Stochastic Polyak Stepsize with a Moving Target

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Abstract

We propose a new stochastic gradient method called MOTAPS (Moving Targetted Polyak Stepsize) that uses recorded past loss values to compute adaptive stepsizes. MOTAPS can be seen as a variant of the Stochastic Polyak (SP) which is also a method that also uses loss values to adjust the stepsize. The downside to the SP method is that it only converges when the interpolation condition holds. MOTAPS is an extension of SP that does not rely on the interpolation condition. The MOTAPS method uses n auxiliary variables, one for each data point, that track the loss value for each data point. We provide a global convergence theory for SP, an intermediary method TAPS, and MOTAPS by showing that they all can be interpreted as a special variant of online SGD. We also perform several numerical experiments on convex learning problems, and deep learning models for image classification and language translation. In all of our tasks we show that MOTAPS is competitive with the relevant baseline method.

1 Introduction

Consider the problem

\[ w^* \in \arg\min_{w \in \mathbb{R}^d} f(w) := \frac{1}{n} \sum_{i=1}^{n} f_i(w), \]  

where each \( f_i(w) \) represents the loss of a model parametrized by \( w \in \mathbb{R}^d \) over a given \( i \)th data point. We assume that there exists a solution \( w^* \in \mathbb{R}^d \). Let \( \mathcal{W}^* \) denote the set of minimizers of \ref{eq:1}.

An ideal method for solving \ref{eq:1} is one that exploits the sum of terms structure, has easy-to-tune hyper-parameters, and is guaranteed to converge. Stochastic gradient descent (SGD) exploits this sum of terms structure by only using a single stochastic gradient (or a batch) \( \nabla f_i(w) \) per iteration. Because of this, SGD is efficient when the number of data points \( n \) is large, and can even be applied when \( n \) is infinite and \ref{eq:1} is an expectation over a continuous random variable.

The issue with SGD is that it is difficult to use because it requires tuning a sequence of step sizes, otherwise known as a learning rate schedule. Indeed, SGD only converges when using a sequence of step sizes that converges to zero at just the right rate. Here we develop methods with adaptive step sizes that use the loss values to set the stepsize.

We derive our new adaptive methods by first exploiting the interpolation equations given by

\[ f_i(w) = 0, \quad \text{for } i = 1, \ldots, n. \]  

We say that the interpolation assumption holds if there exists \( w^* \in \mathcal{W}^* \) that solves \ref{eq:2}. Two well-known settings where the interpolation assumption holds are 1) for binary classification with a linear model where the data can be separated by a hyperplane \cite{Cra+06} or 2) when we know that each \( f_i(w) \) is non-negative, and we have enough parameters in our model so there exists a

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solution that fits all data points. This second setting is often referred to as the overparametrized regime [VBS18], and it is becoming a common occurrence in several sufficiently overparametrized deep neural networks [Zha+17; Bel+19].

Our starting point is to observe that the Stochastic Polyak (SP) method [BZK20; Loi+20] directly exploits and solves the interpolation equations \[4\]. Indeed, the SP method is a subsampled Newton Raphson method [YLG20] as we show next.

The subsampled Newton Raphson method at each iteration samples a single index \(i \in \{1, \ldots, n\}\) and focuses on solving the single equation \(f_i(w) = 0\). This single equation can still be difficult to solve since \(f_i(w)\) can be highly nonlinear. So instead, we linearize \(f_i\) around a given \(w^t \in \mathbb{R}^d\), and set the linearization of \(f_i(w)\) to zero, that is
\[
f_i(w^t) + \langle \nabla f_i(w^t), w - w^t \rangle = 0.
\]
This is now a linear equation in \(w \in \mathbb{R}^d\) that has \(d\) unknowns and thus has infinite solutions. To pick just one solution we use a projection step as follows
\[
w^{t+1} = \arg\min_{w \in \mathbb{R}^d} \|w - w^t\|^2 \quad \text{subject to } f_i(w^t) + \langle \nabla f_i(w^t), w - w^t \rangle = 0. \tag{3}
\]
The solution to this projection step (See Lemma\[B.2\] for details) is given by
\[
w^{t+1} = w^t - \frac{f_i(w^t)}{\|\nabla f_i(w^t)\|^2} \nabla f_i(w^t). \tag{4}
\]

This method \[4\] is known as the Stochastic Polyak method [Loi+20]. The SP has many desirable properties: It is incremental, it adapts its step size according to the current loss function, and it enjoys several invariance properties as we point out later in Remark\[2.2\]. Thus in many senses the SP is an ideal stochastic method. The downside to SP is that to arrive at \[4\] we have to assume that the interpolation assumption holds. The main objective of this paper is to design methods akin to the SP method that do not rely on the interpolation assumption.

Next we highlight some of our contributions.

1.1 Contributions

New perspectives and analysis of Stochastic Polyak. We provide three viewpoints of the SP method: 1) as a subsampled Newton method in Section\[2.1\] 2) as a type of online SGD method in Section\[2.2\] and finally 3) motivated through star-convexity in Section\[4\] which is closely related to Polyak’s original motivation [Pol87].

Moving Targeted Stochastic Polyak. By leveraging the subsampled Newton viewpoint, we develop a new variant of the SP method that does not rely on the interpolation assumption. As an initial step in this direction, we first assume that \(f(w^*)\) is known, and we introduce the Targeted stochastic Polyak Stepsize (TAPS) method. TAPS uses \(n\) auxiliary scalar variables that track the evolution of the individual function values \(f_i(w)\). Of course, \(f(w^*)\) is not known in general. Using the SGD viewpoint of SP, we propose the Moving Targeted Stochastic Polyak (MOTAPS), that does not require knowledge of \(f(w^*)\). Rather, MOTAPS has the same \(n\) auxiliary scalars as TAPS plus one additional variable that tracks the global loss \(f(w)\).

Unifying Convergence Theory. We prove that all three of the methods SP, TAPS and MOTAPS can be interpreted as variants of online SGD, and we use this to establish a unifying convergence theorem for all three of these methods. Furthermore, we show how these variants of online SGD enjoy a remarkable growth property that greatly facilitates a proof of convergence. Indeed, we present Theorems\[5.2\] and\[5.3\] that hold for these three methods by using this online SGD viewpoint. Theorem\[5.2\] uses a star-convexity assumption [LV16; HSS19], which are a class of non-convex functions that includes convex functions, loss functions of some neural networks along the path of SGD [Zha+19; KLY18], several non-convex generalized linear models [LV16], and learning linear dynamical systems [HMR18]. Using the SGD viewpoint of SP, we derive an explicit convergence theory of SP for smooth and star convex loss functions in Sections\[D.2\] and\[H.3\].

1In [BZK20] the authors also observed that the full batch Polyak stepsize in 1D is a Newton Raphson method.
2To be precise, the proof in [HMR18] relies on a quasi-convex assumption that is
\[
h_i(z^*) \geq h_i(z^t) + \gamma \langle \nabla h_i(z^t), z^* - z^t \rangle,
\]
1.2 Related Work

Developing methods that adapt the stepsize using information collected during the iterative process is now a very active area of research. Adaptive methods such as AdaGrad [DHST11] and Adam [KB15] have a step size that adapts to the scaling of the gradient, and thus are generally easier to tune than SGD, and have now become staples in training deep neural networks (DNNs). While the practical success of Adam is undeniable, a fundamental understanding of why these methods work so well remains elusive, particularly on models that interpolate data such as DNNs.

Recently a new family of adaptive methods based on the Polyak step size [Pol87] has emerged, including the stochastic Polyak step size (SP) method [OPT19; Loi+20] and ALI-G [BZK20]. SP is also an adaptive method, since it adjusts its step size depending on both the current loss value and magnitude of the stochastic gradient. Under the interpolation condition, the SP method converges sublinearly under convexity [Loi+20] and star-convexity [GSL21], and linearly under strong convexity and the PL condition [Loi+20; GSL21]. Recently in [Loi+20] the authors proposed the SPS\textsubscript{max} method, which is a variant of SP that caps large stepsizes which greatly helps to stabilize the numeric convergence of SP. Prior to this, the ALI-G method [BZK20] can be interpreted as dampened version of SPS\textsubscript{max} method with follow-up work highlighting the importance of momentum in accelerating these methods in practice [LB21].

Our derivation of SP as a projection in (3) shows that SP can be interpreted as a extension of the passive-aggressive methods to nonlinear models [Cra+06]. Indeed, the passive aggressive methods apply the same projection in (3) but with the constraint \( f_i(w) = 0 \). This projection has a closed form solution when \( f_i \) is a hinge loss over a linear model, which was the setting where passive-aggressive models were first developed and most applied.

Another related set of methods are the model based methods in [AD19], where each new iteration is the result of minimizing the sum of a model of \( f_i(w) \) and the norm squared distance to a prior point, that is

\[
\begin{equation}
\label{eq:5}
w_{t+1} = \arg\min_{w \in \mathbb{R}^d} \left\{ f_{t,i}(w) + \frac{1}{2} \left\| w - w^t \right\|^2 \right\},
\end{equation}
\]

where \( f_{t,i}(w) \) is some local model of \( f_i(w) \) such that \( f_{t,i}(w^t) = f_i(w^t) \). This model includes linearizations of \( f_i(w) \) as a special case. The SPS\textsubscript{max} method [Loi+20] is in fact a special case of the model based methods [5], where-in the model is given by the positive part of a local linearization, that is

\[
\begin{equation}
\label{eq:6}
f_{t,i}(w) = \max\{f_i(w^t) + \langle \nabla f_i(w^t), w - w^t \rangle, 0\},
\end{equation}
\]

as observed in [BZK20]. Using the positive part is justified for non-negative loss functions.

Line search methods that work with stochastic gradients [Vas+19; Zha+20] are another promising and related direction for automatically adapting step-sizes.

2 The Stochastic Polyak Method

We start by presenting the SP (Stochastic Polyak method) through two different viewpoints. First, we show that SP is a special case of the subsampled Newton-Raphson method [YLG20]. Using this first viewpoint, and leveraging results from [YLG20], we then go on to show that SP can also be viewed as a type of online SGD method, which greatly facilitates the analysis of SP.

2.1 The Newton-Raphson viewpoint

As observed in the introduction in Section[1] the SP method is designed for solving interpolation equations. Here we formalize and extend this observation before moving on to our new methods.

We can derive an extended form of the SP method that does not rely on the interpolation assumption. Instead of the interpolation assumptions, let us assume for now that we have access to the loss values \( f_i(w^*) \) for each \( i = 1, \ldots, n \), where \( w^* \in \mathcal{W} \). If we knew the \( f_i(w^*) \)'s then we can solve the optimization problem (1) by solving instead the nonlinear equations

\[
\begin{equation}
\label{eq:11}
\frac{d}{dt} w(t) = -\gamma \nabla f_i(w(t)), \quad w(0) = w^t,
\end{equation}
\]

where \( \gamma > 0 \) is relaxation parameter. For \( \gamma = 1 \) quasi-convex are star-convex functions.
\[ f_i(w) = f_i(w^*), \quad \text{for } i = 1, \ldots, n. \quad (7) \]

Using the same reasoning in Section 1, we can design an adaptive method for solving (7) by sampling the \( i \)th loss, linearizing and projecting as follows

\[ w^{t+1} = \arg\min_{w \in \mathbb{R}^d} \|w - w^t\|^2 \quad \text{subject to } f_i(w^t) + \langle \nabla f_i(w^t), w - w^t \rangle = f_i(w^*). \quad (8) \]

The solution to this projection step (see Lemma B.2 for details) is given by

\[ w^{t+1} = w^t - \frac{f_i(w^t) - f_i(w^*)}{\|\nabla f_i(w^t)\|^2} \nabla f_i(w^t). \quad (9) \]

This method is a minor extension of (4) wherein we now allow \( f_i(w^*) \neq 0 \). Despite this minor change, we will also refer to (9) as the Stochastic Polyak method\(^4\).

The issue with the SP method is that we often do not know \( f_i(w^*) \), except in the case of interpolation where \( f_i(w^*) = 0 \). Outside of this setting, it is unlikely that we would have access to each \( f_i(w^*) \). Thus, we relax this requirement in Sections 3 and 4. But first, we present yet another viewpoint of SP as a type of online SGD method.

### 2.2 The SGD viewpoint

Fix a given \( w^t \in \mathbb{R}^d \) and consider the following auxiliary objective function

\[ \min_{w \in \mathbb{R}^d} h_{i,t}(w) := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \left( f_i(w) - f_i(w^*) \right)^2 \|\nabla f_i(w^t)\|^2. \quad (10) \]

Here we use the pseudoinverse convention that if \( \|\nabla f_i(w^t)\| = 0 \in \mathbb{R} \) then \( \|\nabla f_i(w^t)\|^{-1} = 0 \). Clearly \( w = w^* \) is a minimizer of (10). This suggests that we could try to minimize (10) as a proxy for solving the equations (7). Since (10) is a sum of terms that depends on \( t \), we can use online SGD to minimize (10). To describe this online SGD method let

\[ h_{i,t}(w) := \frac{1}{2} \left( f_i(w) - f_i(w^*) \right)^2 \|\nabla f_i(w^t)\|^2 \quad \text{and thus } \quad \nabla h_{i,t}(w) = \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w^t)\|^2} \nabla f_i(w). \quad (11) \]

The online SGD method is given by sampling \( i_t \in \{1, \ldots, n\} \) and then iterating

\[ w^{t+1} = w^t - \gamma \nabla h_{i_t,t}(w^t) \quad \text{or} \quad w^{t+1} = w^t - \gamma \frac{f_{i_t}(w^t) - f_{i_t}(w^*)}{\|\nabla f_{i_t}(w^t)\|^2} \nabla f_{i_t}(w^t), \quad (12) \]

which is equivalent to the SP method (9) but with the addition of a stepsize \( \gamma > 0 \). This online SGD viewpoint of SP is very useful for proving convergence of SP. Indeed, there exist many convergence results in the literature on online SGD for convex, non-convex, smooth and non-smooth functions that we can now import to analyzing SP. Furthermore, it turns out that (12) enjoys a remarkable growth property that facilitates many SGD proof techniques, as we show in the next lemma.

**Lemma 2.1 (Growth).** The functions \( h_{i,t}(w) \) defined in (11) satisfy

\[ \|\nabla h_{i,t}(w^t)\|^2 = 2h_{i,t}(w^t). \quad (13) \]

Consequently due to (10) we have that

\[ \frac{1}{n} \sum_{i=1}^{n} \|\nabla h_{i,t}(w^t)\|^2 = 2h_t(w^t). \quad (14) \]

**Proof.** The proof follows immediately from (11) and (10) since

\[ \|\nabla h_{i,t}(w^t)\|^2 = \frac{(f_i(w) - f_i(w^t))^2}{\|\nabla f_i(w^t)\|^2} \|\nabla f_i(w^t)\|^2 = \frac{(f_i(w) - f_i(w^*))^2}{\|\nabla f_i(w^t)\|^2} \quad 2h_{i,t}(w^t). \]

In Section 3 we will exploit this SGD viewpoint and the growth property in Lemma 2.1 to prove the convergence of SP. But first we develop new variants of SP that do not require knowing the \( f_i(w^*) \)'s.

---

\(^4\)Using \( f_i(w^*) \) in the numerator is apparently new, and what we call the Stochastic Polyak method here is not the same as the Stochastic Polyak method proposed in \( \text{Loi} \& \text{Wu} \). In Section 4, we detail these differences. In the more common case where \( f_i(w^*) = 0 \), there is consensus that (9) is called the Stochastic Polyak method.  

which we would like the total loss to reach. Alternatively, if we sample (16), projecting the current iterates onto this constraint gives

\[ \|\nabla f_i(w^p)\|^2 = p f_i(w)^{p-1} \nabla f_i(w) \|^2 = p^2 f_i(w)^{2p-2} \|\nabla f_i(w)\|^2. \]

Thus \( f_i(w)^{2p} / \|\nabla f_i(w)^p\|^2 = \frac{1}{p} f_i(w)^2 / \|\nabla f_i(w)\|^2 \) and raising each \( f_i(w) \) to the power \( p \) results in the same optimization problem, albeit scaled by \( p^2 \).

Remark 2.2 (Invariances). This SGD viewpoint also hints as to why SP is invariant to several transformations of the problem (11). Indeed, the reformulation given in (10) is itself invariant to any re-scaling or translations of the loss functions. That is, replacing each \( f_i \) by \( c_i f_i \) or \( f_i + c_i \) where \( c_i \neq 0 \) has no effect on (10). Furthermore, the SP method is invariant to taking powers of the loss functions. We can again see this through the reformulation (10). Indeed, if we assume interpolation with \( f_i(w^*) = 0 \) and replace each \( f_i(w) \) by \( f_i(w)^p \) where \( p > 0 \), then

\[ \|\nabla f_i(w^p)\|^2 = p f_i(w)^{p-1} \nabla f_i(w) \|^2 = p^2 f_i(w)^{2p-2} \|\nabla f_i(w)\|^2. \]

Thus \( f_i(w)^{2p} / \|\nabla f_i(w)^p\|^2 = \frac{1}{p} f_i(w)^2 / \|\nabla f_i(w)\|^2 \) and raising each \( f_i(w) \) to the power \( p \) results in the same optimization problem, albeit scaled by \( p^2 \).

--Proven in Lemma B.1

3 Targeted Stochastic Polyak Steps

Now suppose that we do not know \( f_i(w^*) \) for \( i = 1, \ldots, n \). Instead, we only have a target value for which we would like the total loss to reach.

Assumption 3.1 (Target). There exists a target value \( \tau \geq 0 \) such that every \( w^* \in \mathcal{W}^n \) is a solution to the nonlinear equation

\[ f(w) = \frac{1}{n} \sum_{i=1}^{n} f_i(w) = \tau. \quad (15) \]

Using Assumption 3.1 we develop new variants of the SP method as follows. First we re-write (15) by introducing auxiliary variables \( \alpha_i \in \mathbb{R} \) for \( i = 1, \ldots, n \) such that

\[ \frac{1}{n} \sum_{i=1}^{n} \alpha_i = \tau, \quad (16) \]

\[ f_i(w) = \alpha_i, \quad \text{for } i = 1, \ldots, n. \quad (17) \]

This reformulation exposes the \( i \)th loss function (and thus the \( i \)th data point) as a separate equation. Because each loss (and associated data point) is on a separate row, applying a subsampled Newton-Raphson method results in an incremental method, as we show next.

Let \( w^t \in \mathbb{R}^d \) and \( \alpha^t = (\alpha_1^t, \ldots, \alpha_n^t) \in \mathbb{R}^n \) be the current iterates. At each iteration we can either sample (16) or one of the equations (17). We then apply a Newton-Raphson step using just this sampled equation. For instance, if we sample one of the equations in (17), we first linearize in \( w \) and \( \alpha_i \) around the current iterate and set this linearization to zero, which gives

\[ f_i(w^t) + \langle \nabla f_i(w^t), w - w^t \rangle = \alpha_i. \quad (18) \]

Projecting the previous iterates onto this linear equation gives

\[ w^{t+1}, \alpha_i^{t+1} = \arg\min_{w \in \mathbb{R}^d, \alpha_i \in \mathbb{R}} \|w - w^t\|^2 + \|\alpha_i - \alpha_i^t\|^2 \text{ subject to } f_i(w^t) + \langle \nabla f_i(w^t), w - w^t \rangle = \alpha_i. \]

The solution to the above is given by the updates in lines 8 and 9 in Algorithm 1 when \( \gamma = 1 \). Alternatively, if we sample (16), projecting the current iterates onto this constraint gives

\[ \alpha^{t+1} = \arg\min_{\alpha \in \mathbb{R}^n} \|\alpha - \alpha^t\|^2 \text{ subject to } \frac{1}{n} \sum_{i=1}^{n} \alpha_i = \tau. \quad (19) \]

The closed form solution to (19) is given in line 6 in Algorithm 1 when \( \gamma = 1 \). In Algorithm 1 we give the complete pseudocode of the subsampled Newton-Raphson method applied to (17). We refer to this algorithm as the Target Stochastic Polyak method, or TAPS for short.

Observation thanks to Konstantin Mischenko.

5 Proven in Lemma B.1
Thus we have that (23) and (24) are equal to lines 9 and 8 in Algorithm 1, respectively. Alternatively the proof of this lemma, and all subsequent lemmas are in the appendix in Section C. Due to notational consistency.

3.1 The SGD viewpoint

The TAPS method in Algorithm 1 can also be cast as an online SGD method. To see this, first we re-write (10) as the minimization of an auxiliary function

$$ h_i(w, \alpha) := \frac{1}{n+1} \left( \sum_{i=1}^{n} \frac{1}{2} \left( f_i(w) - \alpha_i \right)^2 + \frac{n}{2} \left( \tau - \alpha \right)^2 \right). $$

In the following lemma we show that minimizing (20) is equivalent to minimizing (1). The natural stopping is a sanity check that SGD does not satisfy.

