Quantum Integrability of Coupled N=1 Super Sine/Sinh-Gordon Theories and the Lie Superalgebra D(2, 1; α).

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Abstract

We discuss certain integrable quantum field theories in (1+1)-dimensions consisting of coupled sine/sinh-Gordon theories with \( N = 1 \) supersymmetry, positive kinetic energy, and bosonic potentials which are bounded from below. We show that theories of this type can be constructed as Toda models based on the exceptional affine Lie superalgebra \( D(2, 1; \alpha) \) (or on related algebras which can be obtained as various limits) provided one adopts appropriate reality conditions for the fields. In particular, there is a continuous family of such models in which the couplings and mass ratios all depend on the parameter \( \alpha \). The structure of these models is analyzed in some detail at the classical level, including the construction of conserved currents with spins up to 4. We then show that these currents generalize to the quantum theory, thus demonstrating quantum-integrability of the models.

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# 1 Introduction

Of the many integrable quantum field theories in (1+1)-dimensions, perhaps the most famous is the sine-Gordon model: a single scalar field $\phi$ of mass $m$ with lagrangian

$$L_{\sin}(\phi; m) = \partial_+ \phi \partial_- \phi + \frac{m^2}{2} \cos 2\phi.$$  

(1.1)

A close relative which is slightly less well-known is the sinh-Gordon model, with lagrangian

$$L_{\sinh}(\phi; m) = \partial_+ \phi \partial_- \phi - \frac{m^2}{2} \cosh 2\phi.$$  

(1.2)

The two are related at a formal level by

$$L_{\sin}(\phi; m) = -L_{\sinh}(i\phi; m)$$  

(1.3)

but despite this they have quite different properties. The sinh-Gordon theory is actually significantly simpler, involving a single species of particle created by the field $\phi$ whose quantum scattering is given by an elastic S-matrix. The sine-Gordon model, by contrast, has a potential with degenerate vacua and hence solitons which interpolate between these. The ‘fundamental’ meson corresponding to $\phi$ now emerges as a bound state which may even disappear from the spectrum completely for certain values of the coupling (we have suppressed the dimensionless coupling constants above).

The sine/sinh-Gordon models are merely the simplest members of a general class of massive integrable field theories, known as Toda theories, which can be constructed from affine Lie algebras. The simple roots can be used to define exponential interactions between scalar fields living in the Cartan subalgebra so as to ensure the existence of a Lax pair which then guarantees classical integrability. The reality properties of the fields can be changed by a trick analogous to (1.3) whenever there is a reflection symmetry of the affine Dynkin diagram, with different choices of reality conditions essentially corresponding to different real forms of a given complex algebra. In the simplest cases above, the algebra is of complex type $A_1^{(1)}$, with the sine-Gordon and sinh-Gordon theories corresponding to the real forms $su(2)^{(1)}$ and $sl(2)^{(1)}$ respectively.

Supersymmetric versions of (1.1) and (1.2) are readily constructed. The $N = 1$ sine-Gordon model can be defined in superspace by a lagrangian

$$\mathcal{L}_{\sin}(\Phi; m) = \frac{i}{2} D_+ \Phi D_- \Phi + m \cos \Phi$$  

(1.4)

where $\Phi$ is a real superfield. Similarly, the $N = 1$ sinh-Gordon theory is

$$\mathcal{L}_{\sinh}(\Phi; m) = \frac{i}{2} D_+ \Phi D_- \Phi - m \cosh \Phi$$  

(1.5)

It is not surprising that these generalizations exist, since any theory of a single scalar field with a bounded potential admits a supersymmetric extension. But for other Toda models, with more than one field, the situation is rather subtle. The most systematic approach is

\footnote{Our superspace conventions will be discussed in detail in section 3; the particular numerical coefficients appearing in the bosonic and superspace lagrangians are consistent with the expansion of superfields in components that we shall consider later.}
to generalize the Lax-pair construction, replacing the underlying affine Lie algebra with an affine Lie superalgebra (see [1] for a review). Even then the theory is supersymmetric only if the system of simple roots used is totally fermionic. The bosonic sector of such a model does not consist simply of a single bosonic Toda theory, but rather of a number of different bosonic Toda theories which are related to the choice of superalgebra simple roots. In other words, it is only these specific combinations of bosonic Toda models which admit supersymmetric extensions.\(^2\)

Just as in the bosonic situation, the superspace lagrangians (1.4) and (1.3) are related by

\[ \mathcal{L}_{\text{sin}}(\Phi;m) = -\mathcal{L}_{\text{sinh}}(i\Phi;m) \]  

(1.6)

but the idea of changing reality conditions in this way actually takes on a new significance for superalgebras. For bosonic Toda models, a maximally non-compact real algebra defines a theory with positive-definite kinetic energy, while other real forms produce theories with indefinite kinetic energy whose physical status is therefore less clear. The single exception to this is the sine/sinh-Gordon correspondence above, where the new lagrangian obtained from the substitution \( \phi \rightarrow i\phi \) happens to be negative-definite, rather than indefinite, so that it is still physically sensible. For superalgebra Toda models, however, there are a number of more complicated ways to generalize the behaviour (1.6) and still arrive at physically sensible theories. In a sense, the bosonic situation is reversed, because the maximally non-compact form of a Lie superalgebra typically gives rise to a Toda model with indefinite kinetic energy, which can then be rendered positive-(or negative-)definite in a limited number of cases by an appropriate change of reality condition.

The first systematic analysis of affine superalgebra theories along these lines was recently carried out in [1], to which we refer the reader for additional background, including more detailed references to earlier work. A number of cases emerged as being of special interest, and it is our aim in this paper to study the most novel and important of these, corresponding to the exceptional Lie superalgebra \( D(2,1;\alpha)^{(1)} \) (the others can actually be recovered in various limits). This gives a one-parameter family of massive, supersymmetric theories with positive kinetic energy, whose bosonic sectors consist of two sine-Gordon models and one sinh-Gordon model, with relations between the masses and couplings which are fixed by the value of the parameter \( \alpha \). It is certainly not obvious that these particular combinations of sine/sinh-Gordon theories can be made to interact in a supersymmetric fashion while maintaining classical integrability, but all this is guaranteed by the superalgebra construction. We shall show at the end of the paper that integrability of this family of models holds quantum-mechanically too.

We give a brief introduction to the Lie superalgebra \( D(2,1;\alpha) \) and its affine extension in section 2, which is sufficient to allow us to write down the classical theory. Thereafter we shall study the model, including its integrability properties, directly from its lagrangian. In section 3 we analyze the classical theory in some detail, describing how to ‘twist’ it along the lines of (1.3) and (1.6) to ensure that it has positive kinetic energy, and showing that the resulting bosonic sub-sector of the model is a sum of two sine-Gordon theories and one sinh-Gordon theory. We find the classical masses and three-point functions of the various particles, and describe the solitonic solutions to the equations of motion. In section 4 we discuss other models which can be obtained from the \( D(2,1;\alpha)^{(1)} \) model in various limits. In section 5 we construct explicitly all classically conserved currents of (super)spins up

\(^2\) It is, however, possible to construct more general theories with non-linearly realized supersymmetry, see [2].
to $7/2$ in the $D(2,1;\alpha)^{(1)}$ family, and in section 4 we show that these currents survive quantization, with suitable modifications which we calculate, thus demonstrating quantum integrability of the theory. Section 7 summarizes our results and discusses some projects for future study.

2 The Exceptional Lie Superalgebra $D(2,1;\alpha)$

We recall that a Lie superalgebra $G$ is a $\mathbb{Z}_2$-graded algebra $G = G_0 \oplus G_1$ equipped with a graded commutator satisfying the graded Jacobi-identity. The even subspace $G_0$ is a Lie algebra (non-semisimple in general), while the odd subspace $G_1$ forms a representation of $G_0$. The basic (or contragredient) Lie superalgebras can be characterized by a Cartan matrix $c_{ij}$ corresponding to a set of simple roots $\alpha_i$. A root $\alpha_i$ is called bosonic (fermionic) if the corresponding generator $E_{\alpha_i}$ belongs to the even (odd) subspace. The basic Lie superalgebras are equipped with an invariant bilinear form which induces an inner-product on the root-space which is generally of indefinite signature. Bosonic roots can thus have positive or negative length squared, while fermionic roots can in addition have zero length squared. In the Dynkin diagram for a Lie superalgebra $G$ the bosonic roots are represented by “white” nodes ◦, null fermionic roots (fermionic roots with zero length) are represented by “grey” nodes ⊗, and fermionic roots with non-zero length are represented by “black” nodes •. Note that, in contrast to the case of Lie algebras, for Lie superalgebras the Cartan matrix is not uniquely defined; there are in general several possible inequivalent choices of simple roots. For more information on Lie superalgebras see e.g. [1, 3].

The exceptional basic Lie superalgebra $D(2,1;\alpha)$ is a deformation of the Lie superalgebra $D(2,1) = osp(4,2)$, with a continuous deformation parameter $\alpha$; $D(2,1)$ corresponds to taking $\alpha = 1$ (or $\alpha = -2$, or $\alpha = -\frac{1}{2}$, as we will see shortly). In general $\alpha$ takes values in $\mathbb{C} \setminus \{-1,0\}$, but we will consider only $\alpha \in \mathbb{R} \setminus \{-1,0\}$, since only in this case do we get a real inner-product on the root-space. The even (bosonic) subalgebra of $D(2,1;\alpha)$ is $G_0 = A_1 \oplus A_1 \oplus A_1$, while the odd subspace transforms as the $(2,2,2)$ representation of $G_0$.

The simple root system of $D(2,1;\alpha)$ which will be relevant for us consists of fermionic roots $\alpha_i$ ($i = 1,2,3$), spanning a three dimensional vector-space $V$. The simple roots are all null, $\alpha_i^2 = 0$, but they have non-vanishing inner-products with each other:

$$\alpha_1 \cdot \alpha_2 = \alpha; \quad \alpha_1 \cdot \alpha_3 = 1; \quad \alpha_2 \cdot \alpha_3 = -1 - \alpha \quad (2.7)$$

A convenient description can be given using orthogonal basis vectors $\varepsilon_i$ ($i = 1,2,3$) for $V$, normalized so that

$$\varepsilon_1^2 = -\frac{1 + \alpha}{2}; \quad \varepsilon_2^2 = \frac{1}{2}; \quad \varepsilon_3^2 = \frac{\alpha}{2}. \quad (2.8)$$

In terms of these we have

$$\alpha_1 = -\varepsilon_1 + \varepsilon_2 + \varepsilon_3; \quad \alpha_2 = \varepsilon_1 - \varepsilon_2 + \varepsilon_3; \quad \alpha_3 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3. \quad (2.9)$$

The full root system of $D(2,1;\alpha)$ contains fermionic elements $\{\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}$ (all combinations of signs allowed) and bosonic elements $\{\pm 2\varepsilon_i\}$. The Cartan matrix and Dynkin diagram are given by:
Note that there is no way to normalize the Cartan matrix such that the entries are integers. In order to draw the Dynkin diagram we therefore cannot follow the usual procedure, to draw \( n_{ij} \) “bonds” between nodes \( i \) and \( j \), where \( n_{ij} = \max(|c_{ij}|, |c_{ji}|) \). Instead we draw a bond between nodes \( i \) and \( j \), and we write the actual value of \( c_{ij} = c_{ji} \) on the bond. The “+” in the diagram stands for a value of +1.

