A DECOMPOSITION THEOREM OF VARADHAN TYPE FOR CO-LOCAL FORMS ON LARGE SCALE INTERACTING SYSTEMS

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ABSTRACT. In our previous article with Yukio Kametani, we investigated the geometric structure underlying a large scale interacting system on infinite graphs, via constructing a suitable cohomology theory called uniformly local cohomology, which reflects the geometric property of the microscopic model, using a class of functions called the uniformly local functions. In this article, we introduce the co-local functions on the geometric structure associated to a large scale interacting system. We may also define the notion of uniformly local functions for co-local functions. However, contrary to the functions appearing in our previous article, the co-local functions reflect the stochastic property of the model, namely the probability measure on the configuration space. We then prove a decomposition theorem of Varadhan type for closed co-local forms. The space of co-local functions and forms contain the space of $L^2$-functions and forms. In the last section, we state a conjecture concerning the decomposition theorem for the $L^2$-case.

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0. Introduction

In our article [1] with Yukio Kametani, we investigated the geometric structure underlying a large scale interacting system on infinite graphs, via constructing a suitable cohomology theory called uniformly local cohomology, which reflects the geometric property of the microscopic model. In this article, as a first step to consider stochastic data on the model, we will equip the configuration space of our model with a probability measure. We will then define a class of

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functions and forms, which we call the co-local functions and forms, reflecting the property of the probability measure.

The precise mathematical object that we investigate is as follows. We define a locale \((X, E)\) to be any locally finite simple symmetric directed graph which is connected. Here, \(X\) denotes the set of vertices of the graph and \(E \subset X \times X\) the set of directed edges. The locale represents the space where the interaction takes place. We let \(S\) be a set of states, which in this article we define to be a finite set with a designated element which we call the base state. The set of states represents all of the possible states on a single vertex. We define an interaction to be a map \(\phi: S \times S \to S \times S\) such that for any pair of states \((s_1, s_2) \in S \times S\) satisfying \(\phi(s_1, s_2) \neq (s_1, s_2)\), we have

\[ \hat{i} \circ \phi \circ \hat{i} \circ \phi(s_1, s_2) = (s_1, s_2), \]

where \(\hat{i}: S \times S \to S \times S\) is the bijection obtained by exchanging the components of \(S \times S\). The interaction represents the possible change of states on adjacent vertices connected by an edge of the locale.

Associated to the data \(((X, E), S, \phi)\), we define the configuration space \(S^X\) to be the set

\[ S^X := \prod_{x \in X} S \]

with the set of transitions \(\Phi := \{(s, s^e) \mid s \in S^X, e \in E\}\), where for any \(e = (o(e), t(e)) \in E \subset X \times X\), the configuration \(s^e = (s^e_x) \in S^X\) is such that \((s^e_{o(e)}, s^e_{t(e)}) = \phi(s_{o(e)}, s_{t(e)})\) and \(s^e_x = s_x\) if \(x \neq o(e), t(e)\). The pair \((S^X, \Phi)\) form a symmetric graph which we call the configuration space with transition structure. This graph represents all possible change of the configuration at a single instant. This structure is independent of the transition rate – stochastic data which encodes the expected frequency of the transitions.

A typical example of a locale is given by the Euclidean lattice \((\mathbb{Z}^d, (\mathbb{Z}^d)^*)\), and the simplest nontrivial example of the set of states is \(S = \{0, 1\}\) with 0 taken to be the base state. We may interpret the elements 0 and 1 respectively to represent the non-existence or existence of a particle. An interaction in this case is given by \(\phi(s_1, s_2) = (s_2, s_1)\). The configuration space \(S^X = \{0, 1\}^{\mathbb{Z}^d}\) expresses all of the possible configuration of particles on \(X\), and the transition structure \(\Phi\) expresses the change of configurations arising from the exchange of particles on an edge of the locale. Typical models, such as the multi-color exclusion process and the generalized exclusion process may be described using this framework. See \[1\] §2.1 for other examples of locales, set of states, and interactions.

In our previous article \[1\], we investigated the geometric property of \((S^X, \Phi)\) through a class of functions and forms which we call uniformly local, and proved a decomposition theorem for uniformly local closed forms, which may be interpreted as a uniformly local variant of the decomposition theorem originally proposed by Varadhan \[6\], which plays a crucial role in proving the hydrodynamic limit for non-gradient systems. The uniformly local functions are certain functions in \(C(S^X)\), where \(S^X\) denotes the subset of \(S^X\) consisting of configurations at base state outside a finite number of vertices, and \(C(S^X)\) denotes the \(\mathbb{R}\)-linear space of \(\mathbb{R}\)-valued functions on \(S^X\). The definition of the set \(C(S^X)\) as well as the definition of uniformly functions are purely algebraic, and is independent of the choice of a probability measure on \(S^X\).
In this current article, we will equip the configuration space $S^X$ with a probability measure $\mu$. We will then define a certain variant of the space $C(S^X)$ which we call the space of co-local functions, as a projective limit of local functions via projections defined via the conditional expectations of the probability measure. The co-local functions are in fact Martingales indexed by the finite subsets of the set of vertices of the locale. As the name suggests, the space of co-local functions is the $\mathbb{R}$-linear dual of the space of local functions on $S^X$. In the case that the probability measure $\mu$ is a product measure of a probability measure on $S$, we will define the notion of closed co-local forms, and prove a Varadhan type decomposition theorem for such forms. The space of co-local functions and forms contain the space of $L^2$-functions and forms. In §5, we consider $L^2$-functions and forms and formulate a conjecture giving the Varadhan decomposition.

1. The Space of Co-local Functions

Let the notations be as in the §0. In this section, we will consider a probability measure on $S^X$ and the conditional expectation. We let $\mathcal{F}$ be the set of all subsets of the state space $S$, which is a finite set since we have assumed that $S$ is finite. Then $\mathcal{F}$ is trivially a $\sigma$-algebra. In other words, $\mathcal{F}$ contains $S$ and is closed under taking complement and countable unions. Note that $\mathcal{F}$ coincides with the Borel $\sigma$-algebra for the discrete topology of $S$. The condition that $X$ is connected and locally finite ensures that the set of vertices is countable. For any $\Lambda \subset X$, we let $\mathcal{F}_\Lambda := \mathcal{F}^\otimes \Lambda$ be the product $\sigma$-algebra on $S^\Lambda$ obtained from $\mathcal{F}$, which coincides with the Borel $\sigma$-algebra for the topological space $S^\Lambda$. In particular, if $\Lambda$ is finite, then $\mathcal{F}_\Lambda$ coincides with the set of all subsets of $S^\Lambda$. For any $\Lambda \subset X$, the pair $(S^\Lambda, \mathcal{F}_\Lambda)$ is a measurable space.

Definition 1.1. For any $\Lambda \subset X$, denote by $C(S^\Lambda)$ the $\mathbb{R}$-linear space of real valued measurable functions on $S^\Lambda$.

In particular, if $\Lambda$ is finite, then $C(S^\Lambda)$ is simply the set of real valued functions on $S^\Lambda$. Any inclusion $\Lambda \subset \Lambda'$ of subsets in $X$, the inclusion induces a projection $S^{\Lambda'} \to S^\Lambda$, which in turn induces the inclusion $C(S^\Lambda) \hookrightarrow C(S^{\Lambda'})$. We let $\mathcal{I}$ be the set of finite subsets $\Lambda \subset X$. Then $\mathcal{I}$ is a directed set for the order given by the inclusion. Indeed, $\subset$ gives a partial order, and for any $\Lambda, \Lambda' \in \mathcal{I}$, if we let $\Lambda'' := \Lambda \cup \Lambda'$, then we have $\Lambda, \Lambda' \subset \Lambda''$. For any $\Lambda, \Lambda' \in \mathcal{I}$ such that $\Lambda \subset \Lambda'$, there exists a natural projection $S^{\Lambda'} \to S^\Lambda$, which induces a natural inclusion $C(S^\Lambda) \hookrightarrow C(S^{\Lambda'})$. This gives $\{C(S^\Lambda)\}_{\Lambda \in \mathcal{I}}$ a structure of a directed system.

Definition 1.2. We define the set of local functions on $S^X$ to be the direct limit

$$C_{\text{loc}}(S^X) := \lim_{\Lambda \in \mathcal{I}} C(S^\Lambda) = \bigcup_{\Lambda \in \mathcal{I}} C(S^\Lambda).$$

This definition of local functions is exactly the definition given in [1].

Let $S^X_*$ be the set of configurations whose components are at base set except for a finite number of vertices, and let $C(S^X_*)$ be the $\mathbb{R}$-linear space of real valued functions on $S^X_*$. In [1], we defined the space of uniformly local functions to be a certain subspace of $C(S^X_*)$ containing the space of local functions $C_{\text{loc}}(S^X)$. The space $C(S^X_*)$ may be interpreted as a projective limit of space
of local functions, given as follows. For any inclusion \( \Lambda \subset \Lambda' \subset X \), consider the projection \( \iota^\Lambda : C(S^{\Lambda'}) \to C(S^\Lambda) \) defined by

\[
\iota^\Lambda f(s) := f(\iota_{\Lambda}(s))
\]

for any \( s \in S^\Lambda \), where \( \iota_{\Lambda}(s) \) is the configuration \( s' = (s'_x) \in S^{\Lambda'} \) such that \( s'_x = s_x \) for any \( x \in \Lambda \) and \( s'_x \) is at base state for any \( x \in \Lambda' \setminus \Lambda \). Then the \( \mathbb{R} \)-linear spaces \( C(S^\Lambda) \) for \( \Lambda \in \mathcal{F} \) form a projective system with respect to the projection \( \iota^\Lambda \).

