Appendix B. A Survey of the Hodge Conjecture for Abelian Varieties by B. Brent Gordon

Introduction

The goal of this appendix is to review what is known about the Hodge conjecture for abelian varieties, with an emphasis on how Mumford-Tate groups have been applied to this problem. In addition to the book in which this appears, other survey or general articles that precede this one are Hodge's original paper [B.53], Grothendieck's modification of Hodge's general conjecture [B.43], Shioda's excellent survey article [B.117], Steenbrink's comments on the general Hodge conjecture [B.120], and van Geemen's pleasing introduction to the Hodge conjecture for abelian varieties [B.35]. Naturally there is some overlap between this appendix and van Geemen's article, but since his emphasis is on abelian varieties of Weil type, we hope that this appendix will be a useful complement.

Since the language of linear algebraic groups and their Lie algebras, which cannot be avoided in any discussion of Mumford-Tate groups and their application to the Hodge conjecture for abelian varieties, may not be familiar to students of complex algebraic geometry and Hodge theory, we begin by recalling the definitions and facts we need and giving some examples. Towards the end of section one we also recall some basic facts about abelian varieties, including the Albert classification of their endomorphism algebras (Theorem 1.12.2), and give some example of abelian varieties to which we refer later. Most readers will find it more profitable to begin with section two, where we discuss the definitions and some general structural properties of the Hodge, Mumford-Tate and Lefschetz groups associated to an abelian variety, or section three, and refer to the first section as needed.

Starting with section three we have tried to be comprehensive in summarizing the known results and indicating the main ideas involved in their proofs, while at the same time selecting some cross-section of proofs to discuss in more detail. In section three we follow Murty's exposition [B.84] of the Hodge \((p,p)\) conjecture for arbitrary products of elliptic curves. In section four we summarize Shioda's results on abelian varieties of Fermat type [B.116], but only briefly consider the issues and results related to abelian varieties of Weil type, since [B.35] treats this topic well. In section five we discuss the work of Moonen and Zarhin on four-dimensional abelian varieties [B.75], for this provides a nice illustration of the interplay between the endomorphism algebra and the Mumford-Tate group of an abelian variety. In section six we look at the work of Tankeev and Ribet on the Hodge conjecture for simple abelian varieties that satisfy some conditions on their dimension or endomorphism algebra [B.124] [B.125] [B.126] [B.93] [B.94]; for example, the Hodge conjecture is true for simple abelian varieties of prime dimension. Here we look more closely at Ribet's approach, where he introduced and used the Lefschetz group of an abelian variety. Then the results of Murty and Hazama discussed in section seven build on and go beyond Ribet's methods to treat abelian varieties not assumed to be simple, but still assumed to satisfy some


conditions on their dimensions or endomorphism algebras \([B.81] [B.82] [B.83] [B.46] [B.47] [B.49]\).

In section eight we shift directions slightly, for here we have collected together examples of exceptional Hodge cycles, i.e., Hodge cycles not accounted for by linear combinations of intersections of divisor classes, and in this section, not known to be algebraic. Largely missing from section eight, but considered in section nine, are the particular problems posed by abelian varieties of complex multiplication type. Dodson \([B.28] [B.29] [B.30]\) and others have constructed numerous examples of such abelian varieties that support exceptional Hodge cycles.

In section ten we examine what is known about the general Hodge conjecture for abelian varieties. The majority of the work on this problem is either a very geometric treatment of special abelian varieties in low dimension, for example \([B.12] [B.104]\), or requires special assumptions about the endomorphism algebra, dimension or Hodge group, as in \([B.127] [B.128] [B.50] [B.4]\).

In the final section eleven we briefly mention three alternative approaches to proving the (usual) Hodge conjecture for arbitrary abelian varieties: First, a method involving the Weil intermediate Jacobian \([B.98]\); then that the Tate conjecture for abelian varieties implies the Hodge conjecture for abelian varieties \([B.88] [B.87]\) and \([B.27]\); and thirdly, that the Hodge conjecture for abelian varieties would follow from knowing Grothendieck’s invariant cycles conjecture \([B.42]\) for certain general families of abelian varieties, and moreover, that for these families, the invariant cycles conjecture would follow from the \(L_2\)-cohomology analogue of Grothendieck’s standard conjecture (A) that the Hodge \(*\)-operator is algebraic \([B.44]\) \([B.4]\). The present state of our knowledge about the Hodge conjecture for abelian varieties is such that any or none of these approaches might ultimately work, or a counterexample might exist.

Preceding the bibliography is a rough chronological table of the work that directly address some aspect of the Hodge conjecture for abelian varieties. I have tried to make sure that this table and this appendix as a whole mention all the relevant references through the end of 1996; if I have omitted something or otherwise not done it justice, that was quite unintentional.

1. Abelian varieties and linear algebraic groups

The purpose of this section is to establish the language we use throughout the rest of this appendix to discuss abelian varieties and certain linear algebraic groups and Lie algebras associated with them. Although abelian varieties and linear algebraic groups are both algebraic groups, the issues surrounding them tend to be of a very different nature. It turns out to be most convenient to begin by recalling some of the definitions and basic properties of linear algebraic groups and their Lie algebras, and introducing some of the examples of these to which we will later refer, and then in the second half of the section review some of the definitions and basic properties of abelian varieties, and introduce some of the examples we will investigate later.

1.0. Notational conventions.

1.0.1. Field of definition. Let \(F\) be a field and \(V\) and algebraic variety. Then we will write \(V_F\) to signify or emphasize that \(V\) is defined over \(F\). When \(V\)
is an algebraic variety defined over $F$ and $K$ is a field containing $F$, then $V_K = V_F \times_{\text{spec } F} \text{spec } K$ is the base change to $K$, i.e., $V$ as a variety defined over $K$. We will generally try to distinguish the abstract variety $V_F$ defined over $F$ from its concrete set of $F$-points $V(F)$, and then $V(K) = V_F(K)$ is the set of $K$-points.

1.0.2. Definition. Suppose $K$ is a separable algebraic extension of $F$ of finite degree $d$, and $V$ is an algebraic variety defined over the larger field $K$. Let $\{\sigma_1, \ldots, \sigma_d\}$ be the set of distinct embeddings of $K$ into the algebraic closure $F_{\text{alg}}$ of $F$. Then the restriction of scalars functor $\text{Res}_{K/F}$ from varieties over $K$ to varieties over $F$ is defined as follows: First let $V_{\sigma_i} = V_K \times_{\text{spec } K, \sigma_i} F_{\text{alg}}$. Then for any variety $W$ defined over $F$ and a morphism $\varphi : W \to V$ defined over $K$ there are morphisms $\varphi_{\sigma_i} : W \to V_{\sigma_i}$. Then if 

$$(\varphi_{\sigma_1}, \ldots, \varphi_{\sigma_d}) : W \to V_{\sigma_1} \times \cdots \times V_{\sigma_d}$$

is an isomorphism, then $W = \text{Res}_{K/F} V$ is the variety obtained from $V$ by restriction of the field of definition from $K$ to $F$. Its uniqueness is a consequence of the universal property that whenever $X$ is any variety defined over $F$ and $\psi : X \to V$ is a morphism defined over $K$, then there exists a unique $\Psi : X \to W$ defined over $F$ such that $\psi = \varphi \circ \psi$. In practice it is often easiest to look at the $K$-points, then

$$\text{Res}_{K/F} V(K) \simeq \prod_{\sigma \in \text{Hom}_F(K, F_{\text{alg}})} V_{K, \sigma}(K)$$

together with the action of $\text{Gal}(F_{\text{alg}}/F)$ permuting the factors according to its action on $\{\sigma_1, \ldots, \sigma_d\}$. For further details see [B.136] 1.3.

1.1. Definition. An algebraic group over $F$ is an algebraic variety $G$ defined over $F$ together with morphisms

$$\text{mult} : G \times G \to G \quad \text{and} \quad \text{inv} : G \to G,$$

both defined over $F$, and an element $e \in G(F)$ such that $G$ is a group with identity $e$, multiplication given by $\text{mult}$, and inverses given by $\text{inv}$. A morphism of algebraic groups is a morphism of algebraic varieties which is also a group homomorphism. As a variety an algebraic group is smooth, since it contains an open subvariety of smooth points and the group of translations $h \mapsto gh$ acts transitively.

1.1.1. Definition. An abelian variety is a complete connected algebraic group. It follows from this definition that an abelian variety is a smooth projective variety and that its group law is commutative, see for example [B.96], [B.121], [B.68], [B.95], [B.69], [B.74], [B.79] or [B.134]. It also follows that every morphism of abelian varieties as varieties can be expressed as a composition of a homomorphism with a translation, though of course only homomorphisms are morphisms of abelian varieties as algebraic groups. In this appendix we will only be dealing with abelian varieties defined over $\mathbb{C}$, that is, complex abelian varieties. When $A$ is a complex abelian variety then the manifold underlying $A(\mathbb{C})$ is a complex torus.
1.1.2. Definition. An affine algebraic group is also called a linear algebraic group. This is justified by the fact that an affine algebraic group is isomorphic, over its field of definition, to a closed subgroup of \( \text{GL}(n) \) for some \( n \), see [B.14], [B.55], [B.52], [B.119], [B.133].

1.2. Definition. Any affine algebraic group that is isomorphic (as an algebraic group) to the diagonal subgroup of \( \text{GL}(n) \) for some \( n \) is called an algebraic torus. For additional basic exposition on algebraic tori see [B.14] III.8 or [B.55] §16.

1.2.1. Example. Our most basic and important example of an algebraic torus is \( \mathbb{G}_m := \text{GL}(1) \). A priori \( \mathbb{G}_m = \mathbb{G}_m / \mathbb{Q} \) is defined over \( \mathbb{Q} \), and thus \( \mathbb{G}_m(F) = F^\times \) for any field \( F \) containing \( \mathbb{Q} \). Similarly, with the conventions of 1.0, \( \mathbb{G}_m / F(K) = K^\times \) when \( K \) is a field containing \( F \).

1.2.2. Example. We may also apply the restriction of scalars functor to an algebraic torus. For the purposes of this appendix, one of the most important examples that we will use later is

\[
S := \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m / \mathbb{C}.
\]

Then \( S(\mathbb{R}) = \mathbb{C}^\times \) and \( S(\mathbb{C}) \simeq \mathbb{C}^\times \times \mathbb{C}^\times \), where these last two factors are interchanged by complex conjugation. In particular \( S(\mathbb{R}) \) embeds as the diagonal in \( S(\mathbb{C}) \).

1.3. Definition. A connected linear algebraic group of positive dimension is said to be semisimple if it has no closed connected commutative normal subgroups except the identity. A (Zariski-connected) linear algebraic group \( G \) is said to be reductive if is the product of two (Zariski-connected) normal subgroups \( G_{ab} \) and \( G_{ss} \), where \( G_{ab} \) is an algebraic torus and \( G_{ss} \) is semisimple, and \( G_{ab} \cap G_{ss} \) is finite. A fortiori any semisimple group is reductive.

1.4. Definition. Recall that a representation of a group \( G \) is a homomorphism \( \rho : G \to \text{GL}(V) \) from \( G \) to the automorphism group of a vector space \( V \). Such a representation may be referred to as \( \langle \rho, V \rangle \) or simply by \( \rho \) or by \( V \). If \( \langle \sigma, W \rangle \) is another representation of \( G \), a map \( \psi : V \to W \) such that \( \sigma(g) \circ \psi = \psi \circ \rho(g) \) for all \( g \in G \) is said to be \( G \)-linear or \( G \)-equivariant. In this case, if \( \psi \) is an isomorphism the representations \( \langle \rho, V \rangle \) and \( \langle \sigma, W \rangle \) are said to be equivalent. We frequently identify equivalent representations.

A subrepresentation of a representation is defined in the natural way, and a representation is said to be irreducible if it contains no nontrivial subrepresentations. Further, given representations \( \langle \rho, V \rangle \) and \( \langle \sigma, W \rangle \) of \( G \) we may form their direct sum or their tensor product. Thus the \( r^{th} \) exterior power \( \langle \wedge^r \rho, \wedge^r V \rangle \) of a representation arises naturally as a subrepresentation of the \( r \)-fold tensor product of the representation \( \langle \rho, V \rangle \) with itself.

Let \( V^\vee = \text{Hom}(V, F) \) denote the dual space to \( V \) (if \( V \) is a vector space over \( F \)), and let \( \langle , \rangle : V \times V \to F \) be the natural pairing. Then \( \rho \) induces a representation \( \rho^\vee : G \to \text{GL}(V^\vee) \), called the dual, or contragredient representation, of \( G \). It is defined by requiring

\[
\langle \rho^\vee(g)v^\vee, \rho(g)v \rangle = \langle v^\vee, v \rangle;
\]

concretely this means that \( \rho^\vee(g) = {}^t\rho(g)^{-1} \).
When $G$ is a subgroup of $GL(V)$, for some vector space $V$, in particular when $G$ comes as a subgroup of $GL(n)$, the group of invertible $n \times n$ matrices, then by the standard representation of $G$ we mean the natural inclusion $G \hookrightarrow GL(V)$.

1.5. Examples of semisimple and reductive groups. The examples that will be of interest to us are all classical groups, defined from the outset as subgroups of $GL(n)$.

1.5.1. Example. The first basic example is $SL(n)$, the subgroup of $GL(n)$ of matrices of determinant 1. For $n \geq 2$, $SL(n)$ is semisimple. It follows that $GL(n)$ is reductive, as it is the product of its subgroup of diagonal matrices and $SL(n)$.

1.5.2. Example. Let $F$ be a subfield of the real numbers, in particular $\mathbb{R}$ itself, let $K$ be an imaginary quadratic extension of $F$, and let $V$ be a vector space over $K$. Then a Hermitian form on $V$ is an $F$-bilinear form $H : V \times V \to K$ such that $H(\sigma(u), \sigma(v)) = \sigma(H(u, v))$, where $\sigma$ is the nontrivial automorphism of $K$ over $F$, the restriction of complex conjugation. Then the unitary group $U(V, H)$ is the subgroup of $g \in GL(V)$ such that $H(gu, gv) = H(u, v)$, and the special unitary group $SU(V, H)$ is the subgroup of $U(V)$ of elements of determinant 1. Note that $U(V, H)$ and $SU(V, H)$ are algebraic groups defined over $F$. When $F = \mathbb{R}$ and $H$ can be represented by a diagonal matrix with $p$ 1's and $q$ $(-1)$'s then we may write $U(p, q)$ for $U(V, H)$ or $SU(p, q)$, respectively; when $q = 0$, that is when $H$ is equivalent to the standard form $(u, v) \mapsto \langle u, v \rangle$, we write $U(n)$ or $SU(n)$. As a particular special case, note that $U(1)$ is defined over $\mathbb{R}$, and $U(1, \mathbb{R})$ is the group of complex numbers of absolute value 1. Similarly as in the previous example, $SU(n)$ is semisimple and $U(n)$ is reductive, for $n \geq 2$.

1.5.3. Example. When $E$ is a skew-symmetric bilinear form on a vector space $V$, that is, $E(v, u) = -E(u, v)$, then the symplectic group $Sp(V, E)$ is the subgroup of $g \in GL(V)$ such that $E(gu, gv) = E(u, v)$. The symplectic similitude group $GSp(V, E)$ is the group of $g \in GL(V)$ such that there is a scalar $\nu(g)$ such that $E(gu, gv) = \nu(g)E(u, v)$. Thus $GSp(V, E)$ contains $Sp(E, V)$ as the subgroup of similitude norm $\nu(g) = 1$. When $E$ can be represented by a matrix of the form

$$
\begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix},
$$

then we may write $Sp(2n)$ for $Sp(V, E)$ and $GSp(2n)$ for $GSp(V, E)$. Moreover, $Sp(2n)$ is semisimple.

1.5.4. Example. When $S$ is a symmetric bilinear form on a vector space $V$, then the special orthogonal group $SO(V, S)$ is the subgroup of $g \in SL(V)$ such that $S(gu, gv) = S(u, v)$. In particular, if $S$ can be represented by an identity matrix $I_n$ then we may write $SO(n)$ instead of $SO(V, S)$. Also $SO(n)$ is semisimple.

1.6. Lie algebras. It will be useful later to have available some of the language of Lie algebras, so we briefly recall some of the definitions here. Our major references for this paragraph (and the next two) are \([B.34]\), \([B.54]\), \([B.14]\), \([B.55]\) and \([B.103]\).
1.6.1. Definition. A Lie algebra is a vector space $\mathfrak{g}$ together with a skew-symmetric bilinear map, the bracket operation,

$$[\ , \ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$  

Example. Let $V$ be a vector space over a field $F$. The fundamental example of a Lie algebra is $\mathfrak{gl}(V)$, which is $\text{End}_F(V)$ as a vector space with the bracket given by $[X, Y] := XY - YX$.

1.6.2. The Lie algebra of an algebraic group. Let $G$ be a linear algebraic group over a field $F$. In order to review how a Lie algebra $\text{Lie}(G)$ is associated to $G$, first recall that when $A$ is any $F$-algebra then the Lie algebra of $F$-derivations from $A$ to $A$ can be described as

$$\text{Der}_F(A, A) := \{X \in \mathfrak{gl}(A) : X(f \cdot g) = (Xf) \cdot g + f \cdot (Xg), \text{ for } f, g \in A\}$$

with the induced bracket $[X, Y] := XY - YX$. Now let $A = F[G]$, the coordinate ring of $G$ (as algebraic variety). Then $G$ acts on $F[G]$ by left translations:

$$(\lambda g)(f) := f(g^{-1}x), \text{ for } f \in F[G] \text{ and } g, x \in G.$$  

Then the set of left invariant derivations

$$L(G) := \{X \in \text{Der}_F(F[G], F[G]) : \lambda_g \circ X = X \circ \lambda_g \text{ for all } g \in G\}$$

is a Lie subalgebra of $\text{Der}_F(A, A)$, and some authors take this as the definition of $\text{Lie}(G)$. Next, recall that when $O_e$ is the local ring at $e$ and $m_e$ its maximal ideal, the tangent space to $G$ at the identity is

$$T(G)_e = \text{Der}_F(O_e, O_e/m_e) \cong \text{Hom}_{F\text{-mod}}(m_e/m_e^2, F).$$

Of course $O_e/m_e$ is the residue field of the local ring at the identity $e$. Then it turns out that evaluation at the identity $e$ of $G$ gives an isomorphism from $L(G)$ to $T(G)_e$ to $G$ at the identity (see references on linear algebraic groups cited above). Thus $\text{Lie}(G)$ can also be defined as $T(G)_e$ with the bracket operation induced by the isomorphism with $L(G)$.

Example. As the notation suggests, $\mathfrak{gl}(V) = \text{Lie}(\text{GL}(V))$.

Remark. One motivation for working with Lie algebras is that for a connected linear group $G$ a homomorphism $\varphi : G \to H$ to another group $H$ is determined by its differential at the identity. In this way Lie algebras linearize some of the problems of representation theory. More generally, when some property of $G$ is determined by an open neighborhood of the identity, it is often more effective work with $\text{Lie}(G)$.

1.6.3. The adjoint representations. In general the differential of a morphism of (irreducible) algebraic varieties $\varphi : X \to Y$ at $x \in X$ is the linear map on tangent spaces $d\varphi_x : T(X)_x \to T(Y)_{\varphi(x)}$ induced by $\varphi^* : O_{\varphi(x)} \to O_x$. In particular, $G$ acts on itself by inner automorphisms

$$\text{Int}_g : h \mapsto ghg^{-1},$$

where $g \in G$. A group automorphism $\varphi : G \to G$ is called inner if there exists $g \in G$ such that $\varphi = \text{Int}_g$. The group $G$ acts on itself by inner automorphisms via the adjoint representation $\text{Ad} : G \to \text{Aut}(G)$, $h \mapsto \text{Ad}_h$, where $\text{Ad}_h(g) = \text{Ad}_h(g) = ghg^{-1}$.
and this action fixes the identity. Then the differential of this map

\[ \text{Ad}(g) := d(\text{Int}_g)_e : T(G)_e \to T(G)_e \]

defines the \textit{adjoint representation} of \(G\)

\[ \text{Ad} : G \to \text{Aut}(T(G)_e) : g \mapsto \text{Ad}(g). \]

If we go one step further and take the differential of the adjoint representation, we get a Lie algebra morphism

\[ \text{ad} := d\text{Ad} : T(G)_e \to \text{End}(T(G)_e), \]

with \(\text{ad}(X)(Y) = [X,Y]\).

1.6.4. \textbf{Definition.} Now let \(G\) be (the real points of) a connected algebraic group over \(\mathbb{R}\), and let \(K\) be a maximal compact subgroup. Then a \textit{Cartan involution} of \(G\) with respect to \(K\) is an involutive automorphism of \(G\) whose fixed point set is precisely \(K\). The differential of a Cartan involution is a Cartan involution of \(\text{Lie}(G)\), and the decomposition

\[ \text{Lie}(G) = \mathfrak{k} + \mathfrak{p} \]

where \(\mathfrak{k}\) is the fixed point set and \(\mathfrak{p}\) is the \((-1)\)-eigenspace, is called a \textit{Cartan decomposition}. It follows that

\[ [\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}. \]

1.6.5. \textbf{Definition.} A semisimple real Lie algebra \(\mathfrak{g}\) with a Cartan decomposition \(\mathfrak{g} = \mathfrak{k} + \mathfrak{p}\) is of \textit{Hermitian type} if there exists an element \(H_0\) in the center of \(\mathfrak{k}\) such that \((\text{ad}(H_0))^2 = -1\) as endomorphisms of \(\mathfrak{p}\). Recall that a reductive group is an extension of a semisimple group by an algebraic torus. We will say that a real reductive algebraic group \(G\) is of \textit{Hermitian type} if the abelian part of \(G\) is compact and the semisimple part of its Lie algebra, \(\text{Lie}(G)_{ss}\), is of Hermitian type in the previous sense.

1.7. \textbf{Examples.} Let \(V\) be a vector space over a field \(F\). We have already noted that \(\text{gl}(V) = \text{Lie}(\text{GL}(V))\) is \(\text{End}(V)\) as a vector space with the bracket \([X,Y] = XY - YX\).

The subgroup of \(\text{gl}(V)\) of endomorphisms with trace 0 is \(\mathfrak{sl}(V) = \text{Lie}(\text{SL}(V))\).

Suppose \(V\) is a symplectic space of dimension \(2n\) whose skew-symmetric form is represented by

\[ \left( \begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array} \right) \]

Then \(\mathfrak{sp}(2n) = \text{Lie}(\text{Sp}(2n))\) consists of matrices of the form

\[ X = \left( \begin{array}{cc} M & N \\ P & Q \end{array} \right) \]

such that \(N\) and \(P\) are symmetric and \(^tM = -Q\).

When \(V\) has dimension \(n\) and comes with a symmetric bilinear form represented by an identity matrix, then \(\mathfrak{so}(n) = \text{Lie}(\text{SO}(n))\) consists of \(n \times n\) skew-symmetric matrices.
1.8. The spin representations of $\mathfrak{so}(n)$. As a subgroup of $\mathfrak{gl}(V) = \text{End}(V)$ any of the Lie algebras above naturally acts on $V$, and we may think of this as the standard representation of the Lie algebra, cf. 1.4. Moreover, up to equivalence, all the representations of $\mathfrak{sl}(n)$, respectively $\mathfrak{sp}(2n)$, occur in some tensor power of the standard representation. However, that is not the case for $\mathfrak{so}(n)$, so here we briefly recall the complex representation(s) that do not.

Let $V$ be an $n$-dimensional vector space with a nondegenerate symmetric bilinear form $Q$. Then the quotient of the tensor algebra of $V$ by the ideal generated by all elements of the form $v \otimes v - Q(v, v)$ for $v \in V$ is called the Clifford algebra $C(V)$ of $V$. Since this ideal preserves the property that an element of the tensor algebra is the product of an even number of vectors, such products generate a subalgebra $C^+(V)$ of $C(V)$ called the even Clifford algebra. For more details about Clifford algebras some good sources are [B.20] [B.31] or [B.19].

Now following [B.34] Chapter 20 we first observe that since $C(V)$ and $C^+(V)$ are associative algebras they determine Lie algebras with $[a, b] = a \cdot b - b \cdot a$. Moreover, $\mathfrak{so}(V)$ embeds in $C^+(V)$ as a Lie subalgebra. Roughly speaking, on the one hand there is an embedding $\psi : \wedge^2 V \to C^+(V)$, given by $\psi(a \wedge b) = a \cdot b - Q(a, b)$, while on the other hand there is an isomorphism $\varphi : \wedge^2 V \xrightarrow{\sim} \mathfrak{so}(V) \subset \mathfrak{gl}(V)$ given by

$$
\varphi(a \wedge b)(v) = 2(Q(b, v)a - Q(a, v)b).
$$

Now suppose $n = 2m$ is even (and the underlying field $F = \mathbb{C}$). Then $V$ can be written as the sum of two $m$-dimensional isotropic subspaces, $V = W \oplus W'$ (meaning that the restriction of $Q$ to $W$, respectively $W'$, is zero). Then the key lemma is that $C(V) \simeq \text{End}(\wedge^* W)$, where $\wedge^* W$ signifies the exterior algebra of $W$. From this it follows that, if we write $\wedge^* W = \wedge^+ W \oplus \wedge^- W$ corresponding to even and odd exterior powers, then

$$
C^+(V) \simeq \text{End}(\wedge^+ W) \oplus \text{End}(\wedge^- W).
$$

Therefore the embedding of $\mathfrak{so}(V)$ into $C^+(V)$ determines two (inequivalent) representations of $\mathfrak{so}(V)$, namely its actions on $\wedge^+ W$ and $\wedge^- W$ respectively. These are referred to as the half-spin representations of $\mathfrak{so}(V)$, and their sum is the spin representation.

When $n = 2m + 1$ is odd, then we may write $V = W \oplus W' \oplus U$, where $W$ and $W'$ are $m$-dimensional isotropic subspaces, as before, and $U$ is 1-dimensional and orthogonal to both $W$ and $W'$. In this case

$$
C(V) \simeq \text{End}(\wedge^* W) \oplus \text{End}(\wedge^* W')
$$

and $C^+(V) \simeq \text{End}(\wedge^* W)$. Thus the embedding of $\mathfrak{so}(V)$ into $C^+(V)$ determines a single spin representation.

1.9. Quaternion algebras. A quaternion algebra over a field $F$ (of characteristic not 2) is a simple $F$-algebra of rank 4 whose center is precisely $F$. Over the complex numbers, or any algebraically closed field of characteristic not equal to 2, there is up to isomorphism only one quaternion algebra, the $2 \times 2$ matrix algebra. Over the real numbers, aside from the $2 \times 2$ matrix algebra there is up to isomorphism only one other quaternion algebra, the Hamiltonian quaternion algebra $\mathbb{H}$,
generated over \(\mathbb{R}\) by 1 and elements \(i\) and \(j\) such that
\[
i^2 = j^2 = -1, \quad ij = -ji.\]

Note that \(\mathbb{H}\) is a division algebra. Over \(\mathbb{Q}\) there are infinitely many non-isomorphic quaternion algebras, all of which except the \(2 \times 2\) matrix algebra are (noncommutative) division algebras; see for example [B.132]. A quaternion algebra over \(\mathbb{Q}\), or more generally a quaternion algebra over a subfield of \(\mathbb{R}\), is said to be definite or indefinite according as its tensor product with \(\mathbb{R}\) is isomorphic to \(\mathbb{H}\) or to \(M_2(\mathbb{R})\). In particular, an indefinite quaternion algebra can be embedded into \(M_2(\mathbb{R})\).

In general, a quaternion algebra over \(F\) has a basis consisting of 1 and elements \(\alpha, \beta\) and \(\alpha \beta\) such that \(\alpha^2\) and \(\beta^2\) are nonzero elements of \(F\) and \(\beta \alpha = -\alpha \beta\). There is also a canonical involution on a quaternion algebra given by
\[
(a + b\alpha + c\beta + d\alpha \beta)' = a - b\alpha - c\beta + d\alpha \beta \quad \text{or} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
\]

Then the reduced trace and reduced norm of an element \(\gamma\) are \(\gamma + \gamma'\) and \(\gamma \cdot \gamma'\) respectively.

A subring of a quaternion algebra that is also a lattice, i.e., a free \(\mathbb{Z}\)-module of rank 4, is called an order in the quaternion algebra. Very roughly, maximal orders play a similar role for quaternion algebras as rings of integers do for number fields, except that maximal orders in quaternion algebras need not be unique.

