NON-COMPACT SUBSETS OF THE ZARISKI SPACE OF AN INTEGRAL DOMAIN

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Abstract. Let $V$ be a minimal valuation overring of an integral domain $D$ and let $\text{Zar}(D)$ be the Zariski space of the valuation overrings of $D$. Starting from a result in the theory of semistar operations, we prove a criterion under which the set $\text{Zar}(D) \setminus \{V\}$ is not compact. We then use it to prove that, in many cases, $\text{Zar}(D)$ is not a Noetherian space, and apply it to the study of the spaces of Kronecker function rings and of Noetherian overrings.

1. Introduction

The Zariski space $\text{Zar}(K|D)$ of the valuation rings of a field $K$ containing a domain $D$ was introduced (under the name abstract Riemann surface) by O. Zariski, who used it to show that resolution of singularities holds for varieties of dimension 2 or 3 over fields of characteristic 0 [32, 33]. In particular, Zariski showed that $\text{Zar}(K|D)$, endowed with a natural topology, is always a compact space [34, Chapter VI, Theorem 40]; this result has been subsequently improved by showing that $\text{Zar}(K|D)$ is a spectral space (in the sense of Hochster [18]), first in the case where $K$ is the quotient field of $D$ [4, 5], and then in the general case [8, Corollary 3.6(3)]. The topological aspects of the Zariski space has subsequently been used, for example, in real and rigid algebraic geometry [19, 31] and in the study of representation of integral domains as intersections of valuation overrings [26, 27, 28]. In the latter context, i.e., when $K$ is the quotient field of $D$, two important properties for subspaces of $\text{Zar}(K|D)$ to investigate are the properties of compactness and of Noetherianess.

In this paper, we concentrate on the case where $K$ is the quotient field of $D$, studying subspaces of $\text{Zar}(K|D) = \text{Zar}(D)$ that are not compact. The starting point is a criterion based on semistar operations, proved in [8, Theorems 4.9 and 4.13] (see also [11, Proposition 4.5] for a slightly stronger version) and integrated, as in [9, Example 3.7], with

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the use of the two-faced definition of the integral closure/b-operation, either through valuation overrings or through equations of integral dependence (see e.g. [20, Chapter 6]). In particular, we analyze sets of the form Zar($D$) \ {V}, where V is a minimal valuation overring of $D$: we show in Section 3 that such a space is compact only if V can be obtained from $D$ in a very specific way (more precisely, as the integral closure of a localization of a finitely generated algebra over $D$), and we follow up in Sections 4 and 5 by showing that this condition implies a bound on the dimension of V in relation with the dimension of $D$ (Proposition 4.3) and a quite strict condition on the intersection of sets of prime ideals of $D$ (Theorem 5.1). Section 6 is dedicated to a brief application of these criteria to the study of Kronecker function rings (the definition will be recalled later).

In Section 7, we consider the set Over($D$) of overrings of $D$ (which is known to be itself a spectral space [7, Proposition 3.5]). Using the result proved in the previous sections, we show that, when $D$ is a Noetherian domain, some distinguished subspaces of Over($D$) (for example, the subspace of overrings of $D$ that are Noetherian) are not spectral.

2. Preliminaries and notation

2.1. Spectral spaces. A topological space $X$ is a spectral space if there is a ring $R$ such that $X$ is homeomorphic to the prime spectrum Spec($R$), endowed with the Zariski topology. Spectral spaces can be characterized in a purely topological way as those spaces that are $T_0$, compact, with a basis of open and compact subset that is closed by finite intersections and such that every irreducible closed subset has a generic point (i.e., it is the closure of a single point) [18, Proposition 4].

On a spectral space $X$ it is possible to define two new topologies: the inverse and the constructible topology.

The inverse topology is the topology on $X$ having, as a basis of closed sets, the family of open and compact subspaces of $X$. Endowed with the inverse topology, $X$ is again a spectral space [18, Proposition 8]; moreover, a subspace $Y \subseteq X$ is closed in the inverse topology if and only if $Y$ is compact (in the original topology) and $Y = Y^{\text{gen}}$ [8, Remark 2.2 and Proposition 2.6], where

$$Y^{\text{gen}} := \{ z \in X \mid z \leq y \text{ for some } y \in Y \} = \{ z \in X \mid y \in \text{Cl}(z) \text{ for some } y \in Y \},$$

with $\text{Cl}(z)$ denoting the closure of the singleton $\{z\}$ (again, in the original topology) and $\leq$ is the order induced by the original topology [17, d-1], which coincides on Spec($R$) with the set-theoretic inclusion.

The constructible topology on $X$ (also called patch topology) is the coarsest topology such that the open and compact subsets of $X$ are both open and closed. Endowed with the constructible topology, $X$
is a spectral space that is also Hausdorff (see [30] Propositions 3 and 5, [29] or [14] Proposition 5), and the constructible topology is finer than both the original and the inverse topology. A subset of \( X \) closed in the constructible topology is said to be a proconstructible subset of \( X \); if \( Y \) is proconstructible, then it is a spectral space when endowed with the topology induced by the original spectral topology of \( X \), and the constructible topology on \( Y \) is exactly the topology induced by the constructible topology on \( X \) (this follows from [3] 1.9.5(vi-vii)).

2.2. Noetherian spaces. A topological space \( X \) is Noetherian if \( X \) verifies the ascending chain condition on the open subsets, or equivalently if every subspace of \( X \) is compact. Examples of Noetherian spaces are finite spaces and the prime spectra of Noetherian rings. If \( \text{Spec}(R) \) is a Noetherian space, then every proper ideal of \( R \) has only finitely many minimal primes (see e.g. the proof of [2] Chapter 4, Corollary 3, p.102) or [1] Chapter 6, Exercises 5 and 7).

2.3. Overrings and the Zariski space. Let \( D \subseteq K \) be an extension of integral domains. We denote the set of all rings contained between \( D \) and \( K \) by Over\((K|D)\); if \( K \) is a field (not necessarily the quotient field of \( D \)), the set of all valuation rings containing \( D \) with quotient field \( K \) is denoted by Zar\((K|D)\), and it is called the Zariski space (or the Zariski-Riemann space) of \( D \).

The Zariski topology on \( \text{Over}(K|D) \) is the topology having, as a subbasis, the sets of the form

\[
B(x_1, \ldots, x_n) := \{ T \in \text{Over}(K|D) \mid x_1, \ldots, x_n \in T \},
\]

as \( \{x_1, \ldots, x_n\} \) ranges among the finite subsets of \( K \). Under this topology, both \( \text{Over}(K|D) \) [2 Proposition 3.5] and its subspace Zar\((K|D)\) [5 4] are spectral spaces, and the order induced by this topology is the inverse of the set-theoretic inclusion. In particular, every \( Y \subseteq \text{Over}(K|D) \) with a minimum element is compact, and, if \( Z \) is an arbitrary subset of \( \text{Over}(K|D) \), then \( Z^{\text{gen}} = \{ T \in \text{Over}(K|D) \mid T \supseteq A \text{ for some } A \in Z \} \).

