Tilting Modules and Exceptional Sequences for a Family of Dual Extension Algebras

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Abstract
We provide a classification of generalized tilting modules and full exceptional sequences for a certain family of quasi-hereditary algebras, namely dual extension algebras of path algebras of uniformly oriented linear quivers, modulo the ideal generated by paths of length two, with their opposite algebra. An important step in the classification is to prove that all indecomposable self-orthogonal modules (with respect to extensions of positive degree) admit a filtration with standard subquotients or a filtration with costandard subquotients. Furthermore, we prove that that every generalized tilting module, not equal to the characteristic tilting modules, admits either a filtration with standard subquotients or a filtration with costandard subquotients. Since the algebras studied in this article admit a simple-preserving duality, combining these two statements reduces the problem to classifying generalized tilting modules admitting a filtration with standard subquotients. To finalize the classification of generalized tilting modules we develop a combinatorial model for the poset of indecomposable self-orthogonal modules with standard filtration with respect to the relation arising from higher extensions. This model is also used for the classification of full exceptional sequences.

Keywords Generalized tilting module · Exceptional sequence · Dual extension algebra · Quasi-hereditary algebra · Extension

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1 Introduction

Since its introduction in [1, 2], tilting theory has become an important part of the representation theory of finite-dimensional algebras. A basic classification problem in tilting theory is to classify all generalized tilting modules over a given algebra. In general, this problem is very difficult. Some instances of where this problem and its generalizations have been studied can be found in [3–7].

Quasi-hereditary algebras were first defined in [8]. In [9], highest weight categories were introduced as a category theoretical counterpart of certain structures in the representation theory of complex semisimple Lie algebras. In the same paper, [9], the quasi-hereditary algebras were characterized as exactly those finite-dimensional algebras whose module categories are highest weight categories. Examples of quasi-hereditary algebras include hereditary algebras, Schur algebras, algebras of global dimension two and algebras describing blocks of BGG category $O$.

The chief protagonists of the representation theory of quasi-hereditary algebras are the standard and costandard modules, as well as the associated subcategories, $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$, of the module category, consisting of those modules which admit a filtration by standard and costandard modules, respectively. Importantly, Ringel showed in [10] that $(\mathcal{F}(\Delta), \mathcal{F}(\nabla))$ is a homologically orthogonal pair. Moreover, Ringel showed that for any quasi-hereditary algebra, there exists a generalized tilting module $T$, called the characteristic tilting module, whose additive closure equals exactly $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.

Originally studied in [11], and related to tilting theory, are exceptional modules and full exceptional sequences over a given algebra. For a quasi-hereditary algebra, the sequences of standard and costandard modules are examples of full exceptional sequences. The papers [6, 12] provide a classification of the full exceptional sequences over the Auslander algebra of $k[x]/(x^n)$ and its quadratic dual, respectively.

A large family of quasi-hereditary algebras is constituted by the dual extension algebras defined by Xi in [13]. These were introduced as an example of a class of BGG algebras. BGG algebras are quasi-hereditary algebras admitting a simple preserving duality on their module categories, see [14]. Xi’s original construction was soon generalized and has been studied in greater detail in [15–21]. The algebras studied in this paper are examples of dual extension algebras.

The following is a brief description of the main results of this article.

(A) Let $A_n$ be the uniformly oriented linear quiver with $n$ vertices, let $k$ an algebraically closed field and set

$$A_n = kA_n/(\text{rad } kA_n)^2.$$  

Let $\Lambda_n$ be the dual extension algebra $A_n(\Lambda_n, A_n^{op})$. We obtain a classification of generalized tilting modules over the algebra $\Lambda_n$. This is achieved through the following steps. First, we show that any generalized tilting module is contained in $\mathcal{F}(\Delta)$ or in $\mathcal{F}(\nabla)$. Using the simple-preserving duality, this reduces the problem to the classification of generalized tilting modules in $\mathcal{F}(\Delta)$. Then, we consider the set of isomorphism classes of indecomposable self-orthogonal modules with a standard filtration. We provide a combinatorial description of the non-zero extensions of positive degree between the modules in this set in terms of a certain transitive relation. This interpretation allows us to classify generalized tilting modules contained in $\mathcal{F}(\Delta)$ as the maximal anti-chains with respect to this fixed relation.

(B) We obtain a classification of full exceptional sequences of $\Lambda_n$-modules. We show that a full exceptional sequence is uniquely determined by going through each index $i$, for
2 \leq i \leq n$, and choosing either the standard module $\Delta(i)$ or the costandard module $\nabla(i)$. More precisely, a full exceptional sequence is of the form
\[
(\nabla(m_1), \nabla(m_2), \ldots, \nabla(m_i), L(1), \Delta(n_1), \Delta(n_2), \ldots, \Delta(n_j))
\]
where
\begin{itemize}
  \item $i + j = n - 1$;
  \item $\{m_1, m_2, \ldots, m_i, n_1, n_2, \ldots, n_j\} = \{2, 3, \ldots, n\}$;
  \item $m_1 > m_2 > \cdots > m_i$ and $n_1 < n_2 < \cdots n_j$.
\end{itemize}
Comparing the results with the first author’s article, [6], in which the same problems are studied for another class of algebras, we see that the classification of generalized tilting modules is more complicated in the current case, while the classification of full exceptional sequences is identical in the two cases, in the sense that the form of the sequences is the same.

The present article is organized as follows. In Section 2, we briefly introduce the algebras $\Lambda_n$, which are our objects of study, and recall some results on their quasi-hereditary structure. In Section 3, we classify the indecomposable $\Lambda_n$-modules using the results from [22, 23] on the classification of indecomposable modules over special biserial algebras. We also introduce some notation which is important for the readability of the subsequent sections. Section 4 contains a classification of the self-orthogonal indecomposable $\Lambda_n$-modules. In Section 5, we classify the generalized tilting modules over $\Lambda_n$. Section 6 contains the classification of full exceptional sequences over $\Lambda_n$.

2 Background

Throughout the rest of the article, let $k$ be an algebraically closed field. Let $\Lambda_n$ be the uniformly oriented linear quiver
\[
1 \rightarrow 2 \rightarrow \cdots \rightarrow n - 1 \rightarrow n,
\]
for some $n \in \mathbb{Z}_{>1}$ and denote by $k\Lambda_n$ the corresponding path algebra. Let $A_n$ be the quotient of $k\Lambda_n$ by the ideal $(\text{rad } k\Lambda_n)^2$. Finally, we define $\Lambda_n$ to be the dual extension algebra of $A_n$ with its opposite algebra, $A_n^{\text{op}}$, that is, $\Lambda_n = \mathcal{A}(A_n, A_n^{\text{op}})$. Then, $\Lambda_n$ is given by the quiver
\[
\begin{array}{cccccccc}
1 & \xrightarrow{\alpha_1} & 2 & \xrightarrow{\alpha_2} & 3 & \cdots & n - 1 & \xrightarrow{\alpha_{n-1}} & n \\
& \xleftarrow{\alpha'_1} & & \xleftarrow{\alpha'_2} & & \xleftarrow{\alpha'_{n-1}} & & \\
\end{array}
\]
subject to the relations
\[
\alpha_{i+1}\alpha_i = 0, \quad \alpha'_j\alpha'_{j+1} = 0 \quad \text{and} \quad \alpha_i\alpha'_i = 0.
\]

Let $\Lambda_n$-mod denote the category of finite-dimensional left $\Lambda_n$-modules. Throughout the rest of the article, we take “module” to mean left module. The algebra $\Lambda_n$ has a simple-preserving duality on its module category, denoted by $\ast$, induced by the anti-automorphism given by swapping the arrows $\alpha_i$ and $\alpha'_i$ in the quiver of $\Lambda_n$. 

\[\text{Springer}\]
Let $L(i)$, where $1 \leq i \leq n$, denote the simple $\Lambda_n$-module corresponding to the vertex $i$. Let $P(i)$ and $I(i)$ denote its projective cover and injective envelope, respectively.

**Definition 1** [9] Let $\Delta$ be a finite-dimensional algebra. Let $\{1, \ldots, n\}$ be an indexing set for the isomorphism classes of simple $\Delta$-modules and let $<$ be a partial order on $\{1, \ldots, n\}$. The algebra $\Delta$ is said to be *quasi-hereditary* with respect to $<$ if there exist modules $\Delta(i)$, where $i \in \{1, \ldots, n\}$, called *standard modules*, satisfying the following.

(QH1) There is a surjection $P(i) \twoheadrightarrow \Delta(i)$ whose kernel admits a filtration with subquotients $\Delta(j)$, where $j > i$.

(QH2) There is a surjection $\Delta(i) \twoheadrightarrow L(i)$ whose kernel admits a filtration with subquotients $L(j)$, where $j < i$.

This is equivalent to the existence of modules $\nabla(i)$, where $i \in \{1, \ldots, n\}$, called *costandard modules*, satisfying the following.

(QH1)$'$ There is an injection $\nabla(i) \hookrightarrow I(i)$ whose cokernel admits a filtration with subquotients $\nabla(j)$, where $j > i$.

(QH2)$'$ There is an injection $L(i) \hookrightarrow \nabla(i)$ whose cokernel admits a filtration with subquotients $L(j)$, where $j < i$.

It is easy to see that $\Lambda_n$ is quasi-hereditary with respect to the natural ordering on $\{1, \ldots, n\}$. Indeed, (QH1) and (QH2) are easily verified with standard and costandard modules as below. Note that results from [13] show that the dual extension algebra $\Lambda_n^\text{op}$ is quasi-hereditary. Throughout, let $\mathcal{F}(\Delta)$ denote the full subcategory of $\Lambda_n$-mod consisting of those modules which admit a filtration by standard modules. Similarly, let $\mathcal{F}(\nabla)$ denote the full subcategory consisting of those modules which admit a filtration by costandard modules. Since $\Lambda_n$ is quasi-hereditary, Theorem 5 from [10] guarantees that there exists a basic module $T$, called the *characteristic tilting module*, such that the additive closure of $T$, denoted by $\operatorname{add} T$, equals $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. Moreover, $T$ has the same number of non-isomorphic indecomposable summands as the number of isomorphism classes of simple modules, and we write

$$T = \bigoplus_{k=1}^{n} T(k).$$

The indecomposable direct summand $T(k)$ of $T$ is uniquely determined (up to isomorphism) by the property that it belongs to $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ and that there exists a monomorphism $\Delta(k) \hookrightarrow T(k)$, whose cokernel admits a filtration by standard modules.

We conclude this section by drawing the Loewy diagrams of the structural modules of $\Lambda_n$ and stating some of their elementary properties.

$$\begin{align*}
P(i) : & \ i-1 \rightarrow i \rightarrow i+1 \quad I(i) : \ i-1 \rightarrow i \rightarrow i+1 \quad \text{for } i = 2, \ldots, n-1, \\
\Delta(i) : & \ i-1 \rightarrow i \rightarrow \nabla(i) : \ i-1 \rightarrow i \quad \text{for } i = 2, \ldots, n.
\end{align*}$$

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The remaining cases are $\Delta(1) = \nabla(1) = L(1)$, $P(n) = \Delta(n)$, $I(n) = \nabla(n)$ and

$$P(1) = I(1) : \begin{array}{c} 1 \\ 2 \end{array}.$$  

Finally, from these pictures we have the following lemma.

**Lemma 1** For every $i = 1, \ldots, n - 1$ we have the following non-split short exact sequences in $\Lambda_n$-mod:

$$\Delta(i + 1) \hookrightarrow P(i) \twoheadrightarrow \Delta(i), \quad \nabla(i) \hookrightarrow I(i) \twoheadrightarrow \nabla(i + 1).$$

**Proposition 2** For $m = j - i > 0$ we have

$$\text{Ext}_{\Lambda_n}^m(\Delta(i), \Delta(j)) \neq 0 \quad \text{and} \quad \text{Ext}_{\Lambda_n}^m(\nabla(j), \nabla(i)) \neq 0.$$  

**Proof** Using the simple-preserving duality, the first inequality implies the second. By applying the functor $\text{Hom}_{\Lambda_n}(\Delta(i), \_)$ to the short exact sequence

$$\Delta(j) \hookrightarrow P(j) \twoheadrightarrow \Delta(j - 1),$$

and considering the resulting long exact sequence, we get $\text{Ext}_{\Lambda_n}^{j-i}(\Delta(i), \Delta(j)) \cong \text{Ext}_{\Lambda_n}^{j-i-1}(\Delta(i), \Delta(j - 1))$. Repeating this argument, we get

$$\text{Ext}_{\Lambda_n}^{j-i}(\Delta(i), \Delta(j)) \cong \text{Ext}_{\Lambda_n}^{j-i-1}(\Delta(i), \Delta(j - 1)) \cong \ldots \cong \text{Ext}_{\Lambda_n}^1(\Delta(i), \Delta(i + 1)) \neq 0,$$

where, in the last step, we use that the short exact sequences in Lemma 1 are non-split. \qed

### 3 Indecomposable $\Lambda_n$-Modules

In order to classify indecomposable $\Lambda_n$-modules, we use the fact that the algebra $\Lambda_n$ is a string algebra. For these algebras, the classification is known.

**Definition 2** [23] Let $\Lambda = kQ/I$ be the quotient of the path algebra of the quiver $Q = (Q_0, Q_1)$ by some admissible ideal $I$. For an arrow $\alpha \in Q_1$, denote by $s(\alpha)$ and $t(\alpha)$ the source and target vertex of $\alpha$, respectively. Then $\Lambda$ is called *special biserial* if the following hold.

(SB1) For each vertex $i$, there are at most two arrows with $i$ as its source, and at most two arrows with $i$ as its target.

(SB2) For $\alpha, \beta, \gamma \in Q_1$ such that $t(\alpha) = t(\beta) = s(\gamma)$ and $\alpha \neq \beta$, we have $\gamma\alpha \in I$ or $\gamma\beta \in I$.

