CONTROLLABILITY OF HARMONIC MAP HEAT FLOW WITH AN EXTERNAL FIELD

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Abstract. We investigate the controllability of harmonic map heat flow by means of an external magnetic field. In contrast to the situation of a parabolic system with internal or boundary control, the magnetic field acts as the coefficients of the lower order terms of the equation. We show that for initial data whose image stays in a hemisphere, with one control acting on a subset of the domain plus a spatial-independent control acting on the whole domain, the state of the system can be steered to any ground state, i.e. any given unit vector, within any short time. To achieve this, in the first step a spatial independent control is applied to steer the solution into a small neighborhood of the peak of the hemisphere. Then under stereographic projection, the original system is reduced to an internal parabolic control system with initial data sufficiently close to 0 such that the existing method for local controllability can be applied. The key process in this step is to give an explicit solution of an underdetermined algebraic system such that the affine type control can be converted into an internal control.

1. Introduction

We investigate the controllability of the following system with homogenous Neumann boundary condition

\[
\begin{aligned}
\partial_t d - \Delta d &= |\nabla d|^2 d + (H \cdot d)H - (H \cdot d)^2 d, & \text{in } Q, \\
\partial_\nu d &= 0, & \text{on } \Sigma,
\end{aligned}
\]  

(1.1)

where \( \Omega \subset \mathbb{R}^3 \) is an open set with \( C^1 \) boundary, \( \nu \) denotes its outer-pointing normal, and \( Q = \Omega \times (0, T), \Sigma = \partial\Omega \times (0, T) \). It is motivated by the analysis and optimal control of a simplified Ericksen-Leslie system describing the dynamics of a liquid crystal (see [7], [10] and [1]) when the hydrodynamic effects are neglected. We recall that the classical mathematical description of the static configuration of liquid crystal material under a magnetic field is to consider the Oseen-Frank model [5]. In the simplest case, the energy functional of such model has the form (see [8])

\[
\mathcal{E}(d) = \frac{1}{2} \int_{\Omega} (|\nabla d|^2 - (H \cdot d)^2) \, dx,
\]  

(1.2)

where \( d : \Omega \to \mathbb{S}^2 \) describes the local orientation of the liquid crystal molecules, and \( H : \Omega \to \mathbb{R}^3 \) denotes the external magnetic field. Here we omit the diamagnetic susceptibility constant in front of the term \( (H \cdot d)^2 \). The orientation \( d \) tends to align along the magnetic field \( H \) for the sake of minimizing the total energy (1.2). By introducing a Lagrange multiplier to penalize the constraint \( |d| = 1 \), we can derive the Euler-Lagrange equation of (1.2)

\[-\Delta d = |\nabla d|^2 d + (H \cdot d)H - (H \cdot d)^2 d, \]  

(1.3)

and thus (1.1) is the corresponding gradient flow. In the last three decades there has been an enormous amount of progress concerning harmonic heat flow, see the comprehensive monograph [9]. Concerning the system with another form of external field compared with (1.1), see [2] for the existence of a classical solution and its large time asymptotic when the initial data lies in a hemisphere. However, to the best of our knowledge, there has been no result concerning the controllability of such system, neither by boundary control nor by magnetic field.

The main result of this work is stated as follows:
Theorem 1.1. For every $T > 0$ and every $p \in \mathbb{S}^2$, let $d_0 \in C^{2+\alpha}(\Omega, \mathbb{S}^2)$ be such that
\[ \inf_{x \in \Omega} d_0(x) \cdot e > 0 \text{ for some } e \in \mathbb{S}^2, \] (1.4)then there exists $\lambda(t)$ and $H_0(x, t)$ such that the system (1.1) with initial data $d|_{t=0} = d_0$ and control
\[ H = \lambda(t)e + \chi\omega H_0 \] (1.5)satisfies $d(\cdot, T) \equiv p$.

Remark 1.2. To steer the system (1.1) to a ground state $p \in \mathbb{S}^2$, we shall first choose $H = \lambda e$ with $\lambda$ being sufficiently large such that it forces the solution $d$ to stay in a small neighborhood of $e$ within $[0, T/4]$. Then we construct $H = \lambda e H_0$ by proving a local controllability result within $[T/4, T/2]$ such that $d(T/2, \cdot) \equiv p$. Finally by applying this local controllability result finitely many times, we can achieve $d(T, \cdot) \equiv e$. However, it is not clear to us how to construct a control $H$ in (1.1) without the component $\lambda(t)e$.

The rest of the paper will be organized as follows. In section 2, we recast (1.1) into a semi-linear parabolic system with internal control. In section 3, we prove the existence and uniqueness of the global in time classical solution to (1.1) with a special choice of the magnetic field, i.e. $H = \lambda(t)e$. This result is based on a Bôcher type estimate. Based on the results in these two sections, we give the proof of Theorem 1.1 in the last section.

Regarding notation, we shall use bold letters to denote vectors or matrices, and use the normal letters with indices to denote their components. For instance, $d = (d_1, d_2, d_3) = (d_i)_{1 \leq i \leq 3}$. We shall adopt the convention in differential geometry that the partial derivatives $\partial_x$ of various tensors are abbreviated by adding $\cdot$ to the corresponding components: $\partial_x d_j = d_{j \cdot}$. Moreover, repeated indices will be summed. The standard basis vectors are denoted by $e_i$ with $1 \leq i \leq 3$. We shall use $a \cdot b = a_i b_i$ for the inner product and colon for the contraction of two matrices $A : B = A_{ij}B_{ij}$.

