ABSTRACT

We describe a deformation of the observable algebra of quantum gravity in which the loop algebra is extended to framed loops. This allows an alternative nonperturbative quantization which is suitable for describing a phase of quantum gravity characterized by states which are normalizable in the measure of Chern-Simons theory. The deformation parameter, $q$, is $e^{i\hbar G^2\Lambda/6}$, where $\Lambda$ is the cosmological constant. The Mandelstam identities are extended to a set of relations which are governed by the Kauffman bracket so that the spin network basis is deformed to a basis of $SU(2)_q$ spin networks. Corrections to the actions of operators in non-perturbative quantum gravity may be readily computed using recoupling theory; the example of the area observable is treated here. Finally, eigenstates of the $q$-deformed Wilson loops are constructed, which may make possible the construction of a $q$-deformed connection representation through an inverse transform.

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1 Introduction

In the past five years a number of striking consequences of diffeomorphism invariance have emerged in the non-perturbative approach to quantum gravity based on the loop representation ([1] - [21] For reviews see [3, 4]). One of these provides a basis of spatially diffeomorphism invariant states labeled by diffeomorphism equivalence classes of embedding of spin networks [5, 6]. In this context, a spin network is a graph with edges labeled by representations of $SU(2)$ and vertices labeled by the ways that the edge representations may be combined into a singlet. This concept of a spin network was first introduced by Penrose [22] in work on the four color problem. Later, he applied spin networks to a combinatorial construction of geometry [23]. The concept independently reappeared in lattice gauge theory where spin networks label bases of states, a property equally useful in non-perturbative quantum gravity where spin networks were introduced following the discovery that this basis diagonalizes two interesting classes of observable, areas and volumes [6].

This work constructs one natural extension of these results to a class of theories in which the role of the spin networks is replaced by a closely related set of combinatorial
and topological networks called quantum spin networks or q-spin nets. These structures have emerged in the investigation of topological quantum field theory, and play a key role in elucidating the connection between Chern-Simons theory and the Kauffman bracket. Closely related to the topological and algebraic structures which underlie conformal field theory and hence perturbative string theory, quantum spin networks turn out also to be central to category-theoretic foundations of topological quantum field theory in three and four dimensions.

The need for a deformation of the loop algebra is evident in Chern-Simons theory as expectation values of loop observables

\[ K[\gamma] = \langle T[\gamma] \rangle_{CS} = \int d\mu[A] \exp \left( \frac{k}{4\pi} S_{CS} \right) T_{\gamma}[A], \]

where \( T_{\gamma}[A] \) is the Wilson loop of the connection \( A \) around \( \gamma \), are not defined. There exist divergences which can be removed only if the loops are framed. Once this is done, the integral defines the Kauffman bracket, which is a diffeomorphism invariant function of the embeddings of framed loops. The expectation values of loops define a set of identities which extends the Mandelstam identities satisfied by Wilson loop observables. This means that the measure \( d\mu[A] \), cannot be one of the diffeomorphism invariant measures constructed in studies of quantum gravity in terms of elements of the completion \( A/G \).

This is relevant for quantum gravity because of the Kodama state

\[ \Psi_{CS}[A] = \exp \left( \frac{3}{\lambda} S_{CS}(A) \right) \]

where \( \lambda = G^2 \Lambda \) is the dimensionless cosmological constant and \( S_{CS}(A) \) is the Chern-Simons invariant of the left handed Ashtekar-Sen connection \( A \). This state is one of the few explicit solutions to the constraints of quantum gravity in the connection representation. Furthermore, for small \( \lambda \), it may be interpreted as a semi-classical states associated with De-Sitter spacetime. It is then interesting to hypothesize that this state gives a non-perturbative description of the vacuum state in the presence of the cosmological constant. To investigate this hypothesis we may study excitations of the Kodama state, of the form

\[ \Psi[A, \phi] = \Psi_{CS}[A] \Xi[A, \phi] \]

where \( \phi \) is a matter degree of freedom. Among these are the states

\[ \Psi[A] = \Psi_{CS}[A] T_{\gamma}[A]. \]

Alternatively, in the presence of boundaries, the Chern-Simons state seems to define a sector of the theory in the loop representation, of states

\[ \Psi_{\rho}[\gamma] = \int d\mu[A, a] \Psi_{CS}[A] T_{\gamma}[A] \rho[a] \]

where \( \rho[a] \) is a state of the Chern-Simons theory of the boundary. These states may be sufficient to span the physical state space as they saturate the Bekenstein bound when the boundary has a fixed, finite area.
For these reasons, it seems likely that in the presence of either a cosmological constant or appropriate boundary conditions quantum gravity will be formulated in terms of the Kodama state. In the loop representation, however, expressions such as Eq. (1) are not defined unless the loops are framed. Thus, we construct an extension of the loop representation to include states which are functionals of framed loops. One way to do this is to construct an extension of the loop algebra. This is the main goal of this paper. We shall see that there is a natural modification of the loop algebra involving framed loops and an extended set of identities that combine the Mandelstam identities with the relations satisfied by the Kauffman bracket. The resulting algebra has a representation which is spanned by a basis labeled by $q$-deformed spin networks.

The deformation parameter, $q$

$$q = e^{i\pi/r}$$  \hspace{1cm} (6)

with $r = k + 2$ arises through the dependence of the Kodama state on the cosmological constant. With Newton’s constant $G$ and $\hbar$ the coupling constant of Eq. (1) is

$$k = \frac{6\pi}{\hbar^2 G^2 \Lambda} + \alpha.$$  \hspace{1cm} (7)

where $\Lambda$ and $\alpha$ are, respectively, the cosmological constant and the value of a $CP$ breaking phase coming from a $\int F \wedge F$ term in the action. The definition of Eq. (6) implies that the cosmological constant must take on discrete values [13]. In addition, the limit in which $k \to \infty$ removes the effects of framing so that quantum spin networks return to ordinary spin networks. As a result, the algebra we describe here may be thought of as a deformation in $\hbar^2 \Lambda$ of the classical loop algebra, which incorporates framing of loops as a quantum effect which goes away in the limit $\hbar \to 0$.

This may seem a bit peculiar, as the cosmological constant is usually expected to only influence the large scale, and to only affect the theory at the level of dynamics. However, since the representation of the ordinary loop algebra leads to the spin network basis, the deformation in $\hbar^2 \Lambda$ must be taken into account in the kinematical algebra of the theory. Indeed, it is common in quantum field theory for the kinematical state space of the theory to be modified to incorporate dynamics. For instance, one discovers in rigorous studies of $\phi^4$-theory in 2 and 3 dimensions that, by Haag’s theorem, we cannot implement dynamics with the Fock space quantization of the associated free field theory. In addition, the structures of the kinematical state spaces know about the mass $m$, which is the parameter of highest dimension in scalar field theory, as $\Lambda$ is the parameter of highest dimension in gravitational theory. We then conjecture that the cosmological constant may play an analogous role in quantum gravity, and so requires a deformation of the observable algebra and representation of the quantum theory at the kinematical level.

