Entanglement, combinatorics and finite-size effects in spin chains

Bernard Nienhuis\textsuperscript{1}, Massimo Campostrini\textsuperscript{2} and Pasquale Calabrese\textsuperscript{2,3}

\textsuperscript{1} Institute for Theoretical Physics, Universiteit van Amsterdam, 1018 XE Amsterdam, The Netherlands
\textsuperscript{2} INFN Pisa, Italy
\textsuperscript{3} Dipartimento di Fisica dell’Università di Pisa, 56127 Pisa, Italy
E-mail: B.Nienhuis@uva.nl, Massimo.Campostrini@df.unipi.it and calabres@df.unipi.it

Received 2 December 2008
Accepted 22 January 2009
Published 26 February 2009

Online at stacks.iop.org/JSTAT/2009/P02063
doi:10.1088/1742-5468/2009/02/P02063

Abstract. We carry out a systematic study of the exact block entanglement in the $XXZ$ spin chain at $\Delta = -1/2$. We present the first analytic expressions for reduced density matrices for $n$ spins in a chain of length $L$ (for $n \leq 6$ and arbitrary but odd $L$) for a truly interacting model. The entanglement entropy and the moments of the reduced density matrix and its spectrum are then easily derived. We explicitly construct the ‘entanglement Hamiltonian’ as the logarithm of this matrix. Exploiting the degeneracy of the ground state, we find the scaling behavior of the entanglement of the zero-temperature mixed state.

Keywords: integrable spin chains (vertex models), quantum integrability (Bethe ansatz), entanglement in extended quantum systems (theory)

ArXiv ePrint: 0808.2741
1. Introduction

Entanglement is a central concept in quantum information science and it is becoming a common tool for use in studying and analyzing extended quantum systems because of its use in detecting the scaling behavior close to a quantum critical point [1]. It is has been pointed out that this scaling behavior is connected to the efficiency of numerical methods such as quantum and density matrix renormalization group (DMRG) ones [1].

Let $\rho$ be the density matrix of a system and let the Hilbert space be written as a direct product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. $A$’s reduced density matrix is $\rho_A = \text{Tr}_B \rho$ and the entanglement entropy is the corresponding Von Neumann entropy

$$S_A = -\text{Tr}_A \rho_A \log \rho_A,$$

and analogously for $S_B$. When $\rho$ corresponds to a pure quantum state, $S_A = S_B$.

When $A$ is a segment of length $n$ of an infinite one-dimensional system in a critical ground state, the corresponding entanglement entropy $S_n$ diverges as the logarithm of the subsystem size [2]–[4]:

$$S_n = \frac{c}{3} \log n + c_1,$$

where $c$ is the central charge of the associated conformal field theory (CFT) and $c_1$ a non-universal constant. Away from the critical point, $S_n$ saturates to a constant [3] proportional to the logarithm of the correlation length [4]. These properties made the entanglement entropy a basic tool for use in analyzing 1D models. While it is impossible to mention here all the important contributions in the field, we refer the interested reader to the reviews [1].

In recent times, it has been remarked by a few authors [5, 6] that the reduced density matrix $\rho_n$ contains much more information than $S_n$. To the best of our knowledge the full reduced density matrix is known only for free systems [7] and is difficult to obtain by numerical methods because these tend to focus on its eigenvalues. In order to go beyond free systems and to study the effect of strong interactions, in this paper we report a first systematic study of the reduced density matrix of the antiferromagnetic XXZ chain at
Δ = −1/2 defined by the Hamiltonian

\[ H = -\sum_{j=1}^{L} [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z], \]  

with periodic boundary conditions (σ_{L+1} = σ_1) and an odd number of sites. Here σ_i^{x,y,z} stand for the Pauli matrices at the site i. This critical model has the unique property that all the components of the ground-state wavefunction are integer multiples of the smallest one [8]. We will argue in the following that this property suffices for getting \( \rho_n(L) \) for \( n \leq 6 \) and arbitrary \( L \): an exceptional result for a truly interacting system. Another unique feature of this spin chain is that the ground-state energy, which is doubly degenerate for any finite \( L \), has no corrections to scaling: \( E_0 = -3L/2 \) exactly. We refer to the two ground states as \( |\Psi_\pm\rangle \) with the upper index being the sign of the total spin in the \( z \) direction.