**Remark 3.2** (TAPS stops at the solution). Algorithm 2 stops when it reaches the solution. That is, if $w^t = w^*$ and $\alpha^t_i = f_i(w^*)$ for all $i$, then both lines 8 and 9 have no effect on $w$ or the $\alpha_i$'s. Furthermore $\tau = \bar{\alpha}^t := \frac{1}{n} \sum_{i=1}^{n} \alpha^t_i$ and consequently the $\alpha_i$'s are no longer updated in line 6. This natural stopping is a sanity check that SGD does not satisfy.

**Lemma 3.3.** Let Assumption 3.1 hold. Every stationary point of (20) is a stationary point of (1). Furthermore, every minimizer $(w^*, \alpha^*) \in \mathbb{R}^{d+n}$ of (20) is such that $w^*$ is a minima of (1) and

$$ \alpha_i^* = f_i(w^*). $$

Consequently we have that $h_i(w^*, \alpha^*) = 0.$

The proof of this lemma, and all subsequent lemmas are in the appendix in Section C. Due to Lemma 3.3 we can focus on minimizing (20). Furthermore, note from Lemma 3.3 we have that the minimizer of (20) does not depend on $i$, despite the dependence of the objective $h_i(w, \alpha)$ on $t$.

Since (20) is an average of $(n+1)$ terms we can apply an online SGD method. To simplify notation, for $i = 1, \ldots, n$ let

$$ h_{i,t}(w, \alpha) := \frac{1}{2} \left( f_i(w) - \alpha_i \right)^2 + \frac{n}{2} \left( \tau - \alpha \right)^2. $$

Note that despite our notation $h_{n+1,t}(w, \alpha)$ does not in fact depend on $t$ or $w$. We do this for notational consistency.

Let $\gamma > 0$ be the learning rate. Starting from any $w^0$ and with $\alpha^0_i = 0$ for all $i$, at each iteration $t$ we sample an index $i_t \in \{1, \ldots, n+1\}$. If $i_t \neq n+1$ then we sample $h_{i_t,t}$ and update

$$ w^{t+1} = w^t - \gamma \nabla w h_{i_t,t}(w^t, \alpha^t) $$

$$ \alpha_{i_t}^{t+1} = \alpha_{i_t}^t - \gamma \nabla \alpha_i h_{i_t,t}(w^t, \alpha^t). $$

Thus we have that (23) and (24) are equal to lines 8 and 9 in Algorithm 1 respectively. Alternatively if $i_t = n+1$ then we sample $h_{n+1,t}$ and our SGD step is given by

$$ \alpha^{t+1} = \alpha^t - \gamma \nabla \alpha h_{n+1,t}(w^t, \alpha^t) $$

$$ \alpha^{t+1} = \alpha^t - \gamma \left( \bar{\alpha}^t - \tau \right). $$
which is equal to line 6 in Algorithm 1.

We rely on this SGD interpretation of the TAPS method to provide a convergence analysis in Section 5 (specialized to TAPS in Section E). Key to this forthcoming analysis, is the following property.

**Lemma 3.4 (Growth).** The functions \( h_{i,t}(w) \) defined in (22) satisfy

\[
\| \nabla h_{i,t}(w^t, \alpha) \|^2 = 2h_{i,t}(w^t, \alpha), \quad \text{for } i = 1, \ldots, n + 1.
\]  

Consequently the function \( h_t(w, \alpha) \) in (20) satisfies

\[
\frac{1}{n + 1} \sum_{i=1}^{n} \| \nabla h_{i,t}(w^t, \alpha) \|^2 = 2h_t(w^t, \alpha).
\]  

In the next section we completely remove Assumption 3.1 to develop a stochastic method that records only function values and needs no prior information on \( f_i(w^*) \) or \( f(w^*) \).

### 4 Moving Targeted Stochastic Polyak Steps

Here we dispense of Assumption 3.1 and instead introduce \( \tau \) as a variable. Our objective is to design a moving target variant of the TAPS method that updates the target \( \tau \) in a such a way that guarantees convergence. To design this moving target variant, we rely on the SGD online viewpoint. Consider the auxiliary objective function

\[
\min_{w \in \mathbb{R}^d, \alpha \in \mathbb{R}^n, \tau \in \mathbb{R}} h_t(w, \alpha, \tau) := \frac{1}{n + 1} \left( \sum_{i=1}^{n} \frac{1 - \lambda}{2} \| \nabla f_i(w^t) \|^2 + 1 + \frac{1}{2}n(\tau - \alpha)^2 + \lambda \tau^2 \right),
\]  

where \( \lambda > 0 \) is a dampening parameter. Note that for \( \lambda = 0 \) we recover the same auxiliary function of the TAPS method in (20).

**Lemma 4.1.** Let

\[
\alpha^*_i := f_i(w^*) \quad \text{and} \quad \tau^* = f(w^*), \quad \text{for } i = 1, \ldots, n.
\]  

It follows that

\[
h_t(w^*, \alpha^*, \tau^*) = \frac{\lambda f(w^*)^2}{2(n + 1)}.
\]  

Furthermore, every stationary point of (28) is a stationary point of (1). Finally if \( f(w) \geq 0 \) and \( (w^*, \hat{\alpha}, \hat{\tau}) \) is a minima of (28) then \( w^* \) is a minima of (1).

Since minimizing (28) is equivalent to minimizing (1), we can focus on solving (28). Following the same pattern from the previous sections, we will minimize the sum of \( (n + 1) \) terms in (28) using SGD. In applying SGD, we partition the function (28) into \( n + 1 \) terms, where the first \( n \) terms are given by

\[
h_{i,t}(w, \alpha, \tau) = \frac{1 - \lambda}{2} \| \nabla f_i(w^t) \|^2 + 1 \quad \text{for } i = 1, \ldots, n.
\]  

The \( (n + 1) \) th term is given by

\[
h_{n+1,t}(w, \alpha, \tau) := \frac{(1 - \lambda)n}{2} (\tau - \alpha)^2 + \frac{\lambda}{2} \tau^2 \quad \text{thus}
\]

\[
\nabla_\tau h_{n+1,t}(w, \alpha, \tau) = (1 - \lambda)n(\tau - \alpha) + \lambda \tau.
\]  

Sampling the \( (n + 1) \) th term and taking a gradient step to update \( \tau^t \in \mathbb{R} \) gives the following update

\[
\tau^{t+1} = \tau^t - \gamma \nabla_\tau h_{n+1,t}(w, \alpha, \tau)|_{(w, \alpha, \tau) = (w^*, \alpha^*, \tau^*)} = (1 - \gamma(\lambda + (1 - \lambda)n)) \tau^t + \gamma(1 - \lambda)n\bar{\alpha}^t
\]  

\[
= (1 - \gamma)\tau^t + \gamma \frac{(1 - \lambda)n}{\lambda + (1 - \lambda)n} \bar{\alpha}^t.
\]  

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where we have introduced a separate learning rate $\gamma_\tau$ for updating $\tau$. We find that a separate learning rate $\gamma_\tau$ is needed for updating $\tau$, otherwise to keep $\tau$ from being negative in (32) we would need to restrict $\gamma$ to be less than $\frac{1}{1-(1-\lambda)n}$, which can be small when $\lambda$ is close to zero. See Algorithm 3 for the resulting method. We refer to this method as the Moving Target Stochastic Polyak Stepsize or MOTAPS for short.

Algorithm 2 MOTAPS: Moving Target Stochastic Polyak Step

1: **Inputs:** Dampening $\lambda \in [0, 1]$ and learning rates $\gamma, \gamma_\tau \in [0, 1]$  
2: **Default:** $\gamma = 0.9, \gamma_\tau = \lambda = 0.1, w^0 = 0, \alpha^0 = \tau^0 = 0 = \tau$ for $i = 1, \ldots, n$  
3: **for** $t = 0, \ldots, T - 1$ **do**  
4: Sample $i \in \{1, \ldots, n+1\}$ according to some law  
5: **if** $i = n + 1$ **then**  
6: $\alpha_{t+1}^i = \alpha_t^i + \gamma (\tau - \tau^t)$, for $j = 1, \ldots, n$.  
7: $\tau = (1 - \gamma_\tau)\tau + \gamma_\tau \frac{1 - \lambda n}{1 - (1 - \lambda) n} \alpha_t^i$  
8: $\tau^{t+1} = \tau^t + \gamma (\tau - \tau^t)$  
9: **else**  
10: $\alpha_{t+1}^i = \alpha_t^i + \gamma \frac{f_i(w^t) - \alpha_t^i}{\|\nabla f_i(w^t)\|^2 + 1}$  
11: $w^{t+1} = w^t - \gamma \frac{\nabla f_i(w^t) - \alpha_t^i}{\|\nabla f_i(w^t)\|^2 + 1} \nabla f_i(w^t)$  
12: $\alpha^{t+1} = \alpha^t + \frac{1}{n}(\alpha_t^{i+1} - \alpha_t^i)$  
13: **Output:** $w^T$

The dampening parameter $\lambda$ controls how fast the stochastic gradients of $h_i(w, \alpha, \tau)$ can grow, as we show next. As a consequence, later on we will see that the $\lambda$ will later control the rate of convergence of MOTAPS.

**Lemma 4.2.** Consider $h_i(w, \alpha, \tau)$ given in (28). If

$$\lambda \leq \frac{2n + 1}{2n + 3} < 1 \quad (34)$$

then

$$\frac{1}{n + 1} \sum_{i=1}^{n+1} \|\nabla h_{i,t}(w^t, \alpha, \tau)\|^2 \leq 2(1 - \lambda)(2n + 1)h_i(w^t, \alpha, \tau). \quad (35)$$

Next we establish a general convergence theory through which we will analyse SP, TAPS and MOTAPS.

5 Convergence Theory

All of our methods presented thus far can be cast as a particular variant of online SGD. Indeed, SP, TAPS and MOTAPS given in (9), Algorithms 1 and 2, respectively, are equivalent to applying SGD to (10), (20) and (28), respectively. We will leverage this connection to provide a convergence theorem for these three methods. Throughout our proofs we use

$$\min_z h_i(z) := \frac{1}{n} \sum_{i=1}^{n} h_{i,t}(z) \quad (36)$$

as the auxiliary function in consideration. Here $z$ represents the variables of the problem. For instance, for the SP method (9) we have that $z = w \in \mathbb{R}^d$, for TAPS in Algorithm 1 we have that $z = (w, \alpha) \in \mathbb{R}^{n+d}$ and finally for MOTAPS in Algorithm 2 we have that $z = (w, \alpha, \tau) \in \mathbb{R}^{n+d+1}$.

Consider the online SGD method applied to minimizing (30) given by

$$z^{t+1} = z^t - \gamma \nabla h_{i,t}(z^t), \quad (37)$$

where $i_t \in \{1, \ldots, n\}$ is sampled uniformly and i.i.d at every iteration and $\gamma > 0$ is a step size. For each method we also proved a growth condition (see Lemmas 2, 3, 4 and 5) that we now state as an assumption.
Assumption 5.1. There exists $G \geq 0$ such that
\[
\mathbb{E} \left[ \| \nabla h_{i,t}(z^t) \|^2 \right] \leq 2G h_{i}(z^t).
\] (38)

5.1 General Convergence Theory

Here we present two general convergence theorems that will then be applied to our three algorithms. The first theorem relies on a weak form of convexity known as star convexity.

Theorem 5.2 (Sublinear). Suppose Assumption 5.1 holds with $G > 0$. Let $\gamma < 1/G$ and suppose there exists $z^*$ such that $h_t$ is star-convex at $z^t$ and around $z^*$, that is
\[
h_t(z^*) \geq h_t(z^t) + \langle \nabla h_t(z^t), z^* - z^t \rangle.
\] (39)

Then we have that
\[
\min_{t=1,\ldots,k} \mathbb{E} \left[ h_t(z^t) - h_t(z^*) \right] \leq \frac{1}{k} \frac{1}{2\gamma(1-G\gamma)} \mathbb{E} \left[ \| z^0 - z^* \|^2 \right] + \frac{G\gamma}{1-G\gamma} \frac{1}{k} \sum_{t=1}^{k} h_t(z^*). \] (40)

Our second theorem relies on a weakened form of strong convexity known as strong star-convexity.

Theorem 5.3 (Linear Convergence). Suppose Assumption 5.1 holds with $G > 0$. Let $\gamma \leq 1/G$. If there exists $\mu > 0$ and $z^*$ such that $h_t$ is $\mu$-strongly star–convex along $z^t$ and around $z^*$, that is
\[
h_t(z^*) \geq h_t(z^t) + \langle \nabla h_t(z^t), z^* - z^t \rangle + \frac{\mu}{2} \| z^* - z^t \|,
\] (41)

then
\[
\mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] \leq (1 - \gamma\mu)^{t+1} \| z^0 - z^* \|^2 + 2G\gamma^2 \sum_{i=0}^{t} (1 - \gamma\mu)^i \mathbb{E} [h_t(z^*)].
\] (42)

In the next three sections we specialize these theorems, and their assumptions, to the SP, TAPS and MOTAPS methods, respectively. In particular, in Section 5.2 we show how two previously known convergence results for SP are special cases of Theorem 5.2 and 5.3. In Section 5.3, we show that the auxiliary functions of TAPS and MOTAPS in (20) and (28) are locally strictly convex under a small technical assumption. In Section 5.4, we finally prove convergence of MOTAPS.

5.2 Convergence of SPS

Before establishing the convergence of SP, we start by stating a slightly more general interpolation assumption as follows.

Assumption 5.4 (Interpolation). We say that the interpolation assumption holds when
\[
\exists w^* \in \mathcal{W}^* \text{ such that } f_i(w^*) = \min_{w \in \mathbb{R}^d} f_i(w), \text{ for } i = 1, \ldots, n.
\] (43)

Here we specialize Theorems 5.2 and 5.3 to the SP method (9). Both of these theorems rely on assuming that the proxy function $h_t$ is star-convex or strongly star-convex. Thus first we establish sufficient conditions for this to hold.

Lemma 5.5. Let the interpolation Assumption 5.4 hold. If every $f_i$ is star convex along the iterates $w^t$ given by (9), that is,
\[
f_i(w^*) \geq f_i(w) + \langle \nabla f_i(w), w^* - w \rangle
\] (44)

then $h_{i,t}(w)$ is star convex along the iterates $w^t$ and around $w^*$, that is,
\[
h_{i,t}(w^*) \geq h_{i,t}(w^t) + \langle \nabla h_{i,t}(w^t), w^* - w \rangle.
\] (45)
Furthermore if \( f_i \) is \( \mu_i \)-strongly convex and \( L_i \)-smooth then \( h_t(w) \) is \( \frac{1}{2L} \sum_{i=1}^n \frac{\mu_i}{L_i} \)-strongly star-convex, that is
\[
 h_t(w^*) \geq h_t(w^t) + \left\langle \nabla w h_t(w^t), w^* - w \right\rangle + \frac{1}{4n} \sum_{i=1}^n \frac{\mu_i}{L_i} \|w^t - w^*\|^2. \quad (46)
\]

Using Lemma 5.5, the convergence of SP is now a corollary of Theorems 5.2 and 5.3.

**Corollary 5.6 (Convergence of SP).** If \( \gamma < 1 \) and every \( f_i(w) \) is star-convex along the iterates \( w^t \) given by (9) then
\[
 \frac{1}{k} \sum_{t=0}^{k-1} \frac{1}{2n} \sum_{i=1}^n \mathbb{E} \left[ \left( \frac{f_i(w^t) - f_i(w^*)}{\|\nabla f_i(w^t)\|} \right)^2 \right] \leq \frac{1}{k2\gamma} \frac{1}{1-\gamma} \mathbb{E} \left[ \|w^0 - w^*\|^2 \right]. \quad (47)
\]

Furthermore if the interpolation Assumption 5.4 holds and if each \( f_i(w) \) is \( L_i \)-smooth then
\[
 \min_{t=0,\ldots,k} \mathbb{E} [f(w^t) - f^*] \leq \frac{1}{k2\gamma} \frac{1}{1-\gamma} \mathbb{E} \left[ \|w^0 - w^*\|^2 \right], \quad (48)
\]
where \( L_{\text{max}} := \max_{i=1,\ldots,n} L_i \).

The resulting convergence in (48) has already appeared in Theorem 4.4 in [GSL21]. In [GSL21], the authors use a proof technique that relies on a new notion of smoothness (Lemma 4.3 in [GSL21]). Here we have that 5.6 is rather a direct consequence of interpreting SP as a type of SGD method.

**Corollary 5.7.** If \( \gamma \leq 1 \), the interpolation Assumption 5.4 holds, and every \( f_i \) is \( L_i \)-smooth and \( \mu \)-strongly star-convex then the iterates \( w^t \) given by (9) converge linearly according to
\[
 \mathbb{E} \left[ \|w^{t+1} - w^*\|^2 \right] \leq \left( 1 - \gamma \frac{1}{2n} \sum_{i=1}^n \frac{\mu_i}{L_i} \right)^{t+1} \|w^0 - w^*\|^2. \quad (49)
\]

This corollary shows that Theorem D.3 in [GSL21] is a special case of Theorem 5.3 and again a direct result of interpreting SP as a type of SGD method. The rate of convergence in (49) is also tighter than the analysis given in Theorem 3.1 in [Loi+20] where the rate is \( 1 - \frac{\gamma}{2 - \frac{1}{2} \sum_{i=1}^n \frac{\mu_i}{L_{\text{max}}}} \).

### 5.3 Convergence of TAPS

Here we explore the consequences of Theorems 5.2 and 5.3 to the TAPS method. To this end, let \( z := (w, \alpha) \) and let
\[
 h_t(z) = h_t(w, \alpha) := \frac{1}{n+1} \left( \sum_{i=1}^n \frac{1}{2} \frac{(f_i(w) - \alpha_i)^2}{\|\nabla f_i(w)\|^2 + 1} + \frac{n}{2} (\alpha - \gamma)^2 \right). \quad (50)
\]

As a first step, we need to determine sufficient conditions for this auxiliary function \( h_t \) of TAPS to be star-convex. The relationship between the convexity or star-convexity of \( f_i \) and \( h_t \) is highly nontrivial. This is because the star-convexity \( h_t \) is not a consequence of \( f_i \) being star-convex, nor the converse. Instead, star-convexity of \( h_t \) translates to new nameless assumptions on the \( f_i \) functions. As an insight into this difficulty, supposing that each \( f_i \) is convex is not enough to guarantee that \( h_t \) is convex. As a simple counterexample, let \( n = 1 \) and \( f_1(w) = w^2 \). Thus \( \tau = 0 \) and from (50) we have
\[
 h_t(z) = \frac{1}{2} \left( \frac{1}{2} \frac{(w^2 - \alpha_1)^2}{\|w^t\|^2 + 1} + \frac{1}{2} \alpha_1^2 \right).
\]

It is easy to show that for \( \alpha_1 \) large enough, the Hessian of \( h_t \) has a negative eigenvalue, and thus \( h_t \) is non-convex. Conversely, \( h_t \) can have local convexity even when the underlying loss function is arbitrarily non-convex, as we show in the next lemma and corollary.
Lemma 5.8 (Locally Convex). Consider the iterates of Algorithm 2. Let \((w, \alpha) \in \mathbb{R}^{d+n}\) and consider \(h_t(w, \alpha)\) defined in (50). Assume that the gradients at \(w\) span the entire space, that is
\[
\text{span} \{\nabla f_1(w), \ldots, \nabla f_n(w)\} = \mathbb{R}^d, \quad \forall w.
\] (51)
If Assumption 3.1 holds, every \(f_i(w)\) for \(i = 1, \ldots, n\) is twice continuously differentiable and
\[
\frac{1}{n+1} \sum_{i=1}^{n} \nabla^2 f_i(w^t) \left( f_i(w^t) - \alpha^t_i \right) \geq 0,
\] (52)
then \(h_t\) is strictly convex at \((w^t, \alpha^t)\) with
\[
\nabla^2 h_t(w^t, \alpha^t) \succ 0.
\]
The condition on the span of the gradients (137) typically holds in the setting where we have more data than dimensions (features). Fortunately this occurs in precisely the setting where TAPS makes most sense since it makes sense to apply TAPS when \(f^* = \tau > 0\). This can only occur in the underparametrized setting, where we have more data than features.

The condition in Lemma 5.8 that is difficult to verify is (52). A sufficient condition for (52) to hold is
\[
\nabla^2 h_t(w^t, \alpha^t) \succ 0.
\] (53)
Since \(\alpha^t_i\) are essentially tracking \(f_i(w^t)\) (see line 8 in Algorithm 1), we can state (53) in words as: if \(\alpha^t_i\) is underestimating \(f_i(w^t)\) then \(f_i\) should be convex at \(w^t\), and conversely if \(\alpha^t_i\) is overestimating \(f_i(w^t)\) then \(f_i\) should be concave at \(w^t\).

There is one point where (52) holds trivially, and that is at every point such that \(\alpha_t = f_t(w)\). This includes every minimizer \((w^*, \alpha^*)\) since by Lemma 3.3 we have that \(\alpha_t^* = f_t(w^*)\). Consequently, as we state in the following corollary, under minor technical assumptions, we have that \(h_t(w, \alpha)\) has no degenerate local minimas. This shows that \(h_t\) has some local convexity.

