Variable Parameter Analysis for Scheduling One Machine

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Abstract

In contrast to the fixed parameter analysis (FPA), in the variable parameter analysis (VPA) the value of the target problem parameter is not fixed, it rather depends on the structure of a given problem instance and tends to have a favorable asymptotic behavior when the size of the input increases. While applying the VPA to an intractable optimization problem with \( n \) objects, the exponential-time dependence in enumeration of the feasible solution set is attributed solely to the variable parameter \( \nu \), \( \nu \ll n \). As opposed to the FPA, the VPA does not imply any restriction on some problem parameters, it rather takes an advantage of a favorable nature of the problem, which permits to reduce the cost of enumeration of the solution space. Our main technical contribution is a variable parameter algorithm for a strongly NP-hard single-machine scheduling problem to minimize maximum job lateness. The target variable parameter \( \nu \) is the number of jobs with some specific characteristics, the “emerging” ones. The solution process is separated in two phases. At phase 1 a partial solution including \( n - \nu \) non-emerging jobs is constructed in a low degree polynomial time. At phase 2 less than \( \nu! \) permutations of the \( \nu \) emerging jobs are considered. Each of them are incorporated into the partial schedule of phase 1. Due to the results of an earlier conducted experimental study, \( \nu/n \) varied from \( 1/4 \) for small problem instances to \( 1/10 \) for the largest tested problem instances, so that that the ratio becomes closer to 0 for large \( n \).

Keywords: fixed and variable parameter analysis, implicit enumeration, algorithm, parameterized time complexity, single machine scheduling, time complexity

1 Introduction

Exact implicit enumeration algorithms for \( NP \)-hard combinatorial optimization problems with \( n \) objects run in time exponential in \( n \). It is assumed that all \( n \) objects contribute to the quality of the constructed solutions, and hence none of them can be omitted in the exponential-time enumeration process. This is true for a number of problems. For example, every object in the knapsack problem may potentially contribute in the total weight and the total profit of a given solution. Similarly, every object in the traveling salesman problem contributes in the cost of any feasible tour, every object in the subset sum problem may potentially contribute in the total sum in a formed solution. For not all optimization problems the objects have such homogeneous characteristics. A particular relationship between parameters of each object and the objective function determines different characteristics of each object. For example, in graph optimization problems, not every vertex has similar homogeneous properties; in particular, some of them may not be part of an optimal solution. For instance, in graph domination problem, which is not
fixed parameter tractable nor approximable, some graph vertices can be assumed to be part of an optimal dominating set, whereas some other vertices cannot be part of it (e.g., all support vertices can be assumed to be part of a minimum dominating set, whereas none of the single degree (non-isolated) vertices can be part of it; these are simple examples, a deeper study of graph structure can reveal a considerable number of vertices with similar properties). This reduces the initial set of vertices to which an exponential-time implicit enumeration process can be applied. Likewise, in many \(NP\)-hard scheduling problems, different jobs have different (non-homogeneous) properties so that some of them may be ignored in an exponential-time enumeration process (we illustrate this point here for one such scheduling problem).

The variable parameter (VP) analysis, that we propose in this paper, restricts an exponential-time enumeration process to a small subset of \(\nu\) objects, where \(\nu\) is a variable parameter. Parameterized complexity analysis, in general, has been successfully used to tackle \(NP\)-hard optimization problems. A fixed parameter algorithm for an input of size \(n\) and a target parameter \(k\) runs in time polynomial in \(n\), whereas the exponential dependence is due to the fixed parameter \(k\). A fixed parameter algorithm gives an advantage over a traditional one whenever parameter \(k\) can be (sufficiently) smaller than \(n\). Whenever this is not the case or/and there is no numerical parameter that can be fixed, one may look for a variable parameter algorithm (VP-algorithm). In contrast to the fixed parameter analysis, in the VP-analysis, the value of the target variable parameter \(\nu\) is not fixed, it rather depends on the structure of a given problem instance. While applying the VP-analysis to an intractable optimization problem with \(n\) objects, the exponential-time dependence in enumeration of the feasible solution set is attributed solely to the VP \(\nu\). Thus the efficiency of a VP-algorithm is determined by the asymptotic behavior of the VP \(\nu\).

The VP-analysis will obviously have an advantage over traditional enumeration algorithms if parameter \(\nu\) has a favorable asymptotic behavior, i.e., if it increases considerably slower than the length of the input (such a favorable asymptotic behavior of parameter \(\nu\) can be established experimentally or even theoretically). As opposed to the fixed parameter analysis, the VP-analysis does not impose any restriction on any problem parameter, it rather takes an advantage of favorable structural properties of the problem. A VP \(\nu\) is a function of different problem instance parameters, including its size and specific structural parameters. Since these parameters may vary from a problem to a problem, we may not explicitly express \(\nu\) as a function. In particular terms, we may let \(\nu\) to be the number of objects in a specially determined subset of the set of all objects of the problem, the so-called kernel. The process of the selection of a kernel, i.e., the kernelization, is an important part of a VP-algorithm (the kernelization has also been used in the traditional fixed parameter analysis). Given a VP \(\nu\), the time complexity of a VP-algorithm with this parameter \(\nu\) is \(f(\nu)(n - \nu)^\Theta(1)\), for some computable function \(f\). Observe that \(f\) is supposed to be an exponential function if the corresponding problem is \(NP\)-hard.

In this paper, we develop VP-algorithm for a scheduling problem, where \(n\) jobs with release times and due dates are to be performed by a single machine. The objective is to minimize the maximum job lateness. The problem is strongly \(NP\)-hard \([7]\). Our VP-algorithm essentially relies on an important feature of the problem (and also other scheduling problems), that a relatively small group of the so-called “emerging” jobs basically contribute to the complexity status of the problem. We give a method that partitions the whole set of jobs into four basic types (1)-(4), the jobs of each type possessing similar properties. Then we show how the exponential-time enumeration process can be restricted solely to the \(\nu\) emerging (type (1)) jobs. If it were possible to bound \(\nu\) for any given problem instance theoretically, a VP-algorithm parameterized with that \(\nu\), would run in polynomial time (note that it is unlikely that this is possible for an \(NP\)-hard
We were able to verify the values of parameter $\nu$ experimentally, for about thousand randomly generated instances with up to 1000 jobs \[1\]. Importantly, for a significant amount of these instances, independently of their size, $\nu$ turned out to be a relatively small integer number; in particular, there was observed no gradual increase of $\nu$ with the increase of the increase of the size of the instances. Among the remained “difficult” instances, for small $n$, $\nu$ was about $n/4$, whereas for larger sized instances, $\nu$ decreased gradually: for the largest instances with around 1000 jobs, the average was less than $n/10$. (We refer the reader to Table 1 in [1] for the detailed data.) Based on this experimental study, considering solely difficult problem instances, i.e., where the dependence of parameter $\nu$ on the length of the input was evident, we believe that, the general tendency is that the ratio $\nu/n$ gradually becomes closer to 0 when the size of the input increases (in fact, there are real-life problem instances, where the parameter $\nu$ is a priory small number, see Section 7).

Our VP-algorithm runs in time $O(\nu!\nu(n - \nu)^2 \log n)$ (although a worst-case factor $\nu!$ does not accurately reflect the real running time of the algorithm: large subsets of permutations of the critical jobs can be discarded during the enumeration process, see Sections 4 and 7). The algorithm consists of the pre-processing kernelization stage, and stages 1 and 2. At the kernelization stage, a given problem instance is partitioned into the “easy” and “difficult” sub-instances, containing non-emerging and emerging jobs, respectively. In this way, the parameter $\nu$ (the number of the emerging jobs) is also determined (note that it is not an explicit part of the input). At stage 1, a partial schedule for the easy sub-instance of the $n - \nu$ non-emerging jobs is constructed in a low degree polynomial time. Stage 2 incorporates an implicit enumeration procedure that deals with the difficult sub-instance. It enumerates some non-dominated permutations of the $\nu$ emerging jobs. Each permutation is incorporated into the partial schedule of stage 1 according to the order of the jobs in that permutation.

We present the necessary preliminary material in the next section. In Section 3 we describe the initial kernelization step and the polynomial-time procedure of phase 1. In Sections 4 and 5 we describe the enumeration procedure of phase 2 and the iterative kernelization step. In Section 6 we prove the correctness of the overall algorithm. In Section 7 we give some concluding observations.

1.1 Problem definition

The single-machine scheduling problem that we consider here is important from both, practical and theoretical standpoints. Besides the real-life applications, its study is essential for a better understanding of more complex optimization problems including multiprocessor and shop scheduling problems. Our single-machine scheduling problem can be formulated as follows. There are $n$ jobs to be scheduled on a single machine. Each job $j$ becomes available at its release time $r_j$ (only from time $r_j$ it can be assigned to the machine). The due date $d_j$ is the desirable time for the completion of job $j$. Job $j$ needs to be processed uninterruptedly on the machine during $p_j$ time units, which is its processing time. The machine can handle at most one job at a time. A feasible schedule $S$ is a mapping that assigns every job $j$ a starting time $t_j(S)$ on the machine so that above stated restrictions are satisfied. $c_j(S) = t_j(S) + p_j$ is the completion time of job $j$ on the machine. The penalty for the late completion of job $j$ is measured by its lateness $L_j = c_j(S) - d_j$. The objective is to find an optimal schedule, a feasible one with the minimum maximum job lateness $L_{\text{max}} = \max_j L_j$. 
This problem can alternatively be viewed by considering job deliveries instead of job due dates and changing the objective to the minimization of the maximum full job completion time, as described below.

