Role of the Brans-Dicke scalar in the holographic description of dark energy

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Abstract

We study cosmological application of the holographic energy density in the Brans-Dicke theory. Considering the holographic energy density as a dynamical cosmological constant, it is more natural to study it in the Brans-Dicke theory than in general relativity. Solving the Friedmann and Brans-Dicke field equations numerically, we clarify the role of Brans-Dicke field during evolution of the universe. When the Hubble horizon is taken as the IR cutoff, the equation of state (\(w_\Lambda\)) for the holographic energy density is determined to be \(\frac{5}{3}\) when the Brans-Dicke parameter \(\omega\) goes infinity. This means that the Brans-Dicke field plays a crucial role in determining the equation of state. For the particle horizon IR cutoff, the Brans-Dicke scalar mediates a transition from \(w_\Lambda = -\frac{1}{3}\) (past) to \(w_\Lambda = \frac{1}{3}\) (future). If a dust matter is present, it determines future equation of state. In the case of future event horizon cutoff, the role of the Brans-Dicke scalar and dust matter are turned out to be trivial, whereas the holographic energy density plays an important role as a dark energy candidate with \(w_\Lambda = -1\).

1 Introduction

Type Ia supernova observations[1] suggest that our universe is in accelerating phase and the dark energy contributes \(\Omega_{DE} \simeq 0.60 - 0.70\) to the critical energy density of the present universe. Also cosmic microwave background observations[2] imply that the standard cosmology is given by inflation and FRW universe [3].

A typical candidate for the dark energy is the cosmological constant in general relativity. Recently Cohen et al[4] showed that in the effective theory of quantum field theory, the UV cutoff \(\Lambda\) is related to the IR cutoff \(L_\Lambda\), due to the limit set by forming a black hole. In other words, if \(\rho_\Lambda\) is the quantum
zero-point energy density caused by the UV cutoff, the total energy density of the system with size $L_A$ should not exceed the mass of the system-sized black hole: $L_A^3 \rho_A \leq L_A / G$. Here the Newtonian constant $G$ is related to the Planck mass by $G = 1 / M_P^2$. The largest IR cutoff $L_A$ is chosen as the one saturating this inequality and the holographic energy density is then given by $\rho_A = 3c^2 M_P^2 / 8\pi L_A^2$ with an appropriate factor $3c^2 / 8\pi$. Comparing with the cosmological constant, we regard it as a dynamical cosmological constant. Taking $L_A$ as the size of the present universe (Hubble horizon $R_{HH}$), the resulting energy density is comparable to the present dark energy density \cite{5}. Even though this holographic approach leads to the data, this description is incomplete because it fails to explain the equation of state for the dark energy-dominated universe\cite{6}. In order to resolve this situation, one introduces other candidates for the IR cutoff. One is the particle horizon $R_{PH}$. This provides $\rho_A \sim a^{-2(1+1/c)}$, which means that the equation of state is given by $\omega_A = 1/3$ for $c = 1$\cite{7}. However, it corresponds to a radiation-dominated universe and it is a decelerating phase. In order to find an accelerating phase, we need to introduce the future event horizon $R_{FH}$. In the case of $L_A = R_{FH}$, one finds $\rho_A \sim a^{-2(1-1/c)}$ which could describe the dark energy with $\omega_A = -1$ for $c = 1$. This is close to the data\cite{1} and the related works appeared in ref.\cite{8, 9, 10, 11}.

On the other hand, it is worthwhile to investigate the holographic energy density in the framework of the Brans-Dicke theory. The reasons are as follows. Because the holographic energy density belongs to a dynamical cosmological constant, we need a dynamical frame to accommodate it instead of general relativity. Further, taking $L_A = R_{HH}$, it fails to determine the equation of state $w_A$ in the general relativity framework. In addition to these, the Brans-Dicke scalar speeds up the expansion rate of a dust matter-dominated era (reduces deceleration), while slows down the expansion rate of cosmological constant era (reduces acceleration)\cite{12, 13}. The Brans-Dicke generalization was first studied by Gong\cite{14}. Since the Brans-Dicke description of gravitation is to replace the Newtonian constant $G$ by a time varying scalar $\Phi(t)$, the holographic energy density is given by $\rho_A = 3\Phi / 8\pi L_A^2$ with $c^2 = 1$. Gong recovered the same results as those in general relativity for a large $\omega$. The present authors studied the same issue by considering a Bianchi identity as a consistency condition\cite{15}. The equation of state for Hubble IR cutoff is determined to be $w_A = 5/3$ when the Brans-Dicke parameter $\omega$ goes infinity. This implies that the Brans-Dicke framework is suitable for studying an evolution of the holographic energy density.