1.10. Definition. A complex structure on a real vector space \(W_\mathbb{R}\) of dimension \(2g\) is given by any of the following equivalent data:

(i) A scalar multiplication by \(\mathbb{C}\) with which \(W_\mathbb{R}\) is a \(g\)-dimensional complex vector space.

(ii) An endomorphism \(J \in \text{End}(W_\mathbb{R})\) such that \(J^2 = -\text{Id}\).

(iii) A homomorphism \(h_1 : U(1) \to \text{GL}(W_\mathbb{R})\) of algebraic groups over \(\mathbb{R}\) such that for \(u \in U(1)\) the action of \(h(u)\) on \(W_\mathbb{C}\) has only \(u^{\pm 1}\)-eigenspaces, each occurring with equal multiplicity \(g\).

(iv) A homomorphism \(h : \mathbb{S} \to \text{GL}(W_\mathbb{R})\) of algebraic groups over \(\mathbb{R}\) such that for \((z, w) \in \mathbb{S}(\mathbb{C})\) the action of \(h(z, w)\) on \(W_\mathbb{C}\) has only \(z^tw^0\) and \(z^0w^1\)-eigenspaces, each occurring with equal multiplicity \(g\).

To see the equivalence of these conditions, if a complex vector space structure is given, let \(J\) be the action of multiplication by \(i = \sqrt{-1}\). Then \(h_1(i) = J\) determines either \(h_1\) or \(J\) in terms of the other. Since \(\mathbb{S} = \mathbb{G}_{m/\mathbb{R}} \cdot U(1)\), then \(h\) is determined by \(h_1\) and \(\mathbb{R}\)-linearity, or by \(h(a + bi) = a\text{Id} + bJ\). And \(h\) in turn defines a scalar multiplication by \(\mathbb{C}\). Also \(h_1\) is the restriction of \(h\) to \(U(1) \subset \mathbb{S}\), on account of which we sometimes simply write \(h\) instead of \(h_1\).

1.10.1. Example. In the notation of 1.6.5, \(\text{ad}(H_0)\) defines a complex structure on \(p\), with which \(p\) becomes a complex vector space.

1.10.2. Example. To give a complex torus \(V/L\), where \(V\) is a \(g\)-dimensional complex vector space and \(L \subset V\) is a lattice, is the same as giving a real \(2g\)-dimensional vector space \(W\) together with a complex structure, say \(J\), and a lattice \(L \subset W\). We can go back and forth between these two points of view by thinking of \(W\) as the real vector space underlying \(V\) and \(J\) as the induced complex structure,
or by thinking of \( V \) as the complex vector space defined by the pair \((W, J)\). In particular it will sometimes be convenient below to present a complex torus as a triple \((W, J, L)\) instead of in the form \(V/L\).

1.11. Complex abelian varieties. After the definition given in 1.1.1, a complex abelian variety \(A\) is a complete, connected algebraic group over \(\mathbb{C}\) whose group law is necessarily commutative. A morphism of abelian varieties will always be taken to mean a morphism in the sense of algebraic groups (see 1.1).

1.11.1. Definition. A Riemann form on a complex torus \(V/L\) is a nondegenerate, skew-symmetric, real-valued, \(\mathbb{R}\)-bilinear form \(E : V \times V \to \mathbb{R}\) such that

(i) \(E(iv, iw) = E(v, w)\),
(ii) \((v, w) \mapsto E(v, iw)\) is symmetric and positive definite, and
(iii) \(E(v, w) \in \mathbb{Z}\) whenever \(v, w \in L\).

For a proof of the following proposition, see the references cited in 1.1.1.

1.11.2. Proposition. A complex torus is the underlying manifold of a complex abelian variety if and only if it admits a Riemann form. \(\square\)

1.11.3. Definition. A morphism of complex abelian varieties is called an isogeny if it is surjective and has a finite kernel. Given an isogeny \(\varphi : A \to A'\) there exists a dual isogeny \(\varphi' : A' \to A\) such that \(\varphi' \circ \varphi = m \text{Id}_A\) and \(\varphi \circ \varphi' = m \text{Id}_{A'}\) for some positive integer \(m\) called the degree of \(\varphi\). Thus two complex abelian varieties are said to be isogenous iff there exists an isogeny from one to the other, and being isogenous is an equivalence relation. An abelian variety is said to be simple iff it is not isogenous to a product of (positive dimensional) abelian varieties.

The following proposition is proved in Lecture 12, 12.25, or see the references on abelian varieties cited above.

1.11.4. Proposition (Poincaré Reducibility Theorem). If \(A\) is an abelian variety and \(A' \subset A\) is an abelian subvariety, then there exists an abelian subvariety \(A'' \subset A\) such that \(A' \cap A''\) is finite and \(A\) is isogenous to \(A' \times A''\). In particular, any abelian variety is isogenous to a product of simple abelian varieties.

1.11.5. Definition. Two Riemann forms \(E\) and \(E'\) are said to be equivalent iff there exist positive integers \(n\) and \(n'\) such that \(nE = n'E'\). A polarization of a complex torus \(T\) is an equivalence class, say \([E]\), of Riemann forms on \(T\). By a polarized abelian variety we mean an abelian variety together with a choice of polarization. In light of 1.10.2 and 1.11.2, a polarized abelian variety is determined by data \((W, J, L, E)\), where \(W\) is an even-dimensional real vector space, \(J\) is a complex structure on \(W\), \(L\) is a lattice in \(W\), and \(E\) is a Riemann form on the complex torus \((W, J, L)\).

1.12. The endomorphism algebra of an abelian variety. For a complex abelian variety \(A\) let \(\text{End}(A)\) denote its endomorphism ring, and let

\[
\text{End}^0(A) := \text{End}(A) \otimes_\mathbb{Z} \mathbb{Q}.
\]
1.12.1. Lemma.
1. When $A$ and $A'$ are isogenous abelian varieties, $\text{End}^0(A) \simeq \text{End}^0(A')$.
2. $\text{End}^0(A)$ is a semisimple $\mathbb{Q}$-algebra with a positive involution.

Recall that an involution $\iota$ of $\text{End}^0(A)$ is said to be positive if for nonzero $\varphi \in \text{End}^0(A)$ the trace $\text{Tr}(\varphi \cdot \varphi^\iota) > 0$.

**Proof.** If $\varphi : A \rightarrow A'$ is an isogeny and $\varphi^{\vee} : A' \rightarrow A$ is the dual isogeny, then $\varphi^\ast : \text{End}^0(A') \rightarrow \text{End}^0(A)$ and $\frac{1}{m}(\varphi^{\vee})^\ast : \text{End}^0(A) \rightarrow \text{End}^0(A')$ are mutually inverse ring homomorphisms.

To prove part 2, first observe that in general the image of a homomorphism of complex tori is a subtorus, and the kernel is a closed subgroup whose connected component of the identity is a subtorus of finite index in the full kernel. Thus Schur’s Lemma implies that $\text{End}^0(A)$ is a division algebra when $A$ is a simple abelian variety, and then the semisimplicity of $\text{End}^0(A)$ for general $A$ follows from the Poincaré Reducibility Theorem.

To see that $\text{End}^0(A)$ has a positive involution $\iota$, let $E$ be a Riemann form on $A$. Then $H(u,v) = E(u,iv) + iE(u,v)$ is a Hermitian form on $W_{\mathbb{R}}$, and if we take $\iota$ to be the antiautomorphism that takes $\varphi \in \text{End}^0(A)$ to its adjoint with respect to $H$ then the conditions making $E$ a Riemann form imply that $\iota$ is a positive involution. \hfill \Box

**Definition.** The involution $\iota$ of $\text{End}^0(A)$ described in the proof above is called the Rosati involution.

Thus when $A$ is simple, $\text{End}^0(A)$ is a division algebra over $\mathbb{Q}$ which admits a positive involution. Such algebras were classified by Albert [B.6], [B.7], [B.8], see also [B.109] and [B.79]. The result is the following.

1.12.2. Theorem (Albert classification). Let $A$ be a simple complex abelian variety. Let $K$ be the center of $\text{End}^0(A)$ and let $K_0$ be the subfield of elements of $K$ fixed by the Rosati involution. Then $\text{End}^0(A)$ is one of the following types:

(I) $\text{End}^0(A) = K = K_0$ is a totally real algebraic number field, and the Rosati involution acts as the identity.

(II) $K = K_0$ is a totally real number algebraic field, and $\text{End}^0(A)$ is a division quaternion algebra over $K$ such that every simple component of $\text{End}^0(A) \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $M_2(\mathbb{R})$; there is an element $\beta \in \text{End}^0(A)$ such that $^t\beta = -\beta$, and $\beta^2 \in K$ is totally negative; and the Rosati involution is given by $\alpha^\iota = \beta^{-1} \cdot ^t\alpha \cdot \beta$.

(III) $K = K_0$ is a totally real number algebraic field, and $\text{End}^0(A)$ is a division quaternion algebra over $K$ such that every simple component of $\text{End}^0(A) \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to the Hamiltonian quaternion algebra $\mathbb{H}$ over $\mathbb{R}$; and $\alpha^\iota = ^t\alpha$.

(IV) $K_0$ is a totally real number field, and $K$ is a totally imaginary quadratic extension of $K$, and $\text{End}^0(A)$ is division algebra with center $K$, and the restriction of the Rosati involution to $K$ acts as the restriction of complex conjugation to $K$. 

Thus we will say that an abelian variety or an abelian manifold $A$ is of type (I), (II), (III) or (IV) if $A$ is simple and $\text{End}^0(A)$ is of that type in the classification above, or when $A$ is not simple, if it is isogenous to a product of simple abelian varieties of that type.

1.13. Examples of abelian varieties. We now introduce some basic constructions of complex abelian varieties with various endomorphism algebras. In the next sections we will more fully analyze their Hodge structures.

1.13.1. Elliptic curves. Let $E$ be a 1-dimensional complex abelian variety, an elliptic curve. Then $E(\mathbb{C}) \cong \mathbb{C}/(\tau \mathbb{Z} + \mathbb{Z})$ for some $\tau \in \mathbb{C}$ with $\text{Im} \tau > 0$. A Riemann form is given by the pairing

$$((a\tau + b), (c\tau + d)) \mapsto \frac{1}{\text{Im} \tau} \text{Im}((a\tau + b)(c\tau + d)) = ad - bc$$

It is an elementary exercise to show that $\text{End}^0(E)$ can only be isomorphic to $\mathbb{Q}$ or to an imaginary quadratic field, say $K$; in the latter case $E$ is said to have complex multiplication by $K$. Moreover $E$ has complex multiplication if and only if $\tau$ is quadratic over $\mathbb{Q}$, in which case $\mathbb{Q}(\tau) = K$. The general elliptic curve, whose period $\tau$ has algebraically independent transcendental real and imaginary parts, has $\text{End}^0(E) \cong \mathbb{Q}$.

1.13.2. Abelian varieties with multiplication by an imaginary quadratic field. We will say that an abelian variety has multiplication by an imaginary quadratic field $K$ if there is an embedding $K \hookrightarrow \text{End}^0(A)$. Following [B.135], to construct a simple complex abelian variety $A$ with multiplication $K$, let $W$ be an $n$-dimensional vector space over $K$. Then $W_{\mathbb{R}} = W \otimes_{\mathbb{Q}} \mathbb{R}$ may be identified with $\mathbb{C}^n$, but we may also twist the complex structure as follows. Write $W_{\mathbb{R}} = W'_R \oplus W''_R$ as the direct sum of two subspaces over $\mathbb{C}$, and define a complex structure $J \in \text{End}_R(W_{\mathbb{R}})$ by $Jw' = iw'$ for $w' \in W'_R$ and $Jw'' = -iw''$ for $w'' \in W''_R$. Then with this complex structure, $\alpha \in K$ acts on $(W_{\mathbb{R}}, J)$ by $w' \mapsto \alpha w'$ for $w' \in W'_R$ and $w'' \mapsto \alpha w''$ for $w'' \in W''_R$. In particular, when $L \subset W$ is a lattice such that $L \otimes_{\mathbb{Z}} \mathbb{Q} = W$, then the set of $\alpha \in K$ such that $\alpha \cdot L \subseteq L$ is a subring of $K$ commensurable with the ring of integers of $K$. Thus $A(\mathbb{C}) = (W_{\mathbb{R}}/L, J)$ is a complex torus with an embedding $K \hookrightarrow \text{End}^0(A)$. To exhibit a Riemann form for $A$, let $H$ be any $\mathbb{Q}$-valued Hermitian form on $W \times W$ which as a $\mathbb{C}$-valued form on $W_{\mathbb{R}} \times W_{\mathbb{R}}$ is positive definite on $W'_R$ and negative definite on $W''_R$, meaning in particular that these two subspaces are orthogonal with respect to $H$. Then $H = S + iE$ for $\mathbb{R}$-valued forms $S$ and $E$, and the imaginary part $E$ of $H$ is a Riemann form for $A$.

Let $n' = \dim_{\mathbb{C}} W'_R$ and $n'' = \dim_{\mathbb{C}} W''_R$. Then $n' + n'' = n$, and by [B.109] Thm.5, when $n \geq 3$ then both $n'$ and $n''$ are positive, and $n = 2$ does not occur. If $n' = n''$ the pair $(A, K)$ is said to be an abelian variety of Weil type. Equivalently, an abelian variety of Weil type is a pair $(A, K)$ consisting of an abelian variety $A$ and an imaginary quadratic field $K$ with an embedding $K \hookrightarrow \text{End}^0(A)$ such that for $\alpha \in K$ the corresponding endomorphism has the eigenvalues $\alpha$ and $\bar{\alpha}$ with equal multiplicity. The example of Mumford in [B.88], see Lecture 7, 7.23–7.28, is one example of an abelian variety of Weil type.
Another case that will arise below is when \( n' \) and \( n'' \) are relatively prime. We will refer to a pair \((A,K)\) satisfying this condition as an abelian variety of Ribet type, see [B.94] Thm.3.

1.13.3. Simple abelian varieties of odd prime dimension. Let \( A \) be a simple complex abelian variety of odd prime dimension \( g \). Then reading off from the tables in [B.85], the cases that occur are: \( \text{End}_0^0(A) \cong \mathbb{Q} \), which is the general case; or \( \text{End}_0^0(A) \) is a totally real number field of degree \( g \) over \( \mathbb{Q} \); or \( \text{End}_0^0(A) \) is an imaginary quadratic field, in which case \( A \) is of Ribet type; or \( A \) is of CM-type, that is, a totally imaginary quadratic extension of a totally real field of degree \( g \) over \( \mathbb{Q} \).

1.13.4. Simple abelian fourfolds. By reading the tables in [B.85], the endomorphism algebras that can occur for a simple abelian fourfold are, by Albert type:

(I) \( \text{End}_0^0(A) \) is \( \mathbb{Q} \), or a real quadratic field, or a totally real quartic field;

(II) \( \text{End}_0^0(A) \) is an indefinite division quaternion algebra over \( \mathbb{Q} \) or a totally indefinite division quaternion algebra over a real quadratic field;

(III) \( \text{End}_0^0(A) \) is a definite division quaternion over \( \mathbb{Q} \);

(IV) \( \text{End}_0^0(A) \) is an imaginary quadratic field, in which case it can only be of Ribet type with \( \{n', n''\} = \{1, 3\} \) or of Weil type with \( n' = n'' = 2 \), or else \( \text{End}_0^0(A) \) is a CM-field of degree 4 or 8 over \( \mathbb{Q} \).

1.13.5. Abelian varieties with real multiplication. An abelian variety of type (I) is sometimes said to have real multiplication. To construct an example of a simple abelian variety with real multiplication, let \( K \) be a totally real number field with \([K : \mathbb{Q}] = g\), and let \( \mathcal{O} \) be the ring of integers of \( K \). Then there are \( g \) distinct embeddings \( \alpha \mapsto \alpha^{(j)} \) of \( K \) into \( \mathbb{R} \). Let \( \tau_j \in \mathbb{C} \) with \( \text{Im} \tau_j > 0 \), for \( 1 \leq j \leq g \). Then the image of \( \mathcal{O} \oplus \mathcal{O} \) under the map

\[
(\alpha, \beta) \mapsto (\alpha^{(1)} \tau_1 + \beta^{(1)}), \ldots, \alpha^{(g)} \tau_g + \beta^{(g)}
\]

is a lattice \( L \subset \mathbb{C}^g \), and \( A = \mathbb{C}^g/L \) is a complex abelian variety. A Riemann form is given by

\[
E(z,w) = \sum_{j=1}^{g} (\text{Im} \tau_j)^{-1} \text{Im}(z_j \bar{w}_j),
\]

where \( z, w \in \mathbb{C}^g \). Then \( K \hookrightarrow \text{End}_0^0(A) \), and this is an isomorphism for general \((\tau_1, \ldots, \tau_g)\). It can be shown that any simple abelian variety \( A \) for which \( \text{End}_0^0(A) \) is a totally real number field is isogenous to one which can be constructed as we have here [B.27].

1.13.6. Abelian varieties of CM-type. Recall that an algebraic number field \( K \) is said to be a CM-field iff it is a totally imaginary quadratic extension of a totally real number field \( K_0 \). The embeddings of a CM-field \( K \) into \( \mathbb{C} \) come in complex conjugate pairs. Then CM-type for \( K \) is a subset \( S \subset \text{Hom}(K, \mathbb{C}) \) containing exactly one from each pair of conjugate embeddings, so that \( \text{Hom}(K, \mathbb{C}) = S \cup \overline{S} \).

A simple abelian variety \( A \) is said to be of CM-type, or to have complex multiplication by \( K \), iff there exists a field \( K \hookrightarrow \text{End}_0^0(A) \) such that \([K : \mathbb{Q}] \geq 2 \text{dim} A\), in which case equality holds, \( K \simeq \text{End}_0^0(A) \) and \( K \) is a CM-field, see [B.115] or
More generally, an abelian variety may be said to be of CM-type if it is isogenous to a product of simple abelian varieties of CM-type, or equivalently if $\text{End}^0(A)$ contains a commutative semisimple $\mathbb{Q}$-algebra $R$ with $[R : \mathbb{Q}] = 2 \dim A$.

To construct a simple abelian variety of CM-type, let $K$ be a CM-field with totally real subfield $K_0$ such that $[K : \mathbb{Q}] = 2g$, and let $S$ be a CM-type for $K$, and let $\mathcal{O}$ be the ring of integers of $K$. Then $K \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}^g$. If we embed $\mathcal{O} \hookrightarrow \mathbb{C}^g$ by $\alpha \mapsto (\sigma \alpha)_{\sigma \in S}$ and let $L$ be the image of this map, then $A = \mathbb{C}^g/L$ is an abelian variety. To construct a Riemann form, choose an element $\beta \in \mathcal{O}$ such that $K = K_0(\beta)$, and $-\beta^2$ is a totally positive element of $K_0$, and $\text{Im}(\sigma \beta) > 0$ for $\sigma \in S$. Then

$$E(z, w) = \sum_{j=1}^g \sigma_j(\beta)(z_j \bar{w}_j - \bar{z}_j w_j)$$

is a Riemann form, where $z, w \in \mathbb{C}^g$. When $z = (\sigma \alpha_1)_{\sigma \in S}$ and $w = (\sigma \alpha_2)_{\sigma \in S}$ with $\alpha_1, \alpha_2 \in K$, then $E(z, w) = \text{Tr}_{K/\mathbb{Q}}(\beta \alpha_1 \bar{\alpha}_2)$. Moreover, $A$ has complex multiplication by $K$, with $\alpha \in \mathcal{O}$ acting by $z_j \mapsto \sigma_j(\alpha)z_j$ for $1 \leq j \leq g$; for more detail see [B.70] §1.4. In addition, it can be shown that any simple abelian variety of CM-type is isogenous to one such as we constructed above [B.27].

1.13.7. Abelian Surfaces with Quaternionic Multiplication. A simple abelian variety of type (II) may be said to have quaternionic multiplication, or sometimes, to be of QM-type. The simplest example is an abelian surface $A$ whose endomorphism algebra is an indefinite division quaternion algebra $D$ over $\mathbb{Q}$. To construct such an abelian surface, let $\mathcal{O}$ be an order in $D$, fix an embedding $j : D \hookrightarrow M_2(\mathbb{R})$, and let $\tau \in \mathbb{C}$ with $\text{Im} \tau > 0$. Then the image of $\mathcal{O}$ under the map $\psi : \alpha \mapsto j(\alpha) \left( \begin{smallmatrix} 1 \\ \tau \end{smallmatrix} \right)$ is a lattice $L \subset \mathbb{C}^2$, and $A = \mathbb{C}^2/L$ is a 2-dimensional abelian variety. In the special case that $D \simeq M_2(\mathbb{Z})$ and $A$ is isogenous to the product of two isogenous elliptic curves. Otherwise $D$ is a division algebra and $A$ is a simple abelian surface. In this latter case there is an element $\beta \in \mathcal{O}$ such that $\beta' = -\beta$ and $\beta^2 < 0$ in $\mathbb{Q}$ (recall that $\beta \mapsto \beta'$ is the canonical involution on $D$). Then a Riemann form on $A$ is given by

$$E((z_1, z_2), (w_1, w_2)) = \frac{1}{\text{Im} \tau} \text{Im} \text{Tr} \left( j(\beta) \cdot \begin{pmatrix} z_1 \bar{w}_2 & \bar{z}_1 w_1 \\ z_2 \bar{w}_2 & \bar{z}_2 w_1 \end{pmatrix} \right) = \text{Tr}(\beta \cdot \alpha_1' \cdot \alpha_2)$$

when $(z_1, z_2) = \psi(\alpha_1)$ and $(w_1, w_2) = \psi(\alpha_2)$ for $\alpha_1, \alpha_2 \in D \otimes_{\mathbb{Q}} \mathbb{R}$. Furthermore, for $\gamma \in \mathcal{O}$, multiplication by $j(\gamma)$ on $(z_1, z_2) \in \mathbb{C}^2$ preserves $L$. Then by tensoring with $\mathbb{Q}$ we get an inclusion $D \hookrightarrow \text{End}^0(A)$, which is an isomorphism for general $\tau$.

We leave it as an exercise for the reader to combine the construction of this example with that of 1.13.5 to obtain an arbitrary simple abelian variety of type (II).

1.13.8. General Abelian Varieties. When $A$ is a $g$-dimensional complex abelian variety then $A(\mathbb{C}) \simeq \mathbb{C}^g/(T\mathbb{Z}^g + \mathbb{Z}^g)$ for some $T$ in the Siegel upper half-space of genus $g$, consisting of symmetric complex $g \times g$ matrices with positive-definite imaginary part. Then generalizing the 1-dimensional case, a Riemann form is given by

$$E(z, w) = \text{Im}(z(\text{Im} T)^{-1} \bar{w}) = a \cdot d - c \cdot b$$
when $z = Ta + b$ and $w = Tc + d$, with $a, b, c, d \in \mathbb{R}^g$. Then for general $T$, that is, when all the real and imaginary parts of the distinct entries of $T$ are algebraically independent real transcendental numbers, $\text{End}^0(A) \simeq \mathbb{Q}$.

2. The Hodge, Mumford-Tate and Lefschetz groups of an abelian variety

2.1. Rational Hodge structures. A real Hodge structure is a natural generalization of a complex structure, and a rational Hodge structure is a real Hodge structure with an underlying $\mathbb{Q}$-structure. For a rational vector space $V = V_{\mathbb{Q}}$ we write $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$ and $V_C = V \otimes_{\mathbb{Q}} \mathbb{C}$.

2.1.1. Definition. A rational Hodge structure of weight $n$ consists of a finite-dimensional $\mathbb{Q}$-vector space $V$ together with any of the following equivalent data:

(i) A decomposition $V_C = \bigoplus_{p+q=n} V^{p,q}$, called the Hodge decomposition, such that $V_{p,q} = V^{q,p}$.

(ii) A decreasing filtration $F^p_H V_C$ of $V_C$, called the Hodge filtration, such that $F^p_H V_C \oplus F^{n-p+1}_H V_C = V_C$.

(iii) A homomorphism $h_1 : U(1) \to \text{GL}(V_{\mathbb{R}})$ of real algebraic groups, and also specifying that the weight of the Hodge structure is $n$.

(iv) A homomorphism $h : \mathbb{S} \to \text{GL}(V_{\mathbb{R}})$ of real algebraic groups such that via the composition $\mathbb{G}_{m/\mathbb{R}} \to \mathbb{S} \to \text{GL}(V_{\mathbb{R}})$ an element $t \in \mathbb{G}_{m/\mathbb{R}}$ acts as $t^{-n} \cdot \text{Id}$.

To see that the data (i)–(iv) are equivalent, the Hodge decomposition and the Hodge filtration are related by $F^p_H V_C = \bigoplus_{p \geq r} V^{p,n-p}$ and $V^{p,n-p} = F^p_H V_C \cap \overline{F^{n-p}_H V_C}$. The homomorphism $h$ and the Hodge decomposition are related by $h(z) \cdot v = z^{-p} z^{-q} \cdot v$ for $v \in V^{p,q}$. The homomorphism $h_1$ can be obtained as the restriction of $h$; conversely, $V^{p,q}$ can be recovered as the subspace of $V_C$ on which $h_1(u)$ acts as $u^{q-p}$, provided $n = p + q$ is specified.

2.1.2. Examples. 1. Let $V$ be a $\mathbb{Q}$-vector space of even dimension, and suppose $h : \mathbb{S} \to \text{GL}(V_{\mathbb{R}})$ defines a complex structure on $V_{\mathbb{R}}$. Then $V$ is a rational Hodge structure of weight $-1$: criterion 1.10(iv) in the definition of complex structure immediately implies criterion 2.1.1(iv). Alternatively, the $z^1 w^0$- and $z^0 w^1$-eigenspaces of $h(z,w)$ acting on $V_C$ are $V^{-1,0}$ and $V^{0,-1}$ respectively, and these are complex conjugate as required by 2.1.1(i) because $h$ is defined over $\mathbb{R}$.

2. When $X$ is (the analytic space underlying) a complex projective variety, or more generally, a compact Kähler manifold, then $H^n(X, \mathbb{Q})$ is a rational Hodge structure of weight $n$.

3. In general, when $V$ is a rational Hodge structure of weight $n$, its dual $V^\vee$ is a rational Hodge structure of weight $-n$, and $V^{s \otimes r} \otimes_{\mathbb{Q}} (V^\vee)^{\otimes s}$ is a rational Hodge structure of weight $(r - s)n$.

Remark. The convention that $(z, w) \in \mathbb{S}(\mathbb{C})$ acts as $z^{-p} w^{-q}$ on $V^{p,q}$ follows [B.25] §1 and [B.27]. But as we will see below it really is the more natural choice in the context of Hodge structures associated to complex abelian varieties.
2.1.3. Definition. The type of a rational Hodge structure $V$ is the set of pairs $(p, q)$ such that $V^{p,q} \neq 0$.

2.1.4. Definition. When $V$ is a rational Hodge structure of even weight $2p$, the subspace of Hodge vectors in $V$ is the subspace of 1-dimensional rational sub-Hodge structures of $V$,

$H(V) := V_Q \cap V^{p,p}$.

2.1.5. Definition. A morphism of rational Hodge structures $\varphi : V_1 \to V_2$ is a $\mathbb{Q}$-vector space map on the underlying vector spaces such that over $\mathbb{C}$

(i) $\varphi(V_1^{p,q}) \subseteq V_2^{p,q}$, for all $p, q$; or

(ii) $\varphi(F^p_HV_1) \subseteq F^p_HV_2$; for all $r$; or

(iii) $\varphi$ commutes with the action of $U(1)$, and preserves the common weight of $V_1$ and $V_2$; or

(iv) $\varphi$ commutes with the action of $S$, preserving the action of $\mathbb{G}_m/\mathbb{Q} \to S$.

2.1.6. Definition. A polarization of a rational Hodge structure $V$ is a morphism of rational Hodge structures $\psi : V \otimes V \to \mathbb{Q}(-n)$ such that the real-valued form $(u, v) \mapsto \psi(u, h(i)v)$ on $V_{\mathbb{R}}$ is symmetric and positive-definite. Here $\mathbb{Q}(m)$ is the is the vector space $\mathbb{Q}$ as a one-dimensional rational Hodge structure of type $(-m, -m)$, for $m \in \mathbb{Z}$.