(SB3) For $\alpha, \beta, \gamma \in Q_1$ such that $s(\alpha) = s(\beta) = t(\gamma)$ and $\alpha \neq \beta$ we have $\alpha\gamma \in I$ or $\beta\gamma \in I$.

If, in addition, the ideal $I$ is generated by zero relations, $\Lambda$ is called a *string algebra*.

We immediately note that $\Lambda_n$ is a string algebra for all $n \in \mathbb{Z}_{>1}$. For special biserial algebras and string algebras the classification of indecomposable modules is known, see [22, 23]. There exist two classes of indecomposable modules, the so-called string modules
and band modules. We will show that in the case of $\Lambda_n$, there are no band modules and therefore a complete list of the indecomposable $\Lambda_n$-modules is given by the string modules.

We follow closely the notation of [23]. Let $L = (L_0, L_1)$ denote the quiver

$$L = 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \ldots \xrightarrow{a_{r-1}} r \xrightarrow{a_r} r + 1, \quad r \geq 0$$

where the $a_i$ are arrows with either orientation. Define a map $\varepsilon : L_1 \to \{ -1, 1 \}$ by

$$\varepsilon(a_i) = \begin{cases} 
1, & \text{if } a_i : i \to i + 1; \\
-1, & \text{if } a_i : i + 1 \to i.
\end{cases}$$

Similarly, we denote by $Z = (Z_0, Z_1)$ the quiver

$$Z = \overbrace{1 \xrightarrow{b_T} 2 \xrightarrow{b_T} \ldots \xrightarrow{b_{r-1}} r}, \quad r \geq 2$$

where the $b_T$ are arrows with either orientation and where $\overline{i}$ denotes the congruence class of $i$ modulo $r$. Again, define $\varepsilon : Z_1 \to \{ -1, 1 \}$ by

$$\varepsilon(b_T) = \begin{cases} 
1, & \text{if } b_T : \overline{i} \to \overline{i + 1}; \\
-1, & \text{if } b_T : \overline{i + 1} \to \overline{i}.
\end{cases}$$

Let $Q$ be some quiver. A quiver homomorphism $w : L \to Q$ is called a walk of length $r$ in $Q$. A walk is called a path if $\varepsilon(a_i) = 1$ for all $i$. Similarly, a homomorphism $u : Z \to Q$ is called a tour in $Q$. A tour is called a circuit if $\varepsilon(b_T) = 1$ for all $i$. The restriction of $v$ (or $w$) to a connected linear subquiver $L'$ of $L$ (or $Z$) is called a subwalk or a subpath (or, a subtour or subcircuit).

**Definition** 3 [23] Fix a quiver $Q$ and an admissible ideal $I \subset \k Q$. A walk $v : L \to Q$ is called a $V$-sequence if the following hold.

1. (VS1) Each subpath of $v$ does not belong to $I$.
2. (VS2) If $\varepsilon(a_i) \neq \varepsilon(a_{i+1})$, then $v(a_i) \neq v(a_{i+1})$.

Similarly, a tour $u : Z \to Q$ is called a primitive $V$-sequence if the following hold.

1. (VS3) The tour $u$ is not a circuit and each subpath of $u$ does not belong to $I$.
2. (VS4) If $\varepsilon(b_i) \neq \varepsilon(b_{i+1})$, then $u(b_i) \neq u(b_{i+1})$.
3. (VS5) There is no automorphism $\sigma \neq \text{id}$ of $Z$, permuting the vertices cyclically such that $u = u \circ \sigma$.

In [23], the authors show that we can obtain all indecomposable modules over a special biserial algebra from $V$-sequences and primitive $V$-sequences. These correspond exactly to the string and band modules, respectively. In Proposition 3, we will see that there are no primitive $V$-sequences $u : Z \to Q$, and consequently, no band modules. To obtain an indecomposable module from a $V$-sequence $v : L \to Q$, consider the following representation of the bound quiver $(Q, I)$. At each vertex $x \in Q_0$, we put the vector space $\k$ if $x$ is in the image of $v$, and the zero space otherwise. At each arrow $x \xrightarrow{\alpha} y \in Q_1$, we put the identity map on the vector space $\k$ if $\alpha$ is in the image of $v$, and the zero map otherwise. This representation is then equivalent to a $\k Q/I$-module.
However, an indecomposable module $M$ does not arise from a unique V-sequence. In fact, the indecomposable module $M$ corresponding to a V-sequence $v : L \to Q$ is isomorphic to the indecomposable module $M'$ corresponding to a V-sequence $v' : L' \to Q$ if and only if there is a quiver isomorphism $\sigma : L' \to L$ such that $v' = v \circ \sigma$. In this situation, we say that the V-sequences $v$ and $v'$ are isomorphic. There are only two possibilities for such an isomorphism $\sigma$. The first possibility is that $L = L'$ and $\sigma = \text{id}$. The second possibility is that $\sigma$ acts on the vertices of $L'$ by $\sigma(i) = r + 2 - i$, that is, $\sigma$ swaps the vertex $1$ and the vertex $r + 1$, the vertex $2$ and the vertex $r$, and so on. Here, we must have an arrow $i \xrightarrow{\alpha'} j$ in $L'$ if and only if we have an arrow $r + 2 - i \xrightarrow{\alpha} r + 2 - j$ in $L$. The corresponding V-sequence $v' : L' \to Q$ must then be given by $v'(i) = v(r + 2 - i)$.

Example 1 Let $L$ be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$. Then $L'$ as described above is the quiver $1 \xleftarrow{\gamma'} 2 \xrightarrow{\beta'} 3 \xrightarrow{\alpha'} 4$. In a picture, the isomorphism $\sigma : L' \to L$ is as follows.

```
1 2 3 4
\sigma
v(4) = v'(1) v(3) = v'(2) v(2) = v'(3) v(1) = v'(4)
```

Proposition 3 Let $Q$ be the quiver of $\Lambda_n$ and let $I \subset \mathbb{k}Q$ be the ideal generated by the relations in Section 1. Then, there are no primitive V-sequences $u : Z \to Q$. Moreover, the only V-sequences $v : L \to Q$, up to isomorphism, are of one of the following forms.

(a) We have $\varepsilon(a_i) \neq \varepsilon(a_{i+1})$ and $v(i + 1) = v(i) + 1$ for all $i$. The V-sequence is given by the following picture.

```
1 \cdots \frac{a_1}{v} \frac{a_2}{v} \frac{a_{r-1}}{v} \frac{a_r}{v} \frac{r}{v} \frac{r + 1}{v}
```

(b) We have $\varepsilon(a_i) = \varepsilon(a_{s+1}) = 1$ for some $s$, $\varepsilon(a_i) \neq \varepsilon(a_{i+1})$ for all $i \neq s$, $v(i + 1) = v(i) + 1$ for $i \leq s$ and $v(i + 1) = v(i) - 1$ for $i > s$. The V-sequence is given by the following picture.

```
1 \cdots \frac{a_1}{v} \frac{a_{s-1}}{v} \frac{s}{v} \frac{a_s}{v} \frac{s + 1}{v} \frac{a_{s+1}}{v} \frac{s + 2}{v} \frac{a_{r+2}}{v} \frac{\ldots}{v} \frac{a_r}{v} \frac{r + 1}{v}
```

\[ v(1) \cdots \frac{a_1}{v} \frac{a_{s-1}}{v} \frac{s}{v} \frac{a_s}{v} \frac{s + 1}{v} \frac{a_{s+1}}{v} \frac{s + 2}{v} \frac{a_{r+2}}{v} \frac{\ldots}{v} \frac{a_r}{v} \frac{v(1) + 2s - r}{v} \]
Proof Let $v : L \to Q$ be a $V$-sequence, where $v(i) = j$, and consider the subquiver

$$i \xrightarrow{a_i} i + 1 \xrightarrow{a_{i+1}} i + 2$$

of $L$. There are four cases, depending on the values of $\varepsilon(a_i)$ and $\varepsilon(a_{i+1})$.

(I) Assume that $\varepsilon(a_i) = \varepsilon(a_{i+1}) = 1$. Then, our subquiver looks like

$$i \xrightarrow{a_i} i + 1 \xrightarrow{a_{i+1}} i + 2$$

By assumption, $v(a_i)$ is an arrow starting in $j$, so that $v(a_i) = \alpha_j$ or $v(a_i) = \alpha'_{j-1}$. If $v(a_i) = \alpha_{j-1}$, then, we immediately get that

$$v(a_{i+1}) = \alpha_{j-1}$$

and in both cases, $v(a_{i+1})v(a_i) = 0$, contradicting (VS1).

If $v(a_i) = \alpha_j$, then, we have

$$v(a_{i+1}) = \alpha_{j+1}$$

If $v(a_{i+1}) = \alpha_{j+1}$, we again contradict (VS1). Therefore, we conclude that the subquiver

$$i \xrightarrow{a_i} i + 1 \xrightarrow{a_{i+1}} i + 2$$

is mapped to the following subquiver.

(II) Assume that $\varepsilon(a_i) = \varepsilon(a_{i+1}) = -1$. Then, our subquiver looks like

$$i \xleftarrow{a_i} i + 1 \xleftarrow{a_{i+1}} i + 2$$

A similar argument as in the previous case shows that this is mapped to the following subquiver.

(III) Assume that $\varepsilon(a_i) = 1$ and $\varepsilon(a_{i+1}) = -1$. Then, our subquiver looks like

$$i \xrightarrow{a_i} i + 1 \xrightarrow{a_{i+1}} i + 2$$

Again, we have $v(a_i) = \alpha_j$ or $v(a_i) = \alpha'_{j-1}$. If $v(a_i) = \alpha_j$, we have $v(a_{i+1}) = \alpha'_{j+1}$ since, by (VS2), $v(a_i) \neq v(a_{i+1})$, and we get the following picture.

$$v(i) = j \xrightarrow{\alpha_j} v(i + 1) = j + 1$$

$$v(i + 2) = j + 2 \xleftarrow{\alpha'_{j+1}} v(i)$$

If, instead, $v(a_i) = \alpha'_{j-1}$, we get $v(a_{i+1}) = \alpha_{j-2}$ and the following picture.

$$v(i + 2) = j - 2 \xrightarrow{\alpha_{j-2}} v(i)$$

$$v(i) = j \xleftarrow{\alpha'_{j-1}} v(i + 1) = j - 1$$
(IV) Assume that $\varepsilon(a_i) = -1$ and $\varepsilon(a_{i+1}) = 1$. Then, our subquiver looks like
\[ i \xrightarrow{a_i} i+1 \xrightarrow{a_{i+1}} i+2. \]
By similar arguments as in the previous case, we get one of the following two possible pictures.

Let $u : Z \to Q$ be a primitive $V$-sequence and let $\bar{a}$ be such that $u(\bar{a}) = j \leq u(\bar{a})$ for all $s \in \mathbb{Z}_0$. It follows that $u(\bar{a} + 1) = j + 1 = u(\bar{a} - 1)$. Then, to not contradict (VS4), the subquiver

\[ \overline{a - 1} \longrightarrow \overline{a} \longrightarrow \overline{a + 1} \]

of $L$ is mapped to the subquiver

of $Q$. But this configuration contradicts (VS3). We conclude that there are no primitive $V$-sequences.

Let $v : L \to Q$ be a $V$-sequence. From (I)–(IV), we know that if $v(i + 1) = v(i) - 1$, then $v(i + 2) = v(i) - 2$, $v(i + 3) = v(i) - 3$ and so on. In this case $\varepsilon(a_s) \neq \varepsilon(a_{s+1})$ for all $s \geq i$. In particular, $\varepsilon(a_s) = \varepsilon(a_{s+1})$ can occur at most once in a $V$-sequence. There are two cases.

(a) We have $\varepsilon(a_s) \neq \varepsilon(a_{s+1})$, for all $s$. Then, either $v(i + 1) = v(i) + 1$ or $v(i + 1) = v(i) - 1$, for all $i$. However, any $V$-sequence of the latter type is isomorphic to one of the former type. This situation corresponds to part (a) of the statement of the proposition.

(b) We have $\varepsilon(a_s) = \varepsilon(a_{s+1})$, for some $s$, $\varepsilon(a_i) \neq \varepsilon(a_{i+1})$, for all $i \neq s$, $v(i + 1) = v(i) + 1$, for $i \leq s$ and $v(i + 1) = v(i) - 1$, for $i > s$. Then, either $\varepsilon(a_s) = \varepsilon(a_{s+1}) = 1$, or $\varepsilon(a_s) = \varepsilon(a_{s+1}) = -1$. However, any $V$-sequence of the latter type is isomorphic to one of the former type. This situation corresponds to part (b) of the statement of the proposition.

\[ \square \]

**Definition 4** Define $\Omega(i, j, k)$, where $i, j \leq k$, to be the (up to isomorphism unique) indecomposable $\Lambda_n$-module with the following Loewy diagram.
(a) If $k \equiv i \mod 2$ and $k \equiv j \mod 2$:

\[
\begin{array}{c}
  i & \rightarrow & i + 1 & \rightarrow & \cdots & \rightarrow & k - 1 & \rightarrow & k \\
  j & \rightarrow & j + 1 & \rightarrow & \cdots & \rightarrow & k - 1 \\
\end{array}
\]

(b) If $k \equiv i \mod 2$ and $k \not\equiv j \mod 2$:

\[
\begin{array}{c}
  i & \rightarrow & i + 1 & \rightarrow & \cdots & \rightarrow & k - 1 & \rightarrow & k \\
  j & \rightarrow & j + 1 & \rightarrow & \cdots & \rightarrow & k - 1 \\
\end{array}
\]

(c) If $k \not\equiv i \mod 2$ and $k \equiv j \mod 2$:

\[
\begin{array}{c}
  i & \rightarrow & i + 1 & \rightarrow & i + 2 & \rightarrow & \cdots & \rightarrow & k - 1 & \rightarrow & k \\
  j & \rightarrow & j + 1 & \rightarrow & \cdots & \rightarrow & k - 1 \\
\end{array}
\]

(d) If $k \not\equiv i \mod 2$ and $k \not\equiv j \mod 2$:

\[
\begin{array}{c}
  i & \rightarrow & i + 1 & \rightarrow & i + 2 & \rightarrow & \cdots & \rightarrow & k - 1 & \rightarrow & k \\
  j & \rightarrow & j + 1 & \rightarrow & j + 2 & \rightarrow & \cdots & \rightarrow & k - 1 \\
\end{array}
\]

Note that, in case (a) and (b), we may have $i = k$, and, in case (a) and (c), we may have $j = k$. For all $i$, $j$, $k$, the simple preserving duality maps the module $\Omega(i, j, k)$ to the module $\Omega(j, i, k)$.