Replacing \( \alpha \) by \(-1 - \alpha\) in the Cartan matrix is equivalent to a reordering of the simple roots; and replacing \( \alpha \) by \( 1/\alpha \) is again equivalent to such a reordering, after multiplying the matrix by an overall factor \( \alpha \). We conclude that there is an isomorphism between algebras with parameters \( \alpha \), \(-1 - \alpha\) and \( 1/\alpha \); in fact the transformations \( \alpha \rightarrow -1 - \alpha \) and \( \alpha \rightarrow 1/\alpha \) generate a group of order 6 (permutations of 3 objects) and the following parameters all correspond to isomorphic algebras:

\[
\alpha \sim -1 - \alpha \sim \frac{1}{\alpha} \sim -\frac{1}{1+\alpha} \sim -\frac{\alpha}{1+\alpha} \sim -\frac{1+\alpha}{\alpha}
\] (2.10)

Note that \( D(2,1;-2) \) and \( D(2,1;-\frac{1}{2}) \) are both isomorphic to \( D(2,1;1) \) and thus to \( D(2,1) \). The equivalences (2.10) connect the three intervals of real numbers \((-\infty, -1), (-1, 0) \) and \((0, \infty) \). This allows us to restrict attention to \( \alpha \) positive if we so wish, since an algebra \( D(2,1;\alpha) \) with a negative parameter \( \alpha \) is always isomorphic to an algebra with a positive value of \( \alpha \).

To pass from \( D(2,1;\alpha) \) to its affine extension \( D(2,1;\alpha)^{(1)} \), we must add an additional generator of non-zero grade, rendering the algebra infinite-dimensional. For our purposes it is sufficient to consider the projection of the new root system onto the horizontal or zero-grade subspace spanned by the original simple roots of \( D(2,1;\alpha) \). In this subspace, the projection of the affine root is simply the lowest root of \( D(2,1;\alpha) \), namely

\[
\alpha_0 = -(\alpha_1 + \alpha_2 + \alpha_3) = - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)
\] (2.11)

The Cartan matrix and Dynkin diagram of the affine Lie algebra are:

\[
\begin{pmatrix}
0 & -1 - \alpha & 1 & \alpha \\
-1 - \alpha & 0 & \alpha & 1 \\
1 & \alpha & 0 & -1 - \alpha \\
\alpha & 1 & -1 - \alpha & 0
\end{pmatrix}
\] (2.12)

### 3 The Toda Models Defined by \( D(2,1;\alpha)^{(1)} \)

#### 3.1 Standard Reality Conditions

To write down an integrable model corresponding to a finite-dimensional Lie superalgebra \( G \) or its affine extension \( G^{(1)} \), we follow the general procedure summarized in [1]. We work in two-dimensional super-Minkowski space with bosonic light-cone coordinates \( x^\pm \) and fermionic light-cone coordinates \( \theta^\pm \) which are real (or Majorana) spinors; covariant
super-derivatives are defined by

\[ D_\pm = \frac{\partial}{\partial \theta^\pm} - i \theta^\pm \partial_\pm, \quad D^2_\pm = -i \partial_\pm, \quad \{D_+, D_-\} = 0. \]

We introduce a superfield \( \Phi \) with values in the real vector space \( V \) of dimension \( \text{rank}(G) \) spanned by the simple roots, which we assume are all odd. An integrable and manifestly supersymmetric theory is then defined by a superspace lagrangian density

\[ L = \frac{i}{2} D_+ \Phi \cdot D_- \Phi + \sum_j \mu_j \exp \alpha_j \cdot \Phi \quad (3.13) \]

where \( \mu \) is a parameter with the dimensions of mass. It is also customary to introduce a dimensionless coupling constant \( \beta \) in the theory as an overall factor \( 1/\beta^2 \) multiplying the lagrangian, but we will refrain from doing this until we discuss quantization in section 6.

The superfield equations of motion following from this lagrangian are

\[ i D_+ D_- \Phi = \sum_j \mu_j \alpha_j \exp \alpha_j \cdot \Phi. \quad (3.14) \]

For a finite-dimensional algebra \( G \), the sums above run over \( j = 1, \ldots, \text{rank}(G) \). This type of model is conformally invariant and the mass parameter \( \mu \) can actually be removed by a shift in the the superfield \( \Phi \). The Toda theory based on the affine algebra \( G^{(1)} \) has an additional term in the potential labelled by \( j = 0 \) corresponding to the affine root, though the fields live in the same space \( V \). This additional term gives a minimum to the potential, breaking conformal invariance. In either case the equations of motion can be written as a zero-curvature condition for a Lax connection in superspace and this is what guarantees the classical integrability of the theory. Although we are interested primarily in massive theories, some knowledge of the conformal models will be important later.

The case of interest to us is \( G = D(2,1; \alpha) \) and to write down the models explicitly we define a convenient set of fields

\[ \Phi_i = \varepsilon_i \cdot \Phi \quad (3.15) \]

where \( \varepsilon_i \) are the orthogonal basis vectors introduced in the last section. It is also useful to introduce the following notation for terms in the potential:

\[ U_j(\Phi) = \exp(\alpha_j \cdot \Phi) \]

or in detail:

\[ \begin{align*}
U_0(\Phi) &= \exp(-\Phi_1 - \Phi_2 - \Phi_3) \\
U_1(\Phi) &= \exp(-\Phi_1 + \Phi_2 + \Phi_3) \\
U_2(\Phi) &= \exp(+\Phi_1 - \Phi_2 + \Phi_3) \\
U_3(\Phi) &= \exp(+\Phi_1 + \Phi_2 - \Phi_3)
\end{align*} \quad (3.16) \]

We can then write the superspace lagrangian for either the conformal or massive models as

\[ L = \frac{-i}{1 + \alpha} D_+ \Phi_1 D_- \Phi_1 + i D_+ \Phi_2 D_- \Phi_2 + \frac{i}{\alpha} D_+ \Phi_3 D_- \Phi_3 + \mu U(\Phi). \quad (3.17) \]

For the conformal theory based on \( D(2,1; \alpha) \) the potential term is:

\[ U(\Phi) = U_1(\Phi) + U_2(\Phi) + U_3(\Phi). \]
For the massive theory based on the affine algebra $D(2, 1; \alpha)^{(1)}$ the potential is

$$U(\Phi) = U_0(\Phi) + U_1(\Phi) + U_2(\Phi) + U_3(\Phi)$$

$$= 4 (\cosh \Phi_1 \cosh \Phi_2 \cosh \Phi_3 - \sinh \Phi_1 \sinh \Phi_2 \sinh \Phi_3).$$

(3.18)

The equations of motion for the massive model are then:

$$2i D_+ D_- \Phi_1 = -\mu (1 + \alpha) (-U_0 - U_1 + U_2 + U_3)$$

$$2i D_+ D_- \Phi_2 = \mu (-U_0 + U_1 - U_2 + U_3)$$

$$2i D_+ D_- \Phi_3 = \mu \alpha (-U_0 + U_1 + U_2 - U_3)$$

(3.19)

Note that in either case the lagrangian depends on the parameter $\alpha$ through its appearance in the inner-product on the target-space $V$; in fact $\alpha$ can be regarded as an additional dimensionless coupling constant.

To understand better the structure of the massive theory, we can check from the equations of motion that any two of the three fields can consistently be set to zero. On doing so, the resulting single-field Lagrangians are:

$$L_{\Phi_2=\Phi_3=0} = -\frac{1}{1 + \alpha} (iD_+ \Phi_1 D_- \Phi_1 - 4\mu (1 + \alpha) \cosh \Phi_1)$$

(3.20)

$$L_{\Phi_1=\Phi_3=0} = (iD_+ \Phi_2 D_- \Phi_2 + 4\mu \cosh \Phi_2)$$

(3.21)

$$L_{\Phi_1=\Phi_2=0} = \frac{1}{\alpha} (iD_+ \Phi_3 D_- \Phi_3 + 4\mu \alpha \cosh \Phi_3)$$

(3.22)

Comparing with (1.5) we see that these are just three $N=1$ sinh-Gordon theories with masses $2\mu(1 + \alpha)$, $-2\mu$ and $-2\mu \alpha$ respectively, and with a relative minus sign between the first and the last two sinh-Gordon lagrangians.

To gain more insight, we pass to the component formulation of the theory by expanding the superfields:

$$\Phi = \phi + i\theta^+ \psi_+ + i\theta^- \psi_- + i\theta^+ \theta^- \sigma.$$  

After eliminating the auxiliary fields in the usual way we obtain a component-field lagrangian of the form

$$L = L_{\text{bos}}(\phi) + L_{\text{term}}(\psi) + L_{\text{int}}(\phi, \psi),$$

where the various terms appearing have the following structure: The bosonic sector takes the form

$$L_{\text{bos}} = -\frac{1}{1 + \alpha} \left( \partial_+ \phi_1 \partial_- \phi_1 - 2\mu^2 (1 + \alpha)^2 \cosh 2\phi_1 \right)$$

$$+ \left( \partial_+ \phi_2 \partial_- \phi_2 - 2\mu^2 \cosh 2\phi_2 \right)$$

$$+ \frac{1}{\alpha} \left( \partial_+ \phi_3 \partial_- \phi_3 - 2\mu^2 \alpha^2 \cosh 2\phi_3 \right)$$

$$= -\frac{1}{1 + \alpha} L_{\text{sinh}}(\phi_1; m(1 + \alpha)) + L_{\text{sinh}}(\phi_2; m) + \frac{1}{\alpha} L_{\text{sinh}}(\phi_3; m\alpha)$$

(3.23)

where $L_{\text{sinh}}$ is defined in (1.2), and we have set $m = 2\mu$. The bosonic part of the lagrangian is thus a sum of three sinh-Gordon models. This is exactly what one should expect from the fact that the bosonic subalgebra of $D(2, 1; \alpha)$ is $A_1 \oplus A_1 \oplus A_1$. These three terms are nothing but the bosonic components of the three super-sinh Gordon lagrangians above.
We emphasize again that one of the sinh-Gordon models appears with an overall minus sign, due to the indefinite signature of the invariant inner-product.

