Magnetogenesis from rotating scalar: 
à la scalar chiral magnetic effect

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Abstract

The chiral magnetic effect is a phenomenon that an electric current parallel to the magnetic fields is induced in the presence of the chiral asymmetry in the fermionic system. In this article we point out that the electric current induced by the dynamics of a pseudo scalar field that anomalously couples to electromagnetic fields can be interpreted as a similar effect in the scalar system. Noting that the velocity of the pseudo scalar field, which is the phase of a complex scalar, represents that the system carries a global U(1) number asymmetry, we see that the induced current is proportional to the asymmetry and parallel to the magnetic field, which is the same to the chiral magnetic effect. We discuss that in a mechanism like the Affleck-Dine mechanism an asymmetry carried by the Affleck-Dine field can induce the electric current and give rise to the instability in the (electro)magnetic field if it is unbroken by the expectation value of the Affleck-Dine field. Cosmological consequences of this mechanism, which is similar to the chiral plasma instability, is investigated.

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I. INTRODUCTION

The chiral magnetic effect (CME) [1, 2], from which electric currents are induced by magnetic fields in the presence of chiral asymmetry, has recently received strong interests in broad range of fields of study. Since it is originated from quantum anomalies [3, 4], which are ubiquitous in quantum systems regardless of their energy scales, it has been noticed that it can play important roles in relativistic heavy-ion collisions [2, 5–12], Weyl semimetals [13–19], astrophysical objects such as neutron stars [20–24] and supernovae [25, 26], and so on. Moreover, in the hot early Universe when the chiral asymmetry is a good conserved quantity [27], it has been argued that the CME can cause a tachyonic instability in the (hyper)magnetic fields [28, 29], now known as the chiral plasma instability, which is studied very recently with a full magnetohydrodynamic simulations [30–33]. In these studies, it has been identified that the maximal transfer from the chiral asymmetry to the magnetic helicity is likely to be accomplished. The generated (hyper)magnetic fields are maximally helical and hence they can be the source of the baryon asymmetry of the Universe [34–40]. ¹

In these phenomena, the chiral asymmetry carried by light fermions is essential since it is the origin of the CME. On the other hand, it has been noticed that the chiral anomaly originated from heavy fermions leaves its traces in the low energy effective theory of axions [49–52] in the form of anomalous coupling between the axion and gauge bosons [53].² Then, the background dynamics of axion field also induces an electric current, similar to the CME. It has also been argued that the chiral asymmetry is interpreted effectively as an axion-like scalar degree of freedom [55–57]. Indeed, the cosmological coherent dynamics of axion-like fields are applied for the generation of cosmological magnetic fields during [58–61] and after inflation [62–64], or later times [65].

Once we would like to interpret the magnetic field amplification from the axion-like fields in a similar way to the CME, the dynamics of the axion-like fields can be identified as the non-vanishing chemical potential of the global U(1)_{PQ} symmetry, which is similar to the chiral chemical potential. Then the difference between the chiral plasma instability and magnetic field amplification from the axion-like fields in the literatures [58–65] lies in the

¹ Due to the baryon overproduction [39, 40], it is impossible for this mechanism to be responsible for the intergalactic magnetic fields suggested by the blazar observations [41–48].
² See also the recent discussions in Ref. [54].
conservation of chirality/global U(1)$_{\text{PQ}}$ charge. In the former case, after the generation of chiral asymmetry, it is assumed that the chiral symmetry gets back a good symmetry of the system in the absence of chiral anomaly so that the maximal transfer from the chiral asymmetry to the magnetic helicity is possible. On the contrary, in the latter case, the axion dynamics is induced by the global U(1)$_{\text{PQ}}$ symmetry breaking potential and hence U(1)$_{\text{PQ}}$ charge is not a conserved charge when the magnetic fields are amplified. Thus, in a sense, the magnetic field amplification from the axion-like fields in the literatures is not maximally efficient.

We here note that the anomalous coupling of the axion-like fields is not limited to the QCD axions or axion-like particles appeared in the string theory but is common to pseudo scalar fields in general. One example that takes advantage of the pseudo scalar dynamics in cosmology is the Affleck-Dine (AD) mechanism for baryogenesis \[66, 67\]. In this mechanism, a complex scalar field with baryonic (or leptonic) charge acquires an expectation value in the early Universe, and an explicit baryon (or lepton) number violating interaction gives once the non-vanishing velocity in the phase direction (or the Nambu-Goldstone mode) or the baryon (or lepton) asymmetry. It is implemented that the baryon (or lepton) number violating interaction gets ineffective quickly so that the baryon (or lepton) number becomes a good conserved quantity after its generation. As a consequence, the complex scalar exhibits a coherent rotation in the field space with a constant angular velocity. In this article, we point out that if a complex scalar field charged under à la global U(1)$_{\text{PQ}}$ symmetry whose phase direction has an anomalous coupling to the U(1) gauge fields evolves in a similar way to the AD mechanism, the magnetic fields are amplified through the induced current from a constant global U(1) asymmetry, in which the maximal transfer from the asymmetry to the magnetic helicity is possible. Thus, the mechanism is more efficient than the previous realizations of the magnetogenesis through the axion-like field dynamics and is similar to the chiral plasma instability.

This mechanism has several important messages on the model building of the early Universe cosmology. On the one hand, if the axion-like fields including the Peccei-Quinn-Weinberg-Wilczek (PQWW) axions \[49, 52\] experiences the cosmological evolution like the AD mechanism, it leads to a new mechanism of magnetogenesis. On the other hand, if the phase direction of the AD field has an anomalous coupling to the unbroken U(1) gauge fields, the magnetic fields amplification can occur even in the usual AD mechanism in the
minimal supersymmetric standard model (MSSM) and other supersymmetric extensions of the Standard Model of particle physics (SM), which has not been noticed before. Indeed, we will show that in some flat directions in the supersymmetric SM, the phase direction of the AD fields has the anomalous coupling to unbroken U(1) gauge symmetry and this new mechanism can be realized along the flat directions. This may change the cosmological consequences of the AD mechanism such as the $Q$-ball formation \cite{68-74}. One may wonder if it may spoil the AD mechanism as the baryogenesis mechanism. Indeed, the baryon or lepton asymmetry carried by the AD field is first transferred to the magnetic helicity and it once gets smaller. But the baryon asymmetry is regenerated at the electroweak symmetry breaking through the transfer from the magnetic helicity, as is shown in Ref. \cite{39}. Thus the AD mechanism can still be responsible for the baryon asymmetry, but it is somehow indirect, like the case discussed in Ref. \cite{40}.

This paper is organized as follows. In the next section we will study the cosmological consequences of the complex scalar fields with the anomalous coupling that experiences the evolution like the AD mechanism and determine the resultant magnetic field properties generated by this new mechanism in terms of the model parameters. In Sec. \textbf{III} we discuss the realization and embedding of the system for this mechanism in well-motivated models beyond the SM. Sec. \textbf{IV} is devoted for our concluding remarks and future prospects of this mechanism.

\section{Magnetogenesis from Rotating Scalar in the Field Space}

\subsection{Axion-induced current as the scalar chiral magnetic effect}

First we study a toy model as a low energy effective theory and investigate its cosmological consequences. In the next section we will discuss the realizations of the scenario in the realistic models of physics beyond the SM. Let us consider a simple model of a complex scalar field (à la AD field) with an approximate global U(1)$_A$ symmetry and a massless U(1) gauge field,

\begin{equation}
- \frac{\mathcal{L}}{\sqrt{-g}} = \partial_\mu \phi^* \partial^\mu \phi + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (m_0^2 - c H^2) |\phi|^2 + \lambda_1 \phi^2 + \text{h.c.} \\
+ \left( a_2 \phi^2 + \text{h.c.} \right) + \frac{|\phi|^{2n-2}}{M^{2n-6}} + c_F \frac{e^2}{16\pi^2} \theta F_{\mu\nu} \tilde{F}^{\mu\nu},
\end{equation}

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motivated by the AD mechanism \cite{66, 67}, which catches the essence of our idea. Note that we take that the scalar field $\phi$ is neutral under the U(1) gauge interaction. Here we adopt the metric convention $g_{\mu\nu} = (-, +, +, +)$ and consider the Friedmann background $ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$ with $H = \dot{a}/a$ being the Hubble parameter. We use the dot as the derivative with respect to the physical time $t$. $m_0$ is the zero-temperature mass, $c_H$ is a numerical coefficient of the order of the unity that parameterizes the negative Hubble induced mass, $b_\phi$ and $a_\phi$ parameterizes the small global U(1)$_A$ symmetry breaking terms ($b_\phi$ and $a_\phi$-terms, respectively), and $M$ is the cutoff scale of the higher-order operators. We assume that the scalar field receives the negative Hubble induced mass during and after inflation before reheating and the value of $c_H$ does not change significantly. $b_\phi$ is taken to be a real while $a_\phi$ is taken to be complex without loss of generality. \( \tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}/2\sqrt{-g} \) is the dual tensor with $\epsilon_{\mu\nu\rho\sigma}$ being the Levi-Civita symbol, $\epsilon^{0123} = 1$, $\theta$ is the phase of the scalar field $\phi$, $e$ is the gauge coupling constant, and $c_F$ is the numerical coefficient of the order of the unity for the anomalous coupling. As the phase $\theta$ is the (pseudo) Nambu Goldstone boson associated with spontaneous symmetry breaking of U(1)$_A$ symmetry, it can also be regarded as an axion-like field.

