On Chern-Moser Normal Forms of Strongly Pseudoconvex Hypersurfaces with High-Dimensional Stability Group

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We explicitly describe germs of strongly pseudoconvex non-spherical real-analytic hypersurfaces $M$ at the origin in $\mathbb{C}^{n+1}$ for which the group of local CR-automorphisms preserving the origin has dimension $d_0(M)$ equal to either $n^2 - 2n + 1$ with $n \geq 2$, or $n^2 - 2n$ with $n \geq 3$. The description is given in terms of equations defining hypersurfaces near the origin, written in the Chern-Moser normal form. These results are motivated by the classification of locally homogeneous Levi non-degenerate hypersurfaces in $\mathbb{C}^3$ with $d_0(M) = 1, 2$ due to A. Loboda, and complement earlier joint work by V. Ezhov and the author for the case $d_0(M) \geq n^2 - 2n + 2$.

1 Introduction

Let $M$ be a strongly pseudoconvex real-analytic hypersurface in $\mathbb{C}^{n+1}$ passing through the origin. In some local holomorphic coordinates $z = (z_1, \ldots, z_n)$, $w = u + iv$ in a neighborhood of the origin, $M$ is given by an equation written in the Chern-Moser normal form (see [CM])

$$v = |z|^2 + \sum_{k,l \geq 2} F_{kl}(z, \bar{z}, u),$$

(1.1)

where $|z|$ is the norm of the vector $z$, and $F_{kl}(z, \bar{z}, u)$ are polynomials of degree $k$ in $z$ and $l$ in $\bar{z}$ whose coefficients are analytic functions of $u$ such that the following conditions hold

$$\text{tr} F_{2\bar{z}} \equiv 0,$$

$$\text{tr}^2 F_{2\bar{z}} \equiv 0,$$

$$\text{tr}^3 F_{3\bar{z}} \equiv 0.$$  

(1.2)

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Here the operator tr is defined as
\[
\text{tr} := \sum_{\alpha=1}^{n} \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\alpha}.
\]

Everywhere below we assume that the equation of \( M \) is given in the normal form.

Let \( \text{Aut}_0(M) \) denote the stability group of \( M \) at the origin, i.e. the group of all local CR-automorphisms of \( M \) defined near the origin and preserving it. Every element \( \varphi \) of \( \text{Aut}_0(M) \) extends to a biholomorphic mapping defined in a neighborhood of the origin in \( \mathbb{C}^{n+1} \) and therefore can be written as
\[
\begin{align*}
z & \mapsto f_\varphi(z, w), \\
w & \mapsto g_\varphi(z, w),
\end{align*}
\]
where \( f_\varphi \) and \( g_\varphi \) are holomorphic. We equip \( \text{Aut}_0(M) \) with the topology of uniform convergence of the partial derivatives of all orders of the component functions on neighborhoods of the origin in \( M \). The group \( \text{Aut}_0(M) \) with this topology is a topological group.

It is shown in [CM] that every element \( \varphi = (f_\varphi, g_\varphi) \) of \( \text{Aut}_0(M) \) is uniquely determined by a set of parameters \( (U_\varphi, a_\varphi, \lambda_\varphi, r_\varphi) \), where \( U_\varphi \) lies in the unitary group \( U_n \), \( a_\varphi \in \mathbb{C}^n \), \( \lambda_\varphi > 0 \), \( r_\varphi \in \mathbb{R} \). These parameters are found from the following relations
\[
\begin{align*}
\frac{\partial f_\varphi}{\partial z}(0) &= \lambda_\varphi U_\varphi, \\
\frac{\partial f_\varphi}{\partial w}(0) &= \lambda_\varphi U_\varphi a_\varphi, \\
\frac{\partial g_\varphi}{\partial w}(0) &= \lambda_\varphi^2, \\
\text{Re} \frac{\partial^2 g_\varphi}{\partial w^2}(0) &= 2\lambda_\varphi^2 r_\varphi.
\end{align*}
\]

For results on the dependence of local CR-mappings on their jets in more general settings see e.g. [BER1], [BER2], [EB], [Z].

