Relative scale separation in orbifolds of $S^2$ and $S^5$

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ABSTRACT: In orbifold vacua containing an $S^q/\Gamma$ factor, we compute the relative order of scale separation, $r$, defined as the ratio of the eigenvalue of the lowest-lying $\Gamma$-invariant state of the scalar Laplacian on $S^q$, to the eigenvalue of the lowest-lying state. For $q = 2$ and $\Gamma$ finite subgroup of SO(3), or $q = 5$ and $\Gamma$ finite subgroup of SU(3), the maximal relative order of scale separation that can be achieved is $r = 21$ or $r = 12$, respectively. For smooth $S^5$ orbifolds, the maximal relative scale separation is $r = 4.2$. Methods from invariant theory are very efficient in constructing $\Gamma$-invariant spherical harmonics, and can be readily generalized to other orbifolds.

KEYWORDS: Flux Compactifications, Superstring Vacua, AdS-CFT Correspondence, Discrete Symmetries

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1 Introduction and summary

Einstein \((p + q)\)-dimensional gravity minimally coupled to a \(q\)-form, admits Freund-Rubin solutions \([1]\) of the form \(\text{AdS}_p \times M_q\), with \(M_q\) a \(q\)-dimensional Einstein manifold of positive curvature. When embedded in supergravity theories arising from low-energy limits of string theory, these solutions exhibit what is called an absence of scale separation: \(L_{\text{AdS}} \sim L_{\text{int}}\) where \(L_{\text{AdS}}, L_{\text{int}}\) is the radius of curvature of \(\text{AdS}_p, M_q\) respectively, see appendix A for our conventions.

More generally, for any compact internal manifold \(M_q\), the order of scale separation of an \(\text{AdS}_p \times M_q\) solution can be defined as,

\[
r_0 \equiv L_{\text{AdS}}^2 m_{KK}^2 ,
\]

where \(m_{KK}^2\) is the lowest eigenvalue of minus the scalar Laplacian on \(M_q\), which is also identified with the square of the lowest-lying Kaluza-Klein (KK) mass. Thus the absence of scale separation can be stated as \(r_0 = \mathcal{O}(1)\).

Other known classes of (possibly warped) \(\text{AdS}_p \times M_q\) solutions, with more complicated form profiles and internal geometries \(M_q\), can have different values of \(r_0\), although the absence of scale separation persists in large classes of “pure” supergravity solutions \([3–6]\), i.e. in the absence of sources such as orientifolds. In \([9]\), a different definition of scale separation was used, that was argued to be excluded under the same assumptions as in the no-go theorems against de Sitter \([11–13]\). In \([10]\) it was shown that, in the absence of orientifolds, for every AdS solution of type II supergravity, there exists a certain parametric limit for which \(L_{\text{AdS}} \to \infty, r_0 \to c\), with \(c\) a positive constant.

In the presence of orientifolds, the vacua of \([15]\) appear to enjoy parametric scale separation. As was pointed out in \([16]\), the ten-dimensional supergravity description of DGKT involves smeared O6 planes, and belongs to the general class of \(\mathcal{N} = 1\) AdS \(4\) SU(3)-structure compactifications of massive IIA \([17]\). A double T-duality then gives an AdS4 solution on the Iwasawa manifold (or a certain T\(^2\) fibration over K3) \([18]\), which was recently argued to also enjoy scale separation \([19]\). The DGKT vacua have been criticized, not least for their use of smeared orientifolds \([20, 21]\). Recently, localized versions of DGKT have been constructed, to leading order in a certain large-flux expansion in \([10, 22]\).

The question of scale separation has been revived within the context of the swampland

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Note however that for \(p, q\) unconstrained, these Freund Rubin solutions exhibit parametric scale separation: \(L_{\text{AdS}}/L_{\text{int}} \to \infty\), for \(p/q \to \infty\), cf. (A.7). The large-dimension behavior of these solutions was recently discussed in \([2]\) in the context of the swampland conjectures.

In the case of warped products of the form \(\text{AdS}_p \times M_q\), the KK square masses are given as eigenvalues of a certain modified Laplacian operator on \(M_q\) \([7, 8]\).

Although different definitions of the order of scale separation are possible, this does not affect the results of the present note, which focuses on the order of relative scale separation, cf. (1.2). In particular, instead of (1.1), \([9]\) uses the quantity \(L_{\text{AdS}}^2 m_{KK}^4\) as the order of scale separation, measured in units of the lower-dimensional Planck mass. As was shown in \([10]\), the bound derived in \([9]\) does not imply absence of parametric scale separation, defined as the condition: \(r_0 \to c\) as \(L_{\text{AdS}} \to \infty\).

This is the same scaling as in the strong AdS distance conjecture \([14]\), however the argument in \([10]\) does not rely on supersymmetry.
program [14, 19, 23–28]. In particular, the strong AdS distance conjecture would imply that the DGKT vacua are inconsistent in a theory of quantum gravity.

Starting from a Freund-Rubin solution of the form $\text{AdS}_p \times M_q$ without scale separation, $L_{\text{int}} \sim L_{\text{AdS}}$, one may try to reduce the size of $M_q$ by orbifolding by a discrete group $\Gamma$. Naively, one might estimate $L_{\text{int}} \sim V^{1/q}$, where $V$ is the volume of $M_q$, so that orbifolding would reduce the radius of curvature by a factor of $|\Gamma|^{1/q}$, where $|\Gamma|$ is the order of $\Gamma$. Moreover, suppose that $\Gamma$ belongs to an infinite series of finite groups parameterized by $n \in \mathbb{N}$, such that $|\Gamma| \to \infty$ for $n \to \infty$. The orbifolding would then seem to offer a straightforward way of obtaining scale-separated vacua.

This naive expectation fails for two reasons. Firstly, the orbifolding can reduce the size of $M_q$ below the curvature radius of $\text{AdS}_p$, but only in a subset of the directions: it is expected that some directions of $M_q$ remain of the order of $L_{\text{AdS}}$, even after orbifolding, excluding parametric scale separation [29]. This phenomenon can be seen very explicitly in the $S^2/\mathbb{Z}_n$ orbifold discussed below: while orbifolding decreases the size of the longitudinal direction of $S^2$ by a factor of $n$, the size of the latitudinal direction remains unchanged, cf. figure 1a below. The other reason is that the action of $\Gamma$ on $M_q$ may have fixed points, in which case the orbifold $M_q/\Gamma$ has singularities, invalidating its description in the supergravity approximation. In certain cases a singular $M_q/\Gamma$ admits a well-defined description within string theory, however the string-theory description includes new light states localized at the singularities, so that not all orbifold modes can be decoupled in the $|\Gamma| \to \infty$ limit.

The main focus of this note is on orbifolds of the form $S^q/\Gamma$, with $\Gamma \subset \text{SO}(q+1)$. It is useful to think of $S^q$ as embedded in an ambient $\mathbb{R}^{q+1}$ space, so that the action of $\Gamma$ on $S^q$ is the same as its action on the angular coordinates of $\mathbb{R}^{q+1}$ (parameterized in spherical coordinates). If the action of $\Gamma$ only has the origin of $\mathbb{R}^{q+1}$ as a fixed point, the $S^q/\Gamma$ orbifold is smooth, otherwise there are orbifold singularities.\footnote{The singular locus of an orbifold is the set of points $p$ whose stabilizers $\Gamma_p \subseteq \Gamma$ are non-trivial [30]. Of course $\Gamma \subset \text{SO}(q+1)$ stabilizes the origin of $\mathbb{R}^{q+1}$, but the latter does not belong to $S^q$.} Since the orbifolding does not change the local properties of the geometry away from the singularities, the orbifolded geometry remains a local solution of (super)gravity, although the orbifolding may break all or part of the supersymmetry the original solution may possess.

Although singular orbifold vacua are not globally well-defined in supergravity, they can be well-defined in string theory [31, 32]. In certain cases an $\text{AdS}_p \times S^q/\Gamma \times M_d$ supergravity solution can be thought of as the near-horizon limit of string-theory backgrounds with branes embedded in $\mathbb{R}^{1,p-2} \times \mathbb{R}^{q+1}/\Gamma \times M_d$, which can be studied with perturbative string-theory methods [33–36].

