Exotic phase diagram of a topological quantum system

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We study the quantum phase transitions (QPTs) in the Kitaev spin model on a triangle-honeycomb lattice. In addition to the ordinary topological QPTs between Abelian and non-Abelian phases, we find new QPTs which can occur between two phases belonging to the same topological class, namely, either two non-Abelian phases with the same Chern number or two Abelian phases with the same Chern number. Such QPTs result from the singular behaviors of the nonlocal spin-spin correlation functions at the critical points.

I. INTRODUCTION

A quantum phase transition (QPT) involves an abrupt change of the ground state in a many-body system due to its quantum fluctuations. Discovering and characterizing new QPTs in a two-dimensional topological quantum system have recently attracted considerable interest. Because the ground states in some topological quantum systems (e.g., the Kitaev spin models on honeycomb and triangle-honeycomb lattices) are exactly solvable, QPTs in these systems can be analytically investigated. In these topological systems, the discovered QPTs include the transition between a gapped Abelian phase and a gapless phase, the transition between Abelian and non-Abelian phases, and the transition between two non-Abelian phases with different Chern numbers. Also, an unconventional QPT between two non-Abelian phases was found in the Kitaev spin model on a triangle-honeycomb lattice by a fermionization method. Nevertheless, to the best of our knowledge, the QPT between two topological phases of the same Chern number (which belong to the same topological class) has not yet been found.

Here we show that, in the Kitaev spin model on a triangle-honeycomb lattice, a QPT can indeed happen between two gapped phases of the same topological class, in addition to the ordinary topological QPT between two phases of different Chern numbers. To demonstrate this, we focus on two parameter regimes as typical examples: (i) When the parameters vary across a critical curve separating two gapped phases of the same Chern number \( \nu = 0 \) or \( \pm 1 \), a first-order QPT occurs; (ii) when the parameters vary across a special critical point where several critical curves meet, a continuous QPT can occur. This is due to the exotic ground-state phase diagram which has either critical curves between two gapped phases (with the same or different Chern numbers) or critical points where four different gapped phases (with Chern numbers 0, 0, 1, and -1) terminate. These results reveal that the Kitaev spin model on a triangle-honeycomb lattice exhibits novel topological properties. Moreover, we find that such QPTs result from the singular behaviors of the nonlocal spin-spin correlations at the critical points.

The paper is organized as follows. In Sec. II, we present the solution for the ground state of the Hamiltonian in the uniform-flux sector. Sections III and IV show two typical phase diagrams of the ground-state wavefunctions and study various QPTs in the considered Kitaev spin model. Finally, a brief conclusion is given in Sec. V.

II. GROUND STATE IN THE UNIFORM-FLUX SECTOR

The Kitaev spin model on a triangle-honeycomb lattice is schematically shown in Fig. 1(a) and the model Hamiltonian is given by

\[
H = J_x \sum_{x \text{-link}} \sigma_i^x \sigma_j^x + J_y \sum_{y \text{-link}} \sigma_i^y \sigma_j^y + J_z \sum_{z \text{-link}} \sigma_i^z \sigma_j^z \\
+ J'_x \sum_{x' \text{-link}} \sigma_i^x \sigma_j^x + J'_y \sum_{y' \text{-link}} \sigma_i^y \sigma_j^y + J'_z \sum_{z' \text{-link}} \sigma_i^z \sigma_j^z,
\]

FIG. 1: (Color online) (a) A schematic illustration of the Kitaev spin model on a triangle-honeycomb lattice. Each unit cell (dotted diamond) contains six spins at sites 1,2,3. The six types of bonds are labeled by \( J_\alpha \) and \( J'_\alpha \), where \( \alpha = x,y,z \). (b) Schematic diagrams of the three plaquette operators \( P_0, P_1, \) and \( P_2 \), which are defined in Eq. (2).
where $\sigma_i^\alpha$, with $\alpha = x, y, z$, are the three Pauli operators at site $i$. Among the six sums in Eq. (1), four involve interactions within each unit cell (see Fig. 1): (i) The $x$-link couples either spins 2 and 3 or spins 5 and 6, (ii) the $y$-link couples either spins 1 and 3 or spins 4 and 6, (iii) the $z$-link couples either spins 1 and 2 or spins 4 and 5, and (iv) the $z'$-link couples spins 3 and 6. Nearest-neighbor unit cells are coupled by the $x'$- and $y'$-links. As found in Ref. 6 the ground state of Hamiltonian (1) is at least (6)-fold degenerate for the Abelian (non-Abelian) phase.

For the three types of hermitian plaquette operators defined by [see Fig. 1(b)]

$$
P_0 = \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x \sigma_5^x \sigma_6^x \sigma_7^x \sigma_8^x \sigma_9^x \sigma_{10}^x \sigma_1^y \sigma_2^y, 
$$

$$
P_1 = \sigma_1^y \sigma_2^y \sigma_3^y, 
$$

$$
P_2 = \sigma_1^z \sigma_2^z \sigma_3^z, 
$$

each has eigenvalues $\pm 1$. These plaquette operators commute with not only each other but also the Hamiltonian (1). As verified in Ref. 6, the ground state of Hamiltonian (1) is either in the sector of the Hilbert space in which all plaquette operators in Eq. (2) have eigenvalue one or in the sector where the time-reversal transformation of each plaquette operator in Eq. (2) has eigenvalue one. The former is denoted as the uniform-flux sector (12).

