Interpreting a conformally flat 
pure radiation space-time

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ABSTRACT

A physical interpretation is presented of the general class of conformally flat pure radiation metrics that has recently been identified by Edgar and Ludwig. It is shown that, at least in the weak field limit, successive wave surfaces can be represented as null (half) hyperplanes rolled around a two-dimensional null cone. In the impulsive limit, the solution reduces to a \textit{pp}-wave whose direction of propagation depends on retarded time. In the general case, there is a coordinate singularity which corresponds to an envelope of the wave surfaces. The global structure is discussed and a possible vacuum extension through the envelope is proposed.

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1. Introduction

Vacuum and Einstein–Maxwell space-times of Kundt’s class, which admit a non-expanding shear-free and twist-free null geodesic congruence, are well known [1]. A null Einstein–Maxwell field can, of course, often be reinterpreted as some other form of pure radiation. However, Wils [2] has found a conformally flat pure radiation space-time of this class that cannot be interpreted as a null electromagnetic, scalar or neutrino field. This solution was subsequently generalised by Koutras and McIntosh [3] who expressed it in the form

$$ds^2 = 2(ax + b)du dv - 2av du dx + [2f(u)(ax + b)(x^2 + y^2) - a^2 v^2]du^2 - dx^2 - dy^2.$$ 

For this, the only non-zero component of the curvature tensor is $\Phi_{22} = 2f(u)/(ax + b)$, so it represents a conformally flat space-time with null radiation. When $a = 0, b = 1$, this is just the familiar plane wave metric. When $a = 1, b = 0$, it is the solution of Wils [2].

In fact, this still does not describe the full class of conformally flat pure radiation metrics. The complete family has been given by Edgar and Ludwig [4] who derived the general solution of this type as a member of Kundt’s class. In a subsequent paper [5], they re-derived the general solution using the GHP formalism. This general metric (excluding the plane wave limit) can be written in the form

$$ds^2 = 2x du dv - 2v du dx + [2f(u)x(x^2 + y^2 + g(u)y + h(u)) - v^2]du^2 - dx^2 - dy^2. \quad (1)$$

This can be shown to include the Koutras–McIntosh solution above (with $a \neq 0$) after a further coordinate transformation. When $g(u) = 0$ and $h(u) = 0$, it is just the solution of Wils [2]. Its explicit conformal factor was determined in [6].

This family of metrics is of particular interest to those who work on the computer-aided classification of exact solutions. As shown in [3], in general, they contain no invariants or Killing or homothetic vectors, and thus provide an interesting exceptional case for classification. It has been conjectured that any space-time can be uniquely characterized using the derivatives of the curvature tensor in which no higher than the fourth derivative is ever required. An invariant classification of the above space-times has recently been provided by Skea [7]. In this, he has shown that, to distinguish the Wils metric within the general Edgar–Ludwig solution, it is necessary to go as far as but no further than the fourth derivative of the curvature tensor, thus supporting the above conjecture.

To our knowledge, a physical interpretation of these conformally flat pure radiation metrics, and indeed of the larger class of Kundt (non-pp)-waves, has not previously been attempted. The purpose of this paper is to make an initial contribution towards understanding the physical properties of these space-times. We consider here only the line element (1) which excludes the very familiar plane-wave solutions.
2. Initial observations

It may immediately be observed from (1) that the surfaces \( u = u_0 = \text{const.} \) are planes spanned by “cartesian” coordinates \( x \) and \( y \). In addition, there is a coordinate singularity when \( x = 0 \). In the following sections, we will investigate the nature of this singularity and the family of successive planes.

Since the solution contains an arbitrary “amplitude” function \( f(u) \) which represents a deviation from a flat background, it is appropriate to consider the cases in which \( f \) is small, or in which it describes an impulsive, step or sandwich wave. Of course these solutions represent matter propagating with the speed of light rather than gravitational or other forms of wave. However the terminology of “waves” is natural and will be adopted, and the surfaces \( u = u_0 = \text{const.} \) will be referred to as “wave surfaces”.

