New Gauge Invariant Formulation of the Chern-Simons Gauge Theory: Classical and Quantal Analysis

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ABSTRACT

Recently proposed new gauge invariant formulation of the Chern-Simons gauge theory is considered in detail. This formulation is consistent with the gauge fixed formulation. Furthermore it is found that the canonical (Noether) Poincaré generators are not gauge invariant even on the constraints surface and do not satisfy the Poincaré algebra contrast to usual case. It is the improved generators, constructed from the symmetric energy-momentum tensor, which are (manifestly) gauge invariant and obey the quantum as well as classical Poincaré algebra. The physical states are constructed and it is found in the Schrödinger picture that unusual gauge invariant longitudinal mode of the gauge field is crucial for constructing the physical wave-functional which is genuine to (pure) Chern-Simons theory. In matching to the gauge fixed formulation, we consider three typical gauges, Coulomb, axial and Weyl gauges as explicit examples. Furthermore, recent several confusions about the effect of Dirac’s dressing function and the gauge fixings are clarified. The analysis according to old gauge independent formulation a’la Dirac is summarized in an appendix.

PACS Nos: 11.10.Ef, 11.10.Lm, 11.15.-q, 11.30.-j
March 1999

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I. Introduction

In general, there are two approaches in quantum field theory: the gauge invariant formulation (GIF) and the gauge fixed formulation (GFF). The latter is the conventional one, where one chooses a gauge. In the former what we are interested in this paper on the other hand, one does not fix the gauge but works with gauge invariant quantities. There are several methods which achieve this former approach depending on what one chooses the gauge invariant variables [1, 2, 3, 4] and his interests. In this paper, we consider the formalism *a la* Dirac [5] which provides one of the simplest formulation and whose validity is independent on the space-time dimensions and the treating models in principle.

The idea of the Dirac’s gauge invariant formalism is to describe all physical effect by the manifestly gauge invariant variables, called *physical variables* by introducing Dirac’s dressing function $c_k(x, y)$: The physical variables are defined as the quantities which commute with all first-class constraints within the Hamiltonian formulation, which are believed to be directly measurable ones, such that the gauge fixing condition needs not be introduced. If this procedure is succeeded, the model is said to be gauge invariant within that formulation; usually this was trivial thing at classical level but non-trivial at quantum level even for the gauge theories which have the gauge invariance of Lagrangian (density) [5].

Recently this gauge invariant method using the Dirac’s gauge invariant variables has been of considerable interest in the pertubative analysis of QED and QCD, especially in relation to the infrared divergence and quark confinement problems [6]. However as for the formalism itself, it is not clear how the results in GIF can be matched to GFF even though this matching has been considered in several recent analyzes [7].

Furthermore, a similar gauge independent Hamiltonian analysis *a la* Dirac [5, 9] has been recently considered for the Chern-Simons (CS) gauge theory with matter fields [10]. Actually after the CS gauge theory was invented, there arose several debates about the gauge dependence of the spin and statistics transmutation phenomena for the charged matter fields, since the analysis was carried out with specific gauge fixing [10, 11, 12]. So, with the formulation without gauge fixing, one can expect to resolve this debate since one is not confined to a specific gauge. But the result of the recent gauge independent analysis for this problem in Ref. [8] was questionable since there was no room for spin transmutation. This was in sharp contrast to the well-known spin transmutation of GFF [10, 11, 12].

This paper is devoted to a detailed study of our new gauge invariant Hamiltonian formulation that has been recently suggested as a resolution of these problems [13]. In Sec. II, we introduce a physically plausible assumption for the Poincaré transformation of the Dirac’s gauge invariant fields that these fields transform conventionally to the *space* and *time* transformation. As a result we find a new set of equations for Dirac dressing function $c_k(x, y)$. However,
these fields do not transform conventionally under the spatial rotation and Lorentz boost. In Sec. III, we consider quantization. It is found that the gauge invariant field operators satisfy the graded commutation (exchange) relations depending on the dressing, and physical states are constructed as the products of the gauge invariant field operators with the gauge invariant vacuum state algebraically. This is compared to the physical wavefunctional in the Schrödinger picture and it is found, as a genuine effect of the (pure) CS theory, that the gauge invariant but longitudinal mode of the gauge field $A_i$ is important as well as the usual gauge varying longitudinal mode which carries full gauge transformation property of $A_i$. Moreover, it is shown that the improved generators, constructed from the symmetric (Belinfante) energy-momentum tensor [14], which are (manifestly) gauge invariant, obey the Poincaré algebra. But we show that canonical (Noether) Poincaré generators are not gauge invariant “even on the constraints surface” and do not satisfy the Poincaré algebra. These results are valid even at the quantum level as well as as the classical ones. The inequivalence of the improved and canonical generators is essentially due to the CS term, and is important for genuine spin transmutation in the relativistic CS gauge theory. Furthermore the fact that only the symmetric energy-momentum tensor, not the canonical one, is meaningful is consistent with Einstein’s theory of gravity. In Sec. IV, we provide and explain our recently proposed method which matches GIF to GFF consistently. The Coulomb, axial and Weyl gauges are considered as explicit examples. Moreover we clarify several confusions which result from the misunderstanding the gauge fixing kernel and the dressing function. Sec. V is devoted to the discussion and summary. As discussion, we have considered the manifestly gauge invariant action and its possible generalization which is connected to the known equivalence of the self-dual and Maxwell-CS theories. In Appendix A, we consider the old Dirac formalism for the determination of the extended Poincaré generators and we find that this formulation is invalid only when one neglects the singular boundary terms. In Appendix B, we derive the physical wavefunction of the Maxwell-CS theory in the Schrödinger picture in our context and with emphasis on the difference to the (pure) CS theory. In Appendix C, we present the proof of the master formula for the matching of GIF and GFF.

II. New gauge invariant formulation
A. Dirac’s gauge invariant variables

Our model is the Abelian CS gauge theory with massive relativistic complex scalars [8, 10] which is described by the Lagrangian density

$$L = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} \partial_\mu A_\rho + (D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi,$$

(1)

where $\epsilon^{012} = 1$, $g_{\mu\nu} = \text{diag}(1, -1, -1)$, and $D_\mu = \partial_\mu + iA_\mu$. This Lagrangian density is invariant up to the total divergence under the gauge transformations $\phi(x) \rightarrow \exp[-i\Lambda(x)]\phi(x), A_\mu(x) \rightarrow$
\[ A_\mu(x) + \partial_\mu \Lambda(x), \] where \( \Lambda \) is a well-behaved function such that \( \epsilon^{\mu\nu\lambda} \partial_\mu \partial_\nu \Lambda = 0 \). As a reflection of this symmetry, there are the first-class constraints

\[
\begin{align*}
T_0 & \equiv \pi_0 \approx 0, \\
T & \equiv J_0 - \kappa B \approx 0,
\end{align*}
\]

which are the primary and secondary constraints, respectively, and there is also a second-class constraint \( T_i \equiv \pi_i - \frac{\kappa}{2} \epsilon_{ij} A^j \approx 0 \) \((i = 1, 2)\), which results from the symplectic structure of (\( \mathbb{P} \)) in the Dirac’s canonical formalism. Here, \( J_0 \) is the time component of the conserved matter current \( J_\mu = i[(D_\mu \phi)^* \phi - \phi^* D_\mu \phi] \) and \( B = \epsilon_{ij} \partial_i A^j \) is the magnetic field. But all of them are not crucial in the development of our formulation: It is found that only the secondary constraints \( T \approx 0 \) is the non-trivial one. Actually, the Faddeev-Jackiw (FJ) (or symplectic) bracket method does the work properly and in this method the basic (equal time) Poisson brackets (called FJ or symplectic) brackets \( [\mathbb{X}] \) become

\[
\begin{align*}
\{ A^i(x), A^j(y) \} & = \frac{1}{\kappa} \epsilon^{ij} \delta^2(x - y), \\
\{ \phi(x), \pi(y) \} & = \{ \phi^*(x), \pi^*(y) \} = \delta^2(x - y), \\
\end{align*}
\]

with \( \pi = (D_0 \phi)^* \), \( \pi^* = D_0 \phi \), and there remains only the (Gauss’ law) constraint \( T(x) \approx 0 \): In this FJ method, the primary constraints of the Dirac bracket method, \( T_0 \approx 0, T_i \approx 0 \), need not be introduced.

Now in order to develop the manifestly gauge invariant Hamiltonian formulation we introduce the following variables

\[
\begin{align*}
\hat{\phi}(x) & \equiv \phi(x) \exp(i W(x)), \\
\hat{\pi}(x) & \equiv \pi(x) \exp(-i W(x)), \\
\mathcal{A}_\mu(x) & \equiv A_\mu(x) - \partial_\mu W(x),
\end{align*}
\]

and their complex conjugates with

\[ W(x) = \int d^2 z \: c_k(x, z) A^k(z). \]

These variables are manifestly gauge invariant, i.e., \( \{ T(x), \mathcal{F}_\alpha(y) \} = 0 \), \( \mathcal{F}_\alpha = (\mathcal{A}_i, \hat{\phi}, \hat{\pi}) \) in the Hamiltonian formulation\(^3\) if the Dirac dressing function \( c_k(x, z) \) satisfies

\[ \partial_z^2 c_k(x, z) = -\delta^2(x - z). \]

\(^3\)In order to include \( A_0 \) also in this category, one could introduce \( \int d^2 x \: (\partial_0 \Lambda) \pi_0 \) as a temporal-gauge transformation with \( \{ A_0(x), \pi_0(y) \} = \delta^2(x - y) \) in addition to (\( \mathbb{P} \)).
Here, we note that there are infinitely many solutions of \( c_k(x, z) \) which satisfy (7) and the gauge invariance of the variables in (5) should be understood on each solution hyper-surface but not on the entire solution space: This will be discussed in detail in Sec. IV.

Moreover, we note that the decomposition of the base gauge fields as

\[ A_i(x) = \mathcal{A}_i(x) + \partial_i W(x) \]  

(8)

is not always the same as the usual decomposition to the transverse and longitudinal components

\[ A_i(x) = A^T_i(x) + A^L_i(x) \]  

(9)

with \( \nabla \cdot A^T = 0 \) and \( \nabla \times A^L = 0 \): Although \( (A_i, \partial_i W) \) is similar to \( (A^T_i, A^L_i) \) in that the first parts do not gauge transform and only the second parts gauge transform, \( \mathcal{A}_i \) does not always satisfy the divergence-free condition:

\[ \partial^i \mathcal{A}_i(x) = \partial_i \mathcal{A}_i(x) + \int d^2 z \nabla^2_x c_k(x - z) A^k(z) \neq 0. \]  

(10)

[The transformation between these two decomposition method will be discussed in detail in Sec. III. B; it will be shown also that \( \mathcal{A}_i \) satisfies more generalized condition, and this generalized condition reduces to the divergence-free condition for a particular form of \( c_k(x - z) \) in Sec. IV.]

On the other hand, in the usual decomposition (9), the gauge invariant variables (5) for the matter fields are found to be

\[ \hat{\phi}(x) = \phi(x) e^{-i \partial^{-1}_i A^L_i(x)} \]
\[ \hat{\pi}(x) = \pi(x) e^{i \partial^{-1}_i A^L_i(x)} \]  

(11)

when the zero-mode of the operator \( \partial_i \partial_i = \nabla^2 \) is unimportant such that \( \nabla^{-2} \nabla^2 = 1 \) [\( \partial^{-1}_i \equiv \partial_i \nabla^{-2} \) and \( \nabla^{-2} \) is defined as the ordered equation \( \nabla^2 \nabla^{-2} = 1 \)] in the Coulomb gauge.

\section*{B. Poincaré Transformation of gauge invariant fields}

In order to give some physical meaning to the gauge invariant fields (5), the transformation properties under the Poincaré generators should be defined. To this end, let us consider

\[ \text{Recently the replacement } \phi(x) \rightarrow \phi(x) e^{i \partial^{-1}_i A^L_i(x)}, \text{ which is the same as what one uses when he wants to remove the gauge dependent } A^L_i, \text{ part in the first order form of the Lagrangian, has been understood as a Darboux transformation in the context of the Hamiltonian reduction} \]  

(13): Although they didn’t use \( \hat{\phi} \) explicitly, their renamed field \( \phi \) in the right-hand side is nothing but \( \hat{\phi} \) in our formulation.
the (manifestly) gauge invariant Poincaré generators which are expressed only by the gauge
invariant fields and \( \mathcal{D}_i \equiv \partial_i + iA_i \):

\[
P^0_s = \int d^2x \left[ |\pi|^2 + |\mathcal{D}^i \hat{\phi}|^2 + m^2 |\hat{\phi}|^2 \right],
\]
\[
P^i_s = \int d^2x \left[ \pi \mathcal{D}^i \hat{\phi} + (\mathcal{D}^i \hat{\phi})^* \pi^* \right],
\]
\[
M^{12}_s = \int d^2x \, \epsilon_{ij} x^i \left[ \pi \mathcal{D}^j \hat{\phi} + (\mathcal{D}^j \hat{\phi})^* \pi^* \right],
\]
\[
M^{0i}_s = x^0 P^i_s - \int d^2x \, x^i \left[ |\pi|^2 + |\mathcal{D}^i \hat{\phi}|^2 + m^2 |\hat{\phi}|^2 \right].
\]  (12)

These are improved generators following the terminology of Callan et al. \[10\] constructed from
the symmetric (Belinfante) energy-momentum tensor \( T_{\mu\nu}^{\text{sym}} \) constructed
from the symmetrized (Belinfante) tensor \( T_{\mu\nu}^{\text{sym}} \) constructed from
\[
T_{\mu\nu}^{\text{sym}}(x) \equiv \frac{\delta I}{\sqrt{g} g_{\mu\nu}(x)} \bigg|_{g_{\mu\nu} \to \eta_{\mu\nu}}
\]
\[
= \left( \mathcal{D}^\mu \hat{\phi} \right)^* \left( \mathcal{D}^\nu \hat{\phi} \right) + \left( \mathcal{D}^\nu \hat{\phi} \right)^* \left( \mathcal{D}^\mu \hat{\phi} \right) - \eta_{\mu\nu} \left[ \left( \mathcal{D}^\rho \hat{\phi} \right)^* \left( \mathcal{D}_\rho \hat{\phi} \right) - m^2 \hat{\phi}^* \hat{\phi} \right],
\]
\[
P^\mu_s = \int d^2x \, T^{0\mu}_s,
\]
\[
M^{\mu\nu}_s = \int d^2x \, \left[ x^\mu T^{0\nu}_s - x^\nu T^{0\mu}_s \right],
\]
where \( \pi^* = \mathcal{D}^0 \hat{\phi} = (\partial^0 + iA_0) \hat{\phi} \) and \( I = \int d^3x \mathcal{L} \). Though the gauge invariance of \( P^\mu_s, M^{\mu\nu}_s \) of (12) is manifest (at least classically), there can be added additional term \( \Gamma \equiv \int d^2x \, [u(x)T(x) + u_0(x)T_0(x)] \), which is proportional to the (first-class) constraints, to generate the correct Poincaré
transformations for the undressed base fields \( \phi \) and \( A^\mu \) \[14\]. However, we note that as far as we
are interested in the dynamics of the physically relevant variables of (12), we do not need this
additional term \( \Gamma \) in the Poincaré generators of (12) \[6\] since \( \Gamma \) has the vanishing Poisson
bracket with the gauge invariant variables \( \mathcal{F}_\alpha \). (See Appendix A for the fixing the constraints
term \( \Gamma \) from the transformation properties of the undressed base fields \( \text{a'la} \) Dirac.) Then, the
generators in (12) should generate the correct transformation for the \( \mathcal{F}_\alpha \); but this will depend
on how the variables \( \mathcal{F}_\alpha \) are defined.

In the followings, we will define the gauge invariant canonical fields \( \mathcal{F}_\alpha = (A_i, \phi, \phi^*) \), which
appear in the Poincaré generators, as what involving a particle-like object and investigate what
information can be obtained from this definition.

