INVARIANTS OF THE BI-LIPSCHITZ CONTACT EQUIVALENCE OF CONTINUOUS DEFINABLE FUNCTION GERMS

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Abstract. We construct an invariant of the bi-Lipschitz contact equivalence of continuous function germs definable in a polynomially bounded o-minimal structure, such as semialgebraic functions. For a germ $f$, the invariant is given in terms of the leading coefficients of the asymptotic expansions of $f$ along the connected components of the tangency variety of $f$.

1. Introduction

Lipschitz geometry of maps is a rapidly growing subject in contemporary Singularity Theory. Recent progress in this area is due to the tameness theorems proved by several authors (see, for example, [11, 6, 8, 9, 10, 17]). However the description of a set of invariants is barely developed (see also [2]). This paper presents a numerical invariant of continuous function germs definable in a polynomially bounded o-minimal structure (e.g., semialgebraic functions) with respect to the bi-Lipschitz contact equivalence. The most important ingredient of the invariant constructed here is the so-called tangency variety. More precisely, let $f: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a continuous function germ, which is definable in a polynomially bounded o-minimal structure. The tangency variety $\Gamma(f)$ of $f$ consists of all points $x$ in some neighborhood of the origin $0 \in \mathbb{R}^n$ such that the fiber $f^{-1}(f(x))$ is tangent to the sphere in $\mathbb{R}^n$ centered at $0$ with radius $\|x\|$. The restriction of $f$ on each connected component of $\Gamma(f) \setminus \{0\}$ defines a definable function $f_k$ of a single variable. Then the invariant of $f$ is given in terms of the leading coefficients of the asymptotic expansions of these functions $f_k$.

The rest of the paper is organized as follows. In Section 2 we present some preliminaries which will be used later. The definition and some properties of tangency varieties are given in Section 3. The main result is provided in Section 4.

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2. Preliminaries

Throughout this work we shall consider the Euclidean vector space \( \mathbb{R}^n \) endowed with its canonical scalar product \( \langle \cdot, \cdot \rangle \) and we shall denote its associated norm \( \| \cdot \| \). The closed ball (resp., the sphere) centered at the origin \( 0 \in \mathbb{R}^n \) of radius \( \epsilon \) will be denoted by \( B_\epsilon \) (resp., \( S_\epsilon \)).

2.1. The bi-Lipschitz contact equivalence. The contact equivalence between (smooth) mappings was introduced by J. Mather [14]. The natural extension of Mather’s definition to the Lipschitz setting in the function case appeared in [1], and to the general case in [17]. Let us start with the following definition.

Definition 2.1. Two map germs \( f,g : (\mathbb{R}^n,0) \to (\mathbb{R}^p,0) \) are called bi-Lipschitz contact equivalent (or \( K \)-bi-Lipschitz equivalent) if there exist two germs of bi-Lipschitz homeomorphisms \( h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0) \) and \( H : (\mathbb{R}^n \times \mathbb{R}^p,0) \to (\mathbb{R}^n \times \mathbb{R}^p,0) \) such that \( H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\} \) and the following diagram is commutative:

\[
\begin{array}{ccc}
(\mathbb{R}^n,0) & \xrightarrow{(id,f)} & (\mathbb{R}^n \times \mathbb{R}^p,0) & \xrightarrow{\pi_n} & (\mathbb{R}^n,0) \\
\downarrow h & & \downarrow H & & \downarrow h \\
(\mathbb{R}^n,0) & \xrightarrow{(id,g)} & (\mathbb{R}^n \times \mathbb{R}^p,0) & \xrightarrow{\pi_n} & (\mathbb{R}^n,0)
\end{array}
\]

where \( id : \mathbb{R}^n \to \mathbb{R}^n \) is the identity map and \( \pi_n : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \) is the canonical projection.

In this paper we consider the case \( p = 1 \), thus the maps \( f, g \) are functions. There is a more convenient way to work with the bi-Lipschitz contact equivalence of functions, due to the following result:

Theorem 2.1 (see [1] Theorem 2.1]). Let \( f,g : (\mathbb{R}^n,0) \to (\mathbb{R},0) \) be two continuous function germs. If \( f \) and \( g \) are bi-Lipschitz contact equivalent, then there exists a bi-Lipschitz homeomorphism germ \( h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0) \), there exist positive constants \( c_1, c_2 \) and a sign \( \sigma \in \{-1,1\} \) such that in a neighbourhood of the origin \( 0 \in \mathbb{R}^n \) the following inequalities hold true

\[
c_1 f(x) \leq \sigma g(h(x)) \leq c_2 f(x).
\]

2.2. O-minimal structures. The notion of o-minimality was developed in the late 1980s after it was noticed that many proofs of analytic and geometric properties of semi-algebraic sets and maps could be carried over verbatim for sub-analytic sets and maps. We refer the reader to [4, 12, 13, 18, 19] for the basic properties of o-minimal structures used in this paper.