**Lemma 1.3.** We have

\[
C(S^\Lambda_x) = \lim_{\Lambda \in \mathcal{F}} C(S^\Lambda).
\]

**Proof.** Suppose \((f^\Lambda) \in \lim_{\Lambda \in \mathcal{F}} C(S^\Lambda)\). For any \( s \in S^\Lambda_x \), let \( \Lambda \in \mathcal{F} \) be sufficiently large such that the components \( s_x \) of \( s = (s_x) \) is at base state outside \( x \in \Lambda \). Then \( f(s) := \iota^\Lambda f(s) \) is independent of the choice of such \( \Lambda \), hence this construction defines a function \( f : S^\Lambda_x \to \mathbb{R} \). This gives an \( \mathbb{R} \)-linear homomorphism

\[
\lim_{\Lambda \in \mathcal{F}} C(S^\Lambda) \to C(S^\Lambda_x).
\]

On the other hand, suppose we have a function \( f : S^\Lambda_x \to \mathbb{R} \). For any \( \Lambda \in \mathcal{F} \), let \( f^\Lambda(s) := f(s|_\Lambda) \) for any \( s \in S^\Lambda_x \), where \( s|_\Lambda \) denotes the configuration in \( S^\Lambda_x \) whose components at \( x \in \Lambda \) coincides with that of \( s \) and is at base state outside \( \Lambda \). Then \( f^\Lambda \) is a function in \( C(S^\Lambda) \), and \((f^\Lambda) \) for \( \Lambda \in \mathcal{F} \) form a projective system with respect to the projection \( \iota^\Lambda \). This gives an \( \mathbb{R} \)-linear homomorphism inverse to that of \((1)\), which proves that \((1)\) is an isomorphism as desired. \(\square\)

The projection \( \iota^\Lambda : C(S^\Lambda) \to C(S^\Lambda) \) maps any function \( f : S^\Lambda_x \to \mathbb{R} \) to a function \( \iota^\Lambda f : S^\Lambda \to \mathbb{R} \). This operation may be interpreted as restricting a function \( f \) on \( S^\Lambda \) to a function which depends only on the local configurations \( S^\Lambda \) on the vertices \( \Lambda \subset X \). One drawback of the projection \( \iota^\Lambda \) is that by construction, \( \iota^\Lambda f \) depends only on the restriction of \( f \) to \( S^\Lambda_x \) and does not reflect the behavior of \( f \) on the entirety of the configuration space \( S^\Lambda \). In order to redress the projection, we introduce a probability measure \( \mu \) on \( S^\Lambda \) encoding the probability of occurrence of the configurations in \( S^\Lambda \).

We fix a probability measure \( \mu \) on \((S^\Lambda, \mathcal{F}_\Lambda)\). For any \( \Lambda \subset X \), we denote again by \( \mu \) the measure on \( S^\Lambda \) obtained as the pushforward of the measure \( \mu \) with respect to the projection \( S^\Lambda \to S^\Lambda \). We assume in addition that \( \mu \) is supported on \( S^\Lambda \) for any finite \( \Lambda \subset X \). In other words, we have \( \mu(s) := \mu(\{s\}) > 0 \) for any \( s \in S^\Lambda \). For any \( f \in C(S^\Lambda) \), we let \( E_\mu[f] := \int_{S^\Lambda} f \, d\mu \) be the expectation value of \( f \) on \( S^\Lambda \) with respect to the measure \( \mu \). For any \( \Lambda \subset X \) and an integrable function \( f \in C(S^\Lambda) \) for \( \Lambda \subset \Lambda' \), we let \( \pi_{\Lambda'}^\Lambda f := E_\mu[f|_{\mathcal{F}_\Lambda}] \) be the *conditional expectation* with respect to the projection \( \text{pr}_\Lambda : S^{\Lambda'} \to S^\Lambda \). More precisely, we define \( \pi_{\Lambda'}^\Lambda f \in C(S^\Lambda) \) to be the integrable function on \( S^\Lambda \) characterized by the property that

\[
E_\mu[(\pi_{\Lambda'}^\Lambda f)g] := \int_{S^\Lambda} (\pi_{\Lambda'}^\Lambda f)g \, d\mu = \int_{S^{\Lambda'}} fg \, d\mu
\]

for any integrable \( g \in C(S^\Lambda) \), where we denote again by \( g \) the function in \( C(S^{\Lambda'}) \) induced by the natural inclusion \( C(S^\Lambda) \hookrightarrow C(S^{\Lambda'}) \). In particular, if \( \Lambda, \Lambda' \in \mathcal{F} \), by taking \( g \) to be the *indicator
function \(1_s\) for \(s \in S^\Lambda\) which is one on \(s\) and zero outside of \(s\), we see that \(\pi^\Lambda_\mu f(s)\) is explicitly given as

\[
\pi^\Lambda_\mu f(s) = \frac{1}{\mu(s)} \sum_{s' \in S^\Lambda' \atop \text{pr}_\Lambda(s') = s} f(s') \mu(s').
\]

In other words, \(\pi^\Lambda_\mu f(s)\) is the expected value of the function \(f\) on the set of configurations in \(S^\Lambda\) which projects to \(s \in S^\Lambda\). Hence if \(f\) is an integrable function in \(C(S^X)\), then \(\pi^\Lambda_\mu f\) is a function in \(C(S^\Lambda)\) which reflects the property of \(f\) on the entire of \(S^X\) weighted by the probability measure \(\mu\). In the case that \(f \in C(S^\Lambda)\), if we view \(f\) as an element in \(C(S^\Lambda')\) for \(\Lambda \subset \Lambda'\) through the natural inclusion \(C(S^\Lambda) \hookrightarrow C(S^\Lambda')\), then we see that \(\pi^\Lambda_\mu f = f\).

The conditional expectation satisfies the tower property, which may be proved for the case \(\Lambda, \Lambda', \Lambda'' \in \mathcal{F}\) such that \(\Lambda \subset \Lambda' \subset \Lambda''\) and integrable \(f \in C(S^\Lambda'')\) by

\[
\pi^\Lambda_\mu(\pi^\Lambda_{\mu'} f)(s) = \frac{1}{\mu(s)} \sum_{s' \in S^\Lambda' \atop \text{pr}_\Lambda(s') = s} \pi^\Lambda_{\mu'} f(s') \mu(s') = \frac{1}{\mu(s)} \sum_{s' \in S^\Lambda' \atop \text{pr}_\Lambda(s') = s} \sum_{s'' \in S^\Lambda'' \atop \text{pr}_\Lambda(s'') = s'} f(s'') \mu(s'') = \pi^\Lambda_\mu f(s).
\]

Note that if \(\Lambda \in \mathcal{F}\), then any function in \(C(S^\Lambda)\) is integrable. Hence for any \(\Lambda, \Lambda' \in \mathcal{F}\) such that \(\Lambda \subset \Lambda'\), the conditional expectation induces an homomorphism \(\pi^\Lambda_\mu : C(S^\Lambda') \to C(S^\Lambda)\), which is a projection since \(\pi^\Lambda_\mu(\pi^\Lambda_{\mu'} f) = \pi^\Lambda_\mu f\) for any \(f \in C(S^\Lambda')\). The tower property shows that \(\{C(S^\Lambda)\}_{\Lambda \in \mathcal{F}}\) form a projective system with respect to the projections \(\pi^\Lambda_\mu\).

**Definition 1.4.** We define the set of co-local functions on \(S^X\) to be the projective limit

\[
C_{\text{col,} \mu}(S^X) := \varprojlim_{\Lambda \in \mathcal{F}} C(S^\Lambda)
\]

of \(C(S^\Lambda)\) with respect to the projections \(\pi^\Lambda_\mu\). We call any element \(f\) in \(C_{\text{col,} \mu}(S^X)\) a co-local function on \(S^X\).

By definition, a co-local function \((f^\Lambda) \in C_{\text{col,} \mu}(S^X)\) is a system of measurable functions \(f^\Lambda \in C(S^\Lambda)\) for \(\Lambda \in \mathcal{F}\) satisfying \(\pi^\Lambda_\mu f^{\Lambda'} = f^\Lambda\) for any \(\Lambda, \Lambda' \in \mathcal{F}\) such that \(\Lambda \subset \Lambda'\). Such a system of random variables related via the conditional expectation is usually referred to as a Martingale with respect to the index set \(\mathcal{F}\). We note that if \(X\) is an infinite locale, then co-local functions do not necessarily define a function on \(S^X\). We believe the space of co-local functions is a natural framework to consider formal infinite sum of functions appearing in the works of Varadhan (see \(\text{[8]}\) below for the case of conserved quantities).

If \(f\) is a function in \(C(S^\Lambda)\) for some \(\Lambda \in \mathcal{F}\), then \(f\) is an integrable function in \(C(S^X)\) through the natural inclusion \(C(S^\Lambda) \hookrightarrow C(S^X)\). By \(\text{[2]}\), we see that \(\pi^\Lambda_\mu f = f\) as a function in \(C(S^\Lambda)\). For any local function \(f \in C_{\text{loc}}(S^X)\), if we let \(f^\Lambda := \pi^\Lambda_\mu f\) for any \(\Lambda \in \mathcal{F}\), then the system \((f^\Lambda)\)
is a co-local function on $S^X$. Thus we have a homomorphism
\[ C_{\text{loc}}(S^X) \to C_{\text{col}, \mu}(S^X), \quad f \mapsto (\pi_{\mu}^A f) \]
which is injective since $f = \pi_{\mu}^A f$ for $\Lambda$ sufficiently large. Hence we may view the space of co-local functions as enlarging the space of local functions.

**Remark.** In what follows, we will denote $\pi_{\mu}^A$ and $C_{\text{col}, \mu}(S^X)$ simply as $\pi^A$ and $C_{\text{col}}(S^X)$. If $X$ is a finite locale, then we simply have $C_{\text{loc}}(S^X) = C_{\text{col}}(S^X) = C(S^X)$. As explained below, $C(S^\Lambda)$ for any $\Lambda \in \mathcal{F}$ has a structure of a Hilbert space with respect to the inner product (4). Since $C_{\text{col}}(S^X)$ is a projective limit of Hilbert spaces, it has a structure of a Fréchet space (see for example [2] Proposition 2.3.7). The space of local functions $C_{\text{loc}}(S^X)$ is dense in $C_{\text{col}}(S^X)$ for this topology.

The term *co-local* is derived from the duality between local and co-local functions, given in Lemma 1.5 below. For any $\Lambda \in \mathcal{F}$, the space $C(S^\Lambda)$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_\mu$ given for any $f, g \in C(S^\Lambda)$ by

\[ \langle f, g \rangle_\mu := E_\mu[f g] = \int_{S^\Lambda} f g d\mu. \]

The Reiz representation theorem gives an isometric isomorphism

\[ C(S^\Lambda) \overset{\cong}{\longrightarrow} (C(S^\Lambda))^\ast \quad f \mapsto (g \mapsto \langle f, g \rangle_\mu), \]

where $(C(S^\Lambda))^\ast$ denotes the space of bounded, or equivalently continuous $\mathbb{R}$-linear functionals on $C(S^\Lambda)$. Since we have assumed that $S$ hence $S^\Lambda$ is finite, $C(S^\Lambda)$ is a finite dimensional $\mathbb{R}$-linear space. Hence any $\mathbb{R}$-linear functional on $C(S^\Lambda)$ is automatically bounded. This shows that we have $(C(S^\Lambda))^\ast = (C(S^\Lambda))^\vee$, where $(C(S^\Lambda))^\vee$ denotes the algebraic dual $(C(S^\Lambda))^\vee := \text{Hom}_\mathbb{R}(C(S^\Lambda), \mathbb{R})$.