This work is complementary to two recent papers. In [9] \( \rho_n \) has been calculated in the thermodynamic (TD) limit (also for \( n \) up to 6). In [10] the connection with loop models has been explored, for a different measure of entanglement. In contrast, here the emphasis is on combinatorial and finite-size scaling aspects. The exact value of some elements of the reduced density matrix for smaller values of \( n \) is also known for general \( \Delta \) [11] and all of them up to \( n = 6 \) for \( \Delta = -1 \) [12].

2. Analytic results for \( \rho_n \)

Since the ground-state energy is exactly proportional to the system size, and the Hamiltonian is represented by a matrix with rational elements, the ground-state vector also has only rational components. Suitably normalized, all ground-state components are integer multiples of the smallest one. The ground state for system sizes up to \( L = 25 \) can be then obtained with absolute precision in very modest computer time. With these ground states we can construct the corresponding density matrices \( \rho_n(L) \) with \( n \leq L \). Any element of \( \rho_n(L) \) is necessarily a rational number and in fact a rational function of \( L \), with numerator and denominator of degree \( \lfloor n^2/2 \rfloor \). The data suffice for guessing the denominator to be \( 2^n L^n \prod_{k=1}^{\lfloor n/2 \rfloor} (L^2 - 4k^2)^{n-2k} \). Antisymmetrized (symmetrized) with respect to the two ground states, \( \rho_n(L) \) turns out to be an even (odd) function of \( L \). As a result it can be determined completely for general \( L \) and for \( n \leq 6 \). For example, for \( n = 1 \) and 2 we obtained

\[ \rho_1(L) = \frac{1}{2L} \begin{pmatrix} L+1 & 0 \\ 0 & L-1 \end{pmatrix}, \]  

\[ \rho_2(L) = \frac{1}{2^4 L^2} \begin{pmatrix} 2(L+2)^2 - 2 & 2(L^2 + 3) & 0 & 0 \\ 0 & 6L^2 - 6 & 5L^2 + 3 & 0 \\ 0 & 5L^2 + 3 & 6L^2 - 6 & 0 \\ 0 & 0 & 0 & 2(L-2)^2 - 2 \end{pmatrix}. \]

The other reduced density matrices are too large to display. We enclose an electronic Mathematica file (the density matrix is \( \text{rho}[L,n] \) with \( L \) odd and \( n = 1, \ldots, 6 \)). In the Mathematica file and in the following, the indices of the matrix are the value (plus 1) of the binary number representing the site product state \( (1 \text{ for } + \text{ and } 0 \text{ for } -) \).
Entanglement, combinatorics and finite-size effects in spin chains

Figure 1. Finite-size scaling of the entanglement entropy \( S_n(L) \) against the CFT prediction \( \frac{1}{3} \log \sin(\pi n/L) + c'_1 \) (full line). We fixed\(^4\) \( c'_1 = 0.7305 \).

We observed the following properties of \( \rho_n \).

(i) The first and the last elements correspond to the probabilities of a string of equal spins, i.e. the emptiness formation probabilities (EFP), \( E_{\pm}(L,n) \), given by

\[
E_{\pm}(L,n) = n^{-1} \prod_{k=0}^{n-1} k! \frac{(3k+1)!}{(2k)!} \frac{(L-k-1)!}{(L+k)!} \frac{((L \pm 1)/2 + k)!}{((L \pm 1)/2 - k - 1)!},
\]

for minority (see [8]) and majority spins respectively. These two elements approach the same limit as \( L \to \infty \), and are equal to \( \rho[1,1] = A_n/2^{n^2} \), where \( A_n \) is the number of \( n \times n \) alternating sign matrices (ASM).

(ii) Some elements satisfy relations that connect \( \rho_n \) to \( \rho_{n-1} \) when summing over one spin (see appendix A in [9]). These equations can be used to derive expressions for some more elements. For example, we have \( \rho_{n+1}[1,1] + \rho_{n+1}[2,2] = \rho_n[1,1] \) that, combined with the ASM sequence for \( \rho_n[1,1] \), gives \( \rho_n[2,2] = (2^n A_{n-1} - A_n)/2^{n^2} \). Similar relations can be derived for a few other elements.

(iii) For \( L \to \infty \) all non-zero elements remain non-zero and reproduce the results of [9].

(iv) The analytic continuation to general \( L \) satisfies \( \rho_n(L)[i,j] = \rho_n(-L)[2^n + 1 - i, 2^n + 1 - j] \).

