A NOTE ON THE EXISTENCE RESULTS FOR SCHRÖDINGER-MAXWELL SYSTEM WITH SUPER-CRITICAL NONLINEARITIE

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Abstract. The paper considers the following Schrödinger-Maxwell system with super-critical nonlinearitie,

\[
\begin{aligned}
-\Delta u + K(x)\phi u &= |u|^{p-1}u + h(x), \quad \text{in } \Omega, \\
-\Delta \phi &= K(x)u^2, \quad \text{in } \Omega, \\
\phi = u = 0, \quad &\text{in } \partial \Omega,
\end{aligned}
\]

where \(\Omega \subset \mathbb{R}^3\) is a bounded domain with smooth boundary, \(1 < p\) and \(K, h \in L^\infty(\Omega)\). We prove the existence of at least one non-trivial weak solution. This result is already known for the subcritical case. In this paper, we extend it to the supercritical values of \(p\) as well. We use a new variational principle to prove our result.

1. Introduction and main results

In the present paper we study the existence of solution for the following electrostatic non-linear Schrödinger-Maxwell equations also known as nonlinear Schrödinger-Poisson system

\[
\begin{aligned}
-\Delta u + K(x)\phi u &= |u|^{p-1}u + h(x), \quad \text{in } \Omega, \\
-\Delta \phi &= K(x)u^2, \quad \text{in } \Omega, \\
\phi = u = 0, \quad &\text{in } \partial \Omega,
\end{aligned}
\]

where \(\Omega \subset \mathbb{R}^N, \ (N = 3),\) is a bounded domain with smooth boundary, \(1 < p\) and \(K, h \in L^\infty(\Omega)\).

Similar system arises in many mathematical physics contexts while looking for existence of standing waves for the nonlinear Schrödinger equations interacting with an unknown electrostatic field. For more details on the physics aspect we refer the reader to [6, 9].

In recent years, a number of papers have contributed to investigate the existence of solutions of (1.1). We can cite [1, 2, 3, 4, 7, 8, 9, 11, 12, 16] and the references therein. For the case where \(\Omega\) is a bounded domain, we would like to cite the papers of Ruiz and Siciliano [17] and Siciliano [18]. In all those papers, the solutions found are in the case where \(1 < p < 5\).

In the unbounded case, Ambrosetti and Ruiz [2] and Ruiz [16] considered problem

\[
\begin{aligned}
-\Delta u + V(x)u + \mu \phi u &= |u|^{p-1}u, \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= 4\pi^2u^2, \quad \text{in } \mathbb{R}^3.
\end{aligned}
\]

By working in the radial functions subspace of \(H^1(\mathbb{R}^3)\) and taking \(1 < p < 5\) and \(V(x) = 1\), they were able to obtain the existence and multiplicity results.

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In [12], Jiang and Zhou have treated the problem (1.2) where $\Omega = \mathbb{R}^3$, $K = \lambda > 0$, $1 < p < 6$ and $V$ change sign. With further assumptions on $V$, the authors have proved that problem (1.1) has at least a positive solution.

If $0 < p < 1$, Bahrouni and Ounaies [4] has treated system (1.1) where $\Omega = \mathbb{R}^3$. By using the variational method, they have proved that problem (1.1) has infinitely many solutions. We also refer to [5, 13, 17, 18, 19].

Motivated by papers above, we are interested in finding solution for system (1.1), by assuming only that $p > 1$. Our methodology is based on a new variational principle established in [14, 15].

In order to state our main results, we give the following assumptions:

\[ (K) \quad K \in L^\infty(\Omega) \text{ and } K(x) \geq 0, \quad \forall x \in \Omega. \]

\[ (H) \quad h \in L^\infty(\Omega) \text{ and } h(x) > 0, \quad \forall x \in \Omega. \]

Now we can state our result.

