SHIFTED MOMENTS OF THE RIEMANN ZETA FUNCTION

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Abstract. In this article, we prove that the Riemann hypothesis implies a conjecture of Chandee on shifted moments of the Riemann zeta function. The proof is based on ideas of Harper concerning sharp upper bounds for the $2k$-th moments of the Riemann zeta function on the critical line.

1. Introduction

This article concerns the shifted moments of the Riemann zeta function

\[ I_k(T, \alpha_1, \alpha_2) = \int_0^T |\zeta(1/2 + i(t + \alpha_1))|^k |\zeta(1/2 + i(t + \alpha_2))|^k dt, \]

where $T \geq 1$ and $\alpha_1 := \alpha_1(T), \alpha_2 := \alpha_2(T)$ are real-valued functions satisfying

\[ |\alpha_1|, |\alpha_2| \leq 0.5T. \]

These are generalisations of the $2k$-th moments of the Riemann zeta function

\[ I_k(T) = \int_0^T |\zeta(1/2 + it)|^{2k} dt, \]

since $I_k(T) = I_k(T, 0, 0)$. The theory of the moments of the Riemann zeta function is an important topic in analytic number theory (see the classic books [20], [7], [11], and [17]). Unconditionally, Heap-Soundararajan [5] (for $0 < k < 1$) and Radziwill-Soundararajan [16] (for $k \geq 1$) proved that

\[ I_k(T) \gg T(\log T)^{k^2}. \]

Assuming the Riemann hypothesis, Harper [4] showed that for any $k \geq 0$,

\[ I_k(T) \ll T(\log T)^{k^2}. \]

Harper’s argument builds on the work of Soundararajan [19], who showed that under the the Riemann hypothesis, for any $\varepsilon > 0$, one has

\[ I_k(T) \ll T(\log T)^{k^2 + \varepsilon}. \]

Based on a random matrix model, Keating and Snaith [8] conjectured that for $k \in \mathbb{N}$,

\[ I_k(T) \sim C_k T(\log T)^{k^2}, \]

for a precise constant $C_k$. By the classical works of Hardy-Littlewood [3] and Ingham [6], the asymptotic (1.5) is known, unconditionally, for $k = 1, 2$. Recently, the first author [13] showed that
a certain conjecture for ternary additive divisor sums implies the validity of (1.5) for $k = 3$. In [14], the authors have shown that the Riemann hypothesis and a certain conjecture for quaternary additive divisor sums imply that (1.5) is true in the case $k = 4$. This work [14] crucially uses the bounds for the shifted moments of the zeta function established in Theorem 1.2 below.

In [1], the more general shifted moments

$$M_k(T, \alpha) = \int_0^T |\zeta(\frac{1}{2} + it + \alpha)|^{2k_1} \cdots |\zeta(\frac{1}{2} + i(t + \alpha_m))|^{2k_m} dt,$$

where $k = (k_1, \ldots, k_m) \in (\mathbb{R}_{>0})^m$ and $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, were introduced. Chandee [1] introduced Theorems 1.1 and 1.2] proved the following upper and lower bounds for $I_k(T, \alpha)$.

**Theorem 1.1** (Chandee). Let $k_i$ be positive real numbers. Let $\alpha_i = \alpha_i(T)$ be real-valued functions of $T$ such that $\alpha_i = o(T)$. Assume that $\lim_{T \to \infty} \alpha_i \log T$ and $\lim_{T \to \infty} (\alpha_i - \alpha_j) \log T$ exist or equal $\pm \infty$. Assume that for $i \neq j$, $\alpha_i \neq \alpha_j$ and $\alpha_i - \alpha_j = O(1)$. Then the Riemann Hypothesis implies that for $T$ sufficiently large, one has

$$M_k(T, \alpha) \ll_{k, \varepsilon} T(\log T)^{k_1^2 + \cdots + k_m^2 + \varepsilon} \prod_{i < j} \left( \min \left\{ \frac{1}{|\alpha_i - \alpha_j|}, \log T \right\} \right)^{2k_i k_j}.$$

Furthermore, if $k_i$ are positive integers, then for $T$ sufficiently large, unconditionally, one has

$$M_k(T, \alpha) \gg_{k, \beta} T(\log T)^{k_1^2 + \cdots + k_m^2} \prod_{i < j} \left( \min \left\{ \frac{1}{|\alpha_i - \alpha_j|}, \log T \right\} \right)^{2k_i k_j},$$

where

$$\beta = \max \left( \{i, j) \ | |\alpha_i - \alpha_j| = O(1/\log T) \} \right) \left\{ \lim_{T \to \infty} |\alpha_i - \alpha_j| \log T \right\}.$$

For the upper bound, Chandee used the techniques of Soundararajan [19]; for the lower bound, Chandee’s argument is based on the work of Rudnick and Soundararajan [15].

Based on Keating and Snaith’s random matrix model [8], Chandee [1, Conjecture 1] made the following conjecture on shifted moments that generalised a conjecture of Kösters [10] as follows:

**Conjecture 1** (Chandee). Let $k \in \mathbb{N}$ and let $\alpha = (\alpha_1, \alpha_2)$ be as in Theorem 1.1. Then one has

$$I_k(T, \alpha_1, \alpha_2) \left\{ \begin{array}{ll}
\asymp_k T(\log T)^{k^2} & \text{if } \lim_{T \to \infty} |\alpha_1 - \alpha_2| \log T = 0, \\
\asymp_{k, \varepsilon} T(\log T)^{k^2} & \text{if } \lim_{T \to \infty} |\alpha_1 - \alpha_2| \log T = c \neq 0, \\
\asymp_k T \left( \frac{\log T}{|\alpha_1 - \alpha_2|} \right)^{k^2} & \text{if } \lim_{T \to \infty} |\alpha_1 - \alpha_2| \log T = \infty.
\end{array} \right.$$

Note that for any positive real $k$, $M_k(T, \alpha) = I_{2k}(T, \alpha_1, \alpha_2)$ for $k = (k, k)$ and $\alpha = (\alpha_1, \alpha_2)$. Therefore, Theorem 1.1 of Chandee has established the conjectured lower bound for $I_k(T, \alpha_1, \alpha_2)$. It remains to prove the sharp upper bound for $I_k(T, \alpha_1, \alpha_2)$ in order to establish Conjecture 1.1. In this article, assuming the Riemann hypothesis, we establish Chandee’s conjecture by proving the following theorem.

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Theorem 1.2. Let \( k \geq 1 \) be real. Let \( \alpha_1 \) and \( \alpha_2 \) be real-valued functions \( \alpha_i = \alpha_i(T) \) of \( T \) which satisfy the bound (1.2) and
\[
|\alpha_1 + \alpha_2| \leq T^{0.6}.
\]
Then the Riemann hypothesis implies that for \( T \) sufficiently large, we have
\[
I_k(T, \alpha_1, \alpha_2) \ll_k T (\log T)^{k^2} F(T, \alpha_1, \alpha_2)^{k^2},
\]
where \( F(T, \alpha_1, \alpha_2) \) is defined by
\[
F(T, \alpha_1, \alpha_2) := \begin{cases} 
\min \left\{ \frac{1}{|\alpha_1 - \alpha_2|}, \log T \right\} & \text{if } |\alpha_1 - \alpha_2| \leq \frac{1}{100}; \\
\log(2 + |\alpha_1 - \alpha_2|) & \text{if } |\alpha_1 - \alpha_2| > \frac{1}{100}.
\end{cases}
\]
We establish this result by following the breakthrough work of Harper [4].

Remarks.

(1) Theorem 1.2 contains Harper’s result (1.3) as a special case. A key point is that Harper’s method can be modified so that the argument of [4] still works when the shifts \( \alpha_1, \alpha_2 \) are introduced in (1.1). One reason the argument works is that we are able to make use of the trigonometric identity
\[
\cos(\theta_1) + \cos(\theta_2) = 2 \cos\left(\frac{\theta_1 + \theta_2}{2}\right) \cos\left(\frac{\theta_1 - \theta_2}{2}\right)
\]
in (3.5) below.

(2) It is natural to ask whether Theorem 1.2 can be extended to the more general moments \( M_k(T, \alpha) \) where the components of \( k = (k_1, \ldots, k_m) \) are not necessarily equal and \( m \geq 2 \).

(3) In this theorem and throughout this article, whenever we write “sufficiently large \( T \)”, we mean that there exists \( T_0 := T_0(k) \) a positive parameter depending on \( k \) such that \( T \geq T_0 \).

Conventions and notation. In this article, given two functions \( f(x) \) and \( g(x) \), we shall interchangeably use the notation \( f(x) = O(g(x)) \), \( f(x) \ll g(x) \), and \( g(x) \gg f(x) \) to mean that there is \( M > 0 \) such that \( |f(x)| \leq M|g(x)| \) for sufficiently large \( x \). Given fixed parameters \( \ell_1, \ldots, \ell_r \), the notation \( f(x) \ll_{\ell_1, \ldots, \ell_r} g(x) \) means that the \( |f(x)| \leq Mg(x) \) where \( M = M(\ell_1, \ldots, \ell_r) \) depends on the parameters \( \ell_1, \ldots, \ell_r \). The letter \( p \) will always denote a prime number. In addition, \( p_i, p_i' \), and \( q_i \) with \( i \in \mathbb{N} \) shall denote prime numbers.