3. The solution in NP notation

The metric (1) can be expressed in the contravariant form

\[
g^{\mu\nu} = \begin{pmatrix}
0 & \frac{1}{x} & 0 & 0 \\
\frac{1}{x} & -\frac{2}{x}H & -\frac{v}{x} & 0 \\
0 & -\frac{v}{x} & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]  

(2)

where \((x^1, x^2, x^3, x^4) = (u, v, x, y)\) and \( H(u, x, y) = f(u)(x^2 + y^2 + g(u)y + h(u)) \). Using the Newman–Penrose formalism, a tetrad can be chosen such that

\[
D = \partial_v, \quad \Delta = \frac{1}{x}(\partial_u - H\partial_v - v\partial_x), \quad \delta = \frac{1}{\sqrt{2}}(\partial_x - i\partial_y).
\]  

(3)

The only non-zero NP quantities are then

\[
\tau = -\frac{1}{\sqrt{2}x}, \quad \nu = \frac{f(u)}{\sqrt{2}x}[2(x + iy) + ig(u)], \quad \Phi_{22} = \frac{2f(u)}{x}.
\]  

(4)

Since \( \kappa = \rho = \sigma = 0 \), it follows that the congruence tangent to the null radiation is geodesic with zero expansion, twist and shear (i.e. the metric is of Kundt’s class). The tetrad that has been chosen is parallelly propagated along the congruence. However, the fact that \( \tau \) is nonzero indicates that the congruence rotates about some direction perpendicular to that in which it propagates.

Since \( \Phi_{22} \) is the only non-zero component of the curvature tensor, the scalar invariants all vanish, and so \( x = 0 \) cannot correspond to a scalar curvature singularity. Moreover, on rescaling the tetrad by putting \( \tilde{\ell}^\mu = (1/\sqrt{x})\ell^\mu, \tilde{n}^\mu = \sqrt{x}n^\mu \), the non-zero Ricci tensor component is given by \( \tilde{\Phi}_{22} = 2f(u) \). If \( f \) is non-zero everywhere, a further rescaling can be used to make it equal to unity, but since we wish to include the possibilities of impulsive, step or sandwich waves, we will here retain \( f(u) \) as an arbitrary amplitude function.
4. The source of the solution

Let us now consider the possible sources of these space-times. These contain pure radiation, which can always be considered phenomenologically as a “null fluid” or “null dust”. However, pure radiation solutions can sometimes be interpreted as representing a null electromagnetic field, a massless scalar field or a neutrino field. In fact, none of these interpretations are possible for this class of space-times. To demonstrate this, we first note that the only non-zero component of the Ricci tensor is $\Phi_{22}$. In this case, the Einstein–Maxwell equations can only be satisfied if a function $\Phi_2$ can be found such that

$$D\Phi_2 = (\rho - 2\epsilon)\Phi_2, \quad \delta\Phi_2 = (\tau - 2\beta)\Phi_2, \quad 8\pi\Phi_2\Phi_2' = \Phi_{22}.$$

Alternatively, the space-time will admit a massless scalar field $\varphi$ if

$$D\varphi = \delta\varphi = \bar{\delta}\varphi = 0, \quad \Delta\varphi\Delta\bar{\varphi} = \Phi_{22}.$$

Finally, the space-time will admit a (two-component) massless neutrino interpretation if a function $\phi$ can be found such that

$$D\phi = (\rho - \epsilon)\phi, \quad \delta\phi = (\tau - \beta)\phi, \quad 8\pi i[\phi\Delta\bar{\phi} - \bar{\phi}\Delta\phi + (\bar{\gamma} - \gamma)\phi\bar{\phi}] = \Phi_{22}.$$

With the non-zero NP quantities (4), it can be shown that none of these sets of equations can be satisfied. This is consistent with the fact that the Wils metric [2] was demonstrated not to be a null electromagnetic, scalar or neutrino field.

We must thus conclude that these space-times represent some form of “null fluid” which must be composed of some form of incoherent null radiation.

5. Wave surfaces in the weak field limit

Let us now investigate the cases in which the arbitrary “amplitude” function $f(u)$ is small, or in which it represents an impulsive, step or sandwich wave. In the latter cases, $f(u)$ is zero ahead of the wave where the space-time is flat and the line element (1) reduces to the form

$$ds^2 = 2xdu dv - 2v du dx - v^2 du^2 - dx^2 - dy^2. \quad (6)$$

This is the background into which the step or impulsive wave propagates. To determine the geometry of the wavefront, we need to express this in standard cartesian coordinates and then describe the front $u = u_0 = \text{const.}$ in these coordinates. This would give a clear geometrical and physical interpretation. When considering the alternative case in which $f$ is small, the metric (1) represents a small perturbation of Minkowski space, and the wave surfaces can similarly be described by considering the null hypersurfaces $u = u_0$ in cartesian coordinates.

