GLOBAL WELL-POSEDNESS FOR EFFECTIVELY DAMPED WAVE MODELS WITH NONLINEAR MEMORY

TAYEB HADJ KADDOUR
Department of Mathematics, Faculty of exact sciences and informatics
University of Chlef, P.O. Box 50, 02000, Ouled-Fares, Chlef, Algeria
Laboratory of mechanic and energetic, University of Chlef, Algeria

MICHAEL REISSIG*
Faculty for Mathematics and Computer Science, TU Bergakademie Freiberg
Pruferstr. 9, 09596, Freiberg, Germany

(Communicated by Alain Miranville)

Abstract. In this paper, we study the Cauchy problem for a special family of effectively damped wave models with nonlinear memory on the right-hand side. Our goal is to prove global (in time) well-posedness results for Sobolev solutions. Due to the effective dissipation the model is parabolic like from the point of view of energy decay estimates of the corresponding linear Cauchy problem with vanishing right-hand side. For this reason there appears a Fujita type exponent as a threshold. Applying modern tools from Harmonic Analysis we prove several results by taking into consideration different regularity properties of the data.

1. Introduction. In this paper, let us consider the following Cauchy problem for a family of effectively damped wave models with a nonlinear memory on the right-hand side. The model we have in mind is

\[
\begin{aligned}
  & u_{tt} - \Delta u + (1 + t)^r u_t = \int_0^t (t - \tau)^{-\gamma} |u(\tau, x)|^p \, d\tau, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
  & u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),
\end{aligned}
\]

(1.1)

where \( r \in (-1, 1) \) and \( \gamma \in (0, 1) \). The corresponding linear Cauchy problem with vanishing right-hand side is

\[
\begin{aligned}
  & u_{tt} - \Delta u + (1 + t)^r u_t = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
  & u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n.
\end{aligned}
\]

Due to the paper [12] this model is for \( r \in (-1, 1) \) parabolic like from the point of view of energy decay estimates. For this reason parabolic like tools are appropriate

2020 Mathematics Subject Classification. Primary: 35G25, 35B40; Secondary: 35B33.

Key words and phrases. Damped wave equation, effective damping, nonlinear memory, Cauchy problem, energy solutions.

The research of this paper is supported by DAAD, Erasmus+ Project between the Hassiba Benbouali University of Chlef and TU Bergakademie Freiberg, 2015-1-DE01-KA107-002026, during the stay of the first author at Technical University Bergakademie Freiberg within the periods April 2016 to June 2016, and a stay of one month April 2017 supported by Hassiba Benbouali University.

* Corresponding author.
to derive global (in time) existence or blow-up results. At first, let us recall some recent results for (1.1) with \( r = 0 \), that is, for the model
\[
\begin{cases}
  u_{tt} - \Delta u + u_t = \int_0^t (t - \tau)^{-\gamma} |u(\tau, x)|^p \, d\tau, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
  u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbb{R}^n.
\end{cases}
\] (1.2)

The basic question is the question for the critical exponent \( p_{\text{crit}} = p_{\text{crit}}(n) \), dividing the range of admissible exponents \( p \) in a set, where, in general, local (in time) small data Sobolev solutions blow-up and a set, where small data yield global (in time) Sobolev solutions. It is clear that the exponent \( \gamma \), the dimension \( n \) and the regularity of the data influence the critical exponent. In [3] the author used decay estimates for Sobolev solutions and their partial derivatives to the corresponding linear Cauchy problem with vanishing right-hand side to prove global (in time) existence of small data Sobolev solutions and local (in time) existence of large data Sobolev solutions as well.

Firstly let us consider the \( L^1 \cap L^2 \) theory, that is, the data are from energy space with additional \( L^1 \) regularity. As critical exponent the author proposed
\[
p(n, \gamma) = \max \left\{ p_\gamma(n); \frac{1}{\gamma} \right\}, \quad \text{where } p_\gamma(n) = 1 + \frac{2(2 - \gamma)}{(n + 2(1 - \gamma)) +},
\] (1.3)

which coincides with the critical exponent for the following classical heat equation with nonlinear memory:
\[
  u_t - \Delta u = \int_0^t (t - \tau)^{-\gamma} (|u|^{p-1}u)(\tau, x) \, d\tau, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n.
\]

Here \( (n + 2(1 - \gamma)) + = \max\{n + 2(1 - \gamma); 0\} \). This was discovered in [1]. In this way the paper [3] filled some gaps of the paper [9], where the same model (1.2) was treated by using the method of weighted spaces to prove the existence of high regular solutions. Moreover, in [9] the author studied blow-up of solutions, too, to verify \( p(n, \gamma) \) as the critical exponent. Let \( \gamma \) tend to 1 in (1.2). Then the right-hand side tends in distributional sense to \(|u|^p\). The critical exponent for
\[
\begin{cases}
  u_{tt} - \Delta u + u_t = |u|^p, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
  u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbb{R}^n,
\end{cases}
\] (1.4)
is the Fujita exponent \( p_{Fuj} = p_{Fuj}(n) = 1 + \frac{2}{n} \), which appears after choosing \( \gamma \to 1 \) in (1.3).

Secondly, let us consider the \( L^2 \) theory, that is, the data are taken from some energy space without additional \( L^1 \) regularity. Then in [3] the author proposed the critical exponent
\[
  \hat{p}(n, \gamma) = \max \left\{ \hat{p}_\gamma(n); \frac{1}{\gamma} \right\}, \quad \text{where } \hat{p}_\gamma(n) = 1 + \frac{4(2 - \gamma)}{(n - 4(1 - \gamma)) +}.
\] (1.5)

If \( \gamma \) tends to 1 in (1.5), then we get \( p_{\text{crit}} = p_{\text{crit}}(n) = 1 + \frac{4}{n} \), the critical exponent in the \( L^2 \) theory for (1.4). We have even for \( m \in [1, 2] \) the critical exponent \( p_{\text{crit}} = p_{\text{crit}}(m, n) = 1 + \frac{2m}{n(2m - 1)} \) in the \( L^m \cap L^2 \) theory for (1.4) (see [4]). The influence of nonlinear memory terms was also studied for other models (see, for example, [5]).

Finally, the following Cauchy problem for the semilinear damped wave equation
\[
\begin{cases}
  u_{tt} - \Delta u + b(t)u_t = |u|^p, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
  u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbb{R}^n,
\end{cases}
\]
was studied in [6] and, more recently, in [7] under the assumption that the term $b(t)u_t$ is a general effective damping term in the sense of [12]. It is no surprise that for this model the critical exponent is the Fujita exponent. Consequently, in the case $\gamma = 1$ the exponent $r \in (-1, 1)$ in $b(t) = (1 + t)^r$ has no influence on the critical exponent. But what happens in the case $\gamma \in (0, 1)$? Let us turn to answer this question in the following considerations. For this reason we come back to the model

\[
\begin{align*}
  &\text{Duhamel’s principle and we reduce the study of the Cauchy problem (1.1) to the}
  \\
  &\text{question in the following considerations. For this reason we come back to the model}
  \\
  &\text{Cauchy problem (1.1) is formally given by}
  \\
  &\text{and the family of parameter-dependent Cauchy problems}
  \\
  &\text{If we denote by $v = v(t, \tau, x)$ the solution of Cauchy problem (1.7), and by $u^{lin} = u^{lin}(t, x)$ the solution of Cauchy problem (1.6), then the solution $u = u(t, x)$ of the Cauchy problem (1.1) is formally given by}
  \\
  &\text{We put}
  \\
  &\text{So, the solution of the Cauchy problem (1.1) can be formally written as}
  \\
  &The solution $u^{lin} = u^{lin}(t, x)$ of Cauchy problem (1.6) is given by
  \\
  &where $E_0 = E_0(t, 0, x)$ and $E_1 = E_1(t, 0, x)$ are the fundamental solutions for the Cauchy problem (1.6), that is, $E_0 = E_0(t, 0, x)$ corresponds to the initial data $(u_0, u_1) = (\delta_0, 0)$ and $E_1 = E_1(t, 0, x)$ corresponds to the initial data $(u_0, u_1) = (0, \delta_0)$, where $\delta_0$ denotes Dirac’s Delta-distribution supported in 0. The solution $v = v(t, \tau, x)$ of the Cauchy problem (1.7) is obviously given by
  \\
  &Using (1.8), (1.9) and (1.10) the solution of the Cauchy problem (1.1) is formally given as a solution of the fixed point equation
  \\
  &u = E_0(t, 0, x) *_x u_0(x) + E_1(t, 0, x) *_x u_1(x) + \int_0^t E_1(t, \tau, x) *_x h(\tau, u) d\tau.
\]
Let us introduce some notations. For all \( T > 0 \) we denote by \( X(T) \) the solution space of Sobolev solutions to the Cauchy problem (1.1). For all \( u \in X(T) \) we define the mapping \( N \) as follows:

\[
N: u \in X(T) \rightarrow N(u) = E_0(t, 0, x) + E_1(t, 0, x) + \int_0^t E_1(t, \tau, x) * _{x}h(\tau, u)d\tau.
\]

Our main strategy is to prove well-posedness results for Sobolev solutions to (1.1) as solutions of the fixed point equation \( u = N(u) \) by proving the following inequalities for all \( u, v \in X(T) \):

\[
\|Nu\|_{X(T)} \leq C\|u_0, u_1\|_{A_{m, \sigma}} + C\|u\|_{X(T)}^p, \quad (1.11)
\]

\[
\|Nu - Nv\|_{X(T)} \leq C\|u - v\|_{X(T)}(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}), \quad (1.12)
\]

where the norm \( \| \cdot \|_{X(T)} \) in \( X(T) \) will be defined later in a suitable way and makes \( X(T) \) as a Banach space. For the further considerations we introduce the scale \( \{A_{m, \sigma}\} \) of Banach spaces with \( \sigma > 0 \) and \( m \in [1, 2] \), where

\[
A_{m, \sigma} = A_{m, \sigma}(\mathbb{R}^n) = (L^m(\mathbb{R}^n) \cap H^\sigma(\mathbb{R}^n)) \times (L^m(\mathbb{R}^n) \cap H^\max(0, \sigma - 1)(\mathbb{R}^n)). \quad (1.13)
\]

Throughout the present paper we write \( \mathbf{f} \lesssim \mathbf{g} \) when there exists a constant \( C > 0 \) such that \( \mathbf{f} \leq C\mathbf{g} \), and \( \mathbf{f} \approx \mathbf{g} \) when \( \mathbf{g} \lesssim \mathbf{f} \lesssim \mathbf{g} \). Nonnegative constants \( C \) or \( C_l, l \in \mathbb{N} \), are always supposed to be independent of \( T > 0 \).

2. Main results for low regular data. In this section we present our main results for the global (in time) well-posedness of small data solutions to the Cauchy problem

\[
\begin{cases}
  u_{tt} - \Delta u + (1 + t)^r u_t = \int_0^t (t - \tau)^{-\gamma}|u(\tau, x)|^p d\tau, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
  u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n,
\end{cases} \quad (2.1)
\]

where \( r \in (-1, 1), \gamma \in (0, 1) \) and the data \((u_0, u_1)\) is supposed to belong to suitable function spaces with low regularity. In general, such results are applicable for low dimensions \( n = 1, 2 \) or 3 (see the examples in Section 4.2). In the following considerations we use the notation

\[
B(t, \tau) := \int_{\tau}^t (1 + s)^{-r} ds = \frac{1}{1 - r}\left((1 + t)^{1-r} - (1 + \tau)^{1-r}\right) \text{ for } (\tau, t) \in [0, t] \times [0, \infty).
\]

2.1. Well-posedness of Sobolev solutions. We assume that the data \((u_0, u_1)\) belongs to \( A_{m, \sigma} \) with \( \sigma \in (0, 1) \) and \( m \in [1, 2] \). Let us introduce, for \( r \in (-1, 1) \) and \( \gamma \in (0, 1) \) the parameters

\[
p_m(\gamma, n, r) = 1 + \frac{2(2 - \gamma) - 2r}{\frac{n}{m}(1 - r) + 2(\gamma - 1) + 2r}.
\]

First we turn to the case \( \gamma + r < 1 \).