When the Hubble parameter is much larger than the zero-temperature mass, the net mass term is negative and the scalar field gets an expectation value,

$$\phi = \frac{\varphi(t)}{\sqrt{2}} e^{-i\theta(x)}.$$  
(2)

Once the phase of the scalar field acquire the non-zero velocity, $\dot{\theta} \neq 0$, or the scalar field rotates in the field space, due to e.g., the U(1)$_A$ breaking $a_\phi$-term, like in the case of the AD mechanism \cite{66, 67}, this represents the U(1)$_A$ asymmetry is induced in the system,

$$n_A = i (\phi^* \phi - \dot{\phi} \phi^*) = \varphi^2 \dot{\theta}.$$  
(3)

Taking this configuration as the background, it can be easily seen that the equation of motion for the gauge field reads

$$\partial_{\mu}(\sqrt{-g} F^{\mu\nu}) + \sqrt{-g} c_F \frac{e^2}{4\pi^2} (\partial_{\mu}\theta) \tilde{F}^{\mu\nu} = 0, \quad \text{or} \quad -\frac{1}{a^2} \frac{d}{dt}(a^2 E^i) + \frac{\epsilon^{ijk}}{a(t)} \frac{\partial B_k}{\partial x^j} - c_F \frac{e^2}{4\pi^2} \dot{\theta} B^i = 0.$$  
(4)

Thus we can identify that we have an induced current by the number density of the U(1)$_A$ asymmetry,

$$J^i_{\text{ind}} = c_F \frac{e^2}{4\pi^2} \dot{\theta} B^i = c_F \frac{e^2}{8\pi^2} \frac{n_A(t)}{a^3(t) \varphi^2(t)} B^i.$$  
(5)
so that this induced current mimics the chiral magnetic effect \[1, 2\] with a correspondence

\[ \mu_5 \leftrightarrow cF - \frac{n_A}{4a^3(t)} \phi^2. \] (6)

Here the physical electric and magnetic fields are defined as

\[ E^i = a(t) F^{0i}, \quad E_i = a^{-1}(t) F_{i0}, \quad B_i = \frac{a^2(t)}{2} \epsilon_{ijk} F^{jk}, \quad B^i = \frac{a^{-2}(t)}{2} \epsilon^{ijk} F_{jk}. \] (7)

This induced electric current is nothing but the axion-induced current in the axion electromagnetism. We should also note that it has been argued that the chiral magnetic effect is understood as an effective axion field in literatures \[55–57\]. We here just simply emphasize that by relating the axion velocity \( \dot{\theta} \) to the number density of the \( U(1)_A \) asymmetry, the correspondence between the chiral magnetic effect and the axion-induced current is clearer.

Note that the number density of the chiral asymmetry at high temperature \( T \) is given in terms of the chiral chemical potential by \( n_5 = \mu_5 T^2/6 \).

B. Generation of \( U(1)_A \) asymmetry and magnetogenesis in the early Universe

The axion-induced current causes the tachyonic instability on the gauge fields, which is the essence of the axionic inflationary magnetogenesis \[58–60\]. In that case, the non-zero \( \dot{\theta} \) is induced by the potential that strongly breaks the \( U(1)_A \) symmetry and hence the corresponding asymmetry \( n_A \) is not a constant during the course of the magnetic field amplification. Especially during the axion oscillation, subsequent to the slow-roll inflation, \( \dot{\theta} \) changes its sign constantly. Thus the magnetic field amplification is less efficient, and the process is somehow different from the chiral plasma instability \[28–32, 40\]. In contrast, if the scalar field rotation in the field space is induced by a \( U(1)_A \) breaking term that gets ineffective just after its onset, the \( U(1)_A \) asymmetry is an approximate conserved quantity and \( n_A \) or \( \dot{\theta} \) can be taken as a constant, until the backreaction becomes important. In that case, the process is quite similar to the chiral plasma instability. In the following, we investigate the mechanism to generate the \( U(1)_A \) asymmetry in the similar way to the AD mechanism \[66, 67\], and the magnetogenesis from that.

Suppose the Universe has experienced inflation and the inflaton oscillation dominated era with the matter domination like evolution of the scale factor, \( a \propto t^{2/3} \), follows. Here we
adopt the model with Eq. (1), assuming \( m_0 \simeq |a_\phi| \gg \sqrt{b_\phi}\). When the Hubble parameter is large during inflation and during the inflaton oscillation dominated era, \( H > m_0/\sqrt{c_H} \), \( \phi \) field follows the (time-dependent) potential minimum generated by the balance between the negative Hubble induced mass term and the \( |\phi|^{2n-6} \) term, \( \varphi \simeq (HM^{n-3})^{1/(n-2)} \), with a spatially homogeneous distribution. Thanks to inflation, we also suppose that the phase direction \( \theta \) also distributes spatially homogeneously and is taken to be a constant. As the Hubble parameter decreases, eventually the potential minimum disappear at \( H_{osc} \simeq m_0/\sqrt{c_H} \) and the \( \phi \) field starts oscillating coherently around the origin. At the onset of oscillation, the \( a_\phi \)-term gives the kick in the phase direction so that non-zero number density of U(1) charge,

\[
n_A \simeq \varphi_{osc}^2 \dot{\theta}, \quad \text{with} \quad \varphi \simeq \varphi_{osc} \equiv (m_0 M^{n-3})^{1/(n-2)}, \quad \dot{\theta} \simeq m_0, \quad (8)
\]

is generated and the trajectory of the scalar fields in the complex field space is an ellipse with a small eccentricity for \( a_\phi \sim m_0 \) [67]. Here the subscript “osc” indicates that the quantity is evaluated at the onset of the scalar field oscillation. Soon after the onset of oscillation, U(1)-breaking \( a_\phi \)-term gets ineffective quickly and \( n_A \) becomes a good conserved quantity. Then the scalar field evolve as

\[
\varphi \propto a^{-3/2}, \quad \dot{\theta} \simeq m_0 = \text{const.} \quad (9)
\]

as long as the \( b_\phi \)-term is negligible. The former comes from the fact that both the real and imaginary part of the scalar fields are the harmonic oscillator in the matter dominated Universe and damp in proportion to \( t^{-1} \) and the latter is derived from the comoving number density conservation, \( a^3 n_A = a^3 \dot{\varphi} \varphi^2 = \text{const.} \)

Now let us examine how the gauge fields are amplified due to the tachyonic instability and how they backreact to the scalar field dynamics. The equations of motion for the phase

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3 Here we do not take into account the thermal effects [75, 76] to show our idea simply and clearly. In principle, there should be thermal corrections to the scalar potential even before the completion of reheating, since the partial decay of inflaton quanta generates a high temperature plasma as a subdominant component of the Universe. The absence of such thermal corrections are valid if, e.g., the inflaton decays mainly into a hidden sector and the SM particles are not produced. If thermal corrections to the scalar field potential exist, they induce an early onset of the scalar field oscillation, whose eccentricity is larger.
direction of the scalar field and gauge fields are given by

\[ \partial_\mu (\sqrt{-g} \varphi^2 (t) \partial^\mu \theta) - \sqrt{-g} c_\mu F_{\mu \nu} \tilde{F}^{\mu \nu} = 0, \]  
(10)

\[ \partial_\mu (\sqrt{-g} F^{\mu \nu}) + \sqrt{-g} c_\mu e^2 (\partial_\mu \theta) \tilde{F}^{\mu \nu} = 0. \]  
(11)

The latter exhibits the instability of the gauge fields for non-zero background \( \dot{\theta} \). It can be explicitly seen as follows. As long as the phase direction evolves with a homogeneous constant velocity, \( \partial_\mu \theta \simeq (\dot{\theta}, 0, 0, 0) \), with a negligible backreaction, we can take them as a background for the evolution of the gauge fields. Switching from the physical time to the conformal time so that the metric is \( ds^2 = a^2 (\tau) (-d\tau^2 + dx^2) \), the equation of motion for the gauge fields reads

\[ -\frac{\partial^2}{\partial \tau^2} A_i + \sum_j \frac{\partial^2}{\partial x_j^2} A_i + c_F \frac{e^2 a(\tau)}{4\pi^2} \dot{\theta} \sum_{j,k} \epsilon_{ijk} \partial_j A_k = 0, \]  
(12)

where we work in the radiation gauge \( \nabla \cdot A = 0, A_0 = 0 \). To solve the equation of motion, it is convenient to work in the momentum space with performing a Fourier transformation,

\[ A_i (\tau, x) = \sum_{\lambda = \pm} \int \frac{d^3 k}{(2\pi)^{3/2}} \left[ A_\lambda (\tau, k) \epsilon_{i,\lambda} (k) e^{ikx} + \text{h.c.} \right], \]  
(13)

with \( \epsilon_{i,\lambda} (k) \) being the circular polarization tensor that satisfies

\[ \epsilon_{i,\lambda} (k) \cdot \epsilon^*_{i,\lambda'} (k) = \delta_{\lambda,\lambda'} \quad k^i \epsilon_{i,\lambda} (k) = 0, \quad i\epsilon^{ijk} k_j \epsilon_{k,\lambda} (k) = \lambda k \epsilon_{i,\lambda} (k). \]  
(14)

With these decomposition the equation motion for the Fourier modes is rewritten as

\[ -\frac{\partial^2}{\partial \tau^2} A_\lambda (\tau, k) - k^2 A_\lambda (\tau, k) + c_F \frac{e^2 a(\tau)}{4\pi^2} \dot{\theta} \lambda k A_\lambda (\tau, k) = 0. \]  
(15)

We can see that the last term acts as a tachyonic mass term for \( \dot{\theta} \lambda > 0 \) and triggers the instability of gauge fields. For the inflaton oscillation dominated Universe with \( a(t) \propto t^{2/3}, a(\tau) \propto \tau^2 \), the \( \pm \) mode of the gauge field feels instability for \( \dot{\theta} \gtrsim 0 \) at \( k_{\text{ins}} / a(\tau_{\text{ins}}) \simeq c_F e^2 \dot{\theta} / 4\pi^2 \) around \( a(\tau_{\text{ins}}) \tau_{\text{ins}} \simeq 4\pi^2 / (c_F e^2 \dot{\theta}) \) or \( H_{\text{ins}} \simeq c_F e^2 \dot{\theta} / 8\pi^2 \simeq c_F e^2 m_0 / 8\pi^2 \) and grows exponentially. As a result, maximally helical gauge fields are produced. Here the subscript “ins” indicates that the quantity is evaluated at the time when the instability starts to grow. We here assumed that there are no light charged degrees of freedom. If they exist, electric currents are induced like the Schwinger effect. Then the magnetic field amplification would get less efficient \([61]\) and the light particles may be thermalized \([77]\). Hence the process
of gauge field amplification gets more involved, which is beyond the scope of the present article. See also the discussion at the end of Sec. III B.

