Optimization Model for a Location-Allocation-Routing in a Periodic Distribution Network

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Abstract. The main concern of this paper is to decide the locations and allocations of facilities that impact not only the profitability of an organization but the fast ability to serve customers. Generally the location-routing problem is to minimize the overall cost by simultaneously selecting a subset of candidate facilities and constructing a set of delivery routes that satisfy some restrictions. In this paper we consider the periodic routing of vehicle to be used to distribute goods. Integer programming model is built to solve the problem. A feasible neighbourhood search is developed to solve the result model.

1. Introduction

The design of a distribution system involves a decision to select and find the best locations for facilities and to allocate customers to the selected facilities. These decision problem can be solved using location-allocation models. The objective of such models is to select the optimal locations of facilities from a list of candidate such that the total transportation cost from facilities to customers is minimized and an optimal number of customers have to be allocated in an area of interest in order to satisfy the customer demands. Therefore determining the locations of facilities within a distribution network is an important decision that impacts not only the profitability of an organization but the ability to serve customers. The term allocation implies rules which specify how demands are allocated to the candidate locations. There are three primary components in the location-allocation models, viz., the customer (or demand) locations, the list for candidate location, and a distance or travelling time between facilities and customer locations.

Facility location problems have attracted many researchers and have been applied to many real world problems. At the beginning, facility location problem is proposed by Weber [3]. He introduced, what is called, a Weber facility location problem, to decide location of a warehouse in such a way to minimize the distance traveled between the warehouse and its customers. Tewari and Jena (1987) used facility location model to improve geographical accessibility to public schools in rural area in India. Location-allocation models play important roles for designing health facilities such as locating the best sites for service facilities in a new area, evaluating the efficiency of the past location decisions and improving existing location patterns (Rahman & Smith, 2000). The authors provide an excellent review of location-allocation literature that employed location-allocation models in planning health care systems in developing nations. Buzai (2013) also used this model to locate the primary health care centers in the city of Lujan, Argentina. Polo et al., (2015) integrate location-allocation model with accessibility in order to improve the spatial planning of public health services.
In location modeling deliveries are made on out-and-back routes visiting a single customer or most frequently the customer who travel individually to the facility site. The consequence is, the cost of delivery is independent of other deliveries made. In many contexts, however, deliveries are made along multiple stop routes visiting two or more customers; in this case, the cost of delivery depends on the other customers on the route and the sequence in which they are visited. In order to capture accurately the cost of multiple stop routes within a location model, the routing problem must be solved at the same time as the location problem. This type of problem is called location-routing problem.

In its most general form, the location-routing problem (LRP) seeks to minimize total cost by simultaneously selecting a subset of candidate facilities and constructing a set of delivery routes that satisfy the following constraints:
- Customer demands are satisfied without exceeding vehicle or facility capacities.
- The number of vehicles, the route lengths and the route durations do not exceed the specified limits and, Each route begins and ends at the same facility.

Location-routing problems are clearly related to both the classical location problem and the vehicle routing problem. In fact, both of the latter problems can be viewed as special cases of the LRP. If we require all customers to be directly linked to a depot, the LRP becomes a standard location problem. If, on the other hand, we fix the depot locations, the LRP reduces to a VRP. From a practical viewpoint, location-routing forms part of distribution management, while from a mathematical point of view, it can usually be modeled as a combinatorial optimization problem. We note that this is an NP-hard problem, as it encompasses two NP-hard problems (facility location and vehicle routing). Since a number of problem versions exist, we cannot reproduce all the formulations here. In the first instance, the reader is referred to [5] for an excellent review of various formulations. Most of the research to date has focused on heuristic methods since LRPs merge two NP-hard problems. The heuristics generally decompose the problem into its three components, facility location, customer allocation to facilities and vehicle routing, and solve a series of well-known problems such as p-median, location-allocation and vehicle routing. Exact methods have been developed for a small number of LRP models that are derived from two-index flow formulations for the vehicle routing problem (VRP). Laporte and Nobert [6] solve a single depot model by a constraint relaxation method. Laporte [5] develops an equivalent model and also extends the model to the case where the number of vehicles used is a variable in the model. Laporte et al. [8] solve a multi-depot problem in which at most p facilities are located by adapting REVERSE algorithm. The largest problems solved have seven candidate facilities and 40 customers. Laporte et al. [7] solve a multi-depot capacitated LRP using a constraint relaxation method. In their work, the largest problem solved to optimality has eight candidate facilities and 20 customers. Laporte et al. [7] use a branch and-bound procedure to solve asymmetric LRPs that include as many as three candidate facilities and 80 customers.

