Topological B-model and $\hat{c} = 1$ String Theory

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Abstract: We study the topological B-model on a deformed $\mathbb{Z}_2$ orbifolded conifold by investigating variation of complex structures via quantum Kodaira-Spencer theories. The fermionic/brane formulation together with systematic utilization of symmetries of the geometry gives rise to a free fermion realization of the amplitudes. We derive Ward identities which solve the perturbed free energy exactly. We also obtain the corresponding Kontsevich-like matrix model. All these confirm the recent conjecture on the connection of the theory with $\hat{c} = 1$ type 0A string theory compactified at the radius $R = \sqrt{\alpha'/2}$.

Keywords: Topological B-Model, Deformed Orbifolded Conifold, 0A String Theory.
1. Introduction

Topological string theories on Calabi-Yau(CY) threefolds were introduced [1] to probe the topological nature of the space. There are two different types, the A-model and the B-model, of topological string theories, which study the Kähler structure moduli space and the complex structure moduli space of the CY threefolds, respectively. These theories have proven to be very useful to study various aspects of critical and noncritical string theories. Among others, they were used to compute F-terms in superstring compactifications to four dimensions and were shown to be equivalent to $c = 1$ non-critical strings. Furthermore, it was shown [2, 3, 4] that the topological B-model on local CY threefolds is large $N$ dual to the Dijkgraaf-Vafa(DV) matrix...
model, from which the computation of nonperturbative superpotentials of $\mathcal{N} = 1$ gauge theories reduces to perturbative computations in a DV matrix model. For the recent review on the topological string theory, see [5, 6, 7].

Recently, an extremely powerful method to solve exactly the topological B-model on an interesting class of local CY threefolds was developed in [8]. It was shown that the model is governed by the $\mathcal{W}$-algebra symmetries of the CY threefolds, namely the holomorphic diffeomorphism. In particular, it was powerful enough to show the full equivalence between the topological B-model on deformed conifold and the $c = 1$ noncritical bosonic string theory at the self-dual radius. Both theories admit the same free fermion description and have the symmetries which characterize the theories completely with the emergence of dual Kontsevich-like matrix model.

In somewhat different context, there have been interesting developments [9, 10] in $\hat{c} = 1$ type 0 noncritical string theory. There are two different types, so-called 0A and 0B, of noncritical string theories, both of which have a matrix model description. Since all the matrix models are believed to be equivalent to the topological B-models on some CY threefolds, it is natural to expect that there is a topological B-model on some CY threefold which is equivalent to $\hat{c} = 1$ type 0 string theory. Indeed, it was suggested in an interesting paper [11] that $\hat{c} = 1$ type 0A string theory at the radius $R = \sqrt{\alpha'/2}$ is equivalent to the topological B-model on the (deformed) $\mathbb{Z}_2$ quotient of the conifold. In [11] the ground ring structure of $\hat{c} = 1$ string theory was identified with certain orbifolded conifold. The partition functions, integrable structures and the associated Ward identities of $\hat{c} = 1$ theory were considered and the corresponding Kontsevich-like matrix model was constructed. From the topological B-model on the $\mathbb{Z}_2$ orbifolded conifold, several results are obtained leading to the stated conjecture. However, because the $\mathbb{Z}_2$ orbifolded conifold has non-isolated singularities, topological strings are not well-defined and thus the computation in [11] is, at best, suggestive.

In this paper, we consider the topological B-model on the deformation of $\mathbb{Z}_2$ orbifolded conifold, which is well-defined since the deformed geometry is smooth. We adopt the method given in [8] to study the topological B-model and confirm that the model is indeed equivalent to the $\hat{c} = 1$ type 0A string theory compactified at the radius $R = \sqrt{\alpha'/2}$ with non-vanishing RR flux background. The key role is played by the $\mathcal{W}$-algebra symmetry which comes from the holomorphic diffeomorphism of the CY threefolds. The $\hat{c} = 1$ type 0 string theory also enjoys the same kind of $\mathcal{W}$-algebra symmetry. As this symmetry is powerful enough to constrain the whole theory, we can regard this, more or less, as a proof of their equivalence. Indeed we will show that the topological B-model on that space has the free fermion description just like the $\hat{c} = 1$ theory. We use the symmetry to derive the Ward identities from which we determine the perturbed partition function of the topological B-model under complex deformations. It matches exactly with the generating functional of $\hat{c} = 1$ matrix model under the perturbation by tachyon momentum modes. This
also guarantees that both theories admit the same Kontsevich-like matrix model description.

As will be clear, the RR flux in $\hat{c} = 1$ 0A string theory corresponds to a deformation parameter of the $Z_2$ orbifolded conifold which makes the space non-singular. Therefore the non-vanishing RR flux is crucial in order to have a well-defined topological B-model on the non-singular space. We can consider the limit where the corresponding deformation parameter vanishes. The resultant formulae we obtain in this paper are well-defined under the limit and correspond to the formulae in $\hat{c} = 1$ theory with vanishing RR flux.

The organization of this paper is as follows: In section 2, we explain the prescription to solve the topological B-model on some class of local CY threefolds. In particular, we take an example of the model on the deformed conifold which is equivalent to the $c = 1$ bosonic string theory compactified at the self-dual radius. Using this example we explain salient features of those models which show their equivalence most clearly. In section 3, we turn to $\hat{c} = 1$ 0A matrix model and describe the relevant features for our study on the topological B-model. In section 4, firstly we review the deformation and the resolution of the orbifolded conifold, which give rise to non-singular geometries on which string theory can be well-defined. And then we explain the basic set-up to solve the topological B-model on the deformation of $Z_2$ orbifolded conifold. Using this, in section 5, we study the integrable structure of the model and show that it is exactly the same as the one of the $\hat{c} = 1$ 0A string theory. We show that both theories have the same Ward identities and thus the same perturbed free energy and, as a result, correspond to the same Kontsevich-like matrix model. In section 6, we draw our conclusions and further comments.

### 2. Topological B-model on the Local Calabi-Yau threefold

In this section we review the topological B-model on non-compact CY geometries and explain general strategies to solve the model on some class of CY geometries following [8]. In particular we focus on the topological B-model on the deformed conifold which was shown to be equivalent to $c = 1$ bosonic string theory compactified at the self-dual radius. These models share a lot of characteristic features with those models we consider in this paper.

#### 2.1 Basic set-up

The topological B-model describes the quantum theory of the complex structure deformation of Calabi-Yau geometries, which corresponds to the quantum Kodaira-Spencer theory of gravity [12]. In the theory we consider maps from a Riemann surface $\Sigma_g$ of genus $g$ to the target CY manifold. For each Riemann surface $\Sigma_g$, we compute the corresponding free energy, $F_g(t_i)$, where $t_i$'s denote the complex structure deformation parameters. Those free energies are summed with weight
$g_s^{2g-2}$ to give the free energy of the B-model topological closed strings for all genus as

$$\mathcal{F}(g_s, t_i) = \sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g(t_i),$$  \hspace{1cm} (2.1)$$

where $g_s$ is the string coupling constant.

Recently, the B-model topological string theory on the local CY threefold of the form

$$uv - H(x, y) = 0$$  \hspace{1cm} (2.2)$$

was studied by considering $\mathcal{W}$-algebra symmetries underlying in this type of CY threefolds $[8]$. These symmetries correspond to the holomorphic diffeomorphisms of the target CY threefold which preserve the equation of CY threefold (2.2) and the holomorphic three-form

$$\Omega = \frac{1}{4\pi^2} \frac{dx \wedge dy \wedge du}{u}.$$  \hspace{1cm} (2.3)$$

The CY geometry (2.2) can be viewed as a fibration over the $(x, y)$ plane with one dimensional fibers. $H(x, y) = 0$ in the base manifold is the locus where the fiber degenerates into two components $u = 0$ and $v = 0$. By integrating along a contour around $u = 0$, the periods of the three-form $\Omega$ over three-cycles $C_i$ become integrals of the two-form

$$\int_{D_i} dx \wedge dy$$  \hspace{1cm} (2.4)$$

over domains $D_i$ in the $(x, y)$-plane whose boundary is one-cycles $c_i$ on the algebraic curve $H(x, y) = 0$. Hence by Stokes’ theorem, the periods reduce to integrals of the one-form

$$\int_{c_i} ydx.$$  \hspace{1cm} (2.5)$$

Therefore the CY geometry is characterized by the algebraic curve $H(x, y) = 0$ and thus the complex deformations of the CY geometry are captured by the canonical one-form, $ydx$.

If non-compact B-branes wrap the fiber, the worldvolume theory of the B-branes is given by a dimensional reduction of holomorphic Chern-Simons theory to one complex dimension $[13]$. It becomes the theory of the Higgs fields $x(u)$ and $y(u)$ whose zero modes are identified with coordinates of the moduli space of these non-compact B-branes, which is nothing but the base manifold. In the action, the Higgs fields $x(u)$ and $y(u)$ play roles as canonically conjugated fields and thus their zero modes have a canonical commutation relation

$$[x, y] = ig_s.$$  \hspace{1cm} (2.6)$$

$1$The normalization of the holomorphic three-form is chosen for later convenience.
All these imply that $x$ and $y$ are conjugate variables which form a symplectic pair with symplectic structure $dx \wedge dy$. After the reduction to the base manifold, $(x, y)$ plane, the holomorphic diffeomorphisms that preserve the equation (2.2) of the CY geometry and the corresponding holomorphic three form $\Omega$ descend to the diffeomorphisms of the plane that preserve the symplectic form $dx \wedge dy$, and therefore are given by general holomorphic canonical transformations on $(x, y)$.

