Quantum Tunneling from the Charged Non-Rotating BTZ Black Hole with GUP

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ABSTRACT: In the present paper, the quantum corrections to the temperature, entropy and specific heat capacity of the charged non-rotating BTZ black hole are studied by generalized uncertainty principle in tunneling formalism. It is shown that quantum corrected entropy would be of the form of predicted entropy in quantum gravity theories like string theory and loop quantum gravity.

KEYWORDS: Particles Tunneling, BTZ Black Hole, Generalized Uncertainty Principle
1 Introduction

In the decade of the seventies, Bekenstein presented some analogies between black hole physics and thermodynamics [1–3], and Hawking with collaborators introduced the four laws of black hole mechanics [4]. In the same decade, Hawking showed that black holes radiate thermally through quantum vacuum fluctuations near their event horizons [5, 6]. After such works, many efforts in theoretical physics had been accomplished to investigate thermodynamic properties of black holes. Despite many efforts, there exist numerous unsolved problems in the black holes physics; namely the microscopic origin of black holes entropy is not fully understood.

In the last half century, the quantum gravity problem has been one of the most important issues for theoretical physicists. In this route, existence of a minimal length and also a quantum corrected entropy,

\[ \hat{S} = S + \alpha \ln S + \beta S^{-1} + \gamma, \]  

have been proposed by quantum gravity candidate theories like string theory and loop quantum gravity [7–11]. In the equation (1.1), the coefficients \( \alpha, \beta \) and \( \gamma \) are parameters related to the theory.

In view of the foregoing, it is natural to ask how to examine a quantum gravity candidate theory. A simple answer would be that an eligible theory should have good descriptions, with some likely corrections, of physical objects such as black holes. Therefore, one may namely investigate the thermodynamic properties of black holes in such a theory. One approach to this end is the generalized uncertainty principle (GUP)\(^1\) given by [22]

\[ \Delta x \Delta p \geq \hbar (1 - \frac{\lambda l_p}{\hbar} \Delta p + \frac{\lambda^2 l_p^2}{\hbar^2} \Delta p^2), \]  

\(^1\)For recent works on GUP and related topics, see [12–21].
where $\lambda$ is a dimensionless positive parameter, $l_p = \sqrt{\frac{\hbar G}{c^4}} = \frac{M_p \sqrt{c^3}}{\ell^2} \approx 10^{-35} (m)$, $M_p = \sqrt{\frac{\hbar c}{G}}$ and $c$ are Planck length, Planck mass and velocity of light, respectively. In obtaining the equation (1.2) from that of [22], it is assumed that $<p> \sim 0$ and consequently $<p^2> \sim \Delta p^2$. From other point of view, due to the Planck length is proportional to Newton coupling constant, the quantum corrections in equation (1.2) may be considered as gravitational effects. Black hole physics and string theory suggest a GUP including a quadratic term in the momenta while doubly special relativity (DSR) suggests one that is linear in the momenta [23]. So, it is appropriate to use the most general form of GUP with linear and quadratic terms, this is the physical motivation for using the specific form of GUP in the equation (1.2). On the other hand, the Hawking radiation may be considered as quantum tunneling across the event horizon [24]. One approach to such an investigation is Hamilton-Jacobi method wherein it has been used from WKB approximation to determine the particles’s tunneling rate [25, 26]. In this point, it would be interesting to calculate the thermodynamic quantities of black holes in tunneling formalism with GUP. Recently, such an investigation has been highly regarded in the literature [27–34].

All above information give us motivation to use tunneling formalism with GUP and determine the thermodynamic properties of the charged non-rotating BTZ black hole. In the section 2, the quantum tunneling phenomena by WKB method in the near horizon of the charged non-rotating BTZ black hole is reviewed. In the section 3, the quantum corrections to thermodynamic quantities of the charged non-rotating BTZ black hole, due to the effects of GUP in tunneling formalism, are investigated. In the section 4, some conclusions and discussions are presented.

