BALANCED EMBEDDING OF DEGENERATING ABELIAN VARIETIES

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Abstract. For a certain maximal unipotent family of Abelian varieties over the punctured disc, we show that after a base change, one can complete the family over a disc such that the whole degeneration can be simultaneously balanced embedded into a projective space by the theta functions. Then we study the relationship between the balanced filling-in and the Gromov-Hausdorff limit of flat Kähler metrics on the nearby fibers.

1. Introduction

Let $X \subset \mathbb{C}P^\nu$ be a $n$-dimensional projective variety. The embedding $X \hookrightarrow \mathbb{C}P^\nu$ is called balanced if

$$\int_X \left( \frac{z_i \bar{z}_j}{|z|^2} - \frac{\delta_{ij}}{\nu+1} \right) \omega_{FS}^{\nu} = 0,$$

where $z_0, \cdots, z_{\nu}$ are the homogenous coordinates of $\mathbb{C}P^\nu$, $|z|^2 = \sum |z_i|^2$, and $\omega_{FS}$ is the Fubini-Study metric. In [30], S. Zhang proved that $X$, as an algebraic cycle in $\mathbb{C}P^\nu$, is Chow polystable if and only if the embedding $X \subset \mathbb{C}P^\nu$ can be translated to a balanced one via an element $u \in \text{SL}(\nu+1)$.

A theorem due to Donaldson [6] shows the connection of the balanced embedding and the existence of Kähler metric with constant scalar curvature. More precisely, let $(X, L)$ be a polarized manifold of dimension $n$ such that the automorphism group $\text{Aut}(X, L)$ is finite. If there is a Kähler metric $\omega$ with constant scalar curvature representing $c_1(L)$, then Donaldson’s theorem asserts that for $k \gg 1$, $L^k$ induces a balanced embedding $\Phi_k : X \hookrightarrow \mathbb{C}P^\nu_k$ with $\Phi_k^* \mathcal{O}_{\mathbb{C}P^\nu_k}(1) = L^k$. Furthermore,

$$\|\omega - k^{-1} \Phi_k^* \omega_{FS}\|_{C^r(X)} \to 0,$$

when $k \to \infty$, in the $C^r$-sense for any $r > 0$. The Kähler metric $k^{-1} \Phi_k^* \omega_{FS}$ is called a balanced metric.

If $(X, L)$ is a polarized Calabi-Yau manifold, Yau’s theorem on the Calabi conjecture says that there exists a unique Ricci-flat Kähler-Einstein metric $\omega$ with $\omega \in c_1(L)$, i.e. the Ricci curvature $\text{Ric}(\omega) \equiv 0$ (cf. [29]). By Donaldson’s theorem, $L^k$ induces a balanced embedding $X \hookrightarrow \mathbb{C}P^\nu_k$ for
\( k \gg 1 \), and the Ricci-flat Kähler-Einstein metric \( \omega \) can be approximated by the balanced metrics. If \((X, L)\) is a principally polarized Abelian variety, it is proven first in \([22]\) (and also independently \([28]\)) that the standard embedding induced by the classical theta functions of level \( k \) is balanced. Moreover, in this case the convergence of balanced metrics to the flat metric can be verified via a complete elementary way without quoting \([6]\).

In \([26]\), Strominger, Yau and Zaslow propose a geometric way of constructing mirror Calabi-Yau manifolds via dual special lagrangian fibration, which is the celebrated SYZ conjecture. Later, a new version of the SYZ conjecture is proposed by Gross, Wilson, Kontsevich, and Soibelman, (cf. \([13, 17, 18]\)) by using the collapsing of Ricci-flat Kähler-Einstein metrics. Let \((\mathcal{X} \to \Delta, \mathcal{L})\) be a maximal unipotent degeneration of polarized Calabi-Yau \( n \)-manifolds, i.e. the relative canonical bundle \( K_{\mathcal{X}/\Delta} \) is trivial, such that \( 0 \in \Delta \) is a large complex limit point, and \( \omega_t \) be the Ricci-flat Kähler-Einstein metric satisfying \( \omega_t \in c_1(\mathcal{L}|_{\mathcal{X}_t}) \) for \( t \in \Delta^o \). The collapsing version of SYZ conjecture asserts that

\[
(X_t, \text{diam}_{\omega_t^2}(X_t)\omega_t) \to (B, d_B)
\]

in the Gromov-Hausdorff sense, when \( t \to 0 \), where \((B, d_B)\) is a compact metric space. Furthermore, there is an open dense subset \( B_0 \) of \( B \), which is smooth, and is of real dimension \( n \), and admits a real affine structure. The metric \( d_B \) is induced by a Monge-Ampère metric \( g_B \) on \( B_0 \), i.e. under affine coordinates \( y_1, \cdots, y_n \), there is a potential function \( \varphi \) such that

\[
g_B = \sum_{ij} \frac{\partial^2 \varphi}{\partial y_i \partial y_j} dy_idy_j, \quad \text{and} \quad \det \left( \frac{\partial^2 \varphi}{\partial y_i \partial y_j} \right) = \text{const}.
\]

Clearly it is true for Abelian varieties. This conjecture is verified by Gross and Wilson for fibred K3 surfaces with only type \( I_1 \) singular fibers in \([13]\), and is studied for higher dimensional HyperKähler manifolds in \([14, 15]\).

Bernd Siebert raises a question to relate the balanced embeddings of \( X_t \) to the metric limit of rescaled Ricci-flat Kähler-Einstein metrics \( \varepsilon_t \omega_t \) for a certain family of constants \( \varepsilon_t \). In the Gross-Siebert program (cf. \([8, 11]\)), theta functions are constructed on certain degenerations of polarized Calabi-Yau manifolds \((\mathcal{X} \to \text{Spec}\mathbb{C}[[t]], \mathcal{L})\) as the canonical basis of the space of sections for \( \mathcal{L} \) (cf. \([9, 12, 10]\)), which is predicted by the Homological Mirror Symmetry conjecture. In particular, these theta functions recover the classical theta functions in the case of principally polarized Abelian varieties. If \((\mathcal{X}, \mathcal{L})\) is an analytic family (cf. \([24]\)), Siebert asks whether the theta functions give the balanced embeddings of polarized Calabi-Yau manifolds, and furthermore whether there is a family version of the Donaldson’s theorem for the degeneration of Calabi-Yau manifolds near large complex limits. In this note, we study this question in the case of principally polarized Abelian varieties, and establish the connection between the limit metric \( g_B \) and the balanced embeddings.
In Section 2, let us recall the basic setup for a maximal unipotent family
of principally polarized Abelian varieties over the punctured disc, and then
state our main result. Theorem 2.1 says that after a certain base change,
one can find a filling-in to complete the family of Abelian varieties to a
degeneration, such that the whole degeneration can be simultaneously bal-
anced embedded in a projective space over a disc by the canonical theta
functions constructed via Gross-Siebert program. Theorem 2.2 studies the
relationship between the balanced filling-in and the Gromov-Hausdorff limit
of flat Kähler metrics on the nearby fibers. In Section 3, we review the
construction of theta functions on degenerations of Abelian varieties in the
Gross-Siebert program. Finally, Theorem 2.1 and Theorem 2.2 are proved
in Section 4.

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2. Set up and Main Theorems

In this paper, we always denote
\[ M \simeq \mathbb{Z}^n, \quad M_\mathbb{R} = M \otimes \mathbb{Z} \mathbb{R}, \quad N = \text{Hom}_\mathbb{Z}(M, \mathbb{Z}), \quad N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R}, \quad T_\mathbb{C} = N \otimes \mathbb{Z} \mathbb{C}^*, \]
\[ \langle \cdot, \cdot \rangle \] the pairing between
\[ N_\mathbb{R} \quad \text{and} \quad M_\mathbb{R}. \]

2.1. A family of Abelian varieties. This subsection gives the basic setup
of this paper, which is a family of Abelian varieties over the punctured disc
approaching to a large complex limit.

