Non-commutative geometry and covariance: from the quantum plane to quantum tensors

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Abstract

Reflection and braid equations for rank two \( q \)-tensors are derived from the covariance properties of quantum vectors by using the \( R \)-matrix formalism.

1 Introduction

Quantum groups may be looked at in various ways. From a mathematical point of view, they may be introduced by making emphasis on their \( q \)-deformed enveloping algebra aspects \([1, 2]\) or by making emphasis in the \( R \)-matrix formalism that describe the deformed group algebra \([3]\). A point of view which is particularly useful in possible physical applications is to look at quantum groups as symmetries of quantum spaces \([4, 5]\). The simplest example of this approach is constituted by the well known quantum plane \( C^2_q \), or associative algebra (a \( q \)-plane is not a manifold) generated by two elements \((x, y) = X \) (a \( q \)-two-vector) subjected to the commutation property \([4]\)

\[
xy = qyx . \tag{1}
\]

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The commutation relation (1) can also be expressed by using the $q$-symplectic metric $\epsilon^q$

$$\epsilon^q = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}, \quad (\epsilon^q)^2 = -I$$

by the equation

$$X^T \epsilon^q X = 0, \quad \epsilon^q_{ij} X_i X_j = 0$$

which reflects that the $q$-symplectic norm of a $q$-two-vector vanishes.

It is also possible to introduce a pair of (odd) variables $(\xi, \eta) = \Omega$ (an odd $q$-two-vector) satisfying

$$\xi \eta = -\frac{1}{q} \eta \xi, \quad \xi^2 = 0 = \eta^2.$$  

If it is required that, after the transformations $X' = TX$, $\Omega' = T\Omega$ the new entities $(x', y')$, $(\xi', \eta')$ satisfy also (1), (4), it is found that the commutation properties of the elements of $T$

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are completely determined. These are the well known relations (the entries of $T$ commute with those of $X$ and $\Omega$)

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad [a, d] = \lambda cb, \quad [b, c] = 0,$$

($\lambda \equiv q - q^{-1}$) which constitute a presentation of the $GL_q(2, C)$ algebra generated by $(a, b, c, d)$. For $q=1$, $(x, y)$ commute and $(\xi, \eta)$ anticommute. In a non-commutative differential calculus this second set of variables are identified with the differentials of $(x, y)$. Here we shall consider the quantum plane (1) only as the representation (co-module) space of the $GL_q(2, C)$ quantum group (8). In terms of the $R$-matrix formalism (3), eqs. (1) and (6) may be written as (see eq. (11) below)

$$R_{12} X_1 X_2 = q X_2 X_1 \iff R_{21}^{-1} X_1 X_2 = q^{-1} X_2 X_1,$$

$$R_{12} T_1 T_2 = T_2 T_1 R_{12} \iff R_{21}^{-1} T_1 T_2 = T_2 T_1 R_{21}^{-1},$$

where the standard notation $T_1 = T \otimes 1$, $T_2 = 1 \otimes T$, $T_{1ij,kl} = T_{ik}\delta_{jl}$, $T_{2ij,kl} = \delta_{ik}T_{jl}$, $i, j, k, l = 1, 2$ has been used, and $X_1 X_2$ and $X_2 X_1$ are, respectively, the four-vectors $(xx, xy, yx, yy)$ and $(xx, yx, xy, yy)$. Both relations in (7) (and in (8)) are equivalent. This is easy to see by using the permutation operator $P$ which gives $(PXP)_{ij,kl} = R_{ji,k} (PR_{12}P = R_{21})$ and $(PX_1 X_2)_{ij} = (X_1 X_2)_{ji}$ i.e., $PX_1 X_2 = X_2 X_1$. In this matrix notation it is obvious that (8) is consistent with the requirement of invariance of (7) under the transformation $X' = TX,$
\[ R_{12}X'_1X'_2 = qX'_2X'_1. \]
Since the elements of \( X \) commute with the entries of \( T \), we obtain
\[ R_{12}X'_1X'_2 = R_{12}(T_1X_1)(T_2X_2) = R_{12}T_1T_2X_1X_2 \]
\[ = T_2T_1R_{12}X_1X_2 = qT_2T_1X_2X_1 = qX'_2X'_1 \]
using (8), and the invariance of (7) follows: the preservation of (7) under the ‘\( q \)-symmetry’ transformation requires (8). In components, (7) reads
\[ R_{ij,kl}X^kX^l = qX^jX^i, \quad \hat{R}_{ij,kl}X^kX^l = qX^iX^j, \]
where \( R_{12} \) and \( \hat{R} = PR, \hat{R}_{ij,kl} = R_{ji,kl} \) are given by
\[ R = \begin{bmatrix} q & 0 \\ \lambda & 1 \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \]