1. Space and time translations

First of all, we consider the spatial translation generated by

\[
\{ \hat{\phi}(x), P^j_s \} = \partial^j \hat{\phi}(x)e^{iW(x)} + i\hat{\phi}(x)e^{iW(x)} \int d^2z \, c_k(x,z) \partial_+^j A^k(z)
\]
\[ \partial^i \hat{\phi}(\mathbf{x}) - i \hat{\phi}(\mathbf{x}) \int d^2 \mathbf{z} \left[ \partial_z^2 c_k(\mathbf{x}, \mathbf{z}) + \partial_z^i c_k(\mathbf{x}, \mathbf{z}) \right] A^i(\mathbf{z}). \]

From the second to third lines, the integration by parts has been performed and we have dropped the terms \( \int d^2 \mathbf{z} \partial_z^i \left( c_k(\mathbf{x}, \mathbf{z}) A^i(\mathbf{z}) \right) \) and \( \int d^2 \mathbf{z} \partial_z^i \left( c_k(\mathbf{x}, \mathbf{z}) A^j(\mathbf{z}) \right) \), which vanish for sufficiently rapidly decreasing integrand. This seems to show the translational anomaly due to the second term. (Following the terminology of Hagen et al. [10, 11, 12], “anomaly” means an unconventional contribution whose origin is at the classical level.) However, this anomaly should not appear in order that \( \hat{\phi} \) responds conventionally to translations, i.e., \( \{ \hat{\phi}(\mathbf{x}), P^i_\alpha \} = \partial^i \hat{\phi}(\mathbf{x}) \) like as the particle-like object which is described by the local fields: The usual local fields have no translational anomaly, even though they have additional terms in the rotation because of their spin or other properties.\(^5\) From this property, we obtain the condition that \( c_k(\mathbf{x}, \mathbf{z}) \) be translation invariant

\[ \partial_z^i c_k(\mathbf{x}, \mathbf{z}) = -\partial_z^i c_k(\mathbf{x}, \mathbf{z}), \quad (13) \]

i.e.,

\[ c_k(\mathbf{x}, \mathbf{z}) = c_k(\mathbf{x} - \mathbf{z}). \quad (14) \]

This results would be also easily anticipated ones if the base fields \( \phi \) and \( A_i \) are translationally invariant as well. But this is not always necessary to derive \((14)\). Actually in this case the base fields are not translationally invariant under the generators of \((12)\).

\[ \{ \phi(\mathbf{x}), P^i_j \} = D^i \phi(\mathbf{x}), \]

\[ \{ A_i(\mathbf{x}), P^i_s \} = -\epsilon_{ij} \frac{1}{K} J_0(\mathbf{x}) \approx F^i_j(\mathbf{x}), \]

\[ \{ A_0(\mathbf{x}), P^i_s \} = 0, \]

but the translationally non-invariant terms cancel each others in the gauge invariant combination \((3)\). Furthermore, the condition \((13)\) or \((14)\) also guarantees the correct spatial translation

\(^5\)Similar assumption has been recently considered independently in a different context by Bagan et al. [17].

\(^6\)These unconventional coordinate transformations on fields can be understood as the “gauge-covariant” coordinate transformation [18, 19] \( \delta f A_\mu \equiv L(f A_\mu - \partial_\mu (f^\alpha A_\alpha) = f^\alpha F_{\alpha\mu}, \quad \delta f \phi \equiv L f \partial_\alpha \phi = f^\alpha (\partial_\alpha + i A_\alpha) \phi \) under the coordinate transformation \( \delta x^\mu = -f^\mu(x) [f^\mu = \text{constant for space-time translation}, f^\mu = \omega^\mu_\nu x^\nu (\omega^\mu_\nu = -\omega^\nu_\mu) \text{ for the space rotation and Lorentz boost}] \), where \( L_f \) denotes the Lie derivative; the discrepancies about \( A_0 \) transformation can be traced back to the (strong) implementation of the constraint \( \pi_0 \approx 0 \) which is the gauge transformation generator of \( A_0 \). By recovering \( \pi_0 \) in the Poincaré generator complete equivalence will be obtained. These results are consistent with the fact that, by introducing the (first-class) constraints terms additionally into the Poincaré generator, one can obtain the correct coordinate transformations as is shown in Appendix A: The additional constraints terms compensate the gauge transformation part with gauge function \( f^\alpha A_\alpha \) in the gauge-covariant coordinate transformation \( \delta f \) and the correct transformations \( \delta f A_\mu = L_f A_\mu, \quad \delta f \phi = L_f \phi \) are obtained. We thank Prof. R. Jackiw for suggesting this way of understanding.
law for all other gauge invariant fields in $\mathcal{F}^\alpha$:

$$\{\mathcal{F}_\alpha(x), P_s^j\} = \partial^i\mathcal{F}_\alpha(x)\ , \ \mathcal{F}_\alpha = (A_i, \hat{\phi}, \hat{\phi}^*)$$

Similarly, by considering the time translation

$$\{\hat{\phi}(x), P_s^0\} = \partial^0\hat{\phi}(x)e^{iW(x)} + i\hat{\phi}(x)e^{iW(x)} \int d^2z \ c_k(x-z)\partial^0A^k(z) = \partial^0\hat{\phi}(x) - i\hat{\phi}(x) \int d^2z \ \partial^0(c_k(x-z))A^k(z)$$

we obtain the correct time translation if the boundary integral $\int d^2z \ \partial^k(c_k(x-z)A^0(z))$ vanishes and $c_k(x-z)$ be time independent. This property is also satisfied for all other gauge invariant fields and the results read in a compact form

$$\{\mathcal{F}_\alpha(x), P_s^0\} = \partial^0\mathcal{F}_\alpha(x).$$

Contrast to this, the transformation for the base fields are not correctly generated, i.e.,

$$\{\phi(x), P_s^0\} = D_0\phi(x),$$
$$\{A_i(x), P_s^0\} = -\epsilon_{ij} \frac{1}{\kappa}J_j(x) = F^0_{ij}(x),$$
$$\{A_0(x), P_s^0\} = 0.$$

2. Space rotations and Lorentz boost

For the space rotation and Lorentz boost, contrast to the translations, the anomalies are present even in the transformation for $\mathcal{F}^\alpha$ since in that case they represent the spin or other properties of $\mathcal{F}_\alpha$. The brackets with the rotation generator are expressed as

$$\{\hat{\phi}(x), M_s^{12}\} = \epsilon_{ij} x_i\partial_j + iA_j(x)\hat{\phi}(x)e^{iW(x)} + i\hat{\phi}(x)e^{iW(x)} \int d^2z \ c_k(x-z)z^k \frac{1}{\kappa}J_0(z)$$

$$= \epsilon_{ij} x_i\partial_j\hat{\phi}(x) - i\Xi^{12}(x)\hat{\phi}(x),$$

$$\{A_i(x), M_s^{12}\} = \epsilon_{jk} x_j\partial_kA_i(x) - \epsilon_{ij} A_j(x) - \partial_i \int d^2z \ c_k(x-z) [\epsilon^{il}z^l\partial^jA^k(z)] - \epsilon_{kj} A^j(z)]$$

$$= \epsilon_{jk} x_j\partial_kA_i(x) - \epsilon_{ij} A_j(x) + \partial_i\Xi^{12}(x),$$

and the brackets with the Lorentz boost are

$$\{\hat{\phi}(x), M_s^{0j}\} = (x^0\partial^j - x^j\partial^0)\hat{\phi}(x)e^{iW(x)} + i\phi(x)e^{iW(x)} \int d^2z \ c_k(x-z) [\epsilon^0\partial^j - z^j\partial^0]A^k(z) + \delta_{kj}A_0(z)$$

\footnote{Correct transformation for $A_0$ also can be obtained by reviving the $\pi_0$ term in the Poincaré generators similar to what was noted in the footnote ‘3’. See Appendix A for detail.}
\[
\{ A_i(x), M_{s}^{0i} \} = (x_0 \partial^j \dot{\phi}(x) - x_j \partial^0 \dot{\phi}(x) - i \Xi^{0j}(x) \dot{\phi}(x),
\]

\[
\{ A_i(x), M_{s}^{12} \} = (x_0 \partial^0 - x^0 \partial^0) A_i(x) - \delta_{ij} A_0 - \frac{1}{K} \int d^2 z \, c_k(x - z) \left[ (\xi^0 \partial^j - z^j \partial^0) A^k(z) + \delta_{kj} A_0(z) \right]
\]

\[
= x^0 \partial^0 A_i(x) - x^j \partial^0 A_i(x) - \delta_{ij} A_0 + \partial_i \Xi^{0j}(x) \phi(x);
\]

or in the compact form these are expressed as follows

\[
\{ F_\alpha(x), M_{s}^{\mu \nu} \} = x^\mu \partial^\nu F_\alpha(x) - x^\nu \partial^\mu F_\alpha(x) + \Sigma^{\mu \nu}_{\alpha \beta} F_\beta(x) + \Omega^{\mu \nu}_\alpha(x),
\]

\[
\Omega^{\mu \nu}_\phi(x) = -i \Xi^{\mu \nu}(x) \dot{\phi}(x),
\]

\[
\Omega^{\mu \nu}_{\phi^*}(x) = i \Xi^{\mu \nu}(x) \dot{\phi^*}(x),
\]

\[
\Omega^{\mu \nu}_{A_i}(x) = \partial_i \Xi^{\mu \nu}(x),
\]

(15)

where

\[
\Xi^{\mu \nu} = -\Xi^{\nu \mu},
\]

\[
\Xi^{12}(x) = \epsilon_{ij} x_i A^j(x) + \frac{1}{K} \int d^2 z \, z_k c_k(x - z) J_0(z),
\]

\[
\Xi^{0i}(x) = -x_i A^0(x) - \frac{1}{K} \int d^2 z \, z_i \epsilon_{kj} c_k(x - z) J^j(z)
\]

with the spin-factors \( \Sigma^{\mu \nu}_{\alpha \beta} = \eta^{\mu \alpha} \eta^{\nu \beta} - \eta^{\mu \beta} \eta^{\nu \alpha} \), \( \Sigma^{\mu \nu}_{\phi^*} = 0 \) for the gauge and scalar fields, respectively. As a comparison, the corresponding brackets for the base fields are as follows:

\[
\{ \phi(x), M_{s}^{12} \} = \epsilon_{ij} x^j D^i \phi(x),
\]

\[
\{ A_i(x), M_{s}^{12} \} = x_i \frac{1}{K} J_0(x) \approx x_i \epsilon_{kj} \partial_k A^j(x),
\]

\[
\{ A_0(x), M_{s}^{12} \} = 0,
\]

and

\[
\{ \phi(x), M_{s}^{0j} \} = (x^0 D^j - x^j D^0) \phi(x),
\]

\[
\{ A_i(x), M_{s}^{0j} \} = x^0 F^j_i(x) - x^j F^i_0(x),
\]

\[
\{ A_0(x), M_{s}^{0j} \} = 0
\]

for the space rotation and Lorentz transformations, respectively, which show the incorrect transformation for the gauge \textit{varying} base fields; we must supplement the constraint term \( \Gamma \) in order to give the correct transformation even for the gauge varying base fields as well as the gauge invariant fields \( F_\alpha \). The anomalous term \( \Omega^{\mu \nu}_\alpha \) for the transformation of \( F_\alpha \) in (15) is gauge invariant because it is expressed only with the \( F_\alpha \). At first, it seems odd that the gauge
invariant variables do have the anomaly, but as will be clear in later section, these variables are nothing but the Hagen’s rotational anomaly term and other gauge restoring terms in GFF. This will be treated in Sec. IV. But here, it will be interesting to note that $A_\mu$ can be re-expressed completely by the matter currents as

\begin{align*}
A_i(x) &= A_i(x) - \int d^2z \ \partial^i c_k(x-z) A^k(z) \\
&= A_i(x) - \int d^2z \ c_k(x-z) (F_{k1}(z) - \partial_k A_i(z)) \\
&= -\int d^2z \ c_k(x-z) F_{k1}(z) \\
&\approx -\frac{1}{\kappa} \int d^2z \ \epsilon_{ik} c_k(x-z) J^0(z), \\
\end{align*}

\begin{align*}
A_0(x) &= A_0(x) - \int d^2z \ c_k(x-z) \partial_0 A^k(z) \\
&= A_0(x) - \int d^2z \ c_k(x-z) \left( F_{k0}(z) + \partial^k A^0(z) \right) \\
&= -\int d^2z \ c_k(x-z) F_{k0}(z) \\
&= -\frac{1}{\kappa} \int d^2z \ \epsilon_{kj} c_k(x-z) J^j(z) \\
\end{align*}

(16)

: The third lines in (16) and (17) are just the results of integration by parts without recourse to the particular properties of CS theory; in the last steps we used the constraint $T \approx 0$ and the Euler-Lagrange equation of (1), $F^{0k} = -\frac{1}{\kappa} \epsilon_{kj} J^j$, respectively, which are genuine to the CS theory. It is remarkable that the expressions (16) and (17) in terms of $J^\mu$, which solves the constraint and equation of motion, were obtained without solving the differential equations but from the simple algebraic manipulation by imposing the constraint and equation of motion. Moreover, the anomalous terms are, then, expressed as

\begin{align*}
\Xi^{ij} &= \frac{1}{\kappa} \int d^2z \ (x^k - z^k)c_k(x-z) J_0(z) \\
\Xi^{0i} &= -\frac{1}{\kappa} \int d^2z \ (x^i - z^i)c_k(x-z) \epsilon_{kj} J^j(z). \\
\end{align*}

(18)

These solutions are similar to the Coulomb gauge solution and hence imply the similarity of GIF to GFF with the Coulomb gauge in particular. (This will be discussed again in Sec. IV. A in a different context.) However it should be noted that the Lorentz anomaly does not occur in the transformation of the current $J^\mu$, even though $A_\mu$, which is expressed by $J^\mu$ as given above, does have the anomaly: This will be connected to the fact that the anomaly depends on the dressing $c_k(x-z)$ but $J^\mu$ is already gauge invariant without recourse to that dressing.

### III. Quantization

*This has the same origin to what has been observed in a different context in Ref. [10].*
The quantization in our gauge invariant formulation is carried out by assuming the (equal
time) quantum commutation relation

\[
\begin{align*}
[A^i_{\text{op}}(x), A^j_{\text{op}}(y)] &= \frac{i\hbar}{\kappa} \epsilon^{ij} \delta^2(x - y), \\
[\phi_{\text{op}}(x), \pi_{\text{op}}(y)] &= [\phi^+_{\text{op}}(x), \pi^+_{\text{op}}(y)] = i\hbar \delta^2(x - y),
\end{align*}
\]

(19)

others vanish

for the operator valued fields \(A^i_{\text{op}}, \phi_{\text{op}}, \pi_{\text{op}} = (D_0\phi)_{\text{op}}\), and their complex conjugates, and the
physical states \(|\Psi_{\text{phys}}\rangle\), which are annihilated by the Gauss’ law constraint \((3)\) (with normal
ordering : :)

\[
T_{\text{op}} |\Psi_{\text{phys}}\rangle = 0.
\]

(20)

Here, we note that the quantum commutation relations in \((19)\) are not gauge independent
because the involved field operators \(A^i_{\text{op}}, \phi_{\text{op}}, \ldots\) etc. are gauge varying ones. So, we first
consider the commutation (or exchange) relations for the gauge invariant variables \(F^\alpha_{\text{op}}\) which are more basic objects in our formulation.

