Remote creation of one-qubit mixed state in a short homogeneous spin-1/2 chain.

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Abstract

We consider a method of remote mixed state creation of one-qubit subsystem (receiver) in a spin-1/2 chain governed by the nearest-neighbor XY-Hamiltonian. Owing to the evolution of the chain along with the variable local unitary transformation of one- or two-qubit sender, a large variety of states of the receiver can be created during some time interval $0 \leq t \leq T$. It is remarkable that, having a two-qubit sender, the same region may be created at a properly fixed time instant using just the variable local unitary transformation of sender. In this case we have a completely local control of the remote state creation. The effectiveness of the remote state creation may be characterized by the density function of the createable states.

PACS numbers:
I. INTRODUCTION

The problem of pure quantum state transfer in spin-1/2 chains was first formulated in the well-known paper by Bose [1]. Many different methods of either perfect [2, 4] or high fidelity (probability) [5–8] state transfer have been proposed and studied along with their possible improvements. Perhaps, the most known systems are the spin chain with properly adjusted coupling constants (the so-called fully engineered spin chain) [2–4] and the homogeneous spin chain with remote end nodes (the so-called boundary controlled spin chain) [7].

It was noted that the high fidelity state transfer requires the very rigorous adjustment of parameters of the chain such as the dipole-dipole coupling constants [2–4] and/or the local distribution of the external magnetic field [9]. That chain is very sensitive to the perturbations of its parameters which lead, in particular, to significant decreasing of the state transfer fidelity [10–14]. Although the boundary-controlled chain is much simpler for realization the price for that is a long time of state transfer which significantly reduces the effectiveness of such a chain [15].

Alternatively, the transfer of the complete information about the initial mixed state of the given subsystem (sender) to another subsystem (receiver) was proposed as a development of the state transfer methods [16] (the state-information transfer). The information transfer is not sensitive to the parameters of the transfer line [16]. After the information about the state of sender is obtained by the receiver at some time instant we may recover the initial state of the sender (if needed) using the local (non-unitary) transformation, namely, solving the system of linear algebraic equations.

In this paper we combine the ideas of the both pure state-transfer and mixed state-information transfer and consider the problem of remote mixed state creation. More precisely, we study the creation problem of possible states of the receiver at some instant \( t \) starting with some initial state of the whole considered system and using only the initial local unitary transformation of the sender, herewith the evolution of the whole system is governed by some Hamiltonian. This gives us a tool of remote state control of the receiver.

The problem of the remote state creation in a spin-1/2 chain using sender and receiver of different dimensionalities \( N^A \) and \( N^B \) is very complicated multi-parameter one. In this paper, after representing some general statements regarding this process, we concentrate on the particular examples of short chains with one-qubit receiver (the end-node of a chain)
and one(two)-qubit sender (the first node(s) of a chain). We consider the state creation during some time interval $0 \leq t \leq T$ with a fixed $T$ and show that, if we use the two-qubit sender $(N^A = 4 > N^B = 2)$, all states createable during that time interval using the variable parameters of the local unitary transformation $U^A$ may be created at some properly fixed time instant $t_0$, $0 \leq t_0 \leq T$, using just the local variable unitary transformation $U^A$. This effect is impossible in the case of one-node sender, i.e. $N^A = N^B = 2$.

The paper is organized as follows. General ideas of the state creation as a map of the variable parameter of the sender to the required parameters of the receiver are formulated in Sec. II. The mixed state creation in a spin-1/2 chain governed by the nearest neighbor XY-Hamiltonian with one-qubit receiver and one- or two-qubit sender is studied in Sec. III. Basic results are briefly discussed in Sec. IV. The auxiliary information regarding the basis of the Lie algebra of $SU(4)$ is given in the Appendix, Sec. V.

II. STATE-CREATION OF END-SUBSYSTEM OF HOMOGENEOUS SPIN-1/2 CHAIN.

Let us consider the problem of states creation of the $N^B$-dimensional subsystem $B$ (the receiver) of a homogeneous spin-1/2 chain using the local unitary transformation $U^A \in SU(N^A)$ of another $N^A$-dimensional subsystem $A$ (the sender). For simplicity, we assume that the sender and the receiver are placed, respectively, in the beginning and in the end of a spin chain. The rest nodes connecting the sender and the receiver are called the transmission line $C$. Let the spin dynamics be governed by the nearest neighbor XY Hamiltonian $H$ (commuting with the $z$-projection of the total spin momentum $I_z$):

$$H = \mathcal{H} + \gamma I_z, \quad [\mathcal{H}, I_z] = 0, \quad I_z = \sum_{i=1}^{N} I_{z;i},$$

$$\mathcal{H} = -\sum_{i=1}^{3} \frac{d}{2}(I_{i+1}^+I_i^- + I_i^-I_{i+1}^+).$$

Here $d$ is the coupling constant between the nearest neighbors, $I_{i}^{\pm} = I_{x;i} \pm iI_{y;i}$, $I_{\alpha;i}$, $\alpha = x, y, z$, are the projection operators of the total spin angular momentum, and $\gamma$ is the Larmor frequency of the global external magnetic field. We put $d = 1$ without loss of generality. To simplify our calculations, we consider only a particular type of the initial states, namely, the
states represented by the tensor product of three blocks:

$$\rho_0 = \rho_0^A \otimes \rho_0^C \otimes \rho_0^B.$$  \(2\)

Here $\rho_0^A$, $\rho_0^B$ and $\rho_0^C$ describe the initial states of, respectively, the sender, the receiver and the transmission line. The remote state creation algorithm can be splitted into three steps.