In the alternative setting, we replace job due dates with job delivery times. The delivery time $q_j$ of job $j$ is the additional amount of time that is required for the full completion of job $j$ after this job is completely processed by the machine. In this way, $C_j(S) = c_j(S) + q_j$ is the full completion time of job $j$ in schedule $S$. The objective here is to find a feasible schedule that minimizes the maximum job full completion time or the makespan

$$|S| = \max_j C_j(S).$$

Job delivery times have an immediate practical sense: Every job $j$ needs to be delivered to the customer by an independent agent immediately after the machine finishes its processing (for example, the delivery of two different jobs can be accomplished in parallel by two independent agents, whereas the machine can process some other job during these deliveries).

To see the equivalence between the two settings, given an instance of the second version, take a suitably large number $K \geq \max_j q_j$ and define due date of job $j$ as $d_j = K - q_j$; this completely defines a corresponding instance in the first setting. Vice-versa, given an instance of the first setting, take a magnitude $D \geq \max_j d_j$ and define job delivery times as $q_j = D - d_j$ (see Bratley et al. \cite{2} for the detail). Note that jobs with larger job delivery times tend to have larger full completion times, hence the larger is job delivery time, the more urgent it is. Similarly, for the first setting, the smaller job due date is, the more urgent it is.

According to the conventional three-field notation for the scheduling problems introduced by Graham et al. \cite{8}, the above two settings are abbreviated as $1|r_j|L_{\text{max}}$ and $1|r_j,q_j|C_{\text{max}}$, respectively. In the first field the single-machine environment is indicated, the second field specifies distinguished job parameters, and in the third field the objective criteria is given (job processing times and job due dates are not explicitly specified as job processing times are always present and the $L_{\text{max}}$ criterion automatically yields jobs with due dates).

### 1.2 Some related work

We give a short overview of some related work in the scheduling area. (We refer the reader to \cite{4,5,6} for general guideline in parameterized analysis.) The first exact implicit enumeration branch-and-bound algorithm for the studied here single-machine scheduling problem was proposed in 70s by McMahon & Florian \cite{15}, and later another algorithm based on similar ideas was described by Carlier \cite{3}. There exist polynomial time approximation schemes for the problem \cite{11,14,23}. As to the polynomially solvable special cases, if all the delivery times (due dates) are equal, then the problem is easily solvable by a venerable greedy $O(n \log n)$ heuristic by Jackson \cite{13}. The heuristic can straightforwardly be adopted for the exact solution of the related version in which all job release times are equal. Jackson’s heuristic, iteratively, includes the next unscheduled job with the largest delivery time (or the smallest due date). An extension of this heuristic, described by Schrage \cite{18}), gives a 2-approximation solution for problem $1|r_j,q_j|C_{\text{max}}$ with job release times (and an exact solution for the preemptive case $1|r_j,q_j,\text{pmtn}|C_{\text{max}}$). Iteratively, at each scheduling time $t$ given by job release or completion time, among the jobs released by that time, the extended heuristic schedules a job with the largest delivery time. This extended heuristic, to which we shall refer to as LDT (Largest Delivery Time) heuristic, has the same time complexity as its
predecessor. As it is observed by Horn [12], it delivers an optimal solution in case all the jobs have unit processing time (given that the job parameters are integers), and it is easy to see that the adaptation of the heuristic for the preemptive version of the problem is also optimal. The (non-preemptive) problem becomes more complicated for equal (not necessarily unit) job processing times, intuitively, because during the execution of a job another more urgent job may now be released. Note that this cannot happen for the unit-time setting, as job release times are integers. The setting with equal (non-unit) length jobs can still be solved in polynomial $O(n^2 \log n)$ time. Garey et al. [9] used a union and find tree with path compression and have achieved to improve the time complexity to $O(n \log n)$ (not an easily accomplished achievement). In [20] an $O(n^2 \log n \log p)$ time algorithm for a more general setting $1|p_j \in \{p, 2p\}, r_j|L_{\text{max}}$ is described. Here a job processing time can be either $p$ or $2p$. It was recently shown that the problem remains polynomial for divisible job processing times [22], whereas it is strongly $NP$-hard if job processing times are from the set $\{p, 2p, 3p, \ldots\}$, for any integer $p$ [21].

The latter work [21] deals with a parametrised setting of problem $1|r_j|L_{\text{max}}$ where the fixed parameters are the maximum job completion time $p_{\text{max}}$ and the maximum job due date $d_{\text{max}}$. The condition when the problem is fixed parameter tractable for parameter $p_{\text{max}}$ is established. This condition implies that the problem is fixed parameter tractable for two parameters $p_{\text{max}}$ and $d_{\text{max}}$. For the setting $1|r_j, q_j|C_{\text{max}}$, better than 2-approximation polynomial-time algorithms exist. Potts [17] showed that by repeated application of LDT-heuristic $O(n)$ times, the performance ratio can be improved to 3/2, resulting in an $O(n^2 \log n)$ time performance. Nowicki and Smutnicki [16] have proposed another 3/2-approximation algorithm with time complexity $O(n \log n)$. Hall and Shmoys [11] illustrated that the application of the LDT-heuristic to the original and a specially-defined reversed problems leads to a further improved approximation of 4/3 in time $O(n^2 \log n)$.

2 Preliminaries

To build our VP-algorithm and determine the variable parameter $\nu$, a closer study of our single-machine scheduling problem is required. In this study, we use existing concepts and definitions for scheduling problems. The reader is referred to [19] and to a more recent reference [22] for detailed descriptions and illustrations. Here we give a brief description of the concepts that we use here. Our presentation, though technical and specific, will allow us to dive deeper into the structure of the problem and determine a desired partition of the set of jobs.

To start with, consider an LDT-schedule $S$ and a longest consecutive job sequence $K$ in it, i.e., a sequence of the successively scheduled jobs without idle-time intervals in between them, such that: (i) for the last job $o$ of that sequence,

$$C_o(S) = \max_i \{C_i(S)\},$$

and (ii) no job from the sequence has the delivery time less than $q_o$. We will refer to $K$ as a kernel in schedule $S$, and to job $o = o(K)$ as the corresponding overflow job. Abusing the terminology, we will refer to a kernel interchangeably as a sequence and as the corresponding job set, and will denote the first kernel in schedule $S$ by $K(S)$. 
The next observations easily follow: (i) the number of kernels and the overflow jobs in schedule $S$ is the same; (ii) no gap (an idle-time interval) within a kernel $K \in S$ exists; (iii) the overflow job $o(K)$ is either succeeded by a gap or it is succeeded by job $j$ with $C_j(S) < C_o(S)$ (hence, $q_j < q_o$).

Suppose job $i$ precedes job $j$ in LDT-schedule $S$. We will say that job $i$ pushes job $j$ in schedule $S$ if LDT-heuristic will reschedule job $j$ earlier if $i$ is forced to be scheduled behind $j$.

A block in an LDT-schedule $S$ is its consecutive part consisting of the successively scheduled jobs without any gap between them preceded and succeeded by a (possibly a 0-length) gap (in this sense, intuitively, a block is an independent part in a schedule). Note that every kernel $K$ in schedule $S$ is contained within the same block $B \in S$. In general, a block may contain one or more kernels.

Let us consider kernel $K \in B$, $B \in S$, and suppose that the first job of that kernel is pushed by job $l \in B$ ($l \notin K$). Then $q_l < q_o$ must hold since otherwise job $l$ would form part of kernel $K$. We shall refer to job $l$ as the delaying emerging job for kernel $K$, and to any job $e \in B$ with $q_e < q_o$ as an emerging job in schedule $S$.

Given LDT-schedule $S$ with the delaying emerging job $l$ for kernel $K$, let

$$
\delta(K) = c_l(S) - \min_{i \in K} \{ r_i \}
$$

be the delay of kernel $K$ in that schedule; i.e., $\delta(K)$ is the forced right-shift imposed by the delaying job $l$ for the jobs of kernel $K$.

The next property implicitly define a lower bound $LB$ on the optimal schedule makespan $OPT$:

**Property 1** $\delta(K) < p_l$, hence $|S| - OPT < p_l$.

### 2.1 Conflicts in LDT-schedules

LDT-heuristic can be used to generate different feasible schedules. Initially, we generate LDT-schedule $\sigma$ by applying LDT-heuristic to the original problem instance. By modifying the originally given problem instance and applying the heuristic repeatedly, alternative LDT-schedules can be obtained, as we will see just a bit later.

During the construction of an LDT-schedule $S$, we iteratively update the current scheduling time $t$ as either the completion time of the job scheduled the last so far or/and the release time of the earliest released yet unscheduled job, whichever magnitude is larger. We will use $S^t$ for the (partial) LDT-schedule constructed by time $t$, and $j_t$ the job that is scheduled at time $t$.

Scheduling time $t$ is said to be a conflict scheduling time in schedule $S^t$ if within the execution interval of job $j_t$ (including its right endpoint) another job $j$ with $q_j > q_{j_t}$ is released; i.e., job $j_t$ is pushing a more urgent job $j$. Then jobs $j$ and $j_t$ are said to conflict between each other.
Lemma 1 If during the construction of LDT-schedule $\sigma$ no conflict scheduling time occurs, then it is optimal. In particular, at any scheduling time $t$, no job released within the execution interval of job $j_t$ can initiate a kernel in schedule $\sigma$ unless it conflicts with job $j_t$.

