A new perspective on the distance problem over prime fields

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Abstract

Let \( \mathbb{F}_p \) be a prime field, and \( \mathcal{E} \) a set in \( \mathbb{F}_p^2 \). Let \( \Delta(\mathcal{E}) = \{||x - y|| : x, y \in \mathcal{E}\} \), the distance set of \( \mathcal{E} \). In this paper, we provide a quantitative connection between the distance set \( \Delta(\mathcal{E}) \) and the set of rectangles determined by points in \( \mathcal{E} \). As a consequence, we obtain a new lower bound on the size of \( \Delta(\mathcal{E}) \) when \( \mathcal{E} \) is sufficiently small, improving a previous estimate due to Lund and Petridis and establishing an approach that should lead to significant further improvements.

1 Introduction

Let \( \mathbb{F}_p \) be the prime field of order \( p \). Given two points \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) in \( \mathbb{F}_p^2 \), the distance between \( x \) and \( y \) is defined by

\[ ||x - y|| := (x_1 - y_1)^2 + (x_2 - y_2)^2. \]

For \( \mathcal{E} \subset \mathbb{F}_p^2 \), define

\[ \Delta(\mathcal{E}) := \{||x - y|| : x, y \in \mathcal{E}\}. \]

Bourgain, Katz, and Tao [2] proved that if \( |\mathcal{E}| = p^\alpha \), \( 0 < \alpha < 2 \), and \( p \equiv 3 \) mod 4, then

\[ |\Delta(\mathcal{E})| \gg |\mathcal{E}|^{\frac{1}{2} + \epsilon} \]  \hspace{1cm} (1)

for some \( \epsilon = \epsilon(\alpha) > 0 \), where here and throughout, \( X \gg Y \) means that there exists a uniform constant \( C \) such that \( X \geq CY \).

The exponent \( \frac{1}{2} + \epsilon \) has been quantified and improved over the years. Stevens and De Zeeuw [15] used a new point-line incidence bound and the framework in [2] to show that

\[ |\Delta(\mathcal{E})| \gg |\mathcal{E}|^{\frac{1}{2} + \frac{1}{10}} = |\mathcal{E}|^{\frac{6}{11}} \]  \hspace{1cm} (2)

under the condition that \( |\mathcal{E}| \ll p^{\frac{15}{17}} \) with \( p \equiv 3 \) mod 4.

Iosevich, Koh, Pham, Shen, and Vinh [3] improved this result by using a lower bound on the number of distinct distances between a line and a set in \( \mathbb{F}_p^2 \), and the additive energy of

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a set on the paraboloid in \( \mathbb{F}_p^3 \). They proved that for \( \mathcal{E} \subset \mathbb{F}_p^2 \) with \( p \equiv 3 \mod 4 \), if \( |\mathcal{E}| \ll p^{3}, \) then
\[
|\Delta(\mathcal{E})| \gtrsim |\mathcal{E}|^{\frac{128}{147}} = |\mathcal{E}|^{\frac{4}{7} + \frac{13}{63}}.
\]

This result has been recently improved by Lund and Petridis in [9], namely, they indicated that
\[
|\Delta(\mathcal{E})| \gtrsim |\mathcal{E}|^{\frac{4}{7} + \frac{13}{63}},
\]
under the condition that \( |\mathcal{E}| \ll p^{8/5} \) with \( p \equiv 3 \mod 4 \).

Here and throughout, \( X \gtrsim Y \) means that for every \( \epsilon > 0 \), there exists \( C_\epsilon \) such that \( X \leq C_\epsilon Y \), and \( X \sim Y \) means that \( c_3 X \leq Y \leq c_4 X \) for some positive constants \( c_3 \) and \( c_4 \), independent of \( p \).

**Remark 1.1.** In the setting of arbitrary finite fields \( \mathbb{F}_q \), Iosevich and Rudnev [6] showed that the conclusion [1] does not hold. For instance, assume that \( q = p^2 \), then one can take \( \mathcal{E} = \mathbb{F}_p^2 \), then it is not hard to see that \( |\Delta(\mathcal{E})| = |\mathbb{F}_p| = |\mathcal{E}|^{1/2} \). Thus, they reformulated the problem in the spirit of the Falconer distance conjecture in the Euclidean space. More precisely, for \( \mathcal{E} \subset \mathbb{F}_q^2 \), how large does \( \mathcal{E} \) need to be to guarantee that \( \Delta(\mathcal{E}) \) covers either the whole field or a positive proportion of all elements in \( \mathbb{F}_q^2 \)?