Success in developing exact methods for solving larger instances of LRPs is likely to come from leveraging the advances in exact methods for solving VRPs and other difficult combinatorial optimization problems. Motivated by the success of set partitioning formulations for a variety of transportation problems, such as the VRP with time windows (e.g. [2]). Guerra et al. [3] propose a heuristic algorithm for solving LRP in a logistic system. Toyoglu et al. [11] consider the LRP using a combination of facility location and vehicle routing problems. The main objective of their paper is to develop LRP with fewer constraints and variables. It can be seen that VRP plays important role in LRP in order to be able to serve customers efficiently and give rise profitability to enterprise. Whenever the delivery for each customer is done periodically, within a given time horizon, the VRP is then called periodic VRP (PVRP). Therefore it is necessary to have a visiting schedules associated with each customer. A fleet of vehicles is available and each vehicle leaves the depot, serves a set of customers, when its work shift or capacity is over, returns to the depot. The problem is to minimize the total length of the routes travelled by the vehicles on the periodic time horizon.

2. Methodology

2.1. Location-Allocation Model
The basic forms of location-allocation models for private sector is the $P$-median problem. The model is to minimize the total of travel distances between the customer points and the nearest servicing facilities. We define set of Notations as follows.

**Set**
- $I$: Set of customer nodes
- $J$: Set of potential facility sites
- $M$: Number of customer points in the considered area
- $N$: Number of potential facility locations

**Parameters**
- $a_i$: Demand at node $i \in I$
- $d_{ij}$: Distance between node $i \in I$ and $j \in J$
- $Q$: Number of facilities to be located

**Variables**
- $X_{ij}$: Binary variable whether customer $i \in I$ is assigned to a facility $j \in J$

The following is how the model is built. Firstly, we formulate the objective function. This is to minimize the total travel distance between customer $i$ and facility location $j$.

\[
\text{Minimize } \sum_{i \in I} \sum_{j \in J} a_i d_{ij} x_{ij} \tag{1}
\]

There are constraints need to be satisfied. We need the following expression to ensure that each customer (or demand) is assigned to a single facility.

\[
\sum_{j \in J} x_{ij} = 1, \quad \forall i \in I \tag{2}
\]

\[
x_{ij} \leq x_{ij}, \quad \forall i \in I, \forall j \in J \tag{3}
\]

The next equation is to limit the number of facilities to be located

\[
\sum_{j \in J} x_{ij} = Q \tag{4}
\]

### 2.2. Location-Routing Periodic Model

We will then present a LRP formula based on set - partitioning with distance constraints. The aim is to select a set of locations and to construct a set of associated delivery routes in a way that minimizes the cost of the facility and the cost of routing. The set of routes must ensure that every customer is visited on a single route exactly once and that the length of each route does not exceed the maximum distance. The deliveries are carried out periodically. In the PVRP, customers are not directly assigned to vehicles, but customers are visited a preset number of times in a schedule selected from a schedule of options. First, we describe $T$ as the horizon for the time period of the days to formulate the model.

Let $K$ mark the set of vehicles. For each vehicle $m \in K$, let $Q_k$ denote the capacity of vehicle $k$. As this is a periodic problem, it is then necessary to define the number of distinct delivery combinations, which is denoted with $ND$. Along with that, let $S_i$ be the set of allowable delivery combinations or menu of schedule for customer $i$. The parameter $a_{st}$ links schedules to days, where $a_{st} = 1$ if day $t \in T$ is in schedule $s \in S$ and $a_{st} = 0$ otherwise. Each schedule $s \in S$ has an associated visit frequency $\gamma^s$ measured by the number of days in the schedule: $\gamma^s = \sum_{t \in T} a_{st}$. For a given schedule option $s$, the headway between visits is defined in terms of the visit frequency as $H^s = 1 / \gamma^s$.

Let $I$ be the set of customer location nodes and $J$ be the set of candidate facility location nodes. We describe the graph $G=(N,A)$, where $N = I \cup J$ is the set of nodes and $A = N \times N$ is the set of arcs. We let $d_{ij}$ for all $(i, j) \in A$ be the distance within nodes $i$ and $j$. The distances correspond to the inequality of the triangle. For applications where the distance constraint covers the route to the last customer and not the time of the return journey to the depot, we set $d_{ij}$ to 0 for all $(i, j)$ with $i \in I$ and $j \in J$. We define a feasible route $k$ associated with facility $j$ as a simple circuit starting at facility $j$, visiting one or
more customer nodes and returning to facility \( j \), which has a total distance of at most the maximum distance, indicated by \( M \). We then let \( P_j \) indicate the set of all feasible routes for all \( j \in J \) associated with the \( j \) facility. The cost of a route \( k \in P_j \) is the sum of the costs of the arcs in the route.