Theorem 2.1. Assume that \( \gamma \in (0, 1), r \in (-1, 1) \) and \( \gamma + r < 1 \). Moreover, \( \frac{2}{m} \leq p \leq \frac{n}{n - 2\gamma}, \frac{2}{m} \leq p < \infty \) if \( n \leq 2\sigma \) (this is valid only for \( n = 1 \) and \( \sigma \in [\frac{1}{2}, 1) \)) with \( n \geq 1, m \in [1, 2] \) and \( \sigma \in (0, 1) \). Moreover, we suppose

\[
\frac{n}{m}(1 - r) + 2(\gamma - 1) + 2r > 0 \quad (c.f. \ with \ (2.2)).
\]
and in the case \( n \geq 3 \) we assume
\[
\frac{n}{2} \left( \frac{1}{m} - \frac{1}{2} \right) + \frac{\sigma}{2} < 1.
\]

Finally, let us suppose the condition
\[
p > p_m(\gamma, n, r).
\]

Then, there exists a positive constant \( \varepsilon_0 \) such that for any initial data
\[
(u_0, u_1) \in \mathcal{A}_{m, \sigma} := (L^m \cap H^\sigma) \times (L^m \cap L^2)
\]
satisfying
\[
\|(u_0, u_1)\|_{\mathcal{A}_{m, \sigma}} \leq \varepsilon\text{ for all } \varepsilon \leq \varepsilon_0,
\]
there is a uniquely determined global (in time) Sobolev solution
\[
u \in C([0, \infty), H^\sigma)
\]
to the Cauchy problem (2.1). Moreover, the solution satisfies the following estimates:
\[
\|u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{3}{2} \left( \frac{1}{m} - \frac{1}{2} \right)(1-r) - \gamma - r} \|(u_0, u_1)\|_{\mathcal{A}_{m, \sigma}},
\]
\[
\|Du(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{3}{2} \left( \frac{1}{m} - \frac{1}{2} \right)(1-r) + (1 - \frac{2}{m})(1-r) - \gamma} \|(u_0, u_1)\|_{\mathcal{A}_{m, \sigma}}.
\]

2.2. Proof of Theorem 2.1. We introduce for \( T > 0 \) the space of Sobolev solutions
\[
X(T) = C([0, T], H^\sigma)
\]
with the norm
\[
\|u\|_{X(T)} = \sup_{0 \leq t \leq T} \left\{ (1 + t)^{\frac{3}{2} \left( \frac{1}{m} - \frac{1}{2} \right)(1-r) + \gamma + r - 1} \|u(t, \cdot)\|_{L^2} + (1 + t)^{-\frac{3}{2} \left( \frac{1}{m} - \frac{1}{2} \right)(1-r) - (1 - \frac{2}{m})(1-r) + \gamma} \|Du(t, \cdot)\|_{L^2} \right\}.
\]

First, let us prove the inequality (1.11). The inequality
\[
\|u^{lin}\|_{X(T)} \lesssim \|(u_0, u_1)\|_{\mathcal{A}_{m, \sigma}}
\]
is an immediate consequence of the Propositions 1 and 3 for \( \tau = 0 \) and \( h(0, u(x)) = u_1(x) \). It remains to show the inequality
\[
\|u^{nl}\|_{X(T)} \lesssim \|u\|_{X(T)}^p.
\]

Taking into account the results of Proposition 3 we get
\[
\|u^{nl}(t, \cdot)\|_{L^2} \lesssim \int_0^T (1 + \tau)^{-\gamma} \left( 1 + B(t, \tau) \right)^{-\frac{3}{2} \left( \frac{1}{m} - \frac{1}{2} \right)} \times \int_0^\tau (\tau - s)^{-\gamma} \|u(s, \cdot)|^p\|_{L^m \cap L^2} \, ds \, d\tau.
\]

Then, we have to estimate the following norms:
\[
\|\|(u(\cdot, \cdot))\|^p\|_{L^m} \quad \text{and} \quad \|\|u(\cdot, \cdot)|^p\|_{L^2}.
\]

For this reason, we apply the classical Gagliardo-Nirenberg inequality (5.2). In this way we obtain for \( j = m, 2 \) the chain of inequalities
\[
\|u(s, \cdot)|^p\|_{L^m} \lesssim \|u(s, \cdot)|^{p(1-\theta(jp))}\|Du(s, \cdot)\|_{L^2}^{\theta(jp)} \lesssim (1 + s)^{p(1-\theta(jp)) \left( \frac{1}{m} - \frac{1}{2} \right)(1-r) - \gamma - r} \times (1 + s)^{p\theta(jp) \left( \frac{1}{m} - \frac{1}{2} \right)(1-r) + (1 - \frac{2}{m})(1-r) - \gamma} \|u\|_{X(T)}^p \lesssim (1 + s)^{-\beta_j} \|u\|_{X(T)}^p,
\]

where
\[
\beta_j = \frac{1}{p} \left( \frac{1}{m} - \frac{1}{2} \right)(1-r) - \gamma - r + \frac{\theta(jp) \left( \frac{1}{m} - \frac{1}{2} \right)(1-r) + (1 - \frac{2}{m})(1-r) - \gamma}{p}.
\]
where
\[ \theta(jp) = \frac{n}{\sigma} \left( \frac{1}{2} - \frac{1}{jp} \right), \]  \hspace{1cm} (2.9)\]
\( \theta(jp) \in [0, 1] \) if and only if
\[ \frac{2}{j} \leq p \leq \frac{2n}{(n - 2\sigma) \cdot j}, \]
that is,
\[ \begin{cases} \frac{2}{m} \leq p \leq \frac{n}{n - 2\sigma} & \text{if } n > 2\sigma, \\ \frac{2}{m} \leq p & \text{if } n = 1 \text{ and } \sigma \in \left[ \frac{1}{2}, 1 \right], \end{cases} \]
and
\[ \beta_j = \left( \frac{n}{2m} (1 - r) + \gamma + r - 1 \right) p - \frac{n}{2j} (1 - r). \]  \hspace{1cm} (2.10)\]

Taking in (2.8) \( j = m \) and \( j = 2 \), respectively, we find
\[ \| u(s, \cdot) \|_{L^p} \lesssim (1 + s)^{-\beta} \| u \|_{X(T)}^p \]  \hspace{1cm} (2.11)\]
and
\[ \| u(s, \cdot) \|_{L^p} \lesssim (1 + s)^{-\beta - \frac{n}{2} (\frac{1}{m} - \frac{1}{2}) (1 - r)} \| u \|_{X(T)}^p \lesssim (1 + s)^{-\beta} \| u \|_{X(T)}^p, \]  \hspace{1cm} (2.12)\]
since \( r < 1 \) and \( m < 2 \), where
\[ \beta = \beta_m = \left( \frac{n}{2m} (1 - r) + \gamma + r - 1 \right) p - \frac{n}{2m} (1 - r). \]  \hspace{1cm} (2.13)\]
So, using (2.11) and (2.12) we may conclude
\[ \| u(s, \cdot) \|_{L^p} + \| u(s, \cdot) \|_{L^2} \lesssim (1 + s)^{-\beta} \| u \|_{X(T)}^p. \]  \hspace{1cm} (2.14)\]
Noting that \( \beta > 1 \) if and only if \( p > p_m(\gamma, n, r) \). Including (2.14) in (2.7) we find
\[ \| u^{nl}(t, \cdot) \|_{L^2} \lesssim \mathcal{J}_n(t) \| u \|_{X(T)}^p, \]  \hspace{1cm} (2.15)\]
where
\[ \mathcal{J}_n(t) = \int_0^t (1 + \tau)^{-\gamma} (1 + B(t, \tau))^{-\frac{n}{2} (\frac{1}{m} - \frac{1}{2})} \int_0^\tau (\tau - s)^{-\gamma} (1 + s)^{-\beta} dsd\tau. \]  \hspace{1cm} (2.16)\]

Since \( \beta > 1, \gamma + r < 1 \) and \( \frac{n}{2} (\frac{1}{m} - \frac{1}{2}) < 1 \) we estimate \( \mathcal{J}_n(t) \) by using Lemma 5.1 as follows:
\[ \mathcal{J}_n(t) \lesssim (1 + t)^{1 - \frac{n}{2} (\frac{1}{m} - \frac{1}{2}) (1 - r) - \gamma - r}. \]  \hspace{1cm} (2.17)\]
Then, introducing the estimate (2.17) into (2.15) we find
\[ \| u^{nl}(t, \cdot) \|_{L^2} \lesssim (1 + t)^{1 - \frac{n}{2} (\frac{1}{m} - \frac{1}{2}) (1 - r) - \gamma - r} \| u \|_{X(T)}^p. \]  \hspace{1cm} (2.18)\]

Now we deal with \( \| D^\sigma u^{nl}(t, \cdot) \|_{L^2} \). Again, taking into account the results of Proposition 3 we conclude for all \( 0 < \sigma < 1 \) the estimate
\[ \| D^\sigma u^{nl}(t, \cdot) \|_{L^2} \lesssim \int_0^t (1 + \tau)^{-\gamma} (1 + B(t, \tau))^{-\frac{n}{2} (\frac{1}{m} - \frac{1}{2}) - \frac{\sigma}{2} } \times \int_0^\tau (\tau - s)^{-\gamma} \| u(s, \cdot) \|_{L^p} \| u^{nl}(t, \cdot) \|_{L^2} dsd\tau. \]  \hspace{1cm} (2.19)\]
Using the estimate (2.14) we get
\[ \| D^\sigma u^{nl}(t, \cdot) \|_{L^2} \lesssim \| u \|_{X(T)}^p \int_0^t (1 + \tau)^{-\gamma} (1 + B(t, \tau))^{-\frac{n}{2} (\frac{1}{m} - \frac{1}{2}) - \frac{\sigma}{2}} \]
Putting (2.22) into (2.20) we derive the estimate \( X \) as follows:

Since

Then we have to estimate \( \| \cdot \| \) and the results of Proposition 3, we have

Putting (2.22) into (2.20) we derive the estimate

Then, the inequality (2.6) is concluded from (2.23), (2.18) and the definition (2.4) of the norm in \( X(T) \).

Now, let us turn to the inequality (1.12).

By definition of the operator \( N \) and the results of Proposition 3, we have

Then we have to estimate

First, we have by Hölder’s inequality for \( j = m, 2 \),

Then, the aim is to estimate the following terms for \( j = m \) and \( j = 2 \):

Let us turn to estimate \( \|u(s, \cdot) - v(s, \cdot)\|_{L^p} \). By using Gagliardo-Nirenberg inequality (5.2) with \( k = \sigma \) and \( q = jp \) we estimate

We use the same tools to estimate \( \|u(s, \cdot)\|_{L^p}^{p-1} \). We get

\[
\|u(s, \cdot)\|_{L^p}^{p-1} \lesssim \|u(s, \cdot)\|_{L^p}^{(p-1)(1-\theta(jp))} \|D\|^{\theta(jp)} \|u(s, \cdot)\|_{L^p}^{\theta(jp)}
\]
\[
\lesssim (1 + s)^{(p-1)(1-\theta(jp))} \times (1 + s)^{(p-1)(1-\theta(jp))(1-\frac{1}{m} - \frac{1}{2} - \gamma - r)} \|u\|_{X(T)}^{p-1}
\]
\[
\lesssim (1 + s)^{p-1} \|u\|_{X(T)}^{p-1}. \tag{2.27}
\]
In the same way we derive
\[
||v(s, \cdot)||^{p-1}_{L^p} \lesssim (1 + s)^{(p-1)(1-\frac{2}{m} - \frac{1}{2}) - \gamma - \tau - (p-1)\frac{\gamma}{2}(1-\tau)} ||v||^{p-1}_{X(T)}.
\] (2.28)

Finally, by (2.28), (2.27) and (2.26) we get from (2.25), for \(j = m, 2\) the estimates
\[
||u(s, \cdot)||^p - ||v(s, \cdot)||^p||_{L^1} \lesssim (1 + s)^{-\beta_j} ||u - v||_{X(T)}(||u||^{p-1}_{X(T)} + ||v||^{p-1}_{X(T)}),
\] (2.29)

where \(\theta(jp)\) and \(\beta_j\) are given by (2.9) and (2.10), respectively. Taking \(j = m\) and \(j = 2\) in (2.29) it follows
\[
||u||^p - ||v||^p||_{L^m} \lesssim (1 + s)^{-\beta_m} ||u - v||_{X(T)}(||u||^{p-1}_{X(T)} + ||v||^{p-1}_{X(T)}),
\] (2.30)

and
\[
||u||^p - ||v||^p||_{L^2} \lesssim (1 + s)^{-\beta_2} ||u - v||_{X(T)}(||u||^{p-1}_{X(T)} + ||v||^{p-1}_{X(T)}).
\] (2.31)