The complex structure deformation of the curve $H(x, y) = 0$ can appear only at ‘infinity’ i.e. only at the boundary if the ‘compactified’ curve is a Riemann surface of genus zero which does not have any complex deformation moduli. Let us introduce a local coordinate $x$ near each boundary such that $x \to \infty$ at the boundary. Near each boundary the canonical one-form is given by $ydx$ which can be identified with the Kodaira-Spencer field $\phi(x)$ as

$$ydx = \partial \phi.$$

The background field $\phi_{cl}(x) = \langle \phi(x) \rangle$ can be determined from the relation $H(y_{cl}(x), x) = 0$ as $y_{cl}(x) = \partial x \phi_{cl}$. Arbitrary deformations of chiral bosonic scalar field $\phi(x)$ which correspond to complex deformations of the curve near the boundaries are given by the reidentification of $y(x)$ or, equivalently, by mode expansion of $\phi(x)$ around $x \to \infty$ of the form

$$y(x) = \partial_x \phi(x) = y_{cl}(x) + t_0 x^{-1} - g_s \sum_{n=1}^{\infty} n t_n x^{n-1} + g_s \sum_{n=1}^{\infty} \partial t_n x^{-n-1}.$$

The quantum free energy $F$ is a function of the infinite set of couplings $t^i$, where $i$ labels the boundaries. Then one can regard the quantum free energy $F$ as a state $|V\rangle$ in the Hilbert space $H^\otimes k$, where $H$ is the Hilbert space of a single free boson and $k$ is the number of boundaries [14]. One convenient representation of the state $|V\rangle$ is a coherent state representation. In terms of the standard mode expansion of a chiral boson

$$\partial_x \phi(x) = -i \sum_{n \in \mathbb{Z}} \alpha_n x^{-n-1}, \quad [\alpha_n, \alpha_m] = ng_s^2 \delta_{n+m,0},$$

the coherent state $|t\rangle$ is defined as

$$|t\rangle \equiv \exp\left[ \frac{i}{g_s} \sum_{n=1}^{\infty} t_n \alpha_{-n} \right] |0\rangle.$$

In this coherent state representation, the partition function can be expressed as

$$Z(t^i) = \exp F(t^i) = \langle t^1 | \otimes \cdots \otimes | t^k | V \rangle,$$

where $k$ is the number of boundaries.
One way to perturb the geometry is to insert B-branes near the boundaries. Non-compact B-branes which wrap the fiber and reside on the locus $H(x, y) = 0$ can be realized in the closed string sector by introducing (anti-)brane creation operator, $\psi(x)$ ($\psi^*(x)$):

$$
\psi(x) = \exp \left[ -\frac{i}{g_s} \phi(x) \right] : \quad \psi^*(x) = \exp \left[ \frac{i}{g_s} \phi(x) \right] :
$$

which is the fermionization of Kodaira-Spencer field $\phi$ [12].

Let us consider two patches with symplectic pairs of coordinates $(x_i, y_i)$ and $(x_j, y_j)$, respectively, and the corresponding fermions. Those two symplectic pairs are related by a canonical transformation preserving the symplectic form $dx_i \wedge dy_i = dx_j \wedge dy_j$ with a generating function $S(x_i, x_j)$. The corresponding fermions transform just like the wave function on the geometry, and hence the transformation of fermions between two patches, $i, j$ is given by [8]

$$
\psi_j(x_j) = \int dx_i e^{\frac{g_s}{i}S(x_i, x_j)} \psi_i(x_i).
$$

In the next subsections we consider an explicit example in which the quantum free energy of the topological B-model can be computed explicitly in this setup.

### 2.2 The $c = 1$ bosonic string

In this subsection we review some aspects of the matrix model description of $c = 1$ bosonic string theory which are relevant for our study. For the review on various aspects of $c = 1$ theory, see [16, 17, 18, 19, 20].

The $c = 1$ bosonic string theory compactified at the self-dual radius is equivalent [21, 22, 23] to the topological B-model on the deformed conifold which is given by the hypersurface (2.2) with

$$
H(x, y) = xy - \mu,
$$

where $\mu$ is the complex deformation parameter. The first indication of this equivalence came from the nonchiral ground ring of $c = 1$ bosonic string at the self-dual radius [21]. The operator product of the spin zero ghost number zero BRST invariant operators is again BRST invariant and gives a commutative and associatiave ring structure, modulo BRST exact terms, as

$$
O(z)O'(0) \sim O''(0) + \{Q, \ldots\}.
$$

This is called the ground ring. The right- and left-moving sectors of $c = 1$ bosonic string give chiral rings and thus, in combination, the corresponding closed string theory has nonchiral ground ring. At the self-dual radius, it is generated by four
operators $u, v, x, y$, which correspond to tachyon momentum and winding states, and they obey the relation
\[ uv - xy = \mu_M, \] (2.16)
where $\mu_M$ is the level of the Fermi sea of the $c = 1$ matrix model.

Noncritical $c = 1$ bosonic string theory has a free-fermion description with inverted harmonic oscillator potential. The Hamiltonian is given by
\[ H = \frac{1}{2}(p^2 - x^2), \] (2.17)
in the usual $\alpha' = 1$ convention of $c = 1$ bosonic string theory. It is convenient to introduce the light-cone variables $x_\pm = (x \pm p)/\sqrt{2}$, in terms of which the Hamiltonian becomes \[ \text{[24, 25, 26, 27]} \]
\[ H = \frac{1}{2}(x_+x_- + x_-x_+). \] (2.18)
In this formulation it is clear that $x_-$ and $x_+$ are conjugate with commutation relation, $[x_-, x_+] = i$ and thus the Schrödinger wave function can be represented by either $x_+$ or $x_-$. Let us denote the wave function in the $x_+$ and $x_-$ representations as $\psi_+$ and $\psi_-$, respectively. The energy eigenfunctions are given by
\[ \psi^E_\pm(x_\pm) = \frac{1}{\sqrt{2\pi}} e^{\pm iE/2}. \] (2.19)
The vacuum state of the system corresponds to the Fermi sea in which fermions are filled in the left-hand side of inverted harmonic oscillator. The correlation functions of tachyon operators in $c = 1$ string theory correspond to the perturbation of the Fermi sea in the matrix quantum mechanics.

In the $S$-matrix formulation of the $c = 1$ matrix model, right-moving modes, $\psi_-(x_-)$, and left-moving modes, $\psi_+(x_+)$, correspond to incoming and outgoing excitations, respectively. These in and out wave functions are related by $S$-matrix as
\[ \psi_+(x_+) = (S\psi_-)(x_+) = \int dx_- K(x_+, x_-)\psi_-(x_-), \] (2.20)
where the kernel, $K(x_-, x_+)$ can be taken as $e^{ix_+x_-}/\sqrt{2\pi}$. Note that one may choose different kernel such as $\sqrt{2\pi}\cos(x_+x_-)$ or $i\sqrt{2\pi}\sin(x_+x_-)$. All these kernels give the same results modulo non-perturbative terms. Since we are concerned about the free energy only at the perturbative level, we can choose any one of them as a kernel. It is convenient to use the exponential kernel for matching the corresponding expression from the B-model topological string side. Also note that, since fermions fill in the left-hand side of the inverted harmonic oscillator potential, it is reasonable to take the integration region along the negative real axis\footnote{When fermions fill the right-hand side of inverted harmonic oscillator, the positive real axis is taken as the integration region \[ \text{[24, 24].} \]}. Reflection coefficient can be
introduced as
\[ R(E)\psi(x_+) = (S\psi)(x_+) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} dx_- e^{ix_+x_-} \psi(x_-), \quad (2.21) \]
from which it can be computed as
\[ R(E) = -\frac{1}{\sqrt{2\pi}} e^{i\frac{\pi}{2}(iE+\frac{1}{2})} \Gamma(-iE+\frac{1}{2}). \quad (2.22) \]

One can euclideanize the time coordinate in the target space and consider the compactification of the theory with compactification radius \( R \). The above reflection coefficient can be used to obtain the ‘free energy’ of the model. The perturbative ‘free energy’ of grand canonical ensemble is defined by
\[ \mathcal{F}(\mu) = \int_{-\infty}^{\infty} dE \rho(E) \ln[1 + e^{2\pi R(E+\mu)}], \quad (2.23) \]
where \( \rho(E) \) is the density of states and \( \mu \) is a chemical potential. The relation between the density of states and the reflection coefficient is known to be \([25, 10, 20]\)
\[ \rho(E) = \frac{1}{2\pi} \left[ \frac{d\phi(E)}{dE} - \ln \Lambda \right], \quad (2.24) \]
where \( \phi(E) = Im \ln R(E) \) and \( \Lambda \) is a cut-off.

In order to get an expression free from cut-off dependence, it is convenient to consider the third order derivative of the free energy
\[ \frac{\partial^3 \mathcal{F}}{\partial \mu^3} = -2\pi R \int_{-\infty}^{\infty} dE \frac{d^2 \rho(E)}{dE^2} \frac{1}{e^{2\pi R(E+\mu)} + 1}. \quad (2.25) \]

In the \( c = 1 \) matrix model, we have
\[ \frac{d^3 \phi_{c=1}(E)}{dE^3} = \text{Im} \left[ (-i)^3 \psi^{(2)}(iE + \frac{1}{2}) \right], \quad (2.26) \]
where the polygamma function \( \psi^{(n)} \) is defined as
\[ \psi^{(n)}(z) \equiv \frac{d^{n+1}}{dz^{n+1}} \ln \Gamma(z) = (-1)^{n+1} \int_{0}^{\infty} dt \frac{t^n e^{-zt}}{1 - e^{-t}}. \quad (2.27) \]
The computation of \( \frac{\partial^3 \mathcal{F}_{c=1}(\mu_M)}{\partial \mu_M^3} \) is straightforward, given by the contour integral in the upper half-plane, and leads to
\[ \frac{\partial^3 \mathcal{F}_{c=1}(\mu_M)}{\partial \mu_M^3} = -2\pi R \int_{-\infty}^{\infty} dE \frac{d^2 \rho_{c=1}(E)}{dE^2} \frac{1}{e^{2\pi R(E+\mu_M)} + 1} \]
\[ = R \text{Im} \left[ \int_{0}^{\infty} dt \, e^{-i\mu_M t} \frac{t/2}{\sinh(t/2) \sinh(t/2R)} \right]. \quad (2.28) \]
The resultant perturbative free energy of $c = 1$ matrix model at the self-dual radius is given by

$$\mathcal{F}_{c=1}(\mu_M) = -\frac{1}{2} \mu_M^2 \ln \mu_M - \frac{1}{12} \ln \mu_M + \sum_{g \geq 2} \frac{|B_{2g}|}{2g(2g-2)} \mu_M^{2-2g},$$  \hspace{1cm} (2.29)

where

$$B_{2g} = (-1)^{g-1} |B_{2g}| = \frac{(-1)^{g-1}2(2n)!}{(2\pi)^{2n} \zeta(2n)}.$$

\hspace{1cm} (2.30)

### 2.3 The Topological B-model on the deformed conifold

As explained in sect. 2.1, the Kodaira-Spencer theory on the deformed conifold which corresponds to $c = 1$ strings at the self-dual radius is described by the chiral boson on the Riemann surface,

$$H(x, y) = xy - \mu = 0.$$  \hspace{1cm} (2.31)

The period integrals over the symplectic basis of the three-cycles in the case of CY geometry of the type (2.2) are given by

$$X_i = \oint_{A_i} \Omega = \frac{1}{2\pi i} \int_{a_i} y dx, \quad F_i = 2\pi i \int_{B_i} \Omega = \int_{b_i} y dx.$$  \hspace{1cm} (2.32)

These pairs of periods are related by the following relation

$$F_i = \frac{\partial F_0}{\partial X_i},$$

where $F_0$ denotes the tree level free energy or prepotential.