Definition 2.2. An o-minimal structure on the real field \( \mathbb{R} \) is a sequence \( S := (S_n)_{n \in \mathbb{N}} \) such that for each \( n \in \mathbb{N} \):
(a) $S_n$ is a Boolean algebra of subsets of $\mathbb{R}^n$.
(b) If $A \in S_m$ and $B \in S_n$, then $A \times B \in S_{m+n}$.
(c) If $A \in S_{n+1}$, then $p(A) \in S_n$, where $p: \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first $n$ coordinates.
(d) $S_n$ contains all algebraic subsets of $\mathbb{R}^n$.
(e) Each set belonging to $S_1$ is a finite union of points and intervals.

A set $A \subset \mathbb{R}^n$ is said to be a **definable set** if $A \in S_n$. A map $f: A \to \mathbb{R}^m$ is said to be a **definable map** if its graph is definable.

The structure $S$ is said to be **polynomially bounded** if for every definable function $f: \mathbb{R} \to \mathbb{R}$, there exist $d \in \mathbb{N}$ and $R > 0$ (depending on $f$) such that $|f(x)| \leq x^d$ for all $x > R$.

Examples of (polynomially bounded) o-minimal structures are

- the semi-linear sets,
- the semi-algebraic sets (by the Tarski–Seidenberg theorem),
- the globally sub-analytic sets, i.e., the sub-analytic sets of $\mathbb{R}^n$ whose (compact) closures in $\mathbb{RP}^n$ are sub-analytic (using Gabrielov’s complement theorem).

2.3. Normals and subdifferentials. Here we recall the notions of the normal cones to sets and the subdifferentials of real-valued functions used in this paper. For more details we refer the reader to [15, 16].

**Definition 2.3.** Consider a set $\Omega \subset \mathbb{R}^n$ and a point $x \in \Omega$.

(i) The **regular normal cone** (known also as the *prenormal* or Fréchet normal cone) $\hat{N}_x \Omega$ to $\Omega$ at $x$ consists of all vectors $v \in \mathbb{R}^n$ satisfying

$$\langle v, x' - x \rangle \leq o(\|x' - x\|) \quad \text{as} \quad x' \to x \quad \text{with} \quad x' \in \Omega.$$

(ii) The **limiting normal cone** (known also as the *basic* or Mordukhovich normal cone) $N_x \Omega$ to $\Omega$ at $x$ consists of all vectors $v \in \mathbb{R}^n$ such that there are sequences $x^k \to x$ with $x^k \in \Omega$ and $v^k \to v$ with $v^k \in \hat{N}_x \Omega$.

If $\Omega$ is a manifold of class $C^1$, then for every point $x \in \Omega$, the normal cones $\hat{N}_x \Omega$ and $N_x \Omega$ are equal to the normal space to $\Omega$ at $x$ in the sense of differential geometry, i.e., $\hat{N}_x \Omega = N_x \Omega$ and $v \perp T_x \Omega$ for all $v \in \hat{N}_x \Omega$, where $T_x \Omega$ stands for the tangent space of $\Omega$ at $x$; see [16] Example 6.8].

For a function $f: \mathbb{R}^n \to \mathbb{R}$, we define the **epigraph** of $f$ to be

$$\text{epi} f := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y \geq f(x)\}.$$

A function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be **lower semi-continuous** at $x$ if it holds that

$$\liminf_{x' \to x} f(x') \geq f(x).$$

Functional counterparts of normal cones are subdifferentials.
Definition 2.4. Consider a function \( f: \mathbb{R}^n \to \mathbb{R} \) and a point \( x \in \mathbb{R}^n \). The limiting and horizon subdifferentials of \( f \) at \( x \) are defined respectively by
\[
\partial f(x) := \left\{ v \in \mathbb{R}^n \mid (v, -1) \in N_{(x, f(x))}\text{epi}f \right\},
\partial^\infty f(x) := \left\{ v \in \mathbb{R}^n \mid (v, 0) \in N_{(x, f(x))}\text{epi}f \right\}.
\]