We denote by Zar\(_{\text{min}}(D)\) the set of minimal elements of Zar\((D)\); since Zar\((D)\) is a spectral space, every \( V \in \text{Zar}(D) \) contains an element \( W \in \text{Zar}_{\text{min}}(D) \).

If \( K \) is the quotient field of \( D \), then we set Over\((K|D) =: \text{Over}(D)\) and Zar\((K|D) =: \text{Zar}(D)\). Elements of \( \text{Over}(D) \) are called overrings of \( D \), elements of \( \text{Zar}(D) \) are the valuation overrings of \( D \) and elements of \( \text{Zar}_{\text{min}}(D) \) are the minimal valuation overrings of \( D \).

The center map is the application

\[
\gamma: \text{Zar}(K|D) \rightarrow \text{Spec}(D)
\]

\[
V \mapsto m_V \cap D,
\]
where $m_V$ is the maximal ideal of $V$. When $\text{Zar}(K|D)$ and $\text{Spec}(D)$ are endowed with the respective Zariski topologies, the map $\gamma$ is continuous ([34 Chapter VI, §17, Lemma 1] or [4, Lemma 2.1]), surjective (this follows, for example, from [1, Theorem 5.21] or [15, Theorem 19.6]) and closed [4, Theorem 2.5].

2.4. Semistar operations. Let $D$ be a domain with quotient field $K$. Let $\mathbf{F}(D)$ be the set of $D$-submodules of $K$, $\mathcal{F}(D)$ be the set of fractional ideals of $D$, and $\mathcal{F}_f(D)$ be the set of finitely generated fractional ideals of $D$.

A semistar operation on $D$ is a map $\star : \mathcal{F}(D) \rightarrow \mathcal{F}(D)$, $I \mapsto I^\star$, such that, for every $I, J \in \mathcal{F}(D)$ and every $x \in K$,

1. $I \subseteq I^\star$;
2. if $I \subseteq J$, then $I^\star \subseteq J^\star$;
3. $(I^\star)^\star = I^\star$;
4. $x \cdot I^\star = (xI)^\star$.

Given a semistar operation $\star$, the map $\star_f$ is defined on every $E \in \mathcal{F}(D)$ by

$$E^{\star_f} = \bigcup \{F^\star | F \in \mathcal{F}_f(D), F \subseteq E\}.$$ 

The map $\star_f$ is always a semistar operation; if $\star = \star_f$, then $\star$ is said to be of finite type. Two semistar operations of finite type $\star_1, \star_2$ are equal if and only if $I^{\star_1} = I^{\star_2}$ for every $I \in \mathcal{F}_f(D)$. See [25] for general informations about semistar operations.

If $\Delta \subseteq \text{Zar}(D)$, then $\wedge_{\Delta}$ is defined as the semistar operation on $D$ such that

$$I^{\wedge_{\Delta}} := \bigcap \{IV | V \in \Delta\}$$

for every $D$-submodule $I$ of $K$; a semistar operation of type $\wedge_{\Delta}$ is said to be a valuative semistar operation. By [11 Proposition 4.5], $\wedge_{\Delta}$ is of finite type if and only if $\Delta$ is compact (in the Zariski topology of $\text{Zar}(D)$). If $\Delta, \Lambda \subseteq \text{Zar}(D)$, then $\wedge_{\Delta} = \wedge_{\Lambda}$ if and only if $\Delta^{\text{gen}} = \Lambda^{\text{gen}}$ [10 Lemma 5.8(1)], while $(\wedge_{\Delta})_f = (\wedge_{\Lambda})_f$ if and only if $\Delta$ and $\Lambda$ have the same closure with respect to the inverse topology [8, Theorem 4.9].

The semistar operation $\wedge_{\text{Zar}(D)}$ is usually denoted by $b$ and called the $b$-operation.

3. The use of minimal valuation domains

The starting point of this paper is the following well-known result.

**Proposition 3.1** (see e.g. [20 Proposition 6.8.2]). Let $I$ be an ideal of an integral domain $D$; let $x \in D$. Then, $x \in IV$ for every $V \in \text{Zar}(D)$ if and only if there are $n \geq 1$ and $a_1, \ldots, a_n \in D$ such that $a_i \in I^i$ and

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$
An inspection of the proof of the previous proposition given in [20] shows that this result does not really rely on the fact that is an ideal of , or on the fact that , and to ever . In the terminology of seminal operations, this means that, for each , is exactly the set of that verifies an equation like (11), with . We are interested in generalizing that proof in a different way; we need the following definitions.

**Definition 3.2.** Let be an integral domain and let , . We say that dominates if, for every and every , there is a such that .

For example, dominates every subset of , while the set of localizations of dominates .

**Definition 3.3.** Let be an integral domain. We denote by the set of finitely generated -algebras of , or equivalently

\[
D[\mathcal{F}_f] := \{ D[I] : I \in \mathcal{F}_f(R) \}.
\]

Even if the proof of the following result essentially repeats the proof of Proposition 6.8.2, we replay it here for clarity.

**Proposition 3.4.** Let be an integral domain, and suppose that dominates . Then, for every finitely generated ideal of , \( I^\Lambda = I^K \).

*Proof.* Clearly, \( I^K \subseteq I^\Lambda \). Suppose thus that \( x \in I^\Lambda \), \( x \neq 0 \), and let \( I = (i_1, \ldots, i_k)D \). Define \( J := x^{-1}I \in \mathcal{F}_f(D) \), and let \( A := D[J] = D[x^{-1}i_1, \ldots, x^{-1}i_k] \); by definition, \( J \subseteq A \).

If \( JA \neq A \), then there is a maximal ideal \( M \) of \( A \) containing \( J \), and thus, by domination, there is a valuation domain \( V \in \Delta \) containing \( A \) whose maximal ideal \( m_V \) is such that \( JV \subseteq m_V \), and thus \( IV \subseteq xm_V \). However, \( x \in I^K \subseteq IV \), which implies \( x \in xm_V \), a contradiction.

Hence, \( JA = A \), i.e., \( 1 = j_1a_1 + \cdots + j_na_n \) for some \( j_t \in J \), \( a_t \in A \); expliciting the elements of \( A \) as elements of \( D[J] \) and using \( J = x^{-1}I \), we find that there must be an \( N \in \mathbb{N} \) and elements \( i_t \in I^t \) such that \( x^N = i_1x^{N-1} + \cdots + i_{N-1}x + i_N \), which gives an equation of integral dependence of \( x \) over \( I \). Therefore, \( x \in I^K \), as requested. \( \square \)

We can now use the properties of valuative semistar operations to study compactness.