We need to introduce a cut-off $\Lambda$ for the integral over noncompact $B$-cycle ($b$-cycle). Then the period, for the curve (2.31), can be computed as

$$X = \frac{1}{2\pi i} \int_a y dx = \mu, \quad F = 2 \int_{\sqrt{\mu}} y dx = \mu \ln \left( \frac{\Lambda}{\mu} \right),$$

from which one can easily read off the tree level free energy as

$$F_0 = \frac{1}{2} XF = \frac{1}{2} \mu^2 \ln \mu + \mathcal{O}(\Lambda).$$

The full free energy of the topological B-model, all order in the string coupling, can be obtained via Dijkgraaf-Vafa (DV) conjecture on the equivalence between the topological B-model and the DV matrix model [2, 3, 4]. The DV matrix model dual to the topological B-model on the deformed conifold is known to be the Gaussian matrix model whose partition function is given by

$$Z = e^F = \frac{1}{\text{Vol}(U(N))} \int \mathcal{D}M \ e^{-\frac{1}{2g_s} \text{Tr} M^2}.$$  \hspace{1cm} (2.36)
In this correspondence, the 't Hooft coupling \( t = g_s N \) is identified with \( i\mu \) in the topological B-model. In the 't Hooft limit, the free energy can be explicitly computed and is given by
\[
F_DV(t) = \frac{1}{2} \left( \frac{t}{g_s} \right)^2 \left( \ln t - \frac{3}{2} \right) - \frac{1}{12} \ln t + \sum_{g \geq 2} \frac{B_{2g}}{2g(2g - 2)} \left( \frac{t}{g_s} \right)^{2-2g} .
\] (2.37)
Therefore we have
\[
F_{\text{top}}(\mu = g_s \mu_M) = F_DV(t = ig_s \mu_M) = F_{c=1}(\mu_M) ,
\] (2.38)
up to irrelevant regular terms.

The Riemann surface (2.31) has two boundaries and can be described by two patches whose coordinate is chosen to be \( x \) and \( y \), respectively. Those two asymptotic regions, which correspond to \( x \to \infty \) and \( y \to \infty \), may describe the incoming and outgoing states in the \( c = 1 \) matrix model, respectively. As explained in the previous subsection, we introduce Kodaira-Spencer fields in each patch as
\[
y = \partial_x \phi(x) , \quad x = -\partial_y \tilde{\phi}(y) .
\] (2.39)
Therefore, from the correspondence with \( c = 1 \) matrix model, \( \partial_x \phi(x) \) can be regarded as “incoming” modes and \( \partial_y \tilde{\phi}(y) \) as “outgoing” ones.

The classical part of \( \phi \) which describes the original background geometry is given by
\[
\partial_x \phi_{cl}(x) = \frac{\mu}{x} , \quad \partial_y \tilde{\phi}_{cl}(y) = -\frac{\mu}{y} .
\] (2.40)
The quantum parts of \( \phi \) describe the complex deformations of the geometry and have mode expansions
\[
\phi_{qu}(x) = -g_s \sum_{n=1}^{\infty} t_n x^n - g_s \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial t_n} x^{-n} ,
\]
\[
\tilde{\phi}_{qu}(y) = -g_s \sum_{n=1}^{\infty} \tilde{t}_n y^n - g_s \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial \tilde{t}_n} y^{-n} ,
\] (2.41)
where, in the classical limit, the couplings are given by periods
\[
g_s t_n = -\frac{1}{n} \oint_{x \to \infty} y x^{-n} , \quad g_s \frac{\partial F_0}{\partial t_n} = \oint_{x \to \infty} y x^n .
\] (2.42)

Non-compact B-brane creation operator in each patch can be introduced as (2.12). Suppose we put branes at positions \( y = y_i \) near the boundary \( y \to \infty \), then the gravitational backreaction is given by
\[
\prod_{i=1}^{N} \tilde{\psi}(y_i) = \prod_{i} : \exp \left[ -\frac{i}{g_s} \tilde{\phi}(y_i) \right] : = \Delta(y) : \exp \left[ -\frac{i}{g_s} \sum_{i=1}^{N} \tilde{\phi}(y_i) \right] : ,
\] (2.43)
where \( \Delta(y) = \prod_{i<j} (y_i - y_j) \) denotes the Vandermonde determinant. The expectation value of \( \partial \tilde{\phi}(y) \) in this perturbed background becomes

\[
\langle \prod_{i=1}^{N} \tilde{\psi}(y_i) \partial \tilde{\phi}(y) \rangle = -ig_s \sum_{i=1}^{N} \frac{1}{y_i - y} = -ig_s \sum_{i=1}^{N} \sum_{n=1}^{\infty} y_i^{-n} y^{n-1} .
\] (2.44)

This tells us that we can use B-branes to deform the curve with the coupling

\[
\bar{t}_n = \frac{i}{n} \sum_{i=1}^{N} y_i^{-n} .
\] (2.45)

Since \( x \) and \( y \) are conjugate each other as shown in (2.6), they play dual roles as a coordinate in one patch and a momentum in the other, and vice versa. Therefore the canonical transformation between two coordinate patches, which leads to the transformation (2.13), is nothing but the Fourier transform:

\[
\tilde{\psi}(y) = (S\psi)(y) = \frac{1}{\sqrt{2\pi}} \int dx \ e^{ixy/g_s} \psi(x) ,
\] (2.46)

which is reminiscent of \( c = 1 \) relation in (2.20).

One can compute the quantum free energy of the topological B-model on the deformed conifold, or the corresponding state \( |V\rangle \), using these B-branes and \( \mathcal{W} \) symmetry of the model. Ward identities associated with \( \mathcal{W} \) symmetry are enough to fix the quantum free energy. The curve \( xy = \mu \) has two punctures and, in this case, the \( \mathcal{W} \) symmetry generators relate an operation on one puncture to the one on the other. For example, consider an action of the \( \mathcal{W}_{1+\infty} \) generator given by the Hamiltonian

\[
f(x_i, p_j) = x_i^n ,
\] (2.47)

which can be written in the fermionic representation as

\[
\mathcal{W}_{n+1}^m = \oint \psi(x_i) x_i^n \psi^*(x_i) .
\] (2.48)

The corresponding Ward identity is given by [8]

\[
\oint_{x \to \infty} dx \ \psi(x) x^n \psi^*(x) |V\rangle = - \oint_{y \to \infty} dy \ \tilde{\psi}(y) (ig_s \partial_y)^n \tilde{\psi}^*(y) |V\rangle .
\] (2.49)

This is identical with the Ward identity [28] of the \( c = 1 \) string amplitude. This ensures that both theories are equivalent, sharing the same integrable structure.

After successive integrations by parts, the Ward identity can have the following alternative form:

\[
\oint_{x \to \infty} dx \ \psi^*(x) x^n \psi(x) |V\rangle = - \oint_{y \to \infty} dy \ \tilde{\psi}^*(y) (ig_s \partial_y)^n \tilde{\psi}(y) |V\rangle .
\] (2.50)

In the following sections, we will use this form of Ward identity to show the equivalence between the topological B-model on the deformed \( \mathbb{Z}_2 \) orbifolded conifold and the compactified \( \bar{c} = 1 \) 0A string theory.
3. The $\hat{c} = 1$ type 0A string theory

The $\hat{c} = 1$ string has $\mathcal{N} = 1$ superconformal symmetry on the worldsheet with one scalar superfield $X$, whose bosonic component, $x$, corresponds to the time coordinate in the target manifold, and one super Liouville field $\Phi$, which comes from the worldsheet supergraviton multiplet. Under the nonchiral GSO projection, it becomes either type 0A or 0B theory. The spectrum of 0A(0B) theory consists of massless tachyon field and R-R vector(scalar) fields. One can euclideanize the time coordinate, $x$, of the target manifold and consider the circle compactification with compactification radius $R$. After the compactification, these two theories, type 0A and type 0B theories are T-dual under $R \rightarrow \frac{\alpha'}{R}$.

In this paper we focus on the type 0A theory, especially at the radius $R = \sqrt{\frac{\alpha'}{2}}$, which must be equivalent to type 0B theory at the dual radius $R = \sqrt{2\alpha'}$. After the compactification, the NS-NS spectrum of type 0A theory consists of tachyon field momentum states with momentum $k = n/R$ and winding states with $k = wR/\alpha'$ where $n, w$ take integer values, while in the R-R sector, the theory has winding modes with $k = wR/\alpha'$, only.

The ground ring is generated by four elements, $u, v, x$ and $y$, among which $u$ and $v$ come from NS-NS momentum modes and $x$ and $y$ from R-R winding modes. It was argued in [10] that the ground ring of $\hat{c} = 1$ type 0A string theory at the radius $R = \sqrt{\frac{\alpha'}{2}}$ is given by

$$uv - (xy - \mu)^2 - \frac{q^2}{4} = uv - (xy - \mu + \frac{i}{2}q)(xy - \mu - \frac{i}{2}q) = 0,$$

where $\mu$ is the cosmological constant and $q$ is the net D0-brane charge in the background. This is the same form as the equation of the deformed $\mathbb{Z}_2$ orbifolded conifold which will be studied in detail using the topological B-model.

In this section we review some aspects of the matrix model description of the $\hat{c} = 1$ 0A string theory, which will be relevant in connection with the topological B-model we will consider.

3.1 Type 0A matrix quantum mechanics

The matrix model description of the $\hat{c} = 1$ 0A theory is given by the world volume theory of $N + q$ D0-branes and $N$ anti-D0-branes, which is $U(N + q) \times U(N)$ matrix quantum mechanics [10]:

$$L = \text{Tr} \left[(D_0 t)\dagger D_0 t + \frac{1}{2\alpha'} t\dagger t\right],$$

where $t$ denotes the tachyon field in the bifundamental representation under $U(N + q) \times U(N)$. This model can be described by a non-relativistic free-fermion in two di-
dimensions with upside-down harmonic oscillator potential. The single particle Hamiltonian is given by

$$H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{4\alpha'}(x^2 + y^2).$$  \hspace{1cm} (3.3)$$

The conserved charge of the angular momentum $J = xp_y - yp_x$ is identified with the net D0-brane charge $q$ \[10\]. Note that in each sector of the angular momentum $J = q$, the model becomes \[10, 29, 30\] effectively one-dimensional model which is known as deformed matrix quantum mechanics \[31\] with the Hamiltonian,

$$H' = -\frac{1}{2} \frac{d^2}{dr^2} - \frac{1}{4\alpha'}r^2 + \frac{q^2 - 1}{2r^2},$$  \hspace{1cm} (3.4)$$

where $r = \sqrt{x^2 + y^2}$.

