Lattice data inspired but Minkowski space calculated QCD fundamental propagator

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Abstract

We study the Dyson-Schwinger equation for the quark propagator in Minkowski space. In order to have analytical behaviour at the timelike axis of momenta under control, we use the Stieltjes and the Hilbert transformation for the interaction kernels and discuss the solution from the perspective of these transformations. In addition, a lattice fit for the gluon propagator and approximation for quark-gluon vertex are employed, and within the model the quark propagator is obtained through the solution of Dyson-Schwinger equation in Minkowski space. The resulting propagators in all studied cases do not show up particle like pole and production thresholds. Instead of, the quark propagator satisfies Hilbert transformation and the associated dynamical mass function becomes complex without a presence of particle like branch point.

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I. INTRODUCTION

The QCD Green’s functions (GFs) can be used as a building blocks when for calculating the properties of hadrons [1–5] at arbitrary scale. In the last decade considerable progress in the lattice evaluation of QCD fundamental GFS have been made [6–9], being in a reasonable numerical agreement with the calculation performed in the continuous framework of Dyson-Schwinger equations (DSEs) [10–12]. The DSEs are the integral equations, i.e. they are based on the utilizing of continuous space time, tacitly, up to a few exceptions, the DSEs studies were performed in the Euclidean space, establishing thus a meaningful comparison with the lattice data. Also the most of the DSEs solutions we restricted to Landau gauge, which is the only one among more general covariant gauges, achieved numerically in the lattice. Mainly due to this reason, the lattice inspired is formally restricted to this gauge only, while the main focus on the analytical structure and related points are quite general and independent on any gauge choice.

In recent agreement with the lattice data, the gluon propagator provides infrared finite, however non vanishing solution. This phenomena has been explained in term of non-Abelian Schwinger mechanism in Background Field method [11]. Quite related, but only recently clarified point is the understanding of significant role of the dressed quark-gluon vertex for a correct description of dynamical chiral symmetry breaking in QCD. There is a wide consensus that without infrared enhancement of quark-gluon vertex the chiral symmetry breaking does not occur.

On the other side many questions remain unsolved, e.g. the relation between confinement of fundamental QCD object -the quarks and gluons- and the dynamical symmetry breaking is not yet completely understand. Recall, the confinement is conventionally understand from the study of Wilsonian loops, and various confining potentials are calculated in the static limit of infinitely heavy quarks. In this paper, we argue that chiral symmetry breaking an the confinement of light quarks, albeit slightly differ from the conventional wisdom, actually happen simultaneously, which fact is directly observed when the DSEs are considered and solved in our real world Minkowski space time. We argue that due to the strong coupling at low energy, the QCD quantum loops lead to the complex quark and gluon dynamical mass functions, without usual appearance of a real valued branch points.

The equations for QCD propagators represent a part of SDEs system. For practical
purpose of solution it is convenient to express them in the the form of irreducible GFs, i.e. in terms of the selfenergies, polarizations and proper vertices. Then SDE equation for the quark propagator reads

\[ S^{-1}(p) = p - m_0 - \Sigma(p), \]  

(1.1)

where selfenergy function \( \Sigma \) therefore completely determines the analytical property of the propagator.

Contrary to the solution of SDEs performed in the Euclidean space, the numerical solution in Minkowski spacetime is not a well defined problem, unless one does not specify analytical boundary condition properly. The problem is that we are looking for solution in terms of (generally non-unique) tempered distribution. It turns that a "physical" solution is given as a limit of complex multivaluable functions at the region we expect the function has an analytical cut. Obviously, as the computer works with columns of the numerical data and a desired analytical property is not directly implemented, the numerical search easily fails for almost any admissible method.

Let us recall here, that the Euclidean space quark and gluon propagators have been quite accurately approximated by the meromorphic functions with complex conjugated poles \[10, 16–26\]. In the first part of presented paper we study analytical structure of a simple correlator made out of the propagators with various analytical properties. We concern on the appearance of the lowest real branch point assuming the propagator satisfy Hilbert or Stieltjes transformation which automatically includes the case of the complex conjugated pole as well. In most of the cases a naive Wick rotation is invalid and the integration in Minkowski momentum spacetime must be carefully reconsidered. In the second part of the paper, the DSE for the quark propagator is solved numerically in Minkowski momentum space assuming the kernel has a special analytical properties discussed in the preceding Section. The Section IV provides the numerical solution for a lattice inspired kernel of the Schwinger-Dyson equation.

II. PEREQUISITIES FROM SCALAR CORRELATOR IN MINKOWSKI SPACE

In nonconfining theory and in the case of stable particle, the selfenergy \( \Sigma \) in Eq. (1.1) should be real bellow the threshold, allowing thus appearance of a real pole in \( S \), which uniquely corresponds with a free propagation of simple field excitation mode -the particle.
More then 50 years ago, hereby for a non-confining theory, the analytical structure of perturbative Feynman graphs have been investigated \[37–39\]. Especially, it is well known, that two point correlators satisfy the integral representation, which allows write down a simple relation between its imaginary and real part. The following spectral representation

\[
\int d\alpha \frac{\rho(\alpha)}{p^2 - \alpha + i\epsilon}
\]

was found to be a very useful tool for solving DSEs nonperturbatatively in many models without confinement phenomena, however it usually fails, when trying to extend its use for a description of the dynamical chiral symmetry breaking phenomena and QCD bound states, e.g. the pion. Quite interestingly, it was found that the residue of propagator where vanishing and the Lehman spectral weight function turned to be sign-changing when approaching critical value of couplings. In other words, for increasing coupling, the associated particle modes have tendencies to disappear from the spectrum and the probability interpretation of spectral density is lost. For a historical suggestions for using spectral representation in nonperturbative context see \[41–46\], for an actual numerical solutions see \[47, 48\].

