{2} m r_0\) the spherical Hamiltonian \(\mathcal{H}\) reduces to the Hamiltonian of the one-dimensional Higgs oscillator [16]:

\[
\mathcal{H}_{osc} = p_\varphi^2 + 2(m r_0)^2 \cot^2 \theta - (m r_0)^2.
\] (3.10)

Its action-angle variables can be easily calculated either directly by putting \(|p_\varphi| = \sqrt{2} m r_0\) in (2.14) (see, e.g. [17]) or by taking the limit \(\lim_{|p_\varphi| \to \sqrt{2} m r_0} I_1\). In both cases we come to the same result,

\[
I_1 = \sqrt{\mathcal{H}_{osc} + 3(m r_0)^2} - \sqrt{2} m r_0, \quad \Phi_1 = \arcsin \sqrt{\frac{\mathcal{H}_{osc} + 3(m r_0)^2}{\mathcal{H} + (m r_0)^2}} \cos \theta.
\] (3.11)

Respectively, the Hamiltonian reads

\[
\mathcal{H}_{osc} = \left( I_1 + \sqrt{2} m r_0 \right)^2 - 3(m r_0)^2.
\] (3.12)

This point, however, does not correspond to any special values of the frequencies. One can see this by taking the limit from (3.8),(3.9):

\[
\Omega_1^{osc} = \lim_{p_\varphi \to \pm \sqrt{2} m r_0} \Omega_1 = \frac{d\mathcal{H}_{osc}}{dI_1} = 2 \sqrt{\mathcal{H} + 3(m r_0)^2} = 2 \left( I_1 + \sqrt{2} m r_0 \right),
\] (3.13)

while the second frequency reads:

\[
\Omega_2^{osc} = \lim_{p_\varphi \to \pm \sqrt{2} m r_0} \Omega_2 = \pm \sqrt{2} m r_0 \left( \frac{15}{4} - \frac{(m r_0)^2}{2(I + 3(m r_0)^2)} - \frac{\sqrt{2}}{m r_0} \sqrt{I + 3(m r_0)^2} \right) =
\]

\[
= \pm \sqrt{2} m r_0 \left( \frac{15}{4} - \frac{(m r_0)^2}{4(I_1 + \sqrt{2} m r_0)} - \frac{\sqrt{2}}{m r_0}(I_1 + \sqrt{2} m r_0) \right).
\] (3.14)

4 Radial dynamics

In previous Sections we have got the complete description of the “spherical sector” of the “near-horizon extreme Kerr particle” in terms of action-angle variables. For the restoring of the radial dynamics of particle, one should simply replace, in (2.4), the \(L(p_\varphi, p_\theta, \varphi, \theta)\) by \(L(I) = L(I_{1,2}) - I_2^2\), and immediately solve the equations of motion.

However, it is more deductive to write down the whole Hamiltonian systems in the “conformal basis”, where the Hamiltonian takes the form of conventional non-relativistic conformal mechanics [18]

\[
H = \frac{P_\theta^2}{2} + \frac{2L(I_{1,2})}{X^2}, \quad \Omega = dP_X \wedge dX + dI_a \wedge d\Phi^a,
\] (4.1)

where

\[
X = \sqrt{2K(r, p_r, I)}, \quad P_X = \frac{D(r, p_r, I)}{\sqrt{2K(r, p_r, I)}}, \quad \Phi^a = \Phi^a + \frac{1}{2} \frac{\partial}{\partial I_a} \int_{s=p_r} ds \log \left( \sqrt{s^2/4 + L(I)} + I_2 \right).
\] (4.2)

The classical properties of this system are well-known [19]. Particularly, for \(\mathcal{I} < 0\) we deal with the phenomenon of falling towards the black hole center, and for \(\mathcal{I} > 0\) we deal with the scattering problem. The case \(\mathcal{I} = 0\) corresponds to the free particle system.

Quantum mechanics of this system is also considered in details [20]. It is well known, that the respective quantum Hamiltonian is not self-conjugated, and does not possess a ground state [21, 22].

For applying this well-known results to our concrete case, we should simply perform the Bohr-Sommerfeld quantization of the spherical part of the system under consideration

\[
I_1 = \hbar(n_1 + \frac{1}{2}), \quad I_2 = \hbar n_2, \quad n_1, n_2 \in \mathbb{N},
\] (4.3)
and consider the quantized $\mathcal{I}(n_1, n_2)$ as an effective coupling constant.