Let \( Z(\cdot, \cdot) : M_\mathbb{R} \times M_\mathbb{R} \to \mathbb{R} \) be a positive definite bilinear form satisfying
\( Z(M, M) \subset \mathbb{Z}. \) If we define the quadratic function
\begin{equation}
\varphi(y) = \frac{1}{2} Z(y, y)
\end{equation}
on \( M_\mathbb{R}, \) and the affine linear function
\begin{equation}
\alpha_\gamma(\cdot) = Z(\gamma, \cdot) + \frac{1}{2} Z(\gamma, \gamma),
\end{equation}
for any \( \gamma \in M, \) then
\begin{equation}
\varphi(y + \gamma) = \varphi(y) + \alpha_\gamma(y).
\end{equation}
The couple \( \{M, Z\} \) determines a family of principally polarized Abelian
varieties over the punctured disc \( (\pi : \mathcal{X}_n \to \Delta^\circ, L_n) \) as the following.

We define an \( M \)-action on \( M \times \mathbb{Z} \) via \((m, r) \mapsto (m, r + Z(\gamma, m))\) for any
\( \gamma \in M, \) which induces an \( M \)-action on \( T_\mathbb{C} \) by
\begin{equation}
Z^m \mapsto Z^m s^{Z(\gamma, m)}, \quad \gamma \in M,
\end{equation}
for any \( s \in \Delta^\circ. \) More explicitly, let
\[ e_1, \ldots, e_n \in M \]
be a basis of \( M, \) we have coordinates \( z_1 = Z^{e_1}, \ldots, z_n = Z^{e_n} \) on \( T_\mathbb{C} \cong (\mathbb{C}^\times)^n, \)
and the \( M \)-action is that \((z_1, \ldots, z_n) \mapsto (z_1 s^{Z(\gamma, e_1)}, \ldots, z_n s^{Z(\gamma, e_n)})\) for any
We claim that the quotient $X_s = T_C/M$ is a principally polarized Abelian variety with period matrix
\[
\begin{bmatrix}
I, & \frac{\log s}{2\pi\sqrt{-1}}Z_{ij}
\end{bmatrix}
\]
where $Z_{ij} = Z(e_i, e_j) \in \mathbb{Z}$.

Denote $e_1^*, \ldots, e_n^*$ the dual basis of $N$, and $N_C = N \times \mathbb{C} \cong \mathbb{C}^n$. We have a natural embedding $N \hookrightarrow N_C$ as the real part, and by abusing notations, we regard $e_1^*, \ldots, e_n^*$ as a $\mathbb{C}$-basis of $N_C$. The universal covering $N_C \rightarrow T_C$ is given by $w_i \mapsto z_i = \exp 2\pi\sqrt{-1}w_i$, $i = 1, \ldots, n$, where $w_1, \ldots, w_n$ are coordinates of $N_C$ respecting to $e_1^*, \ldots, e_n^*$. Then $T_C \cong N_C/N$ where $N$ acts on $N_C$ given by $w_i \mapsto w_i + \langle \mu, e_i \rangle = w_i + \mu_i$ for any $\mu = \sum \mu_i e_i^* \in N$. We have an $N \times M$-action on $\mathbb{C}^n$ by
\[
w_i \mapsto w_i + \langle \mu, e_i \rangle + \frac{\log s}{2\pi\sqrt{-1}}Z(\gamma, e_i),
\]
for any $(\mu, \gamma) \in N \times M$, and we obtain $X_s = T_C/M = \mathbb{C}^n/\Lambda_s$, where the lattice $\Lambda_s = \text{span}_\mathbb{Z}\{e_1^*, \ldots, e_n^*, \frac{\log s}{2\pi\sqrt{-1}}Z(e_1, \cdot), \ldots, \frac{\log s}{2\pi\sqrt{-1}}Z(e_n, \cdot)\}$. Furthermore, we construct a family of Abelian varieties $\mathcal{X}_\eta = (T_C \times \Delta^\circ)/M \rightarrow \Delta^\circ$ over the punctured disc $\Delta^\circ$ with fiber $X_s$.

We extend the $N \times M$-action on $\mathbb{C}^n$ to $\mathbb{C}^n \times \mathbb{C}$ by
\[
(w, \lambda) \mapsto (w + \mu + \frac{\log s}{2\pi\sqrt{-1}}Z(\gamma, \cdot), \exp \pi\sqrt{-1}(\frac{\log s}{2\pi\sqrt{-1}}Z(\gamma, \gamma) + 2\langle w, \gamma \rangle)),
\]
for any $(\mu, \gamma) \in N \times M$, where $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$. The quotient $(\mathbb{C}^n \times \mathbb{C})/(N \times M)$ is the relative ample bundle $\mathcal{L}_\eta$. The classical Riemann theta function (cf. [3])
\[
\vartheta = \sum_{\gamma \in M} \exp \pi\sqrt{-1}\left(\frac{\log s}{2\pi\sqrt{-1}}Z(\gamma, \gamma) + 2\langle w, \gamma \rangle\right)
\]
is the distinguished section of $\mathcal{L}_\eta$. On any $X_s$, the first Chern class $c_1(\mathcal{L}_\eta|X_s)$ is represented by the flat Kähler metric
\[
\omega_s = -\frac{\pi\sqrt{-1}}{\log |s|} \sum_{ij} Z^{ij} dw_i \wedge d\bar{w}_j,
\]
where $Z^{ij}$ denotes the inverse of the matrix $Z_{ij} = Z(e_i, e_j)$.

Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ denote the coordinates on $N_C$ with respect to the basis $e_i^*$, $1 \leq i \leq n$, $(\frac{\log |s|}{2\pi\sqrt{-1}} + \text{arg}(s))Z(e_j, \cdot)$, $1 \leq j \leq n$, where $0 \leq \text{arg}(s) < 2\pi$. We also regard $y_1, \ldots, y_n$ (resp. $x_1, \ldots, x_n$) as coordinates on $M_\mathbb{R}$ (resp. $N_\mathbb{R}$) respecting to $e_1, \ldots, e_n$. Then
\[
w_i = x_i + \langle \text{arg}(s) - \frac{\log |s|}{2\pi}, Z_{ij}y_j \rangle, \quad i = 1, \ldots, n,
\]
and as a symplectic form,
\[ \omega_s = \sum_{ij} dx_i \wedge dy_j. \]

The corresponding Riemannian metric is
\[ g_s = \frac{-2\pi}{\log |s|} \sum_{i,j} Z^{ij} dx_i dx_j - \frac{\log |s|}{2\pi} \sum_{i,j} Z_{ij} dy_i dy_j, \]
and, when \( s \to 0 \),
\[ (X_s, g_s) \to (B, g_B = \sum_{i,j} Z_{ij} dy_i dy_j) \]
in the Gromov-Hausdorff topology, where \( B = M_\mathbb{R}/M \).

