Linear and Non-Linear Stability Analysis for Thermal Convection in A Bidispersive Porous Medium with Thermal Non-Equilibrium Effects

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Abstract
The linear instability and nonlinear stability analyses are performed for the model of bidispersive local thermal non-equilibrium flow. The effect of local thermal non-equilibrium on the onset of convection in a bidispersive porous medium of Darcy type is investigated. The temperatures in the macropores and micropores are allowed to be different. The effects of various interaction parameters on the stability of the system are discussed. In particular, the effects of the porosity modified conductivity ratio parameters, \( \Gamma_p \) and \( \Gamma_s \), with the inter-phase momentum transfer parameters \( \mu_1 \) and \( \mu_2 \), on the onset of thermal Convection are also considered. Furthermore, the nonlinear stability boundary is found to be below the linear instability threshold. The numerical results are presented for free-free boundary conditions.

Keywords: Bidispersive porous medium, Local thermal non-equilibrium, Linear instability, Nonlinear stability, Darcy model.
therein. The fundamental model for thermal convection in a bidispersive porous medium is due to Nield and Kuznetsov [6-11]. Furthermore important work discussed the problem of thermal convection in a bidispersive porous medium is done by Gentile and Straughan in [12,13]. This has also been considered further using a variety of geometries and incorporating various other effects, for example, anisotropic parameter and double diffusion cf. Straughan [14-17], and Saleh and Haddad [18]. The papers by Franchi et al. [19] present the effect of local thermal non-equilibrium on a bidispersive porous medium. These authors work on development a theory of double porosity material, where the solid skeleton, and the fluid in the micro and macropores may have different temperatures. It is worth to mention that the local thermal non-equilibrium have recently raised much interesting subject. See for example Malashetty et al. [20] investigated the effect of thermal non-equilibrium on the onset of convection when the Lapwood–Brinkman model is included for the momentum equation. The same authors in [21] studied the problem of Onset of convection in an anisotropic porous layer using thermal non-equilibrium. Shivakumara et al. [22] considered the effects of the onset of thermal non-equilibrium convection in a viscoelastic fluid saturate sparsely packed porous layer. Straughan [23] analyzed the linear instability and nonlinear stability boundaries using thermal non-equilibrium model. Malashetty et al. [24] discussed the effect of rotation on thermal convection in a fluid-saturated porous layer with thermal nonequilibrium model. Malashetty and Heera [25] considered the rotation and that the local thermal non-equilibrium effect on double diffusive convection in porous media. Malashetty et al. [26] considered the problem of double diffusive thermal convection in a fluid-saturated porous layer using a thermal non-equilibrium model, while Malashetty et al. [27] investigated thermal non-equilibrium effect on the onset of convection in a couple stress fluid saturated porous medium. Furthermore, Gaikwad et al. [28] explained the combined effect of rotation and local thermal non-equilibrium on the double diffusive convection using linear instability and nonlinear stability methods.

Other aspect of the effect of thermal non-equilibrium on the onset of convection porous layer involving effects like rotation, variable viscosity, density maximum, heterogeneous permeability, and viscous dissipation have been studied by many researchers, e.g., Malashetty and Swamy [29], Shivakumara et al. [30, 31], and Barletta and Celli [32]. Further work using thermal non-equilibrium model has been given by Straughan [5,33], Dayananda and Shivakumara [34], and Santamaria-Holek [35].

In the present paper, we provide an accurate numerical calculation for linear instability and nonlinear stability. In particular, we investigate in details the thermal convection in a bidisperse porous medium where two different temperature fields for the porous solid and for the saturating fluid are assumed in order to model the local thermal non-equilibrium. Moreover, we focus our attention on the effect of different interaction parameters on the critical Rayleigh number. We take this opportunity to point out that the effects of various parameters such as couple stress, uniform magnetic field, permeability of the medium, concentration and the thermo-diffusion, variable viscosity, the velocity and temperature, thermal anisotropy and rotating, are investigated in general by many authors cf. Al-Hhafajy and Abdulhadi [36], Al-Khafajy and Labban [37], Kareem and Abdulhadi [38], Khudair and Al-Khafajy [39], and Haddad [40,41].

The paper is organized as follows: In section 2 the basic equations and the nondimensionalized perturbation equations are presented. The linear stability and the nonlinear stability analysis are the subject of sections 3 and 4. In section 5 the numerical results are reported.

**Governing Equations**

The equations for a bidisperse local thermal non-equilibrium flow in a Darcy porous material are derived in Franchi et al. [19], with $U_\lambda$ and $U_p$ being the pore-averaged
velocities in the macro and micropores, $p^f$ and $p^p$ are the pressures in the macro and micropores, $\mu$ is the dynamic viscosity, $g_i$ is the gravity vector, $\rho$ is the density, and $k_f, k_p$ are the permeability in the macro and micropores, and $\zeta$ is an interaction coefficient. We suppose that the saturated porous medium is occupying the three dimensional layer $\{(x, y) | 0 < z < d \}$ with the temperatures $T^s = T^f = T^p = T_L$, at $z = 0$, and $T^s = T^f = T^p = T_U$, at $z = d$, where $T_L, T_U$ are constants with $T_L > T_U$. The governing equations consist of the momentum and Continuity equations and adopting the Boussinesq approximation in the macro and micropores of Darcy type may be found in Nield and Kuznetsov [8], Straughan [5], and Franchi et al. [19].

$$
\frac{-\mu}{k_f} U^f_i - \zeta (U^f_i - U^p_i) - p^f_i - \frac{\rho C_p (1 - \varphi) \varepsilon}{D} T^f_i = 0,
$$
$$
U^f_{i,j} = 0,
$$
$$
\frac{-\mu}{k_p} U^p_i - \zeta (U^p_i - U^f_i) - p^p_i - \frac{\rho C_p (1 - \varphi) \varepsilon}{D} T^p_i = 0,
$$
$$
U^p_{i,j} = 0,
$$

where $D = \varphi + (1 - \varphi) \varepsilon$, $T^f_i$, $T^p_i$, and $T^s_i$, respectively, are reference values of temperature in the macro pore, micropores, and solid skeleton. $\phi$ is the macroporosity, $\varepsilon$ is the microporosity. Standard indicial notation is employed in (1) and throughout. We let $U^f_i = \varphi V^f_i$ and $U^p_i = (1 - \varphi) \varepsilon V^p_i$ to be the fluid velocities in the macro and micropores, $V^f_i$ and $V^p_i$ are the pore average velocities in the macro and micropores. The balance of energy equations for the temperature is established as in Franchi et al. [19], and can be written as

$$
\varepsilon_i (\rho c)_s T^s_i = \varepsilon_i k_s \Delta T^f_i + s_1 (T^f_i - T^s_i) + s_2 (T^p_i - T^s_i),
$$
$$
\varphi (\rho c)_f V^f_i T^f_i = \varphi k_f \Delta T^f_i + h(T^p_i - T^f_i) + s_1 (T^f_i - T^f_i) + s_2 (T^f_i - T^f_i),
$$
$$
\varepsilon_2 (\rho c)_p V^p_i T^p_i = \varepsilon_2 k_p \Delta T^p_i + h(T^f_i - T^p_i) + s_1 (T^f_i - T^p_i),
$$

where $\varepsilon_1 = (1 - \varepsilon)(1 - \varphi)$, $\varepsilon_2 = (1 - \varphi) \varepsilon$. Here $(\rho c)_s, (\rho c)_f$, $(\rho c)_p$ are the products of the density and the specific heat at constant pressure in the solid in the fluid in the macro pores, and in the fluid in the micropores, respectively. The terms $k_s, k_f$, and $k_p$ are thermal conductivities in the solid, and in the fluid in the macro and micropores, respectively. We denote by $s_1, f$, and $p$ the solid, the macro pores, and the micropores the terms $h$, $s_1$, and $s_2$ are interaction coefficients, and we have here assumed that the interactions are linear in the temperature differences.

