Effective field theory methods to model compact binaries

Stefano Foffa\textsuperscript{1} and Riccardo Sturani\textsuperscript{2}

\textsuperscript{1} Département de Physique Théorique, Université de Genève, CH-1211 Geneva, Switzerland
\textsuperscript{2} ICTP South American Institute for Fundamental Research, Instituto de Física Teórica, Universidade Estadual Paulista, Sao Paulo, SP 011040-070, Brasil

E-mail: stefano.foffa@unige.ch and sturani@ift.unesp.br

Received 21 October 2013, revised 18 October 2013
Accepted for publication 20 December 2013
Published 28 January 2014

Abstract
In this short review we present a self-contained exposition of the effective field theory method approach to model the dynamics of gravitationally bound compact binary systems within the post-Newtonian approximation to General Relativity. Applications of this approach to the conservative sector, as well as to the radiation emission by the binary system are discussed in their salient features. Most important results are discussed, as in-depths and details can be found in the referenced papers.

Keywords: post-Newtonian approximation to general relativity, effective field theory, gravitational waves, gravitational 2-body problem
PACS numbers: 04.20.−q, 04.25.Nx, 04.30.Db

1. Introduction
The existence of gravitational waves (GWs) is an unavoidable prediction of General Relativity (GR) and astrophysical objects bound in binary systems are prototypical, even though not exclusive, sources of GWs. The precise evidence of a system emitting GWs comes from the celebrated ‘Hulse–Taylor’ binary pulsar [1], whose orbital decay rate is in agreement with the GR prediction to about one part in a thousand [2], see also [3–6] for more examples of observed GW emission from pulsar binary systems.

A network of earth-based, kilometer-sized GW observatories is currently under development with the goal of detecting GWs: the two Laser Interferometer Gravitational-Wave Observatories (LIGO) in the US and the French–Italian Virgo interferometer in Italy have been taking data at unprecedented sensitivities for several years, see e.g. [7] for recent...
results, and are now undergoing upgrades to their advanced stage, see e.g. [8] for a recent review (another smaller detector belonging to the network is the German–British Gravitational Wave Detector GEO600). The gravitational detector network is planned to be joined by the Japanese detector KAGRA by the end of this decade [9] and by an additional interferometer in India by the beginning of the next decade. The advanced detector era is planned to start in the year 2015 and it is expected that few years will be necessary to reach planned sensitivity, which should allow several detections of GW events per year [10].

Compact binary systems offer a privileged setting where to confront GR with observations, and their dynamics has been the object of intensive studies since the advent of GR. Here we focus on the post-Newtonian (PN) approximation to GR, see e.g. [11] for a review, which consists in a perturbative expansion around Minkowski space. The expansion parameter is the relative velocity $v$ of the binary constituents, or equivalently the gravitational field strength $G_N M/r$ (where $G_N$ is the standard Newton constant, $M$ the total mass of the binary system, and $r$ the orbital separation between its constituents), as by the virial theorem $v^2 \sim G_N M/r$.

The approach to solving for the dynamics of the two-body problem adopted here relies on the non-relativistic formulation of GR originally proposed in [12], see also [13] for a review, which sets the problem in an effective field theory (EFT) framework.

The use of Feynman diagrams or diagrammatic techniques to address the two-body problem in GR is not a novelty, see e.g. [14–16] and [17] respectively for pioneer work in this direction. With respect to these early works, the EFT approach has the merit of recognizing scale separation as an organizational principle for systematic computation at the Lagrangian level.

Indeed the two-body problem exhibits a clear separation of scales: the size of the compact objects $r_s$, like black holes and/or neutron stars, the orbital separation $r$ and the gravitational wavelength $\lambda$. Using again the virial theorem the hierarchy $r_s < r \sim r_s/v^2 < \lambda \sim r/v$ can be established. The EFT approach allows to use the scale separation of the physical problem to arrange a transparent and systematic power counting in the expansion parameter, with physics at different scales related by renormalization group flow. It has many common features with ordinary quantum field theory (QFT) (Feynman diagrams, divergence regularizations, logarithmic running of physical observables) as both the classical effective field theory described here and QFT share properties belonging to any field theory. For this reason the language used here and in the other papers treating classical gravity with effective field theory methods is reminiscent of quantum concepts as the field theory tool involved have been historically developed first to treat quantum phenomena. For instance the Feynman path integral is employed here at a completely classical level in order to solve perturbatively the GR equations of motion.

The interest in the analytic description of gravitationally bound binary systems has been revived in recent times by the activity of the above mentioned GW observatories whose output is particularly sensitive to the GW phase. It is very important to have an accurate description of the waveform, whose shape depends on the source motion, for both maximizing the detection probability and for extracting the highest possible physical content from candidate events. Moreover data analysis techniques involve the generation of several tens of thousands to millions waveforms, thus requiring their analytic knowledge in order to have quick and efficient data analysis pipelines.

In particular the dynamical quantities allowing to determine physical observables like the phase of the GW signal, are the energy of the bound orbit and the emitted flux of GWs. Since signals falling in the detector band sensitivity are in the very last stage of the coalescence,
the binary system orbits are expected to have circularized by then [18, 19], so the analytic quantities to be computed are the energy of circular orbits and the gravitational flux as a function of the relative velocity of the binary constituents.

Moreover recent progress have made available numerical-relativity waveforms emitted in the last $O(10)$ orbits of a binary system (including merger and ring-down) [20]. In order to construct complete hybrid waveforms encompassing all the stages of a coalescence from inspiral to merger and ring-down, the highest possible accuracy on the analytic inspiral phase is necessary to reduce the length of the numerically evolved part of the waveform, which is in general very time consuming [21].

It is then expected that GW observations and numerical modeling will bring new inputs from both the phenomenological and the theoretical numerical side to the two-body problem in GR, whereas the effective field theory approach described here is giving new momentum to the analytic studies on the theoretical side.

2. General theory

The effective field theory approach to the GR two-body problem is analogue to other effective field theory approaches adopted to study specific systems in particle physics, like the heavy quark field theory [22, 23]. We want to study the dynamics of a pair of heavy and compact objects (black holes/neutron stars) interacting through the exchange of gravitational degrees of freedom and emitting GWs.

The effective Lagrangian $S_{\text{eff}}$ of any extended object of size $r_{\text{source}}$ interacting with a gravitational field with characteristic length-scale variation $L \gg r_{\text{source}}$, can be parametrized in terms of its mass $m$, spin tensor $S_{ab}$ and higher order multipoles [24]

$$S_{\text{eff}} \supset \int \! \mathrm{d}t \left( -m - \frac{1}{2} S_{ab} \omega_{a\mu}^b u^\mu + \frac{cQ}{2} I_{ij} E^{ij} + \frac{cJ}{2} I_{ij} B^{ij} + \frac{cO}{2} I_{ijk} \partial_i E_{jk} + \ldots \right),$$  

where $\omega_{a\mu}^b$ is the spin connection coupling to the total angular momentum, while the electric (magnetic) tensor $E_{ij}$ ($B_{ij}$) is defined by

$$E_{ij} = C_{\mu ij} u^\mu u^\nu,$$

$$B_{ij} = \epsilon_{ij\mu\nu} u^\mu C_{\nu\rho}^\alpha u^\rho,$$

decomposing the Weyl tensor $C_{\mu\nu\rho\sigma}$ analogously to the electric and magnetic decomposition of the standard electromagnetic tensor $F_{\mu\nu}$. This amounts to decompose the source motion in terms of a point particle world-line and moments describing its internal dynamics. The $I_{ij}, I_{ijk}, J_{ij}$ tensors are the lowest order in an infinite series of source moments, the $2n$th electric (magnetic) moment in the above action scale at leading order as $m r_{\text{source}}^n$ ($m v_{\text{source}}^n$), and they couple to the Taylor expanded $E_{ij}$ ($B_{ij}$) which scales as $L^{-(1+n)}$, showing that the above multipole expansion is an expansion in terms of $r_{\text{source}}/L$.

Note that the multipoles, beside being intrinsic, can also be induced by the tidal gravitational field or by the intrinsic angular momentum (spin) of the source. For quadrupole moments, the intrinsic case will be explicitly dealt with in subsection 3.2, whereas the tidal induced quadrupole moments $I_{ij}, J_{ij}|_{\text{tidal}} \propto E_{ij}, B_{ij}$ give rise to the following terms in the effective action

$$S_{\text{tidal}} = \int \! \mathrm{d}t \left[ cE_{ij}E^{ij} + cB_{ij}B^{ij} \right].$$

$4$ We adopt the $(-, +, +, +)$ signature, $\tau$ is the proper time running along the source world-line, $u^\mu = \mathrm{d}x^\mu / \mathrm{d}\tau$ is the 4-velocity of the center of mass. Latin indices $i, j, k, l$ denote pure spatial dimensions, ranging from 1 to 3. We will use the arrow notation ($\vec{v}$) for $d$-dimensional spatial vectors, and the boldface ($\mathbf{k}$) to denote their Fourier-transformed counterparts in momentum space.
This is also in full analogy with electromagnetism, where for instance particles with no permanent electric dipole experience a quadratic coupling to an external electric field (see [25] for an EFT study of finite size effects in the electromagnetic case).

Equation (3) can be used to describe a single, spin-less compact object in the field of its binary system companion. Considering that the Riemann tensor generated at a distance \( r \) by a source of mass \( m \) goes as \( m / r^3 \), the finite size effect given by the \( E_{ij}E^{ij} \) term goes as \( c_E m^2 / r^6 \). For dimensional reasons \( c_E \sim G_N r_{\text{source}}^5 \) [12], thus showing that the finite size effects of a spherical symmetric body in the binary potential are \( O(Gm/r)^5 \) times the Newtonian potential, a well known result which goes under the name of effacement principle [26] (the coefficient \( c_E \) actually vanishes for black holes in 3 + 1 dimensions [26–28]).

One may consider the inclusion in \( S_{\text{ext}} \) of monopole terms linear in curvature invariants like \( c_R \int d\tau R \) and \( c_V \int R_{\mu\nu}\tilde{\varepsilon}^\mu\tilde{\varepsilon}^\nu \). However these terms can be safely omitted as they vanish by the Einstein equations outside the source generating them\(^5\). As linear terms in the Ricci tensor and Ricci scalar cannot appear, the terms involving the least number of derivatives are the ones written above in equation (1), in terms of the (traceless part of the) Riemann tensor.

We will focus in the next section onto the derivation of the effective potential of a binary system, obtaining the general relativistic generalization of the Newtonian potential. This is obtained by integrating out the degrees of freedom that mediate the gravitational attraction to obtain a Fokker-type action \( S_{\text{eff}} \) describing the instantaneous interaction between objects parametrized by world-lines \( x_{A,B} \). Formally this is achieved by computing the Feynman path integral

\[
e^{iS_{\text{eff}}} = \int Dh_{\mu\nu} e^{i[S_{\text{bulk}}(\eta_{\mu\nu}+h_{\mu\nu})+S_{\text{ext}}(x_{A,B},\eta_{\mu\nu}+h_{\mu\nu})]},
\]

where \( S_{\text{bulk}} \) involves only the gravitational degrees of freedom and is given by the standard Einstein Hilbert action plus the harmonic gauge fixing term corresponding to the harmonic gauge used in [11]

\[
S_{\text{bulk}} = S_{\text{EH}} + S_{\text{GF}}, \quad S_{\text{GF}} \equiv -\Lambda^2 \int d^d\tau \sqrt{\det g} \Gamma_{\mu} \Gamma^{\mu},
\]

with \( \Lambda = (32\pi G_N)^{-1/2} \) in \( d = 3 \). It is understood that all quantum contributions to the path integral (4) will be discarded in order to keep account only of classical effects. Indeed quantum contributions to the astrophysical processes we are interested in are negligible.

Because of the nonlinearities of the Einstein Hilbert action and of the gravity–matter coupling, the functional integral in equation (4) cannot be performed exactly, but only perturbatively. For instance the leading perturbative order is represented by the diagram in figure 1: it accounts for the potential generated by the exchange of a gravitational degree of freedom. Performing the above functional integral is equivalent to solving the nonlinear equations iteratively around the linear solution at the level of the action: the perturbative solution can then be organized in Feynman diagrams as it is customary done in field theory.