The amplification of gauge fields stops when the backreaction gets non-negligible. Since by taking the spatial average, Eq. (10) can be understood as the conservation law between the asymmetry and the magnetic helicity,

$$\frac{\partial}{\partial \tau} \left( a^3(\tau) \varphi^2 \dot{\theta} + c_F \frac{e^2}{8\pi^2} h \right) = 0,$$

we can estimate that when

$$c_F \frac{e^2}{8\pi^2} h_{\text{sat}} = \frac{c_F}{V} \frac{e^2}{8\pi^2} \int_V d^3 x \varepsilon^{ijk} A_i \partial_j A_k \simeq a^3(\tau_{\text{ins}}) \varphi^2(\tau_{\text{ins}}) \dot{\theta} \simeq a^3(\tau_{\text{osc}}) \varphi^2(\tau_{\text{osc}}) \dot{\theta}$$

$$\Leftrightarrow h_{\text{sat}} = \frac{8\pi^2}{c_F e^2} a^3(\tau_{\text{osc}}) \varphi^2_{\text{osc}} \dot{\theta}$$

(17)

the amplification of gauge fields get saturated. In other words, magnetic field amplification stops when the maximal transfer from the chiral asymmetry to the magnetic helicity is completed. Here the subscript “sat” indicates that the quantity is evaluated at the time when the gauge field amplification gets saturated. Note that since the instability is an exponential grow, we can approximate that

$$\tau_{\text{sat}} \simeq \tau_{\text{ins}}.$$  Focusing on the magnetic fields, by approximating

$$h_{\text{sat}} \simeq a^2(\tau_{\text{sat}}) A_{\text{sat}} B_{\text{sat}} = a^3(\tau_{\text{ins}}) \frac{B_{\text{sat}}^2}{k_{\text{ins}} / a(\tau_{\text{ins}})} ,$$

where

$$B_{\text{sat}} = \frac{k_{\text{ins}}}{a^2(\tau_{\text{ins}})} A_{\text{sat}},$$

(18)

we obtain the physical magnetic field properties, the magnetic field strength $B$ and coherence length $\lambda_B$, at the time when the gauge field amplification gets saturated as

$$B_{\text{sat}} \simeq \sqrt{\frac{8\pi^2}{c_F e^2} \left( \frac{a(\tau_{\text{osc}})}{a(\tau_{\text{ins}})} \right)^{3/2} \varphi_{\text{osc}} \sqrt{\frac{k_{\text{ins}}}{a(\tau_{\text{ins}})}} \simeq \sqrt{2} \left( \frac{a(\tau_{\text{osc}})}{a(\tau_{\text{ins}})} \right)^{3/2} \varphi_{\text{osc}} \dot{\theta}}$$

$$\simeq \sqrt{2} \left( \frac{H_{\text{ins}}}{H_{\text{osc}}} \right) \varphi_{\text{osc}} \dot{\theta} \simeq 2 \times 10^{12} \text{GeV}^2 \left( \varphi_{\text{osc}} \left( \frac{10^{12} \text{GeV}}{10^3 \text{GeV}} \right) \right) \left( \frac{\dot{\theta}}{10^3 \text{GeV}} \right).$$

$$\lambda_{B,\text{sat}} \simeq \lambda_B(\tau_{\text{ins}}) \simeq 2\pi \left( \frac{k_{\text{sat}}}{a(\tau_{\text{sat}})} \right)^{-1} \simeq 2\pi \left( \frac{c_F e^2}{4\pi^2} \dot{\theta} \right)^{-1} \simeq c_F^{-1} \left( \frac{3}{1\text{GeV}} \right) \left( \frac{\dot{\theta}}{10^3 \text{GeV}} \right)^{-1}.$$  

(19)

Here we take $e \simeq 0.3$. It is noted that $B_{\text{sat}}$ is independent of $c_F$ while $\lambda_{B,\text{sat}}$ is inversely proportional to $c_F$. Note also that in the absence of thermal plasma, the electric fields with a similar amount to the magnetic fields are produced at the same time.
C. Cosmological evolution of magnetic fields

Thus far we have not specified the relationship between the fields in the model to the particle contents of the SM and the arguments are also applicable for the hidden U(1) gauge fields. Let us investigate the cosmological consequences in the case if the gauge fields are those of the U(1) gauge symmetry in the SM. After the saturation of gauge field amplification, the physical magnetic field (as well as the electric field) evolves adiabatically, \( B \propto a^{-2} \) and \( \lambda_B \propto a \), until the SM particles are thermalized and the magnetohydrodynamics becomes important for their evolution [37, 80]. Once the SM particles are thermalized, the electric fields are screened due to the thermal effect while magnetic fields keep their properties. The magnetic fields induces the fluid dynamics and the fluid develops a turbulence. Then both magnetic fields and velocity fields start to co-evolve according to the magnetohydrodynamic equations and follow the inverse cascade process once the eddy turnover scale of the fluid catches up the magnetic field coherence length, \( \lambda_B \simeq v_A t \simeq B/\sqrt{\rho H} \), where \( v_A \) is the Alfvén velocity [78, 79]. The magnetic field further evolve until today according to the magnetohydrodynamics, which determines the linear relation between the magnetic field strength and coherence length today as [78]

\[
\lambda_B(t_0) \sim 1 \text{pc} \left( \frac{B(t_0)}{10^{-14} \text{G}} \right),
\]

where \( t_0 \) is the present physical time. On the other hand, thermal plasma induces a large electric conductivity, which makes the comoving magnetic helicity is a good conserved quantity. Since during the adiabatic evolution it is also conserved, we have the relation

\[
a(t_0)^3 \lambda_B(t_0) B^2(t_0) \simeq a^3(\tau_{\text{ins}}) \lambda_B(\tau_{\text{ins}}) B^2(\tau_{\text{ins}}).
\]

Then we have

\[
\lambda_B(t_0) B^2(t_0) = \left( \frac{a_{RH}}{a(t_0)} \right)^3 \left( \frac{a_{\text{ins}}}{a_{RH}} \right)^3 \lambda_{\text{ins}} B_{\text{ins}}^2
\]

\[
= \frac{g_{ss} T_0^3}{g_{\text{RH}}^2 T_{RH}^3} \left( \frac{H_{RH}}{H_{\text{ins}}} \right)^2 \left( \frac{12 \times 10^{24} \text{GeV}^3}{c_F} \right) \left( \frac{\varphi_{\text{osc}}}{10^{12} \text{GeV}} \right)^2 \left( \frac{\dot{\theta}}{10^3 \text{GeV}} \right)
\]

\[
= 9 \times 10^{-68} \left( \frac{T_{RH}}{10^8 \text{GeV}} \right) \left( \frac{H_{\text{ins}}}{\text{GeV}} \right)^{-2} \left( \frac{12 \times 10^{24} \text{GeV}^3}{c_F} \right) \left( \frac{\varphi_{\text{osc}}}{10^{12} \text{GeV}} \right)^2 \left( \frac{\dot{\theta}}{10^3 \text{GeV}} \right)
\]

\[
= \left( \frac{10^{-35} \text{pcG}^3}{c_F} \right) \left( \frac{T_{RH}}{10^8 \text{GeV}} \right) \left( \frac{H_{\text{ins}}}{\text{GeV}} \right)^{-2} \left( \frac{\varphi_{\text{osc}}}{10^{12} \text{GeV}} \right)^2 \left( \frac{\dot{\theta}}{10^3 \text{GeV}} \right),
\]

(23)
where we have used $g_{**} = 3.91$, $T_0 = 2.3 \times 10^{-13} \text{GeV}$, $1 \text{pc} = 1.56 \times 10^{32} \text{GeV}^{-1}$, and $1 \text{G} = 1.95 \times 10^{-20} \text{GeV}^2$ (in natural Lorentz-Heaviside units) and assumed that at $H = H_{\text{RH}}$, the Universe is filled with relativistic particles with effective temperature $T_{\text{RH}}$ with the energy density and entropy is given by $\rho = (\pi^2 g_{**}^{\text{RH}}/30) T_{\text{RH}}^4$, $s = (2\pi^2 g_{**}^{\text{RH}}/45) T_{\text{RH}}^3$. We have also assumed that the Universe is eventually filled with the SM radiation without additional entropy production. Combining it with Eq. (21), and assuming $c_F \simeq 1$, we obtain the present magnetic field properties,

$$B(t_0) \simeq 10^{-16} \text{G} \left( \frac{T_{\text{RH}}}{10^8 \text{GeV}} \right)^{1/3} \left( \frac{H_{\text{ins}}}{10^3 \text{GeV}} \right)^{-2/3} \left( \frac{\dot{\theta}}{10^3 \text{GeV}} \right)^{1/3} \left( \frac{\varphi_{\text{osc}}}{10^{12} \text{GeV}} \right)^{2/3},$$

(24)

$$\lambda_B(t_0) \simeq 10^{-2} \text{pc} \left( \frac{T_{\text{RH}}}{10^8 \text{GeV}} \right)^{1/3} \left( \frac{H_{\text{ins}}}{10^3 \text{GeV}} \right)^{-2/3} \left( \frac{\dot{\theta}}{10^3 \text{GeV}} \right)^{1/3} \left( \frac{\varphi_{\text{osc}}}{10^{12} \text{GeV}} \right)^{2/3}. \quad (25)$$

Thus the detection of intergalactic magnetic fields with maximal helicity can be a trace of this scenario.

Moreover, we note that the set of fiducial values is suitable for baryogenesis \[39\]. This is not surprising because if there is not a magnetic field amplification and the asymmetry is conserved, the asymmetry-to-entropy ratio is

$$\frac{n}{s} = \left( \frac{a_{\text{osc}}}{a_{\text{RH}}} \right)^3 \left( \frac{\dot{\theta}}{2\pi^2 g_{**}^{\text{RH}}/45} T_{\text{RH}}^3 \right) \sim 10^{-9}, \quad (26)$$

for the fiducial values. In this scenario, if the generated magnetic fields are those of hypergauge interaction, the asymmetry produced by the scalar field dynamics is first transferred to the hypermagnetic helicity, and it is eventually transferred back to the baryon asymmetry at the electroweak symmetry breaking without large loss in the sum of magnetic helicity and $U(1)_A$ asymmetry, as is similar to the case studied in Ref. \[40\]. Even if the electroweak symmetry is broken down to the electromagnetism by the expectation values of the scalar field and the electromagnetic fields are produced in this scenario, they transform into the hypermagnetic fields once when the scalar field decays. Then the same process in the above follows.

### D. Comment on the $b_\phi$-term

Thus far we completely omitted the effect of $b_\phi$-term to avoid the time variation of the $U(1)_A$ asymmetry. However, in the phenomenological point of view, this term is unavoidable
in some realizations as we discuss in the next section. We here discuss how small this term should be for this mechanism to work.