1. Create the initial state of the sender, receiver and transmission line, see eq.\(2\).

2. Apply the unitary transformation $U(\varphi)$ to the subsystem $A$ to obtain the new initial density matrix $\rho_0(\varphi)$:

$$\rho_0(\varphi) = (U(\varphi)\rho_0^A U^+(\varphi)) \otimes \rho_0^C \otimes \rho_0^B,$$  \(3\)

where $\varphi$ is the list of parameters of the unitary transformation which may vary in an arbitrary way. However, in principle, there might be constraints imposed on these parameters. The choice of the parameters $\varphi_i$ depends on the needed state of receiver.

3. Switch on the spin-evolution governed by the Hamiltonian \((1)\) in accordance with the Liouville equation

$$\rho(\varphi, t) = e^{-itH} \rho_0(\varphi) e^{itH}.$$  \(4\)

4. Finally, the state of the subsystem $B$ at the time instant $t$ is described by the marginal matrix $\rho^B(t)$,

$$\rho^B(\varphi, t) = \text{Tr}_{A,C} \rho(\varphi, t).$$  \(5\)

Now we shall give two remarks regarding the above algorithm.

**Remark 1.** In the above algorithm, the first and the second steps serve to prepare the initial state $\rho_0(\varphi)$. However, we do not join these two steps. The matter is that the problem of the initial state creation for the whole system (the first step) and the preparation of the local unitary transformation $U^A$ (the second step) have different features and, consequently, different obstacles for their realization. A principal difference is that the initial state creation involves the creation of the eigenvalues of the local density matrices $\rho_0^A$, $\rho_0^C$ and $\rho_0^B$, while the unitary transformation $U^A$ may not effect these eigenvalues. We consider that after the initial states of the receiver $\rho_0^B$ and of the transmission line $\rho_0^C$ are established, we may not
change them. Also we may not change the eigenvalues of the initial state of the sender \( \rho_0^A \). On the contrary, the parameters of the unitary transformation \( U^A \) may vary with the purpose to create a needed state of the receiver.

**Remark 2.** We assume that, for some reasons, the unitary transformations can not be locally applied to the both subsystems \( B \) and \( C \). We also may not handle the Hamiltonian governing the evolution (the third step). But we have a method of changing of the parameters of the local unitary transformation \( U^A \) applied to the sender. Thus we may handle the state of the receiver only through the variable local unitary transformation of the sender which is remote from the receiver. For this reason, we refer to such state creation as a remote state creation.

Next, we shall note that not all \((N^A)^2 - 1\) parameters of the local unitary transformation \( U^A \in SU(N^A)\) can effect on the state of the receiver. In fact, the mixed initial state of the subsystem \( A \) may be represented as a set of eigenvalues and appropriate unitary matrix of eigenvectors. The later may be embedded into the local unitary transformation \( U^A \) (just changing the parameters \( \varphi_i \)), while the eigenvalues are fixed by the initial state of \( A \) and they may not be changed by the unitary transformation. It was shown \([17, 18]\) that \( N^A - 1 \) parameters (the number of independent eigenvalues) disappear from the unitary transformation, so that we stay with

\[
D_{mixed}^A = (N^A)^2 - 1 - (N^A - 1) = N^A(N^A - 1)
\]

parameters \( \{\varphi_1, \ldots, \varphi_{N^A(N^A-1)}\} \). Two more parameters are encoded into the evolution operator. These parameters are the time \( t \) and the Larmor frequency \( \gamma \), see eq.(1). The parameter \( \gamma \) is not effective in the case of the diagonal initial states \( \rho_0^B \) and \( \rho_0^C \) because it can be also embedded into the transformation \( U^A \). To show this we represent the evolution of the density matrix due to the term \( \gamma I_z \) in the Hamiltonian as

\[
e^{-iI_z^A \gamma t} \otimes e^{-iI_z^C \gamma t} \otimes e^{-iI_z^B \gamma t} \rho_0^A(\varphi) e^{iI_z^A \gamma t} \otimes e^{iI_z^C \gamma t} \otimes e^{iI_z^B \gamma t} =
\]

\[
e^{-iI_z^A \gamma t} (U^A(\varphi) \rho_0^A(U^A)^+ (\varphi)) e^{iI_z^A \gamma t} \otimes \rho_0^C \otimes \rho_0^B,
\]

where \( I_z^A, I_z^C \) and \( I_z^B \) are the \( z \)-projections of the total spin angular momentum of subsystems \( A, C \) and \( B \) respectively and we take into account that all of these projections are diagonal matrix operators. Consequently, the parameter \( \gamma \) (more precisely, the product \( \gamma t \)) appears only in the state of the sender and therefore leads just to the redefinitions of the parameters
\(\varphi_i\) of the local unitary transformation \(U^A\). Thus we have \(D_{\text{mixed}}^A + 1 = N^A(N^A - 1) + 1\) variable real parameters of the sender which handle the state of the receiver. This state, in turn, is parametrized by the \(D^B\) real parameters, \(D^B = (N^B)^2 - 1\), which are \(N^B - 1\) independent eigenvalues of the matrix \(\rho^B\) and \(N^B(N^B - 1)\) real parameters of its eigenvectors. In result we have the following map of \(D_{\text{mixed}}^A + 1\) variable parameters of sender into \(D^B\) required parameters of the state of the receiver:

