The Detectability Lemma and Quantum Gap Amplification

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Abstract

The quantum analogue of a constraint satisfaction problem is a sum of local Hamiltonians - each (term of the) Hamiltonian specifies a local constraint whose violation contributes to the energy of the given quantum state. Formalizing the intuitive connection between the ground (minimal) energy of the Hamiltonian and the minimum number of violated constraints is problematic, since the number of constraints being violated is not well defined when the terms in the Hamiltonian do not commute. The detectability lemma proved in this paper provides precisely such a quantitative connection. We apply the lemma to derive a quantum analogue of a basic primitive in classical computational complexity: amplification of probabilities by random walks on expander graphs. We call it the quantum gap amplification lemma. It holds under the restriction that the interaction graph of the local Hamiltonian is an expander. Our proofs are based on a novel structure imposed on the Hilbert space that we call the XY decomposition, which enables a reduction from the quantum non-commuting case to the commuting case (where many classical arguments go through).

The results may have several interesting implications. First, proving a quantum analogue to the PCP theorem is one of the most important challenges in quantum complexity theory. Our quantum gap amplification lemma may be viewed as the quantum analogue of the first of the three main steps in Dinur’s PCP proof [Din07]. Quantum gap amplification may also be related to spectral gap amplification, and in particular, to fault tolerance of adiabatic computation, a model which has attracted much attention but for which no fault tolerance theory was derived yet. Finally, the detectability lemma, and the XY decomposition provide a handle on the structure of local Hamiltonians and their ground states. This may prove useful in the study of those important objects, in particular in the fast growing area of “quantum Hamiltonian complexity” connecting quantum complexity to condensed matter physics.

1 Introduction

There is a close analogy between two fundamental notions from computational complexity theory and quantum physics: constraint satisfaction problems and the ground energy of local Hamiltonians. Each term in the local Hamiltonian specifies a local constraint whose violation contributes to the energy of the given quantum state. Hence the energy of the quantum state corresponds intuitively to the number of violated quantum constraints. A canonical example of this is the correspondence between the classical Cook-Levin theorem and its quantum analogue proved by Kitaev [KSV02]. Kitaev showed that estimating the ground energy of a local Hamiltonian to within inverse polynomial accuracy (the quantum analogue of determining the minimal number of violated constraints) is complete for the quantum analog of NP, namely QMA.

But how accurate is this intuitive correspondence between the energy of a state and the number of violated quantum constraints? The main issue is that in the quantum case, the terms of the

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Hamiltonian do not commute in general. This means that it is not even meaningful to ask: how many constraints are violated by a given state? Or in keeping with the probabilistic nature of quantum physics: what is the probability that the given state violates at least \( k \) constraints?

Our first main result in this paper is the quantum detectability lemma, which provides a way of making sense of and answering these questions. The lemma applies to any local Hamiltonian subject to the mild restrictions that every particle (qubit) in the local Hamiltonian participates in a bounded number of constraints, and each term in the Hamiltonian is chosen from a finite set of possibilities.

To state the detectability lemma, consider partitioning the terms in the Hamiltonian into \( g \) sets, which we call layers, so that in each layer all terms are mutually commuting. Under the restrictions on the Hamiltonian, it is possible to choose the number of layers \( g \) to be a constant. Notice that in every one of the \( g \) layers, it is meaningful to ask how many constraints are violated; every quantum state \( |\psi\rangle \) induces a probability distribution on how many constraints are violated for a given layer. The detectability lemma states that if the ground energy of the system is finite, then the probability that one or more constraints are violated in at least one of the layers is also finite. This is the probability of detecting one or more violations when measuring the constraints in that layer – hence the name detectability lemma. In its most general form, the detectability lemma also ensures that for systems with high ground energy, there exists a layer in which the probability for more than \( \ell \) violations is finite. Here \( \ell \) is some integer that has to be smaller than some normalized version of the ground energy.

To understand the subtleties of the lemma, consider a system with a ground state \( \epsilon_0 > 0 \), in which every layer consists of \( m \) constraints. Now consider the distribution induced by some state \( |\psi\rangle \) on the different sectors of the layers (by sector we mean the subspace corresponding to a certain number of violated constraints in that layer). One can imagine that the induced distribution in every layer has a tiny \( \epsilon_0 / m \) weight in the \( m \) violations part, and the rest is concentrated in the 0 violations part with no weight in the intermediate part (with 1, 2, \ldots, \( m - 1 \) violations). Such a setup would certainly comply with the ground energy condition, but constraint violation would not be detectable. When the different layers commute, it is easy to see that this scenario cannot happen, since it would imply a common ground state for all layers which contradicts \( \epsilon_0 > 0 \). However, when the layers do not commute, the relationship between the distributions becomes non-trivial. The \( \ell = 0 \) detectability lemma shows that even in the non-commuting case, the above scenario cannot happen. It shows that there is at least one layer in which the total weight on one or more violations is larger than some constant that is linear in \( \epsilon_0 \) (for small \( \epsilon_0 \)) and is independent of the system size \( m \).

The heart of the proof is a certain decomposition of the Hilbert space called the XY-decomposition, which is interesting in its own right. It captures a structural relationship between the ground spaces of the different layers of the Hamiltonian. The decomposition first partitions the Hilbert space into a tensor product of local spaces (defined by objects which we call pyramids) and then further decomposes each of these local spaces into commuting and non-commuting parts with respect to the Hamiltonian. Roughly speaking, the commuting parts are dealt with by classical means. In the non-commuting parts we identify an important parameter \( 0 < \theta < 1 \) of the system (characterized by the finite family of constraints allowed in the Hamiltonian) and find a way to point at an exponential decay of the states in terms of that parameter. This exponential decay allows for local analysis of the actions of the individual terms of the Hamiltonian.

Classical gap amplification, first proved in the context of saving random bits in RP and BPP amplification \cite{AKS87, IZ89}, is a basic primitive in complexity theory. The idea is that if one is interested in amplifying the probability of hitting a given subset of the nodes (or edges) in a graph, then if the graph is an expander, a random walk would do almost as well as picking the nodes (or edges) independently. More generally a constraint satisfaction problem is represented by a hypergraph, with each hyperedge corresponding to a constraint. To amplify the gap between the acceptance and rejection probability, one considers the "\( t \)-step walk" on the hypergraph. Now, if the hypergraph is expanding, one can show that the gap gets amplified by a factor of \( \Omega(t) \). This idea has since found
many other important implications, for example, in Dinur’s proof of the PCP theorem [Din07].

In this paper we prove a quantum analogue of the classical amplification lemma: the hyper-constraints are also generated from $t$ terms in the original Hamiltonian which form a walk in the interaction graph. Each new hyper-constraint is the projection on the intersection of the $t$ constraints on the walk. We show that if the original interaction graph was an expander, the average ground energy per term of the new Hamiltonian (consisting of the hyper-constraints) is $\Omega(t)$ times the average ground energy per term of the original Hamiltonian, thus establishing a bound similar to the classical lemma. The proof relies critically on the quantum detectability lemma, along with the classical analysis of walks on expander graphs.

The idea of the proof is that the overall amplification is lower-bounded by the amplification of a single layer. But as the constraints of a layer commute, we can treat them classically and apply the classical amplification lemma to the distribution of violations at that layer. The amplification of a layer therefore depends on its distribution. This is exactly where the detectability lemma is needed, as it ensures us that there is at least one layer with a distribution that allows for substantial amplification.

Discussions and Possible Implications: The results in this paper are related to several important open problems in quantum computation complexity. First, the study of the computational complexity of local Hamiltonians has blossomed over the last few years, and touches upon efficient simulation of quantum systems and theoretical condensed matter physics. The techniques developed in this paper, the $XY$-decomposition and the quantum detectability lemma, can be expected to contribute to our understanding of this new area.

Second, the PCP theorem is arguably the most important development in computational complexity theory over the last two decades. Is there a quantum analogue? One natural formulation is the following: suppose we are given a local Hamiltonian on $n$ qubits with the promise that the ground energy is either 0 or at least $1/p(n)$ for some polynomial $p(n)$. Is there a way to map this to a new local Hamiltonian such that the ground energy is either 0 or $\Omega(n)$? Proving such a quantum PCP theorem is a major challenge in quantum complexity theory; it would have implications for our understanding of inapproximability results of quantum complexity problems, quantum fault tolerance, the understanding of entanglement and notions such as no-cloning, as well as on the basic notion of energy gap amplification in condensed matter physics (see Section 8 for more precise definition and discussion).

Our quantum gap amplification lemma can be viewed as a very weak form of the above statement of quantum PCP. The problem of course is that checking the new $t$-walk constraints requires $t$ queries, which is too large even if we wish to check a single constraint. Dinur’s proof of the classical PCP theorem combines this kind of gap amplification with two other steps - degree reduction and assignment testing. In this sense quantum gap amplification is a possible first step towards emulating the outline of Dinur’s proof in the quantum setting.

Gap amplification is tightly connected (though not the same!) to spectral gap amplification, a notion of interest in adiabatic quantum computation (and in condensed matter physics in general). Adiabatic computation is a model of quantum computation which is equivalent in power to the standard one, and has attracted considerable attention ([FGGS00], [vDMV01], [vDV01], [Rei04], [RC02], [FGG02], [ATS03], [AvDK+04] and more). In adiabatic computation, the system evolves under a Hamiltonian with a non-negligible spectral gap between the ground state and the next excited state. Physical intuition suggests that such a model might be inherently robust to thermal noise [CFP01]. Despite work on the subject [LKS05], [RC05], [SL05], [Lid08], including the development of quantum error correcting codes tailored for adiabatic evolution [JF06], an analogue to the threshold result of the standard model [ABO97], [KLZ98], [Kit03] is still missing. Can the spectral gap in adiabatic computation be amplified to a constant, to provide fault tolerance? This is probably impossible when the system of qubits is arranged on a line, since even though such a system can be adiabatically universal when the gap is inverse polynomial [AGIK07], Hastings has showed [Has07], [Osb07] that adiabatic evolution in
one dimension with constant spectral gap can be simulated efficiently classically. However, it may very well be true that amplification of the spectral gap in some well defined sense is possible, if the underlying geometry is that of an expander. It seems likely that a proof of a quantum PCP theorem would pave the way to such a result, though we should caution that there is no proof showing such an implication.

**Open Problems** In this paper we handle the restricted case in which the local terms in the Hamiltonian are projections. We leave the general case for future work.

As a benchmark open problem, we pose the following question: prove an exponential size quantum PCP, in analogy with the first classical PCP results [AB]. This already seems to require some non-trivial work in quantum information theory. We note that it is possible to prove a quantum PCP theorem where the proof is of doubly exponential size; this seems to show that the no-cloning theorem, which some believe to be an obstacle against quantum PCP, might in fact be possible to bypass.

Proving a quantum analogue of Dinur’s degree reduction, which allows reducing the degree of the graph of interactions in the Hamiltonian, is another major open problem, which is related to the above problem.

Another related problem is to improve the parameters in current perturbation gadgets [KKR06, OT08, BDLT08, JF08] significantly; Perturbation gadgets are objects that allow decomposing Hamiltonians acting on some number of qubits, to sums of terms acting on smaller sets while maintaining some properties of the eigenvalues of the original Hamiltonian (within some small error). These are clearly tightly related to the notion of degree reduction, however the parameters in current perturbation gadgets are not good enough for the purposes of degree reduction, since the harm the gap too much.

## 2 Background - Local Hamiltonians and Local Projections

A $k$-local Hamiltonian $H$ on $n$ qubits is an operator $H : \mathbb{B}^\otimes n \to \mathbb{B}^\otimes n$ that can be written as a sum $H = \sum_{i=1}^{M} H_i$, where $M = \text{poly}(n)$ and every $H_i$ is a Hermitian operator acting on at most $k$ qubits. In this paper we restrict attention to the case where the $H_i$ operators are projections. We will usually denote them by $Q_i$:

$$H = \sum_{i=1}^{M} Q_i. \quad (1)$$

Another assumption that we use is that every projection intersects with only a finite number of other projections. Together with the $k$-locality, this implies that the projections can be partitioned into a constant number, denoted $g_i$, of subsets (which we call layers) such that the projections in each layer are non-intersecting and thus commuting. We denote a system satisfying the above restriction by a $k$-QSAT system.

For a state $|\psi\rangle$, $\langle \psi | H | \psi \rangle = \sum_i \langle \psi | H_i | \psi \rangle$ is called the energy of the state. In our case, this can be any number between 0 and $M$. The minimal energy of the system is the lowest eigenvalue of the Hamiltonian, and is denoted by $\epsilon_0$.

