RIGIDITY OF RANDOM GROUP ACTIONS

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Abstract. We prove that if a finitely generated random group action is robustly expansive and has the shadowing property, then it is rigid. We apply this result to analyze the rigidity of certain iterated function systems or actions of the discrete Heisenberg group.

1. Introduction. A dynamical system is rigid when every nearby system is topologically equivalent to it. This property seems to appear first in Smale's paper [19]. It was impulised by the Lie group actions on manifolds, Mostow rigidity theorem [15] and the classification of abelian Anosov actions on nilmanifolds. On the other hand, a random group action is just an action in which some noise was added. They have been extensively studied during the last decades or so [2], [12].

In this paper we will try to extend the rigidity theory from differentiable group actions to differentiable random group actions. Indeed, we prove that every finitely generated robustly expansive random group action with the shadowing property of a closed manifold is rigid. As application we prove that certain actions of the discrete Heisenberg group on tori are rigid. Let us present our result in a precise way.

Given a map $L : A \times B \to C$, $a \in A$ and $b \in B$ we denote by $L(a) : B \to C$ and $L^b : A \to C$ the maps $L(a)b = L^b(a) = L(a, b)$. The identity map $I : A \to A$ will be denoted by $I_A$ (or simply by $I$ to do not overcharge notations). We denote by $B^A$ the set of maps from $A$ to $B$. The set of subsets of $A$ is denoted by $2^A$.

Hereafter $G$ is a finitely generated group and $\Omega$ is a nonempty set. Denote by $Sym(\Omega)$ the group consisting of all bijective self-maps of $\Omega$ with the composition of maps as the group operation.

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Definition 1.1. A $G$-action on $\Omega$ is a group homomorphism $\theta : G \to \text{Sym}(\Omega)$, so that the following properties hold:

1. $\theta(e) = I$ where $e$ is the neutral element of $G$;
2. $\theta(gh) = \theta(g) \circ \theta(h)$ for every $g,h \in G$.

(If $\Omega$ is a topological space we will assume that $\theta(g)$ is a continuous map for every $g \in G$.)

Let $X$ be a compact metric space. Denote by $C(X)$ the set of continuous maps from $X$ into itself. Replacing $\mathbb{Z}$ or $\mathbb{R}$ by the general group $G$ in Definition 1.1.1 of [2] we obtain the following definition.

Definition 1.2 ([6]). Let $\theta$ be a $G$-action on $\Omega$. A random $G$-action on $X$ driven by $\theta$ is a family of maps $\{T^\omega : G \to C(X)\}_{\omega \in \Omega}$ satisfying

1. $T^\omega(e) = I$ for every $\omega \in \Omega$;
2. $T^\omega(gh) = T^{\theta(h)}(g) \circ T^\omega(h)$ for every $\omega \in \Omega$, $g,h \in G$.

Remark 1.3. It is customary to assume additional hypotheses on the driver $\theta$ likewise continuity or measurability [2, 12, 6] but here we will not do it.

Now we define expansive random group action. The similar concept for group actions is given by Osipov and Tikhomirov [16] generalizing the original definition by Utz [20]. For random $\mathbb{Z}$-action see Corollary 3.2 of [8].

Definition 1.4. An expansivity function of a random $G$-action $T$ is a map $\rho : \Omega \to \mathbb{R}^+$ such that if $x,y \in X$ satisfy $d(T^\omega(g)x, T^\omega(g)y) \leq \rho(\theta(g)\omega)$ for every $g \in G$, and some $\omega \in \Omega$, then $x = y$. A random $G$-action $T$ is expansive if it has an expansivity function.

Next we present the shadowing property for random group actions. The original definition for $\mathbb{Z}$-actions (i.e. homeomorphisms) is due to Bowen [4]. The similar concept for group actions was presented by Osipov and Tikhomirov [16]. For random $\mathbb{Z}$-actions we can mention Definition 3.5 in [8]. For iterated function systems (special examples of random $\mathbb{Z}$-actions, see Proposition 5.1) we can mention [3] and [7].

