EXISTENCE OF SMOOTH SOLUTIONS TO COUPLED CHEMOTAXIS-FLUID EQUATIONS

MYEONGJU CHAE  
Department of Applied Mathematics  
Hankyong National University  
Ansung, Republic of Korea

KYUNGKEUN KANG  
Department of Mathematics  
Yonsei University  
Seoul, Republic of Korea

JIIHOON LEE  
Department of Mathematics  
Sungkyunkwan University  
Suwon, Republic of Korea

(Communicated by Juan Luis Vázquez)

ABSTRACT. We consider a system coupling the parabolic-parabolic chemotaxis equations to the incompressible Navier-Stokes equations in spatial dimensions two and three. We establish the local existence of regular solutions and present some blow-up criterions. For two dimensional chemotaxis-Navier-Stokes equations, regular solutions constructed locally in time are, in reality, extended globally under some assumptions pertinent to experimental observations in [21] on the consumption rate and chemotactic sensitivity. We also show the existence of global weak solutions in spatially three dimensions with stronger restriction on the consumption rate and chemotactic sensitivity.

1. Introduction. In this paper, we consider mathematical models describing the dynamics of oxygen diffusion and consumption, chemotaxis, and viscous incompressible fluids in $\mathbb{R}^d$, with $d = 2, 3$. Bacteria or microorganisms often live in fluid, in which the biology of chemotaxis is intimately related to the surrounding physics. Such a model was proposed by Tuval et al. [21] to describe the dynamics of swimming bacteria, Bacillus subtilis. We consider the following equations in [21] and set $Q_T = (0, T] \times \mathbb{R}^d$ with $d = 2, 3$:

\[
\begin{align*}
\partial_t n + u \cdot \nabla n - \Delta n &= -\nabla \cdot (\chi(c)n\nabla c), \\
\partial_t c + u \cdot \nabla c - \Delta c &= -k(c)n, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p &= -n\nabla \phi, \quad \nabla \cdot u = 0
\end{align*}
\]

where $c(t, x) : Q_T \to \mathbb{R}^+, n(t, x) : Q_T \to \mathbb{R}^+, u(t, x) : Q_T \to \mathbb{R}^d$ and $p(t, x) : Q_T \to \mathbb{R}$ denote the oxygen concentration, cell concentration, fluid velocity, and
scalar pressure, respectively. The nonnegative function \( k(c) \) denotes the oxygen consumption rate, and the nonnegative function \( \chi(c) \) denotes chemotactic sensitivity. Initial data are given by \( (n_0(x), c_0(x), u_0(x)) \). To describe the fluid motions, we use Boussinesq approximation to denote the effect due to heavy bacteria. The time-independent function \( \phi = \phi(x) \) denotes the potential function produced by different physical mechanisms, e.g., the gravitational force or centrifugal force. Thus, \( \phi(x) = axd \) is one example of gravity force, and \( \phi(x) = \phi(|x|) \) is an example of centrifugal force. The authors in [21] suggested that the functions \( k(c) \) and \( \chi(c) \) are constants at large \( c \) and rapidly approach zero below some critical \( c^* \). Hence, in [21], these functions are approximated by step functions, e.g., \( k(c) = \kappa_1 \theta(c - c^*) \) and \( \chi(c) = \kappa_2 \theta(c - c^*) \) for some positive constants \( \kappa_1 \) and \( \kappa_2 \). Also in [2], numerical simulation of plumes was obtained for the same species of bacteria in [21] in two dimensions. Furthermore, they assumed that the functions \( \chi(c) \) and \( k(c) \) are constant multiples of each other, i.e., \( \chi(c) = \mu k(c) \).

The main goals of this paper are to show the local existence of regular solutions in two and three dimensions with general conditions on the oxygen consumption rate and chemotactic sensitivity, and to demonstrate global existence of regular solutions in two dimensions and weak solutions in three dimensions with appropriate assumptions of \( \chi(c) \), \( k(c) \), \( \phi \) and initial data. Here we mention the related works for the result in this paper. If we ignore the coupling of the fluids, we obtain the angiogenesis type system. The classical model to describe the motion of cells was suggested by Patlak[17] and Keller-Segel[11, 12]. It consists of a system of the angiogenesis type system. The classical model to describe the motion of cells was suggested by Patlak[17] and Keller-Segel[11, 12]. It consists of a system of

\[
\begin{cases}
  n_t = \Delta n - \nabla \cdot (n\chi \nabla c), \\
  \alpha c_t = \Delta c - \tau c + n,
\end{cases}
\]

(2)

where \( \chi \) is the sensitivity and \( \tau^{-1/2} \) represents the activation length. The system in (2) has been extensively studied by many authors(see [9, 10, 15, 16, 22] and references therein). For the chemical consumption model by the cell or bacteria, we refer to the following chemotaxis model motivated by angiogenesis.

\[
\begin{cases}
  n_t = \Delta n - \nabla \cdot (n\chi(c) \nabla c), \\
  c_t = -n \nabla n, \\
  n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x),
\end{cases}
\]

(3)

The global existence of a weak solution to the system in (3) was obtained by Corrias, Perthame and Zaag[3, 4] with a small data assumption of \( \|n_0\|_{L^2} \). The bacterial movement toward the concentration gradient model in the absence of the fluid, i.e., \( u = 0 \), was recently studied. When \( u \equiv 0 \), \( \chi(c) \equiv \chi \) and \( k(c) \equiv c \) in (1), it was shown in [19] that there exists a uniquely global bounded solution if \( 0 < \chi \leq 1/6(d + 1)\|c_0\|_{L^\infty} \).

If the flow of fluid is slow, then the Navier-Stokes equations can be simplified to the Stokes equations. For the case \( \chi(c) \equiv \chi \), Lorz[14] showed the local existence of solutions for the chemotaxis-Stokes system in two dimensional bounded domain with the mixed boundary conditions and chemotaxis-Navier-Stokes system in three dimensional bounded domain with boundary conditions \( \partial_n n = \partial_n c = u = 0 \).
Duan, Lorz, and Markowich [6] showed the global existence of a weak solution to the chemotaxis-Stokes equations in \( \mathbb{R}^2 \) with some small data assumptions on either \( \|c_0\|_{L^1} \) and \( \phi \), or \( \|c_0\|_{L^\infty} \) only and the assumptions on the functions such that
\[
\chi(c) > 0, \quad \chi'(c) \geq 0, \quad k'(c) > 0, \quad \frac{d^2}{dc^2} \left( \frac{k(c)}{\chi(c)} \right) < 0.
\]

For the chemotaxis-Navier-Stokes system (1), Duan, Lorz, and Markowich [6] showed the global-in-time existence of the \( H^3(\mathbb{R}^d) \)-solution, near constant states, to (1) in \( \mathbb{R}^3 \), i.e., if initial data \( \| (n_0 - n_{\infty}, c_0, u_0) \|_{H^3} \) is sufficiently small, then there exists a unique global solution. Also, they obtained the time decay rates of the obtained classical solution near steady states. In [13], Liu and Lorz showed the global existence of a weak solution to the chemotaxis-Navier-Stokes equations in \( \mathbb{R}^2 \) with similar assumptions to those in (4) on \( k \) and \( \chi \) such that
\[
\chi(c), \chi'(c), k(c), k'(c) \geq 0, \quad (\chi(c)k(c))' > 0, \quad \text{and} \quad \frac{d^2}{dc^2} \left( \frac{k(c)}{\chi(c)} \right) < 0.
\]

Very recently, Winkler [23] considered the equations (1) on a bounded domain \( \Omega \) in \( \mathbb{R}^2 \) with the boundary condition \( \partial_\nu n = \partial_\nu c = u = 0 \) on \( \partial \Omega \), and proved the global-in-time existence of a classical solution under the assumption such that
\[
\chi(c) > 0, \quad k(c) \geq 0, \quad \left( \frac{k(c)}{\chi(c)} \right)' > 0, \quad (\chi(c)k(c))' \geq 0, \quad \text{and} \quad \frac{d^2}{dc^2} \left( \frac{k(c)}{\chi(c)} \right) \leq 0.
\]

Also in [23], chemotaxis-Stokes system on a bounded domain in \( \mathbb{R}^3 \) is considered and the global existence of a weak solution is shown under the assumption (6).

The equation of \( n \) in (1) could have been replaced by a porous medium equation, i.e., \( \Delta n \) is replaced by \( \Delta n^m \) and the following chemotaxis-Stokes system has been considered in [7].

\[
\begin{align*}
\partial_t n + u \cdot \nabla n - \Delta n^m &= -\nabla \cdot (\chi(c)n \nabla c), \\
\partial_t c + u \cdot \nabla c - \Delta c &= -k(c)n, \\
\partial_t u - \Delta u + \nabla p &= -n \nabla \phi, \quad \nabla \cdot u = 0,
\end{align*}
\]

In [7], Francesco, Lorz and Markowich showed the global existence of a bounded solution to (7) on a bounded domain in \( \mathbb{R}^2 \) with the boundary conditions \( \partial_n n^m = \partial_\nu c = u = 0 \) when \( m \in (\frac{2}{3}, 2] \). In [20], Tao and Winkler extended the result to the case \( m > 1 \) on a bounded domain in \( \mathbb{R}^2 \). In [13], Liu and Lorz proved the global existence of a weak solution to (7) in \( \mathbb{R}^3 \) when \( m = \frac{4}{3} \).

As mentioned earlier, the aim of this paper is to obtain the local-in-time existence of a regular solution in two and three dimensions and the global-in-time existence of a regular solution to (1) in two dimensions under some physically relevant conditions on the consumption rate and chemotactic sensitivity. Now we are ready to state our main results. The first result in this article is the local existence in time of a regular solution to (1). Comparing with the result in [6], we show the local-in-time existence without smallness of the initial data.

**Theorem 1.1.** (Local existence) Let \( m \geq 3 \) and \( d = 2, 3 \). Assume that \( \chi, k \in C^m(\mathbb{R}^+ \cup \{0\}) \) and \( k(0) = 0, \|\nabla^l \phi\|_{L^\infty} < \infty \) for \( 1 \leq |l| \leq m \). There exists \( T > 0 \), the maximal time of existence, such that, if the initial data \( (n_0, c_0, u_0) \in H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d) \times H^m(\mathbb{R}^d) \), then there exists a unique regular solution \( (n, c, u) \) of (1) satisfying for any \( t < T \)
\[
(n, c, u) \in L^\infty(0, t; H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d) \times H^m(\mathbb{R}^d)),
\]
\[
(\nabla n, \nabla c, \nabla u) \in L^2(0, t; H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d) \times H^m(\mathbb{R}^d)), \\
(\partial_t n, \partial_t c, \partial_t u) \in L^\infty(0, t; H^{m-3}(\mathbb{R}^d) \times H^{m-2}(\mathbb{R}^d) \times H^{m-2}(\mathbb{R}^d)).
\]

**Remark 1.** For simplicity, we denote
\[
\| (n(t), c(t), u(t)) \|_{X_m} := \| n(t) \|_{H^{m-1}(\mathbb{R}^d)} + \| c(t) \|_{H^m(\mathbb{R}^d)} + \| u(t) \|_{H^m(\mathbb{R}^d)}.
\]
We remark that if \( T \) is the maximal time of existence with \( T < \infty \) in Theorem 1.1, then
\[
\limsup_{t \to T} \| (n(t), c(t), u(t)) \|_{X_m}^2 + \int_0^T \| (n(t), c(t), u(t)) \|_{X_{m+1}}^2 = \infty, \quad m \geq 3.
\]

Secondly, we obtain two blow-up criteria for the system (1) depending on dimensions.

**Theorem 1.2.** Suppose that \( \chi, \ k, \ \phi \) and the initial data \((n_0, c_0, u_0)\) satisfy all the assumptions presented in Theorem 1.1. If \( T^* \), the maximal time existence in Theorem 1.1, is finite, then one of the following is true in each case of two or three dimensions, respectively:

\[
(2D) \quad \int_0^{T^*} \| \nabla c \|_{L^\infty(\mathbb{R}^2)}^p = \infty.
\]

\[
(3D) \quad \int_0^{T^*} \| u \|_{L^p(\mathbb{R}^3)}^q + \int_0^{T^*} \| \nabla c \|_{L^\infty(\mathbb{R}^3)}^p = \infty, \quad \frac{3}{p} + \frac{2}{q} = 1, \ 3 < p \leq \infty.
\]

**Remark 2.** Theorem 1.2 can be interpreted as follows: If \( \int_0^T \| \nabla c \|_{L^\infty}^2 < \infty \) in two dimensions or if \( \int_0^T (\| u \|_{L^p}^q + \| \nabla c \|_{L^\infty}) dt < \infty \) in three dimensions, then the local solution persists beyond time \( T \), i.e., \((n, c, u) \in L^\infty(0, T + \delta; X_m) \cap L^2(0, T + \delta; X_{m+1}) \) for some \( \delta > 0 \).

The third main result is the global existence of a regular solution in two-dimensional spatial domain \( \mathbb{R}^2 \). Motivated by experiments in [21] and [2], we assume that the oxygen consumption rate \( k(c) \) and chemotactic sensitivity \( \chi(c) \) satisfy the following conditions:

- **(A)** There exists a constant \( \mu \) such that \( \sup | \chi(c) - \mu k(c) | < \epsilon \) for a sufficiently small \( \epsilon > 0 \).
- **(B)** \( \chi(c), k(c), \chi'(c), k'(c) \) are all non-negative, i.e., \( \chi(c), k(c), \chi'(c), k'(c) \geq 0 \).

We remark that, in our analysis, assumption (A) plays a crucial role in obtaining \( L\log L \times H^1 \times L^2 \) type estimates. Comparing with the global-in-time existence results in [23] in a bounded domain in \( \mathbb{R}^2 \), the assumption (A) is quite different from (6), for example, (6) does not include the case that \( k(c) = N_1 c, \chi(c) = N_2 c \), where \( N_1 \) and \( N_2 \) are positive constants.

**Theorem 1.3.** (Global existence in two dimensions) Let \( d = 2 \). Suppose that \( \chi, k, \phi \) and the initial data \((n_0, c_0, u_0)\) satisfy all the assumptions presented in Theorem 1.1 and \( \int_{\mathbb{R}^2} n_0 \ln n_0 dx < \infty \). Assume further that \( \chi \) and \( k \) satisfy the assumptions (A) and (B) and \( \phi \geq 0 \). Then a unique regular solution \((n, c, u)\) exists globally in time and satisfies for any \( T < \infty \)

\[
(n, c, u) \in L^\infty(0, T; H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)), \\
(\nabla n, \nabla c, \nabla u) \in L^2(0, T; H^{m-3}(\mathbb{R}^2) \times H^{m-2}(\mathbb{R}^2) \times H^{m-2}(\mathbb{R}^2)).
\]

\[
(\partial_t n, \partial_t c, \partial_t u) \in L^\infty(0, T; H^{m-3}(\mathbb{R}^2) \times H^{m-2}(\mathbb{R}^2) \times H^{m-2}(\mathbb{R}^2)).
\]
Remark 3. If we approximate Heaviside functions using the smooth functions, then the suggested functional forms of \( \chi \) and \( k \) in [21] satisfy the assumptions (A) and (B). Furthermore, the assumptions on \( \phi \) are satisfied by gravitational and centrifugal forces. Also we note that 2D numerical studies were performed under the assumption \( \chi(c) = \mu k(c) \) in [2].

Our final main result is on the global-in-time existence of weak solutions in three dimensions. The notion of a weak solution of (1) is detailed in section 4 (see Definition 4.1). For existence of global weak solution, we need similar restrictions on \( k(c) \) and \( \chi(c) \) as in Theorem 1.3. More precisely, compared to (A), we impose a stronger assumption, denoted by (AA), which is given as follows:

(AA) There exists a constant \( \mu \) such that \( \chi(c) - \mu k(c) = 0 \).