Deciding whether $\epsilon_0$ is above some threshold $a$ or below a threshold $b$ with $a - b > 1/poly(n)$ is known as the $k$-local Hamiltonian problem (which is complete for Quantum NP). It can be viewed as the quantum analog of the $k-SAT$ problem. We often refer to the projections $Q_i$ as constraints, and when $\langle \psi | Q_i | \psi \rangle > 0$ we say that $Q_i$ is violated (with respect to the state $|\psi\rangle$). Similarly, when $\langle \psi | Q_i | \psi \rangle = 0$, we say that $Q_i$ is satisfied, or that $|\psi\rangle$ is in the accepting subspace of $Q_i$. 


3 The XY decomposition

We consider a $k$-QSAT system over $n$ qubits with $M$ constraints that can be arranged in $g$ layers. Let us start by describing at a high level the XY-decomposition and how it is applied:

We start with a decomposition of the Hilbert space into a tensor product of local spaces, and restrict our attention to only those terms of the Hamiltonian that act non-trivially on exactly one of these spaces. The way the actual decomposition is carried out depends upon an ordering of the layers, and is described in the pyramid construction below. Each of these local spaces can now be further decomposed into a direct sum of subspaces according to whether all the terms of the Hamiltonian acting on this subspace commute ($X$) or not ($Y$). This defines a natural XY-decomposition of any state. The main point of the XY-decomposition is that it allows us to capture some structural relationship between the ground spaces of the different layers of the Hamiltonian. The starting point for this is the observation that in each $Y$ subspace there is some finite angle between the ground spaces of the different layers. Now stepping back, we can decompose the tensor product of all the local spaces into subspaces according to the number of $Y$ components. The actions of the Hamiltonian on the local spaces collectively ensure that if we start from an arbitrary state and successively project it onto the ground spaces of the different layers, then the weight of the resulting state in each subspace decays exponentially in the number of $Y$ components. This is a key property used in the proof of the detectability lemma.

3.1 Pyramids and Pyramid projections

We partition the Hilbert space of the $k$-QSAT system into a product of subspaces by defining the notion of a “pyramid”, a special “connected” subset of the constraints, as follows. First, we arbitrarily order the layers from 1 to $g$. A pyramid is created by picking its apex - a constraint in the first layer - and for each successive layer picking all constraints that intersect with the set of constraints picked in previous layers. We denote the Hilbert space of the qubits which participate in the pyramid by $H_{pyr}$. We now consider any maximal set of disjoint pyramids as illustrated in Fig. 1. Clearly the entire Hilbert space can be written as a tensor product of the pyramid spaces $H_{pyr}$, and constraints from different pyramids commute.

In the next step, we decompose the Hilbert space $H_{pyr}$ of the first pyramid into a direct sum of subspaces $\{X_j\}$ and a a subspace $Y$: every space $X_j$ is made of vectors which are simultaneous eigenvectors of all projections in the pyramid. Moreover, in every such $X_j$, each projection is allowed to take only one value - 0 or 1. Then $Y$ is defined to be the residual subspace, i.e., the subspace that is orthogonal to all the $X_j$ subspaces inside $H_{pyr}$. Clearly, all these spaces are orthogonal to each other. We refer to $Y$ as the “non-commuting” part of the Hilbert space $H_{pyr}$; all other subspaces correspond to the “commuting parts”. Of course, this decomposition can be done for every one of the pyramids.

We denote a sector of the XY decomposition by a string $\nu$. $\nu$ specifies either an $X_i$ space or a $Y$
space at each location, and we define

$$|\nu| \overset{\text{def}}{=} \text{No. of} \ Y \text{ sites in } \nu.$$  \hfill (2)

We also define \( P_\nu \) to be the projection into the tensor product of these spaces. Note that \( P_\nu \) is by itself a product of all the corresponding \( P_X, P_Y \) projections. Every state in \( H, |\psi\rangle \), can therefore be written as

$$|\psi\rangle = \sum_\nu P_\nu |\psi\rangle \overset{\text{def}}{=} \sum_\nu \lambda_\nu |\psi_\nu\rangle.$$  \hfill (3)

This is the \( XY \) decomposition.

We will in fact eventually use a finite number of these \( XY \) decompositions. It is easy to see that there exists a constant \( f(k, g) \) (independent of \( n \)) of \( XY \) decompositions such that every constraint in the top layer appears in one pyramid top in one of the \( XY \) decompositions, and so all top constraints are “covered” by one of the decompositions. But for most of the remainder of the paper, we fix one \( XY \) decomposition and stick to it.

### 3.2 Commutation relations between projections inside the pyramids

For a fixed pyramid, we denote the operators which act on \( H_{\text{pgr}} \) and project of on the subspaces \( \{X_j\}_j \) and \( Y \) by \( \{P_{X_j}\}_j, P_Y \) respectively. It is easy to verify the following properties:

- The projections form a valid decomposition of \( H_{\text{pgr}} \):
  
  \[
  [P_{X_i}, P_{X_j}] = [P_{X_i}, P_Y] = 0, \\
  P_Y + \sum_j P_{X_j} = 1
  \]  \hfill (4)

- Those projections commute with the constraints in the pyramid:
  
  \[
  [Q, P_{X_j}] = [Q, P_Y] = 0.
  \]  \hfill (5)

- For every two constraints \( Q_1, Q_2 \) in the pyramid and every subspace \( X_j \)
  
  \[
  P_{X_j} [Q_1, Q_2] P_{X_j} = 0.
  \]  \hfill (6)

### 3.3 The parameter \( \theta \)

Next, we define the parameter \( 0 < \theta < 1 \), which plays a crucial role in the paper.

**Definition 3.1 (The parameter \( \theta \))** Fix a pyramid. Consider the product \( Q_0 \cdot Q_1 \cdot \ldots \cdot Q_N \) where every \( Q_i \) is either a projection from the pyramid or its complement, and every pyramid projection (or its complement) appears exactly once. \( Y \) does not contain any common eigenvector of all those projections. Hence there exists a constant \( 0 < \theta < 1 \) such that for any possible pyramid in the system, and any order in which the constraints are chosen to appear in the product,

\[
\|P_Y \cdot Q_0 \cdot \ldots \cdot Q_N \cdot P_Y\| \leq \theta.
\]  \hfill (7)

\( \theta \) is a constant that depends only on the family of constraints and on the constant \( g \) (which determines the maximal number of constraints in a pyramid).
3.4 The $\Pi^{(i)}_{\leq \ell}$ projections

The $XY$ decomposition is useful for analyzing the action of the $\Pi^{(i)}_{\leq \ell}$ projections, which play a central role in the exponential decay and the detectability lemma. The $\Pi^{(i)}_{\leq \ell}$ projection projects to the subspace of $\ell$ or less violations of the constraints in the $i$’th layer. For example, for $\ell = 0$, $\Pi^{(i)}_{\leq 0}$ projects into the accepting space of the $i$’th layer.

A central observation is that we can present $\Pi^{(i)}_{\leq \ell}$ according to the pyramids structure. Using the fact that the constraints in the $i$’th layer all commute with each other and defined on non-intersecting qubits, we may write it in terms of violations inside the pyramids and violations outside the pyramids. Specifically, we write it as the following sum of $\ell + 1$ terms:

$$\Pi^{(i)}_{\leq \ell} = \sum_{j=0}^{\ell} \Delta_j^{(i)} \cdot R_{\leq \ell-j}^{(i)} .$$

Here $\Delta_j^{(i)}$ denote the projection into the subspace in which all the constraints in the $i$’th layer inside the pyramid have exactly $j$ violations, and $R_{\leq \ell-j}^{(i)}$ denotes the projection into the subspace in which the constraints of the $i$’th layer outside the pyramids have $j$ or less violations.

The core idea is that due to the pyramid structure, the support of the constraints of pyramids at layer $i$ is included in the support of the constraints of the pyramids at layer $i+1$, the layer beneath it. Therefore we can “pull back” the $\Delta_j^{(i)}$ operators across the $R_{\leq \ell}^{(i)}$ operators as long as $i' < i$, and so

$$\Pi^{(i)}_{\leq \ell} \cdots \Pi^{(i)}_{1} = \sum_{j_1, \ldots, j_\ell} (\Delta_{j_1}^{(i)} \cdots \Delta_{j_\ell}^{(i)}) \cdot (R_{\leq \ell-j_\ell}^{(i)} \cdots R_{\leq \ell-j_1}^{(i)}) .$$

This is a central equation that will be useful for us in what to follow.

3.5 The exponential decay for the 2 layers $\ell = 0$ case

To introduce the behavior of the exponential decay, we start with the simpler case of two layers, which we call “blue” and “red”. The structure in this case is illustrated in Fig. 2.

We define $\Pi_{\text{red}}$ ($\Pi_{\text{blue}}$) to be the $\ell = 0$ projections from Sec. 3.4. This means that $\Pi_{\text{red}}$ projects into the tensor product of the zero (accepting) subspaces of all the terms in the red layer, and similarly the $\Pi_{\text{blue}}$ for the blue layer.

We may now write $\Pi_{\text{red}}$ and $\Pi_{\text{blue}}$ in terms of violations inside and outside the pyramids as defined in Eq. 9. Notice that the $\ell = 0$ case is particularly simple because $\Pi_{\text{red}}, \Pi_{\text{blue}}$ can be written as products. Take for example $\Pi_{\text{blue}}$. It can then be written as the product $\Pi_{\text{blue}} = (1 - Q_1) \cdot (1 - Q_2) \cdots$ with $Q_i$ being the blue constraints. It is therefore clear that we can write

$$\Pi_{\text{blue}} = \Delta_{\text{blue}} R_{\text{blue}} ,$$

where

$$\Delta_{\text{blue}} = \text{terms inside the pyramids} ,$$

$$R_{\text{blue}} = \text{terms outside the pyramids} .$$
Similarly, we define $\Pi_{\text{red}} = \Delta_{\text{red}} R_{\text{red}}$.

As discussed in the previous section, because of the pyramid's structure, the support of $R_{\text{red}}$ and $\Delta_{\text{blue}}$ are non-intersecting (See Fig. 2) and therefore

$$\Pi_{\text{red}} \Pi_{\text{blue}} = \Delta_{\text{red}} \Delta_{\text{blue}} R_{\text{red}} R_{\text{blue}} . \tag{14}$$

We now prove the exponential decay behavior. Let us first coarse-grain the $XY$ decomposition by gathering together all sectors with the same number of $Y$ spaces. In other words, for every integer $0 \leq s \leq M$, define a projection

$$P_s \overset{\text{def}}{=} \sum_{|\nu| = s} P_\nu . \tag{15}$$

Then this is still a valid decomposition as the $P_s$ are orthogonal to each other and $\sum_{s=0}^m P_s = 1$. The exponential decay lemma states that if we apply this decomposition to some state after applying the $\Pi_{\text{blue}}$ and $\Pi_{\text{red}}$ projections, then we can upper bound the weight of the $s$ sector in terms of $\theta^s$.

