KHOVANOV HOMOLOGY AND THE SLICE GENUS

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Abstract. We use Lee’s work on the Khovanov homology to define a knot invariant \(s\). We show that \(s(K)\) is a concordance invariant and that it provides a lower bound for the slice genus of \(K\). As a corollary, we give a purely combinatorial proof of the Milnor conjecture.

1. Introduction

In [6], Khovanov introduced an invariant of knots and links, now widely known as the Khovanov homology. This invariant takes the form of a graded homology theory \(Kh(L)\), whose graded Euler characteristic is the unnormalized Jones polynomial of \(L\). In [9], Lee showed that \(Kh(L)\) is naturally viewed as the \(E_2\) term of a spectral sequence which converges to \(\mathbb{Q} \oplus \mathbb{Q}\). In this paper, we use this spectral sequence to define a knot invariant \(s(K)\). The definition of \(s(K)\) was motivated by a similar invariant \(\tau(K)\) which is defined using knot Floer homology [15], [17]. In fact, the similarities between the two invariants extend far beyond their manner of definition.

Our main result is that the invariant \(s\) gives a lower bound for the slice genus:

Theorem 1. 
\[|s(K)| \leq 2g_s(K)\]
where \(g_s(K)\) denotes the slice genus.

In fact,

Theorem 2. The map \(s\) induces a homomorphism from \(\text{Conc}(S^3)\) to \(\mathbb{Z}\), where \(\text{Conc}(S^3)\) denotes the concordance group of knots in \(S^3\).

For alternating knots, \(s(K)\) does not provide any new information about \(g_s(K)\):

Theorem 3. If \(K\) is an alternating knot, then \(s(K)\) is equal to the classical knot signature \(\sigma(K)\).

There is, however, a class of knots for which \(s(K)\) gives much better — indeed, sharp — information. We say that a knot is positive if it admits a planar diagram with all positive crossings.

Theorem 4. If \(K\) is a positive knot, 
\[s(K) = 2g_s(K) = 2g(K)\]
where \(g(K)\) is the ordinary genus of \(K\).

As a corollary, we get a Khovanov homology proof of following result, which was first proved by Kronheimer and Mrowka using gauge theory [3]:

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Corollary 1. (The Milnor Conjecture) The slice genus of the \((p, q)\) torus knot is \((p - 1)(q - 1)/2\).

As the reader familiar with knot Floer homology will have already noted, the theorems above all hold with \(2\tau(K)\) in place of \(s(K)\). (See [15] for the first three, and [11] and [18] for the final one.) Indeed, the equality \(s(K) = 2\tau(K)\) holds in all cases for which the author knows the value of \(\tau(K)\). Based on these observations, we make the following (perhaps optimistic)

Conjecture. For any knot \(K \subset S^3\), \(s(K) = 2\tau(K)\).

Readers familiar with the Khovanov homology may also have observed that the notation \(s(K)\) has already been used by Bar-Natan [1] to describe an apparent knot invariant which appears in one of his “phenomenological conjectures.” This is no coincidence. Indeed, the author’s interest in the subject was first aroused by the observation that Bar-Natan’s \(s\) appeared to give a lower bound for the slice genus. Although we are unable to prove that the \(s(K)\) defined here is the same as that determined by Bar-Natan’s conjecture, we do give a fairly general condition (at least for small knots) under which the two agree.

The remainder of the paper is organized as follows. In section 2, we review the Khovanov complex and Lee’s construction of a spectral sequence from it. In section 3, we define \(s\) and show that it behaves nicely with respect to the structure of the concordance group. Section 4 is devoted to the proof of Theorem 1. In section 5, we prove Theorems 3 and 4 and discuss the relationship between \(s(K)\) and \(\tau(K)\) in more detail. Finally, section 6 contains proofs of some technical results establishing the invariance of Lee’s spectral sequence, which are needed in section 2.

Finally, we take this opportunity to fix two conventions which we will use throughout. First, we will always work with \(\mathbb{Q}\) coefficients. Although Khovanov’s complex can be defined with coefficients in \(\mathbb{Z}\), Lee’s theorem (Theorem 2.2) does not hold in this context. Second, we will often abuse our notation, letting \(L\) refer both to a planar diagram of a link and to the underlying link itself. The reader should have little trouble determining from context which meaning is intended.

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2. Review of Khovanov homology

In this section, we briefly recall the construction of the Khovanov complex [6] and Lee’s extension of it [9].

2.1. The cube of resolutions. Given a link diagram \(L\) with crossings labeled 1 through \(k\), we can form the cube of all possible resolutions of \(L\). This is a \(k\)-dimensional cube with its vertices and edges decorated by 1-manifolds and cobordisms between them. More specifically, each crossing of \(L\) can be resolved in two different ways, as illustrated in Figure 1. To each vertex \(v\) of the cube \([0, 1]^k\), we associate the planar diagram \(D_v\) obtained by resolving the \(i\)-th crossing of \(L\) according to the \(i\)-th coordinate of \(v\). Then \(D_v\) is a planar diagram without crossings, so it is a disjoint union of circles.

Let \(e\) be an edge of the cube. The coordinates of its two ends differ in one component — say the \(l\)-th. We call the end which has a 0 in this component the initial end, and denote it by \(v_e(0)\). The other end is called the terminal end, written \(v_e(1)\). We assign to \(e\) the cobordism \(S_e : D_{v_e(0)} \to D_{v_e(1)}\), which is a product cobordism except in a neighborhood of the \(l\)-th crossing, where it is the obvious saddle cobordism between the 0 and 1-resolutions.
The Khovanov complex is constructed by applying a 1 + 1 dimensional TQFT $A$ to the cube of resolutions. In other words, one replaces each vertex $v$ by the group $A(D_v)$, and each edge $e$ by the map $A(S_e)$. The underlying group of $CKh(L)$ is the direct sum of the groups $A(D_v)$ for all vertices $v$, and the differential on the summand $A(D_e)$ is a sum of the maps $A(S_e)$ for all edges $e$ which have $v$ as their initial end. More precisely, for $x \in A(D_v)$

$$d(x) = \sum_{i=1}^{c_0(v)} (-1)^{s(e_i)} A(S_{e_i}).$$

Here $c_0(v)$ is the number of crossings in $v$ which have a 0–resolution, and $e_i$ is the edge which corresponds to changing the $i$-th such crossing to a 1-resolution. The signs $(-1)^{s(e_i)}$ are chosen in such a way that $d^2 = 0$. (There are many different ways to do this, but it is easy to see that they all give rise to isomorphic chain complexes.) The homological grading of an element $x \in A(D_v)$ is defined to be $\text{gr}(v) = |v| - n_-$, where $|v|$ is the number of 1’s in the coordinates of $v$ and $n_-$ is the number of negative crossings in the diagram for $L$. Note that $d$ increases the homological grading by 1 — strictly speaking, the Khovanov homology is a cohomology theory!

