ON UNSPANNED LATENT RISKS IN DYNAMIC TERM STRUCTURE MODELS

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ABSTRACT

We explore the importance of unspanned risks to a real-time Bayesian investor seeking to maximize her utility. We propose a novel class of unspanned Dynamic Term Structure Models that embed a stochastic market price of risk specification. We develop a suitable Sequential Monte Carlo inferential and prediction scheme that takes into account all relevant uncertainties. We find that latent factors contain significant predictive power above and beyond the yield curve. Most importantly, they exploit information hidden from the yield curve and generate significant utility gains, out-of-sample. The hidden component associated with slope risk is countercyclical and links with real activity.
1 Introduction

1.1 Unobserved Factors or Macroeconomic Variables?

The presence of information hidden from the yield curve, yet highly relevant for predicting bond excess returns, has attracted particular attention in recent literature. The reason being that several empirical studies (see, Ludvigson and Ng (2009), Cooper and Priestley (2009), Duffee (2011), Wright (2011), Joslin et al. (2014), Cieslak and Povala (2015), Gargano et al. (2019), Bianchi et al. (2021) and Li et al. (2021)) cast doubt on the fundamental and powerful stylised fact of the term structure of interest rates, which suggests that current yields contain all relevant information for forecasting future yields, returns and bond risk premia. In particular, the aforementioned studies have documented evidence of various variables containing significant predictive power above and beyond the yield curve. While the literature has mainly focused on information extracted from macroeconomic time-series, the list of unspanned risk factors can be long and diverse (e.g. Li et al. (2021) and Bakshi et al. (2022)) as argued by Joslin et al. (2014).

More recently, however, the capability of unspanned macroeconomic risks, offering evidence of statistical and economic benefits to bond investors, has come under scrutiny. First, Bauer and Hamilton (2018) question earlier results, concluding that, in many cases, prior evidence of excess return predictability is weaker than initially documented and far from statistically significant. Most importantly, in contrast to the in-sample analysis, they conclude that the addition of extra macroeconomic factors leads to less stable and accurate predictions and, ultimately, to the deterioration of the out-of-sample performance. A similar observation is made by Barillas (2011), who argues that an unspanned factor adds little predictability to the term structure, and Giacoletti et al. (2021), who use Bayesian learning to evaluate the real-time out-of-sample performance of Joslin et al. (2014)'s macro-finance dynamic model and conclude that it actually underperforms the simple, nested, yields-only model.

Second, related literature has recently noted the importance of an investor having access to real-time macroeconomic information as opposed to a fully revised information set. Recent evidence highlights the discrepancy in the out-of-sample predictive performance of bond excess returns when real-time macro variables are used as predictors vis-à-vis fully revised factors. In particular, Ghysels et al. (2018) argue that the evident predictive power of macroeconomic variables is largely due to data revisions and, as such, it diminishes when real-time macro data is used, a conclusion that is also reached by Wan et al. (2021). Most importantly, statistical predictability due to macroeconomic factors is not turned into utility gains for bond investors, in contrast to what was initially argued by Gargano et al. (2019) and Bianchi et al. (2021).

Third, a notable exception to the analysis of unspanned macroeconomic risks is the study of Duffee (2011) (and Barillas (2011)) which introduces an unspanned latent factor framework that is not reliant on macroeconomic data. According to Duffee (2011), estimation of term structure models using directly observable macroeconomic data might be a rather powerful approach, yet it comes with a great risk of misspecification.

1.2 Unspanned Latent Approach

The above-mentioned issues raise important questions on the validity of prior empirical and theoretical results. With this in mind, we propose a novel class of arbitrage-free unspanned Dynamic Term Structure Models (DTSMs), that embeds a stochastic market price of risk specification. Our approach resembles the macro-finance framework of Joslin et al. (2014) and the unspanned latent factor framework of Duffee (2011), in that the model is factorised into a 'spanned' component, where risk factors can be retrieved from the information provided by the current yield curve, as well as an ‘unspanned’ component, that could include factors extracted from macroeconomic variables. It is assumed that

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1E.g. the output gap of Cooper and Priestley (2009), the ‘real’ and ‘inflation’ factors of Ludvigson and Ng (2009), the measures of economic activity and inflation of Joslin et al. (2014), the long-run inflation expectation of Cieslak and Povala (2015), etc.

2A notable exception is the case of Ludvigson and Ng (2009)’s factors which slightly improve the predictive performance of the models both in-sample and out-of-sample.
the latter is not determined by the yield curve data, yet remains highly relevant for inference and, more importantly, prediction purposes. The developed setup is then used to explore how valuable the unspanned information is to a real-time Bayesian investor seeking to forecast future excess bond returns and generate systematic economic gains. Our approach differs from previous studies and offers several advantages.

First, we depart from Joslin et al. (2014) in that the unspanned component is regarded as a latent stochastic process, rather than consisting of directly observable (macroeconomic) variables and as such, our analysis is independent on the debate between ‘fully-revised’ vis-à-vis ‘real-time’ macros. In line with Duffee (2011), we estimate this latent factor using the Kalman filter, which allows us to infer any hidden information from the term structure dynamics. Doing so, the unobserved component may absorb model misspecification or be regarded as the element in the model that captures the dynamics of the wider market environment. There are three main aims to our formulation of unspanned latent factors. One, dimension reduction is pursued, when it comes to risk prices, in order to consider parsimonious models that contain almost all the information in the yield curve data. Two, we aim to incorporate additional latent factors in the model, avoiding their direct involvement in the bond pricing procedure, so that they cannot be fully recovered by the cross section of yields. Hence, such latent factors are meant to capture information which is hidden from the yield curve and may be impossible to summarise using information coming solely from observable macroeconomic time series (see, Ludvigson and Ng (2009)). Three, we seek to investigate whether there is a direct link between information incorporated into the hidden component and macroeconomic activity.

Second, we develop a suitable Sequential Monte Carlo (SMC) inferential and prediction scheme that draws from Chopin (2002, 2004); Del Moral et al. (2006) and takes advantage of the linear state space structure of the model to incorporate the associated Kalman filter. As a computational method, our proposed scheme may be viewed as a simplified version of the SMC² algorithm, introduced by Chopin et al. (2012), where the particle filter is replaced by the Kalman filter that corresponds to the models of this paper. This provides an efficient estimation algorithm that can be updated each time new data become available, without having to rerun the estimation procedure from scratch. Furthermore, it guarantees joint identification of parameters and latent states and takes into account all relevant uncertainties.

Third, we take the perspective of a real-time Bayesian learner, who updates her beliefs, as more information is accumulated, using a DTSM with unspanned latent factors. Our developed sequential estimation scheme directly produces the entire forecasting distribution, through predictive densities, at each observation time point. Thus, it can naturally assess the predictive performance of any model, across different maturities and prediction horizons. While prior studies have investigated the predictability of excess returns in-sample, there is less empirical evidence on the out-of-sample performance of the unspanned (macro) DTSMs. We revisit the out-of-sample performance of our model attempting to assess whether a bond investor can actually exploit the evident statistical predictability when making investment decisions. Is investors’ economic utility improved by making full use of unspanned latent risks?

Our empirical analysis is concentrated on non-overlapping excess bond returns as in Gargano et al. (2019), Andreasen et al. (2021) and Wan et al. (2021). This is in contrast to earlier studies (e.g. Cochrane and Piazzesi (2005), Ludvigson and Ng (2009), Joslin et al. (2014), Cieslak and Povala (2015), Ghysels et al. (2018), Bianchi et al. (2021), etc.), which mostly consider annual holding periods, and as such, work with 12-month overlapping excess returns. The reason being that the presence of overlapping observations, coupled with persistent regressors, induces strong serial correlation in the error terms as evidenced by Bauer and Hamilton (2018). Furthermore, working with annual overlapping returns prevents us from identifying relevant information which is short lived, an important drawback during periods of market turmoil as pointed out by Gargano et al. (2019).

Our evaluation framework consists of three stages. First, we evaluate the predictive performance of models using the out-of-sample $R^2$ metric of Campbell and Thompson (2008). Second, to investigate the economic significance of

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3Duffee (2011) does not explore the econometric identification of such a model, nor does he empirically implement a dynamic term structure model with unspanned risks.

4see also Dai et al. (2020) for some recent work including a survey on sequential schemes.
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Models with unspanned latent risks, we solve a dynamic portfolio choice problem as in Della Corte et al. (2008) and Thornton and Valente (2012), for an investor with power utility preferences, and compute standard metrics such as certainty equivalence returns (CER) (see, Johannes et al. (2014) and Gargano et al., 2019, among others). To assess the robustness of our results and establish a better link with existing studies, we consider three different (allocation) scenarios for investors. The first two prevent them from taking extreme positions (see, Thornton and Valente (2012), Sarno et al., (2016), Gargano et al., 2019), while the third relaxes restrictions and allows for maximum leveraging and short-selling (see, Bianchi et al. (2021) and Wan et al., 2021). Third, we examine whether the hidden component of the unspanned latent risks bears any relationship with economic activity. Inspired by the empirical design of Duffee (2011), we regress the hidden component of risk premia on different macroeconomic variables.

Our findings help us infer a host of interesting conclusions related to the US bond market. First, we find that latent factors substantially improve explanatory power on bond excess returns, especially at shorter maturities, thus capturing information associated with the short end of the maturity spectrum. Second, we provide clear evidence of significant improvement to the out-of-sample predictive performance implied by unspanned latent factor models, compared to models that utilise information coming solely from the yield curve (i.e. ‘yields-only’ models). This further suggests that unspanned latent risks are capable of exploiting information relevant to predicting returns beyond the term structure of interest rates. Third, and quite importantly, those models are capable of translating statistical predictability into economic gains, revealing that latent factors share information relevant not only to predicting returns but also to generating significant portfolio benefits to bond investors seeking to maximise their utility. Fourth, we provide evidence of superior economic performance for those models that exploit information hidden from the yield curve compared to the parsimonious model of Dubiel-Teleszynski et al. (2021) that has been proved to generate the largest portfolio benefits among all ‘yields-only’ models. This finding is in direct contrast to Wan et al. (2021), who find no support of economic benefits coming from revised macroeconomic factors and in line with Gargano et al. (2019) and Bianchi et al. (2021), who argue that predictability is turned into utility gains. The latter, however, exploit information from a fully revised macroeconomic dataset, which is proved to enhance predictive power (see, Ghysels et al. (2018)). Our results are not reliant on macroeconomic data. Fifth, we explore the linkages between information exploited by the hidden component of the latent factors and macroeconomic fundamentals. Our results uncover a direct link between the hidden component associated with slope risk and real activity.

1.3 Outline

The remainder of this paper is organised as follows. Section 2 describes the modelling framework. Section 3 presents our suggested sequential learning with latent processes and the forecasting procedure along with the framework for assessing the predictive and economic performance of models. Section 4 discusses the data and the sample period used and presents the family of models considered. Section 5 discusses the results in terms of predictive performance and economic value, including the associated explanatory power, as well as the links between the latent risks and the macroeconomy. Finally, Section 6 concludes the paper by providing some relevant discussion.

2 Dynamic Term Structure Models with Unspanned Latent Risks

2.1 Standard Case

Consider the no-arbitrage class of discrete-time Affine Term Structure Models (ATSM) (see, Ang and Piazzesi (2003) and Cochrane and Piazzesi (2005)) and, in particular, the Gaussian case. Under the physical probability measure $\mathbb{P}$, let us consider first that an $(N \times 1)$ vector of state variables $X_t$, $t = 0, 1, ..., T$, evolves according to a first-order Gaussian Vector Autoregressive (VAR) process

$$X_t = \mu^p + \Phi^p X_{t-1} + \Sigma \varepsilon_t$$
where $\varepsilon_t \sim N(0, I_N)$, $\Sigma$ is an $(N \times N)$ lower triangular matrix, $\mu^Q$ is an $(N \times 1)$ vector and $\Phi^Q$ is an $(N \times N)$ matrix. Under this initial framework, the one period risk-free interest rate $r_t^Q$ is assumed to be an affine function of the state variables

$$r_t = \delta_0 + \delta_1 X_t$$

where $\delta_0$ is a scalar and $\delta_1$ is an $(N \times 1)$ vector. Absence of arbitrage implies the existence of a pricing kernel $\mathcal{M}_{t+1}$ specified as an exponentially affine function of the state vector

$$\mathcal{M}_{t+1} = \exp(-r_t - \frac{1}{2} \lambda'_t \lambda_t - \lambda'_t \varepsilon_{t+1})$$

with $\lambda_t$ being the time-varying market prices of risk, which is also assumed to be affine in the state vector $X_t$

$$\lambda_t = \Sigma^{-1} (\lambda_0 + \lambda_1 X_t)$$

where $\lambda_0$ is an $(N \times 1)$ vector and $\lambda_1$ is an $(N \times N)$ matrix. If we assume that the pricing kernel $\mathcal{M}_{t+1}$ prices all bonds in the economy and we let $P^n_t$ denote the time-$t$ price of an $n$-period zero-coupon bond, then the price of the bond is computed from $P^n_{t+1} = E_t (\mathcal{M}_{t+1} P^n_{t+1})$. Thus, it follows that bond prices are exponentially affine functions of the state vector (see, Duffie and Kan [1996])

$$P^n_t = \exp(A_n + B'_n X_t)$$

with loadings, $A_n$ being a scalar and $B_n$ an $(N \times 1)$ vector, satisfying the following recursions

$$A_{n+1} = A_n + B'_n (\mu - \lambda_0) + \frac{1}{2} B'_n \Sigma \Sigma' B_n - \delta_0$$

$$B_{n+1} = (\Phi - \lambda_1)' B_n - \delta_1$$

with $A_1 = -\delta_0$ and $B_1 = -\delta_1$. This implies that the $Q$ dynamics of the state vector are given by

$$X_t = \mu^Q + \Phi^Q X_{t-1} + \Sigma \varepsilon_t^Q$$

(1)

where $\mu^Q = \mu - \lambda_0$, $\Phi^Q = \Phi - \lambda_1$ and $\varepsilon_t^Q \sim N(0, I_N)$. The continuously compounded $n$-period yield $y^n_t$ is also an affine function of the state vector

$$y^n_t = \frac{-\log P^n_t}{n} = A_{n,X} + B'_{n,X} X_t$$

(2)

where the loading scalar $A_{n,X}$ and the loading $(N \times 1)$ vector $B_{n,X}$ are calculated using the above recursions as $A_{n,X} = -A_{n}/n$ and $B_{n,X} = -B_{n}/n$.

