Abstract. We prove the existence of tilting bundles on global quotient stacks that are produced by compatible finite group actions on flat families.

1. Introduction

Geometric tilting theory started with the construction of tilting bundles on the projective space by Beilinson [5]. Later Kapranov [27], [28], [29] constructed tilting bundles for certain homogeneous spaces. Further examples can be obtained from certain blow ups and taking projective bundles [14], [15], [39]. A smooth projective $k$-scheme admitting a tilting object satisfies very strict conditions, namely its Grothendieck group is a free abelian group of finite rank and the Hodge diamond is concentrated on the diagonal, at least in characteristic zero [12].

However, it is still an open problem to give a complete classification of smooth projective $k$-schemes admitting a tilting object. In the case of curves one can prove that a smooth projective algebraic curve has a tilting object if and only if the curve is a one-dimensional Brauer–Severi variety. But already for smooth projective algebraic surfaces there is currently no classification of surfaces admitting such a tilting object. It is conjectured that a smooth projective algebraic surface has a tilting bundle if and only if it is rational (see [11], [19], [20], [21], [22], [32] and [41] for results in this direction).

In the present work, we will focus on a certain type of a quotient stack and prove the existence of tilting bundles for their derived category. Several examples of stacks admitting a tilting object are known (see [25], [26], [30], [34], [35], [37] and [38]). But as in the case of schemes, one has to settle for existence criteria for stacks admitting a tilting object.

Assume $k$ is an algebraically closed field of characteristic zero. In [39] it is proved a generalization of the main result of [15]. It is the following theorem:

**Theorem.** ([39], Theorem 4.15) Let $\pi: X \to Z$ be a flat proper morphism between smooth projective $k$-schemes and $E_1, \ldots, E_n$ a set of locally free sheaves in $D^b(X)$ such that for any point $z \in Z$ the collection $E_1^z = E_1 \otimes \mathcal{O}_{X_z}, \ldots, E_n^z = E_n \otimes \mathcal{O}_{X_z}$ of the restrictions to the fiber $\pi^{-1}(z) = X_z$ is a full strongly exceptional collection for $D^b(X_z)$. Suppose $T$ is a tilting bundle on $Z$. Then there exists an ample sheaf $M$ on $Z$ such that $\bigoplus_{i=1}^n \pi^*(T \otimes M^\otimes i) \otimes E_i$ is a tilting bundle on $X$.

Our goal is to give an equivariant version of this theorem and in this way to produce a tilting bundle on the quotient stack $[X/G]$. So assume $k$ is algebraically closed and of characteristic zero. We consider smooth projective $k$-schemes $X$ and $Z$ with a compatible action of a finite group $G$ and investigate the case of $G$-morphisms $\pi: X \to Z$ where the underlying morphism of schemes $X \to Z$ is flat and proper. We call such morphisms...
flat $G$-maps for simplicity. For a smooth projective $k$-scheme $X$, denote by $\text{Coh}_G(X)$ the abelian category of equivariant coherent sheaves and by $D^b_G(X)$ its bounded derived category. Furthermore, $k[G]$ denotes the regular representation of $G$. We then prove the following theorem.

**Theorem.** (Theorem 4.5) Let $\pi: X \to Z$ be a flat $G$-map and $E_1, \ldots, E_n$ a set of locally free sheaves in $D^b_G(X)$ such that, considered as a set of objects in $D^b(X)$, for any point $z \in Z$, the collection $E_1^z = E_1 \otimes \mathcal{O}_{X_z}, \ldots, E_n^z = E_n \otimes \mathcal{O}_{X_z}$ of the restrictions to the fiber $\pi^{-1}(z) = X_z$ is a full strongly exceptional collection for $D^b(X_z)$. Suppose $T \in \text{Coh}_G(Z)$ is a tilting bundle on $Z$. There exists an equivariant ample sheaf $\mathcal{M}$ on $Z$ such that $(\bigoplus_{i=1}^n \pi^*(T \otimes \mathcal{M}^{\otimes i}) \oplus E_i) \otimes k[G]$ is a tilting bundle on $[X/G]$.

If all indecomposable pairwise non-isomorphic direct summands of $\mathcal{T}$ and all $E_i$ are invertible sheaves, we obtain a full strongly exceptional collection.