We suppose that the fluid saturated bidispersive porous medium satisfies equation (1) and equation (2). The velocity boundary conditions are $U^f_i n_i = 0, U^p_i n_i = 0$ at $z = 0, d$. The steady solution in whose stability we are interested has form

$$
\bar{U}^f_i = 0, \bar{U}^p_i = 0, \bar{T}^f_i = \bar{T}^p_i = -\beta z + T_L = 0,
$$

where

$$
\beta = \frac{T_L - T_U}{d}.
$$
To investigate the stability of the steady solution, we introduce perturbation \( \left( u^f, u^p, \pi^f, \pi^p, \Theta^f, \Theta^p \right) \) in such a way that

\[
U^f_i = \bar{U}^f_i + u^f_i, \quad U^p_i = \bar{U}^p_i + u^p_i, \quad p^f = \bar{p}^f + \pi^f, \quad p^p = \bar{p}^p + \pi^p,
\]

\[
T^s = \bar{T}^s + \Theta^s, \quad T^f = \bar{T}^f + \Theta^f, \quad T^p = \bar{T}^p + \Theta^p.
\]

The nonlinear perturbation equations arising from equation (1) and equation (2) are

\[
\frac{-\mu}{k_f} u^f_i - \zeta (u^f_i - u^p_i) - \pi^p_f = -g k_f \rho_o \alpha \phi \Theta^f - g k_f \rho_o \alpha (1 - \phi) \epsilon \Theta^p = 0,
\]

\[
u^f_i = 0,
\]

\[
\frac{-\mu}{k_p} u^p_i - \zeta (u^p_i - u^f_i) - \pi^f_p = -g k_p \rho_o \alpha \phi \Theta^p - g k_p \rho_o \alpha (1 - \phi) \epsilon \Theta^f = 0,
\]

\[
u^p_i = 0,
\]

\[
\epsilon_i (\rho c), \Theta^f = \epsilon_i k_f \Delta \Theta^f + s_i (\Theta^f - \Theta^p) + s_s (\Theta^p - \Theta^s),
\]

\[
\phi (\rho c) \phi^f + (\rho c) u^f_i \Theta^f - (\rho c) \beta \omega^f = \phi k_f \Delta \Theta^f + h (\Theta^p - \Theta^f) + s_i (\Theta^f - \Theta^f),
\]

\[
\epsilon_x (\rho c) \phi^p + (\rho c) u^p_i \Theta^p - (\rho c) \beta \omega^p = \epsilon_x k_p \Delta \Theta^p + h (\Theta^f - \Theta^p) + s_s (\Theta^p - \Theta^p),
\]

where \( w^f = u^f_j \) and \( w^p = u^p_j \), while \( k_i = (0,0,1) \). We then introduce the nondimensionalizations scale with length scale \( d \), time scale \( \tau \), pressure \( p \), velocity \( U \), and temperature scale

\[
\tau = \frac{d^2 (\rho c)}{k_f}, \quad p = d \zeta U, \quad U = \frac{k_f}{(\rho c) d^2}, \quad T^* = u \sqrt{\frac{(\rho c) d^2 \beta \zeta}{g \rho_o \alpha \phi k_f}},
\]

and the Rayleigh number \( R_a = R^2 \) is given by

\[
R^2 = \frac{(\rho c) d^2 \beta g \rho_o \alpha}{\zeta \phi k_f}.
\]

Other nondimensional variables are required and these are given by

\[
\mu_1 = \frac{\mu}{k_f}, \quad \mu_2 = \frac{\mu}{k_p}, \quad L_s = \frac{(\rho c) s}{(\rho c) f}, \quad \xi = L_s \frac{\epsilon_i}{\phi}, \quad \xi_p = \frac{\epsilon_x}{\phi k_f}, \quad S_1 = \frac{s_i d^2}{\phi k_f}, \quad S_2 = \frac{s_s d^2}{\phi k_f},
\]

\[
H = \frac{h d^2}{\phi k_f}, \quad \Gamma_s = \frac{\epsilon_i k_f}{\phi k_f}, \quad \Gamma_p = \frac{\epsilon_x k_p}{\phi k_f}, \quad L_p = \frac{(\rho c) p}{(\rho c) f}, \quad \xi = L_p \frac{\epsilon_x}{\phi},
\]

The nondimensional equations which is Achieved from equation (3) are

\[
\mu_i u^f_i + (u^f_i - u^p_i) = -\pi^f_i + R \frac{\phi k_i}{D} \theta^f + R \frac{(1 - \phi) \epsilon k_i}{D} \theta^p, \quad u^f_i = 0,
\]

\[
\mu_i u^p_i + (u^p_i - u^f_i) = -\pi^p_i + R \frac{\phi k_i}{D} \theta^p + R \frac{(1 - \phi) \epsilon k_i}{D} \theta^f, \quad u^p_i = 0,
\]

\[
\xi \Theta^f = \Gamma_s \Delta \Theta^f + S_1 (\Theta^f - \Theta^p) + S_s (\Theta^p - \Theta^s),
\]

\[
\theta^p = \frac{1}{\phi} u^f_i \theta^f = R w^f + \Delta \theta^f + H (\theta^p - \theta^f) + S_1 (\theta^f - \theta^f),
\]

\[
\xi \Theta^p = \frac{L_p}{\phi} u^p_i \theta^p = R L_p w^p + L_p \Delta \Theta^p + H (\theta^p - \theta^f) + S_s (\theta^p - \Theta^p).
\]
Equation (4) hold on \(\{(x, y) \in \mathbb{R}^2 \mid 0 < z < d\}\) and the boundary conditions

\[
w^f = u^f = 0, \quad w^p = u^p = 0, \quad \phi^f = 0, \quad \phi^p = 0, \quad \theta^f = 0, \quad \theta^p = 0, \quad \text{at} \quad z = 0, 1,
\]

with \(u^f = (u^f, v^f, w^f)\) and \(u^p = (u^p, v^p, w^p)\) are satisfying on plane tiling periodicity in \((x, y)\).

**Solution of Linear Stability Equations**

In this section, we seek to find the critical Rayleigh number of linear theory, we first linearize equation (3) then we write the variables \(u^f, \pi^f, \pi^p, \theta^f, \theta^p\), and \(\phi^p\) by explicitly separating the time dependent parts and we follow the work of Chandrasekhar [42] by imposing a temporal growth rate like \(\epsilon^{\sigma}t\) for solutions of the form

\[
u^f = u^f(x)e^{\sigma t}, \quad u^p = u^p(x)e^{\sigma t}, \quad \pi^f = \pi^f(x)e^{\sigma t}, \quad \pi^p = \pi^p(x)e^{\sigma t}, \quad \theta^f = \theta^f(x)e^{\sigma t}, \quad \theta^p = \theta^p(x)e^{\sigma t}.
\]

Then \(\pi^f\) and \(\pi^p\) are eliminated by taking curl of equation (4)1 and equation (4)2, and then analysis the linear system. The linear system arising from equation (4) is

\[
\begin{align*}
\mu_1 \Delta w^f + (\Delta w^f - \Delta w^p) - R \frac{\partial \phi}{\partial z} \Delta^* \theta^f - R \frac{(1-\phi)}{D} \Delta^* \theta^p &= 0, \\
\mu_2 \Delta w^p + (\Delta w^p - \Delta w^f) - R \frac{\partial \phi}{\partial z} \Delta^* \theta^f - R \frac{(1-\phi)}{D} \Delta^* \theta^p &= 0, \\
\sigma^2 \theta^f &= \Gamma f \Delta \theta^f + S_1 (\theta^f - \theta^f) + S_2 (\phi^p - \phi^f), \\
\sigma^2 \theta^p &= \Gamma p \Delta \theta^p + H (\theta^p - \theta^f) + S_1 (\theta^f - \theta^f) + S_2 (\theta^p - \theta^f),
\end{align*}
\]

where \(\Delta^* = \nabla^2 / \partial x^2 + \nabla^2 / \partial y^2\) is the horizontal Laplacian. This is an eigenvalue problem for \(\sigma\) to be solved subject to the boundary conditions in equation (5).