2. Some tools

We shall require the following tools, which are fundamental for the argument. Firstly, by a minor modification of the main Proposition of [19] (see also [4] Proposition 1), we have the following proposition providing an upper bound for the Riemann zeta function.

Proposition 2.1. Assume that the Riemann hypothesis holds. Let \( \lambda_0 = 0.491 \cdots \) denote the unique positive solution of \( e^{-\lambda_0} = \lambda_0 + \lambda_0^2/2 \). Let \( T \) be large. Then for \( \lambda \geq \lambda_0, 2 \leq x \leq T^2 \) and
Let \( n = p_1^{a_1} \cdots p_r^{a_r} \), where \( p_i \) are distinct primes, and \( a_i \in \mathbb{N} \). Then for \( T \) large, one has
\[
\int_T^{2T} \prod_{i=1}^r (\cos(t \log p_i))^{a_i} dt = Tg(n) + O(n),
\]
where the implied constant is absolute, and
\[
g(n) = \prod_{i=1}^r \frac{1}{2^{a_i} ((a_i/2)!)^2}
\]
if every \( a_i \) is even, and \( g(n) = 0 \) otherwise. Consequently, for \( T \) large and any real number \( \gamma \), we have
\[
\int_T^{2T} \prod_{i=1}^r (\cos((t + \gamma) \log p_i))^{a_i} dt = (T + \gamma)g(n) + O(|\gamma|) + O(n),
\]
where the implied constants are absolute.

We shall also need the following further variant of Lemma 4 of Radziwiłł.

**Lemma 2.3.** Let \( n = p_1^{a_1} \cdots p_r^{a_r} p_{r+1}^{a_{r+1}} \cdots p_s^{a_s} \), where \( p_i \) are distinct primes, and \( a_i \in \mathbb{N} \). Then we have
\[
\int_T^{2T} \prod_{1 \leq i \leq r} (\cos(t \log p_i))^{a_i} \prod_{r+1 \leq i \leq s} (\cos(2t \log p_i))^{a_i} dt = Tg(n) + O((p_1^{a_1} \cdots p_r^{a_r} \cdot p_{r+1}^{2a_{r+1}} \cdots p_s^{2a_s})),
\]
where the implied constant is absolute. Consequently, for any real \( \gamma \), we have
\[
\int_T^{2T} \prod_{i=1}^r (\cos((t + \gamma) \log p_i))^{a_i} \prod_{r+1 \leq i \leq s} (\cos((2(t + \gamma) \log p_i))^{a_i} dt
= (T + \gamma)g(n) + O(|\gamma|) + O((p_1^{a_1} \cdots p_r^{a_r} \cdot p_{r+1}^{2a_{r+1}} \cdots p_s^{2a_s})),
\]
where the implied constants are absolute.

**Proof.** Following Radziwiłł (Lemma 4) of [15], for \( c \in \mathbb{N} \), we can write
\[
(\cos(ct \log p_i))^{a_i} = \frac{1}{2^{a_i}} \left( e^{ict \log p_i} + e^{-ict \log p_i} \right)^{a_i} = \frac{1}{2^{a_i}} \left( \frac{a_i}{a_i/2} + \sum_{a_i/2 \neq \ell_i \leq a_i} \frac{1}{2^{a_i}} \left( \ell_i \right) e^{(a_i-2\ell_i)ct \log p_i} \right),
\]
where \((a_i/2) = 0\) if \(a_i/2\) is not a positive integer. Hence, setting \(c_i = 1\) for \(1 \leq i \leq r\) and \(c_i = 2\) for \(r + 1 \leq i \leq s\), we obtain
\[
\prod_{1 \leq i \leq s} (\cos(c_i t \log p_i))^{a_i} = \prod_{1 \leq i \leq s} \left( \frac{1}{2a_i} \left( \frac{a_i}{a_i/2} \right) + \sum_{a_i/2 \neq \ell_i \leq a_i} \frac{1}{2a_i} \left( \frac{a_i}{\ell_i} \right) e^{i(\ell_i/a_i - 2\ell_i) c_i t \log p_i} \right) = g(n) + \sum'_{\ell_1, \ldots, \ell_s} \prod_{1 \leq i \leq s} \frac{1}{2a_i} \left( \frac{a_i}{\ell_i} \right) e^{i(\ell_i/a_i - 2\ell_i) c_i t \log p_i},
\]
where the primed sum is over \((\ell_1, \ldots, \ell_s) \neq (\frac{a_1}{2}, \ldots, \frac{a_s}{2})\) such that \(1 \leq \ell_j \leq a_j\) for every \(1 \leq j \leq s\). Thus, we deduce
\[
(\text{2.1}) \int_T^{2T} \prod_{1 \leq i \leq r} (\cos(t \log p_i))^{a_i} \prod_{r+1 \leq i \leq s} (\cos(2t \log p_i))^{a_i} dt = T g(n) + \sum'_{\ell_1, \ldots, \ell_s} \prod_{1 \leq i \leq s} \frac{1}{2a_i} \left( \frac{a_i}{\ell_i} \right) t \int_T^{2T} (*) dt.
\]
The integrand \((*)\) is
\[
\exp(it(b_1 \log p_1 + \cdots + b_r \log p_r + 2b_{r+1} \log p_{r+1} + \cdots + 2b_s \log p_s)),
\]
where \(b_i = a_i - 2\ell_i\). (Note that, as later, \(b_1, \ldots, b_s\) cannot be all zero, and \(|b_i| \leq a_i\).) We then see
\[
\left| \int_T^{2T} (*) dt \right| \leq \frac{2}{|b_1 \log p_1 + \cdots + b_r \log p_r + 2b_{r+1} \log p_{r+1} + \cdots + 2b_s \log p_s|}.
\]
(Note that the denominator is non-zero since \((b_1, \ldots, b_s) \neq (0, \ldots, 0)\) and \(p_1, \ldots, p_s\) are distinct.)
Grouping together those terms with \(b_i > 0\) and \(b_i < 0\), respectively, we can write
\[
|b_1 \log p_1 + \cdots + b_r \log p_r + 2b_{r+1} \log p_{r+1} + \cdots + 2b_s \log p_s| = | \log(M/N)|,
\]
where \(M \neq N\) are positive integers. Without loss of generality, we may assume \(M > N\) and obtain
\[
| \log(M/N)| = \log(M/N) \geq \log \left( \frac{N + 1}{N} \right) = \log \left( 1 + \frac{1}{N} \right) \geq \frac{1}{2N} \geq \frac{1}{2\sum_{i=1}^{a_r} (p_i^{a_i} \cdots p_r^{a_r} / p_{r+1}^{2a_{r+1}} \cdots p_s^{2a_s})}.
\]
Therefore, the primed sum in (2.1) is
\[
\ll \sum'_{\ell_1, \ldots, \ell_s} \prod_{1 \leq i \leq s} \frac{1}{2a_i} \left( \frac{a_i}{\ell_i} \right) \cdot (p_1^{a_1} \cdots p_r^{a_r}) \cdot (p_{r+1}^{2a_{r+1}} \cdots p_s^{2a_s}).
\]
Finally, observing that
\[
\sum'_{\ell_1, \ldots, \ell_s} \prod_{1 \leq i \leq s} \frac{1}{2a_i} \left( \frac{a_i}{\ell_i} \right) \leq \prod_{1 \leq i \leq s} \sum_{0 \leq \ell_i \leq a_i} \frac{1}{2a_i} \left( \frac{a_i}{\ell_i} \right) = \prod_{1 \leq i \leq s} \frac{1}{2a_i} (1 + 1)^{a_i} = 1,
\]
we complete the proof. \(\square\)

Lastly, we recall the following variant of Mertens’ estimate (see, e.g., [2, p. 57] or [12, Lemma 2.9]).
Lemma 2.4. Let $a$ and $z \geq 1$ be real numbers. Then one has

\[
\sum_{p \leq z} \frac{\cos(a \log p)}{p} \begin{cases} = \log \left( \min \left\{ \frac{1}{|a|}, \log z \right\} \right) + O(1) & \text{if } |a| \leq \frac{1}{100}; \\ \leq \log \log(2 + |a|) + O(1) & \text{if } |a| > \frac{1}{100}, \end{cases}
\]

where the implied constants are absolute.

