THE SPACE OF 3-MANIFOLDS AND VASSILIEV
FINITE-TYPE INVARIANTS

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Abstract. In this paper we develop the theory of finite-type invariants for homologically nontrivial 3-manifolds. We construct an infinite-dimensional affine space with a hypersurface in it corresponding to manifolds with Morse singularities. Connected components of the complement of this hypersurface correspond to homeomorphism type of spin 3-manifolds. This suggests the natural axiomatics of Vassiliev finite-type invariants for arbitrary closed 3-manifolds. An example of an invariant of order 1 is given.

0. Introduction.

The idea which goes back to V. Arnold is to complete the space of all objects by the degenerate ones so as to get a Euclidean space and then study the topology of the degenerate locus which is related to the topology of its complement via Alexander duality. This approach had a beautiful application in the theory of knots. In 1986 V.Vassiliev constructed so-called finite-type invariants of knots [V]. He completed the space of all knots by adding degenerate ones and got an infinite-dimensional constructible Euclidean space $E$. The degenerate knots form a hypersurface $D$ (discriminant) in this space. Isotopy classes of knots are exactly connected components of the complement to $D$. Alexander duality relates $H^0(E \setminus D)$ to the homology of $D$ which is then studied by using its natural stratification. One would like to apply this approach to the classification of 3-manifolds. This is the subject of the present paper. We construct our space $E$ by a version of Pontryagin-Thom construction. More precisely $E$ is the space of maps $f : \mathbb{R}^8 \to \mathbb{R}^5$, such that $f$, when restricted to the complement of the ball of sufficiently large but not fixed radius is just a projection $\mathbb{R}^8 \to \mathbb{R}^5$. This is obviously an affine space, since $f_1 + (1 - t)f_2$ is also in $E$ if $f_1$ and $f_2$ are. The discriminant $D$ consists of maps $f : \mathbb{R}^8 \to \mathbb{R}^5$ which have critical point with critical value 0. If $f \notin D$, then $f^{-1}(0)$ is a smooth punctured 3-manifold, i.e., a compact manifold from which one point is deleted. Our main result is as follows:

Theorem 1. To each connected component of $E - D$ there corresponds a homeomorphism class of 3-dimensional spin manifolds. For any connected spin manifold there are

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exactly 4 connected components of $E - D$ giving its homeomorphism type. In this case the gauge group $\pi_8(SO(5)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, which acts on $E - D$, permutes the four chambers corresponding to our manifold.

By a spin manifold we understand a pair $(M, \theta)$ where $M$ is an oriented 3-manifold, and $\theta$ is a spin structure on $M$. Two spin manifolds $(M, \theta)$ and $(M', \theta')$ are called homeomorphic, if there exists a homeomorphism $M \to M'$ taking $\theta$ to $\theta'$.

The construction is as follows: given a connected punctured manifold $\hat{M}^3$ there is unique isotopy type of embedding $\hat{M}^3 \hookrightarrow \mathbb{R}^8$. A representation of $M$ as $f^{-1}(0)$ as above gives a framing of $M$, i.e. a trivialization of its normal bundle in $\mathbb{R}^8$. The set of such framings is found to be $\mathbb{Z} \oplus \Sigma(M)$ for connected nonpunctured manifold $M$, where $\Sigma(M)$ is the set of spin structures on $M$. By puncturing manifold we kill the $\mathbb{Z}$-component of the framing and are left only with $\Sigma(M)$, which depends on the surgery presentation of manifold $M$. Given a framing, the question of representing $M$ as $f^{-1}(0)$ with $df$ inducing this framing, is a certain problem of obstruction theory. It turns out that this problem has a nice answer:

**Theorem 2.** Let $M$ be a connected 3-manifold. For any spin structure $\theta$ on $M$ there is $f : \mathbb{R}^8 \to \mathbb{R}^5$ as above such that $f \notin D$ and further, $M = f^{-1}(0)$ and $df$ induces the spin structure $\theta$. The set of isotopy classes of such $f$ consists of 4 elements.

**Acknowledgements.** I want to thank my advisor S. Weinberger who supervised this work, O.Viro who attracted my attention to the importance of spin structures and also S. Ackbulut, R.Kirby, G.Kuperberg, M. Polyak for useful discussions.
1. Proof of Theorem 1.