Noting that
\[
\beta_2 = \beta_m + \frac{n}{2} \left(\frac{1}{m} - \frac{1}{2}\right)(1 - r),
\]
we conclude from (2.30) and (2.31)
\[
||u||^p - ||v||^p||_{L^m \cap L^2} \lesssim (1 + s)^{-\beta} ||u - v||_{X(T)}(||u||^{p-1}_{X(T)} + ||v||^{p-1}_{X(T)}),
\] (2.32)

where \(\beta\) is given by (2.13). Plugging the estimate (2.32) in (2.24) we find
\[
||(Nu - Nv)(t, \cdot)||^p \lesssim J_n(t)||u - v||_{X(T)}(||u||^{p-1}_{X(T)} + ||v||^{p-1}_{X(T)}),
\] where \(J_n(t)\) is defined by (2.16). Then, using (2.17) we get the estimate
\[
||(Nu - Nv)(t, \cdot)||^p \lesssim (1 + t)^{1 - \frac{m}{2}(\frac{1}{n} - \frac{1}{2})} (1 - \gamma) ||u - v||_{X(T)}(||u||^{p-1}_{X(T)} + ||v||^{p-1}_{X(T)}).
\]

Now, let us turn to estimate the term \(||D|\sigma (Nu - Nv)(t, \cdot)||^p \lesssim 2.33\). First, applying the results of Proposition 3, we arrive at
\[
||D|\sigma (Nu - Nv)(t, \cdot)||^p \lesssim \int_0^t (1 + \tau)^{-r} (1 + B(t, \tau))^{-\frac{\gamma}{2}(\frac{1}{n} - \frac{1}{2})} - \frac{\gamma}{2}
\times \int_0^\tau (\tau - s)^{-\gamma} ||u(s', \cdot)||^p - ||v(s', \cdot)||^p||_{L^m \cap L^2} dsdt.
\] (2.33)

Including the estimate (2.32) into (2.33) we find
\[
||D|\sigma (Nu - Nv)(t, \cdot)||^p \lesssim ||u - v||_{X(T)}(||u||^{p-1}_{X(T)} + ||v||^{p-1}_{X(T)}) \int_0^t (1 + \tau)^{-r} (1 + B(t, \tau))^{-\frac{\gamma}{2}(\frac{1}{n} - \frac{1}{2})} - \frac{\gamma}{2}
\times \int_0^\tau (\tau - s)^{-\gamma} (1 + s)^{-\beta} dsdt
\lesssim J_{n, \sigma}(t)||u - v||_{X(T)}(||u||^{p-1}_{X(T)} + ||v||^{p-1}_{X(T)}),
\] (2.34)

where \(J_{n, \sigma}(t)\) is defined by (2.21). Using the estimate (2.22) for \(J_{n, \sigma}(t)\) we find from (2.34) the estimate
\[
||D|\sigma (Nu - Nv)(t, \cdot)||^p \lesssim (1 + t)^{-\frac{m}{2}(\frac{1}{n} - \frac{1}{2})} (1 - \gamma) ||u - v||_{X(T)}(||u||^{p-1}_{X(T)} + ||v||^{p-1}_{X(T)}).
\] (2.35)

Finally, the desired inequality is concluded from the estimates (2.35), (2.33) and the definition (2.4) for the norm in \(X(T)\). This ends the proof of Theorem 2.1. \(\square\)
2.3. Other well-posedness results for low regular data. In the following we discuss the case \( \gamma + r \geq 1 \) in contrary to the statement of Theorem 2.1, where we supposed \( \gamma + r < 1 \).

**Theorem 2.2.** Assume that \( \gamma, r \in (0, 1) \), and \( \gamma + r > 1 \). Moreover, \( \frac{2}{m} \leq p \leq \frac{n}{n-2\sigma} \), \( \frac{2}{m} \leq p < \infty \) if \( n \leq 2\sigma \) with \( n \geq 1 \), \( m \in [1, 2] \) and \( \sigma \in (0, 1) \). Then, there exists a positive constant \( \varepsilon_0 \) such that for any initial data

\[ (u_0, u_1) \in A_{m, \sigma} := (L^m \cap H^\sigma) \times (L^m \cap L^2) \]

satisfying

\[ \|(u_0, u_1)\|_{A_{m, \sigma}} \leq \varepsilon \text{ for all } \varepsilon \leq \varepsilon_0, \]

there is a uniquely determined global (in time) Sobolev solution

\[ u \in C([0, \infty), H^\sigma) \]

to the Cauchy problem (2.1) in the following cases:

1. \( \frac{2}{n} (\frac{1}{m} - \frac{1}{2}) + \frac{2}{r} \leq 1 \), where the exponent \( p \) satisfies the condition

\[ p > 1 + \frac{2m}{n(1-r)} \] \hspace{1cm} (2.36)

and the solution satisfies the following estimates:

\[ \|u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2} (\frac{1}{m} - \frac{1}{2})(1-r)} \|(u_0, u_1)\|_{A_{m, \sigma}}, \]

\[ \|D^\sigma u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2} (\frac{1}{m} - \frac{1}{2})(1-r)-\frac{\sigma}{2}} \|(u_0, u_1)\|_{A_{m, \sigma}}; \]

2. \( \frac{2}{n} (\frac{1}{m} - \frac{1}{2}) < 1 < \frac{2}{n} (\frac{1}{m} - \frac{1}{2}) + \frac{2}{r} \), where the exponent \( p \) satisfies the conditions

\[ p > \begin{cases} 
 1 + \frac{2m}{n(1-r)} & \text{if } (a_m + \frac{2}{r})(1-r) \leq \gamma, \\
 1 + \frac{1 - a_m(1-r) - \frac{2}{r} (\frac{1}{m} - \frac{1}{2})(a_m(1-r) - \gamma)}{[a_m(1-r) - \frac{2\sigma}{n} (a_m(1-r) - \gamma)]} & \text{if } (a_m + \frac{2}{r})(1-r) > \gamma, 
\end{cases} \] \hspace{1cm} (2.37)

and the solution satisfies the following estimates:

\[ \|u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2} (\frac{1}{m} - \frac{1}{2})(1-r)} \|(u_0, u_1)\|_{A_{m, \sigma}}, \]

\[ \|D^\sigma u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\min \left\{ \frac{\sigma}{2} (\frac{1}{m} - \frac{1}{2}) + \frac{n}{2} (1-r) \gamma \right\}} \|(u_0, u_1)\|_{A_{m, \sigma}}; \]

3. \( \frac{2}{n} (\frac{1}{m} - \frac{1}{2}) \geq 1 \), where the exponent \( p \) satisfies the conditions

\[ p > \begin{cases} 
 1 & \text{if } a_m(1-r) \geq \gamma, \\
 1 + \frac{1 - a_m(1-r) - \frac{n}{2} (\frac{1}{m} - \frac{1}{2})(a_m(1-r) - \gamma)}{2m [a_m(1-r) - \frac{2\sigma}{n} (a_m(1-r) - \gamma)]} & \text{if } 0 < \gamma < (a_m + \frac{2}{r})(1-r), \] \hspace{1cm} (2.38)

and the solution satisfies the following estimates:

\[ \|u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\min \left\{ \frac{\sigma}{2} (\frac{1}{m} - \frac{1}{2}) + \frac{n}{2} (1-r) \gamma \right\}} \|(u_0, u_1)\|_{A_{m, \sigma}}, \]

\[ \|D^\sigma u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\min \left\{ \frac{\sigma}{2} (\frac{1}{m} - \frac{1}{2}) + \frac{n}{2} (1-r) \gamma \right\}} \|(u_0, u_1)\|_{A_{m, \sigma}}. \]

Here \( a_m = \frac{n}{2} (\frac{1}{m} - \frac{1}{2}) \).
Theorem 2.3. Let us assume the conditions of Theorem 2.2, but now we suppose \( \gamma + r = 1 \). Then, there exists a positive constant \( \varepsilon_0 \) such that for any initial data 
\[
(u_0, u_1) \in A_{m, \sigma} := (L^m \cap H^\sigma) \times (L^m \cap L^2)
\]
satisfying
\[
\|(u_0, u_1)\|_{A_{m, \sigma}} \leq \varepsilon \text{ for all } \varepsilon \leq \varepsilon_0,
\]
there is a uniquely determined global (in time) Sobolev solution 
\[
u \in C([0, \infty), H^\sigma)
\]
to the Cauchy problem (2.1) in the following cases:
1. \( \frac{n}{2}(1 - \frac{1}{m}) + \frac{n}{2} \leq 1 \), where the exponent \( p \) satisfies the condition
\[
p > 1 + \frac{2m}{n(1 - r)} \tag{2.39}
\]
and the solution satisfies the following estimates:
\[
\|u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2}\left(1 - \frac{1}{m}\right)} \log(2 + t) \|(u_0, u_1)\|_{A_{m, \sigma}},
\]
\[
\|D^\sigma u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2}\left(1 - \frac{1}{m}\right) - \frac{n}{2}(1 - r)} \log(2 + t) \|(u_0, u_1)\|_{A_{m, \sigma}}.
\]
2. \( \frac{n}{2}(1 - \frac{1}{m}) \geq 1 \), where the exponent \( p \) satisfies the condition
\[
p > \frac{1}{\gamma} \tag{2.40}
\]
and the solution satisfies the following estimates:
\[
\|u(t, \cdot)\|_{L^2} \lesssim \ell(t)(1 + t)^{-\gamma} \|(u_0, u_1)\|_{A_{m, \sigma}},
\]
\[
\|D^\sigma u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\gamma} \|(u_0, u_1)\|_{A_{m, \sigma}}.
\]

Here, we introduce
\[
\ell(t) = \begin{cases} 
1 & \text{if } \frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) > 1, \\
\log(2 + t) & \text{if } \frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) = 1.
\end{cases}
\]

Remark 1. In the last result we leave the case \( a_m \in (1 - \frac{n}{2}, 1) \) out. If we put formally \( \frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) + \frac{n}{2} = 1 \) and \( \gamma + r = 1 \) in (2.2), then we obtain the condition (2.39). If we put formally \( \frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) + \frac{n}{2} = 1 \) in the second case of Theorem 2.2, then the second condition in (2.37) has no meaning. Consequently, (2.37) coincides with (2.36). If we put formally \( \frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) = 1 \) in the second case of Theorem 2.2, then the first condition in (2.38) has no meaning. Consequently, (2.37) coincides with (2.38). If we put formally \( \frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) = 1 \) in the second case of Theorem 2.3, then the condition (2.40) coincides with the first condition in (2.38). Finally, if we put \( a_m(1 - r) = \gamma \) in the second condition of (2.38), then it coincides with the first condition of (2.38).

2.4. Proof of Theorems 2.2 and 2.3. The proofs of Theorems 2.2 and 2.3 follow more or less the lines of the previous proofs. For this reason we sketch very briefly the required modifications. As we did in the proof of Theorem 2.1, we define a suitable weighted space of Sobolev solutions as in (2.3) and (2.4). Once the space is fixed, we are going to estimate \( \|u(s, \cdot\|_{L^m \cap L^2} \) which can be realized after using the Gagliardo-Nirenberg inequality (5.2) as in the proof of Theorem 2.1. So we arrive at the integrals (2.16) and (2.21). In order to estimate these integrals we recall Lemma 5.1 and take \( \beta > 1 \) to fix a lower bound for the admissible exponents \( p \).
in both integrals. In this way the values of $\alpha_1 := \frac{2}{m} \left( \frac{1}{m} - \frac{1}{2} \right) + \frac{\epsilon}{2}$ and of $\alpha_2 := \frac{2}{m} \left( \frac{1}{m} - \frac{1}{2} \right)$ come into play. For Theorem 2.2, the fact $\gamma + r > 1$ obstructs, sometimes, to express explicitly $\min\{\alpha_j(1-r)\gamma\}$, $j = 1, 2$. For Theorem 2.3 we, sometimes, interchange between $\gamma$ and $1 - r$, and have used the estimate $(1+t)^{-\beta} \log(2+t) \lesssim (1+t)^{-\beta'}$ for all $\beta' \in (1, \beta)$.

3. Main results for higher regular data. In this section we present our main results for the global (in time) well-posedness of small data solutions to the Cauchy problem

$$u_{tt} - \Delta u + (1+t)^r u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau, x)|^p d\tau, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,$$

where $r \in (-1, 1)$, $\gamma \in (0, 1)$ and the data $(u_0, u_1)$ is supposed to belong to suitable function spaces with higher regularity. In general, such results are applicable for higher dimensions $n$.

