Translational and great Darboux cyclides

Niels Lubbes

June 8, 2023

Abstract

A surface that is the pointwise sum of circles in Euclidean space is either coplanar or contains no more than 2 circles through a general point. A surface that is the pointwise product of circles in the unit-quaternions contains either 2, 3, 4, or 5 circles through a general point. A surface in a unit-sphere of any dimension that contains 2 great circles through a general point contains either 4, 5, 6, or infinitely many circles through a general point. These are some corollaries from our classification of translational and great Darboux cyclides. We use the combinatorics associated to the set of low degree curves on such surfaces modulo numerical equivalence.

Keywords: real surfaces, pencils of circles, singular locus, Darboux cyclides, Clifford torus, Möbius geometry, elliptic geometry, hyperbolic geometry, Euclidean geometry, Euclidean translations, Clifford translations, unit quaternions, weak del Pezzo surfaces, divisor classes, Néron-Severi lattice

MSC2010: 51B10, 51M15, 14J17, 14C20

1 Introduction

In this article, we characterize surfaces in $\mathbb{R}^3$ that contain at least two circles through each point. Such surfaces are algebraic by [22, Theorem 2] and thus with surface we shall mean a real irreducible algebraic surface (see §2).

Surfaces that are a union of circles in two different ways have applications in architecture [20], kinematics [13, 17] and geometric modeling in general [1, 11, 19]. In particular, the “Darboux cyclides” have a long history [5, 12], and its various properties are still a topic of recent research [2, 8, 18, 21, 28, 30]. In order to clarify our main result and its relation to [26], we recall some definitions for the non-expert.
An inversion with respect to a sphere \( O \subset \mathbb{R}^3 \) with center \( c \) and radius \( r \) is the map \( f: \mathbb{R}^3 \setminus \{c\} \rightarrow \mathbb{R}^3 \setminus \{c\} \) such that \( ||x - c|| \cdot ||f(x) - c|| = r^2 \) and the vectors \( x - c \) and \( f(x) - c \) are codirected for all \( x \in \mathbb{R}^3 \setminus \{c\} \). Such a map exchanges the interior and exterior of \( O \) and takes generalized circles to generalized circles, where a generalized circle is either a circle or a line. We call two surfaces in \( \mathbb{R}^3 \) Möbius equivalent if one surface is mapped to the other by a composition of inversions.

Let \( \mu: S^3 \rightarrow \mathbb{R}^3 \) with \( \mu(y) := (y_1, y_2, y_3)/(1-y_4) \) denote the stereographic projection from the point \( (0,0,0,1) \) on the 3-dimensional unit-sphere \( S^3 \subset \mathbb{R}^4 \). The Möbius degree of a surface \( Z \subset \mathbb{R}^3 \) is defined as \( \deg \mu^{-1}(Z) \). A surface \( Z \subset \mathbb{R}^3 \) is called \( \lambda \)-circled if the Zariski closure of \( \mu^{-1}(Z) \) contains at least \( \lambda \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \) circles through a general point. If \( \lambda \in \mathbb{Z}_{\geq 0} \), then we assume that \( Z \) is not \( (\lambda+1) \)-circled. If \( \lambda \geq 2 \), then we call \( Z \) celestial. The Möbius degree and \( \lambda \) are both Möbius invariants.

We may identify the unit-sphere \( S^3 \subset \mathbb{R}^4 \) with the unit quaternions and we denote the Hamiltonian product by \( \star \). We consider the following constructions where \( A \) and \( B \) are curves in \( \mathbb{R}^3 \) or \( S^3 \):

\[
A + B := \{ a + b \in \mathbb{R}^3 \mid a \in A \text{ and } b \in B \},
\]

\[
A \star B := \{ a \star b \in S^3 \mid a \in A \text{ and } b \in B \}.
\]

Suppose that \( Z \subset \mathbb{R}^3 \) is a surface. We call \( Z \) Bohemian or Cliffordian if there exist generalized circles \( A \) and \( B \) such that \( Z \) is the Zariski closure of \( A + B \) and \( \mu(A \star B) \), respectively. A surface that is either Bohemian or Cliffordian is called translational. If \( A \) and \( B \) are great circles such that \( A \star B \subset S^3 \) is a surface, then \( A \star B \) is called a Clifford torus.

A Darboux cyclide in \( \mathbb{R}^3 \) is a surface of Möbius degree four. A Q cyclide is a Darboux cyclide that is Möbius equivalent to a quadric \( Q \). For example, a CH1 cyclide is Möbius equivalent to a Circular Hyperboloid of 1 sheet (see Figure 1), where we

![Figure 1: Examples of Darboux cyclides.](image-url)
used the following abbreviations:

- E = elliptic/ellipsoid
- P = parabolic/paraboloid
- O = cone
- C = circular
- H = hyperbolic/hyperboloid
- Y = cylinder

A CO cyclide and CY cyclide is also known as a spindle cyclide and horn cyclide, respectively. A ring cyclide, Perseus cyclide or Blum cyclide is a Darboux cyclide without real singularities that is 4-circled, 5-circled and 6-circled, respectively (see Figure 1). See Table 5 for a complete list of names for celestial Darboux cyclides.

It follows from [26, Main Theorem 1.1] that a celestial surface in \( \mathbb{R}^3 \) is either a Darboux cyclide or Möbius equivalent to a Bohemian or Cliffordian surface. The following question arises: what are the Bohemian and Cliffordian Darboux cyclides? We shall provide necessary conditions using the combinatorics of divisor classes of curves on such surfaces. This method also allows us restrict the possible candidates for the “great” celestial Darboux cyclides; we call a surface \( Z \subset \mathbb{R}^3 \) great if its inverse stereographic projection \( \mu^{-1}(Z) \) is covered by great circular arcs.

We will use Theorems A and B and therefore build on [15].

**Theorem 1.** Suppose that \( Z \subset \mathbb{R}^3 \) is a \( \lambda \)-circled surface of Möbius degree \( d \) such that \( \lambda \geq 2 \) and \( (d, \lambda) \neq (8, 2) \).

(a) The surface \( Z \) is Bohemian if and only if \( Z \) is either a plane, CY or EY.

(b) If \( Z \) is Cliffordian, then \( Z \) is either a Perseus cyclide, ring cyclide or CH1 cyclide. Conversely, if \( Z \) is a ring cyclide, then \( Z \) is Möbius equivalent to a Cliffordian surface.

(c) If \( Z \) is great, then \( Z \) is either a plane, sphere, Blum cyclide, Perseus cyclide, ring cyclide, EO cyclide or CO cyclide. Conversely, if \( Z \) is either a plane, sphere, ring cyclide, EO cyclide or CO cyclide, then \( Z \) is Möbius equivalent to a great celestial surface.

We summarized Theorem 1 in Table 1. See Examples 25, 35 and 36 for an example for each row and each possible type. In particular, we consider for \( 0 \leq i, j \leq 8 \) the surface \( Z_{ij} \subset \mathbb{R}^3 \), which is defined as the Zariski closure of a stereographic projection of the surface

\[ \{C_i(\alpha) \ast C_j(\beta) \mid 0 \leq \alpha, \beta < 2\pi\} \subset S^3, \]
where the circle parametrizations $C_i(t)$ are defined in Table 2. We show in Example 25 that $Z_{01}$, $Z_{23}$ and $Z_{45}$ are a ring cyclide, Perseus cyclide and CH1 cyclide, respectively. The Cliffordian surfaces $Z_{06}$ and $Z_{78}$ are of degree 8 and illustrated in Figure 2.

**Table 1**: Overview of $\lambda$-circled surfaces in $\mathbb{R}^3$ of Möbius degree $d$ that are either Bohemian, Cliffordian, or great and celestial.

| name            | $d$ | $\lambda$ | possible types       |
|-----------------|-----|-----------|----------------------|
| plane/sphere    | 2   | $\infty$ | Bohemian, great      |
| Blum cyclide    | 4   | 6         | great                |
| Perseus cyclide | 4   | 5         | Cliffordian, great   |
| ring cyclide    | 4   | 4         | Cliffordian, great   |
| CH1 cyclide     | 4   | 3         | Cliffordian          |
| EY              | 4   | 3         | Bohemian             |
| CY              | 4   | 2         | Bohemian             |
| EO cyclide      | 4   | 3         | great                |
| CO cyclide      | 4   | 2         | great                |
| $8$             | 2   |           | Bohemian, Cliffordian, great |

**Table 2**: Parametrizations of circles in $S^3$ with $0 \leq t < 2\pi$. Only $C_0(t)$ and $C_1(t)$ define great circles.

\[
C_0(t) := (\cos(t), \sin(t), 0, 0),
C_1(t) := \frac{1}{7}(5 \cos(t), 4 \sin(t), 3 \sin(t), 0),
C_2(t) := \frac{1}{7}(\cos(t), \sin(t), -2, 2),
C_3(t) := \frac{1}{7}(2 \cos(t), 2 \sin(t), 2, 1),
C_4(t) := \frac{1}{3-2 \cos(t)}(-2 + 2 \cos(t), 2 \sin(t), 0, 1 - 2 \cos(t)),
C_5(t) := \frac{1}{3+2 \cos(t)}(2 + 2 \cos(t), 2 \sin(t), 0, 1 + 2 \cos(t)),
C_6(t) := \frac{1}{17+12 \cos(t)}(12 + 8 \cos(t), 8 \sin(t), 0, 9 + 12 \cos(t)),
C_7(t) := \frac{1}{3-2 \cos(t)}(-2 + 2 \cos(t), -2 \sin(t), 0, 1 - 2 \cos(t)),
C_8(t) := \frac{1}{3+2 \cos(t)}(2 + 2 \cos(t), 2 \sin(t), 0, 1 + 2 \cos(t)).
\]
From Theorem 1 and its proof we recover the following four corollaries.

**Corollary 1.** A Darboux cyclide is not Möbius equivalent to both a Bohemian surface and a Cliffordian surface.

**Corollary 2.** If \( A, B \subset \mathbb{R}^3 \) are circles such that \( A + B \) is a non-planar \( \lambda \)-circled surface of Möbius degree \( d \), then \((\lambda, d) = (2, 8)\).

**Corollary 3.** If \( Z \subset S^n \) with \( n \geq 3 \) is a surface that contains two great circles through a general point and is not contained in a hyperplane section, then \( n = 3 \) and its stereographic projection \( \mu(Z) \) is either a Blum cyclide, Perseus cyclide, or ring cyclide.

See Example 36 for a Blum cyclide, Perseus cyclide and ring cyclide that contain two great circles through each point. See Figures 1 and 9 for renderings of these cyclides.

**Corollary 4.** If \( Z \subset \mathbb{R}^3 \) is a great ring cyclide, then \( Z = \mu(A \ast B) \) for some great circles \( A, B \subset S^3 \). Great Perseus cyclides are not Cliffordian.

Notice that a ring cyclide is by Corollary 4 Möbius equivalent to the stereographic projection of a Clifford torus.

We conjecture the converses of Theorems 1(b) and 1(c):

**Conjecture 1.** If \( Z \subset \mathbb{R}^3 \) is either a Perseus cyclide or CH1 cyclide, then \( Z \) is Möbius equivalent to a Cliffordian surface.

**Conjecture 2.** If \( Z \subset \mathbb{R}^3 \) is either a Blum cyclide or Perseus cyclide, then \( Z \) is Möbius equivalent to a great surface.
Overview

In §2, we propose a projective model for Möbius geometry which is a compactification of \( \mathbb{R}^3 \). In §3, we show that the intersection of Bohemian and Cliffordian surfaces with the boundary of this compactification consist of complex lines and/or base points of pencils of circles. In §4, we state a classification of possible incidences between complex lines and base points in Darboux cyclides. We use these results in §5, §6 and §7 and obtain a list of all possible candidates for Cliffordian, Bohemian and great Darboux cyclides, respectively. Moreover, we show that each candidate is realized by some example. In §8, we conclude the proof for Theorem 1.

1 Introduction
2 A projective model for Möbius geometry
3 Associated pencils and absolutes
4 Divisor classes of curves on Darboux cyclides
5 Cliffordian Darboux cyclides
6 Bohemian Darboux cyclides
7 Great Darboux cyclides
8 Combining the results
9 Acknowledgements
References

2 A projective model for Möbius geometry

We define a real variety \( X \) to be a complex variety together with an antiholomorphic involution \( \sigma: X \rightarrow X \) called the real structure and we denote its real points by

\[ X_R := \{ p \in X \mid \sigma(p) = p \}. \]

Such varieties can always be defined by polynomials with real coefficients (see [25, Section I.1] and [23, Section 6.1]). In what follows, points, curves, surfaces and projective spaces \( \mathbb{P}^n \) are real algebraic varieties and maps between such varieties are compatible with their real structures unless explicitly stated otherwise. In particular, curves and surfaces in this article are by default reduced and irreducible. We assume that the real structure \( \sigma: \mathbb{P}^n \rightarrow \mathbb{P}^n \) sends \( x \) to \( (x_0 : \ldots : x_n) \).

Let \( f: X \rightarrow Y \subset \mathbb{P}^n \) be a rational map that is not defined at \( U \subset X \). By abuse of notation we denote \( f(X \setminus U) \subset Y \) by \( f(X) \). We call \( f \) a morphism if it is everywhere defined and thus \( U = \emptyset \).
We consider the following hyperquadric $S^3 \subset \mathbb{P}^4$ and three different hyperplane sections $U, E, Y \subset S^3$:

- **Möbius quadric:** $S^3 := \{ x \in \mathbb{P}^4 \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \}$,
- **Euclidean absolute:** $U := \{ x \in S^3 \mid x_0 - x_4 = 0 \}$,
- **elliptic absolute:** $E := \{ x \in S^3 \mid x_0 = 0 \}$, and
- **hyperbolic absolute:** $Y := \{ x \in S^3 \mid x_4 = 0 \}$.

The following operators $P, C, R$ and $S$ are used to switch between different affine and projective models:

- If $Z \subset \mathbb{R}^n$, then $P(Z) \subset \mathbb{P}^n$ denotes the Zariski closure of $\iota_n(Z)$, where the embedding $\iota_n: \mathbb{R}^n \hookrightarrow \mathbb{P}^n$ sends $(z_1, \ldots, z_n)$ to $(1 : z_1 : \ldots : z_n)$.
- If $Z \subset \mathbb{R}^n$, then $C(Z) \subset \mathbb{C}^n$ denotes the Zariski closure of the embedding of $Z$ into $\mathbb{C}^n$ via the standard embedding $\mathbb{R}^n \hookrightarrow \mathbb{C}^n$.
- If $C \subset \mathbb{P}^n$, then $R(C) \subset \mathbb{R}^n$ is defined as $\iota_n^{-1}(\{ x \in C_\mathbb{R} \mid x_0 \neq 0 \})$, where $\iota_n^{-1}: \mathbb{P}^n_\mathbb{R} \rightarrow \mathbb{R}^n$ sends $(x_0 : \ldots : x_n)$ to $(x_1, \ldots, x_n)/x_0$.
- If $Z \subset \mathbb{R}^3$, then $S(Z) \subset S^3$ is defined as $P(\mu^{-1}(Z))$, where $\mu: S^3 \rightarrow \mathbb{R}^3$ is the stereographic projection at §1.

If $R(\{a\}) = \{b\}$ for $a \in \mathbb{P}^n$, then we write $R(a) = b$ instead.

**Remark 1.** We observe that, $P(S^3) = S^3$, $R(S^3) = S^3$, $S(\mathbb{R}^3) = S^3$, $R(\mathbb{E}) = \mathbb{E}_\mathbb{R} = \emptyset$, $C(\mathbb{E}) \neq \emptyset$ and $R(\mathbb{Y}) = S^2$. The quadrics $S^3$, $E$ and $Y$ are smooth, and $U$ is a quadratic cone with vertex in $U_\mathbb{R} = \{(1 : 0 : 0 : 0 : 1)\}$. ◄

We consider the following linear projections from $S^3$ to $\mathbb{P}^3$:

- **stereographic projection** $\pi: S^3 \rightarrow \mathbb{P}^3$, $\pi(x) := (x_0 - x_4 : x_1 : x_2 : x_3)$,
- **central projection** $\tau: S^3 \rightarrow \mathbb{P}^3$, $\tau(x) := (x_1 : x_2 : x_3 : x_4)$, and
- **vertical projection** $\nu: S^3 \rightarrow \mathbb{P}^3$, $\nu(x) := (x_0 : x_1 : x_2 : x_3)$.

**Remark 2.** The stereographic projection $\pi$ corresponds via $P$ with $\mu: S^3 \rightarrow \mathbb{R}^3$ as defined in §1. The projection center of $\pi$ lies in $U_\mathbb{R}$. Moreover, $\pi$ defines a biregular isomorphism $S^3 \setminus U \cong \pi(S^3 \setminus U)$ and $\pi(U) = \{ x \in \mathbb{P}^3 \mid x_0 = x_1^2 + x_2^2 + x_3^2 = 0 \}$ is an irreducible conic in $\mathbb{P}^3$ without real points. The central and vertical projections define 2:1 morphisms with ramification locus $E$ and $Y$, respectively. The branching
we remark that a linear projection \( \tau : S^3 \to \mathbb{P}^3 \) corresponds via \( R \) to a 2:1 linear map \( S^3 \to \mathbb{R}^3 \) whose fibers are antipodal points. For intuition, we remark that a linear projection \( S^1 \to \mathbb{R} \) of the unit circle \( S^1 \subset \mathbb{R}^2 \) is via \( P \) a 1-dimensional analogue of \( \pi, \tau \) and \( \nu \), if the center lies either on \( S^1 \), in the interior of \( S^1 \), or in the exterior of \( S^1 \), respectively.

