Abstract

The quon algebra describes particles, “quons,” that are neither fermions nor bosons using a label $q$ that parametrizes a smooth interpolation between bosons ($q = +1$) and fermions ($q = -1$). We derive “conservation of statistics” relations for quons in relativistic theories, and show that in relativistic theories quons must be either bosons or fermions.

There are three reasons to study theories that allow violations of statistics, i.e., that allow particles that are neither bosons nor fermions. One, which may seem frivolous, is to stretch the framework of quantum physics and to find out what possibilities open up when one does so. The second, quite concrete, is to respond to experimental interest in high-precision tests of the symmetrization postulate (that all identical particles occur in one-dimensional representations of the symmetric
group) and of the the connection of spin and statistics (that integer-spin particles must be bosons and odd-half-integer spin particles must be fermions) by providing a theoretical framework to describe such violations together with a parameter (or parameters) that can characterize violations if they are found and allow quantitative bounds as well as the comparison of bounds on violations from different types of experiments if violations are not found. The third is to see how theories can exclude all possibilities except bosons and fermions and thus provide an understanding of the empirical fact that only bosons and fermions have been observed.

In this paper, we will only consider quantum field theories in which the annihilation and creation operators are constrained by a bilinear operator algebra. This includes bosons and fermions, and, from the point of view of the Green ansatz[1], also parabosons and parafermions[1, 2, 3] as well as quons[1, 4]. Since parabosons and parafermions correspond to bosons and fermions with an exact hidden degree of freedom[6, 7, 8], the only case that may be of interest from the standpoint of small violations of statistics is that of quons. We point out that the bilinear operator algebras that we consider give examples of each of the possible types of identical particle statistics in three space dimensions found in the general analysis of Haag and collaborators[7, 8].

Any theory that violates statistics must also violate one or more of the usual conditions of relativistic quantum field theory, otherwise one could prove that the symmetrization postulate and the spin-statistics connection would hold. The condition that quons violate is locality of observables (which, in general, is not the same as locality in the sense of having field products at a single point); however this is not tested directly to high precision.

Let us briefly review what is known about quonic theories. Quon statistics is compatible with Lorentz invariance and the CPT theorem, at least for free quon fields[5]. Nonrelativistic quon theories are valid quantum field theories (with positive squared norms) for $-1 \leq q \leq 1$ both for free quons and for quons that interact with particle conserving interactions[9]. Bound states of $n$ quons with parameter $q$ have the parameter $q^n$[10], a generalization of the Wigner–Ehrenfest-Oppenheimer result for bosons and fermions[11, 12]. In the nonrelativistic context, the quon theory has been used to parametrize bounds on violations of statistics[13]. For example, as a
nonrelativistic description of electrons interacting via the Coulomb interaction, the quon theory allows the bound $-1 \leq q(e) \leq -1 + 3.4 \times 10^{-26}$ to be inferred from the experiment of E. Ramberg and G.A. Snow\cite{[14]}, where $q(e)$ is the $q$-parameter for electrons and, as described just below, $q = -1$ is the fermion limit of quons.

The quon theory uses a bilinear algebra,

$$a(k)a^\dagger(l) - q(a,a)a^\dagger(l)a(k) = \delta(k,l), \quad (1)$$

and the usual Fock-like vacuum condition,

$$a(k)|\Omega\rangle = 0, \quad (2)$$

where $|\Omega\rangle$ is the vacuum. The adjoint of Eq. (1) implies $q(a,a) \equiv q(a)$ is real. It is easy to see that squared norms will be positive only for $-1 \leq q \leq 1$. This follows from the square of the norm of the general two-particle state,

$$\|\psi(k_1,k_2)a^\dagger(k_1)a^\dagger(k_2)|\Omega\\rangle\|^2$$

$$= |\psi(k_1,k_2)|^2 + q(a)\psi^*(k_1,k_2)\psi(k_2,k_1) \quad k_1 \neq k_2, \quad (3)$$

where repeated indices are summed over. For $\psi$ symmetric (antisymmetric) the result is $(1 + q(a))|\psi(k_1,k_2)|^2$ ($(1 - q(a))|\psi(k_1,k_2)|^2$), which shows that $q(a)$ must lie in the closed interval $(-1,1)$ as stated above. It is more difficult to show that for $q$ in this range all norms are positive or zero; proofs of this appear in \cite{[15], [16]} among other places. The limiting cases, $q = -1 (q = 1)$ correspond to Fermi (Bose) statistics.