2.1.7. The rational Hodge structure associated to an abelian variety. When we speak of the rational Hodge structure associated to an abelian variety $A$ we always mean the rational Hodge structure $H^1(A, \mathbb{Q})$. Moreover, any morphism $\varphi : A \to A'$ of abelian varieties induces a morphism of rational Hodge structures $\varphi^* : H^1(A', \mathbb{Q}) \to H^1(A, \mathbb{Q})$. In particular, it is easy to check that when $\varphi$ is an isogeny it induces an isomorphism on the associated rational Hodge structures. Thus, up to isomorphism, the rational Hodge structure associated to an abelian variety depends only on its isogeny class. Similarly, an element of $\text{End}^0(A)$ induces an endomorphism on the rational Hodge structure $H^1(A, \mathbb{Q})$.

2.1.8. Notation. For a complex abelian variety $A$ we will regularly let $W = H_1(A, \mathbb{Q})$ and $V = H^1(A, \mathbb{Q}) = W^\vee$. Further, we denote the Hodge classes of $A$ by

$H^p(A) := H^{2p}(A, \mathbb{Q}) \cap H^{p,p}(A)$ \quad $H(A) := \bigoplus_p H^p(A)$,

and let $\mathcal{D}(A) \subset H^{2p}(A, \mathbb{Q})$ be the $\mathbb{Q}$-linear span of $p$-fold intersections of divisors on $A$, and $\mathcal{D}(A) := \bigoplus_p \mathcal{D}^p(A)$.

2.2. Definition. Let $(V, h : S \to \text{GL}(V_{\mathbb{R}}))$ be a rational Hodge structure. The Hodge group $Hg(V)$ of $V$, also called the special Mumford-Tate group of $V$, is the smallest algebraic subgroup of $\text{GL}(V)$ defined over $\mathbb{Q}$ such that $h(U(1)) \subset Hg(V_{\mathbb{R}})$. The Mumford-Tate group $MT(V)$ of $V$ is the smallest algebraic subgroup of $\text{GL}(V)$ defined over $\mathbb{Q}$ such that $h(\mathbb{S}) \subset MT(V_{\mathbb{R}})$. As a matter of notation we let $hg(V) := \text{Lie}(Hg(V))$ and $mt(V) := \text{Lie}(MT(V))$ denote their respective Lie algebras, as subalgebras of $\text{End}_{\mathbb{Q}}(V)$.

When $A$ is a complex abelian variety then by the Hodge or the Mumford-Tate group of $A$ we mean $Hg(A) := Hg(H^1(A, \mathbb{Q}))$ and $MT(A) := MT(H^1(A, \mathbb{Q}))$ respectively.
Remark. The Hodge group of an abelian variety was introduced in [B.77], see also [B.78]. The general notion of the Mumford-Tate group of a rational Hodge structure seems to appear first in [B.24] §7, and from a rather abstract point of view, in [B.97]. A thorough analysis of the Mumford-Tate groups of Hodge structures that can be generated by the Hodge structures of abelian varieties can be found in [B.25] §1, while the best place to find proofs of the basic properties of Mumford-Tate groups in general is [B.27] §3.

The first properties to follow directly from the definition are the following.

2.3. Lemma. Let $V$ be a rational Hodge structure. Then
(i) $\mathrm{Hg}(V)$ and $\mathrm{MT}(V)$ are connected linear algebraic groups;
(ii) $\mathrm{Hg}(V) \subseteq \mathrm{SL}(V)$;
(iii) $\mathrm{MT}(V) = \mathbb{G}_m \cdot \mathrm{Hg}(V)$.

The vital role of Mumford-Tate groups in analyzing Hodge structures comes from the following fact.

2.4. Proposition. Let $V$ be a rational Hodge structure, and $r, s \in \mathbb{N}$. Then $\mathrm{MT}(V)$ acts on the rational Hodge structure $V^\otimes r \otimes (V^\vee)^\otimes s$, and the rational $\mathrm{MT}(V)$-subrepresentations in $V^\otimes r \otimes (V^\vee)^\otimes s$ are precisely the rational sub-Hodge structures of $V^\otimes r \otimes (V^\vee)^\otimes s$.

The action of $\mathrm{MT}(V)$ on $V$ extends “diagonally” to an action on $V^\otimes r$, and the action of $\mathrm{MT}(V)$ on $V^\vee$ is the contragredient of its action on $V$, as in 1.4.

Proof. To simplify the notation, let $T := V^\otimes r \otimes (V^\vee)^\otimes s$. Then action of $\mathrm{GL}(V)$ on $V$ induces an action of $\mathrm{GL}(V)$, and thus of $\mathrm{MT}(V)$ on $T$. Now suppose first that $W \subset T$ is a $\mathbb{Q}$-rational subspace preserved by the $\mathrm{MT}(V)$-action. Then over $\mathbb{R}$ the composition $h : \mathbb{S} \hookrightarrow \mathrm{MT}(V_{\mathbb{R}}) \to \mathrm{GL}(W_{\mathbb{R}})$ describes the sub-Hodge structure on $W_{\mathbb{R}} \subset T_{\mathbb{R}}$. Conversely, if $W \subset T$ is a rational sub-Hodge structure, then $W$ is a rational subspace of $T$ such that $W_{\mathbb{R}} \subset T_{\mathbb{R}}$ is preserved by the action of $h(\mathbb{S})$. Therefore $W$ is preserved by the action of $\mathrm{MT}(V)$ on $T$. □

Corollary. Let $V$ be a rational Hodge structure of weight $n$. Then for any $r, s \in \mathbb{N}$ such that $(r - s)n = 2p$,
$$\mathcal{H}^p(V^\otimes r \otimes (V^\vee)^\otimes s) = (V^\otimes r \otimes (V^\vee)^\otimes s)^{\mathrm{Hg}(V)}$$

Remark. Hodge and Mumford-Tate groups have proved to be a powerful for studying the Hodge conjecture for abelian varieties. Typically the starting point is this corollary to Proposition 2.4, which implies that the Hodge cycles in $H^*(A^n, \mathbb{Q})$, say, for a complex abelian variety $A$, are precisely the invariants under the action of $\mathrm{Hg}(A)$. Then, the problem is to determine enough about $\mathrm{Hg}(A)$ to be able to describe its invariants, or at least determine their dimension. In many cases it is possible to show this way that the space of Hodge cycles is generated by those of degree 2, in other words, by divisors on $A$, and then in these cases the
Hodge conjecture is verified. In other cases it is possible to show that the space of Hodge cycles is not generated by divisors, but still something can be said about the dimension of the space of Hodge cycles. However, the actual computations are sometimes quite technical. There are many variously narrow results, and a few general results, as we will try to show in the following sections.

Since a polarization of a complex abelian variety induces a polarization on its associated Hodge structure, the following proposition insures that all the Hodge and Mumford-Tate groups with which we will work in this appendix are reductive (definition 1.4); compare [B.25] Principe 1.1.9.

**2.5. Proposition.** Let $V$ be a polarizable rational Hodge structure. Then $\text{MT}(V)$ and $\text{Hg}(V)$ are reductive.

**Proof (after [B.27] Prop.3.6).** Suppose $h : \text{U}(1) \to \text{GL}(V_{\mathbb{R}})$ defines the Hodge structure on $V$, and let $\psi : V \otimes V \to \mathbb{Q}(-n)$ be a polarization. Then for $u, v \in V_{\mathbb{C}}$ and $g \in \text{Hg}(V, \mathbb{C})$ (the complexification, or complex points of $\text{Hg}(V)$)

$$\psi(u, h(i)\bar{v}) = \psi(gu, gh(i)\bar{v}) = \psi(gu, h(i)(h(i)^{-1}gh(i))\bar{v}) = \psi(gu, h(i)g^*\bar{v}),$$

where the first equality holds because $\psi$ is a Hodge structure morphism, and $g^* := h(i)^{-1}gh(i)$. Therefore the positive-definite form on $V_{\mathbb{R}}$ given by $(u, v) \mapsto \psi(u, h(i)v)$ is invariant under the real form $\text{Hg}^*(V_{\mathbb{R}})$ of $\text{Hg}(V, \mathbb{C})$ fixed by the involution $g \mapsto g^*$. It follows that $\text{Hg}^*(V_{\mathbb{R}})$ is a compact real form of $\text{Hg}(V)$, and thus all its finite-dimensional representations are semisimple. This is equivalent to $\text{Hg}^*(V_{\mathbb{R}})$ being reductive, see for example [B.103] I.3. Then since the linear algebraic $\text{Hg}(V)$ possesses a reductive real form, it is reductive (as an algebraic group), and then $\text{MT}(V)$ is reductive as well. \hfill \Box

**2.5.1. Corollary.** When $V$ is a polarizable rational Hodge structure, $\text{Hg}(V)$ is the largest subgroup of $\text{GL}(V)$ fixing all Hodge vectors in all $V^\otimes r \otimes (V^\vee)^\otimes s$, for $r, s \in \mathbb{N}$.

**Proof.** A reductive subgroup of $\text{GL}(V)$ is characterized by its invariants in the extended tensor algebra of $V$ (compare [B.27] Prop.3.1). \hfill \Box

Once we know that $\text{Hg}(V)$ and $\text{MT}(V)$ are reductive, the next corollary follows from Lemma 2.3(iii).

**2.5.2. Corollary.** Let $V$ be polarizable rational Hodge structure. Then $\text{Hg}(V)$ is semisimple if and only if the center of $\text{MT}(V)$ is $\mathbb{G}_m$, i.e., consists only of scalars.

Now we turn specifically to the Hodge structures of complex abelian varieties. When combined with Proposition 2.4, the following lemma says that the endomorphism algebra of a complex abelian variety may be identified with the rational Hodge structure endomorphisms of its associated rational Hodge structure.
2.6. Lemma. Let $A$ be a complex abelian variety. Then
\[ \text{End}^0(A) \cong \text{End}_{\text{MT}(A)}(H_1(A, \mathbb{Q})) = \text{End}_{\text{Hg}(A)}(H_1(A, \mathbb{Q})). \]

Proof. Let $\dim A = g$. Then $A(\mathbb{C}) = \mathbb{C}^g/L$ for some lattice $L$, and let $W = W_\mathbb{Q} := L \otimes \mathbb{Q}$. Then we can identify $W \cong H_1(A, \mathbb{Q})$ and identify the universal covering space $\mathbb{C}^g$ of $A(\mathbb{C})$ as the real $2g$-dimensional space $W_\mathbb{R} = W \otimes \mathbb{R}$ together with the induced complex structure represented as a homomorphism $h : \mathbb{S} \to \text{GL}(W_\mathbb{R})$. Now an element of $\text{End}^0(A)$ is characterized by firstly being a $\mathbb{Q}$-linear endomorphism of $W$ and secondly being a complex-linear endomorphism of $W_\mathbb{R}$, which precisely means that it commutes with $h(\mathbb{S}(\mathbb{R}))$. But a $\mathbb{Q}$-linear endomorphism of $W$ that commutes with $h(\mathbb{S}(\mathbb{R}))$ must commute with all of $\text{MT}(A)$ acting on $W$. \hfill \Box

2.7. Proposition ([B.17]). Let $A$ be a complex abelian variety. If $\text{End}^0(A)$ is a simple $\mathbb{Q}$-algebra with center $\mathbb{Q}$, then $\text{Hg}(A)$ is simple.

Recall that $\text{End}^0(A)$ is a simple $\mathbb{Q}$-algebra precisely when $A$ is simple. Then following [B.142] p.66, the idea of the proof is that the center, say $c$, of $\text{mt}(A)$ contains $\mathbb{Q} \cdot \text{Id}$ and is contained in the center of $\text{End}^0(A)$. So if the center of $\text{End}^0(A)$ is $\mathbb{Q} \cdot \text{Id}$ then $c = \mathbb{Q} \cdot \text{Id}$, and $\text{Hg}(A)$ is semisimple, and for simple $A$, it is simple.

As we have already begun to see, most of what can be said about the structure and classification of the Hodge and Mumford-Tate groups of abelian varieties is a consequence of the presence and properties of a polarization. The most fundamental fact is the following.

2.8. Lemma. Let $A$ be a complex abelian variety, and let $[E]$ be a polarization of $A$ represented by the Riemann form $E$. Further, let $W = W_\mathbb{Q} = H_1(A, \mathbb{Q})$. Then $E$ is a skew-symmetric bilinear form on $W$, and there are natural representations
\[ \text{Hg}(A) \hookrightarrow \text{Sp}(W, E) \quad \text{and} \quad \text{MT}(A) \hookrightarrow \text{GSp}(W, E). \]

Proof. This follows from the observation that the Riemann form $E$ is a polarization on the rational Hodge structure $W$. First, $E(h(i)u, h(i)v) = E(u, v)$, where $u, v \in W_\mathbb{R}$ and $h : \mathbb{S} \to \text{GL}(W_\mathbb{R})$ represents the complex structure. Thus if we write $h(s) = a\text{Id} + bh(i)$, then $E(h(s)u, h(s)v) = [a + bi]^2 E(u, v)$. Therefore $h(\mathbb{S}) \hookrightarrow \text{GSp}(W_\mathbb{R}, E)$, and then by taking the Zariski-closure over $\mathbb{Q}$ we find $\text{MT}(A) \hookrightarrow \text{GSp}(W, E)$. \hfill \Box

The following criterion for the semisimplicity of the Hodge group is linked to whether there are any simple components of Weil type, see 1.13.2; cf. also the discussions in sections four and five, below.
2.9. Proposition ([B.118]). Suppose A is an abelian variety defined over \( \mathbb{C} \). Then the Hodge group of A is not semisimple if and only if for some simple component B of A the center of \( \text{End}^0(B) \) is a CM-field \( K \) such that \((B,K)\), with \( K \) embedded in \( \text{End}^0(B) \) by the identity map, is not of Weil type.

The next proposition is that the real Lie groups \( \text{Hg}(A,\mathbb{R}) \) and \( \text{MT}(A,\mathbb{R}) \) are of Hermitian type (see 1.6.5).

2.10. Proposition ([B.77]). Let A be a complex abelian variety. Then \( \text{Hg}(A,\mathbb{R}) \) is of Hermitian type. Further, letting \( K = K_\mathbb{R} \) denote the centralizer of \( h(i) \), or equivalently of \( h(U(1)) \), then the topologically connected component \( K^+ \) of \( K \) is a maximal compact subgroup of the topologically connected component \( \text{Hg}(A,\mathbb{R})^+ \) of \( \text{Hg}(A,\mathbb{R}) \), and the quotient \( \text{Hg}(A,\mathbb{R})^+/K^+ \) is a Hermitian symmetric space of noncompact type, i.e., a bounded symmetric domain.

Proof (after [B.25], but see also [B.103] and [B.51] Ch.VIII). First, the connected center of \( \text{Hg}(A,\mathbb{R}) \) is compact, since the group itself is generated by compact subgroups, namely the \( \text{Aut}_Q(\mathbb{C}) \)-conjugates of \( h(U(1)) \). From the proof that \( \text{Hg}(A) \) is reductive it follows that \( \text{Ad}(h(i)) \) defines a Cartan involution on \( \text{Hg}(A,\mathbb{R}) \), and thus \( \text{ad}(h(i)) \) is a Cartan involution on \( g = \text{h}(g,\mathbb{R})_{ss} \), the semisimple part of the reductive Lie algebra \( \text{h}(g,\mathbb{R}) \). Let \( \mathfrak{f} + \mathfrak{p} \) be the corresponding Cartan decomposition. Then the restriction \( \tilde{J} \) of \( h(i) \) to \( \mathfrak{p} \) is a derivation of \( g \). Since \( g \) is semisimple, there therefore exists \( H_0 \in \mathfrak{g} \) such that \( \tilde{J} = \text{ad}(H_0) \). And since \( \tilde{J} \) commutes with the Cartan decomposition \( \text{ad}(h(i)) \), we have that \( H_0 \) is in \( \mathfrak{f} \), and in the center of \( \mathfrak{f} \), as required.

The following result points to how the complexified Lie algebras of the Hodge and Mumford-Tate groups of a complex abelian variety fit into the general classification of complex semisimple Lie algebras. This formulation of the result follows [B.142].

2.11. Theorem ([B.25] §1, but see also [B.108] §3 and Appendix and [B.141]). Let A be a complex abelian variety, let \( \mathfrak{g} \) be a simple factor of the complex semisimple Lie algebra \( \mathfrak{m}l(A,\mathbb{C})_{ss} \), and let \( r \) denote the rank of \( \mathfrak{g} \). Further, let \( \mathfrak{w} = \text{H}_1(A,\mathbb{Q}) \), and let \( V \subset \mathfrak{w}_\mathbb{C} \) be an irreducible subrepresentation for the action of \( \mathfrak{g} \) on \( \mathfrak{w}_\mathbb{C} \). Then \( \mathfrak{g} \) and \( V \) must be one of the following:

(A) \( \mathfrak{g} \simeq \mathfrak{s}(r+1) \) and \( V \) is equivalent to the \( s \)-th exterior power of the standard representation of dimension \( r+1 \), for some \( 1 \leq s \leq r \).

(B) \( \mathfrak{g} \simeq \mathfrak{s}0(2r+1) \), and \( V \) is equivalent to the spin representation of dimension \( 2r \).

(C) \( \mathfrak{g} \simeq \mathfrak{sp}(2r) \), and \( V \) is equivalent to the standard representation of dimension \( 2r \).

(D) \( \mathfrak{g} \simeq \mathfrak{s}0(2r) \), and \( V \) is equivalent to the standard representation of dimension \( 2r \), or to one of the two half-spin representations of dimension \( 2^{r-1} \).

We do not give the proof here, but we just mention that the proof depends essentially on the symplectic representation in 2.8.
2.12. **Proposition ([B.78])**. A complex abelian variety is of CM-type if and only if $\operatorname{Hg}(A)$ is an algebraic torus.

**Proof.** Suppose first that $A$ is of CM-type. Then $\operatorname{End}^0(A)$ contains a commutative semisimple $\mathbb{Q}$-algebra of dimension $2\dim A$ over $\mathbb{Q}$. From Lemma 2.6 it follows that $\operatorname{Hg}(A)$ commutes with a maximal commutative semisimple subalgebra $R' \subset \operatorname{End}(W)$, where $W = H_1(A, \mathbb{Q})$. Therefore $\operatorname{Hg}(A)$ is contained in the units of $R'$ and thus must be an algebraic torus. Conversely, if $\operatorname{Hg}(A)$ is an algebraic torus, then it is diagonalizable over $\mathbb{C}$. Therefore its centralizer in $\operatorname{End}(W) \otimes \mathbb{C}$, and thus its centralizer in $\operatorname{End}(W)$, contains a maximal commutative semisimple subalgebra $R' \subset \operatorname{End}(W)$. But then $[R' : \mathbb{Q}] = \dim W = 2\dim A$ and $R' \subset \operatorname{End}^0(A)$, so $A$ is of CM-type. □

**Remark.** If $A$ is an abelian variety with complex multiplication by $K$, and $K_0$ is the maximal totally real subfield of $K$, then a more precise statement is that

$$\operatorname{Hg}(A) \subseteq \ker\{\operatorname{Res}_{K/\mathbb{Q}} \mathbb{G}_m/K \rightarrow \operatorname{Res}_{K_0/\mathbb{Q}} \mathbb{G}_m/K_0\},$$

where the arrow is induced by the norm map from $K$ to $K_0$. The argument above shows that $\operatorname{Hg}(A) \subseteq \operatorname{Res}_{K/\mathbb{Q}} \mathbb{G}_m/K$. Then observe that $h_1(U(1))$ is contained in the real points of the indicated kernel, and recall that $\operatorname{Hg}(A)$ is the smallest algebraic subgroup defined over $\mathbb{Q}$ which over $\mathbb{R}$ contains $h_1(U(1))$.

2.13. **Definition.** Let $K$ be a CM field, $S$ a CM-type for $K$, and let $A$ be the corresponding abelian variety (up to isogeny), as described in 1.5.4. Then the CM-type $(K, S)$ or the abelian variety $A$ with that CM-type is said to be nondegenerate if $\dim \operatorname{Hg}(A) = \dim A = \frac{1}{2}[K : \mathbb{Q}]$.

The Lefschetz group of an abelian variety. The Lefschetz group of an abelian variety was first studied by Ribet [B.94] and further investigated by Murty [B.82] §2.

2.14. **Definition.** Let $A$ be a complex abelian variety, let $W = H_1(A, \mathbb{Q})$, and let $[E]$ be a polarization of $A$ represented by the Riemann form $E$. The **Lefschetz group** of $A$ is the connected component of the identity in the centralizer of $\operatorname{End}^0(A)$ in $\operatorname{Sp}(W, E)$,

$$\operatorname{Lf}(A) := \{g \in \operatorname{Sp}(W, E) : g \circ \varphi = \varphi \circ g \text{ for all } \varphi \in \operatorname{End}^0(A)\}^\circ.$$

In this definition $\operatorname{Lf}(A)$ appears to depend on the choice of polarization, but if $[E']$ is another polarization, then there is an element $\psi \in \operatorname{End}^0(A)$ and positive $m \in \mathbb{Z}$ such that $mE' = E\psi$. To see this, we take the point of view that $E$ and $E'$ define isogenies, say $\varphi$ and $\varphi'$ respectively, from $A$ to its dual $A^\vee$. Then we can take $\psi = \varphi'^\vee \circ \varphi$ and $m = \deg \varphi$. Thus $\operatorname{Lf}(A)$ does not in fact depend on the choice of polarization. Furthermore, it is clear that $\operatorname{Lf}(A)$ is an algebraic group defined over $\mathbb{Q}$, and

$$\operatorname{Hg}(A) \subseteq \operatorname{Lf}(A).$$

The Lefschetz groups also has the following nice multiplicative property, that the Hodge and Mumford-Tate groups in general do not.
2.15. Lemma ([B.82] Lem.2.1). If $A$ is isogenous to a product $B_1^{\nu_1} \times \cdots \times B_r^{\nu_r}$, with the $B_i$ simple and non-isogenous, then

$$\text{Lf}(A) \simeq \text{Lf}(B_1) \times \cdots \times \text{Lf}(B_r).$$

Proof. First let $A_i = B_i^{\nu_i}$, for $1 \leq i \leq r$, and choose polarizations $[E_i]$ of $A_i$. Then $[E_1 \oplus \cdots \oplus E_r]$ is a polarization of $A$, since $W = H_1(A,\mathbb{Q}) \simeq \bigoplus_{i=1}^{r} H_1(A_i,\mathbb{Q})$. Further, since the $B_i$ are non-isogenous, $\text{Hom}(A_i,A_j) = 0$ for $i \neq j$, whence $\text{End}^0(A) = \prod_{i=1}^{r} \text{End}^0(A_i)$. Therefore any automorphism of $W$ that commutes with the action of $\text{End}^0(A)$ must preserve each $H_1(A_i,\mathbb{Q})$, and thus $\text{Lf}(A) \simeq \text{Lf}(A_1) \times \cdots \times \text{Lf}(A_r)$.

Now fix $i$, and let $B = B_i$ and $A = B^n$. Then if $[E]$ is a polarization of $B$, then $[E \oplus \cdots \oplus E]$ is a polarization of $A$. Further, $\text{End}^0(B) \simeq M_n(\text{End}^0(B))$, so the centralizer of $\text{End}^0(A)$ in $\text{Sp}(H_1(A,\mathbb{Q}), E \oplus \cdots \oplus E)$ can be identified with the centralizer of $\text{End}^0(B)$ in $\text{Sp}(H_1(B,\mathbb{Q}), E)$. Therefore $\text{Lf}(A) \simeq \text{Lf}(B)$. \qed

For later reference we state some variants of “Goursat’s Lemma” that turn out to be useful, especially when extending results from simple abelian varieties to products of abelian varieties. The formulations given below come from [B.90], [B.83] and [B.75].

2.16. Proposition (Goursat’s Lemma).

1. Let $G$ and $G'$ be groups and suppose $H$ is a subgroup of $G \times G'$ for which the projections $p : H \to G$ and $p' : H \to G'$ are surjective. Let $N$ be the kernel of $p'$ and let $N'$ be the kernel of $p$. Then $N$ is a normal subgroup of $G$ and $N'$ is a normal subgroup of $G'$, and the image of $H$ in $G/N \times G'/N'$ is the graph of an isomorphism $G/N \simeq G'/N'$.

2. Let $V_1$ and $V_2$ be two finite-dimensional complex vector spaces. Let $s_1, s_2$ be simple complex Lie subalgebras of $\mathfrak{gl}(V_1), \mathfrak{gl}(V_2)$ respectively, of type $A$, $B$ or $C$. Let $s$ be a Lie subalgebra of $s_1 \times s_2$ whose projection to each factor is surjective. Then either $s = s_1 \times s_2$ or $s$ is the graph of an isomorphism $s_1 \simeq s_2$ induced by an $s$-module isomorphism $V_2 \simeq V_1$ or $V_2 \simeq V_1'$.

3. Let $s_1, \ldots, s_d$ be simple finite-dimensional Lie algebras and let $\mathfrak{g}$ be a subalgebra of the product $s_1 \times \cdots \times s_d$. Assume that for $1 \leq i \leq d$ the projection $\mathfrak{g} \to s_i$ is surjective, and that whenever $1 \leq i < j \leq d$ the projection of $\mathfrak{g}$ onto $s_i \times s_j$ is surjective. Then $\mathfrak{g} = s_1 \times \cdots \times s_d$.

4. Let $I$ be a finite set and for each $\sigma \in I$, let $s_\sigma$ be a finite-dimensional complex simple Lie algebra. Let $\mathfrak{g}, \mathfrak{h}$ be two algebras such that

(a) $\mathfrak{g} \subseteq \mathfrak{h}$.

(b) $\mathfrak{h}$ is a subalgebra of $\prod_{\sigma \in I} s_\sigma$ such that the projection to each $s_\sigma$ is surjective.

(c) $\mathfrak{g}, \mathfrak{h}$ have equal images on $s_\sigma \times s_\tau$ for all pairs $(\sigma, \tau) \in I \times I$, $\sigma \neq \tau$.

Then $\mathfrak{g} = \mathfrak{h} = \prod_{\sigma \in I, \tau \in J} s_\sigma$ for some subset $J \subseteq I$.

5. Let $V_1, \ldots, V_n$ be finite-dimensional vector spaces over an algebraically closed field of characteristic zero, and let $\mathfrak{g}$ be a semisimple Lie subalgebra of $\text{End}(V_1) \times \cdots \times \text{End}(V_n)$. For $1 \leq i \leq n$ let $\mathfrak{g}_i \subseteq \text{End}(V_i)$ be the projection of $\mathfrak{g}$ onto the $i$-th factor. Assume that $\mathfrak{g}_i$ is nonzero and simple for
all $i$. Then for any simple Lie algebra $\mathfrak{h}$ let $I(\mathfrak{h}) \subset \{1,\ldots,n\}$ be the set of indices for which $\mathfrak{g}_i \simeq \mathfrak{h}$. Assume that for any $\mathfrak{h}$ with $\#I(\mathfrak{h}) > 1$ the following conditions are satisfied:

(a) All automorphisms of $\mathfrak{h}$ are inner.
(b) For $i \in I(\mathfrak{h})$ the representations $V_i$ are all isomorphic.
(c) $\text{End}_{\mathfrak{g}}(\bigoplus_{i \in I(\mathfrak{h})} V_i) = \prod_{i \in I(\mathfrak{h})} \text{End}_{\mathfrak{g}_i}(V_i)$.

Then $\mathfrak{g} \cong \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$.

### 3. Products of Elliptic curves

Tate seems to be the first to have checked the (usual) Hodge conjecture for powers $E^n$ of an elliptic curve, see [B.130], [B.43] §3, but he never published his proof. In [B.80] Murasaki showed the $\mathcal{H}^p(E^n) = \mathcal{D}^p(E^n)$ for all $p$ by exhibiting explicit differential forms that give a basis for $\mathcal{H}^1(E^n)$ and then carrying out explicit computations with them. In a different direction, Imai [B.58] showed that when $E_1,\ldots,E_n$ are pairwise non-isogenous elliptic curves, then $Hg(E_1 \times \cdots \times E_n) \simeq Hg(E_1) \times \cdots \times Hg(E_n)$. A unified approach to computing the Hodge and Mumford-Tate groups, and verifying the Hodge conjecture, for arbitrary products of elliptic curves can be found in [B.84]. Since Murty’s approach provides a nice example of how the Hodge and Mumford-Tate groups can be used to verify the usual Hodge conjecture, we summarize his exposition here.

**Theorem.** Let $A = E_{1}^{m_1} \times \cdots \times E_{r}^{m_r}$, where the $E_i$ are pairwise non-isogenous elliptic curves. Then

1. $Hg(A) = Hg(E_1) \times \cdots \times Hg(E_r)$.
2. $\mathcal{H}(A) = \mathcal{H}(E_1^{m_1}) \otimes \cdots \otimes \mathcal{H}(E_r^{m_r}) = \mathcal{D}(A)$.