2.2. Khovanov’s TQFT. We now give a more explicit description of the TQFT $A$. Let $V$ be a vector space spanned by two elements, $v_+$ and $v_-$. The vector space associated by $A$ to a single circle is defined to be $V$, so that if $D$ is a diagram composed of $n$ disjoint circles, $A(D) = V^\otimes n$. Thus we can think of $CKh(L)$ as being the vector space spanned by the space of “states” for $L$, where a state consists of a complete resolution of $L$, together with a labeling of each component of the resolution by either $v_+$ or $v_-$. The cobordisms $S_e$ come in two forms: either two circles can merge into one, or one can split into two. In the first case, $A(S_e)$ is given by a map $m: V^\otimes 2 \to V$, where the two factors in the tensor product correspond to the labels on the two circles that merge, and the copy of $V$ in the image corresponds to the label on the single resulting circle. Likewise, in the second case, $A(S_e)$ is given by a map $\Delta: V \to V^\otimes 2$. The formulas for these maps are

$$m(v_+ \otimes v_+) = v_+ \quad \quad \quad \Delta(v_+) = v_+ \otimes v_- + v_- \otimes v_+$$
$$m(v_+ \otimes v_-) = m(v_- \otimes v_+) = v_- \quad \quad \quad \Delta(v_-) = v_- \otimes v_-$$
$$m(v_- \otimes v_-) = 0.$$

For reference, we also record two other maps $\iota$ and $\epsilon$ used to define $A$. These maps are not needed at the moment, but they make an appearance in section 4 when we study cobordisms. Corresponding to the addition of a 0-handle (the birth of a circle in a diagram), there is a map $\iota: Q \to V$, and corresponding to the addition of a two handle (the death of a circle) there is a map $\epsilon: V \to Q$. These maps are given by

$$\epsilon(v_-) = 1 \quad \quad \quad \iota(1) = v_+$$
$$\epsilon(v_+) = 0.$$
\[ A \] is especially nice because it is a graded TQFT. We define a grading \( p \) on \( V \) by setting \( p(v_\pm) = \pm 1 \) and extend it to \( V^\otimes n \) by \( p(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = p(v_1) + p(v_2) + \ldots + p(v_n) \). Then it is easy to see that if \( v \) is a homogenous element of \( V^\otimes n \), \( p(S_\pm(v)) = p(v) - 1 \). Next, we define a grading \( q \) on \( \text{CKh}(L) \) by \( q(v) = p(v) + \text{gr}(v) + n_+ - n_- \), where \( n_\pm \) are the number of positive and negative crossings in the diagram \( L \). (The term \( n_+ - n_- \) is included so that the \( q \)-grading remains invariant for different diagrams of the same knot.) Then \( q(d(v)) = q(v) \), so \( \text{CKh}(L) \) splits into a direct sum of complexes, one for each \( q \)-grading. In fact, its graded Euler characteristic is the unnormalized Jones polynomial of \( L \), but we will not make use of this here.

In [8], Khovanov proves that the homology of \( \text{CKh}(L) \) (thought of as a bigraded group) is an invariant of the underlying link \( L \). We denote this homology group by \( K h(L) \).

2.3. Lee’s TQFT. In [9], Lee considers a similar construction, but with another TQFT \( \mathcal{A}' \) in place of \( A \). The underlying vector spaces for these two TQFT’s are the same, but the maps \( m' : V \otimes V \to V \) and \( \Delta' : V \to V \otimes V \) induced by cobordisms are slightly different. They are given by

\[
\begin{align*}
m'(v_+ \otimes v_-) &= m'(v_- \otimes v_+) = v_+ & m'(v_+ \otimes v_-) &= m'(v_- \otimes v_+) = v_+
\end{align*}
\]

\[
\begin{align*}
\Delta'(v_+) &= v_+ \otimes v_- + v_- \otimes v_+ \quad & \Delta'(v_-) &= v_- \otimes v_- + v_+ \otimes v_+.
\end{align*}
\]

(The maps \( \iota \) and \( \epsilon \) corresponding to the addition of 0 and 2-handles are the same as for \( A \).) We denote the resulting complex by \( \text{CKh}'(L) \) and its homology by \( K h'(L) \).

Using the obvious identification between the underlying groups of \( \text{CKh}(L) \) and \( \text{CKh}'(L) \), we can define a \( q \)-grading on the latter group as well. It is clear from equation 3 that this grading does not behave quite so well with respect to the differential \( d' \). Indeed, \( \Delta'(v_-) \) is not even homogenous. It is easy to see, however, that if \( v \in \text{CKh}'(L) \) is a homogenous element, then the \( q \)-grading of every monomial in \( d'(v) \) is greater than or equal to the \( q \)-grading of \( v \). In other words, the \( q \)-grading defines a filtration on the complex \( \text{CKh}'(L) \).

This fact leads to the following theorem, which is implicit in [9]:

**Theorem 2.1.** There is a spectral sequence with \( E_2 \) term \( K h(L) \) which converges to \( K h'(L) \). The \( E_2 \) and higher terms of this spectral sequence are invariants of the link \( L \).

The first part of the theorem is more or less immediate from the observations above. The filtration on \( \text{CKh}' \) gives rise to a spectral sequence converging to \( K h' \). The differential on its \( E_1 \) term is the part of \( d' \) which preserves (rather than raises) the \( q \)-grading. Comparing equations 2 and 3, we see that the \( E_1 \) term is the complex \( \text{CKh} \).

To prove the second statement, we check that the spectral sequence is invariant under the Reidemeister moves. Suppose \( L \) and \( \tilde{L} \) are two diagrams related by the \( i \)-th Reidemeister move. In [9], Lee defines maps \( \rho_i : \text{CKh}'(L) \to \text{CKh}'(\tilde{L}) \) which induce isomorphisms on homology. In section 6, we show that these maps induce isomorphisms on \( E_2 \) terms of spectral sequences, thus completing the proof of the theorem.

2.4. Calculation of \( K h' \). Lee’s second major result is that the homology group \( K h'(L) \) is surprisingly simple. To show this, she introduces a new basis \( \{ a, b \} \) for \( V \), where \( a = v_- + v_+ \) and \( b = v_- - v_+ \). With respect to this new basis, the maps \( m' \) and \( \Delta' \) are given by

\[
\begin{align*}
m'(a \otimes a) &= 2a & \Delta'(a) &= a \otimes a
\end{align*}
\]

\[
\begin{align*}
m'(a \otimes b) &= m'(b \otimes a) = 0 & \Delta'(b) &= b \otimes b
\end{align*}
\]

\[
\begin{align*}
m'(b \otimes b) &= -2b
\end{align*}
\]
and the maps $\epsilon'$ and $\iota'$ are given by

$$
\epsilon'(a) = \epsilon'(b) = 1 \quad \quad \quad \iota'(1) = (a - b)/2
$$

Using this basis, she proves

**Theorem 2.2.** (Theorem 5.1 of [9]) $Kh'(L)$ has rank $2^n$, where $n$ is the number of components of $L$.

Indeed, Lee exhibits an explicit bijection between the set of possible orientations for $L$ and a set of generators of $Kh'(L)$, which we refer to as **canonical generators**. This bijection may be described as follows. Given an orientation $o$ of $L$, let $D_o$ be the corresponding oriented resolution. We label the circles in $D_o$ with $a$'s and $b$'s according to the following rule. To each circle $C$ we assign a mod 2 invariant, which is the mod 2 number of circles in $D_o$ which separate it from infinity. (In other words, draw a ray in the plane from $C$ to infinity, and take the number of other times it intersects the other circles, mod 2.) To this number, we add 1 if $C$ has the standard (counterclockwise) orientation, and 0 if it does not. Label $C$ by $a$ if the resulting invariant is 0, and by $b$ if it is 1. We denote the resulting state by $s_o$.