Next, we follow the canonical setup of Joslin et al. [2011] and rotate the vector of unobserved state variables $X_t$ such that they are linear combinations of the observed yields. In particular, we rotate $X_t$ to the first $N$ principal components (PCs) of observed yields as

$$\mathcal{P}_t = W y_t = W A_X + W B_X X_t$$

(3)

where $W$ denotes an $(N \times J)$ matrix containing the first $N$ PCs’ loadings and $y_t$ is a $(J \times 1)$ vector of continuously compounded yields. The $(J \times 1)$ vector $A_X$ and the $(J \times N)$ matrix $B_X$ contain model-implied loadings of yields on risk factors, specifically $A_{n,X}$ are elements of vector $A_X$ and $B_{n,X}$ are transposed rows of matrix $B_X$. This allows us to rewrite the model in terms of $\mathcal{P}_t$s, as these are defined by the affine transformation of $X_t$s according to (3). More specifically, we can rewrite the yield equation (2) in vector form

$$y_t = A_X + B_X X_t,$$

(4)

Working with monthly data implies that $r_t$ is the 1-month yield.
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and then as a function of the observable risk factors $P_t$

$$y_t = A_P + B_P P_t, \quad (5)$$

where the loadings $A_P$ and $B_P$, are given below:

$$A_P = A_X - B_X (W B_X)^{-1} W A_X \quad (6)$$
$$B_P = B_X (W B_X)^{-1} \quad (7)$$

The risk-neutral dynamics of $P_t$ are given as

$$P_t = \mu_{P}^Q + \Phi_{P}^Q P_{t-1} + \Sigma_P \varepsilon_P^t \quad (8)$$

where the risk-neutral measure parameters $\mu_{P}^Q$, $\Phi_{P}^Q$ and $\Sigma_P$ are derived similarly from (1). Given the new observable state vector $P_t$, the one period short rate $r_t$ is also an affine function of $P_t$ given as

$$r_t = \delta_{0P} + \delta_{1P}^t P_t \quad (9)$$

and the market price of risk specification becomes accordingly

$$\lambda_t = \Sigma_P^{-1} (\lambda_{0P} + \lambda_{1P}^t P_t)$$

Similarly, the $P$ dynamics of $P_t$ are of equivalent form to (8) with

$$\mu_{P}^P = \mu_{P}^Q + \lambda_{0P} \quad (10)$$
$$\Phi_{P}^P = \Phi_{P}^Q + \lambda_{1P} \quad (11)$$

In the above equations the parameters $\mu_{P}^Q$, $\Phi_{P}^Q$, $\Sigma_P$, $\delta_{0P}$, $\delta_{1P}$, $\lambda_{0P}$ and $\lambda_{1P}$ can be derived by standard calculations based on the affine transformation of (3); for more details see for example Bauer (2018).

Notice also that in (2) and (5), yields are assumed to be observed without any measurement error. Nevertheless, an $N$-dimensional observable state vector cannot perfectly price $J > N$ yields, and as such, we further assume that the $(J - N)$ bond yields used in the estimation are observed with independent $N(0, \sigma_e^2)$ measurement errors. An equivalent way to formulate this is to write

$$y_t = A_P + B_P P_t + e_t \quad (12)$$

and to consider the dimension of $e_t$ as effectively being $(J - N) \times 1$. Letting $W_\perp$ denote a basis of the null space of $W$, the measurement error assumption can be also expressed as

$$W_\perp e_t \sim N(0, \sigma_e^2 I_{J-N})$$

where $(W_\perp e_t)$ is a $(J - N) \times 1$ vector (Bauer, 2018).

In our setting, we also follow one of the identification schemes proposed in Joslin et al. (2011) (see, Proposition 1), where the short rate is the sum of the state variables, given as $r_t = i X_t$ with $i$ being a vector of ones, and the parameters $\mu_Q$ and $\Phi_Q$ of the state vector’s $Q$-dynamics are given as $\mu_Q = [k_Q^0, 0, 0]$ and $\Phi_Q = diag(g^Q)$, where $g^Q$ denotes an $(N \times 1)$ vector containing the real and distinct eigenvalues of $\Phi_Q$.

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6 According to Duffee (2011), outside of knife-edge cases the matrix $(W B_X)$ is invertible, and as such $P_t$ contains the same information as $X_t$.

7 Alternative specifications for the eigenvalues are considered in Joslin et al. (2011), however real eigenvalues are found to be empirically adequate.
2.2 Incorporating Unspanned Latent Components

2.2.1 Model

In this paper we consider an extension of the model presented in the previous section. Our approach resembles the framework of Joslin et al. (2014) in that the model is factorised into a ‘spanned’ component, i.e. risk factors that can be retrieved by the information provided in historical yield curve data, as well as an ‘unspanned’ component that could include background factors such as macroeconomic variables. It is assumed that the latter is not determined by the yield curve data but it remains highly relevant for the inference and, more importantly, prediction purposes.

The point where our approach differs from Joslin et al. (2014) is that the unspanned component is regarded as a latent stochastic process, rather than consisting of directly observable macroeconomic variables. In this way the unobserved component may absorb model misspecification or be regarded as a market environment-specific factor. There are two main aims of this alternative formulation. First, dimension reduction is pursued in order to consider parsimonious models that contain almost all the information in the yield curve data. This is achieved by the Principal Component Analysis (PCA) approach as in (5) but the projection is now made to \( R < N \) observable factors, where \( R < N \). Second, we would like to incorporate additional latent components in the model avoiding their direct involvement in the bond pricing procedure, so that they cannot be fully recovered by the yield data.

The extended model can be regarded as the following transformation of \( X_t \) that decomposes into the spanned component \( P_t \) and the unspanned latent \( Z_t \):

\[
\begin{bmatrix}
    P_t \\
    Z_t
\end{bmatrix} =
\begin{bmatrix}
    W_R A_X & W_R B_X \\
    \gamma_0 & \gamma_1
\end{bmatrix}
X_t
\]

where \( P_t \) are the first \( R \) principal components extracted from the yield data, \( W_R \) consists of the first \( R \) rows of the matrix \( W \) containing the PCA weights, and \( A_X \) and \( B_X \) are the loadings matrices appearing in (4) (and are of dimension \((J \times 1)\) and \((J \times N)\) respectively). The vector \( \gamma_0 \) and matrix \( \gamma_1 \) (of dimension \((N - R) \times 1\) and \((N - R) \times N\)) may contain arbitrary constants subject to the restrictions in Appendix B of Joslin et al. (2014), mainly ensuring that the matrix \([W_R B_X; \gamma_1]\) is invertible.

Given the linear nature of (13) the \( P \) dynamics of \([P_t; Z_t]\) are of the following form

\[
\begin{bmatrix}
    P_t \\
    Z_t
\end{bmatrix} =
\begin{bmatrix}
    \mu_P^P \\
    \mu_P^Z
\end{bmatrix} +
\begin{bmatrix}
    \Phi_P^P & \Phi_P^Z \\
    \Phi_P^Z & \Phi_Z^Z
\end{bmatrix}
\begin{bmatrix}
    P_{t-1} \\
    Z_{t-1}
\end{bmatrix} +
\begin{bmatrix}
    \Sigma_P \epsilon_t
\end{bmatrix}
\]

where \( \mu_P^Z \) and \( \Phi_P^Z \) are defined in (10) and (11) respectively. The covariance matrix \( \Sigma_P \Sigma_P' \) is assumed to be block diagonal with components \( \Sigma P \Sigma_P' \) and \( \Sigma Z \Sigma Z' \), where \( \Sigma_P \) is an \((N - R) \times (N - R)\) diagonal matrix with positive entries. For identification purposes, as well as practical considerations, we proceed by setting \( \mu_P^Z \) and \( \Phi_Z^Z \) to a vector and a matrix of zeros, whereas \( \Phi_P^Z \) is an \((N - R)\)-dimensional identity matrix. Finally, \( \Phi_Z^Z \) is an \((N - R) \times (N - R)\) diagonal matrix with non-zero entries taking values in \((-1, 1)\), to guarantee stability. This lets us write the model as

\[
P_t = \mu_P^P + \Phi_P^P P_{t-1} + Z_{t-1} + \Sigma_P \epsilon_t^P,
\]

\[
Z_t = \Phi_Z^Z Z_{t-1} + \Sigma_Z \epsilon_t^Z,
\]

where

\[
\epsilon_t^P \sim N(0, \text{I}_R),
\]

\[
\epsilon_t^Z \sim N(0, \text{I}_{N-R}).
\]

Compared to the standard model in the previous section, the model of (15) and (16) above has \( \mu_P^Z + Z_{t-1} \) instead of just \( \mu_P^Z \), in the drift. We can therefore view the \( Z_t \)s as random effects operating at each time interval \([t, t+1]\) and being
correlated. To ensure that $Z_t$ is not spanned by $P_t$ we set $r_t$ as in (9) and define the pricing kernel as
\[
\mathcal{M}_{t+1} = \exp(-r_t - \frac{1}{2} \lambda_t(\lambda_t - \lambda_t^{P_t} \varepsilon_t^{P_t}))
\]
where $\varepsilon_t^{P_t}$ consists of the first $R$ components of $\varepsilon_t$ in (14) and the market price of risk $\lambda_t$ is defined conditional on $Z_t$ as
\[
\lambda_t = \Sigma^{-1}_P \left[ \mu_P - \mu_P^Q + (\Phi_P - \Phi^Q_P)\mathcal{P}_t + \Phi^P_{PZ}Z_t \right],
\]
\[
= \Sigma^{-1}_P (\lambda_0P + \lambda_1\mathcal{P}_t + Z_t).
\]
The model is completed by setting the $\mathcal{Q}$ dynamics for $\mathcal{P}_t$ as in (8). Note that since $Z_t$ is not part of the pricing procedure its risk-neutral dynamics are not required.

### 2.2.2 Kalman Filtering

The model of (15) and (16), introduced in the previous section, can be slightly modified to match the general linear Gaussian state-space model, see for example Durbin and Koopman (2012), by working with $\alpha_t = Z_{t-1}$, instead of $Z_{t-1}$, and setting the initial conditions as
\[
\alpha_0 \sim N(a_0 \vert 0, P_0 \vert 0)
\]
with $a_0 \vert 0 = 0$ and $P_0 \vert 0 = \Sigma_{ZZ'}$. This allows for the Kalman filter to be applied to marginalise out $Z_t$ and evaluate the densities $f^P(P_t \vert P_{t-1}, \theta)$, as in (22) below, with $\theta$ denoting the relevant parameters specified in the next section. More specifically, we let
\[
s_t = P_t - \mu_P - \Phi_P^P \alpha_{t-1}
\]
and rewrite (15) and (16) as
\[
s_t = \alpha_t + \Sigma_P \varepsilon_t^P,
\]
\[
\alpha_{t+1} = \Phi^P_Z \alpha_t + \tilde{\Sigma}_Z \varepsilon_t^Z.
\]
Then, as in Durbin and Koopman (2012), starting with $a_0 \vert 0$ and $P_0 \vert 0$ defined above for $t = 0$, the Kalman filter prediction step is
\[
a_{t+1} = \Phi^P_Z a_{t|t},
\]
\[
P_{t+1} = \Phi^P_Z P_{t|t} \Phi_Z^P + \tilde{\Sigma}_Z \tilde{\Sigma}_Z',
\]
for each $t = 1, ..., T - 1$, whereas the Kalman filter update step is
\[
v_t = s_t - a_t
\]
\[
F_t = P_t + \Sigma_P \Sigma_P'
\]
\[
K_t = P_t F_t^{-1}
\]
\[
a_{t|t} = a_t + K_t v_t
\]
\[
P_{t|t} = P_t - K_t P_t
\]
In the above, $a_{t+1}$ is the state prediction and $P_{t+1}$ is its covariance, $v_t$ is the prediction error and $F_t$ is its covariance, $K_t$ is the Kalman gain, and $a_{t|t}$ is the state update and $P_{t|t}$ is its covariance. Consequently, latent $\alpha_t$ (or $Z_{t-1}$) is distributed as $N(a_{t|t}, P_{t|t})$, where the moments are obtained from (20) and (21). We eventually arrive at the log-likelihood representation of $f^P(P_t \vert P_{t-1}, \theta)$, first due to Schweppe (1965), which is following
\[
\log f^P(P_t \vert P_{t-1}, \theta) = -R \log 2\pi - \frac{1}{2} (\log |F_t| + v_t^T F_t^{-1} v_t)
\]
(22)
where $| \cdot |$ is the matrix determinant. The predictive distribution of $P_{T+1}$, based on observations up to time $T$, is also straightforward to obtain from (17), (18) and (19) as

$$P_{T+1}|P_T, \alpha_T \sim N(\mu_P^T + \Phi_P^T P_T + \Phi_Z^T \alpha_T, \Sigma_Z \Sigma_Z^T + \Sigma_P \Sigma_P^T)$$

where $\alpha_T$ is distributed as $N(a_{T|T}, P_{T|T})$, with $a_{T|T}$ and $P_{T|T}$ obtained from the Kalman filter.