The limiting subdifferential \( \partial f(x) \) generalizes the classical notion of gradient. In particular, for \( C^1 \)-smooth functions \( f \) on \( \mathbb{R}^n \), the subdifferential consists only of the gradient \( \nabla f(x) \) for each \( x \in \mathbb{R}^n \). The horizon subdifferential \( \partial^\infty f(x) \) plays an entirely different role—it detects horizontal “normal” to the epigraph—and it plays a decisive role in subdifferential calculus; see [16, Corollary 10.9] for more details.

Theorem 2.2 (Fermat rule). Consider a lower semi-continuous function \( f: \mathbb{R}^n \to \mathbb{R} \) and a closed set \( \Omega \subset \mathbb{R}^n \). If \( \bar{x} \in \Omega \) is a local minimizer of \( f \) on \( \Omega \) and the qualification condition
\[
\partial^\infty f(\bar{x}) \cap N_\bar{x}\Omega = \{0\}
\]
is valid, then the inclusion \( 0 \in \partial f(\bar{x}) + N_\bar{x}\Omega \) holds.

We will also need the following lemma.

Lemma 2.1. Consider a lower semi-continuous definable function \( f: \mathbb{R}^n \to \mathbb{R} \) and a definable curve \( \phi: [a, b] \to \mathbb{R}^n \). Then for all but finitely many \( t \in [a, b] \), the mappings \( \phi \) and \( f \circ \phi \) are \( C^1 \)-smooth at \( t \) and satisfy
\[
v \in \partial f(\phi(t)) \implies \langle v, \phi'(t) \rangle = (f \circ \phi)'(t),
v \in \partial^\infty f(\phi(t)) \implies \langle v, \phi'(t) \rangle = 0.
\]

Proof. (cf. [3, Proposition 4] and [5, Lemma 2.10]). Without loss of generality, assume that the curve \( \phi \) is non-constant. In light of the monotonicity theorem [19, Theorem 4.1], there exists a real number \( \epsilon \in (0, 1) \) such that on the open interval \( (0, \epsilon) \) we have the mappings \( \phi \) and \( f \circ \phi \) are \( C^1 \)-smooth and \( \phi' \) is nonzero. Let
\[
M := \{(\phi(t), f(\phi(t))) \mid t \in (0, \epsilon)\},
\]
which is a subset of the epigraph of \( f \). Clearly, \( M \) is a connected definable \( C^1 \)-manifold of dimension 1. Taking if necessary a smaller \( \epsilon \), we can be sure that there exists a Whitney \( C^1 \)-stratification \( \mathcal{W} \) of \( \text{epi}f \) such that \( M \) is a stratum of \( \mathcal{W} \); see [19, Theorem 4.8], for example.

Take arbitrary (but fixed) \( t \in (0, \epsilon) \) and \( v \in \partial f(\phi(t)) \). By definition, there exist sequences \( \{x^k\} \subset U \) and \( \{ (v^k, t^k) \} \subset \tilde{N}_{(x^k, f(x^k))}\text{epi}f \subset \mathbb{R}^n \times \mathbb{R} \), such that \( x^k \to x := \phi(t) \) and \( (v^k, t^k) \to (v, -1) \) as \( k \to \infty \). Due to the finiteness property of \( \mathcal{W} \), we may suppose that the sequence \( \{(x^k, f(x^k))\} \) lies entirely in some stratum \( S \in \mathcal{W} \) of dimension \( d \). Using the compactness of the Grassmannian manifold of \( d \)-dimensional subspaces of \( \mathbb{R}^n \), we may
assume that the sequence of tangent spaces $T_{(x^k, f(x^k))}S$ converges to some vector space $T$ of dimension $d$. Then the Whitney-(a) property yields that $T_{(x, f(x))}M \subset T$. By definition, for each $k \geq 1$ we have that the vector $(v^k, t^k)$ is Fréchet normal to the epigraph $\text{epi} f$ of $f$ at $(x^k, f(x^k))$; hence, it is also normal (in the classical sense) to the tangent space $T_{(x^k, f(x^k))}S$. By a standard continuity argument, the vector 

$$(v, -1) = \lim_{k \to \infty} (v^k, t^k)$$

must be normal to $T$ and a fortiori to $T_{(x, f(x))}M$. On the other hand, $T_{(x, f(x))}M$ is the vector space generated by the vector $(\phi'(t), (f \circ \phi)'(t)) \in \mathbb{R}^n \times \mathbb{R}$. Consequently, we obtain

$$\langle v, \phi'(t) \rangle = (f \circ \phi)'(t).$$

A similar argument also shows

$$\langle v, \phi'(t) \rangle = 0$$

for all $t \in (0, \epsilon)$ and all $v \in \partial^\infty(\phi(t))$.

