An $n$-in-a-row type game

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Abstract

We consider a Maker-Breaker type game on the plane, in which each player takes $t$ points on their $t^{th}$ turn. Maker wins if he obtains $n$ points on a line (in any direction) without any of Breaker's points between them. We show that, despite Maker’s apparent advantage, Breaker can prevent Maker from winning until about his $n^{th}$ turn. We actually prove a stronger result: that Breaker only needs to play $\omega(\log t)$ points on his $t^{th}$ turn to prevent Maker from winning until this time.

We also consider the situation when the number of points claimed by Maker grows at other speeds, in particular, when Maker claims $t^\alpha$ points on his $t^{th}$ turn.

1 Introduction

The ordinary $n$-in-a-row game is a Maker-Breaker game played on $\mathbb{Z}^2$, where two players, Maker and Breaker, take turns claiming unclaimed points in the plane. Maker wins if he can claim $n$ consecutive points in a row either vertically, horizontally, or diagonally, otherwise Breaker wins. It is known that for $n \leq 4$ this game is a Maker win, and for $n \geq 8$ the game is a Breaker win [3]. It will be convenient to consider the game as consisting of a series of timesteps, each of which consists of one of Maker’s turns and the subsequent turn of Breaker.

Erde [2] considered a variation of this game where, instead of picking one point each turn, the number of points picked by a player on his $t^{th}$ turn is a
function of $t$. In particular, suppose on their first turn Maker and Breaker each claim 1 point, but on their second turn they each claim 2 points, and 3 on their third, and so on. Unlike the $n$-in-a-row game this game is clearly never a Breaker win, since on his $n^\text{th}$ turn Maker can claim an entire winning line. However, Erde showed that Maker cannot win this game in time less than $(1 - o(1))n$ (i.e., before the $(1 - o(1))n^\text{th}$ timestep).

One generalization of the $n$-in-a-row game is to allow the winning lines to have arbitrary slopes; that is we play a Maker-Breaker game on $\mathbb{Z}^2$ where Maker wins if he can claim $n$ consecutive points in a line, with any slope. Clearly this game is easier for Maker than the $n$-in-a-row game. Indeed Beck [1] showed that this game is a Maker win for all $n$ (recall the ordinary game is known to be a Breaker win for all $n \geq 8$). Since this game is easier for Maker it raises the possibility that the analogous modified version where the number of points picked on each turn is increasing can be won by Maker in time less than $(1 - o(1))n$.

In fact, we consider an even easier game for Maker. The players take it in turns to claim points in $\mathbb{Z}^2$, with each player claiming $t^\text{th}$ turn. Maker wins if he gets $n$ points on a straight line (in any direction at all) with no point of Breaker’s between the first and last of these $n$. The points need not be consecutive and there may be other points of Maker or Breaker on this line. (Note this is not a standard Maker-Breaker game as Maker’s winning sets depend on Breaker’s points.) We call such a line segment a winning line segment.

As before Maker can claim the whole of a winning line segment on his $n^\text{th}$ turn and so he can definitely win by time $n$. Our first result is that, even in this game, this is essentially the best Maker can do.

**Theorem 1.** In the above game Breaker can stop Maker winning before time $(1 - o(1))n$.

*Remark.* Throughout the paper we use the standard notations $O$, $o$ and $\Omega$ as well as the common, but less standard, notation $f = \omega(g)$ which denotes the property that $\lim_{n \to \infty} f(n)/g(n) = \infty$.

We also extend Theorem 1 substantially. Fix functions $m, b \colon \mathbb{N} \to \mathbb{N}$ and suppose that on his $t^\text{th}$ turn Maker plays $m(t)$ points and Breaker plays $b(t)$
points in his. For simplicity we shall assume that both $m(t)$ and $b(t)$ are monotone increasing. Roughly, we say Maker wins if he can get $n$ points in a row significantly before the time at which he is playing $n$ points in a single turn. The precise definition is the following.

**Definition.** Define $\tau_n$ to be the earliest time by which Maker can guarantee to have formed a winning line segment of $n$ points. We say the game is a Breaker win if $m(\tau_n) = (1 - o(1))n$, and we say it is a Maker win (with constant $\varepsilon$) if there exists $\varepsilon > 0$ such that $m(\tau_n) < (1 - \varepsilon)n$ for all sufficiently large $n$.

We remark that for some functions $b$ and $m$ neither of the above will hold.

