Stochastic Variational Method as Quantization Scheme II: Field Quantization of Complex Klein-Gordon Equation

T. Koide and T. Kodama

Instituto de Física, Universidade Federal do Rio de Janeiro, C.P. 68528, 21941-972, Rio de Janeiro, Brazil

Stochastic Variational Method (SVM) is the generalization of the variation method to the case with stochastic variables. In this series of papers, we investigate the applicability of SVM as an alternative quantization scheme. In Part II, we discuss the field quantization in the complex Klein-Gordon equation in the framework of SVM. In this scheme, the complete dynamics of the quantized field is described by a set of differential equations for the field configuration, which can be interpreted as the Euler (ideal fluid) equation in the functional space. In this formulation, the Fock state vector is given by the stationary solution of these differential equations and various results in the usual canonical quantization can be reproduced, including the effect of anti-particles. We further propose a systematic procedure to determine one parameter included in SVM which is, so far, given in an ad hoc manner so as to reproduce the Schrödinger equation.

I. INTRODUCTION

Variational approach plays conceptually a fundamental role in elucidating the structure of Classical Mechanics, unifying the origin of dynamics and the relation between symmetries and conservation laws. This is the reason that it is referred to as “principle” rather than the simple re-expression of Newton’s equation of motion. In quantum mechanics, such an aspect is somewhat lost. As a matter of fact, quantum dynamics is given by the Schrödinger equation, and symmetries and conservation laws are formulated as transformation groups in the Hilbert space. Of course, although it is possible to write down an action which reproduces
the Schrödinger equation, such a reexpression does not provide further insight into the structure of quantum mechanics. Rather, the variational approach in quantum mechanics is usually used for practical purposes. Inversely, as seen in the path integral formulation, quantum mechanics is considered to give an explanation for the origin of classical variational principle in the sense that the classical optimized path gives the dominant contribution in the transition amplitude.

It was, however, shown by K. Yasue [1] around three decades ago that the fundamental concepts of quantum mechanics can be formulated in terms of the variational principle, generalizing classical variables to the corresponding stochastic one, whose stochastic property is known as Bernstein (reciprocal) process [2]. This new approach is called the stochastic variational method (SVM) and was originally introduced to reformulate Nelson’s stochastic quantization [3] in terms of the variational principle. In this formulation, classical mechanics and quantum mechanics are unified under the principle of the optimization of actions in such a way that the Schrödinger equation is derived by employing SVM to the action which reproduces Newton’s equation of motion.

The applicability of SVM is not restricted to the derivation of quantum mechanics. It has been shown that the Navier-Stokes equation [4–6], the Gross-Pitaevskii equation [5, 7] and the Kostin equation [8] can be formulated in the framework of SVM. The Noether theorem [9], the uncertainty relation [10] and the applications to the many-body particle systems [7] and continuum media [4, 5] are also possible to be formulated in this scheme. The foundation of this kind of stochastic calculus has been investigated by various authors [11–28]. Considering these interesting aspects of SVM, it is worth exploring the applicability to more complex systems.

In Part II of the series of papers, we investigate the application of SVM for the field quantization. At first glance, this seems to be trivial because it is known that a free field can be regarded as the ensemble of harmonic oscillators as usually done in the canonical quantization. However, there are several aspects which do not appear in quantum mechanics. The existence of anti-particles and the continuum limit are such examples. To investigate these points, we focus on the quantization of the complex Klein-Gordon equation in this work.

We find that the complete dynamics of the quantized field in SVM is described by a set of differential equations for the field configuration, which can be interpreted as the Euler
(ideal fluid) equation in the functional space. Then the Fock state vector is given by the stationary solution of these differential equations and various results in the usual canonical quantization are reproduced, including the effect of anti-particles.

In SVM, the intensity of the noise is a parameter and so far it has been determined in a heuristic way, requiring that the final result reproduces the Schrödinger equation. In this paper, we determine this parameter so as to reproduce the single-particle energy without the knowledge of the existence of the Schrödinger equation. As a result, it is shown that the functional Euler equation is cast into the functional Schrödinger equation in the field configuration space and its time evolution operator must be given by the Hamiltonian operator.

This paper is organized as follows. Before discussing the complex Klein-Gordon equation, we first investigate the quantization of the classical string as a simplest example to illustrate the application of SVM in Sec. II. The quantization of the complex Klein-Gordon equation is discussed in Sec. III. Then we discuss the determination of the intensity of the noise in Sec. IV. Other aspects of SVM such as anti-particles, propagator, the application to the case of the imaginary mass and the continuum limit are discussed in Sec. V. Section VI is devoted to the concluding remarks.

II. CLASSICAL STRING

To introduce the stochastic nature in the classical dynamics, the usual concept of derivatives should be modified. In this section, we illustrate this aspect using the classical string model as an example.

A. SVM quantization: $x$-space

The classical dynamics of the string with a small amplitude $y = y(x, t)$ is described by the following equation of motion,

$$\frac{\partial^2 y}{\partial t^2} - \frac{T}{\sigma} \frac{\partial^2 y}{\partial x^2} = 0,$$

where $\sigma$ and $T$ are the line density of mass and tension, respectively. For the sake of simplicity, we consider the spatially one-dimensional system. The corresponding action is
then given by
\[ I[y] = \int_{t_i}^{t_f} dt \int_0^L dx \left[ \frac{\sigma}{2} \left( \frac{\partial y}{\partial t} \right)^2 - \frac{T}{2} \left( \frac{\partial y}{\partial x} \right)^2 \right], \]  
(2)
where \( t_i \) and \( t_f \) denote the initial and final times, respectively and \( L \) is the total length of the string.

In SVM, the dynamical variable \( y(x, t) \) is extended to the stochastic variable \( \hat{y}(x, t) \). In the following argument, we use the accent symbol \( \hat{\cdot} \) to denote stochastic variables. The time evolution and distribution of the stochastic variables are not smooth and hence the time and space derivatives should be modified from the usual classical ones.

To define the quantity corresponding to the space derivative, we discretize the space into a lattice. Let us divide \( L \) by \( N \) grid points, where \( N \) is the odd integer. The grid interval is given by \( \Delta x = L/N \). We observe the string dynamics on these grid points. Then the amplitude \( y \) on the grid points are expressed as a vector
\[
y(x,t) = \begin{pmatrix}
y(x_0,t) \\
\vdots \\
y(x_n,t) \\
\vdots \\
y(x_{N-1},t)
\end{pmatrix} \rightarrow \underline{y}(t) \equiv \begin{pmatrix}
y_0(t) \\
\vdots \\
y_n(t) \\
\vdots \\
y_{N-1}(t)
\end{pmatrix}, \quad (3)
\]
where \( x_n = x_0 + n\Delta x \). In the following, we use \( \underline{\cdot} \) to indicate a vector quantity for the amplitude configuration space. We further adapt the periodic boundary condition as
\[
y_{n+N}(t) = y_n(t). \quad (4)
\]

On this lattice, we can define two space-derivative matrices: \( \nabla_+ \) and \( \nabla_- \), which are defined by the \( N \times N \) matrices as
\[
\nabla_+ = \frac{1}{\Delta x} \begin{pmatrix}
-1 & 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & 0 & \cdots & 0 \\
\vdots & & \ddots & & \vdots & \vdots \\
0 & \cdots & 0 & -1 & 1 \\
1 & 0 & \cdots & 0 & 0 & -1
\end{pmatrix}, \quad \nabla_- = \frac{1}{\Delta x} \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & -1 \\
-1 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & & \ddots & & \vdots & \vdots \\
0 & \cdots & 0 & -1 & 1 \\
0 & 0 & \cdots & 0 & -1 & 1
\end{pmatrix}, \quad (5)
\]
respectively. Note that $\nabla_+ = -\nabla^T$. Using these definitions, the matrix $\Delta_x$, which corresponds to the Laplacian operator, is expressed as

$$\Delta_x \equiv \nabla_- \nabla_+ = \nabla_+ \nabla_- = \frac{1}{(\Delta x)^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}. \quad (6)$$

Note that these representations are constructed so as to satisfy the partial integration formula as

$$\left( f * \nabla_{\pm} \hat{h} \right)_x = - \left( \nabla_{\mp} f * \hat{h} \right)_x, \quad (7)$$

$$\left( f * \Delta_x \hat{h} \right)_x = - \left( \nabla_+ f * \nabla_+ \hat{h} \right)_x = - \left( \nabla_- f * \nabla_- \hat{h} \right)_x. \quad (8)$$

Here we introduced the notation for the scalar product of two arbitrary $x$-space vectors, $\mathbf{f}$ and $\mathbf{h}$, as

$$\left( \mathbf{f} \cdot \mathbf{h} \right)_x \equiv \Delta x \sum_{n=0}^{N-1} f_n h_n. \quad (9)$$

The factor $\Delta x$ above is introduced to reproduce the inner product of two functions $f(x)$ and $h(x)$ as

$$(f \cdot h)_x = \int dx \ f(x) h(x), \quad (10)$$

in the continuum limit ($L, N \to \infty, \Delta x = L/N \to 0$). In this limit, any $N$-vector $\mathbf{y}$ becomes a functional of $y(x)$.

The usual space derivative appearing in the classical action should be expressed using these matrices. Then we assume that the space derivative is expressed by the average of these two space derivatives $\nabla_+$ and $\nabla_-$ in the lattice representation. Therefore, the lattice representation of the classical action corresponding to Eq.(2) is given by

$$I[y] = \int_{t_i}^{t_f} dt \left[ \frac{\sigma}{2} \left( \frac{\partial y}{\partial t} * \frac{\partial y}{\partial t} \right)_{\Delta x} - \frac{T}{4} (\nabla_+ y * \nabla_+ y)_{\Delta x} - \frac{T}{4} (\nabla_- y * \nabla_- y)_{\Delta x} \right] = \int_{t_i}^{t_f} dt \left[ \frac{\sigma}{2} \left( \frac{\partial y}{\partial t} * \frac{\partial y}{\partial t} \right)_{\Delta x} + \frac{T}{2} (\mathbf{y} \cdot \Delta_x \mathbf{y})_{\Delta x} \right]. \quad (11)$$
As was mentioned, even the time derivative appearing above should be changed when \( y \) is replaced by the stochastic variable. This is discussed when the stochastic variation is introduced.