By using the Jordan-Wigner transformation, Hamiltonian (1) can be converted to a form represented by Majorana fermions. Performing Fourier transform on the Hamiltonian in the uniform-flux sector, we obtain

$$
H_u = \sum_{k \in BZ, j=1} \varepsilon_k^{(j)} \left[ 2A_k^{(j)*}A_k^{(j)} - 1 \right], 
$$

where BZ denotes the first Brillouin zone. In Eq. (3), $\varepsilon_k^{(j)} = \sum_{s=1}^6 c_k^{(s)} w_s$ are fermionic operators, where $c_k^{(s)}$ is the Fourier transform of the Majorana fermionic operator at site $s$ ($s = 1, 2, \cdots, 6$) in a unit cell, and $w_s$ is a function of variable $\varepsilon_k^{(s)}$ (see Appendix A). One can prove that Hamiltonian (3) breaks time reversal symmetry from the property of Majorana fermions $c_k^{(s)}$. This is in sharp contrast to the Kitaev model on a honeycomb lattice in which the time reversal symmetry is preserved. This is due to the difference between the bipartite nature of the honeycomb lattice and the non-bipartite nature of the triangle-honeycomb lattice. Hamiltonian (3) has six energy bands: $\varepsilon_k^{(1)} = -\varepsilon_k^{(6)} = -\varepsilon_k^{(5)}$, $\varepsilon_k^{(2)} = -\varepsilon_k^{(5)}$, and $\varepsilon_k^{(3)} = -\varepsilon_k^{(4)} = -\varepsilon_k^{-}$, with $\varepsilon_k \geq \varepsilon_k^+ \geq \varepsilon_k^- \geq 0$. In each unit cell, six Majorana fermions are defined, but the number of the corresponding fermions are three. Thus, the lowest three bands should be filled for the ground state: $|g\rangle = \prod_k \prod_{j=1}^{3} \sqrt{2} \Lambda_k^{(j)} |0\rangle$, with $\sqrt{2}$ the normalization factor which results from the fact that the fermion $A_k^{(j)}$ is constructed from Majorana fermions in $k$ space, each of whom can only take 1/2 as its occupation number. The ground-state energy per site is given by

$$
E_g = \frac{1}{6N} \sum_k \left[ \varepsilon_k^{(1)} + \varepsilon_k^{(2)} + \varepsilon_k^{(3)} \right], 
$$

where $N$ is the number of unit cells. The energy-band gap, i.e., the minimal energy to excite a fermion from the ground state is $\Delta = \mathcal{M}[\varepsilon_k^{(4)} - \varepsilon_k^{(3)}]$, where $\mathcal{M}[\cdots]$ denotes the minimal value of a function with variable $k$.

Below we show the ground-state phase diagrams in two different parameter regimes. The first diagram contains critical curves separating two phases of either the same or different Chern numbers. This unveils that QPTs can also occur in the same topological class. The second diagram contains critical points where four different phases with Chern numbers 0, 0, 1 and 1 terminate.

### III. CASE A: $J_x = J_y, J_3 = J'_y$, AND $J_zJ'_z = J'_xJ_x$

We first choose parameters $\Lambda_1 \equiv J_3/J_x$, and $\Lambda_2 \equiv J_x/J_z$ to study the ground-state property of the system. In particular, the parameters with $\Lambda_2 = 1$ and $\Lambda_1 > 0$ correspond to the case studied in Ref. 6 where a topological QPT between an Abelian phase and a non-Abelian phase was found at point $\Lambda_1 = \sqrt{3}$. There is also a perturbative study[13] of this model with parameters either $\Lambda_1 \ll 1$ or $\Lambda_1 \gg 1$ when $\Lambda_2 = 1$.

In order to find the ground-state phase diagram of the Kitaev spin model, we should first distinguish the gapless-phase regions from the gapped-phase regions in the phase diagram. Then, we calculate the Chern number[14,15] as a topological index to characterize each gapped-phase region. We note that the six energy bands satisfy the relation $\varepsilon_k^{(j)} = -\varepsilon_k^{(7-j)}$, where $j = 1, 2, 3$, which implies that the closing of the energy-band gap corresponds to either $\mathcal{M}[\varepsilon_k^{(j)}] = 0$ or $\mathcal{M}[\varepsilon_k^{(j)}] = 0$. Combining this condition with the relation $\varepsilon_k^{(j)} = -\varepsilon_k^{(7-j)}$, we can derive that if one has $\mathcal{M}[\prod_{j=1}^{6} \varepsilon_k^{(j)}] = 0$, the
The energy-band gap is zero, i.e., $\Delta = 0$. This gives rise to

$$A_1 = 0; \quad A_2 = 0; \quad A_1^2 = A_2^2 \pm \frac{2}{|A_2|},$$

(4)

where $+ (-)$ applies when $|A_1| > |A_2|$ ($|A_1| < |A_2|$). In Fig. 2(a), each gapped phase determined by Eq. (1) is schematically shown by a solid curve or line. One can see that the $A_1A_2$ plane is divided into 12 distinct regions by these solid curves or lines. Each of these 12 regions is a gapped topological phase and can be characterized by a Chern number.

We can define the Chern number by using the Berry’s phase for a gapped ground state. Among the six bands of Hamiltonian (4), the lower three bands with states $|j, k\rangle = \sqrt{2} A^{(3)}_{j} |0\rangle$, where $j = 1, 2, 3$, are occupied. The Berry’s phase gauge field in momentum space is

$$f_{\alpha}(k) = -i \sum_{j=1}^{3} \langle j, k | \frac{\partial}{\partial k_{\alpha}} | j, k \rangle,$$

(5)

where $\alpha = x, y$. This gives

$$\nu = \frac{1}{\pi} \text{Im} \sum_{s=1}^{6} \sum_{i=1}^{3} \int_{BZ} d^{2}k \left[ \partial w^{*}_{s}(\varepsilon^{(i)}_{k}) / \partial k_{x} + \partial w_{s}(\varepsilon^{(i)}_{k}) / \partial k_{y} \right],$$

(6)

where Im denotes the imaginary part of a complex variable. Numerical results show that each yellow (red) region in Fig. 2(a) corresponds to a gapped non-Abelian phases with Chern number $\nu = 1$ ($-1$). The other 8 white regions correspond to the Abelian phases with $\nu = 0$. A remarkable feature is that a critical curve or line (i.e. gapped phase) can separate two gapped phases with Chern numbers $\nu = (i) 0$ and $\pm 1$, (ii) $0$ and $0$, or (iii) $1$ ($-1$) and $\pm 1$. This reveals that a QPT can occur between two gapped phases belonging to the same topological class.