Considering the remaining terms, the fermionic part of the lagrangian is simply

\[ L_{\text{ferm}} = -\frac{1}{1+\alpha} L_{\text{free}}(\psi_1) + L_{\text{free}}(\psi_2) + \frac{1}{\alpha} L_{\text{free}}(\psi_3) \]

where we use the notation

\[ L_{\text{free}}(\psi) = i\psi_+ \partial_- \psi_+ + i\psi_- \partial_+ \psi_- \]

for the free lagrangian for a massless fermion. Finally the interaction lagrangian is

\[ L_{\text{int}} = i\mu \sum (\alpha_j \cdot \psi_+)(\alpha_j \cdot \psi_-) \exp(\alpha_j \cdot \phi) \]

\[ = m(\bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2 + \bar{\psi}_3 \psi_3) (\cosh \phi_1 \cosh \phi_2 \cosh \phi_3 - \sinh \phi_1 \sinh \phi_2 \sinh \phi_3) \]

\[ + 2m\bar{\psi}_1 \psi_2 (\sinh \phi_1 \sinh \phi_2 \cosh \phi_3 - \cosh \phi_1 \cosh \phi_2 \sinh \phi_3) \]

\[ + 2m\bar{\psi}_1 \psi_3 (\sinh \phi_1 \cosh \phi_2 \sinh \phi_3 - \cosh \phi_1 \sinh \phi_2 \cosh \phi_3) \]

\[ + 2m\bar{\psi}_2 \psi_3 (\cosh \phi_1 \sinh \phi_2 \sinh \phi_3 - \sinh \phi_1 \cosh \phi_2 \cosh \phi_3) \] (3.24)

where the usual Lorentz-invariant inner-product for spinors is \( \bar{x} \chi = i(\psi_+ \chi_- + \chi_+ \psi_-) \).

### 3.2 Twisted Reality Conditions

Having written down the model, our next step is to discuss the imposition of different reality conditions, analogous to our discussion of (1.3) and (1.6) in the introduction. We repeat that our aim is to change the bosonic subsector from a sum of three sinh-Gordon theories with a relative sign between them (coming from the indefinite signature of the inner-product on the target space \( V \)) to a combination of sine/sinh-Gordon theories which all contribute with the same sign.

In [4] it was pointed out that any symmetry of order 2 of the Dynkin diagram for a Lie (super)algebra gives us a choice of reality conditions for the Toda fields consistent with integrability. Specifically, if \( \tau \) is the automorphism acting on simple roots \( \alpha_i \), we can impose the reality condition

\[ \alpha_i \cdot \Phi^* = \tau(\alpha_i) \cdot \Phi \] (3.25)

where * denotes complex conjugation. One can think of this as arising from the Toda theory based on the particular real form of the algebra which is defined as the fixed-point set of the anti-linear conjugation map \( \hat{\tau} \), where

\[ \hat{\tau}(H_{\alpha_i}) = H_{\tau(\alpha_i)} , \quad \hat{\tau}(E_{\pm \alpha_i}) = E_{\pm \tau(\alpha_i)} . \]

Taking \( \tau \) to be the identity corresponds to the trivial symmetry of the Dynkin diagram. Then the Toda fields are all real, and the real form of the algebra is split, or maximally non-compact. For non-trivial symmetries \( \tau \) more interesting possibilities arise, however.

Applying this to the case at hand, the Dynkin diagram of \( D(2, 1; \alpha)^{(1)} \) has symmetry \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) with three non-trivial elements\(^4\) which act on the Dynkin diagram:

\[ \text{(1): } \quad \text{(2): } \quad \text{(3): } \]

\[ ^4\text{For } \alpha = 1, \text{ when } D(2, 1; \alpha) \text{ reduces to } D(2, 1), \text{ there is an additional symmetry corresponding to the permutation of the roots } \alpha_2 \text{ and } \alpha_3 \text{—it is not relevant now but will be discussed again in section 4.} \]
We have omitted the values of the bonds for clarity, but they are as in the Dynkin diagram \((2.12)\). In terms of our orthogonal basis vectors the symmetries depicted above can be written:

\[
\begin{align*}
(1) \quad & \varepsilon_1 \rightarrow \varepsilon_1, \; \varepsilon_2 \rightarrow -\varepsilon_2, \; \varepsilon_3 \rightarrow -\varepsilon_3; \\
(2) \quad & \varepsilon_1 \rightarrow -\varepsilon_1, \; \varepsilon_2 \rightarrow \varepsilon_2, \; \varepsilon_3 \rightarrow -\varepsilon_3; \\
(3) \quad & \varepsilon_1 \rightarrow -\varepsilon_1, \; \varepsilon_2 \rightarrow -\varepsilon_2, \; \varepsilon_3 \rightarrow \varepsilon_3.
\end{align*}
\]

(3.26)

The product of any two of these symmetries gives the third. Naturally these are also symmetries of the lagrangian, as can be seen from equation \((3.17)\).

Taking as an example the symmetry denoted by \((1)\), we can therefore consistently impose reality conditions

\[
(\Phi_1)^* = \Phi_1; \quad (\Phi_2)^* = -\Phi_2; \quad (\Phi_3)^* = -\Phi_3
\]

(3.27)
i.e. we can replace \(\Phi_2\) by \(i\hat{\Phi}_2\) and \(\Phi_3\) by \(i\hat{\Phi}_3\) where \(\hat{\Phi}_j\) are real fields (and correspondingly for the components \(\phi_j\) and \(\psi_j\)). It is clear from e.g. \((3.17)\) that this does indeed give a real lagrangian. Equivalently, we can say that the space of fields \(V\) is now the real span of the vectors

\[
\hat{\varepsilon}_1 = \varepsilon_1; \quad \hat{\varepsilon}_2 = i\varepsilon_2; \quad \hat{\varepsilon}_3 = i\varepsilon_3
\]

(3.28)
The inner-product on \(V\) is clearly changed.

For each of the symmetries \((1), (2), (3)\) we can introduce modified basis vectors for the target space \(V\) as above. The inner-product is diagonal in this basis, and it will be positive (negative) definite if all the squares of the bosonic roots are positive (negative). Examining each case, we find:

| symmetry | \(2\varepsilon_1^2\) | \(2\varepsilon_2^2\) | \(2\varepsilon_3^2\) | inner-product: |
|----------|----------------|----------------|----------------|----------------|
| \((0)\)  | \(-(1 + \alpha)\) | 1              | \(\alpha\)    | indefinite     |
| \((1)\)  | \(-(1 + \alpha)\) | \(-1\)        | \(-\alpha\)   | negative-definite for \(\alpha > 0\) |
| \((2)\)  | \((1 + \alpha)\)  | 1              | \(-\alpha\)   | positive-definite for \(-1 < \alpha < 0\) |
| \((3)\)  | \((1 + \alpha)\)  | \(-1\)        | \(\alpha\)    | negative-definite for \(\alpha < -1\) |

where we have denoted the trivial symmetry, corresponding to standard reality conditions, by \((0)\).

Given our earlier discussion about the isomorphic nature of algebras with parameters \(\alpha\) related by the transformations \((2.10)\), we can without loss of generality assume that \(\alpha\) is positive. Having made this choice, we see that that the reality conditions \((3.27)\) corresponding to the symmetry \((1)\) are the unique choice which result in a lagrangian with definite signature. The theory with standard reality conditions corresponds to the split real form \(D(2, 1; \alpha; 0)\) of \(D(2, 1; \alpha)\), with bosonic subalgebra \(sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})\), while the theory with twisted reality conditions corresponds to a Toda theory based on the real form \(D(2, 1; \alpha; 1)\) with bosonic subalgebra \(su(2) \oplus su(2) \oplus sl(2, \mathbb{R})\). (Note that only the definiteness of the signature is relevant; classically it is not important whether the signature is positive or negative, and when we consider the quantum theory we will simply choose the overall sign of the lagrangian such that the signature of the kinetic part is positive definite.)

For ease of notation we shall now drop the hats on the new real fields, simply replacing our original fields \(\Phi_2 \rightarrow i\Phi_2\) and \(\Phi_3 \rightarrow i\Phi_3\) and similarly for their components \(\phi_2, \phi_3\) and \(\psi_2, \psi_3\). In terms of these new real superfields the lagrangian is

\[
\mathcal{L} = \frac{-i}{1 + \alpha} D_+ \Phi_1 D_- \Phi_1 - iD_+ \Phi_2 D_- \Phi_2 - \frac{i}{\alpha} D_+ \Phi_3 D_- \Phi_3 + \mu U(\Phi) .
\]

(3.29)
where
\[ U(\Phi) = 4(\cosh \Phi_1 \cos \Phi_2 \cos \Phi_3 + \sinh \Phi_1 \sin \Phi_2 \sin \Phi_3) \quad (3.30) \]
The component lagrangian takes the form:
\[
L_{\text{bos}} = -\frac{1}{1 + \alpha} \left( \partial_+ \phi_1 \partial_- \phi_1 - 2\mu^2 (1 + \alpha)^2 \cosh 2\phi_1 \right) \\
- \left( \partial_+ \phi_2 \partial_- \phi_2 + 2\mu^2 \cos 2\phi_2 \right) \\
- \frac{1}{\alpha} \left( \partial_+ \phi_3 \partial_- \phi_3 + 2\mu^2 \alpha^2 \cos 2\phi_3 \right) \\
= -\frac{1}{1 + \alpha} L_{\sinh}(\phi_1; m(1 + \alpha)) - L_{\sin}(\phi_2; m) - \frac{1}{\alpha} L_{\sin}(\phi_3; m\alpha) \quad (3.31)
\]
where the sine-Gordon lagrangian \( L_{\sin} \) is defined in (1.1) and, as before, \( m = 2\mu \). The fermionic part of the lagrangian becomes
\[
L_{\text{ferm}} = -\frac{1}{1 + \alpha} L_{\text{free}}(\psi_1) - L_{\text{free}}(\psi_2) - \frac{1}{\alpha} L_{\text{free}}(\psi_3),
\]
while the interaction lagrangian is
\[
L_{\text{int}} = m(\bar{\psi}_1 \psi_1 - \bar{\psi}_2 \psi_2 - \bar{\psi}_3 \psi_3) (\cosh \phi_1 \cos \phi_2 \cos \phi_3 + \sinh \phi_1 \sin \phi_2 \sin \phi_3) \\
- 2m\bar{\psi}_1 \psi_2 (\sinh \phi_1 \sin \phi_2 \cos \phi_3 - \cosh \phi_1 \cos \phi_2 \sin \phi_3) \\
- 2m\bar{\psi}_1 \psi_3 (\sinh \phi_1 \cos \phi_2 \sin \phi_3 - \cosh \phi_1 \sin \phi_2 \cos \phi_3) \\
+ 2m\bar{\psi}_2 \psi_3 (\cosh \phi_1 \sin \phi_2 \sin \phi_3 + \sinh \phi_1 \cos \phi_2 \cos \phi_3). \quad (3.32)
\]
We see that the lagrangian indeed has definite signature, as desired, with the bosonic part being a sum of sinh-Gordon model and two sine-Gordon models.

### 3.3 Similarities and Differences

The main point is of course the profound difference between (3.23), with indefinite kinetic energy and unbounded potential, and (3.31), with definite kinetic energy and a potential bounded from below. The quantization of our new theory with twisted reality conditions is (at least conceptually) straightforward; whereas the meaning which can be ascribed to the quantization of the original model with standard reality conditions is delicate to say the least.