Finally, let $c$ be the supremum of real numbers $T \in [0, 1]$ such that for all but finitely many $t \in [0, T)$, we have for all $v \in \partial f(\phi(t))$ and all $w \in \partial^\infty(\phi(t))$,

$$\langle v, \phi'(t) \rangle = (f \circ \phi)'(t) \quad \text{and} \quad \langle w, \phi'(t) \rangle = 0.$$

Then $c \geq \epsilon$. We must prove that $c = 1$. Suppose that this is not the case. Replacing the interval $[0, 1)$ by the interval $[c, 1)$ and repeating the previous argument, we find a small real number $\epsilon' > 0$ such that for all $t \in (c, c + \epsilon')$, all $v \in \partial f(\phi(t))$ and all $w \in \partial^\infty(\phi(t))$, 

$$\langle v, \phi'(t) \rangle = (f \circ \phi)'(t) \quad \text{and} \quad \langle w, \phi'(t) \rangle = 0,$$

thus contradicting the definition of $c$. The proof is complete. \hfill \square

3. Tangencies

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a continuous definable function germ. Let us begin with the following definition (see also [7]).

**Definition 3.1.** The tangency variety of $f$ (at 0) is defined as follows:

$$\Gamma(f) := \{ x \in (\mathbb{R}^n, 0) \mid \exists \lambda \in \mathbb{R} \text{ such that } \lambda x \in \partial f(x) \cup \partial(-f)(x) \}.$$

**Remark 3.1.** When $f$ is of class $C^1$ one has

$$\partial f(x) = -\partial(-f)(x) = \{ \nabla f(x) \},$$

and so

$$\Gamma(f) = \{ x \in (\mathbb{R}^n, 0) \mid \exists \lambda \in \mathbb{R} \text{ such that } \lambda x = \nabla f(x) \}.$$
By definition, it is not hard to check that $\Gamma(f)$ is a definable set. Moreover, thanks to the Fermat rule (Theorem 2.2), we can see that for any $t > 0$, the tangency variety $\Gamma(f)$ contains the set of minimizers (and minimizers) of $f$ on the sphere $S_t$; in particular, $0$ is a cluster point of $\Gamma(f)$.

Applying the Hardt triviality theorem (see [19, Theorem 4.11]) for the definable function $\Gamma(f) \to \mathbb{R}$, $x \mapsto \|x\|$, we find a constant $\epsilon > 0$ such that the restriction of this function on $\Gamma(f) \cap B_\epsilon \setminus \{0\}$ is a topological trivial fibration. Let $p$ be the number of connected components of a fiber of this restriction. Then $\Gamma(f) \cap B_\epsilon \setminus \{0\}$ has exactly $p$ connected components, say $\Gamma_1, \ldots, \Gamma_p$, and each such component is a definable set. Moreover, for all $t \in (0, \epsilon)$ and all $k = 1, \ldots, p$, the sets $\Gamma_k \cap S_t$ are connected. Corresponding to each $\Gamma_k$, let

$$f_k: (0, \epsilon) \to \mathbb{R}, \ t \mapsto f_k(t),$$

be the function defined by $f_k(t) := f(x)$, where $x \in \Gamma_k \cap S_t$.