We wish to prove bounds on $m$ and $b$ showing when the game is a Maker win and when it a Breaker win. We concentrate on the case when $m(t) = t^\alpha$. For $\alpha \geq 1$ we have a rather surprising result which is essentially tight.

**Theorem 2.** Suppose that $m(t) = t^\alpha$ for $\alpha \geq 1$. Then if $b(t) = O(\log t)$ the game is a Maker win whereas if $b(t) = \omega (\log t)$ the game is a Breaker win.

For $\alpha < 1$ we have only a weaker upper bound.

**Theorem 3.** Suppose that $m(t) = t^\alpha$ for some $\alpha < 1$. Then if $b(t) = \omega (t^{1-\alpha})$ the game is a Breaker win.

Despite the much weaker upper bound we have no better lower bound for $b$ than in Theorem 2 that is, the best we can say is that Maker has a winning strategy if $b(t) = O(\log t)$. However, our strategy for Breaker in both of the theorems has a special form: Breaker never relies on placing a single point on two of Maker’s lines. More precisely Breaker can still win in all the above cases if whenever he places any point he also has to designate a specific direction and it only breaks Maker’s winning sets through that point in that specific direction. (Breaker may play the same point with a different direction but that counts as an extra point.) We call the version of the game where Breaker has to specify this direction the **directed-Breaker** game. We can prove that Theorem 3 is essentially tight for this game.

**Theorem 4.** Suppose that $m(t) = t^\alpha$ for $\alpha < 1$ and $b(t) = o(t^{1-\alpha})$. Then Maker wins the directed-Breaker game.
2 Elementary remarks

We start with a trivial remark. We can think of the points that Breaker has claimed at any time in the game as having split each line in the plane into a number of \textit{line segments}: blocks of points that are unclaimed or claimed by Maker, either lying in between two of Breaker’s points, or one of Breaker’s points and infinity (i.e., an infinite ray), or a line not containing any of Breaker’s points. We call any such segments that do not contain at least \( n \) integer points in total \textit{inactive}. Inactive line segments are not useful to Maker as they cannot be extended to a winning line segment.

Suppose that Maker wins the game in time \( T \) with \( m(T) \leq (1 - \varepsilon)n \). Then after Breaker’s last turn there must have been an active line segment containing at least \( \varepsilon n \) of Maker’s points. Thus, we see that it is important for Breaker to try to ensure that no such line segments exist at the end of his turn.

\textbf{Definition.} Suppose that \( l \) is a line segment containing some of Maker’s points. We say that Breaker has \( \varepsilon \)-split the line \( l \) if he has placed points on it splitting it into smaller line segments such that, after the split, there are no active segments containing (at least) \( \varepsilon n \) of Maker’s points. In cases where \( \varepsilon \) is clear we will just say that Breaker has \textit{split} the line.

The following lemma provides a simple bound on how many points Breaker needs to split a line.

\textbf{Lemma 5.} Suppose that \( \varepsilon > 0 \) and that \( l \) is an active line segment containing less than \( n \) of Maker’s points. Then Breaker can \( \varepsilon \)-split the line \( l \) using at most \( 2/\varepsilon \) points.

\textbf{Proof.} Breaker starts from one end and counts along \( \varepsilon n - 1 \) of Maker’s points. He would like to play the next integer point, say \( x \), on the line but this may not be possible as \( x \) may already be a Maker point. So instead he plays both the last free integer point before \( x \) on the line segment and the first free integer point after \( x \). He then repeats the process on the remaining line segment after the second of these points. Obviously when this process has finished there is no active line segment containing \( \varepsilon n \) of Maker’s points and Breaker has played at most \( 2/\varepsilon \) points. \( \square \)
Next we describe a simpler related game that we will mention at times. In the game as described so far there are advantages and disadvantages to playing first: if Breaker claims a point then it stops his opponent from claiming it (an advantage) but means Maker knows where he has played (a disadvantage).

In the modified version Maker chooses some number of timesteps $T$ and $\varepsilon > 0$ such that $m(T) = (1 - \varepsilon)n$, and then he gets to play all the points he would have played up to that point in the original game at once. Then Breaker plays all of his, with the additional freedom that he may choose points that Maker has already chosen. Breaker wins if Maker’s largest active line segment of size at the end of the game has size less than $\varepsilon n$. This game is easier for Breaker than the standard game – it is clear that if Breaker can win the standard game then he can also win this game – and it is easier to think about. We will refer to this game as the batched game.

The key tool in our proofs is the Szemerédi-Trotter Theorem [?].