Prominent examples of sphere orbifolds can be obtained starting from the $\text{AdS}_5 \times S^5$ vacua of IIB and the $\text{AdS}_2 \times S^2 \times \text{CY}_3$ vacua of IIA (the $\text{CY}_3$ can also be replaced by $T^6$, or $K3 \times T^2$). As is well known, the former can be viewed as the near-horizon geometry of a stack of D3 branes in flat space, sitting at the origin of the transverse $\mathbb{R}^5$ [37], while the latter can be viewed as the near-horizon geometry of a stack of D0 and three stacks of intersecting D4 branes wrapping four-cycles of the $\text{CY}_3$ and sitting at the origin of the
transverse $\mathbb{R}^3$, see [5, 38] for a recent discussion. In either case, representing the transverse space to the branes, $\mathbb{R}^{q+1}$, as a cone over $S^q$, we may orbifold by a subgroup $\Gamma \subset SO(q+1)$ acting on the base of the cone. This results in a near horizon geometry of the form $AdS_5 \times S^5/\Gamma$ [39] and $AdS_2 \times S^2/\Gamma \times CY_3$, for $q = 5, 2$ respectively.

In this case the string theory spectrum consists of the untwisted and twisted sectors. The former is the projection of the spectrum of the unorbifolded theory to the $\Gamma$-invariant states (which includes the $\Gamma$-invariant supergravity states as a subset). When the discrete group is freely-acting, the masses of all twisted-sector states scale as $L_S/\alpha'$ (as they correspond to strings stretching between different points of $S^q$ that are identified under the action of $\Gamma$). In the supergravity regime, $L_S^2 \gg \alpha'$, the twisted-sector states are thus much heavier than the KK mass, which scales as $1/L_S$. However, if the action of $\Gamma$ has fixed points on $S^q$, the twisted sector includes light states localized at the singular locus, which must be taken into account in the low-energy description of the theory, in addition to the $\Gamma$-invariant supergravity fields. Thus, as already mentioned, in the case of singular orbifolds, the orbifolding does not increase the order of scale separation.

In the case of smooth orbifolds, in a regime where the supergravity low-energy description is valid, in order to determine the scale separation after orbifolding, we must examine the $\Gamma$-invariant states of the spectrum of the scalar Laplacian on $S^q$. Let $m_{KK}^2$ and $m_{KK,\Gamma}^2$ be the lowest eigenvalue of (minus) the scalar Laplacian spectrum on $S^q$ and its $\Gamma$-invariant projection respectively. The relative order of scale separation is determined by the ratio,

$$r \equiv \left( \frac{m_{KK,\Gamma}^2}{m_{KK}^2} \right)^2,$$

so that the total scale separation after orbifolding is given by $r_0 r$, cf. (1.1).

To our knowledge, a systematic calculation of $r$, for different discrete groups $\Gamma$, is not available in the literature. Although, as already mentioned, we do not expect the orbifolding to lead to parametric scale separation, $r \rightarrow \infty$, the exact value of $r$ might be of interest in applications involving effective field theories in AdS space, see e.g. [40] for a recent discussion. In the present note we calculate the relative order of scale separation $r$ for the $q = 2, 5$ cases, for all $\Gamma \subset SO(3)$ and $\Gamma \subset SU(3)$ respectively.

Non-supersymmetric $AdS_5 \times S^5/\Gamma$ vacua have been shown to be perturbatively unstable if $\Gamma$ has fixed points on $S^5$ [41]. If $\Gamma$ is freely-acting, these vacua can be perturbatively stable, however they are unstable non-perturbatively [42] against tunneling to a Witten-type bubble of nothing [43]. These results have been generalized to other non-supersymmetric $AdS_5$ backgrounds in [44, 45]. More generally, it has been conjectured that all non-supersymmetric AdS vacua supported by fluxes are unstable [46–48], and recent evidence from the study of concrete examples does not contradict the conjecture [49–51]. Since it is expected that supersymmetry is broken unless $\Gamma \subset SU(3)$, in this note we will focus on this case.

Our results for $r$ as a function of $\Gamma$ are summarized in the two tables 1 and 2. As anticipated, we confirm the absence of parametric scale separation: the maximal value of $r$ that can be achieved is $r = 21$ and $r = 12$, for the case of $S^2/\Gamma$ and $S^5/\Gamma$ respectively. Interestingly, although both $SO(3)$ and $SU(3)$ contain finite subgroups that come in infinite
Table 1. Orbifolds of $S^2/\Gamma$ for finite groups $\Gamma \subset \text{SO}(3)$ of order $|\Gamma|$, listed with their corresponding relative order of scale separation $r$, and the lowest degree $k$ for which there is a $\Gamma$-invariant spherical harmonic on $S^2$. The explicit form of the invariants is given in section 2. None of these orbifolds admits Killing spinors; all of them contain singular points.

| $\Gamma$ | $|\Gamma|$ | $\mathbb{Z}_m \times \mathbb{Z}_n$ | $T_n$ | $\Delta(3n^2)$ | $\Delta(6n^2)$ |
|----------|----------|-------------------------------|--------|----------------|----------------|
| $\mathbb{Z}_2 \times \mathbb{Z}_2$ | 2 | 2 | 3 | $\leq 3$ | $\leq 4$ |
| $\mathbb{Z}_3 \times \mathbb{Z}_3$ | 3 | 3 | 9 | $\leq 4.2$ | $\leq 6.4$ |

Table 2. Orbifolds of $S^5/\Gamma$ for finite groups $\Gamma \subset \text{SU}(3)$ of order $|\Gamma|$, listed with their corresponding relative order of scale separation $r$, and the lowest degree $k$ for which there is a $\Gamma$-invariant spherical harmonic on $S^5$. The explicit form of the invariants is given in section 4. All of these orbifolds admit Killing spinors; only the $T_n, \mathbb{Z}_n$ orbifolds are smooth.

| $\Gamma$ | $|\Gamma|$ | $\Sigma(60) \times \mathbb{Z}_3$ | $\Sigma(168) \times \mathbb{Z}_3$ | $\Sigma(36\varphi)$ | $\Sigma(72\varphi)$ | $\Sigma(216\varphi)$ | $\Sigma(360\varphi)$ |
|----------|----------|------------------------|------------------------|-----------------|-----------------|-----------------|-----------------|
| $\mathbb{Z}_6 \times \mathbb{Z}_3$ | 180 | 168 | 504 | 108 | 216 | 648 | 1080 |
| $\mathbb{Z}_2 \times \mathbb{Z}_2$ | 2 | 6 | 4 | 6 | 4 | 6 | 6 |
| $\mathbb{Z}_3 \times \mathbb{Z}_3$ | 3 | 12 | 12 | 12 | 12 | 12 | 12 |

series, the maximal relative scale separation is achieved when $\Gamma$ belongs to one of the exceptional subgroups. All of the $S^5/\Gamma$ orbifolds admit Killing spinors, and are thus expected to lead to a supersymmetric theory on $\text{AdS}_5 \times S^5/\Gamma$. In contrast, none of the $S^2/\Gamma$ admit Killing spinors, thus leading to a non-supersymmetric theory on $\text{AdS}_2 \times S^2/\Gamma \times \text{CY}_3$. All of the $S^2$ orbifolds are singular, while the only smooth $S^5/\Gamma$ orbifolds are obtained for $\Gamma = \mathbb{Z}_n$ or $T_n$. In this case the relative scale separation is $r = 2.4$ and $r = 4.2$ respectively.

The outline of the rest of the paper is as follows. In section 2 we present the $S^2/\Gamma$ orbifolds and calculate the relative order of scale separation for each $\Gamma \subset \text{SO}(3)$. We do so first by using brute force, by determining the lowest degree $k$ for which there is a $\Gamma$-invariant spherical harmonic on $S^2$, cf. appendix B. The relative order of scale separation can then be read off of (B.9). This procedure becomes cumbersome rather quickly, so a more efficient method is needed in order to tackle the orbifolds of $S^5$. The relevant tools are provided by invariant theory, and are presented in section 3. Equipped with this technology, we return to the $S^2$ quotient by the icosahedral group in section 3.1, and calculate $r$ in an alternative way, using methods from invariant theory. In section 4 we present the $S^5/\Gamma$ orbifolds, and calculate the relative order of scale separation for each $\Gamma \subset \text{SU}(3)$. In section 5 we discuss the smoothness of the orbifolds, and the existence of Killing spinors. Several technical details can be found in the appendices.
Note added. After this work was completed, I became aware of the recent paper [52] which also contains the results for the smooth $S^5/\Gamma$ orbifolds presented here (cases $\Gamma = Z_n, T_n$ of table 2).

