VIRASORO SINGULAR VECTORS VIA QUANTUM DS REDUCTION *

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Abstract

The BRST quantisation of the Drinfeld - Sokolov (DS) reduction is exploited to recover all singular vectors of the Virasoro algebra Verma modules from the corresponding $A^{(1)}_1$ ones. The two types of singular vectors are shown to be identical modulo terms trivial in the $Q_{BRST}$ cohomology. The main tool is a quantum version of the DS gauge transformation.

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1. In [1] Bauer et al (BFIZ) interpreted any of the Virasoro singular vectors found in [2] as a scalar Lax operator [3] recovered by a matrix valued system, associated to a finite dimensional representation of \( sl(2) \). In [4], [5] it was shown that the BFIZ matrix system emerges naturally from a system generated by the \( sl(2) \) Knizhnik – Zamolodchikov (KZ) equation. Namely the two are related by a quantum DS - type gauge transformation. Furthermore a conjecture was made [5] relating the general singular vectors of the Vir Verma modules to the Malikov – Feigin – Fuks (MFF) [6] \( \tilde{sl}(2) \) singular vectors. An explicit algorithm based on it was shown to reproduce the simplest examples. The technique in [4], [5] was based on a realisation of the generators by differential operators and on a proper basis suggested by general explicit solutions of the KZ equation.

Here we reformulate the problem in purely algebraic terms, implementing the quantum DS reduction in a BRST framework [7] – however for Verma, instead of Fock modules (an approach advocated also in [8]). This enables us in particular to prove the conjecture in [5] exploiting the explicit formula for the Vir singular vectors written by Kent [9], thus demonstrating that the formulae in [6] and [9] are not just analogous in form, but up to BRST trivial terms, simply identical. In a forthcoming paper [10] the approach developed here will be generalised for the \( W_3 \) algebras.

2. For a field \( A(z) \) one has the mode expansion \( A(z) = \sum_{m \in \mathbb{Z}} A_m z^{-m-\Delta} \). The (anti)commutation relations of the modes are equivalent to the singular part of the operator product expansions (OPE) of the fields. The normal ordered product \( (AB)(w) \) of two fields \( A \) and \( B \) is by definition the zero order term in the operator product expansion (OPE) of \( A(z)B(w) \). We will refer to [11] for rules to work with OPEs. With dot or \( \partial \) we denote a derivative; \( (\psi)_m = -(m+\Delta) \psi_m \).

The affine Lie algebra \( g = \hat{sl}(2)_k \) is defined by the OPE

\[
X^+(z) X^-(0) = \frac{k}{z^2} + \frac{2X^0(0)}{z} + \ldots,
\]

\[
X^0(z) X^\pm(0) = \pm \frac{X^\pm(0)}{z} + \ldots,
\]

\[
X^0(z) X^0(0) = \frac{k/2}{z^2} + \ldots,
\]

(1)

assuming \( \Delta_X = 1 \). Next we need a system of fermionic ghosts, i.e., anticommuting fields \( b, c \) with \( \Delta_c = 1 \), \( \Delta_b = 0 \),

\[
b(z) c(0) = \frac{1}{z} + \ldots .
\]

(2)

The constraint \( X^+(z) = 1 \) is introduced in a BRST fashion. The BRST charge \( Q \) is [7]

\[
Q = \oint_{C_0} dz (cX^+ - c)(z) = (cX^+)|_{-1} - c_0 , \quad Q^2 = 0.
\]

(3)

The Sugawara energy momentum tensor \( T^{(\text{sug})} = (X^a X_a)/2(k+2) \) is extended to a field \( T^{(\text{tot})} \) commuting with \( Q \),

\[
T^{(\text{tot})} = T^{(\text{sug})}(z) + \dot{X}^0(z) + (\dot{b} c)(z).
\]

(4)

Let us introduce the field [12]

\[
\dot{X}^0 = X^0 + (b c).
\]

(5)