**Proposition 3.5.** Let be an integral domain, and let be a set that dominates . Then, is compact if and only if it contains \( \text{Zar}_{\min}(D) \).

*Proof.* If \( \Delta \) contains \( \text{Zar}_{\min}(D) \), then \( U \) is an open cover of \( \Delta \) if and only if it is an open cover of \( \text{Zar}(D) \); thus, \( \Delta \) is compact since \( \text{Zar}(D) \) is.
Conversely, suppose $\Delta$ is compact. By Proposition 3.4, $I^{\wedge}\Delta = I^{b}$ for every finitely generated ideal $I$; hence, $(\wedge\Delta)f = b_{f} = b$. By [10, Lemma 5.8(1)], it follows that the closure of $\Delta$ with respect to the inverse topology of $\text{Zar}(D)$ is the whole $\text{Zar}(D)$; however, since $\Delta$ is compact, its closure in the inverse topology is exactly $\Delta^{\text{gen}} = \Delta^{\uparrow} = \{W \in \text{Zar}(D) \mid W \supseteq V \text{ for some } V \in \Delta\}$. Hence, $\Delta$ must contain $\text{Zar}_{\text{min}}(D)$. □

Thus, to find a subset of $\text{Zar}(D)$ that is not compact, it is enough to find a $\Delta$ that dominates $D[\mathcal{F}_{f}]$ but that does not contain $\text{Zar}_{\text{min}}(D)$. The easiest case where this criterion can be applied is when $\Delta = \text{Zar}(D) \setminus \{V\}$ for some $V \in \text{Zar}_{\text{min}}(D)$.

**Theorem 3.6.** Let $D$ be an integral domain and let $V \in \text{Zar}_{\text{min}}(D)$. If $\text{Zar}(D) \setminus \{V\}$ is compact, then $V$ is the integral closure of $D[x_{1}, \ldots, x_{n}]_{M}$ for some $x_{1}, \ldots, x_{n} \in K$ and some $M \in \text{Max}(D[x_{1}, \ldots, x_{n}])$.

**Proof.** If $\Delta := \text{Zar}(D) \setminus \{V\}$ is compact, then by Proposition 3.5, it cannot dominate $D[\mathcal{F}_{f}]$. Hence, there is a finitely generated fractional ideal $I$ such that $\Delta$ does not dominate $A := D[I]$, and so a maximal ideal $M$ of $A$ such that $1 \in MW$ for every $W \in \Delta$. In particular, $A \neq K$ (otherwise $M$ would be $(0)$).

However, there must be a valuation ring containing $A_{M}$ whose center (on $A_{M}$) is $MA_{M}$, and the unique possibility for this valuation ring is $V$: it follows that $V$ is the unique valuation ring centered on $MA_{M}$. However, the integral closure of $A_{M}$ is the intersection of the valuation rings with center $MA_{M}$ (since every valuation ring containing $A_{M}$ contains a valuation ring centered on $MA_{M}$ [15, Corollary 19.7]); thus, $V$ is the integral closure of $A_{M}$. □

4. THE DIMENSION OF $V$

Before embarking on using Theorem 3.6, we prove a simple yet general result.

**Proposition 4.1.** Let $D$ be an integral domain. If $\text{Zar}(D)$ is a Noetherian space, so is $\text{Spec}(D)$.

**Proof.** The claim follows from the fact that $\text{Spec}(D)$ is the continuous image of $\text{Zar}(D)$ through the center map $\gamma$, and that the image of a Noetherian space is still Noetherian. □

Note that the converse of this proposition is far from being true (this is, for example, a consequence of Proposition 5.4 or of Proposition 7.1).

The problem in using Theorem 3.6 is that it is usually difficult to control the behaviour of finitely generated algebras over $D$. We can, however, control the behaviour of the prime spectrum of $D$. 
Figure 1. Rings involved in the proof of Proposition 4.3

Lemma 4.2. Let $D$ be an integral domain, and let $V \in \text{Zar}(D)$ be the integral closure of $D_M$, for some $M \in \text{Spec}(D)$. Then, the set of prime ideals of $D$ contained in $M$ is linearly ordered.

Proof. Let $P, Q$ be two prime ideals of $D$ contained in $M$; then, $PD_M, QD_M \in \text{Spec}(D_M)$. Since $D_M \subseteq V$ is an integral extension, $PD_M = P' \cap D_M$ and $QD_M = Q' \cap D_M$ for some $P', Q' \in \text{Spec}(V)$; however, $V$ is a valuation domain, and thus (without loss of generality) $P' \subseteq Q'$. Hence, $PD_M \subseteq QD_M$ and $P \subseteq Q$, as requested. □

Proposition 4.3. Let $D$ be an integral domain, let $V \in \text{Zar}_{\text{min}}(D)$ and suppose that $\text{Zar}(D) \setminus \{V\}$ is compact. Let $\iota_V : \text{Spec}(V) \rightarrow \text{Spec}(D)$ be the canonical spectral map associated to the inclusion $D \hookrightarrow V$. For every $P \in \text{Spec}(D)$, $|\iota_V^{-1}(P)| \leq 2$; in particular, $\text{dim}(V) \leq 2 \text{dim}(D)$.

Proof. Suppose $|\iota_V^{-1}(P)| > 2$: then, there are prime ideals $Q_1 \subseteq Q_2 \subseteq Q_3$ of $V$ such that $\iota_V(Q_1) = \iota_V(Q_2) = \iota_V(Q_3) =: P$. If $\text{Zar}(D) \setminus \{V\}$ is compact, by Theorem 3.6 there is a finitely generated $D$-algebra $A := D[a_1, \ldots, a_n]$ such that $V$ is the integral closure of $A_M$, for some maximal ideal $M$ of $A$. We can write $A_M$ as a quotient $D[X_1, \ldots, X_n]_b$, where $X_1, \ldots, X_n$ are independent indeterminates and $a, b \in \text{Spec}(D[X_1, \ldots, X_n])$. Since $A_M \subseteq V$ is an integral extension, $Q_i \cap A \neq Q_j \cap A$ if $i \neq j$.

For $i \in \{1, 2, 3\}$, let $q_i$ be the prime ideal of $D[X_1, \ldots, X_n]$ whose image in $A$ is $Q_i$; then, $q_1, q_2$ and $q_3$ are distinct, $q_i \cap D = P$ for each $i$, and the set of ideals between $q_1$ and $q_3$ is linearly ordered (by Lemma 4.2). However, the prime ideals of $D[X_1, \ldots, X_n]$ contracting to $P$ are in a bijective and order-preserving correspondence with the prime ideals of $F[X_1, \ldots, X_n]$, where $F$ is the quotient field of $D/P$; since $F[X_1, \ldots, X_n]$ is a Noetherian ring, there are an infinite number of prime ideals between the ideals corresponding to $q_1$ and $q_3$. This is a contradiction, and $|\iota_V^{-1}(P)| \leq 2$.

