Bosons in fluctuating gauge fields: Bose metal and phase separation

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We study a two-dimensional system of bosons interacting with a fluctuating $U(1)$ gauge field with overdamped dynamics. We find two instabilities of the condensed phase at $T = 0$: one to phase separation and another to a homogeneous non-superfluid (Bose metal). The presence of both instabilities in the model is dependent on the low-energy form of the gauge field propagator. We discuss the relevance of our findings to the $U(1)$ gauge theory of the $t$–$J$ model.

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I. INTRODUCTION

A Bose liquid naturally becomes a superfluid at low temperatures. The existence of a non-superfluid liquid state (the “Bose metal”) down to zero temperature has been a subject of recent controversy. Such a scenario might arise if the bosons interact with fluctuating external fields. A fluctuating vector potential couples directly to the gradient of the phase of the superfluid order parameter. This makes it a natural choice for destroying long-range phase coherence in a superfluid.

In this paper, we study how a vector potential with overdamped dynamics might destroy superfluidity. Our choice of overdamped dynamics is motivated by the gauge theory of the $t$–$J$ model, relevant to the cuprate superconductors. The existence of a Bose metal far below the degeneracy temperature is crucial for the picture of spin-charge separation in that theory.

Our study is the first to consider the damping of the zero-point fluctuations of the gauge field. We find that a metallic phase may exist in the part of the phase diagram relevant to the cuprates. Related models with different dynamics have been studied before. Using a quasistatic approximation for the gauge field, Lee, Lee and Kim showed the loss of superfluidity in a quantum Monte Carlo simulation. On the other hand, a metallic phase was not found with a propagating (“Maxwell”) gauge field. Instead, the system is unstable to phase separation (unless it is stabilised by long-range repulsion.)

In our treatment, we use the perturbative corrections calculated by Feigelman et al. in a simple renormalisation scheme. In Ref. 3, a self-consistent scheme was used to calculate the superfluid density in the presence of fluctuating magnetic fields. It was clear from that work that the superfluid response depends on the compressibility of the Bose liquid, and vice versa: the superfluid expels flux over a penetration depth, but that inevitably reduces the local density, exciting phonons and causing an apparent increase in the compressibility. Since there is the possibility of phase separation (crudely speaking, the bosons and magnetic flux expelling each other), we want to track this inter-dependence in more detail. Our scheme resembles a frequency-dependent mean-field theory. We successively integrate out high-frequency fluctuations, generating at each stage corrections to the low-energy response functions — the superfluid density $n_s$ and the compressibility $\kappa$. We believe that this will give a better picture of the instabilities of the superfluid phase.

The form of the paper is as follows. We start by reviewing the method by which charge dynamics of the $t$–$J$ model can be obtained from a model of bosons interacting with a $U(1)$ gauge field. We then calculate corrections to the compressibility, phase stiffness and Meissner response, in a perturbation expansion about the superfluid state. We use these corrections in a renormalisation scheme, which we apply to systems with both overdamped and propagating gauge fields. Finally, we discuss our results and make some conclusions about the effect of different gauge fields on the bosonic systems.

II. THE MODEL

We will now describe in more detail our model, in particular, its motivation from the gauge theory of the $t$–$J$ model. In this paper, we focus on two-dimensional bosonic systems with (imaginary-time) Lagrangian densities of the form: $(\hbar = c = e = 1)$

$$\mathcal{L}_x = b^\dagger_x (\partial_x - \mu) b_x + \frac{1}{2m} (i\nabla - a_x)^2 b_x^\dagger b_x + \frac{1}{2} U (b_x^\dagger b_x)^2 + \frac{1}{2} \int dx' \sigma_x^\dagger \sigma_{x'} (D_{x,x'})^{-1} \sigma_{x'}^\dagger \sigma_x$$

where $b_x$ is a complex bosonic field with local repulsion $U$ and mass $m$, and $a_x^\dagger$ is a (two-vector) gauge field with propagator $D$. We label both spatial and temporal coordinates by $x = (r, \tau)$.

