A General Memory-Bounded Learning Algorithm

Michal Moshkovitz
Edmond and Lily Safra Center for Brain Sciences
The Hebrew University
Jerusalem 91904, Israel

Naftali Tishby
The Rachel and Selim Benin School of Computer Science and Engineering
The Hebrew University
Jerusalem 91904, Israel

Abstract

In an era of big data there is a growing need for memory-bounded learning algorithms. In the last few years researchers have investigated what cannot be learned under memory constraints. In this paper we focus on the complementary question of what can be learned under memory constraints. We show that if a hypothesis class fulfills a combinatorial condition defined in this paper, there is a memory-bounded learning algorithm for this class. We prove that certain natural classes fulfill this combinatorial property and thus can be learned under memory constraints.

1. Introduction

We are amidst the big data era, which had made the design of learning algorithms that require limited amount of memory crucial. Many works (Shamir, 2014; Steinhardt et al., 2016; Raz, 2016; Kol et al., 2017; Moshkovitz and Moshkovitz, 2017a,b; Raz, 2017; Garg et al., 2017; Beame et al., 2017) have discussed the limitations of bounded-memory learning. In this paper we explore what can be properly learned with bounded-memory algorithms. Specifically, we suggest a general bounded-memory learning algorithm for the case where the examples are sampled from the uniform distribution. We also apply this algorithm to some natural hypothesis classes.

Our general algorithm is first applied to discrete threshold functions, \( \mathcal{H}_{TH} \), where the domain is a discretization of the segment \([0, 1] \) and each hypothesis \( h_\theta \) corresponds to a number \( \theta \in [0, 1] \) and \( h_\theta(x) = 1 \iff x \leq \theta \). There is a simple learning algorithm for this class: save in memory the largest example with label 1. As a sanity check, we show that indeed this class can be learned using our general algorithm. The second class we consider is equal-piece classifiers, \( \mathcal{H}_{EP,p} \), which is a generalization of the class \( \mathcal{H}_{TH} \) but now each hypothesis is defined by a few numbers \( \theta_1 < \ldots < \theta_k \) and \( h_{(\theta_j)}(x) = 1 \iff \exists \theta_i, x \in [\theta_i, \theta_i+p] \). It is unknown how to generalize the simple algorithm for \( \mathcal{H}_{TH} \) to the class \( \mathcal{H}_{EP,p} \); moreover, to the best of our knowledge it remains unclear how to properly learn this class with bounded-memory. However we show that our general algorithm is also applicable for this class.
The third class we consider is decision lists. This class was introduced by Rivest (1987), who also described a learning algorithm for this class. Unfortunately, it is not a bounded-memory algorithm because it saves all the given labeled examples in memory. Nevo and El-Yaniv (2002); Dhagat and Hellerstein (1994) provided learning algorithms for this class but the number of examples used is polynomial only under the assumption that the number of alternations in the decision list is constant. Klivans and Servedio (2006) presented an algorithm that uses a super-polynomial number of examples if the length of the decision list is linear in the number of variables. Long and Servedio (2007) also limited themselves to the uniformly distributed examples scenario but they considered improper learning of this class. Our general algorithm provides a proper bounded-memory learning algorithm.

1.1 Intuition for the General Algorithm
This paper suggests a general bounded-memory learning algorithm in the case where the examples are sampled from the uniform distribution. We also describe a combinatorial condition, called separability, that suffices for the correctness of the algorithm. All of our three applications are hypothesis classes that satisfy separability.

The general bounded-memory algorithm builds upon a basic general learning algorithm which we describe briefly. Let $\mathcal{H}$ be a family of Boolean hypotheses over domain $\mathcal{X}$. The fundamental theorem of statistical learning implies that learning with accuracy $\epsilon$ and constant confidence is possible after observing $m_{\mathcal{H},\epsilon}$ labeled examples and using $O(\log |\mathcal{X}| \cdot m_{\mathcal{H},\epsilon})$ memory bits by saving all $m_{\mathcal{H},\epsilon}$ examples in memory. This can be done by maintaining the version space; i.e., all hypotheses $T \subseteq \mathcal{H}$ that are consistent with the labeled examples seen so far, which means that $T$ is a (preferably small) set that contains the correct hypothesis. As long as there are hypotheses in the version space with an error larger than $\epsilon$, a counting argument implies that there is a large set of examples $S \subseteq \mathcal{X}$ that is able to reduce the size of the version space substantially. The basic learning algorithm that simply keeps the version space in memory uses a large amount of memory. Either one stores $|\mathcal{H}|$ bits in memory, indicating which hypothesis is in the version space, or one stores all the examples seen so far.

In this work we show that in many scenarios the basic learning algorithm can be implemented using bounded memory. The difficulty is that each example $x \in S$ can eliminate a different subset of the version space and thus all the $m_{\mathcal{H},\epsilon}$ labeled examples must be saved in memory. The rationale for the current work is that in many situations there is a large subset of examples that would eliminate the same large part of the version space. Thus, it suffices to only save a few bits of information about the example (indicating that it belongs to the subset $S$) rather than the whole example.

1.2 Informal Summary of our Results
The results are informally summarized below.

1. We introduce the combinatorial condition of separability for hypothesis classes.

2. We present a general memory-bounded proper learning algorithm in the case where the examples are sampled from the uniform distribution and the realizability assumption
holds. We prove the correctness of this algorithm in the case where the classes satisfy the separability condition.

3. We prove that discrete threshold functions satisfy separability.

4. We prove that equal-piece classifiers satisfy separability, and thus can be learned with a proper and bounded memory algorithm.

5. We prove that decision lists satisfy separability, and thus can be learned with a proper and bounded memory algorithm.

1.3 Organization

In Section 2 we formally present the notion of separability. In Section 3 we present the general bounded-memory algorithm and in Section 4 we show that this algorithm can be used to properly learn the three classes presented above with bounded-memory. The technical proofs are presented in the Appendix.

2. Separable Classes

In what follow we fix a bipartite graph \((A, B, E)\). The density between sets of vertices \(S \subseteq A\) and \(T \subseteq B\) is \(d(T, S) = \frac{e(T, S)}{|T||S|}\), where \(e(T, S)\) is the number of edges with a vertex in \(S\) and a vertex in \(T\). Two vertices \(h_1, h_2 \in A\) are \(\epsilon\)-close if \(|N(h_1) \Delta N(h_2)| \leq \epsilon|B|\), where \(N(h)\) denotes the set of neighbors of vertex \(h\). An \(\epsilon\)-ball with center \(h \in A\) is the set

\[B_h(\epsilon) = \{h' \in A \mid h' \text{ and } h \text{ are } \epsilon\text{-close}\}.

We next define the weak separability of a vertex subset and a whole graph.

**Definition 1** (weak-separable). Let \(T \subseteq A\). We say that \(T\) is \((\alpha, \epsilon)\)-weak-separable if for every vertex \(h \in A\) we have \(|T \cap B_h(\epsilon)| < \alpha|T|\).

We say that \(T\) is \(\alpha\)-separable if there are subsets \(S \subseteq B\) and \(T_0, T_1 \subseteq T\) with \(T_0 \cap T_1 = \emptyset\), \(|S| \geq \alpha|B|, |T_0| \geq \alpha|T|, |T_1| \geq \alpha|T|\) such that \(|d(T_0, S) - d(T_1, S)| \geq \alpha\).