Proof. From the condition, there may exist no kernel in schedule $\sigma$ possessing the delaying emerging job. A schedule with this property is known to be optimal (e.g., see [21]).

Thus from now, we assume that the above lemma is not satisfied, i.e., during the construction of schedule $\sigma$ there arises a kernel with the delaying emerging job.

2.2 Creation of alternative LDT-schedules

Let us consider an LDT-schedule $S$ with kernel $K$ and the corresponding delaying emerging job $l$. We create a modified LDT-schedule $S_l$ in which we activate the delaying emerging job $l$ for kernel $K$, that is, we force this job to be scheduled after all jobs of kernel $K$, whereas all the emerging jobs, included after kernel $K$ in schedule $S$ remain to be included after that kernel. As a result, the earliest job of kernel $K$ will be scheduled at its release time in schedule $S_l$.

To construct schedule $S_l$, we apply LDT-heuristic to a modified problem instance, in which the release time of job $l$ and that of all emerging jobs included after kernel $K$ in schedule $S$ becomes no less than that of any job of kernel $K$. Then by LDT-heuristic, job $l$ and any emerging job included after kernel $K$ in schedule $S$ will appear after all jobs of kernel $K$ in schedule $S_l$.

Note that kernel $K^1 = K(S_l)$ can similarly be determined in LDT-schedule $S_l$. If kernel $K^1$ possesses the delaying emerging job, let $l_1$ be that job. Then job $l_1$ is activated for kernel $K^1$ resulting in another LDT-schedule $(S_l)_{l_1}$. We proceed similarly creating the next LDT-schedule $((S_l)_{l_1})_{l_2}$, where $l_2$ is the delaying emerging job for kernel $K^2 = K((S_l)_{l_1})$, and so on. For notational simplicity, we will denote LDT-schedule $(\ldots ((S_l)_{l_1})_{l_2})_{l_k}$ by $S_{l,l_1,l_2,\ldots,l_k}$.

2.3 Example

We give a small problem instance that will be used for the illustrations throughout the paper. Let us consider 13 jobs defined as follows:

Let us consider 13 jobs defined as follows:

$r_1 = 0, p_1 = 12, q_1 = 11,$
$r_2 = 2, p_2 = 2, q_2 = 50,$
$r_3 = 5, p_3 = 3, q_3 = 48,$
$r_4 = 10, p_4 = 5, q_4 = 44,$
$r_5 = 13, p_5 = 4, q_5 = 43,$
$r_6 = 1, p_6 = 7, q_6 = 41,$
$r_7 = 32, p_7 = 10, q_7 = 3,$
$r_8 = 35, p_8 = 7, q_8 = 15,$
$r_9 = 37, p_9 = 4, q_9 = 12,$
$r_{10} = 41, p_{10} = 3, q_{10} = 11,$
$r_{11} = 45, p_{11} = 2, q_{11} = 11,$
$r_{12} = 47, p_{12} = 1, q_{12} = 10,$
The initial LDT-schedule $\sigma$ is depicted in Figure 1. Kernel $K_1 = K(\sigma)$ consists of jobs $(2, 3, 4, 5, 6)$ and possesses the delaying emerging job 1. The overflow job in kernel $K_1$ is job 6 with the full completion time $C_6(\sigma) = 33 + 41 = 74$, the makespan of schedule $\sigma$. As we can see, the scheduling times $t = 0$ and $t = 34$ with $j_0 = 1$ and $j_{34} = 7$ are the conflict scheduling times in schedules $\sigma^0$ and $\sigma^34$, respectively. Hence, $\sigma$ cannot be guaranteed to be optimal.

In Figure 2 an alternative LDT-schedule $\sigma_1$ is depicted. We have a newly arisen kernel $K_2 = K(\sigma_1) = (8, 9, 10, 11, 12)$ possessing the delaying emerging job 7 in schedule $\sigma_1$. The corresponding overflow job is job 12 with $C_{12}(\sigma_1) = 59 + 10 = 69$.

2.4 Decomposition of kernels

In this subsection we overview briefly an important recurrent relationship of LDT-schedules and their kernels. In particular, we will consider jobs of a kernel as an independent set of jobs and construct independent LDT-schedules for these jobs. Some relevant terminology is also briefly introduced. The reader is referred to Section 4 of [22] for a detailed introduction of the relevant definitions and properties and the basic kernel decomposition procedure.

A kernel $K \in S$ possessing the delaying emerging job $l$ is not necessarily a non-split component, in the following sense. Let $K(+l)$ be the fragment of schedule $S$ containing the delaying emerging job $l$ and the kernel $K$; abusing the notation, let $K$ be also the fragment of schedule $K(+l)$ without job $l$. Suppose we omit the delaying emerging job $l$ and apply LDT-heuristic solely to the set of jobs from schedule $K$. This results in a partial LDT-schedule that we denote by $K_l$ (note that, unlike an alternative LDT-schedule $S_l$ obtained from an LDT-schedule $S$, schedule $K_l$ does not contain job $l$). Then the first included job (and possibly the following jobs) of kernel $K$ will be left-shifted in schedule $K_l$ compared to the schedule $K(+l)$. If now the overflow job in schedule $K_l$ remains the same as that in schedule $K(+l)$, then no further processing is required:

**Lemma 2** Suppose the overflow in schedule $K_l$ is the same as the overflow job in schedule $K(+l)$. Then in an optimal solution $S_{OPT}$, the jobs of kernel $K$ are scheduled in the same order as in schedule $K_l$.

Proof. By the condition, there may exist no delaying emerging job for the kernel in schedule $K_l$. Indeed, let $o$ be the overflow job of kernel $K$ in schedule $K(+l)$. By the condition, $o$ is the overflow job also in schedule $K_l$. Recall that (by the definition of a kernel) every job of kernel $K$
has the delivery time, no smaller than that of job \( o \). Hence, none of the jobs can be the delaying emerging job in schedule \( K_l \). Then schedule \( K_l \) may contain no delaying emerging job as it consists of only jobs of kernel \( K \). The lemma follows since a schedule in which no kernel possesses the delaying emerging job is optimal.

Based on Lemma 2, entire schedule fragments that we obtain will be moved into the destiny optimal schedule. If the condition in the lemma is not satisfied, then the procedure proceeds in a number of iterations until the condition in the extended version of Lemma 2 (see below) is satisfied. At iteration 1, a former kernel job, \( \lambda_1 \in K \), becomes the delaying emerging job in schedule \( K_l \). We similarly generate schedule \( (K_l)_{\lambda_1} = K_{l,\lambda_1} \) at iteration 1 and continue in this fashion until the condition in Lemma 2 is satisfied. Let \( l, \lambda_1, \ldots, \lambda_m \) be the sequence of the occurred delaying emerging jobs, where \( \lambda_i \) is the delaying emerging job in schedule \( K_{l,\lambda_1,\ldots,\lambda_{i-1}} \), \( i = 2, \ldots, m \). The procedure halts at iteration \( m + 1 \) where schedule \( K_{l,\lambda_1,\ldots,\lambda_m} \) satisfies the condition in Lemma 3:

**Lemma 3** Suppose the overflow in schedule \( K_{l,\lambda_1,\ldots,\lambda_i} \) is the same as the overflow job in schedule \( K_{l,\lambda_1,\ldots,\lambda_{i-1}} \). Then in an optimal solution, the jobs of kernel \( K(K_{l,\lambda_1,\ldots,\lambda_{i-1}}) \) can be scheduled in the same order as in schedule \( K_{l,\lambda_1,\ldots,\lambda_i} \).

Proof. Similar to that of Lemma 2.

We refer to the above described procedure as the *decomposition* of kernel \( K \). As a result of the decomposition, we obtain a partial schedule \( S^*[K] \) consisting of all jobs of kernel \( K \) except the activated delaying emerging jobs \( \lambda_1, \ldots, \lambda_m \) (note that the latter jobs were omitted). We will incorporate these partial schedules into a complete feasible schedule that we will construct (respecting the absolute time scale to represent these partial schedules). From here on, we will refer to kernel \( K \) and partial schedule \( S^*[K] \) interchangeably. We summarize consequences of kernel decomposition in the following lemma.

**Lemma 4** (i) For any kernel \( K \), the decomposition procedure runs in \( m \) recursive steps in time \( \mathcal{O}(mv \log \nu) \), where \( \nu, m < \nu < n \), is the total number of jobs in kernel \( K \).

(ii) The maximum job full completion time in schedule \( S^*[K] \) is a lower bound on the optimum schedule makespan.

Proof. The number of the recursive calls in the procedure is clearly less than the total number \( m \) of the occurred delaying emerging jobs and Part (i) follows since at every iteration LDT-heuristic with time complexity \( \mathcal{O}(nu \log \nu) \) is applied. For the proof of Part (ii) the reader is referred to Section 4 in [22].