3. Setup and outline of the proof of Theorem 1.2

The goal of this section is to prove Theorem 1.2. To do so, we follow closely Harper [4]. We let $\beta_0 = 0$ and

$$\beta_i = \frac{20^{i-1}}{(\log \log T)^2}$$

for every integer $i \geq 1$. Define $J = J_{k,T} = 1 + \max \{ i \mid \beta_i \leq e^{-1000k} \}$. For $1 \leq i \leq j \leq J$, we set

$$G_{i,j}(t) = G_{i,j,T,\alpha_1,\alpha_2}(t) = \sum_{T^{\beta_{i-1}} < p \leq T^{\beta_i}} \frac{\cos\left(\frac{1}{2} (\alpha_1 - \alpha_2) \log p \right)}{p^{\frac{1}{2} + \frac{1}{2} \log T^{j+\frac{i}{2}(\alpha_1 + \alpha_2)}}} \frac{\log(T^{\beta_j}/p)}{\log T^{\beta_j}}.$$

For $1 \leq i \leq J$, we set

$$F_i(t) = G_{i,J}(t) = \sum_{T^{\beta_{i-1}} < p \leq T^{\beta_i}} \frac{\cos\left(\frac{1}{2} (\alpha_1 - \alpha_2) \log p \right)}{p^{\frac{1}{2} + \frac{1}{2} \log T^{j+\frac{i}{2}(\alpha_1 + \alpha_2)}}} \frac{\log(T^{\beta_j}/p)}{\log T^{\beta_j}}.$$

We define

$$S(0) = S_{T,\alpha_1,\alpha_2}(0) := \left\{ t \in [T, 2T] \mid |\Re G_{1,\ell}(t)| > \beta_1^{-3/4} \text{ for some } 1 \leq \ell \leq J \right\}.$$

For $1 \leq j \leq J - 1$, we define

$$S(j) = S_{k,T,\alpha_1,\alpha_2}(j)$$

as

$$: = \left\{ t \in [T, 2T] \mid |\Re G_{i,\ell}(t)| \leq \beta_i^{-3/4} \text{ for every } (i, \ell) \in \mathbb{N}^2 \text{ such that } 1 \leq i \leq j \text{ and } i \leq \ell \leq J, \right.$$ \hspace{1cm} and 

$$|\Re G_{j+1,\ell'}(t)| > \beta_{j+1}^{-3/4} \text{ for some } j+1 \leq \ell' \leq J \right\}.$$

Finally, we set

$$\mathcal{J} = J_{k,T,\alpha_1,\alpha_2} := \left\{ t \in [T, 2T] \mid |\Re F_i(t)| \leq \beta_i^{-3/4} \text{ for every } 1 \leq i \leq J \right\}.$$

Note that $\beta_{j+1} \leq \beta_j \leq 20e^{-1000k}$ for any $1 \leq j \leq J - 1$.

Observe

$$[T, 2T] = \bigcup_{j=0}^{J-1} S(j) \cup \mathcal{J}.$$
Thus, in order to prove Theorem 1.2 it is sufficient to prove

\[ (3.3) \]
\[
\begin{align*}
\sum_{j=0}^{j=\ell} \int_{t^2 \in \mathbb{S}} |\zeta(\frac{1}{2} + i(t + \alpha_1))|^k |\zeta(\frac{1}{2} + i(t + \alpha_2))|^k \, dt + \int_{t \in \mathbb{T}} |\zeta(\frac{1}{2} + i(t + \alpha_1))|^k |\zeta(\frac{1}{2} + i(t + \alpha_2))|^k \, dt
\end{align*}
\]
\[
\ll T (\log T)^{1/2} F(T, \alpha_1, \alpha_2)^{1/2}.
\]

Applying Proposition 2.1 with \( \lambda = 1 \), for sufficiently large \( T \), \( 2 \leq x \leq T^2 \), and \( t \in [T, 2T] \), we have

\[ \log |\zeta(\frac{1}{2} + i(t + \alpha_i))| \]
\[
\leq \Re \left( \sum_{p \leq x} \frac{1}{p^{1 + \frac{1}{2} \log x} + i(t + \alpha_1)} \log(x/p) \log x + \sum_{p \leq x \min \{ \sqrt{T}, \log T \}} \frac{1}{2p^{1+2i(t+\alpha_1)}} \right) + \log T \log x + O(1).
\]

We further note that the “main term” for the upper bound of \( \log(|\zeta(\frac{1}{2} + i(t + \alpha_1))|^k |\zeta(\frac{1}{2} + i(t + \alpha_2))|^k) \) derived from (3.4) is

\[ (3.5) \]
\[
\begin{align*}
k \Re & \sum_{p \leq x} \frac{1}{p^{1 + \frac{1}{2} \log x} + i(t + \alpha_1)} \log(x/p) \log x + k \Re \sum_{p \leq x} \frac{1}{p^{1 + \frac{1}{2} \log x} + i(t + \alpha_2)} \log(x/p) \\
& = k \sum_{p \leq x} \frac{\cos((t + \alpha_1) \log p) \log(x/p)}{p^{1 + \frac{1}{2} \log x} + i(t + \alpha_1)} \log x + k \sum_{p \leq x} \frac{\cos((t + \alpha_2) \log p) \log(x/p)}{p^{1 + \frac{1}{2} \log x} + i(t + \alpha_2)} \log x \\
& = k \Re \sum_{p \leq x} \frac{\cos((\frac{1}{2} \alpha_1 - \alpha_2) \log p) \log(x/p)}{p^{1 + \frac{1}{2} \log x} + i(t + \frac{1}{2}(\alpha_1 + \alpha_2))} \log x,
\end{align*}
\]

where we have made use of the trigonometric identity \( \Re \). Arguing similarly for the second sum in (3.4), we arrive at

\[ (3.6) \]
\[
\log(|\zeta(\frac{1}{2} + i(t + \alpha_1))|^k |\zeta(\frac{1}{2} + i(t + \alpha_2))|^k)
\]
\[
\leq 2k \Re \left( \sum_{p \leq x} \frac{\cos((\frac{1}{2} \alpha_1 - \alpha_2) \log p) \log(x/p)}{p^{1 + \frac{1}{2} \log x} + i(t + \frac{1}{2}(\alpha_1 + \alpha_2))} \log x + \sum_{p \leq \min \{ \sqrt{T}, \log T \}} \frac{\cos((\alpha_1 - \alpha_2) \log p)}{2p^{1+2i(t+\alpha_1+\alpha_2)}} \right) + 2k \log \frac{T}{\log x} + O(k).
\]

Theorem 1.2 will be deduced from the following three lemmata.

**Lemma 3.1.** In the notation and assumption as above and Theorem 1.2, for any sufficiently large \( T \), we have

\[ (3.7) \]
\[
\int_{t \in \mathbb{T}} \exp \left( 2k \Re \sum_{p \leq 2p_T} \cos((\alpha_1 - \alpha_2) \log p) \log(T^{\beta_1}/p) \log(T^{\beta_2}) \right) \, dt \ll k \, T (\log T)^{1/2} \left( F(T, \alpha_1, \alpha_2) \right)^{1/2},
\]

where \( F(T, \alpha_1, \alpha_2) \) is defined in (1.7).
Lemma 3.2. In the notation and assumption as above, let $T$ be sufficiently large. Then we have
\[
\text{meas}(S(0)) \ll_k T e^{-\left(\log \log T\right)^2/10}.
\]
In addition, for $1 \leq j \leq J - 1$, we have
\[
\int_{t \in S(j)} \exp \left(2k\Re \left( \sum_{p \leq T^{\beta_j}} \frac{\cos \left( \frac{1}{2} (\alpha_1 - \alpha_2) \log p \right) \log (T^{\beta_j} / p)}{p^{1/2 + \beta_j \log T + i(t + \frac{1}{2}(\alpha_1 + \alpha_2))} \log T^{\beta_j}} \right) dt \ll_k T^{(\log T)^2/2} \mathcal{F}(T, \alpha_1, \alpha_2) \frac{t^2}{\beta_j + 1} \exp \left( \frac{\log (1/\beta_j)}{21 \beta_j + 1} \right).
\]

Lemma 3.3. In the notation and assumption as above and Theorem 1.2, we have
\[
\int_{t \in S(j)} \exp \left(2k\Re \left( \sum_{p \leq T^{\beta_j}} \frac{\cos \left( \frac{1}{2} (\alpha_1 - \alpha_2) \log p \right) \log (T^{\beta_j} / p)}{p^{1/2 + \beta_j \log T + i(t + \frac{1}{2}(\alpha_1 + \alpha_2))} \log T^{\beta_j}} + \sum_{p \leq \log T} \frac{\cos \left( (\alpha_1 - \alpha_2) \log p \right)}{2p^{1+i(t+(\alpha_1-\alpha_2))}} \right) dt \ll_k T^{(\log T)^2/2} \mathcal{F}(T, \alpha_1, \alpha_2) \frac{t^2}{\beta_j + 1} \exp \left( \frac{\log (1/\beta_j)}{21 \beta_j + 1} \right).
\]
and for $1 \leq j \leq J - 1$
\[
\int_{t \in S(j)} \exp \left(2k\Re \left( \sum_{p \leq T^{\beta_j}} \frac{\cos \left( \frac{1}{2} (\alpha_1 - \alpha_2) \log p \right) \log (T^{\beta_j} / p)}{p^{1/2 + \beta_j \log T + i(t + \frac{1}{2}(\alpha_1 + \alpha_2))} \log T^{\beta_j}} + \sum_{p \leq \log T} \frac{\cos \left( (\alpha_1 - \alpha_2) \log p \right)}{2p^{1+i(t+(\alpha_1-\alpha_2))}} \right) dt \ll_k T^{(\log T)^2/2} \mathcal{F}(T, \alpha_1, \alpha_2) \frac{t^2}{\beta_j + 1} \exp \left( \frac{\log (1/\beta_j)}{21 \beta_j + 1} \right).
\]