Below we explicitly display the QPTs (see Fig. 3). As in Ref. 16, here we classify the QPTs as two types: The first-order QPT where the first derivative of the ground-state energy $E_g$ with respect to the driving parameter is discontinuous at the transition point, and the continuous QPT where a higher-order derivative of $E_g$ is discontinuous and the derivative(s) with respect to the driving parameter vary along the horizontal dashed (dotted) line in Fig. 2(a). Using the perturbation method in 2, the effective Hamiltonian at $\Lambda_1 \sim 0$ can be obtained, up to third order, as (see Appendix B)

$$H_{\text{eff}} = H_{0}^{(3)} + \frac{6A^3}{A_2} \sum_{n} \left[ P_1(n) \sigma_i^x \sigma_j^y \sigma_k^z + P_2(n) \sigma_i^y \sigma_j^x \sigma_k^z \right],$$

(7)

where $n$ denotes the $n$th unit cell, $H_{0}^{(3)}$ is a term containing none of $P_1$, $P_2$, and the subscripts $i, j,$ and $k$ (i', j', and k') denote the sites linked to plaquette $P_1$ ($P_2$) with x-, y- and z-link, respectively. For the QPT between two phases of the same Chern number, the parameter $\Lambda_1$ changes its sign at the transition point $\Lambda_1 = 0$. This gives rise to different $H_{\text{eff}}$’s at the two sides of the transition point $\Lambda_1 = 0$. Also, the difference between the two phases of the same Chern number but with positive and negative $\Lambda_1$’s can be seen from their wavefunctions (see Appendix A).

Using Feynmann theorem, it can be derived that

$$\frac{\partial E_g}{\partial \Lambda_1} = \frac{1}{6} (C_x + C_y + A_2C_z),$$

(8)

where the spin-spin correlation functions are defined by $C_\alpha = \langle \gamma | \sigma_i^a \sigma_j^a | \gamma \rangle$, with $\alpha = x, y$ and $z$ for $\alpha' = x', y'$ and $z'$. It is clear that the discontinuity of $\partial E_g/\partial \Lambda_1$ at a QPT point results from the discontinuity of $C_\alpha$ there. Here we find that, at the QPT point, the nonlocal spin-spin correlation function on the $x'$, $y'$-, or $z'$-link changes its sign, displaying discontinuity there. As shown in Fig. 3(a) [Fig. 3(b)], $C_z$ indeed has a jump at the QPT point ($\Lambda_1, \Lambda_2$) = (0, 1) [(0, 2)] when $\Lambda_1$ changes from positive to negative. It should be noted that only part of the spin-spin couplings in the Hamiltonian (4) change their sign at these transition points, while the other spin-spin couplings are kept unchanged.

In addition to the first-order QPTs, continuous QPTs can also occur in the Kitaev spin model on a triangle-honeycomb lattice. As shown in Fig. 5(a), $\partial E_g/\partial \Lambda_1$
Below we focus on the QPTs at the points where different gapped phases terminate. When \( \Lambda_x \) varies along the dashed line in Fig. 2(b), a continuous QPT occurs at point \((\Lambda_x, \Lambda_y) = (\pm 1, 0)\), where \( \partial^2 E_g/\partial \Lambda_x^2 \) diverges [see Fig. 2(a)]. Similar to the QPT between two non-Abelian phases belonging to the same topological class, this QPT involves two Abelian phases with the same Chern number \( \nu = 0 \). When \( \Lambda_x \) varies along the dotted line in Fig. 2(b), in addition to the continuous QPTs between Abelian and non-Abelian phases (which occur at \((\Lambda_x, \Lambda_y) \approx (\pm 1.5, 1)\), where \( \partial^2 E_g/\partial \Lambda_x^2 \) are discontinuous), a continuous QPT between two non-Abelian phases happens at point \((\Lambda_x, \Lambda_y) = (0, 1)\), where \( \partial^2 E_g/\partial \Lambda^2 \) diverges. In contrast to the QPT between two non-Abelian phases belonging to the same topological class, this transition involves two non-Abelian phases with \( \nu = \pm 1 \). Analogous to the first-order QPTs occurring in the same topological class (Fig. 2), the continuous QPTs here are also due to the singularity of the nonlocal correlation functions at the critical points (see, e.g., the thick solid curves in Fig. 2).

V. CONCLUSION

We have studied QPTs in the Kitaev spin model on a triangle-honeycomb lattice and revealed the exotic ground-state phase diagram of this model. In addition to the ordinary topological QPTs between Abelian and non-Abelian phases, we find new QPTs that occur between two phases belonging to the same topological class. Moreover, we show that such QPTs are due to the singular behaviors of the nonlocal spin-spin correlation functions at the critical points.
FIG. 5: (Color online) The triangle-honeycomb lattice in Fig. 1(a) is deformed to a topologically equivalent lattice, so as to label each site by row and column indices. In each unit cell, the three sites denoted by 3, 4, and 5 have row and column indices \( m \) and \( n \) with \( m + n \) equal to an even integer, and the other three sites denoted by 1, 2, and 6 have row and column indices with \( m + n \) equal to an odd integer.