In order to show in practice how the iterative solutions can be used to efficiently generate the dynamics of the problem, we find convenient to decompose the metric in the form

\[
g_{\mu\nu} = e^{2\phi/\Lambda} \begin{pmatrix} -1 & A_j/\Lambda \\ A_i/\Lambda & e^{-c_\rho\phi/\Lambda} \delta_{ij} - A_iA_j/\Lambda^2 \end{pmatrix},
\]

\(^5\) Such terms give \( \delta \)-like, unobservable contributions to the classical potential. Equivalently, it can be shown that the field redefinition \( g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu} \) with

\[
\delta g_{\mu\nu} = \int d\tau \frac{\delta(\eta^\mu - \eta^\mu(\tau))}{\sqrt{r}} \left[ (c_R + c_V \Gamma^\mu) g_{\mu\nu} - c_V u^\mu u^\nu \right]
\]

can be used to set to zero the above terms linear in the curvature, see [13] for details.
with \( \gamma_{ij} = \delta_{ij} + \sigma_{ij}/\Lambda \), \( c_d = 2^{(d-1)} \), according to the metric ansatz proposed in [29, 30] and reminiscent of the one first used in [31]. On the previous ansatz \( S_{\text{bulk}} \) reduces to

\[
S_{\text{bulk}} \approx \int \mathrm{d}t \int \frac{d^d \mathbf{x}}{(2\pi)^d} e^{-c_d(\partial \phi)^2 + \cdots},
\]

where only the kinetic term of the gravitational field \( \phi \) has been explicitly written. The \( \phi \) coupling to the source, which is implicit from equation (1), is \( m_\Lambda \int \mathrm{d}t \phi (1 + \mathcal{O}(v^2)) \) and neglecting all interaction terms of the field \( \phi \), the Gaussian integration over \( \phi \) in equation (4) can be done exactly and leads to

\[
S_{\text{eff}} = -m_1 \int \mathrm{d}t_1 - m_2 \int \mathrm{d}t_2 + \frac{m_1 m_2}{2 c_d \Lambda^2} \int \mathrm{d}t_1 \mathrm{d}t_2 G(t_1 - t_2, \mathbf{x}_1(t_1) - \mathbf{x}_2(t_2)),
\]

where \( G(t, \mathbf{x}) \) is the Feynman Green function

\[
G(t, \mathbf{x}) = -i \int \frac{dk}{2\pi} \int \frac{dk}{2\pi} e^{-ik(t, \mathbf{x})} \frac{1}{k^2 - k_0^2 - \imath \epsilon},
\]

with \( k \equiv \int \frac{dk}{2\imath t} \). It is now crucial to take the non-relativistic limit in order to work at a given order in \( v \). This is achieved by observing that the wave-number \( k^\mu = (k^0, \mathbf{k}) \) of the gravitational modes mediating this interaction have \( k^0 \sim v/r, k \sim 1/r \), so in order to have manifest power counting it is necessary to Taylor expand the propagator

\[
G(t, \mathbf{x}) \approx \frac{1 + k_0^2}{k^2} \left( 1 - \frac{\partial t_0}{k^2} + \cdots \right),
\]

where in the last passage the integral over \( k_0 \) has been performed explicitly after trading the \( k_0 \) factors for time derivative operators. Note that we did not write any \( \imath \epsilon \) term in equation (10) as, in the non-relativistic kinematical region we are interested in here where the gravitational mode cannot be on-shell, the pole prescription is inessential. The individual particles can also exchange radiative gravitational modes (i.e. GWs with \( k_0 \sim k \sim v/r \)), but such processes give sub-leading contributions to the effective potential in the PN expansion, and they will be dealt with in subsection 4.3. In other words we are not integrating out the entire gravity field, but the specific off-shell modes in the kinematic region \( k_0 \ll k \).

We are aiming at computing an effective action giving the correct action-at-distance, with retardation effects taken into account by the Taylor expansion in equation (10). After substituting in the effective action (8) the explicit form of the source–gravity coupling one obtains

\[
S_{\text{eff}} \supset m_A m_B \int \mathrm{d}t \left( 1 + \mathcal{O}(v^2) \right)^3 \int \frac{e^{i k \cdot \mathbf{r}}}{k^2} \left( 1 - \frac{\partial_t^2}{k^2} + \cdots \right).
\]

Once the time derivatives act on the exponential they will introduce velocity dependent terms in the effective action, so in order to have a consistent calculation at any given order in \( v^2 \),

![Figure 1. Feynman graph accounting for the Newtonian potential.](image-url)
we have to remember the virial theorem which makes diagrams of the type in figure 2 also potentially of order $v^2$ with respect to the leading one in figure 1. The power counting of diagrams can be made systematic by the rules given in figures 3–5, which can be generalized to higher order interaction vertices.

As said above, no quantum effects will be considered to obtain physical results in this review. However, let us derogate for a moment in order to show that quantum effects are negligibly small in the astrophysical process under considerations. Intermediate massive object lines (like the ones in figure 2) have no propagator associated, as they represent a static source (or sink) of gravitational modes. At the gravitational mode-massive object vertex momentum is not conserved, as the graviton momentum is ultra-soft compared to the massive source. E.g. in the diagram of figure 1, where the massive object emits a single graviton, it recoils by a fractional amount $\frac{dp}{p}$ roughly given by $\frac{dp}{p} = \frac{\hbar k}{mv} \sim \frac{\hbar}{L}$, being $L = mrv$ the macroscopic angular momentum of the binary system. For the phenomenological application we are aiming at, $\frac{\hbar}{L} \sim 10^{-77} (M/M_\odot)^{-2} (v/0.1)$ is ridiculously small.

Consistently with neglecting any quantum effect, diagrams like the one in figure 6 will not be considered. Even explicitly restoring $\hbar$ into the definition of the path integral in equation (4) to establish the correct dimensions of the exponential term, after evaluating only diagrams at tree level (in the quantum language) all physical results will be $\hbar$-independent. According to
the standard rules for taking into account powers of $\hbar$ involved in Feynman diagrams, each vertex brings in an inverse power of $\hbar$ and each internal line a power of $\hbar$. The quantum scaling of diagram is than accounted by $\hbar^{I-V} = \hbar^{I-1}$, using the standard relationship $L = I - V + 1$ among number of loops $L$, vertices $V$ and propagators $I$. Applying this power counting rule to the graph in figure 6, say, shows that it scales as $\hbar/L$ with respect to the Newtonian potential, so it is completely negligible\(^6\).

After integrating out the potential gravitational modes we will be left with an effective action where some of the original Lagrangian terms will be renormalized and new, local ones will be generated in infinite numbers (but finite at each PN order), with the coefficients of the generated terms being the Wilson coefficients. Note that some graphs will be actually divergent like the one in figure 7, which gives a divergent contribution to the effective potential

$$\text{figure 7} \simeq \frac{G^2 m_1^2 m_2}{r} \int \frac{1}{k^2}.$$  \hspace{1cm} (12)

Actually graphs like this can be consistently discarded, and indeed vanish in dimensional regularization, as an effective theory is not supposed to correctly portrait the full theory at arbitrary high energy scales. Divergences like the one of equation (12) can be accounted for by shifting the input parameters in the starting Lagrangian (like the mass of the binary constituents), as we are not aiming at predicting those parameters, but just take them as inputs (see [32] for a thorough discussion along this line). We shall discuss in the following sections three other kinds of divergence, associated to $O(3\text{PN})$ gauge artifacts, to long-distance effects and to short-distance (or ultraviolet (UV)) incompleteness of the effective theory.

\(^6\) Note that adopting a QFT description of a second quantized massive particle coupled to gravity would lead to the same result as here, once the non-relativistic limit is taken, see [16].
As it is standard in perturbative field theory calculations, diagrams contributing to the effective action are only the connected ones, i.e. those in which following Green function’s lines all the vertices can be connected.

The effective theory at the orbital case, in the spin-less case, can treat the binary constituents as point-like until 5PN order, as this is the order at which finite size effects come into play, so the theory can be consider UV complete up to that order (the finite size spinning effects will be discussed in subsection 3.2).

In section 4 we shall describe the binary system as a single extended object coupled to gravity in order to compute observables related to the emission of GWs. The starting point will be the action in equation (1), where the first two terms will not be responsible for radiation, as at leading order they couple the gravitational modes to the conserved mass monopole and to the total angular momentum.

In order to have full predictive power, the effective theory in terms of the multipole moments at the orbital scale will have to be matched to the theory at the orbital scale in order to express the binary multipoles in terms of individual constituent parameters. It will turn out that in computing the radiation back-reaction on the source at the scale \( \lambda \sim r/v \), a logarithmic divergence will appear, showing the UV incompleteness already at \( v^3 \), and requiring that the singularity be resolved by considering the theory at the smaller, orbital scale (in the calculation of the emitted flux the incompleteness will appear at \( v^6 \) order).

While it is possible to absorb power-divergences into bare parameters of the original Lagrangian, as it is usual in field theory, logarithmic divergences will introduce a spurious dependence on an arbitrary scale \( \mu \): in order to cancel the \( \mu \) dependence from physical observables, a compensating dependence of the input parameters has to be imposed, leading to a fully classical implementation of the renormalization group equation, implying that physical parameters running with \( \mu \) will take different values when probed at different length scale, as it will be explicitly shown in subsection 4.3.

3. Conservative

The conservative dynamics of a binary system involves processes characterized by no incoming nor outgoing radiation: in diagrammatic terms, this means absence of external radiative gravity lines. Internal radiative propagators (meaning that the radiation gravitational mode is emitted and then reabsorbed by the system) can in principle be present and indeed appear at 4PN, giving rise to the so-called tail terms studied in [33–35]; we will deal with this peculiar effect in subsection 4.3, while restricting the discussion of this section to diagrams involving potential gravitational modes only.

The GW length \( \lambda \) thus being irrelevant at this stage, the only scales of the problem are the size of compact astrophysical objects like neutron stars/black holes \( r_s \) and the orbital radius \( r \). The main goal here is to determine the dynamics of the system as a function of the orbital parameters and of the internal features of the compact objects, such as mass and spin, and other (like \( c_{E,B} \)) which appear as Wilson coefficients to be fixed by a matching procedure at the scale \( r_s \).

The general strategy consists in

(i) writing down all the relevant vertices of the effective theory and determine their \( v^2 \) and \( G_N \) scaling,
(ii) building all the Feynman diagrams which are relevant to the desired PN order,
(iii) computing the Feynman integrals by Taylor expanding potential gravitational mode propagators around \( k_0 = 0 \).
Power-law divergences arising at this point are automatically reabsorbed by dimensional regularization, while the logarithmic divergences appearing for the first time at 3PN can be eliminated by means of a world-line redefinition.

The pure gravity sector of the theory can be expanded up to the desired order in terms of the Kaluza–Klein variables introduced in equation (6). We report here the expansion up to terms relevant at 4PN\(^7\) (see also [36] for a derivation):

\[
S_{\text{bulk}}^{4\text{PN}} \simeq \int \! dt \, d^4 x \sqrt{-g} \left\{ \frac{1}{4} \left[ (\nabla \sigma)^2 - 2(\nabla \tau)^2 - (\sigma^2 - 2(\sigma \overline{\tau})) e^{-\Gamma_1} \right] 
- c_d [(\nabla \phi)^2 - \dot{\phi}^2 e^{-\Pi_1}] + \left[ \frac{F_{ij}^2}{2} + (\widetilde{\nabla} \cdot \overline{\nabla})^2 - \dot{\widetilde{\nabla}}^2 e^{-\Pi_1} \right] e^\Pi_1 
+ 2 [F_{ij} A^i A^j + \widetilde{A} \cdot \overline{A}(\nabla \cdot \overline{A})] e^{\Pi_1} - c_d \dot{\phi} A \cdot \nabla \phi 
- \frac{2 c_d (\dot{\phi} \nabla \cdot \overline{A} - \dot{\overline{A}} \cdot \nabla \phi) + \dot{\overline{\Sigma}} (\delta_{ij} F_{ij}^{\Gamma_{4\text{PN}}} + 2 A_k \Gamma_{ij}^k - 2 A^i \Gamma^{ij}_k)]}{\Lambda} 
- \frac{1}{\Lambda} \left( \frac{\sigma}{2} (g^{ij} - \sigma^{ij}) \right) (\sigma_{ik} \sigma_{jk}^{\prime} - \sigma_{ik} \sigma_{jk} - \sigma_{ik} \sigma_{jk}^{\prime}) \right\} , \tag{13}
\]

All the bulk vertices and propagators needed up to 4PN order can be derived from equation (13). We write down explicitly the Green function expressions in terms of the space Fourier-transformed variables

\[
W_k^a(t) = \int \! dt \, d^4 x W^a(t, x) e^{-ikx} \quad \text{with} \quad W^a = \{ \phi, A_i, \sigma_{ij} \} ; \tag{14}
\]

\[
P[W_k^a(t_a) W_{k^\prime}^b(t_b)] = \frac{1}{2} P^{\mu \nu} \delta_{ab} (2\pi)^d \delta^d(k + k') P(k^2, t_a, t_b) \delta(t_a - t_b) , \tag{15}
\]

with \(P^{\mu \nu} = - \frac{1}{c_d} P^{AA} = \delta_{ij} \), \(P^{\sigma \sigma} = - (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj} + (2 - c_d) \delta_{ij} \delta_{kl}) \) and

\[
P(k^2, t_a, t_b) = \frac{i}{k^2 - \partial_k \partial_k} \simeq \frac{i}{k^2} \left( 1 + \frac{\partial_k \partial_k}{k^2} + \frac{\partial_k^2 \partial_k^2}{k^2} + \cdots \right) . \tag{16}
\]

As desirable, the three polarization fields \(\phi, A, \sigma\) do not mix at the quadratic level.

