Two-Scale Convergence of Unsteady Stokes Type Equations

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Abstract:
In this paper we study the homogenization of evolution Stokes type equations in the periodic setting. The usual Laplace operator involved in the classical Stokes equations is here replaced by a linear elliptic differential operator of divergence form with periodically oscillating coefficients. Our mean tool is the well known two-scale convergence method.

Keywords:
Periodic Homogenization; Two-scale Convergence; Unsteady Stokes Equations

1. INTRODUCTION

Let \( \Omega \) be a smooth bounded open set in \( \mathbb{R}^N \) (the \( N \)-numerical space \( \mathbb{R}^N \) of variables \( x = (x_1, ..., x_N) \)), where \( N \) is a given positive integer, and let \( T \) and \( \varepsilon \) be real numbers with \( T > 0 \) and \( 0 < \varepsilon < 1 \). We consider the partial differential operator

\[
P^\varepsilon = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_{ij}^\varepsilon \frac{\partial}{\partial x_j} \right)
\]

in \( \Omega \times ]0, T[ \), where \( a_{ij}^\varepsilon (x) = a_{ij}^\varepsilon \left( \frac{x}{\varepsilon} \right) \) (\( x \in \Omega \)), \( a_{ij} \in L^\infty \left( \mathbb{R}^N_y; \mathbb{R} \right) \) (\( 1 \leq i, j \leq N \)) with

\[
a_{ij} = a_{ji}, \tag{1}
\]

and the assumption that there is a constant \( \alpha > 0 \) such that

\[
\sum_{i,j=1}^N a_{ij} (y) \xi_i \xi_j \geq \alpha |\xi|^2 \text{ for all } \xi = (\xi_j) \in \mathbb{R}^N \text{ and } \tag{2}
\]

for almost all \( y \in \mathbb{R}^N \), where \( \mathbb{R}^N_y \) is the \( N \)-numerical space \( \mathbb{R}^N \) of variables \( y = (y_1, ..., y_N) \), and where \( |.| \) denotes the Euclidean norm in \( \mathbb{R}^N \). The operator \( P^\varepsilon \) acts on scalar functions, say \( \varphi \in L^2 \left( 0, T; H^1(\Omega) \right) \). However, we may as well view \( P^\varepsilon \) as acting on vector functions \( u = (u^i) \in L^2 \left( 0, T; H^1(\Omega)^N \right) \) in a \textit{diagonal way}, i.e.,

\[
(P^\varepsilon u)^i = P^\varepsilon u^i \quad (i = 1, ..., N).
\]
For any Roman character such as $i, j$ (with $1 \leq i, j \leq N$), $u^i$ (resp. $u^j$) denotes the $i$-th (resp. $j$-th) component of a vector function $u$ in $L^1_{loc}(\Omega \times [0, T])^N$ or in $L^1_{loc}((\mathbb{R}^N \times \mathbb{R})^N$ where $\mathbb{R}$ is the numerical space $\mathbb{R}$ of variables $\tau$. Further, for any real $0 < \varepsilon < 1$, we define $u^\varepsilon$ as

$$u^\varepsilon(x,t) = u \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) \quad ((x,t) \in \Omega \times [0,T])$$

for $u \in L^1_{loc}(\mathbb{R}^N \times \mathbb{R})$, as is customary in homogenization theory. More generally, for $u \in L^1_{loc}(Q \times \mathbb{R}^N \times \mathbb{R})$ with $Q = \Omega \times [0,T]$, it is customary to put

$$u^\varepsilon(x,t) = u \left( x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) \quad ((x,t) \in \Omega \times [0,T])$$

whenever the right-hand side makes sense (see, e.g., [9, 9]).

After these preliminaries, let $f = (f^i)$ be in $L^2 \left( 0, T; H^{-1}(\Omega; \mathbb{R})^N \right)$. For any fixed $0 < \varepsilon < 1$, we consider the initial boundary value problem

$$\frac{\partial u_\varepsilon}{\partial t} + D^\varepsilon u_\varepsilon + \operatorname{grad} p_\varepsilon = f \text{ in } \Omega \times [0,T], \quad (3)$$

$$\operatorname{div} u_\varepsilon = 0 \text{ in } \Omega \times [0,T], \quad (4)$$

$$u_\varepsilon = 0 \text{ on } \partial \Omega \times [0,T], \quad (5)$$

$$u_\varepsilon(0) = 0 \text{ in } \Omega. \quad (6)$$

We will later see that as in [19], (3)-(6) uniquely define $(u_\varepsilon, p_\varepsilon)$ with $u_\varepsilon \in W(0,T)$ and $p_\varepsilon \in L^2(0,T;L^2(\Omega;\mathbb{R})/\mathbb{R})$, where

$$W(0,T) = \left\{ u \in L^2(0,T;V) : u' \in L^2(0,T;V') \right\}$$

$V$ being the space of functions $u$ in $H^1_0(\Omega;\mathbb{R})^N$ with $\operatorname{div} u = 0$ ($V'$ is the topological dual of $V$) and where

$$L^2(\Omega;\mathbb{R})/\mathbb{R} = \left\{ v \in L^2(\Omega;\mathbb{R}) : \int_\Omega vdx = 0 \right\}.$$

Let us recall that $W(0,T)$ is provided with the norm

$$\|u\|_{W(0,T)} = \left( \|u\|_{L^2(0,T;V)}^2 + \|u'\|_{L^2(0,T;V')}^2 \right)^{\frac{1}{2}} \quad (u \in W(0,T)),$$

which makes it a Hilbert space with the following properties (see [19]): $W(0,T)$ is continuously embedded in $\mathcal{C} \left( [0,T]; L^2(\Omega)^N \right)$ and is compactly embedded in $L^2 \left( 0, T; L^2(\Omega)^N \right)$.

Our aim here is to investigate the asymptotic behavior, as $\varepsilon \to 0$, of $(u_\varepsilon, p_\varepsilon)$ under the assumption that the functions $a_{ij}$ ($1 \leq i, j \leq N$) are periodic in the space variable $y$. The steady state version of this problem (i.e., the homogenization of stationary Stokes type equations) was first investigated by Bensoussan, Lions and Papanicolaou [2]. These authors use the well-known approach of asymptotic expansions combined with Tartar’s energy method. We also mention the paper of Nguetseng and the author [17], on the sigma-convergence of stationary Navier-Stokes type equations.
The present study deals with the periodic homogenization of an evolution problem for Stokes type equations. The very same model equation has been proposed and studied in a paper by Choe and Kim [3]. Using the formal asymptotic expansion, they derived the macroscopic homogenized problem and provided a corrector’s type result, and by some technical results, they justified the convergence of the homogenization process. However, Theorem 7 and Theorem 9 in [3] do not make plainly visible the global homogenized equation satisfied by $u = (u_0, u_1)$ ($u_0$ is the solution of the macroscopic homogenized equations and $u_1$ is the corrector) as in Theorem 3.3 of this paper. Let us mention that the uniqueness of the solution to the global homogenized equation given by Lemma 3.1 of this paper guarantees the uniqueness, and some estimates on the velocity $u$ for $<\varepsilon$ fixed, $0 < \varepsilon < 1$, it is not apparent that the initial boundary value problem (3)-(6) has a solution provided a suitable assumption on the behaviour of coefficients $a_{ij}$ of the viscosity tensor (see, e.g. [17] for such behaviours), whereas the extension of Theorem 7 and Theorem 9 in [3], even to the almost periodic setting is quite a difficult issue.