**Theorem.** (Theorem 4.9) Let $\pi: X \to Z$ be a flat $G$-map and $E_1, \ldots, E_n$ a set of invertible sheaves in $D^b_G(X)$ such that, considered as a set of objects in $D^b(X)$, for any point $z \in Z$, the collection $E_1^z = E_1 \otimes \mathcal{O}_{X_z}, \ldots, E_n^z = E_n \otimes \mathcal{O}_{X_z}$ of the restrictions to the fiber $\pi^{-1}(z) = X_z$ is a full strongly exceptional collection for $D^b(X_z)$. Suppose $T \in \text{Coh}_G(Z)$ is a tilting bundle on $Z$ whose indecomposable pairwise non-isomorphic direct summands are invertible sheaves. Then there is a full strongly exceptional collection for $D^b_G(X)$.

**Conventions.** Throughout this work $k$ is an algebraically closed field of characteristic zero and all locally free sheaves are assumed to be of finite rank.

### 2. Generalities on equivariant derived categories

Let $X$ be a quasiprojective $k$-scheme and $G$ a finite group acting on $X$. A $G$-linearization, also called an equivariant structure, on $\mathcal{F}$ is given by isomorphisms $\lambda_g: \mathcal{F} \to g^*\mathcal{F}$ for all $g \in G$ subject to $\lambda_1 = \text{id}_\mathcal{F}$ and $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$. In the present work, we also call such sheaves equivariant sheaves. Equivariant sheaves are therefore pairs $(\mathcal{F}, \lambda)$, consisting of a sheaf $\mathcal{F}$ on $X$ and a choice of an equivariant structure $\lambda$.

**Remark 2.1.** For a definition of linearization in the case where an arbitrary algebraic group acts on an arbitrary scheme, we refer to [6], [16] or [17].

If $(\mathcal{F}, \lambda)$ and $(\mathcal{G}, \mu)$ are two equivariant sheaves on $X$, the vector space $\text{Hom}(\mathcal{F}, \mathcal{G})$ becomes a $G$-representation via $g \cdot f := (\mu_g)^{-1} \circ g^*f \circ \lambda_g$ for $f: \mathcal{F} \to \mathcal{G}$. The equivariant quasi-coherent respectively coherent sheaves together with $G$-invariant morphisms $\text{Hom}_G(\mathcal{F}, \mathcal{G}) := \text{Hom}(\mathcal{F}, \mathcal{G})^G$ form abelian categories with enough injectives (see [10], [42]) which we denote by $\text{Qcoh}_G(X)$ respectively $\text{Coh}_G(X)$. We put $D_G(\text{Qcoh}(X)) := D(\text{Qcoh}_G(X))$ and $D^b_G(X) := D^b(\text{Coh}_G(X))$.

Let $X$ and $Y$ be quasiprojective $k$-schemes on which the finite group $G$ acts. A $G$-morphism between $X$ and $Y$ is given by a morphism $\phi: X \to Y$ such that $\phi \circ g = g \circ \phi$ for all $g \in G$. Then we have the pullback $\phi^*: \text{Coh}_G(Y) \to \text{Coh}_G(X)$ and the pushforward $\phi_*: \text{Coh}_G(X) \to \text{Coh}_G(Y)$. The functors $\phi^*$ and $\phi_*$ are adjoint; analogously for $L\phi^*$ and $R\phi_*$. For $(\mathcal{F}, \lambda), (\mathcal{G}, \mu) \in \text{Coh}_G(X)$, there is a canonical equivariant structure on $\mathcal{F} \otimes \mathcal{G}$. The functors $\lambda_g$ and $\mu_h$ are $G$-invariant morphisms coming from the maps $\lambda_\phi \otimes \mu_\phi$ (see [3], Proposition 3.46).

By definition, objects of $D^b_G(X)$ are bounded complexes of equivariant coherent sheaves. It is clear that each such complex defines an equivariant structure on the corresponding object of $D^b(X)$. Now let $\mathcal{C}$ be the category of equivariant objects of $D^b(X)$, i.e., complexes $\mathcal{F}^\bullet$ with isomorphisms $\lambda_g: \mathcal{F}^\bullet \to g^*\mathcal{F}^\bullet$ satisfying $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$. This category is in fact triangulated and it is a natural fact that $D^b_G(X)$ and $\mathcal{C}$ are equivalent (see [13], Proposition 4.5 or [14]).

There is also another description of the derived categories needed in the present work. Consider the global quotient stack $[X/G]$, produced by an action of a finite group $G$ on $X$ (see [15], Example 7.17). The quasi-coherent sheaves on $[X/G]$ are equivalent.
to equivariant quasi-coherent sheaves on $X$ (see \[15\], Example 7.21). Henceforth, the abelian categories $\text{Qcoh}(\mathbb{Z} X / G)$ and $\text{Qcoh}_G(X)$ are equivalent and therefore give rise to equivalent derived categories $D_c(\text{Qcoh}(X)) \simeq D_c(\text{Qcoh}(\mathbb{Z} X / G))$. For any two objects $\mathcal{F}^*, \mathcal{G}^* \in D_c(\text{Qcoh}(X))$ we write $\text{Hom}_G(\mathcal{F}^*, \mathcal{G}^*):= \text{Hom}_{D_c(\text{Qcoh}(X))}(\mathcal{F}^*, \mathcal{G}^*)$.