**Lemma 1.5.** For any $(f^\Lambda) \in C_{\text{col}}(S^X)$, consider the bounded linear functional on $C(S^\Lambda)$ given by $g \mapsto \langle f^\Lambda, g \rangle_\mu$. Then this gives an isomorphism

\[ C_{\text{col}}(S^X) \cong (C_{\text{loc}}(S^X))^\vee. \]

**Proof.** For any $\Lambda, \Lambda' \in \mathcal{F}$ such that $\Lambda \subset \Lambda'$, consider $f \in C(S^\Lambda')$ and $g \in C(S^\Lambda)$. By definition of the conditional expectation, we have

\[ \langle \pi^\Lambda f, g \rangle_\mu = \int_{S^\Lambda} (\pi^\Lambda f) g d\mu = \int_{S^{\Lambda'}} f g d\mu = \langle f, g \rangle_\mu. \]

Thus we have a commutative diagram

\[ \begin{array}{ccc}
C(S^{\Lambda'}) & \overset{\cong}{\longrightarrow} & C(S^{\Lambda'})^\vee \\
\pi^\Lambda \downarrow & & \downarrow \iota^\ast \\
C(S^\Lambda) & \overset{\cong}{\longrightarrow} & C(S^\Lambda)^\vee,
\end{array} \]
where \( \iota^* \) is the dual of the natural injection \( \iota: C(S^\Lambda) \hookrightarrow C(S^{\Lambda'}) \). By passing to the projective limit, we have

\[
C_{\text{col}}(S^X) = \lim_{\Lambda \in \mathcal{J}} C(S^\Lambda) \cong \lim_{\Lambda \in \mathcal{J}} C(S^\Lambda)^\vee = (\lim_{\Lambda \in \mathcal{J}} C(S^\Lambda))^\vee = (C_{\text{loc}}(S^X))^\vee
\]

as desired. \( \square \)

2. Uniformly Local Functions

In this section, we give an expansion of co-local functions and define the notion of uniform locality. This is a co-local version of [1, Proposition 3.3]. We first prove some properties concerning the conditional expectation.

**Lemma 2.1.** Consider \( \Lambda, \Lambda' \in \mathcal{J} \) such that \( \Lambda \subset \Lambda' \). Then \( \pi^\Lambda \) is an orthogonal projection of \( C(S^{\Lambda'}) \) to \( C(S^\Lambda) \) with respect to the inner product \( (\cdot, \cdot) \) on \( C(S^{\Lambda'}) \).

**Proof.** The homomorphism \( \pi^\Lambda: C(S^{\Lambda'}) \to C(S^\Lambda) \) is a projection since \( \pi^\Lambda(\pi^\Lambda f) = \pi^\Lambda f \) for any \( f \in C(S^{\Lambda'}) \). For any \( f, g \in C(S^{\Lambda'}) \), the characterization of the conditional expectation shows that

\[
\langle \pi^\Lambda f, \pi^\Lambda g \rangle_\mu = E_{\mu}[\langle \pi^\Lambda f \rangle \mu] = E_{\mu}[\langle \pi^\Lambda g \rangle] = \langle \pi^\Lambda f, g \rangle_\mu,
\]

\[
\langle \pi^\Lambda f, \pi^\Lambda g \rangle_\mu = E_{\mu}[\langle \pi^\Lambda f \rangle \mu] = E_{\mu}[\langle \pi^\Lambda g \rangle] = \langle f, \pi^\Lambda g \rangle_\mu,
\]

which shows that \( \langle \pi^\Lambda f, g \rangle_\mu = \langle f, \pi^\Lambda g \rangle_\mu \) as desired. \( \square \)

We now assume until the end of this section that the probability measure \( \mu \) on \( S^X \) is the product measure \( \mu = \nu^{\otimes X} \) for a probability measures \( \nu \) on \( S \) supported on points, i.e. satisfying \( \nu(\{s\}) > 0 \) for any \( s \in S \). In this case, for any \( \Lambda \subset X \), the push-forward of \( \mu \) to \( S^\Lambda \) coincides with the product measure \( \nu^{\otimes \Lambda} \).

**Lemma 2.2.** Suppose \( \mu \) is the product measure \( \mu = \nu^{\otimes X} \) as above. For any \( \Lambda \in \mathcal{J} \) and \( \Lambda', \Lambda'' \subset \Lambda \), we have

\[
\pi^{\Lambda'}(\pi^{\Lambda''} f) = \pi^{\Lambda' \cap \Lambda''} f
\]

for any function \( f \in C(S^\Lambda) \).

**Proof.** By (2), for any \( s' \in S^{\Lambda'} \), we have

\[
\pi^{\Lambda'} f(s') := \frac{1}{\mu(s')} \sum_{s \in S^\Lambda \atop \text{pr}_{\Lambda'}(s) = s'} f(s) \mu(s) = \sum_{s = (s_x)_{x \in \Lambda \setminus \Lambda'}} f(s) \left( \prod_{x \in \Lambda \setminus \Lambda'} \nu(s_x) \right),
\]
where the second equality is derived from the fact that \( \mu = \nu^X \). Again by (2), we have

\[
(\pi^\Lambda (\pi^\Lambda f))(s') = \sum_{s = (s_x) \in S^\Lambda \atop \text{pr}_{\Lambda'}(s) = s'} \pi^\Lambda f(s) \left( \prod_{x \in \Lambda \setminus \Lambda'} \nu(s_x) \right) = \sum_{s' = (s_x') \in S^{\Lambda''} \atop \text{pr}_{\Lambda'' \cap \Lambda'''}(s') = \text{pr}_{\Lambda'' \cap \Lambda'''}(s')} \pi^{\Lambda''} f(s'') \left( \prod_{x \in \Lambda' \cap \Lambda''} \nu(s_x) \right) \left( \prod_{x \in \Lambda'' \setminus \Lambda'} \nu(s_x) \right),
\]

which coincides with

\[
\pi^{\Lambda' \cap \Lambda''} f(s') = \sum_{s \in S^\Lambda \atop \text{pr}_{\Lambda' \cap \Lambda''}(s) = s'} f(s) \left( \prod_{x \in \Lambda' \setminus (\Lambda' \cap \Lambda'')} \nu(s_x) \right)
\]

as desired. \( \square \)

**Proposition 2.3.** Suppose \( \mu \) is the product measure \( \mu = \nu^X \) on \( S^X \), where \( \nu \) is a probability measure on \( S \) supported on the points of \( S \). For any \( \Lambda \in \mathcal{F} \), let

\[
C_\Lambda(S^X) := \{ f \in C(S^\Lambda) \mid \pi^\Lambda f \equiv 0 \text{ if } \Lambda \not\subseteq \Lambda' \}.
\]

Then for any \( (f^\Lambda) \in C_{\text{col}}(S^X) \), there exists a unique family of functions \( f_\Lambda \in C_\Lambda(S^X) \) such that

\[
f^\Lambda = \sum_{\Lambda'' \subseteq \Lambda} f_{\Lambda''}
\]

for any \( \Lambda \in \mathcal{F} \).

**Proof.** We prove our result by induction on the order of \( \Lambda \). We first let \( f_0 := f^0 \), which is the case for \( \Lambda = \emptyset \). Next, for any \( \Lambda \in \mathcal{F} \), suppose \( f_{\Lambda''} \) is defined for any \( \Lambda'' \subseteq \Lambda \). We let

\[
f_{\Lambda} := f^\Lambda - \sum_{\Lambda'' \subseteq \Lambda} f_{\Lambda''}.
\]

Then for any \( \Lambda' \in \mathcal{F} \) such that \( \Lambda \not\subset \Lambda' \), we have

\[
\pi^\Lambda f_{\Lambda} = \pi^\Lambda f^\Lambda - \sum_{\Lambda'' \subseteq \Lambda} \pi^\Lambda f_{\Lambda''} = f^{\Lambda \cap \Lambda'} - \sum_{\Lambda'' \subseteq \Lambda \cap \Lambda'} f_{\Lambda''} \equiv 0,
\]

where the last equality is from the induction hypothesis. In the calculation, we have used the fact that \( \pi^\Lambda f_{\Lambda} = f^{\Lambda \cap \Lambda'} \) and

\[
\pi^\Lambda f_{\Lambda''} = \pi^{\Lambda'' \cap \Lambda'} f_{\Lambda''} = \begin{cases} f_{\Lambda''} & \Lambda'' \subset \Lambda \cap \Lambda' \\
0 & \Lambda'' \not\subset \Lambda \cap \Lambda' \end{cases}
\]

for any \( \Lambda', \Lambda'' \subseteq \Lambda \), which follows from Lemma 2.2. This proves that we have \( f_{\Lambda} \in C_\Lambda(S^X) \). Hence by induction, there exists unique \( f_{\Lambda} \in C_\Lambda(S^X) \) for any \( \Lambda \in \mathcal{F} \) satisfying (5) as desired. \( \square \)
Remark. By abuse of notation, we will often write the equality (5) as

\[ f = \sum_{\Lambda'' \in \mathcal{F}} f_{\Lambda''} \]

for \( f = (f^\Lambda)_{\Lambda \in \mathcal{F}} \in \text{Col}(S^X) \). If \( f \in \text{C}(S^\Lambda) \) for some \( \Lambda \in \mathcal{F} \), then we have \( f = \sum_{\Lambda'' \subseteq \Lambda} f_{\Lambda''} \).

We define the notion of uniformly local functions for col-local functions as follows. For any \( x, x' \in X \), let \( d_X(x, x') \) be the length of the shortest path from \( x \) to \( x' \) in \( X \), and for any \( \Lambda \subseteq X \), we define the diameter of \( \Lambda \) by \( \text{diam}(\Lambda) := \sup_{x, x' \in \Lambda} d_X(x, x') \).

**Definition 2.4.** We say that a col-local function \( f \) is uniformly local, if there exists \( R > 0 \) such that \( f_{\Lambda} = 0 \) for any \( \Lambda \in \mathcal{F} \) with \( \text{diam}(\Lambda) > R \) in the expansion

\[ f = \sum_{\Lambda \in \mathcal{F}} f_{\Lambda} \]

defined in Proposition 2.3. We denote by \( \text{unif}(S^X) \) the set of all uniformly local functions.

We define the subspaces \( \text{Col}^0(S^X) \) and \( \text{unif}^0(S^X) \) of \( \text{Col}(S^X) \) and \( \text{unif}(S^X) \) by

\[ \text{Col}^0(S^X) := \{ f \in \text{Col}(S^X) \mid f_0 \equiv 0 \}, \quad \text{unif}^0(S^X) := \{ f \in \text{unif}(S^X) \mid f_0 \equiv 0 \}. \]

Furthermore, for any \( \Lambda \in \mathcal{F} \), we let

\[ \text{Col}^0(\Lambda) := \{ f \in \text{C}(\Lambda) \mid f_0 \equiv 0 \}. \]

Since \( S^0 \) is a set consisting of a single point, for any local function \( f \in \text{C}(S^\Lambda) \), the function \( f_0 := \pi^0 f \) is the constant functions with value \( E_{\mu}[f] \). Our condition \( f_0 \equiv 0 \) is equivalent to the condition \( E_{\mu}[f] = 0 \).

The most important example of a uniformly local function is given by the conserved quantities.