This study is motivated by the fact that the homogenization of (3)-(6) is connected with the modelling of heterogeneous fluid flows (see, e.g., [20] for more details about such models).

Our approach is the well-known two-scale convergence method.

Unless otherwise specified, vector spaces throughout are considered over the complex field, $\mathbb{C}$, and scalar functions are assumed to take complex values. Let us recall some basic notation. If $X$ and $F$ denote a locally compact space and a Banach space, respectively, then we write $\mathcal{C}(X; F)$ for continuous mappings of $X$ into $F$, and $\mathcal{B}(X; F)$ for those mappings in $\mathcal{C}(X; F)$ that are bounded. We shall assume $\mathcal{B}(X; F)$ to be equipped with the supremum norm $\|u\|_{\infty} = \sup_{x \in X} |u(x)|$ ($\|\cdot\|$ denotes the norm in $F$). For shortness we will write $\mathcal{C}(X) = \mathcal{C}(X; \mathbb{C})$ and $\mathcal{B}(X) = \mathcal{B}(X; \mathbb{C})$. Likewise in the case when $F = \mathbb{C}$, the usual spaces $L^p(X; F)$ and $L^p_{\text{loc}}(X; F)$ ($X$ provided with a positive Radon measure) will be denoted by $L^p(X)$ and $L^p_{\text{loc}}(X)$, respectively. Finally, the numerical space $\mathbb{R}^N$ and its open sets are each provided with Lebesgue measure denoted by $dx = dx_1...dx_N$.

The rest of the paper is organized as follows. Section 2 is devoted to the preliminary results on existence and uniqueness, and some estimates on the velocity $u_\varepsilon$, the pressure $p_\varepsilon$ and the acceleration $\frac{\partial u_\varepsilon}{\partial t}$ of the fluid, whereas in Section 3 one convergence theorem is established.

## 2. PRELIMINARIES

Let $\Omega$ be a smooth bounded open set in $\mathbb{R}^N$, let $T > 0$ be a real number and let $f = (f^j) \in L^2 \left(0, T; H^{-1}(\Omega)^N \right)$. For $0 < \varepsilon < 1$, it is not apparent that the initial boundary value problem (3)-(6) has a solution $(u_\varepsilon, p_\varepsilon)$, and that the latter is unique. With a view to elucidating this, we introduce, for fixed $0 < \varepsilon < 1$ the bilinear form $a^\varepsilon$ on $H^1_0(\Omega; \mathbb{R})^N \times H^1_0(\Omega; \mathbb{R})^N$ defined by

$$a^\varepsilon (u, v) = \sum_{i,j=1}^N \sum_{k=1}^N \int_{\Omega} a_{ij}^\varepsilon \frac{\partial u^k}{\partial x_i} \frac{\partial v^k}{\partial x_j} \, dx$$

for $u = (u^k)$ and $v = (v^k)$ in $H^1_0(\Omega; \mathbb{R})^N$. By virtue of (1), the form $a^\varepsilon$ is symmetric. Further, in view of (2),

$$a^\varepsilon (v, v) \geq \alpha \|v\|_{H^1_0(\Omega)}^2 \tag{7}$$
for every \( v = (v^k) \in H^1_0(\Omega; \mathbb{R})^N \) and \( 0 < \varepsilon < 1 \), where

\[
\|v\|_{H^1_0(\Omega)^N} = \left( \sum_{k=1}^N \int_{\Omega} |\nabla v^k|^2 \, dx \right)^{\frac{1}{2}}
\]

with \( \nabla v^k = \left( \frac{\partial v^k}{\partial x_1}, \ldots, \frac{\partial v^k}{\partial x_N} \right) \). On the other hand, it is clear that a constant \( c_0 > 0 \) exists such that

\[
|a^\varepsilon(u, v)| \leq c_0 \|u\|_{H^1_0(\Omega)^N} \|v\|_{H^1_0(\Omega)^N}
\]

for every \( u, v \in H^1_0(\Omega; \mathbb{R})^N \) and \( 0 < \varepsilon < 1 \).

We are now in a position to verify the following result.

**Proposition 2.1.**

Suppose \( f \) lies in \( L^2(0, T; L^2(\Omega; \mathbb{R})^N) \). Under the hypotheses (1)-(2), the initial boundary value problem (3)-(6) determines a unique pair \((u_\varepsilon, p_\varepsilon)\) with \( u_\varepsilon \in L^2(0, T; H^1_0(\Omega; \mathbb{R})^N) \) and \( p_\varepsilon \in L^2(0, T; L^2(\Omega; \mathbb{R})) \).

**Proof.** For fixed \( 0 < \varepsilon < 1 \), we consider the Cauchy problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
u_\varepsilon'(t) + A_\varepsilon u_\varepsilon(t) = \ell(t) \text{ in }]0, T[} \\
u_\varepsilon(0) = 0,
\end{array} \right.
\]

(9)

where \( A_\varepsilon \) is the linear operator of \( V \) into \( V' \) defined by

\[
(A_\varepsilon u, v) = a^\varepsilon(u, v) \text{ for all } u, v \in V
\]

and \( \ell \) is the function in \( L^2(0, T; V') \) defined by

\[
(\ell(t), v) = (f(t), v) \text{ for all } v \in V
\]

and for almost all \( t \in ]0, T[ \), and where \((,\) denotes the duality pairing between \( V' \) and \( V \) as well as between \( H^{-1}(\Omega; \mathbb{R})^N \) and \( H^1_0(\Omega; \mathbb{R})^N \). Thanks to (7)-(8) the Cauchy problem (9) admits a unique solution \( u_\varepsilon \) in \( \mathcal{W}'(0, T) \), as is easily seen by following [6, Chap.3, Théorème 1.2, p.116], see also [19, pp.254-260]. Now, let us check that the abstract parabolic problem (9) is equivalent to (3)-(6). Let \( U_\varepsilon(t) = \int_0^t P^\varepsilon u_\varepsilon(s) \, ds \) and \( F(t) = \int_0^t f(s) \, ds \) for \( 0 \leq t \leq T \), where \( u_\varepsilon \) satisfies (9). It is evident that \( U_\varepsilon \) and \( F \) lie in \( \mathcal{W}'([0, T]; H^{-1}(\Omega; \mathbb{R})^N) \). By the first equality of (9) we have

\[
\frac{d}{dt} (u_\varepsilon(t), \phi) = (\ell(t) - A_\varepsilon u_\varepsilon(t), \phi) \text{ for all } \phi \in \mathcal{V},
\]

(10)

where

\[
\mathcal{V} = \left\{ \phi \in \mathcal{D}(\Omega; \mathbb{R})^N : \text{div } \phi = 0 \right\}.
\]

Integrating (10), we have

\[
(u_\varepsilon(t) + U_\varepsilon(t) - F(t), \phi) = 0, \ 0 \leq t \leq T, \ \phi \in \mathcal{V}.
\]
Thus, using a classical argument (see, e.g., [19, p.14]), we get a function \( p_\varepsilon \in C ([0, T]; L^2 (\Omega; \mathbb{R}) / \mathbb{R} ) \) such that
\[ u_\varepsilon + U_\varepsilon + \text{grad} p_\varepsilon = F. \]
Hence \( p_\varepsilon = \frac{\partial p_\varepsilon}{\partial t} \in C^1 (Q) \) and the pair \( (u_\varepsilon, p_\varepsilon) \) verifies (3) (in the distribution sense on \( Q \)), with in addition (4)-(6), of course. Furthermore, by using the fact that \( f \in L^2 \left( 0, T; L^2 (\Omega; \mathbb{R})^N \right) \) we have \( u_\varepsilon \in L^2 \left( 0, T; L^2 (\Omega; \mathbb{R})^N \right) \), as is easily seen by following [19, p.268]. Therefore \( p_\varepsilon \) lies in \( L^2 \left( 0, T; L^2 (\Omega; \mathbb{R}) \right) \) and is unique. Conversely, it is an easy exercise to verify that if \( (u_\varepsilon, p_\varepsilon) \) is a solution of (3)-(6) with \( u_\varepsilon \in \mathcal{H} (0, T) \) and \( p_\varepsilon \in L^2 \left( 0, T; L^2 (\Omega; \mathbb{R}) \right) \), then \( u_\varepsilon \) satisfies (9). The proof is complete.

