LAPLACE AND BI-LAPLACE EQUATIONS FOR DIRECTED NETWORKS AND MARKOV CHAINS

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Abstract. The networks of this – primarily (but not exclusively) expository – compendium are strongly connected, finite directed graphs \( X \), where each oriented edge \((x, y)\) is equipped with a positive weight (conductance) \( a(x, y) \). We are not assuming symmetry of this function, and in general we do not require that along with \((x, y)\), also \((y, x)\) is an edge. The weights give rise to a difference operator, the normalised version of which we consider as our Laplace operator. It is associated with a Markov chain with state space \( X \). A non-empty subset of \( X \) is designated as the boundary. We provide a systematic exposition of the different types of Laplace equations, starting with the Poisson equation, Dirichlet problem and Neumann problem. For the latter, we discuss the definition of outer normal derivatives. We then pass to Laplace equations involving potentials, thereby also addressing the Robin boundary problem. Next, we study the bi-Laplacian and associated equations: the iterated Poisson equation, the bi-Laplacian Neumann and Dirichlet problems, and the “plate equation”. It turns out that the bi-Laplace Dirichlet to Neumann map is of non-trivial interest. The exposition concludes with two detailed examples.

1. Introduction

A finite directed graph is a finite set \( X \) together with a set \( E \subset X^2 \) of directed edges. Thus, we exclude multiple edges. Loops, that is, edges of the form \((x, x)\), play no role in our considerations and are excluded. We assume that \( X \) is strongly connected: for any pair of points \( x, y \in X \), there is a directed path from \( x \) to \( y \). By definition, it consists of vertices \( x = x_0, x_1, \ldots, x_n = y \) such that \((x_{i-1}, x_i) \in E\) for \( i = 1, \ldots, n \), and \( n \) is the length of that path.

We now equip each directed edge with a weight or conductance \( a(x, y) > 0 \), and speak of the resulting weighted graph as a directed network. When \((x, y) \notin E\), we set \( a(x, y) = 0 \). We study the difference operator acting on functions \( u : X \to \mathbb{C} \) by

\[
\mathcal{L}_a u(x) = \sum_y a(x, y) (u(y) - u(x)).
\]

Since nothing of what we are going to consider in this exposition depends substantially on normalisation, we rather pass to

\[
p(x, y) = a(x, y)/m(x), \quad \text{where} \quad m(x) = \sum_y a(x, y)
\]

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and the associated stochastic transition matrix \( P = (p(x, y))_{x,y \in X} \). Then our normalised Laplace operator is
\[
\Delta = P - I, \quad \text{where} \quad Pu(x) = \sum_{y} p(x, y)u(y),
\]
and \( I \) is of course the identity matrix over \( X \), so that \( Iu(x) = u(x) \).

We shall also designate a boundary \( \partial X \) of \( X \). This is defined as an arbitrary non-empty subset \( \partial X \subset X \). It may be natural to require that for every \( y \in \partial X \) there is \( x \in X^o = X \setminus \partial X \) (the interior of \( X \)) such that \( (x, y) \in E \), but mostly we will not need this. In the following, we let \( L(X) \) be the linear space of all functions \( f : X \rightarrow \mathbb{C} \), thought of as column vectors.

Then we are interested in the solutions of the following types of basic problems:

- **Poisson equation**: find \( u \in L(X) \) such that \( \Delta u = f \), where \( f \in L(X) \) is given.
- **Neumann problem**: find \( u \in L(X) \) such that \( \Delta u = f \) on \( X^o \), and the outer normal derivative satisfies \( \partial_n u = g \) on \( \partial X \), where \( f \in L(X^o) \) and \( g \in L(\partial X) \) are given.
- **Dirichlet problem**: find \( u \in L(X) \) such that \( \Delta u = f \) on \( X^o \), and \( u = g \) on \( \partial X \), where \( f \in L(X^o) \) and \( g \in L(\partial X) \) are given.
- **Mixed boundary problem**: decompose \( \partial X = \partial^N \cup \partial^D \). Find \( u \in L(X) \) such that \( \Delta u = f \) on \( X^o \), and \( u = g|_{\partial^D} \) on \( \partial^D \) as well as \( \partial_n u = g|_{\partial^N} \) on \( \partial^N \), where \( f \in L(X^o) \) and \( g \in L(\partial X) \) are given.
- **Robin boundary problem**: Find \( u \in L(X) \) such that \( \Delta u = f \) on \( X^o \), and \( \alpha \cdot u + \beta \cdot \partial_n u = g \) on \( \partial X \), where \( f \in L(X^o) \) and \( g \in L(\partial X) \), as well as \( \alpha, \beta \in \mathbb{C} \setminus \{0\} \) are given.
- **Schrödinger type equations**: in all of the above problems, replace \( \Delta \) by \( \Delta u - v \cdot u \), where \( v \in L(X) \) is a suitable potential.

The first four of these problems are dealt with in §2. Above, one of the important questions is how to define the outer normal derivative at a boundary point. We discuss this with some care, also regarding sub-networks and their boundaries in \( X \).

The problems involving potentials are then considered in §3. This includes the Robin problem.

Subsequently, in §4 we pass to the bi-Laplacian \( \Delta^2 \) and associated problems:

- the iterated Poisson equation,
- the bi-Laplace Neumann problem, and
- the bi-Laplace Dirichlet problem

are as above, with \( \Delta^2 \) in the place of \( \Delta \).

Here, an interesting phenomenon comes up: in passing to \( \Delta^2 \), one gains only one degree of freedom, independently of the size of the boundary. The general plate equation in its first variant is to find \( u \in L(X) \) such that
\[
\Delta^2 u = f \quad \text{on} \quad X^o, \quad \partial_n u = g_1 \quad \text{on} \quad \partial X \quad \text{and} \quad u = g_2 \quad \text{on} \quad \partial X,
\]
where \( f \in L(X^o) \) and \( g_1, g_2 \in L(\partial X) \) are given. It turns out that only under a specific condition on these three functions, there is a solution. This leads to a non-trivial bi-Laplace Dirichlet to Neumann map.
The restrictive condition for solvability of the above plate equation induces us to propose a second variant, which involves the second interior of \( X \) as well as a variant of the outer normal derivative. Using this second interior, one can also solve an iterated Dirichlet problem which can be extended to higher powers of the Laplacian.

In the final section §5 we compute in detail two examples.

The answers to all the problems considered here are obtained by suitable matrix operations. They also have probabilistic interpretations in terms of the underlying Markov chain. The latter is given by a sequence \((Z_n)_{n \geq 0}\) of \( X \)-valued random variables with \( \mathbb{P}[Z_{n+1} = y \mid Z_n = x] = p(x, y) \) for \( x, y \in X \), where of course the Markov assumption holds, namely, conditionally upon the value of \( Z_n \), the past \((Z_j)_{0 \leq j \leq n-1}\) and the future \((Z_k)_{k \geq n+1}\) are independent. The unique stationary distribution of this Markov chain plays an important role. This is one of the reasons why we have chosen to normalise the Laplacian. All results can be easily restated for the operator \( \mathcal{L}_a \) instead of \( \Delta \). (For example, an equation of the form \( \mathcal{L}_a u = f \) transforms into \( \Delta u = \tilde{f} \), where \( \tilde{f}(x) = f(x)/m(x) \).

We mention that with some care one can elaborate the issues considered here also in the case when the graph \( X \) is countably infinite, as long as a stationary probability measure \( \pi \) exists. In probabilistic terms, this means that the underlying Markov chain is positive recurrent.

Motivation for studying the above issues may come from the discretisation of continuous PDEs, but the present note is not written in a spirit of numerical analysis, and algorithmic features are not considered. Classical work is due to Duffin [15], whose finite graphs are subsets of the rectangular lattice \( \mathbb{Z}^d \), considered as discretisations of Euclidean domains; see e.g. also the very recent work of Varopoulos [28], [29]. Another type of motivation comes from “electrical network” theory, which has a long history and a well-known interplay with Markov chain theory, see the beautiful little book by Doyle and Snell [14].

Several of the initial problems presented above are “folklore”, such as the Dirichlet problem (which, however, keeps being “rediscovered”, sometimes by complicated methods). Basic results such as the solution of the Poisson equation for the Laplacian \( P - I \) (in more generality than for finite state spaces) are part of the literature on the potential theory of Markov chains from the 1960ies, see Kemeny and Snell [21] – one of the most significant sources close to the spirit of the present note – and the monograph by Kemeny, Snell and Knapp (in particular, Chapter 9). However, it seems that this has remained secluded from the non-probabilistic world of analysis and smooth potential theory. More recent references from the discrete side are, for example, the lecture notes by Anandam [2], and, among other concerning Schrödinger type equations in the reversible (= self-adjoint) case, Bendito, Carmona and Encinas [5], [6], and, in particular [7], as well as the work of these authors with Gesto [8]. See also the references in those papers.

Related only in part, there is a large body of work on discrete boundary value problems arising for one-dimensional difference equations. Among the many references, we
indicate the monographs by Atkinson \cite{Atkinson} and Agarwal \cite{Agarwal}. There is also a large number of contributions to the spectral theory of (self-adjoint) discrete Laplacians on finite networks, which however is not in the focus of the present work. See e.g. the books by Cvetković, Doob and Sachs \cite{Cvetkovic} and Chung \cite{Chung}, and the beautiful article by Colin de Verdière \cite{Colin}. Furthermore, there are some interesting studies of Laplacians in discrete geometry, see e.g. Kenyon \cite{Kenyon} or Bobenko and Springborn \cite{Bobenko}. (These are only glimpses at the respective parts of the literature.)

On the other hand, the study of the bi-Laplacian in the discrete setting has received only little and partly quite recent consideration in the literature; see e.g. Yamasaki \cite{Yamasaki}, Vanderbei \cite{Vanderbei}, Cohen et al. \cite{Cohen}, Anandam \cite{Anandam}, Picardello and Woess \cite{Picardello}, or Hirschler and Woess \cite{Hirschler}. In the "smooth" literature, there is an ample body of work on this subject, and we point at the lecture notes by Gazzola, Grunau and Sweers \cite{Gazzola}. In this context, the present paper may provide some new insight concerning the finite, discrete setting.

At several points, we have included discussions of possible approaches, where the discrete analogue of the smooth setting might allow different interpretations. We have made an effort to present a comprehensive and coherent exposition, a task which a posteriori may appear easier than it was a priori.

2. The solutions of the basic problems

While we write $L(X)$ for the space of functions $X \rightarrow \mathbb{C}$, of which we think as column vectors, we consider measures as row vectors and write $M(X)$ for the resulting space. Of course, for $\mu \in M(X)$ and $A \subset X$, we have $\mu(A) = \sum_{x \in A} \mu(x)$. In this sense, also $p(x, A) = \sum_{x \in A} p(x, y)$.

Strong connectedness of the graph $X$ amounts to irreducibility of the non-negative matrix $P$, namely: for all $x, y \in X$ there is $n = n_{x,y}$ such that $p^{(n)}(x,y) > 0$, where

\[
P^n = (p^{(n)}(x,y))_{x,y \in X}, \quad P^0 = I
\]

(matrix powers). The following is very well known – see e.g. Seneta \cite{Seneta} or Woess \cite{Woess}. For the sake of completeness we provide an outline of the proof.

\textbf{(2.1) Lemma.} Irreducibility of $P$ implies the following.

(a) Every harmonic function is constant, that is, $h \in L(X)$ satisfies $\Delta h = 0$ if and only if $h$ is constant.

(b) There is a unique positive stationary probability distribution $\pi \in M(X, \mathbb{R})$, that is,

\[
\pi P = \pi, \quad \pi(x) > 0 \text{ for all } x \in X, \quad \text{and } \sum_{x \in X} \pi(x) = 1.
\]

\textbf{Proof.} For (a), if $h$ is harmonic then so are its real and imaginary part. Hence, we may assume that $h$ is real, and the statement follows from the minimum principle: let $x \in X$
such that \( h(x) = \min h \). Then

\[
\sum_y p^{(n)}(x, y) \left( h(y) - h(x) \right) \geq 0,
\]

whence \( h(y) = h(x) \) for each \( x \) with \( p^{(n)}(x) > 0 \), and this holds for any \( n \).