It is again convenient to introduce the light cone variables \[32\]

$$z_\pm = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2\alpha'}}(x + iy) \pm (p_x + ip_y) \right],$$  \hspace{1cm} (3.5)$$

and their complex conjugates $\bar{z}_\pm$. The only nontrivial commutators are

$$[z_+, \bar{z}_-] = [\bar{z}_+, z_-] = -\frac{2i}{\sqrt{2\alpha'}},$$  \hspace{1cm} (3.6)$$

which tells that $(z_+, \bar{z}_+)$ and $(z_-, \bar{z}_-)$ form conjugate pairs. Therefore in terms of these new variables, the wave function can be expressed either in $(z_+, \bar{z}_+)$ representation or in $(z_-, \bar{z}_-)$ representation, denoted by $\psi_+(z_+, \bar{z}_+)$ and $\psi_-(z_-, \bar{z}_-)$, respectively. Furthermore the wave functions in $(z_+, \bar{z}_+)$ representation and those in $(z_-, \bar{z}_-)$ representation should be related by the Fourier transform:

$$\psi_+(z_+, \bar{z}_+) = \frac{1}{2\pi} \int_{-\infty}^{0} dz_-' d\bar{z}_-' e^{iz_+z_- + iz_+\bar{z}_-} \psi_-(z_-, \bar{z}_-),$$  \hspace{1cm} (3.7)$$

where the integration region is taken along the negative real axis, in the same fashion as the $c = 1$ case.

From now on we set $\alpha' = 2$, which is the standard convention for $\hat{c} = 1$ theory. In the $(z_+, \bar{z}_+)$ representation, the Hamiltonian and the angular momentum can be expressed as

$$H = \frac{i}{2} \left( z_+ \frac{\partial}{\partial z_+} + \bar{z}_+ \frac{\partial}{\partial \bar{z}_+} + 1 \right),$$  \hspace{1cm} (3.8)$$

and

$$J = z_+ \frac{\partial}{\partial z_+} - \bar{z}_+ \frac{\partial}{\partial \bar{z}_+}.$$  \hspace{1cm} (3.9)$$
Therefore the wave functions in the sector of angular momentum $q$ are of the form
\[ \psi_{\pm}(z_\pm, \bar{z}_\pm) = \left( \frac{z_\pm}{\bar{z}_\pm} \right)^{q/2} f(z_\pm \bar{z}_\pm). \]
(3.10)

The energy eigenstates with energy $E$ and angular momentum $q$ are given by
\[ \psi_{E,q}^{\pm}(z_\pm, \bar{z}_\pm) = \left( \frac{z_\pm}{\bar{z}_\pm} \right)^{q/2} (z_\pm \bar{z}_\pm)^{\pm iE - \frac{1}{2}}. \]
(3.11)

As will be clear, this is the most natural approach when we compare this theory with the topological B-model on the deformed $\mathbb{Z}_2$ orbifolded conifold. As was the case in the $c = 1$ matrix quantum mechanics, all the results from this formalism agree with the exact results, modulo nonperturbative terms. In the next subsection, we use this formalism to derive the reflection coefficient and the free energy of the theory and to describe the general perturbation of the system.

### 3.2 The reflection coefficient and the free energy

In the $\mathcal{S}$-matrix formulation of the $\hat{c} = 1$ matrix model, $\psi_-$ and $\psi_+$ may be regarded as incoming and outgoing excitations, respectively. These in and out wave functions are related by $\mathcal{S}$-matrix:
\[ \psi_+(z_+, \bar{z}_+)^{(S\psi_-)}(z_+, \bar{z}_+) = (S\psi_-(z_+, \bar{z}_+)), \]
(3.12)

which is nothing but the Fourier transform given in eq. (3.7).

The reflection coefficient $\mathcal{R}(E, q)$ can be introduced in the same way as the one in the $c = 1$ matrix model described earlier as
\[ (S\psi_{E,q}^{-})^{(z_+, \bar{z}_+)} = \mathcal{R}(E, q)\psi_{E,q}^{+}(z_+, \bar{z}_+) \].
(3.13)

Straightforward computation leads to
\[ \mathcal{R}(E, q) = \frac{1}{2\pi} e^{\pi(E+i/2)}\Gamma(-iE + \frac{q}{2} + \frac{1}{2})\Gamma(-iE - \frac{q}{2} + \frac{1}{2}). \]
(3.14)

Note that the exact expression of the reflection coefficient from the deformed matrix model is given by \[10, 33\]
\[ \mathcal{R}(E, q) = \frac{\Gamma(-iE + \frac{q}{2} + \frac{1}{2})}{\Gamma(iE + \frac{q}{2} + \frac{1}{2})}, \]
(3.15)

which differs from the expression, (3.14), in only nonperturbative corrections for $E$.

One can euclideanize the time coordinate in the target space and consider the compactification of the $\hat{c} = 1$ noncritical string theory along that direction. In the context of the $\hat{c} = 1$ matrix quantum mechanics, the Euclidean version of the above reflection coefficient can be obtained by replacing $E$ with $\mu + ip$. Furthermore, since
we are dealing with fermions which is anti-periodic in compactified Euclidean time, the momentum modes should be quantized as \( p = \frac{n+1/2}{R} \) under the compactification with the radius \( R \). As alluded earlier, the topological B-model on the deformed \( \mathbb{Z}_2 \) orbifolded conifold is equivalent to the \( \hat{c} = 1 \) 0A string theory compactified with the radius \( R = 1 \). Indeed, in the topological B-model we will obtain the expression of the form (3.14) after the replacement \( E \to \mu + i(n+1/2) \).

One can obtain the ‘perturbative’ free energy of the \( \hat{c} = 1 \) theory from the ‘perturbative’ reflection coefficient (3.14) in the similar fashion as in the \( c = 1 \) model outlined in the previous section. Now \( \phi_{0A}(E) = \text{Im} \ln \mathcal{R}(E, q) \) is given by

\[
\phi_{0A}(E) = \text{Im} \left[ \ln \frac{\Gamma(-iE + \frac{q}{2} + \frac{1}{2})}{\Gamma(-iE - \frac{q}{2} + \frac{1}{2})} \right],
\]

modulo irrelevant terms, or

\[
\frac{d^3 \phi_{0A}(E)}{dE^3} = \text{Im} \left[ i\psi^{(2)}(-iE + \frac{q}{2} + \frac{1}{2}) + i\psi^{(2)}(-iE - \frac{q}{2} + \frac{1}{2}) \right].
\]

Therefore we have

\[
\frac{\partial^3}{\partial \mu^3} \mathcal{F}_{0A}(\mu, q) = R \text{Im} \left[ \int_0^\infty dt \left\{ e^{-i(\mu+ia/2)t} + e^{-i(\mu-ia/2)t} \right\} \frac{t/2}{\sinh(t/2)} \frac{t/(2R)}{\sinh(t/2R)} \right] = \frac{\partial^3}{\partial \mu^3} \left[ \mathcal{F}_{c=1}(\mu + \frac{i}{2} q) + \mathcal{F}_{c=1}(\mu - \frac{i}{2} q) \right].
\]

One may note that the free energy obtained from the deformed matrix model is given by

\[
\frac{\partial^3}{\partial \mu^3} \mathcal{F}_{0A}(\mu, q) = 2R \text{Im} \left[ \int_0^\infty dt \ e^{-i(\mu-ia/2)t} \frac{t/2}{\sinh(t/2)} \frac{t/(2R)}{\sinh(t/2R)} \right],
\]

which is the same as the above expression (3.18) modulo nonperturbative terms.

We can consider the general perturbation by momentum modes of the tachyon field in the \( \hat{c} = 1 \) 0A string theory. It was suggested in [32] that it can be incorporated in the matrix model by considering the new eigenfunctions

\[
\Psi_{\pm}^{E,q} = e^{\mp i\varphi_{\pm}(z_{\pm}, \bar{z}_{\pm}; E, q)} \psi_{\pm}^{E,q},
\]

where the phases \( \varphi_{\pm} \) have Laurent expansion

\[
\varphi_{\pm}(z_{\pm}, \bar{z}_{\pm}; E, q) = \frac{1}{2} \phi(E, q) + R \sum_{k \geq 1} t_{\pm k}(z_{\pm}, \bar{z}_{\pm})^{k/R} - R \sum_{k \geq 1} \frac{1}{k} \nu_{\pm k}(z_{\pm}, \bar{z}_{\pm})^{-k/R}.
\]

The \( \hat{c} = 1 \) matrix model is the theory of two dimensional free fermions with fixed angular momentum \( q \), which corresponds to the net D0-brane charge. Since
the perturbation by tachyon momentum modes preserves the background net D0-brane charge, it should appear symmetrically in the Hamiltonian under \( z_\pm \leftrightarrow \bar{z}_\pm \). Therefore the perturbed Hamiltonian in the \( (z_+, \bar{z}_+) \) representation may be given by

\[
H_{tot} = H + z_+ \frac{\partial \phi_+}{\partial z_+} + \bar{z}_+ \frac{\partial \phi_+}{\partial \bar{z}_+},
\]

(3.22)

whose eigenfunctions become (3.20). Later we will show that similar structure appears in the deformation of complex moduli in the topological B-model on the deformed \( \mathbb{Z}_2 \) orbifolded conifold.

4. The Topological B model on the deformed orbifolded conifold

Now we are ready to study the topological B model on the deformation of \( \mathbb{Z}_2 \) orbifolded conifold. The CY space we consider is the hypersurface

\[
uv - (xy - \mu_1)(xy - \mu_2) = 0
\]

(4.1)

with the deformation parameters \( \mu_1 \) and \( \mu_2 \). In order to have a non-singular geometry, we should have \( \mu_1 \neq \mu_2 \). Eventually we would like to show that this model is equivalent to the \( \hat{c} = 1 \) type 0A string theory with the compactification radius \( R = \sqrt{\alpha'/2} \) in the background of net D0-brane charge \( q \). In this correspondence, the deformation parameters are related to the cosmological constant \( \mu \) and the net D0-brane charge \( q \) as

\[
\mu_1 = g_s (\mu + \frac{i}{2}q), \quad \mu_2 = g_s (\mu - \frac{i}{2}q).
\]

(4.2)

The Riemann surface

\[
H = (xy - \mu_1)(xy - \mu_2) = 0
\]

(4.3)

is given by the union of two sheets and each sheet corresponds to genus zero surface with two boundaries.

In this section we describe the general set-up to solve the model. In the next section we study the integrable structure of the model and show the equivalence of various models.