We can also define relative $q$ parameters in the bilinear relation between two independent sets of annihilation and creation operators by

$$a(k)b^\dagger(l) - q(a,b)b^\dagger(l)a(k) = 0. \quad (4)$$

The adjoint of this equation,

$$b(l)a^\dagger(k) - q(a,b)^*a^\dagger(k)b(l) = 0, \quad (5)$$

leads to $q(b,a) = q(a,b)^*$. The norm of $(a^\dagger(k)b^\dagger(l) + zb^\dagger(l)a^\dagger(k))|\Omega\rangle$, is $|z|^2 + z^*q(a,b)^* + zq(a,b) + 1$. The minimum of this is $1 - |q(a,b)|^2$, so positivity of norms
requires \(|q(a, b)| \leq 1\), the same condition as for \(q(a)\), except that, so far, \(q(a, b)\) does not have to be real.

We will now study the constraints on this \(q\) parameters due to the conservation of statistics. The key observation is that the Hamiltonian operator must be effectively a bosonic operator. This requirement follows from the condition that the contribution to the energy from subsystems that are widely spacelike separated should be additive \([4]\). The terms appearing in the Hamiltonian are in general products of field operators \(\phi(x)\), which are themselves linear combinations of creation and annihilation operators, \(i.e., \phi_a(x) = \int d^3k/(2\omega_k)[a(k)exp(-ik \cdot x) + \bar{a}(k)^\dagger exp(ik \cdot x)].\) As a result, there will always be a term in the Hamiltonian which is a product of only annihilation operators (or of only creation operators). For example, a trilinear interaction term \(H_I = \phi_a \phi_b \phi_c\) contains a term proportional to \(abc\) (and a term proportional to \(\bar{a}^\dagger \bar{b}^\dagger \bar{c}^\dagger\)). Since the Hamiltonian is bosonic, \(abc\) should also be a bosonic operator and the relative \(q\) factor \(q(x, abc)\) with any annihilation operator \(x\) must be one. Thus

\[
x(k)(a(l_1)b(l_2)c(l_3))^\dagger - q(x, abc)(a(l_1)b(l_2)c(l_3))^\dagger x(k) = \delta(k, l_1)(b(l_2)c(l_3))^\dagger.
\]

(6)

Equation (6) leads to

\[
q(a)q(a, b)q(a, c) = 1,
\]

(7)

and

\[
q(\bar{a}, a)q(\bar{a}, b)q(\bar{a}, c) = 1,
\]

(8)

and cyclic permutations of \(a, b, c\). Since all the \(q\)’s must lie in the closed unit disk this immediately implies

\[
|q| = 1
\]

(9)

for all \(q\)’s. Since \(q(a)\)’s must be real for all particles, the constraint \(|q| = 1\) implies that \(q(a)\) can only be \(\pm 1\); \(i.e.,\) the commutation relations of \(a\) and \(a^\dagger\) must take the standard bosonic and fermionic form. Thus despite the original motivation of quonic statistics, namely to provide a smooth interpolation between bosons \((q = 1)\) and \((q = -1)\), we have come to the conclusion that all quons in relativistic theories are either bosons or fermions. This is the main result of this paper. In addition this result holds in any theory (relativistic or not) that has a term in its Hamiltonian with only creation or only annihilation operators.
The usual rules of conservation of statistics can be recovered by setting \(x(k) = a(k_1)b(k_2)c(k_3)\) in Eq. (1), which gives

\[
q(a, a)q(a, b)q(a, c)q(b, a)q(b, b)q(b, c)q(c, a)q(c, b)q(c, c) = 1. \tag{10}
\]

The constraints \(q(a, b) = q(b, a)^*\) and \(|q| = 1\) imply \(q(a, b)q(b, a) = 1\). Hence the above equation simplifies to

\[
q(a)q(b)q(c) = 1, \tag{11}
\]

which is a restatement of the Wigner–Ehrenfest–Oppenheimer theorem [11, 12] with two possibilities for a three-particle vertex,

Case A: all three particles are bosons,

Case B: one particle is a boson and two are fermions.