First we introduce some definitions. Let’s denote by $E$ the infinite-dimensional affine space of 5-tuples of functions $f = (f_1, ..., f_5)$ on $\mathbb{R}^8$ such that outside a ball of sufficient large radius we have $f_i(x_1, ..., x_8) = x_i$ for $i = 1, ..., 5$.

**Definition 1.** Denote by $D$ and call the discriminant the subset in $E$ which consists of such tuples $f$ that on the submanifold $\{f = 0\}$ there exists a point, s.t. the Jacobian $Jf$ degenerates.

**Definition 2.** Connected components of $E - D$ we will call chambers.

**Definition 3.** Part of the discriminant separating two chambers is called a wall.

For each chamber $V$ and any $f = (f_1, ..., f_5)$ in $V$ the set of zeros $M_f = \{f = 0\}$ is a smooth submanifold in $\mathbb{R}^8$.

Since by Milnor’s theorem any connected 3-manifold can be obtained from the other by the surgery on the sequence of knots, we can assume that to each wall there assigned a knot, over which we have to do surgery to pass to another chamber, containing manifold of different homeomorphism type. For example there is a chamber which contains $S^3$. Walls of this chamber correspond to different knots, which are already combinatorially classified by Vassiliev. To pass from a chamber giving a manifold with two connected components to one that contains the connected sum of them we need not just (2,2)-type surgery as above but also (3,1)-type. In the case of 3-manifolds all three categories PL, TOP, Diff are equivalent [Mo]. For convenience we will work with smooth manifolds. Note that since all 3-manifolds are spin, their tangent bundle is trivial.

**Lemma 1.** The normal bundle to $M^3$ in $S^8$ is trivial.

Proof. Let $r$ denote the 1-dimensional trivial bundle. Note that the normal bundle to $M^3$ in $S^8$ is stably trivial since characteristic classes $w_2, \tau_1$ vanish. It is trivial because having a $k$-bundle $E$ over an $n$-complex ($n < k + 1$) such that $E \oplus r^l$ is trivial we can prove by induction that $E \oplus r$ is trivial. Now, $E \oplus r$ is isomorphic to $r^k + 1$, so we get a map from $M$ into the projective space $P^k$ defined by the position of the fiber of the summand $r$. Since $n < k + 1$, the image of a line can be homotoped to $Id$. Thus $E$ is trivial for $n < k + 1$, which holds in our case.

Now we extend the (trivial) normal $\mathbb{R}^3$-bundle over $\mathbb{R}^8$ and consider our punctured 3-manifold $M$ as the locus of common zeroes of five functions on $R^8$. It is always possible to obtain $M^3$ in this way in some tubular neighborhood $\nu M^3$ of $M^3$ in $R^8$ which we identify with the normal bundle. To do this, we should just consider the tautological bundle whose fiber over a point $(x, v)$ of $\nu M^3$ is the fiber of $\nu M^3$ over $x$. In $\nu M^3$ our manifold $M^3$ can be represented as the zero section of this bundle.

We want to extend this section (i.e., a map $f : \nu M^3 \rightarrow \mathbb{R}^5 - \{0\} \sim S^4$) to a map of all $R^8$ which does not vanish outside $M^3$. By the standard obstruction theory [D-N-F] the
obstruction to extension to the $i$th skeleton in our setting lies in

$$
H^i(B(R)/\nu \hat{M}^3, d\nu(\hat{M}^3) \cup (S^7/\nu(S^2)), \pi_{i-1}(S^4)) =
$$

$$
= H_{8-i}(B(R)/\nu \hat{M}^3; \pi_{i-1}(S^4)) = H_{8-i}(S^8/M^3; \pi i - 1(S^4))
$$

These groups can be easily calculated:

$$
\begin{align*}
H^{i-1}(M^3) &= 0, i > 4 \\
H^i(M^3, \pi_{i-1}(S^4)) &= 0, i \leq 4.
\end{align*}
$$

So the only possible obstruction lies in the group corresponding to $i=0$: $O \in \mathbb{Z}_{12}$.