Let $\omega \in \Omega$ and $A$ be a generating set of $G$, and let $\delta, \epsilon : \Omega \to \mathbb{R}^+$. We say that a map $x : G \to X$ is an $(\omega, \delta, \epsilon)$-pseudo orbit with respect to $A$ if

$$d(T^{\theta(g)}(a)x(g), x(a)) \leq \delta(\theta(a)\omega), \quad \forall a \in A, g \in G.$$ 

We say that $x$ can be $(\omega, \epsilon)$-shadowed if there is $y \in X$ such that

$$d(T^\omega(g)y, x(g)) \leq \epsilon(\theta(g)\omega), \quad \forall g \in G.$$ 

The definition below generalizes both [16] and the one implicitly stated in Proposition 3.6 of [8].

Definition 1.5. We say that $T$ has the shadowing property if there is a finite generating set $A$ such that for every $\epsilon : \Omega \to \mathbb{R}^+$ there is $\delta : \Omega \to \mathbb{R}^+$ such that every $(\omega, \delta, \epsilon)$-pseudo orbit with respect to $A$ can be $(\omega, \epsilon)$-shadowed $(\forall \omega \in \Omega)$.

Now we assume that $X$ is a closed manifold namely a compact connected boundaryless Riemannian manifold. The set of $C^1$ diffeomorphisms of $X$ is denoted by $\text{Diff}^1(X)$. We recall the $C^1$ distance between $r,l \in \text{Diff}^1(X)$ given by

$$d_{C^1}(r,l) = \sup_{v \in T^1X} d(D^1r(v), D^1l(v)).$$

Here $D^1r : T^1X \to TX$ is the restriction of the derivative $Dr : TX \to TX$ to the unit tangent bundle $T^1X$. We extend it to random group actions through a finite
generating set $A$ of $G$. Namely, we say that a random $G$ action $T$ on $X$ is $C^1$ if $T^\omega(g) \in \text{Diff}^1(X)$ for every $\omega \in \Omega$ and $g \in G$. Given two $C^1$ random $G$-actions $T$ and $S$ we define $d_{C^1,A}(T,S) : \Omega \to \mathbb{R}^+$ by

$$d_{C^1,A}(T,S)\omega = \sup_{a \in A} d_{C^1}(T^\omega(a), S^\omega(a)), \quad \forall \omega \in \Omega.$$  

It follows that $d_{C^1,A}$ is an $\mathcal{F}$-metric in the sense of [14] and that different finite generating sets correspond to equivalent $\mathcal{F}$-metrics.

Given $\delta : \Omega \to \mathbb{R}^+$ we define the ball $B_{A}^1(T,\delta)$ as the set of $C^1$ random $G$-actions $S$ such that $d_{C^1,A}(T,S) \leq \delta$. By a $C^1$ neighborhood of a $C^1$ random $G$-action $T$ we mean a subset of $C^1$ random $G$-actions $\mathcal{U}$ containing a ball $B_{A}^1(T,\delta)$ for some finite generating set $A$ and some $\delta : \Omega \to \mathbb{R}^+$.

A local version for diffeomorphism of the following definition was considered by Sambarino and Vieitez [18] (with the name uniform robustly expansivity).

**Definition 1.6.** A random $G$-action $T$ is robustly expansive if there are a function $\rho : \Omega \to \mathbb{R}^+$ and a $C^1$ neighborhood $U$ of $T$ such that $\rho$ is an expansivity function of $S$, for every $S \in U$.

Finally we introduce the notion of rigidity for random group actions.

**Definition 1.7.** A random $G$-action $T$ is rigid if there is a $C^1$ neighborhood $U$ of $T$ such that for every $S \in U$ there is a map $h : \Omega \to C(X)$ such that $h(\omega)$ is a homeomorphism and

$$T^\omega(g) \circ h(\omega) = h(\theta(g)\omega) \circ S^\omega(g), \quad \forall g \in G, \omega \in \Omega.$$  

With these definitions we can state our result.

**Theorem 1.8.** Every finitely generated robustly expansive random group action with the shadowing property of a closed manifold is rigid.

The following corollary can be derived from Theorem 1.8 in the case that $\Omega$ reduces to a single point.

**Corollary 1.9.** Every finitely generated robustly expansive group action with the shadowing property of a closed manifold is rigid.