Let us examine the evolution of the scalar fields in more depth after the onset of oscillation. Taking into account the $b_\phi$-term, the mass of the scalar field in the real and imaginary part differs as

$$m_{re/im} = \sqrt{m_0^2 \pm 2b_\phi} \equiv m_0 \pm \Delta m.$$  \hspace{1cm} (27)

When $b_\phi$ is hierarchically smaller than $m_0^2$, $\Delta m \approx b_\phi/m_0 \ll m_0$. Since the evolution of the scalar fields is given by

$$\phi_R(t) \equiv \text{Re}(\phi(t)) \simeq \sqrt{m_0 M} \left( \frac{\cos(m_0 t)}{m_0 t} \right),$$

$$\phi_I(t) \equiv \text{Im}(\phi(t)) \simeq \sqrt{m_0 M} \left( \frac{\sin(m_0 t)}{m_0 t} \right),$$ \hspace{1cm} (28)

then $\dot{\theta}$ is given by

$$\dot{\theta} = \frac{\dot{\phi}_R(t) \dot{\phi}_I(t) - \dot{\phi}_R(t) \phi_I(t)}{\phi_R^2(t) + \phi_I^2(t)} \simeq \frac{m_0 \cos(2\Delta mt) - \Delta m \cos(2m_0 t)}{1 - \sin(2m_0 t) \sin(2\Delta m t)}.$$ \hspace{1cm} (29)

$\dot{\theta}$ evolves with the combination of the oscillation with a longer period $\Delta t_L \simeq (\Delta m)^{-1}$ and the one with a shorter period $\Delta t_S \simeq (m_0)^{-1}$. This means that even if at the onset of oscillation, the trajectory of the scalar in the field space is complete circle, it eventually gets decoherent and $\dot{\theta}$ cannot be taken as a constant any longer for $t \gg \Delta t_L$. However, we can take it as an approximate constant for a shorter period. Let us adopt an ansatz that $\dot{\theta}$ is regarded as a constant if $0.9m_0 < \dot{\theta} < 1.1m_0$. This is when $\sin(\Delta m t) \simeq 0.1$, which corresponds to the time duration when the denominator of Eq. (29) changes with 10%. Thus we take

$$\Delta t_c = 0.1(\Delta m)^{-1}$$ \hspace{1cm} (30)

for the criteria for the duration during when $\dot{\theta}$ can be regarded as a constant.

Since the magnetic field amplification occurs with the time scale

$$\Delta t \sim H_{\text{ins}}^{-1} \simeq \left( c_F \frac{e^2}{16\pi^2} \dot{\theta} \right)^{-1} \simeq \left( c_F \frac{e^2}{16\pi^2} m_0 \right)^{-1}.$$ \hspace{1cm} (31)

Requiring that this is much shorter than $\Delta t_c$, we obtain the constraint on $\Delta m$ as

$$\Delta m < c_F \frac{e^2}{160\pi^2} m_0 \quad \text{or} \quad b_\phi < c_F \frac{e^2}{160\pi^2} m_0^2 \simeq 5 \times 10^{-5} c_F m_0^2.$$ \hspace{1cm} (32)

This gives a constraint on the $b_\phi$-term in the phenomenological model building, which is discussed in the next section.
III. REALIZATION

In this section, we describe how the low energy effective Lagrangian of the form of Eq (I) is realized in the well-motivated models. The idea is completely analogous to the axions. Namely, for the large value of $\varphi \gg m_0$, $a_\phi = O(0.1 - 1 \text{ TeV})$, $U(1)$ gauge charged fields get heavy, $m \sim \varphi$, with which the triangle diagram induces the anomalous coupling of the form $\sim (e^2 \theta / 16\pi^2) F_{\mu\nu} \tilde{F}^{\mu\nu}$. If we can assign an appropriate global $U(1)$ charge to the $\varphi$ field, while it is neutral under $U(1)$ gauge interaction, after integrating out the heavy matter fields, we are left with the low energy effective Lagrangian of the form Eq. (I). Note that the relevant light degrees of freedom are $\theta$, and $U(1)$ gauge field, $A_\mu$. We will demonstrate several examples in the well-motivated models of the physics beyond the SM as proofs of concept, which suggests that such an anomalous coupling and magnetogenesis from that are general features of the AD mechanism and other similar cosmological scenarios.

A. 2 Higgs Doublet Model

The first (clear) example is the angular direction of the Higgs field in the type-II 2 Higgs doublet model (2HDM). Since it is nothing but the PQWW axion or the CP-odd Higgs field, by mapping the global $U(1)_A$ symmetry to the approximate Peccei-Quinn symmetry, $(U(1)_{PQ} : H_1 H_2 \rightarrow e^{-i\beta} H_1 H_2)$, we have the anomalous coupling between the light CP-odd Higgs and the unbroken $U(1)$ gauge fields, $U(1)_{em}$, when the Higgs fields get large expectation values, while all the $U(1)$ gauge charged SM fermions and $W^\pm$ gauge boson get heavy so that we can expect the effective Lagrangian of the form Eq. (I) at the low energy. Let us see in more depth how to realize the situation of our interest in the type-II 2HDM, especially how to realize the coherent motion of the Higgs fields and the vanishingly small $b$-term as discussed in Sec. [I11].

1. Scalar potential

Let us first investigate how to construct the scalar potential in the type-II 2HDM that allows the Higgs field to develop expectation value during inflation with $|H_1| = |H_2| = \varphi/2$ and allows us to identify the $\varphi$ field as the AD(-like) field. The SM gauge charges as well as
PQ charges for the SM fields in the type-II 2HDM are given as Table I, which allows us to determine the Yukawa couplings as

\[- \frac{\mathcal{L}_{\text{Yuk}}}{\sqrt{-g}} = (y_u)_{ij} Q_{Li} u_{Rj}^\dagger H_1 + (y_d)_{ij} Q_{Li} d_{Rj}^\dagger H_2 + (y_e)_{ij} L_{Li} e_{Rj}^\dagger H_2 + \text{h.c..} \]  

The Lagrangian of the Higgs sector

\[- \frac{\mathcal{L}_{\text{Higgs}}}{\sqrt{-g}} = |D_\mu H_1|^2 + |D_\mu H_2|^2 + V(H_1, H_2), \]  

the form of the scalar potential \( V(H_1, H_2) \) is crucial to realize our setup. Note that the PQ symmetry is anomalous under the hypergauge interaction, which is essential for our scenario, as we will see.

There are eight degrees of freedom of the Higgs fields in total, which we characterize in terms of the four complex scalars as

\[ H_1 = \begin{pmatrix} H_1^+ \\ H_1^0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} H_2^0 \\ H_2^- \end{pmatrix}. \]

Indeed, we can construct a scalar potential with a flat direction along a complex scalar degree of freedom among the four, while other six degrees of freedom are heavy enough along the flat direction. To realize such a feature, the scalar potential inspired by the supersymmetric theories can provide a good example. We can borrow some key ideas of the form of scalar potential in the supersymmetric SM for illustration. In the softly-broken supersymmetric SM, two Higgs doubles are naturally introduced. There are three contributions to the Higgs potential, namely, from the \( D \)-term, \( F \)-term, and soft breaking terms. Assuming that the PQ symmetry of the Higgs sector is broken only by the following higher dimensional superpotential,

\[ W_{\text{PQB}} = \frac{(H_1 H_2)^2}{M}, \]

TABLE I: The SU(2)\(_L\) × U(1)\(_Y\) and PQ charge assignment in the SM.
where $M$ is the cutoff scale, the scalar potential of $H_1$ and $H_2$ is obtained as
\[
V_{AD}(H_1, H_2) = m_1^2|H_1|^2 + m_2^2|H_2|^2 + \left( \frac{a_H}{M} (H_1 H_2) + h.c. \right) + \frac{4(|H_1|^2 + |H_2|^2)|H_1 H_2|^2}{M^2}
+ \frac{g^2 + g'^2}{8} (|H_1|^2 - |H_2|^2)^2 + \frac{g^2}{2} |H_1^0 + H_2^0|^2 + |H_1^0 H_2^*|^2,
\]
where the first three terms are the soft supersymmetry (SUSY)-breaking terms with $m_1 \sim m_2 \sim a_H$ being soft breaking parameters of $\mathcal{O}(0.1 - 1\,\text{TeV})$, and the last term in the first line is the $F$-term contribution. The quartic potential in the second line is the $D$-term potential, which gives the approximate flat direction in the unitary gauge:
\[
(H_1)_{D\text{-flat}} = \begin{pmatrix} 0 \\ \frac{1}{2} \varphi e^{-i\theta} \end{pmatrix}, \quad (H_2)_{D\text{-flat}} = \begin{pmatrix} \frac{1}{2} \varphi e^{-i\theta} \\ 0 \end{pmatrix}.
\]
Note that once the Higgs fields develop the expectation values along the flat direction, the coupled quarks and leptons (as well as scalar quarks and leptons, if any) get heavy and their expectation values vanish, and hence we do not have the $F$-term (as well as the $D$-term) contribution from them. Focusing on the flat direction, parameterized by the fields $\varphi$ and $\theta$, we obtain the effective potential of the à la AD field ($\varphi$ and $\theta$) of the form in Eq. (1) (without the Hubble induced mass).

Let us check if the other six degrees of freedom get sufficiently heavy along the flat direction. Along this direction, taking $\varphi \gg m_0$, we can see the splitting of the mass spectrum into heavy modes with masses of $\mathcal{O}(\varphi)$, and light modes as follows. As SU(2)$_L \times$ U(1)$_Y$ is spontaneously broken to U(1)$_{\text{em}}$, and denoting the fields along the flat direction as $\delta H$, three scalar degrees, $G^0 \equiv \text{Im}(\delta H_1^0 - \delta H_2^0)$ and $G^+ \equiv \delta H_1^+ - \delta H_2^{*-}$, $G^- \equiv G^{*+}$ are eaten by $Z^0$ and $W^\pm$, and become massive with masses $g\varphi/2$ and $\sqrt{g^2 + g'^2}\varphi/2$, respectively. One of the CP-even Higgs degrees of freedom, $H^0 \equiv \text{Re}(\delta H_1^0 - \delta H_2^0)$ and the charged Higgs components, $H^+ \equiv \delta H_1^+ + \delta H_2^{*-}$ and $H^- \equiv H^{*+}$, are also heavy with masses $\sqrt{g^2 + g'^2}\varphi/2$ and $g\varphi/2$ at the leading order. The scalar fields $\varphi$ and $\theta$ get masses only from soft terms and a higher dimensional operator and hence they are much lighter than the above six scalar degrees of freedom. Note that the $\theta$ field is nothing but the CP-odd Higgs or the PQWW axion.