### 2.2.3 Likelihood and Risk Price Restrictions

Statistical inference can proceed using the observations $Y = \{y_t, P_t : t = 0, 1, \ldots, T\}$. The likelihood factorises into two parts stemming from the $P$ and $Q$ respectively. For $R$ observable factors, and not $N$ as in the standard case, the joint likelihood (conditional on the initial point $P_0$) can be written as

$$f(Y|\theta, \tilde{\Sigma}_Z) = \left\{ \prod_{t=0}^{T} f^Q(y_t|P_t, k^{Q}_{t}, g^{Q}, \Sigma_P, \sigma^{2}_t) \right\} \times \left\{ \prod_{t=1}^{T} f^P(P_t|P_{t-1}, k^{Q}_{t}, g^{Q}, \lambda_{0P}, \lambda_{1P}, \Sigma_P, \Phi_P^Z, \tilde{\Sigma}_Z) \right\}$$

where the $Q$-likelihood components $f^Q(\cdot)$ are given by (12) and capture the cross-sectional dynamics of the risk factors and the yields, whereas $P$-likelihood components $f^P(\cdot)$, where the $Z_t$s have been marginalised out by the Kalman filter, are obtained from (22) and capture the time-series dynamics of the observed risk factors. The parameter vector is set to $\theta = (\sigma^{2}_t, k^{Q}_{t}, g^{Q}, \lambda_{0P}, \lambda_{1P}, \Sigma_P, \Phi_P^Z)$. To assure identification, we tune $\Sigma_Z$ in-sample and fix it out-of-sample at $\tilde{\Sigma}_Z$.

Details of the tuning procedure are in Appendix D.

For brevity of further exposition, we let $\lambda_P = [\lambda_{0P}, \lambda_{1P}]$ and $\lambda = \lambda_{1P}$. If all the entries in $\lambda_P$ are free parameters we get the maximally flexible model (model $M_0$ in this paper). Alternative models, where some of these entries are set to zero, have also been proposed in the existing literature. Early studies focused mainly on imposing zero ad hoc restrictions on the parameters governing the dynamics of the risk premia. However, the process of imposing such restrictions in an ad hoc way has been recently criticised (see, Kim and Singleton [2012] and Bauer [2018]) and a few studies (see, Cochrane and Piazzesi [2009], Duffee [2011] and Joslin et al. [2014]) have investigated more systematic approaches on how to impose restrictions on the market price of risk parameters. Generally speaking, in most models the set of unrestricted parameters is usually a subset of $\lambda_P$. In this paper we adopt a restriction set with optimal predictive performance, understood predominantly as economic value and less as statistical predictability, among all possible restriction sets, as evidenced in Dubiel-Teleszynski et al. (2021), based on out-of-sample period of data closely overlapping ours, see Section 4. In that paper a sequential version of the stochastic search variable selection scheme was developed and applied, leading to the conclusion that the optimal restriction set is to only leave $\lambda_{1,2}$ free and set all other elements to 0 (model $M_1$ there and in this paper). The same model has also been recommended earlier by Bauer (2018) on the basis of offline analysis using reversible jump Markov Chain Monte Carlo. Such choice is also to some extent in accordance with the early work on risk dynamics, such as Duffee (2002), which finds that variation in the price of level risk is necessary to capture the failure of the Expectations Hypothesis (EH). This variation was usually captured by linking the price of level risk to the slope of the term structure (Duffee 2011).

Consequently, we can conveniently restate the likelihood specification of (23) as

$$f(Y|\theta, \tilde{\Sigma}_Z) = \left\{ \prod_{t=0}^{T} f^Q(y_t|P_t, k^{Q}_{t}, g^{Q}, \Sigma_P, \sigma^{2}_t) \right\} \times \left\{ \prod_{t=1}^{T} f^P(P_t|P_{t-1}, k^{Q}_{t}, g^{Q}, \Sigma_P, \Phi_P^Z, \lambda_{1,2}, \tilde{\Sigma}_Z) \right\}$$

(24)

where $\theta = (\sigma^{2}_t, k^{Q}_{t}, g^{Q}, \Sigma_P, \Phi_P^Z, \lambda_{1,2})$ is revised accordingly.
3 Sequential Estimation, Filtering, and Forecasting

We begin by providing the main skeleton of the scheme and then elaborate the details of its specific parts, such as the MCMC algorithm including the Kalman filter, and the framework for obtaining and evaluating the economic benefits of predictions.

3.1 Sequential Framework with Latent Processes

Let \( Y_{0:t} = (Y_0, Y_1, \ldots, Y_t) \) denote all the data available up to time \( t \), such that \( Y_{0:T} = Y \). Similarly, the likelihood based on data up to time \( t \) is \( f(Y_{0:t} | \theta, \hat{\Sigma}_Z) \) and is defined in (23). Combined with a prior on the parameters \( \pi(\theta) \), see Appendix A for details, it yields the corresponding posterior

\[
\pi(\theta | Y_{0:t}, \hat{\Sigma}_Z) = \frac{1}{m(Y_{0:t} | \hat{\Sigma}_Z)} f(Y_{0:t} | \theta, \hat{\Sigma}_Z) \pi(\theta)
\]  

where \( m(Y_{0:t} | \hat{\Sigma}_Z) \) is the model evidence based on data up to time \( t \). Moreover, the posterior predictive distribution, which is the main tool for Bayesian forecasting, is defined as

\[
f(Y_{t+h} | Y_{0:t}, \hat{\Sigma}_Z) = \int f(Y_{t+h} | Y_t, \alpha_t, \theta, \hat{\Sigma}_Z) \pi(\theta | Y_{0:t}, \hat{\Sigma}_Z) d\theta
\]  

where \( h \) is the prediction horizon and \( \alpha_t \sim N(a_{\ell|t}, P_{\ell|t}) \) is obtained from the Kalman filter.

Note that the predictive distribution in (26) incorporates parameter uncertainty by integrating out \( \theta \) according to the posterior in (25). Usually, prediction is carried out by expectations with respect to (26), e.g. \( E(Y_{t+h} | Y_{0:t}, \hat{\Sigma}_Z) \) but, since (26) is typically not available in closed form, Monte Carlo can be used in the presence of samples from \( \pi(\theta | Y_{0:t}, \hat{\Sigma}_Z) \). This process may accommodate various forecasting tasks; for example, forecasting several points, functions thereof, and potentially further ahead in the future. A typical forecasting evaluation exercise requires taking all the consecutive points of \( Y \), \( \hat{\Sigma}_Z \) and potentially further ahead in the future. A typical forecasting evaluation exercise requires taking all the consecutive points of \( Y \).

An alternative approach that can handle both model choice and forecasting assessment tasks is to use sequential Monte Carlo (see, Chopin (2002) and Del Moral et al. (2006)) to sample from the sequence of distributions \( \pi(\theta | Y_{0:t}, \hat{\Sigma}_Z) \) for \( t = 0, 1, \ldots, T \). A general description of the Iterated Batch Importance Sampling (IBIS) scheme of Chopin (2002), see also Del Moral et al. (2006) for a more general framework, is provided in Algorithm 1. The degeneracy criterion is usually defined through the Effective Sample Size (ESS) which is equal to

\[
ESS(\omega) = \frac{(\sum_{i=1}^{N_\theta} \omega_i)^2}{\sum_{i=1}^{N_\theta} \omega_i^2}
\]

and is of the form \( ESS(\omega) < \alpha N_\theta \) for some \( \alpha \in (0, 1) \), where \( \omega \) is the vector containing the weights.

The IBIS algorithm provides a set of weighted \( \theta \) samples, or else particles, that can be used to compute expectations with respect to the posterior, \( E[g(\theta) | Y_{0:t}, \hat{\Sigma}_Z] \), for all \( t \) using the estimator \( \sum_i [\omega_i g(\theta_i)] / \sum_i \omega_i \). Chopin (2004) shows consistency and asymptotic normality of this estimator as \( N_\theta \to \infty \) for all appropriately integrable \( g(\cdot) \). The same holds for expectations with respect to the posterior predictive distributions, \( f(Y_{t+h} | Y_t, \alpha_t, \hat{\Sigma}_Z) \); the weighted \( \theta \) samples can be transformed into weighted samples from \( f(Y_{t+h} | Y_t, \alpha_t, \hat{\Sigma}_Z) \) by simply applying \( f(Y_{t+h} | Y_t, \alpha_t, \theta, \hat{\Sigma}_Z) \). A very useful by-product of the IBIS algorithm is the ability to compute \( m(Y_{0:t} | \hat{\Sigma}_Z) = f(Y_{0:t} | \hat{\Sigma}_Z) \), which is the criterion for conducting formal Bayesian model choice. Computing the following quantity in step (a) in Algorithm 1 yields a
Algorithm 1 IBIS algorithm for Gaussian Affine Term Structure Models with unspanned latent factors

Initialise $N_\theta$ particles by drawing independently $[\theta_i, \alpha_0^{(i)}] \sim [\pi(\theta), N(a_0, I)]$ with importance weights $\omega_i = 1$, $i = 1, \ldots, N_\theta$. For $t, \ldots, T$ and each time for all $i$:

(a) Calculate the incremental weights from

$$u_t([\theta_i, \alpha_0^{(i)}]) = f(Y_t|Y_{0:t-1}, [\theta_i, \alpha_0^{(i)}], \hat{\Sigma}_Z)$$

and practically from

$$u_t([\theta_i, \alpha_t^{(i)}]) = f(Y_t|Y_{t-1}, [\theta_i, \alpha_t^{(i)}], \hat{\Sigma}_Z)$$

where instead of conditioning on $\alpha_0^{(i)}$ we condition on $\alpha_{t-1}^{(i)}$, which is distributed as $N\left(\alpha_{t-1|t-1}^{(i)}, P_{t-1|t-1}^{(i)}\right)$, and thus we reduce computational cost of the Kalman filter that is then iterated only once for each particle and we additionally update $\alpha_{t-1}^{(i)}$ to $\alpha_t^{(i)} \sim N\left(\alpha_{t|t}^{(i)}, P_{t|t}^{(i)}\right)$ from the Kalman filter.

(b) Update the importance weights $\omega_i$ to $\omega_i u_t ([\theta_i, \alpha_t^{(i)}])$.

(c) If some degeneracy criterion (e.g. ESS(\omega)) is triggered, perform the following two sub-steps:

(i) Resampling: Sample with replacement $N_\theta$ times from the set of $\theta_i$s according to their weights $\omega_i$. The weights are then reset to one.

(ii) Jittering: Replace $\theta_i$s with $\tilde{\theta}_i$s by running MCMC chains, and thus all iterations of the Kalman filter, with each $\theta_i$ as input and $\tilde{\theta}_i$ as output. Set $[\theta_i, \alpha_t^{(i)}] = [\tilde{\theta}_i, \alpha_t^{(i)}]$.

consistent and asymptotically normal estimator of $f(Y_t|Y_{0:t-1}, \hat{\Sigma}_Z)$

$$m_t = \frac{1}{\sum_{i=1}^{N_\theta} \omega_i} \sum_{i=1}^{N_\theta} \omega_i u_t ([\theta_i, \alpha_t^{(i)}])$$

where $\alpha_{t-1}^{(i)} \sim N\left(\alpha_{t-1|t-1}^{(i)}, P_{t-1|t-1}^{(i)}\right)$ is obtained from the Kalman filter for each $i = 1, \ldots, N_\theta$. An additional benefit provided by sequential Monte Carlo is that it provides an alternative choice when MCMC algorithms have poor mixing and convergence properties and, in general, is more robust when the target posterior is challenging, e.g. multimodal.

To apply IBIS output to models and data in this paper, the following adaptations and extensions are made. We combine the benefits of data tempering and adaptive tempering in a hybrid adaptive tempering scheme which we present in Appendix C. Since the MCMC algorithm used here is an extension of Bauer (2018), and thus it consists of independence samplers that are known to be unstable, we utilise the IBIS output and estimate posterior moments to obtain independence sampler proposals. Within such an adapted framework, we extend the methodology presented in Sections 2.2.1 and 2.2.3 to handle Kalman filtering sequentially. Eventually, we use IBIS output in the construction of a model-driven dynamically rebalanced portfolio of bond excess returns and measure its economic performance.

In applied work, we resort to $N_\theta = 2,000$ particles and 5 MCMC steps, when jittering, while for minimum ESS we set $\alpha = 0.7$. We choose 5 steps at the jittering stage because the mixing behaviour of the underlying MCMC is satisfactory enough. We tracked the correlation between particles before and after that stage to find that performance was already acceptable with this number of iterations.

3.2 Assessing Predictive Performance and Economic Value

Based on the evaluation framework outlined in this section, we then seek to understand whether unspanned latent information extracted from yields-only Gaussian ATSMs, using the Kalman filter, lets us better predict excess returns, compared to corresponding yields-only models. Furthermore, we attempt to explore whether such statistical predictabil-
ity, if any, can be turned into consistent economic benefits for bond investors. Finally, we explore whether making full use (all factors) of such unspanned latent factors benefits them most, or to the contrary (selected factors).