In QCD the only freely moving objects are the bound states - the hadrons. Thus the correlators made from the hadronic currents may obey various sum rules well based on the dispersion relations \[49–53\]. In contrary to hadrons, the colored objects like quarks are not observed, thus the proof or reliable evidence for the absence of singularities associated with quark (gluon) productions would certainly comply with confinement of quarks (and gluons) quanta. This has been translated into the lattice language of quark Wilsonian loop, where the are law represents indirect search for a (an absence of) quark-antiquark threshold \[40\]. A more direct observation based on the study of QCD GFs study in timelike region of Minkowski space-time is barely lacking.

As a first step we write down the Hilbert-Stieltjes transformation for the propagators, as the second step we rewrite this transformation into the form of unbounded Feynman and Dyson integral representations, which will allow us to calculate loop integrals in Minkowski space directly.

In order to understand the feedback of various analytical properties on the selfenergy \(\Sigma\) we will consider the following scalar correlator

\[
\Pi(p^2) = i \int \frac{d^4l}{(2\pi)^3} G_1(l)G_2(l - k)
\]

(2.2)
where the functions $G_{1,2}$ mimic propagators of quarks or gluons in deep infrared. For this purpose we will consider the function $G$, which obey Stieltjes transformation in the forms:

$$G_{S_+}(p) = \int_0^\infty dx \frac{\sigma(x)}{p^2 - x + i\epsilon}; \quad G_{S_-}(p) = G_{S_+}^+(p); \quad G_S(p) = \int_0^\infty dx \frac{\sigma(x)}{p^2 - x} \quad (2.3)$$

noting the spectral representation belongs to $S_+$ function. In addition we will also consider the function which are images of Hilbert transformation, e.g.:

$$G_{H_+}(p) = \int_{-\infty}^{\infty} dx \frac{\sigma(x)}{p^2 - x + i\epsilon}; \quad G_{H_-}(p) = G_{H_+}^+(p); \quad G_H(p) = \int_{-\infty}^{\infty} dx \frac{\sigma(x)}{p^2 - x}. \quad (2.4)$$

Considered spaces classify the functions with respect of the position of the singularities in the whole complex plane, and also with respect to position of possible branch point located at the real axis. Recall for instance that purely meromorphic functions with complex conjugated poles suggested in \[2, 4, 5, 19, 20, 23\] belongs entirely to $H_+$, i.e to the function which can be expresses through the unbounded real valued principal value integral Hilbert transformation (to name the transformation we follow the book \[65\]). It is a matter of the fact that the function with singularities located simultaneously at upper and lower half plane of complex do not satisfy representation of the form \[2.1\], thus it must either from $H$ or $S$, the later implies the presence of positive branch point at the real axis. Actually one can easily check that the presence of positive (negative) epsilon ensures that all singularities are located at the lower (upper) plane of complex $p^2$ plane. Note trivially, the meromorphic function can be written as a combination of $H_+$ and $H_-$ or $S_+$ and $S_-$ because of the identity

$$G_H(p) = P. \int_{-\infty}^{\infty} dx \frac{\sigma(x)}{p^2 - x} = \int_{-\infty}^{\infty} dx \frac{\sigma(x)}{p^2 - x + i\epsilon} + \int_{-\infty}^{\infty} dx \frac{\sigma(x)}{p^2 - x - i\epsilon}. \quad (2.5)$$

As we will discuss in the next section the numerical solutions of quark DSEs actually suggests that the quark propagator is the linear combination of the functions which belongs to $H_+$ and $H_-$. It is plain to say, that one naturally assume that they contribute asymmetrically as we know that only $H^+ (S^+)$ dominates the spacelike ultraviolet as a consequence of asymptotic freedom.

To illustrate of above saying, let us write down a simple example:

$$\int_a^\infty dx \frac{(a-x)g(x, a, b)}{p^2 - x} = g(p^2, a, b) \left[ \frac{\pi b}{2} + \frac{(p^2 - a)}{2} \ln \left( \frac{p^2 - a}{b} \right)^2 \right] \quad (2.6)$$

being a case of $S$ transformation. Whilst the Hilbert transformation of the same weight function is meromorphic

$$\int_{-\infty}^{\infty} dx \frac{(a-x)g(x, a, b)}{p^2 - x} = \pi bg(p^2, a, b), \quad (2.7)$$
and where we have used the shorthand notation for the "Gribov" propagator $g(x; a, b) = [(x - a)^2 + b^2]^{-1}$ with cpx. conjugated poles at $a \pm ib$.

From discussion above it follows that most general case which we will need to consider is the sum of Feynman-Hilbert and Dyson-Hilbert representations for considered propagator

$$G(p^2) = \int dx \frac{\sigma_+(x)}{p^2 - x + i\varepsilon} + \int dx \frac{\sigma_-(x)}{p^2 - x - i\varepsilon}. \quad (2.8)$$

By construction any function given by solely $P$ integral is included by the virtue of the identity (2.5).

Before doing so we mention a well known and easily derivable from the well known:

For the convolution of two functions from two function from $S$ we get $\Pi = 0$ everywhere.

For the product of two function from $S_+$ we get the function from $S_+$ again, which is nothing else but the regular case of perturbation theory Feynman diagram. It can be written in the form of well known dispersion relation

$$\Pi(p^2) = \int_0^\infty dx \frac{\rho_\pi(x)}{p^2 - x + i\varepsilon}$$

$$\rho_\pi(x) = \int_0^\infty dy dz \frac{\sigma_{+1}(y)\sigma_{+2}(z)}{8\pi^2} \sqrt{(x - y - z)^2 - 4yz} \frac{\Theta(x - (y^{1/2} + z^{1/2})^2)}{x} \quad (2.9)$$

where the function $\sigma_{+1}(y)$ and $\sigma_{+2}(y)$ are the weights of the images $G_1$ and $G_2$. The derivation is the textbook example for one loop Feynman diagram (see for its regularized form).

