HILBERT MODULES OVER A PLANAR ALGEBRA AND THE
HAAGERUP PROPERTY

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Abstract. Given a subfactor planar algebra $\mathcal{P}$ and a Hilbert $\mathcal{P}$-module of lowest weight 0 we build a bimodule over the symmetric enveloping inclusion associated to $\mathcal{P}$. As an application we prove diagrammatically that the Temperley-Lieb-Jones standard invariants have the Haagerup property. This provides a new proof of a result due to Popa and Vaes.

1. Introduction and main results

Popa initiated the study of approximation properties of subfactors in $\text{[Pop86, Pop94a, Pop94b]}$. To any finite index subfactor of type $\text{II}_1$ one can associate a combinatorial object called the standard invariant. This invariant has been axiomatized as a paragroup, a $\lambda$-lattice, and a planar algebra respectively by Ocneanu, Popa, and the second author $\text{[Ocn88, Pop95, Jon]}$. An analogue of quantum doubles for a subfactor was introduced by Ocneanu, Longo and Rehren, and Popa $\text{[Ocn88, LR95, Pop94b]}$. The latter construction is called the symmetric enveloping inclusion. For the construction of subfactors of Guionnet et al. in $\text{[GJS10]}$, Curran et al. in $\text{[CJS14]}$ gave a diagrammatic description of the symmetric enveloping inclusion.

Recently, Popa and Vaes introduced a representation theory for subfactors and standard invariants $\text{[PV]}$. They defined the Haagerup property for a subfactor and showed that it depends only on its standard invariant. They then showed that the Temperley-Lieb-Jones standard invariants have the Haagerup property. (Note, this result was already announced in $\text{[Pop06, Remark 3.5.5]}$.) Their proof uses previous work on discrete quantum groups and the equivalence between the bimodule category associated to the Temperley-Lieb-Jones standard invariant and the representation category of the quantum group $\text{PSU}_q(2)$ $\text{[DCFY14]}$. Here we give another proof:

Theorem 1.1. The Temperley-Lieb-Jones standard invariant has the Haagerup property for any loop parameter $\delta \in \{2\cos \frac{\pi}{n}, n \geq 3\} \cup \{2 : \infty\}$.

Our proof only uses planar algebra technology. The idea is that the lowest weight zero annular representations of the planar algebra immediately give ”compact” bimodules (which are obvious in the Curran et al. pictures), which tend to the trivial bimodule in the way required by the Popa-Vaes definition of the Haagerup property.

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2. Preliminaries

2.1. The symmetric enveloping inclusion associated to a subfactor planar algebra. We refer to $\text{[Jon]}$ for more details about planar algebras. We recall the construction of $\text{[CJS14, Section 2]}$. Note, we define the symmetric enveloping inclusion via the product introduced in $\text{[CJS14, Section 2.1]}$ that we call the Bacher product. Let $\mathcal{P} = (\mathcal{P}_n^\pm, n \geq 0)$ be a subfactor

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planar algebra. For any $k, n, m \geq 0$, let $D_k(n, m)$ be a copy of the vector space $\mathcal{P}^+_n \otimes \mathcal{P}^+_m$. We decorate strings with natural numbers to indicate that they represent a given number of parallel strings. The distinguished interval of a box is decorated by a dollar sign if it is not at the top left corner. We will omit unnecessary decorations. Consider the direct sum

$$Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P} := \bigoplus_{n, m \geq 0} D_k(n, m)$$

that we equipped with the Bacher product:

$$x \star_k y = \sum_{a=0}^{\min(2n,2k)} \sum_{b=0}^{\min(2m,2j)}$$

where $x \in D_k(n, m)$ and $y \in D_k(i, j)$. Let $\dagger : Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P} \rightarrow Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ be the anti-linear involution that sends $D_k(n, m)$ to itself and satisfies

$$x^\dagger = \begin{cases} 1 & \text{if } x \text{ is a unit of } D_k(n, m) \text{ and } n, m \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Consider the linear form $\tau : Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P} \rightarrow \mathbb{C}$, which is zero unless $n = m = 0$ and sends the unit of $D_k(0, 0)$ to 1. We have that $(Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}, \star_k, \dagger, \tau)$ is an associative $*$-algebra with a faithful tracial state [CJS14, Corollary 2.3]. Further, $Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ acts by bounded operators on the Gelfand-Naimark-Segal Hilbert space for $\tau$ [CJS14, Theorem 2.1]. Let $M_k \boxtimes M_k$ be its Gelfand-Naimark-Segal completion which is a factor of type II$_1$. Let $M_k$ be the von Neumann subalgebra of $M_k \boxtimes M_k$ generated by elements of the form

$$x \in D_k(n, 0), \; n \geq 0.$$
Hilbert $\mathcal{P}$-modules of lowest weight 0. Consider the Temperley-Lieb-Jones planar algebra $\mathcal{P} = TLJ$ with loop parameter $\delta \geq 2$.

