An analysis of the effects of the orientation angle of a surface crack on the vibration of an isotropic plate

Rainah Ismail and M P Cartmell
School of Engineering, University of Glasgow, Glasgow G12 8QQ, Scotland, UK
Email: rismail@eng.gla.ac.uk

Abstract. This paper proposes a vibration analysis for an isotropic plate containing an arbitrarily orientated surface crack as an enhancement to previous work on cracked plates for which the orientation of the crack angle was not included. The governing equation of motion of the plate with this enhanced crack modelling represents the vibrational response based on classical plate theory into which a developed crack model has been assimilated. The formulation of the angled crack is based on a simplified line-spring model and the cracked plate is subjected to transverse harmonic excitation with arbitrarily chosen boundary conditions. It is found that the vibrational characteristics of the plate structure can be affected significantly by the orientation of the crack in the surface plate. For reasons of comparison and validation a finite element model is used for a further modal analysis in order to corroborate the effect of crack length and crack orientation angle on the modal parameters i.e. the natural frequency and also the vibrational amplitude, as predicted by the analysis. The results show excellent agreement between the two methods.

1. Introduction
A comprehensive background on plates has been provided by Timoshenko and Woinowsky-Krieger [1], and the research of this paper is focused on thin plate structures in order to develop enhanced modelling for reliable, light and efficient structures. This type of structure can lead to unwanted instances of high vibration and then may lead to damaging effects on the structure. Cracks can form and propagate catastrophically with very little warning and then can create a complete unbalance of the structure which leads to ultimate failure. Failure of a structure can result in terrible consequences, economically and most probably and importantly in terms of loss of life. The dynamic responses of rectangular plates with cracks, or minor irregularities under different loading conditions, have been investigated in the past by many researchers for different boundaries conditions [2]. The length, position, and orientation of a crack will affect the vibration characteristics of any host structure, in addition to other effects caused by material properties, plate geometry, and boundary conditions.

Most studies of vibration analysis in cracked plates have investigated cracks which have tended to be located at the centre or edge of the plate, and parallel to one side of the plate. Only a few papers [3-6] have investigated the vibration analysis of a plate with a crack which is not parallel along one side of the plate. However, no literature contains any substantial references to the study of a plate with cracks of variable orientation undergoing forced vibration. The aim of this research is to extend the forced vibration analysis of the cracked plate discussed by Israr [2] and Israr et al. [7] by considering an alternative geometry whereby the crack orientation is variable.
2. Modelling of a plate with a crack of variable angular orientation

An analytical approach is presented for the forced vibration analysis of a plate containing an arbitrarily orientated surface crack, based on three different boundary conditions. The method was based on classical plate theory. Firstly, the equation of motion is derived for an isotropic plate containing the angled surface crack which is arbitrarily orientated at an angle $\beta$ with respect to the $x$ axis of the plate, and this plate is loaded by an arbitrarily located concentrated force, $\bar{q}$, at some point $(x_0, y_0)$ [8]. The crack formulation representing the surface crack of variable angular orientation with length $2a$ is based on a simplified line-spring model [8-9]. The final form of the equation of motion for the forced vibration in a thin plate with a variably oriented surface crack emerges, as follows,

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = -\rho \frac{\partial^2 w}{\partial t^2} + n_x \frac{\partial^2 w}{\partial x^2} + q_{\bar{z}} + \frac{aD}{3} \frac{(1 + \cos 2\beta)}{(\alpha_{w} + \alpha_{v})[1 - v^2]} \frac{(\partial^4 w)}{\partial x^4} + n_y \frac{(\partial^4 w)}{\partial y^4} + \frac{aD}{3} \frac{(1 + \cos 2\beta)}{(\alpha_{w} + \alpha_{v})[1 - v^2]} \frac{2aD sin 2\beta}{(6\alpha_{w} + \alpha_{v})} \frac{(\partial^4 w)}{\partial x^2 \partial y^2} + \frac{2aD sin 2\beta}{(6\alpha_{w} + \alpha_{v})} \frac{\partial^2 w}{\partial x^2 \partial y^2}$$

