Universal Rise of Hadronic Total Cross Sections based on Forward $\pi p$ and $\bar{p}p(pp)$ Scatterings

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Abstract

Recently there are several evidences of the increase of the total cross section $\sigma_{\text{tot}}$ to be $\log^2 s$ consistent with the Froissart unitarity bound, and the COMPETE collaborations in the PDG have further assumed $\sigma_{\text{tot}} \simeq B \log^2(s/s_0)$ to extend its universal rise with a common value of $B$ for all the hadronic scatterings. However, there is no rigorous proof yet based only on QCD. Therefore, it is worthwhile to prove this universal rise of $\sigma_{\text{tot}}$ even empirically. In this letter we attempt to obtain the value of $B$ for $\pi p$ scattering, $B_{\pi p}$, with reasonable accuracy by taking into account the rich $\pi p$ data in all the energy regions. We use the finite-energy sum rule (FESR) expressed in terms of the $\pi p$ scattering data in the low and intermediate energies as a constraint between high-energy parameters. We then have searched for the simultaneous best fit to the $\sigma_{\text{tot}}$ and $\rho$ ratios, the ratios of the real to imaginary parts of the forward scattering amplitudes. The lower energy data are included in the integral of FESR, the more precisely determined is the non-leading term such as $\log s$, and then helps to determine the leading terms like $\log^2 s$. We have derived the value of $B_{\pi p}$ as $B_{\pi p} = 0.311 \pm 0.044 \text{mb}$. This value is to be compared with the value of $B$ for $p\bar{p}, pp$ scattering, $B_{pp}$, in our previous analysis $[11]$, $B_{pp} = 0.289 \pm 0.023 \text{mb}$. Thus, our result appears to support the universality hypothesis.

Key words: sum rules, total cross section, $\rho$ ratio
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Purpose of this Letter

It is well known that the increase of the total cross sections $\sigma_{\text{tot}}$ is at most $\log^2 s$ as the Froissart unitarity bound $[12]$. Recently, there have been several evidences $[3,4,5,6,7]$ to support the increase of $\sigma_{\text{tot}}$ to be $\log^2 s$. The COMPETE collaborations $[4,7]$ have further assumed $\sigma_{\text{tot}} \simeq B \log^2(s/s_0)$ to extend...
the universal rise of all the total cross sections. That is, they took a common value of $B$ to fit all the data of $\bar{p}p(pp)$, $\pi^\pm p$, $K^\pm p$, $\Sigma^- p$, $\gamma p$ and $\gamma \gamma$ scatterings and this resulted in reducing the number of adjustable parameters. The universality of the coefficient $B$ was expected in the paper[8], and other theoretical supports[9,10] based on the arguments describing deep inelastic scattering by gluon saturation in hadron light-cone wavefunction (the Colour Glass Condensate[11] of QCD) were given in recent years. But there has been no rigorous proof yet based only on QCD.

Therefore, it is worthwhile to prove this universal rise even empirically. In the near future, the $pp$ total cross section $\sigma_{pp}^{tot}$ will be measured at the LHC energy ($\sqrt{s} = 14$TeV ) in TOTEM experiment. Therefore, the value of $B$ for $\bar{p}p, pp$ scattering, $B_{pp}$, will be determined with good accuracy. On the other hand, the $\pi p$ total cross sections $\sigma_{\pi p}^{tot}$ have been measured only up to $k=610$GeV, where $k$ is the laboratory momentum of $\pi$ and it corresponds to $\sqrt{s}=33.8$GeV, by the SELEX collaboration[13]. Thus, one might doubt to obtain the value of $B$ for $\pi p$ scattering, $B_{\pi p}$, with reasonable accuracy.