A convenient strategy for building all the Feynman diagrams has two steps (as first done in [37]). At first one determines the shape of the diagram (henceforth called topology), which fixes the powers of \(G_n\) through the following simple rules: a bulk \(n\)-vertex gives \(G_n^{d/2 - n}\), and a matter interaction vertex with \(n\) gravitational mode lines gives \(G_n^{d/2}\), see e.g. figures 3, 5. Starting from the lowest order topology of figure 8, all the higher order ones can be generated iteratively by adding a new propagator with one extremum on one of the two compact objects’ word-lines, and the other to any other element (a bulk vertex, a vertex located on the other object word-line, or in the middle of an other propagator in order to create a new 3-vertex), but not on the same word-line of the first extremum, since as pointed out in section 2, topologies involving propagators that start and end on the same compact object word-line as in figure 7 do not have to be considered.

Then, any given topology is ‘filled’ with the various \(\phi, A, \sigma\) field propagators and with the different matter interaction vertices given by equation (1); these elements determine the

\[\overline{\nabla} \cdot \overline{A} = \gamma^{ij} A_{ij} \quad \text{and} \quad \overline{F}_{ij} = \gamma^{ik} \gamma^{jk} F_{ij} F_{kl}.\]
powers of $v$ characterizing the diagram, and thus its PN order. The simplest example of this procedure is depicted in figure 8.

The advantage of this procedure is that each topology is associated with a specific class of Feynman integrals, so that all the diagrams belonging to the same topology can be computed using the same integration strategy. Moreover, topologies which can be split into sub-topologies do not present any new difficulty from the computational point of view because the corresponding effective potential contributions are given by the product of the sub-topology ones which can thus be evaluated separately.

3.1. The spin-less case

Let us consider first the gravity–matter coupling for the non-spinning case and postpone the more complicate spinning case to the next subsection. In this case the finite size of the binary system components does not enter the dynamics until 5PN order (because of the effacement principle discussed in section 1), so the gravity–source coupling reduces to the mass monopole term, which can be written as

$$S_{pp} = -m \int d\tau = -m \int dt \, e^{\phi/\Lambda} \sqrt{\left(1 - \frac{\vec{A} \cdot \vec{v}}{\Lambda}\right)^2 - e^{-c\phi/\Lambda} \left(v^2 + \frac{\sigma_{ij} v^i v^j}{\Lambda}\right)}.$$  \hspace{1cm} (17)

We have now all the elements to complete step (ii), that is to determine all the relevant graphs at a given PN order. The only diagram contributing at Newtonian level is the first one drawn next to the $O(G_N)$ topology in figure 8, because $\phi$ is the only polarization whose particle interaction vertex does not depend on $v$ at leading order.

The 1PN diagrams can scale as $G_N v^2$ or $G_N^2$; in the first category fall the same diagram as before (which has to be computed at $O(v^2)$ by expanding the particle interaction vertices and the $\phi$ propagator according to equations (17) and (16), respectively), as well as the second diagram in figure 8, which carries two powers of $v$ (one at each particle interaction vertex) at leading order. As to the $G_N^2$ graphs, one has to consider the new topologies shown in figure 9 and take the $v$-independent part of the Feynman diagrams. Only the diagram with a
\[
\phi^2 \text{ source–gravity vertex contributes at this PN order while the diagrams involving a triple bulk interaction vertex can be discarded at this order because they carry at least two powers of } v.\]

Note that the surviving \(G_{2N}^2\) diagram at 1PN order is clearly factorizable in terms of two ‘Newtonian’ topologies, so its calculation is straightforward.

At 2PN we have to consider, as well as the previously analyzed diagrams with the appropriate factors of \(v\) from the expansion of the propagators and vertices, also several new diagrams generated by the topologies already considered (see figure 10 for an example), as well as the ones generated by brand new, \(G_{3N}^3\) topologies. At 2PN, five of them are relevant (each one providing a single diagram), two of which being merely trivial compositions of three ‘Newtonian’ topologies. The three irreducible ones are shown in the upper part of figure 11. The first computations of the effective 2PN Lagrangian within the EFT framework have been done in [37], using the same Kaluza–Klein decomposition adopted here, and in [38].

We conclude the topology and diagram classification before moving to effective potential calculation. At 3PN 63 new diagrams have to be considered, and 6 of them come from the two topologies shown in the lower part of figure 11, which are the only new irreducible topologies needed at this order: in particular, all the 8 \(G_{N}^4\) topologies needed at 3PN are factorizable in terms of simpler ones. The 3PN calculation within the EFT framework has been performed in [39] by means of a semi-automated algorithm thus making the EFT technique match what was the state of the art at the time in this sector of the theory.

At 4PN there are 515 new diagrams, variously distributed among the old topologies, new factorizable ones, as well as 12 new irreducible \(G_{N}^4\) (see figure 12) topologies and

---

\(\phi^2\) source–gravity vertex contributes at this PN order while the diagrams involving a triple bulk interaction vertex can be discarded at this order because they carry at least two powers of \(v\). Note that the surviving \(G_{2N}^2\) diagram at 1PN order is clearly factorizable in terms of two ‘Newtonian’ topologies, so its calculation is straightforward.

At 2PN we have to consider, as well as the previously analyzed diagrams with the appropriate factors of \(v\) from the expansion of the propagators and vertices, also several new diagrams generated by the topologies already considered (see figure 10 for an example), as well as the ones generated by brand new, \(G_{3N}^3\) topologies. At 2PN, five of them are relevant (each one providing a single diagram), two of which being merely trivial compositions of three ‘Newtonian’ topologies. The three irreducible ones are shown in the upper part of figure 11. The first computations of the effective 2PN Lagrangian within the EFT framework have been done in [37], using the same Kaluza–Klein decomposition adopted here, and in [38].

We conclude the topology and diagram classification before moving to effective potential calculation. At 3PN 63 new diagrams have to be considered, and 6 of them come from the two topologies shown in the lower part of figure 11, which are the only new irreducible topologies needed at this order: in particular, all the 8 \(G_{N}^4\) topologies needed at 3PN are factorizable in terms of simpler ones. The 3PN calculation within the EFT framework has been performed in [39] by means of a semi-automated algorithm thus making the EFT technique match what was the state of the art at the time in this sector of the theory.

At 4PN there are 515 new diagrams, variously distributed among the old topologies, new factorizable ones, as well as 12 new irreducible \(G_{N}^4\) (see figure 12) topologies and
Figure 12. The 12 irreducible $G_6^4$ topologies. They all give contribution to the 4PN dynamics.

Figure 13. The 25 irreducible $G_6^5$ topologies that contribute at 4PN. Each of them generate just one 4PN diagram. Other 25 4PN diagrams can be obtained from $G_6^5$ reducible topologies which have not been shown here.

25 $G_6^5$ ones (figure 13). Table 1 gives an overview of the topology and diagram counting. The corresponding Lagrangian has been computed for the first time in [40] up to terms of order $G_6^2$, a sector which was subsequently also covered in the ADM framework [41, 42].

Coming to step (iii), we have to perform perturbatively the functional integration of equation (4). As an illustration we take the contribution given by the second diagram of figure 10. The exponential in the functional integral has to be expanded to the fourth order and the four Lagrangian terms corresponding to the vertices present in the diagram have to be selected:

$$iS_{\text{eff}} \supset -iV_{\text{ex}} \equiv \log \int D\phi DA e^{iS_{\text{bulk}}-iS_{\text{ex}}(\phi,A)}$$

$$\times \frac{1}{2} \int_{\tau_1,\tau_2,\tau_3,\tau_4} \frac{-i\mu_1 \phi_{1a}}{\Lambda} - \frac{-i\mu_1 \phi_{1b}}{\Lambda} \frac{\mu_2 A_2 \cdot \vec{v}_2}{\Lambda} - \frac{2i\epsilon \phi A \cdot \vec{\nabla} \phi}{\Lambda},$$

(18)
where $S_{\text{bulk-free}}$ is the quadratic part of the bulk gravity action, $\phi \equiv \phi(t, x)$ and $\phi_{\text{int}} \equiv \phi(t_{\text{int}}, \vec{x}_1(t_{\text{int}}))$ and so on. Performing the Gaussian gravity action in the above equation (18) boils down to substituting pair of like-fields with Green functions like in equation (8) (indicated below with a contraction, as the procedure is in complete analogy to the Wick theorem in QFT

Expressing the Green functions in the momentum space via equations (14)–(16) one has

$$-iV_{\text{ex}} = -\frac{m_i^2 m_j c d}{\Lambda^4} \int_{t_{\text{fin}}, t_{\text{ini}}} \frac{2v_2^2 \phi_{\text{int}} \phi_{\text{int}}}{A_{m_1 A_{m_2}}}.$$ 

(19)

Expressing the Green functions in the momentum space via equations (14)–(16) one has

$$-iV_{\text{ex}} = -\frac{m_i^2 m_j}{4 c_d \Lambda^2} \int_{t_{\text{fin}}, t_{\text{ini}}} \frac{i\delta'(t - t_{\text{ini}}) \vec{k}_1 \cdot \vec{v}_2(t)}{k^2 (k - k_1)^2 k_1^2} e^{i[(k - k_1) \cdot \vec{x} + k_1 \cdot \vec{x}_1(t) - k \cdot \vec{x}_2(t)]}$$

$$= -\frac{m_i^2 m_j}{4 c_d \Lambda^2} \int_{t_{\text{fin}}, t_{\text{ini}}} \frac{i \vec{k}_1 \cdot \vec{v}_2(t)}{k^2 (k - k_1)^2 k_1^2} e^{ik \vec{r}}.$$ 

(20)

A look at the structure of the denominator tells us that the complexity of this diagram (and of all the diagrams derived from the same topology) is equivalent to 1-loop diagrams in QFT; indeed, this amplitude can be easily evaluated using standard textbook formulae [43] and taking the limit $d \to 3$ (in this case the contribution of the Feynman diagram is finite in dimensional regularization, so $d = 3$ could have been set from the beginning), thus bringing to the following term of the 2PN action

$$V_{\text{ex}} = -\int_{t_{\text{fin}}, t_{\text{ini}}} G_2^2 \frac{m_i^2 m_j}{r^2} \left[ \vec{v}_1^2 \cdot \vec{v}_2 - \frac{r_1 \cdot \vec{v}_1}{r} \right], \quad \vec{v}_f \equiv \frac{\vec{r} \cdot \vec{v}_1}{r}.$$ 

(21)

Naturally, the same diagram contributes also to higher PNs, and the corresponding contribution to the effective potential is obtained as above with the caution of including the appropriate orders in in the $v$ expansion of $S_{\text{pp}}$ from equation (17) and in the propagators expansions, equation (16). The latter may generally bring more and more $k, k_1$ terms in the integrand numerator thus making the evaluation more lengthy, but as the general structure of the denominator does not change, the complexity of momentum integrals remains still comparable to 1-loop ones.

All effective potential contributions can be expressed in terms of (eventually complicated) spatial momentum integrals along the same lines. The actual evaluation strategy of the integrals depends on the topology and, as we have seen, all the topologies up to $G_2^2$ can be computed by directly applying standard textbook formulae. Generally, a $G_n^2$ irreducible topology is expected to involve momentum integrals equivalent to $n$-loops QFT diagrams, but a more careful inspection shows that the situation is actually more favorable. For instance, four of the five irreducible $G_2^2$ topologies in figure 11 involve nested loops integrations, that is integrals where at least one $k_4$ appear just twice in the denominator: in this case this variable can be
integrated out immediately as in the 1-loop case, and the result of the partial integration is, in the $G_N^3$ case, easily integrable in terms of the remaining momentum variables. The only apparent exception to this rule is the H-shaped topology in figure 11, but an appropriate use of Integration by Parts techniques [43] provide the following useful relation
\[
I(\alpha, \beta, \gamma, \delta, \epsilon) = \int_{k_1, k_2} \left[ k_1^{2\alpha} (k - k_1)^{2\beta} k_2^{2\gamma} (k - k_2)^{2\delta} (k_1 - k_2)^{2\epsilon} \right]^{-1}
\]
\[
= \frac{\gamma [I(\alpha-, \gamma+) - I(\epsilon-, \gamma+)] + \delta [I(\beta-, \delta+) - I(\epsilon-, \delta+)]}{2\epsilon + \gamma + \delta - d}.
\]
with the notation $I(\alpha-, \gamma+) \equiv I(\alpha - 1, \beta, \gamma + 1, \delta, \epsilon)$, by means of which the integrals of this topology can be reduced to nested loops ones. Thus, the $G_N^3$ sector does not present new conceptual difficulties with respect to the $G_N^2$ one, although the computational challenge becomes relevant at high PN because of the high number of diagrams involved, see table 1, and of the appearance of more and more $k_\alpha$ factors in the numerators.