(1)

where $D$ is the flexural rigidity of the plate, represented by $D = \frac{E d^3}{12(1 - v^2)}$, with $E$, $h$ and $v$ representing the modulus of elasticity, plate thickness, and Poisson ratio, respectively, and where $\rho(h(\partial^2 w/\partial t^2))$ is the inertia force, with $\rho$ defining the plate density. $n_x$ and $n_y$ are the in-plane forces in the $x$ and $y$ directions, respectively. The non-dimensional compliance coefficients $\alpha_{w}$, $\alpha_{w b}$ and $\alpha_{w b} = \alpha_{w b}$ can be found in [10], and the compliance coefficients $C_{w}$, $C_{w b}$, and $C_{w b} = C_{w b}$ can be seen in [11] and [12]. Next Galerkin’s method is applied in the usual manner to discretise the partial differential equation in (1) and transform the transverse deflection coordinate $w(x, y, t)$ into time dependent modal coordinates. Subsequently, by employing the Berger formulation, the derived governing equation of motion of the cracked plate is converted into a nonlinear ordinary differential equation model in terms of a chosen modal coordinate, and then by considering the system to be under the influence of weak classical linear viscous damping $\mu$, and the load to be harmonic [2], the final equation of motion for the system takes the form of a specialised Duffing equation,

$$\psi_{\gamma}(t) + 2\mu \psi_{\gamma}(t) + \alpha_{\gamma} \psi_{\gamma}(t) + \gamma_{\gamma} \psi_{\gamma}(t) = \frac{\eta_{\gamma}}{D} \frac{q}{q} \cos \Omega_{\gamma} t$$

(2)

where $\alpha_{\gamma} = \frac{K_{\gamma}}{M_{\gamma}}$, $\gamma_{\gamma} = \frac{G_{\gamma}}{M_{\gamma}}$, and $\eta_{\gamma} = \frac{\eta_{\gamma}}{D}$ and

$$M_{\gamma} = \frac{\rho h}{D} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} A_{ij} \int_{0}^{L} \int_{0}^{L} X_{ij} Y_{ij} \; dx \; dy$$

(3)

and

$$K_{\gamma} = \frac{\rho h}{D} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} A_{ij} \int_{0}^{L} \int_{0}^{L} X_{ij} Y_{ij} + 2X_{ij} Y_{ij} + Y_{ij} X_{ij} - \frac{a(1 + \cos 2\beta)}{3(\alpha_{w} + \alpha_{v})} \left(Y_{ij} X_{ij} + \nu X_{ij} Y_{ij}\right)$$

(4)

$$- \frac{2a \sin 2\beta}{3(\alpha_{w} + \alpha_{v})} \left(\nu X_{ij} Y_{ij} + \nu X_{ij} Y_{ij}\right) \; dx \; dy$$

$$G_{\gamma} = \frac{\rho h}{D} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} A_{ij} \int_{0}^{L} \int_{0}^{L} P_{ij} X_{ij} Y_{ij} + \frac{a(1 + \cos 2\beta)}{(\alpha_{w} + \alpha_{v})[1 - v^2]} \; P_{ij} X_{ij} Y_{ij} + \frac{2a \sin 2\beta}{(6\alpha_{w} + \alpha_{v})[1 - v^2]} \; P_{ij} X_{ij} Y_{ij} \; dx \; dy$$

(5)

$$P_{ij} = \frac{6}{h^2 L^2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \left[ \frac{\partial X_{ij}}{\partial x} \right]^2 Y_{ij}^2 + \nu \left[ \frac{\partial Y_{ij}}{\partial y} \right] X_{ij}^2 \; dx \; dy$$

(6)

and

$$P_{ij} = \frac{6}{h^2 L^2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \left[ \frac{\partial X_{ij}}{\partial x} \right]^2 Y_{ij}^2 + \nu \left[ \frac{\partial Y_{ij}}{\partial y} \right] X_{ij}^2 \; dx \; dy$$

(7)
and $q_{ij} = q_x(t) \delta_{ij}$ where $Q_{ij} = X_i(x_0) Y_j(y_0)$ is the integral of the delta function given by Israr et al. [7] in the form,

$$\int X_i(x) \delta(x - x_i) dx = X_i(x_i)$$ \hspace{1cm} (8)

also noting that $\Omega_{ij}$ is the excitation frequency, $A_{ij}$ is the arbitrary amplitude, $\psi_{ij}$ is the time dependent modal coordinate for the system, and, $X_i$ and $Y_j$ represent the characteristic or modal functions in the $x$ and $y$ directions of the cracked rectangular plate, respectively, with $i, j$ being the plate mode designators.