Our goal is to investigate this conjecture by showing that there is a suitable deformation of the algebra which yields cosmological constant corrections to physical observables. Thus, our basic hypothesis is that a sector, or phase, of quantum gravity, given by excitations of the Kodama state, is the physical phase in the presence of a cosmological constant. We call this the “Kodama phase” of quantum gravity. In this phase, purely quantum effects add a degree of freedom to the loops which counts the twisting of loops. It is mathematically described with framed loops and quantum spin networks.
In the next section we define a formal algebra of framed loops. In Section 3 we describe a representation of this algebra in terms of suitable functionals of framed loops and show that it has a basis given by the embeddings of the $q$-deformed spin networks. Sections 4 and 5 describe, respectively, the extension of the algebra to deformations of $T^1$ and $T^2$ operators. The latter allows us to define and compute eigenvalues of the $q$-deformed area operator. Eigenstates of the deformed Wilson loops are constructed in Section 6, and the paper ends with comments on directions for future work.

In closing, we warn the reader that the considerations of this paper are mathematically heuristic. However, the mathematical structures we use here are not new; indeed this paper may be read as a proposal to apply the mathematical structures of Kauffman [25] and Kauffman and Lins [27] to quantum gravity. We establish physical arguments for the application to quantum gravity of these mathematical structures. Interesting questions such as whether there exist measures on $\mathcal{A}/\mathcal{G}$ [9, 10] associated to framed loops or a useful $q$-deformation of the notion of a connection are not treated here. Finally, we mention that work is underway in collaboration with R. Borissov to compute the action of the deformations of operators such as the volume and $H = \int \sqrt{-C}$, where $C$ is the Hamiltonian constraint of quantum gravity [16].

2 The framed commutative loop algebra

We preface this paper with two remarks. First, our hypothesis has an important consequence for diffeomorphism invariant regularization procedures. As is described in [4], naive operator products derive meaning through a limit procedure in which loops - introduced to make point split operators gauge invariant - are shrunk to points. These limits are outside the topology defined by the diffeomorphism invariant states; values of diffeomorphism invariant states on the “shrunk loops” differ discontinuously from values on finite loops. These limits require new topologies which are external to the structure of diffeomorphism invariant state spaces.

The standard definition of these limits assumes they are state independent. However it is clear that this is not always true. This is shown, for example, by the behavior of the loop operators in the limit that a loop is shrunk to a point. According to the standard definitions of the loop representation, if $\beta^\delta$ is a one parameter family of loops such that, in some background euclidean metric, each is a circle of radius $\delta$ then, under standard definitions, the limit

$$\lim_{\delta \to 0} \langle \alpha | \hat{T}[\beta^\delta] \rangle = -2\langle \alpha |$$ \hspace{1cm} (8)

is independent of the relationship between the loops $\beta^\delta$ and the loop $\alpha$ (Here, we use a choice of trace on the group corresponding to “binor notation;” see Section 2.2). On the other hand, in the presence of the Kodama state the actions of loop operators are given by the path integral of Chern-Simons theory [26] or, equivalently, by the Kauffman bracket. In the limit that loops are shrunk down the effect of the loop operator differs from Eq. (8). Instead we have

$$\lim_{\delta \to 0} \int d\mu[A] \Psi_{CS}[A] T_{\beta^\delta}[A] T_{\alpha}[A] = (-q - q^{-1}) \int d\mu[A] \Psi_{CS}[A] T_{\alpha}[A]$$ \hspace{1cm} (9)
for loops $\beta^\delta$ which have vanishing linking number with $\alpha$. If the loops are linked then the limit depends on linking number as well. Thus, in defining a new loop representation to describe the Kodama state and its excitations, the standard assumptions made in the construction of the diffeomorphism invariant regularization procedures must be extended. Happily, these examples suggest how to modify the usual procedure. It is natural to require that, instead of the naive limits such as Eq. (8), loops, in the limit of a regularization procedure, are governed by Kauffman bracket relations. This requirement holds, by definition, for all states of the form of Eq. (4). We call one parameter families of loops which have point limits “sloops” for “shrinking loops.” This hypothesis, which determines the combinatorics of sloops, will be denoted the “sloop hypothesis.”

The second remark is that we could define the deformed loop algebra directly in terms of its action on the $q$-spin network basis. To do this one only needs to compute the action of loop operators in the spin network basis and then deform that action to a basis labeled by $q$-deformed spin networks. While the end result is equivalent to what we do here, we take the less direct course as it is convenient to have deformed equivalence classes of loops in order to verify relations and perform calculations.

2.1 The basic strategy

Our goal is to construct a framed loop algebra, $\mathcal{LA}_f$. We first define a free complex vector space, $\mathcal{FL}_f$ of formal linear combinations of framed multiloops. On this space we define a product and an equivalence class generated by a list of relations which extend and generalize the Mandelstam relations of standard loop observables. These relations realize the hypothesis that the Kauffman bracket relations hold for sloops.

The product on $\mathcal{LA}_f$, $\alpha_f \cup \beta_f$, will be commutative and associative. This extends the usual commutative algebra of $SU(2)$ Wilson loop observables allowing us to define a deformed algebra of framed loop operators denoted $\hat{T}_q[\alpha]$ such that

$$\hat{T}_q[\alpha]\hat{T}_q[\beta] = \hat{T}_q[\alpha \cup \beta]$$

(10)

Once the algebra is defined we find the representation which is a deformation of the usual loop representation. Finally, higher order $\hat{T}$ operators are constructed in this representation.

2.2 Technical note: binor diagramatics

We define $\mathcal{FL}_f$ in terms of its so-called “binor representation.” An element of the vector space is indicated by a two dimensional diagram, which is called the framed loop diagram of $\alpha_f$, indicated $P(\alpha_f)$. The loop in the spatial manifold is indicated by labeling the edges of the diagram. This diagrammatic notation is defined so that the limit in which the deformation parameter $q \rightarrow 1$ takes us to algebra of $SU(2)$ Wilson loop observables, expressed in a diagrammatic notation due to Penrose called the binor notation. The binor notation has built into it two sign rules which come into the correspondence between the diagram of a loop $P(\alpha)$ and the Wilson loop functionals. These correspond to a definition of the trace of a parallel transport so that $Tr[1] = -2$ together with an assignment of $-1$ to every crossing. This notation has the important advantage that it is local and topologically
invariant in the two dimensional plane in which the diagrams live. This greatly simplifies calculations. The Mandelstam identities become

\[ + + = 0. \tag{11} \]

(Diagrams such as these circled by a dashed line represent changes occurring at a point; the parallel transport along edges inside dashed circles are trivial.) Symmetrizations over spinor indices of elements in the connection representation are (due to the added sign) represented by anti-symmetrizations over multiloops in the binor diagram \( P(\alpha) \).

We can express the deformation in terms of a deformation parameter \( A \) such that

\[ A^2 = q. \tag{12} \]

The usual binor representation is then recovered by taking the limit in which \( A \to -1 \).

## 2.3 Framed loops

The motivation for defining framed loops arises from defining operator products through regularization procedures for the Kodama phase. In these regularization procedures new loops are introduced to connect points that are “split apart” in operator products. The resulting operators are defined as limits in which these loops are shrunk down. Ambiguities in these limits, due to the Chern-Simons factor, may be resolved with a finite amount of topological information. This can be encoded in framing. As shown in [23], expectation values such as in Eq. (1) depend on an integer – the self-linking number of the loop. However, to fully define a framed loop it will not be sufficient to append a self-linking number. Additional ambiguities arise when the loops intersect. To resolve these, the definition of a framed loop will involve additional information associated with each intersection point.