Its operator product with \( X^\pm \) is the same as that of \( X^0 \), while

\[
\dot{X}^0(z) \dot{X}^0(0) = \frac{1/2\nu}{z^2} + \ldots ; \quad 1/\nu = k + 2.
\]

(6)
The energy - momentum tensor

\[ T^{(ff)} = \nu(\hat{X}^0 \hat{X}^0) + (1 - \nu)\partial \hat{X}^0, \tag{7} \]

has conformal anomaly \( c_{\nu} = 13 - 6\nu - 6/\nu. \)

By a straightforward computation using the technique of [11] one can show that the tensor \( T^{(\text{tot})} \) can be rewritten in terms of \( \hat{X}^0 \) as

\[ T^{(\text{tot})} = T + \nu\{Q, (bX^-)\}, \tag{8} \]

\[ T = T^{(ff)} + \nu X^- \tag{9} \]

Any of the three tensors \( T^{(\text{tot})}, T^{(ff)}, \) and \( T \) closes a Virasoro algebra with one and the same value of the central charge \( c_{\nu}. \) Furthermore

\[ \left( T^{(\text{tot})}(z) - T^{(ff)}(z) \right) \hat{X}^0(0) = \ldots \text{ (regular)}. \tag{10} \]

The tensor \( T \) appears also in the recent paper [13], where it is argued that the universal enveloping algebra of the Vir algebra of \( T \) appears as the (only nonzero) cohomology of the complex generated by the action of \( Q \) in (the completion of) \( U_g \otimes \text{Cliff}(b,c). \)

3. Now let us turn to representations. Let \( V_0 = V_{\{j,\nu\}} \) be an \( \hat{sl}(2)_k \) Verma module highest weight vector of weight \( r \) (spin \( j = r/2 \)), i.e.,

\[ 2X^0_0 V_0 = r V_0, \quad \text{and} \quad X^+ m V_0 = X^- n V_0 = 0 \quad \text{for} \quad m \geq 0, n > 0. \tag{11} \]

Denote \( V_t = (X^-)^t V_0. \) Obviously \( 2X^0_0 V_t = (r - 2t) V_t \) and \( X^+ V_t = \gamma_t(r) V_{t-1}, \gamma_t(r) = t(r + 1 - t). \) Thus if \( r \) is a non-negative integer \( V_{t+1} \) is a singular vector in the Verma module. For a while we restrict ourselves with this simplest subseries of \( \hat{sl}(2)_k \) singular vectors.

Assume that \( V_0 \) is also a vacuum vector for the \( b, c \) system, i.e.

\[ b_m V_0 = c_n V_0 = 0 \quad \text{for} \quad m > 0, n \geq 0. \tag{12} \]

Obviously \( Q V_t = c_{-1} X^+_0 V_t \) and therefore \( Q \) annihilates exactly the singular vectors \( V_{t+1} \) of the \( \hat{sl}(2) \) Verma module. Furthermore from the explicit expressions (7,8,9) for \( T \) and \( T^{(\text{tot})} \)

\[ L^\text{(tot)}_n V_t = L_n V_t = L^{(ff)}_n V_t = 0, \quad \text{for} \quad n \geq 1, \]

\[ L^\text{(tot)}_0 V_t = L_0 V_t = L^{(ff)}_0 V_t = h_{\{j-t,\nu\}} V_t, \quad h_{\{j,\nu\}} = \nu J(J + 1) - J = h_{\{-J-1+1/\nu,\nu\}}. \tag{13} \]

The basic relation (8) can be solved for \( X^- \) and furthermore recursively for any power of \( X^-_0. \) Namely first splitting

\[ \{Q, (bX^-)_n\} V_p = \left( \{Q, \sum_{m=0}^n b_m X^-_{m-n} \} + X^-_{-1} X^+_0 \right) V_p \]

we rewrite (8) applied to \( V_p \) as
\[ X_{-n} V_p = \left[ \frac{1}{\nu} \left( L_{-n-1}^{\text{tot}} - L_{-n-1}^{\text{ff}} \right) - \{Q, \sum_{m=0}^{n} b_{-m} X_{-m-n}^- \} \right] V_p - X_{-n-1} X_0^+ V_p. \] (14)