2. Reduction to parabolic system with internal control

In this section we shall use the stereographic projection to remove the constrain $|d| = 1$ in (1.1) and reduce it to a parabolic system with internal control whose support lies in an open subset $\omega \subseteq \Omega$. The stereographic projection $\Psi : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{-e_3\}$ is defined via
\[ \Psi(v_1, v_2) = (d_1(v), d_2(v), d_3(v)) := \left( \frac{2v_1}{1 + v_1^2 + v_2^2}, \frac{2v_2}{1 + v_1^2 + v_2^2}, \frac{1 - v_1^2 - v_2^2}{1 + v_1^2 + v_2^2} \right), \] (2.1)

Proposition 2.1. If $d$ be a classical solution (1.1) with
\[ \inf_{\Omega \times (0, T)} |d(x, t) + e_3| > 0. \]
Then $v := \Psi^{-1}(d)$ is a classical solution to the following equation
\[ \begin{cases} \partial_t v - \Delta v - 2\nabla v \cdot \nabla \log h + \frac{2(|\nabla v|^2)}{h}v + \frac{h^2}{4}(H \cdot d)H_i \cdot \nabla v d_i(v), & \text{in } Q, \\ \partial_n v = 0, & \text{on } \Sigma, \end{cases} \] (2.2)
where $h = 1 + |v|^2$. Conversely, if $v$ is a strong solution to (2.2), then $d := \Psi(v)$ is a strong solution to (1.1).

Proof. It follows from (2.1) that
\[ \nabla d = \left( \frac{\partial d_i}{\partial x_k} \right)_{1 \leq i, k \leq 3}, \quad \frac{\partial d_i}{\partial x_k} = \sum_{j=1}^{2} \frac{\partial d_i}{\partial v_j} \frac{\partial v_j}{\partial x_k}, \quad \frac{\partial d_i}{\partial t} = \sum_{j=1}^{2} \frac{\partial d_i}{\partial v_j} \frac{\partial v_j}{\partial t} \] (2.3)
To do this, we first use (2.3) and (2.4) to write

\[ |\nabla d| = \sqrt{\text{tr} \left( \nabla (\nabla d)^T \right)} = \sum_{\ell,k=1}^3 \sum_{j,s=1}^2 \frac{\partial v_j}{\partial x_k} \left( \frac{\partial d_\ell}{\partial v_j} \frac{\partial d_s}{\partial v_s} \right) \frac{\partial v_s}{\partial x_k}, \tag{2.4} \]

\[ \Delta d = \frac{\partial}{\partial x_k} \frac{\partial d_i}{\partial x_k} = \frac{\partial}{\partial x_k} \left( \sum_{j=1}^2 \frac{\partial d_i}{\partial v_j} \frac{\partial v_j}{\partial x_k} \right). \]

Denote

\[ A_{jk}(v) := \frac{\partial d_i}{\partial v_j} \frac{\partial d_i}{\partial v_k}, \tag{2.5} \]

\[ J := -\partial d + \Delta d + |\nabla d|^2 d + (H \cdot d)H - (H \cdot d)^2 d, \tag{2.6} \]

and

\[ M := -\partial v + \Delta v - 2\nabla v \cdot \nabla \log h + \frac{2|\nabla v|^2}{h} v + \frac{h^2}{4} (H \cdot d)H \cdot \nabla v d_i(v), \tag{2.7} \]

with \( h = 1 + |v|^2 \), \( J = \{J_i\}_{1 \leq i \leq 3} \) and \( M = \{M_i\}_{1 \leq i \leq 2} \). Then we need to show the equivalence of the following two conditions:

\[ M = 0 \iff J = 0. \tag{2.8} \]

To do this, we first use (2.3) and (2.4) to write \( J \) component-wise

\[ J_i = -\frac{\partial d_i}{\partial v_j} \frac{\partial d_j}{\partial v_i} \frac{\partial v_j}{\partial x_k} + \frac{\partial}{\partial x_k} \left( \frac{\partial d_i}{\partial v_j} \frac{\partial d_j}{\partial v_i} \right) \frac{\partial v_j}{\partial x_k} + (H \cdot d)H_i \frac{\partial d_i}{\partial v_j}. \]

Multiplying the above equality by \( \frac{\partial d_i}{\partial v_j} \), summing over \( i \) and using \( |d| = 1 \),

\[ J_i \frac{\partial d_i}{\partial v_j} = -A_{ij}(v) \frac{\partial d_i}{\partial v_j} \frac{\partial d_j}{\partial v_i} \frac{\partial v_j}{\partial x_k} + \frac{\partial}{\partial x_k} \left( \frac{\partial d_i}{\partial v_j} \frac{\partial d_j}{\partial v_i} \right) \frac{\partial v_j}{\partial x_k} + (H \cdot d)H_i \frac{\partial d_i}{\partial v_j}. \tag{2.9} \]