Using Fourier analytic methods, Iosevich and Rudnev [6] proved that for \( \mathcal{E} \subset \mathbb{F}_q^d \), if \( |\mathcal{E}| \geq 4q^{d+1/2} \), then \( \Delta(\mathcal{E}) = \mathbb{F}_q \). It has been shown in [3] that the exponent \((d+1)/2\) cannot in general be improved when \( d \) is odd, even if we only want to recover a positive proportion of all the distances. However, in even dimensional spaces, it has been conjectured that the exponent \((d+1)/2\) can be decreasing to \( d/2 \), which is in line with the Falconer distance conjecture in the Euclidean space. Chapman, Erdogan, Koh, Hart and Iosevich [3] proved that if \( \mathcal{E} \subset \mathbb{F}_q^d \), \( q \) is prime, \( q \equiv 1 \mod 4 \) and \( |\mathcal{E}| \geq q^{d/2} \), then \( |\Delta(\mathcal{E})| \gg q \). In the process, they showed that if \( Cq \leq |\mathcal{E}| \leq q^{d/2} \) for a sufficiently large constant \( C \), then \( |\Delta(\mathcal{E})| \gg \frac{|\mathcal{E}|^{d/2}}{q} \). This result was generalized to arbitrary finite fields in [1]. See also [7] for some recent progress on related problems.

The main purpose of this paper is to provide a connection between the distance set \( \Delta(\mathcal{E}) \) and the set of rectangles determined by points in \( \mathcal{E} \) in the plane over prime fields, and use this paradigm to improve the known exponents. In particular, we obtain a new lower bound on the size of \( \Delta(\mathcal{E}) \) when \( \mathcal{E} \) is not too large.

For \( a, b, c \in \mathbb{F}_p^2 \), we say that the triple \( (a, b, c) \) is a *corner* if
\[ (b - a) \cdot (c - a) = 0. \]

For \( a, b, c, d \in \mathbb{F}_p^2 \), we say that the quadruple \( (a, b, c, d) \) forms a *rectangle* if the triples \( (a, b, d), (b, a, c), (c, b, d) \), and \( (d, a, c) \) are corners. The rectangle \( (a, b, c, d) \) is called *non-degenerate* if not all its vertices are on a line. For \( \mathcal{E} \subset \mathbb{F}_p^2 \), let \( \square(\mathcal{E}) \) be the number of non-degenerate rectangles determined by points in \( \mathcal{E} \).

Our main result is the following.

**Theorem 1.1.** Let \( \mathcal{E} \) be a set in \( \mathbb{F}_p^2 \) with \( p \equiv 3 \mod 4 \). Suppose that \( |\mathcal{E}| \ll p^{8/5} \), then
\[
T(\mathcal{E}) \ll |\mathcal{E}|^2 \log |\mathcal{E}| + |\Delta(\mathcal{E})|^{\frac{8}{7}} |\mathcal{E}|^{\frac{12}{7}} + |\mathcal{E}|^{\frac{5}{7}} |\Delta(\mathcal{E})|^{\frac{15}{14}} \square(\mathcal{E})^{\frac{1}{7}},
\]
where \( T(\mathcal{E}) \) denotes the number of isosceles triangles determined by \( \mathcal{E} \) and \( \square(\mathcal{E}) \) is the number of non-degenerate rectangles determined by \( \mathcal{E} \), as above.
Consequently,

$$|\Delta(\mathcal{E})| \gg \min \left\{ |\mathcal{E}|^{\frac{12}{19}}, \frac{|\mathcal{E}|^{19}}{\Box(\mathcal{E})^{11}} \right\}.$$ 

In order to use Theorem 1.1 to full effect, we need the following upper bound on $\Box(\mathcal{E})$ due to Lewko.

**Lemma 1.2** ([8], Theorem 4). Let $\mathcal{E}$ be a set in $\mathbb{F}_p^2$ with $p \equiv 3 \mod 4$. Suppose that $|\mathcal{E}| \ll p^{26/21}$, then we have

$$\Box(\mathcal{E}) \lesssim |\mathcal{E}|^{99/41}.$$ 

As a consequence of Theorem 1.1, we obtain a new lower bound on the size of $\Delta(\mathcal{E})$ when $\mathcal{E}$ is sufficiently small.

**Corollary 1.3.** Let $\mathcal{E}$ be a set in $\mathbb{F}_p^2$ with $p \equiv 3 \mod 4$. Suppose that $|\mathcal{E}| \ll p^{1558/1489}$, then we have

$$|\Delta(\mathcal{E})| \gtrsim |\mathcal{E}|^{1/2 + 69/1558}.$$ 

We note that the condition $|\mathcal{E}| \ll p^{26/21}$ is replaced by $|\mathcal{E}| \ll p^{1558/1489}$, since in the range $p^{1558/1489} \leq |\mathcal{E}| \leq p^{26/21}$, the bound $p^{-1}|\mathcal{E}|^{3/2}$ due to Chapman et al. [3] is better.