We can also easily show that the \(q\) factor of a particle is equal to that of its antiparticle. Note that the Hamiltonian always contains a “pair annihilation” term, which is proportional to the product of the annihilation operator \(a\) and the annihilation operator of its antiparticle \(\bar{a}\). (Both mass terms and kinetic terms fall into this category.) As a result, the relative \(q\) factor between \(a\bar{a}\) and any annihilation operator must be unity. Then it trivially follows that

\[
q(\bar{a}, b) = q(a, b)^{-1} = q(a, b)^*. \tag{12}
\]

Since all these \(q\) are phases (\(|q| = 1\)), Eq. (12) is the mathematical statement that charge conjugation of one of the particles reverses the phase of the relative \(q\) factors. Eq. (12) implies the corollary \(q(\bar{a}, \bar{b}) = q(a, b)\), and for the special case \(a = b\), \(q(\bar{a}) = q(a)\), which is the statement that the antiparticle of a boson (fermion) is also a boson (fermion).

Our results that the diagonal \(q\)'s are plus or minus one and the off-diagonal \(q\)'s have absolute value one are general and hold for theories with interactions of any finite degree. Next we study the trilinear interaction \(H_I = \phi_a\phi_b\phi_c\) to constrain the off-diagonal \(q\)'s further for this case. Since \(\bar{a}\) has the same statistics as \(bc\), we have the following crossed condition:

\[
x(k)(b(l_1)c(l_2))^\dagger - q(x, \bar{a})(b(l_1)c(l_2))^\dagger x(k) = 0, \tag{13}
\]

and similarly, since \(a\) has the same statistics as \(\bar{b}\bar{c}\),

\[
x(k)(\bar{b}(l_1)\bar{c}(l_2))^\dagger - q(x, a)(\bar{b}(l_1)\bar{c}(l_2))^\dagger x(k) = 0, \tag{14}
\]
and again cyclic permutations. In particular, by choosing \( x(k) = \bar{a}(k) \) in Eq. (13), we find

\[
q(\bar{a}, \bar{a}) = q(a) = q(a, b)^* q(a, c)^*,
\]

which relates \( q(a, b) \) to \( q(a, c) \). This relation, and two others from cyclic permutations, relates all the off-diagonal \( q \)'s to each other. Referring to the cases A and B above, we find

Case A: \( q(a) = q(b) = q(c) = 1 \) and \( q(a, b) = q(b, c) = q(c, a) = \exp(iQ) \).

Case B: \( q(a) = -q(b) = -q(c) = 1 \) and \( q(c, a) = q(a, b) = -q(b, c) = \exp(iQ) \).

The phase angle \( Q \) is a characteristic of the trilinear vertex in question and is not constrained by conservation of statistics alone.

However we now point out that, for this case, the phase angle \( Q \) can be rotated away by a generalized Klein transformation[17], so that, after the transformation, \( q(a, b) = q(b, c) = q(c, a) = 1 \) for case A and \( q(c, a) = q(a, b) = -q(b, c) = 1 \) for case B. We note that the true number operator for a field \( x \), which is an infinite series in the annihilation and creation operators,

\[
n_x(k) = x^\dagger(k)x(k) + \sum_t (1 - q^2)^{-1} \sum_t (x^\dagger(t)x^\dagger(k)-qx^\dagger(k)x^\dagger(t))(x(l)x(t)-qx(t)x(l)) + \cdots
\]

obeys

\[
[n_x(k), x^\dagger(l)] = \delta(k, l)x^\dagger(l)
\]

so that the total number operator \( N_x = \sum_k n_x(k) \) obeys

\[
\exp(i\phi N_x) x^\dagger(k) = \exp(i\phi)x^\dagger(k)\exp(i\phi N_x)
\]

and

\[
\exp(i\phi N_x)x(k) = \exp(-i\phi)x(k)\exp(i\phi N_x)
\]

and the number operators for independent fields commute. Define the rephased operators by

\[
a'(k) = a(k)\exp[i(\phi(a, a)N_a + \phi(a, b)N_b + \phi(a, c)N_c)]
\]

and cyclic permutations, with the condition that the product \( a' b' c' = abc \) so that the interaction Hamiltonian is not changed. We also require the standard relative
commutation relations stated above. Straightforward calculations using Eq. (18), (19) and (20) show that the standard forms for both cases A and B result when the phases are chosen as follows

\[ \phi(a, a) = \phi(b, b) = \phi(c, c) = \phi(a, b) = \phi(b, c) = \phi(c, a) = -Q/3. \]  