The necessary step can be achieved using the transformation

$$U = (2x + uv)u \quad \Leftrightarrow \quad u = \frac{X \mp \sqrt{X^2 - UV}}{V}$$
$$V = v \quad \Rightarrow \quad v = V$$
$$X = x + uv \quad \Rightarrow \quad x = \pm \sqrt{X^2 - UV}$$
$$Y = y \quad \Rightarrow \quad y = Y. \quad (7)$$
This takes the background (Minkowski) part (6) of the metric (1) to the obviously flat form
\[ ds^2 = dU \, dV - dX^2 - dY^2. \]

The surfaces \( x = \text{const.} \) can be seen, in this background, to be hyperboloidal, degenerating to a 2-dimensional null cone \( X^2 = UV \) when \( x = 0 \).

It can then be seen that the null surface \( u = u_0 = \text{const.} \) is given by
\[ (1 + u_0^2)T - 2u_0X - (1 - u_0^2)Z = 0, \]
where \( U = T - Z \) and \( V = T + Z \). These surfaces are a family of null hyperplanes whose orientation varies for different values of \( u_0 \). Putting
\[ \sin \alpha = \frac{2u_0}{1 + u_0^2}, \quad \text{or equivalently} \quad u_0 = \tan \frac{\alpha}{2}, \]
the corresponding null normal \( N^\mu \) is given using coordinates \((T, X, Y, Z)\) by
\[ N^\mu = (1, \sin \alpha, 0, \cos \alpha). \]

This clearly demonstrates that, at any time, successive wave surfaces \( u = u_0 \) are rotated about the \( Y \) axis as \( u_0 \) increases from \(-\infty\) to \(+\infty\) (or as \( \alpha \) goes from \(-\pi\) to \(+\pi\)). The rotation of these planes for different values of \( u_0 \) is consistent with the non-zero value of \( \tau \) for these metrics.

This basic geometrical structure of the background enables us to interpret the behaviour of impulsive, shock or sandwich waves in which the profile function \( f(u) \) takes appropriate forms. In these cases \( f \) vanishes for all \( u \) up to some constant \( u_0 \) representing the wavefront. The particular value of \( u_0 \) determines the orientation of the initial wavefront in the Minkowski background. Alternatively, when \( f \) is small, successive wave surfaces can be seen to rotate relative to the approximate Minkowski background.

6. Global structure in the weak field limit

From the transformation (7) it is obvious that, in the weak field limit, the coordinate singularity at \( x = 0 \) occurs on the hypersurface \( T^2 = X^2 + Z^2 \) in Minkowski space-time. On suppressing the \( Y \) coordinate, which may be arbitrary, this is a two-dimensional null cone. In 3-dimensional space, it is a cylinder with radius expanding at the speed of light for \( T > 0 \). (For a complete space-time, we must also include the case when \( T < 0 \) in which the cylinder is contracting.)

The plane wave surfaces \( u = u_0 \) are spanned by \( x \) and \( y \). With (9), these hyperplanes (8) are given by \( T - \sin \alpha X - \cos \alpha Z = 0 \) and are tangent to the null cone or expanding cylinder. For successive wave surfaces with different values of \( \alpha = 2 \arctan u_0 \), the corresponding null hyperplane rolls around the surface of this 2-dimensional null cone as \( \alpha \) increases from \(-\pi\) to \(\pi\) (i.e. \( u_0 \) goes from \(-\infty\) to \(\infty\)).
Of course, the metric (1) is strictly only a local solution of Einstein’s equations. Naturally, we want to interpret it over as wide a range of coordinates as possible. With the above interpretation, from any point in space at any time $T$, there are two planes that are tangent to the expanding null cylinder. But the pure radiation is always directed perpendicular to the planes and, if two distinct plane wave surfaces pass through a point, there will exist radiation with two null components. This is clearly ruled out by our initial assumptions (only $\Phi_{22}$ non-zero). Thus, the plane wave surfaces must be restricted in some way.

In fact, such a restriction is automatically achieved by imposing the condition that $x \geq 0$. From (7) we see that, on a given wave surface $u = u_0$, we have $X - u_0(T + Z) = x$. Substituting $T$ using (8) gives

$$\cos \alpha X - \sin \alpha Z = x.$$  \hspace{1cm} (10)

This is valid on any surface at any time $T$. Imposing the natural condition that $x \geq 0$ effectively restricts the wave surface to a half of each null plane.