Nevertheless, there are also some strong similarities. The classical masses of bosonic and fermionic particles are known to be independent of the choice of reality conditions [4]. For the bosons they can be read off from (3.23) or (3.31):
\[
m_1^2 = m^2 (1 + \alpha)^2; \quad m_2^2 = m^2; \quad m_3^2 = m^2 \alpha^2
\]
The masses of the fermions can be found by expanding the interaction lagrangian to second order in the fields, the results being
\[
m_1 = m(1 + \alpha); \quad m_2 = -m; \quad m_3 = -m\alpha
\]
consistent with supersymmetry. We note that there is a relative sign between the masses for the fermions. However, as in the case of bosons, it is the square of the mass that has
a physical interpretation; the Dirac equation with a negative mass has the same space of solutions as the Dirac equation with positive mass, but with solutions with negative and positive energy interchanged. Notice also that the masses of the various particles in the theory are given by the entries in the Cartan matrix of $D(2, 1; \alpha)$, and that the particles can therefore be directly assigned to the three bonds of the Dynkin diagram of $D(2, 1; \alpha)$; this should be compared to the Toda theory for a bosonic Lie algebra $G$, where the particles in the $G^{(1)}$ Toda theory can be assigned to the nodes of the Dynkin diagram of $G$.

In the last section of this paper we shall show that the integrability of our model extends to the quantum theory, giving hope that its S-matrix could be determined exactly. Any candidate S-matrix must be compared with perturbative scattering calculations, the starting point for which are the classical three-point functions. Note that the symmetries (3.20) of the lagrangian restrict the non-zero three-point couplings $c^{ijk}$ to be $c^{123}$. Expanding the lagrangian with normalized fields to third order, we find the explicit Yukawa interactions:

$$L_{\text{yuk}} = -4\mu\sqrt{\alpha(1 + \alpha)}(\bar{\psi}_1 \psi_2 \phi_3 + \bar{\psi}_1 \psi_3 \phi_2 + \bar{\psi}_2 \psi_3 \phi_1)$$

The last important difference to be emphasized concerns the non-trivial vacuum structure of the sine-Gordon terms appearing in (3.31), as against the unique vacua of the sinh-Gordon theories in (3.23). The sine-Gordon model with lagrangian (1.1) has an infinite set of classical vacua at $\phi = 2p\pi$ and hence soliton and anti-soliton solutions of the classical equations of motion which interpolate between these, with $\phi$ tending to different vacuum values as $x \to \pm\infty$. Explicitly these solitons are given (in their rest-frames) by:

$$\phi^\pm = 4\tan^{-1}[\exp(\pm\frac{m}{2}(x^+ - x^-))]$$

When quantizing the theory (see e.g. [5]) one finds the surprising result that the solitons and anti-solitons are in fact more fundamental than the particles corresponding to the field $\phi$, since the latter can be identified with the breather bound states of a soliton and an anti-soliton.

In the supersymmetric sine-Gordon theory, the classical bosonic solitons are expected on general grounds to give rise to a pair of quantum states with fermion numbers $\pm\frac{1}{2}$ which form a doublet under supersymmetry. The precise way in which these structures appear in the semi-classical spectrum of the $D(2, 1; \alpha)^{(1)}$ model is of crucial importance for determining its S-matrix. At the very least we expect to find two species of solitons and anti-solitons arising from the two sine-Gordon theories in (3.31) as well as the fundamental bosons and fermions of the sinh-Gordon sector. A more detailed analysis of the particle spectrum in the quantum theory is work for the future, however.

4 Other Lie Superalgebras

In this section we take a slight detour in order to consider some related Toda models. In our previous paper [4] we showed that only a very few supersymmetric affine Toda theories exist which share with the $D(2, 1; \alpha)^{(1)}$ family the important property that there are consistent reality conditions for the fields which render the kinetic energy positive-definite. The other Lie superalgebras in question are:

$$C(2)^{(2)} \simeq A(1, 0)^{(2)}, \quad B(1, 1)^{(1)}, \quad A(1, 1)^{(1)}$$
We shall describe how all these can be obtained from the \( D(2,1;\alpha)^{(1)} \) family in various limits. This material is not essential for understanding the \( D(2,1;\alpha)^{(1)} \) example itself, but it does underscore its importance.

### 4.1 The algebras \( C(2)^{(2)} \) and \( B(1,1)^{(1)} \)

To obtain these algebras from \( D(2,1;\alpha)^{(1)} \) we use the procedure of folding, as discussed in e.g. [6]. Given a symmetry \( \tau \) of order 2 of any Dynkin diagram of a Lie superalgebra, the folded algebra is defined to have a Dynkin diagram with simple roots \( \frac{1}{2}(\alpha + \tau(\alpha)) \); this corresponds exactly to the \( \tau \)-invariant subalgebra.

In section 3 we discussed the symmetries of the Dynkin diagram of \( D(2,1;\alpha)^{(1)} \) in some detail in connection with possible reality conditions for the Toda fields. Taking the symmetry (1) in (3.26), for instance, and applying the folding construction we obtain the Dynkin diagram of \( C(2)^{(2)} \):

![Dynkin diagram]

The folded algebra is generated by simple roots \( \frac{1}{2}(\alpha_2 + \alpha_3) \) and \( \frac{1}{2}(\alpha_0 + \alpha_1) = -\frac{1}{2}(\alpha_2 + \alpha_3) \).

In terms of the fields \( \Phi_i \) in the Toda model, this corresponds to setting \( \Phi_2 = \Phi_3 = 0 \) and we see immediately that the result is just the \( N = 1 \) sinh-Gordon model given previously in (3.20). Folding by either of the other symmetries (2) or (3) gives an isomorphic result, corresponding to the other sinh-Gordon sub-theories in (3.21) and (3.22).

Notice that the use of a symmetry \( \tau \) in the folding construction is entirely different from its use in defining twisted reality conditions (3.25). Actually, it makes sense to consider folding by one symmetry and twisting by another simultaneously, provided that the latter operation is based on a symmetry of the original diagram which survives after folding is carried out. Thus, in the example above, either of the symmetries (2) or (3) reduces to the unique reflection symmetry of the folded diagram for \( C(2)^{(2)} \). Twisted reality conditions in the original theory will then descend to the Toda theory based on the folded algebra in an obvious manner. The effect on the super-sinh-Gordon system is of course exactly the change (1.6) which gives the super-sine-Gordon system.

To discuss the next example we first set \( \alpha = 1 \) to obtain \( D(2,1)^{(1)} \). For this special value of the parameter \( \alpha \) there is an additional symmetry of the Dynkin diagram beyond those we discussed earlier in (3.26). When \( \alpha = 1 \) it is clearly possible to exchange the simple roots \( \alpha_2 \leftrightarrow \alpha_3 \) while keeping \( \alpha_0 \) and \( \alpha_1 \) fixed, or equivalently to take \( \epsilon_2 \leftrightarrow \epsilon_3 \) in terms of our orthogonal basis vectors. One may wonder whether this symmetry gives rise to another interesting choice of reality conditions in the \( D(2,1)^{(1)} \) Toda theory itself, but in fact it does not lead to a model with definite kinetic energy, which is the reason we omitted it from our earlier discussion.

By folding the Dynkin diagram of \( D(2,1)^{(1)} \) using this new symmetry we obtain the Dynkin diagram of \( B(1,1)^{(1)} \), as shown\(^4\).

\(^4\) When considering \( \alpha = 1 \) we revert to the standard way of drawing the Dynkin diagram, so the number of bonds between roots \( \alpha_i \) and \( \alpha_j \) is \( \max(|c_{ij}|, |c_{ji}|) \).
The new algebra has simple roots \(\alpha_0, \alpha_1\) and \(\frac{1}{2}(\alpha_2 + \alpha_3)\). In terms of the Toda fields \(\Phi_i\) this corresponds to the identification \(\Phi_2 = \Phi_3\) so that the superspace lagrangian is

\[
\mathcal{L} = -\frac{i}{2} D_+ \Phi_1 D_- \Phi_1 + 2i D_+ \Phi_2 D_- \Phi_2 + 2m(\cosh \Phi_1 \cosh^2 \Phi_2 - \sinh \Phi_1 \sinh^2 \Phi_2) \tag{4.33}
\]

Once again we have the possibility of taking twisted reality conditions corresponding to the symmetry (1) of our original diagram, which descends to the folded diagram by exchanging the two grey nodes. This amounts to replacing \(\Phi_2 \rightarrow i\Phi_2\) to give

\[
\mathcal{L} = \frac{i}{2} D_+ \Phi_1 D_- \Phi_1 + 2i D_+ \Phi_2 D_- \Phi_2 + 2m(\cosh \Phi_1 \cos^2 \Phi_2 + \sinh \Phi_1 \sin^2 \Phi_2) \tag{4.34}
\]

The component content of these theories (which were briefly discussed in [1]) can readily be examined in detail by making the appropriate identifications in the general expressions given earlier in section 3. It was shown in [2] that there are higher-spin quantum conserved currents in this model (assuming standard reality conditions).

4.2 The algebra \(A(1, 1)^{(1)}\)

The case of \(A(1, 1)^{(1)}\) differs from the other algebras we have considered in a number of respects. One reason is that this Toda model generally allows for two independent mass parameters, or equivalently one mass parameter together with an independent dimensionless coupling. Another reason is that the algebra arises as a kind of singular limit of \(D(2, 1; \alpha)^{(1)}\) rather than by through the folding construction.

As explained in section 3, the parameter \(\alpha\) in \(D(2, 1; \alpha)^{(1)}\) takes values in \(\mathbb{C} \setminus \{-1, 0\}\). If we take \(\alpha \in \{-1, 0\}\) various properties like the dimension and rank of the algebra change discontinuously, but we still obtain a sensible Lie superalgebra nevertheless. On choosing either of these values, the Cartan matrix of \(D(2, 1; \alpha)^{(1)}\) reduces to the Cartan matrix of \(A(1, 1)^{(1)}\):

\[
\begin{pmatrix}
0 & -1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0
\end{pmatrix}
\]

where the ordering of rows and columns corresponds to a ‘non-standard’ ordering of nodes in the Dynkin diagram. In terms of the simple roots of \(D(2, 1; \alpha)\), taking \(\alpha = 0\) implies the identification \(\alpha_2 = -\alpha_1\).

The corresponding Toda theory has fields \(\Phi_1\) and \(\Phi_2\) which can be thought of as descending directly from the \(D(2, 1; \alpha)^{(1)}\) model; the new relation amongst the simple roots implies that we must set \(\Phi_3 = 0\). The lagrangian for the \(A(1, 1)^{(1)}\) model is

\[
\mathcal{L} = -i D_+ \Phi_1 D_- \Phi_1 + i D_+ \Phi_2 D_- \Phi_2 + \mu \cosh(\Phi_1 + \Phi_2) + \nu \cosh(\Phi_1 - \Phi_2)
\]
which is slightly more general than those we have considered previously because it involves
two mass parameters, \( \mu \) and \( \nu \). The appearance of a second mass parameter is due to
the fact that the Cartan matrix of \( A(1,1) \) has two null eigenvectors or, in other words,
because there are two linear relations between the four simple roots of \( A(1,1) \).

