The skew Brownian permuton: A new universality class for random constrained permutations

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Abstract
We construct a new family of random permutons, called skew Brownian permuton, which describes the limits of several models of random constrained permutations. This family is parameterized by two real parameters. For a specific choice of the parameters, the skew Brownian permuton coincides with the Baxter permuton, that is, the permuton limit of Baxter permutations. We prove that for another specific choice of the parameters, the skew Brownian permuton coincides with the biased Brownian separable permuton, a one-parameter family of permutons previously studied in the literature as the limit of uniform permutations in substitution-closed classes. This brings two different limiting objects under the same roof, identifying a new larger universality class. The skew Brownian permuton is constructed in terms of flows of solutions of certain stochastic differential equations (SDEs) driven by two-dimensional correlated Brownian excursions in the nonnegative quadrant. We call these SDEs skew perturbed Tanaka equations because they are a mixture of the perturbed Tanaka equations and the equations encoding skew Brownian motions. We prove existence and uniqueness of (strong) solutions for these new SDEs. In addition, we show that some natural permutons arising from Liouville quantum gravity (LQG) spheres decorated with
two Schramm–Loewner evolution (SLE) curves are skew Brownian permutons and such permutons cover almost the whole range of possible parameters. Some connections between constrained permutations and decorated planar maps have been investigated in the literature at the discrete level; this paper establishes this connection directly at the continuum level. Proving the latter result, we also give an SDE interpretation of some quantities related to SLE-decorated LQG spheres.

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1 | INTRODUCTION

1.1 | Limits of random permutations

The study of limits of random permutations is a classical topic in probability theory. The typical question is to determine the behavior of a large random permutation when its size tends to infinity. For many years, the main approach to answering this question has been to study the convergence of relevant statistics, such as the number of inversions, the length of the longest increasing subsequence, the number of cycles, and many others. In the last decade, a more geometric approach has been investigated, mainly to study limits of nonuniform models of random permutations. In this case, the goal is to directly determine the limit of the permutation itself from a global perspective. This theory goes under the name of permuton limits. For a complete introduction to the theory of permuton limits, we refer the reader to [13, section 2.1]. Here we only recall some basic definitions.

A Borel probability measure \( \mu \) on the unit square \([0,1]^2\) is called a permuton if its marginals are uniform, that is, \( \mu([0,1] \times [x,y]) = \mu([x,y] \times [0,1]) = x - y \) for all \( 0 \leq x \leq y \leq 1 \).

Presutti and Stromquist [59] first used permutons, calling them normalized measures, to investigate the packing density of some specific patterns. A more theoretical approach to the study of deterministic permutons (again without using this terminology) was developed by Hoppen, Kohayakawa, Moreira, Rath, and Sampaio [43]. The word permuton appeared for the first time in a work of Glebov, Grzesik, Klimošová, and Král [36]. They adopt this terminology by analogy with graphon in the theory of random graphs [53]. Various results appearing in [43] were generalized to the case of random permutons by Bassino, Bouvel, Féray, Gerin, Maazoun, and Pierrot [8].

To a permutation \( \sigma \) of size \( n \geq 1 \), it is possible to associate a natural permuton \( \mu_\sigma \), given by the sum of Lebesgue area measures

\[
\mu_\sigma(A) = n \sum_{i=1}^{n} \text{Leb} \left( \left[ \frac{(i-1)}{n}, \frac{i}{n} \right] \times \left[ \frac{(\sigma(i) - 1)}{n}, \frac{\sigma(i)}{n} \right] \cap A \right),
\]

where \( A \) is a Borel measurable set of \([0,1]^2\). Note that \( \mu_\sigma \) corresponds to the normalized diagram of \( \sigma \), where each dot has been replaced with a square of dimension \( 1/n \times 1/n \) and total mass \( 1/n \). See the example given in Figure 1.

Let \( \mathcal{M} \) be the set of permutons endowed with the weak topology. A sequence of permutons \( (\mu_n)_{n \in \mathbb{Z}^+} \) converges weakly to \( \mu \), and we write \( \mu_n \rightharpoonup \mu \), if

\[
\int_{[0,1]^2} f \, d\mu_n \to \int_{[0,1]^2} f \, d\mu,
\]

for every continuous function \( f : [0,1]^2 \to \mathbb{R} \). When a sequence of permutons \( (\mu_n)_{n \in \mathbb{Z}^+} \) converges weakly to \( \mu \) we will often say that \( (\mu_n)_{n \in \mathbb{Z}^+} \) converges to \( \mu \) in the permuton sense.

In the past years, permuton limits of various models of nonuniform random permutations have been investigated in the literature. These models can be divided into two main different classes.

- The models that exhibit a random fractal limiting permuton: Here the classical examples are pattern-avoiding permutations (see below for a more detailed discussion).
The models that exhibit a deterministic limiting permuton: Here some examples are Erdős–Szekeres permutations [61], Mallows permutations [63], random sorting networks [28], square and almost square permutations [16, 20], and permutations sorted with the runsort algorithm [1].

The present paper focuses on models that exhibit a limiting random fractal permuton. Our main goal is to introduce a new family of limiting permutons that unifies the various instances of random fractal limiting permutons that appear in the literature and some new ones.

We now quickly review the literature on models of pattern-avoiding permutations that exhibit a random fractal permuton in the limit. If the reader is not familiar with the terminology related to pattern-avoiding permutations, he/she can find a quick introduction in [13, section 1.6.1].

In [7], Bassino, Bouvel, Féray, Gerin, and Pierrot prove that a sequence of uniform random separable permutations† converges to the Brownian separable permuton (see the picture on the left-hand side of Figure 2). This is the first work where a random fractal limiting permuton appears in the literature.

In a second work [8] (see also [15]), the authors prove that the Brownian separable permuton is universal: they consider uniform random permutations in proper substitution-closed classes§ and show that their limits in the permuton sense are a one-parameter deformation of the Brownian separable permuton, called biased Brownian separable permuton (see the picture in the middle of Figure 2).

These new universal permutons are later investigated by Maazoun [54].

In [9], the authors investigate permuton limits for permutations in classes having a finite combinatorial specification for the substitution decomposition. The limit depends on the structure of the specification restricted to families with the largest growth rate. When the specification is strongly connected, two cases occur. If the associated system of equations is linear, the limiting permuton is a deterministic $X$-shape. Otherwise, the limiting permuton is the biased Brownian separable permuton.

† To be precise, we remark that the permuton limits of square and almost square permutations with a finite number of internal points are random, but their randomness can be simply expressed in terms of a beta distributed random variable. On the contrary, almost square permutations with an infinite number of internal points exhibit the same deterministic permuton limit. For these reasons, we prefer to classify square and almost square permutations as permutations that exhibit a deterministic permuton limit.

§ Separable permutations are the permutations avoiding the patterns 2413 and 3142. They are one of the most studied families of pattern-avoiding permutations, see, for instance, [2, 4, 21, 62].

† For an introduction to proper substitution-closed classes, we refer the reader to [13, section 3.2].
Finally, in [19], Maazoun and the author of this paper show that the permuton limit of Baxter permutations is a new random fractal limiting permuton, called the Baxter permuton (see the picture in the right-hand side of Figure 2), that is not included in the biased Brownian separable permuton universality class.

The next two sections review the constructions of the biased Brownian separable permuton and the Baxter permuton. These constructions are fundamental to understand later in Subsection 1.4 our definition of the new family of limiting permutons mentioned before.

### 1.2 The biased Brownian separable permuton

We introduce the biased Brownian separable permuton following [54, sections 1.3–4]. Consider a one-dimensional Brownian excursion (\(e(t)\))\(_{t \in [0,1]}\) on \([0,1]\) and a parameter \(p \in [0,1]\). Conditional on \(e\), consider an independent and identically distributed sequence \((s(\ell))_{\ell \in \mathbb{Z}_{>0}}\) indexed by the local minima of \(e\) and with distribution \(\mathbb{P}(s(\ell) = +1) = p = 1 - \mathbb{P}(s(\ell) = -1)\). We denote by \((\bar{e}, p)\) the pair \((e, (s(\ell))_\ell)\). We define the following random relation \(\prec_{\bar{e}, p}\): conditional on \(e\), if \(x, y \in [0,1]\) and \(x < y\) and \(\min_{[x,y]} e\) is reached at a unique point which is a strict local minimum \(\ell \in [x, y]\) then

\[
\begin{align*}
    x &\prec_{\bar{e}, p} y, & \text{if } s(\ell) = +1, \\
    y &\prec_{\bar{e}, p} x, & \text{if } s(\ell) = -1.
\end{align*}
\]

(1)

Maazoun showed that there exists a random set \(A \subset [0,1]^2\) of almost surely zero Lebesgue measure, that is, \(\mathbb{P}(\text{Leb}(A) = 0) = 1\), such that for every \(x, y \in [0,1]^2 \setminus A\) with \(x < y\) then \(\min_{[x,y]} e\) is reached at a unique point which is a strict local minimum. In particular, the restriction of \(\prec_{\bar{e}, p}\)

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† Baxter permutations were introduced by Glen Baxter in 1964 [5] to study fixed points of commuting functions. Baxter permutations are permutations avoiding the vincular patterns 2413 and 3142, that is, permutations \(\sigma\) such that there are no indices \(i < j < k\) such that \(\sigma(j + 1) < \sigma(i) < \sigma(k) < \sigma(j)\) or \(\sigma(i) < \sigma(k) < \sigma(j + 1)\). These permutations are deeply studied in combinatorics, see, for instance, [23, 24, 27, 33, 55] and references therein.

† Here and throughout the paper, we denote random quantities using bold characters.

† For the technicalities involved in indexing an independent and identically distributed sequence by this random countable set, see [54, section 2.2].
to $[0,1]^2 \setminus A$ is a total order. Setting
\[ \psi_{\mathbf{e},p}(t) := \text{Leb}\left( \{ x \in [0,1] | x \prec_{\mathbf{e},p} t \} \right), \quad t \in [0,1], \]
then the biased Brownian separable permuton is defined (see [54, Theorem 1.3]) as the pushforward of the Lebesgue measure on $[0,1]$ via the mapping $(\text{Id}, \psi_{\mathbf{e},p})$, that is,
\[ \mu^S_p(\cdot) := (\text{Id}, \psi_{\mathbf{e},p})_* \text{Leb}(\cdot) = \text{Leb}\left( \{ t \in [0,1] | (t, \psi_{\mathbf{e},p}(t)) \in \cdot \} \right). \]
Heuristically, $\psi_{\mathbf{e},p}$ is the “continuum permutation” of the elements in the interval $[0,1]$ induced by the order $\prec_{\mathbf{e},p}$ and $\mu^S_p$ is the diagram of $\psi_{\mathbf{e},p}$.

1.3 | The Baxter permuton

We now introduce the Baxter permuton following [19]. To do that, we first define the continuous coalescent-walk process driven by a two-dimensional Brownian excursion.

We start by recalling that a two-dimensional Brownian motion of correlation $\rho \in [-1,1]$, denoted $(\mathbf{W}_\rho(t))_{t \in \mathbb{R}_{\geq 0}} = ((\mathbf{X}_\rho(t), \mathbf{Y}_\rho(t)))_{t \in \mathbb{R}_{\geq 0}}$, is a continuous two-dimensional Gaussian process such that the components $\mathbf{X}_\rho$ and $\mathbf{Y}_\rho$ are standard one-dimensional Brownian motions, and $\text{Cov}(\mathbf{X}_\rho(t), \mathbf{Y}_\rho(s)) = \rho \cdot \min\{t, s\}$. We also recall that a two-dimensional Brownian excursion $(\mathbf{E}_\rho(t))_{t \in [0,1]}$ of correlation $\rho \in (-1,1]$ in the nonnegative quadrant (here simply called a two-dimensional Brownian excursion of correlation $\rho$) is a two-dimensional Brownian motion of correlation $\rho$ conditioned to stay in the nonnegative quadrant $\mathbb{R}_{\geq 0}^2$ and to end at the origin, that is, $\mathbf{E}_\rho(1) = (0,0)$. The latter process was formally constructed in various works (see, for instance, [57, section 3] and [32]).

Let $(\mathbf{E}_{-1/2}(t))_{t \in [0,1]}$ be a two-dimensional Brownian excursion of correlation $-1/2$. Consider the (strong) solutions—which exist and are unique thanks to Theorem 4.6 in [19]—of the following family of stochastic differential equations (SDEs) indexed by $u \in [0,1]$ and driven by $\mathbf{E}_{-1/2}(t) = (\mathbf{X}_{-1/2}(t), \mathbf{Y}_{-1/2}(t))$:
\[
d\mathbf{Z}^{(u)}(t) = \begin{cases} d\mathbf{Y}_{-1/2}(t) - d\mathbf{X}_{-1/2}(t), & t \in (u, 1), \\ 0, & t \in [0, u]. \end{cases}
\]
\[ \mathbf{Z}^{(u)}(t) = 0, \quad t \in [0, u]. \quad (2) \]

Definition 1.1. The continuous coalescent-walk process driven by $\mathbf{E}_{-1/2}$ is the collection of stochastic processes $\{\mathbf{Z}^{(u)}\}_{u \in [0,1]}$ defined by the SDEs in Equation (2).

We can now formally introduce the Baxter permuton (a heuristic explanation is given after Definition 1.2). Consider the following stochastic process:
\[ \varphi_{\mathbf{Z}}(t) := \text{Leb}\left( \{ x \in [0,t] | \mathbf{Z}^{(x)}(t) < 0 \} \cup \{ x \in [t,1] | \mathbf{Z}^{(t)}(x) > 0 \} \right), \quad t \in [0,1], \]
where $\text{Leb}(\cdot)$ denotes the one-dimensional Lebesgue measure.

---

1 We highlight that we excluded the case $\rho = -1$ in the definition of the two-dimensional Brownian excursion. Indeed, if $\rho = -1$ it is not meaningful to condition a two-dimensional Brownian motion of correlation $\rho = -1$ to stay in the nonnegative quadrant.
The Baxter permuton $\mu^B$ is the pushforward of the Lebesgue measure on $[0,1]$ via the mapping $(\text{Id}, \varphi_Z)$, that is,

$$\mu^B(\cdot) := (\text{Id}, \varphi_Z)_* \text{Leb}(\cdot) = \text{Leb}\left\{ (t \in [0,1]) | (t, \varphi_Z(t)) \in \cdot \right\}.$$

The Baxter permuton $\mu^B$ is a random measure on the unit square $[0,1]^2$ and it has uniform marginals [19, Lemma 5.5], hence it is a permuton. Informally, as in the case of the biased Brownian separable permuton, $\varphi_Z$ is the “continuum permutation” of the elements of $[0,1]$ induced by the order

$$t \leq_Z s \quad \text{if and only if} \quad Z^{(i)}(s) < 0,$$

where $0 \leq t < s \leq 1$, and $\mu^B$ is the diagram of $\varphi_Z$. More details are given in Subsection 3.1.

### 1.4 A new family of universal permutons: The skew Brownian permuton

In the previous two sections, we introduced the biased Brownian separable permuton and the Baxter permuton. A natural question is to explain what is the connection between the two. The main goal of this paper is to answer this question by constructing a new family of permutons, called skew Brownian permuton, that includes both the biased Brownian separable permuton and the Baxter permuton. This brings two different limiting objects under the same roof, identifying a new larger universality class.

#### 1.4.1 Definitions and construction

Let $(\mathcal{E}_\rho(t))_{t \in [0,1]}$ be a two-dimensional Brownian excursion of correlation $\rho \in (-1,1]$ and let $q \in [0,1]$ be a further parameter. Consider the solutions (see Subsection 1.4.2 for a discussion on existence and uniqueness) to the following family of SDEs indexed by $u \in [0,1]$ and driven by $\mathcal{E}_\rho = (\mathcal{X}_\rho, \mathcal{Y}_\rho)$:

$$\begin{cases}
   dZ_{\rho,q}^{(u)}(t) = 1_{\{Z_{\rho,q}^{(u)}(t) > 0\}} dY_{\rho}(t) - 1_{\{Z_{\rho,q}^{(u)}(t) \leq 0\}} dX_{\rho}(t) + (2q - 1) \cdot dL_{\rho,q}^{(u)}(t), & t \in (u,1) \\
   Z_{\rho,q}^{(u)}(t) = 0, & t \in [0,u],
\end{cases}$$

where $L_{\rho,q}^{(u)}(t)$ denotes the symmetric local-time process at zero of the process $Z_{\rho,q}^{(u)}$, that is,

$$L_{\rho,q}^{(u)}(t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{Z_{\rho,q}^{(u)}(s) \in [-\varepsilon,\varepsilon]\}} ds.$$

**Definition 1.3.** For all $\rho \in (-1,1]$ and $q \in [0,1]$, we call continuous coalescent-walk process driven by $(\mathcal{E}_\rho, q)$ the collection of stochastic processes $Z_{\rho,q} = \{Z_{\rho,q}^{(u)} \}_{u \in [0,1]}$.

We then consider the following stochastic process:

$$\varphi_{\rho,q}^{(u)}(t) := \text{Leb}\left\{ x \in [0,t] | Z_{\rho,q}^{(u)}(x) < 0 \right\} \cup \left\{ x \in [t,1] | Z_{\rho,q}^{(u)}(x) \geq 0 \right\}, \quad t \in [0,1].$$
Definition 1.4. Fix $\rho \in (-1,1]$ and $q \in [0,1]$. The skew Brownian permuton of parameters $\rho, q$, denoted $\mu_{\rho,q}$, is the pushforward of the Lebesgue measure on $[0,1]$ via the mapping $(\text{Id}, \varphi_{\rho,q})$, that is,

$$\mu_{\rho,q}(\cdot) := (\text{Id}, \varphi_{\rho,q})_*\text{Leb} = \text{Leb}\left(\{t \in [0,1] | (t, \varphi_{\rho,q}(t)) \in \cdot\}\right).$$

Remark 1.5. Note that when $\rho = -1/2$ and $q = 1/2$ the SDEs in Equation (3) are exactly the SDEs considered in Equation (2). Therefore, the Baxter permuton $\mu^B(\cdot)$ coincides with the skew Brownian permuton $\mu_{-1/2,1/2}$.