In this work, we introduce a dust matter to our consideration and solve the equations numerically. Since the holographic energy density is dynamical, it is nontrivial to solve the Friedmann and the Brans-Dicke field equation with three conservation laws. They can not be solved analytically. From this study we investigate the role of the holographic energy density, Brans-Dicke scalar and dust matter for a given IR cutoff. Especially, we wish to show why
a combination of the holographic energy density and future event horizon could describe a dark energy-dominated era.

2 Brans-Dicke cosmology

For cosmological purpose, we introduce the Brans-Dicke (BD) action with a matter

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi} \left( \Phi R - \omega \nabla_\alpha \Phi \nabla^\alpha \Phi \right) + \mathcal{L}_M \right], \quad (1) \]

where \( \Phi \) is the BD scalar which plays the role of an inverse of the Newtonian constant, \( \omega \) is the parameter of BD theory, and \( \mathcal{L}_M \) represents other matter which takes a perfect fluid form. The field equations for metric \( g_{\mu\nu} \) and BD scalar \( \Phi \) are

\[ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T^{BD}_{\mu\nu} + \frac{8\pi}{\Phi} T^M_{\mu\nu}, \quad (2) \]

\[ \nabla_\alpha \nabla^\alpha \Phi = \frac{8\pi}{2\omega + 3} T^M_\alpha, \]

where the energy-momentum tensor for the BD scalar is defined by

\[ T^{BD}_{\mu\nu} = \frac{1}{8\pi} \left[ \frac{\omega}{\Phi^2} \left( \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} (\nabla \Phi)^2 \right) + \frac{1}{\Phi} \left( \nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \nabla_\alpha \nabla^\alpha \Phi \right) \right], \quad (3) \]

and the energy-momentum tensor for other matter takes the form

\[ T^M_{\mu\nu} = p_M g_{\mu\nu} + (\rho_M + p_M) U_\mu U_\nu. \quad (4) \]

Here \( \rho_M(p_M) \) denote the energy density (pressure) of the matter and \( U_\mu \) is a four velocity vector with \( U_\alpha U^\alpha = 1 \).

Assuming that our universe is homogeneous and isotropic, we work with the Friedmann-Robertson-Walker (FRW) spacetime

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (5) \]

We consider a spatially flat spacetime of \( k = 0 \). In the FRW spacetime, the field equations take the forms

\[ H^2 + H \left( \frac{\dot{\Phi}}{\Phi} \right) - \frac{\omega}{6} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 = \frac{8\pi \rho_M}{3 \Phi}, \quad (6) \]

\[ \ddot{\Phi} + 3H \dot{\Phi} = \frac{8\pi}{2\omega + 3} (\rho_M - 3p_M) \]

with the Hubble parameter \( H = \dot{a}/a \). Here we note that the case of \( \omega = -3/2 \) is not allowed when a matter with \( p_M \neq p_M/3 \) comes into the BD theory.
Regarding the BD field as a perfect fluid, its energy and pressure are given by

\[ \rho_{BD} = \frac{1}{16\pi G_0} \left[ \omega \left( \frac{\dot{\Phi}}{\Phi} \right)^2 - 6H \frac{\dot{\Phi}}{\Phi} \right], \]
\[ p_{BD} = \frac{1}{16\pi G_0} \left[ \omega \left( \frac{\dot{\Phi}}{\Phi} \right)^2 + 4H \frac{\dot{\Phi}}{\Phi} + 2\ddot{\Phi} \right], \]

where \( G_0 \) is the present Newtonian constant. Usually, if one does not specify the parameter \( \omega \), one cannot determine the BD equation of state exactly. The Bianchi identity leads to an energy transfer between BD field and other matter

\[ \dot{\rho}_{BD} + 3H(\rho_{BD} + p_{BD}) = \frac{1}{G_0} \frac{\rho_M \dot{\Phi}}{\Phi} \]

and the matter evolves according to its conservation law

\[ \dot{\rho}_M + 3H(\rho_M + p_M) = 0. \]