Analogously, we get $D_b^G(X) \simeq D_b^G(\text{Coh}(\mathbb{Z} X / G))$. Note that for $X = pt$, $\text{Coh}(\mathbb{Z} pt / G) \simeq \text{Coh}_G(\mathbb{Z} pt) \simeq \text{Rep}_G(G)$ is the category of finite-dimensional representations. Moreover, for a finite group $G$, the functor $(-)^G_\text{Coh} : \text{Coh}(\mathbb{Z} pt / G) \rightarrow \text{Coh}(\mathbb{Z} pt), V \mapsto V^G$, is exact (see \[1\], Proposition 2.5). For arbitrary $\mathcal{F}^*, \mathcal{G}^* \in D_b^G(X)$, the finite group $G$ also acts on the vector space $\text{Hom}(\mathcal{F}^*, \mathcal{G}^*):= \text{Hom}_{D_b^G(X)}(\mathcal{F}^*, \mathcal{G}^*)$. The exactness of $(-)^G$ yields $\text{Hom}_G(\mathcal{F}^*, \mathcal{G}^*) \simeq \text{Hom}(\mathcal{F}^*, \mathcal{G}^*)^G$.

The exactness of $(-)^G$ also implies the following fact (see \[1\], Lemma 2.2.8):

**Lemma 2.2.** Let $X$ be smooth quasi-projective k-scheme and $G$ a finite group acting on $X$. For arbitrary $\mathcal{F}^*, \mathcal{G}^* \in D_b^G(X)$ the following holds for all $i \in \mathbb{Z}$:

$$\text{Hom}_G(\mathcal{F}^*, \mathcal{G}^* [i]) \simeq \text{Hom}(\mathcal{F}^*, \mathcal{G}^* [i])^G.$$ 

For $\mathcal{F} \in \text{Coh}_G(X)$ we therefore have $H^i_G(X, \mathcal{F}) \simeq H^i(X, \mathcal{F})^G$. In Section 4 we also need the Leray spectral sequence: For a $G$-morphism $f : X \rightarrow Y$, the spectral sequence is

$$E_2^{p,q} = H^p_G(Y, R^q f_*(\mathcal{F})) \Rightarrow H^{p+q}_{\text{et}}(X, \mathcal{F}).$$

3. **Geometric tilting theory**

In this section we recall some facts of geometric tilting theory. We first recall the notions of generating and thick subcategories (see \[9\], \[43\]).

Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. The subcategory $\mathcal{C}$ is called **thick** if it is closed under isomorphisms and direct summands. For a subset $A$ of objects of $\mathcal{D}$ we denote by $\langle A \rangle$ the smallest full thick subcategory of $\mathcal{D}$ consisting of all objects $M$ such that $\text{Hom}_A(E[i], M) = 0$ for all $i \in \mathbb{Z}$ and all objects $E$ of $A$. We say that $A$ generates $\mathcal{D}$ if $\langle A \rangle = 0$. Now assume $\mathcal{D}$ admits arbitrary direct sums. An object $B$ is called **compact** if $\text{Hom}_\mathcal{D}(B, -)$ commutes with direct sums. Denoting by $\mathcal{D}_c^e$ the subcategory of compact objects we say that $\mathcal{D}$ is **compactly generated** if the objects of $\mathcal{D}_c^e$ generate $\mathcal{D}$. One has the following important theorem (see \[9\], Theorem 2.1.2).

**Theorem 3.1.** Let $\mathcal{D}$ be a compactly generated triangulated category. Then a set of objects $A \subset \mathcal{D}_c^e$ generates $\mathcal{D}$ if and only if $\langle A \rangle = \mathcal{D}_c^e$.

We now give the definition of tilting objects (see \[12\] for a definition of tilting objects in arbitrary triangulated categories).

