THE RADICAL OF THE KERNEL OF A CERTAIN DIFFERENTIAL OPERATOR AND APPLICATIONS TO LOCALLY ALGEBRAIC DERIVATIONS

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Abstract. Let \( R \) be a commutative ring, \( A \) an \( R \)-algebra (not necessarily commutative) and \( V \) an \( R \)-subspace or \( R \)-submodule of \( A \). By the radical of \( V \) we mean the set of all elements \( a \in A \) such that \( a^m \in V \) for all \( m \gg 0 \). We derive (and show) some necessary conditions satisfied by the elements in the radicals of the kernel of some (partial) differential operators, such as all differential operators of commutative algebras; the differential operators \( P(D) \) of \( (\text{noncommutative}) \ A \) with certain conditions, where \( P(\cdot) \) is a polynomial in \( n \) commutative free variables and \( D = (D_1, D_2, \ldots, D_n) \) are either commuting locally finite \( R \)-derivations or commuting \( R \)-derivations of \( A \) such that for each \( 1 \leq i \leq n \), \( A \) can be decomposed as a direct sum of the generalized eigen-subspaces of \( D_i \); etc. In particular, we show that the kernel of certain differential operators of \( A \) is a Mathieu subspace (see [Z2, Z3]) of \( A \). We then apply some results above to study \( R \)-derivations of \( A \), which are locally algebraic or locally integral over \( R \). In particular, we show that if \( R \) is an integral domain of characteristic zero and \( A \) is reduced and torsion-free as an \( R \)-module, then \( A \) has no nonzero locally algebraic \( R \)-derivations. We also show a formula for the determinant of a differential vandemonde matrix over a commutative algebra \( A \). This formula not only provides some information for the elements in the radical of the kernel of all ordinary differential operators of \( A \), but also is interesting on its own right.

1. Background and Motivation

Let \( R \) be a commutative ring and \( A \) an \( R \)-algebra (not necessarily commutative). A derivation \( D \) of \( A \) is a map from \( A \) to \( A \) such that \( D(a + b) = D(a) + D(b) \) and \( D(ab) = D(a)b + aD(b) \) for all \( a, b \in A \). If \( D \) is also \( R \)-linear, we call it an \( R \)-derivation of \( A \).

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For each \( a \in \mathcal{A} \), denote by \( \ell_a \) the map from \( \mathcal{A} \) to \( \mathcal{A} \) that maps \( b \in \mathcal{A} \) to \( ab \). We call the associative algebra generated by \( \ell_a \) \((a \in \mathcal{A})\) and all derivations of \( \mathcal{A} \) the Weyl algebra of \( \mathcal{A} \), and denote it by \( \mathcal{W}(\mathcal{A}) \). The subalgebra of \( \mathcal{W}(\mathcal{A}) \) generated by \( \ell_a \) \((a \in \mathcal{R})\) and all \( \mathcal{R} \)-derivations of \( \mathcal{A} \) will be denoted by \( \mathcal{W}_R(\mathcal{A}) \). We call elements of \( \mathcal{W}(\mathcal{A}) \) the differential operators of \( \mathcal{A} \).

For each \( \Phi \in \mathcal{W}(\mathcal{A}) \), it is well-known and also easy to check that there exist some derivations \( D = (D_1, D_2, \ldots, D_n) \) of \( \mathcal{A} \) and a polynomial \( P(\xi) \in \mathcal{A} \langle \xi \rangle \) (the polynomial algebra over \( \mathcal{A} \) in \( n \) noncommutative free variables \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \)) such that \( \Phi = P(D) \), where \( P(D) \) throughout this paper is defined by first writing all the coefficients of \( P(\xi) \) on the most left of the monomials in \( \xi \), and then replacing \( \xi_i \) by \( D_i \) for all \( 1 \leq i \leq n \). Furthermore, if \( \Phi \in \mathcal{W}_R(\mathcal{A}) \), the same is true with \( D_i \) \((1 \leq i \leq n)\) being \( \mathcal{R} \)-derivations of \( \mathcal{A} \) and \( P(\xi) \in \mathcal{R} \langle \xi \rangle \). We call the differential operator \( \Phi = P(D) \) an ordinary differential operator of \( \mathcal{A} \), if \( P(\xi) \) is univariate, and a partial differential operator of \( \mathcal{A} \) if \( P(\xi) \) is multivariate.

Next, we recall the following two notions of associative algebras that were first introduced in \([Z2, Z3]\).

**Definition 1.1.** An \( \mathcal{R} \)-subspace (or \( \mathcal{R} \)-submodule) \( V \) of an \( \mathcal{R} \)-algebra \( \mathcal{A} \) is said to be a Mathieu subspace (MS) of \( \mathcal{A} \) if for all \( a, b, c \in \mathcal{A} \) with \( a^m \in V \) for all \( m \geq 1 \), we have \( ba^m c \in V \) for all \( m \gg 0 \), i.e., there exists \( N \in \mathbb{N} \) (depending on \( a, b, c \)) such that \( ba^m c \in V \) for all \( m \geq N \).

Note that a MS is also called a Mathieu-Zhao space in the literature (e.g., see \([DEZ, EN, EH, EKC]\), etc.), as first suggested by A. van den Essen \([E2]\). The introduction of this notion is mainly motivated by the studies in \([M, Z1]\) of the well-known Jacobian conjecture (see \([K6, BCW, E1]\)). See also \([DEZ, EKC]\). However, a more interesting aspect of the notion is that it provides a natural but highly non-trivial generalization of the notion of ideals. Currently, this new notion has not been studied (nor understood) for the most of rings including the most of finite rings and finite dimensional algebras over a field.

**Definition 1.2.** \([Z3, p. 247]\) Let \( V \) be an \( \mathcal{R} \)-subspace (or a subset) of an \( \mathcal{R} \)-algebra \( \mathcal{A} \). We define the radical \( \mathfrak{r}(V) \) of \( V \) to be

\[
\mathfrak{r}(V) := \{ a \in \mathcal{A} \mid a^m \in V \text{ for all } m \gg 0 \}.
\]

When \( \mathcal{A} \) is commutative and \( V \) is an ideal of \( \mathcal{A} \), \( \mathfrak{r}(V) \) coincides with the radical of the ideal \( V \), which is defined as \( \mathfrak{r}(V) = \{ a \in \mathcal{A} \mid a^m \in V \text{ for some } m \geq 1 \} \). So this new notion generalizes the radical of ideals and is interesting on its own right. It is also crucial for the study of MSs. For example, it is easy to see that every \( \mathcal{R} \)-subspace \( V \) of an
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R-algebra $A$ with $\mathfrak{r}(V) \subseteq \text{nil}(A)$ (equivalently, $\mathfrak{r}(V) = \text{nil}(A)$, since $\text{nil}(A) \subseteq \mathfrak{r}(V)$ by definition) is a MS of $A$, where $\text{nil}(A)$ denotes the set of all nilpotent elements of $A$. We will frequently use this fact (implicitly) throughout this paper.

Recent studies show that many MSs arise from the images of differential operators, especially, from the images of locally finite or locally nilpotent derivations, of certain associative algebras (e.g., see $Z1, Z2, EWZ, EZ1, EZ2$ and $Z4, Z8$, etc.). Then one natural question is the following:

**Open Problem 1.3.** For which differential operator $L$ of $A$, the kernel $\text{Ker } L$ of $L$ is a MS of $A$?

Note that differential operators are among the most classical and fundamental subjects in mathematics. They have been extensively studied not only in theories of ODE and PDE, $D$-modules, differential or complex manifolds, etc., but also in many other different areas such as general theories of rings and algebras (e.g., see $Kh1$, $Kh2$ and the references therein). Nevertheless, it seems that the question above and the radical of the kernel of differential operators have not been studied before! It is presumably because MSs and the radical of (arbitrary) subspaces are still relatively very new notions. After all, they were introduced in $Z2, Z3$ only about a decade ago.

In this paper we study the open problem above. More precisely, we study the radicals of the kernels of some (ordinary or partial) differential operators of some $R$-algebras $A$, and show that for certain differential operators $\Phi$ of $A$, the kernel $\text{Ker } \Phi$ is indeed a MS of $A$. We also apply some results proved in this paper to study $R$-derivations of $A$ that are locally algebraic or locally integral over $R$ (see Definition 4.1). In particular, we show that if $R$ is an integral domain of characteristic zero and $A$ is reduced and torsion-free as an $R$-module, then $A$ has no nonzero $R$-derivation that is locally algebraic over $R$ (see Theorem 4.6). Finally, we also show a formula for the determinant of a differential vandemonde matrix over commutative algebras (see Proposition 5.1). This formula not only provides some information for the radicals of the kernels of ordinary differential operators of commutative algebras, but also is interesting on its own right.

