Numerical Solution of Free Stochastic Differential Equations

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Abstract

This paper derives a free analog of the Euler-Maruyama method (fEMM) to numerically approximate solutions of free stochastic differential equations (fSDEs). Simply speaking fSDEs are stochastic differential equations in the context of non-commutative random variables (e.g. large random matrices). By applying the theory of multiple operator integrals we derive a free Itô formula from Taylor expansion of operator valued functions. Iterating the free Itô formula allows to motivate and define fEMM. Then we consider weak and strong convergence in the fSDE setting and prove strong convergence order of ½ and weak convergence order of 1. Numerical examples support the theoretical results and show solutions for equations where no analytical solution is known.

Keywords free stochastic differential equations, free probability theory, Euler-Maruyama method, random matrix theory, stochastic differential equations, weak convergence, strong convergence

AMS Codes 46L53, 46L54, 60H10, 65C30

1 Introduction

Nowadays random matrices appear in a broad range of applications (e.g. [11], [14], [1], [34], [3], [21], [19], [16], [35], [29], [25]). Random matrices with certain spectral properties can be obtained as solutions of free stochastic differential equations (fSDEs). A fSDE is an equation of the form

\[ dX_t = a(X_t)dt + b(X_t)dW_t c(X_t), \]  

(1)

where the unknown \( X_t \) is an operator valued process \((X_t)_{t \geq 0}\), \((W_t)_{t \geq 0}\) is a free Brownian motion and \( a, b, c \) are appropriate operator valued functions. At a first sight, one may think about \( X_t \) as random matrices of large dimension. Free Brownian motion can be viewed as the large \( N \) limit of Brownian motions on \( N \times N \) hermitian matrices ([9]). By taking the large \( N \) limit fSDEs are formulated in some appropriate von Neumann algebra \( A \). Free probability theory and free stochastic processes set up the background for a solution theory of fSDEs ([17]). The notion of freeness is the carryover of the notion of independence of random variables to the non-commutative
context. For an introduction to free probability and free stochastic processes we refer to [27], [32], [21], [3], [31], [28]. A short introduction will be given at the beginning of this paper. Free stochastic calculus first appeared in [26] and was further developed by [18], [7] and [9]. The notion of free stochastic processes, free Brownian motion and a free analog of the Itô formula were introduced in [9]. For definition of the free Itô integral, we refer to [4]. Free stochastic processes form an active research area, we refer to [6], [27], [8], [2], [13], [12], [14]. SDEs first appeared in [18], [7] and [10], where such equations are motivated by studying large N quantum field theory and corresponding matrix models. A first existence theory and a variety of SDEs were analytically studied in [17]. Recently in [14] a free variant of the Cox-Ingersoll-Ross model ([15]) is considered in the context of financial mathematics.

To the best of our knowledge the numerical solution of free stochastic differential equations has not yet been studied before. In [10] an Euler-like method was applied to prove the existence of a solution of a special fSDE and furthermore regularity results were obtained in the operator norm.

The purpose of this paper is to develop, analyze and apply a method for numerical approximation of fSDEs. As in the classical case we start by developing a free analog of the Euler-Maruyama method (fEMM). The derivation of the method will be stated and carried out by considering the fSDE in a von Neumann Algebra $\mathcal{A}$ with faithful unital normal trace. The free Itô formula [9, Theorem 4.1.2] will play a central role in this context. The free analog to the classical Itô formula was derived by applying the concept of double operator integrals on $B(\mathcal{H})$. The perturbation theory of operator valued functions and extension to multiple operator integrals was further developed, which gives rise to new conceptual and technical tools. We make use of these developments ([24], [22]) to reformulate the free Itô formula. The formalism defined in [5] allows for a consistent and effective formulation of the free Itô formula. Based on a Taylor polynomial of operator functions ([5], [24]) we derive the Itô formula ([9, Proposition 4.3.4]) directly from the stochastic product rules. The proof of this proposition relies on approximating via polynomials. Alternatively we derive the free Itô formula via Taylor expansions of the corresponding operator valued functions. Then we are in the position to formulate an iterated version of the free Itô formula, which allows the motivation and definition of a free analog of the Euler-Maruyama method (fEMM).

We will give proofs for strong and weak convergence properties. It turns out that strong convergence is of order $\frac{1}{2}$. This is mainly driven by the fact that the $L^2(\varphi)$-norm of the stochastic integral $\int_{\Delta t} a_t dW_t b_t$ is of $O(\sqrt{\Delta t})$. Additionally, the coefficient functions $a, b, c$ of the fSDE need to be operator Lipschitz in $L^2(\varphi)$. The existence theory in [17] requires $a, b, c$ to be Lipschitz in operator norm of $\mathcal{A}$. Weak convergence of order one is proven under the assumptions $a, b, c$ are uniformly bounded in $\mathcal{A}$ and belonging to certain proper spaces $W_n(\mathbb{R})$ (see [5],[24]). Several examples show the capability of fEMM to approximate solutions of fSDEs. We will verify that fEMM can well numerically approximate spectral properties of the solution of the underlying fSDE.

To be able to implement fEMM as a numerical method on a computer, it is necessary to consider the fEMM in a von Neumann algebra of random matrices. We will show that the large $N$ limit of fEMM on matrix level leads to a fEMM defined in a finite von Neumann algebra (just as limits of random matrices end up in an infinite dimensional von Neumann Algebra). Weak and strong convergence properties do hold in any appropriate von Neumann algebra. Since random matrices form itself a von Neumann algebra, the convergence properties of fEMM in $\mathcal{A}$ carry over to the numerical algorithm. Additionally we show that both limits $N \to \infty$ and $\Delta t \to 0$ commute. We give examples by applying fEMM to equations considered in [17] and show, that both limits of step size and matrix size commute and converge in distribution to the distribution of the solution. We will numerically verify the theoretically obtained properties of weak and strong convergence.
The paper is organized as follows. The necessary ingredients to define fSDEs are summarized in section 2 and section 3. In section 4 we formulate the free Itô formula in the context of the framework of multiple operator integrals developed in [5]. We then derive an iterated Itô formula which allows to motivate the free Euler-Maruyama method. Next, the new numerical algorithm is defined in section 5. The main results regarding convergence properties are given in section 6. Section 7 shows examples of the numerical approximation for fSDEs.

2 Free Stochastic Calculus

In this chapter we summarize the main results of free probability theory and free stochastic calculus. Free stochastic calculus was initiated by [26] and developed in a series of papers in [18], [9], [4] and [7]. We will introduce the notion of free Brownian motion, free stochastic calculus which includes a free analog of the classical Itô formula.

2.1 Free Probability Theory

Consider a classical probability space \((\Omega, \mathcal{F}, \mu)\) and random variables as measurable functions \(X : \Omega \rightarrow \mathbb{R}\). By taking an algebraic viewpoint these random variables \(X\) form an (commutative) algebra, where it is possible to assign expectations \(E(X)\) to each random variable. This change of viewpoint allows to consider cases, where the random variables \(X\) do not form an (commutative) algebra, where it is possible to assign expectations \(E(X)\) to each random variable. This change of viewpoint allows to consider cases, where the random variables \(X\) do not form an (commutative) algebra, where it is possible to assign expectations \(E(X)\) to each random variable.

Definition 2.1. A non-commutative probability space is a pair \((\mathcal{A}, \varphi)\), where \(\mathcal{A}\) denotes a von Neumann operator algebra and \(\varphi : \mathcal{A} \rightarrow \mathbb{C}\) a faithful unital normal trace.

