Differential Geometry of the Vortex Filament Equation

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Abstract. Differential calculus on the space of asymptotically linear curves is developed. The calculus is applied to the vortex filament equation in its Hamiltonian description. The recursion operator generating the infinite sequence of commuting flows is shown to be hereditary. The system is shown to have a description with a Hamiltonian pair. Master symmetries are found and are applied to deriving an expression of the constants of motion in involution. The expression agrees with the inspection of Langer and Perline.

Key words: Integrable Hamiltonian system; Vortex filament equation; Hereditary operator; Hamiltonian pair; Master symmetries.

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1 Introduction

The vortex filament equation \( \dot{\gamma} = \kappa b \), \((1)\)

where \( \gamma \) is the curve of the vortex filament parametrized by the arclength, dot stands for the differential with respect to the time, \( \kappa \) is the curvature of \( \gamma \), and \( b \) is the bi-normal vector along \( \gamma \). It is well-known that the vortex filament equation \((1)\) is closely related to the cubic nonlinear Schrödinger (NLS) equation, and the Hasimoto map provides a connection between them \([2]\). The NLS equation is an infinite-dimensional completely integrable Hamiltonian system \([3]\).

Marsden and Weinstein \([4]\) constructed a Hamiltonian description of the vortex filament equation in their study on the moment map for the action of the unimodular diffeomorphism group of \( \mathbb{R}^3 \). Langer and Perline \([5]\) introduced the space BAL — the space of balanced asymptotically linear curves (see Section 2) — as a phase space for the system of vortex filament, and showed that the Hasimoto map is a Poisson map from BAL with the Marsden-Weinstein Poisson structure to (a certain equivalence class of) the phase space of the NLS system with the ‘fourth’ Poisson structure. This result says that the Hasimoto map induces constants of motion in involution for the vortex filament equation as the pull-back of those for the NLS system, hence the vortex filament equation can be understood as a completely integrable system. Further, Langer and Perline found a recursion operator, which generates infinite sequence of commuting Hamiltonian vector fields on BAL.

For some typical integrable Hamiltonian systems, such as the NLS equation, the integrability is studied from various aspects and many remarkable structures are known to exist \([3, 5, 6, 7, 8, 10, 11, 12]\). It is therefore natural to ask whether the same or similar structures exist for the system of vortex filament. In this paper we focus on structures that are described in the language of differential geometry; we will investigate the hereditary property \([1, 11]\) of the recursion operator, Hamiltonian pair \([7, 8]\), and master symmetries \([10]\). For these, the answers are all affirmative; the space BAL admits these structures. The asymptotic boundary condition defining BAL is critical for this result; a different situation is encountered when the curve of a vortex filament is supposed to be a loop \([13]\).

The paper is organized as follows: In Section 2, the definition of BAL is clarified. It
involves introducing a further condition to the conditions defining \( \text{BAL} \) of \([5]\). Also in this section, some basic notions are described, and several useful formulae for the variational calculus on \( \text{BAL} \) are summarized. In Section 3, carefully specifying what are admissible vector fields and what are admissible functionals, we define a differential calculus on \( \text{BAL} \). The calculus provides the framework for the subsequent analysis. Section 4 consists of two subsections. In Subsection 4.1, a recursion operator is shown to be hereditary, and its several consequences are presented. In Subsection 4.2, it is shown with the help of the hereditary recursion operator that certain two operators form a Hamiltonian pair. In Section 5, master symmetries are investigated and are applied to deriving an expression of constants of motion in involution. This proves the inspection of Langer and Parline.

2 Balanced Asymptotically Linear Curves

Let \( \text{APC} \) be the space of infinitely extended, arclength-parametrized smooth curves in the Euclidean space \( \mathbb{R}^3 \) with the standard metric \( \langle \ , \rangle \). We imply by the letter \( \gamma \) an element of \( \text{APC} \) and by \( s \) the parameter for it; \( s \mapsto \gamma(s) \) is a smooth map \( \mathbb{R} \to \mathbb{R}^3 \) such that \( \partial \gamma(s)/\partial s \) is a unit vector in the tangent space \( T_{\gamma(s)}\mathbb{R}^3 \).

A map \( \text{APC} \times \mathbb{R} \to \mathbb{R} \) is referred to as a scalar field on \( \text{APC} \). A map \( x: \text{APC} \times \mathbb{R} \to \bigoplus T_{\gamma(s)}\mathbb{R}^3 \) such that \( x(\gamma, s) \in T_{\gamma(s)}\mathbb{R}^3 \) is referred to as a tangent field (the term ‘vector field’ is reserved for the differential calculus). Here, \( \bigoplus \) stands for the direct-sum with respect to the index \( (\gamma, s) \in \text{APC} \times \mathbb{R} \). Similar terminology is used also for a subset \( \text{BAL} \), a space that we wish to manifest in this section.

The differential operator with respect to \( s \) is denoted by \( \partial_s \) (when acting on scalar fields) or by \( \nabla_s \) (when acting on tangent fields). These operators satisfy

\[
\partial_s(fg) = (\partial_s f)g + f\partial_s g, \tag{2}
\]

\[
\nabla_s(fx) = (\partial_s f)x + f\nabla_s x, \quad \partial_s \langle x, y \rangle = \langle \nabla_s x, y \rangle + \langle x, \nabla_s y \rangle \tag{3}
\]

for all scalar fields \( f, g \) and tangent fields \( x, y \). As in the equations above, we will often surpress the argument \( (\gamma, s) \).

A scalar field \( F \) is called a functional if \( F \) is independent of \( s \), i.e., \( \partial_s F = 0 \).

We say a scalar field \( f \) is asymptotically polynomial-like if there exists a polynomial \( P(s) \in \mathbb{R}[s] \) such that \( f(\gamma, s)/P(s) \to 0 \) in the limit \( s \to \pm\infty \) for every curve \( \gamma \). We say a
scalar field $f$ is rapidly-decreasing if $f(\gamma, s)P(s)$ for every polynomial $P(s) \in \mathbb{R}[s]$ converges to zero in the limit $s \to \pm\infty$ for every curve $\gamma$.

Let $f$ be a scalar field. The scalar field $\partial_s^{-1} f$ (anti-differentiation of $f$) and the functional $\int f$ (definite integration of $f$) are defined by

\[
(\partial_s^{-1} f)(\gamma, s) = \frac{1}{2} \left( \int_{-\infty}^{s} f(\gamma, \bar{s}) d\bar{s} - \int_{s}^{\infty} f(\gamma, \bar{s}) d\bar{s} \right),
\]

\[
(\int f)(\gamma) = \int_{-\infty}^{\infty} f(\gamma, \bar{s}) d\bar{s}
\]

provided that the integrations in the equations above converge. In employing operators $\partial_s^{-1}$ and $\int$ in the following sections, we will ensure the convergence by introducing certain rules. It is easy to see

\[
\partial_s \partial_s^{-1} f = \partial_s^{-1} \partial_s f = f, \quad \partial_s \int f = \int \partial_s f = 0,
\]

\[
\partial_s^{-1}(F f) = F \partial_s^{-1} f, \quad \int (F f) = F \int f,
\]

where $f$ is a rapidly-decreasing scalar field and $F$ is a functional.