For (b), a straightforward compactness argument shows that there are stationary distributions, namely, the accumulation points of the sequence

\[
\frac{1}{n} \sum_{k=0}^{n-1} \mu P^k,
\]

where \( \mu \in M(X) \) is an arbitrary non-vanishing non-negative measure. If \( \pi \) is stationary, then irreducibility implies \( \pi(x) > 0 \) for all \( x \), and we can consider the new transition matrix

\[
(2.3) \quad \hat{P} = (\hat{p}(x, y))_{x,y \in X} \quad \text{with} \quad \hat{p}(x, y) = \frac{\pi(y)p(y, x)}{\pi(x)}.
\]

It is again irreducible, and \( \mu P = \mu \) if and only if \( h(x) = \mu(x)/\pi(x) \) satisfies \( \hat{P}h = h \).

Thus, \( h \) is constant by (a). \( \square \)

For any two non-empty subsets \( A, B \subset X \), we write \( P_{A,B} = (p(x, y))_{x \in A, y \in B} \) for the restriction of \( P \) to \( A \times B \). In particular, we write \( P_A = P_{A,A} \), as well as \( f_A \) for the restriction \( f \) of \( f \in L(X) \) to \( A \). The following is also well-known.

\[
(2.4) \text{Lemma.} \quad \text{If } A \subset X \text{ strictly, then } I_A - P_A \text{ is invertible, and}
\]

\[
G_A := (I_A - P_A)^{-1} = \sum_{n=0}^{\infty} P^n_A,
\]

a convergent series.

\textbf{Proof [Outline].} For each \( x \in A \), let \( n_x \) be the smallest \( n \) such that \( p^{(n)}(x, w) > 0 \) for some \( w \in X \setminus A \). Then

\[
\sum_{y \in A} p^{(n_x)}_A(x, y) < 1.
\]

We deduce that

\[
\sum_{y \in A} p^{(n)}_A(x, y) < 1 \quad \text{for every } n \geq n_x.
\]

Let \( N = \max\{n_x : x \in A\} \). Then there is \( \delta \in (0, 1) \) such that

\[
\sum_{y \in A} p^{(N)}_A(x, y) \leq 1 - \delta \quad \text{for every } x \in A.
\]

Consequently,

\[
\sum_{n=0}^{\infty} p^{(n)}_A(x, y) < \infty \quad \text{for all } x, y \in A.
\]

\( \square \)
While a priori, \( P_A \) and \( G_A \) are matrices over \( A \), it will often be useful to consider them as matrices over the entire set \( X \) with entries \( 0 \) outside of \( A \times A \).

A comment on the probabilistic interpretation: for \( x, y \in A \), the matrix entry \( p_A^{(n)}(x, y) \) is the probability that the Markov chain starting at \( x \) visits \( y \) at time \( n \), without ever leaving \( A \) before that. Thus, the \((x, y)\)-entry of the matrix \( G_A \) is the expected number of visits in \( y \) of the Markov chain starting at \( x \) before it leaves the set \( A \). Indeed, it is straightforward via the Borel-Cantelli Lemma that it must leave \( A \) with probability \( 1 \) at some time.

We choose a “root” vertex \( o \in X \) as a reference point. Let us say that a function \( u \in L(X) \) is grounded, if \( u(o) = 0 \). The following – which can be seen as a consequence of the simple matrix version of the Fredholm alternative – is a basis of almost everything which we are considering in this note. Therefore, in spite of its simplicity, we call it a “theorem”. The measure \( \pi \) is the one of Lemma 2.1. Here and on several occasions further below, we write weighted sums with respect to measures on \( X \), \( X^o \) or \( \partial X \) in the form of integrals in order to stress the analogies with the continuous setting. Of course, we always remain in the realm of finite-dimensional linear algebra.

(2.5) Theorem. Solution of the Poisson equation. For \( f \in L(X) \), the equation \( \Delta u = f \) has a solution \( u \) if and only if the charge \( f \) is balanced, that is,

\[
\int_X f \, d\pi := \sum_{x \in X} \pi(x) f(x) = 0 .
\]

In this case, the unique grounded solution is given by

\[
u = -G_{X\setminus\{o\}} f ,
\]

that is,

\[
u(o) = 0 \quad \text{and} \quad \nu(x) = -\sum_{y \in X\setminus\{o\}} \sum_{n=0}^{\infty} p^{(n)}_{X\setminus\{o\}}(x, y) f(y) \quad \text{for} \ x \in X \setminus \{o\} .
\]

All solutions are given by \( \nu(\cdot) + c \), where \( c \in \mathbb{C} \).

Proof. If \( f \) is such that a solution \( u \) exists, then (since \( \pi P = \pi \))

\[
\int_X f \, d\pi = \pi(P - I)u = \int_X u \, d(\pi P) - \int_X u \, d\pi = 0 ,
\]

so that (2.6) holds.

By Lemma 2.1(a), the kernel of \( \Delta \) consists of the constant functions, whence it is one-dimensional. Thus, the dimension of the range of \( \Delta \) is \( |X| - 1 \). It is contained in the hyperplane \( \{ f \in L(X) : \int_X f \, d\pi = 0 \} \), which itself has dimension \( |X| - 1 \). Thus, the range must be that hyperplane. This proves the first part.

Now let \( f \) satisfy (2.6). By Lemma 2.1 any two solutions of the associated Kirchhoff problem differ by a constant, so that there is a unique grounded solution \( u \). If \( x \in X \setminus \{o\} \)
then, since $u(0) = 0$,

$$f(x) = \Delta u(x) = \left(\sum_y p(x, y)u(y)\right) - u(x) = \left(\sum_{y\neq 0} p(x, y)u(y)\right) - u(x),$$

that is,

$$f_{X\setminus\{0\}} = \left(\tilde{P}_{X\setminus\{0\}} - I_{X\setminus\{0\}}\right)u_{X\setminus\{0\}}.$$

Applying $G_{X\setminus\{0\}}$ to both sides from the left, the result follows via Lemma 2.4.

We can equip $L(X)$ with the inner product $(f, g)_\pi = \int_X f \bar{g} \, d\pi$, so that it becomes a finite-dimensional real Hilbert space, denoted $L(X, \pi)$.

(2.7) Lemma. For all $f, g \in L(X)$,

$$\int_X (f \Delta g - g \hat{\Delta} f) \, d\pi = 0,$$

where $\hat{\Delta} = I - \hat{P}$ with $\hat{P}$ given by (2.3).

Proof. This is immediate from the fact that $\hat{\Delta}$ is the adjoint of $\Delta$ in $L(X, \pi)$.

Given a boundary $\partial X$ as in the Introduction, note that there is no “infinitesimal change of measure” from $\pi$ as the “volume measure” in $X^\circ$ to the same $\pi$ as the “surface measure” on $\partial X$. Therefore it is most natural to define the outer normal derivatives of $f \in L(X)$ on $\partial X$ by

$$\partial_{n^\circ} f(x) = -\Delta f(x) \quad \text{and} \quad \hat{\partial}_{n^\circ} f(x) = -\hat{\Delta} f(x), \quad x \in \partial X. \quad (2.8)$$

(As a matter of fact, this rather is “minus the inner normal derivative”.) Then Lemma 2.7 is equivalent with the following discrete, in general non-self-adjoint version of Green’s second integral identity, justifying our choice of the normal derivative.

$$\int_{X^\circ} (f \Delta g - g \hat{\Delta} f) \, d\pi = \int_{\partial X} \left(\partial_{n^\circ} g - g \hat{\partial}_{n^\circ} f\right) \, d\pi, \quad f, g \in L(X). \quad (2.9)$$

(2.10) Corollary. Solution of the Neumann problem. The solution $u$ of $\Delta u = f$ on $X^\circ$ and $\partial_{n^\circ} u = g$ on $\partial X$ coincides with the solution of the Poisson equation $\Delta u = \tilde{f}$ on $X$, where $\tilde{f}(x) = f(x)$ for $x \in X^\circ$ and $\tilde{f}(y) = -g(y)$ for $y \in \partial X$.

In particular, the Neumann problem is solvable if and only if

$$\int_{X^\circ} f \, d\pi = \int_{\partial X} g \, d\pi.$$

(2.11) Theorem. Solution of the Dirichlet problem. For $f \in L(X^\circ)$ and $g \in L(\partial X)$, there is a unique $u \in L(X)$ such that $\Delta u = f$ on $X^\circ$ and $u = g$ on $\partial X$. The solution is given by

$$u_{X^\circ} = -G_{X^\circ}(f - P_{X^\circ, \partial X} g) \quad \text{and} \quad u_{\partial X} = g.$$
Proof. For \( x \in X^o \), we have
\[
\Delta u(x) = P_{X^o} u_{X^o}(x) - u(x) + P_{X^o, \partial X} u_{\partial X}(x) \\
= f(x) - P_{X^o, \partial X} g(x) + P_{X^o, \partial X} u_{\partial X}(x) = f(x).
\]
So, \( u \) as given is indeed a solution. If \( u_1, u_2 \) are two solutions then \( \Delta(u_1 - u_2) = 0 \) on \( X^o \) and \( u_1 - u_2 \equiv 0 \) on \( \partial X \). By the minimum principle, the real and complex parts of \( u_1 - u_2 \) must assume their minimum and maximum on \( \partial X \). Thus, \( u_1 - u_2 \equiv 0 \) on \( X \). This shows uniqueness.

The above proof follows the classical potential theoretic line. Alternatively, one can also argue by direct operations; second proof: we subdivide into blocks over \( X^o \) and \( \partial X \),
\[
\Delta = \begin{pmatrix}
P_{X^o} - I_{X^o} & P_{X^o, \partial X} \\
P_{\partial X, X^o} & P_{\partial X} - I_{\partial X}
\end{pmatrix}
\quad \text{and} \quad u = \begin{pmatrix} u_{X^o} \\ g \end{pmatrix}.
\]
Then our equation can be written on \( X^o \) as
\[
(P_{X^o} - I_{X^o}) u_{X^o} + P_{X^o, \partial X} g = f,
\]
and multiplying from the left with \( G_{X^o} \) yields the result. \( \square \)

(2.12) **Definition & Remark.** For \( x \in X^o \) and \( y \in \partial X \), let \( \nu_x(y) = G_{X^o, P_{X^o, \partial X}} 1_y(x) \) be the \((x, y)\)-entry of the \( X^o \times \partial X \)-matrix \( G_{X^o} P_{X^o, \partial X} \). This is the probability that the Markov chain starting at \( x \) enters the boundary \( \partial X \) at the point \( y \). For \( x \in \partial X \), we set \( \nu_x = \delta_x \). Thus, \( \nu_x = \nu_{\partial X}^x \) is a probability distribution on \( \partial X \) for each \( x \in X \). Recall that we can view \( G_{X^o} \) as a matrix over \( X \times X \) with entries 0 outside of \( X^o \times X^o \). Then we can write the solution of the Dirichlet problem as
\[
(2.13) \quad u(x) = -G_{X^o} f(x) + \int_{\partial X} g d\nu_x, \quad x \in X.
\]
Let \( H(P, X^o) \) denote the linear space of harmonic functions on \( X^o \), i.e., those \( h \in L(X) \) which satisfy \( \Delta h = 0 \) on \( X^o \). They are all of the form \( h(x) = \int_{\partial X} g d\nu_x \), where \( g \in L(\partial X) \).

We define the exit and the entrance boundary as
\[
\partial_{\text{exit}} X = \{ y \in \partial X : (x, y) \in E \text{ for some } x \in X^o \} \quad \text{and} \quad 
\partial_{\text{entrance}} X = \{ z \in \partial X : (z, x) \in E \text{ for some } x \in X^o \}.
\]
If \( x \in X^o \) then \( \nu_x(y) > 0 \) implies that \( y \in \partial_{\text{exit}} X \). If furthermore \( P_{X^o} \) is irreducible, that is, \( X^o \) is strongly connected, then \( \nu_x(y) > 0 \) for all \( (x, y) \in X^o \times \partial_{\text{exit}} X \).

The Poisson equation is a special case of the Dirichlet problem. Set \( \partial X = \{ o \} \). Given \( f \in L(X^o) \), extend \( f \) to \( X \) by setting
\[
(2.14) \quad f(o) = -\frac{1}{\pi(o)} \int_{X \setminus \{ o \}} f d\pi.
\]
Then the solution of the Poisson equation grounded at \( o \) coincides with the solution of the Dirichlet problem \( \Delta u = f \) on \( X \setminus \{ o \} \), \( u(o) = 0 \).