**Lemma 3.2 (Exponential-decay lemma for $\ell = 0$)** Let $|\psi\rangle$ be an arbitrary (normalized) state, and consider the following normalized state

$$|\Omega\rangle \overset{\text{def}}{=} \frac{1}{x} \Pi_{\text{red}} \Pi_{\text{blue}} |\psi\rangle , \tag{16}$$

and its coarse grained $XY$ decomposition

$$|\Omega\rangle = \sum_s P_s |\Omega\rangle \overset{\text{def}}{=} \sum_s \lambda_s |\Omega_s\rangle . \tag{17}$$

Then there exist weights $\{\eta_s\}$ such that $\sum_s \eta_s^2 \leq 1$, and

$$\lambda_s \leq \frac{1}{x} \theta^s \eta_s . \tag{18}$$

**Proof:** To prove this claim, we take one step backwards, and write $|\Omega\rangle$ in terms of the fine-grained $XY$ decomposition: $|\Omega\rangle = \sum_\nu \lambda_\nu |\Omega_\nu\rangle$. Then

$$\lambda_\nu^2 = \langle \Omega | P_\nu |\Omega\rangle \tag{19}$$

$$= \frac{1}{x^2} \langle \psi | \Pi_{\text{blue}} \Pi_{\text{red}} P_\nu \Pi_{\text{red}} \Pi_{\text{blue}} |\psi\rangle . \tag{20}$$

We now use Eq. (14) and write

$$\Pi_{\text{blue}} \Pi_{\text{red}} P_\nu \Pi_{\text{red}} \Pi_{\text{blue}} = R_{\text{blue}} R_{\text{red}} \Delta_{\text{blue}} \Delta_{\text{red}} P_\nu \Delta_{\text{red}} \Delta_{\text{blue}} R_{\text{red}} R_{\text{blue}} , \tag{21}$$

and as $P_\nu$ commutes with $\Delta_{\text{red}}, \Delta_{\text{blue}}$, this is equal to

$$R_{\text{blue}} R_{\text{red}} P_\nu \Delta_{\text{blue}} \Delta_{\text{red}} \Delta_{\text{blue}} P_\nu R_{\text{red}} R_{\text{blue}} . \tag{22}$$

It follows that

$$\lambda_\nu^2 \leq \frac{1}{x^2} \| P_\nu \Delta_{\text{blue}} \Delta_{\text{red}} \Delta_{\text{blue}} P_\nu \| \cdot \| P_\nu |\Phi\rangle \|^2 , \tag{23}$$

with

$$|\Phi\rangle \overset{\text{def}}{=} R_{\text{red}} R_{\text{blue}} |\psi\rangle . \tag{24}$$
Let us estimate $\|P_\nu \Delta_{blue} \Delta_{red} \Delta_{blue} P_\nu\|$. Every operator in the product factors into a product of operators over every pyramid. Consider a pyramid site which is projected (by $P_\nu$) into a $Y$ subspace. For brevity, call it a $Y$ pyramid. Let $Q_{blue}$ be the blue constraint (the pyramid’s top) and $Q_1, \ldots, Q_N$ the red constraints. We have

$$P_Y \cdot Q_{blue} \cdot Q_1 \cdots Q_N \cdot Q_{blue} \cdot P_Y$$

$$= (P_Y \cdot Q_{blue} \cdot Q_1 \cdots Q_N \cdot P_Y) \cdot (P_Y \cdot Q_N \cdots Q_1 \cdot Q_{blue} \cdot P_Y)$$

where we have used Eq. (6). From Eq. (8), its norm is smaller or equal to $\theta^2$, and since there are $|\nu|$ such $Y$ sites, we deduce that

$$\|P_\nu \Delta_{blue} \Delta_{red} \Delta_{blue} P_\nu\| \leq \theta^{|\nu|}.$$  

(27)

All together, this leads to

$$\lambda^2_\nu \leq \frac{1}{x^2} \theta^{2|\nu|} \langle \Phi | P_\nu | \Phi \rangle,$$

(28)

and summing over all $\nu$ with $|\nu| = s$, we obtain

$$\lambda^2_s = \sum_{|\nu| = s} \lambda^2_\nu \leq \frac{1}{x^2} \theta^{2s} \eta^2_s,$$

(29)

where we have defined

$$\eta^2_s \overset{\text{def}}{=} \langle \Phi | P_s | \Phi \rangle = \sum_{|\nu| = s} \langle \Phi | P_\nu | \Phi \rangle.$$  

(30)

One can also make a similar statement for more than two layers; the proof follows very similar lines. Here we do not state this lemma since the following result implies it as a special case.

### 3.6 Generalizing to many layers and $\ell > 0$

In the general case we consider $g$ layers of constraints and $\ell$ might be larger than 0. The projections $\Pi_{red}, \Pi_{blue}$ are replaced by the $\Pi^{(i)}_{\leq \ell}$ which project into the subspace of $\ell$ or less violations in the $i$'th layer (see Sec. 3.4). This allows us to derive an exponentially decaying bound on the similarly defined coefficients $\lambda_s$, except now the bound will contain some combinatorial factors depending on $\ell$.

**Lemma 3.3 (Exponential decay lemma for general $\ell$)** Consider a $k$-QSAT system with $g$ layers and $M$ projections, drawn from a finite set that is characterized by a parameter $0 < \theta < 1$. Let $0 \leq \ell \leq M$ be an integer and let $|\psi\rangle$ be an arbitrary (normalized) state. Consider the following normalized state

$$|\Omega\rangle \overset{\text{def}}{=} \frac{1}{x} \Pi^{(g)}_{\leq \ell} \cdots \Pi^{(1)}_{\leq \ell} |\psi\rangle,$$

(31)

and its coarse grained $XY$ decomposition

$$|\Omega\rangle = \sum_s P_s |\Omega\rangle \overset{\text{def}}{=} \sum_s \lambda_s |\Omega_s\rangle.$$  

(32)

Then there exist weights $\{\eta_s\}$ such that $\sum_s \eta^2_s \leq 1$, and for every $s \geq \ell$,

$$\lambda_s \leq \frac{1}{x} k^{g^2 \ell} \left( \frac{\ell + 1}{\ell!} \right)^g s^g \theta^s \eta_s.$$  

(33)
The proof of the above claim is more involved than the 2-layers, \( \ell = 0 \) case, and will therefore be given in Appendix A. The main difficulty here is the fact that when \( \ell > 0 \), the projections \( \Pi^{(0)}_{\leq \ell} \) are no longer a simple product of projections as in the \( \ell = 0 \) case. This can already be seen in Eq. (9) that contains \( \ell + 1 \) instead of the one term that we find when we represent \( \Pi_{\text{red}} \) or \( \Pi_{\text{blue}} \) (the \( \ell = 0 \) case). Instead, they can be thought of as huge sum over similar products of projections, where each such product projects into a certain possible configuration of \( \ell \) or less violations. This complicates the analysis, but other than that, the proof follows the same outline of the 2-layers, \( \ell = 0 \) case.

4 The detectability lemma for two layers and \( \ell = 0 \)

In this section we prove the detectability lemma for the special case of two layers and \( \ell = 0 \). The proof is considerably simpler than the general proof, yet it demonstrates the ideas of the general case.

The general setup is similar to the one in Sec. 3.5. We consider a \( k\)-QSAT system with \( \varepsilon_0 > 0 \) that can be arranged in two layers. The first layer is called the “blue layer” and the second layer is the “red layer” (see Fig. 2). We define \( \Pi_{\text{red}} \) as the projection that projects into the accepting (zero) space of the red, and similarly \( \Pi_{\text{blue}} \). In addition, we assume that the constraints are drawn from a finite family of constraints with a parameter \( 0 < \theta < 1 \) (see Sec. 3). Then the 2-layers, \( \ell = 0 \) detectability lemma is:

**Lemma 4.1 (The detectability lemma for two layers and \( \ell = 0 \))**

There exists a function \( f(k) \) such that for every normalized state \( |\psi\rangle \),

\[
\max \left\{ \| (1 - \Pi_{\text{red}}) |\psi\rangle \|^2, \| (1 - \Pi_{\text{blue}}) |\psi\rangle \|^2 \right\} \geq \frac{1}{8} \Delta^2(0) ,
\]

where

\[
\Delta^2(0) \equiv 1 - \frac{1}{(\varepsilon_0/f)(1 - \theta^2)+1} .
\]

To prove this lemma, we will actually prove the following auxiliary lemma.

**Lemma 4.2** For every normalized state \( |\psi\rangle \),

\[
\| \Pi_{\text{red}} \Pi_{\text{blue}} |\psi\rangle \|^2 \leq 1 - \Delta^2(0) .
\]

We will first show that Lemma 4.1 follows from Lemma 4.2.

**Proof of Lemma 4.1 base on Lemma 4.2** Assume Lemma 4.2 and assume by contradiction that both \( \| (1 - \Pi_{\text{red}}) |\psi\rangle \|^2 < \frac{1}{8} \Delta^2(0) \) and \( \| (1 - \Pi_{\text{red}}) |\psi\rangle \|^2 < \frac{1}{8} \Delta^2(0) \). Then we write

\[
\Pi_{\text{blue}} \Pi_{\text{red}} \Pi_{\text{blue}} = 1 + (\Pi_{\text{blue}} - 1) + \Pi_{\text{blue}}(\Pi_{\text{red}} - 1) + \Pi_{\text{blue}} \Pi_{\text{red}}(\Pi_{\text{blue}} - 1) ,
\]

This way, except for the identity, every term on the RHS has a \( \Pi_{\text{blue}} - 1 \) or \( \Pi_{\text{red}} - 1 \) on its right side. We want also the left side to have such a term, so we continue in the same fashion, and write:

\[
\Pi_{\text{blue}}(\Pi_{\text{red}} - 1) = (\Pi_{\text{blue}} - 1)(\Pi_{\text{red}} - 1) + \Pi_{\text{red}} - 1 ,
\]

\[
\Pi_{\text{blue}} \Pi_{\text{red}}(\Pi_{\text{blue}} - 1) = (\Pi_{\text{blue}} - 1) \Pi_{\text{red}}(\Pi_{\text{blue}} - 1) + (\Pi_{\text{red}} - 1)(\Pi_{\text{blue}} - 1) + (\Pi_{\text{blue}} - 1) .
\]

All together we have

\[
\Pi_{\text{blue}} \Pi_{\text{red}} \Pi_{\text{blue}} = 1 + [6 \text{ terms with } (1 - \Pi_{\text{red}}) \text{ or } (1 - \Pi_{\text{red}}) \text{ on both sides}].
\]
When we “sandwich” the above equation with \( \langle \psi | \cdot | \psi \rangle \), the absolute value of each of the 6 terms will be smaller than \( \Delta^2(0)/8 \). This is due to the Cauchy-Schwartz inequality and the assumption that \( \| (1 - \Pi_{\text{red}}) |\psi\rangle \|, \| (1 - \Pi_{\text{blue}}) |\psi\rangle \| < \sqrt{\Delta^2(0)/8} \). Therefore,

\[
\| \Pi_{\text{red}} \Pi_{\text{blue}} |\psi\rangle \|^2 = \langle \psi | \Pi_{\text{blue}} \Pi_{\text{red}} \Pi_{\text{blue}} |\psi\rangle > 1 - \frac{6}{8} \Delta^2(0) > 1 - \Delta^2(0) ,
\]

which is a contradiction.

We now proceed to prove Lemma 4.2.

**Proof of Lemma 4.2:** Using the notation of Sec. 3.5, we define

\[
|\Omega\rangle \equiv \frac{1}{x} \Pi_{\text{red}} \Pi_{\text{blue}} |\psi\rangle , \quad x \equiv \| \Pi_{\text{red}} \Pi_{\text{blue}} |\psi\rangle \| .
\]

We wish to prove an upper bound for \( x \). The idea of the proof is to estimate the total energy of \( \langle \Omega | E | \Omega \rangle \). This energy has no contributions from the red layer since \( |\Omega\rangle \) has been projected by \( \Pi_{\text{red}} \), and so we may write:

\[
\epsilon_0 \leq \langle \Omega | E_{\text{blue}} | \Omega \rangle .
\]

We will find an upper-bound for \( \langle \Omega | E_{\text{blue}} | \Omega \rangle \) in terms of \( \theta \), and this would give us an inequality for \( x, \theta, \epsilon_0 \). Inverting that inequality will give us the desired result.

To estimate \( E_{\text{blue}} \) we consider first one possible \( XY \) decomposition. Let \( E_{\text{top}} \) be the energy of all the blue constraints from the pyramids in this decomposition - the “tops” of the pyramids. The main effort would be to find an upper-bound for \( \langle \Omega | E_{\text{top}} | \Omega \rangle \). Once we do that, we can then repeat this process with other sets of pyramids (namely, other \( XY \) decompositions) until we cover all the blue constraints. All in all, there is a finite number \( f(k) \) of \( XY \) decompositions that are needed for that. Therefore,

\[
\epsilon_0 \leq \langle \Omega | E_{\text{blue}} | \Omega \rangle \leq f(k) \langle \Omega | E_{\text{top}} | \Omega \rangle .
\]

Hence, it remains to bound \( \langle \Omega | E_{\text{top}} | \Omega \rangle \). We start by applying the fine- and coarse-grained \( XY \) decompositions to \( |\Omega\rangle \):  

\[
|\Omega\rangle = \sum_\nu \lambda_\nu |\Omega_\nu\rangle = \sum_s \lambda_s |\Omega_s\rangle .
\]

Then as the \( XY \) projections commute with the projections in \( E_{\text{top}} \), we get

\[
\langle \Omega | E_{\text{top}} | \Omega \rangle = \sum_s \lambda_s^2 \langle \Omega_s | E_{\text{top}} | \Omega_s \rangle .
\]

**Claim 4.3**

\[
\langle \Omega_s | E_{\text{top}} | \Omega_s \rangle \leq s .
\]

**Proof:** We will prove this claim on the fine-grained \( XY \) decomposition, by showing that \( \langle \Omega_\nu | E_{\text{top}} | \Omega_\nu \rangle \leq |\nu| \).