To analyze (6) and (5), assume normal mode with the representations for \(w^f, \quad w^p, \quad \theta^f, \quad \theta^p\), and \(\phi^p\) in the form of

\[
\begin{align*}
\theta^f &= \Theta^f(z) f(x, y), \quad \phi^f = \Theta^f(z) f(x, y), \quad \theta^p = \Theta^p(z) f(x, y), \\
w^f &= W^f(z) f(x, y), \quad w^p = W^p(z) f(x, y),
\end{align*}
\]

where \(f\) is the horizontal plan form, which satisfies \(\Delta^* f = -a^2 f\), \(a\) is a wave number. Then, we allow \(W^f, \quad W^p, \quad \Theta^f, \Theta^p\) and \(\phi^p\) to be composed of \(\sin n\pi z\), for \(n \in \mathbb{N}\) which satisfies the boundary conditions (5). The system (6) can be written in the matrix form.
where \( \Lambda = n^2\pi^2 + a^2 \). Then, one may consider the following two cases.

3.1 Stationary Convection (\( \sigma = 0 \)).

Substituting \( \sigma = 0 \) in (6). The stationary convection boundary is given by

\[
R^2 = \frac{DAk_A + k_B A_2}{a^2[\varphi(k_2 B_3 - k_1 B_1) + (1 - \varphi)\varepsilon(k_1 B_2 - k_2 B_4)]},
\]

(8)

With the coefficients \( k_0, k_1, k_2, A_1, A_2, B_1, B_2, B_3, \) and \( B_4 \) are given by

\[
k_0 = 1 - (\mu_i + 1)(\mu_j + 1),
\]

\[
k_1 = 1 + (\mu_i + 1),
\]

\[
k_2 = 1 + (\mu_i + 1),
\]

\[
A_i = 2S_1S_2H + S_1^2(\Lambda + H + S_1) + H^2(\Gamma_s \Lambda + S_1 + S_2),
\]

\[
A_2 = (\Gamma_s \Lambda + H + S_2)(S_1^2 - (\Lambda + H + S_1)(\Gamma_s \Lambda + S_1 + S_2)),
\]

\[
B_1 = S_1S_2L_p + L_p H(\Gamma_s \Lambda + S_1 + S_2),
\]

\[
B_2 = S_2^2L_p - L_p (\Lambda + H + S_1)(\Gamma_s \Lambda + S_1 + S_2),
\]

\[
B_3 = (S_2^2 - (\Gamma_s \Lambda + S_1 + S_2)(\Gamma_s \Lambda + H + S_2),
\]

\[
B_4 = S_1S_2 + H(\Gamma_s \Lambda + S_1 + S_2).
\]

Then one can show \( \partial R^2 / \partial n^2 > 0 \). Therefore, we select \( n = 1 \) to obtain the lowest instability boundary.

3.2 Oscillatory Convection (\( \sigma = i\sigma_i \), where \( \sigma_i \in \mathbb{R} \)).

To study oscillatory convection put \( \sigma = i\sigma_i \) in equation (6), where \( \sigma_i \in \mathbb{R} \). We solve the determinant equation, then we have

\[
R^2 = \frac{DA[k_0(h_1 + h_2 + h_3)]}{-a^2[k_1 L_p \xi_p (1 - \varphi)\varepsilon + k_2 \phi \xi_p \varepsilon]}.
\]

(9)

Here, the coefficients \( h_1, h_2, \) and \( h_3 \) are defined by

\[
h_1 = \xi_s(\Gamma_s \Lambda + H + S_2),
\]

\[
h_2 = \xi_s(\Gamma_s \Lambda + S_1 + S_2),
\]

\[
h_3 = \xi_s \xi_p(\Lambda + H + S_1).
\]
Numerical techniques are used to find the stationary convection threshold $Ra_{sta}$ and oscillatory convection threshold $Ra_{ost}$, respectively. We minimize $R^2$ in (8) and (9) over $a^2$, as will be presented in the numerical results section.

**Nonlinear Stability Analysis**

Let $V$ be a period cell for the solution to (4) and (5), and let $\| \cdot \|$, $(\cdot,\cdot)$ be the norm and inner product on $L^2(V)$. We commence by multiplying (4)1 by $u_i^t$, (4)2 by $u_i^p$. We also multiply (4)3 by $\theta^t$, (4)4 by $\theta^t$, (4)5 by $\theta^p$ and integrate each over $V$. Thus, we derive the following equations:

\[
\mu_1 \| u^t \|^2 + \| u^t \|^2 - (u^p, u^t) - R \frac{\partial \varphi}{D} (\theta^t, w^t) - R \frac{(1-\varphi)E}{D} (\theta^p, w^t) = 0
\]

\[
\mu_2 \| u^t \|^2 + \| \theta^t \|^2 - (u^p, u^t) - R \frac{\partial \varphi}{D} (\theta^t, w^t) - R \frac{(1-\varphi)E}{D} (\theta^p, w^p) = 0
\]

\[
\frac{1}{2} \frac{d}{dt} \xi_t \| \theta^t \|^2 = -\Gamma \| \nabla \theta^t \|^2 + S_1 (\theta^t, \theta^t) - S_2 (\theta^p, \theta^t) - S_2 \| \theta^t \|^2
\]

\[
\frac{1}{2} \frac{d}{dt} \| \theta^t \|^2 = -\Gamma \| \nabla \theta^t \|^2 + R (w^t, \theta^t) + H (\theta^p, \theta^t) - H \| \theta^t \|^2 + S_2 (\theta^t, \theta^t) - S_2 \| \theta^t \|^2
\]

\[
\frac{1}{2} \frac{d}{dt} \xi_p \| \theta^p \|^2 = -\Gamma \| \nabla \theta^p \|^2 + R L_p (w^p, \theta^t) + H (\theta^p, \theta^t) - H \| \theta^p \|^2 + S_2 (\theta^t, \theta^p) - S_2 \| \theta^p \|^2 .
\]

To determine the parameter $\lambda > 0$ we now form $\lambda(10)_3 + \lambda(10)_4 + \lambda(10)_5$, to obtain an energy identity of form

\[
\frac{dE}{dt} = RI - D
\]

where

\[
E(t) = \frac{\lambda}{2} \left[ \xi_t \| \theta^t \|^2 + \| \theta^t \|^2 + \xi_p \| \theta^p \|^2 \right] .
\]

\[
I = \left( \sqrt{\Lambda} + \frac{\varphi}{D \sqrt{\Lambda}} \right) (w^t, \theta^t) + \left( \sqrt{\Lambda} L_p + \frac{(1-\varphi)E}{D \sqrt{\Lambda}} \right) (w^t, \theta^t) + \frac{\varphi}{D \sqrt{\Lambda}} (\theta^t, w^t) + \frac{(1-\varphi)E}{D \sqrt{\Lambda}} (\theta^p, w^t) ,
\]

\[
D = \Gamma \| \nabla \theta^t \|^2 + \| \nabla \theta^t \|^2 + \Gamma \| \nabla \theta^p \|^2 + \mu_1 \| u^t \|^2 + \mu_2 \| u^t \|^2 + \| u^t - u^p \|^2 + S_1 \| \theta^t - \theta^t \|^2.
\]

Equation (11) is the same as the expression was report by Straughan [5].

Now put

\[
\frac{1}{R^2_e} = \max_{\mathcal{E}} \frac{I}{D} ,
\]

where $\mathcal{E}$ is the space of admissible functions which are given by
\( \mathbf{S} = (u^f, u^p, \theta^e, \theta^f, \theta^p) \mid u^f, u^p \in L^2(V), \theta^e, \theta^f, \theta^p \in \mathcal{N}^1(V) \),
\( \nabla u^f = 0, \nabla u^p = 0, u^f, u^p, \theta^e, \theta^f, \theta^p \), are periodic in \( x, y \),
and \( w^f = w^p = \theta^e = \theta^f = \theta^p = 0 \), on \( z = 0,1 \).