**Proposition 4** The set $\{\Omega(i, j, k) \mid 1 \leq i, j \leq k \leq n\}$ is a complete and irredundant list of isomorphism classes of indecomposable $\Lambda_n$-modules. There are, in total, $\frac{n(n+1)(2n+1)}{6}$ isomorphism classes of indecomposable $\Lambda_n$-modules.

**Proof** By [23], all indecomposable $\Lambda_n$-modules arise from $V$-sequences or primitive $V$-sequences. Using Proposition 3 we know that there are no primitive $V$-sequences, and what the possible $V$-sequences look like. Let $v : L \rightarrow Q$ be a $V$-sequence, of the form (a) in Proposition 3, such that $v(1) = i$. If $\varepsilon(a_r) = 1$, then the indecomposable module corresponding to $v$ is $\Omega(i, i + r, i + r)$. If instead $\varepsilon(a_r) = -1$, then the indecomposable module corresponding to $v$ is $\Omega(i + r, i, i + r)$.
Let \( v : L \to Q \) be a \( V \)-sequence, of the form (b) in Proposition 3, such that \( v(1) = i \). Then the indecomposable module corresponding to \( v \) is \( \Omega(i, i + 2s - r, i + s) \). Thus, we see that any \( V \)-sequence gives rise to a (unique) module of the form \( \Omega(i, j, k) \), and that any such module may be obtained from a \( V \)-sequence. This proves the first part of the proposition.

Every choice of \( i, j, k \) such that \( 1 \leq i, j, k \leq n \) yields a unique module \( \Omega(i, j, k) \). For a fixed \( k \), there are \( k \) choices of \( i \), and \( k \) choices of \( j \), which implies that there are \( k^2 \) non-isomorphic modules \( \Omega(i, j, k) \), with \( k \) fixed. The total number of non-isomorphic indecomposable \( \Lambda_n \)-modules is therefore

\[
\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.
\]

\( \square \)

For any subset \( X \subseteq \{1, 2, \ldots, n\} \), we denote by \( P_X \) the direct sum

\[
P_X = \bigoplus_{i \in X} P(i).
\]

Should \( X \) be the empty set, we define \( P_{\emptyset} := 0 \). We use similar notation for such direct sums of other structural modules. For \( a, b \in \{1, \ldots, n\} \), with \( a \leq b \), we fix the following notation.

- When \( a \equiv b \mod 2 \), we put \( [a, b] = \{a, a + 2, \ldots, b - 2, b\} \).
- We put \( [a, b] = \{c \in \{1, \ldots, n\} \mid a \leq c \leq b \text{ and } c \equiv a \mod 2\} \).
- We put \( (a, b) = \{c \in \{1, \ldots, n\} \mid a \leq c \leq b \text{ and } c \equiv b \mod 2\} \).

For example,

\[
P_{[3,8]} = P(3) \oplus P(5) \oplus P(7) \quad \text{and} \quad P_{[3,8]} = P(4) \oplus P(6) \oplus P(8).
\]

Note that, if \( a \equiv b \mod 2 \), then \([a, b] = [a, b] = (a, b)\).

**Definition 5** Define the **upper arm** of the indecomposable \( \Lambda_n \)-module \( M = \Omega(i, j, k) \), denoted by \( \text{upp}(M) \), as the quotient

\[
\text{upp}(M) = \Omega(i, j, k)/\Omega(k - 1, j, k - 1) \cong \Omega(i, k, k).
\]

Similarly, define the **lower arm** of \( M = \Omega(i, j, k) \), denoted by \( \text{low}(M) \), as the submodule

\[
\text{low}(M) = \Omega(k, j, k) \subset \Omega(i, j, k).
\]

**Example 2** We draw the Loewy diagram of \( M = \Omega(2, 3, 6) \):

```
3 4 5
|   |
|   |
2  → 3  → 4  → 5  → 6
```

Then, the Loewy diagrams of \( \text{upp}(2, 3, 6) \) and \( \text{low}(2, 3, 6) \) are

```
upp(M) :
2 3 4 5 6
```
and

\[ \text{low}(M) : \begin{array}{ccc} 3 & 4 & 5 \\ & & 6 \\ \end{array} \]

respectively.

## 4 Self-Orthogonal Indecomposable Modules

Having described the indecomposable \( \Lambda_n \)-modules, the next step towards describing the generalized tilting \( \Lambda_n \)-modules is determining which indecomposable \( \Lambda_n \)-modules are self-orthogonal, as these will be candidates for inclusion in generalized tilting modules. For details on generalized tilting modules, we refer to Section 5.1. Throughout the following sections we will make frequent use of various dimension shifting arguments. We record the most common one in the following lemma.

**Lemma 5** Let \( M \) and \( N \) be finite-dimensional \( \Lambda_n \)-modules. Consider the following two short exact sequences, where \( K \) is the kernel of the projective cover \( P \rightarrow M \), and \( C \) is the cokernel of the injective envelope \( N \leftarrow I \):

\[
K \leftarrow P \rightarrow M, \quad N \leftarrow I \rightarrow C.
\]

Then,

- (i) \( \dim \text{Ext}^1_{\Lambda_n}(M, N) = \dim \text{Hom}_{\Lambda_n}(M, N) - \dim \text{Hom}_{\Lambda_n}(P, N) + \dim \text{Hom}_{\Lambda_n}(K, N) \);
- (ii) \( \text{Ext}^k_{\Lambda_n}(M, N) \cong \text{Ext}^{k-1}_{\Lambda_n}(K, N) \cong \text{Ext}^{k-1}_{\Lambda_n}(M, C) \), for all \( k \geq 2 \);
- (iii) \( \text{Ext}^k_{\Lambda_n}(M, N) \cong \text{Ext}^{k-2}_{\Lambda_n}(K, C) \), for all \( k \geq 3 \).

**Proof** By applying \( \text{Hom}_{\Lambda_n}(\_ , N) \) to the first sequence, we get the long exact sequence

\[
0 \rightarrow \text{Hom}_{\Lambda_n}(M, N) \rightarrow \text{Hom}_{\Lambda_n}(P, N) \rightarrow \text{Hom}_{\Lambda_n}(K, N) \\
\rightarrow \text{Ext}^1_{\Lambda_n}(M, N) \rightarrow \text{Ext}^1_{\Lambda_n}(P, N) \rightarrow \text{Ext}^1_{\Lambda_n}(K, N) \\
\rightarrow \text{Ext}^2_{\Lambda_n}(M, N) \rightarrow \text{Ext}^2_{\Lambda_n}(P, N) \rightarrow \text{Ext}^2_{\Lambda_n}(K, N) \rightarrow \cdots
\]

and, since \( \text{Ext}^k_{\Lambda_n}(P, N) = 0 \), for all \( k \geq 1 \), we have

\[
\dim \text{Ext}^1_{\Lambda_n}(M, N) = \dim \text{Hom}_{\Lambda_n}(M, N) - \dim \text{Hom}_{\Lambda_n}(P, N) + \dim \text{Hom}_{\Lambda_n}(K, N)
\]

and

\[
\text{Ext}^k_{\Lambda_n}(M, N) \cong \text{Ext}^{k-1}_{\Lambda_n}(K, N),
\]

for all \( k \geq 2 \). If we instead apply \( \text{Hom}_{\Lambda_n}(M, \_ ) \) to the second sequence, a similar argument implies that

\[
\text{Ext}^k_{\Lambda_n}(M, N) \cong \text{Ext}^{k-1}_{\Lambda_n}(M, C),
\]

for all \( k \geq 2 \). By combining these results, we obtain

\[
\text{Ext}^k_{\Lambda_n}(M, N) \cong \text{Ext}^{k-1}_{\Lambda_n}(K, N) \cong \text{Ext}^{k-2}_{\Lambda_n}(K, C),
\]

for all \( k \geq 3 \).

**Lemma 6** The module \( \Omega(i, j, k) \) is contained in \( \mathcal{F}(\Delta) \) if the following conditions are met.

- (i) If \( i \neq 1 \) and \( j \neq 1 \), then \( \Omega(i, j, k) \in \mathcal{F}(\Delta) \) if and only if \( i \equiv k \mod 2 \) and \( j \equiv k \mod 2 \).

\( \implies \) Springer
(ii) If $i = 1$ and $j \neq 1$, then $\Omega(i, j, k) \in \mathcal{F}(\Delta)$ if and only if $j \equiv k \mod 2$.
(iii) If $i \neq 1$ and $j = 1$, then $\Omega(i, j, k) \in \mathcal{F}(\Delta)$ if and only if $i \equiv k \mod 2$.
(iv) If $i = j = 1$, then $\Omega(i, j, k) \in \mathcal{F}(\Delta)$, for all $1 \leq k \leq n$.

Proof We draw the module $\Omega(2, 3, 6)$:

Here it is easy to see the standard filtration, because the standard modules, pictorially, look like

$$\Delta(k) :$$

$$k \quad k - 1$$

In this case, the subquotients of the standard filtration would be $\Delta(6), \Delta(5), \Delta(4), \Delta(3)$. In general, for a module $\Omega(i, j, k)$ with $i \neq 1$, $j \neq 1$, $i \equiv k \mod 2$ and $j \neq k \mod 2$, the subquotients would be $\Delta(i + 1), \Delta(i + 3), \ldots, \Delta(k - 1), \Delta(j + 1), \Delta(j + 3), \ldots, \Delta(k)$.

Let us instead draw the module $\Omega(2, 4, 6)$:

Here, we see that there is no way to remedy the composition factor $L(4)$ contained in the top of $\Omega(2, 4, 6)$, preventing a standard filtration. The rest is similar. ☐

**Lemma 7** The module $\Omega(i, j, k)$ is contained in $\mathcal{F}(\nabla)$ if the following conditions are met.

(i) If $i \neq 1$ and $j \neq 1$, then $\Omega(i, j, k) \in \mathcal{F}(\nabla)$ if and only if $i \neq k \mod 2$ and $j \equiv k \mod 2$.
(ii) If $i = 1$ and $j \neq 1$, then $\Omega(i, j, k) \in \mathcal{F}(\nabla)$ if and only if $j \equiv k \mod 2$.
(iii) If $i \neq 1$ and $j = 1$, then $\Omega(i, j, k) \in \mathcal{F}(\nabla)$ if and only if $i \neq k \mod 2$.
(iv) If $i = j = 1$, then $\Omega(i, j, k) \in \mathcal{F}(\nabla)$, for all $1 \leq k \leq n$.

Proof This follows from Lemma 6 by using the simple-preserving duality. ☐

**Corollary 8** For all $1 \leq k \leq n$, we have $T(k) = \Omega(1, 1, k)$, where $T(k)$ denotes the $k$th indecomposable summand of the characteristic tilting module.

Proof It follows from Lemma 6 and Lemma 7 that $\Omega(1, 1, k) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ for every $k$. The cokernel of the inclusion $\Delta(k) \hookrightarrow \Omega(1, 1, k)$ is equal to $\Omega(k - 1, 1, k - 1) \oplus \Omega(k - 2, 1, k - 2)$. Since both $\Omega(k - 1, 1, k - 1)$ and $\Omega(k - 2, 1, k - 2)$ have a standard
filtration by Lemma 6, so does their direct sum. It is clear that this standard filtration only has subquotients \( \Delta(j) \) with \( j < k \). This implies that \( \Omega(1, 1, k) = T(k) \).

**Lemma 9** Let \( M = \Omega(i, j, k) \) and let \( K \) be the kernel of the projective cover \( P \twoheadrightarrow M \).

(i) If \( k = i \) then \( P \cong P_{(j,k)} \) and if \( k > i \) then \( P \cong P_{(j,k-1)} \oplus P_{(j,k-2)} \).

(ii) The form of the module \( K \) is given by the following table.

| \( i \) | \( j \) | \( k \) | \( K \) |
|---------|---------|---------|---------|
| 1       | 1       | 1       | \( \Delta(2) \) |
| 1       | 1       | \( k > 1 \) | \( \Delta_{(j,k-2)} \oplus \Delta_{(j,k-1)} \) |
| 1       | \( k \) | \( k > 1 \) | \( \Delta_{(j,k)} \oplus \Delta_{(j+1,k-1)} \oplus L(k-1) \) |
| 1       | \( k > j > 1 \), \( j \not\equiv k \mod 2 \) | \( k > j \) | \( \Delta_{(j+2,k-2)} \oplus \Delta_{(j+2,k-1)} \) |
| 1       | \( k > j > 1 \), \( j \equiv k \mod 2 \) | \( k > j \) | \( \Delta_{(j+1,k-2)} \oplus \Delta_{(j+1,k-1)} \oplus L(j-1) \) |
| \( n \) | \( l \) | \( n \) | \( \Delta_{(j+2,n-1)} \oplus L(n-1) \) |
| \( n \) | \( n \) | \( n > j > 1 \), \( j \not\equiv k \mod 2 \) | \( n > j > 1 \) |
| \( n \) | \( n \) | \( n > j > 1 \), \( j \equiv k \mod 2 \) | \( n > j > 1 \) |
| \( k \) | \( l \) | \( n > k > 1 \) | \( \Delta_{(j+2,k+1)} \oplus L(j-1) \) |
| \( k \) | \( k \) | \( n > k > 1 \) | \( \Delta_{(j+2,k+1)} \oplus L(k-1) \) |
| \( k \) | \( k > j > 1 \), \( j \not\equiv k \mod 2 \) | \( n > k > j \) | \( \Delta_{(j+2,k+1)} \) |
| \( k \) | \( k > j > 1 \), \( j \equiv k \mod 2 \) | \( n > k > j \) | \( \Delta_{(j+2,k+1)} \) |
| \( k > i > 1 \), \( i \not\equiv k \mod 2 \) | \( k > i \) | \( \Delta_{(j+2,k-2)} \oplus \Delta_{(j+2,k-1)} \) |
| \( i \equiv k \mod 2 \) | \( i \equiv k \mod 2 \) | \( i \equiv k \mod 2 \) | \( i \equiv k \mod 2 \) |

**Proof** We again consider the example \( M = \Omega(2, 4, 6) \):

![Diagram](image)

We have \( \text{top}(M) = L(3) \oplus L(4) \oplus L(5) \) which gives a projective cover \( P = P(3) \oplus P(4) \oplus P(5) \). Therefore, we have a surjection \( P \twoheadrightarrow M \), whose kernel is equal to \( \Delta(4) \oplus \Delta(5) \oplus \Delta(6) \).
The remaining cases are easily ascertained by drawing the Loewy diagrams of the appropriate modules.