Essentially, the claim follows from the fact that only the \( Y \) sites can contribute energy. Indeed, consider an \( X \) pyramid, and let \( Q \) be its blue constraint. Then by definition, either \( Q |\Omega_\nu\rangle = 0 \) or
$Q|\Omega_\nu\rangle = |\Omega_\nu\rangle$. If the site contributes non-zero energy, the latter must hold. But $|\Omega_\nu\rangle \propto \Pi_\nu |\Omega\rangle \propto P_\nu \Pi_{\text{red}} \Pi_{\text{blue}} |\psi\rangle$, and so we get
\[QP_\nu \Pi_{\text{red}} \Pi_{\text{blue}} |\psi\rangle = \Pi_\nu \Pi_{\text{red}} \Pi_{\text{blue}} |\psi\rangle. \tag{49}\]
We show that the RHS of the above equation must vanish. Indeed, by Eq. (14), the LHS of the equation can be written as
\[QP_\nu \Delta_{\text{red}} \Delta_{\text{blue}} R_{\text{red}} R_{\text{blue}} |\psi\rangle. \tag{50}\]
But as the pyramids’ projections commute with $P_\nu$, we get
\[QP_\nu \Delta_{\text{red}} \Delta_{\text{blue}} = P_\nu Q \Delta_{\text{red}} \Delta_{\text{blue}} P_\nu, \tag{51}\]
and because in the $X$ subspaces the blue and red constraints commute, this is equal to
\[P_\nu \Delta_{\text{red}} Q \Delta_{\text{blue}} P_\nu. \tag{52}\]
This expression must vanish since $\Delta_{\text{blue}}$ contains a $/BD - Q$ term. It follows that the RHS of Eq. (49) must vanish and this proves the claim.

We can now use the above bound inside Eq. (47), together with the bound
\[\lambda_s^2 \leq \frac{1}{x^2} \theta^{2s} \eta_s^2 \tag{53}\]
which follows from the exponential decay lemma. We get:
\[\langle \Omega | E^{\text{top}} | \Omega \rangle = \sum_s s \lambda_s^2 \leq \sum_s \frac{1}{x^2} s \theta^{2s} \eta_s^2. \tag{54}\]

In principle, inserting this into Eq. (45) we could simply bound every $\eta_s^2$ by 1, and, rearranging, get a bound on $x^2$. However, this bound would be bad for very small $\epsilon_0$. Luckily, we can derive a stronger bound on $\eta_s^2$ for $s \geq 1$:

**Claim 4.4** For every $s \geq 1$
\[\eta_s^2 \leq \frac{1 - x^2}{1 - \theta^2}. \tag{55}\]

**Proof:** Summing over Eq. (53), we get
\[1 \leq \frac{1}{x^2} \sum_{s=0}^m \theta^{2s} \eta_s^2, \tag{56}\]
which is equivalent to
\[x^2 \leq \eta_0^2 + \sum_{s=1}^m \theta^{2s} \eta_s^2 \leq \eta_0^2 + \theta^2 \sum_{s=1}^m \eta_s^2. \tag{57}\]
But $\sum_{s=0}^m \eta_s^2 \leq 1$, so $\eta_0^2 \leq 1 - \sum_{s=1}^m \eta_s^2$, and
\[x^2 \leq 1 - \sum_{s=1}^m \eta_s^2 + \theta^2 \sum_{s=1}^m \eta_s^2 = 1 - (1 - \theta^2) \sum_{s=1}^m \eta_s^2, \tag{58}\]
which leads to
\[\sum_{s=1}^m \eta_s^2 \leq \frac{1 - x^2}{1 - \theta^2}, \tag{59}\]
implying the desired bound.
We can now finish the proof of Lemma 4.2. Following Eq. (54) and Eq. (45) we have:
\[
\epsilon_0 \leq \langle \Omega | E_{\text{blue}} | \Omega \rangle \leq \langle \Omega | E_{\text{top}} | \Omega \rangle \leq f(k) \cdot \frac{1 - x^2}{x^2} \cdot \frac{\theta^2}{(1 - \theta^2)^3},
\]
which yields
\[
x^2 \leq \frac{1}{(\epsilon_0/f) \cdot \frac{1 - \theta}{\theta^2} + 1} = 1 - \Delta^2(0).
\]

5 The (general) detectability lemma

The detectability lemma can be generalized for more than 2 layers and for \( \ell > 0 \). This generalization gives us a more detailed picture of the energy distribution. This is important when \( \epsilon_0 \) is much bigger than 1 but is still smaller than its maximal value \( M \). In such a case, the detectability lemma asserts that not only there exists a layer in which some violations are detectable - but that there must be a layer in which \( \ell \) or more violations are detectable. In other words, it forbids a situation in which in all the layers the violations are of only few constraints and there is \( 1/poly \) weight on very high violations (so as to not violate the minimal energy constraint). For the lemma to hold, we need to require that \( \ell \) - the number of violations - does not exceed some normalized version of the minimal energy \( \epsilon_0 \).

**Lemma 5.1 (The general detectability lemma)** Consider a \( k \)-QSAT system with \( g \) layers and a ground energy \( \epsilon_0 > 0 \). Let \( \Pi_{\ell}^{(i)} \) denote a projection into the space of more than \( \ell \) violations in the \( i \)'th layer. Then there exist integer functions \( r(\theta, k, g) \), \( f(k, g) > 1 \) such that for every \( 0 \leq \ell < \frac{1}{g} \left( \frac{\epsilon_0}{f} - \frac{\ell}{g} \right) \) and every normalized state \( |\psi\rangle \) there is at least one layer \( i \) in which:
\[
\| \Pi_{\ell}^{(i)} |\psi\rangle \|^2 \geq \frac{1}{(2g)^2} \Delta^2(\ell).
\]
\( \Delta(\ell) \) is a function of \( \ell, \epsilon_0, \theta, k, g \), and is given by
\[
\Delta^2(\ell) = \begin{cases} 
1 - \frac{1 - \epsilon_0/f}{\theta^2} + 1, & \ell = 0 \\
1 - \frac{1 - \epsilon_0/f}{\ell}, & \ell > 0.
\end{cases}
\]

The proof of the general detectability lemma is deduced from the exponential decay lemma for general \( \ell \), using similar reasoning to how the simpler detectability lemma is deduced from the \( \ell = 0 \) exponential decay lemma. However, the technical details are much more involved due to the same combinatorial factors that appear when moving from \( \ell = 0 \) to \( \ell > 0 \) in the exponential decay lemmas. The full proof in given in Appendix B.

6 Relation of the simple detectability lemma and Kitaev’s lemma

The \( \ell = 0 \) detectability lemma for the two layers can be seen as the converse of a special case of Kitaev’s geometrical lemma, crucial in his proof of the quantum Cook-Levin theorem [KSV02].

**Lemma 6.1 (Kitaev’s lemma (see Ref. [KSV02])** Given finite-dimensional operators \( P \geq 0, Q \geq 0 \) with null eigenspaces, then
\[
P + Q \geq \min \left\{ \Delta(P), \Delta(Q) \right\} \cdot (1 - \cos \alpha),
\]
where \( \Delta(O) > 0 \) is the smallest nonzero eigenvalue of \( O \), and \( \alpha \) the angle between the null spaces of \( P \) and \( Q \).
Therefore if $\Delta(P), \Delta(Q)$ are fixed, by lower-bounding $\alpha$, we can lower-bound the minimal energy of $P + Q$.

The 2-layers, $\ell = 0$ detectability lemma can be seen as the converse of this statement that holds in special case. In such case let the $Q, P$ operators be the $\Pi_{red}$ and $\Pi_{blue}$ projections from Sec. 4. Then the detectability lemma can be used to lower-bound $\alpha$. Indeed, for every state $|\psi\rangle$, Lemma 4.2 asserts that

$$\langle \psi | \Pi_{blue} \Pi_{red} \Pi_{blue} | \psi \rangle \leq 1 - \Delta^2(0) .$$

Then the angle $\alpha$ is given by

$$\cos \alpha = \min_{|\psi\rangle \in H_P} \langle \psi | Q | \psi \rangle = \min_{\|\psi\| = 1} \langle \psi | \Pi_{blue} \Pi_{red} \Pi_{blue} | \psi \rangle \leq 1 - \Delta^2(0) ,$$

where $H_P$ is the null space of $P$. Therefore, $1 - \cos \alpha \geq \Delta^2(\ell = 0, \epsilon_0)$, and combining it with Eqs. (64), we get

$$\Delta^2(\ell = 0, \epsilon_0) \leq 1 - \cos \alpha \leq \epsilon_0 .$$

Moreover, looking at Eq. (63), we see that in the limit $\epsilon_0 \to 0$,

$$\left( \frac{1 - \theta^2}{\theta} \right)^2 (1 - \theta) \frac{\epsilon_0}{f} \leq 1 - \cos \alpha \leq \epsilon_0 .$$

### 7 The quantum gap amplification lemma

Below we describe first the well known classical setting of gap amplification using walks on expander graphs (for completeness we also provide a proof in the appendix). We then define and prove a quantum analogue of this lemma, using the machinery we have developed so far.

#### 7.1 The classical amplification lemma on Expanders

We consider a $d$-regular expander graph $G = (V, E)$ with $n = |V|$ vertices and second largest eigenvalue $0 < \lambda(G) < 1$. With every node of $G$ we associate a variable that takes values in a finite alphabet $\Sigma$. Every edge is associated with a local constraint on the two values of the nodes in the edge. We refer to the set of constraints as a constraint system $C$.

Let $\sigma$ denote an assignment of the variables. We define $\text{UNSAT}_\sigma(G)$ to be the fraction of unsatisfied edges for that under that assignment:

$$\text{UNSAT}_\sigma(G) = \frac{\# \text{ of unsatisfied edges}}{|E|} .$$

In the amplification lemma, we define a new constraint system on $G$ using the notion of a $t$-walk. A $t$-walk on a graph $G$ is a sequence of $t + 1$ adjacent vertices, corresponding to a path of $t$ steps on $G$, starting at the vertex $v_0$ and ending at $v_t$. We denote the edges along the path by $e = (e_1, e_2, \ldots, e_t)$. The new constraint system is defined as follows. Consider all possible $t$-walks on $G$, and for each $t$-walk $e = (e_1, \ldots, e_t)$ we define a constraint that is satisfied if and only if all the constraints along the path are satisfied. Notice that the new constraints are less local than the original constraints, since they are defined on up to $t + 1$ vertices. Moreover, the new constraint system can no longer be thought of as a “constraint-graph” since its constraints are no longer defined on edges but on sets of $t + 1$ nodes. Rather, it is a constraint “hyper-graph”. With some abuse of notation, we will call the new constraint system $G'$.
The UNSAT of $G^t$ is defined by

$$\text{UNSAT}_\sigma(G^t) = \frac{\# \text{ of unsatisfied } t\text{-walks}}{\# \text{ of } t\text{-walks}}.$$ (70)

It seems plausible that $\text{UNSAT}_\sigma(G^t)$ would be significantly larger than $\geq \text{UNSAT}_\sigma(G)$, since if one edge is unsatisfied in $G$, it would appear in many $t$-walks in $G^t$. If the constraints in $G^t$ were chosen by choosing $t$ edges in $G$ independently, then we would have expected an amplification factor of $t$. The fact that we consider walks on an expander means that the behavior is very similar to the completely random case. The amplification lemma thus shows that that by moving from $G$ to $G^t$, the UNSAT is “amplified” by a factor of $t$, provided that $\text{UNSAT}(G)$ is not too close to 1.

**Lemma 7.1 (The classical amplification lemma)** Let $G = (V,E)$ be an expander graph with second largest eigenvalue $0 < \lambda < 1$, and let $\mathcal{C}$ be a constraint system on it using an alphabet $\Sigma$. Let $G^t$ denote the $t$-walk constraint system that was defined above. Define

$$c(\lambda) \overset{\text{def}}{=} \frac{1}{2 + \frac{\lambda}{1-\lambda}}.$$ (71)

Then for every assignment $\sigma$,

$$\text{UNSAT}_\sigma(G^t) \geq \begin{cases} t \cdot c(\lambda) \cdot \text{UNSAT}_\sigma(G), & \text{if } \text{UNSAT}_\sigma(G) \leq \frac{1}{t} \\ \frac{\text{UNSAT}_\sigma(G)}{c(\lambda)}, & \text{if } \text{UNSAT}_\sigma(G) \geq \frac{1}{t}. \end{cases}$$ (72)

The proof is provided in Appendix D.