**Definition 2.5.** We say that a map \( \xi : S \to \mathbb{R} \) is a \( \nu \)-regularized conserved quantity for the interaction \( \phi \), if \( E_{\nu}[\xi] = 0 \) and

\[ \xi(s'_1) + \xi(s'_2) = \xi(s_1) + \xi(s_2) \]

for any \((s_1, s_2) \in S \times S\), where \((s'_1, s'_2) = \phi(s_1, s_2)\). We denote by \( \text{Cons}_\nu^\phi(S) \) the \( \mathbb{R} \)-linear space of \( \nu \)-regularized conserved quantities on \( S \).

The definition of conserved quantities in Definition 2.5 slightly differs from that of [11] since we normalize with the condition \( E_{\nu}[\xi] = 0 \) instead of the condition that \( \xi \) is zero at the base state. For any \( x \in X \), the natural projection \( \text{pr}_{\{x\}} : S^X \to \text{S}^{\{x\}} = S \) induces an inclusion \( \text{C}(S) \hookrightarrow \text{C}(S^X) \). For any \( \xi \in \text{Cons}_\nu^\phi(S) \), we denote by \( \xi_x \) the image of \( \xi \) with respect to this inclusion, and we let \( \xi^\Lambda := \sum_{x \in \Lambda} \xi_x \). The system of local functions \( \{\xi^\Lambda\} \) form a col-local function on \( S^X \) which we denote by \( \xi_X \). Then the expansion of \( \xi_X \) in Proposition 2.3 is simply

\[ \xi_X = \sum_{x \in X} \xi_x \]
which shows that $\xi_X$ is uniformly local for the constant $R = 1$. Due to our normalization $E_r[\xi] = 0$, we see that $\xi_X$ is a function in $C^0_{\text{unif}}(S)$. By associating to $\xi \in \text{Cons}_{\text{unif}}(S)$ the uniformly local function $\xi_X$ in $C^0_{\text{unif}}(S)$, we have an $\mathbb{R}$-linear homomorphism

$$\text{Cons}_{\text{unif}}(S) \to C^0_{\text{unif}}(S),$$

which is injective since $\xi_X$ is zero if and only if $\xi_x$ is zero for any $x \in X$, the last condition equivalent to the condition that $\xi$ is constantly zero.

3. Local and Co-local Forms

In this section, we define the space of local and co-local forms. We let $\mu$ be a probability measure on $S^X$ which is supported on $S^\Lambda$ for any finite $\Lambda \subset X$. For any $\Lambda \subset X$, we let

$$\Phi_\Lambda := \{(s, s^e) \mid s \in S^\Lambda, e \in E_\Lambda, s^e \neq s\}$$

for $E_\Lambda := E \cap (\Lambda \times \Lambda)$, and we define the cotangent bundle of $S^X$ by $T^*S^X := \text{Map}(\Phi_\Lambda, \mathbb{R})$. We call any element in the cotangent bundle a form. Suppose we are given a form $\omega \in T^*S^\Lambda$. For any $e \in E_\Lambda$, we define the function $\omega_e \in C(S^\Lambda)$ by

$$\omega_e(s) := \omega((s, s^e))$$

for any $s \in S^\Lambda$ such that $s^e \neq s$, and $\omega_e(s) = 0$ if $s^e = s$. This gives a natural embedding

$$T^*S^\Lambda \hookrightarrow \prod_{e \in E_\Lambda} C(S^\Lambda), \quad \omega \mapsto (\omega_e). \quad (9)$$

The image of $T^*S^\Lambda$ in $\prod_{e \in E_\Lambda} C(S^\Lambda)$ corresponds to $(\omega_e)$ satisfying $\omega_e(s) = 0$ if $s^e = s$, and $\omega_e(s) = \omega_e(s^e)$ if $s^e = s^e$. Since the system $\{C(S^\Lambda)\}_{\Lambda \in \mathcal{J}}$ form a projective system of $\mathbb{R}$-linear spaces for the maps $\pi^\Lambda: C(S^{\Lambda'}) \to C(S^\Lambda)$ for $\Lambda \subset \Lambda'$ in $\mathcal{J}$, the product $\{\prod_{e \in E_\Lambda} C(S^\Lambda)\}_{\Lambda \in \mathcal{J}}$ also form a projective system.

**Lemma 3.1.** Consider $\Lambda, \Lambda' \in \mathcal{J}$ such that $\Lambda \subset \Lambda'$. The projection $\pi^\Lambda$ on the product induces an $\mathbb{R}$-linear homomorphism $\pi^\Lambda: T^*S^{\Lambda'} \to T^*S^\Lambda$.

**Proof.** By definition of the embedding $(9)$, we first prove that for any $e \in E_\Lambda$, if $\omega_e(s') = 0$ for any $s' \in S^{\Lambda'}$ such that $s'' = s'$, then $\pi^\Lambda \omega_e(s) = 0$ for any $s \in S^\Lambda$ such that $s'' = s$. By calculation of the conditional expectation $(2)$, we have

$$\pi^\Lambda \omega_e(s) = \frac{1}{\mu(s)} \sum_{s' \in A_s} \omega_e(s').$$

Since $s = s^e$, we have $s' = s''$ for any $s' \in A_s$. This proves that $\pi^\Lambda \omega_e(s) = 0$ as desired. Furthermore, if $s' = s''$ for any $e, e' \in E_\Lambda \subset E_{\Lambda'}$, then $s'' = s''$ for any $s' \in A_s$, hence $\omega_e(s') = \omega_{e'}(s')$. This proves that $\pi^\Lambda \omega_e(s) = \pi^\Lambda \omega_{e'}(s)$ as desired. \hfill $\square$

**Definition 3.2.** We define the space of co-local tangent bundle $T^*S^X_{\text{col}}$ by

$$T^*S^X_{\text{col}} := \varprojlim_T T^*S^\Lambda,$$

where the limit is the projective limit with respect to $\pi^\Lambda$. 
We next define the space $C^1$ on $S^\Lambda$. Let
\[ C^1(S^\Lambda) := \text{Map}^{alt}(\Phi_\Lambda, \mathbb{R}), \]
where
\[ \text{Map}^{alt}(\Phi_\Lambda, \mathbb{R}) := \{ \omega \in T^*S^\Lambda \mid \omega(\bar{\varphi}) = -\omega(\varphi) \}. \]
Here $\bar{\varphi} := (t(\varphi), o(\varphi))$ for any $\varphi = (o(\varphi), t(\varphi)) \in \Phi_\Lambda$. We define the differential homomorphism $\partial_\Lambda : C^0(S^\Lambda) \to C^1(S^\Lambda)$ by
\[ \partial_\Lambda f(\varphi) := f(t(\varphi)) - f(o(\varphi)) \]
for any $f \in C(S^\Lambda)$ and $\varphi = (o(\varphi), t(\varphi)) \in \Phi_\Lambda$. For any $\omega \in T^*S^\Lambda$, the image of (9) in $\prod_{e \in E} C(S^\Lambda)$ consists of $(\omega_e)$ satisfying $\omega_e(s) = 0$ if $s^e = s$ and $\omega_e(s) = -\omega_e(s^e)$ if $s^e \neq s$. The differential may be expressed as $\partial_\Lambda f = (\nabla_e f)_{e \in E_\Lambda}$, where
\[ \nabla_e f(s) := f(s^e) - f(s) \]
for any $e_\Lambda \in E$ and $s \in S^\Lambda$.

Throughout the rest of this section, we assume the following.

**Definition 3.3.** In what follows, we say that a measure $\mu$ is ordinary, if for any $\Lambda, \Lambda' \in \mathcal{I}$ such that $\Lambda \subset \Lambda'$ and configurations $s' \in S^{\Lambda'}$, if we let $s := pr_\Lambda(s') \in C(S^\Lambda)$ for the projection $pr_\Lambda : C(S^{\Lambda'}) \to C(S^\Lambda)$, then we have
\begin{equation}
\mu(s^e)\mu(s') = \mu(s)\mu(s^e)
\end{equation}
for any $e \in E_\Lambda$.

**Lemma 3.4.** If $\mu$ is the product measure $\mu = \prod_{x \in X} \nu_x$ for some family of probability measures $\{\nu_x\}_{x \in X}$ on $S$, then $\mu$ is ordinary.

**Proof.** Let the notations be as in Definition 3.3. If we let $s = (s_x) \in S^\Lambda$ and $s' = (s'_x) \in \Lambda'$, then we have $s_x = s'_x$ for $x \in \Lambda$, hence
\[ \mu(s) = \prod_{x \in \Lambda} \nu_x(s_x), \quad \mu(s') = \prod_{x \in \Lambda} \nu_x(s_x) \times \prod_{x \in \Lambda' \setminus \Lambda} \nu_x(s'_x). \]
Our assertion follows from the fact that since $e \in E_\Lambda$, we have $s^e_x = s'_x$ for any $x \in \Lambda' \setminus \Lambda$. $\Box$

If the probability measure $\mu$ is ordinary, then $\pi^{\Lambda}$ preserves the local forms.

**Lemma 3.5.** If the probability measure $\mu$ is ordinary, then the projections $\pi^{\Lambda}$ preserve the space of forms $C^1(S^\Lambda)$ in $T^*S^\Lambda$. In other words, we have a homomorphism
\[ \pi^{\Lambda} : C^1(S^{\Lambda'}) \to C^1(S^\Lambda) \]
for any $\Lambda, \Lambda' \in \mathcal{I}$ such that $\Lambda \subset \Lambda'$.

**Proof.** By Lemma 3.1 and the definition of $C^1(S^\Lambda)$ in Definition 3.2, it is sufficient to prove that if $(\omega_e)$ satisfies $\omega_e(s') = -\omega_e(s^e)$ for any $s' \in S^{\Lambda'}$ such that $s^e \neq s'$, then $\pi^{\Lambda}\omega_e(s) = -\pi^{\Lambda}\omega_e(s^e)$.
for any \( e \in E_\Lambda \). Let \( s \in S^\Lambda \) and \( A_s := \{ s' \in S^{\Lambda'} \mid \text{pr}_\Lambda(s') = s \} \), where \( \text{pr}_\Lambda : S^{\Lambda'} \to S^\Lambda \) is the projection. By calculation \( \square \) of the conditional expectation, we have

\[
\pi^\Lambda \omega_e(s) = \frac{1}{\mu(s)} \sum_{s' \in A_s} \omega_e(s') \mu(s'), \quad \pi^\Lambda \omega_e(s') = \frac{1}{\mu(s')} \sum_{s'' \in A_s} \omega_e(s'') \mu(s'').
\]

Our assertion follows from (10). □

By Lemma 3.5 we define the space of co-local forms \( C^1_{\text{col}}(S^X) \) as follows.

**Definition 3.6.** If the probability measure \( \mu \) is ordinary, then we define the space of co-local forms by

\[ C^1_{\text{col}}(S^X) := \lim_{\rightarrow \mathcal{F}} C^1(S^\Lambda), \]

where the limit is the projective limit with respect to \( \pi^\Lambda \).