2 Orbifolds of $S^2$

The orbifolds of $S^2$ are obtained by orbifolding by a finite subgroup $\Gamma \subset \text{SO}(3)$, viewed as acting on the coordinates of the ambient $\mathbb{R}^3$. In the case of orientation-preserving (also known as “chiral” or “proper”) orbifolds, $\Gamma$ can be obtained from the finite subgroups of $\text{SU}(2)$, cf. appendix C, via the 2:1 map $\text{SU}(2) \to \text{SO}(3)$. They consist of two infinite series and three exceptional cases, corresponding to the symmetry groups of the Platonic solids:

- The cyclic groups $Z_n, n \geq 2$, of order $n$, generated by $R_z(\frac{2\pi}{n})$, where $R_{\vec{u}}(\theta)$ denotes a rotation of angle $\theta$ around the axis $\vec{u}$.
- The dihedral groups $D_n, n \geq 2$, of order $2n$, generated by $R_z(\frac{2\pi}{n})$ and $R_x(\pi)$.
- The tetrahedral group $T$ of order 12 is isomorphic to $A_4$, the set of even permutations of four objects. It is obtained by combining $D_2$, which is generated by $R_z(\pi)$ and $R_x(\pi)$, with the element,

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
$$

which generates cyclic permutations of the coordinates of $\mathbb{R}^3$.
- The octahedral group $O$ of order 24 is isomorphic to $S_4$, the set of permutations of four objects. It is obtained by combining $T$ with $R_z(\frac{\pi}{2})$.
- The icosahedral group $I$ of order 60 is isomorphic to $A_5$, the set of even permutations of five objects. It is generated by $R_z(\frac{2\pi}{5})$ and,

$$
R = \begin{pmatrix}
\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\
0 & -1 & 0 \\
\frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}}
\end{pmatrix}.
$$

In the following we will calculate the first non-zero scalar Laplacian eigenvalue for each of these orbifolds. In the present section, we will do so by using brute force, i.e. by determining the lowest degree $k$ for which there is a $\Gamma$-invariant spherical harmonic on $S^2$, cf. appendix B. The relative order of scale separation can then be read off of (B.9). A more sophisticated method, using invariant theory, will be presented in the following sections.

2.1 $S^2/Z_n$

In the case of the cyclic groups $\Gamma = Z_n$, the fundamental domain of the orbifold is the slice of the sphere bounded by the two meridians at $\phi = 0$ and $\phi = \frac{2\pi}{n}$, with two conical singularities of degree $n$ at the poles. We see that while orbifolding decreases the size of
the longitudinal direction by a factor of $n$, the size of the latitudinal direction remains unchanged, as depicted in figure 1a. This is also reflected in the fact that the lowest non-trivial scalar Laplacian eigenvalue occurs at degree $k = 1$, just as in the unorbifolded case, cf. appendix B. To see this, it suffices to note that the polynomial $p_{k=1}(\vec{x}) = z$, cf. appendix B.1, is invariant under the orbifolding. We thus obtain $r = 1$, i.e. no relative scale separation in this case.

2.2 $S^2/D_n$

The $S^2/D_n$ orbifold can be thought of as modding out $S^2/Z_n$ by a parity transformation in the $y, z$ coordinates, i.e. $\theta \rightarrow \pi - \theta, \phi \rightarrow 2\pi - \phi$ in spherical coordinates. This is half the fundamental domain of $S^2/Z_n$, since the southern half of the slice bounded by the $\phi = 0, \frac{2\pi}{n}$ meridians is identified with the northern half. We now have three conical singularities: two of degree two at $\phi = 0, \frac{2\pi}{n}$ and $\theta = \frac{\pi}{2}$, and a third singularity of degree $n$ at the north pole.

The orbifolding thus decreases the size of the longitudinal direction by a factor of $n$, and the latitudinal direction by a factor of 2, as depicted in figure 1b. This is also reflected in the fact that the lowest non-trivial scalar Laplacian eigenvalue occurs at $k = 2$, instead of $k = 1$. To see this, note that there are no harmonic polynomials at level $k = 1$ which are invariant under the orbifold generators, cf. appendix B.1. At level $k = 2$, on the other hand, $p_2(\vec{x}) = x^2 + y^2 - 2z^2$ is harmonic and invariant under both orbifold generators. We thus obtain an order of scale separation $r = 3$, cf. (B.9).

2.3 $S^2/T$

The $S^2/T$ orbifold is generated by parity transformations in any two coordinates, and cyclic permutations of the coordinates. It can be seen that at levels $k = 1, 2$ no harmonic polynomial is invariant under the orbifold generators, while at level $k = 3$ the polynomial $p_3(\vec{x}) = xyz$ is harmonic and invariant. This results in an order of scale separation $r = 6$. 

Figure 1. The fundamental domains of the $S^2/Z_n$ and $S^2/D_n$ orbifolds, depicted as the shaded regions of the sphere in (1a) and (1b) respectively.
2.4 \( S^2/O \)

It can be seen that the octahedral group does not leave any harmonic polynomials invariant at levels \( k = 1, \ldots, 3 \). At level \( k = 4 \) there is a unique harmonic invariant,

\[
p_4(\vec{x}) = x^4 + y^4 + z^4 - 3x^2y^2 - 3x^2z^2 - 3y^2z^2.
\]  
(2.3)

We thus have an order of scale separation \( r = 10 \).

2.5 \( S^2/I \)

In the icosahedral case, it can be seen that at levels \( k = 1, \ldots, 5 \), no harmonic polynomial is invariant under the orbifold generators, cf. (B.10). At level \( k = 6 \) there is a unique harmonic polynomial \([53]\) given by (up to overall normalization),

\[
p_6(\vec{x}) = (x^2 + y^2)^3 + \frac{42}{5}(x^4 - 10x^2y^2 + 5y^4)xz - 18(x^2 + y^2)^2z^2 + 24(x^2 + y^2)z^4 - \frac{16}{5}z^6.
\]  
(2.4)

This results in an order of scale separation \( r = 21 \).

3 Some invariant theory

In the previous sections we have constructed invariant harmonics using brute force. We will now introduce some more sophisticated machinery which simplifies the task at hand, issued from invariant theory, see e.g. [54]. In section 3.1 we will apply this technology to reproduce the result for \( S^2/I \) from section 2.5.

Starting from any polynomial function \( f(\vec{x}) \), an invariant \( I_f \) (which may be identically zero) can be obtained through the so-called Reynolds operator,

\[
I_f(\vec{x}) = \frac{1}{|R(G)|} \sum_{g \in R(G)} f(g \circ \vec{x}),
\]  
(3.1)

where \( |R(G)| \) denotes the number of elements in \( R(G) \).\(^6\) For a finite matrix group \( G \) and a representation \( R(G) \) thereof, the number of invariant homogeneous polynomials (not necessarily harmonic) in that representation is generated by the Molien function \([55, 56]\), given by,

\[
M_{R(G)}(\lambda) = \frac{1}{|R(G)|} \sum_{g \in R(G)} \frac{1}{\det(1 - \lambda g)}. 
\]  
(3.2)

More specifically, the number \( g_m \) of linearly-independent invariant polynomials of degree \( m \) is the coefficient of \( \lambda^m \) in the expansion of the Molien function,

\[
M_{R(G)}(\lambda) = \sum_{m=0}^{\infty} g_m \lambda^m.
\]  
(3.3)

The subring of polynomials invariant under the action of \( R(G) \) is generated by a minimal set of so-called fundamental invariants. This means that all \( R(G) \)-invariants can be expressed as polynomials of the fundamental invariants. The latter are further divided into

\(^6\)The distinction between \( |R(G)| \) and the order of \( G \) is relevant only if \( R(G) \) is not faithful.
a set of primary invariants $\{I_{m_1}, \ldots, I_{m_r}\}$ of degrees $m_1, \ldots, m_r$, which are algebraically-independent, and a set of secondary invariants $\{1, \bar{I}_{n_1}, \ldots, \bar{I}_{n_s}\}$ of degrees $0, n_1, \ldots, n_s$, which are not. The number $r$ of primary invariants is equal to the dimension of the representation $R(G)$ [57], while the number of secondary invariants is equal to [54],

$$1 + s = \frac{m_1 \cdots m_r}{|G|} .$$

The algebraic relations between primary and secondary invariants are called syzygies, and are of the form,

$$\bar{I}_{n_i}^2 = f(I_{m_1}, \ldots, I_{m_r}) + \sum_j \bar{I}_{n_j} g_j(I_{m_1}, \ldots, I_{m_r}) ,$$

for some polynomial functions $f$, $g_j$ of the primary invariants. The Molien function can be put in the form,

$$M_{R(G)}(\lambda) = \frac{1 + \sum_n a_n \lambda^n}{(1 - \lambda^{m_1}) \cdots (1 - \lambda^{m_r})} ,$$

where $a_n$ is the number of secondary invariants of degree $n > 0$. In particular, $\sum_n a_n = s$, cf. (3.5).