The following regularity result is fundamental for the estimates of the solution \( (u_\varepsilon, p_\varepsilon) \) of (3)-(6).

**Lemma 2.1.**
Suppose \( f, f' \in L^2 (0, T; V') \) and \( f (0) \in L^2 (\Omega; \mathbb{R})^N \). Then the solution \( u_\varepsilon \) of (9) verifies:
\[ u_\varepsilon \in L^2 (0, T; V) \cap L^\infty (0, T; H), \]
where \( H \) is the closure of \( V' \) in \( L^2 (\Omega; \mathbb{R})^N \).

The proof of the above lemma follows by the same line of argument as in the proof of [19, p.299, Theorem 3.5]. So we omit it. We are now able to prove the result on the estimates.

**Proposition 2.2.**
Under the hypotheses of Lemma 2.1, there exists a constant \( c > 0 \) (independent of \( \varepsilon \)) such that the pair \( (u_\varepsilon, p_\varepsilon) \) solution of (3)-(6) in \( \mathcal{H} (0, T) \times L^2 (0, T; L^2 (\Omega; \mathbb{R}) / \mathbb{R} ) \) satisfies:

\[ \| u_\varepsilon \|_{\mathcal{H} (0, T)} \leq c \]
\[ \| \frac{\partial u_\varepsilon}{\partial t} \|_{L^2 (0, T; H^{-1} (\Omega))} \leq c \]
and
\[ \| p_\varepsilon \|_{L^2 (0, T; L^2 (\Omega))} \leq c. \]

**Proof.** Let \( (u_\varepsilon, p_\varepsilon) \) be the solution of (3)-(6). We have
\[ (u_\varepsilon' (t), \nu) + d^\varepsilon (u_\varepsilon (t), \nu) = (f (t), \nu) \quad (\nu \in V) \]
for almost all \( t \in [0, T] \). By taking in particular \( \nu = u_\varepsilon (t) \) in (14), we have for almost all \( t \in [0, T] \)
\[ \frac{d}{dt} | u_\varepsilon (t) |^2 + 2 \alpha \| u_\varepsilon (t) \| ^2 \leq \frac{1}{\alpha} \| f (t) \| _V^2 + \alpha \| u_\varepsilon (t) \| ^2 \]
where |·| and \( \| · \| \) are respectively the norms in \( L^2 (\Omega)^N \) and \( H^1_0 (\Omega)^N \). Hence, for every \( s \in [0, T] \)
\[ | u_\varepsilon (s) |^2 + \alpha \int_s^T \| u_\varepsilon (t) \| ^2 dt \leq \frac{1}{\alpha} \int_0^T \| f (t) \| _V^2 dt. \]
The inequality (18) shows that
\[ u \]
Thus, by (15) and (16) one quickly arrives at (11). Let us show (12). We are allowed to differentiate (14)
\[ v \]
In view of Lemma 2.1, we take in particular
\[ i.e. \]
(12) is immediate. Let us prove (13). For almost all
\[ t \]
Further, since
\[ u^\prime \]
particular
\[ t \]
\[ \alpha \int_0^T \| u^\prime_\varepsilon(t) \|^2 dt \leq \frac{1}{\alpha} \int_0^T \| f(t) \|^2_{V^\prime} dt. \] \hfill (15)
On the other hand, the abstract parabolic problem (9) gives
\[ u^\prime_\varepsilon = f - A_\varepsilon u_\varepsilon. \]
Hence, in view of (8)
\[ \| u^\prime_\varepsilon \|_{L^2(0,T;V^\prime)} \leq \| f \|_{L^2(0,T;V^\prime)} + c_0 \| u_\varepsilon \|_{L^2(0,T;V)} . \] \hfill (16)
Thus, by (15) and (16) one quickly arrives at (11). Let us show (12). We are allowed to differentiate (14) in distribution sense on \([0,T]\), and by virtue of the hypotheses of Lemma 2.1, we get \( u^\prime_\varepsilon \in L^2(0,T;V^\prime) \) and
\[ \left( u^{\prime\prime}_\varepsilon, v \right) + a^\varepsilon \left( u^\prime_\varepsilon, v \right) = \left( f, v \right) \quad (v \in V). \] \hfill (17)
In view of Lemma 2.1, we take in particular \( v = u^\prime_\varepsilon(t) \) in (14). This yields
\[ \left| u^\prime_\varepsilon(t) \right|^2 + a^\varepsilon \left( u(t), u^\prime_\varepsilon(t) \right) = \left( f(t), u^\prime_\varepsilon(t) \right) \quad (t \in [0,T]). \]
Further, since \( u^\prime_\varepsilon \in L^2(0,T;V) \) and \( u^{\prime\prime}_\varepsilon \in L^2(0,T;V^\prime) \), we have \( u^\prime_\varepsilon \in \mathcal{C}([0,T];H) \). Hence, by taking in particular \( t = 0 \) in the preceding equality and using (6) one quickly arrives at
\[ \left| u^\prime_\varepsilon(0) \right|^2 \leq \left| f(0) \right| \left| u^\prime_\varepsilon(0) \right| , \]
i.e.,
\[ \left| u^\prime_\varepsilon(0) \right| \leq \left| f(0) \right| . \] \hfill (18)
The inequality (18) shows that \( u^\prime_\varepsilon(0) \) lies in a bounded subset of \( H \). On the other hand, by taking in particular \( v = u^\prime_\varepsilon(t) \) in (17), we get
\[ \frac{d}{dt} \left| u^\prime_\varepsilon(t) \right|^2 + 2\alpha \left| u^\prime_\varepsilon(t) \right|^2 \leq \frac{1}{\alpha} \left| f(t) \right|^2_{V^\prime} + \alpha \left| u^\prime_\varepsilon(t) \right|^2 \]
for almost all \( t \in [0,T] \). Integrating the preceding inequality on \([0,t] \ (t \in [0,T]) \) leads to
\[ \left| u^\prime_\varepsilon(t) \right|^2 + \alpha \int_0^t \left| u^\prime_\varepsilon(s) \right|^2 ds \leq \frac{1}{\alpha} \left| f(t) \right|^2_{L^2(0,T;V^\prime)} + \left| u^\prime_\varepsilon(0) \right|^2 . \]
It follows from (18) and the preceding inequality that \( u^\prime_\varepsilon \) belongs to a bounded subset of \( L^2 \left( 0,T;L^2(\Omega;\mathbb{R})^N \right) \). Hence (12) is immediate. Let us prove (13). For almost all \( t \in [0,T] \), 
\( p_\varepsilon(t) \in L^2(\Omega;\mathbb{R})/\mathbb{R} \). Thus, by [18, p. 30] there exists \( v_\varepsilon(t) \in H^1_0(\Omega;\mathbb{R})^N \) such that
\[ \text{div} v_\varepsilon(t) = p_\varepsilon(t) \] \hfill (19)
\[ \| v_\varepsilon(t) \| \leq c_1 \| p_\varepsilon(t) \|_{L^2(\Omega)}, \] \hfill (20)
where the constant $c_1$ depends solely on $\Omega$. Multiplying (3) by $v_e(t)$ yields