For the convolution of the function from $S_-$ one gets

$$\Pi(p^2) = -\int dx \frac{\rho_\pi(x)}{p^2 - x - i\varepsilon}, \quad (2.10)$$

with the same $\rho_\pi$ as previously and where the sign is a consequence of the integration over $k_0$ in momentum space (one can use the mirror symmetric contour displayed at the fig. or one can simply conjugate previously considered $\Pi$ multiplied by $1/i$ before.

Let us study the most general case (2.8). In order to get sense to the product of the functionals we rewrite the correlator (2.2) into the form where all products belong to same "type" of distributions

$$\Pi(p^2) = i \int \frac{d^4l}{(2\pi)^4} G_1G_2$$

$$G_1G_2 = \left[ \int \frac{d\omega_1 \sigma_{+1}(\omega_1)}{\omega_1^2 - \omega_1 + i\varepsilon} + \int \frac{d\omega_1 \sigma_{-1}(\omega_1)}{\omega_1^2 - \omega_1 - i\varepsilon} \right]$$
\[
\Pi(p^2) = i \int d\omega_1 \int d\omega_2 \int \frac{d^4 l}{(2\pi)^4} \left[ \frac{\sigma_{+1}(\omega_1)\sigma_{+2}(\omega_2)}{(l^2 - \omega_1 - i\epsilon)(l^2 - \omega_2 - i\epsilon)} + \frac{\sigma_{-1}(\omega_1)\sigma_{-2}(\omega_2)}{(l^2 - \omega_1 + i\epsilon)(l^2 - \omega_2 + i\epsilon)} \right]
\]

If \( G_i \) are from \( S \), then the first two terms correspond with already discussed cases. These two terms then have the lowest branch point located at real positive semi-axis.

The rest can be easily evaluated by using the following identity

\[
\frac{1}{x - a \pm i\epsilon} = P \frac{1}{x - a} \mp i\pi\delta(x - a).
\]

Using a shorthand notations \( \delta_1 = \delta(l^2 - \omega_1), \delta_2 = \delta((l - p)^2 - \omega_2) \) and analogously for principal parts, then the last line can be written as:

\[
\begin{align*}
\sigma_{+1}(\omega_1)\sigma_{-2}(\omega_2) & \left[ P_1 P_2 - i\pi\delta_1 P_2 + i\pi\delta_2 P_1 + \pi^2\delta_1\delta_2 \right] \\
\sigma_{-1}(\omega_1)\sigma_{+2}(\omega_2) & \left[ P_1 P_2 + i\pi\delta_1 P_2 - i\pi\delta_2 P_1 + \pi^2\delta_1\delta_2 \right]
\end{align*}
\]

Now for the function from \( S \)'s, the first terms at each lines are zero separately, the second and the third terms produces purely real pieces in the final result for the correlation function and last terms at each lines are exactly what the Cutkosky rule would give for absorptive part, however with opposite sign and for the continuous mass there. Explicitly

\[
i \int_0^\infty \frac{d\omega_1 d\omega_2}{8\pi} \sigma_+^1(\omega_1)\sigma_+^2(\omega_2) \sqrt{(p^2 - \omega_1 - \omega_2)^2 - 4\omega_1\omega_2} \Theta(p^2 - (\omega_1^{1/2} + \omega_2^{1/2})^2)
\]

for the first line and similar expression is valid for the last term of the second line (with an appropriate weights). Again one can conclude that for the inner momentum integral one gets the usual perturbative branch point \( \omega_1^{1/2} + \omega_2^{1/2} \), which is further smeared by the integrations over the weights \( \sigma \)'s. While a detailed property of the branch point does depend on the Stieltjes weights, the lowest branch point remains bounded at timelike regime (it is equal to \( 4\sigma^2 \) for the example (2.10)). In words, the correlation functions remain real for a spacelike argument for the case of Stieltjes transformation, no matter whether the propagators \( G \) involves complex conjugated poles or not. Note for completeness, that for
FIG. 1: Integration contours in complex momentum plane used for the integration over \( l_0 \) component. Contour filling the first and the third quadrant is used for the first line while the mirror symmetric contour was used to integrate the fourth line of Eq. (2.11). Position of singularities are shown for fixed \( \omega \), the ones close to the real axis (A,C,E,H) correspond to positive value of \( \vec{l}^2 + \omega \), while the ones close to Im axis (B,D,F,G) appear when \( \vec{l}^2 + \omega \) is negative. i.e. for Hilbert transformations only.

As a last but not at least example let us consider the function which involves also Hilbert images. It will dramatically changed the situations when compared to the functions represented by the functions solely from \( S \). We will consider only special case when the both functions are from \( H \) space, then after some effort, one can arrive into a relatively simple expression

\[
\Pi_H(p^2) = \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \sigma_1(\omega_1)\sigma_2(\omega_2)K(p^2,\omega_1,\omega_2)
\]

\[
K_H(p^2;\omega_1,\omega_2) = \frac{-i}{16\pi} \left[ (1 + \frac{\omega_1 - \omega_2}{p^2})\Theta(-\omega_1) + (1 - \frac{\omega_1 - \omega_2}{p^2})\Theta(-\omega_2) \right],
\]

which is valid even for spacelike \( p^2 \). The result is obviously nonzero for general \( \sigma_{1,2} \) and the resulting nontrivial pieces arise from the region where principal integral turns to be an usual regular integral. The result is obviously complex everywhere and also the combination of the function from \( H_+, H_- \) and \( H \) spaces with any other tempered distribution considered in
this section produces the correlator which is complex in the spacelike region of momenta as well.