**Parameters**

\[
a_{ijk} = \begin{cases} 
1, & \text{if route } k \text{ associated with facility } j \text{ visits customer } i, \forall i \in I, \forall j \in J, \forall k \in P_j \\
0, & \text{otherwise}
\end{cases}
\]

\( c_{jk} \) cost of route \( k \) associated with facility \( j \), \( \forall j \in J, \forall k \in P_j \)

\( f_j \) fixed cost associated with selecting facility \( j \), \( \forall j \in J \)

\( \alpha \) object weighted factor

\( T \) Time period

**Decision Variables**

\( Z_{jk}^t \) Binary variable indicating whether vehicle travels from node selected \( j \in J \) on day \( t \in T \) using route \( k \)

\( W_j^s \) Binary variable indicating whether selected facility \( j \in J \) is served with \( s \) delivery combination,

\( V_j^t \) Binary variable stating whether selected facility \( j \) is visited on day \( t \)

\[
X_j = \begin{cases} 
1, & \text{if facility } j \text{ is selected, } \forall j \in J \\
0, & \text{otherwise}
\end{cases}
\]

\[
Y_{jk} = \begin{cases} 
1, & \text{if route } k \text{ associated with facility } j \text{ is selected, } \forall j \in J, \forall k \in P_j \\
0, & \text{otherwise}
\end{cases}
\]

The objective is to minimize cost

\[
\text{Minimize } \alpha \sum_{j \in J} f_j X_j + \sum_{j \in J} \sum_{k \in P_j} c_{jk} Y_{jk} + \sum_{k \in P_j} \beta_k \sum_{j \in J} Z_{jk}
\]

subject to

\[
\sum_{j \in J} \sum_{k \in P_j} a_{ijk} Y_{jk} = 1 \quad \forall i \in I
\]

\[
X_j - Y_{jk} \geq 0 \quad \forall j \in J, \forall k \in P_j
\]

\[
X_j \in \{0,1\} \quad \forall j \in J
\]

\[
Y_{jk} \in \{0,1\} \quad \forall j \in J, \forall k \in P_j
\]

\[
\sum_{j \in J} V_j^t = 1, \quad \forall t \in T
\]

\[
\sum_{j \in J} Z_{jk}^t = 1, \quad \forall t \in T
\]

\[
\sum_{s \in S} W_j^s = 1, \quad \forall j \in J
\]

\[
\sum_{j \in J} \sum_{k \in P_j} Z_{jk}^t = \sum_{j \in J} V_j^t, \quad \forall t \in T
\]

### 2.3. Neighbourhood search

The reduced gradient vector normally used to detect an optimal condition is usually not available in integer programming, despite the fact that the problems are convex. We must therefore ensure that we have the “best” feasible integer solution to guarantee that we make a certain condition for the local search procedure.
Let \([\beta]_k\) be an integer point belongs to a finite set of neighbourhood \(N([\beta]_k)\). We define a neighborhood system linked to \([\beta]_k\), which means that if such an integer point meets the following two requirements

a. If \([\beta]_j \in N([\beta]_k)\) then \([\beta]_k \in [\beta]_j, j \neq k\).

b. \(N([\beta]_k) = [\beta]_k + N(0)\)

In relation to the above mentioned neighborhood system, the proposed integration strategy can be described as follows.

Given a non-integer component, \(x_k\), of an optimal vector, \(x_B\). The adjacent points of \(x_k\), being considered are \([x_k]\) and \([x_k] + 1\). If one of these points meets the constraints and results in a minimum deterioration of the optimal objective value, we shift to another component if we do not have an integrated solution.

Let \([x_k]\) be the feasible integer point that fulfills the above conditions. Then we could say, if \([x_k] + 1 \in N([x_k])\) implies that point \([x_k] + 1\) is either infeasible or yields a lower value than the objective function obtained in relation to \([x_k]\). In this situation, \([x_k]\) is said to be an "optimal" solution for integer programming. In our case, a neighborhood search is obviously carried out through feasible points, so that the integer solution is at least at a distance from the optimal continuous solution.

3. Results and Discussion

3.1. The basic approach

Before we move on to theMINLP problem, the basic process strategy for linear cases should be discussed, i.e., Mixed Integer Linear Programming (MILP) problems.