**Remark 1.2.** We observe that our exponent in Corollary 1.3 is $1/2 + 69/1558 \approx 0.54428$, and the exponent due to Lund and Petridis in [10] is $1/2 + 3/74 \approx 0.54054$. It is important to note that our method is completely different. Moreover, if the Lewko bound $\Box(\mathcal{E}) \lesssim |\mathcal{E}|^{119/108}$ is improved to the conjectured bound $\Box(\mathcal{E}) \lesssim |\mathcal{E}|^2$, then the exponent in Corollary 1.3 would improve to $\approx 0.6315$. Further progress would result from improving the point line incidence bound used in the proof of Theorem 1.1. The conjectured point line incidence bound combined with the conjectured bound on $\Box(\mathcal{E})$ would improve the exponent in Corollary 1.3 to $\frac{3}{4}$. This is the limitation of our method.

**Remark 1.3.** In the case of finite subsets of $\mathbb{R}^2$, the sharp bound on the number of rectangles was established by Pach and Sharir ([10]). This raises the possibility of using this approach in the continuous Euclidean setting. We shall address this issue in the sequel.

Assuming that $\mathcal{E} = A \times A \subset \mathbb{F}_p^2$ has Cartesian product structures, Petridis [11] used the point-plane incidence bound due to Rudnev [13] to prove that

$$|\Delta(A \times A)| = |(A - A)^2 - (A - A)^2| \gg |A|^{3/2},$$

under the assumption $|A| \leq p^{2/3}$. This result has been extended to all dimensions by Pham, Vinh and De Zeeuw [12].

In the following theorem, we will break the exponent $3/2$ for a variant of the distance function, namely, $(A - A)^2 - (A - A)^2$ instead of $(A - A)^2 + (A - A)^2$.

**Theorem 1.4.** For $A \subset \mathbb{F}_p$ with $|A| \ll p^{71/127}$, we have

$$|(A - A)^2 - (A - A)^2| \gtrsim |A|^{7 + 187/127}.$$
In the proof of Theorem 1.4, the following theorem which is interesting on its own plays the key role.

**Theorem 1.5.** Let $A$ be a set in $\mathbb{F}_p$. Suppose that $|A||A - A||A^2 - A^2| \leq p^2$ and $|A - A| = |A|^{1+\epsilon}$ with $0 < \epsilon < 1/54$. Then we have

$$|A^2 - A^2| \gtrsim |A|^{1+\frac{2-2\epsilon}{1+\epsilon}}.$$ 

To prove Theorem 1.5, we use a new sum-product idea which has been introduced recently by Rudnev, Shakan, and Shkredov [14, Theorem 3]. Note that in [14] Rudnev et al. proved that for sufficiently small $A$ in $\mathbb{F}_p$, if $|AA| \leq |A|^{1+\epsilon}$ for some positive small $\epsilon$, then $|A - A| \geq |A|^{\frac{3}{2} + c(\epsilon)}$. As a consequence of this result, they derived that $|AA - AA| \gtrsim |A|^{\frac{3}{2} + \frac{1}{56}}$. However, their method does not imply the same result when we replace $A - A$ by $A + A$. Therefore, breaking the exponent $3/2$ on the size of the set $AA + AA$ or the set $(A - A)^2 + (A - A)^2$ is still an open question. We refer the interested reader to [14] for more discussions.

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## 2 Proof of Theorem 1.1

To prove Theorem 1.1, we will make use of the following lemmas. Let $E$ be a set in $\mathbb{F}_p^2$. For any two points $a, b \in E$ with $\|a - b\| \neq 0$, we define $l_{ab}$ as the line defined by the equation $\|x - a\| = \|x - b\|$; namely,

$$x \cdot 2(b - a) = \|b\| - \|a\|.$$  

(3)

This line is the bisector of the line segment joining points $a$ and $b$ in $\mathbb{F}_p^2$.

We observe that if there is a pair $(c, d) \in E \times E$ such that

$$(d - c, \|d\| - \|c\|) = \lambda \cdot (b - a, \|b\| - \|a\|)$$

for some $\lambda \neq 0$, then the lines $l_{ab}$ and $l_{cd}$ are the same.