$$
(u'_e(t), v_e(t)) + a^2 (u_e(t), v_e(t)) - \int_\Omega p_e(t) \nabla v_e(t) \, dx = (f(t), v_e(t))
$$

for almost all $t \in [0, T]$. Integrating the preceding equality on $[0, T]$ and using (19)-(20) lead to

$$
\|p_e\|^2_{L^2(Q)} \leq c_1 \|u'_e\|^2_{L^2(0,T;L^2(Q))} \|p_e\|_{L^2(Q)} + c_1 \|f\|^2_{L^2(0,T;H^{-1}(\Omega))} \|p_e\|_{L^2(Q)}
+ c_1 c_0 \|u_e\|^2_{L^2(0,T;V)},
$$

where $c$ is the constant in the Poincaré inequality, $c_0$ and $c_1$ are the constants in (8) and (20) respectively. Thus,

$$
\|p_e\|^2_{L^2(Q)} \leq c_1 \|u'_e\|^2_{L^2(0,T;L^2(Q))} + c_1 \|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + c_1 c_0 \|u_e\|^2_{L^2(0,T;V)}. \tag{21}
$$

Combining (21), (11) and (12) leads to (13).

3. A CONVERGENCE RESULT FOR (3)-(6)

We set $Y = (-\frac{1}{2}, \frac{1}{2})^N$, $Y$ considered as a subset of $\mathbb{R}_N$ (the space $\mathbb{R}_N$ of variables $y = (y_1, ..., y_N)$). We set also $Z = (-\frac{1}{2}, \frac{1}{2})$, $Z$ considered as a subset of $\mathbb{R}_\tau$ (the space $\mathbb{R}$ of variables $\tau$). Our purpose is to study the homogenization of (3)-(6) under the periodicity hypothesis on $a_{ij}$.

3.1 Preliminaries

Let us first recall that a function $u \in L^1_{\text{loc}}(\mathbb{R}_N^N \times \mathbb{R}_\tau)$ is said to be $Y \times Z$-periodic if for each $(k, l) \in \mathbb{Z}^N \times Z$ (Z denotes the integers), we have $u(y + k, \tau + l) = u(y, \tau)$ almost everywhere (a.e.) in $(y, \tau) \in \mathbb{R}_N \times \mathbb{R}$. If in addition $u$ is continuous, then the preceding equality holds for every $(y, \tau) \in \mathbb{R}_N \times \mathbb{R}$, of course. The space of all $Y \times Z$-periodic continuous complex functions on $\mathbb{R}_N \times \mathbb{R}_\tau$ is denoted by $C_{\text{per}}(Y \times Z)$; that of all $Y \times Z$-periodic functions in $L^p_{\text{loc}}(\mathbb{R}_N \times \mathbb{R}_\tau)$ ($1 \leq p < \infty$) is denoted by $L^p_{\text{per}}(Y \times Z)$. $C_{\text{per}}(Y \times Z)$ is a Banach space under the supremum norm on $\mathbb{R}_N \times \mathbb{R}$, whereas $L^p_{\text{per}}(Y \times Z)$ is a Banach space under the norm

$$
\|u\|_{L^p(Y \times Z)} = \left( \int_Z \int_Y |u(y, \tau)|^p \, dy \, d\tau \right)^{\frac{1}{p}} (u \in L^p_{\text{per}}(Y \times Z)).
$$

We will need the space $H^1_Y(Y)$ of $Y$-periodic functions $u \in H^1_{\text{loc}}(\mathbb{R}_N^N) = W^{1,2}_{\text{loc}}(\mathbb{R}_N^N)$ such that $\int_Y u(y) \, dy = 0$. Provided with the gradient norm,

$$
\|u\|_{H^1_Y(Y)} = \left( \int_Y |\nabla u|^2 \, dy \right)^{\frac{1}{2}} (u \in H^1_Y(Y)),
$$

where $\nabla u = \left( \frac{\partial u}{\partial y_1}, ..., \frac{\partial u}{\partial y_N} \right)$, $H^1_Y(Y)$ is a Hilbert space. We will also need the space $L^2_{\text{per}}(Z; H^1_Y(Y))$ with the norm

$$
\|u\|_{L^2_{\text{per}}(Z; H^1_Y(Y))} = \left( \int_Z \int_Y |\nabla u(y, \tau)|^2 \, dy \, d\tau \right)^{\frac{1}{2}} (u \in L^2_{\text{per}}(Z; H^1_Y(Y))).
$$
which is a Hilbert space.

Before we can recall the concept of two-scale convergence, let us introduce one further notation. The letter $E$ throughout will denote a family of real numbers $0 < \varepsilon < 1$ admitting 0 as an accumulation point. For example, $E$ may be the whole interval $(0,1)$; $E$ may also be an ordinary sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with $0 < \varepsilon_n < 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. In the latter case $E$ will be referred to as a fundamental sequence.

Let $\Omega$ be a bounded open set in $\mathbb{R}^N$, and $Q = \Omega \times [0,T]$ with $T \in \mathbb{R}^+_*$, and let $1 \leq p < \infty$.

**Definition 3.1.**

A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(Q)$ is said to:

(i) weakly two-scale converge in $L^p(Q)$ to some $u_0 \in L^p(Q;L^\perp_{\text{per}}(Y \times Z))$ if as $E \ni \varepsilon \to 0$,

\[
\int_Q u_\varepsilon(x,t) \psi^\varepsilon(x,t) \, dx \, dt \to \int \int_{Q \times Y \times Z} u_0(x,t,y,\tau) \psi(x,t,y,\tau) \, dx \, dy \, dt \, d\tau
\]

for all $\psi \in L^p(Q;L^\perp_{\text{per}}(Y \times Z)) \left(\frac{1}{p} = 1 - \frac{1}{P}\right)$, where $\psi^\varepsilon(x,t) = \psi(x,t,\frac{x}{\varepsilon},\frac{\tau}{\varepsilon})$ for all $(x,t) \in Q$;

(ii) strongly two-scale converge in $L^p(Q)$ to some $u_0 \in L^p(Q;L^\perp_{\text{per}}(Y \times Z))$ if the following property is verified:

\[
\begin{cases}
\text{Given } \eta > 0 \text{ and } v \in L^p(Q;L^\perp_{\text{per}}(Y \times Z)) \text{ with } \\
\|u_0 - v\|_{L^p(Q \times Y \times Z)} \leq \frac{\varepsilon}{\eta}, \text{ there is some } \alpha > 0 \text{ such that } \\
\|u_\varepsilon - v^\varepsilon\|_{L^p(Q)} \leq \eta \text{ provided } E \ni \varepsilon \leq \alpha.
\end{cases}
\]

We will briefly express weak and strong two-scale convergence by writing $u_\varepsilon \to u_0$ in $L^p(Q)$-weak 2-s and $u_\varepsilon \to u_0$ in $L^p(Q)$-strong 2-s, respectively.

**Remark 3.1.**

It is of interest to know that if $u_\varepsilon \to u_0$ in $L^p(Q)$-weak 2-s, then (22) holds for $\psi \in C(Q；L^\perp_{\text{per}}(Y \times Z))$. See [10, Proposition 10] for the proof.