For the “in particular” statement, take a chain $(0) \subseteq Q_1 \subseteq \cdots \subseteq Q_k$ in $\text{Spec}(V)$. Then, the corresponding chain of the $P_i := Q_i \cap D$ has length at most $\text{dim}(D)$, and moreover $\iota_V^{-1}((0)) = \{(0)\}$. Hence, $k + 1 \leq 2 \text{dim}(D) + 1$ and $\text{dim}(V) \leq 2 \text{dim}(D)$. □

The valuative dimension of $D$, indicated by $\text{dim}_v(D)$, is defined as the supremum of the dimensions of the valuation overrings of $D$; we have always $\text{dim}(D) \leq \text{dim}_v(D)$, and $\text{dim}_v(D)$ can be arbitrarily large.
with respect to $\dim(D)$ [15, Section 30, Exercises 16 and 17]. In particular, with the notation of the previous proposition, the cardinality of $i_D^{-1}(P)$ can be arbitrarily large: for example, if $(D, m)$ is local and one-dimensional, then $|i_D^{-1}(m)| = \dim_v(D)$.

**Corollary 4.4.** Let $D$ be an integral domain such that $\text{Zar}(D)$ is Noetherian. Then, $\dim_v(D) \leq 2 \dim(D)$.

**Proof.** If $\text{Zar}(D)$ is Noetherian, then in particular $\text{Zar}(D) \setminus \{V\}$ is compact for every $V \in \text{Zar}_{\text{min}}(D)$. Hence, $\dim(V) \leq 2 \dim(D)$ for every $V \in \text{Zar}_{\text{min}}(D)$, by Proposition [4.3] since, if $W \supseteq V$ are valuation domain, $\dim(W) \leq \dim(V)$, the claim follows. □

**Proposition 4.5.** Let $D$ be an integral domain, and let $V \in \text{Zar}_{\text{min}}(D)$ be such that $\text{Zar}(D) \setminus \{V\}$ is compact; let $(0) \subsetneq P_1 \subsetneq \cdots \subsetneq P_k$ be the chain of prime ideals of $V$ and let $Q_i := P_i \cap D$. Denote by $ht(P)$ the height of the prime ideal $P$. Then:

(a) for every $0 \leq t \leq \dim(D)$, we have
\[ \dim(V) \leq \dim_v(DQ_t) + 2(\dim(D) - ht(Q_t)); \]

(b) if $DQ_t$ is a valuation domain, then
\[ \dim(V) \leq 2 \dim(D) - ht(Q_t). \]

**Proof.** [a] Let $(0) \subsetneq Q^{(1)} \subsetneq Q^{(2)} \subsetneq \cdots \subsetneq Q^{(s)}$ be the chain $(0) \subseteq Q_1 \subseteq \cdots \subseteq Q_k$ without the repetitions, and let $a$ be the index such that $Q^{(a)} = Q_t$. For every $b > a$, by the proof of Proposition [4.3] there can be at most two prime ideals of $V$ over $Q^{(b)}$; on the other hand, $V_{P_t}$ is a valuation overring of $DQ_t$, and thus $t = \dim(V_{P_t}) \leq \dim_v(DQ_t)$. Therefore,
\[ \dim(V) \leq t + 2(s - a) \leq \dim_v(DQ_t) + 2(\dim(D) - ht(Q_t)) \]

since each ascending chain of prime ideals starting from $Q_t$ has length at most $\dim(D) - ht(Q_t)$.

Point [b] follows, since $\dim(V) = \dim_v(V)$ for every valuation domain $V$. □

**Example 4.6.** A class of integral domain whose Zariski space is Noetherian is constituted by the class of Prüfer domains with Noetherian spectrum. Indeed, if $D$ is a Prüfer domain then the valuation overrings of $D$ are exactly the localizations of $D$ at prime ideals; thus, the center map $\gamma$ establishes a homeomorphism between $\text{Zar}(D)$ and $\text{Spec}(D)$. Thus, if the latter is Noetherian also the former is Noetherian.

In this case, $\dim(D) = \dim_v(D)$.

**Example 4.7.** It is also possible to construct domains whose Zariski space is Noetherian but with $\dim(D) \neq \dim_v(D)$. For example, let $L$ be a field, and consider the ring $A := L + Y L(X)[[Y]]$, where $X$ and $Y$ are independent indeterminates. Then, the valuation overrings
of $A$ different from $F := L(X)((Y))$ are the rings in the form $V + YL(X)[[Y]]$, as $V$ ranges among the valuation rings containing $L$ and having quotient field $L(X)$; that is, $\mathrm{Zar}(A) \setminus \{F\} \simeq \mathrm{Zar}(L(X))L$. By the following Corollary 5.5, $\mathrm{Zar}(A)$ is a Noetherian space.

From this, we can construct analogous examples of arbitrarily large dimension. Indeed, if $R$ is an integral domain with quotient field $K$, and $T := R + \mathcal{X}K[[X]]$, then as above $\mathrm{Zar}(T)$ is composed by $K((X))$ and by rings of the form $V + \mathcal{X}K[[X]]$, as $V$ ranges in $\mathrm{Zar}(R)$; in particular, $\mathrm{Zar}(T) = \{K((X))\} \cup \mathcal{X}$, where $\mathcal{X} \simeq \mathrm{Zar}(R)$. Thus, $\mathrm{Zar}(T)$ is Noetherian if $\mathrm{Zar}(R)$ is. Moreover, $\dim(T) = \dim(R) + 1$ and $\dim_v(T) = \dim_v(R) + 1$.

Consider now the sequence of rings $R_1 := L + YL(X)[[Y]]$, $R_2 := R_1 + Y_2Q(R_1)[[Y_2]]$, \ldots, $R_n := R_{n-1} + Y_nQ(R_{n-1})[[Y_n]]$, where $Q(R)$ indicates the quotient field of $R$ and each $Y_i$ is an indeterminate over $Q(R_{i-1})((Y_{i-1}))$. Recursively, we see that each $\mathrm{Zar}(R_n)$ is Noetherian, while $\dim(R_n) = n \neq n + 1 = \dim_v(R_n)$.

5. Intersections of prime ideals

The results of the previous sections, while very general, are often difficult to apply, because it is usually not easy to determine the valuative dimension of a domain $D$. More applicable criteria, based on the prime spectrum of $D$, are the ones that we will prove next.

**Theorem 5.1.** Let $D$ be a local integral domain, and suppose there is a set $\Delta \subseteq \mathrm{Spec}(D)$ and a prime ideal $Q$ such that:

1. $Q \notin \Delta$;
2. no two members of $\Delta$ are comparable;
3. $\bigcap\{P \mid P \in \Delta\} = Q$;
4. $D_Q$ is a valuation domain.