3.1 $S^2/\mathcal{I}$ again

Let us return to the case of the icosahedral group and reproduce the results of section 2.5 using the machinery of section 3. The group $\mathcal{I}$ acts on the coordinates $(x, y, z)$ of $\mathbb{R}^3$ in the three-dimensional representation. Computing the Molien function (3.2) we obtain,

$$M_3(\lambda) = \frac{1 + \lambda^{15}}{(1 - \lambda^2)(1 - \lambda^6)(1 - \lambda^{10})} = 1 + \lambda^2 + \lambda^4 + 2\lambda^6 + \ldots .$$

This suggests the existence of three primary invariants of degrees two, six and ten and one secondary invariant of degree 15, as can be verified by constructing the syzygies [60]. In addition there is one invariant of degree four and one of degree six which are not independent.

Using the explicit matrix representation of $\mathcal{I}$ given in (D.2), the degree-2 invariant can be constructed by acting with the Reynolds operator (3.1) on the monomial $f = x^2$,

$$I_{x^2} = \frac{1}{3} (x^2 + y^2 + z^2) .$$

---

7The polynomials $p_1, \ldots, p_N$ are called algebraically-dependent if there exists a (not identically vanishing) polynomial in $N$ variables $h(x_1, \ldots, x_N)$, such that $h(p_1, \ldots, p_N) = 0$.

8The algebraic independence of the primary invariants is straightforward to check using the Jacobian criterion [58]. The latter states that the $N$ polynomials $p_1, \ldots, p_N$ in $N$ variables $x_1, \ldots, x_N$ are algebraically independent if and only if,

$$\det \left( \frac{\partial p_i}{\partial x_j} \right) \neq 0 .$$

9We have used a simple Mathematica code to generate all group elements, and compute the Reynolds operator and the Molien function in a series expansion. For some powerful publicly available software see GAP [59] and SUTree [60].
This invariant simply reflects the fact that \( I \) is a subgroup of \( \text{SO}(3) \). Similarly, starting from \( f = x^4 \) we obtain the invariant of degree four, 
\[
I_{x^4} = \frac{1}{5}(x^2 + y^2 + z^2)^2, \quad (3.10)
\]
which, as expected from the discussion below (3.8), is not algebraically independent as it is proportional to \( I_{x^2}^2 \). At degree six we can construct two invariants acting with the Reynolds operator on \( f = x^6 \) and \( f = z^6 \), respectively,
\[
240I_{x^6} = 35(x^2 + y^2)^3 + 6x(x^4 - 10x^2y^2 + 5y^4)z + 90(x^2 + y^2)^2z^2 + 120(x^2 + y^2)z^4 + 32z^6, \quad 75I_{z^6} = 10(x^2 + y^2)^3 - 6x(x^4 - 10x^2y^2 + 5y^4)z + 45(x^2 + y^2)^2z^2 + 15(x^2 + y^2)z^4 + 13z^6. \quad (3.11)
\]
As expected these are not algebraically independent, but satisfy,
\[
16I_{x^6} + 5I_{z^6} = -45I_{x^2}I_{x^4} = 0. \quad (3.12)
\]
It is now easy to verify that neither \( I_{x^2} \), nor \( I_{x^4} \) are harmonic, in agreement with the result of section 2.5. Moreover, the only harmonic linear combination (up to overall normalization) of degree six is given by \( I_{x^6} - I_{z^6} \), which is proportional to the polynomial already determined in (2.4).

### 4 Orbifolds of \( S^5 \)

The orbifolds of \( S^5 \) are obtained by orbifolding by a finite subgroup \( \Gamma \subset \text{SO}(6) \), viewed as acting on the coordinates of the ambient \( \mathbb{R}^6 \). Since it is expected that only quotients for which \( \Gamma \subset \text{SU}(3) \) lead to a supersymmetric theory [34], we will focus on this case. The group \( \Gamma \) acts on the coordinates \((x, y, z)\) of the ambient \( \mathbb{C}^3 \simeq \mathbb{R}^6 \) in the fundamental, three-dimensional representation \( 3 \). We are now interested in invariants which are (complex) polynomials in the six variables \((x, y, z, x^*, y^*, z^*)\),\(^{10}\) on which \( \Gamma \) acts block-diagonally in the \( 3 \otimes \bar{3} \) representation. The relevant Molien function is,
\[
M_{3 \otimes \bar{3}}(\lambda) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \frac{1}{\det(1 - \lambda g) \det(1 - \lambda g^*)}, \quad (4.1)
\]
where \(|\Gamma|\) is the order of \( \Gamma \). On the other hand, the Molien function,
\[
M_3(\lambda) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \frac{1}{\det(1 - \lambda g)}, \quad (4.2)
\]
enumerates the holomorphic invariants of \( \Gamma \) (and of course also the antiholomorphic ones, by complex conjugation). We will therefore refer to it as the “holomorphic Molien function”. Note that the holomorphic and antiholomorphic invariants are automatically harmonic with respect to the Laplacian on \( \mathbb{R}^6 \),
\[
\Delta = \frac{\partial^2}{\partial x \partial x^*} + \frac{\partial^2}{\partial y \partial y^*} + \frac{\partial^2}{\partial z \partial z^*}. \quad (4.3)
\]
\(^{10}\)Real polynomials can be constructed by taking real and imaginary parts thereof.
A classification of finite subgroups of SU(3) appeared over a century ago in [61]. A more detailed analysis, involving character tables and generators, motivated by particle physics applications appeared in [62]. Further additions to the list of [62] appeared subsequently in [63–65]. A comprehensive review of the subject can be found in [66, 67].

Discrete subgroups of SU(3) have been considered in particle physics model-building, going back to the work of [62]. More recently they have been used in flavor physics applications, as a means to constrain Yukawa couplings, mass matrices and mixing angles in the quark and lepton sectors (for reviews and further references see e.g. [68, 69]).

The classification of finite subgroups of SU(3) consists of:

- Abelian groups of diagonal matrices. These are isomorphic to \( \mathbb{Z}_m \times \mathbb{Z}_n \), where \( n \) is a divisor of \( m \).

- Finite subgroups of U(2) of the form,

\[
\begin{pmatrix}
\text{det} \, A & 0 \\
0 & A
\end{pmatrix} ; \quad A \in U(2).
\] (4.4)

- The \( C(n,a,b) \) series generated by,

\[
E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} ; \quad F(n,a,b) = \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^{-a-b} \end{pmatrix},
\] (4.5)

where \( n \in \mathbb{N} - \{0\}; \eta \equiv e^{2\pi i/n}; a, b \in \mathbb{N} \) with \( a, b \leq n - 1 \).

- The \( D(n,a,b;d,r,s) \) series generated by \( E, F(n,a,b) \) given above and,

\[
G(d, r, s) = \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & \delta^s & 0 \\ 0 & -\delta^{r-s} & 0 \end{pmatrix},
\] (4.6)

where \( d \in \mathbb{N} - \{0\}; \delta \equiv e^{2\pi i/d}; r, s \in \mathbb{N} \) with \( r, s \leq d - 1 \).

- The “exceptional”, or “crystallographic” groups,\(^{12}\)

\[
\Sigma(60), \Sigma(60) \times \mathbb{Z}_3, \Sigma(168), \Sigma(168) \times \mathbb{Z}_3, \Sigma(36\varphi), \Sigma(72\varphi), \Sigma(216\varphi), \Sigma(360\varphi),
\] (4.7)

whose explicit generators can be found in e.g. [66].

The relation of the well-known \( \Delta \) and \( T_n \) series of SU(3) subgroups to the above classification is as follows. The \( T_n \) subgroups are special cases of the \( C \)-series. On the other hand, it was shown in [66] that all SU(3) subgroups in the \( C \)-series can be interpreted as three-dimensional irreducible representations of \( \Delta(3n^2) \), which were determined in [71].

\(^{11}\)We follow the presentation of [70] which includes some minor corrections to the list of [61, 62].

\(^{12}\)We consider \( \varphi = 3 \), in the notation of [62]. The case \( \varphi = 1 \) corresponds to subgroups of SU(3)/\( \mathbb{Z}_3 \).
Moreover, \cite{72} showed that every SU(3) subgroup in the D-series is a subgroup of some \Delta(6n^2).\footnote{The subgroups in the D-series cannot all be interpreted as irreducible three-dimensional representations of \Delta(6n^2) \cite{67}.} More explicitly, we have the following subgroup structure (see \cite{60} for the complete finite SU(3)-subgroup tree),

\[
C(n, a, b) \subseteq \Delta(3n^2) ; \quad D(m, a, b; d, r, s) \subseteq \Delta(6n^2) ,
\]  

(4.8)

where in the second relation above \(n\) is equal to the lowest common multiple of \(2, m, d\).

