The surfing effect in the interaction of electromagnetic and gravitational waves. Limits on the speed of gravitational waves.

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Abstract

In the current work we investigate the propagation of electromagnetic waves in the field of gravitational waves. Starting with simple case of an electromagnetic wave travelling in the field of a plane monochromatic gravitational wave we introduce the concept of surfing effect and analyze its physical consequences. We then generalize these results to an arbitrary gravitational wave field. We show that, due to the transverse nature of gravitational waves, the surfing effect leads to significant observable consequences only if the velocity of gravitational waves deviates from speed of light. This fact can help to place an upper limit on the deviation of gravitational wave velocity from speed of light. The micro-arcsecond resolution promised by the upcoming precision interferometry experiments allow to place stringent upper limits on $\epsilon = (v_{gw} - c)/c$ as a function of the energy density parameter for gravitational waves $\Omega_{gw}$. For $\Omega_{gw} \approx 10^{-10}$ this limit amounts to $\epsilon \lesssim 2 \cdot 10^{-2}$.

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I. INTRODUCTION

The detection of gravitational waves is arguable one of the most important outstanding experimental challenges in physics. With the construction of laser interferometric gravitational wave detectors like LIGO, VIRGO, TAMA, GEO600 there are good chances of direct detection in the very near future \[1, 2, 3, 4, 5, 6, 7\]. Alongside the interferometers, which are mainly aimed at detecting gravitational waves of astrophysical origin, the anisotropies in temperature and polarization of the Cosmic Microwave Background (CMB) have a strong potential to discover relic gravitational waves \[8, 9, 10\] (see \[11, 12\] for a recent review).

Most of the current techniques to detect gravitational waves are based on their interaction with electromagnetic fields. In general, the interaction of gravitational waves with electromagnetic radiation leaves imprints on the latter that can be experimentally measured \[13\]. In this work we shall deal with one such interaction effect which we shall call the “surfing effect”, where (figuratively speaking) the electromagnetic wave surfs on a gravitational wave leading to an observable phase change in the electromagnetic wave. This effect was first considered in \[14\]. In the present paper, we shall expand on the results of \[14, 15\] and generalize the effect for an arbitrary gravitational wave field. We shall consider the consequences of this effect for the planned precision radio (or x-ray) interferometric projects \[16, 17\]. As we shall show, due to the transverse nature of gravitational waves, the surfing effect leads to an observable phase change only when the velocity of gravitational waves is different from speed of light. Using this fact, and the micro-arcsecond accuracy promised by the precision interferometry measurements \[16, 17\], we can place significant upper limits on the parameter $\epsilon = (c - v_{gw})/c$ which characterizes the deviation of velocity of gravitational waves from speed of light.

The constraints on the speed of gravitational waves is an interesting experimental challenge. A potentially strong method of constraining $\epsilon$ is to compare the arrival times of a gravitational wave and an electromagnetic wave emitted by a supernova or a gamma ray burst \[18\] (see also \[19\]). Although this method will be able to give very strong constraints, it crucially depends on our ability to model and detect the gravitational wave signal from these sources, which is a significant theoretical and experimental challenge. In \[20\] the speed of gravity, and correspondingly indirectly the speed of gravitational waves, was constrained, by analysis of retarded gravitational potentials in the non-wave zone, by measuring propagation of the quasar’s radio signal past Jupiter. Let us note, without going into detail of ambiguity in the interpretation of this result (see for example \[21\]), that speed of gravity was constrained to $c_g = (1.06 \pm 0.21)c$ corresponding to $\epsilon \lesssim 0.2$. In this
paper we consider the propagation of electromagnetic radiation in the field of gravitational waves, i.e. in the wave zone of the gravitational field, where the velocity parameter $\epsilon$ can be introduced avoiding any ambiguity. Using the surfing effect, we consider an independent method of placing upper limits on the $\epsilon$-parameter, which could potentially give very strong limits on the speed of gravitational waves ($\epsilon \lesssim 10^{-2}$ for $\Omega_{gw} \approx 10^{-10}$).

The plan of the paper is as follows. In Section II we shall consider the propagation of an electromagnetic wave in the field of a single monochromatic gravitational wave. We shall discuss the physical aspects of the surfing effect with a view on the precision interferometric measurements, as well as write down some of the equations that will be used in the following sections. In Section III we generalize the surfing effect for an arbitrary gravitational wave field. Positing statistical properties of the gravitational wave field we derive the consequent statistical properties of the response of an interferometer. In Section IV we use the surfing effect along with the predicted precision level of the interferometry measurements to place upper limits on the velocity parameter $\epsilon$ depending on the energy density of gravitational wave described by the density parameter $\Omega_{gw}$. Finally, in Section V we present a short discussion and summary of the main results of this paper.

II. SINGLE MONOCHROMATIC GRAVITATIONAL WAVE

Let us consider a slightly perturbed flat Friedmann-Lemaitre-Robertson-Walker (FLRW) universe with coordinates $x^\mu \equiv (\eta, x^i)$ and the metric given by [22], [23]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(\eta) \left[ -d\eta^2 + (\delta_{ij} + h_{ij}) dx^i dx^j \right].$$  \hspace{1cm} (1)

Here $\eta$ is conformal time coordinate related to proper time coordinate $t$ through a relation $cdt = a(\eta)d\eta$, and $x^i$ are the spatial coordinates. In the above expression $a(\eta)$ is the scale factor, and $h_{ij}$ is the gravitational wave perturbation. Since, in this work we shall be interested in electromagnetic waves emitted by objects with a redshift $z \lesssim 5$, we will work with a matter dominated Universe model characterized by a scale factor

$$a(\eta) = a_o \left( \frac{\eta}{\eta_o} \right)^2,$$  \hspace{1cm} (2)

where $\eta_o$ corresponds to present time. From the above expression it follows that $a_o$ is the scale factor at the present time. Without loss of generality we can set $a_o = 1$. With this convention, the value of $\eta_o$ is related with the present day Hubble constant $H_o$

$$\eta_o = \frac{2c}{H_o} = 2L_H,$$  \hspace{1cm} (3)
where we have introduced the Hubble length $L_H \equiv c/H_0$.

