ARITHMETIC PROGRESSIONS IN THE TRACE OF
BROWNIAN MOTION IN SPACE

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Abstract. It is shown that the trace of 3 dimensional Brownian motion contains arithmetic progressions of length 5 and no arithmetic progressions of length 6 a.s.

1. Introduction

Given a set $S$ of Hausdorff dimension 1 in the Euclidean plane, 2 dimensional Brownian motion running for unit time will intersect $S$ in a set of Hausdorff dimension 1 as well, with positive probability, see e.g. [1]. This can be used to show that the trace of 2 dimensional Brownian motion a.s. contains arbitrarily long arithmetic progressions starting at the origin and having a fixed difference: look at the unit circle, with positive probability Brownian motion run for unit time intersects the unit circle in a set $S$ of dimension 1. To each point in $S_1$ add it to itself to get $S_2$ a set of dimension 1. Run Brownian motion for an additional unit time. It will be again intersected $S_2$ in a set of dimension 1. To each point in the intersection of the form $2x, x \in S_1$ add $x$ to get arithmetic progression of length 4 in the intersection of this set of dimension 1 and the Brownian motion, running for an additional unit time. By Harnack principle these events are independent up to a constant, their correlations are bounded away from 1. E.g. further condition the Brownian motion to be in a $1/10$ ball around the origin at integer times. Continue in the same manner to get arbitrarily long arithmetic progressions. Scale invariance implies that we arbitrarily long arithmetic progression with probability 1.

In this note we comment that a.s. the trace of 3 dimensional Brownian motion contains arithmetic progressions of length 5, and no arithmetic progressions of length 6.

Same calculation shows that the maximal arithmetic progression in the trace of Brownian motion in $\mathbb{R}^d$ is 3 for $d = 4, 5$ and 2 above that.

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The argument above shows that with positive probability the trace of a unit time two dimensional Brownian motion admits uncountably arithmetic progression of length 3 and difference 1.

Consider $n$ steps simple random walk on the $d$ dimensional square grid $\mathbb{Z}^d$, look at the number of arithmetic progressions of length 3 in the range, study the distribution and large deviations?

**Question:** In the LD regime, is there a deterministic limiting shape possibly a line?

Another natural way to relate arithmetic progressions to Brownian motion, is to look at arithmetic progressions in the time set, in which the Brownian motion is at the same place. Simple dimension heuristic suggests that for 1 dimensional Brownian motion there is a one dimensional set of length 3 arithmetic progressions in time with identical location at these times. Study this set. Are there length 4 arithmetic progressions with positive probability?

2. **Proof**

**Lemma 1.** A 3-dimensional Brownian motion contains no arithmetic progressions of length 6, a.s.

**Proof.** By scaling invariant we may restrict our attention to arithmetic progressions contained in the unit ball $B$, and to spacings at least $\delta$ for some $\delta > 0$. Denote the Brownian motion by $W$. If it contains an arithmetic progression then for every $\varepsilon$ one may find $x_1, \ldots, x_6 \in B \cap \varepsilon \mathbb{Z}^d$ such that $W \cap B(x_i, d\varepsilon) \neq \emptyset$ and such that $x_i$ form an $\varepsilon$-approximate arithmetic progressions, by which we mean that $|x_{i-1} + x_{i+1} - 2x_i| \leq 4d\varepsilon$ for $i = 2, 3, 4, 5$. Further, the $x_i$ are $\delta$-separated in the sense that $|x_i - x_{i+1}| \geq \delta - 2d\varepsilon$ for $i = 1, 2, 3, 4, 5$. Denote the set of such $x_i$ by $X$ and define

$$E(x) = \mathbb{1}\{W \cap B(x_i, \varepsilon) \neq \emptyset \forall i \in \{1, \ldots, 6\}\} \quad x = (x_1, \ldots, x_6)$$

$$X = X(\varepsilon) = \sum_{x \in X} \mathbb{1}\{E(x)\}.$$

We now claim that

$$\mathbb{E}(X) \leq C \quad \mathbb{E}(X^2) \geq c|\log \varepsilon|$$

(1)

where the constants $c$ and $C$ may depend on $\delta$. Both calculations are standard: the first (that of $\mathbb{E}(X)$), is an immediate corollary of the fact that 3$d$ Brownian motion hits a ball of radius $\varepsilon$ with probability $\approx \varepsilon$ and summing over the balls (series where one of the $x_i$ is close to the starting point, 0,
have a higher probability to satisfy $E(x)$, but there are less of them and a calculation shows that they are negligible). The calculation of $E(X^2)$ is similar, we write $E(X^2) = \sum_{x,y \in X} \mathbb{P}(E(x) \cap E(y))$ and estimate the probability directly. We get about constant contribution from every scale of $|x - y|$, hence $|\log \varepsilon|$.