The name “canonical generator” is justified by the following result, whose proof is given in section 6.

**Proposition 2.3.** Suppose $L$ and $\tilde{L}$ are related by the $i$-th Reidemeister move. Then an orientation $o$ on $L$ induces an orientation $\tilde{o}$ on $\tilde{L}$, and $\rho_i'(s_o)$ is a nonzero multiple of $[s_o]$.

We end this section with an elementary but important observation.

**Lemma 2.4.** (Coherent orientations) Suppose there is a region in the state diagram for $s_o$ containing exactly two segments, as shown in Figure 2. Then either the orientations of the two are the same and the labels are different (like part a of the figure) or the orientations are different and the labels are the same (like part b).

**Proof.** We consider three possible cases: either the two segments belong to the same circle in $D_o$, or they belong to two circles, one of which is contained inside the other, or they belong to two circles, neither of which is contained inside the other. In each case, it is easy to verify that the claim holds.

**Corollary 2.5.** If two circles in the state diagram for $s_o$ share a crossing, they have different labels.
3. Definition and Basic Properties of the Invariant

Let $K$ be a knot in $S^3$. By Theorems 2.1 and 2.2, we know that there is a spectral sequence associated to $K$ which converges to $\mathbf{Q} \oplus \mathbf{Q}$. This spectral sequence is a relatively complicated object, but we can extract some simpler invariants of $K$ from it. Let $s_{\text{max}}$ and $s_{\text{min}}$ (with $s_{\text{max}} \geq s_{\text{min}}$) be the $q$-gradings of the two surviving copies of $\mathbf{Q}$ which remain in the $E_\infty$ term of the spectral sequence. Like all $q$-gradings for a knot, $s_{\text{max}}$ and $s_{\text{min}}$ are odd integers. Since the isomorphism type of the spectral sequence is an invariant of $K$, $s_{\text{max}}$ and $s_{\text{min}}$ are invariants as well.

Before making this definition formal, we digress to establish some terminology related to filtrations. Suppose $C$ is a sequence of subcomplexes

$$0 = C_n \subset C_{n-1} \subset C_{n-2} \subset \cdots \subset C_0 = C.$$ 

To such a filtration, we associate a grading defined as follows: $x \in C$ has grading $i$ if and only if $x \in C_i$ but $x \notin C_{i-1}$. If $f : C \to C'$ is a map between two filtered chain complexes, we say that $f$ respects the filtration if $f(C_i) \subset C'_i$. More generally, we say that $f$ is a filtered map of degree $k$ if $f(C_i) \subset C'_{i+k}$.

A filtration $\{C_i\}$ on $C$ induces a filtration $\{S_i\}$ on $H_*(C)$ defined as follows: a class $[x] \in H_*(C)$ is in $S_i$ if and only if has a representative which is an element of $C_i$. If $f : C \to C'$ is a filtered chain map of degree $k$, then it is easy to see that the induced map $f_* : H_*(C) \to H_*(C')$ is also filtered of degree $k$.

A finite length filtration $\{C_i\}$ on $C$ induces a spectral sequence, which converges to the associated graded group of the induced filtration $\{S_i\}$. In other words, the group which survives at grading $i$ in the spectral sequence is naturally identified with the group $S_i/S_{i+1}$.

Let us denote by $s$ the grading on $Kh^i(K)$ induced by the $q$-grading on $CKh^i(K)$. Then the informal definition above is equivalent to

**Definition 3.1.**

$$s_{\text{min}}(K) = \min\{s(x) \mid x \in Kh^i(K), x \neq 0\}$$
$$s_{\text{max}}(K) = \max\{s(x) \mid x \in Kh^i(K), x \neq 0\}$$

Since $Kh$ of the unknot $U$ has rank two and is supported in $q$-gradings $\pm 1$, we have $s_{\text{max}}(U) = 1$, $s_{\text{min}}(U) = -1$.

Another proof that $s_{\text{max}}$ and $s_{\text{min}}$ are knot invariants could be given using

**Proposition 3.2.** The maps $\rho'_{i*}$ and $(\rho'_{i*})^{-1}$ both respect the induced filtration $s$ on $Kh^i$.

The proof may be found in section 4.

3.1. The invariant $s$. Our first task in this section is to prove

**Proposition 3.3.**

$$s_{\text{max}}(K) = s_{\text{min}}(K) + 2$$

which justifies

**Definition 3.4.**

$$s(K) = s_{\text{max}}(K) - 1 = s_{\text{min}}(K) + 1$$

Since $s_{\text{max}}$ and $s_{\text{min}}$ are odd, $s(K)$ is always an even integer.

Before proving the proposition, we need some preliminary results.
Lemma 3.5. Let \( n \) be the number of components of \( L \). There is a direct sum decomposition \( Kh'(L) \cong Kh'_o(L) \oplus Kh'_e(L) \), where \( Kh'_o(L) \) is generated by all states with \( q \)-grading congruent to \( 2 + n \mod 4 \), and \( Kh'_e(L) \) is generated by all states with \( q \)-grading congruent to \( n \mod 4 \). If \( o \) is an orientation on \( L \), then \( s_o + s_{\tau} \) is contained in one of the two summands, and \( s_o - s_{\tau} \) is contained in the other.

Proof. Following Lee [9], we write
\[
m' = m + \Phi_m \\
\Delta' = \Delta + \Phi_\Delta
\]
where \( m \) and \( \Delta \) preserve the \( q \)-grading and \( \Phi_m \) and \( \Phi_\Delta \) raise it by 4. This proves the first statement.

For the second, let \( \iota: CKh'(L) \to CKh'(L) \) be the map which acts by the identity on \( CKh'_o \) and by multiplication by \(-1\) on \( CKh'_e \). We claim that \( \iota(s_o) = \pm s_{\tau} \). To see this, we define a new grading on \( V \) with respect to which \( v_- \) has grading 0 and \( v_+ \) has grading 2. Let \( i: V \to V \) be given by \( i(v_-) = v_- \), \( i(v_+) = -v_+ \), so that \( i(a) = b \) and \( i(b) = a \). Then the induced map \( i^{\otimes n}: V^{\otimes n} \to V^{\otimes n} \) acts as the identity on elements whose new grading is congruent to 0 \mod 4 and as multiplication by \(-1\) on elements whose new grading is congruent to 2 \mod 4. The new grading differs from the \( q \)-grading on \( D_o \) by an overall shift, so
\[
\iota(s_o) = \pm i^{\otimes n}(s_o) = \pm s_{\tau}
\]
It follows that \( s_o + \iota(s_o) = s_o \pm s_{\tau} \) is contained in one summand, while \( s_o - \iota(s_o) = s_o \mp s_{\tau} \) is contained in the other. \( \square \)

Corollary 3.6.
\[
s(s_o) = s(s_{\tau}) = s_{\min}(K)
\]

Corollary 3.7. \( s_{\max}(K) > s_{\min}(K) \).