3. Numerical results

In this section, simulation results are presented for the intact plate and the enhanced cracked plate model with arbitrarily chosen CCFF type boundary conditions. The type of material used in this investigation is an aluminium alloy of 5083 grade, with the modulus of elasticity $E = 7.03 \times 10^{10}$ Nm$^{-2}$, plate density $\rho = 2660$ kgm$^{-3}$, Poisson’s ratio $\nu = 0.33$, and measured damping ratio of $\mu = 0.08$. A convergence study was firstly carried out with [8] in order to verify the correctness of the model within an analytical model of the plate for three different types of boundary conditions [9]. In section 3.1 simulation results are presented for investigation of the natural frequency of the first mode of the intact plate and enhanced cracked plate model for various aspect ratios, and factors that influence changes in the trend of the natural frequency for this boundary condition are discussed in section 3.2.

Table 1. Natural frequencies for the clamped-clamped free-free (CCFF) boundary condition

| Crack angle, $\beta$ (deg) | First Mode Natural Frequency, $\omega_{ij}$ (rad/s) |
|---------------------------|---------------------------------------------------|
|                           | Intact Plate | Cracked Plate a = 3mm | Cracked Plate a = 7.5mm | Intact Plate | Cracked Plate a = 3mm | Cracked Plate a = 7.5mm |
| Square plate ($l_1$=0.3, $l_2$=0.3) | 268.22 | 247.29 | 223.78 | 1072.89 | 989.15 | 989.10 |
| 0° | - | - | - | - | - |
| 20° | 265.40 | 247.30 | 223.78 | 1061.58 | 989.21 | - |
| 40° | 279.11 | 268.65 | 223.78 | 1116.42 | 1074.59 | - |
| 60° | 282.94 | 279.22 | 223.78 | 1131.74 | 1116.86 | - |
| 80° | 275.47 | 268.22 | 223.78 | 1101.90 | 1101.42 | - |
| 90° | 268.22 | 268.22 | 223.78 | 1072.89 | 1072.89 | - |
| Rectangular plate ($l_1$=0.15, $l_2$=0.3) | 770.27 | 770.27 | 770.27 | 770.27 | - | - |
| 0° | 743.88 | 715.78 | 715.78 | 650.48 | 498.48 | - |
| 20° | 765.52 | 742.67 | 742.67 | 703.75 | 586.80 | - |
| 40° | 782.78 | 769.00 | 769.00 | 757.28 | 689.93 | - |
| 60° | 788.05 | 783.03 | 783.03 | 788.12 | 762.82 | - |
| 80° | 779.10 | 778.91 | 778.91 | 784.84 | 782.59 | - |
| 90° | 770.27 | 770.27 | 770.27 | 770.27 | 770.27 | - |

3.1. A plate with a surface crack of variable angular orientation

Dimensions of the plate considered in this investigation are $l_1= 150$ mm, $l_2 = 300$ mm, and plate thickness $h = 3$ mm. A load of 10 N was applied at a distance of 225 mm from the fixed edge on the $x$-axis and 115 mm from the fixed edge on the $y$-axis. Table 1 shows the results for the first mode natural frequencies $\omega_{ij}$ at different lengths of half-crack and for different values of crack orientation angle, $\beta$. The orientation angle is arbitrarily chosen from 0° to 80°, in 20° steps and finishes at 90°. The crack is also located at the centre of the plate with 3mm and 7.5mm half-crack lengths, respectively. The results generally show that the natural frequency reduces with an increase in half-crack length. Furthermore, in terms of the crack orientation angle effect it can clearly be seen that the natural
frequency increases with the increase in the crack angle, up to 60° and then decreases when β exceeds 60°. This similar trend in the crack orientation effect was also found by [3] and [4].

3.2. Factors which influence changes in the natural frequency
In this section a parametric study is performed on the natural frequency equation of the cracked plate for the case of the CCFF boundary condition. This type of boundary condition is selected because the results showed interesting changes in the trend of the natural frequency values at an angle of 60°, and this trend is different to those for SSSS and CCSS. This study is intended to examine configurations to determine the physical parameters that influence the changeover of the maximum value of the crack orientation angle. Here, maximum means that value of crack orientation angle for which the natural frequency reaches a maximum value before it decreases. It is found that all the factors of Figure 1 influence the natural frequency, for example when the half-crack length is 1 mm the natural frequency increases for the crack angle, β up to 50.77° and reduces when β exceeds 50.77°. When the half-crack length increases to 30 mm, the natural frequency increases up to 75.06° and then decreases when β exceeds 75.06°. The results, given in Figure 1(a)-(c) show that the crack orientation angle increases with the crack length and plate aspect ratio, however it decreases with an increase in the plate thickness (Figure 1(b)). For the Poisson ratio it can be seen that in Figure 1(d) the crack angle decreases up to 0.42, and then increases when the ratio exceeds 0.42 while the results of density and modulus of elasticity show a very slight difference for the crack orientation angle for a maximum natural frequency value in which significant changes cannot be observed from Figures 1(e)-(f).