A. Operator exchange relations

At the classical level, the basic brackets between the gauge invariant canonical fields \(F^\alpha\)
become as follows

\[
\begin{align*}
\{\hat{\phi}(x), \hat{\phi}(y)\} &= -\hat{\phi}(x)\hat{\phi}(y) \frac{1}{\kappa} \Delta(x - y), \\
\{\hat{\phi}(x), \hat{\phi}^*(y)\} &= \hat{\phi}(x)\hat{\phi}^*(y) \frac{1}{\kappa} \Delta(x - y), \\
\{\hat{\phi}(x), \hat{\pi}(y)\} &= \delta^2(x - y) + \hat{\phi}(x)\hat{\pi}(y) \frac{1}{\kappa} \Delta(x - y), \\
\{\hat{\pi}(x), \hat{\pi}^*(y)\} &= -\hat{\pi}(x)\hat{\pi}^*(y) \frac{1}{\kappa} \Delta(x - y), \\
\{A_i(x), A_j(y)\} &= \frac{1}{\kappa} \left[ \epsilon_{ij} \delta^2(x - y) + \xi_{ij} (x - y) + \partial_i \partial_j \Delta(x - y) \right], \\
\{A_i(x), \hat{\phi}(y)\} &= -\frac{i}{\kappa} \hat{\phi}(y) \left[ \epsilon_{ik} c_k (y - x) + \partial_k \Delta(x - y) \right], \\
\{A_i(x), \hat{\pi}(y)\} &= \frac{i}{\kappa} \hat{\pi}(y) \left[ \epsilon_{ik} c_k (y - x) + \partial_k \Delta(x - y) \right].
\end{align*}
\]

(21)
Here, we have introduced two functions

$$\Delta(x - y) = \int d^2 z \, e^{ik} c_k(x - z)c_l(y - z),$$

$$\xi_{ij}(x - y) = \epsilon_{ik}\partial_j^y c_k(y - x) - \epsilon_{jk}\partial_i^x c_k(x - y),$$

which are totally antisymmetric under the interchange of all the indices, i.e.,

$$\Delta(x - y) = -\Delta(y - x),$$

$$\xi_{ij}(x - y) = -\xi_{ji}(y - x).$$

Then, the corresponding quantum commutation (exchange) algebras are

$$\hat{\phi}_{op}(x)\hat{\phi}_{op}(y) = \hat{\phi}_{op}(y)\hat{\phi}_{op}(x)e^{-\frac{i\hbar}{\kappa}\Delta(x - y)},$$

$$\hat{\phi}_{op}(x)\hat{\phi}_{op}^\dagger(y) = \hat{\phi}_{op}^\dagger(y)\hat{\phi}_{op}(x)e^{\frac{i\hbar}{\kappa}\Delta(x - y)},$$

$$\hat{\phi}_{op}(x)\hat{\pi}_{op}(y) = \hat{\pi}_{op}(y)\hat{\phi}_{op}(x)e^{i\hbar\kappa\Delta(x - y)},$$

$$\hat{\phi}_{op}(x)\hat{\pi}_{op}^\dagger(y) = \hat{\pi}_{op}^\dagger(y)\hat{\phi}_{op}(x)e^{-i\hbar\kappa\Delta(x - y)},$$

$$\left[ A_{iop}(x), A_{jop}(y) \right] = \frac{i\hbar}{\kappa} \left[ \epsilon_{ij}\delta^2(x - y) + \xi_{ij}(x - y) + \partial_i^x\partial_j^y \Delta(x - y) \right],$$

$$\left[ A_{iop}(x), \hat{\phi}_{op}(y) \right] = -\frac{\hbar}{\kappa} \hat{\phi}(y) \left[ \epsilon_{ik}c_k(y - x) + \partial_i^x \Delta(x - y) \right],$$

$$\left[ A_{iop}(x), \hat{\pi}_{op}(y) \right] = \frac{\hbar}{\kappa} \hat{\pi}(y) \left[ \epsilon_{ik}c_k(y - x) + \partial_i^x \Delta(x - y) \right],$$

where

$$\hat{\phi}_{op}(x) \equiv \phi_{op}(x)exp \left[ iW_{op}(x) \right],$$

$$\hat{\pi}_{op}(x) \equiv \pi_{op}(x)exp \left[ -iW_{op}(x) \right],$$

$$A_{i\mu op}(x) \equiv A_{i\mu op}(x) - \partial_\mu W_{op}(x)$$

with $W_{op}(x) = \int d^2 z \, c_k(x - z)A_{k op}(z)$ and we have used the formula $e^Ae^B = e^{A + B + \frac{1}{2}[A, B]}$ with $[[A, B], A] = [[A, B], B] = 0$. Here, we note that the path-ordering is not needed in defining the exponential factor of (27) although $A_{i op}$’s do not commute by themselves: They are non-commuting only for the same positions such that $A_{i op}$ of the adjacent points of the integration range commutes: Only for the non-Abelian case, the path-ordering is needed\(^9\). These results look like the graded commutation relations of the anyon field\(^{21}\) but it is found that this is not the case always\(^{10, 22}\). This will be treated in the Sec. V in detail.

\(^9\) If we consider the situation where are the crossings of the contours in the line integral representation of $W(x)$, the path-ordering is required\(^{21}\). See also footnote ‘21’ in this paper for this problem.

12
B. Physical states: Algebraic construction

There is well-known way to construct the physical states: If the vacuum state $|0\rangle$ is a physical state which satisfies $\{O, \mathcal{O}\} = 0$, then this state has one unit charge at the operator position $x$ of the operator in addition to the vacuum charge $\mathcal{O}$, which is the power series function $\mathcal{O}$, is also a physical state [this is because $[O, \mathcal{O}] = 0$ is satisfied if $\mathcal{O}$ is gauge-invariant]. As a simplest case, let us consider the state

\[ \hat{\phi}_{op}(x) |0\rangle. \tag{28} \]

Then, it is found that this state has one unit charge at the operator position $x$ of the operator $\mathcal{O}$ (this is because $[O, \mathcal{O}] = 0$ is satisfied if $\mathcal{O}$ is gauge-invariant). As a simplest case, let us consider the state

\[ \hat{\phi}_{op}(x) |0\rangle. \tag{28} \]

Then, it is found that this state has one unit charge at the operator position $x$ of the operator $\mathcal{O}$ (this is because $[O, \mathcal{O}] = 0$ is satisfied if $\mathcal{O}$ is gauge-invariant). As a simplest case, let us consider the state

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Then, it is found that this state has one unit charge at the operator position $x$ of the operator $\mathcal{O}$ (this is because $[O, \mathcal{O}] = 0$ is satisfied if $\mathcal{O}$ is gauge-invariant). As a simplest case, let us consider the state

\[ \hat{\phi}_{op}(x) |0\rangle. \tag{28} \]
with the c-number coefficient $c_a$. Furthermore, more general state with $N$ CFC and $M$ anti-CFC is expressed as $\sum_{b=1}^{M} \sum_{a=1}^{L} c_{ab} \prod_{j=1}^{L} \hat{\phi}_{op}(\mathbf{x}_j) \prod_{i=1}^{L} \hat{\phi}^{i}_{op}(\mathbf{x}_i) |0\rangle$ when $\hat{\phi}^{i}_{op}$'s are placed to the right-hand side of $\hat{\phi}_{op}$'s. In the case of $[31]$, the CFC operators can be re-arranged as a factorized form

$$
\prod_{j=1}^{L} e^{iW_{op}(\mathbf{x}_j)} \prod_{i=1}^{L} \phi_{op}(\mathbf{x}_i) |0\rangle = e^{-\frac{i}{2} \frac{\hbar}{\kappa} \sum_{i=1}^{L-1} \Delta(\mathbf{x}_i - \mathbf{x}_{i+1})} e^{i \sum_{i=1}^{L} W_{op}(\mathbf{x}_i)} \prod_{i=1}^{L} \phi_{op}(\mathbf{x}_i) |0\rangle
$$

in an appropriate order of $\hat{\phi}_{op}$'s and the first exponential factor will show the multi-valuedness under the exchanging any two CFC's if $\Delta(\mathbf{x}_i - \mathbf{x}_{i+1})$ does not vanish.

Let me now consider finally the state $\mathcal{A}_{op}^{i}(\mathbf{x}) |0\rangle$. Then, it is easy to see that this state does not carry the charge nor magnetic flux in addition to the vacuum charge

$$
J_0(y)\mathcal{A}_{op}^{i}(\mathbf{x}) |0\rangle = J_0(y)\mathcal{A}_{op}^{i}(\mathbf{x}) |0\rangle,
$$

$$
B(y)\mathcal{A}_{op}^{i}(\mathbf{x}) |0\rangle = B(y)\mathcal{A}_{op}^{i}(\mathbf{x}) |0\rangle
$$

from the communication relations $[J_0(y), \mathcal{A}_{op}^{i}(\mathbf{x})] = 0, [B(y), \mathcal{A}_{op}^{i}(\mathbf{x})] = 0$. But, it just carries the gauge varying vector field $a^j(y) = -\frac{i\hbar}{\kappa} [\epsilon_{ij} \delta^2(\mathbf{x} - \mathbf{y}) + \epsilon_{kj} \partial^\tau c_k(\mathbf{x} - \mathbf{y})]$ in addition to that of the vacuum:

$$
\mathcal{A}_{op}^{j}(y)\mathcal{A}_{op}^{i}(\mathbf{x}) |0\rangle = \left\{ -\frac{i\hbar}{\kappa} [\epsilon_{ij} \delta^2(\mathbf{x} - \mathbf{y}) + \epsilon_{kj} \partial^\tau c_k(\mathbf{x} - \mathbf{y})] + A_j^i(y)\mathcal{A}_{op}^{i}(\mathbf{x}) \right\} |0\rangle.
$$

The most general state, then, will be

$$
\sum_{c=1}^{N} \sum_{b=1}^{M} \sum_{a=1}^{L} c_{cba} \sum_{k=1}^{c} \mathcal{A}^{k}_{op}(\mathbf{x}_k) \prod_{j=1}^{b} \hat{\phi}_{op}(\mathbf{x}_j) \prod_{i=1}^{a} \hat{\phi}^{i}_{op}(\mathbf{x}_i) |0\rangle
$$

with the c-number coefficient $c_{cba}$. Here, we note that there seems to be no general reason to omit the purely gauge field $\mathcal{A}^{k}_{op}$ part in this construction. However, that part is physically doubtful because it can imply the independent gauge field dynamics contrast to the nature of the CS gauge field. Moreover, the representation of the physical states are not unique: It depends on what gauge invariant operators are fundamental and the physical states with different representations are not equivalent in general. As a more explicit approach, which fixes these problems and shows the more detailed form of the states, we consider the functional Schrödinger picture approach.

**C. Physical wavefunctional in Schrödinger picture**

In the previous section B, we have considered a general algebraic construction for the physical states $|\Psi_{phys}\rangle$. In this section, we consider a more explicit way to construct them, especially in the Schrödinger picture $[23]$. 
To go to the Schrödinger picture, we must choose a representation for the field commutation relation (19). Instead of taking the rotationally non-symmetric representations, which take one (spatial) component of $A_i$ as the conjugate momenta of the other component $A_j$, we take the rotationally symmetric representation which shows the contents of the gauge invariant operators more explicitly, as in the previous section B.

To this end, we note that $A_i$ can be expressed as

$$A_i = \epsilon_{ij} \partial_j^{-1} B + \partial_i \eta$$

which solves the equation $\epsilon_{ij} \partial_i A^j = \epsilon_{ij} \partial_i A^j = B$ with a gauge invariant scalar field $\eta$. The field $\eta$ is determined as

$$\eta(x) = -\nabla^{-2} \left[ \partial^i A_i(x) + \nabla^2 W(x) \right]$$

by considering the divergence (10)

$$\partial^i A_i(x) = \partial^i A_i(x) + \nabla^2 W(x) = -\nabla^2 \eta(x).$$

Using the expressions of (33) and (34), the base field $A_i$ of (8) can be re-arranged as follows

$$A_i = \epsilon_{ij} \partial_j^{-1} B + \partial_i (\eta + W),$$

where the first term corresponds to the transverse component $A_i^T$, and the second term corresponds to the longitudinal component $A_i^L$ of (9) but when the zero-mode can be neglected, one finds that the combined quantity ‘$\eta + W$’ becomes the usual form of the longitudinal mode which appears in (11):

$$\eta + W = -\nabla^{-2} \partial^i A_i - \nabla^{-2} \nabla^2 W + W = \nabla^{-2} \nabla^2 W + W = \partial_i^{-1} A_i = \partial_i^{-1} A_i^L.$$  

Now, let us consider a representation for the Schrödinger picture. To this end, we first note that $W(x)$ is the (unique) canonical conjugate of $B$ while $\eta$ is completely decoupled in the canonical conjugate sector of $B$, and furthermore $\eta$ can not be simultaneously diagonalized with $W$ from the commutation relations [we omit the subscript ‘op’ hereafter]:

$$[W(x,B(y)] = \frac{i\hbar}{\kappa} \delta^2(x-y),$$

$$[\eta(x), B(y)] = 0,$$

$$[W(x), \eta(y)] = \frac{i\hbar}{\kappa} \nabla^{-2} \left[ \epsilon_{ik} \partial_i c_k(x-y) - \nabla^2 \Delta(x-y) \right] \neq 0.$$ 

\(^{10}\text{Here, one finds that all the longitudinal parts are not participated in the gauge transformation but only } \partial_i W \text{ does the work}\)
Hence, for the representation with diagonalized $W(x)$, $B(y)$ will acts as a (functional) operators

$$B(y) |\Psi\rangle \rightarrow \hbar \frac{\delta}{i\kappa \delta W(y)} \Psi(W)$$

(36)

for any state functional $\Psi(W)$. This is a usual step that can be performed, although the detail form of conjugate moment $B$ is different from, theory to theories when $A_i$’s commute with themselves [see Appendix B for the analysis about this usual case in our context], where $\eta$ has no important role in the construction of $\Psi$. Now, here one meets an unusual situation where the naive expectation (36) is not valid and $\eta$ is crucial as well as $W$ in the construction of $\Psi$ : This results from the fact that neither $W(x)$ nor $\eta(y)$ can be taken as an diagonalized base in the Schrödinger picture because of the commutation relations

$$[W(x), W(y)] = \frac{i}{\kappa} \Delta(x - y),$$
$$[\eta(x), \eta(y)] = -i \frac{\hbar}{\kappa} \nabla^{-2} \xi_{kk}(x - y) + i \frac{\hbar}{\kappa} \Delta(x - y),$$

which all are non-vanishing in general. But the combination $\overline{W} = W + \eta$ results a vanishing commutation relation $[\overline{W}(x), \overline{W}(y)] = 0$ for all field points $x$ and $y$ [20] and hence $\overline{W}$ can be a diagonalized base for a representation of the Schrödinger picture but not $W$ alone: Hence, the correct representation is

$$B(y) |\Psi\rangle \rightarrow \hbar \frac{\delta}{i\kappa \delta \overline{W}(y)} \Psi(\overline{W})$$

instead of (36). In this representation, it is easy to see that the physical wavefunctional $\Psi_{\text{phys}}$ (21), which satisfies the Gauss’ law constraint, is any functional made of $\overline{\phi} \equiv \phi e^{i\overline{W}}$ and $\overline{\phi}^\dagger \equiv \phi^\dagger e^{-i\overline{W}}$ [: $J_0 :\equiv i\hbar(\phi \pi - \phi^\dagger \pi^\dagger)$$][21]$

$$[\kappa B(x) - : J_0 (x) :] \Psi_{\text{phys}} = \left[ \frac{\hbar}{i} \frac{\delta}{\delta \overline{W}(x)} - \hbar \phi(x) \frac{\delta}{\delta \phi(x)} + \hbar \phi^\dagger(x) \frac{\delta}{\delta \phi^\dagger(x)} \right] \Psi_{\text{phys}} = 0$$

with the usual representation for the matter parts

$$\pi(x) \left| \Psi_{\text{phys}} \right\rangle \rightarrow \hbar \frac{\delta}{i \delta \phi(x)} \Psi(\phi, \phi^\dagger),$$

11 For the coincident points $x = y$, one can make them vanish within a regularization prescription [24]. But, the general multi-particle states with the distinguishable positions can not be diagonalized with respect to neither $W$ nor $\eta$.

12 In Ref. [23], the forbidden combinations also were introduced explicitly and used to analyze the gauge equivalence.
\[ \phi(x) |\Psi_{\text{phys}}\rangle \to \phi(x)\Psi(\phi, \phi^\dagger), \]
\[ \pi^\dagger(x) |\Psi_{\text{phys}}\rangle \to \frac{\hbar}{i} \frac{\delta}{\delta \phi^\dagger(x)} \Psi(\phi, \phi^\dagger), \]
\[ \phi^\dagger(x) |\Psi_{\text{phys}}\rangle \to \phi^\dagger(x)\Psi(\phi, \phi^\dagger) \]

(37)

; when the zero-mode is neglizable, \( \bar{\phi}, \bar{\phi}^\dagger \) are nothing but the usual ones in (11) from the relation (35). But our setting is more general than the usual one: In particular, in a polynomial representation, it reads

\[ \Psi_{\text{phys}}(\bar{\phi}, \bar{\phi}^\dagger) = \prod_j \bar{\phi}(x_j) \prod_i \bar{\phi}^\dagger(y_i) \]

and there is no factor of the purely gauge field, in this case \( \epsilon_{ij} \partial_j^{-1} B \) instead of \( A_i \). This corresponds to a different representation from (32) and so this wavefunctional is not equivalent to (32) in general: Actually, there is a slight difference in the purely gauge field part compared to (32). For two particles sector, for example it becomes

\[ \Psi_{\text{phys}} = \bar{\phi}(x)\bar{\phi}(y) \]
\[ = e^{i\eta(x)}e^{i\eta(y)}e^{iW(x,y)} e^{i[\eta(x),W(x)] + \frac{1}{2}[\eta(x),W(x)] + \frac{1}{2}[\eta(y),W(y)]} \]
\[ \hat{\phi}(x)\hat{\phi}(y). \]

(38)

by separating \( e^{i\eta} \) and \( \hat{\phi} \) parts: In the functional Schrödinger approach, the full \( A_{op}^k \) part, which has the momenta \( B \) as well as \( \eta \), is not allowed in the form of (32) but only \( \eta \) part is allowed and contributes to the physical states in the representation of (32), in a specific from as as (38).