In the second equality above we employed (2.5). On the other hand, it follows from (2.1) that

\[ \frac{\partial d_i}{\partial v_j} = \begin{pmatrix} 2 \frac{4v_1^2}{(1+v_1^2+v_2^2)^2} - \frac{4v_2^2}{(1+v_1^2+v_2^2)^2} & -\frac{4v_1v_2}{(1+v_1^2+v_2^2)^2} \\ -\frac{4v_1v_2}{(1+v_1^2+v_2^2)^2} & 2 \frac{4v_2^2}{(1+v_1^2+v_2^2)^2} - \frac{4v_1^2}{(1+v_1^2+v_2^2)^2} \end{pmatrix}. \tag{2.10} \]

Recalling that \( h = 1 + v_1^2 + v_2^2 \), we have a precise formula of (2.5),

\[ A_{ij}(v) = \frac{4}{h^2} \delta_{ij}. \tag{2.11} \]

This simplifies (2.9) into

\[ J_i \frac{\partial d_i}{\partial v_j} = -A_{ij}(v) \frac{\partial d_i}{\partial v_j} \frac{\partial d_j}{\partial v_i} \frac{\partial v_j}{\partial x_k} + \frac{\partial}{\partial x_k} \left( \frac{\partial d_i}{\partial v_j} \frac{\partial d_j}{\partial v_i} \right) \frac{\partial v_j}{\partial x_k} + (H \cdot d)H_i \frac{\partial d_i}{\partial v_j} + \frac{4}{h^2} \delta_{ij} \Delta v_j. \tag{2.12} \]

To proceed, we denote

\[ B_{ij} := \frac{\partial^2 d_i}{\partial v_i \partial v_j} \frac{\partial d_i}{\partial v_j}. \]
Notice that
\[ B_{jts} + B_{sdt} = \frac{\partial d_i}{\partial v_j} \frac{\partial^2 d_i}{\partial v_t \partial v_s} + \frac{\partial d_i}{\partial v_s} \frac{\partial^2 d_i}{\partial v_t \partial v_j} = \frac{\partial A_{sj}(v)}{\partial v_t}. \]

By a permutation,
\[ B_{jts} = \frac{1}{2} \left( \frac{\partial A_{sj}(v)}{\partial v_t} + \frac{\partial A_{jt}(v)}{\partial v_s} - \frac{\partial A_{ts}(v)}{\partial v_j} \right) = -\frac{4}{h^3} (h_{tj} \delta_{sj} + h_{st} \delta_{jt} - h_{sj} \delta_{ts}) \]

where \( h_{tj} \) is the abbreviation for \( \frac{\partial h}{\partial v_t} = 2h_t \). Applying this formula to the fourth component of the right hand side in (2.12) gives
\[
\begin{align*}
- \frac{\partial}{\partial x_k} \left( \frac{\partial d_i}{\partial v_t} \right) \left( \frac{\partial d_i}{\partial v_j} \right) = -B_{jts} \frac{\partial v_j}{\partial x_k} \frac{\partial v_t}{\partial x_k} = \frac{4}{h^3} (h_{tj} v_{s,k} v_{s,k} + h_{st} v_{t,k} v_{s,k} - h_{sj} v_{j,k} v_{t,k}) = \frac{4}{h^3} h_{tj} |\nabla v|^2,
\end{align*}
\]

(2.13)

where \( v_{t,j} \) is the abbreviation of \( \frac{\partial h}{\partial x_j} \). Plug (2.13) into (2.12) to get
\[
J_t \frac{\partial d_i}{\partial v_t} = -\frac{4}{h^2} v_{t,j} - \frac{8}{h^3} \nabla h \cdot \nabla v_t + \frac{4}{h^2} \Delta v_t + \frac{8}{h^3} v_t |\nabla v|^2 + (H \cdot d) H_t \frac{\partial d_i}{\partial v_t}
\]

By virtue of (2.7), this is equivalent to
\[
\begin{pmatrix}
\frac{\partial d_1}{\partial v_1} & \frac{\partial d_2}{\partial v_1} & \frac{\partial d_3}{\partial v_1} \\
\frac{\partial d_1}{\partial v_2} & \frac{\partial d_2}{\partial v_2} & \frac{\partial d_3}{\partial v_2} \\
\frac{\partial d_1}{\partial v_3} & \frac{\partial d_2}{\partial v_3} & \frac{\partial d_3}{\partial v_3}
\end{pmatrix}
\begin{pmatrix}
J_1 \\
J_2 \\
J_3
\end{pmatrix}
= \begin{pmatrix}
M_1 \\
M_2
\end{pmatrix}.
\]

Note that \( d \cdot J = 0 \), due to \( |d| = 1 \), the above formula is equivalent to
\[
EJ = (0, M_1, M_2)^T,
\]

(2.14)

where the \( 3 \times 3 \) matrix \( E \) is given by
\[
E := \begin{pmatrix}
d_1 & d_2 & d_3 \\
\frac{\partial d_1}{\partial v_1} & \frac{\partial d_2}{\partial v_1} & \frac{\partial d_3}{\partial v_1} \\
\frac{\partial d_1}{\partial v_2} & \frac{\partial d_2}{\partial v_2} & \frac{\partial d_3}{\partial v_2} \\
\frac{\partial d_1}{\partial v_3} & \frac{\partial d_2}{\partial v_3} & \frac{\partial d_3}{\partial v_3}
\end{pmatrix}.
\]

As a result (2.8) is a consequence of \( \det E \neq 0 \). Actually,
\[
\det E = \det (EE^T) = \det \begin{pmatrix}
1 & 0 & 0 \\
0 & A_{11} & A_{12} \\
0 & A_{21} & A_{22}
\end{pmatrix}^2,
\]

where \( \{A_{ij}\} \) is defined by (2.5), and this combined with formula (2.11) implies that \( \det E \neq 0 \).

