FIBER FANS AND TORIC QUOTIENTS

ALASTAIR CRAW AND DIANE MACLAGAN

Abstract. The GIT chamber decomposition arising from a subtorus action on a quasiprojective toric variety is a polyhedral complex. Denote by Σ the fan that is the cone over the polyhedral complex. In this paper we show that the toric variety defined by the fan Σ is the normalization of the toric Chow quotient of a closely related affine toric variety by a complementary torus.

1. Introduction

Let $X_P$ be the quasiprojective toric variety defined by a full-dimensional polyhedron $P \subseteq \mathbb{Q}^n$, and let $T_L = \text{Spec} \mathbb{k}[L]$ be a $d$-dimensional subtorus of the dense torus $(\mathbb{k}^*)^n$ of $X_P$. The inclusion of $T_L$ into $(\mathbb{k}^*)^n$ induces a linear map $\pi: \mathbb{Q}^n \to \mathbb{Q}^d \cong L \otimes_{\mathbb{Z}} \mathbb{Q}$ and hence a linear projection of polyhedra $\pi: P \to Q$, where $Q = \pi(P)$. Thaddeus [Tha94] proved that $\pi$ induces a cell decomposition of $Q$ that is the GIT chamber decomposition arising from the action of $T_L$ on $X_P$. See Hu [Hu02] for the most general results about toric GIT, including quotients of toric varieties that do not necessarily arise from polyhedra.

By taking the cone over each cell in this GIT chamber decomposition, we obtain a fan $\Sigma$ in $\mathbb{Q}^d \oplus \mathbb{Q}$. It is natural to ask for an explicit description of the toric variety defined by this fan. The main result of this paper provides an answer to this question as follows (see Theorem 3.13 and Section 2.4 for relevant notation):

**Theorem 1.1.** Let $T_L$ be a subtorus acting on a toric variety $X_P$, and let $\Sigma$ be the fan arising from the GIT chamber decomposition associated to the $T_L$-action on $X_P$ as above. Then there is a polyhedral cone $P' := (\tilde{P})^\vee$ and a lattice $L' := \text{im}_{\mathbb{Z}}(\tilde{\pi}^\vee)$ such that $\Sigma$ is the fan of the toric Chow quotient $X_{P'}/T_{L'}$.

The Chow quotient $X_{P'}/T_{L'}$ of a projective toric variety $X_P$ by a subtorus $T_{L'}$ was studied by Kapranov–Sturmfels–Zelevinsky [KSZ92]. We generalize this by constructing the toric Chow quotient $X_P/T_L$ of an arbitrary quasiprojective toric variety $X_P$ by a subtorus $T_L$ (see Definition 3.9).

In order to prove Theorem 1.1, we consider the fiber fan associated to a linear projection of polyhedra $\pi: P \to Q$, denoted $\mathcal{N}(P, Q)$, generalizing the normal fan of a fiber polytope (see Billera–Sturmfels [BS92]). We show that the toric Chow quotient $X_P/T_L$ is a not-necessarily normal toric variety whose fan is the fiber fan $\mathcal{N}(P, Q)$. In the special case where $P$ is a polytope, so $X_P$ is projective, this result is due to Kapranov–Sturmfels–Zelevinsky [KSZ92]. Note however that the statement of Theorem 1.1 is new even in the case where $P$ is a polytope.

If the original polyhedron $P$ is a polyhedral cone, the statement of Theorem 1.1 can be streamlined, and a converse can be added as follows (see Theorem 3.14).

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Theorem 1.2 (Duality for affine toric quotients). Let \( P \subseteq \mathbb{Q}^n \) be a polyhedral cone and \( \pi: P \rightarrow Q \) a linear projection. Set \( P' := P^\vee \) and \( L' := \text{im}_Z(\pi^\vee) \). Then

(i) the fan of the toric Chow quotient \( X_P/T_L \) is equal to the GIT chamber decomposition arising from the action of \( T_{L'} \) on \( X_{P'} \); and

(ii) the GIT chamber decomposition arising from the \( T_L \)-action on \( X_P \) is the fan of the toric Chow quotient \( X_{P'}/T_{L'} \).

The statement of Theorem 1.2 part (i), was partially known to Altmann–Hausen [AH03] in the case where \( P \) is a simplicial cone, though no proof was given.

The main theorems are proved in Section 3.3. Section 2 contains the necessary polyhedral preliminaries, while Sections 3.1 and 3.2 contain background on toric GIT, and introduce the toric Chow quotient, respectively.

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2. Polyhedral Geometry

2.1. Polyhedral Conventions. Let \( P \subseteq \mathbb{Q}^n \) be an \( n \)-dimensional polyhedron. For \( w \in (\mathbb{Q}^n)^* \) we denote by \( \text{face}_w(P) \) the face of \( P \) minimizing \( w \). Given a face \( F \) of \( P \), the inner normal cone \( N_P(F) \) is the set of \( w \in (\mathbb{Q}^n)^* \) such that \( \text{face}_w(P) = F \). The inner normal fan \( \mathcal{N}(P) \) of \( P \) is the polyhedral fan whose cells are the inner normal cones \( \{N_P(F)\} \) as \( F \) varies over the faces of \( P \). Two polyhedra \( P \) and \( P' \) are normally equivalent if they have the same normal fan.

The recession cone \( \text{rec}(P) \) of a polyhedron \( P \) is \( \{u \in \mathbb{Q}^n : x + u \in P \text{ for all } x \in P\} \). If \( P = \{x \in \mathbb{Q}^n : Ax \geq b\} \) for some \( r \times n \) matrix \( A \) and vector \( b \in \mathbb{Q}^r \), then \( \text{rec}(P) = \{x \in \mathbb{Q}^n : Ax \geq 0\} \). The support of a polyhedral fan is the set of vectors lying in some cone of the fan. The fan \( \mathcal{N}(P) \) is supported on the dual cone \( \text{rec}(P)^\vee = \{y \in (\mathbb{Q}^k)^* : y \cdot x \geq 0 \forall x \in \text{rec}(P)\} \) of the recession cone \( \text{rec}(P) \) of \( P \). Indeed, \( y \notin \text{rec}(P)^\vee \) if and only if there exists \( x \in \text{rec}(P) \) such that \( y \cdot x < 0 \), in which case \( y \cdot x' \) is unbounded below for \( x' := \lambda x \in \text{rec}(P) \) with \( \lambda \in \mathbb{Q}_{>0} \).

Let \( \pi: \mathbb{Q}^n \rightarrow \mathbb{Q}^d \) be a linear map, and set \( Q := \pi(P) \). We may assume that \( \pi \) is the projection onto the last \( d \) coordinates. We denote by \( \pi_{n-d} \) the projection onto the first \( n - d \) coordinates \( \mathbb{Q}^{n-d} \), and for \( q \in Q \) we set \( P_q := \pi_{n-d}(\pi^{-1}(q) \cap P) \). For \( w \in (\mathbb{Q}^{n-d})^* \), we write \( \text{face}_w(P_q) \) for the unique smallest face of \( P \) containing the preimage of \( \text{face}_w(P_q) \) under \( \pi_{n-d} \).

2.2. Fiber fans. In this section we define the notion of a fiber fan \( \mathcal{N}(P, Q) \) for a linear projection \( \pi: P \rightarrow Q \) of polyhedra. This is the common refinement of the normal fans of the fibers \( \{\pi^{-1}(q) : q \in Q\} \) of the map \( \pi \). We first show that this definition makes sense.