Proof. Since \( CKh'(K) \) decomposes as a direct sum, its affiliated spectral sequence decomposes too. The homology of each summand is \( \mathbb{Q} \), so each must account for one of the surviving terms in the spectral sequence. The two summands are supported in different \( q \)-gradings, so the surviving terms must have different \( q \)-gradings as well. \( \square \)

Lemma 3.8. For knots \( K_1, K_2 \), there is a short exact sequence
\[
0 \to Kh'(K_1 \# K_2) \overset{p_*}{\to} Kh'(K_1) \otimes Kh'(K_2) \overset{\partial}{\to} Kh'(K_1 \# K_2) \to 0
\]
The maps \( p_* \) and \( \partial \) are filtered of degree \(-1\).
Proof. Consider the diagram for $K_1 \# K_2$ shown in Figure 3. From it, we get a short exact sequence

$$0 \rightarrow CKh'(D_1) \{1\} \rightarrow CKh'(D_2) \xrightarrow{p} CKh'(D_3) \rightarrow 0$$

where $D_1$ and $D_2$ are both diagrams for $K_1 \# K_2$, and $D_3$ is a diagram for the disjoint union $K_1 \coprod K_2$. Since $Kh'(K_1 \# K_2)$ has rank two and $Kh'(K_1 \coprod K_2) \cong Kh'(K_1) \otimes Kh'(K_2)$ has rank four, the resulting long exact sequence must split, giving the short exact sequence of the lemma. It is clear that the maps $p_*$ and $\partial$ are filtered of some degree, which can be worked out by considering (for example) the case $K_1 = K_2 = U$. \hfill $\square$

Proof. (of Proposition 3.9) Consider the exact sequence of the previous lemma with $K_1 = K$ and $K_2$ the unknot. Denote the canonical generators of $K$ by $s_a$ and $s_b$, according to their label near the connected sum point, and the canonical generators of $U$ by $a$ and $b$. Without loss of generality, we may assume that $s(s_a - s_b) = s_{\text{max}}(K)$. From Figure 3, we see that $\partial((s_a - s_b) \otimes a) = s_a$. Since $\partial$ is a filtered map of degree $-1$, we conclude that

$$s((s_a - s_b) \otimes a) \leq s(s_a) + 1$$

$$s_{\text{max}}(K) - 1 \leq s_{\text{min}}(K) + 1$$

Since we already know that $s_{\text{max}}(K) \neq s_{\text{min}}(K)$, this gives the desired result. \hfill $\square$

3.2. Properties of $s$. We check that $s$ behaves nicely with respect to mirror image and connected sum.

**Proposition 3.9.** Let $\overline{K}$ be the mirror image of $K$. Then we have

$$s_{\text{max}}(\overline{K}) = -s_{\text{min}}(K)$$

$$s_{\text{min}}(\overline{K}) = -s_{\text{max}}(K)$$

$$s(\overline{K}) = -s(K)$$

Proof. Suppose that $C$ is a filtered complex with filtration $C = C_0 \supset C_1 \supset \ldots \supset C_n = \{0\}$. Then the dual complex $C^*$ has a filtration $\{0\} = C_0^* \subset C_1^* \subset \ldots \subset C_n^* = C^*$, where $C_{-i}^* = \{x \in C^* \mid \langle x, y \rangle = 0, \forall y \in C_i\}$.

To prove the proposition, we observe that the filtered complex $CKh'(\overline{K})$ is isomorphic to $(CKh'(K))^*$. Indeed, it is easy to see from equation 3.3 that there is an isomorphism

$$r: (V, m', \Delta') \rightarrow (V^*, \Delta'^*, m'^*)$$

which sends $v_\pm$ to $v_\pm^*$. Then if $v$ is a state of the diagram $K$, we define $R(v)$ to be state of $K$ obtained by applying $r$ all the labels of $v$. It is straightforward to check that the map $R : CKh'(\overline{K}) \rightarrow (CKh'(K))^*$ is the desired isomorphism. (Compare with section 7.3 of [6], where it is shown that $CKh(\overline{K}) \cong (CKh(K))^*$.)

We now appeal to the following general result, whose proof is left to the reader:

**Lemma 3.10.** If $C_1$ and $C_2$ are dual filtered complexes over a field, then their associated spectral sequences $E_1^n$ and $E_2^n$ are dual, in the sense that $E_1^n \cong (E_2^n)^*$.

Thus if the two surviving generators in $E_1^n$ have filtration gradings $s_{\text{min}}$ and $s_{\text{max}}$, the surviving generators in $E_2^n$ will have gradings $-s_{\text{max}}$ and $-s_{\text{min}}$. \hfill $\square$

**Proposition 3.11.**

$$s(K_1 \# K_2) = s(K_1) + s(K_2)$$
Proof. We use the short exact sequence of Lemma 3.8. Denote the canonical generators of $K_i$ by $s_i^0$ and $s_i^1$, according to their label near the connected sum point. It is not difficult to see that $Kh'(K_1 \# K_2)$ has a canonical generator $s_0$ which maps to $s_a \otimes s_b$ under $p_*$. Thus

$$s(s_0) - 1 \leq s(s_0^0 \otimes s_0^1)$$

$$s_{min}(K_1 \# K_2) - 1 \leq s_{min}(K_1) + s_{min}(K_2)$$

Applying the same argument to $\overline{K}_1$ and $\overline{K}_2$, and using the fact that $s_{min}(K) = -s_{max}(K)$, we see that

$$s_{max}(K_1 \# K_2) + 1 \geq s_{max}(K_1) + s_{max}(K_2)$$

$$s_{min}(K_1 \# K_2) + 3 \geq s_{min}(K_1) + s_{min}(K_2) + 4$$

Thus

$$s_{min}(K_1 \# K_2) = s_{min}(K_1) + s_{min}(K_1) + 1$$

$$s_{max}(K_1 \# K_2) = s_{max}(K_1) + s_{max}(K_1) - 1.$$

This proves the claim. \qed

4. Behavior under Cobordisms

Let $L_0$ and $L_1$ be two links in $\mathbb{R}^3$. An oriented cobordism from $L_0$ to $L_1$ is a smooth, oriented, compact, properly embedded surface $S \subset \mathbb{R}^3 \times [0,1]$ with $S \cap (\mathbb{R}^3 \times \{i\}) = L_i$. In this section, we define and study a map $\phi_S : Kh'(L_0) \to Kh'(L_1)$ induced by such a cobordism. Our construction follows section 6.3 of [2], where Khovanov describes a similar map for the homology theory $Kh$.

4.1. Elementary cobordisms. Following Khovanov, we decompose the cobordism $S$ into a series of elementary cobordisms, each represented by a single move from one planar diagram to another. (See [2] for a more detailed treatment of this material.) For $i \in [0,1]$, let

$$L_i = S \cap (\mathbb{R}^3 \times \{i\})$$

$$S_i = S \cap (\mathbb{R}^3 \times [0, i]).$$

After a small isotopy of $S$, we can assume that $L_i$ is a link in $\mathbb{R}^3$ for all but finitely many values of $i$. The orientation on $S$ restricts to an orientation on $S_i$, which in turn determines an orientation on $L_i$. We denote this orientation by $\alpha_i$. (Note that with this convention, $\alpha_0$ is the reverse of the orientation induced on $L_0$ by $S_i$.)