Since the trace is finite we may consider \(\mathcal{A}\) as a subset of the predual \(L_1(\varphi)\) of the von Neumann algebra \(\mathcal{A} = L_\infty(\varphi)\). For \(1 \leq p < \infty\) we define \(\|X\|_p = \varphi(|X|^p)^{\frac{1}{p}}\). By \(\|\cdot\|\) we denote the usual operator norm in \(\mathcal{A}\). Although the definition of a non-commutative probability space is rather abstract, once the concepts are stated, they turn into background when working on numerical methods. The notion of independence of classical random variables is extended to the non-commutative setting by the concept of freeness of subalgebras of \(\mathcal{A}\). Let \(\mathcal{A}_1, \ldots, \mathcal{A}_n\) be a family of \(n \in \mathbb{N}\) subalgebras of \(\mathcal{A}\). They are called freely independent (or simply free) in the sense of Voiculescu, if \(\varphi(X_1X_2\ldots X_m) = 0\) whenever the following conditions hold:

1. \(X_j \in \mathcal{A}_{i(j)}\), where \(i(1) \neq i(2), i(2) \neq i(3), \ldots, i(n - 1) \neq i(n)\), \(j = 1, \ldots, m\)
2. \(\varphi(X_i) = 0\) for all \(i = 1, \ldots, n\)

If \(X \in \mathcal{A}\) is a self-adjoint element, then there is a spectral measure \(\mu\) on \(\mathbb{R}\) so that the moments of \(X\) are the same as the moments of the probability measure \(\mu\) defined by \(\varphi(X^k) = \int_x x^k d\mu(x)\). An important role in the subsequent plays the Cauchy transform \(G_X\) of \(\mu\) defined by \(G_X(z) = \int_x \frac{d\mu(x)}{x - z}\), which is an analytic function defined on \(\mathbb{C}^+\) with values in \(\mathbb{C}^+\). The Cauchy transform \(G_X\) is the expectation of the resolvent of \(X\), i.e. \(G_X(z) = \varphi((X - z)^{-1})\). The Cauchy transform carries all the properties of the spectral probability distribution of the self-adjoint operator \(X\). In [9] and [17] it is shown how fSDEs can be handled by a corresponding deterministic
partial differential equations of the Cauchy transform $G_X$. We will strongly depend on these results since it allows us to check the numerical results obtained in section 5 by the free stochastic Euler method defined.

2.2 Free Brownian Motion

Motivated from the concept of classical Brownian motion the definition within non-commutative probability is as follows. Consider a von Neumann Algebra $A$ with a faithful normal tracial state $\phi : A \to \mathbb{C}$. A filtration $\mathbb{F} = (\mathcal{A}_t)_{t \geq 0}$ is a family of subalgebras $\mathcal{A}_t$ of $A$ with $\mathcal{A}_s \subset \mathcal{A}_t$ for $s \leq t$. A free stochastic process is a family of elements $(X_t)_{t \geq 0}$ for which the increments $X_t - X_s$ are free with respect to the subalgebra $\mathcal{A}_s$. A process $(X_t)_{t \geq 0}$ is called adapted to the filtration $\mathbb{F}$ if $X_t \in \mathcal{A}_t$ for all $t \geq 0$.

**Definition 2.2.** A free Brownian motion is a family of self-adjoint elements $(W_t)_{t \geq 0}$ which admit the properties

1. $W_0 = 0$.
2. The increments $W_t - W_s$ are free from $W_s$ for all $0 \leq s < t$. The subalgebra $W_s$ is generated by all $W_\tau$ with $\tau \leq s$.
3. The increment $W_t - W_s$ is semicircle with mean 0 and variance $t - s$ for all $0 \leq s < t$.

We define the filtration $\mathbb{F} = (\mathcal{C}_t)_{t \geq 0}$ where $\mathcal{C}_t$ is generated by all elements $W_s, s < t$.

**Remark 1.** Free Brownian motion $(W_t)_{t \geq 0}$ can be viewed as large $N$ limit of $N \times N$ hermitian random matrices having classical independent Brownian motion entries $b_{ij}(t)$ ([4], [9]). Considering the symmetric $N$-dimensional quadratic random matrix $W_N^t := \frac{1}{\sqrt{N}} (b_{ij}(t))_{N \times N}$, the limit $\lim_{N \to \infty} W_N^t$ defines an element $W_t$ in a von Neumann algebra $A$ with trace $\phi(\cdot) = \lim_{N \to \infty} E(\sqrt{N} \text{tr}(\cdot))$.

2.3 Stochastic Integration with Respect to Free Brownian Motion

Let $(W_t)_{t \geq 0}$ be a free Brownian motion. Let $a, b$ be mappings $[0, T] \to A$ such that $\|a(t)\|\|b(t)\| \in L_2([0, T])$ and $a(t), b(t) \in \mathcal{C}_t$. We shorten the notation $a(t) = a_t, b(t) = b_t$ in the following, if there is no danger of confusion. Under these assumptions it is possible to define an Itô-style free stochastic integration with respect to free Brownian motion ([9], [4]), written as

$$\int_0^t a_s dW_s b_s. \tag{2}$$

For details of the definition and conditions for the existence and properties we refer to [17], [4], [9]. The free stochastic integral fulfills a free analog of Burkholder-Gundy martingale inequalities (Section 3.2. in [9]), i.e.

$$\left\| \int_0^t a_s dW_s b_s \right\| \leq 2\sqrt{2} \left( \int_0^t \|a_s\|^2 \|b_s\|^2 ds \right)^{\frac{1}{2}}. \tag{3}$$

2.4 Free Itô Formula and - Process

An important ingredient in the development of numerical methods for fSDEs and their convergence properties is a free analog of the Itô-formula ([9, Section 4], [18], [4], [17]).
In terms of stochastic integrals the stochastic product rule is given in [9, Theorem 4.1.2] and can simply be written in differential form as ([17])

\[ a dW_t b_t \cdot c_t dW_t = \phi (b_t c_t) a dW_t. \]  

(4)

In the important case \( a = c = d = 1 \) this yields the formal rules \( dW_t b(X_t) dW_t = \phi (b(X_t)) dt \) and \( dW_t dW_t = dt \). In the following we restrict ourselves to self-adjoint elements \( a, b, c, d \in \mathcal{A} \) and denote the set of self-adjoint elements of \( \mathcal{A} \) by \( \mathcal{A}^n \).

**Definition 2.3.** Let \( (W_t)_{t \geq 0} \) be a free Brownian motion and \( \mathbb{F} \) its natural filtration. An adapted mapping \( X_t : [0, T] \rightarrow \mathcal{A}^n \) is called a free Itô-process, if there are operator valued functions \( a, b, c : [0, T] \rightarrow \mathcal{A}^n \) and an element \( X_0 \in \mathcal{A}^n_0 \) so that

\[ X_t = X_0 + \int_0^t a(s) ds + \sum_{i=0}^k \int_0^t b_i(s) dW_i c_i^*(s). \]  

(5)

**Remark 2.** If \( X_0 \) is a self-adjoint element, for \( X_t \) to be self-adjoint, it is required that 

\( a(t) \) and the sum 

\[ S = \sum_{i=0}^k \int_0^t b_i(s) dW_i c_i^*(s) \]

is self-adjoint for each \( t \in [0, T] \).

A simple calculation shows, that the free Itô formula (4) (in integral form see [9, Theorem 4.1.2]) implies the following \( L_2(\varphi) \) isometry (\( \tau < t \)),

\[ \left\| \int_\tau^t b_s dW_s c_s \right\|_2 = \int_\tau^t \left\| c_s \right\|_2 \left\| b_s \right\|_2 ds. \]

Note that this equality implies that

\[ \left\| \int_\tau^t b_s dW_s c_s \right\|_2 = O(\sqrt{t - \tau}). \]

### 3 Free Stochastic Differential Equation (fSDE)

**Definition 3.1.** Let \( X_0 \) be a self-adjoint element in \( \mathcal{A}^n_0 \) and \( a, b', c' : \mathcal{A} \rightarrow \mathcal{A} \) continuous functions in the operator norm such that \( a(\mathcal{A}^n) \subset \mathcal{A}^n \). We call

\[ dX_t = a(X_t) dt + \sum_{i=0}^k b_i'(X_t) dW_i c_i(X_t) \]  

(6)

a (formal) free Stochastic Differential Equation (fSDE). A solution to equation (6) with initial condition \( X(0) = X_0 \) is a process \((X_t)_{t \geq 0}\) with the following properties:

1. \( X(0) = X_0 \) is a self-adjoint element in \( \mathcal{A}^n_0 \)
2. \( X_t \in \mathcal{A}^n_t \) for all \( t \geq 0 \)
3. The equation

\[ X_t = X_0 + \int_0^t a(X_s) ds + \sum_{i=0}^k \int_0^t b_i'(X_s) dW_i c_i(X_s) \]  

(7)

is fulfilled for all \( t \geq 0 \).