Corollary 5.9 (Locally Strictly Convex TAPS). Consider the iterates of the TAPS method given in Algorithm 1. Let \(w^*\) be a minimizer of (1) and let \(\alpha_t^* = f_t(w^*)\) for \(i = 1, \ldots, n\). Assume that the gradients at \(w^*\) span the entire space, that is
\[
\text{span} \{\nabla f_1(w^*), \ldots, \nabla f_n(w^*)\} = \mathbb{R}^d.
\] (54)
If Assumption 3.1 holds and if every \(f_i(w)\) for \(i = 1, \ldots, n\) is twice continuously differentiable then \(\nabla^2 h_t(w, \alpha) \succ 0\) and thus \(h_t(w, \alpha)\) is strictly convex at \((w^*, \alpha^*)\).

Next we specialize Theorem 5.2 to the TAPS method in the following corollary.

Corollary 5.10 (Sublinear Convergence of TAPS). Let \(h_t(z)\) in (20) be star–convex (111) around \(z^* = (w^*, \alpha^*)\) and along the iterates \(z^t = (w^t, \alpha^t)\) of Algorithm 1. If \(\gamma < 1\) and \(f_i(w)\) is \(L_{\text{max}}–\text{Lipschitz} then
\[
\min_{t=1, \ldots, k} \frac{1}{n+1} \left( \sum_{i=1}^{n} \mathbb{E} \left[ \frac{f_i(w^t) - \alpha^t_i}{L_{\text{max}} + 1} \right] + \mathbb{E} \left[ |\alpha^t - \tau|^2 \right] \right) \leq \frac{1}{k \gamma (1-\gamma)} \mathbb{E} \left[ \|w^0 - w^*\|^2 \right].
\] (55)
Alternatively, if \(h_t(z)\) is \(\mu–\text{strongly star–convex} (41) then
\[
\mathbb{E} \left[ \|w^t - w^*\|^2 + \sum_{i=1}^{n} \|\alpha^t_i - f_i(w^*)\|^2 \right] \leq (1 - \gamma \mu)^t \left( \|w^0 - w^*\|^2 + \sum_{i=1}^{n} \|\alpha^t_i - f_i(w^0)\|^2 \right).
\] (56)

5.4 Convergence of MOTAPS

Here we explore the consequences of Theorems 5.2 and 5.3 specialized to Algorithm 2. In this case, the proxy function \(h_t(z) = h_t(w, \alpha, \tau)\) is
\[
h_t(z) := \frac{1}{n+1} \left( \sum_{i=1}^{n} \frac{1 - \lambda}{2} \frac{(f_i(w) - \alpha^t_i)^2}{\|\nabla f_i(w^t)\|^2 + 1} + \frac{n(1 - \lambda)}{2} \frac{(\tau - \alpha^t_i)^2 + \lambda \tau^2}{2} \right). \] (57)
Before applying Theorems 5.2 and 5.3 we should verify when $h_t(z)$ is star-convex or convex. This turns out to be much the same task as verifying that the auxiliary function for TAPS given in (50) is star-convex. This is because the only difference between the two functions is that (57) has an additional $\frac{1}{2}\tau^2$ which adds strong convexity in the new $\tau$ dimension. Thus the discussion and results around Lemma 5.8 and Corollary 5.9 remain largely true for (57). That is, we are only able to establish when $h_t$ is locally convex.

For the remainder of this section we impose that the dampening parameter satisfies

$$\lambda \leq \frac{2n+1}{2n+3} < 1, \quad (58)$$

so that we can apply Lemma 4.2. In our forthcoming corollaries we will prove convergence of MOTAPS to the point $z^* = (w^*, \alpha^*, \tau^*)$ where $w^*$ is a minimizer of (1) and

$$\alpha^*_i := f_i(w^*) \quad \text{and} \quad \tau^* = f(w^*), \quad \text{for} i = 1, \ldots, n. \quad (59)$$

First we develop a corollary based on Theorem 5.2

**Corollary 5.11.** Let $\lambda \in [0, 1]$ satisfy (58) and let $z^t := (w^t, \alpha^t, \tau_t)$ be the iterates of Algorithm 2 when using a stepsize $\gamma = \frac{1}{2(1-\lambda)(2n+1)}$ and $\gamma_t = \gamma (\lambda + (1-\lambda)n)$. If $h_t(z)$ is star convex along the iterates $z^t$ and around $z^* := (w^*, \alpha^*, \tau^*)$ then

$$\min_{i=0,\ldots,k} E \left[ h_t(z^i) - h_t(z^*) \right] \leq \frac{2(1-\lambda)(2n+1)}{k} \| z^0 - z^* \|^2 + \frac{\lambda f(w^*)^2}{2(n+1)}. \quad (60)$$

Furthermore, if $f_i$ is $L_{\max}$–Lipschitz then

$$\frac{1}{n+1} \sum_{t=1}^n \left[ \frac{1}{2} \frac{(f_i(w^t) - \alpha^*_i)^2}{L_{\max} + 1} + \frac{n}{2} (\tau^t - \tau^*)^2 + \frac{\lambda}{2} \left( (\tau^t)^2 - f(w^*) \right) \right] \leq \frac{2(1-\lambda)(2n+1)}{k} \| z^0 - z^* \|^2 + \frac{\lambda f(w^*)^2}{2(n+1)}. \quad (61)$$

This Corollary 5.11 shows that $(f_i(w^t), \tau^t, \tau^t)$ converges to $(\alpha^*_i, \tau^t, f(w^*))$ sublinearly up to an additive error $\frac{\lambda f(w^*)^2}{2(n+1)}$ which is controlled by $\lambda$: When $\lambda$ is very small, this additive error is very small. But $\lambda$ also controls the speed of convergence. Indeed for $\lambda$ close to 1 the method converges faster up to this additive error. Thus $\lambda$ controls a trade-off between speed of convergence and radius of convergence.

The next corollary is based on Theorem 5.3

**Corollary 5.12.** Let $\lambda \in [0, 1]$ satisfy (58) and let $z^t := (w^t, \alpha^t, \tau_t)$ be the iterates of Algorithm 2 when using a stepsize $\gamma = \frac{1}{2(1-\lambda)(2n+1)}$ and $\gamma_t = \gamma (\lambda + (1-\lambda)n)$. If $h_t(z)$ is $\mu$–strongly star-convex along the iterates $z^t$ and around $z^* := (w^*, \alpha^*, \tau^*)$ then

$$\mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] \leq \left( 1 - \frac{\mu}{(1-\lambda)(2n+1)} \right)^{t+1} \| z^0 - z^* \|^2 + \frac{\lambda f(w^*)^2}{\mu(n+1)}. \quad (62)$$

In both Corollary 5.11 and 5.12 the $\lambda$ parameter controls a trade-off between speed of convergence and an additive error term. For example, for the largest value $\lambda = \frac{2n+1}{2n+3}$ (due to (58)) we have that (62), after some simplifications, gives

$$\mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] \leq \left( 1 - \mu/2 \right)^{t+1} \| z^0 - z^* \|^2 + \frac{f(w^*)^2}{\mu(n+1)}. \quad (63)$$

Thus the convergence rates is now $1 - \mu/2$ and independent of $n$. But the additive error term $\frac{f(w^*)^2}{\mu(n+1)}$ is now larger. On the other hand, as $\lambda \to 0$ the rate of convergence tends to $(1 - \frac{n}{2n+3})$, which now depends on $n$, and the additive error term tends to zero.

By controlling this trade-off, next we use Corollary 5.12 to establish a total complexity of Algorithm 2.
Theorem 5.13. Consider the setting of Corollary 5.12. For a given $\epsilon > 0$ it follows that

$$t \geq \frac{(1 - \lambda)(2n + 1)}{\mu} \log\left(\frac{2\|z^0 - z^*\|^2}{\epsilon}\right) \implies \mathbb{E}\left[\|z^{t+1} - z^*\|^2\right] < \frac{\epsilon}{2} + \frac{\lambda f(w^*)}{\mu(n + 1)}. \tag{63}$$

Consequently if we could choose

$$\lambda < \min \left\{ \frac{\mu(n + 1) \epsilon}{f(w^*)^2}, \frac{2n + 1}{2n + 3} \right\} \tag{64}$$

then

$$t \geq \frac{2n + 1}{\mu} \log\left(\frac{2\|z^0 - z^*\|^2}{\epsilon}\right) \implies \mathbb{E}\left[\|z^{t+1} - z^*\|^2\right] < \epsilon. \tag{65}$$

Proof. By standard arguments using properties of the logarithm, we have that

$$t \geq \frac{(1 - \lambda)(2n + 1)}{\mu} \log\left(\frac{2}{\epsilon}\right) \implies \left(1 - \frac{\mu}{(1 - \lambda)(2n + 1)}\right)^t < \frac{\epsilon}{2}.$$  

See for instance Lemma 11 in [Gow16]. Furthermore, by using (64) we have that

$$t \geq \frac{1 - \frac{\mu(n + 1) \epsilon}{f(w^*)^2}}{\frac{2n + 1}{\mu}} \log\left(\frac{2\|z^0 - z^*\|^2}{\epsilon}\right) \geq \frac{2n + 1}{\mu} \log\left(\frac{2\|z^0 - z^*\|^2}{\epsilon}\right) \implies \mathbb{E}\left[\|z^{t+1} - z^*\|^2\right] < \epsilon. \tag{66}$$

Thus by choosing $\lambda$ small enough, we can show that the MOTAPS method converges linearly. This is in stark contrast to SGD where, despite the presence of an additive error when using a constant step size (See Theorem 1 in [MB11]), this additive term only vanishes by setting the stepsize to zero. In contrast for MOTAPS we can set $\lambda$ arbitrarily small without halting the method.

In practice, we would not know how to set $\lambda$ using (64) since we would not know $f(w^*)$. Furthermore, we may not have a particular $\epsilon$ in mind, and instead, prefer to monitor the error and stop when resources are exhausted. To address both of these concerns, the next theorem offers another way to deal with the additive error by eventually decreasing the step size.

Theorem 5.14. Consider the setting of Corollary 5.12. For a given $\epsilon > 0$ if we use an iteration dependent stepsize in Algorithm 2 given by

$$\gamma_t = \begin{cases} 1 & \text{if } t \leq 2(2n + 1) \left\lceil \frac{1 - \lambda}{\mu} \right\rceil, \\ \frac{(t + 1)^2 - t^2}{\mu(t + 1)^2} & \text{if } t \geq 2(2n + 1) \left\lceil \frac{1 - \lambda}{\mu} \right\rceil, \end{cases} \tag{67}$$

and if

$$\lambda \leq \min \left\{ 1 - \frac{2\mu}{2n + 1}, \frac{2n + 1}{2n + 3} \right\},$$

then

$$\mathbb{E}\left[\|z^t - z^*\|^2\right] \leq \frac{(1 - \lambda)\lambda f(w^*)^2}{\mu^2} \frac{16}{t} + \frac{4(2n + 1)^2}{e^2t^2} \left\lceil \frac{1 - \lambda}{\mu} \right\rceil^2 \|z^0 - z^*\|^2. \tag{68}$$

This Theorem 5.14 relies on knowing $\mu$ to set a switching point and the step size in $\{51\}$. In practice it can also be difficult to estimate $\mu$, but this theorem is still useful in that, it suggests that at some point in the execution we should decrease the stepsize

$$\gamma_t = \mathcal{O}\left(\frac{(t + 1)^2 - t^2}{(t + 1)^2}\right) = \mathcal{O}\left(\frac{2t + 1}{(t + 1)^2}\right) = \mathcal{O}\left(\frac{1}{t + 1}\right),$$

much in the same way that SGD is used in practice.
6 Experiments

6.1 Convex Classification

We first experiment with a classification task using logistic regression.

For our experiments on convex classification tasks, we focused on logistic regression. That is

\[ f(w) = \frac{1}{n} \sum_{i=1}^{n} \phi(x_i^Tw) + \frac{\sigma}{2} \|w\|_2^2 \]  \hspace{1cm} (69)

where \( \phi_i(t) = \ln(1+e^{-t}) \), \((x_i, y_i) \in \mathbb{R}^{d+1}\) are the features and labels for \( i = 1, \ldots, n \), and \( \sigma > 0 \) is the regularization parameter. We experimented with the five diverse data sets: leu [Gol99], duke [Wes01], colon-cancer [Alo99], mushrooms [DG17] and phishing [DG17]. Details of these datasets and their properties can be found in Table 1.

| dataset     | \( d \) | \( n \) | \( L_{\max} \) | \( \sigma = 0 \) | \( \gamma^* \) | \( \gamma_\tau^* \) | \( f^* \) | \( \gamma^* \) | \( \gamma_\tau^* \) | \( f^* \) | \( \sigma \) |
|-------------|--------|--------|-------------|-------------|-------------|-------------|--------|-------------|-------------|--------|--------|
| leu         | 7130   | 38     | 824.6       | 1.1         | 10^{-5}    | 0.0         | 0.01   | 0.4        | 0.4495     | 11.74  |
| duke        | 7130   | 44     | 683.2       | 1.1         | 10^{-3}    | 0.0         | 0.1    | 0.4        | 0.4495     | 5.06   |
| colon-cancer| 2001   | 62     | 137.8       | 1.1         | 10^{-5}    | 0.0         | 0.1    | 0.9        | 0.453      | 2.66   |
| mushrooms   | 112    | 8124   | 5.5         | 1.1         | 10^{-4}    | 0.0         | 1.0    | 0.0        | 0.083      | 0.0027 |
| phishing    | 68     | 11055  | 7.75        | 0.01        | 0.5        | 0.142      | 0.01   | 0.9        | 0.188      | 0.0028 |

Table 1: Binary datasets used in the logistic regression experiments together with the best parameters settings for \( \gamma \) and \( \gamma_\tau \) for two different regularization settings.

For the sake of simplicity, here we test the MOTAPS method in Algorithm 2 with \( \lambda = 0.5 \). To determine a reasonable parameter setting for the MOTAPS methods we performed a grid search over the two parameters \( \gamma \) and \( \gamma_\tau \). See Figure 1 for the results of the grid search for an over-parametrized problem colon-cancer and an under-parametrized problem mushrooms. Through these grid searches we found that the determining factor for setting the best stepsize was the magnitude of the regularization parameter \( \sigma > 0 \). If \( \sigma \) was small or zero then \( \gamma = 1 \) and \( \gamma_\tau = 0.001 \) resulted in a good performance. On the other hand, if \( \sigma \) is large then \( \gamma = 0.01 \) and \( \gamma_\tau = 0.9 \) resulted in the best performance. This is most likely due to the effect that \( \sigma \) has on the optimal value \( f(w^*) \), as is also clear in Table 1.

6.2 Comparison to Variance Reduced Methods

We compare our methods against SGD, and two variance reduced gradient methods SAG [SLR17], [DBL14] and SVRG [ZJ13] which are among the state-of-the-art methods for minimizing logistic regression. For setting the parameters for SGD, based on [Gow19], we used the learning rate schedule \( \gamma_t = L_{\max}/t \) where \( L_{\max} \) is the smoothness constant. For SVRG and SAG we used \( \gamma = 1/2L_{\max} \).

For SP and TAPS we used \( \gamma = 1 \) and approximated \( f_0(w^*) = 0 \). Because of this the SP is equivalent to the SPS method given in [Loi20]. Following [Loi20] experimental results, we also implemented
6.3 Deep learning tasks

We performed a series of experiments on three benchmark problems commonly used for testing optimization methods for deep learning. CIFAR10 \cite{Kri09} is a computer vision classification problem and perhaps the most ubiquitous benchmark in the deep learning. We used a large and over-parameterized network for this task, the 152 layer version of the pre-activation ResNet architecture \cite{He+16}, which has over 58 million parameters.

For our second problem, we choose an under-parameterized computer vision task. The street-view house numbers dataset \cite{Net+11} is similar to the CIFAR10 dataset, consisting of the same number of classes, but with a much larger data volume of over 600k training images compared to 50k. To ensure the network can not completely interpolate the data, we used a much smaller ResNet network with 1 block per layer and 16 planes at the first layer, so that there are fewer parameters than data-points.

For our final comparison we choose one of the most popular NLP benchmarks, the IWSLT14 english-german translation task \cite{Cet+14}, consisting of approximately 170k sentence pairs. This task is relatively small scale and so overfitting is a concern on this problem. We applied a modern Transformer network with embedding size of 512, 8 heads and 3/3 encoding/decoding layers.

In each case the minimum loss is unknown so for the TAPS method we assume it is 0. Due to a combination of factors including the use of data-augmentation and L2 regularization, this is only an approximation. The learning rate for each method was swept on a power-of-2 grid on a single training seed, and the best value was used for the final comparison, shown over an average of 10 seeds. Error bars indicate 2 standard errors. L2 regularization was used for each task, and tuned for each problem and method separately also on a power-of-2 grid. We found that the optimal amount of regularization was not sensitive to the optimization method used. Results on held-out test data are shown in Figure\textsuperscript{3} training loss plots can be found in the appendix in Figure\textsuperscript{13}.

Both TAPS and MOTAPS show favorable results compared to SP on all three problems. On the computer vision datasets, neither method quite reaches the generalization performance of SGD with a highly tuned step-wise learning rate schedule (95.2\% for CIFAR10, 95.9\% on SVHN). On the IWSLT14 problem, both TAPS and MOTAPS out-perform Adam \cite{KB14} which achieved a 2.69 test loss and is the gold-standard for this task.

\footnote{In \cite{Loi+20} the authors also recommend the use of a further smoothing trick, but we opted for simplicity and chose not to use this smoothing.}
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Omitted for review.

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Appendix

The Appendix is organized as follows: In Section B, we give some additional lemmas used to establish the closed form update of the methods. In Section C we present the proofs of the lemmas and theorem presented in the main paper. In Sections D, E, and F we discuss the consequences of Theorem 5.2 to the SP, TAPS and MOTAPS method, respectively. In Section G we provide another convergence theorem for SP that shows that for smooth, star-convex loss functions the SP method convergence with a step size $\gamma = 1$. In Section H we present another general convergence theory based on a time dependent smoothness assumption. Finally, in Section I and J we give further details on our implementations of the methods and the numerical experiments.

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### A Comparing SP to the method given in [OP19; Loi+20]

The SP method given in (9) is closely related to the method given in [OP19; Loi+20] which is

\[
    w^{t+1} = w^t - \frac{f_i(w^t) - f_i^*}{\|\nabla f_i(w^t)\|^2} \nabla f_i(w^t),
\]

where \( f_i^* := \inf_w f_i(w) \) for \( i = 1, \ldots, n \). Note that the only difference between (70) and the SP method (9) is that \( f_i(w^*) \) has been replaced by \( f_i^* \). If the Interpolation Assumption 5.4 holds then \( f_i^* = f_i(w^*) \) and the two methods are equal. Outside of the interpolation regime, these two methods are not necessarily the same.
In terms of convergence theory, the difference is only cosmetic, since the method (70) only converges when \( f_\ast^i = f_i(w_\ast^i) \), that is, when the two methods are equal. Indeed, let
\[
\sigma := \frac{1}{n} \sum_{i=1}^{n} (f_i(w_\ast^i) - f_\ast^i).
\]

Note that \( \sigma \geq 0 \) by the definition of \( f_\ast^i \). According to Theorems 3.1 and 3.4 in [Loi+20], the method (70) converges to a neighborhood of the solution with a diameter that depends on \( \sigma \). Thus (70) converges to the solution when \( \sigma = 0 \). This only happens when the interpolation Assumption 5.4 holds. Putting convergence aside, the method (70) has the advantage that for many machine learning problems \( f_i(w_\ast^i) \) is known. This is in contrast to the SP method (9), where \( f_i(w_\ast^i) \) is not known for most applications. In our experiments, we set \( f_i(w_\ast^i) = 0 \), and thus the two methods are equivalent.