(2.14) **Dirichlet to Neumann.** In the smooth setting, the transformation from the Dirichlet to the Neumann problem is delicate, see e.g. BEHRNDT AND LANGER [4].
our discrete setting, it is straightforward: typically, one considers the Dirichlet problem
\[ \Delta u = 0 \text{ on } X^o \text{ and } u = g \text{ on } \partial X, \]
so that the solution is
\[ u(x) = \int_{\partial X} g \, d\nu_x, \quad x \in X. \]
If we then set \( g_1 = \partial_{\bar{\partial}} u = -\Delta u \) on \( \partial X \), then \( u \) solves the Neumann problem for the boundary function \( g_1 \). We have the linear mapping \( g \mapsto g_1 = (I_{\partial X} - Q)g \), where \( Q = P_{\partial X} + P_{\partial X, X^o}G_{X^o}P_{X^o, \partial X} \). We shall examine this stochastic \( \partial X \times \partial X \)-matrix in more detail in \[4\]. The kernel of that mapping consists of the constant functions on \( \partial X \).

The following is immediate from (2.8) and Theorem \[2.11\]

(2.15) Lemma. Solution of the mixed boundary problem. Let \( \partial X = D \cup N \) (both non-empty). For \( f \in L(X^o) \) and \( g \in L(\partial X) \), there is a unique \( u \in L(X) \) such that \( \Delta u = f \) on \( X^o \), and \( u = g_D \) on \( D \) as well as \( \partial_{\bar{\partial}} u = g_N \) on \( N \). The solution is as follows: let \( \tilde{f} \in L(X^o \cup N) \) be given as
\[ \tilde{f}_{X^o} = f \quad \text{and} \quad \tilde{f}_N = -g_N. \]
Then
\[ u_{X^o \cup N} = -G_{X^o \cup N}(\tilde{f} - P_{X^o \cup N, D} g_D) \quad \text{and} \quad u_D = g_D. \]

(2.16) The self-adjoint case. The most intensively studied case is the one when the edge set as well as the conductances are symmetric: \( (x, y) \in E \iff (y, x) \in E \), and \( a(x, y) = a(y, x) \). Then, with our normalisation at the root vertex \( o \), the stationary distribution is given by \( \pi(x) = m(x)/m(X) \), where \( m(x) \) is as in \[1.2\] and \( m(X) = \sum_{y \in X} m(y) \). Thus, \( P \) is reversible, that is, we have \( \pi(x)p(x, y) = \pi(y)p(y, x) \) for all \( x, y \in X \). In this case, \( \Delta = \tilde{\Delta} \) is self-adjoint on \( L(X, \pi) \), and Green’s identity (2.9) assumes the classical form. There is also the discrete version of Green’s first identity. This is present in a variety of textbooks, see e.g. ANANDAM [2] or JORGENSEN and PEARSE [20].

(2.17) Discussion. Normal derivatives. Given \( P \) as above, we have solved the Poisson equation, using the associated stationary distribution \( \pi \). Then we have introduced the outer normal derivatives at \( \partial X \) by (2.5), with justification given by “Green’s identity” (2.9).

As we shall explain below, it may be more natural to define the outer normal derivatives differently, that is, for \( x \in \partial X \), we replace the original transition probabilities \( p(x, y) \) by new ones, \( p'(x, y) \), which again are assumed to satisfy \( \sum_y p'(x, y) = 1 \). Then for \( f \in L(\partial X) \),
\[ \partial_{\bar{\partial}} f(x) = -\sum_y p'(x, y)(f(y) - f(x)), \]
and we can consider the Neumann and mixed boundary problems with this assignment. Of course, this does not change the way how these problems are solved. One just has to replace the original transition matrix \( P \) by \( P' \), whose entries are \( p'(x, y) \) if \( x \in X \), and \( p'(x, y) \) when \( x \in \partial X \).
However, in this case the statement that the solutions of the Neumann and Poisson problems coincide is no more valid. Namely, for the Poisson equation in Theorem 2.5, we use the stationary distribution $\pi$ of the matrix $P$, while for the solution of the Neumann problem, it has to be replaced by the stationary distribution $\pi'$ of the modified transition matrix, and in general $\pi \neq \pi'$.

In particular, this becomes visible in the “classical” self-adjoint case of (2.16). There, the reversibility property $\pi(x) p(x, y) = \pi(y) p(y, x)$ of $P$ is valid on all of $X$, while for the modified matrix $P'$ it will in general fail if one of $x, y$ belongs to $\partial X$. Still, the change will affect the stationary distribution also in $X^o$, that is, in general $\pi'(x) \neq m(x)/m(o)$ even for $x \in X^o$.

Let us also give some additional motivation for the choice of the normal derivatives in (2.8) and for the choice to consider also the non-self adjoint (non-reversible) case.

Suppose we have a bounded open domain $D \subset \mathbb{R}^2$ with smooth boundary curve $\partial D$. We suppose that we can discretise $D$ by inserting a finite square grid $X$, whose span is $h > 0$. Let us also assume that $\partial D$ intersects the grid in a subset $\partial X$ of its vertex set, so that at each vertex $x$ of $X^o = X \cap D$, its four neighbours $x \pm h \cdot e_1$ and $x \pm h \cdot e_2$ belong to the closure of $D$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then we can approximate the second derivatives by symmetric second order differences to approximate the continuous Laplacian $\Delta_D$ at the vertices of $X^o$ as follows:

$$
\Delta_D u(x) \approx \frac{u(x + h \cdot e_1) - 2u(x) + u(x - h \cdot e_1)}{h^2} + \frac{u(x + h \cdot e_2) - 2u(x) + u(x - h \cdot e_2)}{h^2} + \sum_{y \sim x} a(x, y) (u(y) - u(x)) \quad \text{where} \quad a(x, y) = 1/h^2 \; \text{if} \; (x, y) \in E, \; x \in X^o.
$$

The latter is an example of an operator $\mathcal{L}_u$ as defined in the Introduction. The normalisation for passing to our “stochastic” version of $\Delta = \Delta_{[X]}$ means that we have to divide by $m(x)$, which has the constant value $4/h^2$ on $X^o$, while we shall now discuss how to proceed at the boundary. Let us look at a vertex $x \in \partial X$. We can approximate the inward pointing first order partial derivatives by first order differences, which are of the form

$$
\frac{u(y) - u(x)}{h}, \; \text{if} \; (x, y) \in E.
$$
According to the slope of $\partial D$ at $x \in \partial X$ (which the grid does not “see”), the inner normal derivative may be any convex combination of those partial derivatives, that is, of the form
\[
\sum_{y : (x,y) \in E} \lambda(x,y) \frac{u(y) - u(x)}{h},
\]
where $\lambda(x,y) \geq 0$ and $\sum_y \lambda(x,y) = 1$. For the outer normal derivative, we just have to change sign. We then should choose new conductances, which are not symmetric at the boundary vertices. Namely, for any directed edge $(x,y)$, we have
\[
a(x,y) = \begin{cases} 
1/h^2, & \text{if } x \in X^o \\
\lambda(x,y)/h, & \text{if } x \in \partial X.
\end{cases}
\]
This leads to a very natural situation – to be considered in more generality than the above example – where the edge set is symmetric, and the conductances are symmetric along edges within $\partial X$, but symmetry may fail for edges with an endpoint in $\partial X$. Nevertheless, with the associated transition matrix $P'$ and stationary measure $\pi'$, the solution of the Neumann problem in Corollary 2.10 is valid for this very natural general choice of normal derivatives.

At this point we also mention old work – in the perspective of numerical approximation – of Bramble and Hubbard [10] on the discretised Neumann problem on a domain $D \subseteq \mathbb{R}^2$, and its references.

We extend this discussion of normal derivatives to the situation of sub-networks.

(2.19) Remarks on sub-networks. Let $Y \subseteq X$ be a connected sub-network. That is, as an induced (directed) sub-graph of $X$, it is strongly connected. In this case, we define the boundary and interior of $Y$ as
\[
\partial Y = \{x \in Y : (x,z) \in E \text{ for some } z \in X \setminus Y\} \quad \text{and} \quad Y^o = Y \setminus \partial Y.
\]
In our discrete (non-infinitesimal) setting, we need to be careful when speaking about “the” Laplacian on $Y$.

On one hand, we have the restriction $P_Y - I_Y$ of $\Delta = \Delta_{[X]}$ to $Y$. On the other hand, if we start with the edge weights $a(x,y)$ as in [11] as the initial data, then $Y$ becomes a directed network on its own right, and the associated transition matrix $P_{[Y]}$ is given by
\[
p_{[Y]}(x,y) = a(x,y)/m_{[Y]}(x), \quad \text{where } m_{[Y]}(x) = \sum_{w \in Y} a(x,w) \text{ for } x, y \in Y.
\]
In other words, the stochastic matrix $P_{[Y]}$ is obtained from the substochastic matrix $P_Y$ by dividing each row by its row sum. Then we obtain the normalised Laplacian of $Y$ as
\[
\Delta_{[Y]} = P_{[Y]} - I_Y.
\]
Accordingly, the natural choice for the associated outer normal derivative on $\partial Y$ is
\[
\partial \vec{n} f(y) = -\Delta_{[Y]} f(y), \quad y \in \partial Y.
\]
We remark here that the restriction of $P_{[Y]}$ to $Y^o$ coincides with the restriction of $P$ to $Y^o$. Therefore the Green kernel $G_{Y^o}$ is the same for $\Delta$ and $\Delta_{[Y]}$. On the other hand,
observe that the stationary distribution $\pi_{|Y}$ of $P_{|Y}$ is not the restriction of $\pi$ to $Y$. (But in the reversible case, the two are proportional in $Y^o$.)

Coming back to the outer normal derivative, as mentioned, its definition (2.8) on $X$ is in reality the additive inverse of what one can consider as the inner normal derivative. This appears natural in several senses:

- In the smooth setting, the sum of the outer and the inner normal derivative is 0,
- we have the discrete version (2.9) of Green’s second identity, and
- the original network with is boundary does a priori not have an exterior.

But now, our sub-network $Y$ does have an exterior, namely $X \setminus Y$, and there is a second natural choice for the outer normal derivative on $\partial Y$ of a function $f \in L(X)$, namely

\begin{equation}
(2.23) \quad \partial^*_n f(y) = \frac{1}{p(y, X \setminus Y)} \sum_{z \in X \setminus Y} p(y, z) \left( f(z) - f(y) \right).
\end{equation}

(The normalisation by $p(y, X \setminus Y)$ is convenient, but not crucial.) There is no nice analogue of Green’s second identity for this choice, so that we mostly stick to (2.22), but in §4 we shall see an instance where (2.23) gains justification.

In the Poisson equation $\Delta u = f$, the set $\{ x : f(x) \neq 0 \}$ is the set of inhomogeneities of that equation. If we are interested in the solution on the sub-network $Y$ only, we want to shift the inhomogeneities from outside to $\partial Y$. This is balayage, which is also present in the old work [21]. Here, we display it in our setting.

\begin{enumerate}
\item[(2.24)] Balayage. Let $f \in L(X)$ with $\int_X f \, d\pi = 0$, and let $u$ be a solution of the Poisson equation $\Delta u = f$. For a strict sub-network $Y \subset X$, we consider the réduit (reduced function) $u^Y$ of $u$ on $Y$. This is the solution of the Dirichlet problem

$$
\Delta u^Y = 0 \text{ on } X \setminus Y \quad \text{and} \quad u^Y = u \text{ on } Y.
$$

Thus, we have $X \setminus Y$ in the role of $X^o$ and $Y$ in the role of $\partial X$, and the solution is the function

$$
u^Y_x = \int_Y u \, d\nu_x^Y \in H(P, X \setminus Y);$$

see Definition & Remark 2.12. We can also write $u^Y = u - v$, where $v$ solves the Dirichlet problem

$$
\Delta v = f \text{ on } X \setminus Y \quad \text{and} \quad v = 0 \text{ on } Y,
$$

so that $v = G_{X \setminus Y} f$. The balayée (swept out function) of the charge $f$ on $Y$ is then $f^Y = \Delta u^Y$. We find

$$
f^Y(x) = \begin{cases}
  f(x), & \text{if } x \in Y^o, \\
  f(x) - P_{\partial Y, X \setminus Y} G_{X \setminus Y} f(x), & \text{if } x \in \partial Y, \\
  0, & \text{if } x \in X \setminus Y.
\end{cases}
$$


3. Equations including potentials

We now extend the previous equations by adding a potential \( v \in L(X) \). The associated variant of the Poisson equation looks as follows:

(3.1) \[ \text{Given } f \in L(X), \text{ find } u \in L(X) \text{ such that } \Delta u - v \cdot u = f \text{ on } X. \]

Here, we shall assume that \( v(x) \geq 0 \) (real) for all \( x \) and \( v(x) > 0 \) for some \( x \). More general variants will be discussed below. We transform the above equation into

(3.2) \[ \tilde{P} u - u = \tilde{f}, \quad \text{where } \tilde{f}(x) = \frac{f(x)}{1 + v(x)} \text{ and } \tilde{p}(x, y) = \frac{p(x, y)}{1 + v(x)}. \]

If we want to reduce this to one of the problems of §2, then we can add another state to \( X \), setting \( \tilde{X} = X \cup \{\dagger\} \), and extend \( \tilde{P} \) to a stochastic matrix by setting

\[ \tilde{p}(x, \dagger) = \frac{v(x)}{1 + v(x)} \text{ and } \tilde{p}(\dagger, o) = 1. \]

The outgoing probabilities from \( \dagger \) are in reality irrelevant, and it might be more natural to set \( \tilde{p}(\dagger, \dagger) = 1 \), that is, at an \( x \in X \), the original Markov chain “dies” and moves to the “tomb” state \( \dagger \) with probability \( v(x)/(1 + v(x)) \), where it remains forever. The above definition just serves to maintain an irreducible matrix \( \tilde{P} \). The associated Laplacian is now \( \Delta_v = \tilde{P} - I_{\tilde{X}} \) on \( \tilde{X} \).