With standard reality conditions, the superfields \( \Phi_i \) are real and so are the constants
\( \mu \) and \( \nu \). What is of more interest to us, however, is that we can impose twisted reality
conditions to obtain a positive-definite action by using the symmetry (2) in (3.26) of the
original diagram, which amounts to \( \varepsilon_1 \rightarrow -\varepsilon_1 \) in this degenerate limit. This effectively
means that we replace \( \Phi_1 \rightarrow i\Phi_1 \). In addition the mass parameters are now allowed to be
complex too, provided they are related by \( \mu = \nu^* \). It is convenient to set \( \mu = me^{i\gamma} \) and
\( \nu = me^{-i\gamma} \) and then the lagrangian becomes

\[
L = iD_+\Phi_1D_\Phi_1 + iD_+\Phi_2D_\Phi_2 + 2m \cos \gamma \cos \Phi_1 \cosh \Phi_2 - 2m \sin \gamma \sin \Phi_1 \sinh \Phi_2
\]

which is clearly real. In addition to the mass-scale \( m \) it involves the real dimensionless
parameter \( \gamma \).

On the face of it, it seems that we have found another one-parameter family of theories
with positive-definite kinetic energy, similar to our \( D(2,1;\alpha) \) family, though with a
slightly different origin for the continuous parameter. But in fact the models written
above are not really distinct at all because the value of \( \gamma \) can be changed arbitrarily by
a suitable field re-definition. This behaviour also turns out to be related to another very
special property, namely the occurrence of an extra supersymmetry.

To explain how all this comes about, let us examine the component content of the
theory. On expanding the superfields and eliminating the auxiliary fields in the usual way
we find the component lagrangian:

\[
L = L_{\text{bos}} + L_{\text{ferm}} + L_{\text{int}}
\]

with

\[
L_{\text{bos}} = \partial_+\phi_1\partial_-\phi_1 + \frac{1}{2}m^2 \cos 2\phi_1 + \partial_+\phi_2\partial_-\phi_2 - \frac{1}{2}m^2 \cosh 2\phi_2
\]

\[
L_{\text{ferm}} = L_{\text{free}}(\psi_1) + L_{\text{free}}(\psi_2)
\]

where \( L_{\text{free}} \) denotes the free fermion lagrangian as before, and

\[
L_{\text{int}} = m \cos \gamma \left[ (\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2) \cos \phi_1 \cosh \phi_2 - 2\bar{\psi}_1\psi_2 \sin \phi_1 \sinh \phi_2 \right] - m \sin \gamma \left[ (\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2) \sin \phi_1 \sinh \phi_2 + 2\bar{\psi}_1\psi_2 \cos \phi_1 \cosh \phi_2 \right]
\]

In the bosonic sector we clearly have sine-Gordon and sinh-Gordon models, each with mass
\( m \). The lagrangian is supersymmetric, by construction, for any value of the dimensionless
parameter \( \gamma \). The fermions must therefore be degenerate in mass with the bosons, which
can be confirmed by diagonalizing their mass matrix.

Now we claimed above that different values of \( \gamma \) do not really give distinct theories,
and the key to understanding this is that the value of \( \gamma \) can be changed by a rotation of
the fermions. Specifically, the transformation

\[
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} \rightarrow
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix}
\]

13
on the fermions is actually indistinguishable from the shift
\[ \gamma \rightarrow \gamma + 2\theta \]
as can be easily checked from the formula for the interaction lagrangian (the other terms in the lagrangian are manifestly unchanged). This means of course that the value of \( \gamma \) has no physical significance, since it can be changed at will by this field re-definition. It also means incidentally that we can choose \( \theta \) so as to set \( \gamma = 0 \) and thereby read off the fermion masses \( \pm m \), confirming our previous statement.

The final part of the story is the link with \( N = 2 \) supersymmetry. Although the value of \( \gamma \) is unimportant in itself, a change in \( \gamma \) can only be produced by a rotation of the fermions. We stressed above that the lagrangian is supersymmetric for any value of \( \gamma \), but implicit in this statement is the fact that the bosons \( \phi_j \) have the fermions \( \psi_j \) appearing in \( L_{\text{int}} \) as their superpartners. If we change \( \gamma \) and rotate the fermions appropriately to compensate, then the model does not change, but in doing so we obtain new supersymmetry transformations relating the bosons and fermions, so that we actually have a continuous family of supersymmetries \( Q_\gamma \). This is precisely what we mean by \( N = 2 \) supersymmetry, although it is more usually characterized by saying that there are two independent supercharges, in our present notation these correspond simply to \( Q_0 \) and \( Q_{\pi/2} \), for example.

To summarize: The affine Toda theory defined by \( A(1,1)^{(1)} \) allows two independent real mass parameters in general. But for the choice of reality conditions leading to a positive-definite theory, the ratio of these parameters has no physical significance and can be removed by a field re-definition. The resulting model is the \( N = 2 \) super sine/sinh-Gordon theory, which has a single real mass parameter. The disappearance of the additional dimensionless parameter is directly linked to the fact that we have a continuous set of supersymmetries, i.e. \( N = 2 \) rather than \( N = 1 \).

These conclusions can also be expressed very naturally using the language of \( N = 2 \) superspace, in which the rotation of the fermions discussed above becomes an \( R \)-transformation on a complex chiral \( N = 2 \) superfield containing both \( \Phi_1 \) and \( \Phi_2 \). The \( N = 2 \) superpotential term is not invariant under such a transformation, but changes by a phase. This implies that its coefficient, which is ultimately identified with the mass, can always be chosen to be real.

The remarks above are easily generalized to the whole family of \( A(n,n)^{(1)} \) affine Toda theories and they provide a new way of understanding why this family alone amongst \( N = 1 \) affine Toda theories actually admits \( N = 2 \) supersymmetry (see \[\text{ref.}\]). It is only the \( A(1,1)^{(1)} \) example for which the lagrangian has positive kinetic energy, however.

5 Classical Conserved Currents

We now return to our analysis of the \( D(2,1;\alpha)^{(1)} \) Toda model and to the issue of its classical conserved currents.

5.1 Preliminaries and Generalities

The Lax pair construction mentioned earlier gives a systematic way of finding conserved quantities, thereby ensuring the integrability of any Toda model. In particular cases,
however, we can also take a less sophisticated but more direct approach to the construction of conserved quantities. Let us discuss this first in the bosonic situation.

Conformal Toda models have ‘holomorphic’ conservation laws of the form

$$\partial_- J = 0$$

(5.35)

with the quantities $J$ generally forming a $\mathcal{W}$-algebra. The simplest case is the Liouville theory, based on the algebra $sl(2; \mathbb{R})$, with lagrangian

$$L = \partial_+ \phi \partial_\phi - \mu \exp(2\phi).$$

(5.36)

Conservation of the (traceless) energy-momentum tensor corresponds to the fact that the following component is holomorphic:

$$T = \partial_+ \phi \partial_\phi - \partial_\phi^2 \phi$$

(5.37)

We then see immediately that (5.35) holds with $J = T^n$ for any integer $n$.

In massive Toda theories we have instead conservation laws of the form

$$\partial_- J + \partial_+ \bar{J} = 0$$

(5.38)

with both light-cone components of the current non-zero. In passing from a conformally-invariant theory based on $\mathcal{G}$ to a massive theory based on $\mathcal{G}^{(1)}$, we may seek to generalize a holomorphic conservation law by taking the same expression for $J$. Since the equations of motion are modified in the massive theory (by the addition of a term in the potential corresponding to the affine root) $J$ will no longer be holomorphic, but it can often be made to satisfy (5.38) for a suitable choice of $\bar{J}$. The simplest example is again $\mathcal{G} = sl(2; \mathbb{R})$, which entails passing from the Liouville theory above to the sinh-Gordon model (1.2). It is easy to check directly using the equations of motion from (1.2) that (5.38) holds with

$$J = T; \quad \bar{J} = -\frac{m^2}{2} \exp(-2\phi)$$

and

$$J = T^2; \quad \bar{J} = -m^2(\partial_+ \phi)^2 \exp(-2\phi)$$

(5.39) (5.40)

Infinitely many higher-spin currents of this type could be found in a similar fashion. Notice that $\bar{J}$ always involves the term in the potential corresponding to the affine root.

Much of this extends easily to superspace. The simplest superconformal Toda theory (the super-Liouville model) has lagrangian

$$\mathcal{L} = \frac{i}{2} D_+ \Phi D_- \Phi - \mu \exp \Phi$$

(5.41)

with conservation laws of the form

$$D_- J = 0.$$ 

(5.42)

In particular, we find a super-holomorphic spin-3/2 super-energy-momentum tensor

$$T = D_+ \Phi D_+^2 \Phi - D_+^3 \Phi.$$

(5.43)
Since this is fermionic, we cannot take powers of it, but we can obtain additional holomorphic currents through other combinations, such as the spin-7/2 current $TD_+T$.

The affine extension of this model is precisely the super sinh-Gordon model (5.5). Once again, the conformal conservation laws can be promoted to the massive theory in the form

$$D_+J = D_-\bar{J}$$

(5.44)

by choosing $\bar{J}$ in a suitable way. Using the equations of motion from (5.5) it is straightforward to check that (5.44) holds with

$$J = T; \quad \bar{J} = \frac{m}{2}iD_+(\exp(-\Phi))$$

(5.45)

and also with

$$J = TD_+T; \quad \bar{J} = \frac{m}{2}(D_+\Phi(D_+^2\Phi)^2 - D_+^2\Phi D_+^3\Phi)\exp(-\Phi).$$

(5.46)

To recap: we have a general way of searching for conserved currents in the affine theory based on $G^{(1)}$ if we have some knowledge of the conformal currents in the theory based on $G$. This applies in either the bosonic or supersymmetric regimes.

One new feature does arise in the superspace situation, however. We are well-accustomed in conformal field theory to the consideration of superspace conservation equations (5.42) which contain pairs of holomorphic currents related by supersymmetry. Thus we know that the spin-3/2 superfield $T$ in the example above will contain a pair of conventional fields with spins $(\frac{3}{2}, 2)$ which are the generators of supersymmetry and translations. Similarly, the spin-7/2 field will contain component conserved quantities with spins $(\frac{7}{2}, 4)$. It is tempting to expect similar behaviour for conservation laws of the form (5.44) but the following detailed analysis shows that this is unfounded in general.

