Analysis and construction of a parabolic rotor profile for Roots vacuum pumps

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Abstract. Rotor profile is a critical factor affecting the performance of Roots vacuum pumps. Hence, research on rotor profiles of Roots vacuum pumps is important. The rotor profiles are mainly composed of arcs, involutes, cycloids, epicycloid, and combined curves. Moreover, the two rotor profiles form a pair of conjugate curves. To obtain a parabolic rotor profile, transformations of three coordinate systems are established to derive equations and analyze the parabolic curve and its conjugate curve. Additionally, the parameters of the parabolic rotor profile are discussed to determine their ranges and the relationship between the performance of a parabolic rotor profile and the independent parameters. The results indicate that the area efficiency of the parabolic rotor profile is greater than the area efficiency of the arc rotor profile for the same ratio of the top circle radius to pitch circle radius.

1. Introduction

Root vacuum pumps are rotary pumps that are widely used in the chemical, pharmaceutical, and food industries. A pair of rotors rotating in opposite directions is the key component that determines the performance of the pump. Therefore, many studies have been undertaken on the design and analysis of rotor profiles and exploration of new rotor profiles.

Various traditional rotor profiles such as arc, involute, cycloid are used independently or in combination with each other for different applications of Roots vacuum pumps [1-3]. An elliptical rotor profile having three independent geometric parameters and a higher area efficiency compared to the traditional rotor profile was previously studied [4]. The deviation-function method was proposed to reshape the original pitch-curve pairs of circular or noncircular and identical or non-identical curve rotors [5]. A mathematical model was proposed to design a novel rotor profile for multi-stage “IVEC” (Involute and variable Extend Cycloids) type Roots vacuum pump. This rotor profile demonstrated better geometric performance than traditional rotor profiles [6]. A trochoidal rotor profile with a variable trochoid ratio was proposed and studied to generate a smooth trochoid motion using fifth-order polynomial and cubic spline functions [2, 7]. A theoretical analysis of an internal lobe pump was conducted and an integrated automated system for rotor design was developed. The outer rotor was typically characterized by lobes with an elliptical shape, and the inner rotor profile was a conjugate of the outer profile [8]. A geometric approach for the analysis of rotor profiles was proposed based on the principles of instantaneous center and homogeneous coordinate transformation. The inner rotor profile was defined by a combination of two circular arcs. The radius of curvature of the outer rotor was derived
using the relationship between the trochoid ratio and the inner rotor tooth size ratio [9]. A method for designing the rotor profiles of twin-screw compressors was proposed using a rack defined in the normal plane. This method allowed the same hob that is used for screw rotors to be used for manufacturing mating rotors even with different helix angles [10]. To avoid undercutting the rotor profile, the concept of the limit curve was studied. Limit curve allows the designer to obtain significant graphical representations of design limits [11]. Many types of rotor profiles have been studied previously except for the parabolic rotor profile.

The design, analysis, and mathematical modelling of the rotor profile of Roots vacuum pumps as well as determination of the ranges of independent parameters are complicated processes [11]. The rotor profile of a Roots vacuum pump is a center-symmetric and axisymmetric curve. The two rotor profiles maintain continuous tangential contact during operation. Therefore, the rotor profile can be regarded as a pair of conjugate curves rotating in opposite directions. Moreover, the rotor profile of the Roots pump can be obtained by solving the characteristics equations of the conjugate curve. In this paper, we propose the design of a novel parabolic rotor profile using the conjugate curve principle. The equations of the parabolic curve and its conjugate curve are derived to form the parabolic rotor profile. In addition, the effect of independent parameters and their parabolic rotor profile ranges on the performance of a roots vacuum pump are analyzed in our study.

2. Mathematical model of parabolic rotor profile

2.1. Conjugate curves

To obtain the equation of the parabolic rotor profile, equations were derived from the parabolic curve and its conjugate curve. Conjugate curves can be described as a pair of smooth curves that maintains continuous tangential contact with each other in conjugate directions during motion [5]. For a pair of curves to be called conjugate curves in each motion process, the following conditions must be met:

(1) The pair should have smooth regular curves.