Thus, if we define \( \partial \tilde{X} = \{\dagger\} \) then (3.2) becomes the Dirichlet problem

\[ \Delta_v u = \tilde{f} \text{ on } \tilde{X}^o = X \text{ and } u = 0 \text{ on } \partial \tilde{X} = \{\dagger\}. \]

At this point we return to considering \( \tilde{P} \) as a sub-stochastic matrix over \( X \) alone. Then we get from Lemma 2.4 that \( I - \tilde{P} \) is invertible on \( X \), and Theorem 2.11 yields the following.

(3.3) Corollary. Poisson equation with potential. With \( \tilde{P} \) and \( \tilde{f} \) as in (3.2), the unique solution of (3.1) is given on \( X \) by

\[ u = -\tilde{G} \tilde{f}, \quad \text{where } \tilde{G} = (I - \tilde{P})^{-1} = \sum_{n=0}^{\infty} \tilde{P}^n. \]

As a matter of fact, this can be seen as a special case of the following.

(3.4) Proposition. Dirichlet problem with potential. On \( X = X^o \cup \partial X \) consider \( f \in L(X^o) \) and \( g \in L(\partial X) \). The unique solution \( u \) of the problem

\[ \Delta u - v \cdot u = f \text{ on } X^o \text{ and } u = g \text{ on } \partial X \]

is given on \( X^o \) by

\[ u_{X^o} = -\tilde{G}_{X^o}(\tilde{f} - \tilde{P}_{X^o, \partial X} g), \]

where \( \tilde{f} \) on \( X^o \) and \( \tilde{P} \) are as in (3.2) and \( \tilde{G}_{X^o} \) is defined as in Lemma 2.4 with respect to the matrix \( \tilde{P}_{X^o} \).
Proof. This is immediate from the considerations preceding Corollary 3.3, by using the extended network $\tilde{X}$ with enlarged boundary $\partial\tilde{X} = \partial X \cup \{\dagger\}$. Then we have the Dirichlet problem solved in Theorem 2.11, rewritten as

$$\Delta_v u = \tilde{f} \quad \text{on} \quad \tilde{X}^o = X^o \quad \text{and} \quad u = \tilde{g} \quad \text{on} \quad \partial\tilde{X}$$

with boundary function $\tilde{g}$ given by $\tilde{g}_{\partial X} = g$ and $\tilde{g}(\dagger) = 0$. \qed

Recall that we have again the probabilistic interpretation of Definition & Remark 2.12:

$$u(x) = -\tilde{G}_{X^o} \tilde{f} + \int_{\partial X} g \, d\tilde{\nu}_x, \quad x \in X^o,$$

where $\tilde{\nu}_x(y)$ for $x \in X$ is the probability that the Markov chain on $\tilde{X}$ with the extended transition matrix $\tilde{P}$ enters $\partial\tilde{X}$ at $y$. Note that $\tilde{\nu}_x$ is not necessarily a full probability measure, since $\tilde{\nu}_x(\partial X) = 1 - \tilde{\nu}_x(\dagger)$. As $\tilde{g}(\dagger) = 0$, the corresponding part of the boundary integral does not appear.

More generally, we may also consider a general complex valued potential $v \in L(X)$. For our results, we need that

$$(3.5) \quad |1 + v(x)| \geq 1 \quad \text{for all} \quad x \quad \text{and} \quad |1 + v(x)| > 1 \quad \text{for some} \quad x \in X.$$  

We can then define $\tilde{P}$ and $\tilde{f}$ on $X$ as in (3.2). The matrix $\tilde{P}$ is then in general no more sub-stochastic, since it may have negative or complex entries, so that we lose the stochastic interpretation. However,

$$\sum_y |\tilde{p}(x, y)| \leq 1 \quad \text{for all} \quad x \quad \text{and} \quad \sum_y |\tilde{p}(x, y)| < 1 \quad \text{for some} \quad x \in X.$$  

Thus, using Lemma 2.4 once more, we have on $X$ (without the additional point $\dagger$) that

$$(3.6) \quad \tilde{G} = (I - \tilde{P})^{-1} = \sum_{n=0}^{\infty} \tilde{P}^n$$

converges absolutely matrix element-wise, because $|\tilde{P}^n| \leq |\tilde{P}|^n$. After this observation, we see that Corollary 3.3 and Proposition 3.4 remain valid in this more general context.

This comprises the “classical” Schrödinger equation, where $v(x)$ is purely imaginary.

Let us now look at the Robin boundary problem. A bit more generally than in the Introduction, we consider functions $\alpha, \beta \in L(\partial X)$ and want to find $u \in L(X)$ such that $\Delta u = f$ on $X^o$, and the boundary condition is

$$(3.7) \quad \alpha(x) u(x) + \beta(x) \partial n u(x) = g(x) \quad \text{for all} \quad x \in \partial X,$$

where $f \in L(X^o)$ and $g \in L(\partial X)$ are given. We decompose $\partial X = A \cup B$, where $B = \{x \in \partial X : \beta(x) = 0\}$ (which may be empty) and assume naturally that $\alpha(x) \neq 0$ for $x \in B$. Then we redefine the boundary and the interior: $\partial_b X = B$ and $X^o_\beta = X^o \cup A$. 


Laplace equations

Now recall that \( \partial_n u(x) = -\Delta u(x) \) on \( A \). Then we can rewrite the Robin problem as follows. Find \( u \in L(X) \) such that
\[
\Delta u - v \cdot u = f_\beta \quad \text{on} \quad X_\beta^o \quad \text{and} \quad u = g_\alpha \quad \text{on} \quad \partial_\beta X, \quad \text{where}
\]
\[
v(x) = \begin{cases} 
0 & \text{for} \ x \in X^o \\
\alpha(x)/\beta(x) & \text{for} \ x \in A,
\end{cases}
\]
(3.8)
\[
f_\beta(x) = \begin{cases} 
f(x) & \text{for} \ x \in X^o \\
g(x)/\beta(x) & \text{for} \ x \in A,
\end{cases}
\]
and
\[
g_\alpha(x) = g(x)/\alpha(x) \quad \text{for} \ x \in B.
\]

(3.9) Corollary. Solution of the Robin boundary problem. Suppose that \( |\alpha(x) + \beta(x)| \geq |\beta(x)| \) for all \( x \in B \), and that the inequality is strict for some \( x \in B \). Then the solution of the problem with boundary condition (3.7) is given on \( X_\beta^o = X^o \cup A \) by Proposition 3.4 after replacing \( X^o \) by \( X_\beta^o \) and \( \partial X \) by \( \partial_\beta X = B \), as well as \( g \) by \( g_\alpha = g/\alpha \), and setting
\[
\tilde{p}(x, y) = \begin{cases} 
p(x, y)/\beta(x) & \text{for} \ x \in X^o \\
p(x, y)/\alpha(x) & \text{for} \ x \in A,
\end{cases}
\]
and
\[
\tilde{f}(x) = \begin{cases} 
f(x) & \text{for} \ x \in X^o \\
g(x)/\alpha(x) & \text{for} \ x \in A.
\end{cases}
\]
In the case when \( B = \emptyset \), this reduces of course to the solution given in Corollary 3.3.

4. The bi-Laplacian

Without specifying a boundary, the first natural choice for the bi-Laplacian is \( \Delta^2 \), the square of the matrix \( \Delta \). In the associated Poisson equation one does not gain a degree of freedom with respect to the simple Laplacian.

(4.1) Theorem. Iterated Poisson equation. For \( f \in L(X) \), the equation \( \Delta^2 u = f \) has a solution \( u \) if and only if \( \int_X f \, d\pi = 0 \). In this case, the unique solution grounded at \( o \) is given by
\[
u = G_{X\setminus\{o\}}(G_{X\setminus\{o\}} f - \int_X G_{X\setminus\{o\}} f \, d\pi).
\]
All solutions are given by \( u(\cdot) + c \), where \( c \in \mathbb{C} \).

Proof. Set \( \Delta u = v \). Then \( v \) must be a solution of the Poisson equation of Theorem 2.5. That is,
\[
u = -G_{X\setminus\{o\}} f + c, \quad \text{where} \quad c \in \mathbb{C}.
\]
Next, \( u \) must solve the Poisson equation \( \Delta u = v \). Thus, we must have \( \int_X v \, d\pi = 0 \). This forces \( c = \int_X G_{X\setminus\{o\}} f \, d\pi \), so that we get the proposed grounded solution. \( \square \)

In particular, the bi-harmonic functions on \( X \), i.e., the elements of the kernel of \( \Delta^2 \), are the constant functions.

We now consider boundary value problems for the bi-Laplacian. Recall from Definition & Remark 2.12 the family of probability measures \( \nu_x = \nu_{\cdot \mid \partial X} \) (\( x \in X \)) on \( \partial X \). Given the
stationary distribution \( \pi \), we define \( \nu_\pi \) by

\[
\nu_\pi = \sum_{x \in X} \pi(x) \nu_x = \pi G_{X^o} P_{X^o, \partial X}.
\]

(4.2) Theorem. Bi-Laplace Neumann problem. Let \( f \in L(X^o) \) and \( g \in L(\partial X) \). Then the boundary value problem

\[
\Delta^2 u = f \text{ on } X^o \text{ and } \partial_\pi u = g \text{ on } \partial X
\]

is solvable if and only if

\[
\int_{X^o} G_{X^o} f \, d\pi + \int_{\partial X} g \, d\nu_\pi = 0.
\]

In this case, all solutions are given by

\[
G_{X^o \setminus \{o\}} G_{X^o} f + G_{X^o \setminus \{o\}} h + c, \quad \text{where} \quad h(x) = \int_{\partial X} g \, d\nu_x \quad \text{and} \quad c \in \mathbb{C}.
\]

Proof. Recall from (2.8) that \( \partial_\pi u = -\Delta u \) on \( \partial X \). Hence, if we set \( v = \Delta u \) then \( v \) must solve the Dirichlet problem

\[
\Delta v = f \text{ on } X^o \text{ and } v = -g \text{ on } \partial X.
\]

By Theorem 2.11 and Remark 2.12, the unique solution is \( v = -G_{X^o} f - h \), where the harmonic function \( h \in H(P, X^o) \) is as stated. Since \( v = \Delta u \), we must necessarily have \( \int_X v \, d\pi = 0 \), that is, (4.3) must hold. In this case, \( u \) solves the Poisson equation \( \Delta u = v \), which leads to the proposed solution(s). \( \square \)

Before passing to the bi-Laplace Dirichlet problem, we need some preparations, involving the two matrices

\[
R = P_{\partial X, X^o} G_{X^o}^2 P_{X^o, \partial X} \quad \text{and} \quad S = (I_{X^o} - P_{X^o})^2 + P_{X^o, \partial X} P_{\partial X, X^o}
\]

over \( \partial X \) and \( X^o \), respectively. We shall see below that invertibility of these matrices is important in the context of bi-Laplace boundary value problems.