For $E \subset \mathbb{F}_p^2$, we define

$$Q(E) := \{(a, b, c, d) \in E^4 : l_{ab} = l_{cd}\}.$$  

For $E \subset \mathbb{F}_p^2$, let $L$ be the multi-set of lines $l_{ab}$ with $a, b \in E$ and $\|a - b\| \neq 0$. Let $T(E)$ be the number of isosceles triangles determined by points in $E$. In other words, we have

$$T(E) = |\{(a, b, c) \in E^3 : \|a - c\| = \|b - c\| \neq 0\}|.$$  

(4)

It is clear that $T(E)$ is bounded by the number of incidences between $E$ and $L$.

Using the point-line incidence bound due to Stevens and De Zeeuw [15], Lund and Petridis proved the following lemma.
Lemma 2.1 ([9], Proof of Lemma 12). Let $\mathcal{E}$ be a set in $\mathbb{F}_p^2$ with $|\mathcal{E}| \ll p^{8/5}$. Then we have

$$T(\mathcal{E}) \ll |\mathcal{E}|^2 \log |\mathcal{E}| + |\mathcal{E}|^{5/3}|Q(\mathcal{E})|^{4/15}.$$ 

In our next lemma, we give an upper bound of $|Q(\mathcal{E})|$ in terms of $|\Delta(\mathcal{E})|$ and $\square(\mathcal{E})$.

Lemma 2.2. For $\mathcal{E} \subset \mathbb{F}_p^2$, we have

$$|Q(\mathcal{E})| \ll |\Delta(\mathcal{E})| \bigl( \square(\mathcal{E}) + |\mathcal{E}|^2 \bigr).$$

Proof. We partition the set $\{(a, b) \in \mathcal{E} \times \mathcal{E} : ||a - b|| \neq 0\}$ into subsets $\{S_i\}_i$ in a way that in each set $S_i$, if $(a, b), (c, d) \in S_i$, then $l_{ab} = l_{cd}$. Suppose that there are $n$ such sets $S_i$.

It is not hard to see that

$$|Q(\mathcal{E})| = \sum_{i=1}^{n} |S_i|^2.$$ 

We now bound the size of $S_i$ as follows.

Fix an element $(a, b) \in S_i$. We now partition the set $S_i$ into subsets $\{S_{i\lambda}\}_{\lambda \in \Lambda_i \subset \mathbb{F}_p^{2}}$ in a way that if $\lambda \in \Lambda_i$ and $(c, d) \in S_{i\lambda}$, then

$$\lambda \cdot (2(b - a), ||b|| - ||a||) = (2(d - c), ||d|| - ||c||).$$

Note that $\lambda \cdot (2(b - a), ||b|| - ||a||) = (2(d - c), ||d|| - ||c||)$ is equivalent to $\lambda \cdot (b - a, ||b|| - ||a||) = (d - c, ||d|| - ||c||)$

Suppose that for each $i$ there are $m_i$ such subsets. Namely, $|\Lambda_i| = m_i$ for $i = 1, 2, \ldots, n$.

We observe that if $(u, v) \in S_{i\lambda}$ and $(c, d) \in S_{i\lambda'}$, then

$$(u - v, ||v|| - ||u||) = (v, ||v||) - (u, ||u||) = (\lambda/\lambda') \cdot (d - c, ||d|| - ||c||) = (\lambda/\lambda') \cdot (d, ||d||) - (c, ||c||).$$

We now show that $m_i \ll |\Delta(\mathcal{E})|$. Indeed, assume that $||a - b|| = 1$, then for any pair $(c, d) \in S_{i\lambda}$ with $\lambda \in \Lambda_i$, we have $||c - d|| = \lambda^2 \in \Delta(\mathcal{E})$. Therefore, $m_i \ll |\Delta(\mathcal{E})|$.

Hence, it follows by the Cauchy-Schwarz inequality that

$$|S_i|^2 = \left( \sum_{\lambda \in \Lambda_i} |S_{i\lambda}| \right)^2 \ll |\Delta(\mathcal{E})| \sum_{\lambda \in \Lambda_i} |S_{i\lambda}|^2.$$ 

On the other hand, we observe that $\sum_{1 \leq i \leq n} \sum_{\lambda \in \Lambda_i} |S_{i\lambda}|^2$ is equal to the number of quadruples $(a, b, c, d) \in \mathcal{E}^4$ such that $a \neq b, c \neq d$ and $(b - a, ||b|| - ||a||) = (d - c, ||d|| - ||c||)$. We note that the relation

$$(b - a, ||b|| - ||a||) = (d - c, ||d|| - ||c||)$$

is equivalent to

$$(b, ||b||) - (a, ||a||) + (c, ||c||) = (d, ||d||). \quad (5)$$

From this equation, we have

$$||b|| + ||c|| - ||a|| = ||b + c - a||.$$ 

This can be rewritten as

$$(b - a) \cdot (c - a) = 0.$$
Thus, \((a, b, c)\) is a corner. If we switch the roles of \((a, ||a||), (b, ||b||), (c, ||c||),\) and \((d, ||d||)\) in \((5)\), then we will be able to show that \((c, a, d), (d, c, b),\) and \((b, d, a)\) are also corners. This means that \((b, a, c, d)\) is a rectangle.

