Noncommutative solitonic black hole

Ee Chang-Young\(^1\), Kyoungtae Kimm\(^1\), Daeho Lee\(^1\) and Youngone Lee\(^2\)

\(^1\) Department of Physics and Institute of Fundamental Physics, Sejong University, Seoul 143-747, Korea
\(^2\) Research Institute for Natural Science, Hanyang University, Seoul 133-791, Korea

E-mail: cylee@sejong.ac.kr, ktk@theory.sejong.ac.kr, dhlee@theory.sejong.ac.kr and youngone@hanyang.ac.kr

Received 21 January 2012, in final form 26 March 2012
Published 4 May 2012
Online at stacks.iop.org/CQG/29/105008

Abstract

We investigate solitonic black hole solutions in three-dimensional noncommutative spacetime. We do this in gravity with a negative cosmological constant coupled to a scalar field. Noncommutativity is realized with the Moyal product which is expanded up to first order in the noncommutativity parameter in two spatial directions. With numerical simulation we study the effect of noncommutativity by increasing the value of the noncommutativity parameter starting from commutative solutions. We find that even a regular soliton solution in the commutative case becomes a black hole solution when the noncommutativity parameter reaches a certain value.

PACS numbers: 02.40.Gh, 04.25.dg, 04.50.Kd, 05.45.Yv

1. Introduction

Many candidate theories for quantum gravity, such as string theory and loop quantum gravity, suggest that spacetime may not be commutative at sufficiently high energy scales [1, 2]. Meanwhile, black holes in the early universe have been observed recently [3, 4]. Since the energy density of the early universe was very high, it would be interesting to know the effect of noncommutativity on the formation of a black hole. Black holes in three-dimensional spacetime have been extensively studied. One of the reasons is that gravity models in three dimensions are relatively easier to treat than models in four spacetime dimensions. The finding of the BTZ black hole solution [5] has increased a lot of interest in the subject.

In [6], the global vortex solution was studied by considering gravity with a negative cosmological constant coupled to a complex scalar field in three-dimensional commutative spacetime. There, the model Lagrangian with a global $U(1)$ symmetry was given by

$$S = \int d^3x \sqrt{-g} \left[ -\frac{1}{16\pi G_N} (R + 2\Lambda) + \frac{\lambda}{2} \phi^* \phi \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} (\phi^* \phi - v^2)^2 \right],$$

where $\phi(x)$ is a complex scalar field. The obtained solution was a cylindrically symmetric global $U(1)$ vortex solution which smoothly connects the false vacuum at the origin to the true vacuum at spatial infinity. The interest for global vortex solutions has usually arisen from the...
search of nontrivial topological solitons. However, the formation of a black hole by a global vortex is also interesting by itself. This type of black hole solution has been worked out in [6]. Depending upon the ratio of the cosmological constant to the Plank scale, the model supports a spacetime with a regular soliton or a charged black hole.

In this paper, we investigate the effect of noncommutativity on the above model and want to see whether global vortex solutions with black hole configuration are allowed in the noncommutative case. For this purpose, we use the same Lagrangian as in (1) except for the Moyal product between the field variables. For computational purpose, here we use the triad and spin connection instead of the metric as in [7]. Since it is hard to obtain an analytic solution even in the commutative case [6], our approach to find the solution is basically numerical. Our analysis is performed up to first order in the noncommutativity parameter.

Here, we use the noncommutative polar coordinates \((t, \hat{r}, \hat{\varphi})\) defined by the following commutation relation:

\[
[\hat{\theta}, \hat{\varphi}] = 2i\hbar, \quad 0 \text{ otherwise},
\]

where \(\hat{\theta} \equiv \hat{r}^2\). There are two reasons for using the above commutation relation. One is that it is equivalent with the canonical noncommutativity \([\hat{x}, \hat{p}] = i\hbar\) up to the first order in the noncommutativity parameter \(\theta\) [7]. The other is that in the polar coordinates rotational symmetry is more apparent than in the Cartesian coordinates. We note that a would be usual commutation relation for the noncommutative polar coordinates, \([\hat{r}, \hat{\varphi}] = i\hbar\), is not equivalent with the canonical noncommutativity, \([\hat{x}, \hat{p}] = i\hbar\), even up to the first order in \(\theta\) [7].