For simplicity, in this section we shall consider the case of a single monochromatic plane gravitational wave. We shall also restrict our considerations to gravitational waves whose wavelength at the current epoch is small compared to the present Hubble length. This implies that we can consider these waves to be monochromatic waves with an amplitude damping adiabatically with the expansion of the universe

$$h_{ik}(\eta, x^i) = h_0 \left( \frac{a_0}{a(\eta)} \right) p_{ik} e^{i k \cdot x^\mu}, \quad (4)$$

where $h_0$ is the amplitude at the present epoch, $p_{ik}$ is the polarization tensor of the gravitational wave, and $k_\mu = (k_0, k_i)$ is the wave vector of the gravitational wave. It is convenient to introduce the wavenumber $k = (\delta_{ij} k^i k^j)^{1/2}$. The present day wavelength of the gravitational field $\lambda_{gw}$, is related to wavenumber $k$ by the relation $\lambda_{gw} = 2\pi/k$. The present day frequency of the gravitational wave $\omega_{gw}$ is related to the time component of the wavevector $k_0$ by relation $\omega_{gw} = ck_0$.

Let us analyze the electromagnetic wave propagation in the approximation of geometrical optics. In this approximation, the wave equation is written for the quantity $\psi(x^\mu) \equiv \psi(\eta, x^i)$, known as the eikonal. Eikonal has the physical meaning of the phase of an electromagnetic wave field $f(x^\mu) = A(x^\mu)e^{i\psi(x^\mu)}$. (The quantity $A(x^\mu)$ describes the amplitude of the wave field, but shall not interest us in this consideration.) The wave vector $\kappa_\mu$ of the electromagnetic wave is give by $\kappa_\mu = \partial\psi/\partial x^\mu$.

The eikonal equation follows from the isotropy condition $g_{\mu\nu} \kappa^\mu \kappa^\nu = 0$. Substituting the expression for $\kappa^\mu$ in terms of $\psi$ we arrive at the eikonal equation

$$g^{\mu\nu} \frac{\partial\psi}{\partial x^\mu} \frac{\partial\psi}{\partial x^\nu} = 0. \quad (5)$$

We shall seek the solution to equation (5) in a perturbative form

$$\psi(x^\mu) = \psi_0(x^\mu) + \psi_1(x^\mu). \quad (6)$$

Here the zeroth order solution corresponds to the solution in the absence of perturbations, while the first order solution corresponds to the solution in the presence of perturbations $\sim h$. Let us assume that the zeroth order solution corresponds to a plane monochromatic electromagnetic wave:

$$\psi_0 = \omega_E (\eta + e_i x^i) / c, \quad \frac{\partial\psi_0}{\partial \eta} = \omega_E/c, \quad \frac{\partial\psi_0}{\partial x^i} = \omega_E e_i / c, \quad (7)$$

where $e^i (\delta^{ij} e_i e_j = 1)$ is the unit vector along the direction of wave propagation, and $\omega_E$ is the frequency of the electromagnetic wave at the present time. Taking into account the solution (7) of
the zeroth order equation, the first order equation for $\psi_1(x^\mu)$ takes the form
\[ \frac{\partial \psi_1}{\partial \eta} + e^i \frac{\partial \psi_1}{\partial x^i} = \frac{1}{2c} \omega E h_{ik} e^i e^k. \] (8)

The solution to equation (8) is written in terms of the line of sight integral (along the unperturbed light path):
\[ \psi_1(\eta, x^i) = \frac{1}{2c} \omega E e^i e^k \int_{\eta_0 - D}^{\eta_0} ds \ h_{ik} \ (s, x^i + e^i (s - \eta_0)), \] (9)

where $s$ is the $\eta$-time parameter along the light ray path from the emitter to observer. The limits of integration ($\eta_0 - D$) and $\eta_0$ correspond to the time of emission and observation correspondingly. Using (2), the parameter $D$ can be related to the redshift of emission $z$ by
\[ D = 2L_H \left( 1 - \frac{1}{\sqrt{1 + z}} \right). \] (10)

Below, for simplicity of analysis and in order not to obscure the physical interpretation of the surfing effect, we shall consider the problem in flat space-time without cosmological evolution of the scale factor, and correspondingly, assume no cosmological evolution of gravitational wave amplitude. This analysis is equivalent to setting the scale factor in expression (1) to a constant value, i.e. $a(\eta) = a_0 = 1$. Alternatively, this approximation can be viewed as an analysis restricted to small values of redshift, i.e. $z \ll 1$. In this limit, parameter $D$ represents the physical distance to the source, and is related to the redshift by the usual Hubble law $D \approx zL_H$. We shall reintroduce the cosmological evolution in Section III where we shall explain how and why the result modifies. Some of the calculational subtleties that arise when analyzing this situation are considered in Appendix B.

Thus, assuming a plane gravitational wave with constant amplitude (i.e. setting $a(\eta) = a_0$ in (4)), we can explicitly evaluate the integral in (9) to get the resulting phase change due to a gravitational wave
\[ \psi_1(\eta, x^i) = \frac{\omega E h_{0}}{2c} e^{i} e^{k} p_{ik} e^{-i(k_0 \eta - k_i x^i)} \left( \frac{e^{iD(k_0 - k_i e^i) - 1}{i(k_0 - k_i e^i)}}. \right) \] (11)

An interferometer is an experimental device capable of measuring the difference in phase in an electromagnetic wave [17]. The micro-arcsecond precision promised by the upcoming interferometry projects may allow to detect signature of gravitational waves or set upper limits on their magnitude. For this reason, let us switch our attention to the interferometers. The interferometers measure the variation in the phase of the electromagnetic signal received by its base antennae. Let us consider
a long base interferometer setup. In this case there are two antennae separated by a spatial vector \( L^i \). Without losing generality, we can assume that the first of this antennae is located at the coordinate origin. Then, the difference of phase measured by the by the two antennae, due to the gravitational wave influence, is given by

\[
\Delta \psi_1 = \psi_1(\eta, 0) - \psi_1(\eta, L^i). \tag{12}
\]

For the plane electromagnetic wave under consideration, substituting (11) into (12) we arrive at the expression for phase difference measured by the interferometer

\[
\Delta \psi_1 = \frac{\omega_E h_0}{2c} e^{i k^i p_{ik} e^{-ik_0 \eta}} \left[ \frac{e^{iD(k_0 - k^i e^i)} - 1}{i (k_0 - k^i e^i)} \right] \left( 1 - e^{ik_i L^i} \right). \tag{13}
\]

The output of an interferometric measurements is usually quoted in terms of angular resolution. We shall use this convention. The angular resolution of an interferometer corresponding to a phase resolution \( \Delta \psi_1 \) is give by

\[
\Delta \alpha = \frac{c}{L \omega_E} \Delta \psi_1, \tag{14}
\]

where \( L = (\delta_{ij} L^i L^j)^{1/2} \) is the baseline length of the interferometer.