We note that if \((a, b) = (c, d)\) then the rectangle \((b, a, c, d)\) is degenerate. However, the number of such degenerate rectangles is at most \(|E|^2\). In other words, we have proved that
\[
\sum_{1 \leq i \leq n} \sum_{\lambda \in \Lambda_i} |S_{\lambda}|^2 \leq \Box(E) + |E|^2.
\]
This completes the proof of the lemma.

**Proof of Theorem 1.1:** For \(t \in \mathbb{F}_p\), let \(\nu(t)\) be the number of pairs \((x, y) \in E \times E\) such that \(||x - y|| = t\). By the Cauchy-Schwarz inequality (or see the proof of Theorem 1.1 in [5]), we have
\[
\sum_{t \in \mathbb{F}_p} \nu(t)^2 \leq |E|^2 \log |E| + |\Delta(E)| \frac{1}{\sqrt{p}} |E| |E| + |E| \frac{1}{\sqrt{p}} |\Delta(E)| \frac{1}{\sqrt{p}} \Box(E) \frac{1}{\sqrt{p}}.
\]
(6)

It follows from Lemmas 2.1 and 2.2 that
\[
T(E) \ll |E|^2 \log |E| + |\Delta(E)| \frac{1}{\sqrt{p}} |E| |E| + |E| \frac{1}{\sqrt{p}} |\Delta(E)| \frac{1}{\sqrt{p}} \Box(E) \frac{1}{\sqrt{p}}.
\]
(7)

Combining (6) and (7), we have
\[
\sum_{t \in \mathbb{F}_p} \nu(t)^2 \ll |E|^3 \log |E| + |\Delta(E)| \frac{1}{\sqrt{p}} |E| |E| + |E| \frac{1}{\sqrt{p}} |\Delta(E)| \frac{1}{\sqrt{p}} \Box(E) \frac{1}{\sqrt{p}}.
\]

Applying the Cauchy-Schwarz inequality and using the above inequality, we have
\[
|\Delta(E)| \gg \frac{|E|^4}{\sum_{t \in \mathbb{F}_p} \nu(t)^2} \gg \frac{|E|^4}{|E|^3 \log |E| + |\Delta(E)| \frac{1}{\sqrt{p}} |E| |E| + |E| \frac{1}{\sqrt{p}} |\Delta(E)| \frac{1}{\sqrt{p}} \Box(E) \frac{1}{\sqrt{p}}}
\]

Solving this inequality, we get the desired result.

**Proof of Corollary 1.3:** The proof follows directly from Theorem 1.1 and Lemma 1.2.

### 3 Proof of Theorem 1.4

To prove Theorem 1.4 we will make use of the following results. The first lemma was given by Pham, Vinh and De Zeeuw in [12].

**Lemma 3.1** ([12], Corollary 3.1). Let \(A, X\) be sets in \(\mathbb{F}_p\) with \(|X| \gg |A|\). Then we have
\[
|X - (A - A)|^2 \gg \min\{p, |X|^{1/2}|A|\}.
\]

Notice that our next theorem is a more general form of Theorem 1.5. A proof of the following theorem will be given after proving Theorem 1.3.

**Theorem 3.2.** Let \(A\) be a set in \(\mathbb{F}_p\). Suppose that \(|A||A - A||A^2 - A^2| \leq p^2\), \(|A - A| = M|A|\), and \(|A^2 - A^2| = K|A|\). Then we have that \(M^{27}K^{17} \geq |A|^9\) or \(M^{13}K^7 \geq |A|^5\).
Proof of Theorem 1.4: By a translation if necessary, we assume that \(0 \in A\). Let \(\epsilon > 0\) be a parameter chosen at the end of the proof.

If \(|A - A| \geq |A|^{1+\epsilon}\), then it follows from Lemma 3.1 that
\[
|(A - A)^2 - (A - A)^2| \gg |A|^{\frac{3}{2} + \frac{\epsilon}{2}},
\]
provided that \(|A| \leq p^{2/(3+\epsilon)}\).

On the other hand, if \(|A - A| \leq |A|^{1+\epsilon}\), then we now fall into two cases:

Case 1: If \(|A - A||A||A^2 - A^2| \geq p^2\), then we have
\[
|(A - A)^2 - (A - A)^2| \geq |A^2 - A^2| \geq \frac{p^2}{|A - A||A|} \geq \frac{p^2}{|A|^{2+\epsilon}},
\]
where the first inequality above holds since \(0 \in A\).

