Quantum Simulation of Topological Majorana Bound States and Their Universal Quantum Operations Using Charge-Qubit Arrays

Ting Mao and Z. D. Wang

Department of Physics, The University of Hong Kong, Hong Kong, China

(Dated: March 19, 2014)

PACS numbers: 03.67.Lx

Majorana bound states have been a focus of condensed matter research for their potential applications in topological quantum computation. Here we utilize two charge-qubit arrays to explicitly simulate a DIII class one-dimensional superconductor model where Majorana end states can appear. Combined with one braiding operation, universal single-qubits operations on a topological Majorana-based qubit can be implemented by a controllable inductive coupling between two charge qubits at the ends of the arrays. We further show that in a similar way, a controlled-NOT gate for two topological qubits can be simulated in four charge-qubit arrays.

INTRODUCTION

The experimental pursuit of Majorana bound states (MBSs) in one-dimensional (1D) systems has been brought into the limelight since the proposal of Kitaev’s toy lattice model [1]. Based on Kitaev’s original model, many experimental setups have been proposed, regarding to different systems such as solid state systems including a 1D semiconducting wire on an s-wave superconductor [2, 3], a 1D metallic wire on an unconventional superconductor [4] and 1D quantum Ising models when a Jordan-Wigner transformation is performed, such as an array of nonlinear cavities [5] and a superconducting circuit [6]. The non-Abelian statistics of MBSs is also demonstrated by using the braiding operations realized in, for instance, a T-shaped junction [3, 6] or a tunnel-coupled ancillary cavity [5]. However, braiding operations alone are not sufficient to perform universal topological quantum computation [7]. Some topologically unprotected operations have to be introduced to implement other single-qubit gates [8] and two-qubit gates, e.g., a typical one—controlled-NOT (CNOT) gate [9].

Superconducting-qubit circuits have been widely explored in quantum-state engineering and quantum computation due to its capability to control the state of a single qubit and the interqubit couplings [10]. In this paper, we will use an tunable inductively coupled charge-qubit array as the building block for topological qubits. With Jordan-Wigner transformation, we will explicitly map two charge-qubits arrays to the simplest model of the DIII class [11]. 1D topological superconductor, which is two copies of Kitaev’s 1D toy model with different spin species [11] [14]. Using the Kramers doublet ground states of this DIII class model as the basis states of a topological qubit, we will demonstrate that universal single-qubit operations can be achieved by inductively coupling two charge qubits at the ends of the arrays (complemented by the braiding operations), which is conducted in a topologically protected way. Furthermore, we will show that a topological CNOT gate can be realized in four charge-qubits arrays in the same manner. Thus, we may claim that this charge-qubit array system provides an experimentally feasible scheme to simulate universal topological quantum gates.

CHARGE-QUBIT ARRAY

We start with the building block of topological qubits—a charge-qubit array, as shown in Fig. 1. For each charge qubit, the superconducting island is connected to two nearest-neighbor coupling and non-nearest-neighbor coupling between qubits. As stated in Appendix, with a deliberate design of the inductive coils, all the non-nearest-neighbor coupling can be safely neglected. The Hamiltonian of the charge-qubit array reads

\[
H = \frac{-1}{2} \sum_{i=1}^{N} (B_x \sigma_i^x + B_y \sigma_i^y + B_z \sigma_i^z) \\
- \frac{L}{4} \sum_{i=1}^{N-1} (I_x \sigma_i^x (I_{i+1} \sigma_{i+1}^x + I_y \sigma_{i+1}^y)) (I_x \sigma_i^x + I_y \sigma_i^y) (I_{i+1} \sigma_{i+1}^x + I_y \sigma_{i+1}^y) \tag{1}
\]

where

\[
B_x = E_J \cos(\chi_l) + \cos(\chi_r - \phi_{ex}) \\
B_y = -E_J \sin(\chi_l) + \sin(\chi_r - \phi_{ex}) \\
B_z = 4E_J (2n_y - 1) \\
I_x = (\pi E_J / \Phi_0) \sin(\chi_l) - \sin(\chi_r - \phi_{ex}) \\
I_y = (\pi E_J / \Phi_0) \cos(\chi_l) - \cos(\chi_r - \phi_{ex}) \tag{2}
\]

For clarity of discussion, we have assumed that each charge qubit has the same parameters, the inductances of the coils are all equal to L and the Josephson junctions contained in the SQUID loops are identical (each with Josephson energy $E_J$ and capacitance $C_J$). The charging
adiabatically. We consider the charge regime near a charge degeneracy point as the eigenstates of the coupling term of Hamiltonian (1) hereafter.