We can regard \( \overline{\varphi} \) as a multivalued function on \( B = M_\mathbb{R}/M \) by (2.3), and on any small open subset on \( B \), the difference of any two sheets of \( \overline{\varphi} \) is a linear function defined in (2.2). Note that the Hessian matrix of \( \overline{\varphi} \) is well-defined, and \( \frac{\partial^2 \overline{\varphi}}{\partial y_i \partial y_j} = Z_{ij} \). The Riemannian metric \( g_B \) is the Monge-Ampère metric with potential \( \overline{\varphi} \) respecting to the affine coordinates \( y_1, \ldots, y_n \), i.e.
\[ g_B = \sum_{ij} \frac{\partial^2 \overline{\varphi}}{\partial y_i \partial y_j} dy_i dy_j, \quad \text{and} \quad \det \left( \frac{\partial^2 \overline{\varphi}}{\partial y_i \partial y_j} \right) \equiv \det(\omega_{ij}). \]

2.2. Main results. For any \( k \gg 1 \), let \( M_k = kM \), and \( \varphi \) be an \( M_k \)-periodic convex piecewise linear function such that the slopes of \( \varphi \) are in \( N \), and
\[ \varphi(y + \gamma) = \varphi(y) + \alpha_\gamma(y) \]
for any \( \gamma \in kM \), which induces an \( M_k \)-invariant rational polyhedral decomposition \( \mathcal{P} \) of \( M_\mathbb{R} \), i.e. \( \sigma \) is a cell of \( \mathcal{P} \) if and only if \( \varphi \) is linear on \( \sigma \).

The Mumford’s construction (cf. [20]) gives a degeneration of principally polarized Abelian varieties \( (\pi : \mathcal{X} \to \Delta, \mathcal{L}) \) from the data \((M_k, \mathcal{P}, \varphi)\) such that \( \pi : \mathcal{X}_{\Delta^o} \to \Delta^o \) is the base change of \( \mathcal{X}_\eta \) via \( t \mapsto t^k = s \), where \( \mathcal{X}_{\Delta^o} = \pi^{-1}(\Delta^o) \), and \( \mathcal{L}|_{\mathcal{X}_{\Delta^o}} \) is the pull-back of \( \mathcal{L}_k^1 \). The central fiber \( X_0 = \pi^{-1}(0) \) is reduced and reducible with only toric singularities. The intersection complex of \( X_0 \) is \((B_k, \overline{\mathcal{P}})\) where \( B_k = M_\mathbb{R}/M_k \) and \( \overline{\mathcal{P}} \) is the quotient rational polyhedron decomposition of \( \mathcal{P} \). The irreducible components are one to one corresponding to \( n \)-cells in \( \overline{\mathcal{P}} \), and for any \( n \)-cells \( \sigma \in \overline{\mathcal{P}} \), the respective irreducible component is the toric variety \( X_\sigma \) defined by \( \sigma \), where we regard \( \sigma \) as a polytope in \( M_\mathbb{R} \). Furthermore, the restriction of \( \mathcal{L} \) on \( X_\sigma \) is the toric ample line bundle defined by \( \sigma \).

For any \( m \in B_k(\mathbb{Z}) = M/M_k \), a section \( \vartheta_m \) of \( \mathcal{L} \) is constructed in Section 6 of [9] such that the restriction of \( t^\varphi(m) - \varphi(m) \vartheta_m \) on any \( X_t = \pi^{-1}(t), t \neq 0 \), is a classical Riemann theta function (See also [12]). And the restriction of \( \vartheta_m \) on any component \( X_\sigma \) of \( X_0 \) is a monomial section of \( \mathcal{L}|_{\mathcal{X}_\sigma} \).

Note that the choice of \((\mathcal{P}, \varphi)\) is not unique, and different choices give different filling-ins \( X_0 \). However there is a canonical one studied in [2], which
satisfies $\varphi(m) = \overline{\varphi}(m)$ for any $m \in M$. More precisely, let $\varphi : M \rightarrow \mathbb{R}$ be the convex piecewise linear function such that the graph of $\varphi$ is the lower bound of the convex hull of $\{(m, \varphi(m)) | m \in M\} \subset M \times \mathbb{R}$, and let $\mathcal{P}$ be the rational polyhedral decomposition of $M$ induced by $\varphi$, i.e. a cell $\sigma \in \mathcal{P}$ if and only if $\varphi$ is linear on $\sigma$. It is clear that (2.7) is satisfied, and

$$\varphi(m) = \overline{\varphi}(m), \quad \text{for any } m \in M.$$

Let $\overline{\mathcal{P}}$ be the boundary of the convex hull of the lattice points on the graph of $\varphi$, then the polyhedral decomposition $\mathcal{P}$ is obtained in such a way that each cell $\sigma$ of $\mathcal{P}$ is precisely the projection of a face of $\overline{\mathcal{P}}$ onto $M$. The decomposition $\mathcal{P}$ is the mostly divided polyhedral decomposition that one can have. Any cell $\sigma$ of $\mathcal{P}$ intersects with the lattice $M$ only at its vertices, i.e. there is no integral point in the interior of $\sigma$.

We further assume that for any $\sigma \in \mathcal{P}$, the slope of $\varphi|_{\sigma}$ is integral, i.e.

$$d\varphi|_{\sigma} \in N$$

for simplicity. It is not a further restriction, and the reason is as the following. Note that $d\varphi|_{\sigma} \in N \otimes_{\mathbb{Z}} \mathbb{Q}$. For any $m \in M$, there is a $\nu \in \mathbb{N}$ such that $\nu d\varphi|_{\sigma} \in N$ for any cell $\sigma \in \mathcal{P}$ with $m \in \sigma$, i.e. $\sigma$ belongs to the star of $m$. By the $M$-action and (2.7), $\nu d\varphi|_{\sigma} + \nu d\alpha_{\gamma} \in N$ for any $\gamma \in M$, and thus $\nu \varphi$ satisfies (2.9). If we replace $\mathbb{Z}$ by $\nu \mathbb{Z}$, then the new family of Abelian varieties constructed from $\nu \varphi$ is the base change of the original family by $s \mapsto s^\nu$. Hence we assume (2.9).

Before we present the main theorems, we look at some lower dimensional cases. If $\dim_{\mathbb{R}} M = 1$, $B_k$ is a cycle, 1-cells in $\mathcal{P}$ are intervals, and $X_0$ is the Kodaira type $I_k$ fiber, i.e. a cycle of $k$ rational curves. It is well known to experts that $X_0$ can be balanced embedded. When $\dim_{\mathbb{R}} M = 2$, there are only two possible choices of $\mathcal{P}$ by [2]. One is that any 2-cell in $\mathcal{P}$ is a standard simplex, and the other one is that $\mathcal{P}$ consists of cubes. Hence $X_0$ consists either finite many $\mathbb{CP}^2$ or finite many $\mathbb{CP}^1 \times \mathbb{CP}^1$.

The first result of this paper shows that the embedding of the canonical degeneration $X$ of [2] by theta functions constructed in [9] is balanced.

**Theorem 2.1.** For any $k \gg 1$, $B_k = M_k/M_k$, $B_k(\mathbb{Z}) = M/M_k$, and let $(\pi : X \rightarrow \Delta, \mathcal{L})$ be the degeneration of principally polarized Abelian varieties from the triple $(M_k, \mathcal{P}, \varphi)$. The theta functions \( \{\vartheta_m | m \in B_k(\mathbb{Z})\} \) define a relative balanced embedding

$$\Phi_k = [\vartheta_{m_1}, \ldots, \vartheta_{m_k}] : X \rightarrow \mathbb{CP}^{k-1} \times \Delta,$$

i.e. for any $t \in \Delta$, $\Phi_k|_{X_t} : X_t \hookrightarrow \mathbb{CP}^{k-1}$ is a balanced embedding.
Again it was proven in [28] and [22] that any individual Abelian variety $X_t$, $t \neq 0$, can be balanced embedded in certain $\mathbb{CP}^N$ via theta functions. If one set $t \to 0$, the limit variety $X_0$ in the projective space is also balanced. However Theorem 2.1 uses a group of different theta functions that guarantee the balanced embedding varying holomorphically when $t$ approaches to 0. Furthermore, Theorem 2.1 identifies the balanced limit $X_0$ to be the canonical filling-in of $(X_\Delta^\alpha, L_{|X_\Delta^\alpha})$ constructed in [2].