Next, we fix a projection $p : \mathbb{R}^3 \to \mathbb{R}^2$. After a further small isotopy of $S$, we can assume that $p$ defines a regular projection of $L_i$ for all but finitely many values of $i$, and that this set of special values is disjoint from the first set where $L$ failed to be a link. The isotopy type of the oriented planar diagram $L_i$ remains constant except when $L$ passes through one of the two types of special values, where it changes by some well-defined local move. Each of these moves corresponds to an elementary cobordism, so we can write the whole cobordism $S$ as a composition of elementary cobordisms.

The necessary moves may be subdivided into two types: Reidemeister moves and Morse moves. There is one Reidemeister-type move for each of the ordinary Reidemeister moves, as well as one for each of their inverses. These moves do not change the topology of the surface $S_i$. The Morse moves correspond to the addition of a 0, 1 or 2-handle to $S_i$. They are illustrated in Figure 4.
4.2. Induced Maps. Given a cobordism $S$ from $L_0$ to $L_1$, we want to assign to it an induced map $\phi_S : Kh'(L_0) \to Kh'(L_1)$ which respects the filtration on $Kh'$. In addition, we would like this assignment to be functorial, in the sense that if $S$ is the composition of two cobordisms $S_1$ and $S_2$, $\phi_S$ is the composition of $\phi_{S_1}$ and $\phi_{S_2}$. Thus, it suffices to consider the case when $S$ is an elementary cobordism.

Suppose that $S$ is an elementary cobordism corresponding to the $i$-th Reidemeister move or its inverse. Then we define $\phi_S$ to be $\rho^i_*$ or its inverse. By Proposition 3.2, this is a filtered map of degree 0. If $S$ is an elementary cobordism corresponding to a Morse move, then we take $\phi_S$ to be the map induced by $\psi : CKh'(L_0) \to CKh'(L_1)$, where $\psi$ is the result of applying the TQFT $A'$ to the corresponding map of cubes. In other words, if the move corresponds to the addition of a 0-handle or a 2-handle, we apply $\iota'$ or $\epsilon'$, respectively, to the summand at each vertex of the cube. If it corresponds to the addition of a 1-handle, we apply either $m'$ or $\Delta'$, depending on whether the move results in a merge or a split at the vertex in question. It is easy to see that $\phi_S$ is a filtered map of degree 1 for a 0– or 2-handle addition and degree $-1$ for a 1-handle.

In general, given a cobordism $S$, we decompose it as a union of elementary cobordisms: $S = S_1 \cup S_2 \ldots \cup S_k$ and define the induced morphism $\phi_S : Kh'(L_0) \to Kh'(L_1)$ to be the composition $\phi_{S_k} \circ \ldots \circ \phi_{S_1}$, which is a filtered map of degree $\chi(S)$. We expect that the map $\phi_S$ will depend only on the isotopy class of $S$ rel $\partial S$ (c.f [4], where an analogous result is proved for the Khovanov homology), but since we do not need this fact, we will not pursue it here.

4.3. Canonical generators. The maps $\phi_S$ behave nicely with respect to canonical generators.
Proposition 4.1. Suppose $S$ is an oriented cobordism from $L_0$ to $L_1$ which is weakly connected, in the sense that every component of $S$ has a boundary component in $L_0$. Then $\phi_S([s_{o_i}])$ is a nonzero multiple of $[s_{o_1}]$.

Remark: Some sort of connectedness hypothesis is clearly necessary for the proposition to hold. For example, if we take $S$ to be the union of a product cobordism and a trivially embedded sphere, the induced map on $Kk'$ is the zero map.

Proof. In fact, we will prove a slightly stronger statement. Suppose $i$ is a regular value for the cobordism $S$, so that $L_i$ is a link. We divide the components of $S_i$ into two sorts: those of the first type, which have a boundary component in $L_0$, and those of the second type, which do not. We say that an orientation $o$ on $S_i$ is permissible if it agrees with the orientation of $S$ on components of the first type. (Here and in what follows, we use $o_I$ to denote both a permissible orientation on $S_I$ and the orientation it induces on $L_i$.) We claim that

$$\phi_S([s_{o_i}]) = \sum_I a_I [s_{o_I}]$$

where $\{o_I\}$ runs over the set of permissible orientations on $S_i$ and each coefficient $a_I$ is nonzero. Note that the weak connectivity hypothesis implies that there is only one permissible orientation on $S_1$, so the proposition is implied by the claim.

To prove the claim, it suffices to check that if it holds for $S_i$, then it holds for $S_{i'}$ as well, where $S_{i'}$ is the composition of $S_i$ with a single elementary cobordism $S_e$. If this cobordism corresponds to a Reidemeister type move, this is a straightforward consequence of Proposition 2.8. Below, we check that it holds for each of the Morse-type moves as well.

0-Handle Move: In this case, $\phi_{S_i}(s_{o_I}) = s_{o_I} \otimes \frac{1}{2}(a - b)$, where the second factor in the tensor product refers to the labels on the newly created circle. $S_{i'}$ has a new component of the second type — namely, the disk bounded by the new circle — and $s_{o_I} \otimes a$ and $s_{o_I} \otimes b$ are the canonical generators corresponding to the two possible orientations on $S_{i'}$ which agree with $o_I$ on all components other than the new one.

1-Handle Move: Suppose that the orientation $o_I$ is actually the orientation $o_{i'}$ induced by $S_{i'}$. Then the two strands involved in the move have opposite orientations, so by Lemma 2.10, they must have the same label. Since

$$m'(a \otimes a) = 2a \quad \Delta'(a) = a \otimes a
$$

$$m'(b \otimes b) = -2b \quad \Delta'(b) = b \otimes b$$

we see that $\phi_{S_i}(s_{o_I})$ is a nonzero multiple of $s_{o_{i'}}$.

More generally, the orientation $o_I$ is either compatible with some orientation $o_e$ on $S_e$, or it is not. In the former case, the two strands involved in the move point in opposite directions and have the same label, and $\phi_{S_i}(s_{o_I})$ is a nonzero multiple of $s_{o_{i'}}$ where $o_{i'}$ is the orientation induced on $L_{i'}$ by $o_e$. In the latter case, the two strands point in the same direction and have different labels, so $\phi_{S_i}(s_{o_I}) = 0$.

Now we consider what happens to the components of $S_i$ during the move. If the move splits one component of $L_i$ into two components of $L_{i'}$, then the number and type of components of $S_i$ remains constant. In this case, the set of permissible orientations on $S_i$ is naturally identified with the set of permissible orientations on $S_{i'}$. There is always an orientation on $S_i$ compatible with $o_I$, and $\phi_{S_i}(s_{o_I})$ is a nonzero multiple of $s_{o_{i'}}$.

On the other hand, if the move merges two components of $L_i$ into one component of $L_{i'}$, there are several possibilities to consider. If the merge involves only a single component of $S_i$, the situation is like the one above: there is always an orientation on $S_e$ compatible with
some techniques which enable us to efficiently compute impossible to compute by hand for all but the smallest knots. In this section, we describe (of Theorem 2.) If $o_I$ extends to a permissible orientation $o'_I$ on $S_I$, $\phi_{S_i}(s_{o_I}) = s_{o'_I}$, while if it does not, $\phi_{S_i}(s_{o_I}) = 0$.

