FIRST VARIATION OF THE LOG ENTROPY FUNCTIONAL ALONG THE RICCI FLOW

Junfang Li

ABSTRACT. In this note, we establish the first variation formula of the adjusted log entropy functional $\mathcal{Y}_a$ introduced by Ye in [14]. As a direct consequence, we also obtain the monotonicity of $\mathcal{Y}_a$ along the Ricci flow.

Various entropy functionals play crucial role in the singularity analysis of Ricci flow. Let $(M^n, g(t))$ be a smooth family of Riemannian metrics on a closed manifold $M^n$ and suppose $g(t)$ is a solution of Hamilton’s Ricci flow equation. In a recent interesting paper [14], R. Ye introduced a new entropy functional, the adjusted log entropy, as follows

(1.1)\[ \mathcal{Y}_a(g, u, t) = -\int_M u^2 \ln u^2 \text{dvol} + \frac{n}{2} \ln \left( \int_M (|\nabla u|^2 + \frac{R}{4} u^2) \text{dvol} + a \right) + 4at, \]

where the positive function $u \in W^{1,2}(M^n)$ satisfies $\int_M (|\nabla u|^2 + \frac{R}{4} u^2) \text{dvol} + a > 0$, and $R$ denotes the scalar curvature of the metric at time $t$.

The log entropy functional can be used to prove uniform logarithmic Sobolev inequalities along the Ricci flow which also leads to uniform Sobolev inequalities, see Ye’s recent series of papers, [13], [14], etc, and Zhang [15]. This new entropy functional of Ye shares a similar important feature with Perelman’s entropy functionals. Namely, it is nondecreasing under the following coupled system of Ricci flow,

(1.2)\[ \begin{align*}
\frac{\partial}{\partial t} g_{ij} &= -2R_{ij} \\
\frac{\partial}{\partial t} u &= -\Delta u - \frac{|\nabla u|^2}{u} + \frac{R}{2} u.
\end{align*} \]

The first evolution equation is the Ricci flow equation. The second equation ensures $\int_M u^2 d\mu_g = 1$ to be preserved by the Ricci flow. Notice that, if we define $u = e^{-f}$, then the second equation is equivalent to Perelman’s equation $\frac{\partial}{\partial t} f = -\Delta f + |\nabla f|^2 - R$ which instead preserves $\int_M e^{-f} d\mu_g = 1$.

The following statement is obtained by Ye, see Theorem 3.1 in [14].

Research of the author is supported in part by a CRM fellowship.
Assume $a > -\lambda_0(g)$. Then $\mathcal{Y}_a \equiv \mathcal{Y}_a(g(t), u(t), t)$ is nondecreasing. Indeed, we have

$$
\frac{d}{dt} \mathcal{Y}_a = \frac{n}{4\omega} \int_M \left[ \text{Ric} - 2 \frac{\nabla^2 u}{u} + 2 \frac{\nabla u \otimes \nabla u}{u^2} - \frac{4\omega}{n} g \right] u^2 d\mu_g,
$$

where $\omega = \omega(t) = a + \int_M (|\nabla u|^2 + \frac{R}{4} u^2) d\mu_g, \text{ which is positive.}$

Ye used a minimizing procedure and the monotonicity formula of $\mathcal{W}$-entropy of Perelman to show the monotonicity formula of (1.3), see Lemma 4.1, 4.2 in [14].

One question may be interesting is: what is the precise first variation of the functional $\mathcal{Y}_a$? In this short note, instead of giving an inequality for the first variation, we derive a formula of the first variation itself for $\mathcal{Y}_a$. Consequently, we also obtain the monotonicity of $\mathcal{Y}_a$ along the Ricci flow. The proof is a direct approach without appealing to the minimizing procedure and the monotonicity of $\mathcal{W}$ functional. In Remark 1.3, we will show that the first variation formula (1.4) we obtained is equivalent to the righthand side of Ye’s inequality (1.3), see Remark 1.3.