Now we study the connection between the metric limit $(B, g_B)$ and the balanced filling-in when $k \to \infty$. For a fixed $k \gg 1$, $B_k$ has a natural affine structure induced by $M_{\mathbb{R}}$. If $f$ is a convex function on an open subset $U$ of $B_k$ with respect to the affine structure, where we also regard $U$ as a subset of $M_{\mathbb{R}}$ by the quotient, then for any $y_0 \in U$, we let

$$\partial f(y_0) = \{v \in N_{\mathbb{R}}| f(y) \geq \langle v, y - y_0 \rangle + f(y_0) \text{ for all } y \in M_{\mathbb{R}}\},$$

and we define the Monge-Ampère measure

$$\text{MA}(f)(E) = \text{Vol} \left( \bigcup_{y \in E} \partial f(y) \right),$$

for any Borel subset $E \subset U$, where $\text{Vol}$ denotes the standard Euclidean measure on $N_{\mathbb{R}}$ (cf. [3] and [23]). It is well known that $\text{MA}(f + \alpha) = \text{MA}(f)$ for any linear function $\alpha$ on $M_{\mathbb{R}}$, and if $f$ is smooth,

$$\text{MA}(f) = \det \left( \frac{\partial^2 f}{\partial y_i \partial y_j} \right) dy_1 \wedge \cdots \wedge dy_n.$$

Note that we can regard the function $\varphi$ as a multivalue function on $B_k$, and by (2.7), the difference between any two sheets of $\varphi$ is a linear function. Thus we have a well-defined Monge-Ampère measure $\text{MA}(\varphi)$. The next theorem shows that after some rescaling, this Monge-Ampère measure converges to the Monge-Ampère measure of the potential function $\bar{\varphi}$ of $g_B$ when $k \to \infty$, and furthermore, the rescaled potential function $\varphi$ also converges to $\bar{\varphi}$ in the $C^0$-sense, which shows the link between the balanced embeddings and the Gromov-Hausdorff limit.

**Theorem 2.2.** For any $k \gg 1$, the Monge-Ampère measure of $\varphi$ is

$$\text{MA}(\varphi) = \det(Z_{ij}) \sum_{m \in B_k(\mathbb{Z})} \delta_m,$$

where $\delta_m$ is the Dirac measure at $m \in B_k(\mathbb{Z})$. If $\chi_k : B \to B_k$ is induced by the dilation $y_i \mapsto ky_i$, $i = 1, \cdots, n$, of $M_{\mathbb{R}}$, then $\frac{1}{k^2} \chi_k^* \varphi - \bar{\varphi}$ is a well-defined function on $B$, and

$$\sup_B \left| \frac{1}{k^2} \chi_k^* \varphi - \bar{\varphi} \right| \to 0,$$

when $k \to \infty$. Furthermore,

$$\frac{1}{k^n} \chi_k^* \text{MA}(\varphi) \to \text{MA}(\bar{\varphi}) = \det(Z_{ij}) dy_1 \wedge \cdots \wedge dy_n$$

in the weak sense.
In [19], the non-archimedean Monge-Ampère equation is solved for degenerations of Abelian varieties, where the approximation of continuous potential functions by piecewise linear functions are also used. See [4] for the non-archimedean Monge-Ampère equations for more general cases. Here in Theorem 2.2, we are working on the intersection complexes instead of the dual intersection complexes as in [4, 19], and our piecewise linear functions are from the balanced embedded degenerations of Abelian varieties.

We end this section by giving a remark of Calabi-Yau manifolds balanced embedded by theta functions. In Section 3 of [7], it is shown that the Calabi-Yau hypersurfaces

\[ X_t = \{ [z_0, \cdots, z_4] \in \mathbb{CP}^4 | t(z_0^{n+1} + \cdots + z_n^{n+1}) + z_0 \cdots z_n = 0 \} \]

are balanced embedded for any \( t \in \mathbb{C} \). The proof involves two finite group actions on \((X_t, \mathcal{O}_{\mathbb{CP}^n}(1)|_{X_t})\). The first one is \( \text{Ab}_{n+1} = \{(a_0, \cdots, a_n) | a_i \in \mathbb{Z}_{n+1}, a_0 + \cdots + a_n = 0 \}/\mathbb{Z}_{n+1} \), which acts on \( X_t \) by

\[ (z_0, \cdots, z_n) \mapsto (\zeta^{a_0} z_0, \cdots, \zeta^{a_n} z_n) \]

where \( \zeta = \exp \frac{2 \pi \sqrt{-1}}{n+1} \). The second one is the symmetric group \( S_{n+1} \) on \( n+1 \) elements, which acts on \( X_t \) by translating \( z_0, \cdots, z_n \). The same argument as in the proof of Theorem 2.1 also shows the result of \( X_t \) being balanced. On the other hand, \( z_0, \cdots, z_n \) as sections of \( \mathcal{O}_{\mathbb{CP}^n}(1)|_{X_t} \) are theta functions constructed in the Gross-Siebert program for \( |t| \ll 1 \) (at least for \( n = 3 \)) by Example 6.3 in [9], and thus \( X_t \) are balanced embedded by theta functions.

3. Construction of theta functions

We recall the construction of theta functions on degenerations of Abelian varieties, and we follow the arguments in Example 6.1 of [9] and Section 2 of [12] closely. See also [1] for the elliptic curve case.

Let \((P, \varphi)\) be the same as in the above section, i.e. the graph of \( \varphi \) is the lower boundary of the upper convex hull

\[ \text{conv}\{(m, \varphi(m)) | m \in M\} \subset M_{\mathbb{R}} \times \mathbb{R}, \]

and \( P \) be the rational polyhedral decomposition of \( M_{\mathbb{R}} \) induced by \( \varphi \). If we define

\[ \Delta_{\varphi} = \{(m, r) \in M_{\mathbb{R}} \times \mathbb{R} | r \geq \varphi(m)\}, \]

the standard construction for toric degenerations gives a toric variety \( X_{\Delta_{\varphi}} \) with a line bundle \( L_{\Delta_{\varphi}} \).

\[ \Delta_{\varphi} \]

\[ \varphi(y) = y^2 \]

\[ M_{\mathbb{R}} \]
For any \( l \in \mathbb{N} \), \( H^0(X_{\Delta_{\varphi}}, L^l_{\Delta_{\varphi}}) \) is generated by monomial sections
\[
\{Z^{(m, r, l)} | \forall (m, r, l) \in C(\Delta_{\varphi}) \cap (M \times \mathbb{Z} \times \{l\})\}
\]
where
\[
C(\Delta_{\varphi}) = \{(lm', lr', l)| (m', r') \in \Delta_{\varphi}, l \in \mathbb{R}_{>0}\} \subset M_{\mathbb{R}} \times \mathbb{R} \times \mathbb{R}.
\]
Note that we have a canonical regular function
\[
\bar{\pi} = Z^{(0, 1)} = Z^{(0, 1, 0)} : X_{\Delta_{\varphi}} \to \mathbb{C}.
\]
The toric boundary is \( \bar{X}_0 = \{Z^{(0, 1)} = 0\} \), a toric variety with infinite many irreducible components, and \( \bar{X}_t = \{Z^{(0, 1)} = t\} \cong T_C \), for any \( t \neq 0 \). We have a family of toric varieties \( \bar{X}_t \) degenerating to a singular toric varieties \( \bar{X}_0 \).

The degeneration of principally polarized Abelian varieties \( (\pi : X \to \Delta, \mathcal{L}) \) is constructed as the quotient of an \( M_k \)-action on \( (X_{\Delta_{\varphi}}, L_{\Delta_{\varphi}}) \) as the following.