Lemma 2.1. For \( q, q' \in Q \), set \( q \sim q' \) if \( P_q \) is normally equivalent to \( P_{q'} \). Then there are only finitely many equivalence classes for \( \sim \).
Proof. Write \( P = \{ x : Ax \geq b \} \), where \( A \) is an \( r \times n \) matrix for some \( r \), and \( b \in \mathbb{Q}^r \). Then for \( q \in \mathbb{Q}, \pi^{-1}(q) \cap P = \{ (x', q) : x' \in \mathbb{Q}^{n-d}, A(x', q) \geq b \} \). Write \( A \) in block form as \((A'|A'')\), where \( A' \) is an \( r \times (n-d) \) matrix, and \( A'' \) is an \( r \times d \) matrix. Then \( P_q = \{ x' \in \mathbb{Q}^{n-d} : A'x' \geq b - A''q \} \). Now for a given matrix \( A' \) there are only finitely many normal equivalence classes of polyhedra of the form \( \{ x' \in \mathbb{Q}^{n-d} : A'x' \geq c \} \) as \( c \) varies, so it follows that there are only finitely many normal equivalence classes of fibers of \( \pi \).

**Definition 2.2.** Pick \( q_1, \ldots, q_t \in \mathbb{Q} \) such that the \( P_{q_i} \) are representatives of the different normal equivalence classes of fibers of \( \pi \). Let \( F \) be the Minkowski sum of the \( P_{q_i} \). The fiber fan \( \mathcal{N}(P, Q) \) is the inner normal fan of \( F \).

**Lemma 2.3.** The recession cone of \( P_q \) is the same for all \( q \in \mathbb{Q} \).

Proof. Write \( P = \{ x \in \mathbb{Q}^n : Ax \geq b \} \) for some \( r \times n \) matrix \( A \) and \( b \in \mathbb{Q}^r \). Decompose \( A = (A'|A'') \), where \( A' \) is an \( r \times (n-d) \) matrix, and \( A'' \) is an \( r \times d \) matrix. Translating the polyhedron \( P \) by \( a \in \mathbb{Q}^n \) translates \( Q \) by \( \pi(a) \), and \( P_q = (P + a)_{q + \pi(a)} \), so we may assume \( 0 \in P \). Note that \( P_0 = \{ x \in \mathbb{Q}^{n-d} : A'x \geq b \} \). We will show that \( \text{rec}(P_q) = \text{rec}(P_0) \) for all \( q \in \mathbb{Q} \).

Let \( v \in \text{rec}(P_q) \). Then \( u + \lambda v \in P_q \) for all \( u \in P_q \) and \( \lambda > 0 \). Since \( 0 \in P \), we have \( b_i \leq 0 \) for all \( i \), where \( b_i \) is the \( i \)th component of \( b \). Suppose that \((v, 0) \notin P \) for \( 0 \in \mathbb{Q}^d \). Then there is some \( i \) with \( a_i \cdot (v, 0) = b_i < 0 \), where \( a_i \) is the \( i \)th row of \( A \). But then \( a_i \cdot (u + \lambda v, q) = a_i \cdot (u, q) + \lambda a_i \cdot (v, 0) < b_i \) for \( \lambda \) sufficiently large, which means that \((u + \lambda v, q) \notin P \), so \( u + \lambda v \notin P_q \). Therefore we have \( \{ (v, 0) : v \in \text{rec}(P_q) \} \subseteq P \) after all. The above argument shows that \( a_i \cdot (v, 0) = a'_i \cdot v \geq 0 \) for each row \( a'_i \) of \( A' \), so \( v \in \text{rec}(P_0) \).

For the opposite inclusion, note that since \( P_0 = \{ x \in \mathbb{Q}^{n-d} : A'x \geq b \} \), the set \( \{ (v, 0) : v \in \text{rec}(P_0) \} \) lies in \( \text{rec}(P) \). Thus if \( v \in \text{rec}(P_0) \), then \( (u, q) + \lambda (v, 0) \in P \) for all \( \lambda > 0 \) and \( u \in P_q \). This means \( (u + \lambda v, q) \in P \) for all \( \lambda > 0 \) and \( u \in P_q \), so \( u + \lambda v \in P_q \) for all \( \lambda > 0 \) and \( u \in P_q \), giving \( v \in \text{rec}(P_q) \) as required.

If \( \mathcal{F}_1, \ldots, \mathcal{F}_r \) are polyhedral fans with the same support, then their *common refinement* is the fan \( \mathcal{F} \) whose cones are the intersections \( \cap_{i=1}^r \sigma_i \), where \( \sigma_i \) is a cone in \( \mathcal{F}_i \). If \( P \) and \( P' \) are two polyhedra with the same recession cone, then the normal fan of the Minkowski sum of \( P \) and \( P' \) is the common refinement of the normal fans of \( P \) and \( P' \). Thus the fiber fan \( \mathcal{N}(P, Q) \) is the common refinement of the normal fans of the fibers.

**Remark 2.4.** In the case that \( P \) is a polytope, the fiber fan \( \mathcal{N}(P, Q) \) is the normal fan of the *fiber polytope* introduced by Billera and Sturmfels [BS92]. While the fiber fan of a linear projection of polyhedra is still the normal fan of a polyhedron, there is not a canonical choice of a polyhedron with that normal fan if the polyhedron being projected is not a polytope. This motivates our definition of the fiber fan, instead of a more general fiber polyhedron. Note that for applications to toric varieties, a fan suffices to define the variety.

### 2.3. Duality for polyhedral cones

This section establishes a duality result for polyhedral cones that is the key to Theorem 1.2. Versions of Lemmas 2.5 and 2.6 are known to the experts, though we know of no proofs in the literature.
Let $C \subseteq \mathbb{Q}^n$ be a full-dimensional polyhedral cone. For $d < n$, let $\pi : \mathbb{Q}^n \to \mathbb{Q}^d$ be the projection onto the last $d$ coordinates. For $v \in \pi(C)$, consider the polyhedral slice $C_v \subseteq \mathbb{Q}^{n-d}$. As $\mathbb{Q}^{n-d}$ is canonically isomorphic to $\ker(\pi)$, we may regard the normal fan $\mathcal{N}(C_v)$ as lying in the dual vector space $\ker(\pi)^* \cong (\mathbb{Q}^{n-d})^*$. Let $\pi' : (\mathbb{Q}^n)^* \to \ker(\pi)^*$ denote the map dual to the inclusion of $\ker(\pi)$ in $\mathbb{Q}^n$. In the basis of $(\mathbb{Q}^n)^*$ dual to the standard basis of $\mathbb{Q}^n$ this is projection on the first $n - d$ coordinates. For $w \in \pi'(\mathbb{Q}^v)$ we consider the polyhedral slice $C' \cap (\pi')^{-1}(w)$, and write $C'_w := \pi_d(C' \cap (\pi')^{-1}(w)) \subseteq (\mathbb{Q}^d)^*$ for the isomorphic image of $C' \cap (\pi')^{-1}(w)$ under the projection $\pi_d : (\mathbb{Q}^n)^* \to (\mathbb{Q}^d)^*$ onto the last $d$ coordinates. There is a canonical isomorphism $\ker(\pi') \cong (\mathbb{Q}^d)^*$, so the normal fan $\mathcal{N}(C'_w)$ can be regarded as living in either $\mathbb{Q}^d$ or in $(\ker(\pi'))^*$ for all $w \in (\mathbb{Q}^{n-d})^*$.