Let us begin with the definition of non-intersecting framed paths.

An individual framed path, denoted by \( \pi^f \) is a path \( \pi : I \to \Sigma \), with a direction field associated to every point of the path - the “framing.” Framing can be seen as a direction in a plane perpendicular to the tangent vector \( \dot{\pi}^a(s) \) of \( \pi(s) \) i.e. \( \pi^f : I \to \Sigma \times S^1 \). Framed loops, denoted \( \alpha^f, \beta^f, \gamma^f, \ldots \), are closed paths, \( \alpha(0) = \alpha(1) \) with a continuous direction field, denoting the frame of \( \alpha^f \) as \( \theta_\alpha, \theta_\alpha(0) = \theta_\alpha(1) \). A framed multiloop, which following the original loop formulation [1], will be also denoted by greek letters, is a set of individual framed loops. The identity for framed loops is the constant map, \( e \), with \( L(e) = 0 \).

The framing is defined modulo smooth deformations of the direction field. As such all that is relevant to define the framing of a non-self-intersecting loop is the self-linking number, which is the number of times the direction field wraps around the loop. More explicitly, the self-linking number \( L(\gamma^f) \) of a framed loop, \( \gamma^f \), is defined in terms of a linking number.

The linking number \( L(\gamma, \beta) \) of two distinct non-intersecting loops \( \gamma \) and \( \beta \) may be expressed in terms of a two dimensional projection, or diagram, of the loops. These loops, given an orientation, have linking number

\[ L(\gamma, \beta) = \frac{1}{2} \sum c \epsilon(c), \tag{13} \]
Figure 1: Examples of framing: (a.) Two unlinked unknots, \( L = 0 \) (b.) A knot with a direction field in the plane of the diagram - “blackboard framing” - giving a linking number \(-2\) between the knot \( \gamma \) and its frame \( \gamma' \) (c.) A pair of intersecting unknots with linking number \( L = B \) [See Eq. (18)].

where \( c \) sums over all the crossings in the diagram and \( \epsilon = 1 \) is for over crossings, \( \nbigtimes \), and \( \epsilon = -1 \) for under crossings, \( \nbigtimes' \). This is a diffeomorphism invariant quantity.

The self-linking number can be computed using the framing direction field. In some background metric, a framed loop \( \gamma \) is displaced an infinitesimal distance in the direction field to obtain another loop \( \gamma' \). Once an orientation is given, the self-linking number \( L(\gamma) \) is the linking number between these two loops \( L(\gamma, \gamma') \), given by Eq. (13). An example is given in Fig. (1b).

The inverse of a framed loop is defined as reversing the tangent vector of the loop, keeping the self-linking number fixed so that the direction field “reverses” or, is mapped to the antipodal point of \( S^1 \).

When there are intersections, additional information is needed to define a “framed loop.” This can be seen if we regard all intersections and overlapping paths as limits points of sequences of non-intersecting loops. These sequences can approach the intersection in a variety of ways. Framing encodes topological information in the limits. For example, two loop segments can, in the limit, touch “from the top”, or “from the bottom.” These two cases may be diagrammatically represented by \( \nbigtimes \) and \( \nbigtimes' \) which we call “touching from the top” and “touching from the bottom,” respectively, and may be thought of as two, distinct results of a limit of a regularization procedure in which loops are brought together. More generally, we can think of the space of loops with intersections as the completion of the space of non-intersecting loops. These intersections \( \nbigtimes \) and \( \nbigtimes' \) represent distinct points in this space.\(^3\)

As the self-linking numbers of the loops are defined only up to arbitrary smooth deformations of the direction field, they play no role at an intersection. Instead, in a neighborhood of the intersection the direction fields of each loop may be deformed smoothly so that the fields lie in the plane formed by the tangent vectors at the intersection. Given this freedom we can define the intersections in “blackboard” framing in which the plane of projection is determined by the tangent vectors. All intersections may then be expressed as a framing factor times one of these intersections. To summarize, in terms of limits of non-intersecting loops there are precisely two, distinct ways that two framed loops can meet at a point, given by \( \nbigtimes \) and \( \nbigtimes' \). For the purposes of the quantum theory we take these to span a two dimensional space of possible states associated with the intersection.

\(^3\)We thank Carlo Rovelli for suggesting this perspective.
Given $\mathcal{X}$ and $\mathcal{X}'$ as basis elements for the different states associated with the intersection, we may define linear combinations of them that correspond to intermediate cases. These will be of the form

$$z = z + z'$$

where $z$ and $z'$ are complex numbers. Of particular interest is a combination defined by $z = z' = B$

$$z = B$$

in which the coefficient $B$ will be chosen below so that these kinds of intersections satisfy the ordinary Mandelstam identities.

Loops can not only pass through each other at points of intersection, there can be “exchanges of parallel transport” such as in $\mathcal{X}$ and $\mathcal{X}'$. In the case of ordinary loops these are related to the unique intersection state $\mathcal{X}$ by the Mandelstam identity Eq. (11). For framed loops we define $\mathcal{X}$ to be the limit of a sequence in which the two loops meet at a point. The other case $\mathcal{X}'$ is defined similarly. Other kinds of intersections are defined in terms of these by the equivalence relation that will be defined in the next subsection. Furthermore, these equivalence relations will leave us, as in the case of ordinary loops, with only two independent states associated with the routings and framings of a simple intersection, $\mathcal{X}'$. These will be defined so that Kauffman bracket relations are recovered for sloops.

It is useful to extend the notion of linking numbers to cases involving intersections. Since each such case is defined as a limit of a sequence of non-intersecting loops it is straightforward in these cases to define the linking numbers in terms of these sequences. Generally the linking numbers of intersecting loops are found deforming the loops slightly in the direction inverse to the limit that defined the intersection. In the case of an “touch from the top” we deform as

$$\mathcal{X} \rightarrow \mathcal{X}'$$

and then compute the linking number. The linking number associated with other linear combinations are then defined by the condition

$$L(\alpha^{I}, \beta^{I} + \gamma^{I}) = L(\alpha^{I}, \beta^{I}) + L(\alpha^{I}, \gamma^{I})$$

For example,

$$L(\bigcirc \bigcirc) = B \left[ L(\bigcirc \bigcirc) + L(\bigcirc \bigcirc) \right] = B$$

Arbitrary intersection points, at which any number of paths meet, may be described with this principle of completing the space of non-intersecting loops. This is done separately, in [18].

Finally, it is useful to extend the usual definition of the product (or continuation) of two loops to the case of framed loops. Given two loops $\alpha^{I}$ and $\beta^{I}$ coincident only at a single intersection point $p = \alpha^{I}(0) = \beta^{I}(0)$ we can define the framed loop combination $(\alpha \ast \beta)^{I}$ to be the framed loop which is the ordinary product of loops with a continuous direction field, i.e. $\theta_{\alpha}(1) = \theta_{\beta}(0)$. This product, $\ast$, is distinguished from the product on the abstract algebra to be defined in the next section.
2.4 An equivalence relation for framed loops and the algebra $\mathcal{LA}^f$

In the usual loop representation we are not interested in the loops themselves, but only in equivalence classes of loops. For $SU(2)$ Wilson loops, these include the Mandelstam identities which arise from the traces of $2 \times 2$ matrices. We extend these relations for framed loops. Guided by the sloop hypothesis, we construct an algebra of framed loops modulo a set of equivalence relations. This algebra we denote $\mathcal{LA}^f$.