The purpose of this letter is to attack this problem and to compare the values of $B_{pp}$ and $B_{\pi p}$ in a new light. We can use the rich informations of the experimental $\sigma_{tot}$ data in the low energy regions through the finite-energy sum rule (FESR). We adopt the FESR with the integral region between $k = \overline{N}_1$ and $\overline{N}_2$[14] as a constraint between high-energy parameters, and analyze the $\pi^\pm p$ total cross sections $\sigma_{\pi^\pm p}^{tot}$ and $\rho$ ratios $\rho_{\pi^\pm p}$, the ratios of real to imaginary parts of the forward scattering amplitudes. This FESR requires that the low-energy extension of the high-energy asymptotic formula should coincide, roughly speaking, with the average of experimental $\sigma_{tot}$ in the relevant region between $k = \overline{N}_1$ and $\overline{N}_2$. This is called FESR duality. We have already used[12] this sum rule between $\overline{N}_1=10$GeV and $\overline{N}_2=20$GeV. The rich data in $k < 10$GeV were not included in this case, however. The lower energy data are included in the integral of $\sigma_{tot}$, the more precisely determined is the sub-leading term, i.e., the $P'$ term (the term with coefficient $\beta_{P'}$ in Eq. (3)), which is built in the sense of FESR[15,16,17] by the sum of direct channel resonances. Then, it helps to determine the non-leading term such as $\log s$ which then helps to determine the leading term like $\log^2 s$. Thus, in the present work we extend maximally the energy region of the input data to take $\overline{N}_1 \leq 10$GeV, so as to obtain the value of $B_{\pi p}$ as most accurately as possible.

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1 In our previous work[12], we also used the FESR of $P'$ type[15,16,17] which includes the integral of $\sigma_{tot}$ from the $\pi p$ threshold, that is, $\overline{N}_1 = 0$ GeV. However, this sum rule needs one subtraction, and the subtraction term $-F^{(+)}(0)$ should have been added to the LHS of Eq.(9) in ref.[12]. The sum rule (14) for $\pi^\pm p$ in the same reference should be slightly modified since $F^{(+)}(0) = 0$ has been assumed implicitly. In the present analysis we do not use this implicit assumption.
The $\bar{p}p$ scattering has open (meson) channels in the so-called unphysical regions with $\sqrt{s} < 2M$ ($M$ being the proton mass), and it may cause some trouble in applying the FESR. A possible solution for $\bar{p}p$ will be discussed later in Eq. (7). In contrast, there are no such effects in $\pi p$ scattering. Thus, we can take into account more resonances through FESR in order to obtain the low-energy extension from the high-energy side with good accuracy. To obtain a sufficiently small error of $B_{\pi p}$, it appears to be important to include the information of the low-energy scattering data with $0 \leq k \leq 10$GeV through FESR.

We will show that the resulting value of $B_{\pi p}$ is consistent with that of $B_{pp}$, which appears to support the universality hypothesis.

**Analysis of Forward $\pi^\mp p$ Scattering**

In the following, we use the laboratory energy of the incident pion, denoted as $\nu$, instead of the center of mass energy squared, $s$. They are related through

$$s = 2M\nu + M^2 + \mu^2$$  \hspace{1cm} (1)

with each other where $M (\mu)$ is proton(pion) mass. By using the variable $\nu$, a crossing transformation is expressed exactly by $\nu \rightarrow -\nu$ in forward scattering amplitudes.

We take both the crossing-even and crossing-odd forward scattering amplitudes, $F(\pm) (\nu)$, which are defined from forward $\pi^\mp p$ scattering amplitudes $f_{\pi^\mp p} (\nu)$ by

$$F(\pm) (\nu) = (f_{\pi^- p} (\nu) \pm f_{\pi^+ p} (\nu)) / 2.$$  \hspace{1cm} (2)

We assume

$$\text{Im} F(\pm) (\nu) \simeq \frac{\nu}{\mu^2} \left( c_0 + c_1 \log \frac{\nu}{\mu} + c_2 \log^2 \frac{\nu}{\mu} \right) + \frac{\beta_{\nu^\prime}}{\mu} \left( \frac{\nu}{\mu} \right)^{\alpha_{\nu^\prime}}$$ \hspace{1cm} (3)

$$\text{Im} F(-) (\nu) \simeq \frac{\beta_{\nu}}{\mu} \left( \frac{\nu}{\mu} \right)^{\alpha_{\nu}},$$ \hspace{1cm} (4)