**Proof** (after [B.84]). Let $E$ be an elliptic curve. The cases where $E$ has or does not have complex multiplication have to be handled separately. If $E$ has complex multiplication, then $\text{End}^0(E) =: K$ is an imaginary quadratic field, and (the proof of) Proposition 2.12 shows that $\text{MT}(E,\mathbb{Q}) \subseteq K^\times$, as algebraic groups over $\mathbb{Q}$. Since the two-dimensional $\mathbb{S}^1 \subset \text{MT}(E,\mathbb{R})$, we see that $\text{MT}(E) = \text{Res}_{K/\mathbb{Q}}(G_{m,K})$. When $E$ does not have complex multiplication, then $\mathfrak{h}(E)$ is a simple subalgebra of $\mathfrak{sl}(V)$ which is already simple, so $\mathfrak{h}(E) = \mathfrak{sl}(V)$ and thus $Hg(E) = \text{SL}(V)$ and $\text{MT}(E) = \text{GL}(V)$.

Next consider $A = E^n$. Then

$$\mathcal{H}(A) = H^\times(E^n,\mathbb{Q})^{Hg(A)} = \mathfrak{h}(H^1(E,\mathbb{Q}) \oplus \cdots \oplus H^1(E,\mathbb{Q}))^{Hg(A)} = \bigoplus (H^1(E,\mathbb{Q}) \otimes \cdots \otimes H^1(E,\mathbb{Q}))^{Hg(E)}.$$  

Now if $E$ has complex multiplication then $\alpha \in K^\times = \text{MT}(E)$ acts on $H^1(E,\mathbb{Q}) \otimes \mathbb{C}$ by $C \simeq \mathbb{C} \oplus \mathbb{C}$ by $\alpha(z, w) = (\alpha z, \bar{\alpha} w)$. Let $K_1^\times$ denote the elements of $K^\times$ of norm 1. Then

$$(H^1(E,\mathbb{Q}) \otimes \cdots \otimes H^1(E,\mathbb{Q}))^{Hg(E)} \otimes \mathbb{C} = (H^1(E,\mathbb{Q}) \otimes \mathbb{C}) \otimes \cdots (H^1(E,\mathbb{Q}) \otimes \mathbb{C})^{K_1^\times},$$
in which any invariant class arises as a combination of products of elements of
\[ \left( (H^1(E, \mathbb{Q}) \otimes \mathbb{C}) \otimes \left( (H^1(E, \mathbb{Q}) \otimes \mathbb{C}) \right) \right)^{K_1^X} \subseteq (H^2(E \times E, \mathbb{Q}) \otimes \mathbb{C})^{K_1^X}. \]

Therefore the invariants are generated by those in \( H^2(A, \mathbb{Q}) \), which means that \( \mathcal{H}(A) = \mathcal{D}(A) \) in this case.

Next suppose that \( E \) does not have complex multiplication. Then \( \text{Hg}(E) = \text{SL}(2) \) and acts on \( H^1(E, \mathbb{Q}) \) by the standard representation. Now we invoke the well-known fact that the tensor invariants of \( \text{SL}(2) \) are generated by the determinant; see [B.137] or [B.10] App.1 for this. Since the determinant is a representation of degree 2 lying in \( H^1(E, \mathbb{Q}) \otimes H^1(E, \mathbb{Q}) \subset H^2(A, \mathbb{Q}) \), again we find that all Hodge cycles of \( A = E^n \) are generated by divisors.

Finally let \( A = E_1^{m_1} \times \cdots \times E_r^{m_r} \), where the \( E_i \) are pairwise non-isogenous elliptic curves. First suppose all the \( E_i \) have complex multiplication by an imaginary quadratic field \( K_i = \text{End}^0(E_i) \). Since the field \( \text{End}^0(E_i) \) determines the isogeny class of \( E_i \), all the \( K_i \) are distinct. Then
\[ \text{Hg}(A) \subseteq K_1^{X_1} \times \cdots \times K_r^{X_r}, \]
and moreover from the definition, \( \text{Hg}(A) \) surjects onto each factor. Therefore there is a surjection of character groups
\[ \lambda : M \to X(\text{Hg}(A)), \]
where
\[ M := X(K_1^{X_1}) \oplus \cdots \oplus X(K_r^{X_r}). \]
Now to see that \( \lambda \) is an isomorphism and prove the theorem in the case where all the \( E_i \) have complex multiplication, we observe that for each \( i \) the composition
\[ X(K_i^{X_i}) \hookrightarrow M \to X(\text{Hg}(A)) \]
of \( m \mapsto (0, \ldots, 0, m, 0, \ldots, 0) \) with \( \lambda \) is injective. In addition, all of these character groups are \( \mathcal{G} = \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \)-modules and the maps \( \mathcal{G} \)-module maps. Then since the fields are distinct there is some \( \sigma \in \mathcal{G} \) that acts as \( +1 \) on \( X(K_i^{X_i}) \) and \( -1 \) on the other components. Thus if \( m = (m_1, \ldots, m_r) \) is in the kernel of \( \lambda \) then \( \sigma m + m = (2m_1, 0, \ldots, 0) \) must be as well. Then the injectivity of the composition above forces \( m_1 = 0 \), and by induction the kernel of \( \lambda \) is zero.

Next suppose none of the \( E_i \) has complex multiplication. Then we have
\[ \text{h}(A) \subseteq \text{h}(E_1) \times \cdots \times \text{h}(E_r), \]
and mapping surjectively onto each factor. Then by Proposition 2.16.4, if it also maps surjectively onto each pair of factors, it is the entire product. But by Proposition 2.16.2, if \( \text{h}(A) \) does not project onto \( \text{h}(E_i) \times \text{h}(E_j) \) for all pairs \( i \neq j \), then it projects to the graph of an isomorphism between them, which in turn could be used to produce an isogeny between \( E_i \) and \( E_j \), contrary to assumption.

Finally it remains to see that if \( A \) is an abelian variety isogenous to a product \( B \times C \) with \( \text{Hg}(B) \) a torus and \( \text{Hg}(C) \) semisimple, then \( \text{Hg}(A) = \text{Hg}(B) \times \text{Hg}(C) \). However, this is a consequence of Proposition 2.16.1. This completes the proof of the theorem. \( \square \)
4. Abelian varieties of Weil or Fermat type

We have already defined abelian varieties of Weil type (1.13.2), and an abelian variety of Fermat type is one which is isogenous to a product of certain factors of the Jacobian variety of a Fermat curve \( x^m + y^m + z^m = 0 \) [B.116]. The important thing about these examples, insofar as the Hodge conjecture goes, is that they contain the only known examples where the conjecture has been verified for abelian varieties \( \mathcal{H}(A) \neq \mathcal{D}(A) \). However, both types also provide explicit examples of Hodge cycles that are not known to be algebraic. Indeed, as is well-known, Weil has suggested that a place to look for a counterexample to the Hodge conjecture might be among what are now called abelian varieties of Weil type [B.135].

We will begin by summarizing Shioda’s results on abelian varieties of Fermat type. Then, since there is a nice presentation of the issues concerning abelian varieties in [B.35], we will just summarize the salient points here. Finally we will recall the work of Schoen [B.104] and van Geemen [B.36] verifying the Hodge conjecture for special four-dimensional abelian varieties.

Shioda’s results on abelian varieties of Fermat type and Jacobians of hyperelliptic curves ([B.116]). We will attempt to give a careful statement of the results, and refer the reader to the original for the proofs.

4.1. Notation. Fix an integer \( m > 1 \), and for \( a \in \mathbb{Z} \) not congruent to zero modulo \( m \), let \( 1 \leq \bar{a} \leq m - 1 \) be the unique integer such that \( \bar{a} \equiv a \pmod{m} \). Let

\[
\mathfrak{A}_m^n := \{ \alpha = (a_0, \ldots, a_{n+1}) : 1 \leq a_i \leq m - 1, \sum_{i=0}^{n+1} a_i \equiv 0 \pmod{m} \} \\
\mathfrak{B}_m^n := \{ \alpha \in \mathfrak{A}_m^n : |t \cdot \alpha| = (n/2) + 1 \text{ for all } t \in (\mathbb{Z}/m\mathbb{Z})^\times \},
\]

where in the latter case \( n \) must be even, and where for \( t \in (\mathbb{Z}/m\mathbb{Z})^\times \) and \( \alpha \in \mathfrak{A}_m^n \),

\[
t \cdot \alpha := (ta_0, \ldots, ta_{n+1}), \quad |\alpha| := \frac{1}{m} \sum_{i=0}^{n+1} a_i.
\]

For \( \alpha = (a_0, \ldots, a_{r+1}) \in \mathfrak{A}_m^r \) and \( \beta = (b_0, \ldots, b_{s+1}) \in \mathfrak{A}_m^s \) let

\[
\alpha \ast \beta := (a_0, \ldots, a_{r+1}, b_0, \ldots, b_{s+1}) \in \mathfrak{A}_m^{r+s+2}.
\]

Further, let

\[
M_m := \{ \xi = (x_1, \ldots, x_{m-1}; y) : \sum_{\nu=1}^{m-1} t\nu x_\nu = my \text{ for all } t \in (\mathbb{Z}/m\mathbb{Z})^\times, x_\nu, y \in \mathbb{Z}, x_\nu \geq 0, y > 0 \} \\
M_m(d) := \{(x_1, \ldots, x_{m-1}; d) \in M_m \}.
\]
4.2. Definition. An element $\xi \in M_m$ is said to be indecomposable if $\xi \neq \xi' + \xi''$ for any $\xi', \xi'' \in M_m$. An element $\xi \in M_m$ is called quasi-decomposable if there exists $\eta \in M_m(1)$ such that $\xi + \eta = \xi' + \xi''$ for some $\xi', \xi'' \in M_m$ with $\xi', \xi'' \neq \xi$.

It is easy to see that the set $M_m$ is an additive semigroup with only a finite number of indecomposable elements.

4.3. Definition. Let $X_m : x^m + y^m + z^m = 0$ denote the Fermat curve of degree $m$, let $J(X_m)$ be its Jacobian, and let $\mathcal{S}_m = (\mathbb{Z}/m\mathbb{Z})^\times \mathbb{A}_m^1$ be the orbit space. Then there is an isogeny

$$\pi : J(X_m) \rightarrow \prod_{S \in \mathcal{S}_m} A_S$$

where

(i) $A_S$ is an abelian variety of dimension $\varphi(m')/2$ admitting complex multiplication by $\mathbb{Q}(\zeta_{m'})$, where $\zeta_{m'} = e^{2\pi i / m'}$ and $m' = m / \gcd(a, b, c)$ for $(a, b, c) \in \mathbb{A}_m^1$ belonging to the orbit $S$.

(ii) $H^1(A_S, \mathbb{C})$ has the eigenspace decomposition

$$H^1(A_S, \mathbb{C}) = \bigoplus_{\alpha \in S} W(\alpha)$$

for the complex multiplication of (i), where $\dim W(\alpha) = 1$ and such that, if $\pi_S : J(X_m) \rightarrow A_S$ is the composition of $\pi$ and the projection to the $S$ factor, then $\pi_S^* W(\alpha) = U(\alpha)$ with $\alpha \in S$. Here $U(\alpha)$ is defined by

$$H^{1,0}(J(X_m)) = \bigoplus_{\alpha \in \mathbb{A}_m^1 \atop |\alpha|=1} U(\alpha), \quad H^{0,1}(J(X_m)) = \bigoplus_{\alpha \in \mathbb{A}_m^1 \atop |\alpha|=2} U(\alpha)$$

and $U(\alpha)$ is one-dimensional $[\mathbf{B.41}]$ $[\mathbf{B.59}]$.

Then an abelian variety will be said to be of Fermat type of degree $m$ if it is isogenous to a product of a finite number of factors $A_S$ satisfying (i) and (ii) as above.

Thus an abelian variety of Fermat type of degree $m$ is given by

$$(4.3) \quad A = \prod_{i=1}^k A_{S_i}$$

with $S_1, \ldots, S_k \in \mathcal{S}_m$ not necessarily distinct. With this notation we can now state Shioda’s results on the Hodge $(p, p)$ conjecture for these abelian varieties.

4.4. Theorem ($[\mathbf{B.116}]$ Thm.4.3). Let $A$ be an abelian variety of Fermat type of degree $m$. Assume that for any set $\{\alpha_1, \ldots, \alpha_{2d}\}$ of distinct elements in the disjoint union of the $S_i$, $i = 1, \ldots, k$, such that $\alpha_1 \ast \cdots \ast \alpha_{2d} = \gamma \in \mathbb{B}_m^{6d-2}$ there exists some $\beta_1, \ldots, \beta_l \in \mathbb{B}_m^0 \cup \mathbb{B}_m^2 \cup (\mathbb{B}_m^4 \cap \mathbb{A}_m^1 \ast \mathbb{A}_m^1)$ such that $\beta_1 \ast \cdots \ast \beta_l$ coincides with $\gamma$ up to permutation. Then the Hodge $(p, p)$ conjecture is true for $A$ in codimension $d$. 
4.5. **Theorem ([B.116] Thm.4.4).** If every decomposable element of $M_m(y)$ with $3 \leq y \leq 3d$, if any, is quasi-decomposable, then the Hodge $(p,p)$ conjecture is true in codimension $d$ for all abelian varieties of Fermat type of degree $m$. In particular, if $m$ is a prime or $m \leq 20$, then the Hodge conjecture is true in any codimension for all abelian varieties of Fermat type of degree $m$.

4.6. **Theorem ([B.116] Thm.5.6).** For any given $d \geq 2$ there exists some abelian variety of Fermat type $A$ such that the Hodge ring $H^*(A)$ is not generated by $\sum_{r=1}^{d-1} H^r(A)$.

4.7. **Theorem ([B.116] Thm.5.3).** Let $C_m : y^2 = x^m - 1$ be the hyperelliptic curve of genus $g = [(m - 1)/2]$, and let $J(C_m)$ be its Jacobian. If $m > 2$ is a prime number, then the Hodge ring $H^*(J(C_m))$ is generated by $H_1(J(C_m))$, i.e., $H^*(J(C_m)) = D^*(J(C_m))$ and the Hodge conjecture is true for $J(C_m)$. The same result also holds for arbitrary powers of $J(C_m)$.

4.8. **Theorem ([B.116] Thm.5.4).** For any odd $m \geq 3$ the Hodge conjecture is true for $J(C_m)$ in codimension 2.

**Abelian varieties of Weil type.** From 1.13.2, a Weil abelian variety of dimension $g = 2n$ is an abelian variety $A$ together with an imaginary quadratic field $K$ embedded in $\text{End}^0(A)$ such that the action of $\alpha \in K$ has the eigenvalues $\alpha$ and $\bar{\alpha}$ with equal multiplicity $n$. Here we briefly review some of the important points about Weil abelian varieties, mainly following the exposition of [B.35], to which we refer the reader for more detail; the original source is [B.135].

If $(A,K)$ is an abelian variety of Weil type, and $K = \mathbb{Q}(\sqrt{-d})$, then it is possible to choose a polarization $[E]$ on $A$ normalized so that $(\sqrt{-d})^* E = dE$, where here we are viewing $\sqrt{-d} \in O_K \hookrightarrow \text{End}(A)$ as an endomorphism of $A$, so $(\sqrt{-d})^*$ and $(\sqrt{-d})_*$ denote the induced pullback and push-out maps, respectively. Hereafter when we speak of a polarized abelian variety of Weil type we will assume that its polarization is normalized this way. Let $W = H_1(A, \mathbb{Q})$. Then the $K$-valued Hermitian form $H : W \times W \rightarrow K$ associated to $E$ is given by

$$H(u,v) := E(u, (\sqrt{-d})_* v) + (\sqrt{-d}) E(u,v).$$

4.9. **Theorem ([B.135]).** The Hodge group of a general polarized abelian variety $(A,K,E)$ of Weil type is $\text{SU}(W,H)$ (as an algebraic group over $\mathbb{Q}$).

It can be shown that polarized abelian varieties of Weil type of dimension $2n$ are parameterized by an $n^2$-dimensional space which can be described as the bounded symmetric domain associated to $\text{SU}(W,H; \mathbb{R}) \cong \text{SU}(n,n)$. Thus the word “general” in the statement of the theorem refers to a general point in this parameter space, analogously to the usage in 1.13.8.
4.10. Definition. Let \((A, K)\) be an abelian variety of Weil type and dimension \(2n\). Then \(V = H^1(A, \mathbb{Q})\) has the structure of a vector space over \(K\). The space of Weil-Hodge cycles on \(A\) is the subset of \(H^{2n}(A, \mathbb{Q})\)

\[
\mathcal{W}(A) := \bigwedge^{2n}_K H^1(A, \mathbb{Q}),
\]

where \(\bigwedge^{2n}_K\) signifies the 2\(^n\)-th exterior power of \(H^1(A, \mathbb{Q})\) as a \(K\)-vector space.

Lemma. \(\dim_{\mathbb{Q}} \mathcal{W}(A) = 2\), and \(\mathcal{W}(A) \subset \mathcal{H}^n(A)\).

4.11. Theorem (\([B.135]\)). Let \((A, K)\) be an abelian variety of Weil type of dimension \(g = 2n\). If the Hodge groups of \(A\) is \(\text{Hg}(A) = SU(W, H)\), then

\[
\dim \mathcal{H}^p(A) = \begin{cases} 
1, & p \neq n, \\
3, & p = n
\end{cases}
\]

and \(\mathcal{H}^n(A) = D^n(A) \oplus \mathcal{W}(A)\).

Thus abelian varieties of Weil type provide examples of Hodge cycles which do not arise from products of those in codimension one.

In the following the determinant of the Hermitian form \(H\) is well-defined as an element of \(\mathbb{Q}^\times\) modulo the subgroup of norms from \(K^\times\).

4.12. Theorem (\([B.104]\)). The Hodge \((2, 2)\) conjecture is true for a general abelian variety of Weil type \((A, K)\) of dimension 4 with \(K = \mathbb{Q}(\sqrt{-3})\) or \(K = \mathbb{Q}(i)\), when \(\det H = 1\).

A different proof for the case where \(K = \mathbb{Q}(i)\) and \(\det H = 1\) can be found in \([B.36]\). In \([B.35]\) 7.3 it is pointed out that Schoen’s methods together with a result of \([B.32]\) imply the Hodge \((3, 3)\) conjecture for a general 6-dimensional abelian variety of Weil type with \(K = \mathbb{Q}(\sqrt{-3})\) and \(\det H = 1\).

Recently Moonen and Zarhin \([B.76]\) have considered the extent to which Weil’s construction of exceptional Hodge classes can be generalized. For \(K \hookrightarrow \text{End}^0(A)\) a subfield and \(r = 2 \dim A/[K : \mathbb{Q}]\), let

\[
\mathcal{W}_K(A) := \bigwedge^r_K H^1(A, \mathbb{Q}),
\]

and call this the space of Weil classes with respect to \(K\). Let \(V = H^1(A, \mathbb{Q})\). Then \(K\) acts on \(V\) and there is a decomposition

\[
V \otimes \mathbb{C} = \bigoplus_{\sigma \in \text{Hom}(F, \mathbb{C})} V_{\mathbb{C}, \sigma} = \bigoplus_{\sigma \in \text{Hom}(F, \mathbb{C})} (V^{1,0}_{\mathbb{C}, \sigma} \oplus V^{0,1}_{\mathbb{C}, \sigma}).
\]
4.13. Proposition ([B.76]). Let $A$ be a complex abelian variety, $K \hookrightarrow \text{End}^0(A)$ a subfield, and $r = 2 \dim A/[K : \mathbb{Q}]$. With the notation above,

1. If $\dim V_{C,\sigma}^{1,0} = V_{C,\sigma}^{1,0}$ for all $\sigma \in \text{Hom}(F, \mathbb{C})$, where $\sigma$ denotes the complex conjugate of $\sigma$, then $\mathcal{W}_K(A)$ consists entirely of Hodge classes, i.e., $\mathcal{W}_K(A) \subset \mathcal{H}(A)$; if $\dim V_{C,\sigma}^{1,0} \neq V_{C,\sigma}^{1,0}$ for some $\sigma \in \text{Hom}(F, \mathbb{C})$, then the zero class is the only Hodge class in $\mathcal{W}_K(A)$.

2. Suppose $A$ is isogenous to a power $B^m$ of a simple abelian variety $B$, and suppose $K \hookrightarrow \text{End}^0(A)$ is a subfield such that $\mathcal{W}_K(A)$ consists of Hodge classes. Let $F$ be the center of $\text{End}^0(B)$, and $F_0$ the maximal totally real subfield of $F$. Then either $\mathcal{W}_K(A) \subset \mathcal{D}(A)$, or all nonzero classes in $\mathcal{W}_K(A)$ are exceptional; this last possibility occurs precisely in the following cases:
   - (a) $B$ is of type III, $m = 1$ and $K \not\subseteq F$,
   - (b) $B$ is of type III, $m \geq 2$ and $2m[F : \mathbb{Q}]/[K : \mathbb{Q}]$ is odd,
   - (c) $B$ is of type IV, $(\dim_F(\text{End}^0(B)))^{1/2} = 1$, $m = 1$ and $K \not\subseteq F_0$,
   - (d) $B$ is of type IV with $(\dim_F(\text{End}^0(B)))^{1/2} \geq 2$ or $m \geq 2$ and the map
     \[ \text{Lie}(U_F(1)) \hookrightarrow \text{End}_K(V) \xrightarrow{T_K} K \]

     is nonzero. Here $U_F(1) = \text{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F} \cap U(1)$ (see 1.5.2).

5. Simple abelian fourfolds

Since the Hodge $(p,p)$ conjecture is true for any smooth projective complex threefold, it might seem that fourfolds would be the next case to attack. However, as we've just seen, simple abelian fourfolds provide the first examples of abelian varieties of Weil type, for which the Hodge conjecture is mostly not known, and they also provide the first examples of abelian varieties of type (III) in the Albert classification, as well as the first examples of abelian varieties that are not characterized by their endomorphism rings [B.78] §4.

Recently Moonen and Zarhin [B.75] have analyzed the Hodge structures of simple abelian fourfolds, and their work makes an instructive example. The main result is the following, where a subalgebra of $\text{End}^0(A)$ is said to be stable under all Rosati involutions if for every polarization it is stable under the associated Rosati involution. Recall that an exceptional Hodge class is one which is not accounted for by linear combinations of intersections of intersections of divisors.

5.1. Theorem ([B.75] Thm.2.4). When $A$ is a simple abelian fourfold, then $A$ supports exceptional Hodge classes if and only if $\text{End}^0(A)$ contains an imaginary quadratic field $K$ which is stable under all Rosati involutions and such that with the induced action of $K \subseteq \text{End}^0(A)$ the complex Lie algebra $\text{Lie}(A, \mathbb{C})$ of $A$ becomes a free $K \otimes \mathbb{C}$-module.

In the situation of the theorem, $\text{Lie}(A)$ becomes a free $K \otimes \mathbb{C}$-module if and only if $\alpha \in K$ acts as $\alpha$ and as $\bar{\alpha}$ with equal multiplicity 2. Thus a less precise but more simply stated corollary would be the following.

Corollary. When $A$ is a simple abelian fourfold, if $\mathcal{H}^2(A) \neq \mathcal{D}^2(A)$ then $A$ must be an abelian variety of Weil type.
For the abelian fourfolds $A$ for which $\mathcal{H}^2(A) \neq \mathcal{D}^2(A)$ Moonen and Zarhin prove the following theorem.

5.2. **Theorem** ([B.75] Thm.2.12). *Let $A$ be a simple abelian fourfold, and let*

\[
\mathcal{V}(A) := \sum_K \left( \Lambda_K^4 H^1(A, \mathbb{Q}) \right),
\]

*where the sum runs over all imaginary quadratic subfields $K \subset \text{End}^0(A)$ that act on $A$ with multiplicities $(2, 2)$. Then $\mathcal{H}^2(A) = \mathcal{D}^2(A) + \mathcal{V}(A)$.***

This theorem should be compared with Theorem 4.11: Theorem 5.1 applies to fourfolds, whereas in Theorem 4.11 it was assumed that $\text{Hg}(A) = \text{SU}(W, H)$, which is a sort of a generality assumption.

5.3. **Remark:** Earlier work of Tankeev. In 1978 and 1979 Tankeev published two papers [B.122] [B.123] containing results about the Hodge structure and Hodge conjecture for abelian fourfolds. In particular, in [B.123] Thm.3.2 he proved that when $A$ is a simple abelian fourfold of type (I) or type (II), then $\mathcal{H}(A) = \mathcal{D}(A)$. First he showed that if the center of $\text{End}^0(A)$ is a product of totally real fields, then $\text{Hg}(A)$ is semisimple [B.123] Lemma 1.4, and then he proved the Hodge conjecture for simple abelian fourfolds of type (I) or (II) by considering the possible symplectic representations of the complexified Lie algebra $\mathfrak{h}G_{\mathbb{C}}(A)$ and showing that in each case that its invariants in $H^*(A, \mathbb{C})$ are generated by those of degree 2. The earlier paper [B.122] considered the possible pairs $(g, \rho)$, where $g$ is the semisimple part of $\mathfrak{h}G_{\mathbb{C}}(A)$ and $\rho : g \to \text{End}_{\mathbb{C}}(W_{\mathbb{C}})$ denotes its action on $W_{\mathbb{C}} = H_1(A, \mathbb{C})$, under the assumption that that there does not exist an abelian variety $A_0$ defined over $\overline{\mathbb{Q}}$ such that $A_0 \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \cong A$; however, the proof there contains some gaps. Then he derived, under the same assumption that the abelian fourfold $A$ cannot be defined over $\overline{\mathbb{Q}}$, that when $\text{End}^0(A)$ is neither an imaginary quadratic field nor a definite quaternion algebra (type (III)) then $\mathcal{H}(A) = \mathcal{D}(A)$.

We now proceed to sketch the outline of the proof of Theorem 5.1. One direction is covered by the following more general result.

5.4. **Theorem** ([B.75] Thm.3.1). *Let $A$ be a simple abelian variety, and assume that either*

- (a) $A$ is of type (III), or
- (b) $\text{End}^0(A)$ is a CM-field $K$ which contains a CM-subfield $F$ such that the multiplicity with which $\alpha \in F$ acts as $\sigma(\alpha)$ is the same for all $\sigma \in \text{Hom}(F, \mathbb{C})$.

*Then $A$ supports exceptional Hodge classes***

The result for abelian varieties of type (III) is due to Murty [B.82], see 8.6 below, although Moonen and Zarhin give a different proof. For case (b) what they show is that except for the zero element, $\bigwedge^m_F H^1(A, \mathbb{Q})$ consists entirely of exceptional Hodge classes, where $m = 2 \dim A/[F : \mathbb{Q}]$.

The other direction of Theorem 5.2 is proved case by case, running through the different possible endomorphism algebras (see 1.13.4). It turns out that except
When \( \text{End}^0(A) = \mathbb{Q} \), knowing \( \text{End}^0(A) \) together with its action on \( \text{Lie}(A) \) suffices to determine \( \mathfrak{h}_g(A) \), which in turn is enough to identify the absence or presence of exceptional cycles. We will run through the results, giving only some comments on the ingredients of the proofs. As usual, \( W := H_1(A, \mathbb{Q}) \).

5.5. Type I(1). Let \( A \) be a simple abelian fourfold with \( \text{End}^0(A) = \mathbb{Q} \). Then the Lie algebra \( \mathfrak{h}_g(A) \) together with its representation on \( W \) is isomorphic over \( \overline{\mathbb{Q}} \) to one of

(i) \( \mathfrak{sp}_4 \) with the standard representation, or
(ii) \( \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2 \) with the tensor product of the standard representation of each of the three factors.

Both possibilities occur, and in both cases \( \mathfrak{H}(A) = \mathfrak{D}(A) \).

On the assumption that \( \mathfrak{h}_g \) is simple, Theorem 2.11 can be used to show that case (i) is the only possibility. When \( \mathfrak{h}_g = \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_t \) is not simple, then \( W = W_1 \otimes \cdots \otimes W_t \) and at least one \( W_i \) must be 2-dimensional. Then \( \mathfrak{h}_i = \mathfrak{sl}_2 \), and since the representation is symplectic, the complement must be \( \mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2 \). In both cases \( \bigwedge^4 W \) (the subspace of \( \mathfrak{m}t \)-invariants in \( \bigwedge^4 W \)) is computed to be 1-dimensional.