As already mentioned, we study the gauge fields that arise from the $U(1)$ gauge theory of the $t$–$J$ model. The $t$–$J$ model describes electrons with hopping integral $t$ and antiferromagnetic exchange $J$. The Coulomb repulsion is so strong that we assume that there is at most one electron per site. The Hamiltonian is:

$$H = -t \sum_{\langle ij \rangle} \left( c^\dagger_i c_j + \text{h.c.} \right) + J \sum_{\langle ij \rangle} \left( S_i \cdot S_j - \frac{n_i n_j}{2} \right)$$

with a constraint of no double occupancy at each site. The operator $c^\dagger_i \sigma$ creates an electron at site $i$ with spin
σ, n_i is the number of electrons on a site, and S_i = \sum_{\alpha,\beta} c_{i\alpha} \sigma_{\alpha\beta} c^\dagger_{i\beta} is the spin operator at a site. The sums are taken over pairs of nearest neighbours on a square lattice.

A “slave boson” treatment of the constraint of single occupancy reduces this model to a Lagrangian density of the form in equation (3). The electronic operator is split into a spin-half fermion and a charge-e boson: \( c_{i\sigma} = b_{i\sigma} f_{i\sigma} \). The single-occupancy constraint means the fermion and boson currents must be equal and opposite: \( J_1 = -J_0 \). The gauge theory enforces this constraint by making the boson and fermion currents interact indirectly via a vector potential.

After this (exact) decoupling, we may obtain an effective theory for the bosons. The derivation of this (approximate) theory is shown in appendix A. The effective Lagrangian density for the bosons takes the form of equation (3); the theory aims to capture the strong correlations in the \( t-J \) model, which arise from the constraint of no double occupancy.

As explained in appendix A, the scalar part of the gauge field that arises in this treatment has been absorbed into the local repulsion \( U \). The boson density is the charge doping away from half-filling. This appears to be consistent with the charge carrier density from transport measurements. So, we will concentrate on boson densities small compared to integer filling of the lattice and we can ignore the possibility of a Bose Mott insulator in our system.

The excitations of the fermionic subsystem can exchange energy and momentum with the bosons. Integrating out the fermions amounts to treating them as an effective medium through which the gauge field propagates. In fact, the Lagrangian for the gauge field (last term in (3)) is determined by the dynamics of the particle-hole excitations of the fermions. It can be shown that the propagator, \( D_{\omega_{n,q}} \), is directly related to the fermion current response functions. This is consistent with the constraint on the boson and fermion currents.

Let us choose the gauge \( \nabla \cdot a = 0 \) so that only the transverse component, \( a^\perp \), is non-zero and \( D^{\mu\nu} \) has only one degree of freedom which we call \( D \). As shown in appendix A, the spectral density of this propagator is \( A(\omega, q) = 2 \text{Im}[-1/\Pi^R_T(\omega, q)] \) where \( \Pi^R_T = (\delta_{\mu\nu} - q_\mu q_\nu/q^2) [i(\partial_\nu J^\nu)/\omega - (n_i/m_i)\delta_{\mu\nu}] \) is the retarded transverse polarisation of the fermions.

For simplicity, we will model the fermions as an isotropic Fermi liquid with mass \( m_i \) and a density \( n_i \) which we choose to be close to the half-filling density on a lattice model. For the cuprates, this is expected to be reasonable near optimal doping. In other words, we ignore the “pseudogap” spin physics in underdoped materials.

With this choice of isotropic fermions, the polarisation can be derived analytically. This saves a great deal of computational time in the numerical calculations carried out in the next section. However, it ignores the underlying lattice. For this reason we will introduce a high-energy cutoff, \( \Omega \), in the spectral density of the gauge field propagator. We choose \( \Omega = E_F \) so that particle-hole excitations with energies greater than the Fermi energy \( E_F \) are excluded — these would have involved quasiparticles outside the Brillouin zone of the underlying lattice. The form of the polarisation and the effect of the cutoff \( \Omega \) are discussed in appendix A. Taking all this into account, we obtain:

\[
D_{\omega_{n,q}} = \int_{-\Omega}^{\Omega} \frac{d\omega'}{2\pi} \frac{1}{\omega' - i\omega_n} \left( -2\text{Im} \left[ \Pi^R_T(\omega', q) \right] \right)
\]  

At small frequencies and momenta this reduces to the form \( D_{\omega_{n,q}} = (\sigma_g |\omega_n| + \chi q^2)^{-1} \) where \( \chi = (12\pi m_f)^{-1} \) is the diamagnetic susceptibility of a two-dimensional Fermi liquid and \( \sigma_g = 2n_i/k_B q \) is the conductivity due to Landau damping. (However, we do not work with this simple form because we find that strong scattering occurs at large wavevectors.)