Energy is \( E_c = e^2/(2C_g + 8C_J) \). The phase shifts \( \chi_{(r)} \) and \( \phi_{ex} \) are respectively determined by the fluxes through the left (right) SQUID loop inside the charge qubit \( \Phi_{(r)} \) and the external loop \( \Phi_{ex} \) as follows:

\[
\chi_{(r)} = \arctan \left( \frac{\sin(2\pi \Phi_{(r)}/\Phi_0)}{1 + \cos(2\pi \Phi_{(r)}/\Phi_0)} \right)
\]

and \( \phi_{ex} = 2\pi \Phi_{ex}/\Phi_0 \). The dimensionless gate charge is given as \( n_g = C_g V_g/2e \) when the applied fluxes change adiabatically. We consider the charge regime \( E_c \gg E_J \) and use the two lowest charge states \(| n = 0 \rangle, | n = 1 \rangle \) near a charge degeneracy point as the eigenstates of \( \sigma^z \).

The single qubit part of Hamiltonian \( (1) \) can be eliminated by adding some constraints on the controllable parameters \( n_g, \chi_{(r)} \) and \( \phi_{ex} \). Letting \( \chi_{(r)} - \phi_{ex} = \pi + \chi_{(r)} \), \( B_x \) and \( B_y \) go to zero. \( B_z \) vanishes when \( n_g \) is tuned to \( 1/2 \). With these constraints, we only need to consider the coupling term of Hamiltonian \( (1) \) hereafter.

**UNIVERSAL SINGLE-QUBIT OPERATIONS**

As schematically depicted in Fig. 2, we study two such charge-qubit arrays, which are respectively denoted as spin-up \( \uparrow \) and spin-down \( \downarrow \) for discussion convenience. In order to map these two arrays to a DIII class 1D topological superconductor protected by a \( T^2 = -1 \) time reversal symmetry (TRS), we carefully choose a zig-zag path used for the Jordan-Wigner transformation as shown in Fig. 2. The Jordan-Wigner transformation is described as \( f_{i,\uparrow} = 1/2 \prod_{j=1}^{N_i-1} (-\sigma_{i,j}^z)(\sigma_{i,j+1}^{+} - i\sigma_{i,j+1}^{-}) \) and \( f_{i,\downarrow} = 1/2 \prod_{j=1}^{N_i} (-\sigma_{i,j}^z)(\sigma_{i,j}^{+} - i\sigma_{i,j}^{-}) \). Now using the Jordan-Wigner transformation, followed by a \( U(1) \) gauge transformation of the forms \( c_{i,\uparrow} = e^{i\xi} f_{i,\uparrow} \) and \( c_{i,\downarrow} = e^{-i\xi} f_{i,\downarrow} \), we have the Hamiltonian of the simplest model of a DIII class topological superconductor:

\[
H = -t \sum_{s=\uparrow,\downarrow} \sum_{i=1}^{N-1} (c_{i,s}^\dagger c_{i+1,s} - c_{i,s}^\dagger c_{i+1,s})
\]

where \( c_i^\dagger \) and \( c_i \) are the creation and annihilation operators for Dirac fermions and \( t = L(nE_J/\Phi_0)^2 \) when \( \chi_{(r)} \) is tuned to \( \pi/4 \). The TRS can be seen if we define a pseudo
TRS in the spin-up and spin-down charge-qubit arrays as follows:

\[ T \sigma_{x,i} T^{-1} = -i \sigma_{x,i} \quad T \sigma_{y,i} T^{-1} = -i \sigma_{y,i} \quad T \sigma_{z,i} T^{-1} = \sigma_{z,i} \]

\[ T \sigma_{i} T^{-1} = i \sigma_{i} \quad T \sigma_{i} T^{-1} = i \sigma_{i} \quad T \sigma_{i} T^{-1} = -i \sigma_{i} \]

with \( T \sigma_i T^{-1} = -i \). Then the corresponding fermion operators should transform as:

\[ Tc_{i,\uparrow} T^{-1} = P_{y} P_{z} c_{i,\downarrow} \quad Tc_{i,\downarrow} T^{-1} = P_{y} P_{z} c_{i,\uparrow} \]

\[ Tc_{i,\downarrow} T^{-1} = -P_{y} P_{z} c_{i,\uparrow} \quad Tc_{i,\uparrow} T^{-1} = -P_{y} P_{z} c_{i,\downarrow} \]

with \( Tc_i T^{-1} = (-1)^i \). The corresponding fermion parity operators, shown as follows:

\[ \gamma_{i,\uparrow} = c_{i,\uparrow} + c_{i,\downarrow} \quad \gamma_{i,\downarrow} = i(c_{i,\uparrow} - c_{i,\downarrow}) \]

(6)

one can define two Dirac fermion operators \( d_{\uparrow} = 1/2(\gamma_{i,\uparrow} + i\gamma'_{i,\uparrow}) \) and \( d_{\downarrow} = 1/2(\gamma_{i,\downarrow} - i\gamma'_{i,\downarrow}) \) where \( \gamma_{i,s} \equiv \gamma_{i,s} \) are the MBBS at the ends of the chains. Then we have two orthogonal ground states of one Kitaev’s chain \( |0\rangle , |1\rangle \) which satisfy \( d_{\uparrow}|0\rangle = 0 \), \( \{1\rangle = \gamma_{i,\uparrow} \) and \( \gamma_{i,\downarrow} \) are odd fermion parity (e.g., when the number of charge quits in one array \( N \) is even). The four degenerate ground states of Eq. (4) can be denoted as: \( |0\rangle \) and \( |1\rangle \) which we use the convention that \( |0\rangle \) has an even and \( |1\rangle \) an odd fermion parity. It is noted that \( P_{y} P_{z} \) always takes value of \(-1\) in such subspace if we revisit Eq. (5), which guarantees the time reversal invariance of this model.

To perform universal single-qubit operations, we start with a commonly used one, i.e., the braiding operation. For example, in a T-shaped charge-qubit junction as shown in Fig. 2, with turning on/off the interqubit coupling \( t/2 \) in certain sequence \( [i, j, k, l] \), the positions of \( \gamma_{i,\uparrow} \) and \( \gamma_{i,\downarrow} \) can be exchanged, generating a unitary transformation \( U_{z}(\alpha) = \exp(i\alpha \tau_{z}) \) where \( \tau_{\mu} = x, y, z \) are the Pauli matrices in the qubit basis. We further note that the coupling term of two MBBS is commutative with the Hamiltonian since the MBBS do not enter this Hamiltonian. Thus it is natural to use such kind of coupling \( [i, j, k, l] \) to generate the unitary transformation of the qubit state. For instance, \( U_{z}(\alpha) = \exp(i\alpha \tau_{z}) \) can be performed by turning on the coupling \( i\lambda \gamma_{i,\uparrow} \) for a time span \( \Delta t \), where \( \alpha = \lambda \Delta t \). In principle, other transformations like \( U_{x} \) and \( U_{y} \) can also be achieved by observing that \( i\gamma_{i,\uparrow} \) and \( i\gamma_{i,\downarrow} \) correspond to \( \tau_{x} \) and \( \tau_{y} \) in the qubit basis respectively. However, only \( U_{y} \) can be practically realized in the charge-qubit arrays. It can be deduced that an experimentally realizable coupling, i.e., the inductive coupling between two charge qubits at the ends of the arrays, is subject to not only the coupling of MBBS but also the fermion parity operators, shown as follows:

\[ (\sigma_{1,\uparrow}^{x} + \sigma_{1,\downarrow}^{y})(\sigma_{1,\uparrow}^{y} + \sigma_{1,\downarrow}^{x}) = -2P_{y} P_{z} \gamma_{\uparrow} \sim \tau_{y} \]

\[ (\sigma_{1,\uparrow}^{y} + \sigma_{1,\downarrow}^{x})(\sigma_{1,\uparrow}^{x} + \sigma_{1,\downarrow}^{y}) = -2P_{y} P_{z} \gamma_{\downarrow} \sim \tau_{y} \]