Lemma 10 The kernel of the projective cover of \(M\) has a simple direct summand if and only if \(M \notin \mathcal{F}(\Delta)\).

Proof This is easily checked by observing that the combinations of \(i, j\) and \(k\) which yield a simple direct summand of \(K\), according to Lemma 9, are exactly those for which \(M\) does not have a \(\Delta\)-filtration, according to Lemma 6.

Lemma 11 The cokernel of the injective envelope of \(M\) has a simple direct summand if and only if \(M \notin \mathcal{F}(\nabla)\).

Proof This follows from Lemma 10 by using the simple-preserving duality.

Lemma 12 Let \(Q_\bullet\) be the projective resolution of \(L(i)\). The terms \(Q_m\) of \(Q_\bullet\) are given by

\[
Q_m = \begin{cases} 
P_{[i-m,i+m]}, & \text{if } m < i \text{ and } m \leq n - i; \\
P_{[i-m,n]}, & \text{if } n - i < m < i; \\
P_{[m-i+2,i+m]}, & \text{if } i \leq m \leq n - i; \\
P_{[m-i+2,n]}, & \text{if } i \leq m \text{ and } n - i < m.
\end{cases}
\]

Proof This follows from repeated application of Lemma 9.

Lemma 13 Let \(i\) and \(j\) be such that not both are equal to 1. Then \(\text{Ext}^m_{\Lambda_n}(L(i), L(j)) \neq 0\) for

\[
m = \begin{cases} 
| i - j |, & \text{if } i \neq j; \\
2, & \text{if } i = j.
\end{cases}
\]

Proof Assume that \(i < j\). By Lemma 12, the module \(P(j)\) appears at position \(j - i\) of the projective resolution of \(L(i)\), yielding a non-zero extension. The case \(j < i\) follows by using the simple-preserving duality. For \(i = j > 1\), the same lemma tells us that \(P(i)\) appears in the second position of the projective resolution of \(L(i)\), which proves the claim.

Proposition 14 If an indecomposable \(\Lambda_n\)-module \(M\) has neither a standard filtration nor a costandard filtration, then \(M\) is not self-orthogonal.

Proof If \(M = \Omega(i, j, k)\) is not simple (this case was covered in Lemma 13) and has neither a standard nor a costandard filtration, then \(i, j > 1\) and either \(i \equiv k \mod 2\) and \(j \equiv k \mod 2\), or \(i \not\equiv k \mod 2\) and \(j \not\equiv k \mod 2\). This follows from Lemma 6 and Lemma 7.

Let \(K\) denote the kernel of the projective cover \(P \twoheadrightarrow M\) and \(C\) the cokernel of the injective envelope \(M \hookrightarrow I\). By Lemma 10 and Lemma 11 both \(K\) and \(C\) have a simple direct summand.

- If \(i \equiv k \mod 2\) and \(j \equiv k \mod 2\), then \(L(j - 1)\) is a direct summand of \(K\) and \(L(i - 1)\) is a direct summand of \(C\).
- If \(i \not\equiv k \mod 2\) and \(j \not\equiv k \mod 2\), then \(L(i - 1)\) is a direct summand of \(K\) and \(L(j - 1)\) is a direct summand of \(C\).
Unless both $i$ and $j$ are equal to 2, using Lemma 5 together with Lemma 13, we get
\[
\text{Ext}^m_{\Lambda_n}(M, M) \cong \text{Ext}^{m-2}_{\Lambda_n}(K, C) \neq 0,
\]
for some $m > 2$.

If $i = j = 2$, then $M = \Omega(2, 2, k)$ and the kernel $K$ of the projective cover is equal to
\[
\bigoplus_{x=3}^{k-1} \Delta(x) \oplus L(1).
\]
The beginning of the projective resolution of $L(1)$ looks as follows:

\[
\begin{array}{cccc}
\cdots & \rightarrow & P(3) & d_2 \\
& & \Delta(3) & \rightarrow \Delta(2) \\
& & d_1 & \rightarrow P(1) \rightarrow L(1)
\end{array}
\]

Since $L(2)$ is a submodule of $M$ and there is no homomorphism from $P(1)$ to $M$, we have

- a non-zero extension of degree 1 from $L(1)$ to $M$. But this implies that there is a non-

zero extension of degree two from $M$ to itself, as $L(1)$ is a direct summand of $K$ and, by

Lemma 5, we have $\text{Ext}^2_{\Lambda_n}(M, M) \cong \text{Ext}^1_{\Lambda_n}(K, M)$. This proves the claim.

**Lemma 15** Let $M = \Omega(i, j, k) \in \mathcal{F}(\mathbb{V})$. Then $\text{Ext}^m_{\Lambda_n}(L(x), M) = 0$, for all $1 \leq x < \min(i, j)$ and $m \geq 0$.

**Proof** It is clear that $\text{Hom}_{\Lambda_n}(L(x), M) = 0$, for all $1 \leq x < \min(i, j)$, since $M$ does not

have $L(x)$ as a composition factor for such $x$. It is also clear that $\text{Ext}^m_{\Lambda_n}(L(1), M) = 0$, for

$m > 0$, since $L(1) \in \mathcal{F}(\Delta)$ and $M \in \mathcal{F}(\mathbb{V})$. This proves the claim for $x = 1$.

Consider $x$ such that $2 \leq x < \min(i, j)$. In this case, we have the short exact sequence

\[
L(x - 1) \oplus \Delta(x + 1) \rightarrow P(x) \rightarrow L(x),
\]

and by Lemma 5 we have the following equality:

\[
\dim \text{Ext}^1_{\Lambda_n}(L(x), M) = \dim \text{Hom}_{\Lambda_n}(L(x), M) - \dim \text{Hom}_{\Lambda_n}(P(x), M)
\]

\[
+ \dim \text{Hom}_{\Lambda_n}(L(x - 1) \oplus \Delta(x + 1), M).
\]

Since neither $L(x)$ nor $L(x - 1)$ are composition factors of $M$, this equality reduces to

\[
\dim \text{Ext}^1_{\Lambda_n}(L(x), M) = \dim \text{Hom}_{\Lambda_n}(\Delta(x + 1), M).
\]

If $2 \leq x < \min(i, j) - 1$, then clearly

\[
\text{Hom}_{\Lambda_n}(\Delta(x + 1), M) = 0,
\]

since $L(x + 1)$ does not occur as composition factors in $M$. When $x = \min(i, j) - 1$ we also have

\[
\text{Hom}_{\Lambda_n}(\Delta(x + 1), M) = 0.
\]

To see this, note that any non-zero homomorphism $f : \Delta(x + 1) \rightarrow M$ annihilates $\text{rad}(\Delta(x + 1)) = L(x)$ since $L(x)$ does not occur as a composition factor in $M$. This implies that the image of $f$ is isomorphic to $L(x + 1)$. But

$M$ has no such submodule since the (unique) composition factor $L(x + 1)$ is contained in the top of $M$, a contradiction. It follows that $\text{Ext}^1_{\Lambda_n}(L(x), M) = 0$ for all $x < \min(i, j)$.

It remains to show that $\text{Ext}^m_{\Lambda_n}(L(x), M) = 0$ for all $m > 1$. By Lemma 5 and the short

exact sequence above, together with the fact that $M \in \mathcal{F}(\mathbb{V})$, it follows that

\[
\text{Ext}^m_{\Lambda_n}(L(x), M) \cong \text{Ext}^{m-1}_{\Lambda_n}(L(x - 1) \oplus \Delta(x + 1), M) \cong \text{Ext}^{m-1}_{\Lambda_n}(L(x - 1), M).
\]

If $m < x$, then by repeated use of the isomorphism above we have

\[
\text{Ext}^m_{\Lambda_n}(L(x), M) \cong \text{Ext}^1_{\Lambda_n}(L(x - m + 1), M) = 0,
\]

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by the previous case. If \( m \geq x \), then

\[
\Ext_{\Lambda_n}^m(L(x), M) \cong \Ext_{\Lambda_n}^{m-x+1}(L(1), M) = 0,
\]

since \( L(1) \in \mathcal{F}(\Delta) \) and \( M \in \mathcal{F}(\mathcal{V}) \).

**Proposition 16** Let \( \Omega(i, j, k) \in \mathcal{F}(\Delta) \cup \mathcal{F}(\mathcal{V}) \) be such that \( \Omega(i, j, k) \notin \mathcal{F}(\Delta) \cap \mathcal{F}(\mathcal{V}) \). Then \( \Omega(i, j, k) \) is self-orthogonal if and only if \( |i - j| = 1 \).

**Proof** We will show that the claim holds for \( M = \Omega(i, j, k) \) such that \( M \in \mathcal{F}(\mathcal{V}) \), but \( M \notin \mathcal{F}(\Delta) \). It will then follow, by applying the simple-preserving duality, that the claim also holds for \( M \) such that \( M \in \mathcal{F}(\Delta) \), but \( M \notin \mathcal{F}(\mathcal{V}) \). Note that if \( i = j = 1 \), then \( M \in \mathcal{F}(\Delta) \cap \mathcal{F}(\mathcal{V}) \), and if \( i = j > 1 \), then \( M \notin \mathcal{F}(\Delta) \cup \mathcal{F}(\mathcal{V}) \). Thus, we must have \( |i - j| \geq 1 \).

Assume that \( |i - j| = 1 \). Consider the surjection \( T(k) \rightarrow M \) (which is unique up to a scalar) and denote the kernel of this projection by \( K \). If \( i, j > 1 \), then the kernel \( K \) can be written as a direct sum of two indecomposable modules \( U \oplus L \), where \( U = \Omega(1, i-1, i-1) \) and \( L = \Omega(1, j-1, j-1) \). If \( i = 1 \) and \( j = 2 \), or \( i = 2 \) and \( j = 1 \), then the kernel \( K \) is equal to \( L(1) \). This gives us a short exact sequence

\[
K \hookrightarrow T(k) \rightarrow M.
\]

By applying \( \Hom_{\Lambda_n}(\_ , M) \), we get a long exact sequence

\[
0 \rightarrow \Hom_{\Lambda_n}(M, M) \rightarrow \Hom_{\Lambda_n}(T(k), M) \rightarrow \Hom_{\Lambda_n}(K, M) \\
\rightarrow \Ext_{\Lambda_n}^1(M, M) \rightarrow \Ext_{\Lambda_n}^1(T(k), M) \rightarrow \Ext_{\Lambda_n}^1(K, M) \\
\rightarrow \Ext_{\Lambda_n}^2(M, M) \rightarrow \Ext_{\Lambda_n}^2(T(k), M) \rightarrow \Ext_{\Lambda_n}^2(K, M) \rightarrow \cdots
\]

Since \( M \in \mathcal{F}(\mathcal{V}) \) and \( T(k) \in \mathcal{F}(\Delta) \), we have \( \Ext_{\Lambda_n}^m(T(k), M) = 0 \) for all \( m > 0 \). This implies that

\[
\Ext_{\Lambda_n}^m(M, M) \cong \Ext_{\Lambda_n}^{m-1}(K, M),
\]

for all \( m \geq 2 \).

Note that the only common composition factor of \( K \) and \( M \) is \( L(x) \), where \( x = \min(i, j) \). However, \( L(x) \) is not a submodule of \( M \), which means that there are no non-zero homomorphisms from \( K \) to \( M \). Together with the fact that \( \Ext_{\Lambda_n}^1(T(k), M) = 0 \), this implies \( \Ext_{\Lambda_n}^1(M, M) = 0 \).

If \( i = 1 \) and \( j = 2 \), or \( i = 2 \) and \( j = 1 \), then \( \Ext_{\Lambda_n}^m(M, M) \cong \Ext_{\Lambda_n}^{m-1}(K, M) = 0 \), for all \( m \geq 2 \), since \( K = L(1) \in \mathcal{F}(\Delta) \). Now assume that \( i, j > 1 \). By Lemma 9, the kernel \( J \) of the projective cover of \( K \) is equal to \( \Delta(2, i-3) \oplus \Delta(2, j-3) \oplus L(i-2) \oplus L(j-2) \), where \( L(x) \) is interpreted as 0 if \( x < 1 \). Using Lemma 15 and the fact that \( M \in \mathcal{F}(\mathcal{V}) \), this implies that \( \Ext_{\Lambda_n}^{m-2}(J, M) = 0 \). It now follows that

\[
\Ext_{\Lambda_n}^m(M, M) \cong \Ext_{\Lambda_n}^{m-1}(K, M) \cong \Ext_{\Lambda_n}^{m-2}(J, M) = 0,
\]

for all \( m \geq 2 \). Note that the second isomorphism is obtained by applying Lemma 5. This proves the claim that, if \( |i - j| = 1 \), then \( \Omega(i, j, k) \) is self-orthogonal.