(4.5) Lemma. The matrix \( S \) is invertible \( \iff \) \( I_{\partial X} + R \) is invertible.

Proof. We have the following two identities.

\[
P_{\partial X, X^o} G_{X^o}^2 S = (I_{\partial X} + R) P_{\partial X, X^o} \quad \text{and} \quad S G_{X^o}^2 P_{X^o, \partial X} = P_{X^o, \partial X} (I_{\partial X} + R).
\]

If \( (I_{\partial X} + R) \) is invertible then we can multiply the first identity by \( (I_{\partial X} + R)^{-1} \) from the left and expand

\[
G_{X^o}^2 S = I_{X^o} + \left( G_{X^o}^2 P_{X^o, \partial X} (I_{\partial X} + R)^{-1} P_{\partial X, X^o} \right) G_{X^o}^2 S,
\]

which implies that \( G_{X^o}^2 S \) and hence also \( S \) are invertible.

Analogously, if \( S \) is invertible then we can multiply the second identity by \( S^{-1} \) from the right and expand

\[
I_{\partial X} + R = I_{\partial X} + \left( P_{\partial X, X^o} S^{-1} P_{X^o, \partial X} \right) (I_{\partial X} + R),
\]

which implies that \( I_{\partial X} + R \) is invertible. \( \square \)
Let \( \Pi = \text{diag}(\pi(x))_{x \in X} \) and \( \Pi_{X^o} \) and \( \Pi_{\partial X} \) the respective restrictions of this diagonal matrix the the interior and the boundary of \( X \). Recall once more from Definition & Remark \([2.12]\) the hitting distributions

\[
\nu_w(x) = P[\tau < \infty, Z_\tau = x \mid Z_0 = w], \quad w \in X^0, \ x \in \partial X
\]
of our Markov chain on the boundary. Finally, also recall from \([2.3]\) the reversed transition matrix \( \hat{P} = \Pi^{-1}P^t \Pi \). It is irreducible along with \( P \), and there are the associated hitting distributions \( \hat{\nu}_w \), \( w \in X^o \), on \( \partial X \). If \( \hat{\nu}_w(x) > 0 \) for some \( w \in X^o \), then \( x \in \partial_{\text{entrance}} \hat{X} \), the entrance boundary of the original Markov chain of Definition & Remark \([2.12]\).

(4.6) Lemma. Let \( \Upsilon = (\nu_w(x))_{w \in X^o, x \in \partial X} \) and \( \hat{\Upsilon} = (\hat{\nu}_w(x))_{w \in X^o, x \in \partial X} \), two matrices over \( X^o \times \partial X \). Then the matrix \( R \) satisfies

\[
\Pi_{\partial X} R = \hat{\Upsilon}^T \Pi_{X^o} \Upsilon.
\]

In particular, if \( P \) is reversible, i.e. \( \hat{P} = P \), so that \( \Delta \) is self-adjoint on \( L(X, \pi) \), then the matrix \( I_{\partial X} + R \) is invertible.

Proof. We have \( G_{X^o} P_{X^o, \partial X} = \Upsilon \), see Remark \([2.12]\). Therefore \( R = P_{\partial X, X^o} G_{X^o, \Upsilon} \). Furthermore, by the definition of \( \hat{P} \), we have \( \Pi \hat{P} = \hat{P}^T \Pi \). Therefore

\[
\Pi_{\partial X} P_{\partial X, X^o} G_{X^o} = (\hat{G}_{X^o} \hat{P}_{\partial X, X^o})^T \Pi_{X^o} = \hat{\Upsilon}^T \Pi_{X^o}.
\]

This proves the formula for \( \Pi_{\partial X} R \). Now let \( g_1, g_2 \in L(X) \). Set

\[
\hat{h}_1(w) = \int_{\partial X} g_1 \, d\hat{\nu}_w \quad \text{and} \quad h_2(w) = \int_{\partial X} g_2 \, d\nu_w, \quad w \in \partial X.
\]

Then (4.7) implies that

\[
(g_1, Rg_2)_{\pi, \partial X} = (\hat{h}_1, h_2)_{\pi, X^o},
\]

where the subscripts indicate that the inner products are taken with respect to the restriction of \( \pi \) to \( \partial X \), resp. \( X^o \).

Now suppose that \( P \) is reversible. Then \( \hat{\nu}_w = \nu_w \) for all \( w \in X^o \), whence \( \hat{\Upsilon} = \Upsilon \). Assume that \( R \) has a real eigenvalue \(-\alpha < 0\) with associated non-zero eigenfunction \( g \).

Set \( g_1 = g_2 = g \). Then also \( \hat{h}_1 = h_2 =: h \). We get

\[
0 > -\alpha(g, g)_{\pi, \partial X} = (g, Rg)_{\pi, \partial X} = (h, h)_{\pi, X^o} \geq 0,
\]
a contradiction. Therefore, in the reversible case, \( \alpha \cdot I_{\partial X} + R \) is invertible for every \( \alpha > 0 \), in particular, for \( \alpha = 1 \).

\[\square\]

Note that we can factorise

\[ S = (I_{X^o} - P_{X^o})(I_{X^o} - P_{X^o} + \Upsilon P_{\partial X, X^o}). \]

Thus, \( S \) is invertible if and only if the second of those factors is invertible, and then we can write

\[
S^{-1} = K_{X^o} G_{X^o}, \quad \text{where} \quad K_{X^o} = (I_{X^o} - P_{X^o} + \Upsilon P_{\partial X, X^o})^{-1}.
\]

We mention at this point that so far, we did not find a general condition beyond reversibility which guarantees invertibility of \( S \), resp. \( I_{\partial X} + R \).
(4.10) Theorem. Bi-Laplace Dirichlet problem. Let $f \in L(X^o)$ and $g \in L(\partial X)$. If the matrix $S$ of (4.4) is invertible – in particular, in the reversible case – the boundary value problem

$$\Delta^2 u = f \text{ on } X^o \text{ and } u = g \text{ on } \partial X$$

has a unique solution. It is given by

$$u = K_{X^o}G_{X^o}(f + Ug) \text{ on } X^o,$$

where the $X^o \times \partial X$-matrix $U$ is given by

$$U = (I_{X^o} - P_{X^o})P_{X^o,\partial X} + P_{X^o,\partial X}(I_{\partial X} - P_{\partial X}).$$

Proof. We use once more the block decomposition

$$u = \begin{pmatrix} u_{X^o} \\ g \end{pmatrix}, \quad \Delta = \begin{pmatrix} P_{X^o,\partial X} - I_{X^o} & P_{X^o,\partial X} \\ P_{\partial X,X^o} & P_{\partial X} - I_{\partial X} \end{pmatrix} \quad \text{and} \quad \Delta^2 = \begin{pmatrix} S & -U \\ -U^t & S' \end{pmatrix},$$

where $S' = (I_{\partial X} - P_{\partial X})^2 + P_{\partial X,X^o}P_{X^o,\partial X}$ and $U' = (I_{\partial X} - P_{\partial X})P_{\partial X,X^o} + (I_{\partial X} - P_{\partial X})P_{\partial X,X^o}$. Thus, the equation for $u$ on $X^o$ becomes

$$(\Delta^2 u)_{X^o} = Su_{X^o} - Ug,$$

which is equal to $f$ precisely when $u_{X^o}$ has the proposed form. \hfill \Box

While in the “classical” reversible case, the matrices $S$ and $I_{\partial X} + R$ are always invertible, so that the above applies, this is not true in general.

(4.11) Example. Let $X = \{0, 1, \ldots, 2N - 1\}$ and $P$ be given by $p(k, k + 1) = 1$, where $k + 1$ is taken modulo $2N$. All other transition probabilities are $= 0$. Thus, the associated graph is an oriented circle of length $2N$. Now we take $X^o = \{0, 2, 4, \ldots, 2N - 2\}$ and $\partial X = \{1, 3, 5, \ldots, 2N - 1\}$. Then $S$ is the $N \times N$-matrix

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{pmatrix},$$

which is not regular when $N$ is even.

(4.12) Remark. Recall from Definition & Remark 2.12 the space $H(P, X^o)$ of functions which are harmonic on $X^o$. One can transform (4.8) into the following.

The matrices $S$ and equivalently, $I + R$ are not invertible ($-1$ is an eigenvalue of $R$)

$$\iff \text{there is } h_2 \in H(P, X^o), h_2 \neq 0, \text{ such that } (\hat{h}_1, h_2)_\pi = 0 \text{ for every } \hat{h}_1 \in H(\hat{P}, X^o)$$

$$\iff \text{there is } \hat{h}_1 \in H(\hat{P}, X^o), \hat{h}_1 \neq 0, \text{ such that } (\hat{h}_1, h_2)_\pi = 0 \text{ for every } h_2 \in H(P, X^o)$$

$$\iff \text{there are non-zero solutions } u \text{ of the bi-Laplace Dirichlet problem }$$

$$\Delta^2 u = 0 \text{ on } X^o \text{ and } u = 0 \text{ on } \partial X.$$
One sees once more that $I_{\partial X} + R$ and $S$ are invertible in the reversible case. So far, besides Example (4.11), we did not find further (non-reversible) examples where they are non-invertible.

(4.13) Remark. We see that in general we cannot expect to get solutions of the problem to find $u \in L(X)$ such that

\begin{equation}
\Delta^2 u = f \text{ on } X^o, \quad \partial_n u = g_1 \text{ on } \partial X \quad \text{and} \quad u = g_2 \text{ on } \partial X
\end{equation}

for arbitrary $f \in L(X^o)$ and $g_1, g_2 \in L(\partial X)$. Indeed, first of all, $f$ and $g = g_1$ have to satisfy the conservation law (4.3), and after that, we have only one degree of freedom left for the choice of $g_2$. That is, given $g_1$, we can choose $g_2(z)$ for precisely one element $z \in \partial X$. Inserting this value into the solution of the bi-Laplace Neumann problem, we find the constant $c$, after which the other values of $g_2$ on $\partial X$ are determined.

The only case where this is completely satisfactory is the one where $\partial X = \{z\}$ consists of one point only. In this case, $g_1$ and $g_2$ are two constants, and $\nu_x(z) = 1$ for each $x \in X$, so that condition (4.3) becomes

$$\int_{X \setminus \{z\}} G_{X \setminus \{z\}} f \, d\pi + g_1 = 0.$$  

If this holds then there is a unique solution to equation (4.14) for $X^o = X \setminus \{z\}$. Since the choice of the root $o$ for the solution of the Poisson equation in Theorem (2.5) was arbitrary, we may as well use $o = z$ in this case, and then

$$u = G_{X \setminus \{z\}}^2 f + g_1 \cdot G_{X \setminus \{z\}} 1 + g_2,$$

where $1$ is the constant function with value 1.

In spite of what was said in the last remark, it will turn out to be of great interest to derive directly which conditions have to be fulfilled by $g_1$ and $g_2$ (as well as $f$) so that (4.14) can be solved. This will be enhanced by the probabilistic interpretation (which, however, may be skipped by readers who prefer to avoid probability).

(4.15) Definition. Consider the stopping time $\tau = \inf\{n \geq 1 : Z_n \in \partial X\}$. The boundary Markov chain is defined by the transition matrix

$$Q = (q(x,y))_{x,y \in \partial X}, \quad q(x,y) = \mathbb{P}[\tau < \infty, \ Z_\tau = y \mid Z_0 = x].$$

The boundary Laplacian is $\Delta_{\partial X} = Q - I_{\partial X}$.

This gives rise to the induced Markov chain on $\partial X$, i.e., the original Markov chain observed at the successive visits to $\partial X$. The following is a well known consequence of finiteness of $X$ and irreducibility of $P$, see e.g. \[26\] §6.C.

(4.16) Lemma. The stopping time $\tau$ is almost surely finite for any starting point, the matrix $Q$ is stochastic and irreducible, and

$$Q = P_{\partial X} + P_{\partial X, X^o} G_{X^o} P_{X^o, \partial X}.$$
Furthermore, denoting by $\pi_{\partial X}$ the restriction of the stationary distribution $\pi$ of $P$ to $\partial X$, we have that $\pi_{\partial X}$ is stationary for $Q$, that is, $\pi_{\partial X}Q = \pi_{\partial X}$.

