On the Borel transgression in the fibration

\[ G \to G/T \]

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Abstract

Let \( G \) be a semisimple Lie group with a maximal torus \( T \). We present an explicit formula for the Borel transgression \( \tau : H^1(T) \to H^2(G/T) \) of the fibration \( G \to G/T \). This formula corrects an error in the paper [9], and has been applied to construct the integral cohomology rings of compact Lie groups in the sequel works [4, 7].

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1 Introduction

A Lie group is called semisimple if its center is finite; is called adjoint if its center is trivial. In this paper the Lie groups \( G \) under consideration are compact, connected and semisimple. The homology and cohomology are over the ring of integers, unless otherwise stated.

For a Lie group \( G \) with a maximal torus \( T \) let \( \pi : G \to G/T \) be the quotient fibration. Consider the diagram with top row the cohomology exact sequence of the pair \( (G, T) \)

\[
0 \to H^1(G) \to H^1(T) \xrightarrow{\delta} H^2(G, T) \to \tau \xrightarrow{\cong} H^2(G) \to \cdots
\]

where, since the pair \( (G, T) \) is 1–connected, the induced map \( \pi^* \) is an isomorphism. The Borel transgression [11] p.185 in the fibration \( \pi \) is the composition

\[
\tau = (\pi^*)^{-1} \circ \delta : H^1(T) \to H^2(G/T).
\]

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It is irrelevant to the choice of a maximal torus on $G$ since, if $T'$ is another maximal torus then the relation $T' = qTg^{-1}$ holds on $G$ for some $g \in G$.

The transgression $\tau$ is an essential ingredient of the Leray–Serre spectral sequence $\{E_2^{r,s}(G;\mathcal{R}),d_r \mid r \geq 2\}$ of the fibration $\pi$, where the coefficient ring $\mathcal{R}$ is either the ring $\mathbb{Z}$ of integers, the field $\mathbb{R}$ of reals, or the cyclic ring $\mathbb{Z}_p$ of finite order $p$. Precisely, as the quotient manifold $G/T$ is always 1–connected, the Leray–Serre Theorem states that

\begin{align}
\text{(1.1)} & \quad E_2^{1,*}(G;\mathcal{R}) = H^*(G/T) \otimes H^*(T;\mathcal{R}); \\
\text{(1.2)} & \quad \text{the differential } d_2 : E_2^{1,*} \rightarrow E_2^{1,*}, \text{ is determined by } \tau \text{ as }

d_2(x \otimes t) = (x \cup \tau(t)) \otimes 1,
\end{align}

where $x \in H^*(G/T)$, $t \in H^1(T)$. Furthermore, regarding both $E_2^{1,*}(G;\mathcal{R})$ and $H^*(G;\mathcal{R})$ as graded groups, Leray and Reeder [10, 13] have shown that $E_2^{1,*}(G;\mathcal{R}) = H^*(G;\mathcal{R})$;

Kac [9] claimed that, if $p$ is a prime, then $E_2^{1,*}(G;\mathbb{Z}_p) = H^*(G;\mathbb{Z}_p)$.

Marlin [12] conjectured that, if the group $G$ is 1–connected, then $E_2^{1,*}(G;\mathbb{Z}) = H^*(G)$.

Conceivably, an explicit formula for the transgression $\tau$ is requested by the spectral sequence approach to the cohomology theory of Lie groups.

Our main result is Theorem 2.5, where with respect to explicitly constructed bases on $H^2(G/T)$ and $H^1(T)$ a formula for $\tau$ is obtained. This formula corrects an error concerning the differential $d_2$ on $E_2^{1,*}$ occurring in [9]. It has also been applied in our sequel works [11, 11] to construct the integral cohomology rings of compact Lie groups, see Remark 3.4.

2 A formula for the transgression $\tau$

For a semisimple Lie group $G$ with a maximal torus $T$ the tangent space $L(G)$ (resp. $L(T)$) to $G$ at the group unit $e \in G$ (resp. to $T$ at $e \in T$) is also known as the Lie algebra (resp. the Cartan subalgebra) of $G$. The exponential map $\exp$ of $G$ at the unit $e$ builds up the commutative diagram

$$
\begin{array}{ccc}
L(T) & \rightarrow & L(G) \\
\text{exp} \downarrow & & \downarrow \text{exp} \\
T & \rightarrow & G
\end{array}
$$

where the horizontal maps are the obvious inclusions. Equip the real vector space $L(G)$ with an inner product $(,)$ so that the adjoint representation acts as isometries on $L(G)$, and assume that $n = \dim T$.