This is the analog of the triangular KZ equation exploited in [5]. Thus, analogously to what has been done there, one can eliminate recursively all \( X_{-n} \) with \( n > 0 \) obtaining

\[ X_0^- V_p = \sum_{n=0}^{\infty} \left[ \frac{1}{\nu} \left( L_{-n-1}^{\text{tot}} - L_{-n-1}^{\text{ff}} \right) - \{Q, \sum_{m=0}^{n} b_{-m} X_{-m-n}^- \} \right] (-X_0^+)^n V_p. \] (15)

For \( p = r \) the relation (15) expresses the \( \hat{sl}(2) \) singular vector \( V_{r+1} \) in terms of the Vir generators \( T^{\text{tot}} \). To get rid of the remaining dependence on Heisenberg algebra generators present in \( L^{\text{ff}} \) we shall exploit as in [5] a “gauge” transformation – a quantum version of the classical DS gauge transformation. To do that let us first introduce an auxiliary \( sl(2) \) algebra with generators \( t^{\pm,0} \) having the same commutation relations as the zero modes \( X_0^{\pm,0} \). Let \( U_0 \) be a highest weight vector of a finite dimensional, spin \( r/2 \), auxiliary \( sl(2) \) module, thus \( t^+ U_0 = 0 = (t^-)^{r+1} U_0 \).

Consider

\[ I = \sum_{n=0}^{r} (X_0^-)^n V_0 \otimes (t^-)^{r-n} U_0. \] (16)

Obviously (skipping for short direct product notation)

\[ (X_0^+ - t^+) I = 0, \quad (X_0^- - t^-) I = (X_0^-)^{r+1} V_0 \otimes U_0, \quad (X_0^0 + t^0) I = 0. \] (17)

Tensoring (15) by \( \otimes (t^-)^{-p} U_0 \), then summing over \( p = 0, \ldots, r \) and using (17) we obtain

\[ (X_0^-)^{r+1} V_0 \otimes U_0 = \left( -t^- + \frac{1}{\nu} \sum_{n=0}^{r} (-t^+)^n \left( L_{-n-1}^{\text{tot}} - L_{-n-1}^{\text{ff}} \right) \right) I - \{Q, B\} I, \] (18)

where for short

\[ B = \sum_{n=0}^{r} \sum_{m=0}^{n} b_{-m} X_{m-n}^- (-t^+)^n. \] (19)

4. Consider the operator

\[ \mathcal{R}(u) = \circ \exp \left( \int \hat{X}_0^0 (-u) \, du \right) \circ \equiv \sum_{n=0}^{\infty} \mathcal{R}_{-n} (u)^n, \] (20)

where \( \mathcal{R}_{-n} \) are determined recursively according to

\[ k \mathcal{R}_{-k} = \sum_{l=0}^{k-1} (-1)^{k+l+1} \mathcal{R}_{-l} \hat{X}_{-k+l}^0, \quad \mathcal{R}_0 = 1, \] (21)

and the parameter \( u \) will be identified with \( X_0^+ \) or \( t^+ \) in what follows. By \( X_0^0 \) in (20) is denoted the holomorphic part of the field \( X^0(u) = \sum_{n \in \mathbb{Z}} X^0_n u^{n-1} \), i.e. \( (X^0_{-1})_n = X_n^0 \) if \( n \leq -1 \) and zero otherwise. Then for \( u \) commuting with \( \hat{X}_0^0 \), e.g., \( u = t^+ \), (20) is a true exponent and the dots can be omitted.
From
\[ [\hat{X}_0^0, Q] = -[\hat{X}_0^0, c_0] = c_{-n} \]
it follows inductively that
\[ [Q, \mathcal{R}(u)] = -\mathcal{R}(u) uc_{-1}, \] (22)
and hence
\[ Q \mathcal{R}(X_0^+) V_p = 0, \] (23)
showing that \( \mathcal{R}(X_0^+) I = \mathcal{R}(t^+) I = \mathcal{R} I \) is in the kernel of \( Q \).