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We must show inequality (3.3) holds. It suffices to show that each of the two terms on the left hand side of (3.3) is $\ll T^{(\log T)^2/2} \mathcal{F}(T, \alpha_1, \alpha_2) \frac{t^2}{\beta_j + 1}$. By (3.6), we know that $\log(|\zeta(\frac{1}{2} + it + \alpha_1)|^k |\zeta(\frac{1}{2} + it + \alpha_2)|^k)$ is at most
\[
2k\Re \left( \sum_{p \leq T^{\beta_j}} \frac{\cos \left( \frac{1}{2} (\alpha_1 - \alpha_2) \log p \right) \log (T^{\beta_j} / p)}{p^{1/2 + \beta_j \log T + i(t + \frac{1}{2}(\alpha_1 + \alpha_2))} \log T^{\beta_j}} + \sum_{p \leq \log T} \frac{\cos \left( (\alpha_1 - \alpha_2) \log p \right)}{2p^{1+i(t+(\alpha_1-\alpha_2))}} \right) + \frac{2k}{\beta_j} + O(k).
\]
Hence, (3.9) of Lemma 3.3 implies
\[
\int_{t \in \mathcal{F}} |\zeta(\frac{1}{2} + it + \alpha_1)|^k |\zeta(\frac{1}{2} + it + \alpha_2)|^k dt \ll \int_{t \in \mathcal{F}} \exp \left(2k\Re \left( \sum_{p \leq T^{\beta_j}} \frac{\cos \left( \frac{1}{2} (\alpha_1 - \alpha_2) \log p \right) \log (T^{\beta_j} / p)}{p^{1/2 + \beta_j \log T + i(t + \frac{1}{2}(\alpha_1 + \alpha_2))} \log T^{\beta_j}} + \sum_{p \leq \log T} \frac{\cos \left( (\alpha_1 - \alpha_2) \log p \right)}{2p^{1+i(t+(\alpha_1-\alpha_2))}} \right) dt \times e^{2k/\beta_j + O(k)} \ll_k T^{(\log T)^2/2} \mathcal{F}(T, \alpha_1, \alpha_2) \frac{t^2}{\beta_j + 1}.
\]
Similarly, for $1 \leq j \leq J - 1$, we can bound $\log(|\zeta(1/2 + i(t + \alpha_1))|^{k} |\zeta(1/2 + i(t + \alpha_2))|^{k})$ above by

$$2k\Re \left( \sum_{p \leq T^{\beta_j/2}} \cos \left( \frac{1}{2} (\alpha_1 - \alpha_2) \log p \right) \log(T^{\beta_j}/p) \right) + \sum_{p \leq \log T} \frac{\cos((\alpha_1 - \alpha_2) \log p)}{2p^{1+i(1/(\alpha_1+\alpha_2))}} + \frac{2k}{\beta_j} + O(k).$$

It then follows from Lemma 3.3 that

$$\int_{s \in \mathbb{S}(j)} |\zeta(1/2 + i(t + \alpha_1))|^{k} |\zeta(1/2 + i(t + \alpha_2))|^{k} dt \ll e^{2k/\beta_j} \cdot e^{-(21\beta_j+1) \cdot \log(1/(\beta_j+1))} T \cdot (\log T)^{2} \mathcal{F}(T, \alpha_1, \alpha_2)^{1/2}.$$ 

Since $20\beta_j = \beta_{j+1} \leq \beta_1 \leq 20 e^{-1000k}$, $\log(1/(\beta_j+1)) \geq 900k$, and

$$e^{2k/\beta_j} \cdot e^{-(21\beta_j+1) \cdot \log(1/(\beta_j+1))} = e^{2k/\beta_j} \cdot (\log(1/(\beta_j+1)))/200j \ll e^{-0.1k/\beta_j}.$$ 

Observe that $J \leq \frac{2}{\log 20} \log \log T$ and

$$\sum_{j=1}^{J-1} e^{-0.1k/\beta_j} = \sum_{j=1}^{J-1} e^{-2k(\log \log T)^{2}/20^{j}} \leq e^{-2k(\log \log T)^{2}} + \int_{1}^{2/(\log 20)^{2}} e^{-2k(\log \log T)^{2}/20^{x}} dx.$$ 

By the change of variables $20^{-x} = u$ (with $dx = \frac{-1}{\log 20} du$), we see that the integral above equals

$$e^{-2k(\log \log T)^{2}} \int_{1}^{1/20} \frac{du}{u} \ll (\log T)^{2} \int_{1}^{1/20} e^{-2k(\log \log T)^{2} u} du \ll e^{-2k/(2k)}.$$ 

Combining (3.12) and (3.13), we arrive at

$$\sum_{j=1}^{J-1} \int_{s \in \mathbb{S}(j)} |\zeta(1/2 + i(t + \alpha_1))|^{k} |\zeta(1/2 + i(t + \alpha_2))|^{k} dt \ll T(\log T)^{k^{2}} \mathcal{F}(T, \alpha_1, \alpha_2)^{1/2}.$$ 

Finally, for $j = 0$, by the Cauchy-Schwarz inequality and Lemma 3.2 we have

$$\int_{s \in \mathbb{S}(0)} |\zeta(1/2 + i(t + \alpha_1))|^{k} |\zeta(1/2 + i(t + \alpha_2))|^{k} dt \leq \text{meas}(\mathbb{S}(0))^{1/2} \left( \int_{T}^{2T} |\zeta(1/2 + i(t + \alpha_1))|^{2k} |\zeta(1/2 + i(t + \alpha_2))|^{2k} dt \right)^{1/2}.$$ 

Using the Cauchy-Schwarz inequality again and the upper bound (1.4) with $\varepsilon = 1$, we see

$$\int_{T}^{2T} |\zeta(1/2 + i(t + \alpha_1))|^{2k} |\zeta(1/2 + i(t + \alpha_2))|^{2k} dt \ll \left( \int_{T}^{2T} |\zeta(1/2 + i(t + \alpha_1))|^{4k} dt \right)^{1/2} \left( \int_{T}^{2T} |\zeta(1/2 + i(t + \alpha_2))|^{4k} dt \right)^{1/2} \ll T(\log T)^{4k^{2}+1}.$$ 

Hence, combined with (3.15), we derive

$$\int_{s \in \mathbb{S}(0)} |\zeta(1/2 + i(t + \alpha_1))|^{k} |\zeta(1/2 + i(t + \alpha_2))|^{k} dt \ll \sqrt{T} e^{-(\log \log T)^{2}/20} \cdot \sqrt{T}(\log T)^{2k^{2}+1/2} \ll T.$$
Therefore, by combining inequalities (3.3), (3.11), (3.14), and (3.16) the proof of Theorem 1.2 is complete.

\[ \square \]

4. Proof of Lemma 3.1

Observe that

\[ \sum_{p \leq T^{\beta_j}} \frac{\cos(\frac{1}{2}\alpha_1 \log p)}{\frac{1}{2} + \frac{1}{\beta_j} \log \frac{p}{2} + i(t + \frac{1}{2}(\alpha_1 + \alpha_2))} \frac{\log(T^{\beta_j}/p)}{\log T^{\beta_j}} = \sum_{i=1}^{3} F_i(t) \]

where \( F_i \) is defined by (3.1). By (4.1), we have

\[ \int_{t \in \mathcal{T}} \exp \left( \frac{2k\Re}{} \sum_{p \leq T^{\beta_j}} \frac{\cos(\frac{1}{2}(\alpha_1 - \alpha_2) \log p)}{\frac{1}{2} + \frac{1}{\beta_j} \log \frac{p}{2} + i(t + \frac{1}{2}(\alpha_1 + \alpha_2))} \frac{\log(T^{\beta_j}/p)}{\log T^{\beta_j}} \right) dt = \prod_{1 \leq i \leq 3} \int_{t \in \mathcal{T}} \exp(k\Re F_i(t))^2 dt \]

where we recall \( \mathcal{T} \) is defined in (3.2). Next, note that

\[ \int_{t \in \mathcal{T}} \prod_{1 \leq i \leq 3} \exp(k\Re F_i(t))^2 dt \leq \mathcal{J} := \int_{1 \leq i \leq 3} \left( \sum_{0 \leq j \leq 100k\beta_i^{-3/4}} \frac{(k\Re F_i(t))^2}{j!} \right)^2 dt. \]

This inequality establishes that each factor \( \exp(k\Re F_i(t)) \) may be replaced by its Taylor polynomial of length \( 100k\beta_i^{-3/4} \). Full details of this argument can be found in [9, Lemma 5.2, pp. 484-486]. In order to simplify the presentation, we set

\[ \gamma^+ = \frac{1}{2}(\alpha_1 + \alpha_2) \text{ and } \gamma^- = \frac{1}{2}(\alpha_1 - \alpha_2). \]