**Lemma 3.1.** For each $\epsilon > 0$ small enough, all the functions $f_k$ are well-defined and definable.

**Proof.** Fix $k \in \{1, \ldots, p\}$ and take any $t \in (0, \epsilon)$. We will show that the restriction of $f$ on $\Gamma_k \cap S_t$ is constant. To see this, let $\phi: [0, 1] \to \mathbb{R}^n$ be a definable $C^1$-curve such that $\phi(\tau) \in \Gamma_k \cap S_t$ for all $\tau \in [0, 1]$. By definition, we have $\|\phi(\tau)\| = t$ and either $\lambda(\tau)\phi(\tau) \in \partial f(\phi(\tau))$ or $\lambda(\tau)\phi(\tau) \in \partial(-f)(\phi(\tau))$ for some $\lambda(\tau) \in \mathbb{R}$. By replacing $f$ by $-f$, if necessary, we may assume that $\lambda(\tau)\phi(\tau) \in \partial f(\phi(\tau))$. In view of Lemma 2.1 for all but finitely many $\tau \in [a, b]$, the mappings $\phi$ and $f \circ \phi$ are $C^1$-smooth at $\tau$ and satisfy

$$v \in \partial f(\phi(\tau)) \implies \langle v, \phi'(\tau) \rangle = (f \circ \phi)'(\tau).$$

Therefore

$$(f \circ \phi)'(\tau) = \langle \lambda(\tau)\phi(\tau), \phi'(\tau) \rangle = \frac{\lambda(\tau) d\|\phi(\tau)\|^2}{d\tau} = 0.$$

So $f$ is constant on the curve $\phi$.

On the other hand, since the set $\Gamma_k \cap S_t$ is connected definable, it is path connected. Hence, any two points in $\Gamma_k \cap S_t$ can be joined by a piecewise $C^1$-smooth definable curve. It follows that the restriction of $f$ on $\Gamma_k \cap S_t$ is constant and so the function $f_k$ is well-defined. Finally, by definition, $f_k$ is definable. \qed
For each $t \in (0, \epsilon)$, the sphere $S_t$ is a nonempty compact definable set. Hence, the functions

$$
\psi: (0, \epsilon) \to \mathbb{R}, \quad t \mapsto \psi(t) := \min_{x \in S_t} f(x),
$$

$$
\bar{\psi}: (0, \epsilon) \to \mathbb{R}, \quad t \mapsto \bar{\psi}(t) := \max_{x \in S_t} f(x),
$$

are well-defined and definable. The following lemma is simple but useful.

**Lemma 3.2.** For $\epsilon > 0$ small enough, the following equalities

$$
\psi(t) = \min_{k=1,\ldots,p} f_k(t) \quad \text{and} \quad \bar{\psi}(t) = \max_{k=1,\ldots,p} f_k(t)
$$

hold for all $t \in (0, \epsilon)$.

**Proof.** Applying the Curve Selection Lemma (see [19, Property 1.17]) and shrinking $\epsilon$ (if necessary), we find a definable $C^1$-curve $\phi: (0, \epsilon) \to \mathbb{R}^n$ such that for all $t \in (0, \epsilon)$,

$$
\|\phi(t)\| = t \quad \text{and} \quad (f \circ \phi)(t) = \psi(t).
$$

By Lemma 2.1, then we have for any $t \in (0, \epsilon)$,

$$
v \in \partial^\infty f(\phi(t)) \quad \Longrightarrow \quad \langle v, \phi'(t) \rangle = 0.
$$

Observe

$$
\langle \phi(t), \phi'(t) \rangle = \frac{1}{2} \frac{d}{dt} \|\phi(t)\|^2,
$$

and hence the qualification condition

$$
\partial^\infty f(\phi(t)) \cap N_{\phi(t)S_t} = \{0\}
$$

holds for all $t \in (0, \epsilon)$. Consequently, since $\phi(t)$ minimizes $f$ subject to $\|x\| = t$, applying the Fermat rule (Theorem 2.2), we deduce that $\phi(t)$ belongs to $\Gamma(f)$. Therefore,

$$
\psi(t) = \min_{x \in S_t} f(x) = \min_{x \in \Gamma(f) \cap S_t} f(x) = \min_{k=1,\ldots,p} \min_{x \in \Gamma_k \cap S_t} f(x) = \min_{k=1,\ldots,p} f_k(t).
$$

Using the same argument, we also have

$$
\bar{\psi}(t) = \max_{x \in S_t} f(x) = \max_{x \in \Gamma(f) \cap S_t} f(x) = \max_{k=1,\ldots,p} \max_{x \in \Gamma_k \cap S_t} f(x) = \max_{k=1,\ldots,p} f_k(t).
$$

The lemma is proved. \qed
4. The main result

In this section, we fix a polynomially bounded o-minimal structure on \( \mathbb{R} \). The word “definable” will mean definable in this structure.