Instead of repeating here the main results underlying two-scale convergence, we find it more convenient to draw the reader’s attention to a few references, see, e.g., [1], [8], [10] and [21].

However, we recall below two fundamental results. First of all, let

\[
\mathcal{Y}(0,T) = \left\{ v \in L^2(0,T;H^1_0(\Omega;\mathbb{R})) : v' \in L^2(0,T;H^{-1}(\Omega;\mathbb{R})) \right\}.
\]

$\mathcal{Y}(0,T)$ is provided with the norm

\[
\|v\|_{\mathcal{Y}(0,T)} = \left(\|v\|^2_{L^2(0,T;H^1_0(\Omega))} + \|v'\|^2_{L^2(0,T;H^{-1}(\Omega))}\right)^{\frac{1}{2}}, \quad (v \in \mathcal{Y}(0,T))
\]

which makes it a Hilbert space.

**Theorem 3.1.**

Assume that $1 < p < \infty$ and further $E$ is a fundamental sequence. Let a sequence $(u_\varepsilon)_{\varepsilon \in E}$ be bounded in $L^p(Q)$. Then, a subsequence $E'$ can be extracted from $E$ such that $(u_\varepsilon)_{\varepsilon \in E'}$ weakly two-scale converges in $L^p(Q)$.
Theorem 3.2.
Let $E$ be a fundamental sequence. Suppose a sequence $(u_\epsilon)_{\epsilon \in E}$ is bounded in $\mathcal{Y}(0,T)$. Then, a subsequence $E'$ can be extracted from $E$ such that, as $E' \ni \epsilon \to 0$,

\[ u_\epsilon \to u_0 \text{ in } \mathcal{Y}(0,T) \text{-weak}, \]

\[ u_\epsilon \to u_0 \text{ in } L^2(Q) \text{-weak 2-s}, \]

\[ \frac{\partial u_\epsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^2(Q) \text{-weak 2-s \ (1 \leq j \leq N)}, \]

where $u_0 \in \mathcal{Y}(0,T)$, $u_1 \in L^2(Q;L^2_{per}(Z;H^1_Y)).$

The proof of Theorem 3.1 can be found in, e.g., [1], [8], whereas Theorem 3.2 has its proof in, e.g., [10] and [16].

3.2 A global homogenization theorem

Before we can establish a so-called global homogenization theorem for (3)-(6), we require a few basic notation and results. To begin, let

\[ \mathcal{Y}_Y = \left\{ \psi \in C_\infty_{per}(Y;\mathbb{R})^N : \int_Y \psi(y)dy = 0, \ \text{div}_y \psi = 0 \right\}, \]

\[ V_Y = \left\{ w \in H^1_0(Y;\mathbb{R})^N : \text{div}_y w = 0 \right\}, \]

where: $C_\infty_{per}(Y;\mathbb{R}) = C_\infty(\mathbb{R}^N;\mathbb{R}) \cap C_{per}(Y)$, $\text{div}_y$ denotes the divergence operator in $\mathbb{R}^N$. We provide $V_Y$ with the $H^1_0(Y)^N$-norm, which makes it a Hilbert space. There is no difficulty in verifying that $\mathcal{Y}_Y$ is dense in $V_Y$ (proceed as in [15, Proposition 3.2]). With this in mind, set

\[ F^1_0 = L^2(0,T;V) \times L^2(Q;L^2_{per}(Z;V_Y)). \]

This is a Hilbert space with norm

\[ \|v\|_{F^1_0} = \left( \|v_0\|_{L^2(0,T;V)}^2 + \|v_1\|_{L^2(Q;L^2_{per}(Z;V_Y))}^2 \right)^{1/2}, \quad v = (v_0,v_1) \in F^1_0. \]

On the other hand, put

\[ F^\infty_0 = \mathcal{D}(0,T;'Y) \times \left[ \mathcal{D}(Q;\mathbb{R}) \otimes [C_\infty_{per}(Z;\mathbb{R}) \otimes 'Y] \right], \]

where $C_\infty_{per}(Z;\mathbb{R}) = C_\infty(\mathbb{R}^N;\mathbb{R}) \cap C_{per}(Z)$, $C_\infty_{per}(Z;\mathbb{R}) \otimes 'Y$ stands for the space of vector functions $w$ on $\mathbb{R}^N \times \mathbb{R}_\tau$ of the form

\[ w(y,\tau) = \sum_{finite} \chi_i(\tau) \nu_i(y) \quad (\tau \in \mathbb{R}, \ y \in \mathbb{R}^N) \]
with $\chi_i \in C^\infty_{\text{per}}(Z; \mathbb{R})$, $v_i \in \mathcal{V}$, and where $\mathcal{D}(Q; \mathbb{R}) \otimes C^\infty_{\text{per}}(Z; \mathbb{R}) \otimes \mathcal{V}$ is the space of vector functions on $Q \times \mathbb{R}_+^N \times \mathbb{R}$ of the form 

$$
\psi(x,t,y) = \sum_{finite} \phi_i(x,t) w_i(y) \quad ((x,t) \in Q, \ y \in \mathbb{R}_+^N \times \mathbb{R})
$$

with $\phi_i \in \mathcal{D}(Q; \mathbb{R})$, $w_i \in C^\infty_{\text{per}}(Z; \mathbb{R}) \otimes \mathcal{V}$. Since $\mathcal{V}$ is dense in $V$ (see [19, p.18]), it is clear that $\mathcal{F}^\infty_0$ is dense in $\mathcal{F}_0^1$.

Now, let

$$
U = V \times L^2(\Omega; L^2_{\text{per}}(Z; V_Y)).
$$

Provided with the norm

$$
\|v\|_U = \left(\|v_0\|^2 + \|v_1\|^2_{L^2(\Omega; L^2_{\text{per}}(Z; V_Y))}\right)^{\frac{1}{2}} \quad (v = (v_0, v_1) \in U),
$$

$U$ is a Hilbert space. Let us set

$$
\hat{a}_\Omega(u, v) = \sum_{i,j,k=1}^N \int \int_{\Omega \times Y \times Z} a_{ij} \left(\frac{\partial u_0^k}{\partial x_i} + \frac{\partial u_1^k}{\partial y_j}\right) \left(\frac{\partial v_0^k}{\partial x_i} + \frac{\partial v_1^k}{\partial y_j}\right) dy dx dt
$$

for $u = (u_0, u_1)$ and $v = (v_0, v_1)$ in $U$. This defines a symmetric continuous bilinear form $\hat{a}_\Omega$ on $U \times U$. Furthermore, $\hat{a}_\Omega$ is $U$-elliptic. Specifically,

$$
\hat{a}_\Omega(u, u) \geq \alpha \|u\|^2_U \quad (u \in U) \tag{23}
$$

as is easily checked by using (2) and the fact that $\int_Y \frac{\partial u_1^k}{\partial y_j}(x, y, \tau) dy = 0$.

Here is one fundamental lemma.

**Lemma 3.1.**

Under the hypotheses (1)-(2). The variational problem

$$
\begin{cases}
    u_0 \in W^0(0, T) \text{ with } u_0(0) = 0; \\
    u = (u_0, u_1) \in F_0^1; \\
    \int_0^T (u_0'(t), v_0(t)) dt + \int_0^T \hat{a}_\Omega(u(t), v(t)) dt = \int_0^T (f(t), v(t)) dt
\end{cases}
$$

for all $v \in F_0^1 \tag{24}$

has at most one solution.