The following two complex maps will be used for defining “translations” of \( S^3 \):

- \( \zeta_b : \mathbb{P}^4 \to \mathbb{P}^4 \) with \( b = (b_1, b_2, b_3) \in \mathbb{C}^3 \) is the linear transformation corresponding to the following \( 5 \times 5 \) matrix, where \( \Delta := \frac{1}{2}(b_1^2 + b_2^2 + b_3^2) \):
  \[
  \begin{pmatrix}
  1+\Delta & b_1 & b_2 & b_3 & -\Delta \\
  b_1 & 1 & 0 & 0 & -b_1 \\
  b_2 & 0 & 1 & 0 & -b_2 \\
  b_3 & 0 & 0 & 1 & -b_3 \\
  \Delta & b_1 & b_2 & b_3 & 1-\Delta 
  \end{pmatrix}.
  \]

- \( \hat{\gamma}_\ast : S^3 \times S^3 \to S^3 \) is the rational map defined by
  \[
  (x, y) \mapsto (x_0y_0 : x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 : x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3 :
  x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2 : x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1).
  \]

We consider the following complex transformations of \( S^3 \), where \( \text{Aut}_C \mathbb{P}^4 \) denotes the complex projective transformations of \( \mathbb{P}^4 \) and \( H \in \{U, E, Y\} \):

\[
\begin{align*}
\text{Aut}_C S^3 & := \{ \varphi \in \text{Aut}_C \mathbb{P}^4 \mid \varphi(S^3) = S^3 \}, \\
\text{Aut}_H S^3 & := \{ \varphi \in \text{Aut}_C S^3 \mid \varphi(H) = H \}, \\
\text{UT} S^3 & := \{ \zeta_b : \mathbb{P}^4 \to \mathbb{P}^4 \mid b \in \mathbb{C}^3 \}, \\
\text{LT} S^3 & := \{ \varphi : S^3 \to S^3 \mid \varphi(x) = p \hat{\ast} x, \ p \in S^3 \setminus E \}, \text{ and} \\
\text{RT} S^3 & := \{ \varphi : S^3 \to S^3 \mid \varphi(x) = x \hat{\ast} p, \ p \in S^3 \setminus E \}.
\end{align*}
\]

The Möbius transformations are defined as
\[
\text{Aut} S^3 := \{ \varphi \in \text{Aut}_C S^3 \mid \varphi \circ \sigma = \sigma \circ \varphi \},
\]
where \( \sigma : \mathbb{P}^4 \to \mathbb{P}^4 \) denotes the real structure. The Euclidean transformations, elliptic transformations and hyperbolic transformations of \( S^3 \), are defined as

\[
\begin{align*}
\text{Aut}_U S^3 & \cap \text{Aut} S^3, \quad \text{Aut}_E S^3 & \cap \text{Aut} S^3 \quad \text{and} \quad \text{Aut}_Y S^3 & \cap \text{Aut} S^3, \text{ respectively.}
\end{align*}
\]

The Euclidean translations, left Clifford translations and right Clifford translations are defined as

\[
\begin{align*}
\text{UT} S^3 & \cap \text{Aut} S^3, \quad \text{LT} S^3 & \cap \text{Aut} S^3 \quad \text{and} \quad \text{RT} S^3 & \cap \text{Aut} S^3, \text{ respectively.}
\end{align*}
\]

The left generator and right generator that pass through \( p \in E \) are defined as

\[
\begin{align*}
\mathcal{L}_p & := \{ q \hat{\ast} p \mid q \in S^3 \setminus E \} \quad \text{and} \quad \mathcal{R}_p := \{ p \hat{\ast} q \mid q \in S^3 \setminus E \}, \text{ respectively.}
\end{align*}
\]
We shall refer to the complex lines in $\mathbb{U}$ as *generators*.

The following proposition is classical and concerns translations in elliptic geometry (see [4, §7.9 and 7.93]). Our proof is based on [24, Proposition 1]. Recall from §1 that $\_ \star \_ : S^3 \times S^3 \to S^3$ denotes the Hamiltonian product for the unit quaternions.

**Proposition 3.**

(a) $R(x \hat{\ast} y) = R(x) \ast R(y)$ for all $x, y \in S^3$.  
(b) $\mathrm{LT} S^3, \mathrm{RT} S^3 \subset \mathrm{Aut}_E S^3$.

(c) For all $p \in E$, the generators $L_p$ and $R_p$ are the two complex lines in $E$ containing $p$.

(d) For all $\varphi \in \mathrm{LT} S^3$ and $p \in E$, we have $\varphi(L_p) = L_p$. For all $\varphi \in \mathrm{RT} S^3$ and $p \in E$, we have $\varphi(R_p) = R_p$.

**Proof.** We start by introducing some terminology, which is only needed in this proof. The algebra of *quaternions* consist of the vector space $\mathbb{H} := \langle 1, i, j, k \rangle_\mathbb{R}$ together with the associative product $\_ \ast _\mathbb{H} : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ that is defined by

$$i \ast_\mathbb{H} i = j \ast_\mathbb{H} j = k \ast_\mathbb{H} k = i \ast_\mathbb{H} j \ast_\mathbb{H} k = -1$$

with $1 \in \mathbb{H}$ being the multiplicative unit. The algebra of *complex quaternions* is defined as $\mathbb{H}_\mathbb{C} = \mathbb{H} + i \mathbb{H}$, where $i \in \mathbb{C}$ denotes the imaginary unit. Let $\_ \bullet _\mathbb{H} : \mathbb{H}_\mathbb{C} \times \mathbb{H}_\mathbb{C} \to \mathbb{H}_\mathbb{C}$ denote the product induced by $\ast_\mathbb{H}$.

The *conjugate* of a real or complex quaternion $h = h_1 + h_2 i + h_3 j + h_4 k$ is defined as $h^* := h_1 - h_2 i - h_3 j - h_4 k$. A direct calculation shows that

$$h \bullet h^* = h_1^2 + h_2^2 + h_3^2 + h_4^2.$$ 

We observe that $S^3 = \{ h \in S^3 \mid h \ast_\mathbb{H} h^* = 1 \} \subset \mathbb{R}^4 = \mathbb{H}$, and thus the Hamiltonian product $\_ \ast _\mathbb{H} : S^3 \times S^3 \to S^3$ is induced by $\ast_\mathbb{H}$. Similarly,

$$\mathbb{C}(S^3) = \{ h \in \mathbb{H}_\mathbb{C} \mid h \bullet h^* = 1 \} \subset \mathbb{C}^4 = \mathbb{H}_\mathbb{C}.$$ 

A direct calculation shows that the product $\_ \ast_S \_ : \mathbb{C}(S^3) \times \mathbb{C}(S^3) \to \mathbb{C}(S^3)$ induced by $\bullet$ extends to the rational map $\_ \hat{\ast} _\mathbb{P} : S^3 \times S^3 \dashrightarrow S^3$ defined before. This implies that Assertion (a) holds.

If we identify $\mathbb{P}^3$ with the projectivized vector space $\mathbb{P}(\langle 1, i, j, k \rangle_\mathbb{C})$, then $\bullet$ extends to the following rational map $\_ \ast_\mathbb{P} _\mathbb{P} : \mathbb{P}^3 \times \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, where $x = (x_1 : \ldots : x_4)$ and
We are now ready to prove the remaining Assertions (b), (c) and (d).

(b) Since \( q * S q^* = q^* * S q = 1 \), we find that \( \psi_1^{-1}(x) \) is equal to either \( q^* * S x \) or \( x * S q^* \) for all \( x \in C(S^3) \). It follows that \( \psi_2 \in \text{Aut}_C S^3 \). Moreover, \( \iota(C(S^3)) = S^3 \setminus E \), which implies that \( \psi_2(E) = E \). Therefore, \( LT S^3, RT S^3 \subset \text{Aut}_E S^3 \) and thus we concluded the proof of Assertion (b).

We set \( E := \{ h \in \mathbb{H}_C \mid h \cdot h^* = 0 \} \) and for all \( \alpha \in E \) we define

\[
L_\alpha := \{ h \in \mathbb{H}_C \mid h \cdot \alpha = 0 \} \quad \text{and} \quad R_\alpha := \{ h \in \mathbb{H}_C \mid \alpha \cdot h = 0 \}.
\]

(c) Suppose that \( \alpha \in E \setminus \{0\} \) and \( \beta := \alpha^* \). The map \( \mathbb{H}_C \to \mathbb{H}_C \) that sends \( h \) to \( h \cdot \alpha \) is linear with respect to the underlying vector space \( (1, i, j, k)_C \) and has kernel \( L_\alpha \).

Let \( V_\beta := \langle \beta, i \cdot \beta, j \cdot \beta, k \cdot \beta \rangle_C \). By assumption, \( \alpha \cdot \alpha^* \beta = \beta \cdot \beta^* = \beta \cdot \alpha = 0 \) and thus \( V_\beta \subseteq L_\alpha \). Now suppose by contradiction that \( \dim V_\beta < 2 \). In this case,

\[
\beta = c_1 \cdot i \cdot \beta = c_2 \cdot j \cdot \beta = c_3 \cdot k \cdot \beta
\]
we find that there exists $\beta_0, \beta_1, \beta_2, \beta_3 \in \mathbb{C}$ such that
\[
\beta = \beta_0 + \beta_1 \cdot i + \beta_2 \cdot j + \beta_3 \cdot k,
\]
\[
c_1 \cdot i \cdot \beta = -c_1 \cdot \beta_1 + c_1 \cdot \beta_0 \cdot i - c_1 \cdot \beta_3 \cdot j + c_1 \cdot \beta_2 \cdot k,
\]
\[
c_2 \cdot j \cdot \beta = -c_2 \cdot \beta_2 + c_2 \cdot \beta_3 \cdot i + c_2 \cdot \beta_0 \cdot j - c_2 \cdot \beta_1 \cdot k,
\]
\[
c_3 \cdot k \cdot \beta = -c_3 \cdot \beta_3 - c_3 \cdot \beta_2 \cdot i + c_3 \cdot \beta_1 \cdot j + c_3 \cdot \beta_0 \cdot k.
\]
By comparing the coefficients, we see that $\beta_0 = -c_1 \cdot \beta_1, \beta_i = c_i \cdot \beta_0$ and thus $\beta_0 \neq 0$ and $c_i^2 = -1$ for all $1 \leq i \leq 3$. This implies that $\beta_1 = \beta_2 = \beta_3 = \pm i \cdot \beta_0$ and thus
\[
\beta \cdot \beta^* = \beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = -2 \cdot \beta_0^2 = 0.
\]
We arrived at a contradiction as $\beta_0 \neq 0$. We established that
\[
\dim L_\alpha \geq \dim V_\beta \geq 2.
\]
Thus, $\kappa(L_\alpha)$ and $\kappa(R_\alpha)$ are the two complex lines in the projective quadric $\kappa(E)$ that contain the complex point $\kappa(\alpha^*) \in \kappa(L_\alpha) \cap \kappa(R_\alpha)$. Since $\tau(E) = \kappa(E)$ as a direct consequence of the definitions, we may assume without loss of generality that $\kappa(\alpha^*) = \tau(p)$. It follows that $\tau(L_p) = \kappa(L_\alpha)$ and $\tau(R_p) = \kappa(R_\alpha)$. The central projection $\tau$ is a 2:1 morphism with ramification locus $E$ and thus Assertion (c) is true.

(d) We may assume without loss of generality that $\psi_2 = \varphi$ and $\alpha \in E$ such that $\kappa(\alpha^*) = \tau(p)$. Thus, $\psi_2(x) = \iota(q) \cdot x$ for some $q \in \mathbb{C}(S^3)$. If $\beta \in L_\alpha$, then $h \cdot \beta \in L_\alpha$ for all $h \in \mathbb{H}_\mathbb{C}$. This implies that $\psi_4(L_\alpha) = L_\alpha$. As Table 3 commutes, we deduce that $\psi_2(L_p) = \varphi(L_p) = L_p$ as was to be shown. The prove of the second statement is analogous and thus we concluded the proof. \[\square\]

The following proposition shows that the Euclidean translations of $S^3$ correspond to Euclidean translations of $\mathbb{R}^3$ and leave the generators of $U$ invariant.

**Proposition 4.**

(a) $R(\pi \circ \zeta_v(x)) = R(\pi(x)) + v$ for all $x \in S^3_\mathbb{R} \setminus U$ and $v \in \mathbb{R}^3$.

(b) $UTS^3 \subset \text{Aut}_U S^3$.

(c) If $L \subset U$ is a generator, then $\varphi(L) = L$ for all $\varphi \in UT S^3$. 

11
Proof. (a) It follows from a straightforward calculation (see [16, cyclides]) that
\[
(\pi \circ \zeta_b)(x) = (x_{04} : x_1 + x_{04}b_1 : x_2 + x_{04}b_2 : x_3 + x_{04}b_3)
\]
for all \( b \in \mathbb{C}^3 \), where \( x_{04} := x_0 - x_4 \). Thus, \( R(\pi \circ \zeta_b(x)) = R(\pi(x)) + b \) for all \( b \in \mathbb{R}^3 \) and \( x \in \mathbb{S}_2^3 \) such that \( x_{04} \neq 0 \).

(b) Suppose that \( \varphi \in UT \mathbb{S}^3 \) so that \( \varphi = \zeta_b \) for some \( b \in \mathbb{C}^3 \). Let \( M \) be the \( 5 \times 5 \) matrix associated to \( \varphi \) and let \( J \) be the diagonal matrix with \( (-1, 1, 1, 1, 1) \) on its diagonal. We verify that \( M^\top \cdot J \cdot M = c \cdot J \) for some \( c \in \mathbb{C} \) and thus \( \varphi \in \text{Aut}_C \mathbb{S}^3 \). Since \( \varphi(\mathbb{U}) = \mathbb{U} \), it follows that \( \varphi \in \text{Aut}_U \mathbb{S}^3 \). See [16, cyclides] for an automatic verification.

(c) By assumption, \( \varphi = \zeta_b \) for some \( b \in \mathbb{C}^3 \). Let
\[
\iota(z) := (1 : z_1 : z_2 : z_3),
\]
\[
\psi_2(x) := (x_0 : x_1 + x_0 b_1 : x_2 + x_0 b_2 : x_3 + x_0 b_3), \quad \psi_3(z) := (z_1 + b_1, z_2 + b_2, z_3 + b_3).
\]
It follows from Equation (1) that the diagram in Table 4 commutes.

Table 4: See the proof of Proposition 4.

| S^3 | \pi | \varphi | \psi_2 | \psi_3 | \pi \psi_3 | \pi \psi_2 | \psi_3^\top | \pi^\top | \pi \psi_2^\top |
|-----|-----|-------|-------|-------|-------------|-------------|-------------|-------|-------------|
| S^3 | \pi | \psi_2 | \psi_3 | \pi \psi_3 | \pi \psi_2 | \psi_3^\top | \pi^\top | \pi \psi_2^\top |

Let \( H \subset \mathbb{C}^3 \) be a complex plane. Since \( \psi_3(H) \) is parallel to \( H \), we deduce that the Zariski closures of the images \( \iota(H) \) and \( (\iota \circ \psi_3)(H) \) in \( \mathbb{P}^3 \) intersect at a complex line \( K \) at infinity. Recall from Remark 2 that \( \pi(\mathbb{U}) \) is a conic at infinity and thus \( 1 \leq |K \cap \pi(\mathbb{U})| \leq 2 \) by Bézout’s theorem. We find that \( \psi_2(K \cap \pi(\mathbb{U})) = K \cap \pi(\mathbb{U}) \).

Since \( H \) was chosen arbitrary, we deduce that \( \psi_2(q) = q \) for all \( q \in \pi(\mathbb{U}) \). We have \( \pi(L) = q \) for some \( q \in \pi(\mathbb{U}) \), and thus \( \varphi(L) = L \) as asserted. \( \square \)

Remark 5. Notice that \( \mathbb{E}_R = \emptyset \) and that the complex conjugate of a left or right generator in \( \mathbb{E} \) is again left and right, respectively. The generators in \( \mathbb{U} \) are all concurrent. Complex conjugate lines in \( \mathbb{Y} \) intersect in a real point. A hyperplane section of \( \mathbb{S}^3 \) is Möbius equivalent to either \( \mathbb{U}, \mathbb{E} \) or \( \mathbb{Y} \). Since \( \mathbb{U} \) is unlike \( \mathbb{E} \) and \( \mathbb{Y} \) a tangent hyperplane section, we could interpret Euclidean geometry as a limit case of both the hyperbolic and elliptic geometries. \( \triangleright \)

Definition 6. If \( Z \subset \mathbb{R}^3 \) is a Darboux cyclide, then we shall call its Möbius model \( \mathbb{S}(Z) \) in \( \mathbb{S}^3 \) also a Darboux cyclide. Similarly for CH1 cyclide, EY cyclide,
Blum cyclide and so on, and for attributes such as $\lambda$-circled, celestial, Bohemian, Cliffordian and great. We call a conic $C \subset S^3$ a (great/small) circle if $R(C)$ is a (great/small) circle in $S^3 \subset \mathbb{R}^4$.

3 Associated pencils and absolutes

In §2, we considered elliptic and Euclidean geometries as subgroups of the Möbius transformations that preserve some fixed hyperplane section of the Möbius quadric. In this section, we characterize the intersection of Cliffordian and Bohemian surfaces with such hyperplane sections. In particular, we analyze how circles meet the elliptic or Euclidean absolute as they move in their respective pencils.

A pencil on a surface $X \subset S^3$ is defined as an irreducible real hypersurface

$$ F \subset X \times \mathbb{P}^1 $$

such that the 1st and 2nd projections $\pi_1: X \times \mathbb{P}^1 \rightarrow X$ and $\pi_2: X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ are dominant. If $i \in \mathbb{P}^1$ is reached by $\pi_2$, then the Zariski closure of $\pi_1(F \cap X \times \{i\}) \subset X$ is called a member of $F$ and denoted by $F_i$. We call a complex point $p \in X$ a base point of $F$, if $p \in F_i$ for all $i \in \pi_2(F)$. We call $F$ a pencil of conics if $F_i$ is an irreducible complex conic for almost all $i \in \mathbb{P}^1$. We call $F$ a pencil of circles if it is a pencil of conics such that $F_i$ is a circle for infinitely many $i \in \mathbb{P}^1_{\mathbb{R}}$.

Example 7. Suppose that

$$ F := \{(x_0 : x_1 : x_2 : x_3 ; i_0 : i_1) \mid i_0 x_3 = i_1 x_0, \ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\}. $$

Thus $F \subset S^2 \times \mathbb{P}^1$ is defined by the latitudinal circles on a sphere, where $F_i$ is a circle if $i \in \mathbb{P}^1_{\mathbb{R}}$ such that $-1 \leq i_1/i_0 \leq 1$ and otherwise $F_i$ is a complex conic with at most one real point. By definition, $F$ is a pencil of circles.

Remark 8. If $F \subset X \times \mathbb{P}^1$ is a pencil of conics on a celestial Darboux cyclide, then it follows from [22, Theorem 9] that $F$ is the graph of a rational map $f: X \rightarrow \mathbb{P}^1$ whose fibers are complex conics. This implies that the first projection $\pi_1$ is birational and the second projection $\pi_2$ is surjective. The complex points where the map $f$ is not defined correspond to the base points.

The following lemma is used in Lemma 23. We present an elementary proof based on Bézout’s theorem to make our result accessible to a wider audience.
Lemma 9. Suppose that $X \subset \mathbb{S}^3$ is a Darboux cyclide and $F \subset X \times \mathbb{P}^1$ is a pencil of circles.

(a) The member $F_i$ is a complex conic that is not contained in a complex line for all $i \in \mathbb{P}^1$.

(b) If $L \subset X$ is a complex line such that $|F_i \cap L| > 0$ for almost all $i \in \mathbb{P}^1$, then $|\{i \in \mathbb{P}^1 \mid p \in F_i\}| = 1$ for all complex $p \in L$.