The basic examples of group actions satisfying the hypothesis of this corollary are the $\mathbb{Z}$-actions derived from the Anosov diffeomorphisms on closed manifolds. In particular, the above corollary implies the celebrated Anosov structural stability theorem: Every Anosov diffeomorphism of a closed manifold is structurally stable [2]. Different proofs of this theorem can be found in Moser [14], Hirsch-Pugh [10], Mather (see Appendix to part 1 in [19]), Robinson-Verjovsky [17] or Wen [22].

We divide the remainder of this paper as follows. In Section 2 we present some preliminary results. In Section 3 we prove a stability theorem for finitely generated random group actions. In Section 4 we prove our main result Theorem 1.8. In Section 5 we give some applications including iterated function systems and actions of the discrete Heisenberg group on tori.

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**2. Preliminary results.** We will need the following property of expansive random group actions corresponding to Lemma 2.10 in [5].
Lemma 2.1. Let $T$ be an expansive random $G$-action of a compact metric space $X$. If $ho$ is the expansivity function, then every $\omega \in \Omega$ and $\Delta > 0$ there is a finite set $F^\omega \subset G$ such that
\[
\sup_{g \in F^\omega} d(T^\omega(g)x, T^\omega(g)y) \leq \rho(\theta(g)\omega) \quad \text{implies} \quad d(x, y) \leq \Delta.
\]

Proof. Otherwise, by writing
\[
F = \bigcup_{n \in \mathbb{N}} F_n, \quad F_n \subset F_{n+1},
\]
$(F_n$ finite) we get $\Delta > 0$ and sequences $x_n, y_n \in X$ such that
\[
\sup_{g \in F_n} \frac{d(T^\omega(g)x_n, T^\omega(g)y_n)}{\rho(\theta(g)\omega)} < 1 \quad \text{but} \quad d(x_n, y_n) \geq \Delta \quad (\forall n).
\]
By compactness of $X$ we can assume that $x_n \to x$ and $y_n \to y$ for some $x, y \in X$. It follows that $d(x, y) \geq \Delta$.

Now fix $g \in G$. Then, $g \in F_n$ for all $n$ large hence
\[
d(T^\omega(g)x_n, T^\omega(g)y_n) \leq \rho(\theta(g)\omega)
\]
for all $n$ large. Letting $n \to \infty$ we get
\[
d(T^\omega(g)x, T^\omega(g)y) \leq \rho(\theta(g)\omega), \quad \forall g \in G.
\]
Since $\rho$ is an expansivity constant, we get $x = y$ which is absurd. This completes the proof. \hfill \Box

We will use the following auxiliary definition.

Definition 2.2. Let $T$ a random $G$-action of a compact metric space $X$ and $A$ be a finite generating set of $G$. We say that $T$ has the shadowing property with respect to $A$ if for every $\epsilon : \Omega \to \mathbb{R}^+$ there is $\delta : \Omega \to \mathbb{R}^+$ such that every $(\omega, \delta)$-pseudo orbit with respect to $A$ can be $(\omega, \epsilon)$-shadowed $(\forall \omega \in \Omega)$.

In particular, $T$ has the shadowing property if and only if it has the shadowing property with respect to some finite generating set. The following lemma shows that this property does not depend on the finite generating set. We follow the Appendix in Osipov and Tikhomirov [16].

Lemma 2.3. If $A$ and $A'$ are finite generating sets of $G$ and $T$ has the shadowing property with respect to $A$, then $T$ also has the shadowing property with respect to $A'$.

Proof. The generating set $A$ induces the so-called word norm $l_A(g)$ of $g \in G$ as the shorter length of a representation of $g \in G$ as a product of elements of $A$. Length norms corresponding to different generators $A$ and $A'$ are equivalent that is there is $C \geq 1$ such that
\[
\frac{1}{C} l_{A'} \leq l_A \leq C l_{A'}.
\]

Now fix $\epsilon : \Omega \to \mathbb{R}^+$ and let $\delta : \Omega \to \mathbb{R}^+$ be given by the shadowing property with respect to $A$.

