SOLVABILITY OF SOME VOLTERRA TYPE INTEGRAL EQUATIONS IN HILBERT SPACE

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Abstract. We consider an integral equation of Fredholm and Volterra type with spectral parameter depending on time. Conditions of solvability are established when the initial value of the parameter coincides with an eigenvalue of Fredholm operator.

1. Introduction. We consider in an arbitrary Hilbert space $H$ the following integral equation

$$\int_{0}^{t} K(t,s)u(s)ds + Au(t) - \lambda(t)u(t) = f(t), \quad t > 0, \quad (1)$$

where $u : R_+ \to H$ is unknown function, $A : H \to H$ is a linear self-adjoint operator, $K : Q \to R$ - is the kernel, $f(t)$ - is a given function and $\lambda(t)$ - is a function which we may interpret as spectral parameter. We indicated above by $R_+$ the positive half-line $R_+ = \{ t \in R : t \geq 0 \}$ and by $Q$ the set, we suppose also that the kernel $K(t, s)$ is continuous on triangle $Q = \{(t, s) \in R^2 : 0 \leq s \leq t < \infty \}$. Set $M(t) = \sup_{0 \leq s \leq t} |K(t, s)|$. It is clear that $M(t)$ is monotonically increasing function.

The main example is the following integral equation

$$\int_{0}^{t} K(t,s)P(x,s)ds - \int_{0}^{a} R(x,y)P(y,t)dy - \lambda(t)P(x,t) = f(x,t),$$

where $0 \leq x \leq a$ and $t > 0$, which we consider in Hilbert space $H = L_2(0, a)$, where $a > 0$.

The equations of such type are known as partial integral equations and were first considered by Abdus Salam [6] (see also [4], [7]). They arise in the theory of creepage [8] (see [1], [3], [5]). The kernels $K(t, s)$ and $R(x, y)$ are connected with some elastic creeping base and $\lambda(t)$ is the given value which describes the elastic properties of deformable body.

We suppose that $\lambda(t)$ is a continuous function. Denote by $\Lambda(t)$ the range of the function $\lambda(s)$ on the interval $[0, t]$: $\Lambda(t) = \{ \lambda(s), 0 \leq s \leq t \}$.

We consider a function $u : R_+ \to H$. It means that $u(t) \in H$ for any $t \geq 0$. Assume that $u(t)$ is continuous and set $\| u \|_t = \sup_{0 \leq s \leq t} \| u(s) \|$. Denote by $\sigma(A)$ the

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spectrum of the self-adjoint operator $A$. When parameter $\lambda$ is outside this spectrum, i.e. $\lambda \notin \sigma(A)$, there exists the resolvent $R_\lambda = (A - \lambda I)^{-1}$ of the operator $A$.

We consider the auxiliary equation

$$\int_0^t K(t, s)R_\lambda(s)v(s)ds + v(t) = f(t), \quad t > 0. \quad (2)$$

Assume that $v(t)$ is a solution of this equation. Then by putting $u(t) = R_\lambda(t)v(t)$ or $v(t) = (A - \lambda(t)I)u(t)$ we get

$$\int_0^t K(t, s)u(s)ds + (A - \lambda(t)I)u(t) = f(t), \quad t > 0.$$

It means that $u(t)$ is a solution of the equation (1).

At first we suppose that $\lambda(t)$ does not intersect the spectrum of $A$, i.e. $\Lambda(t) \cap \sigma(A) = \emptyset$. Then obviously the norms of resolvent $||R_\lambda||$ are bounded for every $\lambda \in \Lambda(t)$.

Set

$$B(t) = \sup_{\lambda \in \Lambda(t)} ||R_\lambda||. \quad (3)$$

The following simple result shows that outside the spectrum of the operator $A$ there is the solvability of the equation (1) without any additional condition. Theorem 1 was published without proof in [5]

**Theorem 1.** Let the kernel $K(t, s)$ be continuous on each triangle $Q_T = \{(t, s) : 0 \leq s \leq t \leq T\}$. Let $\Lambda(t) \cap \sigma(A) = \emptyset$ for all $t \geq 0$. Then for any continuous function $f(t)$ there exists a unique solution satisfying of equation (1) and the following estimate

$$||u(t)|| \leq ||f||_t B(t) \exp\{tM(t)B(t)\}. \quad (4)$$

To prove this theorem it is sufficient to show that there exists a unique solution of the auxiliary equation (2). This fact evidently follows from the next lemma.