Their equations of states are given by

\[ w_{BD} \equiv \frac{p_{BD}}{\rho_{BD}}, \quad w_M \equiv \frac{p_M}{\rho_M}. \]

3 Brans-Dicke framework with holographic energy density and dust matter

In this section, we investigate how the equation of state for the holographic energy density changes when an interaction between the BD field \( \rho_{BD} \), holographic energy density \( \rho_\Lambda \) and dust matter \( \rho_m \) is included. In this case the Friedmann and BD field equations are

\[ H^2 + H \frac{\dot{\Phi}}{\Phi} - \frac{\omega}{6} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 = \frac{8\pi}{3} \frac{\rho_t}{\Phi}, \]

\[ \ddot{\Phi} + 3H \frac{\dot{\Phi}}{\Phi} = \frac{8\pi}{2\omega + 3} \left( \rho_t - 3p_t \right). \]

Here \( \rho_t = \rho_\Lambda + \rho_m \) and \( p_t = p_\Lambda + p_m \). The holographic energy density \( \rho_\Lambda \) and a dust matter \( \rho_m \) are chosen to be

\[ \rho_\Lambda = \frac{3}{8\pi L_\Lambda^2}, \quad \rho_m = \rho_m^0 a^{-3} \]

with \( p_m = 0 \). In order to solve Eq. (11) with Eqs. (8) and (9), we define

\[ x = \ln a, \quad \varphi = \frac{\Phi'}{\Phi}, \quad \lambda = -\frac{H'}{H}, \quad r = \frac{R'}{R}, \]

where \( G_0 \) is the present Newtonian constant. Usually, if one does not specify the parameter \( \omega \), one cannot determine the BD equation of state exactly. The Bianchi identity leads to an energy transfer between BD field and other matter

\[ \dot{\rho}_{BD} + 3H(\rho_{BD} + p_{BD}) = \frac{1}{G_0} \frac{\rho_M \dot{\Phi}}{\Phi} \]

and the matter evolves according to its conservation law

\[ \dot{\rho}_M + 3H(\rho_M + p_M) = 0. \]

Their equations of states are given by

\[ w_{BD} \equiv \frac{p_{BD}}{\rho_{BD}}, \quad w_M \equiv \frac{p_M}{\rho_M}. \]
where $'$ means the derivative with respect to $x$ and $R \in \{R_{HH}, R_{PH}, R_{FH}\}$. From the definition of $\varphi$ and $\lambda$, it is granted that $H$ and $\Phi$ are taken to be positive\[16\]. Then the Friedmann equation and BD field equation become

$$H^2 \left(1 + \varphi - \frac{\omega}{6} \varphi^2\right) = \frac{8\pi}{3} \frac{\rho_t}{\Phi}, \quad (14)$$

$$H^2 \left(\varphi' - \lambda \varphi + \varphi^2 + 3\varphi\right) = \frac{8\pi}{2\omega + 3} \frac{\rho_t - 3p_t}{\Phi}.$$ 

Also the energy-momentum conservation law leads to the pressure

$$p_\Lambda = -\frac{1}{3}(\varphi - 2r + 3) \rho_\Lambda \quad \text{(15)}$$

whose equation of state is given by

$$w_\Lambda = -\frac{1}{3}(\varphi - 2r + 3). \quad \text{(16)}$$

If other interaction between $\rho_\Lambda$ and $\rho_m$ is included, then the equation of state $w_\Lambda$ takes a different form\[17\].