Stress several things here, first of all it is impossible to have QCD propagators entirely from $H$’s, noting the variable $\omega$ is linearly appearing in the numerator and the appropriate divergences would lead to a deep contradictions with perturbative QCD and the best QCD experiments. It is pertubatively true, that when one starts with all $G$’s form $S^+$ then one finishes with the function from $S^+$ again. This is a basic ingredient of Perturbation Theory Integral Representation derived for more complicated Feynman integrals by Nakanashi [67]. However, the function from $H$ can be generated nonperturbatively, giving important contribution at the infrared $p \simeq \Lambda$ and being thus important for confinement. This is in fact the case of Minkowski space-time study [66], wherein the full quark propagator instead of spectral representation satisfies a Hilbert transformation.

In the all above examples we have silently assumed that the weight functions are real. At the end we we shall mention some attempt to use complex spectral function in the context of the so called spectral quark model [68]. In the paper [68] a many low energy strong QCD hadronic quantities and form form factors have been formally evaluated in the terms of the integral representation with purely imaginary spectral functions as a consequence of phenomenologically guessed contour of integration. One should worry with uniqueness of such prescription since the $\rho$ usually represents the discontinuity of the functions at the cut and thus is not well defined if $\rho$ has a cut there as well. Let us recall that there exist formally infinity number of imaginary spectral $\rho$’s which makes a real $G$ because of the identity:

$$G(p^2) = \frac{i}{2\pi} \left( \int_M^\infty dx \frac{G(x)}{p^2 - x + i\varepsilon} - \int_M^\infty dx \frac{G(x)}{p^2 - x - i\varepsilon} \right),$$  \hspace{1cm} (2.16)$$

which is an exact property of the quark propagator maintained in the spectral quark models (however here $G$ is bounded bellow by $M$). Irrespective of possible phenomenological success, we expect a concept of complex $\rho$ is mismatching a desired usefulness and usual understanding of dispersion relation.
In the paper [66] the confining model based on the model gluon propagator (in position space)
\[ G^{\mu\nu}(x) = i\mu^2 g^{\mu\nu} \quad (3.1) \]
has been considered. This study has been performed directly in the Minkowski space avoiding thus necessity of analytical continuation from Euclidean space. Very interestingly, the authors finished with the solution which is given by Hilbert transformation \[ H \quad (2.4) \] and not a Feynman propagator (Stieltjes \[ S_+ \] transformation). McKay-Munczek quark propagator turns to be a real function for all real momenta \( p^2 \) with the following infrared singularity
\[ P \frac{e^{-m^2/2\mu^2}}{p^2}. \quad (3.2) \]
That a purely imaginary kernel can give a purely real solution is not a big surprise, as long as one is working with distribution \[ (3.1) \]. Obviously the effective kernel is chosen to be purely imaginary, for complex parameter \( \mu^2 \) we can get a complex solutions as well.

In this section we study the quark DSE numerically in momentum Minkowski space. We begin by developing the model in the ladder-rainbow approximation with the kernels, which will be approximated by infrared enhanced function which has or has not complex conjugated singularities. Instead by taking some ad hoc Ansatz we have consider a suited combination of the functions from spaces \( S \) and \( H \). We have use the same weight functions for Stieltjes and Hermite transformation differing thus in analytical properties of constructed the kernel. However we leave the quark propagator undetermined and expect the solution should fall into some of the classes \( S, H \), either the combination is admitted as well. We restrict to the presentation of three simple combinations, noting here the other possible combinations do not differ qualitatively.

For completeness we review basic ingredients for the quark propagator DSE, which reads
\[ S^{-1}(p) = \not{p} - \Sigma(p) = A(p) \not{p} - B(p) \quad (3.3) \]
\[ \Sigma(p) = ig \int \frac{d^4k}{(2\pi)^4} \Gamma^\mu(q,p,k)S(k)\gamma^\nu G_{\mu\nu}(q) \]
\[ G_{\mu\nu}(k) = \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) D(k^2), \]
where the usual conventions were followed: $S$ stands for the quark propagator, which up to a tiny electroweak corrections is well described by two scalar function $A, B$. $G$ is a gluon propagator and $\Gamma$ stands for quark-gluon vertex, which turns to be an important and necessary ingredient for a correct description of confinement and chiral symmetry breaking. We have suppressed all color and Dirac indices for brevity, explicitly written

\[ \Gamma(q, p, k) = g t_a \Gamma^\mu_{\alpha\beta}(q, p, k); \quad \Gamma^\mu_{\alpha\beta}[^{\text{Ladder}}] = \gamma^\mu_{\alpha\beta} Z \]

(3.4)

where $q, p$ are 4-momenta of incoming gluon and quark and $k$ stand for outgoing quark 4-momentum respectively, $Z$ is a Lorentz scalar in the ladder-rainbow approximation, which we will use in this Section, i.e. only a $\gamma$ part out of all twelve component of the vertex $\Gamma$ is considered. Explicitly we get for the function $B$

\[ B(p) = \frac{Tr}{4} i \int \frac{d^4k}{(2\pi)^4} \gamma^\mu t^a S(k) \gamma^\nu t^a G^{[\text{free}]}_{\mu\nu}(q) \alpha(q) \]

\[ = i \int \frac{d^4k}{(2\pi)^4} \frac{B(k)}{k^2 - B(k^2)} K(q) \]

(3.5)

and similarly the equation for $A$ can be derived by projection by $Tr / p$. We will neglect the momentum dependence of $A$, simply noting that variation of $A$ is expected to be quite small in the approximation used \[68\]. In the last DSE $\alpha$ is an effective running coupling made of the $\gamma$ vertex and gluon form factor functions.