**Definition 3.2.** Let $k$ be a field, $X$ a quasiprojective k-scheme and $G$ a finite group acting on $X$. An object $\mathcal{T}^* \in D_c(\text{Qcoh}(X))$ is called **tilting object** on $[X/G]$ if the following hold:

1. Ext vanishing: $\text{Hom}_G(\mathcal{T}^*, \mathcal{T}^*[i]) = 0$ for $i \neq 0$.
2. Generation: If $\mathcal{N}^* \in D_c(\text{Qcoh}(X))$ satisfies $\text{Hom}_G(\mathcal{T}^*, \mathcal{N}^*) = 0$, then $\mathcal{N}^* = 0$.
3. Compactness: $\text{Hom}_G(\mathcal{T}^*, -)$ commutes with direct sums.

Below we state the well-known equivariant tilting correspondence (see \[10\], Theorem 3.1.1). It is a direct application of a more general result on triangulated categories (see \[31\], Theorem 8.5). We denote by $\text{Mod}(A)$ the category of right $A$-modules and by $D^b(A)$ the bounded derived category of finitely generated right $A$-modules. Furthermore, if $\text{perf}(A) \subset D(\text{Mod}(A))$ denotes the full triangulated subcategory of perfect complexes, those quasi-isomorphic to a bounded complexes of finitely generated projective right $A$-modules.
Theorem 3.3. Let $X$ be a quasiprojective $k$-scheme and $G$ a finite group acting on $X$. Suppose we are given a tilting object $T^\bullet$ on $[X/G]$ and let $A = \text{End}_G(T^\bullet)$. Then the following hold:

(i) The functor $\mathbb{R}\text{Hom}_G(T^\bullet, -) : D_G(\text{Qcoh}(X)) \to D(\text{Mod}(A))$ is an equivalence.

(ii) If $X$ is smooth and $T \in D^b_G(X)$, this equivalence restricts to an equivalence $D^b_G(X) \cong \text{perf}(A)$.

(iii) If the global dimension of $A$ is finite, then $\text{perf}(A) \simeq D^b(A)$.

Remark 3.4. If $X$ is a smooth projective $k$-scheme and $G = 1$, the derived category $D(\text{Qcoh}(X))$ is compactly generated and the compact objects are exactly $D^b(X)$ (see [9]). In this case, a compact object $T^\bullet$ generates $D(\text{Qcoh}(X))$ if and only if $\langle T^\bullet \rangle = D^b(X)$. Since the natural functor $D^b(X) \to D(\text{Qcoh}(X))$ is fully faithful (see [22]), a compact object $T^\bullet \in D(\text{Qcoh}(X))$ is a tilting object if and only if $\langle T^\bullet \rangle = D^b(X)$ and $\text{Hom}_{D_G(X)}(T^\bullet, T^\bullet[i]) = 0$ for $i \neq 0$. If the tilting object $T^\bullet$ is a coherent sheaf and $\text{gldim}(\text{End}(T^\bullet)) < \infty$, the above definition coincides with the definition of a tilting sheaf given in [9]. In this case the tilting object is called tilting sheaf on $X$. If it is a locally free sheaf we simply say that $T$ is a tilting bundle. Theorem 3.3 then gives the classical tilting correspondence as first proved by Bondal [7] and later extended by Baer [9].

The next observation shows that in Theorem 3.3 the smoothness of $X$ already implies the finiteness of the global dimension of $A$.

Proposition 3.5. Let $X$, $G$ and $T^\bullet$ be as in Theorem 3.3. If $X$ is smooth and projective, then $A = \text{End}_G(T^\bullet)$ has finite global dimension and therefore the equivalence (i) of Theorem 3.3 restricts to an equivalence $D^b_G(X) \cong D^b(A)$.

Proof. Imitating the proof of Theorem 7.6 in [23], we argue as follows: For two finitely generated right $A$-modules $M$ and $N$, the equivalence $\psi : \mathbb{R}\text{Hom}_G(T^\bullet, -) : D^b_G(X) \to \text{perf}(A)$ (see Theorem 3.3 (ii)) yields

$\text{Ext}^i_A(M, N) \simeq \text{Hom}_G(\psi^{-1}(M), \psi^{-1}(N)[i]) \simeq \text{Hom}(\psi^{-1}(M), \psi^{-1}(N)[i]^G) = 0$

for $i \geq 0$, since $X$ is smooth. Indeed, this follows from the local-to-global spectral sequence, Grothendieck vanishing Theorem and Lemma 2.2. As $X$ is projective, $A = \text{End}_G(T^\bullet)$ is a finite-dimensional $k$-algebra and hence a noetherian ring. But for noetherian rings the vanishing of $\text{Ext}^i_A(M, N)$ for $i \geq 0$ for any two finitely generated $A$-modules $M$ and $N$ suffices to conclude that the global dimension of $A$ has to be finite.

The following fact is folklore. It will be needed in Section 4.