In what follows, we shall be interested in gravitational waves with wavelengths considerably larger that the baseline of the interferometer, i.e. \( L/\lambda_{gw} \ll 1 \). In the case of a space-borne interferometer with base length of \( 10^6 \text{ km} \), this implies \( \nu_{gw} \ll 10^{-1} \text{ Hz} \) for the gravitational wave frequency. For interferometers with a shorter baseline this limit can be significantly larger.

To proceed, let us further introduce unit vectors \( l^i = L^i/L \) and \( \tilde{k}^i = k^i/k \). \( l^i \) is a unit vector pointing in the direction of the interferometer baseline, while \( \tilde{k}^i \) is the unit vector pointing in the direction of the gravitational wave propagation. Assuming \( L/\lambda_{gw} \ll 1 \) in equation (13), we can expand this expression in a series retaining only the lowest order term in \( k^i L^i \). We get

\[
\Delta \alpha = \frac{1}{2} h_0 \ k_i \ l^i \ e^{i k^i p_{ik}} e^{-ik_0 \eta} \left[ \frac{1 - e^{iD(k_0 - k^i e^i)}}{(k_0 - k^i e^i)} \right]. \tag{15}
\]

Up to now we have not posited any relationship between the gravitational wave frequency \( c k_0 \) and the magnitude of the wavenumber \( k \). This relationship (dispersion relation) defines the velocity of a gravitational wave \( v_{gw} = c k_0/k \). In General Relativity this velocity equals the speed of light, but in an alternative theory this might not be the case. In order to analyze the possibility that \( v_{gw} \neq c \), let us use the phenomenological parameter \( \epsilon \), introduced in the previous section

\[
\epsilon = \frac{c - v_{gw}}{c}, \quad \text{where} \quad v_{gw} = \frac{c k_0}{k}. \tag{16}
\]
which characterizes the relative deviation of velocity of gravitational waves from the speed of light. Let us note that, \( \epsilon \) can be related to a non vanishing rest mass of a graviton \( m_g \) through the relation

\[
\epsilon = 1 - \frac{\hbar \omega_{gw}}{\hbar \omega_{gw} + m_g c^2} \approx \frac{m_g c^2}{\hbar \omega_{gw}}. \quad (17)
\]

Returning to equation (15), and substituting the relationship \( k_0 = (1 - \epsilon)k \) into it, we get

\[
\Delta \alpha = \frac{1}{2} h_o e^{ik_p k} \tilde{k}_i l e^{i \epsilon (1-\epsilon) \eta} \left[ \frac{1 - e^{ikD(1-\epsilon)} - \tilde{k}_i e^{i \epsilon}}{(1 - \epsilon - \tilde{k}_i e^{i \epsilon})} \right]. \quad (18)
\]

It is instructive at this point to look more closely at expression (18). We are considering an electromagnetic wave travelling “along” a plane gravitational wave. As follows from this expression, the angular displacement \( |\Delta \alpha| \) is most pronounced when both the waves move in almost parallel directions. This picture is reminiscent of wave surfing, and hence we call this the “surfing” effect. The expression in square brackets becomes large (proportional to \( kD \sim D/\lambda_{gw} \)), when the denominator tends to zero (i.e. \( 1 - \epsilon - \tilde{k}_i e^{i \epsilon} \to 0 \)), leading to a resonance effect. In the case when \( \epsilon = 0 \), this does not lead to significant growth of the angular displacement \( |\Delta \alpha| \), because of the transverse nature of gravitational waves (since \( e^{i \epsilon}p_{ij} \to 0 \) as \( \tilde{k}_i e^{i \epsilon} \to 1 \)). On the other hand, if \( \epsilon \neq 0 \), the expression for \( |\Delta \alpha| \) becomes sufficiently large for a gravitational wave propagating at an angle \( \cos \theta = \tilde{k}_i e^{i \epsilon} \approx (1 - \epsilon) \) to the line of sight. Thus, due to the transverse nature of the gravitational waves this surfing effect is absent if \( \epsilon = 0 \), but can become significant for \( \epsilon \neq 0 \). This effect, as we shall show in the following section, can be used to put stringent constraints on the \( \epsilon \) parameter characterizing the velocity of gravitational waves.

### III. ARBITRARY GRAVITATIONAL WAVE FIELD

In the previous section we considered the case of a single monochromatic gravitational wave. In order to generalize the considerations of the previous section, let us now consider an arbitrary gravitational wave field. This field can be decomposed into spatial Fourier modes

\[
h_{ij}(\eta, x^i) = \int d^3k \sum_{s=1,2} \left[ h_s(k^i, \eta) \tilde{p}_{ij}(k^j) e^{ik_i x^i} + \tilde{h}_s^*(k^i, \eta) \tilde{p}_{ij}(k^j) e^{-ik_i x^i} \right], \quad (19)
\]

where \( d^3k \) denotes the integration over all possible wave vectors, and \( s = 1, 2 \) correspond to the two linearly independent polarizations of a gravitational wave. The mode functions \( h_s(k^i, \eta) \) have the following time evolution

\[
h_s(k^i, \eta) = h_s(k^i) \left( \frac{a_0}{a(\eta)} \right) e^{-ik(1-\epsilon)\eta}, \quad (20)
\]
where \( h_s(k^i) \) is the gravitational wave amplitude at the present time.

Due to the linear nature of the problem, following the decomposition \([19]\), the total angular displacement due to gravitational waves can also be decomposed into Fourier modes in a similar fashion

\[
\Delta \alpha = \int d^3k \sum_{s=1,2} \left[ h_s(k^i) \Delta \tilde{\alpha}_s(k^i) + h_s^*(k^i) \Delta \tilde{\alpha}_s^*(k^i) \right],
\]

(21)

where we have introduced a tilde over \( \alpha \), in the right hand side of the above expression, to indicate the explicit factoring out of the gravitational wave amplitude \( h_s(k^i) \) compared with expression \([18]\).