We will consider three models defined by the kernel $K$, which is chosen to be either from $S$’s or $H$ spaces. As a first very conventional choice we will consider the kernel which satisfies usual dispersion relation, i.e. the function which is from $S^+$. It is chosen in a way the kernel $K$ in Eq. (3.5) is equivalent to the product of two following functions

\[ K = K_{S^+}(\ell^2) = (8\pi^3) C_{S^+} F_{S^+}(\ell^2) G_{S^+}(\ell^2) \]

(3.6)

and where $G$ is the function \[2.6\] with positive Feynman $i\epsilon$ included, i.e.

\[ G_+(x) = \int_a^\infty dy \frac{(a - y)g(y, a, b)}{x - y + i\epsilon} \]

(3.7)

and where $F_+$ is a correct continuation of the following exponential

\[ F(q) = e^{-\sqrt{\frac{q^2}{4}}} ; q^2 < 0, \]

(3.8)

which ensures convergence of DSE and finiteness of the integrals. For all momenta it reads

\[ F_+(x) = e^{-\sqrt{\frac{q^2}{4}}} \Theta(-x) + e^{i\sqrt{\frac{q^2}{4}}} \Theta(-x) \]

(3.9)
and the effective constant is taken such that \((8\pi^3)C_{S+} \simeq 1\).

The second model is defined as

\[
K_S(l^2) = (8\pi^3)C_S F_+(l^2) G_S(l^2) \\
G_S(x) = \int_{-\infty}^{\infty} dy (a - y) g(y, a, b) / (x - y) \tag{3.10}
\]
i.e. \(G\) being from \(S\) implies nontrivial admixture of the function from \(S^-\) as well.

Finally, for the third model we take

\[
K_H(l^2) = (8\pi^3)C_H F_+(l^2) G_H(l^2) \\
G_H(x) = \int_{-\infty}^{\infty} dy (a - y) g(y, a, b) / (x - y) = \pi b g(x, a, b) \tag{3.11}
\]

which means that the kernel in addition to \(S_+\) involves a nontrivial admixture of the functions from \(H_+\) and \(H_-\). The parameters \(a, b\) are taken to be equal \(a = b = \Lambda^2\) for simplicity.

Let us stress once more, the second and the third kernels have complex conjugated singularities, while the first one is a conventional Feynman representation, which is singularity free at the upper half of the first Riemann sheet. There are branch points associated with square root in the exponential and there is cut associated with discontinuity along a real axis. Furthermore, a usual logarithmical cut is presented in the first two cases, this cut starts at \(x = a\) and goes to positive infinity, associated discontinuity of the full kernel appears explicitly in the first kernel only, because of \(i\varepsilon\) presence. For those who are interested, the integral representations are easily derivable for the full kernels \(K\), the example for \(K_{S+}\) is written in the Appendix A for completeness.

We allow the quark propagator is complex everywhere, assuming that the imaginary part will be trivial if the numerical solution of DSE admits. Some of the details of numerics are written in the Appendix B. As a first, we present the numerical solution for the dynamical mass function at spacelike region of momenta, i.e. in the domain where all the Euclidean space results are usually presented. In Fig. 1 the results for small value of quark current mass \(m_0 = 0.01\Lambda\) are presented for the three above considered cases. The letters \(H, S\) and \(S_+\) label the model where the function \(G\) belongs to a given spaces introduced in the previous section and hence to the model defined by the kernel (3.11) and (3.6) respectively. The appropriate effective coupling was chosen to be \(C = 1/90\) for the \(S_+\) and \(S\) model, while in order to achieve better convergence \(C = 1/(\pi 90)\) for the third model (the models with pure meromorphic functions are less stable for larger coupling). The same is shown in Fig.
FIG. 2: Dynamical quark masses as described in the text

2 but for a larger current quark mass $m/\Lambda = 0.5$. Thus the first case can mimic light $u, d$ quarks and the second case can correspond with $-s$ quark- mass if one identifies $\Lambda$ with QCD scale $\Lambda = 230 MeV$ (correctly tuned model could give correct pion mass and the right value of $f_\pi$ in the addition). Note that the definition of the kernels are chosen in a way it provides $K(0)/\Lambda \simeq 1$ for the dynamical mass $M(\Lambda) \simeq \Lambda$ in absolute value, for instance $K_H(0) = 0.88\Lambda^{-1}$.

The whole Minkowski space behavior of the dynamical quark mass function is displayed in the Fig. 4 and in Fig. 5 for log scaled axis of $M$. The imaginary part is oscillating at the
FIG. 4: Dynamical quark masses as described in the text.

timelike domain of $p^2$ irrespective of the current quark mass, while it is more smoothed for the kernel with purely meromorphic function (H). We show only two quite different solutions for better visibility, noting that only positive pieces of the absorptive parts are visible for the solution with $K_S$.

In all presented cases it is obvious that the quark propagator receive nonzero absorptive part everywhere, albeit it is decreasing function and basically turn off at the spacelike scale of several $\Lambda$. We see the indication that there is a branch point located in the vicinity of zero. Although we are not able identify the appropriate integral weights quantitatively,
from the check-list of scalar correlation functions made in the previous Section we expect the infrared part of quark propagator is driven also by the function from $H_+$ and/or $H_-$ spaces. Of course, it must gradually move to the function driven by part from $S^+$ for a high spacelike momenta as it is dictated by QCD perturbative unitarity. By studying these and other cases of $H, S$ functional combinations defining the kernel, we see the evidence that the resulting quark propagator is not entirely from $S^+$ space, unless the interaction strength (coupling $C$) is very small producing thus infrared quark mass much smaller then $\Lambda$ scale. In this case mass-shell physical pole appears in the propagator.

Recall the solution of DSE in the Euclidean space must be real. To this point we should stress that it is easy to switch off the imaginary part of the quark propagator at spacelike region and solve the DSE numerically with ignoring the absorptive part at the spacelike region. We did it in all studied cases and we actually get the real solution there, which is even stable and convergent for all cases considered, however it has never provided trivial absorptive spacelike part of $M$ as an output, which is easy to see by the direct evaluation. We actually did not see any evidence for the solution which provides chiral symmetry breaking solution and is simultaneously purely real at the spacelike. One can only say that the absorptive part is rapidly vanishing at the scale of several $\Lambda$.