Concerning the Neumann boundary condition, we have
\[
\begin{pmatrix}
\partial_\nu d_1 \\
\partial_\nu d_2 \\
\partial_\nu d_3
\end{pmatrix} = \begin{pmatrix}
\frac{\partial d_1}{\partial v_1} & \frac{\partial d_1}{\partial v_2} & \frac{\partial d_1}{\partial v_3} \\
\frac{\partial d_2}{\partial v_1} & \frac{\partial d_2}{\partial v_2} & \frac{\partial d_2}{\partial v_3} \\
\frac{\partial d_3}{\partial v_1} & \frac{\partial d_3}{\partial v_2} & \frac{\partial d_3}{\partial v_3}
\end{pmatrix} \begin{pmatrix}
\partial_\nu v_1 \\
\partial_\nu v_2 \\
\partial_\nu v_3
\end{pmatrix} = \begin{pmatrix}
d_1 \\
d_2 \\
d_3
\end{pmatrix} \begin{pmatrix}
\partial_\nu v_1 \\
\partial_\nu v_2 \\
0
\end{pmatrix}.
\]

This together with \( \det E \neq 0 \) implies the equivalence between boundary conditions \( \partial_\nu d = 0 \) and \( \partial_\nu v = 0 \). So we complete the proof. \( \square \)
In order to reduce (1.1) to an internal control system, we need to solve the following algebraic equations for given \( v = (v_1, v_2) \) and \( f = (f_1, f_2) \):

\[
\frac{2v_1 H_1 + 2v_2 H_2 + (1 - v_1^2 - v_2^2) H_3}{1 + v_1^2 + v_2^2} \begin{pmatrix}
\frac{1}{2}(1 + v_1^2 - v_2^2) H_1 - v_1 v_2 H_2 - v_1 H_3 \\
-v_1 v_2 H_1 + \frac{1}{2}(1 + v_1^2 - v_2^2) H_2 - v_2 H_3 \\
-v_1 H_1 - v_2 H_2 + \frac{1}{2}(-1 + v_1^2 + v_2^2) H_3
\end{pmatrix} = \chi_\omega \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},
\]

(2.15)

where \( \chi_\omega \) is the characteristic function of an open subset \( \omega \subset \Omega \).

**Lemma 2.2.** For every \((v, f) \in C(\Omega, \mathbb{R}^4)\), equation (2.15) has a solution \( H = H(f, v) \) which depends smoothly on \( v \) and \( f \) such that \( \text{supp} \( H \) \subset \omega \).

**Proof.** The equation (2.15) is underdetermined and might have multiple solutions. We look for a special solution by setting

\[
2v_1 H_1 + 2v_2 H_2 + (1 - v_1^2 - v_2^2) H_3 = (1 + v_1^2 + v_2^2) \sqrt{\chi_\omega}.
\]

Then (2.15) can be reduced to the following linear equation about \( H_1 \):

\[
\begin{pmatrix}
\frac{1}{2}(1 + v_1^2 - v_2^2) H_1 - v_1 v_2 H_2 - v_1 H_3 \\
-v_1 v_2 H_1 + \frac{1}{2}(1 + v_1^2 - v_2^2) H_2 - v_2 H_3 \\
-v_1 H_1 - v_2 H_2 + \frac{1}{2}(-1 + v_1^2 + v_2^2) H_3
\end{pmatrix} = \sqrt{\chi_\omega} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.
\]

(2.16)

Denote

\[
A = \begin{pmatrix}
\frac{1}{2}(1 + v_1^2 - v_2^2) & -v_1 v_2 & -v_1 \\
-v_1 v_2 & \frac{1}{2}(1 + v_1^2 - v_2^2) & -v_2 \\
-v_1 & -v_2 & \frac{1}{2}(-1 + v_1^2 + v_2^2)
\end{pmatrix}.
\]

Its eigenvalues and eigenvectors are

\[
\lambda_1 = -\frac{1}{2}(1 + v_1^2 + v_2^2), \quad \mathbf{w}_1 = (v_1, v_2, 1),
\]

\[
\lambda_2 = \frac{1}{2}(1 + v_1^2 + v_2^2), \quad \mathbf{w}_2 = (-\frac{1}{v_1}, 0, 1),
\]

\[
\lambda_3 = \frac{1}{2}(1 + v_1^2 + v_2^2), \quad \mathbf{w}_3 = (-\frac{v_2}{v_1}, 1, 0),
\]

and thus \( A \) is invertible. This shows that (2.16) has a unique solution and the lemma is proved.