**Remark 3.** Due to the continuity of \( a, b', c' \) the integrals in equation (5) are well defined. It should be noted that these function can be taken from more general spaces (see [9]), but for our purposes the continuity requirement is necessary. We only consider the autonomous case, where \( a, b', c' \) do not explicitly depend on \( t \).

**Remark 4.** An existence and uniqueness theorem for fSDEs and several examples are given in [17]. These results rely on locally operator-Lipschitz functions \( a, b', c' \). The existence proofs in [17] can easily be formulated in \( L_2(\varphi) \) by applying section 2.4 instead of the free Burkholder-Gundy inequality.
As an initial example consider the free analog of the Ornstein-Uhlenbeck process ([17]) defined by the fSDE
\[ dX_t = \theta X_t dt + \sigma dW_t, \quad t \geq 0, \quad \theta, \sigma \in \mathbb{R}. \] (8)

Spectral information about the solution can be obtained by taking the Cauchy transform \( G \) of the self-adjoint element \( X_t \). \( G \) fulfills a deterministic partial differential equation ([17, Proposition 3.7]). Applying the Stieltjes inversion formula (see [17]) to their solution it is possible to recover the density of the distribution of \( X_t \). In the case \( \theta < 0 \) it turns out that the density of \( X_t \) is a semicircle distribution with radius
\[ R = \sqrt{\frac{2\sigma^2}{|\theta|}(1 - e^{-2|\theta|t})}. \]

For \( t \to \infty \) the density converges to a semicircle with radius \( \sigma \sqrt{\frac{2}{|\theta|}} \). The case \( \theta \geq 0 \) is treated in the same way. For more examples we refer to [17].

4 Free Itô-Formula in Functional Form

The proof of the free Itô formula in functional form [9, Proposition 4.3.4] is done by first formulating the Itô product rule for polynomials and then taking appropriate limits to operator valued functions with certain properties. Perturbation theory of operator valued functions has been intensely developed in the past decades ([24]). For functions with certain properties, which will be defined below, it is possible to give a Taylor approximation with appropriate remainder term [24, Chapter 5.4] and derive [9, Proposition 4.3.4] from such expansions. Let \( W_n(\mathbb{R}) \) be the set of functions \( f \in C^n(\mathbb{R}), \) such that the \( k \)-th derivative \( f^{(k)}, k = 0, \ldots, n \) is the Fourier transform of a finite measure \( m_f \) on \( \mathbb{R} \). At this point we apply the results in [5, Corollary 5.8] which allow to apply Taylor’s formula to \( f \in W_n(\mathbb{R}) \). Note that \( f \) can be taken from more general spaces (see remark 5), but for our case \( W_n(\mathbb{R}) \) is sufficient. We now derive the free Itô-formula. Consider \([r, t] \subseteq \mathbb{R}, r \geq 0 \) divided into \( n \) intervals. Write
\[ f(X_t) - f(X_r) = \sum_{k=0}^{n-1} f(X_{i+1}) - f(X_i) \] (9)

Applying the Taylor series expansion [5, Corollary 5.8], then for \( f \in W_3(\mathbb{R}) \) we obtain
\[ f(X_{i+1} - X_i) = T f^{[1]} (\Delta X, \Delta X) + T f^{[2]} (\Delta X, \Delta X, \Delta X) + O(\|\Delta X\|^3) \] (10)
writing \( \Delta X = X_{i+1} - X_i \). The definition of the multiple operator integrals \( T f^{[1]}, T f^{[2]} \) is given in [5, Definition 4.1] and [5, Lemma 4.5]. Substituting the process
\[ \Delta X = \int_{t_i}^{t_{i+1}} a(X_s)ds + \int_{t_i}^{t_{i+1}} b(X_s)dW_c(X_s) \]
and applying the product rule [9, Theorem 4.1.2] leads to a simplification of each of the integrals in equation (10). Due to the boundness of the corresponding operator valued functions \( a, b, c \) with and Burkholder-Gundy inequality ([9, Theorem 3.2.1]) the necessary limits can be easily justified. Since this way of deriving the free Itô formula is rather technical in notation, we do not follow this path further in detail. Just as in the classical case, in order to keep the notation as simple as possible, we do
apply differential notation instead and make use of the product rules (4). Applying [5, formula (14) and (15)] and \( dX \) apply differential notation instead and make use of the product rules (4). Applying where rule (4) we proceed by converting each of the derivatives in equation (11).

The multiple operator integrals are given according to [5, Definition 4.1] and [5, Lemma 4.5]. The second order derivative in equation (11) can be simplified by equation (4) to

\[
T_{f^{[1]}}X_0(a_1 dt) = \int_{\Pi^{[2]}} e^{i(s_0-t_1)X_0} a(X_t) e^{i\epsilon_1 X_0} d\nu_f(s_0, s_1) (12)
\]

and

\[
T_{f^{[2]}}X_0(b_1 dW_t c_t) = \int_{\Pi^{[2]}} e^{i(s_0-t_1)X_0} \cdot \int_0^1 b_s dW_s c_s \cdot e^{i\epsilon_2 X_0} d\nu_f(s_0, s_1). (13)
\]

are given according to [5, Definition 4.1] and [5, Lemma 4.5]. The second order derivative in equation (11) can be simplified by equation (4) to

\[
T_{f^{[3]}}X_0X_0(b_1 dW_t c_t, b_2 dW_t c_t) = \int_{\Pi^{[3]}} e^{i(s_0-t_1)X_0} b_1 dW_t c_t e^{i(s_1-t_2)X_0} b_2 dW_t c_t e^{i\epsilon_2 X_0} d\nu_f(s_0, s_1, s_2) =
\]

\[
\int_{\Pi^{[3]}} \varphi(c_t e^{i(s_1-t_2)X_0} b_1) e^{i(s_0-t_1)X_0} b_2 c_t e^{i\epsilon_2 X_0} dt d\nu_f(s_0, s_1, s_2). (14)
\]

Note that the last integral in equation (14) is no longer stochastic. Now we are able to formulate the

**Theorem 4.1** (Free Itô Formula in Integral Form). Suppose \( a, b, c \) are continuous functions \( \mathcal{A} \to \mathcal{A} \) in the operator norm such that \( a(\mathcal{A}^a) \subset \mathcal{A}^a, b(\mathcal{A}^a) \subset \mathcal{A}^a, c(\mathcal{A}^a) \subset \mathcal{A}^a \). Furthermore \( b, c \) are so that the product \( b(X_t) \) is self-adjoint (resp. the sum for \( k > 1 \)). Let \( (X_t)_{t \geq 0} \) be a free Itô-process and \( X_0 \in \mathcal{A}^a \) be a self-adjoint element. Then for functions \( f \in W_3(\mathbb{R}) \) it follows that

\[
f(X_t) = f(X_0) + \int_0^t L^0[f(X_s)] ds + L^1[f(X_s)]_0^t (15)
\]

where the operators \( L^0, L^1 : \mathcal{A}^a \to \mathcal{A}^a \) are introduced as an abbreviation for the expressions

\[
L^0[f(X_s)] = T_{f^{[1]}}X_0(a(X_s)) + T_{f^{[2]}}X_0X_0(b_s dW_s c_s, b_s dW_s c_s) (16)
\]

and

\[
L^1[f(X_s)]_0^t = T_{f^{[3]}}X_0X_0(b_1 dW_t c_t). (17)
\]

The operator integrals are given by equations (12) to (14).