In SVM, the dynamical variable, \( y(t) \) is extended to a stochastic variable,

\[
y(t) \rightarrow \hat{y}(t). \tag{12}
\]

Moreover, as is usual with the variational method, we consider that the stochastic process where the initial and final conditions are fixed, such as the Bernstein (reciprocal) process [2]. To characterize this process, we need to introduce two stochastic differential equations (SDEs) for the amplitudes \( \{ \hat{g}_n(t), \ n = 0, \cdots, N - 1 \} \). One is the forward SDE which describes the forward time evolution \( dt > 0, \)

\[
d\hat{g}_n(t) = u_n(\hat{y}(t), t) dt + \sqrt{\frac{2\nu}{\Delta x}} dW_n(t), \tag{13}
\]

where \( u = \{ u_n, \ n = 0, \cdots, N - 1 \} \) represents the velocity field to be determined as a smooth function of \( \hat{y}(t) \) and \( t \) as a result of the variation. The last term on the right hand side is the noise given by the Wiener process, satisfying the following correlation properties,

\[
E[dW(t)] = 0, \tag{14}
\]

\[
E[dW(t)dW^T(t)] = |dt| I, \tag{15}
\]

where \( I \) denotes the unit matrix and \( E[ \ ] \) represents the event average for the noise.

The other is the backward SDE which describes the backward time evolution \( dt < 0, \)

\[
d\hat{g}_n(t) = \tilde{u}_n(\hat{y}(t), t) dt + \sqrt{\frac{2\nu}{\Delta x}} d\tilde{W}_n(t). \tag{16}
\]

The correlation properties of \( \tilde{W}(t) \) are the same as \( W(t) \), but there is no correlation between \( \tilde{W}(t) \) and \( W(t) \). For the sake of simplicity, we omit \( \tilde{ } \) for \( W(t) \) and \( \tilde{W}(t) \).

From these SDEs, the Fokker-Planck equations are derived. We define the amplitude configuration distribution as

\[
\rho(y, t) = E[\delta^{(N)}(y - \hat{y}(t))]. \tag{17}
\]

Then the Fokker-Planck equations obtained from the forward SDE and the backward SDE are, respectively, given by

\[
\partial_t \rho(y, t) = \nabla_y \cdot \left( -u + \frac{\nu}{\Delta x} \nabla_y \right) \rho(y, t), \tag{18}
\]

\[
\partial_t \rho(\bar{y}, t) = \nabla_y \cdot \left( -\bar{u} - \frac{\nu}{\Delta x} \nabla_y \right) \rho(\bar{y}, t). \tag{19}
\]
Here $\nabla_y = (\partial/\partial y_0, \partial/\partial y_1, \ldots, \partial/\partial y_{N-1})$ and the symbol $\cdot$ represents the product of $N$ vectors as $A \cdot B = \sum_{n=0}^{N-1} A_n B_n$.

For the two equations to be equivalent, the following consistency condition should be satisfied,

$$u(\underline{y}, t) = \tilde{u}(\underline{y}, t) + \frac{2\nu}{\Delta x} \nabla_y \ln \rho(\underline{y}, t).$$  \hfill (20)

See also Ref. [5] for the corresponding argument in particle systems.

Through this condition, the unknown function $\tilde{u}(\underline{y}, t)$ is obtained with this condition when $u(\underline{y}, t)$ is given. The purpose of SVM is to determine the equation for the remaining unknown function $u(\underline{y}, t)$ using the variational procedure.

To implement this operation, we need to express the classical lattice action (11) in terms of the stochastic variables. The problem in this procedure is the treatment of the time derivative term. Correspondingly to the two SDEs, at least, the two different definitions of the time derivative are available: one is the mean forward derivative $D$ and the other the mean backward derivative $\tilde{D}$, which are defined by

$$D\hat{y}(t) = \lim_{dt \to 0^+} E \left[ \frac{\hat{y}(t + dt) - \hat{y}(t)}{dt} \right | \mathcal{P}_t]$$  \hfill (21)

$$\tilde{D}\hat{y}(t) = \lim_{dt \to 0^+} E \left[ \frac{\hat{y}(t) - \hat{y}(t - dt)}{dt} \right | \mathcal{F}_t]$$  \hfill (22)

respectively. These are originally introduced by Nelson [3]. Here $E[g(t')|\mathcal{P}_t]$ denotes the conditional average of a time sequence of stochastic variables $\{g(t'), t_a < t' < t_b\}$ only for $t' > t$, fixing the values of $g(t')$ for $t' \leq t$. Similarly, $E[g(t')|\mathcal{F}_t]$ is the conditional average for the past sequence. For the $\sigma$-algebra of all measurable events, $\{\mathcal{P}_t\}$ and $\{\mathcal{F}_t\}$ represent an increasing and a decreasing family of sub-$\sigma$-algebras, respectively [24]. See also the discussion in Ref. [6].

For the present case, these derivatives are given by

$$D\hat{y}(t) = \underline{u}(\hat{y}(t), t),$$  \hfill (23)

$$\tilde{D}\hat{y}(t) = \underline{\tilde{u}}(\hat{y}(t), t).$$  \hfill (24)

Similarly to the correspondence principle in quantum mechanics, we require that the action which obtained by substituting these new stochastic variables should reduce to the corresponding classical one in the vanishing noise limit, $\nu \to 0$. Moreover, the time reversal symmetry should be satisfied. Thus, as is discussed in Refs. [3, 6, 10], the time derivative
term should be given by the average of the contributions of the mean forward and backward derivatives \[1, 5\]. Then a reasonable stochastic representation of Eq. (11) is given by

\[
I_{\text{SVM}}[\hat{y}] = \int_{t_i}^{t_f} dt \ E \left[ \frac{\sigma}{4} \left\{ (D\hat{y} \ast D\hat{y})_{\Delta x} + (\tilde{D}\hat{y} \ast \tilde{D}\hat{y})_{\Delta x} \right\} + \frac{T}{2} (\hat{y} \ast \Delta_x \hat{y})_{\Delta x} \right].
\] (25)

The stochastic variation is applied to this stochastic action in the following manner. We consider the following variation of the stochastic variable,

\[
\hat{y}_n(t) \rightarrow \hat{y}_n(t) + \eta(\hat{y}_n(t), t),
\] (26)

where \(\eta(x, t)\) is an arbitrary, but infinitesimal smooth function of \(x\) and \(t\), satisfying the boundary condition,

\[
\eta(x, t_{i(f)}) = 0.
\] (27)

The detailed operation of the stochastic variation is the same as that in the particle systems. See Ref. [5] for details. Finally, we obtain the following result,

\[
\left\{ \partial_t + \mathbf{v}(y, t) \cdot \nabla y \right\} \mathbf{v}(y, t) - \frac{2\nu^2}{(\Delta x)^2} \nabla \nabla_y \left\{ \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} \right\} = \frac{T}{\sigma} \Delta_x y(t),
\] (28)

where \(\nabla^2_y = \nabla_y \cdot \nabla_y\). Here we introduced the mean velocity \(\mathbf{v}(y, t)\) as a function of \(y\) by

\[
\mathbf{v}(y, t) = \frac{u(y(t)) + \tilde{u}(y(t))}{2}.
\] (29)

By using this, the two Fokker-Planck equations introduced before are expressed by the following unified form,

\[
\partial_t \rho(\underline{y}, t) = -\nabla_y \cdot (\rho(\underline{y}, t) \mathbf{v}(\underline{y}, t)).
\] (30)

This is nothing but the equation of continuity. The two equations (28) and (30) are the final result of SVM.

It should be mentioned that Eq. (28) is a kind of the Euler equation (ideal fluid equation) in the space of the amplitude configuration. The corresponding equation in quantum mechanics of a particle system is known in the Mandelung-Bohm theory [29–35]. In Eq. (28), the quantum effect is induced only by the last term on the left hand side and this term is the generalization of the so-called quantum potential.

These results can be simplified by introducing the wave function for the amplitude configuration as

\[
\psi(\underline{y}, t) = \sqrt{\rho(\underline{y}, t)} e^{i\theta(\underline{y}, t)},
\] (31)
where the phase is defined by

$$v(y, t) = \frac{2\nu}{\Delta x} \nabla_y \theta(y, t).$$

(32)

In this paper, anticipating the continuum limit of our SVM results, we refer to Eq. (31) as wave functional. Then all other functions of $y$, such as $v$ and $\rho$ become also corresponding functionals.

From Eqs. (28) and (30), the equation for the wave functional is given by

$$i \partial_t \psi(y, t) = \tilde{H} \psi(y, t),$$

(33)

where

$$\tilde{H} = \Delta x \left[ -\frac{\nu}{(\Delta x)^2} \nabla_y^2 + \frac{1}{4\nu \sigma} y^T \cdot \Delta_y y \right].$$

(34)

In the continuum limit, the derivative operators, $\nabla_y$ and $\nabla_y^2$ can be expressed with the respective functional derivatives. See the discussion in Sec. V D for details.

To complete the quantization, we need to specify the intensity of the noise $\nu$ and confirm that $\hbar \tilde{H}$ can be interpreted as the Hamiltonian operator. Thus we cannot yet call this equation as the Schrödinger equation of the amplitude configuration. In the usual SVM, this is determined in a heuristic way, that is, it is fixed so as to reproduce the Schrödinger equation. Then, in the present case, we find

$$\nu = \frac{\hbar}{2\sigma},$$

(35)

because $\sigma$ is the mass of the string.

Later, in the quantization of the complex Klein-Gordon equation, we discuss how this $\nu$ is determined from the direct experimental evidences such as Planck-Einstein-de Broglie’s relation, $E = \hbar \omega$ with $\omega$ being a frequency, without assuming the Schrödinger equation. See Sec. IV B for details.

**B. SVM quantization: $k$-space**

So far, we employed SVM directly to the stochastic lattice action (25), but sometimes it is more convenient to choose another variables instead of $y = \{y_n(t)\}$. Since the Laplacian matrix is symmetric, the base which diagonalizes $\Delta_x$ will be more convenient to use.

Let us denote $N$ eigenvectors of $\Delta_x$ as $u^{(n)}$,

$$\Delta_x u^{(n)} = -k_n^2 u^{(n)},$$

(36)
where \( n = 0, \ldots, N - 1 \) and
\[
k_n = \frac{2}{\Delta x} \sin \left( n \frac{\Delta k \Delta x}{2} \right),
\]
with \( \Delta k \) being \( 2\pi/L \). These eigenvectors are normalized as
\[
(u^{(l)} * u^{(m)})_x = \frac{1}{\Delta k} \delta_{lm}.
\]
The explicit forms of these eigenvectors are shown in Appendix A.