The number of extensions is given by the last homology group:

$$
H^8(S^8/M^3, \pi_8(S^4)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2.
$$

**Lemma 2.** We have an identification $[M : SO(5)] \cong \mathbb{Z} \oplus H^1(M, \mathbb{Z}_2)$.

**Proof.** Consider the fibration

$$
\text{Spin}(5) \to SO(5) \to B\mathbb{Z}_2.
$$

From this fibration we find an exact sequence

$$
[M : \text{Spin}(5)] \to [M : SO(5)] \to [M : B\mathbb{Z}_2],
$$

so

$$
0 \to \mathbb{Z} \xrightarrow{\phi} [M : SO(5)] \to H^1(M, \mathbb{Z}_2) \to 0.
$$

This sequence splits by the map $d$, which takes the map $M \to SO(5)$ into its degree (defined in virtue of equality $\pi_3(SO(5)) = \mathbb{Z}$).

We can write, therefore, our map obstruction for punctured manifold $\hat{M}^3$ in the form

$$
\phi : H^1(M, \mathbb{Z}_2) \to \mathbb{Z}_{12}.
$$

We want to show that every element $\theta \in H^1(M, \mathbb{Z}_2)$ is mapped by $\phi$ to 0.

**Lemma 3.** For any 3-manifold $M$ and any spin structure $\theta$ on $M$ obstruction $O$ vanishes for $(M, \theta)$.

**Proof.** From [Wa] we know that any 3-manifold embeds in 5-space and if $(M, fr)$ represents zero in $\pi_3^s$ then there exists embedding in $R^5$ realizing this framing (i.e. $\pi_3^s = \mathbb{Z}_{24}$ is the obstruction group to getting framed 0-cobordism in $S^5$). By Rohlin [R] we know that the third spin cobordism group is trivial, i.e. each $M^3$ with a given spin structure is spin cobordant to $S^3$. Now we have $\mathbb{Z}$ possibilities to extend a given spin structure to a framing on $M^3$ and those which represent zero in $\pi_3^s$ will provide a framed 0-cobordism. Thus
each spin structure can be extended to a framing in such a way that \((M, fr)\) is given by equations in \(S^5\). This construction can be raised to \(S^8\) in the following way. Suppose \(M^3\) is a complete intersection in \(S^5\) and \(W^4_1\) and \(W^4_2\) are 4-manifolds whose intersection is \(M^3\). In \(S^5\) any closed 4-manifold bounds. Let \(V_1\) and \(V_2\) be 5-manifolds whose boundaries are \(W_1\) and \(W_2\). Now consider \(S^5\) as a submanifold of \(S^6\) and construct ”pushouts” \(V^+_1, V^-_1\), s.t. their common boundary is \(W_1\). Make the same construction for \(V_2\). Construct \(V^+_1 \cup V^-_1 = X_1^5\) and \(V^+_2 \cup V^-_2 = X_2^5\). Now \(M = S^5 \cap X^5_1 \cap X^5_2\). So the construction is raised to \(S^6\). Then raise it to \(S^8\) by induction and to \(R^8\) by puncturing sphere. The obstruction that we calculated measures how much our punctured spin manifold differs from being a complete intersection. Thus obstruction vanishes for all spin structures on \(M\).

**Lemma 4.** The action of the gauge group \(\pi_8(SO(5))\) on the space of 3-manifolds permutes, in a simply transitive way, the chambers corresponding to any given connected spin manifold.

**Proof.** The solution of the obstruction theory problem provided us with four different extensions for every spin 3-manifold, which are given by the last homotopy group \(\pi_8(S^4) = \mathbb{Z}_2 \oplus \mathbb{Z}_2\). This has a natural geometrical explanation. Different trivializations of 5-bundle over \(R^8\) are classified by \(\pi_8(SO(5)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2\). This is a gauge group acting in 5-bundle and the isomorphism

\[
\pi_8(SO(5)) \to \pi_8(S^4)
\]

is induced by the map of the spaces. \(SO(5)\) acts on \(S^4 = R^5 - \{0\}\) and those maps induce the action of a gauge group on the set of all extensions, since all other obstruction groups are trivial. Thus the quotient of the space of 3-manifolds by the action of the gauge group gives unique extension,i.e., chamber for any spin 3-manifold.
2. Axiomatics.