Although the indices in all previous expressions take the values 1, 2, the \( R \)-matrix form of the basic expressions (7) and (8) makes it clear how to generalize them to \( GL_q(n, C) \); all that is needed is the appropriate \( n^2 \times n^2 \) \( R \)-matrix, which is given by
\[ R_{ij,kl} = \delta_{ik}\delta_{jl}(1 + \delta_{ij}(q - 1)) + \lambda\delta_{il}\delta_{jk}\theta(i - j) \quad i, j = 1...n \]
\[ \theta(i - j) = \begin{cases} 0 & i \leq j \\ 1 & i > j \end{cases} \]
With it, the relations defining the ‘quantum hyperplane’
\[ X = (x_1, ..., x_n), \quad x_ix_j = qx_jx_i \quad (i < j) \quad i, j = 1...n \]
are again expressed by (9) and preserved under \( GL_q(n, C) \) because of (8).

All this is, of course, well known. In this report we exhibit how to extend these \( q \)-vector constructions to higher rank quantum tensors (see also [3, 4]). In particular, we shall consider the simplest example of \( q \)-twistors constructed from \( q \)-two-vectors (spinors) \([4], [3], [2]\) and the application to \( q \)-Minkowski space algebras \([2]\).

2 Other covariant objects. Quantum twistors

Consider two isomorphic objects \( X \) and \( Z \), and their hermitian conjugates \( X^\dagger \) and \( Z^\dagger \), transforming under the coaction of two different quantum groups \( T \) and \( T^\dagger \) by
\[ X' = TX, \quad X'^\dagger = X^\dagger T^\dagger, \]
\[ Z' = TZ, \quad Z'^\dagger = Z^\dagger T^\dagger, \]
For instance, in the classical \( SL(2, C) \) case there are two fundamental representations, \( D^{\frac{1}{2}, 0} \) and \( D^{0, \frac{1}{2}} \), realized by complex unimodular matrices \( A \) and
In the quantum case this corresponds to taking two copies $T$ and $\tilde{T}$ of $SL_q(2, C)$, with the obvious ‘reality’ condition added, $\tilde{T}^{-1} = T^\dagger$. In the $q$-case one has to add the commutation relations between elements of $T$ and $T^\dagger$.

The commutation relations in the general situation involve four $R$-matrices $R^{(i)}$, $i = 1, \ldots, 4$,

\[
R^{(1)} T_1 T_2 = T_2 T_1 R^{(1)} ,
\]
\[
T_1^\dagger R^{(2)} T_2 = T_2 R^{(2)} T_1^\dagger ,
\]
\[
T_2^\dagger R^{(3)} T_1 = T_1 R^{(3)} T_2^\dagger ,
\]
\[
R^{(4)} T_1^\dagger T_2^\dagger = T_2^\dagger T_1^\dagger R^{(4)} .
\]