**Arrangement and Content:** In Section 2 we assume that $A$ is commutative and derive some necessarily conditions for the elements in the radical of the kernel of an arbitrary differential operator of $A$ (see Theorem 2.1 and Corollary 2.5). Consequently, for every differential
operator $\Phi \in W(A)$ such that $\Phi 1_A$ is not zero nor a zero-divisor of $A$, the kernel $\text{Ker } \Phi$ is indeed a MS of $A$.

In Section 3, we drop the commutativity assumption on $A$ but assume that $(R,+)$ is torsion-free and $A$ is reduced and torsion-free as an $R$-module. We first show in Theorem 3.1 some necessary conditions satisfied by the elements in the radical of the kernel of a differential operator $P(D)$ of $A$, where $P(\cdot)$ is a polynomial over $R$ in $n$ commutative free variables and $D = (D_1, D_2, \ldots, D_n)$ are $n$ commuting $R$-derivations of $A$ such that for each $1 \leq i \leq n$, $A$ can be decomposed as a direct sum of the generalized eigen-subspaces of $D_i$.

We then show in Proposition 3.6 that if $R$ is an integral domain of characteristic zero, then the conclusions in Theorem 3.1 hold also for the differential operators of $A$ which are multivariate polynomials over $R$ in commuting locally finite $R$-derivations of $A$. Finally, we show in Proposition 3.7 that the similar conclusions as those in Proposition 3.6 (with the same assumptions on $R$ and $A$) hold also for all ordinary differential operators $\Phi$ in Theorem 3.1 and Propositions 3.6 3.7 with $\Phi 1_A \neq 0$, Ker $\Phi$ is indeed a MS of $A$.

In Section 4, we apply some results proved in Sections 2 and 3 to study some properties of $R$-derivation of $A$, which are locally algebraic or locally integral over $R$ (see Definition 4.1). We first show in Theorem 4.3 that if $A$ is commutative and $(A, +)$ is torsion-free, then every locally integral $D$ of $A$ has its image in the nil-radical nil $(A)$ of $A$. We then show in Theorem 4.6 that if $R$ is an integral domain of characteristic zero and $A$ is reduced and torsion-free as an $R$-module (but not necessarily commutative), then $A$ has no nonzero $R$-derivation that is locally algebraic over $R$.

In Section 5, we assume that $A$ is commutative and first show in Proposition 5.1 a formula for the determinant of a differential Vandermonde matrix over $A$. We then apply this formula in Proposition 5.4 to derive more necessary conditions satisfied by the elements in the radicals of the kernels of an ordinary differential operator of $A$. We point out in Remark 5.3 that the formula derived in Proposition 5.1 can also be used to derive formulas for the determinants of several other families of matrices. Therefore the formula is also interesting on its own right.

2. The Commutative Algebra Case

In this section, unless stated otherwise, $R$ denotes a unital commutative ring, $A$ a commutative unital $R$-algebra and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$
n noncommutative free variables. We denote by $\mathcal{A}(\xi)$ the (noncommutative) polynomial algebra in $\xi$ over $\mathcal{A}$, and by $\partial_i$ ($1 \leq i \leq n$) the $\mathcal{A}$-derivation $\partial/\partial\xi_i$ of $\mathcal{A}(\xi)$.

Once and for all, we fix in this section a nonzero $P(\xi) \in \mathcal{A}(\xi)$ and $n$ $\mathcal{R}$-derivations $D_i$ ($1 \leq i \leq n$) of $\mathcal{A}$. Write $D = (D_1, D_2, \ldots, D_n)$ and $P(\xi) = a_0 + \sum_{k=1}^d P_k(\xi)$ for some $a_0 \in \mathcal{A}$, $d \geq 1$ and homogeneous polynomials $P_k(\xi)$ ($1 \leq k \leq d$) of degree $k$ in $\xi$.

For each $u \in \mathcal{A}$, we set $\nabla Du := (D_1 u, D_2 u, \ldots, D_n u)$, and call it the gradient of $u$ with respect to $D$. When $D$ is clear in the context, we will simply write $\nabla Du$ as $\nabla u$.

We define $P(D)$ and $P(\nabla u)$ by first writing $P(\xi)$ with all the coefficients of $P(\xi)$ on the most left of the monomials in $\xi$, and then replacing $\xi_i$ by $D_i$ and $D_i u$, respectively, for each $1 \leq i \leq n$.

Note that every differential operator $\Psi$ in the Weyl algebra $\mathcal{W}_R(\mathcal{A})$ of $\mathcal{A}$ can be written as $\Psi = P(D)$ for some $\mathcal{R}$-derivations $D_i$ ($1 \leq i \leq n$) of $\mathcal{A}$ and $P(\xi) \in \mathcal{A}(\xi)$.

The main result of this section is the following:

**Theorem 2.1.** With the setting as above, let $u \in \mathcal{A}$ be such that $u^m \in \ker P(D)$ for all $1 \leq m \leq d$. Then

$$a_0 u^d = (-1)^d d! P_d(\nabla u).$$

Furthermore, if $u^{d+1}$ also lies in $\ker P(D)$, then

$$a_0 u^{d+1} = 0.$$

To show the theorem above, we need first the following two lemmas. The first lemma can be easily verified by using the mathematical induction, which is similar as the proof for the usual binomial formula. So we here skip its proof.

**Lemma 2.2.** Given $u \in \mathcal{A}$, let $\ell_u : \mathcal{A} \to \mathcal{A}$ be the map such that $\ell_u(a) = au$ for all $a \in \mathcal{A}$. Define $\text{ad}_u : \mathcal{W}(\mathcal{A}) \to \mathcal{W}(\mathcal{A})$ by setting $\text{ad}_u(\Psi) = [\ell_u, \Psi] := \ell_u \Psi - \Psi \ell_u$ for all $\Psi \in \mathcal{W}(\mathcal{A})$. Then for all $\Phi \in \mathcal{W}(\mathcal{A})$ and $k \geq 1$, we have

$$(\text{ad}_u)^k(\Phi) = \sum_{i=0}^k (-1)^i \binom{k}{i} u^{k-i} \Phi \circ \ell_u^i.$$

**Lemma 2.3.** Let $u \in \mathcal{A}$ and $d = \text{deg} P(\xi)$. Then the following statements hold:
1) there exists $Q(\xi) \in \mathcal{A}(\xi)$ with either $Q(\xi) = 0$ or $\deg Q(\xi) \leq d - 2$ such that

$$\text{(2.4)} \quad \text{ad}_u P(D) = \sum_{i=1}^{n} (D_i u)(\partial_i P)(D) + Q(D).$$

2) $(\text{ad}_u)^d P(D) = d! P(\nabla u)$.

Proof: 1) First, if $d = \deg P(\xi) = 0$, then the statement holds trivially, since $\mathcal{A}$ is commutative and hence $\text{ad}_u P(D) = 0$. So we assume $d = \deg P(\xi) \geq 1$. By the linearity of $\text{ad}_u$ and $D_i$’s and also by the commutativity of $\mathcal{A}$ we may assume $P(\xi) = \xi_1 \xi_2 \cdots \xi_d$ with $1 \leq i_j \leq n$ for all $1 \leq j \leq d$.

We use the induction on $d \geq 1$. If $d = 1$, then $\text{ad}_u D_{i_1} = \ell_{D_{i_1} u}$. Hence the statement holds by choosing $Q(\xi) = 0$. Assume that the statement holds for all $1 \leq d \leq m - 1$ and consider the case $d = m$.