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**Appendix A**

1. **Derivation of the ground state in the uniform-flux sector**

The Kitaev spin model on a triangle-honeycomb lattice is schematically shown in Fig. 5 and the model Hamiltonian is given by

\[
H = J_x \sum_{x\text{-link}} \sigma_i^x \sigma_j^x + J_y \sum_{y\text{-link}} \sigma_i^y \sigma_j^y + J_z \sum_{z\text{-link}} \sigma_i^z \sigma_j^z
\]

\[+ J_{x'} \sum_{x'\text{-link}} \sigma_i^x \sigma_j^x + J_{y'} \sum_{y'\text{-link}} \sigma_i^y \sigma_j^y + J_{z'} \sum_{z'\text{-link}} \sigma_i^z \sigma_j^z.\]  

(A1)

To use the Jordan-Wigner transformation, we need to label the sites by row and column indices. We deform the honeycomb lattice [see Fig 1(a)] into the topologically equivalent one in Fig. 5. Now each site can be labeled by \((m, n)\), where \( m = 1, 2, \cdots, M \) and \( n = 1, 2, \cdots, N \) are the row and column indices, respectively.

By performing the Jordan-Wigner transformation\(^{18}\)

\[
\sigma_{m,n}^+ = 2a_{m,n}^\dagger \prod_{n' = 1}^{N} \prod_{m < m'} \sigma_{m',n'}^z \prod_{n'' < n} \sigma_{m,n''}^z
\]  

(A2)

on each spin, and using the Majorana fermions

\[
\epsilon_{m,n}^{(s)} = i(a_{m,n}^\dagger - a_{m,n}), \quad d_{m,n}^{(s)} = a_{m,n}^\dagger + a_{m,n}
\]  

(A3)

for sites \( s = 1, 2, 3 \), and

\[
\epsilon_{m,n}^{(s)} = a_{m,n}^\dagger + a_{m,n}, \quad d_{m,n}^{(s)} = i(a_{m,n}^\dagger - a_{m,n}) \]  

(A4)

for sites \( s = 1, 2, 3 \) in each unit cell [see Fig. 5], the Hamiltonian (A1) is converted to

\[
H = iJ_x \left[ \sum_{x\text{-link}(\triangle)} c_{m,n}^{(3)} c_{m,n+1}^{(2)} + \sum_{x\text{-link}(\triangledown)} c_{m,n}^{(5)} c_{m,n+1}^{(6)} \right]
\]

\[+ iJ_y \left[ \sum_{y\text{-link}(\triangle)} c_{m,n}^{(1)} c_{m,n+1}^{(3)} + \sum_{y\text{-link}(\triangledown)} c_{m,n}^{(6)} c_{m,n+1}^{(4)} \right]
\]

\[+ iJ_x' \sum_{x'\text{-link}, n \neq N} c_{m,n}^{(4)} c_{m,n-1}^{(1)}
\]

\[- iJ_y' \sum_{y'\text{-link}, n \neq N} c_{m,n}^{(2)} c_{m,n+1}^{(5)}, \]

\[
\sum_{M/2} \Phi_{x,2m} \sigma_{2m,N}^z \sum_{M/2} \Phi_{x,2m} \sigma_{2m,N}^z
\]

where \( \Phi_{x,m} = \prod_{n=1}^{N} \sigma_{m,n}^z, m = 1, 2, \cdots, M \), is the global flux operator calculated along a given contour encircling the system in the horizontal direction, \( \alpha\text{-link}(\triangle) \) denotes the \( \alpha\text{-link} \) in an up-pointing triangle, and \( \alpha\text{-link}(\triangledown) \) the \( \alpha\text{-link} \) in a down-pointing triangle.

The global flux operator calculated along a given contour encircling the system in the vertical direction, which does not appear in the Hamiltonian (A5), is given by

\[
\Phi_{y,n} = \prod_{m=0}^{M/2-1} \sigma_{1+2m,n}^\dagger \sigma_{2+2m,n}^\dagger \sigma_{2+2m,n+1}^z \sigma_{2+2m,n+2}^z \sigma_{3+2m,n+1}^z \sigma_{3+2m,n+2}^z \sigma_{3+2m,n+3}^z \sigma_{3+2m,n+4}^z
\]

\[(A6)\]

\[
\otimes \sigma_{2+2m,n+3}^z \sigma_{2+2m,n+4}^z \sigma_{2+2m,n+5}^z \sigma_{2+2m,n+6}^z
\]

\[
\otimes \sigma_{3+2m,n+6}^z \sigma_{3+2m,n+7}^z \sigma_{3+2m,n+8}^z \sigma_{3+2m,n+9}^z
\]

\[
\otimes \sigma_{3+2m,n+10}^z \sigma_{3+2m,n+11}^z \sigma_{3+2m,n+12}^z \sigma_{3+2m,n+13}^z
\]

\[
\otimes \sigma_{3+2m,n+14}^z \sigma_{3+2m,n+15}^z \sigma_{3+2m,n+16}^z \sigma_{3+2m,n+17}^z
\]

\[
\otimes \sigma_{3+2m,n+18}^z \sigma_{3+2m,n+19}^z \sigma_{3+2m,n+20}^z \sigma_{3+2m,n+21}^z
\]

\[
\otimes \sigma_{3+2m,n+22}^z \sigma_{3+2m,n+23}^z
\]

where \( n = 1 \) or \( 6 \), with \( r = 0, 1, \cdots, N/6 - 1 \). These two global flux operators \( \Phi_{x,m} \) and \( \Phi_{y,n} \) commute with Hamiltonian (A1). This leads to a topological degeneracy for the ground state because \( \Phi_{x,m} \) and \( \Phi_{y,n} \) can have eigenvalues either 1 or \(-1\). For the Abelian phase, this degeneracy is 4-fold because \( (\Phi_{x,m}, \Phi_{y,n}) \) can be \((1,1), (1,-1), (-1,1), \) and \((-1,-1)\), while the degeneracy is 3-fold for the non-Abelian phase since \( (\Phi_{x,m}, \Phi_{y,n}) = (-1,-1) \) is not allowed. In addition to the global flux operators, the three types of local plaquette operators \( P_0, P_1, \) and \( P_2 \) also commute with Hamiltonian (A1).
Using the Jordan-Wigner transform, they can be written as