Using the results of previous subsection (see Eq. \([18]\)), ignoring cosmological evolution for now, the contribution from a single Fourier component \( \tilde{\alpha}_s(k^i) \) is given by

\[
\Delta \tilde{\alpha}_s(k^i) = \frac{1}{2} \tilde{k}_i \rho \epsilon^k \rho^s \epsilon^k \rho^s \tilde{e}^{ik(1-\epsilon)} \left[ 1 - e^{ikD(1-\epsilon-\tilde{k}_i \epsilon)} \right].
\]

(22)

In general for an arbitrary gravitational wave field \([19]\) the angular displacement measured by the interferometer is given by expressions \([21]\) and \([22]\). In practice we do not have the precise information about the gravitational wave field, and are restricted to knowledge of only its statistical properties. Let us assume the following statistical properties for the mode functions \( h_s(k^i) \)

\[
\langle h_s(k^i) \rangle = 0, \quad \langle h_s(k^i) h_s^*(k'^i) \rangle = \frac{P_h(k)}{16\pi k^3} \delta_{ss'} \delta^3(k^i - k'^i),
\]

(23)

where the brackets denote ensemble averaging, and \( P_h(k) \) is the metric power spectrum per logarithmic interval of \( k \). These conditions correspond to a stationary statistically homogeneous and isotropic gravitational wave field.

The statistical properties of \( \Delta \alpha \) follow from the statistical properties of the underlying gravitational wave field \([23]\). Using \([21]\), \([22]\) and \([23]\) after straightforward calculations we get following statistical properties for angular displacement \( \Delta \alpha \):

\[
\langle \Delta \alpha \rangle = 0,
\]

(24a)

\[
\langle \Delta \alpha^2 \rangle = \int \frac{dk}{k} P_h(k) \Delta \tilde{\alpha}^2(k),
\]

(24b)

where we have introduced the transfer function

\[
\Delta \tilde{\alpha}^2(k) = \frac{1}{8\pi} \int d\Omega \sum_s |\Delta \tilde{\alpha}_s(k^i)|^2.
\]

(25)

In the above expression \( d\Omega \) represents integration over the possible directions of g.w. wave (i.e. \( d^3k = k^2 dkd\Omega \)).
FIG. 1: The graphical representation of the various vectors and angles used in the text. Vector $e^i$ is the unit vector along the direction of the electromagnetic wave, unit vector $l^i$ is aligned with the base of the interferometer, and $k^i$ is the wave vector of the gravitational wave.

We shall now proceed to calculate the expression (25) explicitly. Let us introduce a spherical coordinate system $(\theta, \phi)$ related to the spatial coordinates $\{x^i\}$ in the usual manner [24]. Without loss of generality, we can assume that we are looking in the north-pole direction, i.e. $e^i = (0, 0, 1)$. Let us also introduce the quantity $\mu = \cos \theta = e_i \tilde{k}^i$, characterizing the angle between a gravitational wave and the direction of observation. Furthermore for an interferometer, for optimal resolution $l^i$ is aligned perpendicular to $e^i$, thus without loss of generality we can assume $l^i = (1, 0, 0)$. The geometry of the problem is presented in Figure 1. The polarization tensors for gravitational waves have the form $\hat{s}_{ij}(k^i) = (e^i_\theta \pm i e^i_\phi)(e^j_\theta \pm i e^j_\phi)/2$, with $\pm$ corresponding to the two independent circularly polarized degrees of freedom $s = 1, 2$ ($e^i_\theta$ and $e^i_\phi$ are the meridian and azimuthal unit vectors perpendicular to the gravitational wave wavevector $k_i$, for a detailed discussion see for example [12], [25]). Taking into account the relations

$$e^i e^j \hat{s}_{ij} = \frac{1}{2} (1 - \mu^2) e^{\pm 2i \phi}, \quad l_i \tilde{k}^i = (1 - \mu^2)^{1/2} \cos \phi,$$

and substituting (22) into (25), after straightforward manipulations, the expression (25) for the transfer function takes the form:

$$\Delta \alpha^2(k) = \frac{1}{16} \int_{-1}^{+1} d\mu \ (1 - \mu^2)^3 \left[ \sin^2 \left\{ \frac{\pi D}{\lambda_{gw}} \left( 1 - \epsilon - \mu \right) \right\} \right] \left( 1 - \epsilon - \mu \right).$$

The terms in the above integral have a clear physical meaning. The factor $(1 - \mu^2)^3$ is due to the transverse nature of the gravitational waves and the geometry of space interferometry ($(1 - \mu^2)^2$ and $(1 - \mu^2)$ terms correspondingly). The quantity in square brackets sharply peaks at values
\( \mu \approx (1 - \epsilon) \), which is the result of a resonance effect, i.e. what we call the surfing effect, for gravitational waves travelling at an angle \( \cos \theta \approx (1 - \epsilon) \) to the line of sight. Due to the pre-factor \( (1 - \mu^2)^3 \), this resonance does not give a significant contribution for the case \( \epsilon = 0 \). The integrand in expression (26) is plotted for the two cases in Figure 2. In the limit \( \epsilon \to 0 \) and \( D/\lambda_{gw} \to \infty \) we can calculate the transfer function (26) explicitly. Referring the reader to appendix A for details of calculation, let us present the final result below:

\[
\Delta \tilde{\alpha}^2(k) \approx \frac{1}{20} \left[ 1 + 5\pi \epsilon^3 kD \right].
\]  

(27)

As was mentioned previously, when deriving expression (27) we had ignored the cosmological evolution of the gravitational wave amplitude. In Appendix B we derive the expression for the transfer function when the cosmological evolution of gravitational waves is properly taken into account. The resulting expression is as follows:

\[
\Delta \tilde{\alpha}^2(k) \approx \frac{1}{20} \left[ \frac{(1 + z)^2 + 1}{2} + 5\pi \epsilon^3 kD (1 + z) \right]
= \frac{1}{20} \left[ \frac{(1 + z)^2 + 1}{2} + 5\pi \epsilon^3 kL_H (1 + z) \left( 2 - \frac{2}{\sqrt{1 + z}} \right) \right].
\]  

(28)

From expression (28) we can quantify the condition for the resonance to occur by comparing the two terms in the square brackets. The resonance occurs when, in the right side of (28), the second term (resonance term) is larger that the first term (non-resonance term), i.e. when \( \epsilon^3 kL_H \gg 1 \).

As we shall show in the next section, given the planned level of sensitivity for interferometric measurements, the resonance effect allows to place significant upper bounds on the parameter \( \epsilon \).

The redshift factors \( z \), occurring in expression (28), have a clear physical interpretation. The factor \( (1 + z)^2 \) in the non-resonance part of the transfer function occurs because, its main contribution comes from epoch when the electromagnetic radiation was emitted, corresponding to a redshift of \( z \). It is thus sensitive to the gravitational wave power spectrum at the epoch of emission, which was a factor \( (1 + z)^2 \) stronger than today. On the other hand, in the case of the resonance term, the contribution to the transfer function is gained along the path from emitter to observer. This leads to a weaker \( z \) dependence in this term compared with the non-resonance term.