Any \( N \)-vector \( f \) in the \( x \)-space can be expanded with these bases, so that we write the amplitude of the string \( y \) as
\[
y = \Delta k \sum_{n=0}^{N-1} c_n u^{(n)}.
\]
The set of coefficients \( c_n \) contains equivalent information to \( y \), and we also denote it in the \( N \)-vector form as
\[
\mathbf{c} = \begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix}.
\]
Thus we can use \( \mathbf{c} \) instead of \( y \) to represent the string dynamics. We refer to this representation as the \( k \)-space representation, because, as is shown in Appendix A, the vector \( \mathbf{c} \) converges to the set of the usual Fourier coefficients in the continuum limit, if the original function \( y(x) \) is smooth almost everywhere.

Then we can diagonalize \( \Delta_x \) with these eigenvectors defining the matrix \( O \) as
\[
O^T \Delta_x O = \frac{1}{\Delta k \Delta x} \Lambda,
\]
where \( O \) is given by
\[
O = (u^{(0)}, u^{(1)}, \ldots, u^{(N-1)}).
\]
Here the diagonal matrix \( \Lambda \) is given by \( \Lambda = \text{diag}(-k_0^2, -k_1^2, -k_2^2, \ldots, -k_{N-1}^2) \).

Due to the orthogonalization with the measure introduced in the definition of the scalar product in the \( x \)-space, we also need to introduce a measure in the scalar product for the \( k \)-space vectors. Then, using Eq. (39), the two \( x \)-space vectors \( f \) and \( h \) are expressed with the corresponding \( k \)-space vectors \( \mathbf{c} \) and \( \mathbf{b} \) as
\[
f = \Delta k \ O \ \mathbf{c},
\]
\[
h = \Delta k \ O \ \mathbf{b}.
\]
respectively. Then the scalar products are invariant for the “orthogonal” transform by $O$,

$$\langle f \star h \rangle_x = \langle \mathcal{C} \star \mathcal{B} \rangle_k,$$

(45)
as far as the scalar product of the $k$-space vectors $\mathcal{C}$ and $\mathcal{B}$ is defined as

$$\langle \mathcal{C} \star \mathcal{B} \rangle_k \equiv \Delta k \sum_{n=0}^{N-1} c_n b_n.$$

(46)

See Appendix $\Delta$ for details.

Using this basis, the $k$-space representation of the classical lattice action given by Eq. $(11)$ is represented as

$$I[c] = \int dt \left[ \frac{\sigma}{2} \left( \frac{\partial c}{\partial t} \right)^2 + \frac{\mathcal{T}}{\Delta k} (\mathcal{C} \star \mathcal{L})_{\Delta k} \right]$$

$$= \int dt \Delta k \sum_{n=0}^{N-1} \left[ \frac{\sigma}{2} \left( \frac{\partial c_n}{\partial t} \right)^2 - \frac{\mathcal{T}}{2} k_n c_n^2 \right],$$

(47)

To implement the stochastic variation, we assume the following forward and backward SDEs for $\mathcal{C}$ as

$$d\hat{c}_n(t) = u_n(\mathcal{C}(t), t)dt + \sqrt{\frac{2\nu}{\Delta k}} dW'_n(t),$$

(48)

$$d\hat{c}_n(t) = \tilde{u}_n(\mathcal{C}(t), t)dt + \sqrt{\frac{2\nu}{\Delta k}} d\tilde{W}'_n(t),$$

(49)

respectively. Here the functional velocity fields $u(\mathcal{C}, t)$ and $\tilde{u}(\mathcal{C}, t)$ are smooth functions of $(\mathcal{C}, t)$, and $W'(t)$ and $\tilde{W}'(t)$ are again two independent Wiener processes with the following correlation properties,

$$E[dW'(t)] = E[d\tilde{W}'] = 0,$$

(50)

$$E[dW'(t)dW'^T(t)] = E[d\tilde{W}'d\tilde{W}'^T] = |dt| I.$$

(51)

The distribution function of the $\mathcal{C}$ configuration is defined by

$$\rho(\mathcal{C}, t) = E[\delta^{(N)}(\mathcal{C} - \mathcal{C}(t))].$$

(52)

Then the self-consistency condition is given by

$$\mathcal{U}(\mathcal{C}, t) = \tilde{\mathcal{U}}(\mathcal{C}, t) + \frac{2\nu}{\Delta k} \nabla \ln \rho(\mathcal{C}, t),$$

(53)
and the Fokker-Planck equation is expressed as

$$\partial_t \rho(c,t) = -\nabla_c \cdot \{ \rho(c,t) v(c,t) \}. \quad (54)$$

Here $\nabla_c = (\partial/\partial c_0, \cdots, \partial/\partial c_{N-1})$ and the mean functional velocity $v$ is defined by

$$v(c,t) = \frac{u(c,t) + \bar{u}(c,t)}{2}. \quad (55)$$

The intensity of the noise $\nu'$ can be calculated in two different ways. One is to compare to the SDEs in the $x$-space by using $\hat{y}(t) = (\Delta k) \Omega \hat{c}(t)$. The other to employ the method which is introduced in Sec. IV B. We obtain the same result in either case and then the intensity is given by

$$\nu' = \nu = \frac{\hbar}{2\sigma}. \quad (56)$$

By applying the stochastic variation to the stochastic action (47), we obtain

\[
\{-\partial_t + v(c,t) \cdot \nabla_c \} \ v(c,t) - \frac{\hbar^2}{2\sigma^2(\Delta k)^2} \nabla_c \left\{ \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} \right\} = \frac{T}{\sigma} \Lambda y(t). \quad (57)
\]

Similarly to the case of the $x$-space, the equation for the wave functional corresponding to the above dynamics can be introduced as

$$i \partial_t \psi(c,t) = \tilde{H} \psi(c,t), \quad (58)$$

where

$$\tilde{H} = \left[ -\frac{\hbar}{2\sigma(\Delta k)} \nabla^2_c + \Delta k \frac{\sigma^2 \omega_n^2}{2\hbar} \right], \quad (59)$$

with $\omega_n^2 = k_n^2 T/\sigma$. Here, the wave functional in the $k$-space is given by

$$\psi(c,t) = \sqrt{\rho(c,t)} e^{i\theta(c,t)}, \quad (60)$$

and the phase is defined with the mean functional velocity as

$$\psi(c,t) = \frac{\hbar}{\Delta k} \nabla_c \theta(c,t). \quad (61)$$

### III. COMPLEX KLEIN-GORDON FIELD

In this section, we apply the SVM quantization developed for the string to the complex Klein-Gordon equation. The ambiguity for the introduction of the wave functional is also discussed.
A. SVM quantization on lattice

The classical Lagrangian of the complex Klein-Gordon equation is

\[ L = \int_V d^3x \left[ \frac{1}{c^2} (\partial_t \phi^*(x, t) (\partial_t \phi(x, t)) - \nabla \phi^*(x, t) \cdot \nabla \phi(x, t) - \mu^2 \phi^*(x, t) \phi(x, t) \right], \]

where \( V = L^3 \) is the space volume and \( \mu = mc/\hbar \) with \( m \) and \( c \) being the mass and the speed of light, respectively. We apply the periodic boundary condition at the space boundary.

To implement the stochastic variation, we need to redefine the system on a spatial lattice as discussed in the previous section, and consider the stochastic dynamics on each grid. Let us express the field on the grid points as \( \phi_x \). Then the stochastic Lagrangian is expressed as

\[ L_{SVM} = E \left[ \frac{1}{2c^2} \left\{ (D^* D_x \phi) + (D^* D_x \phi) \right\} + (\phi^* \Delta \phi) - \mu^2 (\phi^* \phi) \right], \]

where \( x \) indicates the grid points, and we introduced the vector notation \( \phi = \{ \phi_x \} \) with the scalar product extended in the three dimensional \( x \)-space as

\[ (f \ast h)_x \equiv \Delta^3 x \sum_x f_x h_x, \]

where \( \Delta^3 x \) is the volume of unit lattice cube, \((\Delta x)^3\), and \( \sum_x \) indicates the sum over all lattice grids. The Laplacian matrix \( \Delta_x \) is also extended to the three dimensional \( x \)-space. See Appendix A for details.

As was done in the classical string, it is possible to implement SVM directly in the \( x \)-space. We will, however, discuss the relation between SVM and the usual canonical quantization and, for this purpose, should develop the quantization in the \( k \)-space. The result in the \( x \)-space is given in Appendix B.

By using the orthogonal transform which diagonalizes the matrix \( \Delta_x \), the field configuration defined at each grid point \( x \) is transformed to the corresponding amplitude in the \( k \)-space \( \hat{C}_k(t) \) as

\[ \hat{\phi}_x(t) = \sqrt{\Delta^3 k} \sum_k \hat{C}_k(t) e^{i k x}, \]

where \( \Delta^3 k = (\Delta k)^3 = (2\pi)^3 / V \), and \( \hat{C}_k(t) \) is the stochastic complex variable written with two real stochastic variables, \( \hat{C}_{R,k}(t) \) and \( \hat{C}_{I,k}(t) \) as

\[ \hat{C}_k(t) = \frac{\hat{C}_{R,k}(t) + i \hat{C}_{I,k}(t)}{\sqrt{2}}. \]
It should be emphasized once again that the one-to-one correspondence between \( \hat{\phi} \) and \( \hat{\Phi} \) expressed by Eq.\( (65) \) is just an analog of the Fourier transform defined between the two discrete sets of variables. It coincides with the Fourier transform when and only when these variables are smooth and the continuum limit is taken. See Appendix \( \Box \) for details.

With this, the stochastic action is given by

\[
I_{SV \Sigma}[\hat{C}_R, \hat{C}_I] = \frac{1}{2} \int_{t_i}^{t_f} dt \sum_{i=R,I} E \left[ \frac{1}{2c^2} \left\{ (D\hat{C}_i * D\hat{C}_i)_k + (\hat{D}\hat{C}_i * \hat{D}\hat{C}_i)_k \right\} - (\hat{C}_i * \Lambda\hat{C}_i)_k \right]
\]

\[
= \frac{1}{2} \int_{t_i}^{t_f} dt \sum_{i=R,I} \sum_k \Delta^3 k \times E \left[ \frac{1}{2c^2} \left\{ (D\hat{C}_{i,k}(t))(D\hat{C}_{i,k}(t)) + (\hat{D}\hat{C}_{i,k}(t))(\hat{D}\hat{C}_{i,k}(t)) \right\} - (q_k^2 + \mu^2)\hat{C}_{i,k}^2(t) \right],
\]

where

\[
\Lambda_{k,k'} = (q_k^2 + \mu^2)\delta_{k,k'}
\]

with

\[
q_k = \frac{2}{\Delta x} \begin{pmatrix} \sin(\Delta x k_x/2) \\ \sin(\Delta x k_y/2) \\ \sin(\Delta x k_z/2) \end{pmatrix}.
\]

Note that this eigenvalue \( q_k^2 \) reduces simply to \( k^2 \) for \( |(\Delta x)k| \ll 1 \).