Irreducible Hilbert $TLJ$-modules of lowest weight 0 have been fully classified in [Jona] and in [GL98] for the unshaded case. For any $0 < \ell \leq \delta$ there exists a Hilbert $TLJ$-module $V(t) = (V(t)_n^+, n \geq 0)$ such that $V(t)^+_0$ is one dimensional and spanned by a unit vector $\xi(t)$ which satisfies

$$\langle \alpha(\xi(t)), \beta(\xi(t)) \rangle = \delta^{\ell} t^{2d},$$

where $\alpha, \beta$ are annular tangles, $c$ is the number of contractible circles in the $(\pm, \pm)$-annular tangle $\beta^\dagger \circ \alpha$ and $d$ is half the number of non-contractible ones. Those Hilbert $TLJ$-modules will be used to construct unital completely positive maps on the symmetric enveloping inclusion associated to the Temperley-Lieb-Jones planar algebra.

3. Hilbert $\mathcal{P}$-modules give $(M \otimes M^{\text{op}} \subset M \boxtimes M)$-bimodules

Let $V = (V_n^+, n \geq 0)$ be a Hilbert $\mathcal{P}$-module of lowest weight 0. For $i, j \geq 0$, let $\mathcal{H}_{i,j}$ be a copy of the Hilbert space $V_{i+j}^+$. Let $\mathcal{H} = \bigoplus_{i,j \geq 0} \mathcal{H}_{i,j}$ be the Hilbert space equal to the direct sum of the $\mathcal{H}_{i,j}$. In particular, $\mathcal{H}_{i+1,j-1}$ is orthogonal to $\mathcal{H}_{i,j}$ in $\mathcal{H}$. Consider the dense pre-Hilbert subspace $\mathcal{K} \subset \mathcal{H}$ spanned by the union of all $\mathcal{H}_{i,j}$. We put

$$\pi_0(x)\xi = \sum_{a,b} \sum_{i,j} \xi_{a,b} \mathcal{L}_{a,b}(x),$$

for any $x \in D(n,m) \subset \text{Gr}\mathcal{P} \boxtimes \text{Gr}\mathcal{P}$ and $\xi \in \mathcal{H}_{i,j}$. This defines a representation

$$\pi_0 : \text{Gr}\mathcal{P} \boxtimes \text{Gr}\mathcal{P} \longrightarrow \mathcal{L}(\mathcal{K}),$$

where $\mathcal{L}(\mathcal{K})$ is the algebra of endomorphism of the vector space $\mathcal{K}$.

Proposition 3.1. For any $x \in \text{Gr}\mathcal{P} \boxtimes \text{Gr}\mathcal{P}$, $\pi_0(x)$ defines a bounded operator on $\mathcal{H}$. Further, the representation $\pi_0$ extends to a normal $*$-representation

$$\pi : M \boxtimes M \longrightarrow \mathcal{B}(\mathcal{H}).$$

Proof. Consider $x$ in $\text{Gr}\mathcal{P} \boxtimes \text{Gr}\mathcal{P}$. We can prove that $\pi_0(x)$ defines a bounded operator by following a similar argument than [JSW10] Theorem 3.3. We continue to denote by $\pi_0(x)$ its extension to $\mathcal{H}$. Let $\xi \in \mathcal{H}_{0,0}$ be a unit vector and let $\omega_\xi$ be its associated vector state. Note, $\omega_\xi \circ \pi_0(x) = \tau(x)$ for any $x \in \text{Gr}\mathcal{P} \boxtimes \text{Gr}\mathcal{P}$, where $\tau$ is the unique normal tracial state on $M \boxtimes M$. Therefore, $\pi_0$ extends to a normal $*$-representation $\pi : M \boxtimes M \longrightarrow \mathcal{B}(\mathcal{H}).$ 