2 Charged non-rotating BTZ black hole

The Hilbert action in the 2+1 dimensional Einstein-Maxwell gravity with a negative cosmological constant is given by

$$S = \frac{1}{\pi} \int d^3x \sqrt{-g} \left( R - \frac{2\Lambda}{16G_3} - \frac{1}{4} F^2 \right), \quad (2.1)$$

where $G_3$, $\Lambda = -\frac{1}{L^2}$ and $F = dA$ are the Newton’s gravitational constant in 2+1 dimensions, the cosmological constant and the Maxwell strength tensor, respectively. The charged non-rotating BTZ black hole is a static solution to the above action. In units $G_3 = \frac{1}{8}$, the metric is given by [37–39]

$$ds^2 = -(\frac{r^2}{\ell^2} - M - \frac{Q^2}{2} \ln(\frac{r^2}{\ell^2}))dt^2 + \frac{1}{\frac{r^2}{\ell^2} - M - \frac{Q^2}{2} \ln(\frac{r^2}{\ell^2})} dr^2 + r^2 d\phi^2, \quad (2.2)$$

where $M$, $Q$ and $\ell$ are ADM mass, charge of the black hole and AdS radius, respectively. The corresponding non-vanishing component of the electromagnetic field is given by

$$A_t = -Q \ln(\frac{r}{\ell}). \quad (2.3)$$

The authors would like to mention that such an investigation for the BTZ black hole was presented in [35, 36] by an alternative approach.
The horizons are given by
\[ \frac{r_+^2}{\ell^2} - M - \frac{Q^2}{2} \ln \left( \frac{r_+^2}{\ell^2} \right) = 0, \] (2.4)
where \( r_+ \) and \( r_- \) present outer (event) and inner (Cauchy) horizon, respectively. The Hawking temperature and Entropy are, respectively, given by
\[ T_H = T_+ = \frac{\kappa_+}{2\pi} = \frac{1}{2\pi} \left( \frac{r_+}{\ell^2} - \frac{Q^2}{2r_+} \right), \] (2.5)
\[ S_+ = 4\pi r_+. \] (2.6)
In equation (2.5), \( \kappa_+ \) is the surface gravity at the outer horizon.

To study the black hole thermodynamics by Hamilton-Jacobi method, one needs to find the near horizon limit of black hole. So taking the near (outer) horizon limit, the near (outer) horizon metric is obtained as following
\[ ds_{NH}^2 = - \left( \frac{2r_+}{\ell^2} - \frac{Q^2}{r_+} \right) (r - r_+) \quad dt^2 + \left( \frac{2r_+}{\ell^2} - \frac{Q^2}{r_+} \right) (r - r_+)^{-1} \quad dr^2 + r_+^2 d\phi^2. \] (2.7)
Now, one may consider massive scalar particles propagating in the background of (2.7),
\[ - \frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g} g^{\mu\nu} \partial_{\nu}) + m^2 \Phi = 0, \] (2.8)
where \( m \) is the mass of scalar particles. Taking \( \Phi = \exp(iI(t,r,\phi)) \), the above massive Klein-Gordon equation in the background of (2.7) is simplified to
\[ - \frac{1}{(\frac{2r_+}{\ell^2} - \frac{Q^2}{r_+})(r - r_+)} (\partial t I)^2 + (\frac{2r_+}{\ell^2} - \frac{Q^2}{r_+})(r - r_+) (\partial r I)^2 + \frac{1}{r_+^2} (\partial \phi I)^2 + m^2 = 0. \] (2.9)
Taking the following ansatz
\[ I = -Et + W(r) + J_m\phi \] (2.10)
where \( J_m \) is the constant angular momentum of particles, one can easily obtain
\[ - \frac{1}{(\frac{2r_+}{\ell^2} - \frac{Q^2}{r_+})(r - r_+)} (-E)^2 + (\frac{2r_+}{\ell^2} - \frac{Q^2}{r_+})(r - r_+) \left( \frac{dW(r)}{dr} \right)^2 + \frac{1}{r_+^2} (J_m)^2 + m^2 = 0. \] (2.11)
Solving above equation leads to
\[ W(r) = \int \frac{\sqrt{-E^2 + (\frac{2r_+}{\ell^2} - \frac{Q^2}{r_+})(r - r_+) (\frac{J_m}{r_+^2})^2 + m^2}}{\left(\frac{2r_+}{\ell^2} - \frac{Q^2}{r_+}\right)(r - r_+)} dr = \frac{\pi i E}{\kappa_+}. \] (2.12)
The particles tunneling rate is given by \( \Gamma \sim e^{-2\text{Im}(I)} \), so in the present case it’s given by
\[ \Gamma \sim e^{-\frac{2\pi E}{\kappa_+}}. \] (2.13)
Now the Hawking temperature of the charged non-rotating BTZ black hole can be got comparing the equation (2.13) with Boltzmann distribution \( e^{-\frac{E}{T}} \), so the Hawking temperature would be
\[ T_H = \frac{\kappa_+}{2\pi} = \frac{1}{2\pi} \left( \frac{r_+}{\ell^2} - \frac{Q^2}{2r_+} \right). \] (2.14)
3 Quantum corrections to BTZ black hole thermodynamics