(c) If $R \subset X$ is a complex line such that $|F_i \cap R| = 0$ for almost all $i \in \mathbb{P}^1$, then there exists $k \in \mathbb{P}^1$ such that $R \subset F_k$ is a component.

Proof. Let $q := (1 : 0 : 0 : 0 : 1)$ denote the center of the stereographic projection $\pi: \mathbb{S}^3 \to \mathbb{P}^3$ and recall from Remark 2 that $\pi$ defines a biregular isomorphism between $X \setminus U$ and $\pi(X \setminus U)$. We may assume up to Möbius equivalence that $q \in X \cap \mathbb{R}$ is general.

Notice that $\mathbb{S}^3 \cap \mathbb{R}$ does not contain lines. A surface in $\mathbb{S}^3$ that contains infinitely many points and contains a complex line through a general point, must contain two complex conjugate lines through this point and thus be a sphere. Hence, $X$ does not contain a complex line through $q$. The point $q$ is general by assumption and therefore smooth and not a base point of any complex pencil of conics on $X$. It follows that $\deg \pi(X) = 3$ and $\pi(X)$ does not contain a complex line through a general point.

Suppose that $j \in \mathbb{P}^1$ is a general complex point. Since $q$ is not a base point of $F$, the linear projection $\pi(F_j)$ is an irreducible complex conic. We denote by $P_j \subset \mathbb{P}^3$ the complex plane containing $\pi(F_j)$. It follows from Bézout’s theorem and $\deg \pi(X) = 3$ that $P_j \cap \pi(X) = \pi(F_j) \cup M$ for some complex line $M$. This implies that the complex plane sections of $\pi(X)$ that contain the complex line $M$ define a complex pencil of conics on $\pi(X)$ and $\pi(F_j)$ is a member of this complex pencil. We already established that $\pi(X)$ contains only finitely many complex lines, and thus $M$ does not depend on the choice of $j$. Therefore, $\pi(F_j)$ is a general member of the complex pencil defined by $M$. By assumption $F$ is a pencil of circles and thus there exists $i \in \mathbb{P}^1 \cap \mathbb{R}$ such that $\pi(F_i)$ is a conic coplanar with $M$. This implies that the complex line $M$ must be real. Let $C \subset X$ be the complex curve such that $\pi(C) = M$. Since $M$ is a line, we find that $C$ must be a circle passing through the point $q$. 

14
We are now ready to prove Assertions (a), (b) and (c).

(a) By definition, $F_i$ is a complex irreducible conic for almost all $i \in \mathbb{P}^1$. When we project from a general point on the line $M$ we obtain a 2:1 map $\pi(X) \dashrightarrow \mathbb{P}^2$ such that $\pi(F_j)$ is mapped to a complex line in a pencil of lines in $\mathbb{P}^2$ centered at the image of $M$. Each complex line in this pencil on $\mathbb{P}^2$ has a complex curve as preimage in $\pi(X)$ and thus for all $i \in \mathbb{P}^1$ such that $q \notin F_i$ the projection $\pi(F_i)$ is a complex curve of degree at most two that is not equal to $M$. As $q$ is general, we deduce that the member $F_i$ is either a complex line or complex conic for all $i \in \mathbb{P}^1$.

Now suppose by contradiction that $F_c$ is contained in a complex line for some $c \in \mathbb{P}^1$. Notice that $F_c \cup C$ is a complex hyperplane section of $X$ as it is the preimage of the complex hyperplane section $\pi(F_c) \cup M$ of $\pi(X)$. Since $S_3^\mathbb{R}$ does not contain lines, the complex conjugate line $\overline{F_c}$ is contained in $X$ as well. Let $\xi : X \dashrightarrow \mathbb{P}^3$ be the complex linear projection from a general complex point in $F_c$. If $F_c \cap \overline{F_c} \subseteq \text{Sing} X$, then $\deg \xi(X) = 2$ and $\xi(\overline{F_c}) \subset \text{Sing} \xi(X)$. This is a contradiction as $\xi(X)$ must be irreducible. Hence, $\deg \xi(X) = 3$ and $\xi(C \cup F_c)$ is a complex hyperplane section that consists of the complex irreducible conic $\xi(C)$. Notice that $\xi(F_c) \in \xi(C)$ is a complex point. We arrived at a contradiction with Bézout’s theorem, since a complex hyperplane section of $\xi(C)$ must be of degree three when counted with multiplicities. This concludes the proof of Assertion (a).

(b) Recall that $q$ is general and thus $F$ has no base point on $C$. Since $j \in \mathbb{P}^1$ is general as well, we find that $|F_j \cap L \cap C| = 0$. As $X \setminus U \cong \pi(X \setminus U)$, we deduce that $|\pi(F_j) \cap \pi(L) \cap M| = 0$ and $|\pi(F_j) \cap \pi(L)| > 0$. It follows that $|\pi(L) \cap M| = 0$ as $\pi(F_j)$ is coplanar with $M$ by construction. Notice that there exists a unique complex plane in $\mathbb{P}^3$ containing a line and complex point outside the line. This implies that $|\{i \in \mathbb{P}^1 \mid r \in P_i\}| = |\{i \in \mathbb{P}^1 \mid r \in \pi(F_i)\}| = 1$ for all $r \in \pi(L)$. As $X \setminus U \cong \pi(X \setminus U)$, we deduce that $|\{i \in \mathbb{P}^1 \mid p \in F_i\}| = 1$ for all $p \in L \setminus U$.

We apply a Möbius transformation $\varphi \in \text{Aut} \mathbb{S}^3$ such that $q \in \varphi(X_\mathbb{R}) \setminus \{\varphi(q)\}$ is general and repeat the same arguments for $\varphi(X)$ instead of $X$. We deduce that $|\{i \in \mathbb{P}^1 \mid p \in F_i\}| = 1$ for all $p \in L \setminus \varphi^{-1}(U)$. Since $L \cap U \cap \varphi^{-1}(U) = \emptyset$, we conclude that Assertion (b) holds.

(c) Since $|F_j \cap R| = 0$, we have $|\pi(F_j) \cap \pi(R)| = 0$ and thus $|\pi(R) \cap M| = 1$. Let $k \in \mathbb{P}^1$ such that $P_k$ is the complex plane spanned by the union $\pi(R) \cup M$. By
Bézout’s theorem, \( P_k \cap \pi(X) \) is a union of either two or three complex lines. Recall from the proof of Assertion (a) that \( M \neq \pi(F_k) \). Hence, the complex line \( \pi(R) \) must be a component of the reducible complex conic \( \pi(F_k) \). As \( X \setminus U \cong \pi(X \setminus U) \) and \( R, F_k \not\subset U \), we conclude that \( R \subset F_k \) so that Assertion (c) holds.

Suppose that \( A, B \subset S^3 \) are circles such that \( A \star B \subset S^3 \) is a surface and observe that \( P(A) \cong \mathbb{P}^1 \). The left associated pencil of the surface \( A \star B \) is defined as the pencil of circles

\[
F \subset P(A \star B) \times P(A)
\]
such that \( F_a = \varphi_a(P(B)) \) for all \( a \in P(A) \setminus \mathbb{E} \), where \( \varphi_a \in \text{LT} S^3 \) sends \( x \) to \( a \hat{\times} x \).

It follows from Proposition 3(a) that for all \( a \in P(A) \setminus \mathbb{E} \), we have

\[
R(F_a) = \{ R(a) \star b \mid b \in B \}.
\]

We remark that \( \text{LT} S^3 \subset \text{Aut}_E S^3 \) by Proposition 3(b) and for all \( a \in P(A) \setminus \mathbb{E} \), we have \( \{ a \hat{\times} b \mid b \in P(B) \setminus \mathbb{E} \} \subset F_a \). The member \( F_a \) may be reducible for \( a \in P(A) \cap \mathbb{E} \).

Similarly, the right associated pencil of \( A \star B \) is defined as the pencil of circles

\[
G \subset P(A \star B) \times S(B)
\]
such that \( G_b = \varphi_b(P(A)) \) for all \( b \in P(B) \setminus \mathbb{E} \), where \( \varphi_b \in \text{RT} S^3 \) sends \( x \) to \( x \hat{\times} b \).

**Lemma 10.** Suppose that \( A \star B \) is a surface for some circles \( A, B \subset S^3 \).

(a) If the left associated pencil \( F \subset P(A \star B) \times P(A) \) has no base points on \( \mathbb{E} \), then \( P(A \star B) \) contains complex conjugate left generators \( L, \overline{L} \subset \mathbb{E} \) such that \( |F_a \cap L| = |F_a \cap \overline{L}| = 1 \) for almost all \( a \in P(A) \).

(b) If the right associated pencil \( G \subset P(A \star B) \times P(B) \) has no base points on \( \mathbb{E} \), then \( P(A \star B) \) contains complex conjugate right generators \( R, \overline{R} \subset \mathbb{E} \) such that \( |G_b \cap R| = |G_b \cap \overline{R}| = 1 \) for almost all \( b \in P(B) \).

**Proof.** (a) For infinitely many points \( i, j \in P(A) \setminus \mathbb{E} \) the members \( F_i \) and \( F_j \) are circles and these circles are related by a left Clifford translation. We know from Bézout’s theorem that \( F_i \) intersects \( \mathbb{E} \) in two complex conjugate points, since \( \mathbb{E} \) is a hyperplane section of \( S^3 \). It follows from Proposition 3(d) and Remark 5 that the complex conjugate left generators \( L, \overline{L} \subset \mathbb{E} \) that pass through these points are left invariant. As \( F \) is Zariski closed and without base points on \( \mathbb{E} \) we conclude that \( L, \overline{L} \subset P(A \star B) \).
The proof for Assertion (b) is analogous. \qed

Let $A, B \subset \mathbb{R}^3$ be circles and notice that $S(A) \cong \mathbb{P}^1$. The associated pencil of $A + B$ is defined as the pencil of circles $F \subset S(A + B) \times S(A)$ such that for almost all $a \in S(A)$, there exists a unique $c \in C(A)$ such that $F_a = \zeta_c(S(B))$.

As a straightforward consequence of the definitions and Proposition 4(a), we find that $\zeta_a(b) \in S(A + B)$ for all $a \in A$ and $b \in S(B)$. Hence, the circles $\{C_a\}_{a \in A}$ on the surface $A + B \subset \mathbb{R}^3$ that are defined by $C_a := \{a + b \mid b \in B\}$ correspond via $R(\pi(_*))$ to a subset of $\{F_a\}_{a \in S(A)}$.

**Lemma 11.** Suppose that $A, B \subset \mathbb{R}^3$ are circles so that $A + B$ is a surface. If the associated pencil $F \subset S(A + B) \times S(A)$ of $A + B$ has no base points on $\mathbb{U}$, then $S(A + B)$ contains complex conjugate generators $L, \bar{L} \subset \mathbb{U}$ such that $|F_a \cap L| = |F_a \cap \bar{L}| = 1$ for almost all $a \in S(A)$.

**Proof.** Let $i \in S(A)$ be general. Since $F$ has no base points on $\mathbb{U}$, we may assume without loss of generality that $F_i$ does not meet the vertex of the tangent hyperplane section $\mathbb{U}$. Thus it follows from Bézout’s theorem that

$$|F_i \cap \mathbb{U}| = |F_i \cap \{x \in \mathbb{P}^4 \mid x_0 - x_4 = 0\}| = 2.$$  

Recall from Proposition 4(c) that the Euclidean translations of $\mathbb{S}^3$ leave the generators of the hyperplane section $\mathbb{U}$ invariant. Hence, there exist complex conjugate generators $L, \bar{L} \subset \mathbb{U}$ such that $|F_a \cap L| = |F_a \cap \bar{L}| = 1$ for almost all $a \in S(A)$. As $F$ is Zariski closed and without base points on $\mathbb{U}$, we conclude that $L, \bar{L} \subset S(A + B)$. \qed

4 Divisor classes of curves on Darboux cyclides

We recall from [15] the possible sets of divisor classes of complex low degree curves on Darboux cyclides in $\mathbb{S}^3$ (recall Definition 6). Each entry in this classification translates into a diagram that visualizes how complex lines, complex isolated singularities and circles intersect.

A smooth model $O(X)$ of a surface $X \subset \mathbb{P}^n$ is a nonsingular surface such that there exists a birational morphism $\varphi: O(X) \to X$ that does not contract complex $(-1)$-curves. We refer to $\varphi$ as a desingularization.
The smooth model $\mathcal{O}(X)$ is unique up to biregular isomorphisms and there exists a desingularization $\varphi: \mathcal{O}(X) \to X$ (see [10, Theorem 2.16]).

The \textit{Néron-Severi lattice} $N(X)$ is an additive group defined by the divisor classes on $\mathcal{O}(X)$ up to numerical equivalence. This group comes with an unimodular intersection product $\cdot$ and a unimodular involution $\sigma_*: N(X) \to N(X)$ induced by the real structure $\sigma: X \to X$. We denote by $\text{Aut} N(X)$ the group automorphisms that are compatible with both $\cdot$ and $\sigma_*$. 

The \textit{class} $[C]$ of a complex curve $C \subset X$ is defined as the divisor class of $\bar{C}$ in $N(X)$, where $\bar{C} \subset \mathcal{O}(X)$ is the union of complex curves in $\varphi^{-1}(C)$ that are not contracted to complex points by the morphism $\varphi$.

We consider the following subsets of $N(X)$:

- $B(X)$ denotes the set of divisor classes of complex irreducible curves $C \subset \mathcal{O}(X)$ such that $\varphi(C)$ is a complex point in $X$,
- $G(X)$ denotes the set of classes of complex irreducible conics in $X$, and
- $E(X)$ denotes the set of classes of complex lines in $X$.

We call $W \subset B(X)$ a \textit{component} if it defines a maximal connected subgraph of the graph with vertex set $B(X)$ and edge set $\{\{a, b\} \mid a \cdot b > 0\}$. The latter subgraph is called the \textit{graph of the component}.

We write $c \cdot W \succ 0$ for a subset $W \subset N(X)$, if there exists $w \in W$ such that $c \cdot w > 0$.

The \textit{singular locus} of $X$ is denoted by $\text{Sing} X$.

The following proposition is an application of intersection theory on surfaces (see [7, Section V.1]). For its proof we assume some background in algebraic geometry, but its assertions are meant to be accessible to non-experts.

**Proposition 12.** Suppose that $X \subset \mathbb{S}^3$ is a celestial Darboux cyclide. Let

- $W(X)$ be the set of components in $B(X)$,
- $\mathcal{F}(X)$ be the set of pencils of circles on $X$,
- $\mathcal{G}(X) := \{g \in G(X) \mid \sigma_*(g) = g\}$, and
- $\mathcal{E}(X)$ be the set of complex lines in $X$.

Suppose that $C, C' \subset X$ are complex lines and/or circles such that $C \neq C'$.
(a) There exists a bijection $\Gamma \colon W(X) \to \text{Sing } X$ such that for all $W \in W(X)$ the following two properties hold:

- $\Gamma(W) \in \text{Sing } X$ if and only if $\sigma_*(W) = W$,
- if $[C] \cdot W > 0$, then $\Gamma(W) \in C$.

The graph of a component in $B(X)$ is a Dynkin graph of type $A_1$, $A_2$ or $A_3$.

(b) The map $\Lambda \colon F(X) \to G(X)$ that sends $F$ to $[F_{(0:1)}]$ is a bijection that satisfies the following two properties for all $F, G \in F(X)$:

- $\Lambda(F)^2 = 0$, and
- if $\Lambda(F) \cdot \Lambda(G) = 2$, then $|F_i \cap G_j| = 2$ for almost all $i, j \in \mathbb{P}^1$.

In particular, $[F_i] = [F_j]$ for all $i, j \in \mathbb{P}^1$.

(c) The map $E(X) \to E(X)$ that sends $L$ to $[L]$ is bijective.

(d) We have that $C \cap C' \neq \emptyset$ if and only if either $[C] \cdot [C'] \neq 0$, or there exists $W \in W(X)$ such that both $[C] \cdot W > 0$ and $[C'] \cdot W > 0$.

(e) For all $p \in X$ and $F \in F(X)$, the complex point $p$ is a base point of $F$ if and only if there exists $W \in W(X)$ such that $\Lambda(F) \cdot W > 0$ and $\Gamma(W) = p$.

Proof. Let $\varphi : O(X) \to X$ be a desingularization.

Claim 1. The smooth model $O(X)$ is a weak del Pezzo surface of degree four and $X$ is its anticanonical model with at most isolated singularities.

This claim follows from [22, Proposition 1], where we followed the terminology at [6, Definition 8.1.18, Theorems 8.3.2 and 8.6.4].

Recall that we defined the class of a complex curve $U \subset X$ as the divisor class of the strict transform of $U$ in the smooth model $O(X)$. We denote the strict transforms of $C \subset X$ and $C' \subset X$ via $\varphi$ by $D \subset O(X)$ and $D' \subset O(X)$, respectively. If $W \in W(X)$, then we denote by $C_W$ the union of complex curves in $O(X)$ whose divisor class is in $W$.

Claim 2. The morphism $\varphi$ restricted to $O(X) \setminus \varphi^{-1}(\text{Sing } X)$ is an isomorphism, and $p \in \text{Sing } X$ if and only if $p = \varphi(C_W)$ for some component $W$ whose graph is a Dynkin graph of type $A_1$, $A_2$ or $A_3$.

It follows from Claim 1 and [6, Proposition 8.1.10 and Theorem 8.2.27] that $\varphi$ contracts $(-2)$-curves to isolated singularities and is an isomorphism outside the
(−2)-curves. Thus $B(X)$ consist of the divisor classes of the (−2)-curves and $C_W$ is a union of (−2)-curves with classes in $W$. We know from [6, Theorem 8.1.11 and Theorem 8.2.28] that the graph of the component $W$ is a Dynkin graph. By [15, Corollary 5] this graph can only be of type $A_1$, $A_2$ or $A_3$.

**Claim 3.** $D \cap U \neq \emptyset$ if and only if $[D] \cdot [U] > 0$ for all $U \in \{C_W \mid W \in \mathcal{W}(X)\} \cup \{D'\}$.

Curves on rational surfaces are linearly equivalent if and only if the curves are numerically equivalent. Thus, it follows from [7, Theorem V.1.1 and Proposition V.1.4] that $[D] \cdot [U]$ is equal to number of intersections in $D \cap U$, when counted with multiplicity.

**Claim 4.** $C \cap C' \setminus \text{Sing } X \neq \emptyset$ if and only if $[C] \cdot [C'] > 0$.

Notice that $[D] \cdot [D'] = [C] \cdot [C']$ by definition. If $C \cap C' \setminus \text{Sing } X \neq \emptyset$, then $D \cap D' \neq \emptyset$ by Claim 2 and thus $[C] \cdot [C'] > 0$ by Claim 3. If $[C] \cdot [C'] > 0$, then $D \cap D' \neq \emptyset$ by Claim 3 and thus $C \cap C' \setminus \text{Sing } X \neq \emptyset$ by Claim 2.