Case 2: If \(|A - A||A||A^2 - A^2| \leq p^2\), then it follows from Theorem 1.5 that \(|A^2 - A^2| \gg |A|^{1+\frac{2+7\epsilon}{11}}\). Since \(0 \in A\), we have
\[
|(A - A)^2 - (A - A)^2| \geq |A^2 - A^2| \gg |A|^{1+\frac{2+7\epsilon}{11}}.
\]

We choose \(\epsilon = 1/71\). Then we obtain the required conclusion in the cases of (5) and (10). Choosing \(\epsilon = 1/71\), the case of (9) also gives the desirable consequence since we have
\[
|(A - A)^2 - (A - A)^2| \geq \frac{p^2}{|A|^{2+\epsilon}} \geq |A|^{\frac{3}{2} + \frac{\epsilon}{2}},
\]
under the condition \(|A| \leq p^{71/125}\). This completes the proof of the theorem. \(\square\)

3.1 Proof of Theorem 3.2

In the proof of Theorem 3.2, we will use the point-plane incidence bound due to Rudnev [13], but we use a strengthened version of this theorem, proved by de Zeeuw in [16]. Let us first recall that if \(R\) is a set of points in \(\mathbb{F}_p^3\) and \(S\) is a set of planes in \(\mathbb{F}_p^3\), then the number of incidences between \(R\) and \(S\), denoted by \(I(R, S)\), is the cardinality of the set \(\{(r, s) \in R \times S : r \in s\}\).

Theorem 3.3 (Rudnev, [13]). Let \(R\) be a set of points in \(\mathbb{F}_p^3\) and \(S\) be a set of planes in \(\mathbb{F}_p^3\), with \(|R| \leq |S|\). Suppose that there is no line that contains \(k\) points of \(R\) and is contained in \(k\) planes of \(S\). Then
\[
I(R, S) \ll \frac{|R||S|}{p} + |R|^{1/2}|S| + k|S|.
\]

In this section, we assume that \(|A - A| = M|A|\) and \(|A^2 - A^2| = K|A|\). For \(A \subset \mathbb{F}_p\), we define \(E_4(A^2)\) as the number of 8-tuples \((x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in (A^2)^8\) such that
\[
x_1 - y_1 = x_2 - y_2 = x_3 - y_3 = x_4 - y_4.
\]

In the following lemma, we give an upper bound of \(E_4(A^2)\) in terms of \(|A|\) and \(|A - A|\).
Lemma 3.4. Let $A$ be a set in $\mathbb{F}_p$. Suppose that $|A - A||A||A^2 - A^2| \ll p^2$, then we have

$$E_4(A^2) \lesssim |A|^4 M^3.$$  

Proof. For each $x \in \mathbb{F}_p$, we define $r_{A^2 - A^2}(x)$ as the number of pairs $(y, z) \in A^2 \times A^2$ such that $y - z = x$. For $1 \leq k \ll |A|$, let

$$n_k := |X_k| := \{x \in A^2 - A^2 : r_{A^2 - A^2}(x) \geq k\}.$$  

By a dyadic pigeon-hole argument, we have

$$E_4(A^2) \ll \sum_k n_k \cdot k^4.$$  

Thus, it is enough to show that

$$n_k \lesssim \frac{M^3|A|^4}{k^4}.$$  

To this end, we consider the following equation

$$(x + u)^2 - y^2 = t \tag{11}$$  

with $x \in A - A, u \in A, y \in A, t \in X_k$. Let $N$ be the number of solutions of this equation. It is clear that $N \geq k|X_k||A|$. By the Cauchy-Schwarz inequality, we have

$$N \leq |A|^{1/2} \sqrt{|\{(x, u, t, x', u', t') \in ((A - A) \times A \times X_k)^2 : (x + u)^2 - t = (x' + u')^2 - t'\}|}$$

$$=: |A|^{1/2} \sqrt{N'}.$$  

To bound $N'$, we let $\mathcal{R}$ be the set of points of the form $(2x, u', -t + x^2 - u'^2)$ with $x \in A - A, u' \in A, t \in X_k$. Let $\mathcal{S}$ be the set of planes of the form $uX - (2x')Y + Z = x^2 - u^2 - t'$ with $u \in A, x' \in A - A, t' \in X_k$. We have $|\mathcal{R}| = |\mathcal{S}| \sim |A - A||A||X_k|$. It is not hard to see that there are at most $|A - A|$ collinear points in $\mathcal{R}$ except that there are vertical lines supporting $|X_k|$ points, but planes in $\mathcal{S}$ contain no vertical lines. Thus, we can apply Theorem 3.3 with $k = |A - A|$ to get