**Proposition 3.7.** If the probability measure \( \mu \) is ordinary, then the projection \( \pi^\Lambda \) is compatible with the differentials \( \partial_\Lambda \) on \( C^0(S^\Lambda) \). Hence the differential on each \( C^0(S^\Lambda) \) induces the differential

\[
\partial : C^0_{\text{col}}(S^X) \to C^1_{\text{col}}(S^X)
\]

on the space of co-local functions.

**Proof.** It is sufficient to prove that for any \( \Lambda, \Lambda' \in \mathcal{F} \) such that \( \Lambda \subset \Lambda' \), we have \( \pi^\Lambda(\nabla_e f) = \nabla_e(\pi^\Lambda f)(s) \) for any \( e \in E_\Lambda \) and \( f \in C^0(S^{\Lambda'}) \). By definition of the differential and calculation of the conditional expectation \( \square \), we have

\[
\pi^\Lambda(\nabla_e f)(s) = \frac{1}{\mu(s)} \sum_{s' \in A_s} \nabla_e f(s') = \frac{1}{\mu(s)} \sum_{s' \in A_s} f(s') \mu(s') - \frac{1}{\mu(s)} \sum_{s' \in A_s} f(s') \mu(s'),
\]

where \( A_s = \{ s' \in S^{\Lambda'} \mid \text{pr}_\Lambda(s') = s \} \) as before. Similarly, we have

\[
\nabla_e(\pi^\Lambda f)(s) = (\pi^\Lambda f)(s') - (\pi^\Lambda f)(s) = \frac{1}{\mu(s')} \sum_{s'' \in A_s} f(s'') \mu(s'') - \frac{1}{\mu(s')} \sum_{s'' \in A_s} f(s'') \mu(s').
\]

Our assertion now follows from the equality (10) satisfied by ordinary probability measures. □

**Definition 3.8.** If the probability measure \( \mu \) is ordinary, then we let

\[ H^0_{\text{col}}(S^X) := \text{Ker}(\partial : C^0_{\text{col}}(S^X) \to C^1_{\text{col}}(S^X)) \]

for the differential \( \partial : C^0_{\text{col}}(S^X) \to C^1_{\text{col}}(S^X) \) of Proposition 3.7.

Next we define the notion of closed local and co-local forms. For any \( \Lambda \subset X \), we define a path in \( S^\Lambda \) to be a sequence of transitions \( \tilde{\gamma} = (\varphi^1, \ldots, \varphi^N) \) in \( \Phi_\Lambda \) such that \( t(\varphi^i) = o(\varphi^{i+1}) \) for \( 0 < i < N \). We say that the path is closed, if \( t(\varphi^N) = o(\varphi^1) \). For a form \( \omega \in C^1(S^\Lambda) \), we define the integration with respect to a path \( \tilde{\gamma} = (\varphi^1, \ldots, \varphi^N) \) by

\[
\int_{\tilde{\gamma}} \omega := \sum_{i=1}^N \omega(\varphi^i).
\]

As in [1] Definition 2.14] we define the closed forms as follows.
**Definition 3.9.** We say that a form $\omega \in C^1(S^\Lambda)$ is closed, if
\[
\int_{\vec{\gamma}} \omega = 0
\]
for any closed path $\vec{\gamma}$ in $S^\Lambda$.

We denote by $Z^1(S^\Lambda)$ the $\mathbb{R}$-linear space of closed forms in $C^1(S^\Lambda)$. The path $\vec{\gamma}$ may be written as $\vec{\gamma} = (s_0, \ldots, s_N)$ for some $s_0, \ldots, s_N \in S^\Lambda$ and $e_1, \ldots, e_N \in E_\Lambda$ such that with $s_0 = o(\vec{\gamma})$, $s_i = s_{i-1}^e$ for $1 \leq i \leq N$, and $s_N = t(\vec{\gamma})$. The path is closed if $s_0 = s_N$. If $\omega = (\omega_e)$ is a closed form in $C^1(S^\Lambda)$, then we have
\[
\sum_{i=1}^N \omega_e_i(s_i) = 0.
\]

**Lemma 3.10.** If the probability measure $\mu$ is ordinary, then the projection $\pi^\Lambda$ induces an $\mathbb{R}$-linear homomorphism
\[
\pi^\Lambda : Z^1(S^{\Lambda'}) \rightarrow Z^1(S^\Lambda)
\]
for any $\Lambda, \Lambda' \in \mathcal{F}$ such that $\Lambda \subset \Lambda'$.

**Proof.** Let $\omega = (\omega_e)$ be a closed form in $Z^1(S^{\Lambda'})$, and consider the closed form $\pi^\Lambda \omega \in Z^1(S^\Lambda)$. Note that we have $\pi^\Lambda \omega = (\pi^\Lambda \omega_e)$. Consider a closed path $\vec{\gamma} = (s_0, \ldots, s_N)$ in $S^\Lambda$ such that $s_i = s_{i-1}^{e_i} = \cdots = s_0^{e_1^{e_2^{\cdots^{e_N}}}}$ for some $e_1, \ldots, e_N \in E_\Lambda$. Note that for any $s'_0 \in A_{s_0} \subset S^{\Lambda'}$, if we let $s'_i = s_{i}^{e_1^{e_2^{\cdots^{e_i}}}}$ for $1 \leq i \leq N$, then $\vec{\gamma}' := (s'_0, \ldots, s'_N)$ is a closed path in $S^{\Lambda'}$. Then we have
\[
\int_{\vec{\gamma}} \pi^\Lambda \omega = \sum_{i=1}^N \pi^\Lambda \omega_e_i(s_i) = \sum_{i=1}^N \sum_{s'_0 \in A_{s_0}} \omega_e_i(s'_i) \frac{\mu(s'_0)}{\mu(s_0)} = \sum_{s'_0 \in A_{s_0}} \mu(s'_0) \left( \sum_{i=1}^N \omega_e_i(s'_i) \right) = 0,
\]
where the third equality follows from (10), and the last equality follows from the fact that $\omega$ is closed in $S^{\Lambda'}$. This gives our assertion. \qed

We now give the definition of closed co-local forms.

**Definition 3.11.** Assume that the probability measure $\mu$ is ordinary. We define the space of closed co-local forms $Z^1_{col}(S^X)$ by
\[
Z^1_{col}(S^X) := \lim_{\Lambda \in \mathcal{F}} Z^1_{col}(S^\Lambda),
\]
where the limit is the projective limit with respect to $\pi^\Lambda$.

We next consider the integration of closed forms. For any $\Lambda \in \mathcal{F}$, by [1 Lemma 2.14, Lemma 2.15], we have an exact sequence
\[
0 \longrightarrow \text{Ker} \, \partial_\Lambda \longrightarrow C^0(S^\Lambda) \longrightarrow \frac{\partial_\Lambda}{\partial_\Lambda} Z^1(S^\Lambda) \longrightarrow 0.
\]

Assume that the probability measure $\mu$ is ordinary. By Proposition [3.7], since the differential $\partial_\Lambda$ is compatible with the projection $\pi^\Lambda$, we see that $\{ \text{Ker} \, \partial_\Lambda \}_{\Lambda \in \mathcal{F}}$ also form a projective system for the projection $\pi^\Lambda$. The Mittag-Leffler condition for projective systems is given as follows (see for example [3] (13.1.2)).
Definition 3.12. A projective system $\{M_\Lambda\}_{\Lambda \in \mathcal{F}}$ satisfies the Mittag-Leffler condition, if for any $\Lambda \in \mathcal{F}$, if we let $N_\Lambda := \bigcap_{\Lambda' \in \mathcal{F}, \Lambda' \supset \Lambda} \text{Im}(\pi^\Lambda : M_{\Lambda'} \to M_\Lambda)$, then there exists $\Lambda'' \in \mathcal{F}$ satisfying $\Lambda \subset \Lambda''$ such that $N_\Lambda = \text{Im}(\pi^\Lambda : M_{\Lambda''} \to M_\Lambda)$. In other words, the image of $\pi^\Lambda$ stabilizes for $\Lambda''$ sufficiently large.

By [3 Proposition (13.2.2)], if the projective system $\{\ker \partial_\Lambda\}_{\Lambda \in \mathcal{F}}$ satisfies the Mittag-Leffler condition (ML), then the projective limit

$$0 \to \lim_{\leftarrow} \ker \partial_\Lambda \to \lim_{\leftarrow} C^0(S^\Lambda) \to \lim_{\leftarrow} Z^1(S^\Lambda) \to 0$$

of (11) is exact. We will use this fact to prove the following.

Proposition 3.13. If the probability measure $\mu$ on $S^X$ is ordinary, then the projective system $\{\ker \partial_\Lambda\}_{\Lambda \in \mathcal{F}}$ satisfies the Mittag-Leffler condition. This implies that the sequence

$$(12)\quad 0 \to H^0_{\text{col}}(S^X) \to C^0_{\text{col}}(S^X) \to \lim_{\leftarrow} \ker \partial_\Lambda$$

is exact, where $H^0_{\text{col}}(S^X) = \lim_{\leftarrow} \ker \partial_\Lambda$ as in Definition 3.8.

Proof. We first note that by [1 Remark 2.28], any function $f \in C(S^\Lambda)$ satisfies $\partial_\Lambda f = 0$ if and only if $f$ is constant on each of the connected components of the graph $(S^\Lambda, \Phi_\Lambda)$. Since $\Lambda \in \mathcal{F}$ is finite, the configuration space $S^\Lambda$ has only a finite number of connected components. This shows that $\ker \partial_\Lambda$ is finite dimensional. This ensures that $\{\ker \partial_\Lambda\}_{\Lambda \in \mathcal{F}}$ satisfies the Mittag-Leffler condition, since any descending sequence of linear subspaces of a finite dimensional linear space is stable. Our assertion now follows from [3 Proposition (13.2.2)]. \hfill $\Box$

4. Decomposition of Varadhan Type

In this section, we consider a group $G$ and an action of $G$ on the locale $X$, and prove the decomposition theorem of Varadhan type for closed co-local forms. Before going into the details, we first fix notations concerning action of $G$. An action of $G$ on $X$ gives a bijection $\sigma : X \to X$ for any $\sigma \in G$. We define the action of $G$ on $S^X$ given by mapping $s = (s_x)_{x \in X} \in \prod_{x \in X} S$ to $\sigma(s) := (s_{\sigma^{-1}(x)})_{x \in X}$ for any $\sigma \in G$. Then $G$ induces an action on $C(S^X)$ given by any $\sigma \in G$ by

$$\sigma(f)(s) = f(\sigma^{-1}(s))$$

for any $s \in S^X$.