**Proof.** Let $v_* = (v_0, v_1) \in U$ and $\phi \in \mathcal{D}([0, T])$. By taking $v = \phi \otimes v_*$ in (24), we arrive at

$$
(u_0'(t), v_0) + \hat{a}_\Omega(u(t), v_*) = (f(t), v_0) \quad (v_* \in U) \tag{25}
$$

for almost all $t \in (0, T)$. Suppose that $u_*$ and $u_{**}$ are two solutions of (24) with $u_* = (u_{*0}, u_{*1})$ and $u_{**} = (u_{**0}, u_{**1})$. Let $u = u_* - u_{**} = (u_{00}, u_{11})$ with $u_0 = u_{00} - u_{**0}$ and $u_1 = u_{**1} - u_{**1}$. Let us show that $u = 0$. By using (25) we see that $u$ verifies:

$$
(u_0'(t), v_0) + \hat{a}_\Omega(u(t), v_*) = 0 \tag{26}
$$
for all \( v_s \in U \) and for almost all \( t \in (0, T) \). But, by virtue of [19, p. 261]

\[
\frac{d}{dt} |u_0(t)|^2 = 2 (u_0'(t), u_0(t))
\]

for almost all \( t \in (0, T) \). Then, taking \( v_s = u(t) \) in (26), we obtain by (23)

\[
\frac{d}{dt} |u_0(t)|^2 + 2\alpha \| u(t) \|_0^2 \leq 0
\]

(27)

for almost all \( t \in (0, T) \). Integrating (27) on \([0, t]\) \((0 \leq t \leq T)\), we get \( |u_0(t)|^2 \leq 0 \) for all \( t \in [0, T] \) and \( \|u\|_0^2 \leq 0 \), thus \( u = 0 \) and the lemma follows.

We are now able to prove the desired theorem. Throughout the remainder of the present section, it is assumed that \( a_{ij} \) is \( Y \)-periodic for any \( 1 \leq i, j \leq N \).

**Theorem 3.3.**

Suppose that the hypotheses of Lemma 2.1 are satisfied. For \( 0 < \varepsilon < 1 \), let \( u_\varepsilon \) be defined by (3)-(6). Then, as \( \varepsilon \to 0 \) we have

\[
\mathbf{u}_\varepsilon \to u_0 \text{ in } \mathcal{W}'(0, T) \text{-weak},
\]

(28)

\[
\frac{\partial u^i_\varepsilon}{\partial x_j} \to \frac{\partial u^i_0}{\partial x_j} + \frac{\partial u^j_0}{\partial y_j} \text{ in } L^2(Q) \text{-weak 2-s } \ (1 \leq j, k \leq N)
\]

(29)

where \( \mathbf{u} = (u_0, u_1) \) (with \( u_0 = (u^0_0) \) and \( u_1 = (u^1_0) \)) is the unique solution of (24).

**Proof.** By Proposition 2.2, we see that the sequences \((p_\varepsilon)_{0<\varepsilon<1}\) and \((u_\varepsilon)_{0<\varepsilon<1} = (u^0_\varepsilon, ..., u^N_\varepsilon)_{0<\varepsilon<1}\) are bounded respectively in \( L^2(Q) \) and \( \mathcal{W}'(0, T) \). Further, it follows from (11) and (12) that for \( 1 \leq k \leq N \), the sequence \((u^k_\varepsilon)_{0<\varepsilon<1}\) is bounded in \( \mathcal{W}'(0, T) \). Let \( E \) be a fundamental sequence. Then, by Theorems 3.1-3.2 and the fact that \( \mathcal{W}'(0, T) \) is compactly embedded in \( L^2(Q)^N \), there exist a subsequence \( E' \) extracted from \( E \) and functions \( u_0 = (u^k_0)_{1 \leq k \leq N} \in \mathcal{W}'(0, T), u_1 = (u^k_1)_{1 \leq k \leq N} \in L^2(Q; L^2_{\text{per}}(Z; H^1_0(Y; \mathbb{R}^N))) \), and \( p \in L^2(Q; L^2_{\text{per}}(Y \times Z; \mathbb{R})) \) such that as \( E' \ni \varepsilon \to 0 \), we have (28)-(29) and

\[
\mathbf{u}_\varepsilon \to u_0 \text{ in } L^2(Q)^N \text{-strong},
\]

(30)

\[
p_\varepsilon \to p \text{ in } L^2(Q) \text{-weak 2-s}.
\]

(31)

But, by virtue of Lemma 3.1, the theorem will be entirely proved if we show that \( \mathbf{u} = (u_0, u_1) \) verifies (24). In fact, according to (4), we have \( \text{div} u_0 = 0 \) and \( \text{div}_Y u_1 = 0 \). Therefore \( \mathbf{u} = (u_0, u_1) \in \mathbb{P}_1^1 \). Let us recall that \( u_0 \) can be considered as a continuous function of \([0, T]\) into \( H \) since \( \mathcal{W}'(0, T) \) is continuously embedded in \( \mathcal{C}([0, T]; H) \). Let us show that \( u_0(0) = 0 \). For \( v \in V \) and \( \varphi \in \mathcal{C}^1([0, T]) \) with \( \varphi(T) = 0 \) and \( \varphi(0) = 1 \), we have by an integration by part

\[
\int_0^T (u_\varepsilon'(t), v) \varphi(t) \, dt + \int_0^T (u_\varepsilon(t), v) \varphi'(t) \, dt = - (u_\varepsilon(0), v).
\]

According to (6), we have by passing to the limit in the preceding equality as \( E' \ni \varepsilon \to 0 \)

\[
\int_0^T (u'_0(t), v) \varphi(t) \, dt + \int_0^T (u_0(t), v) \varphi'(t) \, dt = 0.
\]
Hence \((u_0(0), v) = 0\) for all \(v \in V\), and as \(V\) is dense in \(H\) we conclude that \(u_0(0) = 0\). Now, let us check that \(u = (u_0, u_1)\) verifies the variational equation of (24). For \(0 < \varepsilon < 1\), let

\[
\Phi_\varepsilon = \psi_0 + \varepsilon \psi_1^\varepsilon \quad \text{with} \quad \psi_0 \in D(Q; \mathbb{R})^N \quad \text{and} \quad \psi_1 \in D(Q; \mathbb{R}) \otimes [\mathcal{E}_\perp^m (Z; \mathbb{R}) \otimes Y].
\]

 biased by \(\Phi_\varepsilon\), yields

\[
\int_0^T (u_e'(t), \Phi_\varepsilon(t)) dt = - \sum_{i=1}^N \int_Q u'_e \left[ \frac{\partial \psi_0^i}{\partial t} + \varepsilon \left( \frac{\partial \psi_1^i}{\partial t} \right)^\varepsilon + \left( \frac{\partial \psi_1^i}{\partial \tau} \right)^\varepsilon \right] dx dt,
\]