### IV. UPPER LIMIT ON THE VELOCITY OF GRAVITATIONAL WAVES

Let us now consider the implications of the surfing effect for the precision interferometry measurements, and the achievable upper limits on \( \epsilon \). When considering stochastic gravitational wave
FIG. 2: The illustration of the resonance effect, present for $\epsilon \neq 0$. The graphs show integrand in expression (26). For the case $\epsilon \neq 0$ the integrand sharply peaks at angle $\mu \approx (1 - \epsilon)$ (solid red line), while for the case $\epsilon = 0$ the effect is absent (dashed blue line). In the case of $\epsilon \neq 0$, the gravitational waves travelling at an angle $\cos \theta = \approx (1 - \epsilon)$ to the line of sight are the predominant contributors to the surfing effect. The figure on the left shows the integrand for the whole region of $\mu$, while the figure on the right zooms into the region around the resonance.

fields, it is customary to introduce the density parameter $\Omega_{gw}$ to characterize the strength of the gravitational wave field [2], [3], [6]. $\Omega_{gw}$ is related to the power spectrum $P_h(k)$ by the relation

$$
\Omega_{gw}(k) = \frac{\pi^2}{3} \left( \frac{k}{k_H} \right)^2 P_h(k)
$$

where $k_H = 2\pi H_o/c$, and $H_o$ is the present day Hubble constant. The quantity $\Omega_{gw}$ is the present day ratio of energy density of gravitational waves (per unit logarithmic interval in $k$) to the critical density of the Universe $\rho_{crit} = 3c^2 H_o^2 / 8\pi G$.

For simplicity, below we shall assume a simple power law behaviour for density parameter $\Omega_{gw} = \Omega_{gw}(k_o) \cdot (k/k_o)^{n_T}$. Although restricted, this form of spectrum is a good approximation for a large variety of models in gravitational wave frequency range of our interest. For example, this type of a power spectrum arises due to the evolution of relic gravitational waves with a (primordial) spectral index equal to $n_T$, (i.e. $P_h(k)|_{prim} \propto k^{n_T}$). The flat, scale invariant power spectrum (also known as Harrison-Zeldovich power spectrum) corresponds to $n_T = 0$. In general the power law spectrum for $\Omega_{gw}$ just assumes the absence of features in the spectrum of gravitational waves at the wavelengths of our interest.

Let us consider electromagnetic radiation from a distant quasar. Expression (24b) allows us to calculate the expected angular fluctuation in the position of this quasar caused by a stochastic background of gravitational waves. In order to proceed we require to specify the limits of integration
\( k_{\text{min}} \) and \( k_{\text{max}} \) in (24b). \( k_{\text{min}} \) and \( k_{\text{max}} \) determine the frequency range of gravitational waves that can be probed by precision interferometry. The lower limit \( k_{\text{min}} \) is determined by the time duration of observations \( T_{\text{obs}} \), \( k_{\text{min}} \approx 2\pi/cT_{\text{obs}} \). The upper limit \( k_{\text{max}} \approx 2\pi/c\delta t \) is determined by the time resolution of the observations \( \delta t \), and we shall assume \( \delta t \ll T_{\text{obs}} \) (i.e. \( k_{\text{max}} \gg k_{\text{min}} \)). Let \( D \) be the distance to the quasar, which we shall assume is comparable to the Hubble length, i.e. \( D \approx L_H = cH_o^{-1} \). We shall be working under the assumption \( kD = 2\pi D/\lambda_{gw} \gg 1 \), corresponding to the reasonable condition that the gravitational waves of our interest have a wavelengths much shorter than \( L_H \).

As can be seen from expression (27) the behaviour of the transfer function \( \Delta \tilde{\alpha}^2 (k) \) depends on value of the quantity \( 5\pi \epsilon^3 kL_H \). In order to analyze the various possibilities let us introduce

\[
\epsilon_* = (5\pi k_{\text{min}} L_H)^{-1/3} \approx 2.3 \cdot 10^{-4},
\]

where we have assumed \( k_{\text{min}} = 2\pi/cT_{\text{obs}} \), and \( T_{\text{obs}} = 10 \text{ yrs} \). In the above expression, and elsewhere below we set \( H_o = 75 \text{ km/sec/Mpc} \) for numerical evaluations.

For the angular displacement \( <\Delta \alpha^2> \), in the case \( \epsilon \ll \epsilon_* \), substituting (28) into (24b), taking into account the definition (29) and integrating in the limits from \( k_{\text{min}} \) to \( k_{\text{max}} \) we get

\[
<\Delta \alpha^2> = \frac{3}{40\pi^2} \frac{\Omega_{gw}(k_o)}{(1-n_T/2)} \left[ \left( \frac{k_H}{k_{\text{min}}} \right)^2 \left( \frac{k_{\text{min}}}{k_o} \right)^{n_T} - \left( \frac{k_H}{k_{\text{max}}} \right)^2 \left( \frac{k_{\text{max}}}{k_o} \right)^{n_T} \right] \left( \frac{1+z}{2} \right)
\]

\[
\approx \frac{3}{40\pi^2} \frac{\Omega_{gw}(k_{\text{min}})}{(1-n_T/2)} \left( \frac{T_{\text{obs}}}{T_H} \right)^2 \left( \frac{(1+z)^2 + 1}{2} \right), \quad \text{for} \ \epsilon \ll \epsilon_*.
\]

In the opposite case of \( \epsilon \gg \epsilon_* \) we get

\[
<\Delta \alpha^2> = \frac{3\Omega_{gw}(k_o) \epsilon^3}{(1-n_T)} \left[ \left( \frac{k_H}{k_{\text{min}}} \right) \left( \frac{k_{\text{min}}}{k_o} \right)^{n_T} - \left( \frac{k_H}{k_{\text{max}}} \right) \left( \frac{k_{\text{max}}}{k_o} \right)^{n_T} \right] \left( 1+z - \sqrt{1+z} \right)
\]

\[
\approx \frac{3\Omega_{gw}(k_{\text{min}}) \epsilon^3}{(1-n_T)} \left( \frac{T_{\text{obs}}}{T_H} \right) \left( 1+z - \sqrt{1+z} \right), \quad \text{for} \ \epsilon \gg \epsilon_*.
\]

In expressions (31) and (32) \( T_H = H_o^{-1} \) is the Hubble time and \( k_H = 2\pi/L_H \). We have assumed \( k_{\text{max}} \gg k_{\text{min}} \) and used \( k_H/k_{\text{min}} \approx T_{\text{obs}}/T_H \). In the above expressions we restrict our analysis to the case \( n_T < 1 \) which covers most of the practically interesting cases, including \( n_T = 0 \) corresponding to a flat primordial spectrum of relic gravitational waves. In further evaluations below we shall set the redshift \( z = 4 \), which corresponds to a redshift with significant amount of quasar sources available for observations.