3.1. Well-posedness of energy solutions with suitable regularity. For all $n \geq 3$, $\sigma > 1$, $m \in [1, 2)$, $\gamma \in (0, 1)$ and $r \in (-1, 1)$, we set

$$p_m(\gamma, n, r, \sigma) := \frac{2\sigma + 2n}{m} \frac{(1-a_m)(1-r)}{(n-2\sigma)(1-a_m)(1-r) + 2\sigma},$$

where here and in the sequel we set, sometimes, $a_m = \frac{n}{2} \left( \frac{1}{m} - \frac{1}{2} \right)$. We are going to prove the following statement.

**Theorem 3.1.** Let us assume $n \geq 3$, $\sigma \in (1, \frac{n}{2})$, $\gamma \in (0, 1)$ and $r \in (-1, 1)$ such that $\gamma + r < 1$. Moreover,

$$\frac{n}{2} \left( \frac{1}{m} - \frac{1}{2} \right) < 1, \quad \frac{n}{2} \left( \frac{1}{m} - \frac{1}{2} \right) + \frac{\sigma}{2} > 1 \text{ with } m \in [1, 2).$$

For the exponent $p$ we suppose the condition

$$p > \max\left\{ [\sigma]; p_m(\gamma, n, r, \sigma) \right\},$$

where $p_m(\gamma, n, r, \sigma)$ is defined by (3.2). Finally, $p$ satisfies

$$p \leq 1 + \frac{2}{n-2\sigma}. \quad (3.5)$$

Then, there exists a positive constant $\varepsilon_0$ such that for any initial data

$$(u_0, u_1) \in \mathcal{A}_{m, \sigma} := (L^m \cap H^{\sigma}) \times (L^m \cap H^{\sigma-1})$$

satisfying

$$\|(u_0, u_1)\|_{\mathcal{A}_{m, \sigma}} \leq \varepsilon \text{ for all } \varepsilon \leq \varepsilon_0,$$

there is a uniquely determined global (in time) energy solution

$$u \in C([0, \infty), H^{\sigma}) \cap C^1([0, \infty), H^{\sigma-1})$$

to the Cauchy problem (3.1). Moreover, the solution satisfies the following estimates:

$$\|u(t, \cdot)\|_{L^2} \lesssim (1+t)^{1-\frac{n}{2} \left( \frac{1}{m} - \frac{1}{2} \right) (1-r) - \gamma - r} \|(u_0, u_1)\|_{\mathcal{A}_{m, \sigma}},$$

$$\|D^{\sigma} u(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}_{m, \sigma}},$$

$$\|u_t(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\gamma - r} \|(u_0, u_1)\|_{\mathcal{A}_{m, \sigma}},$$

$$\|D^{\sigma-1} u_t(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\gamma - r} \|(u_0, u_1)\|_{\mathcal{A}_{m, \sigma}}.$$
3.2. **Proof of Theorem 3.1.** First, let us introduce for $T > 0$ the space of energy solutions with suitable regularity

$$X(T) = C([0, T], H^\sigma) \cap C^1([0, T], H^{\sigma-1})$$

with the norm

$$\|u\|_{X(T)} = \sup_{0 \leq t \leq T} \left\{ (1 + t) \frac{2}{\theta}(\frac{1}{\theta} - \frac{1}{2}(1 - \theta) + \gamma + r - 1) \|u(t, \cdot)\|_{L^2} + (1 + t)^\gamma \|D^\sigma u(t, \cdot)\|_{L^2} + (1 + t)^{\gamma + r} \|D^{\sigma-1} u(t, \cdot)\|_{L^2} \right\}.$$ (3.6)

As it is explained in the proof of Theorem 2.1 we have immediately after applying the Propositions 2 and 4 the estimate

$$\|u^{in}\|_{X(T)} \lesssim \|(u_0, u_1)\|_{A_{m, \sigma}}.$$ 

Then, our aim is to prove the inequality (2.6). Thanks to the results of Proposition 4 we have for $\kappa = 0, \sigma$ (with the convention $|D|^{\kappa} = \text{Id}$ if $\kappa = 0$),

$$\|D^\kappa u^{in}(t, \cdot)\|_{L^2} \lesssim \int_0^t (1 + \tau)^{-r} (1 + B(t, \tau))^\frac{2}{\theta}(\frac{1}{\theta} - \frac{1}{2}) - \frac{r}{2} \times \int_0^\tau (\tau - s)^{-r} \|u(s, \cdot)^\theta\|_{L^\infty} \|u(s, \cdot)\|_{L^2} ds d\tau.$$ (3.7)

For this reason we shall estimate the norms

$$\|u(s, \cdot)^\theta\|_{L^\infty}, \quad \|u(s, \cdot)^\theta\|_{L^2}, \quad \text{and} \quad \|u(s, \cdot)^\theta\|_{H^{\sigma-1}}.$$ 

As in the proof of Theorem 2.1 we verify for $j = m, 2$ the estimates

$$\|u(s, \cdot)^\theta\|_{L^p} \approx \|u(s, \cdot)^{\theta(jp)}\|_{L^p} \lesssim \|u(s, \cdot)^{\theta(jp)}\|_{L^\infty} \|D^\sigma u(s, \cdot)^{\theta(jp)}\|_{L^p} \lesssim (1 + s)^{\theta(jp)(1 - a_m(1 - r) - \gamma - r) - \gamma\theta(jp)} \|u\|_{X(T)}^p \lesssim (1 + s)^{-\beta_j} \|u\|_{X(T)}^p,$$

where $\theta(jp) = \frac{n}{\sigma} \left( \frac{1}{2} - \frac{1}{jp} \right) \in [0, 1]$ if and only if

$$\frac{2}{j} \leq p \leq \frac{2n}{(n - 2\sigma)j} \quad \text{if} \quad n > 2\sigma,$$

and

$$-\beta_j = p(1 - \theta(jp))(1 - a_m(1 - r) - \gamma - r) - \gamma p\theta(jp) = p(1 - a_m(1 - r) - \gamma - r) - p\theta(jp)(1 - a_m)(1 - r)$$

$$= p(1 - a_m(1 - r) - \gamma - r) - p(1 - a_m)(1 - r) \frac{n}{\sigma} \left( \frac{1}{2} - \frac{1}{jp} \right)$$

$$= p(1 - a_m(1 - r) - \gamma - r) - \frac{n}{2\sigma}(1 - a_m)(1 - r) + \frac{n}{\sigma}(1 - a_m)(1 - r)$$

$$= \left( \frac{1 - a_m}{2\sigma} \right)(1 - a_m)(1 - r) - \frac{n}{\sigma}(1 - a_m)(1 - r).$$

We note that $\beta_2 = \beta_m + \frac{n}{\sigma} \left( \frac{1}{2} - \frac{1}{m} \right)(1 - a_m)(1 - r), \quad r < 1$ and $0 < a_m < 1$. Then we may estimate

$$\|u(s, \cdot)^\theta\|_{L^\infty} \lesssim (1 + s)^{-\beta} \|u\|_{X(T)}^p,$$ (3.8)

where

$$\beta = \beta_m = \left( \frac{n}{2\sigma} - 1 \right)(1 - a_m)(1 - r) + \gamma \frac{n}{\sigma}(1 - a_m)(1 - r),$$ (3.9)
under the conditions \( \frac{2}{n\sigma} \leq p \leq \frac{n}{n-2\sigma} \) and \( p > p_m(\gamma, n, r, \sigma) \). Both conditions are satisfied because of the assumptions (3.4) and (3.5). Here we use that due to the assumptions, in particular, \( \sigma \in (1, \frac{n}{2}) \) we conclude
\[
\left( \frac{n}{2\sigma} - 1 \right)(1 - a_m)(1 - r) + \gamma > 0.
\]

To estimate the norm \( \|u(s, \cdot)|^p\|_{\dot{H}^{\sigma-1}} \) we apply the fractional chain rule (5.3) as follows:
\[
\|u(s, \cdot)|^p\|_{\dot{H}^{\sigma-1}} = \|D|^{\sigma-1}|u(s, \cdot)|^p\|_{L^2} \lesssim \|u(s, \cdot)|^{p-1}\|D|^{\sigma-1}u(s, \cdot)\|_{L^2} \quad \text{for} \quad p > [\sigma - 1],
\]
where
\[
\frac{p-1}{q_1} + \frac{1}{q_2} = \frac{1}{2}.
\]

A possible choice is, for example, \( q_1 = n(p - 1) \) and \( q_2 = \frac{2n}{n-\sigma} \) for \( n \geq 3 \). The norm \( \|u(s, \cdot)|^{p-1}\|_{L^2} \) can be estimated by using classical Gagliardo-Nirenberg inequality (5.2). In this way we get
\[
\|u(s, \cdot)|^{p-1}\|_{L^2} \lesssim \|u(s, \cdot)|^{(p-1)(1-\theta_3(q_1))}\|D|^{\sigma-1}u(s, \cdot)\|_{L^2}^{(p-1)\theta_3(q_1)} \lesssim (1 + s)^{(p-1)(1-\frac{2}{n\sigma})(1-\gamma) + \frac{1}{n}(1-a_m)(1-r)\frac{n}{n-\sigma}} \|u\|_{X(T)}^{p-1},
\]
where \( \theta_3(q_1) = \frac{n}{\sigma} \left( \frac{1}{2} - \frac{1}{q_1} \right) \in [0, 1] \) if and only if
\[
2 \leq q_1 \leq \frac{2n}{n-2\sigma} \quad \text{if} \quad n > 2\sigma.
\]

On the other hand, applying the fractional Gagliardo-Nirenberg inequality (5.1) gives
\[
\|D|^{\sigma-1}u(s, \cdot)\|_{L^2} \lesssim \|u(s, \cdot)|^{1-\theta_{\sigma, \sigma-1}(q_2)}\|D|^{\sigma}u(s, \cdot)\|_{L^2}^{\theta_{\sigma, \sigma-1}(q_2)} \lesssim (1 + s)^{(1-\frac{2}{n\sigma})(1-\gamma) + \frac{1}{n\sigma}(1-a_m)(1-r)\frac{n}{n-\sigma}} \|u\|_{X(T)},
\]
where \( \theta_{\sigma, \sigma-1}(q_2) = \frac{n}{\sigma} \left( \frac{1}{2} - \frac{1}{q_2} \right) + \frac{\sigma-1}{\sigma} \in \left[ \frac{\sigma-1}{\sigma}, 1 \right] \) if and only if
\[
2 \leq q_2 \leq \frac{2n}{n-\sigma} \quad \text{if} \quad n \geq 3.
\]

Substituting the estimates (3.13) and (3.12) into (3.10) we obtain after using (3.11) and the facts that
\[
0 \leq a_m < 1, \quad \sigma > 1, \quad r < 1 \quad \text{and} \quad (p-1)\theta_3(q_1) + \theta_{\sigma, \sigma-1}(q_2) \geq p\theta(mp) \quad \text{for} \quad m \in [1, 2]
\]
the estimate
\[
\|u(s, \cdot)|^p\|_{\dot{H}^{\sigma-1}} \lesssim (1 + s)^{-\beta-\beta_0} \|u\|_{X(T)}^p \lesssim (1 + s)^{-\beta} \|u\|_{X(T)}^p,
\]
where \( \beta \) is defined by (3.9) and \( \beta_0 \geq 0 \). Consequently, from the estimates (3.14) and (3.8) we conclude the estimate
\[
\|u(s, \cdot)|^p\|_{L^m \cap L^2 \cap \dot{H}^{\sigma-1}} \lesssim (1 + s)^{-\beta} \|u\|_{X(T)}^P,
\]
Now let us turn to estimate the norms \( \|u\|_{X(T)}^p \) and \( \|u\|_{X(T)}^p \). The estimates (3.18) and (3.19) lead to the following estimates:

\[
\|D^{\kappa}u(t,\cdot)\|_{L^2} \lesssim \|u\|_{X(T)}^p \int_0^t (1 + \tau)^{-r}(1 + B(t, \tau))^{-\frac{2}{\beta} - \frac{1}{2}} - \frac{2}{7}
\times \int_0^\tau (\tau - s)^{-\gamma}(1 + s)^{-\beta} dsd\tau \lesssim J_{n,\kappa}(t)\|u\|_{X(T)}^p, \tag{3.16}
\]

where

\[
J_{n,\kappa}(t) = \int_0^t (1 + \tau)^{-r}(1 + B(t, \tau))^{-\frac{2}{\beta} - \frac{1}{2}} - \frac{2}{7} \int_0^\tau (\tau - s)^{-\gamma}(1 + s)^{-\beta} dsd\tau. \tag{3.17}
\]