Proposition 3.6. Let $X$ be a smooth projective $k$ and $T^\bullet$ a tilting object on $X$. Then for an invertible sheaf $\mathcal{L}$ the object $T^\bullet \otimes \mathcal{L}$ is also a tilting object on $X$.

In the literature, instead of the tilting object $T^\bullet$ one often studies the set $\mathcal{E}^\bullet_1, \ldots, \mathcal{E}^\bullet_n$ of its indecomposable pairwise non-isomorphic direct summands. There is a special case where all the summands form a so-called full strongly exceptional collection. We recall the definition and follow here [10].

Definition 3.7. Let $X$ and $G$ be as in Definition 3.2. An object $\mathcal{E}^\bullet \in D^b_G(X)$ is called exceptional if $\text{Hom}_G(\mathcal{E}^\bullet, \mathcal{E}^\bullet[i]) = 0$ when $i \neq 0$, and $\text{Hom}_G(\mathcal{E}^\bullet, \mathcal{E}^\bullet) = k$. An exceptional collection in $D^b_G(X)$ is a sequence of exceptional objects $\mathcal{E}^\bullet_1, \ldots, \mathcal{E}^\bullet_n$ satisfying $\text{Hom}_G(\mathcal{E}^\bullet_i, \mathcal{E}^\bullet[j]) = 0$ for all $l \in \mathbb{Z}$ if $i > j$. The exceptional collection is called strongly exceptional if in addition $\text{Hom}_G(\mathcal{E}^\bullet_i, \mathcal{E}^\bullet[j]) = 0$ for all $i$ and $j$ when $i \neq 0$. Finally, we say the exceptional collection is full if the smallest full thick subcategory containing all $\mathcal{E}^\bullet_i$ equals $D^b_G(X)$. 
A generalization is the notion of a semiorthogonal decomposition of \( D^n_G(X) \). Recall, a full triangulated subcategory \( \mathcal{D} \) of \( D^n_G(X) \) is called \textit{admissible} if the inclusion \( \mathcal{D} \hookrightarrow D^n_G(X) \) has a left and right adjoint functor.

**Definition 3.8.** Let \( X \) and \( G \) be as in Definition 3.2. A sequence \( \mathcal{D}_1, \ldots, \mathcal{D}_n \) of full triangulated subcategories of \( D^n_G(X) \) is called \textit{semiorthogonal} if all \( \mathcal{D}_i \subset D^n_G(X) \) are admissible and \( \mathcal{D}_i \subset \mathcal{D}^\perp_j = \{ F^\ast \in \mathcal{D}_j(X) \mid \text{Hom}_G(G^\ast, F^\ast) = 0, \forall G^\ast \in \mathcal{D}_i \} \) for \( i > j \).

Such a sequence defines a \textit{semiorthogonal decomposition} of \( D^n_G(X) \) if the smallest full thick subcategory containing all \( \mathcal{D}_i \) equals \( D^n_G(X) \).

For a semiorthogonal decomposition of \( D^n_G(X) \), we write \( D^n_G(X) = \langle D_1, \ldots, D_r \rangle \).

**Example 3.9.** It is an easy exercise to show that a full exceptional collection \( E^1, \ldots, E^r \) in \( D^n_G(X) \) gives rise to a semiorthogonal decomposition \( D^n_G(X) = \langle D_1, \ldots, D_n \rangle \) with \( D_i = \langle E_i^r \rangle \) (see [24], Example 1.60).

Exceptional collections and semiorthogonal decompositions were intensively studied and we know quite a lot of examples of schemes admitting full exceptional collections or semiorthogonal decompositions. For an overview we refer to [8] and [33].

4. Equivariant derived category of flat families

Let \( X \) and \( Z \) be smooth projective \( k \)-schemes on which a finite group \( G \) acts. In the sequel we consider \( G \)-morphisms \( \pi : X \to Z \) where the underlying morphism of schemes \( X \to Z \) is flat and proper. Such morphisms we simply call \textit{flat} \( G \)-maps.

We state some preliminary facts.