The consistency of these equations requires $R^{(2)} = \mathcal{P} R^{(3)} \mathcal{P} = R^{(3)} \dagger$ and $R^{(4)} = (\mathcal{P} R^{(1)} \dagger \mathcal{P}) \dagger$. Notice that $R^{(1)}$ and $R^{(4)}$ are the $R$-matrices of two quantum groups $T$ and $T^\dagger$ related by a $\ast$-operation (and that $R^{(1)}$, e.g., may be taken as $R_{12}$ or $R_{21}$). In contrast, $R^{(2)}$ (and hence $R^{(3)}$) is a matrix defining how the elements of both quantum groups commute and accordingly it is not a priori fixed. In general, one could introduce instead of $T^\dagger$ another matrix $S$; the $q$-matrices $S$ and $T$ need not even have the same dimension. If, say, $T$ and $S$ are $n \times n$ and $m \times m$ matrices, $S_1$ and $T_2$ in the second equation of (15) would be $S_1^\dagger = (S^\dagger \otimes 1_n)$ and $T_2 = (1_m \otimes T)$ and $R^{(2)}$ would be an $(m \times n) \times (m \times n)$ matrix. Similarly, in the third equation $S_2^\dagger = (1_n \otimes S^\dagger)$, $T_1 = (T \otimes 1_m)$ and $R^{(3)}$ would be an $(n \times m) \times (n \times m)$ matrix, while $R^{(4)}$ would be an $m^2 \times m^2$ matrix.

The form of the eqs. (15) is the result of the equations which express the commutation relations among the components of the vectors $X, Z$. Since in principle $T$ and $T^\dagger$ do not commute, we have to allow for possibly non-trivial commutation relations among the components of $X$ and $Z^\dagger$. Thus, the set of commutation relations left invariant is given by

\[
R^{(1)} X_1 X_2 = \kappa_1 X_2 X_1 ,
\]
\[
Z_1^\dagger R^{(2)} X_2 = X_2 Z_1^\dagger ,
\]
\[
Z_2^\dagger R^{(3)} X_1 = X_1 Z_2^\dagger ,
\]
\[
\kappa_2 Z_1^\dagger Z_2^\dagger = Z_2^\dagger Z_1^\dagger R^{(4)} ,
\]

where $\kappa_1$ and $\kappa_2$ are appropriate eigenvalues of the $R$-matrices. The invariance of the first and last equations is proven as in Sec. 1 and the others similarly. For instance, for the second equation we check that

\[
Z_1^\dagger R^{(2)} X_2' = (Z_1^\dagger T_1^\dagger) R^{(2)} (T_2 X_2) = Z_1^\dagger T_2 R^{(2)} T_1^\dagger X_2
\]
\[ T_2 Z_1 R^{(2)} X_2 T_1^\dagger = T_2 X_2 Z_1^\dagger T_1^\dagger = X_2^\prime Z_1^\dagger \]

using the second equations in (15) and (16), respectively, in the second and fourth equalities. In particular, if \( R^{(2)} = I = R^{(3)} \), both quantum groups are independent (commuting), and this is reflected in the fact that the components of \( X \) and \( Z^\dagger \) commute.

Let us use the above construction to introduce another covariant object which generalizes (with some restrictions) the concept of twistor to the \( q \)-deformed case. Let \( X \) and \( Z^\dagger \) satisfy the previous set of commutation relations. In particular, \( X \) and \( Z^\dagger \) may be, for instance, \( q \)-two-vectors (\( q \)-spinors), of \( SL_q(2, C) \); this case will be analyzed in more detail below. Tensoring two \( q \)-vectors we introduce the object

\[ K \equiv X Z^\dagger \quad (K_{ij} = X_i Z_j^\dagger) \]  

Then, the transformation of \( K \) induced by (14) is

\[ \varphi : K \mapsto K' = TKT^\dagger \quad (K'_{ij} = T_{im} K_{mn} T_{nj}^\dagger) \]  

The entries of \( K \) are, of course, non-commuting. We shall see that these commutation relations can be expressed in a closed, elegant and compact equation which permits to extract the algebra generated by the entries of \( K \) without considering its explicit realization in terms of the components of \( X \) and \( Z^\dagger \). Using the above relations we may now derive the equation describing the commutation relations which define the algebra generated by the entries of \( K \). With \( K_1 = X_1 Z_1^\dagger \) (\( K_{1, ij, kl} = (K \otimes 1)_{ij, kl} = X_i Z_k^\dagger \delta_{jl} \)) and \( K_2 = X_2 Z_2^\dagger \) (\( K_{2, ij, kl} = (1 \otimes K)_{ij, kl} = \delta_{ik} X_j Z_l^\dagger \)), we find using (16) that

\[ R^{(1)} K_1 R^{(2)} K_2 = R^{(1)} X_1 Z_1^\dagger R^{(2)} X_2 Z_2^\dagger = R^{(3)} X_1 X_2 Z_1^\dagger Z_2^\dagger = (\kappa_1/\kappa_2) X_2 X_1 Z_1^\dagger Z_2^\dagger R^{(4)} = (\kappa_1/\kappa_2) Z_2^\dagger R^{(3)} X_1 Z_1^\dagger R^{(4)} \]  