The situation is somehow similar in the $G_N^2$ case, as it turns out that the topologies of this order involve, in the most difficult case, 3-loops integrals which are either nested or reducible through integrations by parts to integrals like the one in equation (22). Consequently one has the remarkable result that all the topologies up to $G_N^3$ are basically tractable in terms of 1-loop equivalent QFT diagrams, see also [44] for related work.

At $G_N^3$ however things change, for two reasons: first, the use of integration by parts becomes more complicated and substantially intractable by hand. This problem can be overcome by using automated reduction packages which are routinely used in particle physics multi-loop calculations, see e.g. [45]. Second, and more important, the ‘miracle’ according to which everything could be ultimately reduced to 1-loop integrals does not take place anymore: in the worst cases, that is for the topologies in (row, column) = (3, 2) and (4, 5) in figure 13, one is left even after integration by parts with integrals equivalent to a 4-loop mass-less QFT diagram, which has to be evaluated in $d \sim 3$ by means of ad hoc techniques [46]. A possibly more efficient way to reorganize the diagrams have been proposed in [47], while a radically different computational method has been recently suggested in [48].

Starting from 3PN, divergences appear in the form of $(d - 3)$ poles:
\[
L_{\text{pole}}^{3\text{PN}} = -\frac{11G_N^2m_1^2m_2^2}{2(d-3)} \left[ a_1^2 + 2\vec{a}_1 \cdot \vec{a}_2 \right] + \frac{11G_N^3m_1^3m_2^3}{3(d-3)} \alpha_1' + (1 \leftrightarrow 2),
\]
with $a_{1,2}$ being the accelerations. This divergence is not due to a short-distance incompleteness of the effective field theory approach, and it has been found in all the past treatments at 3PN with different kind of regularizations, see [49–53]. Since a Lagrangian is not an observable we can allow divergent terms in it as long as any relation among observables is given by finite expressions: e.g. this singularity does not appear in the expression for $E(\omega)$ relating the energy $E$ of the system to the orbital angular velocity $\omega$. It is however more practical to deal with a finite quantity also at the Lagrangian level and this can be obtained at 3PN by means of the following word-line shift [54]:
\[
\vec{x}_{1,2} \rightarrow \vec{x}_{1,2} + \frac{G_N^2m_1^3}{3} \vec{a}_{1,2}.
\]

The EFT approach allowed us to compute for the first time the dynamics at 4PN up to $O(G_N^3)$ (while some sectors at higher $G_N$ order have been recently covered in the ADM framework [42]). We write here the expression of the energy in the center of mass frame,
discuss in subsection 4.3, how the logarithmic piece can be derived from radiation reaction computation. The \( \nu \) in [56, 57], and its non-logarithmic part has been analytically computed in [58]. We shall the compact object spins has been computed for the first time in [59–61], triggering a renewed attention on such sector and a healthy competition with more traditional approaches, which EFT methods are giving a relevant contribution to the study of the spin sector of compact objects ([40] for other details: class. Quantum Grav. [31] 2014 043001 T opical Review

...in this context, as well as the expression for the Energy up to 3PN, can be found in [55].

Specializing then to circular orbits, that allows to express both \( v \) and \( G_N M/r \) in terms of \( x \equiv (G_N M \omega)^{2/3} \), at 4PN one has

\[
E(x)_{\text{4PN}} = -\mu \frac{x^5}{2} \left[ -\frac{3969}{128} + \frac{448}{15} \log(x) - \frac{123671}{5760} + \frac{9037}{1536} \pi^2 + \frac{1792}{15} \log 2 + \frac{896}{15} \gamma \right] v^2 \\
+ \left( -\frac{498449}{3456} + \frac{3157}{576} \pi^2 \right) v^2 + \left( \frac{301}{1728} + \frac{77}{31104} \right) v^3, 
\]

(26)

where \( \gamma \approx 0.577 \ldots \) is the Eulero–Mascheroni constant. Equation (25), together with inputs from Lorentz invariance of the 3PN Lagrangian, allows to derive the \( v^3 \) and \( v^4 \) term in the above equation (26), first obtained in [41], while the \( v^5 \) term has been obtained more recently in [42]. The term linear in \( v \) has been obtained within the extreme mass ratio limit approach in [56, 57], and its non-logarithmic part has been analytically computed in [58]. We shall discuss in subsection 4.3, how the logarithmic piece can be derived from radiation reaction computation. The \( v \)-independent part can be derived from the Schwarzschild result.

3.2. Spin

EFT methods are giving a relevant contribution to the study of the spin sector of compact binary systems: the next-to leading order (NLO) dynamics with a quadratic dependence on the compact object spins has been computed for the first time in [59–61], triggering a renewed attention on such sector and a healthy competition with more traditional approaches, which led to the confirmation of the new results and even to the extension to next-to-next-to leading order (NNLO) for the \( S_1 S_2 \) potential [62, 63] and for spin–orbit [62, 64–66].

As the spin of a compact object and the lowest order spin–orbit and spin–spin interactions scale respectively like

\[ S \sim \frac{m v_{rot} R_s}{r^2}, \quad V_{SO} \sim \frac{G_N M}{r^2} \vec{v} \cdot \vec{S}, \quad V_{SS} \sim \frac{G_N}{r^3} \vec{S}_1 \cdot \vec{S}_2, \]

(27)

one deduces that the lowest order (LO) spin–orbit potential is a 1.5PN term for maximally rotating objects \( (v_{rot} \sim 1) \), while the LO spin–spin interaction starts at 2PN.
Spin interactions in GR are introduced by means of two tetrads (for a more detailed discussion, see the papers cited at the beginning of this section, as well as [67–71]): $e^\mu_a$, which transforms the metric into the locally free-falling frame

$$g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab}, \quad (28)$$

and $e_A^\mu$, which is co-rotating with the spinning body, and related to the previous one by a local Lorentz transformation

$$e_\mu^A = \Lambda^A_{\alpha} e^\alpha_\mu.$$  

The transport of $e^\mu_A$ along the word-line of a reference point chosen inside the extended body defines the generalized angular velocity

$$d e^A_\mu \frac{d \tau}{\rho} \equiv u^\rho e^A_\mu; \quad \rho = \Omega^\mu_{\nu\rho} e^\nu_A = \Omega^\mu_{\nu\rho} e^\nu_\mu = \Omega^\nu_{\mu\nu}, \quad (29)$$

where $u^\rho$ is the four velocity which characterizes the word-line. Local coordinate, Lorentz and parametrization invariances require the Lagrangian to be made of invariant contractions of $\Omega^\mu_{\nu\rho}$, $u^\rho$ and eventually of the local curvature tensors, but do not unambiguously fix its form even in the case of flat space-time. However it turns out that if one neglects finite size effects, the variation of any possible Lagrangians w.r.t. to the spinning body local position and tetrad, when expressed in terms of the conjugate momenta

$$p^\mu = \frac{\delta L}{\delta \mu}, \quad S^{\mu\nu} = \frac{\delta L}{\delta \Omega^\mu_{\nu\rho}},$$

gives the same (Mathisson–Papapetrou) equations of motion [72–74]:

$$\frac{d p^\mu}{d \tau} = -\frac{1}{2} R^{\mu\nu\rho\sigma} u^\nu S^{\rho\sigma},$$

$$\frac{d S^{\mu\nu}}{d \tau} = p^\mu u^\nu - p^\nu u^\mu. \quad (30)$$

Since the spin is related to the conjugate momentum $S^{\mu\nu}$ rather than to the fundamental tetrad variables themselves, it is actually more convenient to work with a functional that behaves as an Hamiltonian with respect to the spin, while remaining a Lagrangian with respect to the body position $x^\mu$. Such functional is called a Routhian [75] and one can verify that the following form involving the spin connection $\omega_{ab}^\mu \equiv e^b_\nu e^a_\mu$; $R_0 = -m \sqrt{-u^2} - \frac{1}{2} S^{ab} \omega^\mu_{ab} u^\mu,$  

gives exactly the Mathisson–Papapetrou equations by means of

$$\frac{\delta}{\delta x^\mu} \int d\tau R = 0, \quad \frac{d S^{ab}}{d \tau} = \{R, S^{ab}\}, \quad (32)$$

once the following Poisson bracket is taken into account:

$$\{S^{ab}, S^{cd}\} = \eta^{ac} S^{bd} + \eta^{bd} S^{ac} - \eta^{ab} S^{cd} - \eta^{cd} S^{ab}.$$ \quad (33)$$

The antisymmetric tensor $S^{\mu\nu}$ (which appears above through its locally flat-frame components $S^{ab} \equiv S^{\mu\nu} e^\mu_a e^\nu_b$) is the generalized spin of the body and it contains redundant degrees of freedom. The redundancy corresponds to the ambiguity related the choice of a reference world-line inside the body. One can reduce from six to the three degrees of freedom needed to describe an ordinary spin vector by imposing the spin supplementary condition (SSC), which relates the vector $S^{(\mu}$ to the physical spin components $S_i^\mu$ \textbf{,} $S^{(\mu} = e^{j}_{\nu} S_{ij}$. There is not a unique way to impose such condition and the so-called covariant SSC

$$S^{\mu\nu} p_\nu = 0 \quad (34)$$

will be taken here. The requirement of SSC conservation along the word-line gives the following relation:

$$p^\mu = m \frac{u^\mu}{\sqrt{-u^2}} + \frac{1}{2m} R_{\nu\beta\rho\sigma} S^{\mu\nu} S^{\rho\sigma} \frac{u^\beta}{\sqrt{-u^2}} + O(R_{\nu\beta\rho\sigma}), \quad (35)$$
where the first term in the right hand side (rhs) is SSC-independent and gives the familiar dynamics for a non-spinning body.

Such relation can be enforced at the level of the Routhian by adding

\[ R_{SSC} = -\frac{1}{2m} R_{abcd} S^{a} S^{b} \frac{d^{4}u^{4}}{\sqrt{-u}} + O(R_{ij}^{2}) \]  

at the lhs of equation (31).

It should be remarked that imposition of the SSC implies \( S^{0} \sim S^{ij}v_{j} \) thus providing different scalings for the different components of the spin tensor. Being an algebraic constraint, the SSC can be imposed by direct replacement of \( S^{0} \) indifferently at the level of the fundamental Routhian or in the effective potential or in the equations of motion: the second option will be followed here because it simplifies intermediate calculations, at the price however of some loss of transparency in the results, which will not have a transparent physical interpretation until the SSC will be enforced.

Spin-induced finite size effects become relevant much before than in the spin-less case; the lowest order of these effects is the spin-induced quadrupole moment, which can be taken into account by the following Routhian term

\[ R_{fs} \equiv \frac{C_{E}}{2m} \frac{E_{ab}}{\sqrt{-u}} S^{a} S^{b}, \]  

(37)

where \( E_{ab} \) is the electric part of the Weyl tensor, and \( C_{E} = 1 \) for black holes, while it has to be fixed via a matching procedure in the non-BH case. This term gives an effective contribution to the \( I_{e} E_{ij} \) interaction in equation (1) already at 2PN order.

The spin-dependent part of the Routhian can be expressed as follows in terms of the Kaluza–Klein fields:

\[ R \supset S^{ij} \left\{ \frac{1}{4} F_{ij} (1 + 4\phi + 8\phi^{2}) + \frac{1}{2} A_{i} A_{j} (1 + 3\phi) + \frac{1}{2} A_{i} v_{j} \phi + \frac{1}{4} F_{ijkl} \sigma^{k} + \frac{1}{8} A_{i} A_{j} \right\} \]

\[ + S^{0} \left\{ \frac{1}{2} A_{i} (1 + 3\phi) + \left( \phi_{i} - \frac{1}{4} F_{ij} v_{j} \right) (1 + 2\phi) + \frac{1}{2} \sigma_{ij} v_{j} + \frac{1}{4} F_{ij} v_{j} - \frac{3}{2} A_{i} v_{i} \phi_{j} \right\} \]

\[ + \frac{1}{2} A^{k} \phi_{k} v_{j} + \frac{1}{2} (\phi_{i} A_{j} - A_{i} \phi_{j}) v_{j} + \frac{1}{2} A_{i} \phi - \frac{1}{2} \sigma_{ij} \phi_{j} \right\} + \frac{1}{2m} S^{ik} S^{ij} A_{i} A_{j} v_{j} \]

\[ + \frac{C_{E}}{2} \left\{ (\nabla^{2} \phi)^{2} + \vec{a} \cdot \vec{\nabla} \phi (S^{ij})^{2} + 2 S^{ik} S^{jk} \phi_{i} v_{j} + S^{ij} S^{ij} \phi_{ij} \right\} \]

\[ + \left( \phi_{i} (1 + 2\phi) + 2 \phi_{i} \phi_{j} + A_{i} v_{j} + 2 \phi_{i} v_{j} + \frac{3}{2} \phi_{i} v_{j}^{2} + 2 \phi_{i} v_{j} v_{j} + \frac{1}{4} F_{ik} F_{jk} \right) S^{ik} S^{ij} \],

(38)

where \( d \) has been set to 3 as all the results obtained so far from this Routhian are at most NNLO and thus finite. By analogy to the spin-less case, (gauge-dependent) divergences are expected to appear at next-to-next-to-next-to leading order, corresponding to 4.5PN for spin–orbit, and to 5PN for spin–quadratic interactions.