2-Handle Move: In this case, a permissible orientation $o_I$ on $S_i$ extends to a unique permissible orientation $o'_I$ on $S'_I$. Since $\epsilon'(a) = \epsilon'(b) = 1$, $\phi_{S_i}(s_{o_I}) = s_{o'_I}$. To prove the claim, it suffices to show that two permissible orientations on $S'_I$ cannot induce the same orientation on $L_{o_I}$. But if this were the case, $S_i$ would have a closed component, contradicting the hypothesis that $S$ is weakly connected.

\[\Box\]

Corollary 4.2. If $S$ is a connected cobordism between knots $K_0$ and $K_1$, then $\phi_S$ is an isomorphism.

Proof. Fix an orientation $o$ on $S$. Then \{s_{o_1}, s_{e_1}\} is a basis for $Kh'(K_1)$. Its image under $\phi_S$ is \{s_{k_1o_1}, s_{k_2e_1}\} $(k_1, k_2 \neq 0)$, which is a basis for $Kh'(K_2)$.

\[\Box\]

4.4. The slice genus. We can now prove the first two theorems from the introduction.

Proof. (of Theorem 1) Suppose $K \subset S^3$ bounds an oriented surface of genus $g$ in $B^4$. Then there is an orientable connected cobordism of Euler characteristic $-2g$ between $K$ and the unknot $U$ in $\mathbb{R}^3 \times [0, 1]$. Let $x \in Kh'(K) - \{0\}$ be a class for which $s(x)$ is maximal. Then $\phi_S(x)$ is a nonzero element of $Kh'(U)$. Now $\phi_S$ is a filtered map with filtered degree $-2g$, so

$$s(\phi_S(x)) \geq s(x) - 2g.$$  

On the other hand, $s_{\max}(U) = 1$, so

$$s(\phi_S(x)) \leq 1.$$  

It follows that $s(x) \leq 2g + 1$, so $s_{\max}(K) \leq 2g + 1$ and $s(K) \leq 2g$. To show that $s(K) \geq -2g$, we apply the same argument to $\overline{K}$ (which bounds a surface $S$ of genus $g$) and use the fact that $s(\overline{S}) = -s(K)$.

\[\Box\]

Proof. (of Theorem 2) If $K_1$ and $K_2$ are concordant, then $K_1 \# \overline{K_2}$ is slice, so

$$0 = s(K_1 \# \overline{K_2}) = s(K_1) - s(K_2).$$  

Thus $s$ gives a well-defined map from $\text{Conc}(S^3)$ to $\mathbb{Z}$. That this map is a homomorphism is immediate from Propositions 3.9 and 3.11

\[\Box\]

Corollary 4.3. Suppose $K_+$ and $K_-$ are knots that differ by a single crossing change — from a positive crossing in $K_+$ to a negative one in $K_-$. Then

$$s(K_-) \leq s(K_+) \leq s(K_-) + 1$$  

Proof. In [11], Livingston shows that this skein inequality holds for any knot invariant satisfying the properties of Theorems 1 and 2.

\[\Box\]

5. Computations and Relations with other Invariants

Although the invariant $s(K)$ is algorithmically computable from a diagram of $K$, it is impossible to compute by hand for all but the smallest knots. In this section, we describe some techniques which enable us to efficiently compute $s$. 
5.1. Using $Kh$. For many knots, it is a simple matter to compute $s(K)$ from the ordinary Khovanov homology $Kh(K)$. Although $Kh(K)$ is also hard to compute by hand, there are already a number of computer programs available for this purpose, including Bar-Natan’s pioneering program [1] and a more recent, faster program written by Shumakovitch [19].

In [1], Bar-Natan made the following observation, based on his computations of $Kh$ for knots with 10 and fewer crossings.

**Conjecture.** (Bar-Natan) The graded Poincare polynomial $P_{Kh}(K)$ of $Kh(K)$ has the form

$$P_{Kh}(K) = q^{s(K)}(q + q^{-1}) + (1 + tq^4)Q_{Kh}(K)$$

where $Q_{Kh}(K)$ is a polynomial with all positive coefficients.

In [9], Lee showed that this conjecture holds whenever her spectral sequence for $Kh'$ converges after the $E_2$ term. In this case, it is easy to see that the invariant $s(K)$ is equal to the exponent $s(K)$ which appears in Bar-Natan’s conjecture.

To see how widely applicable this condition is, we introduce the notion of the homological width of a knot.

**Definition 5.1.** If $K$ is a knot, let $\mu(K) = \{a - 2b | q^{a+b}$ is a monomial in $P_{Kh}(K)\}$. The width $W(K)$ is one more than the difference between the maximum and minimum elements of $\mu(K)$.

In other words, $W(K)$ is the number of diagonals in the convex hull of the support of $Kh(K)$.

**Proposition 5.2.** If $W(K) \leq 3$, then the spectral sequence for $Kh'(K)$ converges after the $E_2$ term, and our $s(K)$ is the same as Bar-Natan’s.

**Proof.** Suppose $W(K)$ has width $\leq 3$. Then if $x$ is an element of $Kh'(K)$ with $q$-grading $a$ and homological grading $b$, the minimum possible $q$-grading of an element with homological grading $b - 1$ is $a - 6$. Since the differential $d_n$ on the $E_n$ term of the spectral sequence lowers the $q$-grading by $4(n - 1)$, $d_n$ must be trivial for all $n \geq 3$. \[\square\]

Theorem 3 follows from this fact, since Lee has shown [10] that if $K$ is an alternating knot, then it has width two and Bar-Natan’s $s$ is equal to $\sigma(K)$.

The proposition also applies to many non-alternating knots. Indeed, using Shumakovitch’s tables and a computer, it is straightforward to check that there are only four knots with 13 or fewer crossings whose width is greater than three. Inspecting $Kh$ of these four exceptions, one sees that in each case, the spectral sequence must converge after the $E_2$ term. Thus for all knots with 13 or fewer crossings, the value of $s(K)$ agrees with the value of Bar-Natan’s $s$ tabulated in [1] and [19]. Below, we list those knots of 11 crossings or fewer for which $s(K) \neq \sigma(K)$. There are 22 such knots, and $|s(K)| > |\sigma(K)|$ (and thus provides a better bound on the slice genus) for precisely half of them.

| $K$  | $s(K)$ | $\sigma(K)$ | $K$  | $s(K)$ | $\sigma(K)$ | $K$  | $s(K)$ | $\sigma(K)$ | $K$  | $s(K)$ | $\sigma(K)$ |
|------|--------|-------------|------|--------|-------------|------|--------|-------------|------|--------|-------------|
| $9_{42}$ | 0      | 2           | $11_{n,9}$ | 6      | 4           | $11_{n,70}$ | 2     | 4     |
| $10_{132}$ | -2     | 0           | $11_{n,12}$ | 2      | 0           | $11_{n,77}$ | 8     | 6     |
| $10_{136}$ | 0      | 2           | $11_{n,19}$ | -2     | -4          | $11_{n,79}$ | 0     | 2     |
| $10_{139}$ | 8      | 6           | $11_{n,20}$ | 0      | -2          | $11_{n,92}$ | 0     | -2    |
| $10_{145}$ | -4     | -2          | $11_{n,24}$ | 0      | 2           | $11_{n,96}$ | 0     | 2     |
| $10_{152}$ | -8     | -6          | $11_{n,31}$ | 4      | 2           | $11_{n,138}$ | 0     | 2     |
| $10_{154}$ | 6      | 4           | $11_{n,38}$ | 0      | 2           | $11_{n,183}$ | 6     | 4     |
Knots with 10 or fewer crossings are labeled according to their numbering in Rolfsen, while those with 11 crossings use the Knotscape numbering. The values of the signature are taken from [2]. All of the knots in the table have a homological width of 3, which raises the following question: if $K$ has homological width 2 (i.e. is H-thin in the terminology of [7]), must $s(K) = \sigma(K)$?