Since $\text{ad}_u$ is a derivation of $W(\mathcal{A})$, we have

$$\text{ad}_u P(D) = \sum_{j=1}^{m} D_{i_1} \cdots (\text{ad}_u D_{i_j}) \cdots D_{i_m} = \sum_{j=1}^{m} D_{i_1} \cdots (\ell_{D_{i_j} u}) \cdots D_{i_m}$$

$$= \sum_{j=1}^{m} (\ell_{D_{i_j} u}) D_{i_1} \cdots \widehat{D_{i_j}} \cdots D_{i_m} + \sum_{j=2}^{m} [D_{i_1} \cdots D_{i_{j-1}}, \ell_{D_{i_j} u}] D_{i_{j+1}} \cdots D_{i_m},$$

where $\widehat{D_{i_j}}$ means that the term $D_{i_j}$ is omitted. Thus

$$\text{ad}_u P(D) = \sum_{j=1}^{m} (D_{i_j u}) D_{i_1} \cdots \widehat{D_{i_j}} \cdots D_{i_m}$$

$$+ \sum_{j=2}^{m} (\text{ad}_{-D_{i_j} u}(D_{i_1} \cdots D_{i_{j-1}})) D_{i_{j+1}} \cdots D_{i_m}.$$  

Applying the induction assumption to the terms $\text{ad}_{-D_{i_j} u}(D_{i_1} \cdots D_{i_{j-1}})$ ($2 \leq j \leq m$) in the sum above we see that there exists $Q(\xi) \in \mathcal{A}(\xi)$ with $Q(\xi) = 0$ or $\deg Q(\xi) \leq m - 2$ such that

$$\text{ad}_u P(D) = \sum_{j=1}^{m} (D_{i_j u}) D_{i_1} \cdots \widehat{D_{i_j}} \cdots D_{i_m} + Q(D)$$

$$= \sum_{j=1}^{n} (D_i u)(\partial_i P)(D) + Q(D).$$

Hence by the induction statement 1) follows.

2) First, by statement 1) it is easy to see that $(\text{ad}_u)^d P(D) = (\text{ad}_u)^d P_d(D)$. Then by the linearity of $\text{ad}_u$ and $D_i$’s and also by
the commutativity of \( A \) we may assume \( P_d(\xi) = \xi_{i_1} \xi_{i_2} \cdots \xi_{i_d} \) with \( 1 \leq i_j \leq n \) for all \( 1 \leq j \leq d \). Applying statement 1) \((d \text{ times})\) we have

\[
(ad_u)^d P(D) = \sum_{1 \leq k_1, k_2, \ldots, k_d \leq n} (D_{k_1}u)(D_{k_2}u) \cdots (D_{k_d}u)(\partial_{k_1} \partial_{k_2} \cdots \partial_{k_d} P).
\]

Then by the equation above and the commutativity of \( A \), it is easy to see that the equation in statement 2) follows. \( \blacksquare \)

Now we can prove the main result of this section.

**Proof of Theorem 2.1.** By Eq. (2.3) and Lemma 2.3 2) we have

\[(2.5)\]

\[
d! P_d(\nabla u) = (-1)^d \sum_{i=0}^d (-1)^i \binom{d}{i} u^{d-i} P(D) \circ \ell^i_u.
\]

By applying both sides of the equation above to \( 1 \in A \) and then using the condition \( u^i \in \ker P(D) \) \((1 \leq i \leq d)\), we get \( d! P_d(\nabla u) = (-1)^d u^d P(D) \cdot 1_A \). It is well-known and also easy to check that every derivation of a commutative ring annihilates the identity element of the ring. Hence \( d! P_d(\nabla u) = (-1)^d u^d a_0 \), i.e., Eq. (2.1) follows. Similarly, by applying Eq. (2.5) above to \( u \in A \) and using the condition \( u^i \in \ker P(D) \) \((1 \leq i \leq d + 1)\), we get \( d! P_d(\nabla u) u = 0 \). Then by Eq. (2.1) we get Eq. (2.2). \( \blacksquare \)

Two consequences of Theorem 2.1 are as follows.

**Corollary 2.4.** Let \( D, P(\xi) \) and \( a_0 \) be as in Theorem 2.1. If \( P(D) = 0 \), then \( a_0 = 0 \) and \( d! P_d(\nabla u) = 0 \) for all \( u \in A \).

**Proof:** Since \( P(D) = 0 \), we have \( A = \ker P(D) = r(\ker P(D)) \). Applying Eq. (2.2) to \( u = 1 \) we get \( a_0 = 0 \). Then the corollary follows immediately from Eq. (2.1). \( \blacksquare \)

**Corollary 2.5.** Let \( D, P(\xi), a_0 \) be as in Theorem 2.1, and \( \text{nil}(A) \) the nil-radical of \( A \), i.e., the set of all nilpotent elements of \( A \). Then the following statements hold:

1) \( r(\ker P(D)) \subseteq r(\text{Ann}(a_0)) \), where \( \text{Ann}(a_0) \) is the set of the elements \( b \in A \) such that \( a_0 b = 0 \);
2) if \( a_0 \) is not zero nor a zero-divisor of \( A \), then \( r(\ker P(D)) = \text{nil}(A) \) and \( \ker P(D) \) is a MS of \( A \);
3) if \( a_0 = 0 \), then we have

\[
r(\ker P(D)) \subseteq r\left( \{ u \in A \mid d! P_d(\nabla u) = 0 \} \right).
\]
In particular, if \( n = 1 \), i.e., \( D \) is a single derivation of \( A \), and the leading coefficient of \( P(\xi) \) is not a zero-divisor of \( A \), then
\[
\text{pr}(\text{Ker} \, P(D)) \subseteq \text{pr}\{u \in A \mid d!Du \in \text{nil} \, (A)\}.
\]

Proof: 1) Let \( u \in \text{pr}(\text{Ker} \, P(D)) \). Then there exists \( N \in \mathbb{N} \) such that \( u^m \in \text{Ker} \, P(D) \) for all \( m \geq N \). In particular, \( u^{Nk} = (u^N)^k \in \text{Ker} \, P(D) \) for all \( k \geq 1 \). Then by Theorem 2.1 we have \( a_0u^N = 0 \), and hence \( a_0u^\ell = 0 \) for all \( \ell \geq N \). Thus \( u \in \text{pr}(\text{Ann} \, (a_0)) \), and statement 1) follows.

2) Since \( a_0 \) is not zero nor a zero-divisor of \( A \), we have \( \text{Ann} \, (a_0) = \text{nil} \, (A) \). By Definition 2.2 it is easy to see that \( \text{pr}(\text{nil} \, (A)) \subseteq \text{nil} \, (A) \), and \( \text{nil} \, (A) \subseteq \text{pr}(V) \) for all \( R \)-subspace \( V \subseteq A \). Then by statement 1) we have \( \text{pr}(\text{Ker} \, P(D)) = \text{pr}(\text{nil} \, (A)) = \text{nil} \, (A) \) and \( \text{Ker} \, P(D) \) is a MS of \( A \).

3) Let \( v \in \text{pr}(\text{Ker} \, P(D)) \). Then there exists \( N \in \mathbb{N} \) such that \( v^m \in \text{Ker} \, P(D) \) for all \( m \geq N \). In particular, for all \( m \geq N \) we have \( v^{mk} = (v^m)^k \in \text{Ker} \, P(D) \) for all \( k \geq 1 \). Then by Eq. (2.1) and the condition \( a_0 = 0 \) we have \( v^m \in \{u \in A \mid d!P_d(\nabla u) = 0\} \) (for all \( m \geq N \)). Thus \( v \in \text{pr}\{u \in A \mid d!Du \in \text{nil} \, (A)\} \).

Furthermore, assume that \( n = 1 \) and the leading coefficient of \( P(\xi) \) is not a zero-divisor of \( A \). Then for all \( u \in A \) with \( d!P_d(\nabla u) = 0 \) we have \( d!(Du)^d = 0 \). Hence \( (d!Du)^d = 0 \) and \( d!Du \in \text{nil} \, (A) \). Then in this case we have \( v \in \text{pr}\{u \in A \mid d!Du \in \text{nil} \, (A)\} \). \( \quad \square \)

Example 2.6. Let \( R = \mathbb{C} \) and \( A \) the \( \mathbb{C} \)-algebra of all smooth complex valued functions \( f(x) \) over \( \mathbb{R} \). Let \( D = \frac{d}{dx} \). Then for each nonzero univariate polynomial \( P(\xi) \in \mathbb{C}[\xi] \), \( \text{Ker} \, P(D) \) is the set of solutions \( f(x) \in A \) of the ordinary differential equation \( P(D)f = 0 \).

Let \( \lambda_i \) (\( 1 \leq i \leq k \)) be the set of all distinct roots of \( P(\xi) \) in \( \mathbb{C} \) with multiplicity \( m_i \). Then it is well-known in the theory of ODE (e.g., see 14 or any other standard textbook on ODE) that \( \text{Ker} \, P(D) \) is the \( \mathbb{C} \)-subspace of \( A \) spanned by \( x^je^{\lambda_i x} \) for all \( 1 \leq i \leq k \) and \( 1 \leq j \leq m_i \).