By compactness, and the fact that $T^\omega(g) : X \to X$ is continuous (hence uniformly continuous) for $\omega \in \Omega$ and $g \in G$, we can choose $\delta_1 : \Omega \to \mathbb{R}^+$ with $\delta_1 \leq \frac{1}{C} \delta$ such that
\[
d(T^\omega(g)x, T^\omega(g)y) \leq \frac{1}{C} \delta(\omega),
\]
for all $\omega \in \Omega$, all $g \in G$ with $l_{A'}(g) \leq C$ and $x, y \in X$ with $d(x, y) \leq \delta_1(\omega)$.

Let $x' : G \to X$ be a $(\omega, \delta_1)$-pseudo orbit with respect to the generating set $A'$ i.e. 
\[ d(T^{\theta(g)}\omega(a')x'(g), x'(a'g)) \leq \delta_1(\theta(a'g)\omega), \quad \forall a' \in A', g \in G. \]

Since 
\[
\begin{align*}
&d(T^{\theta(g)\omega}(a'_C \cdots a'_1)x'(g), x'(a'_C \cdots a'_1g)) = \\
&d(x'(a'_C \cdots a'_1g), T^{\theta(a'_C-1 \cdots a'_1)\omega}(a'_C) \circ T^{\theta(a'_C-2 \cdots a'_1)\omega}(a'_C-1) \circ \cdots \circ T^{\theta(a'_1)\omega}(a'_1)) \\
\end{align*}
\]

one has $d(T^{\theta(g)\omega}(h)x'(g), x'(hg)) \leq \delta(\theta(hg)\omega)$ for all $g \in G$ and $h \in G$ with $l_{A'}(h) \leq C$. But by (2.1) every $a \in A$ satisfies $l_{A}(a) \leq C$ so 
\[ d(T^{\theta(g)\omega}(a)x'(g), x'(ag)) \leq \delta(\theta(ag)\omega), \quad \forall a \in A, g \in G. \]

Then, $x'$ is a $(\omega, \delta)$-pseudo orbit with respect to $A$. It follows from the hypothesis that $x'$ can be $\epsilon$-shadowed. As this latter property does not depend on generating sets, we are done.

The $C^0$-distance between maps $l, r : X \to X$ is defined by 
\[ d_{C^0}(l, r) = \sup_{x \in X} d(l(x), r(x)). \]

Given a finite generating set $A$ of $G$ we define the $C^0$ distance between two random $G$-actions $T$ and $S$ with respect to $A$ as the $\mathcal{F}$-metric 
\[ d_{C^0, A}(T, S)\omega = \sup_{a \in A} d_{C^0}(T^{\omega}(a), S^{\omega}(a)), \quad \forall \omega \in \Omega. \]

Below we show that the $\mathcal{F}$-metrics $d_A$ and $d_{A'}$ corresponding to finite generating sets $A$ and $A'$ are equivalent. The proof is derived from that of Lemma 2.2 in Chung and Lee [5].

**Lemma 2.4.** If $A$ and $A'$ are finite generating sets of $G$, then for every $\delta : \Omega \to \mathbb{R}^+$ there is $\delta_1 : \Omega \to \mathbb{R}^+$ such that if $S$ and $T$ are random $G$-actions on $X$ with common driver $\theta$ satisfying $d_{C^0, A'}(T, S) \leq \delta_1$, then $d_{C^0, A}(T, S) \leq \delta$.

**Proof.** Define $m = \max_{a \in A} l_{A}(a)$ and choose $\delta_0 : \omega \to \mathbb{R}^+$ such that 
\[ \delta_0(\omega) + \sum_b \delta_0(\theta(b)\omega) \leq \delta(\omega), \quad \forall \omega \in \Omega, \]
where $b$ runs over all possible products of at most $m$ elements of $A'$. Take $C$ as in (2.1).

As in the proof of Lemma 2.3, we choose $\delta_1 : \Omega \to \mathbb{R}^+$ such that 
\[ d(T^{\theta(g)\omega}(h)x, T^{\theta(g)\omega}(h)y) \leq \delta_0(\omega) \]
for all $\omega \in \Omega$, $g, h \in G$ with $\max\{l_{A'}(g), l_{A'}(h)\} \leq C$ and $x, y \in X$ with $d(x, y) \leq \delta_1(\omega)$.

Now suppose that $d_{C^0, A'}(T, S) \leq \delta_1$.