Finally, we note that the time-independent functional Schrödinger equation becomes

\[ H \Psi_E(\overline{\phi}, \overline{\phi}^\dagger) = \int d^2x \left[ \pi \pi^\dagger + \overline{D^i\phi(D^j\phi)^\dagger} + m^2 \overline{\phi} \overline{\phi}^\dagger \right] : \Psi_E(\overline{\phi}, \overline{\phi}^\dagger) \]
\[ = \int d^2x \left[ -\hbar^2 \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(x)^\dagger} + \overline{\partial^i\phi\partial^i\phi}^\dagger + i \left( \overline{\phi} \partial^i\phi^\dagger - \partial^i\phi \overline{\phi}^\dagger \right) A^{itr} + \overline{\phi} \overline{\phi}^\dagger (A^{itr})^2 \right] \Psi_E(\overline{\phi}, \overline{\phi}^\dagger) \]
\[ = E \Psi_E(\overline{\phi}, \overline{\phi}^\dagger), \]

where the barred variables in the Hamiltonian is the quantities involved \( \overline{\phi}, \overline{\phi}^\dagger \) and \( A^{itr} \equiv \epsilon_{ij} \partial_j^{-1} B = \frac{\hbar}{\kappa} \epsilon_{ij} \partial_j^{-1} \frac{\delta}{\delta W(x)} \) which generates \( (-) \frac{\hbar}{\kappa} \epsilon_{ij} \partial_j^{-1} \delta^2(x - y) = (-) \frac{\hbar}{2\kappa} \epsilon_{ij} \frac{(x-y)(x-y)^\prime}{|x-y|^2} \) for each \( \overline{\phi}(y)(\overline{\phi}^\dagger(y)) \) in the physical wavefunctional \( \Psi_E(\overline{\phi}, \overline{\phi}^\dagger) \); this reflects the relation \( A^{itr} \approx \frac{1}{\kappa} \epsilon_{ij} \partial_j^{-1} : J_0 : \).

D. Poincaré algebra
As an important criterion of the consistency of the model, let us consider the Poincaré algebra. In general, the quantum algebra can have some anomaly term compared to the classical one [24]. But it is found that this is not the case in our model: Classical algebra and quantum algebra are the same with appropriate choice of ordering and prescription. To see this, we first note, after some calculation, that one can obtain the relation which is the most non-trivial one in the Dirac-Schwinger conditions as follows,

\[
\begin{align*}
[T_{s}^{00}(x), T_{s}^{00}(y)] &= (T_{s}^{0i}(x) + T_{s}^{0i}(y)) \partial_{i} \delta^{2}(x - y).
\end{align*}
\]

Using this condition, it is straightforward to find the following quantum Poincaré algebra

\[
\begin{align*}
[P_{s}^{\mu}, P_{s}^{\nu}] &= 0, \\
[P_{s}^{\mu}, M_{s}^{\kappa \lambda}] &= i\hbar(\eta^{\mu \kappa} P^{\nu} - \eta^{\mu \nu} P^{\kappa}), \\
[M_{s}^{\mu \nu}, M_{s}^{\nu \lambda}] &= i\hbar \left( \eta^{\mu \nu} M^{\nu \lambda} - \eta^{\nu \kappa} M^{\mu \kappa} + \eta^{\nu \lambda} M^{\mu \kappa} - \eta^{\mu \lambda} M^{\nu \kappa} \right) \tag{39}
\end{align*}
\]

as well as the classical one. Here, we have considered symmetric ordering for the quantum generators in (39) and used the condition of finite matrix elements of the Poincaré generators.

On the other hand, there are canonical (Noether) Poincaré generators which are usually identical to the improved ones (12) on the constraint surface when one drops the boundary terms. But this is not trivial matter in lower dimensions like as in (2+1)-dimensions since in that case the boundary term should be treated more carefully [26]. Moreover, in the Chern-Simons theory the situation is more serious: The gauge varying “bulk” terms also appear in the canonical generators contrast to the improved ones. Let us describe this in detail. The (classical) canonical Poincaré generators are defined as

\[
\begin{align*}
P_{c}^{\mu} &= \int d^{2}x \; T_{c}^{0\mu}, \\
M_{c}^{\mu \nu} &= \int d^{2}x \left[ x^{\mu} T_{c}^{0\nu} - x^{\nu} T_{c}^{0\mu} + \pi_{\alpha} \Sigma_{\alpha \beta}^{\mu \nu} A^{\beta} \right]
\end{align*}
\]

with the canonical energy-momentum tensor

\[
T_{c}^{\mu \nu} = \sum_{F=\phi,\phi^{*},A_{\mu}} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} F)} \partial^{\nu} F - \eta^{\mu \nu} \mathcal{L}
\]

\[
= (D^{\mu} \phi)^{*} (D^{\nu} \phi) + (D^{\mu} \phi^{*}) (D^{\nu} \phi^{*}) - J^{\mu} A^{\nu} - \eta^{\mu \nu} \left[ (D^{\sigma} \phi)^{*} (D_{\sigma} \phi) - m^{2} \phi^{*} \phi \right]
\]

\[
+ \kappa \frac{1}{2} \eta^{\mu \delta} \epsilon_{\sigma \rho \phi} A^{\sigma} \partial^{\rho} A^{\phi} - \kappa \frac{1}{2} \eta^{\mu \nu} \epsilon_{\sigma \rho \phi} A^{\sigma} \partial^{\rho} A^{\nu}
\]

in the covariant form $[\pi_{\alpha} \equiv (\pi_{0}, \pi_{i}) = (0, \frac{\kappa}{2} \epsilon_{ij} A^{j})] \text{ or}$

\[
\begin{align*}
T_{c}^{00} &= |\pi_{\phi}|^{2} + D^{i} \phi^{*} D_{i} \phi + m^{2} |\phi|^{2} - A_{0} (J^{0} - \kappa B), \\
T_{c}^{0i} &= \pi_{\phi} D^{i} \phi + \pi_{\phi}^{*} D_{i} \phi^{*} - J^{0} A^{i} - \kappa \frac{1}{2} \epsilon_{j k} A^{j} \partial^{k} A^{i}
\end{align*}
\]
in the components form. More explicitly, the canonical Poincaré generators become

\[ P^0_c = \int d^2x \left[ 1 |\pi_0|^2 + |D^i\phi|^2 + m^2|\phi|^2 - A^0(J^0 - \kappa B) \right], \]
\[ P^i_c = \int d^2x \left[ \pi^0 D^i\phi + (D^i\phi)^* \pi^*_0 - J^0 A^i - \frac{\kappa}{2} \epsilon_{ijk} A^j \partial^i A^k \right], \]
\[ M^{i2}_{c} = \int d^2x \left[ \epsilon_{ij} x^j T^{ij}_c + \epsilon_{ij} \pi^i A^j \right], \]
\[ M^{0i}_c = x^0 P^i_c - \int d^2x \left[ x^i T^{00}_c - \pi^0 A^i + \pi^i A^0 \right]. \]

Then, one can easily find the following relations between the canonical and improved generators

\[ P^{0}_c \approx P^{0}_s, \]
\[ P^{i}_c \approx P^{i}_s, \]
\[ M^{i2}_{c} \approx M^{i2}_{s} + \frac{\kappa}{2} \int d^2x \partial^k \left( x^k A^i A^j - x^i A^k A^j \right), \]
\[ M^{0i}_{c} \approx M^{0i}_{s} + \frac{\kappa}{2} \int d^2x A_0 \epsilon^{ij} A^j, \]

where we have dropped the boundary term \( \int d^2x \partial^i \left( \epsilon_{jk} A^i A^k \right) \), which vanishes for the finite gauge field part of \( P^i_c \) or \( M^{i2}_{c} \); however, the boundary term in \( \epsilon^{ij} A^j \) cannot be simply neglected. Here, one can observe that the canonical boost generator \( M^{0i}_{c} \) is not gauge invariant because of term \( \frac{\kappa}{2} \int d^2x A_0 \epsilon^{ij} A^j \) although the term \( M^{i2}_{c} \) is gauge invariant for the rapidly decreasing gauge transformation function \( \Lambda \) asymptotically. Moreover, the commutators involving \( M^{0i}_{c} \) do not satisfy the Poincaré algebra:

\[ \frac{1}{i\hbar} [M^{0i}_{c}, P^0_c] \approx -P^j_c + \frac{\kappa}{2} \epsilon_{ik} \int d^2x \partial^0 \left( A^0 A^k \right), \]
\[ \frac{1}{i\hbar} [M^{0i}_{c}, P^j_c] \approx -\delta_{ij} P^0_c + \frac{\kappa}{2} \epsilon_{ik} \int d^2x \partial^i \left( A^0 A^k \right), \]
\[ \frac{1}{i\hbar} [M^{0i}_{c}, M^{i2}_{c}] \approx -\epsilon_{ij} M^{0j}_{c} - \frac{\kappa}{2} \epsilon_{ij} x^0 \int d^2x \partial^j \left( \epsilon_{ik} A^j A^k \right), \]
\[ \frac{1}{i\hbar} [M^{0i}_{c}, M^{0j}_{c}] \approx -\epsilon_{ij} M^{i2}_{c} - \frac{\kappa}{2} \epsilon_{ij} \int d^2x \partial^k \left( x^k A^i A^j - x^i A^k A^j \right), \]
\[ -\frac{\kappa}{2} \epsilon_{ij} \int d^2x \left[ \frac{5}{2} A_0^2 + A^k A^k + \partial^0 (x^k A^0 A^k) \right]. \]

Here, we have also used the symmetric ordering for the quantum Poincaré generators and used the prescription \( \partial \delta^2(0) \equiv 0 \) in order to remove the undesirable infinities which arise from the non-commuting \( A_i \) at the same points; with these choices the quantum algebras are the same as the classical ones.

In conclusion, it is the improved generators \([12]\), constructed from the symmetric energy-momentum tensor, which are (manifestly) gauge invariant and obey the quantum as well as
classical Poincaré algebra. Hence these improved generators have a unique meaning consistently with Einstein’s theory of gravity\textsuperscript{13}; this will lead to the uniqueness of the anomalous spin of the relativistic matter, which comes only from $M_{s}^{12}$. (Detailed discussion will be presented in Sec. IV.)

IV. Matching of GIF and GFF

So far, we have considered the manifestly gauge invariant formulation by introducing the Dirac dressing function. Now, the interesting question is how the gauge invariant results are matched to gauge fixed results. Actually, there have been some confusions about this issue \textsuperscript{3, 3, 4}. This will be clarified in the subsection D and now we start by describing the correct matching to GFF which has been presented recently \textsuperscript{33}.

In order to perform the matching, we need two things. One is the formula, called master formula

\[ \{L_a, L_b\} \approx \{L_a, L_b\}_{D\Gamma}. \]  

[The proof is presented in the Appendix C and only the interpretation of the result is in order here.] Here $L_a$ is any gauge invariant quantity, where the bracket with the first-class constraint $T$ of (4) vanishes, $\{L_a, T\} \approx 0$. The left-hand side of the formula (41) is the basic bracket of $L_a$’s. The right-hand side of (41) is the Dirac bracket with gauge fixing function $\Gamma = 0$, \textit{det}\{\Gamma, T\} $\neq 0$. Moreover, in the latter case, since $\Gamma = 0$ can be strongly implemented, $L_a$ can be replaced by $L_a|_{\Gamma}$ that represents the projection of $L_a$ onto the surface $\Gamma = 0$. The left-hand side is gauge independent by construction since $L_a$’s and the the basic bracket algebra (4) are introduced gauge independently. On the other hand, the Dirac bracket \textsuperscript{9} depends explicitly on the chosen gauge $\Gamma$ in general. But there is one exceptional case, i.e., when the Dirac bracket is considered for the gauge invariant variables. Our master formula (41) explicitly show this exceptional case: The Dirac bracket for the gauge invariant variables $L_a$ or their projection $L_a|_{\Gamma}$ on the surface $\Gamma = 0$ are still gauge invariant and equal to the basic bracket for the corresponding variables\textsuperscript{14}.

\textsuperscript{13}There may be other differently improved generators depending on what gravity theory is chosen like as in Ref. \textsuperscript{16}. But we do not consider this possibility in this paper. Moreover, the preferred property of the symmetric energy-momentum tensor compared to canonical one by the gauge invariance was examined also by Deser and MaCarthy \textsuperscript{27}. But they missed the important role of the gauge invariance, which is genuine for the CS gauge theory, on the integrated quantities, Poincaré generators.

\textsuperscript{14}For the first-class constraint $T$, it was known in Ref. \textsuperscript{28}; this formula was implicitly included also in the recently developed Batalin-Fradkin-Tyutin formalism \textsuperscript{29}. But, the formula (41) is valid even for the second-class constraint $T$ and this fact has done an important role in the Dirac’s canonical analysis of the boundary CS theory consistently to the symplectic reduction method \textsuperscript{30}. 

20
Another important thing for the matching is to know how the defining equation (7) for $c_k(x - y)$ is modified in GFF. By considering $\hat{\phi}$ (or $A_\mu$) in a specific gauge and the residual gauge transformation of $\phi$ and $A_\mu$, one can find modified (but still make $\hat{\phi}$ be gauge invariant) equation for $c_k(x - y)$. Here we will consider the following three typical cases with the resultant modified equations of $c_k(x - y)$:

a) Coulomb gauge ($\partial_i A_i \approx 0$) : $\int d^2 z c_j(x - z) A_j(z) = 0$,

b) Axial gauge ($A_1 \approx 0$) : $\partial_z^2 c_2(x - z) = -\delta^2(x - z)$,

c) Weyl gauge ($A_0 \approx 0$) : $\partial_j z c_j(x - z) = -\delta^2(x - z)$.

These results are generally valid for any other gauge theories when they are formulated by our gauge invariant formulation. Note that these results are different from recent claims of Ref. [6] except in the case of Coulomb gauge. Moreover, the Weyl gauge does not modify the equation for $c_k$ from (7). Then, using these relations and (41), we can consider the gauge fixed results directly from GIF. However, as can be observed in these examples, gauge fixings restrict the solution space in general. Therefore, all the variables which appear in (15) are gauge invariant for each solution hyper-surface which is selected by gauge fixing, but their functional form may be different depending on the chosen gauges. Let us derive the results of (42) in detail and the matching to GFF using them.