![Figure 1. Factors which influence crack orientation angle for maximum natural frequency.](image)

4. Finite Element Method (ABAQUS/CAE 6.9.1)
In this study the modal characteristics of an intact plate and twelve cracked plates, namely the natural frequencies, mode shapes and displacements, were all evaluated using Abaqus/CAE. In Table 2 the frequency values from frequency extraction analysis for the three first modes of the cracked plate for orientation angle β are shown together with the results for an intact plate. It can be seen that the large crack shifts the frequency values of the 1st, 2nd and 3rd modes downwards, as expected, and is due to reduced plate stiffness. For both crack lengths of 3 mm and 7.5 mm, the frequency values increase monotonously from 0° up to 60°, and then decrease when β is more than 60°. Table 2 also shows the results of the amplitude response of the simulations, and, as expected, the amplitude values increase...
due to the small crack in the plate. These amplitude responses are also affected by the crack orientation angle where the amplitude decreases from 0° up to 60° when the frequency value increases, and then increases again when the frequency value decreases. Figure 2 shows the first three mode shapes of the plates for a crack length of 3 mm and orientation angle of 60°. The dark blue areas in these figures indicate nodal displacements for the first three modes of vibration, representing the areas where the displacement is close to zero.

| Table 2. Frequency extraction analysis and amplitude response |
|-------------------------------------------------------------|
| **FEA Results**                                             |
| Crack Orientation angle, \( \beta \) | Frequency (Hz) | Amplitude (mm) |
|----------------------------------------|----------------|----------------|
| Intact Plate                           |                |                |
| -                                      | 122.94         | 259.80         | 525.16        | 7.372 |
| 0°                                     | 122.73         | 259.32         | 524.18        | 7.483 |
| 20°                                    | 122.77         | 259.49         | 524.37        | 7.379 |
| 40°                                    | 122.84         | 259.82         | 525.00        | 7.263 |
| 60°                                    | 123.05         | 260.79         | 527.47        | 7.200 |
| 80°                                    | 122.82         | 259.54         | 524.38        | 7.727 |
| 90°                                    | 122.73         | 259.36         | 524.22        | 7.739 |
| Cracked Plate 3 mm                      |                |                |
| 0°                                     | 122.69         | 259.13         | 524.00        | 7.564 |
| 20°                                    | 122.75         | 259.37         | 524.18        | 7.227 |
| 40°                                    | 122.81         | 259.74         | 524.79        | 6.969 |
| 60°                                    | 123.04         | 260.72         | 527.34        | 6.796 |
| 80°                                    | 122.81         | 259.57         | 524.29        | 7.309 |
| 90°                                    | 122.69         | 259.33         | 524.12        | 7.614 |
| Cracked Plate 7.5 mm                    |                |                |
| 0°                                     | 122.69         | 259.33         | 524.12        | 7.614 |

**Figure 2.** Vibration mode shapes of a cracked plate with crack orientation angle, \( \beta = 60° \).

5. **Comparative study**
A comparative study of the theoretical modelling and FE approaches is presented here. Both sets of results in Table 3 show a significant change in natural frequency and response amplitude for the different lengths of the crack and the varying orientation angle of the crack. The natural frequencies obtained for both results decrease slightly with an increase in the crack length for every crack inclination angle. In addition, it is apparent that the FE predicted frequency and amplitude trends are similar to the analytical results for which the frequency values increase from 0° up to 60° and then decrease when \( \beta \) is more than 60°, while the amplitude responses behave conversely.

6. **Conclusions**
The results show overall that the cracked plate model is very sensitive to the crack length and orientation angle and that it is able to predict well for the cases tested using the CCFF boundary conditions. Orientation angles for which the natural frequency is maximum are affected by crack
Table 3. Comparison results between theoretical and FE analysis for Mode I only.