Before stating our last main result, we introduce some function spaces. \( H^{-1}(\mathbb{R}^3) \) means the dual space of \( H^1(\mathbb{R}^3) \). We also introduce the function spaces \( \mathcal{V}(\mathbb{R}^3) \), \( \mathcal{V}_r(\mathbb{R}^3) \), \( \mathcal{H}(\mathbb{R}^3) \) defined as follows:

\[
\mathcal{V}(\mathbb{R}^3) = \{ u = (u_1, u_2, u_3) \mid u_i \in H^1(\mathbb{R}^3) \}, \quad \mathcal{V}_r(\mathbb{R}^3) = \{ u \in \mathcal{V}(\mathbb{R}^3) \mid \text{div} u = 0 \},
\]

\[
\mathcal{H}(\mathbb{R}^3) = \text{the closure of } \mathcal{V}_r(\mathbb{R}^3) \text{ in } (L^2(\mathbb{R}^3))^3.
\]

The dual space of \( \mathcal{V}(\mathbb{R}^3) \) is denoted by \( \mathcal{V}'(\mathbb{R}^3) = \{ u = (u_1, u_2, u_3) \mid u_i \in H^{-1}(\mathbb{R}^3) \} \). The duality \( \langle w, v \rangle \) for \( w \in \mathcal{V}'(\mathbb{R}^3) \), \( v \in \mathcal{V}(\mathbb{R}^3) \) is, as usual, given as \( \langle w, v \rangle = \sum_{i=1}^{3} \langle w_i, v_i \rangle_{H^{-1} \times H^1} \) and we denote \( \mathcal{V}_r^0(\mathbb{R}^3) = \{ w \in \mathcal{V}'(\mathbb{R}^3) \mid \langle w, v \rangle = 0 \text{ for all } v \in \mathcal{V}_r(\mathbb{R}^3) \} \). As far as we know, it is not known whether or not solutions to (1) exist globally in time without any assumptions on the smallness of the initial data. Our last main result is global existence in time of a weak solution to chemotaxis-Navier-Stokes system in \( \mathbb{R}^3 \) with the assumptions (AA). More precisely we have the following.

Theorem 1.4. Let \( d = 3 \) and \( (n_0, c_0, u_0) \) satisfy

\[
n_0 \in L^1(\mathbb{R}^3), \quad c_0 \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), \quad u_0 \in \mathcal{H}(\mathbb{R}^3),
\]
\[
n_0 \geq 0, \quad c_0 \geq 0, \quad \int_{\mathbb{R}^3} (1 + |\ln n_0| + |x|) n_0 dx < \infty. \tag{10}
\]

Assume further that \( \chi \) and \( k \in C^1(\mathbb{R}^+) \) satisfy the assumption (AA), (B) and \( k(0) = 0 \) and \( \phi \) satisfies \( \phi \geq 0 \) and \( \| \nabla \phi \|_{L^\infty} < \infty \) for \( 0 \leq |l| \leq 2 \). Then a weak solution \( (n, c, u) \) exists globally in time. Moreover it satisfies the following energy inequality

\[
\int_{\mathbb{R}^3} \left( \frac{|u|^2}{2} + n \phi + n |\ln n| + \frac{|\nabla c|^2}{2} + \langle x \rangle n \right) dx
\]
\[
+ \int_0^T \| \nabla u \|_{L^2}^2 + \| \nabla \sqrt{n} \|_{L^2}^2 + \| \Delta c \|_{L^2}^2 dt \leq C, \tag{11}
\]

with

\[
C = C(T, \| \chi(c) \|_{L^\infty}, \| \langle x \rangle n_0 \|_{L^1}, \| \nabla c_0 \|_{L^2}, \| n_0 \|_{L^1}, \| \Delta \phi \|_{L^\infty}, \| \nabla \phi \|_{L^\infty}, \| \phi \|_{L^\infty}).
\]

The rest of this paper is organized as follows. In Section 2, we prove local-in-time existence of a regular solution for two and three dimensional chemotaxis system with incompressible Navier-Stokes equations and obtain some blow-up criteria for the solution. In Section 3, we show the global in time existence of a regular solution in two dimensions. In Section 4, we establish the existence of a weak solution in three dimensions.
2. Local existence and blow-up criterion.

2.1. Local existence. We first consider the chemotaxis system coupled with the Navier-Stokes equations in two and three dimensions. We show the local existence of solutions \((n, c, u)\) in \(H^{m-1} \times H^m \times H^m\) space with \(m \geq 3\).

**Proof of Theorem 1.1.** We follow similar procedures of an iterative scheme developed in [6]. We construct a solution sequence \((n^j, c^j, u^j)_{j \geq 0}\) by iteratively solving the Cauchy problems on the following linear equations

\[
\begin{align*}
\partial_t n^{j+1} + u^j \cdot \nabla n^{j+1} &= \Delta n^{j+1} - \nabla \cdot (c^j n^{j+1} \nabla c^j), \\
\partial_t c^{j+1} + u^j \cdot \nabla c^{j+1} &= \Delta c^{j+1} - k(c^j) n^j, \\
\partial_t u^{j+1} + u^j \cdot \nabla u^{j+1} + \nabla p^{j+1} &= \Delta u^{j+1} - n^j \nabla \phi, \quad \nabla \cdot u^{j+1} = 0.
\end{align*}
\]  
(12)

We first set \((n^0(x, t), c^0(x, t), u^0(x, t)) = (n_0(x), c_0(x), u_0(x))\). Then, using the same initial data to solve the linear Stokes type equations and the linear parabolic equations, we obtain \(u^1(x, t), c^1(x, t)\) and \(n^1(x, t)\), respectively. Similarly, we define \((n^j(x, t), c^j(x, t), u^j(x, t))\) iteratively. For this, we presume that \(c^j\) and \(n^j\) are nonnegative and show the existence and the convergence of solutions in the adequate function spaces. We show the nonnegativity of \(c^j\) and \(n^j\) at the end of the proof.

To prove the conclusion, i.e., to obtain contraction in adequate function spaces, we show the uniform boundedness of the sequence of functions under our construction via energy estimates.

- **(Uniform boundedness)** We here show that the iterative sequences \((n^j, c^j, u^j)\) are in \(X_m := H^{m-1} \times H^m \times H^m\) space for all \(j \geq 0\). Observing that

\[
\sum_{|\alpha| \leq m-1} \|\partial^\alpha (u^j n^{j+1})\|_{L^2} \leq C(\|u^j\|_{L^\infty} \|n^{j+1}\|_{H^{m-1}} + \|n^{j+1}\|_{L^\infty} \|u^j\|_{H^{m-1}}),
\]

\[
\|\chi(c^j) n^{j+1} \nabla c^j\|_{H^{m-1}} \leq C \|n^{j+1}\|_{H^{m-1}} \left(1 + \|c^j\|_{H^m}^m\right),
\]

\[
\|k(c^j) n^j\|_{H^{m-1}} \leq C \|n^j\|_{H^{m-1}} \left(1 + \|c^j\|_{H^m}^m\right),
\]

we have the following energy estimates:

(i) The estimate of \(n^{j+1}\)

\[
\frac{1}{2} \frac{d}{dt} \|n^{j+1}\|_{H^{m-1}}^2 + \|\nabla n^{j+1}\|_{H^{m-1}}^2 \leq C \|u^j\|_{L^\infty} \|n^{j+1}\|_{H^{m-1}} \|\nabla n^{j+1}\|_{H^{m-1}} + C \|u^j\|_{H^{m-1}} \|n^{j+1}\|_{H^{m-1}} \|\nabla n^{j+1}\|_{H^{m-1}} + C \left(1 + \|c^j\|_{H^m}^m\right) \|n^{j+1}\|_{H^{m-1}} \|\nabla n^{j+1}\|_{H^{m-1}} \leq C \left(1 + \|u^j\|_{H^m}^2 \|c^j\|_{H^m}^{2m}\right) \|n^{j+1}\|_{H^{m-1}}^2 + \frac{1}{2} \|\nabla n^{j+1}\|_{H^{m-1}}^2. 
\]  
(13)

(ii) The estimate of \(c^{j+1}\)

\[
\frac{1}{2} \frac{d}{dt} \|c^{j+1}\|_{H^m}^2 + \|\nabla c^{j+1}\|_{H^m}^2 \leq C \|u^j\|_{L^\infty} \|c^{j+1}\|_{H^m} \|\nabla c^{j+1}\|_{H^m} + C \|u^j\|_{H^{m-1}} \|\nabla c^{j+1}\|_{H^m} + C \left(1 + \|c^j\|_{H^m}^{m-1}\right) \|n^j\|_{H^{m-1}} + C \|u^j\|_{H^m} \|c^{j+1}\|_{H^m}^2 + \frac{1}{2} \|\nabla c^{j+1}\|_{H^m}^2. 
\]  
(14)

(iii) The estimate of \(u^{j+1}\)

\[
\frac{1}{2} \frac{d}{dt} \|u^{j+1}\|_{H^m}^2 + \|\nabla u^{j+1}\|_{H^m}^2 \leq C \|\nabla u^j\|_{L^\infty} \|u^{j+1}\|_{H^m}^2.
\]
from Gronwall’s inequality, we have (16). In short, we have \( n \) interval \( 0 < M > k \) Thus, the

\[ \text{For convenience, we denote} \]

and integrate over spatial varaibles, then we obtain

\[ \text{(16) for small} \]

\[ T > 0. \]

\[ \text{Thus, the} \]

where standard commutator estimates are used. We show that there exists a constant \( M > 0 \) such that, for any \( j \), the following inequality holds for a small time interval \([0, T]\) (\( T \) will be specified later):

\[ \sup_{0 \leq t \leq T} \left( \| n^j \|^2_{H^{-1}} + \| c^j \|^2_{H^m} + \| u^j \|^2_{H^m} \right) \]

\[ + \int_0^T \| \nabla n^j \|^2_{H^{-1}} + \| \nabla c^j \|^2_{H^m} + \| \nabla u^j \|^2_{H^m} \, dt \leq M. \] (16)

Here \( M \) is a number with \( M \geq 4(\| n_0 \|^2_{H^{-1}} + \| c_0 \|^2_{H^m} + \| u_0 \|^2_{H^m}) \).

We prove (16) via an inductive argument. Suppose (16) hold for \( j \leq i \). If we add (13), (14), and (15) and use Young’s inequality, then we have

\[ \frac{d}{dt} \left( \| n^{i+1} \|^2_{H^{-1}} + \| c^{i+1} \|^2_{H^m} + \| u^{i+1} \|^2_{H^m} \right) \]

\[ + \| \nabla n^{i+1} \|^2_{H^{-1}} + \| \nabla c^{i+1} \|^2_{H^m} + \| \nabla u^{i+1} \|^2_{H^m} \leq C \left( 1 + \| u^i \|^2_{H^m} + \| c^i \|^2_{H^m} \right) \left( \| n^{i+1} \|^2_{H^{-1}} + \| c^{i+1} \|^2_{H^m} \right) \]

\[ + C \left( \| u^i \|^2_{H^m} \| c^{i+1} \|^2_{H^m} + C \| u^i \|^2_{H^m} \| u^{i+1} \|^2_{H^m} + C \| n^i \|^2_{H^{-1}} \right) \]

\[ \leq C(1 + M + M^m) \| n^{i+1} \|^2_{H^{-1}} + CM \| c^{i+1} \|^2_{H^m} \]

\[ + CM \| u^{i+1} \|^2_{H^m} + C(1 + M^{m-1})M + CM. \]

In the last inequality, we use the induction hypothesis. Hence, we get

\[ \frac{d}{dt} \left( \| n^{i+1} \|^2_{H^{-1}} + \| c^{i+1} \|^2_{H^m} + \| u^{i+1} \|^2_{H^m} \right) \]

\[ + \| \nabla n^{i+1} \|^2_{H^{-1}} + \| \nabla c^{i+1} \|^2_{H^m} + \| \nabla u^{i+1} \|^2_{H^m} \leq C(1 + M + M^m) \left( \| n^{i+1} \|^2_{H^{-1}} + \| c^{i+1} \|^2_{H^m} + \| u^{i+1} \|^2_{H^m} \right) \]

\[ + C(1 + M^{m-1})M. \] (17)

We choose time \( T \) such that \( \max\{C(1 + M + M^m)T, \, C(1 + M^{m-1})T\} \leq \frac{1}{4} \). Then from Gronwall’s inequality, we have (16). In short, we have \( (n^{i+1}, \, c^{i+1}, \, u^{i+1}) \in L^\infty(0, \, T; \, X_m) \) and \( (\nabla n^{i+1}, \, \nabla c^{i+1}, \, \nabla u^{i+1}) \in L^2(0, \, T; \, X_m) \) and the uniform bound (16) for small \( T > 0 \).

Also if we multiply \((c^{i+1})^{q-1}\) on the both sides of the second equation of (12) and integrate over spatial variables, then we obtain

\[ \frac{1}{q} \frac{d}{dt} \| c^{i+1} \|^q_{L^q} + \frac{4(q - 1)}{q^2} \| \nabla (c^{i+1}) \|^2_{L^2} \leq 0. \]

Thus, the \( L^\infty \) norm of \( c^{i+1} \) is uniformly bounded, which implies that \( \chi(c^j) \) and \( k(c^j) \) are uniformly bounded for all \( j \geq 0 \).

- (Contraction) The estimate of this part is similar to that of the previous one. For convenience, we denote \( \delta f^{i+1} := f^{i+1} - f^j \). Subtracting the \( j \)-th equations
from the \((j + 1)\)-th equations, we have the following equations for \(\delta n^{j+1}\), \(\delta c^{j+1}\) and \(\delta u^{j+1}\):

\[
\begin{align*}
\partial_t \delta n^{j+1} + u^j \cdot \nabla \delta n^{j+1} - \Delta \delta n^{j+1} &= -\delta u^j \cdot \nabla n^j - \nabla \cdot (\chi(c^j) \delta n^{j+1} \nabla c^j) \\
&\quad - \nabla \cdot (\chi(c^j) u^j \nabla c^j) + \nabla \cdot (\chi(c^{j-1}) \mu_j \nabla c^j), \\
\partial_t \delta c^{j+1} + u^j \cdot \nabla \delta c^{j+1} - \Delta \delta c^{j+1} &= -\delta u^j \cdot \nabla c^j - k(c^j) \delta n^j + (k(c^j) - k(c^{j-1})) \mu_j - 1, \\
\partial_t \delta u^{j+1} + u^j \cdot \nabla \delta u^{j+1} + \nabla \delta p^{j+1} - \Delta \delta u^{j+1} &= -\delta u^j \cdot \nabla u^j - \delta n^j \phi, \\
\nabla \cdot \delta n^{j+1} &= 0.
\end{align*}
\]

\((18)\)

(i) The estimate of \(\delta n^{j+1}\).

Using the following standard commutator estimates

\[
\sum_{|\alpha| \leq m-1} \int |\partial^\alpha (u^j \cdot \nabla \delta n^{j+1}) \partial^\alpha \delta n^{j+1} - (u^j \cdot \nabla \partial^\alpha \delta n^{j+1}) \partial^\alpha \delta n^{j+1}| \leq C(L\|n^j\|_{L^\infty} \|\delta n^{j+1}\|^2_{H_m-1} + \|u^j\|_{H_{m-1}} \|\nabla \delta n^{j+1}\|_{L^\infty} \|\delta n^{j+1}\|_{H_{m-1}}),
\]

we have the following estimate:

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\delta n^{j+1}\|^2_{H_m-1} + \|\nabla \delta n^{j+1}\|^2_{H_m-1} &\leq C \|\nabla u^j\|_{L^\infty} \|\delta n^{j+1}\|_{H_{m-1}} \\
&+ C \|u^j\|_{H_{m-1}} \|\delta n^{j+1}\|_{H_{m-1}} + C \|\delta u^j\|_{L^\infty} \|\delta n^{j+1}\|_{H_{m-1}} \\
&+ C \|\delta n^{j+1}\|_{H_{m-1}} \|\nabla u^j\|_{L^\infty} \|\delta n^{j+1}\|_{H_{m-1}} \\
&+ C \|n^j\|_{H_{m-1}} (1 + \|c^j\|_{H_{m-1}} + \|c^{j-1}\|_{H_{m-1}}) \|\delta c^j\|_{H_{m}} \|\nabla \delta n^{j+1}\|_{H_{m-1}}. \\
\end{align*}
\]

(ii) The estimate of \(\delta c^{j+1}\).