Then, thanks to Lemma 5.1, the assumption \( \beta > 1 \), or equivalently, \( p > p_m(\gamma, n, r, \sigma) \), and the assumption (3.3) we may estimate \( J_{n,\kappa}(t) \) for \( \kappa = 0 \) or \( \kappa = \sigma \) as follows:

\[
J_{n,0}(t) \lesssim (1 + t)^{-\frac{2}{\beta} - \frac{1}{2} + (1 - r) - \gamma - r}, \tag{3.18}
\]

\[
J_{n,\sigma}(t) \lesssim (1 + t)^{-\gamma}, \tag{3.19}
\]

respectively. The estimates (3.18) and (3.19) lead to the following estimates:

\[
\|u^{nl}(t,\cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{2}{\beta} - \frac{1}{2} + (1 - r) - \gamma - r}\|u\|_{X(T)}^p, \tag{3.20}
\]

and

\[
\|D^\sigma u^{nl}(t,\cdot)\|_{L^2} \lesssim (1 + t)^{-\gamma}\|u\|_{X(T)}^p. \tag{3.21}
\]

Now let us turn to estimate the norms \( \|u^{nl}(t,\cdot)\|_{L^2} \) and \( \|D^\sigma u^{nl}(t,\cdot)\|_{L^2} \). Again, by Proposition 4, we have for \( \kappa = 1, \sigma \) the estimates

\[
\|D^{\kappa-1}u^{nl}(t,\cdot)\|_{L^2} \lesssim (1 + t)^{-r} \int_0^t (1 + \tau)^{-r}(1 + B(t, \tau))^{-\frac{2}{\beta} - \frac{1}{2} - \frac{2}{7}} - \frac{2}{7} \int_0^\tau (\tau - s)^{-\gamma}\|u(s,\cdot)\|^p_{L^p \cap L^2 \cap H^{\sigma-1}} dsd\tau. \tag{3.22}
\]

Taking into account the estimate (3.15) we get from (3.22) the estimates

\[
\|D^{\kappa-1}u^{nl}(t,\cdot)\|_{L^2} \lesssim (1 + t)^{-r}\mathcal{G}_{n,\kappa}(t)\|u\|_{X(T)}^p, \tag{3.23}
\]

where

\[
\mathcal{G}_{n,\kappa}(t) = \int_0^t (1 + \tau)^{-r}(1 + B(t, \tau))^{-\frac{2}{\beta} - \frac{1}{2} - \frac{2}{7}} - \frac{2}{7} \int_0^\tau (\tau - s)^{-\gamma}(1 + s)^{-\beta} dsd\tau. \tag{3.24}
\]

In both cases \( \kappa = 1 \) and \( \kappa = \sigma \), after using Lemma 5.1 the integral \( \mathcal{G}_{n,\kappa}(t) \) is estimated as follows:

\[
\mathcal{G}_{n,\kappa}(t) \lesssim (1 + t)^{-\gamma}. \tag{3.25}
\]

Including the estimate (3.25) into (3.23) we find for \( \kappa = 1 \)

\[
\|u^{nl}(t,\cdot)\|_{L^2} \lesssim (1 + t)^{-r}\|u\|_{X(T)}^p, \tag{3.26}
\]

and for \( \kappa = \sigma \)

\[
\|D^\sigma u^{nl}(t,\cdot)\|_{L^2} \lesssim (1 + t)^{-\gamma}\|u\|_{X(T)}^p. \tag{3.27}
\]

Finally, the desired inequality (2.6) is concluded after using the estimates (3.20), (3.21) (3.26), (3.27) and the definition (3.6) of the norm in \( X(T) \). Summarizing...
we obtained (1.11). Now, we turn to the inequality (1.12). Applying Proposition 4
implies for $\kappa = 0, \sigma$ the estimates
\[
\| D^{\kappa}(N u - N v)(t, \cdot) \|_{L^2} \lesssim \int_0^T \left( 1 + \tau \right)^{-r} (1 + B(t, \tau))^{-\frac{7}{2} (\frac{1}{m} - \frac{1}{2}) - \frac{r}{2}} \times \int_0^T (\tau - s)^{-\gamma} \| |u(s, \cdot)|^p - |v(s, \cdot)|^p \|_{L^m \cap L^2 \cap H^{\sigma-1}} \, ds \, d\tau.
\] (3.28)

Then, we have to estimate the norms
\[
\| |u(s, \cdot)|^p - |v(s, \cdot)|^p \|_{L^m \cap L^2 \cap H^{\sigma-1}}, \quad \| |u(s, \cdot)|^p - |v(s, \cdot)|^p \|_{H^{\sigma-1}}.
\]

As we did in the proof of Theorem 2.1, we show immediately that
\[
\| |u(s, \cdot)|^p - |v(s, \cdot)|^p \|_{L^m \cap L^2 \cap H^{\sigma-1}} \lesssim (1 + s)^{-\beta} \| u - v \|_{X(T)} (\| u \|_{X(T)}^{p-1} + \| v \|_{X(T)}^{p-1}), \quad (3.29)
\]
with $\beta$ as in (3.9). Let us estimate the norm $\| |u(s, \cdot)|^p - |v(s, \cdot)|^p \|_{H^{\sigma-1}}$. First, applying Leibniz formula from Proposition 6 allows us to conclude for $p > [\sigma]$ the estimate
\[
\| |u(s, \cdot)|^p - |v(s, \cdot)|^p \|_{H^{\sigma-1}} \lesssim \int_0^1 \left| \| D^{\sigma-1} (|u|^p - |v|^p)(s, \cdot) \|_{L^2} \right| \, dw \\
\lesssim \int_0^1 \| D^{\sigma-1} (u - v)(s, \cdot) \|_{L^2} \| u - w(u - v) \|_{L^2} \| |u - w(u - v)|^p - 2(s, \cdot) \|_{L^2} \, dw \\
\lesssim \int_0^1 \| u - v(s, \cdot) \|_{L^3 \cap L^2} \| D^{\sigma-1} (u - v) \|_{L^4} \| |u - w(u - v)|^p - 2(s, \cdot) \|_{L^4} \, dw
\]
with
\[
\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4} = \frac{1}{2}.
\] (3.30)

The term $\| D^{\sigma-1} (u - v)(s, \cdot) \|_{L^1}$ can be estimated by using the fractional Gagliardo-Nirenberg inequality (5.1) in the form
\[
\| D^{\sigma-1} (u - v)(s, \cdot) \|_{L^1} \lesssim \| u(s, \cdot) - v(s, \cdot) \|_{L^2}^{1 - \theta_31(r_1)} \| D^{\sigma} (u - v)(s, \cdot) \|_{L^2}^{\theta_31(r_1)} \\
\lesssim (1 + s)^{1 - \frac{\theta_31(r_1)}{2} (1 - a_m) (1 - r) - \gamma + \frac{\theta_31(r_1)}{2} (1 - a_m) (1 - r) - (1 - a_m) (1 - r)} u - v X(T)
\] (3.32)

with $\theta_31(r_1) = \frac{\theta_21}{2} - \frac{1}{r_1}$, $\frac{\theta_21}{2} + \frac{\sigma - 1}{r_1} \in [\frac{\sigma - 1}{2}, 1] \subseteq [0, 1]$ if and only if
\[
2 \leq r_1 \leq \frac{2n}{n - 2} \quad \text{for } n \geq 3.
\]

In the same way, by using the classical Gagliardo-Nirenberg inequality, we estimate the norm $\| (u - w(u - v)) u - w(u - v) \|_{L^2}^{p-2} (s, \cdot)$ as follows:
\[
\| (u - w(u - v)) u - w(u - v) \|_{L^2}^{p-2} (s, \cdot) \|_{L^2}^{p-1} (s, \cdot) \| \lesssim \| u - w(u - v) \|_{L^2}^{(p-1) \theta_32(r_2)} \| D^{\sigma} (u - w(u - v)) \|_{L^2}^{(p-1) \theta_32(r_2)} \\
\lesssim (1 + s)^{(1 - \frac{\theta_31}{2}) (1 - a_m) (1 - r) - \gamma + \frac{\theta_31}{2} (1 - a_m) (1 - r)} u - w(u - v) X(T)
\] (3.33)
where \( \theta_{32}(r_2) = \frac{\sigma}{\sigma} \left( \frac{1}{2} - \frac{1}{(p-1)r_2} \right) \in [0, 1] \) if and only if
\[
\frac{2}{p-1} \leq r_2 \leq \frac{2n}{(p-1)(n-2\sigma)} \quad \text{if } n > 2\sigma.
\]

Due to the estimates (3.33) and (3.32), we get after using (3.31) and the facts that \( r < 1, \sigma > 1 \) and \( \sigma > 1 \) the estimate
\[
\| D^{\sigma-1}(u - v)(s, \cdot) \|_{L^2} \| (u - w(u - v))(u - w(u - v))^{p-2}(s, \cdot) \|_{L^2} \lesssim (1 + s)^{-\beta} \left( \frac{1}{a_m} \right)^{(1-a_m)(1-r) - \gamma + \frac{1}{r_5}(1-a_m)(1-r)} \| u - v \|_{X(T)}
\]
where \( \beta = \frac{a}{\sigma} \left( \frac{1}{2} - \frac{1}{r_3} \right) \in [0, 1] \) if and only if
\[
2 \leq r_3 \leq \frac{2n}{n - 2\sigma} \quad \text{if } n > 2\sigma.
\]

Let us turn to estimate the norm \( \| D^{\sigma-1}(u - w(u - v))(u - w(u - v))^{p-2} \|_{L^4} \).

Thanks to the fractional chain rule (5.3) we have
\[
\| D^{\sigma-1}(u - w(u - v))(u - w(u - v))^{p-2} \|_{L^4} \lesssim \| u - w(u - v) \|_{L^2}^{p-2} \| D^{\sigma-1}(u - w(u - v)) \|_{L^6}^{p-2}
\]
with
\[
p - \frac{2}{r_5} + \frac{1}{r_6} = \frac{1}{r_4} \quad \text{and } p > [\sigma].
\]

Using the classical Gagliardo-Nirenberg inequality (5.2) gives
\[
\| (u - w(u - v))(s, \cdot) \|_{L^5}^{p-2} \lesssim \| (u - w(u - v))(s, \cdot) \|_{L^2}^{(p-2)(1-\theta_5(r_5))} \| D^{\sigma}(u - w(u - v))(s, \cdot) \|_{L^2}^{p-2} \theta_5(r_5)
\]
\[
\lesssim (1 + s)^{(1-\frac{1}{a_m})(1-r) - \gamma + \frac{1}{r_5}(1-a_m)(1-r)} \| u - w(u - v) \|_{X(T)}
\]
with \( \theta_5(r_5) = \frac{a}{\sigma} \left( \frac{1}{2} - \frac{1}{r_5} \right) \in [0, 1] \) if and only if
\[
2 \leq r_5 \leq \frac{2n}{n - 2\sigma} \quad \text{if } n > 2\sigma.
\]

Again, by applying the fractional Gagliardo-Nirenberg inequality (5.1) we may estimate the norm \( \| D^{\sigma-1}(u - w(u - v)) \|_{L^6} \) as follows:
\[
\| D^{\sigma-1}(u - w(u - v)) \|_{L^6} \lesssim \| u - w(u - v) \|_{L^2}^{1-\theta_6(r_6)} \| D^{\sigma}(u - w(u - v)) \|_{L^2}^{\theta_6(r_6)}
\]
\[
\lesssim (1 + s)^{(1-\frac{1}{a_m})(1-r) - \gamma + \frac{1}{r_6}(1-a_m)(1-r)} \| u - w(u - v) \|_{X(T)}
\]
with \( \theta_6 = \frac{a}{\sigma} \left( \frac{1}{2} - \frac{1}{r_6} \right) + \frac{\sigma - 1}{\sigma} \in [\frac{\sigma - 1}{\sigma}, 1] \) if and only if
\[
2 \leq r_6 \leq \frac{2n}{n - 2} \quad \text{for } n \geq 3.
\]
For \( r_1 \) and \( r_2 \) we may choose \( r_1 = \frac{2n}{n+2} \) and \( r_2 = n \). For \( r_3, \ldots, r_6 \), we may choose \( r_3 = r_5 = n(p-1), r_4 = \frac{2n}{n+2} \) and, consequently, we find \( r_4 = \frac{2n(p-1)}{n(p-1)-2} \).