**Definition 2** ((\(\alpha, \epsilon\))-separable graph). We say that a bipartite graph \((A, B, E)\) is \((\alpha, \epsilon)\)-separable if any \(T \subseteq A\) that is \((\alpha, \epsilon)\)-weak-separable is also \(\alpha\)-separable.

A hypothesis class \(\mathcal{H}\) over domain \(\mathcal{X}\) can be represented as a bipartite-graph in the following way. The vertices are the hypotheses \(\mathcal{H}\) and the examples \(\mathcal{X}\), and the edges connect every hypothesis \(h \in \mathcal{H}\) to the examples \(x \in \mathcal{X}\) if and only if \(h(x) = 1\). We call the appropriate bipartite graph the hypotheses graph of \(\mathcal{H}\).

**Definition 3** ((\(\alpha, \epsilon\))-separable class). A hypothesis class is \((\alpha, \epsilon)\)-separable if its hypotheses graph is \((\alpha, \epsilon)\)-separable.

3. A Bounded Memory Algorithm

An \((m, b, \delta, \epsilon)\)-bounded memory learning algorithm is one that uses at most \(m\) labeled examples sampled from the uniform distribution, \(b\) bits of memory, and returns a hypothesis that is \(\epsilon\)-close to the correct hypothesis with probability at least \(1 - \delta\).
In Algorithm 1 we present the general bounded-memory proper learning algorithm. The algorithm uses the following subroutines:

- Is-close\((h, \epsilon)\) — tests whether \(h\) is \(\epsilon\)-close to the correct hypothesis (Algorithm 2 in the Appendix)
- Estimate\((S, \tau)\) — estimates \(d(S, f)\) up to an additive error of \(\tau\), where \(f\) is the correct hypothesis (Algorithm 3 in the Appendix)

The correctness of the subroutines is stated in the following technical claims and proved in the Appendix.

**Claim 4.** Let \((A, B, E)\) be a bipartite graph. For any \(T \subseteq A\) that is \(\alpha\)-separable there are \(S \subseteq B\) with \(|S| \geq \alpha|B|\), \(T_0, T_1 \subseteq T\) with \(|T_0|, |T_1| \geq \frac{1}{4}\alpha^2|T|\) and \(d_0, d_1 \in \mathbb{R}\) with \(d_1 - d_0 \geq \frac{4}{9}|S|\) such that \(h \in T_0\) implies \(e(h, S) \leq d_0\) and \(h \in T_1\) implies \(e(h, S) \geq d_1\).

**Claim 5.** There is an algorithm such that for any hypothesis \(h\), for any \(\epsilon \in (0, 1)\) and for any integer \(k\) it uses \(k\) labeled examples and

- if \(h\) is \(\epsilon\)-close to the correct hypothesis, then with probability at least \(1 - 2e^{-2k\epsilon^2}\) the algorithm returns True.
- if \(h\) is not \(3\epsilon\)-close to the correct hypothesis, then with probability at least \(1 - 2e^{-2k\epsilon^2}\) the algorithm returns False.

**Claim 6.** Denote by \(f\) the correct hypothesis. There is an algorithm such that for any set of examples \(S \subseteq \mathcal{X}\) with \(|S| \geq \alpha|\mathcal{X}|\) for any \(\tau \in (0, 1)\) and for any integer \(k\) the algorithm uses \(\frac{2K}{\alpha}\) labeled examples and returns \(\hat{Y}\) with \(|\hat{Y} - d(f, S)| < \tau\), with probability at least \(1 - 2(e^{-k\alpha} + e^{-2k\epsilon^2})\).

The next theorem proves the correctness of the algorithm (the details appear in the Appendix); we will omit the \(O\) symbol for simplicity.

**Theorem 7.** Let \(\mathcal{H}\) be \((\alpha, \epsilon)\)-separable hypothesis class and denote \(s = \frac{\log |\mathcal{H}|}{\alpha^2}\). Then for any integer \(k\) there is a \((\frac{k}{\alpha} \cdot s, s + \log \frac{k}{\alpha}, (e^{-k\alpha^2/8} + e^{-2k\epsilon^2}) \cdot s, \epsilon)\)-bounded memory algorithm for \(\mathcal{H}\).

4. Applications

In this section we prove that Discrete Threshold Functions (in Section 4.1), Equal-Piece Classifiers (in Section 4.2), and Decision Lists (in Section 4.3) are separable. This implies, using Theorem 7, that they are properly learnable with bounded memory.

4.1 Threshold functions

The class of threshold functions in \([0, 1]\) is \(\{h_b : [0, 1] \rightarrow \{0, 1\} : b \in [0, 1]\}\) and \(h_b(x) = 1 \iff x \leq b\). The class of discrete thresholds \(\mathcal{H}_{TH,n}\) is defined similarly but over discrete domain \(\mathcal{X}\) of size \(n\) with \(\mathcal{X} = \{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\}\) and \(b \in \{\frac{1}{2n}, \frac{2}{2n}, \ldots, \frac{n-1}{2n}\}\).
Algorithm 1: General Bounded Memory Algorithm

Parameters: class $\mathcal{H}$, $\alpha, \epsilon > 0$, and an integer $k > 0$

$T = \mathcal{H}$

while $|T| > 1$ do

if there exists $h$ with $|T \cap B_h(\epsilon)| \geq \alpha |T|$ then

if Is-close($h, \epsilon$) returns True then

return $h$

else

$T = T \setminus B_h(\epsilon)$

end if

else

$r := \text{Estimate}(S, \alpha/4)$

find $S, T_1, T_0, d_0, d_1$ using Claim 4

if $r > \frac{d_1}{|S|} - \frac{\alpha}{4}$ then

$T := T \setminus T_0$

else

$T := T \setminus T_1$

end if

end if

end while

return $T$

Theorem 8. For any $0 < \alpha < 1/3$, the class $\mathcal{H}_{TH,n}$ is $(\alpha, \alpha)$-separable.

The proof of the theorem appears in the Appendix. Using Theorem 7, with $k = \Theta\left(\frac{\log \log n + \log 1/\alpha}{\alpha^2}\right)$ we can deduce the following corollary.

Corollary 9. For any $\epsilon \in (0, 1/3)$ there is a $\left(\frac{\log n \log \log n + \log n \log 1/\epsilon}{\epsilon^2}, \frac{\log n}{\epsilon^2}, 0.1, \epsilon\right)$-bounded memory learning algorithm for $\mathcal{H}_{TH,n}$.

4.2 Equal-Piece Classifiers

Each hypothesis in $h \in \mathcal{H}_{EP,p}$ corresponds to a (disjoint) union of intervals each of length exactly $p < 1$, that is, $\bigcup [a_i^h, a_i^h + p]$ and $h(x) = 1$ if and only if $x$ is inside one of these intervals. More formally, the examples are the numbers $X$ as defined in Section 4.1 and the hypotheses $h \in \mathcal{H}_{EP,p}$ correspond to the parameters $a_1^h, a_2^h, \ldots, a_k^h$ with $a_1^h + p < a_2^h, \ldots, a_{k-1}^h + p < a_k^h, a_k^h + p < 1$ and they define the intervals

$[a_1^h, a_1^h + p], [a_2^h, a_2^h + p], \ldots, [a_k^h, a_k^h + p]$.