**Lemma 3.1.** There is an \( M \)-action on \((X_{\Delta_{\varphi}}, L_{\Delta_{\varphi}})\) such that the projection \( \bar{\pi} \) is \( M \)-invariant, i.e. \( \bar{\pi}(m \cdot) = \bar{\pi}(\cdot) \) for any \( m \in M \); the induced \( M \)-action on monomial rational functions is given by
\[
Z^{(m, r)} \mapsto Z^{(m, r + Z(\gamma, m))}, \quad \gamma \in M, \quad (m, r) \in M \times \mathbb{Z};
\]
and the induced \( M \)-action on \( H^0(X_{\Delta_{\varphi}}, L_{\Delta_{\varphi}}) \) is given by
\[
Z^{(m, r, 1)} \mapsto Z^{(m + \gamma, r + \alpha(\gamma), 1)}, \quad \gamma \in M, \quad (m, r) \in \Delta_{\varphi}
\]
for monomial sections.

**Proof.** If \( \Sigma \subset N_{\mathbb{R}} \times \mathbb{R} \) denotes the normal fan of \( \Delta_{\varphi} \), then one-dimensional rays of \( \Sigma \) are one-to-one correspondence to the maximal dimensional cells in \( P \), and the primitive generator of a ray has the form \((-d\varphi|_\sigma, 1)\) for an \( n \)-dimensional cell \( \sigma \) of \( P \). Then \( M \) acts on \( N \times \mathbb{Z} \) by \( T_0^\gamma : (\mu, l) \mapsto (\mu - l d\alpha_\gamma, l) \) for any \( \gamma \in M \), which preserves \( \Sigma \), and thus induces an \( M \)-action on \( X_{\Delta_{\varphi}} \). The dual \( M \)-action on \( M \times \mathbb{Z} \) is the transpose, i.e. \( T_0^\gamma : (m, r) \mapsto (m + r + d\alpha_\gamma(m)) \) for any \( \gamma \in M \). Thus the \( M \)-action preserves the regular function \( \bar{\pi} = Z^{(0, 1)} \), and the induced \( M \)-action on monomial rational functions is given by
\[
Z^{(m, r)} \mapsto Z^{T_0^\gamma(m, r)} = Z^{(m + Z(\gamma, m))},
\]
for any \( \gamma \in M \) and \( (m, r) \in M \times \mathbb{Z} \).

For constructing the \( M \)-action on \( L_{\Delta_{\varphi}} \), we consider the \( M \)-action on \( C(\Delta_{\varphi}) \) defined by
\[
T_\gamma : (m, r, l) \mapsto (m + l\gamma, r + d\alpha_\gamma(m) + lc_\gamma, l) \quad \gamma \in M,
\]
where \( d\alpha_\gamma(m) = Z(\gamma, m) \) by \( \Box 22 \) and \( c_\gamma = \frac{1}{2} Z(\gamma, \gamma) \). We have \( T_\gamma(m, \varphi(m), 1) = (m + \gamma, \varphi(m + \gamma), 1) \) by \( \Box 27 \). This action lifts the \( M \)-action on \( X_{\Delta_{\varphi}} \) to an \( M \)-action on \( L^1_{\Delta_{\varphi}} \). More precisely the \( M \)-action on \( L^1_{\Delta_{\varphi}} \) is given by
\[
Z^{(m, r, l)} \mapsto Z^{T_\gamma(m, r, l)} = Z^{(m + l\gamma, r + d\alpha_\gamma(m) + lc_\gamma, l)}
\]
for monomial sections.

For any \( k \in \mathbb{N} \), \( M_k = kM \) is a subgroup, and acts on \((X_{\Delta^0}, L_{\Delta^0})\) induced by the \( M \)-action in the above lemma. Note that the map \( \bar{\pi} \) is \( M_k \)-invariant, and \( M_k \) acts properly and discontinuously on \( \bar{\pi}^{-1}(\Delta) \) for the unit disc \( \Delta \subset \mathbb{C} \). The quotient is the degeneration of principally polarized Abelian varieties \( \pi : \mathcal{X} = \bar{\pi}^{-1}(\Delta)/M_k \to \Delta \), and \( \pi^{-1}(t) = X_t = \bar{X}_t/M_k \).

The central fiber \( X_0 \) of \( \mathcal{X} \) is a union of finite irreducible toric varieties, and the corresponding intersection complex is \( B_k = M_{\mathbb{R}}/M_k \) with rational polyhedron decomposition \( \bar{\mathcal{P}} \) induced by \( \mathcal{P} \). There is a one to one corresponding between the \( n \)-dimensional cells of \( \bar{\mathcal{P}} \) and the irreducible components of \( X_0 \).

More precisely, for any \( n \)-dimensional cell \( \sigma \) of \( \bar{\mathcal{P}} \), we regard it as a rational polytope in \( M_{\mathbb{R}} \), and it defines a polarized toric variety \((X_\sigma, L_\sigma)\). The irreducible component of \( X_0 \) corresponding to \( \sigma \) is isomorphic to \( X_\sigma \). The quotient of \( L_{\Delta^0} \) by the \( M_k \)-action is the relative ample line bundle \( L \) on \( \mathcal{X} \). The restriction of \( L \) on any irreducible component \( X_\sigma \) of \( X_0 \) is \( L_\sigma \). The \( M_k \)-invariant sections descend to sections of \( L \).

We claim that and \( X_{\Delta^0} = \pi^{-1}(\Delta^0) \) is a base change of \( X_\eta \) (cf. Section 2.1) via \( t \mapsto t^k = s \). Note that for any \( t \neq 0 \), \( M_k \) acts on \( \bar{X}_t = T_\mathbb{C} \) by

\[
Z^{(m,r)} := Z^m t^r \mapsto Z^{T_\gamma (m,r)} := Z^{(m,r)} t^{\gamma(m,r)}.
\]

Since \( t^{kZ(k^{-1},1)} \), \( k^{-1} \gamma \in M \), we obtain that \( \bar{X}_t/M_k = X_s \) by (2.4) where \( t^k = s \).

For any \( m \in B_k(\mathbb{Z}) = M/M_k \), we define the theta function

\[
\vartheta_m = \sum_{\gamma \in M_k} Z^T_{\gamma(m,\varphi(m),1)} = \sum_{\gamma \in M_k} Z^{(m+\gamma,\varphi(m+\gamma),1)},
\]

which is a section of \( L \) (cf. Example 6.1 of [9] and Section 2 of [12]). By abusing of notation, we will use \( m \) to denote both a point in \( M \) and its image under the quotient map

\[
M \to B_k(\mathbb{Z}) = M/kM \subset B_k = M_{\mathbb{R}}/kM
\]

without any confusing. We obtain a basis \( \{ \vartheta_m | m \in B_k(\mathbb{Z}) \} \) of \( H^0(\mathcal{X}, L) \).