IV. EDUCATED MINKOWSKI SPACE SOLUTION MOTIVATED BY EUCLIDEAN LATTICE RESULTS

In the previous Section we have presented numerical results for the ladder-rainbow DSE with exponentially suppressed ultraviolet spacelike modes. The kernel with complex conjugated poles as well as the one based on the conventional analytical assumption were used, noting that the all cases lead to a conjecture that quark propagator involves a part consistent with Hilbert transformation providing thus nontrivial imaginary part for the quark propagator at spacelike region.

According to the recent findings in Landau gauge, a nontrivial dressing of quark-gluon vertex is required for a correct description of chiral symmetry breaking via quark DSE. In this section we continue a numerical study by using a kernel, which is more or less motivated by a recent lattice funding. In accordance with previous argumentation we assume to quark-gluon vertex can be complex as well as the quark self energy was.
Let us remind the DSE for quark-gluon vertex for this purpose, which reads
\[ \Gamma(q, p, k) = \Gamma^{[0]} + ig \int \frac{d^4k}{(2\pi^4)} \Gamma_{SSV_{qqqq}} + ig \int \frac{d^4k}{(2\pi^4)} \Gamma_{ggg_{GGV_{qqgg}}} , \]  
where \( V_{qqqq} \) and \( V_{ggqq} \) is a shorthand notation for four-(anti)quarks and 2gluons-2quarks scattering kernels. Note the later one involves ghost-quark scattering kernel, which according to Euclidean studies should be responsible for a main enhancement of chiral symmetry breaking effect in the quark DSE.

For any approximation of the vertex the Minkowski space quark gap equation can be written like
\[ M(p) = i \int \frac{d^4k}{(2\pi^4)} \frac{M(k)}{k^2 - M(k^2)} K(q, p, k) \]  
where we have defined scalar function \( K(q, p, k) \), which arises after the summation and projection of the product of the quark DSE kernel. Note trivially that the definition includes the renormalization wave function \( A \) as well, however it does not include the boundary condition necessary for a given analytical continuation, i.e. one should add \( i\varepsilon \) part in a case of a real pole occurrence.

From Eq. (4.2) it is obvious that one does not need to consider all twelve form factors associated with a various tensorial structure of the quark-gluon vertex when study quark DSE alone. Their role is prominent for calculating of meson properties with various spin. The only necessary hint is that all these form factors lead to the large infrared enhancement of the quark DSE kernel. For an actual quantitative results for a various form factor calculated in the Euclidean space, see the original papers [4, 14, 15]. To achieve similar effect, we will make an Ansatz for the kernel \( K(q, p, k) \) in the following way:
\[ K(q, p, k) = \tilde{C}\tilde{\Gamma}_I(q^2)D(q^2) \]
\[ \tilde{\Gamma}_I(q^2) = \frac{1}{(4\pi^2)q^2}Ln(1 - q^2/M^2) \]
\[ = \frac{1}{2q^2(4\pi)^2}\ln\left(\frac{(q^2 - \Lambda^2)^2 + \Lambda^4}{2\Lambda^4}\right) \]
\[ + \frac{i}{q^2(4\pi)^2}\arctg\left(\frac{q^2}{(q^2 - 2\Lambda^2)}\right) , \]
\[ D(q) = \frac{q^2 - i\Lambda^2}{(q^2 - \Lambda^2)^2 + \Lambda^4} , \]
Where the function \( D \) is the lattice fit -the Gribov gluon propagator, however here with slightly more general complex prefactor. The constant \( \Lambda \approx \Lambda_{QCD} \) is a single dimensionfull parameter of the model.
The choice of the kernel is motivated by an expected appearance of a Lorentz covariant generalization of the linear interquark potential. It is chosen in a way that it exactly corresponds with the scalar triangle approximation of the quark gluon vertex function wherein one inner vertex has a vanishing external momentum and wherein two associated connected internal lines have vanishing masses. In this way the function $\tilde{\Gamma}$ corresponds to the 1 loop convolution of $1/k^4$ function with another ”massive” gluon propagator. According to Ward identities and our experience with the Minkowski space solutions studied in the previous section, we expect the quark -gluon vertex turns to be complex valued function in the infrared as well. As we are not able to solve Minkowski space quark-gluon DSE at recent stage, we do this by taking $M^2 \to (\Lambda^2 + i\Lambda^2)$ in the kernel.

To solve such quark DSE in Minkowski space is not an easy task and we provide some further technical details in the Appendix B. The results are shown in Fig. 6 and in the Fig. 7 for several values of quark-gluon effective coupling strengths $C$. Recall here that at the scale $\Lambda$ the both -real and imaginary- components of quark mass are generated with comparable sizes. This should be viewed as a nontrivial consequence of what actually happen when one assumed complex branch points [2, 4] in Minkowski space instead of in the Euclidean one. It is an another interpretation of $\Lambda_{QCD}$, which is not only the scale where QCD coupling $\alpha_s$ becomes strong but it is the single parameter which also corresponds with the inverse four-vector distance over which the dressed quark and gluons may propagate before losing its identity by absorption/annihilation in hadronization process.

V. CONCLUSION AND FURTHER PROSPECTS

Motivated by some recent lattice fit [19, 20, 23–25] for quark and gluon propagators we have discussed analytical properties of the scalar correlator. A properties of Stieltjes and Hilbert transformation with and without $i\varepsilon$ prescription were used to classify each cases separately. Resulting correlator is manifestly Lorentz invariant and it can be written in the dispersion relation with the lowest branch point bounded by a positive value even if the propagators have complex conjugated poles. The sufficient condition is that the propagators in correlator are Stieltjes transformable. However, the correlator becomes complex in the spacelike region, if one or both propagators satisfy Hilbert transformation. Admitting QCD correlations function in the form of Gribov or Cauchy distribution, they should be complex
FIG. 6: Dynamical quark mass calculated in a ”lattice inspired model” for various effective coupling $C$.