Recall that given a connection the identities satisfied by Wilson loops may be implemented on the free vector space of formal sums of single loops by requiring that if

$$\sum_i c_i T[\alpha_i, A] = 0$$

(19)

for all connections, then these loops are linearly dependent

$$\sum_i c_i \alpha_i = 0$$

(20)

on the free vector space of single loops. (For $SU(2)$ Wilson loops, this formulation in terms of single loops and this equivalence relation is equivalent the loop representation with multiloops.) When the product on the free vector space is defined as

$$\left( \sum_j c_j \alpha_j \right) \cdot \left( \sum_k d_k \beta_k \right) = - \sum_{j,k} c_j d_k \left( \alpha_i * \beta_j + \alpha_i * \beta_j^{-1} \right)$$

(21)

these Mandelstam relations define an ideal, so that the quotient of the vector space by the ideal defines an algebra, which is usually called the “holonomy algebra” as a reminder that the Mandelstam relations are augmented by equivalence under holonomy. We would like to generalize these identities for framed loops. However, as no notion of $q$-deformed holonomy exists (to our knowledge) we must fall back on a purely combinatoric definition and thus have to conjecture that the equivalence relations defines an ideal. Fortunately, the hypothesis that the Kauffman bracket relations hold in the limit of small loops suffices to defines the equivalence relations and the resulting algebra.

We begin by defining a set of equivalence relations on the free vector space $\mathcal{FL}^f$ of framed loops. The first two relations are taken over from the usual loop algebra. The first is retracing, for a single loop

$$\alpha^f * \eta^f * \left( \eta^f \right)^{-1} = \alpha^f.$$ 

(22)

where $\eta^f$ is an arbitrary framed path of the loop $\eta^f * \left( \eta^f \right)^{-1}$ beginning at the base point of the framed loop $\alpha^f$. The second identity results from reparametrization invariance, for any function $f : I \rightarrow I$

$$\gamma^f(s) = \gamma^f(f(s)).$$

(23)

While “accelerating” the parameterization of loops has no effect on framing, if the reparametrization does reverse the orientation of any loop then the direction field must be reversed as well.
The remaining relations have no counterpart in ordinary loops. One set has to do with

twisting of a single loop. It is determined by the sloop hypothesis to be

\[ \sim = -A^{-3} \]  

(24)

In addition, the sloop hypothesis determines that the relations between the different kinds

of touching and exchanges must be given by the skein relations,

\[ = A^{-1} + A \]  

\[ = A + A^{-1} \]  

(25)

(26)

for the different touches and reroutings at a simple intersection, \( \otimes \). We may note that

at \( A = -1 \) the framing must be irrelevant, and hence the two independent intersections

\( \otimes \) and \( \otimes \) reduce to an ordinary intersection. We see that both identities reduce to the

Mandelstam identity, written in the binor notation. Further, there is a linear combination

of \( \otimes \) and \( \otimes \) that does satisfy the ordinary Mandelstam identity. The coefficient \( B \)
of Eq. (15) can easily be computed from Eq. (25) and Eq. (26) to be

\[ B = \frac{1}{A + A^{-1}}. \]  

(27)

Thus, these are an extended, combinatorial form of the equivalence relation on the free

vector space of \( SU(2) \) loops, Eq. (20). All of the relations of the holonomy algebra are

included in the extended set of equivalence relations. This extension arises from the new

elements \( \otimes \) and \( \otimes \) which account for framing of intersections.

We may now define the product of two equivalence classes of framed loops. This product,

which we will denote by \( \cup \) is defined analogously to the product of Eq. (21) on the free

vector space of single loops, so that the ordinary Mandelstam identities are satisfied by the

product. If two single, framed loops \( \alpha \) and \( \beta \) intersect then \( \alpha \cup \beta \) is defined to be the

framed loop in which the state at the intersection is taken to be \( \text{defined by Eq. (15)}. \)

This means that the ordinary Mandelstam identities are satisfied, by \( \cup \), so that

\[ \alpha \cup \beta + \alpha * \beta + \alpha * (\beta)^{-1} = 0 \]  

(28)

This will be sufficient to guarantee that the product \( \cup \) is associative and commutative on

the equivalence classes of framed loops.

Next, we define the product \( \cup \) in the case that two loops trace a common path as in Fig.

(2). Consider two framed loops \( \alpha' = \eta_1' * \eta_2' \) and \( \beta' = \eta_2' * \eta_3' \) constructed from the paths

\( \eta_1', \eta_2' \) and \( \eta_3' \) shown in Fig. (2). Note that because of the global nature of the framing we

can take the framing normal to the common path \( \eta_2' \). (We can change the framing of any

line at will using the identity (24).) There is a linking number associated with the overlap

segment of two loops. We may draw a box around the common path such that the points

\( \eta_2'(0) \) and \( \eta_2'(1) \) are embedded in the floor and ceiling of the box (see Fig. (3)). Closing

\( \eta_2 \) to the tangle \( \eta_2 \), we ask that the linking number vanishes. The vanishing of the linking

number over the common path of \( \alpha' \cup \beta' \) means that it can be thought of as the limit of

a sequence of unlinked paths.
The elements of the “extended holonomy equivalence classes” on \( \mathcal{FL}_f \) defined by the relations Eqs. (22), (23), (24), and (24) will be denoted by \( \tilde{\alpha}^f \). The algebra constructed from these elements \( \tilde{\alpha}^f \) with the product \( \cup \) is an abelian, associative algebra, which we will call the framed loop algebra and denote \( \mathcal{LA}_f \).

The commutativity and associativity of \( \mathcal{LA}_f \) follow, as in the \( SU(2) \) case, directly from the Mandelstam relations. The key observation is that the usual equivalence \( \alpha = \alpha^{-1} \) in the usual holonomy algebra is also true on \( \mathcal{LA}_f \), \( \tilde{\alpha}^f = (\tilde{\alpha}^f)^{-1} \). This follows both from the definition of the direction field framing of the inverse and also from the inclusion of all the equivalence relations for normal, unframed loops in the extended equivalence relations.

### 2.5 The operator algebra on framed multiloops

We have defined a formal algebra, \( \mathcal{LA}_f \). We now construct a representation of this algebra and use it to define the corresponding quantum theory. This first step to do this is to express the algebra as a formal algebra of linear operators. To do this we define an operator \( \hat{T}_q[\alpha] \) associated to each element \( \tilde{\alpha}^f \) of \( \mathcal{LA}_f \). The subscript \( q \) on the operator \( \hat{T}_q[\alpha] \) means that it is associated with framed loop \( \tilde{\alpha}^f \). Thus, since the label \( q \) and the superscript \( f \) are redundant, we drop the \( f \).