### 3.2.1 Bond Excess Returns

Define the observed $h$-holding period return from buying an $n$-year bond at time $t$ and selling it at time $(t + h)$ as

$$ r_{t,t+h}^n = p_{t+h}^{n-h} - p_t^n $$

where $p_{t+h}^{n-h}$ is the log price of the $(n - h)$-period bond at time $(t + h)$ and $p_t^n$ is the log price of the $n$-period bond at time $t$. The latter translates to the corresponding yield in the following manner

$$ y_t^n = -\frac{1}{n} p_t^n $$

Furthermore, define the observed continuously compounded excess return of an $n$-year bond as the difference between the holding period return of the $n$-year bond and the $h$-period yield as

$$ r_{x,t,t+h}^n = -(n - h) y_{t+h}^{n-h} + n y_t^n - h y_t^h $$

If, instead of taking the observed one, we take the model-implied continuously compounded yield $y_t^n$, calculated according to (5), we arrive at the predicted excess return $\tilde{r}_{x,t,t+h}^n$ which becomes

$$ \tilde{r}_{x,t,t+h}^n = A_{n-h,p} - A_{n,p} + A_{h,p} + B'_{n-h,p} P_{t+h} - (B_{n,p} - B_{h,p})' P_t $$

where $P_t$ is observed and $\tilde{P}_{t+h}$ is a prediction from the model. Our developed framework, see Algorithm 1, allows drawing from the predictive distribution of $(\tilde{P}_{t+h}, \tilde{r}_{x,t,t+h}^n)$ based on all information available up to time $t$. More specifically, for each $\theta_i$ particle the $\mathbb{P}$-dynamics of $P_t$ can be used to obtain a particle of $\tilde{P}_{t+h}$, which can then be transformed into a particle of $\tilde{r}_{x,t,t+h}^n$ via equation (27). Detailed steps for the case of $h = 1$ are outlined in Algorithm 2.

**Algorithm 2** Predictive distribution of excess returns for Gaussian Affine Term Structure Models with unspanned latent factors

First, at time $t$, for some $n$ and $h = 1$, using $(\omega, \theta_i), i = 1, \ldots, N_\theta$, from the IBIS algorithm, iterate over $i$:

(a) Given $\theta_i$, compute $A_{i+1,p}$ and $B_{i+1,p}$, for $i_1 \in \{1, n - 1, n\}$, from (6) and (7).

(b) Given $\theta_i$, obtain prediction of $P_{t+1}$ by drawing, first from

$$ \alpha_{t+1}^{(i)} \sim N \left( \Phi_P \alpha_{t+1}^{(i)}, \Sigma_P \right) $$

and then from

$$ \tilde{P}_{t+1}^{(i)} | \alpha_{t+1}^{(i)} \sim N \left( \mu_P + \Phi_P \alpha_{t+1}^{(i)}, \Sigma_P \right) $$

where distribution of $\alpha_{t+1}^{(i)}$ is obtained from the Kalman filter as $N \left( \alpha_{t+1}^{(i)} \Big| \mu_t, \Sigma_{1,t+1}^{(i)} \right)$.

(c) Compute particle prediction of $r_{x,t,t+1}^n$ as

$$ \tilde{r}_{x,t,t+1}^n = A_{n-1,p} - A_{n,p} + A_{1,p} + B'_{n-1,p} \tilde{P}_{t+1}^{(i)} - (B_{n,p} - B_{1,p})' P_t $$

Second, since $(\omega_i, \tilde{P}_{t+1}^{(i)}, \tilde{r}_{x,t,t+1}^n), i = 1, \ldots, N_\theta$, is a particle approximation to predictive distribution of $(P_{t+1}, r_{x,t,t+1})$, compute the point prediction of $r_{x,t,t+1}$ using particle weights $\omega_i$, as

$$ \tilde{r}_{x,t,t+1}^n = \frac{1}{\sum_{i=1}^{N_\theta} \omega_i} \sum_{i=1}^{N_\theta} \omega_i \tilde{r}_{x,t,t+1}^{(i)} $$

Third, repeat the above two steps for different $n$ and $h$. For $h > 1$, use $\tilde{P}_{t+h-1}$ and $\alpha_{t+h-1}^{(i)}$ in place of $P_t$ and $\alpha_{t+1}^{(i)}$, and $i_1 \in \{h, n - h, n\}$ in place of $i_1$. 

---

1. On Unspanned Latent Risks in Dynamic Term Structure Models
The predictive accuracy of bond excess return forecasts is measured in relation to an empirical benchmark. We follow related literature and first adopt the EH as this benchmark, which essentially uses historical averages as the optimal forecasts of bond excess returns. This empirical average is

$$r^n_{x,t+h} = \frac{1}{t-h} \sum_{j=1}^{t-h} r^n_{x,j+h}.$$  

To assess the predictive ability of models considered, we first consider the out-of-sample $R^2 (R^2_{os})$, following [Campbell and Thompson (2008)], defined as:

$$R^2_{os} = 1 - \frac{\sum_{s=t_0}^{t} (r^n_{x,s,s+h} - \tilde{r}^n_{x,s,s+h})^2}{\sum_{s=t_0}^{t} (r^n_{x,s,s+h} - \bar{r}^n_{x,s+s+h})^2}.$$  

for $\tilde{r}^n_{x,s,s+h}$ being the mean of the predictive distribution. Positive values of this statistic mean that model-implied forecasts outperform the empirical averages and suggest evidence of time-varying return predictability. The values of $R^2_{os}$ are aggregated over all prediction times ($t_0$ to $t$) for each maturities. In order to get a feeling for how large the differences from the EH or model $M_1$ benchmarks are, we report the p-values from one-sided Diebold-Mariano test (see, Gargano et al (2019)), with Clark and West (2007) adjustment as in Wan et al (2021), noting that these are viewed as indices rather than formal hypothesis tests.

3.2.2 Economic Performance of Excess Return Forecasts

From a bond investor’s point of view it is of paramount importance to establish whether the predictive ability of a model can generate economically significant portfolio benefits, out-of-sample. The portfolio performance may also serve as a metric to compare models that make more or less use of unspanned latent information, that is include more or fewer unspanned latent factors. To that end, we consider a Bayesian investor with power utility preferences

$$U(W_{t+h}) = U(w^n_t, r^n_{x,t+h}) = \frac{W_t^{1-\gamma}}{1-\gamma},$$

where $W_{t+h}$ is an $h$-period portfolio value, $\gamma$ is the coefficient of relative risk aversion and $w^n_t$ is a portfolio weight on the risky $n$-period bond. If we let $(1 - w^n_t)$ be a portfolio weight of the riskless $h$-period bond, then the portfolio value $h$ periods ahead is given as

$$W_{t+h} = (1 - w^n_t) \exp(r^n_t) + w^n_t \exp(r^n_t + r^n_{x,t+h})$$

where $r^n_t$ is the risk-free rate. Such an investor maximises her expected utility over $h$-periods in the future, based on $x_{1:t} = \{r^n_{x,t}, \mathcal{P}_{0:t}\}$, i.e. the available information up to time $t$, according to

$$E_t[U(W_{t+h})|x_{1:t}] = \int U(W_{t+h}) f(W_{t+h}|x_{1:t})dW_{t+h},$$  

$$= \int U(w^n_t, r^n_{x,t+h}) f(r^n_{x,t+h}|x_{1:t}) dr^n_{x,t+h},$$

where $f(r^n_{x,t+h}|x_{1:t})$ is the predictive density described earlier. At every time $t$, our Bayesian investor solves an asset allocation problem getting optimal portfolio weights by numerically solving

$$\tilde{w}_t^n = \arg \max \frac{1}{\sum_{i=1}^{N_{\theta}} \omega_{\theta_i}} \sum_{j=1}^{N_{\theta}} \omega_{\theta_j} \left\{ [(1 - w^n_t) \exp(r^n_t) + w^n_t \exp(r^n_t + \tilde{r}^n_{x,t+h})]^{1-\gamma} \right\}$$

with $N_{\theta}$ being the number of particles from the predictive density of excess returns, weighted using importance weights $\omega_{\theta_i}, i = 1, ..., N_{\theta}$, which come from the IBIS algorithm.
To obtain the economic value generated by each model, we use the resulting optimum weights to compute the CER as in Johannes et al. (2014) and Gargano et al. (2019). In particular, for each model, we first and foremost define the CER as the value that equates the average utility of each model against the average utility of the EH benchmark specification. Denoting realised utility from the predictive model as \( \tilde{U}_t = U \left( \tilde{w}_t^n, \{ \tilde{r}_t^{n,j} \}_{j=1}^N \right) \) and realised utility from the EH benchmark as \( U_t \), we get
\[
CER = \left( \frac{\sum_{s=t_0}^t \tilde{U}_s}{\sum_{s=t_0}^t U_s} \right)^{\frac{1}{1-\gamma}} - 1
\]

Then, we compare the portfolio performance of each model we develop in this paper against the EH benchmark and model \( M_1 \). Finally, in a similar manner to \( R^2 \), we report p-values of CER results based on one-sided Diebold-Mariano test, which are viewed informally as indices rather than hypothesis tests.

4 Data and Models

In this section we discuss the data we use throughout in detail. Specifically, we elaborate on the US Treasury yields data we employ in this paper, and how we split them into a training and a testing subsample. Furthermore, we describe the macroeconomic variables we use to identify the nature of the hidden components extracted from the latent factors we obtain. We then also explain what models we consider, as distinguished by the different positions unspanned latent factors take in the given model.

4.1 Yields and Macros

The yield data set we use contains monthly observations of US Treasury zero-coupon bond yields with maturities of 1, 2, 3, 4, 5, 7 and 10 years, spanning the period from January 1985 to the end of 2018. We split our sample into two sub-samples, a training one ending at the end of 2007 which, as such, precludes the relevant financial crisis, and a testing one which includes the period after the end of 2007, a period determined by, first, different monetary actions and the establishment of unconventional policies and, second, interest rates hitting the zero-lower bound. The post-2007 global financial crisis period is excluded from our training sub-sample. By doing so we let the models learn about this event merely online and we expect the use of unspanned latent information to facilitate that. As such, we also verify the concerns about the capability of Gaussian ATSMs to deal with the zero-lower bound (see, Kim and Singleton (2012) and Bauer and Rudebusch (2016)). These concerns are explicitly explored in the testing sub-sample, spanning the period from January 2008 to the end of 2018. The data processing involves extracting the first three principal components from the yield curve. This can be done either by calculating the principal component (PC) loadings from the training or the entire period. The correlations between these series reach levels above 0.99, suggesting negligible differences. Nevertheless, in order to prevent any data leaking issues when assessing the predictive performance, the loadings from the training period only are used.

In terms of the macroeconomic variables for the US economy which we use here specifically to identify the nature of the hidden part in the risk premium factor, as described in the next section, we consider several indices, some of which are well covered in the literature. These include core inflation (CPI) as in Cieslak and Povala (2015), the three-month moving average of the Chicago Fed National Activity Index (GRO) from Joslin et al. (2014), which is a measure of current real economic conditions, as well as three variables from Ludvigson and Ng (2009), namely, real activity (\( F_1 \)), its cube (\( F_1^3 \)), and stock market (\( F_8 \)) factors. Moreover, we complement those with the unemployment

Statistical significance is measured using a one-sided Diebold-Mariano statistic with Clark-West adjustment, based on Newey-West standard errors.

Yields used are unsmoothed Fama-Bliss yields constructed by Le and Singleton (2013). We thank Ahn Le for generously providing the data set.

In contrast to Cieslak and Povala (2015), who devise trend inflation by appropriately smoothing core inflation, we use the latter as is. Such a choice is better suited for comparisons with unspanned latent factors, which are a-priori independent, first order autoregressive processes, not necessarily smooth.
rate \((UNR)\) and manufacturing capacity utilisation \((MNF)\). All these macroeconomic time series are at a monthly frequency and the period is such that it corresponds to that for yields, as discussed above.

### 4.2 Models and Rationale Behind

In addition to the previously defined \(M_0\) and \(M_1\) yields only models, we also consider several models that can be viewed as the \(M_1\) model with additional unspanned latent factors. We differentiate these models by the positions the unspanned latent factors take. The naming of these models is set by the following scheme

\[
LF_{ijk} \quad i, j, k \in \{0, 1\}
\]

where, for example, for \(i = 0\) and \(j = k = 1\), \(LF_{011}\) means that we allow for unspanned latent information in the second and third equation in (15), under the assumption that \(R = 3\). We only investigate a subset of the available alternative models in greater detail.

As a result of the risk price restrictions (on \(\lambda_P\)) we adopt in this paper (model \(M_1\)), we define the stochastic risk premium factor as

\[
RP_t^Z = \lambda P_t + Z_t = \begin{bmatrix}
\lambda_{1,2} P_{2,t} + Z_{1,t} \\
Z_{2,t} \\
Z_{3,t}
\end{bmatrix}
\]

where the first element is similar to Duffee (2011), yet in our case its time-varying component \((\lambda_{1,2} P_{2,t})\) is a restricted version of the 'risk premium factor' defined in Duffee’s paper since only \(\lambda_{1,2}\) is free, suggesting that investors require compensation from level risk stemming from changes to slope only. The stochastic component, \(Z_{1,t}\), which is \textit{a priori} unspanned and latent, affects this compensation. In that particular case, \(Z_{1,t}\) is the first element of the unspanned latent variable \(Z_t\). If instead \(i = 0\) and \(j = k = 1\), then

\[
RP_t^Z = \begin{bmatrix}
\lambda_{1,2} P_{2,t} \\
Z_{2,t} \\
Z_{3,t}
\end{bmatrix}
\]

where the first element is related to level risk and equals \(RP_t\). In this element, the stochastic component is now eliminated and as such, the time-varying risk premium factor is equivalent to the corresponding factor in yields only model \(M_1\).

What is important to understand is whether or not the impact of the stochastic component in (30) is distorting information relevant to predicting excess returns, already available in \(RP_t\) alone. One way forward towards achieving this goal is to conduct predictability and economic value exercises for the models considered and make appropriate comparisons, as is done in Section 5. Furthermore, we should examine \textit{a posteriori} to what extent \(Z_t\), assumed unspanned and latent \textit{a priori}, is actually hidden from the yield curve, and if so, whether such hidden information contributes to the predictability and economic value results. Importantly, we should investigate the relationship, if any, between the hidden component of the unspanned latent factors and macroeconomic activity.