In addition,\footnote{Note, however, that \(T_n\) is not defined for all \(n \in \mathbb{N}\), cf. section 4.11.} \(T_n \subset \Delta(3n^2) \subset \Delta(6n^2)\),

\[
T_n \subset \Delta(3n^2) \subset \Delta(6n^2) ,
\]  

(4.9)

where both subgroup relations are maximal. Moreover, within the exceptional series, the following subgroup structure holds,

\[
\Sigma(36 \varphi) \subset \Sigma(72 \varphi) \subset \Sigma(216 \varphi) ; \quad \Sigma(60) \subset \Sigma(60) \times \mathbb{Z}_3 \subset \Sigma(360 \varphi) .
\]  

(4.10)

Consider the case \(H \subset G \subset SU(3)\). Clearly all \(G\)-invariants are also \(H\)-invariants. It follows that the \(G\)-invariant KK spectrum is a subset of the \(H\)-invariant KK spectrum. In particular, \(m_{KK/G} \geq m_{KK/H}\), hence modding out the internal space with \(G\) leads, in general, to higher scale separation than modding out with \(H\), cf. (1.2). Therefore the highest scale separation is achieved for maximal finite subgroups of SU(3).\footnote{A subgroup \(G\) is called maximal if there is no other proper subgroup \(G'\) such that \(G \subset G' \subset SU(3)\).}  

\section{4.1 \(S^5/\mathbb{Z}_m \times \mathbb{Z}_n\)}

These are abelian groups of diagonal matrices. Their elements are of the general form,\footnote{Let \(m\) be the maximal order of an element of the abelian group. Defining \(\mu = e^{2\pi i/m}\), it can be seen that every element of the group can be put in the form of eq. (4.11) with \(a = \mu^c\), \(b = \mu^d\), and \(c, d \in \{0, \ldots, m-1\}\). It follows (cf. Theorem 2.1 and section A of \cite{67}) that the resulting group must be isomorphic to \(\mathbb{Z}_m \times \mathbb{Z}_n\) with \(n\) a divisor of \(m\).}

\[
\begin{pmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & a^*b^*
\end{pmatrix} , \quad a, b \in U(1) ,
\]  

(4.11)

so they manifestly leave invariant the quadratic harmonic polynomials, \(|x|^2 - |z|^2\), \(|y|^2 - |z|^2\), which corresponds to an order of scale separation \(r = 2.4\).

\section{4.2 \(S^5/\Gamma_{U(2)}\)}

The quadratic harmonic polynomial, \(|x|^2 + |y|^2 - 2|z|^2\), is invariant under the action of elements of the general form (4.4), thus resulting in an order of scale separation \(r = 2.4\).
4.3 $S^5/\Sigma(60)$

The group $\Sigma(60)$ turns out to be isomorphic to the icosahedral group: $\Sigma(60) \simeq I \subset SO(3)$, already considered in section 2. Computing the Molien function (4.1) we obtain,

$$M_{3\Sigma3}(\lambda) = 1 + 3\lambda^2 + 6\lambda^4 + 17\lambda^6 + \ldots, \quad (4.12)$$

where we have noted that the fundamental representation of $\Sigma(60)$ is real. Three linearly-independent quadratic invariants are,

$$I_{2,1} = x^2 + y^2 + z^2; \quad I_{2,2} = I_{2,1}^*; \quad I_{2,3} = |x|^2 + |y|^2 + |z|^2, \quad (4.13)$$

constructed by acting with the Reynolds operator on $x^2, x^*^2, |x|^2$ respectively. The first two are $SO(3)$ invariants, while the third one is an $SU(3)$ invariant. More generally, for all $n \in \mathbb{N}$, there is exactly one invariant of $SU(3)$ at order $2n$, proportional to $I_{n,3}^2$. This invariant is never harmonic, except for $n = 0$. On the other hand, $I_{2,1}, I_{2,2}$ are harmonic (since they are holomorphic and antiholomorphic respectively). We thus obtain an order of scale separation $r = 2.4$.

Six linearly-independent quartic invariants are,

$$I_{4,1}^2; \quad I_{4,2}^2; \quad I_{4,3}^2; \quad I_{2,1}I_{2,2}; \quad I_{2,1}I_{2,3}; \quad I_{2,2}I_{2,3}. \quad (4.14)$$

At degree six, we have 17 linearly-independent sextic polynomials, including the $I_{x^6}, I_{z^6}$ of (3.11) and their complex conjugates. The latter four invariants are all harmonic.

4.4 $S^5/\Sigma(60) \times \mathbb{Z}_3$

The group $\Sigma(60) \times \mathbb{Z}_3$ is generated by $A \equiv e^{2\pi i/3}\mathbb{1}$, together with the generators of $\Sigma(60)$. Therefore the invariants of $\Sigma(60) \times \mathbb{Z}_3$ are the subset of the invariants of $\Sigma(60)$ which are also invariant under $A$. Since $A$ multiplies $(x, y, z)$ by $e^{2\pi i/3}$, clearly $I_{2,1}, I_{2,2}$ of (4.13) are not invariant under $A$. On the other hand $I_{2,3}$ is invariant under $A$, however it is not harmonic.

The only quartic invariants of $\Sigma(60)$ which are also invariant under $A$, are $I_{2,1}I_{2,2}, I_{2,3}^2$, cf. (4.14), neither of which is harmonic. However their linear combination $I_{2,3}^2 - 2I_{2,1}I_{2,2}$ is indeed harmonic. We thus obtain an order of scale separation $r = 6.4$.

At degree six, the harmonic invariants $I_{x^6}, I_{z^6}$ and their complex conjugates, already discussed in section 4.3, are also invariant under $A$.

4.5 $S^5/\Sigma(168)$

The group $\Sigma(168)$ is generated by,

$$S' = \frac{i}{\sqrt{7}} \begin{pmatrix} \eta^2 - \eta^5 & \eta - \eta^6 & \eta^4 - \eta^3 \\ \eta - \eta^6 & \eta^4 - \eta^3 & \eta^2 - \eta^5 \\ \eta^4 - \eta^3 & \eta^2 - \eta^5 & \eta - \eta^6 \end{pmatrix}; \quad T' = \frac{i}{\sqrt{7}} \begin{pmatrix} \eta^3 - \eta^6 & \eta^3 - \eta & \eta - 1 \\ \eta^2 - 1 & \eta^6 - \eta^5 & \eta^6 - \eta^2 \\ \eta^5 - \eta^4 & \eta^4 - 1 & \eta^5 - \eta^3 \end{pmatrix}, \quad (4.15)$$

where $\eta \equiv e^{2\pi i/7}$. Computing the Molien function (4.1) we obtain,

$$M_{3\Sigma3}(\lambda) = 1 + \lambda^2 + 3\lambda^4 + 8\lambda^6 + \ldots . \quad (4.16)$$
The quadratic invariant is $I_{2,3}$ of (4.13). Three linearly-independent quartic invariants are $I_{2,3}^2$, together with,

$$I_{4,1} = xy^3 + x^3z + y^3z,$$  \hfill (4.17)

and its complex conjugate. Since $I_{4,1}$ is holomorphic, it is also harmonic. We thus obtain an order of scale separation $r = 6.4$.

4.6 $S^5/\Sigma(168) \times \mathbb{Z}_3$

The group $\Sigma(168) \times \mathbb{Z}_3$ is generated by $A = e^{2\pi i/3}$, together with the generators of $\Sigma(168)$. Therefore the invariants of $\Sigma(168) \times \mathbb{Z}_3$ are the subset of the invariants of $\Sigma(168)$ which are also invariant under $A$. At quadratic and quartic level, only $I_{2,3}$ and $I_{2,3}^2$ respectively are invariant under $A$, cf. section 4.5. However neither of these is harmonic. Hence $S^5/\Sigma(168) \times \mathbb{Z}_3$ does not have any harmonic invariants of degree less than five. Moreover, $\Sigma(168) \times \mathbb{Z}_3$ does not have any invariants of degree five either, as follows from the Molien function (4.16). At degree six, the following polynomial, obtained by acting with the Reynolds operator on $(xyz)^2$, is invariant under the group generators,

$$I_{6,1} = x^5y + y^5z + z^5x - 5x^2y^2z^2.$$  \hfill (4.18)

Since $I_{6,1}$ is holomorphic, it is also harmonic. Hence the order of scale separation is $r = 12$.