4.1 Orbifolded conifolds

In recent years, topological string theory on conifold has been extensively studied [34, 35, 36]. The conifold is three dimensional singularity in \( \mathbb{C}^4 \) defined by

\[
uv - xy = 0.
\]

(4.4)
The conifold can be realized as a holomorphic quotient of $\mathbb{C}^4$ by the $\mathbb{C}^*$ action given by [37, 38]

$$(A_1, A_2, B_1, B_2) \mapsto (\lambda A_1, \lambda A_2, \lambda^{-1} B_1, \lambda^{-1} B_2) \quad \text{for } \lambda \in \mathbb{C}^*.$$ (4.5)

Thus it is a toric variety with a charge vector $Q' = (1, 1, -1, -1)$ and the fan $\Delta = \sigma$ is given by a convex polyhedral cone in $\mathbb{N}_R' = \mathbb{R}^3$ generated by $v_1, v_2, v_3, v_4 \in \mathbb{N}' = \mathbb{Z}^3$ where

$v_1 = (1, 0, 0), \quad v_2 = (0, 1, 0), \quad v_3 = (0, 0, 1), \quad v_4 = (1, 1, -1).$ (4.6)

The isomorphism between the conifold $\mathcal{C}$ and the holomorphic quotient is given by

$$x = A_1 B_1, \quad y = A_2 B_2, \quad u = A_1 B_2, \quad v = A_2 B_1.$$ (4.7)

We take a further quotient of the conifold $\mathcal{C}$ by a discrete group $\mathbb{Z}_k \times \mathbb{Z}_l$. Here $\mathbb{Z}_k$ acts on $A_i, B_j$ by

$$(A_1, A_2, B_1, B_2) \mapsto (e^{-2\pi i/k} A_1, A_2, e^{2\pi i/k} B_1, B_2),$$ (4.8)

and $\mathbb{Z}_l$ acts by

$$(A_1, A_2, B_1, B_2) \mapsto (e^{-2\pi i/l} A_1, A_2, B_1, e^{2\pi i/l} B_2).$$ (4.9)

Thus they will act on the conifold $\mathcal{C}$ by

$$(x, y, u, v) \mapsto (x, y, e^{-2\pi i/k} u, e^{2\pi i/k} v)$$ (4.10)

and

$$(x, y, u, v) \mapsto (e^{-2\pi i/l} x, e^{2\pi i/l} y, u, v).$$ (4.11)

Its quotient is called the hyper-quotient of the conifold or the orbifolded conifold and denoted by $\mathcal{C}_{kl}$. To put the actions (4.5), (4.8) and (4.9) on an equal footing, consider the over-lattice $\mathbb{N}$:

$$\mathbb{N} = \mathbb{N}' + \frac{1}{k} (v_3 - v_1) + \frac{1}{l} (v_4 - v_1).$$ (4.12)

Now the lattice points $\sigma \cap \mathbb{N}$ of $\sigma$ in $\mathbb{N}$ is generated by $(k + 1)(l + 1)$ lattice points as a semigroup (These lattice points will be referred as a toric diagram.). The charge matrix $Q$ will be $(k + 1)(l + 1)$ by $(k + 1)(l + 1) - 3$. The discrete group $\mathbb{Z}_k \times \mathbb{Z}_l \simeq \mathbb{N}/\mathbb{N}'$ will act on the conifold $\mathcal{C}^4//U(1)$ and its quotient will be the symplectic reduction $\mathcal{C}^{(k + 1)(l + 1)}///U(1)^{(k + 1)(l + 1) - 3}$ with the moment map associated with the charge matrix $Q$. The new toric diagram for $\mathcal{C}_{kl}$ will also lie on the plane at a distance $1/\sqrt{3}$ from the origin with a normal vector $(1, 1, 1)$.
Figure 1: Toric Diagrams left: fully resolved geometry middle: partially resolved geometry right: $Z_2$ orbifolded conifold

The action $(\mathbb{I}10), (\mathbb{I}11)$ of $\mathbb{Z}_k \times \mathbb{Z}_l$ on the conifold $C$ can be lifted to an action on $\mathbb{C}^4$ whose coordinates are $x, y, u, v$. The ring of invariants will be $\mathbb{C}[x^l, y^l, xy, u^k, v^k, uv]$ and the orbifolded conifold $C_{kl}$ will be defined by the ideal $(xy-uv)\mathbb{C}[x^l, y^l, xy, u^k, v^k, uv]$. Thus after renaming variables, the defining equation for the orbifolded conifold will be

$$C_{kl} : xy = z^l, \quad uv = z^k.$$  \hspace{1cm} (4.13)

Hence the $Z_2 \cong Z_2 \times Z_1$ orbifolded conifold $C_{k1}$ can be written as

$$uv = (xy)^2.$$  \hspace{1cm} (4.14)

Its toric diagram is shown on the right of Figure 1. The general complex deformation space of this singularity is given by the Milnor ring

$$\frac{\mathbb{C}\{x, y, u, v\}}{(xy^2, x^2y, u, v)} \cong \frac{\mathbb{C}\{x, y\}}{(xy, x^2y)}.$$  \hspace{1cm} (4.15)

As we stated before, the ground ring of the $\hat{c} = 1$ type 0A string theory is a deformed $Z_2$ orbifolded conifold

$$uv - \left(xy - g_s(\mu + \frac{i}{2}q)\right)\left(xy - g_s(\mu - \frac{i}{2}q)\right) = 0.$$  \hspace{1cm} (4.16)

After change of variables, one may rewrite this equation as

$$u^2 + v^2 = (z - a)(z + a), \quad x^2 + y^2 = z, \quad a > 0$$  \hspace{1cm} (4.17)

and regard as a family over $z$-plane with generic fiber $C^* \times C^*$. By rescaling the variable $z$, we may assume that $a$ is very closer to 0. Then over the real segment $[0, a]$ the family is like

$$u^2 + v^2 \sim z - a, \quad x^2 + y^2 \sim z$$  \hspace{1cm} (4.18)

which implies that

$$u^2 + v^2 - x^2 - y^2 \sim -a.$$  \hspace{1cm} (4.19)
Then \((\text{Im} u, \text{Im} v, \text{Re} x, \text{Re} y)\) describes an \(S^3\) cycle. Similarly over \([-a, 0]\), the family is like

\[ u^2 + v^2 \sim -z - a, \quad x^2 + y^2 \sim z \tag{4.20} \]

which implies that

\[ u^2 + v^2 + x^2 + y^2 \sim -a. \tag{4.21} \]

The quadruple \((\text{Im} u, \text{Im} v, \text{Im} x, \text{Im} y)\) describes another \(S^3\) cycle. So there two \(S^3\) cycles.

As observed in [39], the closed string theory on this deformed conifold is dual to a open string theory on the resolved conifold via a geometric transition through a partially resolved conifold. In this situation, the partially resolved conifold is obtained by introducing \(\mathbb{P}^1\) which is shown on the middle of the Figure 1.

Now the partially resolved conifold has two conifold singularities and one can resolve these singularities by small resolution which replaces each singularity by a \(\mathbb{P}^1\) cycle. In the large \(N\) duality, the \(S^3\) cycles on the deformed conifold are shrunken and are replaced by the \(\mathbb{P}^1\) cycles on the resolved conifold whose toric diagram is shown on the left of Figure 1. So if one considers the large \(N\) duality of the open strings on the resolved conifold, the D-branes on \(\mathbb{P}^1\)'s disappear and the fluxes on \(S^3\) will be generated in the closed string picture.

### 4.2 Free energy

The free energy of the topological B-model on the CY geometry given by (4.1) can be obtained in the similar fashion as in the case of the theory on the deformed conifold described in section 2.3. The only difference is that now we have two pairs of periods which correspond to two sheets in \(H = 0\). Again we need to introduce a cut-off \(\Lambda\) for the computation of periods over noncompact \(B_i\)-cycle (\(b_i\)-cycle in \(H = 0\)). For the curve

\[ xy = \mu_i, \tag{4.22} \]

the periods can be computed as

\[ X^i = \oint_{A_i} \Omega = \frac{1}{2\pi i} \oint_{a_i} ydx = \mu_i, \]

\[ F_i = 2\pi i \int_{B_i} \Omega = 2 \int_{\sqrt{\Lambda}} ydx = \mu_i \ln \left( \frac{\Lambda}{\mu_i} \right), \tag{4.23} \]

where the holomorphic three-form \(\Omega\) is given by (2.3). Therefore the tree level free energy is given by the sum of two pieces as (see also [10])

\[ F_0 = \frac{1}{2} \sum_i X^i F_i = -\frac{1}{2} \sum_i \mu_i^2 \ln \mu_i + O(\Lambda). \tag{4.24} \]
All order free energy of the topological B-model on the deformed $\mathbb{Z}_2$ orbifolded conifold can be obtained by using the DV matrix model. As alluded earlier, the deformed $\mathbb{Z}_2$ orbifolded conifold has two $S^3$ cycles. RR-flux along two $S^3$ cycles can be introduced without changing the topological amplitudes [11]. The closed topological B-model on this deformed conifold is dual to the open topological B-model on the resolved conifold via a geometric transition through a partially resolved conifold. The resolved geometry has three $\mathbb{P}^1$, but the B-branes wrap along two disconnected $\mathbb{P}^1$ cycles only (see Figure 2). The open strings connecting separated B-branes are massive and thus decoupled in the low energy limit. Since the worldvolume theory of compact B-branes on each $\mathbb{P}^1$ essentially reduces to the DV matrix model as given by (2.36), the whole theory may be described by the decoupled $U(N_1) \times U(N_2)$ DV matrix model. The ’t Hooft coupling $t_i$ of each matrix model is identified with $i\mu_i/g_s$ in the topological B-model on the deformed orbifolded conifold. Therefore we have

$$\mathcal{F}_{\text{top}}(\mu_1, \mu_2) = \mathcal{F}_{\text{DV}}(t_1 = i\frac{\mu_1}{g_s}, t_2 = i\frac{\mu_2}{g_s}) = \mathcal{F}_{\text{DV}}(t_1 = i\frac{\mu_1}{g_s}) + \mathcal{F}_{\text{DV}}(t_2 = i\frac{\mu_2}{g_s}).$$

(4.25)

Thus, using eqs. (2.38) and (3.18) with the identification as in (4.2), the all order free energy of the topological B-model on the deformed orbifolded conifold is identical with the free energy of the 0A string theory at the compactification radius $R = 1$:

$$\mathcal{F}_{\text{top}}(\mu_1, \mu_2) = \mathcal{F}_{c=1}(\frac{\mu_1}{g_s}) + \mathcal{F}_{c=1}(\frac{\mu_2}{g_s}) = \mathcal{F}_{0A}(\mu = \frac{\mu_1 + \mu_2}{2g_s}, q = -i\frac{\mu_1 - \mu_2}{g_s}).$$

(4.26)

This gives a strong indication that those two theories are indeed equivalent.