Consider a MILP problem with the following form

\[
\begin{align*}
\text{Minimize} & \quad P = c^T x \\
\text{Subject to} & \quad Ax \leq b \\
x & \geq 0 \\
x_j & \text{ integer for some } j \in J
\end{align*}
\]

(14) (15) (16) (17)

A component of the optimal basic feasible vector \((x_B)_k\), to MILP solved as continuous can be written

\[(x_B)_k = \beta_k - \alpha_{k1}(x_N)_1 - \cdots - \alpha_{kj}(x_N)_j - \cdots - \alpha_{kn} - m(x_N)n
\]

(18)

Note that, this expression can be found in the final tableau of Simplex procedure. If \((x_B)_k\) is an integer variable and we assume that \(\beta_k\) is not an integer, the partitioning of \(\beta_k\) into the integer and fractional components is that given

\[
[\beta_k] + f_k, \quad 0 \leq f_k \leq 1
\]

(19)

suppose we wish to increase \((x_B)_k\) to its nearest integer, \([\beta] + 1\). Based on the idea of suboptimal solutions we may elevate a particular nonbasic variable, say \((x_N)_j\), above its bound of zero, provided \(\alpha_{kj}\), as one of the element of the vector \(\alpha_j\), is negative. Let \(\Delta_j\) be amount of movement of the non variable \((x_N)_j\), such that the numerical value of scalar \((x_B)_k\) is integer. Referring to Eqn. (17), \(\Delta_j\) can then be expressed as

\[
\Delta_j = \frac{1 - f_k}{-\alpha_{kj}}
\]

(20)

while the remaining nonbasic stay at zero. It can be seen that after substituting (18) into (19) for \((x_N)_j\) and taking into account the partitioning of \(\beta_k\) given in (18), we obtain \((x_B)_k = [\beta] + 1\). Thus, \((x_B)_k\) is now an integer. It is now clear that a non-basic variable plays an important role in integrating the relevant basic variable. The following results are therefore necessary to verify that a non-integer variable must be used in the integration process.

**Theorem 1.** Suppose the MILP problem (13)-(16) has an optimal solution, then some of the nonbasic variables, \((x_N)_j, j = 1, \ldots, n\), must be non-integer variables.

**Proof.** Solving the problem as a continuous problem of slack variables (which are not integrated, except for equality constraints). If we assume that the vector of the basic variables consists of all slack variables, all integer variables would be evaluated in the non-basic vector \(x_N\).
3.2. Derivation of the method

It is clear that the other components of vector $x_B$, $(x_B)_{i \neq k}$, will also be affected as the numerical value of scalar $(x_N)_{j^*}$ increases to $\Delta j^*$. Therefore, if there is a positive vector $a_j^*$ element, i.e. $a_j^*$ for $i \neq k$, the corresponding element of $x_B$ decreases and eventually passes through zero. However, because of the non-negative restriction, any vector $x$ component must not be below zero. A formula called the minimum ratio test is therefore necessary to see what is the maximum movement of the non-basic $(x_N)_{j^*}$ so that all $x$ components remain feasible. This ratio test would include two cases.

a. A basic variable $(x_B)_{i \neq k}$ decreases to zero (lower bound) first.

b. The basic variable, $(x_B)_k$ increases to an integer.

Specifically, corresponding to each of these two cases above, one would compute

$$\theta_1 = \min_{i \neq k | a_j^* > 0} \frac{\beta_i}{a_j^*}$$

$$\theta_2 = \Delta j^*$$

How far one can release the nonbasic $(x_N)_{j^*}$ from its bound of zero, such that vector $x$ remains feasible, will depend on the ratio test $\theta$ given following $\theta_1 = \min(\theta_1, \theta_2)$

obviously, if $\theta = \theta_1$, one of the basic variable $(x_B)_{i \neq k}$ will hit the lower bound before $(x_B)_k$ becomes integer. If $\theta = \theta_2$, the numerical value of the basic variable $(x_B)_k$ will be integer and feasibility is still maintained. Analogously, we would be able to reduce the numerical value of the basic variable $(x_B)_k$ to its closest integer $[\beta_k]$. In this case the amount of movement of a particular nonbasic variable, $(x_N)_{j^*}$, corresponding to any positive element of vector $a_j^*$, is given by