\[
\mathcal{M}_{\text{mixed}}(\varphi, t; \lambda, \beta) : \{\varphi_1, \ldots, \varphi_{N^A(N^A-1)}, t\} \rightarrow \{\lambda_1, \ldots, \lambda_{N^B-1}, \beta_1, \ldots, \beta_{N^B(N^B-1)}\}, \quad (8)
\]

where \(\lambda = \{\lambda_1, \ldots, \lambda_{N^B-1}\}\). Here all parameters take the values inside of the appropriate intervals, which will be explicitly represented in the examples of Sec III. Moreover, if we deal with the pure initial state of the sender \(A\) (a single non-zero eigenvalue of \(\rho^A_0\)), then the number of variable parameters decreases by \((N^A - 1)(N^A - 2)\) owing to the additional symmetry with respect to the transformation \(\text{diag}\{1, \tilde{U}\}\) with \(\tilde{U} \in SU(N^A - 1)\). Thus, we obtain:

\[
D_{\text{pure}}^A = N^A(N^A - 1) - (N^A - 1)(N^A - 2) = 2(N^A - 1). \quad (9)
\]

Now the map \((8)\) reads:

\[
\mathcal{M}_{\text{pure}}(\varphi, t; \lambda, \beta) : \{\varphi_1, \ldots, \varphi_{2(N^A-1)}, t\} \rightarrow \{\lambda_1, \ldots, \lambda_{N^B-1}, \beta_1, \ldots, \beta_{N^B(N^B-1)}\}. \quad (10)
\]

Comparing maps \((8)\) and \((10)\) we see that the number of variable parameters increases quadratically with \(N^A\) in the case of an arbitrary mixed initial state of the sender and this number increases linearly with \(N^A\) in the case of a pure initial state of the sender.

Obviously, we may hope to create the whole state space of the receiver if

\[
N^A(N^A - 1) + 1 \geq (N^B)^2 - 1 \quad \text{for map } \mathcal{M}_{\text{mixed}} \quad (8),
\]

\[
2(N^A - 1) + 1 \geq (N^B)^2 - 1 \quad \text{for map } \mathcal{M}_{\text{pure}} \quad (10).
\]

Let \(N^B = 2\). Then the equality in both conditions \((11)\) and \((12)\) corresponds to \(N^A = 2\). In this case the time is an important parameter of the both maps \(\mathcal{M}_{\text{mixed}}\) and \(\mathcal{M}_{\text{pure}}\). If \(N^A > 2\), then \(D_{\text{mixed}}^A + 1 \geq 12\) (in condition \((11)\)) or \(D_{\text{pure}}^A + 1 \geq 6\) (in condition \((12)\)), where the equality corresponds to the case \(N^A = 4\) (remember, that \(N^A\) is the natural power of 2). In this case the time \(t\) is not so important because conditions \((11)\) and \((12)\) remain valid even if we disregard the time \(t\) as a parameter of the considered maps, i.e. reduce the left
hand sides of conditions (11) and (12) by one. Consequently, we may expect to create the whole state space of receiver at a properly fixed time instant $t_0$. In general, disregarding the time $t$ in maps (8) and (10), we reduce these maps, respectively, to the following ones:

$$
M_{\text{mixed}}(\varphi; \lambda, \beta) : \{\varphi_1, \ldots, \varphi_{N^A(N^A-1)}\} \rightarrow \{\lambda_1, \ldots, \lambda_{N^B-1}, \beta_1, \ldots, \beta_{N^B(N^B-1)}\}, \quad (13)
$$

$$
M_{\text{pure}}(\varphi; \lambda, \beta) : \{\varphi_1, \varphi_2(N^A-1)\} \rightarrow \{\lambda_1, \ldots, \lambda_{N^B-1}, \beta_1, \ldots, \beta_{N^B(N^B-1)}\}. \quad (14)
$$

Comparing maps (8) and (10) with, respectively, (13) and (14) we see that the former may not be considered as a completely local ones because they have a time as a parameter. In other words, to obtain the required state of receiver, one has to transfer the information about the proper time instant for the state registration. On the contrary, maps (13) and (14) are completely local because the receiver registers the state at a fixed time instant $t_0$, which can be reported in advance.

A. Pure state transfer as a special case of mixed state creation

The arbitrary pure state transfer along the spin chain [1] may be considered as a very special case of the state creation via map (10), when all the parameters of the local unitary transformation $U^A$ are fixed. In this case the parameter $\gamma$ (which was disregarded in all above maps (8), (10), (13) and (14)) becomes important and the map (10) must be replaced with the following one:

$$
M_{\text{pure}}(\gamma; t; \lambda, \beta) : \{\gamma, t\} \rightarrow \{\lambda_1 \equiv \lambda, \beta_1, \beta_2\}, \quad (15)
$$

$$
0 \leq \beta_i \leq 1, \quad i = 1, 2, \quad \lambda = 1.
$$

Herewith the fixed parameters $\varphi_1$ and $\varphi_2$ of the given pure initial state of sender must be equal, respectively, to the parameters $\beta_1$ and $\beta_2$ of the receiver:

$$
\varphi_i = \beta_i, \quad i = 1, 2. \quad (16)
$$

We see that there are only two variable parameters in the map (15) which must create three required values of the parameters $\lambda$, $\beta_1$ and $\beta_2$, which is impossible in the long homogeneous chain. For this reason, to realize the pure state transfer, we need additional efforts, namely, the rigorous adjustment of the parameters of the spin chain, such as the dipole-dipole interaction constants [2, 4] and/or the local Larmor frequencies [9]. Such adjustment, generally speaking, is not required in the mixed state-creation process considered in this paper.
III. EXAMPLES OF THE STATE CREATION IN SHORT SPIN-1/2 CHAINS

