High-Accuracy Numerical Approximations to Several Singularly Perturbed Problems and Singular Integral Equations by Enriched Spectral Galerkin Methods

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Dedicated to Professor Jie Shen on the Occasion of his 60th Birthday

Abstract. Usual spectral methods are not effective for singularly perturbed problems and singular integral equations due to the boundary layer functions or weakly singular solutions. To overcome this difficulty, the enriched spectral-Galerkin methods (ESG) are applied to deal with a class of singularly perturbed problems and singular integral equations for which the form of leading singular solutions can be determined. In particular, for easily understanding the technique of ESG, the detail of the process are provided in solving singularly perturbed problems. Ample numerical examples verify the efficiency and accuracy of the enriched spectral Galerkin methods.

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Key words: Singularly perturbed problems, weakly singular integral equations, boundary layers, enriched spectral Galerkin methods, Jacobi polynomials.

1 Introduction

As we know that the numerical error of the most existed numerical methods strictly rely on the regularity of the solution $u$ of the underlying problem. Especially, for the problems with high regularity solutions, spectral methods are capable of providing highly accurate solutions with significantly less unknowns than using a finite-element or finite difference methods [3, 5, 13, 26]. However, usual spectral methods based on orthogonal polynomials/functions do not have satisfactory convergence rate for singularly perturbed

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problems and singular integral equations due to solutions of those problems usually exhibited boundary layer phenomena or singular behaviours. More precisely,

- **singularly perturbed equations** [7, 11, 15, 17, 23, 28, 34]: Given a tiny perturbed parameters $\varepsilon$. The solutions of singularly perturbed equations usually involve the boundary layer, i.e., the solutions change sharply (not tempered) in a narrow domain, such as $e^{-\frac{x}{\varepsilon}}$, so that the usual spectral method cannot catch the information of the boundary layer functions accurately.

- **Singular integral equations** [4, 6, 10, 16, 20, 24, 32, 33]: The solutions of many integral equations with weakly singular kernel behave as a summation of Müntz polynomials $\sum_{i=0}^{\infty} x^{r_i}, r_i > 0$ (see [4, 6]). Usually the index of the leading singular term $r_0$ is a small positive real number, so spectral methods are inefficient for singular integral equations due to the limited regularity of the solutions.

The results of the spectral approximation to boundary layer functions behaving as $e^{-\frac{x}{\varepsilon}}$ can be considerably improved by using special mapped polynomials [18, 19, 27, 31], where singular mappings are used to establish spectral method with improved algebraic rates of convergence. However, these approximation results are still not uniform in $\varepsilon$. Combining the parameter $\varepsilon$, Schwab and Suri [23] use two-element spectral method (or $p$ version on two elements) to derive a robust exponential rate for boundary layer functions. Different from the singularly perturbed equations, the weak singularity of the solutions of the singular integral equations cannot derive the exponential convergence rate just by two elements due to the derivative of the solution are unbounded. So Wang [32] et al. and Yi [33] et al. use $h-p$ finite element methods, based on the geometric mesh [14], to handle Volterra integro-differential equations with smooth and weakly singular kernels. Meanwhile, by using a special mapping, Hou et al. [16] derive an Müntz spectral method to enhance the convergence rate of the usual spectral method for singular integral equations. In this paper, we adopt the enriched spectral Galerkin method [9] to deal with several singularly perturbed problems and singular integral equations with a few boundary layer functions and leading singular terms which can be determined. The main merit of this method is that the enriched spectral Galerkin method keeps the structure of the usual spectral method. Especially in ESG-II, via a special property of the spectral method, we can obtain an improved numerical approximation just by repeating usual spectral method several times.

The remainder of this paper is organized as follows. In the next section, we provide a general framework of the enriched spectral-Galerkin methods. Moreover, we introduce the classical Jacobi polynomials and their basic properties which will be extensively used in subsequent sections. In Section 3, based on a Jacobi spectral Galerkin scheme and the analysis of the boundary layer functions, we apply ESG-II into several singularly perturbed problems. In Section 4, we study an singular integral equation, derive the form of the singular solutions and then apply ESG-II to obtain accurate solutions. Some concluding remarks are given in Section 5.
2 Preliminary

For the completeness of the statement, we provide a terse description of the enriched spectral Galekin method in the first part. Moreover, the definition and some basic properties of the Jacobi polynomials are listed in the second half of this section.