We denote by $t$, $n$, $b$ the tangent fields forming the Frénét frame, namely, $t(\gamma, s)$, $n(\gamma, s)$ and $b(\gamma, s)$ are orthonormal vectors in $T_{\gamma(s)}\mathbb{R}^3$ satisfying $t(\gamma, s) = \partial \gamma(s)/\partial s$ and the Frénet-Serret relation

\[
\nabla_s t = \kappa n, \quad \nabla_s n = -\kappa t + \tau b, \quad \nabla_s b = -\tau n.
\]

Here, $\kappa$ and $\tau$ are scalar fields characterized by (8), namely, $\kappa(\gamma)$ and $\tau(\gamma)$ are the curvature and torsion, respectively, of the curve $\gamma$. Every tangent field is uniquely written as a linear combination of $t$, $n$, $b$ with the coefficients in scalar fields.

The space $BAL$ introduced in [5] is a subset of $APC$ such that (a) the curvature $\kappa(\gamma)$ of $\gamma \in BAL$ is non-vanishing, (b) $\gamma \in BAL$ is asymptotic to a fixed line, e.g., to $z$-axis, and (c) ambiguity in the parametrization is completely eliminated with imposing balancing condition.

To describe the balancing condition, we need to fix a reference curve $\gamma_0 \in APC$ (or a reference line, $z$-axis) fulfilling the asymptotic condition as in (b). The condition (b) says the existence of functionals $\ell_{\pm}: BAL \to \mathbb{R}$ with which the asymptotic behaviour of $\gamma \in BAL$ in the region $s \to \pm\infty$ is written as $\gamma(s \pm \ell_{\pm}(\gamma)) \to \gamma_0(s)$. With these functionals, the balancing condition (c) for $\gamma \in BAL$ can be written as $\ell_{+}(\gamma) = \ell_{-}(\gamma)$. The functional $\ell := \ell_{+} + \ell_{-}$ referred to as the renormalized length (relative to $\gamma_0$) is well-defined, though the curve $\gamma \in BAL \subset APC$ is of infinite length.
We supplement the condition (b) with prescribing how \( \gamma \in BAL \) converges to the reference curve \( \gamma_0 \); we suppose
\[
\begin{align*}
\kappa & \text{ is rapidly-decreasing, and } \\
\kappa^{-1} \partial_s^n \kappa & \text{ and } \partial_s^n \tau, \ n = 0, 1, \ldots \text{ are all asymptotically polynomial-like,}
\end{align*}
\]  
where \( \kappa^{-1} := 1/\kappa. \)

In the next section we introduce a differential calculus on \( BAL \). The action of vector fields on functionals in this calculus is defined to reproduce the usual variational calculus. Here we make a few remarks on the variational calculus and give several useful formulae.

For more detailed description, we refer to [5].

Let \( x \) be a tangent field written as
\[
\begin{equation}
\begin{aligned}
x = \partial_s^{-1}(\kappa g)t + gn + h b
\end{aligned}
\tag{10}
\end{equation}
\] with certain rapidly-decreasing scalar fields \( g, h \). Below, \( x(\gamma) \) is identified with a variational vector field along \( \gamma \). The restriction on \( x \) mentioned above is to force the variation to keep the arclength-parametrization and conditions (b), (c).

In this paper the variational differential operator associated with \( x \) of the form (10) is denoted by \( \delta_x \) (when acting on scalar fields) or by \( \nabla_x \) (when acting on tangent fields). For calculating the former, the following formulae are useful:
\[
\begin{align*}
\delta_x (fg) &= (\delta_x f)g + f\delta_x g, \\
\delta_x \partial_s f &= \partial_s \delta_x f, \quad \delta_x \partial_s^{-1} f = \partial_s^{-1} \delta_x f, \quad \delta_x ff = f\delta_x f, \\
\delta_x \kappa &= \langle n, \nabla_s \nabla_x x \rangle, \\
\delta_x \tau &= \partial_s \langle \kappa^{-1} b, \nabla_s \nabla_x x \rangle + \langle \kappa b, \nabla_s x \rangle, \\
\delta_x s &= 0, \\
\delta_x \ell &= f(-\kappa n, x),
\end{align*}
\]  
where \( f \) and \( g \) are scalar fields. By abuse of notation, we often use the letter \( s \), by which we mean the scalar field \( \hat{s} \) such that \( \hat{s}(\gamma, s) = s \), as we have done in (15). The latter can be calculated by the following formulae:
\[
\begin{align*}
\nabla_x (fy) &= (\delta_x f)y + f\nabla_x y, \\
\nabla_x t &= \langle n, \nabla_s x \rangle n + \langle b, \nabla_s x \rangle b, \\
\nabla_x n &= \langle \kappa^{-1} b, \nabla_s \nabla_s x \rangle b - \langle n, \nabla_s x \rangle t, \\
\nabla_x b &= -\langle b, \nabla_s x \rangle t - \langle \kappa^{-1} b, \nabla_s \nabla_x x \rangle n,
\end{align*}
\]
where $f$ is a scalar field and $y$ is a tangent field. These satisfy
\[
\delta_x \langle y, z \rangle = \langle \nabla_x y, z \rangle + \langle y, \nabla_x z \rangle.
\] (21)

3 Differential Calculus

Let $A_n$, $n \in \mathbb{Z}$, be the $\partial_s$-invariant space (i.e., $\partial_s f \in A_n$ for all $f \in A_n$) of scalar fields $f$ such that $\kappa^{-n} f$ is asymptotically polynomial-like. The elements of $A_n$ with $n > 0$ are rapidly-decreasing.

We notice the following properties possessed by $A_n$:

a1. $A_n$, $\forall n \in \mathbb{Z}$, is an $\mathbb{R}$-vector space,

a2. $A_n \subset A_{n-1}$ (as $\mathbb{R}$-vector spaces) for all $n \in \mathbb{Z}$,

a3. $A_{-\infty} := A_0 \cup A_{-1} \cup \cdots$ is a commutative associative $\mathbb{R}$-algebra with the unit 1,

a4. $fg \in A_{i+j}$ if $f \in A_i$, $g \in A_j$ for all $i, j \in \mathbb{Z}$,

a5. $\partial_s$ is an $\mathbb{R}$-linear operator such that $\partial_s (A_n) \subset A_n$ for all $n \in \mathbb{Z}$,

a6. $\partial_s^{-1}$ and $f$ are $\mathbb{R}$-linear operators such that $\partial_s^{-1} f \in A_0$ and $ff \in A_0$ for all $f \in A_2$,

a7. $\kappa \in A_1$, $\kappa^{-1} \in A_{-1}$, and $\tau$, $s$, $\ell$, $1 \in A_0$,

b1. $\partial_s$ is a derivation of $A_{-\infty}$, i.e., Eq. (2) hold for all $f$, $g \in A_{-\infty}$,

b2. Eqs. (6) and (7) hold for all $f \in A_2$ and $F \in \text{Ker} \partial_s$,

b3. $f(f\partial_s^{-1}g) = -f(g\partial_s^{-1}f)$ for all $f$, $g \in A_2$,

b4. $\kappa\kappa^{-1} = \partial_s s = 1$, and $\partial_s \ell = 0$.