**Theorem 1.1.** Assume that $(H)$ and $(K)$ hold. Suppose that $p > 1$. Then, there exists $m > 0$ such that if $\|h\|_{L^N(\Omega)} \leq m$, problem (1.1) admits at least one nontrivial solution.

The remainder of our paper is organized as follows. In section 2, some preliminary results are presented. While section 3 is dedicated to the proof of Theorems [14].

### 2. Variational settings and preliminary results

First, we give some notations. For $1 \leq m < +\infty$, $L^m(\Omega)$ is the usual Lebesgue space with the norm

$$\|u\|_{L^m(\Omega)} = \left( \int_{\Omega} |u|^m \, dx \right)^{\frac{1}{m}}.$$

Hereafter, the space $E = H^1_0(\Omega) \cap L^{p+1}(\Omega)$ is endowed with the following norm

$$\|u\| = \|\nabla u\|_2 + \|u\|_{p+1}.$$

We shall now recall some results for the Sobolev space required in the sequel (see [10, 14]).

**Lemma 2.1.** Let $\Omega$ be a bounded $C^{0,1}$ domain in $\mathbb{R}^N$. Then:

i) If $0 \leq m < k - \frac{N}{p} < m + 1$, the space $W^{k,p}(\Omega)$ is continuously imbedded in $C^{m,\alpha}(\overline{\Omega})$, $\alpha = k - \frac{N}{p} - m$, and compactly imbedded in $C^{m,\beta}(\overline{\Omega})$ for any $\beta < \alpha$.

ii) $u \to \|\Delta u\|_{L^N(\Omega)}$ is an equivalent norm on $H^1_0(\Omega) \hookrightarrow W^{2,N}(\Omega) = E \hookrightarrow W^{2,N}(\Omega)$.

An important fact involving system (1.1) is that this class of system can be transformed into a Schrödinger equation (see, for instance [9, 16]), with a nonlocal term. By the Lax-Milgram Theorem, given $u \in E$, there exists a unique $\phi_u \in H^1_0(\Omega)$ such that $-\Delta \phi_u = K(x)u^2$. By using standard arguments, we have that $\phi_u$ verifies the following properties (see [9, 16]):

**Lemma 2.2.** For any $u \in E$, we have

1) there exists $C > 0$ such that $\|\phi_u\| \leq C\|u\|^2$.

2) $\phi_u \geq 0$, $\phi_{tu} = t^2 \phi_u$, $\forall t \geq 0$ and $u \in E$.

3) If $u_n \rightharpoonup u$ in $E$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $E$ and $\lim_{n \to +\infty} \int_{\Omega} \phi_{u_n} u_n^2 \, dx = \int_{\Omega} \phi_u u^2 \, dx$.

4) If $u \in W^{2,N}(\Omega) \hookrightarrow H^1_0(\Omega)$, then $\phi_u \in W^{2,N}(\Omega) \hookrightarrow H^1_0(\Omega)$.
and if \( u / \) the Gateaux derivative of following set-valued operator: if

\[
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{4} \int_{\Omega} K(x)\phi_u u^2 \, dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} \, dx - \int_{\Omega} h(x)u \, dx, \quad \forall u \in E
\]

is \( C^1 \) on \( E \) and

\[
\langle I'(u), \varphi \rangle = \int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\Omega} K(x)\phi_u u \varphi \, dx - \int_{\Omega} |u|^{p-1} u \varphi \, dx - \int_{\Omega} h(x) \varphi \, dx,
\]

for all \( u \in E \) and \( \varphi \in H_0^1(\Omega) \). It is also known that \((u, \phi) \in E \times H_0^1(\Omega)\) is a solution of (1.1) if and only if \( u \in E \) is a critical point of the functional \( I \), and \( \phi = \phi_u \), see for instance [6]. Let us recall that a Palais-Smale sequence for the functional \( I \), for short we write (PS)-sequence, is a sequence \((u_n)\) such that

\[
(I(u_n)) \text{ is bounded in } E \text{ and } \left\| I'(u_n) \right\|_{E'} \to 0.
\]

I is said to satisfy the Palais-Smale condition if any (PS)-sequence possesses a convergent subsequence in \( E \). Now, we recall some important definitions and results from [10].