The last identity can be obtained by direct matrix operations. We insert a small observation concerning the matrix $R$ over $\partial X$ of (4.4). Recall that $G_{X^o} = (I_{X^o} - P_{X^o})^{-1}$, and consider more generally

$$G_{X^o}(\lambda) = \left(\lambda \cdot I_{X^o} - P_{X^o}\right)^{-1} \quad \text{and} \quad Q(\lambda) = P_{\partial X} + P_{\partial X, X^o} G_{X^o}(\lambda) P_{X^o, \partial X}.$$  

It is well-defined and analytic in a neighbourhood of $\lambda = 1$, and as a consequence of the well-known resolvent equation $G_{X^o}(\lambda_1) - G_{X^o}(\lambda_2) = (\lambda_2 - \lambda_1) \cdot G_{X^o}(\lambda_1)G_{X^o}(\lambda_2)$, we get

$$(4.17) \quad \frac{d}{d\lambda}(\lambda \cdot I_{\partial X} - Q(\lambda)) = I_{\partial X} + R(\lambda), \quad \text{where} \quad R(\lambda) = P_{\partial X, X^o} G_{X^o}(\lambda)^2 P_{X^o, \partial X}.$$  

We have $Q = Q(1)$ and $R = R(1)$, so that in the following, we may interpret $-(I_{\partial X} + R)$ as the derivative (at $\lambda = 1$) of $\Delta_{\partial X}$.

**Theorem. Discrete plate equation, Variant 1.** For $f \in L(X^o)$ and $g_1, g_2 \in L(\partial X)$, the problem (4.14) admits a solution if and only if

$$\Delta_{\partial X} g_2 + P_{\partial X, X^o} G_{X^o}^2 f = -(I_{\partial X} + R)g_1,$$

where $\Delta_{\partial X} = Q - I_{\partial X}$ is the boundary Laplacian, and $R$ is given by (4.4).

In this case, the solution is given by

$$u = G_{X^o}^2 f + G_{X^o} h_1 + h_2, \quad \text{where} \quad h_i(x) = \int_{\partial X} g_i d\nu_x \quad (x \in X, i = 1, 2).$$

Equivalently, for any choice of $z \in \partial X$,

$$u = G_{X \setminus \{z\}} G_{X^o} f + G_{X \setminus \{z\}} h_1 + g_2(z).$$

**Proof.** Since we require $\Delta^2 u = f$ on $X^o$, the function $v = \Delta u$ must be given on all of $X$. The function $v$ must solve the Dirichlet problem

$$\Delta v = f \quad \text{on} \quad X^o \quad \text{and} \quad v = -g_1 \quad \text{on} \quad \partial X.$$

By (2.13), this yields

$$v = -G_{X^o} f - h_1.$$

Then $u$ must solve the Dirichlet problem

$$\Delta u = v_{X^o} \quad \text{on} \quad X^o \quad \text{and} \quad u = g_2 \quad \text{on} \quad \partial X,$$

whence

$$u = -G_{X^o} v_{X^o} + h_2.$$

Thus, we get the first formula for the proposed solution, but we still need to check compatibility. In $x \in \partial X$ then we must have $\Delta u(x) = v(x) = -g_1(x)$, that is

$$g_1(x) = u(x) - \sum_{y} p(x, y) u(y)$$

$$= g_2(x) - \sum_{y \in \partial X} p(x, y) g_2(y) - \sum_{w \in X^o} p(x, w) \left( h_2(w) + G_{X^o}^2 f(w) + G_{X^o} h_1(w) \right)$$
Now recall that we consider functions as column vectors, and that \( h_i = Y g_i = G_{X^0} P_{X^0, \delta X} g_i \). Thus, the above means that
\[
g_1 = g_2 - P_{\delta X} g_2 - P_{\delta X, X^0} G_{X^0} P_{X^0, \delta X} g_2 - P_{\delta X, X^0} G_{X^0}^2 f - P_{\delta X, X^0} G_{X^0}^2 P_{X^0, \delta X} g_1.
\]
Reordering the terms, we obtain that condition (4.19) is necessary for the solution. If (4.19) holds then we can read the above arguments backwards and see that indeed \( \Delta u = v \) on the whole of \( X \), as required, so that the solution is feasible.

The second formula for the solution now follows from Theorem 4.2.

A discussion of condition (4.19) is now in place.

(4.20) Dirichlet to Neumann for the Bi-Laplacian. We first observe that the solution of (4.14) is also a solution of the bi-Laplace Neumann problem, so that \( f \) and \( g_1 \) must satisfy (1.3). We may ask where this is “hidden” in the condition (4.19). Given \( f \) and \( g_1 \), the latter is a Poisson equation for the boundary Laplacian for the determination of \( g_2 \). Since the transition matrix \( Q \) is irreducible with invariant measure \( \pi_{0X} \), Theorem 2.5 implies that for admitting solution, it is necessary and sufficient that
\[
(4.21) \quad \int_{\partial X} \left( P_{\delta X, X^0} G_{X^0}^2 f + (I_{\delta X} + R) g_1 \right) d\pi = 0.
\]
With some small effort, this transforms precisely into (1.3). Indeed, (4.7) implies that
\[
\pi(x) P_{\delta X, X^0} G_{X^0}(x, w) = \hat{v}_w(x) \pi(w) \quad \text{for} \quad x \in \partial X, \ w \in X^0.
\]
Therefore
\[
\int_{\partial X} P_{\delta X, X^0} G_{X^0}^2 f \, d\pi = \sum_{x \in \partial X, w \in X^0} \hat{v}_w(x) \pi(w) G_{X^0} f(w) = \int_{X^0} G_{X^0} f \, d\pi,
\]
since each \( \hat{v}_w \) is a probability distribution on \( \partial X \). In the same way, using (1.8),
\[
\int_{\partial X} (I_{\delta X} + R) g_1 \, d\pi = \int_{\partial X} g_1 \, d\pi + (1, \ R \hat{g}_1)_{\pi, \partial X} = \int_{\partial X} g_1 \, d\pi + (1, \ h_1)_{\pi, X^0} = \int_{\partial X} g_1 \, d\nu_x,
\]
since \( \nu_x = \delta_x \) for \( x \in \partial X \).

Thus, if \( f \) and \( g_1 \) are given and such that (1.3), resp. (4.21) hold, then Theorem 2.5 yields that for arbitrary \( z \in \partial X \), the possible choices for \( g_2 \) are
\[
g_2 = \left( I_{\delta X \setminus \{z\}} - Q_{\delta X \setminus \{z\}} \right)^{-1} \left( P_{\delta X, X^0} G_{X^0}^2 f + (I_{\delta X} + R) g_1 \right) + c,
\]
for any \( c \in \mathbb{C} \).

Conversely, if \( f \) and \( g_2 \) are given, and if the matrix \( I_{\delta X} + R \) is invertible – in particular, in the reversible case – the function \( g_1 \) is determined uniquely as
\[
(4.22) \quad g_1 = -(I_{\delta X} + R)^{-1} \left( \Delta_{\delta X} g_2 + P_{\delta X, X^0} G_{X^0}^2 f \right).
\]
In that case, if \( f \equiv 0 \), then we have the linear Dirichlet to Neumann map \( L(\partial X) \rightarrow L(\partial X) \), \( g_2 \rightarrow T g_1 \) with the transfer matrix
\[
(4.23) \quad T = -(I_{\delta X} + R)^{-1} \Delta_{\delta X} = (I_{\delta X} + R)^{-1} (I_{\delta X} - Q).
\]
Its kernel consists once more of the constant functions, and its image is the hyperplane
\[ \left\{ g_1 \in L(\partial X) : \int_{\partial X} g_1 \, d\nu = 0 \right\}. \]

In any case, the situation for Dirichlet & Neumann conditions for the bi-Laplacian in the
 discrete setting is quite different from the smooth case as considered, e.g., by GANDER AND LI \[17\]. See also the Discussion \[4.26\] below.

We now propose a second approach to the “plate equation” via an at first glance slight
modification of the problem \((4.14)\). We decompose \(X^o = Y = Y^o \cup \partial Y\) according to
\((2.20)\). Thus, \(Y^o\) is the “second interior” of \(X\). Recall from Remarks \[2.19\] the definition of \(\Delta[Y]\) and the second way \((2.23)\) of defining the outer normal derivative. Furthermore, we
require that \(Y\) is strongly connected, so that the matrix \(P[Y]\) is irreducible, and we write
\(Q[\partial Y]\) for the transition matrix of the boundary process on \(\partial Y\) according Definition \[4.15\] and Lemma \[4.16\] (with \(Y\) in the place of \(X\)). There is also the associated matrix \(R[\partial Y]\)
corresponding to \((4.4)\) and Lemma \[4.6\]. With these ingredients, we have the following.

\[(4.24)\) Theorem. Discrete plate equation, Variant 2. Let \(f \in L(Y^o), g_1 \in L(\partial Y)\)
and \(g_2 \in L(\partial X)\). If the matrix \(I_{\partial Y} + R[Y]\) is invertible - in particular, in the reversible
case - the problem to find \(u \in L(X)\) such that
\[ \Delta^2[Y] u = f \text{ on } Y^o, \quad \partial_n^* u = g_1 \text{ on } \partial Y \quad \text{and} \quad u = g_2 \text{ on } \partial X \]
has a unique solution. On \(Y\), it is the is the solution of the bi-Laplace Dirichlet problem
\[ \Delta^2[Y] u = f \text{ on } Y^o, \quad u = g \text{ on } \partial Y, \quad \text{where} \]
\[ g(y) = \frac{1}{p(y, \partial X)} \sum_{z \in \partial X} p(y, z) g_2(z) - g_1(y), \quad y \in Y, \]
according to Theorem \[4.10\]

Proof. By \((2.23)\) we must have
\[ \partial_n^* u(y) = \frac{1}{p(y, \partial X)} \sum_{z \in X \setminus Y} p(y, z) g_2(z) - u(y), \quad y \in \partial Y. \]
This yields the values of \(u = g\) on \(\partial Y\). \(\Box\)

We explicitly propose these two variants, which show that the discrete analogue of
typical “smooth” equations may be subject to different interpretations; see the discussion
below.

We conclude this Section with another, simple variant of the Dirichlet problem for the
bi-Laplacian \(\Delta^2\) on \(X\).

\[(4.25)\) Theorem. Iterated Dirichlet problem. With \(X^o = Y = Y^o \cup \partial Y\), let \(f \in L(Y^o), g_1 \in L(\partial Y)\)
and \(g_2 \in L(\partial X)\). Then the boundary value problem
\[ \Delta^2 u = f \text{ on } Y^o, \quad \Delta f(y) = g_1 \text{ on } \partial Y \quad \text{and} \quad u = g_2 \text{ on } \partial X \]
has a unique solution, which is given by
\[ u = G_{X^o}G_{Y^o}f - G_{X^o}h_1 + h_2, \]
where
\[ h_1(y) = \int_{\partial Y} g_1 \, dv_y \quad (y \in Y) \quad \text{and} \quad h_2(x) = \int_{\partial X} g_2 \, dv_x \quad (x \in X). \]

Proof. Define \( v = \Delta u \) on \( X^o \). It must solve the Dirichlet problem
\[ \Delta v = f \quad \text{on} \quad Y^o \quad \text{and} \quad v = g_1 \quad \text{on} \quad \partial Y, \]
whence by Theorem 2.11, \( v = -G_{Y^o}f + h_1 \). Next, \( u \) must solve the Dirichlet problem
\[ \Delta u = v \quad \text{on} \quad X^o \quad \text{and} \quad v = g_2 \quad \text{on} \quad \partial X. \]
The solution is \( u = -G_{X^o}v + h_2 \).

It is clear that one can iterate further, taking \( \Delta^n \) and the \( n \)th interior of \( X \) on which \( f \) is defined (as long as that interior is non-empty), as well as the “onion layers” of successive boundaries on which the respective boundary functions \( g_1, \ldots, g_n \) are specified.

(4.26) Discussion: bi-harmonic Green kernel. The bi-harmonic Green kernel should be the respective kernel which provides the solution of the problems considered in the last three theorems, where all boundary values are set to 0, and one is given only the function \( f \) defined in the interior. In the smooth situation, one of the interesting problems concerns the negative part of that kernel corresponding to the plate equation; see [18].

From our theorems, we see that the choice of that kernel is case-dependent.

Let us start with the last one, from Theorem 4.25. The kernel is \( G_{X^o}G_{Y^o} \), which is non-negative and \( > 0 \) on \( X^o \times Y^o \) (as long as \( X^o \) is strongly connected). This corresponds to the bi-harmonic Green kernel of [30]. It appears not to have a natural counterpart in the classical smooth setting.