Recall, if $T \subset S$ is an inclusion of von Neumann algebras, then a Hilbert $(T \subset S)$-module is a couple $(\mathcal{H}, \xi)$ such that $\mathcal{H}$ is a Hilbert $S$-module and $\xi$ is a $T$-central vector of $\mathcal{H}$. 

where the dollar sign is in a region with shading $\epsilon$ (resp. $\varepsilon$). A tangle in $\text{Ann}\mathcal{P}((m, \varepsilon), (n, \epsilon))$ is called a $((m, \varepsilon), (n, \epsilon))$-annular tangle. Let $A\mathcal{P} = (A\mathcal{P}((m, \varepsilon), (n, \epsilon)), n, m \geq 0, \epsilon, \varepsilon \in \pm)$ be the annular algebroid associated to $\mathcal{P}$. We denote by $\alpha \mapsto \alpha^\dagger$ the anti-linear involution which sends a $((m, \varepsilon), (n, \epsilon))$-annular tangle to a $((n, \epsilon), (m, \varepsilon))$-annular tangle by reflection in a circle half way between the inner and outer boundaries. A Hilbert $\mathcal{P}$-module is a graded vector space $V = (V_n^\pm, n \geq 0)$, where each $V_n^\pm$ is a finite dimensional Hilbert space, $A\mathcal{P}$ acts on $V$, and the inner product is compatible with this action. It means that if $\alpha \in A\mathcal{P}((m, \varepsilon), (n, \epsilon))$ then it defines a linear map from $V_n^\varepsilon$ to $V_n^\varepsilon$ such that

$$\langle \alpha(v), w \rangle = \langle v, \alpha^\dagger(w) \rangle,$$

for any $v \in V_n^\varepsilon, w \in V_n^\varepsilon$.

The lowest weight of a Hilbert $\mathcal{P}$-module $V$ is the smallest natural number $n$ such that $V_n^+ \neq \{0\}$.
Corollary 3.2. Let $V$ be a Hilbert $\mathcal{P}$-module of lowest weight 0. Consider the Hilbert space $\mathcal{H}$ constructed above and let $\xi \in \mathcal{H}_{0,0}$ be a unit vector. Then, $(\mathcal{H}, \xi)$ has a structure of Hilbert $(M \hat{\otimes} M^{\text{op}} \subset M \hat{\otimes} M)$-bimodule where the left action is given by $\pi$ and the right action is defined similarly.

Proof. Proposition 3.1 implies that $\mathcal{H}$ is a $M \hat{\otimes} M$-bimodule with the action described above. Consider $x \otimes y^{\text{op}} \in \text{GrP} \otimes \text{GrP}^{\text{op}}$, where $\text{GrP} \otimes \text{GrP}^{\text{op}} = M \hat{\otimes} M^{\text{op}} \cap \text{GrP} \hat{\otimes} \text{GrP}$. Since $\xi \in \mathcal{H}_{0,0}$, we have

$$(x \otimes y^{\text{op}}) \cdot \xi = \begin{pmatrix} x & \xi \\ y & \xi \end{pmatrix} \quad \text{and} \quad \xi \cdot (x \otimes y^{\text{op}}) = \begin{pmatrix} \xi & x \\ \xi & y \end{pmatrix}.$$ 

Those two pictures are isotopic to each other. Therefore, $(x \otimes y^{\text{op}}) \cdot \xi = \xi \cdot (x \otimes y^{\text{op}})$. By density of $\text{GrP} \otimes \text{GrP}^{\text{op}}$ inside $M \hat{\otimes} M^{\text{op}}$, we obtain that $\xi$ is a $M \hat{\otimes} M^{\text{op}}$-central vector. \qed

4. The Temperley-Lieb-Jones standard invariant has the Haagerup property

In this article, any inclusion of tracial von Neumann algebras will be supposed to be unital and tracial. We recall the definition of the relative Haagerup property due to Boca [Boc93]. Note, Popa defined a very similar property [Pop06]. Those two definitions coincide in the context of Definition 4.2.

Definition 4.1. Consider an inclusion of tracial von Neumann algebras $\mathcal{N} \subset (\mathcal{M}, \tau)$. A completely positive approximation of the identity (CPAI) for $\mathcal{N} \subset (\mathcal{M}, \tau)$ is a sequence of normal $\mathcal{N}$-bimodular trace-preserving unital completely positive maps $(\varphi_l : \mathcal{M} \to \mathcal{M}, l \geq 0)$ such that $\| \varphi_l(x) - x \|_2 \to_l 0$, for any $x \in \mathcal{M}$, and the unique continuous extension $\Theta_l \in B(L^2(\mathcal{M}, \tau))$ of $\varphi_l$ to $L^2(\mathcal{M}, \tau)$ is in the compact ideal space of $(\mathcal{M}, e_{\mathcal{N}})$.