Expanding out all of the \( j \)-th powers and opening the square, we see that

\[ \mathcal{J} \]

\[ \sum_{j, \ell} \left( \prod_{1 \leq i \leq 3} \frac{k_{j_i} \ell_{i}}{j_i \ell_{i}} \right) \sum_{\hat{p}, \hat{q}} C(\hat{p}, \hat{q}) \int_{1 \leq i \leq 3} \prod_{1 \leq i \leq 3} \cos((t + \gamma^+ \log p(i, r)) \cos((t + \gamma^+ \log q(i, s)) dt \]

\[ \times \prod_{1 \leq i \leq 3} \cos(\gamma^- \log p(i, r)) \cos(\gamma^- \log q(i, s)), \]

where the first sum is over all

\[ \tilde{j} = (j_1, \ldots, j_3), \tilde{\ell} = (\ell_1, \ldots, \ell_3) \text{ where } 0 \leq j_i, \ell_i \leq 100k\beta_i^{-3/4}, \]

the second sum is over

\[ \hat{p} = (p(1, 1), \ldots, p(1, j_1), p(2, 1), \ldots, p(2, j_2), \ldots, p(3, j_3)) \text{ and } \]

\[ \hat{q} = (q(1, 1), \ldots, q(1, \ell_1), q(2, 1), \ldots, q(2, \ell_2), \ldots, q(3, \ell_3)) \]
whose components are primes which satisfy
\[ T^{\beta_i} - 1 < p(i, 1), \ldots, p(i, j_i), q(i, 1), \ldots, q(i, \ell_i) \leq T^{\beta_i} \]
for any \( 1 \leq i \leq J \), and
\[
C(\tilde{p}, \tilde{q}) = \prod_{1 \leq i \leq J} \prod_{1 \leq r \leq j_i} \prod_{1 \leq s \leq \ell_i} \frac{1}{p(i, r)^{\frac{1}{\log T^{\beta_i}}}} \frac{1}{\log T^{\beta_i}} \frac{1}{q(i, s)^{\frac{1}{\log T^{\beta_i}}}} \frac{1}{\log T^{\beta_i}}.
\]
Following the argument in [4, p. 10] (see the third displayed equation), we have
\[
(4.5) \prod_{1 \leq i \leq J} \prod_{1 \leq r \leq j_i} \prod_{1 \leq s \leq \ell_i} p(i, r)q(i, s) \leq T^{0.1}.
\]
By Lemma 2.2 and (4.5), it follows that
\[
(4.6) \int_T^{2T} \prod_{1 \leq i \leq J} \prod_{1 \leq r \leq j_i} \prod_{1 \leq s \leq \ell_i} \cos((t + \gamma^+) \log p(i, r)) \cos((t + \gamma^+) \log q(i, s))dt
\]
\[
= (T + \gamma^+)g \left( \prod_{1 \leq i \leq J} \prod_{1 \leq r \leq j_i} \prod_{1 \leq s \leq \ell_i} p(i, r)q(i, s) \right) + O(\gamma^+) + O(T^{0.1}).
\]
Let
\[
D(\tilde{p}, \tilde{q}) = \prod_{1 \leq i \leq J} \prod_{1 \leq r \leq j_i} \prod_{1 \leq s \leq \ell_i} \frac{1}{p(i, r)} \frac{1}{q(i, s)}
\]
and observe that
\[
(4.7) C(\tilde{p}, \tilde{q}) \leq D(\tilde{p}, \tilde{q}).
\]
By (4.6), (4.7), and the bound \(|\cos x| \leq 1\) for real \( x \), it follows that (4.4) equals
\[
\mathcal{J} = (T + \gamma^+) \sum_{j, \ell} \left( \prod_{1 \leq i \leq \gamma} \frac{k_{ji}}{j_i!} \frac{k_{\ell i}}{\ell_i!} \right) \sum_{\tilde{p}, \tilde{q}} C(\tilde{p}, \tilde{q})g \left( \prod_{1 \leq i \leq J} \prod_{1 \leq r \leq j_i} \prod_{1 \leq s \leq \ell_i} p(i, r)q(i, s) \right)
\]
\[
\times \prod_{1 \leq i \leq J} \prod_{1 \leq r \leq j_i} \prod_{1 \leq s \leq \ell_i} \cos(\gamma^- \log p(i, r)) \cos(\gamma^- \log q(i, s))
\]
\[
+ O \left( (\gamma^+)^2 + T^{0.1} \right) \sum_{j, \ell} \left( \prod_{1 \leq i \leq \gamma} \frac{k_{ji}}{j_i!} \frac{k_{\ell i}}{\ell_i!} \right) \sum_{\tilde{p}, \tilde{q}} D(\tilde{p}, \tilde{q})
\).
\]
By the argument of Harper [4, p. 10], it can be shown that the big-O term above is at most \((\gamma^+)^2 + T^{0.1})T^{0.1} \log \log T)^{2k}.\]
The inner summand in (4.8) is

\[ C(\tilde{p}, \tilde{q}) = \left( \prod_{1 \leq i \leq j} \prod_{1 \leq r \leq j} \prod_{1 \leq s \leq \ell_i} p(i, r) q(i, s) \right) \prod_{1 \leq i \leq j} \prod_{1 \leq r \leq j} \prod_{1 \leq s \leq \ell_i} \cos(\gamma - \log p(i, r)) \cos(\gamma - \log q(i, s)). \]

Since \( g \) is supported on squares, this expression is non-zero if and only if

\[ \prod_{1 \leq i \leq j} \prod_{1 \leq r \leq j} \prod_{1 \leq s \leq \ell_i} p(i, r) q(i, s) = p_1^2 \cdots p_N^2 \]

for some \( N \in \mathbb{N} \). In this case, we have

\[ (4.9) \quad \prod_{1 \leq i \leq j} \prod_{1 \leq r \leq j} \prod_{1 \leq s \leq \ell_i} \cos(\gamma - \log p(i, r)) \cos(\gamma - \log q(i, s)) = \cos^2(\gamma - \log p_1) \cdots \cos^2(\gamma - \log p_N) \geq 0. \]

By (4.2), (4.4), (4.8), (4.9), and (4.7), we deduce that

\[ \mathcal{J} \ll T \prod_{1 \leq i \leq j} \sum_{0 \leq j, \ell \leq 100 \beta_i^{-3/4}} \frac{k^{j+\ell}}{j! \ell!} \sum_{T^{\beta_i-1} < p_1, \ldots, p_j, q_1, \ldots, q_\ell < T^{\beta_i}} g(p_1 \cdots p_j q_1 \cdots q_\ell) \sqrt{p_1 \cdots p_j q_1 \cdots q_\ell} \]

\[ \times \cos(\gamma - \log p_1) \cdots \cos(\gamma - \log p_j) \cos(\gamma - \log q_1) \cdots \cos(\gamma - \log q_\ell) + O((|\gamma^+| + T^{0.1}) T^{0.1} (\log \log T)^{2k}) \]

\[ = T \prod_{1 \leq i \leq j} \sum_{0 \leq m \leq 200 \beta_i^{-3/4}} \frac{k^m}{m!} \sum_{0 \leq j, \ell \leq 100 \beta_i^{-3/4}} \frac{1}{j! \ell!} \sum_{T^{\beta_i-1} < p_1, \ldots, p_m < T^{\beta_i}} g(p_1 \cdots p_m) \sqrt{p_1 \cdots p_m} \]

\[ \times \cos(\gamma - \log p_1) \cdots \cos(\gamma - \log p_m) + O((|\gamma^+| + T^{0.1}) T^{0.1} (\log \log T)^{2k}) \]

\[ \leq T \prod_{1 \leq i \leq j} \sum_{0 \leq m \leq 200 \beta_i^{-3/4}} \frac{k^m 2^m}{m!} \sum_{T^{\beta_i-1} < p_1, \ldots, p_m < T^{\beta_i}} g(p_1 \cdots p_m) \sqrt{p_1 \cdots p_m} \cos(\gamma - \log p_1) \cdots \cos(\gamma - \log p_m) \]

\[ + O((|\gamma^+| + T^{0.1}) T^{0.1} (\log \log T)^{2k}), \]

where the last inequality makes use of the non-negativity of the inner summand. Since \( g \) is supported on squares, we must have that \( m \) is even, say \( m = 2n \) with \( n \geq 0 \). By relabelling the prime variables as \( q_1, \ldots, q_{2n} \), we see that

\[ (4.10) \quad \mathcal{J} \ll T \prod_{1 \leq i \leq j} \sum_{0 \leq n \leq 100 \beta_i^{-3/4}} \frac{k^{2n} 2^{2n}}{(2n)!} \sum_{T^{\beta_i-1} < q_1, \ldots, q_{2n} < T^{\beta_i}} g(q_1 \cdots q_{2n}) \sqrt{q_1 \cdots q_{2n}} \cos(\gamma - \log q_1) \cdots \cos(\gamma - \log q_{2n}) \]

\[ + O((|\gamma^+| + T^{0.1}) T^{0.1} (\log \log T)^{2k}). \]
Next, we observe that \( q_1 \cdots q_{2n} \) is a square if and only it equals \( p_1^2 \cdots p_n^2 \) for some primes \( p_u \in [T^{3i-1}, T^{3i}] \) with \( 1 \leq u \leq n \). Grouping terms according to \( q_1 \cdots q_{2n} = p_1^2 \cdots p_n^2 \) gives

\[
\sum_{T^{3i-1} < q_1, \ldots, q_{2n} \leq T^{3i}} \frac{g(q_1 \cdots q_{2n})}{\sqrt{q_1 \cdots q_{2n}}} \cos(\gamma \log q_1) \cdots \cos(\gamma \log q_{2n})
= \sum_{T^{3i-1} < p_1, \ldots, p_n \leq T^{3i}} \sum_{q_1 \cdots q_{2n} = (p_1 \cdots p_n)^2} \frac{g(p_1^2 \cdots p_n^2)}{\sqrt{p_1^2 \cdots p_n^2}} \cos^2(\gamma \log p_1) \cdots \cos^2(\gamma \log p_n)
\times \#\{(p'_1, \ldots, p'_n) \mid p'_1 \cdots p'_n = p_1 \cdots p_n\}^{-1}
= \sum_{T^{3i-1} < p_1, \ldots, p_n \leq T^{3i}} \frac{g(p_1^2 \cdots p_n^2)}{\sqrt{p_1^2 \cdots p_n^2}} \cos^2(\gamma \log p_1) \cdots \cos^2(\gamma \log p_n)
\times \#\{(q_1, \ldots, q_{2n}) \mid q_1 \cdots q_{2n} = (p_1 \cdots p_n)^2\}^{-1}
\times \#\{(p'_1, \ldots, p'_n) \mid p'_1 \cdots p'_n = p_1 \cdots p_n\}.
\]