\section{Auxiliary Lemmas}

\textbf{Lemma B.1.} The solution to
\[
w^{t+1}, \alpha_i^{t+1} = \arg\min_{w \in \mathbb{R}^d, \alpha_i \in \mathbb{R}} \|w - w^t\|^2 + \|\alpha_i - \alpha_i^t\|^2
\]
subject to \( f_i(w^t) + \langle \nabla f_i(w^t), w - w^t \rangle = \alpha_i \) (71)
is given by
\[
\alpha_i^{t+1} = \alpha_i^t + \frac{f_i(w^t) - \alpha_i^t}{\|\nabla f_i(w^t)\|^2 + 1}
\]
\[
w^{t+1} = w^t - \frac{f_i(w^t) - \alpha_i^t}{\|\nabla f_i(w^t)\|^2 + 1} \nabla f_i(w^t).
\] (72)

\textbf{Proof.} Introducing the variable \( x = [w, \alpha_i] \in \mathbb{R}^{d+1} \) we can re-write (73) as
\[
x^{t+1} = \arg\min_x \|x - x^t\|^2
\]
subject to \( \left[ \begin{array}{c} \nabla f_i(w^t) \\ -1 \end{array} \right] x = -f_i(w^t) + \langle \nabla f_i(w^t), w^t \rangle \). (73)

Using Lemma B.2 (just below) we have that the solution to the above is given by
\[
x^{t+1} = x^t + \left[ \nabla f_i(w^t) \right]^{-1} \left[ \begin{array}{c} \frac{1}{\|\nabla f_i(w^t)\|^2 + 1} (-f_i(w^t) + \langle \nabla f_i(w^t), w^t \rangle - \langle \nabla f_i(w^t), w^t \rangle - \alpha_i^t) \end{array} \right]
\]
Substituting out \( x = [w, \alpha_i] \) and simplifying we have
\[
\left[ \begin{array}{c} w^{t+1} \\ \alpha_i^{t+1} \end{array} \right] = \left[ \begin{array}{c} w^t \\ \alpha_i^t \end{array} \right] + \left[ \nabla f_i(w^t) \right]^{-1} \left[ \begin{array}{c} \frac{\alpha_i^t - f_i(w^t)}{\|\nabla f_i(w^t)\|^2 + 1} \end{array} \right].
\]
which is equal to (72). \( \square \)

\textbf{Lemma B.2.} The solution to
\[
x^+ = \arg\min_{x \in \mathbb{R}^d} \|x - x^0\|^2
\]
subject to \( a^\top x = b \) (74)
is given by
\[
x^+ = x^0 + \frac{a}{\|a\|^2} (b - a^\top x^0)
\] (75)

\textbf{Proof.} Substitute \( z = x - x^0 \) and consider the resulting problem
\[
z^+ = \arg\min_{z \in \mathbb{R}^d} \|z\|^2
\]
subject to \( a^\top z = b - a^\top x^0 \) (76)
One of the properties of the pseudo-inverse is that the least norm solution to the linear equation in (76) is given by

$$z^+ = a^+ (b - a^+ x^0),$$

(77)

where $a^+$ is the pseudo-inverse of $a^T$. It is now easy to show that $a^+ = \frac{a}{\|a\|_2}$ is the pseudo-inverse of $a$. Substituting back $x$ and the definition of $a^+$ in (77) gives (75).

B.1 Linear Algebra

Lemma B.3. For any matrices $A, B,$ and $C$ of appropriate dimensions we have that

$$\left\| \begin{bmatrix} A & C \\ C^T & B \end{bmatrix} \right\| \leq \|A\| + 2 \|C\| + \|D\|$$

(78)

Proof. Let $[vw]$ we a vector of unit norm. It follows that

$$\left\| \begin{bmatrix} A & C \\ C^T & B \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \right\| = \sqrt{\|Av + Cw\|^2 + \|C^Tv + Bw\|^2}$$

$$\leq \|Av + Cw\| + \|C^Tv + Bw\|$$

$$\leq \|Av\| + \|Cw\| + \|C^Tv\| + \|Bw\|$$

$$\leq \|A\| \|v\| + \|C\| \|w\| + \|C\| \|v\| + \|B\| \|w\|$$

$$\leq \|A\| + 2 \|C\| + \|D\|,$$

where in the first inequality we used that, for any $a, b > 0$ we have that $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$, and in the last inequality we used that $\|w\|, \|v\| \leq \|[w \ v]\| = 1$.

C Missing Proofs

Here we present the missing proofs from the main text.

C.1 Proof of Lemma 3.3

First note that for the function in (22) we have that

$$\nabla w h_i, t(w, \alpha) = \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} \nabla f_i(w), \quad \nabla \alpha_i h_i, t(w, \alpha) = -\frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1},$$

(79)

and

$$\nabla \alpha_i h_{n+1, i}(w, \alpha) = (\pi - \tau).$$

(80)

Proof. The stationarity conditions of (20) are given by setting the gradients to zero, which from (80) we have that

$$\nabla w h_i(w, \alpha) = 0,$$

$$\nabla \alpha_i h_i(w, \alpha) = 0, \text{ for } i = 1, \ldots, n$$

$$\Downarrow$$

$$\frac{1}{n+1} \sum_{i=1}^{n} \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} \nabla f_i(w) = 0$$

(81)

$$\frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} = (\pi - \tau), \text{ for } i = 1, \ldots, n.$$  

(82)

If $\pi = \tau$ then from (82) we have that $f_i(w) = \alpha_i$ for all $i$, and thus from Assumption 3.1 we have that $w$ must be a minimizer of (1), and thus a stationary point.

7This follows by the definition of pseudo-inverse since $a^+ a^+ = a^+$, $a^+ a^+ a^+ = a^+$ and both $a^+ a^+$ and $a^+ a^+$ are symmetric.
On the other hand, if $\pi \neq \tau$, then by substituting (82) into (81) gives
\[
\frac{1}{n + 1} \sum_{i=1}^{n} (\pi - \tau) \nabla f_i(w) = \frac{n(\pi - \tau)}{n + 1} \left( \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(w) \right) = 0. \tag{83}
\]
Consequently since $\pi \neq \tau$, we have $\frac{1}{n} \sum_{i=1}^{n} \nabla f_i(w) = 0$ and thus $w$ is a stationary point of (1).
Finally, if $(w^*, \alpha^*)$ is a minimizer of (20) then by Assumption 3.1 necessarily $h_i(w^*, \alpha^*) = 0$. Thus $f_i(w^*) = \alpha_i^*$ and $\pi^* = \tau$. Thus again by Assumption 5.1 we have that $w^*$ must be a minimizer of (1).

C.2 Proof of Lemma 3.4

Proof. First note that
\[
\| \nabla_{\alpha_i} h_{i,t}(w^t, \alpha) \|^2 \leq \left( \frac{f_i(w^t) - \alpha_i}{\| \nabla f_i(w^t) \|^2 + 1} \right)^2 \| \nabla f_i(w^t) \|^2. \tag{84}
\]
Furthermore
\[
\| \nabla_{\alpha_i} h_{i,t}(w^t, \alpha) \|^2 \leq \left( \frac{f_i(w^t) - \alpha_i}{\| \nabla f_i(w^t) \|^2 + 1} \right)^2 \| \nabla f_i(w^t) \|^2 + 1. \tag{85}
\]
Consequently adding (84) and (85) gives
\[
\| \nabla h_{i,x}(w^t, \alpha) \|^2 \leq \| \nabla_{\alpha_i} h_{i,t}(w^t, \alpha) \|^2 + \| h_{i,t}(w^t, \alpha) \|^2 + \| \nabla_{\alpha_i} h_{i,t}(w^t, \alpha) \|^2 \tag{86}
\]
Furthermore
\[
\| \nabla h_{n+1,t}(w, \alpha) \|^2 \leq \sum_{i=1}^{n} (\pi - \tau)^2 \tag{87}
\]
Consequently
\[
\frac{1}{n + 1} \sum_{i=1}^{n+1} \| \nabla h_{i,t}(w^t, \alpha) \|^2 \leq \frac{1}{n + 1} \sum_{i=1}^{n+1} 2 h_{i,t}(w^t, \alpha) = 2 h_{i}(w^t, \alpha). \tag{88}
\]

C.3 Proof of Lemma 4.1

Lemma C.1. Let
\[
\alpha_i^* := f_i(w^*) \quad \text{and} \quad \tau^* = f(w^*), \quad \text{for } i = 1, \ldots, n. \tag{89}
\]
It follows that
\[
h_i(w^*, \alpha^*, \tau^*) = \frac{\lambda f(w^*)^2}{2(n + 1)}. \tag{90}
\]
Furthermore, every stationary point of (28) is a stationary point of (1). Finally if $f(w) \geq 0$ and $(w^*, \alpha, \tau)$ is a minimia of (28) then $w^*$ is a minimia of (1).

Proof. Substituting (88) into (28) gives
\[
h_i(w^*, \alpha^*, \tau^*) := \frac{1}{n + 1} \frac{\lambda}{2} (\tau^*)^2 = \frac{\lambda f(w^*)^2}{2(n + 1)}. \tag{91}
\]

Each stationary point of (28) satisfies
\[
\nabla_w h_t(w, \alpha, \tau) = \frac{1 - \lambda}{n + 1} \sum_{i=1}^{n} \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w^t)\|^2 + 1} \nabla f_i(w) = 0, \tag{90}
\]
\[
\nabla_{\alpha} h_t(w, \alpha, \tau) = \frac{1 - \lambda}{n + 1} \frac{\alpha_i - f_i(w)}{\|\nabla f_i(w^t)\|^2 + 1} + \frac{1 - \lambda}{n + 1} (\bar{\tau} - \tau) = 0, \tag{91}
\]
\[
\nabla_{\tau} h_t(w, \alpha, \tau) = (1 - \lambda)n(\tau - \bar{\tau}) + \lambda \tau = 0. \tag{92}
\]

From the last equation we have that
\[
\bar{\tau} - \tau = \frac{\lambda}{(1 - \lambda)n} \tau, \tag{93}
\]
and consequently substituting out \(\bar{\tau} - \tau\) in (91) by using (93) gives
\[
\nabla_{\alpha} h_t(w, \alpha, \tau) = \frac{1 - \lambda}{n + 1} \frac{\alpha_i - f_i(w)}{\|\nabla f_i(w^t)\|^2 + 1} + \frac{1}{n + 1} \frac{\lambda \tau}{\tau} = 0. \tag{94}
\]

Passing the \(\tau\) term to the other side gives
\[
\frac{\lambda}{n} \tau = (1 - \lambda) \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w^t)\|^2 + 1}, \quad \text{for } i = 1, \ldots, n. \tag{95}
\]

This allows us to substitute in (90) giving
\[
\nabla_w h_t(w, \alpha, \tau) = \frac{\lambda \tau}{n + 1} \left( \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(w) \right) = 0. \tag{96}
\]

From this we can conclude that if \((w, \alpha, \tau)\) is a stationary point of (28), then \(w\) is a stationary point of our original objective function. Let \((w, \alpha, \tau)\) be a stationary point. It follows from (93) that \(\tau = \frac{(1 - \lambda)n}{(1 - \lambda)n + \lambda \bar{\tau}}\), and thus after substituting into (28) gives
\[
h_t(w, \alpha, \tau) := \frac{1}{n + 1} \left( \sum_{i=1}^{n} \frac{1 - \lambda}{2} \frac{(f_i(w) - \alpha_i)^2}{\|\nabla f_i(w^t)\|^2 + 1} + \frac{n(1 - \lambda)}{2} \frac{\lambda (\tau - \bar{\tau})^2 + \frac{\lambda}{2} \tau^2}{(n(1 - \lambda) + \lambda \bar{\tau})^2} \right).
\]

\[
h_t(w, \alpha, \tau) = \frac{1}{n + 1} \left( \sum_{i=1}^{n} \frac{1 - \lambda}{2} \frac{(f_i(w) - \alpha_i)^2}{\|\nabla f_i(w^t)\|^2 + 1} + \frac{n(1 - \lambda)}{2} \frac{\lambda}{n(1 - \lambda) + \lambda \bar{\tau}} \right) + \frac{1}{n + 1} \left( \sum_{i=1}^{n} \frac{1}{2} \frac{(f_i(w) - \alpha_i)^2}{\|\nabla f_i(w^t)\|^2 + 1} + \frac{n \lambda}{2(n(1 - \lambda) + \lambda \bar{\tau})^2} \right) \tag{97}
\]

Furthermore, \(\tau = \frac{(1 - \lambda)n}{(1 - \lambda)n + \lambda \bar{\tau}}\) substituting into (94) and multiplying the result by \((n + 1)\) gives
\[
\frac{\alpha_i - f_i(w)}{\|\nabla f_i(w^t)\|^2 + 1} + \frac{\lambda}{n(1 - \lambda) + \lambda \bar{\tau}} = 0, \quad \text{for } i = 1, \ldots, n.
\]

This can be re-arranged and written more compactly as the linear system
\[
\left( \mathbf{D}^{-1} + \frac{\mathbf{1} \mathbf{1}^\top}{n(n(1 - \lambda) + \lambda \bar{\tau})} \right) \alpha = \mathbf{D}^{-1} \mathbf{F}, \tag{98}
\]
where
\[
\mathbf{D} := \text{diag} \left( \|\nabla f_1(w^t)\|^2 + 1, \ldots, \|\nabla f_n(w^t)\|^2 + 1 \right) \quad \text{and} \quad \mathbf{F} = (f_1(w), \ldots, f_n(w)).
\]
Using the Woodbury identity, the solution to the above is given by

$$
\alpha = \left( D^{-1} + \lambda \frac{11^T}{n(n(1-\lambda) + \lambda)} \right)^{-1} D^{-1} F,
$$

(99)

$$
= \left( 1 - D \left( \frac{n(n(1-\lambda) + \lambda)}{\lambda} + \frac{1^T D 1}{1^T} \right)^{-1} 1^T \right) F.
$$

(100)

$$
= \left( 1 - \frac{D 11^T}{n(n(1-\lambda) + 2\lambda) + \lambda \sum_{i=1}^n \|\nabla f_i(w^t)\|^2} \right) F.
$$

(101)

Which reading line by line gives

$$
\alpha_i = f_i(w) - \lambda \frac{D c_i \sum_{j=1}^n f_j(w)}{n(n(1-\lambda) + 2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2} = f_i(w) - \lambda \frac{\left( \frac{\sum_{j=1}^n \|\nabla f_j(w^t)\|^2}{n(n(1-\lambda) + 2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2} + 1 \right) \sum_{j=1}^n f_j(w)}{n(n(1-\lambda) + 2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2}.
$$

(102)

Taking the average over $i$ in the above gives

$$
\overline{\alpha} = f(w) - \lambda f(w) \frac{n + \sum_{j=1}^n \|\nabla f_j(w^t)\|^2}{n(n(1-\lambda) + 2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2} = f(w) \left( 1 - \frac{n + \sum_{j=1}^n \|\nabla f_j(w^t)\|^2}{n(n(1-\lambda) + 2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2} \right) = f(w) \frac{n(n(1-\lambda) + 2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2}{n(n(1-\lambda) + 2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2}.
$$

(103)

Substituting (102) and (103) into (97) gives

$$
h_t(w, \alpha, \tau) \frac{n + 1}{1 - \lambda} = \sum_{i=1}^n \frac{\lambda^2}{2} \left( \frac{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2}{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2} \right)^2 + \frac{1}{2} \frac{n\lambda}{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2} \frac{n\lambda}{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2}
$$

$$
= \sum_{i=1}^n \frac{\lambda^2 n^2}{2} f(w)^2 \frac{\|\nabla f_i(w^t)\|^2}{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2} + \frac{1}{2} \frac{n\lambda}{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2} \frac{n\lambda}{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2}
$$

$$
= \sum_{i=1}^n \frac{\lambda^2 n^2}{2} f(w)^2 \left[ \frac{\|\nabla f_i(w^t)\|^2}{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2} + \frac{1}{2} \frac{n\lambda}{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2} \right]^2
$$

$$
+ \frac{1}{2} \frac{n\lambda}{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2} \left[ f(w) \frac{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2}{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2} \right]^2
$$

$$
= \frac{\lambda}{2} f(w)^2 \frac{n^2}{n(n(1-\lambda)+2\lambda) + \lambda \sum_{j=1}^n \|\nabla f_j(w^t)\|^2}.
$$

where in first equality we used (102) and in the third equality we used (103). Since $w^t$ is fixed, and every minima of (68) is a stationary point, we have that the minima in $w$ of the above is given by

$$
w^* \in \arg \min f(w)^2 = \arg \min f(w),
$$

where we used the positivity of $f(w)$. □
C.4 Proof of Lemma 4.2

Here we prove an extended version of Lemma 4.2 with some additional intermediary results that make the lemma easier to follow.

Lemma C.2. Consider the functions
\[ h_{i,t}(w, \alpha, \tau) := \frac{1 - \lambda}{2} \frac{(f_i(w) - \alpha)^2}{\|\nabla f_i(w^t)\|^2 + 1}, \quad \text{for } i = 1, \ldots, n, \] (104)
and \( h_{n+1,t}(w, \alpha, \tau) \) given in (31). It follows that \( h_t(w, \alpha, \tau) \) defined in (28) is equivalent to
\[ h_t(w, \alpha, \tau) = \frac{1}{n+1} \sum_{i=1}^{n} h_{i,t}(w, \alpha, \tau) \] (105)
Furthermore, if
\[ \lambda \leq \frac{2n+1}{2n+3} < 1 \] (106)
then
\[ \|\nabla h_{i,t}(w^t, \alpha, \tau)\|^2 = 2h_{i,t}(w^t, \alpha, \tau), \] (107)
\[ \|\nabla h_{n+1,t}(w, \alpha, \tau)\|^2 \leq 2(1 - \lambda)(2n+1)h_{n+1,t}(w, \alpha, \tau), \] (108)
and consequently
\[ \frac{1}{n+1} \sum_{i=1}^{n+1} \|\nabla h_{i,t}(w^t, \alpha, \tau)\|^2 \leq 2(1 - \lambda)(2n+1)h_t(w^t, \alpha, \tau). \] (109)

Proof. Using the definitions of \( h_i(w^t, \alpha, \tau) \) in (28) we have that (105) holds.
Furthermore (107) follows from Lemma 3.4 As for \( h_{n+1,t}(w, \alpha, \tau) \) in (31) we have that
\[ h_{n+1,t}(w, \alpha, \tau) = \frac{n(1-\lambda)}{2}(\alpha - \tau)^2 + \frac{\lambda}{2n} \] \[ \nabla_{\tau} h_{n+1,t}(w, \alpha, \tau) = (1-\lambda)n(\tau - \overline{\alpha}) + \lambda \tau. \] \[ \nabla_{\alpha} h_{n+1,t}(w, \alpha, \tau) = (1-\lambda)1(\tau - \overline{\alpha}) \] (110)
Consequently
\[ \|\nabla h_{n+1,t}(w, \alpha, \tau)\|^2 = ((1-\lambda)n(\tau - \overline{\alpha}) + \lambda \tau)^2 + (1-\lambda)^2 \|1\|^2(\overline{\alpha} - \tau)^2 \] \[ \leq 2(1-\lambda)^2n^2(\tau - \overline{\alpha})^2 + 2\lambda^2 \tau^2 + (1-\lambda)^2n(\overline{\alpha} - \tau)^2 \] \[ = 2(1-\lambda)(2n+1)\frac{(1-\lambda)n(\tau - \overline{\alpha})^2}{2} + 4\lambda \tau^2 \] \[ \leq 2 \max\{(1-\lambda)(2n+1), 2\lambda\} h_{n+1,t}(w, \alpha, \tau). \]
Due to (106) we have that
\[ \max\{(1-\lambda)(2n+1), 2\lambda\} = (1-\lambda)(2n+1). \]
This proves (107). As a consequence from (107) and (108) we have that
\[ \frac{1}{n+1} \sum_{i=1}^{n+1} \|\nabla h_{i,t}(w^t, \alpha, \tau)\|^2 \leq 2 \max\{1, (1-\lambda)(2n+1)\} \frac{1}{n+1} \sum_{i=1}^{n+1} h_{i,t}(w^t, \alpha, \tau) \quad \text{(Using (107) and (108))} \] \[ \leq 2(1-\lambda)(2n+1)h_t(w^t, \alpha, \tau). \quad \text{(Using (106) and (28))} \]

C.5 Proof of Theorem 5.2

Here we give the proof of Theorem 5.2 We prove a slightly more general version of Theorem 5.2 by not requiring that the auxiliary function is zero at the optimal. That is \( h_t(z^*) \) may be non-zero. The exact result in Theorem 5.2 follows from applying the following Theorem C.3 with \( h_t(z^*) = 0 \).

25
Taking expectation, re-arranging and summing both sides from $t$ in \[VBS18\], Thereom 4.1 in \[GSL21\] and Theorem 3.1 in \[Gow+19\].

**Proof.** This proof is partially based on 4.10 in \[YLG20\], which in turn is based on Theorem 6 in \[VBST18\], Thereom 4.1 in \[GSL21\] and Theorem 3.1 in \[Gow+19\].

Expanding the squares we have

$$\|z^{t+1} - z^*\|^2 \leq \|z^t - z^*\|^2 - 2\gamma \langle \nabla_w h_t(z^t), z^t - z^*\rangle + \gamma^2 \mathbb{E}_t \|\nabla_wh_t(z^t)\|^2$$

$$\leq \|z^t - z^*\|^2 - 2\gamma \langle \nabla_w h_t(z^t), z^t - z^*\rangle + 2G\gamma^2 h_t(z^t)$$

$$\leq \|z^t - z^*\|^2 - 2\gamma(h_t(z^t) - h_t(z^*)) + 2G\gamma^2 h_t(z^t)$$

$$= \|z^t - z^*\|^2 - 2\gamma(1 - G\gamma)(h_t(z^t) - h_t(z^*)) + 2G\gamma^2 h_t(z^t)$$

Taking expectation, re-arranging and summing both sides from $t = 0, \ldots, k$ we have that

$$\sum_{t=0}^k \mathbb{E}[h_t(z^t) - h_t(z^*)] \leq \frac{1}{2\gamma(1 - G\gamma)} \sum_{t=0}^k \left( \mathbb{E}\|z^t - z^*\|^2 - \mathbb{E}\|z^{t+1} - z^*\|^2 \right) + \frac{G\gamma}{1 - G\gamma} \sum_{t=0}^k h_t(z^*)$$

$$\leq \frac{1}{2\gamma(1 - G\gamma)} \mathbb{E}\|z^0 - z^*\|^2 + \frac{G\gamma}{1 - G\gamma} \sum_{t=0}^k h_t(z^*).$$

Now dividing through by $k$ gives \[112\].