Taking a general component expansion of the current superfields:

$$J = \alpha + \theta^+ j + \theta^- h + i\theta^+\theta^- \beta$$

$$\bar{J} = \bar{\alpha} + \theta^- \bar{j} + \theta^+ \bar{h} + i\theta^-\theta^+ \bar{\beta}$$

(5.47)

we can examine the content of the equation (5.44) explicitly, and we find

$$\partial_- j + \partial_+ \bar{j} = 0$$

(5.48)

$$\partial_- \alpha = \bar{\beta}$$

(5.49)

$$\partial_+ \bar{\alpha} = \beta$$

(5.50)

$$h = \bar{h}$$

(5.51)

(we assume for definiteness that $J$ and $\bar{J}$ are fermionic rather than bosonic, then the component fields defined above are all real). One of these equations is indeed a conservation equation for the bosonic current with components $(j, \bar{j})$. But the other equations give no additional conservation laws in the absence of any extra information about the structure of the currents.

An additional conservation law does arises, however, if

$$J = -iD_+K$$

(5.52)

5These considerations also arose recently in a rather different context [9].
for some superfield $K$, since this implies

$$\bar{\beta} = -\partial_+ \omega, \quad \bar{h} = -\partial_+ k \quad \text{where} \quad K = k + i\theta^+ \bar{\alpha} + i\theta^- \omega + i\theta^+ \theta^- \bar{j} \quad (5.53)$$

and then (5.49) becomes a conservation equation

$$\partial_- \alpha + \partial_+ \omega = 0 \quad (5.54)$$

A simple calculation reveals that the fermionic charge constructed from the current $(\alpha, \omega)$ and the bosonic charge constructed from $(j, \bar{j})$ are indeed related by supersymmetry. It is also simple to show that the condition on the superfield (5.52) is the only way in which we can obtain an additional conservation law of this type. (Assuming (5.54) and examining its change under a supersymmetry transformation restricts the other fields in ways which amount exactly to (5.52).)

The conformal case is a particularly simple way of satisfying (5.52) with $\bar{j} = K = 0$. A less trivial example of how this condition can be met is provided by the conservation of the super-energy-momentum tensor. In a supersymmetric theory it is always possible to define a super-energy-momentum tensor so that (5.52) is fulfilled. It is easy to see this for models of the sort discussed in this paper, in which scalar superfields $\Phi$ interact via a lagrangian of the general form

$$\mathcal{L} = \frac{i}{2} D_+ \Phi \cdot D_- \Phi - V(\Phi) \quad (5.55)$$

where $V$ is some potential. It follows from the equations of motion that

$$D_- (D_+ \Phi \cdot D_+ \Phi) + iD_+^2 V(\Phi) = 0 \quad (5.56)$$

(and similarly with $\pm$ exchanged). Comparing with (5.44) we see that this superspace conservation equation clearly involves a current fulfilling the additional property (5.52). This is only to be expected, since energy-momentum certainly should come together with a conserved superpartner, namely the supersymmetry charge itself.

A special case of this analysis is provided by super-energy-momentum conservation in the sinh-Gordon model (5.45). Notice however that the higher-spin conserved current (5.46) does not meet the condition (5.52). From the analysis above we know that it will produce a bosonic conserved current of spin 4, but we do not expect this to have a superpartner.

5.2 The $D(2, 1; \alpha)$ models

The $\mathcal{W}$-algebra of conserved currents in the conformal Toda theory constructed from $D(2, 1; \alpha)$ contains two generators of (super)spin $\frac{3}{2}$, and one generator of spin 2, see e.g. [10]. Explicit expressions for the currents (with standard reality conditions) are:

$$T = -\frac{1}{1 + \alpha} (D_+ \Phi_1 D_+^2 \Phi_1 - D_+^3 \Phi_1) + (D_+ \Phi_2 D_+^2 \Phi_2 - D_+^3 \Phi_2) + \frac{1}{\alpha} (D_+ \Phi_3 D_+^2 \Phi_3 - D_+^3 \Phi_3)$$

$$F = -\frac{\alpha}{1 + \alpha} (D_+ \Phi_1 D_+^2 \Phi_1 - D_+^3 \Phi_1) + \frac{1}{1 + \alpha} (D_+ \Phi_3 D_+^2 \Phi_3 - D_+^3 \Phi_3)$$

$$- D_+ (D_+ \Phi_1 D_+ \Phi_2 + D_+ \Phi_3 D_+ \Phi_1 + D_+ \Phi_2 D_+ \Phi_3) + 2D_+ \Phi_1 D_+ \Phi_2 D_+ \Phi_3$$

$$W = D_+^4 \Phi_2 + \frac{1}{\alpha} D_+^3 \Phi_1 D_+ \Phi_3 + D_+^3 \Phi_2 \left( \frac{1}{1 + \alpha} D_+ \Phi_1 - D_+ \Phi_2 - \frac{1}{\alpha} D_+ \Phi_3 \right)$$
\[- \frac{1}{1 + \alpha} D_+^3 \Phi_3 D_+ \Phi_1 - (D_+^2 \Phi_2)^2 \]

\[- D_+^2 (\Phi_1 - \Phi_3) \left( \frac{1}{1 + \alpha} D_+ \Phi_1 D_+ \Phi_2 + \frac{1}{\alpha(1 + \alpha)} D_+ \Phi_1 D_+ \Phi_3 - \frac{1}{\alpha} D_+ \Phi_2 D_+ \Phi_3 \right) \]

\[+ D_+^2 \Phi_2 \left( \frac{1}{1 + \alpha} D_+ \Phi_1 D_+ \Phi_2 - \frac{1 + 2\alpha}{\alpha(1 + \alpha)} D_+ \Phi_1 D_+ \Phi_3 + \frac{1}{\alpha} D_+ \Phi_2 D_+ \Phi_3 \right) \quad (5.57)\]

The first of these is simply the energy-momentum tensor of the conformal theory. These three currents satisfy \( D_- T = D_- F = D_- W = 0 \), as can be checked directly using the equations of motion. The expressions are consistent with the ones given for \( D(2,1) \) in \[7\] (though the basis used for the fields in that reference is different from the one used here). Note that if desired we could add a multiple of \( T \) to \( F \), and a combination of derivatives of these two to \( W \) to make the expressions for \( F \) and \( W \) more symmetrical in the three fields, but the resulting expressions are then slightly lengthier.

We now seek conserved currents in the affine Toda theory built on \( D(2,1; \alpha) \) which are of the form \((J, \bar{J})\) with \( J \) some combination of holomorphic currents from the conformal theory.

The energy-momentum tensor must be such a conserved current, and we can easily verify, using the equations of motion, that also \( F \) leads to a conserved current in the affine theory. Explicitly, we find that (5.44) holds with

\[ J = T; \quad \bar{J} = 2i \mu D_+ U_0(\Phi) \]
\[ J = F; \quad \bar{J} = -2i \mu (\alpha D_+ \Phi_1 + (1 + \alpha) D_+ \Phi_3) U_0(\Phi) \quad (5.58) \]

Notice that the super-energy-momentum tensor has explicitly the form (5.52) necessary for the existence of a superpartner, in agreement with our discussion in the last section. For the second current based on \( F \), however, the expression for \( \bar{J} \) above cannot be written in the form \( D_+ K \) and we deduce that has no superpartner.

The conservation of \( W \) does not extend to the affine theory. We find

\[ D_- W \propto \left( -D_+^3 \Phi_2 + \frac{1}{1 + \alpha} D_+ \Phi_1 D_+^2 (\Phi_2 + \Phi_3) - \frac{1}{\alpha} D_+ \Phi_3 D_+^2 (\Phi_1 + \Phi_2) + D_+ \Phi_2 D_+^2 \Phi_2 \right) U_0(\Phi) \]
\[ (5.59) \]

which cannot be written as a total \( D_+ \) derivative.

In order to find higher-spin conserved currents in the classical we look for the most general differential polynomials in the generators \( T, F \) and \( W \) which are holomorphic in the conformal theory. Note that total derivatives are trivially conserved, however, so we are interested only in such polynomials modulo total derivatives. The first non-trivial candidate for a conserved current is one with \( J = TF \), of spin 3, but this turns out not be conserved in the affine model. Turning next to currents where \( J \) has spin 7/2, there are 5 different terms we can construct using \( F, T \) and \( W \) as building blocks (discarding total derivatives):

\[ T D_+ T; \quad F D_+ F; \quad T D_+ F; \quad T W; \quad \text{and} \quad FW \]

Using these terms, we find that we can construct two conserved currents of spin \( \frac{7}{2} \):

\[ J_1 = T D_+ T - \frac{1}{\alpha(1 + \alpha)} F D_+ F + 2i T W \]
\[ J_2 = T D_+ F - \frac{1 + 2\alpha}{2\alpha(1 + \alpha)} F D_+ F + i FW \quad (5.60) \]
The first current reduces to the one found in [1] when we set $\alpha = 1$, while the second has not been found previously to our knowledge. In the calculations we have at no point used the reality conditions of the fields, so the conserved currents found here are clearly conserved also in the twisted theory.

To conclude this discussion of classical currents, let us summarize the component content of what we have found. As shown above, the spins of the superfield conserved currents in the $D(2, 1; \alpha)$ conformal Toda model are $(\frac{3}{2}, \frac{3}{2}, 2)$. If we expand on component fields, we find 6 conserved currents with spins $(\frac{3}{2}, \frac{3}{2}, 2, 2, \frac{5}{2})$. The three bosonic currents with spin 2 are modifications (by fermions) of the energy-momentum tensors of the three independent Liouville models which make up the bosonic sector of the theory. In the affine model, the conserved currents have spins $(\frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2})$ but only one of these, the full energy-momentum tensor of the theory, is expected to have a superpartner. If we expand on components we therefore have conserved currents with spins $(\frac{3}{2}, 2, 2, \frac{5}{2}, 2)$.

At first sight this seems incompatible with the fact that the bosonic sector of the theory is composed of three sinh-Gordon models (for standard reality conditions), each of which has its own conserved energy-momentum tensor of spin 2 when the fermions are set to zero. However, a closer investigation of the current $W$ shows that it contains a spin-2 part which is conserved if we put the fermions to zero, even though the current $W$ itself does not satisfy a conservation equation in the full affine theory. Taking the expression for $D W$ above, expanding in components and setting the fermions to zero we find

$$\partial_+ W_2 \propto -\partial_+ \sigma_2 + \frac{\sigma_1}{1 + \alpha} (\partial_+ \phi_2 + \partial_+ \phi_3) - \frac{\sigma_2}{\alpha} (\partial_+ \phi_1 + \partial_+ \phi_2) + \sigma_2 \partial_+ \phi_2 e^{-\phi_1 - \phi_2 - \phi_3}$$

where $W_2$ is the spin-2 component of $W$. Upon using the equations of motion for the auxiliary fields $\sigma = \sum_j \mu \alpha_j e^{\alpha_j \phi}$ we get

$$\partial_+ W_2 \propto \partial_+ \phi_2 e^{-2\phi_2}$$

which is indeed a total derivative. It is interesting to note that a conserved current in a bosonic theory may become non-conserved in the supersymmetrized theory. This is reminiscent of recent results in $(1, 0)$ supersymmetric generalizations of Toda models [11], where the addition of fermions to a bosonic conformal Toda theory spoils the conservation of higher-spin currents.