4.3 The structure of the curve

As argued in section 2, the study of the topological B-model on the CY space of the type (2.2) boils down to the one of the complex deformations on the Riemann surface, $\mathcal{H} = 0$. Furthermore, if the surface is given by the genus zero surface with punctures, the complex deformations can appear only at the ‘punctures’. In our case
at hand, the geometry belongs to the CY space of the type (2,2) where the Riemann surface, $H = 0$, is given by the union of two curves

$$xy = \mu_1, \quad xy = \mu_2.$$  \hfill (4.27)

Each curve describes a sphere with two punctures and hence the Riemann surface can be regarded as the union of two spheres which are ‘connected’ at the punctures (see Figure 3). The region near each puncture, or boundary can be associated with asymptotic region described by $x \to \infty$ or $y \to \infty$ where the two curves ‘meet’.

This tells us that we need to consider the complex structure deformations on those two curves only. Furthermore the deformations can appear only at the boundaries where two curves are ‘connected’. Therefore the complex deformations at the boundaries influence both curves at the same time. Those deformations near the boundaries, $x \to \infty$ and $y \to \infty$, are generically described, respectively, by

$$\delta y = -g_s \sum_{n=1}^{\infty} n t_n x^{n-1} + g_s \sum_{n=1}^{\infty} \frac{\partial}{\partial t_n} x^{-n-1},$$

$$\delta x = -g_s \sum_{n=1}^{\infty} n \tilde{t}_n y^{n-1} + g_s \sum_{n=1}^{\infty} \frac{\partial}{\partial \tilde{t}_n} y^{-n-1}.$$  \hfill (4.28)

In order to describe these complex deformations (4.28) on the surface (4.29), it is convenient to introduce independent coordinates $x_i, y_i, i = 1, 2$ for each curve in (4.27) and denote each curve as

$$H_i(x_i, y_i) = x_i y_i - \mu_i = 0.$$  \hfill (4.29)

Figure 3: Two-sheet Riemann surface relevant for type 0A string theory
In this description, we study the complex deformations on the curves \( H_i = 0 \) which become those deformations \(^{14.28}\) on the curve \( H = 0 \) after the identifications of the coordinates, \( x_1 = x_2 = x, \ y_1 = y_2 = y \). Then the complex deformations described above are those which deform the curves \( H_i = 0 \), while \( H_1 - H_2 \) fixed.

Alternatively, one may begin with the higher dimensional geometry with

\[
uv - H_1(x_1, y_1)H_2(x_2, y_2) = 0, \tag{4.30}
\]

where \( H_i = x_i y_i - \mu_i, \ i = 1, 2 \). This is a local Calabi-Yau fivefold, which can be regarded as Calabi-Yau threefold for fixed \( x_1 \) and \( y_1 \) (or \( x_2 \) and \( y_2 \)). The geometry we consider corresponds to the subspace with the identification \( x_1 = x_2 \) and \( y_1 = y_2 \). Then one can study the complex deformations on the curves \( H_i = 0 \) which again give the complex deformations on the curve \( H = 0 \) after the identifications. Only the symmetric combinations of \( x_1 \) and \( x_2 \) (\( y_1 \) and \( y_2 \)) among the complex deformations of the curve 1 and 2 survive under the identifications. This is due to \( \mathbb{Z}_2 \) symmetry under \( x_1 \leftrightarrow x_2, \ y_1 \leftrightarrow y_2 \) and \( \mu_1 \leftrightarrow \mu_2 \), which is inherited from the original \( \mathbb{Z}_2 \) symmetry under \( \mu_1 \leftrightarrow \mu_2 \) of the CY space \(^{1.1}\). In this way, the relation between the \( \hat{c} = 1 \) 0A string theory and the topological B-model on the deformed orbifolded conifold becomes mostly clear.

Indeed, one may regard \( H_1 \) and \( H_2 \) as Hamiltonians for two sheets, and they correspond to the \( z_+ \) and \( \bar{z}_+ \) part of the Hamiltonian in the free fermionic description of the type 0A string theory. We can combine these Hamiltonians as

\[
\tilde{H} = \frac{1}{2}(H_1 + H_2) = x_1 y_1 + x_2 y_2 - \mu_1 - \mu_2, \tag{4.31}
\]

and

\[
J = (H_1 - H_2) = x_1 y_1 - x_2 y_2 - \mu_1 + \mu_2. \tag{4.32}
\]

The similarity between eqs \(^{3.8, 3.9}\) and eqs \(^{4.31, 4.32}\) is quite striking! Indeed, we obtain the complex deformations of the surface \(^{4.3}\) by restricting the deformations of \( H_i = 0 \) to the deformations of \( \tilde{H} = 0 \) while \( J = -\mu_1 + \mu_2 \) fixed.

### 4.4 Free fermion description

As in the deformed conifold case, we can parametrize the complex deformations as Laurent expansions of \( y_i(x) \) in the \( x_i \)-patch and \( x_i(y) \) in the \( y_i \)-patch by

\[
y_i = \partial_{x_i} \phi(x_1, x_2), \quad x_i = -\partial_{y_i} \tilde{\phi}(y_1, y_2). \tag{4.33}
\]

The classical part of \( \phi \) is given by

\[
\partial_{x_i} \phi_{cl}(x_1, x_2) = \frac{\mu_i}{x_i}, \quad \partial_{y_i} \tilde{\phi}_{cl}(y_1, y_2) = -\frac{\mu_i}{y_i}. \tag{4.34}
\]
As noted above, the relevant complex deformations are those which survive after the identification, or those which satisfy $J = -\mu_1 + \mu_2$. Therefore the appropriate complex deformations are

\[
\phi(x_1, x_2) = \mu_1 \ln x_1 + \mu_2 \ln x_2 - g_s \sum_{n=1}^{\infty} t_n (x_1 x_2)^n - g_s \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial t_n} (x_1 x_2)^{-n},
\]

(4.35)

\[
\tilde{\phi}(y_1, y_2) = -\mu_1 \ln y_1 - \mu_2 \ln y_2 - g_s \sum_{n=1}^{\infty} \tilde{t}_n (y_1 y_2)^n - g_s \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial \tilde{t}_n} (y_1 y_2)^{-n},
\]

where, in the classical limit, the couplings are simply given by periods

\[
g_s t_n = -\frac{1}{n} \oint_{x_1 \to \infty} \oint_{x_2 \to \infty} y_1 y_2 (x_1 x_2)^{-n},
\]

\[
g_s \frac{\partial F_0}{\partial t_n} = \oint_{x_1 \to \infty} \oint_{x_2 \to \infty} y_1 y_2 (x_1 x_2)^n.
\]

(4.36)

Each term in $t_n$ corresponds to the deformation $y \to y + nx^{2n-1}$ in the deformed $\mathbb{Z}_2$ orbifolded conifold.

Let us consider the deformation of the geometry by inserting non-compact B-branes wrapping the fiber i.e. $u, v$ directions. As we described earlier in the previous subsection, the deformation of the geometry can appear only at the asymptotic region and affect both curves in the same amount. Henceforth, we may consider the B-branes at the boundary affecting both curves simultaneously. In the higher dimensional picture, this corresponds to putting B-branes at the boundaries of both curves, $H_i = 0$, at the same time. From the brane worldvolume theory, one may regard $(x_i, y_i)$ as conjugate pairs, and therefore they play dual roles as a coordinate in one patch and momentum in the other, and vice versa.

In the closed string picture, these B-branes can be incorporated by the brane creation/annihilation operators with two complex variables:

\[
\psi(x_1, x_2) = : \exp \left[ -\frac{i}{g_s} \phi(x_1, x_2) \right] :, \quad \tilde{\psi}(y_1, y_2) = : \exp \left[ -\frac{i}{g_s} \tilde{\phi}(y_1, y_2) \right] :,
\]

(4.37)

which correspond to the deformations of the geometry with (4.35). Since $x_i$ and $y_i$ are conjugate each other, the transformation law between fermions in $(x_1, x_2)$ and $(y_1, y_2)$ patches is given by the Fourier transform:

\[
\tilde{\psi}(y_1, y_2) = \frac{1}{2\pi} \int dx_1 dx_2 \ e^{\frac{i}{g_s} (y_1 x_1 + y_2 x_2)} \psi(x_1, x_2).
\]

(4.38)

The fermions $\psi(x_1, x_2)$ in $x_i$ patch and $\tilde{\psi}(y_1, y_2)$ in $y_i$ patch may correspond to in and out states, respectively, in the $S$-matrix formulation of $\hat{c} = 1$ theory.
It is convenient to perform the following change of variables:\footnote{One may choose \( z' = x_2 \) and \( w' = y_2 \), instead. This gives the same results due to the \( \mathbb{Z}_2 \) symmetry of the model under \( \mu_1 \leftrightarrow \mu_2 \) (or \( q \leftrightarrow -q \)).}

\[
z = x_1 x_2, \quad z' = x_1, \quad w = y_1 y_2, \quad w' = y_1, \quad (4.39)
\]

and to introduce new fermionic functions \( \chi \) and \( \chi_{qu} \), which will be used repeatedly, as follows:

\[
\psi(x_1, x_2) = z'^i (\mu_2 - \mu_1)/g_s \chi(z) = z'^i (\mu_2 - \mu_1)/g_s z^{-i\mu_2/g_s} \chi_{qu}(z),
\]

\[
\tilde{\psi}(y_1, y_2) = w'^i (\mu_1 - \mu_2)/g_s \tilde{\chi}(w) = w'^i (\mu_1 - \mu_2)/g_s w^{i\mu_2/g_s} \tilde{\chi}_{qu}(w). \quad (4.40)
\]

The fermion field \( \chi_{qu} \) represents the quantum part of the brane creation operator. As it depends only on one variable \( z = x_1 x_2 \) (or \( w = y_1 y_2 \)), the treatment of this function is analogous to the one in the case of the deformed conifold, where the Riemann surface is given by single-sheet with two punctures (see section 2.3). As explained earlier, the deformation of the Riemann surface should affect the two sheets at the same time in the same way, thus it is natural that the brane/fermion creation operator \( \chi_{qu} \) depends only on the symmetric combination of \( x_1 \) and \( x_2 \), and thus \( z = x_1 x_2 \).