Thus for the case of a single trilinear interaction the conservation of statistics rules together with the generalized Klein transformations lead to the standard results that fields are either bosons or fermions and the relative commutation relations are bosonic unless both fields are fermions, in which case the relative relation is fermionic. It is plausible that the standard results also hold for theories with several fields and with interactions of finite higher degree. A likely way to study this question is to generalize the technique of H. Araki\[18\] which he used to demonstrate the standard form for relative commutation relations in theories with bosons and fermions. If the phases cannot be removed in the general case, it would be interesting to study their physical significance.

Returning to a more phenomenological note, we apply these results to electrodynamics, \( H_I = e^+ e^- \gamma \), where we choose \( b = e^+, c = e^- \) and \( a = \gamma \). We find

\[ q(e^+) = q(e^-) = q(e^+, e^-) = \pm 1, \]  

and

\[ q(\gamma) = q(e^+, \gamma) = q(e^-, \gamma) = 1. \]  

It is not surprising that although we can prove the symmetrization postulate for electrons and photons and the spin-statistics connection for photons, we cannot prove the spin-statistics connection for electrons since our formalism does not use local commutativity of observables (in the sense that observables commute at spacelike separation) and the spin of the fields did not enter our calculation. In other words, we have not specified whether we are studying electrodynamics with spinor or scalar electrons, both of which would be described by the schematic interaction Hamiltonian \( H_I \) given above.

In passing we discuss the corresponding constraints for a nonrelativistic theory. Since there is no crossing symmetry in a nonrelativistic theory, the fact that the process \( a \rightarrow bc \) is allowed does not imply that crossed processes like \( c \rightarrow b \bar{a} \) can occur;
thus the constraints are weaker. (Note that the process we consider here is not one of the crossed processes we discussed in the relativistic case.) Following the pattern of the arguments given above we find $q(a) = q(a, b)q(a, c)$, $q(a, b) = q(b)q(b, c)$, and $q(a, c) = q(b, c)q(c)$. In contrast to the relativistic case in which all diagonal $q$’s have magnitude one, in the nonrelativistic case only the weaker constraint $|q| \leq 1$ holds. For nonrelativistic electrodynamics with $a = b = e^-$ and $c = \gamma$, the constraints are $q(e^-) = q(e^-)q(e^-, \gamma)$ and $q(e^-, \gamma) = q(e^-, \gamma)q(\gamma)$. Together these imply $q(\gamma) = q(e^-, \gamma) = 1$. The electron $q$-parameter $q(e^-)$ is unconstrained.

We also note that the result $q(bound) = q_n^2$ for a bound state of $n$ quons in a nonrelativistic theory can be found using the technique just given. If the constituents, $a$, obey

$$a(k)a^\dagger(l) - qa^\dagger(l)a(k) = \delta(k, l)$$

then, since the transition $a^n \leftrightarrows b$ is allowed, the bound state, $b$, of $n$ a’s should obey

$$b(k)b^\dagger(l) - q^n b^\dagger(l)b(k) = \delta(k, l) + \text{nonleading terms},$$

$$a(k)b^\dagger(l) - q^n b^\dagger(l)a(k) = 0.$$  

Thus as found earlier we have $q(bound) = q_n^2$.

In this paper we studied the implications of conservation of statistics for theories governed by a generalized commutation relation that involves bilinears in the creation and annihilation operators. Bilinear relations are a natural type of algebra to characterize statistics since they include the bose and fermi cases and with the Green ansatz the parabose and parafermi cases as well as the quons. The demonstration that quons in relativistic theories must be either bosons or fermions is a step in the direction of understanding the experimental absence of statistics other than bose or fermi in three-dimensional space.

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