**Claim 5.** $C \cap C' \cap \text{Sing } X \neq \emptyset$ if and only if there exists $W \in \mathcal{W}(X)$ such that $[C] \cdot W \succ 0$ and $[C'] \cdot W \succ 0$.

It follows from Claim 2 that $C \cap C' \cap \text{Sing } X \neq \emptyset$ if and only if there exists $W \in \mathcal{W}(X)$ such that $C \cap C_W \neq \emptyset$ and $C' \cap C_W \neq \emptyset$. By definition, $[D] \cdot [C_W] > 0$ if and only if $[C] \cdot W \succ 0$, and thus Claim 5 follows from Claim 3.

**Claim 6.** If $C$ and $C'$ meet transversally, then $|C \cap C'| \geq [C] \cdot [C']$.

It follows from [7, Theorem V.1.1] that $|D \cap D'| = [D] \cdot [D']$ and thus Claim 6 follows from Claim 2.

We are now ready to prove the Assertions (a), (b), (c), (d) and (e) of Proposition 12.

(a) We define $\Gamma(W) := \varphi(C_W)$ for all $W \in \mathcal{W}(X)$ and thus $\Gamma$ is well-defined and bijective by Claim 2. Since $\sigma(C_W) = C_W$ if and only if $\sigma(W) = W$, the proof for this assertion is concluded by Claims 2 and 5.

(b) Complex curves on rational surfaces are linearly equivalent if and only if the complex curves are numerically equivalent and thus $[F_i] = [F_j]$ for all $i, j \in \mathbb{P}^1$. We may assume without loss of generality that $F_{(0,1)}$ is a circle and thus $[F_i] \in \mathcal{G}(X)$ for all $i \in \mathbb{P}^1$. It follows that the map $\Lambda$ is well-defined and injective. We know from Claim 1 and [22, Theorem 9] that the strict transform of a conic in $X$ to
the smooth model $O(X)$ belongs to a 1-dimensional base point free complete linear series of curves on $O(X)$. This implies that any conic in $X$ is the member of a pencil of conics on $X$. Hence, if $[U] \in G(X)$ for some irreducible conic $U \subset X$, then $[U]^2 = 0$ and there exists a pencil $T \subset X \times \mathbb{P}^1$ of conics that has $U$ as member. Since the members of $T$ cover a Zariski open set of $X_{\mathbb{R}}$, we deduce that infinitely many members of $T$ are circles, which implies that $T \in F(X)$. We established that $\Lambda(T) = [U]$ so that $\Lambda$ is surjective. Moreover, $[U]^2 = 0$ and thus $\Lambda(F)^2 = 0$. The members $F_i$ and $G_j$ meet transversally for general choice of $i, j \in \mathbb{P}^1$. Since complex conics in $S^3$ intersect in at most two points, the second property follows from Claim 6.

(c) Complex lines $L \subset X$ do not move in a pencil and thus are in 1:1 correspondence with their classes $[L] \in E(X)$.

(d) Direct consequence of Claims 4 and 5.

(e) Let us additionally assume that $C$ and $C'$ are members of $F$. Since $\Lambda(F)^2 = [D] \cdot [D'] = |D \cap D'| = 0$ by Assertion (b) and Claim 3, the $\Rightarrow$ direction follows from Claims 2, 4 and 5. The $\Leftarrow$ direction follows the second property at Assertion (a) and $\Lambda$ being well-defined.

In this article, $N(X) \cong \langle \ell_0, \ell_1, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle \mathbb{Z}$, where the nonzero intersections between the generators are $\ell_0 \cdot \ell_1 = 1$ and $\varepsilon_2^2 = \varepsilon_3^2 = \varepsilon_4^2 = -1$. We use the following shorthand notation; those are going to be elements in $B(X) \cup G(X) \cup E(X)$:

\begin{align*}
  b_1 &:= \varepsilon_1 - \varepsilon_3, & b_{ij} &:= \ell_0 - \varepsilon_i - \varepsilon_j, & b_0 &:= \ell_0 + \ell_1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4, \\
  b_2 &:= \varepsilon_2 - \varepsilon_4, & b'_{ij} &:= \ell_1 - \varepsilon_i - \varepsilon_j, \\
  g_0 &:= \ell_0, & g_2 &:= 2\ell_0 + \ell_1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4, & g_{ij} &:= \ell_0 + \ell_1 - \varepsilon_i - \varepsilon_j, \\
  g_1 &:= \ell_1, & g_3 &:= \ell_0 + 2\ell_1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4, & g'_{ij} &:= b_0 + \varepsilon_i.
\end{align*}

For convenience, we included at [16] a table and graphs that encode the pairwise intersection numbers of the above elements.
We consider the following unimodular involutions $\sigma_*: N(X) \to N(X)$ that are induced by the real structure $\sigma: X \to X$:

$2A_1: \quad \sigma_*(\ell_0) = \ell_0, \; \sigma_*(\ell_1) = \ell_1, \; \sigma_*(\epsilon_1) = \epsilon_2, \; \sigma_*(\epsilon_3) = \epsilon_4$,

$3A_1: \quad \sigma_*(\ell_0) = \ell_1, \; \sigma_*(\epsilon_1) = \epsilon_2, \; \sigma_*(\epsilon_3) = \epsilon_4$,

$D_4: \quad \sigma_*(\ell_0) = g_3, \; \sigma_*(\ell_1) = \ell_1, \; \sigma_*(\epsilon_i) = \ell_1 - \epsilon_i$ for $1 \leq i \leq 4$.

Recall from Proposition 12(a) that the graph of a component $W \subset B(X)$ is a Dynkin graph of type $A_1$, $A_2$ or $A_3$. The corresponding isolated double point is called a node, cusp or tacnode, respectively. If $\sigma_*(W) \neq W$, then $\text{type}(W) \in \{A_1, A_2, A_3\}$ denotes the type of $W$. If $\sigma_*(W) = W$, then $\text{type}(W) \in \{\underline{A_1}, \underline{A_2}, \underline{A_3}\}$ denotes the underlined type of $W$. If $\{W_1, \ldots, W_n\}$ is the set of components in $B(X)$, then we denote the singular type $\text{SingType}_X$ as a formal sum $\text{type}(W_1) + \ldots + \text{type}(W_n)$.

A Darboux cyclide $X \subset S^3$ is called a $S1$ cyclide or $S2$ cyclide, if $R(X)$ is smooth and homeomorphic to a sphere or the disjoint union of two spheres, respectively.

The following theorem follows from [15, Theorem 4 and Corollary 5].

**Theorem A.** If $X \subset S^3$ is a $\lambda$-circled Darboux cyclide such that $\lambda \geq 2$, then

$$N(X) \cong \langle \ell_0, \ell_1, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \rangle \mathbb{Z},$$

and $\sigma_*$, $\text{SingType}_X$, $B(X)$, $E(X)$, $G(X)$, $\lambda$ are up to $\text{Aut} N(X)$ defined by a row in Tables 5 and 6, together with the name of $X$.

**Remark 13.** The $G(X)$, $E(X)$ and $\text{SingType}_X$ are, up to $\text{Aut} N(X)$, uniquely determined by $B(X)$ together with $\sigma_*$ (see [15, Theorem 4 and Corollary 5]). Notice that the 3th, 5th, and 6th columns in Table 5 and the ordering/underlines in Table 6 are immediate corollaries of Theorem A.
Table 5: See Theorem A. A class is send by the unimodular involution $\sigma_*$ to itself if underlined.

| cyclide | $\sigma_*$ | components in $B(X)$ | SingType $X$ | $|E(X)|$ | $|G(X)|$ | $\lambda$ |
|---------|------------|----------------------|-------------|---------|---------|---------|
| Blum    | $2A_1$     | $\emptyset$          | $\emptyset$ | 16      | 10      | 6       |
| Perseus | $2A_1$     | $\{b_1\}, \{b_2\}$  | $2A_1$      | 8       | 7       | 5       |
| ring    | $2A_1$     | $\{b_{13}\}, \{b_{24}\}, \{b'_{14}\}, \{b'_{23}\}$ | $4A_1$      | 4       | 4       | 4       |
| EH1     | $2A_1$     | $\{b_{12}\}$         | $A_1$       | 12      | 8       | 4       |
| CH1     | $2A_1$     | $\{b_{13}\}, \{b_{24}\}, \{b'_{12}\}$ | $2A_1 + A_1$ | 6       | 5       | 3       |
| HP      | $2A_1$     | $\{b_{12}, b'_{34}\}$ | $A_2$       | 8       | 6       | 2       |
| EY      | $2A_1$     | $\{b_{12}, b_1, b_2\}$ | $A_3$       | 4       | 5       | 3       |
| CY      | $2A_1$     | $\{b_{12}, b_1, b_2\}, \{b'_{13}\}, \{b'_{24}\}$ | $A_3 + 2A_1$ | 2       | 2       | 2       |
| EO      | $2A_1$     | $\{b_{12}\}, \{b_{34}\}$ | $2A_1$      | 8       | 7       | 3       |
| CO      | $2A_1$     | $\{b_{12}\}, \{b_{34}\}, \{b'_{13}\}, \{b'_{24}\}$ | $2A_1 + 2A_1$ | 4       | 4       | 2       |
| EE/EH2  | $3A_1$     | $\{b_9\}$            | $A_1$       | 12      | 8       | 2       |
| EP      | $3A_1$     | $\{b_{13}, b'_{24}\}$ | $A_2$       | 8       | 6       | 2       |
| S1      | $3A_1$     | $\emptyset$          | $\emptyset$ | 16      | 10      | 2       |
| S2      | $D_4$      | $\emptyset$          | $\emptyset$ | 16      | 10      | 2       |
Table 6: See Theorem A. A class is send by the unimodular involution \( \sigma_* \) to itself if underlined, and otherwise to its left or right neighbor in the listing, if its position is even or odd, respectively. The dashed row dividers indicate when \( \sigma_* \) is defined by \( 2A_1, 3A_1 \) or \( D_4 \).

| cyclide | \( B(X), E(X), G(X) \) |
|---------|--------------------------|
| Blum    | \{ \}, \{ e_1, e_2, e_3, e_4, e_{01}, e_{02}, e_{03}, e_{04}, e_{11}, e_{12}, e_{13}, e_{14}, e'_3, e'_2, e'_1 \}, \{ g_0, g_1, g_{12}, g_{34}, g_2, g_3, g_{13}, g_{24}, g_{14}, g_{23} \} |
| Perseus | \{ b_1, b_2 \}, \{ e_3, e_4, e_{01}, e_{02}, e_{11}, e_{12}, e'_3 \}, \{ g_{13}, g_{24}, g_0, g_1, g_{12}, g_2, g_3 \} |
| ring    | \{ b_{13}, b_{24}, b'_1, b'_2 \}, \{ e_1, e_2, e_3, e_4, e_{03}, e_{11}, e_{12} \}, \{ g_0, g_1, g_{12}, g_{34} \} |
| EH1     | \{ b_{12} \}, \{ e_1, e_2, e_3, e_4, e_{03}, e_{04}, e_{11}, e_{12}, e_{13}, e_{14}, e'_2, e'_1 \}, \{ g_0, g_1, g_{34}, g_{13}, g_{24}, g_{14}, g_{23} \} |
| CH1     | \{ b_{13}, b_{24}, b'_1 \}, \{ e_1, e_2, e_3, e_4, e_{13}, e_{14} \}, \{ g_{14}, g_{23}, g_0, g_1, g_{14} \} |
| HP      | \{ b_{12}, b'_3 \}, \{ e_1, e_2, e_3, e_4, e_{03}, e_{11}, e_{12} \}, \{ g_0, g_1, g_{13}, g_{24}, g_{14}, g_{23} \} |
| EY      | \{ b_{12}, b_1, b_2 \}, \{ e_3, e_4, e_{11}, e_{12} \}, \{ g_{13}, g_{24}, g_0, g_1, g_3 \} |
| CY      | \{ b_{12}, b_1, b_2, b'_3, b'_2 \}, \{ e_3, e_4 \}, \{ g_0, g_1 \} |
| EO      | \{ b_{12}, b_{24} \}, \{ e_1, e_2, e_3, e_4, e_{11}, e_{12}, e_{13}, e_{14} \}, \{ g_{13}, g_{24}, g_{14}, g_{23}, g_0, g_1, g_3 \} |
| CO      | \{ b_{12}, b'_3, b'_2 \}, \{ e_1, e_2, e_3, e_4 \}, \{ g_0, g_1, g_{14}, g_{23} \} |
| EE/EH2  | \{ h_0 \}, \{ e_1, e_2, e_3, e_4, e_{01}, e_{12}, e_{02}, e_{11}, e_{03}, e_{14}, e_{04}, e_{13} \}, \{ g_0, g_1, g_{12}, g_{34}, g_{13}, g_{24}, g_{14}, g_{23} \} |
| EP      | \{ b_{13}, b'_2 \}, \{ e_1, e_2, e_3, e_4, e_{02}, e_{11}, e_{04}, e_{13} \}, \{ g_0, g_1, g_{12}, g_{34}, g_{14}, g_{23} \} |
| S1      | \{ \}, \{ e_1, e_2, e_3, e_4, e_{01}, e_{12}, e_{02}, e_{11}, e_{03}, e_{14}, e_{04}, e_{13}, e'_3, e'_2, e'_1 \}, \{ g_0, g_1, g_{12}, g_{34}, g_{13}, g_{24}, g_{14}, g_{23}, g_2, g_3 \} |
| S2      | \{ \}, \{ e_1, e_{11}, e_2, e_{12}, e_3, e_{13}, e_4, e_{14}, e_{01}, e'_3, e'_2, e'_1, e_{03}, e'_3, e'_4 \}, \{ g_0, g_3, g_{12}, g_{34}, g_{13}, g_{24}, g_{14}, g_{23}, g_1, g_2 \} |
Figure 3: Incidences between complex lines and isolated singularities on a Darboux cyclide $X$ (see Example 14). Each complex line is represented as a line segment and labeled with its corresponding class in $E(X)$. A real or non-real isolated singularity is represented as a disc with a solid and dashed border, respectively. Each singularity is labeled with the sum of classes in the corresponding component in $B(X)$.

Example 14. The diagrams in Figures 3, 4 and 10 show the incidences between complex lines and isolated singularities in EY cyclide, CY cyclide, Perseus cyclide, CH1 cyclide, ring cyclide, CO cyclide, and EO cyclide. In case the Darboux cyclides are inversions of quadratic surfaces, it is straightforward to compute such incidences via elementary computations. For example, an EY cyclide $X \subset S^3$ is for some $\lambda > 0$, Möbius equivalent to

$$X' := \{ x \in \mathbb{P}^4 \mid x_1^2 + \lambda^2 x_2^2 = (x_0 - x_4)^2, \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_0^2 \}.$$

If $L \subset X'$ is a complex line, then it must be contained in $U = \{ x \in S^3 \mid x_0 = x_4 \}$ and thus there exist $\alpha, \beta \in \{1, -1\}$ such that

$$L = \{ x \in \mathbb{P}^4 \mid x_0 = x_4, \quad x_1 = \alpha \lambda x_2 = \frac{\beta \lambda}{\sqrt{1-\lambda^2}} x_3 \}.$$

For Darboux cyclides $X \subset S^3$ such that $X_\mathbb{R}$ is smooth this approach is less feasible. Instead, we obtained the diagrams by applying Proposition 12 to Theorem A as follows. Since the approach is the same for each case, we suppose again that $X \subset S^3$ is an EY cyclide. We apply Theorem A and find that $\text{SingType} X = A_3$, $B(X) = \{b_1, b_2, b_12\}$ and $E(X) = \{e_{11}, e_{12}, e_3, e_4\}$ (see Tables 5 and 6). A component $W \subset B(X)$ corresponds to an isolated singularity by Proposition 12(a) and a class in $E(X)$ corresponds to a complex line in $X$ by Proposition 12(c). If $\sigma_*(W) = W$, then the isolated singularity is real. Each line segment in the diagram for the EY cyclide in Figure 3 represents a complex line in $X$. Two line segments intersect at a disc if and only if the corresponding complex lines meet at a complex point $p \in X$. If $p$ is real or non-real, then the disc has solid and dashed border, respectively. If $p$
is an isolated singularity of $X$, then the disc is labeled with the sum of the classes in the corresponding component $W \subset B(X)$. In the EY cyclide case, $W = \{b_1, b_2, b_{12}\}$ and thus the label equals $b_1 + b_2 + b_{12} = b_{34}$. We use Propositions 12(a) and 12(d) to determine whether complex lines and/or isolated singularities intersect. If $\sigma_*(a) = b$ and $a \cdot b = 1$ for some $a, b \in E(X)$, then the corresponding complex conjugate lines meet in a real point. The diagrams for the remaining cyclides are obtained analogously, and are automatically verified at [16, cyclides]. Notice that a non-real singularity in an CY cyclide meets only one complex line.

![Diagram](image)

**Figure 4:** Incidences between complex lines and isolated singularities (see Example 14).

**Example 15.** Suppose that $Z \subset \mathbb{R}^3$ is a smooth torus of revolution. The astronomer Yvon Villarceau observed that $Z$ contains through a general point a latitudinal circle, a longitudinal circle and two cospherical Villarceau circles [29, 1848]. Let us identify the classes of these circles and identify those circles that are members of a pencil with base points. For this purpose, we suppose that $\Gamma : W(X) \to \text{Sing } X$ and $\Lambda : \mathcal{F}(X) \to \mathcal{G}(X)$ are the bijections defined at Proposition 12, where $X := S(Z)$. Let $F, G \in \mathcal{F}(X)$ be the two pencils of Villarceau circles, and let $F', G' \in \mathcal{F}(X)$ be the pencils of latitudinal and longitudinal circles, respectively. Since $X_\mathbb{R}$ is smooth and $X$ is covered by four pencils of circles, it follows from Propositions 12(a) and 12(b) that $\sigma_*(W) \neq W$ for all $W \in W(X)$ and $|\mathcal{G}(X)| = 4$. Thus Theorem A implies that the corresponding type entries in the SingType $X$ column of Table 5 are not underlined and the corresponding value in the $\lambda$ columns is equal to 4. We find that $X$ is a ring cyclide so that $\mathcal{G}(X) = \{g_0, g_1, g_{12}, g_{34}\}$ up to $\text{Aut } N(X)$ (see the corresponding row in Table 6). Notice that $\alpha \cdot \beta = 2$ for $\alpha, \beta \in \mathcal{G}(X)$ if and
only if \( \{\alpha, \beta\} = \{g_{12}, g_{34}\} \). It now follows from Proposition 12(b) that without loss of generality \((\Lambda(F), \Lambda(G), \Lambda(F'), \Lambda(G')) = (g_{12}, g_{34}, g_0, g_1)\). Since \( g_0 \cdot \{b'_1\} > 0 \) and \( g_1 \cdot \{b_{13}\} > 0 \), we deduce from Propositions 12(a) and 12(c) and that \( F' \) and \( G' \) each have complex conjugate base points in \( \text{Sing} \ X \).