Then it is readily verified directly from the fact above that \( \text{pr}(\text{Ker} \, P(D)) = \{0\} \) if \( P(0) \neq 0 \); and \( \text{pr}(\text{Ker} \, P(D)) = \mathbb{C} \) if \( P(0) = 0 \). Consequently, Theorem 2.7, Corollary 2.5, and also Proposition 5.4 in Section 5 all hold in this case.

Furthermore, from the example below we get a family of examples of MSs from the solution spaces of linear Partial Differential Equations.

Example 2.7. Let \( R = \mathbb{C} \) or \( \mathbb{R} \), and \( A \) be the \( R \)-algebra of all smooth \( R \)-valued functions \( f(x) \) over an open subset of \( \mathbb{R}^n \) (or let \( A \) be the polynomial algebra in \( n \) commutative free variables over \( R \)). Let \( D_i = \frac{\partial}{\partial x_i} \) (\( 1 \leq i \leq n \)) and \( D = (D_1, D_2, ..., D_n) \). Then for every partial
differential operator $\Phi$ of $A$, there exists a polynomial $P(\xi) \in A[\xi]$ such that $\Phi = P(D)$. Then by Corollary 2.5, we see that $\text{Ker } \Phi$, or equivalently, the solution space in $A$ of the PDE: $\Phi f = 0$, is a MS of $A$ as long as $\Phi 1_A \neq 0$.

We end this section with the following two remarks.

First, we will show in Propositions 3.7 and 5.4 that for the ordinary differential operators $\Psi$ of certain $R$-algebras $A$ (not necessarily commutative), the radical $r(\text{Ker } \Psi)$ also satisfies some other necessarily conditions (other than those in Theorem 2.1 and Corollary 2.5).

Second, Theorem 2.1 and Corollary 2.5 cannot be generalized to differential operators of noncommutative algebras, which can be seen from the following:

Example 2.8. Let $X$, $Y$ be two noncommutative free variables and $R\langle X, Y \rangle$ the polynomial algebra in $X$ and $Y$ over $R$. Let $J$ be the two-sided ideal of $R\langle X, Y \rangle$ generated by $Y^2$ and $A := R\langle X, Y \rangle / J$. Let $D = \partial / \partial X$ and $P(\xi) = 1 - X \xi \in A[\xi]$. Then $P(D) = I - \ell_X D$, where $I$ denotes the identity map of $A$, and $\ell_X$ the multiplication map by $X$ from the left. Let $f = XY \in A$. Then it is readily checked that for all $m \geq 1$, we have $P(D)(f^m) = 0$ but $P(D)(X f^m) = -X f^m \neq 0$. Therefore, $0 \neq f \in r(\text{Ker } P(D))$ and $\text{Ker } P(D)$ is not a MS of $A$.

3. Some Cases for Non-Commutative Algebras

In this section, unless stated otherwise, $R$ denotes a commutative ring such that the abelian group $(R, +)$ is torsion-free, and $A$ an $R$-algebra (not necessarily commutative) that is torsion-free as an $R$-module.

We denote by $I_A$, or simply $I$, the identity map of $A$, and by $\text{nil } (A)$ the set of all nilpotent elements of $A$. We say $A$ is reduced if $\text{nil } (A) = \{0\}$. Furthermore, for each $a \in A$, we denote by $\text{Ann}_\ell(a)$ the set of elements $b \in A$ such that $ab = 0$.

Let $D$ be an $R$-derivation of $A$. We say that $A$ is decomposable w.r.t. (with respect to) $D$ if $A$ can be written as a direct sum of the generalized eigen-subspaces of $D$. More precisely, $A = \bigoplus_{\lambda \in H} A_\lambda$, where $H$ is the set of all generalized eigenvalues of $D$ in $R$ and $A_\lambda = \sum_{i=1}^{\infty} \text{Ker}(D - \lambda I)^i$ for each $\lambda \in H$. It is easy to verify inductively that for all $m \geq 1$, $a, b \in A$ and $\lambda, \mu \in R$, we have

$$(3.6) \quad (D - (\lambda + \mu)I)^m(ab) = \sum_{i=0}^{m} \binom{m}{i} ((D - \lambda I)^ia)((D - \mu I)^{m-i}b).$$
Then by the identity above we have that $A_\lambda A_\mu \subseteq A_{\lambda+\mu}$ for all $\lambda, \mu \in H$. In other words, the decomposition $A = \bigoplus_{\lambda \in H} A_\lambda$ is actually an additive $R$-algebra grading of $A$.

Some examples of $R$-derivations with respect to which $A$ is decomposable are semi-simple $R$-derivations, for which $A_\lambda (\lambda \in H)$ coincides with the eigenspace of $D$ corresponding to the eigenvalue $\lambda$ of $D$, and also locally finite derivations when the base ring $R$ is an algebraically closed field (e.g., see [E1, Proposition 1.3.8]).

Throughout this section $D_i (1 \leq i \leq n)$ stand for $n$ commuting $R$-derivations of $A$, i.e., $D_i D_j = D_j D_i$ for all $1 \leq i, j \leq n$, such that $A$ is decomposable w.r.t. each $D_i$. Then there exists a semi-subgroup $\Lambda$ of the abelian group $(R^n, +)$ such that

$$A = \bigoplus_{\lambda \in \Lambda} A_\lambda,$$  

where for each $\lambda = (k_1, k_2, \ldots, k_n) \in \Lambda$,

$$A_\lambda = \bigcap_{i=1}^n \left( \bigcap_{j=1}^\infty \text{Ker}(D_i - k_i I)^j \right).$$

In particular,

$$A_0 = \bigcap_{i=1}^n \left( \bigcap_{j=1}^\infty \text{Ker} D_i^j \right).$$

Note also that each $A_\lambda (\lambda \in \Lambda)$ is invariant under $D_i (1 \leq i \leq n)$, and $A_\lambda A_\mu \subseteq A_{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda$.

Now, let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be $n$ commutative free variables and $R[\xi]$ the polynomial algebra in $\xi$ over $R$. We set $D := (D_1, D_2, \ldots, D_n)$ and fix a polynomial $0 \neq P(\xi) \in R[\xi]$.

Write $P(\xi) = \sum_{k=0}^d P_k(\xi)$ for some $d \geq 0$ and homogeneous polynomials $P_k(\xi)$ ($1 \leq k \leq d$) of degree $k$ in $\xi$. Let $P(D)$ be the differential operator of $A$ obtained by replacing $\xi_i$ by $D_i$ ($1 \leq i \leq n$). Since $D_i$’s are $R$-derivations and commute with one another, $P(D)$ is well-defined.

The first main result of this section is the following theorem which in some sense extends Theorem 2.1 to the differential operator $P(D)$ of the $R$-algebra $A$ (that is not necessarily commutative).

**Theorem 3.1.** With the setting as above, assume further that $A$ is reduced. Then the following statements hold:

1) if $P(0) = 0$ and $P_k(\xi)$ ($1 \leq k \leq d$) have no nonzero common zeros in $R^n$, then $r(\text{Ker} P(D)) \subseteq A_0$;

2) if $P(0) \neq 0$, then $r(\text{Ker} P(D)) = \{0\}$, and $\text{Ker} P(D)$ is a MS of $A$. 

In order to prove the theorem above, we need first to show some lemmas.

**Lemma 3.2.** Let \( R \) be an arbitrary commutative ring and \( A \) an \( R \)-algebra that is torsion-free as an \( R \)-module. Let \( D \) and \( P(\xi) \) be fixed as above. Then the following statements hold:

1) Ker \( P(D) \) is homogeneous w.r.t. the grading of \( A \) in Eq. (3.7), i.e.,

\[
\text{Ker} P(D) = \bigoplus_{\lambda \in \Lambda} (A_\lambda \cap \text{Ker} P(D)). \tag{3.10}
\]

2) Let \( \mathcal{Z}_\Lambda(P) \) be the set of \( \lambda \in \Lambda \) such that \( P(\lambda) = 0 \). Then

\[
\text{Ker} P(D) \subseteq \bigoplus_{\lambda \in \mathcal{Z}_\Lambda(P)} A_\lambda. \tag{3.11}
\]

**Proof:**

1) Since for each \( \lambda \in \Lambda \), \( A_\lambda \) is preserved by \( D_i \) (1 \( \leq i \leq n \)), and hence is also preserved by \( P(D) \), from which Eq. (3.10) follows.

