SPECTRA OF QUADRATIC VECTOR FIELDS ON $\mathbb{C}^2$: THE MISSING RELATION

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Abstract. Consider a quadratic vector field on $\mathbb{C}^2$ having an invariant line at infinity and isolated singularities only. We define the extended spectra of singularities to be the collection of the spectra of the linearization matrices of each singular point over the affine part, together with all the characteristic numbers (i.e. Camacho-Sad indices) at infinity. This collection consists of 11 complex numbers, and is invariant under affine equivalence of vector fields.

In this paper we describe all polynomial relations among these numbers. There are 5 independent polynomial relations; four of them follow from the Euler-Jacobi, the Baum-Bott, and the Camacho-Sad index theorems, and are well-known. The fifth relation was, until now, completely unknown. We provide an explicit formula for the missing 5th relation, discuss it’s meaning and prove that it cannot be formulated as an index theorem.

1. Introduction

This work deals with generic quadratic vector fields on the affine plane $\mathbb{C}^2$, and the singular holomorphic foliations that these vector fields define on the projective plane $\mathbb{P}^2$.

The space of polynomial vector fields of degree at most $n$ has a natural vector space structure. We will say that a property is generic for vector fields of degree $n$, if it is satisfied by every vector field in some dense Zariski open subset of this vector space. For example, a generic vector field $v$ of degree $n$ has exactly $n^2$ isolated singularities. Also, it is well known that the foliation of $\mathbb{C}^2$ defined by $v$ can be extended to a foliation of $\mathbb{P}^2$ with isolated singularities. This extension $\mathcal{F}_v$ is unique, and generically $\mathcal{F}_v$ has an invariant line at infinity. This means that the line $\mathcal{L} = \mathbb{P}^2 \setminus \mathbb{C}^2$, once a finite number of singularities are removed, is a leaf of the foliation $\mathcal{F}_v$. In the generic case, the number of singularities on $\mathcal{L}$ is exactly $n + 1$.

We deal exclusively with vector fields with these properties.

1.1. The extended spectra of singularities. Let $p$ be an isolated singular point of some vector field $v = \frac{P(x, y)}{x} \frac{\partial}{\partial x} + \frac{Q(x, y)}{y} \frac{\partial}{\partial y}$, and consider the linearization matrix

$$Dv(p) = \begin{pmatrix} P' & P' \\ Q' & Q' \end{pmatrix}_{(x, y) = p}.$$
Analytically equivalent vector fields have conjugate linearization matrices, hence the spectrum of the linearization matrix at each singular point is an analytic invariant.

**Definition 1.1.** Let \( p \) be a singular point of \( v \). We define the *spectrum* of \( v \) at \( p \) as the ordered pair \( \text{Spec}(v, p) = (\text{tr} Dv(p), \text{det} Dv(p)) \). The *finite spectra of singularities* of \( v \) is the set (formally, the multiset)

\[
\text{Spec } v = \{ \text{Spec}(v, p) \mid v(p) = 0 \}.
\]

In order to study the extended foliation in a neighborhood of the line at infinity we introduce the following change of coordinates:

\[
z = \frac{1}{x}, \quad w = \frac{y}{x}.
\]

A simple computation shows that, in these coordinates, a generic degree \( n \) polynomial vector field induces a foliation given by an equation of the form

\[
\frac{dz}{dw} = z^{n+1} \sum_{j=1}^{\lambda_j} \frac{\lambda_j}{w - w_j} + O(z^2).
\]

The line at infinity is given by \( \mathcal{L} = \{z = 0\} \), and the singular points on it correspond to the poles \( w_j \). The *characteristic numbers at infinity* are defined to be the residues \( \lambda_j \), which are precisely the Camacho-Sad indices \( \lambda_j = \text{CS}(F_v, \mathcal{L}, w_j) \).

**Definition 1.2.** The *extended spectra of singularities* of a polynomial vector field \( v \) is the collection of the finite spectra of singularities, together with the characteristic numbers at infinity.

**Remark 1.1.** Since each number in the extended spectra is a local analytic invariant, affine equivalent vector fields have the same extended spectra.

Note that even though we work with local invariants, we only consider the spectra as a collection of these invariants taken over all singularities. Therefore, the questions we deal with are of a global nature.

### 1.2. Relations coming from index theorems.

Let us fix the following class of quadratic vector fields.

**Definition 1.3.** Denote by \( \mathcal{V}_2 \) the space of all quadratic vector fields \( v \) on \( \mathbb{C}^2 \) such that

- \( v \) has exactly 4 isolated singularities;
- the extended foliation \( F_v \) has an invariant line at infinity carrying exactly 3 singular points.

For technical reasons, we pass to a finite cover, and assume that both finite and infinite singularities of \( v \) are enumerated.

Then the extended spectra of a vector field \( v \in \mathcal{V}_2 \) is an ordered tuple of 11 complex numbers: 8 coming from the finite spectra and 3 characteristic numbers at infinity. The object of this paper is to give a complete description of *all the algebraic relations* among these 11 numbers.
These numbers are related by four classical index theorems:

\[ \sum_{v(p)=0} \frac{1}{\det Dv(p)} = 0, \]  
\[ \sum_{v(p)=0} \frac{\text{tr } Dv(p)}{\det Dv(p)} = 0, \]  
\[ \sum_{p \in \text{Sing } F_v} \text{BB}(F_v, p) = 16, \]  
\[ \sum_{p \in \mathcal{L} \cap \text{Sing } F_v} \text{CS}(F_v, \mathcal{L}, p) = 1. \]  

These are, respectively, the Euler-Jacobi relations, the Baum-Bott theorem and the Camacho-Sad theorem, see Sec. 2 for details.

1.3. The new “hidden” relation. Let us do a simple dimension count. On the one hand, the space \( V_2 \) has dimension 12, and the affine group \( \text{Aff}(2, \mathbb{C}) \) has dimension 6. Therefore, the quotient (in the sense of Geometric Invariant Theory) \( V_2 \mod \text{Aff}(2, \mathbb{C}) \) has dimension 6. On the other hand, the extended spectra, which is of dimension 11, modulo the 4 equations above is a space of dimension 7. This gap in the dimensions implies that there must exist at least one more algebraic relation among these numbers. This last hidden relation was, until very recently, completely unknown. In order to describe this relation, let us introduce some notation.

Let \( v \in V_2 \) have singularities \( p_1, \ldots, p_4 \) on \( \mathbb{C}^2 \) and singular points at infinity \( w_1, w_2, w_3 \). Put \( \text{Spec}(v, p_k) = (t_k, d_k) \), and let \( \lambda_j \) be the characteristic number of \( w_j \).