**C.6 Proof of Theorem 5.3**

**Theorem C.4.** Suppose Assumption 5.1 holds with $G > 0$. Let $\gamma \leq 1/G$. If there exists $\mu > 0$ and $z^*$ such that $h_t$ is $\mu$-strongly star-convex along $z^t$ and around $z^*$, that is

$$h_t(z^*) \geq h_t(z^t) + \langle \nabla h_t(z^t), z^* - z^t\rangle + \frac{\mu}{2} \|z^* - z^t\|,$$

then

$$\mathbb{E}\|z^{t+1} - z^*\|^2 \leq (1 - \gamma\mu)^{t+1} \|z^0 - z^*\|^2 + 2G\gamma^2 \sum_{i=0}^t (1 - \gamma\mu)^i \mathbb{E}[h_i(z^*)].$$

Finally, if $h_t(z^*) = 0$ for all $t$ then we have that \[116\] and \[38\] together imply that $\mu \leq G$ and thus \[116\] gives linear convergence.

**Proof.** This proof is partially based on 4.10 in \[YLG20\], which in turn is based on Theorem 6 in \[VBST18\], Thereom 4.1 in \[GSL21\] and Theorem 3.1 in \[Gow+19\].
Expanding the squares we have that
\[
\mathbb{E}_t \left[ \left| z^{t+1} - z^* \right|^2 \right] \leq \left| z^t - z^* \right|^2 - 2\gamma \left\langle \nabla_w h_t(z^t), z^t - z^* \right\rangle + \gamma^2 \mathbb{E}_t \left[ \left| \nabla_w h_{t,i}(z^t) \right|^2 \right]
\]
\[
\leq \left| z^t - z^* \right|^2 - 2\gamma \left\langle \nabla_w h_t(z^t), z^t - z^* \right\rangle + 2G\gamma^2 h_t(z^t)
\]
\[
\leq (1 - \gamma \mu) \left| z^t - z^* \right|^2 - 2\gamma (1 - G) (h_t(z^t) - h_t(z^*)) + 2G\gamma^2 h_t(z^*) \geq 0
\]
where to get to the last line we used that \( (1 - G) (h_t(z^t) - h_t(z^*)) \geq 0 \) which holds because \( \gamma \leq \frac{1}{G} \).
Taking the expectation and applying the above recursively gives
\[
\mathbb{E}_t \left[ \left| z^{t+1} - z^* \right|^2 \right] \leq (1 - \gamma \mu)^{t+1} \left| z^0 - z^* \right|^2 + 2G\gamma^2 \sum_{i=0}^t (1 - \gamma \mu)^i h_i(z^*)
\]
which is the result \( (116) \).

Furthermore, if \( h_t(z^*) = 0 \) we have that \( \mu \leq G \) follows from a small modification of Theorem 4.10 in \cite{YLG20}. Indeed taking expectation over \( (115) \) and using \( (38) \) we have that
\[
\mathbb{E}_t \left[ \left| z^t - z^* \right|^2 \right] \leq \frac{1}{2G} \mathbb{E}_t \left[ \left| \nabla h_t(z^t) \right|^2 \right] + \frac{\mu}{2} \left| z^t - z^* \right|^2
\]
\[
= \frac{G}{2} \mathbb{E}_t \left[ \left| z^t - z^* - \frac{1}{L} \nabla h_{t,i}(z^t) \right|^2 \right] - \frac{G - \mu}{2} \left| z^t - z^* \right|^2.
\]
Rearranging and using that \( h_t(z^*) = 0 \) gives
\[
\frac{G - \mu}{2} \left| z^t - z^* \right|^2 \geq \frac{L}{2} \mathbb{E}_t \left[ \left| z^t - z^* - \frac{1}{G} \nabla h_{t,i}(z^t) \right|^2 \right] \geq 0.
\]
Thus \( \mu \leq G \).

\section{Convergence of The Stochastic Polyak Method}

Here we explore sufficient conditions for the assumptions in Theorems \( 5.2 \) and \( 5.3 \) to hold for the SP method \( (9) \). To this end, let
\[
h_i(w) := \frac{1}{n} \sum_{i=1}^n \frac{1}{2} \left( f_i(w) - f_i(w^*) \right)^2 \frac{1}{\left\| \nabla f_i(w^t) \right\|^2},
\]
\[
h_{i,i}(w) := \frac{1}{2} \frac{1}{\left\| \nabla f_i(w^t) \right\|^2} \left( f_i(w) - f_i(w^*) \right)^2.
\]
We will also explore the consequences of these theorems. In these section we say that \( f_i \) is \( L_i \) smooth if
\[
f_i(z) \leq f_i(w) + \left\langle \nabla f_i(w), z - w \right\rangle + \frac{L_i}{2} \left\| z - w \right\|^2, \quad \forall z, w \in \mathbb{R}^d.
\]
We will also use the interpolation Assumption \( 5.4 \) throughout this section. Thus
\[
f_i(w^*) = \min_{w \in \mathbb{R}^d} f_i(w) \leq f(z), \quad \text{for all } i \in \{1, \ldots, n\}, \ z \in \mathbb{R}^d.
\]
Using smoothness and interpolation, we first establish the following descent lemma.

\begin{lemma}
If the interpolation Assumption \( 5.4 \) holds and each \( f_i(w) \) is \( L_i \)-smooth \( (122) \) then
\[
f_i(w) - f_i(w^*) \geq \frac{1}{2L_i} \left\| \nabla f_i(w) \right\|^2, \quad \forall w \in \mathbb{R}^d, \ i = 1, \ldots, n.
\]
\end{lemma}
where we used \( f \). Now if we assume that \( \mu_i \) is strongly star-convex and \( \tilde{L}_i \)-smooth then we have that

\[
0 \geq \frac{1}{2} \left( \frac{f_i(w^t) - f_i(w^*)}{\|\nabla f_i(w^t)\|} \right)^2 + \left( \frac{f_i(w^t) - f_i(w^*)}{\|\nabla f_i(w^t)\|^2} - \nabla f_i(w^t) \right) w^* - w^t
\]

\[
0 \geq \frac{1}{2} (f_i(w^t) - f_i(w^*)) + \nabla f_i(w^t), w^* - w^t
\]

\[
f_i(w^t) \geq f_i(w^*) + \nabla f_i(w^t), w^* - w^t
\]

where we used \( f_i(w^t) - f_i(w^*) \geq 0 \) which is a consequence of interpolation. This proves \( \text{Lemma D.2.} \)

Now if we assume that \( f_i \) is \( \mu_i \)-strongly star-convex and \( \tilde{L}_i \)-smooth then we have that by

\[
h_{i,t}(w^*) \geq h_{i,t}(w^t) + \nabla h_{i,t}(w^t), w^* - w^t + \frac{\mu}{4} \tilde{L}_i \|w^t - w^*\|^2
\]
we have from (123) that
\[ w^* \]

The result (129) would follow from (128) if
\[ w^* \]

and (129) follows by Theorem 5.2.

Having established when \( h_i \) is star convex and strongly star convex, we can now apply Theorems 5.2 and Theorem 5.3, which when specialized to SP gives the following corollaries. This result has already been established in Theorem 4.4 and Theorem D.3 in [GSL21]. Thus here we have showed and Theorem 5.3, which when specialized to

D.2 Proof of Corollary 5.6 and 5.7

Having established when \( h_i \) is star convex and strongly star convex, we can now apply Theorems 5.2 and Theorem 5.3, which when specialized to SP gives the following corollaries. This result has already been established in Theorem 4.4 and Theorem D.3 in [GSL21]. Thus here we have showed that the results in [GSL21] follow as a direct consequence of the interpretation of SP as a variant of the online SGD method. We also extend the following theorem to allow for \( \gamma = 1 \) in Theorem G.2.

\[ f_i(w^*) \geq f_i(w^t) + \langle \nabla f_i(w^t), w^* - w^t \rangle + \frac{\mu}{4} \| w^t - w^* \|^2 \]

Finally, from smoothness and Lemma D.1, we have that \( 1 \geq \frac{1}{2L_i} \| \nabla f_i(w^t) \|^2 \) consequently

\[ f_i(w^*) \geq f_i(w^t) + \langle \nabla f_i(w^t), w^* - w^t \rangle + \frac{\mu}{4} \| w^t - w^* \|^2 \]

Consequently the above implications hold, and thus \( h_{i,t} \) is \( \frac{1}{4L_i} \)-strongly star convex. Taking the average of (127) over \( i \) gives (126), which concludes the proof.

D.2 Proof of Corollary 5.6 and 5.7

Having established when \( h_i \) is star convex and strongly star convex, we can now apply Theorems 5.2 and Theorem 5.3, which when specialized to SP gives the following corollaries. This result has already been established in Theorem 4.4 and Theorem D.3 in [GSL21]. Thus here we have showed that the results in [GSL21] follow as a direct consequence of the interpretation of SP as a variant of the online SGD method. We also extend the following theorem to allow for \( \gamma = 1 \) in Theorem G.2.

**Corollary D.3.** If \( \gamma < 1 \) and every \( f_i(w) \) is star-convex along the iterates \((w^t)\) given by (9) then

\[ \frac{1}{k} \sum_{t=0}^{k} \frac{1}{2n} \sum_{i=1}^{n} \mathbb{E} \left[ \left( f_i(w^t) - f_i(w^*) \right)^2 \right] \leq \frac{1}{k} \frac{1}{2\gamma(1-\gamma)} \mathbb{E} \left[ \| w^0 - w^* \|^2 \right]. \]  

(129)

Furthermore if the interpolation Assumption 5.4 holds and if each \( f_i(w) \) is \( L_i \)-smooth then

\[ \min_{t=0,\ldots,k} \mathbb{E} \left[ f(w^t) - f^* \right] \leq \frac{1}{k} \frac{L_{\max}}{2\gamma(1-\gamma)} \mathbb{E} \left[ \| w^0 - w^* \|^2 \right], \]  

(130)

where \( L_{\max} := \max_{i=1,\ldots,n} L_i. \)

**Proof.** The proof of (129) follows as a special case of Theorem 5.2 by identifying \( h_i \) with (120) and \( h_{i,t} \) with (121). Indeed, according to (13) we have that \( h_i \) satisfies the growth condition (38) with \( G = 1 \) and according to (125) \( h_i \) is star-convex (111) around \( w^* \). Finally since \( h_i(w^*) = 0 \) the result (129) follows by Theorem 5.2.

The result (130) would follow from (129) if

\[ L_{\max} \frac{1}{n} \sum_{i=1}^{n} \frac{(f_i(w) - f_i(w^*))^2}{\| \nabla f_i(w) \|^2} \geq 2(f(w) - f^*). \]  

(131)

This Assumption has appeared recently in [GSL21] where it was proven that (131) is a consequence of each \( f_i(w) \) being \( L_i \)-smooth. We give a simpler proof next for completeness. That is, assuming that there exists \( w \) such that \( f_i(w) \neq f_i(w^*) \) and thus \( \nabla f_i(w) \neq 0 \) (otherwise (131) holds trivially) we have from (123) that

\[ \frac{1}{\| \nabla f_i(w) \|^2} \geq \frac{1}{2L_i(f_i(w) - f_i(w^*)}. \]  

(132)
Multiplying both sides by \((f_i(w) - f_i(w^*))^2\) and averaging over \(i = 1, \ldots, n\) gives

\[
\frac{1}{n} \sum_{i=1}^{n} (f_i(w) - f_i(w^*))^2 \geq \frac{1}{n} \sum_{i=1}^{n} \frac{f_i(w) - f_i(w^*)}{L_i} \geq \frac{1}{n} \sum_{i=1}^{n} 2f_i(w) - f_i(w^*)
\]

\[
= \frac{2(f(w) - f^*)}{L_{\max}}.
\]

Using (131) and (129) we have

\[
\min_{t=0, \ldots, k} \mathbb{E} \left[ f(w^t) - f^* \right] \leq \frac{1}{k} \sum_{t=0}^{k} \mathbb{E} \left[ f(w^t) - f^* \right]
\]

\[
\leq \frac{1}{k} \sum_{t=0}^{k} \frac{L_{\max}}{2n} \sum_{i=1}^{n} \mathbb{E} \left[ \left( \frac{f_i(w^t) - f_i(w^*)}{\|\nabla f_i(w^t)\|} \right)^2 \right]
\]

\[
\leq \frac{1}{k} \frac{L_{\max}}{2\gamma(1 - \gamma)} \mathbb{E} \left[ \|w^0 - w^*\|^2 \right]
\]

which concludes the proof of \(O(1/n)\).

Corollary D.4. If \(\gamma \leq 1\), the interpolation Assumption 5.4 holds, and every \(f_i\) is \(L_i\)-smooth and \(\mu\)-strongly star-convex then the iterates \(w^t\) given by (9) converge linearly according to

\[
\mathbb{E} \left[ \|w^{t+1} - w^*\|^2 \right] \leq \left( 1 - \gamma \frac{1}{2n} \sum_{i=1}^{n} \frac{\mu_{i}}{L_{i}} \right)^{t+1} \|w^0 - w^*\|^2 \quad (132)
\]

Proof. The proof of (132) follows as a special case of Theorem 5.3 by identifying \(h_t\) with (120) and \(h_{i,t}\) with (121). Indeed, according to (13) we have that \(h_t\) satisfies the growth condition (38) with \(G = 1\). Furthermore \(f_i\) is \(\mu\)-strongly star convex and \(L_i\)-smooth, then from Lemma 13.2 we have that \(h_t\) is \(\frac{1}{k} \sum_{i=1}^{n} \frac{\mu_{i}}{L_{i}}\)-strongly star convex. Finally since \(h_t(w^*) = 0\) the result (129) follows by Theorem 5.3.

E Convergence of the Targeted Stochastic Polyak Stepsize

Here we explore the consequences and conditions of Theorem 5.2 for the TAPS method given in Algorithm 1.

E.1 Proof of Corollary E.1 and more

First we re-state Theorem 5.2 specialized to Algorithm 1.

Corollary E.1. Let \(h_t(z)\) be defined in (20) and suppose that \(h_t(z)\) is star convex (111) around \(z^* = (w^*, \alpha^*)\) and along the iterates \(z^t = (w^t, \alpha^t)\) of Algorithm 1.

If \(\gamma < 1\) and in addition \(f_i(w)\) is \(L_{\max}\)-Lipschitz then

\[
\min_{t=1, \ldots, k} \frac{1}{n+1} \left( \sum_{i=1}^{n} \mathbb{E} \left[ f_i(w^t) - \alpha_i^t \right]^2 + \mathbb{E} \left[ \alpha_i^t - \gamma \right]^2 \right) \leq \frac{1}{k} \frac{1}{\gamma(1 - \gamma)} \mathbb{E} \left[ \|w^0 - w^*\|^2 \right]. \quad (133)
\]

Alternatively, if \(h_t(z)\) is \(\mu\)-strongly star-convex (115) then

\[
\mathbb{E} \left[ \|w^t - w^*\|^2 + \sum_{i=1}^{n} \|\alpha_i^t - f_i(w^*)\|^2 \right] \leq (1 - \gamma \mu)^t \left( \|w^0 - w^*\|^2 + \sum_{i=1}^{n} \|\alpha_i^0 - f_i(w^0)\|^2 \right). \quad (134)
\]

Theorem E.1 provides us with a \(O(1/k)\) convergence in expectation when \(h_t(z)\) is star convex. Indeed, the bound in (133) shows that \(\bar{\pi}\) converges to \(\tau\) at a rate of \(O(1/k)\). Finally from the target assumption (15) we have that \(h_t(z^*) = 0\), thus \(f_i(w^t)\) and \(\alpha_i^t\) converge to \(f_i(w^*)\) at a rate of \(O(1/k)\).
Proof. The proof follows by applying Theorem 5.2. Indeed, by letting \( h_{i,t}(z) = \frac{1}{2n} \left( f_i(w^{(t)}) - \alpha_t^i \right)^2 \) for \( i = 1, \ldots, n \) and \( h_{n+1,t}(z) = \frac{n}{2}(\bar{\alpha} - \tau)^2 \). Thus \( h_t(z) = \frac{1}{n+1} \sum_{i=1}^{n+1} h_{i,t}(z) \). By Lemma 3.4 we have that

\[
\mathbb{E}_{z \sim \frac{1}{n+1}} \left[ \|\nabla h_t(z^*)\|^2 \right] = \frac{1}{n+1} \sum_{i=1}^{n+1} \|\nabla h_{i,t}(z^*)\|^2
\]

\[
= \frac{1}{n+1} \left( \sum_{i=1}^{n} \|\nabla f_{i,w^t}(w^t, \alpha^t)\|^2 + \|\nabla h_{n+1}(\alpha^t)\|^2 \right)
\]

\[
= \frac{1}{n+1} \left( \sum_{i=1}^{n} 2f_{i,w^t}(w^t, \alpha^t) + 2f_{\alpha^t}(\alpha^t) \right)
\]

\[
= 2h_t(z^*).
\]

Consequently \( h_t \) satisfies the growth condition (38) with \( G = 1 \). By assumption \( h_t \) is star convex along the iterates \( z^t \), thus the two condition required for Theorem 5.2 to hold are satisfied, and as a consequence, we have that (112) holds. Substituting out \( h_t(z^t) \) we have that

\[
\frac{1}{k} \sum_{t=0}^{k} \frac{1}{n+1} \left( \sum_{i=1}^{n} \frac{1}{2n} \|\nabla f_{i,w^t}(w^t)\|^2 + \frac{n}{2}(\alpha^t - \tau)^2 \right) \leq \frac{1}{k} \frac{1}{2\gamma(1-\gamma)} \mathbb{E} \left[ \|w^0 - w^*\|^2 \right].
\]

(135)

Furthermore, if \( f_i \) is \( L_{\text{max}} \)-Lipschitz, that is if \( \|\nabla f_i(w^t)\| \leq L_{\text{max}} \) then from (135) we have that

\[
\frac{1}{k} \sum_{t=0}^{k} \frac{1}{n+1} \left( \sum_{i=1}^{n} \frac{1}{2L_{\text{max}}} + \frac{n}{2}(\alpha^t - \tau)^2 \right) \leq \frac{1}{k} \frac{1}{2\gamma(1-\gamma)} \mathbb{E} \left[ \|w^0 - w^*\|^2 \right],
\]

(136)

from which (133) follows by lower bounding the average over \( k \) by the minimum.

Finally, if there exists \( \mu > 0 \) such that \( h_t(z) \) is strongly star-convex (115), then by noting that

\[
\|z - z^*\|^2 = \|w^t - w^*\|^2 + \|\alpha^t - \alpha^*\|^2 = \|w^t - w^*\|^2 + \sum_{i=1}^{n} \|\alpha_t^i - f_i(w^t)\|^2
\]

we have that (116) gives (134).

\[\square\]

E.2 Proof of Lemmas 5.8 and Corollary 5.9

For ease of reference, we first re-state the lemmas.

Lemma E.2 (Locally Convex). Consider the iterates of Algorithm 2. Let \((w, \alpha) \in \mathbb{R}^{d+n}\) and consider \( h_i(w, \alpha) \) defined in (50). Assume that the gradient of \( w \) spans the entire space, that is

\[
\text{span} \{\nabla f_1(w), \ldots, \nabla f_n(w)\} = \mathbb{R}^d, \quad \forall w.
\]

(137)

If Assumption 3.1 holds, every \( f_i \) for \( i = 1, \ldots, n \) is twice continuously differentiable and

\[
\frac{1}{n+1} \sum_{i=1}^{n} \nabla^2 f_i(w^t) \cdot \frac{f_i(w^t) - \alpha_t^i}{\|\nabla f_i(w^t)\|^2 + 1} \geq 0, \quad \forall t,
\]

(138)

then \( h_t \) is strictly convex with at \((w^t, \alpha^t)\) that is

\[
\nabla^2 h_t(w^t, \alpha^t) > 0, \quad \forall t.
\]

Proof. We have \((f_i(w) - \alpha_i)^2\) is locally convex, and thus star convex, iff its Hessian is positive definite around \((w^*, \alpha^*)\). Computing the gradient of \((f_i(w) - \alpha_i)^2\) we have that

\[
\nabla (f_i(w) - \alpha_i)^2 = 2 \left[ \nabla f_i(w) \cdot \frac{f_i(w) - \alpha_i}{1} \right] (f_i(w) - \alpha_i)
\]

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The proof of Corollary 5.9 then follows from Lemma E.2 by plugging in \( \alpha^*_i = f_i(w^*) \) into (139).