Thanks to Proposition 2.1 and Lemma 2.2, the controllability of (1.1) is reduce to the following system with internal control:

\[
\begin{align*}
\partial_t v - \Delta v &= F(v, \nabla v) + \chi_\omega f, & \text{in } Q, \\
v|_{t=0} &= v_0, & \text{in } \Omega, \\
\partial_v v &= 0, & \text{on } \Sigma,
\end{align*}
\]

(2.17)

where

\[
F(v, \nabla v) := -2\nabla v \cdot \nabla \log(1 + |v|^2) + \frac{2|\nabla v|^2}{1 + |v|^2} v.
\]

(2.18)

To proceed, we need the following result in [4]

**Lemma 2.3.** For every \( T > 0 \), the system

\[
\begin{align*}
\partial_t y - \Delta y &= a(x, t)y + \chi_\omega u, & \text{in } Q, \\
y|_{t=0} &= y_0, & \text{in } \Omega, \\
\partial_v y &= 0, & \text{on } \Sigma
\end{align*}
\]

(2.19)

is null-controllable at \( t = T \) with control \( u \in L^\infty(Q) \) such that

\[
\|u\|_{L^\infty(Q)} \leq e^{c_0(T, \omega)K(T, a)} \|y_0\|_{H^2(\Omega)},
\]

(2.20)
where $c_0(\Omega, \omega)$ is a generic constant and

$$K(T, a) = 1 + 1/T + \|a\|_\infty^{2/3} + T(1 + \|a\|_\infty). \quad (2.21)$$

We shall also need Kakutani’s fixed point theorem:

**Proposition 2.4** (Kakutani’s fixed point theorem). Let $Z$ be a non-empty, compact and convex subset of a Hausdorff locally convex topological vector space $Y$. Let $\Phi : Z \to 2^Z$ be upper semi-continuous and $\Phi(x)$ is non-empty, compact, and convex for all $x \in Z$. Then $\Phi$ has a fixed point in the sense that there exists $x \in Z$ such that $x \in \Phi(x)$.

With these preparations, we can prove the local controllability of (2.19), following [3].

**Proposition 2.5.** For any $T > 0$, there exist positive constants $c_2$ such that if

$$v_0 \in W^{2,p}(\Omega), \quad \|v_0\|_{L^2(\Omega)} \leq c_2, \quad (2.22)$$

then system (2.17) is null-controllable at time $T$.

**Proof.** Without loss of generality, we can assume that $T = 1$. Moreover, by parabolic regularity, we can assume

$$\|v_0\|_{W^{2,p}(\Omega)} \leq c_2.$$ 

Actually, it follows from [12, page 317] that, for any $\delta \in (0, T/2)$,

$$\|v(\cdot, \cdot)\|_{W^{2,p}(\Omega)} \leq C(\delta)\|v_0\|_{L^2(\Omega)}.$$ 

So it suffices to set $f \equiv 0$ for $t \in [0, \delta]$ and to control the system for $t \in [\delta, T]$, viewing $y(\cdot, \delta)$ as the initial state.

In order to employ Kakutani’s fixed point theorem, we set

$$Z := \{z \in C^0([0, T]; W^{1,\infty}(\Omega)) \leq R, \quad z(x, 0) = v_0 \} \quad (2.23)$$

with $R > 0$ being determined later on. Clearly, $Z$ is a nonempty convex and compact subset of some negative Sobolev space, say $H^{-1}(Q)$. Given $z \in Z$, consider the linear null-control system

$$\begin{cases}
\partial_t v - \Delta v = g(z, \nabla z)v + \chi_o f, & \text{in } Q, \\
v|_{t=0} = v_0, & \text{on } \Omega, \\
\partial_n v = 0, & \text{on } \Sigma, 
\end{cases} \quad (2.24)$$

where the $2 \times 2$-matrix-valued function $g$ is chosen such that

$$g(v, \nabla v) = F(v, \nabla v).$$

More precisely,

$$g(v, \nabla v) := \left\{ -4\nabla v_i \cdot \nabla v_j + 2|\nabla v|^2\delta_{ij} \right\} \quad (1 \leq i, j \leq 2).$$

Regarding (2.24), we shall look for control $f$ in the class

$$\mathcal{F} = \left\{ f \in L^\infty(Q) \mid \|f\|_{L^\infty(Q)} \leq e^{c_0(\Omega, \omega)(3+R+R^2/3)}\|v_0\|_{L^2(\Omega)} \right\}.$$ 

It follows from Lemma 2.3 that, for every $z \in Z$, there exists $f \in \mathcal{F}$ such that the system (2.24) satisfies $v(\cdot, T) = 0$. In other words, for every $z \in Z$, the following set is not empty:

$$\mathcal{C}(z) := \left\{ f \quad \mid \quad \|f\|_{L^\infty(Q)} \leq e^{c_0(\Omega, \omega)(3+R+R^2/3)}\|v_0\|_{L^2(\Omega)}, \right\}, \quad (2.25)$$

such that the solution to (2.24) satisfies $v(\cdot, T) = 0$.