Let $\text{relint}(F)$ denote the relative interior of a set $F$, which is the interior of $F$ in its affine span.

**Lemma 2.5.** Let $F \subseteq C$ be a face. If $v \in \pi(\text{relint}(F))$ then we have $\pi'(\mathcal{N}_C(F)) = \mathcal{N}_{C_v}(F_v)$.

**Proof.** Write $C = \{ x \in \mathbb{Q}^n : Ax \geq 0 \}$ where $A$ is the $r \times n$ matrix whose rows $a_1, \ldots, a_r \in (\mathbb{Q}^n)^*$ form the facet normals of $C$. We may assume that $F$ is cut out by the first $p$ facet inequalities defining $C$, so $\mathcal{N}_C(F) = \mathbb{Q}_{\geq 0}(a_1, \ldots, a_p)$ and $\pi'(\mathcal{N}_C(F)) = \mathbb{Q}_{\geq 0}(\pi'(a_1), \ldots, \pi'(a_p))$. Note that

$$F \cap \pi^{-1}(v) = \left\{ (u, v) : a_i \cdot (u, v) = 0 \text{ for } i = 1, \ldots, p; \quad a_i \cdot (u, v) \geq 0 \text{ for } i = p+1, \ldots, r \right\}$$

since $F$ is cut out by the first $p$ facet inequalities defining $C$. The assumption that $v \in \pi(\text{relint}(F))$ ensures that each $\geq$ above is a $>$, so

$$F_v = \left\{ u \in \mathbb{Q}^{n-d} : \pi'(a_i) \cdot u = c_i \text{ for } i = 1, \ldots, p; \quad \pi'(a_i) \cdot u > c_i \text{ for } i = p+1, \ldots, r \right\}$$

where $c_i = -a_i \cdot v$ for $i = 1, \ldots, r$. This implies that the rational cone $\mathcal{N}_{C_v}(F_v)$ is also generated by $\{ \pi'(a_1), \ldots, \pi'(a_p) \}$ as claimed. $\square$

**Lemma 2.6.** Fix $v \in \pi(C)$ and $w \in \pi'(C')$. If $F \subseteq C$ is a face then

$$F = \overset{\sim}{\text{face}}_w(C_v) \iff v \in \pi(\text{relint}(F)) \text{ and } w \in \pi'(\text{relint}(\mathcal{N}_C(F))).$$

**Proof.** Suppose $F = \overset{\sim}{\text{face}}_w(C_v)$. Then the preimage of $\text{face}_w(C_v)$ under $\pi_{n-d}$ lies in no proper face of $F$, so the intersection of $\text{relint}(F)$ with this preimage is nonempty. This implies that the set $\pi^{-1}(v) \cap \text{relint}(F)$ is nonempty, so $v \in \pi(\text{relint}(F))$. We have $w \in \text{relint}(\mathcal{N}_{C_v}(\overset{\sim}{\text{face}}_w(C_v)))$ by definition. Applying Lemma 2.5 to the face $F_v = \pi_{n-d}(F \cap \pi^{-1}(v)) = \overset{\sim}{\text{face}}_w(C_v)$ gives $w \in \pi'(\mathcal{N}_C(F)) = \pi'(\text{relint}(\mathcal{N}_C(F)))$.

Conversely, suppose $v \in \pi(\text{relint}(F))$ and $w \in \pi'(\text{relint}(\mathcal{N}_C(F)))$. The first assumption ensures that the face $F_v \subseteq C_v$ is nonempty and its preimage under $\pi_{n-d}$ lies in no proper subface of $F$. Also, $w \in \pi'(\text{relint}(\mathcal{N}_C(F))) = \pi'(\text{relint}(\mathcal{N}_C(F))) = \text{relint} \mathcal{N}_{C_v}(F_v)$ by Lemma 2.5, hence $F_v = \overset{\sim}{\text{face}}_w(C_v)$. Since the preimage of $F_v$ under $\pi_{n-d}$ lies in no proper subface of $F$, we have $F = \overset{\sim}{\text{face}}_w(C_v)$. $\square$

We now present the main result of this section. See Figure 1 for an illustration.
Proposition 2.7. For \( v \in \pi(C) \) and \( w \in \pi'(C') \) we have
\[
\mathcal{N}_C(\text{face}_w(C_v)) = \text{face}_v(C'_w).
\]

Proof. Applying Lemma 2.6 to the face \( F := \text{face}_w(C_v) \) of \( C \) gives \( v \in \pi(\text{relint}(F)) \) and \( w \in \pi'(\text{relint}(\mathcal{N}_C(F))) \). The face \( \mathcal{N}_C(F) \) of \( C' \) satisfies \( \mathcal{N}_C'(\mathcal{N}_C(F)) = F \), so \( w \in \pi'(\text{relint}(\mathcal{N}_C'(\mathcal{N}_C(F)))) \) and \( v \in \pi(\text{relint}(\mathcal{N}_C'(\mathcal{N}_C(F)))) \). Now apply Lemma 2.6 to the face \( \mathcal{N}_C(F) \) of \( C' \) to deduce that \( \mathcal{N}_C(F) = \text{face}_v(C'_w) \) as required. \( \square \)

Remark 2.8. Proposition 2.7 is the key duality result that is required in the proof of Theorem 6.2 of Craw–Maclagan–Thomas [CMT05].

The following definition is a variant for polyhedral cones of a notion for polytopes due to Billera–Sturmfels [BS92, §2].

Definition 2.9. For \( v \in \pi(C) \), the subdivision of \( \pi'(C') \) consisting of the cells \( \{ \pi'(\text{face}_w(C'_w)) : w \in \pi'(C') \} \) is the \( v \)-induced \( \pi'^- \)coherent subdivision of \( \pi'(C') \).

The duality result of Proposition 2.7 leads immediately to the following description of the normal fan of the polyhedral slice \( C_v \) for \( v \in \pi(C) \).

Corollary 2.10. For \( v \in \pi(C) \), the inner normal fan of the polyhedron \( C_v \) is the \( v \)-induced \( \pi'^- \)coherent subdivision of \( \pi'(C') \). In other words,
\[
\mathcal{N}(C_v) = \{ \pi'(\text{face}_w(C'_w)) : w \in \pi'(C') \}.
\]
In particular, for \( v = 0 \) we have \( \mathcal{N}(C_0) = \pi'(C') \).

Proof. By Lemma 2.6 and Proposition 2.7 we have
\[
\mathcal{N}_{C_v}(\text{face}_w(C_v)) = \pi'(\mathcal{N}_C(\text{face}_w(C_v))) = \pi'(\text{face}_v(C'_w)).
\]
The set of cones \( \mathcal{N}_{C_v}(\text{face}_w(C_v)) \) taken over all vectors \( w \in \pi'(C') \) equals the inner normal fan \( \mathcal{N}(C_v) \) as required. For the second statement, observe that \( \text{face}_0(C'_w) = C'_w \) for all \( w \in \pi'(C') \), and the smallest face of the cone \( C'_w \) containing the set \( C'_w \) is \( C'_w \) itself. \( \square \)
Definition 2.11. Given a linear map \( \pi : \mathbb{Q}^n \to \mathbb{Q}^d \) and a fan \( \Sigma \subseteq \mathbb{Q}^n \), the fan induced by \( \pi(\Sigma) \) is the fan obtained by taking the common refinement of the cones \( \{\pi(\sigma) : \sigma \in \Sigma\} \). If \( \Sigma \) consists of the faces of a cone \( P \subseteq \mathbb{Q}^n \), we simply write \( \pi(P) \).