Then, for any minimal valuation overring $V$ of $D$ contained in $D_Q$, $\mathrm{Zar}(D) \setminus \{V\}$ is not compact; in particular, $\mathrm{Zar}(D)$ is not Noetherian.

**Proof.** Note first that, since $V$ is a minimal valuation overring, its center $M$ on $D$ must be the maximal ideal of $D$ [15, Corollary 19.7]. Suppose that $\mathrm{Zar}(D) \setminus \{V\}$ is compact: by Theorem 3.6 there is a finitely generated $D$-algebra $A := D[x_1, \ldots, x_n]$ such that $V$ is the integral closure of $A_M$ for some $M \in \mathrm{Max}(A)$.

Let $I := x_1^{-1}D \cap \cdots \cap x_n^{-1}D \cap D = (D :_D x_1) \cap \cdots \cap (D :_D x_n)$. If $I \subseteq Q$, then $(D :_D x) \subseteq Q$ for some $x := x$; then, since $D_Q$ is flat over $D$,

$$(D_Q :_{D_Q} x) = (D :_D x)D_Q \subseteq QD_Q,$$

and in particular $x \notin D_Q$. However, $V \subseteq D_Q$, and thus $x \notin V$, a contradiction. Hence, we must have $I \notin Q$.

In this case, there must be a prime ideal $P_1 \in \Delta$ not containing $I$. Moreover, $I \cap P_1 \notin Q$, too, and thus there is another prime $P_2 \in \Delta.$
$P_1 \not= P_2$, not containing $I$. By Lemma 4.2, the prime ideals of $A$ inside $M$ are linearly ordered; in particular, we can suppose without loss of generality that $\text{rad}(P_2 A) \subseteq \text{rad}(P_1 A)$.

Let now $t \in P_2 \setminus P_1$; then, $t \in \text{rad}(P_1 A)$, and thus there are $p_1, \ldots, p_k \in P_1$, $a_1, \ldots, a_n \in A$ such that $t^e = p_1 a_1 + \cdots + p_k a_k$ for some positive integer $e$. For each $i$, $a_i = B_i(x_1, \ldots, x_n)$, where $B_i$ is a polynomial over $D$ of total degree $d_i$; let $d := \sup\{d_1, \ldots, d_k\}$, and take an $r \in I \setminus P_1$ (recall that $I \not\subseteq P_1$). Then, $r^d B_i(x_1, \ldots, x_n) \in D$ for each $i$; therefore,

$$r^d t^e = p_1 r^d a_1 + \cdots + p_k r^d a_k \in p_1 D + \cdots + p_k D \subseteq P_1.$$

However, by construction, both $r$ and $t$ are out of $P_1$; since $P_1$ is prime, this is impossible. Hence, $\text{Zar}(D) \setminus \{V\}$ is not compact, and $\text{Zar}(D)$ is not Noetherian.

The first corollaries of this result can be obtained simply by putting $Q = (0)$. Recall that a $G$-domain (or Goldman domain) is an integral domain such that the intersection of all nonzero prime ideals is nonzero. They were introduced by Kaplansky for giving a new proof of Hilbert’s Nullstellensatz (see for example [22, Section 1.3]).

**Corollary 5.2.** Let $D$ be a local domain of finite dimension, and suppose that $D$ is not a $G$-domain. Then, $\text{Zar}(D) \setminus \{V\}$ is not compact for every $V \in \text{Zar}_{\text{min}}(D)$.

**Proof.** Since $D$ is finite-dimensional, every prime ideal of $D$ contains a prime ideal of height 1; since $D$ is not a $G$-domain, it follows that the intersection of the set $\text{Spec}^1(D)$ of the height-1 prime ideals of $D$ is $(0)$. The localization $D_{(0)}$ is the quotient field of $D$, and thus a valuation domain; therefore, we can apply Theorem 5.1 to $\Delta := \text{Spec}^1(D)$. □

**Corollary 5.3.** Let $D$ be a local domain. If $D$ has infinitely many height-1 primes, then $\text{Zar}(D)$ is not Noetherian.

**Proof.** Let $I$ be the intersection of all height-1 prime ideals. If $I \not= (0)$, every height-one prime of $D$ would be minimal over $I$; since there is an infinite number of them, $\text{Spec}(D)$ would not be Noetherian, and by Proposition 4.1 neither $\text{Zar}(D)$ would be Noetherian. Hence, $I = (0)$. But then we can apply Theorem 5.1 (for $Q = I$). □

Note that the hypothesis that $D$ is local is needed in Theorem 5.1 and in Corollary 5.3; for example, $\mathbb{Z}$ has infinitely many height-1 primes, and $\bigcap\{P \mid P \in \text{Spec}^1(D)\} = (0)$, but $\text{Zar}(\mathbb{Z}) \simeq \text{Spec}(\mathbb{Z})$ is a Noetherian space.

**Proposition 5.4.** Let $D$ be an integral domain. If $D$ is not a field, then $\text{Zar}(D[X])$ is a not a Noetherian space.

**Proof.** Since $D$ is not a field, there exist a nonzero prime ideal $P$ of $D$. For any $a \in P$, let $p_a$ be the ideal of $D[X]$ generated by $X - a$;
then, each $p_a$ is a prime ideal of height 1, $p_a \neq p_b$ if $a \neq b$, and $\bigcap\{p_a \mid a \in P\} = \emptyset$.

The prime ideal $m := PD[X] + XD[X]$ contains every $p_a$; by Corollary\(5.3\) Zar($D[X]_m$) is not Noetherian. Therefore, neither Zar($D[X]$) is Noetherian.

**Corollary 5.5.** Let $F \subseteq L$ be a transcendental field extension.

(a) If $\text{trdeg}_F(L) = 1$ and $L$ is finitely generated over $F$ then Zar($L|F$) is Noetherian.

(b) If $\text{trdeg}_F(L) > 1$ then Zar($L|F$) is not Noetherian.

**Proof.** (a) Let $L = F(\alpha_1, \ldots, \alpha_n)$; without loss of generality we can suppose that $\alpha_1$ is transcendental over $F$. Then, the extension $F(\alpha_1) \subseteq L$ is algebraic and finitely generated, and thus finite.

Each $V \in$ Zar($L|F$) must contain either $\alpha_1$ or $\alpha_1^{-1}$; therefore, Zar($L|F$) = Zar($L[F(\alpha_1)] \cup$ Zar($L[F(\alpha_1^{-1})]$)). However, Zar($L|A$) = Zar($A'$) for every domain $A$, where we denote by $A'$ the integral closure of $A$ in $L$; since $F[\alpha_1]$ (respectively, $F[\alpha_1^{-1}]$) is a principal ideal domain and $F(\alpha_1) \subseteq L$ is finite, the integral closure of $F[\alpha_1]$ (resp., $F[\alpha_1^{-1}]$) is a Dedekind domain, and thus Zar($L[F(\alpha_1)]$) = Zar($F(\alpha_1')$) $\simeq$ Spec($F[\alpha_1']$) is Noetherian. Being the union of two Noetherian spaces, Zar($L|F$) is itself Noetherian. 