The entanglement entropy in the TD limit is

\[
S_1 = \log 2, \quad S_2 = 0.950\,749, \quad S_3 = 1.092\,87, \\
S_4 = 1.190\,76, \quad S_5 = 1.265\,88, \quad S_6 = 1.327\,01,
\]

the same as in [9]. We are in position to study the finite-size effects. In figure 1 we plot \( S_n(L) - \frac{1}{3} \log L/\pi \), and we compare it with the CFT prediction \( \frac{1}{3} \log \sin(\pi n/L) + c'_1 \) valid for large enough \( n \). We notice that all the results fall on the same curve, except for small deviations at \( n = 1 \). It is impressive and perhaps unexpected that the asymptotic scaling sets in for such a small value of \( n \).

\(^4\) A precise determination of the non-universal constants \( c'_1 \) and \( c_n \) has been obtained via DMRG [13].

doi:10.1088/1742-5468/2009/02/P02063
3. Combinatorics

It has been noted in the study of Hamiltonian (3) that many physical quantities are sequences of integers or rationals that can be recognized in terms of known ones, with the ASM sequence for the ground-state components the best known [8], but not the only one [8,14,15].

Once a sequence has been recognized, the critical exponents can be derived exactly from its asymptotic behavior, which in part motivates the great interest in such studies. One could have hoped that the eigenvalues of the reduced density matrices would prove rational or at least simple functions of \( L \), but this is not the case. It is then natural to transfer our attention to the elements of the density matrix itself. Because of the conservation of the spin in the \( z \) direction, the density matrix has non-zero entries only between states with the same magnetization (\( \pm n/2, \pm n/2 + 1, \ldots, n/2 - 1, n/2 \)) and organizes into sectors, as is evident from the block structure above. We already reported the formula for the first and last elements of \( \rho_n(L) \), which are the only elements in the sectors with spin \( \pm n/2 \). For the other sectors, the elements grow too quickly with \( n \) for us to recognize any simple behavior. For this reason we explored the possibility that some non-trivial combinations of them could have a simpler structure. Let us start by considering the sectors with total spin \( \pm (n/2 - 1) \). After some tries, we found that reasonable growing sequences are given by the trace of the matrix multiplied with the matrix \((-1)^{i+j} \). In the TD limit, this results in the following sequence:

\[
1, 2, 11, 72, 806, 14352, \quad (8)
\]

for both sectors, up to an overall factor \( 2^{-n^2} \). This sequence grows with \( n \) mildly, but we have been unable to identify it. At finite \( L \), the same trace for \( \rho_n(2n + 1) \) gives

\[
1, 2, 31, 124, 489, 1826, 6843, 25712, 97213, 369478, 1410831, 5408272, \quad (9)
\]

and

\[
2, 2, 2, 2, 2, 2, 2, 2, \ldots \quad (10)
\]

for the two sectors with \( S_z = \pm (n/2 - 1) \) of \( \rho_n(2n + 1) \) respectively, up to the overall factor \( E_-(2n + 1, n) \) from equation (6). The first sequence also grows in a reasonable way, but we still have not been able to guess what it is. For the second one, the recognition is however immediate and provides a non-trivial relation whose physical origin is still unknown. Similar sequences are easily derived for the other sectors. We want to stress here that this generation of sequences is not just an academic game. If we had been able to find a sufficient number combinations of elements of the reduced density matrix that can be recognized, we would have had access to analytic forms of the reduced density matrix for any \( n \) and \( L \). Simply looking at the elements of the density matrix this could seem an impossible task, but we explicitly showed that at least one combination of them is very easy to identify, and that there are other sequences that do not look prohibitive.

The main reason that we reported these series here is to stimulate further studies in this direction, in order to eventually achieve the knowledge of the full density matrix.

\[
\text{The moments of } \rho_n(L) \\
R_n^{(\alpha)}(L) = \text{Tr} \rho_n^{\alpha}(L), \quad (11)
\]
for $\alpha$ integer are sequences of rationals. For critical systems they display the universal asymptotic behavior [2, 4]

$$R_n^{(\alpha)}(L) = c_\alpha \left[ \frac{L \sin \frac{\pi n}{L}}{\pi} \right]^{-c(\alpha-1)/6},$$

(12)

from which one can reconstruct the full spectrum of $\rho_n(L)$ [6], and its behavior is connected with the accuracy of some numerical algorithms based on matrix product states [16]. Despite this universal behavior, $R_n^{(\alpha)}$ has been considered only marginally in the literature [4, 17]. In the TD limit, since all elements of $\rho_n(\infty)$ have a common denominator $2^{n^2}$, all moments can be written as

$$R_n^{(\alpha)} = r_n^{(\alpha)} 2^{-\alpha n^2}$$

(13)

with $r_n^{(\alpha)}$ integers.