2.1 Enriched spectral Galerkin methods (ESG)

Spectral methods are capable of providing highly accurate solutions to smooth problems with significantly less unknowns than using a finite-element or finite difference methods [3, 5, 13, 26]. However, solutions to many singular perturbed/integral problems may deteriorate the convergence rate due to the weakly singular solutions or the boundary layer functions. In order to recover the high-accuracy of the spectral method for singularly perturbed problems and singular integral equations, we introduce the enriched Galerkin spectral methods in the regime of Galerkin method. More precisely, we consider the following weak formulation: Given \( f \in X' \), find \( u \in X \) such that

\[
a(u, v) = (f, v) \quad \forall v \in X,
\]

where \( X \) is a Hilbert space with norm \( \| \cdot \|_X \) and \( X' \) is its dual space, \( a(u, v) \) is a bilinear form in \( X \times X \). Let \( X_N = \text{span}\{\phi_n\}_{n=1}^N \) with \( \phi_n \in X \) being certain smoothly orthogonal polynomials/functions such that the subspace \( X_N \to X \). Then, the classical spectral-Galerkin method is to find \( u_N \in X_N \) such that

\[
a(u_N, v_N) = (f, v_N) \quad \forall v_N \in X_N.
\]

Note that if the solution \( u \) of the problem (2.1) is smooth and changed not so dramatically, then \( \|u_N - u\|_X \) will converge to zero rapidly. However, in many situations, the solutions of the singularly perturbed problems and singular integral equations will not be smooth or tempered due to problems with weakly singular kernel in the integral operator or with a very small perturbed parameter \( \varepsilon \), so the traditional spectral methods with usual basis functions will not lead to accurate approximations.

2.1.1 Enriched spectral Galerkin method-I

For many singularly perturbed problems and singular integral equations, it is possible to determine the forms of a few leading singular terms or boundary layer functions. Assuming that the \( k \) first leading singular terms or boundary layer functions are \( \psi_i, i = 1, \ldots, k \), it is then natural to add those singular terms to the approximation space \( X_N \), leading to the so called enriched spectral method. Precisely, given a set of singular functions \( \psi_i, i = 1, 2, \ldots, k \), for the problem (2.1), we look for \( u^k_N \in X^k_N \) s.t.

\[
a(u^k_N, v^k_N) = (f, v^k_N) \quad \forall v^k_N \in X^k_N.
\]
where the enriched space $X_N^k := X_N \oplus S_k$, $S_k := \text{span}\{\psi_i\}_{i=1}^k$. Note that for some special cases, we should modify the singular term to satisfy the boundary conditions so that the enriched space are consistent with Hilbert space $X$.

By substituting $u_N^k = \sum_{i=1}^N \tilde{u}_i \phi_i + \sum_{i=1}^k \tilde{s}_i \psi_i$ into (2.3) and taking $v_N^k = \phi_n$, $n = 1, 2, \ldots, N$ and $\psi_i$, $i = 1, 2, \ldots, k$ successively, we obtain the following linear system:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{s} \end{bmatrix} = \begin{bmatrix} \tilde{f} \\ \tilde{f} \end{bmatrix},$$

(2.4)

where $\tilde{u} = (\tilde{u}_1 \, \tilde{u}_2 \, \ldots \, \tilde{u}_N)^T$, $\tilde{s} = (\tilde{s}_1 \, \tilde{s}_2 \, \ldots \, \tilde{s}_k)^T$, and

$$A = (a_{nm})_{N \times N}, \quad D = (d_{ij})_{k \times k}, \quad a_{nm} = a(\phi_n, \phi_n), \quad d_{ij} = a(\psi_i, \psi_j),$$

$$B = (b_{ni})_{N \times k}, \quad C = (c_{ni})_{k \times N}, \quad b_{ni} = a(\phi_n, \psi_i), \quad c_{ni} = a(\phi_i, \psi_i),$$

$$\tilde{f} = (f_n)_{N \times 1}, \quad \tilde{f} = (f^k)_{k \times 1}, \quad f_n = (f, \phi_n), \quad f^k = (f, \psi_i).$$

Then the above system can be efficiently solved by forming the Schur-complement matrix $CA^{-1}B - D$, and then we can obtain $\tilde{s}$ and $\tilde{u}$ successively from

$$(CA^{-1}B - D) \tilde{s} = CA^{-1}\tilde{f} - \tilde{f}, \quad A\tilde{u} = \tilde{f} - B\tilde{s}.$$  

(2.6)

ESG-I is very efficient, and alleviates, to some extent, the ill conditioning problem caused by singular functions. However, the numerical results can still be plagued by ill conditioning as $k$ increased, so we prefer to use the following ESG-II in practice.

### 2.1.2 Enriched spectral Galerkin method-II

A special feature of the spectral methods is that, for smooth functions, their spectral expansion coefficients will decay exponentially fast. Based on this property, we can establish ESG-II to solve many singular problems. For the detail of this new numerical method, one can refer to the recent work [9]. Here comes a short description as follows.

Similar to ESG-I, we assume that a few leading singular terms of the underlying problems can be determined and denoted by $\psi_i$, $i = 1, 2, \ldots, k$. Here we only need to repeat the classical spectral scheme (2.2) several times instead of the enriched spectral scheme (2.3). For easily understanding ESG-II, we list some facts as follows:

- There exist some constants $c_1, c_2, \ldots, c_k$ such that $u = u_s + (c_1 \psi_1 + c_2 \psi_2 + \ldots + c_k \psi_k)$, where $u_s$ is the smooth part (compare with $u$ and $\psi_1$). The main work of the ESG-II is to derive the corresponding numerical approximations $c_iN$.