From now on a wave surface will be considered as the half-plane satisfying (8) with this restriction. In this case, for any point outside the cone at any time there exists only one wave surface $u = u_0 \in (-\infty, \infty)$ passing through it, thus resolving the physical problem mentioned above. The entire space outside the null cone is now covered exactly once. The wave surfaces for $T$ and $Y$ constant are illustrated in figure 1. The restriction $x \in [0, \infty)$ also resolves the mathematical ambiguity in the signs in the right hand side of (7) in which upper signs only are now assumed. This ambiguity corresponds to the invariance of the metric (6) under the reflections $x \to -x$, $v \to -v$. To retain this invariance in the full metric (1) we can also transform $f \to -f$ as is required to maintain the sign of $\Phi_{22}$.

The wave surface (8), $u = u_0$ with $x \geq 0$, is tangent to the null cone or expanding cylinder as illustrated in figure 2. Any half-plane wave surface is bounded by the line $x = 0$ where it touches the null cone, i.e. on $X = \tan \alpha Z$, $Y$ arbitrary. For successive wave surfaces with different values of $\alpha = 2 \arctan u_0$, the corresponding null half-hyperplane is rotated around the surface of this 2-dimensional null cone.

This analysis of successive wave surfaces as hyperplanes rolling over a cone with only half of each plane considered is applicable in the weak field limit also to the family of (non-pp-wave) type N solutions of Kundt’s class$^1$. For the conformally flat solutions, this geometrical construction does not distinguish between the Wils solution and the more general Edgar–Ludwig solution.

7. The impulsive limit

Let us now consider the case of an impulsive wave representing a thin shell of null matter on an arbitrary wave surface $u = u_0$ in the space-time described by (1). Putting

\hspace{1cm}  \footnote{Note added in proof: For a general discussion of constructions of this type see [10].}
\[ f(u) = c\delta(u - u_0) \] and using the transformation (7), we obtain

\[
ds^2 = dT^2 - dX^2 - dY^2 - dZ^2 \\
+ \mathcal{A} \delta(T - \sin \alpha X - \cos \alpha Z)(dT - \sin \alpha dX - \cos \alpha dZ)^2
\]

(11)

where the amplitude function \( \mathcal{A} \) is given by

\[
\mathcal{A} = c(1 + u_0^2)[(\cos \alpha X - \sin \alpha Z)^2 + Y^2 + g(u_0)Y + h(u_0)]
\]

in which \( u_0 \) is related to \( \alpha \) by (9).

In this particular case, there is no ambiguity in extending the wave surface \( u = u_0 \) to the entire hyperplane (8). The restriction \( x \geq 0 \) need not be applied. However, it may be recalled that \( \Phi_{22} = 2f(u)/x \), so the natural extension results in a negative energy density over half of the extended wave surface. An alternative extension in which the additional half-plane is simply a reflection of the original so that \( \Phi_{22} = 2f(u)/|x| \) may also be considered.

The resulting complete metric (11) is clearly an impulsive pp-wave propagating in the direction \( \sin \alpha X + \cos \alpha Z \) corresponding to \( N^\mu \) in a Minkowski background. This reduction to the familiar pp-wave is as expected since, at any time, the wavefront is plane. It may also be noted that this reduction is similar to the reduction of vacuum type N Kundt waves to pp-waves in the impulsive limit [8].

Comparing the family of such impulsive solutions having different values for \( u_0 \), the relative rotations of the different wave surfaces is clearly illustrated. For example, when \( u_0 = 0 \), the impulse propagates in the \( Z \) direction while, for \( u_0 = \pm 1 \), it propagates in the positive or negative \( X \) directions. It can further be seen that the two additional arbitrary functions \( g(u) \) and \( h(u) \) play the role of free parameters in the amplitude function \( \mathcal{A} \) on different wave surfaces. Although these may be removed by a coordinate transformation on any particular null hypersurface \( u = u_0 \), they represent a non-trivial freedom in the full solution given by (1).

8. The coordinate singularity

We now consider the character of the singularity at \( x = 0 \) in more detail. This is clearly not a curvature singularity as the polynomial curvature scalars all vanish. According to the classification scheme of Ellis and Schmidt [9], it may be described as a quasi-regular singularity. In the weak field limit, we have seen that the singular surface \( x = 0 \) is a 2-dimensional null cone. It is evident that it can be understood as an envelope of the wave surfaces (see figure 1).