6 Quantum Conserved Currents

Until now, we have been considering only the classical theory. Although we will not embark on a full-scale investigation of the quantum theory in this paper, it is of considerable interest to decide whether the theory is not only classically integrable, but also quantum integrable. General arguments show that a theory with just one conserved current of spin higher than 2 (or in the supersymmetric case of spin higher than 3/2) is quantum integrable [12], and to show that the theory is quantum integrable it is therefore sufficient to show that the classical conserved currents found above are also conserved (with quantum corrections) in the quantum theory.

It has been shown by Gualzetti et al [7] that in the case of the affine supersymmetric Toda theory defined by $D(2, 1)^{(1)}$ a quantum conserved current of spin 7/2 does indeed exist, and it is natural to expect that the corresponding current exists also in the deformed
theory defined by $D(2,1;\alpha)$. However, one cannot reject a priori the possibility that a quantum anomaly could spoil the conservation of this current for general values of $\alpha$, with $\alpha = 1$ (and $-\frac{1}{2},-2$) being a critical value where the anomaly happens to vanish. In addition, it is of interest to investigate the quantum behaviour of the new current of spin-$7/2$ found in this paper. We therefore need to show explicitly that quantum conserved currents exist, and to do this we will apply the method used in [7]. We will use standard reality conditions for the fields; but, as we will explain later, the choice of reality conditions does not affect the outcome of the calculations.

We wish to show that currents $J$ exist such that $D_\pm\langle J \rangle$ is a total $D_\mp$-derivative. We will use massless perturbation theory, i.e. we treat the interaction part of the Lagrangian

$$U(\Phi) = \exp(-\Phi_1-\Phi_2-\Phi_3) + \exp(-\Phi_1+\Phi_2+\Phi_3) + \exp(\Phi_1+\Phi_2-\Phi_3) + \exp(\Phi_1-\Phi_2+\Phi_3)$$  \hspace{1cm} (6.61)

as a perturbation to the free theory; for simplicity we have set $\mu$ equal to one. A potential anomaly would appear as a local term in

$$D_\mp\langle J(X^+,X^-) \rangle = D_\mp \left\langle J(X^+,X^-) \exp \left( \frac{i}{\beta^2} \int d^2y d^2\eta U(\Phi) \right) \right\rangle_0$$  \hspace{1cm} (6.62)

where we have introduced a coupling constant $\beta$ by rescaling the lagrangian by $\frac{1}{\beta^2}$. $\langle \cdot \rangle_0$ denotes the expectation value in the free theory, and normal ordering of the exponential terms in $U(\Phi)$ is implicit. $X^\pm = (x^\pm,\theta^\pm)$ and $Y^\pm = (y^\pm,\eta^\pm)$ are superspace coordinates.

Using the symmetries (3.26) of the lagrangian, any one of the four terms in (6.61) can be transformed into any of the other terms, and therefore we need only consider one of the four exponential terms in $U(\Phi)$; furthermore, since we are looking for local terms we need expand the exponential in (6.62) only to first order. For this reason we consider

$$D_\mp \left\langle J(X^+,X^-) \left( \frac{i}{\beta^2} \int d^2y d^2\eta \exp(-\Phi_1-\Phi_2-\Phi_3) \right) \right\rangle_0$$

The massless superspace propagators are

$$\langle \Phi_i(X)\Phi_j(Y) \rangle = -i\delta_{ij}b_iD_\mp D_\mp \Delta(x,y)\delta(2)\delta(\theta-\eta)$$

where $\Delta(x,y)$ is the usual bosonic propagator satisfying $\partial_\mp \partial_\mp \Delta(x,y) = -\frac{i\varepsilon^2}{2}\delta(x-y)$ and $b_i = 2\varepsilon_i^2$ (the basis vectors $\varepsilon_i$ are defined in section 2). The bosonic propagator is explicitly given by the expression

$$\Delta(x,y) = -\frac{\beta^2}{4\pi} \log(2(x^+ - y^+)(x^- - y^-))$$

with $\partial_\mp \left( \frac{1}{x^+ - y^+} \right) = 2\pi i\delta(2)(x-y)$. It is convenient to carry out first the integration over the anti-commuting variables. In order to do this, we simplify the expressions such that each term has exactly one delta function in the anticommuting variables, using

$$D_\mp D_\pm \delta(2)\delta(\theta-\eta) = -i \quad \text{and} \quad D_\mp \delta(2)\delta(\theta-\eta)D_+ \delta(2)\delta(\theta-\eta) = i\delta(2)(\theta-\eta).$$

Terms which cannot be reduced to a single fermionic delta function using these rules will disappear. We do not intend to find explicit expressions for $J$; we intend only to show that the result is a total derivative, and we can therefore freely discard total derivatives to simplify the calculations.
We write the current $J$ as an (implicitly normal ordered) polynomial of derivatives of the fields. It is useful to note that when considering a term which is of order $m + 1$ in the fields, the $m$-loop calculation (i.e. $m + 1$ contractions with the exponential) gives a total derivative as result, and therefore need not be considered. Furthermore, a brief calculation shows that any contraction involving two fields both with an odd order of derivatives gives a vanishing result, since it involves higher powers of the delta function in the anti-commuting fields which cannot be removed using the rules given above.

A simple calculation shows that the two classically conserved currents $T$ and $F$ are both conserved in the quantum theory without any quantum corrections. It is evident that any quantum conserved current is conserved also in the classical limit, so the first candidates for higher-spin quantum conserved currents are the classical spin 7/2-currents $J_i$ given in equation (5.60). It turns out to be most convenient to consider a completely general current of spin 7/2, and to find coefficients such that this current is conserved in the quantum theory. It is then a non-trivial check of the calculations that the quantum currents which are found in this manner do indeed reduce to the known classical currents in the limit $\beta \to 0$.

The most general (modulo total derivatives) current of spin 7/2, which respects the symmetries of the lagrangian, can be written as:

$$J = k_1 D^2_+ \Phi_1 D^2_+ \Phi_1 + k_2 D^3_+ \Phi_2 D^2_+ \Phi_2 + k_3 D^4_+ \Phi_3 D^2_+ \Phi_3 + k_4 D_+ \Phi_1 D^3_+ \Phi_2 \Phi_3 + k_5 D_+ \Phi_2 D^3_+ \Phi_1 D^3_+ \Phi_3 + k_6 D_+ \Phi_3 D^3_+ \Phi_1 D^3_+ \Phi_2 + k_7 D^2_+ \Phi_1 D^2_+ \Phi_2 D^3_+ \Phi_3 + k_8 D^2_+ \Phi_2 D^2_+ \Phi_2 D^3_+ \Phi_3 + k_9 D_+ \Phi_1 D^2_+ \Phi_1 D^2_+ \Phi_3 D^2_+ \Phi_3 + k_{10} D_+ \Phi_1 D^3_+ \Phi_1 D^2_+ \Phi_2 D^2_+ \Phi_2 + k_{11} D_+ \Phi_2 D^3_+ \Phi_1 D^2_+ \Phi_3 D^2_+ \Phi_2 + k_{12} D_+ \Phi_2 D^3_+ \Phi_2 D^2_+ \Phi_1 D^2_+ \Phi_1 + k_{13} D_+ \Phi_3 D^3_+ \Phi_3 D^2_+ \Phi_2 D^2_+ \Phi_2 + k_{14} D_+ \Phi_4 D^3_+ \Phi_3 D^2_+ \Phi_3 D^2_+ \Phi_3 + k_{15} D_+ \Phi_2 D^2_+ \Phi_1 D^2_+ \Phi_1 D^2_+ \Phi_1 + k_{16} D_+ \Phi_4 D^2_+ \Phi_2 D^2_+ \Phi_2 D^2_+ \Phi_2 + k_{17} D_+ \Phi_3 D^2_+ \Phi_3 D^2_+ \Phi_3 D^2_+ \Phi_3 + k_{18} D_+ \Phi_4 D^2_+ \Phi_1 D^2_+ \Phi_2 D^2_+ \Phi_2 + k_{19} D_+ \Phi_5 D^2_+ \Phi_1 D^2_+ \Phi_3 D^2_+ \Phi_3 + k_{20} D_+ \Phi_3 D^2_+ \Phi_3 D^2_+ \Phi_2 D^2_+ \Phi_2 + k_{21} D_+ \Phi_4 D^2_+ \Phi_1 D^2_+ \Phi_2 D^2_+ \Phi_1 + k_{22} D_+ \Phi_5 D^2_+ \Phi_2 D^2_+ \Phi_3 D^2_+ \Phi_3 + k_{23} D_+ \Phi_5 D^2_+ \Phi_1 D^2_+ \Phi_3 D^2_+ \Phi_3$$

Using the observations given above, we see that we need do the calculations only up to two loops (triple contractions). As an example of the structure of the results of the calculations, we give explicitly a partial result:

$$D_+ \left\langle D_+ \Phi_1 D^2_+ \Phi_1 D^2_+ \Phi_2 D^2_+ \Phi_3 \frac{i}{\beta^2} \int d^2y d^2\eta \exp(-\Phi_1 - \Phi_2 - \Phi_3) \right\rangle_0 = \frac{1}{\beta^2} \left\{ \frac{1}{8\pi} (2b_j D_+ (D_+ \Phi_1 D^2_+ \Phi_1 D^2_+ \Phi_j) + b_i D_+ (D_+ \Phi_1 D^2_+ \Phi_2 D^2_+ \Phi_j) + b_j D_+ (D_+ \Phi_2 D^2_+ \Phi_1 D^2_+ \Phi_j)) \right\} \exp(-\Phi_1 - \Phi_2 - \Phi_3) + \text{total derivatives}$$

After calculating the relevant part of $D_+ \langle J \rangle$, we look for a differential polynomial in the fields $P(\Phi)$ of spin 5/2 such that the result of the calculation is equal to $D_+ (P(\Phi) \exp(-\Phi_1 - \Phi_2 - \Phi_3))$. This gives a set of equations for the unknown coefficients $k_1$ to $k_{23}$, as well as
for the coefficients of the polynomial $P$. The equations are rather cumbersome, and we have used the algebraic manipulation program Mathematica to solve them.

After solving the equations, we find that two of the parameters $k_i$ are still free. One of these parameters gives the overall scale, but the fact that a free parameter remains after fixing the scale shows that there are two conserved quantum currents of spin $7/2$ in the theory, corresponding to the two classically conserved currents (5.60). We fix the remaining two independent parameters by demanding that the currents reduce to $J_1$ and $J_2$ in the classical limit (modulo total derivatives). The resulting expressions for the quantum conserved currents are rather long and are given in the appendix. In the case $\alpha = 1$, $J_1$ corresponds to the conserved current found in [7]. The classical and quantum currents denoted by $J_2$ have not been found before to our knowledge, even in the case $\alpha = 1$.