Equation (3) completes our definition of the model in (3) by specifying the spectrum of gauge field fluctuations. In the following sections we discuss instabilities of the superfluid state in this model.

## III. THE RENORMALISATION SCHEME

In this section, we investigate instabilities of the superfluid state in the general model given by equation (3). We discuss the dependence of these instabilities on the form of the gauge field propagator. The interaction between the bosonic excitations and the gauge field affects the compressibility of the bosons and their phase stiffness, as well as the Meissner response of the gauge field. These corrections can be calculated in perturbation theory, but near instabilities of the superfluid state a more advanced treatment is necessary, so we introduce a simple renormalisation scheme.

We first expand about a uniform condensate by writing: \( \Psi_x = \sqrt{n + \rho_x e^{i\theta_x} \phi} \), where \( n \) is the mean boson density, \( \rho_x \) represents density fluctuations, and \( \theta \) represents fluctuations in the phase of the bosonic field. The Lagrangian density becomes \( \mathcal{L}_x = \mathcal{L}^0_x + \mathcal{L}^\text{int}_x \), where

\[
\mathcal{L}^0_x = \frac{i}{2} \rho_x \partial_x \theta_x + \frac{U}{2} \rho_x^2 + \frac{1}{8m} |\nabla \rho_x|^2
\]  
\[+ \frac{n}{2m} |\nabla \theta_x|^2 + \frac{1}{2} \int dx' a^\dagger_x (D^0_{x' - x})^{-1} a_x \]  

and

\[ \mathcal{L}^\text{int}_x = \frac{1}{2m} (|\nabla \theta_x + a^\dagger_x|^2 - \rho_x) - \frac{\rho_x}{8mn(n + \rho_x)} |\nabla \rho_x|^2 \]

The propagator \( (D^0_{x' - x})^{-1} = \left( \frac{n}{m} \delta(x - x') + D^{-1}_{x' - x} \right) \) where the first term is the Meissner response which modifies the bare propagator.

The Lagrangian density \( \mathcal{L}^0 \) describes a superfluid Bose liquid with compressibility \( \kappa = U^{-1} \) and a superfluid
density $n_s = n$. We are interested in how the interaction terms in $L^\text{int}$ affect these quantities. The system will remain superfluid as long as both of these quantities are finite.

Setting $a = 0$ in the action above corresponds to the neutral superfluid. The resulting action is not quadratic, but neither of the remaining interaction terms modify the superfluid density (this must be equal to the total density for a Galilean invariant system). We therefore drop these terms, keeping only interaction terms that contain the gauge field, $a$. If the discarded terms have an effect on the compressibility then this can be absorbed into $U$.

Any effect on the superfluid response must therefore come from the terms involving the gauge field. Simple perturbation theory is not suitable for discussing the instabilities at small $n_s$ and small $U$, since strong renormalisation of these quantities renders it invalid. Feigelman et al.\cite{Feigelman1991} used a self-consistent method to discuss the reduction of the superfluid response in a system with long-ranged repulsion (so the compressibility is not renormalised).

We use a different method, which allows us to treat the corrections to both the gauge field propagator and the bosonic propagator on an equal basis. We integrate out the highest frequency fields, which gives an effective theory for the remaining fields, with renormalised superfluid fraction and compressibility. This process is then repeated until only the very smallest frequencies are left.

As the bandwidth is reduced, three scenarios are possible; these are shown schematically in figure 1. If both $n_s$ and $\kappa$ remain finite after all the fields have been integrated then the system will remain superfluid. If $\kappa$ diverges at finite $n_s$ then there will be an instability to phase separation, but if $n_s$ vanishes at finite $\kappa$ then there will be an instability to a homogeneous non-superfluid, which we call a metal in the rest of this paper.

To proceed, we generalise our model to allow for a superfluid density that is not equal to the mean boson density and a compressibility that is not equal to $U^{-1}$.