Let \( E \) be a real Banach space. Let \( \psi : E \to \mathbb{R} \sim \{\infty\} \) be a proper (i.e. \( \text{Dom}(\psi) = \{u \in E; \psi(u) < \infty\} \neq \emptyset \)) convex function. The subdifferential \( \partial \psi \) of \( \psi \) is defined to be the following set-valued operator: if \( u \in \text{Dom}(\psi) \), set

\[
\partial \psi(u) = \{u' \in E'; \langle u', v - u \rangle + \psi(u) \leq \psi(v), \quad \forall v \in E\},
\]

and if \( u \notin \text{Dom}(\psi) \), set \( \partial \psi(u) = \emptyset \). If \( \psi \) is Gateaux differentiable at \( u \), denote by \( D\phi(u) \) the Gateaux derivative of \( \psi \) at \( u \). In this case \( \partial \psi(u) = \{D\psi(u)\} \).

The restriction of \( \psi \) to \( K \subset E \) is denoted by \( \psi_{K} \) and defined by

\[
\psi_{K}(u) = \psi(u) \quad \text{if} \quad u \in K \quad \text{and} \quad \psi_{K}(u) = +\infty \quad \text{if} \quad u \notin K.
\]

Let \( J \) be a function on \( E \) satisfying the following hypothesis:

(R): \( J = \psi - \phi \), where \( \phi \in C^1(E, \mathbb{R}) \) and \( \psi : E \to (-\infty, +\infty] \) is proper, convex and lower semi continuous.

**Definition 2.3.** A point \( u \in E \) is said to be a critical point of \( I = \psi - \phi \) if \( u \in \text{Dom}(\psi) \) and if it satisfies the inequality \( (D\phi(u), u - v) + \psi(v) - \psi(u) \geq 0, \quad \forall v \in E \), where \( D\phi(u) \) stands for the derivative of \( \phi \) at \( u \).

**Lemma 2.4.** If \( I \) satisfies (R), then each local minimum of \( I \) is necessarily a critical point of \( I \).

*Proof.* See [14].

Now, we define the functionals \( \phi, \psi : E \to \mathbb{R} \) by

\[
\phi(u) = -\frac{1}{4} \int_{\Omega} K(x)\phi_u u^2 \, dx + \frac{1}{p+1} \int_{\Omega} |u|^{p+1} \, dx + \int_{\Omega} h(x)u \, dx,
\]

and

\[
\psi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx \quad \text{and} \quad I_{K}(u) = \psi_{K}(u) - \phi(u).
\]
3. Proof of Theorem 1.1

We now give the following variational principle version applicable to problem (1.1).

**Theorem 3.1.** Let $K \subset E$ be a convex and weakly closed subset of $E$. If the following two assertions hold:

(i) The functional $I_K$ has a critical point $u_1 \in E$ as in Definition 2.3 and;

(ii) there exists $u_2 \in K$ such that

$$
\int_{\Omega} \nabla u_2 \nabla \varphi dx = - \int_{\Omega} K(x)\phi_{u_1} u_1 \varphi dx + \int_{\Omega} |u_1|^{p-1} u_1 \varphi dx + \int_{\Omega} h(x)\varphi dx, \quad \forall \varphi \in E.
$$

Then $u_1 \in K$ is a weak solution of system (1.1).