Remark 1.2. The fact that all 7 singular points are different implies that all four determinants \( d_k \), and all characteristic numbers \( \lambda_j \) are non-zero numbers.

Let \( \Lambda \) denote the product \( \Lambda = \lambda_1 \lambda_2 \lambda_3 \) and define \( \mathcal{R} \) be the graded polynomial ring \( \mathcal{R} = \mathbb{C}[t_1, \ldots, t_4, d_1, \ldots, d_4, \Lambda] \), where the generators \( t_k \) are of degree 1, \( d_k \) are of degree 2, and \( \Lambda \) is of degree zero.

Theorem 1.1. There exists a polynomial \( H \in \mathcal{R} \) such that

1. the extended spectra of any quadratic vector field in \( V_2 \) satisfies
   \[ H(t; d; \Lambda) = 0; \]
2. the above equality is independent from \( \text{EJ1} \)–\( \text{CS} \);
3. if \( F \in \mathbb{C}[t; d; \lambda] \) is another polynomial vanishing on the extended spectra of every vector field in \( V_2 \), then \( F = 0 \) follows from \( \text{EJ1} \)–\( \text{CS} \) together with the equalities \( \text{EJ1} \)–\( \text{CS} \) and inequalities \( d_k \neq 0, k = 1, \ldots, 4 \), and \( \lambda_j \neq 0, j = 1, 2, 3 \).

The polynomial \( H \) in Theorem 1.1 is not uniquely defined. To get a uniquely defined polynomial (up to rescaling), we can eliminate \( t_4 \) and \( d_4 \) using \( \text{EJ1} \) and \( \text{EJ2} \). Let \( S \) be the subring of \( \mathcal{R} \) consisting of polynomials not depending on \( t_4 \), \( d_4 \) or \( \Lambda \).

Theorem 1.2. There exist polynomials \( H_0, H_1, H_2 \in S \) homogeneous of weighted degree 14 such that the polynomial \( H = H_2 \Lambda^2 + H_1 \Lambda + H_0 \) is irreducible in \( \mathcal{R} \), and satisfies the assertions of Theorem 1.1. The polynomial \( H \) is uniquely defined up to rescaling by a non-zero complex number.
The explicit expression for $H$ (available at [KR]) was obtained using a computer algebra system, see Sec. 5.4 for details. Unfortunately, the polynomial $H$ has a very long expression: it consists of 996 monomials.

Another natural goal is to find a polynomial $H$ which is diagonal symmetric, i.e. invariant under applying the same permutation to $t_k$'s and $d_k$'s. Geometrically, such permutation corresponds to reenumeration of the singular points $p_k$.

**Theorem 1.3.** There exist diagonal symmetric polynomials $\tilde{H}_0, \tilde{H}_1, \tilde{H}_2 \in S$ homogeneous of weighted degree 10 such that for a generic quadratic vector field we have

$$ (d_1 d_2 d_3)^2 \tilde{H}_k = d_4^3 H_k, $$

where $H_k$ are the polynomials from Theorem 1.2. The polynomial $\tilde{H} = \tilde{H}_2 \Lambda^2 + \tilde{H}_1 \Lambda + \tilde{H}_0$ satisfies the assertions of Theorem 1.1.

The polynomial $\tilde{H}$ is not uniquely defined. We provide an explicit formula for one of such polynomials in Appendix A.

1.4. **Lack of new index theorems.** Despite the length of the formula for $H$, we have used its explicit expression to show that (2) does not come from an index theorem. Moreover, we show that any possible “index-theorem-like identity” can be deduced from the four classical index theorems, hence concluding the lack of existence of new index theorems that constrain the extended spectra of quadratic vector fields.

The lack of existence of new index theorems is discussed in Sec. 6 but it follows from the next theorem.

**Theorem 1.4.** There exists no pair $(R, r)$ consisting of a rational function $R$ on $\mathbb{C}^8$ and a symmetric rational function $r$ on $\mathbb{C}^3$ with the property that every quadratic vector field with non-degenerate singularities satisfies the relation

$$ R(t; d) = r(\lambda), $$

except for those that can be derived from the previously known relations (EJ1)–(CS).

**Remark 1.3.** The question of finding the hidden relations on the extended spectra was inspired by a very similar question on the hidden relations between the spectra of the derivatives at the fixed points of a rational endomorphism $f : \mathbb{P}^n \to \mathbb{P}^n$ posed by Adolfo Guillot in [Gui04]. In fact, the case of vector fields may be understood as a sub-case of Guillot’s problem [Ram16].

2. **The classical index theorems**

2.1. **The Euler-Jacobi relations.** Let us recall, in the particular case relevant to us, a classical result known as the Euler-Jacobi formula [GH94, Chpt. 5, Sec. 2].

**Theorem 2.1.** If $P, Q$ are polynomials in $\mathbb{C}[x, y]$ of degree $n$ whose divisors intersect transversely at $n^2$ different points $p_1, \ldots, p_{n^2} \in \mathbb{C}^2$ and $g(x, y)$ is a polynomial of degree at most $2n - 3$, then

$$ \sum_{k=1}^{n^2} g(p_k) J(p_k) = 0, $$

where $J(x, y)$ is the Jacobian determinant $J(x, y) = \det \frac{\partial(P, Q)}{\partial(x, y)}$. 

Consider a polynomial vector field \( v = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} \) of degree \( n \geq 2 \). Substituting \( g(x, y) = 1 \) or \( g(x, y) = \text{tr} Dv(x, y) \) into (4), we obtain (EJ1) or (EJ2), respectively.

**Corollary 2.1.** A quadratic vector field \( v \) having four non-degenerate singularities \( p_1, \ldots, p_4 \in \mathbb{C}^2 \) satisfies (EJ1) and (EJ2).

We call these equations the Euler-Jacobi relations on spectra.

**Remark 2.1.** In the end, the relations (EJ1) and (EJ2) come from the residue theorem [GH94, Chpt. 5, Sec. 1]: the local indices are nothing more than the residues of the rational 2-forms \( dx \wedge dy \) \( PQ \), and \( (\text{tr} Dv) dx \wedge dy \) \( PQ \) at the points \( p_k \). The residue theorem then implies the total sum is zero.