**Example.** We may observe the decomposition of kernel \( K_1 \) from our example in Figures 1, 2 and 3. Schedule \( K_1(+1) \) consisting of jobs 1 through 6 and extending through the time interval \( [0,33] \) forms part of the complete LDT-schedule \( \sigma \) from Figure 1. This partial schedule corresponds to the initial iteration 0 in the decomposition procedure. The delaying emerging job is \( l = 1 \). At iteration 1, schedule \( (K_1)_{1} \) is the fragment of the complete schedule \( \sigma_1 \) from Figure 2 from time 1 to time 22. Here \( \lambda_1 = 6 \). Hence, the procedure continues with iteration 2 generating partial schedule \( (K_1)_{1,6} \) with the time interval \( [2,17] \) that consists of jobs 2,3,4 and 5, see Figure 3. The overflow job in partial schedules \( (K_1)_{1} \) and \( (K_1)_{1,6} \) is the same job 5, and the kernel in schedule \( (K_1)_{1,6} \) consisting of jobs 3,4 and 5, possesses no delaying emerging job. Hence the decomposition procedure halts at iteration 2 and outputs partial schedule \( S^*[K_1] \) of Figure 3.
3 The VP-algorithm

In this section we describe stages 0-2 of our VP-algorithm. At stage 0 the initial job partition, and hence the initial set of the emerging (type (1)) jobs, is formed. Based on this partition, a partial schedule without the emerging jobs is constructed. At stages 1 and 2, a given permutation of the emerging jobs is incorporated into the partial schedule of stage 0. The algorithm keeps track of the state of current computations by repeatedly updating the current configuration which consists of the set of kernels and the corresponding job partition.

3.1 Stage 0: Construction of partial schedules without type (1) jobs

In this subsection, first we specify how the initial partition of the whole set of jobs in four disjoint subsets is done. Then we determine the initial “easy” sub-instance and construct one or more partial schedules for that sub-instance.

3.1.1 The initial partition of the set of jobs

The initial job partition is created using the following auxiliary procedure that also forms the initial set of kernels. The procedure creates schedule $\sigma^0 = \sigma$ at iteration 0. Iteratively, at iteration $h > 0$, schedule $\sigma^h$ of iteration $h$ is $\sigma^h = \sigma^{h-1}_{l_h} = \sigma_{l_1,...,l_h}$, where $l_i$ is the delaying emerging job in the LDT-schedule $\sigma^{i-1}$ (given that kernel $K^h = K(\sigma^{h-1})$ of iteration $i = 0,...,h$ possesses the delaying emerging job $l_i$). The procedure halts at iteration $\xi$ if there is no kernel in schedule $\sigma^{\xi-1}$ with the delaying emerging job or/and any kernel in schedule $\sigma^{\xi-1}$ contains the jobs of the kernel of some earlier iteration (clearly, $\xi < n$). The procedure returns the formed set of kernels $\{K_1, K_2, ..., K_{\xi-1}\}$.

In the following, we will $K^-$ ($K^+$, respectively) for the kernel immediately preceding (immediately succeeding, respectively) kernel $K$. Once the initial set of kernels $\{K^1, K^2, ..., K^{\xi-1}\}$ is so formed, the initial partition of the set of jobs is determined as follows.

Type (1) jobs are the emerging jobs, divided into three categories. The first two categories are defined below, and the third category of the type (1) jobs will be defined later.

1. A type (1.1) job is an emerging job for any of the kernels $K^1, K^2, ..., K^\xi$.

2. A set of the type (1.2) jobs, associated with kernel $K \in \{K^1, K^2, ..., K^\xi\}$, is formed from the delaying emerging jobs $\lambda_1, ..., \lambda_m$ (omitted in schedule $S^*[K]$ during the decomposition of kernel $K$).

3. The type (2) jobs associated with kernel $K \in \{K^1, K^2, ..., K^\xi\}$ are the jobs of the last kernel $K(K_{l_1,\lambda_1,...,\lambda_{m-1}})$ occurred in the decomposition of kernel $K$ (the one for which Lemma 2 (Lemma 3) was satisfied).
4. The type (3) jobs associated with kernel $K \in \{K^1, K^2, \ldots, K^\xi\}$ are formed by the remaining jobs of schedule $S^*[K]$.

5. All the remaining, i.e., non-type (1)-(3), jobs are the type (4) jobs (these are non-emerging, non-kernel jobs).

Thus with every kernel $K$ its own type (1.1), type (2) and type (3) jobs are associated, whereas a type (1.1) job can be associated with one or more (successive) kernels (this will happen if that type (1.1) job is an emerging job for these kernels).

Similarly, we associate a type (4) job with a particular kernel or with a pair of the neighboring kernels depending on its position: if a type (4) job $j$ is scheduled between two adjacent kernels $K^i$ and $K^{i+1}$ (before the first kernel $K^1$ or after the last kernel $K^\xi$), then it cannot be an emerging job for none of the kernels. Hence, it must be scheduled in between these two kernels (before or after the first and the last kernel, respectively).

Note that the last partial schedule (kernel) $K(K_1, \lambda_1, \ldots, \lambda_{m-1})$ created during the decomposition of kernel $K$ consists of the type (2) jobs associated with kernel $K$; equivalently, these are type (2) jobs of kernel $K(S^*[K])$ of partial schedule $S^*[K]$. Note also that any type (3) job is included before these type (2) jobs in schedule $S^*[K]$. The corresponding type (1.2) jobs (which do not belong to schedule $S^*[K]$) will be included immediately before, within or immediately after the jobs of schedule $S^*[K]$, as we will see in Section 4.

Example. First, we illustrate how the initial set of kernels is formed. Initially at iteration 0, kernel $K_1$ is the one from schedule $\sigma$. At iteration 1 schedule $\sigma_1$ of Figure 2 is constructed and a new kernel $K_2$ with the delaying emerging job $l_1 = 7$ and the overflow job 12 with $C_{12}(\sigma_1) = 59 + 10 = 69$ arises in that schedule. At iteration 2, the kernel in the LDT-schedule $\sigma_{1,7}$ of Figure 4 contains the jobs of kernel $K_1$ and hence the procedure halts at iteration 2 and outputs the initial set of kernels $\{K_1, K_2\}$.

Once the initial set of kernels is so formed, the initial job partition is obtained. We easily observe that type (1.1) jobs are 1 and 7, the type (1.2) job associated with kernel $K_1$ is job 6. Note that there is no type (1.2) job associated with kernel $K_2$. Type (2) jobs associated with kernel $K_1$ are jobs 3, 4 and 5. Type (2) jobs associated with kernel $K_2$ are jobs 8, 9, 10, 11 and 12. There is a single type (3) job 2 associated with kernel $K_1$, and there is no type (3) job associated with kernel $K_2$. There is only one type (4) job 13.

3.1.2 Construction of partial schedules of the type (2)-(4) jobs

Based on the initial job partition, the first partial schedule of the type (2)-(4) jobs is created by the following procedure. The procedure first merges the partial schedules $S^*[K^t]$, $t = 1, \ldots, \xi$, on the time axes (the time interval of each partial schedule being considered in the absolute time scale). This creates partial schedule $\sigma(2, 3)$ including all the type (2) and the type (3) jobs (since
no two adjacent partial schedules \( S^*[K'] \) and \( S^*[K''] \) \((i \neq i')\) may overlap in time, this schedule is well-defined).

Next, schedule \( \sigma(2, 3) \) is augmented by the type (4) jobs. A type (4) job is included in the time interval between the corresponding two kernels (before kernel \( K^1 \) or after kernel \( K^{\xi} \)) according to its position in schedule \( \sigma \). The type (4) jobs, to be included within the same time interval, are scheduled by LDT-heuristic. Let \( \sigma(2, 3, 4) \) be the resultant partial schedule of the type (2)-(4) jobs.

A natural question is, if there arises a new kernel in schedule \( \sigma(2, 3, 4) \), i.e., if it possesses a kernel containing a job which does not belong to any of the schedules \( S^*[K], \ K \in \{K^1, K^2, \ldots, K^{\xi-1}\} \). Note that such a possibility is not excluded, since the maximum full completion time of a job of each of the kernels from the initial set of kernels \( \{K^1, K^2, \ldots, K^{\xi-1}\} \) was reduced in schedule \( \sigma(2, 3, 4) \); as a result, a former type (4) job may become part of a new kernel in that schedule.

**Configuration updates.** If a new kernel \( K \) in schedule \( \sigma(2, 3, 4) \) arises, then the current set of kernels needs to be updated by including kernel \( K \) in it. Note that the current job partition is also changed. Hence, stage 0 creates another partial schedule of the type (2)-(4) jobs in which the schedule segment containing kernel \( K \) is replaced by schedule \( S^*[K] \). The current job partition is updated correspondingly: the rise of kernel \( K \) yields new type (2) and type (3) jobs, whereas the current set of the type (4) jobs is reduced respectively. Hence, the current job partition is updated by transferring the former type (4) jobs of kernel \( K \) to the new sets of type (1.2), type (2) and type (3) jobs, correspondingly. At the same time, every type (4) job from the current job partition, which is an emerging job for kernel \( K \), is transferred to the set of the type (1.1) jobs.

**Description of stage 0.** Stage 0 returns schedule \( \sigma(2, 3, 4) \) if there arises no new kernel in it. Otherwise, \( \tilde{\sigma}^0 := \sigma(2, 3, 4) \) is set to be the LDT-schedule of iteration 0. Iteratively at iteration \( h \), if \( K \) is the newly arisen kernel in the LDT-schedule \( \tilde{\sigma}^h \) of iteration \( h \), then \( \tilde{\sigma}^{h+1} := \tilde{\sigma}^h(S^*[K]) \), where \( \tilde{\sigma}^h(S^*[K]) \) is the LDT-schedule obtained from schedule \( \tilde{\sigma}^h \) by replacing its fragment consisting of the jobs of kernel \( K \) by partial schedule \( S^*[K] \). The update of the set of kernels and the current job partition is carried out as just described. Stage 0 halts at iteration \( h \) and returns schedule \( \tilde{\sigma}^h \) if no new kernel in it arises.