We assume that \( M \) is non-spherical at the origin, i.e. \( M \) in a neighborhood of the origin is not CR-equivalent to an open subset of the sphere \( S^{2n+1} \subset \mathbb{C}^{n+1} \). In this case for every element \( \varphi = (f_\varphi, g_\varphi) \) of \( \text{Aut}_0(M) \) we have \( \lambda_\varphi = 1 \) and the parameters \( a_\varphi, r_\varphi \) are uniquely determined by the matrix \( U_\varphi \); moreover, the mapping
\[
\Phi : \text{Aut}_0(M) \to GL_n(\mathbb{C}), \quad \Phi : \varphi \mapsto U_\varphi
\]
is a topological group isomorphism between $\text{Aut}_0(M)$ and $G_0(M) := \Phi(\text{Aut}_0(M))$, with $G_0(M)$ being a real algebraic subgroup of $GL_n(\mathbb{C})$ (see [CM], [B], [L1], [BV], [VK]). We pull back the Lie group structure from $G_0(M)$ to $\text{Aut}_0(M)$ by means of $\Phi$ and denote by $d_0(M)$ the dimension of $\text{Aut}_0(M)$. Clearly, $d_0(M) \leq n^2$.

We are interested in characterizing hypersurfaces for which $d_0(M)$ is large (certainly positive). We show that in some normal coordinates the equations of such hypersurfaces take a very special form. As will be explained below, results of this kind potentially can be applied to the classification problem for locally CR-homogeneous strongly pseudoconvex hypersurfaces. For $n = 1$ this problem was solved by E. Cartan in [C]. For $n = 2$ with $d_0(M) > 0$ an explicit classification was obtained in [L3], [L2]. For $n \geq 3$ there is no such classification even for hypersurfaces with high-dimensional stability group. Note, however, that globally homogeneous hypersurfaces have been extensively studied (see e.g. [AHR] and references therein). We also mention that locally homogeneous hypersurfaces in $\mathbb{C}^3$ with non-degenerate indefinite Levi form and 2-dimensional stability group were classified in [L4] and that recently in [FK] Fels and Kaup have determined all locally homogeneous 5-dimensional CR-manifolds with certain degenerate Levi forms.

For a non-spherical hypersurface $M$ the group $\text{Aut}_0(M)$ is known to be linearizable, i.e. in some normal coordinates every $\varphi \in \text{Aut}_0(M)$ can be written in the form

$$
\begin{align*}
  z & \mapsto U\varphi z, \\
  w & \mapsto w,
\end{align*}
$$

(see [KL]). If all elements of $\text{Aut}_0(M)$ in some coordinates have the above form, we say that $\text{Aut}_0(M)$ is linear in these coordinates. Thus, in order to describe hypersurfaces $M$ with a particular value of $d_0(M)$, one needs to: (a) write $M$ in normal coordinates in which $\text{Aut}_0(M)$ is linear, (b) determine all closed subgroups $H$ of $U_n$ of dimension $d_0(M)$, and (c) find all $H$-invariant real-analytic functions of $z$, $\overline{z}$ and $u$, homogeneous of fixed degrees in each of $z$ and $\overline{z}$. Then every $F_{kl}(z, \overline{z}, u)$ in (1.1) is a function of the kind found in (c), and one obtains the general form of $M$.

In [EI] we considered the case $d_0(M) \geq n^2 - 2n + 2$ for $n \geq 2$. It turned out that if $d_0(M) \geq n^2 - 2n + 3$, then $d_0(M) = n^2$, that is, $G_0(M) = U_n$. Clearly, every $U_n$-invariant real-analytic function is a function of $|z|^2$ and $u$, and thus the equation of $M$ in any normal coordinates in which $\text{Aut}_0(M)$ is
linear has the form
\[ v = |z|^2 + \sum_{p=4}^{\infty} C_p(u) |z|^{2p} \]  
(1.3)

where \( C_p(u) \) are real-valued analytic functions of \( u \), and for some \( p \) we have \( C_p(u) \neq 0 \). Here the condition \( p \geq 4 \) comes from identities (1.2).