Then from (12), we deduce
\[
\frac{dE}{dt} \leq -D(1 - \frac{R}{R_E}).
\]
and then, from the Poincaré’s inequality on \( D \), we have
\[
\frac{dE}{dt} \leq -c'\pi^2 \| \theta \|^2,
\]
which integrates to
\[
E(t) \leq E(0)e^{-ct}.
\]
Thus, \( E(t) \) tends to 0 as \( t \to \infty \) at least exponentially. Therefore, \( \| \theta^f \| \to 0, \| \theta^p \| \to 0 \) and
\( \| \theta^e \| \to 0 \) at least exponential.

To obtain the decay of \( \| u^f \| \) and \( \| u^p \| \), we have to employ the arithmetic-geometric mean inequality in (10) and (10)
\[
(1 + \mu_1)\| u^f \|^2 \leq \frac{R\varphi}{2D\alpha_1} \| \theta^f \|^2 + \frac{R\varphi\alpha_1}{2D} \| w^p \|^2 + \frac{R(1 - \varphi)e\beta_1}{2D} \| \theta^p \|^2,
\]
\[
(1 + \mu_2)\| u^p \|^2 \leq \frac{R\varphi}{2D\alpha_2} \| \theta^p \|^2 + \frac{R\varphi\alpha_2}{2D} \| w^f \|^2 + \frac{R(1 - \varphi)e\beta_2}{2D\beta_2} \| \theta^f \|^2 + \frac{R(1 - \varphi)e\beta_2}{2D\beta_2} \| \theta^p \|^2.
\]
\[
\frac{1}{2} \left\{ (1 + \mu_1)\| u^f \|^2 + (1 + \mu_2)\| u^p \|^2 \right\} \leq \left\{ \frac{1}{(1 + \mu_1)} + \frac{1}{(1 + \mu_2)} \right\} \frac{R^2}{D^2} \left( \varphi^2 \| \theta^f \|^2 + (1 - \varphi)^2 e^2 \| \theta^p \|^2 \right).
\]

We let
\[
\alpha_1 = \frac{(1 + \mu_1)D}{2R\varphi}, \quad \alpha_2 = \frac{(1 + \mu_2)D}{2R\varphi}, \quad \beta_1 = \frac{2R(1 - \varphi)e}{D(1 + \mu_1)}, \quad \beta_2 = \frac{2R(1 - \varphi)e}{D(1 + \mu_2)},
\]
\[
\frac{1}{2} \left\{ (1 + \mu_1)\| u^f \|^2 + (1 + \mu_2)\| u^p \|^2 \right\} \leq \left\{ \frac{1}{(1 + \mu_1)} + \frac{1}{(1 + \mu_2)} \right\} \frac{R^2}{D^2} \left( \varphi^2 \| \theta^f \|^2 + (1 - \varphi)^2 e^2 \| \theta^p \|^2 \right),
\]

where the decay of \( u^f \) and \( u^p \) follow. We now turn our attention to the maximization problem
(12) with \( R_E > 1 \).

In order to determine \( R_E \), we have to derive the Euler–Lagrange equations and to maximize in the coupling parameter \( \lambda \). The Euler–Lagrange equations arising from equation (12) are determined from
\( R_E \delta I - \delta D = 0 \).

Let us define \( u^f, u^p, \theta^f, \theta^p, \), and \( \theta^e \) in terms of arbitrary \( C^2(0,1) \) functions \( h^f, h^p, \eta^f, \eta^p, \)
and \( \eta^e \) with
\[
h^f(0) = h^f(1) = h^p(0) = h^p(1) = 0 \quad \text{and} \quad \eta^f(0) = \eta^f(1) = \eta^p(0) = \eta^p(1) = \eta^e(0) = \eta^e(1) = 0.
\]
Hence we have that

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\[
\delta D = \int_{\gamma} \left[ (\Gamma_\sigma \nabla (\theta' + \epsilon \eta'))^2 + \nabla (\theta' + \epsilon \eta')^2 + \Gamma_\sigma \nabla (\theta^\rho + \epsilon \eta^\rho)^2 + \mu_1 (u_1' + \epsilon h_1')^2 + \mu_2 (u_2' + \epsilon h_2')^2 \\
+ (u_1' + \epsilon h_1' - (u_1^\rho + \epsilon h_1^\rho))^2 + S_1 (\theta' + \epsilon \eta' - (\theta' + \epsilon \eta'))^2 + S_2 (\theta^\rho + \epsilon \eta^\rho - (\theta' + \epsilon \eta'))^2 \\
+ H(\theta^\rho + \epsilon \eta^\rho - (\theta' + \epsilon \eta'))^2 \right] dV|_{\gamma=0},
\]

\[
\delta D = \int_{\gamma} \left[ 2\Gamma_\sigma \nabla (\theta' + \epsilon \eta') \nabla \eta' + 2\nabla (\theta' + \epsilon \eta') \nabla \eta' + 2\Gamma_\sigma \nabla (\theta^\rho + \epsilon \eta^\rho) \nabla \eta^\rho + 2\mu_1 (u_1' + \epsilon h_1') h_i^\rho \\
+ 2\mu_2 (u_2' + \epsilon h_2') h_i^\rho + 2(u_1' + \epsilon h_1') (h_i^\rho - h_i^\rho) + 2S_1 (\theta' + \epsilon \eta' - (\theta' + \epsilon \eta')) (\eta^\rho - \eta') \\
+ 2S_2 (\theta^\rho + \epsilon \eta^\rho - (\theta' + \epsilon \eta')) + 2H(\theta^\rho + \epsilon \eta^\rho - (\theta' + \epsilon \eta')) (\eta^\rho - \eta') \right] dV|_{\gamma=0},
\]

and

\[
\delta I = \int_{\gamma} \left[ \frac{\phi}{D\sqrt{\lambda}} (w' + \epsilon h_i') \eta' + (\theta' + \epsilon \eta') h_i^\rho + \frac{\phi (1-\phi)\epsilon}{D\sqrt{\lambda}} (w^\rho + \epsilon h_2^\rho) (w' + \epsilon h_i') (u_1' + \epsilon h_1') \pi^\rho (x) - (u_2' + \epsilon h_2') \pi^\rho (x) \right] dV|_{\gamma=0},
\]

we have included the constraint \( u_1' \) and \( u_2' \) by using a Lagrange multiplier \( \pi^\rho (x) \), and \( \epsilon \) is a positive constant. Furthermore, after some integrations by parts and using the boundary conditions, we find that

\[
\delta D = \int_{\gamma} \left[ 2\Gamma_\sigma \nabla \eta' + \nabla \theta' \nabla \eta' + \Gamma_\sigma \nabla \theta^\rho \nabla \eta^\rho + \mu_1 (u_1' + \epsilon h_1') + \mu_2 (u_2' + \epsilon h_2') \\
+ (u_1' - u_1^\rho) (h_i^\rho - h_i^\rho) + S_1 (\theta' - \theta^\rho) (\eta^\rho - \eta') \\
+ S_2 (\theta^\rho - \theta^\rho) (\eta^\rho - \eta') + H(\theta^\rho - \theta^\rho) (\eta^\rho - \eta') \right] dV,
\]