**Remark 5.** The function \( f \) in theorem 4.1 can be taken from the Besov space \( \mathcal{B}^2_{a,1}(\mathbb{R}) \) for which \( W_n(\mathbb{R}) \subset \mathcal{B}^2_{a,1}(\mathbb{R}) \). For a definition of \( \mathcal{B}^2_{a,1}(\mathbb{R}) \) we refer to [24, pp. 9]. For the purpose of this paper it is sufficient to consider \( W_n(\mathbb{R}) \).
5 Free analog of Euler-Maruyama Method (fEMM)

We are now going to define a method for the numerical solution of the fSDE (6). For simplicity we assume $d = 1$ in the following. Consider the free Itô process (5) over the time interval of length $\Delta t$,

$$X_{t+\Delta t} = X_t + \int_t^{t+\Delta t} a(X_s)ds + \int_t^{t+\Delta t} b(X_s)dW_s, c(X_s).$$ (18)

Assuming $a, b, c \in W_3(\mathbb{R})$ we can apply the free Itô Formula (15) for $f = a, b, c$ in equation (18). This yields an iterated free Itô formula which allows to motivate and define a free analog of the Euler-Maruyama method. Using the abbreviations $a(X_t) = a_t$ (similar notation for $b, c$) and $t_1 = t + \Delta t$ we obtain

$$\int_t^{t_1} a_t ds + \int_t^{t_1} \int_s^{t_1} L^0[a_s]du ds + \int_t^{t_1} L^1[a_s]ds +$$

$$\int_t^{t_1} \left\{ \left( b_t + \int_t^s L^0[b_u]du + L^1[b_u]ds \right) \right\} dW_s \left( c_t + \int_t^s L^0[c_u]du + L^1[c_u]ds \right).$$ (19)

Since $a_t, b_t, c_t$ do not depend on the integration variable $s$, we rewrite equation (19) as

$$X_{t_1} - X_t = a_t \Delta t + b_t(W_{t_1} - W_t) + c_t + \rho.$$ (20)

with the remainder

$$\rho = \int_t^{t_1} \int_s^{t_1} L^0[a_s]du ds + \int_t^{t_1} L^1[a_s]ds +$$

$$+ \int_t^{t_1} b_t dW_s \left( \int_t^s L^0[c_u]du + L^1[c_u]ds \right) +$$

$$+ \int_t^{t_1} \int_t^s L^0[b_u]du \right\} dW_s \left( c_t + \int_t^s L^0[c_u]du + L^1[c_u]ds \right) +$$

$$+ \int_t^{t_1} \int_t^s L^1[b_u]du \right\} dW_s \left( c_t + \int_t^s L^0[c_u]du + L^1[c_u]ds \right).$$ (21)

By the boundedness and continuity of the involved functions $a, b, c$ the above integrals are well defined. In the case $d > 1$ we can simply put the sum-sign in front of each integral in $\rho$ which contains either $b$ or $c$. The free Euler-Maruyama method can now be motivated from equation (20) by simply skipping the remainder $\rho$.

**Definition 5.1 (fEMM).** Given $T > 0$, consider a partition of $[0, T]$ into $L \in \mathbb{N}$ intervals $[t_{k-1}, t_k], k = 1, \ldots, L$ with constant step size $\Delta t = \frac{T}{L}$. Define the one-step free Euler-Maruyama approximation (fEMM) $\overline{X}_k$ of the solution $X_t$ of equation (6) on $[0, T]$ by

$$\overline{X}_{k+1} = \overline{X}_k + a(\overline{X}_k)\Delta t + b(\overline{X}_k)\Delta W_k, c(\overline{X}_k), \ k = 0, 1, \ldots, L - 1$$ (22)

with starting value $X_0 = \overline{X}_0 \in \mathcal{A}^{\text{sa}}$ and $\Delta W_k = W_{k+1} - W_k$. $\overline{X}_k$ denotes the numerical approximation to $X_t$ at timepoint $t_k$.

In general the fSDE and fEMM act in a finite unital faithful von Neumann algebra $\mathcal{A}$. For the implementation on a computer it is necessary to consider fEMM in the von Neumann algebra $\mathcal{M}_{\mathcal{N}}^\vee(\mathbb{R})$ of random matrices (section 2.1). This leads to the situation shown in figure 1. Given a solution $X_t \in \mathcal{A}^{\text{sa}}$ of the fSDE (6) at $t = k\Delta t \in [0, T], k \in \{0, \ldots, L - 1\}$. Applying fEMM in $\mathcal{A}^{\text{sa}}$ we obtain an approximation $\overline{X}_k \in \mathcal{A}^{\text{sa}}$. Considering the implementation of fEMM on a computer, we obtain an approximation $\overline{X}_k \in \mathcal{M}_{\mathcal{N}}^\vee(\mathbb{R})$ to $\overline{X}_k$. To judge the quality of the approximation we
\[
X_t \in A^{sa} \xrightarrow{\Delta t \to 0} X_k \in A^{sa}
\]

\[
N \to \infty \quad N \to \infty
\]

\[
X^N_t \in \mathcal{M}_N^R(\mathbb{R}) \xrightarrow{\Delta t \to 0} X^N_k \in \mathcal{M}_N^R(\mathbb{R})
\]

Figure 1: Diagram of the approximation scheme of fEMM. Implementation of 5.1 is realized in \(\mathcal{M}_N^R(\mathbb{R})\) (bottom right), which gives an approximation to the solution \(X_t\) of equation (6) (top left). The limits \(N \to \infty\) according to the size of random matrices and the step size limit \(\Delta t\) do commute and give convergence of \(X^N_k\) in distribution to \(X_t \in A^{sa}\).

6 Convergence Results

This section gives two theorems regarding strong and weak convergence properties of fEMM. The results will be numerically verified in section 7.

6.1 Strong convergence of fEMM

**Definition 6.1.** The numerical approximation fEMM (definition 5.1) is said to converge strongly to the solution \(X_t\) of equation (6) with order \(p > 0\), if there is a constant \(C > 0\) independent of \(\Delta t\), so that

\[
\sup_{0 \leq t_k \leq T} \varphi \left( |X_k - X_k| \right) \leq C(\Delta t)^p.
\]

for any fixed time point \(t_k = k\Delta t \in [0, T]\), \(k = 0, \ldots, L\). \(X_k\) denotes the solution \(X_t\) evaluated at \(t_k\) and \(X_k = X(t_k)\). At \(t = 0\) we have \(X(0) = X_0 = X^N_0\).

**Theorem 6.1.** Consider the fSDE (6) and \(L_a > 0\). Let \(a: \mathcal{A} \to \mathcal{A}\) be an operator function with \(a(A^{sa}) \subset A^{sa}\). Additionally let the function \(a\) be operator Lipschitz in \(L_2(\varphi)\), i.e.

\[
\|a(X) - a(Y)\|_2 \leq L_a \|X - Y\|_2.
\]
for arbitrary elements $X, Y \in \mathcal{A}^{sa}$. Analog conditions hold for functions $b$ and $c$. Then the fEMM approximation (5.1) has strong convergence order of $p = \frac{1}{2}$, i.e.

$$\sup_{0 \leq t_k \leq T} \varphi \left( |\mathbf{X}_k - X_k| \right) \leq C(\Delta t)^{\frac{1}{2}}.$$  \hspace{1cm} (24)

The constant $C$ is independent of step size $\Delta t$.

**Remark 6.** In section 5 we mentioned, that for the implementation on a computer we use fEMM in $\mathcal{A}^{sa} = \mathcal{M}_N^{sa}(\mathbb{R})$. The definition of fEMM definition 5.1, the strong convergence property of $p = \frac{1}{2}$, and weak order of convergence $p = 1$ (see section section 6.2) can be directly carried over to the von Neumann algebra of $N \times N$ random matrices $\mathcal{M}_N^{sa}(\mathbb{R})$.

The proof of theorem 6.1 closely follows the proof of strong convergence of the Euler-Maruyama method for commutative stochastic differential equations, see [15]. The differences lie in estimating the free stochastic integrals in $L_2(\varphi)$ (see section 2.4).