Next we prove the analog of the result in [5]:

**The Virasoro singular vectors of Benoit Saint-Aubin (BS-A) for \( T^{(\text{tot})} \) are equivalent, up to \( Q \) exact terms, to the \( \hat{sl}(2) \) singular vectors \( V_{r+1} \).**

Since the BS-A vector (see (29) below) is recovered by the BFIZ system it is enough to show that
\[ V_{r+1} \otimes U_0 = \left( -t^{-} + \frac{1}{\nu} \sum_{n=0}^{\nu} L_{-n-1}^{(\text{tot})} (-t^+) \right) \mathcal{R} I + Q \cdots . \] (24)

**Proof:** Apply on both sides of the equation (18) the gauge transformation \( \mathcal{R} = \mathcal{R}(t^+) \) from the left. The l.h.s. remains unchanged. According to (10) the gauge transformation commutes with the second term in (18). The result for the first term is collected in the following relation
\[ [\mathcal{R}(t^+), -t^{-}] I = \frac{1}{\nu} \sum_{n=0}^{\nu} (-t^+) L_{-n-1}^{(\text{tot})} \mathcal{R} I. \] (25)
Thus the corresponding term in (18) is cancelled leaving only the piece with the generators \( L_{-n-1}^{(\text{tot})} \).

The proof of (25) consists of several steps: 1) perform the commutator of the auxiliary \( \hat{sl}(2) \) generators, 2) use twice the defining recursion relation (21) to create a quadratic in \( \hat{X}^0 \) term, 3) trade the auxiliary generator \( t^0 \) for \( X_0^0 \) exploiting the last equation in (17), 4) group the various terms to recover the normal product in (7), taking also into account (6).

Finally one checks that \( \mathcal{R} \) maps the last term in (18) into the image of \( Q \). Indeed from the definitions it follows that \( Q I = c_{-1} X_0^+ I = c_{-1} t^+ I \), and taking into account (22), (19) we have
\[ \mathcal{R} \{ Q, B \} I = [\mathcal{R}, Q] B I + \mathcal{R} B Q I + Q \mathcal{R} B I \]
\[ = \mathcal{R} \{ c_{-1}, B \} t^+ I + Q \mathcal{R} B I = Q \mathcal{R} B I. \] (26)

Neglecting the terms in the image of \( Q \) turns (24) into the system of [1] with basis vectors which can be recovered from \( \mathcal{R} I \) using (16,17).

5. There is yet another way of recovering the BS-A vectors from the \( \hat{sl}(2) \) singular vector \( V_{r+1} \). Obviously (24) remains true if the generators \( L_{-n-1}^{(\text{tot})} \) are replaced by \( L \) from (9) since \( \mathcal{R} I \) is annihilated by \( Q \). Recall that \( T \) also commutes with \( Q \) and provides the same Vir algebra as \( T^{(\text{tot})} \). More than that, this alternative expression can be derived directly in terms of the simpler tensor \( T \) - avoiding the iteration of (8) described in (14-15, 18).

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1 In [5] the analog of (24) was written in terms of the improved tensor instead of \( T^{(\text{tot})} \). The price of avoiding the ghosts was that only “half” (the nonnegative modes) of the Vir algebra was recovered.
Indeed identify the parameter $u$ with $X_0^+$, instead of $t^+$, i.e., from now on $\mathcal{R} \equiv \mathcal{R}(u) = \mathcal{R}(X_0^+)$. Then one has for any vector $V$ annihilated by all positive grade generators as well as by the ghost zero mode $c_0$

$$\mathcal{R} X_0^- V = \frac{1}{\nu} \sum_{p=0}^1 L_{-p-1} \mathcal{R} (-X_0^+)^{p+1} V. \quad (27)$$