Assume that \( |i - j| > 1 \). The projective cover of \( M \) will be \( P_{[i-1,k-1]} \oplus P_{(j,k-2)} \) and the corresponding kernel will be \( \Delta(i+1,k-2) \oplus \Delta(j+1,k-1) \oplus L(i-1) \oplus L(j-1) \), where \( L(x) \) is interpreted as 0 if \( x < 1 \). This, together with Lemma 5 and the fact that \( M \in \mathcal{F}(\mathcal{V}) \), implies that

\[
\Ext_{\Lambda_n}^m(M, M) \cong \Ext_{\Lambda_n}^{m-1}(L(i - 1), M) \oplus \Ext_{\Lambda_n}^{m-1}(L(j - 1), M),
\]

for all \( m \geq 1 \).
We will prove that either $\text{Ext}^1_{\Lambda_n}(L(i-1), M) \neq 0$, or $\text{Ext}^1_{\Lambda_n}(L(j-1), M) \neq 0$, depending on whether $j \leq i - 2$ or $i \leq j - 2$. We will consider the case when $j \leq i - 2$. The case $i \leq j - 2$ is proven using the exact same arguments, but with $i$ replaced by $j$.

The beginning of the projective resolution of $L(i-1)$ looks as follows:

$$
\cdots \xrightarrow{d_2} P(i-2) \oplus P(i) \xrightarrow{d_1} P(i-1) \xrightarrow{d_0} L(i-1) \oplus \Delta(i)
$$

Since $L(i-2)$ is a submodule of $M$, there is a homomorphism $f : P(i-2) \oplus P(i) \to M$ such that the image of $f$ is isomorphic to $L(i-2)$. Furthermore, the composition $f \circ d_2$ is equal to the zero homomorphism. However, we cannot have $f = g \circ d_1$ for any homomorphism $g : P(i-1) \to M$. Indeed, the kernel of the unique (up to scalar) non-zero map from $P(i-1)$ to $M$ is equal to $L(i-1)$. The image of $g \circ d_1$ therefore contains the submodule $L(i)$. Since the image of $f$ is isomorphic to $L(i-2)$ the two maps cannot be equal. This means that $\text{Ext}^1_{\Lambda_n}(L(i-1), M) \neq 0$, which implies that $\text{Ext}^2_{\Lambda_n}(M, M) \neq 0$. 

5 Generalized Tilting Modules

5.1 Background of Generalized Tilting Modules

The main goal of this article is to classify all generalized tilting $\Lambda_n$-modules. Classical tilting modules were first introduced by [1, 2] and were later generalized by Miyashita in [24]. Recall that for a $\Lambda_n$-module $M$, $\text{add} M$ denotes the full subcategory of $\Lambda_n$-mod consisting of direct summands of finite direct sums of $M$.

**Definition 6** [24] Let $\Lambda$ be an algebra and let $T$ be a $\Lambda$-module. Then, $T$ is called a generalized tilting module if

(T1) $T$ has finite projective dimension;
(T2) $\text{Ext}_\Lambda^m(T, T) = 0$ for all $m > 0$;
(T3) there is an exact sequence

$$
0 \to \Lambda \to Q_0 \to Q_1 \to \cdots \to Q_r \to 0
$$

such that $Q_i \in \text{add} T$ for all $0 \leq i \leq r$.

Recall that every quasi-hereditary algebra has finite global dimension, so (T1) is satisfied for every $\Lambda_n$-module.

**Theorem 17** [25] Let $\Lambda$ be an algebra of finite representation type and let $T$ be a $\Lambda$-module satisfying the first two properties of a generalized tilting module:

(T1) $T$ has finite projective dimension;
(T2) $\text{Ext}_\Lambda^m(T, T) = 0$, for all $m > 0$.

Then, there is a $\Lambda$-module $S$, such that $T \oplus S$ is a generalized tilting module.

**Corollary 18** [25] Let $\Lambda$ be an algebra of finite representation type and let $T$ be a $\Lambda$-module satisfying (T1), (T2) and
(T3’) \( T \) has \( n \) non-isomorphic indecomposable direct summands.

Then, \( T \) is a generalized tilting module.

Corollary 18 implies that to classify generalized tilting modules, it is enough to classify all collections of \( n \) indecomposable self-orthogonal modules such that all extensions (of positive degree) between each pair of modules vanish.

5.2 Non-zero Extensions Between Modules in \( \mathcal{F}(\mathcal{V}) \) and \( \mathcal{F}(\mathcal{D}) \)

The aim of this subsection is to prove that for any self-orthogonal modules \( M \in \mathcal{F}(\mathcal{V}) \) and \( N \in \mathcal{F}(\mathcal{D}) \), there is a non-zero extension, from \( M \) to \( N \), of positive degree. This reduces the problem of classifying all generalized tilting \( \Lambda_n \)-modules to finding all generalized tilting modules in \( \mathcal{F}(\mathcal{D}) \) and then, by using the simple-preserving duality, obtaining all generalized tilting modules in \( \mathcal{F}(\mathcal{V}) \).

**Proposition 19** Let \( M \in \mathcal{F}(\mathcal{V}) \) and \( N \in \mathcal{F}(\mathcal{D}) \) be indecomposable \( \Lambda_n \)-modules such that \( M \) and \( N \) are self-orthogonal and \( M, N \notin \mathcal{F}(\mathcal{D}) \cap \mathcal{F}(\mathcal{V}) \). Then \( \text{Ext}^m_{\Lambda_n}(M, N) \neq 0 \) for some \( m \geq 1 \).

**Proof** By Proposition 16, \( M \) and \( N \) must be of one of the following forms.

- \( M = \Omega(i, i + 1, k) \) if \( i \equiv k \mod 2 \);
- \( M = \Omega(i + 1, i, k) \) if \( i \not\equiv k \mod 2 \);
- \( N = \Omega(j + 1, j, \ell) \) if \( j \equiv \ell \mod 2 \);
- \( N = \Omega(j, j + 1, \ell) \) if \( j \not\equiv \ell \mod 2 \).

By Lemma 9, the kernel \( K \) of the projective cover of \( M \) is equal to \( \bigoplus_{x=1}^{k-1} \Delta(x) \oplus L(i-1) \oplus L(i) \), where \( L(x) \) is interpreted as zero if \( x < 1 \). By dualizing Lemma 9, the cokernel \( C \) of the injective envelope of \( N \) is equal to \( \bigoplus_{x=j+1}^{j-1} \nabla(x) \oplus L(j-1) \oplus L(j) \), where \( L(x) \) is interpreted as zero if \( x < 1 \).

Unless \( i = j = 1 \), using Lemma 5 together with Lemma 13, we get \( \text{Ext}^m_{\Lambda_n}(M, N) \cong \text{Ext}^{m-2}_{\Lambda_n}(K, C) \neq 0 \), for some \( m > 0 \). Assume that \( i = j = 1 \). Since \( i = 1 \), as mentioned before, the kernel \( K \) of the projective cover of \( M \) is equal to \( \bigoplus_{x=2}^{k-1} \Delta(x) \oplus L(1) \). The beginning of the projective resolution of \( L(1) \) looks as follows:

\[
\cdots \rightarrow P(3) \xrightarrow{d_2} P(2) \xrightarrow{d_1} P(1) \rightarrow L(1)
\]

If \( \ell = 2 \), then \( N = \Delta(2) \) and we a non-split short exact sequence

\[
\Delta(2) \hookrightarrow P(1) \twoheadrightarrow L(1).
\]

This, together with Lemma 5, implies that there is a non-zero extension of degree two from \( M \) to \( N \).

If \( \ell > 2 \), there is a homomorphism \( f : P(2) \to N \), whose image is isomorphic to \( L(2) \) and annihilates \( \text{rad} P(2) \). Since the image of the differential \( d_2 \) is contained in \( \text{rad} P(2) \), we have \( f \circ d_2 = 0 \). However, the unique (up to scalar) homomorphism \( g : P(1) \to N \) is such that \( L(2) \) does not occur as a composition factor in the image, so that we must have
Therefore, we have a non-zero extension of degree one from $L(1)$ to $N$. But this, together with Lemma 5, implies that there is a non-zero extension of degree two from $M$ to $N$. This proves the claim.

### 5.3 A Strict Partial Order on $\mathcal{F}(\Delta)$

We know that the only generalized tilting module contained in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ is the characteristic tilting module. Moreover, we have shown that if $M \in \mathcal{F}(\nabla)$, $N \in \mathcal{F}(\Delta)$ and $M, N \notin \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$, then $\text{Ext}^m_{\Delta_n}(M, N) \neq 0$ for some $m \geq 1$. This implies that any generalized tilting module must be contained in either $\mathcal{F}(\Delta)$ or $\mathcal{F}(\nabla)$. Together, these statements imply that it is enough to find the basic generalized tilting modules contained in $\mathcal{F}(\Delta)$, which do not equal the characteristic tilting module. All basic generalized tilting modules will then be these modules, their duals (with respect to the simple preserving duality), and the characteristic tilting module.

Let $\mathcal{D}_n$ denote the set of indecomposable self-orthogonal modules in $\mathcal{F}(\Delta)$. Next, we define the relation $\prec_E$ on $\mathcal{D}_n$ given by $M \prec_E N$ if and only if $\text{Ext}^m(M, N) \neq 0$ for some $m \geq 1$. Note that it is not clear from the definition whether this relation is transitive.

Let $Q_n$ be the set of pairs $(i, k)$, with $1 \leq i \leq k \leq n$. We want these pairs to encode the indecomposable self-orthogonal modules contained in $\mathcal{F}(\Delta)$. Let $M(i, k)$ be the module

$$M(i, k) = \begin{cases} \Omega(1, 1, k), & \text{if } i = 1; \\ \Omega(i, i - 1, k), & \text{if } i > 1 \text{ and } i \equiv k \mod 2; \\ \Omega(i - 1, i, k), & \text{if } i > 1 \text{ and } i \not\equiv k \mod 2. \end{cases}$$

This defines a bijection $\varphi : Q_n \to \mathcal{D}_n$ given by $(i, k) \mapsto M(i, k)$.

We define the following relation on $Q_n$:

$$(i, i) \prec_0 (i + 1, \ell), \text{ for } \ell = i + 1, \ldots, n.$$  

For $k = 1, 2, \ldots$ we have the additional relations

$$(i, i + 2k) \prec_0 (i + 1, i + 1 + 2\ell), \text{ for } \ell = 0, 1, \ldots, k - 1;$$  

$$(i, i + 2k) \prec_0 (i + 1, \ell), \text{ for } \ell = i + 2k + 1, i + 2k + 2, \ldots, n;$$  

$$(i, i + 2k + 1) \prec_0 (i + 1, i + 1 + 2\ell), \text{ for } \ell = 0, 1, \ldots, k - 1.$$  

Now define $\prec$ to be the transitive closure of $\prec_0$. This defines a strict partial order on $Q_n$.

By a **strict partial order** on a set, we mean a relation which is transitive, asymmetric and irreflexive. The set $Q_n$ is naturally graded by $\deg(i, k) = i$, and from the definition it is clear that this grading, together with the strict partial order $\prec$, makes $Q_n$ into a graded poset.

Note that, since we have the relations $(i, k) \prec_0 (i + 1, i + 1), (i + 1, i + 1) \prec_0 (i + 2, \ell)$, for all $\ell \geq j + 2$, it follows that, if $k \neq i + 1$, then

$$(i, k) \prec (j, \ell),$$

for every $j \geq i + 2$ and every $\ell \geq j$.

The aim of the next subsection is to prove the following theorem:

**Theorem 20** The bijection $\varphi : Q_n \to \mathcal{D}_n$ defined above is an order isomorphism between $(Q_n, \prec)$ and $(\mathcal{D}_n, \prec_E)$. 

\( \square \) Springer
Before proving the theorem, let us briefly consider its implications. Let \( n = 4 \). We have the following Hasse diagram for the graded poset \((Q_n, \prec)\):

\[
\begin{align*}
(1, 1) & \quad (1, 2) & \quad (1, 3) & \quad (1, 4) \\
(2, 2) & \quad (2, 3) & \quad (2, 4) \\
(3, 3) & \quad (3, 4) \\
(4, 4)
\end{align*}
\]

We view the above picture as a graph, with the vertices \((i, k)\) corresponding to indecomposable self-orthogonal modules in \(\mathcal{F}(\Delta)\). We draw an edge \((i, k) \to (j, \ell)\) if and only if \((i, k) \prec (j, \ell)\), which, as we will show, holds if and only if there is a non-zero extension between the corresponding modules.

With the theorem established, we will be able to find all generalized tilting modules over \(\Lambda_n\) which are contained in \(\mathcal{F}(\Delta)\) by finding all anti-chains of length \(n\) in the above graph. To see this, note that an anti-chain of length \(n\) corresponds exactly to a self-orthogonal module of finite projective dimension, having \(n\) indecomposable summands. Such a module is a generalized tilting module by Corollary 18.

In the picture above, we can simply read off the anti-chains from the picture. We see that

\[
\{(1, 1), (1, 2), (1, 3), (1, 4)\}, \quad \{(1, 2), (1, 3), (1, 4), (2, 3)\}, \\
\{(1, 2), (2, 3), (2, 4)\}, \quad \{(1, 2), (1, 4), (2, 3), (2, 4)\}, \\
\{(1, 2), (2, 3), (3, 3), (3, 4)\}, \quad \{(1, 2), (2, 3), (2, 4), (3, 4)\} \\
\text{and} \quad \{(1, 2), (2, 3), (3, 4), (4, 4)\}
\]

are all possible anti-chains. The first anti-chain corresponds to the characteristic tilting module, and the last anti-chain corresponds to the module \(P(1) \oplus P(2) \oplus P(3) \oplus P(4)\).