We have not used explicitly the reality conditions of the fields in these calculations, and so the calculations are equally valid in the twisted version: if we define a field $\hat{\Phi}_i$ by $\Phi_i = i\hat{\Phi}_i$ and take as propagator

$$\langle \hat{\Phi}_i(X)\hat{\Phi}_i(Y)\rangle = ib_iD_+D_-\Delta(x,y)\delta^{(2)}(\theta - \eta)$$

i.e. the opposite sign of the propagator of $\Phi_i$, the calculations are clearly not affected.

7 Conclusion and Discussion

We have considered in this paper certain supersymmetrized combinations of sine/sinh-Gordon theories corresponding to Toda models based on the family of exceptional affine Lie superalgebras $D(2,1;\alpha)\,(^{(1)}$) with certain reality conditions for the fields. Theories based on Lie superalgebras have indefinite kinetic energy in general, but the models we have constructed have positive kinetic energy and potentials which are bounded from below. We have confirmed that these models are both classically and quantum-mechanically integrable by finding explicit higher-spin conserved currents of (super)spin $7/2$.

Undoubtedly one of the most interesting features is the appearance of the continuous parameter $\alpha$. The models have a bosonic sector consisting of two sine-Gordon theories and one sinh-Gordon theory but with masses and couplings to fermions which depend on this continuous parameter. We have also seen how all other Toda models with positive-definite lagrangians, e.g. the $N = 2$ sine-Gordon theory, can be obtained as some kind of limit of this continuous family.

Having established quantum integrability, the most obvious question for future study is the determination of an exact S-matrix. An essential step will be a more detailed investigation of the semi-classical spectrum, incorporating both sine-Gordon solitons and sinh-Gordon fundamental particles together with the effects of supersymmetry. We hope to return to these fascinating questions in the future.

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A. Explicit Expressions for the Quantum Conserved Currents

The quantum conserved current which reduces to $J_1$ in the classical limit $\beta \to 0$ (modulo total derivatives) is:

$$
- \frac{1}{(1 + \alpha)^3} \left[ 1 + \frac{(1 + \alpha)(1 + 4\alpha + \alpha^2)}{\alpha} \frac{\beta^2}{8\pi} - 35(1 + \alpha)^2 \frac{\beta^4}{64\pi^2} \right] D_1^3 \Phi_1 D_1^4 \Phi_1 +
$$

$$
\left[ 1 + \frac{1 - 2\alpha - 2\alpha^2}{\alpha(1 + \alpha)} \frac{\beta^2}{8\pi} - 35\frac{\alpha^2}{64\pi^2} \right] D_1^3 \Phi_2 D_1^4 \Phi_2 +
$$

$$
\frac{1}{\alpha^3} \left[ 1 + \frac{\alpha(-2 - 2\alpha + \alpha^2)}{1 + \alpha} \frac{\beta^2}{8\pi} - 35\frac{\alpha^2}{64\pi^2} \right] D_1^3 \Phi_3 D_1^4 \Phi_3 +
$$

$$
\frac{2(1 - \alpha)}{\alpha^3} \left[ 1 - \frac{3\alpha}{1 + \alpha} \frac{\beta^2}{8\pi} \right] D_1^3 \Phi_1 D_1^4 \Phi_2 D_1^3 \Phi_3 - \frac{2(1 + 2\alpha)}{\alpha^2(1 + \alpha)^2} \left[ 1 - 3\alpha(1 + \alpha) \frac{\beta^2}{8\pi} \right] D_1^3 \Phi_2 D_1^4 \Phi_1 D_1^3 \Phi_3 +
$$

$$
\frac{4(1 - \alpha)}{\alpha(1 + \alpha) 8\pi} D_1^2 \Phi_1 D_1^3 \Phi_2 D_1^3 \Phi_3 + \frac{4(2 + \alpha)}{\alpha(1 + \alpha) 8\pi} D_1^2 \Phi_2 D_1^4 \Phi_2 D_1^3 \Phi_3 D_1^3 \Phi_1 +
$$

$$
\frac{1 - \alpha}{\alpha(1 + \alpha)^2} \left[ 1 + \frac{12(1 + \alpha)}{1 - \alpha} \frac{\beta^2}{8\pi} \right] D_1^3 \Phi_1 D_1^4 \Phi_1 D_1^3 \Phi_3 D_1^3 \Phi_3 +
$$

$$
\frac{1 - \alpha}{\alpha(1 + \alpha)^2} \left[ 1 - \frac{12\alpha(1 + \alpha)}{1 - \alpha} \frac{\beta^2}{8\pi} \right] D_1^3 \Phi_1 D_1^3 \Phi_1 D_1^2 \Phi_2 D_1^3 \Phi_2 +
$$

$$
\frac{1 + 2\alpha}{\alpha(1 + \alpha)} \left[ 1 - \frac{18(1 + \alpha)}{1 + 2\alpha} \frac{\beta^2}{8\pi} \right] D_1^2 \Phi_2 D_1^3 \Phi_2 D_1^3 \Phi_3 D_1^3 \Phi_3 +
$$

$$
\frac{1 + 2\alpha}{\alpha(1 + \alpha)} \left[ 1 - \frac{18\alpha}{1 + 2\alpha} \frac{\beta^2}{8\pi} \right] D_1^2 \Phi_2 D_1^3 \Phi_2 D_1^2 \Phi_1 D_1^3 \Phi_1 +
$$

$$
\frac{2 + \alpha}{\alpha^2(1 + \alpha)} \left[ 1 - \frac{12\alpha(1 + \alpha)}{2 + \alpha} \frac{\beta^2}{8\pi} \right] D_1^3 \Phi_3 D_1^3 \Phi_3 D_1^2 \Phi_2 D_1^2 \Phi_2 +
$$

$$
\frac{2 + \alpha}{\alpha^2(1 + \alpha)} \left[ 1 - \frac{18\alpha}{2 + \alpha} \frac{\beta^2}{8\pi} \right] D_1^3 \Phi_3 D_1^3 \Phi_3 D_1^2 \Phi_1 D_1^2 \Phi_1 +
$$

$$
\frac{1}{(1 + \alpha)^3} \left[ 1 - 3(1 + \alpha) \frac{\beta^2}{8\pi} \right] D_1^3 \Phi_1 D_1^2 \Phi_1 D_1^2 \Phi_1 D_1^2 \Phi_1 +
$$

$$
\left[ 1 + \frac{3\beta^2}{8\pi} \right] D_1^3 \Phi_2 D_1^2 \Phi_2 D_1^2 \Phi_2 D_1^2 \Phi_2 + \frac{1}{\alpha^3} \left[ 1 + 3\alpha \frac{\beta^2}{8\pi} \right] D_1^3 \Phi_3 D_1^2 \Phi_3 D_1^2 \Phi_3 D_1^2 \Phi_3 +
$$

$$
- \frac{6}{1 + \alpha} \frac{\beta^2}{8\pi} D_1^3 \Phi_1 D_1^2 \Phi_1 D_1^2 \Phi_1 D_1^2 \Phi_2 - \frac{6}{\alpha(1 + \alpha)} \frac{\beta^2}{8\pi} D_1^3 \Phi_1 D_1^2 \Phi_1 D_1^2 \Phi_3 D_1^2 \Phi_3 +
$$

$$
\frac{6}{\alpha} \frac{\beta^2}{8\pi} D_1^3 \Phi_3 D_1^2 \Phi_3 D_1^2 \Phi_2 D_1^2 \Phi_2 +
$$

$$
\frac{2}{\alpha(1 + \alpha)} D_1^3 \Phi_1 D_1^2 \Phi_1 D_1^2 \Phi_3 \left[ 1 + \frac{\beta^2}{1 + \alpha} D_1^2 \Phi_1 D_1^2 \Phi_1 - (1 + 2\alpha) D_1^2 \Phi_2 D_1^2 \Phi_2 + \frac{2 + \alpha}{\alpha} D_1^2 \Phi_3 D_1^2 \Phi_3 \right].
$$

(A.65)
while the current which reduces to $J_2$ in the classical limit is:

\[
- \left[ \frac{\alpha}{(1 + \alpha)^3} + \frac{(1 + 5\alpha) \beta^2}{\alpha(1 + \alpha)^2} \frac{8\pi}{\alpha} + \frac{5(3 - 10\alpha) \beta^4}{(1 + \alpha)} \frac{64\pi^2}{D_3^2} \Phi_1 D_4^2 \Phi_1 + \frac{1 + \alpha}{\alpha} \frac{4 + 5\alpha \beta^2}{(1 + \alpha)^2} \frac{5(13 + 10\alpha) \beta^4}{\alpha} \frac{64\pi^2}{D_3^2} \Phi_3 D_4^2 \Phi_3 + \frac{2}{\alpha^2} \frac{4\Phi_1 D_2^2 \Phi_3 D_4^2 \Phi_3 + 4(2 + 3\alpha) \beta^2}{\alpha} \Phi_2 D_3^2 \Phi_3 D_4^2 \Phi_3 + \frac{12}{\alpha} \frac{1 + 3\alpha}{1 + \alpha} \frac{6(3 + 2\alpha) \beta^2}{\alpha} \frac{8\pi}{D_4^2} \Phi_3 D_3^2 \Phi_1 D_4^2 \Phi_2 + \frac{1 + 3\alpha}{\alpha^2} \frac{6(1 + 2\alpha) \beta^2}{1 + \alpha} \frac{8\pi}{D_4^2} \Phi_2 D_3^2 \Phi_3 D_1^2 \Phi_1 + \frac{3}{\alpha} \frac{1 + 2\alpha}{(1 + \alpha)^3} \frac{3(1 + 2\alpha) \beta^2}{\alpha (1 + \alpha)^2} \frac{8\pi}{D_4^2} \Phi_1 D_2^2 \Phi_1 D_4^2 \Phi_1 + \frac{1 + \alpha}{\alpha} \frac{6(1 + 4\alpha) \beta^2}{\alpha} \frac{8\pi}{\Phi_2} - \frac{1 + 3\alpha}{\alpha (1 + \alpha)^2} \frac{6(1 + 3\alpha) \beta^2}{\alpha} \frac{8\pi}{\Phi_2} + \frac{2 + 3\alpha}{\alpha^2} \frac{3(1 + 2\alpha) \beta^2}{\alpha} \frac{8\pi}{\Phi_3} D_3^2 \Phi_3 D_4^2 \Phi_3 + \frac{1 + \alpha}{\alpha} \frac{3(1 + 2\alpha) \beta^2}{\alpha} \frac{8\pi}{\Phi_3} + \frac{6}{\alpha} \frac{\beta^2}{8\pi} D_4^2 \Phi_3 D_2^2 \Phi_3 D_4^2 \Phi_3 + \frac{6}{\alpha} \frac{\beta^2}{8\pi} D_4^2 \Phi_3 D_2^2 \Phi_3 D_4^2 \Phi_3 + \frac{2(2 + 3\alpha)}{\alpha^2} \frac{24 \beta^2}{8\pi} D_4^2 \Phi_3 D_2^2 \Phi_3 D_4^2 \Phi_3} \tag{A.66}
\]
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