**2.2. The Baum-Bott theorem.** The Euler-Jacobi indices are well defined for singularities of vector fields, but not for foliations. One of the most important invariants of an isolated singularity of a planar foliation is the Baum-Bott index. Suppose the germ of a foliation \( (F, p) \) with an isolated singularity is given by a holomorphic 1-form \( \omega \). The Baum-Bott index of \( F \) at \( p \) is defined as
\[
\text{BB}(F, p) = \frac{1}{(2\pi i)^2} \int_{\Gamma} \beta \wedge d\beta,
\]
where \( \Gamma \) is the boundary of a small ball centered at \( p \), and \( \beta \) is a smooth \((1, 0)\)-form that satisfies \( d\omega = \beta \wedge \omega \) in a neighborhood of \( \Gamma \). In the particular case where \( F \) is locally given by a non-degenerate vector field \( v \), the index can be easily computed as
\[
\text{BB}(F, p) = \frac{\text{tr}^2 Dv(p)}{\det Dv(p)}.
\]

The Baum-Bott theorem, originally proved in [BB70] in a more general setting, can be stated in our particular case as follows [Bru15]:

**Theorem 2.2.** Let \( F \) be a foliation of projective degree \( d \) on \( \mathbb{P}^2 \). Then
\[
\sum_{p \in \text{Sing } F} \text{BB}(F, p) = (d + 2)^2.
\]

**Corollary 2.2.** Let \( v \in \mathcal{V}_2 \) have finite spectra \( \{(t_k, d_k)\} \) and characteristic numbers at infinity \( \{\lambda_j\} \). Then
\[
\sum_{k=1}^{4} t_k^2 d_k + \sum_{j=1}^{3} \frac{(\lambda_j + 1)^2}{\lambda_j} = 16.
\]

**2.3. The Camacho-Sad theorem.** The Camacho-Sad theorem concerns singularities of a foliation along an invariant curve. Suppose \( C \) is a smooth curve invariant by a foliation \( F \) (see [Bru15] for the general case). If \( p \) is an isolated singularity of \( F \) on \( C \), we can choose a local holomorphic 1-form \( \omega \) generating \( F \) and a local equation \( f \) for \( C \) to obtain a decomposition
\[
\omega = h df + f \eta,
\]
where \( h \) is a holomorphic function and \( \eta \) is a holomorphic 1-form. In this case, the Camacho-Sad index is defined as follows:
\[
\text{CS}(F, C, p) = -\frac{1}{2\pi i} \int_{\gamma} \frac{\eta}{h},
\]
where \( \gamma \subset C \) is the boundary of a small disk centered at \( p \).

**Theorem 2.3 (CS82).** Let \( F \) be a foliation on a complex surface \( S \) and let \( C \subset S \) be a compact \( F \)-invariant curve. Then

\[
\sum_{p \in C \cap \text{Sing } F} CS(F, C, p) = C \cdot C,
\]

where \( C \cdot C \) denotes the self intersection number of \( C \) in \( S \).

Note that (1) implies that \( CS(F, \mathcal{L}, w_j) = \lambda_j \), so we have the following corollary.

**Corollary 2.3.** The characteristic numbers at infinity of a vector field \( v \in V_2 \) satisfy the relation

\[
\lambda_1 + \lambda_2 + \lambda_3 = 1,
\]

which is clearly equivalent to (CS).

This well-known relation may also be verified directly cf. [Remark 5.1]

### 3. Twin vector fields

In a previous paper of the second author, the question of whether or not a generic quadratic vector field (up to affine equivalence) is completely determined by its spectra (finite or extended) was studied. The answer is that the finite spectra does not determine the vector field completely, there is a finite ambiguity coming from the existence of twin vector fields.

**Definition 3.1.** We will say that two vector fields \( v \) and \( v' \) are twin vector fields, if they are not equal yet they have exactly the same singular locus and, for each point \( p \) in the common singular set, the matrices \( Dv(p) \) and \( Dv'(p) \) have the same spectrum.

**Theorem 3.1 (Ram17).** A generic quadratic vector field has exactly one twin. Moreover, if two vector fields from the class \( V_2 \) have the same finite spectra (no assumption on the position of the singularities) then, after transforming one of them by a suitable affine map, they are either identical or a pair of twin vector fields.

The above theorem implies that given a generic vector field \( v \in V_2 \), there exist exactly two disjoint orbits of the action of \( \text{Aff}(2, \mathbb{C}) \) on \( V_2 \) consisting of vector fields having the same finite spectra: the orbit of \( v \) and the orbit of its twin. Note that this result is consistent with the dimensional count done before: \( V_2 \sslash \text{Aff}(2, \mathbb{C}) \) has dimension 6 and the space of finite spectra, which consists of 8 complex numbers constrained by two Euler-Jacobi relations, has dimension 6 as well.

A similar statement about the Baum-Bott index for foliations of \( \mathbb{C}P^2 \) of projective degree 2 was proved by Lins Neto: in [LN12] it is proved that the generic fiber of the Baum-Bott map consists of exactly 240 orbits of the natural action of \( \text{Aut}(\mathbb{C}P^2) \) on the space of foliations.

**Remark 3.1.** Given a generic vector field \( v \in V_2 \) we can compute its twin \( v' \) by solving a simple system of algebraic equations (cf. [Ram17]). The coefficients defining \( v' \) are expressed as rational functions on the coefficients defining \( v \). Thus, we obtain a rational map (which is an involution) \( \tau: V_2 \dashrightarrow V_2 \). This map is equivariant with respect to the action of the affine group on \( V_2 \), and so descends to the quotient \( V_2 \sslash \text{Aff}(2, \mathbb{C}) \) as a rational involution that we also denote by \( \tau \).
As the next theorem shows, twin vector fields have the same finite spectra but necessarily unequal characteristic numbers at infinity.

**Theorem 3.2 ([Ram17]).** Two generic quadratic vector fields are affine equivalent if and only if their extended spectra of singularities coincide.

### 4. Predicting the form of the hidden relation

#### 4.1. Preliminaries

We seek for an algebraic relation among the 11 numbers in the extended spectra. We shall work only with the following 7 variables: $t_1, t_2, t_3, d_1, d_2, d_3, \Lambda$ (recall that $\Lambda$ is the product of characteristic numbers $\lambda_1 \lambda_2 \lambda_3$), from these we can recover the full extended spectra using the previously known equations (EJ1)–(CS).

A first hope would be to explicitly write $\Lambda$ as a function of $t_1, \ldots, d_3$. However, this would be impossible: if this was the case then the finite spectra would completely determine the extended spectra and by Theorem 3.2 it would completely determine the vector field (up to affine equivalence). This contradicts Theorem 3.1.

In this section we explain how we can achieve the next best thing. Namely, we can implicitly express $\Lambda$ in terms of $t_1, \ldots, d_3$ through a quadratic equation whose two roots correspond to the two values that $\Lambda$ may take (one for each of the two twins that realize the finite spectra $t_1, \ldots, d_3$).

If the coefficients of the above quadratic expression are polynomial, we obtain an equation $H_2 \Lambda^2 + H_1 \Lambda + H_0 = 0$ as in Theorem 1.2.