\[ P_0(m, n) = \sum_{\Delta} c_{m,n}^d d_{m+1,n}^d d_{m,n+1}^d + \sum_{\nabla} c_{m,n}^c m_{m,n+1} c_{m,n+1}^c \]

where \((m, n), (m', n'), (m'', n'')\) correspond, respectively, to the sites denoted by 3, 5, and 1 in each unit cell. As shown in Ref. 5, the ground state lies in the sector of the Hilbert space where either (i) \(P_0(m, n) = 1\) and \(P_1(m', n') = P_2(m'', n'') = 1\) or (ii) \(P_0(m, n) = 1\) and \(P_1(m', n') = P_2(m'', n'') = -1\). In each of these two sectors, the time reversal symmetry of Hamiltonian \(A1\) is broken. The former is called as the uniform-flux sector. This means that the ground state has another 2-fold degeneracy due to this broken time reversal symmetry, in addition to the topological degeneracy discussed above. Thus, we have 8-fold (6-fold) degeneracy for the ground state in the Abelian (non-Abelian) phase, corresponding to 8 (6) flux configurations for \(\Phi_{x,m}, \Phi_{y,n}, P_0, P_1, P_2\).

Here we focus on the ground state in the uniform-flux sector with \((\Phi_{x,m}, \Phi_{y,n}) = (1, 1)\), which allow both Abelian and non-Abelian phases. Note that only one ground state exists in this sector. Now, Hamiltonian \(A5\) is reduced to

\[ H_u = iJ_x \sum_{x-link(\Delta)} c_{m,n}^d c_{m+1,n}^d + \sum_{z-link(\nabla)} c_{m,n}^c c_{m,n+1}^c \]

\[ -iJ_y \sum_{y-link(\Delta)} c_{m,n}^d c_{m+1,n}^d + \sum_{y-link(\nabla)} c_{m,n}^c c_{m,n+1}^c \]

\[ +iJ_z \sum_{z-link(\Delta)} c_{m,n}^d c_{m+1,n}^d + \sum_{z-link(\nabla)} c_{m,n}^c c_{m,n+1}^c \]

\[ = H_k \begin{pmatrix}
0 & iJ_z & -iJ_y & -iJ_x e^{ik \cdot e_2} & 0 
-iJ_z & 0 & -iJ_x & 0 & -iJ_y e^{-ik \cdot e_1} 
iJ_y & iJ_z & 0 & 0 & 0 
iJ_x e^{-ik \cdot e_2} & 0 & 0 & iJ_y & 0 
iJ_x e^{ik \cdot e_1} & 0 & -iJ_z & 0 & iJ_y 
\end{pmatrix} \]

where \(c_{s}^d = c_{s}^d\) and the two basis vectors of the unit cell are \(e_1 = e_x/2 - \sqrt{3}e_y/2\) and \(e_2 = e_x/2 + \sqrt{3}e_y/2\). To obtain the quasi-particle spectrum, we write Eq. \(A11\) as

\[ H_u = \sum_{k \in BZ} \Phi_k^d H_k \Phi_k^c \]

where \(\Phi_k^d = \left( c_k^d, c_k^{d(2)} c_k^{c(3)} c_k^{c(4)} c_k^{c(5)} c_k^{c(6)} \right)\), and

\[ e_k^6 - a e_k^4 + b e_k^2 - c = 0 \]

where

\[ a = 2(J_x^2 + J_y^2 + J_z^2) + J_x^2 + J_y^2 + J_z^2 \]

\[ b = 2(J_x^2 J_y^2 + J_y^2 J_z^2 + J_z^2 J_x^2 + J_x^2 J_z^2 + J_y^2 J_z^2 + J_z^2 J_x^2) \]

\[ c = J_x^2 J_y^2 + J_y^2 J_z^2 + J_z^2 J_x^2 + J_x^2 J_y^2 + J_y^2 J_z^2 + J_z^2 J_x^2 - 2J_x J_y J_z \cos k_x \]

\[ -2J_x J_y J_z \cos k_x \]

\[ \Phi_k^c = \Phi_k^d \]

\[ +J_x^2 J_y^2 + J_y^2 J_z^2 + J_z^2 J_x^2 + J_x^2 J_y^2 + J_y^2 J_z^2 \]

\[ -2J_x J_y J_z \cos k_x \]
\[-2J_z'J_x' (J_x'^2 - J_x^2) \cos k_1, \quad \text{(A15)}\]

with

\[
k_1 = \frac{k_x - \sqrt{3}k_y}{2}, \quad k_2 = \frac{k_x + \sqrt{3}k_y}{2}.
\]

The solution of Eq. (A14) reads

\[
\begin{align*}
\varepsilon_k^{(1)} &= -\varepsilon_k^{(6)} = -\sqrt{\frac{a}{3} + 2p \cos \varphi}, \\
\varepsilon_k^{(2)} &= -\varepsilon_k^{(5)} = -\sqrt{\frac{a}{3} - p} \left( \cos \varphi - \sqrt{3} \sin \varphi \right), \\
\varepsilon_k^{(3)} &= -\varepsilon_k^{(4)} = -\sqrt{\frac{a}{3} - p} \left( \cos \varphi + \sqrt{3} \sin \varphi \right),
\end{align*}
\]

(17)

where

\[
\varphi = \frac{1}{3} \arccos \left( \frac{q}{2p^3} \right), \quad 0 \leq \varphi \leq \frac{\pi}{3}
\]