We write $L^\text{s} = L^\text{s}_x + L^\text{S}_1$, where

$$L^\text{s}_x = \frac{i}{2}\rho_x \partial_x \theta_x + \frac{1}{2\kappa^2} \rho_x^2 + \frac{1}{8m n_s} |\nabla \rho_x|^2 + \frac{n_s}{2m} |\nabla \theta_x|^2 + \frac{1}{2} a_x^+ (D^\text{s}_x)^{-1} a_x^\dagger$$

and

$$L^\text{S}_1 = -\frac{1}{m} \rho_x (a_x \cdot \nabla \theta_x) + \frac{1}{2m} (a_x^+)^2 \rho_x$$

with $(D^\text{s}_x)^{-1} = (n_s/m) \delta(x-x') + D^{-1}$. Note that the phase stiffness of the bosons and the Meissner response of the gauge field are characterised by the same $n_s$. This is required by the underlying $U(1)$ gauge symmetry. In the perturbational calculation the correction to the phase stiffness comes from the first term in $L_1$ and the correction to the Meissner response comes from the second term. However, both corrections are given by the same polarisation insertion (the first bubble in figure 2); this ensures that they are equal, as required.

We start the renormalisation process with $\kappa^{-1} = U$ and $n_s = n$. At each stage, we integrate out all boson excitations with frequencies whose magnitudes lie between the maximum $\omega_n$ and $\omega_n - \delta \omega$. We then average over fast fluctuations in the gauge field, as done in Refs.\cite{Fradkin1989, Fradkin1991}. As each frequency shell is integrated out, we correct the values of $n_s$ and $\kappa$ and use these values, denoted as $n_s(\omega_n)$ and $\kappa(\omega_n)$, for the next shell.

The one-loop corrections to $n_s$ and $\kappa$ are given by:

$$\delta n_s = - (\delta \omega) \Pi_{\omega_n}^{(1)}, \quad \delta \kappa^{-1} = - (\delta \omega) K_{\omega_n}^{(1)}$$

where

$$K_{\omega_n}^{(1)} = \frac{1}{2\pi m^2} \int_0^\infty \frac{dq}{2\pi} (D^s_{\omega_n q})^2$$

and

$$\Pi_{\omega_n}^{(1)} = \frac{1}{\pi m^2} \int_0^\infty \frac{dq}{2\pi} D^s_{\omega_n q} C_{\omega_n q}$$

These depend on the gauge field propagator $D$ (about which we have so far assumed nothing) and the propagator for phonons in the Bose liquid:

$$C_{\omega_n q} = \frac{m}{2n_s} \langle \rho_{\omega_n q} \rho_{-\omega_n - q} \rangle$$

$$= \frac{q^2}{\omega_n^2 + n_s q^2 (\kappa^{-1} + m^2 / 4m^2 n_s)}$$

These corrections correspond to the diagrams in figure 2. To reiterate, the renormalisation scheme is implemented by using the renormalised values of $n_s$ and $\kappa$ when evaluating $C_{\omega_n}$ and $D_{\omega_n}$ at each stage of the procedure. Perturbation theory corresponds to performing the frequency integrals keeping $n_s$ and $\kappa$ constant. Our renormalisation scheme resembles a frequency dependent mean-field theory in that these parameters are constant with respect to momentum, but dependent on frequency. We see that $\delta n_s$ is negative and $\delta \kappa$ is positive: the perturbations reduce the superfluid correlations and increase the density fluctuations.
and \( \tilde{L} \) turbative corrections to \( \lambda \) at constant define dimensionless propagators \( \tilde{f} \) reasonable for small \( m \) between the bosons and the gauge field. The limit \( m \) energies of the bosons.

The separation boundary in figure 3 is a rough guide only. instabilities of the superfluid state. The metal/phase-

The processes affect the phase stiffness and Meissner response (left) and the compressibility (right).

### A. Overdamped propagator

We now carry out this renormalisation scheme using the gauge field propagator given in equation (3) and investigate the resulting phase diagram. Let us now identify four independent dimensionless parameters for the system. We choose \( \lambda_0^2 = (n_t/n)(m/m_t) \), \( U_r = mU \), \( m_r = m_t/m \) and \( \Omega_r = \Omega/E_F \). \( \lambda_0 \) is the penetration depth of the Bose superfluid at \( n_s = n \) in units of the fermion separation (i.e. lattice spacing at \( n_t = 1 \)). With the other dimensionless parameters fixed, varying \( \lambda_0 \) corresponds to changing the boson density \( n \): \( \lambda_0 \propto 1/n^{1/2} \). \( U_r \) measures the relative size of the repulsive and kinetic energies of the bosons. \( U_r \) should be at least \( O(1) \) if we want to model the hard-core repulsion in the slave-boson model. The parameter \( m_r \) determines the coupling between the bosons and the gauge field. The limit \( m_r \to 0 \) at constant \( \lambda_0 \) corresponds to the vanishing of the perturbative corrections to \( L^* \). We expect our results to be reasonable for small \( m_r \).