Therefore, by using the estimates (3.38) and (3.39) we get from (3.36) after using (3.37) the estimate
\[
\|D\|^{-1}[(u-w(u-v))|u-w(u-v)|^{p-2}] \|_{L^q_4} \\
\lesssim (1+s)^{(1-a)(1-r)-(p-1)/(p-2) + \frac{3}{p-2}}(1-a)(1-r) \|u-w(u-v)\|^{-1}_{X(T)},
\]
Hence, both estimates (3.40) and (3.35) imply after using the facts \( 0 \leq a_m < 1, \sigma > 1 \) and \( r < 1 \) together
\[
\|u-v\|_{L^2} \|D\|^{-1}[(u-w(u-v))|u-w(u-v)|^{p-2}] \|_{L^q_4} \\
\lesssim (1+s)^{-\beta} \|u-v\|_{X(T)} \int_0^1 \|u-w(u-v)\|^{-1}_{X(T)} dw \\
\lesssim (1+s)^{-\beta} \|u-v\|_{X(T)} \left( \|u\|^{-1}_{X(T)} + \|v\|^{-1}_{X(T)} \right),
\]
where \( \beta \) is defined by (3.9). Then, plugging (3.34) and (3.40) into (3.30) we get
\[
\|u-v\|_{L^2} \|D\|^{-1}[(u-w(u-v))|u-w(u-v)|^{p-2}] \|_{L^q_4} \\
\lesssim (1+s)^{-\beta} \|u-v\|_{X(T)} \left( \|u\|^{-1}_{X(T)} + \|v\|^{-1}_{X(T)} \right).
\]
Finally, thanks to the estimates (3.42) and (3.29) we conclude that
\[
\|u(s,\cdot)|^p - |v(s,\cdot)|^p \|_{L^m \cap L^2 \cap H^{\sigma-1}} \lesssim (1+s)^{-\beta} \|u-v\|_{X(T)} \left( \|u\|^{-1}_{X(T)} + \|v\|^{-1}_{X(T)} \right).
\]
Now, including the estimate (3.43) into (3.28) we get
\[
\|D\|^\kappa (Nu-Nv)(t,\cdot) \|_{L^2} \lesssim J_{\kappa,\sigma}(t) \|u-v\|_{X(T)} \left( \|u\|^{-1}_{X(T)} + \|v\|^{-1}_{X(T)} \right),
\]
where \( J_{\kappa,\sigma}(t) \) is given by (3.17). Hence, due to the estimates (3.18) and (3.19) we find
\[
\|D\|^\kappa (Nu-Nv)(t,\cdot) \|_{L^2} \\
\lesssim (1+t)^{\frac{3}{2}(\frac{1}{2} - \frac{3}{4})(1-r)-\gamma} \|u-v\|_{X(T)} \left( \|u\|^{-1}_{X(T)} + \|v\|^{-1}_{X(T)} \right),
\]
and
\[
\|D\|^{-1} \|D\|^\kappa (Nu-Nv)(t,\cdot) \|_{L^2} \lesssim (1+t)^{-\gamma} \|u-v\|_{X(T)} \left( \|u\|^{-1}_{X(T)} + \|v\|^{-1}_{X(T)} \right),
\]
respectively. Now, we turn to estimate the norms
\[
\|\partial_t (Nu-Nv)(t,\cdot) \|_{L^2} \quad \text{and} \quad \|D\|^{-1} \|D\|^\kappa \partial_t (Nu-Nv)(t,\cdot) \|_{L^2}.
\]
Again by using Proposition 4 we have for \( \kappa = 1 \) and \( \kappa = \sigma \) the estimates
\[
\|D\|^{-1} \|D\|^\kappa \partial_t (Nu-Nv)(t,\cdot) \|_{L^2} \\
\lesssim (1+t)^{-\gamma} \int_0^t (1+\tau)^{-\gamma} (1+B(t,\tau))^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{4}) - \frac{3}{2} - 1} \\
\times \int_0^\tau (\tau-s)^{-\gamma} \left( \|u(s,\cdot)|^p - |v(s,\cdot)|^p \right) \| \in L^m \cap L^2 \cap H^{\sigma-1} \| ds d\tau.
\]
Including the estimate (3.43) into (3.46) we get
\[
\|D\|^{-1} \|D\|^\kappa \partial_t (Nu-Nv)(t,\cdot) \|_{L^2} \lesssim (1+t)^{-\gamma} J_{\kappa,\sigma}(t) \|u-v\|_{X(T)} \left( \|u\|^{-1}_{X(T)} + \|v\|^{-1}_{X(T)} \right),
\]
where $G_{n,k}(t)$ is defined by (3.24). Due to (3.25) we obtain for $k = 0$ the estimate
\[
\|\partial_t (Nu - Nv)(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\gamma - r} \|u - v\|_{X(T)} (\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}),
\]
and for $k = \sigma$ the estimate
\[
\|D^{\sigma-1} \partial_t (Nu - Nv)(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\gamma - r} \|u - v\|_{X(T)} (\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}).
\]
Finally, the estimates (3.48), (3.47), (3.45), (3.44) and the definition of the norm in $X(T)$ yield the desired inequality (1.12). This completes the proof of Theorem 3.1.

3.3. Other well-posedness results for energy solutions with suitable regularity

In this section we present further results for global (in time) existence of small data energy solutions with suitable regularity. The proofs are only small modifications of the proof to Theorem 3.1.

**Theorem 3.2.** Let us assume $n \geq 3$, $\sigma \in (1, \frac{7}{2})$, $\gamma \in (0, 1)$ and $r \in (-1, 1)$ such that $\gamma + r \geq 1$. Moreover,
\[
n(\frac{1}{2} - \frac{1}{m}) < 1, \quad n(\frac{1}{m} - \frac{1}{2}) + \frac{\sigma}{2} > 1 \quad \text{with} \quad m \in [1, 2),
\]
and $p$ satisfies the condition
\[
p > \max \left\{ \lfloor \sigma \rfloor; p_m(\gamma, n, r, \sigma) \right\},
\]
where $p_m(\gamma, n, r, \sigma)$ is defined as follows:
\[
p_m(\gamma, n, r, \sigma) = \begin{cases} 
2\sigma - \frac{2n}{m}(a_m(1 - r) - \gamma) & \text{if } \gamma + r > 1 \text{ and } \left(\frac{n}{2} \left(\frac{1}{m} - \frac{1}{2}\right) + \frac{\sigma}{2}\right)(1 - r) > \gamma, \\
\frac{2m}{n(1 - r)} \left[2n\gamma + (2\sigma - n)a_m(1 - r)\right]_+ & \text{if } \gamma + r > 1 \text{ and } \left(\frac{n}{2} \left(\frac{1}{m} - \frac{1}{2}\right) + \frac{\sigma}{2}\right)(1 - r) \leq \gamma, \\
2\sigma - \frac{2n}{m}(a_m(1 - r) - \gamma) & \text{if } \gamma + r = 1.
\end{cases}
\]
Finally, $p$ satisfies
\[
p \leq 1 + \frac{2}{n - 2\sigma}.
\]
Then, there exists a positive constant $\varepsilon_0$ such that for any initial data
\[
(u_0, u_1) \in A_{m, \sigma} := \left(L^m \cap H^\sigma\right) \times \left(L^m \cap H^{\sigma-1}\right)
\]
satisfying
\[
\|(u_0, u_1)\|_{A_{m, \sigma}} \leq \varepsilon \text{ for all } \varepsilon \leq \varepsilon_0,
\]
there is a uniquely determined global (in time) energy solution
\[
u \in C([0, \infty), H^\sigma) \cap C^1([0, \infty), H^{\sigma-1})
\]
to the Cauchy problem (3.1). Moreover, the solution satisfies the following estimates:
\[
\|u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{\sigma}{2}(\frac{1}{m} - \frac{1}{2})(1 - r)} \|(u_0, u_1)\|_{A_{m, \sigma}},
\]
\[
\|D^{\sigma}u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\min\left(\frac{\sigma}{2}(\frac{1}{m} - \frac{1}{2}) + \frac{\sigma}{2}, \gamma\right)} \|(u_0, u_1)\|_{A_{m, \sigma}},
\]
\[
\|u_t(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-r - \min\left((a_m + 1)(1 - r); \gamma\right)} \|(u_0, u_1)\|_{A_{m, \sigma}},
\]
\[
\|D^{\sigma-1}u_t(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-r - \min\left((a_m + \frac{\sigma}{2} + 1)(1 - r); \gamma\right)} \|(u_0, u_1)\|_{A_{m, \sigma}}.
\]
if \( \gamma + r > 1 \), where \( a_m = \frac{2}{m} \left( \frac{1}{m} - \frac{1}{2} \right) \), and
\[
\| u(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\frac{\gamma}{2} \left( \frac{1}{m} - \frac{1}{2} \right) (1 - r) \log(2 + t)} \|(u_0, u_1)\|_{A_{m, \sigma}},
\]
\[
\| |D|^{\sigma} u(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\gamma} \|(u_0, u_1)\|_{A_{m, \sigma}},
\]
\[
\| u_t(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-1} \|(u_0, u_1)\|_{A_{m, \sigma}},
\]
\[
\| |D|^{\sigma-1} u_t(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-1} \|(u_0, u_1)\|_{A_{m, \sigma}},
\]
if \( \gamma + r = 1 \).

**Theorem 3.3.** Let us assume \( n \geq 3, \sigma \in (1, \frac{n}{2}) \), \( \gamma \in (0, 1) \) and \( r \in (-1, 1) \) such that \( \gamma + r < 1 \). Moreover,
\[
n \left( \frac{1}{m} - \frac{1}{2} \right) \geq 1 \text{ with } m \in [1, 2).
\]

For the exponent \( p \) we suppose the condition
\[
\max \left\{ \left[ \sigma \right]; \frac{1}{\gamma} \right\} < p \leq 1 + \frac{2}{n - 2\sigma}.
\]
(3.50)

Then, there exists a positive constant \( \varepsilon_0 \) such that for any initial data \( (u_0, u_1) \in A_{m, \sigma} := (L^m \cap H^\sigma) \times (L^m \cap H^{\sigma - 1}) \)

satisfying
\[
\|(u_0, u_1)\|_{A_{m, \sigma}} \leq \varepsilon \text{ for all } \varepsilon \leq \varepsilon_0,
\]
there is a uniquely determined global (in time) energy solution
\[
u \in C([0, \infty), H^\sigma) \cap C^1([0, \infty), H^{\sigma - 1})
\]
to the Cauchy problem (3.1). Moreover, the solution satisfies the following estimates:
\[
\| u(t, \cdot) \|_{L^2} \lesssim \left\{ \begin{array}{ll}
(1 + t)^{-\gamma} \|(u_0, u_1)\|_{A_{m, \sigma}} & \text{if } n \left( \frac{1}{m} - \frac{1}{2} \right) > 1,
(1 + t)^{-\gamma} \log(2 + t) \|(u_0, u_1)\|_{A_{m, \sigma}} & \text{if } n \left( \frac{1}{m} - \frac{1}{2} \right) = 1,
\end{array} \right.
\]
\[
\| |D|^{\sigma} u(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\gamma} \|(u_0, u_1)\|_{A_{m, \sigma}},
\]
\[
\| u_t(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\gamma - r} \|(u_0, u_1)\|_{A_{m, \sigma}},
\]
\[
\| |D|^{\sigma-1} u_t(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\gamma - r} \|(u_0, u_1)\|_{A_{m, \sigma}}.
\]

**Remark 2.** If we formally put \( a_m = \frac{2}{m} \left( \frac{1}{m} - \frac{1}{2} \right) = 1 \) in Theorem 3.1, then condition (3.4) coincides with condition (3.50) from Theorem 3.3. If we formally put \( \gamma + r = 1 \) in Theorem 3.1, then condition (3.4) coincides with condition (3.2) from Theorem 3.2.