Remark 1.6. We highlight that for different values of $u \in [0,1]$, the SDEs in Equation (3) are defined using the same two-dimensional Brownian excursion $(\mathcal{E}_{\rho}(t))_{t \in [0,1]}$. Therefore, the coupling of $\mathcal{Z}_{\rho,q}^{(u)}$ for different values of $u \in [0,1]$ is highly nontrivial.

To guarantee that the skew Brownian permuton is well-defined for all $\rho \in (-1,1]$ and $q \in [0,1]$, we need first to discuss existence and uniqueness of solutions to the SDEs in Equation (3) and then to check that $\mu_{\rho,q}$ is indeed a permuton, that is, it has uniform marginals.

1.4.2 The skew perturbed Tanaka equations and correctness of the definition

We now focus on the SDEs in Equation (3), which we call skew perturbed Tanaka equations. The terminology is inspired by the names of some SDEs already studied in the literature. Indeed, when $q = 1/2$, that is, when the local time term cancels in Equation (3), we obtain the perturbed Tanaka equations studied in [26,60]. As here we are adding a local time term, we adopt the terminology skew perturbed Tanaka equations by analogy with the case of skew Brownian motions: A skew Brownian motion of parameter $q \in [0,1]$ is a standard one-dimensional Brownian motion where each excursion is flipped independently to the positive side with probability $q$ (see, for instance, [46, Theorem 6]). It is shown in [42] that the unique strong solution to the SDE

$$\mathcal{Z}(t) = B(t) + (2q - 1)d\mathcal{L}\mathcal{Z}(t), \quad t \in \mathbb{R}_{\geq 0},$$

where $B(t)$ is a standard one-dimensional Brownian motion, is a skew Brownian motion of parameter $q$.

It is also convenient to distinguish the following two cases:

- when $\rho \in (-1,1)$ (and $q \in [0,1]$), we will refer to the SDEs in Equation (3) as skew pure perturbed Tanaka equations;
- when $\rho = 1$ (and $q \in [0,1]$), we will refer to the SDEs in Equation (3) simply as skew Tanaka equations.

---

1 For a discussion on the possible construction of the skew Brownian permuton when $\rho = -1$, we refer the reader to the open problems in Subsection 1.6.

1 To be precise, the articles [26,60] study the perturbed Tanaka equations driven by a correlated Brownian motion instead of a correlated Brownian excursion as in Equation (3). In [26,60], it is proved that there exists a unique (strong) solution to the perturbed Tanaka equation driven by a correlated Brownian motion. Then in [19, Theorem 4.6], pathwise uniqueness and existence of a strong solution to the SDE in Equation (3) was deduced from the results in [26, 60] using absolute continuity arguments.
The skew pure perturbed Tanaka equations have not been investigated so far in the literature. Our first result guarantees existence and uniqueness of a strong solution.

**Theorem 1.7.** Fix \( \rho \in (−1, 1) \) and \( q \in [0, 1] \). Pathwise uniqueness and existence of a strong solution to the SDEs in Equation (3) hold for all \( u \in (0, 1) \).

More precisely, denoting by \( \mathbb{P}_\rho \) the law of \( \mathbf{E}_\rho \), we consider the sigma-algebra \( \mathcal{F}_t^{(u)} \) generated by \( \mathbf{E}_\rho(s) − \mathbf{E}_\rho(u) \) for \( s \in [u, 1] \) and completed by negligible events of \( \mathbb{P}_\rho \). For every \( u \in (0, 1) \), there exists a continuous \( \mathcal{F}_t^{(u)} \)-adapted stochastic process \( Z^{(u)}_{\rho, q} \) on \([u, 1)\), such that

1. the function \((\omega, u) \mapsto Z^{(u)}_{\rho, q}\) is jointly measurable;
2. for every \( u, r \in (0, 1) \), \( u < r \), \( Z^{(u)}_{\rho, q} \) satisfies Equation (3) almost surely on the interval \([u, r]\);
3. if \( u, r \in (0, 1) \), \( u < r \), and \( \tilde{Z} \) is a continuous \( \mathcal{F}_t^{(u)} \)-adapted stochastic process that satisfies Equation (3) almost surely on the interval \([u, r]\), then \( \tilde{Z} = Z^{(u)}_{\rho, q} \) almost surely on \([u, r]\).

**Remark 1.8.** We remark that Theorem 1.7 is also a fundamental tool to prove permuton convergence toward the skew Brownian permuton for various models of random constrained permutations, as shown in [14]. For more details, see, for instance, the proof of [14, Theorem 4.4].

We highlight that the case \( \rho = 1 \) is excluded in Theorem 1.7. Indeed, when \( \rho = 1 \), existence of a strong solution to the SDEs in Equation (3) fails (see Subsection 2.2 for a precise discussion). Note that when \( \rho = 1 \), the two-dimensional Brownian excursion \( (\mathbf{E}_1(t))_{t \in [0, 1]} \) rewrites as \( \mathbf{E}_1(t) = (e(t), e(t)) \), where \( (e(t))_{t \in [0, 1]} \) is a one-dimensional Brownian excursion on \([0, 1]\). Therefore, the SDEs in Equation (3) take the simplified form

\[
\begin{align*}
    dZ^{(u)}(t) &= \text{sgn}(Z^{(u)}(t))de(t) + (2q - 1) \cdot dL^{(u)}(t), \quad t \in [u, 1), \\
    Z^{(u)}(t) &= 0, \quad t \in [0, u],
\end{align*}
\]

where \( \text{sgn}(x) := 1_{\{x > 0\}} - 1_{\{x \leq 0\}} \).

Solutions to the SDEs in Equation (5) are not measurable functions of the driving process \((e(t))_{t \in [0, 1]} \) but can be constructed using some external randomness as follows. Conditional on \( e \), consider an independent and identically distributed sequence of random variables \((s(\ell))_{\ell} \) indexed by the local minima\(^1\) of \( e \) and with distribution \( \mathbb{P}(s(\ell) = +1) = q = 1 - \mathbb{P}(s(\ell) = -1) \). For \( 0 \leq u \leq t \leq 1 \), set \( m^{(u)}(t) := \inf_{[u, t]} e \) and \( \xi^{(u)}_{q}(t) := s(\sup\{r \in [u, t] : e(r) = m^{(u)}(t)\}) \). Then, define for \( 0 \leq u \leq t \leq 1 \),

\[
Z^{(u)}_{q}(t) := (e(t) - m^{(u)}(t)) \cdot \xi^{(u)}_{q}(t).
\]

\(^1\)For the technicalities involved in indexing an independent and identically distributed sequence by this random countable set, see again [54, section 2.2].
**Proposition 1.9.** The family \( \{ \mathcal{Z}_q^{(u)}(t) \}_{u \in [0,1]} \) defined in Equation (6) is a family of solutions to the SDEs in Equation (5).

**Remark 1.10.** The construction in Equation (6) is inspired by the works of Hajri [39] and Le Jan and Raimond [49, 51] on stochastic flow of maps. We will discuss this connection in Subsection 2.2. For a discussion on the uniqueness of the solution in Proposition 1.9, we refer the reader to Remark 2.8.

Theorem 1.7 and Proposition 1.9 therefore imply that the continuous coalescent-walk process \( \mathcal{Z}_{\rho,q} = \{ \mathcal{Z}_{\rho,q}^{(u)} \}_{u \in [0,1]} \) introduced in Definition 1.3 is well-defined for all \((\rho, q) \in (-1,1] \times [0,1]\). Checking that the marginals of skew Brownian permuton \( \mu_{\rho,q} \) are uniform (as done in Subsection 3.1), we obtain the following result.

**Theorem 1.11.** The skew Brownian permuton \( \mu_{\rho,q} \) is well-defined for all \((\rho, q) \in (-1,1] \times [0,1]\). That is, \( \mu_{\rho,q} \) is a permuton for all \((\rho, q) \in (-1,1] \times [0,1]\).

Simulations of the skew Brownian permuton \( \mu_{\rho,q} \) for various values of \((\rho, q) \in (-1,1] \times [0,1]\) can be found in Figure 3. How these simulations were obtained is explained in Appendix B.

### 1.4.3 Relations with the biased Brownian separable permuton and the Baxter permuton

As already highlighted in Remark 1.5, the Baxter permuton coincides with the skew Brownian permuton of parameters \( \rho = -1/2 \) and \( q = 1/2 \). The relation between the skew Brownian permuton and the biased Brownian separable permuton is less trivial and will be investigated in Subsection 3.2, where we will prove the following result.

**Theorem 1.12.** For all \( p \in [0,1] \), the biased Brownian separable permuton \( \mu_p^S \) has the same distribution as the skew Brownian permuton \( \mu_{1,1-p} \).

As mentioned in the abstract, our new construction of the skew Brownian permuton brings two different limiting objects under the same roof, identifying a new larger universality class and explaining the connection between the Baxter permuton and the biased Brownian separable permuton.

The unsatisfactory feature of the instances of the skew Brownian permuton mentioned above is that either \( \rho = 1 \) (biased Brownian separable permuton) or \( q = 1/2 \) (Baxter permuton), and in both cases the SDEs in Equation (3) take a simplified form: either the driving process is a one-dimensional Brownian excursion, as in Equation (5), or the local time term cancels, as in Equation (2) (and so we simply have a perturbed Tanaka equation instead of a skew perturbed Tanaka equation). In a companion paper [14], we show that this is not the case for the family of strong-Baxter permutations. We recall this result in the next section.

### 1.4.4 A model of random permutations converging to the skew Brownian permuton with nondegenerate parameters

We start by defining strong-Baxter permutations.
We collect here some simulations of the skew Brownian permuton $\mu_{\rho,q}$ for various values of $(\rho, q) \in (-1, 1] \times [0, 1]$. In every row, there are five simulations of $\mu_{\rho,q}$ and at the end of each row, there is the corresponding two-dimensional Brownian excursion of correlation $\rho$ in the nonnegative quadrant (the specific value of $\rho$ is indicated at the beginning of every row). Each column corresponds to different values of the parameter $q$ (the specific value of $q$ is indicated at the top of every column). We highlight that permutons in the same row are driven by the same Brownian excursion plotted at the end of the row and so they are coupled. Note that when $\rho = 1$ (this is the case of the last row) the corresponding two-dimensional Brownian excursion is simply a one-dimensional Brownian excursion and it is plotted using the standard diagram for real-valued functions.

**Definition 1.13.** **Strong-Baxter permutations** are permutations avoiding the three vincular patterns $2 \, 4 \, 1 \, 3$, $3 \, 1 \, 4 \, 2$, and $3 \, 4 \, 1 \, 2$, that is, permutations $\sigma$ such that there are no indices $1 \leq i < j < k-1 < n$ such that $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j) < \sigma(k) < \sigma(i) < \sigma(j+1)$ or $\sigma(j+1) < \sigma(k) < \sigma(i) < \sigma(j)$. 
In [14], we proved the following permuton convergence result.

**Theorem 1.14** [14, Theorem 1.6]. Let $\sigma_n$ be a uniform strong-Baxter permutation of size $n$. The following convergence in distribution in the permuton sense holds:

$\mu_{\sigma_n} \overset{d}{\to} \mu_{\rho, q}$

where $\rho \approx -0.2151$ is the unique real root of the polynomial

$1 + 6\rho + 8\rho^2 + 8\rho^3,$

and $q \approx 0.3008$ is the unique real root of the polynomial

$-1 + 6q - 11q^2 + 7q^3.$

In the same paper, we also investigated the permuton limit of semi-Baxter permutations. We first recall their definition.

**Definition 1.15.** Semi-Baxter permutations are permutations avoiding the vincular pattern 2413, that is, permutations $\sigma$ such that there are no indices $1 \leq i < j < k - 1 < n$ such that $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$.

**Theorem 1.16** [14, Theorem 1.8]. Let $\sigma_n$ be a uniform semi-Baxter permutation of size $n$. The following convergence in distribution in the permuton sense holds:

$\mu_{\sigma_n} \overset{d}{\to} \mu_{\rho, q}$

where

$\rho = -\frac{1 + \sqrt{5}}{4} \approx -0.8090$ and $q = \frac{1}{2}.$

A summarizing table of the various models of random constrained permutations that are currently known to converge to the skew Brownian permuton is given in Figure 4. We hope that this table will be enlarged in future research projects (see possible research directions in Subsection 1.6).

### 1.5 The skew Brownian permuton in Liouville quantum gravity theory

In the previous section, we saw that the skew Brownian permuton is the permuton limit of various families of constrained permutations. In this section, we explain how the skew Brownian permuton also arises from a Liouville quantum gravity (LQG) sphere decorated with two coupled Schramm–Loewner evolution (SLE) curves. This result will explain directly at the continuum level various connections between decorated planar maps and constrained permutations investigated in the literature at the discrete level. More precisely, in the combinatorial literature, several bijections between families of constrained permutations and decorated planar maps have been investigated, see, for instance, [11, 12, 30, 35, 45].
We highlight that the connection between these two (apparently unrelated) objects has proven to be a relevant tool to establish some results on the skew Brownian permuton using SLE/LQG techniques (see [17, 18]). In addition, we believe that building on this new connection, some convergence results available for discrete models of random permutations might be transferred to convergence results for the corresponding discrete models of planar maps (we refer to Subsection 1.6 for more details on open problems).

Finally, we mention that our results give a new description of the interactions between two SLE-curves (coupled in the imaginary geometry sense; see below for more details) decorating an LQG cone or sphere in terms of the skew perturbed Tanaka equations (see Lemmas 4.2 and 4.3). To the best of our knowledge, this connection with SDEs was not been explicitly exploited so far in the LQG literature.

We introduce various classical objects related to LQG. We do not give precise definitions of all these objects, but we provide precise references for each of them.†

We fix $\gamma \in (0, 2)$ and we introduce the following parameters defined in terms of $\gamma$,\n\begin{align*}
\kappa &= \gamma^2, & \kappa' &= 16/\gamma^2, & \rho &= -\cos(\pi\gamma^2/4), & \chi &= 2/\gamma - \gamma/2. \tag{7}
\end{align*}

We start by recalling the construction of space-filling SLE$_{\kappa'}$ curves on $\mathbb{C} \cup \{\infty\}$. One way of constructing them (as proved in [56]) is from the flow lines of the vector field $e^{i(\hat{h}/\chi + \theta)}$, where $\hat{h}$ is a whole-plane Gaussian free field (modulo a global additive multiple of $2\pi\chi \gamma$) and $\theta \in [0, \pi]$. Flow lines of the vector field $e^{i(\hat{h}/\chi + \theta)}$ are constructed in the imaginary geometry sense and are shown to be whole-plane SLE$_{\kappa}(2 - \kappa)$ processes from the starting point to $\infty$. As shown in

\[\text{FIGURE 4} \quad \text{A table with the various models of random constrained permutations that are known to converge to the skew Brownian permuton. The last column shows the values of the parameters $\gamma$ and $\kappa'$ for the corresponding SLE-decorated LQG spheres. These values are obtained using the relations in Equation (7) and Theorem 1.17. For the case $\rho = 1$ we conjecture that the corresponding $\gamma$-parameter is $\gamma = 2$; for further explanations see Remark 1.18.}\]
[56, Theorem 1.9], flow lines of $e^{i(\hat{h}/x+\theta)}$ started at different points of $\mathbb{Q}^2$ merge into each other upon intersecting and form a tree. The space-filling SLE$_{\kappa'}$ counterflow line from $\infty$ to $\infty$ generated by $\hat{h}$ and of angle $\theta$, denoted $\eta_0$, is the Peano curve of this tree, that is, it is the curve which visits the points of $\mathbb{C}$ in chronological order, where a point $x \in \mathbb{C}$ is visited before a point $y \in \mathbb{C}$ if the flow line of angle $\theta$ from $x$ merges into the flow line of angle $\theta$ from $y$ on the left side of the latter flow line. For $\theta \in [0, \pi]$, we consider the pair $(\eta_0, \eta_\theta)$ of space-filling SLE$_{\kappa'}$ counterflow lines from $\infty$ to $\infty$ generated by the same Gaussian free field $\hat{h}$ of angle 0 and $\theta$, respectively.

Let now $(\mathbb{C} \cup \{\infty\}, h, \infty)$ be a $\gamma$-LQG sphere, independent of $\hat{h}$, with quantum area one and one marked point at $\infty$ (see [38, Definition 3.20]). We parameterize the pair $(\eta_0, \eta_\theta)$ by the $\mu_h$-LQG area measure (see [38, section 3.3]) so that $\eta_0(0) = \eta_0(1) = \eta_\theta(0) = \eta_\theta(1) = \infty$ and $\mu_h(\eta_0([0, t])) = \mu_h(\eta_\theta([0, t])) = t$ for each $t \in [0, 1]$.

**Theorem 1.17.** Fix $\gamma \in (0, 2)$ and $\theta \in [0, \pi]$. Let $(\mathbb{C} \cup \{\infty\}, h, \infty)$ and $(\eta_0, \eta_\theta)$ be the $\gamma$-LQG sphere and the pair of space-filling SLE$_{\kappa'}$ introduced above. For $t \in [0, 1]$, let $\psi_{\gamma, \theta}(t) \in [0, 1]$ denote the first time when $\eta_\theta$ hits the point $\eta_0(t)$.† Then the random measure

$$(\text{Id}, \psi_{\gamma, \theta})_\ast \text{Leb}$$

is a skew Brownian permuton of parameter $\rho = -\cos(\pi \gamma^2/4)$ and $\bar{q} = \bar{q}_\gamma(\theta) \in [0, 1]$.