For any subset $\Lambda \subset X$, the element $\sigma \in G$ induces a bijection $\sigma : \Lambda \cong \sigma(\Lambda)$. Hence on $S^\Lambda$, this induces a bijection

$$(13)\quad \sigma : S^\Lambda \cong S^{\sigma(\Lambda)}.$$

Note that for any $s \in S^\Lambda$, we have $\sigma(s) \in S^{\sigma(\Lambda)}$. In terms of components, if we let $s = (s_x)_{x \in \Lambda} \in S^\Lambda$, then we have $\sigma(s) = (s_{\sigma^{-1}(x)})_{x \in \Lambda} = (s_x)_{x \in \sigma(\Lambda)} \in S^{\sigma(\Lambda)}$. Hence (13) induces a bijection

$$(14)\quad \sigma : C(S^\Lambda) \cong C(S^{\sigma(\Lambda)}),$$

which maps any function \( f \in C(S^\Lambda) \) to the function \( \sigma(f) \in C(S^{\sigma(\Lambda)}) \). For any \( \Lambda \in \mathcal{F} \), we have \( \sigma(\Lambda) \in \mathcal{F} \) for any \( \sigma \in G \). Clearly, the action of \( G \) is compatible with the natural inclusion \( C(S^\Lambda) \hookrightarrow C(S^X) \), hence we have an action of \( G \) on \( C_{\text{loc}}(S^X) \).

The action of the group \( G \) on \( X \) defines a map of graphs \( \sigma: (S^\Lambda, \Phi_\Lambda) \to (S^{\sigma(\Lambda)}, \Phi_{\sigma(\Lambda)}) \), which induces a natural actions on \( C(\Phi_\Lambda) \) by \( \sigma(\omega)(\varphi) = \omega(\sigma^{-1}(\varphi)) \). We have the following.

**Lemma 4.1.** For any \( \Lambda \in \mathcal{F} \), the action of \( G \) is compatible with the differential

\[
\partial_\Lambda: C^0(S^\Lambda) \to C^1(S^\Lambda).
\]

**Proof.** For \( f \in C(S^\Lambda) \), we have

\[
\sigma(\partial f)(\varphi) = \partial f(\sigma^{-1}(\varphi)) = f(\sigma^{-1}(t(\varphi))) - f(\sigma^{-1}(o(\varphi))) = \sigma(f)(t(\varphi)) - \sigma(f)(o(\varphi)) = (\partial \sigma(f))(\varphi).
\]

This shows that \( \partial_\Lambda \) is compatible with the action of the group \( G \) as desired. \( \square \)

Note that if we let \( \mathcal{F}_X \) be the Borel \( \sigma \)-algebra for \( S^X \), then we have \( \sigma(A) \in \mathcal{F}_X \) for any \( A \in \mathcal{F}_X \) and \( \sigma \in G \). We define a \( G \)-invariant probability measure on \( S^X \) as follows.

**Definition 4.2.** Suppose \( \mu \) is a probability measure on \( S^X \). We say that \( \mu \) is invariant with respect to the action of \( G \), if we have

\[ \mu(\sigma(A)) = \mu(A) \]

for any \( A \in \mathcal{F}_X \).

If \( \mu = \prod_{x \in X} \nu_x \) for some family of probability measures \( \{\nu_x\} \) on \( S \), and if \( \nu_{\sigma(x)} = \nu_x \) for any \( \sigma \in G \), then the probability measure \( \mu \) is invariant with respect to the action of \( G \). In particular, the product measure \( \mu = \nu^\otimes X \) for a measure \( \nu \) on \( S \) is invariant with respect to the action of \( G \).

For the rest of this section, we assume that \( G \) is a probability measure which is invariant with respect to the action of \( G \). For any \( f \in C(S^X) \) and \( \Lambda \in \mathcal{F} \), if we let \( \pi^\Lambda f \in C(S^\Lambda) \) be the conditional expectation, then we have \( \sigma(\pi^\Lambda f) \in C(S^{\sigma(\Lambda)}) \) for any \( \sigma \in G \). The integration over any \( A \subset S^{\sigma(\Lambda)} \) is given as

\[
\int_A \sigma(\pi^\Lambda f) d\mu = \int_{\text{pr}^{-1}_{\sigma(\Lambda)}(A)} \sigma(\pi^\Lambda f)(s) d\mu(s) = \int_{\text{pr}^{-1}_{\sigma(\Lambda)}(A)} \pi^\Lambda f(\sigma^{-1}(s)) d\mu(s) = \left(\int_{\text{pr}^{-1}_{\sigma(\Lambda)}(A)} \pi^\Lambda f(s) d\mu(s)\right) = \left(\int_{\text{pr}^{-1}_{\sigma(\Lambda)}(A)} \sigma(f)(s) d\mu(s)\right).
\]

where we have used the fact that \( d\mu(s) = d\mu(\sigma(s)) \) for the equalities \((*)\). Then the uniqueness of the conditional expectation gives the equality \( \sigma(\pi^\Lambda f) = \pi^{\sigma(\Lambda)}(\sigma(f)) \). This shows that the projections \( \pi^\Lambda \) are compatible with the action of \( G \), hence we have an action of \( G \) on the co-local functions \( C_{\text{col}}(S^X) = \lim_{\mathcal{F}} C(S^\Lambda) \). The action \([14]\) gives a mapping

\[ \sigma: C_\Lambda(S^X) \to C_{\sigma(\Lambda)}(S^X) \]
on $C_\Lambda(S^X)$. By Proposition 2.3, we have a unique expansion

$$f = \sum_{\Lambda \in \mathcal{J}} f_\Lambda$$

for any $f \in C_{\text{col}}(S^X)$, where $f_\Lambda \in C_\Lambda(S^X)$ for any $\Lambda \in \mathcal{J}$. Then we have

$$\sigma(f) = \sum_{\Lambda \in \mathcal{J}} \sigma(f_\Lambda).$$

Since $f_\Lambda \in C_\Lambda(S^X)$, we have $\sigma(f_\Lambda) \in C_{\sigma(\Lambda)}(S^X)$, hence the uniqueness of expansion gives

$$\sigma(f_\Lambda) = \sigma(f)_{\sigma(\Lambda)}$$

for any $\Lambda \in \mathcal{J}$. This proves the following assertion.

**Lemma 4.3.** The action of the group $G$ on the locale $X$ gives a natural action of $G$ on the space of uniformly local functions $C^0_{\text{unif}}(S^X)$.

Suppose now that $\mu = \nu^\otimes X$ for some probability measure $\nu$ on $S$ which is supported on $S$. Then $\mu$ is invariant by the action of $G$. In addition, by Lemma 3.4, the measure $\mu$ is ordinary. By Proposition 3.13 we have an exact sequence

$$\cdots \rightarrow C^0_{\text{col}}(S^X)^G \rightarrow Z^1_{\text{col}}(S^X)^G \overset{\partial}{\rightarrow} H^1_{\text{col}}(G, C^0_{\text{col}}(S^X)) \rightarrow H^1_{\text{col}}(G, C^0_{\text{col}}(S^X)) \rightarrow \cdots,$$

where $C^0_{\text{col}}(S^X)^G$ and $Z^1_{\text{col}}(S^X)^G$ denote the $G$-invariant subgroups of $C^0_{\text{col}}(S^X)$ and $Z^1_{\text{col}}(S^X)$, and $H^1_{\text{col}}(G, C^0_{\text{col}}(S^X))$ and $H^1_{\text{col}}(G, C^0_{\text{col}}(S^X))$ are group cohomology of $G$ with coefficients in $C^0_{\text{col}}(S^X)$ and $C^0_{\text{col}}(S^X)$. This gives an injective homomorphism

$$(15) \quad Z^1_{\text{col}}(S^X)^G / \partial(C^0_{\text{col}}(S^X)^G) \hookrightarrow H^1(G, C^0_{\text{col}}(S^X)).$$

The boundary morphism $\partial$ is given explicitly by mapping any $\omega \in Z^1_{\text{col}}(S^X)^G$ to the cocycle $\rho$ given by $\rho(\sigma) = (1 - \sigma)\theta$ for any $\sigma \in G$, where $\theta \in C^0_{\text{col}}(S^X)$ is a function satisfying $\partial \theta = \omega$.

Next, suppose the group $G$ is torsion free and the action of $G$ on $X$ is free. Then the action of $G$ on the set $\mathcal{J} \setminus \{\emptyset\}$ is free. We denote by $\mathcal{J}_0$ a representative of the equivalence classes of $\mathcal{J} \setminus \{\emptyset\}$ with respect to the action of $G$. This implies that for any nonempty $\Lambda \subset X$, there exists an unique $\tau \in G$ such that $\tau^{-1}(\Lambda) \in \mathcal{J}_0$. For any $f \in C^0_{\text{col}}(S^X)$, the canonical decomposition

$$f = \sum_{\Lambda \in \mathcal{J}} f_\Lambda, \quad f_\Lambda \in C_\Lambda(S^X)$$

with $f_\emptyset = 0$ may be written as

$$f = \sum_{\tau \in G} \sum_{\Lambda_0 \in \mathcal{J}_0} f_{\tau(\Lambda_0)}.$$
For any \( \tau \in G \), if we let
\[
  f_\tau := \sum_{\lambda_0 \in \mathcal{I}_0} f_{\tau(\lambda_0)},
\]
then we have
\[
f = \sum_{\tau \in G} f_\tau.
\]

**Theorem 4.4.** Suppose \( \mu = \nu^X \), and that the group \( G \) is torsion free and the action of \( G \) on \( X \) is free. Then the boundary morphism \( \partial \) induces an isomorphism
\[
Z_\text{col}^1(S^X)^G / \partial(C^0_{\text{col}}(S^X)^G) \cong H^1(G, H^0_{\text{col}}(S^X)).
\]
In particular, the choice of \( \mathcal{I}_0 \) gives a splitting of the boundary morphism, hence a decomposition
\[
Z_\text{col}^1(S^X)^G \cong \partial(C^0_{\text{col}}(S^X)^G) \oplus H^1(G, H^0_{\text{col}}(S^X)).
\]

**Proof.** It is sufficient to construct a section of the boundary morphism
\[
\delta: Z_\text{col}^1(S^X)^G \to H^1(G, H^0_{\text{col}}(S^X)).
\]
Let \( \rho \in Z^1(G, H^0_{\text{col}}(S^X)) \) be a group cocycle representing a class in \( H^1(G, H^0_{\text{col}}(S^X)) \). Then \( \rho \) is a map from \( G \) to \( H^0_{\text{col}}(S^X) = \text{Ker} \partial \subset C^0_{\text{col}}(S^X) \) satisfying \( \rho(\sigma \tau) = \sigma \rho(\tau) + \rho(\sigma) \) for any \( \sigma, \tau \in G \). We let
\[
\theta_\rho := \sum_{\tau \in G} \rho(\tau)_{\tau} \in C^0_{\text{col}}(S^X),
\]
where \( \rho(\tau)_{\tau} \) is the function \( f_\tau \) in (17) for \( f = \rho(\tau) \). Then we have
\[
\sigma(\theta_\rho) = \sum_{\tau \in G} \sigma(\rho(\tau)_{\tau}) = \sum_{\tau \in G} (\sigma(\rho)_{\tau})(\sigma \tau) = \sum_{\tau \in G} (\rho(\sigma \tau)_{\sigma \tau} - \rho(\sigma))_{\sigma \tau} = \sum_{\tau \in G} (\rho(\tau) - \rho(\sigma))_{\tau} = \theta_\rho - \rho(\sigma).
\]
This shows that \( (1 - \sigma)\theta_\rho = \rho(\sigma) \). Since \( \rho(\sigma) \in H^0_{\text{col}}(S^X) = \text{Ker} \partial \), the compatibility of \( \partial \) with the action of \( G \) gives
\[
(1 - \sigma)\partial \theta_\rho = \partial((1 - \sigma)\theta_\rho) = \partial \rho(\sigma) = 0
\]
for any \( \sigma \in G \), which shows that \( \omega_\rho := \partial \theta_\rho \in Z_\text{col}^1(S^X)^G \). By the definition of the boundary morphism, we see that \( \delta(\omega_\rho) = \rho as an element in \( H^1(G, H^0_{\text{col}}(S^X)) \). This shows that the map \( \rho \mapsto \omega_\rho \) gives a section of (20) as desired. This proves that (18) is in fact an isomorphism and gives the decomposition (19) for our choice of \( \mathcal{I}_0 \).