### 7.2 The quantum amplification lemma

The setting of the quantum amplification lemma is a natural generalization of the classical setting. We consider a $d$-regular expander graph $G = (V,E)$ with a second-largest eigenvalue $0 < \lambda(G) < 1$. On top of $G$ we define a $k$-QSAT system as follows. We identify every vertex with a qudit of dimension $q$. Every edge $e \in E$ is identified with a projection $Q_e$ on the two qudits that are associated with the vertices of the edge. This defines $k$-QSAT system with $k = 2 \log(q)$ and a Hamiltonian

$$H = \sum_{e \in E} Q_e.$$ (73)

For any state $|\psi\rangle$, we define the quantum UNSAT of the system to be the average energy of the edges:

$$\text{QUNSA T}_\psi(G) \overset{\text{def}}{=} \frac{1}{|E|} \langle \psi | H | \psi \rangle = \frac{1}{|E|} \sum_{e \in E} \langle \psi | Q_e | \psi \rangle.$$ (74)

To define a new – “amplified” – constraint system, we use a construction similar to the classical case. We consider all possible $t$-walks ($t$ is fixed) $e = (e_1, \ldots, e_t)$ and for each such walk, we define a $t \log(q)$-local projection $Q_e$ as follows. We take the intersection of all the accepting spaces along the path and define it to be the accepting space of $Q_e$. In other words, $Q_e$ projects into the orthogonal complement of that space. We refer to the new system as $G^t$, and define

$$\text{QUNSA T}_\psi(G^t) \overset{\text{def}}{=} \frac{\sum_{e} \langle \psi | Q_e | \psi \rangle}{\# \text{ of } t\text{-walks}},$$ (75)

$$\text{QUNSA T}(G^t) \overset{\text{def}}{=} \min_{\psi} \text{QUNSA T}_\psi(G^t).$$ (76)

As in the classical case, the quantum amplification lemma shows how $\text{QUNSA T}(G^t)$ is amplified with respect to $\text{QUNSA T}(G)$. The amplification is linear in $t$ when $\text{QUNSA T}(G)$ is far enough from 1, and then becomes saturated, just like in the classical case.
Lemma 7.2 (The quantum amplification lemma) Consider a $k$-QSAT system on an expander graph $G = (V, E)$ with a second largest eigenvalue $0 < \lambda < 1$ as defined above. Then

$$\text{QUNSA}T(G^t) \geq c(\lambda) \cdot K(q, d, \theta) \cdot \min \left\{ t \cdot \text{QU}NSA\T(G), 1 \right\},$$

(77)

Where $K(q, d, \theta)$ is independent of the graph size and $c(\lambda)$ is given by Eq. (71).

Proof: By definition, $\text{QUNSA}T(G) = \epsilon_0/|E|$ where $\epsilon_0$ is the ground energy of $G$. Let $|\psi\rangle$ be a state for which $\text{QUNSA}T(G^t) = \text{QUNSA}T_\psi(G^t)$.

We first notice that our $k$-QSAT system can be written with at most $g = 2d$ layers. We choose a layer $i$ and expand $|\psi\rangle$ in terms of its violations in that layer: $|\psi\rangle = \sum_{j=0}^{|E|} \alpha_j |\psi_j\rangle$. (78)

Here $|\psi_j\rangle$ is the projection of $|\psi\rangle$ to the space with $j$ violations in the $i$’th layer. Thus $|\psi\rangle$ is a superposition of states in which the number of violated constraints of the $i$’th level have a well-defined value.

We consider an auxiliary $k$-QSAT system $G_i$ which has same underlying graph $G$ and the same constraints of the $i$’th layer - but the rest of the constraints are null - i.e. they are always satisfied. It is clear that for every state $|\psi\rangle$, $\text{QUNSA}T_\psi(G^t_i) \geq \text{QUNSA}T_\psi(G^t_i)$. Moreover, as all the projections in $G^t_i$ commute within themselves and with the original projections of the $i$’th layer, we have

$$\text{QUNSA}T_\psi(G^t_i) = \sum_j \alpha_j^2 \cdot \text{QUNSA}T_\psi(G^t_i).$$

(79)

We will now show:

Claim 7.3

$$\text{QUNSA}T_\psi(G^t_i) \geq \begin{cases} t \cdot c(\lambda) \cdot \frac{j}{|E|} & \text{for } j \leq \frac{|E|}{t} \\ c(\lambda) & \text{for } j > \frac{|E|}{t} \end{cases}.$$ (80)

Proof: This follows from the classical amplification lemma. We expand $|\psi_j\rangle$ as a superposition $|\psi_j\rangle = \sum_\nu \beta_\nu |\psi_\nu\rangle$, where $|\psi_\nu\rangle$ has a well-defined value (1 or 0, namely violating or not) at each edge of $G_i$, with the total number of violations being exactly $j$. Moreover, it is easy to see that as the projection into the state $|\psi_\nu\rangle$ commutes with the projections of $G_i$, then

$$\text{QUNSA}T_\psi(G^t_i) = \sum_\nu \beta_\nu^2 \cdot \text{QUNSA}T_\psi(G^t_i),$$

(81)

$$\text{QUNSA}T_\psi(G^t_i) = \sum_\nu \beta_\nu^2 \cdot \text{QUNSA}T_\psi(G^t_i),$$

(82)

hence it is sufficient to prove Eq. (80) for $\text{QUNSA}T_\psi(G^t_i)$. This, however, follows directly from the classical amplification lemma since under the state $|\psi_\nu\rangle$ the constraints of $G_i$ have a well-defined, classical values. We can therefore treat the situation as a classical system $G_\nu$ with some assignment $\sigma$ and $\text{UNSAT}_\sigma(G_\nu) = j/|E|$. According to the classical amplification lemma, if $j/|E| \leq 1/t \leftrightarrow j \leq |E|/t$ then $\text{UNSAT}_\sigma(G^t_i) \geq t \cdot c(\lambda) \cdot \frac{j}{|E|}$, otherwise, $\text{UNSAT}_\sigma(G^t_i) \geq c(\lambda)$. But as everything is classical for $G_i$ and $G^t_i$ in the $\nu$ sector then,

$$\text{UNSAT}_\sigma(G^t_i) = \text{QUNSA}T_\psi(G^t_i)$$

(83)

and this proves the claim.
Let us now use this claim to estimate the amplification. Combining Eq. (80) with Eq. (79), we find
\[
\text{QUNSAT}(G') = \text{QUNSAT}_\psi(G') \geq \text{QUNSAT}_\psi(G_i') \\
\geq t \frac{c(\lambda)}{|E|} \left( \alpha_1^2 + 2\alpha_2^2 + 3\alpha_3^2 + \ldots + \frac{|E|}{t}\alpha_t^2 \right) + c(\lambda) \left( \alpha_1^2|E|/t + \ldots + \alpha_t^2 \right). \tag{84}
\]
Therefore, as $\text{QUNSAT}(G) = \frac{\epsilon_0}{|E|}$, the amplification ratio we are looking for is
\[
\frac{\text{QUNSAT}(G')}{\text{QUNSAT}(G)} \geq t \frac{c(\lambda)}{\epsilon_0} \left( \alpha_1^2 + 2\alpha_2^2 + 3\alpha_3^2 + \ldots + \frac{|E|}{t}\alpha_t^2 \right) + c(\lambda) \frac{|E|}{\epsilon_0} \left( \alpha_1^2|E|/t + \ldots + \alpha_t^2 \right). \tag{85}
\]
The above equation is central and can be derived for any layer (namely, for any $i$). However, without additional information, it cannot be used to show amplification of $\text{QUNSAT}(G')$. The reason is that the weights $\alpha_j^2$ can theoretically conspire in such a way that no amplification would occur. For example, $1/poly(|E|)$ of the weight can be concentrated on $\alpha_{|E|}^2$ and the rest on $\alpha_0^2$, and then there is no amplification since in these two sectors there is no amplification (one is completely satisfied and the other is completely saturated). Fortunately, we can use the detectability lemma to rule out the possibility that this sort of non-amplifying distribution appears simultaneously in all layers.

The idea is to consider two possible cases: \( \frac{\epsilon_0}{\epsilon_0} - \frac{1}{1-\theta} \leq 2r \) (the low-energy case) and \( \frac{\epsilon_0}{r} - \frac{1}{1-\eta} > 2r \) (the high-energy case). For the former we use the $\ell = 0$. In the former, we use the $\ell > 0$ detectability lemma. Let us start with the low energy case.

7.3 The low energy case: $\frac{\epsilon_0}{\epsilon_0} - \frac{4}{1-\theta} \leq 2r$

Here we estimate the amplification using the $\ell = 0$ detectability. Specifically, Lemma 5.1 ensures us that there a layer $i$ in which,
\[
\alpha_i^2 + \alpha_2^2 + \ldots + \alpha_{|E|}^2 \geq \frac{1}{(2g)^2} \Delta^2(0). \tag{87}
\]
On the other hand, it is easy to see that Eq. (80) implies
\[
\frac{\text{QUNSAT}(G')}{\text{QUNSAT}(G)} \geq t \frac{c(\lambda)}{\epsilon_0} \left( \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \ldots + \alpha_{|E|}^2 \right). \tag{88}
\]
Therefore,
\[
\frac{\text{QUNSAT}(G')}{\text{QUNSAT}(G)} \geq t \cdot c(\lambda) \cdot (2g)^{-2} \cdot \frac{\Delta^2(0)}{\epsilon_0}. \tag{89}
\]
Let us now lower bound the expression $\frac{\Delta^2(0)}{\epsilon_0}$. $\Delta^2(0)$ is a continuous function of $\epsilon_0$ that is bounded between 0 and 1 for $\epsilon_0 \geq 0$. We have to worry about to things: (i) if $\epsilon_0$ becomes too large, the ratio might become small, and (ii) as $\epsilon_0 \to 0$, also $\Delta^2(0) \to 0$. The first worry is taken cared by fact that in the low-energy case $\epsilon_0$ is upper bounded by $\frac{\epsilon_0}{\epsilon_0} - \frac{4}{1-\theta} \leq 2r$. The second one is taken cared by noticing the approach of $\Delta^2(0)$ to 0 as $\epsilon_0$ is linear in $\epsilon_0$ (see Eq. (83)). Therefore as $\epsilon_0 \to 0$, the ratio approaches some positive constant. All in all, we conclude that in the low-energy case,
\[
\frac{\text{QUNSAT}(G')}{\text{QUNSAT}(G)} \geq t \cdot c(\lambda) \cdot K_1(q, d, \theta). \tag{90}
\]
7.4 The high energy case: $\frac{\epsilon_0}{f} - \frac{4}{1-\theta} \geq 2r$

In the high-energy case, we use the detectability lemma with a particular $\ell$ to show the amplification. We choose $\ell$ as large as possible so that $(\ell + 1)/\epsilon_0$ will be lower-bounded by a positive function of $q, d, \theta$. Specifically, the high energy condition implies $\frac{\epsilon_0}{f} - \frac{2}{1-\theta} \geq 2r$, and so we choose

$$\ell = \left\lceil \frac{1}{r} \left( \frac{\epsilon_0 f}{2} - \frac{2}{1-\theta} \right) \right\rceil \geq 2 .$$

Then on one hand,

$$\ell < \frac{1}{r} \left( \frac{\epsilon_0 f}{2} - \frac{2}{1-\theta} \right) ,$$

and so $(1-\theta) \left( \frac{\epsilon_0}{f} - r\ell \right) > 2$, yielding a finite detectability in Lemma 5.1

$$\Delta^2(\ell) > 1 - \frac{1}{2} = \frac{1}{2} \quad \Downarrow \quad \alpha_{\ell+1}^2 + \alpha_{\ell+2}^2 + \ldots + \alpha_{|E|}^2 \geq \frac{1}{(2g)^2} \Delta^2(\ell) \geq \frac{1}{8g^2} .$$

On the other hand, Eq. (91) also implies

$$\ell + 1 \geq \frac{1}{r} \left( \frac{\epsilon_0 f}{2} - \frac{2}{1-\theta} \right) ,$$

and so

$$\frac{\ell + 1}{\epsilon_0} \geq \frac{1}{r} \left( \frac{1}{f} - \frac{2}{\epsilon_0(1-\theta)} \right) .$$

But $\frac{\epsilon_0}{f} - \frac{4}{1-\theta} > 0$, therefore

$$\frac{\ell + 1}{\epsilon_0} \geq \frac{1}{2fr} .$$

Let us now return to Eq. (86). By omitting all the $\alpha_i^2$ terms with $i \leq \ell$, we obtain

$$\frac{\text{QUNSAT}(G')}{\text{QUNSAT}(G)} \geq t \frac{c(\lambda)}{\epsilon_0} (\ell + 1) \left( \alpha_{\ell+1}^2 + \ldots + \alpha_{|E|/t}^2 \right) + c(\lambda) \cdot \frac{|E|}{\epsilon_0} \left( \alpha_{|E|/t+1}^2 + \ldots + \alpha_{|E|}^2 \right) .$$