Let us consider the objects $\mathcal{E}_n$ that are fully characterized by the rules a1–a7 above; regarding a1–a7 (in which $A_n$ should be read as $\mathcal{E}_n$) as the axioms for $\mathcal{E}_n$, we define $\mathcal{E}_n$, $n \in \mathbb{Z}$, as a family of objects generated by the symbols or indeterminates $\{\kappa, \kappa^{-1}, \tau, s, \ell, 1\}$ with the algebraic operations. Here and in the following paragraph, by algebraic operations we mean addition, scaling by a real number, multiplication, $\partial_s$, $\partial_s^{-1}$ and $f$. It is the implication of a6 that $\partial_s^{-1}$ and $f$ cannot act on $\mathcal{E}_n$, $n < 2$. By definition, rules (such as b1–b4) not following from a1–a7 are not available for $\mathcal{E}_n$. 
Let \( g_1, \ldots, g_r \) be independent variables running over \( \mathcal{E}_{n_1}, \ldots, \mathcal{E}_{n_r} \), respectively. We say \( f(g_1, \ldots, g_r) \) is an \( \mathcal{E}_n \)-valued variable algebraically depending on \( g_1, \ldots, g_r \) if \( f(g_1, \ldots, g_r) \) is an expression written in terms of \( \{g_1, \ldots, g_r, \kappa, \kappa^{-1}, \tau, s, \ell, 1\} \) with use of the algebraic operations and if the rules a1–a7 supplemented with the condition \( g_i \in \mathcal{E}_n \) conclude \( f(g_1, \ldots, g_r) \in \mathcal{E}_n \).

We denote by \( \mathcal{A}_n \) the space of scalar fields on \( \text{BAL} \) having an expression that belongs to \( \mathcal{E}_n \). It is easy to see that \( \mathcal{A}_n \) is a subset of \( \mathcal{A}_n \). The statements a1–a7 and b1–b4 remain true even if every \( \mathcal{A}_n \) is read as \( \mathcal{A}_n \). Moreover, these together with the positive-definiteness or at least nondegeneracy of the bi-linear form (33) are all of the fundamental setting we need in constructing the theory developed in this paper.

**Proposition 1.** Let \( \Phi \) be an \( \mathbb{R} \)-linear map \( \mathcal{A}_1 \to \mathcal{A}_0 \) induced from an \( \mathbb{R} \)-linear map \( \mathcal{E}_1 \to \mathbb{R}(\mathcal{E}_2) \) in the apparent way. If this map is written as \( \Phi(g) = \int f(g) \) with an \( \mathcal{E}_2 \)-valued variable \( f(g) \) algebraically depending on \( g \in \mathcal{E}_1 \), then there exists \( h \in \mathcal{A}_1 \) with which one can write \( \Phi(g) = \int gh \forall g \in \mathcal{A}_1 \) as an equation in \( \mathcal{A}_0 \).

**Proof.** We note the formulae

\[
\int (f\partial_s g) = -\int (g\partial_s f), \quad \int (f\partial^{-1}_s g) = -\int (g\partial^{-1}_s f), \quad \int (f) = \int (g), \quad \int (g f) = \int (g f),
\]

(22)
each of which is valid as an equation in \( \mathcal{A}_0 \) if the left-hand side is given as an \( \mathcal{E}_0 \)-valued variable algebraically depending on \( f \) and \( g \). From bi-\( \mathbb{R} \)-linearity of multiplication and \( \mathbb{R} \)-linearity of \( \partial_s, \partial^{-1}_s \) and \( f \), we see the existence of an expression \( \Phi(g) = \sum_i \int f_i(g) \) with \( f_i(g) \) being \( \mathcal{E}_2 \)-valued variables algebraically depending on \( g \in \mathcal{E}_1 \) such that no additions are used in the expression of \( f_i(g) \). Further, it is possible to suppose \( g \) appears in each expression \( \int f_i(g) \) only once because of the \( \mathbb{R} \)-linearity of \( \Phi \). For such expressions, it is apparent how to apply successively the formulae (22) to \( \int f_i(g) \) to rewrite it into the form \( \int gh_i \). This process is justified if one considers the equations in \( \mathcal{A}_0 \), while consideration in \( \mathcal{E}_0 \) is useful in verifying that the expressions \( \int (\cdots) \) appearing in each step make sense as \( \mathcal{E}_0 \)-valued variables and eventually in deducing \( h_i \in \mathcal{A}_1 \).

Let \( \mathcal{T}_n, n \in \mathbb{Z} \), be the \( \mathbb{R} \)-vector space of tangent fields defined by \( \mathcal{T}_n := \{ft + gn + hb \mid f \in \mathcal{A}_{n-1}, g, h \in \mathcal{A}_n\} \). It is easy to see that \( \mathcal{T}_n \) is \( \nabla_s \)-invariant, i.e., \( \nabla_s(\mathcal{T}_n) \subseteq \mathcal{T}_n \).

We denote by \( \varphi \) the surjection associated with the identification \( ft \sim 0 \) in \( \mathcal{T}_n \), namely, putting \( N = \varphi(n), B = \varphi(b) \), we write

\[
\varphi(ft + gn + hb) = gN + hB
\]

(23)
for scalar fields $f, g, h$. The vector spaces $\varphi(T_n)$ are left $A_0$-modules with $f(gN + hB) = (fg)N + (fh)B \ \forall f \in A_0, \forall g, h \in A_n$.

Let $\mathcal{X} := \varphi(T_1) = \{gN + hB \mid g, h \in A_1\}$. Each element of $\mathcal{X}$ is referred to as a vector field on $BAL$. Through the injection $\wp: \mathcal{X} \rightarrow T_n$ defined by

$$\wp(gN + hB) := \partial_{n}^{-1}(\kappa g)t + gN + hB,$$

a vector field $X$ induces a derivation — variational differential associated with $\varphi(X)$. This derivation acting on $A_n$ or $T_n$ can be evaluated with the formulae (11)–(20).

**Proposition 2.** The vector spaces $A_n$ and $T_n$ are invariant under the action of the vector fields, namely, $\delta_{\wp(X)}(A_n) \subset A_n$ and $\nabla_{\wp(X)}(T_n) \subset T_n \ \forall X \in \mathcal{X}, \forall n \in \mathbb{Z}$.