Take $x \in X$, $a \in A$ and $\omega \in \Omega$. By (2.1) we have $l_{A'}(a) \leq C$ so we can write $a = a'_1 \cdots a'_n$ for $n = l_{A'}(a) \leq C$ and $a'_n \cdots a'_1 \in A'$. It follows from the definitions that $n \leq m$.

Since $d_{C^0, A'}(T, S) \leq \delta_1$, one has
\[
d(T^{\omega}(a'_1)z, S^{\omega}(a'_1)z) \leq \delta_1(\omega)
\]
and
\[
d(T^{\theta(a'_1 \cdots a'_n)}(a'_n \cdots a'_{n+1})z, S^{\theta(a'_1 \cdots a'_n)}(a'_n \cdots a'_{n+1})z) \leq \delta_1(\theta(a'_1 \cdots a'_n)\omega),
\]
for all $\omega \in \Omega$, $z \in X$, $i = 1, \ldots, n - 1$.

Therefore,
\[
d(T^{\omega}(a) x, S^{\omega}(a) x) \leq d(T^{\theta(a'_1)}(a'_1) x, T^{\theta(a'_1)}(a'_1) x) + \left( T^{\theta(a'_1 \cdots a'_3)}(a'_3) \circ S^{\omega}(a'_1) x, T^{\theta(a'_1 \cdots a'_3)}(a'_3) \circ S^{\omega}(a'_1) x \right) + \
\cdots +
\left( T^{\theta(a'_{n-1} \cdots a'_n)}(a'_n) \circ S^{\theta(a'_{n-1} \cdots a'_n)}(a'_n) \circ S^{\omega}(a'_1) x, S^{\theta(a'_{n-1} \cdots a'_n)}(a'_n) \circ S^{\theta(a'_{n-1} \cdots a'_n)}(a'_n) \circ S^{\omega}(a'_1) x \right)
\]
\[
\leq \delta_0(\omega) + \delta_0(\theta(a'_1)\omega) + \cdots + \delta_0(\theta(a'_{n-1} \cdots a'_n)\omega) \quad (n \leq m) \leq \delta(\omega), \quad \forall \omega \in \Omega,
\]
proving the result.

3. **Topological stability for random group actions.** The following concept combines the topological stability for group actions [5, 16] and Example 2 in [8].

**Definition 3.1.** We say that a random $G$-action $T$ on $X$ is **topologically stable** if there is a finite generating set $A$ such that for every $\epsilon : \Omega \to \mathbb{R}^+$ there is $\delta : \Omega \to \mathbb{R}^+$ such that for every random $G$-action $S$ with $d_{C^0, A}(T, S) \leq \delta$ there is $h : \Omega \to \mathcal{C}(X)$ such that
\[
d_{C^0}(h(\omega), I_X) \leq \epsilon(\omega) \quad \text{and} \quad T^{\omega}(g) \circ h(\omega) = h(\theta(g)\omega) \circ S^{\omega}(g),
\]
for every $g \in G, \omega \in \Omega$.

**Remark 3.2.** By Lemma 2.4 we have that the above definition does not depend on the finite generating set $A$.

With this definition we can prove the following result. It extends the Chung-Lee stability theorem [5] which in turn generalizes Walters [21].

**Theorem 3.3.** Every expansive finitely generated random action with the shadowing property of a compact metric space is topologically stable.
Proof. Let $T$ be a expansive finitely generated random $G$-action with the shadowing property $T$ of a compact metric space $X$. Denote by $\theta$ the driver of $T$ which by definition is a $G$-action on $\Omega$. Denote by $\rho : \Omega \to \mathbb{R}^+$ the expansivity function of $T$. By definition $T$ has the shadowing property with respect to some finite generating set $A$. Take any function $\epsilon : \Omega \to \mathbb{R}^+$ and define $\epsilon' = \frac{1}{2} \min \{ \rho, \epsilon \}$. This gives us a map $\epsilon' : \Omega \to \mathbb{R}^+$ from which we take the function $\delta : \Omega \to \mathbb{R}^+$ given by the shadowing property of $T$ with respect to $A$. Take a random $G$-action $S$ on $X$ with driver $\theta$ such that $d_{C^0, A}(T, S) \leq \delta$.