The determination of the effective potential proceeds along the same lines of the spin-less case, with the new Feynman rules dictated by (38). Spin insertions in the diagrams introduce PN penalty factors, making the integrals to be computed easier than the ones without spin at the same PN order, while the physical interpretation of the results is made less transparent in the spinning case.

To illustrate the latter point, let us consider the lowest order spin–orbit interaction. According to the scaling rules (and reminding that \( S^{0} \sim v_{j} S^{ij} \)), the effective potential is
a 1.5PN contribution that can be derived from the two graphs in figure 14 and their mirror images. The computation is straightforward and gives

$$V_{SO}^{LO} = -\frac{G m_2}{r^3} \left[ \vec{S}_1 \cdot (\vec{v}_1 - 2\vec{v}_2) \times \vec{r} + S_1^0 r_1 \right] + (1 \leftrightarrow 2).$$  \hfill (39)

The non-physical degrees of freedom represented by $S^0$ must now be eliminated through a SSC, as for instance the covariant one in equation (34). By taking such condition at leading order in $v$ one gets

$$V_{SO}^{LO} = -\frac{2G m_2}{r^3} \vec{S}_1 \cdot \vec{v} \times \vec{r} + (1 \leftrightarrow 2),$$  \hfill (40)

which however does not correspond to the canonical result, see e.g. [76]:

$$V_{SO}^{LO} = 2\frac{G m_2}{r^3} \vec{S}_1 \cdot \vec{v} \times \vec{r} + \frac{1}{2} \vec{S}_1 \cdot \vec{v}_1 \times \vec{a}_1 + (1 \leftrightarrow 2).$$  \hfill (41)

The mismatch does not lead to any difference in physical observables, as it can be cured by means of the following spin-dependent change of variables

$$\vec{x}_{1,2} \rightarrow \vec{x}_{1,2} + \frac{1}{2m_{1,2}} \vec{S}_{1,2} \times \vec{v}_{1,2}.$$  \hfill (42)

Alternatively, an expression matching exactly equation (41) may be obtained by imposing the so-called Newton–Wigner SSC, $S_{\mu\nu} (p_{\nu} + m_0 e_{\nu}) = 0$, as well as the Newtonian equations of motion for the accelerations [77].

To summarize, the choice of working with $S^0$ at the effective potential level makes the results formally SSC-dependent, but the difference vanishes on observables. Clearly when going at higher PN orders one should not forget to include effects coming from the higher order terms in the SSC relation eventually inherited from lower PNs. An alternative procedure would be to impose the SSC directly at the level of the fundamental Routhian, as done for example in [69]. Whatever choice is made, it is not straightforward to compare results derived within different approaches, like the EFT method and the ADM approach (a problem somehow addressed for instance in [78]). Not surprisingly, such difficulties become computationally more relevant at higher PN order, as is the case for the $S_1 S_2$ 4PN sector, where a full comparison between the two approaches has not yet been carried on.

4. Radiation

In the previous section we have shown how to obtain an effective action à la Fokker describing the dynamics of a binary system at the orbital scale $r$ in which gravitational degrees of freedom have been integrated out, resulting in a series expansion in $v^2$, as in a conservative system odd powers of $v$ are forbidden by invariance under time reversal.
The gravitational tensor in 3+1 dimensions has six physical degrees of freedom (ten independent entries of the symmetric rank two tensor in 3+1 dimensions minus four gauge choices): four of them are actually constrained, non-radiative physical degrees of freedom, responsible for the gravitational potential, and the remaining two are radiative, or GWs.

In order to compute interesting observables, like the average energy flux emitted by or the radiation reaction on the binary system, it will be useful to ‘integrate out’ also the radiative degrees of freedom, with characteristic length scale $\lambda = r/v$, as it will be shown in the next subsections.

We aim now at writing the coupling of an extended source appearing in equation (1) in terms of the energy–momentum tensor $T^{\mu\nu}(t, \vec{x})$ moments. Here we use $T^{\mu\nu}$, as in [24], to denote the term relating the effective action $S_{1g}$ to the gravitational mode perturbation $h_{\mu\nu}$

$$
S_{1g} = \frac{1}{2} \int \, dt \int d^d x T^{\mu\nu}(t, \vec{x}) h_{\mu\nu}(t, \vec{x}).
$$

(43)

With this definition $T^{\mu\nu}$ receives contribution from both matter and the gravity pseudo-tensor appearing in the traditional GR description of the emission processes.

Given that the variation scale of the energy–momentum tensor and of the radiation field are respectively $r_{\text{source}}$ and $\lambda$, by Taylor-expanding the standard term $T^{\mu\nu} h_{\mu\nu}$

$$
\sum_{n} \frac{1}{n!} \partial_{i_1} \ldots \partial_{i_n} h_{\mu\nu}(t, \vec{x}) \int d^d y T^{\mu\nu}(t, \vec{y}) y^{i_1} \ldots y^{i_n},
$$

(44)

we obtain a series in $r_{\text{source}}/\lambda$, which for binary systems gives $r_{\text{source}} = r \ll \lambda = r/v$.

The results of the integral in equation (44) are source moments that, following standard procedures not exclusive of the effective field theory approach described here, are traded for mass and velocity multipoles. For instance, the integrated moment of the energy–momentum tensor can be traded for the mass quadrupole $Q^{ij}(t)$

$$
Q^{ij}(t) = \int d^d x T^{00}(t, \vec{x}) \chi^i \chi^j,
$$

(45)

by repeatedly using the equations of motion under the form $T^{\mu\nu}_{\nu} = 0$:

$$
\int d^d x [T^{0k} x^j + T^{0j} x^k] = \int d^d x T^{0k}(x^i x^j), \quad k
$$

$\int d^d x T^{0k} x^j = - \int d^d x T^{0j} x^k$

$$
= \int d^d x T^{00} x^j = \vec{Q}^j
$$

(46)

$$
2 \int d^d x T^{ij} = \int d^d x [T^{ik} x^{j} + T^{kj} x^{i}] = \int d^d x [T^{0k} x^j + T^{0j} x^k] = \int d^d x T^{00} x^j = \vec{Q}^j.
$$

(47)

The above equations also show that as for a composite binary system $T^{00} \sim O(v^0)$, then $T^{0k} \sim O(v^1)$ and $T^{ij} \sim O(v^2)$.

Taking as the source of GWs the composite binary system, the multipole series is an expansion in terms of $r/\lambda = v$, so when expressing the multipoles in terms of the parameters of the individual binary constituents, powers of $v$ have to be tracked in order to arrange a consistent expansion. At lowest order in the multipole expansion and at $v^0$ order

$$
S_{\text{ext}} |_{v^0} = - \frac{1}{\Lambda} \int \, dt \int d^d x T^{00} |_{v^0} \phi = - \frac{M}{\Lambda} \int \, d\phi.
$$

(48)
where in the last passage the explicit expression
\[ T^{00}(t, \vec{x})|_v = \sum_A m_A \delta(\vec{x} - \vec{x}_A(t)), \]
has been inserted. At order \( v \) the contribution from the first order derivative in \( \phi \) have to be added to the contribution of \( T_{\mu\nu}|_v \), which gives
\[ S_{\text{ext}}|_v = \frac{1}{\Lambda_1} \int dt \, d^d x ( - T^{00}|_v \phi_i + T^{0i}|_v A_i ), \]
with
\[ T^{0i}(t, \vec{x})|_v = \sum_A m_A v^i_A \delta(\vec{x} - \vec{x}_A(t)), \]
and neither \( T^{00} \) nor \( T^{ij} \) contain terms linear in \( v \). Since the total mass appearing in equation (48) is conserved (at this order) and given that in the center of mass frame \( \sum_A m_A \vec{x}_A = 0 = \sum_A m_A \vec{v}_A \), there is no radiation up to order \( v \). From order \( v^2 \) on, following a standard procedure, see e.g. [19], it is useful to decompose the source coupling to the gravitational fields in irreducible representations of the \( SO(3) \) rotation group, to obtain
\[ S_{\text{ext}}|_{v^2} = \frac{1}{2 \Lambda_1} \int dt \, d^d x T^{ij}|_v \sigma^{ij} \]
\[ \text{and using the decomposition [19]} \]
\[ \int d^d x T^{ij}|_v = \frac{1}{6} \int d^d x T^{00}|_v \sigma^{ij} + \frac{1}{3} \int d^d x (T^{0j}|_v \sigma^{ik} - 2 T^{0k}|_v \sigma^{ij} - 2 T^{0i}|_v \sigma^{jk}), \]
we can re-write
\[ S_{\text{ext}}|_{v^2} = \int dt \left( \frac{1}{6} Q_{ijkl}|_v E_{ij,k} - \frac{2}{3} P_i j |_v B_{ij} \right), \]
where
\[ P_i j = \int d^d x (\epsilon^{ijkl} T^{0l}|_v \sigma^{ij} + \epsilon^{ijkl} T^{0i}|_v \sigma^{jk}), \]
and
\[ Q^{ijk} = \int d^d x T^{00}|_v \sigma^{ij} \sigma^{jk}. \]
allowing to identify $P^i \leftrightarrow P_{ij}$ and $I_{ijk} \leftrightarrow Q_{ijk}$ at leading order (with $c_I = -4/3$ and $c_O = 1/3$).

At $v^4$ order the $T^{ij|kl} x^l \sigma_{ijkl}$ term, beside giving the leading hexadecapole term (or $2^{4\text{th}}$-pole), also gives a $v^2$ correction to the leading quadrupole interaction $P^i E_{ij}$, which can be written as

$$S_{\text{ext}|v^4} \supset \int d^{d+1}x \left[ T^{00|v^2} + T^{kk|v^2} - \frac{4}{3} T^{00|v^2} \frac{\delta v}{d} \frac{x^2}{d} \right] E_{ij}.$$

(59)

For the systematics at higher orders see [79] or the standard textbook [19].

4.1. Matching between the radiation and the orbital scale

In the previous subsection we have spelled out the general expression of the effective multipole moments in terms of the energy–momentum tensor moments. However we have only used two ingredients from the specific binary problem

- $T_{00} \sim m v^0$
- the source size is $r$ and the length variation of the background is $\lambda \sim r/v$.

Now we are going to match the coefficients appearing in equation (1) with the parameters of the specific theory at the orbital scale.

At leading order $Q_{ij|v^0} = \sum_{A} m_{A} x_{A} x_{A}$ and the $v^2$ corrections to $T_{00}$ can be read from diagrams in figures 15, 16. Such diagrams account for the contribution to the energy–momentum tensor of the gravitational field and are obtained by computing the effective action
with the background field method [80], and picking the term in the resulting effective action linearly coupled to the background gravity field [12].

As \( \phi \) couples to \( T^{00} + T^{ik} / (d - 2) \) and \( \sigma_{ij} \) to \( T^{ij} \), from the diagrams one obtains:

\[
\int d^d x \left( T^{00} + \frac{1}{d - 2} T^{ik} \right) \bigg|_{x^2} = \sum_A \frac{1}{2} \int d^d x A_n^2 m_A^2 \frac{m_A v^2}{g(d)} \sum_{B \neq A} \frac{G_N m_A m_B}{p^{d-2}},
\]

\[
\int d^d x T^{ik} \bigg|_{x^2} = \sum_A \frac{1}{2} \int d^d x A_n^2 m_A^2 \frac{m_A v^2}{g(d)} \sum_{B \neq A} \frac{G_N m_A m_B}{p^{d-2}},
\]

where \( g(d) = (d - 2) \Gamma (d/2 - 1) / [\pi^{d/2 - 1} 2^{d-4} (d - 1)] \). The calculation can be iterated for all higher multipoles, and it does not contain any fundamental difference if framed within the effective field theory approach or traditional methods.