4.7 $S^5/\Sigma(36\varphi)$

The group $\Sigma(36\varphi)$ is be generated by,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad C = \frac{1}{\omega - \omega^2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$  \hfill (4.19)

where $\omega = e^{2\pi i/3}$. Computing the Molien function (4.1) we obtain,

$$M_{36\varphi}(\lambda) = 1 + \lambda^2 + 2\lambda^4 + 2\lambda^5 + 8\lambda^6 + \ldots.$$  \hfill (4.20)

At quadratic and quartic level, we have the $I_{2,3}$ and $I_{2,3}^2$ invariants respectively. There is one additional quartic invariant,

$$I_{4,2} = (x^2 - yz)(y^2 - xz) x^6 z^6 + (z^2 - xy)(x^2 y^2 + c.c. \quad (4.21)

The combination $2I_{4,2} + I_{2,3}^2$ is harmonic, so we obtain an order of scale separation $r = 6.4$.

4.8 $S^5/\Sigma(72\varphi)$

The group $\Sigma(72\varphi)$ is obtained by combining $\Sigma(36\varphi)$ with the generator,

$$D = \frac{1}{\omega - \omega^2} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}.$$  \hfill (4.22)
Computing the Molien function (4.1) we obtain,

\[ M_{\mathbb{S}_3}(\lambda) = 1 + \lambda^2 + \lambda^4 + 2\lambda^5 + 4\lambda^6 + \ldots. \]  

(4.23)

At quadratic and quartic level, we have the \( I_{2,3} \) and \( I_{2,3}^2 \) invariants respectively, which are not harmonic. Two quintic linearly-independent invariants can be constructed,

\[ I_{5,1} = |x|^2(y^3 - z^3) + |y|^2(z^3 - x^3) + |z|^2(x^3 - y^3); \quad I_{5,2} = I_{5,1}^*, \]  

(4.24)

by acting with the Reynolds operator on \( |x|^2 y^3 \), \( |x|^2 y^* z^3 \), respectively. Both of these invariants are harmonic, so we obtain an order of scale separation \( r = 9 \).

4.9 \( S^5/\Sigma(216\varphi) \)

The group \( \Sigma(216\varphi) \) can be generated by the following two generators [66],

\[ A = \frac{1}{\omega - \omega^2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}; \quad B = \epsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \]  

(4.25)

where \( \epsilon \equiv e^{4\pi i/9}, \omega \equiv e^{2\pi i/3} \). Computing the Molien function (4.1) we obtain,

\[ M_{\mathbb{S}_3}(\lambda) = 1 + \lambda^2 + \lambda^4 + 2\lambda^5 + 3\lambda^8 + 4\lambda^{10} + \ldots. \]  

(4.26)

At quadratic, quartic and sextic order, we have the \( I_{2,3}, I_{2,3}^2 \) and \( I_{2,3}^3 \) invariants respectively, which are not harmonic. At order six, there is one additional invariant,

\[ I_{6,2} = 18|xyz|^2 + (x^3 + y^3 + z^3)(x^{*3} + y^{*3} + z^{*3}), \]  

(4.27)

which can be constructed by acting with the Reynolds operator on \( |xyz|^2 \). This invariant is not harmonic either, however it can be seen that the linear combination \( 5I_{6,2} - 3I_{2,3}^3 \) is indeed harmonic. We thus obtain an order of scale separation \( r = 12 \).

4.10 \( S^5/\Sigma(360\varphi) \)

The generators of \( \Sigma(360\varphi) \) can be taken to be,

\[ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad C = \frac{1}{2} \begin{pmatrix} -1 & \mu_2 & \mu_1 \\ \mu_2 & \mu_1 & -1 \\ \mu_1 & -1 & \mu_2 \end{pmatrix}; \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & -\omega^2 & 0 \end{pmatrix}, \]  

(4.28)

where \( \mu_1 = \frac{i}{2}(-1 + \sqrt{5}), \mu_2 = -\frac{i}{2}(1 + \sqrt{5}), \omega \equiv e^{2\pi i/3} \). On the other hand \( \Sigma(60) \) is generated by \( A, B \) and \( C \), so that \( \Sigma(60) \subset \Sigma(360\varphi) \). Moreover,

\[ (A \cdot D)^2 = \omega \Pi, \]  

(4.29)

which generates \( \mathbb{Z}_3 \). It follows that \( \Sigma(60) \times \mathbb{Z}_3 \subset \Sigma(360\varphi) \), and therefore all \( \Sigma(360\varphi) \) invariants are also \( \Sigma(60) \times \mathbb{Z}_3 \) invariants. From section 4.4 we then conclude that \( \Sigma(360\varphi) \) does not have any harmonic invariants of degree less than four. To check whether or not
\( \Sigma(360 \varphi) \) has a degree-four harmonic invariant, it suffices to check whether \( I_{2,3}^2 - 2I_{2,1}I_{2,2} \) of section 4.4 is invariant under the generator \( D \) of (4.28). It can be seen that it is not, therefore \( \Sigma(360 \varphi) \) does not have any harmonic invariants of degree less than five. Moreover, \( \Sigma(360 \varphi) \) does not have any invariants of degree five either, since its subgroup \( \Sigma(60 \varphi) \) does not, as follows from the Molien function (4.12). On the other hand, \( \Sigma(360 \varphi) \) has one holomorphic (and therefore harmonic) invariant of degree six, constructed explicitly in [60]. We thus obtain an order of scale separation \( r = 12 \), cf. (B.9).

\subsection{4.11 \( T_n \)}

The series \( T_n \) is generated by \( E \) and \( F(n, 1, b) \) of (4.5), with \( 1 + b + b^2 \equiv 0 \pmod{n} \), \( n \geq 2 \). The smallest group in this series is obtained for \( n = 7 \), \( b = 2 \). All \( T_n \)'s admit a cubic harmonic invariant, \( xyz \), which corresponds to an order of scale separation \( r = 4.2 \).

\subsection{4.12 \( \Delta(3n^2) \)}

This series is generated by \( E \) and \( F(n, 0, 1) \) of (4.5), with \( n \geq 2 \). It admits a cubic harmonic invariant, \( xyz \), which corresponds to an order of scale separation \( r \leq 4.2 \). For \( n = 2 \) we have \( \Delta(12) \simeq A_4 \), and the additional quadratic harmonic invariants \( I_{1,2} \) of (4.13), which gives an order of scale separation \( r = 2.4 \).

\subsection{4.13 \( \Delta(6n^2) \)}

This series is generated by \( E, F(n, 0, 1) \) and \( G(2, 1, 1) \) of (4.5), (4.6), with \( n \geq 2 \). It does not admit any cubic harmonic invariants, but it admits the quartic harmonic invariant, \( 2(|x|^4 + |y|^4 + |z|^4) - I_{2,3}^2 \), which corresponds to an order of scale separation \( r \leq 6.4 \). For \( n = 2 \) we have \( \Delta(24) \simeq S_4 \), and the additional quadratic harmonic invariants \( I_{2,1}, I_{2,2} \) of (4.13), which corresponds to an order of scale separation \( r = 2.4 \).

\section{Smoothness and Killing spinors}

For the background \( \text{AdS}_p \times S^q / \Gamma \times M_d \) to preserve supersymmetry, we would typically need the existence of Killing spinors on the orbifolded sphere. On the other hand, it is well known that Killing spinors on \( S^q \) are in one-to-one correspondence with (covariantly) constant spinors on the ambient \( \mathbb{R}^{q+1} \), so that each Killing spinor is the restriction to \( S^q \) of a constant spinor of \( \mathbb{R}^{q+1} \) [73]. Therefore, if \( \tilde{\Gamma} \) is a lift of \( \Gamma \) to \( \text{Spin}(q+1) \) corresponding to the spin structure of \( S^q / \Gamma \), Killing spinors of the latter would correspond to \( \tilde{\Gamma} \)-invariant spinors. In particular, for \( q = 5 \), taking \( \Gamma \subset \text{SU}(3) \subset \text{SU}(4) \simeq \text{Spin}(6) \) thus guarantees the existence of Killing spinors on \( S^5 / \Gamma \).

The subgroups \( \Gamma \in \text{SO}(6) \) for which \( S^5 / \Gamma \) is smooth, were classified in [74]. Moreover, there is an infinite number of smooth quotients possessing Killing spinors [75, 76]. Rephrasing the results presented in [76], by appropriately choosing the complex structure of the ambient \( \mathbb{C}^3 \), there are two infinite series of smooth \( S^5 / \Gamma \) orbifolds possessing Killing spinors,
• \( \Gamma \) isomorphic to \( \mathbb{Z}_n \), generated by,

\[
\begin{pmatrix}
\eta & 0 & 0 \\
0 & \eta^a & 0 \\
0 & 0 & \eta^b
\end{pmatrix},
\]

where \( \eta \equiv e^{2\pi i/n}, \, n \in \mathbb{N} - \{0\} \); \( a, b \in \mathbb{Z} \) with \( a + b + 1 \equiv 0 \pmod{n} \), \( 1 \leq |a|, |b| \leq n - 1 \), and \( (a, n) = (b, n) = 1 \), where \( (p, q) \) denotes the greatest common divisor of \( p, q \).