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\delta c^{j+1}\|^2_{H_m} + \|\nabla \delta c^{j+1}\|^2_{H_m} &\leq C \|\nabla u^j\|_{L^\infty} \|\delta c^{j+1}\|_{H_{m-1}} \|\nabla \delta c^{j+1}\|_{H_{m}} \\
&+ C \|u^j\|_{H_{m-1}} (1 + \|c^j\|_{H_{m-1}} + \|c^{j-1}\|_{H_{m-1}}) \|\delta c^j\|_{H_{m}} \|\nabla \delta c^{j+1}\|_{H_{m}} \\
&+ C \|c^j\|_{H_{m}} \|\nabla u^j\|_{H_{m-1}} \|\nabla \delta c^{j+1}\|_{H_{m}} + C \|\delta u^j\|_{L^\infty} \|\nabla \delta c^{j+1}\|_{H_{m}} \\
&+ C \|n^j\|_{H_{m-1}} (1 + \|c^j\|_{H_{m-1}} + \|c^{j-1}\|_{H_{m-1}}) \|\delta c^j\|_{H_{m}} \|\nabla \delta c^{j+1}\|_{H_{m}} \\
&\leq C \|\nabla u^j\|^2_{H_{m-1}} \|\delta c^{j+1}\|^2_{H_{m-1}} + C \|c^j\|^2_{H_{m-1}} \|\delta u^j\|^2_{H_{m-1}} + C \|c^{j-1}\|^2_{H_{m-1}} \|\delta n^j\|^2_{H_{m-1}} \\
&+ C \|n^j\|_{H_{m-1}}^2 (1 + \|c^j\|^2_{H_{m-1}} + \|c^{j-1}\|^2_{H_{m-1}}) \|\delta c^j\|^2_{H_{m}} + \frac{1}{2} \|\nabla \delta c^{j+1}\|^2_{H_{m}}.
\end{align*}
\]

where we used the Mean Value Theorem for the last term.

(iii) The estimate of \(\delta u^{j+1}\).

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\delta u^{j+1}\|^2_{H_m} + \|\nabla \delta u^{j+1}\|^2_{H_m} &\leq C \|\nabla u^j\|_{L^\infty} \|\delta u^{j+1}\|_{H_{m-1}} \|\nabla \delta u^{j+1}\|_{H_m} \\
&+ C \|\nabla \delta u^j\|_{L^\infty} \|\delta u^{j+1}\|_{H_{m-1}} + C \|\delta u^j\|_{L^\infty} \|\delta u^{j+1}\|_{H_{m+1}} \\
&+ C \|\delta n^j\|_{H_{m-1}} \|\nabla u^j\|_{L^\infty} \|\delta u^{j+1}\|_{H_{m+1}} + C \|\delta n^j\|_{H_{m-1}} \|\delta u^{j+1}\|_{H_{m+1}} \\
&\leq C \|\nabla u^j\|_{H_m} \|\delta u^{j+1}\|_{H_{m-1}} + C \|\nabla u^j\|^2_{H_m} \|\delta u^j\|^2_{H_{m-1}} + C \|\delta n^j\|^2_{H_{m-1}} + \frac{1}{2} \|\delta u^{j+1}\|^2_{H_{m+1}}.
\end{align*}
\]
where similar standard commutator estimates are used as in the case of \( \delta u^{j+1} \).

Using Young’s inequality, we have

\[
\frac{d}{dt}(\|\delta n^{j+1}\|_{L^2}^2 + \|\delta c^{j+1}\|_{L^2}^2 + \|\delta u^{j+1}\|_{L^2}^2) \\
+ \|\nabla \delta n^{j+1}\|_{L^2}^2 + \|\nabla \delta c^{j+1}\|_{L^2}^2 + \|\nabla \delta u^{j+1}\|_{L^2}^2 \\
\leq C(\|\delta n^{j+1}\|_{L^2}^2 + \|\delta c^{j+1}\|_{L^2}^2 + \|\delta u^{j+1}\|_{L^2}^2) \\
+ C\|\delta n^{j+1}\|_{L^2}^2 + C\|\delta c^{j+1}\|_{L^2}^2 + C\|\delta u^{j+1}\|_{L^2}^2,
\]

where \( C \) depend on the \( L^2 \times L^2 \times L^2 \) norm of \( (n^j, c^j, u^j) \) and \((n^{j-1}, c^{j-1}, u^{j-1})\) and the maximum values of \( \chi^{(i)} \) and \( \kappa^{(i)} \). Gronwall’s inequality gives us

\[
\sup_{0 \leq t \leq T} (\|\delta n^{j+1}\|_{L^2}^2 + \|\delta c^{j+1}\|_{L^2}^2 + \|\delta u^{j+1}\|_{L^2}^2) \\
\leq CT \exp(CT) \sup_{0 \leq t \leq T} (\|\delta n^{j}\|_{L^2}^2 + \|\delta c^{j}\|_{L^2}^2 + \|\delta u^{j}\|_{L^2}^2).
\]

From the above inequality, we find that \((n^j, c^j, u^j)\) is a Cauchy sequence in the Banach space \( C(0, T; X_m) \) for some small \( T > 0 \), and thus we have the limit in the same space. The regularity of time derivatives of the limit is direct from the equations.

- **Uniqueness** To show the uniqueness of the above local-in-time solution, we assume that there exist two local-in-time solutions \((c_1(x, t), n_1(x, t), u_1(x, t))\) and \((c_2(x, t), n_2(x, t), u_2(x, t))\) of (1) with the same initial data over the time interval \([0, T]\), where \( T \) is any time before the maximal time of existence. Let \( \tilde{c}(x, t) := c_1(x, t) - c_2(x, t) \), \( \tilde{n}(x, t) := n_1(x, t) - n_2(x, t) \), and \( \tilde{u}(x, t) := u_1(x, t) - u_2(x, t) \). Then \((\tilde{c}, \tilde{n}, \tilde{u})\) solves

\[
\begin{align*}
\partial_t \tilde{n} + u_1 \cdot \nabla \tilde{n} - \Delta \tilde{n} &= -\tilde{u} \cdot \nabla n_2 - \nabla \cdot \left( (\chi(c_1) - \chi(c_2))n_1 \nabla c_1 \right) \\
- \nabla \cdot \left( \chi(c_2) n_2 \nabla \tilde{c} \right), \\
\partial_t \tilde{c} + u_1 \cdot \nabla \tilde{c} - \Delta \tilde{c} &= -\tilde{u} \cdot \nabla c_2 - (k(c_1) - k(c_2))n_1 - k(c_2)\tilde{n}, \\
\partial_t \tilde{u} + u_1 \cdot \nabla \tilde{u} - \Delta \tilde{u} + \nabla \tilde{p} &= -\tilde{u} \cdot \nabla u_2 - \tilde{n} \nabla \phi, \\
\nabla \cdot \tilde{u} &= 0,
\end{align*}
\]

Multiplying \( \tilde{n} \) to both sides of the first equation of (19) and integrating over \( \mathbb{R}^d \), we have

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{n}\|_{L^2}^2 + \|\nabla \tilde{n}\|_{L^2}^2 \leq \|\tilde{n} \cdot \nabla n_2\|_{L^2} \|\nabla \tilde{n}\|_{L^2} + \|\chi(c_1) - \chi(c_2)\|_{L^2} \|n_1 \nabla c_1\|_{L^2} \|\nabla \tilde{n}\|_{L^2} \\
+ \|\chi(c_2)\|_{L^2} \|n_2 \nabla \tilde{c}\|_{L^2} \|\nabla \tilde{n}\|_{L^2} + \|\chi(c_2)\|_{L^2} \|n_2 \nabla \tilde{c}\|_{L^2} \|\nabla \tilde{n}\|_{L^2} \\
\leq C\|\tilde{n}\|_{L^2}^2 \|n_2\|_{L^2}^2 + C\|\tilde{c}\|_{L^2}^2 \|n_1\|_{L^\infty} + C\|\tilde{n}\|_{L^2} \|\nabla c_1\|_{L^\infty} \|\tilde{n}\|_{L^2} \\
+ C\|\nabla \tilde{n}\|_{L^2} \|n_2\|_{L^\infty} + C\|\nabla \tilde{n}\|_{L^2} \|\nabla \tilde{n}\|_{L^2}.
\]

Multiplying \( \tilde{c} \) and \(-\Delta \tilde{c}\) to both sides of the second equation of (19), we have

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{c}\|_{L^2}^2 + \|\nabla \tilde{c}\|_{L^2}^2 \leq \|\tilde{n} \cdot \nabla c_2\|_{L^2} \|\tilde{c}\|_{L^2} + C\|\tilde{c}\|_{L^2} \|n_1\|_{L^\infty} + C\|\tilde{n}\|_{L^2} \|\tilde{c}\|_{L^2} \\
\leq C\|\tilde{n}\|_{L^2} \|c_2\|_{L^\infty} + C\|\tilde{c}\|_{L^2} \|n_1\|_{L^\infty} + C\|\tilde{n}\|_{L^2} \|\tilde{c}\|_{L^2},
\]

and

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \tilde{c}\|_{L^2}^2 + \|\Delta \tilde{c}\|_{L^2}^2 \leq \|u_1 \cdot \nabla \tilde{c}\|_{L^2} \|\Delta \tilde{c}\|_{L^2} + \|\tilde{u} \cdot \nabla c_2\|_{L^2} \|\Delta \tilde{c}\|_{L^2} \\
+ \|k(c_1) - k(c_2)\|_{L^2} \|n_1\|_{L^\infty} \|\Delta \tilde{c}\|_{L^2} + C\|\tilde{n}\|_{L^2} \|\Delta \tilde{c}\|_{L^2} \\
\leq C\|\nabla \tilde{c}\|_{L^2} \|u_1\|_{L^\infty} + C\|\tilde{u}\|_{L^2} \|\nabla c_2\|_{L^\infty} + C\|\tilde{c}\|_{L^2} \|n_1\|_{L^\infty} + C\|\tilde{n}\|_{L^2} \|\nabla \tilde{c}\|_{L^2} + \|\Delta \tilde{c}\|_{L^2}.
\]
Multiplying $\tilde{u}$ to both sides of the third equation of (19) and integrating over $\mathbb{R}^d$, we have
\[ \frac{1}{2} \frac{d}{dt} \|\tilde{u}\|_{L^2}^2 + \|\nabla \tilde{u}\|_{L^2}^2 \leq \|\tilde{u}\|_{L^2} \|u_2\|_{L^2} \|\nabla \tilde{u}\|_{L^2} + C \|\tilde{n}\|_{L^2} \|\tilde{u}\|_{L^2} \]
\[ \leq C \|\tilde{u}\|_{L^2}^2 \|u_2\|_{L^\infty} + C \|\tilde{n}\|_{L^2}^2 + C \|\tilde{u}\|_{L^2}^2 + \epsilon \|\nabla \tilde{u}\|_{L^2}^2. \]
Summing the above estimates, we obtain
\[ \frac{1}{2} \frac{d}{dt} \|\tilde{n}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2 + \|\nabla \tilde{c}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 \]
\[ \leq C(\|\nabla c_1\| + \|\nabla c_2\|) \|\tilde{n}\|_{L^2}^2 + \|\tilde{n}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + 1 \]
\[ \times (\|\tilde{n}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2 + \|\nabla \tilde{c}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2). \]
Since all $L^\infty$ norms of $(n_1, c_1, u_1)$ are controlled by $H^{m-1} \times H^m \times H^m$ norm of $(n_1, c_1, u_1)$ with $m \geq 3$, and the initial data of $(\tilde{c}, \tilde{n}, \tilde{u})$ are all zero, $(\tilde{c}, \tilde{n}, \tilde{u})$ are all zero for $T > 0$. That implies the uniqueness of the local classical solution.

• (Nonnegativity) For completeness, we briefly show that $n^j$ and $c^j$ are nonnegative for all $j$. To use induction, we assume $c^j$ and $n^j$ are nonnegative. If we apply the maximum principle to the equation of $c^{j+1}$ in (12), we find that $c^{j+1}$ is nonnegative $(k(c^j)n^j)$ is nonnegative. Let us decompose $n^{j+1} = n^{j+1}_+ - n^{j+1}_-$, where
\[ n^{j+1}_+ = \begin{cases} n^{j+1} & n^{j+1} \geq 0 \\ 0 & n^{j+1} < 0 \end{cases}, \quad n^{j+1}_- = \begin{cases} -n^{j+1} & n^{j+1} \leq 0 \\ 0 & n^{j+1} > 0 \end{cases}. \]
Recall that the weak derivative of $n^{j+1}_-$ is $-\nabla n^{j+1}$ if $n^{j+1}_- < 0$, otherwise zero. It holds that
\[ \int_0^t \int_{\mathbb{R}^3} \partial_t n^{j+1}_-(n^{j+1}_- - 0) \cdot dx \, ds = \frac{1}{2} \| \nabla(n^{j+1}_- - 0) \|_{L^2} - \| \nabla(n^{j+1}_- - 0) \|_{L^2} \]
since $n^{j+1}_-, \partial_t n^{j+1}_+ \in L^2(0, T; L^2(\mathbb{R}^2))$ (see e.g. [18]). Now multiplying the negative part $(n^{j+1}_-) -$ on both sides of the first equation of (12) and integrating over $[0, t] \times \mathbb{R}^2$, we have
\[ \frac{1}{2} \int_0^t \| \nabla(n^{j+1}_-) \|_{L^2}^2 + \| \nabla(n^{j+1}_-) \|_{L^2}^2 \, ds \]
\[ \leq C \int_0^t \| \nabla(n^{j+1}_-) \|_{L^2}^2 \| \nabla c^j \|_{L^\infty}^2 + \frac{1}{2} \| \nabla(n^{j+1}_-) \|_{L^2}^2 \, ds. \]
Using Gronwall’s inequality, we have
\[ \| \nabla(n^{j+1}_-) - (0) \|_{L^2}^2 \leq \| \nabla(n^{j+1}_-) - (0) \|_{L^2}^2 \exp \left( C \int_0^t \| \nabla c^j \|_{L^\infty}^2 \, ds \right). \]
Since the initial data $n^{j+1}_0$ is nonnegative, we conclude that $n^{j+1}$ is nonnegative. This completes the proof.

2.2. Blow-up criterion. Next, we observe a blow-up criterion for the fluid chemotaxis equations. In the following, we use a notation $X^q_{a}L^r_0$, for simplicity, which denotes $L^r(0, T; X)$.