- Owing to the fact that the resource term $f$ is known and the underlying singular terms $\psi_i$, $i = 1, 2, \ldots, k$ can be determined, we can use the spectral scheme (2.2) to derive the numerical solutions that

$$u_N = \sum_{n=1}^N \tilde{u}_n \phi_n, \quad \psi_{i,N} = \sum_{n=1}^N \tilde{\psi}_{i,n} \phi_n.$$
• We cannot derive the numerical solution \( u_{s,N} = \sum_{n=1}^{N} \tilde{u}_{s,n} \phi_n \) of the smooth part \( u_s \) due to constants \( c_i, i = 1, 2, \ldots, k \) are unknown. However, we have the relation

\[
\tilde{u}_N = u_{s,N} + (c_1 \Psi_{1,N} + c_2 \Psi_{2,N} + \ldots + c_k \Psi_{k,N}).
\]

Then, for \( n = 1, 2, \ldots, N \), the corresponding coefficients hold

\[
\tilde{u}_{s,n} = \tilde{u}_n - (c_1 \Phi_{1,n} + c_2 \Phi_{2,n} + \ldots + c_k \Phi_{k,n}).
\]

• Without loss of generality, we can derive the numerical approximations \( c_i, N \) by setting \( \tilde{u}_{s,n} = 0 \) for \( n = N - k + 1, \ldots, N \) due to the coefficients \( \tilde{u}_{s,n} \) of the smooth part \( u_s \) converge to zero at a high rate of speed. Then we have a numerical approximation to \( u_s \) below

\[
u^{II}_{s,N} = u_N - (c_1 N \Psi_{1,N} + c_2 N \Psi_{2,N} + \ldots + c_k N \Psi_{k,N}).
\]

The above facts exhibits that we have the numerical approximations for constants \( c_j \) and the smooth part \( u_s \). Combining the singular terms \( \psi_i \), we obtain the numerical solution of ESG-II that

\[
u^{II}_N = u^{II}_{s,N} + (c_1 N \psi_{1,N} + c_2 N \psi_{2,N} + \ldots + c_k N \psi_{k,N}). \tag{2.7}
\]

### 2.2 Jacobi polynomials

Let \( \alpha, \beta > -1 \) and \( \omega^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta} \). The Jacobi polynomials \( \{P_n^{(a,b)}(x)\}_{n=0}^{\infty} \) are mutually orthogonal and satisfy

\[
\int_{-1}^{1} P_n^{(a,b)}(x) P_m^{(a,b)}(x) \omega^{a,b}(x) \, dx = \gamma_n^{(a,b)} \delta_{nm}, \tag{2.8}
\]

where

\[
\gamma_n^{(a,b)} = \frac{2^{a+b+1} \Gamma(n+a+1) \Gamma(n+b+1)}{(2n+a+b+1) n! \Gamma(n+a+b+1)}. \tag{2.9}
\]

In particular, for \( \alpha = \beta = 0 \) and \( \alpha = \beta = -1/2 \), Jacobi polynomials \( P_n^{(a,b)}(x) \) are the Legendre polynomials \( L_n(x) \) and the Chebyshev polynomials \( T_n(x) \) up to a constant, respectively.

Jacobi polynomials are extensively investigated and there are abundant properties (see [1, 5, 13, 25] and their references therein). Here we list some important properties as follows:

1. **Sturm-Liouville problem**

\[
\mathcal{L}^{a,\beta} P_n^{(a,b)}(x) = \lambda_n^{a,\beta} P_n^{(a,b)}(x), \quad \lambda_n^{a,\beta} = n(n+a+b+1),
\]

where the singular Sturm-Liouville operator

\[
\mathcal{L}^{a,\beta} u = -(1-x)^{-\alpha}(1+x)^{-\beta} \partial_x \left((1-x)^{a+1}(1+x)^{\beta+1}\partial_x u \right).
\]
2. **Closed form**

\[
P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(n+\alpha+\beta+1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n+k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)} (\frac{x-1}{2})^k.
\]

3. **Three-term recurrence relation**

\[
P_{n+1}^{(\alpha,\beta)}(x) = (a_{n+1}^{\alpha,\beta} x - b_{n+1}^{\alpha,\beta}) P_n^{(\alpha,\beta)}(x) - c_n^{\alpha,\beta} P_{n-1}^{(\alpha,\beta)}(x), \quad n \geq 1,
\]

\[
P_0^{(\alpha,\beta)}(x) = 1, \quad P_1^{(\alpha,\beta)}(x) = (\alpha + \beta + 2)x / 2 + (\alpha - \beta) / 2,
\]

where

\[
a_{n+1}^{\alpha,\beta} = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)},
\]

\[
b_{n+1}^{\alpha,\beta} = \frac{(\beta^2 - \alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)},
\]

\[
c_n^{\alpha,\beta} = \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}.
\]