Let \( f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0) \) be a continuous definable function germ. As in the previous section, we associate to the function \( f \) a finite number of (definable) functions \( f_1, \ldots, f_p \) of a single variable. Let

\[
K_0 := \{ k \mid f_k \text{ is constant} \}.
\]

By the Growth Dichotomy Lemma (see [19, Theorem 4.12]), we can write for each \( k \in \{1, \ldots, p\} \setminus K_0 \),

\[
f_k(t) = a_k t^{\alpha_k} + o(t^{\alpha_k}) \quad \text{as} \quad t \to 0^+,
\]

where \( a_k \in \mathbb{R}, a_k \neq 0 \), and \( \alpha_k \in \mathbb{R}, \alpha_k > 0 \). Put

\[
K_- := \{ k \notin K_0 \mid a_k < 0 \},
\]

\[
K_+ := \{ k \notin K_0 \mid a_k > 0 \}.
\]

Finally we let

\[
\text{Inv}(f) := \begin{cases}
(0, \min_{k \in K_+} \alpha_k) & \text{if } K_0 \neq \emptyset, K_- = \emptyset \text{ and } K_+ \neq \emptyset, \\
(- \min_{k \in K_-} \alpha_k, 0) & \text{if } K_0 \neq \emptyset, K_- \neq \emptyset \text{ and } K_+ = \emptyset, \\
(- \min_{k \in K_-} \alpha_k, \min_{k \in K_+} \alpha_k) & \text{if } K_- \neq \emptyset \text{ and } K_+ \neq \emptyset, \\
(\min_{k \in K_+} \alpha_k, \max_{k \in K_+} \alpha_k) & \text{if } K_0 = K_- = \emptyset \text{ and } K_+ \neq \emptyset, \\
(- \min_{k \in K_-} \alpha_k, - \max_{k \in K_-} \alpha_k) & \text{if } K_0 = K_+ = \emptyset \text{ and } K_- \neq \emptyset, \\
(0, 0) & \text{if } K_- = K_+ = \emptyset.
\end{cases}
\]

If \( \text{Inv}(f) = (a, b) \), we follow the convention that \(-\text{Inv}(f) := \text{Inv}(-f) = (-b, -a)\).

We now arrive to the main result of this paper.

**Theorem 4.1.** Let \( f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0) \) be two continuous definable function germs. If \( f \) and \( g \) are bi-Lipschitz contact equivalent then

\[
\text{Inv}(f) = \pm \text{Inv}(g).
\]

**Proof.** Since \( f \) and \( g \) are bi-Lipschitz contact equivalent, it follows from Theorem 2.1 that there exist a bi-Lipschitz homeomorphism germ \( h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0) \) and some positive constants \( c_1, c_2 \) and a sign \( \sigma \in \{ \pm 1 \} \) such that

\[
c_1 f(x) \leq \sigma (g \circ h)(x) \leq c_2 f(x) \quad \text{for all} \quad \|x\| \ll 1. \tag{1}
\]

Assume that \( \sigma = 1 \). (The case \( \sigma = -1 \) is proved similarly.) Consider the definable functions

\[
\psi_f : [0, \epsilon) \rightarrow \mathbb{R}, \quad t \mapsto \psi_f(t) := \min_{x \in S_t} f(x), \quad \overline{\psi}_f : [0, \epsilon) \rightarrow \mathbb{R}, \quad t \mapsto \overline{\psi}_f(t) := \max_{x \in S_t} f(x),
\]

\[
\psi_g : [0, \epsilon) \rightarrow \mathbb{R}, \quad t \mapsto \psi_g(t) := \min_{x \in S_t} g(x), \quad \overline{\psi}_g : [0, \epsilon) \rightarrow \mathbb{R}, \quad t \mapsto \overline{\psi}_g(t) := \max_{x \in S_t} g(x),
\]

where \( S_t = S_t(0, \epsilon) \) denotes the set of \( \epsilon \)-stips of \( f \) and \( g \) at \( x \).
where \( \epsilon \) is a positive number and small enough so that these functions are either constant or strictly monotone. Assume that we have proved the following relations:

\[
\psi_f \simeq \psi_g \quad \text{and} \quad \overline{\psi}_f \simeq \overline{\psi}_g,
\]

where \( A \simeq B \) means that \( A/B \) lies between two positive constants. These, together with Lemma 3.2, imply easily that \( \text{Inv}(f) = \text{Inv}(g) \), which is the desired conclusion.