(18)

\[
p = \frac{1}{3} \sqrt{a^2 - 3b}, \quad q = \frac{2a^3}{27} - \frac{ab}{3} + c.
\]

With the six energy bands in Eq. (A17), the Hamiltonian (A19) can be written as

\[
H_u = \sum_{j=1}^{6} \varepsilon_k^{(j)} \left[ 2A_k^{(j)*}A_k^{(j)} - 1 \right],
\]

(19)

where \(A_k^{(j)*} = \sum_{s=1}^{6} c_k^{(s)^*} w_s(c_k^{(j)}) \) is a fermionic operator, with \( c_k^{(s)} \) the Fourier transform of the Majorana fermionic operator at site \( s \) (\( s = 1, 2, \ldots, 6 \)) of the unit cell, and

\[
w_s(c_k^{(j)}) = \frac{w_s' (\varepsilon_k^{(j)})}{\sqrt{\sum_{i=1}^{6} w_s' (\varepsilon_k^{(j)})^2}}.
\]

(20)

In Eq. (A20),

\[
w_s' = \frac{iJ_x}{W} \left[ (\varepsilon_k^{(j)})^2 - J_x^2 \right] e^{i\phi} w_s' + \frac{iJ_y}{W} \left( iJ_z c_k^{(j)} + J_z c_k^{(j)} \right) w_s', \]

\[
w_2 = \frac{iJ_z}{W} \left[ J_x J_y - iJ_z c_k^{(j)} \right] e^{i\phi} w_s' + \frac{iJ_y}{W} \left( iJ_z c_k^{(j)} + J_z c_k^{(j)} \right) w_s',
\]

\[
w_3 = \frac{iJ_x}{W} \left[ J_x J_z + iJ_z c_k^{(j)} \right] e^{i\phi} w_s' + \frac{iJ_y}{W} \left( iJ_z c_k^{(j)} + J_z c_k^{(j)} \right) w_s',
\]

\[
w_4 = \frac{iJ_x}{W} \left[ J_x J_y + iJ_z c_k^{(j)} \right] e^{i\phi} w_s' + \frac{iJ_y}{W} \left( iJ_z c_k^{(j)} + J_z c_k^{(j)} \right) w_s',
\]

\[
w_5 = \frac{iJ_y J'_x J'_y - \varepsilon_k^{(j)} J'_x J'_z}{\varepsilon_k^{(j)} W + (\varepsilon_k^{(j)})^2 / 2} w_s', \quad w_6 = \frac{iJ_y - iJ_z}{\varepsilon_k^{(j)} W + (\varepsilon_k^{(j)})^2 / 2} w_s'.
\]

(21)

In order to find the ground-state wavefunction for Hamiltonian (A19), we should first define the vacuum \(|0\rangle\) by

\[
a_{m,n}|0\rangle = 0,
\]

(23)

where \( a_{m,n} \) is the fermionic operator defined in Eq. (A22), and the index \((m,n)\) runs over every site of the lattice. From Eqs. (A3) and (A4), we have

\[
\langle 0 | c_k^{(s)*} c_k^{(s')} \rangle = \frac{1}{2} \delta_{k',-k} \delta_{s',s}.
\]

(24)

Through Fourier transform, it leads to

\[
\langle 0 | \sqrt{2} A_k^{(j)*} | 0 \rangle = \frac{1}{2} \delta_{k',-k} \delta_{s',s}.
\]

(25)

For state \(|j,k\rangle = \sqrt{2} A_k^{(j)*} |0\rangle\) in the occupied band, it is normalized as

\[
\langle j,k | j,k \rangle = \langle 0 | \sqrt{2} A_k^{(j)*} | 0 \rangle = \frac{1}{2} \sum_{s,s'=1}^{6} w_s (c_k^{(j)}) w_s' (c_k^{(j)}) \cdot \frac{1}{2} \delta_{s',s} = 1,
\]

(26)

which clarifies that the normalization factor \(\sqrt{2}\) is due to the fact that the occupation number can only take \(1/2\) for the Fourier transform of each Majorana fermion [see Eq. (A25)]. Hence we can write the ground-state wavefunction as

\[
|g\rangle = \prod_{k \in BZ} N k = 1 \prod_{k \in BZ} \sqrt{2} A_k^{(j)*} |0\rangle.
\]

(27)
The corresponding ground-state energy per site is
\[ E_g = \frac{1}{6M} \sum_{\mathbf{k} \in \text{BZ}} \left( \varepsilon_k^{(1)} + \varepsilon_k^{(2)} + \varepsilon_k^{(3)} \right) \]
\[ = \frac{\sqrt{3}}{48\pi^2} \int_{\text{BZ}} d^2k \left( \varepsilon_k^{(1)} + \varepsilon_k^{(2)} + \varepsilon_k^{(3)} \right). \]  
(A28)

2. Two different ground states in the same topological class

Below we show that the quantum phase transition (QPT) between two phases belonging to the same topological class is nontrivial. For example, in the case of the jumps \(J_x = J_y, J'_x = J'_y\), and \(J_z' = J_z'J_x\), the QPT that occurs when \(\Lambda_1 \equiv J_x'/J_x\) varies across the critical line \(\Lambda_1 = 0\) does not change the topological class of the ground state, i.e., the QPT occurs between either two non-Abelian phases with Chern number \(\nu = \pm 1\) or two Abelian phases with \(\nu = 0\). By using the parameters \(\Lambda_1 \equiv J_x'/J_x\) and \(\Lambda_2 \equiv J_z/J_x\), Eq. (A21) can be reduced to