In this section we consider the system $A - C - B$ with one-qubit receiver $B$ and transmission line $C$, whereas the sender $A$ may be either one- or two-qubit. Below we consider both these cases in details.

Of course, a one-qubit transmission line $C$ is very short, but, nevertheless, it enriches the features of the state-creation process. In particular, the four node homogeneous chain with two-qubit sender does not preserve the conditions for the perfect pure state transfer.

A. Three-node chain with a one-qubit sender

We proceed with the three node chain having the one-qubit subsystems $A$, $B$ and $C$, thus $N^A = N^B = 2$. The numerical simulations show that the most promising is the initial state of the subsystem $A$ with a single nonzero eigenvalue (a pure initial state). We also consider the pure initial state of the subsystem $C$ and a mixed initial state of the receiver $B$. Thus, the initial state of the whole spin chain is given by expression (2) with

$$\rho_0^A = \text{diag}(1, 0), \quad \rho_0^C = \text{diag}(1, 0), \quad \rho_0^B = \text{diag}(\lambda^B, 1 - \lambda^B).$$

The unitary $SU(2)$ transformation responsible for the state creation with the diagonal initial matrix $\rho_0^A$ is the two-parameter one:

$$U^A(\varphi) = e^{-i\varphi_2\sigma_3} e^{-i\frac{\varphi_1}{2}\sigma_2} e^{i\varphi_2\sigma_3} = \begin{pmatrix} \cos \frac{\varphi_1}{2} & -e^{-i2\varphi_2} \sin \frac{\varphi_1}{2} \\ e^{i2\varphi_2} \sin \frac{\varphi_1}{2} & \cos \frac{\varphi_1}{2} \end{pmatrix},$$

$$0 < \varphi_i < 1, \quad i = 1, 2, \quad \varphi = \{\varphi_1, \varphi_2\}.$$

The evolution of this chain is described by formula (4) in accordance with the Liouville equation. Finally, the state of the subsystem $B$ at some instant $t$ is described by the marginal matrix $\rho^B(t)$:

$$\rho^B(t) = Tr_{A,C} \rho(t) = Tr_{A,C} e^{-itH} \left( U^A \rho_0^A (U^A)^+ \otimes \rho_0^C \otimes \rho_0^B \right) e^{itH},$$

which can be represented in the form

$$\rho^B(t) = U(t) \Lambda(t) U^+(t)$$

(21)
Since $\beta$ here the parameters with $\phi$ parameter. It is remarkable that map (23) admits a simplification due to the linear relation between the $\beta$ with $A$ After calculation of the trace with respect to the subsystems (28) we obtain $\rho$ Thus we obtain $\rho$ and, consequently, $\rho^B$ in the form (28) we obtain $\rho^B(t)$ in the form (21,22) with $U(t) = e^{-i\pi\varphi_2\sigma_2(t)}e^{i\pi\varphi_2\sigma_3}$ and, consequently, $B(2)$ and $C(2)$, which is a linear relation between $\beta_2$ and $\varphi_2$ mentioned above.
Thus, varying the parameter $\varphi_2$ we may obtain any required value of the parameter $\beta_2$ at the needed time instant $t$. This allows us to disregard the parameters $\varphi_2$ and $\beta_2$ in the map (23) and replace this map with the following one:

$$M_{\text{pure}}(\varphi_1, t; \lambda, \beta_1) : \{\varphi_1, t\} \rightarrow \{\lambda, \beta_1\},$$

(30)

$$0 \leq \varphi_1 \leq 1, \quad 0 \leq t \leq \pi \sqrt{2},$$

(31)

$$0 \leq \beta_1 \leq 1, \quad 1/2 \leq \lambda \leq 1,$$

(32)

where we consider $\lambda \geq 1/2$ without loss of generality. No new states may be created at $t > \pi \sqrt{2}$, which follows from the periodicity of the spin-dynamics. Map (30-32) is numerically studied in the following subsection.