By the Weyl–Moyal correspondence [8], the physics in the noncommutative spacetime can be described by the physics in the commutative spacetime with the Moyal product. The Moyal product corresponding to the commutation relation (2) is given by

\[
(f \star g)(\rho, \varphi) = e^{i\hbar(\frac{\rho'}{\rho} + \frac{\varphi'}{\varphi})} f(\rho, \varphi) g(\rho', \varphi') \Bigg|_{(\rho, \varphi) = (\rho', \varphi')}.
\]

In the following section, we investigate black holes supported by soliton solutions of the noncommutative action extended from the commutative action (1). After expanding the noncommutative equations via the Moyal star product up to first order in \(\theta\), we look for the solutions for these commutative equations of motion and examine how the noncommutativity changes physics from the known commutative black hole solutions. In section 3, we conclude with discussion.

2. Noncommutative solitonic solutions

The pure gravity action with a negative cosmological constant can be written as the Chern–Simons action [9], and one can express the first two terms in the action (1) in terms of the triad \(e^a\) and the spin connection \(\omega^a(a = 0, 1, 2)\). Now we write the noncommutative version for the action (1) as

\[
\hat{S} = \frac{1}{8\pi G_N} \int \left( \hat{e}_a \wedge \hat{R}^a + \frac{\Lambda}{6} \epsilon^{abc} \hat{e}_a \wedge \hat{e}_b \wedge \hat{e}_c \right) + \int \hat{d}^3x \hat{\theta} \hat{L}[\hat{\phi}],
\]

where \(\hat{e}\) is the determinant of \(\hat{e}_a\) and \(\hat{R}^a\) is the curvature 2-form, \(\hat{R}^a = d\hat{\omega}^a + \frac{1}{2}\epsilon^{abc} \hat{\omega}_b \wedge \hat{\omega}_c\), and the Lagrangian \(\hat{L}[\hat{\phi}]\) for the scalar field is given by

\[
\hat{L}[\hat{\phi}] = -\frac{1}{4} (\partial_{\mu} \hat{\phi} \star g^{\mu\nu} \star \partial_{\nu} \hat{\phi} + \partial_{\mu} \hat{\phi} \star g^{\mu\nu} \star \partial_{\nu} \hat{\phi}) - \frac{\Lambda}{4} (\hat{\phi} \star \hat{\phi} - v^2) \star (\hat{\phi} \star \hat{\phi} - v^2).
\]

Here we use the following definition of the noncommutative metric:

\[
\hat{g}_{\mu\nu} = \frac{1}{2} \eta_{ab} \left( \hat{e}_a^\mu \star \hat{e}_b^\nu + \hat{e}_b^\mu \star \hat{e}_a^\nu \right).
\]
which is Hermitian and symmetric. Then the equations of motion are obtained as follows:

\[
\frac{\epsilon^{\mu\nu\rho}}{8\pi G_N} \left[ \hat{R}_\mu + \frac{\Lambda}{2} \epsilon_{abc} \hat{e}^b \wedge \hat{e}^c \right]_\nu + \frac{1}{6} \epsilon^{\mu\nu\rho} \epsilon_{abc} \left( \hat{e}^b \wedge \hat{e}^c \wedge \hat{L} \right)_\nu + \frac{1}{2} \epsilon^{\mu\nu\rho} \epsilon_{abc} \left( \hat{L} \wedge \hat{e}^b \wedge \hat{e}^c \right)_\nu + \frac{1}{8} \eta^{\mu\nu} \left[ \hat{e}_\mu \wedge \hat{e}_\nu \wedge \hat{\partial}_\rho \hat{\phi} \right] = 0.
\]