**Proof.** It is essential that every vector field $X = gN + hB$ is written with $g, h \in A_1$. Taking notice of this situation, we find that the formulae (11)–(20) ensure the invariance of $A_n$ and $T_n$ under the action of vector fields.

The space $\mathcal{X}$ of vector fields is a Lie algebra, and $A_n$ are $\mathcal{X}$-modules. This is an immediate consequence of the following theorem.

**Theorem 3.** The $\mathbb{R}$-vector space $\varphi(\mathcal{X})$ is a Lie algebra with the product

$$[x, y] := \nabla_x y - \nabla_y x \ \forall x, y \in \varphi(\mathcal{X}).$$

For every $n \in \mathbb{Z}$, the algebra $A_n$ is a $\varphi(\mathcal{X})$-module with the action $\delta_x$, $x \in \varphi(\mathcal{X})$, namely,

$$\left(\delta_x \delta_y - \delta_y \delta_x - \delta_{[x, y]}\right)f = 0 \ \forall x, y \in \varphi(\mathcal{X}), \forall f \in A_n.$$  

**Proof.** The statements are verified by using (11)–(20). A convenient procedure is as follows: First, verify that $[x, y]$ belongs to $\varphi(\mathcal{X})$ for all $x, y \in \varphi(\mathcal{X})$. Second, show the equation (26) for $f = \kappa, \tau, s, \ell$ and then extend (26) to the whole $A_{-\infty} = A_0 \cup A_{-1} \cup \cdots$. Finally, verify the Jacobi identity in $\varphi(\mathcal{X})$ with the help of (23).

The theorem above is quite similar to Theorem 1 of [13] in particular in the proof, though the considered objects are different.

To simplify expressions, we put

$$\tilde{\nabla}_X := \varphi \circ \nabla_{\wp(X)} \circ \varphi$$
for $\forall X \in \mathcal{X}$, so that we can write the commutator product of $\mathcal{X}$ as

$$[X, Y] = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X \quad \forall X, Y \in \mathcal{X}. \tag{28}$$

Likewise we put

$$\langle g_1 N + h_1 B, g_2 N + h_2 B \rangle \perp := g_1 g_2 + h_1 h_2, \tag{29}$$

which defines a bi-linear form $\varphi(T_i) \times \varphi(T_j) \rightarrow A_{i+j}$. Then, we have

$$\tilde{\nabla}_X Y = (\delta_{\varphi(X)} \langle N, Y \rangle \perp) N + (\delta_{\varphi(X)} \langle B, Y \rangle \perp) B + (\partial_s^{-1} \langle \kappa N, Y \rangle \perp) \varphi \nabla_s \varphi(X)$$

$$- \langle \kappa^{-1} B, \varphi \nabla_s \varphi(X) \rangle \perp (\langle B, Y \rangle \perp N - \langle N, Y \rangle \perp B) \tag{30}$$

for all $X, Y \in \mathcal{X}$.

Let $\mathcal{F}$ be the subalgebra of $A_0$ generated by $1$, $\ell$ and the elements of $\int(A_2)$. The vector space $\mathcal{F}$ is an $\mathcal{X}$-submodule of $A_0$, i.e., $\delta_{\varphi(X)}(\mathcal{F}) \subset \mathcal{F} \forall X \in \mathcal{X}$. We denote the action of $\mathcal{X}$ on $\mathcal{F}$ by the left-action, namely, $XF = \delta_{\varphi(X)} F \forall X \in \mathcal{X}, \forall F \in \mathcal{F}$. This action is a derivation:

$$X(FG) = (XF)G + F(XG) \quad \forall X \in \mathcal{X}, \forall F, G \in \mathcal{F}. \tag{31}$$

Since $\mathcal{F} \subset A_0$, we see that $\mathcal{X}$ is a left $\mathcal{F}$-module. Taking notice of $\partial_s F = 0 \forall F \in \mathcal{F}$ and referring to (11)—(20), we easily find $\varphi_{(F,X)} g = F \delta_{\varphi(X)} g$, $\tilde{\nabla}_{FX} Y = F \tilde{\nabla}_X Y$ and $\tilde{\nabla}_X (FY) = (\delta_{\varphi(X)} F) Y + F \tilde{\nabla}_X Y \forall F \in \mathcal{F}$, $\forall X, Y \in \mathcal{X}$, $\forall g \in A_n$. Hence we see

$$(FX)G = F(XG) \quad \forall F, G \in \mathcal{F}, \forall X \in \mathcal{X}, \tag{32}$$

$$\mathcal{L}_X (FY) = (\mathcal{L}_X F) Y + F \mathcal{L}_X Y \quad \forall X, Y \in \mathcal{X}, \forall F \in \mathcal{F}, \tag{33}$$

where

$$\mathcal{L}_X Y := [X, Y], \quad \mathcal{L}_X F := XF \quad \forall X, Y \in \mathcal{X}, \forall F \in \mathcal{F}. \tag{34}$$

Below, we construct in the usual, algebraic manner a differential calculus, in which the algebra $\mathcal{F}$ consisting of functionals on $BAL$ plays the role of the algebra of functions. The construction is based on the pair $(\mathcal{F}, \mathcal{X})$ of commutative algebra and Lie algebra. It is essential for this construction that $\mathcal{F}$ is a left $\mathcal{X}$-module, $\mathcal{X}$ is a left $\mathcal{F}$-module, and the equations (31)—(33) hold. We would like to make a further remark. Let $g: \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{F}$ be a symmetric form defined by

$$g(X, Y) := \int \langle X, Y \rangle \perp \tag{35}$$

with (29). This is bi-$\mathcal{F}$-linear and positive-definite. We refer to $g$ as the Riemannian structure on $BAL$. Given $F \in \mathcal{F}$, the vector field $X \in \mathcal{X}$ such that $YF = g(X, Y) \forall Y \in \mathcal{X}$ is called
the gradient of $F$ and is denoted by $\text{grad } F$. The existence of the gradient for every element of $\mathcal{F}$ can be verified by virtue of Propositions 1 and 2. In contrast to the differential calculus on finite dimensional Riemannian manifolds, this seems to be quite nontrivial. This situation is necessary for realizing the space of 1-forms as a space identifiable with $\mathcal{X}$.

Let $D^p$ denote the vector space of maps $\eta: \mathcal{X}^p \to \mathcal{F}$ such that $F := \eta(U_1, \ldots, U_p)$ with $U_i = g_iN + h_iB \in \mathcal{X}$ is $\mathcal{F}$-linear in each $U_i$, skew-symmetric (if $p \geq 2$) under the exchange of $U_i$ and $U_j$, $i \neq j$, and $F$ can be expressed as an $\mathcal{E}_0$-valued variable algebraically depending on $g_1, h_1, \ldots, g_p, h_p$. Such a map $\eta \in D^p$ is referred to as a $p$-th order differential form or $p$-form on $BAL$. From Proposition 1 and the nondegeneracy of (35), we see that for every 1-form $\xi$ there uniquely exists a vector field $X$ such that $\xi(Y) = g(X, Y) \forall Y \in \mathcal{X}$.