The measurement of \( <\Delta \alpha^2> \) for distant quasars by the planned interferometric projects [16], [17] would be able to constrain either \( \Omega_{gw} \) or \( \Omega_{gw} \epsilon^3 \), depending on the value of \( \epsilon \) compared with
A null result in the measurement of $\langle \Delta \alpha^2 \rangle$, in the case $\epsilon \ll \epsilon_*$, would place the following limit on the energy density of gravitational waves $\Omega_{gw}$ (using expression (31))

$$
\Omega_{gw} \leq 4.0 \cdot 10^{-4} \left( \frac{1 - n_T/2}{10 \text{ yrs}} \right)^2 \left( \frac{\Delta \alpha_{rms}}{1 \mu\text{as}} \right)^2 \left( \frac{10 \text{ yrs}}{T_{obs}} \right)^2.
$$

(33)

The above expression serves as the (weakest) upper limit on $\Omega_{gw}$ that can be set by precision interferometry measurements, irrespective of the value $\epsilon$ (since, as will become clearer from the expression below, for values of $\epsilon \gg \epsilon_*$ this upper limit only becomes more stringent). In the case $\epsilon \gg \epsilon_*$, from (32), we get the following upper limit for the quantity $\Omega_{gw}^3$

$$
\Omega_{gw}^3 \leq 3.7 \cdot 10^{-15} \left( \frac{1 - n_T}{10^{-10} \Omega_{gw}} \right) \left( \frac{\Delta \alpha_{rms}}{1 \mu\text{as}} \right)^2 \left( \frac{10 \text{ yrs}}{T_{obs}} \right)^2.
$$

(34)

In expressions (33) and (34) we have introduced $\Delta \alpha_{rms} = \sqrt{\langle \Delta \alpha^2 \rangle}$, which is the root mean square of the angular resolution of the interferometer. This precision is around 1 $\mu\text{as}$ for the currently planned interferometers, and reaches 0.4 $\mu\text{as}$ for the proposed MAXIM x-ray interferometer. It is worth noting that, at this angular resolution, the upper limit on $\Omega_{gw} \lesssim 6.4 \cdot 10^{-5}$ that can be achieved by precision interferometric measurements is comparable to the current limits set by LIGO \cite{27, 28}.

Figure 3 shows the constraints on energy density parameter $\Omega_{gw}$ and the velocity parameter $\epsilon$ achievable with an angular resolution of $\Delta \alpha_{rms} = 0.4 \mu\text{as}$ promised by the MAXIM project \cite{17}. The figure also shows the current constraints on the $\Omega_{gw}$ parameter \cite{27, 28, 29, 30}, along with sensitivity levels of some of the planned experiments \cite{31, 26, 32}.

An independent measurement of $\Omega_{gw}$ (at a level below $\Omega_{gw} \lesssim 10^{-5}$) by ground based interferometers \cite{3}, planned space borne interferometer LISA \cite{26}, or Cosmic Microwave Background anisotropy and polarization measurements \cite{11} would allow to place direct constraints on velocity parameter $\epsilon$. In this case, using (32), we can calculate the upper limit on $\epsilon$ achievable by precision interferometric measurements

$$
\epsilon \leq 3.3 \cdot 10^{-2} \left( 1 - n_T \right) \left( \frac{10^{-10}}{\Omega_{gw}} \right) \left( \frac{\Delta \alpha_{rms}}{1 \mu\text{as}} \right)^2 \left( \frac{10 \text{ yrs}}{T_{obs}} \right)^{\frac{1}{2}}.
$$

(35)

This constraint corresponds to the region $\epsilon > \epsilon_*$ on Figure 3.

In Table I we summarize the predictions for $\Omega_{gw}$, and the corresponding upper limits on $\epsilon$, for some of the viable models that generate a considerable amount of stochastic gravitational wave backgrounds \cite{33, 34, 12}. 

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FIG. 3: The upper limit on the energy density $\Omega_{gw}$ and velocity parameter $\epsilon$ achievable by an interferometric observation of a source at redshift $z = 4$ with resolution $\Delta \alpha_{rms} = 0.4 \mu as$ and $T_{obs} = 10 \ years$. The shaded area shows the region that can be ruled out by precision interferometry together with other existing constraints. The horizontal lines on the graph show various constraints on the density parameter $\Omega_{gw}$ (solid hairy lines - current upper bounds, dashed line - future sensitivity levels).

V. CONCLUSION

Although gravitational waves are yet to be detected, there are currently upper limits placed by observations. The CMB places the most stringent limits on gravitational waves of cosmological origin of $\Omega_{gw} \leq 10^{-14}$ extrapolated to laboratory scale frequencies. It is crucial that this limit depends on extrapolation of data measured at the very long wavelengths (comparable to the Hubble length) down to wavelengths of the order of one light year and much lesser. Further more, these extrapolations assume a spectral index close to zero, i.e. a flat (scale invariant) spectrum of primordial gravitational waves. At frequencies relevant for precision interferometry observations, the most stringent constraint is placed by the Pulsar Timing measurements $\Omega_{gw} < 2 \times 10^{-8}$ at frequencies $10^{-9} - 10^{-7} \ Hz$ [30], and LIGO results $\Omega_{gw} < 6.5 \times 10^{-5}$ at frequencies 51-150 Hz [27],
Theoretical Model | Predicted $\Omega_{gw}$ | Upper Limit on $\epsilon$ | Upper Limit on $m_g$
--- | --- | --- | ---
Relic gravitational waves, $n_T = 0$ | $\Omega_{gw} \approx 10^{-14}$ | $\epsilon \lesssim 0.4$ | $m_g \lesssim 5.2 \cdot 10^{-23}$ eV
Relic gravitational waves, $n_T = 0.2$ at $\nu = 1$ Hz | $\Omega_{gw} \approx 10^{-10}$ | $\epsilon \lesssim 1.8 \cdot 10^{-2}$ | $m_g \lesssim 2.3 \cdot 10^{-24}$ eV
Local Strings | $\Omega_{gw} \approx 10^{-8}$ | $\epsilon \lesssim 3.9 \cdot 10^{-3}$ | $m_g \lesssim 5.1 \cdot 10^{-25}$ eV
Global strings | $\Omega_{gw} \approx 10^{-12}$ | $\epsilon \lesssim 8.4 \cdot 10^{-2}$ | $m_g \lesssim 1.1 \cdot 10^{-23}$ eV
Extended Inflation | $\Omega_{gw} \approx 10^{-9}$ | $\epsilon \lesssim 8.4 \cdot 10^{-3}$ | $m_g \lesssim 1.1 \cdot 10^{-24}$ eV
$1^{st}$ order EW transitions | $\Omega_{gw} \approx 10^{-9}$ | $\epsilon \lesssim 8.4 \cdot 10^{-2}$ | $m_g \lesssim 1.1 \cdot 10^{-24}$ eV