4. **Concluding remarks.**

4.1. **What about the assumption** \( \beta > 1 ? \) In all our proofs we apply Lemma 5.1 only. This corresponds to the case \( \beta = b > 1 \). The interested reader may ask what happens if instead we apply Lemma 5.2. We take as a toy model the model from Theorem 2.1 with \( \gamma + r < 1 \). We try to avoid the condition \( \beta > 1 \) in the proof of Theorem 2.1 and to apply the third estimate from Lemma 5.2 instead. At first we have to fix for \( T > 0 \) the space of Sobolev solutions
\[
X(T) = C([0, T], H^\sigma)
\]
with the norm
\[
\| u \|_{X(T)} = \sup_{0 \leq t \leq T} \left\{ (1 + t)^{d_1} \| u(t, \cdot) \|_{L^2} + (1 + t)^{d_1 + \kappa_1} \| |D|^{\sigma} u(t, \cdot) \|_{L^2} \right\}
\]
with a real \( d_1 \) and a positive \( \kappa_1 \), because in our model we have a parabolic effect, higher order derivatives decay faster and faster. Then we should try to close the circle, that is, to prove the inequalities (1.11) and (1.12) by using of course Lemma 5.2. Following the approach from the proof of Theorem 2.1 we get the following conditions in the case \( b := \beta < 1 \):

1. The condition
   \[
   \frac{n}{m}(1-r) + 2(\gamma - 1) + 2r > 2(1-\beta)
   \]
   so, this condition is more restricted than
   \[
   \frac{n}{m}(1-r) + 2(\gamma - 1) + 2r > 0
   \]
   in (2.2).

2. The parameters \( d_1 \) and \( \kappa_1 \) are determined as follows:
   \[
   d_1 = \frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right)(1-r) + \beta + \gamma + r - 2, \quad \kappa_1 = \frac{\sigma}{2}(1-r).
   \]

3. Finally, we get the following condition for the exponent \( p \):
   \[
   p > p_\beta := 1 + \frac{2(2-\gamma) - 2r}{\frac{2\sigma}{m}(1-r) + 2(\gamma - 1) + 2r + 2(\beta - 1)}.
   \]
   This condition is more restricted than the condition (2.2).

Summarizing we see that for our toy model the choice \( \beta > 1 \) is optimal.

4.2. Examples.

4.2.1. The case of spatial dimension \( n = 1 \). We may apply Theorem 2.1. If \( \sigma \in [\frac{1}{2},1) \), then we have the conditions \( \frac{2}{m} \leq p < \infty, \gamma + r < 1 \) and \( \frac{2}{m}(1-r) + 2(\gamma + r - 1) > 0 \). Then due to this theorem we have global (in time) Sobolev solutions for
   \[
   p > 1 + \frac{2(2-\gamma - r)}{\frac{1}{m}(1-r) + 2(\gamma + r - 1)} \quad \text{and} \quad p \geq \frac{2}{m}.
   \]
   If \( \sigma \in (0, \frac{1}{2}) \), then we have global (in time) Sobolev solutions if the condition
   \[
   1 + \frac{2(2-\gamma - r)}{\frac{1}{m}(1-r) + 2(\gamma + r - 1)} < \frac{1}{1 - 2\sigma}
   \]
   is satisfied. We may apply Theorem 2.2, part 1. If \( \sigma \in [\frac{1}{2},1) \), then we have the conditions \( \frac{2}{m} \leq p < \infty \) and \( \gamma + r > 1 \). Then due to this theorem we have global (in time) Sobolev solutions for
   \[
   p > 1 + \frac{2m}{1-r} \quad \text{and} \quad p \geq \frac{2}{m}.
   \]
   If \( \sigma \in (0, \frac{1}{2}) \), then we have global (in time) Sobolev solutions if the condition
   \[
   1 + \frac{2m}{1-r} < \frac{1}{1 - 2\sigma}
   \]
   is satisfied. In a similar way we may apply Theorem 2.3, part 1. The results of Section 3 are not applicable because of the condition \( \sigma \in (1, \frac{n}{2}) \). But, one can try, by using similar tools as in the proofs of the results from Section 3, to prove well-posedness in energy spaces with suitable higher regularity.
4.2.2. The case of spatial dimension \( n = 2 \). We may apply Theorem 2.1. We have the conditions \( \frac{2}{m} \leq p \leq \frac{2}{n-2\sigma} \), \( \gamma + r < 1 \) and \( \frac{2}{m}(1-r) + 2(\gamma + r - 1) > 0 \). If \( \sigma \in (0,1) \), then we have due to Theorem 2.1 global (in time) Sobolev solutions if the condition

\[
1 + \frac{2(2-\gamma-r)}{2m(1-r) + 2(\gamma + r - 1)} < \frac{2}{2 - 2\sigma}
\]

is satisfied. We may apply Theorem 2.2, part 1. If \( \sigma \in (0,1) \), then we have the conditions \( \frac{2}{m} \leq p \leq \frac{2}{n-2\sigma} \) and \( \gamma + r > 1 \). Then due to this theorem we have global (in time) Sobolev solutions if the condition

\[
1 + \frac{2m}{2(1-r)} < \frac{2}{2 - 2\sigma}
\]

is satisfied. In a similar way we may apply Theorem 2.3, part 1. The results of Section 3 are not applicable because of the condition \( \sigma \in (1, \frac{n}{2}) \). But, one can try, by using similar tools as in the proofs of the results from Section 3, to prove well-posedness in energy spaces with suitable higher regularity.

4.2.3. The case of spatial dimension \( n \geq 3 \). We proved several results in the case \( n \geq 3 \). But, in all these results one has to understand the admissible range of exponents \( p \) satisfying some restrictions in the results. We may apply Theorem 2.1 to the case \( n = 3 \) as we explained in the last two subsections. Now we can apply to the case \( n = 3 \) the statement 1 of Theorem 2.2, too. So, we get global (in time) Sobolev solutions if the condition

\[
1 + \frac{4}{3(1-r)} < \frac{3}{3 - 2\sigma}
\]

is satisfied. We may apply for \( n \geq 3 \) the results of Theorem 3.1. First we have to guarantee the condition

\[
[\sigma] < 1 + \frac{2}{n - 2\sigma}.
\]

This implies in the case of large \( n \) that the regularity \( \sigma \) should be close to \( \frac{n}{2} \). Moreover, we have to guarantee

\[
1 + \frac{n}{m\sigma}(1 - a_m)(1 - r) < 1 + \frac{2}{n - 2\sigma}.
\]

If

\[
\frac{1}{\gamma} < 1 + \frac{2}{n - 2\sigma},
\]

then one can choose \( m \) in such a way that \( a_m \) is close to 1 to verify the above condition. We may apply Theorem 3.3. One has among other things to guarantee the conditions

\[
\frac{n}{2} \left( \frac{1}{m} - \frac{1}{2} \right) \geq 1 \quad \text{and} \quad \max \left\{ [\sigma]; \frac{1}{\gamma} \right\} < 1 + \frac{2}{n - 2\sigma}.
\]

So, it is reasonable to suppose the regularity parameter \( \sigma \) close to \( \frac{n}{2} \). For reason of continuous dependence of supposed conditions Theorem 3.2 is applicable to some models, too.
4.3. Comparison with blow-up results. In the paper [10] the authors proved the following blow-up result for Sobolev solutions to (1.1).

Theorem 4.1. Let $0 < \gamma < 1$ and $p \in (1, \infty)$. Assume that the data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and satisfies the conditions
\[
\int_{\mathbb{R}^n} u_0(x) \, dx > 0 \quad \text{and} \quad \int_{\mathbb{R}^n} u_1(x) \, dx > 0.
\] (4.1)
Then, if $p$ satisfies for $r \in (-1, 0)$ or $r \in (0, 1)$ the condition
\[
\frac{p}{p-1} > \inf_{d>0} \max \left\{ \frac{nd+1}{1-\gamma+d}; \sqrt{\frac{nd+1}{(1-\gamma)(1-r)} + \left(1-\gamma-(1-r)\frac{nd+1}{2(1-\gamma)(1-r)}\right)^2 - \frac{1-\gamma-(1-r)\frac{nd+1}{2}}{2(1-\gamma)(1-r)}} \right\},
\] (4.2)
or for $r = 0$ the condition
\[
\frac{p}{p-1} > \frac{2}{2-\gamma},
\] (4.3)
then Sobolev solutions (defined in a special sense) of the Cauchy problem (1.1) do not exist globally in time.

At least in the case $r = 0$ we found for low dimensions $n$ the critical exponent
\[
p_{\text{crit}} = p_{\text{crit}}(n) = 1 + \frac{2-\gamma}{\frac{n}{2} + \gamma - 1}.
\]
The above result yields for $r = 0$ a blow-up behavior of Sobolev solutions for $p \in (1, p_{\text{crit}}(n))$ if $n + 2(\gamma - 1) > 0$ is supposed. On the contrary, the statement of Theorem 2.1 yields at least for $m = 1$ and $r = 0$ the global (in time) existence of small data Sobolev solutions if
\[
\max \left\{ 2; p_{\text{crit}}(n) \right\} < p \leq \frac{n}{n-2\sigma}.
\]
5. Appendix.

5.1. Some auxiliary estimates for integrals. The following lemma we apply in the proof of global existence of small data solutions to nonlinear models.

Lemma 5.1. Assume that $0 < \gamma < 1$, $a \geq 0$, $b > 1$ and $r \in (-1, 1)$. Then we have
\[
\int_0^t (1+s)^{-r} (1+B(t,s))^{-a} \int_0^s (s-\tau)^{-\gamma}(1+\tau)^{-b} \, d\tau \, ds
\]
\[
\leq C \begin{cases} 
(1+t)^{-\min\{a(1-r)\gamma\}} & \text{if } \max\{a; \gamma + r\} > 1, \\
(1+t)^{-\min\{a(1-r)\gamma\} \log(2+t)} & \text{if } \max\{a; \gamma + r\} = 1, \\
(1+t)^{1-a(1-r)-\gamma-r} & \text{if } \max\{a; \gamma + r\} < 1.
\end{cases}
\]
Proof. Thanks to Lemma 4.1 from [2] the interior integral is estimated as follows:
\[
\int_0^s (s-\tau)^{-\gamma}(1+\tau)^{-b} \, d\tau \lesssim (1+s)^{-\gamma}.
\]
Then we estimate the integral
\[
I(t) := \int_0^t (1+s)^{-(\gamma+r)} (1+B(t,s))^{-a} \, ds.
\]
In order to use suitable properties of the function $B = B(t, s)$ we split the integral into two parts, namely,

$$I(t) = \int_0^{\frac{t}{2}} \cdots ds + \int_0^t \cdots ds := I_1(t) + I_2(t).$$

To estimate $I_1(t)$ we use the property $B(t, s) \approx B(t, 0)$ for $s \in [0, \frac{t}{2}]$ to get

$$I_1(t) \lesssim (1 + B(t, 0))^{-a} \int_0^{\frac{t}{2}} (1 + s)^{-(\gamma + r)} ds$$

$$\lesssim (1 + B(t, 0))^{-a} \begin{cases} 1 & \text{if } \gamma + r > 1, \\
\log(2 + t) & \text{if } \gamma + r = 1, \\
(1 + t)^{1-\gamma-r} & \text{if } \gamma + r < 1. \end{cases}$$

To estimate $I_2(t)$ we use the fact that $B(t, s) \approx (1 + t)^{1-r} - (1 + s)^{1-r}$ for $s \in \left[\frac{t}{2}, t\right]$ to get

$$I_2(t) \lesssim \left(1 + \frac{t}{2}\right)^{-\gamma} \int_{\frac{t}{2}}^t (1 + s)^{-r} (1 + (1 + t)^{1-r} - (1 + s)^{1-r})^{-a} ds$$

$$\lesssim \left(1 + \frac{t}{2}\right)^{-\gamma} \begin{cases} 1 & \text{if } a > 1, \\
\log\left(1 + (1 + t)^{1-r} - (1 + \frac{t}{2})^{1-r}\right) & \text{if } a = 1, \\
\left(1 + (1 + t)^{1-r} - (1 + \frac{t}{2})^{1-r}\right)^{1-a} & \text{if } a < 1. \end{cases}$$

Then the statements of Lemma 5.1 are concluded after using

$$(1 + B(t, 0))^{-a} \lesssim (1 + t)^{-a(1-r)} \quad \text{and} \quad (1 + t)^{1-r} - \left(1 + \frac{t}{2}\right)^{1-r} \lesssim (1 + t)^{1-r}.$$  

The proof is complete.

The reader may ask for a corresponding result in the case $b \in (0, 1]$. One can prove in a similar way to the proof of Lemma 5.1 the following result.