An example $x \in X$ has $h(x) = 1$ if and only if there is $1 \leq i \leq k$ such that $x \in [a_i^h, a_i^h + p]$.

Note that the class $\mathcal{H}_{EP,p}$ is quite complex since it is easy to verify that it has a VC-Dimension of at least $1/p$ (partition $[0, 1]$ into $p$ consecutive equal parts and take one point from each part — this set is shattered by $\mathcal{H}_{EP,p}$).
Theorem 10. For any $\alpha, \epsilon \in (0, 1)$ with $\epsilon < 24/p$ and $\frac{2}{|\mathcal{X}|} < \alpha < \frac{\epsilon^2}{24}$ the class $\mathcal{H}_{EP,p}$ is $(\alpha, \epsilon)$-separable.

Proof. Fix $\frac{2}{|\mathcal{X}|} < \alpha < \frac{\epsilon^2}{24}$ and $T \subseteq \mathcal{H}_{EP,p}$ that is $(\alpha, \epsilon)$-weak-separable. To prove that $\mathcal{H}_{EP,p}$ is $(\alpha, \epsilon)$-separable, we will show that $T$ is $\alpha$-weak-separable.

Assume by contradiction that $T$ is not $\alpha$-weak-separable. We will show that there is a set $S \subseteq \mathcal{X}$ with $|S| \geq \alpha|\mathcal{X}|$ and $T_1, T_2 \subseteq \mathcal{H}$ with $|T_1|, |T_2| \geq \alpha|T|$ such that $d(S, T_1) = 1$ and $d(S, T_2) = 0$; thus, in particular $|d(S, T_1) - d(S, T_2)| = 1 > \alpha$ which will prove the claim. We will in fact prove that there is an open interval $I \subseteq [0, 1]$ of length $\alpha' := 2\alpha$ such that the sets $T_1 = \{h \in T| \exists k, I \subseteq [a_k^h, a_k^h + p]\}$ and $T_2 = \{h \in T| \forall k, I \cap [a_k^h, a_k^h + p] = \emptyset\}$ satisfying $|T_1|, |T_2| \geq 2\alpha|T|$. We call such a $I$ separating. This will prove that $\mathcal{H}_{EP,p}$ is $(\alpha, \epsilon)$-separable since for $S = I \cap \mathcal{X}$ we have that $|S| \geq \alpha|\mathcal{X}|$ (from the assumption in the claim regarding the upper bound on $\mathcal{X}$), $d(S, T_1) = 1$, and $d(S, T_2) = 0$. For ease of notation we replace $\alpha' = 2\alpha$ by $\alpha$ from now on.

Next, we will prove something even stronger by showing that there is a sequence $0 = u_0 \leq u_1 \leq \cdots$ and a sequence of sets $T = T^0 \supseteq T^1 \supseteq \cdots$ such that for all $i \geq 0$ if there is no separating $I$ in the “window”

$$W_i := [u_i, u_i + p - i\alpha]$$

and there is no separating $I$ in the previous windows either, then the following four properties are satisfied:

1. $|T^i| \geq (1 - 2i\alpha)|T|$
2. $u_i \geq ip - \alpha \sum_{j=1}^{i} j$
3. every $h_1, h_2 \in T^i$ is similar up to the current window:

$$|\{x \in \mathcal{X} \cap [0, u_i]|h_1(x) \neq h_2(x)\}| \leq \alpha \sum_{j=1}^{i+2} j|\mathcal{X}|$$

4. no hypothesis $h \in T^i$ has an endpoint in the current window: for all $h \in T^i$ and $k$ it holds that

$$a_k^h + p \notin [u_i, u_i + p - i\alpha].$$

Assume by contradiction that there is no separating $I$. Hence, we deduce that the four properties hold for all windows $W_1, W_2, \ldots$. By Property 2, for $\ell \leq \lceil \frac{2}{p} \rceil < \frac{3}{p}$ it holds that $u_\ell \geq 1$. Fix $h \in T^\ell$. By Property 1 we know that

$$|T^\ell| \geq |T| \left(1 - \frac{6\alpha}{p}\right)$$

(since $\alpha < p/18$)

$$> |T| \cdot \frac{2}{3}$$

$$> \alpha|T|$$
Take any \( h' \in T^i \). Since \( \mathcal{X} \subseteq [0, 1] \), Property 3 implies that \( h' \in B_h(\epsilon) \) (since \( \frac{12\alpha}{p} < \epsilon \)), which is a contradiction to weak-separability of \( T \).

To complete the proof we prove by induction on \( i \) that if there are no separating \( I \) in the windows up to \( W_i \) then there are \( T^i, u_i \) that have the previous four properties.

**Induction basis:** Since \( T^0 = T \) and \( u_0 = 0 \), Properties 1 and 2 hold. Since \( 0 \notin \mathcal{X} \), Property 3 holds. There is no endpoint in the interval \( W_0 = [0, p] \) (since the length of each interval in any hypothesis in \( \mathcal{H}_{EP, p} \) is \( p \)) and since \( 0 \notin \mathcal{X} \) we can assume that for each \( h \) and \( k \) it holds that \( a^h_k > 0 \) thus proving Property 4 holds.

**Induction step:** By the induction hypothesis there is no separating \( I \). To use it we intuitively move a small sliding-window \( I \) of length \( \alpha \) in the current window \( W_i \). For each such \( I \) we can calculate the number of hypotheses in \( T^i \) that contain \( I \),

\[
c_i^I = |\{h \in T^i \exists k, I \subseteq [a^h_k, a^{h+1}_k]\}|
\]

and the number of hypotheses in \( T^i \) that do not intersect \( I \),

\[
c^I_0 = |\{h \in T^i \forall k, I \cap [a^h_k, a^{h+1}_k] = \emptyset\}|
\]

Note that \( I \) is separating if and only if \( c_i^I \geq \alpha|T| \) and \( c^I_0 \geq \alpha|T| \). Thus, our assumption is that for every \( I \subseteq W_i \) either \( c_i^I < \alpha|T| \) or \( c^I_0 < \alpha|T| \). Observe that by Property 4 there are no endpoints of \( T^i \) in \( W_i \), which immediately implies that \( c_i^I \) can only increase as we slide \( I \) within \( W_i \). We next consider two cases depending on whether there is \( I \subseteq W_i \) with \( c_i^I \geq \alpha|T| \) or not. In each case we need to define \( u_{i+1}, T^{i+1} \) and prove that the four properties hold for \( i + 1 \) which will complete the proof.