\[ (\sigma_{1,\uparrow}^{y} + \sigma_{1,\downarrow}^{x})(\sigma_{1,\uparrow}^{x} + \sigma_{1,\downarrow}^{y}) = 2P_{y} P_{z} \gamma_{\uparrow} \sim \tau_{0} \]

where \( P_{y} P_{z} \) acts as \( \pm \tau_{z} \). Now from the Eq. (7), it is straightforward to see that \( U_{y} \) can be realized simply by inductively coupling the charge qubits \( C_{1,\uparrow} \) and \( C_{1,\downarrow} \). \( U_{0} \) can also be obtained in the same way which will be used in the implementation of CNOT gate. By a combination of the braiding operations \( U_{x}(\alpha) \) and \( U_{y} \), universal single-qubit operations can be achieved: \( U_{y}(\beta) U_{z}(\frac{\pi}{2}) U_{y}(\alpha) \).

A possible scheme to realize a controllable inductive coupling between two charge qubits \( C_{1,\uparrow} \) and \( C_{1,\downarrow} \) is to use a superconducting circuit containing one inductive coil (its inductance \( L_{c} \) should be chosen smaller than \( L \) in order to avoid the fusion of two MBBS \( \gamma_{\uparrow} \) and \( \gamma_{\downarrow} \)) and several superconducting switches as shown in Fig. 2. The ability of turning on/off the coupling is provided by the superconducting switches which allow switching between insulating and superconducting states in a very short switching time under the control of external field (e.g., \( L_{c} \)). We can see that this scheme is fully conducted in a topologically protected way.

**CONTROLLED-NOT GATE**

Among two-qubit gates, the CNOT gate is the most commonly used one and proved as an essential element, together with all single-qubit operations, forms
a universal set which is sufficient for any quantum computation [17]. Since we have already shown that one pair of charge-qubit arrays can be used as a topological qubit, it is natural to perform CNOT gate in two pairs (denoted by A and B) of the arrays, as depicted in Fig. 3. Following the same mapping as done before, we come to the basis states of two qubits \( B_1 = \{ \{ 0 \} A, \{ 0 \} B, \{ 1 \} A, \{ 1 \} B, \{ 10 \} A, \{ 10 \} B, \{ 01 \} A, \{ 01 \} B \} \). To realize the CNOT gate, it is conventional to firstly generate the unitary operation \( U_{yy}(\alpha) = \exp(i\alpha \sigma_y^A \tau_y^B) \) by turning on the Ising-type coupling \( \sigma_y^A \tau_y^B \) which, however, is not applicable in our system. Instead, our goal is to generate the unitary transformation \( U_{yy}(\alpha) = \exp(i\alpha \omega_0 \tau_y^A \tau_y^B) \) which acts on the space spanned by the extended basis \( B_8 = \{ B_4, \{ 00 \} A, \{ 00 \} B, \{ 11 \} A, \{ 11 \} B \} \) where we introduce the Pauli matrices \( \omega_{ij}(\mu = x, y, z) \) to represent the fermion parity space spanned by two ground states of Eq. [4] ( \( \{ 01 \} \) or \( \{ 10 \} \) ) and \( \omega_y \) is the unit matrix. Following the same scheme for single-qubit operations, we have two controllable couplings between qubit A and qubit B (as shown in Fig. 3):

\[
\sigma_{x,y,b}^A + \sigma_{y,b}^N \sigma_{x,y,b}^N = -2P_B \gamma_B \gamma_A \sim \omega_y \tau_y^A \tau_y^B
\]

\[
\sigma_{x,y,b}^A + \sigma_{y,b}^N \sigma_{x,y,b}^N = 2i \gamma_B \gamma_A \sim \omega_y \tau_y^A \tau_y^B
\]

where \( P_B \) is the coupling between the two charge qubits at the ends of the nearest-neighbor qubits, the inductive coupling becomes increasingly lower. Thus, in general case, for any two non-nearest-neighbor qubits, the \( L_e \) describing the strength of the coupling is quite small, so that the coupling is negligible. In the same way, for any two nearest-neighbor qubits, the effects of remaining circuits placed on them are also negligible. It should be also noted that in the controllable coupling circuit, \( L_e \) used is smaller than \( L \). Then the corresponding mutual inductance \( M_e \) should be adjusted to ensure that we only need to consider the coupling between the two charge qubits at the ends of arrays.