**Theorem** (Szemerédi-Trotter). Suppose $P$ is a set of points, $L$ is a set of line segments in $\mathbb{R}^2$ such that any two line segments meet in at most one point, and let $I$ be the set of incidences (an incidence is a point-line segment pair with the line segment containing that point). Then

$$|I| = O\left(|P|^{2/3}|L|^{2/3} + |P| + |L|\right)$$

and its corollary

**Corollary** (Szemerédi-Trotter). Let $P, L$ be as above. Then the number of line segments containing at least $k$ points is

$$O\left(\frac{|P|^2}{k^3} + \frac{|P|}{k}\right)$$

We remark that this theorem is normally stated in terms of lines rather than line segments, however it is a folklore result that the line segment version is implied as a simple consequence.

As we will be using the Szemerédi-Trotter Theorem it is convenient to make the following definition.
Definition. For any $n$ and $k$ define $\text{SzT}(n, k)$ to be the maximum number of incidences that can occur between $n$ points and $k$ (non-overlapping) line segments.

We start by illustrating our methods by applying them to the batched game.

Proposition 6. Suppose $m(t) = t$ and $b(t) = \omega(1)$. Then Breaker can win the batched game.

Proof. We aim for a contradiction: suppose Maker can win with $T = (1 - \varepsilon)n$. Since $m(t) = t$ the total number of points Maker plays up to the $T^{th}$ timestep is less than $Tn$. By the corollary to the Szemerédi-Trotter Theorem the number of line segments Maker can create containing more than $\varepsilon n$ points is at most

$$O \left( \frac{(Tn)^2}{(\varepsilon n)^3} + \frac{Tn}{\varepsilon n} \right) = O(T).$$

Some of these line segments might contain significantly more than $\varepsilon n$ points, and would therefore take more than $O(1)$ points to $\varepsilon/2$-split. However, we can count a line segment containing $\sim k\varepsilon n$ points as $\lceil k \rceil$ line segments each containing $\leq \varepsilon n$ points and since we are using the line segment version of the Szemerédi-Trotter Theorem this doesn’t change our bound.

Since $b(t) = \omega(1)$, Breaker has $\omega(T)$ points to play. Therefore, by (the ideas of) Lemma 5, Breaker can $\varepsilon$-split each of these segments. This guarantees that there are no segments left with length greater than $\varepsilon n$ and, thus, Breaker wins.

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3 A weighted bin game

In this section we introduce a weighted bin game. This is a simpler game that is easy to analyse but our main proofs will compare the real game with this simple game. It is a single player (who we will call Maker) game.

Definition. Suppose $T$ is a constant and $b, M : \mathbb{N} \to \mathbb{N}$ are functions. The weighted bin game $(b, M, T)$ is the following game. The game is played with $1 + \sum_{t=1}^{T} b(t)$ bins. On turn $t$ Maker places adds some weight to some bins
subject to some constraints given below. Then the $b(t)$ largest bins are killed. The game lasts $T$ turns after which there is a single remaining live bin. Maker’s aim is to maximise the weight of this remaining live bin.

The constraint for Maker is the following: for any $s > 0$ the total weight added during the last $s$ turns is at most $M(s)$.

**Lemma 7.** Suppose that Maker plays the game above playing weight $w_i$ on his $i^{th}$ turn for each $i$. Then the weight remaining in the last bin is at most

$$
\sum_{s=1}^{T} \frac{w_s}{\sum_{t=s}^{T} b(t) + 1}.
$$

**Proof.** Suppose at the start of his $s^{th}$ turn the $\sum_{t=s}^{T} b(t) + 1$ remaining live bins have average weight $a$. Clearly, after Maker adds this turn’s weight the average weight is

$$
a + \frac{w_s}{\sum_{t=s}^{T} b(t) + 1},
$$

and this average weight does not increase when the $b(t)$ largest bins are killed.

Hence, at the end of the game there is a single bin with (average) weight at most

$$
\sum_{s=1}^{T} \frac{w_s}{\sum_{t=s}^{T} b(t) + 1}
$$

as claimed. \qed

**Remark.** Obviously Maker can obtain the bound given in this lemma, but we shall not make use of that.

**Lemma 8.** Suppose that $T$ is fixed, that $M(s)$ is any increasing function $\mathbb{N} \to \mathbb{N}$ with $M(0) = 0$, and that $b(t)$ has the property that, for any $s$, we have $\sum_{t=s}^{T} b(t) \geq b(T)(T - s + 1)/2$. Let $\Delta M$ be the function given by $\Delta M(s) = M(s) - M(s - 1)$. Then any strategy for the weighted bin game $(b, M, T)$ finishes with at most weight

$$
\frac{2}{b(T)} \sum_{t=1}^{T} \frac{\Delta M(t)}{t}
$$

in the final bin.
Remark. The constraint on $b$ is roughly requiring that $b$ not be super-linear in growth.