For convenience, we use a rescaled frequency, \( w = \omega m t/n_t \), and rescaled momentum \( u = p n_t^{-1/2} \). We also define dimensionless propagators \( \tilde{D}_{w,u} = n_t D_{w,q}/m_t \), and \( \tilde{C}_{w,u} = n_t C_{w,q}/m_t^2 \).

The result is that:

\[
(\delta \omega) \left( mK_{\omega_n}^{(1)} \right) = \frac{(\delta w) m_r}{2 \pi} \int_0^\infty \frac{u \, du}{2 \pi} \left( \tilde{D}_{w,u}^* \right)^2 \tag{12}
\]

and

\[
(\delta \omega) \Pi_{\omega_n}^{(1)} = \frac{(\delta w) m_r^2}{\pi} \int_0^\infty \frac{u \, du}{2 \pi} \tilde{D}_{w,u}^* \tilde{C}_{w,u} \tag{13}
\]

As mentioned above, we can see from these expressions that \( m_r \) can be regarded as a measure of the coupling strength: the effect of the gauge field on the bosons is small if \( m_r \) is small. Our choice of renormalised variables means that \( \tilde{D}^* \) is independent of \( m_r \) and \( U_r \).

Performing the scaling process numerically (at \( \Omega/E_F = 1 \)) gives a phase diagram as shown in figure 3. Our expansion (3) about the superfluid state in means that these results are most reliable in the superfluid phase, up to the phase boundaries, which represent the onset of the instabilities of the superfluid state. The metal/phase-separation boundary in figure 3 is a rough guide only.

We find that, at small \( U_r \), there is an instability to phase separation. This is because the correction to \( \kappa^{-1} \), and hence \( U \), from (3) is not directly dependent on \( U \). On the other hand, at large \( U_r \), we find that the superfluid is stable. This is expected because \( \Pi^{(1)} \) is small compared to unity and \( K^{(1)} \) is small compared to \( U \), so that the corrections that we calculate are small.

For intermediate values of \( U_r \), we find that there may be a metallic state: this is most likely at small doping (i.e., a large bare penetration depth \( \lambda_0 \)). However, because \( \Pi^{(1)} \) is a fractional correction and not an absolute one, we note that it is not guaranteed that reducing the doping will result in a metal even as \( n \to 0 \) (\( \lambda_0 \to \infty \)).

The form of the phase diagram depends on \( m_r \) (see figure 4). There is always a Meissner effect in the limit \( m_r \to 0 \) (at constant \( \lambda_0 \)). For small \( m_r \), only the instability to phase separation occurs. A metallic region appears in the phase diagram at \( m_r \approx 0.3 \). Increasing \( m_r \) above 0.3 favours both instabilities at the expense of the superfluid.

In terms of doping, the boson density decreases from
left to right in Fig. 5. We see that the maximal doping for which the metal exists is approximately 10% for \( m_r = 1 \) and 5% for \( m_r = 5 \), provided that \( U_r \) is not too small.

**B. Propagating gauge field**

We now compare the results for an overdamped gauge field with the results for a propagating (Maxwell) gauge field, as used by Feigelman et al. [1]. The propagator is \( D_{\omega_n,q} = g^2 (\omega_n^2 + c^2 q^2)^{-1} \). As before, we rescale the boson mass and the penetration depth, using the parameters in the gauge field propagator. The relevant dimensionless parameters in this case are \( U_r = mU \) as before; \( \lambda_{M/0} = \left( mg^2 / n \right) \) which is the rescaled penetration depth, and \( \alpha = g^2 / (mc^2) \): the coupling constant for the gauge field. Rescaling frequency and momentum via \( w = (\omega_n / g^2) \), \( u = cp / g^2 \), the equations (2) and (3) still apply, the only changes being in the form of \( D \), and in the replacement of \( m_e \) by \( \alpha \) and \( \lambda_0 \) by \( \lambda_M \). In this case the momentum integrals in equations (1) and (2) can be done analytically, yielding:

\[
(mK_{\omega_n}^{(1)}) = \frac{g^2}{8\pi^2 mc^2 g} \left[ \frac{1}{\omega_n^2} + m^{-1}n_s(\omega_n) \right] \tag{14}
\]