#### 4.2. Building the hidden relation

There are three maps that we need to build up the hidden relation, the fact that all three maps are rational is the key fact that allow us to recover a polynomial equation in the end. The maps are the following:

1. The rational involution $\tau: \mathcal{V}_2 \sslash \text{Aff}(2, \mathbb{C}) \dashrightarrow \mathcal{V}_2 \sslash \text{Aff}(2, \mathbb{C})$ in Remark 3.1 that assigns to each vector field its unique twin,
2. The rational map $\Lambda: \mathcal{V}_2 \dashrightarrow \mathbb{C}$ that assigns to each vector field its product of characteristic numbers $\Lambda(v) = \lambda_1 \lambda_2 \lambda_3$ (see Sec. 5.2 for a detailed discussion on this map),
3. The birational isomorphism $\psi: \mathbb{C}^6 \dashrightarrow \mathcal{V}_2 \sslash \text{Aff}(2, \mathbb{C}) \sslash \tau$ defined as the birational inverse of the map $[v] \mapsto (t_1, \ldots, d_3)$.

We remark that the map $\mathcal{V}_2 \dashrightarrow \mathbb{C}^6 \sslash \text{Aff}(2, \mathbb{C})$ which maps $v$ to $(t_1, \ldots, d_3)$ is a dominant rational map whose generic fiber is two points corresponding to a pair of twins (cf. [Ram17]), thus once we pass to the quotient by the involution $\tau$ we obtain a birational isomorphism.

We’re now ready to construct the hidden relation. The map $\mathcal{V}_2 \sslash \text{Aff}(2, \mathbb{C}) \dashrightarrow \mathbb{C}$ given by $v \mapsto \Lambda(v) + \Lambda(\tau(v))$ is invariant with respect to $\tau$ and so descends to the quotient $\mathcal{V}_2 \sslash \text{Aff}(2, \mathbb{C}) \sslash \tau$. Precomposing with $\psi$ we obtain a map

$$R_1: \mathbb{C}^6 \dashrightarrow \mathbb{C}$$

$$(t_1, \ldots, d_3) \mapsto \Lambda(v) + \Lambda(\tau(v))$$

In a similar way we construct a map $R_2(t_1, \ldots, d_3) = \Lambda(v) \Lambda(\tau(v))$. It follows now that the spectra of a generic quadratic vector field satisfies the equation

$$\Lambda^2 - R_1(t; d) \Lambda + R_2(t; d) = 0.$$
From this discussion we deduce the existence of a new relation, and we see why it ought to be a quadratic equation on \( \Lambda \). An algorithm to obtain the hidden relation following the above arguments is described in Sec. 5.4.2. However, in order to compute the hidden relation in a more efficient way, we’re going to follow a different strategy (Sec. 5.4.1).

5. Computing the hidden relation

5.1. Plan of the proof. In this section we are going to prove in detail Theorems 1.1–1.3. The heuristic idea behind the proof diverges from Sec. 4 but is straightforward. Consider the map \( \mathcal{M} : \mathbb{C}^2 / \text{Aff}(2, \mathbb{C}) \to \mathbb{C}^7 \) that assigns to each vector field \( v \) the tuple \((t_1, t_2, t_3, d_1, d_2, d_3, \Lambda)\), where \( \Lambda = \lambda_1 \lambda_2 \lambda_3 \). In terms of Sec. 4, \( \mathcal{M} = (\psi^{-1}, \Lambda) \). Our goal is to describe the image of \( \mathcal{M} \). It turns out that \( \mathcal{M} \) is a rational map, and we shall find an explicit formula for this map. Finally, (the closure of) the image of \( \mathcal{M} \) is the projection of the graph of \( \mathcal{M} \) to the codomain, so we can use standard Gröbner basis algorithms to find the vanishing ideal of this projection.

In Sec. 5.2 we will provide a useful way to express \( \Lambda \) in terms of the coefficients of \( P \) and \( Q \). Then in Sec. 5.3 we provide explicit coordinates on \( \mathbb{C}^2 / \text{Aff}(2, \mathbb{C}) \). The explicit formulas for the finite spectra in these coordinates together with the relation from Sec. 5.2 give an explicit formula for \( \mathcal{M} \). Finally, in Sec. 5.4 we will use this formula to establish a new relation between the finite spectra, and the number \( \Lambda \).

In Sec. 5.5 we will prove that the new relation satisfies the assertions of Theorem 1.2 and in Sec. 5.6 we will use it to prove Theorem 1.3.

5.2. The product of the characteristic numbers at infinity. Consider a polynomial vector field \( P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y} \) of degree \( n \). In the coordinates \( z = \frac{1}{x}, \ w = \frac{y}{x} \), it takes the form

\[
\frac{dz}{dw} = z \frac{\tilde{P}(z, w)}{w P(z, w) - Q(z, w)},
\]

where \( \tilde{P}(z, w) = z^n P \left( \frac{1}{z}, \frac{w}{z} \right) \), \( \tilde{Q}(z, w) = z^n Q \left( \frac{1}{z}, \frac{w}{z} \right) \). In the generic case \( w \tilde{P}(0, w) - \tilde{Q}(0, w) \) is a non-zero polynomial of degree \( n + 1 \), therefore

\[
\frac{dz}{dw} = z \frac{F(w)}{G(w)} + O(z^2),
\]

where \( F(w) = \tilde{P}(0, w) \) and \( G(w) = w \tilde{P}(0, w) - \tilde{Q}(0, w) \).

Remark 5.1. Note that the leading coefficients of \( F \) and \( G \) are both equal to the leading coefficient of \( \tilde{P}(0, w) \), hence \( \lim_{w \to \infty} w \frac{F(w)}{G(w)} = 1 \). Comparing this equality and (7) to (1), we get another proof of (CS).

Comparing (7) to (1), we see that the characteristic numbers \( \lambda_k \) are the residues of the rational function \( \frac{F(w)}{G(w)} \) at the zeroes \( w_k \) of its denominator, hence

\[
\lambda_k = \frac{F(w_k)}{G'(w_k)}.
\]

The following lemma allows us to find the product \( \Lambda = \prod_{k=1}^{n+1} \lambda_k \) as a rational function of the coefficients of \( P \) and \( Q \).
Lemma 5.1. Let $\Lambda, F, G$ be as above. Then
\[ \Lambda = \frac{\text{Res}(F, G)}{\text{Res}(G', G)}. \] (9)

Here and below $\text{Res}(...)$ stands for the resultant of its arguments.