Inspired by the empirical design of Duffee (2011), we define the hidden part of the stochastic risk premium factor as the part unspanned by \(P_t\) in the following way

\[
\bar{RP}_t^Z = RP_t^Z - E[RP_t^Z|P_t] = Z_t - E[Z_t|P_t]
\]

---

We collect those data sets from St. Louis FRED (https://fred.stlouisfed.org)

In Duffee (2011), the time-varying risk premium factor, \(RP_t\), is a linear combination of the state vector and as such, investors demand compensation to face shocks to all factors.
where $E[Z_t|\mathcal{P}_t]$ is the projection of unspanned latent factors on principal components obtained from observed yields, which is thus spanned and consequently not a hidden part of this factor. After Duffee (2011), $\tilde{R}P^Z_t$ can be estimated as residual from a regression of the former on the latter

$$
\tilde{R}P^Z_t \equiv Z_t - a - b'\mathcal{P}_t
$$

where we take $Z_t$ as the mean from its posterior distribution and $a$ and $b$ are the underlying Ordinary Least Squares parameter estimates.

### 5 Empirical Results

This section presents the main results of the statistical and economic performance exercises described in Sections 3.2.1 and 3.2.2. In particular, we seek to explore whether predictability of bond excess returns, as well as economic benefits to bond investors, can be enhanced by integrating information hidden from the yield curve. Prior to that, we assess the dynamics of the estimated latent factors and explore their ability to improve explanatory power on excess returns. Eventually, we seek to uncover a link between latent risks and macroeconomic activity.

#### 5.1 Observing the Unobserved

The yield data contain important economic events, such as the 2008-2009 recession period, and therefore it is interesting to examine the trajectories of unobserved factors during such periods. In particular, this episode started from a flat yield curve during the pre-recession period, following an inversion of the curve reflecting an increase in short rates due to expectations about the Fed tightening its policy, and then a steepening as a reaction to policy adjustments by the Fed reflecting strong growth and inflation expectations.

Models with unspanned latent factors have picked up those adjustments differently, as depicted in Figure 1. In particular, economic uncertainty arising immediately in 2008 and lasting until about 2012, a year approximately marking the end of the crisis, is well reflected by a decrease, from positive to negative, in the posterior mean of the latent factor $Z_{2,t}$ (models $LF_{111}$ and $LF_{011}$), affecting the slope of the yield curve, following a rebound to zero thereafter. This picture is quite different when we look at the posterior mean of the unobserved factor $Z_{1,t}$ (model $LF_{111}$), which has an impact on the level of the yield curve. In this case, we only observe a steep decrease from zero in the beginning of 2008, followed by a W-shaped recovery lasting until early 2009. A look at the unobserved dynamics behind the curvature of the yield curve, namely the posterior mean of $Z_{3,t}$ (models $LF_{111}$, $LF_{011}$ and $LF_{001}$), reveals increased oscillations around zero starting in 2008 and lasting approximately until 2012, which are definite shocks, though nontrivial to interpret. It seems that the estimated latent factors capture important effects associated with the yield curve. Of particular importance, however, is the source of information and whether it can contribute to the explanatory power of excess returns.

#### 5.2 Explanatory Power of Unspanned Latent Factors

Table 1 presents adjusted $R^2$ for the projection of the three unspanned latent factors on the three principal components. Results reveal a large explanatory power of the unspanned latent factor $Z_{2,t}$, associated with slope risk. Quite interestingly, regardless of the modelling specification, a very large fraction (ranging between 45% and 54%) of the variation of $Z_{2,t}$ is explained by the information spanned by the yield curve. Nevertheless, an important question that arises is whether there is also information hidden from the yield curve that, as such, cannot be captured by the three PCs, but which is useful for prediction purposes. We seek to answer the aforementioned question in what follows below.

Table 2 reports adjusted $R^2$ for model $LF_{001}$, which features a latent factor $Z_{3,t}$ in addition to the three yield factors. When adding an unspanned latent factor on the third PC, the explanatory power across the maturity spectrum remains, which is different from Duffee (2011). We obtain the state vector from observed yields directly.
in general, intact. There are a few small, albeit statistically significant, increases mainly at longer maturities and short prediction horizons.

The situation is reversed when adding $Z_{2,t}$ to the information set. More specifically, the latent factor substantially improves explanatory power on excess returns, especially at shorter maturities and longer horizons, as shown in Table 3. In particular, at 2-year maturity, adjusted $R^2$ increases from 13.54%, in the yields-only case, at the 6-month horizon, up to 16.15%. At the 12-month horizon, the associated $R^2$ jumps from 17.3% up to 23.58%, a statistically significant increase of 7.59 percentage points. Those increases do not hold at longer maturities, implying that the predictive power of $Z_{2,t}$ is stronger at short maturities only, which reveals that unspanned $Z_{2,t}$ captures information associated with the short end of the yield curve. This stands in line with prior evidence that different maturities do not move in response to a single factor.

Beside the fact that adding a latent factor on the first PC, on top of the other latent factors in model $LF_{111}$, has no impact on the explanatory power, as measured by $R^2$, results for this model and also for $LF_{011}$ are quantitatively and qualitatively similar, and thus not shown. Moving forward, it is of paramount importance to assess whether the evident explanatory power of the unspanned factors on excess returns is enough to offer any statistical and economic evidence of return predictability.

5.3 Bond Return Predictability and Economic Performance

5.3.1 Bond Return Predictability

This subsection presents results on the statistical performance of bond excess return forecasts generated by models with unspanned latent risks. Table 4 reports out-of-sample $R^2$ values, for all models across bond maturities. Our empirical results focus on a one-month holding period and as such target non-overlapping excess bond returns. The in-sample period is 1985-2007 and the out-of-sample analysis spans the period 2008-2018. We, first, consider evidence in relation to the EH benchmark (i.e. Panel A). With the notable exception of the maximally flexible model, $M_0$, which performs poorly out-of-sample, most models attain good predictive performance, with positive $R^2_{os}$ across bond maturities, revealing strong evidence of out-of-sample bond return predictability. This is important considering that all models utilise information coming solely from the cross section of yields (either ‘yield-only’ or ‘yield-plus’ in Duffee (2011) terminology).

Furthermore, our results suggest that models with latent factors (on the second and third PC) tend to produce more accurate forecasts and generate higher $R^2_{os}$, especially at shorter maturities. In particular, for model $LF_{010}$ ($LF_{011}$), which embeds a single unspanned latent factor on the second (and third) PC, $R^2_{os}$ is 5.86% (5.65%) and highly significant for the 2-year maturity bond and 3.42% (2.57%) for the 10-year maturity bond. It seems that the latent factor on the second PC, which captures variation associated with the steepness of the curve, adds considerable improvement to the predictive power of models, revealing that investors demand compensation for carrying slope risk.

To investigate this further, we then ask whether the unspanned latent factors share information to predict returns that goes beyond the cross section of yields. Panel B of Table 4 displays the out-of-sample $R^2$ values for all models in comparison to model $M_1$, which imposes heavy restrictions on the dynamics of risk compensation and has been proved to offer the best predictive performance among all yield-only models. Results provide clear evidence of significant improvement to the out-of-sample predictive performance, suggesting that unspanned latent risks are capable of exploiting information relevant to predicting returns beyond the term structure of interest rates. In particular, for model $LF_{010}$, which features a single latent factor on the second PC and offers the best performance in terms of out-of-sample predictability, relative to EH, $R^2_{os}$ is 4.62% (3.31%) for the 2-year (3-year) maturity bond, while for models with two unspanned factors, such as models $LF_{011}$ and $LF_{110}$, $R^2_{os}$ are 4.40% (2.47%) and 4.31% (2.47%) respectively. Qualitatively similar (albeit weaker) predictive performance is observed for models $LF_{011}$ and $LF_{111}$. Finally, comparing across bond maturities, it seems apparent that such improvements are particularly pronounced at shorter maturities, suggesting that unspanned latent factors share information associated with the short end of the maturity spectrum.
5.3.2 Economic Performance

Turning to economic performance, we ask whether models with unspanned latent risks offer any economic evidence of return predictability in the US bond market. We therefore seek to understand whether the improved statistical performance, evidenced in Section 5.3.1, is indeed translated into better economic performance. To establish a better link with the existing literature, we investigate the economic evidence considering three different scenarios for investors. The first two prevent them from taking extreme positions, while the third relaxes restrictions and allows for maximum leveraging and short-selling. In particular, in the first scenario we follow Thornton and Valente (2012), Sarno et al. (2016) and Gargano et al. (2019) and restrict portfolio weights to range in the interval $[-1, 2]$, thus imposing maximum short-selling and leveraging of 100% respectively. Second, we follow Huang et al. (2020) and restrict portfolio weights to the interval $[-1, 5]$ which keeps maximum short-selling at 100% while increasing the upper bound, which now amounts to a maximum leveraging of 400%. Third, we follow Bianchi et al. (2021) and impose no allocation restrictions to investors, allowing for portfolio weights to be unbounded. Finally, we set the coefficient of risk aversion to $\gamma = 5$, across scenarios.

Table 5 reports results for the out-of-sample economic value exercise described in Section 3.2.2. In particular, it displays annualised CER values, for different models, generated using forecasts of bond excess returns. Positive values indicate that models perform better than the EH benchmark. Panel A presents evidence under the first investment scenario. As expected, the only model that fails to translate predictability into economic value is the maximally flexible model, $M_0$, which is not capable of producing any positive out-of-sample economic benefits, mostly generating CERs with negative sign across maturities and investment scenarios. Notably, highly restricted model $M_1$, which has been proved to offer the best predictive and economic performance among those models that consider information coming solely from the cross section of yields, seems to struggle to generate any gains to bond investors, as evidenced by CER values which are either close to zero or marginally positive but still statistically insignificant. The situation is even more pronounced for the second (Panel B) and third (Panel C) investment scenarios.

The situation is reversed, however, for models with unspanned latent factors, where our results tell a consistent story. No matter the model specification considered, corresponding CER values are positive and statistically significant, revealing strong economic performance especially at longer maturities. Importantly, models are capable of translating the evident statistical predictability into economic gains for bond investors seeking to maximise their utility. Comparing across models, $LF_{001}$ and $LF_{010}$ generate the largest portfolio gains with CER values ranging between 0.09% and 1.85% for the former model specification and 0.12% to 1.49% for the latter.

Much greater improvements are observed under the second investment scenario (i.e. Panel B), where corresponding CERs increase in magnitude up to 2.96% (2.85%) and 2.48% (1.84%) respectively for bonds with maturity of 7-years (10-years). Interestingly, the economic performance of models $LF_{100}$, $LF_{110}$ and $LF_{111}$ appears to deteriorate, with CER values that are mostly (marginally) positive but statistically insignificant.

Results are quite pronounced when bounds on portfolio allocations are completely relaxed (i.e. Panel C). In particular, models $LF_{001}$, $LF_{010}$ and $LF_{001}$, which offer the largest portfolio gains, generate CER values that range between 2.26% and 2.42% for a 2-year maturity bond and between 1.90% and 2.12% for a medium-term 5-year maturity bond. For a 10-year bond, CERs range between 1.25% (but insignificant) for model $LF_{001}$ to 2.85% and significant for model $LF_{011}$. In contrast, models $LF_{100}$, $LF_{110}$ and $LF_{111}$ fail to offer any statistically significant gains, out-of-sample.

Results provide evidence that models with latent risks are capable of exploiting information relevant not only to predict returns but also to offer value and generate significant out-of-sample portfolio benefits to bond investors seeking to maximise their utility. This is quite an important finding, considering that investors only have access to yield curve data, and no other information is utilised, in contrast to studies which seek to detect further information from macroeconomic sources (see, Gargano et al. (2019), Bianchi et al. (2021) and Wan et al. (2021), among many others).
Next, we move to investigate in greater depth, whether models with unspanned latent risks offer any meaningful utility gains to bond investors when compared to model $M_1$, which has been proved to generate the largest portfolio benefits among all yields-only models (see, Dubiel-Teleszynski et al. (2021)). As such, we repeat the economic value exercise by computing the corresponding CERs relative to model $M_1$. Table 6 summarises the results, which offer interesting insights. We, first, consider the results associated with the first scenario (i.e. Panel A), which show evidence in favour of positive out-of-sample economic gains for bond investors. In particular, for most models, corresponding CERs are positive and statistically significant, especially for mid- to long-term investments. Generated utility gains are up to 1.07% across models and maturities, revealing evidence of economic improvement offered by latent factors that is marginal, but nevertheless still of value to investors.

Further improvements on the magnitude of CERs are observed under the second investment scenario where we relax the upper bound, allowing for maximum leveraging of 400%. In particular, model $LF_{001}$ delivers CER values that range between 0.17% and 1.64%, while model $LF_{010}$, (which features a latent factor on the second PC) generates statistically significant CERs that range between 0.27% and 1.37%. CERs increase with maturity, suggesting more profitable investments for long maturity bonds. Those results reveal that investors who utilise information beyond the yield curve are capable of generating larger gains, compared to those that consider information coming solely from the term structure of interest rates. Notably, the situation is reversed for models $LF_{100}$, $LF_{110}$ and $LF_{111}$, where performance appears to deteriorate, as showcased by CER values that mostly have negative signs or are either close to zero or marginally positive but still statistically insignificant.

Results are even more pronounced under the unconstrained allocation scenario as shown in Panel C. We find that CER values are positive and statistically significant for the best performing models, $LF_{001}$, $LF_{010}$ and $LF_{011}$, while the performance of models $LF_{100}$, $LF_{110}$ and $LF_{111}$ is inferior, generating CERs which are mostly negative. In particular, for model $LF_{001}$ CER values range between 1.00% and 1.64%, with the most profitable investment being the 7-year bond, while for model $LF_{010}$, statistically significant CERs range between 0.97% and 1.35%, with the 5-year maturity bond being the most profitable. Results provide evidence that non-overlapping excess bond returns not only are predictable, but also offer significant economic benefits to investors when compared to the best-performing yield-only model $M_1$.