Corollary 2.12. The fiber fan \( \mathcal{N}(C, \pi(C)) \) is the fan induced by \( \pi^\vee(C^\vee) \).

Proof. The fan \( \mathcal{N}(C, \pi(C)) \) is the common refinement of the normal fans of the fibers \( C_v \) for \( v \in \pi(C) \). As \( v \) varies in \( \pi(C) \), \( \text{face}_{w}(C_v^\vee) \) varies over all faces of \( C^\vee \). Thus \( \mathcal{N}(C, \pi(C)) \) is the common refinement of all \( \pi^\vee(F) \) for \( F \) a face of \( C^\vee \). \( \square \)

2.4. Duality for general polyhedra. We now generalize the results of the previous subsection from polyhedral cones to general polyhedra.

Definition 2.13. For any polyhedron \( P \subseteq \mathbb{Q}^n \), let \( \bar{P} \subseteq \mathbb{Q}^n \oplus \mathbb{Q} \) be the polyhedral cone obtained as the closure in \( \mathbb{Q}^n \oplus \mathbb{Q} \) of the set \( \{(\lambda \mathbf{x}, \lambda) : \mathbf{x} \in P, \lambda \in \mathbb{Q}_{>0}\} \). Note the \( \bar{P} \cap (\mathbb{Q}^n + \{0\}) \cong \text{rec}(P) \).

Given \( \pi : \mathbb{Q}^n \to \mathbb{Q}^d \), we write \( \bar{\pi} : \mathbb{Q}^n \oplus \mathbb{Q} \to \mathbb{Q}^d \oplus \mathbb{Q} \) for the map sending \((\mathbf{x}, \lambda)\) to \((\pi(\mathbf{x}), \lambda)\). The first projection \( p_1 : \mathbb{Q}^n \oplus \mathbb{Q} \to \mathbb{Q}^n \) fits into the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{ker}(\bar{\pi}) \xrightarrow{\pi} \mathbb{Q}^n \oplus \mathbb{Q} \xrightarrow{\bar{\pi}} \mathbb{Q}^d \oplus \mathbb{Q} \longrightarrow 0 \\
0 & \longrightarrow & \text{ker}(\pi) \xrightarrow{\iota} \mathbb{Q}^n \xrightarrow{\pi} \mathbb{Q}^d \longrightarrow 0
\end{array}
\]

and canonically identifies \( \text{ker}(\pi) \) with \( \text{ker}(\bar{\pi}) \). Write \( \pi^\vee : (\mathbb{Q}^n)^* \to \text{ker}(\pi)^* \) and \( \bar{\pi}^\vee : (\mathbb{Q}^n+1)^* \to \text{ker}(\bar{\pi})^* \) for the maps obtained by pullback via \( \iota \) and \( \bar{\iota} \) respectively. These maps fit into the dual diagram

\[
\begin{array}{ccc}
0 & \leftarrow & \text{ker}(\bar{\pi})^* \xleftarrow{\bar{\pi}^\vee} (\mathbb{Q}^n+1)^* \leftarrow (\mathbb{Q}^d)^* \leftarrow 0 \\
0 & \leftarrow & \text{ker}(\pi)^* \xleftarrow{\pi^\vee} (\mathbb{Q}^n)^* \leftarrow (\mathbb{Q}^d)^* \leftarrow 0
\end{array}
\]

Since \( P \subseteq \mathbb{Q}^n \) and \( \bar{P} \subseteq \mathbb{Q}^n \oplus \mathbb{Q} \), the normal fans satisfy \( \mathcal{N}(P) \subseteq (\mathbb{Q}^n)^* \) and \( \mathcal{N}(\bar{P}) \subseteq (\mathbb{Q}^n \oplus \mathbb{Q})^* \) respectively.

Proposition 2.14. After identifying \( \text{ker}(\pi)^* \) with \( \text{ker}(\bar{\pi})^* \) via the vertical map \( p_1^* \) from diagram (2.2), the following fans coincide:

1. the fiber fan \( \mathcal{N}(\bar{P}, \pi(\bar{P})) \);
2. the fiber fan \( \mathcal{N}(P, \pi(P)) \);
3. the fan induced by \( \bar{\pi}^\vee(\mathcal{N}(\bar{P})) \);
4. the fan induced by \( \pi^\vee(\mathcal{N}(P)) \).

Proof. We first show that fans (1) and (2) coincide. For \( \mathbf{q} \in \mathbb{Q}^d \), set \( \bar{\mathbf{q}} := (\mathbf{q}, 1) \in \mathbb{Q}^d \oplus \mathbb{Q} \). Note that \( (\bar{P})_{\bar{\mathbf{q}}} = P_{\mathbf{q}} \), since \( P \) is the intersection of \( \bar{P} \) with \( \mathbb{Q}^d \oplus \{1\} \). Since \( \bar{P} \) is a cone, the polyhedron \( (\bar{P})_{(\mathbf{q}, j)} \) is normally equivalent to \( (\bar{P})_{(\mathbf{q}, j,1)} \) for \( j > 0 \). It follows that the fan \( \mathcal{N}(P, \pi(\bar{P})) \) is equal to the common refinement of (the images under \( p_1^* \) of) the normal fans of fibers \( (\bar{P})_{(\mathbf{q}, j)} \) for \( j > 0 \) and \( \mathbf{q} \in \mathbb{Q}^d \).
Since \((\tilde{P})_{(q,j)} = \emptyset\) for \(j < 0\), the statement that the fiber fans (1) and (2) coincide follows once we show that \(\mathcal{N}(P, \pi(P))\) is a refinement of the normal fan of \((\tilde{P})_{(q,0)}\) for all \((q,0) \in \pi(\tilde{P})\). Since \((\tilde{P})_{(q,0)} = \text{rec}(P)_q\), we have \((q,0) \in \pi(\tilde{P})\) if and only if \(q \in \pi(\text{rec}(P))\). Thus, it is enough to show that \(\mathcal{N}(P, \pi(P))\) refines \(\mathcal{N}(\text{rec}(P)_q)\) for all \(q \in \pi(\text{rec}(P))\). We show below that for all \(q \in \pi(\text{rec}(P))\) there is \(\lambda > 0\) such that \(\lambda q \in \pi(P)\), and \(\mathcal{N}(P_{\lambda q})\) refines \(\mathcal{N}(\text{rec}(P)_{\lambda q})\). Since these normal fans equal the corresponding ones with \(\lambda q\) replaced by \(q\), this shows that \(\mathcal{N}(P_q)\) and hence \(\mathcal{N}(P, \pi(P))\) is a refinement of the normal fan of \(\text{rec}(P)_q\).