Proof. Multiplying (8) for all $k = 1, 2, \ldots, n + 1$ we obtain the formula
\[ \Lambda = \prod_{k=1}^{n+1} F(w_k) \prod_{k=1}^{n+1} G'(w_k). \]

The numerator in the above expression is the product of $F(w)$ at each of the roots of $G$. This product is equal to $n + 1 \prod_{k=1}^{n+1} F(w_k) = \text{LC}(G)^{-n} \text{Res}(F, G)$, where $\text{LC}(G)$ is the leading coefficient of $G$. Similarly, the denominator equals $\text{LC}(G)^{-n} \text{Res}(G', G)$ and so we obtain (9). \□

5.3. Normal form. Note that a quadratic vector field from the class $V_2$ cannot have three collinear singularities. Indeed, otherwise it would vanish on the line passing through these singularities. Hence we have the following lemma.

Lemma 5.2. Every quadratic vector field with four isolated singularities is affine equivalent to a vector field with singularities at $p_1 = (0, 0), p_2 = (1, 0), p_3 = (0, 1)$. Any such vector field $v = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$ is defined by polynomials
\[
\begin{align*}
P(x, y) &= a_0 x^2 + a_1 xy + a_2 y^2 - a_0 x - a_2 y, \\
Q(x, y) &= a_3 x^2 + a_4 xy + a_5 y^2 - a_3 x - a_5 y,
\end{align*}
\] (10)
for some complex numbers $a_0, \ldots, a_5$.

Under this “normal form” we can immediately compute the explicit expressions for the traces $t_k$ and determinants $d_k$ at each singular point $p_k$, for $k = 1, 2, 3$. Also, Lemma 5.1 provides us an expression for $\Lambda = \lambda_1 \lambda_2 \lambda_3$ as a rational function of $a_j$. Explicit expressions for all these values are provided in Lemma A.1 in Appendix A.

5.4. The hidden relation. We have 7 polynomial relations on 13 variables $t_k, d_k, k = 1, 2, 3, a_j, j = 0, \ldots, 5, \Lambda$. The common zero locus of these relations is the graph of the map $\mathcal{M}$ introduced in Sec. 5.1. Our goal is to eliminate $a_j$ from these equations. Geometrically, this corresponds to projecting the graph of $\mathcal{M}$ to the codomain of $\mathcal{M}$, thus finding (the closure of) its image.

There are (at least) two ways to achieve this goal.

5.4.1. Fast computation. Consider the ideal $J$ generated by our relations, and use a computer algebra system to eliminate $a_j$ from this ideal. It turns out that the resulting ideal $I$ is generated by a single polynomial $H \in \mathbb{C}[t; d; \Lambda]$. This polynomial will be the “hidden” relation. We conclude that (the closure of) the image of $\mathcal{M}$ is a hypersurface in $\mathbb{C}^7$. This agrees with the dimensional count: we had 7 equations, then we eliminated 6 variables, so we have $7 - 6 = 1$ equation left.
The polynomial $H$ has degree 2 in $\Lambda$, and has weighted degree 14, cf. Theorem 1.2. We have done this computation in CoCoA 5 [ABL] and Macaulay2 [GS], getting the same polynomial $H$ in both cases. These computations are available at [KR].

5.4.2. Slow computation. Before implementing the fast algorithm described above, we have obtained the hidden relation following a different method which is closer to the ideas presented in Sec. 4.

The first step is to invert the map that sends a vector field $v$ to its finite spectra $(t; d)$. Consider the explicit formulas for $t_k, d_k$, see Appendix A, as equations on $a_j$. The formulas for $t_k$ are linear in $a_j$, and the formulas for $d_k$ have degree 2. As in [Ram17], Theorem 2.1 implies that the fourth zero of a vector field given by (10) is the point $(-\frac{d_4}{d_2}, -\frac{d_4}{d_3})$. This fact adds two more linear equations to our system, and we need to solve a quadratic equation on the line in $\mathbb{C}^6$ given by 5 linear equations. Let $D \in \mathbb{C}[t; d]$ be the discriminant of this quadratic equation. Then $a_j$ are rational functions of $t_k, d_k, \sqrt{D}$. Then we substitute these expressions into (9), and get a formula for $\Lambda$ as a rational function of $t_k, d_k, \sqrt{D}$. This formula can be easily transformed into a polynomial equation $H = 0$ quadratic in $\Lambda$ (cf. Sec. 4.2). We have implemented this approach in GiNaC [Joh], and the result agrees with the results of the “fast” method (cf. [KR]).

5.5. Proof of Theorem 1.2. Let us prove that the polynomial $H$ constructed above satisfies all assertions of Theorem 1.2.

The facts that $H$ has weighted degree 14 and is irreducible were verified both in CoCoA 5 and Macaulay2. By construction, the spectra of a quadratic vector field in $V_2$ satisfies $H$, so assertion 1 of Theorem 1.1 is clear.

Let us prove that $H$ satisfies assertion 2.

Proposition 5.1. The identity $H = 0$ is independent from the identities (EJ1) – (CS) coming from the classical index theorems.

Proof. Denote by $\sigma_j(\lambda)$ the $j$-th elementary symmetric polynomial on $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. Then (CS) takes the form $\sigma_1(\lambda) = 1$, and (BB) takes the form

$$\frac{\sigma_2(\lambda)}{\sigma_3(\lambda)} = -\sum_{k=1}^{4} \frac{t_k^2}{d_k} + 9,$$

(11) cf (5). Finally, $\sigma_3(\lambda)$ is the $\Lambda$ from the hidden relation (2).

Note that we can consider both classical relations, and (2) as relations on $t_k, d_k, k = 1, \ldots, 4, \text{ and } \sigma_j, j = 1, 2, 3$. After the birational coordinate change

$$(t; d; \sigma_1, \sigma_2, \sigma_3) \mapsto (t; d; \sigma_1, \frac{\sigma_2}{\sigma_3}, \sigma_3),$$

(12) the hidden relation (2) is the only relation that includes $\sigma_3 = \Lambda$. Therefore, it is independent from the classical identities. \qed

Finally, let us show that $H$ satisfies assertion 3. Consider a polynomial $F \in \mathbb{C}[t; d; \lambda]$ vanishing at the extended spectrum of every quadratic vector field in $V_2$. It is easy to see that for each vector field $v \in V_2$ with enumerated singularities, there exists a path in $V_2$ joining $v$ to any other element of $V_2$ that corresponds to the same vector field but with another prescribed order of singularities at infinity (and the same order for finite zeros). Hence, without loss of generality, we may assume that
F is symmetric in \( \lambda \), and so it can be rewritten as \( F(t; d; \lambda) = \tilde{F}(t; d; \sigma(\lambda)) \). Using the classical relations, we can get rid of \( t_4, d_4, \sigma_1, \sigma_2 \) to obtain a rational expression on \( t_1, \ldots, d_3, \Lambda \). The inequalities \( d_k \neq 0, \lambda_j \neq 0 \) allow us to lift denominators and recover a polynomial expression.