Note that the total number of iterations is bounded by \( n \), since obviously, no more than \( n \) new kernels may occur. Note also that the update of schedule \( \tilde{\sigma}^h \) at iteration \( h \) yields no conflicts since the interval of the newly created partial schedule \( S^*[K] \) does not overlap with that of schedule \( S^*[K'] \), for any kernel \( K' \) from the current set of kernels (note that if the newly arisen kernel \( K \) possesses no delaying emerging job, then partial schedule \( S^*[K] \) just coincides with kernel \( K \)). This observation, together with point (a) in Lemma 5 ensures, in particular, that every partial schedule generated at stage 0 is a well-defined feasible schedule. For notational simplicity, the last generated schedule returned by stage 0 will again be denoted by \( \sigma(2, 3, 4) \). The next observation follows.

**Observation 1**

(i) All partial schedules generated a stage 0 are well-defined feasible schedules.

(ii) If a new kernel in such partial LDT-schedule arises, then it consists of some type (4) jobs from the current job partition; if this kernel possesses the delaying emerging job then it is a type (4) job from the current job partition.

(iii) The partial schedule \( \sigma(2, 3, 4) \) returned by stage 0 contains no kernel with the delaying emerging job.
Lemma 5 There is an optimal schedule $S_{OPT}$ in which:

(a) Any type (4) job is included between the intervals of partial schedules $S^*[K^i]$ and $S^*[K^{i+1}]$, before the interval of the first partial schedule $S^*[K]$ or after that of the last partial schedule $S^*[K^\lambda]$. In particular, there is no intersection of the execution interval of a type (4) job in schedule $\sigma(2,3,4)$ with the time interval of any type (2) and type (3) job from that schedule.

(b) If a type (1.1) job is an emerging job for two or more kernels $S^*[K^i], \ldots, S^*[K^{i+l}]$, then it is scheduled either before or after partial schedules $S^*[K^i], \ldots, S^*[K^{i+l}]$ in schedule $S_{OPT}$; a type (1.1) job which is not an emerging job for kernel $K^\lambda$ does not appear after the jobs of partial schedule $S^*[K^\lambda]$ in that schedule.

(c) A type (1.2) job associated with kernel $K^i$ is scheduled within the interval of partial schedule $S^*[K^i]$ or after that interval but before the interval of partial schedule $S^*[K^{i+1}]$.

(d) Any type (2) and type (3) job associated with a kernel $K$ is scheduled within the interval of partial schedule $S^*[K^i]$ or after that time interval. In the later case, it is pushed by a type (1.2) job associated with kernel $K$ or/and by a corresponding type (1.1) job.

Proof. Part (a) holds as no type (4) job can be an emerging job for the corresponding kernel: a type (4) job scheduled between partial schedules $S^*[K^i]$ and $S^*[K^{i+1}]$ is less urgent than any job from schedule $S^*[K^i]$ but it is more urgent than any job from schedule $S^*[K^{i+1}]$ (as otherwise it would be a type (1.1) job for kernel $K^{i+1}$). Hence, it cannot be included after partial schedule $S^*[K^{i+1}]$ in schedule $S_{OPT}$. At the same time, there can be no benefit in including such jobs in between the jobs of these partial schedules. Likewise, the type (4) jobs included after partial schedule $S^*[K^\lambda]$ in schedule $\sigma(2,3,4)$ can be included after all jobs of that sequence (since they are less urgent than all jobs from schedule $S^*[K^\lambda]$ and hence there will be no benefit in rescheduling them earlier). This proves part (a). Part (b), stating that the type (1.1) jobs can be dispelled in between the corresponding kernel sequences easily follows. As to part (c), note that no type (1.2) job associated with kernel $K^i$ is released before the interval of partial schedule $S^*[K^i]$ and it cannot be scheduled after the original execution interval of that kernel without causing the increase in the makespan. Part (d) similarly follows.

Lemma 6 The makespan of partial schedule $\sigma(2,3,4)$ of type (2)-(4) jobs returned at stage 0 is a lower bound on the optimal schedule makespan.

Proof. By Lemma 5 and the fact that two kernel segments do not overlap in schedule $\sigma(2,3)$, $\sigma(2,3)$ is a feasible partial schedule such that the maximum full job completion time in it is a lower bound on the optimal schedule makespan. We show that this magnitude cannot be surpassed by any other job from schedule $\sigma(2,3,4)$. Indeed, any job $j \in \sigma(2,3,4)$ is either from partial schedule $S^*[K^i]$, for some kernel $K$ from the set of kernels delivered at stage 0 or $j$ is a type (4) job from the corresponding job partition. In the latter case, our claim follows since job $j$ could not belong to any newly kernel at stage 0. Consider now the former case. We show that no type (4) job may push job $j$ in schedule $\sigma(2,3,4)$, and hence the full completion time of job $j$ is a lower bound on the optimum makespan. Indeed, job $j$ may potentially be pushed only by a type (4) job in schedule $\sigma(2,3,4)$. But any type (4), job originally scheduled before the delaying emerging job of kernel $K$, completes before the starting time of that job. But the latter job is omitted in schedule $\sigma(2,3,4)$ and hence no job can push job $j$. 

Example. The first created schedule of the type (2)-(4) jobs, \( \sigma(2, 3, 4) \), is represented in Figure 5. There arises a new kernel \( K_3 \) consisting of a single job 13, with \( C_{13}(\sigma(2, 3, 4)) = 60 + 2 = 62 \) in that schedule. The iterative subroutine updates the current set of kernels and job partition, respectively. The updated schedule is identical to that of Figure 5 since \( S^i[K_3] = K_3 \). Stage 0 returns this schedule since kernel \( K_3 \) possesses no delaying emerging job.

3.2 Stage 1: Generating a complete schedule respecting a permutation of the type (1) jobs

At stages 1 and 2, the type (1) jobs are incorporated into the partial schedule \( \sigma(2, 3, 4) \) of stage 0. Let \( \pi = \{i_1, \ldots, i_\nu\} \) be a permutation of the type (1) jobs from the job partition created by stage 0. At stages 1 and 2, we aim is to find, among all feasible schedules respecting the order of the type (1) jobs in permutation \( \pi \), one with the minimum makespan. For that, will extend schedule \( \sigma(2, 3, 4) \) to one or more complete feasible schedules respecting permutation \( \pi \).

3.2.1 Filtering inconsistent and dominated permutations

We may avoid a brutal enumeration of all possible \( \nu! \) permutations of the type (1) jobs. In this subsection we show how potentially inconsistent and dominated permutations can be discarded. Later in Section 7 we argue that by considering the remaining permutations in a special priority order, the number of the enumerated permutations can further be reduced.

Filtering inconsistent permutations. We may discard permutations which cannot be consistent with solution \( S_{OPT} \). Recall that type (1.2) jobs associated with a particular kernel \( K \) are to be scheduled either immediately before or within or immediately after the time interval of schedule \( S^*[K] \). In particular, these jobs cannot be scheduled before any type (1.2) job associated with a kernel preceding kernel \( K \) and after any type (1.2) job associated with a kernel succeeding kernel \( K \), i.e., no other type (1) job is to be included in between these type (1.2) jobs (see point (c) in Lemma 5). Hence, in any permutation, consistent with any optimal solution, these precedence relations are respected. We will refer to a permutation of the type (1) jobs in which the corresponding restrictions are respected for the type (1.2) jobs associated with every kernel as a consistent permutation.

Dominated permutations. The order of the type (1) jobs imposed by a consistent permutation \( \pi = \{i_1, \ldots, i_\nu\} \) may yield the creation of a dominated (non-active) complete schedule, in which case permutation \( \pi \) will again be discarded. The order of the type (1) jobs imposed by a consistent permutation \( \pi \) may yield the creation of an avoidable gap. Such a gap may potentially occur at iteration \( \iota \) if job \( i_\iota \) is released earlier than the previously included job \( i_{\iota-1} \). Then job \( i_\iota \) can potentially be included before job \( i_{\iota-1} \) without causing any non-permissible delay. In this case, a permutation in which job \( i_\iota \) comes after job \( i_{\iota-1} \) can be neglected.

More formally, let \( \sigma(\pi, \iota) \) be the partial schedule of iteration \( \iota \). Suppose at iteration \( \iota \) there...
is a gap in schedule $\sigma(\pi, \iota - 1)$ before time $r_i$, within which job $i$ may feasibly be included. If there are several such gaps, consider the earliest occurred one, say $g$. Let schedule $\sigma(\pi, \iota, \iota - 1)$ be an extension of schedule $\sigma(\pi, \iota - 1)$ in which job $i$ is included at the beginning of gap $g$ or at time $r_i$, whichever magnitude is larger and the following jobs from schedule $\sigma(\pi, \iota - 1)$ are correspondingly right-shifted (so job $i$ will appear before job $i_{-1}$ in that schedule). If now the makespan of schedule $\sigma(\pi, \iota, \iota - 1)$ is no larger than that of schedule $\sigma(\pi, \iota - 1)$ then the latter schedule (the corresponding permutation $\pi' = \{i_1, \ldots, i_{-1}, \ldots, i_k\}$) dominates the former schedule (permutation $\pi$), where job $i$ is said to be damped by job $i_{-1}$.

It follows that a schedule respecting a dominated permutation can be neglected as the schedule corresponding to a corresponding dominant permutation will be created, unless the latter permutation gets dominated by another permutation. But since the dominance relation is transitive, if the latter even occurs, the above first permutation is also dominated by the third one. The next lemma follows.