Fix $x \in X$ and $\omega \in \Omega$. Because $d_{C^0, A}(T, S) \leq \delta$ one has

$$d(T^{\theta(\omega)}(a)S^\omega(g)x, S^\omega(\rho g)x) = d(T^{\theta(\omega)}(a)S^\omega(g)x, S^\theta(\omega)(a)S^\omega(g)x) \leq \delta(\omega), \quad \forall a \in A, g \in G,$$

proving that the map $g \in G \mapsto S^\omega(g)x$ is an $(\omega, \delta)$-pseudo orbit of $T$. It follows from the shadowing property that there is $\hat{x}(\omega) \in X$ such that

$$d(T^\omega(g)\hat{x}(\omega), S^\omega(\rho g)x) \leq \epsilon'(\theta(g)\omega), \quad \forall g \in G.$$

Since $\epsilon' \leq \frac{\epsilon}{2}$, we can easily see that $\hat{x}(\omega)$ is the unique point satisfying this inequality for given $x$ and $\omega$. This produces a $\Omega$-parametrized family of maps $h(\omega) : X \to X$ satisfying

$$d(T^\omega(h(\omega)x, S^\omega(x)) \leq \epsilon'(\theta(g)\omega), \quad \forall g \in G, \omega \in \Omega, x \in X. \quad (3.1)$$

By putting $g = e$ (the neutral element of $G$) in (3.1) we get $d(h(\omega)x, x) \leq \epsilon(\omega)$ for every $\omega \in \Omega$ and $x \in X$ Therefore,

$$d_{C^0}(h(\omega), Ix) \leq \epsilon(\omega), \quad \forall \omega \in \Omega.$$

Now, replacing $g$ by $gr$ in (3.1) we get

$$d(T^\omega(gr)h(\omega)x, S^\omega(gr)x) \leq \epsilon'(\theta(gr)\omega)$$

hence

$$d(T^{\theta(r)\omega}(g)T^\omega(r)h(\omega)x, S^{\theta(r)\omega}(g)S^\omega(r)x) \leq \epsilon'(\theta(g)\theta(r)\omega).$$

Replacing $\omega$ and $x$ by $\theta(r)\omega$ and $S^\omega(r)x$ respectively in (3.1) we get

$$d(T^{\theta(r)\omega}(g)h(\theta(r)\omega)x, S^{\theta(r)\omega}(g)S^\omega(r)x) \leq \epsilon'(\theta(g)\theta(r)\omega).$$

Then, the triangle inequality yields

$$d(T^{\theta(r)\omega}(g)(T^\omega(r)h(\omega)x, T^{\theta(r)\omega}(g)(h(\theta(r)\omega)S^\omega(r)x)) \leq \rho(\theta(g)\theta(r)\omega)$$

for every $g \in G$. Then, by expansiveness,

$$T^\omega(r)h(\omega)x = h(\theta(r)\omega)S^\omega(r)x, \quad \forall x \in X$$

proving

$$T^\omega(r) \circ h(\omega) = h(\theta(r)\omega) \circ S^\omega(r), \quad \forall r \in G, \omega \in \Omega.$$

To finish we prove the continuity of $h(\omega)$ for every $\omega \in \Omega$. Fix such an $\omega$. Let $\Delta > 0$ and $F^\omega$ given by Lemma 2.1. Since $F^\omega$ is finite, there is $\gamma > 0$ such that

$$d(x, y) \leq \gamma \quad \text{implies} \quad \sup_{g \in F^\omega} d(S^\omega(g)x, S^\omega(g)y) \leq \frac{\rho(\theta(g)\omega)}{3}.$$
Then, if \( d(x, y) \leq \gamma \) and \( g \in F^\omega \) one has
\[
d(T^\omega(g)h(\omega)x, T^\omega(g)h(\omega)y) = d(h(\theta(g)\omega)S^\omega(g)x, h(\theta(g)\omega)S^\omega(g)y) \\
\leq d(h(\theta(g)\omega)S^\omega(g)x, S^\omega(g)x) + \\
d(S^\omega(g)x, S^\omega(g)y) + \\
d(h(\theta(g)\omega)S^\omega(g)y, S^\omega(g)y) \\
\leq 2\epsilon'(\theta(g)\omega) + \frac{\rho(\theta(g)\omega)}{3} \\
\leq \rho(\theta(g)\omega).
\]