2) Let \( 0 \neq u \in \text{Ker} P(D) \) and write \( u = \sum_{i=1}^{\ell} u_{\lambda_i} \) for some distinct \( \lambda_i \in \Lambda \) and \( u_{\lambda_i} \in A_{\lambda_i} \) (1 \( \leq i \leq \ell \)). Then by Eq. (3.10) we have that \( u_{\lambda_i} \in \text{Ker} P(D) \) for all 1 \( \leq i \leq \ell \). So we may assume \( \ell = 1 \) and \( u \in A_\lambda \) for some \( \lambda \in \Lambda \).

Write \( \lambda = (k_1, k_2, \ldots, k_n) \). For each 1 \( \leq j \leq n \), we define a non-negative integer \( r_j \) as follows. First, let \( r_n \) be the greatest non-negative integer such that \( (D_n - k_n I)^{r_n} u \neq 0 \), and inductively, for each 1 \( \leq j \leq n - 1 \), let \( r_j \) be the greatest non-negative integer such that

\[
(D_j - k_j I)^{r_j} \left( \prod_{s=j+1}^{n} (D_s - k_s I)^{r_s} \right) u \neq 0.
\]

Set \( \tilde{u} := \left( \prod_{j=1}^{n} (D_j - k_j I)^{r_j} \right) u \). Then \( 0 \neq \tilde{u} \in A_\lambda \), \( \tilde{u} \in \text{Ker} P(D) \), and \( D_j \tilde{u} = k_j \tilde{u} \) for all 1 \( \leq j \leq n \). Hence \( 0 = P(D)\tilde{u} = P(\lambda)\tilde{u} \). Since \( A \) is torsion-free as an \( R \)-module, we have \( P(\lambda) = 0 \), as desired. \( \square \)

**Definition 3.3.** Let \( B \) be a subset of \( R^n \) and \( \lambda \in A \). We say \( \lambda \) is an extremal element of \( B \) if for all \( m \geq 1 \), \( m\lambda \) can not be written as a linear combination of other elements of \( B \) with positive integer coefficients whose sum is less than or equal to \( m \).

The following lemma should be known. But for the sake of completeness, we include here a direct proof.

**Lemma 3.4.** Let \( R \) be a commutative ring such that the abelian group \((R, +)\) is torsion free. Then every nonempty finite subset \( B \) of \( R^n \) has at least one extremal element.
Proof: Write $B = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $\lambda_i \neq \lambda_j$ for all $1 \leq i \neq j \leq n$. We use induction on $n$. If $n = 1$, there is nothing to show. So we assume $n \geq 2$.

Consider first the case $n = 2$ with $\lambda_2 \neq 0$. If the lemma fails, then $m_1\lambda_1 = k_1\lambda_2$ and $m_2\lambda_2 = k_2\lambda_1$ for some $m_i, k_i \geq 1$ with $k_i \leq m_i$. Then $m_1m_2\lambda_2 = m_1(k_2\lambda_1) = k_2(m_1\lambda_1) = k_1k_2\lambda_2$. Hence $m_1m_2 = k_1k_2$, for $\lambda_2 \neq 0$ and $(R, +)$ is torsion-free, from which we have $m_1 = k_1$ (and $m_2 = k_2$). By the assumption that $(R, +)$ is torsion-free again, we have $\lambda_1 = \lambda_2$. Contradiction.

Now assume the lemma holds for all $2 \leq n \leq k$ and consider the case $n = k + 1$. If $\lambda_{k+1}$ is an extremal point of $A$, then there is nothing to show. Assume otherwise. Then there exist $m \geq 1$ and $c_i \in \mathbb{N}$ ($1 \leq i \leq k$) such that

$$m\lambda_{k+1} = \sum_{i=1}^{k} c_i\lambda_i,$$

(3.12)

$$1 \leq \sum_{i=1}^{k} c_i \leq m.$$

(3.13)

By the induction assumption the set $B' := \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ has an extremal element, say, $\lambda_1$. We claim that $\lambda_1$ is also an extremal point of the set $B$. Otherwise, there exist $q \geq 1$ and $c'_j \in \mathbb{N}$ ($2 \leq j \leq k + 1$) such that

$$q\lambda_1 = c'_{k+1}\lambda_{k+1} + \sum_{j=2}^{k} c'_j\lambda_j,$$

(3.14)

$$1 \leq c'_{k+1} + \sum_{j=2}^{k} c'_j \leq q.$$

(3.15)

Then by Eqs. (3.14) and (3.12) we have

$$mq\lambda_1 = mc'_{k+1}\lambda_{k+1} + m\sum_{j=2}^{k} c'_j\lambda_j$$

$$= c'_{k+1}\sum_{i=1}^{k} c_i\lambda_i + m\sum_{j=2}^{k} c'_j\lambda_j.$$

(3.16)
For the sum of all the coefficients of the linear combination on the right hand side of the equation above, by Eqs. (3.13) and (3.15) we have

\[ 1 \leq c_{k+1}^l \sum_{i=1}^k c_i + m \sum_{j=2}^k c_j^l \leq c_{k+1}^l m + m \sum_{j=2}^k c_j^l \]

(3.17) \[ = m(c_{k+1}^l + \sum_{j=2}^k c_j^l) \leq mq. \]

Then by Eqs. (3.16) and (3.17), \( \lambda_1 \) is not an extremal element of \( B' \), which contradicts the choice of \( \lambda_1 \). Therefore \( \lambda_1 \) is an extremal point of \( B \), and the lemma follows. \( \square \)

**Lemma 3.5.** Let \( 0 \neq u \in \mathfrak{r}(\text{Ker} \ P(D)) \) and write \( u = \sum_{i=1}^\ell u_{\lambda_i} \) for some distinct \( \lambda_i \in \Lambda \) \((1 \leq i \leq \ell)\) and \( 0 \neq u_{\lambda_i} \in A_{\lambda_i} \). Then for each extremal element \( \lambda_j \) of the set \( \{ \lambda_i | 1 \leq i \leq \ell \} \), either \( u_{\lambda_j} \) is nilpotent, or \( P_k(\lambda_j) = 0 \) for all \( 0 \leq k \leq d \).

**Proof:** Assume that \( u_{\lambda_j} \) is not nilpotent. Since \( \lambda_j \) is an extremal element of the set \( \{ \lambda_i | 1 \leq i \leq \ell \} \), it is easy to see that for each \( m \geq 1 \), the homogeneous component of \( u^m \) in \( A_{m\lambda_j} \) is equal to \( u_{\lambda_j}^m \). Since \( u^m \in \text{Ker} \ P(D) \) when \( m \gg 0 \), by Lemma 3.2, 1) and 2) we have \( u_{\lambda_j}^m \in \text{Ker} \ P(D) \) and \( P(m\lambda_j) = 0 \) for all \( m \gg 0 \). More explicitly, for all \( m \gg 0 \), we have

\[ 0 = P(m\lambda_j) = \sum_{k=0}^d m^k P_k(\lambda_j). \]

Since \((R, +)\) is torsion-free, by the vandemonde determinant we have \( P_k(\lambda_j) = 0 \) for all \( 0 \leq k \leq d \). \( \square \)

**Proof of Theorem 3.1** Let \( 0 \neq u \in \mathfrak{r}(\text{Ker} \ P(D)) \) and write \( u = \sum_{i=1}^\ell u_{\lambda_i} \) for some distinct \( \lambda_i \in \Lambda \) \((1 \leq i \leq \ell)\) and \( 0 \neq u_{\lambda_i} \in A_{\lambda_i} \). Let \( B \) be the set of all nonzero \( \lambda_i \) \((1 \leq i \leq \ell)\).

If \( B \neq \emptyset \), then by Lemma 3.4, \( B \) has at least one extremal element, say \( \lambda_j \). Then by Definition 3.3, \( \lambda_j \) is also an extremal element of the set \( B \cup \{0\} \). Since \( A \) is reduced, \( u_{\lambda_j} \) is not nilpotent. Then by Lemma 3.5, \( P_k(\lambda_j) = 0 \) for all \( 0 \leq k \leq d \).

If \( P(0) = 0 \), and \( P_k(\xi) \) \((1 \leq k \leq d)\) have no nonzero common zero in \( R^n \), then we have \( \lambda_j = 0 \), which is a contradiction. Therefore, in this case we have \( B = \emptyset \) and \( u \in A_0 \), whence the statement 1) follows.