The negative Hubble induced mass terms for $H_1$ and $H_2$ can be added as
\[
\Delta V_{\text{Hubble}} = -c_1 H_1^2|H_1|^2 - c_2 H_2^2|H_2|^2,
\]
by supposing, e.g., the non-minimal couplings to gravity, $-\xi_1 R|H_1|^2 - \xi_2 R|H_2|^2$ with $R$ being the Ricci scalar, or non-trivial Kähler potential between the inflaton and the Higgs.
doublets in the supersymmetric case \[67, 81\]. Note that the Ricci scalar is \( R = O(H^2) \) during inflation and matter dominated Universe.

In Eq. (37), we did not address the \( b_H \)-term potential presented in Eq. (1) (i.e. \( \Delta V = b_H H_1 H_2 + \text{h.c.} \)). A sizable \( b_H \)-term is dangerous for generation of the magnetic fields as discussed in Sec. II D. However, if \( b_H \) is much smaller than \( m_1^2 + m_2^2 \), as required (see Eq. (32)), the value of \( \langle |H_2^0| \rangle \) at the present Universe is too small to be realistic because \( \langle |H_2^0| \rangle / \langle |H_1^0| \rangle \approx |b_H| / (m_1 + m_2)^2 \). This leads to non-perturbatively large Yukawa couplings to obtain correct masses of down quarks and charged leptons, \( m_{d/e} = y_{d/e} \langle |H_0^2| \rangle \). One way to avoid this problem and give more freedom to the \( b_H \)-term is to consider the case where the \( b_H \)-term in the present Universe is dominated by the vacuum expectation value of a scalar field as \( b_H \sim \langle S^2 \rangle = O(m_1^2 + m_2^2) \), by introducing a gauge singlet PQ charged complex scalar field, \( S \), while the soft-breaking \( b_H \)-term contribution is vanishingly small. Let us consider the following potential for the \( S \) field,

\[
\Delta V_{b-term} = (m_S^2 + \kappa_1 |H_1|^2 + \kappa_2 |H_2|^2)|S|^2 + (\kappa H_1 H_2 S^2 + a_S^2 S + \text{h.c.}) + \frac{\lambda_S S^4}{4} |S|^4. \tag{40}
\]

Here \( |m_S| \sim |a_S| = O(m_1^2 + m_2^2) \), \( \kappa, \kappa_1, \kappa_2 \), and \( \lambda_S \) are parameters of the order of the unity, and \( a_S \) is the soft PQ breaking parameter, which allows \( S \) field develops an expectation value of order of 0.1 – 1 TeV in the vacuum to give the \( b_H \)-term to the Higgs doublets. When the Higgs field develops the expectation values along the flat direction, \( H_1 \simeq H_2 \sim \varphi \), \( S \) becomes heavy with a mass of \( O(\varphi) \), and its vacuum value shifted by \( a_S \)-term is quite suppressed as \( \langle S \rangle \sim a_S^2 / \varphi^2 \ll m_0 = \sqrt{(m_1^2 + m_2^2)/2} \). The resulting \( b_H = \kappa S^2 \sim a_S^2 / \varphi^4 \) is much smaller than \( m_0^2 \) so that an effective magnetic field generation is allowed. As the \( \varphi \) field value decreases and becomes \( O(m_0) \), then \( \langle S \rangle \sim m_0 \), and \( b_H \sim m_0^2 \), so the PQWW axion becomes heavy with a mass of \( O(m_0) \), and can be safe from various constraints at the present Universe.

We would like to emphasize that the scalar potential we suggest in this section is a proof of concept, in which a flat direction (\( |H_1| = |H_2| \)) exists and \( b_H \)-term is dynamical, which is suitable for our magnetogenesis scenario. Clever ideas are welcome and desirable in order to provide more natural set-up for our mechanism. See App. A for a concrete example to realize the \( H_1 H_2 \) flat direction without a bare \( b_H \)-term in a supersymmetric extension of the SM (\( H_1 \rightarrow H_u \), and \( H_2 \rightarrow H_d \)).
2. Effective action with light degrees of freedom

Let us now see how the anomalous coupling $\sim (e^2/16\pi^2)\theta F_{\mu\nu}\tilde F^{\mu\nu}$ is obtained in the low energy effective Lagrangian. Here we focus on the non-supersymmetric theory although we use the SUSY-inspired potential. When the Higgs fields obtain large field values along the flat direction $\varphi \gg m_0$, we can divide the fields, not only the Higgs field described in the above but also the matter and gauge fields, into the heavy fields whose masses are proportional to $\varphi$, and the light fields which are massless or obtain masses at most with the soft breaking scales. The former includes the quarks, leptons except for the neutrinos, weak gauge bosons, and heavy Higgs fields, as well as the singlet scalar $S$, if any, and the latter includes the gluons, (electromagnetic) photon, neutrinos, and light Higgs field (the à la AD field). In the unitary gauge, the Lagrangian density for the light fields is

$$-\frac{\mathcal{L}_{\text{light}}}{\sqrt{-g}} = \frac{1}{2} \text{Tr} G_{\mu\nu} G^{\mu\nu} + i \bar{\psi}_L \sigma^\mu \partial_\mu \psi_L + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{\varphi^2}{2} (\partial_\mu \theta)^2$$

$$+ \frac{1}{2} (m_0^2 - c_H H^2) \varphi^2 + \frac{a_\phi}{8M} \varphi^4 \cos(4\theta - \theta_A) + \frac{\varphi^6}{8M^2},$$

where $a_\phi = |a_\phi| e^{i\theta_A}$, $m_0^2 = (m_1^2 + m_2^2)/2$, $c_H = -(c_1 + c_2)/2$. $m_0^2$ can be naturally positive even if $m_1^2 m_2^2 < 0$ in order for electroweak symmetry breaking at the present Universe. The Lagrangian density for the heavy fields up to quadratic order is given as

$$-\frac{\mathcal{L}_{\text{heavy}}}{\sqrt{-g}} = \frac{1}{2} (\partial_\mu H^0)^2 + \frac{1}{2} \left( \frac{g^2}{4} \right) (H^0)^2 + (\partial_\mu H^+) (\partial^\mu H^-) + \frac{(g^2 + g^2) \varphi^2}{4} H^+ H^-$$

$$+ \frac{1}{2} (\partial_\mu S_R)^2 + \frac{1}{2} \left( m_S^2 + \frac{(\kappa_1 + \kappa_2 + \kappa)}{4} \right) S_R^2 + \sqrt{2} a_S S_R$$

$$+ \frac{1}{2} (\partial_\mu S_I)^2 + \frac{1}{2} \left( m_S^2 + \frac{(\kappa_1 + \kappa_2 - \kappa)}{4} \right) S_I^2 + \bar{\psi}_{ui} \left( i \gamma^\mu D_\mu + \frac{y_{ui}}{2} \varphi e^{i\gamma_5 \theta} \right) \psi_{ui}$$

$$+ \bar{\psi}_{di} \left( i \gamma^\mu D_\mu + \frac{y_{di}}{2} \varphi e^{i\gamma_5 \theta} \right) \psi_{di} + \bar{\psi}_{ei} \left( i \gamma^\mu D_\mu + \frac{y_{ei}}{2} \varphi e^{i\gamma_5 \theta} \right) \psi_{ei}$$

$$+ \frac{1}{2} W^+_{\mu\nu} W^{\mu\nu} - \frac{g^2 \varphi^2}{4} W^+_{\mu\nu} W^{\mu\nu} - \frac{1}{4} Z^0_{\mu\nu} Z^{\mu\nu} + \frac{(g^2 + g^2) \varphi^2}{8} Z^0_{\mu\nu} Z^{\mu\nu},$$

where Dirac fermions are constructed as

$$\psi_{ui} = \begin{pmatrix} u_{Li} \\ u_{Ri} \end{pmatrix}, \quad \psi_{di} = \begin{pmatrix} d_{Li} \\ d_{Ri} \end{pmatrix}, \quad \psi_{ei} = \begin{pmatrix} e_{Li} \\ e_{Ri} \end{pmatrix},$$

with using the chiral representation for the Dirac matrices, $S = (S_R + i S_I)/\sqrt{2}$, and for simplicity $\kappa$ and $a_S$ are taken to be real. The unbroken gauge group is SU(3)$_C \times$U(1)$_{em}$, and
the corresponding covariant derivative is given by

\[ D_\mu = \partial_\mu - ig_\psi T^a_\psi G^a_\mu - ieq_\psi A_\mu, \]

where \( T^a_\psi \) is the generator of SU(3)_C, and \( q_\psi \) is the EM charge for a given fermion \( \psi \). For quarks, \( T^a_{u,d} = \lambda^a/2 \), where \( \lambda^a \) are Gell-Mann matrices, and for charged leptons, \( T^a_e = 0 \). The EM charges \( (q_\psi) \) are \( q_u = 2/3 \), \( q_d = -1/3 \), and \( q_e = -1 \). We ignore the interaction between \( \theta \) and \( S \) because it does not cause any effects of our interest.

For the low energy scale much less than \( \varphi \), the effective action can be obtained by integrating out heavy fields. Since the expectation values of heavy fields vanish due to the heavy mass from the flat direction, basically they do not leave any traces, but expect for the anomalous coupling and the threshold correction. While the latter can be absorbed by the redefinition of model parameters, the former should be added explicitly to the Lagrangian. It is derived by calculating one-loop triangle diagrams mediated by heavy fermions \( (\psi^u_i, \psi^d_i, \psi^e_i) \) so that

\[
-\mathcal{L}_{\text{anom}} = \frac{2N_f g_\psi^2}{16\pi^2} \theta \text{Tr} G^\mu_\nu \tilde{G}^{\mu\nu} + \frac{N_f e^2}{16\pi^2} \left( 3q_u^2 + 3q_d^2 + q_e^2 \right) \theta F^\mu_\nu \tilde{F}^{\mu\nu}
\]

\[
= \frac{3g_\psi^2}{8\pi^2} \theta \text{Tr} G^\mu_\nu \tilde{G}^{\mu\nu} + \frac{e^2}{2\pi^2} \theta F^\mu_\nu \tilde{F}^{\mu\nu}.
\]

Here \( N_f \) is the number of heavy families, and we take \( N_f = 3 \) for the SM. The appearance of such anomalous terms can be understood by noting that the flat direction is charged under PQ symmetry, which is anomalous under the U(1)_em and SU(3)_C, and all the PQ charged fermions are heavy along the flat direction. Now we reach at the low energy effective Lagrangian of the form Eq. \( \text{[11]} \) and hence we conclude that the magnetogenesis from the rotating flat direction in the type-II 2HDM with a AD-like mechanism can take place. Note that all the U(1)_em charged particles are massive and hence we do not have to worry about the induced current by the Schwinger effect.