\[ w'_1 = \frac{i\Lambda_1}{W'} \left\{ \left( \varepsilon_k^{(j)} \right)^2 - 1 \right\} e^{i\varepsilon_k^{(j)}} w_4 + \left( i\Lambda_2 \varepsilon_k^{(j)} + 1 \right) e^{-i\varepsilon_k^{(j)}} w'_5 \]
\[ + \Lambda_2 \left( \varepsilon_k^{(j)} - 1 \right) w_6, \]
\[ w'_2 = \frac{i\Lambda_1}{W'} \left\{ \left( \varepsilon_k^{(j)} \right)^2 - 1 \right\} e^{i\varepsilon_k^{(j)}} w_3 + \left( i\Lambda_2 \varepsilon_k^{(j)} + 1 \right) e^{-i\varepsilon_k^{(j)}} w'_5 \]
\[ + \Lambda_2 \left( \varepsilon_k^{(j)} - 1 \right) w_6, \]
\[ w'_3 = \frac{i\Lambda_1}{W'} \left\{ \left( \varepsilon_k^{(j)} \right)^2 - 1 \right\} e^{i\varepsilon_k^{(j)}} w_4 + \left( i\Lambda_2 \varepsilon_k^{(j)} + 1 \right) e^{-i\varepsilon_k^{(j)}} w'_5 \]
\[ + \Lambda_2 \left( \varepsilon_k^{(j)} - 1 \right) w_6, \]
\[ w'_4 = \frac{i\Lambda_2}{W'} \left\{ \left( \varepsilon_k^{(j)} \right)^2 - 1 \right\} e^{i\varepsilon_k^{(j)}} w_5 \]
\[ + \Lambda_2 \left( \varepsilon_k^{(j)} - 1 \right) w_6, \]
\[ w'_5 = \frac{j\Lambda_1}{W'} \left\{ \left( \varepsilon_k^{(j)} \right)^2 - 1 \right\} e^{i\varepsilon_k^{(j)}} w_4 + \left( i\Lambda_2 \varepsilon_k^{(j)} + 1 \right) e^{-i\varepsilon_k^{(j)}} w'_5 \]
\[ + \Lambda_2 \left( \varepsilon_k^{(j)} - 1 \right) w_6, \]
\[ w'_6 = \frac{j\Lambda_1}{W'} \left\{ \left( \varepsilon_k^{(j)} \right)^2 - 1 \right\} e^{i\varepsilon_k^{(j)}} w_5 + \left( i\Lambda_2 \varepsilon_k^{(j)} + 1 \right) e^{-i\varepsilon_k^{(j)}} w'_5 \]
\[ + \Lambda_2 \left( \varepsilon_k^{(j)} - 1 \right) w_6. \]

From Eq. (A31), it follows that
\[ |g(-\Lambda_1, \Lambda_2)\rangle = \prod_{k \in \text{BZ}} F \left( -\varepsilon_k^{(1)}, -\varepsilon_k^{(2)}, -\varepsilon_k^{(3)}, -\varepsilon_k^{(4)}, -\varepsilon_k^{(5)}, -\varepsilon_k^{(6)} \right) |0\rangle. \]  
(A33)

Equations (A32) and (A33) show that the ground state \(|g(\Lambda_1, \Lambda_2)\rangle\) is mapped to \(|g(-\Lambda_1, \Lambda_2)\rangle\) when \(\varepsilon_k^{(s)}\), \(s = 1, 2,\) and 3, are changed to \(-\varepsilon_k^{(s)}\). Because \(\varepsilon_k^{(s)}\), \(s = 1, 2,\) and 3, are the Fourier transform of the Majorana fermions \(c_k^{(s)}\) defined on sites 1, 2, and 3 in each unit cell (i.e., all the triangles related to plaquette operator \(P^{(2)}\)). The transformation \(c_k^{(s)} \rightarrow -c_k^{(s)}\) corresponds to a \(\pi\) phase change to each of the Majorana fermions \(c_k^{(s)}\) at sites \(s = 1, 2,\) and 3. Note that the two ground states \(|g(\Lambda_1, \Lambda_2)\rangle\) and \(|g(-\Lambda_1, \Lambda_2)\rangle\) are different [see Eqs. (A32) and (A33)], although the energy spectrum of the Hamiltonian \(H(\Lambda_1, \Lambda_2)\) is symmetric about the critical line \(\Lambda_1 = 0\).

Appendix B: Effective Hamiltonian

In order to show that both the QPTs between two Abelian phases and those between two non-Abelian phases are nontrivial, we use the perturbation method in Ref. 3 to derive the effective Hamiltonian at \(\Lambda_1 \approx 0\). In this case with \(J_x = J_y, J'_x = J'_y,\) and \(J_z' = J_z'J_x\), we characterize the phase diagram by the two parameters \(\Lambda_1 \equiv J_x'/J_x,\) and \(\Lambda_2 \equiv J_z/J_x\). We assume \(J_x > 0\) and take it as the unit of energy.

The Hamiltonian of the system can be written as
\[ H = H_0 + V, \]  
(B1)

with
\[ H_0 = \sum_{\text{x-link}} \sigma_z^{x} \sigma_z^{y} + \sum_{\text{y-link}} \sigma_z^{y} \sigma_z^{y} + \sum_{\text{z-link}} \sigma_z^{y} \sigma_z^{z}, \]
\[ V = \Lambda_1 \sum_{\text{x'-link}} \sigma_z^{x} \sigma_z^{y} + \Lambda_1 \sum_{\text{y'-link}} \sigma_z^{y} \sigma_z^{y} + \Lambda_1 \sum_{\text{z'-link}} \sigma_z^{y} \sigma_z^{z}, \]  
(B2)

where \(V\) is a perturbation.
As in Ref. 3, we calculate the effective Hamiltonian as

\[ H_{\text{eff}} = \mathcal{Y}^\dagger (V + VG_0'V + VG_0'VG_0'V + \cdots) \mathcal{Y} \]
\[ = H^{(1)} + H^{(2)} + H^{(3)} + \cdots, \quad (B3) \]

where \( \mathcal{Y} \) maps the effective Hilbert space onto the ground-state subspace of \( H_0 \) and \( G_0' \) is the Green function for excited states of \( H_0 \). The ground state of \( H_0 \) is a direct-product state consisting of the ground states of the following Hamiltonians:

\[ H_{p_1} = \sigma_1^x \sigma_3^x + \sigma_2^y \sigma_3^y + \Lambda_2 \sigma_1^z \sigma_2^z, \]
\[ H_{p_2} = \sigma_2^x \sigma_3^x + \sigma_1^y \sigma_3^y + \Lambda_2 \sigma_1^z \sigma_2^z, \quad (B4) \]

defined on down-pointing and up-pointing triangles [see Fig. 1(b)].