Fix \(q \in \pi(\text{rec}(P))\). The condition that \(\lambda q \in \pi(P)\) is satisfied for all \(\lambda > 0\), or all \(\lambda\) if \(0 \in P\). Write \(P = \{x \in \mathbb{Q}^n : Ax \geq b\}\), where \(A\) is an \(r \times n\) matrix, and \(b \in \mathbb{Q}^r\). Write \(A\) in block form as \(A = (A'\mid A'')\), where \(A'\) is a \(r \times (n-d)\) matrix, and \(A''\) is a \(r \times d\) matrix. Then \(P_{\lambda q} = \{x \in \mathbb{Q}^{n-d} : A'x \geq -A''\lambda q\}\), while \(\text{rec}(P)_{\lambda q} = \{x \in \mathbb{Q}^{n-d} : A'x \geq -A''\lambda q\}\). We first consider the case that \(A''q \neq 0\). Since \(\lambda q \in \pi(P)\), \(P_{\lambda q}\) and \(\text{rec}(P)_{\lambda q}\) are nonempty for all \(\lambda\), so \(b - A''\lambda q\) and \(-A''\lambda q\) lie in some chambers of the chamber complex of the Gale dual of \(A'\) (see [BFS90]). For most generic choices of \(q\) the vector \(-A''q\) lies in the interior of a top-dimensional chamber, so for \(\lambda \gg 0\) the vectors \(-A''q\) and \(b - A''\lambda q\) lie in the same chamber, and thus \(P_{\lambda q}\) and \(\text{rec}(P)_{\lambda q}\) are normally equivalent. If \(q\) is not generic, then for \(\lambda \gg 0\) the vector \(b - A''\lambda q\) either lies in the same chamber as \(-A''\lambda q\), or in a larger dimensional chamber that has the chamber of \(-A''\lambda q\) as a face. In the first case \(P_{\lambda q}\) and \(\text{rec}(P)_{\lambda q}\) are normally equivalent, while in the second the normal fan of \(P_{\lambda q}\) is a refinement of that of \(\text{rec}(P)_{\lambda q}\). Finally, if \(A''q = 0\), then \(\text{rec}(P)_{\lambda q} = \text{rec}(P_{\lambda q})\) is the recession cone of \(P_{\lambda q}\), so the refinement is automatic in this case.

To see that fan (2) equals fan (3), apply Corollary 2.12 with \(C = \tilde{P}\) and \(\pi\) in place of \(\pi\). Thus, it remains to show that fans (3) and (4) coincide. The proof is a simple trick. The normal fan \(\mathcal{N}(P)\) equals the fiber fan \(\mathcal{N}(P, g(P))\) of the trivial linear map \(g : \mathbb{Q}^n \to \mathbb{Q}^0\), for which the lift \(\tilde{g}\) is the second projection \(p_2 : \mathbb{Q}^n \oplus \mathbb{Q} \to \mathbb{Q}\). The equality of the fans (1) and (2) established earlier in this proof holds for any surjective linear map, and in particular it holds the trivial map \(g\). Thus, \(p_1^*\) identifies \(\mathcal{N}(P) = \mathcal{N}(P, g(P))\) with \(\mathcal{N}(\tilde{P}, p_2(\tilde{P}))\). The equality of fans (2) and (3) then implies that \(p_1^*\) identifies \(\pi^*(\mathcal{N}(P))\) with \(\pi^*(\mathcal{N}(\tilde{P}))\). Finally, apply \(\pi^*\) to see that \(p_1^*\) identifies \(\pi^*(\mathcal{N}(P))\) with \((\pi^* \circ p_2^*)(\mathcal{N}(\tilde{P}))\). If the upward-pointing arrow in diagram 2.2 representing \(p_1^* : (\mathbb{Q}^n)^* \to (\mathbb{Q}^n \oplus \mathbb{Q})^*\) is replaced by the downward-pointing arrow representing \(p_2^* : (\mathbb{Q}^n \oplus \mathbb{Q})^* \to (\mathbb{Q}^n)^*\), the resulting diagram commutes giving \(\pi^* = p_1^* \circ \pi^* \circ p_2^*\). This proves that \(p_1^*\) identifies fans (3) and (4) as required.

\[\square\]

**Lemma 2.15.** Let \(\pi : P \to Q\) be a linear projection of polyhedra. Then \(\pi(\tilde{P}) = \tilde{Q}\), so \(\pi(\tilde{P})\) is the cone over \(Q\). Moreover, the fan induced by \(\pi(\tilde{P})\) is the cone over the subdivision of \(Q\) induced by \(\pi(P)\), so it is obtained by taking the cone over each cell in that subdivision.

**Proof.** It is easy to see that \(\pi(\text{rec}(P)) = \text{rec}(\pi(P)) = \text{rec}(Q)\), so the equality of sets \(\pi(\tilde{P}) = \tilde{Q}\) follows from the fact that \(\pi(\lambda x, \lambda) = (\lambda \pi(x), \lambda)\) for \(x \in P\). For the second statement, note first that the faces of \(\tilde{P}\) divide naturally into two disjoint
sets: those arising from faces of the recession cone of $P$; and those of the form $\tilde{E}$ for some face $E \subseteq P$. The faces in the former set lie in $Q^n \oplus \{0\} \subseteq Q^n \oplus \mathbb{Q}$, so it is enough to show that the given decomposition of $Q$ is the intersection of the slice $Q^d \oplus \{1\} \subseteq Q^d \oplus \mathbb{Q}$ with the fan obtained as the common refinement of the cones \{\(\tilde{\pi}(E) \subseteq \tilde{Q} : E \subseteq P\) is a face\}. The result follows since for every face $E \subseteq P$, the intersection of $Q^d \oplus \{1\}$ with the cone $\tilde{\pi}(E) \subseteq \tilde{Q}$ is the cell $\pi(E) \subseteq Q$. \hfill \Box

**Proposition 2.16.** Let $\pi : P \to Q$ be a linear projection of polyhedra, and set $C := (\tilde{P})^\vee$. Then the fiber fan $\mathcal{N}(C, \tilde{\pi}^\vee(C))$ is the fan induced by $\tilde{\pi}(P)$, and thus is the cone over the subdivision of $Q$ induced by $\pi(P)$.

**Proof.** Applying Corollary 2.12 to the map $\tilde{\pi}^\vee : C \to \tilde{\pi}^\vee(C)$ gives $\mathcal{N}(C, \tilde{\pi}^\vee(C)) = (\tilde{\pi}^\vee)^\vee((C^\vee))$. Since $\tilde{P}$ is a polyhedral cone, we have $C^\vee = ((\tilde{P})^\vee)^\vee = P$. The first statement follows from the fact that $(\tilde{\pi}^\vee)^\vee = \tilde{\pi}$. The second statement follows from Lemma 2.15 \hfill \Box

3. On Quotients of Toric Varieties by Subtori

3.1. GIT quotients of quasiprojective toric varieties. We first recall the polyhedral construction of quasiprojective toric varieties that are projective over affine toric varieties (see Thaddeus [Tha94, §2.8]). We work over an algebraically closed field $\mathbb{k}$.

Let $P \subseteq \mathbb{Q}^n$ be an $n$-dimensional lattice polyhedron and let $\tilde{P}$ be the closure in $\mathbb{Q}^n \oplus \mathbb{Q}$ of the cone over $P$ as given in Definition 2.14. The semigroup algebra $R := \mathbb{k}[\tilde{P} \cap (\mathbb{Z}^n \oplus \mathbb{Z})]$ defines the affine toric variety $\text{Spec } R$. The second projection determines a grading of the ring $R$. Let $R_j$ be the $\mathbb{k}$-vector subspace spanned by the elements of $R$ of degree $j \in \mathbb{N}$. Then $X_P := \text{Proj } \bigoplus_{j \geq 0} R_j$ is the $n$-dimensional quasiprojective toric variety whose fan is $\mathcal{N}(P)$, the inner normal fan of the polyhedron $P$. Note that $X_{\tilde{P}} \cong \text{Spec } R$ is the affine cone over $X_P$. Also, since $R_0$ is isomorphic to the semigroup algebra $\mathbb{k}[\text{rec}(P) \cap \mathbb{Z}^n]$ determined by the recession cone $\text{rec}(P) \subseteq \mathbb{Q}^n$, we see that $X_P$ is projective over the affine toric variety $\text{Spec } R_0$. Thus, $X_P$ is projective if $P$ is a polytope.