A. Coulomb gauge ($\partial_i A_i \approx 0$)

It is important to note that in this gauge, there is no residual gauge symmetry: If we consider the gauge transformation, i.e., $A_i \rightarrow A_i + \partial_i \Lambda$, $\Lambda$ should satisfy the Laplace equation $\nabla^2 \Lambda(x) = 0$ over all space-times and so this equation has only one trivial solution $\Lambda(x) = 0$. On the other hand, since the gauge transformation of $\phi$ is defined as what makes $D_i = \partial_i + i A_i$ as a covariant derivative when acted upon $\phi$, $\phi$ does not transform (modular unimportant global phase transformation), either. Hence, it is found that the additional factor $W$, which cancels the gauge transformations of $A_i$ and $\phi$, is unnecessary or it can be made to be zero simply, i.e.,

$$W(x) = \int d^2 z c_j(x, z) A^j(z) = 0.$$  

In this case, the gauge invariant variables in (3) and the corresponding base fields are equivalent, i.e., $A_\mu = A_\mu$, $\hat{\phi} = \phi$, $\hat{\pi} = \pi$ and thus the solution of $A_\mu$ in this gauge can be directly read from

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\footnote{Authors of Refs. [6, 7] considered $\int d^2 z c_j(x - z) A^j(z) = 0$ even “b)” and “c)” cases. But then, the manifestly gauge invariant fields in (3) are not gauge invariant under the residual gauge symmetries $\phi \rightarrow e^{-i A} \phi$, $A_\mu \rightarrow A_\mu + \partial_\mu A$ with $x^1$ and $x^0$ independent $\Lambda$ for “b)” and “c)”, respectively.}
\( A_i(x) \approx -\frac{1}{2\pi \kappa} \int d^2z \, \epsilon_{ik} \frac{(x-z)_k}{|x-z|^2} j^0(z) \),

\[ A_0(x) = -\frac{1}{2\pi \kappa} \int d^2z \, \epsilon_{kj} \frac{(x-z)_k}{|x-z|^2} J^j(z). \]

Now, in order to find the solution \( c_j \) of (43), let us define

\[ c_j(x-z) = \partial_j^z \chi(x-z) \] (44)

then, it is easy to see

\[ \int d^2z \, c_j(x-z) A^j(z) = \int d^2z \, \partial_j^z \chi(x-z) A^j(z) = -\int d^2z \, \chi(x-z) \partial_j^z A^j(z) = 0 \]

when one neglects the boundary term, i.e.,

\[ \int d^2z \, \partial_j^z \left( \chi(x-z) A^j(z) \right) = 0. \] (45)

Now, to find the solution of \( \chi \) with this property, let us note that \( \chi \) satisfies the Poisson equation

\( \nabla^2 \chi(x-z) = \delta^2(x-z) \) according to (7). The well-known solution of this equation is

\[ \chi(x-z) = \frac{1}{4\pi} \ln|x-z|^2 \] (46)

up to unimportant constant term. It is easy to find that this solution satisfies (43) as

\[ \int d^2z \, \partial_j^z \left( \chi(x-z) A^j(z) \right) = \oint_{S_R \to \infty} Rd\hat{r} \cdot \left( \chi(x-z) A^j(z) \right) = \frac{Q}{4\pi^2 \kappa} \oint_{S_R \to \infty} d\theta \hat{\theta} \cdot \hat{\theta} \ln R = 0 \]

“geometrically” though not negligible in the naive asymptotic \( r \)-dependence. [Here the integration is evaluated on a circle with infinite radius \( R \), polar angle \( \theta \), and their corresponding (orthogonal) unit vectors \( \hat{r}, \hat{\theta} \).] Hence, one finds the solution, by using (44) and (46), as follows

\[ c_j(x-z) = -\frac{1}{2\pi} \frac{(x-z)_j}{|x-z|^2}. \] (47)

\(^{16}\)Here, one should be careful in order not to obtain a wrong result by applying \( A_\mu \to A_\mu \) before applying the constraint or equations of motion: Only for the final formula in (16) and (17), one can get the correct results.
Then, we find the anomalous transformation (15) with
\[
\Xi^{12}(x) = \frac{1}{2\pi \kappa} Q, \\
\Xi^{0i}(x) = \frac{1}{2\pi \kappa} \int d^2 z \frac{(x - z)^i(x - z)^k}{|x - z|^2} \epsilon_{jk} J^j(z),
\]
where \( Q = \int d^2 z J_0 \), which can be directly obtained from the expressions of (18) by substituting (17). These are exactly Hagen’s rotational anomaly and Coulomb gauge restoring term in the Lorentz transformation, respectively [10].

Next, let us consider the two functions \( \Delta(x - y) \) and \( \xi_{ij}(x - y) \) which characterize the commutation relations. About \( \Delta(x - y) \) (22), one finds that, upon using (44),
\[
\Delta(x - y) = \int d^2 z \epsilon^{kj} \partial^z \chi(x - y) \partial^z \chi(y - z) \\
= -\int d^2 z \epsilon^{kj} \nabla^z \chi(x - y) \chi(y - z) \\
= -\oint_{S^1_\infty} \hat{d} \hat{\theta} \cdot \hat{r} \ln R \\
= 0
\]
from the geometric reason, by performing the integration by parts in the second line. Now, about \( \xi_{ij}(x - y) \) (23), one finds that, upon using the antisymmetry \( c_j(x - z) = -c_j(z - x) \) for the solution (47)
\[
\xi_{ij}(x - y) = -\epsilon_{ik} \partial^y c_k(y - x) + \epsilon_{kj} \partial^x c_k(x - y) \\
= (\epsilon_{ik} \partial^y + \epsilon_{kj} \partial^x) c_k(x - y)
\]
which becomes, for each indices, as follows:

\( a \) \( (\epsilon_{ik} \partial^y + \epsilon_{ki} \partial^x) c_k(x - y) = 0 \) (for \( i = j \)),

\( b \) \( (\epsilon_{1k} \partial^2 + \epsilon_{k2} \partial^1) c_k(x - y) = \partial_2^2 c_1(x - y) + \partial_1^x c_1(x - y) \\
= -\delta^2(x - y) \) (for \( i = 1, j = 2 \)),

\( c \) \( (\epsilon_{2k} \partial^1 + \epsilon_{k1} \partial^2) c_k(x - y) = -\partial_1^x c_1(x - y) - \partial_2^2 c_2(x - y) \\
= \delta^2(x - y) \) (for \( i = 2, j = 1 \)).

In a compact form, it becomes
\[
\xi_{ij}(x - y) = -\epsilon_{ij} \delta^2(x - y).
\]
Using these results, one finds that the basic brackets defined in (2 1) are the usual Dirac brackets in the Coulomb gauge

\[ \{ A_i(x), A_j(y) \}_{D(Coulomb)} \approx \{ A_i(x), A_j(y) \}_{D(Coulomb)} = 0, \]

\[ \{ \phi(x), \phi(y) \}_{D(Coulomb)} \approx \{ \phi(x), \phi(y) \}_{D(Coulomb)} = 0, \]

\[ \{ A_i(x), \phi(y) \}_{D(Coulomb)} \approx \{ A_i(x), \phi(y) \}_{D(Coulomb)} = -\frac{i}{2\pi\kappa} \epsilon_{ik} \frac{(x - z)_k}{|x - z|^2} \phi(y). \]

Furthermore, these two results imply that the gauge invariant operator \( \hat{\phi}_{op} \) satisfies the boson commutation relation, \( [\hat{\phi}_{op}(x), \hat{\phi}_{op}(y)] = 0 \) in this case instead of the generic graded commutation relations (26). Here, we note the special importance of the Coulomb gauge in that the original fields \( \phi, \phi^*, A_{\mu} \) themselves are already gauge invariant fields because of the result (43) and hence they already have the full anomaly structures of (43). Furthermore, this gauge is the simplest one to obtain the anomalous spin of the original matter field \( \phi \) as \( \Xi^{12} \) of (48) since this does not have other gauge restoring terms as in the rotationally non-symmetric gauge. This is made clear by noting the relation of (40)

\[ M_{12}^{s} \approx M_{12}^{c} - \frac{\kappa}{2} \int d^2z \partial^k \left( z^k A^l A^l - z^l A^l A^k \right), \]

where \( M_{12}^{c} \) is the canonical angular momentum

\[ M_{12}^{c} = \int d^2z \left[ \epsilon_{lk} z^l \left( \pi \partial^k \phi + (\partial^k \phi)^* \pi^* \right) - \kappa z^l A^l (\partial^k A^k) + \frac{\kappa}{2} \partial^k (z^l A^l A^k) \right]. \]

The surface terms in \( M_{12}^{s} - M_{12}^{c} \) and \( M_{12}^{c} \), which are gauge invariant for the rapidly decreasing gauge transformation function \( \Lambda \), give the gauge independent spin terms “\((1/4\pi\kappa)Q^2\)” [10, 11, 12] (unconventional) and “0” (conventional) in \( M_{12}^{s} \), respectively: Because of the gauge independence of the surface terms, only the calculation in one simple gauge, e.g., Coulomb gauge is sufficient\(^{17}\) to get this general result and in that case one obtains explicitly the boundary integrals as follows

\[ \int d^2z \partial^k \left( z^k A^l A^l \right) = - \int_{S_R^{k} \rightarrow \infty} R d\theta \hat{r} \cdot r A^l A^l \]

\[ = - \frac{1}{4\pi\kappa^2} \int_{S_R^{k} \rightarrow \infty} R^2 d\theta \epsilon_{lk} \frac{r^k}{|r|^2} Q \epsilon_{lj} \frac{r^j}{|r|^2} Q \]

\[ = - \frac{1}{\kappa^2} Q^2, \]

\[ \int d^2z \partial^k \left( z^l A^l A^k \right) = - \int_{S_R^{k} \rightarrow \infty} R d\theta \hat{r} \cdot r \cdot A z^l A^l \]

\[ = 0. \]

\(^{17}\)Explicit manipulations of the gauge independence of the unconventional term have been established only for some limited class of gauges [10, 11, 12]. But these results will be generalized to the case of general gauges due to the gauge invariance of the term.
From which the anomalous spin $\frac{Q}{2\pi} \pi$ of (48) for the base matter field, which is defined as the surface term $M_{s}^{12} - M_{c}^{12}$ for the base matter field is readily seen to follow for general gauges. The commutation relation $[Q, \phi(x)] = \phi(x)$ which is a basic ingredient in the derivation, is gauge independent relation because it expresses the gauge independent fact that $\phi$ carries the unit charge. On the other hand, the second term of $M_{c}^{12}$, which vanishes only in the Coulomb gauge, gives the gauge restoring contribution to the rotation transformation for the matter field for the general gauges; actually, this is the case for the rotation non-invariant gauge since the Coulomb gauge only is the “rotation invariant” one. Finally, we note that the anomalous spin, which comes only from $M_{s}^{12}$, has a unique meaning because of the uniqueness of the improved generators in that they are gauge invariant on the constraints surface and obey the Poincaré algebra though this is not the case for the canonical ones.

Furthermore, this uniqueness of anomalous statistics is in contrast to the anomalous statistics, which has only artificial meaning in this case. This is because we can obtain in any field theories any arbitrary statistics by constructing gauge invariant exotic operators of the form of Semenoff and its several variations. In this sense the relativistic CS gauge theory does not respect the spin-statistics relation in agreement with Hagen’s result. Here, we would like to comment that the situation of non-relativistic CS gauge theory is not better than the relativistic case. This is because even though the anomalous statistics is uniquely defined by removing the gauge field (in this case the gauge field is pure gauge due to point nature of the sources in non-relativistic quantum field theory) the anomalous spin has no unique meaning: The anomalous spin can be removed by redefining the angular momentum generator without distorting the Poincaré algebra.

B. Axial gauge ($A_{1} \approx 0$)

In this gauge, the residual gauge symmetry is $A_{i} \rightarrow A_{i} = \partial_{i} \Lambda$, $\partial_{1} \Lambda = 0$ which preserves the chosen gauge $A_{1} \approx 0$. Under this transformation, the matter field $\hat{\phi}$ is gauge invariant when the dressing satisfies the equation

$$\partial_{z}^{2}c_{2}(x - z) = -\delta^{2}(x - z), \quad (50)$$

and by comparing to the original equation (4) one obtains furthermore another equation

$$\partial_{z}^{1}c_{1}(x - z) = 0. \quad (51)$$

---

18 This can be explicitly checked by considering the general gauges $\int d^{2}z K_{\mu}(x, z) A_{\mu}(z) = 0$ with a kernel $K_{\mu}(x, z)$ as will be discussed in subsection D in a different context.

19 The other factor $z^{1}A^{1}$ will not be zero for all space region.

20 These gauge should be “translation” invariant also such that the each gauge is defined over all space.
Now, let us consider the several solutions of the equations (50) and (51). First the simplest solution is
\[ c_1(x - z) = 0, \]
\[ c_2(x - z) = -\delta(x_1 - z_1)\epsilon(x_2 - z_2) \] (52)
with the step function \( \epsilon(x) \)
\[ \epsilon(x) = \begin{cases} 
0 & (x < 0) \\
1 & (x > 0) 
\end{cases} . \]
This gives for \( W \) a line integral
\[ W(x) = \int_{-\infty}^{x_2} dz_2 A^2(x_1, z_2). \]
In this case, \( \phi_{op}(x) \) carries the vector potential at the position \( y \), \( a^1(y) = -\frac{\hbar}{\kappa} \delta(x_1 - y_1)\epsilon(x_2 - y_2), a^2(y) = 0 \), which is orthogonal to and non-vanishing only along the integration path; this can be considered as a shrink of the space where the gauge field lives for the Coulomb gauge into one-dimensional (straight) lineal space. Moreover, since this solution (52) corresponds to a different solution hyper-surface to the Coulomb gauge and therefore, its related anomalous terms in (15) have different functional form to (48) even though they are gauge invariant on their own hyper-surfaces:
\[ \Xi^{01}(x) = 0, \]
\[ \Xi^{02}(x) = \frac{1}{\kappa} \int_{-\infty}^{\infty} dy_2 J_1(x_1, y_2), \]
\[ \Xi^{12}(x) = -\frac{1}{\kappa} \int_{-\infty}^{\infty} dy_2 J_0(x_1, y_2). \]
Here, we have used the formula (18).
Furthermore, one finds that the two characteristic functions of (22) and (23) become as follows
\[ \Delta(x - y) = 0, \]
\[ \xi_{11}(x - y) = -\partial_1^r \delta(x_1 - y_1), \]
\[ \xi_{12}(x - y) = -\xi_{21}(x - y) = \delta^2(x - y), \]
\[ \xi_{22}(x - y) = 0, \] (53)
\[ ^{21} \text{Here, we are considering the line-integral representation of } W \text{ in the context of gauge fixing which is well-defined over all space, the crossings in the contour of the integral is not considered. Hence, in our case, there is no path-ordering even when we consider the quantum theory contrast to the case of [20].} \]
and thus the basic brackets of (21), which are found to be the Dirac bracket in the axial gauge 
\[ A_1 \approx 0 \], become as follows

\[
\{ A_1(x), A_1(y) \}_{D(axial)} = -\frac{1}{\kappa} \partial_1^x \delta(x_1 - y_1),
\]

\[
\{ \hat{\phi}(x), \hat{\pi}(y) \}_{D(axial)} = \delta^2(x - y),
\]

\[
\{ A_1(x), \hat{\phi}(y) \}_{D(axial)} = -\frac{i}{\kappa} \hat{\phi}(y) \delta(y_1 - x_1) \epsilon(y_2 - x_2),
\]

\[
\{ A_1(x), \hat{\pi}(y) \}_{D(axial)} = \frac{i}{\kappa} \hat{\phi}(y) \delta(y_1 - x_1) \epsilon(y_2 - x_2),
\]

where

\[
A_1(x) = -\partial_1^x \int d^2 z \ c_2(x - z) A^2(z)
\]

\[
= -\partial_1^x \int^{x_2} d z_2 \ A^2(x_1, z_2),
\]

\[
A_2(x) = A_2(x) - \partial_2^x \int d^2 z \ c_2(x - z) A^2(z)
\]

\[
= 0,
\]

\[
\hat{\phi}(x) = \phi(x) e^{i \int^{x_2} d z_2 \ A^2(x_1, z_2)},
\]

\[
\hat{\pi}(x) = \pi(x) e^{-i \int^{x_2} d z_2 \ A^2(x_1, z_2)}.
\]

(54)

However, it is not straightforward to obtain the Dirac bracket for the base fields \( \phi \) and \( A^i \) themselves. Moreover, for more general solution

\[
c_1(x - z) = G(x_2 - z_2),
\]

\[
c_2(x - z) = \delta(x_1 - z_1) \epsilon(x_2 - z_2) + F(x_1 - z_1)
\]

(55)

with the functions \( F \) and \( G \) which have the dependence only along the 1 and 2 directions, respectively, and the gauge invariant variables (54) are not changed except \( A_1(x) = \int d^2 z \ [\partial_1^x F(x_1 - z_1)] A^2(z) \) but all other variables \( \Xi^{\mu \nu}, \Delta, \xi_{ij} \) have the explicit dependence of two functions \( F \) and \( G \):

\[
\Xi^{12}(x) = -x^2 \int d^2 z \ [\partial_1^x F(x_1 - z_1)] A^2(z) - \frac{1}{\kappa} \int^{x_2} d z_2 \ J_0(x_1, z_2),
\]

\[
\Xi^{01}(x) = \frac{1}{\kappa} \int d^2 z \ (x - z)_1 \ [G(x_2 - z_2) J_2(z) - F(x_1 - z_1) J_1(z)],
\]

\[
\Xi^{02}(x) = \frac{1}{\kappa} \int d^2 z \ (x - z)_2 \ [G(x_2 - z_2) J_2(z) - F(x_1 - z_1) J_1(z)] + \frac{1}{\kappa} \int^{x_2} d z_2 \ J_1(x_1, z_2),
\]

\[
\Delta(x - y) = \int^{y_2} d z_2 \ G(x_2 - z_2) - \int^{x_2} d z_2 \ G(y_2 - z_2)
\]
\[ + \int d^2z \left[ G(x_2 - z_2)F(y_1 - z_1) - F(x_1 - z_1)G(y_2 - z_2) \right], \]
\[ \xi_{11}(x - y) = -\partial_1^x \delta(x_1 - y_1) - \partial_1^y \left[ F(y_1 - x_1) + F(x_1 - y_1) \right], \]
\[ \xi_{12}(x - y) = -\xi_{21}(x - y) = \delta^2(x - y). \]

It is interesting to note that \( \xi_{ij}(x - y) \) is the same as (53) if \( F(x_1 - y_1) \) is an anti-symmetric function, i.e., \( F(x_1 - y_1) = -F(y_1 - x_1) \). Moreover, we note that \( \Delta(x - y) \) is not zero in general and hence the commutation relations of \( \hat{\phi}'s \) are not the bosonic ones and we have additional contribution in the commutation relation involving \( A_i(x) \) according to (27).