$$N' = I(\mathcal{R}, \mathcal{S}) \ll |A - A|^{3/2}|A|^{3/2}|X_k|^{3/2} + |A - A|^2|A||X_k|, \tag{12}$$

where we also use the assumption $|A - A||A||A^2 - A^2| \ll p^2$ and the fact that $|X_k| \leq |A^2 - A^2|$. If the second term of the RHS of the inequality (12) dominates, then we get

$$|X_k| \leq \frac{|A - A|}{|A|} = M.$$  

So, $E_4(A^2) \lesssim M|A|^4$.

If the first term dominates, then from the inequalities $k|X_k||A| \leq N \leq |A|^{1/2} (N')^{1/2}$ we have

$$|X_k| \leq \frac{|A|^4 M^3}{k^4}.$$  

With this upper bound of $|X_k|$, we have

$$E_4(A^2) \lesssim |A|^4 M^3.$$  

This completes the proof of the lemma. \qed
Let $P$ be the subset of $A^2 - A^2$ such that for any $x \in P$, we have $r_{A^2 - A^2}(x) \geq \frac{|A|}{2K}$. For any $w \in A^2 - A^2$, let $P_w := (A^2 - A^2) \cap (P - w)$. One can follow the first paragraph of the proof of [14, Theorem 3] to prove the following lemma.

**Lemma 3.5.** For $A \subset \mathbb{F}_p$, we have

$$|A|^4 \ll \sqrt{E_4(A^2)}\sqrt{X},$$

where

$$X = \sum_{w \in A^2 - A^2} \left| \{(u, v) \in P_w \times P_w : u - v \in P\} \right|.$$

**Proof.** We consider the following equation:

$$x - u = y - v = w,$$

with $x, y \in P$, $u, v, w \in A^2 - A^2$. It follows from the definition of $P$ that

$$|\{(a_1, a_2) \in A \times A : a_1^2 - a_2^2 \in P\}| \gg |A|^2.$$

One can use the dyadic pigeon-hole to show that there exists a subset $A' \subset A$ with $|A'| \gtrsim |A|$ such that for any $y \in A'$ the number of $x \in A$ such that $x^2 - y^2 \in P$ is at least $|A|$. For each $y \in A'$, we denote the set of such $x$ by $N_y$.

For any $c \in A$ and $b \in A'$, we have

$$c^2 - b^2 = (a^2 - b^2) - (a^2 - c^2) = (d^2 - b^2) - (d^2 - c^2).$$

(14)

Thus, $(x, y, u, v) = (a^2 - b^2, d^2 - b^2, a^2 - c^2, d^2 - c^2)$, with $b \in A'$, $a, d \in N_b, c \in A$, is a solution of (13). In other words, there are at least $|A|^4$ tuples $(a, b, c, d) \in A^4$ which gives us solutions of (13) in the form of (14).

For each tuple $(a, b, c, d) \in A^4$, we define $[a^2, b^2, c^2, d^2]$ as the set of tuples $(a', b', c', d') \in A^4$ such that $(a^2, b^2, c^2, d^2) = (a^2, b^2, c^2, d^2) + (t, t, t, t)$ for some $t \in \mathbb{F}_p$. It is not hard to check that this defines an equivalence class by translation. On the other hand, each equivalence class gives us an unique solution $(x, y, u, v)$ of (13). Therefore, by the Cauchy-Schwarz inequality, we have

$$|A|^4 \ll \sqrt{\sum_{[a^2, b^2, c^2, d^2]} \left| \{(x, y, u, v) \in P^2 \times (A^2 - A^2)^2 : x - u = y - v = w\} \right|} \cdot \sqrt{E_4(A^2)\sqrt{X}}.$$

This completes the proof of the lemma.

In the next step, we need to bound $X$. To this end, we need the following lemmas.