**Proposition 1.** (A Blow-up criterion) Suppose that $\chi$, $k$, $\phi$ and the initial data $(n_0, c_0, u_0)$ satisfy all the assumptions presented in Theorem 1.1. If $T < \infty$ is the maximal time of existence, then
\[ \int_0^T \left( \| \nabla u(t) \|_{L^\infty(\mathbb{R}^d)} + \| \nabla c(t) \|_{L^\infty(\mathbb{R}^d)}^2 \right) \, dt = \infty. \]
Proof. At first, we consider the $L^2$ estimate of $n$. Multiplying $n$ to both sides of the equation of $n$ and integrating, we have

$$\frac{1}{2} \frac{d}{dt} \|n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 \leq C \|\chi(c)n\nabla c\|_{L^2} \|\nabla n\|_{L^2}.$$ 

Since $\chi$ is continuous and $c$ is uniformly bounded until the maximal time of existence, we have

$$C \|\chi(c)n\nabla c\|_{L^2} \|\nabla n\|_{L^2} \leq \frac{1}{4} \|\nabla n\|_{L^2}^2 + C \|\nabla c\|_{L^\infty}^2 \|n\|_{L^2}^2.$$ 

For the estimates of $c$, we use the calculus inequality

$$\|\nabla (u \cdot \nabla c) - (u \cdot \nabla) \nabla c\|_{L^2} \leq C \|\nabla u\|_{L^\infty} \|\nabla c\|_{L^2}.$$ 

Multiplying $-\Delta c$ to both sides of the equation of $c$ and integrating, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla c\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 \leq C \|\Delta u\|_{L^\infty} \|\nabla c\|_{L^2}^2 + C \|\chi(c)n\nabla c\|_{L^2}^2 + C \|\nabla (k(c)n)\|_{L^2}^2 + \frac{1}{4} \|\Delta c\|_{L^2}^2.$$ 

For the equations of $u$, multiplying $-\Delta u$ to both sides of the equations and integrating by parts, we have

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \leq C \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2}^2 + C \|n\|_{L^2} \|\Delta u\|_{L^2}.$$ 

Collecting all the estimates, we obtain

$$\frac{d}{dt} (\|n\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) + (\|\nabla n\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 + \|\Delta u\|_{L^2}^2) + C (\|\nabla c\|_{L^\infty} \|\nabla u\|_{L^\infty}) (\|n\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 + \|\nabla u\|_{L^2}^2).$$ 

From Gronwall’s inequality, we have

$$\sup \left(\|n\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) + \int_0^T (\|\nabla n\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 + \|\Delta u\|_{L^2}^2) \, dt \leq C \left(\|n_0\|_{L^2}^2 + \|\nabla c_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2\right) \exp \left(\int_0^T \|\nabla u\|_{L^\infty} + \|\nabla c\|_{L^\infty} \, dt\right).$$ 

Note that $\|n\|_{L^\infty(0,T;L^2)}$ and $\|\nabla n\|_{L^2(0,T;L^2)}$ are uniformly bounded if $\int_0^T \|\nabla u\|_{L^\infty} + \|\nabla c\|_{L^\infty} \, dt$ is bounded. Moreover, we see that $n \in L^q T \cap L^\infty T$ and $\nabla n^{q/2} \in L^2 T^2$ for all $2 < q < \infty$. Indeed,

$$\frac{d}{dt} \|n\|_{L^q}^q + \|\nabla n\|_{L^2}^2 \leq C \int \left|n \nabla c n^{-1}\right| \, dx \leq C \left\|\nabla c\|_{L^\infty} \|n\|_{L^q} \|\nabla n\|_{L^2} + \frac{1}{2} \left\|\nabla n\|_{L^2}^2 \right\|_{L^2}.$$ 

From the above inequality, we have $\|n(t)\|_{L^q} \leq C$, where $C$ is independent of $q$. Letting $q \to \infty$, we have $n \in L^\infty T.$ 

Next, we consider the estimate in the space $(n, c) \in H^1 \times H^2$. We have

$$\frac{1}{2} \frac{d}{dt} \|\nabla n\|_{L^2}^2 + \|\Delta n\|_{L^2}^2 \leq C \|\nabla u\|_{L^\infty} \|\nabla n\|_{L^2}^2 + C \|\nabla n\|_{L^2} \|\nabla c\|_{L^\infty} \|\nabla^2 n\|_{L^2}^2 + C \|n\|_{L^\infty} \|\Delta c\|_{L^2} \|\nabla^2 n\|_{L^2} + C \|n\|_{L^\infty} \|\nabla c\|_{L^\infty} \|\nabla c\|_{L^2} \|\nabla^2 n\|_{L^2}.$$ 

From Young’s inequality and Gronwall’s inequality, we have

$$\sup \|\nabla n\|_{L^2}^2 + \int_0^T \|\nabla^2 n\|_{L^2}^2 \, dt \leq \left(\|\nabla n_0\|_{L^2}^2 + C \|n\|_{L^\infty(0,T;L^2)} \left(\int_0^T \|\Delta c\|_{L^2}^2 + \|\nabla c\|_{L^\infty(0,T;L^2)} \int_0^T \|\nabla c\|_{L^\infty}^2 \right) \right).$$
\[
\times \exp \left( \int_0^T \|\nabla u\|_{L^\infty} + \|\nabla c\|_{L^\infty}^2 \, dt \right).
\]

Hence, \( n \in H^1_T L^\infty \cap H^2_T L^2 \). For the \( H^2 \) estimate of \( c \), we have
\[
\frac{1}{2} \frac{d}{dt} \|\Delta c\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 \leq C \|\nabla u\|_{L^\infty} \|\Delta c\|_{L^2}^2 + C \|\Delta u\|_{L^2} \|\nabla u\|_{L^\infty} \|\nabla c\|_{L^2}^2 + C \|\nabla c\|_{L^2} \|n\|_{L^\infty} \|\nabla \Delta c\|_{L^2}.
\]
By Gronwall’s inequality, we have \( c \in H^2_T L^\infty \cap H^3_T L^3 \). Similarly, \( u \in H^2_T L^\infty \cap H^3_T L^3 \).

Then, we consider the estimate in the space \((n, c, u) \in H^2 \times H^3 \times H^3\). Proceeding similarly to the above, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|n\|_{H^2}^2 + \|\nabla n\|_{H^2}^2 \leq C \|u\|_{L^\infty} \|n\|_{H^2} \|\nabla n\|_{H^2} + C \|\nabla u\|_{L^\infty} \|n\|_{H^1} \|\nabla n\|_{H^2} + \frac{1}{4} \|\nabla n\|_{H^2}^2 + C \|\chi(c)\|_{H^2} \|\nabla c\|_{H^2}^2.
\]
In the above, the last term can be controlled by
\[
\|\chi(c)\|_{H^2} \|\nabla c\|_{H^2} \leq C \|n\|_{H^2} \|\chi(c)\|_{H^2} \|\nabla c\|_{H^2},
\]
and
\[
\|\nabla^2 (\chi(c)\nabla c)\|_{L^2} \leq C \|\nabla^2 c\|_{L^2} + C \|\nabla^2 c\|_{L^\infty} + C \|\nabla^3 c\|_{L^\infty},
\]
We already obtained \( c \in H^2_T L^\infty \cap H^3_T L^3 \). Hence, if we use Young’s inequality and Gronwall’s inequality, we have
\[
\sup_n \|n\|_{H^2}^2 + \int_0^T \|\nabla n\|_{H^2}^2 \, dt \leq \|n_0\|_{H^2}^2 \exp \left( C + \int_0^T \|\nabla u\|_{L^\infty} + \|\nabla c\|_{L^\infty}^2 \, dt \right).
\]
In the above, we can control the term \( \|\Delta(k(c)n)\|_{H^2} \) by \( C \|\Delta(k(c)n)\|_{H^2} \) if we control \( \|\nabla c\|_{L^\infty} \). For the estimate of \( u \), we have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{H^3}^2 + \|\nabla^2 u\|_{H^3}^2 \leq C \|\nabla u\|_{L^\infty} \|\nabla c\|_{H^3} \|\nabla u\|_{H^3} + C \|\nabla u\|_{H^3} \|\nabla c\|_{L^\infty} \|\nabla c\|_{H^3} + C \|\chi(c)\|_{H^3} \|\nabla c\|_{H^3}.
\]
Using Gronwall’s inequality, we have \((c, u) \in (H^2_T L^\infty \cap H^3_T L^3) \times (H^3_T L^\infty \cap H^4_T L^4)\).

Let us consider \( H^{m-1} \times H^m \times H^m \) estimates. The case \( m = 2, 3 \) and 4 are proved in the above, hence we consider the \( m \geq 5 \) case. Taking \( \partial^n \) \((n_0) \leq m - 1 \) and multiplying \( \partial^n n \) to both sides of the equation \( n \) and integrating and summing, we have
\[
\frac{1}{2} \frac{d}{dt} \|n\|_{H^{m-1}}^2 + \|\nabla n\|_{H^{m-1}}^2 \leq C \|\nabla u\|_{L^\infty} \|n\|_{H^{m-1}} \|\nabla n\|_{H^{m-1}} + C \|\chi(c)\|_{H^{m-1}} \|\nabla c\|_{H^{m-1}}.
\]
We already obtained the estimate for the case \( m = 4 \), thus \( \|\nabla c\|_{L^\infty(0,T;L^\infty)} \) is bounded. Also we have
\[
\|\chi(c)\|_{H^{m-1}} \leq C \|\chi(c)\|_{H^{m-1}} \|n\|_{H^{m-1}},
\]
and
\[
\|\chi(c)\|_{H^{m-1}} \leq C \|\chi(c)\|_{H^m} + K_{m-1},
\]
where \( K_{m-1} = C||c||_{H^{m-1}} \|\nabla c\|_{L^\infty} + C||c||_{H^m-1} \|\nabla c\|_{L^\infty} + \ldots \), which is known to be bounded in the previous step.

For the \( H^m \) estimate of \( c \), we proceed similarly to have
\[
\frac{1}{2} \frac{d}{dt} \|c\|_{H^m}^2 + \|\nabla c\|_{H^m}^2 \leq C\|\nabla u\|_{L^\infty} \|c\|_{H^m} \|\nabla c\|_{H^m} + C\|u\|_{H^m} \|\nabla c\|_{L^\infty} \|\nabla c\|_{H^m} \\
+ C\|((k(c)n))\|_{H^{m-1}}^2 + \frac{1}{4} \|\nabla c\|_{H^m}^2.
\]

As is shown for the term \( \|\chi(c)n\nabla c\|_{H^{m-1}} \), we control the term \( \|((k(c)n))\|_{H^{m-1}}^2 \) by \( C\|n\|_{H^{m-1}}^2 \). For the estimate of \( u \), we have
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^m}^2 + \|\nabla u\|_{H^m}^2 \leq C\|\nabla u\|_{L^\infty} \|u\|_{H^m} \|\nabla u\|_{H^m} + \frac{1}{4} \|\nabla u\|_{H^m}^2 + C\|n\|_{H^{m-1}}^2.
\]

Thus, by collecting all the above estimates and using Gronwall’s inequality, we have \((n, c) \in (H^{m-1}_2 \cap H^m_2 L^4_t) \times (H^m_2 L^\infty_t \cap H^{m+1}_2 L^4_t) \times (H^m_2 L^\infty_t \cap H^{m+1}_2 L^4_t) \). This completes the proof. \( \square \)

We are ready to present the proof of Theorem 1.2.

**Proof of Theorem 1.2.** In the proof of Proposition 3, we notice that \( \|\nabla c\|_{L^\infty} \) is solely responsible for \( n \in L^2_t L^\infty \) and \( \nabla n \in L^2_t L^2 \). Indeed,
\[
\frac{d}{dt} \|n\|_{L^2_t}^2 + \|\nabla n\|_{L^2_t}^2 \leq C \int_{\mathbb{R}^d} |n\nabla c\nabla n| \, dx \leq C \|\nabla c\|_{L^\infty}^2 \|n\|_{L^2_t}^2 + \frac{1}{2} \|\nabla n\|_{L^2_t}^2 .
\]

This implies \( u \in L^2_t L^\infty \) and \( \nabla u \in L^2_t L^2 \) by
\[
\frac{d}{dt} \|u\|_{L^2_t}^2 + \|\nabla u\|_{L^2_t}^2 \leq C\|n\|_{L^2} \|u\|_{L^2} .
\]

Moreover, we have \( n \in L^2_t L^\infty_t \) and \( \nabla n^{\alpha/2} \in L^2_t L^2 \) for all \( 2 < \alpha < \infty \);
\[
\frac{d}{dt} \|n\|_{L^q_t}^q + \|\nabla n^{\alpha/2}\|_{L^2_t}^2 \leq C \int_{\mathbb{R}^d} |n\nabla c\nabla n| \, dx \leq C \|\nabla c\|_{L^\infty} \|n\|_{L^q_t} \|\nabla n^{\alpha/2}\|_{L^2_t}^2 .
\]

Next, we see that \( \nabla c \in L^2_t L^\infty_t \) and \( \nabla^2 c \in L^2_t L^2 \). Indeed,
\[
\frac{d}{dt} \|\nabla c\|_{L^2_t}^2 + \|\nabla^2 c\|_{L^2_t}^2 \leq C \|\nabla c\|_{L^\infty} \|u\|_{L^2} \|\nabla^2 c\|_{L^2_t} + C \|n\|_{L^2} \|\nabla^2 c\|_{L^2_t} .
\]

We first consider the two-dimensional case.

- **(2D case)** For convenience, we denote vorticity as \( \omega := \nabla \times u; \) that is, \( \omega = \partial_1 u_2 - \partial_2 u_1 \) in two dimensions. Next, we consider the vorticity equation
\[
\omega_t - \Delta \omega + u \nabla \omega = -\nabla^\perp n \nabla \phi,
\]
where \( \nabla^\perp n = (-\partial_2 n, \partial_1 n) \). We note that \( \omega \in L^2_t L^\infty \) and \( \nabla \omega \in L^2_t L^2 \), since
\[
\frac{d}{dt} \|\omega\|_{L^2_t}^2 + \|\nabla \omega\|_{L^2_t}^2 \leq C \|\nabla n\|_{L^2} \|\omega\|_{L^2} .
\]

Furthermore, we observe that \( \nabla \omega \in L^2_t L^\infty \) and \( \nabla^2 \omega \in L^2_t L^2 \). Indeed, testing by \( -\Delta \omega \), we get
\[
\frac{d}{dt} \|\nabla \omega\|_{L^2_t}^2 + \|\nabla^2 \omega\|_{L^2_t}^2 \leq \|u\|_{L^4} \|\nabla \omega\|_{L^4} \|\Delta \omega\|_{L^2} + \|\nabla n\|_{L^2} \|\Delta \omega\|_{L^2} \\
\leq C \|u\|_{L^2} \|\nabla \omega\|_{L^2} + \|\nabla^2 \omega\|_{L^2} + \|\nabla n\|_{L^2} \|\Delta \omega\|_{L^2} .
\]
Therefore, via embedding, we have
\[
\int_0^T \|\nabla u\|_{L^\infty} \, dt \leq \int_0^T \|\nabla u\|_{H^2} \, dt \leq C \int_0^T \|\omega\|_{H^2} \, dt < \infty.
\]
This completes the proof of the 2D case.