(b) Suppose $\text{trdeg}_F(L) > 1$. Then, there are $X, Y \in L$ such that \{X, Y\} is an algebraically independent set over $F$; in particular, we have a continuous surjective map Zar($L|F$) $\longrightarrow$ Zar($F(X, Y)|F$) given by $V \mapsto V \cap F(X, Y)$. However, Zar($F(X, Y)|F$) contains Zar($F[F(X, Y)]$); by Proposition\(5.4\) the latter is not Noetherian, since $F[X, Y]$ is the polynomial ring over $F[X]$, a domain of dimension 1. Thus, Zar($L|F$) is not Noetherian.

The condition that $\bigcap\{P \mid P \in \Delta\} = Q$ of Theorem\(5.1\) can be slightly generalized, requiring only that the intersection is contained in $Q$. However, doing so we can only prove that Zar($D$) is not Noetherian, without always finding a specific $V$ such that Zar($D \setminus \{V\}$ is not compact.

**Proposition 5.6.** Let $D$ be a local integral domain, and suppose there is a set $\Delta \subseteq$ Spec($D$) and a prime ideal $Q$ such that:

1. $Q \notin \Delta$;
2. no two members of $\Delta$ are comparable;
3. $\bigcap\{P \mid P \in \Delta\} \subseteq Q$;
4. $D_Q$ is a valuation domain.

Then, Zar($D$) is not Noetherian.

**Proof.** If Spec($D$) is not Noetherian, by Proposition\(4.1\) neither is Zar($D$); suppose that Spec($D$) is Noetherian.

Let $I := \bigcap\{P \mid P \in \Delta\}$; since an overring of a valuation domain is still a valuation domain, we can suppose that $Q$ is a minimal prime
of $I$. Since $D$ has Noetherian spectrum, the radical ideal $I$ has only a
finite number of minimal primes, say $Q := Q_1, Q_2, \ldots, Q_n$; let $\Delta := \{p \in \Delta \mid Q_i \subseteq p\}$ and $I_i := \bigcap\{p \mid p \in \Delta_i\}$. By standard properties of
minimal primes, $\Delta = \Delta_1 \cup \cdots \cup \Delta_n$ and $I = I_1 \cap \cdots \cap I_n$.

In particular, $I_1 \cap \cdots \cap I_n \subseteq Q$; hence, $Q_k \subseteq Q$ for some $k$. However,
$Q_k \subseteq I_k$, and thus $Q_k \subseteq Q$; since different minimal primes of the same
ideal are not comparable, $k = 1$ and $Q \subseteq I_1 \subseteq Q$, i.e., $I_1 = Q$. Then,
$\Delta_1$ is a family of primes satisfying the hypothesis of Theorem 5.1; in
particular, Zar$(D)$ is not Noetherian.

An essential prime of a domain $D$ is a $P \in \text{Spec}(D)$ such that
$D_P$ is a valuation domain. $D$ is an essential domain if it is equal
to the intersection of the localizations of $D$ at the essential primes.
If, moreover, the family of the essential primes is compact, then $D$
can be called a Prüfer $v$-multiplication domain ($\text{Pr}v\text{MD}$ for short) [12, Corollary 2.7]; note that the original definition of $\text{Pr}v\text{MD}s$ was given
through star operations (more precisely, $D$ is a $\text{Pr}v\text{MD}$ if and only if
$D_P$ is a valuation ring for every $t$-maximal ideal $P$ [16, 21]).

**Proposition 5.7.** Let $D$ be an essential domain. Then, Zar$(D)$ is
Noetherian if and only if $D$ is a Prüfer domain with Noetherian spec-
trum.

**Proof.** If $D$ is a Prüfer domain with Noetherian spectrum, then Zar$(D) \simeq$
Spec$(D)$ is Noetherian (see Example 4.6). Conversely, suppose Zar$(D)$
is Noetherian: by Proposition 4.1, Spec$(D)$ is Noetherian. Let $E$ be
the set of essential prime ideals of $D$: since Spec$(D)$ is Noetherian, $E$
is compact, and thus $D$ is a $\text{Pr}v\text{MD}$.

Suppose by contradiction that $D$ is not a Prüfer domain. Then, there
is a maximal ideal $M$ of $D$ such that $D_M$ is not a valuation domain;
since the localization of a $\text{Pr}v\text{MD}$ is a $\text{Pr}v\text{MD}$ [21, Theorem 3.11], and
Zar$(D_M)$ is a subspace of Zar$(D)$, without loss of generality we can
suppose $D = D_M$, i.e., we can suppose that $D$ is local.

Since $E$ is compact, every $P \in E$ is contained in a maximal element
of $E$; let $\Delta$ be the set of such maximal elements. Clearly, $D = \bigcap\{D_P \mid P \in \Delta\}$. If $\Delta$ were finite, $D$ would be an intersection of finitely many
valuation domains, and thus it would be a Prüfer domain [15, Theorem
22.8]; hence, we can suppose that $\Delta$ is infinite. Let $I := \bigcap\{P \mid P \in \Delta\}$.

Each $P \in \Delta$ contains a minimal prime of $I$; however, since Spec$(D)$
is Noetherian, $I$ has only finitely many minimal primes. It follows that
there is a minimal prime $Q$ of $I$ that is not contained in $\Delta$; in particular,
$\bigcap\{P \mid P \in \Delta\} \subseteq Q$, and thus we can apply Proposition 5.6. Hence,
Zar$(D)$ is not Noetherian, which is a contradiction. □

**Remark 5.8.** The previous proof can be interpreted using the termin-
ology of the theory of star operations. Indeed, any essential prime $P$
is a $t$-ideal, i.e., $P = P^t$, where (for any ideal $J$ of $D$) $J^t := \bigcup\{(D :
Corollary 5.9. Let $D$ be a Krull domain. Then, $\text{Zar}(D)$ is Noetherian if and only if $\dim(D) = 1$, i.e., if and only if $D$ is a Dedekind domain.

Proof. If $\dim(D) = 1$ then $D$ is Noetherian and so is $\text{Zar}(D)$. If $\dim(D) > 1$, then $D$ is not a Prüfer domain; since each Krull domain is a PvMD, we can apply Proposition 5.7. □

Note that this corollary can also be proved directly from Corollary 5.3 since, if $D$ is Krull, and $P \in \text{Spec}(D)$ has height 2 or more, then $D_P$ has infinitely many height-1 primes.