The following Lemmas 16 and 17 are needed in §5.

**Lemma 16.** Suppose that \( X \subset S^3 \) is a celestial Darboux cyclide.

(a) \( X \) is covered by a pencil of circles with complex conjugate base points if and only if \( X \) is either a Perseus cyclide, ring cyclide, CH1 cyclide, CY cyclide or CO cyclide.

(b) If \( X \) is covered by at least two pencils of circles with complex conjugate base points, then \( X \) is a ring cyclide.

(c) If \( X \) is a CY cyclide or CO cyclide, then \( X \cap \mathcal{E} \) does not contain complex lines.

**Proof.** Suppose that \( \Gamma : \mathcal{W}(X) \to \text{Sing} \ X \) and \( \Lambda : \mathcal{F}(X) \to \mathcal{G}(X) \) are the bijections defined at Proposition 12 and let \( \mathcal{M}(X) := \{W\in \mathcal{W}(X) \mid \sigma_4(W) \neq W\} \).

**Claim 1.** Each of the following items hold up to \( \text{Aut} \ N(X) \):

- If \( X \) is a Perseus cyclide, then \( \mathcal{G}(X) = \{g_0, g_1, g_{12}, g_2, g_3\} \),
  \[ \mathcal{M}(X) = \{\{b_1\}, \{b_2\}\}, \quad g_{12} \cdot \{b_1\} > 0, \quad g_{12} \cdot \{b_2\} > 0, \]
  and \( g \cdot b = 0 \) for all \( g \in \{g_0, g_1, g_2, g_3\} \) and \( b \in \{b_1, b_2\} \).

- If \( X \) is a ring cyclide, then \( \mathcal{G}(X) = \{g_0, g_1, g_{12}, g_{34}\} \),
  \[ \mathcal{M}(X) = \{\{b_{13}\}, \{b_{24}\}, \{b'_1\}, \{b'_2\}\}, \]
  \[ g_0 \cdot \{b'_1\} > 0, \quad g_0 \cdot \{b'_2\} > 0, \quad g_1 \cdot \{b_{13}\} > 0, \quad g_1 \cdot \{b_{24}\} > 0, \]
  and \( g \cdot b = 0 \) for all \( g \in \{g_{12}, g_{34}\} \) and \( b \in \{b_{13}, b_{24}, b'_1, b'_2\} \).

- If \( X \) is a CH1 cyclide, then \( \mathcal{G}(X) = \{g_0, g_1, g_{34}\} \),
  \[ \mathcal{M}(X) = \{\{b_{13}\}, \{b_{24}\}\}, \quad g_1 \cdot \{b_{13}\} > 0, \quad g_1 \cdot \{b_{24}\} > 0, \]
  and \( g \cdot b = 0 \) for all \( g \in \{g_0, g_{34}\} \) and \( b \in \{b_{13}, b_{24}\} \).

- If \( X \) is a CY cyclide or CO cyclide, then \( \mathcal{G}(X) = \{g_0, g_1\} \),
  \[ \mathcal{M}(X) = \{\{b'_1\}, \{b'_2\}\}, \quad g_0 \cdot \{b'_1\} > 0, \quad g_0 \cdot \{b'_2\} > 0, \]
  and \( g_1 \cdot b'_{13} = g_1 \cdot b'_{24} = 0 \).
This claim follows from Theorem A (see Tables 5 and 6 for $\mathcal{M}(X)$ and $\mathcal{G}(X)$, respectively). See [16, cyclides] for a table with entries $g \cdot b$ for all $g \in G(X)$ and $b \in B(X)$.

**Claim 2**  The pencil $F \in \mathcal{F}(X)$ has complex conjugate base points if and only if $\Lambda(F) \cdot W > 0$ for some $W \in \mathcal{M}(X)$.

This claim follows from Propositions 12(a) and 12(b).

**Claim 3**  If $|\text{Sing} X| - |\text{Sing} X_\mathbb{R}| > 0$, then $X$ is either a Perseus cyclide, ring cyclide, CH1 cyclide, CY cyclide or CO cyclide.

This claim follows from Theorem A (see the SingType $X$ column in Table 5).

(a) It follows from Propositions 12(a) and 12(e) that a non-real base point of a pencil $F \in \mathcal{F}(X)$ is contained in $\text{Sing} X$. Hence, the $\Rightarrow$ direction follows from Claim 3. The $\Leftarrow$ direction follows from Proposition 12(b) together with Claims 1 and 2.

(b) Suppose that the pencil $F \in \mathcal{F}(X)$ has complex conjugate base points. It follows from Claims 1 and 2 that if $X$ is a Perseus cyclide, CH1 cyclide, CY cyclide or CO cyclide, then $\Lambda(F)$ equals $g_{12}$, $g_1$, $g_0$ and $g_0$, respectively. Moreover, if $X$ is a ring cyclide, then $\Lambda(F) \in \{g_0, g_1\}$ (see also Example 15). Hence, this assertion follows from Assertion (a) and the injectivity of $\Lambda$.

(c) Recall from Example 14 that the incidences between complex lines and real isolated singularities in a CY cyclide and CO cyclide are illustrated in the diagrams of Figure 3 (right) and Figure 10 (left), respectively. We observe that each line segment in these two diagrams meet a disc with solid border. Thus each complex line in $X$ meets some real isolated singularity. Since $\mathbb{E}_\mathbb{R} = \emptyset$, we conclude that $X \cap \mathbb{E}$ does not contain complex lines. \hfill $\Box$

**Lemma 17.** Suppose that $X \subset \mathbb{S}^3$ is either a Perseus cyclide, CH1 cyclide or ring cyclide, and suppose that $C \subset X$ is a hyperplane section.

(a) If $X$ is a ring cyclide and $\text{Sing} X \subset C$, then $C$ consists of four complex lines.

(b) If $R, \overline{R} \subset C$ are complex conjugate lines such that $|R \cap \overline{R}| = 0$, then there exist complex conjugate lines $L$ and $\overline{L}$ such that $C = L \cup \overline{L} \cup R \cup \overline{R}$ and $|L \cap R| = |L \cap \overline{R}| = |\overline{L} \cap R| = |\overline{L} \cap \overline{R}| = 1$.  

28
Proof. (a) Recall from Example 14 that the incidences of complex lines and isolated singularities in \( X \) are depicted in the rightmost diagram of Figure 4. By assumption, the singular points are contained in the hyperplane section \( C \). A complex line in \( X \) that meets a hyperplane in \( \mathbb{P}^4 \) in more than one complex point must be contained in this hyperplane. Hence, \( C \) consists by Bézout’s theorem of four complex lines.

(b) By Proposition 12(c), two complex lines in \( X \) are complex conjugate if and only if their classes in \( E(X) \) are related via \( \sigma_* \). By assumption, \( X \) is either a Perseus cyclide, CH1 cyclide or ring cyclide. Hence, \( \sigma_* \) is of type 2A1 by Theorem A (see the second column of Table 5). In particular, \( \sigma_*(e_1) = e_2, \sigma_*(e_3) = e_4, \sigma_*(e'_3) = e'_4, \sigma_*(e_{01}) = e_{02}, \sigma_*(e_{11}) = e_{12} \) and \( \sigma_*(e_{13}) = e_{14} \).

First, suppose that \( X \) is a Perseus cyclide. Let us consider the incidences between the complex lines in \( X \) as depicted in the leftmost diagram of Figure 4. Notice that \( R \) and \( \overline{R} \) are presented by line segments that are labeled with \([R]\) and \(\sigma_*(\overline{R})\), respectively. Since \(|R \cap \overline{R}| = 0\), we observe that \(([R], [\overline{R}])\) is equal to either \((e_3, e_4), (e'_3, e'_4), (e_{11}, e_{12})\), or \((e_{01}, e_{02})\). If \(([R], [\overline{R}]) = (e_3, e_4)\), then the complex conjugate lines \( L \) and \( \overline{L} \) such that \(([L], [\overline{L}]) = (e'_3, e'_4)\), each meet \( R \cup \overline{R} \) (and thus the hyperplane \( C \)) in two complex points. A complex line in \( X \) that meets a hyperplane in \( \mathbb{P}^4 \) in more than one complex point must be contained in this hyperplane and thus \( L, \overline{L} \subset C \). Therefore, \( C = L \cup \overline{L} \cup R \cup \overline{R} \) by Bézout’s theorem and \(|L \cap R| = |L \cap \overline{R}| = |\overline{L} \cap R| = |\overline{L} \cap \overline{R}| = 1\).

The remaining three cases for \(([R], [\overline{R}])\) are symmetric, and thus Assertion (b) holds for the Perseus cyclide.

If \( X \) is a CH1 cyclide or ring cyclide, then Assertion (b) is shown analogously using the corresponding diagrams in Figure 4, except that \(([R], [L]) \in \{(e_3, e_{13}), (e_{13}, e_3)\}\) and \(([R], [L]) \in \{(e_1, e_3), (e_3, e_1)\}\), respectively, where \(\overline{R} = \sigma_*(\overline{R})\) and \(\overline{L} = \sigma_*(\overline{L})\). In particular, if \( X \) is a CH1 cyclide, then \(([R], [\overline{R}]) \neq (e_1, e_2)\), since \(|R \cap \overline{R}| = 0\) and the line segments labeled with \(e_1\) and \(e_2\) in the middle diagram of Figure 4 represent complex conjugate lines that meet at an isolated singularity.

\[\square\]

5 Cliffordian Darboux cyclides

We develop a necessary condition for a Darboux cyclide \( X \) to be Cliffordian in terms of the sets \( B(X), E(X) \) and \( G(X) \) in Table 6.
Suppose that $X \subset \mathbb{S}^3$ is a Darboux cyclide. For $a, b \in N(X)$, we set $a \odot b := 1$ if either $a \cdot b = 1$ or if $a \neq b$ and there exists a component $W \subset B(X)$ such that both $a \cdot W > 0$ and $b \cdot W > 0$; in all other cases we set $a \odot b := 0$. Notice that if $L, L' \subset X$ are different complex lines, then $|L \cap L'| = [L] \odot [L']$ by Proposition 12(d) and thus the operator $\odot$ provides an algebraic criterion for complex lines to intersect.

A Clifford quartet is defined as a subset $\{a, b, c, d\} \subset E(X)$ of cardinality four such that $\sigma_*(a) = b$, $\sigma_*(c) = d$, $a \odot b = c \odot d = 0$ and $a \odot c = c \odot b = b \odot d = d \odot a = 1$.

Example 18. If $X$ is a ring cyclide, then $E(X) = \{e_1, e_2, e_3, e_4\}$ by Theorem A. Since $|E(X)| = 4$, $\sigma_*(e_1) = e_2$, $\sigma_*(e_3) = e_4$, $e_1 \odot e_2 = e_3 \odot e_4 = 0$ and $e_1 \odot e_4 = e_3 \odot e_2 = e_2 \odot e_4 = 1$, we find that $E(X)$ forms a Clifford quartet. Recall from Example 14 that the diagram for the ring cyclide in Figure 4 represents each class $a \in E(X)$ in terms of a line segment, and $a \odot b = 1$ for $a, b \in E(X)$ if and only if the two corresponding line segments in the diagram meet at a disc. Hence, we can use such diagrams, together with the specification of $\sigma_*$, to recognize Clifford quartets.

Recall from Remark 1 that $P(S^3) = S^3$ and thus $P(A \ast B) \subset S^3$ for all $A, B \subset S^3$.

Lemma 19. If $P(A \ast B)$ is a Darboux cyclide for some circles $A, B \in S^3$, then $P(A \ast B) \cap E$ consists of two left generators and two right generators whose classes form a Clifford quartet.

Proof. Let $F \subset P(A \ast B) \times P(A)$ and $G \subset P(A \ast B) \times P(B)$ be the left and right associated pencils of $A \ast B$, respectively. We set $X := P(A \ast B)$.

If neither $F$ nor $G$ has base points in $E$, then it follows from Lemma 10 and Bézout’s theorem that $X \cap E = \{x \in X \mid x_0 = 0\}$ consist of two left generators and two right generators. By Proposition 12(d), the classes of these generators form a Clifford quartet and thus the proof is concluded for this case.

In the remainder of the proof we assume that $F$ has a base point in $E$. Since $E_\mathbb{R} = \emptyset$, we find that $F$ has two complex conjugate base points in $E$. Recall from Propositions 12(a) and 12(e) that each base point corresponds to a complex isolated singularity of $X$.

First suppose that $G$ has base points in $E$ as well. These base points must be complex conjugate and thus $X$ is a ring cyclide by Lemma 16(b). It follows from
Lemma 17(a) that the hyperplane section \( E \cap X \) consists of four complex lines. Recall from Example 18 that \( E(X) \) defines a Clifford quartet and thus we concluded the proof.

Finally, suppose that \( G \) does not have base points in \( E \). In this case, the hyperplane section \( X \cap E \) contains two right generators \( R \) and \( \overline{R} \) by Lemma 10. We apply Lemmas 16(a) and 16(c) and find that \( X \) is either a Perseus cyclide, ring cyclide or CH1 cyclide. The main assertion now follows from Lemma 17(b).

Now we introduce an algebraic necessary condition for a Darboux cyclide to be Cliffordian.

**Definition 20.** Suppose that \( X \subset S^3 \) is a celestial Darboux cyclide. We call \( (A, a, g, U) \) a Clifford data if

- \( A = \{a, b, c, d\} \) is a Clifford quartet for \( X \) with distinguished element \( a \),
- \( g \in G(X) \) such that \( \sigma_*(g) = g \) and \( g \cdot a \neq 0 \), and
- \( U = \{e \in E(X) \mid e \cdot a = 1 \text{ and } e \odot b = e \odot c = e \odot d = 0\} \).

We call the Clifford data \( (A, a, g, U) \) a certificate if \( g \cdot u \neq 0 \) for all \( u \in U \). We say that \( X \) satisfies the Clifford criterion if there exist at least one certificate.

**Example 21.** It follows from Theorem A that \( (\{e_1, e_2, e_3, e_4\}, e_1, g_{12}, \emptyset) \) is a certificate for a ring cyclide \( X \subset S^3 \), and thus a ring cyclide satisfies the Clifford criterion. Similarly, a Perseus cyclide and CH1 cyclide satisfy the Clifford criterion with certificates \( (\{e_{01}, e_{02}, e_{11}, e_{12}\}, e_{01}, g_1, \emptyset) \) and \( (\{e_3, e_{13}, e_{14}\}, e_3, g_{34}, \emptyset) \), respectively. In contrast, the Clifford data \( (\{e_1, e_2, e_3', e_4\}, e_1, g_{12}, \{e_{01}, e_{11}, e_2'\}) \) for a Blum cyclide is not a certificate, since \( g_{12} \cdot e_{11} = 0 \) (see Figure 5). In fact, the Blum cyclide does not satisfy the Clifford criterion. See [16, cyclides] for a software implementation that computes for each case in Table 6 all possible Clifford data, and checks whether they are certificates. For the Blum cyclide see alternatively Figure 11.

**Remark 22.** The following lemma can be seen as a generalization of [4, 7.94]. Donald Coxeter refers to [9, Chapter X] and Felix Klein attributes these insights to William Kingdon Clifford (1845–1879 CE). Clifford passed away at an early age and his theories in elliptic geometry were only partially published. Klein saw it as a duty to workout and disseminate these theories [9, page 238]. To my mind, Clifford...
taught us that by combining most elementary curves we gain essential insights into
the geometry of space.

\textbf{Lemma 23.} \textit{If }\mathbf{X} = \mathbf{P}(A \ast B) \subset \mathcal{S}^3 \text{ is a Darboux cyclide for some circles } A, B \in \mathcal{S}^3, \text{ then } \mathbf{X} \text{ satisfies the Clifford criterion.}

\textbf{Proof.} Let } F \subset \mathbf{P}(A \ast B) \times \mathbf{P}(A) \text{ and } G \subset \mathbf{P}(A \ast B) \times \mathbf{P}(B) \text{ be the left and right associated pencils of } A \ast B, \text{ respectively.}

First suppose that } F \text{ or } G \text{ has base points on } \mathcal{E}. \text{ These base points must be complex conjugate as } \mathcal{E}_R = \emptyset. \text{ Lemmas 19 and 16(c) imply that } \mathbf{X} \text{ is not a CY cyclide or CO cyclide. Thus, it follows from Lemma 16(a) that } \mathbf{X} \text{ is either a Perseus cyclide, ring cyclide or CH1 cyclide and the main assertion holds for these three cases by Example 21.}

In the remainder of the proof we assume that neither } F \text{ nor } G \text{ has base points on } \mathcal{E}. \text{ Notice that } \mathbf{X} \cap \mathcal{E} = \{x \in \mathbf{X} \mid x_0 = 0\} \text{ defines a hyperplane section. Hence, the intersection } \mathbf{P}(A) \cap \mathcal{E} \text{ consists by Bézout’s theorem of the complex conjugate points } \{a, \bar{a}\}. \text{ We obtain for all complex } \alpha \in \mathbf{P}(A) \setminus \{a, \bar{a}\} \text{ the complex left Clifford translation } \varphi_\alpha \in \text{LT}\mathcal{S}^3 \text{ such that } \varphi_\alpha(x) = \alpha \ast x \text{ for all } x \in \mathcal{S}^3 \setminus \mathcal{E}. \text{ By definition, } \varphi_\alpha(\mathbf{P}(B)) \text{ is a member of the pencil } F \text{ for all } \alpha \in \mathbf{P}(A) \setminus \{a, \bar{a}\}. \text{ We know from Proposition 3(b) that } \varphi_\alpha \in \text{Aut}_\mathcal{S}\mathcal{S}^3 \text{ and thus } \varphi_\alpha(\mathbf{P}(B)) \text{ is an irreducible complex conic for all } \alpha \in \mathbf{P}(A) \setminus \{a, \bar{a}\}. \text{ We know from Lemma 19 that } \mathbf{X} \cap \mathcal{E} \text{ consists of two left generators } \mathcal{L}, \bar{\mathcal{L}} \subset \mathcal{E} \text{ and two right generators } \mathcal{R}, \bar{\mathcal{R}} \subset \mathcal{E} \text{ intersecting in four complex points } p, \bar{p}, q, \bar{q} \in \mathcal{E} \text{ (see Figure 6). It follows from Lemmas 9(b) and 10(a) that } |\{i \in \mathbb{P}^1 \mid p \in F_i\}| = 1 \text{ for all } p \in \mathcal{L} \cup \bar{\mathcal{L}}. \text{ Since } \mathcal{E} \subset \mathcal{S}^3 \text{ is a hyperplane section and } |F_i \cap (\mathcal{L} \cup \bar{\mathcal{L}})| = 2, \text{ we deduce from Bézout’s theorem that } |F_i \cap \mathcal{R}| = 0 \text{ for almost all } i \in \mathbb{P}^1. \text{ Thus, we know from Lemma 9(c) that the unique member of } F \text{ that contains } p \in \mathcal{L} \text{ is a complex reducible conic with } \mathcal{R} \text{ as component. Similarly, the unique member

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example.png}
\caption{Incidences between 7 of the 16 lines in a Blum cyclide.}
\end{figure}
of \( F \) that contains \( q \in L \) has \( \overline{R} \) as component. Since the complex left Clifford translations of \( P(B) \) are irreducible, it follows that \( \varphi_{\alpha}(P(B)) \cap \{p, q\} = \emptyset \) for all \( \alpha \in P(A) \setminus \{a, \overline{a}\} \). We established the following complex continuous map:

\[
\xi: P(A) \setminus \{a, \overline{a}\} \to L \setminus \{p, q\}, \quad \alpha \mapsto \varphi_{\alpha}(P(B)) \cap L.
\]

In fact, \( \xi \) is an complex isomorphism as each point on \( L \) is reached by exactly one member of \( F \) by Lemma 9(b). In other words, the complex left Clifford translations of the circle \( P(B) \) trace out \( L \setminus \{p, q\} \).

\[ L \]

\[ \varphi_{\alpha}(P(B)) \]

\[ \overline{R} \]

\[ q \]

\[ \overline{p} \]

\[ R \]

\[ p \]

\[ \overline{q} \]

**Figure 6:** The incidences between \( L, \overline{L}, R, \overline{R} \) and \( \varphi_{\alpha}(P(B)) \) for some \( \alpha \in P(A) \setminus \{a, \overline{a}\} \).