In all, our results provide two important and novel conclusions. First, they suggest that the unspanned latent risks share important and relevant information beyond that conveyed by the cross section of yields and this information contributes substantially not only to the out-of-sample predictive performance of models, but also to investors’ utility. This is not entirely surprising, however, given that the latent factors evolve according to an autoregressive process of order one, thus improvements, if any, are expected at such a short investment horizon. It further supports the evident ability of the models to exploit information hidden from the yield curve. Second, such unspanned information is not contributed equally, but rather varies across the bond maturity spectrum.

Those findings are in direct contrast to the conclusions of Wan et al. (2021), who argue that statistical predictability is not turned into economic benefits, and in line with the studies of Gargano et al. (2019) and Bianchi et al. (2021), which find support of economically significant predictability in bond returns. The latter, however, exploit information from a fully revised macroeconomic dataset, which is proved to enhance predictive power (see, Ghysels et al. (2018)). Our results are not reliant on macroeconomic data.

### 5.4 Linking Unspanned Latent Risks with the Macroeconomy

Driven by the statistical and economic performance of models with unspanned risks, we seek to investigate their link with macroeconomic forces. In particular, we are interested in investigating further the relationship, if any, between the hidden component of the latent factors, as defined in (31), and macroeconomic (rather than pure financial) variables proven to forecast variation in bond excess returns. Inspired by the empirical design of Duffee (2011), we follow the procedure described in Section 4.2. Table 7 presents adjusted $R^2$ from regressing the hidden component of risk premia
on different macroeconomic variables. Looking at model $LF_{010}$ (i.e. second panel), our results reveal that particular macros are capable of explaining substantial amounts of variation in the hidden component of the latent factor $Z_{2,t}$, which is related to slope risk. In particular, $GRO$ explains 11% while $MNF$ ($UNR$) captures up to 19% (15%) of the variation in $\tilde{RP}_{Z_{2,t}}$. Even more importantly, the hidden component of bond risk premia associated with the second latent factor seems to be countercyclical, as revealed by the negative sign on the (statistically significant) coefficients of the real activity indicators, $GRO$ and $F_1^{15}$ displayed in Table 8. This further implies that shocks to $GRO$ (and $F_1$) induce counter-cyclical movements in the risk premium associated with slope risk. In contrast, the loading on unemployment ($UNR$) is positive and statistically significant.

Explanatory power is even more pronounced for models $LF_{011}$ and $LF_{111}$. In the former case, 13% and 11% of the variation in $\tilde{RP}_{Z_{2,t}}$ is captured by measures of economic activity (i.e. $GRO$ and $F_1$), whereas $MNF$ leads to a large $R^2$ of 21%. In the latter model, the fraction of the hidden component of $Z_{2,t}$ that is explained by macros ($MNF$ and $UNR$) is even higher, with adjusted $R^2$ being 23% and 27% respectively. Corresponding results on the cyclicity of the hidden component for this latent factor in these two models are qualitatively similar as evidenced by the negative (statistically significant) coefficients, see Table 8.

Overall, we document that the hidden component of bond risk premia associated with the slope factor is counter-cyclical and mostly related to variables that proxy real activity. As such, our results uncover a direct link between macroeconomic activity and excess return predictability in the US bond market.

6 Conclusions

This paper explores the importance of information hidden from the yield curve and assesses the capability of unspanned risks to offer evidence of statistical and economic benefits to bond investors. We propose a novel class of arbitrage-free unspanned DTSM that embeds a stochastic price of risk specification. The model is factorised into a 'spanned' component, which exploits information provided by the yield curve, and an 'unspanned' component, which integrates information that is hidden from the cross section of yields yet relevant for prediction purposes. The latter is estimated using the associated Kalman Filter via developing a tailored inferential and prediction SMC scheme that takes into account all relevant uncertainties. The developed setup is then used to explore how valuable the unspanned information is to a real-time Bayesian investor seeking to forecast future excess bond returns and generate systematic economic gains.

Empirical results provide clear evidence of out-of-sample bond return predictability compared to the EH benchmark. Comparisons relative to the restricted yields-only model, $M_1$, reveal improvement to the out-of-sample predictive performance of models that allow for unspanned latent risks, suggesting that those risks contain significant predictive power above and beyond the yield curve. Such improvements are particularly pronounced at shorter maturities revealing that latent factors carry information associated with the short end of the maturity spectrum.

Most importantly, these models generate CER values which are positive and statistically significant, and as such, they are capable of translating the evident statistical predictability into economic gains for bond investors seeking to maximise their utility. Comparisons relative to model $M_1$, which generates the largest portfolio benefits among all yields-only models, help us infer further important conclusions. We find that CER values are positive and statistically significant, revealing that non-overlapping returns not only are predictable, but also offer significant economic gains to investors, out-of-sample. These results further support the ability of the models to exploit information hidden from the yield curve; in particular, information which is short lived. This finding is in contrast to existing literature (see, Wan et al. (2021), etc.).

\footnote{We follow Duffee (2011) and normalise the real activity factor of Ludvigson and Ng (2009), $F_1$ (and consequently its cube $F_1^3$), to positively co-vary with $GRO$.}
Finally, we examine whether movements in excess returns bear any relationship with the economy. As such, we explore the linkages between the hidden component of the unspanned latent factors and macroeconomic variables. We find that particular macros explain substantial variation of the hidden component, revealing a direct line between macroeconomic activity and excess return predictability in the US bond market. In particular, the hidden component associated with slope risk is countercyclical and relates to real activity.

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A Specification of Priors

We first transform all restricted range parameters so that they have unrestricted range. Specifically, we work in the log-scale of $k_Q^\infty$ and consider a Cholesky factorization of $\Sigma$ where, again, the diagonal elements are transformed to the real line. In order to preserve the ordering of the eigenvalues $g_Q^Q$ we apply a reparametrisation and work with their increments that are again transformed to the real line. Next, independent Normal distributions with zero means and large variances are assigned, with the exception of $\lambda_{1,2}$ where the variance is set to the number of observations and $\sigma^2_e$ where we assign the conjugate Inverse-Gamma prior as in Bauer (2018).

For $\Phi_Z$ and $\Sigma_P$, we first transform their restricted range components, so that they have unrestricted range. We also scale those of their elements, which typically take very small values. Specifically, we work with logit transformation to constrain diagonal elements of $\Phi_Z$ in $(-1, 1)$ and consider a Cholesky factorisation of $\Sigma_P$ where the diagonal elements are transformed to the real line and off-diagonal elements are scaled by $10^4$. Next, independent normal distributions with zero means are assigned to each of their components. Large variances are assigned to each element of $\Sigma_P$ and for the diagonal elements of $\Phi_Z$ respective variances are set to 2.

B Markov Chain Monte Carlo Scheme

Following from (24) and (25), and given a prior $\pi(\theta)$ as described in Appendix [A], the posterior can be written in a more detailed manner as

$$
\pi(\theta|Y, \tilde{\Sigma}_Z) = \left\{ \prod_{t=0}^T f_Q(y_t|P_t, k_Q^\infty, g_Q^Q, \Sigma_P, \sigma^2_e) \right\} \times \
\left\{ \prod_{t=1}^T f_P(P_t|P_{t-1}, k_Q^\infty, g_Q^Q, \Sigma_P, \Phi_Z^P, \lambda_{1,2}, \tilde{\Sigma}_Z) \right\} \times \pi(\theta).
$$

The above posterior is not available in closed form. Thus, methods such as MCMC can be used to draw samples from it using Monte Carlo. Nevertheless, the MCMC output is not guaranteed to lead to accurate Monte Carlo calculations in cases of chains with poor mixing and convergence properties. It is therefore recommended to construct a suitable MCMC algorithm that does not exhibit such unfavourable characteristics. Such an MCMC scheme is outlined in Algorithm [5].
Algorithm 3 MCMC scheme for Gaussian affine term structure models with unspanned latent factors

Initialise all values of $\theta$. Then at each iteration of the algorithm:

(a) Update $\sigma^2_t$ from its full conditional distribution that can be shown to be an Inverse Gamma distribution with parameters $\alpha/2$ and $\beta/2$, such that $\alpha = \alpha + T(J - R)$ and $\beta = \beta + \sum_{t=0}^{T} \| \hat{e}_t \|^2$, where $\alpha = \beta = 0$, since the prior is assumed diffuse, $\hat{e}_t$ is a time-$t$ residual from (12), and $\| . \|^2$ is Euclidean norm squared.

(b) Update $\Sigma_{\theta}$ using an independence sampler based on the maximum likelihood estimate and the Hessian obtained before running the MCMC, using multivariate t-distribution with 5 degrees of freedom as proposal distribution.

(c) Update $(k_{\theta_{i}}^\alpha, g_{\theta_{i}}^\alpha)$ in a similar manner to (b).

(d) Update $(\Phi_{2}, \lambda_{1,2})$ in a similar manner to (b).

C Adaptive Tempering

Adaptive tempering serves the goal of smoothing steep transitions between posteriors as the data accumulate. Implementation of the IBIS scheme with hybrid adaptive tempering steps is presented in Algorithm 4. Unlike the standard IBIS case shown in Algorithm 1, we initialise the particles by drawing from the posterior $\pi(\theta|Y_{0:t-1}, \widehat{\Sigma}_{z})$ instead of the prior $\pi(\theta)$, and $N(\theta_{0},\Sigma_{0})$ for the latent part. This is done for practical reasons since we are only interested in the posteriors time $t$ onwards.

It is straightforward to implement step 4(b) iv in Algorithm 3 for an independence sampler; however, adjustments are necessary for a full Gibbs step. This is the case for $\sigma^2_t$ in step (a) in Algorithm 3, see Appendix B. Technical details are the same as in the corresponding appendix to the paper we refer to above.

Algorithm 4 IBIS algorithm with hybrid adaptive tempering for Gaussian Affine Term Structure Models with unspanned latent factors

Initialise $N_\theta$ particles by drawing independently $[\theta_i, a_{t-1}^{(i)}] \sim \pi(\theta|Y_{0:t-1}, \widehat{\Sigma}_{z})$ with importance weights $\omega_i = 1, i = 1, \ldots, N_\theta$. For $t, \ldots, T$ and each time for all $i$:

1. Set $\omega_i' = \omega_i$.
2. Calculate the incremental weights from $u_t([\theta_i, a_{t-1}^{(i)}]) = f(Y_t|Y_{t-1}, a_{t-1}^{(i)}, \theta_i, \widehat{\Sigma}_{z})$ where $a_{t-1}^{(i)} \sim N \left( a_{t-1|t-1}^{(i)}, P_{t-1|t-1}^{(i)} \right)$, and update $a_{t}^{(i)}$ to $a_{t}^{(i)} \sim N \left( a_{t|t}^{(i)}, P_{t|t}^{(i)} \right)$ from the Kalman filter.
3. Update the importance weights $\omega_i$ to $\omega_i u_t([\theta_i, a_{t-1}^{(i)}])$.
4. If degeneracy criterion $\text{ESS}(\omega)$ is triggered, perform the following sub-steps:
   (a) Set $\phi = 0$ and $\phi' = 0$.
   (b) While $\phi < 1$
      i. If degeneracy criterion $\text{ESS}(\omega''')$ is not triggered, where $\omega_{i}''' = \omega_i' [u_t([\theta_i, a_{t-1}^{(i)}])]^{1-\phi'}$, set $\phi = 1$, otherwise find $\phi \in [\phi', 1]$ such that $\text{ESS}(\omega''')$ is greater than or equal to the trigger, where $\omega_{i}'' = \omega_i' [u_t([\theta_i, a_{t-1}^{(i)}])]^{\phi-\phi'}$, for example using bisection method, see [Kantas et al., 2014].
      ii. Update the importance weights $\omega_i$ to $\omega_i' [u_t([\theta_i, a_{t-1}^{(i)}])]^{\phi-\phi'}$.
      iii. Resample: Sample with replacement $N_\theta$ times from the set of $\theta_i$s according to their weights $\omega_i$. The weights are then reset to one.
      iv. Jitter: Replace $\theta_i$s with $\hat{\theta}_i$s by running MCMC chains with each $\theta_i$, as input and $\hat{\theta}_i$ as output, using likelihood given by $f(Y_{0:t-1}|\theta, \widehat{\Sigma}_{z})/f(Y_t|a_{t}^{(i)}, \theta, \widehat{\Sigma}_{z})^\phi$. Set $[\theta_i, a_{t-1}^{(i)}, a_{t}^{(i)}] = [\hat{\theta}_i, a_{t-1|t-1}^{(i)}, a_{t|t}^{(i)}]$.
      v. Calculate the incremental weights from $u_t([\theta_i, a_{t-1}^{(i)}]) = f(Y_t|Y_{t-1}, a_{t-1}^{(i)}, \theta_i, \widehat{\Sigma}_{z})$ where $a_{t-1}^{(i)} \sim N \left( a_{t-1|t-1}^{(i)}, P_{t-1|t-1}^{(i)} \right)$.
      vi. Set $\omega_i' = \omega_i$ and $\phi' = \phi$. 