If \( P(0) \neq 0 \), then we also have \( B = \emptyset \) and \( u \in A_0 \), for \( P_0(\lambda_j) = P(0) \neq 0 \). Furthermore, since \( P(0) \neq 0 \), by Lemma 3.2, 2) we have
0 \not\in \mathcal{Z}_R(P) \text{ and } A_0 \cap \ker P(D) = \{0\}, \text{ whence } u = 0. \text{ Contradiction. Therefore } \tau(\ker P(D)) \text{ in this case contains no nonzero element and statement 2) follows. } \square

Next, we show that Theorem 3.1 with some extra conditions holds also for commuting locally finite $R$-derivations. Recall that an $R$-derivation $D$ of an $R$-algebra $\mathcal{A}$ is \textit{locally finite} (over $R$) if for each $u \in \mathcal{A}$, the $R$-submodule of $\mathcal{A}$ spanned by elements $D^ku \ (k \geq 0)$ over $R$ is finitely generated as an $R$-module.

**Proposition 3.6.** Assume that $R$ is an integral domain of characteristic zero and $\mathcal{A}$ is a reduced $R$-algebra that is torsion-free as an $R$-module. Denote by $K_R$ the field of fractions of $R$ and $\bar{K}_R$ the algebraic closure of $K_R$. Let $P(\xi) \in R[\xi]$ and $D = (D_1, D_2, \ldots, D_n)$ be $n$ commuting locally finite $R$-derivations of $\mathcal{A}$. Write $P(\xi) = \sum_{k=0}^{d} P_k(\xi)$ with $P_k(\xi) \ (0 \leq k \leq d)$ being homogeneous of degree $k$. Then the following statements hold:

1) if $P(0) = 0$ and $P_k(\xi) \ (1 \leq k \leq d)$ have no nonzero common zeros in $\bar{K}_R^n$, then we have $\tau(\ker P(D)) \subseteq A_0$, where $A_0 = \bigcap_{i=1}^{n} (\sum_{m=1}^{\infty} \ker D_i^m)$;

2) if $P(0) \neq 0$, then $\tau(\ker P(D)) = \{0\}$, and $\ker P(D)$ is a MS of $A$.

**Proof:** Set $\bar{\mathcal{A}} = \bar{K}_R \otimes_R \mathcal{A}$. Since $\mathcal{A}$ is torsion-free as an $R$-module, the standard map $\mathcal{A} \simeq R \otimes_R \mathcal{A} \to K_R \otimes_R \mathcal{A}$ is injective, for by [AM Prop. 3.3] $K_R \otimes_R \mathcal{A}$ is isomorphic to the localization $S^{-1} \mathcal{A}$ with $S = R \setminus \{0\}$. Since every field is absolutely flat, the standard map $K_R \otimes_R \mathcal{A} \to \bar{K}_R \otimes_R \mathcal{A}$ is also injective. Therefore, we may view $\mathcal{A}$ as an $R$-subalgebra of $\bar{\mathcal{A}} = \bar{K}_R \otimes_R \mathcal{A}$ in the standard way and extend $D$ $K_R$-linearly to $\bar{\mathcal{A}}$, which we denote by $\bar{D} = (\bar{D}_1, \bar{D}_2, \ldots, \bar{D}_n)$.

Note that $\bar{D}_i \ (1 \leq i \leq n)$ are $n$ commuting $\bar{K}_R$-derivations of $\bar{\mathcal{A}}$, which are also locally finite over $\bar{K}_R$. Then $\bar{\mathcal{A}}$ by [E1 Proposition 1.3.8]) is decomposable w.r.t. $\bar{D}_i$ for each $1 \leq i \leq n$. By applying Theorem 3.1 to $P(\bar{D})$ and using the fact $\bar{A}_0 \cap \bar{\mathcal{A}} = \bar{A}_0$ we see that the proposition follows. \square

Next, we use the proposition above to show that Corollary 2.5 with some extra conditions can be extended to the ordinary differential operators of some noncommutative algebras.

**Proposition 3.7.** Let $R$, $\mathcal{A}$ be as in Proposition 3.6 and let $D$ be an arbitrary (single) $R$-derivation of $\mathcal{A}$. Then for every univariate polynomial in $0 \neq P(\xi) \in R[\xi]$, the following statements hold:
1) if $P(0) = 0$, then we have $\mathfrak{r}(\operatorname{Ker} P(D)) \subseteq \mathfrak{r}(A_0)$, where $A_0 = \sum_{j=1}^{\infty} \operatorname{Ker} D^j$;
2) if $P(0) \neq 0$, then $\mathfrak{r}(\operatorname{Ker} P(D)) = \{0\}$, and $\operatorname{Ker} P(D)$ is a MS of $A$.

Proof: The case $\deg P(\xi) = 0$ is trivial. So we assume $\deg P(\xi) \geq 1$.
Let $K_R$ be the field of fractions of $R$ with the algebraic closure $\bar{K}_R$, and set $\bar{A} = K_R \otimes_R A$. As pointed out in the proof of Proposition 3.6 we may view $A$ as an $R$-subalgebra of $\bar{A}$ in the standard way and extend $D$ $K_R$-linearly to a $K_R$-derivation of $\bar{A}$, which we denote by $\bar{D}$.

Let $V = \operatorname{Ker} P(D)$. Then $V$ is an $R$-subspace of $A$ preserved by $D$.
Set $\bar{V} = \bar{K}_R \otimes_R V$. Then $D|_{\bar{V}}$ as a $K_R$-linear map from $\bar{V}$ to $\bar{V}$ is algebraic over $K_R$, for $P(D|_{\bar{V}}) = P(D)|_{\bar{V}} = 0$ and hence $P(D|_{\bar{V}}) = 0$. It is well-known (e.g., see [Hu, Proposition 4.2]) that $\bar{V}$ can be decomposed as a direct sum of the generalized eigenspaces of $D|_{\bar{V}}$. Let $\bar{B}$ be the $K_R$-subalgebra of $\bar{A}$ generated by elements of $V$. Then $\bar{B}$ is $D$-invariant. Furthermore, by Eq. (3.6) it is easy to see that $\bar{B}$ is decomposable w.r.t. $D|_{\bar{B}}$.

Now let $u \in \mathfrak{r}(\operatorname{Ker} P(D))$. Then there exists $N \geq 1$ such that $u^m \in \operatorname{Ker} P(D)$, and hence is also in $\bar{B}$, for all $m \geq N$. Consequently, $u^m \in \mathfrak{r}(\operatorname{Ker} P(D|_{\bar{B}}))$ for all $m \geq N$. Write $P(\xi) = \sum_{k=0}^{d} P_k(\xi)$ (as before) with $P_k(\xi)$ ($0 \leq k \leq d$) homogeneous $k$ in $\xi$. Then $P_k(\xi)$ ($1 \leq k \leq d$) have no nonzero common zero in $K_R$, for $P(\xi)$ is a univariate polynomial of degree $d \geq 1$.

If $P(0) = 0$, then by applying Proposition 3.6 1) to $P(D|_{\bar{B}})$ (as a differential operator of $\bar{B}$), we have $u^m \in \sum_{j=1}^{\infty} \operatorname{Ker} \bar{D}^j$ for all $m \geq N$. Since $\operatorname{Ker} \bar{D}^j = A \cap \operatorname{Ker} \bar{D}^j$ for all $j \geq 1$, we further have $u^m \in A_0 = \sum_{j=1}^{\infty} \bar{D}^j$ for all $m \geq N$. Hence $u \in \mathfrak{r}(A_0)$ and statement 1) follows.

If $P(0) \neq 0$, then by applying Proposition 3.6 2) to $P(\bar{D}|_{\bar{B}})$ (as a differential operator of $\bar{B}$) we have $u^m = 0$, and hence $u = 0$, for $A$ is reduced. Therefore statement 2) also holds. \qed

We end this section with the following open problem which we believe is worthy of further investigations.

Open Problem 3.8. Let $R$ be an arbitrary commutative ring and $A$ an arbitrary unital noncommutative $R$-algebra. Let $D = (D_1, D_2, \ldots, D_n)$ be $n$ $R$-derivations of $A$ and let $Q(\xi)$ be a polynomial in $n$ noncommutative free variables $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ over $R$. Set $a_0 := Q(0)$ and denote by $\operatorname{Ann}_\ell(a_0)$ the set of all elements $b \in A$ such that $a_0 b = 0$. Is it always true that $\mathfrak{r}(\operatorname{Ker} Q(D)) \subseteq \mathfrak{r}(\operatorname{Ann}_\ell(a_0))$?
4. Some Applications to Locally Algebraic Derivations

In this section we use some results proved in the last two sections to derive some properties of locally algebraic derivations and locally integral derivations.