In the above the factor \( \#\{(p'_1, \ldots, p'_n) \mid p'_1 \cdots p'_n = p_1 \cdots p_n\}^{-1} \) accounts for possible repetitions when counting squares \( p_1^2 \cdots p_n^2 \). With this observation, by following the argument in \([4\text{, p. 11}])\), we have that the first term in \( (4.11) \) equals

\[
(4.11)
T \prod_{1 \leq i \leq 2} \sum_{0 \leq n \leq 100\beta_i^{-3/4}} \frac{(2k)^{2n}}{(2n)!} \sum_{T^{3i-1} < p_1, \ldots, p_n \leq T^{3i}} \frac{g(p_1^2 \cdots p_n^2)}{p_1 \cdots p_n} \frac{\#\{(q_1, \ldots, q_{2n}) \mid q_1 \cdots q_{2n} = p_1^2 \cdots p_n^2\}}{\#\{(q_1, \ldots, q_n) \mid q_1 \cdots q_n = p_1 \cdots p_n\}}
\times \cos^2(\gamma \log p_1) \cdots \cos^2(\gamma \log p_n),
\]

where each \( q_i \) again denotes a prime in \((T^{3i-1}, T^{3i})\).

By \([4\text{, Eq. (4.2)}]\), we know

\[
g(p_1^2 \cdots p_n^2) = \frac{1}{2^{2n}} \prod_{j=1}^r \frac{(2\alpha_j)!}{(\alpha_j)!^2}
\]

and

\[
\frac{\#\{(q_1, \ldots, q_{2n}) \mid q_1 \cdots q_{2n} = p_1^2 \cdots p_n^2\}}{\#\{(q_1, \ldots, q_n) \mid q_1 \cdots q_n = p_1 \cdots p_n\}} = \frac{(2n)!}{\prod_{j=1}^r (2\alpha_j)!} \left( \frac{n!}{\prod_{j=1}^r \alpha_j!} \right)^{-1}
\]

whenever \( p_1 \cdots p_n \) is a product of \( r \) distinct primes with multiplicities \( \alpha_1, \ldots, \alpha_r \) (in particular, \( \alpha_1 + \cdots + \alpha_r = n \)). Therefore, the expression \( (4.11) \) is equal to

\[
T \prod_{1 \leq i \leq 2} \sum_{0 \leq n \leq 100\beta_i^{-3/4}} \frac{k^{2n}}{n!} \sum_{T^{3i-1} < p_1, \ldots, p_n \leq T^{3i}} \frac{\cos^2(\gamma \log p_1) \cdots \cos^2(\gamma \log p_n)}{p_1 \cdots p_n} \frac{1}{\prod_{j=1}^r \alpha_j!}
\leq T \prod_{1 \leq i \leq 2} \sum_{0 \leq n \leq 100\beta_i^{-3/4}} \frac{1}{n!} \left( \frac{k^2}{n!} \sum_{T^{3i-1} < p \leq T^{3i}} \frac{\cos^2(\gamma \log p)}{p} \right)^n
\leq T \exp \left( k^2 \sum_{p \leq T^{3i}} \frac{\cos^2(\gamma \log p)}{p} \right).
\]

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Hence, we arrive at

\[(4.12) \quad \mathcal{I} \ll T \exp \left( k^2 \sum_{p \leq T^{\beta_2}} \frac{\cos^2(\frac{1}{2}(\alpha_1 - \alpha_2) \log p)}{p} \right) + (|\gamma^+| + T^{0.1})T^{0.1}(\log \log T)^{2k}. \]

Since \( \beta_2 < 1 \) and \( \cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta)) \), from (2.2), it follows that

\[
\sum_{p \leq T^{\beta_2}} \frac{\cos^2(\frac{1}{2}(\alpha_1 - \alpha_2) \log p)}{p} \leq \sum_{p \leq T} \frac{\cos^2(\frac{1}{2}(\alpha_1 - \alpha_2) \log p)}{p}
\]

\[(4.13) \quad = \frac{1}{2} \sum_{p \leq T} \frac{1}{p} + \frac{1}{2} \sum_{p \leq T} \frac{\cos((\alpha_1 - \alpha_2) \log p)}{p}
\]

\[
\leq \frac{1}{2} \log \log T + \frac{1}{2} \log(\mathcal{F}(T, \alpha_1, \alpha_2)) + O(1),
\]

where \( \mathcal{F}(T, \alpha_1, \alpha_2) \) is defined in (1.7). Therefore, by (4.12), (4.13), and (1.6),

\[\mathcal{I} \ll T(\log T)^{\frac{k^2}{2}} \mathcal{F}(T, \alpha_1, \alpha_2)^{\frac{k^2}{2}} + T^{0.8}(\log \log T)^{2k} \ll T(\log T)^{\frac{k^2}{2}} \mathcal{F}(T, \alpha_1, \alpha_2)^{\frac{k^2}{2}}.\]

This combined with (4.2) and (4.3) completes the proof of Lemma 3.1.

5. PROOF OF LEMA 3.2

Following Harper, by the definition \( S(j) \), we can bound the left hand side of (4.8) by

\[(5.1) \quad \sum_{\ell=j+1}^{3} \int_{T}^{2T} \left( \prod_{1 \leq i \leq j} (\exp(k \text{Re} G_{i,j}(t)))^2 \right) \mathbb{1}_{A_{j,\ell}}(t) dt, \]

where

\[A_{j,\ell} := \left\{ t \in \mathbb{R} \mid |\text{Re} G_{i,j}(t)| \leq \beta_{j+1}^{-3/4}, \forall 1 \leq i \leq j, \text{ but } |\text{Re} G_{j+1,\ell}(t)| > \beta_{j+1}^{-3/4} \right\}.\]

From the definition of \( A_{j,\ell} \), it follows that

\[\mathbb{1}_{A_{j,\ell}}(t) \leq (\beta_{j+1}^{3/4} |\text{Re} G_{j+1,\ell}(t)|)^M \]

for any positive integer \( M \). From this point on, we set

\[M = 2[1/(10\beta_{j+1})].\]

Therefore the integral in (5.1) is

\[(5.2) \quad \leq \int_{T}^{2T} \prod_{1 \leq i \leq j} (\exp(k \text{Re} G_{i,j}(t)))^2 (\beta_{j+1}^{3/4} |\text{Re} G_{j+1,\ell}(t)|)^M dt \]

\[\ll (\beta_{j+1}^{3/4})^{[1/(10\beta_{j+1})]} \int_{T}^{2T} \prod_{1 \leq i \leq j} \left( \sum_{0 \leq n \leq 100k \beta_{j+1}^{-3/4}} \frac{(k \text{Re} G_{i,j}(t))^n}{n!} \right)^2 \text{Re} G_{j+1,\ell}(t)^M dt, \]

The second estimate can be established similar to the proof of Lemma 5.2 of [9]. Arguing as in the proof of Lemma 3.1 and [11, p. 13], since \( g(p_1 \cdots p_m) \cos(\gamma \log p_1) \cdots \cos(\gamma \log p_m) \geq 0 \), we

\[1\text{For } S \subset \mathbb{R}, \mathbb{1}_S(t) = 1 \text{ if } t \in S \text{ and } \mathbb{1}_S(t) = 0 \text{ if } t \notin S \text{ (the indicator function of } S).]
deduce

\[(5.3)\]

\[
\int_T^{2T} \prod_{1 \leq i \leq j} \left( \sum_{0 \leq n \leq 100k_{\alpha - 3/4}} \frac{(k\Re G_{i,j}(t))^n}{n!} \right)^2 (\Re G_{j+1,t}(t))^M dt
\]

\[
\ll T \prod_{1 \leq i \leq j} \sum_{0 \leq m \leq 20k_{\beta_0 - 3/4}} \frac{k^m m!}{m!} \sum_{T^{\beta_0-1} \leq p_1 \ldots \leq p_M \leq T^{\beta_0}} \frac{g(p_1 \ldots p_M)}{\sqrt{p_1 \ldots p_M}} \cos(\gamma^- \log p_1) \cdots \cos(\gamma^- \log p_M)
\]

\[
\times \sum_{T^{\beta_j} \leq p_1 \ldots \leq p_M \leq T^{\beta_{j+1}}} \frac{g(p_1 \ldots p_M)}{\sqrt{p_1 \ldots p_M}} \cos(\gamma^- \log p_1) \cdots \cos(\gamma^- \log p_M)
\]

\[
+ O(\vert\gamma^+\vert + T^{0.3}T^{0.3}(\log \log T)^{2k}).
\]