- **Case 1:** there is no \( I \subseteq W_i \) with \( c_i^I \geq \alpha|T| \), i.e., \( c_i^I \) is always smaller than \( \alpha|T| \) as we slide \( I \). Define

\[
T^{i+1} = \{h \in T^i \forall I \subseteq W_i \forall k, I \notin [a^h_k, a^{h+1}_k]\} \quad \text{and} \quad u_{i+1} = u_i + |W_i|
\]

Property 1 holds since by definition of \( T^{i+1} \) and by Property 1 of the induction hypothesis

\[
|T^{i+1}| > |T^i| - \alpha|T| \geq (1 - 2\alpha)\alpha|T| - \alpha|T| \geq (1 - 2(i + 1)\alpha)|T|.
\]

From the definition of \( u_{i+1} \) and the induction hypothesis

\[
u_{i+1} = u_i + p - i\alpha \\
\geq ip - \alpha \sum_{j=1}^{i} j + p - i\alpha \\
\geq (i + 1)p - \alpha \sum_{j=1}^{i+1} j
\]

Property 2 holds. Before we prove that Properties 3 and 4 hold we prove the following auxiliary claim.
Claim 11. For each \( h \in T^{i+1} \) and \( x \in \mathcal{X} \cap [u_i, u_{i+1} - \alpha] \) it holds that \( h(x) = 0 \).

Proof. Assume by contradiction that there is \( h \in T^{i+1} \) and \( x \in \mathcal{X} \cap [u_i, u_{i+1} - \alpha] \) with \( h(x) = 1 \). This means that there is \( k \) with \( x \in [a_k^h, a_k^h + p] \), which implies that \( a_k^h \leq u_{i+1} - \alpha \) and \( a_k^h + p \geq u_i \). From Property 4 there is no end point in the current window \([u_i, u_{i+1}]\); hence \( a_k^h + p > u_{i+1} \), which is a contradiction to the definition of \( T^{i+1} \) with \( I = (u_{i+1} - \alpha, u_{i+1}) \subseteq [a_k^h, a_k^h + p] \). \( \square \)

To prove that Property 3 holds for \( \mathcal{W}_{i+1} \): we get that for each \( h_1, h_2 \in T^{i+1} \), by Claim 11 and the induction hypothesis we have that

\[
\begin{align*}
|\{x \in \mathcal{X} \cap [0, u_{i+1}] | h_1(x) \neq h_2(x)\}| &= |\{x \in \mathcal{X} \cap [0, u_i] | h_1(x) \neq h_2(x)\}| \\
&+ |\{x \in \mathcal{X} \cap (u_i, u_{i+1} - \alpha] | h_1(x) \neq h_2(x)\}| \\
&+ |\{x \in \mathcal{X} \cap (u_{i+1} - \alpha, u_{i+1}] | h_1(x) \neq h_2(x)\}| \\
&\leq \alpha \sum_{j=1}^{i+2} j|\mathcal{X}| + \alpha \sum_{j=1}^{i+3} j|\mathcal{X}|
\end{align*}
\]

To prove that Property 4 holds for \( \mathcal{W}_{i+1} \): if there is an endpoint in \( \mathcal{W}_{i+1} \) then, since the length of \( \mathcal{W}_{i+1} \) is smaller than \( p - \alpha \), its start point is before \( u_{i+1} - \alpha \). This contradicts Claim 11.

- **Case 2**: there is \( I \subseteq \mathcal{W}_i \) with \( c_I^f \geq \alpha|I| \). Since \( c_I^f \) increases as we slide \( I \), we focus on the first sliding-window \( I = (i_1, i_2) \subseteq \mathcal{W}_i \) such that \( c_I^f \geq \alpha|I| \). Since there is no separating \( I \) in \( \mathcal{W}_i \) we get that \( c_0^f < \alpha|I| \). There are again two cases depending on whether \( I \) is at the beginning of \( \mathcal{W}_i \) or not.

- **Case 2.1**: if \( i_1 = u_i \), we define

\[
T^{i+1} = \{ h \in T^i | \exists k, I \cap [a_k^h, a_k^h + p] \neq \emptyset \} \quad \text{and} \quad u_{i+1} = u_i + p + \alpha.
\]

To prove that Property 1 holds for \( \mathcal{W}_{i+1} \): follows from the induction hypothesis and the fact that \( T^{i+1} = T^i \setminus c_0^f \) and \( |c_0^f| < \alpha|I| \).

To prove that Property 2 holds for \( \mathcal{W}_{i+1} \): follows from the induction hypothesis and the definition of \( u_{i+1} \).

Before we prove that Properties 3 and 4 hold we need the following auxiliary claim.

**Claim 12.** For each \( h \in T^{i+1} \) and \( x \in \mathcal{X} \cap [u_i + \alpha, u_i + p - i\alpha] \) it holds that \( h(x) = 1 \).

Proof. For each \( h \in T^{i+1} \) there is \( k \) such that \( (u_i, u_i + \alpha) \cap [a_k^h, a_k^h + p] \neq \emptyset \). Hence \( a_k^h \leq u_i + \alpha \) and \( a_k^h + p \geq u_i \). Since there is no end point in the current window \( a_k^h + p \notin [u_i, u_i + |\mathcal{W}_i|] \) we have that \( a_k^h + p > u_i \). To sum up, we have \( [u_i + \alpha, u_i + p - i\alpha] \subseteq [a_k^h, a_k^h + p] \), which proves the claim. \( \square \)
To prove that Property 3 holds for $W_{i+1}$: note that $|\{x \in X \cap [0, u_{i+1}] | h_1(x) \neq h_2(x)\}|$ is equal to
\[
|\{x \in X \cap [0, u_i] | h_1(x) \neq h_2(x)\}| + |\{x \in X \cap (u_i, u_i + \alpha) | h_1(x) \neq h_2(x)\}| + |\{x \in X \cap [u_i + \alpha, u_i + p - i\alpha] | h_1(x) \neq h_2(x)\}| + |\{x \in X \cap (u_i + p - i\alpha, u_i + p + \alpha) | h_1(x) \neq h_2(x)\}| \\
\leq \alpha \sum_{j=1}^{i+2} j,
\]
where the inequality follows from the induction hypothesis and Claim 12.

To prove Property 4 holds for $W_{i+1}$: if there is an $h \in T^{i+1}$ with an end-point in
\[
[u_{i+1}, u_{i+1} + |W_{i+1}|] = [u_{i+1}, u_{i+1} + p - (i + 1)\alpha]
\]
then its start-point is in
\[
[u_{i+1} - p, u_{i+1} - (i + 1)\alpha] = [u_i + \alpha, u_i + p - i\alpha] \subseteq [u_i, u_i + |W_i|],
\]
which is a contradiction to Claim 12 and Property 4 for $W_i$.

- **Case 2.2:** If $i_1 > u_i$, we define
\[
T^{i+1} = \{h \in T^i | \exists k : a_k^h \in I\} \quad \text{and} \quad u_{i+1} = i_2 + p.
\]

To prove that Property 1 holds for $W_{i+1}$: Since $I$ is the first sliding window with $c_1^I \geq \alpha |T|$ there are at most $\alpha |T|$ of the hypotheses in $T^i$ that start before $I$. Since $c_0^I < \alpha |T|$ there are at most $\alpha |T|$ of the hypotheses in $T^i$ that start after $I$. In other words there are at least $1 - 2\alpha$ hypotheses that intersect $I$; i.e., $|T^{i+1}| \geq |T^i| - 2\alpha |T|$. By Property 1 for $W_i$ we have $|T^{i+1}| \geq (1 - 2(i + 1)\alpha) |T|$. To prove that Property 2 holds for $W_{i+1}$: simply note that $i_2 \geq u_i$.