4. **Derivative relation**

\[
\partial_x^k P_n^{(\alpha,\beta)}(x) = d_{n,k}^{\alpha,\beta} P_{n-k}^{(\alpha+k,\beta+k)}(x), \quad n \geq k,
\]

where

\[
d_{n,k}^{\alpha,\beta} = \frac{\Gamma(n+k+\alpha+\beta+1)}{2^k \Gamma(n+\alpha+\beta+1)}.
\]

In particular, in order to meet the homogeneous boundary condition arisen in many cases, we need to use a special generalized Jacobi polynomial

\[
P_n^{(-1,-1)}(x) = \frac{x-1}{2} P_{n-1}^{(1,1)}(x), \quad n = 2,3,\ldots
\]

(2.10)

Similar to the classical Jacobi polynomials, it holds the following derivative relation

\[
\partial_x P_n^{(-1,-1)}(x) = \frac{n-1}{2} L_{n-1}(x), \quad n = 2,3,\ldots,
\]

(2.11)

where \(L_{n-1}(x) = P_n^{(0,0)}(x), n = 0,1,\ldots\) are the well-known Legendre polynomials.

### 3 Applications to singularly perturbed equations

In this section, we will apply the ESG-II to the following singularly perturbed equations

\[
\begin{align*}
-\epsilon u''(x) + bu'(x) + cu(x) &= f(x), \quad x \in I := (-1,1), \\
\frac{u}{\epsilon}(-1) &= u(1) = 0,
\end{align*}
\]

(3.1)
where \( \varepsilon \in (0, 1) \) and constants \( b \in \mathbb{R}, c \geq 0. \)

The variational formulation of the singularly perturbed equations can be read as: Find \( u \in H_{0}^{1}(I) \) such that
\[
A_\varepsilon(u, v) := \varepsilon (u', v') + b(u', v) + c(u, v) = < f, v > \quad \forall v \in H_{0}^{1}(I). \tag{3.2}
\]
The well-posedness of the variational formulation can be derived by Lax-Milgram lemma and the following coercivity and continuity
\[
A_\varepsilon(u, u) \geq \min \{ \varepsilon, c \} \| u \|_{1}^{2}, \quad A_\varepsilon(u, v) \leq \max \{ \sqrt{\varepsilon^2 + 2b^2}, \sqrt{2c^2} \} \| u \|_{1} \| v \|_{1}.
\]

Then, we can find a numerical solution \( u_N \) in the suitable finite dimensional subspace \( P_{0}^N(I) := P_{N}(I) \cap H_{0}^{1}(I) \), i.e., to find \( u_N \in P_{0}^N(I) \) such that
\[
A_\varepsilon(u_N, v) = < f, v > \quad \forall v \in P_{0}^N(I). \tag{3.3}
\]
where \( P_{0}^N(I) \) can be spanned by generalized Jacobi polynomials \( \{ \tilde{P}_{n}^{(-1, -1)} \}_{n=2}^{N} \), i.e., the numerical solution \( u_N \) can be expanded as \( u_N = \sum_{n=2}^{N} \tilde{a}_n \tilde{P}_{n}^{(-1, -1)} \), \( m=2,3,...,N \) successively, we obtain the matrix system
\[
(\varepsilon S + bD + cM)\tilde{u} = \tilde{f}, \tag{3.4}
\]
where \( \tilde{u} = [\tilde{a}_0, \tilde{a}_1, ..., \tilde{a}_N]^T \) and \( \tilde{f}(m) = (f, L_m), \ m=0,1,...,N \). Thanks to the orthogonality of Legendre polynomials, and (2.10)-(2.11), it's easy to derive the exact entries of the matrices \( S, D \) and \( M \).

### 3.1 SPPs with one side boundary layer

We consider the following special case \( b \neq 0, c = 0 \) as the start point,
\[
\begin{cases}
-\varepsilon u''(x) + bu'(x) = f(x), \quad x \in I = (-1, 1), \\
u(-1) = u(1) = 0.
\end{cases} \tag{3.5}
\]

Note that the homogeneous linear equation \( -\varepsilon u''(x) + bu'(x) = 0 \) has the fundamental solutions \( c_1 + c_2 e^{bx/\varepsilon} \). We can see that the sign of the parameter \( b \) determine the location of the boundary layer. More precisely, for \( b < 0 \) the boundary layer will arises near the left end point; for \( b > 0 \) the boundary layer will arise near the right end point.