We now make a somewhat stronger claim on the interaction between different $x$. We claim that there exists $\lambda > 0$ such that
$$
\mathbb{P}(E(x) \cap \{X \leq \lambda|\log \varepsilon|\}) \leq \frac{C}{|\log \varepsilon|} \mathbb{P}(x).
$$
To see this let, for each scale $k \in \{1, \ldots, |\log \varepsilon|\}$,
$$
X_k = \sum_{y} \{1(E(y)) : y \in \mathcal{X} \text{ s.t. } 2^k \varepsilon \leq |y_i - x_i| < 2^{k+1} \varepsilon \ \forall i \in \{1, \ldots, 6\}\}
$$
A simple calculation shows that $E(X_k|E_x) \geq c$ and $E(X_k^2|E_x) \leq C$ so
$$
\mathbb{P}(X_k > 0|E_x) \geq c.
$$
Further, the events $X_k > 0$ (still conditioned on $E_x$) are approximately independent in the following sense:
$$
\text{cov}(X_k > 0, X_l > 0|E_x) \leq 2e^{-c|k-l|}.
$$
This follows from Harnack’s inequality: assume for concreteness that $k < l$. Then the event $X_k$ depends only on what happens in the balls $B(x_i, 2^{k+1} \varepsilon)$ while the event $X_l$ depends only on what happens outside the balls $B(x_i, 2^l \varepsilon)$.

Applying Harnack’s inequality in each of the annuli $B(x_i, 2^{m+1} \varepsilon) \setminus B(x_i, 2^m \varepsilon)$ for $m \in \{k, \ldots, l - 1\}$ we see that the correlation decays exponentially, showing (3).

With (3) we can easily see (2), by summing over $k$ and using a second moment estimate for the variable $\#\{k : X_k > 0\}$ (in fact, it is not difficult to get a much better estimate than $C/|\log \varepsilon|$, an $\varepsilon^c$ is also possible. But we will not need it).

Summing (2) over all $x$ gives
$$
\mathbb{P}(X \in (0, \lambda|\log \varepsilon|]) \leq \frac{C}{|\log \varepsilon|}.
$$
This, with $E(X) \leq C$ shows that $\mathbb{P}(X > 0) \leq C/|\log \varepsilon|$, proving the lemma.

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**Lemma 2.** A 3-dimensional Brownian motion contains arithmetic progressions of length 5, a.s.
Proof. Let $\varepsilon$ and $X = X(\varepsilon)$ be as in the proof of the previous lemma (except we now fix $\delta$ to be, say, $\frac{1}{10}$). It is straightforward to calculate

$$\mathbb{E}(X(\varepsilon)) \geq \frac{c}{\varepsilon} \quad \mathbb{E}(X(\varepsilon)^2) \leq \frac{C}{\varepsilon^2}$$

which show that $\mathbb{P}(X(\varepsilon) > 0) \geq c$. A simple calculation shows that for some $\lambda > 0$ we have that $X(\lambda \varepsilon) > 0 \implies X(\varepsilon) > 0$. Hence $\{X(\lambda^k) > 0\}$ is a sequence of decreasing events with probabilities bounded below. This implies that

$$\mathbb{P}\left(\bigcap_k \{X(\lambda^k) > 0\}\right) > 0.$$

The event of the intersection can be described in words as follows: for every $k$ there exists $x_1^{(k)}, \ldots, x_6^{(k)} \in B$ which are $\frac{1}{10}$-separated and $\lambda^k$-approximate arithmetic progression such that $W \cap B(x_i^{(k)}, \lambda^k) \neq \emptyset$ for $i \in \{1, \ldots, 6\}$. Taking a subsequential limit we get $x_i^{(k_n)} \to x_i$ and these $x_i$ will be $\frac{1}{10}$-separated, will form an arithmetic progression, and will be on the path of $W$. So we conclude

$$\mathbb{P}(W \text{ contains a 5-term arithmetic progression in } B) > 0.$$  

Scaling invariance now shows that the probability is in fact $1$. \qed 

References

[1] P. Mörters, and Y. Peres, Brownian motion. Cambridge Series in Statistical and
Probabilistic Mathematics, 30. Cambridge University Press, Cambridge, 2010.
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