### 5.4 Proof of Theorem 20

Let \(M, N\) be two \(\Lambda_n\)-modules. Consider the following two exact sequences, where \(K\) is the kernel of the projective cover \(P \twoheadrightarrow M\) and \(C\) is the cokernel of the injective envelope \(N \hookrightarrow I\):

\[
K \hookrightarrow P \twoheadrightarrow M, \quad N \hookrightarrow I \twoheadrightarrow C.
\]

If \(M\) is the module corresponding to \((i, k)\), where \(k > i\), then, by Lemma 9, we have \(P = \bigoplus_{x=i}^{k-1} P(x)\) and \(K = \bigoplus_{x=i+1}^{k-1} \Delta(x)\). The module corresponding to \((i, i)\) is \(\Delta(i)\) and, in this case, \(P = P(i)\) and \(K = \Delta(i + 1)\). If \(N\) is the module corresponding to \((j, \ell)\), then, by looking at \(N^*\) and using Lemma 9, we find that \(I = \bigoplus_{x=j-1}^{\ell-1} I(x)\) and \(C = \bigoplus_{x=j}^{\ell-1} \nabla(x) \oplus L(j-2) \oplus L(j-1)\), where \(L(x)\) is interpreted as zero if \(x < 1\).

Recall that, by Lemma 5, we have the following.

(i) \(\dim \text{Ext}_{\Lambda_n}^1(M, N) = \dim \text{Hom}_{\Lambda_n}(M, N) - \dim \text{Hom}_{\Lambda_n}(P, N) + \dim \text{Hom}_{\Lambda_n}(K, N)\);

(ii) \(\text{Ext}_{\Lambda_n}^k(M, N) \cong \text{Ext}_{\Lambda_n}^{k-1}(K, N) \cong \text{Ext}_{\Lambda_n}^{k-1}(M, C)\), for all \(k \geq 2\);

(iii) \(\text{Ext}_{\Lambda_n}^k(M, N) \cong \text{Ext}_{\Lambda_n}^{k-2}(K, C)\), for all \(k \geq 3\).
Using the above, together with Lemma 9, we get the following equations for the dimensions of extension spaces from $M$ to $N$:

$$\dim \text{Ext}^1_\Lambda_n(M, N) = \dim \text{Hom}_\Lambda_n(M, N) - \dim \text{Hom}_\Lambda_n(P_X, N) + \dim \text{Hom}_\Lambda_n(\Delta Y, N),$$

where $X = \{i, i + 1, \ldots, k - 1\}$, $Y = \{i + 1, i + 2, \ldots, k - 1\}$, if $k > i$ and $X = \{i\}$, $Y = \{i + 1\}$, if $k = i$.

$$\dim \text{Ext}^2_\Lambda_n(M, N) = \dim \text{Hom}_\Lambda_n(\Delta X, N) - \dim \text{Hom}_\Lambda_n(P_X, N) + \dim \text{Hom}_\Lambda_n(\Delta Y, N),$$

where $X = \{i + 1, i + 2, \ldots, k - 1\}$, $Y = \{i + 2, i + 3, \ldots, k\}$, if $k > i$ and $X = \{i + 1\}$, $Y = \{i + 2\}$, if $k = i$.

$$\dim \text{Ext}^m_\Lambda_n(M, N) = \dim \text{Ext}^{m-2}_\Lambda_n(\Delta X, L(j - 2) \oplus L(j - 1)),$$

where $m \geq 3$, $X = \{i, i + 1, \ldots, k - 1\}$, if $k > i$, and $X = \{i\}$, if $k = i$. Note that for the last equality we have used the fact that $\text{Ext}^m_\Lambda_n(\Delta(x), \nabla(y)) = 0$, for all $m > 0$.

These equations indicate that we need to find the dimensions of various homomorphism spaces, as well as determining when we have a non-zero extension of positive degree from a standard module to a simple module, or not.

**Lemma 21** We have the following dimensions for the homomorphism spaces from a projective module to $M(i, k)$ and from a standard module to $M(i, k)$, respectively:

$$\dim \text{Hom}_\Lambda_n(P(x), M(i, k)) = \begin{cases} 1, & \text{if } x = i - 1 \lor x = k; \\ 2, & \text{if } i \leq x \leq k - 1; \\ 0, & \text{if } x < i - 1 \lor x > k. \end{cases}$$

$$\dim \text{Hom}_\Lambda_n(\Delta(x), M(i, k)) = \begin{cases} 1, & \text{if } i - 1 \leq x \leq k; \\ 0, & \text{if } x < i - 1 \lor x > k. \end{cases}$$

**Proof** Since $\dim \text{Hom}_\Lambda_n(P(x), M(i, k))$ is equal to the number of composition factors $L(x)$ in $M(i, k)$, the first result follows immediately from the definition of $M(i, k)$. If $x < i - 1$, or $x > k$, then $\Delta(x)$ and $M(i, k)$ do not have any common composition factors, and there cannot exist any non-zero homomorphisms from $\Delta(x)$ to $M(i, k)$. For $x = k$, we note that $\Delta(k)$ is a submodule of $M(i, k)$, and, since there is only one composition factor $L(k)$ in $M(i, k)$, it is clear that the inclusion of $\Delta(k)$ into $M(i, k)$ is the only homomorphism.

Now assume that $i - 1 \leq x \leq k - 1$. We claim that, for each such $x$, there is exactly one homomorphism from $\Delta(x)$ to $M(i, k)$ (up to a scalar), namely the homomorphism that sends the top of $\Delta(x)$ to the (unique) submodule $L(x) \subset M(i, k)$ and annihilates the radical of $\Delta(x)$. If $x \neq 1$, or equivalently, if $\Delta(x)$ is not simple, the only other possibility would be an inclusion of $\Delta(x)$ into $M(i, k)$, but for such $x$, the module $\Delta(x)$ is not a submodule of $M(i, k)$.

The following lemma can be proven using the main theorem in [26], but for pedagogical reasons we give a more elementary proof.

**Lemma 22** The space $\text{Hom}_\Lambda_n(M(i, k), M(j, \ell))$, where $i + 2 \leq k$, has the following dimension, depending on $i$, $j$, $k$, $\ell$: 

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\begin{itemize}
  \item \( \dim \text{Hom}_{A_n}(M(i, k), M(j, \ell)) = 0 \), if \( i > \ell \) or \( j > k \).
  \item \( \dim \text{Hom}_{A_n}(M(i, k), M(j, \ell)) = 1 \), if \( j = k \) (and \( i \leq \ell \)).
  \item \( \dim \text{Hom}_{A_n}(M(i, k), M(j, \ell)) = \min(k, \ell) - \max(i, j - 1) \), if \( i \leq \ell, j < k \) and, in addition, one of the following hold:
    \begin{itemize}
      \item \( j \geq i + 2 \);
      \item \( j = i + 1 \) and \( \ell = k \);
      \item \( j = i + 1, \ell < k \) and \( i \equiv \ell \mod 2 \);
      \item \( j = i + 1, \ell > k \) and \( i \not\equiv \ell \mod 2 \).
    \end{itemize}
  \item \( \dim \text{Hom}_{A_n}(M(i, k), M(j, \ell)) = \min(k, \ell) - \max(i, j - 1) + 1 \), if \( i \leq \ell, j < k \) and, in addition, one of the following hold:
    \begin{itemize}
      \item \( j = i + 1, \ell < k \) and \( i \not\equiv \ell \mod 2 \);
      \item \( j = i + 1, \ell > k \) and \( i \equiv \ell \mod 2 \);
      \item \( j \leq i \).
    \end{itemize}
\end{itemize}

**Proof**
First we note that we have the following formulas for the top of \( M(i, k) \) and the socle of \( M(j, \ell) \):

\[
\text{top}(M(i, k)) = \bigoplus_{x=i}^{k-1} L(x),
\]

\[
\text{soc}(M(j, \ell)) = \begin{cases}
  \bigoplus_{x=j-1}^{\ell-1} L(x), & \text{if } j > 1; \\
  \bigoplus_{x=1}^{\ell-1} L(x), & \text{if } j = 1.
\end{cases}
\]

Observe that, if \( i > \ell \) or \( j > k \), then no direct summand of \( \text{top}(M(i, k)) \) is a composition factor of \( M(j, \ell) \), so \( \dim \text{Hom}_{A_n}(M(i, k), M(j, \ell)) = 0 \). If \( j = k \) (and thus \( i \leq \ell \)), then \( \dim \text{Hom}_{A_n}(M(i, k), M(j, \ell)) = 1 \), and a basis vector in \( \text{Hom}_{A_n}(M(i, k), M(j, \ell)) \) is given by the homomorphism defined by mapping \( L(k-1) = L(j-1) \subset \text{top}(M(i, k)) \) to the unique submodule \( L(j-1) \subset \text{soc}(M(j, \ell)) \).

Assume throughout the rest of the proof that \( i \leq \ell \) and \( j < k \). Then, each composition factor \( L(x) \), belonging to the top of \( M(i, k) \), can be mapped to the composition factor \( L(x) \), contained in the socle of \( M(j, \ell) \), for each \( x \) such that \( \max(i, j-1) \leq x \leq \min(k-1, \ell-1) \). This means that

\[
\dim \text{Hom}_{A_n}(M(i, k), M(j, \ell)) \geq \min(k, \ell) - \max(i, j - 1).
\]

We will now investigate whether or not there is an additional homomorphism \( f \) that does not belong to the subspace spanned by the homomorphisms previously mentioned. To get such a homomorphism \( f \), some composition factor \( L(x) \), contained in the top of \( M(i, k) \), must be mapped to the (unique) composition factor \( L(x) \) not contained in the socle of \( M(j, \ell) \). This means that either \( L(x) \) is contained in the top of \( M(j, \ell) \), or that \( x = \ell \).

Assume that \( f \) is such a homomorphism. Since the image of \( f \) is a submodule, the composition factors \( L(x-1) \) and \( L(x+1) \), contained in the radical of \( M(j, \ell) \), must also belong to the image of \( f \), assuming that they are composition factors of \( M(j, \ell) \). This means that the composition factors \( L(x-1) \) and \( L(x+1) \), contained in the radical of \( M(i, k) \), do not belong to the kernel of \( f \). But the kernel is a submodule, so any composition factors having arrows to \( L(x-1) \) or \( L(x+1) \) in the Loewy diagram of \( M(i, k) \) also do not belong to the kernel of \( f \). Repeating these two arguments we will either reach a contradiction,
meaning that there are no more homomorphisms, or come to the conclusion that there is (up to scalar) exactly one more homomorphism.

We will investigate whether or not there exists $x$ such that a homomorphism $f$ could map $L(x)$, contained in the top of $M(i, k)$, to the (unique) composition $L(x)$ not contained in the socle of $M(j, \ell)$. This will depend on the parameters $i, j, k, \ell, x$.

- Assume that $j \geq i + 2$. Repeating the arguments above we find that one of the composition factors $L(j - 1)$ or $L(j - 2)$, depending on the parity of $x$ and $j$, both contained in the top of $M(i, k)$, are not contained in the kernel of $f$. This is a contradiction. Indeed, $L(j - 2)$ is not a composition factor of $M(j, \ell)$, so $L(j - 2)$ must be contained in the kernel of $f$. The other case corresponds to $x$ and $j$ having different parity. In this case the (unique) composition factor $L(j - 1)$ of $M(j, \ell)$ does not belong to the same arm as the composition factor $L(x)$ in question. This implies that even if $L(j - 1)$ would be contained in the image of $f$, the map would not commute with the action of $\Lambda_n$, and is therefore not a homomorphism. Therefore, in this case, we have $\dim \text{Hom}_{\Lambda_n}(M(i, k), M(j, \ell)) = \min(k, \ell) - \max(i, j - 1)$.

- Assume that $j = i + 1$. First we consider $x$ such that $x \equiv i \mod 2$. Then the composition factor $L(x)$ that belongs to the top of $M(i, k)$ is contained in the same arm as the (unique) composition factor $L(i - 1)$ of $M(i, k)$. Similarly, the composition factor $L(x)$ that does not belong to the socle of $M(j, \ell)$ is contained in the same arm as the composition factor $L(i + 1)$ that belongs to the socle of $M(j, \ell)$. If $f$ would map $L(x)$ contained in the top of $M(i, k)$ to the (unique) composition factor $L(x)$ not contained in the socle of $M(j, \ell)$, using previous arguments leads to a similar contradiction as in the case when $j \geq i + 2$. This implies that $f$ must map the composition factor $L(x)$, for $x \equiv i \mod 2$, that belongs to the top of $M(i, k)$, to the socle of $M(j, \ell)$. Next we consider $x$ such that $x \not\equiv i \mod 2$. Then the composition factor $L(x)$ that belongs to the top of $M(i, k)$ is contained in the same arm as the composition factor $L(i)$ that belongs to the socle of $M(i, k)$. Similarly, if $x \not\equiv \ell$, the composition factor $L(x)$ that does not belong to the socle of $M(j, \ell)$ is contained in the same arm as the composition factor $L(i)$ that belongs to the socle of $M(j, \ell)$. If $x = \ell$, then it follows that $L(i) \subset \text{soc}(M(j, \ell))$ belongs to the lower arm of $M(j, \ell)$. We have five cases.

1. Suppose $\ell < k$ and $i \equiv \ell \mod 2$. This implies that $L(x) \subset \text{upp}(M(j, \ell))$ and $x \not\equiv \ell$. If $f$ would map $L(x)$ contained in the top of $M(i, k)$ to the (unique) composition factor $L(x)$ not contained in the socle of $M(j, \ell)$, then, in the same way as before, we see that $L(\ell + 1) \subset M(i, k)$ is not contained in the kernel of $f$, which is a contradiction since $L(\ell + 1)$ is not a composition factor of $M(j, \ell)$. Therefore, in this case, we have $\dim \text{Hom}_{\Lambda_n}(M(i, k), M(j, \ell)) = \min(k, \ell) - \max(i, j - 1)$.

2. Suppose $\ell < k$ and $i \not\equiv \ell \mod 2$. Then $\text{low}(M(j, \ell))$ occurs as a quotient of $M(i, k)$, giving us a homomorphism. Using the same argument as above we see that this is the only possibility for $f$. Therefore, in this case, we have $\dim \text{Hom}_{\Lambda_n}(M(i, k), M(j, \ell)) = \min(k, \ell) - \max(i, j - 1) + 1$.