Class. Quantum Grav. 29 (2012) 105008

In terms of \((t, \rho, \phi)\) coordinates compatible with the commutation relation (2) as follows:

\[
\hat{T}_\mu \equiv \frac{d\hat{e}_\mu}{\rho} + \frac{1}{2} \epsilon_{abc} \left( \hat{\omega}_b \wedge \hat{e}_c + \hat{e}_b \wedge \hat{\omega}_c \right) = 0,
\]

\[
\hat{\partial}_\mu \left( \hat{\tilde{g}}^{\mu\nu} \wedge \hat{\partial}_\nu \hat{\tilde{e}} + \hat{\partial}_\nu \hat{\tilde{e}} \wedge \hat{\partial}_\mu \hat{\tilde{e}} + \hat{\partial}_\mu \hat{\tilde{e}} \wedge \hat{\partial}_\nu \hat{\tilde{e}} + \hat{\partial}_\nu \hat{\tilde{e}} \wedge \hat{\partial}_\mu \hat{\tilde{e}} + \hat{\partial}_\nu \hat{\tilde{e}} \wedge \hat{\partial}_\mu \hat{\tilde{e}} \right) = \hat{\lambda} \hat{\tilde{e}} \hat{\tilde{e}} + \hat{\partial}_\nu \hat{\tilde{e}} \hat{\tilde{e}} + \hat{\partial}_\mu \hat{\tilde{e}} \hat{\tilde{e}} = -2 \hat{\lambda} \hat{\tilde{e}}\hat{\tilde{e}}.
\]

In the commutative limit, \(\theta \to 0\), equations (7)–(9) reduce to the commutative ones [10, 6] as expected.

In the commutative case, a static and rotationally symmetric metric can be put into the following form [6]:

\[
ds^2 = -e^{2A(r)} B(r) \, dt^2 + \frac{dr^2}{B(r)} + r^2 \, d\phi^2.
\]

The triad and spin connection corresponding to this line element are given by [10]

\[
e^0 = e^A \sqrt{B} \, dt, \quad e^1 = \frac{1}{\sqrt{B}} \, dr, \quad e^2 = r \, d\phi
\]

\[
\omega^0 = -\sqrt{B} \, d\phi, \quad \omega^1 = 0, \quad \omega^2 = -\sqrt{B} \frac{d}{dr} (e^A \sqrt{B}) \, dr.
\]

In terms of \(A, B\) and \(\phi\), where \(\phi = |\phi(r)| e^{i\omega}\), the commutative equations of motion reduced from (7) to (9) are given by

\[
A' = 8\pi G_N r (|\phi'|^2),
\]

\[
B' = 2\Lambda r - 8\pi G_N r \left[ B (|\phi'|^2) + \frac{n^2}{r^2} |\phi|^2 + \frac{\lambda}{2} (|\phi|^2 - v^2)^2 \right],
\]

\[
|\phi|^2 + \left( A' + \frac{B'}{B} + \frac{1}{r} \right) |\phi'| = \frac{1}{B} \left( \frac{n^2}{r^2} + \frac{\lambda}{2} (|\phi|^2 - v^2) \right) |\phi|,
\]

and these are exactly the same equations that appeared in [6].

In line with the Weyl–Moyal correspondence and taking a hint from the commutative metric (10), we now make an ansatz for the noncommutative metric expressed in the \((t, \rho, \phi)\) coordinates compatible with the commutation relation (2) as follows:

\[
d\hat{s}^2 = -e^{\hat{A}(\rho)} \hat{B}(\rho) \, dt^2 + \frac{d\rho^2}{4\hat{B}(\rho)} + \rho \, d\phi^2.
\]

A noncommutative triad for the above metric compatible with the definition of the metric (6) can be chosen as

\[
\hat{e}^0 = e^{\hat{A}(\rho)} \sqrt{\hat{B}(\rho)} \, dt, \quad \hat{e}^1 = \frac{d\rho}{\sqrt{4\hat{B}(\rho)}}, \quad \hat{e}^2 = \sqrt{\hat{B}} \, d\phi.
\]

With the above choice of the triad, the noncommutative spin connection can be determined from the noncommutative torsion free condition (8):

\[
\hat{\omega}^0 = -\sqrt{\hat{B}(\rho)} \, d\phi, \quad \hat{\omega}^1 = 0, \quad \hat{\omega}^2 = -\sqrt{4\hat{B}(\rho)} \frac{d}{d\rho} (e^{\hat{A}(\rho)} \sqrt{\hat{B}(\rho)}) \, dt.
\]
Now we expand the metric functions $\hat{A}$ and $\hat{B}$, and the scalar field $\hat{\phi}$ for static global vortices with vorticity $n$ in terms of $\theta$ up to first order as follows:

$$e^{i\hat{A}(\theta)}\hat{B}(\rho) = e^{i\hat{A}(\theta)}\hat{B}(\rho) + \theta \hat{F}(\rho) + O(\theta^2),$$

$$\hat{B}(\rho) = \hat{B}(\rho) + \theta \hat{G}(\rho) + O(\theta^2),$$

$$\hat{\phi}(\rho, \varphi) = (\hat{\phi}(\rho) + \theta \hat{\Phi}(\rho))e^{in\varphi} + O(\theta^2).$$