which are expected to be valid in the asymptotically high-energy region above some energy $\nu \geq N$. $\nu$ is related with $k$ by $\nu = \sqrt{k^2 + \mu^2}$, and the momentum corresponding to $\nu = N$ is represented by the quantity with overline such as $k = \overline{N}$ in this letter. The imaginary parts are related to the total cross sections $\sigma_{\text{tot}}^{(\pm)}$ by the formula $\text{Im} F(\pm) (\nu) = \frac{k}{4\pi} \sigma_{\text{tot}}^{(\pm)}$, and $\sigma_{\pi^\mp p}^{\pi^\mp p}$ is given by $\sigma_{\text{tot}}^{\pi^\mp p} = \sigma_{\text{tot}}^{(+) \pm} \sigma_{\text{tot}}^{(-) \mp}$. These formulas (3) and (4) are derived by traditional
Pomeron-Reggeon exchange model except for the terms with coefficients $c_2$ and $c_1$. The coupling coefficients $\beta_{P'}, c_0, \beta_V$ are the unknown parameters in the Regge theory. The $\alpha_{P'}, \alpha_V$ are determined phenomenologically by the intercepts of Regge trajectories of $f_2(1275), \rho(770)$. The $c_2, c_1$ terms are introduced consistently with Froissart bound to describe the rise of $\sigma_{tot}$ in high-energy regions.

By using the crossing property $F^{(\pm)}(-\nu) = \pm F^{(\pm)}(\nu)^*$, the real parts are given by

$$\text{Re}F^{(\nu)}(\nu) \simeq \frac{\pi \nu}{2 \mu^2} \left( c_1 + 2c_2 \ln \frac{\nu}{\mu} \right) - \frac{\beta_{P'}}{\mu} \left( \frac{\nu}{\mu} \right)^{\alpha_{P'}} \cot \frac{\pi \alpha_{P'}}{2} + F^{(+)}(0), \quad (5)$$

$$\text{Re}F^{(-)}(\nu) \simeq \frac{\beta_V}{\mu} \left( \frac{\nu}{\mu} \right)^{\alpha_V} \tan \frac{\pi \alpha_V}{2}, \quad (6)$$

where $F^{(+)}(0)$ is a subtraction constant. The equations (5) and (6) are used in fit to $\rho$ ratios, $\rho_{\pi^+p} = \text{Re} f_{\pi^+p}/\text{Im} f_{\pi^+p}$.

We fit the experimental $\sigma_{tot}^{\pi^+p}$ and $\rho_{\pi^+p}$ ratios simultaneously. The $c_{2,1,0}, \beta_{P',V}$ and $F^{(+)}(0)$ are parameters, while the $\alpha_{P'} \simeq \alpha_V$ is taken to be the empirical value $\simeq 0.5$. The $B_{\pi p}$ is related with the dimensionless parameter $c_2$ by $B_{\pi p} = \frac{4\pi}{\mu^2}c_2$ in unit of mb.

The FESR is used as a constraint between these parameters \cite{14,12}.

$$\frac{2}{\pi} \int_{N_1}^{N_2} \frac{\nu}{k^2} \text{Im}F^{(+)}(\nu) \, d\nu = \frac{1}{2\pi^2} \int_{N_1}^{N_2} \sigma_{tot}^{(+)}(k) \, dk, \quad (7)$$

where the laboratory energies $N_{1,2}$ are related to the corresponding momenta $\overline{N}_{1,2}$ by $N_{1,2} = \sqrt{\overline{N}_{1,2}^2 + \mu^2}$ as explained above. The value of $N_2$ should be selected to be reasonably high momentum above which no resonance structures are observed, while $N_1$ may be taken to be in the resonance energy region in the sense of FESR duality.

The integrand of the LHS of Eq. (7) is the low-energy extension of Eqs. (3). The RHS is the integral of experimental $\sigma_{tot}^{(+)}\simeq (\sigma_{tot}^{\pi^+p} + \sigma_{tot}^{\pi^-p})/2$ in the resonance energy regions. This shows up several peak and dip structures corresponding to a number of $N$ and $\Delta$ resonances, in addition to the non-resonating background. Thus, Eq. (7) means the FESR duality, that is, the average of these resonance structures plus the non-resonating background in $\sigma_{tot}^{(+)}$ should coincide with the low-energy extension of the asymptotic formula. Practically, the RHS can be estimated from the experimental $\sigma_{tot}^{\pi^+p}$ very accurately with
Table 1
Values of parameters in the best fit with five-parameters, using FESR as a constraint, where the value of $\beta_{P'}$ is obtained from FESR constraint and $(\alpha_{P'},\alpha_V)$ is fixed to be $(0.500,0.497)$. The statistical errors of $c_2$ are also given. The result of six-parameter fit without using FESR is also shown in the last row as No SR.