• \( \Gamma \) isomorphic to \( T_n \), generated by \( E \) and \( F(n, 1, b) \) of (4.5), with \( 1 + b + b^2 \equiv 0 \pmod{n} \). In addition we must impose \( (3(b - 1), n) = 1 \). These conditions only admit solutions for very specific integers \( n \).

Both of these series are of the form \( \Gamma \subset \text{SU}(3) \), which of course was expected in view of what was mentioned at the beginning of this section. The first of the two series admits quadratic harmonic invariants, linear combinations of \( |x|^2 - |z|^2 \) and \( |y|^2 - |z|^2 \), which corresponds to an order of scale separation \( r = 2.4 \). The second series admits a cubic harmonic invariant, \( xyz \), which corresponds to an order of scale separation \( r = 4.2 \).

Let us now come to the \( S^2 \) orbifolds. None of the these obifolds is smooth: their singularity types are well understood and have been classified, see e.g. Ch.13 of [30]. Moreover, \( S^2/T \) does not admit Killing spinors, since there are no \( \text{SU}(2) \)-invariant spinors for any of the subgroups \( \tilde{\Gamma} \subset \text{SU}(2) \), as can be easily verified using the generators listed in appendix C.

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A Conventions

The \( p \)-dimensional AdS space can be defined as a the hyperboloid,

\[
x_0^2 + x_p^2 - \sum_{i=1}^{p-1} x_i^2 = L_{\text{AdS}}^2,
\]

in an ambient \( \mathbb{R}^{2,p-1} \) space, with a standard flat metric, parameterized by \( x_0, \ldots, x_p \). The constant \( L_{\text{AdS}} \) is the \textit{radius of curvature} of AdS. The induced AdS\(_p \) metric \( (g_{\mu\nu}) \) obeys,

\[
R_{\mu\nu} = -\frac{p-1}{L_{\text{AdS}}^2} g_{\mu\nu},
\]

The general prime factor decomposition of such \( n \) was determined in [77]. In particular, if 3 is not a divisor of \( n \), as is the case here, \( n \) must be a product of prime numbers, each of which is of the form \( 6k+1 \), for positive integers \( k \), and \( b \neq 1 \pmod{3} \). Ref. [76] also lists the conditions: \( n \) odd, and \( b \neq b^3 \equiv 1 \pmod{n} \). These follow from the conditions already imposed in the main text.
where $R_{\mu\nu}$ is the Ricci tensor of AdS$_p$. In local coordinates, covering half the hyperboloid, the metric takes the form,

$$ds^2(\text{AdS}_p) = L_{\text{AdS}}^2 \left[ dp^2 + e^{2p} ds^2(\mathbb{R}^{1, p-2}) \right]. \quad (A.3)$$

Similarly, the round $q$-dimensional sphere $S^q$ of radius $L_S$ is defined by,

$$\sum_{i=1}^{q+1} x_i^2 = L_S^2, \quad (A.4)$$

in an ambient $\mathbb{R}^{q+1}$ space, with a standard flat metric, parameterized by $x_1, \ldots, x_{q+1}$. The induced $S^q$ metric $(g_{mn})$ obeys,

$$R_{mn} = \frac{q-1}{L_S^2} g_{mn}, \quad (A.5)$$

where $R_{mn}$ is the Ricci tensor of $S^q$.

Einstein gravity in $(p + q)$ dimensions, minimally coupled to a $q$-form,

$$S = \int d^{p+q}x \sqrt{g} \left( R + \frac{1}{2q} F_{m_1 \ldots m_q} F^{m_1 \ldots m_q} \right), \quad (A.6)$$

admits Freund-Rubin solutions [1] of the form AdS$_p \times M_q$, where $M_q$ is a $q$-dimensional Einstein manifold. These solutions obey [78],

$$\frac{L_{\text{AdS}}}{L_{\text{int}}} = \frac{(p-1)}{(q-1)}, \quad (A.7)$$

where $L_{\text{int}}$ is the radius of curvature of $M_q$,

$$R_{mn} = \frac{q-1}{L_{\text{int}}^2} g_{mn}, \quad (A.8)$$

with $R_{mn}$ the Ricci tensor of $M_q$. In the special case where $M_q$ is the $q$-dimensional sphere, $L_{\text{int}}$ reduces to the radius $L_S$.

### B Spherical harmonics

Consider the $q$-dimensional unit sphere $S^q$,

$$S^q = \left\{ \vec{x} \in \mathbb{R}^{q+1} \left| \sum_{i=1}^{q+1} (x^i)^2 = 1 \right. \right\}. \quad (B.1)$$

The basis of spherical harmonics on $S^q$ is inherited from the space of degree-$k$ homogeneous polynomials of $\mathbb{R}^{q+1}$,

$$p_k(\vec{x}) = c_{i_1 \ldots i_k} x^{i_1} \ldots x^{i_k} ; \quad k \in \mathbb{N}, \quad (B.2)$$

where the coefficients are totally symmetric and traceless (for $k \geq 2$),

$$c_{i_1 \ldots i_k} = c_{(i_1 \ldots i_k)} ; \quad c_{i_1 i_2 \ldots i_k} \delta^{i_1 i_2} = 0. \quad (B.3)$$
Equivalently, these are the polynomials which are harmonic with respect to the scalar Laplacian of $\mathbb{R}^{q+1}$,

$$\Delta_{\mathbb{R}^{q+1}} p_k(\vec{x}) = 0 \ . \tag{B.4}$$

Restricting to $S^q$, it follows that they are eigenstates of the scalar Laplacian of $S^q$,

$$\Delta_{S^q} p_k(\vec{x}) = -k(k + q - 1)p_k(\vec{x}) \ . \tag{B.6}$$

For a $q$-sphere of radius $L_S$, the KK mass-squared is minus the lowest-order non-vanishing Laplacian eigenvalue,

$$m_{KK}^2 = \frac{q}{L_S^2} \ . \tag{B.7}$$

Suppose that orbifolding $S^q$ by a finite subgroup $\Gamma \subset \text{SO}(q+1)$ projects out the first $k - 1$ non-trivial eigenmodes. It follows that

$$m_{KK/\Gamma}^2 = \frac{k(k + q - 1)}{L_S^2} \ , \tag{B.8}$$

so that from (1.2) we obtain the relative order of scale separation,

$$r = \frac{k(k + q - 1)}{q} \ . \tag{B.9}$$

B.1 Spherical harmonics on $S^2$

We list a basis of the first few harmonic polynomials on $\mathbb{R}^3$. Restricted to the unit sphere $S^2 \subset \mathbb{R}^3$ they provide a basis of spherical harmonics on $S^2$.

$$k=0 : \ 1$$

$$k=1 : \ x, y, z$$

$$k=2 : \ xy, xz, yz, x^2 - z^2, y^2 - z^2$$

$$k=3 : \ xyz, x^3 - 3xy^2, x^3 - 3xz^2, y^3 - 3yx^2, y^3 - 3yz^2, z^3 - 3zx^2, z^3 - 3zy^2$$

$$k=4 : \ x^3y - xy^3, y^3z - yz^3, z^3x - xz^3$$

$$x^3z - 3xyz^2, y^3x - 3xy^2z, z^3y - 3yz^2x, x^4 - 6x^2y^2 + y^4, y^4 - 6y^2z^2 + z^4, x^4 - 6x^2y^2 + z^4$$