For example, the values of $r_n^{(2)}$ up to $n = 6$ are

$$
\begin{align*}
    r_1^{(2)} &= 2, & r_2^{(2)} &= 130, & r_3^{(2)} &= 107468, \\
    r_4^{(2)} &= 1796678230, & r_5^{(2)} &= 413605561988912, \\
    r_6^{(2)} &= 176825650530249935380.
\end{align*}
$$

(14)

We are not able to recognize this fast growing sequence. Increasing $\alpha$, the growth is even faster.

The numerical values of $R_n^{(2)}$:

$$
\begin{align*}
    R_1 &= 0.5, & R_2 &= 0.507813, & R_3 &= 0.409958, \\
    R_4 &= 0.418322, & R_5 &= 0.367356, & R_6 &= 0.374443
\end{align*}
$$

(15)

clearly display even–odd oscillations that prevent us from carrying out any precise scaling analysis like the previous one for the entropy. Multiplying this by $n^{1/4}$ (see equation (12)) we get

$$\begin{align*}
    0.5, & \quad 0.604, & \quad 0.540, & \quad 0.592, & \quad 0.549, & \quad 0.586,
\end{align*}
$$

(16)

values that tend to approach a constant value, confirming the CFT scaling. A systematic study of these oscillations requires the analysis of larger values of $n$, that are not accessible to the present method and for which we performed DMRG calculations that appear elsewhere [13]. For periodic boundary conditions, these oscillations are only present for $\alpha \neq 1$, and not for the entanglement entropy. Thus they have a different origin to those found elsewhere for $S_n$ that are due to boundary effects [18, 19]. The same kinds of oscillations have been found numerically for the multi-interval entanglement [20]. It is rather natural that all the elements of $\rho_n(L)$ oscillate as a consequence of the tendency to antiferromagnetic order of the $XXZ$ chain at $\Delta = -1/2$, and consequently most of the averages that are calculated from them are expected to oscillate as the moments do. Why and how these oscillations mutually cancel only for the von Neumann entropy remains mysterious.

$R_n^{(\alpha)}$ is not the only scaling quantity that can be represented as a sequence of rationals. A good alternative is given by the central values $Q_n^{(\alpha)} \equiv R_n^{(\alpha)}(2n+1)$, which grows more slowly. In fact, in the matrix $\rho_n(2n+1)$ the common denominator is the inverse of
Figure 2. Spectrum of $\rho_n$ in the TD limit; the dots represent the first and last elements of equation (6). Note that these dots sit at $3/4$ ($2/3$) of the spectrum for odd (even) $n$.

$E_-(2n+1,n)$ in equation (6). As a typical example we report $Q_n^{(2)} = q_n E_-(2n+1,n)$:

\[
q_1 = 5, \quad q_2 = 327, \quad q_3 = 159,502, \quad q_4 = 680,263,760, \\
q_5 = 22,821,555,833,635, \quad q_6 = 6,408,136,183,930,928,388.
\]

Unfortunately, although it grows slower than $r_n^{(2)}$, we are also unable to recognize the sequence, but the guessing seems less prohibitive.

4. Spectrum of $\rho_n$ and effective Hamiltonian

The spectrum of $\rho_n$ in the TD limit is shown in figure 2. Several interesting features are evident in the plot. For odd $n$ all the eigenvalues are doubly degenerate, while for even $n$ only some of them are. The spectrum at $n$ is roughly repeated at the bottom and at the top of the spectrum at $n+2$ with some new structure in between. This is also what happens for free fermions [7] ($\Delta = 0$ in equation (3)). Thus the interaction appears to change only quantitative features of the spectrum and not the qualitative ones. This is highly non-trivial because non-zero $\Delta$ introduces well-known strong non-perturbative effects. The smallest eigenvalue scales like const$^n$ (i.e. the top of figure 2 is almost a parabola). Such scaling is known to be true for the ‘all up’ eigenvalue $\rho[1,1]$, i.e. for the EFP [15], marked as a dot in the figure.