The forward and backward SDEs for these stochastic variables, \( \hat{C}_{R,I}(t) \) and \( \hat{C}_{I,R}(t) \) are, respectively, given by

\[
d\hat{C}_{i,k}(t) = u_{i,k}(\hat{C}_R(t), \hat{C}_I(t), t) dt + \sqrt{\frac{2\nu}{\Delta^3 k}} dW_{i,k}(t),
\]

\[
d\hat{C}_{i,k}(t) = \bar{u}_{i,k}(\hat{C}_R(t), \hat{C}_I(t), t) dt + \sqrt{\frac{2\nu}{\Delta^3 k}} d\bar{W}_{i,k}(t),
\]

and the correlation properties of the noises are

\[
E[dW_{i,k}(t)] = E[d\bar{W}_{i,k}(t)] = 0,
\]

\[
E[dW_{j,k}(t)dW_{i,k}(t)] = E[d\bar{W}_{j,k}(t)d\bar{W}_{i,k}(t)] = \delta_{jk}\delta_{k,k'}|dt|,
\]

Here the indices \( i,j \) denote \( R \) or \( I \). As before, there is no correlation between \( dW_{i,k}(t) \) and \( d\bar{W}_{i,k}(t) \).

The distribution for the \( k \)-space configuration \( (\hat{C}_R(t), \hat{C}_I(t)) \) is then defined by

\[
\rho(\hat{C}_R, \hat{C}_I, t) = E \prod_k \delta(C_{R,k} - \hat{C}_{R,k}(t))\delta(C_{I,k} - \hat{C}_{I,k}(t))
\]

\[
= \prod_k \delta(C_{R,k} - \hat{C}_{R,k}(t))\delta(C_{I,k} - \hat{C}_{I,k}(t)).
\]

(74)
The self-consistency condition is derived from the equivalence of the two Fokker-Planck equations of the two SDEs introduced above as

\[ u_{i,k}(C_R, C_I, t) = \tilde{u}_{i,k}(C_R, C_I, t) + \frac{2\nu}{\Delta^3 k} \frac{\partial}{\partial C_{i,k}} \ln \rho(C_R, C_I, t). \]  

(75)

Using this condition the Fokker-Planck equation is expressed as

\[ \partial_t \rho(C_R, C_I, t) = -\sum_{j=R,I} \sum_k \frac{\partial}{\partial C_{j,k}} \left\{ \rho(C_R, C_I, t) v_{j,k}(C_R, C_I, t) \right\}. \]  

(76)

Here we used the mean functional velocity defined by

\[ v_{i,k}(C_R, C_I, t) = \frac{u_{i,k}(C_R, C_I, t) + \tilde{u}_{i,k}(C_R, C_I, t)}{2}. \]  

(77)

With these definitions, the stochastic variation leads to the following functional Euler equation for the complex Klein-Gordon equation,

\[
\begin{align*}
\left( \partial_t + \sum_{j=R,I} \sum_{k'} v_{j,k'}(C_R, C_I, t) \frac{\partial}{\partial C_{j,k'}} \right) v_{i,k}(C_R, C_I, t) \\
- \frac{2\nu^2}{(\Delta^3 k)^3} \frac{\partial}{\partial C_{i,k}} \left\{ \rho^{-1/2}(C_R, C_I, t) \sum_{j=R,I} \sum_{k'} \left( \frac{\partial}{\partial C_{j,k'}} \right)^2 \rho^{1/2}(C_R, C_I, t) \right\} \\
= -\frac{1}{2} \frac{\partial}{\partial C_{i,k}} \sum_{j=R,I} \sum_{k'} \omega_{k'}^2 C_{j,k'}^2,
\end{align*}
\]

(78)

where

\[ \omega_k = c \sqrt{q_k^2 + \mu^2}. \]  

(79)

The two equations (76) and (78) compose a closed system for our unknown variables \( \rho \), \( v_{R,k} \) and \( v_{I,k} \) which determine completely the dynamics of the system in terms of the field configuration.

We now introduce the wave functional for the field configuration in the \( k \)-space as

\[ \psi(C_R, C_I, t) = \rho(C_R, C_I, t)e^{i\theta(C_R, C_I, t)}, \]  

(80)

where the phase is defined by

\[ v_{i,k}(C_R, C_I, t) = \frac{2\nu}{\Delta^3 k} \frac{\partial}{\partial C_{i,k}} \theta(C_R, C_I, t). \]  

(81)

The dynamics of this wave functional is determined by the following functional equation,

\[ i\partial_t \psi(C_R, C_I, t) = \hat{H} \psi(C_R, C_I, t), \]  

(82)
with the time evolution operator

$$\hat{H} = \Delta^3 k \sum_k \sum_{j=R,I} \left[ -\frac{\nu}{(\Delta^3 k)^2} \left( \frac{\partial}{\partial C_{j,k}} \right)^2 + \frac{1}{4\nu} \omega_k^2 C_{j,k}^2 \right].$$

(83)

As was already mentioned in the discussion of the string, to complete the procedure of the quantization, we need to specify the value of the noise intensity and confirm that the time-evolution operator $\hat{H}$ is determined by the Hamiltonian operator.

**B. Ambiguity for definition of phase and divergent vacuum energy**

For the definition of the phase, there is an ambiguity which has not yet been discussed so far. Because of the definition of the phase in Eq. (81), it is possible to add an arbitrary function $f(t)$, which is independent of the field $C_{i,k}$, in the evolution equation of the phase as

$$\partial_t \theta = -\sum_{i=R,I} \sum_k \frac{\nu}{(\Delta^3 k)} \left[ \left( \frac{\partial}{\partial C_{i,k}} \right)^2 - \rho^{-1/2} \frac{\partial^2}{\partial C_{i,k}^2} \rho^{1/2} \right] - \frac{\Delta^3 k}{4\nu} \sum_{i=R,I} \sum_k \omega_k^2 C_{i,k}^2 + f(t).$$

(84)

By using this ambiguity, we can eliminate the divergent vacuum energy. In other word, this divergence is attributed to the artificial introduction of the phase and does not appear in solving the coupled equation of the Fokker-Planck equation and the functional Euler equation directly. See also the argument around Eq. (93).

The same ambiguity appears even in quantum mechanics. See the discussion around Eq. (45) in Ref. [5].

**IV. DETERMINATION OF NOISE INTENSITY**

In this section, we discuss the intensity of the noise $\nu$. We determine this value so as to reproduce the single-particle energy. To introduce this energy, the Fock state vector should be defined in this framework.

**A. Fock state vector as stationary solution**

In SVM, the dynamics of the quantized fields are described by Eq. (82). This is the functional differential equation and the stationary solution can be obtained explicitly. As
will be shown soon later, this stationary solution corresponds to the state vector in the Fock space.

The explicit form of this wave functional is obtained in the same manner as the calculation of the harmonic oscillator potential in quantum mechanics. For the free field, the dynamics of each \( k \) component is separated and hence we introduce the following factorized ansatz for the stationary state,

\[
\psi(C_R, C_I, t) = e^{-i\Omega t} \prod_k \psi_{k,R}(C_R,k)\psi_{k,I}(C_I,k). \tag{85}
\]

Then the stationary eigenvalue equation becomes one unique form for all the factorization index, \( \alpha = (k, i = R, I) \) as

\[
\partial_\xi^2 \psi_\alpha(\xi) + (\epsilon_\alpha - \xi^2) \psi_\alpha(\xi) = 0, \tag{86}
\]

where

\[
\xi = C_{i,k} \sqrt{\frac{\omega_k}{2\nu} \Delta^3 k}. \tag{87}
\]

And the set of eigenvalues \( \{\epsilon_\alpha\} \) gives the frequency \( \Omega \) of the stationary state as

\[
\Omega = \frac{1}{2} \sum_{\alpha = (k,i)} \epsilon_\alpha \omega_k. \tag{88}
\]

The solution of this differential equation is, as is well-known, expressed with the Hermite polynomial, \( H_n(\xi) \) as

\[
\Phi_n(\xi) = H_n(\xi)e^{-\xi^2/2}. \tag{89}
\]

In this case, \( \epsilon_\alpha \) in Eq.\((86)\) is given by

\[
\epsilon_\alpha = 2n_\alpha + 1, \tag{90}
\]

where \( n_\alpha = 0, \cdots, \infty. \)

Finally, the stationary solution or the eigenstate for the component \( k \) is given by

\[
\psi_k(C_{R,k}, C_{I,k}, t) = N_k e^{-i\omega_k(n+1)t} \sum_{m=0}^n H_{n-m}(C_{R,k}\sqrt{\omega_k/2\nu})H_m(C_{I,k}\sqrt{\omega_k/2\nu})
\times e^{-\omega_k(C_{R,k}^2+C_{I,k}^2)/2\nu}, \tag{91}
\]

with \( N_k \) being the normalization factor which is given by initial conditions. In the above, we fixed the total frequency as

\[
\epsilon_k = \epsilon_{(k,R)} + \epsilon_{(k,I)} = \omega_k(n + 1). \tag{92}
\]
The summation over \( m \) is related to the two degrees of freedom, \( i = R, I \).