Suppose that \( U \subset E(X) \) is the set of classes of complex lines \( M \subset X \) such that \( [M] \cdot [L] = 1 \) and \( [M] \odot [\overline{L}] = [M] \odot [\overline{R}] = [M] \odot [\overline{R}] = 0 \). By Proposition 12(d), we have \( M \cap L = \{m\} \) and \( m \in L \setminus \{p, q\} \). We established that there exists \( \alpha \in P(A) \) such that \( m \in F_\alpha \) and thus \( F_\alpha \cap M \neq \emptyset \). Suppose by contradiction that \( [F_\alpha] \cdot [M] = 0 \). By Proposition 12(d), there exists a component \( W \subset B(X) \) such that \( [F_\alpha] \cdot W > 0 \) and \( [M] \cdot W > 0 \). It follows from Propositions 12(a) and 12(e) that \( m \) is a base point of \( F \). We arrived at a contradiction as \( F \) does not have base points on \( E \). Therefore, we require that \( [F_\alpha] \cdot [M] \neq 0 \) for all \( [M] \in U \). We conclude from Propositions 12(b), 12(c) and 12(d) that \( ([L], [\overline{L}], [\overline{R}], [\overline{R}]), [L], [F_\alpha], U \) is a certificate and thus \( X \) satisfies the Clifford criterion. \( \square \)

**Proposition 24.** A Cliffordian Darboux cyclide \( X \subset S^3 \) is either a Perseus cyclide, ring cyclide or CH1 cyclide.

**Proof.** The Darboux cyclide \( X \) satisfies the Clifford criterion by Lemma 23. We apply Theorem A and consider the 14 triples \( (B(X), E(X), G(X)) \) in Table 6. For each such triple we go through all possible Clifford quartets in \( E(X) \). For each such Clifford quartet \( A \) we consider all possible Clifford data \( (A, a, g, U) \). We verify that only a Perseus cyclide, ring cyclide or CH1 cyclide admits a Clifford data \( (A, a, g, U) \).
that is a certificate. We used [16, cyclides] to do the verification automatically. In particular, we find that a Clifford quartet exists only if $X$ is a Blum cyclide, Perseus cyclide, ring cyclide, EH1 cyclide, CH1 cyclide, HP cyclide, or S1 cyclide.

**Example 25.** We show that the surfaces $Z_{01}$, $Z_{23}$ and $Z_{45}$ defined at §1 are a ring cyclide, Perseus cyclide and CH1 cyclide, respectively. Moreover, we show that $Z_{06}$ and $Z_{78}$ are Cliffordian surfaces of degree 8. The required computations are done automatically at [16, cyclides]. See [14, orbital] for an alternative implementation of these methods. Suppose that $0 \leq i \leq 8$. Let $M_i := M_1$ be the corresponding $5 \times 5$ matrix in Table 7.

**Table 7:** $5 \times 5$ matrices that represent elements in $\text{Aut} \mathbb{S}^3$.

| $M_0$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $(1, 0, 0, 0, 0)$ | $(0, 1, 0, 0, 0)$ | $(0, 0, 1, 0, 0)$ | $(0, 0, 0, 1, 0)$ | $(0, 0, 0, 0, 1)$ | $(1, 0, 0, 0, 0)$ | $(0, 1, 0, 0, 0)$ | $(0, 0, 1, 0, 0)$ | $(0, 0, 0, 1, 0)$ |
| $(3, 0, 0, 0, 0)$ | $(0, 3, 0, 0, 0)$ | $(0, 0, 3, 0, 0)$ | $(0, 0, 0, 3, 0)$ | $(0, 0, 0, 0, 3)$ | $(3, 0, 0, 0, 0)$ | $(0, 3, 0, 0, 0)$ | $(0, 0, 3, 0, 0)$ | $(0, 0, 0, 3, 0)$ |
| $(5, 0, 0, 0, 0)$ | $(0, 5, 0, 0, 0)$ | $(0, 0, 5, 0, 0)$ | $(0, 0, 0, 5, 0)$ | $(0, 0, 0, 0, 5)$ | $(5, 0, 0, 0, 0)$ | $(0, 5, 0, 0, 0)$ | $(0, 0, 5, 0, 0)$ | $(0, 0, 0, 5, 0)$ |
| $(7, 0, 0, 0, 0)$ | $(0, 7, 0, 0, 0)$ | $(0, 0, 7, 0, 0)$ | $(0, 0, 0, 7, 0)$ | $(0, 0, 0, 0, 7)$ | $(7, 0, 0, 0, 0)$ | $(0, 7, 0, 0, 0)$ | $(0, 0, 7, 0, 0)$ | $(0, 0, 0, 7, 0)$ |
| $(9, 0, 0, 0, 0)$ | $(0, 9, 0, 0, 0)$ | $(0, 0, 9, 0, 0)$ | $(0, 0, 0, 9, 0)$ | $(0, 0, 0, 0, 9)$ | $(9, 0, 0, 0, 0)$ | $(0, 9, 0, 0, 0)$ | $(0, 0, 9, 0, 0)$ | $(0, 0, 0, 9, 0)$ |
| $(11, 0, 0, 0, 0)$ | $(0, 11, 0, 0, 0)$ | $(0, 0, 11, 0, 0)$ | $(0, 0, 0, 11, 0)$ | $(0, 0, 0, 0, 11)$ | $(11, 0, 0, 0, 0)$ | $(0, 11, 0, 0, 0)$ | $(0, 0, 11, 0, 0)$ | $(0, 0, 0, 11, 0)$ |
| $(13, 0, 0, 0, 0)$ | $(0, 13, 0, 0, 0)$ | $(0, 0, 13, 0, 0)$ | $(0, 0, 0, 13, 0)$ | $(0, 0, 0, 0, 13)$ | $(13, 0, 0, 0, 0)$ | $(0, 13, 0, 0, 0)$ | $(0, 0, 13, 0, 0)$ | $(0, 0, 0, 13, 0)$ |
| $(15, 0, 0, 0, 0)$ | $(0, 15, 0, 0, 0)$ | $(0, 0, 15, 0, 0)$ | $(0, 0, 0, 15, 0)$ | $(0, 0, 0, 0, 15)$ | $(15, 0, 0, 0, 0)$ | $(0, 15, 0, 0, 0)$ | $(0, 0, 15, 0, 0)$ | $(0, 0, 0, 15, 0)$ |

Let $J$ be the diagonal matrix with $(-1, 1, 1, 1, 1)$ on its diagonal. We verify that there exists $\lambda \in \mathbb{Q} \setminus \{0\}$ such that $M_i^T \cdot J \cdot M_i = \lambda J$, and thus $M_i$ defines a Möbius transformation $\varphi_i: \mathbb{S}^3 \to \mathbb{S}^3$. The curve parametrization $C_i(t)$ in Table 2 is related to the matrix $M_i$ as follows, where $\psi(t) := (1 : \cos(t) : \sin(t) : 0 : 0)$:

$$P(\{C_i(t) \ | \ 0 \leq t \leq 2\pi\}) = \{(\varphi_i \circ \psi)(t) \ | \ 0 \leq t \leq 2\pi\}.$$ 

Since $\psi(t)$ parametrizes a great circle, we observe that $C_i(t)$ parametrizes a circle as well. Let $\vec{c} := (1, 0, 0, 0, 0)$ and notice that $R((1 : 0 : 0 : 0 : 0))$ is the center of $\mathbb{S}^3$. If $i \in \{0, 1\}$, then we verify that there exists $\lambda \in \mathbb{Q} \setminus \{0\}$ such that $M_i \cdot \vec{c} = \lambda \vec{c}$. Hence, $C_0(t)$ and $C_1(t)$ parametrize great circles. For all $(i, j) \in \{(0, 1), (2, 3), (4, 5), (0, 6), (7, 8)\}$, we implicitize the surface

$$X_{ij} := P(\{C_i(\alpha) \times C_j(\beta) \ | \ 0 \leq \alpha, \beta < 2\pi\}) \subset \mathbb{S}^3$$

and find the following (we refer to [16, cyclides] for the details):

$$(\deg X_{01}, \deg X_{23}, \deg X_{45}, \deg X_{66}, \deg X_{78}) = (4, 4, 4, 8, 8) \quad \text{and} \quad (|\text{Sing } X_{01}|, |\text{Sing } X_{23}|, |\text{Sing } X_{45}|) = (4, 2, 3).$$

Since $X_{ij} = S(Z_{ij})$ by definition, it follows from Proposition 24 and Theorem A (see the SingType $X$ column in Table 5) that $Z_{01}$, $Z_{23}$ and $Z_{45}$ are a ring cyclide, Perseus cyclide and CH1 cyclide, respectively. As $\deg X_{06} = \deg X_{78} = 8$, we find that $Z_{06}$

34
and $Z_{78}$ are Cliffordian surfaces of Möbius degree 8. Since $C_0(t)$ parametrizes a great circle, it follows that the surfaces $Z_{01}$ and $Z_{06}$ are great.

\[ \text{Lemma 26.} \text{ Suppose that } X \subset S^3 \text{ is a Darboux cyclide that contains two circles through a general point that do not meet in two points. If some pair of complex conjugate lines in } X \text{ intersect, then these complex lines meet at an isolated singularity.} \]

Proof. Let $G(X) := \{ g \in G(X) \mid \sigma_*(g) = g \}$ and suppose that $L, \overline{L} \subset X$ are complex conjugate lines such that $L \cap \overline{L} \neq \emptyset$. We know from Proposition 12(b) that there exist different $f, f' \in G(X)$ such that $f \cdot f' \neq 2$. By Proposition 12(c), both $[L]$ and $\sigma_*([L]) = [\overline{L}]$ belong to $E(X)$. We apply Theorem A and verify for each of the 14 cases in Table 6 that the following statement holds: If there exist different $f, f' \in G(X)$ such that $f \cdot f' \neq 2$, then $e \cdot \sigma_*(e) = 0$ for all $e \in E(X)$. See [16, cyclides] for an automatic verification of this statement. It follows from Proposition 12(d) that there exists a component $W \subset B(X)$ such that $[L] \cdot W > 0$ and $\sigma_*([L]) \cdot W > 0$. By Proposition 12(a) such a component $W$ corresponds to an isolated singularity in $L \cap \overline{L}$ and thus we concluded the proof.

\[ \text{Proposition 27.} \text{ If } A, B \subset \mathbb{R}^3 \text{ are generalized circles such that } S(A + B) \text{ is a Darboux cyclide, then } A + B \text{ is either a CY or EY.} \]

Proof. We first consider the quadratic case:

Claim 1. If $\deg(A + B) \leq 2$, then $A + B$ is either a CY or EY and either $A$ or $B$ is a line.

Since $\deg S(A+B) = 4$, we find that $\deg(A+B) = 2$. We go through the well-known classification of quadratic surfaces up to Euclidean similarity (see [15, Proposition 4] and Figure 1) and conclude that Claim 1 holds.

Now let us assume by contradiction that $\deg A = \deg B = 2$. Notice that $A + B$ is not a plane by Claim 1 and thus the circles $A$ and $B$ are not coplanar.
Let \( C_v := \{v\} + B \) and \( D_v := A + \{v\} \) for \( v \in \mathbb{R}^3 \).

**Claim 2.** \( |C_a \cap D_b| = 1 \) for almost all \( a \in A \) and \( b \in B \).

Let \( H_a \subset \mathbb{R}^3 \) denote the spanning plane of \( C_a \). Since \( a \) and \( b \) are general, there exists \( q \in A \setminus \{a\} \) such that the circle \( D_b \) meets \( H_a \) transversally at the two points \( a + b \in A \) and \( q + b \). As we translate the circle \( D_b \) along \( C_a \), the incidence points \( a + b \) and \( q + b \) trace out the coplanar circles \( C_a \subset H_a \) and \( C_q \subset H_a \), respectively. Since \( |C_a \cap C_q| < \infty \) and \( b \) is general, we deduce that \( q + b \notin C_a \). It follows that \( C_a \cap D_b = \{a + b\} \) and thus Claim 2 holds true.

**Claim 3.** \( |C_i \cap C_j| = 0 \) for almost all \( i, j \in A \).

The spanning planes of \( C_i \) and \( C_j \) are parallel, but not equal. This implies the assertion of Claim 3.

Let \( F \subset S(A + B) \times S(A) \) and \( G \subset S(B + A) \times S(B) \) be the associated pencils of \( A + B \) and \( B + A \), respectively, where \( A + B = B + A \). It follows from Claim 2 that \( |F_a \cap G_b| = 1 \) for almost all \( a \in S(A) \) and \( b \in S(B) \). We deduce from Claim 3 that a base point of \( F \) or \( G \) must lie in \( \mathbb{U} \). We make a case distinction on the base points of \( F \) and \( G \).

First, we suppose that either \( F \) or \( G \) has no base points. We know from Lemma 11 that \( X \) contains complex conjugate lines that meet at the vertex of \( \mathbb{U} \). General members of \( F \) and \( G \) meet in one point and thus \( X \) has an isolated singularity at this vertex by Lemma 26. Hence, \( \deg \pi(X) = \deg(A + B) = 2 \) with \( \deg A = \deg B = 2 \).

We arrived at a contradiction with Claim 1.

Next, we suppose that \( F \) has a real base point in \( \mathbb{U}_\mathbb{R} \). In this case \( \deg \pi(X) = \deg(A + B) = 2 \) with \( \deg A = \deg B = 2 \), since \( \mathbb{U}_\mathbb{R} \subset \text{Sing} X \) by Propositions 12(a) and 12(e). We arrived at a contradiction with Claim 1.

Finally, suppose by contradiction that both \( F \) and \( G \) have non-real base points in \( \mathbb{U} \). Then \( X \) must be a ring cyclide by Lemma 16(b) and \( \text{Sing} X \subset \mathbb{U} \) by Propositions 12(a) and 12(e). It follows from Lemma 17(a) that the hyperplane section \( X \cap \mathbb{U} \) consists of four lines. We arrived at a contradiction with the diagram in Figure 4 (see Example 14), as all lines in \( \mathbb{U} \) should be concurrent.

We arrived at a contradiction at all three cases and thus we established either \( A \) or \( B \) is a line. Therefore, \( A + B \) is covered with lines, so that either \( F \) or \( G \) has a base
point in $\mathbb{R}^n$. Hence, by Propositions 12(a) and 12(e) the center of stereographic projection $\pi$ is in $\text{Sing} X$ so that $\deg(A + B) \leq 2$. The proof is now concluded by Claim 1.

\section{Great Darboux cyclides}

In this section, we show that a great celestial Darboux cyclide is either a Blum cyclide, Perseus cyclide, ring cyclide, EO cyclide or CO cyclide. Moreover, we show that great ring cyclides are Cliffordian and that great Perseus cyclides are not Cliffordian. The hyperbolic and Euclidean analogues of great Darboux cyclides are considered as well.

We call a Darboux cyclide $X \subset S^3$ \textit{elliptic} or $\delta$-\textit{elliptic} for some $\delta \in \{1, 2\}$, if there exists a Möbius transformation $\varphi \in \text{Aut} S^3$ such that $(\tau \circ \varphi)(X)$ is a hyperquadric in $\mathbb{P}^3$ that is covered by $\delta > 0$ pencils of lines. The $\delta$-\textit{hyperbolic} and $\delta$-\textit{Euclidean} Darboux cyclides are defined analogously, but with $\tau$ replaced with $\nu$ and $\pi$, respectively (recall Remark 2).

Notice that an elliptic Darboux cyclide is Möbius equivalent to a great Darboux cyclide, since great circles are centrally projected to lines.

\begin{lemma}
For all points $p \in \mathbb{P}^4$ there exists a Möbius transformation $\varphi \in \text{Aut} S^3$ such that $\varphi(p)$ coincides with the projection center of either $\tau$, $\nu$ or $\pi$.
\end{lemma}

\begin{proof}
A point $p \in \mathbb{P}^4$ corresponds via the hyperquadric $S^3$ uniquely to its polar hyperplane section $H_p \subset S^3$. Since $H_p$ is Möbius equivalent to either $E$, $Y$ or $U$, there exists $\varphi \in \text{Aut} S^3$ such that $\varphi(p)$ is equal to $(1 : 0 : 0 : 0 : 0)$, $(0 : 0 : 0 : 0 : 1)$ and $(1 : 0 : 0 : 0 : 1)$, respectively. These are the projection centers of $\tau$, $\nu$ and $\pi$, respectively.
\end{proof}

The following classical result is essentially [3, Chapter VII, Theorem 20, page 296], but we followed the proof of [26, Theorem 5.30 in the updated arXiv version]. See [21, Section 2.3] for an alternative proof strategy for Lemma 29 under the additional assumption that the ideal of $X$ is generated by two quadratic forms.
Lemma 29. If $X \subset S^3$ is a surface such that $\deg X \neq 2$ and $F, G \subset X \times \mathbb{P}^1$ are pencils of circles such that $|F_i \cap G_j| = 2$ for almost all $i, j \in \mathbb{P}^1$, then $X$ is a Darboux cyclide that is either elliptic, hyperbolic or Euclidean. Moreover, if $F$ is base point free, then $X$ is either 2-elliptic or 2-hyperbolic.