\textbf{• (3D case)} We will show this case by contradictory arguments. We suppose that the condition \((9)\) is not true. We first recall the vorticity equation
\[
\omega_t - \Delta \omega + u \nabla \omega = \omega \nabla u - \nabla \times n \nabla \phi.
\]
Under the condition \((9)\) we have \(\omega \in L^2_t L^\infty_x\) and \(\nabla \omega \in L^2_t L^2_x\) as follows. We denote \(Q^* = \mathbb{R}^3 \times (T^* - \delta, t)\) for \(T^* - \delta < t < T^*\). For any given \(p, q\) satisfying \(3/p + 2/q = 1\), \(3 < p \leq \infty\), we choose \(l, m\) such that \(1/p + 1/l = 1/2\) and \(1/q + 1/m = 1/2\). We then remind that, due to the Gagliardo-Nirenberg's inequality,
\[
\|u\|_{L^{l,m}_{t,x}} \leq C \|u\|_{L^2_{t,x}}^{\frac{1}{2}} \|\nabla u\|_{L^{3,1}_{t,x}}^{\frac{1}{2}}, \quad 2 \leq l \leq 6, \quad \frac{3}{l} + \frac{2}{m} = \frac{3}{2},
\]
where \(\theta = (6 - l)/2l\) and \(1 - \theta = (3l - 6)/2l\). Then we have
\[
\frac{d}{dt} \|\omega\|_{L^2_t}^2 + \|\nabla \omega\|_{L^2_{t,x}}^2 \leq \|u\|_{L^p_{t,x}} \|\omega\|_{L^l_{t,x}} \|\nabla \omega\|_{L^2_{t,x}} + C \|n\|_{L^2_{t,x}} \|\nabla \omega\|_{L^2_{t,x}}.
\]
Next, integrating in time over \((T^* - \delta, t)\),
\[
\|\omega(t)\|_{L^2_t}^2 + \int_{T^* - \delta}^t \|\nabla \omega\|_{L^2_{t,x}}^2 \, ds \\
\leq \|\omega(T^* - \delta)\|_{L^2_t}^2 + C \|u\|_{L^{p,q}_{t,x}} \|\omega\|_{L^{l,m}_{t,x}} \|\nabla \omega\|_{L^2_{t,x}} + C \|n\|_{L^2_{t,x}} \|\nabla \omega\|_{L^2_{t,x}}.
\]
Note that \(\theta > 0\). By Young's inequality, we have
\[
\|\omega(t)\|_{L^2_t}^2 + \int_{T^* - \delta}^t \|\nabla \omega\|_{L^2_{t,x}}^2 \, ds \\
\leq \|\omega(T^* - \delta)\|_{L^2_t}^2 + C \|u\|_{L^{p,q}_{t,x}} \|\omega\|_{L^{l,m}_{t,x}} \|\nabla \omega\|_{L^2_{t,x}} + C \|n\|_{L^2_{t,x}} \|\nabla \omega\|_{L^2_{t,x}} + \frac{1}{2} \|\nabla \omega\|_{L^2_{t,x}}^2.
\]
Since \(\|u\|_{L^{p,q}_{t,x}}\) can be sufficiently small in \((T^* - \delta, T^*) \times \mathbb{R}^3\) by decreasing \(\delta\), we have
\[
\|\omega(t)\|_{L^2_t}^2 + \frac{1}{2} \int_{T^* - \delta}^t \|\nabla \omega\|_{L^2_{t,x}}^2 \leq \|\omega(T^* - \delta)\|_{L^2_t}^2 + C \|n\|_{L^2_{t,x}}^2, \quad (24)
\]
which is bounded by \((20)\). Since \(t\) is arbitrary for all \(t < T^*\), this estimate is uniform.

Next, we observe that \(\nabla^2 c \in L^2_t L^\infty_x\) and \(\nabla^3 c \in L^2_t L^2_x\). Indeed, we estimate
\[
\frac{d}{dt} \|\nabla^2 c\|_{L^2_t}^2 + \|\nabla^3 c\|_{L^2_t}^2 \\
\leq C(\|\nabla n\|_{L^2_t} + \|\nabla c\|_{L^\infty_t} \|\nabla u\|_{L^2_t} + \|\nabla^2 c\|_{L^2_t} \|\nabla^3 c\|_{L^2_t}) \\
\leq C(\|\nabla n\|_{L^2_t} + \|\nabla c\|_{L^\infty_t} \|\nabla u\|_{L^2_t} + \|\nabla^2 c\|_{L^2_t} \|\nabla^3 c\|_{L^2_t}),
\]
and use \((20), (21)\). Similarly, we show that \(n \in L^\infty_t H^1_x \cap L^2_t H^2_x\) by estimating
\[
\frac{d}{dt} \|\nabla n\|_{L^2_t}^2 + \|\nabla^2 n\|_{L^2_t}^2 \leq \|u\|_{L^6_t} \|\nabla n\|_{L^3_t} \|\nabla^2 n\|_{L^2_t} + \|\nabla c\|_{L^\infty_t} \|\nabla n\|_{L^2_t} \|\Delta n\|_{L^2_t} \\
+ \|u\|_{L^6_t} \|\nabla^2 c\|_{L^3_t} \|\nabla^2 n\|_{L^2_t} + \|\nabla c\|_{L^\infty_t} \|n\|_{L^6_t} \|\nabla c\|_{L^3_t} \|\Delta n\|_{L^2_t}.
\]
Finally, we show that $\omega \in H^1 L^1_T \cap H^2 L^2_T$. Testing $-\Delta \omega$ to the equations, we have
\[
\frac{d}{dt} \|\nabla \omega\|_{L^2}^2 + \|\nabla^2 \omega\|_{L^2}^2 \leq \|u\|_{L^p} \|\nabla \omega\|_{L^2} \big||\nabla^2 \omega\|_{L^2} + C \|\nabla n\|_{L^2} \|\nabla^2 \omega\|_{L^2},
\]
where $3 < p \leq \infty$ and $1/p + 1/l = 1/2$. Note that, via the Gagliardo-Nirenberg’s inequality,
\[
\|\nabla u\|_{L^4} \|\omega\|_{L^4} \|\nabla^2 \omega\|_{L^2} \leq C \|\omega\|_{L^4}^2 \big||\nabla^2 \omega\|_{L^2} \leq C \|\omega\|_{L^2}^2 \big||\nabla^2 \omega\|_{L^2}^7.
\]
We treat the term $\|u\|_{L^p} \|\nabla \omega\|_{L^4} \|\nabla^2 \omega\|_{L^2}$ similarly to $\|u\|_{L^p} \|\omega\|_{L^4} \|\nabla \omega\|_{L^2}$ in the estimation of (24). Therefore, since $\nabla^2 \omega \in L^2 L^2_T$, we have
\[
\int_0^T \|\nabla u\|_{L^\infty} dt \leq \int_0^T \|\nabla u\|_{H^2} dt \leq C \int_0^T \|\omega\|_{H^2} dt < \infty.
\]
This completes the proof. 

3. Global solutions in two dimensions. In this section, we provide the proof of global existence of smooth solutions in time with large initial data in two dimensions. For the proof of Theorem 1.3, we show some a priori estimates, which are uniform until the maximal time of existence. Moreover, such estimates imply that the blow-up condition quantity in Theorem 1.2 is uniformly bounded up to the maximal time of existence. Therefore, the maximal time cannot be finite. Now we present the proof of Theorem 1.3.

Proof of Theorem 1.3. We first present the following estimates for the solutions to the two-dimensional chemotaxis system coupled with the Navier-Stokes equations.
\[
n(1 + |x| + |\ln n|) \in L^\infty(0, T; L^1(\mathbb{R}^2)), \quad \nabla \sqrt{n} \in L^2(0, T; L^2(\mathbb{R}^2)), \tag{25}
\]
\[
c \in L^\infty(0, T; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)), \quad \nabla c \in L^2(0, T; L^2(\mathbb{R}^2)), \tag{26}
\]
\[
u \in L^\infty(0, T; L^2(\mathbb{R}^2)), \quad \nabla \nu \in L^2(0, T; L^2(\mathbb{R}^2)). \tag{27}
\]
We have the mass conservation for $n(t, x)$ as
\[
\int_{\mathbb{R}^2} n(t, x) dx = \int_{\mathbb{R}^2} n_0(x) dx. \tag{28}
\]
Multiplying $c^{q-1}(t, x)$ to both sides of the second equation of (1) and integrating over $\mathbb{R}^2$, we have
\[
\frac{1}{q} \frac{d}{dt} \|c\|_{L^q}^q + \frac{4(q - 1)}{q^2} \|\nabla c\|_{L^2}^2 + \int_{\mathbb{R}^2} k(c) n \epsilon^{q-1} dx = 0. \tag{29}
\]
Hence, we have $c \in L^\infty(0, T; L^q)$ for any $1 < q \leq \infty$ and $\nabla c^2 \in L^2(0, T; L^2)$ for any $1 < q < \infty$. Considering the equation of $n \ln n$, we have
\[
\partial_t (n \ln n) = -((u \cdot \nabla)n)(\ln n + 1) + \Delta (n \ln n) - \frac{\nabla n^2}{n} - \nabla \cdot (\chi(c)n \nabla c)(\ln n + 1). \tag{30}
\]
Taking the integration over $\mathbb{R}^2$, we have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} n \ln n dx + 4 \int_{\mathbb{R}^2} \nabla \sqrt{n}^2 dx + \int_{\mathbb{R}^2} \chi'(c) |\nabla c|^2 n dx = -\int_{\mathbb{R}^2} \chi(c) \Delta c n dx. \tag{31}
\]
Multiplying $-\Delta c$ to both sides of (1) and integrating over $\mathbb{R}^2$, we obtain
\[
\frac{d}{dt} \|\nabla c\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 = \int_{\mathbb{R}^2} k(c) \Delta c n dx + \sum_{j, k} \int_{\mathbb{R}^2} c \partial_k u_j \partial_j \partial_k c dx
\]
Multiplying $\mu$ to both sides of (32) and then adding (31), we have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} n \ln n + \mu |\nabla c|^2 + \int_{\mathbb{R}^2} 4|\nabla \sqrt{n}|^2 + \mu |\Delta c|^2 \, dx
\leq c \|\Delta c\|_{L^2}^2 \sqrt{n}_t^2 + C_2 |c|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 + \frac{1}{4} \mu \|\Delta c\|_{L^2}^2
\]
where we used the condition (A). Here we choose $\epsilon$ to be so small that $\epsilon C_3 < 2$ and, for convenience, set $\lambda_2 := C_2 |c_0|_{L^\infty}^2$. On the other hand, multiplying $u$ to both sides of the third equations of (1) and integrating over $\mathbb{R}^2$, we have
\[
\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 L^2 + \|\nabla u\|^2_{L^2} = - \int_{\mathbb{R}^2} n \nabla u dx.
\]  
Multiplying $\phi$ to both sides of the first equation of (1) and integrating over $\mathbb{R}^2$, we have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} n \phi dx = - \int_{\mathbb{R}^2} u \cdot \nabla n \phi dx - \int_{\mathbb{R}^2} n \cdot \nabla \phi dx + \int_{\mathbb{R}^2} \phi (c)n \nabla c \cdot \nabla \phi dx
\leq - \int_{\mathbb{R}^2} u \cdot \nabla n \phi dx + C_4 \|\nabla \sqrt{n}\|_{L^2} \|\nabla \sqrt{n}\|_{L^2} + C_5 \|n\|_{L^1} \|\nabla c\|_{L^2}.
\]  
Summing (34) and (35), we have
\[
\frac{d}{dt} \left( \frac{1}{2} \|u\|_{L^2}^2 + \int_{\mathbb{R}^2} n \phi dx \right) + \|\nabla u\|_{L^2}^2
\leq C_5 \|\nabla \sqrt{n}\|_{L^2} \|\nabla \sqrt{n}\|_{L^2} + C_6 \|\nabla \sqrt{n}\|_{L^2} \|\nabla \sqrt{n}\|_{L^2} \|\nabla c\|_{L^2}.
\]  
Multiplying $\lambda_1$ to both sides of (36) and adding (33), we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^2} n \ln n + \mu |\nabla c|^2 + \frac{\lambda_1}{2} |u|^2 + \lambda_1 n \phi dx + \int_{\mathbb{R}^2} 2|\nabla \sqrt{n}|^2 \frac{\mu}{2} |\Delta c|^2 + \frac{\lambda_1}{2} |\nabla u|^2 dx
\leq \lambda_1 C_5 \|n_0\|_{L^1} \|\nabla \sqrt{n}\|_{L^2} + \lambda_1 C_6 \|n_0\|_{L^1} \|\nabla \sqrt{n}\|_{L^2} \|\nabla c\|_{L^2}
\leq \frac{\lambda_1^2 C_6^2}{2} \|n_0\|_{L^1} + \frac{\lambda_1^2 C_6^2}{2} \|n_0\|_{L^1} \|\nabla c\|_{L^2}^2 + \|\nabla \sqrt{n}\|_{L^2}.
\]  
Using Gronwall's inequality, we have
\[
\sup_{0 \leq t \leq T} \left( \int_{\mathbb{R}^2} n \ln n + \mu |\nabla c|^2 + \frac{\lambda_1}{2} |u|^2 + \lambda_1 n \phi dx \right)
+ \int_0^T \int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \frac{\mu}{2} |\Delta c|^2 + \frac{\lambda_1}{2} |\nabla u|^2 dx \, dt \leq C(T).
\]  
Next, we show that $n \ln n \in L^\infty (0, T; L^2(\mathbb{R}^2))$, following a typical argument for dealing with kinetic entropy (see e.g. [5]). We first note that
\[
\int_{\mathbb{R}^2} n (\ln n)^- \leq C + C \int_{\mathbb{R}^2} n (x),
\]
where \((\ln n)_-\) is a negative part of \(\ln x\) and \(\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}\). Indeed, setting \(D_1 = \{ x : n(x) \leq e^{-|x|} \}\) and \(D_2 = \{ x : e^{-|x|} < n(x) \leq 1 \}\), we have

\[
\int_{\mathbb{R}^2} n(\ln n)_- = - \int_{D_1} n \ln n - \int_{D_2} n \ln n \\
\leq C \int_{D_1} \sqrt{n} + \int_{D_2} n \langle x \rangle \leq C \int_{\mathbb{R}^2} e^{-|x|} + \int_{\mathbb{R}^2} n \langle x \rangle.
\]

(39)

This deduces the estimate (38). Next, integrating (37) in time \(t\), we get

\[
\int_{\mathbb{R}^2} n(\cdot, t) \ln n(\cdot, t) + \mu \|\nabla c(t)\|_{L^2}^2 + \frac{\lambda_1}{2} \|u(t)\|_{L^2}^2 + \lambda_1 \|n(t)\|_{L^1} \\
+ \int_0^t \int_{\mathbb{R}^2} 2 \left( |\nabla \sqrt{n}|^2 + \frac{\mu}{2} |\Delta c|^2 + \frac{\lambda_1}{2} |\nabla u|^2 \right) dx dt \\
\leq C_7 + C_8 t + C_0 \int_0^t \|\nabla c\|_{L^2}^2 dx dt,
\]

(40)

where \(C_7 = \int_{\mathbb{R}^2} n_0 \ln n_0 + \mu \|\nabla c_0\|_{L^2}^2 + \frac{\lambda_1}{2} \||u_0\|_{L^2}^2 + \lambda_1 \|n_0\|_{L^1}\). Remembering (38), we compute

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \langle x \rangle n dx = \int_{\mathbb{R}^2} n u \nabla \langle x \rangle dx + \int_{\mathbb{R}^2} n \Delta \langle x \rangle dx + \int_{\mathbb{R}^2} \chi(c) n \nabla c \nabla \langle x \rangle dx.
\]

(41)

The term \(\int_{\mathbb{R}^2} n u \nabla \langle x \rangle dx\) is bounded as follows:

\[
\left| \int_{\mathbb{R}^2} n u \nabla \langle x \rangle dx \right| \leq \|\nabla \sqrt{n}\|_{L^2}^2 \|u\|_{L^2} \leq \frac{1}{2} \|\nabla \sqrt{n}\|_{L^2}^2 + C \|n\|_{L^1} \|u\|_{L^2}^2.
\]

Noting that \(|\nabla \langle x \rangle| + |\Delta \langle x \rangle| \leq C\), we get

\[
\left| \int_{\mathbb{R}^2} n \Delta \langle x \rangle dx + \int_{\mathbb{R}^2} \chi(c) n \nabla c \nabla \langle x \rangle dx \right| \leq C + C \|\nabla \sqrt{n}\|_{L^2} \|\nabla c\|_{L^2},
\]

where we used that \(\|n\|_{L^2} \leq C \|n_0\|_{L^1}^\frac{1}{2} \|\nabla \sqrt{n}\|_{L^2}^\frac{1}{2}\). In summary, we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \langle x \rangle n dx \leq \delta \|\nabla \sqrt{n}\|_{L^2}^2 + C \|u\|_{L^2}^2 + C_\delta \|\nabla c\|_{L^2}^2 + C,
\]

(42)

where \(\delta\) is sufficiently small, which will be specified later. Therefore, integrating (42) in time,

\[
\int_{\mathbb{R}^2} \langle x \rangle n(\cdot, t) dx \leq \int_{\mathbb{R}^2} \langle x \rangle n_0 dx + \delta \int_0^t \|\nabla \sqrt{n}\|_{L^2}^2 \\
+ C \int_0^t \|u\|_{L^2}^2 + C_\delta \int_0^t \|\nabla c\|_{L^2}^2 + C t.
\]

(43)

Now adding \(2 \int n(\ln n)_-\) to both sides of (40), we obtain

\[
\int_{\mathbb{R}^2} n(\cdot, t) \ln n(\cdot, t) + \mu \|\nabla c(t)\|_{L^2}^2 + \frac{\lambda_1}{2} \|u(t)\|_{L^2}^2 + \lambda_1 \|n(t)\|_{L^1} \\
+ \int_0^t \int_{\mathbb{R}^2} 2 \left( |\nabla \sqrt{n}|^2 + \mu |\Delta c|^2 + \lambda_1 |\nabla u|^2 \right) dx dt \\
\leq C + C t + C \int_0^t \|\nabla c\|_{L^2}^2 dx dt + C \int_0^t \|u\|_{L^2}^2 dx dt,
\]

(44)

where \(\delta\) in (42) is so small that term \(\int_0^t \|\nabla \sqrt{n}\|_{L^2}^2\) is absorbed to the left hand side of (40). Since (44) holds for all \(t\) until the maximal time of existence, due to
Gronwall’s inequality, we obtain \( n \in L^\infty(0, T; L^2(\mathbb{R}^2)) \). Moreover, again via the inequality (44), we deduce (25)-(27).