**Proof of theorem 6.1.** From the fEMM approximation $\mathbf{X}_k$ at the time point $t_k$, $k = 1, \ldots, L$ we define a step process $\mathbf{X}(t) = \mathbf{X}_k$ for $t_{k-1} \leq t < t_k$. We use the short notation $\mathbf{X}(s) = \mathbf{X}_s$, $X(s) = X_s$, $a(X(s)) = a_s$, $a(\mathbf{X}(s)) = \mathbf{a}_s$.

Analog for $b, c$. Consider a point $t \in [0, T]$. Let $n_1 \in \mathbb{N}$ such that $t \in [t_{n_1}, t_{n_1+1}]$. Then

$$\mathbf{X}_1 - X_1 = \mathbf{X}_{n_1} - X_1 = \mathbf{X}_{n_1} - \left( X_0 + \int_{t_0}^{t_1} a(X_s)ds + \int_{t_0}^{t_1} b(X_s)dW_s, c(X_s) \right) = \sum_{k=0}^{n_1-1} (\mathbf{X}_{k+1} - \mathbf{X}_k) - \int_{t_0}^{t_1} a_s ds - \int_{t_0}^{t_1} b_s dW_s c_s = \sum_{k=0}^{n_1-1} \mathbf{a}_k \Delta t + \sum_{k=0}^{n_1-1} \mathbf{b}_k \Delta W_s \mathbf{c}_k - \int_{t_0}^{t_1} a_s ds - \int_{t_0}^{t_1} b_s dW_s c_s \hspace{1cm} (25)$$

Due to the definition of the step-wise process $\mathbf{X}(t)$ we can reformulate the terms $\mathbf{a}_k \Delta t$ and $\mathbf{b}_k dW_s \mathbf{c}_k \Delta t$ as an integrals as follows. We deduce

$$\mathbf{a}_k \Delta t = a(\mathbf{X}_k) \Delta t = a(\mathbf{X}(t_k)) \Delta t = \int_{t_k}^{t_{k+1}} a(X(s))ds = \int_{t_k}^{t_{k+1}} a_s ds$$

and

$$\mathbf{b}_k dW_s \mathbf{c}_k \Delta t = \int_{t_k}^{t_{k+1}} b(X(s))dW_s c(X(s))ds = \int_{t_k}^{t_{k+1}} b_s dW_s c_s .$$

Note that $\mathbf{a}_s, \mathbf{b}_s, \mathbf{c}_s$ are constant over $[t_k, t_{k+1}]$. Continuing from the last line of equation (25) we obtain

$$\mathbf{X}_t - X_t = \int_{t_0}^{t_{n_1}} \mathbf{a}_s ds + \int_{t_0}^{t_{n_1}} \mathbf{b}_s dW_s \mathbf{c}_s - \int_{t_0}^{t_{n_1}} a_s ds - \int_{t_0}^{t_{n_1}} b_s dW_s c_s$$

and further

$$\mathbf{X}_1 - X_1 = \int_{t_0}^{t_{n_1}} (\mathbf{a}_s - a_s) ds - \int_{t_0}^{t_{n_1}} a_s ds + \int_{t_0}^{t_{n_1}} \mathbf{b}_s dW_s \mathbf{c}_s - \int_{t_0}^{t_{n_1}} b_s dW_s c_s.$$
Then the square of the $L_2(\varphi)$-norm of the difference $\bar{X}_t - X_t$ is
\[
\varphi \left( |\bar{X}_t - X_t|^2 \right) = \varphi \left( \left| \int_0^{t_{nt}} (\bar{\pi}_s - a_s) \, ds - \int_0^{t} a_s \, ds + \int_0^{t_{nt}} \bar{b}_s dW_s \bar{\pi}_s - \int_0^{t_{nt}} b_s dW_s c_s + \int_0^{t} b_s dW_s c_s \right|^2 \right). \tag{26}
\]

By applying the inequality
\[
\|X_1 + X_2 + X_3 + X_4\|_2^2 \leq 4 \left( \|X_1\|_2^2 + \|X_2\|_2^2 + \|X_3\|_2^2 + \|X_4\|_2^2 \right)
\]
for $X_1, X_2, X_3, X_4 \in \mathcal{A}^n$, we deduce from equation (26)
\[
\varphi \left( |\bar{X}_t - X_t|^2 \right) \leq 4\varphi \left( \left| \int_0^{t_{nt}} (\bar{\pi}_s - a_s) \, ds \right|^2 \right) + 4\varphi \left( \left| \int_0^{t} a_s \, ds \right|^2 \right) +
4\varphi \left( \left| \int_0^{t_{nt}} \bar{b}_s dW_s \bar{\pi}_s - \int_0^{t_{nt}} b_s dW_s c_s \right|^2 \right) + 4\varphi \left( \left| \int_0^{t} b_s dW_s c_s \right|^2 \right). \tag{27}
\]

Using the abbreviation
\[
v(t) = \|\bar{X}_t - X_t\|_2^2
\]
and applying Jensen’s inequality it follows from equation (27) that
\[
v(t) \leq 4T \int_0^{t_{nt}} \varphi \left( |(\bar{\pi}_s - a_s)|^2 \right) \, ds + 4\Delta t \int_{t_{nt}}^{t} \varphi \left( |a_s|^2 \right) \, ds +
4\varphi \left( \left| \int_0^{t_{nt}} \bar{b}_s dW_s \bar{\pi}_s - \int_0^{t_{nt}} b_s dW_s c_s \right|^2 \right) + 4\varphi \left( \left| \int_0^{t} b_s dW_s c_s \right|^2 \right). \tag{28}
\]

Estimating the first integral in (28) gives
\[
\int_0^{t_{nt}} \varphi \left( |(\bar{\pi}_s - a_s)|^2 \right) \, ds = \int_0^{t_{nt}} \|\bar{\pi}_s - a_s\|_2^2 \, ds \leq \int_0^{t_{nt}} \bar{L}_a^2 \|\bar{X}_s - X_s\|_2^2 \, ds =
\leq \bar{L}_a^2 \int_0^{t_{nt}} \varphi \left( |\bar{X}_s - X_s|^2 \right) \, ds = \bar{L}_a^2 \int_0^{t_{nt}} v(s) \, ds. \tag{29}
\]

The second integral in (28) is an $O(\Delta t)$, since
\[
\int_{t_{nt}}^{t} \varphi \left( |a_s|^2 \right) \, ds = \int_{t_{nt}}^{t} \|a_s\|_2^2 \, ds \leq C_a \int_{t_{nt}}^{t} (1 + \|X_s\|_2^2) \, ds \leq
\leq C_1 (t - t_{nt}) \leq C_1 \Delta t. \tag{30}
\]

The constant $C_1 \in \mathbb{R}$ does not depend on $\Delta t$. The third integral in (28) is estimated as follows.
\[
\varphi \left( \left| \int_0^{t_{nt}} \bar{b}_s dW_s \bar{\pi}_s - \int_0^{t_{nt}} b_s dW_s c_s \right|^2 \right) =
\leq \varphi \left( \left| \int_0^{t_{nt}} (\bar{b}_s - b_s) dW_s \bar{\pi}_s \right|^2 \right) + \left| \int_0^{t_{nt}} b_s dW_s (\bar{\pi}_s - c_s) \right|^2 =
\leq \left\| \int_0^{t_{nt}} (\bar{b}_s - b_s) dW_s \bar{\pi}_s \right\|_2^2 + \left\| \int_0^{t_{nt}} b_s dW_s (\bar{\pi}_s - c_s) \right\|_2^2 \tag{31}
\]
Applying to the $L_2(\phi)$ isometry of the stochastic integral, the Lipschitz conditions on $a, b, c$ and the Cauchy-Schwarz inequality we continue from the last line of equation (31) to get

$$
\varphi \left( \int_0^{t_{n+1}} b_s dW_s - \int_0^{t_{n+1}} b_s dW_s c_s \right)^2 \leq K \int_0^{t_{n+1}} \| \cdot \|_2^2 ds = K \int_0^{t_{n+1}} v(s) ds.
$$