6. An application: Kronecker function rings

Let $D$ be an integrally closed integral domain with quotient field $K$. For every $V \in \text{Zar}(D)$, let $V(X) := V[X]_{m_V[X]} \subseteq K(X)$, where $m_V$ is the maximal ideal of $V$. If $\Delta \subseteq \text{Zar}(D)$, the Kronecker function ring of $D$ with respect to $\Delta$ is

$$\text{Kr}(D, \Delta) := \bigcap \{V(X) \mid V \in \Delta\};$$

equivalently,

$$\text{Kr}(D, \Delta) = \{f/g \mid f, g \in D[X], g \neq 0, c(f) \subseteq (c(g))^{\wedge_\Delta}\},$$

where $c(f)$ is the content of $f$ and $\wedge_\Delta$ is the semistar operation defined in Section 2.4. See [15, Chapter 32] or [13] for general properties of Kronecker function rings.

The set of Kronecker function rings is exactly the set of overrings of the basic Kronecker function ring $\text{Kr}(D, \text{Zar}(D))$; this set is in bijective correspondence with the set of finite-type valuative semistar operations [15, Remark 32.9], or equivalently with the set of nonempty subsets of $\text{Zar}(D)$ that are closed in the inverse topology [8, Theorem 4.9].

Let $\mathcal{K}(D)$ be the set of Kronecker function rings $T$ of $D$ such that $T \cap K = D$. Then, $\mathcal{K}(D)$ is in bijective correspondence with the set of finite-type valuative star operations, or equivalently with the set of inverse-closed representation of $D$ through valuation rings, i.e., the sets $\Delta \subseteq \text{Zar}(D)$ that are closed in the inverse topology and such that $\bigcap \{V \mid V \in \Delta\} = D$ [27, Proposition 5.10].

It has been conjectured [23] that $\mathcal{K}(D)$ is either a singleton (in which case $D$ is said to be a vacant domain; see [6]) or infinite, and this has been proved to be the case for a wide class of pseudo-valuation domains [6, Theorem 4.10]. As a consequence of the following proposition, we will prove this conjecture for another class of domains.
Proposition 6.1. Let $D$ be an integrally closed integral domain such that $1 < |K(D)| < \infty$. Then, there is a minimal valuation overring $V$ of $D$ such that Zar$(D) \setminus \{V\}$ is compact.

Proof. Suppose $|K(D)| > 1$. Then, there is an inverse-closed representation $\Delta$ of $D$ different from Zar$(D)$; let $\Lambda := \text{Zar}(D) \setminus \Delta$. For each $W \in \Lambda$, let $\Delta(W) := \Delta \cup \{W\}$; then, every $\Delta(W)$ is an inverse-closed representation of $D$, and $\Delta(W) \neq \Delta(W')$ if $W \neq W'$ (since, without loss of generality, $W \not\subseteq W'$, and thus $W \notin \Delta(W')$). Hence, each $W \in \Lambda$ give rise to a different member of $K(D)$; since $|K(D)| < \infty$, it follows that $\Lambda$ is finite.

If now $V$ is minimal in $\Lambda$, then $\text{Zar}(D) \setminus \{V\} = \Delta \cup (\Lambda \setminus \{V\})$ is closed by generizations; since $\Lambda$ is finite, it follows that $\text{Zar}(D) \setminus \{V\}$ is the union of two compact subspaces, and thus it is itself compact. □

Corollary 6.2. Let $D$ be an integrally closed local integral domain, and suppose there exist a set $\Delta \subseteq \text{Spec}(D)$ of incomparable nonzero prime ideals such that $\bigcap\{P \mid P \in \Delta\} = (0)$. Then, $|K(D)| \in \{1, \infty\}$.

Proof. By Theorem 5.1, each $\text{Zar}(D) \setminus \{V\}$ is noncompact. The claim now follows from Proposition 6.1. □

7. OVERRINGS OF NOETHERIAN DOMAINS

If $D$ is a Noetherian domain, Theorem 3.6 admits a direct application, without using any of the results proved in Sections 4 and 5. Indeed, if $D$ is Noetherian with quotient field $K$, then it is the same for any localization of $D[x_1, \ldots, x_n]$, for arbitrary $x_1, \ldots, x_n \in K$; thus, the integral closure of $D[x_1, \ldots, x_n]_M$ is a Krull domain for each maximal ideal $M$ of $D[x_1, \ldots, x_n]$ ([24] (33.10)) or [20] Theorem 4.10.5). Since a domain that is both Krull and a valuation ring must be a field or a discrete valuation ring, Theorem 3.6 implies that $\text{Zar}(D) \setminus \{V\}$ is not compact as soon as $V$ is a minimal valuation overring of dimension 2 or more.

We can actually say more than this; the following is a proof through Proposition 6.1 of an observation already appeared in [9, Example 3.7].

Proposition 7.1. Let $D$ be a Noetherian domain with quotient field $K$, and let $\Delta$ be the set of valuation overrings of $D$ that are Noetherian (i.e., $\Delta$ is the union of $\{K\}$ with the set of discrete valuation overrings of $D$). Then, $\Delta$ is compact if and only if $\dim(D) = 1$.

Proof. If $\dim(D) = 1$, then $\Delta = \text{Zar}(D)$, and thus it is compact.

On the other hand, for every ideal $I$ of $D$, $I^{\Delta} = I^b$ [20] Proposition 6.8.4]; however, if $\dim(D) > 1$, then $\text{Zar}(D)$ contains elements of dimension 2, and thus $\Delta$ cannot contain $\text{Zar}_{\text{min}}(D)$. The claim now follows from Proposition 6.5. □

Remark 7.2.
(1) The equality $I^\Delta = I^b$ holds also if we restrict $\Delta$ to be the set of discrete valuation overrings of $D$ whose center is a maximal ideal of $D$ [20, Proposition 6.8.4]. For each prime ideal of height 2 or more, by passing to $D_P$, we can thus prove that the set of discrete valuation overrings of $D$ with center $P$ is not compact (and in particular it is infinite).

(2) The previous proposition also allows a proof of the second part of Corollary 5.5 without using Theorem 5.1, since $F[X,Y]$ is a Noetherian domain of dimension 2.

By Proposition 7.1 in particular, the space $\Delta$ of Noetherian valuation overrings of $D$ (where $D$ is Noetherian and $\dim(D) \geq 2$) is not a spectral space, since it is not compact. Our next purpose is to see $\Delta$ as an intersection $X \cap \text{Zar}(D)$, for some subset $X$ of $\text{Over}(D)$, and use this representation to prove facts about $X$. We start with using the inverse topology.

**Proposition 7.3.** Let $D$ be a Noetherian domain with quotient field $K$, and let:
- $X_1$ be the set of all overrings of $D$ that are Noetherian and of dimension at most 1;
- $X_2$ be the set of all overrings of $D$ that are Dedekind domains ($K$ included).

For $i \in \{1, 2\}$, the following are equivalent:
- (i) $X_i$ is compact;
- (ii) $X_i$ is spectral;
- (iii) $X_i$ is proconstructible in $\text{Over}(D)$;
- (iv) $\dim(D) = 1$.