### 4.2. Spin contribution to the source moments

In the case of spinning individual sources, in order to add the spin contributions to the energy–momentum tensor we start from the spin-world-line term in equation (31) to obtain

\[
\sqrt{-g} T^{\mu
u}(t, \vec{x}) = \frac{1}{2} \sum_A \partial_\mu (\vec{x} - \vec{x}(t)) \left( S^{\mu
u}_A u^A + S^{\nu}_A u^A \right),
\]

from which it is possible to derive [81] the leading order energy–momentum tensor components linear in the spins:

\[
T^{00}(t, \vec{k}) \big|_{S^2} = \sum_A \frac{1}{2} S^{00}_A |\vec{k}| e^{-i k \cdot \vec{c}_A},
\]

\[
T^{ij}(t, \vec{k}) \big|_{S^2} = \sum_A \frac{1}{2} S^{ij}_A |\vec{k}| e^{-i k \cdot \vec{c}_A},
\]

where a mixed coordinate-momentum space has been adopted, and the leading \( O(S^2) \) are given by

\[
T^{00}(t, \vec{k}) \big|_{S^2} = - \sum_A \frac{S^{(A)}_{00}}{2 m_A} S^{ij}_A \vec{s}_A \cdot \vec{k} e^{-i k \cdot \vec{c}_A},
\]

with \( T^{ij} \big|_{S^2} \sim v T^{00} \big|_{S^2} \sim v^2 T^{00} \). Since \( S^{ij} k \sim S^{ij} v k \sim m v^3 \) (we recall that \( k \sim 1/r \) is the wave-number exchanged between binary constituents), the above components of the energy–momentum tensor can be used to compute the source moments necessary to derive physical observables, as discussed in the next subsections. At leading order in spin and \( v \), the electric and magnetic quadrupole moments read (using the covariant SSC)

\[
I_{ij} \big|_{S^2} \supset \sum_A 8 \epsilon^{ijk} \left( v_{Ai} S_i x_j - \frac{4}{3} x_{Ai} S_i v_j - \frac{4}{3} x_{Ai} S_i x_j + i \leftrightarrow j \right),
\]

\[
J_{ij} \big|_{S^2} \supset \sum_A S_{Ai} x_j + S_{Aj} x_i,
\]

Note that since only \( \int T^{ik} \) is needed, and not \( T^{ik} \) itself, it could have been computed from equation (47) instead of from the diagram in figure 16.
where $S^i = \epsilon^{ijk} S_{jk}$. For nonlinear terms one has to add diagrams at the orbital scale analogous to figures 15, 16 with spin insertion at the vertices, as well as the $O(S^2)$ term in the worldline energy–momentum tensor in equation (62), which translates to quadratic terms in the quadrupole moments given by

$$I_{ij}\bigg|_{S^2} = \sum_A \frac{c^{(A)}}{m_A} S_A^i S_A^j + S_A^i S_A^j,$$

$$J_{ij}\bigg|_{S^2} = \sum_A \frac{c^{(A)}}{m_A} \epsilon^{ikl} v^k S_A^i S_A^j + i \leftrightarrow j).$$

(65)

4.3. Integrating out gravitational waves: radiation reaction

We have now built an effective theory for extended objects in terms of the source moments and also shown how to match the orbital scale with the theory describing two point particles experiencing mutual gravitational attraction. We can further use the extended object action in equation (1) to integrate out the gravitational radiation to obtain an effective action $S_{\text{mult}}$ for the source multipoles alone.

In order to perform such computation, boundary conditions asymmetric in time have to be imposed, as no incoming radiation at past infinity is required. Using the standard Feynman propagator, which ensures a pure in-(out-)going wave at past (future) infinity, would lead to a non-causal evolution as it can be shown by looking at the following toy model [82], which is defined by a scalar field $\psi$ coupled to a source $J$:

$$S_{\text{toy}} = \int d^{d+1}x \left[ -\frac{1}{2} \left( \partial \psi \right)^2 + \psi J \right].$$

(66)

We may recover the field generated by the source $J$ as

$$\psi(t, \vec{x}) = \int d^{d+1}x G(t - t', \vec{x} - \vec{x}') J(t', \vec{x}'),$$

(67)

where the Feynman propagator given by equation (10) can also be written as

$$G(t, \vec{x}) = \theta(t) \Delta_+ (t, \vec{x}) + \theta(-t) \Delta_- (t, \vec{x}),$$

(68)

with $\Delta_\pm = \frac{e^{\pm i\omega t}}{i\omega k} / k$, which is clearly a-causal because of the $\theta(-t)$ term. In a causal theory $\psi$ would be given by the same equation (67) but with the Feynman propagator replaced by the retarded one $G_{\text{Ret}}(t, \vec{x})$, given by:

$$G_{\text{Ret}}(t, \vec{x}) = -\int \frac{d\omega}{2\pi} \frac{e^{-i\omega t + i\vec{k} \cdot \vec{x}}}{k^2 - (\omega + i\epsilon)^2} = -\frac{1}{4\pi^2} \delta(t - x)$$

$$= -i \theta(t) \left[ \Delta_+ (t, \vec{x}) - \Delta_- (t, \vec{x}) \right] = G_{\text{Adv}}(-t, -\vec{x}).$$

(69)

However it is not possible to naively use the retarded propagator in the action (66), as it would still yield non-causal equations of motions [83]. This problem was not present in the conservative dynamics described in section 3 as the Feynman Green function with symmetric boundary conditions is the appropriate one to describe a conservative system.

However there is a consistent way to define an action for non-conservative system with asymmetric time boundary condition: by adopting a generalization of the Hamilton’s variational principle similar to the closed-time-path, or in–in formalism (first proposed in [84], see [85] for a review) as described in [83], which requires a doubling of the field variables.
For instance the toy model in equation (66) is modified so that the generating functional for
correlated functions in the in–in formalism has the path integral representation
\[ e^{iS_{\text{eff}}[J_1, J_2]} = \int \mathcal{D}\psi_1 \mathcal{D}\psi_2 \exp \left\{ i \int d^{d+1}x \left[ -\frac{1}{2} (\partial\psi_1)^2 + \frac{1}{2} (\partial\psi_2)^2 - J_1 \psi_1 + J_2 \psi_2 \right] \right\}. \] (70)

In this toy example the path integral can be performed exactly, and using the Keldysh
representation [86] defined by
\[ \Psi_1^- \equiv \Psi_1 - \Psi_2, \quad \Psi_1^+ \equiv (\Psi_1 + \Psi_2)/2, \]
one can write
\[ S_{\text{eff}}[J^+, J^-] = \frac{i}{2} \int d^{d+1}x d^{d+1}y J_B(x) G^{BC}(x-y) J_C(y), \] (71)
where the \( B, C \) indices take values \{+, -\} and
\[ G^{BC}(t, \vec{x}) = \begin{pmatrix} 0 \\ iG_{\text{Adv}}(t, \vec{x}) \end{pmatrix}, \]
where \( G^{++} = 0 \) and \( G_{\text{Adv, Ret}, H} \) are the usual advanced, retarded propagators and Hadamard
function respectively, with \( G_H = \Delta_+ + \Delta_- \). In our case, the lowest order expression of the
quadrupole in terms of the binary constituents world-lines \( x_A \), i.e.
\[ Q^{ij}(v_0) = 2 \sum_{A=1} m_A \left( x^{Ai} x^{Aj} - \delta^{ij} x_A^2 \right), \] (73)
is doubled to
\[ Q^{ij}(v_0) = 2 \sum_{A=1} m_A \left[ x^{A+}_- x^{A+}_+ + x^{A+}_- x^{A+}_- - \frac{2}{d} \delta^{ij} x^{A+}_+ x^{A+}_- \right], \]
\[ Q^{ij}(v_0) = 2 \sum_{A=1} m_A \left[ x^{A+}_+ x^{A+}_- - \frac{1}{d} \delta^{ij} x^{A+}_+ x^{A+}_- + O(x_A^2) \right]. \] (74)
The word-line equations of motion that properly include radiation reaction effects are given by
\[ 0 = \frac{\delta S_{\text{eff}}[\vec{x}_1^\pm, \vec{x}_2^\pm]}{\delta \vec{x}_A} \Big|_{\vec{x}_A = \vec{x}_A^\pm}. \] (75)

At lowest order, by integrating out the radiation gravitational modes, i.e. by computing
the diagram in figure 17, one obtains the Burke–Thorne [87] potential term in the effective
action \( S_{\text{mult}} \)
\[ S_{\text{mult}} \big|_{\text{figure 17}} = -\frac{G_N}{5} \int d\tau Q^{ij}(\tau) Q^{ij}(\tau), \] (76)
where \( A^{(n)}(\tau) \equiv \frac{d^n A(\tau)}{d\tau^n} \), which has been derived in the EFT framework in [82]. Corrections
to the leading effect appears when considering as in the previous subsection higher orders in
the multipole expansion: the 1PN correction to the Burke–Thorne potential were originally computed in [34, 88] and re-derived with effective field theory methods in [89].

The genuinely nonlinear effect, computed originally in [33, 34] and within effective field theory methods in [90], appears at relative 1.5PN order and it is due to the diagram in figure 18. The result turns out to have a short-distance singularity which introduces a logarithmic contribution to the effective action (by virtue of equation (75) only terms linear in $Q_{ij}^{\pm}$ are kept)

$$S_{\text{mult}}|_{\text{figure 18}} = -\frac{1}{5}G_{N}^{2}M\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^{6}\left[\frac{1}{\epsilon} - \frac{41}{30} + i\pi \text{sgn}(\omega) - \log\pi + \gamma + \log\frac{\omega^{2}}{\mu^{2}}\right]$$

$$\times [Q_{ij}^{+}(\omega)Q_{+ij}(\omega) + Q_{ij}^{+}(\omega)Q_{+ij}(\omega)],$$

where $\text{sgn}(\omega) = \pm 1$ for $\omega < 0$. We note the presence of the logarithmic term which is non-analytic in $k$-space and non-local (but causal) in direct space: after integrating out a mass-less propagating degree of freedom the effective action is not expected to be local [91]. A local mass counter term $M_{ct}$ defined by

$$M_{ct} = -\frac{2}{5}G_{N}^{2}M\left(\frac{1}{\epsilon} + \gamma - \log\pi\right)Q_{ij}(\omega)Q_{ij}^{(0)}$$

can be straightforwardly added to the world-line effective action to get rid of the divergence appearing as $\epsilon \to 0$. According to the standard renormalization procedure, one can define a renormalized mass $M^{(R)}(t, \mu)$ for the monopole term in the action (1), depending on time (or frequency) and on the arbitrary scale $\mu$ in such a way that physical quantities (like the energy or the radiation reaction force) will be $\mu$-independent.10

The derivation of the $\mu$ dependence of the renormalized mass was first obtained in [92] by evaluating $Q_{ij}^{(n)}$ corrections to the energy–momentum tensor of a binary system. Here we give a simplified version of such derivation, following [90], by deriving the logarithmic corrections to the equations of motion from equation (77)

$$\delta \ddot{x}_{A}(t)|_{\log} = -\frac{8}{5}v_{A}(t)G_{N}^{2}M\int_{-\infty}^{t}dt'Q_{ij}^{(7)}(t')\log[(t - t')\mu].$$

Separating the logarithm argument into a $t$-dependent and a $t$-independent part, one gets a logarithmic term not-involving time which gives a conservative contribution to the force in equation (79) which shifts logarithmically the mass of the binary system. The logarithmic mass-shift $\delta M^{(R)}$ can be determined by requiring that its time derivative balance the acceleration shift given by equation (79) [56]

$$\frac{d(\delta M^{(R)})}{dt} = -\sum_{A}m_{A}\delta \ddot{x}_{A} \cdot \vec{v}_{A}.$$  

10 Note that at the order required in the diagram in figure 18, $M^{(R)}(t, \mu)$ can be safely treated as a constant $M$ on both its arguments $t$ and $\mu$. 

---

**Figure 18.** Next-to-leading diagram in the radiation back-reaction process.
Substituting equation (79) into equation (80) and using the leading order quadrupole moment expression in equation (73) allows to turn the rhs of equation (80) into a total time derivative, enabling to identify the logarithmic mass-shift as [56]

$$\delta M^{(R)} |_{log} = \frac{2G_5^2 M}{5}(2Q^{(5)}_{ij}Q^{(1)}_{ij} - 2Q^{(4)}_{ij}Q^{(2)}_{ij} + Q^{(3)}_{ij}Q^{(3)}_{ij}) \log(\mu).$$  

Equation (81) can be rewritten as a renormalization group flow equation [92]

$$\mu \frac{d}{d\mu} M^{(R)}(t, \mu) = -\frac{2G_5^2 M}{5}(2Q^{(5)}_{ij}Q^{(1)}_{ij} - 2Q^{(4)}_{ij}Q^{(2)}_{ij} + Q^{(3)}_{ij}Q^{(3)}_{ij}).$$  

This classical renormalization of the mass monopole term (which can be identified with the Bondi mass of the system, that does not include the energy radiated to infinity) is explained in [92] by considering that the emitted radiation is scattered by the curved space and then absorbed, hence observers at different distance from the source would not agree on the value of the mass.