Moreover, due to $\|z(x, t)\|_{L^\infty(0, T; W^{1,\infty})} \leq R$ and standard estimates of parabolic equations, there exists $c_1 = c_1(\Omega)$ such that

$$\|v\|_{C^0([0, T]; W^{1,\infty})} \leq e^{c_1(\Omega, R)} \left( \|v_0\|_{W^{2, p}} + \|f\|_{L^\infty} \right) \leq e^{c_1(\Omega, R)} \left( \|v_0\|_{W^{2, p}} + e^{c_0(\Omega, \omega)(3+R+R^2/3)}\|v_0\|_{L^2(\Omega)} \right)$$
We denote \( \lambda \) and \( v \). Note that (3.2) is the variation of \( \lambda \). This is a small but crucial improvement we made is to show that the gradient estimate is independent of time. We shall assume continuity of operator \( g \), i.e.,

\[
\left\{ \begin{array}{l}
\partial_t v - \triangle v = |\nabla v|^2 v + \partial V(d), & \text{in } Q, \\
\partial_\nu v = 0, & \text{on } \Sigma.
\end{array} \right.
\]

(3.1)

Then choosing \( \lambda > 1 \), then (3.1) holds. It remains to verify the hypothesis of Kakutani’s fixed point theorem for \( \Phi \). It is clear that \( z \) is a closed, compact, convex subset of a negative Sobolev space. For each \( z \in Z \), the compactness of \( \Phi(z) \) comes from (2.26) and compact embedding:

\[
L^\infty(0, T; W^{2,p}) \cap W^{1,\infty}(0, T; L^p) \Rightarrow C^0([0, T]; W^{1,\infty}),
\]

see for instance [11]. The continuity of \( \Phi(z) \) follows from the linearity of (2.24) and local continuity of operator \( g(z, \nabla z) : W^{1,\infty}(\Omega) \rightarrow L^\infty(\Omega) \). This completes the proof of the result.

3. Classical Solution to Harmonic Map Heat Flow

In this section we investigate (1.1) with \( H = \lambda(t) e \) for some \( e \in S^2 \) and smooth function \( \lambda(t) \), i.e.

\[
\left\{ \begin{array}{l}
\partial_t d - \Delta d = |\nabla d|^2 d + \partial V(d), & \text{in } Q, \\
\partial_\nu d = 0, & \text{on } \Sigma.
\end{array} \right.
\]

(3.1)

where we denote

\[
\partial V(d) := (H \cdot d) H - (H \cdot d)^2 d = \lambda^2 (e \cdot d) e - \lambda^2 (e \cdot d)^2 d.
\]

(3.2)

Note that (3.2) is the variation of \( V(d) := -(H \cdot d)^2 / 2 = \lambda^2 \mu^2 / 2 \) under the constraint \( |d| = 1 \). The main result of this section is given below, which is essentially due to [6] and [2]. The small but crucial improvement we made is to show that the gradient estimate is independent of \( \lambda \). Then choosing \( \lambda \) sufficiently large will force the solution \( d \) to approach \( e \) within any short time. We shall assume \( \lambda(t) \in C^1[0, T] \).

**Proposition 3.1.** For arbitrary \( T > 0 \), assume \( d_0 \in C^{2+\alpha}(\bar{\Omega}, S^2) \) fulfill \( \partial_\nu d_0 = 0 \) on the boundary \( \partial \Omega \) and

\[
\epsilon_0 := \min_{x \in \Omega} d_0(x) \cdot e > 0.
\]

(3.3)

Then (3.1) has a unique solution \( d(x, t) \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega} \times [0, T], S^2) \) with initial data \( d_0 \). Moreover,

\[
\sup_{\Omega \times [0, T]} |\nabla d(x, t)| \leq \frac{2}{\epsilon_0} \sup_{x \in \Omega} |\nabla d_0|,
\]

(3.4)

and

\[
d(x, t) \cdot e \geq \epsilon_0 > 0, \quad \forall (x, t) \in \Omega \times [0, T].
\]

(3.5)

We start with a lemma saying that the projection of equation (1.1) to direction \( e \in S^2 \) satisfies a parabolic equation where the maximum principle applies:

**Lemma 3.2.** Under the assumption of Proposition 3.1, if \( d \in C^{2+\alpha, 1+\alpha/2}(\bar{Q}, S^2) \) is a solution to (3.1) with initial data \( d_0 \), then (3.5) holds.
Lemma 3.3. Under the assumption of Proposition 3.1, if $\mathbf{d} \in C^{2+\alpha,1+\beta}(\overline{Q}, S^2)$ is a solution to (3.1) with initial data $\mathbf{d}_0$, then (3.4) holds.

Proof. We choose any $\delta_0 \in (0, \epsilon_0)$ and denote

$$A(x, t) := \frac{e(\mathbf{d})}{f^2(\mathbf{d})}$$

where

$$e(\mathbf{d}) := \frac{1}{2} \|
abla \mathbf{d}(x, t)\|^2, \quad f(\mathbf{d}) := \mathbf{d}(x, t) \cdot \mathbf{e} - \delta_0.$$  

So we know from Lemma 3.2 that

$$f(\mathbf{d}) \geq \epsilon_0 - \delta_0 > 0.$$  

Tedious calculation gives

$$(\partial_t - \Delta)e(\mathbf{d})$$

$$= \nabla(d_t - \Delta \mathbf{d}) : \nabla \mathbf{d} - |\nabla^2 \mathbf{d}|^2$$

$$= \nabla|\nabla \mathbf{d}|^2 \mathbf{d} : \nabla \mathbf{d} + |\nabla \mathbf{d}|^2 \nabla \mathbf{d} : \nabla \mathbf{d} + \nabla \partial V(\mathbf{d}) : \nabla \mathbf{d} - |\nabla^2 \mathbf{d}|^2$$

$$= |\nabla \mathbf{d}|^4 + \nabla \partial V(\mathbf{d}) : \nabla \mathbf{d} - |\nabla^2 \mathbf{d}|^2,$$

and

$$(\partial_t - \Delta)f(\mathbf{d}) = \mathbf{d} \cdot \nabla|\nabla \mathbf{d}|^2 + \partial V(\mathbf{d}) \cdot \mathbf{e}.$$