The exponential map $\exp : L(T) \rightarrow T$ defines a set $S(G) = \{L_1, \ldots, L_m\}$ of $m = \frac{1}{2}(\dim G - n)$ hyperplanes on $L(T)$, namely, the set of singular hyperplanes through the origin in $L(T)$ [9, p.168]. The map $\exp$ carries the normal line $l_k$ to $L_k$ through the origin $0 \in L(T)$ onto a circle subgroup on $T$. Let $\pm \alpha_k \in l_k$ be the non–zero vectors with minimal length so that
Definition 2.1. The subset $\Delta = \{ \pm \alpha_k \in L(T) \mid 1 \leq k \leq m \}$ will be called the root system of $G$.

In addition, the planes in $S(G)$ divide $L(T)$ into finitely many convex regions, called the Weyl chambers of $G$. Fix a regular point $x_0 \in L(T)$, and let $\mathcal{F}(x_0)$ be the closure of the Weyl chamber containing $x_0$. Assume that $L(x_0) = \{ L_1, \cdots, L_n \} \subset S(G)$ is the subset consisting of the walls of $\mathcal{F}(x_0)$, and let $\alpha_i \in \Phi$ be the root normal to $L_i$ and pointing toward $x_0$.

Remark 2.2. Our geometric approach to the simple roots, as well as the Cartan matrix of $G$, is dual to those that are commonly used in literatures, e.g. [1, 8]. This allows us to perform subsequent construction and calculation coherently on the Euclidean space $L(T)$ without referring to its dual space $L(T)^\ast$. □

On the Euclidean space $L(T)$ there are three distinguished lattices. Firstly, the set $\Delta = \{ \alpha_1, \cdots, \alpha_n \}$ of simple roots generates over the integers the root lattice $\Lambda_r$ of $G$. Next, the pre–image of the exponential map $\exp : L(T) \rightarrow T$ at the group unit $e \in T$ gives rise to the unit lattice $\Lambda_e := \exp^{-1}(e)$ of $G$. Thirdly, using simple roots one defines the set $\Omega = \{ \phi_1, \cdots, \phi_n \} \subset L(T)$ of fundamental dominant weights of $G$ by the formula

$$2(\phi_i, \alpha_j)/\langle \alpha_j, \alpha_j \rangle = \delta_{ij}$$

that generates over the integers the weight lattice $\Lambda_\omega$ of $G$. It is known that (see [3 (3.4)]):

Lemma 2.3. On the Euclidean space $L(T)$ one has

$$\Lambda_r \subseteq \Lambda_e \subseteq \Lambda_\omega.$$  

In addition

i) the group $G$ is 1–connected if and only if $\Lambda_r = \Lambda_e$;

ii) the group $G$ is adjoint if and only if $\Lambda_e = \Lambda_\omega$;

iii) the basis $\Delta$ on $\Lambda_r$ can be expressed by the basis $\Omega$ on $\Lambda_\omega$ by

$$\begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{pmatrix} = A \begin{pmatrix}
\phi_1 \\
\vdots \\
\phi_n
\end{pmatrix},$$

where $A$ is the Cartan matrix of $G$. □

Granted with the notion introduced above we turn to the construction of a canonical basis of the second cohomology $H^2(G/T)$ of the base manifold $G/T$.