Proof. The members of $F$ and $G$ are complex plane sections of $S^3$ and thus for almost all $t \in \mathbb{P}^1$ there exist linear forms $f_t, \tilde{f}_t, g_t, \tilde{g}_t \in \mathbb{C}[x_0, \ldots, x_4]$ such that

$$F_t = \text{ZeroSet}(f_t, \tilde{f}_t, s) \quad \text{and} \quad G_t = \text{ZeroSet}(g_t, \tilde{g}_t, s),$$

where $s := -x_0^3 + x_1^2 + x_2^2 + x_3^2 + x_4^2$. Let $p \in \text{ZeroSet}(f_u, \tilde{f}_u, f_v, \tilde{f}_v)$ for some general $u, v \in \mathbb{P}^1_{\mathbb{R}}$ so that the point $p \in \mathbb{P}^4$ is the intersection of the spanning planes of the conics $F_u$ and $F_v$. Since $|F_i \cap G_j| = 2$ by assumption, we deduce that both $F_u \cup G_j$ and $F_v \cup G_j$ are each contained in some complex hyperplane section of $S^3$. Hence, we may assume without loss of generality $g_j$ and $\tilde{g}_j$ lie in the ideals $\langle f_u, \tilde{f}_u \rangle$ and $\langle f_v, \tilde{f}_v \rangle$ of the ring $\mathbb{C}[x_0, \ldots, x_4]$, respectively. This implies that $p \in \text{ZeroSet}(g_j, \tilde{g}_j)$ and thus the complex spanning plane of $G_j$ contains $p$. Repeating the same argument with $F$ and $G$ interchanged shows that the complex spanning plane of $F_i$ contains the point $p$ as well.

By Lemma 28, there exists a Möbius transformation $\varphi \in \text{Aut} S^3$ such that $\varphi(p)$ coincides with the projection center of either $\tau$, $\nu$ or $\pi$.

First suppose that $\varphi(p) = (1 : 0 : 0 : 0 : 0)$. In this case, the general members $F_i$ and $G_j$ are 2:1 projected to complex lines via the map $\tau \circ \varphi$. Since $\deg X \neq 2$ by assumption, we deduce that $\deg(\tau \circ \varphi)(X) = 2$ and thus $X$ is an elliptic Darboux cyclide. If $F$ is base point free, then two general members of $F$ are disjoint. This implies that the 2:1 projections of two general members of $F$ are disjoint lines in the quadric $(\tau \circ \varphi)(X)$. Hence, the ruled quadric $(\tau \circ \varphi)(X)$ must be smooth and thus doubly ruled. This implies that $X$ is 2-elliptic.

If $\varphi(p) = (0 : 0 : 0 : 0 : 1)$, then $X$ must be a hyperbolic Darboux cyclide and if $F$ is base point free, then $X$ is 2-hyperbolic. The proof is analogous as in the elliptic case.

Finally, suppose that $\varphi(p) = (1 : 0 : 0 : 0 : 1)$. In this case, $F$ and $G$ have a common base point at $p$. Thus, the general members $F_i$ and $G_j$ are via the map $\pi \circ \varphi$ birationally projected to complex lines. Since $\deg X \neq 2$, we find that $X$ must
be an Euclidean Darboux cyclide.

We considered all three cases and thus the proof is concluded. \qed

**Lemma 30.** Suppose that $X \subset S^3$ is a celestial Darboux cyclide.

(a) If $X$ is elliptic, then $|\text{Sing } X_R| \in \{0, 2\}$ and $\text{Sing } X_R = (\text{Sing } X) \setminus \mathbb{E}$.

(b) If $X$ is 2-elliptic, then complex conjugate lines in $X$ do not intersect.

(c) The surface $X$ is 1-elliptic if and only if $|\text{Sing } X_R| = 2$ and $X$ is covered by a pencil of circles with two real base points.

**Proof.** (a) We may assume up to Möbius equivalence that $\tau(X)$ is a ruled quadric. Notice that the central projection $\tau$ is a 2:1 covering that defines, with respect to the complex analytic topology, locally a complex isomorphism outside the ramification locus $\mathbb{E}$. This implies that $\tau((\text{Sing } X) \setminus \mathbb{E}) \subset \text{Sing } \tau(X)$. If $|\text{Sing } X_R| = 0$, then $\tau(X)$ must be smooth and thus $\text{Sing } X \subset \mathbb{E}$ so that $\text{Sing } X_R = (\text{Sing } X) \setminus \mathbb{E} = \emptyset$. Now suppose that $|\text{Sing } X_R| > 0$. In this case, $\tau(X)$ must be singular and thus $(\text{Sing } X) \setminus \mathbb{E}$ consist of two real antipodal points that are send via $\tau$ to the vertex of the quadratic cone $\tau(X)$. It follows that $|\text{Sing } X_R| = 2$ and $\text{Sing } X_R = (\text{Sing } X) \setminus \mathbb{E}$ as asserted.

(b) We may assume up to Möbius equivalence that $\tau(X)$ is a doubly ruled quadric. Suppose by contradiction that there exist complex conjugate lines $L, \overline{L} \subset X$ that intersect at some point $p$. Complex conjugate lines in the doubly ruled quadric $\tau(X)$ do not intersect and thus $\tau(L) = \tau(\overline{L})$ so that $p$ is contained in the ramification locus $\mathbb{E}$. We arrived at a contradiction since $p \in X_R$ and $\mathbb{E}_R = \emptyset$.

(c) First, we show the $\Rightarrow$ direction. We may assume without loss of generality that $\tau(X)$ is a quadratic cone and thus $\text{Sing } \tau(X)_R = \{v\}$ for some point $v$. We deduce that there exist antipodal points $p, q \in \text{Sing } X_R$ such that $\tau(p) = \tau(q) = v$. Moreover, there exists a pencil of circles $F \subset X \times \mathbb{P}^1$ with base points $p$ and $q$, and its members are 2:1 centrally projected to lines in $\tau(X)$ that meet at $v$. We know from Assertion (a) that $|\text{Sing } X_R| = 2$.

Next, we show the $\Leftarrow$ direction. We may assume up to Möbius equivalence that the base points $p, q \in X_R$ are antipodal so that the circles in the pencil are 2:1 projected by $\tau$ to lines in $\tau(X)$ that pass through the point $\tau(p) = \tau(q)$. Hence, $\tau(X)$ is a
quadratic cone with vertex \( \tau(p) \), which implies that \( X \) is 1-elliptic.

**Lemma 31.** A ring cyclide \( X \subset S^3 \) is 2-elliptic and not 2-hyperbolic.

**Proof.** By Theorem A (see Table 6) there exist \( g_{12}, g_{34} \in \{ g \in G(X) \mid \sigma_*(g) = g \} \) such that \( g_{12} \cdot g_{34} = 2 \) and \( g_{12} \cdot b = 0 \) for all \( b \in B(X) \), where \( B(X) = \{ b_{13}, b_{24}, b_{14}', b_{23}' \} \).

Hence, we know from Propositions 12(b) and 12(e) that there exist pencils \( F, G \subset X \times \mathbb{P}^1 \) such that \( |F_i \cap G_j| = 2 \) for general \( i, j \in \mathbb{P}^1 \) and \( F \) is base point free. It follows from Lemma 29 that \( X \) is either 2-elliptic or 2-hyperbolic.

Now suppose by contradiction that \( X \) is 2-hyperbolic. We may assume without loss of generality that \( \nu(X) \subset \mathbb{P}^3 \) is a doubly ruled quadric. The restriction of the 2:1 covering \( \nu \) to \( X \) defines outside the ramification locus \( Y \subset S^3 \) a local complex analytic isomorphism on each of the two sheets. Since \( \nu(X) \) is smooth, it follows that \( \text{Sing} \ X \subset Y \). Recall from Example 14 (see the rightmost diagram of Figure 4) that there exist skew complex lines \( L, L' \subset X \) such that \( [L] = e_1, [L'] = e_2, |L \cap L'| = 0 \) and \( |L \cap X| = |L' \cap X| = 2 \). Since \( \mathbb{Y} \cap X \) is a hyperplane section of \( X \) and \( |L \cap \mathbb{Y}| = |L' \cap \mathbb{Y}| = 2 \), it follows from Bézout’s theorem that \( L, L' \subset \mathbb{Y} \). Notice that \( L \) and \( L' \) are complex conjugate lines as \( \sigma^*(\lfloor L \rfloor) = \lfloor L' \rfloor \). We arrived at a contradiction, because \( \mathbb{Y}_R \cong S^2 \) and thus complex conjugate lines in \( \mathbb{Y} \) are not disjoint (see Remark 5). This concludes the proof.

**Proposition 32.** Suppose that \( X \subset S^3 \) is a celestial Darboux cyclide.

(a) If \( X \) is elliptic, then \( X \) is either a Blum cyclide, Perseus cyclide, ring cyclide, EO cyclide, or CO cyclide.

(b) If \( X \) is either a ring cyclide, EO cyclide, or CO cyclide, then \( X \) is elliptic.

**Proof.** (a) First, we suppose that \( |\text{Sing} \ X_R| > 0 \). In this case, \( |\text{Sing} \ X_R| = 2 \) by Lemma 30(a) and thus \( X \) is either a EO cyclide or CO cyclide by Theorem A (see Table 5).

Next, we suppose that \( |\text{Sing} \ X_R| = 0 \). By Theorem A (see Table 5), \( X \) is either a S1, S2, Blum, Perseus or ring cyclide. If \( X \) is a S1 cyclide, then we know from Theorem A (see Table 6) that \( e_{01}, e_{12} \in E(X), \sigma_*(e_{01}) = e_{12} \) and \( e_{01} \cdot e_{12} = 1 \). Similarly, if \( X \) is a S2 cyclide, then \( e_1, e_{11} \in E(X), \sigma_*(e_1) = e_{11} \) and \( e_1 \cdot e_{11} = 1 \).

We apply Propositions 12(c) and 12(d) and find that S1 cyclides and S2 cyclides
contain two complex conjugate lines that intersect. Such cyclides are not elliptic by Lemmas 30(b) and 30(c). Thus if $X_R$ is smooth, then $X$ must be either a Blum, Perseus or ring cyclide.

(b) If $X$ is a ring cyclide, then the proof is concluded by Lemma 31. Now suppose that $X$ is either an EO cyclide or CO cyclide. We know from Theorem A that $|\text{Sing } X_R| = 2$, $b_{12}, b_{34} \in B(X)$, $g_1 \in G(X)$, $\sigma_*(g_1) = g_1$, $\sigma_*(b_{12}) = b_{12}$, $\sigma_*(b_{34}) = b_{34}$ and $g_1 \cdot b_{12} = g_1 \cdot b_{34} = 1$. Hence, $X$ is by Propositions 12(b) and 12(e) covered by a pencil of circles with two real base points. It follows from Lemma 30(c) that $X$ is elliptic.

Proposition 33. If $X \subset S^3$ is a great ring cyclide, then there exist great circles $A, B \subset S^3$ such that $X = P(A \ast B)$.

Proof. We fix a point $e \in X_R$. First, suppose that $R(e)$ equals the identity quaternion in $S^3$.

We know from Theorem A that $|\text{Sing } X_R| = 0$ and $|\text{Sing } X| = 4$. It follows from Lemma 30(c) and Lemma 30(a) that $\tau(X)$ is a doubly ruled quadric and $\text{Sing } X \subset \mathbb{E}$. We apply Lemma 17(a) and find that $X \cap \mathbb{E}$ consist of two left generators $L, \widetilde{L}$ and two right generators $R, \widetilde{R}$. As $\tau(X)$ is doubly ruled, there exist two great circles $A, B \subset R(X)$ such that $e \in P(A) \cap P(B)$. We may assume without loss of generality that the line $\tau(P(A))$ does not belong to the pencil of lines containing $\tau(R)$ and $\tau(\widetilde{R})$. Hence, each circle in the pencil containing $P(A)$ meets both right generators $R, \widetilde{R} \subset \mathbb{E}$.

We assume by contradiction that $X \neq P(A \ast B)$. Let $F \subset P(A \ast B) \times P(B)$ and $G \subset P(A \ast B) \times P(A)$ be the right and left associated pencils of $A \ast B$. Notice that $R(c)$ with $c := (1 : 0 : 0 : 0 : 0)$ is the center of $S^3$ and that $p \ast c = c \ast p$ for all $p \in S^3$. It follows that the left or right Clifford translation of a great circle in $S^3$ is again great. Thus, both $F$ and $G$ have infinitely many members that are great circles on $P(A \ast B)$. These great circles are centrally projected to lines on the doubly ruled quadric $\tau(P(A \ast B))$. This implies that both $F$ and $G$ are base point free. It now follows from Lemma 10(b) that $|F_b \cap R'| = |F_b \cap \widetilde{R'}| = 1$ for almost all $b \in B$ and some complex conjugate right generators $R', \widetilde{R'} \subset \mathbb{E}$. Since $F_e = P(A)$ and $|P(A) \cap R'| = |P(A) \cap \widetilde{R'}| = 1$, we deduce from Lemma 9(b) that $R' = R$ and $\widetilde{R'} = \widetilde{R}$. 

41
Notice that $F_e = \mathbf{P}(A)$ and $G_e = \mathbf{P}(B)$ and thus $\mathbf{P}(A), \mathbf{P}(B) \subset X \cap \mathbf{P}(A \star B)$. We fix some general point $b \in B$. Let $C \subset X$ be the great circle that passes through $b$ and belongs to the same pencil on $X$ as $\mathbf{P}(A)$. Recall that $|C \cap R| = |C \cap \overline{R}| = 1$ as is illustrated in Figure 7. By assumption, $F$ does not cover $X$ and thus $C$ is not a member of $F$. We arrived at a contradiction as the lines $\tau(F_b)$ and $\tau(C)$ span a plane so that the complex lines $\tau(R)$ and $\tau(\overline{R})$ cannot be skew. We established that $X = \mathbf{P}(A \star B)$ for great circles $A, B \subset \mathbf{R}(X)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7}
\caption{See the proof of Proposition 33. The incidences between the great circles $\mathbf{P}(A), \mathbf{P}(B), C, F_b$ and the right generators $R, \overline{R} \subset \mathbb{E}$, under the assumption that $\mathbf{R}(e)$ is the identity quaternion in $S^3$.}
\end{figure}

Finally, suppose that $\mathbf{R}(e)$ is not equal to the identity quaternion in $S^3$. There exists a right Clifford translation $\varphi \in \mathbf{R}T\mathbb{S}^3$ such that $\mathbf{R}(\varphi(e))$ is equal to the identity quaternion. Recall from Proposition 3(d) that $\varphi$ leaves the right generators of $\mathbb{E}$ invariant. The right Clifford translation of a great circle is again great and thus $\varphi(X) = \mathbf{P}(A \star B)$ for some great circles $A, B \subset \varphi(X)$. The unit quaternions $S^3$ form a group and thus there exists $r \in S^3$ such that $\mathbf{R}(\varphi(p)) = \mathbf{R}(p) \star r$ for all $p \in \mathbb{S}^3_\mathbb{R}$. Therefore, $X = \mathbf{P}(A \star B')$, where $B' := \{b \star r^{-1} \mid b \in B\}$. This concludes the proof as $X = \mathbf{P}(A \star B')$ for some great circles $A, B' \subset S^3$.

**Proposition 34.** If $X \subset \mathbb{S}^3$ is a great Perseus cyclide, then $X$ is not Cliffordian, $\tau(X)$ is a doubly ruled quadric, $\text{Sing} \ X \subset \mathbb{E}$ and $X \cap \mathbb{E}$ does not contain lines.

**Proof.** Recall from Example 14 and the leftmost diagram in Figure 4 that the incidences between all the complex lines and complex conjugate isolated singularities are as depicted in Figure 8, where $L, \overline{L}, R, \overline{R}, M, \overline{M}, T, \overline{T} \subset X$ denote the complex lines and $p, \overline{p} \in \text{Sing} \ X$ are the complex conjugate isolated singularities. We know from Lemmas 30(c) and 30(a) that $\tau(X)$ is a doubly ruled quadric and $\text{Sing} \ X = \{p, \overline{p}\} \subset \mathbb{E}$. We claim that none of the complex lines in $X$ are contained in
First suppose by contradiction that $L \subset \mathbb{E}$. In this case the complex conjugate line $\overline{L}$ must also be contained in $\mathbb{E}$. Therefore, $L, \overline{L}, R, \overline{R} \subset \mathbb{E}$ by Bézout’s theorem. It follows again from Bézout’s theorem that $T, \overline{T}, M, \overline{M} \not\subset \mathbb{E}$. We arrived at a contradiction, since $\tau(X)$ contains three complex lines $\tau(R), \tau(L)$ and $\tau(T)$ through the complex point $\tau(p)$ instead of two. We established that $L \not\subset \mathbb{E}$, and by using the same arguments we find that $L, R, \overline{R}, M, \overline{M}, T, \overline{T} \not\subset \mathbb{E}$ as well. Since $X \cap \mathbb{E}$ does not contain complex lines, we conclude from Lemma 19 that $X$ is not Cliffordian.

![Figure 8: Incidences between complex conjugate lines and isolated singularities in a great Perseus cyclide, where $p, \overline{p} \in \mathbb{E}$.](image)

We proceed in Examples 35 and 36 to provide implicit equations for some great celestial Darboux cyclides. This section is concluded with Remarks 37 to 39, namely an analysis of the geometries of great celestial Darboux cyclides by using the introduced methods. The reader may opt to jump directly to §8 at this point.

**Example 35** (great EO/CO cylinders). We consider the following surface

$$X := \tau^{-1}(\{y \in \mathbb{P}^3 \mid \alpha y_0^2 + \beta y_1^2 - y_2^2 = 0\}) = \{x \in S^3 \mid \alpha x_1^2 + \beta x_2^2 - x_3^2 = 0\},$$

for some $\alpha, \beta \in \mathbb{R}_{>0}$. Notice that $\{y \in \mathbb{P}^3 \mid \alpha y_0^2 + \beta y_1^2 - y_2^2 = 0\}$ is a ruled quadric and thus $X$ is great. Suppose that $X' \subset S^3$ is a CO cyclide or EO cyclide. We claim that there exist $\alpha, \beta \in \mathbb{R}_{>0}$ such that $X'$ is Möbius equivalent to $X$ and $\alpha = \beta$ if and only if $X$ is a CO cyclide. We may assume up to Möbius equivalence that the center $p$ of $\pi$ lies in $\text{Sing} X_{\mathbb{R}}$. The Möbius transformations that leave $p$ invariant correspond via $R(\pi(\_))$ to Euclidean similarities in $\mathbb{R}^3$. Hence, it follows from the classical result [15, Proposition 4] that there exists $\varphi \in \text{Aut} S^3$ and $\alpha, \beta \in \mathbb{R}_{>0}$ such that $\varphi(p) = p$ and $(\pi \circ \varphi)(X') = \{y \in \mathbb{P}^3 \mid \alpha y_1^2 + \beta y_2^2 - y_3^2\}$. Moreover, $\alpha = \beta$ if and only if $X'$ is a CO cyclide. Since $\pi(x) = (x_0 - x_4 : x_1 : x_2 : x_3)$, we deduce that $\varphi(X') = \{x \in S^3 \mid \alpha x_1^2 + \beta x_2^2 - x_3^2 = 0\}$ as was to be shown.