Hence, the commuting properties of the quantum twistor are given by

\[ R^{(1)} K_1 R^{(2)} K_2 = (\kappa_1/\kappa_2) K_2 R^{(3)} K_1 R^{(4)} \]  

Eq. (21) is (with \( \kappa_1/\kappa_2 = 1 \)) nothing else than the reflection equation with no spectral parameter dependence (see [4, 7] and references therein and [9] in the context of braided algebras) which follows by imposing the invariance of the commuting properties of the entries of \( K \) by the coaction (14). As shown here, eq. (21) also follows from interpreting \( K \) as an object made out of two \( q \)-vectors, in general not necessarily of the same dimension so that in general \( K \) is not a squared matrix.

Let \( X, Z \) be two \( q \)-two-vectors (spinors). Then, \( K = XX^\dagger \) is a (null) quantum twistor: as we shall see, its quantum determinant (\( \det_q K \)) is necessarily zero (as it is as well for \( XZ^\dagger \)). In contrast, the \( q \)-twistor \( K = XZ^\dagger + ZX^\dagger \) has \( \det_q K \neq 0 \).
Notice that, in general, there are four possibilities to write (21) (obviously, related in between) since there are two possibilities for $R^{(1)}$ and for $R^{(4)}$ in (15) and in (16) (see (6) and (8)). However, this freedom is reduced when covariant objects $K$ constructed out of four vectors are considered since covariance requires to introduce commutation relations between $Z$ and $X$ and between $Z^\dagger$ and $X^\dagger$ using $R^{(1)}$ and $R^{(4)}$. Let us consider the hermitian matrix

$$K = X Z^\dagger + Z X^\dagger \quad (22)$$

($Z$ and $X$ have the same number of components). To compute the commutation properties of $K$, the complete set of relations among $X$, $Z$, $X^\dagger$ and $Z^\dagger$ are required. Thus, besides (16), we need to introduce the following set of covariant relations

$$R^{(1)}X_1Z_2 = Z_2X_1$$
$$Z_1^\dagger R^{(2)}Z_2 = Z_2 Z_1^\dagger$$
$$X_2^\dagger R^{(3)}X_1 = X_1X_2^\dagger$$
$$X_1^\dagger Z_2^\dagger = Z_2^\dagger X_1^\dagger R^{(4)}$$

the structure of which is again dictated from (13) by covariance. From the first and the last eqs. in (23) we obtain (supposing $R^{(1)}$ real)

$$R^{(4)} = (R^{(1)})^t \quad (24)$$

which implies that the eigenvalues are equal, $\kappa_1 = \kappa_2$. Then, the commutation relation for the entries of $K$ (in matrix form) are easily computed using (16) and (23)

$$R^{(1)}K_1R^{(2)}K_2 = R^{(1)}(X_1Z_1^\dagger + Z_1X_1^\dagger)R^{(2)}(X_2Z_2^\dagger + Z_2X_2^\dagger)$$
$$= R^{(1)}(X_2Z_2^\dagger + Z_2X_2^\dagger)R^{(2)}(X_1Z_1^\dagger + Z_1X_1^\dagger)R^{(4)}$$

$$= K_2R^{(3)}K_1R^{(4)} \quad (25)$$

where $R^{(1)} = R^{(4)}$ and $R^{(2)} = R^{(3)}\mathcal{P}$ and Hecke’s condition for the $R$-matrix has been used. Now, we have only one reflection equation for $K$, since the two possibilities for $R^{(1)}$ produce two equations which are identical after a similarity transformation with $\mathcal{P}$.