For a simple root $\alpha \in \Delta$ let $K(\alpha) \subset G$ be the subgroup with Lie algebra $l_\alpha \oplus L_\alpha$ ([2 p.238, Exercise 6]), where $l_\alpha \subset L(T)$ is the 1–dimensional subspace spanned by $\alpha$, and $L_\alpha \subset L(G)$ is the root space (viewed as an oriented real 2–plane)
belonging to the root $\alpha$ ([8, p.35]). Then the circle subgroup $S^1 = \exp(t_\alpha)$ is a maximal torus on $K(\alpha)$, while the quotient manifold $K_\alpha/S^1$ is diffeomorphic to the 2–dimensional sphere $S^2$. Moreover, the inclusion $(K_\alpha, S^1) \subset (G, T)$ of subgroups induces an embedding

$$(2.1) \quad s_\alpha : S^2 = K_\alpha/S^1 \to G/T$$

whose image is known as the Schubert variety associated to the root $\alpha$ [6]. By the basis theorem of Chevalley [3] the maps $s_\alpha$ with $\alpha \in \Delta$ represent a basis of the second homology $H_2(G/T)$. As a result, if one lets $\omega_i \in H^2(G/T)$ be the Kronecker dual of the homology class represented by the map $s_\alpha_i$, $1 \leq i \leq n$, then one has that

**Lemma 2.4.** The set $\{\omega_1, \cdots, \omega_n\}$ is a basis of the cohomology group $H^2(G/T)$. □

On the other hand let $\Theta = \{\theta_1, \cdots, \theta_n\}$ be an ordered basis of the unit lattice $\Lambda_e$. It defines $n$ oriented circle subgroups on the maximal torus

$$(2.2) \quad \tilde{\theta}_i : S^1 = \mathbb{R}/\mathbb{Z} \to T \text{ by } \tilde{\theta}_i(t) := \exp(t\theta_i), \quad 1 \leq i \leq n,$$

that represent also an ordered basis of the first homology $H_1(T)$. As result if we let $t_i \in H^1(T)$ be the class Kronecker dual to the map $\tilde{\theta}_i$, then

$$(2.3) \quad H^*(T) = \Lambda(t_1, \cdots, t_n) \text{ (i.e. the exterior ring generated by } t_1, \cdots, t_n).$$

In view of the relation $\Lambda_r \subseteq \Lambda_e$ by Lemma 2.3 there exists an unique integer matrix $C(\Theta) = (c_{i,j})_{n \times n}$ expressing the ordered basis $\Delta$ of $\Lambda_r$ by the ordered basis $\Theta$ of $\Lambda_e$. That is, the relation

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = C(\Theta) \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}$$

holds on $L(T)$. Our main result is

**Theorem 2.5.** With respect to the basis $\{t_1, \cdots, t_n\}$ on $H^1(T)$ and and the basis $\{\omega_1, \cdots, \omega_n\}$ on $H^2(G/T)$, the transgression $\tau$ is given by the formula

$$\begin{pmatrix} \tau(t_1) \\ \vdots \\ \tau(t_n) \end{pmatrix} = C(\Theta)^* \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix},$$

where $C(\Theta)^*$ is the transpose of the matrix $C(\Theta)$.

**Proof.** We begin with the simple case where the group $G$ is 1–connected. Then a basis $\Theta$ of the unit lattice $\Lambda_e = \Lambda_r$ can be taken to be $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ by i) of Lemma 2.3. Since $C(\Theta)$ is now the identity matrix we are bound to show that

$$\tau(t_i) = \omega_i, \quad 1 \leq i \leq n.$$
For a simple root \( \alpha_i \in \Delta \) the inclusion \((K(\alpha_i), S^3) \subset (G, T)\) of subgroups induces the following bundle map over \( s_{\alpha_i} \):

\[
\begin{array}{ccc}
S^1 & \to & T \\
\downarrow & & \downarrow \\
K(\alpha_i) = S^3 & \to & G \\
\pi_1 \downarrow & & \pi \\
K(\alpha_i)/S^1 = S^2 & \to & G/T \\
\end{array}
\]

where, since the group \( G \) is 1–connected, the subgroup \( K(\alpha_i) \) is isomorphic to the 3–sphere \( S^3 \) (i.e. the group of unit quaternions), while the map \( \pi_i \) is the Hopf fibration over \( S^2 \). This implies that, in the homotopy exact sequence of the fibration

\[
\cdots \to \pi_2(G) \overset{\partial}{\to} \pi_2(G/T) \overset{\partial}{\to} \pi_1(T) \to \cdots,
\]

the connecting homomorphism \( \partial \) satisfies the relation

(2.5) \( \partial[s_{\alpha_i}] = [\tilde{\alpha}_i], \ 1 \leq i \leq n. \)