The logarithm of the density matrix can be interpreted as an effective Hamiltonian for the subsystem through the natural definition $\rho_n(L) = e^{-H_n(L)}$ at a fictitious temperature $T = 1$ (as observed independently [5]). This effective Hamiltonian can be written down exactly only for free fermions [7], but it is extremely difficult to obtain even part of it for interacting systems, because it requires the knowledge of the full density matrix. The present study gives this unique opportunity (it is enough to diagonalize the density matrix, taking the logarithm of the eigenvalues, and then use the diagonalizing transformation to go back to the original spin basis). In the TD limit we find that the largest terms are a diagonal interaction, $J_{zz} S_j^z S_{j+1}^z$, and a nearest-neighbor exchange $J_{xx} S_j^x S_{j+1}^x + h c$ with a ratio $\sim -0.55$ that is almost the same as in the original model, $-0.5$. All other
terms (exchanges over larger distance, exchange of more than one pair of spins, multi-spin interaction) are more than one order of magnitude smaller. The couplings depend on the position roughly quadratically:
\[ J^z_j(n) \approx A \left( \frac{n-j}{n} \right)^2, \]  
with \( A \sim 0.6 \). The parabolic dependence of the coupling constants is shown in figure 3.

5. The symmetrized density matrix

The degeneracy of the ground state at finite \( L \) gives another unique opportunity: the symmetrized density matrix
\[ \rho^s = \frac{1}{2}(|\Psi_+\rangle\langle\Psi_+| + |\Psi_-\rangle\langle\Psi_-|) \]  
corresponds to minimum energy and it is a zero-temperature mixed state. \( \rho^s \) has no interpretation in CFT, and so it is a new quantity. In the TD limit, \( S(\rho^s_n) = S(\rho_n) \). In figure 4 we report the exact results for the entanglement entropy \( S^s_n(L) \) of this mixed state. Up to \( n/L \approx 0.5 \) they are well described by \( S^s_n(L) = (\log n)/3 + c'_1 \) (independent of \( L \) and with the same \( c'_1 \) as before), as shown by the full line in the plot. Note that a finite-size correction in the form of a sine (dashed line) does not work for \( x > 0.2 \). However, a good collapse is observed for all \( x \). (The last points that do not fall on the master curve correspond to \( n = L - 1 \) and are not expected to scale.) We find that the corrections to the \( \log n \) behavior are of the order of \( (n/L)^4 \). The collapsed data are fitted remarkably well by \( S^s_n(L = n/x) \approx 1/3 \log[n(1 - A \sinh^4 x)] \) with \( A \approx 0.48 \). We do not claim that this form has any justification.

In a mixed state the entropy (1) is not a good measure of entanglement because it mixes classical and quantum correlations. From quantum information we know that an appropriate measure is the mutual information
\[ M_n = S_n + S_{L-n} - S_L. \]  

Figure 3. Dependence of the coupling constants of the entanglement Hamiltonian on the position in the block. Squares (circles) refer to the \( XY \) (two times \( ZZ \)) couplings, while the full lines are only guides to the eye, for connecting points at the same \( n \).
Entanglement, combinatorics and finite-size effects in spin chains

Figure 4. Entanglement entropy of the symmetrized density matrix \( S_n^s(L) \) versus \( n/L \) and its asymptotic behavior. The full line is \((\log n)/3 + c_1'\) which gives an effective description up to \( n/L \sim 0.5\). The dashed line is the finite-size CFT prediction for the pure state \((\log L/\pi \sin \pi n/L)/3 + c_1'\) that clearly does not work for \( x \geq 0.2\). The dashed–dotted curve is the heuristic formula \( S_n^{(s)}(L = n/x) \approx 1/3 \log[n(1 - A \sinh^4 x)] \) with \( A \approx 0.48\) that works well for all values of \( n \) (except very small \( n \) and \( L - n \)). Inset: scaling of the mutual information \( M_n \) as a function of \( n/L \) compared with the symmetrized heuristic guess.

The calculation of \( M_n \) is more difficult because even for small \( n \) we need \( S_{L-n} \) that can correspond to a large block. The data available from the exact ground state (up to \( n = 13, L = 21 \) plotted in the inset of figure 4) collapse and define a universal function that is described by the symmetrized version of the heuristic \( S_n^{(s)}(L) \) introduced before, plotted as a full line in figure 4.

Such mixed zero-temperature states are present every time that the ground state is degenerate at finite \( L \). Among these, supersymmetric lattice models [21] are very interesting and they could be understood more deeply in this framework.