Therefore, in order to check assertion 3, it is enough to consider polynomials \( F \in \mathbb{C}[t_1, \ldots, d_3, \Lambda] \). If such polynomial \( F \) vanishes on all extended spectra of vector fields \( v \in \mathcal{V}_2 \), then it belongs to the vanishing ideal \( I \) of (the closure of) the image of \( \mathcal{M} \). Due to Sec. 5.4.1, \( I \) is the principal ideal generated by \( H \). Since \( H \) is irreducible, it follows that \( F \) is either identically zero or a multiple of \( H \). Hence, \( F = 0 \) follows from \( H = 0 \) and thus \( H \) satisfies assertion 3.

Finally, \( H \) satisfies all the assertions of Theorem 1.1.

5.6. Reducing the degree of \( H \). Consider the polynomial \( H \) as a function of the spectra of all four finite zeros. It turns out that this function is not symmetric under permutations of the zeros. More precisely, if we substitute into \( H \) formulas for \( t_4, d_4 \) instead of, e.g., \( t_3, d_3 \), the resulting function will be proportional to \( H \), but not equal to it.

Among components of \( H \), the “free term” \( H_0 \) has the simplest formula. Looking at this formula, we guessed that the function \( (d_1d_2d_3)^{-2}d_4^4H \) is diagonal symmetric, i.e., it survives under permutations of the finite zeros of \( v \in \mathcal{V}_2 \). Moreover, this function is equal to a polynomial \( \tilde{H} \in \mathbb{C}[t_1, \ldots, t_4; d_1, \ldots, d_4] \) on the submanifold given by \( \{EJ_1\} \) and \( \{EJ_2\} \).

Formally, one can check that \( d_4^4H \) belongs to the ideal generated by the numerators of \( \{EJ_1\}, \{EJ_2\} \), and \( (d_1d_2d_3)^2 \), hence \( d_4^4H - (d_1d_2d_3)^2\tilde{H} \) belongs to the ideal \( EJ \) generated by the numerators of \( \{EJ_1\}, \{EJ_2\} \) for some polynomial \( \tilde{H} \), and we can find an explicit formula for \( \tilde{H} \). It is easy to see that this polynomial \( \tilde{H} \) satisfies the assertions of Theorem 1.3.

The polynomial \( \tilde{H} \) is not uniquely defined. Indeed, we can replace it by any other polynomial \( \tilde{H}' \) such that \( (d_1d_2d_3)^2(\tilde{H} - \tilde{H}') \) belongs to the ideal \( EJ \). All the polynomials \( \tilde{H} \) we have found have more than 1,500 terms, but some of them are diagonal symmetrizations of relatively short polynomials. The explicit formula for one of these polynomials is given in Appendix A.

6. Index theory

In this section we prove Theorem 1.4. In Sec. 6.1, we explain how this theorem relates to the index theory. Next, in Sec. 6.2, we prove this theorem modulo the main lemma whose proof is postponed till Sec. 6.3.

6.1. Motivation for Theorem 1.4. The study of indices and index theorems is fundamental in the development of geometry and topology. Theorems like the Poincaré-Hopf index theorem, the Gauss-Bonnet theorem or the Lefschetz fixed-point theorem are a few important examples. These local-to-global theorems, which relate the local behavior of some geometric object around “special” points to some global invariant (usually measured in some cohomology space) are indeed powerful and fascinating.

The theorems we previously knew that relate the extended spectra of a polynomial vector field are of all this type, where a sum of local contributions (an index) taken over all singular points equals a fixed number that depends only on the degree of the vector field. The indices in question have been defined as residues of
meromorphic forms, and can also be understood as localizations of characteristic classes (see for example [Suw98]).

It was proved in [Ram16] that all four classical relations can be realized as particular cases of the so-called Woods Hole trace formula, also known as the Atiyah-Bott fixed point theorem (a generalization of the Lefschetz fixed point theorem due to Atiyah and Bott in the complex analytic case, and to Verdier in the algebraic case), and one could be tempted to think that the hidden relation would also be of this type. This is not the case. As it will be shown in this section, there are no more index theorems that relate the extended spectra other than those which can be derived from the classical ones. In particular, the hidden relation does not come from an index theorem.

**Remark 6.1.** For quadratic vector fields we had only one hidden relation. However, as the degree $n$ of the vector fields grows, the number of hidden relations grows asymptotically as $n^2$. Indeed, on one hand a generic polynomial vector field of degree $n$ has exactly $n^2$ zeroes and $n + 1$ singular points at infinity. Thus, the extended spectra consists of $2n^2 + n + 1$ complex numbers. On the other hand, the space of degree $n$ polynomial vector fields is of dimension $n^2 + 3n + 2$. It is hard to imagine that there would be a countable number of index theorems to account for all these hidden relations. Therefore, in this way, it is not surprising that the hidden relations do not come from index theorems.

Below we formalize the statement that the hidden relation for quadratic fields does not come from an index theorem.

An index ought to be a number we assign to an isolated singularity of a vector field, foliation, space or map. This index should only depend on the local behavior of our geometric object around such singular point. Moreover, this index should be invariant under analytic equivalence. We know from Poincaré that a complex hyperbolic singularity of a planar vector field is analytically linearizable. Analytic invariance of the index implies that the index cannot depend on anything but the spectra.

**Remark 6.2.** An index theorem for generic polynomial vector fields of degree $n$ on $\mathbb{C}^2$ should be such that the local-to-global equation is of the following form:

$$\sum_{k=1}^{n^2} \text{ind}_{\mathbb{C}^2}(v, p_k) + \sum_{j=1}^{n+1} \text{ind}_L(F_v, w_j) = L(n),$$

where $\text{ind}_{\mathbb{C}^2}(v, p_k)$ is a rational function on the spectrum of the linearization matrix $Dv(p_k)$, $\text{ind}_L(F_v, w_j)$ is a rational function on the characteristic number $\lambda_k$ of the singularity at infinity $w_k$, and $L(n)$ is a number that depends only on the degree $n$ of the vector field.