**Lemma 1.** Let $v_0(t) = f(t)$, and

$$v_k(t) = \int_0^t K(t, s)R_\lambda(s)v_{k-1}(s)ds, \quad k = 1, 2, \ldots \quad (5)$$

Then

$$||v_k||_t \leq ||f||_t \frac{[M(t)B(t)]^k}{k!} t^k. \quad (6)$$

**Proof.** (see [5])

Using this lemma we may complete the proof of the Theorem 1

**Proof. 1. Existence.** We consider a sequence $\{v_k(t)\}$ and introduce the function

$$v(t) = \sum_{k=0}^\infty (-1)^k v_k(t). \quad (7)$$
Hence, according to Lemma 1. If we take proved below in [2] there exists \( C_q > 0 \) so that \( \frac{F_q(t)}{t^{q}} \leq C_q \exp\{t^q\} \), \( t \geq 0 \). Then according to Lemma 1. If we take \( q = 1 \) we have
\[
F_1(t) \leq C_1 \exp\{t\} \quad t \geq 0 \quad \text{and} \quad C_1 \geq 1
\]  
(7)

\[
\|v(t)\| \leq \sum_{k=0}^{\infty} \|v_k(t)\| \leq \|f\|_t \sum_{k=0}^{\infty} \left[ \frac{M(t)B(t)}{k!} \right]^k t^k = \|f\|_t F_1(M(t)B(t))
\]

Consequently, the series (6) converges absolutely and according to (4),
\[
\int_0^t K(t, s)R_{\lambda(s)} v(s) ds = \sum_{k=0}^{\infty} (-1)^k \int_0^t K(t, s) R_{\lambda(s)} v(s) ds
\]
\[
= \sum_{k=0}^{\infty} (-1)^k v_{k+1}(t) = v_1(t) - v_2(t) + v_3(t) - ...
\]
\[
= (v_0(t) - v_0(t)) + v_1(t) - v_2(t) + v_3(t) - ...
\]

Hence the function \( v(t) \) satisfies equation (2). Further, as mentioned above, the function
\[
u(t) = R_{\lambda(t)} v(t)
\]
satisfies equation (1). Obviously,
\[
\|u(t)\| \leq ||R_{\lambda}|| \|v(t)\| \leq ||R_{\lambda(t)}|| \|f\|_t F_1(M(t)B(t)) \leq ||f||_t B(t) F_1(M(t)B(t))
\]

The required estimate in Theorem 1 follows from this inequality and from inequality (7).

2. Uniqueness. Assume that there are two solutions \( u_1(t) \) and \( u_2(t) \) of the equation (1). Denote \( u(t) = u_1(t) - u_2(t) \). Then \( \int_0^t K(t, s) u(s) ds + A u(t) - \lambda(t) u(t) = 0 \), \( t \geq 0 \), consider the function \( v(t) = (A - \lambda(t)I) u(t) \). Then \( u(t) = R_{\lambda(t)} v(t) \) and \( \int_0^t K(t, s) R_{\lambda(s)} v(s) ds + v(t) = 0 \), or taking into consideration the following notation we may write \( v(t) = -W_{\lambda(t)} v(t) \), where \( W_{\lambda(t)} v(t) = \int_0^t K(t, s) R_{\lambda(s)} v(s) ds \). Obviously, \( v(t) = (-1)^k W_{\lambda(t)}^k v(t), \quad k = 1, 2, ... \) and according to Lemma 1.
\[
\|v(t)\| \leq ||v||_t [M(t)B(t)]^k \left( \frac{t^k}{k!} \right) \to 0, \quad k \to \infty
\]

Hence, \( v(t) = 0 \) and consequently \( u(t) = 0 \), it means that \( u_1(t) = u_2(t) \).

The problem is more complicated when \( \Lambda(t) \) has a common point with spectrum by of \( A \), which is the main idea of our consideration. We suppose that operator \( A \) is self-adjoint, i.e. the kernel \( R(x, y) \) is symmetric:
\[
R(x, y) = R(y, x)
\]
Suppose that \( \lambda(0) \) coincides with one of the isolated points of the spectrum of the operator \( A = A^* \). We suppose also that \( \lambda(t) \notin \sigma(A) \) for all \( t > 0 \). It will be proved below that in this case the following value is finite:

\[
B_1(t) = \sup_{0 < s \leq t} s | R_{\lambda(s)} | < \infty, \quad t > 0.
\]

We introduce the class of kernels, which are vanishing at point \((0, 0) \in Q\) with some order \( \alpha > 0 \).

We suppose that the kernel \( K(t, s) \) belongs to the Hölder space \( C^\alpha(Q) \) for some \( \alpha \), \( 0 < \alpha < 1 \).