The exterior derivative is a map $d: D^p \to D^{p+1}$ defined by

$$
(d\eta)(U_0, \ldots, U_p) = \sum_{i=0}^{p} (-1)^i U_i(\eta(U_0, \ldots, \check{U}_i, \ldots, U_p))
$$

$$
+ \sum_{i<j} (-1)^{i+j} \eta([U_i, U_j], U_0, \ldots, \check{U}_i, \ldots, \check{U}_j, \ldots, U_p) \quad (36)
$$

$$
\forall \eta \in D^p, \forall U_i \in \mathcal{X},
$$

where $\check{U}_i$ stands for the absense of $U_i$. This is a coboundary operator, i.e., $d \circ d = 0$. The interior product $\iota_X : D^{p+1} \to D^p$ with $X \in \mathcal{X}$ is defined by

$$
(\iota_X \eta)(U_1, \ldots, U_p) = \eta(X, U_1, \ldots, U_p).
$$

(37)

The Lie derivative $\mathcal{L}_X$ in the direction of $X \in \mathcal{X}$ is an operator acting on each $\mathcal{X}$-module consisting of certain $\mathcal{F}$-vectors, so-called tensor fields, and $X \mapsto \mathcal{L}_X$ provides a representation of the Lie algebra $\mathcal{X}$. The definition was already given both on $\mathcal{F}$ and $\mathcal{X}$ in (34). The extension to other tensor fields is made by imposing Leibnitz rule. For example, if $R$ is an $\mathcal{F}$-linear map $\mathcal{X} \to \mathcal{X}$, then $\mathcal{L}_X (RY) = (\mathcal{L}_X R)Y + R\mathcal{L}_X Y \forall X, Y \in \mathcal{X}$. For a $p$-form $\eta$, the formula

$$
\mathcal{L}_X \eta = \iota_X d\eta + d\iota_X \eta
$$

(38)

is available. It is possible to introduce the exterior product, which is, however, not used in this paper.
4 Recursion Operator and Hamiltonian Pair

An $\mathcal{F}$-linear map $K: \mathcal{X} \to \mathcal{X}$ is called a skew-adjoint operator if $g(KX, Y) = -g(X, KY)$. With a skew-adjoint operator $J$ defined by

$$J(X) = \langle B, X \rangle_N - \langle N, X \rangle_B,$$

(39)

the Marsden-Weinstein Poisson structure [4] can be written as $\{F, G\} = (J \text{ grad } F)G \forall F, G \in \mathcal{F}$. The operator $J$, $J^2 = -1$, is a complex structure; it can be shown by a direct calculation that the Nijenhuis torsion (see Eq. (43)) of $J$ vanishes, though this fact is not used in this paper. The vortex filament equation (1) can be understood as a Hamiltonian equation with the Hamiltonian functional $\ell$; indeed

$$\kappa b = \varphi(J \text{ grad } \ell).$$

(40)

Making use of the Hasimoto map, Langer and Perline [5] found that the vector fields $\kappa B$, $KJ^{-1}(\kappa B)$, $(KJ^{-1})^2(\kappa B)$, ... are Hamiltonian flows (see Subsection 4.2) associated with the constants of motion in involution, where $K$ is another skew-adjoint operator defined by

$$K(X) = J\varphi \nabla_s \varphi J(X).$$

(41)

The operator

$$R := KJ^{-1} = J \circ \varphi \circ \nabla_s \circ \varphi$$

(42)

is referred to as the recursion operator. It should be emphasized that the definition of $\mathcal{X}$ given in the preceding section is consistent with $J$, $K$ and $R$, namely, these operators make sense as $\mathcal{F}$-linear maps $\mathcal{X} \to \mathcal{X}$.

Below, after giving several results on the recursion operator $R$ (Subsection 4.1), we shall show that $J$ and $K$ form a Hamiltonian pair (Subsection 4.2). The approach pursued in this section is similar to that of [11].

4.1 The hereditary property

Let $R$ be an $\mathcal{F}$-linear map $\mathcal{X} \to \mathcal{X}$. In most statements of this subsection, we need not suppose $R$ is the recursion operator defined by (42); the exception is Theorem 4.

The Nijenhuis torsion $N_R$ of an $\mathcal{F}$-linear operator $R: \mathcal{X} \to \mathcal{X}$ is an $\mathcal{F}$-linear map $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ defined by

$$N_R(X, Y) := (\mathcal{L}_R X R - R \mathcal{L}_X R) Y$$

$$= [RX, RY] - R[RX, Y] - R[X, RY] + R^2[X, Y].$$

(43)
An $\mathcal{F}$-linear operator $R$ such that $N_R = 0$ is said to be hereditary [6, 9]; see also [7, 8, 11]. For a hereditary operator $R$, it is easy to see that $\mathcal{L}_{RX}(R^n) = R\mathcal{L}_X(R^n)$ and further $\mathcal{L}_{R^nX}(R^n) = R^m\mathcal{L}_X(R^n)$, namely,

$$[R^mX, R^nY] - R^n[R^mX, Y] = R^m[X, R^nY] - R^{m+n}[X, Y]$$ (44)

for all vector fields $X, Y$.

**Theorem 4.** The recursion operator $R$ defined by (42) is hereditary.

**Proof.** For all $X \in \mathcal{X}$, define $\tilde{\nabla}_X R: \mathcal{X} \to \mathcal{X}$ by

$$(\tilde{\nabla}_X R)(Y) = \tilde{\nabla}_X (RY) - R(\tilde{\nabla}_X Y) \quad \forall Y \in \mathcal{X},$$ (45)

which is found to be

$$(\tilde{\nabla}_X R)(Y) = (\partial_s^{-1} \langle \kappa N, RY \rangle_\perp) J^{-1} RX + (\partial_s^{-1} \langle RX, RY \rangle_\perp) \kappa B$$ (46)

by using (30). Substituting the formula above into

$$N_R(X, Y) = (\tilde{\nabla}_{RX} R)(Y) - R((\tilde{\nabla}_X R)(Y) - (X \leftrightarrow Y),$$

we find $N_R = 0$.