As a result,

$$(\partial_t - \Delta)A(x, t)$$

$$= \frac{(\partial_t - \Delta)e(\mathbf{d})}{f^2} - \frac{2e(\mathbf{d})(\partial_t - \Delta)f}{f^3} + \frac{4\nabla e(\mathbf{d}) \cdot \nabla f}{f^3} - \frac{6e(\mathbf{d})|\nabla f|^2}{f^4}$$

$$= \frac{|\nabla \mathbf{d}|^4 + \nabla \partial V(\mathbf{d}) : \nabla \mathbf{d} - |\nabla^2 \mathbf{d}|^2}{f^3} - \frac{|\nabla \mathbf{d}|^4 (\mathbf{d} \cdot \mathbf{e}) + |\nabla \mathbf{d}|^2 \partial V(\mathbf{d}) \cdot \mathbf{e}}{f^3}$$

$$+ \frac{4\nabla e(\mathbf{d}) \cdot \nabla f}{f^3} - \frac{6e(\mathbf{d})|\nabla f|^2}{f^4}$$

$$= -\frac{\delta_0 |\nabla \mathbf{d}|^4}{f^3} + \frac{\nabla \partial V(\mathbf{d}) : \nabla \mathbf{d}}{f^2} - \frac{|\nabla^2 \mathbf{d}|^2}{f^2} - \frac{|\nabla \mathbf{d}|^2 \partial V(\mathbf{d}) \cdot \mathbf{e}}{f^3}$$

$$+ \frac{4\nabla e(\mathbf{d}) \cdot \nabla f}{f^3} - \frac{6e(\mathbf{d})|\nabla f|^2}{f^4} =: I_1 + I_2,$$

or simply

$$((\partial_t - \Delta)A(x, t) = I_1 + I_2,$$  

(3.7)
where
\[
\begin{align*}
I_1 &= -\left(\nabla^2 d\right)^2 f^2 + 4\nabla e(d) \cdot \nabla f - \frac{6e(d)|\nabla f|^2}{f^4}, \\
I_2 &= -\frac{\delta_0|\nabla d|^4}{f^4} + \frac{\nabla \partial V(d) \cdot \nabla d}{f^3} - \frac{|\nabla d|^2 \partial V(d) \cdot e}{f^3}.
\end{align*}
\tag{3.8}
\]

It remains to treat \(I_1\) and \(I_2\):
\[
\begin{align*}
I_1 &= -\left(\frac{|\nabla d|^2}{f^2} + \frac{2\nabla e(d) \cdot \nabla f}{f^4} - \frac{2e(d)|\nabla f|^2}{f^4} + \frac{2\nabla e(d) \cdot \nabla f}{f^4} - \frac{4e(d)|\nabla f|^2}{f^4}\right) \\
&\leq -\left(\frac{|\nabla d|^2}{f^2} + \frac{2|\nabla d||\nabla f|}{f^4} - \frac{|\nabla d|^2|\nabla f|^2}{f^4} + \frac{2A \cdot \nabla f}{f^4}\right) \\
&= -\left(\frac{|\nabla d|^2}{f^2} - \frac{|\nabla d||\nabla f|}{f^4}\right)^2 + \frac{2A \cdot \nabla f}{f^4} \leq \frac{2A \cdot \nabla f}{f^4}. \\
\end{align*}
\tag{3.9}
\]

To treat \(I_2\), we employ (3.2) to get
\[
\partial V(d) \cdot e = \lambda \left(\mu - \mu^3\right) \geq 0 \text{ since } \mu = d \cdot e \in [0, 1],
\tag{3.10}
\]

and since we choose \(\lambda = \lambda(t)\), this leads to
\[
\nabla \partial V(d) : \nabla d = \lambda^2|\nabla \mu|^2 - \lambda^2\mu^2|\nabla d|^2. 
\tag{3.11}
\]

Employing (3.6), (3.11) and (3.10) together yields the estimate for \(I_2\):\[
I_2 = f^{-3} \left((\mu - \delta_0)^2|\nabla \mu|^2 + (\delta_0 - 1)\mu^2|\nabla d|^2 - \delta_0|\nabla d|^4\right) \\
\leq \delta_0(\mu^2 - 1)\lambda^2|\nabla d|^2 - \delta_0|\nabla d|^4 \leq 0.
\tag{3.12}
\]

In the second step above we used \(|\nabla \mu| \leq |\nabla d|\). Now plugging (3.9) and (3.12) into (3.7) leads to
\[
(\partial_t - \Delta)A(x, t) \leq \frac{2A \cdot \nabla f}{f^4}. 
\tag{3.13}
\]

Concerning the boundary condition, we claim that
\[
\partial_\nu A(x, t) = 0, \text{ on } \Sigma.
\]