Namely, we assume that for all \((t, s) \in Q \) and \((\tau, \sigma) \in Q\) for some constant \( K_\alpha > 0 \) the following estimate

\[
| K(t, s) - K(\tau, \sigma) | \leq K_\alpha [(t - \tau)^2 + (s - \sigma)^2]^{\alpha/2}
\]

is valid.

Further, we say that \( f \in N^\alpha(R_+) \) if \( f : R_+ \to H \) is continuous on the half-line \( t \geq 0 \), and

\[
||f||_(\alpha) = \sup_{t > 0} t^{-\alpha} ||f(t) - f(0)|| < \infty.
\]

We suppose that \( f \in N^\alpha \) and prove the following statement.

**Theorem 2.** Let the function \( \lambda(t) \) be continuously differentiable on \([0, T]\) and \( \lambda(0) = \lambda_0 \neq 0 \) be an isolated eigenvalue of \( A \). Let

\[
\lambda'(0) \neq 0,
\]

\[
K(0, 0) = 0.
\]

If the condition

\[
f(0) \perp \ker(A - \lambda_0 I)
\]

is fulfilled then there exists a unique solution \( u(t) \) of the equation (1) so that the estimate

\[
||u(t)|| \leq C(T)||f||_(\alpha)
\]

holds.

The constant \( C(T) \) depends on function \( B_1(T) \) and constant \( K_\alpha \), which are defined by equality (8) and estimate (9) correspondingly.

The condition (11) is essential, and without it we can not prove Theorem 2. This condition means that spectral parameter \( \lambda(t) \) has to move away from the point \( \lambda(0) \) with non-zero velocity. We show also that the conditions (12) and (13) are important.

At first we give some lemmas.

**Lemma 2.** Suppose that \( \lambda(t) \) is a continuously differentiable function for \( t \geq 0 \). Let \( \lambda(0) = \lambda_0 \) be an isolated point of the spectrum \( \sigma(A) \) and \( \lambda(t) \notin \sigma(A) \) for all \( t > 0 \). If condition (11) is fulfilled then for every \( T > 0 \) the inequality
\[ |\lambda(t) - \lambda_0| \geq c(T)t, \quad 0 \leq t \leq T, \quad c(T) > 0 \]  
(14) 

is valid.

**Proof.** (see [5])

Let \( R_{\lambda(t)}(A) \) be the resolvent of \( A \), i.e. \( R_{\lambda(t)}(A) = (A - \lambda I)^{-1} \). Next lemma proves the required estimate of resolvent.

**Lemma 3.** Under the conditions of Lemma 2 for an arbitrary \( T > 0 \) the following estimate

\[ t ||R_{\lambda(t)}(A)|| \leq C(T) < \infty, \quad 0 < t \leq T, \]  
(15)

is valid.

**Proof.** (see [5])

Now we can prove the following lemma.

**Lemma 4.** Let the functions \( v_k(t) \) be defined by \( v_0(t) = f(t), \)

\[ v_k(t) = \int_0^t K(t, s)R_{\lambda(s)}v_{k-1}(s)ds, \quad k = 1, 2, ... \]  
(16)

If all of conditions of the theorem 2 are fulfilled then

\[ ||v_k|| \leq ||f||_{(\alpha)}[C(t)]^k t^{\alpha(k+1)}k!, \]  
(17)

where \( C(T) = K_\alpha B_1(t)/\alpha \).

**Proof.** (see [5])

First, we prove the existence of a solution to (1) when \( f(0) = 0 \).

**Lemma 5.** Suppose that all conditions of Theorem 2 are fulfilled and let \( f(0) = 0 \). Then there exists a solution to (1) such that the following estimate is valid:

\[ ||u(t)|| \leq C(T)||f||_{(\alpha)}, \]  

where \( ||f||_{(\alpha)} \) is defined by (9).

**Proof.** According to Lemma 4, the series \( v(t) = \sum_{k=0}^{\infty} (-1)^k v_k(t) \), defined by (16) converges absolutely and the sum \( v(t) \) satisfies the equation

\[ v(t) = \int_0^t K(t, s)R_{\lambda(s)}v(s)ds = f(t). \]  
(18)