(2) At every moment, both the curves should only have a single point contact with each other on the tangent of the curves.

(3) Both curves remain in contact only for a moment during a cycle.

As shown in Figure 1, three coordinate systems were established for the conjugate curve: $x_1O_1y_1$, $x_2O_2y_2$, and $xOy$ referred to as coordinate systems $S1$, $S2$, and $S$, respectively. $S1$ and $S2$ were movable coordinate systems connected to rotor 1 and rotor 2, respectively. $S$ was an unmovable coordinate system. The coordinates of $M$ in the three coordinate systems were $(x_1, y_1)$, $(x_2, y_2)$, and $(x, y)$. Moreover, the transformation relationship between each coordinate system was analyzed as follows [3-6]:

We assumed that at a certain moment, the two rotors of the Roots vacuum pump rotated to the position shown in Figure 2, where rotor 1 rotated counterclockwise and rotor 2 rotated clockwise. Therefore, the relationship between the coordinate systems $S$ and $S1$ was expressed as

$$\begin{align*}
    x &= x_1 \cos(\varphi_1) - y_1 \sin(\varphi_1) \\
    y &= x_1 \sin(\varphi_1) + y_1 \cos(\varphi_1) - d_1
\end{align*}$$

(1)

Where $\varphi_1$ was the angle of the coordinate system $S1$ at a certain position from its initial position and $d_1$ was the distance from $O$ to $O_1$. 

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Figure 1. Three coordinate systems for the conjugate curve.

Figure 2. Coordinates of $M$ as per the three different coordinate system.
Similarly, the relationship between the coordinate systems S and S2 was obtained as

\[
\begin{align*}
    x_2 &= x_1 \cos \varphi_2 + y_1 \sin \varphi_2 \\
    y_2 &= -x_1 \sin \varphi_2 + y_1 \cos \varphi_2 + d_2
\end{align*}
\] (2)

Where \( \varphi_2 \) was the angle of coordinate system S2 at a certain position from its initial position and \( d_2 \) was the distance from \( O \) to \( O_2 \).

The relationship between the coordinate systems S1 to S2 was expressed as:

\[
\begin{align*}
    x_2 &= x_1 \cos (\varphi_1 + \varphi_2) - y_1 \sin (\varphi_1 + \varphi_2) + d \sin \varphi_2 \\
    y_2 &= x_1 \sin (\varphi_1 + \varphi_2) + y_1 \cos (\varphi_1 + \varphi_2) - d \cos \varphi_2
\end{align*}
\] (3)

Where \( d \) was the distance from \( O_1 \) to \( O_2 \), and \( d = d_1 + d_2 \). The two rotors of the Roots vacuum pump are of the same size and rotate in opposite directions at the same speed. Therefore, \( \varphi_1 = \varphi_2, d_1 = d_2 \).

2.2. Conjugate curve equations

The rotor profile of a root vacuum pump was axisymmetric and centrosymmetric. Therefore, we only needed to design 1/4th of the entire rotor profile as follows:

(1) Determining \( R \) or \( d \): As shown in Figure 2, \( R \) was the pitch circle radius of the rotor profile and was given by the formula \( R = d / 2 \). The pitch circle equation corresponding to coordinate system S1 was:

\[
x_1^2 + y_1^2 = R^2
\] (4)

(2) Determining the range of \( x_1 \): We drew a line \( y_1 = x_1 \). The angle between the line and the \( x_1 \) axis was \( \pi / 4 \). The known curve was located in the first quadrant of coordinate system S1 above the line. As shown in Figure 3, the line and the pitch circle intersect at point \( A(\sqrt{2}R/2, \sqrt{2}R/2) \). Therefore, \( x_1 \in (0, \sqrt{2}R/2) \).

(3) Determining the known partial curve equations: The profile equation of the known curve of the rotor is established using coordinate system S1. For example, a parabolic curve is represented by \( y_1 = -ax_1^2 + b \ (a > 0, b > 0) \). The parabolic curve passed through point \( A \), satisfying the following equation:

\[
\frac{\sqrt{2}}{2} R = -\frac{a}{2} R^2 + b
\] (5)

Generally, the value of radius of the rotor top circle \( b \) was provided during designing. Therefore, \( a \) was obtained by solving Eq (5).