From now on we focus on the change of $w_\Lambda$ by choosing an IR cutoff $L_\Lambda$. Firstly, we take Hubble horizon as the IR cutoff $(L_\Lambda = R_{HH} = 1/H)$. We have $\lambda = r$ and then eliminate $\lambda$ to obtain

$$\varphi' = \frac{\omega(\omega + 1)\varphi}{6} \left(\varphi - \frac{6}{\omega}\right) \left(\varphi - \frac{1}{\omega + 1}\right),$$

$$r = \frac{1}{2} \left(\frac{\omega \varphi}{3} - 1\right) \left[(\omega + 1) \varphi - 1\right] + \frac{\varphi + 3}{2}. \quad \text{(17)}$$

One can solve the above equation numerically. We have a plot for $w_\Lambda$ as is shown in Fig. 1. In Fig. 1 an upper dotted line represents a graph for $w_\Lambda = \frac{5}{3}$ without a dust matter in the limit of $\omega \to \infty$. This case was already found in\[15\]. Solid lines represent the change of equation of state $w_\Lambda$ with a dust matter. When a dust matter is present, the BD theory allows two solutions in the far past: $w_\Lambda = \frac{5}{3}$ for a large $\omega$ and $w_\Lambda = \frac{1}{3}$, independent of $\omega$. As the BD field evolves, the equation of state for the holographic energy density converges that of dust matter. The universe behaves as a dust matter-dominated phase in the far future, irrespective of where it starts. If the BD scalar is turned off, one cannot determine the equation of state for the holographic energy density only\[6\]. In this sense, although we do not obtain a dark energy era, the BD framework is essential for determining $w_\Lambda$ and it goes well with $L_\Lambda = 1/H$.

For particle horizon IR cutoff with $L_\Lambda = R_{PH} \equiv a \int_0^\alpha \frac{da}{H a^2}$ and future event horizon IR cutoff $L_\Lambda = R_{FH} \equiv a \int_\alpha^\infty \frac{da}{H a^2}$, $r$ is given by

$$r = 1 \pm \frac{1}{\Omega_\Lambda \left(1 + \varphi - \frac{\omega}{6} \varphi^2\right)} \quad \text{(18)}$$
Figure 1: A plot of equation of state $w_\Lambda$ versus $x = \ln a$ for Hubble horizon. Here BD denotes the Brans-Dicke framework.

with $\Omega_\Lambda \equiv \rho_\Lambda / \rho_t$. Here + denotes particle horizon, while − represents future event horizon. Eliminating $\lambda$ leads to two coupled equations

$$\varphi' = -\frac{1 + \varphi - \frac{\omega}{3} \varphi^2}{2\omega + 3} \left[ 3 \left( (\omega + 1) \varphi - 1 \right) + (\varphi - 2r + 3)(\omega \varphi - 3)\Omega_\Lambda \right], \quad (19)$$

$$\Omega'_\Lambda = (\varphi - 2r + 3)\Omega_\Lambda (1 - \Omega_\Lambda).$$

Here $\lambda$ is related to $r$ via

$$\lambda = r + \frac{1 - \frac{\omega}{3} \varphi}{2(1 + \varphi - \frac{\omega}{6} \varphi^2)} \varphi'. \quad (20)$$

We solve the coupled equations numerically and plot $w_\Lambda$ in Fig. 2 and Fig. 3 for particle horizon and future event horizon, respectively.

For particle horizon IR cutoff $L_\Lambda = R_{PH}, w_\Lambda = \frac{1}{3}$ is found for the holographic energy density solely when using general relativity. A thin line stands for $w_\Lambda$ in the general relativity framework together with a dust matter. A medium line results from the DB framework without matter and a thick line represents $w_\Lambda$ in the BD framework with a dust matter. In the general relativity framework, the equation of state of holographic energy density starts with $w_\Lambda = \pm \frac{1}{3}$ in the far past and ends with $w_\Lambda = 0$ like as a dust matter. However, in the BD framework, equation of state of the holographic energy density starts with $w_\Lambda = -\frac{1}{3}$ at the far past and then, transits to $w_\Lambda = \frac{1}{3}$. This implies that without matter, the holographic energy density becomes a radiation. In the BD framework together with a dust matter, the equation of state of holographic energy density starts with $w_\Lambda = -\frac{1}{3}$ in the far past.
The BD field makes a transition to a radiation phase and finally, $w_{\Lambda}$ transits to a dust matter in the far future. In this case, the dust matter determines a future equation of state for the holographic energy density. This means that a dust matter dominates in the holographic energy density with particle horizon. Finally we mention that the BD scalar plays a role of the mediator between $w_{\Lambda} = -\frac{1}{3}$ and $w_{\Lambda} = \frac{1}{3}$.