\[
\delta I = \int_{\gamma} \left[ \frac{\phi}{D\sqrt{\lambda}} (w' \eta' + \theta' h_i^\rho) + \frac{\phi (1-\phi)\epsilon}{D\sqrt{\lambda}} (w^\rho \eta^\rho + \theta^\rho h_2^\rho) + \frac{\phi}{D\sqrt{\lambda}} (\theta^\rho h_2^\rho + w^\rho \eta^\rho) \\
+ \frac{(1-\phi)\epsilon}{D\sqrt{\lambda}} (\theta^\rho h_i^\rho + w^\rho \eta^\rho) - h_i^\rho \pi^\rho - h_2^\rho \pi^\rho \right] dV.
\]
Then the Euler–Lagrange equations for the maximum problem (12) are
\[
\mu_i u_i^f + (u_i^f - u_i^p) - R_e (\frac{\sqrt{\lambda}}{2} + \frac{\phi}{2D \sqrt{\lambda}}) \theta^f k_i - R_e (\frac{(1-\phi)\varepsilon}{2D \sqrt{\lambda}}) \theta^p k_i = \pi_i^f,
\]
\[
\mu_p u_p^f + (u_p^f - u_p^p) - R_e \frac{\phi}{2D \sqrt{\lambda}} \theta^f k_i - R_e (L_p \frac{\sqrt{\lambda}}{2} + \frac{(1-\phi)\varepsilon}{2D \sqrt{\lambda}}) \theta^p k_i = \pi_i^p,
\]
\[
\Gamma_s \Delta \theta^f + S_1 (\theta^f - \theta^p) + S_2 (\theta^p - \theta^f) = 0,
\]
(14)
\[
R_e (\frac{\sqrt{\lambda}}{2} + \frac{\phi}{2D \sqrt{\lambda}}) w^f + R_e \frac{\phi}{2D \sqrt{\lambda}} w^p + \Delta \theta^f + S_1 (\theta^f - \theta^p) + H (\theta^p - \theta^f) = 0,
\]
\[
R_e (\frac{(1-\phi)\varepsilon}{2D \sqrt{\lambda}}) w^f + R_e (L_p \frac{\sqrt{\lambda}}{2} + \frac{(1-\phi)\varepsilon}{2D \sqrt{\lambda}}) w^p + \Gamma_p \Delta \theta^p + S_2 (\theta^p - \theta^f) + H (\theta^f - \theta^p) = 0,
\]
where \( \pi^f \) and \( \pi^p \) are Lagrange multipliers. Further progress is made by taking curl of equation (14)_1 and equation (14)_2, we obtain
\[
\mu_i \Delta w^f + (\Delta w^f - \Delta w^p) - R_e (\frac{\sqrt{\lambda}}{2} + \frac{\phi}{2D \sqrt{\lambda}}) \Delta^* \theta^f - R_e (\frac{(1-\phi)\varepsilon}{2D \sqrt{\lambda}}) \Delta^* \theta^p = 0,
\]
\[
\mu_p \Delta w^p + (\Delta w^p - \Delta w^f) - R_e \frac{\phi}{2D \sqrt{\lambda}} \Delta^* \theta^f - R_e (L_p \frac{\sqrt{\lambda}}{2} + \frac{(1-\phi)\varepsilon}{2D \sqrt{\lambda}}) \Delta^* \theta^p = 0,
\]
\[
\Gamma_s \Delta \theta^f + S_1 (\theta^f - \theta^p) + S_2 (\theta^p - \theta^f) = 0,
\]
\[
R_e (\frac{\sqrt{\lambda}}{2} + \frac{\phi}{2D \sqrt{\lambda}}) w^f + R_e \frac{\phi}{2D \sqrt{\lambda}} w^p + \Delta \theta^f + S_1 (\theta^f - \theta^p) + H (\theta^p - \theta^f) = 0,
\]
\[
R_e (\frac{(1-\phi)\varepsilon}{2D \sqrt{\lambda}}) w^f + R_e (L_p \frac{\sqrt{\lambda}}{2} + \frac{(1-\phi)\varepsilon}{2D \sqrt{\lambda}}) w^p + \Gamma_p \Delta \theta^p + S_2 (\theta^p - \theta^f) + H (\theta^f - \theta^p) = 0.
\]
(15)

We again use a normal mode representation as for the linear stability analysis in section 3. These results are to solve the determinant equation
\[
\begin{vmatrix}
-(\mu_1 + 1)\Lambda & \Lambda & R_e (\frac{\sqrt{\lambda}}{2} + c) a^2 & R_e b a^2 & 0 \\
\Lambda & -(\mu_1 + 1)\Lambda & R_e c a^2 & R_e (L_p \frac{\sqrt{\lambda}}{2} + b) a^2 & 0 \\
0 & 0 & S_1 & S_2 & -(\Gamma_s \Lambda + S_1 + S_2) \\
R_e (\frac{\sqrt{\lambda}}{2} + c) & R_e c & -(\Lambda + S_1 + H) & H & S_1 \\
R_e b & R_e (L_p \frac{\sqrt{\lambda}}{2} + b) & H & -(\Gamma_p \Lambda + S_2 + H) & S_2
\end{vmatrix} = \begin{bmatrix} W^f \\ W^p \\ \Theta^f \\ \Theta^p \\ \Theta^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]
(16)
where \( c = \frac{\varphi}{2D\sqrt{\lambda}} \) and \( b = \frac{(1-\varphi)\varepsilon}{2D\sqrt{\lambda}} \). The corresponding boundary conditions are
\[
W' = 0, \quad W'' = 0, \quad \Theta' = 0, \quad \Theta'' = 0, \quad \Theta' = 0, \quad \text{at} \quad z = 0,1.
\]
We follow the same procedure to that used in the linear instability. Set the determinant of the matrix to zero, we arrive at the equation in \( R_x \)
\[
AR_x^4 + BR_x^2 + C = 0, \quad (17)
\]
with coefficients
\[
A = -a^4\gamma_1, \quad B = a^2\Lambda(\Omega_1 - \Omega_2 + \Omega_3) , \quad C = -\Lambda^2k_0(\gamma_2 + \gamma_3) = 0,
\]
where
\[
\Omega_1 = (\Gamma_x + S_2 + S_3)H[2(\mu_1 + 1)(L_p\sqrt{\lambda} + b)c + 2(\mu_2 + 1)(\sqrt{\lambda} + c) + b] \\
+ (\Gamma_x + S_1 + S_2)(\Gamma_x + S_1 + S_2 + H)[(\mu_1 + 1)c^2 + (\mu_2 + 1)(\sqrt{\lambda} + c)^2] \\
+ (\Gamma_x + S_1 + S_2)(\Lambda + S_1 + H)[(\mu_1 + 1)(L_p\sqrt{\lambda} + b)^2 + (\mu_2 + 1)b^2],
\]
\[
\Omega_2 = (\mu_1 + 1)\left(S_1 \left(L_p\sqrt{\lambda} + b\right) + S_2 c\right)^2 + (\mu_1 + 1)\left(S_1 b - S_2 \left(\sqrt{\lambda} + c\right)\right)^2,
\]
\[
\Omega_3 = (\sqrt{\lambda} + c)(L_p\sqrt{\lambda} + b + bc)(2H(\Gamma_x + S_1 + S_2) + 2S_1 S_2) \\
+ (L_p\sqrt{\lambda} + b)b(2(\Gamma_x + S_1 + S_2)(\Lambda + S_1 + H) - 2S_1^2) \\
+ (\sqrt{\lambda} + c)c(2(\Gamma_x + S_1 + S_2)(\Gamma_x + S_2 + H) - 2S_2^2),
\]
and
\[
\gamma_1 = (\Gamma_x + S_1 + S_2)((\sqrt{\lambda} + c)(L_p\sqrt{\lambda} + b) - bc)^2, \\
\gamma_2 = (\Gamma_x + S_1 + S_2)(H^2 - (\Lambda + S_1 + H)(\Gamma_p + S_2 + H)), \\
\gamma_3 = 2S_1 S_2(2H + S_2(\Gamma_p + S_2 + H) + S_2(\Lambda + S_1 + H)).
\]
The energy Rayleigh number \( R_x \) is then given by
\[
R_x^2 = \frac{\Lambda(\Omega_1 - \Omega_2 + \Omega_3)\pm \sqrt{(\Omega_1 - \Omega_2 + \Omega_3)^2 - 4\gamma_1^2 k_0(\gamma_2 + \gamma_3)}}{2a^2\gamma_1}, \quad (18)
\]
We require to find the critical nonlinear Rayleigh number \( R_{a\omega} \), such that
\[
Ra = \max_{a} \min_{\omega} \min_{\omega} R_x^2 \left(a^2, \lambda\right).
\]
Numerical results for the nonlinear energy approach are presented in section 5.