To investigate quantum corrections to the BTZ black hole thermodynamics, one may consider the GUP effects in the tunneling formalism. To this end, one can consider the GUP relation (1.2) and write it as follows

$$\Delta p \geq \frac{\hbar (\Delta x + \lambda \ell_p)}{2 \lambda^2 \ell_p^2} \left( 1 - \sqrt{1 - \frac{4 \lambda^2 \ell_p^2}{(\Delta x + \lambda \ell_p)^2}} \right).$$  \hfill (3.1)

Taking Taylor expansion around $\frac{\lambda \ell_p}{\Delta x}$, the inequation (3.1) would become as follows

$$\Delta p \geq \frac{\lambda \ell_p}{\Delta x} \left( 1 - \frac{2 \lambda^2 \ell_p^2}{\Delta x^2} \right) + \cdots$$  \hfill (3.2)

where $\hbar = 1$ has been chosen. Now using the saturated form of the uncertainty principle $E \Delta x \geq 1$, which follows from the saturated form of the Heisenberg uncertainty principle $\Delta x \Delta p \geq 1$ [28], in the inequation (3.2), one can get

$$E_{GUP} \geq E \left( 1 - \frac{2 \lambda^2 \ell_p^2}{\Delta x^2} \right)$$  \hfill (3.3)

where written up to the second order in $\ell_p$. Here $E$ is the energy of the tunneling particles and $E_{GUP}$ is the corrected energy of them. Like section 2, the tunneling rate of particles with energy $E_{GUP}$ reads

$$\Gamma \sim e^{-\frac{2 \pi E_{GUP}}{\kappa^+}}.$$  \hfill (3.4)

As discussed in section 2, comparing the equation (3.4) with the Boltzmann distribution $e^{-\frac{E}{T}}$, the quantum corrected Hawking temperature can be obtained

$$T_{GUP} = \frac{\kappa^+}{2 \pi} \left( 1 - \frac{\lambda \ell_p}{\Delta x} + \frac{2 \lambda^2 \ell_p^2}{\Delta x^2} \right)^{-1}.$$  \hfill (3.5)

Now it may be chosen $\Delta x = 2r_+$, by this fact that the uncertainty in the position of the particle near the black hole horizon is of order of the horizon radius [40]. Therefore, the quantum corrected Hawking temperature can be rewritten as

$$T_{GUP} = T_H \left( 1 - \frac{\lambda \ell_p}{2r_+} + \frac{\lambda^2 \ell_p^2}{2r_+^2} \right)^{-1},$$  \hfill (3.6)

or

$$T_{GUP} \simeq T_H \left( 1 + \frac{\lambda \ell_p}{2r_+} - \frac{\lambda^2 \ell_p^2}{2r_+^2} \right).$$  \hfill (3.7)