### B. LH_u flat direction in supersymmetric SM

In the previous section, we utilize some of the properties of supersymmetric SM just to justify a part of the scalar potential in the type-II 2HDM but do not take into account any SUSY fields. In this section, we shall consider the supersymmetric extension of the SM more seriously, as is adopted in the AD mechanism. In the MSSM, or
extended supersymmetric SMs, there are many scalar fields, namely, the SUSY partners of the SM fermions such as squarks and sleptons, which exhibit many flat directions \(^{82}\), along which the scalar potential vanishes except for the SUSY-breaking effects and contributions from non-renormalizable operators. Scalar fields can develop expectation values along a flat direction to cause the AD mechanism.

Let us focus on the \(LH_u\) flat direction as a proof of concept, which has been often used for the AD leptogenesis \(^{83}\). In order to make the scalar dynamics simpler, we will consider a flat direction only governed by a slepton with a single flavor \(f\), \(\tilde{L}_{L_f}\), and \(H_u^4\), while \(H_d\) and other scalar fields do not develop non-zero field values. Hereafter we use the tilde for supersymmetric partners. Such a condition can be easily realized, e.g., in next-to-minimal-supersymmetric Standard Model (NMSSM) with a superpotential, 

\[
W_{\text{NMSSM}} = y_u Q_L^c u^c_R H_u + y_d Q_L^c d^c_R H_d + y_e L_L^c e^c_R H_d + \lambda S H_u H_d + \frac{1}{2} m_S S^2 + \frac{1}{3} \kappa S^3. \tag{46}
\]

It can be easily seen that with the configuration

\[
(H_u)_{D\text{-flat}} = \begin{pmatrix}
0 \\
\frac{1}{2} \varphi e^{-i\theta}
\end{pmatrix}, \quad (\tilde{L}_{L_f})_{D\text{-flat}} = \begin{pmatrix}
\frac{1}{2} \varphi e^{-i\theta} \\
0
\end{pmatrix}, \tag{47}
\]

the \(D\)-term potential as well as the \(F\)-term potential for the \(\varphi\) and \(\theta\) fields vanishes and their potential is lifted only from the SUSY-breaking effect as well as the Hubble induced terms which give them “light” masses of orders of the soft SUSY-breaking mass, \(m_{\text{soft}} = \mathcal{O}(\text{TeV})\), and the Hubble scale, respectively. The scalar fields, \(H_d\) and \(S\), acquire “heavy” masses along the flat direction from the \(F\)-term as \(\sim \lambda \varphi\) and are trapped at the origin so that we can integrate them out from the low energy effective theory. Taking \(H_d = S = 0\) while keeping the \(H_u\) and \(\tilde{L}_{L_f}\) fields explicitly, the form of their scalar potential is the same as Eq. (37) by replacing \((H_1, H_2)\) to \((H_u, \tilde{L}_{L_f})\). Since \(H_d\) and \(L_L\) have the same SM gauge charges, this clearly shows that the expectation value of the \(\varphi\) field breaks the SM \(SU(2)_L \times U(1)_Y\) symmetry down to the \(U(1)_{\text{em}}\) symmetry so that three scalar modes in the \(H_u\) and \(\tilde{L}_{L_f}\) fields other than the \(\varphi\) and \(\theta\) fields are absorbed by vector bosons. Similarly, one CP

\(^4\) Note that multiple flavors of sleptons \(^{84}\) as well as the \(H_d\) field \(^{85}\) can co-exist with the \(L_{L_f}H_u\) flat direction, which lead to possible multiple field dynamics in the AD mechanism. Indeed, in the MSSM, due to the \(\mu\)-term and \(B\mu\)-term, the expectation value of \(H_d\) field is induced along the \(L_{L_f}H_u\) flat direction. See App. B for the detail.
even and one complex field become also heavy from $D$-term potentials, $\approx g\varphi$. As a result their field values can be safely set to be zero, and, again, they can be integrated out from the low-energy effective theory. The low energy effective scalar potential along the $D$-flat direction, parameterized by the $\varphi$ and $\theta$ fields, is same as that of Eq. (41). The difference compared to the non-supersymmetric type-II 2HDM studied in the previous section is the additional fermionic degrees of freedom: Higgsinos ($\tilde{H}_u, \tilde{H}_d$) and gauginos ($\tilde{W}^a, \tilde{B}$), and a pattern of the fermion mass splitting. While all the charged fermions get massive in the 2HDM case, there remain massless charged fermions in the $L_{Lf}H_u$ flat direction case. The Yukawa interactions of the charged fermions which get masses from the expectation values of $H_u$ and $\tilde{L}_{Lf}$ are given by

\begin{equation}
\left( y_{ui}H_u^0u_Li\gamma^5R_i + g(H_u^0)^*\tilde{W}^-\tilde{H}_u^+ + y_{ef}\tilde{L}_{Lf}^0\tilde{H}_d^+e_{Rf}^c + g(\tilde{L}_{Lf}^0)^*e_{Lf}\tilde{W}^+ \right) + \text{h.c.} \tag{48}
\end{equation}

In the unitary gauge, the corresponding Lagrangian density for the heavy fermions can be written as

\begin{equation}
-\frac{\mathcal{L}_{\text{heavy}}}{\sqrt{-g}} = \tilde{\psi}_{ui} \left( i\gamma^\mu D_\mu + \frac{y_{ui}}{2}\varphi\epsilon^{i\gamma_5}\theta \right) \psi_{ui} + \tilde{\psi}_{H_u} \left( i\gamma^\mu D_\mu + \frac{g}{2}\varphi\epsilon^{-i\gamma_5}\theta \right) \psi_{H_u} + \tilde{\psi}_{H_d} \left( i\gamma^\mu D_\mu + \frac{y_{ef}}{2}\varphi\epsilon^{i\gamma_5}\theta \right) \psi_{H_d} + \tilde{\psi}_W \left( i\gamma^\mu D_\mu + \frac{g}{2}\varphi\epsilon^{-i\gamma_5}\theta \right) \psi_W \tag{49}
\end{equation}

where the Dirac fermions $\psi$ are defined as

\begin{equation}
\psi_{ui} = \begin{pmatrix} u_{Li} \\ u_{Ri}^c \end{pmatrix}, \quad \psi_{H_u} = \begin{pmatrix} \tilde{W}^- \\ \tilde{H}_u^+ \end{pmatrix}, \quad \psi_{H_d} = \begin{pmatrix} \tilde{H}_d^- \\ e_{Rf}^c \end{pmatrix}, \quad \psi_W = \begin{pmatrix} e_{Lf} \\ \tilde{W}^+ \end{pmatrix}. \tag{50}
\end{equation}

Note that $\psi_{H_d}$ and $\psi_W$ become heavy due to the non-zero $\langle \tilde{L}^f \rangle = \varphi/2$. They have same electromagnetic charges ($q_e = -1$), but couple to the axion oppositely, so integrating them out does not yield the low energy coupling between the axion and photons. This is consistent with the fact that lepton number is not anomalous under $U(1)_{em}$. There is no such a kind of cancellation between $\psi_{ui}$ and $\psi_{H_u}$ ($q_u = 2/3, q_{H_u} = q_e = -1$), which yields the low energy couplings as

\begin{equation}
-\frac{\mathcal{L}_{\text{anom}}}{\sqrt{-g}} = \frac{N_fg_s^2}{16\pi^2} \eta\text{Tr}G_{\mu\nu}\tilde{G}^{\mu\nu} + \frac{e^2}{16\pi^2} \left( N_f 3q_u^2 - q_e^2 \right) \theta F_{\mu\nu}\tilde{F}^{\mu\nu} \nonumber
\end{equation}

\begin{equation}
= \frac{3g_s^2}{16\pi^2} \eta\text{Tr}G_{\mu\nu}\tilde{G}^{\mu\nu} + \frac{3e^2}{16\pi^2} \theta F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{51}
\end{equation}

\footnote{The absence of other flavor of sleptons can be justified by supposing the positive Hubble induced mass for them.}
Once more, we have used \( N_f = 3 \). Since \( \langle H_d \rangle = 0 \) during evolution of the \( \varphi \) field, three \( d \)-quark pairs \( \psi_{d_{i=1,2,3}} = (d_L d_R^i) \), and two charged lepton pairs \( \psi_{e_{i\neq f}} = (e_L e_R^i) \) are massless. Because in our field basis those light charged fermions only couple to \( H_d \), not \( H_u \) and \( \tilde{L}_f \), there is no coupling between the axion and massless fermions. This can be seen by assigning \( U(1)_{A'} \) charges to the fermion fields including the axion \( \theta \) as

\[
Q_\psi \equiv -\frac{5}{4}B - \frac{7}{4}L - Y - q_{em} + \frac{1}{4}q_{PQ}, \tag{52}
\]

so that the charge of axion is one, \( Q_\theta = 1 \), but the electromagnetic charged massless fermions are neutral. Here the PQ charge assignments \( q_{PQ} \) are given in Table II. Since this \( U(1)_{A'} \) contains the PQ charge, it is clear that it is anomalous under \( SU(3)_C \times U(1)_{em} \).

| Fields | \( Q_L \) | \( u_R \) | \( d_R \) | \( L_L \) | \( e_R \) | \( H_u \) | \( H_d \) |
|--------|---------|--------|--------|--------|--------|--------|--------|
| \( SU(2)_L \) | 2       | 1      | 1      | 2      | 1      | 2      | 2      |
| \( U(1)_Y \) | 1/6     | -2/3   | 1/3    | -1/2   | 1      | 1/2    | -1/2   |
| \( U(1)_{PQ} \) | 1       | 1      | 1      | 1      | -2     | -2     | 4      |