#### 3.1.1 For \( b < 0 \)

If the parameter \( b < 0 \), the boundary layer will arise near the left end point \(-1\), and the boundary layer function behave as \( e^{bx/\varepsilon} \). In order to meet the homogeneous boundary conditions, we take the singular term
\[
\psi(x) = 2e^{\frac{b(x+1)}{\varepsilon}} - 2 + (1 - e^{\frac{2b}{\varepsilon}})(1 + x).
\]

We state the detail of the ESG-II as follows:
1. Substituting the singular term $\psi$ into the singularly perturbed problem (3.5), it’s straightforward to know that $\psi$ is the solution of the problem

$$
\begin{align*}
-\varepsilon \psi''(x) + b \psi'(x) &= b(1 - e^{2b}), \\
\psi(-1) &= \psi(1) = 0.
\end{align*}
$$

Then we solve the above problem by the process (3.2)-(3.4) and derive the numerical solution

$$
\psi_N = \sum_{n=2}^{N} \tilde{\psi}_n P_n^{(-1,-1)}(x).
$$

2. Given $f \in L^2(I)$. Following the same process (3.2)-(3.4), the numerical approximation $u_N$ of the solution $u = u_* + c\psi$ can be derived that

$$
\begin{align*}
\psi_N &= \sum_{n=2}^{N} \tilde{\psi}_n P_n^{(-1,-1)}(x), \\
u_N &= \sum_{n=2}^{N} (\tilde{u}_n + c\tilde{\psi}_n) P_n^{(-1,-1)}(x),
\end{align*}
$$

where $u_{*,N}$ is the numerical approximation to the smooth part $u_*$. By assuming the last one coefficient $\tilde{u}_{*,N} = 0$, we have the numerical approximation

$$
c_N = \tilde{u}_N / \tilde{\psi}_N.
$$

Note that the numerical solution $u_{*,N}$ is unknown, we just used the property that the coefficients $\tilde{u}_{*,n}$ decay to zero very fast due to $u_*$ is smooth. But we can approximate the smooth part $u_*$ by

$$
u_{*,N} = u_N - c_N \psi_N.
$$

3. With $\psi, c_N$ and $\nu_{*,N}$ in hands, we can install a new approximation to $u$ below

$$
u_k = \nu_{*,N} + c_N \psi, \quad k = 1.
$$

In order to test the validity of the ESG-II, we take $f \equiv 1 - \pi \cos(\pi x) + \varepsilon \pi^2 \sin(\pi x)$ and $b = -1$. Then the related exact solution $u$ can be detected that

$$
u(x) = \frac{2e^{-\frac{1+x}{\varepsilon}} - 2 + (1 - e^{-\frac{2}{\varepsilon}})(1 + x)}{e^{-\frac{2}{\varepsilon}} - 1} + \sin(\pi x).
$$

With fixed $N = 35$, we plot the error curves of the $u - u_N, \psi_N - \psi$ and $u_k^N - u$ in Figure 1. The numerical results show that the ESG-II is effective comparing with the usual spectral method. Furthermore, to show the efficiency of the ESG-II, we draw the convergence curves in the left of Figure 2, in which the errors measured by discrete $L^2$ norm below

$$
\|v\|_{L^2} \approx \left( \sum_{j=1}^{N} (v(x_j))^2 \omega_j \right)^{1/2}
$$
where \( \{x_j, \omega_j\} \) are the related Gauss nodes and weights of the Legendre polynomials \( L_N(x) \).

Note that the numerical solutions \( u_N \) and \( \psi_N \), derived from the usual spectral method, are inefficient to approximate solutions \( u \) and \( \psi \) due to boundary layer functions changed sharply. However, we can derive the high-accuracy numerical solutions \( u^k_N \) from the low-efficiency numerical solutions \( u_N \) and \( \psi_N \) via the special techniques (3.7)-(3.11). In a sense, ESG-II can recovers the high efficiency for singularly perturbed equations just by repeating the usual spectral method several times.

\[\begin{align*}
\psi(x) &= 2e^{\frac{b(x-1)}{\epsilon}} - 2 + (e^{-\frac{2b}{\epsilon}} - 1)(x-1),
\end{align*}\]

and the related auxiliary problem

\[\begin{align*}
-\epsilon \psi''(x) + b \psi'(x) &= b(e^{-\frac{2b}{\epsilon}} - 1), \\
\psi(-1) &= \psi(1) = 0.
\end{align*}\]

Then we can follow the process (3.7)-(3.11) to solve the singularly perturbed equation (3.5) as before. We take the solution \( u \) and \( f \) as follows:

\[\begin{align*}
u(x) &= \frac{2e^{\frac{b(x-1)}{\epsilon}} - 2 + (e^{-\frac{2b}{\epsilon}} - 1)(x-1) + (1-x^2)e^x}{b(e^{-\frac{2b}{\epsilon}} - 1)}, \\
f(x) &= 1 + \epsilon(1 + 4x + x^2)e^x + b(1 - 2x - x^2)e^x.
\end{align*}\]

The numerical results plotted in Figure 2 illustrate that ESG-II is efficient for singularly perturbed equations with \( b > 0 \) and small parameters \( \epsilon \).