TABLE I: An upper limit on the velocity parameter $\epsilon$, for some viable models that predict a considerable gravitational wave background, that can be placed by interferometric observations with resolution $\Delta \alpha_{rms} = 0.4 \mu$as, and observation time $T_{obs} = 10$ yrs. The last column shows the upper limits on the mass of the graviton $m_g$ (see equation (17)) for a fiducial wavelength $\lambda_{gw} \approx 1$ ly.

As can be seen from Table I there are a host of viable theoretical models that predict gravitational wave backgrounds above the sensitivity levels of planned experiments like Advanced LIGO, LISA and SKA-PTA. If these experiments detect a gravitational wave background, the precision interferometry observations would be able to place strong constraints on $\epsilon \lesssim 10^{-2}$. Let us finally note that, these interferometric measurements would be directly sensitive to gravitational waves with frequencies $\nu_{gw} \lesssim 2\pi/T_{obs} \approx 2 \cdot 10^{-8}$ Hz, which is the frequency region that would be probed by LISA and SKA-PTA.

Along side the candidates for a statistically isotropic gravitational wave background, we can also expect a significant stochastic background of gravitational waves from galactic white dwarf binaries [4] which are expected to be a dominant contribution to “noise” in LISA. Consideration of anisotropic gravitational wave backgrounds requires an approach slightly different from considerations of Section [11], but it is reasonable to assume that these sources could also place considerable upper limits on $\epsilon$.

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APPENDIX A: EVALUATION OF THE TRANSFER FUNCTION

Let us evaluate the integral in expression (26) in the physically interesting case when $\epsilon \to 0$ and $kD \gg 1$. In this case the expression for the transfer function can be separated into two distinctive contributions

$$\Delta \tilde{\alpha}^2(k) = \Delta \tilde{\alpha}^2_{NR}(k) + \Delta \tilde{\alpha}^2_R(k), \quad (A1)$$

where $\Delta \tilde{\alpha}^2_{NR}(k)$ is the non-resonance contribution

$$\Delta \tilde{\alpha}^2_{NR}(k) = \frac{1}{16} \int_{-1}^{1-\epsilon-\Delta\mu} d\mu \left(1 - \mu^2\right)^3 \sin^2\left(\frac{\pi D}{\lambda_{gw}} \frac{1 - \epsilon - \mu}{1 - \epsilon - \mu^2}\right), \quad (A2)$$

and $\Delta \tilde{\alpha}^2_R(k)$ is the resonance contribution

$$\Delta \tilde{\alpha}^2_R(k) = \frac{1}{16} \int_{1-\epsilon+\Delta\mu}^{1-\epsilon+\Delta\mu} d\mu \left(1 - \mu^2\right)^3 \sin^2\left(\frac{\pi D}{\lambda_{gw}} \frac{1 - \epsilon - \mu}{1 - \epsilon - \mu^2}\right), \quad (A3)$$

The quantity $\Delta\mu$ occurring in the limits of integration in the above expressions is fixed by the condition for the resonance to occur. This condition corresponds to the region, around $\mu = 1 - \epsilon$, where the sine function undergoes a few oscillations. Thus $\Delta\mu = N\lambda_{gw}/D = 2\pi N/kD$, where $N$ is the number of oscillations of the sine function, around the point $\mu = 1 - \epsilon$, included in evaluation of the resonance. The value of $N$ is limited by the condition $\Delta\mu = 2\pi N/kD \ll \epsilon$, implying $N \ll \epsilon kD/2\pi$. Since in all our considerations we assume $\epsilon \ll 1$, and $\epsilon^3 kD \gg 1$, the condition imposed on $N$ is consistent with an additional condition $N \gg 1$ that we shall assume.

When evaluating (A2), in the case of $\epsilon \to 0$, we can neglect the second integral in comparison...
with the first. In evaluation of the remaining integral we set $\epsilon = 0$. Thus, we get

$$
\Delta \tilde{\alpha}^2_{NR}(k) \approx \frac{1}{16} \int_{-1}^{1} d\mu \ (1 - \mu) (1 + \mu)^3 \sin^2 \left( \frac{kD}{2} (1 - \mu) \right)
$$

$$
= \frac{1}{32} \int_{-1}^{1} d\mu \ (1 - \mu) (1 + \mu)^3 \left( 1 - \cos (kD (1 - \mu)) \right)
$$

$$
\approx \frac{1}{32} \int_{-1}^{1} d\mu \ (1 - \mu) (1 + \mu)^3 = \frac{1}{20}, \quad (A4)
$$

where, assuming $kD \gg 1$, we have explicitly separate out the rapid oscillatory part and neglect it.

In order to evaluate (A3), in the case of $\epsilon \to 0$ and $kD \gg 1$, it is helpful to notice that the factor $(1 - \mu^2)^3$ in the right side of (A3) is a slowly varying function over the range of integration. Taking this factor (evaluated at $\mu = 1 - \epsilon$) outside the integral we get the following approximation for the resonance part of the transfer function

$$
\Delta \tilde{\alpha}^2_R(k) \approx \frac{1}{2} \epsilon^3 \int_{1-\epsilon+\Delta\mu}^{1-\epsilon+\Delta\mu} d\mu \left[ \sin^2 \left\{ \frac{kD}{2} (1 - \epsilon - \mu) \right\} \right] = \frac{1}{4} \epsilon^3 kD \int_{-N\pi}^{+N\pi} dx \frac{\sin^2 x}{x^2}
$$

$$
\approx \frac{1}{4} \pi \epsilon^3 kD \left( 1 - O \left( \frac{1}{N^2} \right) \right) \approx \frac{1}{4} \pi \epsilon^3 kD. \quad (A5)
$$