Now let \( \mathbf{I}_n \in \mathbb{R}^{n \times n} \) be the identity matrix in \( \mathbb{R}^{n \times n} \), let

\[
\mathbf{D}_t := \text{diag} \left( \frac{1}{\|\nabla f_1(w^t)\|^2 + 1}, \ldots, \frac{1}{\|\nabla f_n(w^t)\|^2 + 1} \right) \in \mathbb{R}^{n \times n}
\]

and let

\[
\mathbf{H}_t(w, \alpha) := \sum_{i=1}^{n+1} \frac{\nabla^2 f_i(w) - \alpha_i}{\|\nabla f_i(w^t)\|^2 + 1}\]

Using (139) and by the definition of \( h_t \) in (50) we have that

\[
\nabla^2 h_t(w, \alpha) = \frac{1}{n+1} \left[ DF(w)\mathbf{D}_t DF(w)^\top - DF(w)\mathbf{D}_t \right] - \mathbf{H}_t(w, \alpha) \mathbf{I}_n (1 + \frac{1}{n}) + \left[ \mathbf{H}_t(w, \alpha) 0 0 \right] =: \mathbf{M}_t(w)
\]

where we used the \( \nabla^2 n (\pi - \tau)^2 = \frac{1}{n} \mathbf{I}_n \). Thus the matrix (141) is a sum of two terms. By the assumption (138) we have that the second part that contains \( \mathbf{H}_t(w, \alpha) \) is positive definite. Next we will show that the first matrix \( \mathbf{M}_t(w) \) is symmetric positive definite. Indeed, left and right multiplying the above by \([x, a] \in \mathbb{R}^{d+n}\) gives

\[ [x \ a] \mathbf{M}_t(w) [x \ a]^\top = \|DF(w)^\top x - a\|^2_{\mathbf{D}_t} + \|a\|^2_{\mathbf{I}_n + \frac{1}{n}\mathbf{D}_t} - \|DF(w)^\top x - a\|^2_{\mathbf{D}_t} \]

or short

\[ [x \ a] \mathbf{M}_t(w) [x \ a]^\top = \|DF(w)^\top x - a\|^2_{\mathbf{D}_t} + \|a\|^2_{\mathbf{I}_n + \frac{1}{n}\mathbf{D}_t} \]

Next we show that (142) is strictly positive for every \((x, a) \neq 0\). To this end, first note that the matrix \((1 + \frac{1}{n})\mathbf{I}_n - \mathbf{D}_t \) is positive definite, which follows since the \( i \)th diagonal element is positive with

\[
[(1 + \frac{1}{n})\mathbf{I}_n - \mathbf{D}_t]_{ii} = 1 + \frac{1}{n} - \frac{1}{\|\nabla f_i(w^t)\|^2 + 1} > 0.
\]

Consequently if \( a \neq 0 \) we have that (142) is strictly positive. On the other hand, if \( a = 0 \) let us prove by contradiction that (142) is still positive for \( x \neq 0 \). Indeed suppose that \( x \neq 0 \) and

\[ \|DF(w)^\top x\|_{\mathbf{D}_t}^2 = 0 \implies \sum_{i=1}^{n} \nabla f_i(w)^\top x = 0. \]

But due to our assumption (54), we have that \( DF(w)^\top \) has full column rank, and thus \( x = 0 \), which is a contradiction. Thus (142) is positive for every \((x, a) \neq 0\) from which we conclude that the Hessian \( \nabla^2 h_t(w, \alpha) \) in (141) is positive definite.

The proof of Corollary 5.9 then follows from Lemma E.2 by plugging in \( \alpha^*_i = f_i(w^*) \) into (139).
Convergence of the Moving Target Stochastic Polyak Stepsize

Here we explore the consequences of Theorems 5.2 and 5.3 specialized to Algorithm 2. Throughout this section let
\[ \lambda \leq \frac{2n + 1}{2n + 3} < 1 \]  
and let \( z^t := (w^t, \alpha^t, \tau_t) \) be the iterates of Algorithm 2 when using a stepsize \( \gamma = \gamma_t \). Let
\[ h_t(z) := \frac{1}{n + 1} \left( \sum_{i=1}^{n} \frac{1}{2} \left( f_i(w) \right) - \frac{(f_i(w) - \alpha_t)^2}{L_{\text{max}} + 1} + \frac{n(1 - \lambda)}{2} \right). \]  
and let \( w^* \) be a minimizer of \( (1) \) and let
\[ \alpha_t^* := f_i(w^*) \quad \text{and} \quad \tau^* = f(w^*), \quad \text{for } i = 1, \ldots, n. \]  

F.1 Proof of Corollary 5.11

**Corollary F.1.** If \( \gamma = \gamma_t = \frac{1}{2(1 - \lambda)(2n + 1)} \) and if \( h_t(z) \) is star convex along the iterates \( z^t \) and around \( z^* := (w^*, \alpha^*, \tau^*) \) then
\[ \min_{t = 0, \ldots, k} \mathbb{E}[h_t(z^t) - h_t(z^*)] \leq \frac{2(1 - \lambda)(2n + 1)}{k} \| z^0 - z^* \|^2 + \frac{\lambda f(w^*)^2}{2(n + 1)}. \]  

Furthermore, if \( f_i \) is \( L_{\text{max}} \)-Lipschitz then
\[ \frac{1}{n + 1} \mathbb{E} \left[ \sum_{i=1}^{n} \frac{1}{2} \left( f_i(w^t) - \alpha_t^* \right)^2 \right] \leq \frac{2(1 - \lambda)(2n + 1)}{k} \| z^0 - z^* \|^2 + \frac{\lambda f(w^*)^2}{2(n + 1)}. \]  

**Proof.** The proof follows by applying Theorem 5.2 and Lemmas 4.2 and 4.1. Indeed \( h_t(z) \) satisfies the growth condition \( [38] \) with \( G = (1 - \lambda)(2n + 1) \). By assuming that \( h_t(z) \) is star convex along the iterates \( z^t \) we have satisfied the two condition required for Theorem 5.2 to hold, which when substituting in \( G \) and \( \gamma = \frac{1}{2(1 - \lambda)(2n + 1)} \) gives
\[ \min_{t = 1, \ldots, k} \mathbb{E}[h_t(z^t) - h_t(z^*)] \leq \frac{2(1 - \lambda)(2n + 1)}{k} \mathbb{E} \left[ \left\| z^0 - z^* \right\|^2 \right] + \frac{1}{k} \sum_{t=1}^{k} h_t(z^*). \]  

Furthermore using the bound \( [30] \) in Lemma 4.1 we have that
\[ \frac{1}{k} \sum_{t=1}^{k} h_t(z^*) = \frac{\lambda f(w^*)^2}{2(n + 1)} \]  
and thus \( (146) \) holds. Finally, if \( f_i \) is \( L_{\text{max}} \)-Lipschitz, that is if \( \| \nabla f_i(w^t) \| \leq L_{\text{max}} \), then using the definition of \( h_t(z) \) in \( (144) \) we can lower bound \( h_t(z^t) - h_t(z^*) \) by the left-hand side of \( (147) \). \( \square \)

F.2 Proof of Corollary F.2

**Corollary F.2.** If \( \gamma = \gamma_t = \frac{1}{(1 - \lambda)(2n + 1)} \) and if \( h_t(z) \) is \( \mu \)-strongly star–convex along the iterates \( z^t \) and around \( z^* := (w^*, \alpha^*, \tau^*) \) then
\[ \mathbb{E} \left[ \left\| z^{t+1} - z^* \right\|^2 \right] \leq \left( 1 - \frac{\mu}{(1 - \lambda)(2n + 1)} \right)^{t+1} \left\| z^0 - z^* \right\|^2 + \frac{\lambda f(w^*)^2}{\mu(n + 1)}. \]  

**Proof.** The proof follows by applying Theorem 5.3 and Lemmas 4.2 and 4.1. Indeed by Lemma 4.2 \( h_t(z) \) satisfies the growth condition \( [38] \) with \( (1 - \lambda)(2n + 1) \). By assuming that \( h_t(z) \) is \( \mu \)-strongly star
convex along the iterates \( z^t \) we have satisfied the two condition required for Theorem 5.3 to hold. Finally, using (30) in Lemma 4.1, we have that
\[
h_t(z^*) = \frac{\lambda f(w^*)^2}{2(n+1)}
\]
and as a consequence Theorem 5.3 gives
\[
E \left[ \left\| z^{t+1} - z^* \right\|^2 \right] \leq (1 - \frac{\mu}{(1-\lambda)(2n+1)})^{t+1} \left\| z^0 - z^* \right\|^2 + \frac{2}{(1-\lambda)(2n+1)} \sum_{i=0}^{t} (1-\gamma \mu)^i \frac{\lambda f(w^*)^2}{2(n+1)}.
\]
Finally, using (30) in Lemma 4.1, we have that
\[
E \left[ \left\| z^{t+1} - z^* \right\|^2 \right] \leq (1 - \frac{\mu}{(1-\lambda)(2n+1)})^{t+1} \left\| z^0 - z^* \right\|^2 + \frac{2}{(1-\lambda)(2n+1)} \frac{1}{\gamma \mu} \frac{\lambda f(w^*)^2}{2(n+1)}.
\]
and consequently Theorem 5.3 gives
\[
E \left[ \left\| z^{t+1} - z^* \right\|^2 \right] = (1 - \frac{\mu}{(1-\lambda)(2n+1)})^{t+1} \left\| z^0 - z^* \right\|^2 + \frac{\lambda f(w^*)^2}{\mu(n+1)},
\] where in the last equality we used that \( \gamma = \frac{1}{(1-\lambda)(2n+1)}. \)

**F.3 Proof of Theorem F.3**

**Theorem F.3.** Let \( h_t(z) \) be \( \mu \)-strongly star–convex along the iterates \( z^t \) and around \( z^* := (w^*, \alpha^*, \tau^*) \). Let \( \epsilon > 0 \). If we use an iteration dependent stepsize in Algorithm 2, given by
\[
\gamma_t = \begin{cases} 
\frac{1}{(1-\lambda)(2n+1)} & \text{if } t \leq 2(2n+1) \left\lceil \frac{1-\lambda}{\mu} \right\rceil \\
\frac{(t+1)^2 - t^2}{\mu(t+1)^2} & \text{if } t \geq 2(2n+1) \left\lceil \frac{1-\lambda}{\mu} \right\rceil,
\end{cases}
\]
and if
\[
\lambda \leq \min \left\{ 1 - \frac{2\mu}{2n+1}, 2n+1 \right\},
\]
then
\[
E \left[ \left\| z^t - z^* \right\|^2 \right] \leq \frac{(1-\lambda)\lambda f(w^*)^2}{\mu^2 \epsilon^2 t^2} \frac{16}{t} + \frac{4(2n+1)^2}{e^2 t^2} \frac{1}{\mu} \left( 1 - \frac{1-\lambda}{\mu} \right)^2 \left\| z^0 - z^* \right\|^2.
\]

**Proof.** Following the proof of Theorem 5.3 up to (117), we have that for \( \gamma \leq \frac{1}{(1-\lambda)(2n+1)} \) and
\[
h_t(z^*) = \frac{\lambda f(w^*)^2}{2(n+1)}
\]
that
\[
E_t \left[ \left\| z^{t+1} - z^* \right\|^2 \right] \leq (1 - \gamma \mu) \left\| z^t - z^* \right\|^2 + 2\gamma^2 (1-\lambda)(2n+1) \frac{\lambda f(w^*)^2}{2(n+1)}
\]
\[
\leq (1 - \gamma \mu) \left\| z^t - z^* \right\|^2 + 4\gamma^2 (1-\lambda) \lambda f(w^*)^2.
\]
Taking expectation and using the abbreviations
\[
r_t := E \left[ \left\| z^{t} - z^* \right\|^2 \right] \quad \text{and} \quad \sigma^2 := 2(1-\lambda) \lambda f(w^*)^2,
\]
gives that
\[
r_{t+1} \leq (1 - \gamma \mu)r_t + 2\gamma^2 \sigma^2.
\]
With this notation, this is now identical to the setting of Theorem 3.2 in [Gow+19] Using the notation of Theorem 3.2 in [Gow+19], we have that \( 2\mathcal{L} = (1-\lambda)(2n+1) \) and consequently
\[
\mathcal{K} = \frac{\lambda}{\mu} = \frac{1}{2}(2n+1) \left\lceil \frac{1-\lambda}{\mu} \right\rceil. \]
As a result of Theorem 3.2 in [Gow+19], we have that
\[
r_t \leq \frac{\sigma^2}{\mu^2 t} + \frac{16(\mathcal{K})^2}{e^2 t^2} r_0.
\]
With this stepsize, the resulting update is given by
\[ \gamma \]

We would like to choose \( \gamma \) such that the following equation holds
\[ \lambda \leq 1 - \frac{2\mu}{2n + 1}. \]

\[ \Box \]

G Convergence of SP Through Star Convexity with \( \gamma = 1 \)

For completeness, we present yet another viewpoint of the SP method that is closely related to Polyak’s original motivation. We also prove convergence of SP with a large stepsize of \( \gamma = 1 \). This complements both our convergence result for the SP method in Corollary 5.6, which holds for \( \gamma < 1 \).

Consider the stochastic gradient method given by
\[ w^{t+1} = w^t - \gamma^t \nabla f_i(w^t), \]  
(157)

where \( \gamma^t > 0 \) is a step size which we will now choose. Expanding the squares we have that
\[ \|w^{t+1} - w^*\|^2 = \|w^t - w^*\|^2 - 2\gamma^t \nabla f_i(w^t), w^t - w^*\) + \(\gamma^t \nabla f_i(w^t)\|^2. \]  
(158)

We would like to choose \( \gamma^t \) so as to give the best possible upper bound in the above. Unfortunately we cannot directly minimize the above in \( \gamma^t \) since we do not know \( w^* \). However, if each loss function is star convex, then there is hope.

**Assumption G.1.** We say that \( f_i \) is star convex if
\[ f_i(w^*) \geq f_i(w) + \langle \nabla f_i(w), w^* - w \rangle, \quad \text{for } i = 1, \ldots, n. \]  
(159)

Using star convexity (159) in the above gives
\[ \|w^{t+1} - w^*\|^2 \leq \|w^t - w^*\|^2 - 2\gamma^t (f_i(w^t) - f(w^*)) + (\gamma^t)^2 \|\nabla f_i(w^t)\|^2. \]  
(160)

We can now minimize the righthand side in \( \gamma^t \), which gives exactly the Polyak stepsize
\[ \gamma^t = \frac{f_i(w^t) - f_i(w^*)}{\|\nabla f_i(w^t)\|^2}. \]  
(161)

With this stepsize, the resulting update is given by
\[ w^{t+1} = w^t - \frac{f_i(w^t) - f_i(w^*)}{\|\nabla f_i(w^t)\|^2} \nabla f_i(w^t). \]  
(162)

The iterative scheme (162) is now completely scale invariant. That is, the iterates are invariant to replacing \( f_i(w) \) by \( c_i f_i(w) \) where \( c_i > 0 \).

**Theorem G.2** (Convergence for \( \gamma^t \equiv 1 \)). Let Assumptions G.1 and 5.4 hold. The iterates (9) satisfy
\[ \|w^{t+1} - w^*\|^2 \leq \|w^t - w^*\|^2 - \frac{f_i(w^t)^2}{\|\nabla f_i(w^t)\|^2}. \]  
(163)

Furthermore, if we assume that there exists \( L > 0 \) such that the following expected smoothness bound holds
\[ \mathbb{E}_{x \sim p_i} \left[ \frac{f_i(w^t)^2}{\|\nabla f_i(w^t)\|^2} \right] \geq \frac{2}{L} f(w^t), \]  
(164)

then
\[ \min_{j=1,\ldots,k-1} \mathbb{E} [f(w^j) - f^*] \leq \frac{L}{2k} \|w^0 - w^*\|^2. \]  
(165)

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This smoothness assumption (168) is unusual in the literature since on the left hand side we have Taking expectation and using (164) in (166) gives
\[ \sum_{t=0}^{k-1} \frac{f_{i_t}(w^t)^2}{\|\nabla f_{i_t}(w^t)\|^2} \leq \|w^0 - w^*\|^2. \] (166)

Proof. Substituting (161) into (160) gives (163). Summing up both sides of (163) from \( t = 0, \ldots, k-1 \), using telescopic cancellation and re-arranging gives
\[ \sum_{t=0}^{k-1} \frac{f_{i_t}(w^t)^2}{\|\nabla f_{i_t}(w^t)\|^2} \leq \|w^0 - w^*\|^2. \] (167)

Consequently
\[ \min_{j=1,\ldots,k-1} \mathbb{E} \left[ f(w^j) - f^* \right] \leq \sum_{t=0}^{k-1} \mathbb{E} \left[ f(w^t) - f^* \right] \leq \frac{1}{k} \mathcal{L} \|w^0 - w^*\|^2. \]

Note that the Expected Smoothness bound in (164) has been proven to be a consequence of standard smoothness and interpolation in Lemma D.1.

## H Convergence using \( t \)-Smoothness

Here we present an alternative convergence theorem for our variant of online SGD (37) that is based on a smoothness type assumption.

**Theorem H.1 (\( t \)-Smoothness).** If there exists \( L > 0 \) such that
\[ \mathbb{E} \left[ h_{t+1}(z^{t+1}) \right] \leq \mathbb{E} \left[ h_t(z^t) \right] + \mathbb{E} \left[ \langle \nabla h_t(z^t), z^{t+1} - z^t \rangle \right] + \frac{L}{2} \mathbb{E} \left[ \|z^{t+1} - z^t\|^2 \right], \] (168)
and if \( \gamma \leq \frac{1}{\sqrt{LG}} \) then
\[ \min_{t=0,\ldots,T-1} \mathbb{E} \left[ \|\nabla h_t(z^t)\|^2 \right] \leq 3 \sqrt{\frac{LG}{T}} h_0(z^0). \] (169)

Furthermore,
\[ T > \frac{9L^2G}{\epsilon^2} h_0(z^0)^2 \implies \min_{t=0,\ldots,T-1} \mathbb{E} \left[ \|\nabla h_t(z^t)\|^2 \right] < \epsilon. \] (170)

If (168) holds and there exists \( \mu_{PL} > 0 \) such that
\[ \|\nabla h_t(z^t)\|^2 \geq 2\mu_{PL} h_t(z^t), \] (171)
and if the stepsize satisfies
\[ \gamma \leq \frac{\mu_{PL}}{LG}, \] (172)
then the iterates converge linearly according to
\[ \mathbb{E} \left[ h_{t+1}(z^{t+1}) \right] \leq \left( 1 - \mu_{PL} \gamma \right) \mathbb{E} \left[ h_t(z^t) \right]. \] (173)

This smoothness assumption (168) is unusual in the literature since on the left hand side we have \( h_{t+1} \), the auxiliary function at time \( t+1 \), and on the right we have \( h_t \). In Appendix (D) we show that (168) does holds for the SP auxiliary functions when the underlying loss functions satisfy a property that is similar to self-concordancy. But first, the proof.