**Numerical results**

In this section, we present new numerical computations for the linear and nonlinear stability analyses. Our analysis supports the work of Franchi et al [19] by computing the stationary convection instability threshold equation (8), and the oscillatory convection threshold equation (9). Both cases are studied by using golden section search to minimise
over $a^2$ and find the critical values of $R^2$ for linear instability. Moreover, the critical Rayleigh numbers of nonlinear energy stability $Ra_E$ for fixed $a^2$ and $\lambda$ is determined using equation (18). Then, we employ golden section search to minimise in $a^2$ and then maximise in $\lambda$ to determine $Ra_E$, where for $R^2 < Ra_E$ we have stability for the best values of the coupling parameter $\lambda$. We have to do the minimization in equation (8), equation (9) and equation (18) numerically by using Matlab routines. For the parameters that are employed in this article, we have to be very careful when minimize $R^2$ in equation (8) and equation (9) over $a^2$ and then maximise in $\lambda$ to determine $Ra_E$, where for $Ra < Ra_E$ we have stability for the best values of the coupling parameter $\lambda$. We have to do the minimisation in equation (8), equation (9) and equation (18) numerically by using Matlab routines. For the parameters that are employed in this article, we have to be very careful when minimize $R^2$ in equation (8) and equation (9) over $a^2$. In all cases, we found that the stationary curve always lies below the oscillatory convection one.

In the present study, Tables 1, 2 and Tables 3, 4 display the numerical results for values of $\varepsilon = 0.00001$, $\varphi = 0.9999$, $\mu_1 = \mu_2 = 0.1$, $L_p = 15$ and $\Gamma_s = 0.1$ with fixed value of the inter-phase heat transfer parameter $H = 0.00001$, 0.01, and the porosity modified conductivity ratio $\Gamma_p = 0.1, 1$ with various values of the porosity modified interaction coefficients $S_1$ and $S_2$. We also present in Figures 1, 2 and Figures 3, 4 the critical Rayleigh number $Ra$ which is plotted against $S_1$ and $S_2$. From Table 1, the values of $\varepsilon, \varphi, \Gamma_p$ and $H$ are suggested by Nield and Kuznetsov [10], it is observed that when $S_1 = 0.01$ and $\Gamma_p = 0.1$ the stationary convection boundary, $Ra_{sta} = 39.4815$, is similar to that found in Nield and Kuznetsov [10]. For an increase in the value of $S_1$ from 0.01 to 1, the critical values of $Ra_{sta}$ and $Ra_E$ is increased as seen in Table1 and Table2. However, from Figures 1 and 2 we observe that as $\Gamma_p$ increases from $\Gamma_p = 0.1$ to $\Gamma_p = 1$, the values of $Ra$ are increasing this shows the stabilizing effect of the porosity modified conductivity ratio $\Gamma_p$. It is clear from Figures 1 and 2, and from Tables 1 and 2 that when $\Gamma_p = 0.1$ the stationary convection boundary $Ra_{sta}$ decreases with increasing $H$, which shows the destabilizing effect. In addition, the distance between $Ra_{sta}$ and $Ra_E$ increases with decreasing $H$. Thus, with small values of $H$ we have wide subcritical regions.

Table 3 and Table 4 show that for fixed value of $\Gamma_p$, as $S_2$ increases, the critical Rayleigh number increases. Thus, an increasing in $S_2$ causes the system becomes more stable. It is also observed that for a fixed value of $\Gamma_p$ the effect of increasing the value of $S_2$ is to increase the critical wave number for the onset of linear instability. For example, for $\Gamma_p = 0.1$ and $S_2 = 1.01$ the critical wave number is $a_L = 11.0525$, when $S_2 = 8.01$, the critical wave number $a_L = 22.9938$. This means the cells become narrower due to the intense increasing of the critical wave number. Further, it is evident from Figures 3 and 4 that as the value of $\Gamma_p$ decreases, the difference between the linear instability and nonlinear stability thresholds increases. As a result, the region of potential subcritical instabilities between the linear and nonlinear stability thresholds considerable. It is also noted that as $\Gamma_p$ increases the linear instability and nonlinear stability thresholds become closer. This is, in a sense, the best
possible agreement between the two thresholds since the region of potential subcritical instabilities decreases.

Figure 5 and Table 5 display the effect of $\Gamma_s = 0.1, 1$ with $\mu_1 = 0.5, 0.1$ and $\mu_2 = 0.1, 0.5$ respectively for fixed value of $S_2 = 0.01, \Gamma_p = 0.1$ and $\mu = 0.01, 0.1$ and for various values of $S_1$. It is found that the effect of increasing $S_1$ is to increases the critical Rayleigh number $Ra$. However, Figure 5 shows that as $\Gamma_s$ and $\mu_2$ increase with the decreasing $\mu_1$, $Ra$ increases. Therefore, the parameter $\Gamma_s$ have a stabilizing effect on the stability of the system.

Furthermore, the effect of increasing $\Gamma_s$ is to increases the value of wave number. For example, for $S_1 = 6$ and $\Gamma_s = 0.1$ we see from Table 5 that the wave number $a_L = 9.9314$, whereas, when $\Gamma_s = 1$ for the same parameter $S_1 = 6$, the wave number $a_L = 11.3416$. This indicates that the cell width decreases with increasing the parameter $\Gamma_s$, which corresponds to the narrower convection cells. It is also observed that, for a fixed value of $\mu_1 = 0.5$, $\mu_2 = 2$ and $\Gamma_s = 0.1$, the effect of increasing $S_2$ is to increases the wave number. For example, for $S_2 = 4$, we see from Table 6 that the wave number $a_L = 12.6119$, whereas, when $S_2 = 8$, the wave number $a_L = 15.3255$, which leads to cells becoming narrower.

Table 1-Critical Rayleigh and wave numbers of linear and energy theory, vs. $S_1$, for $H = 0.00001$, $\varepsilon = 0.00001$, $\varphi = 0.9999$, $\mu = \mu_2 = 0.1$

| $S_1$  | $Ra_{sta}$ | $a_L$   | $Ra_E$ | $a_E$   | $\lambda$ |
|--------|------------|---------|--------|---------|-----------|
| 0.01   | 39.4815    | 9.8672  | 22.8990| 10.3919 | 0.98978   |
| 1.01   | 39.7958    | 9.6768  | 23.6458| 10.5924 | 0.9901    |
| 2.01   | 39.9430    | 9.5673  | 24.0238| 10.6205 | 0.9903    |
| 3.01   | 40.0281    | 9.4981  | 24.2527| 10.6205 | 0.9903    |
| 4.01   | 40.0832    | 9.4503  | 24.0600| 10.5925 | 0.9906    |
| 5.01   | 40.1219    | 9.4166  | 24.5160| 10.5699 | 0.9905    |
| 6.01   | 40.1505    | 9.3908  | 24.5986| 10.5612 | 0.9906    |
| 7.01   | 40.1725    | 9.3703  | 24.6631| 10.5838 | 0.9906    |
| 8.01   | 40.1900    | 9.3546  | 22.8990| 10.3919 | 0.98978   |

| $S_1$  | $Ra_{sta}$ | $a_L$   | $Ra_E$ | $a_E$   | $\lambda$ |
|--------|------------|---------|--------|---------|-----------|
| 0.01   | 39.4966    | 9.8738  | 23.7482| 10.3692 | 0.883711  |
| 1.01   | 40.0793    | 10.0459 | 24.4926| 10.5838 | 0.892342  |
| 2.01   | 41.3136    | 10.0550 | 24.8693| 10.6046 | 0.894236  |
| 3.01   | 41.6760    | 10.0368 | 25.0972| 10.6046 | 0.895142  |
| 4.01   | 41.9175    | 10.0154 | 25.2498| 10.5838 | 0.895356  |
| 5.01   | 42.0898    | 9.9946  | 25.3591| 10.5611 | 0.896608  |
| 6.01   | 42.2190    | 9.9767  | 25.4412| 10.5471 | 0.897087  |
| 7.01   | 42.3193    | 9.9619  | 25.5050| 10.5331 | 0.897382  |
| 8.01   | 42.3995    | 9.9487  | 25.5562| 10.5244 | 0.897647  |
Table 2 - Critical Rayleigh and wave numbers of linear and energy theory, vs. $S_1$, for $H = 0.01$, $\varepsilon = 0.00001$, $\varphi = 0.9999$, $\mu_1 = \mu_2 = 0.1$