Notice that $K$ in (22) is constructed from two parts, each one of them satisfying the same algebra relations (25):

$$K = K^{(1)} + K^{(2)}$$
$$K^{(1)} = X Z^\dagger$$
$$K^{(2)} = Z X^\dagger \quad (26)$$

These two pieces have specific commutation properties among themselves. Indeed, the (mixed) commutation relations (23) lead to the following non-commuting property between the matrices $K^{(1)}$ and $K^{(2)}$ (non-symmetric under the interchange of $K^{(1)}$ and $K^{(2)}$)

$$R^{(1)}K^{(1)}_1R^{(2)}K^{(2)}_2 = K^{(2)}_2 R^{(3)} K^{(1)}_1 (\mathcal{P} R^{(4)} \mathcal{P})^{-1} \quad (27)$$
Here $R^{(4)} = R^{(1) t}$ and the two possibilities for $R^{(1)}$ produce two different equations for $K^{(1)}$ and $K^{(2)}$ which transform into each other by exchanging $(1) \leftrightarrow (2)$ in $K^{(i)}$. Both had to be possible since $K = K^{(1)} + K^{(2)}$ is symmetric under this exchange. Equation (27), here obtained from the commutation relations (23), is known as ‘braiding equation’ [9]. The commutation properties among the elements of $K^{(1)}$ and $K^{(2)}$ are such that the sum of two objects satisfying (25) verifies also the same relation. Within this terminology, the ‘mixed’ eqs. (23) are the braiding relations for $q$-vectors.

From now on, we shall restrict ourselves to the two-dimensional case which will be useful in the application to $q$-Minkowski space [8]. We shall start by discussing the

**$q$-determinant of $K$:**

Let the quantum group matrices $T$ and $T^\dagger$ be $2 \times 2$ matrices. There exists an invariant quadratic element from $K$, the $q$-determinant of $K$. It is defined by [4]

$$
det_q K P_- \equiv P_- K_1 \hat{R}^{(3)} K_1 P_-
$$

(28)

where $\hat{R}^{(3)} = \mathcal{P} R^{(3)}$ and $P_-$ is the $q$-antisymmetrizer of the $R$-matrix corresponding to the quantum groups $T$ and $T^\dagger$. The $q$-determinant of $T$ and $T^\dagger$ are given by [3]

$$
det_q T P_- = P_- T_1 T_2 \quad , \quad det_q T^\dagger P_- = T_2^\dagger T_1^\dagger P_-
$$

(29)

and the projector $P_-$ can be expressed in terms of the $q$-epsilon tensor (2)

$$
P_{- ij,kl} = [2]^{-1} \epsilon^q_{ij} \epsilon^q_{kl} \quad , \quad P_- = \frac{1}{[2]} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & q^{-1} & -1 & 0 \\
0 & -1 & q & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} .
$$

(30)

When $(\det_q T)(\det_q T^\dagger) = 1$, $\det_q K$ is invariant under the coaction (19). Using the third eq. in (13) and (29)

$$
det_q (TKT^\dagger) = P_- (T_1 K_1 T_1^\dagger) \hat{R}^{(3)} (T_1 K_1 T_1^\dagger) P_-
$$

$$
= P_- T_1 T_2 K_1 \hat{R}^{(3)} K_1 T_2^\dagger T_1^\dagger P_-
$$

$$
= (\det_q T) (\det_q K) (\det_q T^\dagger) .
$$

(31)

Thus, if $(\det_q T)(\det_q T^\dagger) = 1$ we obtain that $\det_q (TKT^\dagger) = \det_q K$. The centrality of $\det_q K$ requires some YBE-like conditions on the $R^{(i)} (i = 1, 2, 3, 4)$ matrices in (21).