We next consider the property that an interaction is faithfully quantified, which was originally proposed in [1, Definition 2.22].

**Definition 4.5.** We say that the interaction \((S, \phi)\) is **faithfully quantified**, if for any locale \((X, E)\) whose set of vertices is **finite**, the associated configuration space with transition structure \(S^X\) satisfies the following property. For any \( s, s' \in S^X \), if \( \xi_X(s) = \xi_X(s') \) for any conserved quantity \( \xi \in \text{Consv}_\nu^X(S) \), then there exists a path \( \tilde{\gamma} \) from \( s \) to \( s' \) in \( S^X \). Here, \( \xi_X(s) := \sum_{x \in X} \xi(s_x) \) for any \( s = (s_x) \in S^X \), which is a finite sum since we have assumed that \( X \) is finite.
Since we have assumed that $S$ is finite, the dimension $c_\phi := \dim_S \text{Consv}_\phi^G(S)$ is finite. Hence our definition of faithfully quantified interactions coincides with that in [1, Definition 2.22]. The following result follows from [1, Lemma 2.28] and the definition of faithfully quantified interactions. We will include the proof for the sake of completeness.

**Lemma 4.6.** Let $(X, E)$ be a finite locale, and suppose $(S, \phi)$ is an interaction which is faithfully quantified. Suppose that $f \in C(S^X)$ satisfies $\partial_X f = 0$. Then we have $f(s) = f(s')$ for any $s, s' \in S^X$ satisfying $\xi_X(s) = \xi_X(s')$ for any conserved quantity $\xi \in \text{Consv}_\phi^G(S)$.

**Proof.** Let $s, s' \in S^X$, and suppose that $\xi_X(s) = \xi_X(s')$ for any conserved quantity $\xi \in \text{Consv}_\phi^G(S)$. Since the pair $(S, \phi)$ is faithfully quantified, there exists a path $\gamma$ from $s$ to $s'$ in $S^X$. Write $\gamma = (\varphi^1, \ldots, \varphi^N)$ with transitions $\varphi^i = (s^{i-1}, s^i) \in \Phi$ for $i = 1, \ldots, N$ such that $s^i = (s^{i-1})^{e_i}$ for some $e_i \in E$, and $s^0 = s, s^N = s'$. Since $\partial_X f = 0$, we have

$$\partial_X f(s^{i-1}) = f((s^{i-1})^{e_i}) - f(s^{i-1}) = f(s^i) - f(s^{i-1}) = 0$$

for any $i = 1, \ldots, N$. This proves that $f(s) = f(s^0) = f(s^N) = f(s')$ as desired. \hfill $\square$

Using Lemma 4.6, we may prove the following.

**Lemma 4.7.** Suppose the interaction $\phi$ is faithfully quantified. Then we have

$$H^0_{\text{col}}(S^X) \subset C^0_{\text{col}}(S^X)^G.$$

**Proof.** Let $f = (f^\Lambda) \in H^0_{\text{col}}(S^X)$. Note that for any $\sigma \in G$, we have $\sigma(f) = (\sigma(f^\Lambda))$, where $\sigma(f^\Lambda) \in C(S^{\sigma(\Lambda)})$. It is sufficient to prove that $\sigma(f^\Lambda) = f^{\sigma(\Lambda)}$. By definition of $H^0_{\text{col}}(S^X)$ given in Definition 3.8, we have $f^\Lambda \in \text{Ker} \partial_\Lambda$. Let $\sigma \in G$, and take $\Lambda' \in \mathcal{F}$ sufficient large so that $\Lambda, \sigma(\Lambda) \subset \Lambda'$. We fix an arbitrary bijection $\nu: \Lambda' \rightarrow \Lambda$ satisfying $\nu(x) = \sigma^{-1}(x)$ if $x \in \sigma(\Lambda)$. We define a map $T_{\nu, \sigma}: S^{\Lambda'} \rightarrow S^\Lambda$ by

$$(T_{\nu, \sigma}(s))_x := s_{\nu(x)}.$$

By construction, $\xi_{\Lambda'}(s) = \xi_{\Lambda'}(T_{\nu, \sigma}(s))$. Since $\partial_\Lambda f^{\Lambda'} = 0$, by Lemma 4.6 and the fact that the interaction $\phi$ is faithfully quantified, we have $f^{\Lambda'}(s) = f^{\Lambda'}(T_{\nu, \sigma}(s))$ for any $s \in S^{\Lambda'}$. This shows that

$$\sigma(f^\Lambda) = \sigma(\pi^\Lambda f^{\Lambda'}) = \sigma(\pi^\Lambda (f^{\Lambda'} \circ T_{\nu, \sigma})) = \pi^{\sigma(\Lambda)} f^{\Lambda'} = f^{\sigma(\Lambda)},$$

which gives the desired result. \hfill $\square$

Using this fact, we may deduce the following from Theorem 4.4.

**Corollary 4.8.** Assume the conditions of Theorem 4.4 and suppose in addition that the interaction $\phi$ is faithfully quantified. Then we have an isomorphism

$$Z^1_{\text{col}}(S^X)^G / \partial(C^0_{\text{col}}(S^X)^G) \cong \text{Hom}_\mathbb{Z}(G, H^0_{\text{col}}(S^X)).$$

In particular, if $G$ is finitely generated of rank $d$, then we have a decomposition

$$Z^1_{\text{col}}(S^X)^G \cong \partial(C^0_{\text{col}}(S^X)^G) \oplus \bigoplus_{j=1}^d H^0_{\text{col}}(S^X).$$
Proof. By Lemma [4.7], the action of $G$ on $H^0_{\text{col}}(S^X)$ is trivial. Hence we have
\[ H^1(G, H^0_{\text{col}}(S^X)) \cong \text{Hom}_{\mathbb{Z}}(G, H^0_{\text{col}}(S^X)), \]
which combined with Theorem [4.4] gives the first isomorphism. If we fix a generator $\tau_1, \ldots, \tau_d$ of $G$, then an element of $\text{Hom}_{\mathbb{Z}}(G, H^0_{\text{col}}(S^X))$ is determined by the images of $\tau_i$ in $H^0_{\text{col}}(S^X)$. This fact and the decomposition (19) of Theorem [4.4] gives the second isomorphism as desired. \hfill \Box

5. The $L^2$-Case: A Conjecture

The decomposition theorem for Varadhan that is necessary for proving the hydrodynamic limit is a decomposition for $L^2$-forms. In this section, we give the definition of $L^2$-forms and formulate a conjecture concerning Varadhan’s decomposition in this case. Let $\mu$ be a probability measure on $S^X$ supported on $S^\Lambda$ for any $\Lambda \in \mathcal{F}$. We let
\[ \|f\|_\mu := \langle f, f \rangle_\mu^{1/2} = E_\mu[f^2]^{1/2} \]
for any $f \in C(S^X)$.

**Definition 5.1.** We define the space of $L^2$-functions $L^2(\mu)$ to be the quotient space
\[ L^2(\mu) := \{ f \in C(S^X) \mid \|f\|_\mu < \infty \}/\{ f \in C(S^X) \mid \|f\|_\mu = 0 \}. \]

By standard facts concerning $L^2$-spaces, $L^2(\mu)$ is known to be a Hilbert space for the inner product $\langle f, g \rangle_\mu = E_\mu[f \bar{g}]$. In particular, $L^2(\mu)$ is complete for the topology given by the norm $\| \cdot \|_\mu$.

**Remark.** It is well-known that if $f \in L^2(\mu)$, then $f$ is integrable for $\mu$. This may be seen from the fact that for any $f \in C(S^X)$, the variance of $|f|$ defined by $V(|f|) := E_\mu[|f|^2 - E_\mu[|f|]^2]$ satisfies $V(|f|) \geq 0$. The linearity of the expected value shows that we have
\[ V(|f|) = E_\mu[(|f| - E_\mu[|f|])^2] = E_\mu[f^2 - 2|f|E_\mu[|f|] + E_\mu[|f|^2]] \\
= E_\mu[f^2] - 2E_\mu[|f|E_\mu[|f|]] + E_\mu[|f|^2] = E_\mu[f^2] - E_\mu[|f|^2]. \]
This shows that if $f \in L^2(\mu)$, then $E_\mu[|f|^2] \leq E_\mu[f^2] = \|f\|_\mu^2 < \infty$ as desired.

For any $f \in L^2(\mu)$, the system $(\pi^\Lambda f)_{\Lambda \in \mathcal{F}}$ defines a co-local function in $C_{\text{col}}(S^X)$, hence we have a natural homomorphism
\[ L^2(\mu) \rightarrow C_{\text{col}}(S^X), \quad f \mapsto (\pi^\Lambda f)_{\Lambda \in \mathcal{F}}. \]
Furthermore, if $f \in C(S^\Lambda)$ for $\Lambda \in \mathcal{F}$, then the function $f$ viewed as an element in $C(S^X)$ satisfies $\|f\|_\mu < \infty$, hence $f$ defines an element in $L^2(\mu)$. This gives a natural homomorphism
\[ C_{\text{loc}}(S^X) \rightarrow L^2(\mu), \]
which is injective since the composite $C_{\text{loc}}(S^X) \rightarrow L^2(\mu) \rightarrow C_{\text{col}}(S^X)$ gives the natural inclusion. Note that since $\pi^\Lambda$ is an orthogonal projection (see Lemma [2.1]), we have
\[ \|\pi^\Lambda f\|_\mu = \langle \pi^\Lambda f, \pi^\Lambda f \rangle_\mu \leq \langle f, f \rangle_\mu = \|f\|_\mu \]
for any $f \in L^2(\mu)$. The following is the Martingale Convergence Theorem for our case.
Theorem 5.2. We let
\[
C_{L^2}(S^X) := \{ (f^\Lambda) \in C_{\text{col}}(S^X) \mid \sup_{\Lambda \in \mathcal{F}} \| f^\Lambda \|_\mu < \infty \}.
\]
Consider a family of sets \( \{ \Lambda_n \}_{n \in \mathbb{N}} \) in \( \mathcal{F} \) such that \( \Lambda_n \subset \Lambda_{n+1} \) and \( X = \bigcup_{n \in \mathbb{N}} \Lambda_n \). For any \( f \in C_{L^2}(S^X) \), if we let \( f_n := f^{\Lambda_n} \) for any \( n \in \mathbb{N} \), then the sequence of functions \( (f_n)_{n \in \mathbb{N}} \) converges strongly to a function \( f_\infty \) in \( L^2(\mu) \).