**Proof.**

(i) $\Rightarrow$ (iii) In both cases, $X = X^{\text{gen}}$: for $X_1$ see [22, Theorem 93], while for $X_2$ see e.g. [15, Theorem 40.1] (or use the previous result and [15, Corollary 36.3]). (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) always holds.

(iv) $\Rightarrow$ (i) If $\dim(D) = 1$, then $X_1 = \text{Over}(D)$, while $X_2 = \text{Over}(D')$, where $D'$ is the integral closure of $D$, and both are compact since they have a minimum.

(iii) $\Rightarrow$ (iv) If $X_i$ is proconstructible, so is $X_i \cap \text{Zar}(D)$ (since $\text{Zar}(D)$ is also proconstructible), and in particular $X_i \cap \text{Zar}(D)$ is compact. However, in both cases, $X_i \cap \text{Zar}(D)$ is exactly the set of Noetherian valuation overrings of $D$; by Proposition 7.1, $\dim(D) = 1$. □

**Remark 7.4.** The equivalence between the first three conditions of Proposition 7.3 holds for every subset $X \subseteq \text{Over}(D)$ such that $X = X^{\text{gen}}$ (and every domain $D$). In particular, it holds if $X$ is the set of overrings of $D$ that are principal ideal domains, and, with the same proof of the other cases, we can show that if $D$ is Noetherian and these conditions hold, then $\dim(D) = 1$. However, it is not clear if, when $D$ is Noetherian and $\dim(D) = 1$, this set is actually compact.
Another immediate consequence of Proposition 7.1 is that the set \( \text{NoethOver}(D) \) of Noetherian overrings of \( D \) is not proconstructible as soon as \( D \) is Noetherian and \( \dim(D) \geq 2 \): indeed, if it were, then \( \text{NoethOver}(D) \cap \text{Zar}(D) = \Delta \) would be proconstructible, against the fact that \( \Delta \) is not compact. However, this is also a consequence of a more general result. We need a topological lemma.

**Lemma 7.5.** Let \( Y \subseteq X \) be spectral spaces. Suppose that there is a subbasis \( B \) of \( X \) such that, for every \( B \in B \), both \( B \) and \( B \cap Y \) are compact. Then, \( Y \) is a proconstructible subset of \( X \).

**Proof.** The hypothesis on \( B \) implies that the inclusion map \( Y \hookrightarrow X \) is a spectral map; by [3, 1.9.5(vii)], it follows that \( Y \) is a proconstructible subset of \( X \).

**Proposition 7.6.** Let \( D \) be an integral domain with quotient field \( K \), and let \( D[\mathcal{F}_f] \) be the set of finitely generated \( D \)-algebras contained in \( K \).

(a) \( D[\mathcal{F}_f] \) is dense in \( \text{Over}(D) \), with respect to the inverse topology.

(b) Let \( X \) such that \( D[\mathcal{F}_f] \subseteq X \subseteq \text{Over}(D) \). Then, \( X \) is spectral in the Zariski topology if and only if \( X = \text{Over}(D) \).

**Proof.** (a) A basis of the constructible topology is given by the sets of type \( U \cap (X \setminus V) \), as \( U \) and \( V \) ranges in the open and compact subsets of \( \text{Over}(D) \). Such an \( U \) can be written as \( B_1 \cup \cdots \cup B_n \), where each \( B_i = B(x_1^{(i)}, \ldots, x_n^{(i)}) \) is a basic open set of \( \text{Over}(D) \); thus, we can suppose that \( U = B(x_1, \ldots, x_n) \). Suppose \( \Omega := U \cap (X \setminus V) \) is nonempty; we claim that \( A := D[x_1, \ldots, x_n] \in \Omega \cap D[\mathcal{F}_f] \). Clearly \( A \in D[\mathcal{F}_f] \) and \( A \subseteq U \); let \( T \in \Omega \). Then, \( T \in U \), and thus \( A \subseteq T \); therefore, \( A \) is in the closure \( \text{Cl}(T) \) of \( T \), with respect to the Zariski topology. But \( X \setminus V \) is closed, and thus \( \text{Cl}(T) \subseteq X \setminus V \); i.e., \( A \in X \setminus V \). Hence, \( A \in \Omega \cap D[\mathcal{F}_f] \), which in particular is nonempty, and \( D[\mathcal{F}_f] \) is dense.

(b) Suppose \( X \) is spectral. For every \( x_1, \ldots, x_n \), the set \( X \cap B(x_1, \ldots, x_n) \) has a minimum (i.e., \( D[x_1, \ldots, x_n] \)), so it is compact. Since the family of all \( B(x_1, \ldots, x_n) \) is a basis, by Lemma 7.5 it follows that \( X \) is proconstructible. By the previous point, we must have \( X = \text{Over}(D) \).

**Corollary 7.7.** Let \( D \) be a Noetherian domain. The spaces

- \( \text{NoethOver}(D) := \{ T \in \text{Over}(D) \mid T \text{ is Noetherian} \} \), and
- \( \text{KrullOver}(D) := \{ T \in \text{Over}(D) \mid T \text{ is a Krull domain} \} \)

are spectral if and only if \( \dim(D) = 1 \).

**Proof.** If \( \dim(D) = 1 \), then the claim follows by Proposition 7.6.

If \( \dim(D) \geq 2 \), then \( \text{NoethOver}(D) \) is not spectral by Proposition 7.4(b) and the Hilbert Basis Theorem; the case of \( \text{KrullOver}(D) \) follows in the same way, since \( \text{KrullOver}(D) \cap B(x_1, \ldots, x_n) \) always has a minimum (i.e., the integral closure of \( D[x_1, \ldots, x_n] \)).
More generally, consider a property \( P \) of Noetherian domains such that every field and every discrete valuation ring satisfies \( P \); for example, \( P \) may be the property of being regular, Gorenstein or Cohen-Macaulay. Let \( X_P(D) \) be the set of overrings of \( D \) satisfying \( P \); then, \( X_P(D) \cap \text{Zar}(D) \) is not compact, and thus \( X_P(D) \) is not proconstructible. On the other hand, if \( X_P(T) \) is compact for every overring of \( D \) that is finitely generated as a \( D \)-algebra, then by Lemma 7.5 it follows that \( X_P(D) \) cannot be a spectral space. Thus, the assignment \( D \mapsto X_P(D) \) cannot be “too good”: either some \( X_P(T) \) is not compact, or \( X_P(D) \) is not spectral.

**Question.** Let \( P \) be the property of being regular, the property of being Gorenstein or the property of being Cohen-Macaulay. Is it possible to characterize for which Noetherian domains \( D \) there is a \( T \in \text{Over}(D) \) such that \( X_P(T) \) is not compact and for which \( X_P(D) \) is not spectral?

8. **Acknowledgments**

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