(4) The known partial curve equation was substituted in Eq (3) to solve for the conjugate curve.

(5) We integrated the known partial curves from coordinate system S1 and its conjugate curve from coordinate system S2 into one coordinate system to obtain 1/4th profile of the rotor. The entire rotor profile is obtained by mirroring the 1/4th profile along the two coordinate axes.
Figure 3. Parabolic curve.

3. Parabolic rotor profile

As shown in Figure 3, the parabolic curve $AC$, line $O_1A$ ($y_1 = x_1$), and the pitch circle intersected at $A$. Moreover, the parabolic curve $AC$ and the $y_1$ axis intersected at $C (0, b)$. Therefore, the conjugate curve was designed in coordinate system $S_2$ as follows:

Assuming a point $M (x_1, y_1)$ on the parabolic curve $AC$, the slope of the tangent to the parabolic curve at this point was

$$\tan(\gamma) = -2ax_1$$  \hfill (6)

Where $\gamma$ was the angle between the slope line and the $x_1$ axis. Line $MP$ was normal to the parabolic curve at point $M$, and point $P$ was the intersection between the normal line and the pitch circle. We connected points $O_1$ and $P$. The angle between line $O_1P$ and $y_1$ axis was $\phi$. The point $M$ contacted its conjugate point when line $O_1P$ rotated $\phi$ counterclockwise about $O_1$. We drew a line $O_1L$ perpendicular to $MP$. The length of line $O_1L$ in RT$\Delta O_1LP$ is

$$O_1L = R \cos \left( \gamma + \phi - \frac{\pi}{2} \right)$$  \hfill (7)

Using coordinates of $M$ from coordinate system $S_1$, $O_1L$ was expressed as
Combining Eqs (7) – (8), the relationship between $\gamma$ and $\phi$ was expressed as

$$\phi = -\gamma + \arccos \left( \frac{x_1 \cos \gamma + y_1 \sin \gamma}{R} \right) + \frac{\pi}{2}$$  \hspace{1cm} (9)$$

Substituting the coordinates of $M$ on the parabolic curve $AC$ into Eq (4), the conjugate curve of the parabolic rotor profile is obtained as

$$\begin{cases} 
    x_2 = x_1 \cos (2\phi) - y_1 \sin (2\phi) - d \sin (\phi) \\
    y_2 = x_1 \sin (2\phi) + y_2 \cos (2\phi) + d \cos (\phi)
\end{cases}$$  \hspace{1cm} (10)$$

Eq (10) is the conjugate curve corresponding to the parabolic curve $AC$. $\phi$ was determined from Eq (6) and (9), and $x_1$ and $y_1$ were determined using the parabolic equation.

For the above analysis, we assumed that the pitch circle radius is $R = 15$, the top circle radius is $b = 20$. Therefore, $b / R = 1.333$ for the parabolic rotor profile. The parabolic curve was obtained from Eq (5) and its conjugate curve was obtained from Equations (6) and (10). The two curves are shown in Figure 4. As shown in Figure 5, the two curves were integrated by rotating the conjugate curve $\pi / 2$ degrees clockwise about $O_2$ and moving it to coordinate system $S1$ to obtain a 1/4th curve of the rotor profile. The entire parabolic rotor profile was obtained by mirroring the symmetrical 1/4th curve along the $x$ and $y$ axes.