**Lemma 5.2.** Assume that $0 < \gamma < 1$, $a \geq 0$, $b \in (0, 1]$, and $r \in (-1, 1)$. Then we have

$$\int_0^t (1 + s)^{-r} (1 + B(t, s))^{-a} \int_0^s (s - \tau)^{-\gamma} (1 + \tau)^{-b} d\tau ds$$

$$\leq C \begin{cases} (1 + t)^{-\min\{a(1-r); \gamma + b - 1\}} & \text{if } \max\{a; \gamma + r + b - 1\} > 1, \\
(1 + t)^{-\min\{a(1-r); \gamma + b - 1\}} \log(2 + t) & \text{if } \max\{a; \gamma + r + b - 1\} = 1, \\
(1 + t)^{-a(1-r) - b - \gamma - r} & \text{if } \max\{a; \gamma + r + b - 1\} < 1. \end{cases}$$

### 5.2. Decay estimates for solutions to auxiliary Cauchy problems

The following results can be found in [12, 7, 6].

**Proposition 1.** Let us consider the Cauchy problem (1.6) with $u_1 \equiv 0$. If the data $u_0$ belongs to $A_{m, \sigma}$ with $\sigma \in (0, 1)$ and $m \in [1, 2]$, then we have the following Matsumura type decay estimates for the Sobolev solutions:

$$\|u(t, \cdot)\|_{L^2} \leq C(1 + t)^{-\frac{\sigma}{2} + \frac{\sigma}{2}(1-\frac{1}{r})} \|u_0\|_{A_{m, \sigma}},$$

$$\|D^\alpha u(t, \cdot)\|_{L^2} \leq C(1 + t)^{-\frac{\alpha}{2} + \frac{\alpha}{2}(1-\frac{1}{r}) - \frac{\alpha}{2}(1-\frac{1}{r})} \|u_0\|_{A_{m, \sigma}}.$$
Proposition 2. Let us consider the Cauchy problem (1.6) with \( u_1 \equiv 0 \). If the data \( u_0 \) belongs to \( A_{m,\sigma} \) with \( \sigma \geq 1 \) and \( m \in [1,2] \), then we have the following Matsumura type decay estimates for the energy solutions:

\[
\|u(t,\cdot)\|_{L^2} \leq C(1 + t)^{-\frac{n}{2}(1+\frac{1}{p})} \|u_0\|_{A_{m,\sigma}},
\]

\[
\|D^\sigma u(t,\cdot)\|_{L^2} \leq C(1 + t)^{-\frac{n}{2}(1+\frac{1}{p})} \|u_0\|_{A_{m,\sigma}},
\]

\[
\|u_t(t,\cdot)\|_{L^2} \leq C(1 + t)^{-\frac{n}{2}(1+\frac{1}{p})} \|u_0\|_{A_{m,\sigma}},
\]

\[
\|D^\sigma u_t(t,\cdot)\|_{L^2} \leq C(1 + t)^{-\frac{n}{2}(1+\frac{1}{p})} \|u_0\|_{A_{m,\sigma}}.
\]

Proposition 3. Let us consider the Cauchy problem (1.7) with \( h = h(\tau, u) \in L^m \cap L^2 \) for some \( m \in [1,2] \). Then the Sobolev solutions \( v = v(t, x) \) to (1.7) belong for \( \sigma \in (0,1) \) to

\[
C([\tau, \infty), H^\sigma (\mathbb{R}^n))
\]

and satisfy the following estimates:

\[
\|v(t,\cdot)\|_{L^2} \leq C(1 + \tau)^{-\sigma} (1 + B(t,\tau))^{-\frac{n}{2}(1+\frac{1}{p})} \|h(\tau,\cdot)\|_{L^m \cap L^2},
\]

\[
\|D^\sigma v(t,\cdot)\|_{L^2} \leq C(1 + \tau)^{-\sigma} (1 + B(t,\tau))^{-\frac{n}{2}(1+\frac{1}{p})} \|h(\tau,\cdot)\|_{L^m \cap L^2}.
\]

Proposition 4. Let us consider the Cauchy problem (1.7) with \( h = h(\tau, u) \in L^m \cap H^{\sigma-1} \) for some \( m \in [1,2] \). Then the energy solutions \( v = v(t, x) \) to (1.7) belong for \( \sigma \geq 1 \) to

\[
C([\tau, \infty), H^{\sigma} (\mathbb{R}^n)) \cap C([\tau, \infty), H^{\sigma-1} (\mathbb{R}^n)).
\]

The energy solutions satisfy the following estimates:

\[
\|v(t,\cdot)\|_{L^2} \leq C(1 + \tau)^{-\sigma} (1 + B(t,\tau))^{-\frac{n}{2}(1+\frac{1}{p})} \|h(\tau,\cdot)\|_{L^m \cap H^{\sigma-1}},
\]

\[
\|D^\sigma v(t,\cdot)\|_{L^2} \leq C(1 + \tau)^{-\sigma} (1 + B(t,\tau))^{-\frac{n}{2}(1+\frac{1}{p})} \|h(\tau,\cdot)\|_{L^m \cap H^{\sigma-1}},
\]

\[
\|v_t(t,\cdot)\|_{L^2} \leq C(1 + t)^{-\sigma} (1 + \tau)^{-\sigma} (1 + B(t,\tau))^{-\frac{n}{2}(1+\frac{1}{p})} \|h(\tau,\cdot)\|_{L^m \cap H^{\sigma-1}},
\]

\[
\|D^\sigma v_t(t,\cdot)\|_{L^2} \leq C(1 + t)^{-\sigma} (1 + \tau)^{-\sigma} (1 + B(t,\tau))^{-\frac{n}{2}(1+\frac{1}{p})} \|h(\tau,\cdot)\|_{L^m \cap H^{\sigma-1}}.
\]

5.3. Main inequalities-tools from harmonic analysis. The following results can be found among other things in [8] or [11].

5.3.1. Fractional Gagliardo-Nirenberg inequality.

Proposition 5. Let \( 1 < p, p_0, p_1 < \infty \) and \( \kappa \in [0, \sigma) \). Then the following fractional Gagliardo-Nirenberg inequality holds for all \( u \in L^{p_0} \cap H^{\sigma} : \)

\[
\|u\|_{H^\kappa} \lesssim \|u\|_{L^p}^{1-\theta} \|u\|_{H^\sigma}^\theta \quad \text{for} \quad \frac{\kappa}{\sigma} \leq \theta \leq 1,
\]

(5.1)

where

\[
\theta = \frac{\frac{1}{p} - \frac{1}{p_0} + \frac{\kappa}{\sigma}}{\frac{1}{p} - \frac{1}{p_1} + \frac{\kappa}{\sigma}}.
\]

The following corollary is a particular case of Proposition 5 after choosing \( p_0 = p_1 = 2, p = q, \kappa = 0 \) and \( \sigma = k \) by taking account of the condition \( \theta \in [0,1] \).

Corollary 1. Let \( u \in L^2 \cap H^k \). Then the following inequality holds:

\[
\|u\|_{L^q} \lesssim \|u\|_{L^2}^{1-\theta_k(q)} \|u\|_{H^k}^{\theta_k(q)} \quad \text{for} \quad \theta_k(q) = \frac{n}{k} \left( \frac{1}{2} - \frac{1}{q} \right)
\]

(5.2)
for any $k \in (0, \frac{n}{2})$ and any $q$ such that
\[ 2 \leq q \leq \frac{2n}{n-2k}. \]

The case $q = \frac{2n}{n-2k}$ reduces the inequality (5.2) to a well-known statement in the frame of Sobolev embeddings.

**Corollary 2.** For $u \in \dot{H}^k$, where $q \in [2, \infty)$ and $k = n(\frac{1}{2} - \frac{1}{q})$, the following inequality holds:
\[ \|u\|_{L^q} \lesssim \|u\|_{\dot{H}^k}. \]

Interpolation formulas are sometimes used to obtain suitable estimates. Here we recall the relation
\[ \|u\|_{\dot{H}^q} \leq \|u\|_{\dot{H}^{k_1}}^{1-q\theta}\|u\|_{\dot{H}^{k_2}}^q \quad \text{for some } \theta \in [0, 1] \quad \text{with } k_1(1-\theta) + k_2\theta = \sigma. \]

**5.3.2. Fractional Leibniz rule.**

**Proposition 6.** Let $\sigma > 0$, $1 \leq r \leq \infty$ and $1 < p_1, p_2, q_1, q_2 \leq \infty$ satisfying
\[ \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}. \]
Then it holds the following fractional Leibniz rule:
\[ \|D|^{\sigma}(fg)\|_{L^r} \lesssim \|D|^{\sigma}f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{q_1}} \|D|^{\sigma}g\|_{L^{q_2}} \]
for any $f \in \dot{H}^{\sigma}_{p_1} \cap L^{p_2}$ and $g \in \dot{H}^{\sigma}_{q_2} \cap L^{q_2}$.

**5.3.3. Fractional chain rule.**

**Proposition 7.** Let $\sigma \in (0, 1), 1 < r, r_1, r_2 < \infty$ and $F$ a $C^1$ function satisfying for any $\tau \in [0, 1]$ and $u, v \in \mathbb{R}$ the inequality
\[ |F'(\tau u + (1-\tau)v)| \leq \mu(\tau)|G(u) + G(v)|, \]
for some continuous nonnegative function $G$ and $\mu \in L^1[0, 1]$. Then,
\[ \|F(u)\|_{\dot{H}^{\sigma}_{r_2}} \lesssim \|G(u)\|_{L^{r_1}} \|u\|_{\dot{H}^{\sigma}_{r_2}}, \]
for any $u \in \dot{H}^{\sigma}_{r_2}$ such that $G(u) \in L^{r_1}$, provided that
\[ \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}. \]

In particular, to estimate norms like $\|u|^{p}\|_{\dot{H}^{\sigma}_{r_2}}$ or $\|\pm u|^{p-1}\|_{\dot{H}^{\sigma}_{r_2}}$, we use the fractional chain rule and the Gagliardo-Nirenberg inequality. In this way we may conclude at first for $s \in (1, 2)$ and then by a straight-forward step to $s \geq 2$ the estimate
\[ \|\pm u|^{p-1}\|_{\dot{H}^{\sigma}_{r_2}} + \|u|^{p}\|_{\dot{H}^{\sigma}_{r_2}} \lesssim \|u\|^{p-1}_{L^{q_1}} \|D|^{s-1}u\|_{L^{q_2}}, \quad (5.3) \]
where
\[ \frac{p - 1}{q_1} + \frac{1}{q_2} = \frac{1}{r}, \quad s > 1. \]

**Acknowledgements.** The authors thank both referees for their proposals to improve the readability of the paper. The first author would like also to thank the DGRSDT in Algeria for supporting the Laboratory of Mechanic and energetic LME.
REFERENCES

[1] T. Cazenave, F. Dickstein and F. D. Weissler, An equation whose Fujita critical exponent is not given by scaling, *Nonlinear Anal.*, **68** (2008), 862–874.

[2] S. Cui, Local and global existence of solutions to semilinear parabolic initial value problems, *Nonlinear Anal.*, **43** (2001), 293–323.

[3] M. D’Abbicco, The influence of a nonlinear memory on the damped wave equation, *Nonlinear Anal.*, **95** (2014), 130–145.

[4] M. D’Abbicco, G. Girardi and M. Reissig, A scale of critical exponents for semilinear waves with time-dependent damping and mass terms, *Nonlinear Anal.*, **179** (2019), 15–40.

[5] M. D’Abbicco and S. Lucente, The beam equation with nonlinear memory, *Z. Angew. Math. Phys.*, **67** (2016), 18 pp.

[6] M. D’Abbicco, S. Lucente and M. Reissig, Semilinear wave equations with effective damping, *Chin. Ann. Math., Serie B*, **34** (2013), 345–380.

[7] A. Djaouti and M. Reissig, Weakly coupled systems of semilinear effectively damped waves with time-dependent coefficient, different power nonlinearities and different regularity of the data, *Nonlinear Anal.*, **175** (2018), 28–55.

[8] M. R. Ebert and M. Reissig, *Methods for Partial Differential Equations. Qualitative Properties of Solutions, Phase Space Analysis, Semilinear Models*, Birkhäuser, Cham, 2018.

[9] A. Fino, Critical exponent for damped wave equations with nonlinear memory, *Nonlinear Anal.*, **74** (2011), 5495–5505.

[10] T. Hadj Kaddour and M. Reissig, Blow-up results for effectively damped wave models with nonlinear memory, 21 pp., *accepted for publication in CPAA*.

[11] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, De Gruyter series in nonlinear analysis and applications, Walter de Gruyter & Co., Berlin, 1996.

[12] J. Wirth, Wave equations with time-dependent dissipation II. Effective dissipation, *J. Differ. Equ.*, **232** (2007), 74–103.

Received October 2020; revised February 2021.

E-mail address: hkttn2000@yahoo.fr
E-mail address: reissig@math.tu-freiberg.de