Indeed, \( v(t) + \sum_{k=0}^{\infty} (-1)^k \int_0^t K(t, s)R_{\lambda(s)}v_k(s)ds = v(t) + \sum_{k=0}^{\infty} (-1)^k v_{k+1}(t) = v_0(t) = f(t) \). Set \( u(t) = R_{\lambda(t)}v(t), \quad 0 < t \leq T \). Then obviously \( v(t) = [A - \lambda(t)I]u(t) \) and according to (18) \( ||f||_{(\alpha)} t^\alpha \sum_{k=0}^{\infty} \frac{|C_1^{(k)}|}{k!} = ||f||_{(\alpha)} t^\alpha e^{C_1 t} 0 \leq t \leq T \), Hence \( ||v(t)|| \leq \)
Here \( C_\alpha \) and \( C_1 \) are some positive constant. Since \( u(t) = R_{\lambda(t)} v(t) \) obviously, \( \| u(t) \| \leq \| R_{\lambda(t)} \| \| v(t) \| \leq \| R_{\lambda(t)} \| C_\alpha(T) \| f \|_{(\alpha)} = C(T) \| f \|_{(\alpha)} \). Hence \( \| u(t) \| \leq C(T) \| f \|_{(\alpha)} \).

**Proof of Theorem 2.**

1. **Existence.**

Set
\[
(Bu)(t) = \int_0^t K(t, s) u(s) ds + Au(t). 
\]

Using (19) we may rewrite equation (1) as
\[
(Bu)(t) - \lambda(t) u(t) = f(t). 
\]

Lemma 5 states that this equation has a solution under the assumption that \( f(0) = 0 \). To consider the general case note that for an arbitrary \( f_0 \in H \) such that \( f_0 \perp \ker(A - \lambda_0 I) \) the equation
\[
Au - \lambda_0 u = f_0 
\]
has a solution \( u_0 \).

Hence, according to (19) \((Bu_0)(t) = k(t) u_0 + Au_0 = k(t) u_0 + \lambda_0 u_0 + f_0 \), where \( k(t) = \int_0^t K(t, s) ds \). Consequently,
\[
Bu_0 - \lambda(t) u_0 = f_0 + g(t),
\]
where \( g(t) = [k(t) + \lambda_0 - \lambda(t)] u_0 \). It is clear that \( g(0) = 0 \).

Let \( f_0 = f(0) \). Consider an auxiliary equation
\[
Bv(t) - \lambda(t) v(t) = f_1(t)
\]
where \( f_1(t) = f(t) - f_0 - g(t) \), and it is clear that \( f_1(0) = 0 \).

We remark also that \( f_1 \in N^\alpha \). Hence, according to Lemma 5, there exists the solution \( v(t) \) of the equation (23). Then setting \( u(t) = v(t) + u_0 \) and taking into consideration equalities (23) and (22) we get \( Bu(t) - \lambda(t) u(t) = f(t) \).

Consequently, \( u(t) \) is a solution of the equation (1).

2. **Uniqueness.** Assume that there exists two solutions \( u_1 \) and \( u_2 \) of the equation (1). Then the difference \( u = u_2 - u_1 \) satisfies the homogeneous equation
\[
Au + \int_0^t K(t, s) u(s) ds - \lambda(t) u(t) = 0.
\]

Set for \( 0 \leq t \leq T \)
\[
v(t) = [A - \lambda(t) I] u(t).
\]

Using definition (25) and the fact that \( u \) is a solution of the equation (24) we get
\[
v(t) = - \int_0^t K(t, s) u(s) ds.
\]
It follows from (26) that $v(0) = 0$ and $||v(t)|| = ||\int_{0}^{t} K(t,s)u(s)ds|| \leq ||u||_{T} \int_{0}^{t} |K(t,s)|ds = O(t)$. Hence, the function $v(t)$ satisfies the condition (10) for all $\alpha \leq 1$ and thus $v \in N^\alpha$, $0 < \alpha \leq 1$. Further, it is clear that the function $v(t)$ defined by (25) satisfies equation

$$v(t) + \int_{0}^{t} K(t,s)R_{\lambda(s)}v(s)ds = 0.$$  

(27)

Indeed, since $\lambda(t) \notin \sigma(A)$ for $0 < t \leq T$ we have $u(t) = R_{\lambda(t)}v(t)$, $0 < t \leq T$, and after inserting this value into the right-hand side of (26) we get (27). Using definition $Q_{\lambda}v(t) = \int_{0}^{t} K(t,s)R_{\lambda(s)}v(s)ds = 0$, we may rewrite equation (27) as follows: $v(t) = -Q_{\lambda}v(t)$, $0 \leq t \leq T$. Consequently,

$$v(t) = (-1)^{k}Q_{\lambda}^{k}v(t), \quad 0 \leq t \leq T,$$  

(28)

and according to Lemma 4 these equations are correctly defined for all $k = 1, 2, \ldots$. The same Lemma 4 shows that the right-hand side of (28) tends to 0 when $k \to \infty$. Thus, $v(t) \equiv 0$.

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