Actually, using \(\partial_\nu d = 0\) on \(\Sigma\), we can calculate
\[
\partial_\nu A(x, t) = \frac{\nabla_\nu \partial_\nu d : \nabla d}{(d \cdot e - \delta_0)^2} + \frac{|\nabla d|^2}{2} \nabla_\nu (|d \cdot e - \delta_0|^{-2}) \\
= \frac{(\nabla_\Gamma \partial_\nu d + \nu \partial_\nu^2 d) : (\nabla_\Gamma d + \nu \partial_\nu d)}{(d \cdot e - \delta_0)^2} \text{ on } \Sigma,
\]

where \(\nabla_\Gamma\) denotes the tangential derivative along the boundary \(\partial \Omega\). This, together with (3.13) and the maximum principle, leads to
\[
A(x, t) \leq \sup_{x \in \Omega} \frac{|\nabla d_0|^2}{f^2(d_0)} \leq \frac{1}{(\epsilon_0 - \delta_0)^2} \sup_{x \in \Omega} |\nabla d_0|^2.
\]

This implies the desired result by choosing \(\delta_0 = \epsilon_0/2\). \(\square\)

With the aid of the above lemmas, we can give the proof of Proposition 3.1:

**Proof of Proposition 3.1.** Since (3.1) is a semi-linear parabolic system, the existence and uniqueness of the local in time solution follow from standard theory (see for instance [12, Chapter 15]): there exists \(T > 0\) such that
\[
\|d\|_{C^{2+\alpha,1+\frac{4}{2}(\bar{Q},\mathbb{R}^2)}(T, d_0, \lambda)} \leq C(T, d_0, \lambda).
\]
Applying Lemma 3.2 gives
\[ \mu(x, t) = d(x, t) \cdot e \geq \epsilon_0 > 0 \text{ in } Q. \] (3.14)
In order to extend the solution to every \( T > 0 \), we need to bound \( \|d\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega \times [0, T/2])} \) in terms of \( \|d_0\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega)} \) up to a constant that is independent of \( T \), and this is a consequence of Lemma 3.3. More precise, the right hand side of (3.1) is bounded in \( L^\infty(\Omega) \) by a constant depending on \( \sup_{x \in \Omega} |\nabla d_0|, \epsilon_0, \lambda \) and \( \Omega \) but not on \( T \). So parabolic regularity theory implies \( \|d\|_{C^{1+\alpha, 1/2+\alpha/2}(\Omega)} \) is bounded by a constant that is independent of \( T \). Consequently the right hand side of (3.1) lies in \( C^{\alpha, \alpha/2}(\Omega) \), and thus the Schauder’s estimate implies
\[ \|d\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega)} \leq C, \]
where \( C \) is independent of \( T \). This completes the proof of existence of global in time classical solution to (3.1). The uniqueness of the solution follows from the standard energy method.

4. PROOF OF THEOREM 1.1

By the assumption (1.4) and Proposition 3.1, there exists a unique solution \( d(x, t) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega \times [0, T/2]) \) to (3.1) with initial data \( d_0 \) such that
\[ \sup_{\Omega \times [0, T/2]} |\nabla d(x, t)| \leq \frac{16}{\epsilon_0^2} \sup_{x \in \Omega} |\nabla d_0|^2, \]
and
\[ \mu(x, t) = d(x, t) \cdot e \geq \epsilon_0 > 0, \quad \forall (x, t) \in \Omega \times (0, T/2). \]
On the other hand, \( \mu \) satisfies
\[ \partial_t \mu - \Delta \mu = |\nabla d|^2 \mu + \lambda^2 (\mu - \mu^3). \]
So if we denote \( \phi = 1 - \mu \), since \( \mu \in [\epsilon_0, 1] \) for some \( \epsilon_0 \in (0, 1) \), we have
\[ \partial_t \phi - \Delta \phi \leq -\lambda^2 (1 - \phi) \phi (2 - \phi) \leq -\lambda^2 \epsilon_0 \phi. \]
So the maximum principle implies the decay
\[ 1 - d(x, t) \cdot e = \phi(x, t) \leq e^{-\lambda^2 \epsilon_0 t}. \]
If we choose \( \lambda(t) \) to be sufficiently large over \( [0, T/2] \), it will make \( \sup_{\Omega} |d(x, T/2) - e| \ll 1 \) such that \( v_0(x) = \Psi^{-1}(d(x, T/2)) \) satisfies (2.22). Then we consider the control system
\[
\begin{cases}
\partial_t v - \Delta v = F(v, \nabla v) + \chi(\epsilon f), & \text{in } \Omega \times (T/2, 3T/4), \\
v|_{t=T/2} = v_0, & \text{in } \Omega, \\
\partial_v v = 0, & \text{on } \partial \Omega \times (T/2, 3T/4).
\end{cases}
\]
We can apply Proposition 2.5 to obtain a control \( f \in L^\infty(\Omega \times (T/2, 3T/4)) \) such that \( v(\cdot, 3T/4) \equiv 0 \). According to Proposition 2.1 and Lemma 2.2, \( d = \Psi(v), H = H(f, v) \) satisfies (1.1) on \( \Omega \times (0, 3T/4) \) and \( d(\cdot, 3T/4) = e \). To drive the system to the prescribed state \( p \in S^2 \), we denote \( p_0 = e, p_N = p \) and choose finitely many points \( \{p_i\}_{1 \leq i \leq N-1} \subset S^2 \) such that \( |p_i - p_{i-1}| \ll 1 \) provided that \( N \) is sufficiently large. Then inductively, we let the system evolve with initial data \( p_{i-1} \) and drive it to \( p_i \), due to Proposition 2.5. So we have \( d(\cdot, T) = p \).

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