The constant $K$ depends on the Lipschitz constants $L_b, L_a$ and the $L_2(\phi)$ norm of $\|X(s)\|_2^2$ and $\|\cdot\|_2$. The constant $K$ does not depend on $\Delta t$. The last stochastic integral in (28) is handled by the $L_2(\phi)$ isometry of the stochastic integral, i.e.

$$
\varphi \left( \int_{t_{n+1}}^{t} b_s dW_s c_s \right)^2 = \int_{t_{n+1}}^{t} \| b_s \|_2^2 \| c_s \|_2^2 ds \leq C_3 \int_{t_{n+1}}^{t} (1 + \| X(s) \|_2^2)^2 ds \leq C_4 (t_{n+1} - t) \leq C_4 \Delta t.
$$

Again, since $\sup_{s \in [0, T]} \| X(s) \|_2 < \infty$ by definition, we have $C_4 < \infty$ and does not depend on $\Delta t$. Inserting (29), (30), (31), (33) into (28) yields

$$
v(t) \leq 4(TL_b^2 + K) \int_0^{t_{n+1}} v(s) ds + 4C_1L_a \Delta t^2 + 4C_4 \Delta t.
$$

For $\Delta t$ small enough $v(t)$ fulfills the inequality

$$
v(t) \leq D\Delta t + E \int_0^{t_{n+1}} v(s) ds.
$$

The Gronwall inequality implies

$$
v(t) \leq F\Delta t, F < \infty, t \in [0, T].
$$

The supremum of the $L_1(\phi)$-norm over $[0, T]$ of the error $\overline{X}(t) - X(t)$ is first estimated by

$$
\sup_{0 \leq s \leq T} \varphi(\overline{X}(s) - X(s)) \leq \sup_{0 \leq s \leq T} \varphi((\overline{X}(s) - X(s))^2)^\frac{1}{2} \leq \sqrt{T} \sqrt{\Delta t}.
$$

Since $\overline{X}(t_k) = \overline{X}_k$ for all $0 \leq t_k \leq T$ we have

$$
\sup_{0 \leq t_k \leq T} \varphi((\overline{X}_k - X_k)) \leq C \sqrt{\Delta t}.
$$

\[\square\]

### 6.2 Weak convergence of fEMM

The main content of this section is theorem 6.3, which states weak convergence of order $p = 1$ under certain assumptions on the coefficients functions $a, b, c$. First we give the definition of weak convergence in the context of fSDEs. To prove theorem 6.3 we need some preparatory statements. At first, lemma 6.1 states that the expectation of the remainder in equation (21) of the iterated Itô formula (20) is $O(\Delta t^4)$. This allows to formulate theorem 6.2, which states weak order $p = 2$ for one single fEMM step. It is then possible to take over the proof in [20, Theorem 2.2.1] to obtain the desired result of weak convergence order $p = 1$.

In the sequel, we use the abbreviations $\Delta = X_{t+\Delta t} - X_t$ and $\overline{\Delta} = \overline{X}_{t+\Delta t} - X_t$. Note that $\rho = X_{t+\Delta t} - \overline{X}_{t+\Delta t} = (X_{t+\Delta t} - X_t) - (\overline{X}_{t+\Delta t} - X_t) = \Delta - \overline{\Delta}$. 

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Definition 6.2. The numerical approximation \( f_{\text{EMM}} \) defined by (22) is said to converge weakly to the solution \( X_t \) of (6) with order \( p > 0 \), if there is a constant \( C_f > 0 \) independent of \( \Delta t \), so that for \( f \) from a sufficiently large class of functions

\[
\sup_{0 \leq t \leq T} |\varphi(f(X_t)) - \varphi(f(X_{\Delta}))| \leq C_f T \Delta^p \tag{34}
\]

as \( \Delta t \to 0 \).

Proof. Applying the trace \( \varphi \) to the iterated Itô formula (21) we have to consider in total 10 integrals (by resolving the brackets). All integrals in (21) except \( t_1 = t + \Delta t \)

\[
\varphi(p) = \varphi \left( \int_t^{t_1} \int_t^s L^0[a_u] du \, ds \right) \tag{36}
\]

are zero due to freeness property of the free Brownian motion and zero trace of the stochastic integral. We start using equation (16) and the definition of the multiple operator integrals equations (12) and (13) and proceed as

\[
\left| \varphi \left( \int_t^{t_1} \int_t^s L^0[a_u] du \, ds \right) \right| \leq \\
\leq \left| \varphi \left( \int_t^{t_1} \int_t^s \int_{\Pi} e^{i(x_0-s_1)X_1} a(X_u) e^{i\lambda_1 X_1} du_\alpha(s_0, s_1) du \, ds \right) \right| + \\
+ \left| \varphi \left( \int_{\Pi} e^{i(x_0-s_1)X_1} b_u c_u \varphi(c_u b_u) e^{i\lambda_2 X_1} du_\alpha(s_0, s_1, s_2) du \, ds \right) \right| = I_1 + I_2. \tag{37}
\]

Due to the freeness of the factors in the integrand of \( I_1 \) we obtain the estimation

\[
I_1 \leq \int_t^{t_1} \int_t^s \int_{\Pi} \left| \varphi \left( e^{i(x_0-s_1)X_1} a(X_u) e^{i\lambda_1 X_1} \right) \right| du_\alpha(s_0, s_1) du \, ds = \\
= \int_t^{t_1} \int_t^s \int_{\Pi} \left| \varphi \left( e^{i(x_0-s_1)X_1} \right) \varphi \left( a(X_u) \right) e^{i\lambda_1 X_1} \right| du_\alpha(s_0, s_1) du \, ds = \\
= \int_t^{t_1} \int_t^s \int_{\Pi} \left| \varphi \left( a(X_u) \right) \right| du_\alpha(s_0, s_1) \leq K_1 \Delta t^2.
\]

where \( K_1 \) independent of \( \Delta t \). The last equality follows because \( \left| \varphi \left( e^{i(x_0-s_1)X_1} \right) \right| = 1 \) and \( a(X_u) \) is uniformly bounded in \( A \). Furthermore \( (\Pi^{(2)}, \nu_u) \) is a finite measure space \((5)\). The second integral \( I_2 \) in the last line of equation (37) is estimated as

\[
\left| \varphi \left( \int_t^{t_1} \int_t^s e^{i(x_0-s_1)X_1} b_u c_u \varphi(c_u b_u) e^{i\lambda_2 X_1} du_\alpha(s_0, s_1, s_2) du \, ds \right) \right| \leq \\
\leq \int_{\Pi} \left| \varphi^2(b_u c_u) \right| du_\alpha(s_0, s_1, s_2) \leq K_2 \Delta t^2 \tag{38}
\]

The last inequality follows, since \( b_u, c_u \) are uniformly bounded in \( A \).

Now we are fully prepared to formulate and prove
Theorem 6.2. Consider one single step of fEMM (see ??) with start value $X_t \in \mathcal{A}^\omega$ at time point $t \in [0,T]$. Let $f \in W_2(\mathbb{R})$. If $a, b, c \in W_3(\mathbb{R})$ and uniformly bounded in $\mathcal{A}$, then one single fEMM step of size $\Delta t$ with starting value $X_t$ has weak convergence of order $p = 2$, i.e.,

$$|\varphi(f(X_{t+\Delta t})) - \varphi(f(X_t))| \leq K \Delta t^2.$$  

(39)

Proof. $\Delta = X_{t+\Delta t} - X_t$. $\Xi = X_{t+\Delta t} - X_t$. According to [5, Corollary 5.8] we develop $f$ into a Taylor Series for $\Delta$, resp. $\Xi$. Let $f \in W_2(\mathbb{R})$, then

$$f(X_{t+\Delta t}) = f(\Delta + X_t) = f(X_t) + T_{f[t]}^{X_t}(\Delta) + R_{\Delta}$$

and

$$f(\Xi_{t+\Delta t}) = f(\Xi + X_t) = f(X_t) + T_{f[t]}^{X_t}(\Xi) + R_{\Xi}.$$ 

Subtracting yields

$$f(X_{t+\Delta t}) - f(\Xi_{t+\Delta t}) = T_{f[t]}^{X_t}(\Delta - \Xi) + R_{\Delta} - R_{\Xi}.$$ 