Let $\eta$ be a $p$-form, $p \geq 1$. We say $\eta$ is compatible with $R$ if

$$\iota_{RY} \iota_X \eta = \iota_Y \iota_{RX} \eta \quad \forall X, Y \in \mathcal{X}.$$ (47)

By definition, every 1-form is compatible with $R$. Suppose $\eta$ is a $p$-form compatible with $R$. Then, it is possible to define a $p$-form $\eta \circ R$ such that $\iota_X (\eta \circ R) = \iota_{RX} \eta$ for all $X \in \mathcal{X}$. Further, $\eta \circ R$ is again compatible with $R$. Hence a $p$-form compatible with $R$ induces $p$-forms $\eta \circ R^n$ compatible with $R$, $n = 0, 1, 2, \ldots$, such that

$$\iota_X (\eta \circ R^n) = \iota_{R^nX} \eta \quad \forall X \in \mathcal{X}.$$ (48)

For a $p$-form $\eta$ compatible with $R$, we have the formulae

$$\mathcal{L}_{R^nX} \eta = \iota_{R^nX} d \eta + \mathcal{L}_X (\eta \circ R^n) - \iota_X d (\eta \circ R^n),$$ (49)

$$\iota_Y \mathcal{L}_X (\eta \circ R) = \iota_{RY} \mathcal{L}_X \eta + \iota_{(\mathcal{L}_X R)Y} \eta,$$ (50)
where \( n \) is an arbitrary non-negative integer and \( X \) is an arbitrary vector field. The former \((49)\) is an immediate consequence of \((38)\). The latter \((50)\) is nothing but the Leibnitz rule
\[
(L_X(\eta \circ R))(Y, U_2, \ldots, U_p) = (L_X\eta)(RY, U_2, \ldots, U_p) + \eta((L_X R)Y, U_2, \ldots, U_p).
\]

For a 2-form \( \omega \) compatible with \( R \), we have
\[
\{ (L_{R^n} X \omega)(Y, Z) - (\omega \circ R^n)(X, [Y, Z]) \} + \text{cycle}(X, Y, Z) \\
= \{ d\omega(R^n X, Y, Z) + \text{cycle}(X, Y, Z) \} - 2d(\omega \circ R^n)(X, Y, Z),
\]
where \( n \) is an arbitrary non-negative integer and \( X, Y, Z \) are arbitrary vector fields. The formula \((51)\) is verified as follows: Starting with the substitution of \((49)\), we calculate the left-hand side of \((51)\) as
\[
\begin{align*}
d\omega(R^n X, Y, Z) + (L_X(\omega \circ R^n))(Y, Z) - d(\omega \circ R^n)(X, Y, Z) \\
- (\omega \circ R^n)(X, [Y, Z]) + \text{cycle}(X, Y, Z) \\
= \{ d\omega(R^n X, Y, Z) + (L_X(\omega \circ R^n))(Y, Z) - (\omega \circ R^n)(Y, [Z, X]) \\
+ \text{cycle}(X, Y, Z) \} - 3d(\omega \circ R^n)(X, Y, Z) \\
= \{ d\omega(R^n X, Y, Z) + X((\omega \circ R^n))(Y, Z)) - (\omega \circ R^n)([X, Y], Z) \\
+ \text{cycle}(X, Y, Z) \} - 3d(\omega \circ R^n)(X, Y, Z).
\end{align*}
\]

Then, recalling the definition \((34)\) of exterior derivative, we see this equals to the right-hand side of \((51)\).

**Lemma 5.** Let \( R \) be an \( F \)-linear map \( \mathcal{X} \to \mathcal{X} \). If \( \eta \) is a \( p \)-form compatible with \( R \), then
\[
\iota_Y \iota_X d(\eta \circ R^2) - \iota_{RY} \iota_X d(\eta \circ R) - \iota_Y \iota_{RX} d(\eta \circ R) + \iota_{RY} \iota_{RX} d\eta = -\iota_{N_R(X, Y)} \eta
\]
for all vector fields \( X, Y \).

**Proof.** That the left-hand side of \((52)\) equals to
\[
\iota_Y \mathcal{L}_X(\eta \circ R^2) - \iota_{RY} \mathcal{L}_X(\eta \circ R) - \iota_Y \mathcal{L}_{RX}(\eta \circ R) + \iota_{RY} \mathcal{L}_{RX} \eta
\]
can be shown by the substitution of the identity \( \iota_Y \iota_X d = \iota_Y \mathcal{L}_X - \mathcal{L}_Y \iota_X + d\iota_Y \iota_X \forall X, Y \in \mathcal{X} \) following from \((38)\). Using \((34)\), we can rewrite the expression above into \( \iota_{(L_X R)Y} (\eta \circ R) - \iota_{(L_{RX} R)Y} \eta \), which obviously equals to the right-hand side of \((52)\).
4.2 Schouten bracket and Hamiltonian pair

Let us recall the notion of a Hamiltonian operator/pair [1, 2].

Let \( H_i, i = 0, 1 \) be \( \mathcal{F} \)-linear maps \( D^1 \to X \). Suppose \( H_i \) are skew-symmetric, i.e.,

\[
\xi(H_i \eta) = -\eta(H_i \xi) \quad \forall \xi, \eta \in D^1.
\] (53)
The Schouten bracket \([H_0, H_1]\) between \( H_0 \) and \( H_1 \) is a skew-symmetric tri-\( \mathcal{F} \)-linear map \( D^1 \times D^1 \times D^1 \to \mathcal{F} \) defined by

\[
[H_0, H_1](\xi, \eta, \zeta) = \xi(H_0 \mathcal{L}_{H_1} \eta \zeta) + \mathcal{L}_{H_0}(H_1) + \text{cycle}(\xi, \eta, \zeta).
\] (54)

A skew-symmetric \( \mathcal{F} \)-linear map \( H_0: D^1 \to X \) is referred to as a Hamiltonian operator if \([H_0, H_0] = 0\). The vector field \( H_0 \) associated with \( F \in \mathcal{F} \) via Hamiltonian operator \( H_0 \) is called the Hamiltonian vector field of \( F \). A Hamiltonian operator \( H_0 \) induces the Poisson structure

\[
\{F, G\}_{H_0} = (H_0 \eta) F + \{H_0, H_1\} \quad \forall F, G \in \mathcal{F}
\] (55)
and \( H_0 \circ d \) is a morphism \( \mathcal{F} \to X \) of Lie algebras. Hamiltonian operators \( H_0 \) and \( H_1 \) are said to be a Hamiltonian pair if \([H_0, H_1] = 0\).

The existence of a Hamiltonian pair with certain conditions implies the integrability — the existence of a sequence of functionals in involution or Poisson-commutative functionals.

Returning to the case of \( BAL \), we define \( H_n: D^1 \to X \), \( n = 0, 1, \ldots \) by

\[
H_n = R^n H_0, \quad g(X, H_0 \eta) = -\eta(JX) \quad \forall X \in X, \forall \eta \in D^1
\] (56)
with \( J \) and \( R \) defined by (39) and (42). Under the identification caused by the Riemannian structure \( g \), we see that \( H_0 \) and \( H_1 \) are nothing but the operators \( J \) and \( K \), respectively. Indeed, \( H_n \circ d = R^n J \circ \text{grad} \).