In particular, the wave functional of the lowest eigenstate corresponds to \( n_\alpha = 0 \) for all \( \alpha \), and is given by

\[
\psi_0(C_R, C_I) = \prod_k \left( \frac{\omega_k}{\pi \nu} \right)^{1/2} e^{-\omega_k(C_R^2 + C_I^2) / 2 \nu}. \tag{93}
\]

Here we dropped the phase factor \( e^{-i\omega_k t} \) by using the ambiguity \( f(t) \) which appears in the definition of the phase as was discussed in Sec. III B. As for the normalization, we adopted

\[
\int \mathcal{D}[C_R] \mathcal{D}[C_I] |\psi_0(C_R, C_I)|^2 = 1. \tag{94}
\]

We now introduce the creation-annihilation operators defined by

\[
a_k + b_{-k} = \frac{A_k}{\sqrt{2}} C_{R,k} + \frac{B_k}{\sqrt{2}} \frac{\partial}{\partial C_{R,k}}, \tag{95}
\]

\[
a_k^\dagger + b_{-k}^\dagger = \frac{A_k}{\sqrt{2}} C_{R,k} - \frac{B_k}{\sqrt{2}} \frac{\partial}{\partial C_{R,k}}, \tag{96}
\]

\[
a_k - b_{-k} = i \left( \frac{A_k}{\sqrt{2}} C_{I,k} + \frac{B_k}{\sqrt{2}} \frac{\partial}{\partial C_{I,k}} \right), \tag{97}
\]

\[
a_k^\dagger - b_{-k}^\dagger = -i \left( \frac{A_k}{\sqrt{2}} C_{I,k} - \frac{B_k}{\sqrt{2}} \frac{\partial}{\partial C_{I,k}} \right), \tag{98}
\]

where

\[
A_k = \sqrt{\frac{\omega_k}{2\nu} \Delta^3 k}, \quad B_k = \sqrt{\frac{2\nu}{\omega_k} \frac{1}{\Delta^3 k}}. \tag{99}
\]

One can easily confirm that

\[
[a_k, a_{k'}^\dagger] = [b_k, b_{k'}^\dagger] = \delta^{(3)}_{k,k'}. \tag{100}
\]

Then the lowest eigenstate satisfies

\[
a_k \psi_0 = b_k \psi_0 = 0, \tag{101}
\]

for \( \forall k \). Moreover, as seen from Eqs. (88) and (90), all the eigenstates of the wave functional can be specified by a set of integers \( \{n_\alpha\} \) so that we interpret \( n_\alpha \) as the number of quanta with \( \alpha \). As is well-known in the harmonic oscillator case, these states are obtained from the lowest eigenstate by applying the creation operators \( a_k^\dagger \) and \( b_k^\dagger \), which is related to the Rodrigues formula,

\[
(\xi - \partial_\xi)^n e^{-\xi^2 / 2} = H_n(\xi) e^{-\xi^2 / 2}. \tag{102}
\]

Thus the Fock space vector constructed from \( \phi_0 \) by operating the creation-annihilation operators are equivalent to the eigenstate of Eq. (82).
In the lowest eigenstate, the width of the distribution in the field configuration is proportional to $\nu$. Thus, in the vanishing noise limit $\nu \to 0$, this has a strong peak around $\{C_{R,k}, C_{I,k}\} = \{0, 0\}$ where the energy of the system disappears. This corresponds to the classical limit as we will see below.

Additionally, the lowest eigenstate is not normalisable for the massless case $\mu = 0$ because $\omega_{k=0} = 0$. Thus, to construct a consistent description in terms of the Fock state vectors, this mode should be excluded from the definition of the lowest eigenstate as

$$\psi(C_{R}, C_{I}, t) = \prod_{k \neq 0} \psi_k(C_{R,k}, C_{I,k}, t).$$  \hspace{1cm} (103)

**B. Noise intensity and Hamiltonian as the time evolution operator**

By using the arguments developed so far, we can determine the noise intensity.

First we identify the quanta introduced above with physical particles and hence the lowest eigenstate as the vacuum state. Then we can argue the single-particle energy.

As shown in Ref. [10], the Hamiltonian $H$ in SVM is obtained by the Legendre transform of the stochastic Lagrangian. In our case, it is given by

\[
H = \frac{1}{2c^2} \sum_{i=R,I} \sum_{k} \left[ -\frac{4\nu^2}{\Delta^3 k} \left( \frac{\partial^2}{\partial C_{i,k}^2} \right)^2 + \omega_k^2 C_{i,k}^2 \right] + \frac{1}{2c^2} \int D[C_R]D[C_I] \rho \sum_{i=R,I} \sum_{k} \left[ -\frac{4\nu^2}{\Delta^3 k} \left( \frac{\partial^2}{\partial C_{i,k}^2} \right)^2 + \omega_k^2 C_{i,k}^2 \right]. \hspace{1cm} (104)
\]

Note that the above result can be reexpressed as

\[
H = \langle \psi(t)|H|\psi(t) \rangle, \hspace{1cm} (105)
\]

where the Hamiltonian operator $H$ is defined by

\[
H = \frac{1}{2c^2} \sum_{i=R,I} \sum_{k} \left[ -\frac{4\nu^2}{\Delta^3 k} \frac{\partial^2}{\partial C_{i,k}^2} + \omega_k^2 C_{i,k}^2 \right], \hspace{1cm} (106)
\]

and we introduced the expectation value of a functional $A(C_{R}, C_{I}, t)$ by

\[
\langle \psi(t)|A|\psi(t) \rangle = \int \prod_k dC_{R,k} dC_{I,k} \psi^*(C_{R}, C_{I}, t) A(C_{R}, C_{I}, t) \psi(C_{R}, C_{I}, t). \hspace{1cm} (107)
\]
It should be noted that the Hamiltonian operator defined here has a different form from the time evolution operator $\tilde{H}$. Thus, in general, these are different operators.

The same Hamiltonian can be obtained from the invariance of the stochastic action for the time translation by using the stochastic Noether theorem. See Ref. \[9\]

Let us consider a single-particle state with a small wavenumber $k$ which is given by operating a single $a_k^\dagger$ to the vacuum,

$$
\psi(C_R, C_I, t) = e^{-i \omega_k t} \frac{1}{\sqrt{\Delta^3 k}} a_k^\dagger \psi_0,
$$

where, from Eq. (68), $\omega_k$ reduces to $c \sqrt{k^2 + \mu^2}$ at least for small $k$. As for the factor $1/\sqrt{\Delta^3 k}$, see the discussion in Sec. V D. Substituting this into Eq. (104), or equivalently Eq. (105), the energy is given by

$$
\mathcal{H} = \frac{2 \nu}{c} \sqrt{k^2 + \mu^2}.
$$

Here the contribution from the vacuum polarization is subtracted. On the other hand, from the Planck-Einstein-de Broglie relation, the single-particle energy is known as $\hbar c \sqrt{k^2 + \mu^2}$. To reproduce this result, the intensity of the noise should be given by

$$
\nu = \frac{\hbar c^2}{2}.
$$

It should be noted that, substituting this result into the time evolution operator $\tilde{H}$, we can find that the following relation is finally satisfied,

$$
H = \hbar \tilde{H}.
$$

That is, Eq. (82) can be interpreted as the functional Schrödinger equation and the Hamiltonian operator is the generator of the time evolution of this dynamics.

This relation is satisfied for other quantizations which we have studied so far and very general. Therefore, we will propound Eq. (111) as the condition to determine the intensity $\nu$ in the SVM quantization, instead of calculating the single-particle energies for each case.

Finally, the functional Schrödinger equation is expressed as

$$
\begin{align*}
    i\hbar \partial_t \psi(C_R, C_I, t) &= H \psi(C_R, C_I, t) \\
    &= \Delta^3 k \sum_k \sum_{j=R,I} \left[ -\frac{\hbar^2 c^2}{2(\Delta^3 k)^2} \left( \frac{\partial}{\partial C_{j,k}} \right)^2 + \frac{1}{2c^2} \omega_k^2 C_{j,k}^2 \right] \psi(C_R, C_I, t).
\end{align*}
$$

(112)
In addition, by using the creation-annihilation operators, the Hamiltonian operator is expressed as

$$H = \sum_{k} \hbar \omega_k (a_k^\dagger a_k + b_k^\dagger b_k + 1).$$  \hspace{1cm} (113)

V. OTHER ASPECTS IN SVM QUANTIZATION

We completed the quantization of the complex Klein-Gordon equation in SVM. In this section, we discuss other properties of the results of SVM.

A. Noether charge

From Eq. (113), we can reasonably observe that the energy is expressed by the summation of the particle and anti-particle contributions. However, this is not yet sufficient to conclude that the contribution from the anti-particles is precisely included. To verify the equivalence between SVM and the canonical quantization, it is necessary also to check the Noether charge, which should be given by the subtraction of the contributions from particles and anti-particles. The Noether theorem in SVM is discussed for the case of particle systems in Ref. 9. We show that this can be generalized for the case of fields.

The Lagrangian for the complex Klein-Gordon field is symmetric for the global phase transform,

$$\hat{\phi} \rightarrow \hat{\phi} e^{i\alpha} \approx (1 + i\alpha) \hat{\phi} = \hat{\phi} + \delta\hat{\phi},$$  \hspace{1cm} (114)

where $\alpha$ is a small number. Then the stochastic action is modified as

$$\delta I_{sto} = \frac{1}{2c^2} \int_{t_i}^{t_f} dt \frac{d}{dt} E \left[ (\delta \hat{\phi}^* \ast D\hat{\phi})_x + (\delta \hat{\phi} \ast D\hat{\phi}^*)_x + (\delta \hat{\phi}^* \ast \tilde{D}\hat{\phi})_x + (\delta \hat{\phi} \ast \tilde{D}\hat{\phi}^*)_x \right].$$  \hspace{1cm} (115)

In this derivation, we used the stochastic partial integration formula 36,

$$E[(D\tilde{X}(t))\tilde{Y}(t) + \tilde{X}(t)\tilde{D}\tilde{Y}(t)] = \frac{d}{dt} E[\tilde{X}(t)\tilde{Y}(t)],$$  \hspace{1cm} (116)

assuming that the expectation value $E[\ ]$ vanishes at the infinite distance.

Thus the Noether charge is expressed in SVM as

$$Q = \frac{ic}{\hbar c^2} E \left[ \left\{ \left\{ D\hat{\phi}_x + \tilde{D}\hat{\phi}_x \right\} \ast \hat{\phi}_x \right\}_x - \left\{ D\hat{\phi}_x + \tilde{D}\hat{\phi}_x \right\} \ast \hat{\phi}_x \right]_x.$$  \hspace{1cm} (117)
Here we choose $\alpha = 2e/\hbar$. This expectation value is expressed with the wave functional introduced above as

$$Q = 2\varepsilon \int \mathcal{D}[C_R] \mathcal{D}[C_I] \rho(C_R; C_I, t) \sum_k \left[ \frac{\partial \theta(C_R; C_I, t)}{\partial C_{I,k}} C_{R,k} - \frac{\partial \theta(C_R; C_I, t)}{\partial C_{R,k}} C_{I,k} \right].$$

(118)

By using the creation-annihilation operators introduced before, the above expression is rewritten as

$$Q = e \langle \psi(t) | \sum_k (a_k^\dagger a_k - b_{-k}^\dagger b_{-k}) | \psi(t) \rangle.$$  

(119)

This is equivalent to the result of the canonical quantization.

Thus the SVM quantization developed so far can deal in the anti-particle degrees of freedom correctly.

**B. Propagator**

We have shown that the theory obtained from SVM corresponds to the Schrödinger picture in quantum field theory. On the other hand, the propagator plays an important role in the Heisenberg and interaction pictures. This can be introduced as follows.