**Remark 1.18.** The explicit expression of the function $\bar{q}_\gamma(\theta)$ is unknown. It holds that $\bar{q}_\gamma(\pi/2) = 1/2$. Moreover, for every fixed $\gamma \in (0, 2)$, the function

$$\bar{q}_\gamma(\theta) : [0, \pi] \to [0, 1]$$

is a homeomorphism and therefore has an inverse function $\theta_\gamma(\bar{q})$. Finally, for all $\theta \in [0, \pi/2)$, it holds that $\bar{q}_\gamma(\theta) + \bar{q}_\gamma(\pi - \theta) = 1$. All these properties are consequences‡ of [52, Lemma 4.3]. We highlight that the fact that $\rho = -\cos(\pi \gamma^2/4)$ and $\bar{q}_\gamma(\theta)$ is a homeomorphism guarantees that the range of parameters for the skew Brownian permuton appearing in Theorem 1.17 is $\rho \in (-1, 1)$ and $q \in [0, 1]$. That is, all the possible skew Brownian permutons appear in Theorem 1.17, except the biased Brownian separable permuton $\mu_{1,q}$ for $q \in [0, 1]$. We conjecture that the latter case can be obtained from the critical 2-LQG because there are several analogies between the construction of $\mu_{1,q}$ and the results in [3].

The proof of Theorem 1.17 is given in Section 4 building on results of [31] and [37]. We also invite the curious reader to look at [17, Lemma 2.8] where a simple consequence of Theorem 1.17 is derived (giving another description of the skew Brownian permuton in terms of SLEs and LQG) and where it is also shown that our specific choice of $\psi_{\gamma, \theta}(t) \in [0, 1]$ in the statement of Theorem 1.17 can be replaced by any measurable function $\bar{\psi}_{\gamma, \theta}(t) \in [0, 1]$ such that $\eta_\theta(\bar{\psi}_{\gamma, \theta}(t)) = \eta_0(t)$ for all $t \in [0, 1]$.

† We recall that space-filling SLE curves have multiple points. Nevertheless, for each $z \in \mathbb{C}$, almost surely $z$ is not a multiple point of $\eta_0$, that is, $\eta_0$ hits $z$ exactly once. As $h$ is independent from $(\eta_0, \eta_\theta)$ and $\eta_0$ and $\eta_\theta$ are parameterized by $\mu_h$-mass, almost surely the set of times $t \in [0, 1]$ such that $\eta_0$ is a multiple point of $\eta_0$ has zero Lebesgue measure.

‡ The fact that the function $\bar{q}_\gamma(\theta)$ in Theorem 1.17 is the same function considered in [52, Lemma 4.3] follows from Lemma 4.2.
1.6 \ Open problems

In this final section of the introduction, we collect a list of open questions and problems that we plan to address in future research projects.

(1) **Intensity measure and Hausdorff dimension of the skew Brownian permuton.** The skew Brownian permuton $\mu_{\rho,q}$ is a new fractal random measure of the unit square and we plan to investigate some of its properties in the future. For instance, two natural questions are as follows.

(a) What is the density of the intensity measure $\mathbb{E}[\mu_{\rho,q}]$? How does it depend on the two parameters $\rho$ and $q$?

(b) What is the Hausdorff dimension of the support of $\mu_{\rho,q}$? And again, how does it depend on the two parameters $\rho$ and $q$?

We highlight that the questions above were answered in [54] for the biased Brownian separable permuton $\mu_{1,q}$. For instance, it was shown that almost surely, the support of $\mu_{1,q}$ is totally disconnected, and its Hausdorff dimension is 1 (with one-dimensional Hausdorff measure bounded above by $\sqrt{2}$). For an expression for the intensity measure see [54, Theorem 1.7].

We believe that a key tool to answer the two questions above would be the new connection with SLE-decorated LQG spheres described in Theorem 1.17.

(2) **Properties of the stochastic process $\varphi_{Z_{\rho,q}}$.** Recall that the stochastic process $\varphi_{Z_{\rho,q}}$ was defined in Equation (4) as follows

$$
\varphi_{Z_{\rho,q}}(t) := \text{Leb} \left( \{ x \in [0,t) | Z_{\rho,q}^{(x)}(t) < 0 \} \cup \{ x \in [t,1] | Z_{\rho,q}^{(y)}(x) \geq 0 \} \right), \quad t \in [0,1].
$$

Heuristically speaking, the skew Brownian permuton $\mu_{\rho,q}$ is the graph of the function $\varphi_{Z_{\rho,q}}$. Therefore, determining some properties of the process $\varphi_{Z_{\rho,q}}$ might be a useful step in the investigation of the permuton $\mu_{\rho,q}$. Our intuition suggests that $\varphi_{Z_{\rho,q}}$ might be a particular (conditioned) fragmentation process in the sense of [10]. We also believe that a key step to understanding the process $\varphi_{Z_{\rho,q}}$ will be to investigate the joint law of two processes $Z_{\rho,q}^{(x)}$ and $Z_{\rho,q}^{(y)}$. We remark that similar questions have been investigated in [25] for pair of solutions to the SDEs

$$
\begin{align*}
\frac{dZ^{(u)}}{t} &= dB(t) + (2q-1) \cdot dE_{u}Z^{(u)}(t), \quad t \in \mathbb{R}_{>u}, \\
Z^{(u)}(t) &= 0, \quad t \in [0,u],
\end{align*}
$$

where $B(t)$ is a standard one-dimensional Brownian motion.

(3) **Proportions of patterns in the skew Brownian permuton.** We first define the permutation induced by $k$ points in the square $[0,1]^2$. Take a sequence of $k$ points $(X, Y) = ((x_1, y_1), ..., (x_k, y_k))$ in $[0,1]^2$ with distinct $x$ and $y$ coordinates. The $x$-reordering of $(X, Y)$ is the unique reordering of the sequence $(X, Y)$ such that $x_{(1)} < ... < x_{(k)}$, and is denoted by $((x_{(1)}, y_{(1)}), ..., (x_{(k)}, y_{(k)}))$. The values $(y_{(1)}, ..., y_{(k)})$ are then in the same relative order as the values of a unique permutation of size $k$, that we call the permutation induced by $(X, Y)$.

Let $\mu$ be a permuton and $((X, Y)) \in Z_{>0}$ be an independent and identically distributed sequence with distribution $\mu$. We denote by $\text{Perm}_k(\mu)$ the random permutation induced by $((X, Y)) \in [1, k]$.
Finally, if \( \sigma \) is a permutation of size \( n \) and \( \pi \) a pattern\(^1\) of size \( k \leq n \), then we denote by \( \text{occ}(\pi, \sigma) \) the number of occurrences of \( \pi \) in \( \sigma \). Moreover, we denote by \( \tilde{\text{occ}}(\pi, \sigma) \) the proportion of occurrences of \( \pi \) in \( \sigma \), that is, \( \tilde{\text{occ}}(\pi, \sigma) := \frac{\text{occ}(\pi, \sigma)}{\binom{n}{k}} \).

**Question 1.19.** Let \( \mu_{\rho, q} \) be a skew Brownian permuton with parameters \( (\rho, q) \in (-1,1] \times [0,1] \). For all \( k \in \mathbb{Z}_{>0} \) and patterns \( \pi \) of size \( k \) is it possible to determine the limiting distribution of the random variables

\[
\lim_{n \to \infty} \tilde{\text{occ}}(\pi, \text{Perm}_n(\mu_{\rho, q})) = \mathbb{P}(\text{Perm}_k(\mu_{\rho, q}) = \pi | \mu_{\rho, q})? 
\]

We point out that the fact that the latter limit exists and satisfies the expression above is a consequence of the fact that \( \text{Perm}_n(\mu_{\rho, q}) \) converges in distribution in the permuton sense to \( \mu_{\rho, q} \) (see [8, Lemma 2.3 and Theorem 2.5]).

A simpler question is to compute, for all \( k \in \mathbb{Z}_{>0} \) and patterns \( \pi \) of size \( k \), the probabilities

\[
\mathbb{P}(\text{Perm}_k(\mu_{\rho, q}) = \pi) = \lim_{n \to \infty} \mathbb{E}[\tilde{\text{occ}}(\pi, \text{Perm}_n(\mu_{\rho, q}))].
\]

Our intuition suggests the following statement.

**Conjecture 1.20.** For all \((\rho, q) \in (-1,1) \times (0,1)\) and for all \( k \in \mathbb{Z}_{>0} \) and patterns \( \pi \) of size \( k \), it holds that

\[
\mathbb{P}(\text{Perm}_k(\mu_{\rho, q}) = \pi) > 0.
\]

We remark that as \( \mu_{1, q} \) can be obtained as the limit of pattern-avoiding permutations then we know that \( \mathbb{P}(\text{Perm}_k(\mu_{\rho, q}) = \pi) \) is zero for some \( q \in (0,1) \) and some pattern \( \pi \).

(4) **The length of the longest increasing subsequence in the skew Brownian permuton.**

This question is motivated by the recent work [6]. The authors show that the length of the longest increasing subsequence in a sequence of permutations converging to the biased Brownian separable permuton has sublinear size. Their proof builds on some self-similarity properties of the biased Brownian separable permuton.

**Question 1.21.** Let \( (\sigma_n)_{n \in \mathbb{Z}_{\geq 0}} \) be a sequence of random permutations converging in distribution in the permuton sense to the skew Brownian permuton \( \mu_{\rho, q} \). What is the length of the longest increasing subsequence in \( \sigma_n \)?

We expect, under some regularity conditions on the sequence \( \sigma_n \), a formula only depending on \( \rho, q \).

(5) **Properties of the parameters \( \rho \) and \( q \) defining the skew Brownian permuton.** The skew Brownian permuton \( \mu_{\rho, q} \) is defined in terms of the two parameters \( \rho \in (-1,1] \) and \( q \in [0,1] \). It would be interesting to find some natural statistics on permutons (i.e., a map from the space of permutons \( \mathcal{M} \) to \( \mathbb{R} \) describing some natural quantity) determined by these two parameters. For instance, we expect that \( q \) controls the proportion of inversions in \( \mu_{\rho, q} \). More precisely, we conjecture the following.

---

\(^1\) We refer the reader to [13, section 1.6.1] for an introduction to permutation patterns.
Conjecture 1.22. Set \( P(\text{Perm}_2(\mu_{\rho, q}) = 21) = f(\rho, q) \). Then for every fixed \( \rho \in (-1, 1] \) the function \( f(\rho, q) \) is increasing in \( q \).

It would be even more interesting to derive an explicit expression for \( f(\rho, q) \). It would be also remarkable to find some natural statistics \( \text{stc}(\cdot) \) on permutons such that \( \text{stc}(\mu_{\rho, q}) = g(\rho, q) \) almost surely, for some function \( g \). This would imply that the laws of \( \mu_{\rho, q} \) are singular for different values of the parameters \( \rho \in (-1, 1] \) and \( q \in [0, 1] \).

(6) The skew Brownian permuton for \( \rho = -1 \). The skew Brownian permuton \( \mu_{-1, q} \) has not been defined yet, because when \( \rho = -1 \), it is meaningless to condition a two-dimensional Brownian motion of correlation \( \rho = -1 \) to stay in the nonnegative quadrant. It would be interesting to investigate if there is a natural way of constructing \( \mu_{-1, q} \). A possible way would be to consider the limit of the permutons \( \mu_{\rho_n, q} \) for an appropriate sequence \( \rho_n \to -1 \). Looking at the simulations in Figure 3, we also believe that a natural candidate model for defining \( \mu_{-1, q} \) is the Mallows permuton \([63, 64]\).

(7) Critical 2-LQG and biased Brownian separable permuton. Theorem 1.17 connects the skew Brownian permuton \( \mu_{\rho, q} \) to SLE-decorated LQG spheres in the range of parameters \((\rho, q) \in (-1, 1) \times [0, 1] \). It would be interesting to show that the biased Brownian separable permuton \( \mu_{1, q} \) for \( q \in [0, 1] \) arises from the critical 2-LQG sphere (in the same spirit of Theorem 1.17). Here some new ideas are needed, as it is not clear how to define two SLE curves decorating a 2-LQG sphere (see [3]).

(8) Models of planar maps associated with semi-Baxter permutations and separable permutations. We believe that there are two natural models of decorated planar maps associated with semi-Baxter permutations and separable permutations that might converge to SLE-decorated LQG spheres. This belief is justified by Theorem 1.17 and the results in [19]. More precisely:

• We conjecture that bipolar posets (introduced in [35]) are in bijection with semi-Baxter permutations and converge to a \( \gamma \)-LQG sphere decorated with two SLE curves of angle \( \theta(q) \), where \( \gamma \) and \( \theta(q) \) are related to the parameters \( \rho, q \) appearing in Theorem 1.16 through the relations given in Theorem 1.17.

• We further conjecture that rooted series-parallel maps (studied in [11], Proposition 6), where it is also shown that they are in bijection with separable permutations) converge to the critical 2-LQG sphere. This would be the first discrete model of planar maps shown to converge to the critical 2-LQG sphere.

We point out that the results in this paper give three different possible approaches to answer the questions above: the permutation approach, the SDE approach, and the SLE-decorated LQG approach.†

2 THE SKEW PERTURBED TANAKA EQUATIONS

In this section, we focus on the skew perturbed Tanaka equations introduced in Equations 3, 5. In particular, in the following two subsections, we prove Theorem 1.7 and Proposition 1.9. In both cases, we will first investigate the SDEs in Equations 3, 5 when they are driven by a correlated

† Note added in revision: The SLE-decorated LQG approach has been first used in [18] to compute the intensity measure of the Baxter permuton (giving a partial answer to the Open Problem 1a) and to prove Conjectures 1.20 and 1.22; and recently in [17] to solve the Open Problem 1b and to give a partial answer to Question 1.21.
two-dimensional Brownian motion instead of a correlated two-dimensional Brownian excursion, and then we will transfer the results to our specific cases using absolute continuity arguments.

These results are fundamental to then show in Subsection 3.1 that the skew Brownian permuton is well-defined, proving Theorem 1.11.

2.1 | The skew pure perturbed Tanaka equation

2.1.1 | The Brownian motion case

Let \((\mathcal{W}_\rho(t))_{t \in \mathbb{R}_0^+} = (\mathcal{X}_\rho(t), \mathcal{Y}_\rho(t))_{t \in \mathbb{R}_0^+}\) be a two-dimensional Brownian motion of correlation \(\rho \in (-1, 1)\) and let \(q \in [0, 1]\) be a parameter. We consider the following SDE

\[
d\mathcal{Z}_{\rho,q}(t) = 1_{\{\mathcal{Z}_{\rho,q}(t) > 0\}} d\mathcal{Y}_\rho(t) - 1_{\{\mathcal{Z}_{\rho,q}(t) \leq 0\}} d\mathcal{X}_\rho(t) + (2q - 1) \cdot d\mathcal{L}^{\mathcal{Z}_{\rho,q}}(t), \quad t \in \mathbb{R}_0^+, \tag{8}
\]

where we recall that \(\mathcal{L}^{\mathcal{Z}_{\rho,q}}(t)\) is the symmetric local-time process at zero of \(\mathcal{Z}_{\rho,q}\). From now on, to simplify notation, we write all the involved processes forgetting the indices \(\rho\) and \(q\). We also denote by \(C(I)\) the set of continuous functions from an interval \(I\) of \(\mathbb{R}_0^+\) to \(\mathbb{R}\). Recall also the definition of skew Brownian motion from the beginning of Subsection 1.4.2. We prove the following result.

**Theorem 2.1.** Fix \(\rho \in (-1, 1)\) and \(q \in [0, 1]\). Pathwise uniqueness and existence of a strong solution to the SDE in Equation (8) hold. In addition, the solution is a skew Brownian motion of parameter \(q\).

More precisely, if \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) is a filtered probability space satisfying the usual conditions, and assuming that \(\mathcal{W}_\rho\) is an \(\mathcal{F}_t\)-Brownian motion of correlation \(\rho\),

1. if \(\mathcal{Z}, \tilde{\mathcal{Z}}\) are two \(\mathcal{F}_t\)-adapted continuous processes that solve Equation (8) almost surely, then \(\mathcal{Z} = \tilde{\mathcal{Z}}\) almost surely;
2. there exists an \(\mathcal{F}_t\)-adapted continuous process \(\mathcal{Z}\) that solves Equation (8) almost surely.

In particular, for every \(t \in \mathbb{R}_0^+\) there exists a measurable solution map \(F_t : C([0, t]) \to C([0, t])\) such that

1. \(F_t(\mathcal{W}_\rho|_{[0,t]})\) satisfies Equation (8) almost surely on the interval \([0, t]\);
2. for every \(s, t \in \mathbb{R}_0^+\) with \(s \leq t\), then \(F_t(\mathcal{W}_\rho|_{[0,t]})|_{[0,s]} = F_s(\mathcal{W}_\rho|_{[0,s]})\) almost surely.

From now on we fix \(\rho \in (-1, 1)\) and \(q \in (0, 1)\). The case \(q \in \{0, 1\}\) will be considered at the end of this section. We introduce the function \(r(x) = x/(1-q) \cdot 1_{x > 0} + x/q \cdot 1_{x \leq 0}\) and the SDE

\[
d\mathcal{R}(t) = (1-q)1_{\{\mathcal{R}(t) > 0\}} d\mathcal{Y}(t) - q1_{\{\mathcal{R}(t) \leq 0\}} d\mathcal{X}(t), \quad t \in \mathbb{R}_0^+. \tag{9}
\]

We have the following result.

**Proposition 2.2.** Let \(\mathcal{R} = (\mathcal{R}(t))_{t \in \mathbb{R}_0^+}\) and \(\mathcal{Z} = (\mathcal{Z}(t))_{t \in \mathbb{R}_0^+}\) be two stochastic processes such that \(\mathcal{Z}(t) = r(\mathcal{R}(t))\) for all \(t \in \mathbb{R}_0^+\). The process \(\mathcal{R}\) is a strong solution to Equation (9) if and only if the process \(\mathcal{Z}\) is a strong solution to Equation (8).