$$\Delta j^* = \frac{f_k}{a_{kj^*}}$$

in Linear Programming (LP) terminology the operation conducted in Eqns. (17) and (18) is called the pricing operation. The reduced cost vector $d_j$ is used to calculate the deterioration in the objective value of the function due to the release of a non-basic variable. Therefore, when deciding which non-basic should be released in the integration process for the sake of minimizing deterioration, vector $d_j$ must be taken into account. Evoke that the minimum continuous solution offers a lower bound to any integer-feasible solution. Nonetheless, the amount of movement of particular nonbasic variable as given in Eqns. (11) or (15), depends in some way on the corresponding element of vector $a_j$. As a result, it can be observed that the deterioration of the objective function value due to releasing a nonbasic variable $(x_N)_{j^*}$ so as to integerize a basic variable $(x_B)_k$ may be measured by the ratio

$$\left| \frac{d_k}{a_{kj^*}} \right|$$

where $|a|$ means the absolute value of scalar $a$.

To minimize the detonation of the optimal continuous solution, the following strategy is used to determine which non-basic variable can be increased from its zero limit, that is,

$$\min_j \left\{ \left| \frac{d_k}{a_{kj^*}} \right| , \quad j = 1, ..., n - m \right\}$$

From the “active constraint” strategy and the partitioning of the constraints corresponding to basic(B), superbasic (S) and nonbasic (N) variables we can write

$$\begin{bmatrix} B & S & N \end{bmatrix} \begin{bmatrix} x_B \\ x_S \\ x_N \end{bmatrix} = \begin{bmatrix} b \\ b_S \\ b_N \end{bmatrix}$$

or

$$Bx_B + Sx_N + Nx_S = b$$

$$x_N = b_N$$

The basis matrix $B$ is assumed to be square and nonsingular, we get

$$x_B = \beta - Wx_N - ax_N$$

Where

$$\beta = B^{-1} b$$
Expression (23) implies that the non-basic variables are kept in equal measure. The integration strategy discussed in the previous section, designed for the MILP problem, is evident from the "nearly" basic expression of Eqn. (24). In particular, we could release from its bound, Eqn. (23), a non-basic variable and exchange it with a corresponding basic variable in the integration process, although the solution would be degenerated. In addition, for the MINLP problem, the above theorem (1) can be extended.

**Theorem 2.** Suppose the MINLP problem has a bounded optimal continuous solution, then we can always get a non-integer $y_j$ in the optimum basic variable vector.

**Proof.**

a. If these variables are nonbasic, they will be at their bound. Therefore they have integer value.

b. If a $y_j$ is superbasic, it is possible to make $y_j$ basic and bring in a nonbasic at its bound to replace it in the superbasic.

However, the ratio test addressed in (14) cannot be used as a tool to ensure that the optimal gill found in the integer solution remains in the feasible area. Instead, we use Minos' feasibility test to check whether or not the integer solution is feasible.

### 3.3. Pivoting

Currently, we are in a position where particular basic variable, $(x_B)_k$ is being integerized, thereby a corresponding nonbasic variable, $(x_N)_j$, is being released from its bound of zero. Suppose the maximum movement of $(x_N)_j$ satisfies $\theta_* = \Delta_j$.

such that $(x_B)_k$ is integer valued to exploit the manner of changing the basis in linear programming, we would be able to move $(x_N)_j$ into $B$ (to replace $(x_B)_k$) and integer-valued $(x_B)_k$ into $S$ in order to maintain the integer solution. We now have a degenerate solution since a basic variable is at its bound.

The integerizing process continues with a new set $[B, S]$. In this case, eventually we may end up with all of the integer variables being superbasic.

**Theorem 3.** A suboptimal solution exists to the MILP and MINLP problem in which all of the integer variables are superbasic.

**Proof.**

a. If all of the integer variables are in $N$, then they will be a bound.

b. If an integer variable is basic it is possible to either

- Interchange it with a superbasic continuous variable, or
- Make this integer variable superbasic and bring in a nonbasic at its bound to replace it in the basis which gives a degenerate solution.

The other case which can happen is that a different basic variables $(x_B)_{i \neq k}$ may hit its bound before $(x_B)_k$ becomes integer. Or in other words, we are in a situation where $\theta_* = \Delta_1$.

In this case we move the basic variable $(x_B)_j$ into $N$ and its position in the basic variable vector would be replaced by nonbasic $(x_B)_j$. Note $(x_B)_k$ is still a non-integer basic variable with a new value.

### 4. Conclusion

This paper presents a LRP model with some prohibited route. The model framework is based on VRP with time windows with forbidden route. The forbidden route is then excluded from the previous route assigned. We use a feasible neighborhood search to solve the model.

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