For any irreducible component \( X_\sigma \subset X_0 \), \( \vartheta_m|_{X_\sigma} \) is a monomial section of \( L_\sigma \), and it is not a zero section if and only if \( m \in \sigma \). For any \( t \neq 0 \),

\[
\vartheta_m(w) = \sum_{\gamma \in M_k} Z^{m+\gamma} t^{\varphi(m+\gamma)}
\]

\[
= \sum_{\gamma \in M_k} \exp \pi \sqrt{-1} \left( 2(w, m+\gamma) + \frac{\log t}{2\pi \sqrt{-1}} Z(m+\gamma, m+\gamma) \right),
\]

by \( \varphi(m+\gamma) = \bar{\varphi}(m+\gamma) \), where \( w = w_1 e_1^* + \cdots + w_n e_n^* \). Thus it is the classical theta function

\[
\vartheta_m(w) = \vartheta \left[ \begin{array}{c} m \\ 0 \end{array} \right] (w, \frac{\log t}{2\pi \sqrt{-1}} Z_{ij})
\]
on \(X_t\). If we regard \(\vartheta_m\) as on \(X_s\), \(s = t^k\), then
\[
\vartheta_m(w) = \sum_{\gamma' \in M} \exp \frac{\pi \sqrt{-1}}{2} (2k\langle w, \frac{m}{k} + \gamma' \rangle + k \cdot \log \frac{s}{2\pi \sqrt{-1}} Z(\frac{m}{k} + \gamma', \frac{m}{k} + \gamma')), 
\]
and a direct calculation shows
\[
\vartheta_m(w + \mu + \frac{\log s}{2\pi \sqrt{-1}} Z(p, \cdot)) = \vartheta_m(w) \exp k \pi \sqrt{-1} (-2\langle w, p \rangle - \frac{\log s}{2\pi \sqrt{-1}} Z(p, p)),
\]
for any \(\mu \in N\) and \(p \in M\). Thus \(\mathcal{L}|_{X_t} \cong \mathcal{L}^k|_{X_s}\), i.e. \(\mathcal{L}|_{X_\Delta}\) is the pull-back of \(\mathcal{L}^k\).

**Remark 3.2.** Notice that, in particular, the monodromy action \(\log t \rightarrow \log t + 2\pi \sqrt{-1}\) acts trivially on \(\vartheta_m\) via
\[
\vartheta \left[ \begin{array}{c} m \\ 0 \end{array} \right] (w, \left( \frac{\log t}{2\pi \sqrt{-1}} + 1 \right) Z_{ij}) = \sum_{\gamma \in M} \exp \frac{\pi \sqrt{-1}}{2} (2\langle w, m + \gamma \rangle + (\frac{\log t}{2\pi \sqrt{-1}} + 1) Z(m + \gamma, m + \gamma))
\]
\[
= \vartheta \left[ \begin{array}{c} m \\ 0 \end{array} \right] (w, \frac{\log t}{2\pi \sqrt{-1}} Z_{ij})
\]
since \(Z(m + \gamma, m + \gamma) = 2\varphi(m + \gamma) \in 2\mathbb{Z} \) for \(m \in B_k(\mathbb{Z})\).

**Example 3.3.** We illustrate the explicit formula of the theta function \(\vartheta_m\) in (8.3) in local coordinates for a special 1-dimensional family. Let \(M \cong \mathbb{Z}\), \(k = 3\), and \(\varphi(y) = y^2\) on \(M_{\mathbb{R}}\). Note that \((0, 0)\) is a vertex of \(\Delta_{\varphi}\), and the \(\mathbb{C}\)-algebra \(\mathbb{C}[T_{(0, 0)} \Delta_{\varphi} \cap M]\) is generated by \(z_1 = \mathcal{Z}^{(1, 1)}\), \(z_2 = \mathcal{Z}^{(-1, 1)}\) and \(t = \mathcal{Z}^{(0, 1)}\), where \(T_{(0, 0)} \Delta_{\varphi}\) is the tangent cone of \(\Delta_{\varphi}\) at \((0, 0)\). The toric variety \(Y_0 = \text{Spec}(\mathbb{C}[T_{(0, 0)} \Delta_{\varphi} \cap M])\) is defined in \(\mathbb{C}^3\) by equation \(z_1 z_2 = t^2\), and an open subset of \(Y_0\) is biholomorphic to a neighborhood \(U_0\) of the zero strata of \(X_0\) in \(X\) corresponding to the vertex \((0, 0)\). Let us fix a trivialization of \(\mathcal{L}|_{U_0} \cong \mathcal{O}_{Y_0}|_{U_0}\) via the identification \(\mathcal{Z}^{(m,r,1)} \rightarrow \mathcal{Z}^{(m,r)}\) for any \((m, r) \in \Delta_{\varphi} \cap M\). Then we have
\[
\vartheta_0 = 1 + \sum_{\nu \in \mathbb{Z}, \nu > 0} (z_1^{3\nu} + z_2^{3\nu}) t^{9\nu^2 - 3\nu},
\]
\[
\vartheta_1 = z_1 + \sum_{\nu \in \mathbb{Z}, \nu > 0} (z_1^{1+3\nu} t^{9\nu^2 + 3\nu} + z_2^{3\nu-1} t^{9\nu^2 - 9\nu + 2}),
\]
\[
\vartheta_2 = z_2 + \sum_{\nu \in \mathbb{Z}, \nu > 0} (z_1^{3\nu-1} t^{9\nu^2 - 9\nu + 2} + z_2^{3\nu+1} t^{9\nu^2 + 3\nu}),
\]
by (3.5). In particular, \(\vartheta_i\) extends to the central fiber.

Notice that the central extension of the product group \(B_k(\mathbb{Z}) \times T_k\) with \(T_k := N/N_k\) is precisely the finite Heisenberg group \(\mathbb{H}_k = \mu_k \times B_k(\mathbb{Z}) \times T_k\)
(cf. [21, Section 3]) with the multiplication rule:

\[(3.7) \quad (\mu, a, b) \cdot (\mu', a', b') = (\mu \mu' \exp \frac{2\pi\sqrt{-1} \langle b, a \rangle}{k}, a + a', b + b')\]

for any \((\mu, a, b), (\mu', a', b') \in \mu_k \times B_k(\mathbb{Z}) \times T_k\), where \(\mu_k\) is the cyclotomic group of order \(k\).

**Lemma 3.4.** The group \(B_k(\mathbb{Z}) \times T_k\) acts on \((\mathcal{X}, \mathcal{L})\) and induces a representation of \(\mathbb{H}_k\) on the \(H^0(\mathcal{X}, \mathcal{L}) = \text{Span}_H \{\vartheta_m\}_{m \in B_k(\mathbb{Z})}\) via

i) For any \(a \in B_k(\mathbb{Z})\)

\[\mathfrak{T}_a \vartheta_m = \vartheta_{a+m}\]

ii) For any \(b \in T_k\)

\[\mathcal{G}_b \vartheta_m = \vartheta_m \exp \frac{2\pi\sqrt{-1} \langle b, m \rangle}{k},\]

for all \(m \in B_k(\mathbb{Z})\), where \(H\) denotes the ring of holomorphic functions on \(\Delta\). In particular, the representation of \(\mathbb{H}_k\) on \(H^0(\mathcal{X}, \mathcal{L})\) is irreducible.

**Proof.** The \(M\)-action on \((X_{\Delta, \varphi}, L_{\Delta, \varphi})\) in Lemma 3.1 induces the \(B_k(\mathbb{Z})\)-action on \((\mathcal{X}, \mathcal{L})\), which acts on \(H^0(\mathcal{X}, \mathcal{L})\) given by

\[\mathfrak{T}_a \vartheta_m = \sum_{\gamma \in \mathbb{M}_k} \mathcal{Z} T_{\alpha}(m+\gamma, \varphi(m+\gamma), 1)\]

for any \(m \in B_k(\mathbb{Z})\) and \(a \in B_k(\mathbb{Z})\) by (3.5). We obtain i), and next we prove ii).