Before we prove that Properties 3, 4 hold we need the following auxiliary claims.

**Claim 13.** For each $h \in T^{i+1}$ and $x \in X \cap [i_2, i_1 + p]$ it holds that $h(x) = 1$.

**Proof.** Since for each $h \in T^{i+1}$ it holds that that there is $k$ such that $a_k^h \in I$ then it holds that $i_1 \leq a_k^h \leq i_2$. Hence, $[i_2, i_1 + p] \subseteq [a_k^h, a_k^h + p]$. \(\square\)

**Claim 14.** For each $h \in T^{i+1}$ and $x \in X \cap (u_i, i_1)$ it holds that $h(x) = 0$.

**Proof.** From Property 4 for $W_i$ we know that there is no end point in the current window $[u_i, u_i + p - i\alpha]$ and specifically in $(u_i, i_2) \subseteq [u_i, u_i + p - i\alpha]$ (because $I = (i_1, i_2) \subseteq W_i = [u_i, u_i + p - i\alpha]$). Since for each $h \in T^{i+1}$ it holds that there is $k$ such that $a_k^h \in I = (i_1, i_2)$ then the claim follows. \(\square\)
To prove that Property 3 holds for $W_{i+1}$: take $h_1, h_2 \in T^{i+1}$ and note that

$$\left| \{ x \in X \cap [0, u_{i+1}] | h_1(x) \neq h_2(x) \} \right| = \left| \{ x \in X \cap [0, u_i] | h_1(x) \neq h_2(x) \} \right| + \left| \{ x \in X \cap (u_i, i_1) | h_1(x) \neq h_2(x) \} \right| + \left| \{ x \in X \cap [i_1, i_2) | h_1(x) \neq h_2(x) \} \right|$$

$$+ \left| \{ x \in X \cap [i_2, i_1 + p) | h_1(x) \neq h_2(x) \} \right| + \left| \{ x \in X \cap [i_1 + p, u_{i+1}] | h_1(x) \neq h_2(x) \} \right| \quad \text{for } i, i+1, i_1, i_2, i_1 + p, i_2 \in \mathbb{N}.$$ 

By the induction assumption the first term is at most $\sum_{j=1}^{i+2} j |X|$, from Claims 13, 14.

To prove Property 4 holds for $W_{i+1}$: note that if there is a hypothesis $h \in T^{i+1}$ with an end point in the window $W_{i+1} = [u_{i+1}, u_{i+1} + p - (i + 1)\alpha]$, its corresponding start point is in $[u_{i+1} - p, u_{i+1} - (i + 1)\alpha] = [i_2, (i_1 + \alpha) - (i + 1)\alpha + p] = [i_2, i_1 + p - i\alpha]$. By construction of $T^{i+1}$ there is no $h \in T^{i+1}$ with a start point in the interval $[i_2, i_1 + p]$.

Proof.

Using Theorem 7, with $k = \Theta \left( \max \{ \epsilon^{-2}, \alpha^{-2} \} \cdot \log \frac{\log |H_{EP,p}|}{\alpha} \right)$ we can deduce the following corollary.

**Corollary 15.** For any $\alpha, \epsilon \in (0, 1)$ with $\frac{2}{|X|} \alpha^{-1} \log |H_{EP,p}| \epsilon^{-1} > \frac{\alpha}{2}$ there is a learning algorithm for $\mathcal{H}_{EP,p}$ that is

$$(k \cdot \log \frac{|H_{EP,p}|}{\alpha^3}, \log \frac{|H_{EP,p}|}{\alpha^2}, 0, 1, \epsilon)$$-bounded memory.

### 4.3 Decision Lists

A decision list is a function defined over $n$ Boolean inputs of the following form:

$$\text{if } \ell_1 \text{ then } b_1 \text{ else if } \ell_2 \text{ then } b_2 \text{ else } \ldots \text{ if } \ell_k \text{ then } b_k \text{ else } b_{k+1},$$

where $\ell_1, \ldots, \ell_k$ are literals over the $n$ Boolean variables and $b_1, \ldots, b_{k+1}$ are bits in $\{0, 1\}$. We say that the $i$-th level in the last expression is the part "if $\ell_i$ then $b_i$" and the literal $\ell_i$ leads to the bit $b_i$. Given some assignment to the $n$ Boolean variables we say that the literal $\ell_i$ is true if it is true under this assignment. Note that there is no need to use the same variable twice in a decision list. In particular, we can assume without loss of generality that $k = n$. Denote the set of all decision lists over $n$ Boolean inputs by $\mathcal{H}_{DL,n}$.

**Theorem 16.** For any $\epsilon \in (2^{-n}, 1)$ the class $\mathcal{H}_{DL,n}$ is $(\min \{ \frac{1}{200n^3}, \epsilon \}, \epsilon)$-separable.

**Proof.** Fix $\epsilon \in (0, 1)$ and $T \subseteq \mathcal{H}_{DL,n}$ that is $(\min \{ \frac{1}{200n^3}, \epsilon \}, \epsilon)$-weak-separable. To prove that $\mathcal{H}_{DL,n}$ is $(\min \{ \frac{1}{200n^3}, \epsilon \}, \epsilon)$-separable we will show that $T$ is $\min \{ \frac{1}{200n^3}, \epsilon \}$-weak-separable. To show that we will find two literals $\ell^0, \ell^1$ and $T_0, T_1 \subseteq T$ with $|T_0|, |T_1| \geq \frac{1}{200n^3}$ that have the following properties ($\ast$):

1. For all hypotheses in $T_0$:
• $\ell^0$ leads to 0
• $\ell^0$ appears at level $i_0 \leq \log \frac{1}{\epsilon} + 1$
• $\ell^0$ is in a lower level than $\ell^1$

2. Similarly for all hypotheses in $T_1$:

• $\ell^1$ leads to 1
• $\ell^1$ appear at level $i_1 \leq \log \frac{1}{\epsilon} + 1$
• $\ell^1$ is in a lower level than $\ell^0$

3. There is a level $j \leq \min\{i_0, i_1\}$ and a bit $b \in \{0, 1\}$ such that all hypotheses in $T_0 \cup T_1$

(a) are identical up to level $j$
(b) leads to the same value $b$ in levels $j + 1$ to $\max\{i_0, i_1\} - 1$

Note that for any decision list permuting consecutive literals that all lead to the same bit creates an equivalent decision list; thus, when we write “identical decision lists”, we mean identical up to this kind of permutation.