**Lemma 3.6.** For any $w \in A^2 - A^2$, suppose that $|A - A||A^2 - A^2| \ll p^2$. Then we have

$$T_w := \left| \{(u, v) \in P_w \times P_w : u - v \in P\} \right| \ll M^{3/2}K|P_w|^{3/2} + M^2K|P_w|.$$
Proof. For any \( p \in P \), we have \( r_{A^2-A^2}(p) \gg \frac{|A|}{K} \), so

\[
T_w \cdot \frac{|A|}{K} \cdot |A|^2 \ll |\{(x, y, z, t, u, v) \in (A - A)^2 \times A^2 \times P_w^2 : (x + z)^2 + (y + t)^2 = u - v\}|.
\]

By repeating the argument of the proof of Lemma 3.4 with \( P_w \) in the place of \( X_k \), we have the number of such tuples \((x, y, z, t, u, v)\) bounded by

\[
|A - A|^{3/2}|P_w|^{3/2}|A|^{3/2} + |A||P_w||A - A|^{2} \ll M^{3/2}|A|^{3/2}|P_w|^{3/2} + M^2|A|^{3}|P_w|.
\]

This gives us

\[
T_w \ll M^{3/2}K|P_w|^{3/2} + M^2K|P_w|.
\]

Suppose we sort the set \( A^2 - A^2 \) in the following order \( A^2 - A^2 := \{w_1, \ldots, w_{|A^2 - A^2|}\} \) with \( r_{P_{-(A^2-A^2)}}(w_i) \geq r_{P_{-(A^2-A^2)}}(w_j) \) for any \( i \geq j \).

**Lemma 3.7.** Suppose that \(|A^2 - A^2||A - A||A| \leq p^2\). For any \( 1 \leq n \leq |A^2 - A^2| \), we have

\[
|P_{w_n}| \leq M^{3/2}K^2|A|n^{-1/2}.
\]

**Proof.** Set \( W_t := \{w \in A^2 - A^2 : r_{P_{-(A^2-A^2)}}(w) \geq t\} \), i.e. \( W_t \) is the set of elements \( w \) such that there are at least \( t \) pairs \((x, y) \in P \times (A^2 - A^2)\) such that \( w = x - y \). It is not hard to check that

\[
t|W_t| \cdot \frac{|A|}{K} |A|^2 \leq |\{(w, x, y, z, t, u) \in W_t \times (A-A)^2 \times A^2 \times (A^2 - A^2) : w = (x+z)^2 + (y+t)^2 - (u-v)^2\}|.
\]

We repeat the argument of the proof of Lemma 3.3 with \( \mathcal{R} := \{(2x, v, x^2 + v^2 - w) : x \in A - A, v \in A, w \in W_t\} \) and \( \mathcal{S} := \{zX + 2yY + Z = u : z \in A, y \in A - A, u \in A^2 - A^2\} \) to bound the number of such tuples \((w, x, y, z, t, u)\) by

\[
|A - A|^{3/2}|A|^{3/2}|W_t|^{1/2}|A^2 - A^2| + |A - A|^{2}|A^2 - A^2||A|.
\]

If the second term dominates, we have

\[
|W_t| \leq \frac{|A - A|}{|A|} = M \leq \frac{K^4M^3|A|^2}{t^2}.
\]

Otherwise, we get

\[
|W_t|^{1/2} \leq \frac{K^2M^{3/2}|A|}{t}.
\]

We observe that \( r_{P_{-(A^2-A^2)}}(w_n) = |P_{w_n}| \). So,

\[
n = \left| \{w \in A^2 - A^2 : r_{P_{-(A^2-A^2)}}(w) \geq |P_{w_n}| \} \right| \leq \frac{K^4M^3|A|^2}{|P_{w_n}|^2}.
\]

In other words,

\[
|P_{w_n}| \leq M^{3/2}|A|K^2n^{-1/2}.
\]

This completes the proof of the lemma. \( \square \)
Lemma 3.8. Let $A$ be a set in $\mathbb{F}_p$. Suppose that $|A^2 - A^2||A - A||A| \leq p^2$. Then we have

$$X \leq M^{15/4} K^{17/4} |A|^{7/4} + M^{7/2} K^{7/2} |A|^{3/2}.$$ 

Proof. Applying Lemmas 3.6 and 3.7 we have

$$X \leq \sum_{w_n \in A^2 - A^2} |T_{w_n}| \ll M^{3/2} K \sum_{w_n} |P_{w_n}|^{3/2} + M^2 K \sum_{w_n} |P_{w_n}|$$

$$\leq M^{15/4} K^{17/4} |A|^{7/4} + M^{7/2} K^{7/2} |A|^{3/2}.$$ 

This completes the proof of the lemma. □

Proof of Theorem 3.2: It follows from Lemmas 3.4 and 3.5 that $|A|^2 \lessapprox M^{7/8} X^{1/2}$. Combining Lemma 3.8 with the above inequation, we obtain

$$|A|^2 \lessapprox M^{27/8} K^{-1/8} |A|^{7/8} + M^{13/4} K^{7/4} |A|^{3/4}.$$ 

This implies that $M^{27} K^7 \gtrsim |A|^9$ or $M^{13} K^7 \gtrsim |A|^5$, as desired. □

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