Figure 4. Parabolic curve and its conjugate curve.
4. Discussion of the parabolic rotor profile

4.1. Top circle radius and pitch circle radius

In coordinate system \( S1 \), the normal line equation of any point \((x_0, y_0)\) on the parabolic curve \( AC \) was expressed as follows:

\[
y_1 - y_0 = \frac{1}{2ax_0}(x_1 - x_0)
\]  

(11)

The intersection of the normal line and the \( y_1 \)-axis was at point \( E (0, y_1 \) \), as shown in Figure 3. The normal line \( ML \) to the parabolic curve of the rotor profile intersected with arc \( AO \) at point \( P \) and with \( y_1 \)-axis at point \( E \). Point \( E \) was located below the arc \( AO \). If the point of intersection between the normal line \( ML \) and \( y_1 \)-axis was located above arc \( AO \), \( \phi \) would not have belonged in the range \( 0 \) to \( \pi / 4 \). Therefore,

\[
y_1 - \frac{1}{2a} \leq R
\]  

(12)

In the parabolic curve \( AC \), the range of \( y_1 \) was:

\[
\frac{R}{\sqrt{2}} \leq y_1 \leq b
\]  

(13)

Combining Eqs (12) – (13), the parameters of parabolic curve must satisfy the Eq (14):

\[
b - \frac{1}{2a} \leq R
\]  

(14)

Substituting Eq (5) into Eq (15), Eq (15) can be rewritten as follows:

\[
R \cdot a \leq 1.335
\]  

(15)
Replacing $a$ with $b$ based on Eq (6), Eq (16) can be expressed:

$$\frac{b}{R} \leq 1.374$$  \hspace{1cm} (16)

From Eq (16), it was observed that the ratio of the top circle radius to the pitch circle radius could not be greater than 1.374 to avoid the undercutting phenomenon and to ensure $\phi$ lies between 0 to $\pi / 4$ for the parabolic rotor profile. The top circle radius $b$ and pitch circle radius $R$ are independent parameters.

4.2. Comparison of results for variable $\lambda$ designs

The ratio of top circle radius to the pitch circle radius is given by

$$\lambda = \frac{b}{R}$$  \hspace{1cm} (17)

Based on the mathematical models of parabolic rotor profile, $\lambda$ could be not greater than 1.374. Four parabolic Root rotor profiles were plotted for $\lambda = 1.25$, $\lambda = 1.30$, $\lambda = 1.35$, $\lambda = 1.37$ and $R=15$. The results were shown in Figure 6.

![Figure 6. Design results of the parabolic rotor profile.](image-url)
The area efficiency $\eta$ is defined as:

$$\eta = \frac{A_1 - A_2}{A_1} \quad (18)$$

Where, $A_1$ is area of top circle, $A_2$ is the sum of the cross-sectional areas of rotor.

The variation in area efficiency with $\lambda$ is shown in Figure 7. As $\lambda$ increases, area efficiency increases. The area efficiency of the parabolic rotor profile was 0.445 as $\lambda=1.37$. Therefore, the area efficiency of the parabolic rotor profile is not necessarily higher than the efficiency of the traditional rotor profiles. For example, the area efficiency of the involute rotor profile is 0.518 at $\lambda=1.50$. However, the area efficiencies of involute rotor profile and arc rotor profile are 0.443 and 0.394, respectively, for $\lambda=1.37$. Therefore, the area efficiency of the parabolic profile can be considered equal to the involute rotor profile and greater than the arc rotor profile for $\lambda=1.37$. This indicates that the parabolic rotor profile had an area efficiency advantage as $\lambda$ was not greater than 1.374 in our study.

![Figure 7. Area efficiency varying with ratio of the top circle radius to the pitch circle radius.](image)

5. Conclusions

In this study, we proposed novel parabolic rotor profile for a Root vacuum pump based on the conjugate curves principle. We derived a geometric model, profile equation, and relation between the independent parameters and rotor profile. The ratio of the top circle radius to the pitch circle radius was analyzed to determine the range of $\lambda$.

The proposed parabolic rotor profile adds diversity to the rotor design of Roots vacuum pumps. The proposed parabolic rotor profile displays higher area efficiency than the involute rotor profile and the arc rotor profile for $\lambda = 1.374$. However, the disadvantage of the proposed parabolic rotor profile is that the value of $\lambda$ cannot exceed 1.374. This value of $\lambda$ is relatively smaller compared to the values of $\lambda$ for other rotor profiles. Therefore, further research needs to be conducted to extend the range of $\lambda$ for the parabolic rotor profile.

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