1. Numerical one-qubit state creation

We consider map (30-32) with initial condition (17). The region of createable states in the space $(\lambda, \beta_1)$ (32), is depicted in Fig.1a-d for the following set of $\lambda^B$:

$$\lambda^B = 1, \frac{3}{4}, \frac{1}{4}, 0.$$

(33)

Each line in these figures correspond to a particular time instant of map (30) with the parameter $\varphi_1$ varying inside of the interval specified in (31). In the case of pure initial state, $\lambda^B = 1, 0$, the lines cover the whole space $(\lambda, \beta_1)$ (see Fig.1a,d). The vertical lines $\lambda = 1$ in these figures are associated with the time $t = \pi \sqrt{2}$ corresponding to the perfect pure state transfer from the first to the third node [2]. The case $\lambda^B = 1$ (Fig.1a) is of the most interest because the map (30) is mutually unique. Moreover, the lines are time-ordered in this case: the time prescribed to each of these lines increases in the direction of the arrow from 0 to $\pi \sqrt{2}$. So, in principle, we may construct the one-to-one relation between the pairs $(\varphi_1, t)$ and $(\lambda, \beta_1)$. Thus, having a particular pair $(\lambda, \beta_1)$, we may restore the parameter of sender $\varphi_1$ and the time instant $t$ when the state was sent.

Regarding the mixed initial states $\lambda^B = 3/4, 1/4$, see Fig.1b,c, not any state of the receiver may be created by the local unitary transformations $U^A$, which is indicated by the unavailable regions in these figures. In addition, the map (30) is not mutually unique, because a particular state $(\lambda, \beta_1)$ can be created by more then one pair $(\varphi_1, t)$.

We shall also note the case $\lambda^B = \frac{1}{2}$ when we may create only the states with $\frac{1}{2} \leq \lambda \leq 1$ and $\beta_1 = 0, \frac{\pi}{2}$, i.e. arbitrary diagonal states.
FIG. 1: The two-parameter receiver state-space \((\lambda, \beta_1)\) of the map (30-32). We consider the three node spin chain with one-qubit sender and set (33) of \(\lambda^B\). This figure demonstrates the non-uniform distribution of the createable states. (a) \(\lambda^B = 1\), the pure initial state; the whole space \((\lambda, \beta_1)\) is createable and the map (30) is mutually unique; the time \(t\) increases in the direction of the arrow. (b,c) \(\lambda^B = 3/4\) and \(\lambda^B = 1/4\), respectively; the mixed initial state, the map (30) is not mutually unique and the unavailable region appears. (d) \(\lambda^B = 0\), the pure initial state; the whole space \((\lambda, \beta_1)\) is createable, but the map (30) is not mutually unique.

The overall disadvantage of the proposed algorithm of the state creation with equal dimensionalities of the sender and receiver is that we have to involve the time \(t\) as a parameter of map (30) in order to cover a valuable region of the space \((\lambda, \beta_1)\). Consequently, this map is not completely governed by the local unitary transformation \(U^A\). This disadvantage is compensated in the case \(N^A = 4\) considered in the next subsection. Notice that the map (30) with a pure initial state covers the complete state space \((\lambda, \beta_1)\) only in the case of two- and
three node chains with the nearest neighbor interactions, which resonates with the condition for the perfect pure state transfer. Involving the dipole-dipole remote node interactions, the unavailable region appears even in the case of pure initial states. Remember that the perfect pure state transfer is impossible in a long homogeneous chain. The remote state creation in the long non-homogeneous chain with parameters providing the perfect pure state transfer is not considered here.

2. Density function as a characteristics of createable state-space

The distribution of createable states in Fig. 1a-b is non-uniform. If we fix some small area in space \((\lambda, \beta_1)\), then the more points are in this area, the more points from the space \((\varphi_1, t)\) are transferred into it. This means that the areas in Fig. 1 with high density of points are simpler for realization. To better visualize this effect we introduce the so-called density function as follows:

\[
S(\lambda_i, \beta_{1j}) = \lim_{N_\lambda, N_{\beta_1} \to \infty} \frac{s(\lambda_i, \beta_{1j})}{N^{st}},
\]

where \(s(\lambda_i, \beta_{1j})\) is the number of states in the rectangle

\[
(\lambda_i, \lambda_i + \varepsilon \lambda], \ (\beta_j, \beta_j + \varepsilon \beta],
\]

\(N^{st}\) is the total number of states in the rectangle (32). The function \(S\) is normalized as follows:

\[
\sum_{i=0}^{N_\lambda-1} \sum_{j=0}^{N_{\beta_1}-1} S(\lambda_i, \beta_{1j}) \varepsilon \lambda \varepsilon \beta_1 = 1.
\]

We represent the contour plot of the density function (34) in Fig.2 for \(\varepsilon_{\beta_1} = 2\varepsilon_\lambda = \varepsilon\) with \(\varepsilon = 0.01\) and \(N^{st} = 960000\). This value of \(N^{st}\) is related with the uniform splitting of the variation intervals (31) corresponding to the parameters \(\varphi_1\) and \(t\) into, respectively, 399 segments (400 points) and 2399 segments (2400 points). These figures show the dependence of the density function on \(\lambda^B\). The choice of \(\lambda^B\) must be defined by the particular area of states which we need to create. The bright areas are simpler for their realization. The maximal values \(S_{max}\) of the density function together with their coordinates \(\lambda_{max}\) and \(\beta_{1;max}\) for different values of \(\lambda^B\) from list (33) are collected in Table II.
FIG. 2: The density function of the two-parameter receiver state-space \((\lambda, \beta_1)\) of the map (30-32). The three node chain with one-qubit sender is considered for set (33) of \(\lambda^B\), compare with Fig. 1. The bright regions are most simple for realization. The maximal values of the density function \(S_{\text{max}}\) together with the appropriate values of the parameters \(\lambda_{\text{max}}\) and \(\beta_{1;\text{max}}\) are collected in Table 1. (a, d) \(\lambda^B = 1\) and 0 respectively, the pure initial state; the whole state space of the receiver may be created. (b, c) \(\lambda^B = 3/4\) and 1/4 respectively, the mixed initial state; the whole state space of the receiver may not be created; the unavailable regions are indicated by the black areas in both figures.