We adapt the notations in [14]. If we let $u = e^{-f}$, then $\mathcal{F} = \int_M (R + |\nabla f|^2) e^{-f} d\mu_g = 4 \int_M (|\nabla u|^2 + \frac{R}{4} u^2) d\mu_g$ which implies $4\omega = 4a + \mathcal{F}$. We note that $\mathcal{F}$ is one of the entropy functionals introduced by Perelman [12].

Now we introduce the main theorem of this paper.

**Theorem 1.1.** Assume $a > -\lambda_0(g) (=- \frac{1}{4} \lambda_0(\mathcal{F}))$. Then $\mathcal{Y}_a \equiv \mathcal{Y}_a(g(t), u(t), t)$ is nondecreasing. Indeed, we have

$$
\frac{d}{dt} \mathcal{Y}_a = \frac{n}{4\omega} \int_M \left[ \text{Ric} - 2 \frac{\nabla^2 u}{u} + 2 \frac{\nabla u \otimes \nabla u}{u^2} - \frac{4(\omega - a)}{n} g \right] u^2 d\mu_g + \frac{4a^2}{\omega},
$$

where $\omega = a + \int_M (|\nabla u|^2 + \frac{R}{4} u^2) d\mu_g, \text{ which is positive. The monotonicity is strict, unless the manifold is a gradient shrinking soliton with positive first eigenvalue } \lambda_0(g) > 0 \text{ and also } a = 0.$

**Proof.** By the notations of $u$ and $f$, we observe that $\mathcal{Y}_a = -\mathcal{S} + \frac{n}{2} \ln \frac{1}{4} (\mathcal{F} + 4a) + 4at$, where $\mathcal{S} = -\int_M fe^{-f} d\mu_g$ is the differential Shannon Entropy. Along the coupled system of Ricci flow, we have

$$
\frac{d}{dt} \mathcal{Y}_a = \frac{n}{2(\mathcal{F} + 4a)} \left[ \frac{d}{dt} \mathcal{F} + \frac{n}{2} (4a - \mathcal{F})(4a + \mathcal{F}) \right]
$$

$$
= \frac{n}{2(\mathcal{F} + 4a)} \left[ \int_M |\nabla_i \nabla_j f|^2 e^{-f} d\mu_g + \frac{1}{n} (4a - \mathcal{F})(4a + \mathcal{F}) \right]
$$

$$
= \frac{2n}{2} \left[ \int_M |\nabla_i \nabla_j f|^2 - \frac{\mathcal{F}}{n} R g_{ij}^2 e^{-f} d\mu_g + \frac{(4a)^2}{n} \right] \geq 0.
$$
Change of variables from $f$ to $u$ completes the proof.

In the above, we have used the properties that $\frac{d}{dt} S = F$ and $\frac{d}{dt} F = 2 \int |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu_g$ under the coupled system (1.2) which can be proved by direct computations. Related references can be found in [1], [2], or original paper of Perelman [12] and [11], [6]. Recall that, $\lambda_0(g)$ denotes the first eigenvalue of $-\Delta + \frac{R}{4}$ in [14], i.e.

$$\lambda_0(g) = \inf \int_M (|\nabla u|^2 + \frac{R}{4} u^2) d\mu_g,$$

where the infimum is taken over all $u$ satisfying $\int_M u^2 d\mu_g = 1$. Hence, the second property ensures that $\lambda_0(g(t))$ is nondecreasing and $\int_M (|\nabla u|^2 + \frac{R}{4} u^2) d\mu_g + a > 0$ for all time $t$.

Instead of working on $W$-functional, dealing directly with $S$ and $F$-functionals makes the proof rather elementary. Below a few remarks are in order.

**Remark 1.2.** From the first variation formula, we know precisely when the monotonicity is strict. Only when $a = 0$ the monotonicity can be non-strict. In this case, we know the manifold must be a shrinking gradient Ricci soliton with $\lambda_0(g) > 0$.

**Remark 1.3.** Simple computations yields that (1.4) we obtained is equivalent to the righthand side of Ye’s inequality (1.3).