Using the definition (28) and the $R$-matrix property $\hat{R}_{ab,cd}^{(3)} = \hat{R}_{ba,cd}^{(3)}$, we can compute explicitly the $q$-determinant of $K$ in the following realizations
1. for the matrix $K = XZ^\dagger$ (and hence for the $q$-twistor $K = XX^\dagger$)

\[
(det_q K) P_{-i_j,k_l} = P_{-i_j,ab} K_{ac} \hat{R}_{c,b,mn}^{(3)} K_{mp} P_{-pn,k_l} \\
\propto \epsilon_{ij}^q \epsilon_{ab}^q X_a Z_c^\dagger \hat{R}_{bc,mn}^{(3)} X_m Z_p^\dagger \epsilon_{pn}^q \epsilon_{kl}^q \\
= \epsilon_{ij}^q \epsilon_{ab}^q X_a X_b Z_c^\dagger Z_p^\dagger \epsilon_{pn}^q \epsilon_{kl}^q \\
= \epsilon_{ij}^q (X^\dagger X)(Z^\dagger Z)^\dagger \epsilon_{kl}^q = 0
\]

since $(X^\dagger e^q X) = 0 = (Z^\dagger e^q Z)$. This reflects the well-known fact in non-deformed twistor theory that twistors constructed out of two spinors determine null length vectors;

2. for the $q$-twistor $K = XZ^\dagger + ZX^\dagger$, a similar calculus to the previous one gives

\[
(det_q K) P_{-i_j,k_l} \propto \epsilon_{ij}^q \epsilon_{ab}^q (X_a Z_c^\dagger + Z_a X_c^\dagger) \hat{R}_{bc,mn}^{(3)} (X_m Z_p^\dagger + Z_m X_p^\dagger) \epsilon_{pn}^q \epsilon_{kl}^q \\
= \epsilon_{ij}^q [X^\dagger e^q Z](X^\dagger e^q Z)^\dagger + (Z^\dagger e^q X)(Z^\dagger e^q X)^\dagger] \epsilon_{kl}^q \neq 0
\]

then, to get twistors with non-null $q$-determinant we need four spinors in the definition of $K$ (notice that $X$, $X^\dagger$, $Z$ and $Z^\dagger$ are algebraically independent objects). If the scalar products $(X^\dagger e^q Z)$ and $(Z^\dagger e^q X)$ are central elements in the algebra generated by $X, Z, X^\dagger$ and $Z^\dagger$ the $q$-determinant of $K$ is also central.

3 An application: $q$-Minkowski space

The classical construction of a Minkowski vector uses two (dotted and undotted) spinors,

\[
K_{\alpha\beta} = \xi_{\alpha} \xi_{\beta} = (\sigma_{\mu} x^\mu)_{\alpha\beta} \quad \alpha, \beta = 1, 2 \quad , \quad (34)
\]

and $K'_{\alpha\beta} = A_{\alpha} \cdot \gamma K_{\gamma\delta} (\tilde{A}^{-1})^\delta_{\beta}$, where $A$ and $\tilde{A} = (A^{-1})^\dagger$ are the two fundamental representations of $SL(2,C)$. A $q$-deformation of the Lorentz group may be obtained [11]-[12] by replacing $A$ and $\tilde{A}$ by two copies $T$ and $\tilde{T}$ of $SL_q(2,C)$. Applying the pattern described above we now have two pairs of $q$-spinors

\[
X \rightarrow X' = TX \quad Z \rightarrow Z' = TZ \quad , \quad (35)
\]

\[
X^\dagger \rightarrow X'^\dagger = X^\dagger \tilde{T}^{-1} \quad Z^\dagger \rightarrow Z'^\dagger = Z^\dagger \tilde{T}^{-1} \quad , \quad (36)
\]

obviously, the reality condition $T^\dagger = \tilde{T}^{-1}$ must be considered to have that $(X)^\dagger = X^\dagger$, from which we may construct the following hermitian objects ($q$-twistors)

\[
K = XX^\dagger \quad \text{or} \quad K = XZ^\dagger + ZX^\dagger \quad , \quad (37)
\]
and find their transformation properties. When the reality condition $T^\dagger = \tilde{T}^{-1}$ is imposed, the coaction
\[ K' = TK\tilde{T}^{-1} = TKT^\dagger \] (38)
preserves the hermiticity property of $K$. Since, by assumption, $T$ and $\tilde{T}$ are $SL_q(2, C)$ matrices, i.e.,
\[ R_{12}T_1T_2 = T_2T_1R_{12} \quad \text{and} \quad R_{12}\tilde{T}_1\tilde{T}_2 = \tilde{T}_2\tilde{T}_1R_{12}, \] (39)
the first and last equations of (15) are fulfilled if
\[ R(1) = R_{12} \quad \text{or} \quad R_{21}^{-1} \quad (\kappa_1 = q \quad \text{or} \quad q^{-1}) , \] (40)
\[ R(4) = R_{21} \quad \text{or} \quad R_{12}^{-1} \quad (\kappa_2 = q^{-1} \quad \text{or} \quad q) . \] (41)