If such a sequence exists we say that $\mathcal{N} \subset (\mathcal{M}, \tau)$ has the relative Haagerup property.

Definition 4.2. [PV] A subfactor $N \subset M$ has the Haagerup property if its symmetric enveloping inclusion has the relative Haagerup property. A standard invariant $\mathcal{G}$ has the Haagerup property if there exists a subfactor $N \subset M$ with standard invariant isomorphic to $\mathcal{G}$ which has the Haagerup property.

Recall that if two subfactors have isomorphic standard invariants, then one of them has the Haagerup property if and only if the other one has the Haagerup property, see [Pop06] Remark 3.5.5 or [PV].

Lemma 4.3. Let $\mathcal{P}$ be a subfactor planar algebra. Then $\mathcal{P}$ has the Haagerup property if and only if its associated symmetric enveloping inclusion $M \hat{\otimes} M^{\text{op}} \subset M \hat{\otimes} M$ has the relative Haagerup property.

Proof. Consider the subfactor $M_0 \subset M_1$ defined in Section 2. Its planar algebra is equal to $\mathcal{P}$. Popa’s symmetric enveloping inclusion associated to $M_0 \subset M_1$ is isomorphic to $M_1 \hat{\otimes} M^{\text{op}} \subset M \hat{\otimes} M_1$. Consider the inclusion $M_0 \hat{\otimes} M_0^{\text{op}} \subset M_1 \hat{\otimes} M_1$. Let $e$ be the Jones projection $e = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. Note, the compression $e(M_0 \hat{\otimes} M_0^{\text{op}}) e \subset e(M_1 \hat{\otimes} M_1) e$ is isomorphic to $M \hat{\otimes} M^{\text{op}} \subset M \hat{\otimes} M$. Therefore, by [Pop06] Proposition 2.3 and Proposition 2.4, $M \hat{\otimes} M^{\text{op}} \subset M \hat{\otimes} M$ has the relative Haagerup property if and only if $M_1 \hat{\otimes} M_1^{\text{op}} \subset M_1 \hat{\otimes} M_1$ has the relative Haagerup property. \qed
Lemma 4.4. Let $TLJ$ be the Temperley-Lieb-Jones planar algebra with a loop parameter $\delta \geq 2$ and let $M \otimes M^{op} \subset M \boxtimes M$ be its associated symmetric enveloping inclusion. Consider the $2n$th-Jones-Wenzl idempotent $g_n \in TLJ^+_n$ that we identity with its associated element in $D(n, n) \subset M \boxtimes M$. Let $L_n \subset L^2(M \boxtimes M)$ be the $M \otimes M^{op}$-bimodule generated by $g_n$. Then $L_n$ is isomorphic to $X_n \otimes X_n^{op}$, where $X_n$ is the irreducible $M_0$-bimodule corresponding to the $2n$th vertex in the principal graph of the subfactor $M_0 \subset M_1$. Further, $L^2(M \boxtimes M)$ is equal to the direct sum of the bimodule $L_n$.

Proof. We follow an argument in [CJS14, pp. 120-122]. Let us show that $L_n$ is orthogonal to $L_m$ if $n \neq m$. This is equivalent to show that for any $x, y \in TLJ$, we have $x g_n y \perp g_m$ in the planar algebra $TLJ$. But this is obvious. Observe, the $*$-algebra $GrP \boxtimes GrP$ is generated by the set of Jones-Wenzl idempotents and $GrP \otimes GrP^{op}$. Therefore, $L^2(M \boxtimes M)$ is equal to the direct sum of the bimodules $L_n$. Consider the $M$-bimodule $X_n \subset L^2(M_n)$ equal to the image of $g_n$ viewed as an element of $TLJ^+_n = M^+ \cap M_n \subset B(L^2(M_n))$. We have an isomorphism from $X_n \otimes X_n^{op}$ onto $L_n$ given by the tangle which connects the $2n$ side strings of an elements of $X_n$ (resp. $X_n^{op}$) to the top strings of $g_n$ (resp. the bottom strings of $g_n$).