Finally, the total transfer function, given by the sum of the non-resonance (A3) and resonance parts (A5), has the following form

$$
\Delta \tilde{\alpha}^2(k) \approx \frac{1}{20} \left[ 1 + 5 \pi \epsilon^3 kD \right]. \quad (A6)
$$

APPENDIX B: THE SURFING EFFECT IN THE PRESENCE OF COSMOLOGICAL EVOLUTION

In Section II when evaluating expression (9) for simplicity and clarity we had neglected cosmological evolution of the gravitational wave amplitude, i.e. we set $a(\eta) = 1$. When we incorporate the cosmological expansion, it no longer becomes possible to evaluate the integral in expression (9) in terms of elementary functions. Although this complicates the calculational aspects of the problem, much of the physical aspects remain the same as discussed in Sections III and III. Introducing

$$
\gamma \equiv k(1 - \epsilon - \tilde{k}_i \epsilon^i), \quad (B1)
$$
the expression (9) (in the case of a matter dominated universe governed by the scale factor (2))
can be rewritten in the following form
\[
\psi_1(\eta, x^i) = \omega E h_o e^{i k p i k} \gamma \eta_o^2 \int_{\gamma(\eta_o - D)} dx \frac{e^{ix}}{x^2}, \tag{B2}
\]
The expression (15) for the angular displacement due to a gravitational wave modifies correspondingly to
\[
\Delta \alpha = -\frac{i}{2} h_o e^{i k p i k} \tilde{k}_r l^r \gamma k \eta_o^2 \int_{\gamma(\eta_o - D)} dx \frac{e^{ix}}{x^2}. \tag{B3}
\]
At this point it is convenient to consider the non-resonance and resonance contributions separately. Let us consider each in turn, beginning with the non-resonance contribution. For this case, \(\gamma \eta_o \gtrsim \gamma(\eta_o - D) \gg 1\), and the integral in expression (B3) can be evaluated asymptotically to give
\[
\Delta \alpha_{NR} \approx \frac{1}{2} h_o e^{i k p i k} \tilde{k}_r l^r \gamma k \eta_o^2 \left( \frac{e^{ix}}{x^2} + O \left( \frac{1}{x^3} \right) \right) \bigg|_{\gamma(\eta_o - D)}.
\]
\[
\approx \frac{1}{2} h_o e^{i k p i k} \tilde{k}_r l^r e^{i \gamma \eta_o} \left( \frac{\eta_o}{\eta_o - D} \right)^2 \left[ \frac{k}{\gamma} \left( e^{i \gamma D} - \left( \frac{\eta_o - D}{\eta_o} \right)^2 \right) \right],
\]
\[
= \frac{1}{2} h_o (1 + z) e^{i k p i k} \tilde{k}_r l^r e^{2i \gamma L H} \left[ e^{2i \gamma L H (1 - \gamma (1 + z)^{-1/2}) - (1 + z)^{-1}} \right], \tag{B4}
\]
where we have used \(\eta_o = 2L_H\) and expression (10) to relate \(D\) and redshift \(z\). The key difference between (B4) and (15) is the presence of the cosmological redshift factor \((1+z)\). This is the reflection of the fact that the non-resonance part of \(\Delta \alpha\) probes the gravitational wave field at emission when the field was a factor \((1+z)\) stronger than at present. The non-resonance contribution to the transfer function (25) can be evaluated in a fashion similar to the evaluation of \(\Delta \tilde{\alpha}_{NR}^2\) in Appendix A. The result is given by
\[
\Delta \tilde{\alpha}_{NR}^2(k) = \frac{1}{4\pi} \int d\Omega \left| \Delta \tilde{\alpha}(k^i) \right|^2 \bigg|_{i=0} = \frac{1}{20} \left[ \frac{(1 + z)^2 + 1}{2} \right]. \tag{B5}
\]
Let us now turn to the resonance contribution in the transfer function. This contribution can be estimated by setting \(\gamma(\eta_o - D) \lesssim \gamma \eta_o \ll 1\). In this limit we can approximate the expression (B3) in the following way
\[
\Delta \alpha_R \bigg|_{\gamma \eta_o \rightarrow 0} \approx \frac{i}{2} h_o e^{i k p i k} \tilde{k}_r l^r \gamma k \eta_o^2 \left( \frac{1}{x} \right) \bigg|_{\gamma(\eta_o - D)}.
\]
\[
= -\frac{i}{2} h_o k D \sqrt{1 + z} e^{i k p i k} \tilde{k}_r l^r. \tag{B6}
\]
FIG. 4: The integrand figuring in the evaluation of the transfer function \( \Delta \hat{\alpha}_R^2(k) \) for the case of resonance, i.e. \( \mu \approx 1 - \epsilon \). The dashed line shows the approximation used in (B7), while the solid line shows the integrand in the exact evaluation. The value of redshift is set to \( z = 4 \). The overall (common to both the curves) normalization has been chosen arbitrarily.

Thus, in comparison with expression (18) taken in the limit \( \gamma D \to 0 \), the expression (B6) has an extra factor \( \sqrt{1 + z} \). Using this approximation we can estimate the resonance part of the transfer function as follows

\[
\Delta \hat{\alpha}_R^2(k) \approx \frac{1}{16}(1 + z) \int_{1 - \epsilon - \Delta \mu}^{1 - \epsilon + \Delta \mu} d\mu \left( 1 - \mu^2 \right)^3 \left[ \sin^2 \left( \frac{\pi D (1 - \epsilon - \mu)}{\omega_{gw}} \right) \right] \left( 1 - \epsilon - \mu \right) \left( 2 - \frac{2}{\sqrt{1 + z}} \right),
\]

(B7)

where the resonance integral has been evaluated in the same manner as in Appendix A. Figure 4 shows the comparison of the approximate integrand used in (B7) and the exact integrand calculated numerically. As can be seen from the figure, expression (B7) gives a good approximation to the exact result.

Combining expressions (B4) and (B7) we get the expression for the total transfer function

\[
\Delta \hat{\alpha}^2(k) \approx \frac{1}{20} \left[ \left( \frac{(1 + z)^2 + 1}{2} \right) + 5\pi (1 + z) \epsilon^3 k D \right]
\]

\[
= \frac{1}{20} \left[ \left( \frac{(1 + z)^2 + 1}{2} \right) + 5\pi \epsilon^3 k L_H (1 + z) \left( 2 - \frac{2}{\sqrt{1 + z}} \right) \right].
\]

(B8)