**Remark 3.1.** Any quasiprojective toric variety $X$ that is projective over an affine toric variety arises in this way once a relatively ample torus-invariant divisor $D$ is chosen on $X$, in which case $R_j \cong H^0(X, \mathcal{O}_X(jD))$ for all $j \in \mathbb{N}$.

Let $(\mathbb{k}^*)^n = \text{Spec } \mathbb{k}[\mathbb{Z}^n]$ denote the dense torus of $X_P$, and let $T_L := \text{Spec } \mathbb{k}[L]$ be a $d$-dimensional algebraic subtorus of $(\mathbb{k}^*)^n$. Choose generators for $L \cong \mathbb{Z}^d$ and write $\pi_\mathbb{Z} : \mathbb{Z}^n \to \mathbb{Z}^d$ for the map of character lattices induced by the inclusion $T_L \hookrightarrow (\mathbb{k}^*)^n$. Composing $\pi_\mathbb{Z}$ with the $\mathbb{Z}^n$-grading of the ring $R$ arising from the action of $(\mathbb{k}^*)^n$ on $X_P$ determines a $\mathbb{Z}^d$-grading of $R$. Given a character $\nu \in \mathbb{Z}^d$ of the torus $T_L$, the GIT quotient of $X_P$ by the action of $T_L$ linearized by $\nu$ is the quotient of the affine variety $\text{Spec } R$ by the lift of $L$. Specifically, begin by defining $\pi_\mathbb{Z} : \mathbb{Z}^n \oplus \mathbb{Z} \to \mathbb{Z}^d \oplus \mathbb{Z}$ by sending $(x, \lambda)$ to $(\pi_\mathbb{Z}(x), \lambda)$. This lattice map induces an inclusion of the $(d + 1)$-dimensional torus $T_L := \text{Spec } \mathbb{k}[\tilde{L}]$ into the dense torus $(\mathbb{k}^*)^{n+1} = \text{Spec } \mathbb{k}[\mathbb{Z}^n \oplus \mathbb{Z}]$ of $X_P$, where $\tilde{L} = L \oplus \mathbb{Z}$. Characters
Proof. We know that $\wedge \mapsto$ in the diagram (2.1) sends the polyhedron $j \in v$ in the dual diagram (2.2) identifies the normal fans $\tilde{P}$ Zelevinsky [KS92] when $t$ respectively by extending scalars. The following result is due to Kapranov–Sturmfels–Zelevinsky [KS92] when $P$ is a polytope, and to Thaddeus [Tha94] in general.

Lemma 3.2. For $v \in \pi(P)$, the GIT quotient $X_P / \!/ v T_L$ is the toric variety with fan $\mathcal{N}(P_v)$.

Proof. We know that $X_P / \!/ v T \cong X_\tilde{P} / \!/ \tilde{v} T_L$ from above. The left-most vertical arrow in the diagram (2.1) sends the polyhedron $\tilde{P}_v$ isomorphically onto $P_v$, so the left-most vertical map in the dual diagram (2.2) identifies the normal fans $\mathcal{N}(\tilde{P}_v)$ and $\mathcal{N}(P_v)$. The result follows from the fact that for $j \in \mathbb{N}$, the graded piece $R_{j\tilde{v}}$ is isomorphic to the $\mathbb{k}$-vector space spanned by the lattice points of the slice $\tilde{P} \cap \tilde{v}^{-1}(j\tilde{v})$.

□

Proposition 3.3. For $v \in \pi(P)$, the fan $\mathcal{N}(P_v)$ of $X_P / \!/ v T$ is isomorphic to the $\tilde{v}$-induced $\tilde{v}^\vee$-coherent subdivision of the cone $\tilde{v}^\vee(\tilde{P}_v) \subseteq \ker(\tilde{v})^\ast$.

Proof. Corollary 2.10 shows that $\mathcal{N}(\tilde{P}_v)$ is the $\tilde{v}$-induced $\tilde{v}^\vee$-coherent subdivision of the cone $\tilde{v}^\vee(\tilde{P}_v)$. The result follows since $\mathcal{N}(P_v)$ is isomorphic to $\mathcal{N}(\tilde{P}_v)$.

□

Definition 3.4. A character $v \in \pi(P)$ is generic if every $v$-semistable point of $X_P$ is $v$-stable in the sense of GIT (see Dolgachev–Hu [DH98 (0.2.2)]). For generic characters $v, v' \in \pi(P)$, we set $v \sim v'$ if every $v$-stable point of $X_P$ is $v'$-stable and vice-versa. This equivalence relation gives a polyhedral decomposition of $\pi(P)$, called the GIT chamber decomposition associated to the action of $T_L$ on $X_P$.

Remark 3.5. Thaddeus [Tha94] showed that the GIT chamber decomposition is the subdivision of $\mathbb{Q}$ induced by $\pi(P)$ in the sense of Definition 2.11. For analogous results for general GIT quotients, see Thaddeus [Tha96] or Dolgachev–Hu [DH98].

3.2. The toric Chow quotient. In this section we construct the toric Chow quotient of a quasiprojective toric variety by a subtorus. We first introduce a subcategory of the category of $\mathbb{k}$-schemes. Fix $T := (\mathbb{k}^\ast)^n$ for $n \in \mathbb{N}$.

Definition 3.6. Consider the category $\mathcal{C}_T$ whose objects are Noetherian $\mathbb{k}$-schemes $X$ with a $T$-action $\sigma_X : T \times X \to X$ and an irreducible $T$-invariant component $X_0 \subseteq X$, such that the restriction of the $T$-action to $X_0$, denoted $\sigma_{X_0} : T \times X_0 \to X_0$, gives $X_0$ the structure of a not-necessarily normal toric variety with dense torus $T$. The morphisms of $\mathcal{C}_T$ are $T$-equivariant proper morphisms $f : X \to X'$ over $\mathbb{k}$ that induce a birational morphism $f|_{X_0} : X_0 \to X'_0$ between the toric components.

Lemma 3.7. Fiber products exist in the category $\mathcal{C}_T$. Moreover, the fiber product in $\mathcal{C}_T$ coincides with the fiber product in the category of $\mathbb{k}$-schemes.
Proof. For an object $S$ in $\mathcal{C}_T$, let $f : X \to S$ and $f' : X' \to S$ be morphisms in $\mathcal{C}_T$. We must show that the fiber product in the category of $k$-schemes $Z := X \times_S X'$ is an object of $\mathcal{C}_T$, and that the canonical projections $p : Z \to X$ and $q : Z \to X'$ are morphisms in $\mathcal{C}_T$.

To construct the component $Z_0 \subseteq Z$, note that the schemes $X$, $X'$ and $S$ contain open subvarieties $T_X$, $T_X'$ and $T_S$ respectively, each isomorphic to the torus $T$. The fiber product $T_Z := T_X \times_{T_S} T_{X'}$ is canonically isomorphic to the subscheme $p^{-1}(T_X) \cap q^{-1}(T_{X'})$ of $Z$ by Chap. I, Coro 3.2.3, so it is open in $Z$. Furthermore, the $T$-equivariant birational morphisms $f|_{T_0} : X_0 \to S_0$ and $f'|_{T_0} : X'_0 \to S_0$ restrict to give isomorphisms $T_X \cong T_S$ and $T_{X'} \cong T_S$, so $T_Z$ is isomorphic to $T \times_T T \cong T$ and hence is irreducible. In particular, the $n$-torus $T_Z \subseteq Z$ is dense in some component of $Z$ that we denote $Z_0$.