**Lemma 4.1.** Let \( \pi : X \to Z \) be a flat proper morphism and \( E_1, \ldots, E_n \) a set of locally free sheaves in \( D^b(X) \) such that for any point \( z \in Z \) the collection \( E^1_p = E_1 \otimes O_{X_z}, \ldots, E^r_p = E_n \otimes O_{X_z} \) of the restrictions to the fiber \( \pi^{-1}(z) = X_z \) is a full strongly exceptional collection for \( D^b(X_z) \). Then the following holds:

\[
\mathbb{R}^s \pi_\ast(E_q \otimes E^r_p) = \begin{cases} 0 & \text{for } s > 0 \\ 0 & \text{for } s = 0 \text{ and } q < p \\ \pi_\ast(E_q \otimes E^r_p) & \text{for } s = 0 \text{ and } q \geq p 
\end{cases}
\]

**Proof.** If \( \pi : X \to Z \) is a locally trivial fibration with typical fiber \( F \) and \( E_1, \ldots, E_n \) are invertible sheaves this is exactly the claim of [15], p.430. As \( \pi : X \to Z \) is flat and proper, flat base change holds (see [24], (3.18)). Carrying out the same arguments as in the proof of the claim of [15], p.430, we see that the above statement also holds for \( \pi : X \to Z \) flat and proper and \( E_1, \ldots, E_n \) being arbitrary locally free sheaves. \( \square \)

**Lemma 4.2.** Let \( X \) be a smooth projective \( k \)-scheme on which a finite group \( G \) acts. Then there exists an equivariant ample sheaf \( N \).

**Proof.** Let \( L \) be an ample invertible sheaf on \( X \), then \( g^\ast L \) is ample for any \( g \in G \). Now the tensor product \( \bigotimes_{g \in G} g^\ast L \) is ample and has a natural equivariant structure \( \lambda \). Take \( (N, \lambda) = (\bigotimes_{g \in G} g^\ast L, \lambda) \). \( \square \)

**Lemma 4.3.** Let \( \pi : X \to Z \) and \( E_i \) be as in Lemma 4.1. Suppose \( \mathcal{A}^\ast \) is a compact object with \( \langle \mathcal{A}^\ast \rangle = D^b(Z) \), then \( \bigoplus_{i=1}^n \pi^\ast(\mathcal{A}^\ast) \otimes E_i = D^b(X) \) and therefore \( \bigoplus_{i=1}^n \pi^\ast(\mathcal{A}^\ast) \otimes E_i \) generates \( D(Qcoh(X)) \).

**Proof.** In [14], Theorem 3.1 it is proved that the functor \( \pi^\ast(-) \otimes E_i : D^b(Z) \to D^b(X) \) is fully faithful and that \( D^b(X) = \langle \pi^\ast D^b(Z) \otimes E_1, \ldots, \pi^\ast D^b(Z) \otimes E_n \rangle \) is a semiorthogonal decomposition. Here the full subcategories \( \pi^\ast D^b(Z) \otimes E_i \) consist of objects of the form \( \pi^\ast(F^\ast) \otimes E_i \), where \( F^\ast \in D^b(Z) \). Therefore, the functor \( \pi^\ast(-) \otimes E_i \) from above induces
an equivalence between $D^b(Z)$ and $\pi^*D^b(Z) \otimes E_i$. Since $\langle A^* \rangle = D^b(Z)$, we immediately get $(\bigoplus_{i=1}^n \pi^*(A^*) \otimes E_i) = D^b(X)$. Note that the compact objects of $D(Qcoh(X))$ are all of $D^b(X)$ (see [33]). The rest follows from Theorem 3.1 as $(\bigoplus_{i=1}^n \pi^*(A^*) \otimes E_i$ is a compact object of $D(Qcoh(X))$.

To prove Theorem 4.4 below, we apply the following result that we only cite (see [33, Theorem 4.1 and 4.2])

**Theorem 4.4.** Let $X$ be a smooth projective $k$-scheme and $G$ a finite group acting on $X$. Suppose there is a $T^* \in D_G(Qcoh(X))$ which, considered as an object in $D(Qcoh(X))$, is a tilting object on $X$. Let $k[G] = \bigoplus_i W_i^{\dim(W_i)}$ be the regular representation of $G$, then $T^* \otimes k[G]$ and $\bigoplus_i T^* \otimes W_i$ are tilting objects on $[X/G]$.

**Theorem 4.5.** Let $\pi: X \rightarrow Z$ be a flat $G$-map and $E_1, \ldots, E_n$ a set of locally free sheaves in $D^b_G(X)$ such that, considered as a set of objects in $D^b(X)$, for any point $z \in Z$ the collection $E_i^z = E_i \otimes \mathcal{O}_{X_z}, \ldots, E_n^z = E_n \otimes \mathcal{O}_{X_z}$ of the restrictions to the fiber $\pi^{-1}(z) = X_z$ is a full strongly exceptional collection for $D^b(X_z)$. Suppose $T \in Coh_G(Z)$ is a tilting bundle on $Z$. There is an equivariant ample sheaf $M$ on $Z$ such that $(\bigoplus_{i=1}^n \pi^*(T \otimes M^{\otimes i}) \otimes E_i) \otimes k[G]$ is a tilting bundle on $[X/G]$.