| $S_1$ | $Ra_{sa}$ | $a_s$ | $Ra_{x}$ | $a_x$ | $\lambda$ |
|-------|-----------|-------|---------|-------|-----------|
| 0.01  | 36.69807  | 9.215543 | 22.91493 | 10.37787 | 0.989563 |
| 1.01  | 37.03958  | 9.073001 | 23.66157 | 10.60646 | 0.989909 |
| 2.01  | 37.19873  | 8.989991 | 24.03958 | 10.62912 | 0.990123 |
| 3.01  | 37.29039  | 8.930459 | 24.26847 | 10.64312 | 0.990123 |
| 4.01  | 37.34988  | 8.89176  | 24.42182 | 10.66046 | 0.990205 |
| 5.01  | 37.39158  | 8.864721 | 24.53175 | 10.58381 | 0.990205 |
| 6.01  | 37.42242  | 8.84296  | 24.61438 | 10.56115 | 0.990337 |
| 7.01  | 37.44614  | 8.826617 | 24.67874 | 10.54715 | 0.990337 |
| 8.01  | 37.46496  | 8.81436  | 24.73028 | 10.53315 | 0.990337 |

| $S_1$ | $Ra_{sa}$ | $a_s$ | $Ra_{x}$ | $a_x$ | $\lambda$ |
|-------|-----------|-------|---------|-------|-----------|
| 0.01  | 39.21878  | 9.805174 | 23.76454 | 10.36921 | 0.888025 |
| 1.01  | 40.42548  | 9.975719 | 24.50874 | 10.59246 | 0.891996 |
| 2.01  | 41.02581  | 9.984855 | 24.88625 | 10.62912 | 0.990123 |
| 3.01  | 41.35551  | 9.965987 | 25.11328 | 10.73909 | 0.895142 |
| 4.01  | 41.62501  | 9.944595 | 25.26594 | 10.58381 | 0.895834 |
| 5.01  | 41.79588  | 9.924763 | 25.37527 | 10.58381 | 0.896394 |
| 6.01  | 41.92388  | 9.906491 | 25.45736 | 10.54715 | 0.896822 |
| 7.01  | 42.02333  | 9.892674 | 25.52126 | 10.54715 | 0.897168 |
| 8.01  | 42.10282  | 9.879452 | 25.57236 | 10.52449 | 0.897382 |

Table 3 - Critical Rayleigh and wave numbers of linear and energy theory, vs. $S_2$, for $H = 0.00001$, $\varepsilon = 0.00001$, $\varphi = 0.9999$, $\mu_1 = \mu_2 = 0.1$

| $S_2$ | $Ra_{sa}$ | $a_s$ | $Ra_{x}$ | $a_x$ | $\lambda$ |
|-------|-----------|-------|---------|-------|-----------|
| 0.01  | 39.4815   | 9.8672 | 22.8990 | 10.3919 | 0.9898   |
| 1.01  | 43.9789   | 11.5295 | 22.9283 | 10.3779 | 0.9865   |
| 2.01  | 51.4937   | 13.7849 | 22.9430 | 10.3779 | 0.9847   |
| 3.01  | 59.3218   | 15.8437 | 22.9517 | 10.3779 | 0.9837   |
| 4.01  | 67.1161   | 17.7086 | 22.9575 | 10.3779 | 0.9830   |
| 5.01  | 74.8110   | 19.4203 | 22.9615 | 10.3779 | 0.9825   |
| 6.01  | 82.3987   | 21.0078 | 22.9645 | 10.3779 | 0.9822   |
| 7.01  | 89.8857   | 22.4913 | 22.9667 | 10.3692 | 0.9820   |
| 8.01  | 97.2816   | 23.8909 | 22.9685 | 10.3692 | 0.9818   |

| $S_2$ | $Ra_{sa}$ | $a_s$ | $Ra_{x}$ | $a_x$ | $\lambda$ |
|-------|-----------|-------|---------|-------|-----------|
| 0.01  | 39.4966   | 9.8738 | 23.7482 | 10.3692 | 0.8884   |
| 1.01  | 40.117    | 10.1284 | 23.7820 | 10.3779 | 0.8842   |
| 2.01  | 41.4161   | 10.5795 | 23.7990 | 10.3779 | 0.8822   |
| 3.01  | 42.9383   | 11.0534 | 23.8092 | 10.3779 | 0.8810   |
| 4.01  | 44.5496   | 11.5188 | 23.8159 | 10.3779 | 0.8801   |
| 5.01  | 46.1978   | 11.9719 | 23.8206 | 10.3692 | 0.8794   |
| 6.01  | 47.8594   | 12.4093 | 23.8240 | 10.3692 | 0.8791   |
| 7.01  | 49.5228   | 12.8318 | 23.8266 | 10.3692 | 0.8787   |
| 8.01  | 51.1819   | 13.2422 | 23.8286 | 10.3692 | 0.8785   |

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Table 4-Critical Rayleigh and wave numbers of linear and energy theory, vs. $S_2$, for $H = 0.01$, $\varepsilon = 0.00001$, $\varphi = 0.9999$, $\mu = \mu_2 = 0.1$

| $S_2$ | $Ra_{an}$ | $a_e$ | $Ra_e$ | $a_e$ | $\lambda$ |
|-------|-----------|-------|--------|-------|-----|
| 0.1   | 36.6980   | 9.2155| 22.9149| 10.3779| 0.9896 |
| 1.01  | 41.7939   | 11.0525| 22.9442| 10.3919| 0.9862 |
| 2.01  | 49.3713   | 13.2610| 22.9590| 10.3919| 0.9845 |
| 3.01  | 57.1666   | 15.2537| 22.9677| 10.3919| 0.9834 |
| 4.01  | 64.9004   | 17.0529| 22.9735| 10.3919| 0.9828 |
| 5.01  | 72.5255   | 18.6989| 22.9775| 10.3919| 0.9823 |
| 6.01  | 80.0404   | 20.2252| 22.9805| 10.3919| 0.9819 |
| 7.01  | 87.4544   | 21.6503| 22.9827| 10.3779| 0.9816 |
| 8.01  | 94.7780   | 22.9938| 22.9844| 10.3779| 0.9814 |