\[
\Pi_{\omega_n}^{(1)} = \frac{g^2}{4\pi^2 mc^2} \frac{\kappa(\omega_n)}{n} \left[ \frac{1}{\gamma(\gamma - 1) + B} \right] \times \left[ \gamma \log \left( \frac{\gamma^2}{B} \right) + \frac{2B - \gamma}{\gamma} \log \left( \frac{1 + X}{1 - X} \right) \right] \tag{15}
\]

where \( \gamma = n_s(\omega_n) \left[ \omega_n^2 + \left( n_s(\omega_n) g^2 / m \right) \right] / (4nmc^2), B = \omega_n^2 \kappa^2(\omega_n) / (4n_s(\omega_n)), \) and \( X = \sqrt{1 - 4B} \) are all dimensionless and are defined only for ease of writing.

As discussed by Feigelman et al. [1], the propagating gauge field combined with short-range repulsion tends to cause phase separation. This is especially apparent at small \( n_s \): the correction \( K_{\omega_n}^{(1)} \) diverges at small \( n_s \) and \( \omega \). This means that the instability to phase separation will tend to dominate over the loss of superfluid response, unless \( n_s \) disappears at rather a high frequency. The phase diagram for moderate coupling strength, \( \alpha = 1 \), is shown in figure 5. The instability to phase separation is dominant at this coupling, so there is no metallic phase.

At larger values of the coupling, around \( \alpha \simeq 2 \), a metallic phase does appear; however, our one-loop perturbation theory is not reliable at these couplings. A phase diagram is shown in figure 6. It is also clear from figure 6 that the metal requires very large values of the parameter \( mU \) to stabilise the system against phase separation. So, in contrast to the overdamped case, the parameters of the propagating field need to be pushed to rather unphysical values to see a metallic phase.

Another consequence of the singularity in \( K_{\omega_n}^{(1)} \) is that \( \kappa \) is very strongly renormalised near any metal-superfluid phase boundary. There is evidence that \( \kappa^{-1} \) will always vanish before \( n_s \) in a very small region near to the boundary: see figure 6. The phase-separated region between the metal and superfluid regions narrows with increasing \( \lambda_M \). At large values of \( \lambda_M \), an apparent superfluid-metal transition does appear, but the numerical calculations show a discontinuity in the compressibility at the transition. This may be a symptom of an intervening phase separated region, which is too narrow to resolved numerically.

As mentioned previously, the results of the renormalisation scheme are likely to be most valid in the superfluid phase, and on the boundaries of that phase. The Maxwell case is distinct from the overdamped case in that \( K_{\omega_n}^{(1)} \) diverges as \( n_s, \omega_n \to 0 \), which has a strong effects on behaviour of the system very close to the phase boundary.

**IV. CONCLUSION**

To summarise, we have analysed the effect of both overdamped and propagating gauge fields on the ground state of a Bose liquid. In the overdamped case, we find...
two different instabilities of the superfluid phase. Specifically, the penetration depth for the gauge field may diverge. This is interpreted as an instability towards a homogeneous metallic phase. Alternatively, we find that its compressibility may diverge. We interpret this to be phase separation. We cannot comment on any spatial structures in the new ground state because our treatment cannot identify any characteristic length scale for this instability. A more careful treatment of lattice effects and the compressibility at different wavevectors is needed.

If the overdamped gauge field is replaced by a propagating one, then the instability to phase separation dominates over the divergence of the penetration depth for all systems with local repulsion. This is in agreement with Feigelman et al.\[2\]

Applying these results to the slave bosons that describe the charge dynamics of the $t$–$J$ model, both instabilities are found at dopings up to around 10%. For $U_r > 1$, a metallic phase is favoured at these densities. (A large $U_r$ is natural for the hard-core slave bosons.) Although precise numerical values may be affected by the various approximations in our treatment, this Bose metal appears to be in the regime in parameter space relevant to a non-Fermi-liquid description of the normal state of the cuprates.