Since \(g\) is supported on squares, by following an argument similar to the proof of Lemma 3.1, we find that the previous expression is bounded by

\[
\ll T \exp \left( k^2 \sum_{p \leq T^{\beta_j}} \frac{\cos^2(\gamma^- \log p)}{p} \right) \times \frac{M!}{2M(M/2)!} \left( \sum_{T^{\beta_j} \leq p \leq T^{\beta_{j+1}}} \frac{\cos^2(\gamma^- \log p)}{p} \right)^{M/2}
\]

\[
+ O(\vert\gamma^+\vert + T^{0.3}T^{0.3}(\log \log T)^{2k})
\]

\[(5.4)\]

\[
\ll T \exp \left( k^2 \sum_{p \leq T^{\beta_j}} \frac{\cos^2(\gamma^- \log p)}{p} \right) \times \left( \frac{1}{20^j} \sum_{T^{\beta_j} \leq p \leq T^{\beta_{j+1}}} \frac{\cos^2(\gamma^- \log p)}{p} \right)^{1/(10^{j+1})}
\]

\[
+ O(\vert\gamma^+\vert + T^{0.3}T^{0.3}(\log \log T)^{2k}).
\]

Hence, by (5.1), (5.2), (5.3), and (5.4), we arrive at

\[
\int_{t \in S(j)} \exp \left( 2k\Re \sum_{p \leq T^{\beta_j}} \frac{\cos(\frac{1}{2}(\alpha_1 - \alpha_2) \log p)}{p^{1/2 + \beta_j \log T + i(1/2)(\alpha_1 + \alpha_2)}} \log(T^{\beta_j}/p) \right) dt
\]

\[
\ll (j - 1)T \exp \left( k^2 \sum_{p \leq T^{\beta_j}} \frac{\cos^2(\frac{1}{2}(\alpha_1 - \alpha_2) \log p)}{p} \right)
\]

\[
\times \left( \frac{\beta_{j+1}^{1/2}}{20} \sum_{T^{\beta_j} \leq p \leq T^{\beta_{j+1}}} \frac{\cos^2(\frac{1}{2}(\alpha_1 - \alpha_2) \log p)}{p} \right)^{1/(10^{j+1})}
\]

\[
+ (j - 1)(\vert\gamma^+\vert + T^{0.3}T^{0.3})(\log \log T)^{2k}.
\]

Recall that \(J \leq \log \log \log T\), \(\beta_0 = 0\), \(\beta_1 = \frac{1}{(\log \log T)^2}\), and

\[
\sum_{p \leq T^{\beta_1}} \frac{1}{p} \leq \log \log T.
\]

Observe that for \(j = 0\), the left of (5.5) is \(\operatorname{meas}(S(0))\). Therefore, by the trivial bound \(\cos^2(\frac{1}{2}(\alpha_1 - \alpha_2) \log p) \leq 1\) and the assumption \(\vert\gamma^+\vert = \vert\alpha_1 + \alpha_2\vert \ll T^{0.6}\), we derive \(\operatorname{meas}(S(0)) \ll T e^{-\frac{1}{(\log \log T)^2}/10}\).
For $1 \leq j \leq J - 1$, we have $J - j \leq \frac{\log(1/\beta_j)}{\log 2}$ and

$$\sum_{T^{\beta_j} < p \leq T^{\beta_{j+1}}} \frac{\cos^2 \left( \frac{1}{2} (\alpha_1 - \alpha_2) \log p \right)}{p} \leq \sum_{T^{\beta_j} < p \leq T^{\beta_{j+1}}} \frac{1}{p} = \log \beta_{j+1} - \log \beta_j + o(1) \leq 10.$$ 

Thus, by (2.2) and the assumption $|\gamma^+| \ll T^{0.6}$, we see that the left of (5.5) is

$$\ll_k T (\log T) \frac{k^2}{2} \mathcal{J}(T, \alpha_1, \alpha_2) \frac{k^2}{T} \exp \left( - \frac{\log (1/\beta_{j+1})}{21 \beta_{j+1}} \right)$$

as desired.

6. Proof of Lemma 3.3

The proof of Lemma 3.3 is similar to the proofs of Lemmas 3.1 and 3.2 One key difference is that we need to invoke Lemma 2.3 in place of Lemma 2.2. In this section, we shall establish the estimate (3.9). As the proof of (3.10) is similar, the details shall be omitted. The integral in (3.9) shall be denoted $\int_T \exp(\varphi(t)) \, dt$ where $\exp(\varphi(t))$ is the integrand in (3.9). First, we decompose this integrand in terms of integer parameters $m$ satisfying $0 \leq m \leq \frac{\log \log T}{\log 2}$. For each such $m$, we define

$$P_m(t) = \sum_{2^m < p \leq 2^{m+1}} \frac{\cos((\alpha_1 - \alpha_2) \log p)}{2p^{1+i(2t+(\alpha_1+\alpha_2))}}.$$ 

and the sets

$$\mathcal{P}(m) := \left\{ t \in [T, 2T] \mid |\Re P_m(t)| > 2^{-m/10}, \text{ but } |\Re P_n(t)| \leq 2^{-n/10} \text{ for every } m + 1 \leq n \leq \frac{\log \log T}{\log 2} \right\}.$$ 

Observe that we have the identity

$$\sum_{p \leq \log T} \frac{\cos((\alpha_1 - \alpha_2) \log p)}{2p^{1+i(2t+(\alpha_1+\alpha_2))}} = \sum_{0 \leq m \leq \frac{\log \log T}{\log 2}} P_m(t).$$

We now have the decomposition

$$\int_T \exp(\varphi(t)) \, dt = \sum_{0 \leq m \leq \frac{\log \log T}{\log 2}} \int_{\mathcal{P}(m)} \exp(\varphi(t)) \, dt + \int_{T \setminus (\mathcal{P}(m)^c)} \exp(\varphi(t)) \, dt.$$ 

In order to establish (3.9), we shall bound each of the integrals on the right side of (6.4).

If $t$ does not belong to any $\mathcal{P}(m)$, then $|\Re P_n(t)| \leq 2^{-n/10}$ for all $n \leq \frac{\log \log T}{\log 2}$. (Indeed, for those $t$ belonging to none of $\mathcal{P}(m)$, $0 \leq m \leq \frac{\log \log T}{\log 2}$, if $|\Re P_m(t)| > 2^{-m/10}$ for some $0 \leq m \leq \frac{\log \log T}{\log 2}$, then $|\Re P_k(t)| > 2^{-L/10}$ for some $m + 1 \leq L \leq \frac{\log \log T}{\log 2}$ as $t \notin \mathcal{P}(m)$. Choosing $L$ to be maximal, we then have $|\Re P_n(t)| \leq 2^{-n/10}$ for every $L + 1 \leq n \leq \frac{\log \log T}{\log 2}$, which means $t \in \mathcal{P}(L)$, a contradiction.)

For such an instance, $\Re \sum_{p \leq \log T} \frac{\cos((\alpha_1 - \alpha_2) \log p)}{2p^{1+i(2t+(\alpha_1+\alpha_2))}} = O(1)$. Hence, the contribution of such $t$ to the
Thus, we derive

$$\int_{t \in \mathcal{T} \cap \mathcal{P}(m)} \exp(\varphi(t)) \, dt$$

(6.5)

$$\ll \int_{t \in \mathcal{T} \cap \mathcal{P}(m)} \exp\left(2k\Re \sum_{p \leq T^{\beta_2}} \cos\left(\frac{1}{2}(\alpha_1 - \alpha_2) \log p\right) \frac{\log(T^{\beta_2}/p)}{p^{2 + \beta_2 \log T + i(t + \frac{1}{2}(\alpha_1 + \alpha_2))}} \right) \, dt$$

$$\ll_k T (\log T) \frac{e^2}{2} (\mathcal{F}(T, \alpha_1, \alpha_2))^\frac{e^2}{2}.$$

It remains to estimate the contribution from $t \in \mathcal{T} \cap \mathcal{P}(m)$, with $0 \leq m \leq \frac{\log \log T}{\log 2}$, to $\int_{\mathcal{T}}$ (more precisely, the first integral on the right of (6.4)). To do so, we first consider the case that $0 \leq m \leq \frac{2 \log \log T}{\log 2}$. In this case, we have

$$\left| \Re \sum_{p \leq \log T} \frac{\cos((\alpha_1 - \alpha_2) \log p)}{2p^{1+i(2t+(\alpha_1 + \alpha_2))}} \right| \leq \sum_{0 \leq n \leq \frac{\log \log T}{\log 2}} |\Re P_n(t)|$$

$$\leq \sum_{0 \leq n \leq m} |\Re P_n(t)| + \sum_{m+1 \leq n \leq \frac{\log \log T}{\log 2}} |\Re P_n(t)|$$

$$\leq \sum_{p \leq 2m+1} \frac{1}{2p} + \sum_{m+1 \leq n \leq \frac{\log \log T}{\log 2}} \frac{1}{2n/10}.$$ 