Table 5-Critical Rayleigh and wave numbers of linear and energy theory, vs. $S_1$, for $S_2 = 0.01$, $\varepsilon = 0.00001$, $\varphi = 0.9999$, $\Gamma_p = 0.1$

| $\mu$ = 0.5 | $\mu$ = 0.1 | $\Gamma_s$ = 0.1 | $H = 0.01$, $L_p = 15$, $\Gamma_p = 0.1$ |
|-------|-----------|-------|--------|--------|-------|--------|--------|--------|--------|--------|
| $S_1$ | $Ra_{an}$ | $a_e$ | $Ra_e$ | $a_e$ | $\lambda$ | $S_1$ | $Ra_{an}$ | $a_e$ | $Ra_e$ | $a_e$ | $\lambda$ |
| 1.01  | 40.4902   | 9.9940| 24.6603| 10.5838| 0.8783 |
| 2.01  | 41.0995   | 10.0057| 25.0361| 10.6065| 0.8805 |
| 3.01  | 41.4648   | 9.9883| 25.2633| 10.6065| 0.8818 |
| 4.01  | 41.7081   | 9.9666| 25.4152| 10.5838| 0.8829 |
| 5.01  | 41.8818   | 9.9471| 25.5238| 10.5612| 0.8835 |
| 6.01  | 42.0119   | 9.9314| 25.6052| 10.5471| 0.8840 |
| 7.01  | 42.1130   | 9.9150| 25.6683| 10.5245| 0.8844 |
| 8.01  | 42.1938   | 9.9018| 25.7188| 10.5245| 0.8854 |

| $\mu$ = 0.1 | $\mu$ = 0.5 | $\Gamma_s$ = 1 | $H = 0.01$, $L_p = 15$, $\Gamma_p = 0.1$ |
|-------|-----------|-------|--------|--------|-------|--------|--------|--------|--------|
| $S_1$ | $Ra_{an}$ | $a_e$ | $Ra_e$ | $a_e$ | $\lambda$ | $S_1$ | $Ra_{an}$ | $a_e$ | $Ra_e$ | $a_e$ | $\lambda$ |
| 1.01  | 41.0188   | 10.2211| 33.5235| 10.8535| 0.9193 |
| 2.01  | 42.6773   | 10.5575| 34.8189| 11.1823| 0.9225 |
| 3.01  | 44.1767   | 10.8250| 35.9930| 11.4702| 0.9253 |
| 4.01  | 45.5436   | 11.0377| 37.0659| 11.6934| 0.9275 |
| 5.01  | 46.7979   | 11.2067| 38.0525| 11.8767| 0.9295 |
| 6.01  | 47.9551   | 11.3416| 38.9645| 12.0320| 0.9313 |
| 7.01  | 49.0273   | 11.4480| 39.8112| 12.1506| 0.9328 |
| 8.01  | 50.0243   | 11.5305| 40.6000| 12.2466| 0.9342 |
Table 6- Critical Rayleigh and wave numbers of linear and energy theory, vs. $S_2$, for $S_1=0.01$ 
$\epsilon = 0.00001, \varphi = 0.9999, \Gamma_s = 0.1, 1$

| $\mu_1 = 0.5$ | $\mu_2 = 2$ | $\Gamma_s = 0.1$ |
|---------------|-------------|------------------|
| $\mu_1 = 0.5$ | $\mu_2 = 2$ | $\Gamma_s = 0.1$ |
| 1.01 | 40.4870 | 10.3061 | 32.8520 | 10.3552 | 0.8616 |
| 2.01 | 42.6969 | 11.0707 | 32.8797 | 10.3552 | 0.8593 |
| 3.01 | 45.2245 | 11.8548 | 32.8946 | 10.3552 | 0.8579 |
| 4.01 | 47.8633 | 12.6119 | 32.9031 | 10.3552 | 0.8572 |
| 5.01 | 50.5382 | 13.3353 | 32.9078 | 10.3412 | 0.8568 |
| 6.01 | 53.2175 | 14.0272 | 32.9102 | 10.3412 | 0.8564 |
| 7.01 | 55.8869 | 14.6913 | 32.9110 | 10.3412 | 0.8563 |
| 8.01 | 58.5396 | 15.3255 | 32.9107 | 10.3326 | 0.8563 |

| $\mu_1 = 0.2$ | $\mu_2 = 0.05$ | $\Gamma_s = 1$ |
|---------------|-------------|------------------|
| 1.01 | 39.3037 | 9.8505 | 25.8696 | 10.3919 | 0.8931 |
| 2.01 | 39.5780 | 9.9742 | 25.9125 | 10.4059 | 0.8882 |
| 3.01 | 39.9777 | 10.144 | 25.9520 | 10.4145 | 0.8836 |
| 4.01 | 40.4714 | 10.3442 | 25.9884 | 10.4285 | 0.8794 |
| 5.01 | 41.0363 | 10.5642 | 26.0221 | 10.4372 | 0.8756 |
| 6.01 | 41.6559 | 10.7979 | 26.0533 | 10.4512 | 0.8720 |
| 7.01 | 42.3178 | 11.0387 | 26.0824 | 10.4512 | 0.8686 |
| 8.01 | 43.0127 | 11.2847 | 26.1096 | 10.4512 | 0.8656 |

Figure 1- Critical Rayleigh number $Ra$ is plotted against $S_1$ with $\epsilon = 0.00001, \varphi = 0.9999, \Gamma_s = 0.1, 1, L_p = 15, S_2 = 0.01$. The solid curves are for linear instability and the dotted curves are for nonlinear stability, for $\Gamma_p = 0.1, 1, H = 0.00001, \mu_1 = \mu_2 = 0.1$. 
Figure 2-Critical Rayleigh number $Ra$ is plotted against $S_1$ with $\varepsilon = 0.00001$, $\varphi = 0.9999$, $\Gamma_s = 0.1$, $L_p = 15$, $S_2 = 0.01$. The solid curves are for linear instability and the dotted curves are for nonlinear stability, for $\Gamma_p = 0.1, 1$, $H = 0.01$, and $\mu_1 = \mu_2 = 0.1$.

Figure 3-Critical Rayleigh number $Ra$ is plotted against $S_2$ with $\varepsilon = 0.00001$, $\varphi = 0.9999$, $\Gamma_s = 0.1$, $L_p = 15$, $S_1 = 0.01$. The solid curves are for linear instability and the dotted curves are for nonlinear stability, for $\Gamma_p = 0.1, 1, H = 0.00001$, and $\mu_1 = \mu_2 = 0.1$. 
Figure 4-Critical Rayleigh number $Ra$ is plotted against $S_2$ with $\varepsilon = 0.00001$, $\varphi = 0.9999$, $\Gamma_s = 0.1$, $L_p = 15$, $S_1 = 0.01$. The solid curves are for linear instability and the dotted curves are for nonlinear stability, for $\Gamma_p = 0.1, 1$, $H = 0.01$, and $\mu_1 = \mu_2 = 0.1$.

Figure 5-Critical Rayleigh number $Ra$ is plotted against $S_1$ with $\varepsilon = 0.00001$, $\varphi = 0.9999$, $\Gamma_p = 0.1$, $L_p = 15$, $S_2 = 0.01$. The solid curves are for linear instability and the dotted curves are for nonlinear stability, for $\Gamma_p = 0.1, 1$, $H = 0.01$. 

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Figure 6 - Critical Rayleigh number $Ra$ is plotted against $S_2$ with $\varepsilon = 0.00001$, $\varphi = 0.9999$, with $\Gamma_p = 1$, $L_p = 15$, $S_1 = 0.01$. The solid curves are for linear instability and the dotted curves are for nonlinear stability, for $\Gamma_p = 0.1$, $1$, $H = 0.01$.

Conclusion

The onset of convection in a fluid saturated bidisperse porous medium is investigated when the temperature of the fluid and solid phases is in local thermal non-equilibrium. The linear instability threshold and nonlinear one is derived analytically in case of free surfaces. The stationary convection boundary, $Ra_{sta} = 39.4815$, is similar to that found by Nield and Kwznetsove for particular values of the porosity modified interaction coefficient and the porosity modified conductivity ratio, $\varepsilon = 0.00001$, $\varphi = 0.9999$, $H = 0.00001$, $S_2 = 0.01$, $L_p = 15$, $\Gamma_s = 0.1$ and $\Gamma_p = 0.1$. Through investigation we found that the onset of convection is by stationary convection for various interaction parameters. It can be argued from the results that the critical Rayleigh numbers $Ra$ and the critical wave numbers $\alpha_i$ are greater in the case of $\mu_1 = \mu_2 = 0.1$, $L_p = 15$ and $S_2$ increases. Also, we observed that the subcritical instability region decreases as the parameter $\Gamma_s$ increases. The effects of increasing $\Gamma_p$ and $\Gamma_s$ as well as increasing the value of $S_1$ were seen to stabilize the system. Also, we observed that for small values of $\Gamma_p$ and $\Gamma_s$, the effect of increasing $S_2$ is to stabilize the system. This indicates that the thermal convection occurs more easily.

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