Since both of the Hurewicz homomorphisms

\[\pi_1(T) \to H_1(T) \text{ and } \pi_2(G/T) \to H_2(G/T)\]

are isomorphisms, and since the transgression \( \tau \) is Kronecker dual to \( \partial \) in the sense that

\[\tau = \text{Hom}(\partial, 1) : \text{Hom}(\pi_1(T), \mathbb{Z}) \to \text{Hom}(\pi_2(G/T), \mathbb{Z})\]

one obtains from (2.5) that \( \tau(t_i) = \omega_i, \ 1 \leq i \leq n. \)

Turning to a general situation assume that the group \( G \) is semisimple. Let \( d : (G_0, T_0) \to (G, T) \) be the universal cover of \( G \) with \( T_0 \) the maximal torus on \( G_0 \) that corresponds to \( T \) under \( d \). Then, with respect to the canonical identifications (induced by the tangent map of \( d \) at the group unit)

\[L(G_0) = L(G) \text{ and } L(T_0) = L(T),\]

the exponential map of \( G \) admits the decomposition

\[\exp = d \circ \exp_0 : (L(G_0), L(T_0)) \to (G_0, T_0) \to (G, T),\]

where \( \exp_0 \) is the exponential map of \( G_0 \). It follows that, if we let

\[p(\Lambda_r, \Lambda_e) : T_0 = L(T_0)/\Lambda_r \to T = L(T_0)/\Lambda_e\]

be the covering map induced by the inclusion \( \Lambda_r \subseteq \Lambda_e \) of the lattices, then

(2.6) \( d | T_0 = p(\Lambda_r, \Lambda_e) : T_0 \to T. \)

Note that the induced map \( p(\Lambda_r, \Lambda_e)_* \) on \( \pi_1(T_0) \) is determined by the matrix

\[C(\Theta) = (c_{ij})_{n \times n} \text{ as }\]

(2.7) \( p(\Lambda_r, \Lambda_e)_*[\tilde{\alpha}_i] = c_{i,1}[\tilde{\theta}_1] + \cdots + c_{i,n}[\tilde{\theta}_n]. \)
On the other hand, by the naturality of homotopy exact sequence of fibrations, the restriction \( d | T_0 \) fits into the commutative diagram

\[
\begin{array}{ccc}
\pi_2(G_0/T_0) & \xrightarrow{\partial_0} & \pi_1(T_0) \\
\| & & \downarrow (d | T_0)_* \\
\pi_2(G/T) & \xrightarrow{\partial} & \pi_1(T)
\end{array}
\]

where the vertical identification on the left comes from the fact that the covering \( d : (G_0, T_0) \to (G, T) \) induces a diffeomorphism \( G_0/T_0 \cong G/T \), and where \( \partial_0, \partial \) are the connecting homomorphisms in the homotopy exact sequences of the bundles \( G_0 \to G_0/T_0, G \to G/T \), respectively. It follows that, for a simple root \( \alpha_i \in \Delta \), one has

\[
\partial [s_{\alpha_i}] = (d | T_0)_* \circ \partial_0 [s_{\alpha_i}] \quad \text{(by the diagram (2.8))}
\]

\[
= (d | T_0)_* [\tilde{\alpha}_i] \quad \text{(by the proof of the previous case)}
\]

\[
= p(\Lambda_r, \Lambda_e)_*([\tilde{\alpha}_i]) \quad \text{(by (2.6)).}
\]

The proof is now completed by (2.7), together with the fact (again) that the map \( \tau \) is Kronecker dual to \( \partial \). □

3 Applications

In a concrete situation formula (2.4) is ready to apply to evaluate the transgression \( \tau \) (henceforth, the differential \( d_2 \) on \( E^*_2(G) \)) associated to the fibration \( G \to G/T \). We present below three examples.