We note that from the blow-up criterion in two dimensions in Theorem 1.2, it suffices to show that \( \nabla \psi \in L^2(0, T; L^2(\mathbb{R}^2)) \) for global existence of smooth solutions in \( \mathbb{R}^2 \). We first consider the vorticity equation of velocity fields. Taking curl, we have

\[
\partial_t \omega + (u \cdot \nabla) \omega - \Delta \omega = -\nabla^\perp n \cdot \nabla \phi,
\]

where \( \nabla^\perp = (-\partial_2, \partial_1) \). If we multiply \( \omega \) to both sides of the above equation and integrate over \( \mathbb{R}^2 \), then we have

\[
\frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 = \int_{\mathbb{R}^2} n \nabla \phi \nabla^\perp \omega dx \leq C \|n\|_{L^4} \|\nabla^\perp \omega\|_{L^2}.
\]

Hence, we have

\[
\|\omega\|_{L^\infty(0, T; L^2)}^2 + \|\nabla \omega\|_{L^4(0, T; L^2)}^2 \leq C \|n\|_{L^2(0, T; L^2)}^2.
\]

Since \( \|n\|_{L^2} \leq \|\sqrt{\nabla^\perp n}\|_{L^4} \leq C \|\sqrt{\nabla^\perp n}\|_{L^2} \), we have \( \omega \in L^\infty(0, T; L^4) \cap L^2(0, T; H^1) \). Next we consider the equation of \( n \). Multiplying \( n \) and integrating over \( \mathbb{R}^2 \), we have

\[
\frac{1}{2} \frac{d}{dt} \|n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 = \int_{\mathbb{R}^2} \chi(c)n \nabla c \nabla n dx
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}^2} \nabla \cdot (\chi(c) \nabla c) n^2 dx \leq C \int_{\mathbb{R}^2} |\nabla^2 c| n^2 dx + C \int_{\mathbb{R}^2} |\nabla^3 c| n^2 dx
\]

\[
\leq C(\|\nabla^2 c\|_{L^2} + \|\nabla^2 c\|_{L^4}) \|n\|_{L^4} \leq C \|\nabla^2 c\|_{L^2} \|n\|_{L^2} \|\nabla n\|_{L^2},
\]

where we used that \( \chi \) is \( C^1 \) and \( c \in L^\infty(0, \infty; L^\infty) \), i.e., \( \chi(c) \) and \( \chi'(c) \) are bounded. Due to Young’s inequality, we have

\[
\frac{d}{dt} \|n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 \leq C \|n\|_{L^2}^2 \|\nabla^2 c\|_{L^2}^2.
\]

Therefore, via Gronwall’s inequality, we have \( n \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \). Multiplying \( \Delta^2 c \) to both sides of the equation of \( c \) and integrating over \( \mathbb{R}^2 \), we have

\[
\frac{1}{2} \frac{d}{dt} \|\Delta c\|_{L^2}^2 + \|\nabla \Delta c\|_{L^2}^2 \leq \|\nabla u\|_{L^4} \|\nabla c\|_{L^4} \|\nabla \Delta c\|_{L^2}
\]

\[
+ \|u\|_{L^\infty} \|\nabla^2 c\|_{L^2} \|\nabla \Delta c\|_{L^2} - \int_{\mathbb{R}^2} k(c) n \Delta^2 c dx.
\]

We note that the last term above is controlled as follows:

\[
\left| \int_{\mathbb{R}^2} k(c) n \Delta^2 c dx \right| \leq \left| \int_{\mathbb{R}^2} k'(c) \nabla c \cdot (\nabla \Delta c) dx \right| + \left| \int_{\mathbb{R}^2} k(c) \nabla n \cdot (\nabla \Delta c) dx \right|
\]

\[
\leq C \|\nabla \Delta c\|_{L^2} \|n\|_{L^2} \|\nabla c\|_{L^4} + C \|\nabla n\|_{L^2} \|\nabla \Delta c\|_{L^2}
\]

\[
\leq \epsilon \|\nabla \Delta c\|_{L^2}^2 + C \|n\|_{L^2}^4 \|\nabla c\|_{L^4}^2 + C \|\nabla n\|_{L^2}^2.
\]

Hence, we have

\[
\frac{1}{2} \frac{d}{dt} \|\Delta c\|_{L^2}^2 + \|\nabla \Delta c\|_{L^2}^2 \leq C \|\nabla u\|_{L^4} \|\nabla c\|_{L^4}^2
\]

\[
+ C \|u\|_{L^\infty} \|\Delta c\|_{L^2}^2 + C \|n\|_{L^2}^4 \|\nabla c\|_{L^4}^2 + C \|\nabla n\|_{L^2}^2.
\]

\[
\leq C \|\omega\|_{L^2} \|\nabla \omega\|_{L^2} \|\nabla c\|_{L^2} \|\nabla^2 c\|_{L^2} + C \|\nabla \omega\|_{L^2} \|\Delta c\|_{L^2}^2
\]

\[
+ C \|n\|_{L^2} \|\nabla n\|_{L^2} \|\nabla c\|_{L^2} \|\nabla^2 c\|_{L^2} + C \|\nabla n\|_{L^2}^2.
\]
Gronwall’s inequality gives \( c \in L^\infty(0, T; H^2_0) \cap L^2(0, T; H^3) \), which implies via embedding that \( \nabla c \in L^2(0, T; L^\infty) \). This completes the proof.

4. **Global weak solution in three dimensions.** In this section we will show the global existence of the weak solutions for (1) in three dimensions. We start with the notion of a weak solution for the system (1).

**Definition 4.1.** Let \( 0 < T \leq \infty \). A triple \((n, c, u)\) is called a weak solution to the Cauchy problem (1) in \( \mathbb{R}^3 \times [0, T) \) if the following conditions are satisfied:

(a) The functions \( n \) and \( c \) are non-negative and \((n, c, u)\) satisfy
\[
n(1 + |x| + |\ln n|) \in L^\infty(0, T; L^1(\mathbb{R}^3)), \quad \nabla \sqrt{n} \in L^2(0, T; L^2(\mathbb{R}^3)),
\]
\[
c \in L^\infty(0, T; L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)), \quad \nabla c \in L^2(0, T; L^2(\mathbb{R}^3)),
\]
\[u \in L^\infty(0, T; L^2(\mathbb{R}^3)), \quad \nabla u \in L^2(0, T; L^2(\mathbb{R}^3)).\]

(b) The functions \( n, c, u \) solve the chemotaxis-fluid equations (1) in the sense of distributions, namely for any \( \Psi \in C^1([0, T]; (C^\infty_c(\mathbb{R}^3))^3) \) with \( \nabla \cdot \Psi = 0 \)
\[
\int_{\mathbb{R}^3} (u \cdot \Psi)(T) + \int_0^T \int_{\mathbb{R}^3} u \cdot (\partial_t \Psi + \nabla \Psi) + \int_0^T \int_{\mathbb{R}^3} u \otimes u \cdot \nabla \Psi
\]
\[= \int_0^\infty \int_{\mathbb{R}^3} n \nabla \phi \cdot \Psi + \int_{\mathbb{R}^3} u_0 \cdot \Psi(0, x) = 0,
\]
where \( u \otimes u : \nabla \Psi = \sum_{j,k=1}^{3} w_j u_j \partial_j \Psi^k \) and
\[
\int_0^\infty \int_{\mathbb{R}^3} n(\partial_t \varphi + \Delta \varphi) + \int_0^\infty \int_{\mathbb{R}^3} n u \cdot \nabla \varphi + \int_0^\infty \int_{\mathbb{R}^3} \chi(c) n \nabla c \cdot \nabla \varphi + \int_{\mathbb{R}^3} n_0(0, x) \varphi(0, x) = 0,
\]
\[
\int_0^\infty \int_{\mathbb{R}^3} c(\partial_t \varphi + \Delta \varphi) + \int_0^\infty \int_{\mathbb{R}^3} c u \cdot \nabla \varphi - \int_0^\infty \int_{\mathbb{R}^3} k(c) n \varphi + \int_{\mathbb{R}^3} c_0(0, x) \varphi(0, x) = 0
\]
for any \( \varphi \in C^1([0, T]; (C^\infty_c(\mathbb{R}^3))) \) with \( \varphi(\cdot, T) = 0 \).

We assume for a moment that the solution \((n, c, u)\) of (1) is sufficiently regular and proceed to compute a priori estimate of an energy inequality under the Assumption (AA) and (B). The purely formal computation in this section can be justified applied to a regularized solution \((n^{k,\epsilon}, c^{k,\epsilon}, u^{k,\epsilon})\) of (58), (59).

We note first, by maximum principle, that
\[n(t, x) \geq 0, \quad c(t, x) \geq 0, \quad \|c(t)\|_{L^p} \leq \|c_0\|_{L^p} \quad \text{for} \quad t \geq 0, \quad 1 \leq p \leq \infty.
\]
It is straightforward that \( \|n(t)\|_{L^1} = \|n_0\|_{L^1} \) for \( t \geq 0 \) and
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^3} \frac{|u|^2}{2} + \int_{\mathbb{R}^3} n \phi \right) + \int_{\mathbb{R}^3} |\nabla u|^2 = \int_{\mathbb{R}^3} n \Delta \phi + \int_{\mathbb{R}^3} \chi(c) n \nabla c \nabla \phi, \quad (45)
\]
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \ln n dx + \int_{\mathbb{R}^3} \frac{|\nabla n|^2}{n} dx + \int_{\mathbb{R}^3} \chi'(c)|\nabla c|^2 n dx = -\int_{\mathbb{R}^3} \chi(c) \Delta n dx, \quad (46)
\]
\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla c|^2 dx + \int_{\mathbb{R}^3} |\Delta c|^2 dx = \int_{\mathbb{R}^3} k(c) \Delta n dx + \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} c \partial_i \partial_j c \partial_i u_j dx. \quad (47)
\]
Multiplying \( \mu \) to the last equation (47) and adding it to the second equation (46), we have
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^3} \ln n + \mu |\nabla c|^2 + \int_{\mathbb{R}^3} |\nabla \sqrt{n}|^2 + \mu \int_{\mathbb{R}^3} |\nabla c|^2 + \int \chi'(c)|\nabla c|^2 n
\]
\[ \leq - \int_{\mathbb{R}^3} (\chi(c) - \mu k(c)) \Delta c dx + \mu \|c_0\|_{L^\infty} \|\nabla c\|_{L^2} \Delta c_{\|c\|_{L^2}} \]

\[ \leq C_1^4 \|\nabla u\|_{L^2}^2 + \frac{\mu}{4} \|\Delta c\|_{L^2}^2 \]  
(48)

for some \(C_1\), which can be taken bigger than 1, i.e. \(C_1 > 1\). Also it holds that

\[ \frac{d}{dt} \int_{\mathbb{R}^3} (x) dx = \int_{\mathbb{R}^3} n u \nabla(x) dx + \int_{\mathbb{R}^3} n \Delta(x) dx + \int_{\mathbb{R}^3} \chi(c) n \nabla c \nabla(x) dx. \]  
(49)

Since the term \(\int_{\mathbb{R}^3} n u \nabla(x) dx\) is bounded as follows:

\[ \|n\|_{L^\frac{3}{2}} \|u\|_{L^6} \leq C \|n\|_{L^1} \|\nabla \sqrt{n}\|_{L^2} \|\nabla u\|_{L^2} \leq \frac{1}{2} \|\nabla \sqrt{n}\|_{L^2}^2 + \frac{1}{2} \|\nabla u\|_{L^2}^2 + C(\|n_0\|_{L^1}), \]
we can have

\[ \frac{d}{dt} \int_{\mathbb{R}^3} (x) dx \leq \frac{1}{2} \|\nabla \sqrt{n}\|_{L^2}^2 + \frac{1}{2} \|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^3} \chi(c) n \nabla c \nabla(x) dx + C. \]  
(50)

We estimate the term \(\int_{\mathbb{R}^3} \chi(c) n \nabla c \nabla(x) dx\) similarly as above.

\[ \int_{\mathbb{R}^3} \chi(c) n \nabla c \nabla(x) dx \leq C ||\nabla \sqrt{n}\|_{L^2} ||\nabla c\|_{L^3} \leq C ||\nabla \sqrt{n}\|_{L^2} ||\nabla \sqrt{n}\|_{L^2} ||\nabla c\|_{L^2} \Delta c_{\|c\|_{L^2}} \]

\[ \leq C ||\nabla c\|_{L^2} ||\Delta c\|_{L^2} + \frac{1}{4} ||\nabla \sqrt{n}\|_{L^2}^2 \leq C ||\nabla c\|_{L^2}^2 + \frac{1}{4} ||\Delta c\|_{L^2}^2 + \frac{1}{4} ||\nabla \sqrt{n}\|_{L^2}^2. \]  
(51)

Multiplying \(C_1\) to \((45)\) and adding it together with \((48)\) and \((50)\), we have

\[ \frac{d}{dt} \left( \int_{\mathbb{R}^3} C_1 \left( \frac{|u|^2}{2} + n \phi \right) + n \ln n + \frac{|\nabla c|^2}{2} + (x) dx \right) \]

\[ + \frac{C_1 - 1}{2} \|\nabla u\|_{L^2}^2 + \frac{1}{4} \|\nabla \sqrt{n}\|_{L^2}^2 + \frac{1}{4} \|\Delta c\|_{L^2}^2 \leq C (||\nabla c||_{L^2}^2 + ||u||_{L^2}^2) + C. \]  
(52)

Then, by Gronwall’s inequality, we have

\[ \int_{\mathbb{R}^3} \left( \frac{|u|^2}{2} + n \phi + n \ln n + \frac{|\nabla c|^2}{2} + (x) dx \right) \]

\[ + \int_0^T \|\nabla u\|_{L^2}^2 + \|\nabla \sqrt{n}\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 dt \leq C, \]  
(53)

where \(C(T, ||\chi(c)||_{L^\infty}, ||n_0||_{L^1}, ||n||_{L^1}, ||\Delta \phi||_{L^\infty}, ||\sqrt{\phi}||_{L^\infty})\). By same reasoning for treating \(n(\ln n)\) term in \((39)\), it follows that

\[ \int_{\mathbb{R}^3} \left( |u|^2 + n \phi + n |\ln n| + |\nabla c|^2 + (x) dx \right) \]

\[ + \int_0^T \|\nabla u\|_{L^2}^2 + \|\nabla \sqrt{n}\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 dt \leq C. \]  
(54)

Streamline of constructing global weak solutions, as in usual steps for the Navier-Stokes equations, is the following:

- regularizing the system for which we prove the existence of smooth solutions
- finding uniform estimates for the solutions of the regularized system
- passing to the limit on the regularized parameters.
4.1. **Regularization.** In this subsection, we intend to construct approximate solutions of the system.