Further, for \( d_0(M) = n^2 - 2n + 2 \) we showed that the equation of \( M \) in some normal coordinates in which \( \text{Aut}_0(M) \) is linear has the form
\[ v = |z|^2 + \sum_{p+q \geq 2} C_{pq}(u) |z_1|^{2p} |z|^{2q}, \]  
(1.4)

where \( C_{pq}(u) \) are real-valued analytic functions of \( u \), \( C_{pq}(u) \neq 0 \) for some \( p, q \) with \( p > 0 \), and \( C_{pq} \) for \( p + q = 2, 3 \) satisfy certain conditions arising from identities (1.2). Equation (1.4) is the most general form of a hypersurface with \( d_0(M) = n^2 - 2n + 2 \) and cannot be simplified any further without additional assumptions on \( M \). This equation is a consequence of our description of closed connected subgroups of \( U_n \) of dimension \( n^2 - 2n + 2 \) obtained earlier in [IK].

In [L3] A. Loboda classified strongly pseudoconvex locally CR-homogeneous hypersurfaces in \( \mathbb{C}^3 \) with \( d_0(M) = 2 \) (here \( n = 2 \)) by means of normal form techniques (see also [L4]). Using the homogeneity of \( M \) and the condition \( d_0(M) = 2 \) he showed that the equation of \( M \) must significantly simplify, which eventually yielded the classification. His arguments avoid using the explicit form of closed connected 2-dimensional subgroups of \( U_2 \) (every such subgroup is conjugate to \( U_1 \times U_1 \)) and, as a result, special normal form (1.4). It seems that (1.4) can be utilized to simplify the proof of the main result of [L3]. Further, equation (1.4) may be a useful tool for describing locally CR-homogeneous strongly pseudoconvex hypersurfaces with \( d_0(M) = n^2 - 2n + 2 \) for arbitrary \( n \geq 2 \). Overall, the introduction of algebraic arguments into the analysis of normal forms seems to be a fruitful idea.

Observe for comparison that every locally CR-homogeneous strongly pseudoconvex hypersurface with \( d_0(M) \geq n^2 - 2n + 3 \) and \( n \geq 2 \) is spherical, since by (1.3) the origin is an umbilic point of \( M \). This is in contrast with hypersurfaces whose Levi form is non-degenerate and indefinite (see [E1]) for

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1In [E1] we erroneously stated that identities (1.2) imply that \( C_{pq} = 0 \) for \( p + q = 2, 3 \). This is in general not the case (see the erratum to [E1]).
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a description of such hypersurfaces with \( d_0(M) \geq n^2 - 2n + 3 \) and \([L2]\) for the homogeneous case with \( n = 2 \).

In this paper we consider the cases \( d_0(M) = n^2 - 2n + 1 \) with \( n \geq 2 \), and \( d_0(M) = n^2 - 2n \) with \( n \geq 3 \). Our result is the following theorem.

**THEOREM 1.1** Let \( M \) be a strongly pseudoconvex real-analytic non-spherical hypersurface in \( \mathbb{C}^{n+1} \) passing through the origin.

(A) If \( d_0(M) = n^2 - 2n + 1 \) and \( n \geq 2 \), then in some normal coordinates near the origin in which \( \text{Aut}_0(M) \) is linear the equation of \( M \) takes one of the following three forms:

\[
v = |z|^2 + \sum_{p+q \geq 2} C_{pqrs}(u) z_1^p z_2^q z_1^r z_2^s, \quad (1.5)
\]

where \( k_1, k_2 \) are non-zero integers with \( (k_1, k_2) = 1 \) and \( k_2 > 0 \), \( C_{pqrs}(u) \) are real-analytic functions of \( u \), and \( C_{pqrs}(u) \neq 0 \) for some \( p, q, r, s \) with either \( p \neq r \) or \( q \neq s \) (here \( n = 2 \));

\[
v = |z|^2 + \sum_{2p+q \geq 2} C_{pq}(u) |z_1^2 + z_2^2 + z_3^2|^2 |z|^{2q}, \quad (1.6)
\]

where \( C_{pq}(u) \) are real-valued analytic functions of \( u \), and \( C_{pq}(u) \neq 0 \) for some \( p, q \) with \( p > 0 \) (here \( n = 3 \));

\[
v = |z|^2 + \sum_{p+r, q+r \geq 2} C_{pqr}(u) z_1^p z_2^r |z|^{2r}, \quad (1.7)
\]

where \( C_{pqr}(u) \) are real-analytic functions of \( u \), and \( C_{pqr}(u) \neq 0 \) for some \( p, q, r \) with \( p \neq q \).