It follows from Lemma 2.1 that \( d(h(\omega)x, h(\omega)y) \leq \Delta \) proving the continuity of \( h(\omega) \). This completes the proof. \( \square \)

4. **Proof of Theorem 1.8.** Let \( T \) be a finitely generated robustly expansive random action with the shadowing property of a closed manifold \( X \). Let \( G \) be the corresponding group, and \( \theta \) the driver. We have that \( T \) is expansive and has the shadowing property and so it is topologically stable by Theorem 3.3. By hypothesis, there are a finite generating set \( A \) and functions \( \delta_0, \rho : \Omega \to \mathbb{R}^+ \) such that \( \rho \) is an expansivity function of every \( S \in \mathcal{U}_0 = B^1_A(T, \delta_0) \). Since topological stability does not depend on the finite generating set (Remark 3.2), we can assume that \( A \) is the generating set in the corresponding definition.

We can choose \( \epsilon > 0 \) such that every continuous map \( r : X \to X \) with \( d_{C^0}(r, Id_X) \leq \epsilon \) is onto \([9]\). Let \( \epsilon' = \frac{\min(\rho, \epsilon)}{2} \) and \( \delta : \Omega \to \mathbb{R}^+ \) be given by the topological stability of \( T \) for this \( \epsilon' \). We also choose a \( C^1 \) neighborhood \( \mathcal{U} \subset \mathcal{U}_0 \) such that \( d_{C^0}(A(T, S)) \leq \delta \) for every \( S \in \mathcal{U} \).

Now take a random \( G \)-action \( S \in \mathcal{U} \). Then, \( S \in \mathcal{U}_0 \) and so \( \rho \) is an expansivity function of \( S \). Also \( d_{C^0}(A(T, S)) \leq \delta \) and so, by topological stability, there is \( h : \Omega \to C(X) \) such that \( d_{C^0}(h(\omega), Id_X) \leq \epsilon'(\omega) \) and \( T^\omega(g) \circ h(\omega) = h(\theta(g)\omega) \circ S^\omega(g) \) for every \( \omega \in \Omega \) and \( g \in G \). Since \( \epsilon' \leq \epsilon, d_{C^0}(h(\omega), Id_X) \leq \epsilon \) hence \( h(\omega) \) is onto for every \( \omega \in \Omega \). We finish by proving that \( h(\omega) \) is injective for every \( \omega \in \Omega \). Take \( x, y \in X \) and \( \omega \in \Omega \) such that \( h(\omega)x = h(\omega)y \). It follows that
\[
d(S^\omega(g)x, S^\omega(g)y) \leq d(S^\omega(g)x, h(\theta(g)\omega)S^\omega(g)x) + \\
d(h(\theta(g)\omega)S^\omega(g)x, h(\theta(g)\omega)S^\omega(g)y) + \\
d(S^\omega(g)y, h(\theta(g)\omega)S^\omega(g)y) \\
\leq 2d_{C^0}(h(\theta(g)\omega), Id_X) + \\
d(T^\omega(g)h(\omega)x, T^\omega(g)h(\omega)y) \\
= 2d_{C^0}(h(\theta(g)\omega), Id_X) \\
\leq 2\epsilon'(\theta(g)\omega) \\
\leq \rho(\theta(g)\omega), \quad \forall g \in G.
\]

Since \( \rho \) is an expansivity function of \( S \), we conclude that \( x = y \). Then, \( h(\omega) \) is injective and so a homeomorphism for every \( \omega \in \Omega \). This completes the proof. \( \square \)

5. **Applications.** In this section we present some applications of Theorem 1.8. A source of examples is given by the following proposition.

**Proposition 5.1.** For every collection \( F \) of homeomorphisms of a compact metric space \( X \) there is a unique random \( \mathbb{Z} \)-action \( T \) on \( X \) satisfying the following properties:
1. $T$ is driven by the shift namely the $\mathbb{Z}$-action defined by $\theta : \mathbb{Z} \times \mathcal{F}^\mathbb{Z} \to \mathcal{F}^\mathbb{Z}$, 
\[ \theta(n)\omega(m) = \omega(m + n), \quad \forall n, m \in \mathbb{Z}, \omega \in \mathcal{F}^\mathbb{Z}. \]

2. $T$ is normalized, i.e., $T^\omega(1) = \omega(0)$ for every $\omega \in \mathcal{F}^\mathbb{Z}$.