5.6. Notation. Before treating the remaining type (I) cases, suppose in general that \( \text{End}^0(A) \) contains a totally real field \( F \), and suppose a polarization \( [E] \) on \( A \) is given. Then there is a unique \( F \)-bilinear alternating form \( \psi : W \times W \to F \) whose trace \( \text{Tr}_{F/\mathbb{Q}}(\psi(u,v)) = E(u,v) \). Then from the uniqueness of \( \psi \) it follows that \( \mathfrak{h}_g(A) \) is contained in

\[
\mathfrak{sp}_F(W,\psi) := \{ \varphi \in \text{End}_F(W) : \psi(\varphi(u),v) + \psi(u,\varphi(v)) = 0 \text{ for all } u,v \in W \},
\]

regarded as a Lie algebra over \( \mathbb{Q} \).

5.7. Type I(2). Let \( A \) be a simple abelian fourfold with \( \text{End}^0(A) = F \) a real quadratic field. Then in the notation above, \( \mathfrak{h}_g(A) \cong \mathfrak{sp}_F(W,\psi) \). In particular, \( \mathfrak{H}(A^n) = \mathfrak{D}(A^n) \) for all \( n \).

Sketch of proof. The representation \( \mathfrak{h}_g_{\mathbb{C}} \) on \( W_{\mathbb{C}} \) splits as a direct sum \( W_{\mathbb{C}} = W_1 \oplus W_2 \) with \( \dim W_1 = \dim W_2 = 4 \). Further, the restriction of \( \psi \) to \( W_i \) is a nondegenerate skew-symmetric bilinear form \( \psi_i : W_i \times W_i \to \mathbb{C} \), and \( \mathfrak{h}_g_{\mathbb{C}} \subseteq \mathfrak{sp}(W_1,\psi_1) \times \mathfrak{sp}(W_2,\psi_2) \). Then the projection of \( \mathfrak{h}_g_{\mathbb{C}} \) onto \( \mathfrak{sp}(W_i,\psi_i) \) acting on \( W_i \) must be on the list of Theorem 2.11, and since it is irreducible, symplectic, 4-dimensional, it must be equal \( \mathfrak{sp}(W_i,\psi_i) \). Then since all automorphisms of \( \mathfrak{sp}_{4\mathbb{C}} \) are inner, Proposition 2.16.5 implies that \( \mathfrak{h}_g_{\mathbb{C}} = \mathfrak{sp}(W_1,\psi_1) \times \mathfrak{sp}(W_2,\psi_2) \), and thus \( \mathfrak{h}_g = \mathfrak{sp}_F(W,\psi) \).

5.8. Type I(4). Let \( A \) be a simple abelian fourfold with \( \text{End}^0(A) = F \) a totally real field with \( [F: \mathbb{Q}] = 4 \). Then in the notation of 5.6, \( \mathfrak{h}_g(A) \cong \mathfrak{sp}_F(W,\psi) \cong \mathfrak{sl}_2 F \). In particular, \( \mathfrak{H}(A^n) = \mathfrak{D}(A^n) \) for all \( n \).
The method of proof is similar to and easier than the previous case. Over \( \overline{\mathbb{Q}} \) or \( \mathbb{C} \) the representation of \( \mathfrak{h} \) on \( W \) splits into a sum of four mutually nonisomorphic, irreducible, symplectic, 2-dimensional representations, and Proposition 2.16.3 applies.

5.9. Type II. Let \( A \) be a simple abelian fourfold of type (II), i.e., \( \text{End}^0(A) \) is an indefinite quaternion algebra \( D \) over a totally real field \( F \) of degree \( e \in \{1, 2\} \) over \( \mathbb{Q} \). Then \( \mathfrak{h} \) is the centralizer of \( D \) in \( \mathfrak{sp}(W, E) \). In particular, \( \mathcal{H}(A^n) = \mathcal{D}(A^n) \) for all \( n \).

For both \( e = 1 \) and \( e = 2 \) this is a special case of [B.21] Thm.4.10 and [B.22] Thm.7.4. Compare also with the method of Ribet [B.94], 6.3 below.

5.10. Type III. Let \( A \) be a simple abelian fourfold of type (III), i.e., \( \text{End}^0(A) \) is a definite quaternion algebra \( D \) over \( \mathbb{Q} \). Then \( \mathfrak{h} \) is the centralizer of \( D \) in \( \mathfrak{sp}(W, E) \), which is a \( \mathbb{Q} \)-form of \( \mathfrak{so}_4 \). Moreover, \( \dim \mathcal{H}^2(A) = 6 \), and \( \dim \mathcal{D}^2(A) = 1 \), and \( \mathcal{H}^2(A) = \mathcal{D}^2(A) + \mathcal{V}(A) \), where \( \mathcal{V}(A) \) is as in 5.2.

5.11. Type IV(1,1). Let \( A \) be a simple abelian fourfold such that \( \text{End}^0(A) = K \) is an imaginary quadratic field.

(i) If \( K \) acts with multiplicities \( \{1, 3\} \) then \( \mathfrak{h}(A) = \mathfrak{u}(W/K) \), and \( \mathcal{H}(A^n) = \mathcal{D}(A^n) \) for all \( n \).

(ii) If \( K \) acts with multiplicities \( \{2, 2\} \). In this case \( \mathfrak{h}(A) = \mathfrak{su}(W/K) \), and \( \dim \mathcal{H}^2(A) = 3 \), and \( \dim \mathcal{D}^2(A) = 1 \), and \( \mathcal{H}^2(A) = \mathcal{D}^2(A) + \mathcal{V}(A) \), where \( \mathcal{V}(A) \) is as in 5.2.

In case (i) \( A \) is of Ribet type (1.13.2), see 6.3 below. In case (ii) the equality \( \mathfrak{h}(A) = \mathfrak{su}(W/K) \) must be proved, and then Theorem 4.11 applies.

5.12. Type IV(2,1). Let \( A \) be a simple abelian fourfold such that \( \text{End}^0(A) = K \) is a CM-field of degree 4 over \( \mathbb{Q} \). Then \( \mathfrak{h}(A) \simeq \mathfrak{u}_K(W, \psi) \). In particular, \( \mathcal{H}(A^n) = \mathcal{D}(A^n) \) for all \( n \).

Similarly as in 5.6, here \( \psi : W \times W \to K \) is the unique \( K \)-Hermitian form such that \( \text{Tr}_{K/\mathbb{Q}}(\alpha \cdot \psi(u, v)) \) is a Riemann form for \( A \), for \( \alpha \in K \) such that \( \overline{\alpha} = -\alpha \) (The uniqueness is proved in [B.27]). In this case \( \mathfrak{h}(A) \) is contained in \( \mathfrak{u}_K(W, \psi) := \{ \varphi \in \text{End}_K(W) : \psi(\varphi(u), v) + \psi(u, \varphi(v)) = 0 \text{ for all } u, v \in W \} \), regarded as a Lie algebra over \( \mathbb{Q} \).

5.13. Type IV(4,1). Let \( A \) be a simple abelian fourfold such that \( \text{End}^0(A) = K \) is a CM-field of degree 8 over \( \mathbb{Q} \).

(i) If \( K \) does not contain an imaginary quadratic field \( F \) acting on \( A \) with multiplicities \( \{2, 2\} \), then \( \mathfrak{h}(A) = \mathfrak{u}_K \), which is a commutative Lie algebra of rank 4, and \( \mathcal{H}(A^n) = \mathcal{D}(A^n) \) for all \( n \).
(ii) If $K$ does contain an imaginary quadratic field $F$ acting on $A$ with multiplicities $(2, 2)$, then $h_1(A) = s_{K/F}$. In this case $\dim H^2(A) = 8$, and $\dim D^2(A) = 6$, and $H^2(A) = D^2(A) + V(A)$.

We discuss abelian varieties with complex multiplication further below, see section nine.

6. Simple abelian varieties with conditions on dimension or endomorphism algebra

After Tankeev's early work on simple abelian fourfolds [B.122] [B.123], the next progress on the Hodge $(p, p)$ conjecture was the work of Tankeev [B.124] [B.125] [B.126] and Ribet [B.93] [B.94] on simple abelian varieties of types (I), (II) or (IV) whose dimension and endomorphism algebras satisfy various conditions. More precisely, Tankeev proved the following.

6.1. Theorem ([B.125] [B.126]). Let $A$ be a simple abelian variety of dimension $d$. Then if

1. $A$ is of nondegenerate CM-type (2.13), or
2. $\text{End}^0(A)$ is a totally real field of degree $e$ over $\mathbb{Q}$, and $d/e$ is odd, or
3. $\text{End}^0(A)$ is a totally indefinite division quaternion algebra over a totally real field $K$ of degree $e$ over $\mathbb{Q}$, and $d/2e$ is odd, or
4. $d$ is a prime,

then $H(A) = D(A)$.

However, soon afterward Ribet extended some of those results by first observing the following basic criterion, which he says was used implicitly in Tankeev's work, and then identifying some instances where it is satisfied.

6.2. Theorem ([B.94] Theorem 0). Let $A$ be an abelian variety, and suppose

(a) $\text{End}^0(A)$ is a commutative field, and
(b) $H_1(A) = L_1(A)$ (the Lefschetz group, see 2.14).

Then $H(A^n) = D(A^n)$ for $n \geq 1$.

6.3. Theorem ([B.94] Theorems 1–3). Let $A$ be an abelian variety of dimension $d$, and suppose

1. $\text{End}^0(A)$ is a totally real field of degree $e$ over $\mathbb{Q}$, and $d/e$ is odd, or
2. $d$ is prime and $A$ is of CM-type, or
3. $\text{End}^0(A)$ is an imaginary quadratic field $K$, and the multiplicities $n'$ and $n''$ with which $\alpha \in K$ acts as $\alpha$ and $\bar{\alpha}$ respectively are relatively prime.

Then $H_1(A) = L_1(A)$ and thus $H(A^n) = D(A^n)$ for $n \geq 1$.

Corollary. When $A$ is a simple abelian variety of prime dimension, then $H(A^n) = D(A^n)$ for $n \geq 1$. 

For if $A$ is simple and of prime dimension, then one of the conditions of Theorem 6.3 must be satisfied, see 1.13.3.

**Remark.** In [B.139] Yanai showed that a prime-dimensional abelian variety of simple CM-type is nondegenerate (2.13).

In a similar spirit as 6.3.2 above, Hazama proved the following.

6.4. **Theorem** ([B.45]). Let $A$ be a simple abelian variety of CM-type. Then $H(A^n) = D(A^n)$ for all $n$ if and only if $\dim H(A) = \dim A$.

**Sketch of proof of Theorem 6.2** (after [B.94]). In order to give some flavor of the techniques involved, consider the proof of Theorem 6.2. Since it is always the case that $H(A) \subseteq Lf(A)$, see 2.14, the condition that these two groups are equal should be thought of as saying that $H(A)$ is as large as possible, whence has as few invariants as possible. Then the proof is separated into two cases, according as $\text{End}^0(A) = K$ is a totally real or a CM-field.

Consider first the case where $K$ is totally real. Then similarly as in 5.6 there is a unique $K$-bilinear alternating form $\psi : W \times W \rightarrow K$ whose trace $\text{Tr}_{K/Q}(\psi(u,v))$ is a Riemann form $E(u,v)$ representing the chosen polarization on $A$, where $W = H_1(A,\mathbb{Q})$ [B.27] 4.7. The uniqueness of $\psi$ implies that a $K$-automorphism of $W$ preserves $\psi$ if and only if it preserves $E$. Then from the definition 2.14 we get that $H(A) = Lf(A)$ is the symplectic group of $\psi$ acting on $W$ as a $K$-vector space, i.e.,

$$H(A) = \text{Res}_{K/Q} \text{Sp}(W_0, \psi),$$

where $W_0$ is $W$ as a $K$-vector space.

Now what we need to prove is that $(\bigwedge^n(W^\vee))^H(A)$ is generated by its elements of degree 2, and for this question it suffices to extend scalars to $\mathbb{C}$. Then $W \otimes \mathbb{C}$ is a free $K \otimes \mathbb{Q} \mathbb{C}$-module of rank $r = 2 \dim A/[K : \mathbb{Q}]$, and thus

$$W \otimes \mathbb{C} \cong \bigoplus_{\sigma \in \text{Hom}(K,\mathbb{C})} U_\sigma,$$

where $U_\sigma$ is a complex vector space of dimension $r$. Therefore

$$H(A,\mathbb{C}) \cong \prod_{\sigma \in \text{Hom}(K,\mathbb{C})} \text{Sp}(U_\sigma, \psi_\sigma),$$

from which we see that

$$(\bigwedge^n(W^\vee))^H(A) \otimes \mathbb{C} \cong \bigotimes_{\sigma} (\bigwedge^n(U_\sigma^\vee))^\text{Sp}(U_\sigma, \psi_\sigma).$$

So it suffices to know that this algebra of invariants is generated by its elements of degree 2, which is the case; see [B.94], or derive this fact using the methods of [B.10], [B.137] or [B.34].

The proof for the case where $\text{End}^0(A) = K$ is a CM-field, either of degree 2 $\dim A$ or of degree 2 over $\mathbb{Q}$ follows a very similar pattern. As in 5.12 there is an element $\alpha \in K$ such that $\overline{\alpha} = -\alpha$ and a unique Hermitian form $\psi : W \times W \rightarrow K$ such that a Riemann form representing a polarization on $A$ is given by $E(u,v) = \text{Tr}_{K/Q}(\alpha \psi(u,v))$. Then the centralizer $Lf(A)$ of $K$ in $\text{Sp}(W, E)$, which by hypothesis
coincides with $\text{Hg}(A)$, is $\text{Res}_{K_0/Q} U(U_0, \psi)$, where $K_0$ is the maximal totally real subfield of $K$ and $W_0$ is $W$ as a $K$-vector space.

Now when we extend scalars to $\mathbb{R}$,

$$W \otimes_{\mathbb{Q}} \mathbb{R} \simeq \bigoplus_{\sigma \in \text{Hom}(K_0, \mathbb{R})} U_{\sigma}.$$  

Moreover, $\psi$ induces a nondegenerate Hermitian form $\psi_\sigma$ on each $U_\sigma$, from which

$$\text{Hg}(A, \mathbb{R}) \simeq \prod_{\sigma \in \text{Hom}(K_0, \mathbb{R})} U(U_\sigma, \psi_\sigma).$$

Thus in this case we need to know that for each $\sigma$ and for all $n$ the algebra of invariants $\left(\bigwedge^n(U_\sigma)^*)^{U(U_\sigma, \psi_\sigma)}\right)$ is generated by elements of degree $2$, which is the case. The first step towards proving this is to extend scalars to $\mathbb{C}$, so that the unitary group $U(U_\sigma, \psi_\sigma)$ becomes a general linear group; we omit the invariant theory arguments here, see op. cit. for more details. \hfill \Box

**Sketch of proof of Theorem 6.3.3.** We are assuming that $\text{End}^0(A) = K$ is an imaginary quadratic field, and the multiplicities $n', n''$, with which $\alpha \in K$ acts as $\alpha$ and acts as $\bar{\alpha}$ are relatively prime, and we want to show that $\text{Hg}(A) = \text{Lf}(A)$. In fact, it turns out to be more convenient to show that $\text{MT}(A) = \mathbb{G}_m \cdot \text{Hg}(A)$ coincides with $G = \mathbb{G}_m \cdot \text{Lf}(A)$. This group $G$ may also be described as the largest connected subgroup of the symplectic similitude group $\text{GSp}(W, E)$ that commutes with $K$, and in the present case $G = \mathbb{G}_m \cdot \text{Res}_{K'\mathbb{Q}} U(W, \psi)$. It is clear that $\text{MT}(A) \subseteq G$.  

Now, if $d = \dim A$, then $W$ is free of rank $d$ as a vector space over $K$, and thus $W \otimes_{\mathbb{Q}} \mathbb{C}$ is free of rank $d$ over $K \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$, where the two copies are naturally indexed by the embeddings of $K$ into $\mathbb{C}$. Therefore we may write

$$W \otimes_{\mathbb{Q}} \mathbb{C} = W' \oplus W'' ,$$

and this decomposition is compatible with the Hodge decomposition of and the action of $\text{MT}(A)$ on $W \otimes_{\mathbb{Q}} \mathbb{C}$, since it is induced by endomorphisms of $A$. In particular,

$$W' = (W' \cap H^{-1,0}(A)) \oplus (W' \cap H^{0,-1}(A)).$$

Now the action of $\text{MT}(A, \mathbb{C})$ on $W \otimes_{\mathbb{Q}} \mathbb{C}$ induces an action of $\text{MT}(A, \mathbb{C})$ on $W'$, and the next step of the proof is to see that the induced map $\text{MT}(A, \mathbb{C}) \to \text{GL}(W')$ is surjective. However, this follows from [B.106] Prop.5; it is here that the relative primality of $n'$ and $n''$ is a required hypothesis. It follows that the commutator subgroup of $\text{MT}(A, \mathbb{C})$ maps onto $\text{SL}(W')$. What this means is that when we write $G$ as the product of its center $C$ and its semisimple part $G_{ss}$, then $\text{MT}(A) \supset G_{ss}$. Thus it remains to show that $C \subseteq \text{MT}(A)$ as well. And since $\dim C = 2$, it will suffice to show that the dimension of the center of $\text{MT}(A)$ is at least $2$, which in turn would follow from showing that $\text{MT}(A)$ maps onto a 2-dimensional torus.

Since $\text{MT}(A) \subseteq \text{Res}_{K/\mathbb{Q}} \text{GL}(W_0)$, where $W_0$ is $W$ considered as a vector space over $K$, we may consider the determinant map $\Theta : \text{MT}(A) \to \text{Res}_{K/\mathbb{Q}} \mathbb{G}_m$ $= : T$. Then from the fact that $\text{MT}(A)$ contains $\mathbb{G}_m/\mathbb{Q}$ acting as homotheties on $W$, the image of $\Theta$ contains $\mathbb{G}_m/\mathbb{Q} \subseteq T$. If we extend scalars to $\mathbb{C}$, then $T_{\mathbb{C}} \simeq \mathbb{G}_m/\mathbb{C} \times \mathbb{G}_m/\mathbb{C}$, and the image of $\mathbb{G}_m/\mathbb{Q}$ in $T$ becomes the diagonal. On the other hand, $\text{MT}(A)_{\mathbb{C}}$ also contains $h(S_\mathbb{C}) \simeq \mathbb{G}_m/\mathbb{C} \times \mathbb{G}_m/\mathbb{C}$. Then $\Theta(h(z, 1)) = (z(n'), z(n''))$, and because
n' \neq n'', this generates a torus distinct from the diagonal. Therefore Θ is surjective, which completes the proof.

7. More abelian varieties with conditions on dimension or endomorphism algebra

During the 1980’s Hazama, Murty and others continued to generate results related to the Hodge conjecture by examining the interactions among the dimension, endomorphism algebra, and Hodge or Mumford-Tate group, in a spirit akin to the work of Tankeev and Ribet described in section 6. In particular, Hazama and Murty, working at about the same time but using different methods, produced a number of overlapping results about the Hodge conjecture for not-necessarily-simple abelian varieties extending the results of Tankeev and Ribet.

Abelian varieties with generalized real multiplication. A first set of results can be loosely grouped together as dealing with abelian varieties with generalized real multiplication.

7.1. Definition. Several different definitions of what is meant by real multiplication appear in the literature. The narrowest would be that the abelian variety A is of type (I), i.e., every simple factor A_s of A has \text{End}\,^0(A_s) equal to a totally real field. A slightly broader definition would be to require that \text{End}\,^0(A) contains a product R of totally real fields such that [R : \mathbb{Q}] = \text{dim} A [B.46]. Zarhin [B.142] calls a g-dimensional abelian variety of RM-type if it contains a commutative semisimple \mathbb{Q}\text{-}algebra of degree g over \mathbb{Q}, and notes that this means that any abelian variety of CM-type is also automatically of RM-type. Murty variously considers the cases where a commutative semisimple \mathbb{Q}\text{-}algebra R \subseteq \text{End}\,^0(A) is its own centralizer in \text{End}\,^0(A) [B.81], or R is maximal among commutative semisimple subalgebras of \text{End}\,^0(A) and is a product of totally real fields [B.83]. Altogether the most useful general definition might be to say that an abelian variety A has generalized real multiplication if \text{End}\,^0(A) contains a commutative semisimple subalgebra R with [R : \mathbb{Q}] = \text{dim} A, and A is not of CM-type, i.e., \text{End}\,^0(A) does not contain a commutative semisimple subalgebra of degree 2 \text{dim} A over \mathbb{Q}. To avoid ambiguity we will try to give precise statements of results without using this terminology.

The following theorem tries to summarize the main results concerning the Hodge \((p,p)\) conjecture for abelian varieties with some generalized real multiplication.

7.2. Theorem ([B.46] [B.81] [B.83]). Let A be an abelian variety.
1. Suppose \text{End}\,^0(A) contains a product R of totally real fields with the property that [R : \mathbb{Q}] = \text{dim} A, and no simple component of A of CM-type has dimension greater than 1. Then \mathcal{H}(A) = \mathcal{D}(A).
2. Suppose \text{End}\,^0(A) contains a commutative semisimple subalgebra R that is its own centralizer in \text{End}\,^0(A), and \text{H}^0(A, \Omega^1) is free of rank 1 over \text{R} \otimes_{\mathbb{Q}} \mathbb{C}. Then \mathcal{H}(A) = \mathcal{D}(A).
3. Suppose that a maximal commutative semisimple subalgebra R of \text{End}\,^0(A) is a product of totally real fields, and that W = \text{H}_1(A, \mathbb{Q}) is free over R of
rank $2m$, where $m$ is odd. Then $\text{Hg}(A) = \text{Lf}(A)$ and thus $\mathcal{H}(A^k) = \mathcal{D}(A^k)$ for all $k \geq 1$.

7.3.1. Remarks on Theorem 7.2.1 ([B.46]). The first observation about an abelian variety $A$ satisfying the conditions of 7.2.1 is that its simple isogeny factors must be of type (I), or type (II), or elliptic curves with complex multiplication, as elliptic curves without complex multiplication are included in type (I) [B.37]. In particular, the condition is stable under taking products or abelian subvarieties.

Without going into the proof at too great a length, some of the main ingredients include firstly a lemma, due to Tankeev [B.123] Lemma 1.4, that if the center of $\text{End}_0^0(A)$ is a product of totally real fields, then $\text{Hg}(A)$ is semisimple. Then Hazama proceeds to work out the Hodge Lie algebra $\mathfrak{h}(B)$, where $B$ is the isogeny factor of $A$ containing the simple factors of type (I) or (II). After complexifying and applying Goursat’s Lemma he finds that $\mathfrak{h}(B, \mathbb{C}) \simeq \mathfrak{sl}_2 \times \cdots \times \mathfrak{sl}_2$. Finally, he observes, similarly as we did in section 3 on elliptic curves, that when $B$ is an abelian variety whose Hodge group is semisimple and $C$ is an abelian variety of CM-type, then $\text{Hg}(B \times C) \simeq \text{Hg}(B) \times \text{Hg}(C)$. Then Theorem 7.2.1 follows from the invariant theory of $\mathfrak{sl}_2$ and the known results for elliptic curves.

7.3.2. Remarks on Theorem 7.2.2 ([B.81]). Murty calls a pair $(A, R)$ consisting of an abelian variety and a commutative semisimple subalgebra $R \subset \text{End}^0(A)$ of type (H) when the hypotheses of Theorem 7.2.2 are satisfied. He observes that the product of two abelian varieties of type (H) is again of type (H), and further proves that in general when a commutative semisimple subalgebra of $\text{End}^0(A)$ of degree $\dim A$ over $\mathbb{Q}$ is a product $R$ of totally real fields, then $H^0(A, \Omega^1)$ is free of rank 1 over $R \otimes \mathbb{C}$. Thus some examples of abelian varieties of type (H) include $(E, \mathbb{Q})$, where $E$ is an elliptic curve without complex multiplication, and $(A, F)$ where $A$ is an abelian surface with quaternionic multiplication by a quaternion algebra $B$ over $\mathbb{Q}$, as in 1.13.7, and $F$ is any real quadratic subfield of $B$ which splits $B$.

Although the analysis is somewhat different, some of the main ideas of the proof of Theorem 7.2.2 are similar to some of the main points of the proof of Theorem 7.2.1. In particular, under the hypothesis of type (H) (and making use of Goursat’s Lemma), Murty shows that

$$\mathfrak{h}(A) = \{ m \in \text{End}_{\text{End}^0(A)}(W) : \text{tr}_R m = 0 \},$$

and that not only is this semisimple, but over $\mathbb{C}$ it is a product of $\mathfrak{sl}_2$’s. Thus it is possible to deduce that all invariants are generated by those of degree 2.

7.3.3. Example: Jacobians of elliptic modular curves. Among the motivating examples for both Hazama and Murty were Jacobians of elliptic modular curves. To recall briefly, for $N \geq 3$ let

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) : c \equiv 0 \text{ and } a \equiv d \equiv 1 \pmod N \right\},$$

and let $\mathfrak{H} = \{ z \in \mathbb{C} : \text{Im} z > 0 \}$ denote the upper half-plane. Then the quotient

$$X_1(N)(\mathbb{C}) := \Gamma_1(N) \backslash (\mathfrak{H} \cup \mathbb{Q} \cup \{i\infty\})$$
can be identified with the complex points of a nonsingular projective algebraic curve which can, in fact, be defined over \( \mathbb{Q} \). Moreover, Shimura has shown that in the Jacobian \( J_1(N) := \text{Jac}(X_1(N)) \) all the isogeny factors with complex multiplication are products of elliptic curves, and all the isogeny factors without complex multiplication are of real multiplication type in the sense that the endomorphism algebras of the simple factors contain a totally real number field whose degree over \( \mathbb{Q} \) is the dimension of that factor \([B.112]\) \([B.113]\) \([B.114]\) (see also \([B.89]\)).

7.3.4. Remark on Theorem 7.2.3. Theorem 7.2.3 is an artifact of Murty’s study \([B.83]\) of the semisimple parts of the Hodge groups of abelian varieties, and their relationship with the Lefschetz groups. We will return to this again briefly below.

Stably nondegenerate abelian varieties. A next group of results concerns conditions under which \( \mathcal{H}(A^k) = \mathcal{D}(A^k) \) for all \( k \geq 1 \). Again we combine closely related results of Murty and Hazama; but first we need a definition.

7.4. Definition \(([B.47])\). When \( A \) is a simple abelian variety, the reduced dimension of \( A \) is defined by

\[
\text{rdim} \ A := \begin{cases} 
\dim A, & \text{for } A \text{ of type (I) or of type (III)}, \\
(\dim A)/2, & \text{for } A \text{ of type (II)}, \\
(\dim A)/d, & \text{for } A \text{ of type (IV)}, \end{cases}
\]

By \( C(\text{End}^0(A)) \) here we mean the center of \( \text{End}^0(A) \). When \( A \) is isogenous to \( \prod_i A_i^{m_i} \) with the \( A_i \) simple and nonisogenous, then the reduced dimension of \( A \) is

\[
\text{rdim} \ A := \sum_i \text{rdim} \ A_i.
\]

7.5. Theorem \(([B.82], [B.47])\). For an abelian variety \( A \), the following are equivalent.

1. \( \mathcal{H}(A^k) = \mathcal{D}(A^k) \) for all \( k \geq 1 \).
2. \( A \) has no factor of type (III), and \( \text{Hg}(A) = \text{Lf}(A) \).
3. \( \text{rank} \ \text{Hg}(A)_C = \text{rdim} \ A \).

7.6. Definition. An abelian variety satisfying the conditions of Theorem 7.5 may be called stably nondegenerate.

7.6.1. Remarks. Hazama makes the following elementary observations about stable nondegeneracy \([B.47]\):

1. If \( A \) is stably nondegenerate, and \( B \) is an abelian subvariety of \( A \), then \( B \) is stably nondegenerate. For up to isogeny \( A \simeq B \times B' \), and thus if stable nondegeneracy (in the sense of 7.5.1) failed for \( B \) it would fail for \( A \).
2. For any \( k \geq 1 \), \( A \) is stably nondegenerate if and only if \( A^k \) is stably nondegenerate. This follows from the definition 7.5.1 and the previous observation.
3. For abelian varieties \( A_i \) and integers \( k_i \), the product \( \prod_i A_i^{k_i} \) is stably nondegenerate if and only if \( \prod_i A_i \) is stably nondegenerate. Observe that

\[
\prod_i A_i^{k_i} \subset (\prod_i A_i)^{\max k_i}.
\]
He also proves the following.