We define the operator product so that

\[
\hat{T}_q[\alpha] \hat{T}_q[\beta] \equiv T_q[\alpha \cup \beta].
\]

The algebra of the operators \( T_q[\alpha] \) is associative and commutative by virtue of the properties of \( \cup \). This also means that the \( T_q[\alpha] \) will satisfy the ordinary Mandelstam identities (modulo framing factors associated with the twistings of the loops.) However, not all of the relations satisfied by the \( \hat{T}_q[\alpha] \) agree with the relations defined for Wilson loops of smooth \( SU(2) \) connections. In particular, in the case in which limits are taken in which loops are shrunk down, we find a deformation of the usual relations satisfied by \( SU(2) \) holonomies. This is forced by the requirement that the Kauffman bracket identities are satisfied for sloops. To see this let \( \beta^f(s, t) \) be a one parameter family of unknots such that \( \beta^f(s, 0) = \beta^f(s) \) and \( \beta(s, 1) = e \), the identity loop at the base point, for all \( s \in I \). If the framing is such that
Figure 3: For loops contract able to a point the linking number changes the limit. In this case, the limit as $\beta$ shrinks to a point is $-A^4 - A^{-4}$.

$L[\beta^f(s, t)] = 0$ and $L[\alpha^f, \beta^f(s, t)] = 0$ for all $t \in I$ and loops $\alpha^f$ we have,

$$\lim_{t \to 1} \hat{T}_q[\beta(s, t)]\hat{T}_q[\alpha] = d\hat{T}_q[\alpha]$$  \hspace{1cm} (30)

The identities given already determine

$$d = -q - q^{-1}.$$  \hspace{1cm} (31)

The cases in which $L[\alpha^f, \beta^f(s, t)] \neq 0$ are also determined by the identities, for example in the case $L[\alpha, \beta^f(s, t)] = 1$, shown in Fig. (3) we have [27]

$$\lim_{t \to 1} \hat{T}_q[\beta(s, t)]\hat{T}_q[\alpha] = (-A^4 - A^{-4})\hat{T}_q[\alpha].$$  \hspace{1cm} (32)

2.6 The $q$-spin net basis

An independent basis for the algebra $\mathcal{L}\mathcal{A}^f$ is given by linear combinations of framed loops labeled $q$-spin nets of $SU(2)_q$. A q-spin net is a labelled graph with a vertex set of arbitrary valence. Each edge is labeled by an integer $j$ taken from the set $1, 2, ..., r-1$. Vertices are labeled by additional sets of integers, describing how the singlet representation may be extracted from the product of incident edge representations. For each valence there must be at least one way to extract the singlet, which leads to certain admissibility conditions. For the trivalent case, there is a unique way and the admissibility conditions for $(l, m, n)$ require that $l + m - n, l + n - m$ and $m + n - l$ are positive and even and that $l + m + n \leq 2r - 4$ [27].

Given a q-spin net we may construct a representation of $\mathcal{L}\mathcal{A}^f$ by a simple prescription. Each edge labeled by an integer $n$ is written as a linear combination of terms in which $n$ lines transverse the same curve, with possible braidings. These are given by the formula [27]

$$\begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} = \frac{A^{2n-2}}{[n]!} \sum_{\sigma \in S_n} (A^{-3})^{t(\sigma)}$$  \hspace{1cm} (33)

where the “quantum integer” $[n]$ is defined by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}},$$  \hspace{1cm} (34)
Figure 4: The trivalent vertex (a.) is decomposed into three projectors as in (b.) with
\[ a = (j + k - l)/2, \quad b = (k + l - j)/2, \quad \text{and} \quad c = (j + l - k)/2. \]

the factorial is defined as \([n]! = [n][n-1], \ldots\), and \(\sigma\) is an element of the permutation \(S_n\)
with a minimal braid representation \(\bar{\sigma}\) consisting of the minimum number of over crossing elements \(\sigma\) of the braid group. For instance, for \(n = 2\) we have
\[
\begin{align*}
2! &= \frac{A^2}{[2]!} \left( \big| + A^{-3} \big| \right) \\
&= \big| - \frac{1}{d} \big| \big| \big| \\
\end{align*}
\]
in which the over crossing on the first line represents the “touching from the top,” \(\bar{\sigma}\). We may note that these satisfy some simple identities \(^2\).

\[
\begin{align*}
\begin{array}{r}
\parbox{1cm}{\begin{tikzpicture}
\begin{scope}
\tikzstyle{line}=[draw=black, very thick]
\tikzstyle{vertex}=[draw,fill=white,circle]
\node (v1) at (0,0) [vertex] {};
\node (v2) at (-0.5,0.5) [vertex] {};
\node (v3) at (0.5,0.5) [vertex] {};
\draw (v1) -- (v2) -- (v3);
\end{scope}
\end{tikzpicture}}
\end{array}
& = \begin{array}{c}
\parbox{1cm}{\begin{tikzpicture}
\begin{scope}
\tikzstyle{line}=[draw=black, very thick]
\tikzstyle{vertex}=[draw,fill=white,circle]
\node (v1) at (0,0) [vertex] {};
\node (v2) at (-0.5,0) [vertex] {};
\node (v3) at (0.5,0) [vertex] {};
\draw (v1) -- (v2) -- (v3);
\end{scope}
\end{tikzpicture}}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{r}
\parbox{1cm}{\begin{tikzpicture}
\begin{scope}
\tikzstyle{line}=[draw=black, very thick]
\tikzstyle{vertex}=[draw,fill=white,circle]
\node (v1) at (0,0) [vertex] {};
\node (v2) at (-0.5,0.5) [vertex] {};
\node (v3) at (0.5,0.5) [vertex] {};
\draw (v1) -- (v2) -- (v3);
\end{scope}
\end{tikzpicture}} \quad \begin{array}{c}
\parbox{1cm}{\begin{tikzpicture}
\begin{scope}
\tikzstyle{line}=[draw=black, very thick]
\tikzstyle{vertex}=[draw,fill=white,circle]
\node (v1) at (0,0) [vertex] {};
\node (v2) at (-0.5,0) [vertex] {};
\node (v3) at (0.5,0) [vertex] {};
\draw (v1) -- (v2) -- (v3);
\end{scope}
\end{tikzpicture}}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{r}
\parbox{1cm}{\begin{tikzpicture}
\begin{scope}
\tikzstyle{line}=[draw=black, very thick]
\tikzstyle{vertex}=[draw,fill=white,circle]
\node (v1) at (0,0) [vertex] {};
\node (v2) at (-0.5,0.5) [vertex] {};
\node (v3) at (0.5,0.5) [vertex] {};
\draw (v1) -- (v2) -- (v3);
\end{scope}
\end{tikzpicture}}
\end{array}
& = 0 \\
\begin{array}{r}
\parbox{1cm}{\begin{tikzpicture}
\begin{scope}
\tikzstyle{line}=[draw=black, very thick]
\tikzstyle{vertex}=[draw,fill=white,circle]
\node (v1) at (0,0) [vertex] {};
\node (v2) at (-0.5,0.5) [vertex] {};
\node (v3) at (0.5,0.5) [vertex] {};
\draw (v1) -- (v2) -- (v3);
\end{scope}
\end{tikzpicture}}
\end{array}
& = (-1)^n[n + 1] \\
\begin{array}{c}
\parbox{1cm}{\begin{tikzpicture}
\begin{scope}
\tikzstyle{line}=[draw=black, very thick]
\tikzstyle{vertex}=[draw,fill=white,circle]
\node (v1) at (0,0) [vertex] {};
\node (v2) at (-0.5,0.5) [vertex] {};
\node (v3) at (0.5,0.5) [vertex] {};
\draw (v1) -- (v2) -- (v3);
\end{scope}
\end{tikzpicture}}
\end{array}
& = 0.
\end{align*}
\]