5.2. **Positive knots.** If $K$ is a positive knot, $s(K)$ can be computed directly from the definition. To see this, consider a canonical generator $s_o$ for a positive diagram of $K$. Since each crossing of $K$ is positive, its oriented resolution is the 0-resolution. Thus the state $s_o$ lives in the extreme corner of the cube of resolutions: it has homological grading 0, and there are no generators in $CKh^+(K)$ with homological grading $-1$. It follows that the only class homologous to $s_o$ is $s_o$ itself, so

$$s_{\min}(K) = s([s_o]) = q(s_o)$$

To compute $q(s_o)$, we change back to the basis $\{v_-, v_+\}$. In the expansion of $s_o$ with respect to this basis, there is a unique state with minimal $q$-grading, namely, the state in which every circle of the oriented resolution is labeled with a $v_-$. If the positive diagram of $K$ has $n$ crossings, and its oriented resolution has $k$ circles, then

$$q(s_o) = p(s_o) + gr(s_o) + n_+ - n_- = -k + 0 + n - 0$$

so

$$s(K) = -k + n + 1$$

On the other hand, Seifert’s algorithm gives a Seifert surface $S$ for $K$ with euler characteristic $k - n$, so

$$2g(K) \leq 2g(S) = n - k + 1 = s(K) \leq 2g_*(K)$$

Since $g_*(K) \leq g(K)$, the inequalities above must all be equalities. This completes the proof of Theorem 4.

5.3. **Comparison with $\tau$.** We end this section by commenting on the conjecture relating $s$ and $\tau$ which was stated in the introduction. In addition to the fact that the two invariants share the properties of Theorems 1 through 4, there is a good deal of numerical evidence supporting the conjecture. Recently, a fair amount of work has been done on the problem of computing $\tau$ for knots with 10 and fewer crossings. Combining the results of [4], [11], [13], [14], and [15] with some unpublished computations of the author, it appears that the value of $\tau$ has been determined for all but two knots of 10 crossings and fewer. (The exceptions are 10_141 and 10_150.) For all of these knots, $s = 2\tau$. The equality can also be checked on certain special classes of knots, such as the pretzel knots of [16]. If the conjecture were true, it would make many computations in knot Floer homology easier. (For example, with our current technology, it seems like quite a laborious project to compute $\tau$ for all 11-crossing non-alternating knots.) Even if it is not true, we hope that the remarkable similarity between the two theories will have an enlightening explanation.

6. **Reidemeister Moves**

In this section, we prove the results involving Reidemeister moves which were stated in section 2 and 3.

Proof. (of Theorem 2.1) The proof that the desired spectral sequence exists was sketched in section 2. To prove its invariance, we use the following basic lemma, whose proof may be found in [12], Proposition 3.2.
Lemma 6.1. Suppose $F: C_1 \to C_2$ is a map of filtered complexes which respects the filtrations. Then $F$ induces maps of spectral sequences $F_n: E_1^n \to E_2^n$, and if $F_n$ is an isomorphism, $F_m$ is an isomorphism for all $m \geq n$.

In section 4 of [9], Lee proves the invariance of $Kh'$ by checking its invariance under the three Reidemeister moves. For each move, she exhibits a chain map between the complexes associated to the link diagram before and after the move. To prove the theorem, it suffices to check that these maps respect the $q$-filtration, and that they induce isomorphisms on the $E_2$ terms. The latter claim is straightforward, since in each case the induced maps on the $E_1$ terms are identical to the maps used in section 5 of [9] to prove invariance of $Kh$. Below, we sketch the proof of invariance for each move and explain why the maps in question respect the filtrations. For full details, we refer the reader to [6] and [9].

Reidemeister I Move: Let $\tilde{L}$ be the diagram $L$ with an additional left-hand curl added in. Then $CKh' (\tilde{L})$ can be decomposed as a direct sum $X_1 \oplus X_2$, where $X_2$ is acyclic and $X_1$ is isomorphic to $CKh' (L)$ via the map $\rho_1': CKh' (L) \to X_1$ illustrated in Figure 5. In terms of the basis $\{v_{\pm}\}$, we have

$$\rho_1'(v_-) = v_- \otimes v_- - v_+ \otimes v_+$$
$$\rho_1'(v_+) = v_+ \otimes v_- - v_- \otimes v_+$$

The corresponding map $\rho_1$ in [6] is given by

$$\rho_1(v_-) = v_- \otimes v_-$$
$$\rho_1(v_+) = v_+ \otimes v_- - v_- \otimes v_+$$

so $\rho_1'$ is filtration non-decreasing, and its induced map on $E_1$ terms is $\rho_1$.

Remark: There is another version of the first Reidemeister move, corresponding to the addition of a right-hand curl. Although it is not difficult to define an appropriate map $\rho_1''$, for this move directly, for the sake of brevity we adopt the solution of [1] and [9] and define

Figure 5. The Reidemeister I move and the map $\rho_1'$. 
it to be the composition of maps induced by an appropriate Reidemeister II move followed by a Reidemeister I move.

**Reidemeister II Move:** Let \( L \) and \( \tilde{L} \) be as shown in Figure 6. In this case, \( \text{CKh}'(\tilde{L}) \) can be decomposed as a direct sum \( X_1 \oplus X_2 \oplus X_3 \), where \( X_2 \) and \( X_3 \) are acyclic and there is an isomorphism \( \rho'_2 : \text{CKh}'(L) \to X_1 \), which is given by

\[
\rho'_2(z) = (-1)^{gr(z)}(z + \iota(d'_{01\to11}(z)))
\]

The maps \( \iota \) and \( d'_{01\to11} \) are shown in the figure. The isomorphism \( \rho_2 \) in Figure 6 has the same form, but with \( d_{01\to11} \) in place of \( d'_{01\to11} \). Since \( d - d' \) is strictly filtration increasing, it follows that \( \rho'_2 \) is filtration non-decreasing, and its induced map on \( E_1 \) terms is \( \rho_2 \).