Regarding the two variants concerning the plate equation, we already saw that the solution of Theorem 4.18 is more restrictive. It may be natural to consider only the “first” boundary. On the other hand, \( \Delta \) is not an infinitesimal operator, and in \( \Delta(\Delta u) \), already the first application of \( \Delta \) reaches out to the boundary and involves the boundary values directly. If we set \( g_1 = g_2 = 0 \) in (4.14), then by (4.19), we only get a solution if \( f \in L(X^o) \) satisfies
\[ P_{\partial X, X^o}G_{X^o}^2 f = 0. \]
The associated kernel is then \( G_{X^o}^2 \), which is positive, while it is the function \( f \) whose real as well as complex parts (unless they vanish) must have positive as well as negative values.

In the second variant, the one of Theorem 4.24, we have \( f \in L(Y^o) \), and with the respective boundary values set to zero, the solution is the one of the bi-Laplace Dirichlet problem \( \Delta^2_{\partial Y}|u = f \) with \( u = 0 \) on \( \partial Y \). We may equivalently replace this by the bi-Laplace Dirichlet problem on \( X \),
\[ \Delta^2 u = f \quad \text{on} \quad X^o, \quad \text{where} \quad f \in L(X^o), \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial X. \]
We suppose that the matrices $I_{\partial X} + R$ and $S$ are invertible; see Lemma 4.6. From Theorem 4.10 we get with $K_{X^0}$ given by (4.3)

$$u = K_{X^0}G_{X^0}f.$$  
Thus, $K_{X^0}G_{X^0}$ is our bi-harmonic Green kernel for the plate equation in variant 2 (when rewritten in terms of $\Delta_Y$ and $Y^0$ instead of $\Delta_{[X]}$ and $X^0$). In general, it is not positive everywhere on $X^0 \times X^0$. It appears to be a reasonable analogue of the kernel for the smooth plate equation as in [18].

5. Examples

A. Simple random walk on an integer interval

In our first example, $X = \{0, 1, \ldots, N\}$, the symmetric edges are between successive integers, and we start with symmetric edge weights $a(k, k \pm 1) = 1$. The associated Markov chain is reversible with $p(k, k \pm 1) = 1/2$ for $k = 1, \ldots, N-1$ and $p(0, 1) = p(N, N−1) = 1$, while all other transition probabilities are 0. The stationary probability measure is

$$\pi(k) = \frac{1}{N} \text{ for } k = 1, \ldots, N-1, \quad \text{and} \quad \pi(0) = \pi(N) = \frac{1}{2N}.$$  

We set $o = 0$ and $\partial X = \{0, N\}$. Then it is quite easy to compute the Green kernels $G_{X\{0\}}$ and $G_{X^0}$. We can use for example the computations of [26, §5.A]. (Be careful when using Lemma 5.5 of that reference: the $R_k$ there is the $R_{k−1}$ of the subsequent page 119).

Our $G_{X\{0\}}$ corresponds to the case when state 0 is absorbing and state $N$ is reflecting, and one computes $F_{X\{0\}}(k, m)$, the probability to reach state $m$ when starting at $k$:

$$F_{X\{0\}}(k, m) = \frac{k}{m} \text{ for } k \leq m, \quad \text{and} \quad F_{X\{0\}}(k, m) = 1 \text{ for } k \geq m.$$  

Using the equation of [26, Thm. 1.38], we compute for $m = 1, \ldots, N−1$

$$G_{X\{0\}}(m, m) = \frac{1}{1 - \frac{1}{2}F_{X\{0\}}(m - 1, m) - \frac{1}{2}F_{X\{0\}}(m + 1, m)} = \frac{2}{m}, \quad \text{and} \quad G_{X\{0\}}(N, N) = \frac{1}{1 - F_{X\{0\}}(N-1, N)} = \frac{1}{N}.$$  

Now using that $G_{X\{0\}}(k, m) = F_{X\{0\}}(k, m)G_{X\{0\}}(m, m)$, we get

$$G_{X\{0\}}(k, m) = 2 \min\{k, m\} \text{ for } m < N, \quad \text{and} \quad G_{X\{0\}}(k, N) = k.$$  

Next, $G_{X^0}$ corresponds to the case when both states 0 and $N$ are absorbing. Then, by the same methods, for $k, m \in \{1, \ldots, N−1\}$,

$$F_{X^0}(k, m) = \frac{k}{m} \text{ for } k \leq m, \quad \text{and} \quad F_{X^0}(k, m) = \frac{N-k}{N-m} \text{ for } k \geq m.$$  

We get

$$G_{X^0}(k, m) = G_{X^0}(m, k) = \frac{2k(N-m)}{N} \text{ for } 1 \leq k \leq m \leq N-1.$$  

(5.3)
Poisson equation. Let \( f \in L(X) \) with \( f(0) + 2(f(1) + \cdots + f(N - 1)) + f(N) = 0 \). Then the solution of the Poisson equation grounded at 0 is

\[
(5.4) \quad u_{\text{Kirch}}(k) = -G_{X \setminus \{0\}} f(k) = -2 \sum_{m=1}^{k-1} m f(m) - 2k \sum_{m=k}^{N-1} f(m) - k f(N),
\]

\( k \in \{1, \ldots, N\} \).

We skip the Neumann problem, which is equivalent with the Poisson equation.

Dirichlet problem. Let \( f \in L(X^o) = L(\{1, \ldots, N - 1\}) \) and \( g \in L(\partial X) = L(\{0, N\}) \). We first find the distributions on the boundary, using that \( \nu_k(N) = F_{X \setminus \{0\}}(k, N) \):

\[
(5.5) \quad \nu_k(0) = \frac{N - k}{N} \quad \text{and} \quad \nu_k(N) = \frac{k}{N},
\]

whence

\[
h(k) = \int_{\partial X} g d\nu_k = \frac{N-k}{N} g(0) + \frac{k}{N} g(N).
\]

Then the solution of the Dirichlet problem is

\[
u_{\text{Dir}}(k) = h(k) - G_{X^o} f(k) = h(k) - \frac{2(N-k)}{N} \sum_{m=1}^{k-1} m f(m) - \frac{2k}{N} \sum_{m=k}^{N-1} (N-m) f(m).
\]

Mixed problem. The simplest mixed problem is when in addition to \( f \in L(X^o) \) we require that the solution satisfies

\[
u(0) = g(0) \quad \text{and} \quad \partial_\nu u(N) = g(N), \quad \text{where} \quad g(0), g(N) \in \mathbb{C}.
\]

In this case, we extend \( f \) to \( \{1, \ldots, N\} \) by setting \( f(N) = -g(N) \). Then the solution is

\[
u(k) = g(0) + u_{\text{Kirch}}(k), \quad \text{where the latter is given by (5.4)}.
\]

Poisson and Dirichlet problem with potential. We only consider the easiest case, when the potential \( v \) is constant. We set \( \lambda = 1 + v \), so that our assumption is \( |\lambda| > 1 \). Then \( \tilde{P} = \frac{1}{X} P \) on \( X \). Computing \( \tilde{G} = \sum_{n=0}^{\infty} \frac{1}{\lambda^n} P^n \) amounts to invert a tri-diagonal matrix. There are various ways. We used once more \([26, 5.4]\). Let \( Q_k(\lambda) \) and \( R_k(\lambda) \) be the \( k^{th} \) Chebyshev polynomials of the first and second kind, respectively, that is,

\[
Q_k(\cos \varphi) = \cos k \varphi \quad \text{and} \quad R_k(\cos \varphi) = \frac{\sin(k + 1) \varphi}{\sin \varphi}.
\]

It will be convenient to set \( R_{-1}(\lambda) = 0 \). After some manipulations, setting \( \epsilon_m = 2 \) for \( m \in \{1, \ldots, N - 1\} \) and \( \epsilon_0 = \epsilon_N = 1 \), we get for \( k, m \in X \)

\[
\tilde{G}(k, m) = \begin{cases} \frac{\lambda}{\lambda^2 - 1} \frac{Q_k(\lambda)Q_{N-m}(\lambda)}{R_{N-1}(\lambda)}, & \text{if } k \leq m, \\ \frac{\lambda}{\lambda^2 - 1} \frac{Q_m(\lambda)Q_{N-k}(\lambda)}{R_{N-1}(\lambda)}, & \text{if } k \geq m. \end{cases}
\]
Then, for \( f \in L(\{0, \ldots, N\}) \), the unique solution \( u \) of the Poisson problem with constant potential \( \Delta u - (\lambda - 1)u = f \)
is
\[
u_{\text{Kirch}}(k) = - \sum_{k=0}^{N} \frac{G(k, m) f(m)}{\lambda}.
\]
Next, with \( X^{\circ} = \{1, \ldots, N-1\} \), we can also compute \( \tilde{G}_{X^{\circ}}(k, m) \) for \( k, m \in X^{\circ} \) as follows.
\[
\tilde{G}_{X^{\circ}}(k, m) = \begin{cases} 
2\lambda \frac{R_{k-1}(\lambda) R_{N-m-1}(\lambda)}{R_{N-1}(\lambda)}, & \text{if } k \leq m, \\
2\lambda \frac{R_{m-1}(\lambda) R_{N-k-1}(\lambda)}{R_{N-1}(\lambda)}, & \text{if } k \geq m.
\end{cases}
\]
Next we compute the measures \( \tilde{\nu}_k \). (In [26], 5.A], \( \tilde{\nu}_k(0) \) corresponds to the quantity \( F(k, 0|z) \) in the middle of p. 119, with \( z = 1/\lambda \).
\[
\tilde{\nu}_k(0) = \frac{R_{N-k-1}(\lambda)}{R_{N-1}(\lambda)} \quad \text{and} \quad \tilde{\nu}_k(N) = \frac{R_{k-1}(\lambda)}{R_{N-1}(\lambda)}.
\]
These are in general not probability measures on \( \{0, N\} \), and indeed not even necessarily positive, unless \( \lambda > 1 \) is real. Given \( f \in L(\{1, \ldots, N-1\}) \) and \( g \in L(\{0, N\}) \), the unique solution \( u \) of the Dirichlet problem with constant potential \( \Delta u - (\lambda - 1)u = f \) on \( X^{\circ} \), \( u = g \) on \( \partial X \) is now
\[
u_{\text{Dir}}(k) = g(0) \tilde{\nu}_k(0) + g(N) \tilde{\nu}_k(N) - \sum_{m=1}^{N-1} \tilde{G}_{X^{\circ}}(k, m) \frac{f(m)}{\lambda}.
\]
Robin problem. The simplest case is \( \alpha = \beta \neq 0 \), constant on \( \partial X = \{0, N\} \). Referring to Corollary \( \ref{cor:6.9} \) \( \bar{B} = \emptyset \) and \( X^{\circ}_{\beta} = X \) with \( \bar{p}(k, k \pm 1) = 1/2 \) whenever \( k, k \pm 1 \in \{0, \ldots, N\} \).