| Crack Orientation angle, β | Frequency (Hz) | Error (%) | Amplitude (mm) | Error (%) |
|----------------------------|----------------|-----------|----------------|-----------|
|                            | Frequency (Hz) | Error (%) | Amplitude (mm) | Error (%) |
|                            | Analy.         | FEA       | Analy.         | FEA       |
| Intact Plate               | -              |           | -              |           |
| 0°                         | 122.58         | 122.94    | 0.29           | 7.979     | 7.372 | 7.61 |
| 20°                        | 121.82         | 122.77    | 0.78           | 7.220     | 7.379 | 2.20 |
| 40°                        | 124.57         | 122.84    | 1.39           | 6.993     | 7.263 | 3.86 |
| 60°                        | 125.41         | 123.05    | 1.88           | 6.898     | 7.200 | 4.38 |
| 80°                        | 123.98         | 122.82    | 0.94           | 7.524     | 7.727 | 2.70 |
| 90°                        | 122.58         | 122.73    | 0.12           | 7.979     | 7.739 | 3.01 |
| Cracked Plate 3 mm         |                |           |                |           |
| 0°                         | 113.91         | 122.69    | 7.71           | 8.066     | 7.564 | 6.22 |
| 20°                        | 118.18         | 122.75    | 3.86           | 6.951     | 7.227 | 3.97 |
| 40°                        | 122.37         | 122.81    | 0.36           | 6.659     | 6.969 | 4.66 |
| 60°                        | 124.61         | 123.04    | 1.26           | 6.548     | 6.796 | 3.79 |
| 80°                        | 123.95         | 122.81    | 0.92           | 7.336     | 7.309 | 0.37 |
| 90°                        | 122.58         | 122.69    | 0.09           | 7.979     | 7.614 | 4.57 |
| Cracked Plate 7.5 mm       |                |           |                |           |
| 0°                         | 113.91         | 122.69    | 7.71           | 8.066     | 7.564 | 6.22 |
| 20°                        | 118.18         | 122.75    | 3.86           | 6.951     | 7.227 | 3.97 |
| 40°                        | 122.37         | 122.81    | 0.36           | 6.659     | 6.969 | 4.66 |
| 60°                        | 124.61         | 123.04    | 1.26           | 6.548     | 6.796 | 3.79 |
| 80°                        | 123.95         | 122.81    | 0.92           | 7.336     | 7.309 | 0.37 |
| 90°                        | 122.58         | 122.69    | 0.09           | 7.979     | 7.614 | 4.57 |

length, plate thickness, plate aspect ratio, Poisson ratio, plate density and modulus of elasticity. Simulation with standard parameter properties showed changes in the natural frequency values at an angle of approximately 57° which is reasonably close to 60°. Very close agreement was obtained between the analytical and FE results with the maximum error in the prediction of the frequency value at about 7.71% and the amplitude response at 7.61%.

References

[1] S.P.Timoshenko, S.Woinowsky-Krieger, Theory of plates and shells, McGraw-Hill, New York (1959).
[2] A.Israr, Vibration analysis of cracked aluminium plates, PhD Thesis, University of Glasgow (2008).
[3] K.Maruyama, O.Ichinomiya, Experimental study of free vibration of clamped rectangular plates with straight narrow slits, JSME International Journal, Ser. 3, Vibration, control engineering, engineering for industry 32 (2) (1989) 187-193.
[4] D.Wu, S.S.Law, Anisotropic damage model for an inclined crack in thick plate and sensitivity study for its detection, International Journal of Solids and Structures 41 (2004) 4321-4336.
[5] C.S.Huang, A.W.Leissa, Vibration analysis of rectangular plates with side cracks via the Ritz Method, Journal of Sound and Vibration 323 (2009) 974-988.
[6] C.S.Huang, A.W.Leissa, C.W.Chan, Vibrations of rectangular plates with internal cracks or slits, International Journal of Mechanical Sciences 53 (2011) 436-445. Another reference
[7] A.Israr, M.P.Cartmell, E.Manooch, I.Trendafilova, W.M.Ostachowicz, M.Krawczuk, A.Zak, Analytical modelling and vibration analysis of partially cracked rectangular plates with different boundary conditions and loading, Journal of Applied Mechanics 76 (2009) 1-9.
[8] Ismail, R., and Cartmell, M.P., An investigation into the vibration analysis of a plate with a surface crack of variable angular orientation. Journal of Sound and Vibration, 331 (2012) 2929-2948.
[9] Z.J.Zeng, S.H.Dai, Stress intensity factors for an inclined surface crack under biaxial stress state, Engineering Fracture Mechanics 47 (2) (1994) 281-289.
[10] J.R.Rice, N.Levy, The part-through surface crack in an elastic plate, Journal of Applied Mechanics 39 (1972) 185-194.
[11] P.F.Joseph, F.Erdogan, Surface crack in a plate under antisymmetric loading conditions, International Journal Solid Structures 27 (6) (1991) 725-750.
[12] Y.C.Lu, Y.G.Xu, Line-spring model for a surface crack loaded antisymmetrically, National University of Defense Technology Technical paper (1986).