Thanks to Proposition 2.2, pathwise uniqueness and existence of a strong solution to the SDEs in Equations 8, 9 are equivalent. Thanks to the Yamada–Watanabe theorem† (see

† If further explanations are needed, the reader can look at the discussion at the beginning of [26, section 2.1].
Proposition 2.3. Pathwise uniqueness holds for Equation (9).

Proposition 2.4. There exists a weak solution \( \mathcal{R} = (\mathcal{R}(t))_{t \in \mathbb{R}_{\geq 0}} \) to Equation (9) such that \( r(\mathcal{R}) \) is a skew Brownian motion of parameter \( q \).

Note that the last two propositions together with Proposition 2.2 prove Theorem 2.1 for \( \rho \in (-1,1) \) and \( q \in (0,1) \). We now proceed with the proof of these three propositions.

We start by recalling the symmetric Itô–Tanaka formula for convex functions (see, for instance, [46, section 5.1]) because we will use it repeatedly in this section. Let \( f \) be a function from \( \mathbb{R} \) to \( \mathbb{R} \) which is the difference of two convex functions. Then

\[
f'(x) := \lim_{\varepsilon \to 0, \varepsilon > 0} \varepsilon^{-1}(f(x + \varepsilon) - f(x)) \quad \text{and} \quad f'_\varepsilon(x) := \lim_{\varepsilon \to 0, \varepsilon < 0} \varepsilon^{-1}(f(x + \varepsilon) - f(x))
\]

exist for almost every \( x \in \mathbb{R} \). In addition, there exists a signed measure \( \nu \) on \( \mathbb{R} \), called the second derivative measure, such that

\[
\int_{\mathbb{R}} g(x) \nu(dx) = - \int_{\mathbb{R}} g'(x)f'_{\varepsilon}(x)dx
\]

for every piecewise \( C^1 \) function \( g \) with compact support on \( \mathbb{R} \). If \( f \) has a second derivative, then \( f'' \) is the density of \( \nu \) with respect to the Lebesgue measure.

Let \( \mathcal{M} = (\mathcal{M}(t))_{t \in \mathbb{R}_{\geq 0}} \) be a real-valued semi-martingale. The symmetric Itô–Tanaka formula for \( f(\mathcal{M}) \) states that

\[
f(\mathcal{M}(t)) = f(\mathcal{M}(0)) + \int_0^t \frac{1}{2} (f'(\mathcal{M}(s)) + f'_{\varepsilon}(\mathcal{M}(s)))d\mathcal{M}(s) + \frac{1}{2} \int_{\mathbb{R}} \mathcal{L}_x^\mathcal{M}(t)\nu(dx), \quad t \in \mathbb{R}_{\geq 0},
\]

where \( \mathcal{L}_x^\mathcal{M}(t) \) denotes the symmetric local time process at \( x \) of \( \mathcal{M} \), that is,

\[
\mathcal{L}_x^\mathcal{M}(t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{\mathcal{M}(s) \in [x-\varepsilon,x+\varepsilon]\}} ds.
\]

We can now proceed with the proof of Proposition 2.2.

Proof of Proposition 2.2. We start by recalling that

\[
r(x) = x/(1-q) \cdot 1_{x>0} + x/q \cdot 1_{x<0}.
\]

We also introduce some additional functions that are used in the proof. We set

\[
r'(x) = 1/(1-q) \cdot 1_{x>0} + 1/(2q(1-q)) \cdot 1_{x=0} + 1/q \cdot 1_{x<0},
\]

\[
s(x) = (1-q)x \cdot 1_{x>0} + qx \cdot 1_{x<0},
\]

\[
s'(x) = (1-q) \cdot 1_{x>0} + \frac{1}{2} \cdot 1_{x=0} + q \cdot 1_{x<0}.
\]
Note that \( r(s(x)) = s(r(x)) = x \). We first assume that the process \( \mathcal{R} \) is a strong solution to Equation (9) and we show that the process \( \mathcal{Z} \), defined by \( \mathcal{Z}(t) = r(\mathcal{R}(t)) \), is a strong solution to Equation (8). Applying the Itô–Tanaka formula (Equation (10)), we have that

\[
d\mathcal{Z}(t) = r'(\mathcal{R}(t))d\mathcal{R}(t) + \frac{\gamma}{2}d\mathcal{L}^{\mathcal{R}}(t),
\]

where \( \gamma := 1/(1 - q) - 1/q \). Using Equation (9), we obtain that

\[
d\mathcal{Z}(t) = 1_{\{\mathcal{R}(t) > 0\}}d\mathcal{Y}(t) - 1_{\{\mathcal{R}(t) < 0\}}d\mathcal{X}(t) - \frac{1}{2(1 - q)}1_{\{\mathcal{R}(t) = 0\}}d\mathcal{X}(t) + \frac{\gamma}{2}d\mathcal{L}^{\mathcal{R}}(t)
\]

(11)

where in the last equality we used that \( \mathcal{Z}(t) = r(\mathcal{R}(t)) \) and that \( \int_0^t 1_{\{\mathcal{R}(s) = 0\}}d\mathcal{X}(s) \) is identically zero. Indeed, this stochastic integral has zero mean and

\[
\text{Var} \left[ \int_0^t 1_{\{\mathcal{R}(s) = 0\}}d\mathcal{X}(s) \right] = \mathbb{E} \left[ \int_0^t 1_{\{\mathcal{R}(s) = 0\}}ds \right] = 0,
\]

(12)

as \( \int_0^t 1_{\{\mathcal{R}(s) = 0\}}ds = 0 \) by [44, Exercise 3.7.10].

It remains to find a relation between \( \mathcal{L}^{\mathcal{R}}(t) \) and \( \mathcal{L}^{\mathcal{Z}}(t) \). Define \( \tilde{\text{sgn}}(x) := 1_{x > 0} - 1_{x < 0} \) (note that \( \tilde{\text{sgn}}(0) = 0 \) by definition). By Itô–Tanaka formula (Equation (10)) and Equation (11),

\[
|\mathcal{Z}(t)| = \int_0^t \tilde{\text{sgn}}(\mathcal{Z}(s))d\mathcal{Z}(s) + \mathcal{L}^{\mathcal{Z}}(t)
\]

\[
= \int_0^t \tilde{\text{sgn}}(\mathcal{Z}(s))(1_{\{\mathcal{Z}(s) > 0\}}d\mathcal{Y}(s) - 1_{\{\mathcal{Z}(s) < 0\}}d\mathcal{X}(s)) + \frac{\gamma}{2} \int_0^t \tilde{\text{sgn}}(\mathcal{Z}(s))d\mathcal{L}^{\mathcal{R}}(s) + \mathcal{L}^{\mathcal{Z}}(t)
\]

\[
= \int_0^t \tilde{\text{sgn}}(\mathcal{Z}(s))(1_{\{\mathcal{Z}(s) > 0\}}d\mathcal{Y}(s) - 1_{\{\mathcal{Z}(s) < 0\}}d\mathcal{X}(s)) + \mathcal{L}^{\mathcal{Z}}(t),
\]

(13)

where in the last equality we used that \( \int_0^t \tilde{\text{sgn}}(\mathcal{Z}(s))d\mathcal{L}^{\mathcal{R}}(s) = \int_0^t \tilde{\text{sgn}}(\mathcal{R}(s))d\mathcal{L}^{\mathcal{R}}(s) = 0 \) because \( \tilde{\text{sgn}}(0) = 0 \) and \( \mathcal{L}^{\mathcal{R}}(s) \) increases only when \( \mathcal{R}(s) = 0 \). Again by Itô–Tanaka formula (Equation (10)), setting \( \gamma' := 1/(1 - q) + 1/q \), we have that

\[
|r(\mathcal{R}(t))| = \int_0^t \tilde{\text{sgn}}(r(\mathcal{R}(s)))r'(\mathcal{R}(s))d\mathcal{R}(s) + \frac{\gamma'}{2}d\mathcal{L}^{\mathcal{R}}(t)
\]

(14)

where in the last equality we used that \( \tilde{\text{sgn}}(r(\mathcal{R}(s))) = \tilde{\text{sgn}}(\mathcal{Z}(s)) \) and Equation (9). Comparing Equations 13, 14, we obtain that

\[
\mathcal{L}^{\mathcal{Z}}(t) = \frac{\gamma'}{2}d\mathcal{L}^{\mathcal{R}}(t).
\]

Substituting the latter expression in Equation (11) we can conclude that \( \mathcal{Z} \) is a strong solution to Equation (8).

Assume now that the process \( \mathcal{Z} \) is a strong solution to Equation (8). Recalling that \( r(s(x)) = s(r(x)) = x \), we show that the process \( \mathcal{R} \), defined by \( \mathcal{R}(t) = s(\mathcal{Z}(t)) \) is a strong solution to
Equation (9). By Itô–Tanaka formula (Equation (10)),

$$dR(t) = s'(Z(t))dZ(t) + \frac{1-2q}{2}d\mathcal{L}^Z(t).$$

From Equation (8), using similar arguments as before, we obtain that

$$dR(t) = s'(Z(t))(1_{\{Z(t)>0\}}dY(t) - 1_{\{Z(t)\leq0\}}d\mathcal{X}(t)) + \left((2q-1)s'(0) + \frac{1-2q}{2}\right)d\mathcal{L}^Z(t)$$

$$= (1-q)1_{\{R(t)>0\}}dY(t) - \frac{1}{2}1_{\{R(t)=0\}}d\mathcal{X}(t) - q1_{\{Z(t)<0\}}d\mathcal{X}(t).$$

The latter SDE is equivalent to the SDE in Equation (9) because as before $\int_0^t 1_{\{R(s)=0\}}d\mathcal{X}(s)$ is identically zero. □

We now move to the proof of Proposition 2.3.

**Proof of Proposition 2.3.** Our strategy to show that pathwise uniqueness holds for the SDE

$$dR(t) = (1-q)1_{\{R(t)>0\}}dY(t) - q1_{\{R(t)\leq0\}}d\mathcal{X}(t)$$  \hspace{1cm} (15)

is to apply [34, Theorem 8.1] that guarantees pathwise uniqueness in the following setting: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function of finite variation. Consider two continuous local martingales $(M(t))_{t \in \mathbb{R}_{\geq0}}$ and $(\mathcal{N}(t))_{t \in \mathbb{R}_{\geq0}}$, started at zero, which almost surely satisfy for some constant $c > 0$,

$$\langle M, \mathcal{N} \rangle(t) = 0, \quad d\langle M \rangle(t) \leq c \cdot d\langle \mathcal{N} \rangle(t), \quad t \in \mathbb{R}_{\geq0}.$$  \hspace{1cm} (16)

Then pathwise uniqueness holds for the SDE

$$dZ(t) = f(Z(t))dM(t) + d\mathcal{N}(t).$$  \hspace{1cm} (17)

The rest of the proof is devoted to rewriting the SDE in Equation (15) in the form described above. We first construct the continuous local martingales $(M(t))_{t \in \mathbb{R}_{\geq0}}$ and $(\mathcal{N}(t))_{t \in \mathbb{R}_{\geq0}}$. It is straightforward to check that defining $x$ such that $\sin(2x) = \rho$, then the process

\[
\begin{pmatrix}
\cos(x) & -\sin(x) \\
-\sin(x) & \cos(x)
\end{pmatrix}
\begin{pmatrix}
\mathcal{X}(t) \\
\mathcal{Y}(t)
\end{pmatrix}
\]

is a standard two-dimensional Brownian motion, in particular, it has uncorrelated coordinates. Now, setting $\Theta = (1 + 2(q-1)q - 2(-1 + q)q \sin(2x))^{-1/2}$ and

$$p = (-q \cos(x) + (q-1)\sin(x))\Theta, \quad t = ((q-1)\cos(x) - q \sin(x))\Theta,$$

$$q = (\cos(x) - q \cos(x) + q \sin(x))\Theta, \quad u = (-q \cos(x) + (q-1)\sin(x))\Theta,$$

\[\text{The parameters } p, q, t, u \text{ are chosen so that the process } (M(t), \mathcal{N}(t)) \text{ defined in Equation (16) is a standard two-dimensional Brownian motion and Equation (17) holds.}\]
it is simple to check that the matrix \( \begin{pmatrix} p & t \\ q & u \end{pmatrix} \) is orthogonal. We finally define \( A, B, C, D \) such that
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} p & t \\ q & u \end{pmatrix} \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix}.
\]
With these definitions, the process
\[
(\mathcal{M}(t), \mathcal{N}(t)) := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{X}(t) \\ \mathcal{Y}(t) \end{pmatrix}
\]
is a standard two-dimensional Brownian motion. Indeed, it is a linear isometric transformation of another standard two-dimensional Brownian motion. Additionally, it holds that
\[
A(1-q) = Bq.
\]
Note that setting \( S = (AD - BC)^{-1} \), we have that
\[
\mathcal{X}(t) = S(D\mathcal{M}(t) - B\mathcal{N}(t)) \quad \text{and} \quad \mathcal{Y}(t) = S(A\mathcal{N}(t) - C\mathcal{M}(t)).
\]
Therefore, the SDE \( d\mathcal{R}(t) = (1-q)1_{\{\mathcal{R}(t) > 0\}}d\mathcal{Y}(t) - q1_{\{\mathcal{R}(t) \leq 0\}}d\mathcal{X}(t) \) can be written in the following equivalent form
\[
d\mathcal{R}(t) = S(1-q)1_{\{\mathcal{R}(t) > 0\}}(A \cdot d\mathcal{N}(t) - C \cdot d\mathcal{M}(t)) - Sq1_{\{\mathcal{R}(t) \leq 0\}}(D \cdot d\mathcal{M}(t) - B \cdot d\mathcal{N}(t))
\]
\[
= S(-C(1-q)1_{\{\mathcal{R}(t) > 0\}} - qD1_{\{\mathcal{R}(t) \leq 0\}})d\mathcal{M}(t) + qSBd\mathcal{N}(t),
\]
where in the last equality we used Equation (17). Note that we can write the latter SDE as follows
\[
d\mathcal{R}(t) = f(\mathcal{R}(t))d\mathcal{M}(t) + d\mathcal{N}(t),
\]
where \( f(x) = S(-C(1-q)1_{\{x > 0\}} - qD1_{\{x \leq 0\}}) \) is a bounded variation function and \( \mathcal{N}(t) = qSB\mathcal{N}(t) \). In addition, the continuous martingales \( \mathcal{M}(t) \) and \( \mathcal{N}(t) \) satisfy all properties required to apply [34, Theorem 8.1]. Therefore, we can conclude that pathwise uniqueness holds for the SDE in Equation (15).

We finally prove Proposition 2.4.

Proof of Proposition 2.4. Let \( (\mathcal{B}(t))_{t \in \mathbb{R}_{\geq 0}} \) be a standard one-dimensional Brownian motion. Consider the SDE
\[
d\mathcal{R}(t) = ((1-q)1_{\{\mathcal{R}(t) > 0\}} - q1_{\{\mathcal{R}(t) \leq 0\}})d\mathcal{B}(t).
\]
As we assumed \( q \in (0,1) \), according to a result due to Nakao [58] (see also [47, 48]), the latter SDE has a unique strong solution. Note that
\[
d(\mathcal{R})(t) = ((1-q)^21_{\{\mathcal{R}(t) > 0\}} + q^21_{\{\mathcal{R}(t) \leq 0\}})dt.
\]
We now consider two standard one-dimensional Brownian motions \( (\mathcal{X}(t))_{t \in \mathbb{R}_{\geq 0}}, (\mathcal{Y}(t))_{t \in \mathbb{R}_{\geq 0}} \) such that
\[
\langle \mathcal{B}, \mathcal{X} \rangle(t) = \langle \mathcal{B}, \mathcal{Y} \rangle(t) = \rho t.
\]
This is possible (for instance) by considering two additional independent Brownian motions \( B_1 \) and \( B_2 \), both also independent of \( B \), and setting \( \bar{\mathbf{X}} = \sqrt{\rho B} + \sqrt{1 - \rho} B_1 \) and \( \bar{\mathbf{Y}} = \sqrt{\rho B} + \sqrt{1 - \rho} B_2 \). We define the following two processes

\[
\bar{\mathbf{X}}(t) := \int_0^t 1_{\{\mathbf{R}(s) > 0\}} d\bar{\mathbf{Y}}(s) - \frac{1}{q} \int_0^t 1_{\{\mathbf{R}(s) \leq 0\}} d\mathbf{R}(s),
\]

\[
\bar{\mathbf{Y}}(t) := \frac{1}{1 - q} \int_0^t 1_{\{\mathbf{R}(s) > 0\}} d\mathbf{R}(s) - \int_0^t 1_{\{\mathbf{R}(s) \leq 0\}} d\bar{\mathbf{X}}(s).
\]

Note that

\[
(1 - q)1_{\{\mathbf{R}(t) > 0\}} d\mathbf{Y}(t) - q 1_{\{\mathbf{R}(t) \leq 0\}} d\bar{\mathbf{X}}(t) = d\mathbf{R}(t),
\]

and so the triplet \((\mathbf{R}, \bar{\mathbf{X}}, \bar{\mathbf{Y}})\) solves Equation (9). To show that \((\mathbf{R}, \bar{\mathbf{X}}, \bar{\mathbf{Y}})\) is a weak solution to Equation (9), it remains to show that \((\bar{\mathbf{X}}, \bar{\mathbf{Y}})\) is a two-dimensional Brownian motion with correlation \( \rho \). By definition, \( \bar{\mathbf{X}} \) and \( \bar{\mathbf{Y}} \) are continuous local martingales. In addition, we have that

\[
\langle \bar{\mathbf{X}} \rangle(t) = \int_0^t 1_{\{\mathbf{R}(s) > 0\}} ds + \frac{1}{q^2} \int_0^t 1_{\{\mathbf{R}(s) \leq 0\}} d\langle \mathbf{R} \rangle(s) = \int_0^t 1_{\{\mathbf{R}(s) > 0\}} ds + \int_0^t 1_{\{\mathbf{R}(s) \leq 0\}} ds = t,
\]

where in the second inequality we used Equation (19). Similarly,

\[
\langle \bar{\mathbf{Y}} \rangle(t) = \frac{1}{1 - q^2} \int_0^t 1_{\{\mathbf{R}(s) > 0\}} d\langle \mathbf{R} \rangle(s) + \int_0^t 1_{\{\mathbf{R}(s) \leq 0\}} ds = \int_0^t 1_{\{\mathbf{R}(s) > 0\}} ds + \int_0^t 1_{\{\mathbf{R}(s) \leq 0\}} ds = t.
\]

Therefore, by Lévy’s characterization theorem [44, Theorem 3.3.16], \( \bar{\mathbf{X}} \) and \( \bar{\mathbf{Y}} \) are standard one-dimensional Brownian motions. It only remains to check that \( \bar{\mathbf{X}} \) and \( \bar{\mathbf{Y}} \) have the desired correlation. Note that

\[
\langle \bar{\mathbf{X}}, \bar{\mathbf{Y}} \rangle(t) = \frac{1}{1 - q} \int_0^t 1_{\{\mathbf{R}(s) > 0\}} d\langle \bar{\mathbf{Y}} \rangle(s) + \frac{1}{q} \int_0^t 1_{\{\mathbf{R}(s) \leq 0\}} d\langle \bar{\mathbf{X}} \rangle(s).
\]

From Equations ((18)) and ((20)), we have that

\[
d\langle \bar{\mathbf{X}}, \mathbf{R} \rangle(s) = d\langle \bar{\mathbf{Y}}, \mathbf{R} \rangle(s) = \rho \left((1 - q)1_{\{\mathbf{R}(s) > 0\}} ds + q 1_{\{\mathbf{R}(s) \leq 0\}} ds\right),
\]

and substituting this expression in Equation (21) we can conclude that \( \langle \bar{\mathbf{X}}, \bar{\mathbf{Y}} \rangle(t) = \rho t \), as desired.