**TABLE II: The \( SU(2)_L \times U(1)_Y \) and PQ charge assignment in the NMSSM**

By supposing higher dimensional operators

\[
W_{NM} = \frac{(L_{Lf}H_u)^2}{M_f} + \cdots , \tag{53}
\]

where \( \cdots \) denotes the higher dimensional operators for other lepton flavors (but with omitting them) and the negative Hubble induced mass term for \( \tilde{L}_f \) and \( H_u \) in the same way to the 2HDM case, the final low energy Lagrangian density for the light fields is given by

\[
-\frac{\mathcal{L}_{eff}}{\sqrt{-g}} = \frac{1}{2} \text{Tr} G_{\mu \nu} G^{\mu \nu} + i \bar{\nu} \sigma^\mu \partial_\mu \nu + \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} (\partial_{\mu} \phi)^2 + \frac{\varphi^2}{2} (\partial_{\mu} \theta)^2 \\
+ \frac{1}{2} (m_0^2 - c_H H^2) \varphi^2 + \frac{|a_\phi|}{8M_f} \varphi^4 \cos(4\theta - \theta_A) + \frac{\varphi^6}{8M_f^2} \\
+ \sum_{i=1}^3 \bar{\psi}_{di} i \gamma^\mu D_\mu \psi_{ui} + \sum_{i=1}^2 \bar{\psi}_{ei} i \gamma^\mu D_\mu \psi_{ei} - \frac{\mathcal{L}_{\text{anom}}}{\sqrt{-g}}, \tag{54}
\]

where we have imposed the \( a_\phi \)-term while the \( b_\phi \)-term is absent since lepton number breaking is prohibited at the renormalizable level. Thus we reach at the effective Lagrangian of the form of Eq. (1), but massless \( U(1)_{em} \) charged particles also exist.
We can take different field basis, by $\theta$ dependent chiral transformation of $d$-quarks and charged leptons. Then the axion photon couplings can be removed through the chiral anomaly. Instead, the axion-current interactions are generated.\textsuperscript{6} Therefore this coupling is important for both generation of gauge fields and helicity of the fermions, which has also been discussed in the context of inflationary magnetogenesis in Ref. [61]. Note that the fermion production through the axion-current interaction has also been studied recently in Refs. [89–92].

Note that since $W_{NM}$ breaks a lepton number and becomes the source of the neutrino masses as the Weinberg operator, there is the lower bound on $M$ from the upper bound on the neutrino masses $\sum m_\nu \approx \mathcal{O}(0.1$ eV). On the other hand, since we do not know the lower bound on the lightest neutrino mass, a very large value of $M_f$ is allowed. For example, in order to have the fiducial value for magnetogenesis, $\varphi_{\text{osc}} \simeq 10^{12}$ GeV, studied in Sec. \textsection III for $m_0 \simeq 10^4$ GeV, $M_f \simeq 10^{20}$ GeV, we require a tiny neutrino mass $m_{\nu_f} \sim 10^{-7}$ eV.

Let us comment on the effects of the existence of massless charged particles on magnetogenesis. Through the chiral anomaly, once helical magnetic fields are generated from the dynamics of the rotating scalar, fermions with chiral asymmetry will be also generated, by satisfying $\Delta h \simeq (e^2/16\pi^2)\Delta n_5$, with $n_5$ being the number density of the chiral asymmetry. Moreover, through the Schwinger effect, non-chiral particles can be also generated, which can lead to thermalization of the charged particles [77]. As is discussed in Ref. [61], these effects will suppress the efficiency of magnetogenesis. Thus we might not have magnetic helicity as much as evaluated in Sec. \textsection III. However, in the case of standard chiral plasma instability, the numerical MHD studies have shown that the full transfer of chiral asymmetry to the magnetic helicity is possible even in the fully thermalized system [30–33]. From these observations, we expect that the full transfer of the scalar asymmetry to the magnetic helicity can be accomplished even in our case in the existence of the light particles as well as the thermal plasma. For the concrete conclusion, nevertheless further investigation is needed, which is left for the future study.

In this subsection we have focused on the LH\textsubscript{u} flat direction just for a concrete example as a proof of concept, but we expect that similar effects can be seen in other flat direction in

\textsuperscript{6} The coupling between the phase of the AD field and currents has been used for the realization of spontaneous baryogenesis in Refs. [86–88]. Our discussion suggests that magnetic fields are also produced in these setups.
the supersymmetric SM including the MSSM because it is often the case that there remain an unbroken U(1) gauge symmetry along a flat direction. For example, in the case of $udd$ flat direction, a linear combination of the hyper gauge field, and the third and eighth gluons is unbroken and its anomalous coupling to the phase direction of the flat direction is expected.

In this section, we show that the new mechanism of magnetogenesis studied in Sec. II can be easily realized in the PQWW axion dynamics as well as the usual AD mechanism. As described in the introduction, our findings have two important messages. Namely, 1) by supposing a cosmic history like the AD mechanism, axions can generate magnetic fields efficiently. 2) In some cases the AD mechanism experiences the magnetic field generation, which requires to consider the scenario carefully. Since we have studied only some of simplified situations to show the proof of concept of this idea, further studies are needed to give precise and quantitative consequences of this effect.

IV. DISCUSSION

In this work, we studied the evolution of the U(1) gauge fields that have an anomalous coupling to the phase of a rotating complex scalar field, which is often realized in cosmology in the context of the AD mechanism. The existence of such an anomalous coupling is not surprising since the phase of the AD field can be identified as an axion. Compared to the magnetogenesis from axion dynamics, where the axion oscillates around the CP-violating potential, our new mechanism of magnetogenesis is remarkable in a sense that the CP-violating effect are important only at the onset of the dynamics in the phase direction and are absent during most of its dynamics. As a result, only one helicity mode of the gauge fields receives tachyonic instability continuously, which is the source of efficient magnetogenesis so that the complete transfer from the asymmetry carried by the scalar fields to the magnetic helicity is possible. This is in contrast to the magnetogenesis from the oscillating axions, where the asymmetry carried by the axions are not conserved and hence the complete transfer from the asymmetry to the magnetic helicity is not possible. The similarity between the chiral magnetic effect and the axion-photon coupling has been pointed out, but the mechanism studied in this work has a much closer analogy to the chiral plasma instability, where the chiral magnetic effect induces the instability of the magnetic fields.
It is not trivial if such a situation can be realized in the well motivated models of physics beyond the SM. As a proof of concept, we identified that the PQWW axion in the type-II 2HDM as well as the phase of the $LH_u$ flat direction, often adopted in the AD leptogenesis, can play the role of the phase of the rotating scalar for this new magnetogenesis scenario. In order to avoid the problems caused by the $b_\phi$-term, we adopted a singlet extension of the (MS)SM, but we expect that the magnetogenesis from the phase of a rotating scalar field is unavoidable general phenomena of the AD mechanism even in the MSSM and other similar mechanisms, which has not been recognized before.

In order to evaluate the consequences of magnetogenesis, we employed relatively simplified setup, namely, we assumed that there is no thermal plasma during the scalar field dynamics and omitted the effects of possibly existing light charged particles. The former triggers the early onset of the scalar field rotation, which makes $\dot{\theta}$ is not a constant during the oscillation. The latter implies the induction of the electric current, which correspond to the Schwinger effect in the vacuum and just the Ohm’s current in the thermal plasma. It will screen the electric field and suppresses the efficiency of the magnetogenesis. Due to the chiral anomaly the estimate of the induced current would be quite involved. Since the purpose of the present work is demonstrate the existence of such a magnetogenesis process in the AD mechanism, a popular scenario in the early Universe, here we do not go into the detail but postpone them for the future study.

One may wonder if the anomalous coupling of the AD field can play important role in later times. Especially, one may expect it can cause a new channel of $Q$-ball decay, since this process breaks the global U(1) symmetry that guarantees the stability of $Q$ balls. However, while the size of a $Q$ ball is inverse of the phase velocity of the AD field, the instability scale is larger than that by a factor of inverse of the fine structure constant. Therefore we conclude the $Q$-ball decay triggered by the anomalous coupling is not so efficient, but it may be interesting to explore in depth.

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**Appendix A: \(H_u H_d\) flat direction in supersymmetric SM**

In this appendix, we construct a non-minimal supersymmetric extension of the SM where the \(H_u H_d\) field configuration exhibits a flat direction with an induced \(B\mu\)-term \((i.e., \text{the potential of the form } B\mu H_u H_d + \text{h.c.})\). Note that in the NMSSM discussed in Sec. IIB the \(H_u H_d\) field configuration is no longer flat at the renormalizable level due to the \(F\)-term of \(S\).

In supersymmetric extension of the SM, \(\mu\)-term can be given by the vacuum value of the scalar field, \(S\). Let us consider the following superpotential:

\[
W = y_u Q_u \bar{c}^c R_u + y_d Q_d \bar{c}^c R_d + \frac{S^2 H_u H_d}{M_1} + \frac{S^3 S_c}{M_2} + \frac{(H_u H_d)^2}{M_3},
\]

where we have introduced two gauge singlets \(S\) and \(S_c\). Here \(M_1 M_2,\) and \(M_3\) are very large values compared to the weak scales. The PQ charge assignment of \(S\) and \(S_c\) are given by Table III. Note that \((H_u H_d)^2\) term breaks the PQ symmetry explicitly. The scalar potential

\[
V = \left( m_s^2 + \frac{4|H_u H_d|^2}{M_1^2} \right) |S|^2 + 9|S|^6 + \left( m_{sc}^2 + \frac{|S|^4}{M_2^2} \right) |S_c|^2 + \left[ \frac{A_2}{M_2} S^3 \right] S_c + \text{h.c.} + \left[ \frac{A_1}{M_1} S^2 + \frac{6|S|^2 (SS_c)^*}{M_1 M_2} + \frac{2(|h_u|^2 + |h_d|^2) (S^2)^*}{M_1 M_3} \right] h_u^0 h_d^0 + \text{h.c.} + \left( m_{H_d}^2 + \frac{|S|^4}{M_1^2} \right) |h_d^0|^2 + \left( m_{H_u}^2 + \frac{|S|^4}{M_1^2} \right) |h_u^0|^2 + \frac{A_3}{M_3} (h_u^0 h_d^0)^2 + 4\left( \frac{|h_u|^2 + |h_d|^2| h_u^0 h_d^0|^2}{M_3^2} \right) \frac{g^2 + g'^2}{8} \left( |h_u^0|^2 - |h_d^0|^2 \right)^2.
\]
At the present Universe, assuming $m_S^2 < 0$, $m_{S^c}^2 > 0$ and neglecting the contribution from the Higgs, we get a large vacuum value of $S$ and $S^c$ by the following relevant scalar potentials:

$$V(S) = -|m_S|^2|S|^2 + \frac{9}{M_2^2}|S|^6 + m_{S^c}^2|S^c|^2 + \frac{2A_2(|S|^3)\cos(3\theta_S + \theta_{S^c})}{M_2}|S^c| + \ldots$$

$$\Rightarrow \langle |S| \rangle = \mathcal{O}(\sqrt{|m_S|/M_2}), \quad \langle |S^c| \rangle \sim \mathcal{O} \left( \frac{A_2|m_S|}{m_{S^c}^2} \langle |S| \rangle \right). \quad (A3)$$

Here the vacuum expectation value of $S$ is induced by the negative mass and that of $S^c$ is induced by the $A_2$-term. The large value of $M_2$ ensures the large expectation values of these fields, which justify the omission of the Higgs fields in evaluating them. Then the expectation value of $S$ dynamically induces $\mu$-term and $B\mu$-term as

$$\mu = \frac{\langle S^2 \rangle}{M_1} \sim |m_S| \frac{M_3}{M_1}, \quad B\mu \sim A_1 m_S \frac{M_3}{M_1}. \quad (A4)$$

We naturally assume $M_1 \sim M_2$, so that $\mu \sim |m_S|$, $B\mu \sim A_1 |m_S|$ is realized in the vacuum.