The UV nature of the divergence points to the incompleteness of the effective theory in terms of multipole moments: the terms analytic in $\omega$ in equation (77) are sensitive to the short-distance physics and their actual value should be obtained by going to the theory at orbital radius.

The tail term radiation reaction force is responsible for a conservative force at 4PN (as the leading radiation reaction acts at 2.5PN and the tail term is a 1.5PN correction to it), so it must be added to the conservative dynamics coming from the calculation of the effective action not involving gravitational radiation, and indeed it is responsible for the logarithmic term in equation (26).

### 4.4. Emitted flux

We have now shown how to perform the matching between the theory of extended objects with multipoles and the theory at the orbital scale. Taking the action for extended bodies in equation (1) as a starting point, the emitted GW-form and the total radiated power can be computed in terms of the source multipoles by evaluating the coupling term $A(k)$ in the effective Lagrangian between a GW $\sigma_{ij}(k)$ and its source, using Feynman diagrams with one external radiating gravitational particle. At leading order such term is given by the diagram in figure 19 and results in

$$A(k) = -\frac{k^2}{4\lambda} Q^{(j)}(k)\sigma^{(j)}(k).$$  

---

**Figure 19.** Diagram representing the emission of a GW from a quadrupole source.
In standard field theory language, $A(k)$ is the term linear in the gravity background field and can be straightforwardly obtained from the second of equations (52) after gauge fixing to the TT gauge.

The GW-form can be computed using

$$
h_{ij}(t, \vec{x}) = A_{ij,k} \int \mathcal{D}h_{\mu\nu} \mathcal{D}h_{\mu\nu} + \sigma_{kl}(t, x) e^{S_{\text{bulk}} + S_{\text{ext}}},$$

where we have introduced the TT-projector $A_{ij,k}$ defined as

$$A_{ij,k}(k) = P_{ik} P_{jl} - \frac{1}{d-1} P_{ij} P_{kl},$$

being $n \equiv k/k$ the unit vector in the propagation direction, and we have considered only the linear interactions between the multipoles and the gravity field and neglected terms from gravity self-interactions.

Analogously to what shown in the previous subsection, we have to take into account the GW interaction with the space-time curvature produced by the source itself. Including such effect give rise to a tail effect, accounted by the diagram in figure 20, which gives a quadrupole contribution to the GW amplitude and phase \[35, 93\]

$$h_{ij}(t, \vec{x}) \supset A_{ij,k} \int \frac{d\omega}{2\pi} e^{i(\omega - r) + iG_{\chi} M_{0}} \left[ \frac{1}{2} + \log \left( \frac{\omega}{\mu} \right)^2 + \gamma - \frac{1}{2} \right] (1 + G_{N} m |\omega| \pi) I_{kl}.$$  

The infra-red singularity in the phase of the emitted wave is un-physical as it can be absorbed in a redefinition of time in equation (86). Moreover any experiment, like LIGO and Virgo for instance, can only probe phase differences (e.g. the GW phase difference between the instants when the wave enters and exits the experiment sensitive band) and the un-physical dependencies on the regulator $\epsilon$ and on the subtraction scale $\mu$ drops out of any observable.

The contribution from the magnetic quadrupole is analogous to the one in equation (86), and it is \[93\]

$$h_{ij}(t, \vec{x}) \supset A_{ij,k} \int \frac{d\omega}{2\pi} e^{i(\omega - r) + iG_{\chi} M_{0}} \left[ \frac{1}{2} + \log \left( \frac{\omega}{\mu} \right)^2 + \gamma - \frac{1}{2} \right] (1 + G_{N} m |\omega| \pi) J_{kl}.$$  

Figure 20. Emission of a GW from a quadrupole source with post-Minkowskian correction represented by the scattering off the background curved by the presence of binary system.
where the finite number associated with the logarithm is still un-physical, as it depends on the choice of the arbitrary scale \( \mu \), but the difference between the terms in the phase in equations (86), (87) is physical, as \( \mu \) can be chosen only once [93]. Spin effects can be included straightforwardly by using the appropriate multipole expression.

The total emitted flux can be computed once the amplitude of the GW has been evaluated, via the standard formula

\[
P = \frac{r^2}{32\pi G_N} \int d\Omega \langle h_{ij} h_{ij} \rangle, \tag{88}
\]

but there is actually a shortcut, as the emission energy rate can be computed directly from \( A(k) \) in equation (83) without explicitly solving for \( h_{ij} \) as done in equation (84):

\[
dP(k) = \frac{1}{T} \frac{d^3k}{(2\pi)^3} k^4 Q_{ij}(k)Q_{ij}(k) \Lambda_{ij,kl}(k), \tag{89}
\]

which in standard field theory formalism can be derived by applying the optical theorem (see e.g. section 7.3 of [94] for its derivation) to the diagram in figure 17. Without using the optical theorem, we note that equation (89) can be explicitly derived by substituting in equation (88) the expression (84) and using the direct space representation of \( G_{\text{Ret}} \) given in equation (69).

Integrating equation (89) one gets [24]

\[
P \simeq \frac{G_N}{5\pi T} \int_0^\infty d\omega \omega^6 \left[ |I_{ij}(\omega)|^2 + \frac{16}{9} |J_{ij}(\omega)|^2 + \frac{5}{189} \omega^2 |I_{jk}(\omega)|^2 + \cdots \right], \tag{90}
\]

which, once averaged over time, recovers at the lowest order the standard Einstein quadrupole formula \( P = G_N \langle I_{ij}^2 \rangle / 5 \). There are however corrections to this result for any given multipole, due to the scattering of the GW off the curved space-time because of the presence of the static potential due to the presence of the massive binary system. The first of such corrections scale as \( G_NM \int (d^3k)^2 \delta^3(k) \sim G_NMk \sim v^3 \) (for radiation \( k \sim v/r \)), that is a 1.5PN correction with respect to the leading order. The tail process is described by the diagram in figure 20 and it adds up to the leading order to give a contribution to the flux going as

\[
|A|^{2} = [1 + 2\pi G_NM\omega + O(v^6)]|A|^{2}. \tag{91}
\]

The diagrams quadratic in the background curvature are portrayed in figures 21 and they give an UV divergence, with a logarithmic term [24]

\[
|A|^{2} = \left[ -(G_NM\omega)^2 \frac{214}{105} \ln \frac{\omega^2}{\mu^2} + \cdots \right]|A|^{2}. \tag{92}
\]
depending on the arbitrary subtraction scale $\mu$, where finite contributions have been omitted. This short-distance singularity represents a failure of the effective theory at the radiation scale to correctly describe short-distance physics: in order to fix the omitted numerical quantity analytic in $\omega$ one should match the multipole theory to the theory in which the binary constituents are at a finite distance $r$.

However the coefficient of the logarithm is physical and we can then proceed to renormalize the theory at the radiation scale, which is done in the usual fashion as in QFT, although here the effect is completely classical. Since $|A|^2$ (stripped of the $h_{ij}$ term) enters physical results like energy emission, it should be independent of the arbitrary scale $\mu$: this can only happen if we assume a $\mu$ dependence on the renormalized multipole moments $I_{ij}$ of the type:

$$\mu \frac{dI_{ij}^{(R)}}{d\mu} = -\frac{214}{105} (G_NM\omega)^2 I_{ij}^{(R)}.$$  (93)

Assuming that $A$ is expressed in terms of the $I_{ij}^{(R)}$, the total dependence of $|A|^2$ on $\mu$ cancels out (it makes no difference if using $I_{ij}^{(R)}$ or the ‘bare’ $I_{ij}$ in $A|_\nu^\nu$, as the difference is higher order in $\nu$). The background curvature has the effect of ‘smearing’ the multipole source which cannot be considered perfectly localized at the origin of the coordinates: the value of the $I_{ij}^{(R)}$ will depend on the scale at which the observer will measure it.

A consequence of this result is that equation (93) admits a solution

$$I_{ij}^{(R)}(\omega, \mu) = \left(\frac{\mu}{\mu_0}\right)^2 I_{ij}(\omega, \mu_0),$$  (94)

that constrains the patterns of logarithms that can appear at higher orders. Once the multipole is known at some scale, like the orbital scale separation, then it can be known at any other scale by virtue of equation (94).

Finally one could consider the scattering of the emitted GW wave off another GW, as in figure 22. This process is known as nonlinear memory effect, it represents a 2.5PN correction with respect to the leading emission amplitude [95–97] and it has not yet been computed within the effective field theory formalism.

Combined tail and memory effects enter at 4PN order in the emitted radiation, i.e. double scattering of the emitted radiation off the background curvature and off another GW.
The divergences describing such process have been analyzed in [92], leading to the original derivation of the mass renormalization described in subsection 4.3. The renormalization group equations allow a resummation of the logarithmic term making a non-trivial prediction for the pattern of the leading UV logarithms appearing at higher orders [24, 92].

5. Conclusions

This Topical Review aims at giving an overview of the basic ideas of effective field theory (EFT) methods proposed in [12] to model gravitationally bound, inspiralling compact binary systems. The study of such systems has both phenomenological and theoretical motivations, due to the forthcoming observational campaign of the large interferometric detectors LIGO and Virgo (and eventually KAGRA and Indigo) on one side, and on the development of efficient numerical methods to solve Einstein equations on the other side.

The post-Newtonian (PN) investigation of the compact binary inspiral problem has a long history in analytical perturbative solutions of the Einstein equations, but EFT methods have allowed a new field theory insight into it. The problem admits a description in terms of well separated scales (the individual source size, the binary component distance and the radiation wavelength), with a single dimension-less perturbative parameter (at least in the binary black-hole case), represented by the relative velocity of the individual components of the binary system. The EFT methods allow to treat in a single, powerful framework both conservative and dissipative effects and provide efficient tools to compute observable quantities. They give an organizational principle for performing a systematic expansion in the PN perturbative parameter. The scale factorization is already evident at the level of the action, which allows a considerable computational simplification with respect to methods working at the level of the equations of motion. The effective field theory approach reviewed here has much in common with standard quantum field theory techniques because of the common underlying field theory structure and it is completely classic.

Physics at different scales are related by renormalization group flow, and all kind of divergences, arising from incomplete knowledge of the underlying short-distance physics as well as from long-distance effects and from gauge artifacts, are technically treated on equal footing via dimensional regularization. Indeed the use of field theory since several decades has allowed the development of powerful tools to address all the technical problems (like handling of divergences and computation of Feynman integrals) on the computational side.

Finally, the existence of an additional independent method to compute physical observables of the binary problem in General Relativity is welcome \textit{per se}, as it allows an independent check of computations of formidable complexity.

Acknowledgments

It is a pleasure to thank Luc Blanchet and the referee for providing useful comments. SF is supported by the Fonds National Suisse, RS is supported by the FAPESP grant 2013/04538-5. RS wishes to thank the CERN theory division for hospitality and support during the last stage of this work.