$\triangleleft$
Example 36 (great ring/Persues/Blum cyclide). We consider the following surface $X := \tau^{-1}(\{ y \in \mathbb{P}^3 \mid \alpha y_0^2 + y_1^2 - y_2^2 - \beta y_3^2 = 0 \}) = \{ x \in S^3 \mid \alpha x_1^2 + x_2^2 - x_3^2 - \beta x_4^2 = 0 \}$, for some $\alpha, \beta \in \mathbb{R}_{>0}$. We claim that $X \subset S^3$ is a great ring cyclide, great Persues cyclide or great Blum cyclide if $(\alpha, \beta)$ is equal to $(1, 1)$, $(1, 2)$ and $(2, 2)$, respectively. Since $\tau(X)$ is a doubly ruled quadric it follows that $X$ is great. To identify $X$, let us first compute $\text{Sing} \ X$. The Jacobian matrix of the generators of the ideal $\langle \alpha x_1^2 + x_2^2 - x_3^2 - \beta x_4^2, -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 \rangle$ is up to scaling of the rows as follows:

$$
\begin{pmatrix}
0 & \alpha x_1 & x_2 & -x_3 & -\beta x_4 \\
-x_0 & x_1 & x_2 & x_3 & x_4
\end{pmatrix}.
$$

If $(\alpha, \beta) = (1, 1)$, then the Jacobian matrix has rank one at the four complex points $(0 : 1 : \pm i : 0 : 0)$ and $(0 : 0 : 0 : 1 : \pm i)$ in $X \cap E$. Hence, $X$ is a ring cyclide by Theorem A (see the SingType $X$ column in Table 5). If $(\alpha, \beta) = (1, 2)$, then the Jacobian matrix has rank one at the complex points $(0 : 1 : \pm i : 0 : 0)$ in $X \cap E$. Hence, $X$ is a Perseus cyclide by Theorem A. If $(\alpha, \beta) = (2, 2)$, then the Jacobian matrix has rank two at all complex points in $X$ so that $\text{Sing} \ X = \emptyset$. Hence, $X$ is a Blum cyclide by Theorem A and Proposition 32. In Figure 9 we depicted its stereographic projection $R(\pi(X)) = \{ z \in \mathbb{R}^3 \mid (x^2 + y^2 + z^2)^2 - 6x^2 - 4y^2 + 1 \}$. ◄

**Figure 9:** Stereographic projection of a great Blum cyclide (see Example 36).

Remark 37 (great CO cyclide). Suppose that $X \subset S^3$ is a great CO cyclide. Recall from Example 14 that we encoded the corresponding row in Table 6 in terms of the diagram in Figure 10 (left), where $G(X) = \{ g_0, g_1, g_{14}, g_{23} \}$. By Proposition 12(a), the components $\{ b_{34} \}$ and $\{ b_{12} \}$ correspond to real antipodal singularities of $X$ and are centrally projected to the vertex of the quadratic cone $\tau(X)$. The great circles in $X$ have class $g_1$ and form a pencil with real antipodal base points in $\text{Sing} \ X$. The components $\{ b_{13}' \}$ and $\{ b_{24}' \}$ correspond to the complex isolated singularities that lie in the ramification locus $E$. The central projection of these complex singular points are smooth complex branching points in $\tau(X)$. ◄
Figure 10: Incidences between complex lines and isolated singularities on Darboux cyclides (see Example 14). If the Darboux cyclide is great, then a red line is centrally projected to a blue line (see Remark 37).

Remark 38 (great Blum cyclide). Suppose that $X$ is the great Blum cyclide in Example 36 with $(\alpha, \beta) = (2, 2)$. Our goal is to identify, up to $\text{Aut} \ G(X)$, the classes of great and small circles, and pairs $([L],[L'])$ of classes such that $L,L' \subset X$ are complex lines such that $\tau(L) = \tau(L')$. As a byproduct, we recover the compact diagram in Figure 11 from which we can read off the Clifford quartets and the incidences between the complex lines in Blum cyclides. By Theorem A, we may assume up to $\text{Aut} \ G(X)$ that $B(X), E(X)$ and $G(X)$ are as in Table 6. Since $\tau(X)$ is a doubly ruled quadric by Lemma 30(c), there exist great circles $C,C' \subset X$ such that $|C \cap C'| = 2$. By Proposition 12(b), we may assume without loss of generality that $([C],[C']) = (g_{12},g_{34})$ as $g_{12},g_{34} \in \{g \in G(X) \mid \sigma_*(g) = g\}$ and $g_{12} \cdot g_{34} = 2$. If $D,D' \subset X$ are antipodal small circles, then $|D \cap D' \cap \mathbb{E}| = 2$ and thus $([D],[D']) \in \{(g_0,g_3),(g_1,g_2)\}$ by Proposition 12(b). Since the ramification locus $X \cap \mathbb{E}$ of $\tau$ does not contain complex lines, it follows that for all complex lines $L \subset X$, there exists a complex line $L' \subset X$ such that $\tau(L) = \tau(L')$. The ramification locus $\mathbb{E}$ is a hyperplane section, which implies that $|L \cap L' \cap \mathbb{E}| = 1$ and thus $|L \cap L'| = |L \cdot [L']| = 1$ by Proposition 12(d). Let $a := g_{12} \cdot ([L] + [L'])$ and $b := g_{34} \cdot ([L] + [L'])$. Notice that $\tau(L)$ belongs to one of the two rulings of $\tau(X)$ and thus $(a,b)$ is equal to either $(0,2)$ or $(2,0)$. If $(a,b) = (0,2)$, then $([L],[L']) \in \{(e_1,e'_2),(e_2,e'_1),(e_{13},e_{04}),(e_{14},e_{03})\}$ and if $(a,b) = (2,0)$, then $([L],[L']) \in \{(e_{01},e_{12}),(e_{02},e_{11}),(e'_3,e_4),(e'_4,e_3)\}$. See Figure 11 for a diagrammatic representation of these pairs such that two line segments represent complex lines in the same pencil on the doubly ruled quadric $\tau(X)$ if and only if the line segments
are both horizontal or both vertical. The details are left to the reader.

Figure 11: Each line segment is labeled with \([L], [L']\), where \(L\) and \(L'\) are complex lines in a great Darboux cyclide \(X \subset \mathbb{S}^3\) such that \(\tau(L) = \tau(L')\). For each such label, we have \(|L \cap L'| = 1\). If two line segments labeled with \([L], [L']\) and \([M], [M']\) are both horizontal or both vertical, then \(U \cdot V = 0\) for all \(U \in \{[L], [L']\}\) and \(V \in \{[M], [M']\}\). The four Clifford quartets are up to \(\text{Aut} N(X)\) given by \(\{e_1, e_2, e_3', e_4\}, \{e_2', e_1', e_3, e_4\}, \{e_{13}, e_{14}, e_{01}, e_{02}\}, \{e_{04}, e_{03}, e_{12}, e_{11}\}\).

Remark 39 (great Perseus cyclide). Let us describe the geometry of a great Perseus cyclide \(X \subset \mathbb{S}^3\), by identifying the classes of great circles, small circles and complex lines, and the components corresponding to base points. By Theorem A, we may assume up to \(\text{Aut} N(X)\) that
\[
\{g \in G(X) \mid \sigma_*(g) = g\} = \{g_0, g_1, g_{12}, g_2, g_3\}.
\]
By Proposition 12(c), there exist complex lines \(L, \overline{L}, R, \overline{R}, M, \overline{M}, T, \overline{T} \subset X\) such that
\[
[L] = e_{11}, \ [\overline{L}] = e_{12}, \ [R] = e_{01}, \ [\overline{R}] = e_{02}, \ [M] = e_3, \ [\overline{M}] = e_4, \ [T] = e_3', \ [\overline{T}] = e_4'.
\]
By Proposition 12(a), the complex conjugate isolated singularities \(p\) and \(\overline{p}\) in \(\text{Sing} X\) correspond to the components \(\{b_1\}\) and \(\{b_2\}\), respectively. The incidences between the complex lines and isolated singularities are illustrated in Figure 8 (see also Figure 4).

We know from Proposition 34 that \(\tau(X)\) is a doubly ruled quadric, \(p, \overline{p} \in \mathbb{E}\) and \(L, \overline{L}, R, \overline{R}, M, \overline{M}, T, \overline{T} \not\in \mathbb{E}\). It follows that either \(\tau(L) = \tau(R), \tau(L) = \tau(M)\) or \(\tau(L) = \tau(T)\).

First, suppose by contradiction that \(\tau(L) = \tau(R)\). In this case \(\tau(\overline{L}) = \tau(\overline{R})\). This is
a contradiction, since \( \tau(L) \) intersects its complex conjugate line \( \tau(\overline{L}) \) and thus \( \tau(X) \) must be an ellipsoid instead of being a doubly ruled quadric.

Next, we suppose that \( \tau(L) = \tau(M) \). In this case, the complex conjugate lines \( \tau(L) = \tau(M) \) and \( \tau(\overline{L}) = \tau(\overline{M}) \) belong to the first pencil of lines on the doubly ruled quadric \( \tau(X) \). The complex conjugate lines, \( \tau(R) = \tau(T) \) and \( \tau(\overline{R}) = \tau(\overline{T}) \) belong to the second pencil of lines on \( \tau(X) \). By Proposition 12, a circle with class \( g_1 \) meets each of the lines in \( \{L, \overline{L}, M, \overline{M}\} \) and belongs to a base point free pencil. Similarly, a circle with class \( g_2 \) meets each of the lines in \( \{R, \overline{R}, T, \overline{T}\} \) and belongs to a base point free pencil. From this we establish that \( g_1 \) and \( g_2 \) correspond to pencils of great circles that are centrally projected the first and second ruling of the quadric \( \tau(X) \), respectively. A circle \( C' \subset X \) such that \( [C] = g_0 \) or \( [C'] = g_3 \) meets each of the lines in \( \{L, \overline{L}, T, \overline{T}\} \) and \( \{R, \overline{R}, M, \overline{M}\} \), respectively. Therefore, each circle \( C \subset X \) such that \( [C] = g_0 \) is a small circle whose antipodal points form a small circle \( C' \subset X \) such that \( [C'] = g_3 \) and \( \tau(C) = \tau(C') \). A circle with class \( g_{12} \) passes through the complex conjugate isolated singularities \( p \) and \( \overline{p} \). Hence, each circle \( C \subset X \) such that \( [C] = g_{12} \) is a small circle whose antipodal points form a small circle \( C' \subset X \) such that \( [C'] = g_{12} \) and \( \tau(C) = \tau(C') \).

The case \( \tau(L) = \tau(T) \) is analogous to the previous case: \( \{g_0, g_3\} \) are classes of great circles, \( \{g_1, g_2\} \) are the classes of antipodal little circles, and \( g_{12} \) is the class of a small circle that meets \( p \) and \( \overline{p} \). The details are left to the reader. \( \square \)

### 8 Combining the results

In order to prove Theorem 1 we use the following theorem from [15, Theorem 1].

**Theorem B.** If \( X \subset S^3 \) is a \( \lambda \)-circlad surface of degree \( d \) such that \( \lambda \geq 2 \), then either \( X \) is either a Darboux cyclide or \((\lambda, d) \in \{(\infty, 2), (2, 8)\}\).

**Lemma 40.** If \( X \subset S^3 \) is a surface such that \( R(X) \) is a 2-dimensional sphere, then \( X \) is not Cliffordian.

**Proof.** Suppose by contradiction that \( X = P(A \ast B) \) for some circles \( A, B \subset S^3 \). The left Clifford translations correspond to isoclinic rotations of \( S^3 \). Thus the infinitesimal left Clifford translations of points on the circle \( B \) define a nowhere vanishing
vector field on $\mathbb{R} (X) \subset S^3$. We arrived at a contradiction, since a 2-dimensional sphere does not admit such a vector field by the hairy ball theorem.

**Proof of Theorem 1.** Since $Z \subset \mathbb{R}^3$ is $\lambda$-circled and of Möbius degree $d$ such that $(d, \lambda) \neq (8, 2)$, it follows from Theorem B that $d \in \{2, 4\}$. If $d = 2$, then $Z$ is either a plane or a 2-dimensional sphere.

(a) A CY or EY is always Bohemian as it can be obtained by translating a circle along a line (see Figure 1). A plane is the translation of a line along a line. Hence the proof for this assertion is concluded by Proposition 27.

(b) By Lemma 40 we have $d = 4$ and thus the first part follows from Proposition 24, where $X = S(Z)$. It follows from Propositions 32 and 33 that ring cyclides are Möbius equivalent to Cliffordian surfaces.

(c) Direct consequence of Proposition 32, where $X = S(Z)$.

Corollaries 1 and 2 are direct consequences of Theorem 1.

**Proof of Corollary 3.** The central projection of a surface $Z \subseteq S^n$ that is covered by two pencils of great circles is a doubly ruled quadric. Therefore, $n = 3$ and $Z$ has no real singularities. In particular, the stereographic projection $\mu(Z)$ is not a CO cyclide or EO cyclide. Thus the proof is concluded by Theorem 1(c).

**Proof of Corollary 4.** Direct consequence of Propositions 33 and 34.

## 9 Acknowledgements

I would like to thank Mikhail Skopenkov for many insightful and detailed remarks and corrections. The surface figures were generated using [27, Sage]. This work was supported by the Austrian Science Fund (FWF) project P33003.
References

[1] B. Bastl, B. Jüttler, M. Lávicka, T. Schulz, and Z. Šír. On the parameterization of rational ringed surfaces and rational canal surfaces. *Mathematics in Computer Science*, 8:299–319, 2014.

[2] R. Blum. Circles on surfaces in the Euclidean 3-space. In *Geometry and differential geometry (Proc. Conf., Univ. Haifa, 1979)*, volume 792 of *Lecture Notes in Math.* pages 213–221. Springer, 1980.

[3] J. Coolidge. *A Treatise on the Circle and Sphere*. Oxford University Press, 1916. ISBN 3-540-51563-1.

[4] H. S. M. Coxeter. *Non-Euclidean geometry*. Spectrum. MAA, sixth edition, 1998. ISBN 0-88385-522-4.

[5] G. Darboux. Sur le contact des coniques et des surfaces. *Comptes Rendus*, (91):969–971, 1880.

[6] I. V. Dolgachev. *Classical algebraic geometry: A modern view*. Cambridge University Press, Cambridge, 2012.

[7] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[8] T. Ivey. Surfaces with orthogonal families of circles. *Proc. Amer. Math. Soc.*, 123(3):865–872, 1995.

[9] F. Klein. *Nicht-Euklidische Geometrie, II, Vorlesung*. Göttingen, 1893.

[10] J. Kollár. *Lectures on resolution of singularities*, volume 166. Princeton University Press, 2007. ISBN 978-0-691-12923-5; 0-691-12923-1.

[11] R. Krasauskas and S. Zube. Rational bezier formulas with quaternion and clifford algebra weights. *SAGA - Advances in ShApes, Geometry, and Algebra, Geometry and Computing*, 10:147–166, 2014.

[12] E.E. Kummer. Über die Flächen vierten Grades, auf welchen Schaaren von Kegelschnitten liegen. *J. Reine Angew. Math.*, 64(11):66–76, 1863.

[13] P. Lopez-Custodio and J. Dai. Design of a Variable-Mobility Linkage Using the Bohemian Dome. *Journal of Mechanical Design*, 141, 02 2019.

[14] N. Lubbes. Sage library for constructing and visualizing curves on surfaces, 2017. github.com/niels-lubbes/orbital.
[15] N. Lubbes. Surfaces that are covered by two pencils of circles. *Math. Z.*, 299(3-4):1445–1472, 2021.

[16] N. Lubbes. Cyclides, 2022. [github.com/niels-lubbes/cyclides](https://github.com/niels-lubbes/cyclides).

[17] N. Lubbes and J. Schicho. Kinematic generation of Darboux cyclides. *Comput. Aided Geom. Des.*, 64:11–14, 2018.

[18] E. Morozov. Surfaces containing two isotropic circles through each point. *Comput. Aided Geom. Des.*, 90:15, 2021.

[19] M. Peternell. Generalized dupin cyclides with rational lines of curvature. *Lecture Notes in Computer Science*, 6920:543–552, 2012.

[20] H. Pottmann, A. Asperl, M. Hofer, and A. Kilian. *Architectural Geometry*. Bentley Institute Press, 2007. ISBN 978-1-934493-04-5.

[21] H. Pottmann, L. Shi, and M. Skopenkov. Darboux cyclides and webs from circles. *Comput. Aided Geom. Des.*, 29(1):77–97, 2012.

[22] J. Schicho. The multiple conical surfaces. *Beitr. Alg. Geom.*, 42(1):71–87, 2001.

[23] J-P. Serre. *Topics in Galois theory*. Jones and Bartlett Publishers, 1992. ISBN 0-86720-210-6.

[24] J. Siegele, D.F. Scharler, and H.-P. Schröcker. Rational motions with generic trajectories of low degree. *Comput. Aided Geom. Des.*, 76:10, 2020.

[25] R. Silhol. *Real algebraic surfaces*, volume 1392 of *Lecture Notes in Mathematics*. Springer-Verlag, 1989.

[26] M. Skopenkov and R. Krasauskas. Surfaces containing two circles through each point. *Math. Ann.*, 373(3–4), 2018.

[27] W.A. Stein et al. *Sage Mathematics Software*. The Sage Development Team, 2012. [www.sagemath.org](http://www.sagemath.org).

[28] N. Takeuchi. Cyclides. *Hokkaido Math. J.*, 29(1):119–148, 2000.

[29] Y. Villarceau. Theoreme sur le tore. *Nouvelles annales de mathematiques*, 7:345–347, 1848. [eudml.org/doc/95880](http://eudml.org/doc/95880).

[30] M. Zhao, X. Jia, C. Tu, B. Mourrain, and W. Wang. Enumerating the morphologies of non-degenerate Darboux cyclides. *Comput. Aided Geom. Des.*, 75:15, 2019.

**address of author:** Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences

**email:** info@nielslubbes.com