The proof of (27) is similar to that of (24). The $L^{(f)}$ - piece of $L$ (cf. (9)) is recovered by the commutator of $X_0^-$ with $(X_0^+)^k$. Its derivation repeats that of (25), replacing everywhere $t^{\pm}$ with $X_0^{\pm}$. The change of sign is compensated by the reshuffling of the generators. The negative modes of $X^-$ come from the commutator of $X_0^-$ with $\mathcal{R} - k$. One has to exploit the following relation proved by induction

$$[\mathcal{R} - k, X_{-n}] = \sum_{l=0}^{k-1} (-1)^{l+k} X_{-k-n+l} \mathcal{R} - l. \quad (28)$$

The relation (27) (and the related to it expression written in terms of the auxiliary $sl(2)$ algebra) can be interpreted as the quantum version of the DS gauge fixing transformation of the constrained system, leading to the classical (gauge invariant) counterpart of the current $T(z)$.

Now choose $V = V_t$. Then the power of $X_0^+$ in the r.h.s. of (27) produces $X_0^- V_{t-p-1}$ times the constant $(-1)^p \gamma_t(r) \ldots \gamma_{t-p+1}(r)$ and we can move again $\mathcal{R}$ to the right using (27) - repeating this until $V_0 = \mathcal{R} V_0$ is reached. This reproduces the vectors $F_{r+1} = \mathcal{R} V_{r+1}$ of [1] expressed as functions of the generators $L$, and for $t = r$ we immediately “generate” in this way directly the BS-A singular vector, i.e.,

$$V_{r+1} = \mathcal{R} (X_0^-)^{r+1} V_0 = \mathcal{O}_{(r+1;\nu)} V_0$$

$$= \prod_{i=1}^r \gamma_i(r) \sum_{s=1}^{r+1} (-\nu)^s \sum_{k_i \geq 0; \sum_i k_i = r+1-s} \frac{L_{-1-k_1} \ldots L_{-1-k_1}}{\gamma_{k_1+\ldots+k_{r+1-s-1}(r)} \ldots \gamma_{k_1+k_2+2(r)} \gamma_{k_1+1}(r)} V_0. \quad (29)$$

e.g., for $t = 1$

$$\mathcal{R} V_2 = \frac{1}{\nu} L_{-2} \mathcal{R} V_1 - \frac{1}{\nu} L_{-2} X_0^+ V_1 = \left(\frac{1}{\nu}\right)^2 \left(\left(L_{-2}\right)^2 - \nu \gamma_1(r) L_{-2}\right) = F_2,$$

which recovers the Vir singular vector if $r = 1$. More generally this converts any element in the kernel of $Q$ of the type $\mathcal{R} V_t$, $t = 1,2,\ldots,r, r+1,\ldots$ – into an element of the Vir module generated on $V_0$.

Since $F_0 = \mathcal{R} V_0 = V_0$, $F_{r+1} = \mathcal{R} V_{r+1} = V_{r+1}$, the proof that (29) is a Vir singular vector is an immediate consequence of (13), i.e., we have

$$L_n \mathcal{O}_{(r+1;\nu)} V_0 = 0 = L_n F_0, \text{ for } n \geq 1, \quad L_0 F_0 = h_{(j;\nu)} F_0, \quad L_0 \mathcal{O}_{(r+1;\nu)} V_0 = h_{(-j-1;\nu)} \mathcal{O}_{(r+1;\nu)} V_0. \quad (30)$$

Given (27) and the auxiliary $sl(2)$ algebra (16,17), one can get back the BFIZ system written as in (24), but with $L^{(tot)}$ replaced by $L$ and the $Q$ - exact term in the r.h.s. of (24) dropped.

Obviously the second method of deriving the singular vectors of the Virasoro algebra is technically simpler since it does not require the knowledge of the more complicated $T^{(tot)}$ (nor its recasted form (8)). This is important for generalisations to other $W_N$ algebras – especially in view of the recently proposed general algorithm of [13] for constructing the analogs of $T$. This will be demonstrated for the $W_3$ - algebras in [10].
6. Up to now we have considered only a subclass of the \( \hat{sl}(2) \) singular vectors. Let us now give an idea how one can recover the general Vir singular vectors in the form proposed by Kent [9].