**Definition 4.1.** Let $R$ be a unital commutative ring, $A$ an $R$-algebra and $D$ an $R$-derivation of $A$.

1) We say $D$ is algebraic over $R$ if there exists a nonzero polynomial $p(t) \in R[t]$ such that $p(D) = 0$.

2) We say $D$ is locally algebraic over $R$ if for each $a \in A$, there exists a $D$-invariant $R$-subalgebra $A_1$ of $A$ containing $a$, and a nonzero polynomial $p_a(t) \in R[t]$ such that $p_a(D) \mid A_1 = 0$.

If $p(t)$ in statement 1) (resp., $p_a(t)$ in statement 2) for all $a \in A$) of the definition above can be chosen to be a monic polynomial, we say $D$ is integral (resp., locally integral) over $R$.

An example of a derivation that is locally algebraic but not algebraic is as follows.

**Example 4.2.** Let $x_i (i \geq 1)$ be a sequence of free commutative variables and $\mathbb{C}[x_i \mid i \geq 1]$ the polynomial algebra over $\mathbb{C}$ in $x_i (i \geq 1)$. Let $I$ be the ideal generated by $x_i^{i+1} (i \geq 1)$ and $A = \mathbb{C}[x_i \mid i \geq 1]/I$. Then it can be readily verified that $D := \sum_{i=1}^{\infty} x_i \partial/\partial x_i$ is a well-defined $\mathbb{C}$-derivation of $A$, which is locally algebraic but not (globally) algebraic over $\mathbb{C}$.

**Theorem 4.3.** Let $R$ be a commutative ring and $A$ a commutative $R$-algebra such that the abelian group $(A, +)$ is torsion-free. Then for every $R$-derivation $D$ of $A$, which is locally integral over $R$, the image $\text{Im } D := D(A) \subseteq \text{nil}(A)$, where $\text{nil}(A)$ denotes the nil-radical of $A$, i.e., the set of nilpotent elements of $A$.

**Proof:** Let $a \in A$, and let $A_1$ be a $D$-invariant $R$-subalgebra of $A$ and $p_a(t)$ a monic polynomial in $R[t]$ such that $a \in A_1$ and $p_a(D) \mid A_1 = 0$. Then by Corollary 2.4 we have $d!(Da)^d = 0$, where $d = \deg p_a(t)$. Since $(A, +)$ as an abelian group is torsion-free, we have $(Da)^d = 0$, whence $Da \in \text{nil}(A)$ and the theorem follows. $\square$

Since every nilpotent $R$-derivation of $A$ is locally integral over $R$, by Theorem 4.3 we immediately have the following:

**Corollary 4.4.** Let $R$, $A$ be as in Theorem 4.3 and let $D$ be a nilpotent $R$-derivation of $A$. Then $\text{Im } D \subseteq \text{nil}(A)$.

Furthermore, from the proof of Theorem 4.3 it is also easy to see that we have the following:
**Corollary 4.5.** Let $R$ and $A$ be as in Theorem 4.3. Assume further that $A$ is torsion-free as an $R$-module. Then for every $R$-derivation $D$ of $A$, which is locally algebraic over $R$, we have $\text{Im } D \subseteq \text{nil } (A)$.

Next, we consider the $R$-derivations of some reduced $R$-algebra $A$ (not necessarily commutative), which are locally algebraic over $R$.

**Theorem 4.6.** Let $R$ be a unital integral domain of characteristic zero and let $A$ be a unital reduced $R$-algebra (not necessarily commutative) that is torsion-free as an $R$-module. Then $A$ has no nonzero $R$-derivations that are locally algebraic over $R$. In particular, $A$ has no nonzero nilpotent $R$-derivations.

**Proof:** Let $D$ be an $R$-derivation of $A$ that is locally algebraic over $R$. Let $a \in A$, and $A_1$ be a $D$-invariant $R$-subalgebra of $A$ and $0 \neq p_a(t) \in R[t]$ such that $a \in A_1$ and $p_a(D)|_{A_1} = 0$. Then $a^m \in A_1 \subseteq \text{Ker } p_a(D)$ for all $m \geq 1$, whence $a \in \mathfrak{r}(\text{Ker } p_a(D))$.

Replacing $p_a(t)$ by $t p_a(t)$ we assume $p_a(0) = 0$. Then by applying Proposition 3.4 to the differential operator $p_a(D)$, we have $a \in \mathfrak{r}(A_0)$, where $A_0 = \sum_{i=1}^{\infty} \text{Ker } D^i$. Consequently, $\mathfrak{r}(A_0) = A$. Then by [Z2] Lemma 2.4 we have $A_0 = A$, i.e., $D$ is locally nilpotent.

Let $K_R$ be the field of fractions of $R$ and $B := K_R \otimes_R A$. As pointed out in the proof of Proposition 3.6 we may view $A$ as an $R$-subalgebra of $B$ in the standard way and extend $D K_R$-linearly to a $K_R$-derivation of $B$, which we denote by $\bar{D}$.

Let $a$, $p_a(t)$ be fixed as above, and $N \geq 1$ such that $D^N a = 0$. Write $p_a(t) = t^k h(t)$ for some $k \geq 1$ and $h(t) \in K_R[t]$ with $h(0) \neq 0$. Then $p_a(D)a = 0$ and $D^N a = 0$. Since $\gcd (p_a(t), t^N) = t^\ell$ in $K_R[t]$ with $\ell = \min \{ k, N \}$, we have $D^\ell a = 0$. Hence $D^k a = D^\ell a = 0$. Since $a$ is an arbitrary element of $A$, we have $D^k = 0$. Then by [Z2] Lemma 6.1 we have $D = 0$, whence the theorem follows. \qed

One remark on Theorem 4.6 is that, without the characteristic zero condition, the theorem may be false, which can be seen from the following example. For more integral derivations of algebras over a field of characteristic $p > 0$, see [N].

**Example 4.7.** Let $K$ be a field of characteristic $p > 0$, $A = K[x]$ and $D = d/dx$. Then $D^p = 0$. Hence $D$ is a nonzero $K$-derivation of $A$ that is algebraic over $K$.

One immediate consequence of Theorem 4.3, Corollary 4.5 and Theorem 4.6 is the following corollary which in some sense gives an affirmative answer to the so-called LNED conjecture proposed in [Z2] for
nilpotent, or locally integral, or locally algebraic derivations of certain
algebras.

**Corollary 4.8.** 1) Let $R, A$ be as in Theorem 4.3 and let $D$ be an
$R$-derivation of $A$. If $D$ is locally integral over $R$, then $D$ maps every
$R$-subspace of $A$ to a MS of $A$.

2) Let $R, A$ be as in Corollary 4.5 and let $D$ be an $R$-derivation of
$A$. If $D$ is locally algebraic over $R$, then $D$ maps every $R$-subspace of
$A$ to a MS of $A$.

We end this section with the following proposition which is not
needed elsewhere in this paper but is interesting on its own for the
study of the radical of the kernel of a derivation.

**Proposition 4.9.** Let $R$ be a commutative ring and $A$ a reduced
$R$-algebra (not necessarily commutative) such that $(A, +)$ is torsion-free.
Let $r \geq 1$, $a \in A$ and $D$ be an $R$-derivation of $A$ such that
$D^r a^m = 0$ for all $1 \leq m \leq 2^{r-1}$. Then $a \in \text{Ker } D$. Consequently, $\text{Ker } P(D) \subseteq \mathfrak{r}(\text{Ker } D) = \mathfrak{r}(\text{Ker } D^k)$ for all $k \geq 1$.

Note that when $A$ is commutative, the lemma follows easily from
Theorem 2.1 with $P(D) = D^r$. Here we give a proof without the com-
mutativity of $A$.

**Proof of Proposition 4.9** We use induction on $r$. The case $r = 1$
is obvious. So assume $r \geq 2$. Then $2r - 2 \geq r$ and for each $1 \leq k \leq 2^{r-2}$,
by the Leibniz rule we have

$$0 = D^{2r - 2} a^{2k} = \sum_{i=0}^{2r-2} \binom{2r - 2}{i} (D^i a^k)(D^{2r-2-i} a^k).$$

Since $D^j a^k = 0$ for all $j \geq r$, there is only one term in the sum above
that may not be equal to 0, namely, the term with $i = r - 1$. Therefore
$(2r - 2)(D^{r-1} a^k)^2 = 0$. Since $A$ is reduced and $(A, +)$ is torsion-free,
we have $D^{r-1} a^k = 0$ for all $1 \leq k \leq 2^{(r-1)-1}$. Then by the induction
assumption we have $a \in \text{Ker } D$.