C. Weyl gauge \((A_0 \approx 0)\)

In this case, the residual symmetry is the time-independent gauge transformation \( A_\mu \to A_\mu + \delta_\mu \partial_i \Lambda \), with the time-independent gauge function \( \Lambda \). However, because of the form of Dirac dressing as (3) which is spatially non-local but temporally local, the gauge transformation of the \( \hat{\phi}, A_\mu \) fields (3) is formally the same as that of gauge unfixed case and this gives the same equation for the dressing as (7).

D. Clarification of previous confusions

Up to now, there have been several confusions about the gauge fixing (kernel-) function \( K_\mu(x, z) \) in the gauge condition
\[ \int d^2z \ K_\mu(x, z)A^\mu(z) = 0 \]
(56)
and the Dirac dressing function \( c_k(x - z) \) which satisfies (4) [4, 3, 1]. The main source of these confusions is the identity [36, 5]
\[ \int d^2x \ c_i(y - x)A^i(x) = 0, \]
(57)
where the function \( c_i(y - x) \) rapidly decreases in the asymptotic region so that a boundary integral \( \int d^2x \ \partial_x^i [c_i(y - x)W(x)] \) can be neglected. The identity (57) looks similar to (56), but there is an important difference. In (56), \( A^\mu \) denotes the gauge varying field and hence (56) restricts the gauge symmetry of \( A^\mu \), while \( A^i \) in (57) is gauge invariant by definition and hence (57) does not restrict the gauge symmetry; the identity might be involved with some other symmetries if there are. Actually, it is found that a new symmetry has been introduced implicitly with the introduction of \( c^i(y - x) \) [4].

\[ c_k(x - z) \to c_k^*(x - z) = c_k(x - z) + \epsilon_{kj} \partial_j b(x - z) \]
(58)

\footnote{It is interesting to compare this symmetry to the BRST symmetry [31]. These two symmetries seem to be an extremely different ones; since one (BRST symmetry) is about GFF and the other (\( c_k \)-symmetry) is about GIF. But they are rather very close in that the fundamental variables (\( F_\alpha \) in the former and \( A_\mu, \phi, \phi^* \) in the latter) are not gauge transformed: In the former case, the variable \( F_\alpha \) are not gauge transformed by definition; in the latter case, the variables \( (A_\mu, \phi, \phi^*) \) are not allowed to be gauge transformed because of the introduction of a complete gauge fixing term.}
because the transformed quantity \( c'_k(x - z) \) also satisfies the defining equation (6) for the well-behaved function \( b(x - z) \) with \( \nabla \times \nabla b = 0 \). However, this transformation is not trivial one because it makes \( W \) transform as follows

\[
W(x) \to W(x) + \int d^2z \, b(x - z)B(z)
\]

by neglecting the boundary term \( \int d^2z \, \partial \frac{\partial}{\partial z} \left[ \epsilon_{kj}b(x - z)A^k(z) \right] \) for rapidly decreasing function \( b(x - z) \). Hence one finds a strange situation that the gauge invariant variables \( F^\alpha \) transform as follows

\[
\hat{\phi}(x) \to \hat{\phi}'(x) = e^{i\Lambda(x)}\hat{\phi}(x),
\]

\[
\mathcal{A}_\mu(x) \to \mathcal{A}'_\mu(x) = \mathcal{A}_\mu(x) - \partial_\mu \Lambda(x),
\]

which looks like as a gauge transformation with a transformation function \( \Lambda(x) = -\int d^2z \, b(x - z)J_0(z) \approx -\frac{1}{\kappa} \int d^2z \, b(x - z)J_0(z) \). But the existence of the relation (58), (59), (60) saves this situation. Let us consider the transformation of the condition (58), (59), (60):

\[
0 = \int d^2x \, c_i(y - x)A^i(x) \to 0 = \int d^2x \, c'_i(y - x)A'^i(x)
\]

\[
= \int d^2x \, c_i(y - x)A^i(x) - 2\int d^2x \, b(y - x)B(z)
\]

\[
\approx -\frac{2}{\kappa} \int d^2x \, b(y - x)J_0(x)
\]

where we have neglected the boundary terms in the third line for the rapidly decreasing functions \( c_i(y - x) \) and \( b(y - x) \). Then, one can find that \( 'b(y - x) = 0' \) is the only solution for \( b(y - x) \) which retains the condition (57) when \( x \) is located at the source points where \( J_0(x) \) does not vanishes. It is interesting to note that the identity (57) corresponds to the divergence-free condition \( \nabla \cdot \mathbf{A}^T = 0 \) for the gauge invariant transverse component \( \mathbf{A}^T \) in (3): As the divergence-free condition defines \( \mathbf{A}^T \), the condition (57) can be also considered as a defining equation for \( \mathbf{A}_i \). On the other hand, it is easy to see that only for the Coulomb gauge, the identity (57) is reduced to the divergence-free condition which is consistent to the fact of the equivalence of \( \mathbf{A}_i \) and \( A_i \) in this gauge.

There is one more interesting effect of the condition (57). To see this, let us consider the condition (57) with \( \mathbf{A}_i \) expressed by the solution of (16)

\[
0 = \int d^2x \, c_i(y - x)A^i(x)
\]

\[
= \int d^2z \, B(z) \int d^2x \, \epsilon_{ki}c_i(y - x)c_k(z - x).
\]

In general, the function \( c_k(z - x) \) is sum of the parity even and odd parts. However, if one restricts only one part, i.e., \( c_i(y - x) = \pm c_i(x - y) \) for the parity even or odd parts, respectively,
\[ (61) \] becomes
\[
0 = \mp \int d^2z \ B(z) \Delta(z - y)
\approx \mp \frac{1}{\kappa} \int d^2z \ J_0(z) \Delta(z - y)
\]
and one finds finally
\[
\Delta(z - y) \approx 0
\]
when \( z \) or \( y \) is the position of the sources. [Here, the former and latter coordinates \( z \) and \( y \), respectively have no absolute meaning because of the symmetry (24). Moreover, in this case the coordinates in the function \( b(y - x) \) are also equal footing because of the symmetry of (58) which should be preserved by \( b(y - x) \) (actually with opposite parity).] This result provides an simple interpretation of the result (49) for the solution (47) which has odd parity. On the other hand, the fact of \( \Delta = 0 \) in (53) even for the solution (52) which doesn’t have the definite parity is the result of the particular form of (52): For more general solution (55), non-vanishing \( \Delta \) is expected.

Now, finally we note that there is an identity for \( A_k \)-field as a dual to (57). To this end, let us assume that we can write \( A_\mu \) as
\[
A_\mu(x) = A_\mu(x) - \partial_\mu^h \int d^2z \ K_k(x - z) A^k(x) \quad (62)
\]
with\[ \partial_\mu^h K_k(x - z) = -\delta^2(x - z). \]
Then, it is easy to show that \( A_k \) in (52) satisfies
\[
\int d^2z \ K_k(x - z) A^k(z) = 0 \quad (64)
\]
when \( K_k(x - z) \) is rapidly decreasing asymptotically such that the boundary term \( \int d^2x \ \partial_\nu^h (K_i(y - x) \int d^2z \ K_k(x - z) A^k(z)) \) can be neglected. However, we note that there is no direct connection between the gauge-fixing kernel \( K_k(x - z) \) and the dressing function \( c_k(x - z) \) but only the condition, \( \int d^2x \ \partial_\nu^j (K_i(y - x)W(x)) = 0 \) such that we obtain the gauge condition (54) consistently. [This can be easily obtained by expressing \( A_\mu \) in the right-hand side of (52) in terms of \( A_\mu \) using (1) and comparing the left and right-hand sides.]

\[ ^{23} \Delta = 0 \] is retained under (58) only for the case where \( b(x - y) = b(y - x) \), \( \int d^2z \ \partial_\nu^j [b(x - z)c_j(y - z) - b(y - z)c_j(x - z)] = 0 \) are satisfied; actually these are the condition for the invariance of \( \Delta \) and \( \xi \) in general.

\[ ^{24} \] Note that \( A_k \) does not gauge transform by the definition of (53): (14) is just an identity for the completely gauge fixed quantity \( A_k \). In this sense (64) is not the gauge fixing condition which restricts the gauge variations: If we consider (14) as a gauge fixing, the representation of \( K_i \) as (53) is possible only for the Coulomb gauge as we have studied in Sec. A. (See (52) for comparison)
V. Discussion and summary

A. Gauge invariance in action

In this paper, we have studied the manifestly gauge invariant “Hamiltonian” formulation where the energy momentum tensors are gauge invariant. But this does not imply the gauge invariance of the action in general: The non-Abelian CS gauge theory is an example [34, 35]. Hence, if we might find a representation corresponding to (5) in the non-Abelian CS theory, it will be a problem to get the manifestly gauge invariant action from the original action in terms of the manifestly gauge invariant fields. However, in our case of Abelian theory, there will be no general obstacle to do this. Actually it is found that this is the case: When the gauge field $A_{\mu}$ in the CS term of the original action (1) is replaced by the gauge invariant fields $A_{\alpha}$, it reads

$$
\int d^3 x \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho = \int d^3 x \left[ \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \frac{\kappa}{2} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho \partial_\mu W \right]
$$

$$
= \int d^3 x \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \int d^3 x \frac{\kappa}{2} \partial_\mu \left[ \epsilon^{\mu\nu\rho} \partial_\nu A_\rho W \right]
$$

$$
\approx \int d^3 x \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \int d^3 x \frac{1}{2} \partial_\mu \left[ J^\mu W \right]
$$

; for the well-localized fields one finds the equivalence of the original CS action of the base fields to the CS action of gauge invariant field $A_\alpha$. Furthermore, since the matter parts can be made to be manifestly gauge invariant trivially, the total action integral is invariant manifestly. Actually, in this derivation the coefficient $\kappa$ does not have any role and actually this fact has been considered as the signal of the no-quantization of $\kappa$ in a slightly different context [34]. According to this interpretation, it is expected that the complete transformation of the original action into the action which is expressed by the manifestly gauge invariant fields in the non-Abelian CS theory, if it does exist, at least up to the total derivative terms, is not trivial matter depending on the coefficient $\kappa$. If the interpretation is a correct one, we suspect that

$$
\int d^3 x \kappa \epsilon^{\mu\nu\rho} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) + 8\pi^2 \kappa \omega = \int d^3 x \kappa \epsilon^{\mu\nu\rho} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right),
$$

[\langle \cdot \cdot \cdot \rangle \text{ denotes trace}] up to the total derivatives term, where $\omega$ is an integer number which is involved to the winding number for the homotopically non-trivial, large gauge transformations of a non-Abelian gauge group whose $\pi_3$ is $\mathbb{Z}$: Only in this form, it is consistent with the

\[\text{Gauge invariance of Lagrangian (density) is a requirement which is stronger than that of action integral. Appreciation of this subtlety becomes necessary recently, in discussions of the radiatively induced Lorentz and CPT violating CS term in QED [38].}\]
well-known quantization of $\kappa$ [35]. But, it is unclear whether the manifestly gauge invariant variables corresponding to (5) for all gauge transformation, i.e., large as well as small gauge transformations, exist or not.

**B. Kinetic mass term and dual connection to Maxwell-CS theory**

In our original theory (1) there is $U(1)$ gauge symmetry and so the kinetic mass term $\frac{1}{2}\mu^2 A_\mu A^\mu$, which breaks the symmetry manifestly, can not be introduced in this context. However, in contrast, within the gauge invariant fields context, the mass like term

$$\frac{1}{2}\mu^2 A_\mu A^\mu$$

(65)
can not be discarded generally [36, 37]. To see what this term implies, let us re-express this term as follows, using the formulas in the third steps in (16) and (17) which are valid model independently

$$A_\mu A^\mu(x) = \int d^2 z \int d^2 y \ c_k(x - z)c_l(x - y) \left[ F_{k0}(z)F^{l0}(y) + F_{ki}(z)F^{li}(y) \right]$$

$$= \int d^2 z \int d^2 y \ c_\mu(x - z)c_\nu(x - y)F_{\mu\nu}(z)F^{\mu\nu}(y)$$

(66)

with $c_0(x - z) \equiv 0$, without using the equations of motion. The resulted expression is similar to the Maxwell’s kinetic term, but in a more generalized form, which is absent in the original theory (1). So, in order to introduce the mass term (65), we must also consider an extension from the (pure) CS gauge theory (1). On the other hand, since the result (66) was obtained without using the equation of motion, i.e., model independently, the extended theory need not have exactly the same form as the final form of (66): These two theories may be related only after using the equations of motion or the constraints. The most natural candidate for the extended theory will be, of course, the Maxwell-CS (MCS) theory which has the Maxwell term in addition to the CS theory (1) [33, 34]:

$$L \_{MCS}[A] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{2} \epsilon^{\mu\nu\rho\sigma}A_\mu \partial_\nu A_\rho + (D_\mu \phi)^*(D^\mu \phi) - m^2 \phi^* \phi.$$

(67)

The theory has the equations of motion, $\partial_\mu F^{\mu\nu} + \frac{\kappa}{2} \epsilon^{\mu\nu\rho\sigma}F_{\mu\rho} = J^\nu$. For our purpose, let us consider the special type of dressing (44) and then the gauge invariant fields (5) become [here, we start from the model independent formulas in (14) and (17)]

$$A_i(x) = -\int d^2 z \ c_k(x - z)F_{ki}(z)$$

$$= \int d^2 z \ \chi(x - z)\partial_k F_{ki}(z)$$

$$= -\kappa \int d^2 z \ \chi(x - z) \left( \epsilon^{ij}F_{j0} - \frac{1}{\kappa} \tilde{j}^i \right)(z),$$

(68)
\begin{align*}
\mathcal{A}_0(x) &= -\kappa \int d^2z c_k(x - z) F_{k0}(z) \\
&= \int d^2z \chi(x - z) \partial_k^z F_{k0}(z) \\
&\approx -\kappa \int d^2z \chi(x - z) \left( B + \frac{1}{\kappa} J^0 \right)(z), \quad (69)
\end{align*}

where we have neglected the boundary term \( \int d^2z \partial_k^z \left[ \chi(x - z) F^{k\mu}(z) \right] \). Here, we have introduced the “convection current”, \( \tilde{J}^i \equiv J^i - \partial_0 F_{i0} \) whose divergence is generated only from the CS part: \( \nabla \cdot \tilde{J} = \kappa \dot{B} \); the covariant form \( \tilde{J}^\mu \equiv J^\mu - \partial_0 F_{\mu0} \) is also available from \( J^0 = J^0 \).

Then, by using (68) and (69), one can express the mass term (68) in terms of \( F_{\mu\nu} \) as follows [we neglect the boundary term in the same ways as (68) and (69)]

\begin{align*}
\mathcal{A}_\mu A^\mu(x) &= \kappa^2 \int d^2z \int d^2y \chi(x - z) \chi(x - y) \left[ (B + \frac{1}{\kappa} J^0)^2 - (F_{j0} - \frac{1}{\kappa} \epsilon^{i\ell j} \tilde{J}^i)^2 \right] \\
&= \kappa^2 \int d^2z \int d^2y \chi(x - z) \chi(x - y) \left[ (B^2 - F_{j0} F_{j0}) - \frac{4}{\kappa} \epsilon^{\mu\nu\rho} F_{\nu\rho} \tilde{J}_\mu + \tilde{J}_\mu \tilde{J}_\mu \right] \\
&= \frac{\kappa^2}{2} \int d^2z \int d^2y \chi(x - z) \chi(x - y) F_{\mu\nu}(z) F^{\mu\nu}(y) \\
&\quad + (\tilde{J}_\mu - \text{dependent terms}).
\end{align*}

The final form of the right-hand side looks like the Maxwell term of (67), but it is still different by the non-local expression through \( \chi \)-functions. Moreover, because of the explicit appearance of the function \( \chi \) which is absent in the Lagrangian (67), this term has no counter parts in (67).