Define

$$A = \alpha_{\ell+1}^2 + \ldots + \alpha_{|E|/t}^2 ,$$

$$B = \alpha_{|E|/t+1}^2 + \ldots + \alpha_{|E|}^2 .$$

Then $A + B \geq \frac{1}{8g^2}$ and

$$\frac{\text{QUNSAT}(G')}{\text{QUNSAT}(G)} \geq t \frac{c(\lambda)}{\epsilon_0} (\ell + 1) A + c(\lambda) \cdot \frac{|E|}{\epsilon_0} B .$$

---

Note that by assumption $\epsilon_0$ is larger than $2r$, which can only happen when $|E|$ – the total number of constraints in the system – satisfies $|E| > 2r$, therefore the $\ell$ we choose makes sense.
If \( A \geq \frac{1}{16g^2} \) then from the above equation and by Eq. (102),
\[
\frac{\text{QUNSAT}(G^d)}{\text{QUNSAT}(G)} \geq t \cdot c(\lambda) \cdot \frac{\ell + 1}{\epsilon_0} A \geq t \cdot c(\lambda) \cdot \frac{1}{2fr} \cdot \frac{1}{16g^2} \overset{\text{def}}{=} t \cdot c(\lambda) \cdot K_2(q, d, \theta) .
\]
(103)

If, on the other hand, \( B \geq \frac{1}{16g^2} \) then we can use Eq. (103) to conclude that
\[
\text{QUNSAT}(G^d) \geq c(\lambda) \cdot \frac{1}{16g^2} \overset{\text{def}}{=} c(\lambda) \cdot K_3(d) .
\]
(104)

Combining all these 3 results, it is straightforward to define a function \( K(q, d, \theta) \) such that
\[
\text{QUNSAT}(G^d) \geq c(\lambda) \cdot K(q, d, \theta) \cdot \min \left\{ t \cdot \text{QUNSAT}(G), 1 \right\} .
\]
(105)

8 Discussion regarding quantum PCP

We discuss here quantum PCP in the context of gap amplification. To this end we define what we mean by a quantum PCP theorem.

The classical PCP theorem can be viewed as a strong characterization of the \( \text{NP} \) class. One way to state it is by first defining the class \( \text{PCP}[r, q] \). This is the class of all languages \( L \) for which there is a polynomial verifier that uses \( O(r) \) random bits and has the following properties. It reads an instance \( x \) and has an oracle access to \( O(q) \) bits of some proof \( \pi \). If \( x \in L \) there is a witness for which the verifier accepts with probability 1. Otherwise, for every proof, the acceptance probability is smaller than \( 1/2 \). The PCP theorem then states that \( \text{NP} = \text{PCP}[\log(n), 1] \).

To state the quantum PCP conjecture we first recall the quantum analogous of the \( \text{NP} \) class - the \( \text{QMA} \) class.

**Definition 8.1 (The class QMA)** A language \( L \) is in QMA if there exists a quantum polynomial verifier \( V \) and a polynomial \( p(\cdot) \) such that

- If \( x \in L \), there exists a witness \( |\xi\rangle \in \mathbb{B}^{\otimes p(|x|)} \) such that \( \Pr[V(x, |\xi\rangle) \text{ accepts}] \geq 2/3 \)
- If \( x \notin L \), then for every \( |\xi\rangle \in \mathbb{B}^{\otimes p(|x|)} \) we have \( \Pr[V(x, |\xi\rangle) \text{ accepts}] \leq 1/3 \)

Therefore, a natural definition for a QPCP class is

**Definition 8.2 (The class QPCP[q])** A language \( L \) is in QPCP[q] if there exists a quantum polynomial verifier \( V \) and a polynomial \( p(\cdot) \) with the following properties. \( V \) receives as input a classical string \( x \) and a state \( |\xi\rangle \in \mathbb{B}^{\otimes p(|x|)} \). However, it has only access to \( O(q) \) random qubits from \( |\psi\rangle \). In other words, it has access only to a a density matrix \( \rho \) which is the tracing out of all but the \( O(q) \) random qubits in \( |\xi\rangle\langle\xi| \). The random choice of the qubits is performed according to a probability distribution which is computed by the quantum verifier. We denote its action on \( (x, |\xi\rangle) \) by \( V(x, |\xi\rangle) \).

Then the condition for \( L \) to be in QPCP[q] is that

- If \( x \in L \), there exists a witness \( |\xi\rangle \in \mathbb{B}^{\otimes p(|x|)} \) such that \( \Pr[V(x, |\xi\rangle) \text{ accepts}] \geq 2/3 \).
- If \( x \notin L \), then for every \( |\xi\rangle \in \mathbb{B}^{\otimes p(|x|)} \) we have \( \Pr[V(x, |\xi\rangle) \text{ accepts}] \leq 1/3 \).

Notice that we did not give the quantum verifier any random bits, since it is quantum and can generate randomness by itself. The above definition can have various variants; for example, we might require that the probability distribution, which defines which qubits the verifier sees, is uniform. We do not dwell on the differences between these definitions; they are subtle, and at this stage the subject is not understood well enough (to us) in order to determine the best definition.

A quantum PCP theorem would read:
Conjecture 8.3 (Quantum PCP)

\[ \text{QPCP}[1] = \text{QMA} \quad (106) \]

An essentially equivalent way of formulating the quantum PCP theorem is in terms of local Hamiltonians: is it possible to efficiently transform any \(k\)-\text{QSA T} system with \(1/\text{poly}\) promise gap into a \(k\)-\text{QSA T} system with constant promise gap. In the classical world, this corresponds to the inapproximability of max-3\text{SA T}.

Recently, Dinur gave a beautiful new proof of the classical PCP theorem \[\text{Din07}\], which works directly in this setting. She starts with a classical SAT system with a \(1/\text{poly}\) promise gap and successively amplifies the gap by repeated doubling. This doubling is accomplished by gap amplification followed by alphabet reduction and degree reduction to control the size and locality.

It is tempting to try to apply Dinur’s proof to the quantum case, with \(k\)-\text{QSA T} replacing the \(k\)-SAT problem. As mentioned in Sec. 7, the quantum UNSAT is the ground energy of the system divided by the number of constraints; it is \(\text{QMA}\) complete\(^2\) to decide between the cases when it is zero or larger than some threshold (called the promise gap) which is inverse polynomial. A quantum version of Dinur’s approach would state that this hardness holds even when the promise gap is constant. Formally, this is stated as

Conjecture 8.4 (Quantum PCP by gap amplification) There exists an efficient classical transformation that takes a \(k\)-\text{QSA T} system with a promise gap of \(1/\text{poly}\) and transforms it into a new \(k\)-\text{QSA T} system with a constant promise gap such that the original system has a zero ground energy iff the new system has a zero ground energy.

By the \(\text{QMA}\)-completeness of the \(k\)-\text{QSA T} problem, it is easy to deduce that Conjecture 8.3 follows from Conjecture 8.4. Our quantum gap amplification lemma can be seen as a step towards emulating Dinur’s approach in the quantum setting.

We mention that it has been speculated that a quantum version of the PCP theorem is impossible to achieve, at least along the lines of Dinur’s proof: Dinur’s proof relies heavily on copying the values of the nodes in the graph, whereas in the quantum setting such a copying is impossible due to the no-cloning theorem, which asserts that there is no unitary transformation that copies an unknown state. This argument seems problematic to formalize. One of the reasons is that the argument assumes that the transformation on the Hamiltonian which amplifies the gap must be unitary. However, there is no such requirement on the Hamiltonian map. In fact, we were able to use this observation, and derive a quantum PCP theorem, albeit with a \textit{doubly} exponential long proof, by a straightforward discretization of the problem. The resultant map on Hamiltonians, and consequently on the eigenstates, is non-unitary (not even a unitary embedding). On the other hand, it is not even clear that unitary PCP transformations are ruled out.

We pose as an open problem to reduce the doubly exponential proof to a singly exponential long proof quantum PCP; such a result would be the quantum analogue of the early classical PCP results, in which the proofs were of exponential size \[\text{AB}\].

9 Acknowledgments

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\(^2\)in this discussion we omit the important subtle distinction between the notions of \(\text{QMA}\) and that of \(\text{QMA}^1\), namely, \(\text{QMA}\) with one sided errors. This will be explained in a later version.
A Proving the exponential decay in the general case

In this section we prove the exponential decay in the general case, which is stated in Lemma 3.3 in Sec. 3.6. The proof follows essentially the path of the 2-layers, \( \ell = 0 \) case. We start by proving the decay in the fine-grained \( X Y \) decomposition. Consider then a given \( X Y \) decomposition and some sector \( \nu \) with \( |\nu| \geq \ell \).

**Claim A.1** There exist \((\ell + 1)g\) states \( |\Phi_j\rangle, j = 1, \ldots, (\ell + 1)g \) with \( \|\Phi_j\| \leq 1 \), such that the weight of every \( X Y \) sector \( \nu \) with \( |\nu| \geq \ell \), is bounded by

\[
\lambda^2_{\nu} \leq \frac{1}{x^2} \frac{(\ell + 1)^g}{(\ell!)^2g} \left(|\nu| g^\ell k^g \theta^{|\nu|}\right)^2 \sum_{j=1}^{(\ell+1)g} \|P_{\nu}(\Phi_j)\|^2.
\]

**Proof:** By definition,

\[
\lambda^2_{\nu} = \langle \Omega | P_{\nu} | \Omega \rangle = \frac{1}{x^2} \langle \psi | \Pi^{(1)}_{\leq \ell} \cdots \Pi^{(g)}_{\leq \ell} \cdot P_{\nu} \cdot \Pi^{(1)}_{\leq \ell} \cdots \Pi^{(g)}_{\leq \ell} | \psi \rangle = \frac{1}{x^2} \|P_{\nu} \cdot \Pi^{(g)}_{\leq \ell} \cdots \Pi^{(1)}_{\leq \ell} | \psi \rangle \|^2.
\]

Let us estimate \( \|P_{\nu} \cdot \Pi^{(g)}_{\leq \ell} \cdots \Pi^{(1)}_{\leq \ell} | \psi \rangle \|. Using Eq. (8), we find

\[
\|P_{\nu} \cdot \Pi^{(g)}_{\leq \ell} \cdots \Pi^{(1)}_{\leq \ell} | \psi \rangle \| \leq \sum_{j_1, \ldots, j_g} \|P_{\nu} (\Delta^{(g)}_{j_g} \cdots \Delta^{(1)}_{j_1}) P_{\nu} \| \cdot \|P_{\nu} (R^{(g)}_{\leq \ell - j_g} \cdots R^{(1)}_{\leq \ell - j_1}) | \psi \rangle \|.
\]

We will upper-bound \( \|P_{\nu} (\Delta^{(g)}_{j_g} \cdots \Delta^{(1)}_{j_1}) P_{\nu} \| \). Every projection \( \Delta^{(i)}_{j_i} \) can be written as a sum of products of the form \( Q \cdot Q \cdot (1 - Q) \cdot \ldots \) that work on the projections of the \( i \)-th layer that are inside the pyramid, such that there are exactly \( j \) projections of the form \( Q \) and the rest is of the form \( 1 - Q \) - corresponding to exactly \( j \) violations.