In SVM, the expectation value of a quantity $A(C_R; C_I)$ at time $t$ is calculated by

$$\langle A(t) \rangle = \int \mathcal{D}[C_R] \mathcal{D}[C_I] \rho(C_R; C_I, t) A(C_R; C_I).$$

(120)

It should be noted that the quantity $A(C_R; C_I)$ appearing in the Schrödinger picture in the SVM quantization is always given by the functional of $(C_R; C_I)$ and is not an operator.

On the other hand, the Heisenberg operator of $A$ is introduced by

$$A^{(H)}(C_R; C_I; t) = e^{iHt/\hbar} A(C_R; C_I) e^{-iHt/\hbar}.$$  

(121)

This is an operator because of the derivative operators contained in $H$. For example, when $A(\{C_R; C_I\}) = C_{R,k}$, the corresponding Heisenberg operator is

$$C^{(H)}_{R,k}(t) = \cos(\omega_k t) C_{R,k} - i \frac{\hbar^2}{2\omega_k} \sin(\omega_k t) \frac{\partial}{\partial C_{R,k}}.$$  

(122)

By using this operator, the expectation value of a general observable $A$ given by Eq. (120) is, in the Heisenberg picture, expressed as

$$\langle A \rangle = \int \mathcal{D}[C_R] \mathcal{D}[C_I] \psi^*(C_R; C_I, 0) A^{(H)}(C_R; C_I; t) \psi(C_R; C_I, 0).$$

(123)
FIG. 1: The dynamics of the symmetry breaking. Due to the change of external parameters, the initial potential denoted by the dashed line is transformed to the solid line.

In this form, the SVM expression for a propagator is written as

\[ i\hbar c \Delta F(t, x) = \langle \phi^{(H)}(x, t)\phi^*(0) \rangle \theta(t) + \langle \phi^*(0)\phi^{(H)}(x, t) \rangle \theta(-t) \]

\[ = \int \mathcal{D}[C_R] \mathcal{D}[C_I] \psi^*_0(C_R, C_I) \langle \phi^{(H)}(x, t)\phi^*(0)\theta(t) + \phi^*(0)\phi^{(H)}(x, t)\theta(-t) \rangle \psi_0(C_R, C_I), \]

where \( \theta \) is the Heaviside step function and \( \psi_0 \) is the vacuum wave functional given by Eq. (93). Using the expression (122) and performing the Gaussian integrals, we obtain finally

\[ \Delta F(t, x) = \Delta^3k \sum_k \int \frac{dk^0}{(2\pi)^4} \frac{e^{ikx}}{(k^0)^2 - q_k^2 - \mu^2 + i\epsilon}, \]

(125)

where \( kx = k^\mu x_\mu = k^0ct - k \cdot x \). Taking the continuum limit \( \Delta x \rightarrow 0 \) where the upper limit of the momentum integration goes to \( \infty \), \( q_k \) reduces to \( k \) and the above expression converges to the well-known Lorentz covariant result,

\[ \Delta_F(t, x) = \int \frac{dk^4}{(2\pi)^4} \frac{e^{ikx}}{k^2 - \mu^2 + i\epsilon}. \]

(126)

See Sec. [V.D] for more details of the continuum limit.

### C. Imaginary mass

We have so far shown that the quantization in SVM reproduces the results of the usual canonical quantization, when the Fock space vectors are introduced. This justifies the validity of the field quantization of the complex Klein-Gordon equation in SVM.

However, it should be noted that we need not introduce the Fock space in the procedure of the quantization, differently from the canonical quantization. In principle, the quantum
dynamics can be discussed by solving the functional Schrödinger equation Eq. (112), or equivalently the functional Euler equation Eq. (78), with Eq. (76) directly with an initial condition.

Furthermore, there exist situations where the construction of the Fock space is not well-known. As an example of such a situation, let us consider the Klein-Gordon equation with the imaginary mass, \( \mu = i|\mu| \). This is an interesting model because it is known that the Klein-Gordon equation with the imaginary mass can be mapped to the telegraph equation which is an important equation in classical physics but the corresponding quantum equation is not known.

It is also somewhat related to the dynamics of symmetry breaking. As is shown in Fig. 1, the state should move from the unstable local maximum of an effective potential to the local minimum during the transition between two phases: one is the restored phase (dashed line) and the other the broken phase (solid line). It is considered that, at least, the early stage of this time evolution will be approximately described by the imaginary mass Klein-Gordon equation, for example, in the linear \( \sigma \) model. One can easily see that there is no stationary solution in Eq. (86) since \( \omega_k \) becomes pure imaginary for \( |q_k| < |\mu| \). That is, the Fock state vector cannot be constructed as was discussed in Sec. IV A.

There are several approaches to discuss the similar early-stage dynamics of the phase transition [37]. As far as the authors know, most of approaches are based on the semiclassical approximation of the full quantum dynamics. Here we investigate this dynamics by solving the functional Schrödinger equation.

For the sake of simplicity, we ignore the \( C_{I,k} \) variable and consider the solution for \( C_{R,k} \). We further drop the suffix \( k \) since the adimensional form of the equation is universal for all \( k \) values. Then the functional Schrödinger equation to be solved is

\[
i\partial_\tau \psi(\xi) = \partial^2_\xi \psi(\xi) + \xi^2 \psi(\xi), \tag{127}\]

with \( \tau = ct\sqrt{|\mu|^2 - q_k^2}/2 = \bar{\omega}t/2 \) and \( \xi = C_{R,k}\sqrt{\bar{\omega}(\Delta^2 k)/(\bar{h}c^2)} \). This is nothing but the adimensional Schrödinger equation with the inverted harmonic oscillator potential.

As the initial condition, we consider a Gaussian distribution around \( \xi = 1/2 \),

\[
\psi(\xi, 0) = e^{-(\xi-1/2)^2/2}. \tag{128}\]

The time evolutions of the real and imaginary parts of \( \psi(\xi, t) \) at \( t = 2/\bar{\omega} \) are shown
FIG. 2: The real part (the left panel) and the imaginary part (the right panel) of $\psi(\xi, \tau)$ at $\tau = 1$. The initial condition is given by Eq. (C1).

FIG. 3: The time evolution of the density $|\psi(\xi, \tau)|^2$. The dashed and solid lines represent the initial condition and the result at $\tau = 1$, respectively. One can see that the initial Gaussian distribution diffuses exhibiting oscillation.

The corresponding configuration distribution $|\psi(\xi, t)|^2$ is shown in Fig. 3. The initial density is denoted by the dashed line and the result at $t = 2/\bar{\omega}$ is given by the solid line. The configuration distribution shows a monotonic diffusion without oscillation, and the
initial distribution of the wave functional expands indefinitely. The motion of the center of the wave functional is the same as the classical motion of the field with the initial condition $\xi = 1/2$, and it seems that the fluctuation of the field configuration evolves uniformly in positions anterior and posterior of this classical path.

D. Continuum limit

We have developed the quantization in SVM by introducing the discretized variables. The continuum limit of these variables are given by $L, N \to \infty$ keeping $\Delta^3 x = (L/N)^3 \to 0$. Then the momentum sum appearing in our formulation is replaced by the corresponding integral as

$$\Delta^3 k \sum_k \longrightarrow \int d^3 k.$$  \hfill (129)

And the commutation relations in this limit are expressed as

$$[a_k, a_{k'}^\dagger] = \delta^{(3)}_{k,k'} \longrightarrow [a(k), a^\dagger(k')] = \delta^{(3)}(k - k'),$$ \hfill (130)

$$[b_k, b_{k'}^\dagger] = \delta^{(3)}_{k,k'} \longrightarrow [b(k), b^\dagger(k')] = \delta^{(3)}(k - k'),$$ \hfill (131)

where

$$\lim_{V \to \infty} \sqrt{\frac{1}{\Delta^3 k}} a_k = a(k),$$ \hfill (132)

$$\lim_{V \to \infty} \sqrt{\frac{1}{\Delta^3 k}} b_k = b(k),$$ \hfill (133)

and

$$\lim_{V \to \infty} \frac{1}{\Delta^3 k} \delta^{(3)}_{k,k'} = \delta^{(3)}(k - k').$$ \hfill (134)

One can confirm that the quantities calculated from the functional Schrödinger equation such as the wave functional and the propagator have the well-defined continuum limits by using the above results.

On the other hand, by introducing the functional derivative as

$$\lim_{\Delta^3 k \to 0} \frac{1}{\Delta^3 k} \frac{\partial}{\partial C_{i,k}} = \frac{\delta}{\delta C_{i,k}}.$$ \hfill (135)

the functional Schrödinger equation is expressed as

$$ih \partial_t \psi(\mathbf{C}_R, \mathbf{C}_I, t) = \sum_{j=R,I} \int \frac{d^3 k}{(2\pi)^3} \left[ -\frac{\hbar^2 c^2}{2} \left( \frac{\delta}{\delta C_{j,k}} \right)^2 + \frac{\omega_{k}^2}{2c^2} C_{j,k} \right] \psi(\mathbf{C}_R, \mathbf{C}_I, t).$$ \hfill (136)
Note that the higher order functional derivative such as $(\delta/\delta C_{j,k})^2$ appearing here is known to be able to provoke a singular behavior. However, as is discussed in Refs. [46, 47], this singularity does not induce any problem for the calculations of various observables when it is regularized by, for example, introducing the lattice. Thus we need to keep in mind that the above continuum forms are defined in the lattice representations per se. That is, the lattice representation which was originally introduced to deal in stochastic variables is essential to write down quantum dynamics in SVM.

There is another remark for the ultraviolet behavior of the single-particle energy. In fact, because of $q_k$, the single-particle energy $\omega_k = c\sqrt{q_k^2 + \mu^2}$, which agrees with the pole of the propagator (125), starts to deviate from the usual expression $c\sqrt{k^2 + \mu^2}$ near the momentum cutoff

$$|k_{\text{max}}| = \frac{2\pi}{\Delta x}. \quad (137)$$

Thus, to be consistent with the usual results in quantum field theory, $\Delta x$ must be sufficiently small.

VI. CONCLUDING REMARKS

We discussed the field quantization in the complex Klein-Gordon equation in the framework of the stochastic variational method. In this scheme, the dynamics of the quantized field is determined by the Euler equation in the functional space of the field configuration. In this representation, the quantum effect is expressed by the functional quantum potential and this is the natural generalization of the Mandelung-Bohm theory in quantum mechanics. It is also possible to re-express the functional Euler equation as the Schrödinger equation by introducing the wave functional for the field configuration. Note that the functional Euler equation can be expressed in the form of the the Hamilton-Jacobi equation. This equation for quantum field theory is known for the massless real Klein-Gordon equation. See Eqs. (12.4.9-12.4.11) in Ref. [33]. As for the various approaches for the hydrodynamic representations of quantum dynamics, see Refs. [33, 35, 38, 40].