Proof. Suppose Maker plays weight $w_t$ on the $t^{th}$ turn. By the previous lemma the weight in the final bin is at most

$$\sum_{s=1}^{T} \frac{w_s}{\sum_{t=s}^{T} b(t) + 1}.$$  \hfill (\ast)

The constraint on Maker is that, for any $s$,

$$\sum_{t=s}^{T} w_t \leq M(T - s + 1).$$

To maximise the bound (\ast) we push weight onto the $w_t$ with $t$ large. Thus the weight of the final bin is at most what Maker would get if he played weight $\Delta M(s)$ on step $T - s + 1$.

To conclude the proof we just have to bound (\ast) in this case. We have

$$\sum_{s=1}^{T} \frac{w_s}{\sum_{t=s}^{T} b(t) + 1} \leq \sum_{s=1}^{T} \frac{\Delta M(T - s + 1)}{\frac{1}{2}b(T)(T - s + 1) + 1} \leq \frac{2}{b(T)} \sum_{t=1}^{T} \frac{\Delta M(t)}{t},$$

as claimed. \hfill \Box

From the weighted bin game to the real game

In this section we show why the weighted bin game is relevant: we show that if Maker can win the real game (with certain parameters) then he can obtain a certain weight in the weighted game (with certain related parameters). Since we know exactly what weight Maker can obtain in the weighted game this enables us to deduce results about the real game.

Lemma 9. Suppose that $m(t) = t^\alpha$ and $b(t)$ are such that the $n$-in-a-row game is a Maker win with constant $\varepsilon$. Let $\hat{b} = \hat{b}(t)$ be the function $\frac{1}{4}b(t)$. Then for all sufficiently large $n$ there exists $T = T(n)$, with $T^\alpha \leq n$, such that there is a Maker strategy for the weighted bin game $(\hat{b}, M, T)$, where $M(s) = SzT(T^\alpha s, \hat{b}(T)s + 1)$ giving at least weight $\varepsilon n/2$ in the last remaining bin.
Proof. Consider the following Breaker strategy. As in the statement of the lemma let \( \hat{b} = \frac{\varepsilon}{4}b \). On his turn Breaker \( \varepsilon/2 \)-splits the \( \hat{b} \) active line segments containing the most points of Maker’s. Breaker can do this since, assuming Maker has not already won, there was no line segment with more than \( n \) points at the end of Maker’s turn. Thus, by Lemma 5 these \( \hat{b} \) line segments can be split using at most \( b \) points.

Since Maker has a winning strategy it must win against this particular Breaker strategy. Suppose Maker wins in some number of turns \( T + 1 \). We show that this \( T \) satisfies the conclusion of the theorem. Trivially, from the definition of a win, we have \( T^\alpha \leq (T + 1)^\alpha \leq n \). Note that, at the end of Breaker’s \( T^{th} \) turn Maker must have had an active line segment containing at least \( \varepsilon n \) points.

Let \( M \) be as in the statement of the theorem and define \( B(s) = \sum_{t=T-s+1}^{T} \hat{b}(t) \).

Consider the \( B(T) + 1 \) line segments given by the \( B(T) \) segments that Breaker splits during the game together with Maker’s winning line segment.

We create a bin \( Q_i \) corresponding to each line segment \( L_i \). We map a situation in the point-line game to a situation in the weighted bin game as follows: place weight \( l_i - \varepsilon n/2 \) in bin \( Q_i \) when \( l_i \) is the number of points on line \( L_i \), placing weight zero if this is negative.

Note we do not consider the line segment as being ‘created’ until Breaker has played it’s endpoints: in particular if Maker plays some points on a line segment it only adds weight to the bin corresponding to the current segment, not sub-segments that will be created later. This does not matter because, by the definition of splitting a line, all the newly created line segments have at most \( \varepsilon n/2 \) points so map to empty bins. Thus Breaker’s move under this strategy corresponds to killing the \( \hat{b} \) heaviest bins in the weighted bin game.

After the \( T^{th} \) timestep in the point-line game the remaining live line segment has at least \( \varepsilon n \) points so the corresponding bin has at least weight \( \varepsilon n/2 \).