**Proof.** To prove the existence define \( \{T^\omega : X \to X\}_{\omega \in \mathcal{F}^\mathbb{Z}} \) by
\[
T^\omega(n) = \begin{cases} 
\omega(n – 1) \circ \cdots \circ \omega(0), & \text{if } n \geq 1 \\
I_X, & \text{if } n = 0 \\
(\omega(n))^{-1} \circ (\omega(n+1))^{-1} \circ \cdots \circ (\omega(-1))^{-1}, & \text{if } n \leq -1
\end{cases}
\]

It follows that $T$ is a random $\mathbb{Z}$-action driven by the shift (see p. 118 in [8]).

To prove the unicity let $T$ and $S$ be two random $\mathbb{Z}$-actions on $X$ satisfying items (1) and (2) in the statement of the proposition. It follows that $T^\omega(1) = \omega(0) = S^\omega(1)$ for every $\omega \in \mathcal{F}^\mathbb{Z}$. Also,
\[
T^\omega(-1) = (T^{\theta(-1)}\omega(1))^{-1} = (\theta(-1)\omega(0))^{-1} = (\omega(-1))^{-1},
\]

and likewise $S^\omega(-1) = (\omega(-1))^{-1}$ so $T^\omega(-1) = S^\omega(-1)$ for every $\omega \in \mathcal{F}^\mathbb{Z}$. Since \{-1,1\} generates $\mathbb{Z}$, we conclude from the axioms of random group action that $T = S$. 

The random $\mathbb{Z}$-action in this proposition is called the **iterated function system** (IFS) generated by $\mathcal{F}$.

**Example 5.1.** Let $f : M \to M$ be an Anosov diffeomorphism of a closed manifold. If $\mathcal{F}$ is a family of $C^1$ diffeomorphisms of $M$ whose members are $C^1$ close to $f$, then the IFS generated by $\mathcal{F}$ in Proposition 5.1 is robustly expansive and has the shadowing property (see Proposition 3.6 in [8]) and so it is rigid by Theorem 1.8. This is precisely the structural stability result described in Example 2 of [8].

Now we present an application to the rigidity of discrete Heisenberg group actions on compact manifolds [11]. The **discrete Heisenberg group** is the unique group $H$ generated by three elements $a, b, c$ satisfying
\[ ac = ca, \quad bc = cb \quad \text{and} \quad ab = bac. \]

It is well known that $H$ is virtually nilpotent namely has a nilpotent normal subgroup of finite index.

Let us present explicit actions of this group on closed manifolds. Given $n \in \mathbb{N}$ and matrizes $x, y \in SL(n, \mathbb{Z})$ we define the $(3n \times 3n)$-matrices
\[
a = \begin{pmatrix} x & I_n & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}, \quad b = \begin{pmatrix} y & 0 & 0 \\ 0 & y & I_n \\ 0 & 0 & y \end{pmatrix}, \quad c = \begin{pmatrix} I_n & 0 & x^{-1}y^{-1} \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix}
\]

We can see that these matrices satisfy the above relations for the Heisenberg group. Since they also define diffeomorphisms of the torus $X = \mathbb{T}^{3n}$, we obtain a $C^1$ $H$-action $T$ on $\mathbb{T}^{3n}$. If $x$ has no eigenvalues of modulus 1, then $a$ also has no eigenvalues of modulus 2. For $n = 2$ we can take for instance
\[
x = y = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}
\]

In such a case we see that $T(a)$ is an Anosov diffeomorphism and so $T$ is robustly expansive. Since $H$ is nilpotent, and the Anosov diffeomorphisms have the shadowing property, we can combine lemmas 2.12 and 2.13 in [5] to obtain that $T$ has the
shadowing property too. It then follows from Corollary 1.9 that $T$ is rigid (or $C^{1,1,0}$ locally rigid in [11]'s terminology). This represents a rigidity result of the discrete Heisenberg group actions on tori (see also Theorem E in [11]).

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