The fact that \( r(\mathbf{R}(t))_{t \in \mathbb{R}_0} \) is a skew Brownian motion of parameter \( q \) follows (for instance) using the same arguments as in [46, section 5.2] and recalling that \( \mathbf{R} \) is a solution to Equation (18). \( \square \)

We complete the proof of Theorem 2.1, considering the case \( \rho \in (-1, 1) \) and \( q = 0 \) (the case \( q = 1 \) follows with similar arguments). We consider the functions \( g(x) = x \cdot 1_{x>0} \) and \( g'(x) = 1_{x>0} + \frac{1}{2} \cdot 1_{x=0} \). We assume that \( \mathbf{Z} \) is a strong solution to Equation (8) and we define the process \( \mathbf{R} \) by \( \mathbf{R}(t) = g(\mathbf{Z}(t)) \). By Itô–Tanaka formula (Equation (10)) and Equation (8), we have that

\[
d\mathbf{R}(t) = g'(\mathbf{Z}(t))d\mathbf{Z}(t) + \frac{1}{2} d\mathcal{L}^{\mathbf{Z}}(t) = 1_{\{\mathbf{R}(t) > 0\}} d\mathbf{Y}(t) - \frac{1}{2} 1_{\{\mathbf{R}(t) = 0\}} d\bar{\mathbf{X}}(t).
\]
By definition, the process \( R \) is nonnegative, continuous, and started at zero. The last equation shows that \( R \) is also a martingale, and so \( R \) is almost surely identically zero. This implies that almost surely \( \mathcal{Z}(t) \leq 0 \) for all \( t \in \mathbb{R}_{\geq 0} \) and so \( \mathcal{Z} \) solves the SDE

\[
d\mathcal{Z}(t) = -1_{\{\mathcal{Z}(t) \leq 0\}}d\mathcal{X}(t) - d\mathcal{L}\mathcal{Z}(t), \quad t \in \mathbb{R}_{\geq 0}.
\]

From [42], we know that the latter SDE has a unique strong solution \( \mathcal{Z}(t)_{t \in \mathbb{R}_{\geq 0}} \) that is a skew Brownian motion of parameter 0. This completes the proof of Theorem 2.1.

2.1.2 The Brownian excursion case

Building on Theorem 2.1, it is straightforward to prove Theorem 1.7 using absolute continuity arguments. We include the following (short) proof for making the article as self-contained as we can, but we highlight that the arguments are similar to the one used in [19, Theorem 4.6].

**Proof of Theorem 1.7.** The strategy is to consider the solution mappings \( F_t \) defined in Theorem 2.1 and for all \( r \in (u, 1) \) to define the following measurable (with regard to \( \mathcal{F}(u) \)) process \( S_{u,r} \in \mathcal{C}([u, r]) \):

\[
S_{u,r}(t) := F_{r-u}(\mathcal{E}_\rho(u + s) - \mathcal{E}_\rho(u))_{s \in [0, r-u]}(t - u), \quad t \in [u, r].
\]

As by Proposition A.1, the Brownian excursion \((\mathcal{E}_\rho(u + s) - \mathcal{E}_\rho(u))_{s \in [0, r-u]}\) is absolutely continuous with regard to a two-dimensional Brownian motion of correlation \( \rho \) on \([0, r-u]\), using the items 3–4 in Theorem 2.1 we have that

1. \( S_{u,r} \) almost surely satisfies Equation (3) on interval \([u, r]\);
2. for \( r, r' \in (u, 1) \), we have \( S_{u,r} = (S_{u,r'})_{|u,r] \) almost surely.

In addition, by construction, the map \((u, r, \omega) \mapsto S_{u,r}\) is a measurable function. Now, noting that the two statements above hold simultaneously for all \( r, r' \in (u, 1) \cap \mathbb{Q} \), we have that almost surely there exists a process \( \mathcal{Z}^{(u)} \in \mathcal{C}([u, 1]) \) such that \( \mathcal{Z}^{(u)}_{|u,r]} = S_{u,r} \) for every \( r \in (u, 1) \cap \mathbb{Q} \). Hence \( \mathcal{Z}^{(u)} \) is \( \mathcal{F}^{(u)} \)-adapted and almost surely satisfies Equation (3), proving the existence of a strong solution.

For pathwise uniqueness, let \( \mathcal{Z}^{(u)}, \tilde{\mathcal{Z}}^{(u)} \) be two \( \mathcal{F}^{(u)} \)-adapted solutions to the SDE in Equation (3) and \( r < 1 \). There exist two functionals \( G, \tilde{G} : \mathcal{C}([u, r]) \to \mathcal{C}([u, r]) \) such that almost surely,

\[
\mathcal{Z}^{(u)} = G((\mathcal{E}_\rho(s) - \mathcal{E}_\rho(u))_{s \in [u, r]}) \quad \text{and} \quad \tilde{\mathcal{Z}}^{(u)} = \tilde{G}((\mathcal{E}_\rho(s) - \mathcal{E}_\rho(u))_{s \in [u, r]}).
\]

Using again the absolute continuity (in the other direction) in Proposition A.1, for a two-dimensional Brownian motion \( \mathcal{W}_\rho \) of correlation \( \rho \), the stochastic processes \( G(\mathcal{W}_\rho) \) and \( \tilde{G}(\mathcal{W}_\rho) \) are two solutions to the SDE in Equation (8). As by Theorem 2.1 pathwise uniqueness holds for Equation (8), \( G(\mathcal{W}_\rho) = \tilde{G}(\mathcal{W}_\rho) \) almost surely Therefore, by absolute continuity, \( \mathcal{Z}^{(u)} = \tilde{\mathcal{Z}}^{(u)} \) almost surely This ends the proof. \( \square \)
2.2 The skew Tanaka equation

The goal of this section is to prove Proposition 1.9. Fix \( q \in [0, 1] \). We recall that we want to construct solutions to the following SDEs, defined for all \( u \in [0, 1] \) by

\[
\begin{align*}
  d \mathcal{Z}^{(u)}(t) &= \text{sgn}(\mathcal{Z}^{(u)}(t))de(t) + (2q - 1) \cdot d\mathcal{L}^{(u)}(t), & t \in [u, 1), \\
  \mathcal{Z}^{(u)}(t) &= 0, & t \in [0, u],
\end{align*}
\]

(22)

where \( \text{sgn}(x) := 1_{\{x > 0\}} - 1_{\{x \leq 0\}} \) is a one-dimensional Brownian excursion on \([0, 1]\) and \( \mathcal{L}^{(u)}(t) \) is the symmetric local time at zero of \( \mathcal{Z}^{(u)} \).

Equation (22) when \( q = 1/2 \) and \( (e(t))_{t \in [0, 1]} \) is replaced by a standard one-dimensional Brownian motion \( (B(t))_{t \in \mathbb{R} \geq 0} \) is the well-known Tanaka’s SDE:

\[
\begin{align*}
  d \mathcal{Z}^{(u)}(t) &= \text{sgn}(\mathcal{Z}^{(u)}(t))dB(t), & t \in \mathbb{R} > u, \\
  \mathcal{Z}^{(u)}(t) &= 0, & t \in [0, u].
\end{align*}
\]

(23)

The striking feature of this equation is the absence of pathwise uniqueness: solutions cannot be measurable functions of the driving process \( B \) and must also incorporate additional randomness (see, for instance, [44, Example 5.3.5]).

Similarly, there is absence of pathwise uniqueness also for the following SDEs (for a proof, see, for instance, the discussion between Equations ((4)) and ((6)) in [39]):

\[
\begin{align*}
  d \mathcal{Z}^{(u)}_q(t) &= \text{sgn}(\mathcal{Z}^{(u)}_q(t))dB(t) + (2q - 1) \cdot d\mathcal{L}^{(u)}_q(t), & t \in \mathbb{R} > u, \\
  \mathcal{Z}^{(u)}_q(t) &= 0, & t \in [0, u].
\end{align*}
\]

(24)

This absence of pathwise uniqueness (both for the SDEs in Equations (23) and (24)) raises many questions when we want to couple solutions to Equation (23) (or Equation (24)) for several starting times \( u \in \mathbb{R} \). An elegant solution was developed by Le Jan and Raimond [49, 51] using the notion of stochastic flow of maps. The same authors proved in [50, Theorem 2.1] that there exists a stochastic flow of maps solving Equation (23) and explicitly constructed this flow (which was also studied in [65]). Later, Hajri [39, Theorem 2] extended the ideas of Le Jan and Raimond and explicitly constructed another stochastic flow of maps solving Equation (24).

Using our notation, the construction of Hajri gives the following solution \( \{\mathcal{Z}^{(u)}_q(t)\}_{u \in \mathbb{R} \geq 0} \) to the SDEs in Equation (24). (We suggest comparing the following explanation with Figure 5.)

\[\text{We remark that the construction given in [39, Theorem 2] is presented in a much more general setting. Specifically, Hajri considered general flow evolving on graphs and some generalized SDEs (mixing Tanaka's SDE and the skew Brownian motion SDE). Furthermore, he gave a discrete approximation of this flow in [40]. We do not need to consider this general setting in this paper. We just mention that our specific case corresponds (following the notation in [39, Theorem 2]) to } N = p = 2, \alpha_1 = q \text{ and } \alpha_2 = 1 - q \text{ (so that } \alpha^+ = 1).\]
Conditional on $B$, consider an independent and identically distributed sequence $(s(\ell))$, indexed by the local minima of $B$, and with distribution $P(s(\ell) = +1) = q = 1 - P(s(\ell) = -1)$.

For $u, t \in [0, 1]$ with $u \leq t$, set $m(u)(t) := \inf_{[u, t]} B$ and $\xi_q^{(u)}(t) := s(\sup_{r \in [u, t]} : B(r) = m(u)(t))$. Then the solutions $\{Z_q^{(u)}(t)\}_{u \in \mathbb{R}_{\geq 0}}$ are defined as follows

$\begin{align*}
Z_q^{(u)}(t) &:= (B(t) - m(u)(t))\xi_q^{(u)}(t), & t \in \mathbb{R}_{> u}, \\
Z_q^{(u)}(t) &:= 0, & t \in [0, u].
\end{align*}$

Remark 2.5. Note that the solutions $\{Z_q^{(u)}(t)\}_{u \in \mathbb{R}_{\geq 0}}$ defined in Equation (25) are constructed using the same Brownian motion $B$ and the same sequence of independent and identically distributed signs $(s(\ell))$. In particular, there is a coupling between different solutions.

Remark 2.6. For every fixed $u \in \mathbb{R}_{\geq 0}$, the process $Z_q^{(u)}$ defined in Equation (25) is a skew Brownian motion of parameter $q$ (and in particular is measurable) as shown in [39, Proposition 1].

The fact that the processes $\{Z_q^{(u)}(t)\}_{u \in \mathbb{R}_{\geq 0}}$ defined in Equation (25) form a family of solutions to the SDEs in Equation (24) is a consequence of the more general result stated in [39, Theorem 2], as already mentioned above. As here we do not need such generality, we include a simple self-contained proof of this result.

**Proposition 2.7.** The family $\{Z_q^{(u)}(t)\}_{u \in \mathbb{R}_{\geq 0}}$ defined in Equation (25) is a family of solutions to the SDEs in Equation (24). Moreover, for every $u \in \mathbb{R}_{\geq 0}$, $\{Z_q^{(u)}(t)\}_{t \in \mathbb{R}_{\geq u}}$ is a skew Brownian motion of parameter $q$ started at zero at time $u$.

**Proof.** Fix $u \in \mathbb{R}_{\geq 0}$. Recall from Remark 2.6 that $Z_q^{(u)}(t)$ is a skew Brownian motion of parameter $q$. From [42], there exists a one-dimensional Brownian motion $(W(t))_{t \in \mathbb{R}_{\geq u}}$ started at zero at time $u$.

---

1 We remark again that it is enough to consider the case $N = 2, \alpha_1 = q$ and $\alpha_2 = 1 - q$ in [39, Proposition 1] for our purposes.
\[ \mathbb{Z}_q^{(u)}(t) = \mathcal{W}(t) + (2q - 1)\mathcal{L}_q^{(u)}(t), \quad t \in \mathbb{R}_{\geq u}. \] (26)

From Itô–Tanaka formula (Equation (10)), we have that

\[
\left| \mathbb{Z}_q^{(u)}(t) \right| = \int_{u}^{t} \left( 1_{\{ \mathbb{Z}_q^{(u)}(s) > 0 \}} - 1_{\{ \mathbb{Z}_q^{(u)}(s) < 0 \}} \right) d \mathbb{Z}_q^{(u)}(s) + \mathcal{L}_q^{(u)}(t)
\]

\[
= \int_{u}^{t} \left( 1_{\{ \mathbb{Z}_q^{(u)}(s) > 0 \}} - 1_{\{ \mathbb{Z}_q^{(u)}(s) < 0 \}} \right) d \mathcal{W}(s) + \mathcal{L}_q^{(u)}(t),
\]

where in the last equality we used Equation (26) and the fact that

\[
\int_{u}^{t} \left( 1_{\{ \mathbb{Z}_q^{(u)}(s) > 0 \}} - 1_{\{ \mathbb{Z}_q^{(u)}(s) < 0 \}} \right) d \mathcal{L}_q^{(u)}(s) = 0,
\]

because \( \mathcal{L}_q^{(u)}(s) \) increases only if \( \mathbb{Z}_q^{(u)}(s) = 0 \). Noting that \( \int_{u}^{t} 1_{\{ \mathbb{Z}_q^{(u)}(s) = 0 \}} d \mathbb{W}(s) = 0 \) (this follows, for instance, using the same arguments as in Equation (12)), we obtain that

\[
\left| \mathbb{Z}_q^{(u)}(t) \right| = \int_{u}^{t} \text{sgn}(\mathbb{Z}_q^{(u)}(s)) d \mathcal{W}(s) + \mathcal{L}_q^{(u)}(t). \] (27)

Now by construction (see Equation (25)), we also have that

\[
\left| \mathbb{Z}_q^{(u)}(t) \right| = B(t) - m^{(u)}(t). \] (28)

Recall that every continuous semimartingale can be uniquely decomposed into a continuous local martingale and a continuous finite variation process started at zero. Therefore, comparing Equations 27, 28, and noting that \( \mathcal{L}_q^{(u)}(t) \) and \( -m^{(u)}(t) \) are both increasing (and so with finite variation) continuous processes started at zero, we obtain that

\[
\mathcal{L}_q^{(u)}(t) = -m^{(u)}(t) \quad \text{and} \quad B(t) = \int_{u}^{t} \text{sgn}(\mathbb{Z}_q^{(u)}(s)) d \mathcal{W}(s).
\]

Using the equality on the right-hand side of the last equation, we get

\[
\int_{u}^{t} \text{sgn}(\mathbb{Z}_q^{(u)}(s)) d B(s) = \int_{u}^{t} \text{sgn}(\mathbb{Z}_q^{(u)}(s))^2 d \mathcal{W}(s) = \int_{u}^{t} d \mathcal{W}(s) = \mathcal{W}(t),
\]

where in the last equality we used that \( \left( \mathcal{W}(t) \right)_{t \in \mathbb{R}_{\geq u}} \) is a Brownian motion started at zero at time \( u \). Substituting the last expression in Equation (26), we conclude that \( \mathbb{Z}_q^{(u)}(t) \) is a solution to the SDE in Equation (24). \( \square \)

The result stated in Proposition 1.9 follows from Proposition 2.7 using standard absolute continuity arguments between one-dimensional Brownian excursions and one-dimensional Brownian
motions. As these arguments are very similar to the ones used for the proof of Theorem 1.7 in Subsection 2.1.2, we skip the details here.

We conclude this section with a quick discussion on uniqueness of the solutions to Equation (24) given in Proposition 2.7.