The correctness of the last three properties (⋆) will finish the proof since we can take $S$ to consist of all the assignments where the literals $\ell^0$ and $\ell^1$ are true. In this case it holds that $|S| \geq |\mathcal{X}|/4$, the disjoint subsets $T_0, T_1$ are large (i.e., $|T_0|, |T_1| \geq \frac{\epsilon}{200n^*}$). To bound $|d(T_1, S) - d(T_0, S)|$ from below, we partition $S$ into two parts: $S_1$ all assignments such that at least one of the literals $\ell_1, \ldots, \ell_j$ are true (recall that level $j$ is defined in Item 3), and $S_2 = S \setminus S_1$. Assume without loss of generality that bit $b$ defined in Item 3b is equal to 0. Note that

$$|d(T_1, S) - d(T_0, S)| = \sum_{a \in S} \frac{e(T_1, a)}{|T_1||S|} - \frac{e(T_0, a)}{|T_0||S|}$$

$$= \sum_{a \in S_1} \frac{e(T_1, a)}{|T_1||S|} - \frac{e(T_0, a)}{|T_0||S|} + \sum_{a \in S_2} \frac{e(T_1, a)}{|T_1||S|} - \frac{e(T_0, a)}{|T_0||S|}$$

$$= \sum_{a \in S_2} \frac{e(T_1, a)}{|T_1||S|} - \frac{e(T_0, a)}{|T_0||S|}$$

$$\geq 2^{-\max\{i_0, i_1\} + 1} \geq \epsilon,$$

where the third equality follows from Item 3a, the fourth equality follows from Item 3b, and the first inequality follows from Item 2 since for each assignment that is false in all literals that appear before level $i_1$ we have that $e(T_1, a) = |T_1|$ and these assignments constitute a fraction $2^{-(i_1 - 1)}$ out of the assignments in $S$. The last inequality follows from Items 1, 2.

To prove that there are literals $\ell^0, \ell^1$ and subsets $T_0, T_1$ as desired in (⋆), we will prove by induction on level $i$ that if there are not literals $\ell^0, \ell^1$ up to level $i$, then there is a subset
$T^i \subseteq T$ with $|T^i| \geq (1 - \frac{1}{4n^2})^i|T|$, a bit $b \in \{0, 1\}$ and $j \leq i$ such that for all hypotheses in $T^i$

(\dagger\dagger)

- are identical up to level $j$
- all literals in levels $j + 1$ to $i$ lead to the same value $b$

Proving (\dagger\dagger) will finish the proof because if we do not find $\ell^0, \ell^1$ up to level $i \leq \log \frac{1}{\epsilon} \leq n$ then we have that

$$|T^i| \geq \left(1 - \frac{1}{4n^2}\right)^i |T| \geq \left(1 - \frac{1}{4n^2}\right)^n |T| \geq \left(1 - \frac{1}{4n}\right) |T|, \quad (\dagger\dagger\dagger)$$

where in the second inequality we used the fact that $i \leq n$ and in the third inequality we used the fact that for any natural number $n$, the inequality $(1 - x)^n \geq 1 - nx$ is true. Thus, we have at least $(1 - \frac{1}{4n}) |T|$ hypotheses in $T^i \subseteq T$ that are $\epsilon$-close as we explain next, which is a contradiction to the assumption that $T$ is weak-separable.

Take the decision list $h$ which is exactly the same as all hypotheses in $T^i$ up to level $j$ and it returns $b$ afterwards. We will prove that all the hypotheses in $T^i$ are $\epsilon$-close to $h$. Take any hypothesis $h' \in T^i$. All examples that cause one of the literals up until level $i$ to be true have the same label as $h$. Thus, these hypotheses agree on $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \geq 1 - \epsilon$ fraction of the examples.

We will prove that (\dagger\dagger) is true by induction at level $i$. The basis $i = 0$ is vacuously true. If we do not find $\ell^0, \ell^1$ at level $i + 1$ and the induction hypothesis holds for any $j \leq i$ then first recall that $i \leq \log \frac{1}{\epsilon}$.

To continue, there are a few cases depending on the number of literals in level $i + 1$ that lead to $\bar{b}$, where $\bar{b}$ denotes the opposite value of the bit $b$ (i.e., $\bar{0} = 1, \bar{1} = 0$).

**Case 1:** If for at least $(1 - \frac{1}{8n^2})|T^i|$ of the hypotheses in $T^i$ the literal in level $i + 1$ leads to $\bar{b}$ (i.e., the same bit $b$ as in level $i$) then define $T^{i+1}$ as $T^i$ minus all the $(1 - \frac{1}{8n^2})|T^i|$ hypotheses that the literal in level $i + 1$ does not lead to the bit $b$. The induction claim will follow since $1 - \frac{1}{8n^2} \geq 1 - \frac{1}{4n^2}$.

**Case 2:** If there are at least $\frac{1}{8n^2}|T^i|$ hypotheses in $T^i$ such that the literal in level $i + 1$ leads to $\bar{b}$. Since there are $2n$ literals, there is a literal $\ell$ that is at level $i + 1$ in at least $\frac{1}{16n^3}|T^i|$ of the hypotheses in $T^i$ and lead to $\bar{b}$. Denote this set of hypotheses by $W$.

Call a literal *useful* if in at least $\frac{1}{16n^3}|T^i|$ of the hypotheses in $T^i$ it appears in levels $j + 1$ to $i + 1$ and leads to $b$. Note that there must be at least $i - j$ useful literals (using Claim 17 with $p = |T^i|, m = i - j$, and the fact that $\frac{|T^i|}{2n} \geq \frac{|T^i|}{16n^3}$). Again there are a few cases depending on whether there are $i - j$ useful literals or more.

**Case 2.1:** If there are more than $i - j$ useful literals then there must be a useful literal $\ell'$ that appear in at most $(1 - \frac{1}{3n})|W|$ of the hypotheses in $W$ at a level smaller than $i$ (see Claim 18 with $W$ of size $p = |W|$ and $m = i - j$). In this case we will show how to choose $\ell^0, \ell^1, T_0, T_1$ that will fulfill (**). Pick $\ell^0 = \ell, \ell^0 = \ell'$ and $T_b$ the $\frac{1}{16n^3}|T^i|$ hypotheses that
make the literal $\ell$ useful and $T_b$ the hypotheses where $\ell'$ does not appear before $\ell$. Notice that

$$|T_b| \geq \frac{1}{3n} |W| = \frac{1}{48n^2} |T^i| \geq \frac{1}{48n^4} \left(1 - \frac{1}{4n}\right) |T| \geq \frac{1}{200n^4} |T|,$$

where the second inequality follows from $(\star \star \star)$.

**Case 2.2:** If there are exactly $i-j$ useful literals we will define $T^{i+1}$ by removing all hypotheses in $T^i$ that are one of the following types

- contains a literal that is not useful at some level from $j$ up to $i$.
- contains a literal that leads to $b$ at level $i+1$.

To prove the induction claim we need to prove that $T^{i+1}$ is large. If there are $\frac{1}{16n^2} |T^i|$ hypotheses that the literal in level $i+1$ leads to $b$, then we would have that there are more than $i-j$ useful literals (see Claim 17). Thus,

$$|T^{i+1}| \geq \left(1 - \frac{1}{16n^2} \cdot 2n - \frac{1}{16n^3} \cdot 2n\right) |T^i| \geq \left(1 - \frac{1}{4n^2}\right) |T^i|.$$

Note that the hypotheses in $T^{i+1}$ all contain the same $i-j$ useful literals in levels $j$ to $i$. Hence all of the hypotheses are identical (up to permutation) from levels $j$ to $i$. From the induction hypothesis we get that all the hypotheses are identical up to level $i$. Note also that for all hypotheses in $T^{i+1}$, the literal in level $i+1$ leads to the same value $\bar{b}$. Hence, we proved the induction hypothesis.