**Figure 1**: Left: \( \epsilon = 10^{-3}, N = 35 \); Right: \( \epsilon = 10^{-6}, N = 35 \).

**3.1.2 For \( b > 0 \)**

If the parameter \( b > 0 \), the boundary layer will arise near the right end point 1. We can follow the same process as the case \( b < 0 \) but the singular term

\[\psi(x) = 2e^{\frac{b(x-1)}{\epsilon}} - 2 + (e^{-\frac{2b}{\epsilon}} - 1)(x-1),\]

and the related auxiliary problem

\[\begin{align*}
-\epsilon \psi''(x) + b \psi'(x) &= b(e^{-\frac{2b}{\epsilon}} - 1), \\
\psi(-1) &= \psi(1) = 0.
\end{align*}\]

Then we can follow the process (3.7)-(3.11) to solve the singularly perturbed equation (3.5) as before. We take the solution \( u \) and \( f \) as follows:

\[\begin{align*}
u(x) &= \frac{2e^{\frac{b(x-1)}{\epsilon}} - 2 + (e^{-\frac{2b}{\epsilon}} - 1)(x-1) + (1-x^2)e^x}{b(e^{-\frac{2b}{\epsilon}} - 1)}, \\
f(x) &= 1 + \epsilon(1 + 4x + x^2)e^x + b(1 - 2x - x^2)e^x.
\end{align*}\]

The numerical results plotted in Figure 2 illustrate that ESG-II is efficient for singularly perturbed equations with \( b > 0 \) and small parameters \( \epsilon \).
3.2 SPPs with two sides boundary layer

We consider another interesting case: \( b = 0 \), namely, the following singularly perturbed problem

\[
\begin{aligned}
-\varepsilon u''(x) + cu(x) &= f(x), \quad x \in I = (0,1), \\
\quad u(0) = u(1) &= 0
\end{aligned}
\]  

(3.14)

where \( \varepsilon > 0 \) and \( c > 0 \). We can see from Schwab and Suri [23] that, for given smooth \( f \), the solution of the problem (3.14) decomposed into a smooth part \( u_r(x) \) and the boundary layer functions \( e^{-\sqrt{c/\varepsilon}(x+1)} \) and \( e^{\sqrt{c/\varepsilon}(x-1)} \). To meet the boundary conditions, we take the following two boundary layer functions (BLF) and the related auxiliary problems (AP)

**BLF1:**

\[
\begin{aligned}
\psi_1(x) &= 2e^{-\sqrt{c/\varepsilon}(x+1)} - 2\left(1- e^{-2\sqrt{c/\varepsilon}} \right)(1+x), \\
\psi_2(x) &= 2e^{\sqrt{c/\varepsilon}(x-1)} - 2\left(1- e^{2\sqrt{c/\varepsilon}} \right)(1-x)
\end{aligned}
\]  

(3.15)

**AP1:**

\[
\begin{aligned}
\quad -\varepsilon \psi_1''(x) + c\psi_1(x) &= c\left(1- e^{-2\sqrt{c/\varepsilon}} \right)(1+x) - 2c, \\
\psi_1(-1) &= \psi_1(1) = 0
\end{aligned}
\]

and

**BLF2:**

\[
\begin{aligned}
\psi_2(x) &= 2e^{\sqrt{c/\varepsilon}(x-1)} - 2\left(1- e^{-2\sqrt{c/\varepsilon}} \right)(1-x), \\
\psi_2(-1) &= \psi_2(1) = 0
\end{aligned}
\]  

(3.16)

**AP2:**

\[
\begin{aligned}
\quad -\varepsilon \psi_2''(x) + c\psi_2(x) &= c\left(1- e^{2\sqrt{c/\varepsilon}} \right)(1-x) - 2c, \\
\psi_2(-1) &= \psi_2(1) = 0
\end{aligned}
\]

Repeating the spectral scheme (3.3) for approximating \( u, \psi_1 \) and \( \psi_2 \), we can derive the related numerical approximations \( u_N, \psi_{1,N} \) and \( \psi_{2,N} \), respectively. Obviously we can expand those numerical solutions as

\[
\begin{aligned}
u_N &= \sum_{n=2}^{N} \tilde{u}_n P_n(-1,-1)(x), & \psi_{1,N} &= \sum_{n=2}^{N} \tilde{\psi}_{1,n} P_n(-1,-1)(x), & \psi_{2,N} &= \sum_{n=2}^{N} \tilde{\psi}_{2,n} P_n(-1,-1)(x)
\end{aligned}
\]  

(3.17)
Similar to the previous cases, we assume that the smooth part $u_*$ can be approximated by $u_{s,N} = \sum_{n=2}^{N} \tilde{u}_{s,n} P_n^{(-1,-1)}(x)$, then $u_N = u_{s,N} + c_1 \psi_1 + c_2 \psi_2$ implies that

$$\tilde{u}_n = \tilde{u}_{s,n} + c_1 \psi_{1,n} + c_2 \psi_{2,n}, \quad n = 2, 3, \ldots, N.$$ 