The functional Euler equation obtained in SVM is not completely equivalent to the functional Schrödinger equation, because the divergence associated with the vacuum polarization does not appear in the former approach. However, the Casimir effect should be still contained in this hydrodynamic form, because the ambiguity of the definition of the phase in
the wave functional cannot remove the spatially inhomogeneous vacuum energy.

We further discussed that the noise intensity is determined by calculating the single-particle energy defined by the Lagrange transform of the stochastic Lagrangian. Then it was found that the time evolution of the wave functional must be induced by the Hamiltonian operator. It should be emphasized that this property is one of the postulates in the canonical quantization, while it is obtained as the consequence of the variational principle in SVM.

The Fock state vectors in SVM are given by the stationary solutions of the functional Euler equation or, equivalently, the eigenstates of the functional Schrödinger equation. By using the stochastic Noether theorem, we further confirmed that the anti-particle degrees of freedom are correctly reproduced on this Fock space. Thus we can conclude that SVM functions as the quantization scheme not only particles but also fields.

The Noether charge in the SVM quantization is obtained from the invariance of the stochastic action, while the quantum dynamics is derived from the stationary condition of the same action for the variation. That is, dynamics and symmetry is closely connected in the SVM quantization, as is the case for classical mechanics.

The functional Schrödinger equation in the $x$-space is known as the Schrödinger representation of quantum field theory, which is derived by replacing the conjugate filed with the functional derivative so as to reproduce the commutation relation, for example, 

\[
[\phi(x), \pi(x')] = [\phi(x), -i\hbar \delta/\delta \phi(x')] = i\hbar \delta(x - x').
\]

This approach is expected to have advantages in, for example, the study of the ultraviolet behavior of the Casimir force and the quantization of the gravity comparing with the canonical quantization. See for example, Refs. [46–48]. In these studies, it is known that some regularization scheme should be introduced to give a mathematically well-defined meaning to the functional equation. In the SVM quantization, the introduction of such a regularization is interpreted as the existence of the discontinuous behaviors of the stochastic field variables. As another interesting aspect, the functional Schrödinger equation is expressed in the form of the differential equation and may furnish an alternative method in a numerical study of quantum field theory.

It is known that, differently from quantum mechanics, infinitely many unitary non-equivalent Fock spaces are contained in quantum field theory and hence it is not trivial which one should be chosen. See, for example, Refs. [41, 42]. So far, this choice of the Fock space is known only in the scattering theory and in the case where Landau’s Fermi-liquid ansatz is applicable. While it causes several difficulties, the presence of these unitary non-
equivalent Fock spaces manifests a rich structure of vacua in quantum field theory, such as symmetry breaking. We confirmed that the unitary non-equivalent representations appear in the continuum limit in our formulation. See Appendix C for details.

To establish SVM as the alternative quantization scheme, the applicability to the gauge field and the Dirac field should be studied. Our formulation may be considerably modified, in particular, in the application to the fermionic field. For example, the Wiener process used so far in the SDEs may not be applicable in this case, because the stochastic properties of the Dirac equation seems to be the Poisson process as is discussed in Refs. [43–45].

The stochastic quantization by Parisi and Wu is another quantization method based on the stochastic dynamics [49]. The principal difference between SVM and Parisi-Wu comes from the origin of the noise. In the stochastic quantization of Parisi-Wu, an additional time dimension is introduced to quantize systems and the stochastic dynamics is considered for this time. More detailed comparison is an issue in the future.

In SVM approach, all physical quantities are expressed by the expectation value with the wave functional of the field configuration. This structure make us remember the random matrix theory where the probability is introduced for matrix elements. The relation between quantum physics and the random matrix theory from the perspective of SVM will be an interesting problem.

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**Appendix A: Eigenvectors of Laplacian Matrix**

Let us consider the following eigenvalue problem,

$$\Delta x u^{(n)} = \lambda_n u^{(n)},$$  \hspace{1cm} (A1)

where $n = 0, \ldots, N - 1$. As the eigenvector, we assume the following form,

$$u = \begin{pmatrix} e^{i\alpha} \\ e^{i2\alpha} \\ \vdots \\ e^{iN\alpha} \end{pmatrix}$$ \hspace{1cm} (A2)
with the periodic boundary condition as

$$e^{i(N+l)\alpha} = e^{i\alpha},$$  \hspace{1cm} (A3)

for any integer $l$. Then $\alpha$ is determined by

$$\alpha_n = \frac{2\pi}{N}n,$$  \hspace{1cm} (A4)

where $n = 0, \cdots, N-1$. Substituting this assumption, the eigenvalue equation gives the eigenvalue as

$$\lambda_n = -\left[ \frac{2}{\Delta x} \sin \left( \frac{k_n \Delta x}{2} \right) \right]^2,$$  \hspace{1cm} (A5)

with

$$k_n = \frac{2\pi}{L}n.$$  \hspace{1cm} (A6)

Note that the pair of eigenvalues $\lambda_n$ and $\lambda_{N-n}$ are degenerated,

$$\lambda_{N-n} = \lambda_n.$$  \hspace{1cm} (A7)

Thus, for the sake of convenience, we assign a number to the eigenvectors as follows,

$$u^{(n)} = \sqrt{\frac{1}{\pi}} \begin{pmatrix} \sin (k_n x_0) \\ \sin (k_n x_1) \\ \vdots \\ \sin (k_n x_{N-1}) \end{pmatrix},$$  \hspace{1cm} (A8)

$$u^{(n+1)} = \sqrt{\frac{1}{\pi}} \begin{pmatrix} \cos (k_n x_0) \\ \cos (k_n x_1) \\ \vdots \\ \cos (k_n x_{N-1}) \end{pmatrix},$$

where $n = 1, 3, \ldots, N-2$ and the corresponding eigenvalues are degenerated,

$$\lambda_n = \lambda_{n+1}.$$  \hspace{1cm} (A9)

For $n = 0$, we set

$$u^{(0)} = \sqrt{\frac{1}{2\pi}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$  \hspace{1cm} (A10)

These eigenvectors form an orthogonal basis,

$$\left( u^{(n)} \ast u^{(l)} \right)_x = \frac{1}{\Delta k} \delta_{n,l},$$  \hspace{1cm} (A11)
with the definition of the scalar product,

\[ (u^{(n)} \ast u^{(l)})_{\Delta x} \equiv \Delta x \sum_{m=0}^{N-1} (u^{(n)})_m (u^{(l)})_m. \]  
\[ \text{(A12)} \]

With this basis, any vector \( \underline{f} \) in the \( x \)-space can be expanded as

\[ \underline{f} = \Delta k \sum_{n=0}^{N-1} c_n u^{(n)}, \]  
\[ \text{(A13)} \]
with

\[ c_n = \left( u^{(n)} \ast \underline{f} \right)_x. \]  
\[ \text{(A14)} \]

Then this vector \( \underline{\varphi}^T = (c_0, c_1, ..., c_{N-1}) \) forms the conjugate vector of \( \underline{f} \) in the \( k \)-space.

Substituting this expression into Eq. (A13), we obtain the following relation,

\[ \sum_{n=0}^{N-1} u^{(n)} u^{(n)T} = \frac{1}{\Delta k \Delta x} \mathbf{I}, \]  
\[ \text{(A15)} \]
where \( \mathbf{I} \) is the \( (N \times N) \) identity matrix. If we introduce \( (N \times N) \) matrix \( \mathbf{O} \) as

\[ \mathbf{O} = \left( \begin{array}{cccc} u^{(0)} & \cdots & u^{(N-1)} \end{array} \right), \]  
\[ \text{(A16)} \]
Equations (A11) and (A15) are expressed more compactly as,

\[ \mathbf{O}^T \mathbf{O} = \mathbf{O} \mathbf{O}^T = \frac{1}{\Delta k \Delta x} \mathbf{I}. \]  
\[ \text{(A17)} \]

With this matrix, the transform between \( \underline{f} \) and \( \underline{\varphi} \) is expressed as

\[ \underline{\varphi} = (\Delta x) \mathbf{O}^T \underline{f}, \]  
\[ \text{(A18)} \]
or

\[ \underline{f} = (\Delta k) \mathbf{O} \underline{\varphi}. \]  
\[ \text{(A19)} \]

By writing \( b_0 = c_0/\sqrt{2\pi}, a_n = c_{2n-1}/\sqrt{\pi}, b_n = c_{2n}/\sqrt{\pi} \), Eq. (A19) is expressed as

\[ f_l = b_0 \frac{2\pi}{L} + \frac{2\pi}{L} \sum_{n=1}^{(N-1)/2} \left\{ a_n \sin k_n x_l + b_n \cos k_n x_l \right\}. \]  
\[ \text{(A20)} \]

Therefore, in the limit of \( N \to \infty \), Eqs. (A18) and (A19) coincide with the Fourier series of a continuous function \( f(x) \), with, at most, a finite number of discontinuities. However, in our case, \( \{f_l = f(x_l)\} \) are defined only on discrete grid points so that we cannot use the right-hand side of Eq. (A20) to construct continuous function for arbitrary \( x \).
Another convenient basis can be used by introducing the (complex) linear transformation among the two degenerate eigenstates as

\[ \begin{align*}
\mathbf{u}^{(n)'} &= \frac{1}{\sqrt{2}} \left( \mathbf{u}^{(n)} + i\mathbf{u}^{(n+1)} \right), \\
\mathbf{u}^{(-n)'} &= \frac{1}{\sqrt{2}} \left( \mathbf{u}^{(n)} - i\mathbf{u}^{(n+1)} \right)
\end{align*} \tag{A21, A22} \]

Then the new vector is expressed as

\[ \mathbf{u}^{(n)'} = \frac{1}{\sqrt{N}} \begin{pmatrix}
    e^{-i\frac{N-1}{2}\alpha_n} \\
    \vdots \\
    1 \\
    \vdots \\
    e^{i\frac{N-1}{2}\alpha_n}
\end{pmatrix}, \tag{A23} \]

for \( n = -\frac{N-1}{2}, \ldots, 0, \ldots, \frac{N-1}{2} \).