(B) If \( d_0(M) = n^2 - 2n \) and \( n \geq 3 \), then in some normal coordinates near the origin in which \( \text{Aut}_0(M) \) is linear the equation of \( M \) takes one of the following three forms:

\[
v = |z|^2 + \sum_{2p+r \geq 2, 2q+r \geq 2} C_{pqr}(u) (z_1^2 + z_2^2 + z_3^2)^p (\bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2)^q |z|^{2r}, \quad (1.8)
\]
where $C_{pqr}(u)$ are real-analytic functions of $u$, and $C_{pqr}(u) \neq 0$ for some $p, q, r$ (here $n = 3$);

$$v = |z|^2 + \sum_{p+q+r \geq 2} C_{pqr}(u)|z_1|^{2p}|z_2|^{2q}|z_3|^{2r}, \quad (1.9)$$

where $C_{pqr}(u)$ are real-valued analytic functions of $u$, and $C_{pqr}(u) \neq 0$ for some $p, q, r$ (here $n = 3$);

$$v = |z|^2 + \sum_{p+q \geq 2} C_{pq}(u)|z'|^{2p}|z''|^{2q}, \quad (1.10)$$

where $z' := (z_1, z_2)$, $z'' := (z_3, z_4)$, $C_{pq}(u)$ are real-valued analytic functions of $u$, and $C_{pq}(u) \neq 0$ for some $p, q$ (here $n = 4$).

**Corollary 1.2** Let $M$ be a strongly pseudoconvex real-analytic non-spherical hypersurface in $\mathbb{C}^{n+1}$ passing through the origin. Assume that $n \geq 5$ and $d_0(M) \geq n^2 - 2n$. Then $d_0(M) \geq n^2 - 2n + 1$. Furthermore, in some normal coordinates near the origin in which $\text{Aut}_0(M)$ is linear the equation of $M$ has the form

$$v = |z|^2 + \sum_{p+q+r \geq 2} C_{pqr}(u)|z_1|^{2p}|z_2|^{2q}|z_3|^{2r},$$

where $C_{pqr}(u)$ are real-analytic functions of $u$, and $C_{pqr}(u) \neq 0$ for some $p, q, r$.

Locally CR-homogeneous hypersurfaces in $\mathbb{C}^3$ with $d_0(M) = 1$ (here $n = 2$) were classified in [L5] and we believe that Part (A) of Theorem 1.1 can be used to simplify Loboda’s arguments.

## 2 Proof of Theorem 1.1

The main ingredient of the proof of Theorem 1.1 is the following proposition.

**Proposition 2.1** Let $H$ be a connected closed subgroup of $U_n$ with $n \geq 2$.

If $\dim H = n^2 - 2n + 1$, then $H$ is conjugate in $U_n$ to one of the following subgroups:
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(i) $e^{i\mathbb{R}}SO_3(\mathbb{R})$ (here $n = 3$);

(ii) $U_1 \times SU_{n-1}$ realized as the subgroup of all matrices

$$
\begin{pmatrix}
e^{i\theta} & 0 \\
0 & A
\end{pmatrix},
$$

where $\theta \in \mathbb{R}$ and $A \in SU_{n-1}$, for $n \geq 3$;

(iii) the subgroup $H_{k_1,k_2}^n$ of all matrices

$$
\begin{pmatrix}
a & 0 \\
0 & A
\end{pmatrix}, \quad (2.1)
$$

where $k_1,k_2$ are fixed integers such that $(k_1,k_2) = 1$, $k_2 > 0$, and $A \in U_{n-1}$, $a \in (\det A)^{\frac{k_1}{k_2}} := \exp(k_1/k_2 \ln(\det A))$.