**Remark 2.8.** In this paper, we decided to avoid using the formalism of stochastic flow of maps and so we did not include any claim in Proposition 2.7 related to uniqueness of the solutions to Equation (24) defined in Equation (25) (and we made the same choice for Proposition 1.9 in the introduction). Nevertheless, in [39, Theorem 2] it is also showed that the stochastic flow of maps corresponding to the processes \( \{ Z_u^{(u)}(t) \}_{u \in \mathbb{R}_0} \) defined in Equation (25) is unique, that is, the unique flow adapted to the filtration generated by the driving Brownian motion. We also remark that the proof of this uniqueness result has been later simplified in [41].

## 3 | THE SKEW BROWNIAN PERMUTON

In the previous section, we investigated solutions to the skew perturbed Tanaka equations driven by a two-dimensional correlated Brownian excursion. Building on these results, we can now deduce that the skew Brownian permuton is well-defined, proving Theorem 1.11. Later, we will also show that the biased Brownian separable permuton is a particular case of the skew Brownian permuton, proving Theorem 1.12. These are the two goals of the next two sections.

### 3.1 | The skew Brownian permuton is well-defined

We prove here Theorem 1.11, showing that the skew Brownian permuton \( \mu_{\rho, q} \) is well-defined for all \((\rho, q) \in (-1,1] \times [0,1]\). Before doing that we also give a slightly different (but equivalent) definition of the skew Brownian permuton that will be also helpful later to prove Theorem 1.12.

We fix \((\rho, q) \in (-1,1] \times [0,1]\) and consider the continuous coalescent-walk process\( \{ Z_{\rho, q}^{(u)} \}_{u \in \mathbb{R}_0} \) defined in Equation (25). We first define a random binary relation \( \leq_{Z_{\rho, q}} \) on \([0,1]^2\) as follows:

\[
\begin{cases}
  t \leq_{Z_{\rho, q}} s & \text{for all } 0 < s < 1, \\
  t \leq_{Z_{\rho, q}} s & \text{for all } 0 < t < s < 1 \\
  s \leq_{Z_{\rho, q}} t & \text{for all } 0 < t < s < 1
\end{cases}
\]  

We note that the stochastic process \( \varphi_{Z_{\rho, q}}(t) \) introduce in Equation (4) satisfies

\[
\varphi_{Z_{\rho, q}}(t) = \text{Leb} \left( \left\{ x \in [0,1] \mid x \leq_{Z_{\rho, q}} t \right\} \right), \quad t \in [0,1],
\]

and we recall that the skew Brownian permuton \( \mu_{\rho, q} \) is defined by

\[
\mu_{\rho, q}(\cdot) := (\text{Id}, \varphi_{Z_{\rho, q}})_* \text{Leb}(\cdot) = \text{Leb} \left( \left\{ t \in [0,1] \mid (t, \varphi_{Z_{\rho, q}}(t)) \in \cdot \right\} \right).
\]

\footnote{Actually \( Z_{\rho, q}^{(u)} \) was not defined for \( u \in \{0,1\} \) (see Definition 1.1). As what happens on a negligible subset of \([0,1]\) is irrelevant to the arguments to come, this causes no problems.}
We restrict for the moment to the case \((\rho, q) \in (-1, 1) \times [0, 1]\) (the case \(\rho = 1\) will be treated separately in Subsection 3.2). Using pathwise uniqueness (Item 3 in Theorem 1.7) for the SDEs defining the continuous coalescent-walk process \(Z_{\rho, q}\), it is simple to obtain the following result (for a proof see [19, Proposition 5.2]).

**Proposition 3.1.** Fix \((\rho, q) \in (-1, 1) \times [0, 1]\). There exists a random set \(A \subset [0, 1]^2\) of almost surely zero Lebesgue measure, that is, \(\mathbb{P}(\text{Leb}(A) = 0) = 1\), such that the restriction of the relation \(\preceq_{Z_{\rho, q}}\) to \([0, 1]^2 \setminus A\) is almost surely a total order.

We can now prove Theorem 1.11 (using similar ideas as in [19, Lemma 5.5]).

**Proof of Theorem 1.11 for \(\rho \neq 1\).** We start by proving that for \(t, s \in (0, 1)\) with \(t < s\), \(Z_{\rho, q}^{(t)}(s) \neq 0\) almost surely. Let \(\varepsilon > 0\) be such that \(s < 1 - \varepsilon < 1\). Thanks to Theorem 2.1, \((Z_{\rho, q}^{(t)}(t + r))_{r \in [0, 1 - t - \varepsilon]}\) is absolutely continuous with regard to a skew Brownian motion of parameter \(q\) on \([0, 1 - t - \varepsilon]\).

As the time spent at zero by a skew Brownian motion almost surely has zero Lebesgue measure, we can conclude that \(Z_{\rho, q}^{(t)}(s) \neq 0\) almost surely.

We can now prove that \(\mu_{\rho, q}\) is a permuton. By definition, \(\mu_{\rho, q}\) is a random probability measure on the unit square, and its first marginal is almost surely uniform (see Equation (30)). Therefore, it is enough to verify that also its second marginal is almost surely uniform, that is, that \((\varphi_{Z_{\rho, q}})_* \text{Leb} = \text{Leb}\) almost surely.

Let \((U_i)_{i \in \mathbb{Z}^+}\) be a sequence of independent and identically distributed uniform random variables on \([0, 1]\). We define for \(k \in \mathbb{Z}^+\),

\[
U_{1, k} := \frac{1}{k - 1} \# \left\{ i \in [2, k] \mid U_i \preceq_{Z_{\rho, q}} U_1 \right\} = \frac{1}{k - 1} \sum_{i \in [2, k]} 1 \{ U_i \preceq_{Z_{\rho, q}} U_1 \}.
\]

The random variables \(1 \{ U_i \preceq_{Z_{\rho, q}} U_1 \}_{i \in \mathbb{Z}^+}\) are independent and identically distributed Bernoulli random variables of parameter \(\varphi_{Z_{\rho, q}}(U_1)\), conditioned on \((E_{\rho}, U_1)\). Therefore, from the law of large numbers, \(U_{1, k}\) almost surely converges to \(\varphi_{Z_{\rho, q}}(U_1)\) as \(k\) tends to infinity.

On the other hand, by the exchangeability of the random variables \((U_i)_{i \in \mathbb{Z}^+}\), and using that \(Z_{\rho, q}^{(t)}(s) \neq 0\) almost surely, the random variable \(U_{1, k}\) is uniform in \(\{0, \ldots, k - 1\}\), conditioned on \(E_{\rho}\). Therefore, \(U_{1, k}\) converges in distribution to a uniform random variable on \([0, 1]\) as \(k\) tends to infinity. Thus, we can conclude that \(\varphi_{Z_{\rho, q}}(U_1)\) is uniform on \([0, 1]\) conditionally on \(E_{\rho}\). This concludes the proof.

We state and prove a final lemma useful for later purposes.

**Lemma 3.2.** Fix \((\rho, q) \in (-1, 1) \times [0, 1]\). Almost surely, for almost every \(0 \leq t < s \leq 1\), either \(Z_{\rho, q}^{(t)}(s) > 0\) and \(\varphi_{Z_{\rho, q}}(s) < \varphi_{Z_{\rho, q}}(t)\), or \(Z_{\rho, q}^{(t)}(s) < 0\) and \(\varphi_{Z_{\rho, q}}(s) > \varphi_{Z_{\rho, q}}(t)\).

**Proof.** Consider two independent uniform random variables \(U\) and \(V\) on \([0, 1]\), also independent of \(E_{\rho}\). It is immediate from Proposition 3.1 that if \(U < V\) and \(Z_{\rho, q}^{(U)}(V) > 0\) then \(\varphi_{Z_{\rho, q}}(U) > \varphi_{Z_{\rho, q}}(V)\) almost surely. Similarly, if \(U < V\) and \(Z_{\rho, q}^{(U)}(V) < 0\) then \(\varphi_{Z_{\rho, q}}(U) < \varphi_{Z_{\rho, q}}(V)\) almost surely. The case \(Z_{\rho, q}^{(U)}(V) = 0\) is almost surely excluded by the first part of the previous proof, and the case \(\varphi_{Z_{\rho, q}}(U) = \varphi_{Z_{\rho, q}}(V)\) is almost surely excluded by the fact that \(\varphi_{Z_{\rho, q}}(U)\) and \(\varphi_{Z_{\rho, q}}(V)\) are two independent uniform random variables, thanks to the second part of the previous proof. This is enough to complete the proof of the lemma.
3.2  The biased Brownian separable permuton is a particular case of the skew Brownian permuton

Recalling the construction of the biased Brownian separable permuton from Subsection 1.2 and the construction of the skew Brownian permuton presented in Subsection 3.1, to prove Theorem 1.12 (and also to complete the proof of Theorem 1.11 when \( \rho = 1 \)) it is enough to prove the following result.

**Proposition 3.3.** Fix a one-dimensional Brownian excursion \((e(t))_{t \in [0,1]}\) on \([0,1]\) and a parameter \(p \in [0,1]\). Conditional on \(e\), consider an independent and identically distributed sequence \((s(\ell))_{\ell \in \mathbb{Z}^+}\) indexed by the local minima of \(e\) and with distribution \(\mathbb{P}(s(\ell) = +1) = p = 1 - \mathbb{P}(s(\ell) = -1)\). Let \(\preceq_{e,p}\) be the relation defined in Equation (1) constructed from the pair \((e(t), (s(\ell))_{\ell})\) and \(\preceq_{Z_{1,1-p}}\) be the relation defined in Equation (29), constructed from the pair \((e(t), (-s(\ell))_{\ell})\) (note that the two relations \(\preceq_{e,p}\) and \(\preceq_{Z_{1,1-p}}\) now have a specific coupling).

Then there exists a random set \(A \subset [0,1]^2\) of almost surely zero Lebesgue measure, that is, \(\mathbb{P}(\text{Leb}(A) = 0) = 1\), such that the relations \(\preceq_{e,p}\) and \(\preceq_{Z_{1,1-p}}\) restricted to \([0,1]^2 \setminus A\) are the same total order.

**Proof.** Recall from Subsection 1.2 that there exists a random set \(A \subset [0,1]^2\) of almost surely zero Lebesgue measure such that for every \(x, y \in [0,1]^2 \setminus A\) with \(x < y\) then \(\min_{[x,y]} e\) is reached at a unique point which is a strict local minimum. In addition, \(\preceq_{e,p}\) is a random total order on \([0,1]^2 \setminus A\).

Fix now \(x, y \in [0,1]^2 \setminus A\) with \(x < y\) and assume that \(s(\ell)\) is the sign corresponding to the unique local minimum where \(\min_{[x,y]} e\) is reached. By definition (see Equation (1)),

\[
\begin{cases}
  x \preceq_{e,p} y, & \text{if } s(\ell) = +1, \\
  y \preceq_{e,p} x, & \text{if } s(\ell) = -1.
\end{cases}
\]  

(31)

Now note that as the excursion \(e\) has a unique local minimum on the interval \([x, y]\), say at time \(t\), then by construction (see Equation (6)) the process \(Z_{1,1-p}^{(x)}\) is positive at time \(y\) if \(s(\ell) = +1\) and negative if \(-s(\ell) = -1\), therefore by definition (see Equation (29)),

\[
\begin{cases}
  x \preceq_{Z_{1,1-p}} y, & \text{if } s(\ell) = +1, \\
  y \preceq_{Z_{1,1-p}} x, & \text{if } s(\ell) = -1.
\end{cases}
\]  

(32)

Comparing Equations 31, 32, we can conclude that \(\preceq_{e,p}\) and \(\preceq_{Z_{1,1-p}}\) restricted to \([0,1]^2 \setminus A\) are the same total order. \(\square\)

4  THE SKEW BROWNIAN PERMUTON AND SLE-DECORATED LIOUVILLE QUANTUM GRAVITY SPHERES

This section is devoted to establishing the connection between the skew Brownian permuton and SLE-decorated LQG spheres described in Theorem 1.17.
Recall that \((\mathbb{C} \cup \{\infty\}, \mathbf{h}, \infty)\) and \((\eta_0, \eta_\theta)\) denote the \(\gamma\)-LQG sphere and the pair of space-filling SLE\(_{\kappa'}\) introduced in Subsection 1.5. Theorem 1.17 immediately follows from the result in Proposition 4.1. To state this proposition, we need another construction.

For \(t \in [0, 1]\), let \(X_{\rho}(t)\) be the \(\nu_\mathbf{h}\)-LQG length measure (see [38, section 3.3]) of the left outer boundary of \(\eta_0([0, t])\) and \(Y_{\rho}(t)\) be the \(\nu_\mathbf{h}\)-LQG length measure of the right outer boundary of \(\eta_0([0, t])\). From [57, Theorem 1.1] (see also [38, Theorem 4.10]), the process \(E_{\rho}(t) = (X_{\rho}(t), Y_{\rho}(t))\) defined above has (up to time reparameterization) the law of a two-dimensional Brownian excursion of correlation \(\rho\) in the nonnegative quadrant. In addition, the process \((E_{\rho}(t))_{t \in [0, 1]}\) almost surely determines \(((\mathbb{C} \cup \{\infty\}, \mathbf{h}, \infty), \eta_0)\) as a curve-decorated quantum surface.

**Proposition 4.1.** Fix \(\gamma \in (0, 2)\) and \(\theta \in [0, \pi]\). Recall that \(\rho = -\cos(\pi \gamma^2/4)\). Let \(E_{\rho}(t) = (X_{\rho}(t), Y_{\rho}(t))\) be the two-dimensional Brownian excursion of correlation \(\rho\) defined above from the curve-decorated quantum surface \(((\mathbb{C} \cup \{\infty\}, \mathbf{h}, \infty), \eta_0)\). Recall that, for \(t \in [0, 1]\), \(\psi_{\gamma, \theta}(t) \in [0, 1]\) denotes the first time when \(\eta_\theta(t)\) hits the point \(\eta_0(t)\).

Let \(Z_{\rho, q} = \{Z_{\rho, q}(u)\}_{u \in \mathbb{R}}\) be the continuous coalescent-walk process driven by \((E_{\rho}, q)\) and \(\varphi_{Z_{\rho, q}}\) be the associated stochastic process defined in Equation (4). There exist a constant \(\bar{q} = \bar{q}_{\gamma}(\theta) \in [0, 1]\) such that almost surely for almost all \(t \in (0, 1)\)

\[
\psi_{\gamma, \theta}(t) = \varphi_{Z_{\rho, \bar{q}}}(t).
\]

The proof of Proposition 4.1 builds on the next lemma (see Lemma 4.2) whose proof is postponed to the end of the section. We need some additional notation.

For \(t \in [0, 1]\), let \(\hat{\phi}_{\theta}^{(l)}\) denote the flow line of the vector field \(e^{i(h/\chi+\theta)}\) started at \(\eta_0(t)\). We point out that there is a unique flow line of the vector field \(e^{i(h/\chi+\theta)}\) started from \(\eta_0(t)\) for almost all \(t \in [0, 1]\) and this is enough for our purposes. Finally, we denote by \(\hat{\phi}_{\theta}^{(l)}\) the union of the flow lines \(\hat{\phi}_{\theta}^{(l)}\) and \(\hat{\phi}_{\theta+\pi}^{(l)}\) (followed in the same direction of \(\hat{\phi}_{\theta}^{(l)}\)). In Figure 6, we show the various curves that we are considering and we explain what we mean when we refer to left and right in Lemma 4.2.

**Lemma 4.2.** There exists a constant \(\bar{q} = \bar{q}_{\gamma}(\theta) \in [0, 1]\) such that almost surely for almost all \(t \in (0, 1)\)

\[
\begin{align*}
\{ x \in [t, 1) : Z_{\rho, \bar{q}}^{(l)}(x) \geq 0 \} &= \{ x \in [t, 1) : \eta_0(x) \text{ is weakly on the left of } \hat{\phi}_{\theta}^{(l)} \}, \\
\{ x \in [t, 1) : Z_{\rho, \bar{q}}^{(l)}(x) \leq 0 \} &= \{ x \in [t, 1) : \eta_0(x) \text{ is weakly on the right of } \hat{\phi}_{\theta}^{(l)} \}.
\end{align*}
\]

We can now prove Proposition 4.1 using Lemma 4.2.

**Proof of Proposition 4.1.** Recall that \(\psi_{\gamma, \theta}(t) \in [0, 1]\) denotes the first time when \(\eta_\theta(t)\) hits the point \(\eta_0(t)\). Fix \(t \in (0, 1)\) such that the relations in Lemma 4.2 hold (this is true for almost all \(t \in (0, 1)\)). We want to show that almost surely

\[
\psi_{\gamma, \theta}(t) = \text{Leb} \left( \left\{ x \in [0, t) : Z_{\rho, q}^{(l)}(t) < 0 \right\} \cup \left\{ x \in [t, 1) : Z_{\rho, q}^{(l)}(x) \geq 0 \right\} \right).
\]

In particular, there will be two possible choices of the flow line of the vector field \(e^{i(h/\chi+\theta)}\) started at \(\eta_0(t)\) if \(\eta_0(t)\) is a double point of \(\eta_\theta\).
FIGURE 6 A chart for the various curves considered in this section. In particular, the region covered by $\eta_0([0,t])$ is highlighted in green and the region covered by $\eta_\theta([0,\psi_\theta(t)])$ is highlighted in dashed red. By construction, the boundaries of $\eta_0([0,\psi_\theta(t)])$ are given by $\hat{\phi}_\theta(t)$ and $\hat{\phi}_\theta(t) + \pi$.

Left and right in Lemma 4.2 are defined with respect to the flow line $\phi_\theta^{(i)} = \hat{\phi}_\theta^{(i)} \cup \hat{\phi}_\theta^{(i)}$, followed in the direction of $\hat{\phi}_\theta^{(i)}$.