Therefore the brane/fermion creation operator \( \chi_{qu} \) can be mode-expanded as

\[
\chi_{qu}(z) = \sum_{n \in \mathbb{Z}} \chi_{n + \frac{1}{2}} z^{-n - 1}. \quad (4.41)
\]

Also for later convenience, we introduce \( \phi_\chi \) and \( \phi_{qu} \) which come from the bosonization of \( \chi \) and \( \chi_{qu} \), respectively, as follows:

\[
\chi(z) = : \exp \left[ -\frac{i}{g_s} \phi_\chi(z) \right]:, \quad \phi_\chi(z) = \mu_2 \ln z + \phi_{qu}(z), \quad (4.42)
\]

\[
\chi_{qu}(z) = : \exp \left[ -\frac{i}{g_s} \phi_{qu}(z) \right]:, \quad \phi_{qu}(z) = -g_s \sum_{n=1}^{\infty} t_n z^n - g_s \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial t_n} z^{-n}. \quad (4.43)
\]

The transformation law between fermions \( \chi \) in \( x_i \) patch and \( \tilde{\chi} \) in \( y_i \) patch can be read from (4.38):

\[
\tilde{\chi}(w) = \frac{1}{2\pi} \int dz \int \frac{ds}{s} s^{\frac{(\mu_2 - \mu_1)}{g_s}} e^{\frac{1}{gs}(s + zw/s)} \chi(z). \quad (4.43)
\]

5. Integrable structure of topological B-model

In this section we describe the integrable structure of the topological B-model on the deformation of \( \mathbb{Z}_2 \) orbifolded conifold using the fermionic description introduced in
the previous section. First of all, we describe the \( W \) algebra and the associated Ward identity which is identical with the one in the \( \hat{c} = 1 \) string theory compactified at the radius \( R = 1 \). Then we obtain the state \( |V\rangle \) which corresponds to the quantum free energy and reproduce the reflection coefficient of the corresponding \( \hat{c} = 1 \) string theory. Finally we obtain the Kontsevich-like matrix model which is related to the topological B-model on the deformed orbifolded conifold and the \( \hat{c} = 1 \) string theory.

5.1 Ward identities

We can solve this topological B-model using the symmetry underlying the geometry, in the similar fashion as in the case of the deformed conifold. To this end, let us consider the operator

\[
\oint P \oint P (x_1 x_2)^n \partial_{x_1} \partial_{x_2} \phi(x_1, x_2).
\]

It corresponds to the \( W \)-symmetry generator which gives the transformation

\[
(x, y) \rightarrow (x, y + n x^{2n-1})
\]

The associated Ward identity can be read as

\[
\oint_{x_1 \to \infty} \oint_{x_2 \to \infty} \psi^*(x_1, x_2)(x_1 x_2)^n \psi(x_1, x_2)|V\rangle = -\oint_{y_1 \to \infty} \oint_{y_2 \to \infty} \tilde{\psi}^*(y_1, y_2) \left( (-i g_s)^2 \partial_{y_1} \partial_{y_2} \right)^n \tilde{\psi}(y_1, y_2)|V\rangle.
\]

It is convenient to perform the change of variables as in (4.39), under which the differential operator becomes

\[
g_s^2 \partial_{y_1} \partial_{y_2} = g_s^2 \left[ w^2 \frac{\partial^2}{\partial w^2} + \frac{\partial}{\partial w} + w' \frac{\partial^2}{\partial w \partial w'} \right].
\]

After the trivial contour integration over \( z' \) and \( w' \), we obtain

\[
\oint_{z \to \infty} \chi^*(z) z^n \chi(z)|V\rangle = -\oint_{w \to \infty} \tilde{\chi}^*(w)(-g_s^2)^n \left[ w \partial_w^2 + \left( 1 + i \frac{\mu_1 - \mu_2}{g_s} \right) \partial_w \right]^n \tilde{\chi}(w)|V\rangle.
\]

This shows that, in the computation of the Ward identity, two coordinates \( x_i \), are effectively reduced to one coordinate \( z \), which corresponds to \( x^2 \) after the identification.

One may note that

\[
\tilde{\mathcal{W}}_n \equiv \oint_{w \to \infty} \tilde{\chi}^*(w)(-g_s^2)^n \left[ w \partial_w^2 + \left( 1 + i \frac{\mu_1 - \mu_2}{g_s} \right) \partial_w \right]^n \tilde{\chi}(w)
\]
commute among themselves and form Cartan subalgebra of $\mathcal{W}_{1+\infty}$ algebra \cite{12, 13, 14}. They are responsible for the integrability of the topological B-model on the deformed orbifolded conifold as for the integrability of the $\hat{c} = 1$ 0A string theory.

Since

$$(w\partial_w^2 + A\partial_w)^n = \sum_{k=0}^{n} \binom{n}{k} [A + n - 1]_k w^{n-k} \partial_w^{2n-k},$$

where

$$[x]_k \equiv x(x-1) \cdots (x-k+1), \quad [x]_0 = 1,$$

the Ward identity (5.3) can be expressed as

$$\oint_{z \to \infty} dz \chi^*(z) z^n \chi(z)|V\rangle = -(g_s^2)^n \sum_{k=0}^{n} \binom{n}{k} [A + n - 1]_k \oint_{w \to \infty} dw \, w^{n-k} \tilde{\chi}^*(w) \partial_w^{2n-k} \tilde{\chi}(w) |V\rangle,$$

where $A = 1 + i(\mu_1 - \mu_2)/g_2 = 1 - q$.

In the $S$-matrix formulation \cite{8} with coherent state basis, the perturbed partition function is given by

$$Z(\tilde{t}_n, t_n) \equiv \langle \tilde{t}|V\rangle = -\langle \tilde{t}|S|t\rangle, \quad |t\rangle \equiv e^{\frac{i}{g_s} \sum_{n=1}^{\infty} t_n \tilde{\alpha} - n}|0\rangle.$$  

This leads us to the alternative form of the Ward identity:

$$\frac{1}{i} \partial_{\tilde{t}_n} Z(\tilde{t}_n, t_n) = -(g_s^2)^n \sum_{k=0}^{n} \binom{n}{k} [n - q]_k \frac{1}{2n + 1 - k} \times$$

$$\times \oint dw \frac{2\pi i}{w^{n-k}} e^{\frac{i}{g_s} \tilde{\phi}_\chi(w)} \partial_w^{2n+1-k} \frac{1}{e^{-\frac{i}{g_s} \tilde{\phi}_\chi(w)}} : Z(\tilde{t}_n, t_n) :.$$  

This is exactly the same as the Ward identity which appears in the generating functional of the tachyon momentum mode perturbation in the type 0A theory at the compactification radius $R = 1$, given in \cite{11}.

5.2 The quantum free energy and the reflection coefficient

As explained earlier, the quantum free energy of the topological B-model can be represented by the state $|V\rangle$ in the Hilbert space $H^{\otimes k}$. If there is no deformation at the boundaries, the state $|V\rangle$ is given by the fermionic vacuum state. After the deformations by a set of $t^i_m$, the state $|V\rangle$ may be given by the Bogoliubov transformation of the fermionic vacuum by the quantum part of fermionic operators

$$|V\rangle = \exp \left[ \sum_{m,n \geq 0} a_{mn} \chi_{m-\frac{1}{2}} \chi^*_{n-\frac{1}{2}} + \sum_{m,n \geq 0} \tilde{a}_{mn} \tilde{\chi}_{m-\frac{1}{2}} \tilde{\chi}^*_{n-\frac{1}{2}} \right]|0\rangle.$$  

(5.10)
As it turns out, the coefficients of the Bogoliubov transformation correspond to the reflection coefficients of the $c = 1$ theory.

In order to determine $|V\rangle$, let us consider the two-point function $\langle 0|\tilde{\psi}(y_1, y_2)\psi^*(x_1, x_2)|V\rangle$. As usual, the transformation law between fermions of $(x_1, x_2)$ and $(y_1, y_2)$ patches is given by the Fourier transform, (4.33) and thus the two-point function can be written as

$$
\langle 0|\tilde{\psi}(y_1, y_2)\psi^*(x_1, x_2)|V\rangle = \frac{1}{2\pi} \int du_1 du_2 \, e^{\frac{i}{\pi} (y_1 u_1 + y_2 u_2)} \langle 0|\psi(u_1, u_2)\psi^*(x_1, x_2)|V\rangle.
$$

(5.11)

This two-point function can be computed in two different ways. By using $\chi$ introduced in (4.40), the left-hand side of (5.11) can be expressed as

$$
\langle 0|\tilde{\psi}(y_1, y_2)\psi^*(x_1, x_2)|V\rangle = (u'z')^{\frac{i}{\pi} (\mu_1 - \mu_2)} (wz)^{\frac{i}{\pi} \mu_2} \langle 0|\tilde{\chi}_{qu}(w)\chi_{qu}^*(z)|V\rangle
$$

(5.12)

$$
= (x_1 y_1)^{\frac{i}{\pi} \mu_1} (x_2 y_2)^{\frac{i}{\pi} \mu_2} \sum_{m, n \geq 0} a_{mn} (x_1 x_2)^{-m-1} (y_1 y_2)^{-n-1}.
$$

On the other hand, by using the standard operator product expansion of $\chi$, which is the function of one variable $z$, the two-point function in the right-hand side of (5.11) can be computed as

$$
\langle 0|\tilde{\psi}(y_1, y_2)\psi^*(x_1, x_2)|V\rangle = \left( \frac{z'}{u'} \right)^{\frac{i}{\pi} (\mu_1 - \mu_2)} \left( \frac{z}{u} \right)^{\frac{i}{\pi} \mu_2} \langle 0|\chi_{qu}(u)\chi_{qu}^*(z)|V\rangle
$$

(5.13)

$$
= \left( \frac{z'}{u'} \right)^{\frac{i}{\pi} (\mu_1 - \mu_2)} \left( \frac{z}{u} \right)^{\frac{i}{\pi} \mu_2} \frac{1}{u - z},
$$

where $u \equiv u_1 u_2$ and $u' \equiv u_1$. After performing an expansion over $\frac{u}{z}$, the right-hand side of (5.11) becomes

$$
-(x_1 y_1)^{\frac{i}{\pi} \mu_1} (x_2 y_2)^{\frac{i}{\pi} \mu_2} \sum_{n = 0}^{\infty} (x_1 x_2)^{-n-1} (y_1 y_2)^{-n-1} \times
$$

$$
\times \frac{1}{2\pi} \int d\eta' \, e^{\frac{i}{\pi} \eta' - \frac{i}{\pi} \mu_1 \eta'} \int d\eta \, e^{\frac{i}{\pi} \eta - \frac{i}{\pi} \mu_2 \eta}.
$$

(5.14)

By comparing both sides of (5.11) from (5.12) and (5.14), the coefficients of Bogoliubov transformation can be determined as $a_{mn} = -R_n \delta_{mn}$ and $\tilde{a}_{mn} = -R_n^* \delta_{mn}$ where

$$
R_n = \frac{1}{2\pi} \int d\eta' \, e^{\frac{i}{\pi} \eta' - \frac{i}{\pi} \mu_1 \eta'} \int d\eta \, e^{\frac{i}{\pi} \eta - \frac{i}{\pi} \mu_2 \eta}.
$$

(5.15)

By integrating along the negative real axis, just as we did in the $c = 1$ case, we obtain the expression for the coefficient $R_n$ as

$$
R_n = \frac{1}{2\pi} e^{\pi \mu} e^{i \pi (n+1)} \Gamma \left(-i \mu + n + 1 + \frac{q}{2}\right) \Gamma \left(-i \mu + n + 1 - \frac{q}{2}\right).
$$

(5.16)

It is identical with the Euclidean version of ‘the perturbative’ reflection coefficient of the type $0A$ theory at the radius $R = 1$, given in (3.14).
5.3 Kontsevich-like matrix model

In this section we derive the Kontsevich-like matrix model corresponding to the \( \hat{c} = 1 \) 0A string theory. Let us consider the deformation of the geometry through only at one boundary, say \( x \to \infty \). In this situation, only \( t_n \), the deformation parameters in one patch, are turned on, while those in the other patch, \( \tilde{t}_n \) remain zero. All the amplitudes remain trivial as far as \( \tilde{t}_n = 0 \), which can be easily understood from the fact that the corresponding amplitudes in the \( \hat{c} = 1 \) 0A string theory vanish due to momentum conservation.