With the following redefinitions of functions,

$$\hat{A}(\rho) \equiv A(r), \quad \hat{B}(\rho) \equiv B(r), \quad \hat{\phi}(\rho) \equiv |\phi|(r), \quad \hat{F}(\rho) \equiv F(r), \quad \hat{G}(\rho) \equiv G(r), \quad \hat{\Phi}(\rho) \equiv \Phi(r),$$

and from (7)–(9), we obtain the original commutative equations (11) in the zeroth order of $\theta$, and the following three equations for $F(r)$, $G(r)$ and $\Phi(r)$ in the first order of $\theta$:

$$a_1(r)F(r) + a_2(r)F'(r) + a_3(r)G(r) + a_4(r)G'(r) + a_5(r)\Phi(r) + a_6(r)\Phi'(r) + a_7(r)\Phi''(r) + a_8(r) = 0,$$

$$b_1(r)G(r) + b_2(r)G'(r) + b_3(r)\Phi(r) + b_4(r)\Phi'(r) + b_5(r) = 0,$$

$$c_1(r)F(r) + c_2(r)F'(r) + c_3(r)G(r) + c_4(r)G'(r) + c_5(r)\Phi(r) + c_6(r)\Phi'(r) + c_7(r) = 0.$$

All the coefficients in these equations are functions of the known commutative solution, $|\phi|$, $A$ and $B$, and are given in the appendix.

The numerical analysis was performed up to first order in the noncommutativity parameter $\theta$ for the vorticity $n = 1$. The cosmological constant, the Newton constant and the radial coordinate are scaled to $\Lambda_v = |\Lambda|/\lambda v^2$, $G_v = 8\pi G v^2$ and $\sqrt{\lambda} vr$, respectively.

In order to solve (18), we impose the Dirichlet conditions $F(0) = f_0$ and $G(0) = g_0$. The constants $f_0$ and $g_0$ are fixed in such a manner that the solution $\Phi(r)$ satisfies the following boundary conditions:

$$\Phi(0) = \Phi(\infty) = 0.$$  

We find the solutions by applying the iterative numerical procedure.

The effects of turning on $\theta$ on the global vortex solutions are shown in figure 1. This shows that the scalar field concentrates inward as $\theta$ grows. The gradient of the scalar field around $r = 0$ becomes steeper.

Figure 1. The noncommutative vortex solutions with $\Lambda_v = 0.1$ and $G_v = 1.33$ are plotted for the scalar field $|\phi|/v$. The dashed line is for the case of $\theta = 0$. The thin and thick solid lines are for the cases of $\theta = 0.1$ and 0.15, respectively.
Now, in order to investigate whether our solitonic configuration can develop a black hole solution, we first look for the zeros of the metric function $g_{tt}(r)$, or equivalently, $B(r)$. We then check the obtained zeros by evaluating the Kretschmann scalar $K = \tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}$. When the Kretschmann scalar is regular, then the corresponding point in the space is nonsingular. Thus one can say that the zeros of $B$ having regular values of the Kretschmann scalar are not singularities, but black hole horizons.

In figure 2, the metric function $\tilde{B}$ corresponding to the global vortices in figure 1 is drawn. Note that the commutative solution ($\theta = 0$) has no horizon. On the other hand, when the value of the noncommutativity parameter $\theta$ reaches 0.057, the solution becomes an extremal black hole solution. When the value of $\theta$ becomes larger than this value, for instance $\theta = 0.2$, it becomes a nonextremal black hole solution. When the starting commutative solution is a nonextremal black hole solution with $\Lambda_v = 0.08$, the corresponding noncommutative solution for $\theta = 0.2$ is shown in figure 3. The effect of turning on $\theta$ for already black holes in
Figure 4. The plot of the noncommutative Kretschmann scalar with $\Lambda_v = 0.1$ and $G_v = 1.33$. The dashed line is for the case of $\theta = 0$. The thin and thick solid lines are for the cases of $\theta = 0.057$ and $0.2$, respectively.