So we are left with showing (2). We will prove the first relation; the second one is proved similarly. Indeed, if \( \psi_f \equiv 0 \), then \( \psi_g \equiv 0 \) because of (1) and there is nothing to prove. So assume that \( \psi_f \not\equiv 0 \). Since \( h \) is a bi-Lipschitz homeomorphism germ, there exists a positive constant \( L \) such that

\[
L^{-1} \| x - x' \| \leq \| h(x) - h(x') \| \leq L \| x - x' \| \quad \text{for all} \quad \| (x, x') \| \ll 1.
\]

In particular, we get

\[
L^{-1} \| x \| \leq \| h(x) \| \leq L \| x \| \quad \text{for all} \quad \| x \| \ll 1.
\]

This, together with (1), implies that for all sufficiently small \( t \geq 0 \),

\[
c_2 \psi_f(t) = c_2 \min_{x \in S_t} f(x) \geq \min_{x \in S_t} (g \circ h)(x) \geq \min_{L^{-1}t \leq \| h(x) \| \leq Lt} (g \circ h)(x) = \min_{L^{-1}t \leq \| y \| \leq Lt} g(y).
\]

Let \( \phi: [0, \epsilon) \to \mathbb{R}^n \) be a definable curve such that

\[
g(\phi(t)) = \min_{L^{-1}t \leq \| y \| \leq Lt} g(y).
\]

Reducing \( \epsilon \) if necessary, we may assume that \( \phi \) is of class \( C^1 \) and that either \( L^{-1}t < \| \phi(t) \| < Lt \) or \( \| \phi(t) \| = L^{-1}t \) or \( \| \phi(t) \| = Lt \) for all \( t \in [0, \epsilon) \).

If \( L^{-1}t < \| \phi(t) \| < Lt \), then \( \phi(t) \) is a local minimizer of the function \( g \) on the open set \( \{ y \in \mathbb{R}^n | L^{-1}t < \| y \| < Lt \} \). By the Fermat rule (Theorem 2.2), we get \( 0 \in \partial g(\phi(t)) \). This, together with Lemma 2.1, implies that for all but finitely many \( t \in [0, \epsilon) \),

\[
(g \circ \phi)'(t) = \langle 0, \phi'(t) \rangle = 0.
\]

Consequently, \( (g \circ \phi)(t) = (g \circ \phi)(0) = 0 \) for all \( t \in [0, \epsilon) \), which is a contradiction.

Therefore, we have \( \| \phi(t) \| \equiv rt \), where either \( r = L^{-1} \) or \( r = L \). Moreover, it holds that

\[
\min_{L^{-1}t \leq \| y \| \leq Lt} g(y) = \min_{y \in S_{rt}} g(y) = \psi_g(rt) \simeq \psi_g(t).
\]

Combining this with (3) and (4), we can find a constant \( c > 0 \) such that

\[
c \psi_f(t) \geq \psi_g(t) \quad \text{for all} \quad 0 \leq t \ll 1.
\]

Applying the above argument again and using the first inequality in (1), we also obtain

\[
c' \psi_g(t) \geq \psi_f(t) \quad \text{for all} \quad 0 \leq t \ll 1
\]

for some \( c' > 0 \). Therefore, \( \psi_f \simeq \psi_g \). \( \square \)
Remark 4.1. (i) Notice that, in the above proof, we do not assume that the homeomorphism \( h \) is definable.

(ii) When \( f \) is of class \( C^1 \), it is not hard to see that the exponents \( \alpha_k \) belong to the set of characteristic exponents defined by Kurdyka, Mostowski, and Parusiński [11], and moreover, the latter set is preserved by bi-Lipschitz homeomorphisms (see [9]). On the other hand, we do not know whether the set of the exponents \( \alpha_k \) is an invariant of the bi-Lipschitz contact equivalence or not.

We conclude the paper with some examples illustrating our results. For simplicity we consider the case where \( f \) is a \( C^1 \)-function in two variables \((x,y) \in \mathbb{R}^2\). By definition, then

\[
\Gamma(f) := \left\{ (x,y) \in \mathbb{R}^2 \mid y \frac{\partial f}{\partial x} - x \frac{\partial f}{\partial y} = 0 \right\}.
\]

In view of Theorem 4.1 the four functions given below are not bi-Lipschitz contact equivalent to each other.