Note that there is a natural injective homeomorphism \(i_k : T_k \hookrightarrow T_\mathbb{C}\) such that

\[\mathcal{Z}^m(i_k(b)) = \exp \frac{2\pi\sqrt{-1} \langle b, m \rangle}{k}\]

for any \(b \in T_k\) and \(m \in \mathbb{M}\). The standard \(T_\mathbb{C}\)-action on \((X_{\Delta, \varphi}, L^t_{\Delta, \varphi})\) induces a \(T_k\)-action, which preserves the regular function \(\bar{\pi} = \mathcal{Z}^{(0,1)}\), acts on monomial rational functions by

\[\mathcal{Z}^{(m, r)} \mapsto \mathcal{Z}^{(m, r)} \exp \frac{2\pi\sqrt{-1} \langle b, m \rangle}{k}\]

for any \((m, r) \in \mathbb{M} \times \mathbb{Z}\), and acts on monomial sections of \(L^t_{\Delta, \varphi}\) via

\[\mathcal{Z}^{(m, r, l)} \mapsto \mathcal{Z}^{(m, r, l)} \exp \frac{2\pi\sqrt{-1} \langle b, m \rangle}{k}\]
for any \((m, r, l) \in C(\Delta)\). The induced \(T_k\)-action on \(\bigoplus_{t=0}^{\infty} H^0(X_{\Delta^t}, L^t_{\Delta^t})\) commutes with the \(M_k\)-action by

\[
Z_{T_k(m, r, l)} \exp \frac{2\pi \sqrt{-1}(b, m)}{k} = Z^{(m + t\gamma, r + da, s(l) + t\gamma)} \exp \frac{2\pi \sqrt{-1}(b, m + l\gamma)}{k},
\]

for any \(\gamma \in M_k\) by \((3.3)\). Thus the \(T_k\)-action commutes with the \(M_k\)-action on \((X_{\Delta^t}, L_{\Delta^t})\), which induces a \(T_k\)-action on \((\pi : X \to \Delta, \mathcal{L})\). We denote \(\mathcal{G}\) the induced \(T_k\)-action on \(H^0(\mathcal{X}, \mathcal{L})\), which satisfies

\[
\mathcal{G}_b \vartheta_m = \sum_{\gamma \in M_k} Z^{(m + \gamma, \varphi(m + \gamma), 1)} \exp \frac{2\pi \sqrt{-1}(b, m + \gamma)}{k} \vartheta_m \exp \frac{2\pi \sqrt{-1}(b, m)}{k},
\]

for any \(b \in T_k\) and any \(m \in B_k(\mathbb{Z})\), by \((3.5)\).

**Remark 3.5.** The direct calculations show that on any \(X_t, t \neq 0\), the action of \(B_k(\mathbb{Z}) \times T_k\) on \((X_t, \mathcal{L}|X_t)\) is given by \(w \mapsto w + \frac{b}{k} + \frac{\log t}{2\pi \sqrt{-1}} Z(\cdot, a),\)

\[
(\Xi_a \vartheta_m)(w) = \exp \pi \sqrt{-1}(2(w, a) + \frac{\log t}{2\pi \sqrt{-1}}Z(a, a)) \vartheta_m(w + \frac{\log t}{2\pi \sqrt{-1}}Z(\cdot, a)) = \vartheta_{a+m}(w),
\]

\[
(\mathcal{G}_b \vartheta_m)(w) = \vartheta_m(w + \frac{b}{k}) = \exp \frac{2\pi \sqrt{-1}(b, m)}{k} \vartheta_m(w),
\]

for any \(m \in B_k(\mathbb{Z})\) and \((a, b) \in B_k(\mathbb{Z}) \times T_k\). In particular, there the finite torus \(B_k(\mathbb{Z}) \times T_k\) acts on the image of the projective embedding \(\Phi_k(X_t)\).

### 4. Proofs of Main Theorems

Before we start the proof, let us recall the Hermitian metric on \(\mathcal{L}|X_t \to X_t\) with \(s = t^k\) is given by

\[
(4.1) \quad h(w) := \exp \left( \frac{2\pi}{\log |t|} Z^{ij} y_i y_j \right) = \exp k \left( \frac{2\pi}{\log |s|} Z^{ij} y_i y_j \right)
\]

with

\[
w_i = x_i + (\arg(s) - \sqrt{-1} \frac{\log |s|}{2\pi}) \sum_{j=1}^{n} Z_{ij} y_j, \quad i = 1, \ldots, n.
\]

On \(X_t \cong X_s\), we have

\[
\omega_t = -\sqrt{-1} \partial \bar{\partial} \log h = k \omega_s,
\]

by \((2.5)\).

**Lemma 4.1.** \(\forall a \in M, b \in N\) and function \(f : N_\mathbb{C} \to \mathbb{C}\) we have

\[
(4.2) \quad h(w)|\mathcal{G}_b f|^2 = (h|f|^2)(w + \frac{b}{k}); \quad h(w)|\Xi_a f|^2 = (h|f|^2) \left( w + \frac{\log s}{2\pi \sqrt{-1}} Z(\cdot, a) \right).
\]

**Proof.** See \((28)\) Proposition 3]. \qed
As a consequence, for \( a \in B_k(\mathbb{Z}) \), \( b \in T_k \) and \( f \in H^0(X_t, \mathcal{L}|_{X_t}) \) we have
\[
\|(\mathcal{G}_b f)(w)\|_{FS}^2 = \int \frac{|(\mathcal{G}_b f)(w)|^2}{\sum_m |\vartheta_m(w)|^2} \Phi_k^* \omega_{FS}^n = \int \frac{h|(\mathcal{G}_b f)(w)|^2}{\sum_m h|\vartheta_m(w)|^2} \Phi_k^* \omega_{FS}^n = \int \frac{(h|f|^2)(w + \frac{b}{\bar{k}})}{(\sum_m h|\vartheta_m|^2)(w + \frac{b}{\bar{k}})} (\mathcal{G}_b \circ \Phi_k)^* \omega_{FS}^n = \int \left( \frac{|f|^2}{\sum_m |\vartheta_m|^2} \Phi_k^* \omega_{FS}^n \right) \left( w + \frac{b}{\bar{k}} \right) = \|f(w)\|_{FS}^2.
\]
and
\[
\|(\mathbb{I}_a f)(w)\|_{FS}^2 = \int \frac{|(\mathbb{I}_a f)(w)|^2}{\sum_m |\vartheta_m|^2} \Phi_k^* \omega_{FS}^n = \int \frac{h|(\mathbb{I}_a f)(w) + \frac{\log s}{2\pi\sqrt{-1}} Z(\cdot, \frac{a}{\bar{k}}))|^2}{\sum_m h|\vartheta_a|^2 Z(\cdot, \frac{a}{\bar{k}}))^2} (\mathbb{I}_a \circ \Phi_k)^* \omega_{FS}^n = \int \left( \frac{|f|^2}{\sum_m |\vartheta_m|^2} \Phi_k^* \omega_{FS}^n \right) \left( w + \frac{\log s}{2\pi\sqrt{-1}} Z(\cdot, \frac{a}{\bar{k}}) \right) = \|f(w)\|_{FS}^2.
\]
Hence the finite group generated by image of 
\[
\{ \mathbb{I}_a, \mathcal{G}_b \mid (a, b) \in B_k(\mathbb{Z}) \times T_k \} \subset \text{GL}(H^0(X_t, \mathcal{L}|_{X_t}))
\]
actually lies in \( U(k^n) \) with respect to the Fubini-Study metric induced via the embedding
\[
\Phi_k = [\vartheta_{m_1}, \ldots, \vartheta_{m_k}] : \mathcal{X} \longrightarrow \mathbb{CP}^{k^n-1}.
\]

**Proof of Theorem 2.1.** It follows from above that the action generated by \( B_k(\mathbb{Z}) \times T_k \) via \( \mathcal{I} \) and \( \mathcal{G} \) lies in \( U(k^n) \). And Lemma 3.4 implies that sections \( \{\vartheta_m\}_{m \in B_k(\mathbb{Z})} \) forms an orthonormal basis with respect to the pull back of Fubini-Study metric via the map \( \Phi_k \), that is, the embedding \( \Phi_k \) is balanced for each \( t \in \Delta^\circ \).