**Lemma 7** There is an optimal solution $S_{OPT}$ which respects a consistent non-dominated permutation $\pi$ of the type (1) jobs.

### 3.2.2 The construction procedure of stage 1

Stage 1, invoked for a (consistent non-dominated) permutation $\pi = \{i_1, \ldots, i_\nu\}$, generates a complete feasible schedule $\sigma(\pi, \nu)$ respecting that permutation (unless the offsprings of this permutation are created, see the end of this subsection); i.e., the type (1) jobs from the current job partition are included in the order of permutation $\pi$ in that schedule. Stage 1 works in at most $\nu$ iterations, where job $i_\iota$ is included in $\iota$th iteration, for $\iota = 1, \ldots, \nu$. Initially at iteration 0, $\sigma(\pi, 0) := \sigma(2, 3, 4)$; iteratively, if job $i_\iota$ is not damped by job $i_{-1}$ and the consistency restrictions are not violated, schedule $\sigma(\pi, \iota)$ is obtained from schedule $\sigma(\pi, \iota - 1)$ by including job $i_\iota$ at the earliest idle-time interval at or after time $r_i$ in schedule $\sigma(\pi, \iota - 1)$; if the overlapping with an earlier included job from schedule $\sigma(\pi, \iota - 1)$ occurs, then this job and the jobs following it are right-shifted by the required amount of time (the processing order of these jobs in schedule $\sigma(\pi, \iota - 1)$ being respected). If there is no such idle-time interval, then job $i_\iota$ is scheduled at the completion time of the last scheduled job of schedule $\sigma(\pi, \iota - 1)$. If at iteration $\iota$ it is established that either permutation $\pi$ is not consistent or job $i_\iota$ is damped by job $i_{-1}$, then permutation $\pi$ is discarded. If such an event does not occur, then the procedure halts once it schedules the last job $i_\nu$ at iteration $\nu$.

**Lemma 8** Suppose there is a kernel $K$ in schedule $\sigma(\pi, \nu)$ with no delaying emerging job and it includes no type (1) job from the current job partition. Then schedule $\sigma(\pi, \nu)$ is optimal.

Proof. By the condition, kernel $K$ is a kernel from schedule $\sigma(2, 3, 4)$. But then the full completion time of the overflow job of this kernel is a lower bound on the optimum schedule makespan by Lemma 6, and the lemma obviously follows. 

If there arises a new kernel in schedule $\sigma(\pi, \nu)$, then one cannot assure that, among all feasible schedules respecting permutation $\pi$, it is one with the minimum makespan. In this case, similarly as at stage 0, one or more additional complete schedules respecting permutation $\pi$ can be created, the current set of kernels and the current job partition being updated respectively (as described
in Section 3.1.2). At the same time, since the updated job partition may contain new type (1) jobs, the current set of permutations of the type (1) jobs is augmented with the corresponding new permutations, as we describe below.

Suppose that a former type (4) job $j$ turns out to be an emerging job for a newly arisen kernel $K$ in schedule $\sigma(\pi, \nu)$ (observe that if a type (4) job is pushed by a type (1.1) job newly included in schedule $\sigma(\pi, \nu)$, then it may be converted to a type (1.1) job). Note that job $j$ is included in between partial schedules $S^*[K^-]$ and $S^*[K^+]$ in optimal schedule $S_{OPT}$. Furthermore, as any type (1) job, job $j$ is not included in between the jobs of partial schedule $S^*[K]$ in schedule $S_{OPT}$. We distinguish the above kind of type (1) job (a former type (4) job) from type (1.1) jobs, and will refer to it as a type (1.3) job associated with kernel $K$. The next lemma complements Lemma 5:

**Lemma 9** A type (1.3) job associated with kernel $K$ is included in between partial schedules $S^*[K^-]$ and $S^*[K^+]$ and not in between the jobs of partial schedule $S^*[K]$ in schedule $S_{OPT}$.  

Accordingly, we require any consistent permutation to satisfy the restrictions from the above lemma. Note that the newly arisen kernel yields new type (1.2) and (1.3) jobs, associated with that kernel. Hence, the current set of permutations needs to be complemented. We create a set of new permutations, the offsprings of permutation $\pi$. Due to the positioning restrictions for the type (1.2) and the type (1.3) jobs from Lemmas 5 and 9, the total number of the offsprings of permutation $\pi$ is easily seen to be $\beta!\alpha!$, where $\alpha$ ($\beta$, respectively) is the number of the newly arisen type (1.2) (type (1.3), respectively) jobs associated with kernel $K$. It is also easy to see that no complete schedule respecting permutation $\pi$ needs to be generated since the offsprings of that permutation are created.

Now we can summarize stage 1. It distinguishes three basic cases in schedule $\sigma(\pi, \nu)$. In case (1) it returns schedule $\sigma(\pi, \nu)$, and it may also halt the whole algorithm. In case (2), it forms the offsprings of permutation $\pi$, and in case (3) it invokes stage 2:

1. If there is a kernel $K$ in schedule $\sigma(\pi, \nu)$ with no delaying emerging job, then return that schedule (see Lemma 10 in the next sub-section); if, in addition, kernel $K$ contains no type (1) job, then halt the algorithm ($\sigma(\pi, \nu)$ is an optimal schedule by Lemma 8).

2. If all kernels in schedule $\sigma(\pi, \nu)$ possess the delaying emerging job and there arises a new kernel in that schedule, then update the current set of kernels and job partition and create the offsprings of permutation $\pi$ (in this case no complete schedule respecting permutation $\pi$ will be created).

3. If all kernels in schedule $\sigma(\pi, \nu)$ possess the delaying emerging job and no new kernel in that schedule arises, then call stage 2 {create additional complete schedule(s) respecting permutation $\pi$, see the next sub-section}.
4 Stage 2: Generating additional complete schedules respecting permutation $\pi$

Recall that stage 1 invokes stage 2 if no new kernel in schedule $\sigma(\pi, \nu)$ arises and all kernels in that schedule possess the delaying emerging job. We will distinguish two kernels containing the same set of jobs if the schedule fragments corresponding to these kernels are not identical: Given kernel $K \in \{K^1, K^2, \ldots, K^\xi\}$, we call kernel $\tilde{K}$ the secondary kernel of kernel $K$ if kernels $K$ and $\tilde{K}$ contain the same set of jobs but the corresponding (partial) schedules are different.

Observation 2 Let $\sigma(\pi)$ be a schedule respecting permutation $\pi$ in which no new kernel arises and such that all kernels in that schedule possess the delaying emerging job. Then any kernel in schedule $\sigma(\pi)$ is a secondary kernel of some kernel from the current configuration.

Proof. Recall that the fragment containing the jobs of any kernel $K$ from the current configuration is substituted by partial schedule $S^*[K]$. The latter partial schedule does not contain the type (1.2) and type (1.3) jobs associated with kernel $K$. Neither schedule $S^*[K]$ may contain any type (1.1) job since otherwise it would form a newly arisen kernel. It follows that any kernel in schedule $\sigma(\pi)$ is the secondary kernel of some kernel from the current configuration.

Stage 2, starting with schedule $\sigma(\pi, \nu)$, may generate one or more LDT-schedules respecting permutation $\pi$ by repeatedly activating the delaying emerging job for the secondary kernel in the last generated LDT-schedule (see Observation 2):

(0) Initially, the LDT-schedule of iteration 0 is $\tilde{\sigma}^0 = \sigma(\pi, \nu)$.

(1) Iteratively, at iteration $i > 0$, if in the schedule $\tilde{\sigma}^{i-1}$ of iteration $i - 1$ there is a kernel with a type (1) job or with no delaying emerging job, then stage 2 halts and returns a generated LDT-schedule with the smallest makespan (Lemma 8, see also Lemma 10 below); if, in addition, $i = 0$ then stage 2 returns schedule $\sigma(\pi, \nu)$ and halts the whole algorithm (no more permutation of the type (1) jobs will be considered, see again Lemma 10).

(2) If a new kernel in schedule $\tilde{\sigma}^i$ arises, stage 1 is newly invoked (it will update the current set of kernels and the current job partition, and will create the offsprings of permutation $\pi$).

(3) Else, there is a secondary kernel in schedule $\tilde{\sigma}^i$. Let $l'_i$ be the delaying emerging (type (1.1)) job of the earliest secondary kernel $\tilde{K}^{i-1}$ of schedule $\tilde{\sigma}^{i-1}$ of iteration $i - 1$. The LDT-schedule of iteration $i$ is then

$$\tilde{\sigma}^i = \tilde{\sigma}_{l'_i}^{i-1} = \sigma_{l'_1, l'_2, \ldots, l'_i}(\pi, \nu).$$

Lemma 10 Let $\sigma(\pi)$ be a complete LDT-schedule respecting permutation $\pi$ generated at stage 1 or at stage 2. If this schedule contains a kernel with a type (1) job, then a schedule with the minimum makespan respecting permutation $\pi$ is among the already generated LDT-schedules respecting permutation $\pi$. In particular, if there is a kernel in schedule $\sigma(\pi, \nu)$ including a type (1) job, then it has the minimum makespan among all feasible schedules respecting permutation $\pi$.