Then, the basic relations which define the non-commutative algebra generated by the entries of $K$ are given by eq. (21), which gives the following possibilities
\[ R_{12}K_1R^{(2)}_{12}K_2 = K_2R^{(3)}_{12}K_1R_{21} \quad . \] (42)
As we have already discussed the second possibility (42) is not valid for the twistor with four spinors (second expression in (37)) since it does not correspond to $R^{(1)} = R^{(4)}$. However, it is easy to check that the algebra generated by the entries of $K$ satisfying eq. (42) coincides with the algebra determined by (41) with the additional condition $det_qK = 0$. To see it, the following consequences of the eigenvalue decomposition of $\hat{R}$ ($\hat{R} \equiv \mathcal{P}R = qP_+ - q^{-1}P_-$) are useful
\[ P_-\hat{R} = \hat{R}P_- = -q^{-1}P_- , \quad P_-\hat{R}^{-1} = \hat{R}^{-1}P_- = -qP_- \quad , \] (43)
\[ q^2\hat{R}^{-1} = \hat{R} - (q^3 - q^{-1})P_- . \] (44)
Multiplying now eq. (42) by $P_-\mathcal{P}$ from the left and by $\mathcal{P}P_-$ from the right and using (43) we get
\[ -q^{-1}P_-K_1R^{(2)}_{12}K_2\mathcal{P}P_- = -q^3P_-\mathcal{P}K_2R^{(3)}_{12}K_1P_- \] (45)
as $\hat{R}^{(i)} = \mathcal{P}R^{(i)}$, $R^{(2)} = \mathcal{P}R^{(3)}\mathcal{P}$ and $K_1 = \mathcal{P}K_2\mathcal{P}$
\[ (q^3 - q^{-1})P_-K_1\hat{R}_{12}^{(3)}K_1P_- = 0 \quad . \] (46)
Thus, (if $q^4 \neq 1$) we obtain that $det_qKP_- = P_-K_1\hat{R}_{12}^{(3)}K_1P_- = 0$.

Now, using (44) the RE (42) can be expressed in the following way
\[ R_{12}K_1R^{(2)}_{12}K_2 = K_2R^{(3)}_{12}K_1R_{21} - (q^3 - q^{-1})P_-K_2R^{(3)}_{12}K_1P_- \] (47)
and using the definition of the $q$-determinant
\[ R_{12}K_1R^{(2)}_{12}K_2 = K_2R^{(3)}_{12}K_1R_{21} - (q^3 - q^{-1})\mathcal{P}(det_qK)P_-\mathcal{P} \] (48)
as $\text{det}_q K = 0$, we just obtained eq. (44); thus, as the algebra is the same eq. (12) may be discarded.

The matrices $R^{(2)}, R^{(3)}$ are not determined, and characterize the mixed commutation relations between quantum group elements and conjugated elements in (16), (23) and

$$
\tilde{T}^{-1}_1 R^{(2)} T_2 = T_2 R^{(2)} \tilde{T}^{-1}_1, \quad T_1 R^{(3)} \tilde{T}^{-1}_2 = \tilde{T}^{-1}_2 R^{(3)} T_1 .
$$

(49)

Two particularly special cases arise

a) **Commuting case:** if the two quantum group copies are independent, the quantum matrices commute

$$
T_1 \tilde{T}_2 = \tilde{T}_2 T_1 ,
$$

(50)

here, $R^{(2)} = R^{(3)} = I$, and then, eq. (11) gives the reflection equation, which is equivalent to the ‘$RTT$’ relation (18) (see below)

$$
R_{12} K_1 K_2 = K_2 K_1 R_{21}
$$

(51)