Thus, upon consideration of the scattering problem, the dependence of the angular Hamiltonian $\mathcal{I}$ on action variables $I_{1,2}$ will be reflected at the dynamics of radial variables. In the noncritical regimes the scattering of the particle will be asymmetric (and depending on both quantum numbers $n_1, n_2$), and at the critical point $n_2 = \sqrt{2mr_0}/\hbar$ we shall deal with the symmetric scattering. At the critical point we have an exact coincidence of the spectra obtained by canonical and Bohr-Sommerfeld quantization. The expression for the energy levels cannot be found analytically, but for given values of parameters $mr_0$ one can calculate them numerically from (3.4).

5 Concluding remarks

In the present paper we constructed the action-angle variables of the angular sector of the dynamics of a particle moving near an extreme Kerr throat. These variables can be expressed via initial ones in terms of elliptic functions, so the procedure is not very convenient for analyzing the system.

Due to the dynamical conformal symmetry, the “angular part” of the system accumulates the whole information on the initial dynamics. Moreover, it allows us to present the system in the form of conventional non-relativistic conformal mechanics, where the Casimir of the conformal algebra (“spherical Hamiltonian”) $\mathcal{I}$ appears as an effective coupling constant. Respectively, for negative values of the Casimir, the effective radial dynamics corresponds to the falling on the center, and for positive Casimir values it corresponds to the scattering problem.

The given formulation allows us to immediately construct the $D(1,2|\alpha)$ superconformal extension of the particle near the extreme Kerr throat. Indeed, in action-angle variables we can construct a (formal) $\mathcal{N} = 4$ superextension of the system ((2.9))

$$Q_1 = \sqrt{\mathcal{I}}(\cos \lambda e^{i\kappa} n_1 - \sin \lambda e^{-i\kappa} n_2), \quad Q_2 = \sqrt{\mathcal{I}}(\cos \lambda e^{-i\kappa} n_2 + \sin \lambda e^{i\kappa} n_1):$$

$$\{Q_a, Q_b\} = \delta_{ab}(I + ...), \quad \{Q_a, Q_b\} = 0, \quad a, b = 1, 2$$

where $\lambda(I, \Phi)$ and $\kappa(I, \Phi)$ are some, still undefined functions, the Grassmann variable $\eta_a$ obeys the canonical Poisson bracket relations $\{\eta_a, \eta_b\} = \delta_{a,b}$, and "..." denotes terms containing fermionic degrees of freedom, which we do not present explicitly. By an appropriate choice of $\lambda$ and $\kappa$ one can, seemingly, obtain a physically relevant supersymmetric Hamiltonian, in analogy with the $\mathcal{N} = 2$ case [23] (here we should exploit, in fact, the freedom in the supersymmetrization of mechanical systems, see e.g. [24, 10]). Then, with the $\mathcal{N} = 4$ supersymmetric extension of the angular Hamiltonian $\mathcal{I}$ at hands, one can immediately construct the $D(1,2|\alpha)$ superconformal extension of the whole conformal mechanics [8].

This looks as a contradiction of the customary statement that it is not possible to construct the $\mathcal{N} = 4$ superconformal extension of particle systems near the extreme Kerr throat, which reflects the lack of supersymmetry of the Kerr solution. The matter is that the transition to the action-angle variables has been done by a canonical transformation mixing initial coordinates and momenta. Obviously, the respective field transformation is forbidden for the field-theoretical equations (which admit extreme Kerr solution). Similarly, no go statements on $\mathcal{N} = 4$ (non)supersymmerization of the particle near the extreme Kerr throat (and of the two-dimensional spherical system ((2.9)) concern the $\mathcal{N} = 4$ superextension in terms of existing one-dimensional supermultiplets constructed over the initial spatial variables.

Finally, let us notice that the action-angle variables yield a ground for developing classical perturbation theory. Thus, by their use we can describe the dynamics of a particle in the field of nearly-extreme Kerr black holes, which seemingly, have been observed recently [7]. Moreover, action variables, being adiabatically invariant, allow one to evaluate the particle dynamics near the extreme Kerr throat with the slow time-dependent parameters $m$ and $r_0$.