(1.6)

$$\frac{d}{dt} Y_a = \frac{n}{\omega} \left[ \int_M |R_{ij} + \nabla_i \nabla_j f - \frac{4(\omega - a)}{n} g_{ij}|^2 e^{-f} d\mu_g + \frac{(4a)^2}{n} \right]$$

$$= \frac{n}{\omega} \left[ \int_M |R_{ij} + \nabla_i \nabla_j f - \frac{4\omega}{n} g_{ij}|^2 e^{-f} d\mu_g + 2 \int_M g^{ij} (R_{ij} + \nabla_i \nabla_j f - \frac{4\omega}{n} g_{ij}) \frac{4a}{n} e^{-f} d\mu_g + \frac{(4a)^2}{n} + \frac{(4a)^2}{n} \right]$$

The splitting sum we had in (1.4) clearly shows that when $a \neq 0$, the monotonicity is strict.

**Remark 1.4.** There are intensive study on various entropy functionals which are related to or motivated by Perelman’s entropy functionals in [12]. For example, entropy functionals for linear heat equations in [10], expanding $W$-entropy in [3], Log entropy in [14], entropy on fiber bundles in [9], entropy on an extended Ricci flow system in [8], and various generalized entropy functionals in [7] and [5]. Also we note that there is an interesting entropy functional on the evolution equation for $p$-harmonic functions appeared in [4] recently. All these entropies share one common feature : they have monotonicity properties under geometric evolution equations and the monotonicity is strict unless the metric is a soliton.
Acknowledgement. I would like to thank professor B. Chow for discussions.

REFERENCES

[1] Chow, Bennett; Lu, Peng; Ni, Lei. Hamilton’s Ricci flow. Graduate Studies in Mathematics, 77. American Mathematical Society, Providence, RI; Science Press, New York, 2006.

[2] Chow, Bennett; Chu, Sun-Chin; Glickenstein, David; Guenther, Christine; Isenberg, Jim; Ivey, Thomas; Knopf, Dan; Lu, Peng; Luo, Feng; Ni, Lei. The Ricci flow: techniques and applications. Part I. Geometric aspects. Mathematical Surveys and Monographs, 135. American Mathematical Society, Providence, RI, 2007.

[3] Feldman, Mikhail; Ilmanen, Tom; Ni, Lei. Entropy and reduced distance for Ricci expanders. J. Geom. Anal. 15 (2005), no. 1, 49-62.

[4] Kotschwar, Brett; Ni, Lei. Local gradient estimates of p-harmonic functions, 1/H flow, and an entropy formula. 2007, arXiv:0711.2291v1 [math.AP]

[5] Kuang, Shilong; Zhang, Qi S. A gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow. 2006, arXiv:math/0611298

[6] Li, Junfang. Geometric evolution equations and p-harmonic theory with applications in differential geometry, PhD thesis, 2006.

[7] Li, Junfang. Eigenvalues and energy functionals with monotonicity formulae under Ricci flow. Mathematische Annalen, 338, 2007, 1432-1807.

[8] List, Bernhard. Evolution of an extended Ricci flow system. PhD thesis, 2005.

[9] Lott, John. Dimension reduction and the long-time behaviour of Ricci flow. 2007, arXiv:0711.4063

[10] Ni, Lei. The Entropy Formula for Linear Heat Equation. The Journal of Geometric Analysis, 14, no 1, 2004.

[11] Ni, Lei. Addenda to “The Entropy Formula for Linear Heat Equation”. The Journal of Geometric Analysis, 14, no 2, 2004.

[12] Perelman, Grisha. The entropy formula for the Ricci flow and its geometric applications. arXiv:math.DG/0211159.

[13] Ye, Rugang. The Logarithmic Sobolev Inequality Along The Ricci Flow. 2007, arXiv:0707.2424

[14] Ye, Rugang. The Log Entropy Functional Along the Ricci Flow. 2007, arXiv:0708.2008v3

[15] Zhang, Qi S. A uniform Sobolev inequality under Ricci flow. 2007, arXiv:0706.1594

Department of Mathematics, McGill University, Montreal, Quebec. H3A 2K6, Canada.

E-mail address: jli@math.mcgill.ca