If $\dim H = n^2 - 2n$, then $H$ is conjugate in $U_n$ to one of the following subgroups:

(iv) $SO_3(\mathbb{R})$ (here $n = 3$);

(v) $U_1 \times U_1 \times U_1$ realized as diagonal matrices in $U_3$ (here $n = 3$);

(vi) $U_2 \times U_2$ realized as block-diagonal matrices in $U_4$ (here $n = 4$);

(vii) $SU_{n-1}$ realized as the subgroup of all matrices

$$
\begin{pmatrix}
1 & 0 \\
0 & A
\end{pmatrix}, \quad A \in SU_{n-1}.
$$

Proof: Suppose first that $\dim H = n^2 - 2n + 1$. Since $H$ is compact, it is completely reducible, i.e. $\mathbb{C}^n$ splits into the sum of $H$-invariant pairwise orthogonal complex subspaces, $\mathbb{C}^n = V_1 \oplus \ldots \oplus V_m$, such that the restriction $H_j$ of $H$ to each $V_j$ is irreducible. Let $n_j := \dim \mathbb{C}V_j$ (hence $n_1 + \ldots + n_m = n$) and let $U_{n_j}$ be the group of unitary transformations of $V_j$. Clearly, $H_j \subset U_{n_j}$, and therefore $\dim H \leq n_1^2 + \ldots + n_m^2$. On the other hand $\dim H = n^2 - 2n + 1$, which shows that $m \leq 2$.

Let $m = 2$. Then there exists a unitary change of coordinates in $\mathbb{C}^n$ such all elements of $H$ take the form $(2.1)$, where $A \in U_{n-1}$ and $a \in U_1$.

§For $k_1 \neq 0$ the group $H_{k_1,k_2}^n$ is a $k_2$-sheeted cover of $U_{n-1}$. 
If \( \dim H_1 = 0 \), then \( H_1 = \{1\} \), and therefore \( H_2 = U_{n-1} \). In this case we obtain the group \( H_{0,1} \). Suppose next that \( \dim H_1 = 1 \), i.e. \( H_1 = U_1 \). Then 
\[
 n^2 - 2n \leq \dim H_2 \leq n^2 - 2n + 1.
\]
If \( \dim H_2 = n^2 - 2n \), then \( H_2 = SU_{n-1} \), and hence \( H \) is conjugate to \( U_1 \times SU_{n-1} \) for \( n \geq 3 \) and to \( H_{0,1} \) for \( n = 2 \).

Now let \( \dim H_2 = n^2 - 2n + 1 \), i.e. \( H_2 = U_{n-1} \). Consider the Lie algebra \( \mathfrak{h} \) of \( H \). Up to conjugation, it consists of matrices of the form

\[
\begin{pmatrix}
 l(\mathfrak{A}) & 0 \\
 0 & \mathfrak{A}
\end{pmatrix},
\]

where \( \mathfrak{A} \in \mathfrak{u}_{n-1} \) and \( l(\mathfrak{A}) \neq 0 \) is a linear function of the matrix elements of \( \mathfrak{A} \) ranging in \( i\mathbb{R} \). Clearly, \( l(\mathfrak{A}) \) must vanish on the derived algebra of \( \mathfrak{u}_{n-1} \), which is \( \mathfrak{su}_{n-1} \). Hence matrices (2.2) form a Lie algebra if and only if \( l(\mathfrak{A}) = c \cdot \text{trace} \mathfrak{A} \), where \( c \in \mathbb{Q} \setminus \{0\} \). Such an algebra can be the Lie algebra of a closed subgroup of \( U_1 \times U_{n-1} \) only if \( c \in \mathbb{Q} \setminus \{0\} \). Hence \( H \) is conjugate to \( H_{k_1,k_2} \) for some \( k_1, k_2 \in \mathbb{Z} \), where one can always assume that \( k_2 > 0 \) and \( (k_1,k_2) = 1 \).

Now let \( m = 1 \). We shall proceed as in the proof of Lemma 1.4 in [I]. Let \( \mathfrak{h}^C := \mathfrak{h} \oplus i\mathfrak{h} \subset \mathfrak{gl}_n \) be the complexification of \( \mathfrak{h} \), where \( \mathfrak{gl}_n := \mathfrak{gl}_n(\mathbb{C}) \). The algebra \( \mathfrak{h}^C \) acts irreducibly on \( \mathbb{C}^n \) and by a theorem of E. Cartan, \( \mathfrak{h}^C \) is either semisimple or the direct sum of the center \( \mathfrak{c} \) of \( \mathfrak{gl}_n \) and a semisimple ideal \( \mathfrak{t} \). Clearly, the action of the ideal \( \mathfrak{t} \) on \( \mathbb{C}^n \) is irreducible.