**Proof.** Since $u_1$ is a critical point of $I_K$, then

$$
\frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \frac{1}{2} \int_{\Omega} |\nabla u_1|^2 dx \geq \int_{\Omega} (-K(x)\phi_{u_1} u_1 + |u_1|^{p-1} u_1 + h(x))(v - u_1)dx, \quad \forall v \in K.
$$

Invoking assumption (ii) in the theorem, we deduce that

$$
\int_{\Omega} \nabla u_2 \nabla (u_1 - u_2) dx = \int_{\Omega} (-K(x)\phi_{u_1} u_1 + |u_1|^{p-1} u_1 + h(x))(u_1 - u_2)dx.
$$

Now by substituting $v = u_2$ in (3.1) and taking into account (3.2), we obtain

$$
\frac{1}{2} \int_{\Omega} |\nabla u_2|^2 dx - \frac{1}{2} \int_{\Omega} |\nabla u_1|^2 dx \geq \frac{1}{4} \int_{\Omega} (-K(x)\phi_{u_1} u_1 + |u_1|^{p-1} u_1 + h(x))(u_1 - u_2)dx
$$

$$
= \int_{\Omega} \nabla u_2 \nabla (u_2 - u_1) dx.
$$

On the other hand, in view of the convexity of $\psi$, we infer that

$$
\frac{1}{2} \int_{\Omega} |\nabla u_1|^2 dx - \frac{1}{2} \int_{\Omega} |\nabla u_2|^2 dx \geq \int_{\Omega} \nabla u_2 \nabla (u_1 - u_2) dx.
$$

Using the above pieces of informations, we obtain that

$$
\frac{1}{2} \int_{\Omega} |\nabla u_2|^2 dx - \frac{1}{2} \int_{\Omega} |\nabla u_1|^2 dx = \int_{\Omega} \nabla u_2 \nabla (u_2 - u_1) dx.
$$

This shows that

$$
\frac{1}{2} \int_{\Omega} |\nabla u_2 - \nabla u_1|^2 dx = 0.
$$

Thus,

$$
u_2 = u_1,
$$

for a.e. $x \in \Omega$.

This ends the proof. \qed

We shall use the above theorem to prove our main result in Theorem 1.1. The convex subset $K \subset E$ required in Theorem 3.1 is defined as follows

$$
K(r) = \{u \in E : \|u\|_{W^{2,p}(\Omega)} \leq r\},
$$

for some $r > 0$ to be determined later.
**Lemma 3.2.** Let $r > 0$ be fixed. The set
\[
\{ u \in E : \| u \|_{W^{2,N}(\Omega)} \leq r \},
\]
is a weakly closed in $E$.

**Proof.** See [14].

In the sequel, we need the following technical lemmas.

**Lemma 3.3.** Let $r > 0$ be fixed. Then, there exists $C_1, C_2 > 0$ such that
\[
\| - K(x) \phi_u u + |u|^{p-1} u + h(x) \|_{L^N(\Omega)} \leq C_1 r^3 + C_2 r^p + \| h \|_{L^N(\Omega)}, \quad \forall u \in K(r).
\]

**Proof.** Let $u \in K(r)$. Then, using Lemmas 2.1 and 2.2 and Hölder’s inequality, we get
\[
\| - K(x) \phi_u u + |u|^{p-1} u \|_{L^N(\Omega)} \leq \| K \|_{\infty} \| \phi_u u \|_{L^N(\Omega)} + \| |u|^{p-1} u \|_{L^N(\Omega)}
\]
\[
\leq \| K \|_{\infty} \| u \|_{L^{2N}(\Omega)} \| \phi_u \|_{L^2(\Omega)} + \| u \|_{L^{pN}(\Omega)}^p
\]
\[
\leq c_1 \| u \|_{W^{2,N}(\Omega)} \| \phi_u \|_{W^{2,N}(\Omega)} + c_2 \| u \|_{W^{2,N}(\Omega)}^p
\]
\[
\leq C_1 \| u \|_{W^{2,N}(\Omega)}^3 + C_2 \| u \|_{L^N(\Omega)}^{p-1} \leq C_1 r^3 + C_2 r^p,
\]
where $c_1, c_2, C_1, C_2 > 0$. This ends the proof.