The fact that this is merely a coordinate singularity may be confirmed by noting that on applying the transformation (7) to the exact line element (1), we obtain

\[
ds^2 = dT^2 - dX^2 - dY^2 - dZ^2 + \mathcal{B} \left[(1 + u^2)dt - 2udx - (1 - u^2)d\zeta\right]^2
\]

(12)
where
\[ u(T, X, Z) = \frac{X - \sqrt{X^2 + Z^2 - T^2}}{T + Z} \]
and
\[ B = f(u) \frac{X^2 + Y^2 + Z^2 - T^2 + g(u)Y + h(u)}{2\sqrt{X^2 + Z^2 - T^2}}. \]
It may be observed that the line element (12) is exactly of the Kerr–Schild form, and that this is non-singular (\( \det g_{\mu\nu} = -1 \)) everywhere even on \( x = \sqrt{X^2 + Z^2 - T^2} = 0 \). It can also be seen that (12) reduces to (11) in the impulsive limit.

9. A possible extension of the space-time

In interpreting the solution (1) we made the restriction \( x \geq 0 \) with upper signs taken in (7). This was necessary for its physical interpretation as a pure radiation field. This restricted the wave surfaces to a family of half planes. In this way, we have interpreted a “local” solution up to its natural boundary on the coordinate singularity at \( x = 0 \). We now consider the possibility of extending the solution through this hypersurface.

It has been argued that the coordinate singularity arises as an envelope of the null hyperplanes rolling around a 2-dimensional null cone, and that the metric (1) applies only to the exterior region. Moreover, the envelope \( x = 0 \) is a null hypersurface. In these circumstances, it is possible to include the appropriate section of Minkowski space inside the cone. In this case, a complete solution (for \( T > 0 \)) can be constructed which represents outgoing null matter leaving a cylindrical void in the region \( X^2 + Z^2 < T^2 \). Such an extension, of course, is not unique.

10. Further remarks

Since the coordinate range in (1) has been restricted such that \( x \in [0, \infty) \), and since \( \Phi_{22} = 2f(u)/x \), to ensure that the energy density is non-negative everywhere it is necessary that the arbitrary function \( f(u) \) is chosen such that \( f(u) \geq 0 \).

Let us also consider how the radiation would appear to an observer at a fixed point \( (X, Y, Z) \) in space-time. Assuming that this is initially \( (T = T_0 > 0) \) way outside the null cone and that \( f > 0 \) for all \( u \), the radiation would appear to come from a direction perpendicular to the null hyperplane \( u = u_0 \) which is tangent to the boundary of a cylindrical void of radius \( T_0 \) centred at the origin and which passes through \( (X, Y, Z) \). Then, as \( T \) increases, the direction of the radiation will rotate. This will continue until the boundary reaches the observer at which point the direction of the radiation is exactly radial. After this event, the radiation will cease (provided the space-time has been extended with an interior void).

Of course the magnitude of the radiation detected by the observer will depend on the arbitrary “amplitude” function \( f(u) \). The other arbitrary functions \( g(u) \) and \( h(u) \) cannot be measured directly as they do not appear in the Ricci tensor and there is no gravitational radiation. However, they cannot be removed by a coordinate transformation unless they
are constants. To detect the effect of these terms, it would be necessary to employ a family of observers equipped to determine the higher derivatives of the curvature tensor. As shown by Skea [7], the functions $g(u)$ and $h(u)$ appear explicitly in the second derivatives of the curvature tensor, although the fourth derivatives are required to distinguish between the Edgar–Ludwig solutions with $g$ and $h$ nonconstant and the Wils subclass.

It may finally be observed that pure radiation solutions which can be expressed in the Kerr–Schild form have been discussed in [1]. In fact, the solution (1) is implicitly contained in the general form given in §28.4.3 of [1] for type N pure radiation solutions. However, the conformally flat case described here and in [4–5] is not given explicitly.

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Captions for figures

Figure 1

For constant $T$ and $Y$, the wave surfaces $u = u_0 = \text{const.}$ in the weak field limit can be represented as a family of half lines at a perpendicular distance $T$ from an origin as indicated. The envelope of the lines forms a circle corresponding to the coordinate singularity at $x = 0$. As $T$ increases, the circle expands and the null matter on each wave surface propagates perpendicular to that surface.

Figure 2

In space-time, the half-plane wave surfaces are tangent to a 2-dimensional null cone on which $x = 0$. The pictures represent two sections at constant $Y$ and at constant $T$. 
$x = 0$

$u = u_0$

$x > 0$

$Y = \text{const.}$

$T = \text{const.}$