---

18This can be seen by rewriting the Laplacian of $\mathbb{R}^{q+1}$ in spherical coordinates,

$$\Delta_{\mathbb{R}^{q+1}} = \frac{\partial^2}{\rho^2} + \frac{\partial}{\rho} + \frac{1}{\rho^2} \Delta_{S^q} p_k(\vec{x}) \ , \tag{B.5}$$

where $\rho$ is the radial coordinate. Taking into account that $p_k$ is homogeneous of degree $k$: $\rho \frac{\partial}{\partial \rho} p_k = k p_k$, and restricting to $\rho = 1$, we obtain (B.6).
\[ k=5 : \quad x^5 - 10x^3y^2 + 5xy^4, \quad x^5 - 10x^3z^2 + 5xz^4, \quad y^5 - 10x^2y^3 + 5x^4y, \quad y^5 - 10y^3z^2 + 5yz^4, \]
\[ z^5 - 10x^2z^3 + 5x^4z, \quad z^5 - 10y^2z^3 + 5y^4z, \quad x^3y - xz^3, \quad x^3yz - xy^3z, \]
\[ x^4y - 6x^2yz^2 + yz^4, \quad xy^4 - 6xy^2z^2 + xz^4, \quad x^4z - 6x^2y^2z + y^4z \]

\[ k=6 : \quad x^6 - 15x^4y^2 + 15x^2y^4 - y^6, \quad x^6 - 15x^4z^2 + 15x^2z^4 - z^6, \]
\[ y^6 - 15y^4z^2 + 15y^2z^4 - z^6, \quad x^5z - 10x^3y^2z + 5xy^4z, \quad y^5z - 10x^2y^3z + 5x^4yz, \]
\[ x^5y - 10y^3z^2 + 5xyz^4, \quad y^5x - 10x^2z^3 + 5x^4yz, \quad xz^5 - 10xy^2z^3 + 5xy^4z, \]
\[ 3x^5y - 10x^3y^3 + 3xy^5, \quad 3y^5z - 10y^3z^3 + 3yz^5, \quad x^6 - 15x^4y^2 + y^6 + 90x^2y^2z^2 - 15y^4z^2 - 15x^2z^4 + z^6. \]

(B.10)

C Finite subgroups of SU(2)

The SU(2) subgroups were classified over a century ago [79]. They consist of two infinite series and three exceptional cases [80]:

- The cyclic groups \( \mathbb{Z}_n \), \( n \geq 2 \), generated by,
  \[
  \begin{pmatrix}
  e^{\frac{2\pi i}{n}} & 0 \\
  0 & e^{-\frac{2\pi i}{n}}
  \end{pmatrix},
  \quad (C.1)
  \]

- The binary dihedral groups \( \mathbb{D}_n \), obtained by combining \( \mathbb{Z}_n \) with the generator \( i \sigma^1 \).
- The binary tetrahedral group, obtained by combining \( \mathbb{D}_2 \) with the generator,
  \[
  \frac{1}{\sqrt{2}} \begin{pmatrix}
   \varepsilon^7 & \varepsilon^7 \\
   \varepsilon^5 & \varepsilon
  \end{pmatrix},
  \quad (C.2)
  \]
  where \( \varepsilon \equiv e^{\frac{2\pi i}{3}} \).
- The binary octahedral group obtained by combining the binary tetrahedral with,
  \[
  \frac{1}{\sqrt{2}} \begin{pmatrix}
   \varepsilon & 0 \\
   0 & \varepsilon^7
  \end{pmatrix},
  \quad (C.3)
  \]
- The binary icosahedral group generated by,
  \[
  - \begin{pmatrix}
   \eta^3 & 0 \\
   0 & \eta^2
  \end{pmatrix} \quad \text{and} \quad \frac{1}{\eta^2 - \eta^4} \begin{pmatrix}
   \eta + \eta^4 & 1 \\
   1 & -\eta - \eta^4
  \end{pmatrix},
  \quad (C.4)
  \]
  where \( \eta = e^{\frac{2\pi i}{5}} \).
Figure 2. The icosahedron. Only eight of the twelve vertices are depicted: the top and bottom vertices \((V_1, V_2)\), three of the upper pentagon vertices \((U_{1,2,5})\) and three of the lower pentagon vertices \((L_{1,2,5})\). The distance \(L\) from each of the vertices of the upper and lower pentagons to the \(z\)-axis, is also equal to the vertical distance between the upper and lower pentagons.

D The icosahedral group

The rotational (chiral) icosahedral group \(I\) of order 60 is isomorphic to \(A_5\), the alternating group of five elements. \(I\) can be obtained from the binary icosahedral subgroup of \(SU(2)\), cf. appendix C, via the 2:1 map \(SU(2) \to SO(3)\). The preimage in \(SU(2)\) of the generator (2.2) is the second generator in (C.4). The latter can be put in the standard \(SU(2)\) form,

\[
\begin{pmatrix}
a & b \\
-b^* & a^*
\end{pmatrix}; \quad a = e^{-\frac{i}{2}(\alpha+\gamma)} \cos \frac{\beta}{2}, \quad b = -e^{-\frac{i}{2}(\alpha-\gamma)} \sin \frac{\beta}{2},
\]

with Euler angles: \(\sin \frac{\beta}{2} = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{1}{\sqrt{5}}}, \cos \frac{\beta}{2} = \frac{1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}, \alpha = 0, \gamma = \pi\). It follows that the image of this element in \(SO(3)\) is \(R_y(\beta) \cdot R_z(\pi)\), where \(\sin \beta = \frac{2}{\sqrt{5}}, \cos \beta = -\frac{1}{\sqrt{5}}\). As for any finite group, the elements of \(I\) can be generated by a set of elements satisfying certain relations. This is known as a “presentation” of the group. Several different presentations of \(I\) exist with either two or three basis elements [81]. The presentation given in [82], which was argued in [83] to be more suitable for flavor model building, uses two elements \(S, T\) satisfying \(S^2 = T^5 = (T^2ST^3ST^{-1}ST^{-1})^3 = I\). It can be verified that \(R_z\left(\frac{2\pi}{5}\right)\) and the element in (2.2) provide an explicit representation of \(T, S\) respectively.

One can also explicitly construct the icosahedron which is invariant under these generators, cf. figure 2. In Cartesian coordinates of \(\mathbb{R}^3\), its twelve vertices are given by: the top and
bottom vertices \((0, 0, \pm R)\); the upper pentagon vertices \(R_z \left( \frac{2n\pi}{5} \right) \cdot (-L, 0, \frac{1}{2}L), n = 0, \ldots, 4\); the lower pentagon vertices \(-R_z \left( \frac{2(n+1)\pi}{5} \right) \cdot (L, 0, \frac{1}{2}L)\). The constants \(R = \sin \frac{2\pi}{5}\) and \(L = \frac{1}{2\sin \frac{\pi}{5}}\) represent respectively the radius of the circumscribing sphere and the distance from each of the vertices of the upper and lower pentagons to the \(z\)-axis; the length of the pentagon edges is equal to one. It can be seen that \(L\) is also equal to the vertical distance between the upper and lower pentagons. Note that the upper and lower pentagons are rotated by \(\frac{\pi}{5}\) relatively to each other. Let \(V_1, V_2\) be the top, bottom vertices, \(U_1, \ldots, U_5\) the vertices of the upper pentagon in the order listed above, and \(L_1, \ldots, L_5\) the vertices of the lower pentagon. To show the invariance of the icosahedron under the group \(I\), it suffices to note that the generator \(R_z \left( \frac{2\pi}{5} \right)\) acts on the vertices as the permutation \(V_1, V_2, U_2, \ldots, U_5, U_1, L_2, \ldots, L_5, U_1, U_2, L_1\), while the generator (2.2) acts as the permutation \(L_3, U_1, V_2, U_4, U_3, L_2, L_5, U_5, V_1, U_2, L_1\).

Let \(W\) be any of the 12 vertices of the icosahedron. Thinking of the group elements of \(I\) as permutations of the 12 vertices, we shall denote by \(g(W)\) an element which corresponds to a permutation of the form \((W, \ldots)\). I.e. \(g(W)\) is a rotation which brings the vertex \(W\) in the place of the top vertex \(V_1\). Given 12 elements \(g(W), W \in \{V_1, V_2, U_1, \ldots, U_5, L_1, \ldots, L_5\}\), the 60 elements of \(I\) can be obtained from \(g(W)\) by successive \(\frac{2\pi}{5}\) rotations along the vertical axis: \(R_z \left( \frac{2n\pi}{5} \right) \cdot g(W), n = 0, \ldots, 4\). Explicitly we take,

\[
\begin{align*}
g(V_1) &= 1 \\
g(V_2) &= R R_z R R_z^{-1} R R_z \\
g(U_1) &= R R_z^3 \\
g(U_2) &= R R_z^{-1} \\
g(U_3) &= R \\
g(U_4) &= R R_z \\
g(U_5) &= R R_z^2 \\
g(L_1) &= R R_z R R_z^{-1} \\
g(L_2) &= R R_z R \\
g(L_3) &= R R_z R R_z \\
g(L_4) &= R R_z R R_z R \\
g(L_5) &= R R_z R R_z^3,
\end{align*}
\]

where \(R_z \equiv R_z \left( \frac{2\pi}{5} \right)\) and \(R\) is the generator of (2.2).

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