**Proof.** This first part of the proof is adapted from [KR20]. The second part of the proof that uses the PL condition is based on [GSL21]. From (168) and (37) we have
\[ \mathbb{E} \left[ h_{t+1}(z^{t+1}) \right] \leq \mathbb{E} \left[ h_t(z^t) \right] - \gamma \mathbb{E} \left[ \langle \nabla h_t(z^t), \nabla h_{i_t}(z^t) \rangle \right] + \frac{L\gamma^2}{2} \mathbb{E} \left[ \|\nabla h_{i_t}(z^t)\|^2 \right]. \] (174)
By the law of total expectation we have that
\[ E \left[ \langle \nabla h_t(z^t), \nabla h_{i_t, t}(z^t) \rangle \right] = E \left[ E_t \left[ \langle \nabla h_t(z^t), \nabla h_{i_t, t}(z^t) \rangle \right] \right] \]
and since \( E_t \left[ \nabla h_{i_t, t}(z^t) \right] = \nabla h_t(z^t) \) we have that
\[ E \left[ h_{t+1}(z^{t+1}) \right] \leq E \left[ h_t(z^t) \right] - \gamma E \left[ \| \nabla h_t(z^t) \|^2 \right] + \frac{L \gamma^2}{2} E \left[ \| \nabla h_{i_t, t}(z^t) \|^2 \right] \leq E \left[ h_t(z^t) \right] - \gamma E \left[ \| \nabla h_t(z^t) \|^2 \right] + L \gamma^2 E \left[ h_t(z^t) \right]. \tag{175} \]

Re-arranging (175) we have that
\[ \gamma E \left[ \| \nabla h_t(z^t) \|^2 \right] \leq (1 + L \gamma^2) E \left[ h_t(z^t) \right] - E \left[ h_{t+1}(z^{t+1}) \right]. \tag{176} \]

We now introduce a sequence of weights \( w_{-1}, w_1, w_2, \ldots, w_k \) based on a technique developed by [St19]. Let \( w_{-1} > 0 \) be arbitrary and fixed. We define the remaining weight recurrently
\[ w_t = \frac{w_{t-1}}{1 + L \gamma^2} = \frac{w_{-1}}{(1 + L \gamma^2)^{t+1}}. \]

Multiplying (176) by \( w_t / \gamma \) gives
\[ w_t E \left[ \| \nabla h_t(z^t) \|^2 \right] \leq \frac{w_t}{\gamma} (1 + L \gamma^2) E \left[ h_t(z^t) \right] - \frac{w_t}{\gamma} E \left[ h_{t+1}(z^{t+1}) \right] = \frac{w_{t-1}}{\gamma} E \left[ h_t(z^t) \right] - \frac{w_t}{\gamma} E \left[ h_{t+1}(z^{t+1}) \right]. \]

Summing up both sides for \( t = 0, \ldots, T - 1 \) gives
\[ \sum_{i=0}^{T-1} \frac{w_t}{\gamma} E \left[ \| \nabla h_t(z^t) \|^2 \right] \leq \frac{w_{-1}}{\gamma} h_0(z^0) - \frac{w_{T-1}}{\gamma} E \left[ h_T(z^T) \right] \leq \frac{w_{-1}}{\gamma} h_0(z^0). \tag{177} \]

Now dividing both sides by \( W_T := \sum_{t=0}^{T-1} w_t \) we have that
\[ \min_{t=0, \ldots, T-1} E \left[ \| \nabla h_t(z^t) \|^2 \right] \leq \sum_{t=0}^{T-1} \frac{w_t}{W_T} E \left[ \| \nabla h_t(z^t) \|^2 \right] \leq \frac{w_{-1}}{\gamma W_T} h_0(z^0). \tag{178} \]

To conclude the proof, note that
\[ W_T = \sum_{t=0}^{T-1} w_t \geq T w_{T-1} = \frac{T w_{-1}}{(1 + L \gamma^2)^T}, \tag{179} \]
where we used a standard integral test. Inserting (179) into (178) gives
\[ \min_{t=0, \ldots, T-1} E \left[ \| \nabla h_t(z^t) \|^2 \right] \leq \frac{(1 + L \gamma^2)^T}{\gamma T} h_0(z^0). \tag{180} \]

Now let \( \gamma \leq 1 / \sqrt{LGT} \). Using that \( 1 + a \leq e^a \) we have that
\[ (1 + L \gamma^2)^T \leq e^{LGT \gamma^2} \leq e^1 \leq 3. \]

Using this in (180) gives
\[ \min_{t=0, \ldots, T-1} E \left[ \| \nabla h_t(z^t) \|^2 \right] \leq 3 \sqrt{\frac{L \gamma T}{T}} h_0(z^0). \tag{181} \]

We can now ensure that the left hand side is less than a given \( \epsilon > 0 \) so long as
\[ T > \frac{9L \gamma}{\epsilon^2} h_0(z^0)^2 \sim O \left( \frac{1}{\epsilon^2} \right). \]

Finally, if we assume the PL condition (171) holds, then from (175) we have that
\[ E \left[ h_{t+1}(z^{t+1}) \right] \leq h_t(z^t) - \gamma E \left[ \| \nabla h_t(z^t) \|^2 \right] + L \gamma^2 E \left[ h_t(z^t) \right] \leq (1 - \gamma(2\mu_L - L \gamma)) E \left[ h_t(z^t) \right]. \tag{172} \]
Restricting \( \gamma \) according to (172), taking full expectation and unrolling the recurrence gives (173). \( \square \)
H.1 Sufficient Conditions for SPS

Here we give sufficient conditions on the $f_i$ functions for Theorem H.1 to hold where $h_t$ is given by (120). In particular, we need to establish when the auxiliary function $h_t$ (11) is $t$-smooth.

**Lemma H.2.** If there exists $L_2 > 0$ and $L_3 > 0$ such that
\[
\frac{|f_i(w) - f_i(w^*)|}{\|\nabla f_i(w)\|^2} \|\nabla^2 f_i(w)\| \leq L_2, \tag{183}
\]
\[
\frac{(f_i(w) - f_i(w^*))^2}{\|\nabla f_i(w)\|^3} \|\nabla^3 f_i(w)\| \leq L_3 \tag{184}
\]
then $h_t$ is $L$-smooth with $L = (1 + L_2)(1 + 2L_2) + L_3 + \frac{NL_2}{\gamma}$.

Note that $L_2$ and $L_3$ are independent of the scaling of $f_i$. That is, if we multiply $f_i$ by a constant, it has no affect on the bounds in (184).

**Proof.** The proof has two steps. 1) We show that if the auxiliary function $\phi(w) := h_w(w)$ is $\mathcal{L}$-smooth then $h_t(w)$ satisfies (168) after which 2) we show that $\phi(w)$ is $\mathcal{L}$-smooth.

**Part I.** If $\phi(w) := h_w(w)$ is smooth then $h_t(w)$ satisfies (168).

Note that $\phi(w^{t+1}) = h_t(w^{t+1})$, and that if $\phi(w)$ is $\mathcal{L}$-smooth then
\[
h_{t+1}(w^{t+1}) = \phi(w^{t+1})
\]
\[
\leq \phi(w^t) + \langle \nabla \phi(w^t), w^{t+1} - w^t \rangle + \frac{\mathcal{L}}{2} \|w^{t+1} - w^t\|^2
\]
\[
= h_t(w^t) + \langle \nabla h_t(w^t) |_{y = y^t}, w^{t+1} - w^t \rangle + \langle \nabla_y h_y(w^t) |_{y = y^t}, w^{t+1} - w^t \rangle
\]
\[
+ \frac{\mathcal{L}}{2} \|w^{t+1} - w^t\|^2. \tag{185}
\]

Consequently if we could show that there exists $C > 0$ such that
\[
\langle \nabla_y h_y(w^t) |_{y = y^t}, w^{t+1} - w^t \rangle \leq \frac{C}{2} \|w^{t+1} - w^t\|^2,
\]
then we could establish that (168) holds with $L = \mathcal{L} + C$. Let us show that this $C > 0$ exists. First note that
\[
\nabla_y h_y(w^t) = \nabla_y \frac{1}{n} \sum_{i=1}^n \frac{1}{2} \frac{(f_i(w) - f_i(w^*))^2}{\|\nabla f_i(y)\|^2} = -\frac{1}{n} \sum_{i=1}^n \frac{1}{2} \frac{(f_i(w) - f_i(w^*))^2}{\|\nabla f_i(y)\|^4} \nabla^2 f_i(y) \nabla f_i(y). \tag{186}
\]

From (9) we have that
\[
E_t [\|w^{t+1} - w^t\|] = \frac{\gamma}{n} \sum_{i=1}^n \frac{|f_i(w^t) - f_i(w^*)|}{\|\nabla f_i(w^t)\|}. \tag{187}
\]

Now we can bound the gradient given in (186) as follows
\[
\|\nabla_y h_y(w^t) |_{y = y^t}\| \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{2} \frac{(f_i(w) - f_i(w^*))^2}{\|\nabla f_i(w^t)\|^4} \|\nabla^2 f_i(w^t) \nabla f_i(w^t)\|
\]
\[
\leq \frac{1}{n} \sum_{i=1}^n \frac{1}{2} \frac{|f_i(w) - f_i(w^*)|}{\|\nabla f_i(w^t)\|^2} \|\nabla^2 f_i(w^t)\| \frac{|f_i(w^t) - f_i(w^*)|}{\|\nabla f_i(w^t)\|}
\]
\[
\leq \frac{1}{n} \sum_{i=1}^n \frac{1}{2} \frac{|f_i(w) - f_i(w^*)|}{\|\nabla f_i(w^t)\|^2} \|\nabla^2 f_i(w^t)\| \sum_{j=1}^n \frac{\gamma |f_j(w^t) - f_j(w^*)|}{n} \|\nabla f_j(w^t)\|
\]
\[
= \frac{1}{\gamma} \sum_{i=1}^n \frac{1}{2} \frac{|f_i(w) - f_i(w^*)|}{\|\nabla f_i(w^t)\|^2} \|\nabla^2 f_i(w^t)\| E_t [\|w^{t+1} - w^t\|]. \tag{188}
\]
Consequently

$$
\mathbb{E}_t \left[ \langle \nabla_y h_y(w^t) \rangle_{y=w^t, w^{t+1} - w^t} \right] \leq \frac{1}{\gamma} \sum_{i=1}^n \frac{1}{2} \left| \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w^*)\|^2} \right| \|\nabla^2 f_i(w^t)\| \mathbb{E}_t \left[ \|w^{t+1} - w^t\|^2 \right] \\
\leq \frac{n L_2}{2 \gamma} \mathbb{E}_t \left[ \|w^{t+1} - w^t\|^2 \right].
$$

Thus from the above and (185) we have that

$$
h_{t+1}(w^{t+1}) \leq h_t(w^t) + \langle \nabla h_t(w^t), w^{t+1} - w^t \rangle + \frac{1}{2} \left( \frac{n L_2}{\gamma} + \mathcal{L} \right) \|w^{t+1} - w^t\|^2 (189)
$$

**Part II.** Verifying that $\phi(w) = h_w(w)$ is an $\mathcal{L}$-smooth function.

We will first verify that $\phi_1(w) := \frac{1}{2} \left( \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \right)^2$ is a smooth function, then use that $\phi(w) = \frac{1}{n} \sum_{i=1}^n \phi_i(w)$. To do this, we will examine the Hessian of $\phi_i(w)$ and determine that it is bounded.

$$
\nabla \phi_i(w) = \nabla^2 \frac{1}{2} \left( \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w^*)\|^2} \right)^2 \\
= \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \nabla f_i(w) - \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \nabla^2 f_i(w) \nabla f_i(w) \\
= \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \left( \nabla f_i(w) - \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \nabla^2 f_i(w) \nabla f_i(w) \right). (190)
$$

The second derivative has two terms

$$
\phi_i''(w) = \nabla^2 \frac{1}{2} \left( \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \right)^2 = I + II
$$

where

$$
I = \frac{1}{\|\nabla f_i(w)\|^2} \left( \nabla f_i(w) - \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \nabla^2 f_i(w) \nabla f_i(w) \right) \left( \nabla f_i(w) - \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \nabla^2 f_i(w) \nabla f_i(w) \right)^\top \\
= \frac{1}{\|\nabla f_i(w)\|^2} \left( I - \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \nabla^2 f_i(w) \right) \nabla f_i(w) \nabla f_i(w) ^\top \left( I - \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \nabla^2 f_i(w) \right) (191)
$$

39
and
\[
II = \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \left( \nabla^2 f_i(w) - \frac{1}{\|\nabla f_i(w)\|^2} \nabla^2 f_i(w) \nabla f_i(w) \nabla f_i(w)^\top \right) \\
+ f_i(w) - f_i(w^*) \frac{\nabla^2 f_i(w) \nabla f_i(w) \nabla f_i(w)^\top}{\|\nabla f_i(w)\|^2} - f_i(w) - f_i(w^*) \left( \nabla^2 f_i(w)^2 + \nabla^3 f_i(w) \nabla f_i(w) \right) \\
= \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \left( \nabla^2 f_i(w) \left( I - \frac{1}{\|\nabla f_i(w)\|^2} \nabla f_i(w) \nabla f_i(w)^\top \right) \right) \\
+ f_i(w) - f_i(w^*) \frac{\nabla^2 f_i(w) \left( \frac{1}{\|\nabla f_i(w)\|^2} \nabla f_i(w) \nabla f_i(w)^\top - I \right) \nabla^2 f_i(w)}{\|\nabla f_i(w)\|^2} \\
- \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \nabla^3 f_i(w) \nabla f_i(w) \\
= \frac{(f_i(w) - f_i(w^*))^2}{\|\nabla f_i(w)\|^4} \nabla^3 f_i(w) \nabla f_i(w) \\
(192)
\]

For \( I \) we have from (183) that
\[
\|I\| \leq \left\| I - \frac{f_i(w) - f_i(w^*)}{\|\nabla f_i(w)\|^2} \nabla^2 f_i(w) \right\| \leq 1 + L_2. \tag{193}
\]

Furthermore, note that
\[
\left\| \left( \frac{1}{\|\nabla f_i(w)\|^2} \nabla f_i(w) \nabla f_i(w)^\top - I \right) \right\| = 1 \tag{194}
\]

since \( \frac{1}{\|\nabla f_i(w)\|^2} \nabla f_i(w) \nabla f_i(w)^\top \) is a projection matrix onto Range \( \nabla f_i(w) \).

Thus finally we have that
\[
\|I + II\| \leq \|I\| + \|II\| \leq (1 + L_2)^2 + L_2(1 + L_2) + L_3 \tag{195}
\]

Using Lemma [H.2] and Theorem [H.1] we can establish the following convergence theorem for the SP method.

**Theorem H.3 (\( \ell \)-Smoothness).** Suppose that there exists \( L_2, L_3 \) such that (183) and (184) holds. Consider the iterates \( w^\ell \) of SPS [9] and let \( h_\ell \) be given by (120). If
\[
\gamma \leq \frac{nL_2}{2\ell} \left( \sqrt{1 + \frac{4\ell}{T(nL_2)^2}} - 1 \right) \tag{196}
\]

where \( \ell := (1 + L_2)(1 + 2L_2) + L_3 \) then
\[
\min_{t=0,\ldots,T-1} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} f_i(w^t) - f_i(w^*) \|\nabla f_i(w^t)\|^2 \nabla f_i(w^t) \right\|^2 \right] \leq 3\sqrt{\frac{L}{T} h_0(z^0)} \tag{197}
\]
Proof. Consider the statement of Theorem [H.1] First note that $G = 1$ from Lemma [2.1] According to Lemma [H.2] we have that $h_t$ satisfies (168) with

$$L = (1 + L_2)(1 + 2L_2) + L_3 + \frac{nL_2}{\gamma}.$$ 

Let $\ell := (1 + L_2)(1 + 2L_2) + L_3$ Furthermore from from Theorem [H.1] we need $\gamma < \frac{1}{\sqrt{T}}$ in other words

$$\gamma < \frac{1}{\sqrt{T}} \left( \frac{\ell + \frac{nL_2}{\gamma}}{\gamma} \right) \iff \gamma < \sqrt{T} \frac{\sqrt{\ell} + nL_2}{\gamma} \iff \sqrt{T} \gamma + nL_2 < 1 \iff \gamma^2 T \ell + \gamma T n L_2 - 1 < 0.$$ 

The roots of the above quadratic are given by

$$\gamma = \frac{-T n L_2}{2T \ell} \pm \sqrt{\frac{(T n L_2)^2 + 4T \ell}{2T \ell}}$$

Thus

$$\gamma < \frac{\sqrt{(T n L_2)^2 + 4T \ell} - T n L_2}{2T \ell} \iff \gamma \leq \frac{\sqrt{(T n L_2)^2 + 4T \ell} - T n L_2}{2T \ell}$$

$$\iff \gamma \leq \frac{\sqrt{T} (nL_2)^2 + 4T - \sqrt{T} n L_2}{2T \ell} \iff \gamma \leq \frac{\sqrt{T} n L_2 \sqrt{1 + 4 T (n L_2)^2} - 1}{2T \ell},$$

which after cancellation is equal to (196)

\[\square\]

H.1.1 Examples of scaled smoothness

Now we provide a class of functions for which our sufficient conditions given in Lemma [H.2] hold.

**Example H.4** (Monomials). Let $\phi_i(t) = a_i (t - b_i)^{2r}$ where $r, a_i > 0$ and $b_i \in \mathbb{R}$ for $i = 1, \ldots, n$. It follows that

$$\frac{\| \phi_i(t) \|}{\| \phi_i'(t) \|^2} \cdot \| \phi_i''(t) \| \leq 1,$$

(198)

$$\frac{\phi_i(t)^2}{\| \phi_i'(t) \|^3} \cdot \| \phi_i'''(t) \| \leq 1 + \frac{1}{2r^2}.$$ 

(199)

**Proof.** Verifying the conditions in Lemma [H.2] we have that

$$\frac{\| \phi_i(t) \|}{\| \phi_i'(t) \|^2} \cdot \| \phi_i''(t) \| \leq \frac{2r (2r - 1)(t - b_i)^{2r}(t - b_i)^{2r-2}}{4r^2(t - b_i)^{4r-2}} = \frac{(4r^2 - 2r)}{4r^2} = 1 - \frac{1}{2r} \leq 1.$$ 

Furthermore

$$\frac{\phi_i(t)^2}{\| \phi_i'(t) \|^3} \cdot \| \phi_i'''(t) \| \leq \frac{2r (2r - 1)(2r - 2)(t - b_i)^{2r}(t - b_i)^{2r-3}}{8r^3(t - b_i)^{6r-3}} = \frac{2r (2r - 1)(2r - 2)}{8r^3} = \frac{(2r - 1)(2r - 2)}{4r^2} \leq 1 + \frac{1}{2r^2}.$$ 

\[\square\]

Note that for $r < 1$ we have that $\phi_i(t) = a_i (t - b_i)^{2r}$ is a non-convex function. In the following example we generalize the above example to a non-convex generalized linear model.
**Example H.5** (Generalized Linear model). Let \(x_1, \ldots, x_n \in \mathbb{R}^d\) be \(n\) given data points. Let \(f_i(w) = \phi_i(x_i^T w)\) where \(\phi_i : \mathbb{R} \rightarrow \mathbb{R}\) is a \(C^3\) real valued loss function. Furthermore, suppose that there exists a hyperplane that separates the datapoints. In other words, the problem is over-parametrized so that the solution \(w^* \in \mathbb{R}^d\) is such that \(\phi_i(x_i^T w^*) = 0\). It follows that

\[
\frac{|f_i(w) - f_i(w^*)|}{\|\nabla f_i(w)\|^2} \|\nabla^2 f_i(w)\| \leq \max_t \frac{\phi_i(t)\phi_i''(t)}{\phi_i'(t)^2},
\]

(200)

\[
\frac{(f_i(w) - f_i(w^*))^2}{\|\nabla f_i(w)\|^3} \|\nabla^3 f_i(w)\| \leq \max_t \frac{\phi_i(t)^2\phi_i'''(t)}{\phi_i'(t)^3}.
\]

(201)

Thus if \(\phi_i(t) = a_i(t - b_i)^{2r}\) where \(r > 0\) then according to Example H.4 we have that

\[
\frac{|f_i(w) - f_i(w^*)|}{\|\nabla f_i(w)\|^2} \|\nabla^2 f_i(w)\| \leq 1,
\]

(202)

\[
\frac{(f_i(w) - f_i(w^*))^2}{\|\nabla f_i(w)\|^3} \|\nabla^3 f_i(w)\| \leq 1 + \frac{1}{2r^2}.
\]

(203)

Consequently Theorem H.1 holds with \(L_2 = 1\) and \(L_3 = 1 + \frac{1}{2r^2}\).

**Proof.** Indeed since

\[
\begin{align*}
\nabla f_i(w) &= x_i\phi'(x_i^T w) \\
\nabla^2 f_i(w) &= x_i x_i^T \phi''(x_i^T w) \\
\nabla^3 f_i(w) &= x_i \otimes x_i \otimes x_i \phi'''(x_i^T w)
\end{align*}
\]

Consequently

\[
\frac{|f_i(w) - f_i(w^*)|}{\|\nabla f_i(w)\|^2} \|\nabla^2 f_i(w)\| \leq \frac{\phi(x_i^T w)}{|\phi'(x_i^T w)|^2} |\phi''(x_i^T w)|,
\]

\[
\frac{(f_i(w) - f_i(w^*))^2}{\|\nabla f_i(w)\|^3} \|\nabla^3 f_i(w)\| \leq \frac{\phi(x_i^T w)^2}{|\phi'(x_i^T w)|^3} |\phi'''(x_i^T w)|
\]

The result now follows by taking the max over the arguments on the left hand side. \(\square\)

### H.2 Sufficient conditions on TAPS

Here we explore sufficient conditions for the smoothness assumption in Theorem H.1 to hold for the TAPS method given in Algorithm 1.

First we provide a sufficient condition for the \(t\)–smoothness assumption in Theorem H.1 to hold.

**Lemma H.6.** If \(f_i\) is \(L_2\)–scaled smooth, that is

\[
\frac{|f_i(w^t) - \alpha_i^t|}{1 + \|\nabla f_i(w^t)\|^2} \|\nabla^2 f_i(w^t)\| \leq L_2,
\]

(204)

\[
\frac{(f_i(w^t) - \alpha_i^t)^2}{(\|\nabla f_i(w^t)\|^2 + 1)^2} \|\nabla^3 f_i(w^t)\| \nabla f_i(w)\| \leq L_3
\]

(205)

then \(h_t\) is \(L\)–\(t\)–smooth with \(L = (3 + L_2)(1 + 2L_2) + 1 + L_3 + \frac{nL_2}{\gamma}\).

**Proof.** The proof has two step 1) we show that if the auxiliary function \(\phi(z) := h_z(z)\) is \(L\)–smooth then \(h_z(z)\) satisfies (168) and then 2) show that \(\phi(z)\) is smooth.