Proof. Let $K$ be the (earliest) kernel in schedule $\sigma(\pi)$ containing a type (1) job $e \in K$ (note that $K$ is not a secondary kernel). Either (i) job $e$ was activated in one of the generated LDT-schedules
respecting permutation \( \pi \) or (ii) not. In the latter case (ii), job \( e \) should have been included in the first available gap after the type (1) job immediately preceding it in permutation \( \pi \), in schedule \( \sigma(\pi, \nu) \) (unless \( e \) is the first job in the permutation). Since job \( e \) is part of kernel \( K \), for any job \( j \) succeeding job \( e \) in kernel \( K \), \( q_j \leq q_e \). The lemma (the second part of it) follows if \( e \) is the first job in permutation \( \pi \); otherwise, the makespan of schedule \( \sigma(\pi) \) can only potentially be reduced by rescheduling a job preceding job \( e \) in permutation \( \pi \) behind kernel \( K \) (the improvement may only occur if that job is an emerging job for kernel \( K \)). But such a schedule would not respect permutation \( \pi \).

Consider now case (i) above. Let \( \sigma'(\pi) \) be the first generated LDT-schedule respecting permutation \( \pi \), in which job \( e \) was activated, say, for kernel \( K' \) (by our construction, schedule \( \sigma'(\pi) \) exists). Note that there is a gap immediately before kernel \( K' \) in schedule \( \sigma'(\pi) \) and in any further generated LDT-schedule respecting permutation \( \pi \). Hence, an emerging job, say \( \epsilon \), for kernel \( K \), may only potentially be scheduled in between kernel \( K' \) and job \( e \) in schedule \( \sigma'(\pi) \). Due to the restrictions on the processing order of the jobs from permutation \( \pi \), none of the jobs scheduled within the above time interval can be a type (1) job from that permutation. Hence, by LDT-heuristic, \( q_\epsilon \geq q_e \) and \( \epsilon \) cannot be an emerging job for kernel \( K \). We showed that there may exist no LDT-schedule respecting permutation \( \pi \) with the makespan, less than the current best makespan and the lemma proved.

Observation 3 In schedule \( \bar{\sigma}^i \), the maximum full completion time of a job from partial schedule \( S^*[K^{i-1}] \) is a lower bound on the optimal schedule makespan.

Proof. By the construction, the first job of the secondary kernel \( \bar{K}^{i-1} \) starts at its release time in schedule \( \bar{\sigma}^i \). Then in that schedule, the jobs of kernel \( \bar{K}^{i-1} \) will be rescheduled as in the partial schedule \( S^*[\bar{K}^{i-1}] \) and the observation follows from Lemma 4.

Lemma 11 Stage 2 invoked for permutation \( \pi \) works in less than \((n - \nu)\nu \) iterations.

Proof. It suffices to show that an LDT-schedule with a kernel with no delaying emerging job will be created in less than \( n\nu \) iterations. There may exist at most \( n - \nu - 1 \) kernels in the LDT-schedule of each iteration, whereas the same delaying emerging job may be activated at most once for the same kernel. Since the number of the delaying emerging jobs is bounded from above by the total number of the type (1) jobs, the total number of the activations at stage 2 cannot exceed \((n - \nu)\nu \). Then after less than \((n - \nu)\nu \) iterations, a complete schedule with no delaying emerging job will be created.

Example. We have six possible permutations of the 3 type (1) jobs 1,6 and 7. Permutations \((1,7,6), (7,1,6) \) and \((7,6,1) \) are dominated by permutation \((1,6,7) \), and permutation \((6,7,1) \) is dominated by permutation \((6,1,7) \). So we have only two permutations \((1,6,7) \) and \((6,1,7) \) to consider.

Permutation \((1,6,7) \) yields a complete schedule \( \sigma(\pi, \nu) = \sigma((1,6,7), 3) \) generated at stage 1. This schedule coincides with the schedule of Figure 1. The kernel \( K_1 \) in that schedule possesses

Figure 6: LDT-schedule \( \sigma_1((1,6,7), 3) \)
the delaying emerging job. Hence, stage 2 is invoked and schedule $\sigma_1((1, 6, 7), 3)$ is generated, see Figure 6. There arises a new kernel in that schedule consisting of a single type (1) job 6. Then no further schedule respecting permutation $(1, 6, 7)$ is created due to Lemma 10.

The other permutation $(6, 1, 7)$ yields a complete schedule $\sigma(\pi, \nu) = \sigma((6, 1, 7)), 3)$ generated at stage 1. This schedule coincides with the schedule of Figure 2. Since kernel $K_2$ of that schedule possesses the delaying emerging job, stage 2 is again invoked and schedule $\sigma_7((6, 1, 7), 3)$ is generated, see Figure 4. In that schedule, a kernel consisting of jobs 3, 4 and 5 with the overflow job 5 with $c_5(\sigma_7((6, 1, 7), 3)) = 22 + 43 = 65$ arises. A type (1.2) job 6 is the delaying emerging job. Hence, stage 2 repeatedly generates the next LDT-schedule $\sigma_{7,6}((6, 1, 7), 3)$ respecting permutation $(6, 1, 7)$, see Figure 7. The kernel of that schedule consists of all jobs except job 2, and the corresponding overflow job is job 13. This kernel possesses no emerging job, but it contains a type (1) job. Hence, stage 2 halts by Lemma 10. The algorithm also halts as all consistent non-dominated permutations were already considered.

Summarizing our example, from $13! = 6227020800$ possible permutations of the 13 jobs, the algorithm tested only two permutations of the 3 type (1) jobs from 6 possible permutations, where $\nu = 3$. Five complete solutions for these two permutations were enumerated. Two of the generated solutions turned out to be optimal (Figures 6 and 7).

## 5 Correctness of the algorithm and its time complexity

In this section we prove the soundness of our algorithm incorporating stages (0)-(2), and give an explicit exponential expression that bounds its running time.

**Theorem 1** At least one complete schedule generated for permutation $\pi$ has the minimum makespan among all feasible schedules respecting that permutation. All feasible schedules respecting a given permutation are created in time $O(\nu n^2 \log n)$. Hence, the algorithm generates an optimal solution to problem $1|r_j, q_j|C_{\text{max}}$ in time $O(\nu! \nu^2 n^2 \log n)$.

**Proof.** We start with a brief overview of the earlier established facts and show that the algorithm creates optimal solution $S_{\text{OPT}}$. Recall that for the initial configuration, we constructed partial schedule $\sigma(2, 3, 4)$ and showed that its makespan is a lower bound on the optimal schedule makespan Lemma 6. The type (2)-(4) jobs from the initial configuration are distributed in schedule $\sigma(2, 3, 4)$ according to the points (a) and (d) from Lemma 5. To incorporate the type (1) jobs, we consider all possible ways to distribute them in into schedule $\sigma(2, 3, 4)$ respecting point (b) from Lemma 3. Likewise, type (1.2) jobs, associated with some kernel $K$, are scheduled within or after the partial schedule $S^*[K]$ and before the next partial schedule $S^*[K^+]$ (point (c) in Lemma 5), and type (1.3) jobs, associated with kernel $K$ are included in between schedules $S^*[K^-]$ and $S^*[K^+]$, according to Lemma 6.

In Lemma 7 we showed that only complete schedules respecting consistent non-dominated permutations of the type (1) jobs need to be considered. Let us now consider any such permutation $\pi$. The algorithm returns schedule $\sigma(\pi, \nu)$ (which is the first complete schedule respecting
permutation $\pi$ created at stage 1), and halts if it contains a kernel without a type (1.1) job with no delaying emerging job Lemma $[8]$. Otherwise, let $\sigma(\pi)$ be the last complete schedule generated either at stage 1 or at stage 2. Either a new kernel in schedule $\sigma(\pi)$ arises or all kernels in this schedule are secondary (see Observation $[2]$). In the first case, if there is a newly arisen kernel with a type (1) job in schedule $\sigma(\pi)$, then one of the created complete schedules respecting permutation $\pi$ is one with the minimum makespan (Lemma $[10]$), and hence no more complete schedule respecting that permutation needs to be created. Otherwise, the offsprings of permutation $\pi$ are generated. Clearly, in this case no complete schedule respecting permutation $\pi$ needs to be generated since all offsprings of permutation $\pi$ will individually be considered.

It remains to consider the case where all kernels in schedule $\sigma(\pi)$ are secondary. Note that, since there is no new kernel in schedule $\sigma(\pi)$, each of these kernels possess the delaying emerging job. Let $K$ be the earliest secondary kernel and $l$ the corresponding delaying emerging job in schedule $\sigma(\pi)$. Remind that job $l$ cannot be scheduled in between the jobs of partial schedule $S^*\lbrack K \rbrack$. But the makespan of any feasible schedule respecting permutation $\pi$ in which job $l$ is scheduled before the jobs of partial schedule $S^*\lbrack K \rbrack$ cannot be less than that of schedule $\sigma(\pi)$, since the right-shift caused by job $l$ is unavoidable in any feasible schedule respecting permutation $\pi$. Suppose schedule $\sigma(\pi)$ is not one with the minimum makespan among all complete schedules respecting permutation $\pi$. Then job $l$ can only be included after the jobs of partial schedule $S^*\lbrack K \rbrack$ in an optimal solution $S_{\text{OPT}}$ (see again Lemma $[5]$). The corresponding schedule $\sigma_l(\pi)$ is created at stage 2. If there is a non-secondary kernel in schedule $\sigma_l(\pi)$ we are clearly done. Otherwise, we continue in this fashion repeatedly applying similar reasoning to schedule $\sigma_l(\pi)$ and to schedules of the following iterations of stage 2.

We showed that any non-enumerated schedule respecting permutation $\pi$ has the makespan no-less than that of some enumerated one. Since all consistent non-dominated permutations of type (1) jobs are considered, an optimal schedule must have been enumerated.