The explicit expression for \( O_{(r+1;\nu)} \) can be reordered [9], pushing all generators \( L_{-1} \) to the left, as a sum of terms of decreasing powers of \( L_{-1} \), i.e., terms of the type

\[
\frac{1}{(\nu)^{r+1}} c_{i_1, i_2, \ldots, i_s}(\nu, r) L_{-1}^{r+1-l} L_{-i_1} \cdots L_{-i_s}, \quad \sum_{t=1}^{s} i_t = l, \ i_t \geq 2, \ l = 0, 2, 3, \ldots, r+1.
\]

The coefficients \( c_{i_1, i_2, \ldots, i_s}(\nu, r) \) depend polynomially on the parameters \( \{\nu, r\} \), and so they admit an analytical continuation to arbitrary values of \( r \), when the series representing \( O_{(r+1;\nu)} \) becomes infinite. Then \( O_{(r+1;\nu)} V_0 \) is formally a singular vector (i.e., it satisfies the conditions (30)), or more precisely it is a singular vector in the (generalised) Verma module of the algebra obtained by extending Vir with the powers \( (L_{-1})^n \) with arbitrary value of \( a \). It was shown in [9] that proper compositions of such operators provide true singular vectors in the Vir Verma modules.

The construction in [9] was inspired by the analogous expressions for the general singular vectors in the Verma modules of \( A_1^{(1)} \) in [6]. Let \( V_0 = V_{(J;\nu)} \), for \( k \neq 2 \), be a \( \hat{sl}(2)_k \) Verma module h.w. state of spin \( J \) and assume that it furthermore obeys the conditions (12). Suppose that the weight \( 2J+1 \) can be written (not uniquely in general) as \( 2J+1 = m - (m' - 1)/\nu \), with \( m \) and \( m' \) being positive integers. Denote for short \( f_1 = X_0^- \) and \( f_0 = X_+^1 \). The vector

\[
\mathcal{P}_{(m,m';\nu)} V_{(J;\nu)} = f_1^{m+(m'-1)/\nu} f_0^{m+(m'-2)/\nu} \cdots f_1^{m-(m'-3)/\nu} f_0^{m-(m'-2)/\nu} f_1^{m-(m'-1)/\nu} V_{(J;\nu)},
\]

is a singular vector in the module generated on \( V_{(J;\nu)} \) [6]. It can be cast in a canonical way into an integral powers polynomial of \( X_0^- \) and \( X_a \cdot \). Furthermore all the vectors obtained by dropping from the left the first one, or the first two, etc., of the factors in the r.h.s. of (31), i.e.,

\[
\mathcal{P}_{(m,m';\nu)} V_{(J;\nu)} = V_{(J-m;\nu)} = f_1^{m+(m'-1)/\nu} V_{(-J+1+m;\nu)}
= f_1^{m+(m'-1)/\nu} f_0^{m+(m'-2)/\nu} V_{(J-m+1;\nu)} = \ldots,
\]

etc., formally have the properties of singular vectors. Hence they are annihilated by \( Q \), by the positive modes of \( T(z) \), and furthermore they are kept invariant by the gauge transformation \( R \).

We now apply \( R \) from the left on both sides of (31) using (29) and identifying subsequently \( V_0 \) with the subfactors. After the first step we have (simplifying for short the notation \( \mathcal{O}_m \equiv \mathcal{O}_{(m;\nu)} \))

\[
\mathcal{P}_{(m,m';\nu)} V_{(J;\nu)} = R \mathcal{P}_{(m,m';\nu)} V_{(J;\nu)} = \mathcal{O}_{m+(m'-1)/\nu} f_1^{m+(m'-2)/\nu} V_{(J-m+1;\nu)}.
\]