On the other hand, for $|h_u^0| = |h_d^0| = \varphi/2 \sim \sqrt{m_0 M_3} \gg m_{\text{soft}}$ when the scalar field dynamics in the phase direction takes place, the mass term of $S$ is dominated by the induced term $\varphi^4|S|^2/M_1^2 \sim (M_3/M_1)^2 m_0^2 |S|^2$ over the negative zero-temperature mass $m_S^2 |S|^2$ and gets positive if $M_1^2 \sim M_2^2 \ll M_3^2$. Then it is trapped at the origin so that the $B\mu$-term is absent. Eventually, $S$ will roll to its non-zero vacuum value once $\varphi$ field value gets small enough and the induced mass gets smaller than the negative zero-temperature mass well after the period of our interest, that is, the generation of magnetic fields.

**Appendix B: $H_d$ field configuration along the $LH_u$ flat direction in the MSSM**

In this appendix, we study the configuration of $H_d$ field along the $LH_u$ flat direction in the MSSM, in which $\mu$ and $B\mu$-terms are given by constants. We here show that $H_d$ field gets expectation value induced by the $LH_u$ flat direction, which makes the anomalous coupling of the phase direction of the $LH_u$ flat direction to the photon vanish.

We consider the following superpotential of the MSSM and lepton number violating higher dimensional operators,

$$W = y_u Q_L u^c_R H_u + y_d Q_L d^c_R H_d + y_e Q_L e^c_R H_d + \mu H_u H_d + \frac{(L_L f_R H_u)^2}{M}. \quad (B1)$$

Keeping in mind that $LH_u$ flat direction and $H_u H_d$ flat direction can coexist [82, 85], let us
parameterize the relevant scalar degrees of freedom along the \( LH_u \) flat direction as

\[
\tilde{L}_{L_f} = \begin{pmatrix} \tilde{\nu} \\ 0 \end{pmatrix}, \quad H_u = \begin{pmatrix} 0 \\ \tilde{h}_u^0 \end{pmatrix}, \quad H_d = \begin{pmatrix} \tilde{h}_d^0 \\ 0 \end{pmatrix}.
\]  

(B2)

Then the scalar potential from \( D \)-terms, \( F \)-terms, soft breaking terms, and the Hubble induced mass term is

\[
V = (m_L^2 + c_L H^2)|\tilde{\nu}|^2 + (m_{H_u}^2 + \mu^2 + c_u H^2)|h_u^0|^2 + (m_{H_d}^2 + c_d H^2)|h_d^0|^2 \\
+ \left( \frac{A_u}{M} \tilde{\nu}^2 h_u^0 + B \mu h_u^0 h_d^0 \right) + \frac{4|h_u^0|^4|\tilde{\nu}|^2}{M^2} + \frac{2\tilde{\nu}^2 h_u^0 + \mu h_d^0}{M^2} \\
+ \frac{g^2 + g'^2}{8} \left( |h_u^0|^2 - |h_d^0|^2 - |\tilde{\nu}|^2 \right)^2,
\]

(B3)

where \( |c_u|, |c_d|, |c_L| = O(1) \). We can easily see that the field configuration of the pure \( LH_u \) flat direction (\(|h_u^0| = |\tilde{\nu}|\)) with \(|h_d^0| = 0\) is impossible since there is the tadpole potential for \( h_d^0 \), induced by SUSY breaking \((B\mu\langle h_u^0 \rangle h_d^0)\) and supersymmetric \((\mu\langle \tilde{\nu}^2 h_u^0 \rangle h_d^0/M)\) contributions. Therefore we have to estimate how \( \langle h_d \rangle \) can be large along the \( LH_u \) direction.

For the large values of \(|h_u^0|\) and \(|\tilde{\nu}|\) compared to soft SUSY breaking masses, the \( D \)-term potential makes one scalar degree heavy, so we can integrate out the corresponding field through the equations of motion. Parameterizing the scalar field amplitudes as

\[
|h_u^0| = \varphi_u, \quad |h_d^0| = \varphi_d, \quad |\tilde{\nu}| = \varphi_t.
\]

(B4)

for \( m^2, H^2 \ll \varphi_u^2 \ll \varphi_d^2 \ll \varphi_t^2 \ll M^2 \), which is realized for the negative Hubble induced mass for \( L \) and \( H_u \), \( \varphi_t \) is determined by the \( D \)-flat condition,

\[
\varphi_t^2 \simeq \varphi_u^2 - \varphi_d^2.
\]

(B5)

By imposing this \( D \)-flat condition, the potential for \( \varphi_u \) and \( \varphi_d \) as well as the gauge invariant phase fields, \( \theta_H \) and \( \theta_L \), defined as

\[
h_u^0 h_d^0 = \varphi_u \varphi_d e^{-i\theta_H}, \quad \tilde{\nu} h_u^0 = \varphi_u \varphi_t e^{-i\theta_L},
\]

(B6)
is given by

$$
V_{\text{eff}}(\varphi_u, \varphi_d) = \left[ m^2_{H_d} + \mu^2 - m^2_L + (c_d - c_L)H^2 \right] \varphi_d^2 \\
+ \left[ 2B\mu \varphi_d \cos \theta_H - \frac{4\mu \varphi_d^3 \cos(\theta_H - 2\theta_L)}{M} \right] \varphi_u \\
+ \left[ m^2_L + m^2_{H_u} + \mu^2 + (c_L + c_u)H^2 - \frac{2A\nu \varphi_d^2 \cos(2\theta_L)}{M} + \frac{4\varphi_d^4}{M^2} \right] \varphi_d^2 \\
+ \left[ \frac{4\mu \varphi_d \cos(\theta_H - 2\theta_L)}{M} \right] \varphi_u^3 + \left[ \frac{2A\nu \cos 2\theta_L}{M} - \frac{12\varphi_d^2}{M^2} \right] \varphi_u^4 + \frac{8\varphi_u^6}{M^2}.
$$

(B7)

Here we have assumed that all constant parameters are real for simplicity.

For $m \ll H$, with a reasonable assumption:

$$
m^2_L + m^2_{H_u} + \mu^2 + (c_L + c_u)H^2 < 0,
$$

(B8)

$\varphi_u$ gets a finite vacuum value as

$$
V(\varphi_u) \sim (c_L + c_u)H^2 \varphi_u^2 + \frac{8}{M^2} \varphi_u^6 \Rightarrow \langle \varphi_u \rangle = c\sqrt{HM},
$$

(B9)

with $c = \mathcal{O}(1)$ whereas $\varphi_d \ll \varphi_u$. By inserting this to the potential, supposing $c_d - c_L - 12c^4 > 0$, the dominant contribution for the vacuum value of $\varphi_d$ is given by

$$
V(\varphi_d) \sim (c_d - c_L - 12c^4)H^2 \varphi_d^2 + \frac{4\mu \langle \varphi_u^3 \rangle}{M} \cos(\theta_H - 2\theta_L) \varphi_d \Rightarrow \langle \varphi_d \rangle = \mathcal{O} \left( \frac{\mu}{H} \langle \varphi_u \rangle \right).
$$

(B10)

Note that the contributions from the $\mu$-term is stronger than those from the $B\mu$-term. The angular field, $\theta_H$ also get a mass squared of $\mathcal{O}(\mu H \langle \varphi_u \rangle/\langle \varphi_d \rangle) \sim H^2$, so that $\theta_H$ is also heavy and follows the slow-rolling $\theta_L$ as $\langle \theta_H \rangle = 2\theta_L + \pi$.

As $H$ decreases and crosses the value of $\mathcal{O}(\mu)$, the field value of $\varphi_u$ gets around $\sqrt{\mu M}$. Then the contribution of $B\mu$-term is no longer negligible for the potential of the $\varphi_d$ field so that

$$
V(\varphi_d) \sim (m^2_{H_d} + \mu^2 - m^2_L)^2 \varphi_d^2 - (B\mu \varphi_u(t) \cos \theta_H) \varphi_d \Rightarrow \langle \varphi_d \rangle \sim \varphi_u(t).
$$

(B11)

Thus we find that $\langle \varphi_d \rangle$ becomes same order of $\varphi_u$. Now the dynamics of $\theta_H$ is governed by the $B\mu$-term, which gives a constant heavy mass of $\mathcal{O}(\sqrt{B\mu})$. Then the $\theta_H$ will exhibits the damped oscillation around $\pi$. Therefore while the phase of $LH_u$ rotates in the same way as the usual AD leptogenesis, $H_uH_d$ rotation will be quickly damped away. Since all the electromagnetic charged fermions that are massless in the pure $LH_u$ flat direction case, such
as $d$ quarks, acquire heavy masses from the $H_d$ field value, the anomalous coupling between the phase of $LH_u$ flat direction and photons is cancelled in the low energy effective theory. Now we have found that the dynamical phase $\theta_L$ does not have the anomalous coupling to photons and another phase $\theta_H$, which has the anomalous coupling, no longer shows the constant velocity, we conclude that in the MSSM with a bare $B\mu$-term the magnetogenesis does not happen unless the $B\mu$-term is sufficiently suppressed as discussed in Sec. II D. Note that in Ref. [85] the $B\mu$-term is not taken into account. This is the reason why $\dot{\theta}_H$ becomes constant but not is damped after the onset of scalar field oscillations around the origin there.

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