Now we turn to the time complexity part. The procedure from Section 3.1. to form the initial set of kernels (and the initial job partition) works in at most $n - \nu$ iterations since each next iteration is invoked only if a new kernel in the LDT-schedule of the previous iteration arises. Clearly, there may arise at most $n - \nu$ different kernels. Since at each iteration LDT-heuristic with cost $O(n \log n)$ is applied, the total cost of the procedure is $O((n - \nu)n \log n)$. For each kernel, the decomposition procedure with cost $O(i^2 \log i)$ is invoked to create schedule $\sigma(2, 3, 4)$, where $i$ is the total number of jobs in the that kernel (Lemma $[4]$). Assume, for the purpose of this estimation, that each of the at most $n - \nu$ kernels have the same number of jobs since in this case the maximum overall cost will be attained. Then we easily obtain that the total cost of all the calls of the decomposition procedure is bounded from above by $O(n^2 \log n)$ (maintaining the jobs of each type in separate binary search tree, the total cost of all job partition updates will be $O(n \log n)$, but this is not required in our estimation). Thus the overall cost of stage 0 for creating the initial job partition and schedule $\sigma(2, 3, 4)$ is $O(n^2 \log n) + O(n^2 \log n)) = O(n^2 \log n)$.

For each (consistent non-dominated) permutation of the type (1) jobs, stage 1 works in at most $\nu$ iterations, and the cost of the insertion of each next type (1) job at each iteration is bounded from above by the number of the corresponding right-shifted jobs. Hence, the total cost of the generation of schedule $\sigma(\pi, \nu)$ is $O(\nu n)$ since the verification of the dominant and consistency conditions and configuration updates imply no extra factor. Stage 2 generates less than $(n - \nu)\nu$ additional LDT-schedules for each permutation (Lemma $[11]$), whereas the cost of the activation of the delaying emerging job $l_i$ at each iteration $i$ is that of LDT-heuristic, i.e., it is $O(n \log n)$. Hence for permutation $\pi$, stage 2 runs in time $O(\nu(n - \nu)n \log n)$, which can be
simplified to \( O(\nu n^2 \log n) \). Since no more than \( \nu! \) permutations are considered, the cost for all the permutations is \( O(\nu! n^2 \log n) \) and the overall cost is \( O(n^2 \log n + \nu! n^2 \log n) = O(\nu! n^2 \log n) \).

6 Concluding notes

Our variable parameter algorithm carries out an exponential-time enumeration for the \( \nu \) type (1) jobs. By considering only consistent non-dominated permutations, the number of the considered permutations are reduced. We can further reduce this number by first initiating with a specially constructed steady permutation \( \pi^* \) of the type (1) jobs. It is obtained by the creation of a particular complete schedule. The later schedule in turn, is obtained by augmenting schedule \( \sigma(2, 3, 4) \) with the type (1) jobs, as follows. We basically use LDT-heuristic to include released over time type (1) jobs within schedule \( \sigma(2, 3, 4) \): Starting with schedule \( \sigma(2, 3, 4) \), iteratively, the current partial schedule is extended with the next yet unscheduled and already released by the current scheduling time \( t \) job \( i \) with the maximum delivery time, ties being broken by selecting any shortest job. The scheduling time \( t \), at which job \( i \) is scheduled, is iteratively determined as the earliest idle-time moment in the current augmented schedule such that there is yet unscheduled job released by that time; in case job \( i \) overlaps with the following (non-type (1)) job from the current augmented schedule, this, and the the following jobs are right-shifted correspondingly. The order of the type (1) jobs in so generated schedule defines the steady permutation \( \pi^* \). For our sample example, the corresponding LDT-schedule coincides with that from Figure 1, and the resultant steady permutation is \((1, 6, 7)\). Not surprisingly, it yielded an optimal solution to the instance (Fig. 6).

As we showed in Lemma 8, if in a complete LDT-schedule there is a kernel without the delaying emerging job and it contains no type (1) job, then the schedule is optimal. At the same time, we may observe that any kernel \( \pi \) possessing the delaying emerging job must “interact” with a type (1) job: a type (1.1)/(1.3) job either is the delaying emerging job for that kernel or this kernel includes a type (1.2) job (since no kernel in schedule \( \sigma(2, 3, 4) \) possesses the delaying emerging job, see also Lemma 10). While generating a complete schedule for a permutation \( \pi \), a type (1.2) job associated with some kernel is be included within the execution interval of that kernel or immediately after that interval (see Lemma 5). If this makes that permutation inconsistent, then it can be discarded. Let \( e \) be a type (1.1)/(1.3) delaying emerging job for kernel \( K \) such that it becomes part of another newly arisen kernel. Then it is clear that no more complete schedule in which the delaying emerging job of kernel \( K \) has the delivery time equal or larger than \( q_e \) needs to be created. Any permutation yielding such a complete schedule can respectively be discarded.

As it is easily observable, complete schedules respecting two “neighboring” permutations have certain segments in common given that these two schedules have the same kernel \( K \). The common parts are formed by the segments corresponding to any block, different from the block containing kernel \( K \). Unchanged segments can obviously be copied from one schedule to another whereas only the part of the critical block behind the corresponding delaying emerging job needs to be rescheduled. This part can be rescheduled by right-shifting the corresponding jobs, similarly as in the above described procedure for the creation of the first steady permutation.

Our variable parameter algorithm may serve for the construction of approximation algorithms (for example, [23] describes a polynomial time approximation scheme based on the proposed here framework). Our framework can be extended to other scheduling problems: Recall that
the approach is based on the idea of partitioning of the set of jobs into different types. This partitioning, in turn, relies on the basic concepts from Section 2, that are extendable for different machine environments (for example, the notions from Sections 2.1 and 2.2 were introduced for the parallel identical machine environment [19]). Hence a future work may be directed to variable parameter exact and approximation algorithms, including polynomial time approximation schemes for related scheduling problems.

Our algorithmic framework is flexible in the sense that it permits to incorporate different approaches to solve related scheduling problems exactly or approximately. As an example, let us give an intuitive informal analysis that leads to a pseudo-polynomial time exact solution method of our scheduling problem under some conditions. Recall that the worst-case exponential time dependence is due to the factor $\nu!$; roughly, considering all possible permutations of the type (1) jobs, we can generate all possible distributions of these jobs in between the kernels (the critical fragments). In the optimal solution $S_{\text{OPT}}$, the remaining fragments in between the kernels are filled out by the type (1) and type (4) jobs. If the intervals of these fragments are packed in some “compact” way, then the kernel jobs will be pushed by “appropriate” amount of time units by the type (1) jobs. (A compact packing might be unavoidable also because a type (1) job may become a kernel job if it is rescheduled “too late”). Consider now a complete schedule $\sigma(\pi)$ created for permutation $\pi$ of the type (1) jobs and the corresponding two non-critical fragments in that schedule, the first one consisting of the jobs included before the kernel $K = K(S(\sigma(\pi)))$ and the second one consisting of the jobs included after that kernel in schedule $\sigma(\pi)$. If in schedule $\sigma(\pi)$ the first job of kernel $K$ starts at its release time and no type (1) or type (4) job realizes the maximum full job completion time in that schedule, then it is optimal. Otherwise, assume that these fragments consist of only type (1) jobs, i.e., they contain no type (4) jobs. As above specified, in schedule $S_{\text{OPT}}$ the first above non-critical fragment is filled out by the jobs of permutation $\pi$ in some compact way so that the jobs of kernel $K(S_{\text{OPT}})$ are pushed by an appropriate amount of time units. Let $r = \min_{j \in K} r_j$ and assume that all jobs from permutation $\pi$ are released at time 0. It is easy to see that the time interval of the first fragment in solution $S_{\text{OPT}}$ is one of the following time intervals $[0, r], [0, r + 1], [0, r + 2], \ldots, [0, r + \delta(K)]$, i.e., the first job of kernel $K$ is pushed by an integer magnitude $\Delta \in [1, \delta(K)]$. Now it is not difficult to see that a solution of a well-known (weakly NP-hard) SUBSET SUM problem for the items in permutation $\pi$ and with the threshold $r + \Delta$, for each $\Delta = 0, 1, \ldots, \delta(K)$, gives a desired packing of the first and hence also of the second non-critical fragments for that $\Delta$, if such a packing exists. The application of a standard dynamic programming algorithm for SUBSET SUM yields time complexity $O((r + \Delta)\nu)$, for a given $\Delta$. Using binary search we restrict the possible values for $\Delta$ and obtain an overall cost $O((r + \Delta)\nu \log(\delta(K)))$ for a pseudo-polynomial procedure that creates the corresponding schedules and selects one with the minimum makespan.

Finally, there are real-life problems where the parameter $\nu$ is a priory small number. As an example, consider an airline agent (a machine) serving transit passengers. Each passenger (a job) has a well predictable release time and due date dictated by the corresponding flight arrival and departure times. For the airline, it is non-profitable to have passengers that wait too long in the airport (extra expenses, limited space in the airport, etc.). As a consequence, the most of the passengers have tight enough schedules, i.e., their release and due times are close enough to each other. It terms of our scheduling problem, most of the passengers correspond to non-emerging jobs forming the corresponding kernels or are released and scheduled in between the kernels (solitary tight passengers which were served almost at their arrival time since there were not enough urgent passengers at that time waiting to be served). The remaining few passengers
(ones with a considerable difference between their arrival and departure times) form the set of emerging jobs. The number of these passengers is precisely the parameter $\nu$.

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