The product \( \Delta^{(g)}_{j_g} \cdots \Delta^{(1)}_{j_1} \), therefore, contains a huge number of such products. However, when we “sandwich” it between two \( P_{\nu} \) projections, only few survive - those that are compatible with the \( X \) portion of \( P_{\nu} \). Let us estimate how many survive in a given layer. The \( X \) part is completely fixed, and therefore we have to choose from all the \( Y \) projections at most \( \ell \) violations. There are \( |\nu| \) \( Y \) sites and at each site there are at most \( k^g \) constraints, so overall, for \( \ell \leq |\nu| \), the number of surviving constraints in a single layer is bounded by

\[
\left( \frac{|\nu| k^g}{\ell} \right) \leq \frac{1}{\ell!}(|\nu| k^g)^\ell.
\]

Considering all \( g \) layers, the total number of surviving terms is therefore bounded by \( \left( \frac{|\nu| k^g}{\ell} \right)^g \). The norm of each term is bounded by \( \theta^{|\nu|} \) as there are \(|\nu| \) \( Y \) sites. Therefore, the overall norm is bounded by

\[
\|P_{\nu} (\Delta^{(g)}_{j_g} \cdots \Delta^{(1)}_{j_1}) P_{\nu} \| \leq \frac{1}{(\ell!)^g} |\nu| g^\ell k^g \theta^{|\nu|}.
\]

Thus far, we got

\[
\|P_{\nu} \cdot \Pi^{(g)}_{\leq \ell} \cdots \Pi^{(1)}_{\leq \ell} | \psi \rangle \| \leq \frac{1}{(\ell!)^g} |\nu| g^\ell k^g \theta^{|\nu|} \sum_{j_1, \ldots, j_g} \|P_{\nu} (R^{(g)}_{\leq \ell - j_g} \cdots R^{(1)}_{\leq \ell - j_1}) | \psi \rangle \|.
\]
There are \((\ell + 1)^g\) terms in that sum, and so using standard Cauchy-Schwartz argument we get
\[
\|P_{\nu} \Pi_{\leq \ell}^{(g)} \Pi_{\leq \ell}^{(g)} |\psi\rangle\|^2 \leq \frac{(\ell + 1)^g}{(\ell!)^2g} \left( |\nu|^{g^2 \ell^2} \theta^{|\nu|} \right)^2 \sum_{j_1, \ldots, j_g} \|P_{\nu}(R_{\leq \ell - j_g} \cdots R_{\leq \ell - j_1}) |\psi\rangle\|^2 .
\] (115)

Finally, grouping all the indices \((j_1, \ldots, j_g)\) into one big index \(j\), and defining the un-normalized states
\[
|\Phi_j\rangle \defeq R_{\leq \ell - j_g} \cdots R_{\leq \ell - j_1} |\psi\rangle ,
\] (116)
whose norm is smaller than or equal to 1, we get that for \(|\nu| \geq \ell\),
\[
\lambda_{\nu}^2 = \langle \Omega | P_{\nu} |\Omega\rangle \leq \frac{1}{x^2} \frac{(\ell + 1)^g}{(\ell!)^2g} \left( |\nu|^{g^2 \ell^2} \theta^{|\nu|} \right)^2 \sum_{j=1}^{(\ell+1)^g} \|P_{\nu} |\Phi_j\rangle\|^2 .
\] (117)

To prove Lemma 3.3 pass to the coarse grained XY decomposition by grouping together all the XY sectors with the same number of \(Y\)'s. Then
\[
\lambda_s^2 = \sum_{|\nu| = s} \lambda_{\nu}^2 \leq \frac{1}{x^2} \frac{(\ell + 1)^g}{(\ell!)^2g} \left( s^{g^2 \ell^2} \theta^s \right)^2 \sum_{j=1}^{(\ell+1)^g} \sum_{|\nu| = s} \|P_{\nu} |\Phi_j\rangle\|^2 .
\] (118)

Defining
\[
\eta_s^2 \defeq \frac{1}{(\ell + 1)^g} \sum_{j=1}^{(\ell+1)^g} \sum_{|\nu| = s} \|P_{\nu} |\Phi_j\rangle\|^2 = \frac{1}{(\ell + 1)^g} \sum_{j=1}^{(\ell+1)^g} \|P_{\nu} |\Phi_j\rangle\|^2 ,
\] (119)
we find that \(\sum_s \eta_s^2 \leq 1\) (recall that \(\|\Phi_j\| \leq 1\)) and by Eq. (118), for every \(s \geq \ell\),
\[
\lambda_s \leq \frac{1}{x^2} k^{g^2 \ell} \left( \frac{\ell + 1}{\ell!} \right)^g s^{g^2 \ell^2} \theta^s \eta_s.
\] (120)

\section{B Proving general detectability lemma, Lemma 5.1}

To prove this lemma, we will prove the following auxiliary lemma

\textbf{Lemma B.1} Let \(\Pi_{\leq \ell}^{(g)} = 1 - \Pi_{> \ell}^{(g)}\) denote the projection into \(\ell\) or less violations in the \(i\)'th layer as in Sec. 3.6. Then
\[
\|\Pi_{\leq \ell}^{(g)} \cdots \Pi_{> \ell}^{(g)} |\psi\rangle\|^2 \leq 1 - \Delta^2(\ell) ,
\] (121)
where \(\Delta(\ell)\) is defined in Lemma 5.1 and in the \(\ell > 0\) case we assume that \((\epsilon_0/f) - r\ell > \frac{1}{1 - g}\).

The proof of Lemma B.1 would be given later in Sec. B.2. Based on it, we can prove Lemma 5.1 as follows

\textbf{Proof of Lemma 5.1} Given the state \(|\psi\rangle\) and an integer \(\ell \geq 0\), assume that Eq. (121) holds and yet for every layer,
\[
\|\Pi_{\leq \ell}^{(g)} |\psi\rangle\|^2 < \frac{1}{(2g)^2} \Delta^2(\ell) .
\] (122)
For brevity, we denote

\[ x \defeq \| \Pi_{\leq \ell}^{(g)} \cdots \Pi_{\leq \ell}^{(1)} \psi \| . \tag{123} \]

Then

\[ x^2 = \langle \psi | \Pi_{\leq \ell}^{(1)} \cdots \Pi_{\leq \ell}^{(g-1)} \Pi_{\leq \ell}^{(g)} \Pi_{\leq \ell}^{(g-1)} \cdots \Pi_{\leq \ell}^{(1)} | \psi \rangle . \tag{124} \]

Every product of \( N \) operators can be written as:

\[ O_1 \cdots O_N = \mathbb{1} + (O_1 - \mathbb{1}) + O_1 (O_2 - \mathbb{1}) + O_1 O_2 (O_3 - \mathbb{1}) + \cdots + (O_1 \cdots O_{N-1}) (O_N - \mathbb{1}) . \tag{125} \]

\[ + \cdots + (O_1 \cdots O_{N-1}) (O_N - \mathbb{1}) . \tag{126} \]

Expanding Eq. (124) this way, we get

\[ x^2 = 1 + \langle \psi | (\Pi_{\leq \ell}^{(1)} - \mathbb{1}) | \psi \rangle + \langle \psi | (\Pi_{\leq \ell}^{(2)} - \mathbb{1}) | \psi \rangle + \langle \psi | (\Pi_{\leq \ell}^{(2)} \Pi_{\leq \ell}^{(3)} - \mathbb{1}) | \psi \rangle + \cdots \tag{127} \]

The RHS of the above equation contains \( 2g - 1 \) terms of the form \( \langle \psi | \Pi_{\leq \ell}^{(1)} \cdots \Pi_{\leq \ell}^{(g-1)} (\Pi_{\leq \ell}^{(g)} - \mathbb{1}) | \psi \rangle \). Let us estimate their magnitude. By an expansion similar to Eq. (125), we write

\[ \Pi_{\leq \ell}^{(1)} \cdots \Pi_{\leq \ell}^{(i)} = (1 - \Pi_{\leq \ell}^{(i)}) + (1 - \Pi_{\leq \ell}^{(i-1)}) \Pi_{\leq \ell}^{(i)} + \cdots . \tag{128} \]

Therefore \( \langle \psi | \Pi_{\leq \ell}^{(1)} \cdots \Pi_{\leq \ell}^{(i)} (\Pi_{\leq \ell}^{(i+1)} - \mathbb{1}) | \psi \rangle \) can be written as a sum of at most \( 2g \) terms, each of them is an inner product of \( \langle \psi | (1 - \Pi_{\leq \ell}^{(i)}) \) times some projections, times \( (1 - \Pi_{\leq \ell}^{(i)}) | \psi \rangle \). By our assumption, the norm of the ket and bra is smaller than \( \Delta(\ell)/(2g) \) and as the norms of the projections are smaller than or equal to unity we find

\[ | \langle \psi | \Pi_{\leq \ell}^{(1)} \cdots \Pi_{\leq \ell}^{(i)} (\Pi_{\leq \ell}^{(i+1)} - \mathbb{1}) | \psi \rangle | \leq 2g \frac{\Delta^2(\ell)}{(2g)^2} = \frac{\Delta^2(\ell)}{2g} . \tag{129} \]

Therefore, overall,

\[ x^2 \leq 1 + (2g - 1) \frac{\Delta^2(\ell)}{2g} < 1 + \Delta^2(\ell) , \tag{130} \]

contradicting Eq. (121).

We now turn to the proof of Lemma B.1. The outline of the proof is very similar to the simple case of 2-layers, \( \ell = 0 \), and was discussed in Sec. 5. The main goal of the proof is to estimate the energy of the normalized state \( \frac{1}{x} \Pi_{\leq \ell}^{(g)} \cdots \Pi_{\leq \ell}^{(1)} | \psi \rangle \), which has contributions from all layers. For every layer we will find a crude upper bound of its energy as a function of \( x \) (as well as \( \ell, k, g, \theta \)). Summing all these bounds together, we will get an upper bound to the total energy. This energy is lower bounded by \( \epsilon_0 \), the ground energy of the system, and this gives us an inequality. We then reverse it and extract an upper bound for \( x \).

We start by using the XY decomposition to upper bound the energy of the first layer.

**B.1 Estimating the energy of the first layer**

Consider then an XY decomposition, and let \( E^{\text{top}} \) denote the energy of all the constraints of the first layer (the top layer in Fig. 1) that belong to the pyramids of the decomposition. We define \( | \Omega \rangle \) to be the following normalized state:

\[ | \Omega \rangle \defeq \frac{1}{x} \Pi_{\leq \ell}^{(g)} \cdots \Pi_{\leq \ell}^{(1)} | \psi \rangle . \tag{131} \]

The entire section will be dedicated to proving the following lemma:
Lemma B.2 For $\ell = 0$, 
\[ \langle \Omega | E^{top} | \Omega \rangle \leq \frac{1 - x^2}{x^2} \frac{\theta^2}{(1 - \theta^2)^3} , \] (132)
and for $\ell > 0$ there is a positive function function $r(\theta, k, g)$ (independent of $|\Omega\rangle$) such that 
\[ \langle \Omega | E^{top} | \Omega \rangle \leq r \ell + \frac{1}{x^2} \frac{1}{1 - \theta} . \] (133)

Proof:

The $\ell = 0$ case was essentially already proved in the 2-layers case in Sec. 4 (specifically, see Eq. (35)). The difference between the 2-layers case and the $g$-layers case are semantic and therefore we will only consider the $\ell > 0$ case.

Consider the coarse- and fine-grained $XY$ decomposition of $|\Omega\rangle$,
\[ |\Omega\rangle = \sum_\nu \lambda_\nu |\Omega_\nu\rangle = \sum_s \lambda_s |\Omega_s\rangle . \] (134)

Since $E^{top}$ a sum of the inverses of pyramid projections from the first layer, it must commute with the $XY$ projections $P_\nu$. Therefore,
\[ \langle \Omega | E^{top} | \Omega \rangle = \sum_s \lambda_s^2 \langle \Omega_s | E^{top} | \Omega_s \rangle . \] (135)

Our first claim is

Claim B.3 For every $s$ with non-zero weight $\lambda_s$,
\[ \langle \Omega_s | E^{top} | \Omega_s \rangle \leq \ell + s . \] (136)

Proof: It is sufficient to prove that for every sector $\nu$ with non-zero weight, $\langle \Omega_\nu | E^{top} | \Omega_\nu \rangle \leq \ell + |\nu|$. 

$E^{top}$ has one contribution from every pyramid top. Consider a sector $\nu$ of the fine-grained $XY$ decomposition. It contains $|\nu|$ $Y$ spaces and the rest are $X$ spaces. The maximal energy contribution from the $Y$ pyramids is therefore $|\nu|$. We will now show that the contribution from the $X$ pyramids is at most $\ell$. Essentially, the proof boils down to the fact that the projections commute on the $X$ sectors and therefore if $|\Omega\rangle$ has more than $\ell$ violations on the $X$ sectors then also $\Pi_{\leq \ell}^y |\psi\rangle$ has - which is impossible. The following argument shows this more formally.