By definition and time parameterization of $\eta_0$ and $\eta_\theta$, it holds that

$$\psi_{\theta'}(t) = \text{Leb} \left( \left\{ x \in [0,1] : \eta_0(x) \text{ is weakly on the left of } \phi_\theta^{(i)} \right\} \right).$$

Noting that for $x \in [0,t)$ the point $\eta_0(x)$ is weakly on the left of $\phi_\theta^{(i)}$ if and only if $\eta_0(t)$ is weakly on the right of $\phi_\theta^{(x)}$, we obtain that almost surely

$$\psi_{\theta'}(t) = \text{Leb} \left( \left\{ x \in [t,1] : \eta_0(x) \text{ is weakly on the left of } \phi_\theta^{(i)} \right\} \right) + \text{Leb} \left( \left\{ x \in [0,t] : \eta_0(t) \text{ is weakly on the right of } \phi_\theta^{(x)} \right\} \right)
= \text{Leb} \left( \left\{ x \in [t,1] : \mathbb{Z}_{\rho,\bar{q}}^{(i)}(x) \geq 0 \right\} \right) + \text{Leb} \left( \left\{ x \in [0,t] : \mathbb{Z}_{\rho,\bar{q}}^{(x)}(t) \leq 0 \right\} \right),$$

where in the last equality we used Lemma 4.2. Finally, from Lemma 3.2 it holds that

$$\text{Leb} \left( \left\{ x \in [0,t] : \mathbb{Z}_{\rho,\bar{q}}^{(x)}(t) = 0 \right\} \right) = 0$$

and so we can conclude the proof.

It remains to prove Lemma 4.2. We first state and prove a result similar to Lemma 4.2 (see Lemma 4.3) when the pair of space-filling SLEs $(\eta_0, \eta_\theta)$ is parameterized by an LQG cone instead of an LQG sphere.
Let $(C, \tilde{h}, 0, \infty)$ be a $\gamma$-quantum cone ([38, Definition 3.10]) independent of $\tilde{h}$. We denote by $(\tilde{n}_0, \tilde{n}_\theta)$ the pair $(n_0, n_\theta)$ parameterize by the $\mu_{\tilde{h}}$-LQG area measure so that $\tilde{n}_0(0) = \tilde{n}_\theta(0) = 0$ and $\mu_{\tilde{h}}(\tilde{n}_0([s, t])) = \mu_{\tilde{h}}(\tilde{n}_\theta([s, t])) = t - s$, for each $s, t \in \mathbb{R}$ with $s < t$.

For $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}_-$), let $\tilde{X}_\rho(t)$ be the net change of the $\nu_{\tilde{h}}$-LQG length measure of the left outer boundary of $\tilde{n}_0([0, t])$ (respectively, $\tilde{n}_\theta([-t, 0])$) relative to time 0 and $\tilde{Y}_\rho(t)$ be the net change of the $\nu_{\tilde{h}}$-LQG length measure of the right outer boundary of $\tilde{n}_0([0, t])$ (respectively, $\tilde{n}_\theta([-t, 0])$) relative to 0 (see [38, section 4.2.1]). From [31, Theorems 1.9 and 1.11] (see also [38, Theorem 4.6]) the process $(\tilde{X}_\rho(t), \tilde{Y}_\rho(t))$ defined above has (up to time reparameterization) the law of a two-dimensional Brownian motion of correlation $\rho$. In addition, the process $(\tilde{X}_\rho(t), \tilde{Y}_\rho(t))_{t \in \mathbb{R}}$ almost surely determines $(C, \tilde{h}, -\infty, \tilde{n}_0)$ as a curve-decorated quantum surface.

For $u \in \mathbb{R}$, let $\tilde{\varphi}_\theta^{(u)}$ denote the union of the two flow lines of the vector fields $e^{i(\tilde{h}/\chi + \theta)}$ and $e^{i(\tilde{h}/\chi + \theta + \pi)}$ started at $\tilde{n}_0(u)$ (followed in the direction of the flow line of $e^{i(\tilde{h}/\chi + \theta)}$). As before these flow lines are unique for almost all $u \in \mathbb{R}$.

Lemma 4.3. Let $\tilde{Z}_{\rho,q} = \{\tilde{Z}_{\rho,q}^{(u)}\}_{u \in \mathbb{R}}$ denote the collection of (strong) solutions to the following SDEs indexed by $u \in \mathbb{R}$ and driven by $\tilde{W}_\rho$,

$$
\begin{align*}
\begin{cases}
d\tilde{Z}_{\rho,q}^{(u)}(t) = & 1_{\{\tilde{Z}_{\rho,q}^{(u)}(t) > 0\}} d\tilde{Y}_\rho(t) - 1_{\{\tilde{Z}_{\rho,q}^{(u)}(t) < 0\}} d\tilde{X}_\rho(t) + (2q - 1) \cdot dL_{\tilde{Z}_{\rho,q}^{(u)}}(t), \quad t \in \mathbb{R}_{> u} \\
\tilde{Z}_{\rho,q}^{(u)}(u) = & 0.
\end{cases}
\end{align*}
$$

(33)

There exists a constant $\tilde{q} = \tilde{q}_\gamma(\theta) \in [0, 1]$ such that almost surely for almost all $u \in \mathbb{R}$

$$
\begin{align*}
&\left\{ t \in \mathbb{R}_{> u} : \tilde{Z}_{\rho,q}^{(u)}(t) > 0 \right\} = \left\{ t \in \mathbb{R}_{> u} : \tilde{n}_0(t) \text{ is weakly on the left of } \tilde{\varphi}_\theta^{(u)} \right\}, \\
&\left\{ t \in \mathbb{R}_{> u} : \tilde{Z}_{\rho,q}^{(u)}(t) \leq 0 \right\} = \left\{ t \in \mathbb{R}_{> u} : \tilde{n}_0(t) \text{ is weakly on the right of } \tilde{\varphi}_\theta^{(u)} \right\}.
\end{align*}
$$

Proof. We start by recalling that existence and uniqueness of solutions to the SDE in Equation (33) are guaranteed by Theorem 2.1.

We now fix $u = 0$ and we set $\tilde{\varphi}_\theta := \tilde{\varphi}_\theta^{(0)}$. We assume that the two flow lines of the vector fields $e^{i(\tilde{h}/\chi + \theta)}$ and $e^{i(\tilde{h}/\chi + \theta + \pi)}$ started at $\tilde{n}_0(0)$ are unique. The proof for $u \neq 0$ is similar. For $t \in \mathbb{R}_{\geq 0}$, we consider the event

$$
E_t = \{ \tilde{n}_0(t) \text{ is on the left of } \tilde{\varphi}_\theta \},
$$

and the random variable

$$
\tilde{\tau}_t = \sup \{ s \in \mathbb{R}_{\leq t} : \tilde{n}_0 \text{ crosses } \tilde{\varphi}_\theta \text{ at time } s \}.
$$

We also consider the process

$$
Q(t) := (\tilde{Y}_\rho(t) - \tilde{Y}_\rho(\tilde{\tau}_t))1_{E_t} - (\tilde{X}_\rho(t) - \tilde{X}_\rho(\tilde{\tau}_t))1_{E_t}, \quad t \in \mathbb{R}_{> 0},
$$

(34)
where we recall that \( \overline{W}_\rho(t) = (\overline{X}_\rho(t), \overline{Y}_\rho(t)) \) is the two-dimensional Brownian motion of correlation \( \rho \) encoding \( ((C, \tilde{h}, -\infty), \tilde{\eta}_h) \). In [37, Proposition 3.2], it was shown that there exists a constant \( \overline{q} = \overline{q}_\rho(\theta) \in [0, 1] \) such that \( Q(s) \) is a skew Brownian motion of parameter \( \overline{q} \) and

\[
\{ t \in \mathbb{R}_{\geq 0} : Q(t) \geq 0 \} = \{ t \in \mathbb{R}_{\geq 0} : \tilde{\eta}_h(t) \) is weakly on the left of \( \tilde{\varphi}_h \} ,
\]
\[
\{ t \in \mathbb{R}_{\geq 0} : Q(t) \leq 0 \} = \{ t \in \mathbb{R}_{\geq 0} : \tilde{\eta}_h(t) \) is weakly on the right of \( \tilde{\varphi}_h \} .
\] (35)

Therefore in order to complete the proof, it is enough to show that \( Q(s) \) solves the SDE in Equation (33) for \( u = 0 \). Indeed, thanks to pathwise uniqueness (Theorem 2.1), then we have that \( Q = \overline{Z}_{0,q}^{(0)} \) almost surely.

As \( Q(t) \) is a skew Brownian motion of parameter \( \overline{q} \) then, as shown in [42], there exists a standard one-dimensional Brownian motion \( (\tilde{B}(t))_{t \in \mathbb{R}_{\geq 0}} \) such that

\[
Q(t) = B(t) + (2\overline{q} - 1)L^Q(t), \quad t \in \mathbb{R}_{\geq 0}.
\] (36)

Setting

\[
U(t) := (1 - \overline{q})Q(t)1_{\{Q(t) > 0\}} + \overline{q}Q(t)1_{\{Q(t) \leq 0\}},
\] (37)

then from [42, Equation (11)], we have that

\[
dU(t) = (1 - \overline{q})1_{\{U(t) > 0\}}dB(t) - \overline{q}1_{\{U(t) \leq 0\}}dB(t).
\] (38)

We also introduce the following stochastic process \( \overline{W}_\rho(t) = (\overline{X}_\rho(t), \overline{Y}_\rho(t)) \), defined for all \( t \in \mathbb{R}_{\geq 0} \) by

\[
\overline{X}_\rho(t) := \int_0^t 1_{\{U(s) > 0\}}d\overline{X}_\rho(s) + \int_0^t 1_{\{U(s) \leq 0\}}dB(s),
\]
\[
\overline{Y}_\rho(t) := \int_0^t 1_{\{U(s) > 0\}}dB(s) + \int_0^t 1_{\{U(s) \leq 0\}}d\overline{Y}_\rho(s).
\] (39)

Note that

\[
(1 - \overline{q})1_{\{U(t) > 0\}}d\overline{Y}_\rho(t) - \overline{q}1_{\{U(t) \leq 0\}}d\overline{X}_\rho(t) = (1 - \overline{q})1_{\{U(t) > 0\}}dB(t) - \overline{q}1_{\{U(t) \leq 0\}}dB(t) = dU(t),
\]

where in the last equality we used Equation (38). Setting, \( r(x) = x/(1 - \overline{q}) \cdot 1_{x>0} + x/\overline{q} \cdot 1_{x\leq 0} \), and using Equation (37), we have that \( r(U(t)) = Q(t) \). Therefore, from Proposition 2.2, we obtain that \( Q(t) \) satisfies

\[
dQ(t) = 1_{\{Q(t) > 0\}}d\overline{Y}_\rho(t) - 1_{\{Q(t) \leq 0\}}d\overline{X}_\rho(t) + (2\overline{q} - 1) \cdot dL^Q(t).
\]

Note that if we show that \( \overline{X}_\rho = \overline{X}_\rho \) almost surely and \( \overline{Y}_\rho = \overline{Y}_\rho \) almost surely, that is, \( \overline{W}_\rho = \overline{W}_\rho \) almost surely, then we can conclude the proof. From Equations 39, 37, we have that

\[
\overline{X}_\rho(t) - \overline{X}_\rho(\tau_i) = \int_{\tau_i}^t 1_{\{Q(s) > 0\}}d\overline{X}_\rho(s) + \int_{\tau_i}^t 1_{\{Q(s) \leq 0\}}dB(s).
\] (40)
As \( \int_0^t 1_{\{Q(s) = 0\}} dB(s) \) is identically zero (this follows, for instance, using the same arguments as in Equation (12)) and \( dL^Q(s) = 0 \) when \( Q(s) < 0 \), using Equation (36) we have that
\[
\int_{\tilde{t}_1}^t 1_{\{Q(s) < 0\}} d\mathbb{B}(s) = \int_{\tilde{t}_1}^t 1_{\{Q(s) < 0\}} dQ(s).
\]
In addition, noting that \( \{Q(s) < 0\} \subseteq E_\delta^\circ \) (this follows from Equation (35)) and using Equation (34), we obtain that
\[
\int_{\tilde{t}_1}^t 1_{\{Q(s) < 0\}} dQ(s) = \int_{\tilde{t}_1}^t 1_{\{Q(s) < 0\}} d\tilde{\mathbb{X}}(s).
\]
Substituting the latter two expressions in Equation 40 we conclude that for all \( t \in \mathbb{R}_{\geq 0} \),
\[
\tilde{\mathbb{X}}(t) - \tilde{\mathbb{X}}(\tilde{t}_1) = \tilde{\mathbb{X}}(t) - \tilde{\mathbb{X}}(\tilde{t}_1).
\]
Similarly, \( \overline{\mathbb{Y}}(t) - \overline{\mathbb{Y}}(\tilde{t}_1) = \overline{\mathbb{Y}}(t) - \overline{\mathbb{Y}}(\tilde{t}_1) \) for all \( t \in \mathbb{R}_{\geq 0} \). From Equation 39 and Lévy’s characterization theorem [44, Theorem 3.3.16], we have that both \( \tilde{\mathbb{X}}(t) \) and \( \overline{\mathbb{Y}}(t) \) are standard one-dimensional Brownian motions. Therefore, to conclude that \( \overline{\mathbb{Y}}(t) = \tilde{\mathbb{X}}(t) \) almost surely, it is enough to show that for \( t > \tilde{t}_1 \)
\[
\mathbb{E}\left[ (\overline{\mathbb{Y}}(t) - \overline{\mathbb{Y}}(\tilde{t}_1)) | F_{\tilde{t}_1}\right] = \overline{\mathbb{Y}}(s) - \overline{\mathbb{Y}}(s),
\]
(42)
where \( F_{\tilde{t}_1} := \sigma((\overline{\mathbb{Y}}(t), \overline{\mathbb{Y}}(s))|_{0, \tilde{t}_1}) \). Indeed, the latter equation implies that \( \overline{\mathbb{Y}}(t) - \overline{\mathbb{Y}}(s) \) is a \( F_{\tilde{t}_1} \)-martingale and then in [37, Lemma 3.17] it is shown that if Equation 41 holds then the quadratic varion of \( \overline{\mathbb{Y}}(t) - \overline{\mathbb{Y}}(s) \) must be zero and so \( \overline{\mathbb{Y}}(t) = \overline{\mathbb{Y}}(s) \) almost surely.

We proceed with the proof of Equation 42 by showing that the law of \( (\overline{\mathbb{Y}}(t) - \overline{\mathbb{Y}}(s))|_{[s, \infty)} \) (respectively, \( (\overline{\mathbb{Y}}(t) - \overline{\mathbb{Y}}(s))|_{[s, \infty)} \)) given \( F_{\tilde{t}_1} \) is the unconditional law of \( \overline{\mathbb{Y}}(t) \) (respectively, \( \overline{\mathbb{Y}}(t) \)). We set \( \overline{\mathbb{X}}(t) := \overline{\mathbb{Y}}(t) - \overline{\mathbb{Y}}(\tilde{t}_1) \). From [37, Proposition 3.4 and Lemma 3.16], we have that:

\* the processes \( \overline{\mathbb{X}}|_{[0, t]} \) and \( \overline{\mathbb{Y}}|_{[0, t]} \) (respectively, \( \overline{\mathbb{X}}|_{[0, t]} \) and \( \overline{\mathbb{Y}}|_{[0, t]} \) determine each other. In particular, \( (\overline{\mathbb{X}}(t) - \overline{\mathbb{X}}(0))|_{[s, \infty)} \) is independent of \( \overline{\mathbb{X}}|_{[0, s]} \);
\* \( \overline{\mathbb{X}}|_{[0, s]} \) (respectively, \( \overline{\mathbb{X}}|_{[0, s]} \) determines \( \overline{Q}|_{[0, s]} \) (respectively, \( \overline{Q}|_{[0, s]} \)).

Now from the definition of \( \overline{\mathbb{Y}} \) in Equation 39 and the relations in Equations 36, 37, we also have that:

\* the process \( \overline{\mathbb{X}}|_{[0, s]} \) (respectively, \( \overline{\mathbb{X}}|_{[0, s]} \) is almost surely determined by \( (\overline{\mathbb{X}}(t), \overline{Q})|_{[0, s]} \) (respectively, \( (\overline{\mathbb{X}}(t) - \overline{\mathbb{X}}(s), \overline{Q}(t) - \overline{Q}(s))|_{[s, \infty)} \)). Therefore, from the items above, \( \overline{\mathbb{X}}|_{[0, s]} \) (respectively, \( \overline{\mathbb{X}}|_{[0, s]} \) is almost surely determined by \( \overline{\mathbb{X}}|_{[0, s]} \) (respectively, \( \overline{\mathbb{X}}|_{[0, s]} \)).

From the items above, we can conclude that that the law of \( (\overline{\mathbb{Y}}(t) - \overline{\mathbb{Y}}(s))|_{[s, \infty)} \) (respectively, \( (\overline{\mathbb{Y}}(t) - \overline{\mathbb{Y}}(s))|_{[s, \infty)} \)) given \( F_{\tilde{t}_1} \) is the unconditional law of \( \overline{\mathbb{Y}}(t) \) (respectively, \( \overline{\mathbb{Y}}(t) \)). This ends the proof. \( \square \)
It remains to deduce Lemma 4.2 from Lemma 4.3.

**Proof of Lemma 4.3.** Note that the only difference between the pairs $(\tilde{\eta}_0, \tilde{\eta}_\theta)$ and $(\eta_0, \eta_\theta)$ is in their time parameterization. More precisely,

- $(\tilde{\eta}_0, \tilde{\eta}_\theta)$ are parameterized using the $\gamma$-quantum cone $(C, \tilde{h}, 0, \infty)$ and therefore the left and right boundary measures of $\tilde{\eta}_0$ are encoded by the Brownian motion $\tilde{W}_\rho(t) = (\tilde{\chi}_\rho(t), \tilde{\psi}_\rho(t))$ of correlation $\rho$;
- $(\eta_0, \eta_\theta)$ are parameterized using the $\gamma$-LQG sphere $(C \cup \{\infty\}, h, \infty)$ and therefore the left and right boundary measures of $\eta_0$ are encoded by the Brownian excursion $E_\rho(t) = (\chi_\rho(t), \psi_\rho(t))$ of correlation $\rho$.