We shall show that \( H_m \) and \( H_n \) form a Hamiltonian pair. For this aim, it is useful to introduce the sequence of 2-forms \( \Omega_n \), \( n = 0, 1, \ldots \),

\[
\Omega_n = \Omega_0 \circ R^n; \quad \Omega_0(X, Y) = g(J^{-1}X, Y) \quad \forall X, Y \in X.
\] (57)
The well-definedness of \( \Omega_n \) as 2-forms is explained as follows: As was mentioned, \( J \) and \( K \) are skew-adjoint operators. From the skew-adjointness of \( J \), we see that \( \Omega_0 \) is well-defined as a 2-form. From the skew-adjointness of \( J \) and \( K \), we see that \( \Omega_0 \) is compatible with \( R \).
Hence, \( \Omega_n = \Omega_0 \circ R^n \) are well-defined as 2-forms. These 2-forms are related to \( H_n \) in the following way:

\[
\Omega_m(H_n \xi, Y) = \xi(R^m \xi Y) \quad \forall \xi \in D^1, \forall Y \in \mathcal{X}.
\] (58)

It is possible to show that \( d\Omega_0 = d\Omega_1 = 0 \) by using (50). Since \( R \) is hereditary, we find

\[
d\Omega_0 = d\Omega_1 = d\Omega_2 = \cdots = 0
\] (59)
as a consequence of Lemma 5. We note that \( d\Omega_0 = 0 \) (and Theorem 6 for the case \( m = n = 0 \)) is implied in [4], because \( \Omega_0 \) is the symplectic structure corresponding to the Hamiltonian operator \( J \). It should be mentioned that \( \Omega_n, n \geq 1 \), is not the symplectic structure corresponding to \( H_n \).

**Theorem 6.** Two operators arbitrarily chosen from the sequence \( H_n \) defined by (56) form a Hamiltonian pair, i.e., \( [H_m, H_n] = 0 \) for all non-negative integers \( m \) and \( n \).

This theorem follows immediately from (59) and the lemma below.

**Lemma 7.** Let \( R: \mathcal{X} \to \mathcal{X} \) be a hereditary operator and \( \Omega_0 \) a 2-form compatible with \( R \). Suppose the map \( \mathcal{X} \to D^1, X \mapsto \iota_X \Omega_0 \) is invertible, so that an \( \mathcal{F} \)-linear map \( H_0: D^1 \to \mathcal{X} \) is defined by \( \Omega_0(H_0 \xi, Y) = \xi(Y) \) \( \forall \xi \in D^1, \forall Y \in \mathcal{X} \). Then, \( H_n := R^n H_0, n = 0, 1, 2, \ldots \) are skew-symmetric \( \mathcal{F} \)-linear maps \( D^1 \to \mathcal{X} \) and the Schouten brackets between them are written as

\[
[H_m, H_n](\xi, \eta, \zeta) = 4d\Omega_{m+n}(H_0 \xi, H_0 \eta, H_0 \zeta) - \{d\Omega_m(H_n \xi, H_0 \eta, H_0 \zeta) + (m \leftrightarrow n) + \text{cycle}(\xi, \eta, \zeta)\}
\] (60)

with \( \Omega_n := \Omega_0 \circ R^n \).

**Proof.** The \( \mathcal{F} \)-linearity of \( H_n \) is apparent. Further, the calculation

\[
\xi(H_n \eta) = \Omega_0(H_0 \xi, H_n \eta) = (\Omega_0 \circ R^n)(H_0 \xi, H_0 \eta) = -\Omega_0(R^n \xi, H_0 \eta) = -\xi(H_n \xi)
\]

shows that \( H_n \) are skew-symmetric. The Schouten bracket \( [H_m, H_n] \) is therefore well-defined and is calculated as follows: First, we notice

\[
\xi(H_m L_{H_n \eta} \xi) = -(L_{H_m \eta} \xi)(H_n \xi) = -(H_m \eta)(\xi(H_n \xi)) + \xi([H_m \eta, H_n \xi]),
\]
so that we have

\[ [H_m, H_n](\xi, \eta, \zeta) = \{ -(H_m \eta)(\zeta(H_n \xi)) + \zeta([H_m \eta, H_n \xi]) \} + (m \leftrightarrow n) \text{ + cycle}(\xi, \eta, \zeta). \]

Using (44), we see

\[ [H_m, H_n](\xi, \eta, \zeta) = \{ -(H_m \eta)(\Omega_n(H_0 \zeta, H_0 \xi)) + \Omega_n(H_0 \zeta, [H_m \eta, H_0 \xi]) \]
\[ + \Omega_m(H_0 \zeta, [H_0 \eta, H_n \xi]) - \Omega_m(H_0 \zeta, [H_0 \eta, H_0 \xi]) \}
\[ + (m \leftrightarrow n) \text{ + cycle}(\xi, \eta, \zeta). \]

Taking notice of the symmetry under the exchange \( m \leftrightarrow n \) and the cyclic permutation, we see that the first three terms in the right-hand side sum up to \( -L_{H_m \xi} \Omega_n(H_0 \eta, H_0 \zeta) \). Then, with the help of (51) we obtain (60).

5 Symmetries and Master Symmetries

Let \( X_n, n = 0, 1, 2, \ldots \) be the vector fields

\[ X_n = R^n(\kappa B) \quad (61) \]

and \( Y_n, n = 1, 2, 3, \ldots \) the vector fields

\[ Y_n = R^{n-1}(s \kappa B), \quad (62) \]

where \( R \) is the recursion operator defined by (42). The vector fields \( X_n \) are those given in [12] with a difference in their index (shifted by 2).

Lemma 8. The vector fields \( X_0 \) and \( Y_1 \) act on the recursion operator \( R \) of (42) as follows:

\[ \mathcal{L}_{X_0} R = 0, \quad \mathcal{L}_{Y_1} R = -R^2. \quad (63) \]

Proof. This is shown through a somewhat tedious calculation. It is easy to see \( (\mathcal{L}_X R)(Y) = (\tilde{\nabla}_X R)(Y) - \tilde{\nabla}_{RY} X + R \tilde{\nabla}_Y X \) for all vector fields \( X, Y \). We continue the calculation with substitution of (30) and (46), and finally arrive at the lemma.

As a corollary of the lemma, it is immediate to see

\[ \mathcal{L}_{X_0} R^n = 0, \quad \mathcal{L}_{Y_1} R^n = -n R^{n+1}. \quad (64) \]
Proposition 9. The vector fields $X_0$, $X_1$, \ldots and $Y_1$, $Y_2$, \ldots form a Lie subalgebra of $\mathcal{X}$ such that

\begin{align*}
[X_n, X_m] &= 0, \quad (65) \\
[X_n, Y_m] &= (n + 2)X_{n+m}, \quad (66) \\
[Y_n, Y_m] &= (n - m)Y_{n+m}. \quad (67)
\end{align*}

Proof. As was stated in Theorem \[4\] the recursion operator $R$ is hereditary, hence the formula (44) is available. This formula can be written also in the following form:

\[ [R^m X, R^n Y] = -R^n(L_Y R^m)X + R^m(L_X R^n)Y + R^{m+n}[X, Y] \quad \forall X, Y \in \mathcal{X}. \]

Substituing (64) to the formula above, we obtain (65) and (67). We can show (66) in much the same way with the help of

\[ [X_0, Y_1] = 2X_1. \quad (68) \]

This equation can be verified by using (30).