References

[1] Hulse R A and Taylor J H 1975 Discovery of a pulsar in a binary system \textit{Astrophys. J.} \textbf{195} L51
[2] Weisberg J M and Taylor J H 1984 Observations of post-Newtonian timing effects in the binary pulsar PSR 1913+16 \textit{Phys. Rev. Lett.} \textbf{52} 1348
[3] Burgay M et al 2003 An increased estimate of the merger rate of double neutron stars from observations of a highly relativistic system *Nature* 426 531

[4] Kramer M and Wex N 2009 The double pulsar system: a unique laboratory for gravity *Class. Quantum Grav.* 26 073001

[5] Wolszczan A 1991 A nearby 37.9-ms radio pulsar in a relativistic binary system *Nature* 350 688

[6] Stairs I H, Thorsett S E, Taylor J H and Wolszczan A 2002 Studies of the relativistic binary pulsar psr b1534+12: I. timing analysis *Astrophys. J.* 581 501

[7] Aasi J et al 2013 Search for gravitational waves from binary black hole inspiral, merger and ringdown in LIGO-Virgo data from 2009–2010 *Phys. Rev. D* 87 022002

[8] Sturani R 2012 LIGO/Virgo/GEO/KAGRA science 9th LISA Symposium, Paris (ASP Conference Series) vol 467 (San Francisco, CA: Astronomical Society of the Pacific)

[9] Aso Y et al 2013 Interferometer design of the KAGRA gravitational wave detector arXiv:1306.6747

[10] Abadie J et al (LIGO Scientific and Virgo Collaborations) 2010 Predictions for the rates of compact binary coalescences observable by ground-based gravitational-wave detectors *Class. Quantum Grav.* 27 173001

[11] Blanchet L 2013 Gravitational radiation from post-Newtonian sources and inspiralling compact binaries arXiv:1310.1528

[12] Goldberger W D and Rothstein I Z 2006 An effective field theory of gravity for extended objects *Phys. Rev. D* 73 104029

[13] Goldberger W D 2007 Les houches lectures on effective field theories and gravitational radiation *Les Houches Summer School—Session 86: Particle Physics and Cosmology: The Fabric of Spacetime* p 351

[14] Bertotti B and Plebanski J 1960 Theory of gravitational perturbations in the fast motion approximation *Ann. Phys.* 11 169

[15] Dass N D H and Somi V 1982 Feynman graph derivation of Einstein quadrupole formula *J. Phys. A: Math. Gen.* 15 473

[16] Donoghue J F 1994 General relativity as an effective field theory: the leading quantum corrections *Phys. Rev. D* 50 3874

[17] Damour T and Esposito-Farese G 1996 Testing gravity to second post-Newtonian order: a field theory approach *Phys. Rev. D* 53 5541–78

[18] Peters P C and Mathews J 1963 Gravitational radiation from point masses in a Keplerian orbit *Phys. Rev. D* 131 435–9

[19] Maggiore M 2008 *Gravitational Waves* (Oxford: Oxford University Press)

[20] Hinder I et al 2013 Error-analysis and comparison to analytical models of numerical waveforms produced by the NRAR collaboration arXiv:1307.5307

[21] Hannam M, Ohme F and Husa S 2011 Reliability of complete gravitational waveform models for compact binary coalescences *Phys. Rev. D* 84 064029

[22] Georgi H 1990 An effective field theory for heavy quarks at low-energies *Phys. Lett. B* 240 447

[23] Isgur N and Wise M B 1991 Spectroscopy with heavy quark symmetry *Phys. Rev. Lett.* 66 1130

[24] Goldberger W D and Ross A 2010 Gravitational radiative corrections from effective field theory *Phys. Rev. D* 81 124015

[25] Galley C R, Leibovich A K and Rothstein I Z 2010 Finite size corrections to the radiation reaction force in classical electrodynamics *Phys. Rev. Lett.* 105 094802

[26] Damour T 1983 Gravitational radiation and the motion of compact bodies *Gravitational Radiation* ed N Deruelle and T Piran (Amsterdam: North-Holland) pp 59–144

[27] Binnington T and Poisson E 2009 Relativistic theory of tidal Love numbers *Phys. Rev. D* 80 084018

[28] Kol B and Smolkin M 2012 Black hole stereotyping: induced gravito-static polarization *J. High Energy Phys.* JHEP02(2012)010

[29] Kol B and Smolkin M 2008 Non-relativistic gravitation: from Newton to Einstein and back *Class. Quantum Grav.* 25 145011

[30] Kol B and Smolkin M 2008 Classical effective field theory and caged black holes *Phys. Rev. D* 77 064033

[31] Blanchet L and Damour T 1989 Post-Newtonian generation of gravitational waves *Ann. Inst. Henri Poincare A* 50 377–408

[32] Rothstein I Z 2003 *Tasi Lectures on Effective Field Theories* arXiv:hep-ph/0308266

[33] Blanchet L and Damour T 1988 Tail transported temporal correlations in the dynamics of a gravitating system *Phys. Rev. D* 37 1410

[34] Blanchet L 1993 Time asymmetric structure of gravitational radiation *Phys. Rev. D* 47 4392–420

31
[35] Blanchet L and Schafer G 1993 Gravitational wave tails and binary star systems Class. Quantum Grav. 10 2699–721
[36] Kol B and Smolkin M 2012 Einstein’s action and the harmonic gauge in terms of Newtonian fields Phys. Rev. D 85 044029
[37] Gilmore J B and Ross A 2008 Effective field theory calculation of second post-Newtonian binary dynamics Phys. Rev. D 78 124021
[38] Chu Y-Z 2009 The n-body problem in general relativity up to the second post-Newtonian order from perturbative field theory Phys. Rev. D 80 044031
[39] Foffa S and Sturani R 2011 Effective field theory calculation of conservative binary dynamics at third post-Newtonian order Phys. Rev. D 84 044031
[40] Foffa S and Sturani R 2013 The dynamics of the gravitational two-body problem in the post-Newtonian approximation at quadratic order in the Newton’s constant Phys. Rev. D 87 064011
[41] Jaranowski P and Schafer G 2013 Towards the 4th post-Newtonian Hamiltonian for two-point-mass systems Phys. Rev. D 86 061503
[42] Jaranowski P and Schfer G 2013 Dimensional regularization of local singularities in the 4th post-Newtonian two-point-mass Hamiltonian Phys. Rev. D 87 081503
[43] Smirnov V A 2006 Feynman Integral Calculus (Berlin: Springer)
[44] Kol B and Shir R 2013 Classical 3-loop 2-body diagrams J. High Energy Phys. JHEP09(2013)069
[45] von Manteuffel A and Studerus C 2012 Reduze 2-distributed Feynman integral reduction arXiv:1201.4330
[46] Foffa S, Mastrolia P, Sturani R and Sturm C in preparation
[47] Kol B and Smolkin M 2009 Dressing the post-Newtonian two-body problem and classical effective field theory Phys. Rev. D 80 124044
[48] Neill D and Rothstein I Z 2013 Classical space-times from the S matrix Nucl. Phys. B 877 177
[49] Damour T, Jaranowski P and Schaefer G 2001 Dimensional regularization of the gravitational interaction of point masses Phys. Lett. B 513 147–55
[50] Itoh Y 2004 Equation of motion for relativistic compact binaries with the strong field point particle limit: third post-Newtonian order Phys. Rev. D 69 064018
[51] Blanchet L and Faye G 2000 Equations of motion of point particle binaries at the third post-Newtonian order Phys. Lett. A 271 58
[52] Blanchet L and Faye G 2001 General relativistic dynamics of compact binaries at the third post-Newtonian order Phys. Rev. D 63 062005
[53] de Andrade V C, Blanchet L and Faye G 2001 Third post-Newtonian dynamics of compact binaries: Noetherian conserved quantities and equivalence between the harmonic coordinate and ADM Hamiltonian formalisms Class. Quantum Grav. 18 753–78
[54] Blanchet L, Damour T and Esposito-Farese G 2004 Dimensional regularization of the third post-Newtonian dynamics of point particles in harmonic coordinates Phys. Rev. D 69 124007
[55] Blanchet L and Iyer B R 2003 Third post-Newtonian dynamics of compact binaries: equations of motion in the center-of-mass frame Class. Quantum Grav. 20 755
[56] Le Tiec A, Blanchet L, Detweiler S L and Whiting B F 2010 High-order post-Newtonian fit of the gravitational self-force for circular orbits in the schwarzschild geometry Phys. Rev. D 81 084033
[57] Le Tiec A, Blanchet L and Whiting B F 2012 The first law of binary black hole mechanics in general relativity and post-Newtonian theory Phys. Rev. D 85 064039
[58] Bini D and Damour T 2013 Analytical determination of the two-body gravitational interaction potential at the 4th post-Newtonian approximation arXiv:1305.4884
[59] Porto R A and Rothstein I Z 2006 The hyperfine Einstein–Infeld–Hoffmann potential Rev. Lett. 97 021101
[60] Porto R A 2007 New results at 3PN via an effective field theory of gravity Proc. the MG11 Meeting on General Relativity pp 2493–6
[61] Porto R A and Rothstein I Z 2008 Next to leading order Spin(1)Spin(1) effects in the motion of inspiralling compact binaries Phys. Rev. D 78 044013
[62] Hartung J and Steinhoff J 2011 Next-to-next-to-leading order post-Newtonian spin(1)–spin(2) Hamiltonian for self-gravitating binaries Ann. Phys. 523 919–24
[63] Levi M 2012 Binary dynamics from spin1–spin2 coupling at fourth post-Newtonian order Phys. Rev. D 85 064043
[64] Marsat S, Bohé A, Faye G and Blanchet L 2013 Next-to-next-to-leading order spin-orbit effects in the equations of motion of compact binary systems Class. Quantum Grav. 30 055007

32
[65] Bohé A, Marsat S, Faye G and Blanchet L 2013 Next-to-next-to-leading order spin-orbit effects in the near-zone metric and precession equations of compact binaries Class. Quantum Grav. 30 075017

[66] Bohé A, Marsat S and Blanchet L 2013 Next-to-next-to-leading order spin-orbit effects in the gravitational wave flux and orbital phasing of compact binaries Class. Quantum Grav. 30 135009

[67] Porto R A 2006 Post-Newtonian corrections to the motion of spinning bodies in NRGR Phys. Rev. D 73 104031

[68] Porto R A and Rothstein I Z 2008 Spin(1)Spin(2) effects in the motion of inspiralling compact binaries at third order in the post-Newtonian expansion Phys. Rev. D 78 044012

[69] Porto R A 2010 Next to leading order spin-orbit effects in the motion of inspiralling compact binaries Class. Quantum Grav. 27 205001

[70] Levi M 2010 Next to leading order gravitational spin–orbit coupling in an effective field theory approach Phys. Rev. D 82 104004

[71] Levi M 2010 Next to leading order gravitational Spin1–Spin2 coupling with Kaluza–Klein reduction Phys. Rev. D 82 104031

[72] Mathisson M 1937 Neue mechanik materieller systemes Acta Phys. Pol. 6 163–2900

[73] Papapetrou A 1951 Spinning test particles in general relativity: part 1 Proc. R. Soc. Lond. A 209 248–58

[74] Dixon W G 1970 Dynamics of extended bodies in general relativity: I. Momentum and angular momentum Proc. R. Soc. Lond. A 314 499–527

[75] Goldstein H, Poole C P Jr and Safko J L 2001 Classical Mechanics (Reading, MA: Addison-Wesley)

[76] Damour T, Jaranowski P and Schaefer G 2008 Hamiltonian of two spinning compact bodies with next-to-leading order gravitational spin–orbit coupling Phys. Rev. D 77 064032

[77] Barausse E, Racine E and Buonanno A 2009 Hamiltonian of a spinning test-particle in curved spacetime Phys. Rev. D 80 104025

[78] Hergt S, Steinhoff J and Schaefer G 2012 On the comparison of results regarding the post-Newtonian approximate treatment of the dynamics of extended spinning compact binaries arXiv:1205.4530

[79] Ross A 2012 Multipole expansion at the level of the action Phys. Rev. D 85 125033

[80] DeWitt B S 1967 Quantum theory of gravity: 2. The manifestly covariant theory Phys. Rev. 162 1195–239

[81] Ross A, Porto R A and Rothstein I Z 2011 Spin induced multipole moments for the gravitational wave flux from binary inspirals to third post-Newtonian order J. Cosmol. Astropart. Phys. JCAP03(2011)009

[82] Tiglio M and Galley C R 2009 Radiation reaction and gravitational waves in the effective field theory approach Phys. Rev. D 79 124027

[83] Galley C R 2013 The classical mechanics of non-conservative systems Phys. Rev. Lett. 110 174301

[84] Schwinger J S 1961 Brownian motion of a quantum oscillator J. Math. Phys. 2 407–32

[85] DeWitt B 1986 Effective action for expectation values Quantum Concepts in Space and Time ed R Penrose and C J Isham (Oxford: Clarendon)

[86] Keldysh L V 1964 Diagram technique for nonequilibrium processes Zh. Eksp. Teor. Fiz. 47 1515–27

[87] Burke W L and Thorne K S 1970 Gravitational radiation damping Relativity ed S I Fickler, M Carmeli and L Witten (New York: Plenum) pp 209–28

[88] Iyer B R and Will C M 1993 Post-Newtonian gravitational radiation reaction for two-body systems Phys. Rev. Lett. 70 113

[89] Galley C R and Leibovich A K 2012 Radiation reaction at 3.5 post-Newtonian order in effective field theory Phys. Rev. D 86 044027

[90] Foffa S and Sturani R 2013 Tail terms in gravitational radiation reaction via effective field theory Phys. Rev. D 87 044056

[91] Appelquist T and Carazzone J 1975 Infrared singularities and massive fields Phys. Rev. D 11 2856

[92] Ross A, Goldberger W D and Rothstein I Z 2012 Black hole mass dynamics and renormalization group evolution arXiv:1211.6095 [hep-th]

[93] Ross A, Porto R A and Rothstein I Z 2012 Spin induced multipole moments for the gravitational wave amplitude from binary inspirals to 2.5 post-Newtonian order J. Cosmol. Astropart. Phys. JCAP09(2012)028
[94] Peskin M E and Schroeder D V 1995 An Introduction To Quantum Field Theory (Reading, MA: Addison-Wesley)
[95] Christodoulou D 1991 Nonlinear nature of gravitation and gravitational wave experiments Phys. Rev. Lett. 67 1486–9
[96] Blanchet L and Damour T 1992 Hereditary effects in gravitational radiation Phys. Rev. D 46 4304–19
[97] Blanchet L 1998 Gravitational wave tails of tails Class. Quantum Grav. 15 113–41