Finally, we check the bound on \( M \). Consider the last \( s \) timesteps in the point-line game. In these turns the weight added in the weighted bin game is (at most) the number of new incidences in the original game. The total number of points added in these turns is at most \( T^\alpha s \) and there are \( B(s) + 1 \leq \hat{b}(T)s + 1 \) remaining live line segments. Since we are only counting
the current line segments these segments only meet in single points, so the Szemerédi-Trotter Theorem does apply.

The number of points added to these \( \hat{b}(T) s + 1 \) lines is at most the number of points on the largest \( \hat{b}(T) s + 1 \) lines through these (at most) \( T^\alpha s \) points. By definition this is at most \( \text{SzT}(T^\alpha s, \hat{b}(T) s + 1) \) which concludes the proof. \( \square \)

Next we combine Lemma 9 with our bounds for when Maker can win the weighted bin game to give an explicit bound on how quickly \( b(t) \) can grow, if Maker wins the \( n \)-in-a-row game.

**Lemma 10.** Suppose that \( m(t) = t^\alpha \) and \( b(t) \) are such that the \( n \)-in-a-row game is a Maker win with constant \( \varepsilon \). Further suppose that, for all sufficiently large \( t \) and any \( s \), we have

\[
\sum_{i=s}^t b(i) \geq b(t)(t - s + 1)/2.
\]

Then there exists \( T = T(n) \) with \( T^\alpha \leq n \) such that

\[
\frac{1}{b(T)} \left( (T^{2\alpha+1}/3) b(T)^{2/3} + T^\alpha \log T + b(T) \log T \right) = \Omega(n).
\]

*Proof.* By Lemma 9 Maker has a strategy for a weighted bin game \( (\hat{b}, M, T) \) where \( \hat{b}, M \) and \( T \) are as in that lemma finishing with at least weight \( \varepsilon n/2 \) in the last remaining bin. By the Szemerédi-Trotter Theorem we have

\[
M(s) = \text{SzT}(T^\alpha s, \hat{b}(T) s + 1) \leq C' \left( (T^\alpha s)^{2/3} (\hat{b}(T) s + 1)^{2/3} + T^\alpha s + \hat{b}(T) s + 1 \right).
\]

Let \( M'(s) \) be the right hand side of this equation. Since \( M' \geq M \) Maker can also play the weighted bin game \( (\hat{b}, M', T) \) finishing with at least weight \( \varepsilon n/2 \) in the last remaining bin. Now

\[
\Delta M'(s) = C'(T^\alpha)^{2/3} \left( ((s + 1)(\hat{b}(T)(s + 1) + 1))^{2/3} - \left( s(\hat{b}(T)s + 1) \right)^{2/3} \right)
\]

\[
+ C' T^\alpha + C' \hat{b}(T)
\]

\[
= O((T^\alpha \hat{b}(T))^{2/3} s^{1/3} + T^\alpha + \hat{b}(T)).
\]

Thus, Lemma 8 implies that

\[
\varepsilon n/2 = O \left( \frac{1}{\hat{b}(T)} \sum_{t=1}^T \frac{(T^\alpha \hat{b}(T))^{2/3} t^{1/3} + T^\alpha + \hat{b}(T)}{t} \right)
\]

\[
= O \left( \frac{1}{\hat{b}(T)} \left( T^{(2\alpha+1)/3} \hat{b}(T)^{2/3} + T^\alpha \log T + \hat{b}(T) \log T \right) \right).
\]
Since \( \hat{b} = \frac{4}{3} b \) and \( \varepsilon \) is a constant this rearranges to give the result. \( \qed \)

## 4 Upper bounds

*Proof of Theorem 2.* Suppose \( \alpha \geq 1 \) and \( b(t) = \omega(\log t) \) and the game is a Maker win with constant \( \varepsilon \). Since replacing \( b \) by a smaller function which is also \( \omega(\log t) \) only makes things harder for Breaker, we may additionally assume that, for all sufficiently large \( t \) and any \( s \), we have \( \sum_{i=s}^{t} b(i) \geq b(t)(t-s+1)/2 \). Thus by Lemma 10 there exists some \( T \), with \( T^\alpha \leq n \), such that

\[
\frac{1}{b(T)} \left( T^{(2\alpha+1)/3} b(T)^{2/3} + T^\alpha \log T + b(T) \log T \right) = \Omega(n).
\]

However, this is a contradiction, since \( T^{(2\alpha+1)/3} \leq T^\alpha \leq n \) and \( b(T) = \omega(\log T) \).