24
D Tuning the Latent Process

The parameter $\Sigma Z$ was treated as a tuning parameter, as our experience suggests that is weakly identified, and was just fixed to a plausible value $\hat{\Sigma} Z$ in the testing time periods. Details about the underlying data, in-sample and testing periods are provided in Section 4.1. To that end, we follow a three-step process which is entirely based on in-sample data. First, as in the standard case in Section 2.1, we estimate by maximum likelihood the yields-only DTSM, where $N = 3$ and, out of $\lambda P$, only $\lambda_1, 2$ is unrestricted, to match risk price restrictions we adopt in this paper. Second, we plug in the resulting maximum likelihood estimates (MLEs) in Equation (17) to obtain $\hat{s}_t$, $t = 1,..., \tilde{T}$, where $\tilde{T}$ refers to in-sample period endpoint. Third, we parametrise the matrix $\Sigma Z$ in (33) as

$$vec(\Sigma Z) = c \sqrt{(I_{(N-R)^2} - \Phi^P Z \otimes \Phi^P Z)} vec(\text{diag}[Var(\hat{s})])$$  \hspace{1cm} (32)

where $c > 0$ is an unknown scalar and $\text{diag}[Var(\hat{s})]$ is an $(N - R) \times (N - R)$ diagonal matrix with diagonal elements of the covariance matrix $Var(\hat{s})$ on its diagonal. We then formulate an amended version of the likelihood in (24), using only its $P$-likelihood components $f^p(\cdot)$ and the MLEs $\hat{k}_Q, \hat{g}_Q, \hat{\Sigma}_P$ and $\hat{\lambda}_1, 2$, to get

$$f(\tilde{Y}|(\Phi^P Z, c, \hat{k}_Q, \hat{g}_Q, \hat{\Sigma}_P, \hat{\lambda}_1, 2) = \prod_{t=1}^{\tilde{T}} f^p(\mathcal{P}_t|\mathcal{P}_{t-1}, \Phi^P Z, c, \hat{k}_Q, \hat{g}_Q, \hat{\Sigma}_P, \hat{\lambda}_1, 2)$$  \hspace{1cm} (33)

where $\tilde{Y}$ refers to in-sample data and the only unknown parameters are $\Phi^P Z$ and $c$. We proceed with the Kalman filter, as in Section 2.2.3, to maximise (33) and thus obtain $(\Phi^P Z, \hat{c})$. Using (32), we then obtain $\hat{\Sigma}_Z$. This procedure is applied only until the end of the in-sample period $\tilde{T}$, after which $\Sigma Z$ is fixed to $\hat{\Sigma} Z$ and remains unchanged.
Figure 1: Unspanned latent factors filtered from the yield curve. Focus is given on the period between January 2008 and the end of 2017 for which predictions of the model were also evaluated, but the factors are based on all the data from January 1985. Left column presents the factors from model $LF_{111}$. Right column shows the factors from models $LF_{011}$, first and second row, and $LF_{001}$, third row. Throughout, solid lines represent posterior means and dashed lines are 95% credible intervals.
On Unspanned Latent Risks in Dynamic Term Structure Models

Table 1: Explanatory power of principal components when fitting latent factors, measured via $\bar{R}^2$ - period: January 1985 - end of 2017.

$$R^2 : Z_t = a + b'P_t + e_t$$

| $LF_{001}$ | $Z_{1,t}$ | $Z_{2,t}$ | $Z_{3,t}$ |
|------------|-----------|-----------|-----------|
| 0.09       |           |           |           |
| 0.45       |           |           |           |
| 0.54       | 0.08      |           |           |
| 0.05       | 0.52      | 0.06      |           |

This table reports in-sample $\bar{R}^2$ across alternative regression specifications. The explained variables are individual latent factors $Z_{j,t}$, $j \in \{1, 2, 3\}$, in $Z_t$, from models $LF_{001}$, $LF_{010}$, $LF_{011}$ and $LF_{111}$. The explanatory variables are the principal components $P_t$. The sample period is January 1985 to end of 2017.

Table 2: Explanatory power gains from latent factor estimated using model $LF_{001}$, when fitting excess bond returns, measured via $\bar{R}^2$ over multiple prediction horizons - period: January 1985 - end of 2017.

| $h \backslash n$ | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y |
|------------------|----|----|----|----|----|-----|
|                  | $R^2(\%) : r_{x_{t+h}} = a + b'P_t + e_t$ |
| 1                | 4.26 | 3.29 | 3.08 | 2.74 | 2.77 | 3.55 |
| 3                | 9.50 | 7.93 | 8.69 | 7.45 | 8.60 | 8.53 |
| 6                | 13.54 | 12.70 | 14.10 | 13.75 | 14.42 | 13.70 |
| 9                | 14.97 | 14.16 | 15.76 | 16.59 | 18.35 | 19.03 |
| 12               | 17.30 | 14.77 | 16.37 | 17.83 | 20.47 | 22.90 |

| $LF_{001}$ | $\Delta(3) R^2(\%) : r_{x_{t+h}} = a + b'P_t + cZ_{3,t} + e_t$ |
|------------|--------------------------------------------------|
| 1          | 0.30 0.72* 0.81* 0.99** 1.62** 1.98**            |
| 3          | 0.13* 0.29* 0.42* 0.86** 1.02** 0.77**           |
| 6          | -0.03 -0.13 -0.21 -0.25 -0.11 -0.04             |
| 9          | -0.12 -0.19 -0.22 -0.24 -0.24 -0.23             |
| 12         | -0.17 -0.24 -0.25 -0.25 -0.21 -0.19             |

| $LF_{001}$ | $R^2(\%) : r_{x_{t+h}} = a + b'P_t + cZ_{3,t} + e_t$ |
|------------|--------------------------------------------------|
| 1          | 4.55 3.99 3.86 3.71 4.34 5.45                  |
| 3          | 9.62 8.19 9.07 8.24 9.53 9.23                  |
| 6          | 13.49 12.58 13.91 13.53 14.33 13.67            |
| 9          | 14.87 13.99 15.56 16.39 18.15 18.84            |
| 12         | 17.16 14.57 16.16 17.82 20.30 22.74            |

This table reports in-sample $R^2$ in % across alternative regression specifications and at different prediction horizons of $h = 1$-month, 3-month, 6-month, 9-month and 12-month. The explained variables are different (by maturities) excess bond returns. The explanatory variables are the principal components $P_t$ and the estimated (via posterior mean) latent factor $Z_{3,t}$ from model $LF_{001}$. In-sample $R^2$ values are obtained in a similar manner to the out-of-sample $R^2$ measure of Campbell and Thompson (2008) but using in-sample fit instead of out-of-sample forecasts, and incorporating penalty adjustment. In particular, $R^2$ in the top panel (all highly statistically significant hence not denoted) measures explanatory power gains from using principal components on top of the in-sample average to fit excess bond returns, whereas $R^2$ in the bottom panel (all highly statistically significant hence not denoted) measures explanatory power gains from using principal components and $Z_{3,t}$ on top of the in-sample average to fit excess bond returns. Further, $\Delta(3)$ next to $R^2$ in the mid panel means that the latter measures explanatory power gains from using latent factor $Z_{3,t}$ estimated using model $LF_{001}$ on top of the in-sample average and the principal components to do the same. Positive values of this statistic imply that there is explanatory power gain from adding extra variables. Statistical significance is measured using a one-sided Diebold-Mariano statistic with Clark-West adjustment, based on Newey-West standard errors. * denotes significance at 10%, ** significance at 5% and *** significance at 1% level. The sample period is January 1985 to end of 2017.
On Unspanned Latent Risks in Dynamic Term Structure Models

Table 3: Explanatory power gains from latent factor estimated using model $LF_{010}$, when fitting excess bond returns, measured via $R^2$ over multiple prediction horizons - period: January 1985 - end of 2017.

| $h$, $n$ | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y |
|----------|----|----|----|----|----|-----|
| -        | $R^2$ (%) | $\Delta_t R^2$ (%) | $\Delta_t R^2$ (%) |
| 1        | 4.26 | 3.29 | 3.08 | 2.74 | 2.77 | 3.55 |
| 3        | 9.50 | 7.93 | 8.69 | 7.45 | 8.60 | 8.53 |
| 6        | 13.54| 12.70| 14.10| 13.75| 14.42| 13.70|
| 9        | 14.97| 14.16| 15.76| 16.59| 18.35| 19.03|
| 12       | 17.30| 14.77| 16.37| 17.83| 20.47| 22.90|

This table reports in-sample $R^2$ in % across alternative regression specifications and at different prediction horizons of $h = 1$-month, 3-month, 6-month, 9-month and 12-month. The explanatory variables are different (by maturities) excess bond returns. The explanatory variables are the principal components $\bar{\eta}$ incorporating penalty adjustment. In particular, $\bar{\eta}$ on top of the in-sample average to fit excess bond returns, whereas $\bar{\eta}$ period starts in January 2008 and ends at end of 2018.

Table 4: Out-of-sample statistical performance of bond excess return forecasts against the EH, measured via $R^2$ at $h = 1$-month prediction horizon - period: January 1985 - end of 2018.

| $h = 1m \& n$ | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y |
|---------------|----|----|----|----|----|-----|
| Panel A: forecasts against the EH benchmark | $M_0$ | $M_1$ | $LF_{010}$ | $LF_{011}$ | $LF_{100}$ | $LF_{110}$ |
| $M_0$ | -3.84 | -5.54 | -4.74 | -3.60 | -1.54 | -1.45** |
| $M_1$ | 1.30 | 2.90** | 2.52* | 1.98 | 2.49* | 3.86** |
| $LF_{010}$ | 2.91** | 4.35** | 3.64** | 3.22** | 3.48** | 4.70*** |
| $LF_{011}$ | 5.86*** | 6.12*** | 4.61*** | 3.71** | 3.00** | 3.42*** |
| $LF_{100}$ | 5.65*** | 5.30*** | 3.87*** | 2.79** | 2.10* | 2.57** |
| $LF_{110}$ | 2.52* | 4.00** | 2.58* | 2.16 | 2.63 | 4.03* |
| $LF_{111}$ | 5.66*** | 5.30*** | 3.13** | 2.40* | 2.11* | 2.56** |
| $LF_{111}$ | 5.05*** | 4.53*** | 2.84** | 2.08* | 1.52* | 2.16** |

Panel B: forecasts against yields-only model $M_1$

| $LF_{010}$ | 1.62* | 1.49*** | 1.15** | 1.26** | 1.01** | 0.87* |
| $LF_{011}$ | 4.62*** | 3.31*** | 2.14*** | 1.77*** | 0.53 | -0.46 |
| $LF_{110}$ | 4.40*** | 2.47*** | 1.38** | 0.83 | -0.40 | -1.34 |
| $LF_{111}$ | 4.04*** | 2.47*** | 1.38** | 0.83 | -0.40 | -1.34 |
| $LF_{110}$ | 1.24 | 1.13 | 0.06 | 0.18 | 0.15 | 0.18 |
| $LF_{111}$ | 4.31** | 2.47* | 0.62 | 0.42 | -0.39 | -1.35 |
| $LF_{111}$ | 3.80** | 1.68 | 0.32 | 0.11 | -1.00 | -1.77 |

This table reports out-of-sample $R^2$ across alternative models at $h = 1$-month prediction horizon. The six forecasting models used are ATSM with different numbers of latent factors. Panel A presents $R^2$ relative to the EH benchmark, while panel B presents forecasts against the yield-only model $M_1$. $R^2$ values are generated using the out-of-sample $R^2$ measure of Campbell and Thompson [2008]. In particular, out-of-sample $R^2$ measures the predictive accuracy of bond excess return forecasts relative to the EH or $M_1$ benchmark. The EH implies the historical mean being the optimal forecast of excess returns, while $M_1$ represents the best performing yield-only model. Positive values of this statistic imply that the forecast outperforms the historical mean being the optimal forecast of excess returns, while $M_1$ represents the best performing yield-only model. Positive values of this statistic imply that the forecast outperforms the historical mean being the optimal forecast of excess returns, while $M_1$ represents the best performing yield-only model.
Table 5: Out-of-sample economic performance of bond excess return forecasts against the EH, measured via certainty equivalent returns (%) at $h = 1$-month prediction horizon. - period: January 1985 - end of 2018.

| $h = 1m \cdot n$ | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y |
|------------------|----|----|----|----|----|-----|
| Panel A: $w = [-1,2]$ |
| $M_0$ | -1.09 | -1.61 | -2.43 | -2.63 | -2.09 | -4.46 |
| $M_1$ | 0.00 | 0.00 | -0.04 | 0.10 | 1.23** | 0.78 |
| $LF_{001}$ | 0.00 | 0.00 | 0.09** | 0.13 | 1.36** | 1.85* |
| $LF_{010}$ | 0.00 | 0.00 | 0.12** | 0.37** | 1.49** | 1.38 |
| $LF_{011}$ | 0.00 | 0.00 | 0.10* | 0.25* | 1.24** | 0.76 |
| $LF_{100}$ | 0.14 | 0.23** | 0.28** | 0.23 | 1.13** | 0.93 |
| $LF_{110}$ | 0.17 | 0.28* | 0.41* | 0.61* | 1.63** | 0.87 |

| Panel B: $w = [-1,5]$ |
| $M_0$ | -2.04 | -3.52 | -3.38 | -3.20 | -1.82 | -3.85 |
| $M_1$ | -0.09 | 0.02 | 0.80 | 0.46 | 1.32 | 1.51 |
| $LF_{001}$ | 0.08* | 0.16 | 1.40* | 1.83 | 2.96* | 2.85** |
| $LF_{010}$ | 0.19* | 0.42 | 1.54* | 1.83 | 2.48* | 1.84* |
| $LF_{011}$ | 0.21* | 0.35 | 1.68** | 1.50 | 2.08 | 1.25 |
| $LF_{100}$ | 0.23 | 0.40 | 0.81 | -0.07 | -0.02 | 0.11 |
| $LF_{110}$ | 0.25 | 0.22 | 0.77 | 0.74 | 1.59 | 0.24 |
| $LF_{111}$ | 0.47* | 0.57 | 1.10 | 0.15 | 1.10 | 0.17 |

| Panel C: $w = [-\infty,\infty]$ |
| $M_0$ | -6.37 | -7.05 | -6.13 | -4.82 | -2.16 | -4.50 |
| $M_1$ | 1.42 | 1.16 | 0.83 | 0.48 | 1.32 | 1.51 |
| $LF_{001}$ | 2.42** | 2.31** | 2.12* | 1.96 | 2.96* | 2.85** |
| $LF_{010}$ | 2.40** | 2.31** | 2.15** | 1.83 | 2.48 | 1.84* |
| $LF_{011}$ | 2.26** | 2.01* | 1.90* | 1.47 | 2.08 | 1.25 |
| $LF_{100}$ | 0.22 | 0.35 | -0.20 | -0.42 | -0.02 | 0.11 |
| $LF_{110}$ | 0.61 | 0.99 | 0.80 | 0.75 | 1.59 | 0.24 |
| $LF_{111}$ | 0.02 | 0.28 | 0.25 | 0.13 | 1.10 | 0.17 |