Now we put \( N \) non-compact B-branes at positions \( y_1 = y_{1,l} \) and \( y_2 = y_{2,l} \), \( l = 1, \ldots, N \) near the boundary \( y_i \to \infty \). As explained earlier, this deforms the background geometry with deformation parameters given by

\[
\tilde{t}_n = \frac{i}{n} \text{Tr} A^{-n}, \quad A = \text{diag}(w_1, w_2, \ldots, w_N),
\]

where \( w_l = y_{1,l} y_{2,l} \) as introduced in (4.39).

The perturbed partition function in terms of new variables \( w_l \) and \( w'_l \) as in (4.39) can be written as

\[
Z(\tilde{t}_n, t_n) = \frac{1}{\Delta(w)} \langle N | \prod_{l=1}^{N} \tilde{\chi}_{qu}(w_l) | V \rangle = \frac{(\det A)^{-\frac{i}{g_s} \mu_2}}{\Delta(w)} \langle N | \prod_{l=1}^{N} \tilde{\chi}(w_l) | V \rangle,
\]

where \( |V\rangle \) is the state corresponding to the deformed geometry with \( t_n \) only and \( |N\rangle \) denotes the \( N \) fermion state. The normalization can be determined from the condition \( Z(\tilde{t}_n, t_n)_{t_n=0} = 1 \).

In order to determine this perturbed partition function, first of all, we perform the transformation from \( \tilde{\chi} \) to \( \chi \) using (4.43). Then by using (2.43), we obtain

\[
\langle N | \chi(z_1) \chi(z_2) \cdots \chi(z_N) | V \rangle = \Delta(z) e^{\frac{i}{g_s} \sum_{l=1}^{N} \left( -\mu_2 \ln z_l + g_s \sum_{n>0} t_n z_l^n \right)}.
\]

By collecting pieces together, the partition function can be expressed as

\[
Z(\tilde{t}_n, t_n) = \frac{(\det A)^{-\frac{i}{g_s} \mu_2}}{\Delta(w)} \int \prod_{l=1}^{N} \frac{dz_l}{2\pi} \Delta(z) e^{\frac{i}{g_s} \sum_{l} \left( -\mu_2 \ln z_l + g_s \sum_{n>0} t_n z_l^n \right)} \times
\]

\[
\times \int \prod_{l=1}^{N} ds_l s_l^{q-1} e^{\frac{i}{g_s} \sum_{l>0} \left( s_l + w_l z_l / s_l \right)}.
\]

In this form, it is straightforward to show that the perturbed partition function satisfies the Ward identity. Using the brane position variables (5.17) which has been known as the Miwa-Kontsevich transform [45, 46] and performing the residue
integrals analogous to the $c = 1$ matrix model case \cite{17}, the Ward identity \eqref{5.9} becomes

\[
\frac{1}{i} \frac{\partial Z(\tilde{t}_n, t_n)}{\partial t_n} = (-g_s^2)^n \sum_{k=0}^{n} \binom{n}{k} [n-q]_k \left\{ \sum_{l=1}^{N} \frac{w_l^{n-2i\mu_2/g_s}}{\prod_{m\neq l}(w_m-w_l)} \right\} \times \nonumber \\
\times w_l^{-n-k} \left( \frac{\partial}{\partial w_l} \right)^{2n-k} w_l^{i\mu_2/g_s} \prod_{m\neq l}(w_m-w_l) \right\} Z(\tilde{t}_n, t_n) \nonumber \\
= (-g_s^2)^n (\det A)^{-i\mu_2/g_s} \sum_{l=1}^{N} \frac{1}{\Delta(w)} \nonumber \\
\times \left[ w_l \frac{\partial^2}{\partial w_l^2} + (1-q) \frac{\partial}{\partial w_l} \right] Z(\tilde{t}_n, t_n). \quad (5.21) 
\]

Note that the perturbed partition function in \eqref{5.20} contains the function $F_q(x)$ defined as

\[
F_q(x) = \int ds \ s^{q-1} e^{\frac{i}{gs}(s+x/s)}, \quad (5.22)
\]

which satisfies the Bessel equation:

\[
g_s^2 [w \frac{\partial^2}{\partial w} + (1-q) \frac{\partial}{\partial w}] F_q(wz) = -z F_q(wz). \quad (5.23)
\]

This function can be written in terms of (modified) Bessel (or Hankel) functions as $F_q(x) = z^{\nu/2} Z_{-\nu}(2\sqrt{z}/g_s)$ up to constant. If the integration region is taken over the negative real axis, $Z_{-\nu}$ is given by the Hankel function $H_{-\nu}^{(2)}$. This shows clearly that the perturbed partition function \eqref{5.20} satisfies the Ward identity of the form \eqref{5.21}.

Through the change of variables $s_l \rightarrow w_l s_l$, the perturbed partition function \eqref{5.20} can be rewritten as

\[
Z(\tilde{t}_n, t_n) = \frac{(\det A)^{-i\mu_1}}{\Delta(w)} \prod_{l=1}^{N} ds_l \ e^{\frac{i}{gs} \sum_{l} \left\{ s_l w_l - (\mu_1 - \mu_2) \ln s_l \right\}} \nonumber \\
\times \int \prod_{l=1}^{N} dz_l \ \Delta(z) \ e^{\frac{i}{gs} \sum_{l} \left\{ (z_l/s_l) - \mu_2 \ln z_l + g_s \sum_{n>0} t_n z_l^n \right\}}. \quad (5.24)
\]

After using the Harish-Chandra-Itzykson-Zuber-Mehta integral \cite{18, 19, 20}

\[
\int dU \ e^{iTr(UXU^{\dagger})} = \text{const.} \cdot \frac{\det e^{ixiy_j}}{\Delta(x)\Delta(y)}, \quad (5.25)
\]

and

\[
\Delta\left(\frac{1}{s}\right) = \frac{\Delta(s)}{\prod_{l} s_l^{N-1}}, \quad (5.26)
\]

one can see that the perturbed partition function becomes the one of Kontsevich-like matrix model:

\[
Z(\tilde{t}_n, t_n) = (\det A)^{-i\mu_1} \int \mathcal{D}S \ e^{\frac{i}{gs} \left\{ Tr(AS) - (\mu_1 - \mu_2 - ig_s N) Tr \ln S \right\}} \nonumber \\
\times \int \mathcal{D}Z \ e^{\frac{i}{gs} \left\{ Tr(S^{-1}Z) - \mu_2 Tr \ln Z + g_s \sum_{n>0} t_n \ln Z^n \right\}}. \quad (5.27)
\]
Note that by inserting the B-branes at the boundary the deformation parameter $\mu_i$ is shifted to $\mu_i - i g_s N$ \[8\]. After this shift, it is the same Kontsevich-like matrix model as the one dual to the $\hat{c} = 1$ type 0A theory.

6. Conclusion

In this paper we studied the topological B-model on the deformation of $\mathbb{Z}_2$ orbifolded conifold and show that it is equivalent to the $\hat{c} = 1$ theory compactified at the radius $R = \sqrt{\alpha'/2}$ with nonzero background D0-brane charge. Via the DV matrix model, we obtained the B-model free energy and showed it is identical with the $\hat{c} = 1$ free energy. Most notably, we showed that the topological B-model on that geometry admits free fermion description, exactly the same way as the $\hat{c} = 1$ theory. Furthermore we derived the Ward identities of the model. This led us to obtain the perturbed free energy of the model and the corresponding Kontsevich-like matrix model. All these confirm the equivalence between those two models.

While in the $\hat{c} = 1$ theory side, $q = 0$ case is well-defined, the corresponding case in the topological B-model is not well-defined due to non-isolated singularities. Nevertheless one may notice that all the quantities we obtained in the topological B-model are smooth in the limit $\mu_1 \rightarrow \mu_2 (q \rightarrow 0)$. This suggests that the equivalence of those two theories in the $q = 0$ case should be understood as a limiting process. Namely, in the topological B-model side, we first regularize the singularity of the geometry by the infinitesimal deformation with $\delta = \mu_1 - \mu_2$ and then take the smooth limit $\delta \rightarrow 0$.

In the compactified string theory, we also need to consider the winding modes. Indeed the perturbation by winding modes only is also integrable \[31, 32\]. But it is not clear in the context of free fermion description how to incorporate the perturbation of the compactified $\hat{c} = 1$ theory by both momentum and winding modes. The same problem arises in the equivalence of the topological B-model on the deformed conifold and $c = 1$ bosonic strings at the self-dual radius.

Since the compactified $\hat{c} = 1$ 0A theory is T-dual to the $\hat{c} = 1$ 0B theory on the dual radius, our results suggest that the topological B-model on that geometry is equivalent to $\hat{c} = 1$ 0B string theory at the radius $R = \sqrt{2\alpha'}$. It would be interesting to prove this directly by finding different fermionic realization of the topological B-model.

In this paper we found the equivalence of two theories only at the perturbative level. This is natural as the topological string theory can be defined only perturbatively. One may consider the $\hat{c} = 1$ theory at the radius $R = \sqrt{\alpha'/2}$ as the non-perturbative completion of the topological B-model on the deformed $\mathbb{Z}_2$ orbifolded conifold. For the recent discussion on the nonperturbative approach to topological strings, see \[33, 54, 55, 56, 57, 58\].
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