We next show that $Z_0$ is reduced. Since this is a local question, and $T_Z \subseteq Z_0$ is reduced, we can reduce to the case where $Z_0 = \text{Spec}(A)$ is an irreducible Noetherian $k$-scheme with a nonempty open subscheme $W$ for which the induced scheme structure on $W$ is reduced. We may assume that $W = \text{Spec}(A_f)$ for some $f \in A$. Writing $A = \mathbb{k}[x_1, \ldots, x_m]/I$ for some $I$, we note that $I$ is primary since $Z_0$ is irreducible, and since $W$ is nonempty $f \not\in \sqrt{I}$. Suppose $Z_0$ is not reduced, so there exists $g \in I$ with $g^k \in I$ for some $I$. Then $(g/1)^k \in I_f$, so since $W$ is reduced we have $g/1 \in I_f$. But then $g/1 = y/f^l$ for some $l$ and some $y \in I$. This means that $gf^l \in I$, which contradicts $I$ being primary, $g \not\in I$, and $f \not\in \sqrt{I}$. We thus conclude that $Z_0$ is reduced.

We now describe the torus action. The fiber product $Z$ embeds as a closed subscheme of the product $X \times_k X'$. Since the morphisms $f$ and $f'$ are $T$-equivariant, it can be shown that the product $T$-action on $X \times_k X'$ restricts to give an action $\sigma_Z : T \times Z \to Z$. This restricts to give an action $T \times T_Z \to T_Z$ that coincides with the multiplicative structure on the torus $T_Z$ after identifying $T_Z \cong T$. Thus, $\sigma_Z : T \times Z \to Z$ extends the natural multiplicative structure of $T_Z$. To see that $Z_0$ is $T$-invariant we argue by continuity as follows. The component $Z_0 \subseteq Z$ is closed and contains $T_Z$, so $\sigma_Z^{-1}(Z)$ is a closed subscheme of $T \times Z$ that contains $\sigma_Z^{-1}(T_Z) = T \times T_Z$. In particular, $\sigma_Z^{-1}(Z)$ contains the closure $T \times Z_0$ of $T \times T_Z$, so the image of $T \times Z_0$ under $\sigma_Z$ lies in $Z_0$. The resulting action $\sigma_{Z_0} : T \times Z_0 \to Z_0$ extends the multiplicative structure of $T_Z$ on itself by the above, so $Z_0$ is a not-necessarily-normal toric variety.

It remains to prove that the projections $p$ and $q$ are morphisms in $\mathcal{C}_T$. Properness of $p$ and $q$ follows from properness of $f$ and $f'$ by base extension. The $T$-action on $Z$ was constructed to ensure that $p$ and $q$ are $T$-equivariant. Their restrictions $p|_{T_Z} : T_Z \to T_S$ and $p'|_{T_Z} : T_Z \to T_{S'}$ are isomorphisms, so both $p$ and $q$ are birational on the toric components as required. 

We now return to the GIT set-up from Section 5.1, where $\pi : P \to Q$ is a linear projection of polyhedra, with $L \cong \mathbb{Z}^d$ the image of the corresponding map $\pi_Z : \mathbb{Z}^n \to \mathbb{Z}^d$. Set $M := \ker(\pi_Z)$. For $v \in Q$, the GIT quotient $X_P \#_v T_k = X_{P_v}$ is the toric variety with dense torus $T_M$ defined by the fan $N(P_v)$.

Let $\mathcal{P}$ be the face poset of the subdivision of $Q$ induced by $\pi(P)$, with the faces on the boundary of $Q$ removed. Thus $\tau \prec \sigma$ if $\tau$ is a face of $\sigma$. To each $\sigma \in \mathcal{P}$ we
associate the toric variety $X_\sigma := X_{P_\nu}$ for any $\nu \in \sigma$. If $\tau$ is a face of $\sigma$, then there is a proper, birational toric morphism from $X_\sigma$ to $X_\tau$ by Thaddeus [Tha94, Theorem 3.11 and Corollary 3.12]. Therefore $P$ gives a directed system of $k$-schemes. We are interested in the inverse limit of this system in the category of $k$-schemes.

**Proposition 3.8.** Let $T_M$ be the dense torus in $X_\sigma$ for $\sigma \in P$. Then the inverse limit in the category of $k$-schemes

$$Z := \lim_{\sigma \in P} X_\sigma$$

exists and is an object of the category $C_{T_M}$. In particular, $Z$ has an irreducible component $Z_0$ that is a not-necessarily-normal toric variety with dense torus $T_M$.

**Proof.** For each $\sigma \in P$, the toric variety $X_\sigma$ is an object of the category $C_{T_M}$. The proof is by induction on the number of elements of $P$. The base case is when $P$ has only one maximal element $\sigma$. Then $Z = X_\sigma$ exists and is an object of $C_{T_M}$. Suppose now that the inverse limit exists, and is an object in $C_{T_M}$, for any such poset with fewer elements than $P$ whose elements are toric varieties with dense torus $T_M$, and whose maps are proper birational toric morphisms. Fix a maximal element $\sigma$ of $P$, and let $P' = P \setminus \sigma$, and let $P''$ be the subposet of all $\tau \in P$ with $\tau \prec \sigma$. Let $Z_{P'}$ and $Z_{P''}$ be the corresponding inverse limits. Define

$$Z' := Z_{P'} \times_{Z_{P''}} X_\sigma.$$

Lemma 3.7 and induction show that $Z'$ is an object in $C_{T_M}$. It remains to show that $Z'$ is the inverse limit (3.2) in the category of $k$-schemes. Let $Y$ be any $k$-scheme with maps $f_\tau$ to each $X_\tau$ for $\tau \in P$ that commute appropriately with the maps in $P$. By the universal property of the inverse limits $Z_{P'}$ and $Z_{P''}$ we get a unique map from $Z_{P'}$ to $Z_{P''}$ that commutes with the maps in $P$, and a unique map from $Y$ to $Z_{P'}$ and $Z_{P''}$ whose compositions with the maps from $Z_{P'}$ to $Z_{P''}$ and from $X_\sigma$ to $Z_{P''}$ agree. Then the universal property of fiber products gives a unique map $\phi$ from $Y$ to $Z'$ such that the composition of $\phi$ with the maps from $Z'$ to each $X_\tau$ equals the corresponding $f_\tau$, so $Z'$ is the inverse limit. \(\square\)

**Definition 3.9.** The toric Chow quotient of the toric variety $X_P$ by the subtorus $T_L$ is the not-necessarily-normal toric variety $X_P/T_L := Z_0$ arising as the toric component of the inverse limit $Z$ as in Proposition 3.8.

Corollary 3.11 below justifies the choice of terminology; see also [HS04]. First, we explain the link between the toric Chow quotient and fiber fans.

**Proposition 3.10.** Let $\overline{X_P/T_L}$ be the normalization of the toric Chow quotient. Then $\overline{X_P/T_L}$ is the toric variety with fan $\mathcal{N}(P, \pi(P))$.