In order to resolve these problems, let us consider \( \Box \mathcal{A}_\mu \) instead of \( \mathcal{A}_\mu \):

\begin{align*}
\Box \mathcal{A}_0(x) &= -\kappa \int d^2z \Box^z \chi(x - z) \left( B + \frac{1}{\kappa} J^0 \right)(z) - \kappa \int d^2z \partial_k^z \left( \Box^z \chi(x - z) F_{k0}(z) \right) \\
&= \kappa \left( B + \frac{1}{\kappa} J^0 \right)(x) + \kappa \int d^2z \partial_k^z \left( \delta^2(x - y) F_{k0}(z) \right), \\
\Box \mathcal{A}_i(x) &= -\kappa \int d^2z \Box^z \chi(x - z) \left( \epsilon^{ij} F_{j0} + \frac{1}{\kappa} \partial_0 F_{i0} - \frac{1}{\kappa} J^i \right) - \kappa \int d^2z \partial_k^z \left( \Box^z \chi(x - z) F_{ki}(z) \right) \\
&= \kappa \left( \epsilon^{ij} F_{j0} - \frac{1}{\kappa} \tilde{J}^i \right)(x) + \kappa \int d^2z \partial_k^z \left( \delta^2(x - y) F_{ki}(z) \right). \quad (70)
\end{align*}

Then one obtains

\begin{align*}
- \frac{1}{2\kappa^2} \Box \mathcal{A}_\mu \Box \mathcal{A}^\mu &= \frac{1}{2} (B + \frac{1}{\kappa} J^0)^2 + \frac{1}{2} (F_{j0} - \frac{1}{\kappa} \epsilon^{i\ell j} \tilde{J}^i)^2 \\
&= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\tilde{J}_\mu - \text{dependent terms}), \quad (71)
\end{align*}

where we have neglected the singular boundary terms in (70) and ‘\( \tilde{J}_\mu \)-dependent terms’ is \( \frac{1}{2\kappa} \epsilon^{\mu\nu\rho} \tilde{J}^\mu F_{\nu\rho} - \frac{1}{2\kappa^2} \tilde{J}_\mu \tilde{J}_\mu \). On the other hand, by noting the wave equation for \( F^{\mu\nu} \) [31], \( (\Box + \text{...
\[ F_{\mu\nu} = \kappa \epsilon^{\mu\nu\rho} J_{\rho} + \partial^\mu J^\nu - \partial^\nu J^\mu \]

and so the wave equation for the gauge invariant fields, before using (44),

\[
\square A_\mu(x) = \int d^2 z \, \Box^2 c_\ell(x - z) F_{\kappa\mu}(z) = -\int d^2 z \, c_\ell(x - z) \Box^2 F_{\kappa\mu}(z) = -\kappa^2 A_\mu(x) - \int d^2 z \, c_\ell(x - z) \left( \kappa \epsilon^{\kappa\mu\rho} J_{\rho} + \partial^k J^\mu - \partial^\mu J^k \right)(z)
\]

[we have neglected the boundary terms in the second line] one finds that the left-hand side of (71) becomes the usual mass term beside the \( J^\mu \)-dependent terms

\[
-\frac{1}{2\kappa^2} \Box A_\mu \Box A^\mu = -\frac{1}{2\kappa^2} A_\mu A^\mu + (J^\mu - \text{dependent term}).
\] (72)

Hence, by combining (71) and (72) together, the mass term (65) corresponds to the Maxwell term in the MCS when we neglect both \( J^\mu \) and \( \tilde{J}^\mu \)-dependent terms and the singular boundary terms. On the other hand, we note that the neglected \( J^\mu \) and \( \tilde{J}^\mu \)-dependent terms and the singular boundary terms are order of \( \frac{1}{\kappa} \) with respect to the first terms in (71) and (72). Hence, it is found that the so-called “self-dual” action \([38, 34]\) with respect to the gauge invariant fields \( A_\mu \)

\[
I_{SD}[A] = \int d^3 x \left[ \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \frac{\kappa^2}{2} A_\mu A^\mu \right]
\] corresponds to the MCS theory (or topologically massive gauge theory \([35]\)) without “dynamical matter” parts when we neglect the \( O\left(\frac{1}{\kappa}\right) \) terms:

\[
I_{SD}[A] = I_{MCS}[A] + O\left(\frac{1}{\kappa}\right)
\]

; the result is similar to the previous work \([34, 39]\) in that the SD and MCS theory are equivalent only when we neglect the dynamical matters\([26]\); but the discrepancy of the order of \( O\left(\frac{1}{\kappa}\right) \), which looks like a perturbative correction in the path-integral approach of a model \([40]\), is not understood. Moreover, we have considered the special case of (44) in the proof and it is unclear whether the similar equivalence can be proved in more general cases or not; this special case of (44) which is involved with the Coulomb gauge in GFF, might be connected to the results of the phase space path-integral approach of Ref. \([39]\), where the Coulomb gauge was crucial for the equivalence of Lagrangians, but complete connection is not known\([27]\). Finally, we note that our method provides a new framework for finding the corresponding dual theory compared to the previously known methods \([34, 39, 41, 42]\).

26It is not understood why the two different types of currents are involved in SD and MCS sides each other.

27In the configuration space analysis, in contrast, the Lorentz gauge is crucial for the equivalence \([34, 39]\); in this case, the Lorentz gauge in the self-dual frame is nothing but the Bianchi identity in the MCS frame.
In summary of this paper, we have considered a new GIF consistent with GFF. Our formalism is new in the following three points. (A) We introduced the assumption that there be no translation transformation anomaly for gauge invariant variables $F_\alpha$. From this assumption, we obtained several new conditions for the dressing function $c_k(x, z)$, which are crucial in our development. (B) We introduced the master formula (41), which allowed matching to the gauge fixed system. (C) We found the manner how the equation of the dressing function $c_k(x, z)$ are modified after gauge fixing. Using this formulation, we have obtained a novel GIF, which is consistent with the conventional GFF: The former formulation provides exactly the rotational anomaly of the latter. Hence, in our formulation there is no inconsistency, as in the previous gauge independent formulation of Ref. [8]. As a byproduct, we explicitly found that the anomalous spin of the charged matter has a unique meaning. This is due to the uniqueness of the Poincaré generators when constructed from the symmetric energy-momentum tensor because it is the improved generator which are (manifestly) gauge invariant and obey the quantum as well as classical Poincaré algebra, but this is not the case for the canonical Poincaré generators. Moreover, we have constructed the physical states in the algebraic construction and also in the Schrödinger picture. In the latter method, we found that the “gauge invariant” scalar field $\eta$ of the longitudinal mode of $A_i$ is crucial for constructing the physical wavefunctional which is a genuine effect of (pure) CS theory. The existing confusion about the gauge condition and dressing function have been clarified.

We would like to conclude with two additional comments. First, in our formulation, there is no gauge non-invariance problem of Poincaré generators on the physical states. This is essentially due to absence of additional terms proportional to constraints in the generators of (12), in contrast to the old formulation of Dirac [3]. Second, the master formula (41), which guarantees the classical Poincaré covariance of our CS gauge theory in all gauges, also works in all other gauge theories. Hence as far as the gauge dependent operator ordering problem does not occur, the quantum Poincaré covariance for one gauge guarantees also the covariance for all other gauges. The gauge independent proof of quantum covariance has been an old issue in quantum field theory, and now it is reduced to the solvability of the problem of the gauge dependent operator ordering.

Acknowledgements

One of us (M.-I. Park) would like to thank Prof. Roman Jackiw for reading a draft paper of this work and giving several valuable comments and suggestions, and Profs. Doochul Kim, Choonkyu Lee, Jae Hyung Lee, and Hee Sung Song for warm hospitality of providing a financial support and researching facilities. He also thank Prof. Stanley Deser, Drs. Chanju Kim, Yong-Wan Kim, and Hyun Seok Yang for discussions. This work was supported in part by the Korea
Appendix A. Dirac’s extended Poincaré generators

In this Appendix, we will consider the extension of the generators \((12)\) by including the constraints terms and we will show that the correct transformation law can be obtained for the undressed base fields \(F_\alpha = (A_\mu, \phi, \phi^*)\) as well as the gauge invariant fields \(\mathcal{F}_\alpha = (A_\mu, \hat{\phi}, \hat{\phi}^*)\). This method has been widely used after the formulation by Dirac [8, 9, 28] and has been considered as what having a general validity. But in this appendix we will show that this is valid only when we neglect the singular boundary terms [36].

To this end, let us consider the extended generators by the first-class constraints \(T \approx 0\) and \(T_0 \approx 0\) [here, we included the constraint \(T_0 \approx 0\) in order to give the correct transformation to \(A_0\) also] [3, 4]

\[
\begin{align*}
P^0_{s(E)} &= \int d^2x \left[ |\pi_\phi|^2 + |D^i\phi|^2 + m^2|\phi|^2 + v_0^0 T_0 + v^0 T \right], \\
P^i_{s(E)} &= \int d^2x \left[ \pi_\phi D^i\phi + (D^i\phi)^* \pi_\phi^* + v_0^i T_0 + v^i T \right], \\
M_{12}^{ij}_{s(E)} &= \int d^2x \left[ \epsilon_{ij} x^i \left( \pi_\phi D_j\phi + (D_j\phi)^* \pi_\phi^* \right) + u_0^i T_0 + u^i T \right], \\
M_{0i}^{ij}_{s(E)} &= x^0 P^i - \int d^2x \left[ x^i \left( |\pi_\phi|^2 + |D^i\phi|^2 + m^2|\phi|^2 \right) + u_0^i T_0 + u^i T \right],
\end{align*}
\]

(73)

and consider the transformation property of the undressed base fields \(A_\mu, \phi, \phi^*\); the transformation for the gauge invariant fields are the same as in Sec. II.

Firstly, for the time translational generator \(P^0_{s(E)}\), this produces the following transformation

\[
\begin{align*}
\{A_0(x), P^0_{s(E)}\} &\approx v^0_0(x), \\
\{A_i(x), P^0_{s(E)}\} &\approx \epsilon_{ij} \frac{1}{\kappa} J_j(x) + \partial_i v^0(x), \\
\{\phi(x), P^0_{s(E)}\} &\approx \pi_\phi^*(x) + iv^0 \phi(x), \\
\{\phi^*(x), P^0_{s(E)}\} &\approx \pi_\phi(x) - iv^0 \phi^*(x)
\end{align*}
\]

such that \(P^0_{s(E)}\) can be made to produce the correct transformation \(\{F_\alpha(x), P^0_{s(E)}\} \approx \partial^0 F_\alpha(x)\), if the coefficients are

\[
\begin{align*}
v^0_0 &\approx \partial^0 A_0, \\
\partial_i v^0 &\approx \epsilon_{ij} \frac{1}{\kappa} J_j - \partial_i A_i, \\
v^0 &\approx -A^0.
\end{align*}
\]

(74)
Note that these solutions are consistent with the classical equation of motion \( F^{0k} = \frac{1}{k} \varepsilon_{kj} J^j \) which can be reproduced by combining the second and the third equations in (74) and the fact of \( \pi_{\phi} = (D_0 \phi)^* \).

Secondly, for the space translation generators \( P^i_{s(E)} \), this produces the transformations

\[
\{ A_0(x), P^i_{s(E)} \} \approx v^i_0(x), \\
\{ A_i(x), P^j_{s(E)} \} \approx -\varepsilon_{ij} \frac{1}{k} J^j_0(x) + \partial_i v^j(x), \\
\{ \phi(x), P^j_{s(E)} \} \approx \left( \partial^j + i A^j \right) \phi(x), \\
\{ \phi^*(x), P^j_{s(E)} \} \approx \left( \partial^j - i A^j \right) \phi^*(x)
\]

such that \( P^j_{s(E)} \) can be made to produce the desired transformation \( \{ F_\alpha(x), P^j_{s(E)} \} \approx \partial^j F_\alpha(x) \) if the coefficients are

\[
v^i_0 \approx \partial^i A_0, \\
\partial_i v^j \approx -\varepsilon_{ij} \frac{1}{k} J^j_0 + \partial_j A_i, \\
v^j \approx -A^j.
\]

Note that these solutions are also consistent with the constraint \( B = \frac{1}{k} J^j_0 \).

Thirdly, for the space-rotation generator \( M^1_{s(E)} \), this produces the transformations

\[
\{ A_0(x), M^1_{s(E)} \} \approx u_0(x), \\
\{ A_i(x), M^1_{s(E)} \} \approx x_i \frac{1}{k} J^j_0(x) - \partial_i u(x) + \int d^2 z \partial_i \left( u(z) \delta^2 x - z \right), \\
\{ \phi(x), M^1_{s(E)} \} \approx \left[ \varepsilon_{ij} x^i \left( \partial^j + i A^j \right) + i u \right] \phi(x), \\
\{ \phi^*(x), M^1_{s(E)} \} \approx \left[ \varepsilon_{ij} x^i \left( \partial^j - i A^j \right) - i u \right] \phi^*(x)
\]

and because of the singular boundary term for \( \{ A_i(x), M^1_{s(E)} \} \), it is impossible to get the desired transformation

\[
\{ F_\alpha(x), M^1_{s(E)} \} = (x^1 \partial^2 - x^2 \partial^1) F_\alpha(x) + \Sigma^1_{\alpha\beta} F_\beta(x),
\]

for all base fields \( F_\alpha \) unless we exclude the boundary positions in the field point \( x \). In the case when \( \phi \) (or \( \phi^* \)) has no anomalous transformation, the coefficients are found to be

\[
u_0 \approx \varepsilon_{ij} x^i \partial^j A_0, \quad u \approx -\varepsilon_{ij} x^i A^j
\]

with the transformations

\[
\{ A_0(x), M^1_{s(E)} \} \approx \varepsilon_{jk} x^j \partial^k A_0(x),
\]

37
It seems that there is no other solution which is better than (75). It is not clear how this additional boundary term is related to the anomaly for the gauge invariant fields $F_a$.

Finally, to consider the transformations generated by the Lorentz-boost generators $M^0_{s(E)}$, it is more convenient to re-express $M^0_{s(E)}$ in (73) as

$$M^0_{s(E)} = x^0 P^i_{s(E)} - \int d^2 x \left[ x^i \left( |\pi_\phi|^2 + |D^i \phi|^2 + m^2 |\phi|^2 \right) + u_0^i T_0 + u_3^i T \right],$$

and this produces the transformations

\begin{align*}
\{A_0(x), M^0_{s(E)}\} &\approx x^0 \partial^i A_0(x) - u_0^i(x), \\
\{A_i(x), M^0_{s(E)}\} &\approx (x^0 \partial^j - x^j \partial^0) A_i(x) + \delta_{ij} A_0(x) + \partial_i (x^j A_0 + u^i)(x) \\
&- \int d^2 z \partial_i \left( u^i(z) \delta^2(x - y) \right), \\
\{\phi(x), M^0_{s(E)}\} &\approx (x^0 \partial^j - x^j \partial^0) \phi(x), \\
\{\phi^*(x), M^0_{s(E)}\} &\approx (x^0 \partial^j - x^j \partial^0) \phi^*(x).
\end{align*}

Similar to the rotation transformation, there is a solution for the coefficients

$$u_0^i \approx x^0 \partial^0 A_0 + A_i, \quad u^i \approx -x^i A_0,$$

which produce the desired transformations

$$\{F_a(x), M^0_{s(E)}\} = (x^0 \partial^j - x^j \partial^0) F_a(x) + \Sigma^0_{\alpha\beta} F_\beta(x),$$

for the matter field $F_\alpha$ as follows

\begin{align*}
\{A_0(x), M^0_{s(E)}\} &\approx (x^0 \partial^j - x^j \partial^0) A_0(x) - A_j(x), \\
\{A_i(x), M^0_{s(E)}\} &\approx (x^0 \partial^j - x^j \partial^0) A_i(x) - \delta_{ij} A_0(x) + \int d^2 z \partial^z \left( z^j A_0(z) \delta^2(x - z) \right), \\
\{\phi(x), M^0_{s(E)}\} &\approx (x^0 \partial^j - x^j \partial^0) \phi(x), \\
\{\phi^*(x), M^0_{s(E)}\} &\approx (x^0 \partial^j - x^j \partial^0) \phi^*(x).
\end{align*}

In conclusion, the Dirac’s idea, which introduces the extended Poincaré generator as a correct generator without choosing the gauge condition, can be applied in the CS theory only when we neglect the singular boundary terms. This will be the first example of this phenomena as far as we know.
Appendix B

In this Appendix, we explain the usual case where $W(x)$ itself can be diagonalized and the physical states can be constructed without recourse to $\eta$ field.