Thus, for any \( \varepsilon > 0 \) the game is not a Maker win with constant \( \varepsilon \) for any sufficiently large \( n \), and thus the game is a Breaker win. \( \qed \)

*Proof of Theorem 3.* Suppose \( \alpha < 1 \) and \( b(t) = \omega(t^{1-\alpha}) \) and the game is a Maker win with constant \( \varepsilon \). As in the previous proof we may additionally assume that, for all sufficiently large \( t \) and any \( s \), we have \( \sum_{i=s}^{t} b(i) \geq b(t)(t-s+1)/2 \). Again, Lemma 10 implies that there exists some \( T \), with \( T^\alpha \leq n \), such that

\[
\frac{1}{b(T)} \left( T^{(2\alpha+1)/3} b(T)^{2/3} + T^\alpha \log T + b(T) \log T \right) = \Omega(n).
\]

As before, since \( T^\alpha \leq n \) and \( b(T) = \omega(T^{1-\alpha}) \), this is a contradiction. So, as in the proof of Theorem 2, the game is a Breaker win. \( \qed \)

## 5 Lower bounds

To complete the proof of Theorem 2 we need to prove that if \( m(t) = t^\alpha \) and \( b(t) = O(\log t) \) then the game is a Maker win.

The rough idea is that Maker can follow the (implicit) strategy given for the weighted bin game by choosing several parallel lines, one corresponding to each bin. If at any point Breaker plays a point on one of these
lines then Maker views that line as ‘dead’. The ideas of Lemma sug-

gest that in time $T$ Maker should be able to make a set of size roughly
$m(T) \log T/b(T) = \Omega(m(T))$. Thus, for some $\varepsilon > 0$ Maker should be able to
get $\varepsilon n$ points by time $m(t) = (1 - \varepsilon)n$; i.e., Maker wins.

There are two problems with this argument: the first is that Lemma is
only an upper bound and the second is that in the weighted bin game Maker
can place arbitrary weights in bins, but in the n-in-a-row game he has to
place an integer number of points on each line.

A short calculation (which we do below) solves the first problem, and a
little care with the rounding solves the second.

Proof of Lower bound in Theorem. Suppose that $\alpha > 0$ is fixed, $m(t) = t^\alpha$
and $b(t) \leq C \log t$ for some constant $C$. Fix $0 < \varepsilon < 1/4$ to be chosen later
Let $t_1$ be minimal such that $m(t_1) > (1 - \varepsilon)n$. Let $r$ be the largest power of 2
such that $m(t_1 - r) > n/2$, and let $t_0 = t_1 - r$. We prove that Maker can win
just using his moves between times $t_0$ and $t_1$. During this period Breaker is
playing at most $C \log n$ points each turn.

In fact, we show the stronger statement: Maker can ensure that there is
a line segment with at least $\varepsilon n$ points after Breaker’s go at the end of these
$r$ timesteps while only playing $n/2$ points each turn and allowing breaker to
play $C \log n$ (rather than $C \log t$) each turn. Obviously if Maker can do this
then he can win the n-in-a-row-game on his next turn by playing all of his
at least $(1 - \varepsilon)n$ points on this line segment.

With a slight abuse of notation let $m = n/2$ and $b = C \log n$. Maker’s
strategy is as follows. He picks $rb + 1$ parallel lines (not through any point
that has already been played). At any time during the next $r$ timesteps we
call a line live if it does not contain any of Breaker’s points.

During the first $r/2$ timesteps Maker places his $mr/2$ points as uniformly
as possible on the $rb + 1$ lines regardless of what Breaker does during these
turns. Each line receives at least

$$\left\lfloor \frac{mr}{2(rb + 1)} \right\rfloor \geq \frac{m}{4b}$$

points where the inequality holds for $n$ sufficiently large.
Now during these \( r/2 \) timesteps breaker has killed at most \( rb/2 \) lines. If Breaker has killed less than this number then Maker arbitrarily designates some lines killed until there are exactly this many killed lines. Thus after this period there are exactly \( rb/2 + 1 \) live lines.

Then in the next \( r/4 \) timesteps Maker places his \( mr/4 \) points as uniformly as possible in these \( rb/2 + 1 \) remaining live lines. Each line receives at least

\[
\left\lfloor \frac{mr}{4(rb/2 + 1)} \right\rfloor \geq \frac{m}{4b}.
\]

This time Breaker has killed at most \( rb/4 \) lines and so at least \( rb/4 + 1 \) live lines remain. As before Maker artificially designates lines killed until exactly this many live lines remain.