**Proof.** Below we show that there exists an equivariant ample sheaf $M$ on $Z$ such that $(\bigoplus_{i=1}^n \pi^*(T \otimes M^{\otimes i}) \otimes E_i) \otimes k[G]$ is a tilting bundle on $[X/G]$. Note that by construction, $(\bigoplus_{i=1}^n \pi^*(T \otimes M^{\otimes i}) \otimes E_i) \otimes k[G]$ is a compact object of $D_G(Qcoh(X))$. To apply Theorem 4.4, we only have to show that $(\bigoplus_{i=1}^n \pi^*(T \otimes M^{\otimes i}) \otimes E_i)$ is a tilting bundle on $X$ that in addition admits an equivariant structure. For this, we take an equivariant ample sheaf $N$ on $Z$. Such a $N$ exists according to Lemma 4.2. Now let $M = N^{\otimes m}$ for $m \gg 0$. So we have to find a natural number $m \gg 0$ such that

$$\text{Ext}^l(\pi^*(T \otimes M^{\otimes i}) \otimes E_i, \pi^*(T \otimes M^{\otimes j}) \otimes E_j) = 0, \text{ for } l > 0.$$ But this is equivalent to

$$H^l(X, \pi^*(T \otimes T^v \otimes M^{\otimes (j-i)}) \otimes E_j \otimes E_j^\vee) = 0, \text{ for } l > 0.$$ Applying the Leray spectral sequence (1) to the morphism $\pi: X \rightarrow Z$, we obtain

$$H^r(Z, \mathbb{R}^s \pi_*(\pi^*(T \otimes T^v \otimes M^{\otimes (j-i)}) \otimes E_j \otimes E_j^\vee)) \implies H^{r+s}(X, \pi^*(T \otimes T^v \otimes M^{\otimes (j-i)}) \otimes E_j \otimes E_j^\vee).$$ With the projection formula we find

$$\mathbb{R}^s \pi_*(\pi^*(T \otimes T^v \otimes M^{\otimes (j-i)}) \otimes E_j \otimes E_j^\vee) \simeq T \otimes T^v \otimes M^{\otimes (j-i)} \otimes \mathbb{R}^s \pi_*(E_j \otimes E_j^\vee).$$ Now from Lemma 4.1 we know that $\mathbb{R}^s \pi_*(E_j \otimes E_j^\vee)$ is non-vanishing only for $s = 0$ and $j \geq i$ and that in this case we have $\mathbb{R}^s \pi_*(E_j \otimes E_j^\vee) \simeq \pi_*(E_j \otimes E_j^\vee)$. Thus for $j < i$ we get $\mathbb{R}^s \pi_*(E_j \otimes E_j^\vee) = 0$ and therefore

$$H^r(Z, T \otimes T^v \otimes M^{\otimes (j-i)} \otimes \mathbb{R}^s \pi_*(E_j \otimes E_j^\vee)) = 0.$$ Therefore we find

$$H^l(X, \pi^*(T \otimes T^v \otimes M^{\otimes (j-i)}) \otimes E_j \otimes E_j^\vee) = 0,$$ for $l > 0$ by above spectral sequence. It remains the case $j \geq i$. For $j = i$ we have $\mathbb{R}^s \pi_*(E_i \otimes E_i^\vee) \simeq \mathcal{O}_Z$ (see [44], p.5 right after (3.10)). Hence

$$H^r(Z, T \otimes T^v \otimes M^{\otimes (i-i)} \otimes \mathbb{R}^s \pi_*(E_i \otimes E_i^\vee)) \simeq H^r(Z, T \otimes T^v \otimes \mathcal{O}_Z) \simeq \text{Ext}^r(T, T) = 0$$ for $r > 0$, as $T$ is a tilting bundle on $Z$. From the above spectral sequence we conclude
for $l > 0$. Finally, it remains the case $j > i$. Again we consider the above spectral sequence and notice that it becomes

$$H^l(X, \pi^*(T \otimes T^\vee \otimes \mathcal{M}^{\otimes(j-i)} \otimes \mathcal{E}_i \otimes \mathcal{E}_j^\vee)) = 0,$$

and therefore

$$\text{Ext}^l(\pi^*(T \otimes \mathcal{M}^{\otimes(i)} \otimes \mathcal{E}_i) \otimes \mathcal{E}_j) = 0 \text{ for } l > 0.$$
Proposition 4.8. Let $X$ be a smooth projective $k$-scheme on which a finite group $G$ acts. Suppose that $E_1, \ldots, E_n \in \text{Coh}_G(X)$, considered as a set of objects in $D^b(X)$, is a full strongly exceptional collection for $D^b(X)$. If we denote by $W_1, \ldots, W_m$ the irreducible representation of $G$, then the collection

\[ \{ [E_i \otimes W_1], [E_i \otimes W_2], \ldots, [E_i \otimes W_m] \}_{1 \leq i \leq n} \]

is a full strongly exceptional collection for $D^b_G(X)$.