For any $\varepsilon > 0$, we consider the following curve-decorated quantum surfaces.

- The curve-decorated quantum surface $(\tilde{S}_\varepsilon, \tilde{\eta}_0|_{[\varepsilon, 1-\varepsilon]})$, where $\tilde{S}_\varepsilon$ denotes the quantum surface obtained by restricting the quantum cone field $\tilde{h}$ to $\tilde{\eta}_0([\varepsilon, 1-\varepsilon])$.
- The curve-decorated quantum surface $(\tilde{S}_\varepsilon, \eta_0|_{[\varepsilon, 1-\varepsilon]})$, where $\tilde{S}_\varepsilon$ denotes the quantum surface obtained by restricting the quantum sphere field $h$ to $\eta_0([\varepsilon, 1-\varepsilon])$.

From [37, Lemma 3.12], the curve-decorated quantum surface $(\tilde{S}_\varepsilon, \tilde{\eta}_0|_{[\varepsilon, 1-\varepsilon]})$ is almost surely determined by $(\tilde{W}_\rho(\varepsilon + t) - \tilde{W}_\rho(\varepsilon))_{0 \leq t \leq 1-2\varepsilon}$, while the curve-decorated quantum surface $(\tilde{S}_\varepsilon, \eta_0|_{[\varepsilon, 1-\varepsilon]})$ is almost surely determined by $(E_\rho(\varepsilon + t) - E_\rho(\varepsilon))_{0 \leq t \leq 1-2\varepsilon}$.

The law of $(E_\rho(\varepsilon + t) - E_\rho(\varepsilon))_{0 \leq t \leq 1-2\varepsilon}$ is absolutely continuous with respect to the law of $(\tilde{W}_\rho(\varepsilon + t) - \tilde{W}_\rho(\varepsilon))_{0 \leq t \leq 1-2\varepsilon}$ (see Proposition A.1). This implies that curve-decorated quantum surface $(\tilde{S}_\varepsilon, \eta_0|_{[\varepsilon, 1-\varepsilon]})$ is absolutely continuous with respect to the curve-decorated quantum surface $(\tilde{S}_\varepsilon, \tilde{\eta}_0|_{[\varepsilon, 1-\varepsilon]})$.

From [37, Lemma 3.10], it follows that the flow lines $\{\tilde{\phi}_\rho^{(t)}(\varepsilon)|_{\{t\leq\varepsilon\}}\}$ (respectively, $\{\phi_\rho^{(t)}(\varepsilon)|_{\{t\leq\varepsilon\}}\}$) run until they exit $\tilde{\eta}_0([\varepsilon, 1-\varepsilon])$ (respectively, $\eta_0([\varepsilon, 1-\varepsilon])$) almost surely determined by $\tilde{S}_\varepsilon$ (respectively, $S_\varepsilon$).

Finally, as by [37, Lemma 3.6], $\tilde{\eta}_0$ (respectively, $\eta_0$) hits points on $\tilde{\phi}_\rho^{(t)}$ (respectively, $\phi_\rho^{(t)}$) in chronological order. $\tilde{\phi}_\rho^{(t)}$ (respectively, $\phi_\rho^{(t)}$) cannot revisit $\tilde{\eta}_0([\varepsilon, 1-\varepsilon])$ (respectively, $\eta_0([\varepsilon, 1-\varepsilon])$) after exiting this region. Hence, it is possible to determine from $\tilde{S}_\varepsilon$ (respectively, $S_\varepsilon$) what points of $\tilde{\eta}_0$ (respectively, $\eta_0$) are to the left or right of $\tilde{\eta}_0$ (respectively, $\eta_0$).

On the other hand, thanks to Theorem 1.7 (respectively, Theorem 2.1) the processes $\{Z_{\rho, q}^{(t)}|_{\{t\leq\varepsilon\}}\}_{t \in [\varepsilon, 1-\varepsilon]}$ in the statement of Lemma 4.2 (respectively, $\{\tilde{Z}_{\rho, q}^{(t)}|_{\{t\leq\varepsilon\}}\}_{t \in [\varepsilon, 1-\varepsilon]}$) are almost surely determined through the same solution map $F_{1-\varepsilon}$—by $(E_\rho(\varepsilon + t) - E_\rho(\varepsilon))_{0 \leq t \leq 1-2\varepsilon}$ (respectively, $(\tilde{W}_\rho(\varepsilon + t) - \tilde{W}_\rho(\varepsilon))_{0 \leq t \leq 1-2\varepsilon}$).

Therefore, by absolute continuity, we can deduce from Lemma 4.3 that almost surely for almost all $t \in [\varepsilon, 1-\varepsilon]$

$$\left\{ x \in [t, 1-\varepsilon] : Z_{\rho, q}^{(t)}(x) \geq 0 \right\} = \left\{ x \in [t, 1-\varepsilon] : \eta_0(x) \text{ is weakly on the left of } \phi_\rho^{(t)} \right\},$$

$$\left\{ x \in [t, 1-\varepsilon] : Z_{\rho, q}^{(t)}(x) \leq 0 \right\} = \left\{ x \in [t, 1-\varepsilon] : \eta_0(x) \text{ is weakly on the right of } \phi_\rho^{(t)} \right\}.$$

As $\varepsilon > 0$ can be made arbitrarily small, this proves Lemma 4.2. \(\Box\)
APPENDIX A: ABSOLUTE CONTINUITY BETWEEN CORRELATED BROWNIAN EXCURSIONS IN CONES AND CORRELATED BROWNIAN MOTIONS

Let $\mathcal{W}_\rho$ be a two-dimensional Brownian motion of correlation $\rho \in (-1, 1)$ and $\mathcal{E}_\rho$ a two-dimensional Brownian excursion of correlation $\rho$.

**Proposition A.1.** For every $\varepsilon > 0$, the distribution of $(\mathcal{E}_\rho(\varepsilon + t) - \mathcal{E}_\rho(\varepsilon))_{t \in [0, 1-2\varepsilon]}$ is absolutely continuous with respect to the distribution of $(\mathcal{W}_\rho(t))_{t \in [0, 1-2\varepsilon]}$. In particular, for every $0 < \varepsilon < 1/2$ and for every integrable function $h : C([0, 1-2\varepsilon], \mathbb{R}^2) \to \mathbb{R}$,

$$
\mathbb{E}[h((\mathcal{E}_\rho(\varepsilon + t) - \mathcal{E}_\rho(\varepsilon))_{t \in [0, 1-2\varepsilon]})] = \mathbb{E}
\left[
\left(h((\mathcal{W}_\rho(t))_{t \in [0, 1-2\varepsilon]})\alpha_\varepsilon \left( -\inf_{[0, 1-2\varepsilon]} \mathcal{W}_\rho, \mathcal{W}_\rho(1-2\varepsilon) \right) \right)
\right],
$$

(A.1)

where $\alpha_\varepsilon$ is a bounded positive continuous function on $(\mathbb{R}_+)^2 \times \mathbb{R}^2$.

In addition, as $\alpha_\varepsilon > 0$, we have that the two measures are equivalent.

The theorem above was proved in [19, Proposition A.1] in the specific case when $\rho = -1/2$ building on some specific results on a family of discrete two-dimensional walks called tandem walk [22]. In what follows we prove the general case $\rho \in (-1, 1)$ building on the more general results of [29, 32].

**Proof.** We prove the proposition by considering a two-dimensional random walk $(\mathcal{W}_n)_{n \in \mathbb{Z} \geq 0}$.

Setting $(X, Y) = \mathcal{W}_1 - \mathcal{W}_0$, we assume that $X$ and $Y$ have finite moments, $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1$ and Cov$(X, Y) = \rho$. Under these assumptions, [32, Theorem 4] guarantees that setting $Q := \mathbb{Z}^2_{\geq 0}$,

$$
\mathbb{P}_x \left( \left( \frac{1}{\sqrt{2n}} \mathcal{W}_{\lfloor n t \rfloor} \right)_{t \in [0, 1]} \in \cdot | \mathcal{W}_{[0,n]} \subset Q, \mathcal{W}_n = y \right) \xrightarrow{n \to \infty} \mathbb{P}( (\mathcal{E}_\rho(t))_{t \in [0,1]} \in \cdot ),
$$

(A.2)

for all $x, y \in Q$. In [19, Lemma A.2] it was proved that if $h : (\mathbb{Z}^2)^{n-2m+1} \to \mathbb{R}$ is a bounded measurable function, $x, y \in Q$ and $1 \leq m < n/2$, then

$$
\mathbb{E}_x \left[ h(\mathcal{W}_{i+m} - \mathcal{W}_m)_{0 \leq i \leq n-2m} | \mathcal{W}_{[0,n]} \subset Q, \mathcal{W}_n = y \right] = \mathbb{E}_0 \left[ h(\mathcal{W}_i)_{0 \leq i \leq n-2m} \cdot \alpha_{n,m}^{x,y} \left( -\inf_{0 \leq i \leq n-2m} \mathcal{W}_i, \mathcal{W}_{n-2m} \right) \right],
$$

(A.3)

where

$$
\alpha_{n,m}^{x,y}(a, b) := \sum_{z \in Q: z-a \in Q} \frac{\mathbb{P}_x(\mathcal{W}_m = z, \mathcal{W}_{[0,m]} \subset Q)^{\mathbb{P}_y(\mathcal{W}_m = z + \delta, \mathcal{W}_{[0,m]} \subset Q)}}{\mathbb{P}_x(\mathcal{W}_n = y, \mathcal{W}_{[0,n]} \subset Q)}
$$

(A.4)

The results in [32] are stated under the assumption that Cov$(X, Y) = 0$. This does not restrict the generality of the results and they remain valid (with the obvious adaptations) when Cov$(X, Y) = \rho$. See the bottom part of page 996 in [29] for more explanations on this fact.

Note that the proof of [19, Lemma A.2] does not use any specific property of the tandem walk $(\mathcal{W}_n)_{n \in \mathbb{Z} \geq 0}$. 
and $(i, j) = (j, i)$. Note that if we show that for fixed $x, y \in Q$ and for all $\varepsilon \in (0, 1/2)$, there exists a bounded positive continuous function $\alpha_\varepsilon$ on $(\mathbb{R}_+)^2 \times \mathbb{R}^2$ such that

$$\lim_{n \to \infty} \sup_{a \in \mathbb{Z}_{20}, b \in \mathbb{Z}^2} \left| \alpha_{n, \lfloor n \varepsilon \rfloor}^{x,y}(a, b) - \alpha_\varepsilon \left( \frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}} \right) \right| = 0,$$

(A.5)

then Equation (A.1) follows from Equations (A.2, A.3, A.5). Therefore it just remains to prove Equation (A.5).

Fix $x \in Q$. From [29], there exists a positive function $V$ on $Q$ such that as $n \to \infty$ the following asymptotics hold

$$\mathbb{P}_x(W_{[0,n]} \subset Q) \sim c_1 V(x) n^{-p/2} \text{ as } n \to \infty,$$

(A.6)

$$\delta_1(x, n) := \sup_{y \in Q} n^{p/2+1} \mathbb{P}_x(W_n = y, W_{[0,n]} \subset Q) - c_2 V(x) g \left( \frac{y}{\sqrt{2n}} \right) \to 0,$$

(A.7)

$$\mathbb{P}_x(W_n = y, W_{[0,n]} \subset Q) \sim c_3 \cdot \frac{V(x) V(y)}{n^{p+1}},$$

(A.8)

where $c_1, c_2, c_3$ are three constants, $g$ is a positive bounded integrable function on $\mathbb{R}_+^2$ and $p$ is a parameter depending only on $\rho$ and $Q$. (We highlight that all these constants and the function $g$ can be explicitly computed in specific cases, see, for instance, [19, Lemma A.5].) More precisely, Equation A.6 is [29, Theorem 1], Equation A.7 is [29, Theorem 5], and Equation A.8 is [29, Theorem 6].

In what follows, $m = \lfloor n \varepsilon \rfloor$ for some $\varepsilon > 0$. Let us consider $\alpha_{n,m}^{x,y}(a, b)$ defined in Equation A.4. By Equation A.8, the denominator (which is independent of $a, b$) is of order $n^{-p-1}$.

We now look at the numerator of $\alpha_{n,m}^{x,y}(a, b)$. We first cut the sum at $t \sqrt{n}$ for some $t > 0$ and bound the rest of the sum. Using Equation A.7 for the first factor (recall that $g$ is bounded) and Equation A.6 for the second one, we can guarantee that there exists a constant $C > 0$ depending only on $x, y$ such that

$$R_n := \sum_{|z| > t \sqrt{n}} \mathbb{P}_x(W_m = z, W_{[0,m]} \subset Q) \mathbb{P}_y(W_m = \hat{z} + \hat{b}, W_{[0,m]} \subset Q)$$

$$\leq C n^{-p/2} n^{-p/2-1} \sum_{|z| > t \sqrt{n}} \mathbb{P}_x(W_m = z | W_{[0,m]} \subset Q)$$

$$= C n^{-p-1} \mathbb{P}_x(|W_{\lfloor n \varepsilon \rfloor}| > t \sqrt{n} | W_{[0, \lfloor n \varepsilon \rfloor]} \subset Q).$$

Using [29, Theorem 3], it is possible to find a function $\delta_2(x, y, \varepsilon, n, t)$ independent of $a, b$ such that

$$n^{b+1} R_n \leq \delta_2(x, y, \varepsilon, n, t) \quad \text{ and } \quad \lim_{t \to \infty} \limsup_{n \to \infty} \delta_2(x, y, \varepsilon, n, t) = 0.$$
Now set $S_{n,m} := \sum_{z:z-a\in Q,|z|\leq t} \mathbb{P}_x(W_m = z, W_{[0,m]} \subset Q) \cdot \mathbb{P}_y(W_m = \hat{z} + \hat{b}, W_{[0,m]} \subset Q)$. Using Equation A.7, we have for fixed $x$ and $y$ that

$$S_{n,m} = m^{-p-2} \cdot \mathbb{E}(V(x)V(\hat{y})) \sum_{z:z-a\in Q,|z|\leq t} g\left(\frac{z}{\sqrt{2\varepsilon n}}\right) g\left(\frac{\hat{z} + \hat{b}}{\sqrt{2\varepsilon n}}\right) + O(1) \left(t/\sqrt{n}\right)^2 m^{-p-2} (\delta_1(x,m) + \delta_1(\hat{y},m)).$$

Putting together the latter estimate for the numerator of Equation A.4 with the estimate in Equation A.8 for the denominator, both uniform in $(a,b)$, we have

$$\alpha_{x,y}^{n,\lfloor \varepsilon n \rfloor} (a,b) = O(1)n^{p+1}R_n + o(1) + \frac{1}{\varepsilon^{p+1}} \left(\frac{c_2^2}{c_3} + o(1)\right) \int_{w: w-a \in \mathbb{R}^2_{\geq 0}} g\left(\frac{\hat{w} + \hat{b}/\sqrt{n}}{\sqrt{2\varepsilon}/\sqrt{2\varepsilon}}\right) dw + o(1),$$

where the final $o(1)$ error term comes from the summation approximation. We highlight that all the error terms are uniform in $a$ and $b$. Finally, setting

$$\alpha_x(a,b) := \frac{1}{\varepsilon^{p+1}} \cdot \frac{c_2^2}{c_3} \cdot \int_{w: w-a \in \mathbb{R}^2_{\geq 0}} g\left(\frac{\hat{w} + \hat{b}/\sqrt{n}}{\sqrt{2\varepsilon}/\sqrt{2\varepsilon}}\right) dw,$$

we obtain that

$$\left|\alpha_{n,\lfloor \varepsilon n \rfloor} (a,b) - \alpha_x\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}\right)\right| = O(1)n^{p+1}R_n + o(1) + O(1) \int_{|w|>t} g(w/\sqrt{2\varepsilon}).$$

As $g$ is integrable and we have the bounds in Equation A.9, we can conclude that the latter term tends to zero first taking $n \to \infty$ and then $t \to \infty$. 

**APPENDIX B: SIMULATIONS OF THE SKEW BROWNIAN PERMUTON**

We briefly explain how we obtained the simulations of the skew Brownian permuton $\mu_{\rho,q}$ given in Figure 3.

- The first step is to sample an approximation of a two-dimensional Brownian excursion $E_\rho$ with correlations $\rho$. To do that,
  - we start with (any) two-dimensional walk in the nonnegative quadrant, started and ended at $(0,0)$, with (say) 10 points and linear interpolating among these points. Then we run the following Glauber dynamics: we resample each point of the walk in such a way that the conditional law of each point given the neighboring points is that of the Gaussian distribution with correct mean and variance to correspond to the desired Brownian excursion, then conditioned to be in the nonnegative quadrant.
To get a new walk with double points, we consider the walk obtained in the previous step and then we add a new point in the middle point of each linear segment. Then we run again the same Glauber dynamics used in the previous step.

Iterating the previous step, we always get a new walk with double points, which is a better approximation of two-dimensional Brownian excursion $E_\rho$.

The second step is to construct approximations of the solutions of the SDEs in Equation (3) driven by the approximation of $E_\rho$ obtained in the previous step. This can be done by using some well-chosen discrete coalescent-walk processes (see [14, section 2.2]) whose walks converge to the solutions of the SDEs in Equation (3) (in the same spirit of [14, Proposition 4.8]).

The final step is the simplest one: once we have the approximated solutions of the SDEs in Equation (3), it is simple to construct the process $\varphi_{Z_{\rho,q}}$ defined in Equation (4). Finally, the skew Brownian permuton $\mu_{\rho,q}$ is well-approximated by the graph of the function of the process $\varphi_{Z_{\rho,q}}$.

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