7.6.2. Theorem ([B.49]). If $A$ and $B$ are stably nondegenerate abelian varieties and contain no factors of type (IV), then $A \times B$ is also stably nondegenerate.

The difficulty with type (IV) arises in taking products of, or with, abelian varieties of CM-type, see section nine below. Or from a different point of view, there is the theorem of Tankeev that if all the simple factors of an abelian variety $A$ are of types (I), (II) or (III), then $\text{Hg}(A)$ is semisimple [B.123], and this may fail for type (IV). What can be said is that if $A$ is stably nondegenerate and has no factors of types (IV), and $B$ is stably nondegenerate and of CM-type, then $A \times B$ is stably nondegenerate [B.49]. On the other hand, since by 7.5.2 no abelian variety with a factor of type (III) can be stably nondegenerate, Theorem 7.6.2 applies when all simple factors of $A$ and $B$ are of type (I) or (II).

7.7. Some remarks on Theorem 7.5. In [B.82] Murty proves (1) if and only if (2). Much of the paper is devoted to a careful analysis of the structure of $\text{Lf}(A)$. Given the multiplicativity of $\text{Lf}(A)$, Lemma 2.15, we may assume $A$ is simple. Then fix a maximal commutative subfield $F \subset \text{End}^0(A)$ which is totally real for type (I) and a CM-field in the other three cases, and let $F_0$ be the maximal totally real subfield of $F$. Now extending scalars to $\mathbb{R}$, there is a decomposition of $\text{Lf}(A)$ into factors indexed by the embeddings $F_0 \hookrightarrow \mathbb{R}$. Then these factors are of the form: for type (I), a symplectic group; for type (II), the intersection of a unitary group and a symplectic group; for type (III), the intersection of a unitary group and a special orthogonal group; for type (IV), a unitary group [B.82] Lemma 2.3. Moreover, after complexifying, these act on the corresponding components of $W \otimes \mathbb{R}$ as: for type (I), as a standard symplectic representation; for type (II), two copies of the standard representation of the complex symplectic group; for type (III), two copies of the standard representation of the complex special orthogonal group; for type (IV), the sum of a standard representation of the complex general linear group and its contragredient. Using this structural analysis, Murty is then able to prove the following.

7.7.1. Proposition ([B.82]). If $A$ contains no simple factors of type (III), then for all $k \geq 1$

$$H^*\left(A^k, \mathbb{Q}\right)^{\text{Lf}(A)} = \mathcal{D}(A^k).$$

To complete the proof that 7.5.1 is equivalent to 7.5.2, Murty shows that a simple abelian variety of type (III) supports an exceptional Hodge class, see 8.6 below for more discussion of this.

Hazama’s proof in [B.47] that 7.5.1 is equivalent to 7.5.3 is based on a careful type by type analysis of the Lie algebra $\mathfrak{h}g(A)_\mathbb{C}$ and its action on $W_\mathbb{C}$ using that all the possibilities are as listed in Theorem 2.11. For example, for a simple abelian variety of type (I), the action of a simple component $\mathfrak{g}_i$ of $\mathfrak{h}g(A)_\mathbb{C}$ on the corresponding component $W_i$ of $W_\mathbb{C}$ is a symplectic representation, and indeed $\mathfrak{g}_i \simeq \text{sp}(W_i, \mathbb{C})$. A similar result holds for type (II), whereas for type (IV), the simple components of $\mathfrak{h}g(A)_\mathbb{C}$ are of the form $\mathfrak{sl}_d$. In all these cases, careful invariant theory arguments using [B.137] show that the invariants of $H^*\left(A^k, \mathbb{C}\right)$ are generated by those
of degree 2 if and only if the rank is as claimed. On the other hand, for type (III) Hazama finds that
\[ \text{rank } \text{Hg}(A)_C \leq (\dim A)/2 < \text{rdim } A, \]
i.e., equality never holds, and an abelian variety with a factor of type (III) fails to be stably nondegenerate. It comes about in the proof that in general
\[ \text{rank } \text{Hg}(A) \leq \text{rdim } A. \]

So both criteria 7.5.2 and 7.5.3 can be understood philosophically as saying that \( A \) is stably nondegenerate when \( \text{Hg}(A) \) is as large as possible.

**Further work on Hodge and Mumford-Tate groups.** We conclude this section with certain additional results derived from close study of Hodge and Mumford-Tate groups.

In [B.83] Murty examines the semisimple part of the Hodge group of an abelian variety, and finds the following. As usual, \( W = H_1(A, \mathbb{Q}) \).

7.8. **Theorem ([B.83]).** If a maximal commutative subalgebra \( R \) of \( \text{End}^0(A) \) is a product of CM-fields, and \( W \) is free over \( R \) of odd rank, and if \( \mathcal{H}(A) = \mathcal{D}(A) \), then \( \text{Hg}(A)_{ss} = Lf(A)_{ss} \).

This together with 7.2.3 implies the following.

**Corollary ([B.83]).** When \( A \) is simple and of odd dimension, then \( \mathcal{H}(A) = \mathcal{D}(A) \) implies that \( \text{Hg}(A)_{ss} = Lf(A)_{ss} \).

In [B.57] Ichikawa studies groups of Mumford-Tate type, and extending [B.94] [B.126] and his own earlier work [B.56], he obtains the following result. First we need some notation.

7.9. **Definition.** Let \( A \) be a simple abelian variety of dimension \( g \), let \( K \) the center of \( \text{End}^0(A) \), let \( e = [K : \mathbb{Q}] \) and let \( d^2 = [\text{End}^0(A) : K] \). Then the relative dimension of \( A \) is defined by
\[
\text{rel dim}(A) := \begin{cases} 
  g/e, & \text{if } A \text{ is of type (I)}, \\
  g/2e, & \text{if } A \text{ is of type (II) or type (III)}, \\
  2g/de, & \text{if } A \text{ is of type (IV)}. 
\end{cases}
\]

7.10. **Theorem ([B.57]).** Let \( A \) be an abelian variety all of whose simple factors have odd relative dimension.

1. When \( A \) is isogenous to \( A' \times A'' \), where each simple factor of \( A' \) is of type (I), (II) or (III) and each simple factor of \( A'' \) is of type (IV), then all Hodge cycles on \( A \) are generated by the Hodge cycles on \( A' \) and \( A'' \).

2. When \( A \) is isogenous to \( \prod_j A_j^{n_j} \), where the \( A_j \) are simple and mutually non-isogenous, then all Hodge cycles on \( A \) are generated by the Hodge cycles on the \( A_j \).
8. Exceptional Hodge cycles

Thus far we have looked mainly at examples and conditions under which the Hodge \((p,p)\) conjecture is true. Now we consider the known examples of Hodge cycles that are not known to be algebraic, and thus might be considered potential counterexamples to the conjecture.

8.1. Definition. By an exceptional Hodge cycle on \(A\) we mean an element of \(\mathcal{H}^p(A) = H^{2p}(A,\mathbb{Q}) \cap H^{p,p}(A)\), for some \(p\), which is not in \(\mathcal{D}^p(A)\), that is to say, which cannot be written as a \(\mathbb{Q}\)-linear combination of classes of \(p\)-fold intersections of divisors.

8.2. Mumford’s CM fourfold. Perhaps the first example of an abelian variety where \(\mathcal{H}(A) \neq \mathcal{D}(A)\) was Mumford’s example of the abelian fourfold with complex multiplication corresponding to a particular CM-type (see 1.13.6) for the splitting field of \((3X^4 - 6X^2 + X + 1)(X^2 + 1)\) [B.88]. This example is described in Lecture 7, 7.23–7.28.

8.3. Abelian varieties of Weil type. It was Weil’s observation, however, that the crucial feature of Mumford’s example was not that it was of CM-type, but rather that there was an imaginary quadratic field \(F\) acting on \(A\) in such a way that \(\text{Lie}(A)\) becomes a free \(K \otimes \mathbb{C}\)-module, or equivalently, such that \(\alpha \in K\) acts as \(\alpha\) and as \(\bar{\alpha}\) with equal multiplicity [B.135]. Moreover, as we saw in Theorem 4.11, the general such abelian variety, what we now refer to as an abelian variety of Weil type, has a 2-dimensional space of exceptional Weil-Hodge cycles in \(\mathcal{H}^n(A)\), where \(\dim A = 2n\). In Theorem 4.12 we recalled Schoen’s examples of general abelian fourfolds with \(K = \mathbb{Q}(i)\) or \(K = \mathbb{Q}(\sqrt{-3})\) where he showed that the Weil-Hodge cycles are algebraic [B.104], and little else is known.

8.4. Abelian varieties of Fermat type. Shioda’s work on abelian varieties of Fermat type, see 4.1–4.8 above, provides examples of abelian varieties \(A\) where, at least for some \(p\), the space of Hodge cycles \(\mathcal{H}^p(A) \supsetneq \mathcal{D}^p(A)\) but is nonetheless generated by classes of algebraic cycles [B.116], see Theorems 4.4 and 4.5. In the same work he also shows the existence, for any \(d \geq 2\), of an abelian variety \(A\) of Fermat type whose Hodge ring \(\mathcal{H}^*(A)\) is not generated by \(\sum_{r=1}^{d-1} \mathcal{H}^r(A)\), let alone by \(\mathcal{H}^1(A)\).

8.5. Exceptional cycles in codimension 2. In [B.124] Tankeev produced a family of abelian varieties of dimension \(4^m\) with exceptional cycles in codimension 2 when \(m \geq 2\).

**Theorem ([B.124] Thm.5.6).** For any \(m \geq 1\) there exist abelian varieties \(A\) such that

1. \(\dim A = 4^m\), and
2. \(\text{End}^0(A) = \mathbb{Q}\), and
3. \(\mathfrak{g}_C(A) \simeq (\mathfrak{s}_2 \mathbb{C})^{2m+1}\), acting on \(H^1(A,\mathbb{C}) \simeq (\mathbb{C}^2)^{\otimes (2m+1)}\) as the tensor product of a standard representation of each factor.

Moreover, for any abelian variety satisfying these conditions, \(\dim_{\mathbb{Q}} \mathcal{H}^2(A) = (4^m - 1)/3\). In particular, if \(m \geq 2\) then \(\mathcal{H}^2(A)\) is not generated by classes of intersections of divisors.
The existence part of this theorem is obtained by generalizing Mumford’s example in [B.78] (not the example in [B.88] mentioned in 8.2 above) of an abelian fourfold $A$ with $\text{End}^0(A) = \mathbb{Q}$ and thus not characterized by its endomorphism ring. The computation of $\dim_\mathbb{Q} \mathcal{H}^2(A)$ is proved by induction, and a computation with the roots of $\mathfrak{h}_c(A)$ and the character of its representation.

8.6. Abelian varieties of type (III). In the same paper where he proved that $\mathcal{H}(A) = \mathcal{D}(A)$ for all $k \geq 1$ if and only if $A$ has no factor of type (III) and $\text{Hg}(A) = \text{Lf}(A)$ [B.82], Murty also proved the existence of an exceptional Hodge cycle on abelian varieties of type (III).

**Theorem ([B.82]).** If an abelian variety $A$ has a factor of type (III), then it supports an exceptional Hodge class $\omega$ with the property that $\pi_1^*(\omega) \otimes \pi_2^*(\omega) \in \mathcal{D}(A^2)$, where $\pi_1$ and $\pi_2$ are the projections from $A^2 = A \times A$ to its first and second factors respectively.

**Remark.** In [B.135], where he presented abelian varieties of Weil type as a place to look for counterexamples of the Hodge conjecture, Weil also asked whether a weaker statement might be true, that is, whether the presence of a Hodge cycle on $A$ might imply the presence of an algebraic cycle on some power of $A$. This result of Murty is the first example where the Hodge conjecture itself is not known to be true, but Weil’s question is answered affirmatively.

To get a flavor of the proof, suppose $A$ is simple and of type (III), and let $F$ be the center of $\text{End}^0(A)$, let $m = \dim_{\text{End}^0(A)} W$, and let $d = (\dim A)/[F : \mathbb{Q}]$. Then $d = 2m$, and by [B.109] Prop.15, $m \geq 2$. Then as a consequence of his analysis of $\text{Lf}(A)$, Murty shows that

$$(\wedge^* W^\vee)^{\text{Lf}(A)} \otimes_\mathbb{Q} \mathbb{R} = \bigotimes_{\sigma \in \text{Hom}(F, \mathbb{R})} (\wedge^* X^\vee)^{\text{Lf}(A), \sigma}$$

with $\dim_\mathbb{R} X_\sigma = 4m$. Then $X_\sigma \otimes \mathbb{C}$ becomes isomorphic to two copies of a standard representation of $\text{SO}(V, \psi)$ for a suitable $V$ and $\psi$. Then by [B.137] p.53 the covariant tensors of $\text{SO}(V, \psi)$ are generated by $\psi$ and the determinant, say $\Delta$. Then $\Delta$ cannot be written as a polynomial in the degree 2 invariant $\psi$, but $\Delta^2$ can. Take $\omega$ to be the class corresponding to $\Delta$.

8.7. Determinant cycles. In [B.57] Ichikawa uses the idea of Murty’s construction to develop a certain extension of the work of Tankeev and Ribet on simple abelian varieties [B.124] [B.126] [B.94], see section 6. Firstly he observes that on any abelian variety of type (I), (II) or (III) there exist Hodge cycles that are $\mathbb{C}$-linear combinations of the determinant forms on the spaces $V$ as in the last paragraph. He calls these determinant Hodge cycles. In this language, Murty’s result above is that on an abelian variety of type (III) no determinant cycle is generated by classes of divisors. Then Ichikawa proves the following result. Recall the definition of relative dimension from 7.9.

**Theorem ([B.57]).** Let $A$ be an abelian variety whose simple factors are all of odd relative dimension, and suppose $A$ is isogenous to $A' \times A''$ where each simple
factor of $A'$ is of type (I), (II) or (III) and each simple factor of $A''$ is of type (IV). Further, assume that the relative dimension of any simple factor of $A$ of type (III) is not equal to $\frac{1}{2} \binom{2k}{k}$ for any power $k$ of 2, and that $A''$ is a power of a simple abelian variety of odd prime dimension. Then any Hodge cycle on $A$ is generated by classes of divisors and determinant Hodge cycles.

8.8. Definition ([B.47]). Recall (definition 7.6) that a stably nondegenerate abelian variety is one which satisfies the conditions of Theorem 7.5, in particular, $\mathcal{H}(A^k) = \mathcal{D}(A^k)$ for all $k \geq 1$. Then a stably degenerate abelian variety $A$ is one which is not stably nondegenerate, that is, $\mathcal{H}^p(A^n) \nsubseteq \mathcal{D}^p(A^n)$ for some $p, n$. Then the least $n$ for which this occurs is called the index of degeneracy, which we will denote by $\text{ind}(A)$.

8.9. Stably degenerate abelian varieties. Hazama has given two examples of stably degenerate abelian varieties of type (I) having index of degeneracy 2.

8.9.1. Theorem ([B.47]). There exists a simple abelian variety $A$ of dimension 4 with the following properties:
(a) $A$ is of type (I),
(b) $\mathcal{H}(A) = \mathcal{D}(A)$,
(c) $\mathcal{H}(A^2) \nsubseteq \mathcal{D}(A^2)$.

To give a rough idea of the construction, let $B$ be a totally real number field of degree 3, let $B$ be a quaternion algebra over $K$ such that $B \otimes \mathbb{R} \cong M_2(\mathbb{R}) \times \mathbb{H} \times \mathbb{H}$, and let $G = \text{Res}_{K/Q} \text{SL}(1, B)$. Then $G(\mathbb{R}) \cong \text{SL}(2, \mathbb{R}) \times \text{SU}(2) \times \text{SU}(2)$, and there exists an 8-dimensional $\mathbb{Q}$-rational symplectic representation $\rho : G \rightarrow \text{Sp}(8)$ satisfying the necessary analyticity conditions so that the induced map $\tau : X \rightarrow \mathcal{H}_{\mathcal{A}}$ of Hermitian symmetric domains pulls back the universal family $\mathcal{A} \rightarrow \mathcal{H}_{\mathcal{A}}$ of polarized abelian fourfolds over the Siegel upper half-space to an analytic family of abelian fourfolds $A \rightarrow X$ [B.64]. Moreover, if $A_0$ denotes a generic member of the family $A \rightarrow X$, then $\mathcal{H}^p(A_0^k) = H^{2p}(A_0, \mathbb{Q})^G$ for all $k \geq 1$. Then computations with the complexified Lie algebra $\text{Lie}(G, \mathbb{C}) \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ show firstly that $\dim \mathcal{H}^1(A_0) = 1$, from which follows that $A_0$ is simple, of type (I), and $\mathcal{H}(A) = \mathcal{D}(A)$, and secondly that $\mathcal{H}^2(A^2)$ is not generated by the elements of $\mathcal{H}^1(A^2)$.

8.9.2. Theorem ([B.48]). There exists a simple abelian variety $A$ of dimension 10 with the following properties:
(a) $\mathfrak{h}(A, \mathbb{C}) \cong \mathfrak{sl}(6, \mathbb{C})$,
(b) the representation $\mathfrak{h}(A, \mathbb{C}) \rightarrow \text{End}(H^1(A, \mathbb{C}))$ is equivalent to the representation $\mathfrak{sl}(6, \mathbb{C}) \rightarrow \text{End}(\wedge^3 \mathbb{C}^6)$ induced by the natural action of $\mathfrak{sl}(6, \mathbb{C})$ on $\wedge^3 \mathbb{C}^6$,
(c) $\mathcal{H}(A) = \mathcal{D}(A)$,
(d) $\mathcal{H}(A^2) \nsubseteq \mathcal{D}(A^2)$.

The existence is worked out similarly as in the previous case, except that here $G$ is a $\mathbb{Q}$-form of $\text{SU}(5, 1)$. Again $\dim \mathcal{H}^1(A) = 1$, and $\text{End}^0(A) = \mathbb{Q}$. However, the actual computations are based on using Young diagrams and branching rules, see [B.48] for the details.
8. Exceptional Hodge Cycles

Remark. In [B.48] Hazama constructs a family $A_n$ of simple abelian varieties of dimension $\frac{3}{2}(4n^2+2)$, with $\text{End}^0(A_n) = \mathbb{Q}$, and $\mathfrak{h}(A_n, \mathbb{C}) \simeq \mathfrak{sl}(4n+2, \mathbb{C})$ and $H^1(A_n, \mathbb{C}) \simeq \bigwedge^{2n+1}C^{4n+2}$ as a representation of $\mathfrak{h}(A_n, \mathbb{C})$. The abelian variety of Theorem 8.9.2 is the $A_1$ in this family. Then he also shows that the index of degeneracy $\text{ind}(A_n) \leq 2$ for $n \geq 2$, where the theorem shows that $\text{ind}(A_1) = 2$.

8.10. Invariants of partially indefinite quaternion algebras. In the 1960’s Kuga asked which semisimple algebraic groups $G$ defined over $\mathbb{Q}$ together with which of their symplectic representations $\rho : G \to \text{Sp}(W, \beta)$ satisfy the necessary and sufficient analyticity conditions to allow the construction of an algebraic family of polarized abelian varieties parameterized by $\Gamma \setminus X$, where $\Gamma$ is a discrete subgroup of $G$ and $X$ is the Hermitian symmetric domain associated to $G$ [B.61] [B.62] [B.63]. Shortly thereafter Satake answered Kuga’s question under the assumption that for each $\mathbb{Q}$-simple factor $\rho$ comes from an absolutely irreducible representation of an absolutely simple factor of $G$ [B.99] [B.100] [B.102] [B.103]. It turned out that the list was quite small, and nearly all cases had been considered by Shimura in his analysis of families of abelian varieties characterized by polarization, endomorphism ring and level structure [B.109] [B.110] [B.112]; one more case was treated in [B.101]. Some time later Addington considered Kuga’s question without Satake’s assumption, and for the groups corresponding to units of norm 1 in a partially indefinite quaternion algebras $B$ over a totally real field $F$, i.e.,

$$B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})^n \oplus \mathbb{H}^m$$

and $G = \text{Res}_{F/\mathbb{Q}} \text{SL}(1, B)$,

she developed a combinatorial scheme (called “chemistry”) to describe which symplectic representations give rise to an algebraic family of abelian varieties [B.5]. Then Tjioj [B.131] and Abdulali [B.1] [B.2] [B.3] showed that under certain reasonable hypotheses (“rigidity” or “condition (H2)”) the space of Hodge cycles in a generic fiber $A_0$ of the family is the space of $G$-invariants, $H^{2r}(A_0, \mathbb{Q})^G = (\bigwedge^{2r} W^\vee)^G$. Thus, for the purposes of this appendix, where the issue is Hodge cycles on abelian varieties, statements about Hodge cycles on the generic fiber of such a family can be understood as statements about Hodge cycles on an abelian variety $A_0$ with specified semisimple Hodge group $G$. Then the problem is to describe the invariants of $G$ in the exterior algebra $\bigwedge W^\vee$.

This is the problem taken up by Kuga in [B.64], [B.65] and the series of papers [B.66] [B.67], and Lee in [B.71]. The results are rather involved to state precisely. In [B.64] Kuga looks at some simple examples of the situation just described, and finds conditions (“totally disconnected triangular polymer”) where all the Hodge cycles in the abelian variety $A_0$ are generated by those of degree two or where only the Hodge cycles in codimension 2 or 4 are generated by those of degree two (“triangular polymer without double bond, short cycle or Hexatramm”), which is to say that there are exceptional cycles in higher codimensions. At the end of [B.65] is an example of a 16-dimensional abelian variety $A$ for which the dimensions of the spaces $\mathcal{H}^r(A)$ of Hodge cycles are determined, where $\dim \mathcal{H}^2(A) = 82$ and $\dim \mathcal{H}^2(A) = 10$. In [B.66] and [B.67] the focus is more on the complicated invariant theory in the exterior algebra for the groups and representations under consideration, in particular the latter papers examine the asymptotic behavior of
The problem of complex multiplication

In this section we look at what is known about the Hodge \((p,p)\) conjecture for abelian varieties with complex multiplication. We have already seen that the Hodge group of such an abelian variety is an algebraic torus, necessarily contained in \(\text{Res}_{K/Q} \mathbb{G}_m\), where \(K\) is the field of complex multiplication (Proposition 2.12). For the general theory of complex multiplication, see [B.115] [B.70] and parts of [B.112].

9.1. Definition. Recall (1.13.6) that a CM-type \((K,S)\) consists of a CM-field \(K\) and a subset \(S \subset \text{Hom}(K,\mathbb{C})\) containing exactly one from each pair of conjugate embeddings. Moreover, given a CM-type \((K,S)\), there is a natural construction of an abelian variety \(A\) with that CM-type, that is, with \(\text{End}^0(A) = K\) and \(K\) acting on \(H^{1,0}(A)\) as \(\bigoplus_{\varphi \in S} \varphi\). Then we define the rank of the CM-type \((K,S)\) by

\[
\text{rank}(K,S) := \dim \text{MT}(A).
\]

Remark. The rank of a CM-type seems to have originally been defined by Kubota [B.60], who defined it as \(\dim \{\sum_{\varphi \in S} \varphi(x) : x \in K\}\). The equality of this with the dimension of the Mumford-Tate group follows from the methods in [B.91], see also [B.26] [B.27].

9.2. Pohlmann’s criterion. One of the first results about Hodge cycles on abelian varieties with complex multiplication is a theorem of Pohlmann [B.88] that describes \(\mathcal{H}(A)\) in terms of the Galois theory of \(K\). To state the theorem we need some notation. Let \(A\) be an abelian variety with CM-type \((K,S)\), let \(S = \{\varphi_1, \ldots, \varphi_g\}\) and let \(\overline{S} = \{\overline{\varphi}_1, \ldots, \overline{\varphi}_g\}\), so \(\text{Hom}_Q(K,\mathbb{C}) = S \cup \overline{S}\). Then \(\alpha \mapsto (\varphi_1(\alpha), \ldots, \varphi_g(\alpha))\), for \(\alpha \in K\), induces an isomorphism of \(K\) onto \(H_1(A,\mathbb{Q})\) (in 1.13.6 we mapped \(\mathbb{O}_K\) onto \(H_1(A,\mathbb{Z})\)), via which \(H^1(A,\mathbb{C})\) can be identified with \(\text{Hom}_Q(K,\mathbb{C})\). Further, without loss of generality we may assume \(K \subset \mathbb{C}\), and let \(L\) be the Galois closure of \(K\) in \(\mathbb{C}\) and \(G = \text{Gal}(L/\mathbb{Q})\). Then we let \(\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})\) act on \(f \in H^r(A,\mathbb{C})\), with \(f : \wedge^r K \to \mathbb{C}\), by \((\sigma f)(\lambda) = \sigma(f(\lambda))\) for \(\lambda \in \wedge^r K\). Finally, for an ordered subset \(\Delta \subset S\), let \(|\Delta|\) denote the cardinality of \(\Delta\) and let \(|\Delta| := \wedge_{\varphi \in S} \varphi\).

Theorem ([B.88] Thm.1). When \(A\) is an abelian variety with CM-type \((K,S)\), then \(\mathcal{H}(A) \otimes \mathbb{C}\) has a basis consisting of those \(|\Delta| \in H^{2p}(A,\mathbb{C})\) such that

\[
|\tau \Delta \cap S| = |\tau \Delta \cap \overline{S}|
\]

for every \(\tau \in G\). Thus \(\dim \mathcal{H}(A)\) is the number of ordered subsets \(\Delta \subset S\), with \(|\Delta| = 2p\), that satisfy the condition (9.2.1).
Proof. If \( f = \sum_i c_i \langle \Delta_i \rangle \in \mathcal{H}^p(A) \) with \( c_i \in \mathbb{C} \), then \( \sum_i \sigma(c_i)(\sigma \Delta_i) = \sigma f = f \) is in \( H^{p,p}(A) \) for all \( \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}) \), hence \( \Delta_i \) satisfies (9.2.1). Then every element of \( \mathcal{H}^p(A) \) is a linear combination of \( \Delta_i \) satisfying (9.2.1). Conversely, let \( \Delta \) be such that \( |\Delta| = 2p \) and (9.2.1) is satisfied. Let \( \{u_1, \ldots, u_s\} \) be a basis for \( L \) over \( \mathbb{Q} \), and let \( f_i = \sum_{\tau \in G} \tau(u_i)(\tau \Delta) \) for \( 1 \leq i \leq s \). Then \( \sigma f_i = f_i \) for all \( \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}) \), and by (9.2.1) \( f_i \in H^{p,p}(A) \), so \( f_i \in \mathcal{H}^p(A) \). Further, since \( \text{det}(\tau(u_i))_{\tau \neq i} \neq 0 \), we can solve the system of linear equations \( f_i = \sum_{\tau \in G} \tau(u_i)(\tau \Delta) \) and find that \( \langle \tau \Delta \rangle \in \mathcal{H}^p(A) \otimes \mathbb{C} \) for \( \tau \in G \). Thus \( \langle \Delta \rangle \in \mathcal{H}^p(A) \otimes \mathbb{C} \), as required. \( \square \)

Pohlmann’s theorem give a criterion for exceptional cycles, also see 9.3 below.

9.2.2. Corollary ([B.138]). \( \dim \mathcal{H}^p(A) - \dim \mathcal{D}^p(A) \) is the number of subsets \( \Delta \subset \text{Hom}(K, \mathbb{C}) \) such that
(a) \( \Delta - \overline{\Delta} \neq \emptyset \),
(b) \( \Delta \cap gS = p \) for all \( g \in G \).

9.3. Sporadic cycles. In [B.138] White observes that Pohlmann’s criterion shows that when a CM abelian variety \( A \) is nondegenerate, as defined in 2.13, then \( \mathcal{H}(A) = \mathcal{D}(A) \). He then recounts that between 1977 and 1978 Ribet asked if these two conditions were equivalent, and that Lenstra was quickly able to show that they are, for a simple abelian variety of simple CM-type \((K,S)\), under the additional hypothesis that the CM-field \( K \) is abelian over \( \mathbb{Q} \) (see [B.138] Thm.3).

Then later Hazama showed that a simple abelian variety is nondegenerate if and only if \( \mathcal{H}(A^k) = \mathcal{D}(A^k) \) for all \( k \geq 1 \), see Theorem 6.4 [B.45] and Theorem 7.5 [B.47]. Only recently, however, White showed the following.

Theorem ([B.138] Thm.1). There exists an abelian variety of CM-type with \( \mathcal{H}(A) = \mathcal{D}(A) \) and \( \dim \text{Hg}(A) \leq \dim A \).