Example 4.1. (i) Let \( f(x,y) := x^3 + y^6 \). The tangency variety \( \Gamma(f) \) is given by the equation:

\[
3x^2 y - 6xy^5 = 0.
\]

Hence, for \( \epsilon > 0 \) the set \( (\Gamma(f) \cap B_\epsilon) \setminus \{0\} \) has six connected components:

\[
\begin{align*}
\Gamma_{\pm 1} & := \{(0, \pm t) \mid 0 < t < \epsilon\}, \\
\Gamma_{\pm 2} & := \{(2t^4, \pm t) \mid 0 < t < \epsilon\}, \\
\Gamma_{\pm 3} & := \{(-t, 0) \mid 0 < t < \epsilon\}.
\end{align*}
\]

Consequently,

\[
\begin{align*}
f|_{\Gamma_{\pm 1}} & = t^6, \\
f|_{\Gamma_{\pm 2}} & = t^6 + 8t^{12}, \\
f|_{\Gamma_{\pm 3}} & = \pm t^3.
\end{align*}
\]

It follows that \( K_0 = \emptyset, K_- = \{-3\}, K_+ = \{\pm 1, \pm 2, 3\} \) and \( \text{Inv}(f) = \{-3, 3\} \).

(ii) Let \( f(x,y) := (x^2 - y^3)^2 \). The tangency variety \( \Gamma(f) \) is given by the equation:

\[
2yx(3y^2 - 2)(x^2 - y^3) = 0.
\]

Hence, for \( 0 < \epsilon < \frac{2}{3} \), the set \( (\Gamma(f) \cap B_\epsilon) \setminus \{0\} \) has six connected components:

\[
\begin{align*}
\Gamma_{\pm 1} & := \{(0, \pm t) \mid 0 < t < \epsilon\}, \\
\Gamma_{\pm 2} & := \{(-t^3, t^2) \mid 0 < t < \epsilon\}, \\
\Gamma_{\pm 3} & := \{(\pm t, 0) \mid 0 < t < \epsilon\}.
\end{align*}
\]
Consequently,

\[ f|_{\Gamma_{\pm 1}} = t^6, \]
\[ f|_{\Gamma_{\pm 2}} = 0, \]
\[ f|_{\Gamma_{\pm 3}} = t^4. \]

It follows that \( K_0 = \{ \pm 2 \}, K_- = \emptyset, K_+ = \{ \pm 1, \pm 3 \} \) and \( \text{Inv}(f) = \{0, 4\} \).

(iii) Let \( f(x, y) := x^2 + y^4 \). The tangency variety \( \Gamma(f) \) is given by the equation:

\[ 2xy - 4xy^4 = 0. \]

Hence, for \( 0 < \epsilon < \sqrt{\frac{1}{3}} \), the set \((\Gamma(f) \cap \mathbb{B}_\epsilon) \setminus \{0\}\) has four connected components:

\[ \Gamma_{\pm 1} := \{(0, \pm t) \mid 0 < t < \epsilon\}, \]
\[ \Gamma_{\pm 2} := \{(\pm t, 0) \mid 0 < t < \epsilon\}. \]

Consequently,

\[ f|_{\Gamma_{\pm 1}} = t^4, \]
\[ f|_{\Gamma_{\pm 2}} = t^2. \]

It follows that \( K_0 = K_- = \emptyset, K_+ = \{ \pm 1, \pm 2 \} \) and \( \text{Inv}(f) = \{2, 4\} \).

(iv) Let \( f(x, y) := -x^2 - 2y^6 \). The tangency variety \( \Gamma(f) \) is given by the equation:

\[ -2xy + 6xy^5 = 0. \]

Hence, for \( 0 < \epsilon < \sqrt[4]{\frac{1}{6}} \), the set \((\Gamma(f) \cap \mathbb{B}_\epsilon) \setminus \{0\}\) has four connected components:

\[ \Gamma_{\pm 1} := \{(0, \pm t) \mid 0 < t < \epsilon\}, \]
\[ \Gamma_{\pm 2} := \{(\pm t, 0) \mid 0 < t < \epsilon\}. \]

Consequently,

\[ f|_{\Gamma_{\pm 1}} = -2t^6, \]
\[ f|_{\Gamma_{\pm 2}} = -t^2. \]

It follows that \( K_0 = K_+ = \emptyset, K_- = \{ \pm 1, \pm 2 \} \) and \( \text{Inv}(f) = \{-2, -6\} \).

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