Now let $k \geq 1$ and $u \in \mathfrak{r}(\text{Ker } D^k)$. Then there exists $N \geq 1$ such that
$u^m \in \text{Ker } D^k$ for all $m \geq N$. Applying the result proved above to $u^m$
($m \geq N$) we have $u^m \in \text{Ker } D$ for all $m \geq N$. Hence $u \in \mathfrak{r}(\text{Ker } D)$, and
$\mathfrak{r}(\text{Ker } D^k) \subseteq \mathfrak{r}(\text{Ker } D)$. Conversely, since $\text{Ker } D$ as an $R$-subalgebra of
$A$ is closed under the multiplication and $\text{Ker } D \subseteq \text{Ker } D^k$, we also have
$\text{Ker } D \subseteq \mathfrak{r}(\text{Ker } D)$ and $\mathfrak{r}(\text{Ker } D) \subseteq \mathfrak{r}(\text{Ker } D^k)$. Hence the proposition
follows. □
5. A Differential Vandermonde Determinant

Throughout this section $A$ stands for a commutative ring and $D$ for a derivation of $A$.

**Proposition 5.1.** Let $A$ and $D$ be fixed as above. Then for all $f \in A$ and $n \geq 1$, we have

$$\det \begin{pmatrix} f & f^2 & \cdots & f^n \\ D(f) & D(f^2) & \cdots & D(f^n) \\ D^2(f) & D^2(f^2) & \cdots & D^2(f^n) \\ \vdots & \vdots & \ddots & \vdots \\ D^{n-1}(f) & D^{n-1}(f^2) & \cdots & D^{n-1}(f^n) \end{pmatrix} = \alpha_n (Df)^{(n-1)} f^n,$$

(5.1)

where $\alpha_n = \prod_{k=1}^{n-1} k!$.

The main idea of the proof of the proposition above is to show that the matrix in Eq. (5.1) can be transformed by some elementary column operations to an upper triangular matrix whose $i$-th diagonal entry is equal to $(i-1)! (Df)^{i-1} f$ for all $1 \leq i \leq n$. For example, for the case $n = 2$, by subtracting from the second column the multiple of the first column by $f$ we get

$$\begin{pmatrix} f & f^2 \\ D(f) & D(f^2) \end{pmatrix} \Rightarrow \begin{pmatrix} f & 0 \\ D(f) & fD(f) \end{pmatrix}.$$ 

(5.2)

To see that this can be achieved for all $n \geq 2$, it suffices to show the following lemma, from which Proposition 5.1 immediately follows.

**Lemma 5.2.** Let $D$ and $f$ be as in Proposition 5.1 and $k \geq 2$. Then there exist $\alpha_{k,j} \in A \ (1 \leq j \leq k - 1)$ such that for each $0 \leq i \leq k - 1$, we have

$$D^i(f^k) - \sum_{j=1}^{k-1} \alpha_{k,j} f^{k-j} D^i(f^j) = \delta_{i,k-1} (k-1)! (Df)^{k-1} f,$$

(5.3)

where $\delta_{i,k-1}$ is the Kronecker delta function.

**Proof:** We use induction on $k$. If $k = 2$, then $\alpha_{2,1} = 1$ solves the equations in Eq. (5.3), as already pointed out in Eq. (5.2) above.

Assume that the lemma holds for some $k \geq 2$ and consider the case $k+1$. By writing $f^{k+1}$ as $f \cdot f^k$ and applying the Leibniz rule, we have for each $0 \leq i \leq k$

$$D^i(f^{k+1}) - fD^i(f^k) = \sum_{\ell=0}^{i-1} \begin{pmatrix} i \\ \ell \end{pmatrix} (D^{i-\ell} f) D^{\ell}(f^k)$$
Applying the induction assumption to $D^\ell(f^k)$ and noticing that $\ell = k - 1$, if and only if $i = k = \ell + 1$, since $\ell \leq i - 1 \leq k - 1$:

$$
= \delta_{i,k}k!(Df)^k f + \sum_{\ell=0}^{i-1} \binom{i}{\ell} (D^{i-\ell} f) \left( \sum_{j=1}^{k-1} \alpha_{k,j} f^{k-j} D^\ell(f^j) \right)
$$

$$
= \delta_{i,k}k!(Df)^k f + \sum_{j=1}^{k-1} \alpha_{k,j} f^{k-j} \sum_{\ell=0}^{i-1} \binom{i}{\ell} (D^{i-\ell} f) D^\ell(f^j)
$$

$$
= \delta_{i,k}k!(Df)^k f + \sum_{j=1}^{k-1} \alpha_{k,j} f^{k-j} \left( D^j(f^{j+1}) - f D^j(f^j) \right)
$$

$$
= \delta_{i,k}k!(Df)^k f + \alpha_{k,k-1} f D^i(f^k) - \alpha_{k,1} f^k D(f)
$$

$$
+ \sum_{j=2}^{k-1} (\alpha_{k,j-1} - \alpha_{k,j}) f^{k+1-j} D^i(f^j).
$$

Set

$$
\alpha_{k+1,j} := \begin{cases} 
-\alpha_{k,1} & \text{if } j = 1; \\
\alpha_{k,j-1} - \alpha_{k,j} & \text{if } 2 \leq j \leq k-1; \\
1 + \alpha_{k,k-1} & \text{if } j = k.
\end{cases}
$$

Then $\alpha_{k+1,j}$ ($1 \leq j \leq k$) solve the equations in Eq. (5.3) for the case $k + 1$ and $0 \leq i \leq k$. Hence, by induction the lemma follows. \(\square\)

**Remark 5.3.** One application of the formula in Eq. (5.1) is as follows. We first apply the formula to some special function $f(x)$ and derivative $D$, and then evaluate $x$ at a fixed point $c$. By doing so, we may get formulas for the determinants of several families of matrices, e.g., letting $f = x^d$, $D = x^m \frac{d}{dx}$ and $c = \pm 1$ for all $d,m \in \mathbb{Z}$. In particular, if we choose $d = -1$, $m = 0$ and $c = 1$, then with a little more argument we get the following formula with $0! := 1$ for all $n \geq 1$:

$$
\det \left( (i + j - 2)! \right)_{1 \leq i,j \leq n} = \left( \prod_{k=1}^{n-1} k! \right)^2.
$$

Another consequence of Proposition 5.1 is the following:

**Proposition 5.4.** Let $D$ be a derivation of $A$, $\xi$ be a free variable and $0 \neq P(\xi) = \sum_{i=0}^{d} c_i \xi^i \in A[\xi]$. Let $f \in A$ be such that $f^m \in \text{Ker} P(D)$ for all $1 \leq m \leq d + 1$. Then for each $0 \leq i \leq d$, we have

$$
\alpha_{d+1,c_i(Df)} \frac{1}{2(d+1)} f^{d+1} = 0,
$$
where $\alpha_{d+1} = \prod_{k=1}^{d} k!$

Proof: Let $B$ be the transpose of the matrix in Eq. (5.1) with $n-1 = d$. Since $P(D)(f^m) = 0$ for all $1 \leq m \leq d + 1$, we have $Bv = 0$, where $v$ denotes the column vector $(c_0, c_1, \ldots, c_d)^T$. Then the proposition follows from Eq. (5.1).

Corollary 5.5. Let $D, f, P(\xi)$ be as in Proposition 5.4. Assume further that $(\mathcal{A}, +)$ is torsion-free and that $c_i$ is not zero nor a zero-divisor of $\mathcal{A}$ for some $0 \leq i \leq d$. Then $fDf$ is nilpotent.

Proof: By Proposition 5.4 we have $\alpha_{d+1} c_i (Df)^{\frac{d}{2}(d+1)} f^{d+1} = 0$, and hence, $f^{d+1}(Df)^{\frac{d}{2}(d+1)} = 0$, for $(\mathcal{A}, +)$ is torsion-free and $c_i$ is not zero nor a zero-divisor of $\mathcal{A}$. Then $(fDf)^m = 0$ for all $m \geq \max\{d + 1, \frac{d}{2}(d + 1)\}$, whence the corollary follows.

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