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**APPENDIX A: SLAVE-BOSON DECOUPLING OF THE $t$–$J$ MODEL**

In this section we review the derivation of a bosonic model of the form given in equation (1) from the $t$–$J$ model. Our notation follows that of Lee and Nagaosa.\[3\]

The $t$–$J$ model is defined in equation (2). We apply the ‘slave-boson’ decoupling\[A1\] to this problem and write

$$c_{i\sigma} = f_{i\sigma}b_i$$

where $f$ is a fermionic spin-$\frac{1}{2}$ field and $b$ is a bosonic field. The bosons represent charged ‘holes’, and describe the charge dynamics of the system. Neglecting terms quartic in the bosonic fields (valid at small doping) we arrive at

$$H \simeq \sum_{\langle ij \rangle} \left[ -\frac{t}{2} \left( f_{i\sigma}^\dagger f_{j\sigma}^\dagger b_j + b_i^\dagger b_i \right) + i \sum_i \lambda_i \left( f_{i\sigma}^\dagger f_{i\sigma} + b_i^\dagger b_i - 1 \right) \right]$$ \hspace{1cm} (A1)

where the complex field $\lambda_i$ is a Lagrange multiplier enforcing the constraint of no double occupancy.

Following Lee and Nagaosa\[2\] we make the mean field decoupling $\sum_r (f_{i\sigma}^\dagger f_{j\sigma}) = \chi_0 e^{i\lambda_{ij}}$. We again neglect terms quartic in the boson operators and we arrive at the imaginary time Lagrangian:

$$L_\tau = \sum_{\langle i j \rangle} f_{i\sigma}^\dagger \left[ \partial_\tau - \mu_\tau + i a_\tau^0 \right] f_{i\sigma} + \sum_i b_i^\dagger \left[ \partial_\tau - \mu_\tau - i a_\tau^0 \right] b_i - \frac{1}{2} \chi_0^J \sum_{\langle i j \rangle, \sigma} \left[ f_{i\sigma}^\dagger f_{j\sigma} e^{i a_{ij,\tau}} + c.c \right] - \chi_0^t \sum_{\langle i j \rangle} \left[ b_i^\dagger b_j e^{i a_{ij,\tau}} + c.c \right]$$ \hspace{1cm} (A2)

where $a_\tau^0$ is the real part of $\lambda_i$. The gauge invariance of the original electron operators under the transformation $f_{i\sigma} \rightarrow f_{i\sigma} e^{i\theta_i}$, $b_i \rightarrow b_i e^{i\theta_i}$, allows us to identify the $a_{ij}$ as the $U(1)$ gauge field associated with this gauge symmetry.

If we assume that the relevant fields are slowly varying in space and make the continuum approximation, we obtain a Lagrangian density given by:

$$\mathcal{L}_x = \sum_{\sigma} f_{\sigma,x}^\dagger \left[ \partial_\tau - \mu_\tau + i a_\tau^0 \right] f_{\sigma,x} + b_x^\dagger \left[ \partial_\tau - \mu_\tau + i a_\tau^0 \right] b_x + \frac{1}{2 m_x} \sum_{\sigma} f_{\sigma,x}^\dagger \left( i \nabla - a_x \right)^2 f_{\sigma,x} + \frac{1}{2 m_b} b_x^\dagger \left( i \nabla - a_x \right)^2 b_x$$ \hspace{1cm} (A3)

where the parameters $m_b$ and $m_x$ are effective boson and fermion masses respectively. They are related to the original $t$ and $J$, and to $\chi_0$. The real 2–vector field $a_x$ is defined by $a_{ij,\tau} = (\mathbf{r}_j - \mathbf{r}_i) \cdot \mathbf{a}_{ij,\tau}$. In making the continuum approximation, we move from an original model with square symmetry to an isotropic model.

The fermions and bosons interact only via the gauge field, so integrating out the fermions will give an effective propagator for the gauge field. The effective gauge field action is, to quadratic order:

$$S_g = \frac{1}{2 B L^2} \sum_{\omega_n, q} a_{\omega_n}^* (\omega_n, q) \Pi_{\omega_n}^f (\omega_n, q) a_{\omega_n, q} + \frac{1}{2 B L^2} \sum_{\omega_n, q} a_{\perp \omega_n}^* (\omega_n, q) \Pi_{\perp}^f (\omega_n, q) a_{\perp \omega_n, q}$$ \hspace{1cm} (A4)