For the remainder we choose the integral form (see [23, Theorem 1.43])

$$R_{\Delta} = \frac{1}{2} \int_0^1 (1-\tau)T_{f[t]}^{X_t+\tau \Delta}(\Delta, \Delta) d\tau$$

and analog for $\Xi$. Applying the definition of multiple operator integrals (see [5, Lemma 4.5]) and the trace $\varphi$ yields

$$|\varphi \left( T_{f[t]}^{X_t}(\Delta - \Xi) \right)| = |\int_{\Pi[2]} \varphi \left( e^{i(s_0 - s_1)X_t}(\Delta - \Xi)e^{i(s_1 - s_2)X_t} \right) d\nu(s_0, s_1)| =$$

$$\leq K \Delta t^2 \|m_{f[t]}\| = K_3 \Delta t^2, \quad (40)$$

due to freeness of the factors in the integrand. The last line follows by lemma 6.1 (for $f \in W_n(\mathbb{R})$ the measure $m_{f[t]}$ is finite). We turn to the remainder $R_{\Delta}$.

$$|\varphi(R_{\Delta}) = \left| \int_0^1 (1-\tau) \varphi \left( T_{f[t]}^{X_t+\tau \Delta}(\Delta, \Delta) \right) d\tau \right| =$$

$$= \left| \int_0^1 (1-\tau) \int_{\Pi[1]} \varphi \left( e^{i(s_0 - s_1)(X_t+\tau A)} \varphi(\Delta) \right) \varphi(\Delta) e^{i(s_2 - s_1)(X_t+\tau A)} d\nu(s_0, s_1) \right| \leq$$

$$\leq C_6 \int_0^1 (1-\tau) |\varphi(\Delta)|^2 d\tau \leq C_6 |\varphi(\Delta)|^2 =$$

$$= C_6 \left( \int_{\Delta t} a_t ds + \int_{\Delta t} b_t dW_s c_s \right)^2 = C_6 \left( \int_{\Delta t} a_t ds \right)^2 \leq$$

$$\leq C_7 \varphi \left( \int_{\Delta t} |a_t| ds \right)^2 = C_7 \left( \int_{\Delta t} \varphi(|a_t|) ds \right)^2 \leq C_8 \Delta t^2, \quad (41)$$

In similar consideration it follows that

$$|\varphi(R_{\Xi})| \leq C_9 \Delta t^2. \quad (42)$$

Collecting equations (40) to (42) reveals the statement. \qed
Theorem 6.3. Let $T > 0$ and consider the free Euler Maruyama Method (22) with starting value $X(0) = X_0 \in A^{\infty}$. Under the assumptions of theorem 6.2 the method (22) is weakly convergent with order $p = 1$, i.e.

$$\sup_{0 \leq t_s \leq T} |\varphi(f(X_s)) - \varphi(f(X_t))| \leq C_{f,T} \Delta t$$

(43)

for all functions $f \in W_2(\mathbb{R})$.

Proof. The proof copies from [20], Theorem 2.1. \hfill \Box

Remark 7. In the classical setting of commutative stochastic differential equations the weak order of convergence $p$ is valid for functions $f \in C^{2(p+1)}(\mathbb{R})$ ([20]). In the non-commutative setting for $p = 1$ we require $f \in W_2(\mathbb{R})$.

7 Examples

In this section we consider the numerical solution of several free differential equations taken from [17] and [14]. We compare the numerically determined spectral distribution with theoretical results and numerically verify strong and weak convergence properties of fEMM.

Before we start with examples it is necessary to note some details of the implementation of fEMM and the realization of the free Brownian motion. The implementation of fEMM requires generation of increments of the free Brownian motion. We first generate matrices $\Delta W_i$ with independent and standard normally distributed elements $\Delta W_i$. These matrices $\Delta W_i$ are interpreted as the increments $\Delta W_t$, $i = 1, \ldots, L$. Implementation of fEMM starts by dividing the interval $[0, T]$ into $L = 2^l$, $l \in \mathbb{N}$ intervals with stepsize $\Delta t = T/L$ ($T > 0$). A free Brownian motion $(W_t)_{t \geq 0}$ is then realized on each time point $t_i = i\Delta t$, $i = 0, \ldots, L$. The implementation of fEMM requires generation of increments of the free Brownian motion. We generate $L$ matrices $\Delta W_i = \sqrt{\frac{2}{N\Delta t}}(A + AT)$, $i = 1, \ldots, L$, where $A = (a_{ij})$ is an $N \times N$ Matrix with independent and standard normally distributed elements $a_{ij} = N(0,1)$. These matrices $\Delta W_i$ are interpreted as the increments $W(t_i) - W(t_{i-1})$, $i = 1, \ldots, L$ of the free Brownian motion $(W_t)_{t \geq 0}$ on the interval $[t_{i-1}, t_i]$, $i = 1, \ldots, L$. The increments $\Delta W_i$ are free from each other and have variance $\Delta t$. To determine the order of strong convergence of fEMM numerically, we first generate $M \in \mathbb{N}$ number of paths and then evaluate the $L_1(\varphi)$-norm of $X_L^N - X_T^N$ for each path at the end point $T = L\Delta t > 0$. Calculating the expectation over the number $M > 0$ of paths by $\mathbb{E}\left(\frac{\text{tr}\left([X_L^N - X_T^N]\right)}{N}\right)$, this value is taken as an approximation to the strong error defined by definition 6.1 on matrix level $M_{\infty}(\mathbb{R})$. To overcome the limitation that the exact $M_{\infty}(\mathbb{R})$-valued solution $X_T^N$ (as $\Delta t \to 0$) is in general unknown, we choose a minimal time step $\Delta t_{\text{min}}$ and calculate $X_L^N (T = L\Delta t_{\text{min}})$, where the free Brownian motion realized with increments $\Delta W_i^{\text{min}}$ of variance $\Delta t_{\text{min}}$. Then we take $X_L^N$ as an approximation to the unknown matrix-valued solution $X_T^N$. To check the strong convergence properties we choose larger time steps $\Delta t_R = R\Delta t_{\text{min}}$, with $L/R \in \mathbb{N}$ and $R < L$ and employ fEMM with a corresponding free Brownian motion generated by the increments $\Delta W_j^R = \sum_{i=jR}^{(j+1)R} \Delta W_i^{\text{min}}$, $j = 1, \ldots, L/R$. Since the increments $\Delta W_i^{\text{min}}$ are free, the variance of $\Delta W_j^R$ sum up to $\Delta t_R$. The expression

$$e_s(\Delta t) = \mathbb{E}\left(\frac{\text{tr}\left([X_L^N/R - X_L^N]\right)}{N}\right)$$

(44)

is then taken as the strong error at time point $T = L\Delta t_{\text{min}} = \Delta t_R L/R$. The weak error (43) is numerically evaluated for $f = \text{id}$ by

$$e_w(\Delta t) = \mathbb{E}\left(\varphi\left(X_T^N\right) - \varphi(X_T)\right).$$

(45)
Note that on matrix level we have $\varphi\left(\mathbf{X}_T\right) = \text{tr}(\mathbf{X}_T)/N$. The equations considered in the following allow the exact calculation of $\varphi(X_T)$ (for $X_t \in \mathcal{A}$ that is, including $N \to \infty$).

### 7.1 Free Ornstein-Uhlenbeck Equation

We start with the free variant of the Ornstein-Uhlenbeck equation

$$dX_t = \theta X_t dt + \sigma dW_t, \quad X_0 = 0, \ t \geq 0$$

(46)

where $\theta, \sigma \in \mathbb{R}$. This equation was studied analytically in [17] by deriving and solving a partial differential equations for the Cauchy transform of $X_t$. It turns out that the solution $X_t$ is semicircle at each time point $t \geq 0$. For $\theta > 0$ the time dependent radius is given by

$$R(t) = \sqrt{2\sigma^2/\theta \left(e^{2\theta t} - 1\right)}.$$

For the cases $\theta \leq 0$ we refer to [17].