Maker repeats this \( \log_2 r \) times. At the end of these \( r \) timesteps the single remaining live line will have at least \( \frac{m}{4b} \log_2 r \) points on it.

However, since \( m = n/2 \) and \( b = C \log n \), and

\[
r \geq \frac{1}{2} \left( \left( \frac{3n^\alpha}{4} \right)^{1/\alpha} - \left( \frac{n}{2} \right)^{1/\alpha} \right) = kn^{1/\alpha}
\]

for some constant \( k \) depending only on \( \alpha \). Thus,

\[
\frac{m}{4b} \log_2 r \geq \frac{n}{8C \log n} \log_2(kn^{1/\alpha}) = \frac{n}{8C \alpha \log 2} (1 - o(1))
\]

Hence, if we set \( \varepsilon < \frac{1}{8C \alpha \log 2} \), by the end of these \( r \) timesteps Maker has placed at least \( \varepsilon n \) points in the last line, and so Maker wins. \( \square \)

**Remark.** This proof shows that, for any fixed \( \alpha \), there is a constant \( K_1 \) such that if \( b(t) \leq C \log n \) then Maker can win with constant \( K_1/C \). In contrast, in the proof of the upper bound of Theorem 2 we saw that, provided \( \alpha \geq 1 \), there exists \( K_2 \) such that if \( b(t) \geq C \log n \) Breaker can prevent Maker from winning with constant \( K_2/\sqrt{C} \). Thus, in this more precise formulation we do have a slight gap between our upper and lower bounds even in the case \( \alpha \geq 1 \).
The directed and batched games for $\alpha < 1$

In this subsection we prove Theorem 4 giving the lower bound for directed-Breaker version of the game.

Proof of Theorem 4. In fact we show the stronger result: with $m$ and $b$ as in the statement of the theorem Maker can win the batched version of the directed-Breaker game.

Indeed, Maker chooses to play until time $m(t) = n/2$; i.e., until time $t = (n/2)^{1/\alpha}$. During this time Maker plays $\Omega(n^{1+1/\alpha})$ points in total. He plays these as the integer points in a rectangle with sides $n$ and $\Omega(n^{1/\alpha})$. This sets contains $\Omega(n^{1/\alpha}n^{1/\alpha-1}) = \Omega(n^{2/\alpha-1})$ lines containing $n$ of Maker’s points ($\Omega(n^{1/\alpha})$ starting points and $\Omega(n^{1/\alpha-1})$ gradients).

Breaker has at most

$$b(n^{1/\alpha})n^{1/\alpha} = o(n^{1/\alpha-1}n^{1/\alpha}) = o(n^{2/\alpha-1})$$

points to play so at least one of Maker’s lines does not receive a point and, thus, Maker wins. \qed

We have seen in Theorem 3 that we can match this lower bound for the directed-Breaker game. However, in the ordinary batched game (i.e., not the directed-Breaker version) Breaker can win whenever $b(t) = \omega(\log t)$.

Proposition 11. Suppose that $\alpha < 1$, $m(t) = t^\alpha$ and $b(t) = \omega(\log t)$. Then Breaker can win the batched game.

Proof. Suppose Maker chooses $T$ and $\varepsilon > 0$ such that $m(T) = (1 - \varepsilon)n$, and has chosen his points. Note that $T^\alpha \leq n$ and that $T \to \infty$ as $n \to \infty$. We show that Breaker can prevent Maker from forming any line segment with $\varepsilon T^\alpha \leq \varepsilon n$ points. We use probabilistic methods to construct Breaker’s set. Let $A$ be a subset of Maker’s points chosen independently at random with probability $p = \frac{2 \log T}{\varepsilon T^\alpha}$.

Maker plays at most $T^{1+\alpha}$ points so, by the Szemerédi-Trotter Theorem, Maker has at most

$$O\left(\frac{(T^{1+\alpha})^2}{(\varepsilon T^\alpha)^3} + \frac{T^{1+\alpha}}{\varepsilon T^\alpha}\right) = O(T^{2-\alpha})$$
(since $\alpha < 1$) line segments with more than $\varepsilon T^\alpha$ points in them. (As in Section \[2\] we are considering any line segments containing $\sim k\varepsilon T^\alpha$ points as $\lceil k \rceil$ line segments each containing at most $\varepsilon T^\alpha$ points.)