For future event horizon cutoff $L_{\Lambda} = R_{FH}$, one finds $w_{\Lambda} = -1$ with $c^2 = 1$ for $\rho_{\Lambda}$ only in the general relativity framework. A thin line stands for a graph of $w_{\Lambda}$ with a dust matter in general relativity. A medium/thick lines correspond to the BD theory framework without/with a dust matter. A general relativistic analysis was carried out by Li[7]. Equation of state of the holographic energy density starts with $w_{\Lambda} = -\frac{1}{3}$ in the far past and becomes a cosmological constant with $w_{\Lambda} = -1$ in the far future. For this IR cutoff, the holographic energy density serves as a dark energy and leads to an accelerating era. As is shown in Fig. 3, this feature persists even in the BD framework with or without a dust matter. This means that the holographic energy density goes well with future event horizon $L_{\Lambda} = R_{FH}$. On the other hand, the role of dust matter and BD scalar is trivial when comparing with the holographic energy density.

### 4 Summary

We study cosmological application of the holographic energy density in the Brans-Dicke framework. Considering the holographic energy density as a
Figure 3: A plot for $w_\Lambda$ as a function of $x = \ln a$ for future event horizon. Here GR (BD) denote the general relativity (Brans-Dicke) frameworks.

Table 1: Summary for future equation of state. Here three combinations for holographic energy density ($\rho_\Lambda$), Brans-Dicke scalar ($\rho_{BD}$), and dust matter ($\rho_m$) are evaluated for IR cutoff ($L_\Lambda$) as Hubble horizon ($R_{HH}$), particle horizon ($R_{PH}$) and future event horizon ($R_{FH}$), respectively.

| matter         | $R_{HH}$ | $R_{PH}$ | $R_{FH}$ |
|----------------|----------|----------|----------|
| $\rho_\Lambda + \rho_{BD}$ | $w_\Lambda = 5/3$ | $w_\Lambda = 1/3$ | $w_\Lambda = -1$ |
| $\rho_\Lambda + \rho_m$     | $w_\Lambda = 0$  | $w_\Lambda = 0$  | $w_\Lambda = -1$  |
| $\rho_\Lambda + \rho_{BD} + \rho_m$ | $w_\Lambda = 0$  | $w_\Lambda = 0$  | $w_\Lambda = -1$  |

dynamical cosmological constant, it is more natural to study it in the Brans-Dicke theory than in general relativity. Solving the Friedmann and Brans-Dicke field equations numerically, we investigate the role of Brans-Dicke field during evolution of the universe.

We summarize future equation of state for the holographic energy density in Table 1. When the Hubble horizon is taken as the IR cutoff, the equation of state for the holographic energy density ($w_\Lambda$) is determined to be $\frac{5}{3}$ when the Brans-Dicke parameter $\omega$ goes infinity. This means that the Brans-Dicke scalar is crucial for determining the equation of state when comparing to the case of $\rho_\Lambda$. However, if a dust matter is turned on, its future equation of state is determined by $w_\Lambda = 0$, irrespective of the presence of the Brans-Dicke scalar. Actually, the equation of state for $\rho_\Lambda + \rho_m$ can be determined to be $w_\Lambda = 0$ by the Friedmann equation\[6\].

For particle horizon IR cutoff, the Brans-Dicke scalar mediates the transition from $w_\Lambda = -1/3$ (past) to $w_\Lambda = 1/3$ (future). However, if a dust matter
is present, it determines future equation of state. Hence a dust matter plays an important role in the holographic description with particle horizon.

In the case of future event horizon cutoff, the role of the Brans-Dicke scalar and dust matter are trivial, whereas the holographic energy density plays an important role as a dark energy candidate with $w_A = -1$.

Consequently, we find the major roles in the holographic description of an evolving universe: BD scalar in Hubble horizon; dust matter in particle horizon; holographic energy density in future event horizon. The BD scalar plays a role of the mediator, but it does not determine the future equation of state if a dust matter or the holographic energy density is present.

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