It is apriori not obvious that one cannot write Vassiliev-Ohtsuki-type axioms for arbitrary 3-manifolds without any additional structure. However, the following argument which belongs to O.Viro shows that in this situation all invariants of finite type will be trivial.

Suppose we have an invariant \( \alpha \) of order \( n \), i.e., its \( n+1 \)st “Vassiliev derivative” is zero. Then we show that it should be constant (for \( M^3 \) without additional structure).

Indeed, take the last wall over which we are taking the derivative, and write the corresponding difference.

\[
W_{L_1,\ldots,L_{n-1}}^M - W_{L_1,\ldots,L_{n-1}}^{M' = M_{L_n}} = 0.
\]

Thus the derivative over the last wall equals the derivative over all previous walls and doesn’t depend on it. Since everything is invariant under Kirby moves, adding a small unknotted component to the link in the last wall won’t change the invariant (but will change the link). Doing it inductively we can change the surgery coefficient on each component of the link and make the components unlinked (by application of Kirby moves). By a sequence of such moves we can get from any 3-manifold \( M \) to \( S^3 \), simplifying the link. This will imply that \( \alpha(M) = \alpha(S^3) \), so \( \alpha \) is constant. We show that for manifolds with spin structures the above argument doesn’t work.

In the case of manifolds with spin structures Kirby moves were described in [K-M]. Suppose \( M \) is obtained from \( S^3 \) by surgery on a link \( L \). Suppose \( C \) is a characteristic sublink defining the spin structure. Recall that \( C \) is called characteristic, if \((C, L_i) \equiv (L_i, L_i)(\text{mod} 2)\) for any \( L_i \subset L \). The moves are as follows.

1. Add or delete a disjoint unknotted component \( K \) with framing \( \pm 1 \), and set \( C' = C + K \).
2. If \( i \neq j \), slide \( L_i \) over \( L_j \), i.e., set \( L'_i = L_i + L_j \) and

\[
C' = \begin{cases} 
C, & L_i \notin C \\
C - (L_i + L_j) + L'_i, & L_i, L_j \notin C \\
C - L_i + (L_j + L'_i), & L_i \subset C, L_j \notin C.
\end{cases}
\]

These moves are restrictive comparing to original Kirby moves. For example if we have two characteristic strands with + -crossing and we put an unknotted component over them to change sign of the crossing, then we cannot take this component off the link after performing surgery since it becomes characteristic.

Thus the previous argument (for manifolds without spin structures) which implied triviality of all Vassiliev invariants, does not work in the spin case.
In 1985 A. Casson introduced a new invariant $\lambda$ of oriented homology 3-spheres. It has two different descriptions. One is as the intersection number of two Lagrangian sub-varieties parametrizing representations of the fundamental groups of handlebodies in a Heegard decomposition of $M$, the intersection takes in the symplectic variety of representations of the fundamental group of the Riemann surface, the common boundary of the two handlebodies.

The other definition of the Casson invariant is by the surgery formula:

$$\lambda(S^3) = 0, \quad \lambda(M(K_n)) - \lambda(M(K_{n-1})) = (1/2)\Delta_K'(1),$$

where $K$ is a knot in a homology 3-sphere, $M(K_n)$ is the $(1,n)$th Dehn surgery on $K$, and $\Delta_K(t)$ is the Alexander polynomial of $K$.

This formula suggests that Casson’s invariant should be a Vassiliev type invariant of homology spheres: the difference of values of $\lambda$ for $M(K_n), M(K_{n-1})$ (Vassiliev’s discrete derivative) is expressed in terms of the knot $K$, which corresponds to the wall in the space of 3-manifolds separating $M(K_n)$ and $M(K_{n-1})$. This suggests definition which was given by S. Garoufalidis [G] and H. Ohtsuki [O].