| $N_1$(GeV) | $c_2 \times 10^5$ | $c_1$ | $c_0$ | $F^{(+)}(0)$ | $\beta_V$ | $\beta_{P'}$ |
|------------|------------------|-------|-------|---------------|------------|-------------|
| 10         | 126±30           | -0.0125 | 0.117 | $-0.321$      | 0.0389     | 0.136       |
| 7          | 128±26           | -0.0128 | 0.118 | $-0.384$      | 0.0389     | 0.132       |
| 5          | 127±24           | -0.0128 | 0.118 | $-0.333$      | 0.0388     | 0.133       |
| 4          | 126±22           | -0.0125 | 0.117 | $-0.239$      | 0.0388     | 0.137       |
| 3.02       | 123±21           | -0.0120 | 0.115 | $-0.043$      | 0.0388     | 0.126       |
| 2.035      | 119±20           | -0.0112 | 0.111 | 0.252         | 0.0388     | 0.137       |
| 1.476      | 118±19           | -0.0111 | 0.110 | 0.285         | 0.0388     | 0.139       |
| 0.9958     | 119±18           | -0.0112 | 0.111 | 0.247         | 0.0388     | 0.137       |
| 0.818      | 124±18           | -0.0122 | 0.115 | $-0.069$      | 0.0388     | 0.125       |
| 0.723      | 129±17           | -0.0131 | 0.120 | $-0.347$      | 0.0388     | 0.114       |
| 0.475      | 143±17           | -0.0155 | 0.131 | $-1.111$      | 0.0387     | 0.084       |
| 0.281      | 126±16           | -0.0124 | 0.116 | $-0.123$      | 0.0388     | 0.122       |
| No SR      | 95±45            | -0.0069 | 0.091 | 1.643         | 0.0390     | 0.209       |

errors less than 0.5%, so we can use Eq. (7) as an exact constraint.

In case of $\bar{p}p, pp$ scattering, if we take too small value of $N_1$ close to the threshold $\nu = M$, the FESR (7) is affected strongly by a contribution from the unphysical region $\nu < M$, and often does not work well. Thus, we must take $N_1$ to be fairly larger than $M$. In contrast, there is no such problem in $\pi p$ scattering. The lower the value of $N_1$ is taken, the more the information of low energy scattering data are included. Then, the more accurately estimated value of $c_2$ is obtained. We try to take $N_1$ as small value as possible in the present analysis.

Result of the analyses

The data of $\sigma_{tot}^{\pi p}$ for $k \geq 20$ GeV and $\rho_{tot}^{\pi p}$ for $k \geq 5$ GeV are fitted simultaneously. In the FESR, Eq. (7), $N_2$ is taken to be 20 GeV. The values of $N_1$ are taken to be 10, 7, 5, 4, 3.02, 2.035, 1.476, 0.9958, 0.818, 0.723, 0.475, 0.281 GeV. Except for the first three values, they correspond to the energies of peak and dip positions of experimental $\sigma_{tot}^{\pi^- p}$ or $\sigma_{tot}^{\pi^+ p}$. For each value of $N_1$, the FESR is derived. It is used as a constraint between the parameters, $c_{2,1,0}$
Table 2
Values of the best-fit $\chi^2$ for each case. The FESR is used as a constraint, and five-parameter ($N_P=5$) fit is performed. Both total $\chi^2$ and respective $\chi^2$ for each data with the number of data points are given. The $\chi^2$ of six-parameter ($N_P=6$) fit without using FESR is also shown in the last row as No SR.