The argument involves a rather technical analysis of the \( \mathbb{Q} \)-group ring of a non-abelian group. Eventually, however, the counterexample is a CM-type for a CM-field whose Galois group is
\[ Z/2Z \times Z/2Z \times Z/5Z \times D_5, \]
where the last factor is the dihedral group and the first factor corresponds to complex conjugation. The splitting field of
\[ X^5 - 10X^4 - 70X^3 - 25X^2 + 190X + 12 \]
is a totally real field with Galois group \( D_5 \), and it is easy to make disjoint totally real quadratic and quintic extensions, and then a totally imaginary quadratic extension. For the abelian variety \( A \) with the requisite CM-type for this field, \( \dim \text{Hg}(A) = 84 < 100 = \dim A \).

9.4. Degenerate CM types. Recall from definitions 2.13 and 9.1 that a CM-type \((K,S)\) is said to be nondegenerate if \( \text{rank}(K,S) = \dim A + 1 \), and is called degenerate otherwise, if \( \text{rank}(K,S) \leq \dim A \), where \( A \) an abelian variety CM-type \((K,S)\). A number of examples of degenerate CM-types and lower bounds for the
rank as a function of \( \dim A \) have been given by Ribet [B.92], Dodson [B.28] [B.29] [B.30], Mai [B.72] and Yanai [B.140].

The following proposition of Kubota can be a useful way of measuring the rank of a CM-type. Let \( c \) denote complex conjugation.

**9.4.1. Proposition ([B.60]).**

\[
\text{rank}(K, S) = 1 + \# \{ \chi : \text{Gal}(K/\mathbb{Q}) \to \mathbb{C} : \chi(c) = -1 \ & \sum_{s \in S} \chi(s) \neq 0 \},
\]

where only irreducible \( \chi \) are included.

**9.4.2. Examples ([B.92]).** First, let \( p \geq 5 \) be a prime, let \( K = \mathbb{Q}(\zeta_p) \) be the field of \( p^{\text{th}} \) roots of unity, and identify \( G = \text{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^\times \). For \( g \in G \) let \( (g) \equiv g \pmod{p} \) with \( 1 \leq (g) \leq p - 1 \). Then for \( 1 \leq a \leq p - 2 \) with \( a^3 \not\equiv 1 \pmod{p} \) the set

\[
S_a = \{ g \in G : (g) + (ag) < p \}
\]

is a simple CM-type. It is nondegenerate when \( a = 1 \), but is degenerate for \( p = 67 \) and \( a = 10, 19, 47, 56, 60 \) [B.40]. Lenstra and Stark also noticed that for \( p \equiv 7 \pmod{12} \) and sufficiently large there always exists a number \( a \) for which \( S_a \) is degenerate, loc. cit.

Next let \( K = \mathbb{Q}(\zeta_{32}) \) and, identifying \( \text{Gal}(K/\mathbb{Q}) \) with \( (\mathbb{Z}/32\mathbb{Z})^\times \), let

\[
S = \{ 1, 7, 13, 21, 23, 27, 29 \}, \quad S' = \{ 1, 7, 9, 11, 13, 15, 27, 29 \}.
\]

Then \((K, S)\) and \((K, S')\) are both degenerate (simple) CM-types. This example is due to Lenstra.

Let \( K = \mathbb{Q}(\zeta_{19}) \) and, identifying \( \text{Gal}(K/\mathbb{Q}) \) with \( (\mathbb{Z}/19\mathbb{Z})^\times \), let

\[
S = \{ 1, 3, 4, 5, 6, 7, 8, 10, 17 \}.
\]

Then again \((K, S)\) is a degenerate CM-type. This example is due to Serre.

Finally, let \( p, q, r \) be odd primes, let \( G = \mathbb{Z}/2pqr\mathbb{Z} \) as cyclic group, and let \( K \) be an extension of \( \mathbb{Q} \) with \( \text{Gal}(K/\mathbb{Q}) \simeq G \). Then let \( S \) be the subset of elements having order 1, \( pqr, 2p, 2q, 2r, 2pq, 2pr \) or \( 2qr \). Then \((K, S)\) is a simple CM-type and

\[
\text{rank}(K, S) = 1 + pqr - (p - 1)(q - 1)(r - 1).
\]

This example is due to Lenstra.

All of these examples are verified in [B.92] using Proposition 9.4.1 and exhibiting odd characters \( \chi \) such that \( \sum_{s \in S} \chi(s) = 0 \).

**9.4.3. Degenerate CM-types in composite dimension.** In Theorem 6.3.2 we saw that an abelian variety of CM-type with prime dimension is always non-degenerate. Dodson has proved a number of theorems exhibiting the existence of degenerate abelian varieties in composite, i.e., non-prime, dimensions [B.28] [B.29] [B.30], and more recently Yanai has come up with a method for generating degenerate CM-types that encompasses some of the previous examples [B.140].
Theorem ([B.28] Thm.3.2.1). When $n$ is composite, $n > 4$, then there exist abelian varieties of CM-type with $\dim A = n$ and rank $n - l + 2$ for a divisor $l \geq 2$ of $n$ such that $n/l > 2$.

Theorem ([B.28] Thm.3.2.2). Let $p$ be a prime and suppose the ideal $(2)$ decomposes in the cyclotomic field $\mathbb{Q}(\zeta_p)$ into $g > 1$ factors of degree $f$. Note that $p \equiv \pm 1 \pmod{8}$ is sufficient but not necessary to insure $g > 1$. Then there exist abelian varieties of CM-type having dimension $2fs$ and rank $p - 1$, for $0 \leq s \leq g - 1$. In particular, for $s > 0$ these abelian varieties are simple and degenerate.

In the following $K_0^{\text{Gal}}$ denotes the Galois closure of the field $K_0$, and $[(\mathbb{Z}/2\mathbb{Z})^m]^+$ is the even subgroup of $(\mathbb{Z}/2\mathbb{Z})^m$.

Theorem ([B.28] Thm.3.3.1). (A) Suppose there exists a totally real field $K_0^{\text{Gal}}$ with Galois group isomorphic to the wreath product $(\mathbb{Z}/k\mathbb{Z}) \wr (\mathbb{Z}/l\mathbb{Z})$, with $k > 2$ and $l \neq 1$. Then there exist simple degenerate abelian varieties with complex multiplication by a non-Galois CM-field such that the varieties have dimension $kl$ and rank $n - l + 2$.

(B) Under the same hypotheses except that $k = 2$ is allowed, there exist simple abelian varieties of CM-type with dimension $n' = k^2l$ having rank $\leq kl + 1$.

(C) Further, in the even case, the existence of totally real fields with Galois groups $[(\mathbb{Z}/2\mathbb{Z})^m]^+ \times \mathbb{Z}/m\mathbb{Z}$ with $m \geq 3$, respectively $D_m$ with $m > 5$ and odd, supplies simple abelian varieties of CM-type with dimension $n = 4m$, respectively $n = 2m$, and rank $\leq \frac{n}{2} + 1$.

Theorem ([B.29] Thm.3.1). Let $d$ be a composite number. Then there exist simple abelian varieties of CM-type with dimension $d$ and rank $\leq (d/2) + 1$ whenever $d$ is

1. even;
2. divisible by a square;
3. of the form $d = \binom{p^m}{m}$ with $0 \leq m \leq (p - 1)/2$; or
4. of the form $d = pt$ with $t \mid (p - 1)$ and $t < p - 1$.

Further, the rank $p + 1$ occurs in dimension $d$ at least in the following cases:

1. when $d$ and $p$ are as in 3 or 4 above;
2. when $d = 2^{(p-1)/2}t$ with $t \mid (p - 1)/2$;
3. when $d = pq^2$ where $q$ is an odd prime and $p = q^2 + q + 1$.

The following is a nondegeneracy result.

Theorem ([B.30] Thm.2.1). Let $n$ be odd and suppose that $K$ is a CM-field of degree $2n$ such that the maximal totally real subfield $K_0$ has $\text{Gal}(K_0^{\text{Gal}}/\mathbb{Q})$ isomorphic to the symmetric group or the alternating group on $n$ letters. Then every primitive CM-type $(K, S)$ is nondegenerate.

Recently Yanai has developed a method for generating degenerate CM-types in higher dimension starting with degenerate CM-types in lower dimension.
Theorem ([B.140]). Let \( K \) be a CM-field with \( [K : \mathbb{Q}] = 2d \) and let \( K_1 \) be a proper subfield of \( K \) with \( [K_1 : \mathbb{Q}] = 2d_1 \). Further, let \( \pi : X_K \to X_{K_1} \) be the canonical surjection from the character group of \( \text{Res}_{K/\mathbb{Q}} \mathbb{G}_m/K \) to the character group of \( \text{Res}_{K_1/\mathbb{Q}} \mathbb{G}_m/K_1 \). Suppose that the CM-types \( (K, S) \) and \( K_1, S_1, \) satisfy the condition

\[
\pi\left( \sum_{\sigma \in S} \sigma \right) = a \sum_{\sigma \in S_1} \sigma + b \sum_{\sigma \in S_1} \bar{\sigma}
\]

with some nonnegative integers \( a \) and \( b \) such that \( a + b = [K : K_1] \). Then

\[
d + 1 - \text{rank} S \geq d_1 + 1 - \text{rank} S_1.
\]

Moreover, if \( a = b \) then

\[
d + 1 - \text{rank} S \geq d_1.
\]

In particular, if the CM-type \( (K_1, S_1) \) is degenerate or if \( a = b \) then the CM-type \( (K, S) \) is degenerate.

9.4.4. Lower bounds for CM-types. Ribet [B.92] and Mai [B.72] have given some lower bounds for the rank of a CM-type, and particularly the latter discusses how sharp these might be.

Proposition ([B.92]). \( \text{rank}(K, S) \geq 2 + \log_2(\dim A) \)

Mai considers the case where the CM-filed \( K \) is Galois over \( \mathbb{Q} \). For the next proposition, note that when \( V = \mathbb{Z}[\text{Gal}(K/\mathbb{Q})] \otimes \mathbb{C} \) is considered as a Gal\((K/\mathbb{Q})\)-module, there is a decompositon \( V = \bigoplus \pi d_\pi V_\pi \), where \( \pi \) ranges over the irreducible representations of \( \text{Gal}(K/\mathbb{Q}) \) and \( d_\pi = \dim V_\pi \). A representation \( \pi \) is called odd if the value of its character at complex conjugation is \(-1\).

Proposition ([B.72] Prop.1). When \( K/\mathbb{Q} \) is a Galois extension and \( (K, S) \) is a simple CM-type, then

\[
\text{rank}(K, S) \geq 1 + \sum d_\pi,
\]

where the sum ranges only over those odd irreducible representations \( \pi \) such that \( \pi(\sum_{s \in S} s) \neq 0 \).

Proposition ([B.72] Prop.2). When \( K/\mathbb{Q} \) is a Galois extension and \( (K, S) \) is a simple CM-type, then

\[
\text{rank}(K, S) \leq \max \left\{ \frac{(p - 1)^2 \alpha}{p} : p \text{ an odd prime, and } p^\alpha \mid ([K : \mathbb{Q}]^2) \right\}.
\]

The notation \( p^\alpha \mid N \) means \( p^\alpha \) exactly divides \( N \), i.e., \( p^\alpha \mid N \) and \( p^{(\alpha+1)} \nmid N \).

In the following \( S_a \) is the same CM-type that occurred the first paragraph of 9.4.2. Such CM-types occur among factors of the Jacobians of Fermat curves.
Proposition ([B.72] Prop.3). Let $K = \mathbb{Q} (\zeta_p)$ and identify $\text{Gal}(K/\mathbb{Q})$ with $(\mathbb{Z}/p\mathbb{Z})^\times$. For $1 \leq a \leq p - 2$ let $S_a$ be the CM-type defined by
\[
S_a = \{ g \in \text{Gal}(K/\mathbb{Q}) : 1 \leq \langle g \rangle + \langle ag \rangle < p \},
\]
where $1 \leq \langle g \rangle \leq p - 1$ and $\langle g \rangle \equiv g \pmod{p}$. Then
\[
\text{rank}(K, S) \geq 1 + \frac{19}{21}d, \quad d = \frac{p - 1}{2}.
\]

9.5. André’s description of CM-Hodge cycles as Weil cycles. Finally, we turn to a recent result of André [B.9] that every Hodge cycle on an abelian variety $A$ of CM-type is a linear combination of inverse images under morphisms $A \to B_J$ of Weil-Hodge cycles on various abelian varieties $B_J$ of CM-type. The following definition should be compared with 1.13.6, 4.10, and the discussion preceding Theorem 4.9.

9.5.1. Definition. Let $A$ be an abelian variety, let $F$ be a CM-field contained in $\text{End}^0(A)$, and let $V = H^1(A, \mathbb{Q})$. Then $A$ or $V$ is said to be of Weil type relative to $F$ if there exists an $F$-Hermitian form $\psi$ on $V$ admitting a totally isotropic subspace whose dimension over $F$ is $\frac{1}{2} \text{dim}_F V$, and there exists a purely imaginary element $\alpha \in F$ such that $\text{Tr}_{F/\mathbb{Q}}(\alpha \cdot \psi(u, v))$ defines a polarization on $A$. Then the elements of $\bigwedge_{2p} V$ are called Weil-Hodge cycles relative to $F$.

9.5.2. Theorem ([B.9]). Let $A$ be an abelian variety of CM-type, and $p$ a positive integer. Then there exists a CM-field $F$, a finite number abelian varieties $A_J$ with complex multiplication of Weil type relative to $F$, and morphisms $A \to A_J$, such that every Hodge cycle $\xi \in \mathcal{H}^0(A)$ is a sum of inverse images of Weil-Hodge cycles $\xi_J \in \mathcal{H}^0(A_J)$.

Proof. We sketch André’s proof. Up to isogeny write $A = \prod_i A_i$ as a product of simple CM-abelian varieties, where $A_i$ is of CM-type $(K_i, S_i)$. Let $V = H^1(A, \mathbb{Q})$, let $V_{S_i} = H^1(A_i, \mathbb{Q})$, and let $F$ be the Galois closure of the compositum of all the $\text{End}^0(A_i)$. Then $V = \bigoplus_{i \in I} V_{S_i}$ and $V_{S_i} \otimes F = \bigoplus_{\sigma \in \text{Hom}(F, \mathbb{C})} V_{S_i, \sigma}$. Then
\[
(\bigwedge_{2p} \otimes F \simeq \sum_{d_i = 2p} \left( \bigotimes_{i \in I} V_{S_i}^{d_i} \right) \otimes F \\
\simeq \sum_{\sum d_i \sigma = 2p, (i, \sigma) \in I \times \text{Hom}(F, \mathbb{C})} \bigotimes_{d_i, \sigma \in \{0, 1\}} V_{S_i, \sigma}^{d_i, \sigma}.
\]
Let $T_F = \text{Res}_{F/\mathbb{Q}} \mathfrak{g}_{\text{in}/F}$. Then the action of $(T_F)^I$ on $\bigwedge_{2p} V$ commutes with the action of $\text{Hg}(V)$. Further, the action of $T_F$ can be extended by $F$-linearity to an action on $\bigotimes V_{S_i, \sigma}^{d_i, \sigma}$. It follows that every Hodge cycle $\xi \in \bigwedge_{2p} V$ can be written as $\xi = \sum \lambda_J \theta_J$, where $J$ indexes the set of sequences $(d_i, \sigma)_{(i, \sigma) \in I \times \text{Hom}(F, \mathbb{C})}$ with $d_i, \sigma \in \{0, 1\}$ and $\sum d_i, \sigma = 2p$, and where $\lambda_J \in F$, and $\theta_J \in \bigotimes_{(d_i, \sigma) \in J} V_{S_i, \sigma}^{d_i, \sigma}$, and the restriction of $\text{Hg}(V)$ acting on $F$ fixes $\theta_J$. 
Now observe that each $\tau \in \text{Aut}(F)$ induces an isomorphism of rational Hodge structures $V_{S_i,\sigma} \to V_{\tau S_i,\sigma \tau}$, although this isomorphism does not respect the action of $T_F$. Therefore we may write $V_J = \sum_{(d_i,\sigma) \in J} V_{S_i,\sigma}^{d_i,\sigma}$, where $S_i,\sigma = \sigma^{-1} S$, in such a way that via the isomorphisms induced by $\text{Aut}(F)$ we get a morphism of rational Hodge structures of CM-type $V_J \to V$ and $\theta_J$ comes from an element $\zeta_J \in \bigotimes_{j \in J} V_{S_i,\sigma}^{p-\sigma} \subset \Lambda^{2p} F_V$ which is invariant under the action of $\text{Hg}(V_J)$ on $F$.

Note also that there is a natural basis $\chi_{(j,\sigma)}$ of characters of $T_{F,J}$, where $(j,\sigma)$ runs over $J \times \text{Hom}(F,\mathbb{C})$. Let $\gamma_{(j,\sigma)}$ denote the dual basis of cocharacters. Then if $\zeta_J \neq 0$, it generates the character $\sum_{j \in J} j \chi_{(j,\sigma)}$ of $T_{F,J}$.

On the other hand, the Hodge structure of $V_J$ is determined by the cocharacter $h : U(1) \to (T_{F,J})/\mathbb{R}$, whose complexification may be written out as

$$h_\mathbb{C} = \sum_{(j,\sigma) \in J \times \text{Hom}(F,\mathbb{C})} j(2S_j(\sigma) - 1)\gamma_{(j,\sigma)},$$

where $S_j(\sigma)$ is 1 or 0 according as $\sigma \in S_j$ or not.

Then the fact that $\zeta_J$ is $\text{Hg}(V_J)$-invariant implies that $\sum_{j \in J} j \langle \tau h, \chi_{(j,\text{id})} \rangle = 0$ for all $\tau \in \text{Gal}(F/\mathbb{Q})$. Then expanding this expression, we find

$$\sum_{j \in J} j(2S_j(\sigma) - 1) = 0.$$ 

And since $\sum_{j \in J} j = 2p$, it follows that

$$\sum_{j \in J} jS_j(\bullet) = p,$$

which implies that $V_J$ is a rational Hodge structure of Weil type. Moreover,

$$\xi_J := [F : \mathbb{Q}]^{-1} \sum_{\tau \in \text{Gal}(F/\mathbb{Q})} \lambda^T \zeta_J^\tau \in \Lambda^{2p} F_V$$

is a Weil-Hodge cycle, and $\xi$ is a sum of images of $\xi_J$ under the maps $V_J \to V$, since $\xi = [F : \mathbb{Q}]^{-1} \sum_{\tau \in \text{Gal}(F/\mathbb{Q})} \lambda^T \theta_J^\tau$. \hfill $\Box$

10. The general Hodge conjecture

In this section, when we speak of the general Hodge conjecture we always mean the Grothendieck amended version as in (7.12) of the text, that the $r$th step of the arithmetic filtration $F^r\alpha H^i(A, \mathbb{Q})$ is the largest rational Hodge structure contained in $F^r H^i(Z, \mathbb{C}) \cap H^i(A, \mathbb{Q})$, where $F^r H^i(A, \mathbb{C})$ is the Hodge filtration. In those cases where the stronger statement that $F^r\alpha H^i(A, \mathbb{Q}) = F^r H^i(Z, \mathbb{C}) \cap H^i(A, \mathbb{Q})$ we will speak of Hodge’s original conjecture, or the strong form of the general Hodge conjecture. Based on the results assembled below, it would seem that when it is true, this stronger version is more amenable to being proved.
10.1. The general abelian variety. The earliest results about the general Hodge conjecture for abelian varieties are those of Comessatti [B.23] and Mattuck [B.73], which show that Hodge’s original conjecture is true for the general abelian variety described in 1.13.8. Mattuck’s proof proceeds by induction and explicit computation with period matrices. Since the general $g$-dimensional abelian variety $A$ has

$$\text{Hg}(A) = \text{Sp}(H^1(A, \mathbb{Q}), E) = \text{Lf}(A),$$

(see 2.14) and also $\text{End}^0(A) = \mathbb{Q}$, it satisfies the hypotheses of Theorem 6.2. Thus $\mathcal{H}(A^k) = \mathcal{D}(A^k)$ for all $k \geq 1$, which is to say that the general abelian variety is stably nondegenerate, and Theorem 10.9 below applies.

**Abelian varieties of low dimension.**

10.2. Various abelian threefolds. The first interesting case of the general Hodge conjecture is $\text{GHC}(1,3,X)$ for a threefold $X$, and indeed, it was a special abelian threefold, the product of three copies of an elliptic curve whose period satisfies a cubic relation, that Grothendieck exhibited to show that Hodge’s original conjecture needed to be modified, see (7.5) in the text or [B.43]. In [B.12] Bardelli took up the question of whether Grothendieck’s counterexample to Hodge’s original conjecture satisfies Grothendieck’s amended version, and at the same time he considered a number of other abelian threefolds.

10.2.1. **Theorem** ([B.12] Prop.3.8). The Grothendieck generalized Hodge conjecture holds for:

1. The generic abelian threefold;
2. The generic member of the family of Jacobians of smooth genus three curves admitting a morphism onto some elliptic curve;
3. The generic member of the family of Jacobians of smooth genus three curves admitting a morphism onto some genus two curve;
4. The generic product of an abelian surface and an elliptic curve;
5. The generic product of three elliptic curves;
6. All products of three copies of the same elliptic curve, in particular when the period $\tau$ is quadratic over $\mathbb{Q}$ (the CM case) or cubic over $\mathbb{Q}$ (as in Grothendieck’s counterexample).

Bardelli’s arguments are very geometric in nature. To give a hint of their flavor, let $J_H(A)$ denote the maximal subtorus of the intermediate Jacobian of an abelian threefold $A$ that is orthogonal, with respect to cup product, to $H^{3,0}(A)$. Then the key lemma has the following form.

10.2.2. **Lemma** ([B.12] Lem.2.2). Let $T$ be an irreducible analytic subvariety of the Siegel upper half-space of genus three and let $p : A \to T$ be the restriction of the universal family of principally polarized abelian varieties over the Siegel half-space. Let $t_0 \in T$ be a generic point of $T$ at which $T$ is smooth. Then

$$\dim_{\mathbb{C}} J_H(A_{t_0}) \leq 9 - \dim T.$$
Recall (1.13.8) that the Siegel half-space of genus $g$ consists of complex symmetric $g \times g$ matrices with positive-definite imaginary part, and thus represents the possible complex structures for an abelian variety of dimension $g$.

Another example of abelian threefold for which the general Hodge conjecture is true comes up in [B.86]. In the course of constructing a counterexample to a conjecture of Xiao, Pirola finds the following.

10.2.3. Proposition. The general Hodge conjecture is true for the generic member of the family of abelian threefolds of the form $W = \text{Jac}(C)/f^*E$, where $C$ is a smooth genus 4 curve, $E$ is an elliptic curve, and $f : C \to E$ is a 3 to 1 cyclic Galois covering.

10.3. An abelian fourfold of Weil type. In [B.105] Schoen proves the general Hodge conjecture for abelian fourfolds $A$ of Weil type with multiplication by $\text{End}^0(A) = K = \mathbb{Q}(i)$ (and determinant 1, associated to a Hermitian form of signature $(3,1)$). Previously, in [B.104], see Theorem 4.12, he had proved the Hodge $(2,2)$ conjecture for these fourfolds. Thus in [B.105] it only remained to verify GHC(1,4,A). Therefore the focus is on the rational Hodge substructre $U' \subset \bigwedge^4 H^4(A,\mathbb{Q})$, where $U'$ is the unique $\text{Res}_{K/\mathbb{Q}} \mathbb{G}_m/K$ subrepresentation of $H^4(A,\mathbb{Q})$ which after tensoring with $\mathbb{C}$ becomes isomorphic to the sum of weight spaces $\alpha^4 \oplus \bar{\alpha}^4$. The Hodge type of $U'$ is $\{(3,1), (1,3)\}$. The highly geometric arguments are lengthy and intricate, so we will not go into them here, except to say that one of the main points is that the generic $A$ as above is a generalized Prym variety associated to a cyclic 4-fold covering $\pi : C \to X$ of curves. See [B.105] for the details.

10.4. Powers of elliptic curves or abelian surfaces with quaternionic multiplication. The simplest elliptic curves to deal with are those which entirely avoid the kind of problem in Grothendieck’s counterexample to Hodge’s original conjecture, namely where the period $\tau$ is either quadratic over $\mathbb{Q}$, in which case the elliptic curve has complex multiplication, or a general elliptic curve, whose period $\tau$ is transcendental over $\mathbb{Q}$.

10.4.1. Proposition ([B.117]). If $E$ is an elliptic curve with complex multiplication, then Hodge’s original conjecture is true for $E^k$, for all $k \geq 1$.

10.4.2. Proposition ([B.38]). If $E$ is a general elliptic curve, then Hodge’s original conjecture is true for $E^k$, for all $k \geq 1$.

10.4.3. Proposition ([B.39]). If $A$ is a general abelian surface with quaternionic multiplication, Hodge’s original conjecture is true for $A^k$, for all $k \geq 1$. 


10.4.4. **Proposition ([B.4] Thm.6.1).** When \( A \) is a product of elliptic curves, then the general Hodge conjecture is true for \( A \).

The first of these results is discussed in the text (7.18)–(7.20), and the next two are superseded by Theorem 10.9 below. For the last, the multiplicativity of the Lefschetz group (Lemma 2.15) reduces the problem to powers of a single elliptic curve, see section three. Then the case where the curve is of CM-type is covered by 10.4.1 above, whereas the case where the curve is not of CM-type is a special case of Theorems 10.9 or 10.12 below.

**Abelian varieties with conditions on endomorphisms, dimension or Hodge group.** It is only quite recently that results about the general Hodge conjecture for abelian varieties of comparable generality to what has been proved for the usual Hodge conjecture have begun to appear. Here we collect together the main results before discussing some of what is involved in proving them.

10.5. **Theorem ([B.127] Thm.1).** If \( A \) be a simple abelian variety of type (I) such that \( \dim A/\text{End}^0(A) : \mathbb{Q} \) is odd, then Hodge’s original conjecture holds for \( A \).

10.6. **Theorem ([B.127] Thm.2).** Let \( A \) be a simple abelian variety of CM-type, with \( \text{End}^0(A) = K \) and \( K_0 = \text{the maximal totally real subfield of } K \). If \( [K^\text{Gal} : K_0^\text{Gal}] = 2^{\dim A} \), then Hodge’s original conjecture holds for \( A \).

10.7. **Theorem ([B.128] Thm.1).** Let \( A \) be an abelian variety with \( \text{End}^0(A) = \mathbb{Q} \). If

1. \( \dim A \neq 4^l \),
2. \( \dim A \neq \frac{1}{2}(4^l+2)^{2m-1} \),
3. \( \dim A \neq 2^{8^l+4l-4m-3} \),
4. \( \dim A \neq 4^l(m+1)^{2l+1} \),
5. \( \dim A \neq 2^{8^l+2m-4l-2(8^l+4)^m-1} \),

for any positive integers \( l, m, n, \) then the general Hodge conjecture holds for \( A \). Furthermore, \( \mathcal{H}(A) = \mathcal{D}(A) \), and \( \text{Hg}(A) = \text{Sp}(H^1(A, \mathbb{Q}), E) \).

10.8. **Theorem ([B.129] Thm.1.1).** Let \( A \) be a simple complex abelian variety of dimension \( g \) with Hodge group \( \text{Hg}(A) \), let \( \mathfrak{h}g(A, \mathbb{C}) = \text{Lie} \text{Hg}(A) \otimes \mathbb{C} \), and let \( \mathfrak{h}g(A, \mathbb{C})_{ss} \) be the semisimple part of the reductive Lie algebra \( \mathfrak{h}g(A, \mathbb{C}) \). Consider the following sets of natural numbers:

\[
\text{Ex}(1) := \left\{ 4^l, \frac{1}{2}(4^l+2)^{2m-1}, 2^{8^m+4^l-4m-3}, 4^l(m+1)^{2l+1} : l, m \in \mathbb{Z}_+ \right\};
\]

\[
\text{Ex}(3) := \left\{ 46^l+1, 6^{l+1}, \left(\frac{4^m+4^l}{2m+2}\right)^{2l}, \left(\frac{4^m+2}{2m+1}\right)^{2l} : l, m \in \mathbb{Z}_+ \right\};
\]

\[
4^l(m+2)^{2l}, 2^{l+1}(m+4)^{l+1} : l, m \in \mathbb{Z}_+ \right\};
\]
Ex(4) := \{ \binom{l+2}{m} \text{ for } 1 < m < (l+2)/2, \\
\binom{l+2}{m}^{n+1} \text{ for } 1 \leq m < (l+2)/2 : l,m,n \in \mathbb{Z}_+ \}. \\

1. If \( \text{End}(A) \otimes \mathbb{R} = \mathbb{R} \) and \( g \notin \text{Ex}(1) \), then \( hg(A,\mathbb{C}) = \text{sp}(2g) \) and the general Hodge conjecture is true for \( A^k \), for \( k \geq 1 \).