Although the density function \(S\) demonstrates the distribution of the createable states, this distribution is also understandable from Fig. 1 because the createable states are arranged in lines. The case of two-qubit sender is different. The createable states are not arranged
TABLE I: The maximal values \( S_{max}(\lambda_{max}, \beta_{1; max}) \) of the density function of createable states for the one-qubit sender (the three node chain) in the well-structured lines, so that the density function \( S \) becomes very important in that case.

### B. Four-node spin chain with a two-qubit sender

Let us consider the four node chain with the two-qubit sender (the first and the second nodes), while the receiver (the 4th node) and the transmission line (the 3rd node) remain one-qubit subsystems. In this case the general form of the 12-parameter transformation \( U^A \in SU(4) \) reads \([17]\):

\[
U^A(\varphi) = e^{i\pi\varphi_1}e^{i\gamma_2\frac{\varphi_2}{2}}e^{i\pi\varphi_3}e^{i\gamma_5\frac{\varphi_4}{2}}e^{i\pi\varphi_5}e^{i\gamma_{10}\frac{\varphi_6}{2}}e^{i\pi\varphi_7}e^{i\gamma_2\frac{\varphi_8}{2}}e^{i\pi\varphi_9}e^{i\gamma_5\frac{\varphi_10}{2}}e^{i\pi\varphi_{11}}e^{i\gamma_2\frac{\varphi_{12}}{2}},
\]

\( \varphi = \{\varphi_1, \ldots, \varphi_{12}\}, \ 0 \leq \varphi_i \leq 1, \ i = 1, \ldots, 12 \), \( i = 1, \ldots, 12 \).  

The explicit matrix representation of \( \gamma_i \) is given in the Appendix A, Sec.\( \text{V} \) \([17, 19]\).

In accordance with Sec.\( \text{III} \) in this case we may disregard the time as a varying parameter of the state creation and use the map \( \text{II} \) which now reads:

\[
\mathcal{M}_{pure}(\varphi, \beta) : \{\varphi_1, \ldots, \varphi_{12}\} \rightarrow \{\lambda, \beta_1, \beta_2\},
\]

\( 0 \leq \varphi_i \leq 1, \ i = 1, \ldots, 12, \) \( i = 1, \ldots, 12 \), \( 0 \leq \beta_i \leq 1, \ i = 1, 2, \ \frac{1}{2} \leq \lambda \leq 1, \)

with a fixed time instant \( t_0 \) inside of some interval, \( 0 < t_0 \leq T \).

Here \( T \) is some time-parameter which is chosen by convention. In our case we take \( T = 10 \) because this value reasonably exceeds the state transfer time for the 4-node chain.
We consider the diagonal initial density matrix of form (2) and fix the pure state of the subsystems $A$ and $C$, while the initial state of the receiver $B$ is arbitrary diagonal one, similar to Sec III A. In other words, our initial matrices are the following ones:

$$
\rho_A^0 = \text{diag}(1, 0, 0, 0), \quad \rho_C^0 = \text{diag}(1, 0), \quad \rho_B^0 = \text{diag}(\lambda^B, 1 - \lambda^B).
$$

The density matrix evolves in accordance with eq.(4). Finally, the marginal matrix $\rho_B^B(t)$ describing the receiver’s state may be calculated using formulas (20 - 22) of Sec III A, where the Hamiltonian $H$ is the same (see eq. (1)), while $U^A$ and $\rho_A^0$ are given by eqs.(37) and (43) respectively.

The proposed structure of the initial density matrix $\rho_A^0$ decreases the number of the effective arbitrary parameters [18] which is 6 (instead of 12) owing to the symmetry of the $\rho_A^0$, see eq.(9). Since we need to create three parameters of the receiver’s state, it might be enough to take just three parameters $\varphi_i$ in the map (39). The preferable choice of these parameters is not evident, and the problem of the optimal parametrization of the considered map remains beyond the scope of this paper. Instead, we propose the following parametrization which yields large (and, maybe, the maximal) region of the createable states inside of the time interval specified above.

Let, for instance, the parameters $\varphi_{10}$, $\varphi_{11}$ and $\varphi_{12}$ vary independently, while all other parameters are linearly expressed in terms of the single parameter $\phi$:

$$
\begin{align*}
\varphi_{2n-1} &= \phi, \quad n = 1, \ldots, 5, \\
\varphi_{2n} &= \frac{\phi}{2}, \quad n = 1, \ldots, 4.
\end{align*}
$$

Thus, the map (39-41) simplifies to

$$
\mathcal{M}_{\text{pure}}(\varphi, \beta) : \{\varphi_{10}, \varphi_{11}, \varphi_{12}, \phi\} \to \{\lambda, \beta_1, \beta_2\},
$$

$$
0 \leq \varphi_i \leq 1, \quad i = 10, 11, 12, \quad 0 \leq \phi \leq 1,
$$

$$
0 \leq \beta_i \leq 1, \quad i = 1, 2, \quad \frac{1}{2} \leq \lambda \leq 1,
$$

which is numerically studied in the following subsection.