Without loss of generality, we can derive the approximation values $c_{1,N}$, $c_{2,N}$ of the constants $c_1$, $c_2$ by setting $\tilde{u}_{s,N-1} = 0$ and $\tilde{u}_{s,N} = 0$. Then we have the new approximation to the smooth part $u_*$ below

$$u_{s,N}^{II} = u_N - c_{1,N} \psi_1 - c_{2,N} \psi_2.$$  

(3.18)

Finally, we obtain the numerical solution

$$u_N^k = u_N^{II} + c_{1,N} \psi_1 + c_{2,N} \psi_2, \quad k = 2.$$ 

(3.19)

Similar to the previous cases, for validating the numerical scheme, we first take

$$f(x) = (\epsilon \pi^2 + c) \sin \pi x - c(1 + e^{-2\sqrt{c/\epsilon}})$$

and the corresponding exact solution

$$u(x) = e^{-\sqrt{c/\epsilon} (x+1)} + e^{\sqrt{c/\epsilon} (x-1)} - (1 + e^{-2\sqrt{c/\epsilon}}) \sin(\pi x).$$

The left of the graph demonstrates that the numerical solutions of the ESG-II exponentially converge to the solutions, where the parameters $c = 1$ and $\epsilon = 10^{-p}$.

Next, for given $f = e^x \cos \pi x$, the right of Figure 3 shows that the numerical errors decay to zero exponentially, in which the exact solutions are replaced by the reference solutions $u_{N,N}^k$, $N = 100$ due to the exact solutions can not be detected.
4 Applications to singular integral equations

In this section, we are devoted to applying the enriched spectral method to deal with an integral equation with weakly singular kernel \( K(x,s) := (x-s)^{\mu-1} q(s) \) below

\[
u(x) + \int_{-1}^{1} K(x,s) u(s) ds = f(x), \quad f \in L^2(I), \quad I = (-1,1).
\] (4.1)

Spectral Galerkin method

For \( a,b \in \mathbb{R} \) and \( \rho \in \mathbb{R}^+ \), the left and right fractional integrals are respectively defined as (see e.g., [21, 22]):

\[
\begin{align*}
\mathcal{I}^\rho_x v(x) &= \frac{1}{\Gamma(\rho)} \int_a^x \frac{v(y)}{(x-y)^{1-\rho}} dy, \quad \text{(4.2a)} \\
\mathcal{I}_x^\rho v(x) &= \frac{1}{\Gamma(\rho)} \int_x^b \frac{v(y)}{(y-x)^{1-\rho}} dy, \quad x \in (a,b).
\end{align*}
\]

The weak formulation of the integral equation is to find \( u \in L^2(I) \) such that

\[
\langle a(u,v) := (u,v) + \Gamma(\mu)(q u, \mathcal{I}_x^\mu v) = (f,v), \quad \forall v \in L^2(I),
\]

where the second term of the left hand side owes to \( \langle \mathcal{I}_x^\rho u, v \rangle = \langle u, \mathcal{I}_x^\rho v \rangle \) for all \( u, v \in L(I) \).

Then we can approximate the weak solution in polynomials space. In order to derive the simple and well-conditioned matrix system, we propose the Legendre spectral scheme as follows: Find \( u_N \in P_N(I) \) such that

\[
a(u_N, v) = (f, v) \quad \forall v \in P_N(I),
\] (4.4)

where we can expand the numerical solution \( u_N(x) = \sum_{n=0}^N \tilde{u}_n L_n(x) \).

By substituting \( v(x) = L_m(x), \ m = 0,1,\ldots,N \) successively into the scheme (4.4), we obtain the matrix system

\[
(M + I^\mu) \tilde{u} = \tilde{f},
\] (4.5)

where \( \tilde{u} = [\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_N]^T \) and \( \tilde{f}(m) = (f, L_m), \ m = 0,1,\ldots,N \), the matrices

\[M(m,n) = (L_m, L_n), \quad I^\mu(m,n) = \Gamma(\mu)(q L_m, \mathcal{I}_x^\mu L_n), \quad n,m = 0,1,\ldots,N.\]

Owe to the orthogonality of Legendre polynomials, it’s easy to know that \( M \) is a diagonal matrix with entries \( 2 \delta_{mm} / (2n+1) \). Moreover, via the relation [8, Lemma 2.4]

\[
\mathcal{I}_x^\rho \{ L_m(x) \} = \frac{\Gamma(m+1)}{\Gamma(m+1+\rho)} \frac{1}{(1-x)^{\rho}} P_m^{\rho,-\rho}(x),
\]

we can derive the matrix \( I^\mu \) by Jacobi Gauss quadrature formula with \( (a,\beta) = (\mu,0) \).
Enriched spectral Galerkin method

We can see from the previous section that the prerequisites of ESG-II are: i) an easily implemented spectral scheme; ii) the determined leading singular terms. With the spectral scheme (4.4), the remainder is to determine the singular terms. Fortunately, owe to the analysis of singularity in [4, 6] and the mapping \( x = 2t - 1, \ t \in (0,1) \), we have a consequence of the solution’s singularity of the integral problem (4.1) below.