Our considering model is the Maxwell-CS theory [the action will be described in (67)] and they produce the Gauss’ law constraint

$$\left(\partial_i E^i + \kappa B - J^0 : \right) | \Psi_{\text{phys}} \rangle = 0 \quad (76)$$

where $E^i = F^{i0}$. The non-vanishing commutation relations are $[A_i(x), E^j(y)] = i\hbar \delta_{ij} \delta^2(x - y)$ and so, one can consider the representation where $A_i$ is diagonalized:

$$E^i(x) | \Psi \rangle \rightarrow \frac{\hbar}{i} \frac{\delta}{\delta A_i(x)} \Psi(A),$$

$$A_i(x) | \Psi \rangle \rightarrow A_i(x) \Psi(A). \quad (77)$$

Then, together with the representation for the matter parts (37), the representation (77) make the Gauss’ law (76) become a differential equation

$$\left[ \frac{\hbar}{i} \frac{\partial}{\delta A_i(x)} + \kappa B(x) - \hbar \phi(x) \frac{\delta}{\delta \phi(x)} + \hbar \phi^\dagger(x) \frac{\delta}{\delta \phi^\dagger(x)} \right] \Psi_{\text{phys}} = 0. \quad (78)$$

On the other hand, since the first part becomes

$$\partial_i \frac{\delta}{\delta A_i(x)} = \int d^2y \left[ \partial_i^x \left( \frac{\delta A_j(y)}{\delta A_i(x)} \right) \frac{\delta}{\delta A_j(y)} + \partial_i^x \left( \frac{\delta W(y)}{\delta A_i(x)} \right) \frac{\delta}{\delta W(y)} \right]$$

from the relation

$$\partial_i^x \left( \frac{\delta A_j(y)}{\delta A_i(x)} \right) = \partial_j \delta^2(x - y) + \partial_j \delta^2(x - y) = 0,$$

$$\partial_i^x \left( \frac{\delta W(y)}{\delta A_i(x)} \right) = -\partial_i^x c_i(y - x) = -\delta^2(y - x)$$

(78) becomes, finally

$$\left[ \frac{\hbar}{i} \frac{\delta}{\delta W(x)} + \kappa B(x) - \hbar \phi(x) \frac{\delta}{\delta \phi(x)} + \hbar \phi^\dagger(x) \frac{\delta}{\delta \phi^\dagger(x)} \right] \Psi_{\text{phys}} = 0. \quad (79)$$

Then, it is easy to see that the solution of (79) is the form of (35)

$$\Psi_{\text{phys}}(B, W, \phi, \phi^\dagger) = e^{-i\kappa \int d^2x B(x)W(x) \Phi(B) \varphi(\hat{\phi}, \hat{\phi}^\dagger)}, \quad (80)$$

39
when $\Phi(B)$ and $\varphi(\hat{\phi}, \hat{\phi}^\dagger)$ are any functionals of $B$ and $\hat{\phi}$, $\hat{\phi}^\dagger$ respectively. Because of the first exponential factor, the physical wavefunctional are not gauge invariant and actually the factor produces the $1 - cocyle$ for the MCS theory. Furthermore, we note that $\eta$ field commutes with $W$ and $\partial_i E^i$, the $\eta$ part of $\Psi_{\text{phys}}$ is not determined in any way and only has a redundant role. Finally, we note that this model has a well defined weak coupling limit $\kappa \to 0$, i.e, the three-dimensional QED limit and the wavefunctional is found to be

$$\Psi_{\text{phys}}(B, W, \phi, \phi^\dagger) = \Phi(B) \varphi(\hat{\phi}, \hat{\phi}^\dagger)$$

from (80); however, the strong coupling limit $\kappa \to \infty$, i.e., pure CS limit is not well-defined.

Appendix C. Proof of master formula

In this Appendix, we present the proof of the master formula (41). There are two class of the constraints system largely: The first-class constraint system and the second-class constraint one. In the former, one choose the gauge conditions to remove the redundancies which makes the quantum theory is well-defined; in the latter one does not introduce additional (gauge) conditions since the quantum theory is well-defined (at least formally) without it. These two systems are closely related but the exact equivalence is generally unclear. So, for the definite-ness, we will consider these two cases separately.

First, let us consider the former case, i.e., the case when there is a first-class constraints systems $T \approx 0$ and corresponding gauge condition $\Gamma \approx 0$: $\{T, T\} \approx T \approx 0, \{T, \Gamma\} \neq 0$. [Here, it does not matter what the bracket algebra for $\Gamma$ itself $\{\Gamma, \Gamma\}$ is.] Then, the Poisson bracket matrices $\Delta_{\alpha\beta} \equiv \{\Theta_\alpha, \Theta_\beta\}$ ($\Theta_1 \equiv T \approx 0, \Theta_2 \equiv \Gamma \approx 0$) becomes

$$\Delta_{\alpha\beta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a = \{T, T\} \approx 0, b = \{T, \Gamma\} \neq 0, c = \{\Gamma, T\} \neq 0$, and $d = \{\Gamma, \Gamma\}$ with non-vanishing determinant: $\det \Delta_{\alpha\beta} = ad - bc \approx -bc$. [Here, we are considering only the discrete indices for convenience. Generalization to the continuous indices will be straightforward.] Then, the inverse of $\Delta_{\alpha\beta}$ becomes

$$\Delta^{-1} = \frac{1}{\det \Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \approx -\frac{1}{bc} \begin{pmatrix} d & -b \\ -c & 0 \end{pmatrix}.$$ 

Here, the fact of $\Delta_{22}^{-1} \approx 0$ is crucial in the proof. Then, the Dirac bracket [3] for the gauge invariant $L_\alpha$, which has a vanishing Poisson bracket with respect to the first-class constraints,
i.e., \( \{L_a, T\} \approx 0 \) is found to be

\[
\{L_a, L_b\}_D \equiv \{L_a, L_b\} - \{L_a, \Theta_a\}\Delta^{-1}_{\alpha\beta}\{\Theta_{\beta}, L_b\} \\
\approx \{L_a, L_b\} - \{L_a, \Theta_2\}\Delta^{-1}_{22}\{\Theta_2, L_b\} \\
\approx \{L_a, L_b\},
\]

where we have used the conditions \( \{L_a, T\} \approx 0 \) and \( \Delta^{-1}_{22} \approx 0 \) in the second and third lines, respectively. This proof can be generalized to the case of several first-class constraints and corresponding gauge conditions; even with the (partial) gauge conditions which are involved only for a part of the first-class constraints, this proof is applied for the partially gauge invariant variables which commute only with those parts of the first-class constraints. This proves the master formula (41) for the former case.

Secondly, let us consider the later case, i.e., the case when there is only one second-class constraint \( \chi \approx 0 \): \( \Delta \equiv \{\chi, \chi\} \neq 0 \). Then, the Dirac bracket for the quantity \( L_a \), which commutes with the second-class constraint \( \chi \), i.e., \( \{L_a, \chi\} \approx 0 \) is found to be

\[
\{L_a, L_b\}_D = \{L_a, L_b\} - \{L_a, \chi\}\Delta^{-1}_{22}\{\chi, L_b\} \\
\approx \{L_a, L_b\},
\]

where we have used the fact of \( \{L_a, \chi\} \approx 0 \). This proof can be generalized to the case of several second-class constraints together with the first-class constraints which do not have the involved gauge conditions when \( L_a \) commutes with the second-class constraints. This latter case is rather unusual case compared to the former case [28] and has been studied only recently in the CS theories with boundary [30].
References

[1] J. Goldstone and R. Jackiw, Phys. Lett. B74, 81 (1978); R. Jackiw, “Non-Yang-Mills Gauge Theories”, hep-th/9705028; J. Pachos, “(1+1)-Dimensional SU(N) Static Sources in E and A representations”, hep-th/9801172 (to appear in Phys. Rev. D); “Interactions in Abelian and Yang-Mills Theories”, hep-th/9811249; K. Zarembo, Phys. Lett. B421, 325 (1998).

[2] P. E. Haagensen and K. Johnson, Nucl. Phys. B439, 597 (1995); hep-th/9702204; P. E. Haagensen and K. Johnson and C. S. Lam, Nucl. Phys. B477, 273 (1996).

[3] D. Karabali and V. P. Nair, Nucl. Phys. B464, 135 (1996); Phys. Lett. B379, 141 (1996); D. Karabali, C. Kim and V. P. Nair, Nucl. Phys. B524, 661 (1998); Phys. Lett. B434, 103 (1998).

[4] M. Bellon, L. Chen and K. Haller, Phys. Lett. B 373, 185 (1996); Phys. Rev. D55, 2347 (1996); L. Chen and K. Haller, “Quark confinement and color transparency in a gauge-invariant formulation of QCD”, hep-th/9803250.

[5] P. A. M. Dirac, Can. J. Phys. 33, 650 (1955); J. Schwinger, Particles and Sources (Gordon and Breach Press, 1969).

[6] M. Lavelle and D. McMullan, Phys. Lett. B312, 211 (1993); 329, 68 (1994); Phys. Rep. 279, 1 (1997); Phys. Lett. B436, 339 (1998).

[7] P. Gaete, Z. Phys. C76, 355 (1997); T. Kashiwa and N. Tanimura, Phys. Rev. D56, 2281 (1997).

[8] R. Banerjee, Phys. Rev. Lett. 69, 17 (1992); Phys. Rev. D48, 2905 (1993); R. Banerjee and A. Chatterjee, Ann. Phys. (N.Y.) 247, 188 (1996).

[9] P. A. M. Dirac, Lecture on Quantum Mechanics (Belfer Graduate School of Science, Yeshiva University Press, New York, 1964); A. Hanson, T. Regge and C. Teitelboim, Constrained Hamiltonian Systems (Accad. Naz. dei Lincei press, Rome, 1976).

[10] C. R. Hagen, Ann. Phys. (N.Y.) 157, 342 (1984); Phys. Rev. D31, 2135 (1985); G. W. Semenoff, Phys. Rev. Lett. 61, 517 (1988); R. Jackiw and S. Y. Pi, Phys. Rev. D42, 3500 (1990), (E) 3500 (1990); C. Kim, C. Lee, P. Ko, B.-H. Lee, and H. Min, Phys. Rev. D48, 1821 (1993); Graziano and K. D. Rothe, ibid., D49, 5512 (1994).

[11] C. R. Hagen, Phys. Rev. D31, 331 (1985); D31, 848 (1985).
[12] H. Shin, W.-T. Kim, J.-K. Kim, and Y.-J. Park, Phys. Rev. \textbf{D46}, 2730, (1992); H. S. Yang and B.-H. Lee, \url{hep-th/9809134} (to appear in Phys. Rev \textbf{D}).

[13] M.-I. Park and Y.-J. Park, Phys. Rev. \textbf{D58} (R.C.), 101702 (1998).

[14] F. J. Belinfante, Physica (Utrechet) \textbf{7}, 449 (1940).

[15] L. Faddeev and R. Jackiw, Phys. Rev. Lett. \textbf{60}, 1692 (1988); R. Jackiw, “(Constrained) Quantization Without Tears”, \url{hep-th/9306075}.

[16] C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. \textbf{59}, 42 (1970).

[17] E. Bagan, M. Lavelle and D. McMullan, Phys. Rev. \textbf{D57}, 4521 (1998).

[18] R. Jackiw, Phys. Rev. Lett. \textbf{41}, 1635 (1978).

[19] R. Jackiw, \textit{Diverse Topics in Theoretical and Mathematical Physics} (World Scientific, Singapore, 1995).

[20] G. V. Dunne, R. Jackiw, and C. A. Trugenberger, Ann. Phys. (N.Y.) \textbf{194}, 197 (1989).

[21] F. Wilczek, Phys. Rev. Lett. \textbf{48}, 1144 (1982); \textbf{49}, 957 (1982).

[22] C. R. Hagen, Phys. Rev. Lett. \textbf{63}, 1025 (1989); \textbf{70}, 3518 (1993).

[23] A related review article can be found in \cite{19}.

[24] D. Bak, R. Jackiw and S. Y. Pi, Phys. Rev. \textbf{D49}, 6778 (1994); M.-I. Park and Y.-J. Park, Phys. Rev. \textbf{D50}, 7584 (1994).

[25] S. K. Kim, W. Namgung, K. S. Soh, and J. H. Yee, Phys. Rev. \textbf{D41}, 3792 (1990).

[26] M.-I. Park, “(2+1)-dimensional QED, anomalous surface-term contributions and superconductivity”, \url{hep-th/9805033}.

[27] S. Deser and J. G. MaCarthy, Nucl. Phys. \textbf{B344}, 763 (1990).

[28] M. Henneaux and C. Teitelboim, \textit{Quantization of Gauge Systems} (Princeton Univ. Press, Princeton, New Jersey, 1992).

[29] I. A. Batalin and I. V. Tyutin, Int. J. Mod. Phys. \textbf{A6}, 3245 (1991); Y.-W. Kim, M.-I. Park, Y.-J. Park, and S.-J. Yoon, \textit{ibid.}, \textbf{A12}, 4217 (1997); M.-I. Park and Y.-J. Park, \textit{ibid.}, \textbf{A13}, 2179 (1998).
[30] P. Oh and M.-I. Park, “Symplectic reduction and symmetry algebra in boundary Chern-Simons theory”, hep-th/9805178; M.-I. Park, Phys. Lett. B440, 275 (1998); “Symmetry algebras in Chern-Simons theories with boundary: Canonical approach”, hep-th/9811033 (to appear in Nucl. Phys. B).

[31] C. Becchi, A. Rouet, and R. Stora, Ann. Phys. (N.Y.) 98, 287 (1976); I. V. Tyutin, Lebedev Institute preprint N39 (1975).

[32] R. Banerjee, A. Chatterjee and V. V. Streedhar, Ann. Phys. (N.Y.) 222, 254 (1993).

[33] R. Jackiw and V. A. Kostelecký, “Radiatively induced Lorentz and CPT violation in electrodynamics”, hep-ph/9901358; S. Coleman and S. Glashow, “High-energy tests of Lorentz invariance”, hep-ph/9812413; J.-M. Chung and P. Oh, “Lorentz and CPT violating Chern-Simons term in the derivative expansion of QED”, hep-th/9812132. See also R. Jackiw, “When radiative corrections are finite but undetermined”, hep-th/9903044 for a review.

[34] S. Deser and R. Jackiw, Phys. Lett. B139, 371 (1984).

[35] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48, 975 (1982); Ann. Phys. (N.Y.) 140, 372 (1982); D. Gonzales and A. N. Redlich, Ann. Phys. (N.Y.), 169, 104 (1986).

[36] M.-I. Park, Ph. D. thesis (Sogang Univ., 1997).

[37] T. Kashima and Y. Takahashi, “Gauge Invariance in Quantum Electrodynamics”, hep-th/9401097.

[38] P. K. Townsend, K. Pilch, P. van Nieuwenhuizen, Phys. Lett. B136, 38 (1984).

[39] R. Banerjee, H. J. Rothe, and K. D. Rothe, Phys. Rev. D52, 3750 (1995); R. Banerjee, H. J. Rothe, Nucl. Phys. B447, 183 (1995).

[40] E. Fradkin and F. A. Schaposnik, Phys. Lett. B338, 253 (1994).

[41] N. Banerjee, R. Banerjee, Mod. Phys. Lett. A11, 1919 (1996); R. Banerjee, H. J. Rothe, and K. D. Rothe, Phys. Rev. D55, 6339 (1997).

[42] E. Witten, “On S-duality in Abelian Gauge theory”, hep-th/9505186.