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A Elliptic Integrals

For the convenience of the reader we present here the notions of the elliptic integrals [15, 14]
Appel’s hypergeometric function

\[ F_1(\alpha, \rho, \lambda, \alpha + \beta; ua, va) = a^{1-\alpha-\beta} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \int_0^a x^{\alpha-1} (a - x)^{\beta-1} (1 - ux)^{-\rho} (1 - vx)^{-\lambda} dx. \]

Elliptic integral of the first kind

\[ F(\phi, k) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \int_0^x \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}, \quad k^2 < 1, \ x = \sin \phi \]

Elliptic integral of the second kind

\[ E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi} d\phi = \int_0^x \sqrt{1 - k^2 x^2} \sqrt{1 - x^2} dx, \quad x = \sin \phi \]

Elliptic integral of the third kind

\[ \Pi(\phi, n, k) = \int_0^\phi \frac{d\phi}{(1 + n \sin^2 \phi) \sqrt{1 - k^2 \sin^2 \phi}} = \int_0^x \frac{dx}{(1 + nx^2) \sqrt{1 - x^2} \sqrt{1 - k^2 x^2}} dx, \quad x = \sin \phi \]

Complete elliptic integral

\[ K = F(\pi/2, k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}. \]

The detailed description of elliptic integrals can be found, e.g. in [15, 14].

References

[1] R. P. Kerr, Phys. Rev. Lett. 11 (1963) 237.
[2] B. Carter, Phys. Rev. Lett. 26, 331 (1971).
[3] B. Carter, Phys. Rev. 174, 1559 (1968).
[4] J. M. Bardeen, G.T. Horowitz, Phys. Rev. D 60, 104030 (1999) [hep-th/9905099].
[5] M. Guica, T. Hartman, W. Song, A. Strominger, Phys. Rev. D 80, 124008 (2009) [arXiv:0809.4266].
[6] A. Castro, A. Maloney, A. Strominger, Phys. Rev. D 82, 024008 (2010) [arXiv:1004.0996].
[7] J. E. McClintock, R. Shafee, R. Narayan, R. A. Remillard, S. W. Davis, Li-Xin Li, Astrophys. J. 652, 518 (2006)[astro-ph/0606076]
[8] T. Hakobyan, S. Krivonos, O. Lechtenfeld, A. Nersessian, Phys. Lett. A 374 (2010) 801 [arXiv:0908.3290];
[9] A. Galajinsky, JHEP 1011, 126 (2010) [arXiv:1009.2341].
[10] A. Galajinsky and K. Orekhov, Nucl. Phys. B 850, 339 (2011) [arXiv:1103.1047 [hep-th]].
[11] S. Bellucci and S. Krivonos, JHEP 1110, 014 (2011) [arXiv:1106.4453 [hep-th]].
[12] M. Walker, R. Penrose, Commun. Math. Phys. 18, 265 (1970).
[13] V. I. Arnold, Mathematical methods in classical mechanics, Nauka Publ., 1973.
[14] A. P. Prudnikov, Yu. A. Brychkov, O. I. Mar’ichev, Integrals and series (in Russian), Nauka Publ., Moscow, 1981
[15] H. B. Dwight, Tables of integrals and other mathematical data., 4th edition, The Macmillan Company, N.Y., 1961
[16] P. W. Higgs, J. Phys. A 12, 309 (1979).
    H. I. Leemon, J. Phys. A 12, 489 (1979).

[17] T. Hakobyan, O. Lechtenfeld, A. Nersessian, A. Saghatelian, V. Yeghikyan, Phys. Lett. A376 679 (2012) [arXiv:1108.5189][math-ph]
    S. Bellucci, A. Nersessian, A. Saghatelian, V. Yeghikyan, J. Comput. Theor. Nanosci. 8, 769 (2011) [arXiv:1008.3865].

[18] A. Galajinsky and A. Nersessian, JHEP 1111, 135 (2011) [arXiv:1108.3394][hep-th]

[19] L. D. Landau, E. M. Lifshits, Mechanics. 5th edition, Nauka Publ., Moscow, 2004

[20] L. D. Landau, E. M. Lifshits, Quantum Mechanics. Non-relativistic theory., 4-th edition, Nauka Publ., Moscow, 1989

[21] K. M. Case, Phys. Rev., 80 (1950) 797

[22] V. de Alfaro, S. Fubini and G. Furlan, Nuovo Cim. A34(1976) 569.

[23] O. Lechtenfeld, A. Nersessian, V. Yeghikyan Phys. Lett. A 374 (2010) 4647 [arXiv:1005.0464]

[24] S. Bellucci,A. Nersessian, Phys. Rev. D 73 (2006) 107701 [arXiv:hep-th/0512165].