In Eq. (B3), \( H^{(1)} = \mathcal{Y}^\dagger V \mathcal{Y} \) is obtained as

\[ H^{(1)} = \Lambda_1 \left( \sum_{x'-\text{link}} \sigma_i^x \sigma_j^x + \sum_{y'-\text{link}} \sigma_i^y \sigma_j^y + \Lambda_2 \sum_{z'-\text{link}} \sigma_i^z \sigma_j^z \right). \quad (B5) \]

As in the perturbation method used for the Kitaev model on a honeycomb lattice in the presence of a weak magnetic field, it is interesting to note that though \( H^{(2)} \) involves even number of spin operators, it is not invariant under a time reversal transformation in the uniform-flux sector. Take one sum in the second line of Eq. (B6) as an example:

\[ H^{(2)}_p = \frac{\Lambda_1^2}{\Lambda_2^2} \sum_{(j,k) = z'-\text{link}} (\sigma_j^x \sigma_k^y \sigma_j^z) \quad (B6) \]

It is interesting to note that though \( H^{(2)} \) involves even number of spin operators, it is not invariant under a time reversal transformation in the uniform-flux sector. Take one sum in the second line of Eq. (B6) as an example:

\[ H^{(2)}_p = \frac{\Lambda_1^2}{\Lambda_2^2} \sum_{(j,k) = z'-\text{link}} (\sigma_j^x \sigma_k^y \sigma_j^z) \quad (B7) \]

where \((j,k,m)\) denotes the three sites in either a down-pointing or up-pointing triangle, which is labeled by either \((2,1,3)\) or \((1,3,2)\) in Fig. 1(b). Namely, \( \sigma_1^x \sigma_2^y \sigma_3^z \) is either the plaquette operator \( P_1 \) or \( P_2 \). Since the eigenvalues of \( P_1 \) and \( P_2 \) are both chosen to be 1 in the uniform-flux sector, \( H^{(2)}_p \) can be reduced to involve only three spin operators in this sector and the time reversal symmetry is now broken. Thus, in the uniform-flux sector, one has

\[ TH^{(2)}_p T^{-1} \neq H^{(2)}_p, \]

where \( T \) is the time reversal transformation. Similarly, \( H^{(3)} = \mathcal{Y}^\dagger VG_0'VG_0'V \mathcal{Y} \) can be obtained as

\[ H^{(3)} = \frac{\Lambda_1^3 \Lambda_2^3}{\Lambda_2^2} \sum_{(i_1,i_2,i_3) \neq \Delta, \nabla} \sigma_{i_1}^x \sigma_{i_2}^y \sigma_{i_3}^y \sigma_{i_4}^z \sigma_{i_5}^z \]
\[ + H^{(3)}_p. \quad (B9) \]

Here \((i_1,j_1)\) are the two sites connected by an \( x'-\text{link}, \)
\((i_2,j_2)\) are the two sites connected by a \( y'-\text{link}, \)
\((i_3,j_3)\) are the two sites connected by a \( z'-\text{link}. \)

\[ H^{(3)}_p = \frac{6 \Lambda_1^3}{\Lambda_2^2} \sum \rho_{i,j} (m,n) \sigma_{m,n-1}^y \sigma_{m,n+3}^y \sigma_{m-1,n+1}^y \]
\[ + \frac{6 \Lambda_1^3}{\Lambda_2^2} \sum \Delta (m,n) \sigma_{m,n-1}^y \sigma_{m,n+3}^y \sigma_{m+1,n+1}^y, \quad (B10) \]

where \( \nabla(\Delta) \) runs over all down-pointing (up-pointing) triangles with \((m,n)\) denoting the site 5 (1) in each unit cell (see Fig. 5). By summing over \( H^{(1)}, H^{(2)} \) and \( H^{(3)} \), the effective Hamiltonian, up to the third order, is given by

\[ H_{\text{eff}} = H_0 + \frac{6 \Lambda_1^3}{\Lambda_2^2} \sum \rho_{i,j} (m,n) \sigma_{m,n-1}^y \sigma_{m,n+3}^y \sigma_{m-1,n+1}^y \]
\[ + \frac{6 \Lambda_1^3}{\Lambda_2^2} \sum \Delta (m,n) \sigma_{m,n-1}^y \sigma_{m,n+3}^y \sigma_{m+1,n+1}^y, \quad (B11) \]

where \( H_0^{(3)} \) is a term involving none of the plaquette operators \( P_i, i = 0, 1 \) and 2.

For the QPT between two phases of the same Chern number, the parameter \( \Lambda_1 \) changes its sign at the transition point. This yields the change of signs in certain terms in the effective Hamiltonian. Therefore, the effective Hamiltonian \( H_{\text{eff}} \) is different at the two sides of the transition point \( \Lambda_1 = 0 \).
It is shown in Ref. 5 that when both $J_\alpha$ and $J'_\alpha$ are positive, the ground state is either in the uniform-flux sector or in the sector related by the time reversal transformation. Based on this, one can easily prove that this applies to the cases with arbitrary $J_\alpha$ and $J'_\alpha$. 

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