Assume first that \( \mathfrak{h}^C \) is semisimple, and let \( \mathfrak{h}^C = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k \) be its decomposition into the direct sum of simple ideals. Then the natural irreducible \( n \)-dimensional representation of \( \mathfrak{h}^C \) (given by the embedding of \( \mathfrak{h}^C \) in \( \mathfrak{gl}_n \)) is the tensor product of some irreducible faithful representations of the \( \mathfrak{h}_j \). Let \( n_j \) be the dimension of the corresponding representation of \( \mathfrak{h}_j \), \( j = 1, \ldots, k \). Then \( n_j \geq 2 \), \( \dim \mathfrak{c} \mathfrak{h}_j \leq n_j^2 - 1 \), and \( n = n_1 \cdot \ldots \cdot n_k \).

It is straightforward to show that if \( n = n_1 \cdot \ldots \cdot n_k \) with \( k \geq 2 \) and \( n_j \geq 2 \) for \( j = 1, \ldots, k \), then \( \sum_{j=1}^{k} n_j^2 \leq n^2 - 2n \). Since \( \dim \mathfrak{c} \mathfrak{h}^C = n^2 - 2n + 1 \), it then follows that \( k = 1 \), i.e. \( \mathfrak{h}^C \) is simple. The minimal dimensions of irreducible faithful representations \( V \) of complex simple Lie algebras \( \mathfrak{s} \) are well-known and shown in the following table (see e.g. [OV]).
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| $\mathfrak{s}$ | $\dim V$ | $\dim \mathfrak{s}$ |
|----------------|----------|----------------------|
| $\mathfrak{sl}_k$ $k \geq 2$ | $k$ | $k^2 - 1$ |
| $\mathfrak{so}_k$ $k \geq 7$ | $k$ | $k(k - 1)/2$ |
| $\mathfrak{sp}_{2k}$ $k \geq 2$ | $2k$ | $2k^2 + k$ |
| $\mathfrak{e}_6$ | 27 | 78 |
| $\mathfrak{e}_7$ | 56 | 133 |
| $\mathfrak{e}_8$ | 248 | 248 |
| $\mathfrak{f}_4$ | 26 | 52 |
| $\mathfrak{g}_2$ | 7 | 14 |

It is straightforward to see that none of these dimensions is compatible with the condition $\dim C^h = n^2 - 2n + 1$. Therefore, $h^C = c \oplus t$, where $\dim t = n^2 - 2n$. Then, if $n = 2$, we obtain that $H$ coincides with the center of $U_2$ which is impossible since its action on $C^2$ is then not irreducible. Assuming that $n \geq 3$ and repeating the above argument for $t$ in place of $h^C$, we see that $t$ can only be isomorphic to $\mathfrak{sl}_{n-1}$. But $\mathfrak{sl}_{n-1}$ does not have an irreducible $n$-dimensional representation unless $n = 3$.

Thus, $n = 3$ and $h^C \simeq C \oplus \mathfrak{sl}_2 \simeq C \oplus \mathfrak{so}_3$. Further, we observe that every irreducible 3-dimensional representation of $\mathfrak{so}_3$ is equivalent to its defining representation. This implies that $H$ is conjugate in $GL_3(C)$ to $e^{iR}SO_3(R)$. Since $H \subset U_3$ it is straightforward to show that the conjugating element can be chosen to belong to $U_3$. This completes the proof of the proposition in the case $\dim H = n^2 - 2n + 1$.

For $\dim H = n^2 - 2n$ we argue analogously and see that either $m \leq 2$, or, for $n = 3$ we have $m = 3$. In the latter case $H$ is conjugate in $U_3$ to $U_1 \times U_1 \times U_1$.