On a general unbounded domain \( \Omega \) the global weak solutions for the incompressible Navier-Stokes equations are constructed using the spectral projections \( (P_k)_{k \in \mathbb{Z}} \), associated to the inhomogeneous Stokes operator ([1, Chapter 2]). Let us introduce some preliminaries needed for this section. We apply expositions in [1] in the case of \( \Omega = \mathbb{R}^3 \). See the end of Introduction for definitions of \( \mathcal{H}, \mathcal{V}, \mathcal{V}', \mathcal{V}_\sigma \).

**Definition 4.2.** (Definition 2.3, 2.4 in [1])

1. The operator \( B : \mathcal{H} \to \mathcal{V}_\sigma \) by
   \[
   Bf = u, \text{ where } u \text{ satisfies } u - \Delta u - f \in \mathcal{V}_\sigma.
   \]

2. The operator \( A : \text{Ran}(B) \to \mathcal{H} \) is defined by
   \[
   Au = f, \text{ where } f \text{ satisfies } Bf = u.
   \]

We have the following proposition applying to the Stokes operator \( A - \text{Id} \) the spectral theorem for unbounded self-adjoint operator.

**Proposition 2.** (Theorem 2.2, Corollary 2.2 in [1]) There exists a family of orthogonal projection on \( \mathcal{H}(\mathbb{R}^3) \), denoted by \( (P_k)_{k \in \mathbb{Z}} \), which commutes with \( A - \text{Id} \) and satisfied the following properties. For any \( u \in \mathcal{H}(\mathbb{R}^3) \),

\[
P_k P_k' u = P_{\text{min}(k,k')} u, \quad \lim_{k \to \infty} \|P_k u - u\|_{\mathcal{H}} = 0, \tag{55}
\]

\[
\|\nabla P_k u\|_{L^2} \leq \sqrt{k} \|u\|_{L^2}, \quad \|\Delta P_k u\|_{L^2} \leq k \|u\|_{L^2}, \tag{56}
\]

\[
\|(1 - P_k)u\|_{L^2} \leq \frac{1}{\sqrt{k}} \|u\|_{\mathcal{V}_\sigma}. \tag{57}
\]

In particular, (56) implies \( P_k u \in L^\infty(\mathbb{R}^3) \) for \( u \in L^2(\mathbb{R}^3) \) in three dimensions.

**Definition 4.3.** The bilinear map \( Q \) is defined by

\[
Q : \mathcal{V} \times \mathcal{V} \to \mathcal{V}', \quad (u, v) \mapsto -\text{div}(u \otimes v).
\]

From now on we denote by \( \mathcal{H}_k(\mathbb{R}^3) \) the space \( P_k \mathcal{H}(\mathbb{R}^3) \). We regularize (1) by a frequency cut-off operator \( P_k \) and a mollifier \( \sigma^\varepsilon \):

\[
\begin{aligned}
\partial_t n^{k,\varepsilon}(t) &= -u^{k,\varepsilon} \cdot \nabla n^{k,\varepsilon} + \Delta n^{k,\varepsilon} - \nabla \cdot (n^{k,\varepsilon} (\chi(\sigma^\varepsilon) \nabla c^{k,\varepsilon}) \ast \sigma^\varepsilon)), \\
\partial_t c^{k,\varepsilon}(t) &= -u^{k,\varepsilon} \cdot \nabla c^{k,\varepsilon} + \Delta c^{k,\varepsilon} - k(c^{k,\varepsilon}) (n^{k,\varepsilon} \ast \sigma^\varepsilon), \\
\partial_t u^{k,\varepsilon}(t) &= -P_k Q(u^{k,\varepsilon}, u^{k,\varepsilon}) + P_k \Delta u^{k,\varepsilon} - P_k (n^{k,\varepsilon} \nabla \phi),
\end{aligned}
\]

with initial data

\[
n_0^{k,\varepsilon}, c_0^{k,\varepsilon}, u_0^{k,\varepsilon} = (n_0 \ast \sigma^\varepsilon, c_0 \ast \sigma^\varepsilon, P_k u_0 \ast \sigma^\varepsilon),
\]

where \( n_0, c_0, u_0 \) is the initial data of (1) satisfying the condition (10) in Theorem 1.4. The mollifier is defined as usual such that \( \sigma^\varepsilon(x) = \varepsilon^{-3} \sigma(\varepsilon^{-1} x) \) for \( \sigma \in C_0^\infty(\mathbb{R}^3) \). Apart from the frequency cut-off the regularization is same one for a chemotaxis-fluid model studied in [13]. Repeating similar arguments in Theorem 1.1, we obtain the local solution of (1) in the class

\[
\begin{aligned}
n^{k,\varepsilon} &\in L^\infty(0,T; H^{m-1}((\mathbb{R}^3))) \cap L^2(0,T; H^m((\mathbb{R}^3))) \\
c^{k,\varepsilon} &\in L^\infty(0,T; H^{m-1}((\mathbb{R}^3))) \cap L^2(0,T; H^m((\mathbb{R}^3))) \\
u^{k,\varepsilon} &\in L^\infty(0,T; H^{m-1}((\mathbb{R}^3)) \cap \mathcal{H}_k(\mathbb{R}^3)) \cap L^2(0,T; H^m((\mathbb{R}^3)))
\end{aligned}
\]

\]

for some time $T$ and for all $m > 3$. It turns out that due to the regularization of nonlinear terms and smoothing properties of $P_k$ (see (56)), the local solution of (1) can be extended up to infinite time.

**Proposition 3.** The regularized system (58) has the unique global solution $(n^{k,\epsilon}, c^{k,\epsilon}, u^{k,\epsilon})$ in a class (59) for any time $T < \infty$.

Before presenting the proof we observe that the approximating solution $(n^{k,\epsilon}, c^{k,\epsilon}, u^{k,\epsilon})$ of (58) satisfies an energy inequality.

**Proposition 4.** The solution $(n^{k,\epsilon}, c^{k,\epsilon}, u^{k,\epsilon})$ of (58) satisfies the following inequality.

\[
\int_{\mathbb{R}^3} \left( \frac{|u^{k,\epsilon}|^2}{2} + n^{k,\epsilon} \phi + n^{k,\epsilon} \ln n^{k,\epsilon} + \frac{|\nabla c^{k,\epsilon}|^2}{2} \right) + \langle x \rangle \nabla c^{k,\epsilon} \, dx \\
+ \int_0^T \|\nabla u^{k,\epsilon}\|_{L^2}^2 + \|\nabla \sqrt{n^{k,\epsilon}}\|_{L^2}^2 + \|n^{k,\epsilon}\|_{L^2}^2 \, dt \leq C,
\]

where $C = C(T, \|\chi(c)\|_{L^\infty}, \|\langle x \rangle n_0\|_{L^1}, \|\nabla c_0\|_{L^2}, \|n_0\|_{L^1}, \|\Delta \phi\|_{L^\infty}, \|\nabla \phi\|_{L^\infty}, \|\phi\|_{L^\infty}).$

**Proof.** We note that the same cancellation as in (48) holds for the regularized system (58), hence $(n^{k,\epsilon}, c^{k,\epsilon}, u^{k,\epsilon})$ satisfying (59) satisfy the energy inequalities (45)-(49).

Moreover the following moment bound holds by similar estimates as (50), (51),

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \langle x \rangle n^{k,\epsilon} \, dx = \int_{\mathbb{R}^3} n^{k,\epsilon} u^{k,\epsilon} \nabla \langle x \rangle \, dx + \int_{\mathbb{R}^3} n^{k,\epsilon} \Delta \langle x \rangle \, dx \\
+ \int_{\mathbb{R}^3} n^{k,\epsilon} [\chi(c) \nabla c^{k,\epsilon}] \sigma^k \nabla \langle x \rangle \, dx \\
\leq C \|u^{k,\epsilon}\|_{L^2} \|n^{k,\epsilon}\|_{L^2} + C \|\nabla c^{k,\epsilon}\|_{L^2} \|\Delta c^{k,\epsilon}\|_{L^2} + \frac{1}{2} \|\nabla \sqrt{n^{k,\epsilon}}\|_{L^2}^2 + \|n_0\|_{L^1}.
\]

Then we have (60) with $T$ depending on $\|\nabla c_0\|_{L^2}, \|\langle x \rangle n_0\|_{L^1}, \|n_0\|_{L^1}.$

It is immediate to have

\[
\|\nabla c_0^{k,\epsilon}\|_{L^2} + \|\langle x \rangle n_0^{k,\epsilon}\|_{L^1} \leq \|\nabla c_0\|_{L^2} + \|\langle x \rangle n_0\|_{L^1}.
\]

Note that $x \ln x$ is convex and $d\mu = \sigma^\epsilon(y) \, dy$ provide a probability measure. Then by Jensen’s inequality, we have

\[
n_0^{k,\epsilon} (\ln n_0^{k,\epsilon})_+ \leq (n_0(\ln n_0)_+)^* \sigma^\epsilon.
\]

Integrating the above in $x$ and observing that $\lim_{\epsilon \to 0} \|n_0(\ln n_0)\|_{L^1} = \|n_0\|_{L^1}$, we have

\[
n_0^{k,\epsilon} (\ln n_0^{k,\epsilon})_+ \|L^1 \leq \|n_0\|_{L^1} \ln n_0\|_{L^1}.
\]

For the $\|n_0^{k,\epsilon} (\ln n_0^{k,\epsilon})_+ \|_{L^1}$, proceeding similarly as (39), we have

\[
\|n_0^{k,\epsilon} (\ln n_0^{k,\epsilon})_+ \|_{L^1} \leq C + \int_{\mathbb{R}^3} n^{k,\epsilon}(x) \, dx \leq C \left(1 + \int_{\mathbb{R}^3} n(x) \, dx\right),
\]

from which we deduce the proposition.

Now we give the proof of Proposition 3.
Proof of Proposition 3. We first observe that the regularity criterion in Theorem 1.2 hold true for the system (58). Since its verification is tedious repetition of that of Theorem 1.2, we omit its details. If we consider the second equation of (58), then we have the following energy estimates.

\[
\frac{1}{2} \frac{d}{dt} \|e^{k,\epsilon}\|_{L^2}^2 + \|\nabla e^{k,\epsilon}\|_{L^2}^2 \leq C \|\nabla u^{k,\epsilon}\|_{L^2}^2 \|e^{k,\epsilon}\|_{L^2}^2 + C \|\nabla e^{k,\epsilon}\|_{L^2}^2 \|n^{k,\epsilon}\|_{L^2}^2 \frac{1}{2} \|\nabla e^{k,\epsilon}\|_{L^2}^2.
\]

By using Gronwall’s inequality, we have \(\|\nabla e^{k,\epsilon}\|_{L^\infty L^2} < \infty\). Since \(\|u^{k,\epsilon}(t)\|_{L^2}\) is bounded and \(\|\nabla u^{k,\epsilon}\|_{L^2} \leq C \sqrt{t}\|u^{k,\epsilon}\|_{L^2}\), we can also demonstrate that the Serrin condition in Theorem 1.2 is satisfied for \(u^{k,\epsilon}\). This completes the proof.

4.2. Global weak solutions. In this subsection, we give the proof of Theorem 1.4.

Proof of Theorem 1.4. We consider an approximating sequence \((n^{l,\epsilon}_0, c^{l,\epsilon}_0, u^{l,\epsilon}_0)\) to \((n_0, c_0, u_0)\). Note that

\[
\int_{\mathbb{R}^3} \left| n^{l,\epsilon}_0 - n_0 \right| + |\nabla c^{l,\epsilon}_0 - \nabla c_0| dx + \int_{\mathbb{R}^3} |u^{l,\epsilon}_0 - u_0|^2 dx \to 0,
\]

and

\[
\int_{\mathbb{R}^3} \langle x \rangle n^{l,\epsilon}_0 dx + \int_{\mathbb{R}^3} n^{l,\epsilon}_0 \ln n^{l,\epsilon}_0 \leq C \int_{\mathbb{R}^3} \langle x \rangle n_0 dx + \int_{\mathbb{R}^3} n_0 \ln n_0 dx + C.
\]

We denote by \((n^{l,\epsilon}, c^{l,\epsilon}, u^{l,\epsilon})\) the approximating solution constructed in the previous section for the system (58) with initial data \((n^l(0,\cdot), c^l(0,\cdot)) = (n^l_0(0,\cdot), c^l_0(0,\cdot))\) and \(u^l(0,\cdot) = P u_0(\cdot)\). Several uniform estimates hold for the approximating solutions:

\[
\|e^{l,\epsilon}\|_{L^\infty(0,T;L^p(\mathbb{R}^3))} \leq C \text{ for } 1 \leq p \leq \infty, \tag{62}
\]

\[
\|e^{l,\epsilon}\|_{L^\infty(0,T;H^1(\mathbb{R}^3))} + \|\Delta e^{l,\epsilon}\|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq C, \tag{63}
\]

\[
\|\sqrt{n^{l,\epsilon}}\|_{L^\infty(0,T;L^p(\mathbb{R}^3))} + \|\nabla \sqrt{n^{l,\epsilon}}\|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq C, \tag{64}
\]

\[
\|u^{l,\epsilon}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \|\nabla u^{l,\epsilon}\|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq C. \tag{65}
\]

Then there exists subsequences \((n^{l,\epsilon}, c^{l,\epsilon}, u^{l,\epsilon})\) and some functions \(n, c, u\) such that

\[
\sqrt{n^{l,\epsilon}} \to \sqrt{n} \text{ weak*}, \ L^\infty(0,T;L^2(\mathbb{R}^3)) - \text{weak*}, \tag{66}
\]

\[
c^{l,\epsilon} \to c \text{ in } L^\infty(0,T;L^p(\mathbb{R}^3)) \cap L^\infty(0,T;H^1(\mathbb{R}^3)) - \text{weak*}, \tag{67}
\]

\[
u^{l,\epsilon} \to u \text{ in } L^\infty(0,T;L^2(\mathbb{R}^3)) - \text{weak* in } L^2(0,T;\mathcal{V}_p(\mathbb{R}^3)) - \text{weak}, \tag{68}
\]

for \(1 \leq p \leq \infty\). Let us show that \(n, c, u\) is a weak solution in the sense of Definition 4.1. By Gagliardo-Nirenberg inequality and (64), we have

\[
\int_{\mathbb{R}^3} |n^{l,\epsilon}|^d dx \leq C \|n_0\|_{L^1(\mathbb{R}^3)}^{\frac{2p}{3p-1}} \|\nabla \sqrt{n^{l,\epsilon}}\|_{L^2(\mathbb{R}^3)}^{3(p-1)},
\]

and therefore

\[
\|n^{l,\epsilon}\|_{L^q(0,T;L^p(\mathbb{R}^3))} < C(T), \quad 1 \leq q \leq \frac{2p}{3(p-1)}, \tag{66}
\]