Let $Q_i$ be the projection in the first layer in the $i$'th pyramid, where $\nu$ has an X sector. Then either $Q_i |\Omega_\nu\rangle = 0$ or $Q_i |\Omega_\nu\rangle = |\Omega_\nu\rangle$. If the total contribution from all $X$ sectors is larger than $\ell$, there are $\ell + 1$ pyramids in which $Q_i |\Omega_\nu\rangle = |\Omega_\nu\rangle$. For brevity, assume that these appear in the first $\ell + 1$ pyramids. Then
\[ \left( \prod_{i=1}^{\ell+1} Q_i \right) |\Omega_\nu\rangle = |\Omega_\nu\rangle . \] (137)

Assuming that $\lambda_\nu \neq 0$, we get
\[ \left( \prod_{i=1}^{\ell+1} Q_i \right) P_\nu |\Omega\rangle = P_\nu |\Omega\rangle . \] (138)

Using the definition of $|\Omega\rangle$ in Eq. (134) and Eq. (10), the LHS of the above equation is equal to
\[ \frac{1}{x} \sum_{j_1, \ldots, j_g} \left( \prod_{i=1}^{\ell+1} Q_i \right) P_\nu \cdot (\Delta_{j_g}^{(g)} \cdots \Delta_{j_g}^{(1)}) \cdot (R_{\leq \ell-j_g}^{(g)} \cdots R_{\leq \ell-j_g}^{(1)}) |\psi\rangle . \] (139)
The above expression vanishes. The reason is that it is a sum over terms which all contain
\[(\prod_{i=1}^{\ell+1} Q_i) P_\nu \cdot (\Delta_{jy}^{(\ell+1)} \Delta_{jy}^{(\ell)} \cdots)\] (140)

Using the fact that \(P_\nu\) commutes with the constraints inside the pyramids, and that within an \(X\) sector the constraints of the pyramids commute within themselves, this is equal to
\[P_\nu (\prod_{i=1}^{\ell+1} Q_i) \cdot (\Delta_{jy}^{(\ell+1)} \Delta_{jy}^{(\ell)} \cdots) \cdot (\Delta_{jy}^{(\ell)})\] (141)

But \((\prod_{i=1}^{\ell+1} Q_i) \cdot (\Delta_{jy}^{(\ell)})\) is identically zero as it must contain at least one term of the form \(Q(1-Q)\). It follows that \(P_\nu |\Omega\rangle = 0\) which can only happen when \(\lambda_\nu = 0\). \(\blacksquare\)

Next, we bound the weights \(\lambda_s\) using the exponential decay of Sec. 3.6, which is proved in Sec. A. According to Lemma 3.3, there exists a set of weights \(\eta^2_s\) such that \(\sum_s \eta^2_s \leq 1\) and for every \(s \geq \ell\),
\[\lambda_s \leq \frac{1}{x} k g^2 \ell \left(\frac{\ell + 1}{\ell!}\right) g^{\ell} \theta^s \eta_s\] (142)

Bounding \(\eta_s\) by 1, we get that
\[\lambda_s \leq \frac{1}{x} k g^2 \ell \left(\frac{\ell + 1}{\ell!}\right) g^{\ell} \theta^s\] (143)

We are now in position to prove the main result of this section, Eq. (133). In Appendix C we use the above equation to show that it is possible to find a constant \(r(\theta, k, g)\) such that for every \(s > r\ell\),
\[\lambda^2_s \leq \frac{1}{x^2} \theta^s\] (144)

Consequently, from Claim B.3 it follows that
\[\langle \Omega | E_{\text{top}} | \Omega \rangle \leq \ell + \sum_{s=0}^{r\ell} \lambda^2_s + \frac{1}{x^2} \sum_{s=r\ell+1}^{\infty} \theta^s\] (145)
\[\leq (r + 1) \ell + \frac{1}{x^2} \frac{\theta^{(r\ell+1)}}{1 - \theta}\] (146)

By redefining \(r(\theta, k, g) = r(\theta, k, g) + 1\), and using the fact that \(\theta^{(r\ell+1)} < 1\), we recover Eq. (133).

Finally, we can now prove Lemma B.1

B.2 Proof of Lemma B.1

To prove Lemma B.1 we use Lemma B.2 to estimate the total energy of the system. To estimate the energy of the first layer, we apply Lemma B.2 several times using different \(XY\) decompositions. The \(XY\) decompositions are chosen such that every constraint in the first layer appears in exactly one \(XY\) decomposition. One can easily verify that the total number of such decompositions that is needed for this task is upper bounded by some constant \(f_1(k, g)\). Therefore, the total energy of the first layer is bounded by
\[\langle \Omega | E_1 | \Omega \rangle \leq \begin{cases} f_1 \frac{1-x^2}{x^2 (1-\theta^2)} & , \ell = 0 \\ f_1 \left[r \ell + \frac{1}{x^2} \frac{\theta^2}{1-\theta}\right] & , \ell > 0 \end{cases}\] (147)
To bound the energy of the second layer, we can apply the derivation of the first layer, with some trivial modifications:

\[ g \to g - 1, \]
\[ |\psi\rangle \to \Pi^{(g)}_{\ell} |\psi\rangle. \tag{148} \]

In addition, we need to update the functions \( r(\theta, k, g) \) and \( f_1(k, g) \). It is easy to see that both of them can be decreased. Therefore, it is not surprising to see that the upper bound of the \( \langle \Omega | E_2 | \Omega \rangle \) is smaller than the upper bound of \( \langle \Omega | E_1 | \Omega \rangle \), and this true for all the other layers. Consequently, by setting \( f(k, g) \overset{\text{def}}{=} gf_1(k, g) \), we get:

\[ \epsilon_0 \leq \langle \Omega | E | \Omega \rangle \leq \begin{cases} f \frac{1 - x^2}{x^2} \left( \frac{\theta^2}{1 - \theta^2} \right)^s & , \ell = 0 \\ f \left[ r\ell + \frac{1}{x^2} \right] & , \ell > 0 \end{cases}, \tag{150} \]

Here \( \epsilon_0 \) is the ground energy of the system. Lemma B.1 is now proved by inverting this inequality:

For \( \ell = 0 \):

\[ x^2 \leq \frac{1}{(\epsilon_0/f)(1 - \theta^2)} = 1 - \Delta^2(0), \tag{151} \]

For \( \ell > 0 \):

\[ x^2 \leq \frac{1}{1 - \theta} \cdot \frac{1}{(\epsilon_0/f) - r\ell} = 1 - \Delta^2(\ell). \tag{152} \]

Note, of course, that the \( \ell > 0 \) inequality is only valid for \((\epsilon_0/f) - r\ell > 0\).

\section{Finding \( r(\theta, k, g) \)}

In this section we prove that it is possible to find a constant \( r(\theta, k, g) \) such that for every \( s > r\ell \),

\[ \frac{1}{x^2} \left( \frac{\ell + 1}{\ell!} \right) 2^g k^{2g} \epsilon s^{2s} \theta^{2s} \leq \frac{1}{x^2} \theta^s. \tag{153} \]

Eliminating a factor \( \frac{\theta^s}{x^2} \) and taking a log of the equation, we find the following sufficient condition

\[ 2g [\log(\ell + 1) - \log(\ell!) + 2g^2 \ell \log(k) + (2g\ell + 1) \log(s) + s \log(\theta) < 0 \tag{154} \]

which is equivalent to

\[ \frac{2g}{s} [\ell \log(s) - \log(\ell!)] + \frac{2g}{s} \log(s) + \frac{\log(s)}{s} < \log(1/\theta). \tag{155} \]

Re-arranging it gives

\[ \frac{2g}{s} [\ell \log(s) - \log(\ell!)] + \frac{2g}{s} \log(s) + \frac{\log(r)}{s} < \log(1/\theta). \tag{156} \]

On the LHS we have the sum of 4 terms. For \( s > 3 \), \( \log(s)/s \) is monotonically decreasing (\( \log(\cdot) \) is the natural logarithm). So if the above condition holds for \( s = r\ell \) with \( r > 3 \), it would hold for any \( s > r\ell \). Therefore, a sufficient condition is

\[ \frac{2g}{s} [\ell \log(s) - \log(\ell!)] + \frac{2g}{s} \log(s) + \frac{\log(r)}{s} < \log(1/\theta). \tag{157} \]

Next, for every \( \ell \geq 1 \), the term \( \frac{\log(r+1)}{\ell} \), which appears in the second element is smaller than 1, therefore a sufficient condition is

\[ \frac{2g}{s} [\ell \log(s) - \log(\ell!)] + \frac{2g}{s} \log(s) + \frac{1}{s} < \log(1/\theta). \tag{158} \]
Let us now analyze the first term. Using Sterling’s approximation, we get
\[
\frac{2g}{s/s/\ell - \log(s) - \log(\ell)} \leq \frac{2g}{s/s/\ell - \log(s) - \log(\ell) + \ell}
\]
\[
= \frac{2g}{s/\ell} \log(s/\ell) + \frac{2g}{s/\ell}.
\]
Again, using the assumption that \(s/\ell > r > 3\), then \(\log(s/\ell) < \log(r)/r\), and \(\frac{2g}{s/\ell} < 2g/r\). So overall, we find that as long as \(r > 3\), a sufficient condition for Eq. 153 is
\[
(2g + 1)\frac{\log(r)}{r} + \frac{4g + 2g^2\log(k)}{r} < \log(1/\theta).
\]
The LHS of the above inequality approaches zero as \(r \to +\infty\), hence we can find an \(r(\theta, k, g) > 3\) that satisfies it.

\section{Proof of Classical Amplification lemma}

\textbf{Proof:} (of Lemma 7.1) Given the assignment \(\sigma\), we let \(F \subseteq E\) denote the set of unsatisfied edges in \(G\). Obviously, \(\text{UNSAT}_\sigma(G) = |F|/|E|\). Consider the homogeneous probability distribution over all \(t\)-walks. We define a random variable \(Z(e)\) that counts the number of unsatisfied edges in the \(t\)-walk \(e = (e_1, \ldots, e_t)\). Then
\[
\text{UNSAT}_\sigma(G^t) = \Pr[Z(e) > 0].
\]
Moreover, since \(Z(e)\) is a non-negative random variable that is not identically 0,
\[
\Pr[Z(e) > 0] \geq \frac{\mathbb{E}[Z(e)]}{\mathbb{E}[Z^2(e)]}.
\]
In what follows, we will lower-bound \(\mathbb{E}[Z(e)]\) and upper-bound \(\mathbb{E}[Z^2(e)]\). To do that, we write \(Z(e) = \sum_{i=1}^t Z_i(e)\), where \(Z_i(e)\) is the random variable that is equal to 1 if the \(i\)'th edge of \(e\) is unsatisfied and to 0 otherwise. It is easy to see that for every \(i\), \(\mathbb{E}[Z_i(e)] = |F|/|E|\), and therefore
\[
\mathbb{E}[Z(e)] = t|F|/|E|.
\]
To bound \(\mathbb{E}[Z^2(e)]\), we write
\[
\mathbb{E}[Z^2(e)] = \sum_{i,j} \mathbb{E}[Z_i(e)Z_j(e)] = \sum_{i=1}^t \mathbb{E}[Z_i^2(e)] + 2 \sum_{i<j} \mathbb{E}[Z_i(e)Z_j(e)].
\]
Note that \(Z_i^2(e) = Z_i(e)\) and so
\[
\mathbb{E}[Z^2(e)] = t|F|/|E| + 2 \sum_{i<j} \mathbb{E}[Z_i(e)Z_j(e)].
\]
To estimate \(\mathbb{E}[Z_i(e)Z_j(e)]\), we can use the expansion properties of \(G\), which imply that as \(i, j\) grow apart, \(Z_i(e)\) and \(Z_j(e)\) become more and more independent, and so \(\mathbb{E}[Z_i(e)Z_j(e)] \to \mathbb{E}[Z_i(e)] \cdot \mathbb{E}[Z_j(e)]\). The exact statement is that for \(i > j\),
\[
\mathbb{E}[Z_i(e)Z_j(e)] \leq \frac{|F|}{|E|} \left( \frac{|F|}{|E|} + |\lambda|^{i-j-1} \right).
\]
The proof of this fact is standard, and is given in Ref. [Din07], and will therefore be omitted. Inserting this into Eq. (166), we arrive to

\[ \mathbb{E}[Z^2(e)] = t \frac{|F|}{|E|} + 2 \frac{|F|}{|E|} \sum_{i=1}^{t} \sum_{j=i+1}^{t} \left( \frac{|F|}{|E|} + |\lambda|^{i-j-1} \right) \] (168)

\[ \leq t \frac{|F|}{|E|} + t(t-1) \left( \frac{|F|}{|E|} \right)^2 + \frac{2t}{1-\lambda} \frac{|F|}{|E|} \] (169)

\[ = t \frac{|F|}{|E|} \left( 1 + \frac{|F|}{|E|} (t-1) + \frac{2}{1-\lambda} \right). \] (170)

Using Eq. (163), we get

\[ \text{UNSAT}_{\sigma}(G^t) \geq \frac{t \text{UNSAT}_{\sigma}(G)}{1 + t \text{UNSAT}_{\sigma}(G) + \frac{2}{1-\lambda}} \overset{\text{def}}{=} F(\text{UNSAT}_{\sigma}(G)), \] (171)

where \( F(x) = \frac{tx}{1+tx+\frac{1}{1-\lambda}} \). If \( x \leq 1/t \), \( F(x) \geq \frac{tx}{2+\frac{1}{1-\lambda}} \). On the other hand, as \( F(x) \) is monotonically increasing for \( x > 0 \), then for \( x \geq 1/t \), \( F(x) \geq F(1/t) = \frac{1}{2+\frac{1}{1-\lambda}} \). Setting \( x = \text{UNSAT}_{\sigma}(G) \) and using Equation (7) completes the proof. \( \blacksquare \)

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