Proof. For \( m > n \), since \( f_n = \pi^{\Lambda_n} f_m \), we see from (22) that \( \| f_m \|^2 \geq \| f_n \|^2 \), hence \( \| f_n \|^2 \) is monotonously increasing for \( n \geq 0 \). Let
\[
M := \sup_{n \in \mathbb{N}} \| f_n \|_\mu \leq \sup_{\Lambda \in \mathcal{F}} \| f^\Lambda \|_\mu < \infty.
\]
By Lemma 2.1, the projection \( \pi^{\Lambda_n} \) is orthogonal. This shows that we have
\[
\| f_m \|^2 = \| f_n \|^2 + \| f_m - f_n \|^2
\]
for any \( m > n \), which gives the equality \( \| f_m - f_n \|^2 = \| f_m \|^2 - \| f_n \|^2 \). This shows that
\[
\lim_{m,n \to \infty} \| f_m - f_n \|_\mu = \lim_{m \to \infty} \| f_m \|_\mu - \lim_{n \to \infty} \| f_n \|_\mu = M - M = 0,
\]
hence \((f_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \( L^2(\mu) \). Since \( L^2(\mu) \) is complete for the \( L^2 \)-norm, the sequence \((f_n)_{n \in \mathbb{N}}\) converges strongly to an element \( f_\infty \) in \( L^2(\mu) \) as desired. \( \square \)

Remark. Since \( \mathcal{F} \) is a directed set, the system of functions \((f^\Lambda) \in C_{L^2}(S^X)\) may be interpreted as a net with set of indices \( \mathcal{F} \). As a generalization of Theorem 5.2, we may prove that for any \((f^\Lambda) \in C_{L^2}(S^X)\), we have
\[
\lim_{\Lambda \in \mathcal{F}} f^\Lambda = f_\infty \in L^2(\mu),
\]
where the limit is the convergence in terms of nets.

Theorem 5.2 gives the following corollary.

Corollary 5.3. The space \( C_{L^2}(S^X) \) of Theorem 5.2 coincides with the image of \( L^2(\mu) \) in \( C_{\text{col}}(S^X) \) with respect to the homomorphism (21). Moreover, the space of local functions \( C_{\text{loc}}(S^X) \) is dense in \( C_{L^2}(S^X) \) for the topology defined by the norm
\[
\| (f^\Lambda) \|_\mu := \sup_{\Lambda \in \mathcal{F}} \| f^\Lambda \|_\mu
\]
on \( C_{L^2}(S^X) \).

Proof. Suppose \( f \in L^2(S^X) \). Then for any \( \Lambda \in \mathcal{F} \), by (22), we have
\[
\| f^\Lambda \|_\mu \leq \| f \|_\mu,
\]
hence we see that \( (\pi^\Lambda f) \) satisfies \( \sup_{\Lambda \in \mathcal{F}} \| f^\Lambda \|_\mu \leq \| f \|_\mu < \infty \). This shows that \( (\pi^\Lambda f) \in C_{L^2}(S^X) \) as desired. On the other hand, suppose \((f^\Lambda) \in C_{L^2}(S^X)\). We fix a family \( \{ \Lambda_n \}_{n \in \mathbb{N}} \) of sets in \( \mathcal{F} \) such that \( \Lambda_n \subset \Lambda_{n+1} \) and \( X = \bigcup_{n \in \mathbb{N}} \Lambda_n \), and let \( f_n := f^{\Lambda_n} \) for any \( n \in \mathbb{N} \). Then Theorem 5.2 shows that \((f_n)\) converges strongly to a function \( f_\infty \) in \( L^2(\mu) \). Fix an \( n \in \mathbb{N} \) and let \( g_n \in C(S^{\Lambda_n}) \).
Then for any $m > n$, the orthogonal property of the conditional expectation in Lemma 2.1 implies that $\langle f_m - f_n, g_n \rangle_\mu = 0$, hence we have

$$\langle f_m, g_n \rangle_\mu = \langle f_n + (f_m - f_n), g_n \rangle_\mu = \langle f_n, g_n \rangle_\mu.$$ 

This shows that

$$\langle f_\infty, g_n \rangle_\mu = \lim_{m \to \infty} \langle f_m, g_n \rangle_\mu = \langle f_n, g_n \rangle_\mu.$$

Furthermore, the definition of conditional expectation shows that we have

$$\langle \pi^{\Lambda_n} f_\infty, g_n \rangle_\mu = \langle f_\infty, g_n \rangle_\mu.$$

This proves that $\pi^{\Lambda_n} f_\infty = f_n$ in $C(S^{\Lambda_n})$. For any $\Lambda \in \mathcal{F}$, taking $n$ sufficiently large so that $\Lambda \subset \Lambda_n$, we see that $f^\Lambda = \pi^\Lambda f_n = \pi^\Lambda(\pi^{\Lambda_n} f_\infty) = \pi^\Lambda f_\infty$ by the tower property (3), which shows that $(f^\Lambda)$ is the image of $f_\infty$ by (21) as desired. This gives our first assertion.

Next, consider $f \in L^2(\mu)$ and the associated $(\pi^\Lambda f) \in C_{L^2}(S^X)$. Again, Theorem 5.2 shows that the functions $f_n = \pi^{\Lambda_n} f \in C(S^{\Lambda_n}) \subset C_{\text{loc}}(S^X)$ for $n \in \mathbb{N}$ strongly converge to $f_\infty \in L^2(\mu)$. By the previous argument, since $f^\Lambda = \pi^\Lambda f$ for any $\Lambda \in \mathcal{F}$, we see that the image of $f$ and $f_\infty$ coincide in $C_{L^2}(S^X)$. Hence we have

$$\lim_{n \to \infty} f_n = f$$

in $C_{L^2}(S^X)$, which proves that $C_{\text{loc}}(S^X)$ is dense in $C_{L^2}(S^X)$ as desired. \qed

**Remark.** The space $C_{L^2}(S^X)$ is usually referred to as the space of Martingales bounded in $L^2$ (see for example [5, Chapter 12]). The inclusions

$$C_{\text{loc}}(S^X) \subset C_{L^2}(S^X) \subset C_{\text{col}}(S^X)$$

give what is known as a **Gelfand triple**.

Next, we define the space of $L^2$-forms. Assume that $\mu = \nu^\otimes X$ for some probability measure $\nu$ supported on $S$. For any $\Lambda \in \mathcal{F}$, we equip the space $T^*S^\Lambda$ with a norm given by

$$\|\omega\|^2_\mu := \frac{1}{|E_\Lambda|} \sum_{e \in E_\Lambda} \|\omega_e\|^2_\mu$$

for any $\omega = (\omega_e) \in \prod_{e \in E_\Lambda} C(S^\Lambda)$, where $\|\omega_e\|^2_\mu := E_\mu[\omega_e^2]$. Following Theorem 5.2 we define the space of $L^2$-forms in

$$C_{\text{col}}^1(S^X) = \lim_{\mathcal{F}} C^1(S^\Lambda)$$

as follows.

**Definition 5.4.** We define the space $C_{L^2}^1(S^X)$ of $L^2$-forms on $S^X$ by

$$C_{L^2}^1(S^X) := \{(\omega^\Lambda) \in C_{\text{col}}^1(S^X) \mid \sup_{\Lambda \in \mathcal{F}} \|\omega^\Lambda\|_\mu < \infty\} \subset C_{\text{col}}^1(S^X).$$
As in [4] consider an action of a group $G$ on the locale $X$. We again assume that $G$ is torsion free and that the action of $G$ on $X$ is free. We let

$$\mathcal{G} := C_{L^2}^1(S^X) \cap Z^1_{\text{col}}(S^X)^G \subset Z^1_{\text{col}}(S^X)^G$$

be the space of $G$-invariant closed $L^2$-forms, and we let $\mathcal{F} := \partial (C_{\text{unif}}^0(S^X)^G)$ be the space of exact forms, where the bar denotes the closure of $\partial (C_{\text{unif}}^0(S^X)^G)$ in $\mathcal{C}$ for the topology induced from the norm $\|(\omega^\lambda)\|_\mu := \sup_{\Lambda \in \mathcal{I}} \|\omega^\lambda\|_\mu$ for any $(\omega^\lambda) \in C_{L^2}^1(S^X)^G$.

By definition of the differential $\partial$, for any $\xi \in \text{Consv}_\nu^\phi(S)$, the function $\xi_X \in C_{\text{unif}}^0(S^X) \subset C_{\text{col}}^0(S^X)$ satisfies $\partial \xi_X = 0$. This shows that we have a natural inclusion $\text{Consv}_\nu^\phi(S) \hookrightarrow H^0_{\text{col}}(S^X)$ given by mapping $\xi$ to $\xi_X$. We conjecture the following.

**Conjecture 5.5.** Assume the conditions of Theorem 4.4 and suppose in addition that the interaction $\phi$ is faithfully quantified. Then the isomorphism of Corollary 4.8 induces the isomorphism

$$\mathcal{C}/\mathcal{G} \cong \text{Hom}_\mathbb{Z}(G, \text{Consv}_\nu^\phi(S)).$$

In particular, if $G$ is finitely generated of rank $d$, then we have a decomposition

$$\mathcal{C} \cong \mathcal{G} \oplus \bigoplus_{j=1}^d \text{Consv}_\nu^\phi(S).$$

The above conjecture is the decomposition which is at the heart of the method proposed by Varadhan to prove the hydrodynamic limit in the non-gradient case. In the case that the locale $(X, E)$ is the Euclidean lattice $(\mathbb{Z}^d, (\mathbb{Z}^d)^\dagger)$, the group $G = \mathbb{Z}^d$ with action on $(X, E)$ by translation, the set of states $S = \{0, 1\}$, the interaction is given by $\phi(s_1, s_2) = (s_2, s_1)$ for $(s_1, s_2) \in S \times S$, and $\nu$ is the probability measure on $S$ given by $\nu(\{1\}) = p$ for $p \in (0, 1)$, Conjecture 5.5 was proved by Funaki, Uchiyama and Yau [4].

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