| $\overline{N_1}$(GeV) | $\chi^2_{N_P-N_P}$ | $\chi^2_{N_P-N_P}$ | $\chi^2_{N_P-N_P}$ | $\chi^2_{N_P-N_P}$ | $\chi^2_{N_P-N_P}$ | $\chi^2_{N_P-N_P}$ |
|------------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 10                     | 72.58                | 12.47                | 40.77                | 6.66                 | 12.68                | 8                    |
| 7                      | 72.56                | 12.44                | 40.76                | 6.69                 | 12.67                | 8                    |
| 5                      | 72.49                | 12.53                | 40.65                | 6.67                 | 12.64                | 8                    |
| 4                      | 72.42                | 12.64                | 40.51                | 6.64                 | 12.63                | 8                    |
| 3.02                   | 72.29                | 12.87                | 40.26                | 6.56                 | 12.60                | 8                    |
| 2.035                  | 72.12                | 13.22                | 39.85                | 6.46                 | 12.55                | 8                    |
| 1.476                  | 72.10                | 13.27                | 39.85                | 6.45                 | 12.55                | 8                    |
| 0.9958                 | 72.12                | 13.24                | 39.88                | 6.46                 | 12.54                | 8                    |
| 0.818                  | 72.29                | 12.91                | 40.24                | 6.56                 | 12.57                | 8                    |
| 0.723                  | 72.46                | 12.64                | 40.57                | 6.66                 | 12.60                | 8                    |
| 0.475                  | 73.08                | 11.96                | 41.48                | 6.98                 | 12.66                | 8                    |
| 0.281                  | 72.32                | 12.89                | 40.29                | 6.58                 | 12.56                | 8                    |
| No SR                  | 71.79                | 14.94                | 38.35                | 6.09                 | 12.41                | 8                    |

and $\beta_{P'}$, and the fitting is performed. The number of fitting parameters is five, including $\beta_V$ and $F^{(+)}(0)$. The ($\alpha_{P'}, \alpha_V$) are fixed to be (0.500, 0.497) [12] in all the fitting procedures. The values of parameters and $\chi^2$ in the best fits in respective cases are given in Tables 1 and 2.

It is remarkable that the values of the parameters in the best fits are almost independent of $\overline{N_1}$ (except for the case of 0.475GeV), as can be seen in Table 1. The results are surprisingly stable, although there are many resonant structures observed and $\sigma_{\text{tot}}$ show sharp peak and dip structures in this energy region. The lower the value of $\overline{N_1}$ is taken, the smaller the statistical errors of $c_2$ become in the best fits. We can adopt the case of $\overline{N_1}=0.818$GeV as the representative of our results. The value of $c_2$ in the best fit is

$$c_2 = (124 \pm 18) \times 10^{-5} \ .$$ (8)

The central value of Eq. (8) is almost the same as (126±30) in the case of $\overline{N_1}=10$GeV, but the error is much improved. This shows that the data with $^2$ The $c_2$ $(\log \nu)^2 + c_1 \log \nu$ with $c_2 > 0$ shows the shape of parabola as a function.
$k \leq 10\text{GeV}$ give very important information to determine the high-energy parameters such as $c_2$ through the FESR duality.

**Concluding Remarks**

Using the value of $c_2$ in Eq. (8), we can derive the value of $B_{\pi p}$ as

$$B_{\pi p} = \frac{4\pi}{\mu^2} c_2 = 0.311 \pm 0.044\text{mb} \ .$$

(9)

This value is to be compared with the value of $B_{pp}$ in our previous analysis, $B_{pp} = 0.289 \pm 0.023\text{mb}$[12]. The $B_{\pi p}$ in Eq. (9) is consistent with this $B_{pp}$. Thus, our result appears to support the universality hypothesis for the values of $B$ parameters.

In case of six parameter fit without using FESR, we obtain $c_2 = (95 \pm 45) \cdot 10^{-5}$, shown in the last row of Table 1. This value corresponds to $B_{\pi p} = 0.24 \pm 0.11\text{mb}$. From this value, we would not be able to say anything about the universality due to its large statistical error. The role of FESR is crucially important to obtain a definite conclusion.

It is to be noted that our value of $B_{\pi p}$ is consistent with the value of $B$ by COMPETE collab.[17], $0.308 \pm 0.010\text{mb}$, which is obtained by assuming the universality of $B$ for various processes.

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