Assume firstly that the group \( G \) is 1-connected. By i) of Lemma 2.3 one can take the set \( \Delta = \{ \alpha_1, \cdots, \alpha_n \} \) of simple roots as a preferable basis of the unit lattice \( \Lambda_e \). The transition matrix \( C(\Theta) \) is then the identity matrix. Theorem 2.5 implies that

**Corollary 3.1.** If the group \( G \) is 1-connected, there exists a basis \( \{ t_1, \cdots, t_n \} \) on \( H^1(T) \) so that \( \tau(t_i) = \omega_i, 1 \leq i \leq n \). □

Suppose next that the group \( G \) is adjoint. According to ii) of Lemma 2.3 the set \( \Omega = \{ \phi_1, \cdots, \phi_n \} \) of fundamental dominant weights is a basis of \( \Lambda_e \), and the transition matrix \( C(\Theta) \) from \( \Lambda_e \) to \( \Lambda_r \) is the Cartan matrix \( A \) of \( G \) by iii) of Lemma 2.3. We get from Theorem 2.5 that

**Corollary 3.2.** If the group \( G \) is adjoint, there exists a basis \( \{ t_1, \cdots, t_n \} \) on \( H^1(T) \) so that

\[
\begin{pmatrix}
\tau(t_1) \\
\vdots \\
\tau(t_n)
\end{pmatrix} = A^\top
\begin{pmatrix}
\omega_1 \\
\vdots \\
\omega_n
\end{pmatrix},
\]

where \( A \) is the Cartan matrix of the group \( G \). □

Since both of the groups \( H^1(T) \) and \( H^2(G/T) \) are torsion free, formula (2.4) is also applicable to evaluate the transgression.
\[ \tau : H^1(T; \mathbb{Z}_p) \rightarrow H^2(G/T; \mathbb{Z}_p) \]

for the cohomologies with coefficients in the cyclic ring \( \mathbb{Z}_p \) of order \( p > 1 \). Given a Lie group \( G \) denote by \( PG := G/\mathbb{Z}(G) \) the associated Lie group of the adjoint type, where \( \mathbb{Z}(G) \) is the center of \( G \). In what follows we shall assume that \( G \) is one of the 1–connected Lie groups \( Sp(n), E_6 \) or \( E_7 \), and shall adopt the following conventions:

i) a set \( \Delta = \{ \alpha_1, \ldots, \alpha_n \} \) of simple roots of \( G \) is given and ordered as the vertex of the Dykin diagram of \( G \) pictured on [8, p.58];

ii) the preferable basis of the unit lattice \( \Lambda_e \) of the group \( PG \) is taken to be \( \Omega \).

Granted with the Cartan matrix of \( G \) presented on [8, p.59] Corollary 3.2 implies that

**Corollary 3.3.** For each of the pairs \( (G, p) = (Sp(n), 2), (E_6, 3) \) and \( (E_7, 2) \), the kernel and cokernel of the transgression

\[ \tau : H^1(T; \mathbb{Z}_p) \rightarrow H^2(PG/T; \mathbb{Z}_p), \]

are both isomorphic to \( \mathbb{Z}_p \), whose generators are specified in the following table

| \((G, p)\) | Generator of \( \ker \tau = \mathbb{Z}_p \) | Generator of \( \text{coker} \tau = \mathbb{Z}_p \) |
|------------|----------------------------------|----------------------------------|
| \((Sp(n), 2)\) | \( t_n \) | \( \omega_1 \) |
| \((E_6, 3)\) | \( t_1 - t_3 + t_5 - t_6 \) | \( \omega_1 \) |
| \((E_7, 2)\) | \( t_2 + t_5 + t_7 \) | \( \omega_2 \) |

**Remark 3.4.** Let \( G \) be a compact semisimple Lie group and let \( p \) be prime. In [9, formula (4)] Kač stated a formula for the differential \( d_2 \) on \( E^*_2(G; \mathbb{Z}_p) \) which implies that the transgression \( \tau \) in the characteristic \( p \) is always an isomorphism. This contradicts to Corollary 3.3.

On the other hand, in the context of Schubert calculus a unified presentation of the cohomology ring of the quotient manifolds \( G/T \) has been obtained in [6, Theorem 1.2]. Combining this result with Theorem 2.5 of the present paper, method to construct the integral cohomology rings of compact connected Lie groups using the spectral sequence \( \{ E^*_r(G), d_r \} \) has been developed in the works [4, 7].

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