3. Suppose $\ell > k$ and $i \equiv \ell \mod 2$. Then, $\text{upp}(M(i, k))$ is isomorphic to a submodule of $M(j, \ell)$, giving us a homomorphism. Using the same argument as above we see that this is the only possibility for $f$. Therefore, in this case, we have $\dim \text{Hom}_{\Lambda_n}(M(i, k), M(j, \ell)) = \min(k, \ell) - \max(i, j - 1) + 1$.

4. Suppose $\ell > k$ and $i \not\equiv \ell \mod 2$. This implies that $L(x) \subset \text{low}(M(i, k))$. Then, we find that $L(k + 1) \subset M(j, \ell)$ is contained in the image of $f$. This is a
contradiction since $L(k+1)$ is not a composition factor of $M(i, k)$. Therefore, in this case, we have $\dim \text{Hom}_{A_n}(M(i, k), M(j, \ell)) = \min(k, \ell) - \max(i, j - 1)$.

- Suppose $\ell = k$. If $L(x) \subset \text{upp}(M(i, k))$, then also $L(x) \subset \text{upp}(M(j, \ell))$. Using the same arguments as before leads to a similar contradiction as in the case $j \geq i + 2$. If, instead, $L(x) \subset \text{low}(M(i, k))$ then also $L(x) \subset \text{low}(M(j, \ell))$ and the situation is similar to the previous case. Therefore, in this case, we have $\dim \text{Hom}_{A_n}(M(i, k), M(j, \ell)) = \min(k, \ell) - \max(i, j - 1)$.

- Assume that $j \leq i$. We have three cases.

  - Suppose $\ell < k$. Depending on the parity of $\ell - i$, either $\Omega(\ell, i - 1, \ell)$ or $\Omega(\ell, i, \ell)$ is a quotient of $M(i, k)$ and this quotient is isomorphic to the lower arm of $M(j, \ell)$. This defines a homomorphism $f$ from $M(i, k)$ to $M(j, \ell)$. Using the same arguments as above, we see that this is the only possibility for $f$. Therefore, in this case, we have $\dim \text{Hom}_{A_n}(M(i, k), M(j, \ell)) = \min(k, \ell) - \max(i, j - 1) + 1$.

  - Suppose $\ell > k$. In this case, we see that the upper arm of $M(i, k)$ is isomorphic to a submodule of $M(j, \ell)$, and $f$ can be chosen as (a scalar multiple of) the obvious embedding. Using the same argument as above, we see that this is the only possibility for $f$. Therefore, in this case, we have $\dim \text{Hom}_{A_n}(M(i, k), M(j, \ell)) = \min(k, \ell) - \max(i, j - 1) + 1$.

  - Suppose $\ell = k$. In this case, we see that $M(i, k) \subset M(j, \ell)$ is a submodule, and consequently, $f$ can be chosen as (a scalar multiple of) the obvious embedding. Using the same argument as above, we see that this is the only possibility for $f$. Therefore, in this case, we have $\dim \text{Hom}_{A_n}(M(i, k), M(j, \ell)) = \min(k, \ell) - \max(i, j - 1) + 1$.

\begin{lemma}
If $y \leq x$, then $\text{Ext}^m_{A_n}(\Delta(x), L(y)) = 0$, for all $m > 0$. If $y > x$, then $\text{Ext}^m_{A_n}(\Delta(x), L(y)) \neq 0$, if and only if $m = y - x$.
\end{lemma}

\begin{proof}
Splicing the short exact sequences $\Delta(x + 1) \hookrightarrow P(x) \twoheadrightarrow \Delta(x)$, it follows that, at position $m$ in the projective resolution of $\Delta(x)$, we find the projective module $P(x + m)$. This means that, if $y \leq x$, then $\text{Ext}^m_{A_n}(\Delta(x), L(y)) = 0$, for all $m > 0$. At position $y - x$, we find the projective module $P(y)$, which surjects onto $L(y)$, giving rise to a non-zero extension. It is also clear that $\text{Ext}^m_{A_n}(\Delta(x), L(y)) = 0$, for $m \neq y - x$, since there is no homomorphism from $P(m + x)$ to $L(y)$ for such $m$.
\end{proof}

With these results in hand, we can now determine between which pairs of modules in $\mathcal{D}_n$ all extensions of positive degree vanish.

\begin{lemma}
$\text{Ext}^m_{A_n}(M(i, k), M(i + 1, \ell)) = 0$ for all $m \geq 2$.
\end{lemma}

\begin{proof}
By Eq. 2, we have
\[
\dim \text{Ext}^2_{A_n}(M(i, k), M(i + 1, \ell)) = \\
= \dim \text{Hom}_{A_n}(\Delta_X, M(i + 1, \ell)) - \dim \text{Hom}_{A_n}(P_X, M(i + 1, \ell)) \\
+ \dim \text{Hom}_{A_n}(\Delta_Y, M(i + 1, \ell)),
\]

\end{proof}
where $X, Y \subset \{1, \ldots, n\}$ are subsets depending on $i$ and $k$. Lemma 21 then allows us to compute the dimensions

$$\dim \text{Hom}_{\Lambda_n}(\Delta X, M(i + 1, \ell)), \quad \dim \text{Hom}_{\Lambda_n}(P_X, M(i + 1, \ell)),$$

and

$$\dim \text{Hom}_{\Lambda_n}(\Delta Y, M(i + 1, \ell)),$$

which, of course, depend on $i, k$ and $\ell$. Summing these numbers, with signs prescribed by Eq. 2, in each case, then yields $\dim \text{Ext}^2_{\Lambda_n}(M(i, k), M(i + 1, \ell)) = 0$. This proves the case $m = 2$.

For $m \geq 3$, we use Eq. 3, which says that

$$\dim \text{Ext}^m_{\Lambda_n}(M(i, k), M(i + 1, \ell)) = \dim \text{Ext}^{m-2}_{\Lambda_n}(\Delta X, L(i - 1) \oplus L(i)),$$

where $X = \{i, i + 1, \ldots, k - 1\}$, if $k > i$, and $X = \{i\}$, if $k = i$. Since $x \geq i$, for all $x \in X$, Lemma 23 guarantees that $\dim \text{Ext}^m_{\Lambda_n}(M(i, k), M(i + 1, \ell)) = \dim \text{Ext}^{m-2}_{\Lambda_n}(\Delta X, L(i - 1) \oplus L(i)) = 0$, for all $m \geq 3$.

**Proposition 25** If $j \leq i$, then $\text{Ext}^m_{\Lambda_n}(M(i, k), M(j, \ell)) = 0$, for all $m \geq 1$.

**Proof** This can be proved using the same strategy as for Lemma 24, by using Lemma 21, 22 and 23 as well as Eqs. 1, 2 and 3.

**Proposition 26** Let $(i, k)$ and $(i + 1, \ell)$ be such that $(i, k) < (i + 1, \ell)$. Then,

$$\text{Ext}^1_{\Lambda_n}(M(i, k), M(i + 1, \ell)) \neq 0.$$

**Proof** This can be proved using the same strategy as for Lemma 24, by using Lemma 21 and 22 as well as Eq. 1.

**Proposition 27** Let $(i, k)$ and $(i + 1, \ell)$ be such that $(i, k) \neq (i + 1, \ell)$. Then,

$$\text{Ext}^m_{\Lambda_n}(M(i, k), M(i + 1, \ell)) = \text{Ext}^m_{\Lambda_n}(M(i + 1, \ell), M(i, k)) = 0,$$

for all $m \geq 1$.

**Proof** By Lemma 24 and Proposition 25 all extensions of degree two or higher from $M(i, k)$ to $M(i + 1, \ell)$ vanish, as well as all extensions of positive degree from $M(i + 1, \ell)$ to $M(i, k)$. That the remaining extensions vanish can be proved using the same strategy as for Lemma 24, by using Lemma 21 and 22 as well as Eq. 1.

**Proposition 28** If $k \neq i + 1$ and $j \geq i + 2$, then $\text{Ext}^m_{\Lambda_n}(M(i, k), M(j, \ell)) \neq 0$, for some $m \geq 1$.

**Proof** For $j \geq i + 3$ the result follows from Eq. 3 together with Lemma 23. For $j = i + 2$, we will use the isomorphism

$$\text{Ext}^2_{\Lambda_n}(M(i, k), M(j, \ell)) \cong \text{Ext}^1_{\Lambda_n}(K, M(j, \ell)),$$

where $K$ is the kernel of the projective cover of $M(i, k)$. Note that $K$ always contains $\Delta(i + 1)$ as a direct summand, and that $\Delta(i + 1) = M(i + 1, i + 1)$. Since $(i + 1, i + 1) < (i + 2, \ell)$, for all $\ell$, Proposition 26 implies that $\text{Ext}^2_{\Lambda_n}(M(i, k), M(j, \ell)) \cong \text{Ext}^1_{\Lambda_n}(K, M(j, \ell))$ is non-zero.
Proof of Theorem 20 By applying Propositions 25–28, it is now clear that \((i, k) < (j, \ell)\) if and only if there is a non-zero extension from \(M(i, k)\) to \(M(j, \ell)\).

5.5 Characterization of Generalized Tilting Modules

Using Theorem 20 we can now characterize the generalized tilting \(\Lambda_n\)-modules in \(\mathcal{F}(\Delta)\) in terms of anti-chains with respect to the order \(<\).

Observe that, for any \(n \geq 2\), there is an epimorphism of algebras \(\Lambda_n \to \Lambda_{n-1}\), given by quotienting out the two-sided ideal generated by \(e_n\), the idempotent corresponding to the vertex \(n\). This epimorphism induces a functor \(F : \Lambda_{n-1}\text{-mod} \to \Lambda_n\text{-mod}\), which is well-known to be fully faithful and exact. From this, one can deduce the following.

Lemma 29 \([27, 28]\) For any \(\Lambda_{n-1}\)-modules \(M\) and \(N\), we have

\[
\text{Ext}_{\Lambda_{n-1}}^k(M, N) \cong \text{Ext}_{\Lambda_n}^k(M, N),
\]

for all \(k \geq 0\).

Proposition 30 Assume that \(T\) is a generalized tilting \(\Lambda_n\)-module contained in \(\mathcal{F}(\Delta)\). Then there exists an index \(i\), where \(1 \leq i \leq n - 1\), such that every indecomposable direct summand of \(T\) is isomorphic to either \(M(i, k)\), \(M(i + 1, \ell)\) or \(P(j)\), for some \(i \leq k \leq n\), \(i + 1 \leq \ell \leq n\) or \(j < i\).

In this case, we say that \(T\) belongs to the \(i\)th tier.

Proof Let \(i\) be the least index such that \(T\) has a non-projective summand \(M(i, k)\). Consider the module \(M(j, \ell)\) for some \(j \geq i + 2\). Then, there exists a non-zero extension from \(M(i, k)\) to \(M(j, \ell)\), by Proposition 28. By assumption, if \(j < i\), then any non-projective module \(M(j, \ell)\) is not a summand of \(T\). This leaves as possible summands of \(T\) the modules \(M(i, k)\), \(M(i + 1, \ell)\) or \(P(j)\), for \(j < i\).

Lemma 31 Assume that \(T \in \mathcal{F}(\Delta)\) is a generalized tilting \(\Lambda_n\)-module in the \(i\)th tier. Then \(\bigoplus_{j=1}^{n-1} P(j)\) is a direct summand of \(T\).

Proof Assume towards a contradiction that \(P(j)\) is not a direct summand of \(T\), for some \(j < i\). Let \(M\) be an indecomposable direct summand of \(T\). Then \(M\) is projective or isomorphic to either \(M(i, k)\) or \(M(i + 1, \ell)\). This means that \(\text{Ext}_{\Lambda_n}^m(P(j), M) = 0\) trivially and \(\text{Ext}_{\Lambda_n}^m(M, P(j)) = 0\) for any \(j < i\), either because \(M\) is projective or by Proposition 25. This implies that the module \(T \oplus P(j)\) is a generalized tilting module with \(n + 1\) non-isomorphic direct summands, which is a contradiction, since any such module must have exactly \(n\) non-isomorphic indecomposable direct summands.

Example 3 Let \(n = 5\). To find generalized tilting modules in the first tier, we look for anti-chains consisting of vertices in the first and second rows.
We find the following anti-chains.

\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5)\}, \quad \{(1, 2), (2, 2), (2, 3), (2, 4), (2, 5)\},

\{(1, 2), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5)\} \quad \text{and} \quad \{(1, 2), (1, 4), (2, 3), (2, 4), (2, 5)\}.

Lemma 32 Assume that \(T \in \mathcal{F}(\Delta)\) is a generalized tilting \(\Lambda_n\)-module in the \(i\)th tier. Then, at least one of the modules \(M(i, n)\) or \(M(i + 1, n)\) occurs as a direct summand of \(T\).

Proof Assume towards a contraction that \(T\) contains neither \(M(i, n)\) nor \(M(i + 1, n)\) as a summand. The modules \(M(i, n)\) and \(M(i + 1, n)\) are the only modules in the rows \(i\) and \(i + 1\) which have a composition factor \(L(n)\). Furthermore, no projective module \(P(j)\) with \(j < n - 1\) has \(L(n)\) as a composition factor. This means that \(T\) restricts to a \(\Lambda_{n-1}\)-module, and by Lemma 29, we have \(\text{Ext}^k_{\Lambda_{n-1}}(T, T) \cong \text{Ext}^k_{\Lambda_n}(T, T) = 0\), for all \(k \geq 0\). Then, \(T\) is a generalized tilting \(\Lambda_{n-1}\)-module with \(n\) summands, a contradiction. \(\square\)

Let \(1 \leq i \leq n - 1\) and \(i < x \leq n\). We let \(Z(i, x)\) denote the following module:

\[ Z(i, x) = \bigoplus_{j=1}^{i-1} P(j) \bigoplus_{k \in [i+1, x-1]} M(i, k) \bigoplus_{\ell \in [i+2, x-1]} M(i + 1, \ell) \oplus M(i, x) \oplus M(i + 1, x). \]

Lemma 33 The module \(Z(i, n)\) is a generalized tilting module, for \(1 \leq i \leq n - 1\).