It is immediate from (1) to see that $X_0$ is the flow of the vortex filament equation. A vector field commuting with the flow $X_0$ of an evolution equation is, generally, called a symmetry (of the equation). Thus $X_n$ of (61) can be described as symmetries of the vortex filament equation. Since these symmetries are generated from $X_0$ by the action of $Y_n$ as in (68), vector fields $Y_n$ are referred to as master symmetries \[10\]. Sometimes the term ‘recursion operator’ is used for meaning an $\mathcal{F}$-linear operator $R: \mathcal{X} \to \mathcal{X}$ such that $L_{X_0} R = 0$, i.e., an operator that maps a symmetry into another symmetry \[3\]. The operator $R$ of (12) is a recursion operator also in this sense.

Proposition 10. By means of the Lie derivative, symmetries (61) and master symmetries (62) act on the 2-forms $\Omega_n$ defined by (27) as follows:

\begin{align*}
\mathcal{L}_{X_{m-1}} \Omega_n &= 0, \quad (69) \\
\mathcal{L}_{Y_m} \Omega_n &= (3 - m - n)\Omega_{m+n} \quad (70)
\end{align*}

where $m = 1, 2, \ldots$ and $n = 0, 1, \ldots$.

Proof. Making use of (60), we can show

\[ \mathcal{L}_{X_0} \Omega_0 = 0, \quad \mathcal{L}_{Y_1} \Omega_0 = 2\Omega_1 \quad (71) \]
with a somewhat tedious calculation. Using (64), the equation above and Leibnitz rule, we can derive Eqs. (69) and (70) for the case $m = 1$. Further, we see

$$L_{X}^{\Omega} \equiv L_{X}^{\Omega_{m+n}} \forall X \in \mathcal{X}$$

as a specific case of (49), since $d\Omega_{n} = 0$ as in (59).

Since $\Omega_{n}$ are closed 2-forms, the proposition says

$$d\iota_{X_{m-1}} \Omega_{n} = 0,$$  \hspace{1cm} (72)

$$d\iota_{Y_{m}} \Omega_{n} = (3 - m - n)\Omega_{m+n}.$$  \hspace{1cm} (73)

Apparently, $\Omega_{n}$, $n = 1, 2, 4, 5, \ldots$ are exact.

Let $\zeta_{n}$, $n = 0, 1, \ldots$ be the 1-forms

$$\zeta_{n} := \iota_{X_{n}} \Omega_{0} = (\iota_{X_{0}} \Omega_{0}) \circ R^{n}.$$  \hspace{1cm} (74)

As expressed in (72), 1-forms $\zeta_{n}$ are closed. Since we already know the Lie derivative of $X_{n}$ and $\Omega_{0}$ in the direction of $X_{m-1}$ and $Y_{m}$, we can easily verify that

$$\mathcal{L}_{X_{m-1}} \zeta_{n} = 0,$$  \hspace{1cm} (75)

$$\mathcal{L}_{Y_{m}} \zeta_{n} = (1 - m - n)\zeta_{m+n},$$  \hspace{1cm} (76)

where $m = 1, 2, \ldots$ and $n = 0, 1, \ldots$.

**Theorem 11.** The 1-forms $\zeta_{n}$, $n = 0, 2, 3, \ldots$ defined by (74) are exact and are written as $\zeta_{0} = dI_{0}$, $I_{0} := \ell$ and

$$\zeta_{n} = dI_{n}, \hspace{1cm} I_{n} := \frac{1}{1-n} \zeta_{n-1}(Y_{1}), \hspace{1cm} n = 2, 3, \ldots$$  \hspace{1cm} (77)

The functionals $I_{n}$, $n = 0, 2, 3, \ldots$ are in involution with respect to the Poisson bracket associated with $H_{k}$ with arbitrary $k$. The vector fields $X_{n}$, $n = 0, 2, 3, \ldots$ of (61) are Hamiltonian vector fields of $I_{n}$ with respect to $H_{0}$, namely, $X_{n} = J \text{ grad } I_{n}$.

**Proof.** Since $\zeta_{n}$ are closed 1-forms, (76) can be written as $d\iota_{Y_{m}} \zeta_{n} = (1 - m - n)\zeta_{m+n}$, from which we see that $\zeta_{n}$, $n = 2, 3, \ldots$ are exact 1-forms written as in (77). The same is easy for the case $n = 0$. Using (53) and (58), we see

$$\{I_{i}, I_{j}\}_{H_{k}} = dI_{j}(H_{k}dI_{i}) = \zeta_{j}(H_{k}\zeta_{i}) = \Omega_{0}(X_{j}, H_{k}\zeta_{i}) = -\zeta_{i}(R^{k}X_{j}) = -\Omega_{i+j+k}(X_{0}, X_{0}) = 0,$$
namely, the functionals $I_n$ are in involution with respect to the Poisson bracket associated with $H_k$. Using (57), we see

$$\Omega_0(J \grad I_n, Y) = g(\grad I_n, Y) = dI_n(Y) = \zeta_n(Y) = \Omega_0(X_n, Y)$$

for all $Y \in \mathcal{X}$. This implies $X_n = J \grad I_n$.

Although we gave it in the proof, an explanation for the statements of the theorem other than $\zeta_n = dI_n$ can be found in the general theory [4, 8].

The missing piece, functional $I_1$ such that $X_1 = J \grad I_1$ does not exist within $\mathcal{F}$ (the possible candidate for $I_1$ in other treatments is the total torsion $\mathcal{F}$, [3, 13]).

The theorem above proves the inspection of Langer and Perline [5] saying $I_n = 1 - \int_{-\infty}^{\infty} ds \, \partial^{-1}s \langle \kappa N, X_n^{-1} \rangle \bot$.

Partly, the same is given in [13]. The proof is as follows: Inserting $\partial^{-1}s = 1$ into the integrand and then making partial integration, we see that the right-hand side of the equation above equals to

$$\frac{1}{n-1} \int_{-\infty}^{\infty} ds \, \langle -s \kappa N, X_{n-1} \rangle \bot = \frac{1}{n-1} \langle J^{-1}(s \kappa B), X_{n-1} \rangle = \frac{1}{n-1} \Omega_0(Y_1, X_{n-1}),$$

which obviously coincides with (77). The surface term in this partial integration is absent, since $\partial^{-1}s \langle \kappa N, X_{n1} \rangle \bot \in \mathcal{A}_2$ follows from $\langle \kappa N, X_{n1} \rangle \bot \in \mathcal{A}_2$ by virtue of the fact [3] that $\partial^{-1}s \langle \kappa N, X_{n-1} \rangle \bot$ can be written as polynomials in $\partial^s \kappa$ and $\partial^s \tau$.

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