On the other hand, \( \Phi_k(X_t) \) being balanced for each \( t \neq 0 \) implies that the Chow point for \( \Phi_k(X_t) \) lies on the 0-level set the moment map
\[
\mu_{SU} : \text{Chow}_{\mathbb{CP}^{k^n-1}}(d, n) \longrightarrow \mu_{SU}(k^n) \quad X_t \quad \longrightarrow \quad \sqrt{-1} \cdot \int_{X_t} \left( \sum_m |\vartheta_m|^2 - \frac{\delta_{mm'}}{k^n} \right) \Phi_k^* \omega_{FS}^n \frac{n!}{n!}
\]
of the \( SU(k^n) \)-action on the Chow variety of \( n \)-dimensional degree \( d \) cycles in \( \mathbb{CP}^{k^n-1} \) (c.f. \[27\] Proposition 17 and \[30\] Theorem 1.4), which is proper via standard Kirwan-Kempf-Ness theory in \[10\] (c.f. also in \[25\]). Notice
that \{\vartheta_m(\cdot, \frac{\log}{2\pi N^2(\cdot)} Z)\}_m vary holomorphically with respect to \(t\) and can be extended to \(X_0\) by the construction of theta functions in Section 3, these imply that \(\Phi_k\) has bounded image in \(\mathbb{C}^{k-1}\), by Riemann mapping Theorem, the unique continuous extension \(\Phi_k(X_0)\) must lies in \(\mu_{SU}(\cdot, \cdot)\), that is, the embedding \(\Phi_k(X_0)\) is balanced as well.

\[\text{Proof of Theorem 2.2.}\] For any \(m \in M\), we denote \(\bar{m} \subset N_R\) the dual polytope of \(m\) with respect to \(\varphi\). More precisely, if \(\hat{m}^0 \subset N_R \times \mathbb{R}\) denotes the dual cone of the tangent cone \(T_{(m, \varphi(m))}\Delta_{\varphi} \subset M \times \mathbb{R}\) of \(\Delta_{\varphi}\) at the vertex \((m, \varphi(m))\), then \(\bar{m} = \hat{m}^0 \cap (N_R \times \{1\})\). The Monge-Ampère measure of \(\varphi\) is

\[
\text{MA}(\varphi) = \sum_{m \in B_k(\mathbb{Z})} \text{Vol}(\bar{m}) \delta_m,
\]

where \(\text{Vol}(\bar{m})\) is the Euclidean volume of \(\bar{m}\) (cf. Proposition 2.7.4 in \([5]\)). Since the \(M\)-action \(T^0\) on \(N_R \times \mathbb{R}\) preserves the fan \(\Sigma\) of \(\Delta_{\varphi}\), we obtain \(\bar{T}^0_{m \rightarrow m'}(\bar{m}') = \bar{m}\) for any two \(m\) and \(m' \in M\), and thus \(\text{Vol}(\bar{m}) = V\) is independent of \(m\).

Let \(D\) be the fundamental domain of the \(M\)-action \(T^0\) on \(N_R \times \{1\}\). Since \(\bar{T}^0_{m}(0, l) = (-d\alpha_y, l) = (-Z(\gamma, \cdot), 1)\) for any \(\gamma \in M\), we let \(D\) be the convex hull of \((0, 1), (-Z(ke_1, \cdot), 1), \ldots, (-Z(ke_n, \cdot), 1)\) in \(N_R \times \{1\}\), where \(e_1, \ldots, e_n\) is a basis of \(M\). The Euclidean volume of \(D\) is \(\text{Vol}(D) = k^n \det(Z_{ij})\), where \(Z_{ij} = Z(e_i, e_j)\), and we have

\[
\text{Vol}(D) = \sum_{m \in B_k(\mathbb{Z})} \text{Vol}(\bar{m}) = k^n V.
\]

We obtain \(V = \det(Z_{ij})\), and the conclusion

\[
\text{MA}(\varphi) = \det(Z_{ij}) \sum_{m \in B_k(\mathbb{Z})} \delta_m.
\]

Let \(\chi_k : B \rightarrow B_k\) be the diffeomorphism induced by the dilation \(y_i \mapsto ky_i, i = 1, \ldots, n, \) of \(M_R\). For any smooth function \(f\) on \(B\),

\[
\frac{1}{k^n} \int_B f \chi_k^*(\sum_{m \in B_k(\mathbb{Z})} \delta_m) = \frac{1}{k^n} \sum_{m' \in (\frac{1}{k^n})^M/M} f(m') \rightarrow \int_B f dy_1 \wedge \cdots \wedge dy_n
\]

when \(k \rightarrow \infty\), which implies that

\[
\frac{1}{k^n} \chi_k^* \text{MA}(\varphi) \rightarrow \det(Z_{ij}) dy_1 \wedge \cdots \wedge dy_n = \text{MA}(\varphi)
\]

in the weak sense.

By \((2.3)\) and \((2.7)\), \((\varphi - \bar{\varphi})(y + \gamma) = (\varphi - \bar{\varphi})(y)\) for any \(y \in M_R\) and any \(\gamma \in M\), and \(\varphi - \bar{\varphi}\) is a well-defined function on \(B_k\). Since

\[
\sup_{M_R} |\varphi - \bar{\varphi}| = \sup_{y \in M_R/M} |\varphi - \bar{\varphi}|(y),
\]

we have

\[
\text{MA}(\varphi) = \sum_{m \in B_k(\mathbb{Z})} \text{Vol}(\bar{m}) \delta_m.
\]

This completes the proof.

\[\square\]
we obtain
\[
\sup_B \left| \frac{1}{k^2} \chi_k^* \varphi - \frac{1}{k^2} \chi_k^* \bar{\varphi} \right| = \frac{1}{k^2} \sup_{M_k} |\varphi - \bar{\varphi}| \to 0,
\]
when \(k \to \infty\), and the conclusion by \(\frac{1}{k^2} \bar{\varphi}(ky) = \bar{\varphi}(y)\). \(\square\)

5. Appendix

In this section, we state the following Theorem which unified the proof of balanced embedding for projective space and principally polarized Abelian varieties.

**Theorem 5.1.** Let \(X \subset \mathbb{P}^N\) be a subvariety and \(G < SU(N + 1)\) be a compact subgroup which leaves the embedding \(X \hookrightarrow \mathbb{P}^N\) invariant. Suppose the centralizer of \(c_G < SU(N + 1)\) of \(G\) inside \(SU(N + 1)\) is trivial. Then the embedding \(X \subset \mathbb{P}^N\) is balanced, i.e.

\[
(5.1) \quad \int_X \mu_{\mathbb{P}^N} \frac{\omega_{FS}^n}{n!} = 0.
\]

**Proof.** Notice that the moment map of \(SU(N+1)\)-action on the Chow variety \(\text{Chow}_{\mathbb{P}^N}(n, d)\) of dimension \(n\) and degree \(d\) cycle in \(\mathbb{P}^N\) is precisely given by (5.2). In particular, it is \(SU(N+1)\), and hence \(G\)-equivariant. This implies that for any \(g \in G\)

\[
\int_X \mu_{\mathbb{P}^N} \frac{\omega_{FS}^n}{n!} = \int_{g \cdot X} \mu_{\mathbb{P}^N} \frac{\omega_{FS}^n}{n!} = \int_X \mu_{\mathbb{P}^N} \circ g \frac{\omega_{FS}^n}{n!} = \text{Ad}_g \left( \int_X \mu_{\mathbb{P}^N} \frac{\omega_{FS}^n}{n!} \right).
\]

By our assumption, we have \(\int_X \mu_{\mathbb{P}^N} \frac{\omega_{FS}^n}{n!} \in \mathfrak{c}_G = 0\), where \(\mathfrak{c}_G = \text{Lie}(c_G)\) is the Lie algebra. And our proof is thus completed. \(\square\)

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