Computing the associated Green kernel \( \tilde{G}_{\text{Rob}} = (I - \bar{P})^{-1} \) is analogous to computing \( G_{X^{\circ}} \) in \( \ref{eq:5.3} \). Indeed, instead of adding one “tomb” state, consider the extended space \( \tilde{X} = \{-1, 0, \ldots, N, N+1\} \) with the simple random walk as before and the new boundary \( \{-1, N+1\} \) and interior \( X^{\circ} = X \). Thus, when we shift the elements by 1 and replace \( N \) with \( N + 2 \), we are back to the computation of \( G_{X^{\circ}} \). Hence, for \( k, m \in \{0, \ldots, N\} \),
\[
\tilde{G}_{\text{Rob}}(k, m) = \frac{2(k+1)(N+1-m)}{N+2} \quad \text{for } 0 \leq k \leq m \leq N.
\]
Therefore, given \( f \in L(\{1, \ldots, N-1\}) \) and \( g \in L(\{0, N\}) \), the unique \( u = u_{\text{Rob}} \in L(\{0, \ldots, N\}) \) such that \( \Delta u = f \) on \( \{1, \ldots, N-1\} \) and \( \alpha \cdot (u + \partial_{\bar{r}}u) = g \) on \( \{0, N\} \) is
\[
u_{\text{Rob}}(k) = \frac{g(0)}{2\alpha} + \frac{g(N)}{2\alpha} - \sum_{m=1}^{N-1} \tilde{G}_{\text{Rob}}(k, m) f(m) .
\]
The bi-Laplacian. We have computed the Green kernels \( G_{X^{\circ}\setminus\{0\}} \) and \( G_{X^{\circ}} \) in \( \ref{eq:5.2} \) and \( \ref{eq:5.3} \), as well as the hitting distributions \( \nu_k \) in \( \ref{eq:5.5} \). On this basis, the solutions of the iterated Poisson equation and the bi-Laplace Neumann problem can be written down immediately. So we next compute the transition matrix \( Q \) of the boundary chain of Definition \( \ref{def:4.15} \) and Lemma \( \ref{lem:4.16} \) and the matrix \( R \) of \( \ref{eq:4.4} \).
We have $q(0, 0) = p(0, 1)\nu_1(0) = \nu_1(0) = \frac{N-1}{N} = q(N, N)$, and we get

$$Q = \frac{1}{N} \begin{pmatrix} N - 1 & 1 \\ 1 & N - 1 \end{pmatrix}.$$  

With $\pi$ given by (5.1), we have for $i, j \in \{0, N\}$

$$r(i, j) = 2 \sum_{k=1}^{N-1} \nu_k(i)\nu_k(j).$$

We get

$$R = \frac{N-1}{3N} \begin{pmatrix} 2N - 1 & N + 1 \\ N + 1 & 2N - 1 \end{pmatrix}$$

and

$$(I + R)^{-1} = \frac{1}{N^3 + 2N} \begin{pmatrix} 2N^2 + 1 & 1 - N^2 \\ 1 - N^2 & 2N^2 + 1 \end{pmatrix}.$$  

We now want to consider the bi-Laplace Dirichlet problem with $f \equiv 0$ on $X^o$ and boundary function $g = g_2 \in L\{0, N\}$. Since the boundary has only 2 elements, we prefer to use the transfer matrix for the bi-Laplace Dirichlet to Neumann map according to (4.23), which is

$$T = \frac{3}{N^2 + 2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$  

In other words, if the Dirichlet boundary values are $g(0)$ and $g(N)$, then the solution must have the Neumann boundary values $g_1(0) = -g_1(N) = 3((g(0) - g(N))/(N^2 + 2)$. We get

$$h_1(k) = \int_{\partial X} g_1 \, dv_k = \frac{3N - 6k}{N^3 + 2N} (g(0) - g(N))$$

and

$$h_2(k) = \int_{\partial X} g \, dv_k = \frac{N - k}{N} g(0) + \frac{k}{N} g(N).$$

The unique solution $u$ of the bi-Laplace Dirichlet problem

$$\Delta u = 0 \hspace{1em} \text{on} \hspace{1em} \{1, \ldots, N\}, \hspace{1em} u(0) = g(0), \hspace{1em} u(N) = g(N)$$

is now given via (5.3) as

$$u(k) = h_2(k) + \frac{2(N - k)}{N} \sum_{m=1}^{k-1} m h_1(m) - \frac{2k}{N} \sum_{m=k}^{N-1} (N - m) h_1(m).$$

The other bi-Laplace equations are obtained along the same lines: The plate equation of Theorem 4.24 (with $f \equiv 0$) is a variant of what we have just computed, replacing $\{0, \ldots, N\}$ with $Y = \{1, \ldots, N - 1\}$ and $\partial Y = \{1, N - 1\}$. The iterated Dirichlet problem of Theorem 4.25 means that one has to apply the Green kernel $G_{X^o}$ computed in (5.3) as well as the Green kernel $G_{Y^o}$, which is computed in the same way (shifting down by 1 and replacing $N$ by $N - 2$).

**B. A non-reversible example.**

We set $X = \{1, \ldots, N\}$, where $N \geq 3$, and choose probabilities $p_1, \ldots, p_N > 0$ with sum 1. The transition probabilities are then

$$p(1, k) = p_k \hspace{1em} \text{for} \hspace{1em} k \in X, \hspace{1em} \text{and} \hspace{1em} p(k, k - 1) = 1 \hspace{1em} \text{for} \hspace{1em} k \geq 2.$$
As the boundary, we choose $\partial X = \{N - 1, N\}$, and as the root, we choose $o = 1$. The stationary probability distribution is

$$\pi(k) = \frac{\sum_{m=k}^N p_m}{\sum_{m=1}^N mp_m}. \tag{5.6}$$

Poisson equation. It is easy to compute $G_{X \setminus \{1\}}$:

$$G_{X \setminus \{1\}}(k, m) = 1 \text{ for } 2 \leq m \leq k, \text{ and } G_{X \setminus \{1\}}(k, m) = 0 \text{ otherwise}. \tag{5.7}$$

Therefore, given $f \in L(X)$ with $\int_X f \, d\pi = 0$, the unique solution of the Poisson equation grounded at 1 is

$$u(k) = \sum_{m=2}^k f(m).$$

For the remaining issues, we compute the Green kernel $G(k, m|z) = \sum_{n=0}^\infty p^{(n)}(k, m)z^n$, following the methods of [26]. The computations are also valid when $p_1 + \cdots + p_N < 1$. First, consider $F(k, m|z) = G(k, m|z)/G(m, m|z)$, the generating function of the first hitting probability at $m$, when the Markov chain starts at $k$. We have

$$F(k, m|z) = z^{k-m} \text{ for } k \geq m \text{ and } F(k, m|z) = z^{k-1}F(1, m|z) \text{ for } k < m.$$

Next,

$$F(1, m|z) = \sum_{j=1}^N p_j z^j F(k, m|z) = \sum_{j=1}^{m-1} p_j z^j F(1, m|z) + \sum_{j=m}^N p_j z^{1+j-m} \text{ for } m > 1.$$

We conclude that

$$F(k, m|z) = \sum_{j=m}^N p_j z^{j+k-m} \left/ \left(1 - \sum_{j=1}^{m-1} p_j z^j\right)\right. \text{ for } k \geq m, \text{ and } F(k, m|z) = z^{k-1}F(1, m|z) \text{ for } k < m.$$

Next, the general formula $G(m, m|z) = 1/ \left(1 - \sum_j p(m, j)z F(j, m|z)\right)$ yields

$$G(m, m|z) = \left(1 - \sum_{j=1}^{m-1} p_j z^j\right)/\left(1 - \sum_{j=1}^N p_j z^j\right).$$

Altogether,

$$G(k, m|z) = \left\{\begin{array}{ll}
\left(z^{k-m} - \sum_{j=1}^{m-1} p_j z^{j+k-m}\right)/\left(1 - \sum_{j=1}^N p_j z^j\right) & \text{for } k \geq m, \\
\left(\sum_{j=m}^N p_j z^{j+k-m}\right)/\left(1 - \sum_{j=1}^N p_j z^j\right) & \text{for } k < m.
\end{array}\right. \tag{5.8}$$

Dirichlet problem. Let $f \in L(X^o) = L(\{1, \ldots, N - 2\})$ and $g \in L(\partial X) = L(\{N - 1, N\})$. The Green kernel $G_{X^o}$ is obtained from (5.8) by replacing $N$ with $N - 2$ and setting $z = 1.$
A natural mixed problem is when in addition to

Again, we only consider the easiest case, when the po-

Again, the solutions of the iterated Poisson problem and the bi-Laplace

problem to the Poisson equation. Instead, we are lead to the above Dirichlet problem

and

while of course

We next compute the hitting distributions \( \nu_k \) on the boundary. It is clear that For all

Thus, given \( g \in L(\{N-1,N\}) \), the associated harmonic function on \( X^o \) is

while of course \( h(j) = g(j) \) for \( j \in \{N-1,N\} \). If now in addition \( f \in L(\{1,\ldots,N-2\}) \),

then the solution of the Dirichlet problem is

Mixed problem. A natural mixed problem is when in addition to \( f \in L(X^o) \) we require

that the solution satisfies

Then we must have \( u(N) = g(N) + g(N-1) \), and this time we do not re-conduct the problem to the Poisson equation. Instead, we are lead to the above Dirichlet problem with boundary function \( \tilde{g}(N-1) = g(N-1) \) and \( \tilde{g}(N) = g(N-1) + g(N) \).

Poisson equation with potential. Again, we only consider the easiest case, when the potential \( v \) is constant, with \( \lambda = 1 + v \), so that \( |\lambda| > 1 \). Again, \( \bar{P} = \frac{1}{\lambda} P \) on \( X \), and the associated Green kernel \( \tilde{G} \) is the one of (5.8), with \( z = 1/\lambda \). This leads to the solution.

We skip the Dirichlet problem with potential and the Robin problem.

The bi-Laplacian. Again, the solutions of the iterated Poisson problem and the bi-Laplace

Neumann problem can be written down immediately via the Green kernel \( G_{X\setminus\{1\}} \) computed in (5.7). We also have \( G_{X^o} \) and the hitting distributions on the boundary.

It is easy to compute the transition matrix of the boundary process: \( q(N,N-1) = 1 \)

and \( q(N-1,j) = \nu_1(j), j \in \{N-1,N\} \). In this example, the exit boundary is \( \{N-1,N\} \),

but the entrance boundary is only \( \{N-1\} \), that is, \( \nu_k(N-1) = 1 \) and \( \nu_k(N) = 0 \) for all \( k \in \{1,\ldots,N-2\} \).

With \( \pi \) given by (5.6), we have for \( i,j \in \{N-1,N\} \)

Therefore, the exit boundary is only

Then

\[
G_{X^o}(k,m) = \begin{cases} \frac{\pi(m)}{\pi(N-1)} & \text{for } 1 \leq k \leq m \leq N-2, \text{ and} \\ \frac{\pi(N-1)}{(\pi(m) - \pi(N-1))} & \text{for } 1 \leq m < k \leq N-2. \end{cases}
\]

We next compute the hitting distributions \( \nu_k \) on the boundary. It is clear that For all

Thus, given \( g \in L(\{N-1,N\}) \), the associated harmonic function on \( X^o \) is

while of course \( h(j) = g(j) \) for \( j \in \{N-1,N\} \). If now in addition \( f \in L(\{1,\ldots,N-2\}) \),

then the solution of the Dirichlet problem is

\[
u_k(j) = \nu_1(j) = G_{X^o}(1,1) p_j = \frac{\pi(1)}{\pi(N-1)} p_j = \frac{p_j}{p_{N-1} + p_N}.
\]

Mixed problem. A natural mixed problem is when in addition to \( f \in L(X^o) \) we require
Thus,
\[ Q = \begin{pmatrix} \frac{\pi(1)}{\pi(N-1)} p_{N-1} & \frac{\pi(1)}{\pi(N-1)} p_N \\ \frac{\pi(N-1)}{N-1} p_{N-1} & \frac{\pi(N-1)}{N-1} p_N \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} C p_{N-1} & C p_N \\ 0 & 0 \end{pmatrix}, \]
where
\[ C = \frac{\pi(1)(1 - \pi(N-1) - \pi(N))}{\pi(N-1)^2}. \]

Now the transfer matrix for the Bi-Laplace Dirichlet to Neumann map is
\[ T = \begin{pmatrix} D & -D \\ -1 & 1 \end{pmatrix}, \]
where
\[ D = \frac{\pi(N)^2 - \pi(N) + 2\pi(N)\pi(N-1)}{\pi(N)^2 - \pi(N) + \pi(N-1)}. \]

We remark that \( D < 0 \). If the Dirichlet boundary values are \( g(N-1) \) and \( g(N) \), then the solution must have the Neumann boundary values \( g_1(N-1) = D(g(N-1) - g(N)) \) and \( g_1(N) = g(N) - g(N-1) \). We get for \( k \in \{1, \ldots, N-2\} \)
\[ h_1(k) = \int_{\partial X} g_1 \, d\nu_k = h_1(1) = (g(N) - g(N-1)) \frac{p_N - Dp_{N-1}}{p_{N-1} + p_N} \]
and
\[ h_2(k) = \int_{\partial X} g \, d\nu_k = h_2(1) = \frac{(g(N-1)p_{N-1} + g(N)p_N)}{p_{N-1} + p_N}. \]

The unique solution \( u \) of the bi-Laplace Dirichlet problem
\[ \Delta u = 0 \quad \text{on} \quad \{1, \ldots, N-2\}, \quad u(N-1) = g(N-1), \quad u(N) = g(N) \]
is now given via (5.9) as
\[ u(k) = \left( \frac{1 - \pi(N-1) - \pi(N)}{\pi(N-1)} - (k - 1) \right) h_1(1) + h_2(1). \]

With this computation, we end the second example.

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