Figure 5. The plot of the gravitational mass increase versus $\theta$ with $\Lambda_v = 0.1$ and $G_v = 1.33$.

commutative spacetime is shown. As the noncommutativity parameter $\theta$ gets bigger, the area of the outer horizon increases while the separation between the inner and outer horizons grows. This effect is what we would expect when the mass of a nonextremal black hole increases in the commutative case.

In order to see whether the singularities of the metric at zeros are coordinate artifacts or true physical singularities, the Kretschmann scalar $K$ for the solutions having one or two zeros in figure 2 is plotted in figure 4. This shows that the Kretschmann invariant for the regions around these zeros behaves regularly. From this result, we can say that the zeros are not genuine singularities; rather they correspond to the horizons of black holes.

The increase in gravitational mass for different values of $\theta$ by using the Hamiltonian formalism of [11] is plotted in figure 5. The increase in gravitational mass for a given $\theta$ is defined by

$$\Delta M \equiv H(\theta \neq 0) - H(\theta = 0),$$

where $H$ denotes the Hamiltonian. The result shows that the gravitational mass increases as the noncommutativity parameter $\theta$ increases. This linearity shown in figure 5 is due to our analysis.
which was performed up to first order in \( \theta \). The gravitational mass defined here depends only on the asymptotic geometrical quantities at spatial infinity. The metric is fully determined by the scalar field and its derivatives which are constant at spatial infinity. Since the coefficient of \( \theta \) is constant, it implies the linear dependence of mass on \( \theta \).

3. Conclusion

Our result sums up as follows. The ‘inward’ behavior of the global vortex in figure 1 results in higher peaks of the Kretschmann scalar near \( r = 0 \) as shown in figure 4. These peaks correspond to gravitational energy concentrations. We may interpret this as the noncommutativity of space makes the scalar soliton to have higher concentration near the center than in the commutative case. Thus we can conclude that the above-concentrated scalar soliton behaves as the concentration of gravitational energy to such an extent that a black hole can form.

For other related aspects, we discuss two things. First, in the space of solutions a solution with no singularity changes into a black hole solution as \( \theta \) increases. In other words, a regular solution becomes a black hole solution at a certain value of the noncommutativity parameter \( \theta \) as we increase the value of \( \theta \). This is not what we expected before. A ‘phase change’ in the solution space happens as the noncommutativity parameter changes.

Second, comparing with the result obtained in [7] for the noncommutative BTZ black hole, the role of the scalar field in this work on forming a black hole is similar to that of the magnetic flux \( \theta \) there. The shift of the horizons of the noncommutative black holes in that paper depends upon both \( \theta \) and the magnetic flux \( \theta \) at the origin. The shift of horizons obtained in this work depends on both \( \theta \) and on the scalar field. Thus, one may infer that the global vortex around the origin takes over the role of the magnetic flux in [7].

Acknowledgments

This work was supported by the National Research Foundation (NRF) of Korea grants funded by the Korean government (MEST) R01-2008-000-21026-0 and NRF-2009-0075129 (EC-Y, KK and DL), and The Korea Research Foundation Grant funded by the Korea Government (MOEHRD), KRF-2008-314-C00063 (YL)

Appendix

The coefficients appearing in equations (18) for \( F(r) \), \( G(r) \), \( \Phi(r) \) are given as follows:

\[
\begin{align*}
    a_1(r) &= -r[(n^2 - \lambda v^2 r^2)\phi + \lambda r^2 \phi^3 - rB((1 - rA')\phi' + r\phi'')] \\
    a_2(r) &= r^3 B\phi' \\
    a_3(r) &= r e^{2A}[(n^2 - \lambda v^2 r^2)\phi + \lambda r^2 \phi^3 + rB((1 + rA')\phi' + r\phi'')] \\
    a_4(r) &= e^{2A} r^3 B\phi' \\
    a_5(r) &= -2 e^{2A} r B(n^2 - \lambda v^2 r^2 + 3\lambda r^2 \phi^2) \\
    a_6(r) &= 2 e^{2A} r^2 B(B + rB' + rB') \\
    a_7(r) &= 2 e^{2A} r^3 B^2 \\
    a_8(r) &= n B e^{2A} [(n^2 - 2\lambda v^2 r^2)\phi' - 2\lambda r^2 \phi^3 A' - r^2(A' B' \phi' + B (A' B' \phi' + 2A' \phi' + 2A' \phi'))] \\
    b_1(r) &= \frac{1}{2} [r(B' - 2Ar) + 4\pi G_N (2(n^2 - \lambda v^2 r^2)\phi^2 + \lambda r^2 \phi^3 + r^2(\lambda v^2 - 2B\phi^2))] \\
    b_2(r) &= -rB \\
    b_3(r) &= -16\pi G_N B\phi(n^2 - \lambda v^2 r^2 + \lambda r^2 \phi^2)
\end{align*}
\]
\[ b_4(r) = -16\pi G_N e^2 B^2 \phi', \]
\[ b_5(r) = 8\pi G_N n r B \phi (\phi^2 - v^2) \phi', \]
\[ c_1(r) = \frac{1}{2} r (2 \Lambda r + 2 B A' + B') - 4\pi G_N (2(n^2 - \lambda v^2 r^2) \phi^2 + \lambda r^2 \phi^4 + r^2(\lambda v^4 - 2B \phi'^2)), \]
\[ c_2(r) = -r B, \]
\[ c_3(r) = -r e^{2\Lambda} B (B' + 2B(A' - 4\pi G_N r \phi'^2)), \]
\[ c_4(r) = -16\pi G_N e^2 B \phi (n^2 - \lambda v^2 r^2 + \lambda r^2 \phi^2), \]
\[ c_5(r) = 16\pi G_N e^{2\Lambda} r^2 B^2 \phi', \]
\[ c_6(r) = 8\pi G_N e^2 n \lambda r B \phi (\phi^2 - v^2) \phi'. \]

References

[1] Seiberg N and Witten E 1999 String theory and noncommutative geometry J. High Energy Phys. JHEP09(1999)032 (arXiv:hep-th/9908142)
[2] Freidel L and Livine E R 2006 3D Quantum gravity and effective noncommutative quantum field theory Phys. Rev. Lett. 96 221301 (arXiv:hep-th/0512113)
[3] Fan X et al 2001 High-redshift quasars found in Sloan Digital Sky Survey Commissioning Data IV: luminosity function from the fall equatorial stripe sample Astron. J. 121 54 (arXiv:astro-ph/0008123)
[4] Willott C J et al 2007 Four quasars above redshift 6 discovered by the Canada–France High-z Quasar Survey Astron. J. 134 2435 (arXiv:0706.0914)
[5] Banados M, Teitelboim C and Zanelli J 1992 The black hole in three dimensional space time Phys. Rev. Lett. 69 1849 (arXiv:hep-th/9204099)
[6] Kim N, Kim Y and Kimm K 1997 Global vortex and black cosmic string Phys. Rev. D 56 8029 (arXiv:gr-qc/9707056)
[7] Ee C-Y, Lee D and Lee Y 2009 Noncommutative BTZ black hole in polar coordinates Class. Quantum Grav. 26 185001 (arXiv:0808.2330)
[8] Groenewold H J 1946 On the principles of elementary quantum mechanics Physica 12 405
Moyal J E 1949 Quantum mechanics as a statistical theory Proc. Camb. Phil. Soc. 45 99
[9] Achucarro A and Townsend P K 1986 A Chern–Simons action for three-dimensional anti-de Sitter supergravity theories Phys. Lett. B 180 89
Witten E 1988 (2+1)-dimensional gravity as an exactly soluble system Nucl. Phys. B 311 46
[10] Carlip S, Gegenberg J and Mann R B 1995 Black holes in three dimensional topological gravity Phys. Rev. D 51 6854 (arXiv:gr-qc/9410021)
[11] Hawking S W and Horowitz G T 1996 The gravitational Hamiltonian, action, entropy, and surface terms Class. Quantum Grav. 13 1487 (arXiv:gr-qc/9501014)