**Claim 17.** For any matrix $A$ of size $m \times p$ where each cell in $A$ is an integer in $[n]$ and each integer in $[n]$ does not appear in $A$ more than $p$, then there must be at least $m$ integers in $[n]$ that appear at least $\frac{p}{2n}$ times in $A$.

**Proof.** Assume by contradiction that there are at most $m-1$ integers in $[n]$ that appear at least $\frac{p}{2n}$ times in $A$. By the assumption in the claim, each of these integers can appear at most $p$ times in $A$. All of the other $n-(m-1)$ integers appear at most $\frac{p}{2n}$ times in $A$. Thus, we have that all the numbers occupy at most

$$(m-1)p + (n-(m-1)) \frac{p}{2n} = \left(m - 1 + \frac{n-m+1}{2n}\right) p < mp,$$

which is a contradiction to the assumption that only integers in $[n]$ appear in the $mp$ cells of $A$.

**Claim 18.** For any matrix $A$ of size $m \times p$ where each cell in $A$ is in an integer in $[n]$, $m \leq n$ then the number of integers in $[n]$ that appear at least $\left(1 - \frac{1}{3n}\right)p$ times in $A$ is at most $m$.

**Proof.** Assume by contradiction that there are at least $m+1$ integers in $[n]$ where each appear at least $\left(1 - \frac{1}{3n}\right)p$ times in $A$. Then these integers cover at least $(m+1)\left(1 - \frac{1}{3n}\right)p > mp$ cells in $A$ which is a contradiction to the size of $A$. 


We can deduce the following corollary using Theorem 7, with \( k = \Theta \left( \frac{\log n + \log \frac{1}{\epsilon}}{\epsilon^3} \right) \) and by noticing that
\[
|\mathcal{H}_{DL,n}| \leq n! \cdot 4^n \Rightarrow \log |\mathcal{H}_{DL,n}| \leq 2n^2 \log n
\]
(the first inequality is true since at each level we need to choose a variable that was not used before, to decide whether it will appear with a negation or not and if true, whether it will lead to 0 or 1).

Corollary 19. For any \( n \geq 2 \) and \( \epsilon \in (2^{-n}, \frac{1}{200n^4}) \) there is a learning algorithm for \( \mathcal{H}_{DL,n} \) that is
\[
\left( \frac{\log n + \log \frac{1}{\epsilon}}{\epsilon^3} \cdot n^2 \log n, n^2 \log n + \log \frac{1}{\epsilon}, 0.1, \epsilon \right) - \text{bounded memory}
\]

5. Discussion and Open Questions

In this paper we suggested a general bounded memory algorithm and proved its correctness for classes that are separable. We then derived bounded memory algorithms for three natural classes: Threshold Functions, Equal-Piece Classifiers, and Decision Lists.

Several questions remain open. One question is to prove that other classes are separable and thus can be learned with our general bounded memory algorithm. A second open problem is to extend our work to the case of an unknown distribution over the examples.

Another interesting open problem is to prove a theorem similar to the fundamental theorem of statistical learning but for bounded memory learning. One can view our results as the positive side for this kind of theorem. In a complementary line of work, the negative side has been investigated (Shamir, 2014; Steinhardt et al., 2016; Raz, 2016; Kol et al., 2017; Moshkovitz and Moshkovitz, 2017a,b; Raz, 2017; Garg et al., 2017; Beame et al., 2017). However, there are some classes where neither our algorithm nor the lower bounds can be used. Specifically, if for some \( T \subseteq \mathcal{H} \) there are \( S \subseteq \mathcal{X}, T_0, T_1 \subseteq T \) as in Definition 1 but for other \( T \) there are none such \( S, T_0, T_1 \), then neither our algorithm nor the lower bounds that were investigated are applicable. An open problem is to bridge this gap between the positive and negative sides.

Acknowledgments

This work was partially supported by the Gatsby Charitable Foundation, the Israel Science Foundation, and the Intel ICRI-Cl center. M.M. is grateful to the Harry and Sylvia Hoffman Leadership and Responsibility Program.

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Appendix A. Technical Proofs

A.1 Proofs of Claims from Section 3

Proof. (of Claim 4) By the separable property we know that there are $S, T_0', T'_1$ as in Definition 2. Sort all $t \in T$ by $e(S, t)$ in an ascending order. Define $T_1 \subseteq T$ as the $\alpha |T|$-largest members in $T$ and $T_0 \subseteq T$ as the $\alpha |T|$ smallest members in $T$. Note that $|d(S, T'_1) - d(S, T'_0)| \leq |d(S, T_1) - d(S, T_0)|$. Assume by way of contradiction that the $(\alpha^2/2)|T|$-largest member in
T, denote it by \( t_1 \), and the \((\alpha^2/2)|T|\)-smallest member in \( T \), denote it by \( t_0 \), are too close; i.e., \( e(S, t_1) - e(S, t_0) < \alpha/2|S| \). Let us calculate
\[
|d(T_1, S) - d(T_0, S)| = \frac{e(S, T_1)}{|S||T_1|} - \frac{e(S, T_0)}{|S||T_0|}
\leq \frac{\alpha^2|T|}{2} \cdot \frac{|S|}{|S||T_1|} + \left( |T_1| - \frac{|T|\alpha^2}{2} \right) \frac{e(S, t_1)}{|S||T_1|} - \left( |T_0| - \frac{|T|\alpha^2}{2} \right) \frac{e(S, t_0)}{|S||T_0|}
\]
(recall: \(|T_1| = |T_0| = \alpha|T|\))
\[
\leq \frac{\alpha}{2} + \left( 1 - \frac{\alpha}{2} \right) \frac{e(S, t_1)}{|S|} - \left( 1 - \frac{\alpha}{2} \right) \frac{e(S, t_0)}{|S|}
< \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.
\]
Which is a contradiction to the separable property.

**Algorithm 2: Is-close\((h, \epsilon)\)**

1: **Input:** \( h \in \mathcal{H}, \epsilon > 0 \),
2: **Parameter:** integer \( k \)
3: **Returns:** True if \( h \) is \( \epsilon \)-close to the correct hypothesis and False if \( h \) is not \( 3\epsilon \)-close to the correct hypothesis
4: \( j = 0 \)
5: **for** \( i := 1 \) to \( k \) **do**
6: get labeled example \((x, y)\)
7: **if** \( h(x) \neq y \) **then**
8: \( j+ = 1 \)
9: **end if**
10: **end for**
11: **if** \( j/k \leq 2\epsilon \) **then**
12: return True
13: **else**
14: return False
15: **end if**

*Proof. (of Claim 5)* We will show that Algorithm 2 has the desired properties. Denote by \( X_i \) the random variable that is 1 if the labeled example \((x, y)\) used in the \( i \)-th step of Algorithm 2 has \( h(x) \neq y \), otherwise \( X_i = 0 \). Denote \( \bar{X} = \frac{1}{k} \sum X_i \). Let us consider the two cases presented in the claim

- if \( h \) is \( \epsilon \)-close to the correct hypothesis, then
  \[ \mathbb{E}[\bar{X}] \leq \epsilon. \]
  From Hoeffding’s inequality we know that
  \[ \Pr[\bar{X} - \mathbb{E}[\bar{X}] \geq \epsilon] \leq 2e^{-2k\epsilon^2}. \]

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16
Algorithm 3: Estimate$(S, \tau)$

1: **Input:** $S \subseteq \mathcal{X}, \tau > 0,$
2: **Parameter:** integer $k$
3: **Returns:** $d(S, f) \pm \tau$, where $f$ is the correct hypothesis
4: for $i := 1$ to $2k/\tau$ do
5: get labeled example $(x, y)$
6: if $x \in S$ then
7: counter$S :=$ counter$S + 1$
8: if $y = 1$ then
9: counter$1 :=$ counter$1 + 1$
10: end if
11: end if
12: end for
13: return counter$1$/counter$S$

And the last two inequalities imply that

$$\Pr[\bar{X} \geq 2\alpha] \leq 2e^{-2k\epsilon^2}.$$ 

Note that line 11 in the algorithm tests whether $\bar{X} \leq 2\epsilon$ or not. This means that with probability at least $1 - 2e^{-2k\epsilon^2}$ the algorithm returns True.