Theorem 4.5. Let $TLJ$ be the Temperley-Lieb-Jones planar algebra with any loop parameter $\delta \in \{2 \cos (\frac{\pi}{n}), n \geq 3\} \cup [2 : \infty)$. Then $TLJ$ has the Haagerup property.

Proof. If $\delta = 2 \cos (\frac{\pi}{n})$ for some $n \geq 3$, then $TLJ$ has finite depth. Therefore, its symmetric enveloping inclusion is a subfactor of finite index. This implies that $TLJ$ has the Haagerup property. We assume that $\delta \geq 2$. We write $T = M \otimes M^{op}$ and $S = M \boxtimes M$. Consider $0 < t < \delta$ and the pointed Hilbert $TLJ$-module $(V(t), \xi(t))$ of section 2.3 where $\xi(t) \in V(t)^+_0$ is a unit vector. Let $(H^t, \xi^t)$ be its associated $(T \subset S)$-bimodule as constructed in section 3. Let $Z_t : L^2(S) \longrightarrow H^t$ be the continuous linear map densely defined as follows $Z_t(x \Theta) = \xi^t \cdot x$, for any $x \in S$. Define the normal $T$-bimodular unital completely positive map $\phi_t : S \longrightarrow S$ by the formula $\phi_t(x) = Z_t^* \pi_t(x) Z_t$, where $\pi_t : S \longrightarrow B(H^t)$ is the left action of $S$ on $H^t$. We will show that the net $(\phi_t, 0 < t < \delta)$ is the desired approximation of the identity.

Note, the $T$-bimodules $L_n$ are isomorphic to $X_n \otimes X_n^{op}$ for any $n \geq 0$. Hence, they are irreducible and pairwise non-isomorphic. By Schur’s Lemma, there exists a scalar valued function $c_t : \mathbb{N} \longrightarrow \mathbb{C}$ such that $\Theta_t = \sum_{n \geq 0} c_t(n) s_n$, where $\Theta_t$ is the unique continuous extension of $\phi_t$ to $L^2(S)$ and $s_n$ is the orthogonal projection from $L^2(S)$ onto $L_n$. We have the formula

$$c_t(n) = \frac{\langle \phi_t(g_n), g_n \rangle}{\langle g_n, g_n \rangle}, \text{ for any } n \geq 0.$$ 

Let $\tau_{2n}$ be the non-normalized trace of the C$^*$-algebra $TLJ^+_n$. Remark, $\tau_{2n}(g_n) = \langle g_n, g_n \rangle$, for any $n \geq 0$. Let $q$ be the unique real number bigger than 1 satisfying $q + q^{-1} = \delta$. It is well known that $\tau_{2n}(g_n) = [2n + 1]_q$, where

$$[2n + 1]_q = \frac{q^{2n+1} - q^{-2n-1}}{q - q^{-1}}$$

is the 2n+1th quantum integer with parameter $q$ [Jon83, Section 5.1].

We claim that

$$\langle \phi_t(g_n), g_n \rangle = [2n + 1]_q, \text{ if } n \geq 1,$$
where $\omega$ is a complex number satisfying $\omega + \omega^{-1} = t$. Observe,
\[
\langle \phi_t(g_n), g_n \rangle = \langle g_n \cdot \xi^t, \xi^t \cdot g_n \rangle = \langle g_n \cdot \xi^t \cdot g_n, \xi^t \rangle
\]
\[
= \langle g_n \xi^t, \xi^t \rangle = \langle g_n \xi^t, \xi^t \rangle.
\]
Hence, proving the equality \( \square \) is a routine computation using the induction formula of \cite{Jon83, Section 5.1} or \cite{Wen87}. Therefore,
\[
c_t(n) = \frac{[2n + 1]}{[2n + 1]} \omega \text{ for any } n \geq 0.
\]
Observe,
\[
c_t(n) \to 1, \text{ as } t \to \delta, \text{ for any } n \geq 0, \text{ and}
\]
\[
c_t(n) \to 0, \text{ as } n \to \infty, \text{ for any } 0 < t < \delta.
\]
Note, $\tau \circ \phi_t = a_0 \tau = \tau$. Hence, any sequence of real numbers $(0 < t_n < \delta, \ n \geq 0)$ that converges to $\delta$ defines a CPAI $(\phi_{t_n}, \ n \geq 0)$. Therefore, $T \subset S$ has the relative Haagerup property. \( \square \)

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