The last inequality makes use of the definition of $P_m(t)$ in (6.2). Therefore,

$$\left| \Re \left( \sum_{p \leq 2m+1} \frac{\cos\left(\frac{1}{2}(\alpha_1 - \alpha_2) \log p\right)}{p^{1+i(t+\frac{1}{2}(\alpha_1 + \alpha_2))}} \log(T^{\beta_2}/p) + \sum_{p \leq \log T} \frac{\cos((\alpha_1 - \alpha_2) \log p)}{2p^{1+i(2t+(\alpha_1 + \alpha_2))}} \right) \right|$$

$$\leq \sum_{p \leq 2m+1} \frac{1}{\sqrt{p}} + \sum_{p \leq 2m+1} \frac{1}{2p} + \sum_{m+1 \leq n \leq \frac{\log \log T}{\log 2}} \frac{1}{2n/10}$$

$$\ll 2^{m/2}.$$ 

Thus, we derive

(6.6)

$$\int_{t \in \mathcal{T} \cap \mathcal{P}(m)} \exp\left(2k\Re \left( \sum_{p \leq T^{\beta_2}} \frac{\cos\left(\frac{1}{2}(\alpha_1 - \alpha_2) \log p\right)}{p^{2 + \beta_2 \log T + i(t+\frac{1}{2}(\alpha_1 + \alpha_2))}} \log(T^{\beta_2}/p) + \sum_{p \leq \log T} \frac{\cos((\alpha_1 - \alpha_2) \log p)}{2p^{1+i(2t+(\alpha_1 + \alpha_2))}} \right) \right) \, dt$$

$$\leq e^{O(k^{2m/2})} \int_{t \in \mathcal{T} \cap \mathcal{P}(m)} \exp\left(2k\Re \left( \sum_{2m+1 < p \leq T^{\beta_2}} \frac{\cos\left(\frac{1}{2}(\alpha_1 - \alpha_2) \log p\right)}{p^{1+i(t+\frac{1}{2}(\alpha_1 + \alpha_2))}} \log(T^{\beta_2}/p) \right) \right) \, dt$$

$$\leq e^{O(k^{2m/2})} \int_{t \in \mathcal{T}} \exp\left(2k\Re \left( \sum_{2m+1 < p \leq T^{\beta_2}} \frac{\cos\left(\frac{1}{2}(\alpha_1 - \alpha_2) \log p\right)}{p^{1+i(t+\frac{1}{2}(\alpha_1 + \alpha_2))}} \log(T^{\beta_2}/p) \right) \right) \times (2^{m/10} \Re P_m(t))^{2^{[2m/4]}} \, dt.$$
Let \( N = 2^{2^{3m/4}} \). Arguing as in the proof of Lemma 3.1 while applying Lemma 2.3 instead of Lemma 2.2, we see the above integral is

\[
\ll T \exp \left( k^2 \sum_{2^m < p \leq T^{\beta_3}} \frac{\cos^2 \left( \frac{1}{2} (\alpha_1 - \alpha_2) \log p \right)}{p} \right) \times 2^{mN/10} \sum_{2^m < p_1, \ldots, p_N \leq 2^{m+1}} \frac{g(p_1 \cdots p_N)}{p_1 \cdots p_N}
\]

\[+ (|\gamma^+| + T^{0.1 + o(1)}) T^{0.1} (\log \log T)^{2k}, \]

(6.7)

\[
\ll T \exp \left( k^2 \sum_{2^m < p \leq T^{\beta_3}} \frac{\cos^2 \left( \frac{1}{2} (\alpha_1 - \alpha_2) \log p \right)}{p} \right) \times 2^{mN/10} N! \left( \sum_{2^m < p_1, \ldots, p_N \leq 2^{m+1}} \frac{1}{p_1 \cdots p_N} \right)^{N/2}
\]

\[+ T^{0.7} (\log \log T)^{2k}, \]

as \(|\gamma^+| = |\frac{\alpha_1 + \alpha_2}{2}| \ll T^{0.6}. Indeed, recall that \( 2^m \leq \log T \), and note that if \( 2^m < p_1, \ldots, p_N \leq 2^{m+1} \), then

\[
p_1^2 \cdots p_N^2 \leq 2^{2(m+1)N} \leq 2^{6(\log \log T)(\log T)^{3/4}}.
\]

Therefore, the additional contribution of \((\Re P_m(t))^N \) enlarges the last big-O term (4.8) as

\[
(|\gamma^+| + T^{0.1} 2^{6(\log \log T)(\log T)^{3/4}}) \sum_{2^m < p_1, \ldots, p_N \leq 2^{m+1}} \frac{1}{p_1 \cdots p_N} T^{0.1} (\log \log T)^{2k},
\]

where the sum is equal to

\[
 \left( \sum_{2^m < p \leq 2^{m+1}} \frac{1}{p} \right)^N \leq \left( \frac{2^{m+1} - 2^m}{2^m} \frac{1}{2^m} \right)^N = 1.
\]

Now, by (6.7), the left of (6.6) is

\[
\ll e^{O(k2^{-m/2})} \left( 2^{m/5} \cdot 2^{3m/4} \cdot 2^{-m} \right)^{[3m/4]} T^{(\log T)^{\frac{k^2}{2}}} \mathcal{F}(T, \alpha_1, \alpha_2) \frac{k^2}{2}
\]

\[\ll e^{O(k2^{-m/2})} - 2^{3m/4} T^{(\log T)^{\frac{k^2}{2}}} \mathcal{F}(T, \alpha_1, \alpha_2) \frac{k^2}{2}.
\]

Secondly, we evaluate the contribution from \( t \in \mathcal{J} \cap \mathcal{P}(m) \) with \( \frac{2 \log \log T}{\log 2} < m \leq \frac{\log \log T}{\log 2} \). We shall consider

\[
\int_{t \in \mathcal{J} \cap \mathcal{P}(m)} 1 dt \leq \int_{t \in \mathcal{J}} \left( 2^{m/10} \Re P_m(t) \right)^{2^{[3m/4]}} dt.
\]

Following the previous argument in (6.6) where the exponential factor is replaced by 1, one can show that \( \text{meas}(\mathcal{J} \cap \mathcal{P}(m)) \ll T e^{-2^{3m/4}} \). So, for \( 2^m \geq (\log \log T)^2 \), we see \( \text{meas}(\mathcal{J} \cap \mathcal{P}(m)) \ll
\[ T e^{-\left(\log \log T\right)^{3/2}}. \] In addition, the Cauchy-Schwarz inequality tells us that
\begin{equation}
\int_{t \in T \cap P(m)} \exp \left( 2k \Re \left( \sum_{p \leq T^{\beta_3}} \frac{\cos\left(\frac{1}{2} (\alpha_1 - \alpha_2) \log p\right)}{p^{\frac{1}{2} + \frac{1}{2} \log \log T + i(t + \frac{1}{2} (\alpha_1 + \alpha_2))}} \log \frac{T^{\beta_3}}{p} \right) + \sum_{p \leq T} \frac{\cos\left( (\alpha_1 - \alpha_2) \log p \right)}{2p^{1+i(2t+(\alpha_1+\alpha_2))}} \right) \right) \right) dt 
\end{equation}

\[ \ll e^{k \log \log \log T} \int_{t \in T \cap P(m)} \exp \left( 2k \Re \left( \sum_{p \leq T^{\beta_3}} \frac{\cos\left(\frac{1}{2} (\alpha_1 - \alpha_2) \log p\right)}{p^{\frac{1}{2} + \frac{1}{2} \log \log T + i(t + \frac{1}{2} (\alpha_1 + \alpha_2))}} \log \frac{T^{\beta_3}}{p} \right) \right) dt 
\end{equation}

\[ \ll (\log T)^{k} \left( \int_{t \in T \cap P(m)} \exp \left( 4k \Re \sum_{p \leq T^{\beta_3}} \frac{\cos\left(\frac{1}{2} (\alpha_1 - \alpha_2) \log p\right)}{p^{\frac{1}{2} + \frac{1}{2} \log \log T + i(t + \frac{1}{2} (\alpha_1 + \alpha_2))}} \log \frac{T^{\beta_3}}{p} \right) dt \right)^{\frac{1}{2}} 
\end{equation}

\[ \times \left( \text{meas}(T \cap P(m)) \right)^{\frac{3}{2}}. \]

As Lemma 3.1 gives
\begin{equation}
\int_{t \in T \cap P(m)} \exp \left( 4k \Re \sum_{p \leq T^{\beta_3}} \frac{\cos\left(\frac{1}{2} (\alpha_1 - \alpha_2) \log p\right)}{p^{\frac{1}{2} + \frac{1}{2} \log \log T + i(t + \frac{1}{2} (\alpha_1 + \alpha_2))}} \log \frac{T^{\beta_3}}{p} \right) dt \ll T (\log T)^{4k^2},
\end{equation}

we see that (6.10) is bounded by

\begin{equation}
\ll_k T e^{-\frac{1}{4} (\log \log T)^{3/2}}. \end{equation}

Finally, we complete the proof of the Lemma by combining (6.1) with the bounds (6.8) (0 \leq m \leq \frac{2 \log \log \log T}{\log 2}) and (6.11) (\frac{2 \log \log T}{\log 2} < m \leq \frac{\log \log T}{\log 2}) for \int_{T \cap P(m)} \exp(\varphi(t)) dt and the bound (6.5).

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