Next \( f_0 = X_+^1 \) can be rewritten as \( f_0 = 1 + \{b_0, Q\} \). Using that \( Q^2 = 0 \), any power of \( f_0 \) can be written as \( (f_0)^n = 1 + QA + AQ \) for some \( A \). Representing the power of \( f_0 \) in (32) in this way, the \( Q \) on the right will act on \( V_{(J-m+1;\nu)} \) giving zero, while the \( Q \) on the left passes through \( O_{m+(m'-1)/\nu} \) producing \( Q \)-exact terms. Thus we get for (32)

\[
= \mathcal{O}_{m+(m'-1)/\nu} f_1^{m+(m'-3)/\nu} f_0^{m+(m'-4)/\nu} \cdots f_1^{m-(m'-1)/\nu} V_{(J;\nu)} + Q \ldots
= \mathcal{O}_{m+(m'-1)/\nu} R f_1^{m+(m'-3)/\nu} V_{(-J-1+m-1;\nu)} + Q \ldots
\]

We repeat this step until \( V_{(J;\nu)} \) is reached, turning each power of \( f_1 \) in the MFF expression into the \( L \) - depending factors \( \mathcal{O} \), while getting rid of the powers of \( f_0 \) at the price of \( Q \) exact terms, i.e.,
\[ P_{(m,m';\nu)}V_{\{J;\nu\}} + Q \ldots = \mathcal{O}_{m+(m'-1)/\nu} \mathcal{O}_{m+(m'-3)/\nu} \ldots \mathcal{O}_{m-(m'-1)/\nu} V_{\{J;\nu\}} \equiv \mathcal{O}_{(m,m';\nu)} |h_{\{J;\nu\}}\rangle. \] (33)

We have identified the vector \( V_{\{J;\nu\}} \) with a Vir Verma module highest weight state \( |h_{\{J;\nu\}}\rangle \). The composition \( \mathcal{O}_{(m,m';\nu)} |h_{\{J;\nu\}}\rangle \) in (33) as well as any of its subfactors are annihilated by positive grade Vir generators \( L_n \), \( n \geq 1 \), while \( L_0 \) reproduces the correct eigenvalues.

Thus we have proved the statement

**The general Vir singular vectors** \( \mathcal{O}_{(m,m';\nu)} |h_{\{J;\nu\}}\rangle \) **written in the form of** [9], **are BRST equivalent to the MFF \( \hat{sl}(2) \) vectors** \( P_{(m,m';\nu)}V_{\{J;\nu\}} \).

Note that the \( \hat{sl}(2) \) singular vectors of the type \( f_0^m V_{\{J;\nu\}} \), \( 2J + 1 = -m + 1/\nu \), are BRST trivial, i.e., equivalent to \( \hat{V}_{\{J;\nu\}} \), in agreement with the structure of the corresponding Vir modules.

The Vir singular vector \( \mathcal{O}_{(m,m';\nu)} |h_{\{J;\nu\}}\rangle \) can be also recovered from a singular vector of another \( \hat{sl}(2) \) module labelled by the weight \( 2J' + 1 = 2/\nu - 2J - 1 = -m + (m' + 1)/\nu \). It is given [6] by an expression like (31) but everywhere \( f_1 \) is replaced by \( f_0 \) and vice versa, while the powers are kept the same as in (31) with only \( m' \) changed to \( m' + 1 \) and the h.w. state \( \hat{V}_{\{J;\nu\}} \) replaced by \( \hat{V}_{\{1/\nu - J - 1;\nu\}} \). On the other hand the invariance of \( \{ h_{\{J;\nu\}}, c_\nu \} \) under the change \( \{ J, \nu \} \rightarrow \{ -J, 1/\nu \} \) of parameters implies that one and the same Vir singular vector admits different representations inherited from its \( \hat{sl}(2) \) counterparts. Finally when the \( \hat{sl}(2) \) vectors are true compositions (of factors with positive integral powers, see [6]) they lead to Vir vectors expressible as a product of true BS-A factors.

We conclude with the remark that the approach developed here can be further exploited to derive more explicit formulae for the general Virasoro singular vectors, starting directly from the integral powers versions of the MFF vectors.

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