Proof It follows from Theorem 20 that \(Z(i, n)\) is self-orthogonal. Since \(Z(i, n)\) contains \(n\) non-isomorphic indecomposable direct summands, it follows from Corollary 18 that it is a generalized tilting module. \(\square\)

Proposition 34 Assume that \(T \in \mathcal{F}(\Delta)\) is a basic generalized tilting \(\Lambda_n\)-module in the \(i\)th tier. If both of the modules \(M(i, n)\) and \(M(i + 1, n)\) are direct summands of \(T\), then

\[ T = Z(i, n). \]

Proof There is a non-zero extension from the module \(M(i, n)\) to the modules \(M(i + 1, k)\), for \(k \in [i + 1, n]\). This excludes \(\left\lfloor \frac{n-i-1}{2} \right\rfloor\) modules from occurring as summands in \(T\). Similarly, there is a non-zero extension from \(M(i, k)\) to \(M(i + 1, n)\), for \(k \in [i, n]\). This excludes \(\left\lfloor \frac{n-i+1}{2} \right\rfloor\) modules from occurring as summands in \(T\). In total, this excludes \(n - i\) modules from occurring as summands in \(T\). Since we have \(2n - i\) indecomposable modules in total to choose from, and have excluded \(n - i\) of them, the direct sum of the \(n\) remaining indecomposable modules must be our module \(T\). However, this direct sum is precisely \(Z(i, n)\), which proves the claim. \(\square\)

Theorem 35 Assume that \(T \in \mathcal{F}(\Delta)\) is a basic generalized tilting \(\Lambda_n\)-module which is not equal to the characteristic tilting module. Then, there exists an integer \(i\), where \(1 \leq i \leq n - 1\), and an integer \(x\), where \(i < x \leq n\), such that:

\[ T = Z(i, x) \oplus \bigoplus_{k=x+1}^{n} M(i, k), \]
if $x \equiv i \mod 2$, and

$$T = Z(i, x) \oplus \bigoplus_{k=x+1}^{n} M(i + 1, k),$$

if $x \not\equiv i \mod 2$.

In particular, there are $\frac{n(n-1)}{2}$ basic generalized tilting $\Lambda_n$-modules in $F(\Delta)$ which are not equal to the characteristic tilting module. In total, there are $n(n-1) + 1$ generalized tilting $\Lambda_n$-modules.

**Proof** It is easily verified that Theorem 20 implies that a module of this form is self-orthogonal and that it contains $n$ non-isomorphic indecomposable direct summands. By Lemma 18, it follows that such a module is a generalized tilting module. Furthermore, for each pair $(i, x)$, we obtain $n$ non-isomorphic modules, which gives us $\sum_{i=1}^{n-1} n - i = \frac{n(n-1)}{2}$ basic generalized tilting $\Lambda_n$-modules in $F(\Delta)$ which are not equal to the characteristic tilting module. To obtain the basic generalized tilting $\Lambda_n$-modules in $F(\nabla)$, which are not equal to the characteristic tilting module, we use the simple preserving duality.

To prove that every basic generalized tilting module in $F(\Delta)$, not equal to the characteristic tilting module, is of this form we proceed by induction on $n$. For $n = 2$, we have two basic generalized tilting modules contained in $F(\Delta)$, namely the characteristic tilting module and the module $P_{\Lambda_2}(1) \oplus P_{\Lambda_2}(2) = Z(1, 2)$. Thus, the claim holds for $n = 2$.

Now, let $T$ be a basic generalized tilting module over $\Lambda_n$. By Proposition 30 there exists an integer $i$, where $1 \leq i \leq n - 1$, such that $T$ belongs to the $i$th tier. If $T = Z(i, n)$ there is nothing to prove. Otherwise, according to Proposition 34, we may decompose $T$ as $T = T' \oplus M(i, n)$ or as $T = T' \oplus M(i + 1, n)$, where, in both cases, $T'$ is a basic generalized tilting module over $\Lambda_{n-1}$, according to Lemma 29.

Assume first that $i \equiv n \mod 2$.

- Suppose $T = T' \oplus M(i, n)$. If $T'$ is the characteristic tilting module over $\Lambda_{n-1}$, then $T$ would be the characteristic tilting module over $\Lambda_n$, contradicting our assumption. If

$$T' = Z(i, x) \oplus \bigoplus_{k=x+1}^{n-1} M(i + 1, k),$$

for some $x \not\equiv i \mod 2$, then $T$ contains the summand $M(i + 1, n - 1)$, which is a contradiction as there is a non-zero extension from $M(i, n)$ to $M(i + 1, n - 1)$. Therefore, we must have

$$T' = Z(i, x) \oplus \bigoplus_{k=x+1}^{n-1} M(i, k),$$

for some $1 \leq i \leq n - 2$ and $i < x \leq n - 2$ such that $x \equiv i \mod 2$.

- Suppose $T = T' \oplus M(i + 1, n)$. If $T'$ is the characteristic tilting module over $\Lambda_{n-1}$, then $i = 1$ and we have a non-zero extension from $M(1, n - 2)$ to $M(2, n)$.

If

$$T' = Z(i, x) \oplus \bigoplus_{k=x+1}^{n-1} M(i, k),$$

for some $x \equiv i \mod 2$, then $T$ contains the summand $M(i, n - 1)$, which is a contradiction as there is a non-zero extension from $M(i, n - 1)$ to $M(i + 1, n)$. Therefore, we
must have
\[ T' = Z(i, x) \oplus \bigoplus_{k=x+1}^{n-1} M(i + 1, k), \]
for some \( 1 \leq i \leq n - 2 \) and \( i < x \leq n - 1 \) such that \( x \not\equiv i \mod 2 \).

Assume instead that \( i \not\equiv n \mod 2 \).

- Suppose \( T = T' \oplus M(i, n) \). If \( T' \) is the characteristic tilting module over \( \Lambda_{n-1} \), then \( T \) would be the characteristic tilting module over \( \Lambda_n \), contradicting our assumption. If
\[ T' = Z(i, x) \oplus \bigoplus_{k=x+1}^{n-1} M(i + 1, k), \]
for some \( x \equiv i \mod 2 \), then \( T \) contains the summand \( M(i + 1, n - 2) \), which is a contradiction as there is a non-zero extension from \( M(i, n) \) to \( M(i+1, n-2) \). Therefore, we must have
\[ T' = Z(i, x) \oplus \bigoplus_{k=x+1}^{n-1} M(i, k), \]
for some \( 1 \leq i \leq n - 2 \) and \( i < x \leq n - 1 \) such that \( x \equiv i \mod 2 \).

- Suppose \( T = T' \oplus M(i + 1, n) \). If \( T' \) is the characteristic tilting module over \( \Lambda_{n-1} \), then \( i = 1 \) and we have a non-zero extension from \( M(1, n - 1) \) to \( M(2, n) \).
  If
\[ T' = Z(i, x) \oplus \bigoplus_{k=x+1}^{n-1} M(i, k), \]
for some \( x \equiv i \mod 2 \), then \( T \) contains the summand \( M(i, n - 2) \), which is a contradiction as there is a non-zero extension from \( M(i, n - 2) \) to \( M(i + 1, n) \). Therefore, we must have
\[ T' = Z(i, x) \oplus \bigoplus_{k=x+1}^{n-1} M(i + 1, k), \]
for some \( 1 \leq i \leq n - 2 \) and \( i < x \leq n - 1 \) such that \( x \not\equiv i \mod 2 \).

This finishes the proof. \( \square \)

We remark that \( L(1) \) is a direct summand in exactly one basic generalized tilting module, namely the characteristic tilting module.

### 6 Exceptional Sequences

Let \( \Lambda \) be a finite-dimensional \( \mathbb{k} \)-algebra. Recall, see [11], that an indecomposable \( \Lambda \)-module \( M \) is called exceptional provided that

- \( \text{End}_\Lambda(M) \cong \mathbb{k} \);
- \( \text{Ext}^i_\Lambda(M, M) = 0 \), for all \( i > 0 \).

A sequence \( M = (M_1, \ldots, M_k) \) of \( \Lambda \)-modules is called an exceptional sequence provided that

- each \( M_i \) is exceptional;
- \( \text{Ext}^i_\Lambda(M_x, M_y) = 0 \), for all \( 1 \leq y < x \leq k \) and all \( i \geq 0 \).
An exceptional sequence is called full (or complete) if it generates the derived category. In particular, this means that it must contain at least $n$ modules, where $n$ is the number of isomorphism classes of simple $\Lambda$-modules. Indeed, suppose an exceptional sequence $\mathbf{N}$ contains $k < n$ modules. Let $\mathcal{A} \subset D^b(\Lambda)$ be the triangulated subcategory generated by $\mathbf{N}$. Passing to the Grothendieck group $K_0(\mathcal{A})$, we see that any mapping cone of a homomorphism between modules in $\mathbf{N}$ equals a linear combination of modules in $\mathbf{N}$, implying that $\text{rank} \ K_0(\mathcal{A}) \leq k < n = \text{rank} \ K_0(D^b(\Lambda))$. Thus, such an exceptional sequence cannot be full.

**Proposition 36** The only exceptional $\Lambda_n$-modules are the standard modules $\Delta(i)$ and the costandard modules $\nabla(i)$, for $i = 1, 2, \ldots, n$.

**Proof** By Proposition 16, we know that $\Omega(i, j, k)$ is self-orthogonal exactly when $i = j = 1$ or $|i - j| = 1$. For such $i, j$, the module $\Omega(i, j, k)$ is neither standard nor costandard if and only if $k > \max(i, j)$. In this case, there is one composition factor $L(k - 1)$ in the top of $\Omega(i, j, k)$ and another composition factor $L(k - 1)$ in the socle of $\Omega(i, j, k)$. This means that there is an endomorphism of $\Omega(i, j, k)$ sending the composition factor $L(k - 1)$ in the top to the composition factor $L(k - 1)$ in the socle. Thus $\dim \text{End}_{\Lambda_n}(\Omega(i, j, k)) > 1$ and $\Omega(i, j, k)$ is therefore not an exceptional module.

For any quasi-hereditary algebra both standard and costandard modules are self-orthogonal and have trivial endomorphism algebras, and are therefore exceptional modules. □

**Theorem 37** Let $\mathbf{M}$ be a full exceptional sequence of $\Lambda_n$-modules. Then $\mathbf{M}$ is of the form

$$\left(\nabla(m_1), \nabla(m_2), \ldots, \nabla(m_i), L(1), \Delta(n_1), \Delta(n_2), \ldots, \Delta(n_j)\right)$$

(\ast)

where

- $i + j = n - 1$;
- $\{m_1, m_2, \ldots, m_i, n_1, n_2, \ldots, n_j\} = \{2, 3, \ldots, n\}$;
- $m_1 > m_2 > \cdots > m_i$ and $n_1 < n_2 < \ldots n_j$.

**Proof** By Proposition 36, the standard and costandard modules are the only exceptional modules, so no other modules may be included in an exceptional sequence. We observe that the only non-zero homomorphisms, not equal to a scalar multiple of the identity, between standard and costandard modules are the following.

- We may map the top of $\Delta(k)$ to the socle of $\nabla(k)$.
- We may map the top of $\nabla(k)$ to the socle of $\Delta(k)$.
- We may map the top of $\Delta(k)$ to the socle of $\Delta(k + 1)$.
- We may map the top of $\nabla(k)$ to the socle of $\nabla(k - 1)$.

This implies that an exceptional sequence cannot contain both $\Delta(k)$ and $\nabla(k)$ for $k \neq 1$. Further, as a full exceptional sequence must contain $n$ modules, it has to be of the form (\ast), up to ordering.

By Proposition 19, there is a non-zero extension from each costandard module to any standard module. This implies that any costandard module must come before any standard module in an exceptional sequence. By Theorem 20, or more specifically, by Proposition 26 and Proposition 28, there is a non-zero extension from $\Delta(i)$ to $\Delta(j)$, for every $i < j$. Using the simple preserving duality, we find that there is a non-zero extension from $\nabla(i)$ to $\nabla(j)$.
for every \( i > j \). This implies that any full exceptional sequence must be of the form \((\star)\), as it must contain at least \( n \) modules.

Left to prove is that a sequence \( M = (M_1, \ldots, M_n) \) of the form \((\star)\) actually is a full exceptional sequence. That \( \text{Hom}_{A_n}(M_x, M_y) = 0 \), for \( x > y \), is clear from the first part of the proof. Furthermore, for every quasi-hereditary algebra, the following equality holds.

\[
\text{Ext}^m(\Delta(i), \Delta(j)) \cong \text{Ext}^m(\nabla(j), \nabla(i)) \cong \text{Ext}^m(\Delta(k), \nabla(\ell)) = 0,
\]

for \( i > j \), any \( k, \ell \) and \( m > 0 \). From this it follows that any sequence of the form \((\star)\) is exceptional.

Left to show is that a sequence of the form \((\star)\) generates the derived category (as a triangulated category). We recall that any triangulated category is closed under taking kernels of epimorphisms and cokernels of monomorphisms. As the set of simple modules generate the derived category, it suffices to show that we can obtain the simple modules \( L(i) \) by performing these operations on the modules in our sequence.

We proceed by induction on \( i = 1, 2, \ldots, n \). The basis for the induction is clear, since \( L(1) \) is included in any sequence of the form \((\star)\). Next, assume that \( L(i-1) \) can be obtained from our sequence. By assumption, the sequence contains either \( \Delta(i) \) or \( \nabla(i) \). In the first case, \( L(i) \) is the cokernel of the inclusion \( L(i-1) \hookrightarrow \Delta(i) \). In the second case, \( L(i) \) is the kernel of the projection \( \nabla(i) \twoheadrightarrow L(i-1) \). This shows that any sequence of the form \((\star)\) is a full exceptional sequence.  

\[ \blacksquare \]

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