Trivalent intersections are decomposed according to Fig. (4). A vertex of higher valence
requires an additional label because there is more than one way to combine the \(SU(2)_q\) rep-
representations of its incident edges into an \(SU(2)_q\) singlet. Thus there is a finite dimensional linear space to each \(n\) valent vertex \((n > 3)\) with incident edges labeled by the \(j_i\). A basis for these vertices may be constructed in the following way. One first picks an arbitrary ordering of the edges which are incident on the vertex. One then decomposes the \(n\) valent vertex into a combination of trivalent vertices as illustrated in Fig. (5). The number of internal vertices is \(l = 1 + (n - 4) = n - 3\). A set of linearly independent states associated with the \(n\) valent vertex are label sets \(l\) of the ordering of external lines and the internal
Figure 5: The decomposition of a higher valent intersection into trivalent intersections at a point. The first two incident edges are joined to a new internal edge $i_1$ at the first vertex. Then $i_1$ and $e_3$ are joined into a trivalent vertex with a new internal line $i_2$. The process continues until there are two external vertices left which are joined into the last three vertex with the last internal line, in this case $i_2$.

representations $i_1, i_2, \ldots, r - 1$ on the internal lines so that the trivalent vertices created by this procedure are admissible.

Three comments should be made about this labeling. First internal edges have zero length in the manifold $\Sigma$, so that all the trivalent vertices in this “blowing up at the vertex” are at the same point of $\Sigma$ as the original $n$-valent vertex. Second, given a different labeling of the external edges, the same procedure will yield a different, orthogonal basis. Each relabeling of the edges of the graph thus is represented by a unitary transformation in each of the spaces associated with the vertices. Finally, a decomposition of the vertices of a spin network may be given by arbitrarily labeling all of its edges, which induces a labeling of the edges of each vertex.

Given a q-spin net $\Gamma^q$, we have, after expanding the terms, an element of $\mathcal{L}\mathcal{A}^f$. Conversely, it is straightforward to show that given any framed multiloop $\gamma^f$ we can construct a unique formal linear combination of quantum spin networks $\Gamma^q_i$ so that

$$\gamma^f = \sum_i c_i \Gamma^q_i$$

The construction follows an algorithmic procedure, which extends (because of the extension of the Temperley-Lieb algebra to finite loop segments representing holonomies) the algorithm of Kauffman and Lins [27]. We proceed by labeling abstract edges by framed loops. Each edge may carry a number of segments representing framed loops with common support between vertices. Vertices occurs where the support of the loops changes. Given a labeling of the edges the framing dependence of the loops may be expanded as a sum of $q$-symmetrized lines, defined by Eq. (33). Independent routings involving $n$ segments transversing a single edge are elements of the “extended” (in the sense of holonomy) Temperly-Lieb algebra $\mathcal{T}_n$. But, as described in Kauffman and Lins, a basis for $\mathcal{T}_n$ is described in terms of projection operators and retracing elements such as $\mathcal{X}$; which, because of equivalence under retracings, pull back to the adjacent vertex. The result is an expansion of the framed multiloop as a sum of terms of $q$-symmetrized lines incident on a vertex - a q-spin network.

We have sketched a demonstration that the q-spin nets provide a representation of the framed loop algebra $\mathcal{L}\mathcal{A}^f$. It remains to show that the q-spin nets are independent under the identities of Section 2.4. The demonstration will not be given in full here, but we note that it is an extension of Proposition 2 of Kauffman and Lins. The basic step uses the fact,
already mentioned, that to each edge with $n$ common segments we may associate elements 
of the Temperly-Lieb algebra $T_n$ such that the different $q$-spins label orthogonal projection 
operators.

Finally, we note again that in the case of valences $n > 3$ the uniqueness of the $q$-spin 
network basis is only up to arbitrary relabeling of the edges of the graph, as different 
labelings induce unitary changes of basis at each vertex of valence $n > 3$.

3 The framed loop representation in the $q$-spin net basis

Define $\mathcal{H}^q$ to be the space of functionals on $\mathcal{L}A^f$. Introducing “bra” states $\langle \alpha^f |$ for 
$\tilde{\alpha}^f \in \mathcal{L}A^f$ this may be written as,

$$\Psi[\alpha^f] = \langle \alpha^f | \Psi \rangle. \quad (39)$$

On this space of states we may define a representation of the framed commutative 
loop algebra by

$$\langle \alpha^f | \hat{T}_q[\beta] = \langle \alpha^f \cup \beta^f | \quad (40)$$

By the product properties (30) this defines a faithful representation of the algebra. One 
of the key results of the loop representation is the existence of the spin network basis.\[5\]

We showed above an analogous result, which is that the algebra $\mathcal{L}A^f$, has an independent 
basis given by the $q$-spin nets. We may now apply this directly to show that $\mathcal{H}^q$ has an 
independent basis given by the $q$-spin nets.

Given the decomposition of a $q$-spin net $\Gamma^q = \sum_i c_i \gamma_i^q$ in terms of framed multiloops $\gamma_i^f$ 
we define

$$\langle \Gamma^q | = \sum_i c_i \langle \gamma_i^q | \quad (41)$$

It follows from the independence of the $\Gamma^q$ in $\mathcal{L}A^f$ that these states are independent and 
thus provide a basis (again up to unitary transformations at each higher than trivalent 
node induced by relabeling the edges.)

We may now define the action of the $\hat{T}_q[\alpha]$ directly on the $q$-spin net basis. Given a 
$q$-spin net $\Gamma^q$ and a loop $\tilde{\alpha}^f$, we can define a unique decomposition of the framed loop 
product, of a framed loop and a spin network

$$\Gamma^q \cup \tilde{\alpha}^f = \sum_i c_i [\Gamma \cup \alpha_i]_i^q \quad (42)$$

where the $[\Gamma \cup \alpha_i]_i^q$ are the spin networks produced by iterating the edge addition identity

$$\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{edge_addition_identity}
\end{array}
\end{array}
- \frac{[n]}{[n+1]} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{edge_addition_identity}
\end{array}
\end{array}.
\quad (43)$$

When adding two edges labeled by one (frequently used in the action of the $T$ operators) 
this edge addition identity is simply

$$| = 2 | + \frac{1}{d} \bigcap \bigcup. \quad (44)$$

\[4\]This result has been made mathematically rigorous in the context of diffeomorphism invariant measures 
on the connection representation by Baez [17].
Figure 6: The action of the $T^a[\alpha]$ on a spin network

Note that $\tilde{\alpha}^f$ may intersect $\Gamma^q$ either in isolated points or in common edges. We use the edge addition identity in Eq. (43) along every edge on which $\tilde{\alpha}^f$ and $\Gamma^q$ overlap. If we label the remaining edges of $\tilde{\alpha}^f$ which have no common segment with $\Gamma^q$ with one then we have a sum of graphs with labeled edges. The isolated intersection points of $\tilde{\alpha}^f$ and $\Gamma^q$ are decomposed as $n$-valent vertices. When they have a transverse intersection we have a new 4-valent vertex with internal edges, $1_1, 1_j, j_j$. When the loop and network part at a $n$-valent vertex then we have an $n+2$ valent vertex. We apply the same technique to each beginning or end of an overlapped edge, which either is a new vertex or is a change in vertex labeling when the overlapped edge ends on an existing vertex. We may then conclude that

$$\langle \Gamma^q | \hat{T}_q[\alpha] = \sum_i c_i([\Gamma \cup \alpha]^q_i) \rangle$$

This gives us the action of the loop operator directly in the spin network basis. We may note that this formula applies at all $A$ including $A = -1$ that corresponds to the classical case. Thus it applies equally well to ordinary spin network states.