where we have fixed the gauge such that $\nabla \cdot \mathbf{a} = 0$; only the transverse component of $\mathbf{a}$ is non-zero and is denoted $a_{\perp}$. The size of the system is $L$ and the inverse temperature is $\beta$. The scalar part of the gauge field has a propagator determined by $\Pi_{\omega_n}^f (\omega_n, q) = \langle \rho_{\perp}^f (\omega, q) \rho_{\perp}^f (\omega, q) \rangle$ where $\rho_{\perp} (\omega, q)$ is the Fourier transform of the fermion density. At small energy and momentum $\Pi_{\omega_n}^f (\omega_n, q)$ tends to a constant, as predicted by Thomas-Fermi screening: we
model this field by a local repulsion $U$. This is consistent with the introduction of the scalar field to implement the constraint of exactly one fermion or one boson per site. The propagator of the vector part of $a$ is determined by $\Pi^\perp(\omega, q) = T^\mu_{\nu}(J^\mu(\omega, q), J^\nu(\omega, q))$ where $J^\mu(\omega, q)$ is the Fourier transform of the fermion current and $T^\mu_{\nu} = \delta_{\mu\nu} - \frac{\omega q_{\mu}}{q^2}$ selects the transverse part.

### APPENDIX B: POLARISATION OPERATOR OF AN ISOTROPIC FREE FERMI GAS

In this section we discuss the overdamped gauge field propagator in a little more detail. We first state the form of the free fermion polarisation that we use and then discuss the effect of the cutoff $\Omega$.

The current-current correlation function of the free fermions is given by

$$\langle J^\mu(\omega, q) J^\nu(-\omega, -q) \rangle = \frac{2}{m_f} \times \int \frac{d\omega}{2\pi} \sum_{\omega, p} G^0_{\omega+p}(p + \frac{q}{2}) \delta_{\mu\nu}$$

where $G^0$ is the single-particle Green's function for a free electron gas.

The imaginary part of this propagator is non-zero only in the particle-hole continuum, where $q(q - 2k_F) < 2m_f |\omega| < q(q + 2k_F)$. Within this region, we have:

$$\text{Im} \Pi^\perp(\omega, q) = \frac{1}{3\pi m_f q} \left[ \left( k^2 - \left( \frac{m_f |\omega|}{q} - \frac{q}{2} \right)^2 \right) \right]^{k_F}$$

where $k_F$ is the Fermi momentum and

$$k_{\text{min}} = \max \left( \sqrt{k_F^2 - 2m_f |\omega|}, m_f |\omega| - \frac{q}{2} \right)$$

The real part of the polarisation is given by:

$$\text{Re} \Pi^\perp(\omega, q) = \frac{12m_f^2 k^2 + q^4}{12\pi m_f q^2}$$

for $2m_f |\omega| < q(2k_F - q)$, and by:

$$\text{Re} \Pi^\perp(\omega, q) = \frac{12m_f^2 k^2 + q^4}{12\pi m_f q^2}$$

in the rest of the particle-hole continuum.

As mentioned in section II, we introduce an high-energy cutoff, $\Omega = E_F$, to the spectral function of the gauge field propagator. The effect of $\Omega$ on the fluxes through various areas is shown in figure [7]. The mean square flux through an area $R^2$ is calculated from:

$$\langle \Phi^2 \rangle_R = \int_{R^2} d^2r d^2r' \langle (\nabla \times \mathbf{a})_{r, \tau} (\nabla \times \mathbf{a})_{r', \tau} \rangle \simeq R^4 \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{d\tau p d\tau}{2\pi} p^2 D_{p, \tau=0}$$

If no cutoff is used areas much smaller than one plaquette still have large fluxes through them, which is not compatible with the underlying lattice in the $t$-$J$ model; introducing the cutoff at the Fermi energy strongly reduces fluxes through areas smaller than one plaquette, leaving the behaviour at larger lengthscales unchanged.

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1. D. Das and S. Doniach, Phys. Rev. B 60, 1261 (1999).
2. D. Dalidovich and P. Phillips, cond-mat/0109269.
3. D. Dalidovich and P. Phillips, Phys. Rev. B 64, 052507 (2001).
4. L. B. Ioffe and A. I. Larkin, Phys. Rev. B 39, 8988 (1989).
5. P. A. Lee and N. Nagaosa, Phys. Rev. B 46, 5621 (1992).
6. M. V. Feigelman, V. B. Geshkenbein, L. B. Ioffe, and A. I. Larkin, Phys. Rev. B 48, 16641 (1993).
7 D. H. Kim, D. K. K. Lee, and P. A. Lee, Phys. Rev. B 55, 591 (1997).
8 P. Coleman, Phys. Rev. B 29, 3035 (1984).
9 N. Read and D. Newns, J. Phys. C 16, 3273 (1983).