COUPLED CHEMOTAXIS-FLUID EQUATIONS 2293
for $1 \leq p \leq 3$. Some strong convergences are necessary. We note that \((66)\) implies the source term of the Navier Stokes equation $n^{l,\varepsilon}\nabla \phi$ is in $L^2([0,T]; \mathcal{V}_r'(\mathbb{R}^3))$ uniformly with respect to $l$; for any $w \in L^2([0,T]; \mathcal{V}_r(\mathbb{R}^3))$, it holds that

$$\int_0^T \int_{\mathbb{R}^3} P_l(n^{l,\varepsilon}\nabla \phi) w \, dx \, dt \leq \|\nabla \phi\|_{L^\infty(\mathbb{R}^3)} \|n^{l,\varepsilon}\|_{L^2([0,T]; L^p_{\text{loc}}(\mathbb{R}^3))} \|w\|_{L^2(\mathbb{R}^3)}.$$  

It proves that $\partial_t u^{l,\varepsilon}$ is uniformly bounded in $L^2(0,T; \mathcal{V}_r'(\mathbb{R}^3))$. Note that $u_k$ is uniformly bounded in $L^\infty(0,T; \mathcal{H}(\mathbb{R}^3)) \cap L^2(0,T; \mathcal{V}_r(\mathbb{R}^3))$ due to \((60)\). Combining these facts and \((56)\), \((57)\) we have compactness result for $(u^{l,\varepsilon})$ (see [1, Proposition 2.7] for detailed proof): there exists $u$ in $L^2(0,T; \mathcal{V}_r(\mathbb{R}^3))$ such that up to subsequence

$$\lim_{l \to \infty} \int_0^T \int_{\mathbb{R}^3} |u^{l,\varepsilon}(t,x) - u(t,x)|^2 \, dx \, dt = 0, \quad (67)$$

for any $T > 0$ and compact subset $K$ of $\mathbb{R}^3$. In addition, for $\Psi \in L^2([0,T]; \mathcal{V}(\mathbb{R}^3))$ and $\Phi \in L^2([0,T] \times \mathbb{R}^3)$

$$\lim_{l \to \infty, \varepsilon \to 0} \int_0^T \int_{\mathbb{R}^3} \nabla u^{l,\varepsilon}(t,x) \nabla \Psi(t,x) \, dx \, dt = \int_0^T \int_{\mathbb{R}^3} \nabla u(t,x) \nabla \Psi(t,x) \, dx \, dt,$$

$$\lim_{l \to \infty, \varepsilon \to 0} \int_0^T \int_{\mathbb{R}^3} u^{l,\varepsilon}(t,x) \Phi(t,x) \, dx \, dt = \int_0^T \int_{\mathbb{R}^3} u(t,x) \Phi(t,x) \, dx \, dt. \quad (68)$$

Furthermore, for any $\psi \in C^1(\mathbb{R}^+; \mathcal{V}_r(\mathbb{R}^3))$

$$\lim_{l \to \infty, \varepsilon \to 0} \sup_{t \in [0,T]} \left| \int_{\mathbb{R}^3} (u^{l,\varepsilon}(t,x) - u(t,x)) \psi(t,x) \, dx \right| = 0. \quad (69)$$

Applying a test function $\Psi$ in $C^1([0,T]; \mathcal{V}_r(\mathbb{R}^3))$, we obtain

$$\frac{d}{dt} (u^{l,\varepsilon}(t), \Psi(t)) = \langle \Delta u^{l,\varepsilon}(t), P_l \Psi(t) \rangle + \langle Q(u^{l,\varepsilon}(t), u^{l,\varepsilon}(t)), P_l \Psi(t) \rangle + \langle (n^{l,\varepsilon}\nabla \phi), P_l \Psi(t) \rangle + \langle u^{l,\varepsilon}(t), \frac{d}{dt} \Psi(t) \rangle. \quad (70)$$

Following the arguments in [1], that is, using \((67)-(69)\) and the fact

$$\lim_{l \to \infty} \sup_{t \in [0,T]} \|P_l \Psi(t) - \Psi(t)\|_{\mathcal{V}(\mathbb{R}^3)} = 0, \quad (71)$$

we can pass to the limit with respect to $l$ so that

$$\int_{\mathbb{R}^3} u(t,x) \Psi(T,x) \, dx + \int_0^T \int_{\mathbb{R}^3} (\nabla u : \nabla \Psi - u \otimes u : \nabla \Psi - u \cdot \partial_t \Psi)(s,x) \, dx \, ds$$

$$= \int_{\mathbb{R}^3} u_0(x) \Psi(0,x) \, dx + \lim_{l \to \infty, \varepsilon \to 0} \int_0^T \langle n^{l,\varepsilon}\nabla \phi, \Psi(t) \rangle \, dt.$$ 

For the strong convergence of $(n^{l,\varepsilon})$ we have $\sqrt{n^{l,\varepsilon}} \to \sqrt{n}$ strongly in $L^2_{\text{loc}}(\mathbb{R}^3)$ for a.e. $t \in [0,T]$ by Sobolev embedding. Since $\|\sqrt{n^{l,\varepsilon}}(t)\|_{L^2(\mathbb{R}^3)}$ is continuous in time, we redefine $n(t)$ such that $\|\sqrt{n^{l,\varepsilon}} - \sqrt{n}\|_{L^2(\mathbb{R}^3)} \to 0$ for all $t \in [0,T]$. Then by \((66)\) and Lebesgue Dominated convergence theorem, it follows that

$$\|n^{l,\varepsilon} - n\|_{L^q(0,T; L^p_{\text{loc}}(\mathbb{R}^3))} \to 0, \quad 1 \leq q \leq \frac{2p}{3(p-1)}. \quad (72)$$
Thus we have
\[ \|c^{l,\varepsilon} - c\|_{L^p_{loc}((0,T) \times \mathbb{R}^3)} \to 0, \quad 1 \leq p < \infty \] (73)
by the uniform boundedness (62). Moreover we have
\[ \|\nabla c^{l,\varepsilon} - \nabla c\|_{L^2(0,T;L^p_{loc}(\mathbb{R}^3))} \to 0, \quad 1 \leq p < 6. \] (74)
By (63), \( \|\nabla c^{l,\varepsilon}\|_{L^2(0,T;H^1(\mathbb{R}^3))} \) is uniformly bounded. For any \( \nabla g \in L^3(0,T;L^2(\mathbb{R}^3)) \),
we have
\[ \int_0^T \int_{\mathbb{R}^3} \partial_t \nabla c^{l,\varepsilon} g dx dt \leq \int_0^T \int_{\mathbb{R}^3} u^{l,\varepsilon} \nabla c^{l,\varepsilon} \nabla g + \Delta c^{l,\varepsilon} \nabla g + k(c^{l,\varepsilon})(n^{l,\varepsilon} + \sigma_\varepsilon) \nabla g dx dt. \]
We estimate
\[
\begin{align*}
\int_0^T \int_{\mathbb{R}^3} u^{l,\varepsilon} \nabla c^{l,\varepsilon} \nabla g dx dt & \leq C \int_0^T \|u^{l,\varepsilon}\|_{L^3(\mathbb{R}^3)} \|\nabla c^{l,\varepsilon}\|_{L^2(\mathbb{R}^3)} \|\Delta c^{l,\varepsilon}\|_{L^2(\mathbb{R}^3)} \|\nabla g\|_{L^2(\mathbb{R}^3)} dt \\
& \leq C \int_0^T \|\nabla u^{l,\varepsilon}\|_{L^2(\mathbb{R}^3)} \|\Delta c^{l,\varepsilon}\|_{L^2(\mathbb{R}^3)} \|\nabla g\|_{L^2(\mathbb{R}^3)} dt \\
& \leq C \|\nabla u^{l,\varepsilon}\|_{L^2(0,T;L^2(\mathbb{R}^3))} \|\Delta c^{l,\varepsilon}\|_{L^2(0,T;L^2(\mathbb{R}^3))} \|\nabla g\|_{L^2(0,T;L^2(\mathbb{R}^3))},
\end{align*}
\]
\[ \int_0^T \int_{\mathbb{R}^3} k(c^{l,\varepsilon})(n^{l,\varepsilon} + \sigma_\varepsilon) \nabla g dx dt \leq C \|n^{l,\varepsilon}\|_{L^3(0,T;L^2(\mathbb{R}^3))} \|\nabla g\|_{L^2(0,T;L^2(\mathbb{R}^3))}. \]
Thus we have \( \partial_t c^{l,\varepsilon} \in L^2(0,T;H^{-1}(\mathbb{R}^3)) \). The strong convergences (72)-(74) are enough to pass to the limit for nonlinear terms in the chemotaxis part. For instance, testing a \( \Psi \in C^\infty_c(\mathbb{R}^3) \) to the worst nonlinear term \( \nabla \cdot (n^{l,\varepsilon}(c^{l,\varepsilon}) \nabla c^{l,\varepsilon} + \sigma_\varepsilon) \), we have
\[
\begin{align*}
\int_0^T \int_{\mathbb{R}^3} \nabla \cdot (n^{l,\varepsilon} & (\chi(c^{l,\varepsilon}) \nabla c^{l,\varepsilon} + \sigma_\varepsilon)) \Psi - \nabla \cdot (n\chi(c) \nabla c) \Psi dx dt \\
= & \int_0^T \int_{\mathbb{R}^3} (n^{l,\varepsilon} - n)(\chi(c^{l,\varepsilon}) \nabla c^{l,\varepsilon} + \sigma_\varepsilon) \nabla \Psi dx dt \\
+ & \int_0^T \int_{\mathbb{R}^3} n(\chi(c^{l,\varepsilon}) \nabla c^{l,\varepsilon} + \sigma_\varepsilon - \chi(c) \nabla c) \nabla \Psi dx dt.
\end{align*}
\]
The second integral is
\[
\begin{align*}
\int_0^T \int_{\mathbb{R}^3} [(n \nabla \Psi) & * \sigma_\varepsilon - n \Psi] \chi(c^{l,\varepsilon}) \nabla c^{l,\varepsilon} + n \nabla \Psi (\chi(c^{l,\varepsilon}) \nabla c^{l,\varepsilon} - \chi(c) \nabla c) dx dt \\
= & \int_0^T \int_{\mathbb{R}^3} [(n \nabla \Psi) & * \sigma_\varepsilon - n \Psi] \chi(c^{l,\varepsilon}) \nabla c^{l,\varepsilon} \\
+ & n \nabla \Psi ([\chi(c^{l,\varepsilon}) - \chi(c)](\nabla c^{l,\varepsilon} + \chi(c)(\nabla c^{l,\varepsilon} - \nabla c)) dx dt.
\end{align*}
\]
The integrals go to zero by the uniform estimates (62)-(65) and (72)-(74) with the Lipschitz continuous assumption on \( \chi(\cdot) \).
Lastly, we consider the approximated energy inequality (60) replacing \( n^{l,\epsilon} |\ln n^{l,\epsilon}| \) with \( n^{l,\epsilon} \ln n^{l,\epsilon} \). Taking the limit and using the convexity of \( x \ln x \) we deduce

\[
\int_{\mathbb{R}^3} \left( \frac{|u|^2}{2} + n\phi + n \ln n + \frac{|\nabla c|^2}{2} + (x)n \right) dx + \int_0^T \|\nabla u\|_{L^2}^2 + \|\nabla \sqrt{n}\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 dt \leq C,
\]

with

\[
C = C(T, ||\chi(c)||_{L^\infty}, ||(x)n_0||_{L^1}, \|\nabla c_0\|_{L^2}, ||n_0||_{L^1}, ||\Delta \phi||_{L^\infty}, \|\nabla \phi||_{L^\infty}, ||\phi||_{L^\infty}).
\]

By the same reasoning for treating \( n(\ln n) \) term in (39) we show the weak solutions \((n, c, u)\) satisfy the energy inequality (11). This completes the proof of Theorem 1.4.

Acknowledgments. M. Chae’s work was supported by the National Research Foundation of Korea(NRF-2011-0028951). K. Kang’s work was partially supported by KRF-2008-331-C00024 and NRF-2009-0088692. J. Lee’s work was partially supported by NRF-2011-0006697. We appreciate professor Dmitry Vorotnikov for valuable comments. We also appreciate anonymous referee for his/her careful reading and valuable suggestions.

REFERENCES

[1] J. Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, “Mathematical Geophysics,” Oxford university press, London, 2006.

[2] A. Chertock, K. Fellner, A. Kurganov, A. Lorz and P. A. Markowich, Sinking, merging and stationary plumes in a coupled chemotaxis-fluid model: A high-resolution numerical approach, to appear in J. Fluid Mech.

[3] L. Corrias, B. Perthame and H. Zaag, A chemotaxis model motivated by angiogenesis, C. R. Math. Acad. Sci. Paris, 336 (2003), 141–146.

[4] L. Corrias, B. Perthame and H. Zaag, Global solutions of some chemotaxis and angiogenesis systems in high space dimensions, Milan J. Math., 72 (2004), 1–28.

[5] R. DiPerna and P. L. Lions, On the Cauchy problem for Boltzmann equations: Global existence and weak stability, Ann. Math., 139 (1989), 321–366.

[6] R. Duan, A. Lorz and P. Markowich, Global solutions to the coupled chemotaxis-fluid equations, Comm. Partial Diff. Equations, 35 (2010), 1635–1673.

[7] M. D. Francesco, A. Lorz and P. Markowich, Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior, Discrete Contin. Dynam. Systems - A, 28 (2010), 1437–1453.

[8] Y. Giga and H. Sohr, Abstract \( L^p \) estimates for the Cauchy problem with applications fo Navier-Stokes equations in exterior domains, J. Funct. Analysis, 102 (1991), 72–94.

[9] M. A. Herrero and J. L. L. Velazquez, A blow-up mechanism for chemotaxis model, Ann. Sc. Norm. Super. Pisa, 24 (1997), 633–683.

[10] D. Horstman and G. Wang, Blow-up in a chemotaxis model without symmetry assumptions, European J. Appl. Math, 12 (2001), 159–177.

[11] E. F. Keller and L. A. Segel, Initiation of slide mold aggregation viewed as an instability, J. Theor. Biol., 26 (1970), 399–415.

[12] E. F. Keller and L. A. Segel, Model for chemotaxis, J. Theor. Biol., 30 (1971), 225–234.

[13] J.-G. Liu and A. Lorz, A coupled chemotaxis-fluid model, I, H. Poincaré, Analyse Non Linéaire, 28 (2011), 643–652.

[14] A. Lorz, Coupled chemotaxis fluid model, Math. Models and Meth. in Appl. Sci., 20 (2010), 987–1004.

[15] T. Nagai, T. Senba and K. Yoshida, Applications of the Trudinger-Moser inequality to a parabolic system of chemotaxis, Funkcial Ekvac., 40 (1997), 411–433.

[16] K. Osaki and A. Yagi, Finite dimensional attractors for one-dimensional Keller-Segel equations, Funkcial Ekvac., 44 (2001), 441–469.

[17] C. S. Patlak, Random walk with persistence and external bias, Bull. Math. Biol. Biophys., 15 (1953), 311–338.
[18] L. Tartar, “Topics in Nonlinear Analysis,” Publicatons mathématiques de l’Université de Paris-Sud(Orsay), Paris, 1978.

[19] Y. Tao, Boundedness in a chemotaxis model with oxygen consumption by bacteria, J. Math. Anal. Appl., 381 (2011), 521–529.

[20] Y. Tao and M. Winkler, Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion, Discrete Continuous Dynam. Systems - A, 32 (2012), 1901–1914.

[21] I. Tuval, L. Cisneros, C. Dombrowski, C. W. Wolgemuth, J. O. Kessler and R. E. Goldstein, Bacterial swimming and oxygen transport near contact lines, PNAS, 102 (2005), 2277–2282.

[22] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, J. Differential Equations, 248 (2010), 2889–2905.

[23] M. Winkler, Global large data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops, Comm. Partial Diff. Equations, 37 (2012), 319–351.

Received February 2012; revised July 2012.

E-mail address: mchae@hknu.ac.kr
E-mail address: kkang@yonsei.ac.kr
E-mail address: jihoonlee@skku.edu