**Part I.** If \(\phi(z) := h_z(z)\) is smooth then \(h_z(z)\) satisfies (168).
Note that $\phi(z^t) = h_t(z^t)$, and that if $\phi(z)$ is $\mathcal{L}$-smooth then
\[
h_{t+1}(z^{t+1}) = \phi(z^{t+1})
\leq \phi(z^t) + \langle \nabla \phi(z^t), z^{t+1} - z^t \rangle + \frac{\mathcal{L}}{2} \|z^{t+1} - z^t\|^2
= h_t(z^t) + \langle \nabla_z h_t(z)|_{z=z^t}, z^{t+1} - z^t \rangle + \frac{\mathcal{L}}{2} \|z^{t+1} - z^t\|^2 + \langle \nabla yh_y(z^t)|_{y=w^t}, w^{t+1} - w^t \rangle.
\] (206)

Using the SGD interpretation on TAPS given in Section 3.1 we have that $z^{t+1} = z^t - \gamma \nabla h_t(z^t)$ where $\mathbb{E}[\nabla h_t(z^t)] = \nabla h_t(z^t)$, thus the above gives
\[
\mathbb{E}_t[h_{t+1}(z^{t+1})] \leq h_t(z^t) - \gamma \|\nabla h_t(z^t)\|^2 + \frac{\gamma^2 \mathcal{L}}{2} \mathbb{E}_t[\|\nabla h_t(z^t)\|^2] + \mathbb{E}_t[\langle \nabla yh_y(z^t)|_{y=w^t}, w^{t+1} - w^t \rangle].
\] (207)

Consequently if we could show that there exists $C > 0$ such that
\[
\langle \nabla yh_y(z^t)|_{y=w^t}, w^{t+1} - w^t \rangle \leq \frac{C}{2} \|z^{t+1} - z^t\|^2,
\]
then we could establish that $168$ holds with $L = \mathcal{L} + C$. For this, we have that
\[
\nabla yh_y(z^t)|_{y=w^t} = \nabla_y \frac{1}{n+1} \sum_{i=1}^n \frac{1}{2} \left( f_i(w^t) - \alpha_i \right)^2 = \frac{1}{n+1} \sum_{i=1}^n \frac{1}{2} \left( f_i(w^t) - \alpha_i \right)^2 \nabla^2 f_i(w^t) \nabla f_i(w^t).
\] (208)

Furthermore, from the SGD viewpoint in Section 3.1 and (24) we have that
\[
\begin{align*}
\mathbb{E}_t[\|w^{t+1} - w^t\|] &= \frac{\gamma}{n} \sum_{i=1}^n \frac{|f_i(w^t) - \alpha_i|}{\|\nabla f_i(w^t)\|^2 + 1} \|\nabla f_i(w^t)\| \quad \text{(209)} \\
\mathbb{E}_t[\|z^{t+1} - z^t\|^2] &= \mathbb{E}_t[\|w^{t+1} - w^t\|^2] + \mathbb{E}_t[\|\alpha^{t+1} - \alpha^t\|^2]
\end{align*}
\] (209) + 25
\[
= \frac{\gamma^2}{n+1} \sum_{i=1}^n \frac{(f_i(w^t) - \alpha_i)^2}{(\|\nabla f_i(w^t)\|^2 + 1)^2} \|\nabla f_i(w^t)\|^2
+ \frac{\gamma^2}{n+1} \left( \sum_{i=1}^n \frac{(f_i(w^t) - \alpha_i)^2}{(\|\nabla f_i(w^t)\|^2 + 1)^2} + (\bar{x}^t - \tau)^2 \right)
= \frac{\gamma^2}{n+1} \left( \sum_{i=1}^n \frac{(f_i(w^t) - \alpha_i)^2}{(\|\nabla f_i(w^t)\|^2 + 1)^2} + (\bar{x}^t - \tau)^2 \right).
\] (210)

Now we can bound the gradient given in (186) as follows
\[
\|\nabla yh_y(w^t, \alpha^t)|_{y=w^t}\| \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{2} \left( \|\nabla f_i(w^t)\|^2 + 1 \right) \|\nabla^2 f_i(w^t) \nabla f_i(w^t) \| \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{2} \left( \|\nabla f_i(w^t)\|^2 + 1 \right) \|\nabla^2 f_i(w^t) \nabla f_i(w^t) \| \|f_i(w^t) - \alpha_i\| \|\nabla f_i(w^t)\|^2 \| + 1 \leq \frac{1}{\gamma} \sum_{i=1}^n \frac{1}{2} \left( \|\nabla f_i(w^t)\|^2 + 1 \right) \|\nabla^2 f_i(w^t) \nabla f_i(w^t) \| \sum_{j=1}^n \frac{\gamma}{n} \|\nabla f_i(w^t)\|^2 \| \frac{1}{\|\nabla f_i(w^t)\|^2 + 1} \|\nabla f_i(w^t)\| \leq \frac{1}{\gamma} \sum_{i=1}^n \frac{1}{2} \left( \|\nabla f_i(w^t)\|^2 + 1 \right) \|\nabla^2 f_i(w^t) \nabla f_i(w^t) \| \mathbb{E}_t[\|w^{t+1} - w^t\|] \quad \text{(209)}
\] (209) + (211)
\[
\leq \frac{1}{\gamma} \sum_{i=1}^n \frac{1}{2} \left( \|\nabla f_i(w^t)\|^2 + 1 \right) \|\nabla^2 f_i(w^t) \nabla f_i(w^t) \| \mathbb{E}_t[\|w^{t+1} - w^t\|].
\] (211)
Consequently

\[ \mathbb{E}_t \left[ \langle \nabla_y h_y (w^t, \alpha^t) \rangle_{y=w^t} , w^{t+1} - w^t \right] \leq \| \nabla_y h_y (w^t, \alpha^t) \|_{y=w^t} \mathbb{E}_t \left[ \| w^{t+1} - w^t \| \right] \]

\[ \leq \frac{1}{\gamma} \sum_{i=1}^{n} \frac{1}{2} \frac{|f_i(w) - \alpha_i^t|}{\| \nabla f_i(w^t) \|^2 + 1} \| \nabla^2 f_i(w^t) \| \mathbb{E}_t \left[ \| w^{t+1} - w^t \| \right]^2 \]

\[ \leq \frac{nL_2}{2\gamma} \mathbb{E}_t \left[ \| w^{t+1} - w^t \| \right]^2 \]

Jensen’s Ineq.

\[ \leq \frac{nL_2}{2\gamma} \mathbb{E}_t \left[ \| z^{t+1} - z^t \| \right]^2 \]

Thus from the above and (206) we have that

\[ h_{t+1}(z^{t+1}) \leq h_t(z^t) + \langle \nabla h_t(z^t), z^{t+1} - z^t \rangle + \frac{1}{2} \left( \frac{nL_2}{\gamma} + \mathcal{L} \right) \| z^{t+1} - z^t \|^2. \] (212)

**Part II.** Verifying that \( \phi(z) = h_z(z) \) is an \( \mathcal{L} \)-smooth function. To this end, note that

\[ \nabla^2 \phi(w, \alpha) = \begin{bmatrix} \nabla^2_{w, \phi}(w, \alpha) & \nabla^2_{w, \alpha, \phi}(w, \alpha) \\ \nabla^2_{w, \alpha, \phi}(w, \alpha) & \nabla^2_{\alpha, \phi}(w, \alpha) \end{bmatrix}. \] (213)

To show that \( \| \nabla^2 \phi(w, \alpha) \| \) is bounded we will use that

\[ \| \nabla^2 \phi(w, \alpha) \| \leq \| \nabla^2_{w, \phi}(w, \alpha) \| + 2 \| \nabla^2_{w, \alpha, \phi}(w, \alpha) \| + \| \nabla^2_{\alpha, \phi}(w, \alpha) \|, \] (214)

which relies on Lemma B.3 proven in the appendix.

We will first verify that \( \phi_i(w, \alpha_i) := \frac{1}{2} \left( f_i(w) - \alpha_i \right)^2 \) is a smooth function, then use that \( \phi(w, \alpha) = \frac{1}{n} \sum_{i=1}^{n} \phi_i(w, \alpha_i) \). To do this, we will examine the Hessian of \( \phi_i(w, \alpha_i) \) and determine that it is bounded.

\[ \nabla_w \phi_i(w, \alpha_i) = \nabla_w \frac{1}{2} \left( f_i(w) - \alpha_i \right)^2 \]

\[ = \frac{f_i(w) - \alpha_i}{\| \nabla f_i(w) \|^2 + 1} \nabla f_i(w) - \frac{(f_i(w) - \alpha_i)}{\| \nabla f_i(w) \|^2 + 1} \nabla^2 f_i(w) \nabla f_i(w) \]

\[ = \frac{f_i(w) - \alpha_i}{\| \nabla f_i(w) \|^2 + 1} \left( \nabla f_i(w) - \frac{f_i(w) - \alpha_i}{\| \nabla f_i(w) \|^2 + 1} \nabla^2 f_i(w) \nabla f_i(w) \right). \] (215)

The second derivative has two terms

\[ \nabla^2_w \phi_i(w, \alpha_i) = \nabla^2_w \frac{1}{2} \left( f_i(w) - \alpha_i \right)^2 \| \nabla f_i(w) \|^2 + 1 = I + II \]

where

\[ I = \frac{1}{\| \nabla f_i(w) \|^2 + 1} \left( \nabla f_i(w) - \frac{f_i(w) - \alpha_i}{\| \nabla f_i(w) \|^2 + 1} \nabla^2 f_i(w) \nabla f_i(w) \right) \left( \nabla f_i(w) - \frac{f_i(w) - \alpha_i}{\| \nabla f_i(w) \|^2 + 1} \nabla^2 f_i(w) \nabla f_i(w) \right) \]

\[ = \frac{1}{\| \nabla f_i(w) \|^2 + 1} \left( I - \frac{f_i(w) - \alpha_i}{\| \nabla f_i(w) \|^2 + 1} \nabla^2 f_i(w) \right) \nabla f_i(w) \nabla f_i(w) \left( I - \frac{f_i(w) - \alpha_i}{\| \nabla f_i(w) \|^2 + 1} \nabla^2 f_i(w) \right) \] (216)

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Thus we have that
\[ \nabla f_i(w) = f_i(w) - \alpha_i \frac{\nabla^2 f_i(w) - \frac{1}{\|\nabla f_i(w)\|^2 + 1} \nabla^2 f_i(w) \nabla f_i(w) \nabla f_i(w)^\top}{\|\nabla f_i(w)\|^2 + 1} \]
\[ \quad + \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} \nabla^2 f_i(w) \nabla f_i(w) \nabla f_i(w)^\top - \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} (\nabla^2 f_i(w) + \nabla^3 f_i(w) \nabla f_i(w)) \]
\[ = \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} \nabla^2 f_i(w) \left( \mathbf{I} - \frac{\nabla f_i(w) \nabla f_i(w)^\top}{\|\nabla f_i(w)\|^2 + 1} \right) \]
\[ \quad + \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} \nabla^2 f_i(w) \left( \frac{1}{\|\nabla f_i(w)\|^2 + 1} \nabla f_i(w) \nabla f_i(w)^\top - \mathbf{I} \right) \nabla^2 f_i(w) \]
\[ \quad - \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} \nabla^3 f_i(w) \nabla f_i(w) \]
\[ = \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} \nabla^2 f_i(w) \left( \mathbf{I} - \frac{\nabla f_i(w) \nabla f_i(w)^\top}{\|\nabla f_i(w)\|^2 + 1} \right) \left( \mathbf{I} - \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} \nabla^2 f_i(w) \right) \]
\[ \quad - \frac{(f_i(w) - \alpha_i)^2}{\|\nabla f_i(w)\|^2 + 1} \nabla^3 f_i(w) \nabla f_i(w) \]
\[ \tag{217} \]

For \( I \) we have that
\[ \| I \| \leq \left\| \mathbf{I} - \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} \nabla^2 f_i(w) \right\| \leq 1 + L_2. \tag{218} \]

Furthermore, note that
\[ \left\| \left( \frac{1}{\|\nabla f_i(w)\|^2 + 1} \nabla f_i(w) \nabla f_i(w)^\top - \mathbf{I} \right) \right\| = 1 \tag{219} \]

since \( \frac{1}{\|\nabla f_i(w)\|^2 + 1} \nabla f_i(w) \nabla f_i(w)^\top \) is a projection matrix onto Range \( \nabla f_i(w) \).

Thus we have that
\[ \| \nabla^2_{w,\phi_i(w,\alpha_i)} \| = \| I + II \| \leq \| I \| + \| II \| \]
\[ \leq (1 + L_2)^2 + L_2(1 + L_2) + L_3 \tag{220} \]

\[ \nabla^2_{\alpha_i,\phi_i(w,\alpha_i)} = \frac{1}{2} \nabla^2_{\alpha_i} \frac{(f_i(w) - \alpha_i)^2}{\|\nabla f_i(w)\|^2 + 1} \]
\[ = -\nabla_{\alpha_i} \frac{(f_i(w) - \alpha_i)}{\|\nabla f_i(w)\|^2 + 1} = \frac{1}{\|\nabla f_i(w)\|^2 + 1} \leq 1. \tag{221} \]

\[ \nabla^2_{\alpha_i,\alpha_i,\phi_i(w,\alpha_i)} \quad \text{and} \]
\[ \nabla^2_{\alpha_i,\alpha_i,\phi_i(w,\alpha_i)} = \frac{1}{\|\nabla f_i(w)\|^2 + 1} \left( \nabla f_i(w) - \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} \nabla^2 f_i(w) \nabla f_i(w) \right) \]
\[ \quad - \frac{1}{\|\nabla f_i(w)\|^2 + 1} \left( \nabla f_i(w) - \frac{f_i(w) - \alpha_i}{\|\nabla f_i(w)\|^2 + 1} \nabla^2 f_i(w) \nabla f_i(w) \right) \]
\[ \quad + \frac{1}{\|\nabla f_i(w)\|^2 + 1} \left( \frac{2}{\|\nabla f_i(w)\|^2 + 1} \nabla^2 f_i(w) \nabla f_i(w) - \nabla f_i(w) \right) \]
\[ \tag{222} \]
I Convex Classification: Additional Experiments

I.1 Grid search and Parameter Sensitivity

To investigate how sensitive MOTAPS is to setting its two parameters $\gamma \in [0, 1]$ and $\gamma_\tau \in [0, 1]$ we did a parameter sweep. We searched over the grid given by $\gamma \in \{0.01, 0.1, 0.4, 0.7, 0.9, 1.0, 1.1\}$ and $\gamma_\tau \in \{0.00001, 0.0001, 0.001, 0.01, 0.1, 0.5, 0.9\}$ and ran MOTAPS for 50 epochs over the data, and recorded the resulting norm of the gradient. See Figures 4 and 5 for the results of the grid search on the datasets mushrooms, phishing, colon-cancer and duke respectively. In Table 1 we resume the results of the parameter search, together with the details of each data set.

Figure 4: colon-cancer $(n, d) = (62, 2001)$ Left: $\sigma = 0.0$. Right: $\sigma = \min_{i=1,...,n} \|x_i\|^2 / n = 2.66$.

Figure 5: Logistic Regression with data set phishing $(n, d) = (11055, 68)$ and regularization. Left: $\sigma = 0.0$ and Right: $\sigma = \min_{i=1,...,n} \|x\|^2 / n$.

Thus

$$\|\nabla^2_{\alpha_i, w} \phi(w, \alpha_i)\| \leq 2L_2 + 1 \quad (223)$$

Finally, by (214) we have that

$$\|\nabla^2 \phi(w, \alpha)\| \leq \frac{1}{n} \sum_{i=1}^{n} \left( \|\nabla^2 \phi_i(w, \alpha_i)\| + 2 \|\nabla^2_{w, \alpha_i} \phi(w, \alpha_i)\| + \|\nabla^2_{\alpha_i} \phi(w, \alpha_i)\| \right) \leq (1 + L_2)^2 + L_2(1 + L_2) + L_3 + 2(1 + 2L_2) + 1 = (3 + L_3)(1 + 2L_2) + 1 + L_3. \quad (224)$$

I Convex Classification: Additional Experiments

I.1 Grid search and Parameter Sensitivity

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$$\gamma \in \{0.01, 0.1, 0.4, 0.7, 0.9, 1.0, 1.1\}$$

and

$$\gamma_\tau \in \{0.00001, 0.0001, 0.001, 0.01, 0.1, 0.5, 0.9\}$$

and ran MOTAPS for 50 epochs over the data, and recorded the resulting norm of the gradient. See Figures 4, 5 and 11a for the results of the grid search on the datasets mushrooms, phishing, colon-cancer and duke respectively. In Table 1 we resume the results of the parameter search, together with the details of each data set.

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Ultimately the determining factor for finding the best parameter was the magnitude of the optimal value $f(w^*)$. Since this quantity is unknown to us a priori, we used the size of the regularization parameter as an proxy. Based on these parameter results we devised the following rule-of-thumb for setting $\gamma$ and $\gamma_\tau$ with

$$\gamma = 1.0/(1 + 0.25\sigma e^\sigma) \quad \text{and} \quad \gamma_\tau = 1 - \gamma$$

(225)

where $\sigma$ is regularization parameter.

### 1.2 Comparing to Variance Reduced Gradient Methods

In Figures 6, 7, 8 and 9 we present further comparisons between SP, TAPS and MOTAPS against SGD, SAG and SVRG. We found that the variance reduced gradients methods were able to better exploit strong convexity, in particular for problems with a large regularization, and problems that were under-parameterized, with the phishing problem in Figure 9 being the most striking example. For problems with moderate regularization, and that were over-parametrized, the MOTAPS performed the best. See for example the left of Figure 7 and Figure 8. When the regularization is very small, and the problem is over-parameterized, thus making interpolation much more likely, the SP converged the fastest. See for example the right of Figure 7 and 8

### 1.3 Momentum variants

We also found that adding momentum to SP and MOTAPS could speed up the methods. To add momentum we used the iterate averaging viewpoint of momentum [SGD21] where by we replace the updates in $w^t$ by a weighted average over past iterates. For TAPS and MOTAPS this is equivalent to
where $\eta = \gamma \left(1 + \frac{\beta}{1 - \beta}\right)$ is the adjusted stepsize. See Figures 10, 11 and 12 for the results of our experiments with momentum as compared to ADAM [KB15]. We found that in regimes of moderate regularization ($\sigma = 1/n$) the MOTAPS method was the fastest among all methods, even faster than TAPS despite not having access to $f^*$, see the left side of Figures 10, 11 and 12. Yep when using moderate regularization, adding on momentum gave no benefit to SP, TAPS, and MOTAPS. Quite the opposite, for momentum $\beta = 0.5$, we see that MOTAPS-0.5, which is the MOTAPS method with momentum and $\beta = 0.5$, hurt the convergence rate of the method.

In the regime of small regularization $\sigma = 1/n$, we found that momentum sped up the convergence of our methods, see the right of Figures 10, 11 and 12. On the under-parameterized problem mushrooms, the gains from momentum were marginal, and the ADAM method was the fastest overall, see the right of Figure 10. On the over-parameterized problem colon-cancer, adding momentum to SP gave a significant boost in convergence speed, see the right of Figure 11. Finally on the most over-parametrized problem duke, adding momentum offered a significant speed-up for MOTAPS, but still the ADAM method was the fastest, see the right of Figure 12.

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8 See Proposition 1.6 in [SGD21] for the details of form of momentum and parameter settings.
Figure 12: Experiments on momentum with duke \((n, d) = (44, 7130)\). Left: \(\sigma = 1/n\) and Right: \(\sigma = 1/n^2\).

Figure 13: Deep learning experiments training loss

J Deep learning experimental setup details

In this section we detail the specific implementation choices for each environment. Across all environments, minibatching was accomplished by treating each minibatch as a single data-point. Since per-datapoint values are tracked across epochs, our training setup used minibatches which contain the same set of points each epoch.

J.1 CIFAR10

We trained for 300 epochs using batch size 256 on 1 GPU. Momentum 0.9 was used for all methods. The pre-activation ResNet used has 58,144,842 parameters. Following standard practice we apply data augmentation of the training data; horizontal flipping, 4 pixel padding followed by random cropping to 32x32 square images.

J.2 SVHN

We trained for 150 epochs on a single GPU, using a batch size of 128. Momentum 0.9 was used for each method. Data augmentations were the same as for our CIFAR10 experiments. The ResNet-1-16 network has a total of 78,042 parameters, and uses the classical, non-preactivation structure.

J.3 IWSLT14

We used a very simple preprocessing pipeline, consisting of the Spacy de_core_news_sm/en_core_web_sm tokenizers and filtering out of sentences longer than 100 tokens to fit without our GPU memory constraints. Training used batch-size 32, across 1 GPU for 25 epochs. Other hyper-parameters include momentum of 0.9, weight decay of 5e-6, and a linear learning rate warmup over the first 5 epochs.