Then by virtue of (30) we have

\[
\int_0^T (u_e'(t), \Phi_\varepsilon(t)) dt \rightarrow - \sum_{i=1}^N \int_Q u'_0 \frac{\partial \psi_0^i}{\partial t} dx dt = \int_0^T (u_0'(t), \psi_0(t)) dt
\]

as \(E' \ni \varepsilon \rightarrow 0\). In fact, on one hand

\[
\sum_{i=1}^N \int_Q u'_e \left[ \frac{\partial \psi_0^i}{\partial t} + \varepsilon \left( \frac{\partial \psi_1^i}{\partial t} \right)^\varepsilon + \left( \frac{\partial \psi_1^i}{\partial \tau} \right)^\varepsilon \right] dx dt
\]

\[
\rightarrow \sum_{i=1}^N \int_Q u'_0 \frac{\partial \psi_0^i}{\partial t} dx dt + \int \int_{Q \times Z} u'_0 \frac{\partial \psi_1^i}{\partial \tau} dx dt dy d\tau
\]

as \(E' \ni \varepsilon \rightarrow 0\), on the other hand

\[
\int \int_{Q \times Z} u'_0 \frac{\partial \psi_1^i}{\partial \tau} dx dt dy d\tau = \int_Q u'_0 \left( \int \int_{Y \times Z} \frac{\partial \psi_1^i}{\partial \tau} dy dt \right) dx dt = 0
\]

by virtue of the \(Y \times Z\)-periodicity. The next point is to pass to the limit in (33) as \(E' \ni \varepsilon \rightarrow 0\). To this end, we note that as \(E' \ni \varepsilon \rightarrow 0\),

\[
\int_0^T \varepsilon \left( u_e(t), \Phi_\varepsilon(t) \right) dt \rightarrow \int_0^T \tilde{a}_\Omega (u(t), \Phi(t)) dt,
\]

where \(\Phi = (\psi_0, \psi_1)\) (proceed as in the proof of the analogous result in [14, p.179]). Now, based on (31), there is no difficulty in showing that as \(E' \ni \varepsilon \rightarrow 0\),

\[
\int_Q p_e \text{div} \Phi_e dx dt \rightarrow \int \int_{Q \times Y \times Z} p \text{div} \psi_0 dx dt dy d\tau.
\]

On the other hand, let us check that as \(\varepsilon \rightarrow 0\)

\[
\int_0^T (f(t), \Phi_\varepsilon(t)) dt \rightarrow \int_0^T (f(t), \psi_0(t)) dt.
\]
Indeed, if $f \in L^2 \left( 0, T; L^2(\Omega; \mathbb{R}) \right)$, (35) is immediate by using the classical fact that $\Phi_e \to \Phi_0$ in $L^2(Q)^N$ weak and $\frac{\partial \Phi_e}{\partial x_j} \to \frac{\partial \Phi_0}{\partial x_j}$ in $L^2(Q)^N$. weak (1 $\leq$ $j \leq$ $N$) as $\varepsilon \to 0$. In the general case, (35) follows by the density of $L^2 \left( 0, T; L^2(\Omega; \mathbb{R}) \right)$ in $L^2 \left( 0, T; H^{-1}(\Omega; \mathbb{R}) \right)$.

Having made this point, we can pass to the limit in (33) when $E' \ni \varepsilon \to 0$, and result is that

$$
\int_0^T (u_0'(t), \psi_0(t)) dt + \int_0^T \tilde{a}_\Omega (u(t), \Phi(t)) dt
- \int_Q p_0 div \psi_0 dx dt = \int_0^T (f(t), \psi(t)) dt,
$$

(36)

where $p_0$ denotes the mean of $p$, i.e., $p_0 \in L^2 \left( 0, T; L^2(\Omega; \mathbb{R}) \right)$ and $p_0(x,t) = \int \int_{y \times Z} p(x,t,y,\tau) dy d\tau$ a.e. in $(x,t) \in Q$, and where $\Phi = (\psi_0, \psi_1)$, $\psi_0$ ranging over $\mathcal{D}(Q; \mathbb{R})^N$ and $\psi_1$ ranging over $\mathcal{D}(Q; \mathbb{R}) \otimes \mathcal{D}'(Z; \mathbb{R}) \otimes \mathcal{D}'(\mathcal{V}'; \mathcal{V})$. Taking in particular $\psi_0$ in $\mathcal{D}(0, T; \mathcal{V}')$ and using the density of $\mathcal{D}_0^\infty$ in $\mathcal{D}'_0$, one quickly arrives at (24). The unicity of $u = (u_0, u_1)$ follows by Lemma 3.1. Consequently, (28) and (29) still hold when $E \ni \varepsilon \to 0$. Hence when $0 < \varepsilon \to 0$, by virtue of the arbitrariness of $E$. The theorem is proved. \Box

Now, we wish to give a simple representation of the vector function $u_1$ in Theorem 3.3 for further uses. For this purpose we introduce the bilinear form $\tilde{a}$ on $L^2_{per}(Z; V_Y) \times L^2_{per}(Z; V_Y)$ defined by

$$
\tilde{a}(u, v) = \sum_{i,j,k=1}^N \int_{Y \times Z} a_{ij} \frac{\partial u^k}{\partial y_j} \frac{\partial v^j}{\partial y_i} dy d\tau
$$

for $u = (u^k)$ and $v = (v^k) \in L^2_{per}(Z; V_Y)$. Next, for each pair of indices $1 \leq i, k \leq N$, we consider the variational problem

$$
\begin{cases}
\chi_{ik} \in L^2_{per}(Z; V_Y) : \\
\tilde{a}(\chi_{ik}, w) = \sum_{l=1}^N \int_{Y \times Z} a_{ij} \frac{\partial u^l}{\partial y_j} dy d\tau \\
\text{for all } w = (w^j) \in L^2_{per}(Z; V_Y),
\end{cases}
$$

(37)

which determines $\chi_{ik}$ in a unique manner.

**Lemma 3.2.**
Under the hypotheses and notation of Theorem 3.3, we have

$$
u_1(x,t,y,\tau) = - \sum_{i,j,k=1}^N \frac{\partial u_0^k}{\partial x_i} (x,t) \chi_{ik}(y,\tau)
$$

(38)

almost everywhere in $(x,t,y,\tau) \in Q \times Y \times Z$.

**Proof.** In (24), we choose the test functions $v = (v_0, v_1)$ such that $v_0 = 0$ and $v_1(x,t,y,\tau) = \varphi(x,t) w(y,\tau)$ for $(x,t,y,\tau) \in Q \times Y \times Z$, where $\varphi \in \mathcal{D}(Q; \mathbb{R})$ and $w \in L^2_{per}(Z; V_Y)$. Then for almost every $(x,t)$ in $Q$, we have

$$
\begin{cases}
\tilde{a}(u_1(x,t), w) = - \sum_{i,j,k=1}^N \frac{\partial u_0^k}{\partial x_i} (x,t) \int_{Y \times Z} a_{ij} \frac{\partial w^j}{\partial y_i} dy d\tau \\
\text{for all } w \in L^2_{per}(Z; V_Y).
\end{cases}
$$

(39)

But it is clear that $u_1(x,t)$ (for fixed $(x,t) \in Q$) is the unique function in $L^2_{per}(Z; V_Y)$ solving the variational equation (39). On the other hand, it is an easy exercise to verify that $z(x,t) = - \sum_{i,k=1}^N \frac{\partial u_0^k}{\partial x_i} (x,t) \chi_{ik}$ solves also (39). Hence the lemma follows immediately. \Box
3.3 Macroscopic homogenized equations

Our aim here is to derive a well-posed initial boundary value problem for \((u_0,p_0)\). To begin, for \(1 \leq i,j,k,h \leq N\), let

\[
q_{ijkh} = \delta_{ik} \int_Y a_{ij}(y) \, dy - \sum_{p=1}^{N} \int_Y \int_Z a_{il}(y) \frac{\partial \chi_{jkh}^p}{\partial y_l}(y,\tau) \, dy \, d\tau,
\]

where: \(\delta_{ik}\) is the Kronecker symbol, \(\chi_{jkh} = \chi_{jkh}^1\) is defined by (37). To the coefficients \(q_{ijkh}\) we associate the differential operator \(\mathcal{Q}\) on \(Q\) mapping \(\mathcal{D}'(Q)^N\) into \(\mathcal{D}'(Q)^N\) \((\mathcal{D}'(Q)\) being the usual space of complex distributions on \(Q\)) as

\[
(\mathcal{Q}z)^k = -\sum_{i,j,k,h=1}^{N} q_{ijkh} \frac{\partial^2 z^h}{\partial x_i \partial x_j} \quad (1 \leq k \leq N) \text{ for } z = (z^h), \quad z^h \in \mathcal{D}'(Q).
\]

\(\mathcal{Q}\) is the so-called homogenized operator associated to \(P^\varepsilon\) \((0 < \varepsilon < 1)\).