Note that in particular the above equation may be rewritten as

$$R(t; d) = r(\lambda),$$

where $R$ is a rational function on the finite spectra $\{(t_k, d_k)\}$, and $r$ is a rational function on the characteristic numbers at infinity $\{\lambda_j\}$, and they are invariant under permutations of finite or infinite singularities. Moreover, all classical relations [EJ1]–[CS] are of this form. Recall that Theorem 1.4, which we will prove below, states that any relation of the form [13] follows from the classical relations.
6.2. Lack of new index theorems. In order to prove [Theorem 1.4] we will first use the classical relations to eliminate some variables. Since \( r \) is symmetric, we can rewrite it in terms of \( \sigma_1(\lambda), \sigma_3(\lambda) \) and \( \Lambda = \sigma_3(\lambda) \), cf. [12], then substitute \( \sigma_1(\lambda) = 1 \), see (CS):

\[
r(\lambda) = \tilde{r} \left( \frac{\sigma_2(\lambda)}{\sigma_3(\lambda)}, \sigma_3(\lambda) \right).
\]

Note that for twin vector fields \( v, v' \) with characteristic numbers \( \lambda, \lambda' \) we have \( r(\lambda) = r(\lambda') \), hence

\[
\tilde{r} \left( \frac{\sigma_2(\lambda)}{\sigma_3(\lambda)}, \sigma_3(\lambda) \right) = \tilde{r} \left( \frac{\sigma_2(\lambda')}{\sigma_3(\lambda')}, \sigma_3(\lambda') \right).
\]  

(14)

Twin vector fields have the same finite spectra, hence (11) implies that \( \frac{\sigma_2(\lambda)}{\sigma_3(\lambda)} = \frac{\sigma_2(\lambda')}{\sigma_3(\lambda')} \). Therefore, (14) holds whenever \( \tilde{r} \) does not depend on its second argument. The following lemma states that this is the only possibility.

**Lemma 6.1.** Let \( \tilde{r} \) be a rational function such that for twin vector fields \( v, v' \) with characteristic numbers at infinity \( \lambda, \lambda' \) we have (11). Then \( \tilde{r} \) depends only on its first argument, \( \tilde{r}(\xi, \chi) = \tilde{r}(\xi) \).

We shall prove this lemma in Sec. 6.3. Now we apply it to \( \tilde{r} \), and rewrite (13) in the form

\[
R(t; d) = \tilde{r} \left( \frac{\sigma_2(\lambda)}{\sigma_3(\lambda)} \right).
\]

Substituting (11), we get

\[
R(t; d) = \tilde{r} \left( -\sum_{k=1}^{4} \frac{t_k^2}{d_k} + 9 \right).
\]

Note that this relation does not involve \( \lambda_j \). Finally, we use (11.1) and (11.2) to get rid of \( t_4 \) and \( d_4 \), and get a relation of the form \( \tilde{R}(t_1, t_2, \ldots, t_3, d_3) = 0 \). Now we have applied all the relations (EJ1)–(CS), so we need to show that \( \tilde{R} \equiv 0 \).

It was proved in [Ram17 Theorem 1] that the map that takes each vector field in the normal form described in Lemma 6.2 to the set \( (t_1, d_1, \ldots, t_3, d_3) \in \mathbb{C}^6 \) is a dominant map (i.e. the image is dense in the Zariski topology). This implies that there are no non-trivial polynomials that vanish on every tuple \( (t_1, d_1, \ldots, t_3, d_3) \). In particular, \( \tilde{R} = 0 \) for every generic vector field \( v \) implies that \( \tilde{R} \equiv 0 \).

This completes the proof of [Theorem 1.4] modulo the main Lemma 6.1.

6.3. Proof of the main Lemma 6.1. In order to prove [Lemma 6.1] for each pair of twin vector fields \( v, v' \), consider the triple \( \left( \frac{\sigma_2(\lambda)}{\sigma_3(\lambda)}, \sigma_3(\lambda), \sigma_3(\lambda') \right) \). Denote by \( X \subset \mathbb{C}^3 \) the set of these triples for all pairs of twin vector fields. Then for \( (\xi, \Lambda, \Lambda') \in X \) we have \( \tilde{r}(\xi, \Lambda) = \tilde{r}(\xi, \Lambda') \). Now, it is enough to show that \( X \) has an inner point in \( \mathbb{C}^3 \). Indeed, this would imply that \( X \) is Zariski dense in \( \mathbb{C}^3 \), thus \( \tilde{r}(\xi, \Lambda) = \tilde{r}(\xi, \Lambda') \) for all \( (\xi, \Lambda, \Lambda') \in \mathbb{C}^3 \), hence \( \tilde{r} \) does not depend on its second argument.
Note that $\Lambda = \sigma_3(\lambda)$ and $\Lambda' = \sigma_3(\lambda')$ are the roots of (2). Hence, Vieta’s formulas imply

\[
\frac{H_1}{H_2} = -\Lambda - \Lambda' \\
\frac{H_0}{H_2} = \Lambda \Lambda'.
\]

Consider the rational map $\Phi: \mathbb{C}^8 \to \mathbb{C}^3$ given by

\[
\Phi(t; d) = \left( -\sum_{k=1}^{4} \frac{t_k^2}{d_k} + 9, -\frac{H_1}{H_2}, -\frac{H_0}{H_2} \right) = \left( \frac{\sigma_2(\lambda)}{\sigma_3(\lambda)}, \lambda + \lambda', \lambda \lambda' \right).
\]

Recall that we know explicit formula for $H$, hence we have an explicit formula for $\Phi$. Let $\tilde{\Phi}$ be the restriction of $\Phi$ to the affine submanifold given by (EJ1), (EJ2). We computed the rank of $\tilde{\Phi}$ at the point $(1, 1, 2; 1, 2, -1)$ in CoCoA. This rank equals 3, hence $\tilde{\Phi}(1, 1, 2; 1, 2, -1)$ is an inner point of the image of $\tilde{\Phi}$.

Since (EJ1), (EJ2) are the only relations on the finite spectra, the set

\[
\tilde{X} = \{ (\xi, \lambda + \lambda', \lambda \lambda') | (\xi, \lambda, \lambda') \in X \} \subset \text{im } \tilde{\Phi}
\]

has an inner point as well. Therefore, $X$ has an inner point. This completes the proof of Lemma 6.1, hence the proof of Theorem 1.4.

**Acknowledgements**

An early version of this work appeared as part of the doctoral dissertation of the second author, who would like to thank Yulij Ilyashenko, John Hubbard, and John Guckenheimer for their guidance throughout his Ph.D. We are also grateful to Dominique Cerveau, Étienne Ghys, Adolfo Guillot, and Frank Loray for fruitful conversations on this project. We would also like to thank Nataliya Goncharuk for proofreading the preliminary versions of the article.

**Appendix A. Explicit formulas**

In this appendix we include the explicit expressions for the spectra and the product of characteristic numbers $\Lambda$ announced in Sec. 5.3.

As in Lemma 5.2 put $p_1 = (0, 0), p_2 = (1, 0), p_3 = (0, 1)$. Consider a quadratic vector field $v$ having isolated singularities at $p_k, k = 1, 2, 3$. This vector field is given by (10).