The probability that the set $A$ contains no point from a line segment of length $\varepsilon T^\alpha$ is $(1 - \frac{2\log T}{\varepsilon T^\alpha})^{\varepsilon T^\alpha} \approx T^{-2}$. Hence the probability that there exists such a line segment that does not receive a point from $A$ is $O(T^{2-\alpha} \times T^{-2}) = O(T^{-\alpha})$ which is less than $1/10$ for $n$, and thus $T$, sufficiently large.

The probability that the set $A$ contains at most $pT^{1+\alpha} = \frac{2}{\varepsilon} T \log T$ points is approximately $1/2$. Thus, with positive probability, the set $A$ contains at most $\frac{2}{\varepsilon} T \log T$ points and contains a point from every one of Maker’s line segments of length at least $\varepsilon T^\alpha$. In particular, there exists a set $A'$ satisfying both these conditions.

Now, Breaker gets to play $B = \sum_{t=1}^{T} b(t)$ points. Since $b(t) > \frac{1}{\varepsilon} \log t$ for all sufficiently large $t$ we see that $B > \frac{2}{\varepsilon} T \log T$ for all sufficiently large $T$ (and thus for all sufficiently large $n$).

Thus, Breaker’s strategy is to pick the set $A'$ given above. This shows that he can stop Maker forming a line of length $\varepsilon T^\alpha < \varepsilon n$ and so Breaker wins.

Remark. We note that, by a similar argument to Proposition \[6\] Breaker can win the batched game with $m(t) = t^\alpha$ and $b(t) = \omega(1)$ for all $\alpha \geq 1$.

6 Extensions of the results.

Although we have stated and proved the results in $\mathbb{Z}^2$ they apply in rather more generality. Indeed, since the proofs of the upper bounds only rely on the Szemerédi-Trotter Theorem they apply anywhere that theorem holds. In particular, they apply in higher dimensions (with the winning sets being lines), in $\mathbb{Z}^d$, $\mathbb{Q}^d$ and $\mathbb{R}^d$, and in cases where the winning sets are more general curves: for example solutions to polynomial equations, or any other curves any two of which only intersect in a bounded number of places. In particular if $\alpha \geq 1$ then Breaker can win any of these games whenever $b(t) = \omega(\log t)$.
7 Open questions

In the case where \( \alpha \geq 1 \) there is a clear threshold between a Maker win and a Breaker win, however when \( \alpha < 1 \) the bounds are still very far apart. Our key question is to find the correct bound in this case. Define the threshold function

\[
\beta_c(\alpha) = \sup \{ \beta : \text{The game with } m(t) = t^\alpha \text{ and } b(t) = t^\beta \text{ is a Maker win.} \}
\]

**Question 1.** Find \( \beta_c(\alpha) \) for \( \alpha < 1 \).

Our lower bound for \( b(t) \) in this case is only logarithmic so we do not even know the answer to the following simpler question.

**Question 2.** Is \( \beta_c(\alpha) > 0 \) for any \( \alpha \)?

The form of our bounds seem to suggest that as \( \alpha \) increases Breaker can win with fewer and fewer points. Indeed our upper bound for \( b \) is monotone decreasing. Perhaps the actual threshold is also monotonic?

**Question 3.** Is \( \beta_c(\alpha) \) monotone decreasing?

We saw in the previous section that the results generalise to many other settings. In many senses the most natural setting for our result is \( \mathbb{Q}^2 \) rather than \( \mathbb{Z}^2 \). We have seen that for Maker to win with constant \( \varepsilon \) Maker must guarantee to have a line segment with at least \( \varepsilon n \) points on it after Breaker has played. In \( \mathbb{Q}^2 \) in the case \( m(t) = b(t) \) we do not even know that Maker can ever guarantee to have such a line segment.

**Question 4.** Fix \( n \) and suppose that the game is played on \( \mathbb{Q}^2 \) with \( m(t) = b(t) = t \). Can Maker guarantee to have a line segment containing \( n \) points after Breaker’s move? If so, by what time can he guarantee to have such a line segment?

Another extension where we don’t know the answer is the following: define Maker’s winning set to be sets of \( n \) points whose convex hull contains none of Breaker’s points. This obviously generalises the \( n \) points on a line with none of Breaker’s points between them that we have been considering in this paper.
**Question 5.** Let Maker’s winning set be sets of \( n \) points in \( \mathbb{Z}^d \) whose convex hull contains none of Breaker’s points. What is the threshold for \( b(t) \) when \( \alpha = 1 \)? In particular can Maker win this game with \( m(t) = t \) and \( b(t) = t^\varepsilon \) for some \( \varepsilon > 0 \)?

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