In the case of homologically nontrivial 3-manifolds because of the argument above nontrivial Vassiliev theory can be built only for manifolds with additional structure. The construction of the space of 3-manifolds suggests the only natural axiomatics in this situation:

**Definition.** A map $v : \{\text{homeo types of spin 3-manifolds}\} \to \mathbb{C}$ is called a finite type invariant of at most order $k$ if it satisfies the condition:

\[
\sum_{\text{char } L' \subseteq L} (-1)^{\#L'} v(M_{L'}) = 0
\]

where $L'$ is a characteristic sublink of $L$, as well as the following axioms:

\[
(2) \quad I(\dot{M}) = I(M).
\]

\[
(3) \quad I(M_1 \# M_2) = I(M_1) + I(M_2).
\]

\[
(4) \quad I(S^3) = 0.
\]

The above axiomatics suggests that one should consider spinor modifications of known invariants. It was shown that Rozanskii-Witten invariant [R-W] and universal Vassiliev invariant introduced by Thang Le [Le] restrict to C.Lescop’s [L] generalization of Casson’s invariant. This invariant vanishes for manifolds with first Betti number greater that four. Perhaps spinor modifications of this invariant or it’s perturbations will be nontrivial for manifolds with higher Betti numbers.

Now we introduce the first simple example of Vassiliev invariant of finite order. Start with $M^3$ with spin structure. By Rohlin’s theorem all 3-manifolds are spin cobordant. Consider Euler characteristic of spin 0-cobordism minus 1 modulo 2. Denote it $I(M, spin)$. 

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Lemma 5. $I(M, \text{spin})$ is an invariant of spin manifold $M^3$. It equals Rohlin’s invariant modulo 2.

Proof. We know that if $M^3$ bounding $W^4$ is connected then $W^4$ can be assumed to have only one 0-handle and some 2-handles [K]. Thus $I(M, \text{spin})$ gives us the rank of second cohomology modulo 2. Since closed spin 4-manifold has a unimodular, symmetric and even intersection form, it’s rank is even and our invariant is zero, thus it is well-defined. Note that the signature of the intersection matrix modulo 2 equals its rank modulo 2. Rohlin’s invariant is defined as the signature of the intersection matrix modulo 16. Thus the constructed invariant $I(M, \text{spin})$ equals Rohlin’s invariant modulo 2.

Lemma 6. Invariant $I(M, \text{spin})$ is finite type Vassiliev of order 1.

Proof. By making surgery over a knot we add a handle to spin cobordism, i.e. invariant increases by 1, i.e. its derivative is constant. Taking alternated sum over four chambers adjacent to selfintersection of discriminant of codimension two we get zero. Thus second order Vassiliev derivative is zero and we get an order one Vassiliev invariant.

It is important to understand which singular manifolds form the discriminant D.

Lemma 7. Discriminant D consists of 3-manifolds with Morse singularities. Codimension n selfintersection of discriminant corresponds to a manifold with n singular points. Singular points have a type of a cone over torus for (2,2)-type surgeries and a cone over sphere for (3,1) and (1,3)-type surgeries.

Proof. The whole picture can be described in terms of spin 4-cobordisms connecting our 3-manifolds. Surgery over a knot corresponds to a Morse decomposition of signature (2,2). It is well known how quadratic form of signature (2,2) looks like. It has 2 generators and after projection to $RP^3$ is given by quadratic homogeneous equation. So, it is a cone over the product of projective spaces, i.e. a cone over a torus. There are also two other surgeries of signature (1,3) and (3,1) (attaching and eliminating handles). Those correspond to a cone over sphere. Passing from one chamber to another we are making Dehn surgery over a knot which is assigned to the wall.

Another interesting question which arises in connection with the space of 3 manifolds is understanding of the topology of the chambers:

Conjecture. The homotopy type of the chamber corresponding to $M$ is $K(\pi_0(Diff(M)), 1)$.

Several cases are already known. For example for $S^3$ the diffeomorphism group is homotopy equivalent to $O(4)$. Also, $Diff(S^1 \times S^2) = O(2) \times O(3) \times \Omega SO(3)$, see [H1]. For M Haken $\pi_0 Diff(M) = Out(\pi_1(M)$ and the components of Diff(M) are usually contractible. The exception is when M is Seifert-fibred, with all fibers coherently orientable, and in this case the components of Diff(M) are homotopy equivalent to $S^1$, [H2]. Anonagous theorem in the case of knots have been recently proved by A. Hatcher.
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