**Reidemeister III Move:** Let \( L \) and \( \tilde{L} \) be as shown in Figure 7. Then there are direct sum decompositions

\[
\text{CKh}'(L) \cong X_1 \oplus X_2 \oplus X_3
\]

\[
\text{CKh}'(\tilde{L}) \cong \tilde{X}_1 \oplus \tilde{X}_2 \oplus \tilde{X}_3
\]
Figure 7. The Reidemeister III move. The relevant components of the differentials ($d'_{100 \to 110}$ and $d'_{010 \to 110}$) are marked in bold.
where $X_2, X_3, \tilde{X}_2$, and $\tilde{X}_3$ are acyclic and there is an isomorphism $\rho'_3: X_1 \to \tilde{X}_1$. To describe $X_1$ and $\tilde{X}_1$, we first define maps

$$\beta': CKh'(L(*100)) \to CKh'(L(*010))$$
$$\tilde{\beta}' : CKh'(\tilde{L}(010)) \to CKh'(\tilde{L}(100))$$

by

$$\beta' = l \circ d_{100 \to 110}$$
$$\tilde{\beta}' = l \circ d_{010 \to 110}$$

Then

$$X_1 = \{ x + \beta'(x) + y \mid x \in CKh'(L(*100)), y \in CKh'(L(*1))) \}$$

$$\tilde{X}_1 = \{ x + \tilde{\beta}'(x) + y \mid x \in CKh'(\tilde{L}(010)), y \in CKh'(\tilde{L}(1*)) \}$$

and

$$\rho'_3(x + \beta'(x) + y) = x + \tilde{\beta}'(x) + y.$$  

The isomorphism $\rho_3$ in (6) is defined similarly, except that it uses $d$ instead of $d'$ to define maps $\beta$ and $\beta'$. Since $d'$ does not increase the $q$-grading, we clearly have $q(\beta'(x)) \geq q(x)$. From this, it follows that $\rho'_3$ does not decrease the $q$-grading. Since $d - d'$ strictly increases the $q$-grading, the map induced on $E_1$ terms by $\rho'_3$ is equal to $\rho_3$. To finish the proof, we apply Lemma 6.1 three times: first to the inclusions $X_1 \hookrightarrow CKh'(L)$ and $\tilde{X}_1 \hookrightarrow CKh'(\tilde{L})$, and then to the map $\rho'_3$.

Proof. (of Proposition 2.3) We check the claim directly for each Reidemeister move:

Reidemeister I Move: In this case, it is easy to see that $\rho'_1(s_o) = s_o$.

Reidemeister II Move: Suppose that the two strands in $L$ point in the same direction. Then by Lemma 4.4, they have different labels, so $d_{01 \to 11}(s_o) = 0$. The oriented resolution of $\tilde{L}$ is contained in $CKh'(\tilde{L}(0*1)) \cong CKh'(L)$, so $\rho'_2(s_o) = (-1)^0(s_o) = s_o$.

Now suppose the two strands point in different directions, so that they have the same label. Let us assume for the moment that this label is $a$. Then we define $s_{ij} \in Kh'(\tilde{L}(ij))$ be the state which is identical to $s_o$ outside the area where the move takes place and has all components inside the area of the move labeled with an $a$. Then a direct computation shows that either

$$\rho'_2(s_o) = s_{01} + \frac{1}{2}(s_{10} - s_o)$$
$$= -\frac{1}{2}(s_o + d'(s_{00}))$$

if the two strands belong to the same component, or

$$\rho'_2(s_o) = s_{01} + (s_{10} - s_o)$$
$$= -(s_o + d'(s_{00}))$$

if they belong to different components. This proves the claim in the case where both strands are labeled with an $a$. We leave it to the reader to check that a similar argument applies when they are both labeled with a $b$.

Reidemeister III Move: Here there are three cases to consider. First, suppose that the two overlying strands in $L$ are oriented as shown in Figure 8. Then $s_o \in CKh'(L(*1))$, and it is easy to see that $\rho'_3(s_o) = s_o$.  

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Next, suppose that the three strands are oriented as shown in Figure 8b. Then \( s_\circ \in CKh'(L(\ast 100)) \) and \( \widetilde{s}_\circ \in CKh'(\tilde{L}(\ast 010)) \). Clearly \( \beta'(s_\circ) = \beta'(\widetilde{s}_\circ) = 0 \), so \( s_\circ \in X_1 \) and \( \widetilde{s}_\circ \in \tilde{X}_1 \). Again, it follows that \( \rho^{1'}(s_\circ) = \widetilde{s}_\circ \).

Finally, suppose the strands are oriented as shown in Figure 8c. In this case, the oriented resolution of \( L \) is in \( L(\ast 010) \), and the oriented resolution of \( \tilde{L} \) is in \( \tilde{L}(\ast 100) \). Inside the region under consideration, \( s_\circ \) looks like the state of Figure 8c (perhaps with \( a \)'s and \( b \)'s reversed.) Our first step is to exhibit some \( t_\circ \in X_1 \) which is homologous to \( s_\circ \). As before, we denote by \( s_{ijk} \) the unique state of \( L(\ast ijk) \) which is the same as \( s_\circ \) outside the area of the Reidemeister move and has all its components inside this area labeled by \( a \)'s.

Assume for the moment that all three strands shown in \( L(\ast 000) \) belong to different components. In this case, we can take

\[
t_\circ = s_\circ - 2s_{100} - s_{010} - 2s_{001} = s_\circ - d'(s_{000}).
\]

Indeed, \( \beta'(-2s_{100}) = s_\circ - s_{010} \) and \( s_{001} \in CKh'(L(\ast 1)) \), so \( t_\circ \in X_1 \). Then

\[
\rho^{1'}(t_\circ) = -2s_{010} - 2\beta'(s_{010}) - 2s_{001} = -2s_{010} - 2s_{100} + 2s_\circ - 2s_{001} = 2s_\circ - d'(s_{000})
\]

which proves the claim.

We leave it to the reader to check that a similar argument applies to each of the four other ways in which the segments outside the area of the move can be connected, as well as when the roles of \( a \) and \( b \) are reversed. In each case, it is not difficult to verify that \( \rho^{1'}([s_\circ]) \) is one of \( \pm [s_b], \pm 2[s_b] \), or \( \pm [s_b] \).

\[\square\]

Proof. (of Proposition 3.2) In the case of \( \rho^{1'} \) and \( \rho^{2'} \), the claim is immediate, since these maps are induced by filtered chain maps. For the others, we use the following
Lemma 6.2. Suppose \( f : C_1 \to C_2 \) is a map of filtered chain complexes with the property that the induced map of spectral sequences \( f_2 : E_2^1 \to E_2^2 \) is an isomorphism. Then \( f_*^{-1} \) is a filtered map with respect to the induced filtrations on \( H_*(C_1) \) and \( H_*(C_2) \).

Proof. Since \( f_2 \) is an isomorphism, \( f_\infty \) (the induced map on filtered gradeds) is as well. It follows that \( f_* \) is an isomorphism. Suppose \( f_*^{-1} \) does not respect the filtration. Then there must be some \( v \in H_*(C_1) \) whose filtration is strictly increased by \( f_* \). But this contradicts the fact that \( f_\infty \) is an isomorphism. □

The remaining cases now follow easily from the results used in the proof of Theorem 2.1. Indeed, \( \rho'_1 \) and \( \rho'_2 \) both induce isomorphisms of \( E_2 \) terms, and \( \rho'_3 = \tau_1 \circ \psi_* \circ \tau_2^{-1} \), where \( \tau_1, \tau_2, \) and \( \psi \) all induce isomorphisms of \( E_2 \) terms. □

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