By using this, Eq. \( \text{(A13)} \) is given by

\[ \mathbf{f} = \Delta k \sum_{n=-(N-1)/2}^{(N-1)/2} c_n \mathbf{u}^{(n)'} . \tag{A24} \]

These discretisation scheme can be extended even in the higher dimensional space in a similar manner by introducing the direct product space. For example, for the two space-dimensional case \((x, y)\), the eigenvector is expressed as

\[ \mathbf{u}^{(k)} = \mathbf{u}^{(l)} \otimes \mathbf{u}^{(m)} \tag{A25} \]

where the index \( k \) denotes the combination of \((l, m)\). Then, the normalization \( \text{(A11)} \) is given by

\[ \left( \mathbf{u}^{(k)} \right. \mathbf{u}^{(l)} \right)_x = \frac{1}{(\Delta k)^2} \delta_{k,l}. \tag{A26} \]

By using this basis, a function \( f(x, t) \) in the two spatial dimension is expanded as

\[ \mathbf{f} = (\Delta k)^2 \sum_k c_k \mathbf{u}^{(k)}. \tag{A27} \]

On the other hand, the spatial gradient matrices are extended as

\[ \begin{align*}
\nabla_+ &= \left( \nabla_+^{(x)} \otimes I^{(y)} \right) \mathbf{e}_x + \left( I^{(x)} \otimes \nabla_+^{(y)} \right) \mathbf{e}_y, \\
\nabla_- &= \left( \nabla_-^{(x)} \otimes I^{(y)} \right) \mathbf{e}_x + \left( I^{(x)} \otimes \nabla_-^{(y)} \right) \mathbf{e}_y,
\end{align*} \tag{A28, A29} \]
where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the spatial unit vector for the $x$ and $y$ directions, respectively. Then the corresponding Laplacian matrix is expressed by

$$
\Delta_x = \nabla_x^2 \otimes I^{(y)} + I^{(x)} \otimes \nabla_y^2.
$$

(A30)

**Appendix B: Quantization in x-Space Representation**

We can apply the SVM quantization directly to the field $\phi(\mathbf{x}, t)$ defined on the discretized lattice $x$-space, in the same way discussed in Sec.II.

To introduce the SDEs for the real quantities, we re-express the stochastic action Eq. (63) in terms of the real scalar field as

$$
L_{SV M} = \frac{1}{2} \sum_{j=1,R} E \left[ \frac{1}{c^2} \frac{1}{2} (D_{\tilde{\phi}}(t) * D_{\tilde{\phi}}(t)) \Delta x + (\dot{D}_{\tilde{\phi}}(t) * \dot{D}_{\tilde{\phi}}(t)) \Delta x 
+ (\tilde{\phi}_j(t) * \Delta x \tilde{\phi}_j(t)) \Delta x - \mu^2 (\tilde{\phi}_j(t) * \tilde{\phi}_j(t)) \Delta x \right],
$$

(B1)

where

$$
\tilde{\phi}_x(t) = \frac{\tilde{\phi}_{R,x}(t) + i \tilde{\phi}_{I,x}(t)}{\sqrt{2}},
$$

(B2)

$$(f * h) \Delta x = \Delta^3 x \sum_x f_x h_x,
$$

(B3)

with $\Delta^3 x$ being $(\Delta x)^3 = (L/N)^3$.

As was done for the classical string, the SDEs used in the $x$-space are related to the ones in the $k$-space through the orthogonal matrix which diagonalizes the three dimensional Laplacian $\Delta_x$. Then, we obtain

$$
d\tilde{\phi}_{i,x}(t) = u_{i,x}(\tilde{\phi}_{-R}(t), \tilde{\phi}_{-I}(t), t) dt + \sqrt{\frac{2\nu}{\Delta x^3}} dW_{i,x}(t),
$$

(B4)

$$
d\tilde{\phi}_{i,x}(t) = \tilde{u}_{i,x}(\tilde{\phi}_{-R}(t), \tilde{\phi}_{-I}(t), t) dt + \sqrt{\frac{2\nu}{\Delta x^3}} d\tilde{W}_{i,x}(t),
$$

(B5)

where

$$
E[dW_{i,x}(t)] = E[d\tilde{W}_{i,x}(t)] = 0,
$$

(B6)

$$
E[dW_{j,x'}(t)dW_{i,x}(t)] = E[d\tilde{W}_{j,x'}(t)d\tilde{W}_{i,x}(t)] = \delta_{jk} \delta_{x,x'} |dt|,
$$

(B7)

Here the index $i$ denotes $R$ or $I$. There is no correlation between $dW_{i,x}(t)$ and $d\tilde{W}_{i,x}(t)$. 
The configuration distribution is then defined by
\[ \rho(\phi_R, \phi_I, t) = E \left[ \prod_x \delta(\phi_{R,x} - \hat{\phi}_{R,x}(t)) \delta(\phi_{I,x} - \hat{\phi}_{I,x}(t)) \right]. \] (B8)

The self-consistency condition is derived from the equivalence of the two Fokker-Planck equations as
\[ u_{i,x}(\phi_R, \phi_I, t) = \tilde{u}_{i,x}(\phi_R, \phi_I, t) + 2\nu \frac{\partial}{\Delta^3x} \partial\phi_{i,x} \ln \rho(\phi_R, \phi_I, t). \] (B9)

Using this condition, the Fokker-Planck equation is expressed as
\[ \partial_t \rho(\phi_R, \phi_I, t) = -\sum_{j=I,R} \sum_x \partial_{\phi_{j,x}} \{ \rho(\phi_R, \phi_I, t) v_{j,x}(\phi_R, \phi_I, t) \}. \] (B10)

Here the mean functional velocity is defined by
\[ v_{i,x}(\phi_R, \phi_I, t) = \frac{u_{i,x}(\phi_R, \phi_I, t) + \tilde{u}_{i,x}(\phi_R, \phi_I, t)}{2}. \] (B11)

Then the functional Euler equation is obtained as the result of the stochastic variation,
\[ \left( \partial_t + \sum_{j=I,R} \sum_{x'} v_{j,x'}(\phi_R, \phi_I, t) \frac{\partial}{\partial\phi_{j,x'}} \right) v_{i,x}(\phi_R, \phi_I, t) \]
\[ - \frac{2\nu^2}{(\Delta^3x)^2} \frac{\partial}{\partial\phi_{i,x}} \left\{ \rho^{-1/2}(\phi_R, \phi_I, t) \sum_{j=I,R} \sum_{x'} \left( \frac{\partial}{\partial\phi_{j,x'}} \right)^2 \rho^{1/2}(\phi_R, \phi_I, t) \right\} \]
\[ = c^2 ((\Delta_x - \mu^2)\phi_i)_x. \] (B12)

This is expressed in the form of the functional Schrödinger equation,
\[ i\hbar \partial_t \Psi(\phi_R, \phi_I, t) = -\sum_{i=I,R} \sum_x \left[ -\frac{\hbar^2 c^2}{(\Delta^3x)^2} \frac{\partial^2}{\partial\phi_{i,x}^2} - \phi_{i,x} \Delta_x \phi_{i,x} + \mu^2 \phi_{i,x}^2 \right] \Psi(\phi_R, \phi_I, t). \] (B13)

Here the wave functional is given by
\[ \Psi(\phi_R, \phi_I, t) = \sqrt{\rho(\phi_R, \phi_I, t)} e^{i\theta(\phi_R, \phi_I, t)}, \] (B14)

where
\[ v_{i,x}(\phi_R, \phi_I, t) = \frac{2\nu}{(\Delta^3x)^2} \partial_{\phi_{i,x}} \theta(\phi_R, \phi_I, t). \] (B15)

The vacuum wave functional is
\[ \Psi_0(\phi_R, \phi_I) = N \exp \left\{ -\frac{1}{2\hbar c} \sum_{j=I,R} \int d^3x \phi_j(x) \sqrt{-\Delta_x + \mu^2 \phi_j(x)} \right\}, \] (B16)
where
\[ N = \left( \det \left[ \frac{\Delta^3 x \sqrt{-\Delta x + \mu^2}}{\hbar c \pi} \right] \right)^{1/2}. \] (B17)

Here the operation of the operator \( \sqrt{-\Delta x + \mu^2} \) is defined in the orthogonal basis of \( \Delta x \).

One can easily check that this solution is the same as the functional vacuum wave function obtained in the \( k \)-space, (93), by noting that \( \phi_{i,x} \) and \( C_{i,k} \) are connected through the orthogonal transform.

**Appendix C: Unitary Non-equivalent Representations in Quantum Field Theory**

The expression of the vacuum wave functional (93) resembles the ground state wave function of the solution of the Schrödinger equation with the harmonic oscillator potential in quantum mechanics. In the quantum mechanical case, the stationary states form the complete set and any time evolution of state vectors can be expressed by the linear combination of these states. However, this is not applicable in quantum field theory because of the property of the unitary non-equivalent representation [41, 42].

To see this, let us consider, for example, the following initial condition,
\[ \psi_{\text{ini}}(\phi_R, \phi_I) = \Pi_k \left( \frac{2\Omega_k}{\pi \hbar c^2} \right)^{1/2} e^{-\Omega_k (C^2_{R,k} + C^2_{I,k})/(\hbar c^2)}, \] (C1)

where \( \Omega_k = c\sqrt{q_k^2 + M^2} \) with \( M > m \). Suppose the time evolution of this initial state can be expressed by the linear combination of the eigenstates which are obtained in Sec. IV A.

Then, for example, the overlap with the vacuum wave functional is given by
\[
\int D[C_R] D[C_I] \psi^*_\text{ini}(\bar{\phi}_R, \bar{\phi}_I) \psi_0(\phi_R, \phi_I) = \Pi_k \frac{2\sqrt{\Omega_k \omega_k^m}}{\omega_k + \Omega_k} = e^{-\frac{1}{(\Delta k)^2} \int d^3 k \ln \frac{\omega_k \Omega_k}{2\sqrt{\omega_k \Omega_k}}} = 0. \] (C2)

In the last line, we take the limit of \( V \to \infty \) noting that \( (\omega_k + \Omega_k) / 2\sqrt{\omega_k \Omega_k} \geq 1 \). Similarly, we can show that there is no overlap between \( \psi_{\text{ini}}(\bar{\phi}_R, \bar{\phi}_I) \) and any Fock state vector constructed from \( \psi_0(\phi_R, \phi_I) \). Thus, the time evolution should be calculated by directly solving the functional Schrödinger equation (82) with the initial state (C1).

As another example, let us consider a uniform translation of the amplitude by a \( c \)-number, say \( \phi_0 \),
\[ \phi(x) \to \phi(x) + \phi_0, \] (C3)
which is a coherent transform and often appears to introduce the spontaneous breaking of symmetry. In the $k$-space representation, this corresponds to a shift by a amount of $\phi_0$ in the $k = 0$ mode. It is easy to verify that the overlap between the original and shifted vacua is given by

$$\exp \left\{ -\frac{1}{\Delta^3 k} \frac{\mu}{2\hbar c} \phi_0^2 \right\}.$$  \hspace{1cm} (C4)

This disappears in the continuum limit, $\Delta^3 k \to 0$. 