**Proposition 4.1.** Let \( r \) be a nonnegative integer. Suppose that \( q \in C^r(\bar{I}) \) and \( f \) has the form

\[
    f(x) = \sum_{j+i\mu \leq m} \tilde{f}_{ij}(1+x)^{j+i\mu} + f_m(x), \quad x \in \bar{I},
\]

where \( \tilde{f}_{ij} \) are constants and function \( f_m \in C^m([0,T]) \) for a fixed integer \( m \) with \( 0 \leq m \leq r \).

Let \( u \) be the solution of (4.1). Then there exist constants \( \tilde{u}_{ij} \) such that

\[
    u(x) = \sum_{j+i\mu \leq m} \tilde{u}_{ij}(1+x)^{j+i\mu} + u_m(x), \quad x \in \bar{I},
\]

where function \( u_m \in C^m(\bar{I}) \).

Based on the above singularity analysis, we collect several leading singular terms as a singular space and denote by

\[
    S_{lk} := \text{span}\{ \psi_l : \psi_l(x) = (1+x)^{j+i\mu}, \ l = 1,2,\ldots,l_k \},
\]

where positive integer index \( l \) labelled by the increasing order of all the terms \( (1+x)^{j+i\mu} \) such that \( j+i\mu \notin \mathbb{N} \) and \( j+i\mu < k \). Now we are ready to solve the integral equation by ESG-II.

**Numerical examples**

We first verify the validity of the numerical scheme by taking \( q(x) \equiv 1 \) and an exact solution

\[
    u(x) = (1+x)^{1-\mu} P_{M}^{\mu-1,1-\mu}(x) = (1+x)^{1-\mu} \sum_{n=0}^{M} c_n (1+x)^n, \quad M \in \mathbb{N}.
\]

Then via the fractional integral relation (see [8]), the right hand side term \( f \) can be explicitly derived that

\[
    f(x) = (1+x)^{1-\mu} \int_M^{\mu-1,1-\mu}(x) + B(\mu,M+2-\mu)(1+x)P_{M}^{\mu-1,1}(x),
\]

where \( B(\mu,M+2-\mu) = \Gamma(\mu)\Gamma(M+2-\mu)/\Gamma(M+2) \) is the classical Beta function and the negative integer parameter Jacobi polynomial (generalized Jacobi polynomial) \( J_M^{\mu-1,1}(x) \) is defined by hypergeometric function in [30]. In view of the structure of the exact solution, the singular terms can be ordered by its regularity

\[
    (1+x)^{1-\mu}, \ (1+x)^{2-\mu}, \ldots, (1+x)^{M-\mu}.
\]
Then, using the enriched spectral Galerkin method with \( \mu = 0.7, M = 10 \), it can be observed from the left of Figure 4 that the convergence rate can be enhanced by subtracting singular terms step by step.

Next, let \( q(x) = e^x, \mu = 0.57 \). We consider a smooth function \( f(x) = \sin(x) \) as the data which can be approximated by polynomial efficiently as follows

\[
f(x) = \sum_{n=0}^{m} f_n(1+x)^n + f_m(x),
\]

where \( f_m \in C^m(\bar{I}), m \to \infty \). According to the consequence of the Proposition 4.1, the solution \( u \) has the form

\[
u(x) = \sum_{j+i\mu < m} \tilde{u}_{ij}(1+x)^{j+i\mu} + u_m(x), \quad x \in \bar{I}.
\]

The right of Figure 4 depicts the convergence rate of the usual spectral method and the enriched spectral Galerkin method with \( k = 1, 2, 3 \), where we deem \( u_3, N = 200 \) as the solution \( u \) due to there is no exact solution can be detected.

![Figure 4](image)

**Figure 4**: Left: \( u = (1+x)^{1-\mu} \mu^{-1,1-\mu}(x), \mu = 0.7 \); Right: \( f = \sin x, \mu = 0.57 \)

## 5 Conclusion

In this work we consider several singularly perturbed problems and singular integral equations with a few boundary layer functions/leading singular terms which can be determined. In order to recovery the high-efficiency of the spectral method, the enriched spectral Galerkin methods (ESG), in the same spirit of extended or generalized finite element method \([2, 12, 29]\), are applied to derive the accurate solutions. Successful implementations of ESG rely on three ingredients: (i) analyse the underlying problem and determine a few leading singular terms (ii) modify the singular functions to meet the
boundary conditions of the underlying problem; (iii) use ESG-II, which is based on a special property of the spectral methods, to approximate the solution in the enriched spectral space. In particular, the detail of the process (iii) combing the singularly perturbed problems are provided in Section 3 for easily understanding the algorithm. Ample numerical examples showed that ESG-II is capable of producing significantly improved results over the usual spectral-Galerkin methods with adding only a few boundary layer/singular functions.

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