**Lemma A.1.** The spectra $(t_k, d_k)$ of $v$ at $p_k, k = 1, 2, 3$, is given by

\[
t_1 = -a_0 - a_5; \quad d_1 = -a_2a_3 + a_0a_5;
\]
\[
t_2 = a_0 + a_4 - a_5; \quad d_2 = -a_1a_3 + a_2a_3 + a_0a_4 - a_0a_5;
\]
\[
t_3 = -a_0 + a_1 + a_5; \quad d_3 = a_2a_3 - a_2a_4 - a_0a_5 + a_1a_5.
\]
The numerator and the denominator in (9) are given by

\[ a_2^{-1} \text{Res}(F, G) = -(d_1 d_2 + d_2 d_3 + d_1 d_3) = \frac{d_1 d_2 d_3}{d_4}, \]

\[ a_2^{-1} \text{Res}(G', G) = 4a_3^2 a_2 - a_2^2 a_1^2 + 2a_2^2 a_1 a_5 - 12a_2^2 a_2 a_4 - a_0^2 a_1^2 + 2a_0 a_1^2 a_4 + \]

\[ 18a_0 a_1 a_2 a_3 - 4a_0 a_1 a_2 a_5 - 18a_0 a_2 a_3 a_5 + 12a_0 a_2 a_4^2 + 2a_0 a_4 a_5^2 - \]

\[ 4a_3^3 a_3 - 12a_2^2 a_3 a_5 - a_1^2 a_4^2 - 18a_1 a_2 a_3 a_4 - 12a_1 a_3 a_5^2 + 2a_1 a_2^2 a_5 + \]

\[ 27a_2^2 a_3^2 + 18a_2 a_3 a_4 a_5 - 4a_2 a_4^3 + 4a_3 a_5^3 - a_1^2 a_5^2. \]

Their ratio is equal to \( \Lambda = \lambda_1 \lambda_2 \lambda_3 \).

The second equality in the expression for \( \text{Res}(F, G) \) follows from (E.11), and the last conclusion follows from [Lemma 5.1]. The rest of the lemma can be proved by a direct computation. Though the formula for \( \text{Res}(F, G) \) is much shorter than the formula for \( \text{Res}(G', G) \), we have no geometric interpretation for neither of these formulas.

Finally, we provide explicit formulas for the polynomials \( \tilde{H}_j \) from [Theorem 1.3]. In order to save space, we write the formula for polynomials \( \tilde{H}_j \) such that \( \tilde{H}_j \) are their symmetrizations with respect to the diagonal action of \( S_4 \),

\[ \tilde{H}_j = \sum_{\sigma \in S_4} \tilde{H}_j(t_{\sigma(1)}, t_{\sigma(2)}, t_{\sigma(3)}, t_{\sigma(4)}, d_{\sigma(1)}, d_{\sigma(2)}, d_{\sigma(3)}, d_{\sigma(4)}). \]

Here are the explicit formulas.

\[ \tilde{H}_0 = -4d_1^3 d_2 d_3 + 4d_1^3 d_2^2 d_3 \]

\[ \tilde{H}_1 = 2d_1^2 d_3^3 t_2 + 2d_2^3 d_4^3 t_2 - 6d_1 d_3^3 t_1 t_2 - 2d_3^3 d_4^2 t_1^2 - 2d_2 d_4^2 t_1 t_2 + 6d_1 d_3^2 t_2 t_3 + 36d_2^2 d_3 d_4 t_1^3 + 12d_1 d_2 d_3 d_4^2 t_1 + 36d_2 d_3 d_4^2 t_1^2 + 24d_2^3 d_4^2 t_1^2 t_2 + 24d_1 d_2 d_3 d_4^2 t_1 t_2 - 24d_1 d_2 d_3^2 t_1 t_2 - 36d_2^3 d_4 t_1 t_2 t_3 + 216d_1 d_2 d_3 d_4 t_1 t_2 - 216d_1 d_2^2 d_3^2 d_4 \]

\[ \tilde{H}_2 = 4d_2^2 d_1^3 t_2^2 + 8d_2^3 d_4^3 t_2^2 - 4d_1^3 d_3^2 t_1^2 t_2 - 8d_2 d_1^3 d_3^2 t_1^2 t_2 - 16d_1^3 d_3^2 t_1^2 t_2 + 8d_3 d_4 t_1^3 t_2^2 + 8d_1 d_3^3 t_1^3 t_2 + 16d_2^3 d_1^3 t_1^3 t_2 - 8d_3 d_4 t_1^3 t_2^2 - 2d_2 d_3 d_4 t_1^3 t_2 t_3 + 2d_1 d_2 d_3^2 t_1^3 t_2 t_3 + 2d_1 d_2 d_3^2 t_1^3 t_2 t_3 + 30d_2 d_3 d_4 t_1^3 t_2 t_3 - 8d_1 d_4 t_1^3 t_2^3 - 2d_2 d_4 t_1^3 t_2^2 t_3 + 2d_1 d_3^3 t_1^3 t_2 t_3 + 2d_2 d_3^2 t_1^3 t_2 t_3 + 30d_2 d_3 d_4 t_1^3 t_2 t_3 - 8d_1 d_4 t_1^3 t_2^3 - 2d_2 d_4 t_1^3 t_2^2 t_3 + 30d_1 d_4 t_1^3 t_2^2 t_3 + 30d_1 d_4 t_1^3 t_2^2 t_3 + 54d_2 d_1^3 d_3^2 t_2 - 2d_2 d_3 t_2 t_3 t_4 + 15d_2 d_3 d_4 t_1^3 t_2 t_3 + 15d_2 d_3 d_4 t_1^3 t_2 t_3 - 81d_1^2 d_3^2 t_1^2 t_2 + 135d_2 d_3 d_4 t_1^3 t_2 t_3 + 54d_2^2 d_3^2 t_1^2 t_3 - 40d_1^2 d_3^2 t_2 t_3 t_4 + 162d_2 d_3 d_4 t_1^3 t_2 t_3 + 162d_2 d_3 d_4 t_1^3 t_2 t_3 + 16d_1^2 d_3^2 t_1^2 t_3 t_4 - 972d_1^2 d_3^2 t_2 t_3 t_4 - 324d_1 d_3^2 t_2 t_3 t_4 - 972d_1^2 d_3^2 t_2 t_3 t_4 - 648d_2^2 d_3 d_4 t_1^3 t_2 t_3 + 1296d_2^2 d_3 d_4 t_1^3 t_2 t_3 - 648d_1 d_3 d_4 t_1^3 t_2 t_3 + 648d_1 d_3 d_4 t_1^3 t_2 t_3 + 972d_2^2 d_3 d_4 t_1^3 t_2 t_3 - 2916d_1 d_3 d_4 t_1^3 t_2 t_3 + 2916d_1 d_3 d_4 t_1^3 t_2 t_3.
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