Finally, following the usual procedure for spin network states [5] we can impose an inner product on $H^q$ extending the inner product on spin networks to q-spin nets.

$$< S^q | \Gamma^q > = \delta_{S^q \Gamma^q}. \quad (46)$$

4 The quantum deformed $\hat{T}^a_q$ operators

To complete the definition of the deformed loop algebra we need to give a definition of the “$T^{11}$” operators acting on $H^q$ and show that the result is a closed algebra. First, we work out the action of ordinary $T^a[\alpha](s)$ operators on the spin network basis. By extending this formula, the action of a $q$-deformed $\hat{T}^a_q[\alpha](s)$ is defined for $q$ at a root of unity. We then check that $\hat{T}_q[\alpha]$ and $\hat{T}^a_q[\beta](s)$ form a closed algebra.

The action of an operator $\hat{T}^a[\alpha](s)$ on a spin network state $\{ \Gamma \}$ is illustrated in Fig. 4. When the hand at the point $\alpha(s)$ coincides with a point on an edge of $\Gamma$ with spin $n$ a new four valent vertex is created with incoming edges $(n, n, 1_1)$ as shown. The particular vertex is shown in Fig. 6, it may be decomposed into a trivalent $(n, n, 2)$ vertex connected through an “internal” 2 line to a $(2, 1, 1)$ vertex. The result is multiplied by a factor of $nl \delta^q_1$ and $\Delta^a[\Gamma^q, \alpha](s)$, the distributional factor

$$\Delta^a[\Gamma^q, \alpha](s) = \sum_T \int dt \delta^3(e_I(t), \alpha(s)) \delta^q_I(s) \quad (47)$$
where the sum is over the edges, $e_I$ of the network.

On q-spin nets we will then define the operator $\hat{T}_q^a[\alpha](s)$ to act by exactly the same diagram (6), where the vertex is now a q-spin net vertex. The factors we multiply by of Eq. (47) and $j l^2 P_l$ are taken to be the same. There is an ambiguity in the definition of the action of a hand that is not present in the ordinary $A = -1$ case. This arises when the four valent intersection is defined in terms of trivalent vertices according to Fig. (6b). There may be a phase factor depending on whether the one line crosses over or under the $n$ line. When the vertex is created by the action of an operator, the ambiguity is resolved by careful definition of the regularized operator. We will see below that there is a natural choice in the case of the area operator.

We may now show that the quantum deformed loop operators $\hat{T}_q^a[\alpha]$ and $\hat{T}_q^a[\alpha](s)$ define a closed algebra. We first note that the $\hat{T}_q^a[\alpha]$’s commute, by Eq. (28). As in the case of the ordinary loop algebra, it is straightforward to compute the commutator and verify

$$[\hat{T}_q^a[\alpha], \hat{T}_q^a[\beta](s)] = j l^2 P_l \Delta^a[\alpha, \beta](s) \hat{T}[\alpha\#s\beta]$$  \hspace{1cm} (48)$$

where $\alpha$ is an arbitrary q-spin net, $\beta^I$ is a single framed loop and the combination $\alpha\#s\beta$ is a q-spin net constructed according to the following prescription: Break the loop $\beta^I$ at $s$ and break the edge of the spin network $\alpha$ at the point $p = \beta(s)$. Let the valence of the edge which coincides with $p$ be $n$. We then reconnect the lines with a four valent vertex with a line labeled by two connecting the broken $n$ line and the broken 1 line as shown in Fig. (6). In the case that more than one edge of the spin network $\alpha$ coincides with the point $p$, the result of the commutator is the sum of the actions on each edge. We may note that given the definition of the operator on a $n = 1$ line the action for arbitrary $n$ can also be recovered from the algebra. Also, as in the original case, we could express this in terms of the strip-loop algebra [4], so that the coefficients of the algebra are non-singular.

It remains to verify the commutation relations of the $\hat{T}_q^a[\beta](s)$. To do this it is helpful to define a more general notion of these operators. Let $\beta$ now be a general q-spin net, and let $t$ be any parameterization of the edges. (For example, all the edges may be ordered and the $i$’th each may be parameterized by $s$ running between $i - 1$ and $i$.) We then can define an operator $\hat{T}_q^a[\beta](s)$ for every $s$ such that $\beta(s)$ is on a single (or a “one”) line. The definition in this case is taken from Eq. (48)

$$\langle \alpha^I | \hat{T}_q^a[\beta](s) = \sum_I j I \Delta^a[\alpha, \beta](s) \langle \alpha\#I\beta^I \rangle,$$  \hspace{1cm} (49)$$

where $I$ labels the intersection points where the action is non-vanishing.

It is then straightforward to verify by direct computation that

$$[\hat{T}_q^a[\alpha](s), \hat{T}_q^b[\beta](t)] = l^2 P_l \left( j \Delta^b[\alpha, \beta](t) \hat{T}_q^a[\alpha\#t\beta](s) - j' \Delta^a[\beta, \alpha](s) \hat{T}_q^b[\beta\#s\alpha](t) \right)$$  \hspace{1cm} (50)$$

where $j$ and $j'$ are the q-spin of the lines on which the “hands” act in each case. In verifying these relations, it is convenient to use the various identities which organize the results into the spin network basis only in the last steps of calculation.

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5 Higher loop operators and the $q$-deformed area operator

Once we have defined the action of the first two loop operators the definition may be extended for loop operators with any number of “hands” that correspond to points of insertion of the conjugate electric field $\hat{E}^{ai}$ in the classical loop algebra. We do this by requiring that each hand acts according to the usual definitions of loop operators, but where the combinatorics and framing at each hand is given by Fig. (6). For example, the action of $\hat{T}^{ab}[\gamma](s, t)$, for $\gamma$ a q-spin net with $\gamma(s)$ and $\gamma(t)$ on one segments, may be defined as

$$\langle \alpha^f | \hat{T}^{ab}[\gamma](s, t) = \sum_{IJ} j_{IJ} \alpha^I \Delta^a_\alpha(s) \Delta^b_\alpha(s) \langle \alpha^f \# s \# t \gamma^f |$$

where $\alpha_I$ are edges of the q-spin net and $\alpha^f \# s \# t \gamma^f$ is constructed by implementing the action described above at the coincident points of $\alpha^f$, $\gamma^f(s)$, and $\gamma^f(t)$. There are, however, framing choices for the operators. Defined as the limit of a sequence of loops, these operators have the same ambiguity that arises for intersecting framed loops. As an example, we give a definition of the area operator. 