6. Conclusions and perspective

We presented explicit analytic expressions for the reduced density matrix of the \( XXZ \) spin chain at \( \Delta = -1/2 \). From these matrices we built several sequences of integers that encode the scaling behavior. From the exact density matrix, we constructed the entanglement Hamiltonian, a result that is not easily accessible with other approaches. We found the remarkable property that this Hamiltonian is dominated by nearest-neighbor terms of the same form as the original Hamiltonian. Finally, because the ground state is doubly degenerate, we were able to study the entanglement of a zero-temperature mixed state, showing a behavior very different from a pure one.

doi:10.1088/1742-5468/2009/02/P02063
These results are unique in two ways: they concern the full density matrix rather than just its entropy, and they are completely exact rather than asymptotic. They suggest many questions that can also be posed numerically for more general systems. They show explicitly that even for modest block size, asymptotic behavior is evident, eventually allowing for the study of further finite-size corrections. Finally, since the ground state is known to have strong connections with combinatorial problems of current interest [8,10,14,15], it is expected that further properties of the density matrix for arbitrarily large blocks could be found by combinatorial methods.

Acknowledgment

PC benefited of a travel grant from ESF (INSTANS activity).

References

[1] Amico L, Fazio R, Osterloh A and Vedral V, 2008 Rev. Mod. Phys. 80 517
Cardy J, 2008 Eur. Phys. J. B 64 321
Eisert J, Cramer M and Plenio M B, 2008 arXiv:0808.3773
Holzhey C, Larsen F and Wilczek F, 1994 Nucl. Phys. B 424 443
[2] Vidal G, Latorre J I, Rico E and Kitaev A, 2003 Phys. Rev. Lett. 90 227902
Latorre J I, Rico E and Vidal G, 2004 Quantum. Inf. Comput. 4 048
[3] Calabrese P and Cardy J, 2004 J. Stat. Mech. P06002
Calabrese P and Cardy J, 2006 Int. J. Quantum. Inf. 4 429
[4] Li H and Haldane F D M, 2008 Phys. Rev. Lett. 101 010504
[5] Stroganov Y G, 2001 J. Phys. A: Math. Gen. 34 5335
Razumov A V and Stroganov Y G, 2001 J. Phys. A: Math. Gen. 34 3185
Stroganov Y G, 2001 J. Phys. A: Math. Gen. 34 L179
[6] Sato J and Shiroishi M, 2007 J. Phys. A: Math. Gen. 40 4839
Sato J, Shiroishi M and Takahashi M, 2006 J. Stat. Mech. P12017
[7] Tsuchiya H and Takahashi M, 2006 J. Stat. Mech. P12017
[8] Calabrese P et al., 2009 at press
[9] Jacobsen J L and Saleur H, 2008 Phys. Rev. Lett. 100 087205
[10] Sato J, Shiroishi M and Takahashi M, 2006 J. Stat. Mech. P12017
[11] Zhang M-C and Peschel I, 2001 Phys. Rev. B 64 064412
Peschel I, Kaulke M and Legeza O, 1999 Ann. Phys., Lpz. 8 153
[12] Mitra S, Nienhuis B, de Gier J and Batchelor M T, 2008 Phys. Rev. B 78 024410
Schuch N, Wolf M M, Verstraete F and Cirac J I, 2008 Phys. Rev. Lett. 100 030504
Verstraete F and Cirac J I, 2006 Phys. Rev. B 73 094423
[13] Jin B-Q and Korepin V E, 2004 J. Stat. Phys. 116 79
Zhou H-Q, Barthel T, Fjørerød J O and Schollwoeck U, 2006 Phys. Rev. A 74 050305(R)
[14] Franchini F, Its A R and Korepin V E, 2008 J. Phys. A: Math. Theor. 41 025302
Cardy J L, Castro-Alvaredo O A and Doyon B, 2008 J. Stat. Phys. 130 129
[15] Franchini F, Its A R and Korepin V E, 2008 J. Phys. A: Math. Theor. 41 025302
Cardy J L, Castro-Alvaredo O A and Doyon B, 2008 J. Stat. Phys. 130 129
[16] Laflorencie N, Sorensen E S, Chang M-S and Affleck I, 2006 Phys. Rev. Lett. 96 100603
[17] De Chiara G, Montanaro S, Calabrese P and Fazio R, 2006 J. Stat. Mech. P03001
[18] Furukawa S, Pasquier V and Shiraishi J, 2008 arXiv:0809.5113
[19] Fendley P, Schoutens K and de Boer J, 2003 Phys. Rev. Lett. 90 120402
Fendley P, Nienhuis B and Schoutens K, 2003 J. Phys. A: Math. Gen. 36 12399
Huijse L and Schoutens K, 2008 Eur. Phys. J. B 64 543

doi:10.1088/1742-5468/2009/02/P02063