**Lemma 3.4.** Assume that $C_1$ and $C_2$ are given in Lemma 3.3. Then, there is $r_1 > 0$ such that $C_1 r^3 + C_2 r^p \leq \frac{r}{2}$, $\forall r \in (0, r_1]$. Moreover, if $\| h \|_{L^N(\Omega)} \leq \frac{r}{2}$, we have
\[
C_1 r_1^3 + C_2 r_1^p + \| h \|_{L^N(\Omega)} \leq r_1.
\]

**Proof.** The proof follows by a straightforward computation.

**Lemma 3.5.** Suppose that conditions of Theorem 7.1 are fulfilled. Let $r_1$ be given in Lemma 3.4. Moreover, assume that $\| h \|_{L^N(\Omega)} \leq \frac{r}{2}$. Then for each $u \in K(r_1)$ there exists $v \in K(r_1)$ such that
\[
\int_{\Omega} \nabla v \nabla \varphi dx + \int_{\Omega} K(x) \phi_u u \varphi dx = \int_{\Omega} |u|^{p-1} u \varphi dx + \int_{\Omega} h(x) \varphi dx,
\]
for all $u \in E$ and $\varphi \in H^1_0(\Omega)$. In particular, $v \in W^{2,N}(\Omega) \hookrightarrow H^1_0(\Omega)$, and
\[
- \Delta v = - K(x) \phi_u u + |u|^{p-1} u + h(x), \quad \text{for a.e } x \in \Omega.
\]

**Proof.** Using a standard argument, there exists $v \in H^1_0(\Omega)$ which satisfies (3.5). Since the right hand side is an element in $L^N(\Omega)$, it follows from the standard regularity results that $v \in W^{2,N}(\Omega) \hookrightarrow H^1_0(\Omega)$ and (3.6) holds. Therefore, using Lemmas 3.3 and 3.4, we deduce that
\[
\| v \|_{W^{2,N}(\Omega)} = \| \Delta v \|_{L^N(\Omega)} = \| - K(x) \phi_u u + |u|^{p-1} u + h(x) \|_{L^N(\Omega)} \leq r_1,
\]
the lemma is proven.
Proof of Theorem 1.1 completed:
Let $r_1 > 0$ be as in Lemma 3.4 and define $K = K(r_1)$. We suppose that $\|h\|_{L^N(\Omega)} \leq \frac{r_1}{2}$. Consider the following minimizing problem
\[
\beta = \inf_{u \in E} I_K(u).
\]
Hence, by definition of $\psi_K$, we deduce that
\[
\beta = \inf_{u \in K} I_K(u).
\]
On the other hand, using Lemma 3.3, we infer that $\beta > -\infty$. Take $0 < e \in K$. For $t \in [0, 1]$, we have that $te \in K$ and therefore
\[
I_K(te) \leq t(t \int_{\Omega} |\nabla e|^2 dx + t^3 \int_{\Omega} \phi_e e^2 dx - t^p \int_{\Omega} |e|^{p+1} dx - \int_{\Omega} h(x) edx).
\]
Since $h, e > 0$, we can conclude that $\beta < 0$. Now suppose that $(u_n)$ is a sequence in $E$ such that $I_K(u_n) \to \beta$. So the sequence is bounded and we can conclude by the definition of $I_K$ that the sequence is bounded in $W^{2,N}(\Omega)$. Using standard results in Sobolev spaces, after passing to a subsequence if necessary, there exists $u_1 \in K$ such that $u_n \rightharpoonup u_1$ in $W^{2,N}(\Omega)$ and strongly in $E$. Therefore,
\[
\beta = I_K(u_1) < 0.
\]
Then, by Lemma 2.3, we conclude that $u_1$ is a nontrivial critical point of $I_K$. Now, by Lemma 3.5 together with the fact that $u_1 \in K(r_2)$ we obtain that there exists $u_2 \in K$ such that
\[
-\Delta u_2 = -\phi u_1 + |u_1|^{p-1}u_1 + h.
\]
Combining the above pieces of informations and applying Theorem 3.1, we conclude that $u_1$ is a nontrivial solution of problem (1.1).

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