A $q$-area operator, $A^q[S]$, which measures the area of a surface $S$, is constructed by the procedure described in [6], where the operator we have just defined replaces the usual $\hat{T}^{ab}[\gamma](s, t)$. One discovers as before that the simultaneous eigenstates of all these operators are given by the q-spin networks, or by linear combinations of them involving the different routings in higher valent vertices. Following the same reasoning as in [?] one sees directly that the spectrum is discrete. The eigenvalues of $A^q[S]$ may be calculated in a similar manner as before. Writing the operator as

$$A^q | \Gamma^q n \rangle = \sum_I \sqrt{\frac{1}{\mathcal{S}^q_I(n)}} | \Gamma^q n \rangle$$

were the sum is over intersections of the surface $S$ and the q-spin net. When the q-spin net $\Gamma^q$ intersects $S$ at a single edge, the combinatorial operator $A^q$ “grasps” this edge

$$8 \left( A^q \right)^2 | \Gamma^q n \rangle = n^2 \left( \begin{array}{c} \includegraphics{fig4b} \end{array} \right) = n^2 \left( \begin{array}{c} \includegraphics{fig53} \end{array} \right)$$

(53)

The first line defines the area operator supported on a framed loop with vanishing self linking, represented here as an edge labeled by 2. The second line expresses the first in the explicit form of the vertex given in Fig. (4b). We may then use the identity of Eq. (44) to continue

$$8 \left( A^q \right)^2 | \Gamma^q n \rangle = n^2 \left( \begin{array}{c} \includegraphics{fig54} \end{array} \right) - \frac{1}{d}$$

(54)
which, using identity (36) reduces to, with constants restored,

$$\hat{A}^q | \Gamma^q n \rangle = i^2 \frac{n}{2} \sqrt{\frac{1}{2} \left[ \frac{n+1}{n} \right] - \frac{1}{2} \left[ \frac{n}{n} \right]} | \Gamma^q n \rangle.$$  \hspace{1cm} (55)

One can easily verify that in the limit $A \to -1$ the usual eigenvalues proportional to $\sqrt{j(j+1)}$ (where $j = n/2$) are recovered. We may note that this result is not equal to the square root of the $q$-deformed Casimir operator $[j][j+1]$.

6 Eigenstates of the $\hat{T}_q[\alpha]$

$q$-spin nets have an interesting property for $q$ at a root of unity, which is that there are only a finite number of representations possible on each edge of a graph. We can exploit this fact to arrive at finite expressions for eigenstates of the $\hat{T}_q[\alpha]$ operators. This may allow us to define an inverse transform that will enable us to define a notion that corresponds to the conjugacy classes of connections in the $q$-deformed case.

We consider an eigenstate of $\hat{T}_q[\alpha]$ associated with a simple, un-twisted unknot $\tilde{\alpha}_f$. This will be of the form

$$\langle \alpha | = \sum_{i=1}^{k} c_i \langle \alpha, i |$$  \hspace{1cm} (56)

where $\langle \alpha, i |$ is the spin network state associated to the $i$ representation traced on the framed loop $\tilde{\alpha}_f$. We want to find the coefficients $c_i$ such that

$$\langle \alpha | \hat{T}_q[\alpha] = \lambda \langle \alpha |.$$  \hspace{1cm} (57)

We may use the identity shown in Eq. (43) to find,

$$\langle \alpha, i | \hat{T}_q[\alpha] = \langle \alpha, i + 1 | + \langle \alpha, i - 1 |$$  \hspace{1cm} (58)

where we use implicitly also that $\langle \alpha, r - 1 | = 0$. That is, the solution for the eigenstate uses crucially the fact that the representations of $SU(2)_q$ extend only from spin 0 to spin $r - 1$. It is easy to extract the relations,

$$c_{r-2} = \lambda c_{r-1}, \quad \lambda c_i = c_{i+1} + c_{i-1}, \quad 1 < i < r - 1 \hspace{1cm} (59)$$

$$c_2 = \lambda c_1.$$  \hspace{1cm} (60)

These may be solved in all cases to find a polynomial in $\lambda$. For a given $k$, and thus $r$, there are a finite number of eigenvalues, which are given by solutions to

$$\lambda W^{r-2}(\lambda) = 1$$  \hspace{1cm} (61)

where $W^i(\lambda)$ are defined by

$$W^i(\lambda) = \lambda - \frac{1}{W^{i-1}(\lambda)}$$  \hspace{1cm} (62)

and

$$W^1(\lambda) = \lambda.$$  \hspace{1cm} (63)
7 Discussion

We close with a few comments on directions which may be explored. First, the framed multiloop product $\cup$ may be extended to the general case including arbitrary intersections. This is done in [18]. The combinatorial argument for the uniqueness and independence of the spin network basis that we sketched above should be completed. We expect that this involves only a careful iteration of cases, as it is a simple extension of known results about the Kauffman bracket. This would also be interesting as it would provide an alternative proof of the independence of the spin network basis even in the classical ($q = 1$) case, which would not rely on the connection representation.

There remains the question of whether the formalism we have defined here represents a departure from the notion that the state space of quantum gravity should be defined in terms of measures. It is interesting to conjecture that there may be a completion of this space built on framed rather than ordinary loops. Alternatively, there may exist a framework in which an extended holonomies seen a maps from a framed loop group (in the sense of Gambini and collaborators[2] to $SU(2)_q$ or a related structure. One way to such a construction could be through an inverse transform[19] constructed using the results of Section 6.

We may also remark that the structures we have described here must bear some relationship to the general notion of extended loops developed in[20]. The construction of extended loops was motivated by the similar considerations which gave rise to framed loops. They have the advantage that they are defined in terms of functionals of connections and do allow a definition of integrals related to Eq. (1). However, examples are known in which framed loops are not gauge invariant[21], and thus are not well defined on functionals on $A/G$. Although the question has not, to our knowledge, been settled, it is possible that there is a restricted class of extended loops that are gauge invariant. Such a notion of restricted extended loops may be related to the notion of framed loops that we have used here.

At the present it is not clear if such links between the $q$-deformed loop algebra and the connection representation will emerge. However, even if the results of these investigations were negative, it would not mean that the $q$-deformed loop representation is not useful for quantum gravity. Instead, it may be that an aspect of the quantum world expressible in terms of non-local observables based on framed loops is not captured in terms of the classical description based on connections. As the quantum world is prior to its classical approximation, this may reflect only the necessity of leaving behind the fiction of deriving a quantum theory from its classical limit.

Finally, we may note that the $q$-deformed loop representation may have practical value in calculations in quantum gravity. As will be described elsewhere, the most efficient procedure for computing with spin network states, which employs the recoupling theory extends to the $q$-deformed case[14, 27]. The main difference is that because of the restriction to $j \leq r - 1$, one cannot concentrate more than a fixed amount of area on the edge of one graph, or too much volume on a vertex of a graph. This means that for fixed $k$ the infinite volume limit must be a limit in which graphs become larger and more complex. This may mean that both perturbative and path integral calculations at finite $k$ may be better behaved with respect to possible infrared divergences than the classical $q = 1$ case. Even
if the limit of large $k$, and hence small cosmological constant is to taken in the end, the $q$-deformation may then serve as a natural, diffeomorphism invariant infrared regulator for non-perturbative quantum gravity.

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