- if $h$ is not $3\epsilon$-close to the correct hypothesis, then

$$\mathbb{E}[\bar{X}] > 3\epsilon.$$ 

From Hoeffding’s inequality we know that

$$\Pr[\mathbb{E}[\bar{X}] - \bar{X} \geq \epsilon] \leq 2e^{-2k\epsilon^2}.$$ 

And the last two inequalities imply that

$$\Pr[2\epsilon > \bar{X}] \leq 2e^{-2k\epsilon^2}.$$ 

This means that with probability at least $1 - 2e^{-2k\epsilon^2}$ the algorithm returns False.

**Proof.** (of Claim 6) We will show that Algorithm 3 has the desired properties. Let $m = \frac{2k}{\alpha}$. Denote by $X_i$ the random variable that is 1 if the labeled example $(x, y)$ used in the $i$-th step of Algorithm 2 has $x \in S$, otherwise $X_i = 0$. Denote $\bar{X} = \frac{1}{m} \sum X_i$. Note that

$$\mathbb{E}[\bar{X}] = \frac{|S|}{|\mathcal{X}|} \geq \alpha.$$ 

From Hoeffding’s inequality we know that

$$\Pr[|\mathbb{E}[\bar{X}] - \bar{X}| \geq \alpha/2] \leq 2e^{-2\frac{2k}{m} \frac{\alpha^2}{4}}.$$ 

In particular,

$$\Pr[\bar{X} \leq \alpha/2] \leq 2e^{-k\alpha}.$$ 

17
This implies that with probability at least $1 - 2e^{-k\alpha}$ there are at least $k$ examples $(x, y)$ with $x \in S$. Let us now focus on these $k$ examples. Among them, denote by $Y_i$ the random variable that is 1 if the $i$-th labeled example has $y = 1$, otherwise $Y_i = 0$. Denote $\bar{Y} = \frac{1}{k} \sum Y_i$. Note that $E[\bar{Y}] = d(f, S)$. From Hoeffding’s inequality we know that

$$\Pr[|\bar{Y} - E[\bar{Y}]| \geq \tau] \leq 2e^{-2k\tau^2},$$

i.e., with probability at least $1 - 2e^{-2k\tau^2}$ the algorithm returns an answer $\bar{Y}$ in line 13 of the algorithm with $|\bar{Y} - d(f, S)| < \tau$.

Proof. (of Theorem 7) Description of the algorithm: At each iteration there will be a candidate set of hypotheses $T \subseteq \mathcal{H}$ that contains the correct hypothesis with high probability. If $T$ is not $\alpha$-separable then there is a center $h$ such that $|T \cap B_h(\epsilon)| \geq \alpha|T|$. If Algorithm 2 returns True then $h$ must be $3\alpha$-close (according to Claim 5) and we are done. Otherwise, the correct hypothesis is not in $T \cap B_t(\epsilon)$ and we can move on to the next iteration while removing a large fraction of the hypotheses from $T$.

If $T$ is $\alpha$-separable then using Claims 4, 6 we know that we can remove either $T_0$ or $T_1$. We then move to the next iteration.

Number of iterations: At each iteration we remove at least a fraction of $\alpha^2/2$ of the hypotheses in $T$. Thus, the number of iterations the algorithm makes is at most

$$s := \log \frac{1}{1 - \alpha^2/2} |\mathcal{H}| \leq \frac{\log |\mathcal{H}|}{\alpha^2/2},$$

where the inequality follows from the known fact $1 - 1/x \leq \ln x$.

Number of examples: At each iteration, the algorithm receives at most $\frac{2k}{\alpha}$ examples using Claims 5, 6. Thus, the total number of examples used is at most

$$\frac{2k}{\alpha} \cdot s.$$

Number of memory bits: the memory is composed of two types; one that describes the set of hypotheses $T$ that is currently being examined and $O\left(\log \frac{k}{\alpha}\right)$ bits for all the counters used in the subroutines. We can describe $T$ by the sets that are removed at each iteration, thus we need $s$ bits. Hence, the total number of memory bits is at most $(s + \log \frac{k}{\alpha})$.

A.2 Proof of the Theorem in Section 4.1

Proof. (proof of Theorem 8) Fix $\alpha \in (0, 1)$ and $T \subseteq \mathcal{H}_{TH,n}$ which is $(\alpha, \alpha)$-weak-separable. We want to find $S \subseteq \mathcal{X}$ with $|S| \geq \alpha|\mathcal{X}|$ and $T_0, T_1 \subseteq T$ with $|T_0|, |T_1| \geq \alpha|T|$ such that

$$|d(T_1, S) - d(T_0, S)| \geq \alpha \quad (1)$$

Take $S = \mathcal{X}$, which immediately implies that $|S| \geq \alpha|\mathcal{X}|$. Denote by $t_0 \in [0, 1]$ the minimal value such that for $\alpha|T|$ of the hypotheses $h_b \in T$ it holds that $b \leq t_0$ and by $t_1 \in [0, 1]$ the maximal value such that for $\alpha|T|$ of the hypotheses $h_b \in T$ it holds that $b \geq t_1$. Take $T_0 = \{h_b \in T : b \leq t_0\}$ and $T_1 = \{h_b \in T : b \geq t_1\}$, which immediately implies that $|T_0|, |T_1| \geq \alpha|\mathcal{H}_{TH,n}|$. 

18
Notice that for any $h_b$ in the class it holds that $e(h_b, S) = b$. To prove (1), note that

$$|d(T_1, S) - d(T_0, S)| = \frac{e(T_1, S) - e(T_0, S)}{|T_1||S| - |T_0||S|} \geq \frac{t_1 - t_0}{|S|}$$

Assume by contradiction that $t_1 - t_0 < \alpha |\mathcal{X}|$. Then, since for each $h_b \in T \setminus (T_0 \cup T_1)$ it holds that $b \in (t_0, t_1)$ we get that

$$T \setminus (T_0 \cup T_1) \subseteq B_{\frac{t_1 + t_0}{2}} \left( \frac{\alpha}{2} \right)$$

Since $\alpha \leq 1/3$ we get that $1 - 2\alpha \geq \alpha$, thus $|T \setminus (T_0 \cup T_1)| \geq \alpha |T|$ which is a contradiction to the $(\alpha, \alpha)$-weak-separability of $T$. □