Now, let us consider the initial boundary value problem

\[
\frac{\partial u_0}{\partial t} + \mathcal{Q} u_0 + \nabla p_0 = f \quad \text{in } Q = \Omega \times ]0,T[, \\
divergence u_0 = 0 \text{ in } Q, \\
u_0 = 0 \text{ on } \partial \Omega \times ]0,T[, \\
u_0(0) = 0 \text{ in } \Omega.
\]

**Lemma 3.3.**

The initial boundary value problem (41)-(44) admits at most one weak solution \((u_0,p_0)\) with \(u_0 \in \mathcal{W}(0,T)\) and \(p_0 \in L^2(0,T;L^2(\Omega;\mathbb{R}))\).

**Proof.** If \((u_0,p_0) \in \mathcal{W}(0,T) \times L^2(0,T;L^2(\Omega;\mathbb{R}))\) verifies (41)-(44), then we have

\[
\int_0^T (u_0'(t),v_0(t)) \, dt + \sum_{i,j,k,h=1}^{N} \int_Q q_{ijkh} \frac{\partial u_0}{\partial x_i} \frac{\partial u_0}{\partial x_j} \, dx \, dt
= \int_0^T (f(t),v_0(t)) \, dt
\]

for all \(v_0 \in L^2(0,T;V)\). From the previous equality, one quickly arrives at

\[
\int_0^T (u_0'(t),v_0(t)) \, dt + \sum_{i,j,k,h=1}^{N} \int_Q q_{ijkh} \frac{\partial u_0}{\partial x_i} \frac{\partial u_0}{\partial x_j} \, dx \, dt
= \int_0^T (f(t),v_0(t)) \, dt
\]

where \(u_0^k(x,t,\tau) = -\sum_{i,j,k,h=1}^{N} \frac{\partial u_0^k}{\partial x_i}(x,t) \chi_{jkh}^p(y,\tau)\) for \((x,t,y,\tau) \in Q \times Y \times Z\). Let us check that \(u = (u_0,u_1)\) (with \(u_1(x,t,\tau) = -\sum_{i,j,k,h=1}^{N} \frac{\partial u_1^k}{\partial x_i}(x,t) \chi_{jkh}^p(y,\tau)\) for \((x,t,y,\tau) \in Q \times Y \times Z\)) satisfies (24). Indeed, we have

\[
\sum_{i,j,k,h=1}^{N} \int_Q \int_Y a_{ij}(x,t) \frac{\partial u^k}{\partial x_j} + \frac{\partial u^k}{\partial y_i} \frac{\partial u^k}{\partial y_i} \, dx \, dy \, d\tau = 0
\]

for all \(v_1 = (v_1^k) \in L^2(\Omega \times Y, \mathcal{D}'(Q)\) ), since \(u_1(x,t)\) verifies (39) for \((x,t) \in Q\). Thus, by (45)-(46), we see that \(u = (u_0,u_1)\) verifies (24). Hence, the unicity in (41)-(44) follows by Lemma 3.1.
This leads us to the following theorem.

**Theorem 3.4.**  
Suppose that the hypotheses of Theorem 3.3 are satisfied. For each \( 0 < \varepsilon < 1 \), let \((u_\varepsilon, p_\varepsilon) \in W(0,T) \times L^2(0,T;L^2(\Omega;\mathbb{R})/\mathbb{R})\) be defined by (3)-(6). Then, as \( \varepsilon \to 0 \), we have \( u_\varepsilon \to u_0 \) in \( W(0,T) \)-weak and \( p_\varepsilon \to p_0 \) in \( L^2(0,T;L^2(\Omega)) \)-weak, where the pair \((u_0, p_0)\) lies in \( W(0,T) \times L^2(0,T;L^2(\Omega;\mathbb{R})/\mathbb{R}) \) and is the unique solution of (41)-(44).

**Proof.** Let \( E \) be a fundamental sequence. As in the proof of Theorem 3.3, there exists a subsequence \( E' \) extracted from \( E \) such that as \( E' \ni \varepsilon \to 0 \), we have (28)-(29) and (31) with \( u = (u_{0\varepsilon}, u_0) \in \mathbb{R}^N \) and \( u_0(0) \). Then, from (31) we have \( p_\varepsilon \to p_0 \) in \( L^2(0,T;L^2(\Omega)) \)-weak when \( E' \ni \varepsilon \to 0 \), where \( p_0 \) is the mean of \( p \). Hence, it follows that \( p_0 \in L^2(0,T;L^2(\Omega;\mathbb{R})/\mathbb{R}) \). Further, (36) holds for all \( \Phi = (\psi_0, \psi_1) \in \mathcal{D}(Q;\mathbb{R})^N \times \mathcal{D}(Q;\mathbb{R}) \otimes [C^p_{p,m}(Z;\mathbb{R}) \otimes Y_f] \). Then, substituting (38) in (36) and choosing therein the \( \Phi \)'s such that \( \psi_1 = 0 \), a simple computation leads to (41) with evidently (42)-(44). Hence the Theorem follows by Lemma 3.3 since \( E \) is arbitrarily chosen.

**Remark 3.2.**  
The operator \( \mathcal{D} \) is elliptic, i.e., there is some \( \alpha_0 > 0 \) such that

\[
\sum_{i,j,k,h=1}^{N} q_{ijkh} \xi_i \xi_j \xi_k \xi_h \geq \alpha_0 \sum_{k,h=1}^{N} |\xi_k \xi_h|^2
\]

for all \( \xi = (\xi_i) \) with \( \xi_i \in \mathbb{R} \). Indeed, by following a classical line of argument (see, e.g., [2]), we can give a suitable expression of \( q_{ijkh} \), viz.

\[
q_{ijkh} = a(\chi_{ik} - \pi_{ik}; \chi_{jh} - \pi_{jh}),
\]

where, for each pair of indices \( 1 \leq i,k \leq N \), the vector function \( \pi_{ik} = (\pi_{ik}^1, ..., \pi_{ik}^N) : \mathbb{R}^N \to \mathbb{R} \) is given by \( \pi_{ik}^r(y) = y_i \delta_{kr} \) \((r = 1, ..., N)\) for \( y = (y_1, ..., y_N) \in \mathbb{R}^N \). Hence, the above ellipticity property follows in a classical fashion.

4. CONCLUSION

In our study, one convergence result has been proved for the homogenization process of a non-stationary Stokes type flow. Further, our convergence theorem makes plainly visible the global homogenized equation. It is not the case for the work of Choe and Kim in [3], where they use the formal asymptotic expansion to solve a similar problem. Moreover, we have derived the macroscopic homogenized model, which is of the Stokes type.

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