1. Numerical two-qubit state creation

Thus, we consider map (45-47) with the initial condition (43) where $\lambda^B$ takes the values from set (33). First of all, we would like to realize the createable state distribution in the
FIG. 3: The three-parameter receiver state-space \((\lambda, \beta_1, \beta_2)\) (the black points) of the map \((45, 47)\). The four node chain with two-qubit sender and the pure initial state \(\lambda^B = 1\) is considered at \(t_0 = 6.4\). Here we combine the parameters \(\beta_1\) and \(\beta_2\) in the single parameter \(z(\beta) = 10[10\beta_1] + [10\beta_2]\). Apparently, the region of the available states is uniform in \(\beta_2\), which is conformed by the step-like behavior of the right boundary of the createable region. (a) The whole region of createable states of the receiver. (b) The hardly available area of createable states (the vertical stripe to the left from the dashed line). (c) The right boundary of the region of createable states.

space of three parameters \((\lambda, \beta_1, \beta_2)\). For this purpose we fix \(\lambda^B = 1\) as an example (a pure initial state) and vary the parameters \(\varphi_i, i = 10, 11, 12\) and \(\phi\) inside of the region \((46)\) calculating the corresponding values of the parameters \(\lambda\) and \(\beta_i, i = 1, 2\). The family of all receiver’s states createable in this way is represented by the black points in Fig.3, where the parameters \(\beta_i, i = 1, 2\), are depicted along the ordinate axis as a ”combined” parameter \(z(\beta) = 10[10\beta_1] + [10\beta_2]\) and \([\cdot]\) means the integer part of a number. This figure demonstrates that the state distribution is uniform with respect to the parameter \(\beta_2\). In particular, this uniformity is confirmed by the step-like right boundary of the createable region. Thus, the essential parameters are \(\lambda\) and \(\beta_1\) (similar to Sec. III A 1). Perhaps, this means that there is a linear relation between \(\beta_2\) and a particular parameter of \(U^A\), similar to the relation between the parameters \(\beta_2\) and \(\varphi_2\) in Sec. III A. However, we do not establish such relation in this paper. We only remove \(\beta_2\) from the right side of map \((45)\) and study the two-parameter state-space \((\lambda, \beta_1)\) of the receiver. We chose the time instant \(t_0 = 6.4\) inside of interval \((42)\), because, at this time instant, all states createable during the above time interval for the initial state with \(\lambda^B = 1\) may be created using the map \((45, 47)\).

In this figure we see, that there is a region in the state-space which may not be created by the local transformations of the subsystem \(A\) (right upper corner in Figs.3a,c). However,
this statement is justified only for the initial state (43). Other types of the initial states might have different unavailable regions. There is another region in Fig.3 which is hardly createable. Conditionally, this region may be taken as a vertical stripe \( \frac{1}{2} \leq \lambda \leq 0.52 \), see the left side from the dotted line in Figs.1a,b. We shall give a following remark. Although we select the parameters \( \varphi_{10}, \varphi_{11} \) and \( \varphi_{12} \) in map (45), the similar results concerning the state transfer with \( \lambda^B = 1 \) are obtained for other selected triad of the parameters: \( (\varphi_1, \varphi_2, \varphi_3), (\varphi_4, \varphi_5, \varphi_6), (\varphi_7, \varphi_8, \varphi_9), (\varphi_9, \varphi_{10}, \varphi_{11}) \). This fact supports our assumption that simplified map (45) creates the maximal possible region for the given choice of the initial conditions.

Now, for the selected time instant \( t = 6.4 \), we represent a set of state-creating experiments for different initial conditions (43) corresponding to a set of \( \lambda^B \) given in (43). In all these cases, there is a boundary in the space \( (\lambda, \beta_1) \) separating the createable and unavailable regions, similar to Fig.3. The presence of the unavailable areas for all \( \lambda^B \) resonates with the fact that the perfect pure state transfer is impossible along the homogeneous four node chain. The boundaries corresponding to different values of \( \lambda^B \) are collected in Fig.4. Herewith the createable region is to the left from the appropriate boundary line, while the unavailable region is to the right. We see that the largest createable region corresponds to the pure initial states \( (\lambda^B = 1, 0) \). It is important that there is a small region which may not be created in the chain with initial state (43) and any choice of \( \lambda^B \). This absolutely unavailable region is indicated in Fig.4a and shown in Fig.4b using the proper scale. However, maybe, this region is createable in chains with different initial states. It is interesting to note that the curves in Fig.4b can be approximated as

\[
0.9914 - 2.0034(0.9999 - x)^{0.28} \quad \text{the upper curve,} \quad (48) \\
-0.0100 + 2.0369(1 - x)^{0.28} \quad \text{the lower curve.}
\]

Similar to Sec III.A1 the case \( \lambda^B = \frac{1}{2} \) allows us to create only the arbitrary diagonal states of the receiver, i.e. \( \frac{1}{2} \leq \lambda \leq 1 \) and \( \beta_1 = 0, \frac{\pi}{2} \).