Figure 2 shows the empirical probability density function of the eigenvalues of $X_{500}$ calculated by fEMM for $T = 1$ with a time step $\Delta t = 2^{-10}$ and matrix size of $N = 500$. The red line in figure 2 shows the semicircle distribution for the case $N \to \infty$. In figure 3 shows strong and weak convergence properties of fEMM applied to the free Ornstein-Uhlenbeck equation (46) at $T = 1$.

![Figure 2](image2.png)

Figure 2: Distribution of the eigenvalues of the solution $X_L^N$ at $T = 1$ of equation (46) for $\theta = \sigma = 1$, $L = 1024$ and $N = 500$. The exact solution $X_T$ at $T = 1$ is semicircle with $R \approx 3.575$.

![Figure 3](image3.png)

Figure 3: Strong and weak convergence properties of fEMM applied to the free Ornstein-Uhlenbeck equation (46) at $T = 1$.

As discussed above we employ a minimum time step of $\Delta t_{\text{min}} = 2^{-16}$ and calculate the strong error as $e^{OU}_{\theta}(\Delta t) = \mathbb{E}\left(\text{tr}\left(\mathbf{X}_T^N/R - \mathbf{X}_T^N\right)\right)/N$ for $L = 2^{16}$ ($T = 1$) and $\Delta t = R\Delta t_{\text{min}}$ with $R = 6,8,10$. Figure 3a shows strong convergence order of $p = 1$. This is not a contradiction to the expected value of $p = 0.5$. If the coefficients $b,c$ of the free Brownian motion in the fSDE are constant, the fEMM shows a higher convergence order. This is an analog to the commutative case ([15]). Considering weak convergence of fEMM applied to equation (46) is shown in figure 3b. The weak error (43) is numerically evaluated for $f = id$ by $e^{OU}_{\omega}(\Delta t) = \mathbb{E}\left(\varphi\left(\mathbf{X}_T^N\right)\right)|$ with $L = 2^{12}$.
Note that on matrix level we have $\varphi \left( X^N_L \right) = \text{tr}(X^N_L)/N$ and $\varphi(X_T) = 0$, since the eigenvalue distribution of the solution $X_T$ of equation (46) is a centered semicircle distribution. The numerically estimated convergence order corresponds very well the theoretical value of $p = 1$.

### 7.2 Geometric Brownian Motion I

Suppose that $X_t \in A$ satisfies the following equation

$$dX_t = \theta X_t dt + X_t^\frac{1}{2} dW_t, X_0 = I.$$  \hfill (47)

In [17, Proposition 3.8] it is stated that the spectral distribution of $X_t$ is supported on the interval $[I_1(t), I_2(t)]$, where $I_i(t) = r_i(t) \exp(\theta - 1 - r_i(t))$, $r_i(t) = -1 \pm \sqrt{1 + 4/t}$, $i = 1, 2$. By applying the trace $\varphi$ to equation (47) it follows that $\varphi(X_t) = e^{\theta t}$. The variance of the spectral distribution is $te^{2\theta t}$ and the ratio of the standard deviation to the expectation of $X_t$ is $\sqrt{t}$ ([17]).

Figure 4 shows the empirical spectral distribution of $X^N_t$ of equation (47) approximated by fEMM at different time points. The red line is the spectral distribution of the exact solution $X_t$ recovered from it’s Cauchy transform.

on the interval $[I^1(t), I^2(t)]$, where $I^i(t) = \frac{r_i(t)}{1 + r_i(t)} e^{(\theta - 1 - r_i(t))}$, $r_i(t) = \frac{-1 \pm \sqrt{1 + 4/t}}{2}$, $i = 1, 2$. By applying the trace $\varphi$ to equation (47) it follows that $\varphi(X_t) = e^{\theta t}$. The variance of the spectral distribution is $te^{2\theta t}$ and the ratio of the standard deviation to the expectation of $X_t$ is $\sqrt{t}$ ([17]).

Figure 4 shows the empirical spectral distribution of $X^N_t$ for $N = 100$, $\Delta t = 2^{-10}$ and $\theta = 1$ and different time points. The red line in Figure 4 is the recovery of the spectral distribution for $N \to \infty$ (see [17]).

Figure 5 shows that the time development of the supporting interval of the spectral distribution of $X^N_t$ correspond very well to the theoretical values given by $I^1(t)$ and $I^2(t)$.

Figure 5: Comparison of boundaries of support interval $[R1 = I^1(t), R2 = I^2(t)]$ of the density of spectral distribution of equation (47) for $\theta = 1$, $N = 100$.

Strong and weak convergence properties of fEMM applied to equation (47) is shown in figure 6. The graph in figure 6a shows the approximation of the strong error $\epsilon^\text{GeoI}_s(\Delta t) = \mathbb{E} \left( \text{tr} \left( \frac{X^N_L - X^N_L_{-R}}{N} \right) /N \right)$, where $X^N_L$ is calculated by a minimal time step of $\Delta t = 2^{-16}$ and $X^N_{L/R}$ by time steps $R\Delta t_{\text{min}}$ with $R = 6, 8, 10, 12$. The expected
value is determined over $M = 250$ different paths. The slope of the straight line in figure 6a shows numerically convergence order of $p = 0.5$ which is in accordance to Theorem theorem 6.1. fEMM applied to equation (47) shows weak convergence order $p = 1$ at $T = 1$ as shown in figure 6b. The weak error is calculated on matrix level as $e_{\text{GeoI}}(\Delta t) = \frac{\mathbb{E} \left( \text{tr} \left( \overline{X}_M \right) \right)}{N - e^{\theta T}}$ for 6 different time steps $\Delta t = R/L \ (T = 1)$ for $L = 2^{12}$ and $R = 1, 2, 4, 8, 16, 32$. Again, we have good correspondence between numerical and theoretical results.

7.3 Free CIR-Process

Consider the equation
\[ dX_t = (a - bX_t)dt + \sigma \sqrt{X_t} dW_t + \sigma dW_t \sqrt{X_t}, \quad X_0 = I, \quad (48) \]
where $a, b, \sigma > 0$ are such that $2a^2 \geq \sigma^2$ (see [14]). The expected value can be calculated by applying $\varphi$ to equation (48) which gives $\varphi(X_t) = e^{-bt} (b + a(e^{bt} - 1))$. So far, no further spectral properties are known. The time development the spectral distribution of the numerical solution is shown in figure 7. Strong and weak convergence is shown in figure 8.

8 Conclusions

In this paper we developed a free analog of the well known Euler-Maruyama method. Up to the knowledge of the author the numerical treatment of fSDEs are considered for
the first time. From Taylor series expansion of operator valued functions we derived an iterated free Itô formula. This offers to motivate and define the free Euler-Maruyama method (fEMM) and proved strong and weak convergence properties. We considered the implementation of the method by approximating the elements in von Neumann algebras by self-adjoint random matrices. The method recovers well known analytical results and convergence properties where numerically verified.

A Pointwise Convergence of the Cauchy transform

Lemma A.1. Let $(X_k)$ be a sequence in a von Neumann Algebra $A^{sa}$. Assume that $(X_k)$ converges in the $L_1(\varphi)$-norm to $X \in A^{sa}$. Then the sequence $(G_k)$ of Cauchy transforms of $X_k$ converges on $\mathbb{C}^+$ pointwise to the Cauchy transform $G_X$.

Proof. Let $z \in \mathbb{C}^+$. Since $X_k \in A^{sa}$ it follows that $\| (z - X_k)^{-1} \| \leq \frac{1}{\Im(z)}$ by applying functional calculus to the normal element $(z - X_k)^{-1}$. The same holds for $X$. The statement follows by the estimation $|\varphi((z - X_k)^{-1}) - \varphi((z - X)^{-1})| \leq \|X_k - X\|_1 \| (z - X_k)^{-1} \| \| (z - X)^{-1} \|$. □

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