FIG. 7: Dynamical quark mass calculated in a ”lattice inspired model” for various effective coupling $C$ (low $q^2$ view).

valued when evaluated in Minkowski space. In this respect the oldfashionable idea of quark-hadron duality is violated.

In the rest of the paper the gap equation solutions were presented for the quark propagator employing several approximations of the kernel. As a warm up a simple kernels with different analytical properties were considered. We end up with the Minkowski study of the lattice inspired model. It is mere of the fact that for the interaction strong enough, the dynamical
symmetry breaking is accompanied by generation of complex mass function. The real pole is absent as a natural consequence of complex dynamical mass in agreement with confinement.

It is needless to say that avoiding a numerical ultraviolet regulators is impossible if one would like to use the propagators with a correct perturbative ultraviolet asymptotic. Also, how to include analytical boundary conditions into the more complicated Minkowski space SDE systems, is recently not obvious to the author. The author is aware that the solution of quark SDE in Minkowski space was actually possible due to the simplicity of quark DSE, especially since we have analytically well defined DSE kernel and the quark propagator was the only unknown. We expect the same numerical method is plainly not working when two or more unknown functions is under the numerical search (the most urgent could be system of SDEs for the gluonic polarization function). We expect that any new and successful method of solution would require an analytical boundary conditions. In other words, it should be a human decision again, which class of the functional should be chosen for the treatment of DSEs system in Minkowski space. Educated "analytical" continuation of Euclidean data onto the non-analytical cuts of physical momenta is always possible, although the success of the procedure is quite unlike for the function which are oscillating in Minkowski space-time.

In modern approaches to the bound state problem in QCD, such us DSEs, bound states are built from the propagators of their constituents. Hadrons therefore share and exhibit the features realized in the quarks propagators, albeit some of these are washout since the constituents propagators are always integrated over in various hadronic (Bethe-Salpeter, Fadeev,...) kernels. On of the additional core of performance in the first section is to find a new possibility to transform the original Minkowski momentum space DSEs into a slightly different integral equations, which will be numerically tractable in the entire Minkowski space. This program, which very likely requires to go beyond a naive use of the function from $S^+$ space, remains to be done.

Actually, using the symmetry preserving truncation of DSE system the system of Ward-Takahashi identities between the QCD GFS is expected to be valid in both formulation of the theory: Euclidean and Minkowski as well. Let us consider the pion, which is described by homogeneous Bethe-Salpeter equation. For instance the following well known quark-level Goldberger-Treiman relation [55]:

\[ f_\pi \Gamma_A(k, 0) = B(k^2) \]  \hspace{1cm} (5.1)
remains valid in the chiral limit in Minkowski space as well. The only difference now, is that
the Bethe-Salpeter vertex $\Gamma$ and the pion decay constant

$$f_\pi P_\mu = Z_2 Tr_{C_D} i \int \frac{d^4 k}{(2\pi)^4} \gamma_5 \gamma_\mu S_f(k + P/2)\Gamma(k, P)S_g(k - P/2)$$

(5.2)

are complex functions in a way that their product reproduces the phase of the complex
quark function $B$ correctly.

There are further simple questions I believe can be easily answered in the future. Up to
now, our world is well described by the Standard model and gravity. The Standard Model
has field content where only QCD parts exhibit confinement, the other fields do not - they are
the leptons and electroweak gauge vector bosons. These two sectors interact together already
at classical level since the quarks are charged under the electroweak group. Consequently,
at quantum level there are always contributions that unavoidably mix the GFs of confined
fields with unconfined ones together. More concretely, the quark loops, which contribute to
the photon vacuum polarization, must leave the photon freely propagating. It remains to
be shown by an explicit calculation that the quark propagator of the form $2.8$ does not lead
to any unexpected badness in the resorts where perturbation theory and usual dispersion
theory already successfully works. We leave these for a future performance.

Within a new precise measurement of the pion electromagnetic form factor at the up-
graded JLab facility [56], there is renewed effort [57] in calculation of pion form factor at low
spacelike $Q^2$, e.g. including the domain where the perturbative QCD fails. It is notable, the
recent QCD calculation has been achieved with a complex conjugated poles for representa-
tion of quark propagator [58]. It would be challenging to perform the similar calculations
directly in Minkowski space, where the $\rho$-meson peak should be automatically generated in
quark-photon vertex for timelike photon momenta $q^2$.

The concept of complex confinement should allow the calculation for excited hadrons
as well. The fail in achievement the consistent description of highly excited states is long-
standing trouble of QCD practitioners, the habit is to use nonrelativic quantum mechanical
description for this purpose [59–61]. In this respect suggested Minkowski space tools can
be partially useful when one goes beyond light-front relativistic quark models description
usually used for calculation of transition that includes excited nucleons as well [62, 63]. To
this point, in Ref. [64] there exists a first reliable Minkowski space calculation of the excited
charmonium spectra based on the use of Bethe-Salpeter equation with complex valued charm
quark propagator. The numerics we have described in presented paper has been actually used for this purpose, avoiding thus well known problem of Euclidean- Minkowski space continuation for the first time for the Bethe-Salpeter equation.

VI. APPENDIX A: ASSORTED INTEGRALS

The spectral representation for the function $F_+$ reads

$$F_+(q^2) = \int_0^\infty \sin \sqrt{\frac{y}{\Lambda^2}} \frac{dy}{q^2 - y + i\varepsilon},$$  \hspace{1cm} (6.1)

which for spacelike $\sqrt{-q^2\Lambda^2} = x$ reads

$$F_+(q^2) = -\pi e^{-x}$$  \hspace{1cm} (6.2)

and for timelike argument $t = \sqrt{q^2\Lambda^2}$ gives

$$F_+(q^2) = -\pi \cos t - i\pi \sin t$$  \hspace{1cm} (6.3)

Spectral representation for the kernel $K_{S+}$ reads

$$K(q^2) = \int_0^\infty \rho_K(y) \frac{dy}{q^2 - y + i\varepsilon},$$

$$\rho_K(y) = Konst \frac{\pi \Lambda^2}{\lambda^2 + \Lambda^2} \left[ \cos \left(\frac{\sqrt{y}}{\Lambda}\right) \lambda \Theta(y - \Lambda^2) + \sin \left(\frac{\sqrt{y}}{\Lambda}\right) \left(-\frac{1}{2} + \frac{\lambda}{\pi} \ln |\lambda|\right) \right]$$  \hspace{1cm} (6.4)

where $\lambda = 1 - y/\Lambda^2$ and where we have used the well known algebraic identity \footnote{47} valid for the product of two functions from $S^+$. 