2. If \( \text{End}(A) \otimes \mathbb{R} = M_2(\mathbb{R}) \) and \( g \notin 2 \cdot \text{Ex}(3) \), then \( hg(A,\mathbb{C}) = \text{sp}(g) \) and the general Hodge conjecture is true for \( A^k \), for \( k \geq 1 \).

3. If \( \text{End}(A) \otimes \mathbb{R} = \mathbb{H} \), the Hamiltonian quaternion algebra, and \( g \notin \text{Ex}(3) \), then \( hg(A,\mathbb{C}) = so(g) \) and for \( 0 \leq r \leq g \) we have

\[
\text{dim}_\mathbb{Q} \mathfrak{H}^r = \begin{cases} 
1 & \text{if } r \neq g/2, \\
g + 2 & \text{if } r = g/2
\end{cases}
\]

(in particular, if \( r \neq g/2 \), then \( \mathfrak{H}^r = D^r \)).

4. If \( \text{End}(A) \otimes \mathbb{R} = \mathbb{C} \) and \( g \notin \text{Ex}(4) \), then \( hg(A,\mathbb{C})_{\text{ss}} = sl(g) \) and for all integers \( r \neq g/2 \) we have \( \mathfrak{H}^r = D^r \); in the case \( rg/2 \) we have the relations

\[
\text{dim}_\mathbb{Q} \mathfrak{H}^r = \begin{cases} 
1 & \text{if } hg(A,\mathbb{C}) \text{ is not semisimple}, \\
3 & \text{if } hg(A,\mathbb{C}) \text{ is semisimple}.
\end{cases}
\]

10.9. **Theorem** ([B.50] Thm.5.1). If \( A \) is a stably nondegenerate abelian variety (see 7.5 and 7.6) all of whose simple components are of type (I) or (II), the general Hodge conjecture holds for \( A \) and all powers \( A^k \) of \( A \), for \( k \geq 1 \).

10.10. **Remarks.**

1. It follows from Theorem 6.3.1 that a simple abelian variety of type (I) such that \( \dim A/\text{End}^0(A) : \mathbb{Q} \) is odd is stably nondegenerate, so Theorem 10.5 is a special case of Theorem 10.9. As remarked already, Propositions 10.4.2 and 10.4.3 are included in 10.9, as well. Other examples of stably nondegenerate abelian varieties include arbitrary products of elliptic curves (section 3), simple abelian varieties of prime dimension (Theorem 6.3), or simple abelian varieties of odd dimension without complex multiplication (Theorems 6.3 and 7.5). Also an abelian variety \( A \) is stably nondegenerate if and only if \( A^k \) is stably nondegenerate, for any \( k \geq 1 \), by 7.6.1.2.

2. Proposition 10.4.1 is a special case of Theorem 10.6.

3. The methods used in the proof of Theorem 10.6 are similar to those in the proof of Theorem 9.2 [B.88].

4. The proof of Theorem 10.7 uses the classification result Theorem 2.11. Since \( \text{End}^0(A) = \mathbb{Q} \), Theorem 2.7 applies, and over \( \mathbb{Q} \) (or \( \mathbb{C} \)) the universal cover of \( Hg(A) \) is isomorphic to some number of copies of an almost simple \( \mathbb{Q} \)-group, say \( G_1 \). Then if \( G_1 \) is any of the types in Theorem 2.11 other than \( sp(2d) \), where \( d = \dim A \), then \( d \) is one of the forbidden dimensions. Then the known representation theory of \( sp(2d) \) can be used to control the level of sub-Hodge structures, in a similar spirit though by a different argument as in the next paragraph. The proof of Theorem 10.8 applies similar ideas to the semisimple part of the Hodge group.

The proof of Theorem 10.9 provides an illustrative example of how the representation theory of the symplectic group comes into proving the Hodge conjecture.
10.11. Sketch of proof of Theorem 10.9 (following [B.50]). Recall that the level \( l(W) \) of a rational Hodge structure \( W \), in particular a sub-Hodge structure of \( H^m(A, \mathbb{Q}) \), is the maximum of \( |p - q| \) for which \( W^p \cdot q \neq 0 \). Then it will suffice to prove that for any irreducible rational sub-Hodge structure \( W \) of \( H^m(A, \mathbb{Q}) \) with \( l(W) = m - 2p \) there exists a Zariski-closed subset \( Z \) of codimension \( p \) in \( A \) such that

\[
W \subset \text{Ker}\{H^m(A, \mathbb{Q}) \rightarrow H^m(A - Z, \mathbb{Q})\}.
\]

Now, it is a basic fact from the representation theory of \( \mathfrak{sp}(2n, \mathbb{C}) \) that there is a one-to-one correspondence between its irreducible (finite-dimensional) representations and \( n \)-tuples \((\lambda_1, \ldots, \lambda_n)\) of nonnegative integers with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), see [B.33], [B.34]. Such an \( n \)-tuple is called a Young diagram of length \( n \). Then the crucial proposition, whose proof we omit here, is the following.

10.11.1. Proposition ([B.50]). Let \( A \) be an abelian variety with \( \mathfrak{h}(A, \mathbb{C}) \cong \mathfrak{sp}(2n, \mathbb{C}) \), let \( W \) be an irreducible rational sub-Hodge structure of \( H^m(A^k, \mathbb{Q}) \), and let \((\lambda_1, \ldots, \lambda_n)\) be the associated Young diagram. Then

\[
l(W) = \sum_{i=1}^{n} \lambda_i.
\]

For a Young diagram \((\lambda_1, \ldots, \lambda_n)\) the number \( \sum_i \lambda_i \) is often referred to as the number of boxes, for the traditional representation of a Young diagram as \( n \) rows with \( \lambda_i \) boxes in the \( i \)th row. As a matter of notation, it is convenient to write \((1^a)\) for the Young diagram with \( \lambda_1 = \cdots = \lambda_a = 1 \) and \( \lambda_{a+1} = \cdots = \lambda_n = 0 \), and similarly \((2^c, 1^d)\) for the diagram with \( \lambda_1 = \cdots = \lambda_c = 2 \) and \( \lambda_{c+1} = \cdots = \lambda_{c+d} = 1 \) and \( \lambda_{c+d+1} = \cdots = \lambda_n = 0 \). By convention \((1^0)\) is the Young diagram for the trivial representation. Then we recall the following facts from representation theory.

10.11.2. Lemma.  
1. Let \( V = \mathbb{C}^{2^n} \) as a standard representation of \( \mathfrak{sp}(2n, \mathbb{C}) \). Then for \( 1 \leq i \leq n \)

\[
\bigwedge^i V \cong (1^1) \oplus (1^{i-2}) \oplus \cdots \oplus (1^{i-2[i/2]}),
\]

where the inclusion of \( (1^a) \) into \( \bigwedge^i V \) is defined by taking the exterior product \((i - a)/2 \) times with

\[
\Omega = \sum_{j=1}^{n} e_j \wedge e_{n+j},
\]

where \( \{e_1, \ldots, e_{2n}\} \) is a standard symplectic basis.

2. For nonnegative integers \( a, b \) with \( a \geq b \),

\[
(1^a) \otimes (1^b) \cong \{(1^{a+b}) \oplus (2, 1^{a+b-2}) \oplus \cdots \oplus (2^b, 1^{a-b})\}
\]
\[
\oplus \{(1^{a+b-2}) \oplus (2, 1^{a+b-4}) \oplus \cdots \oplus (2^{b-1}, 1^{a-b})\}
\]
\[
\oplus \cdots \oplus \{(1^{a-b})\},
\]

with the convention that Young diagrams on the right-hand side with more than \( n \) rows are omitted.
Now the proof of the theorem is divided into three steps. First, consider the case where $A = B^k$, where $\mathfrak{h}_g(B, \mathbb{C}) \simeq \mathfrak{sp}(2n, \mathbb{C})$ acting on $V = H^1(B, \mathbb{C}) \simeq \mathbb{C}^{2n}$ as a standard representation. Then any irreducible rational sub-Hodge structure $W$ in $H^m(A, \mathbb{Q})$, say of level $l(W) = m - 2p$, corresponds over $\mathbb{C}$ to an irreducible $\mathfrak{h}_g(B, \mathbb{C})$ representation (see Proposition 2.4) occurring in one of the terms on the right-hand side of

$$H^m(A, \mathbb{C}) \simeq \bigoplus_{m_1 + \ldots + m_r = m} (\bigwedge^{m_1} V \otimes \ldots \otimes \bigwedge^{m_r} V).$$

By Proposition 10.11.1 the number of boxes in the Young diagram associated to $W_\mathbb{C}$ is $m - 2p$. Then Lemma 10.11.2 implies that the contraction, i.e., the reduction in the number of boxes, comes about only by taking the exterior product with $\Omega^p$. However, in the dictionary between the representation theory and the cohomology, $\Omega$ corresponds to a divisor, say $D$, and taking the exterior product with it corresponds to intersecting with $D$. Thus $W$ is the cup product of a rational sub-Hodge structure in $H^{m-2p}(A, \mathbb{Q})$ with $D^p$, which verifies the general Hodge conjecture in this case.

Secondly, consider the case where $A = B_1^{k_1} \times B_2^{k_2}$ with $B_1$ not isogenous to $B_2$, and as in the first case, $\mathfrak{g}_i := \mathfrak{h}_g(B_i, \mathbb{C}) \simeq \mathfrak{sp}(2n_i, \mathbb{C})$ acting via a standard representation on $V_i = H^1(B_i, \mathbb{C})$. Then $\mathfrak{h}_g(A) \simeq \mathfrak{g}_1 \times \mathfrak{g}_2$, and any irreducible rational sub-Hodge structure $W_{\mathbb{Q}}$ of $H^m(A, \mathbb{Q})$ must correspond to an irreducible $\mathfrak{h}_g(A, \mathbb{C})$-representation of the form $W_1 \otimes W_2$ for some irreducible $\mathfrak{h}_g(B_i, \mathbb{C})$-representation $W_i \subset H^{m_i}(B_i^{k_i}, \mathbb{C})$, with $m_1 + m_2 = m$. Moreover, as in Proposition 10.11.1, $l(W) = l(W_1) + l(W_2)$. However, by the previous case, if $l(W_i) = m_i - 2p_i$, then $W_i$ is supported on a Zariski-closed subset $Z_i$ of codimension $p_i$ on $B_i^{k_i}$. Then $W_1 \otimes W_2$ is supported on $Z_1 \times Z_2$ of codimension $p_1 + p_2$ on $A$, which verifies the general Hodge conjecture in this case.

Finally, if $A$ is an arbitrary abelian variety which satisfies the hypotheses of the theorem, then from [B.47] and Theorem 2.11 it follows that

$$\mathfrak{h}_g(A, \mathbb{C}) \simeq \mathfrak{sp}(2n_1, \mathbb{C}) \times \ldots \times \mathfrak{sp}(2n_r, \mathbb{C})$$

acting in the standard way on

$$V_1^{\otimes k_1} \oplus \ldots \oplus V_r^{\otimes k_r}.$$ 

Now, for each $i$ the fundamental form $\Omega_i \in \bigwedge^2 V_i$ is $\mathfrak{sp}(2n_i, \mathbb{C})$-invariant, thus by Lefschetz’s theorem corresponds to a linear combination of divisor classes. Then arguing similarly as in the previous paragraph for $r = 2$ shows that the general Hodge conjecture holds for $A$, as was to be shown. $\square$
Theorem ([B.4]). Let $A$ be an abelian variety whose Hodge group is semisimple and equal to the derived group of the Lefschetz group of $A$, i.e., $\text{Hg}(A) = \text{Lf}(A)^{\text{der}}$. Further suppose that for every simple factor $B$ of $A$ of type (III) the dimension of $H^1(B, \mathbb{Q})$ as a vector space over $\text{End}^0(B)$ is odd. Then if the usual Hodge conjecture is true for $A^k$ for all $k \geq 1$, then the general Hodge conjecture is also true for $A$, and all $A^k$ for $k \geq 1$.

10.12.1. Remarks. In the presence of the assumption that $\text{Hg}(A)$ is semisimple, the hypothesis that $\text{Hg}(A) = \text{Lf}(A)^{\text{der}}$ can be alternatively formulated as follows: There is a natural embedding $\text{Hg}(A) \hookrightarrow \text{Sp}(H^1(A, \mathbb{Q}))$ which induces a holomorphic embedding of the symmetric domain $D$ of $\text{Hg}(A, \mathbb{R})$ into the symmetric domain $\mathfrak{H}$ of $\text{Sp}(H^1(A, \mathbb{R}))$. Then the pull-back to $D$ of the universal family of polarized abelian varieties of dimension $\text{dim} A$ naturally lying over $\mathfrak{H}$ determines a family of abelian varieties of Hodge type in the sense of [B.78]. Then the hypothesis that $\text{Hg}(A) = \text{Lf}(A)^{\text{der}}$, or in the absence of the assumption that $\text{Hg}(A)$ is semisimple, an assumption that $\text{Hg}(A)^{\text{der}} = \text{Lf}(A)^{\text{der}}$, is equivalent to requiring that the family of Hodge type be a family of abelian varieties of PEL-type, in the sense of [B.109] [B.110] [B.111]. That is to say, a family of abelian varieties that is determined by polarization, endomorphism algebra and level structures.

The essential use of this hypothesis in the proof of the theorem is in the multiplicativity of $\text{Lf}(A)$. If $A$ is isogenous to $A_1^{k_1} \times \cdots \times A_r^{k_r}$, then $\text{Lf}(A) = \text{Lf}(A_1) \times \cdots \times \text{Lf}(A_r)$, see Lemma 2.15, and thus under the assumptions at hand, $\text{Hg}(A) = \text{Hg}(A_1) \times \cdots \times \text{Hg}(A_r)$. This makes it possible to reduce the proof to the case where $A$ is isogenous to $A_0^k$ for a simple abelian variety $A_0$. The proof then proceeds by cases, according to whether a simple factor of $\text{End}^0(A_0) \otimes \mathbb{R}$ is $\mathbb{R}$, or $\mathbb{C}$ or $\mathbb{H}$.

10.12.2. Applications. What abelian varieties satisfy the hypotheses of Theorem 10.12? A stably nondegenerate abelian variety with a semisimple Hodge group cannot have factors of type (III) or (IV), so this is the same class as covered by Theorem 10.9. However, Abdulali observes that whenever the usual Hodge conjecture is true for an abelian four-fold $A$ of Weil type, then it is true for all powers $A^k$ of $A$ [B.4]. Thus there is the following consequence of Theorem 4.12.

10.12.3. Corollary. The general Hodge conjecture is true for all powers of a general abelian fourfold of Weil type $(A, K)$ with $K = \mathbb{Q}(\sqrt{-3})$ or $K = \mathbb{Q}(i)$, when the determinant of the associated Hermitian form is 1.

11. Other approaches to the Hodge conjecture

In this section we look at three conditional results on the Hodge conjecture.

11.1. Higher Jacobians. In [B.98], Sampson outlines one possible approach to proving the Hodge conjecture for arbitrary abelian varieties. Given an abelian variety $A$ over $\mathbb{C}$, let $J^p(A)$ denote its $p^{th}$ Weil intermediate Jacobian, for odd $p$ with $1 < p \leq \text{dim} A$. Then Sampson gives an explicit but complicated construction of a surjective homomorphism $\pi : J^p(A) \to A$ which induces an isomorphism $f : \mathfrak{H}^p(A) \to \mathcal{D}^1(J^p(A))$. By the Poincaré Reducibility Theorem 1.11.4, $J^p(A)$
invariant cycles conjecture and thus the Hodge conjecture for abelian varieties. The families of abelian varieties being considered that this conjecture implies the follows. He also formulates the \(L^\ast\) conjecture (A) that the Hodge type in the sense of Grothendieck’s invariant cycles conjecture is a corollary.

For an abelian variety \(A\) of CM-type, the Hodge \((p, p)\) conjecture would follow. However, the highly transcendental nature of the construction makes the connection between \(f^\ast(\varphi)\) and \(\varphi\) rather obscure, as well as apparently making it very difficult to determine whether \(f^\ast\) is injective.

11.2. The Tate conjecture. The references for this are [B.88] [B.87] [B.27], and see also [B.15] [B.16] for related results. If \(A\) is a complex abelian variety, then there is a subfield \(F \subset \mathbb{C}\) finitely generated over \(\mathbb{Q}\) and a model \(A_0\) of \(A\) over \(F\), meaning that \(A = A_0 \otimes_F \mathbb{C}\). Then the \(\ell\)-adic étale cohomology of \(A_0\) over the algebraic closure \(\mathbb{F}^{\text{alg}}\) of \(F\), that is, \(H^2_{\text{ét}}(A_0 \otimes \mathbb{F}^{\text{alg}}, \mathbb{Q}_\ell(p))\), is naturally a \(\text{Gal}(\mathbb{F}^{\text{alg}}/F)\)-module. In [B.130] Tate conjectured that the elements of \(H^2_{\text{ét}}(A_0 \otimes \mathbb{F}^{\text{alg}}, \mathbb{Q}_\ell(p))\) fixed by some open subgroup of \(\text{Gal}(\mathbb{F}^{\text{alg}}/F)\), or equivalently by (the Zariski-closure of) the \(\ell\)-adic Lie subalgebra \(g_\ell \subset \text{End}(H^2_{\text{ét}}(A_0 \otimes \mathbb{F}^{\text{alg}}, \mathbb{Q}_\ell(p)))\) generated by the image of \(\text{Gal}(\mathbb{F}^{\text{alg}}/F)\), is precisely the \(\mathbb{Q}_\ell\)-span of the classes of algebraic cycles. In [B.130] Tate himself observed that this conjecture has an air of compatibility with the Hodge conjecture, and already in [B.77], with the introduction of the Hodge group, Mumford reported that Serre conjectured that

\[ g_\ell = \text{mt}(A) \otimes \mathbb{Q}_\ell. \]

For an excellent early introduction to this conjecture and the relationships between the Tate and Hodge conjectures, see [B.108]; the literature in the 20 years since then is extensive, it would take another appendix at least the size of this one to survey it.

The main result of [B.88] is that for abelian varieties of CM-type, the Hodge and Tate conjectures are equivalent. Then that the validity of the Tate conjecture for an abelian variety \(A\) implies the validity of the Hodge conjecture for \(A\) has been proved by Piatetskii-Shapiro [B.87], Deligne (unpublished) and [B.27]. [B.15] extends the result of [B.87], and [B.16] contains a weaker version of the main theorem of [B.27], from which Tate implies Hodge for abelian varieties follows as a corollary.

11.3. Standard conjectures. In [B.3] Abdulali shows that if one assumes Grothendieck’s invariant cycles conjecture [B.42] for families of abelian varieties of Hodge type in the sense of [B.78], then the Hodge conjecture for abelian varieties follows. He also formulates the \(L_2\)-cohomology analogue of Grothendieck’s standard conjecture (A) that the Hodge \(\ast\) operator is algebraic [B.44], and shows that for the families of abelian varieties being considered that this conjecture implies the invariant cycles conjecture and thus the Hodge conjecture for abelian varieties.

splits, up to isogeny, as \(\text{Ker}(\pi) \times A'\). Thus a cycle class \([Z]\) of codimension \(r\) on \(\text{Ker}(\pi)\) determines a cycle class \(\text{proj}_A((Z \times A') \cdot f(\varphi))\) of codimension \((r + 1)\) on \(A\), with \(\varphi \in \mathcal{H}(A)\). Now if we fix \(Z = H^{p-1}\) to be the \((p-1)\)-fold self-intersection of a fixed hyperplane section, then

\[ \varphi \mapsto f^\ast(\varphi) := \text{proj}_A((H^{p-1} \times A') \cdot f(\varphi)) \]

defines a homomorphism from \(\mathcal{H}(A)\) into the group of cohomology classes of algebraic cycles on \(A\) of codimension \(p\). Then it is not hard to show that if \(f^\ast\) were injective, then the Hodge \((p, p)\) conjecture would follow. However, the highly transcendental nature of the construction makes the connection between \(f^\ast(\varphi)\) and \(\varphi\) rather obscure, as well as apparently making it very difficult to determine whether \(f^\ast\) is injective.
11.3.1. Conjecture (Invariant cycles conjecture [B.42]). Let $f : A \to V$ be a smooth and proper morphism of smooth quasiprojective varieties over $\mathbb{C}$. Let $P \in V$ and let $\Gamma := \pi_1(V, P)$. Then the space of $s \in H^0(V, R^b f_* \mathbb{Q}) \simeq H^b(A_P, \mathbb{Q})^\Gamma$ that represent algebraic cycles in $H^b(A_P, \mathbb{Q})^\Gamma$ is independent of $P$.

11.3.2. Families of abelian varieties of Hodge type ([B.78]). Let $A_0$ be a polarized abelian variety, let $W = H^1(A_0, \mathbb{Q})$, let $L = H^1(A_0, \mathbb{Z})$, and let $E$ be a Riemann form on $W$ representing the polarization. Also, let $h : U(1) \to \text{GL}(W_\mathbb{R})$ be the complex structure on $W_\mathbb{R}$, let $K^+$ be the connected component of the centralizer of $h(U(1))$ in $\text{Hg}(A_0, \mathbb{R})$, and let $D = \text{Hg}(A_0, \mathbb{R})^+/K^+$ be the bounded symmetric domain associated to $\text{Hg}(A_0)$, as in 2.10. Then to each point $x \in D$ we can associate the polarized abelian variety $A_x = (W_\mathbb{R}/L, ghg^{-1}, [E])$, where $x = gK^+$. Further, if $\Gamma \subset \text{Hg}(A_0)$ is a torsion-free arithmetic subgroup that preserves $L$, then $\gamma \in \Gamma$ induces an isomorphism between $A_x$ and $A_{\gamma x}$. Thus we get a family $\{A_x : x \in V\}$ of polarized abelian varieties parameterized by $V = \Gamma \backslash D$, which may be glued together into an analytic space $A \to V$ fibered over $V$. Such a family of abelian varieties is said to be of Hodge type [B.78]. Furthermore, $V$ has a canonical structure as a smooth quasiprojective algebraic variety [B.11], and the analytic map $A \to V$ is an algebraic morphism [B.13].

11.3.3. Theorem ([B.3] Thm.6.1). If Conjecture 11.3.1 is true for all families of abelian varieties of Hodge type, then the Hodge conjecture is true for all abelian varieties.

An outline of the proof may be sketched as follows. The first step is to deduce from Conjecture 11.3.1 that all Weil-Hodge cycles are algebraic. To do this, Abdulali shows that any abelian variety $A_1$ of Weil type is a member of a Hodge family whose general member $A_0$ has Hodge group equal to the full symplectic group. Then since $\mathfrak{h}(A_0) = \mathcal{D}(A_0)$, the invariant cycles conjecture implies that all Weil cycles become algebraic in this family. The next point is to observe that Theorem 9.5.2 implies that if all Weil-Hodge cycles are algebraic, then the Hodge conjecture is true for all abelian varieties of CM-type. However, Mumford showed that every family of abelian varieties of Hodge type contains members of CM-type [B.78]. Then the invariant cycles conjecture can be used again to deduce that a Hodge cycle on any member of the family is algebraic. See [B.3] for more details.
## Chronological listing of work on the Hodge conjecture for abelian varieties

| Year | Author        | Topic                                                      |
|------|---------------|------------------------------------------------------------|
| 1950 | Hodge         | Presented conjecture                                       |
| 1958 | Mattuck       | GHC for general abelian variety                            |
| 1966 | Mumford       | Introduced Hodge group                                     |
| 1968 | Pollmann      | Hodge if and only if Tate for CM-type                      |
| 1969 | Mumford       | Families of Hodge type                                     |
| 1969 | Grothendieck  | Amended general Hodge conjecture                            |
| 1969 | Murasaki       | Elliptic curves                                            |
| 1971 | Piatetskii-Shapiro | Tate implies Hodge                                      |
| 1974 | Borovoï       | Tate implies Hodge                                         |
| 1976 | Imai          | Elliptic curves                                            |
| 1977 | Serre         | Connections between Hodge and Tate conj.                  |
| 1977 | Borovoï       | Absolute Hodge cycles                                      |
| 1977 | Weil          | Weil type                                                  |
| 1978 | Tankeev       | 4-dimensional abelian varieties                            |
| 1979 | Deligne       | Classification of semisimple part of $h_g$                 |
| 1979 | Serre         | Classification of semisimple part of $h_g$                 |
| 1979 | Tankeev       | 4- and 5-dimensional                                       |
| 1981 | Borovoï       | Simplicity of Hodge group                                  |
| 1981 | Shioda        | Fermat type                                                |
| 1981 | Tankeev       | Simple abelian varieties                                   |
| 1981 | Tankeev       | Simple (5-dimensional) abelian varieties                   |
| 1982 | Tankeev       | Simple abelian varieties, prime dimension                  |
| 1982 | Deligne       | Absolute Hodge cycles, Tate implies Hodge                  |
| 1982 | Kuga          | Exceptional cycles                                         |
| 1982 | Sampson       | Alternate approach                                         |
| 1982 | Ribet         | Simple abelian varieties                                   |
| 1983 | Shioda        | Survey                                                     |
| 1983 | Ribet         | Simple abelian varieties, Lefschetz group                  |
| 1983 | Hazama        | Nondegenerate CM-type                                      |
| 1983 | Murty         | Non-simple abelian varieties                               |
| 1983 | Hazama        | Non-simple abelian varieties                               |
| 1984 | Kuga          | Exceptional cycles                                         |
| 1984 | Hazama        | Non-simple abelian varieties                               |
| 1984 | Murty         | Non-simple abelian var., exceptional cycles               |
| 1984 | Dodson        | Degenerate CM-types                                        |
| 1985 | Zarhin        | Classification of $h_g$, survey                            |
| 1985 | Yanai         | Nondegenerate CM-types                                     |
| 1986 | Dodson        | Degenerate CM-types                                        |
| Year | Author(s)          | Contribution                                                                 |
|------|-------------------|-------------------------------------------------------------------------------|
| 1987 | Steenbrink        | General Hodge conjecture, survey                                              |
| 1987 | Dodson            | Degenerate CM-types                                                          |
| 1987 | Bardelli          | GHC, low dimension                                                            |
| 1987 | Ichikawa          | Non-simple abelian varieties, MT groups                                       |
| 1988 | Hazama            | Stably degenerate                                                            |
| 1988 | Murty             | Lefschetz group, semisimple part of Hg(A)                                     |
| 1988 | Schoen            | Weil type                                                                     |
| 1988 | Gordon            | GHC for powers of general QM-surfaces                                         |
| 1989 | Kuga              | Exceptional cycles                                                            |
| 1989 | Hazama            | Non-simple abelian varieties                                                  |
| 1989 | Schoen            | GHC Weil type                                                                  |
| 1989 | Mai               | Degenerate CM-types                                                           |
| 1990 | Murty             | Survey, Hodge group                                                           |
| 1990 | Kuga, Perry, Sah  | Exceptional cycles                                                            |
| 1991 | Ichikawa          | Non-simple abelian varieties, Hodge groups                                    |
| 1992 | Pirola            | GHC special threefolds                                                         |
| 1992 | André             | CM-type and Weil cycles                                                       |
| 1993 | White             | Degenerate CM-type                                                            |
| 1993 | Gordon            | GHC powers of general elliptic curve                                          |
| 1993 | Tankeev           | GHC                                                                           |
| 1994 | Zarhin            | Survey, connections with arithmetic                                           |
| 1994 | van Geemen        | Survey, Weil type                                                             |
| 1994 | Yanai             | Degenerate CM-types                                                           |
| 1994 | Hazama            | GHC                                                                           |
| 1994 | Abdulali          | Alternate approach                                                            |
| 1994 | Tankeev           | GHC                                                                           |
| 1995 | Moonen & Zarhin   | Abelian 4-folds                                                               |
| 1996 | Lee               | Exceptional cycles                                                            |
| 1996 | Abdulali          | GHC                                                                           |
| 1996 | Tankeev           | GHC                                                                           |
| 1996 | Silverberg and Zarhin | Hodge group, connection to arithmetic                                     |
| 1996 | Moonen & Zarhin   | Exceptional Weil cycles                                                       |
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APPENDIX B. THE HODGE CONJECTURE FOR ABELIAN VARIETIES

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