This table reports annualised certainty equivalent returns ($CER$) across alternative models at $h = 1$-month prediction horizon. The coefficient of risk aversion is $\gamma = 5$. Panel A presents CERs under the first scenario, where portfolio weights are restricted to range in the interval $[-1, 2]$, such that investors are prevented from extreme investments. Panel B presents CERs under the second scenario, where, portfolio weights are restricted to range in the interval $[-1, 5]$, which amounts to a maximum short-sale of 100% and a maximum leverage of 400%. Panel C, reports CER values under the third scenario, where extreme investments are allowed and, as such, portfolio weights are free of constraints. CERs are generated by out-of-sample forecasts of bond excess returns and are reported in %. At every time step $t$, an investor with power utility preferences evaluates the entire predictive density of bond excess returns and solves the asset allocation problem, thus optimally allocating her wealth between a riskless bond and risky bonds with maturities 2, 3, 4, 5, 7 and 10-years. $CER$ is then defined as the value that equates the average utility of each alternative model against the average utility of the EH benchmark. The eight forecasting models used are ATSM with alternative risk price restrictions or different numbers of latent factors. Positive values indicate that the models perform better than the EH benchmark. Statistical significance is measured using a one-sided Diebold-Mariano statistic computed with Newey-West standard errors. * denotes significance at 10%, ** significance at 5% and *** significance at 1% level. The in-sample period is January 1985 to end of 2007, and the out-of-sample period starts in January 2008 and ends at end of 2018.
Table 6: Out-of-sample economic performance of bond excess return forecasts against model $M_1$, measured via certainty equivalent returns (%) at $h = 1$-month prediction horizon. - period: January 1985 - end of 2018.

| $h = 1 \text{m,} \, n$ | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y |
|------------------------|----|----|----|----|----|-----|
| **Panel A: w = [-1,2]** |    |    |    |    |    |     |
| $LF_{001}$             | 0.00 | 0.00 | 0.13** | 0.03 | 0.13 | 1.07** |
| $LF_{010}$             | 0.00 | 0.00 | 0.16** | 0.27** | 0.26 | 0.60 |
| $LF_{011}$             | 0.00 | 0.00 | 0.14** | 0.23** | 0.00 | -0.03 |
| $LF_{100}$             | 0.00 | 0.07 | 0.23 | 0.12 | 0.10 | -0.04 |
| $LF_{110}$             | 0.14 | 0.23* | 0.32 | 0.13 | -0.10 | 0.14 |
| $LF_{111}$             | 0.17 | 0.28* | 0.45** | 0.51** | 0.40 | 0.08 |
| **Panel B: w = [-1,5]** |    |    |    |    |    |     |
| $LF_{001}$             | 0.17* | 0.14 | 0.59* | 1.37** | 1.64*** | 1.34** |
| $LF_{010}$             | 0.27** | 0.40* | 0.74** | 1.37*** | 1.17** | 0.33 |
| $LF_{011}$             | 0.30** | 0.33 | 0.88** | 1.04* | 0.76 | -0.26 |
| $LF_{100}$             | 0.32* | 0.38 | 0.01 | -0.53 | -1.33 | -1.39 |
| $LF_{110}$             | 0.33 | 0.20 | -0.03 | 0.28 | 0.28 | -1.27 |
| $LF_{111}$             | 0.56* | 0.55 | 0.29 | -0.31 | -0.22 | -1.33 |
| **Panel C: w = [-inf, +inf]** |    |    |    |    |    |     |
| $LF_{001}$             | 1.00*** | 1.15*** | 1.28** | 1.48*** | 1.64*** | 1.34** |
| $LF_{010}$             | 0.97* | 1.15** | 1.31*** | 1.35** | 1.17** | 0.33 |
| $LF_{011}$             | 0.83* | 0.85* | 1.07** | 0.98* | 0.76 | -0.26 |
| $LF_{100}$             | -1.20 | -0.81 | -1.03 | -0.90 | -1.33 | -1.39 |
| $LF_{110}$             | -0.81 | -0.17 | -0.03 | 0.26 | 0.28 | -1.27 |
| $LF_{111}$             | -1.40 | -0.88 | -0.58 | -0.36 | -0.22 | -1.33 |

This table reports annualised certainty equivalent returns ($CER$s) across alternative models at $h = 1$-month prediction horizon. The coefficient of risk aversion is $\gamma = 5$. Panel A presents CERs under the first scenario, where portfolio weights are restricted to range in the interval [-1, 2], such that investors are prevented from extreme investments. Panel B presents CERs under the second scenario, where portfolio weights are restricted to range in the interval [-1, 5], which amounts to a maximum short-sale of 100% and a maximum leverage of 400%. Panel C, reports CER values under the third scenario, where extreme investments are allowed and, as such, portfolio weights are free of constraints. CERs are generated by out-of-sample forecasts of bond excess returns and are reported in %. At every time step $t$, an investor with power utility preferences evaluates the entire predictive density of bond excess returns and solves the asset allocation problem, thus optimally allocating her wealth between a riskless bond and risky bonds with maturities 2, 3, 4, 5, 7 and 10-years. CER is then defined as the value that equates the average utility of each alternative model against the average utility of restricted yields-only model $M_1$ from Dubiel-Teleszynski et al. [2021]. The six forecasting models used are ATSM with different numbers of latent factors. Positive values indicate that the models perform better than the benchmark model $M_T$. Statistical significance is measured using a one-sided Diebold-Marino statistic computed with Newey-West standard errors. * denotes significance at 10%, ** significance at 5% and *** significance at 1% level. The in-sample period is January 1985 to end of 2007, and the out-of-sample period starts in January 2008 and ends at end of 2018.
Table 7: Explanatory power of macroeconomic variables when fitting latent factors and their components, measured via $\bar{R}^2$ - period: January 1985 - end of 2018.

| $LF_{\text{001}}$ | CPI | GRO | F1 | F1$^3$ | F8 | UNR | MNF | $M^I$ | $M^{II}$ | $M^{III}$ |
|------------------|-----|------|----|--------|----|-----|-----|-------|---------|---------|
| $Z_{1,t}$        | 0.00| 0.00 | 0.00| 0.00   | 0.00| 0.00| 0.00| 0.00  | 0.00    | 0.00    |
| $E[Z_{1,t}|P_t]$ | 0.00| 0.17 | 0.17| 0.14   | 0.00| 0.04| 0.00| 0.17  | 0.18    | 0.06    |
| $\tilde{R}P_{1,t}$ | 0.00| 0.01 | 0.01| 0.01   | 0.00| 0.00| 0.00| 0.00  | 0.01    | 0.00    |

| $LF_{\text{010}}$ | CPI | GRO | F1 | F1$^3$ | F8 | UNR | MNF | $M^I$ | $M^{II}$ | $M^{III}$ |
|------------------|-----|------|----|--------|----|-----|-----|-------|---------|---------|
| $Z_{2,t}$        | 0.05| 0.03 | 0.03| 0.00   | 0.02| 0.20| 0.08| 0.08  | 0.05    | 0.21    |
| $E[Z_{2,t}|P_t]$ | 0.09| 0.01 | 0.00| 0.00   | 0.01| 0.06| 0.00| 0.11  | 0.01    | 0.17    |
| $\tilde{R}P_{2,t}$ | 0.00| 0.11 | 0.09| 0.02   | 0.01| 0.15| 0.19| 0.11  | 0.10    | 0.20    |

| $LF_{\text{011}}$ | CPI | GRO | F1 | F1$^3$ | F8 | UNR | MNF | $M^I$ | $M^{II}$ | $M^{III}$ |
|------------------|-----|------|----|--------|----|-----|-----|-------|---------|---------|
| $Z_{3,t}$        | 0.06| 0.01 | 0.02| 0.00   | 0.02| 0.19| 0.07| 0.07  | 0.04    | 0.20    |
| $E[Z_{3,t}|P_t]$ | 0.09| 0.02 | 0.01| 0.01   | 0.04| 0.00| 0.12| 0.02  | 0.13    |        |
| $\tilde{R}P_{3,t}$ | 0.00| 0.13 | 0.11| 0.03   | 0.01| 0.18| 0.21| 0.13  | 0.12    | 0.23    |

| $LF_{\text{111}}$ | CPI | GRO | F1 | F1$^3$ | F8 | UNR | MNF | $M^I$ | $M^{II}$ | $M^{III}$ |
|------------------|-----|------|----|--------|----|-----|-----|-------|---------|---------|
| $Z_{1,t}$        | 0.05| 0.12 | 0.17| 0.01   | 0.00| 0.16| 0.15| 0.18  | 0.21    | 0.18    |
| $E[Z_{1,t}|P_t]$ | 0.10| 0.00 | 0.00| 0.00   | 0.02| 0.09| 0.00| 0.10  | 0.01    | 0.24    |
| $\tilde{R}P_{1,t}$ | 0.02| 0.14 | 0.17| 0.01   | 0.00| 0.11| 0.17| 0.16  | 0.22    | 0.18    |

| $LF_{\text{111}}$ | CPI | GRO | F1 | F1$^3$ | F8 | UNR | MNF | $M^I$ | $M^{II}$ | $M^{III}$ |
|------------------|-----|------|----|--------|----|-----|-----|-------|---------|---------|
| $Z_{2,t}$        | 0.05| 0.12 | 0.17| 0.01   | 0.00| 0.16| 0.15| 0.18  | 0.21    | 0.18    |
| $E[Z_{2,t}|P_t]$ | 0.10| 0.00 | 0.00| 0.00   | 0.02| 0.09| 0.00| 0.10  | 0.01    | 0.24    |
| $\tilde{R}P_{2,t}$ | 0.02| 0.14 | 0.17| 0.01   | 0.00| 0.11| 0.17| 0.16  | 0.22    | 0.18    |

This table reports in-sample $\bar{R}^2$ across alternative regression specifications. The explained variables are individual latent factors $Z_{j,t}$ and their components $E[Z_{j,t}|P_t]$ and $\tilde{R}P_{j,t}$, $j \in \{1, 2, 3\}$, see from models $LF_{\text{001}}, LF_{\text{010}}, LF_{\text{011}}$ and $LF_{\text{111}}$. The explanatory variables are individual macroeconomic variables or groups thereof, namely $M^I = [CPI, GRO]'$, $M^{II} = [F1, F1^3, F8]'$, $M^{III} = [UNR, MNF]'$. The sample period is January 1985 to end of 2017.
Table 8: Signs and significance of coefficients from explanatory power regressions of latent factors and their components on macroeconomic variables - period: January 1985 - end of 2018.

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\begin{array}{|l|c|c|c|c|c|c|c|}
\hline
\text{sign}(b_j) & Z_{j,t} & E[Z_{j,t}|P_t] & RP_{j,t} & a_j & b_{j,M_t} & e_{j,t}, & j \in \{1, 2, 3\} \\
\hline
L_{F1001} & CPI & GRO & F1 & F1^3 & F8 & UNR & MNF \\
\hline
Z_{3,t} & - & - & - & - & + & + & - \\
E[Z_{3,t}|P_t] & - & - & - & - & + & + & - \\
RP_{3,t} & - & + & + & * & + & - & - \\
\hline
L_{F1010} & CPI & GRO & F1 & F1^3 & F8 & UNR & MNF \\
\hline
Z_{2,t} & + & + & + & * & * & * & - \\
E[Z_{2,t}|P_t] & + & + & + & * & * & + & + \\
RP_{2,t} & + & - & - & - & * & + & + \\
\hline
L_{F111} & CPI & GRO & F1 & F1^3 & F8 & UNR & MNF \\
\hline
Z_{1,t} & - & - & - & - & + & + & + \\
E[Z_{1,t}|P_t] & - & - & - & - & + & + & - \\
RP_{1,t} & - & + & + & + & * & + & + \\
\hline
Z_{2,t} & + & + & + & * & * & + & + \\
E[Z_{2,t}|P_t] & + & + & + & * & * & + & + \\
RP_{2,t} & + & - & - & - & * & + & + \\
\hline
Z_{3,t} & - & - & - & - & + & + & + \\
E[Z_{3,t}|P_t] & - & - & - & - & + & + & - \\
RP_{3,t} & - & + & + & + & * & + & + \\
\hline
\end{array}
\]

This table reports signs and statistical significance of coefficients \( b_j, j \in \{1, 2, 3\} \), across alternative regression specifications. The explained variables are individual latent factors \( Z_{j,t} \) and their components \( E[Z_{j,t}|P_t] \) and \( RP_{j,t} \), \( j \in \{1, 2, 3\} \), see Equation 31, from models \( L_{F1001}, L_{F1010}, L_{F111} \) and \( L_{F111} \). The explanatory variables are macroeconomic variables. Statistical significance is measured using the t-statistic computed with Newey-West standard errors. * denotes significance at 10%, ** significance at 5% and *** significance at 1% level. The sample period is January 1985 to end of 2017.