**Proof.** Write $\overline{Z_0} = \overline{X_P/T_L}$. For each $\sigma \in P$ there is a proper, birational toric morphism $\overline{Z_0} \to X_\sigma$, so the fan of $\overline{Z_0}$ refines the fan of $X_\sigma$ by [Ful93, Proposition 2.4]. In particular, the fan of $\overline{Z_0}$ refines the common refinement $\mathcal{N}(P, \pi(P))$ of these fans.

The proposition follows once we show that $\mathcal{N}(P, \pi(P))$ refines the fan of $\overline{Z_0}$. It is enough to construct a proper, birational toric morphism $X_{CR} \to \overline{Z_0}$, where $X_{CR}$ is
the toric variety with fan $\mathcal N(P, \pi(P))$. Since $\mathcal N(P, \pi(P))$ refines the fan of each $X_\sigma$, there is a proper, birational toric morphism $X_{CR} \to X_\sigma$. The universal property of the inverse limit $Z$ in the category $\mathcal C_T$, then gives a proper $T_M$-equivariant morphism $\phi: X_{CR} \to Z$ that is birational onto $Z_0$. Since $X_{CR}$ is normal, the universal property of normalization gives a morphism $\overline{\phi}: X_{CR} \to \overline{Z_0}$ that is both proper (since $\phi$ is proper and the normalization is separated) and birational (since $\phi$ is birational onto $Z_0$ and the normalization is finite). It remains to show that $\overline{\phi}$ is a toric morphism.

Since $\phi$ is a dominant birational morphism of not-necessarily-normal toric varieties, it is given locally by an inclusion of subsemigroups $S \to S'$ of $M$, where $\text{Spec} \ k[S]$ is a chart on $Z_0$, $\text{Spec} \ k[S']$ is a chart on $X_{CR}$ and $T_M = \text{Spec} \ k[M]$. Note that $S'$ is saturated since $X_{CR}$ is normal. The normalization $\overline{Z_0} \to Z_0$ is given locally by the inclusion of $S$ in its saturation $S_{sat}$. Since $S_{sat}$ is the smallest saturated subsemigroup of $M$, the inclusion $S \hookrightarrow S'$ factors through $S_{sat}$. The induced semigroup morphism $S_{sat} \to S'$ gives the local description of the induced map $\overline{\phi}: X_{CR} \to \overline{Z_0}$, so $\overline{\phi}$ is a toric morphism. □

**Corollary 3.11.** Let $P$ be a polytope. Then the toric Chow quotient $X_P/T_L$ is equal to the Chow quotient in the sense of Kapranov [Kap93, Section 0.1].

**Proof.** Write $\text{Chow}(X_P/T_L)$ for the Chow quotient. For a polytope $P$, Kapranov Sturmfels-Zelevinsky [KSZ92, Proposition 2.3] proved that the normalization of the Chow quotient is the toric variety defined by the fan $\mathcal N(P, \pi(P))$, so $X_P/T_L$ and $\text{Chow}(X_P/T_L)$ share the same normalization. To see that they coincide, recall from [KSZ92, Corollary 4.3] that $\text{Chow}(X_P/T_L)$ is a subvariety of the inverse limit $Z$ defined in equation (3.2). Since $\text{Chow}(X_P/T_L)$ is irreducible and contains the dense torus $T_Z$, it is a subvariety of the component $Z_0 = X_P/T_L$ of $Z$. If this inclusion $\text{Chow}(X_P/T_L) \subseteq X_P/T_L$ were strict, $\text{Chow}(X_P/T_L)$ and $X_P/T_L$ could not share the same normalization, hence $\text{Chow}(X_P/T_L) = X_P/T_L$ as claimed. □

**Remark 3.12.** The statement of Corollary 3.11 was known to the authors of [KSZ92, Section 4], though they did not supply a proof. Thus, for a polyhedron $P$, the toric Chow quotient $X_P/T_L$ is the appropriate generalization of the Chow quotient $\text{Chow}(X_P/T_L)$.

3.3. **Proof of the main theorems.** We now prove the main theorems. Recall that the fan of a not-necessarily-normal toric variety, such as the toric Chow quotient, is the fan of its normalization.

**Theorem 3.13.** Let $\pi: P \to Q$ be a linear projection of polyhedra. Set $P' := (\overline{P})^\vee$ and $L' := \text{im}\ likeness(\overline{\pi}^\vee)$. The cone over the GIT chamber decomposition arising from the $T_L$-action on $X_P$ is the fan of the toric Chow quotient $X_{P'}/T_{L'}$.

**Proof.** The fan defining the toric Chow quotient $X_{P'}/T_{L'}$ is $\mathcal N(P', \overline{\pi}^\vee(P'))$ by Proposition 3.10. Now Proposition 2.16 implies that the fan defining the toric Chow quotient $X_{P'}/T_{L'}$ is equal to the cone over the subdivision of $Q$ induced by $\pi(P)$. The result follows from Remark 3.5 since the subdivision of $Q$ induced by $\pi(P)$ is the GIT chamber decomposition arising from the $T_L$-action on $X_P$. □
The case when $P$ is a polyhedral cone (so $X_P$ is affine) is of particular interest. For example, Cox [Cox95] showed that every simplicial, projective toric variety can be constructed as a GIT quotient of affine space by a subtorus. In the affine case, Theorem 3.13 can be strengthened as follows:

**Theorem 3.14 (Duality for affine toric quotients).** Let $P \subseteq \mathbb{Q}^n$ be a polyhedral cone and $\pi: P \to Q$ a linear projection. Set $P' := P^\vee$ and $L' := \text{im}_\mathbb{Z}(\pi^\vee)$. Then

(i) the fan of the toric Chow quotient $X_P/T_L$ is equal to the GIT chamber decomposition arising from the action of $T_L$ on $X_{P'}$; and

(ii) the GIT chamber decomposition arising from the $T_L$-action on $X_P$ is the fan of the toric Chow quotient $X_{P'}/T_{L'}$.

**Proof.** For part (i), note that the fan of $X_P/T_L$ is the fiber fan $\mathcal{N}(P, \pi(P))$ by Proposition 3.10. Corollary 2.12 shows that $\mathcal{N}(P, \pi(P))$ equals the fan induced by $\pi^\vee(P^\vee)$. This is precisely the GIT chamber decomposition arising from the action of $T_L$ on $X_{P'}$ according by Remark 3.5.

For part (ii), note that the fan of the toric Chow quotient $X_{P'}/T_{L'}$ is the fiber fan $\mathcal{N}(P^\vee, \pi^\vee(P^\vee))$. This equals $(\pi^\vee)^\vee((P^\vee)^\vee) = \pi(P)$ by applying Corollary 2.12 with $C = P^\vee$ and $\pi^\vee$ playing the role of $\pi$. □

**Remark 3.15.** If $X$ is a nonnormal quasiprojective toric variety then one can give similar combinatorial descriptions of both GIT quotients by subtori $X//\sqrt[\vee]T$, and of the normalization of the toric Chow quotient; normality played no essential role in either construction. However the description of the GIT chamber complex in Definition 3.4 is now a priori only a coarsening of the true GIT chamber complex, as the quotient toric varieties are no longer determined solely by their fans.

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Department of Mathematics, Stony Brook University, Stony Brook, NY 11794
E-mail address: craw@math.sunysb.edu

Department of Mathematics, Hill Center-Busch Campus, Rutgers University, 110 Frelinghuysen Rd, Piscataway, NJ 08854
E-mail address: maclagan@math.rutgers.edu