\[
H_c = -\frac{L}{4} (I_x \sigma_x^A + I_y \sigma_y^A)(I_x \sigma_x^B + I_y \sigma_y^B)
\]

Appendix

Here we only focus on the coupling part of the Hamiltonian and start with a simple case. When there are only two charge qubits in Fig. 1, following the result in [15], it is straightforward to write the coupling Hamiltonian as

\[
H_c = \frac{L}{4} x \sigma_x^A + I_y \sigma_y^A(I_x \sigma_x^B + I_y \sigma_y^B)
\]

with \( I_x, I_y \) defined as in [2]. When the number of the qubits is increased to 3, the loop in the middle can be approximately viewed as a parallel connection of two inductive coils with a mutual inductance \( M \) between them. In order to get a small value of the equivalent inductance, the two identical inductive coils are deliberately connected in an opposite configuration. We also make an applicable assumption that the values of \( M \) and \( L \) are very close. Then the equivalent inductance \( L_e = \frac{L^2-M^2}{2L+2M} \) would be considerably small. Under these configurations, it is easy to know that as the number of connected inductive coils increases, the value of \( L_e \) would be increasingly lower. Thus, in general case, for any two non-nearest-neighbor qubits, the \( L_e \) describing the strength of the coupling is quite small, so that the coupling is negligible. In the same way, for any two nearest-neighbor qubits, the effects of remaining circuits placed on them are also negligible. It should be also noted that in the controllable coupling circuit, \( L_e \) used is smaller than \( L \). Then the corresponding mutual inductance \( M_e \) should be adjusted to ensure that we only need to consider the coupling between the two charge qubits at the ends of arrays.

Conclusion and Acknowledgments

We have shown that universal single-qubit gates and the CNOT gate performed on topological Majorana-based qubits can be simulated in the charge-qubit array system by simply controlling the inductive coupling between charge qubits, owing to the design and control flexibilities of the superconducting circuit. This scheme can also be generalized to the multi-topological qubit case, which provides a possible route to realize topological quantum computation.

We would like to thank D. B. Zhang and Y. X. Zhao for helpful discussions. This work was supported by the GRF (HKU7058/11P&HKU7045/13P), the CRF (HKU/11G) of Hong Kong, and the URC fund of HKU.

[1] A.Y. Kitaev, Phys. Usp. 44, 131 (2001).
[2] R.M. Lutchyn, J.D. Sau, and S. Das Sarma, Phys. Rev. Lett. 105, 077001 (2010).
[3] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and P.A. Fisher, Nature Phys. 7, 412 (2011).
[4] S. Nakosai, J.C. Budich, Y. Tanaka, B. Trauzettel, and N. Nagaosa, Phys. Rev. Lett. 110, 117002 (2013).
[5] C.-E. Bardyn and A. Imamoğlu, Phys. Rev. Lett. 109, 253606 (2012).
[6] J.Q. You, Z.D. Wang, W.X. Zhang, and F. Nori, arXiv: 1108.3712 (2011).
[7] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
[8] T.L. Schmidt, A. Nunnenkamp, and C. Bruder, Phys. Rev. Lett. 110, 107006 (2013).
[9] Z.Y. Xue, Eur. Phys. J. D. 67, 89 (2013).
[10] Y. Makhlin, G. Schön, and A. Shnirman, Rev. Mod. Phys. 73, 357 (2001).
[11] A. Schnyder, S. Ryu, A. Furusaki, and A.W.W. Ludwig, Phys. Rev. B 78, 195125 (2008).
[12] A.Y. Kitaev, AIP Conf. Proc. 1134, 22 (2009).
[13] Y.X. Zhao and Z.D. Wang, arXiv: 1305. 3791 (2013).
[14] Z.C. Gu, arXiv: 1308. 2488 (2013).
[15] C. Hutter, Ph.D. thesis, Universität Karlsruhe, 2007.
[16] D. Gupta, W.R. Donaldson, K. Kortkamp, and A.M. Kadin, IEEE Trans. Appl. Supercond. 3, pp.2895-2898 (1993).
[17] A. Barenco, C.H. Bennett, R. Cleve, D.P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J.A. Smolin, and H. Weinfurter, Phys. Rev. A 52, 3457 (1995).
[18] R. Li, X.D. Hu and J.Q. You, Phys. Rev. B 86, 205306 (2012).