To determine the quantum corrected black hole entropy, one can use the first law of black hole thermodynamics given by

$$S_{GUP} = \int \frac{dM}{T_{GUP}}.$$  \hfill (3.8)
Putting (3.6) into (3.8) and carrying out the integral leads to

\[ S_{\text{GUP}} = 4\pi r_+ - 2\pi \lambda_p \ln(r_+) - \frac{2\pi \lambda^2 l_p^2}{r_+} + \text{constant}. \]  

(3.9)

The quantum corrected black hole entropy can be rewritten as

\[ S_{\text{GUP}} = S - 2\pi \lambda_p \ln\left(\frac{S}{4\pi}\right) - \frac{8\pi^2 \lambda^2 l_p^2}{S} + \text{constant}, \]  

(3.10)

or equivalently

\[ S_{\text{GUP}} = S - 2\pi \lambda_p \ln S - \frac{8\pi^2 \lambda^2 l_p^2}{S} + \text{constant}. \]  

(3.11)

As it is seen, the quantum corrected entropy of the BTZ black hole in tunneling formalism with GUP, (3.11), is of the form of the predicted entropy by string theory and loop quantum gravity, (1.1). The modified entropy (3.11), which has been obtained by the most general form of GUP (1.2), is the same with that of [9] which was obtained by GUP including just a quadratic term in the momenta; but two points should be noted. First, it is expected that all of GUPs lead to the same result, hence it depends on the objectives of the study and the problem. For example, in the present study, it is intended to work with the general form of GUP which is compatible with all well-known theories of quantum gravity, such as string theory, loop quantum gravity and DSR. Second, the expansion (3.2), which governs the result, leads to the same terms in both GUP approaches. So, it is acceptable that these two different versions of GUP present the same result.

The heat specific capacity at constant charge, \( C_Q \), is given by

\[ C_Q = T_H \left( \frac{\partial S}{\partial T_H} \right)_Q, \]  

(3.12)

calculating above equation leads to

\[ C_Q = \frac{8\pi^2 \ell^2 T_H}{1 + \frac{Q^2 \ell^2}{2r_+^2}}. \]  

(3.13)

Note that putting \( Q = 0 \), the equation (3.13) gives the specific heat capacity of non-charged non-rotating BTZ black hole. In analogy with classical case, the quantum corrected specific heat capacity at constant charge, \( (C_{\text{GUP}})_Q \), is given by

\[ (C_{\text{GUP}})_Q = T_{\text{GUP}} \left( \frac{\partial S_{\text{GUP}}}{\partial T_{\text{GUP}}} \right)_Q. \]  

(3.14)

Finally, doing some calculation and keeping correction terms up to second order in Planck length, the quantum corrections to the specific heat capacity at constant charge is obtained

\[ (C_{\text{GUP}})_Q \simeq \frac{1 - \left( \frac{\lambda_p}{2r_+} - \frac{\lambda^2 l_p^2}{2r_+^2} \right)}{1 - \left( \frac{\lambda_p}{2r_+} - \frac{3\lambda^2 l_p^2}{4r_+^2} \right) \varepsilon} C_Q, \]  

(3.15)
where $\varepsilon$ is given by

$$\varepsilon = \frac{1 - \frac{Q^2 \ell^2}{2r^2}}{1 + \frac{Q^2 \ell^2}{2r^2}}. \quad (3.16)$$

As expected, the quantum corrections are so small while $\frac{M}{\Delta x} \ll 1$.

4 Conclusions

Using quantum tunneling formalism in presence of generalized uncertainty principle, thermodynamics quantities of the charged non-rotating BTZ has been investigated. It has been shown that the black hole entropy is of the form of the predicted entropy by string theory and loop quantum gravity. It should be mentioned, since it is assumed that $\frac{M}{\Delta x} \ll 1$, thermodynamics of the black hole can not be studied for very small $r_+$. In that case, when the event horizon radius approaches zero, the quantum corrections become significant, however, there would exist a point such that the approximation (3.2) would not be valid anymore. As final note, It would be of interest to study the charged non-rotating BTZ black hole in the noncommutative spacetime via tunneling formalism with GUP. Such an investigation may lead to more correction terms to the thermodynamics of the charged non-rotating BTZ black hole.

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