Proof. We have canonical isomorphisms

\[ \text{Ext}^1(E_i \otimes W_p, E_j \otimes W_q) \simeq \text{Ext}^1(E_i, E_j) \otimes \text{Hom}(W_p, W_q) \]

on $X$. Lemma 2.2 gives

\[ \text{Ext}^1_G(E_i \otimes W_p, E_j \otimes W_q) \simeq (\text{Ext}^1(E_i, E_j) \otimes \text{Hom}(W_p, W_q))^G. \]

From Schur’s Lemma and the fact that $E_1, \ldots, E_n$, considered as a set of objects in $D^b(X)$, is a strongly exceptional collection for $D^b(X)$, it easily follows that (2) is a strongly exceptional collection in $D^b_G(X)$. The generating property of this collection can be seen as follows: As $\mathcal{T} = \bigoplus_{i=1}^n E_i \in D^b_G(X)$, considered as an object in $D^b(X)$, is a tilting bundle on $X$, we conclude from Theorem 4.4 that $\bigoplus_{i=1}^m \mathcal{T} \otimes W_i$ is a tilting bundle on $[X/G]$. In particular, $\bigoplus_{i=1}^m \mathcal{T} \otimes W_i$ generates $D^b_G(\text{Qcoh}(X))$. Note that the compact objects of $D^b_G(\text{Qcoh}(X))$ are all of $D^b_G(X)$ (see [10], p.39). Since $\bigoplus_{i=1}^m \mathcal{T} \otimes W_i$ is compact, Theorem 3.1 yields $\big(\bigoplus_{i=1}^m \mathcal{T} \otimes W_i\big) = D^b_G(X)$. Thus our collection (2) is full. □

We are now able to prove the following:

Theorem 4.9. Let $\pi: X \to Z$ be a flat $G$-map and $E_1, \ldots, E_n$ a set of invertible sheaves in $D^b_G(X)$ such that, considered as a set of objects in $D^b(X)$, for any point $z \in Z$ the collection $E_1 = E_1 \otimes \mathcal{O}_{X_z}, \ldots, E_n = E_n \otimes \mathcal{O}_{X_z}$ of the restrictions to the fiber $\pi^{-1}(z) = X_z$ is a full strongly exceptional collection in $D^b(X_z)$. Suppose $\mathcal{T} \in \text{Coh}_G(Z)$ is a tilting bundle on $Z$ whose indecomposable pairwise non-isomorphic direct summands are invertible sheaves. Then there is a full strongly exceptional collection for $D^b_G(X)$.

Proof. By assumption $\mathcal{T}$ is a tilting bundle on $Z$ whose indecomposable pairwise non-isomorphic direct summands $\mathcal{L}_1, \ldots, \mathcal{L}_m$ are invertible sheaves. Proposition 4.7 ensures that the set $\mathcal{L}_1, \ldots, \mathcal{L}_m$ can be reordered in such a way that the reordered set forms a full strongly exceptional collection. Let us denote this full strongly exceptional collection by $\mathcal{L}_1, \ldots, \mathcal{L}_m$. Now [10], Theorem 2.8 states that there is an ample sheaf $\mathcal{M}$ on $Z$ such that

\[ \{ \pi^*(\mathcal{L}_1 \otimes \mathcal{M}) \cap \mathcal{E}_1\}_{1 \leq i \leq m}, \ldots, \{ \pi^*(\mathcal{L}_1 \otimes \mathcal{M}^{\otimes n}) \cap \mathcal{E}_n\}_{1 \leq i \leq m} \]

is a full strongly exceptional collection in $D^b(X)$. As the proof of this fact follows exactly the lines of the proof of Theorem 4.5, we see that the ample sheaf $\mathcal{M}$ on $Z$ can be chosen to be equivariant. In this way we get that all members of the collection (3) are objects in $\text{Coh}_G(X)$ and Proposition 4.8 provides us with a full strongly exceptional collection for $D^b_G(X)$. □

Remark 4.10. Theorem 4.9 in connection with Example 3.9 also gives a semiorthogonal decomposition for $D^b_G(X)$.

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