VII. APPENDIX B: NUMERICAL METHOD

The lattice theory is without any doubt the most prominent tool recently used for non-perturbative study of Quantum Field Theory. By the construction, it cannot provide complex valued GFS and form factors for purely Euclidean (spacelike) arguments. Discretized Minkowski space analogue of lattice theory is known to be unpractical and according to author knowledge, has never led to a data harvest.

Also trying to solve DSEs in Minkowski space is always accomplished with obstacles related with a number of un-proper principal value integration over the $k_o$ variable. Let
us recall trivial fact, that considering loop integral with perturbative propagators without Feynman $i\varepsilon$ prescription must give trivial result. This is valid for the convolution of any function from the $S$-space discussed in the first section and it should be true even for the integrals which are ultraviolet divergent otherwise. In practice, a numerical subtraction of several opposite sign infinite pieces is never performed with high accuracy. In case of the DSEs, it usually makes the iteration procedure never-relaxing when the propagators with usual high momentum behavior is considered and certainly, many non-perturbative QCD calculations represent ill defined numerical problem in Minkowski space, Nevertheless this fact should not lead to a headlong conclusions that our real world, i.e. Minkowski space solutions, should be read from the analytical continuations of counterpartners obtained in the Euclidean space (for a traditional discussion see [3, 31]).

In this Appendix we review the method which is working at least in the cases of our simple models. For the numerical solutions we have chosen the method of iteration. As the iterations do not automatically relax for given numerics the author keep all the (convergent) numerical codes public in the following url [32]. In order to avoid numerically uncovenient interpolations we convert gap equation into the form

$$M(p^2) = \frac{i}{8\pi^3} \int_{0}^{\infty} dk_0 \int_{-\infty}^{k_0^2} dk^2 \int_{-1}^{1} dz \sqrt{k_0^2 - k^2} K(q^2, p^2, k^2) \frac{M(k^2)}{k^2 - M^2(k^2)}; \quad (7.1)$$

with $q^2 = k^2 + p^2 + 2\sqrt{k_0^2 - k^2} \sqrt{-p^2} z$ for a spacialke argument $p^2 < 0$, where obvious substitution $\vec{k} \rightarrow k^2$ has been made. The Eq. (7.1) is 3dim integral equation for the spacialke argument, where variable $z$ is a cosine between $\vec{k}$ and $\vec{p} = (0, 0, p_z))$. For a timelike arguments we are free to choose $p = (p_0, 0, 0, 0)$ for, which allows to write

$$M(p^2) = \frac{i}{8\pi^3} \sum_{\pm} \int_{0}^{\infty} dk_0 \int_{-\infty}^{k_0^2} dk^2 \sqrt{k_0^2 - k^2} K(q_{\pm}^2, p^2, k^2) \frac{M(k^2)}{k^2 - M^2(k^2)}; \quad (7.2)$$

and the DSE is reduced to 2-dim for the timelike external argument in Eq.(7.2). We also reduced the number of integration points by a simple substitution $k_o \rightarrow -k_o$ for a negative $k_o$. For zero $p^2$, the both equations above become identical for the Lorentz invariant mass function $M(p^2)$. The quark mass function $M$ is the function of single scalar $k^2$ in the kernel.

We have discretized $k^2$ by using Gaussian method and we have checked our Gaussian
integrator in the case of complex mass valued Feynman integral which is known exactly. The following integral

$$i \int \frac{d^4 l}{(2 \pi)^4} \frac{1}{(l^2 - a + ib)^3} = \frac{1}{2(4 \pi)^2} \frac{a + ib}{a^2 + b^2} \quad (7.3)$$

was actually used for this purpose. Note that the Eq. (7.3) corresponds to the scalar triangle with all zero external momenta, wherein all the internal lines correspond to a propagators with the common single complex valued mass $m = \sqrt{(a - ib)}$. Assuming the real numbers $a, b$ satisfy condition $a, b > 0$, it allows to perform usual Wick rotation and analytical integration. Using also the numerical integration in Eq. (7.2) explicitly we have reproduced rhs. of Eq. (7.3) within a few promile accuracy.

We have check the method is working for more general kernels than have been presented in the paper here, however the offer for more complicated cases is very limited. For instance, we did not find stable and precise solution to the strong coupling models where both propagators are unknown in the approximated expressions for selfenergy (we considered a simple version of cubic and Yukawa models). To get the solution a real poles must be avoided, i.e. if an effective coupling considered is too weak the confinement is lost and we usually get a trivial or oscillating unstable solution. The singularity of propagator at the real axis is the avoided for chirality breaking nontrivial solution only. Furthermore, a nodal solutions were obtained in some cases. At least three solutions exist at chiral symmetry breaking phase with zero current quark mass: one which is trivial and the other two differ by the sign. We presented the results as they have been obtained numerically, i.e without changing the sign in any case.

The concept of analytical confinement is actually not new, however there is only quite indirect evidence of it, when the Euclidean metric is used as definite one (see [2, 4, 33–36] for related topics). We argue it is related with the complexity of QCD GFs, which however does not contradict any of basic principles of quantum field theory, e.g. unitarity, reality of energy spectra, etc. since this complexification is own property of GFs of confined objects - those that do not exhibit themselves as a real poles and henceforth do not appear in the
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