Let $m = 2$. Then either $n = 4$ and $H$ is conjugate in $U_4$ to $U_2 \times U_2$, or there exists a unitary change of coordinates in $C^n$ such all elements of $H$ take the form (2.1), where $A \in U_{n-1}$ and $a \in U_1$. If $\dim H_1 = 0$, then $H_1 = \{1\}$, and therefore $H_2 = SU_{n-1}$. Assume now that $\dim H_1 = 1$, i.e. $H_1 = U_1$. Then $n \geq 3$ and $n^2 - 2n - 1 \leq \dim H_2 \leq n^2 - 2n$. Lemma 1.4 of [1] shows that the possibility $\dim H_2 = n^2 - 2n - 1$ cannot in fact occur, and thus we have $\dim H_2 = n^2 - 2n$. Then $H_2 = SU_{n-1}$, and hence $H$ is conjugate to a codimension 1 subgroup of the group of all matrices of the form (2.1) with $A \in SU_{n-1}$. Consider the Lie algebra $h$ of $H$. Up to conjugation, it consists of matrices of the form (2.2), where $A \in \mathfrak{su}_{n-1}$ and $l(A) \neq 0$ is a linear function of the matrix elements of $A$ ranging in $iR$. Clearly, $l(A)$ must vanish on the derived algebra of $\mathfrak{su}_{n-1}$, which is $\mathfrak{su}_{n-1}$ itself. This contradiction shows that
the possibility \( \dim H_1 = 1 \) does not in fact realize.

In the case \( m = 1 \) we argue as in the case \( \dim H = n^2 - 2n + 1 \). If \( \mathfrak{h}^\mathbb{C} \)

is semisimple, it follows as before that \( \mathfrak{h}^\mathbb{C} \) is in fact simple. A glance at the

table of minimal dimensions of irreducible faithful representations of complex

simple Lie algebras now yields that \( n = 3 \) and \( \mathfrak{h}^\mathbb{C} \cong \mathfrak{sl}_2 \cong \mathfrak{so}_3 \), and hence

\( H \) is conjugate in \( U_3 \) to \( SO_3(\mathbb{R}) \). If, finally, \( n \geq 3 \) and \( \mathfrak{h}^\mathbb{C} = \mathfrak{c} \oplus \mathfrak{t} \), where

\( \dim \mathfrak{t} = n^2 - 2n - 1 \), we see that \( \mathfrak{t} \) must be simple and obtain a contradiction

with the above table.

The proof of the proposition is complete. \( \Box \)

To finalize the proof of Theorem 1.1 we now need to determine polynomials in \( z, \overline{z} \) with coefficients depending on \( u \), invariant under each of the
groups listed in (i)-(vii) of Proposition 2.1. This is not hard to do. Indeed,
every \( SO_3(\mathbb{R}) \)-invariant polynomial is a function of \( z_1^2 + z_2^2 + z_3^2 \), \( \overline{z_1}^2 + \overline{z_2}^2 + \overline{z_3}^2 \)

and \( |z|^2 \). If, in addition, such a polynomial is \( e^{iR} \)-invariant, it depends only

on \( |z_1^2 + z_2^2 + z_3^2|^2 \) and \( |z|^2 \). These observations lead to forms (1.6) and (1.8).

Next, \( U_1 \times SU_{n-1} \)-invariant polynomials for \( n \geq 3 \) are in fact \( U_1 \times U_{n-1} \)
invariant and therefore lead to hypersurfaces with \( d_0(M) \geq n^2 - 2n + 2 \). Further,
every \( H_{n,0}^a \)-invariant polynomial is a function of \( z_1, \overline{z_1} \) and \( |z|^2 \), which

leads to form (1.7). Every \( H_{k_1,k_2}^a \)-invariant polynomial for \( k_1 \neq 0 \) and \( n \geq 3 \)
is in fact \( U_1 \times U_{n-1} \)-invariant; such polynomials lead to hypersurfaces with
\( d_0(M) \geq n^2 - 2n + 2 \). Observe also that invariance under the group \( H_{k_1,k_2}^2 \)
with \( k_1 \neq 0 \) (here \( n = 2 \)) leads to form (1.5). Next, \( U_1 \times U_1 \times U_1 \)-invariant polynomials are functions of \( |z_1|^2, |z_2|^2, |z_3|^3 \) and lead to form (1.9). Similarly,
\( U_2 \times U_2 \)-invariant polynomials are functions of \( |z'|^2, |z''|^2 \), where \( z' := (z_1, z_2), z'' := (z_3, z_4) \), and lead to form (1.10). Finally, \( SU_{n-1} \)-invariant polynomials
for \( n \geq 3 \) are in fact \( U_{n-1} \)-invariant and hence lead to hypersurfaces with
\( d_0(M) \geq n^2 - 2n + 1 \).

The proof of Theorem 1.1 is complete. \( \Box \)

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