Eq. (42), in this particular case, produces the RE

$$
R_{12} K_1 K_2 = q^2 K_2 K_1 R_{12}^{-1}
$$

(52)

however, as we have just shown, this possibility leads to the same commutation relations (11) for the entries of $K$ plus the additional condition $\text{det}_q K = 0$ (8). The $q$-Minkowski algebra (81) is isomorphic to the quantum group algebra $GL_q(2)$, by (8)

$$
T = K \sigma^1 , \quad R_{12} T_1 T_2 = T_2 T_1 R_{12} ,
$$

(53)

where $\sigma^1$ is the usual Pauli matrix. Then, it is not possible to define a linear central element in the algebra generated by the entries $(\alpha, \beta, \gamma, \delta)$ of matrix $K$, and the quadratic one is given by the $q$-determinant (28) with $\tilde{R}^{(3)} = \mathcal{P} R^{(3)} = \mathcal{P}$

$$
\text{det}_q K P_- = P_- K_1 \mathcal{P} K_1 P_- = (-q^{-1})(\alpha \delta - q \gamma \beta) P_- .
$$

(54)

b) **Non-commuting case:** now, assuming the non-trivial commutation relations between the two copies of $SL_q(2, C)$

$$
R_{12} T_1 \tilde{T}_2 = \tilde{T}_2 T_1 R_{12}
$$

(55)

we see that (49) is fulfilled for $R^{(2)} = R_{21}$. Then, eq. (13) leads to the RE

$$
R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}
$$

(56)

Again, (12) produces an equation

$$
R_{12} K_1 R_{21} K_2 = q^2 K_2 R_{12} K_1 R_{12}^{-1}
$$

(57)
which leads to the same commutation relations as \(5\) with the restriction 
\[ det_q K = 0. \]

These equations \(8\) define the quantum Minkowski algebra of \([10-13]\), in which the linear central term is identified with the time coordinate and the 
\[ q \]-determinant, defined by \(28\) where \( R^{(3)} = \hat{R} \)

\[
det_q K P_+ = P_- \hat{K}_1 \hat{R} K_1 P_- = (q^{-1})(\alpha \delta - q^2 \beta \gamma) P_+ ,
\]
\[
(58)
\]
gives the quadratic central element which is identified with the invariant \(q\)-Minkowski length.

Having a \(q\)-vector \(X \mapsto TX\) and a \(q\)-matrix \(K \mapsto TKT^\dagger\), it is natural to 
construct higher rank tensors transforming as 
\[
\varphi : L \mapsto T^\otimes n (T^\dagger)^\otimes n ;
\]
\[
(59)
\]
they are invariant subspaces of the \(q\)-Minkowski algebra for the coaction \(\varphi\). 
The generators of the \(q\)-tensors \(L\) may be extracted from matrices of higher 
dimensions, \(e.g.\)

\[
L^2 \sim K^\otimes q^2 = K_1 R_{21} K_2 \mapsto T_1 T_2 K^\otimes q^2 T_1^\dagger T_2^\dagger ,
\]
\[
L^3 \sim K^\otimes q^3 = K_1 R_{21} K_2 R_{31} R_{32} K_3 ,
\]
\[
(60)
\]
\[
L^n \sim K^\otimes q^n = K_1 \prod_{j=2}^{n} (R_{j1} R_{j2} \ldots R_{j-1 j} K_j) .
\]

These subspaces (as in the non-deformed theory) are reducible (for instance, 
\(K^\otimes q^2\) has \(det_q K\) as an invariant element). One can apply to \(T^\otimes 2 = T_1 T_2\) 
the appropriate projector \(P^{(1)}\) to the spin 1 representation (and the same for 
\((T^\dagger)^\otimes 2\) to get a tensor of generators transforming according \(D^{1,1}\) \textit{irrep} of the \(q\)-Lorentz group. Quantum tensors transforming according \(D^{j,s}\) \textit{irreps} could 
be constructed in the same manner. We find additional \(R\)-matrix factors in 
the tensor products of \(K\) \((\oplus)\) (cf. \((28)\)). This construction is useful for a 
description of higher spin \(q\)-wave equations (see also \([13]\)).

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