2. Density function of createable state-space

To characterize the effectiveness of the state creation, we use the density function \( S(\lambda, \beta_1) \) introduced in Sec III.A2. Herewith, for each \( \lambda^B \) from set (43), we construct the family of all
FIG. 4: The two-parameter space \((\lambda, \beta_1)\) of the map (45-47). The four node chain with two-qubit sender is considered for set (33) of \(\lambda^B\) at \(t_0 = 6.4\). We represent the boundary curves separating the createable and unavailable regions of the receiver’s state space. The createable region is to the left from the appropriate boundary curve. There is a region which may not be created by any local transformation of the subsystem \(A\) (the absolutely unavailable region). The vertical stripe of states with \(\frac{1}{2} \leq \lambda \leq 0.52\) is hardly createable, see also Fig. (a) The createable and unavailable regions of the state space of receiver for \(\lambda^B = 1, 3/4, 1/4, 0\). (b) The absolutely unavailable region is bounded by the boundaries corresponding to the pure initial states \(\lambda^B = 1, 0\); solid lines represent the analytical curves (48).

createable states varying the parameters of the local transformation \(U^A\) inside of the region (46). We represent the contour plot of the density function in Figs. 5 for \(\varepsilon_{\beta_1} = 2\varepsilon_{\lambda} = \varepsilon\) with \(\varepsilon = 0.01\) and \(N^{st} = 1758276\). This value of \(N^{st}\) is related with the uniform splitting of the variation intervals (46) corresponding to the parameters \(\varphi_{10,12}\) and \(\varphi_{11}, \phi\), respectively, into 50 segments (51 points) and into 25 segments (26 points). Apparently, the density function is more uniform in comparison with that depicted in Fig. 2. This observation may be explained, in particular, by the fact that now we use four parameters (\(\varphi_i, i = 10, 11, 12, \phi\)) in map (45-47) instead of two parameters (\(\varphi_1\) and \(t\)) in map (30-32), which smooths the density function. The createable region is maximal for the pure initial states \((\lambda^B = 1, 0)\), as is shown in Figs. 2a,d. The maximal values \(S_{max}\) of the density function together with their coordinates \(\lambda_{max}\) and \(\beta_{1,\text{max}}\) for different \(\lambda^B\) from set (33) are collected in Table II.

The unavailable regions (the black right sides) and the hardly createable regions (the dark left sides) are well depicted in these figures. The most simple for realizations are the
TABLE II: The maximal values $S_{max}(\lambda_{max}, \beta_{1;max})$ of the density function of createable states for the two-qubit sender (the four node chain)

| $\lambda^B$ | 1 | $\frac{3}{4}$ | $\frac{1}{4}$ | 0 |
|-------------|---|---------------|---------------|---|
| $S_{max}$   | 36.96 | 45.36 | 43.36 | 35.28 |
| $\lambda_{max}$ | 0.8525 | 0.8475 | 0.8275 | 0.9975 |
| $\beta_{1;max}$ | 0.065 | 0.025 | 0.035 | 0.995 |

...
the state-creation. The meaning and possible application of such spots is not clear yet.

The numerical experiments (which are not represented in this paper) demonstrate that the increase of the length of the chain leads to the increase of the unavailable region of the state-space of the receiver. The same happens if we involve the remote node interactions. These phenomenon has to be studies with the purpose to reveal the family of states createable over the long distances.

Notice also that the absence of the unavailable regions in the three node chain (see Figs.1a,d) resonates with the fact that the considered case corresponds to the perfect pure state transfer. But it is not clear whether considering the longer chain with the parameters providing the perfect pure state transfer we may avoid the appearance of the unavailable region in the state-space of the receiver.

Author thanks Prof. E.B.Fel’dman and Dr. S.I.Doronin for useful comments. This work is partially supported by the Program of the Presidium of RAS No.8 ”Development of methods of obtaining chemical compounds and creation of new materials” and by the RFBR grant No.13-03-00017
V. APPENDIX: EXPLICITE FORM OF THE MATRICES $\gamma_i$ IN EQ.[37]

Below we give the list of matrices $\gamma_i$ representing the basis of the Lie algebra of $SU(4)$ [19]:

\[
\begin{align*}
\gamma_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\gamma_2 &= \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\gamma_3 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
\gamma_4 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\gamma_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\gamma_6 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
\gamma_7 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\gamma_8 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\gamma_9 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
\gamma_{10} &= \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \\
\gamma_{11} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\
\gamma_{12} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
\gamma_{13} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
\gamma_{14} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \\
\gamma_{15} &= \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.
\end{align*}
\]

These matrices are used in the general expression for the unitary transformation $U^A$, see eq.[37]

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FIG. 5: The density function of the two-parameter receiver state-space \((\lambda, \beta_1)\) of the map (45-47). The four node chain with the two-qubit sender is considered for set (\ref{eq:lambda}) of \(\lambda^B\). The bright regions to the left from the boundaries are most simple for the realization. The maximal values of the density function \(S_{\text{max}}\) together with the appropriate values of the parameters \(\lambda_{\text{max}}\) and \(\beta_{1,\text{max}}\) are collected in Table II. The family of bright spot embedded into the dark area appears for all \(\lambda^B\). (a, d) \(\lambda^B = 1\) and 0 respectively, the pure initial state; unlike the chain with the one-qubit sender, the unavailable region (the black area) appears even in this case. (b, c) \(\lambda^B = 3/4\) and 1/4 respectively, the mixed initial state; the unavailable regions are indicated by the black areas and these regions are similar to those shown in Figs. 2b, c.