RADIUS OF STARLIKENESS FOR SOME CLASSES CONTAINING NON-UNIVALENT FUNCTIONS

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Abstract. A starlike univalent function $f$ is characterized by the function $zf'(z)/f(z)$; several subclasses of these functions were studied in the past by restricting the function $zf'(z)/f(z)$ to take values in a region $\Omega$ on the right-half plane, or, equivalently, by requiring the function $zf'(z)/f(z)$ to be subordinate to the corresponding mapping of the unit disk $\mathbb{D}$ to the region $\Omega$. The mappings $w_1(z) := z + \sqrt{1 + z^2}, w_2(z) := \sqrt{1 + z}$ and $w_3(z) := e^z$ maps the unit disk $\mathbb{D}$ to various regions in the right half plane. For normalized analytic functions $f$ satisfying the conditions that $f(z)/g(z), g(z)/zp(z)$ and $p(z)$ are subordinate to the functions $w_i, i = 1, 2, 3$ in various ways for some analytic functions $g(z)$ and $p(z)$, we determine the sharp radius for them to belong to various subclasses of starlike functions.

1. Introduction

Though complex numbers is not ordered field, the inequalities in the real line can be extended to complex plane in a natural way using the concept of subordination. Let $f, g$ be two analytic functions defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The function $f$ is subordinate to the function $g$, written $f \prec g$, if $f = g \circ w$ for some analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ (and such a function $w$ is known as a Schwartz function). A univalent function $f : \mathbb{D} \to \mathbb{C}$ is always locally univalent or, in other words, it has non-vanishing derivative. Therefore, the study of univalent functions can be restricted to the functions normalized by $f(0) = f'(0) - 1 = 0$. We let $\mathcal{A}$ denote the class of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Since $f'(0) \neq 0$ for functions $f \in \mathcal{A}$, the functions in the class $\mathcal{A}$ are univalent in some disk centred at the origin. The largest disk centered at the origin in which $f$ is univalent is called as the radius of univalence of the function $f$. Consider a subset $\mathcal{M} \subset \mathcal{A}$ and a property $P$ (such as univalence, starlikeness, convexity) that the image of the functions in $\mathcal{M}$ may or may not have. It often happens that the image of $\mathbb{D}_r = \{z \in \mathbb{D} : |z| < r\}$ for some $r \leq 1$ has the property $P$; the largest $\rho_f$ such that the functions has the property $P$ in each disk $\mathbb{D}_r$ for $0 < r \leq \rho_f$ is the radius of the property $P$ of the function $f$. The number $\rho := \inf \{\rho_f : f \in \mathcal{M}\}$ is the radius of the property $P$ of the class $\mathcal{M}$; if $\mathcal{G}$ is the class of all $f \in \mathcal{A}$ characterized by the property $P$, then $\rho$ is called the $\mathcal{G}$-radius of the class $\mathcal{M}$. The $\mathcal{G}$-radius of the class $\mathcal{M}$ is denoted by $R_\mathcal{G}(\mathcal{M})$ or simply by $R_\mathcal{G}$ if the class $\mathcal{M}$ is clear from the context.

The class $\mathcal{M}$ we are interested in is characterized by the ratio between functions $f$ and $g$ belonging to $\mathcal{A}$; several authors [1, 2, 6, 9, 10, 11, 15, 17] have studied such classes. The
classes which we are considering are as follows.

\[
\mathcal{T}_1 = \{ f \in \mathcal{A} : \frac{f}{g} \prec e^z, \frac{g}{zp} \prec e^z \text{ for some } g \in \mathcal{A} \text{ and } p < \sqrt{1+z} \}
\]

\[
\mathcal{T}_2 = \{ f \in \mathcal{A} : \frac{f}{g} \prec \sqrt{1+z}, \frac{g}{zp} \prec \sqrt{1+z} \text{ for some } g \in \mathcal{A} \text{ and } p < e^z \}
\]

\[
\mathcal{T}_3 = \{ f \in \mathcal{A} : \frac{f}{g} \prec z + \sqrt{1+z^2}, \frac{g}{zp} \prec z + \sqrt{1+z^2} \text{ for some } g \in \mathcal{A} \text{ and } p < z + \sqrt{1+z^2} \}.
\]

These classes are motivated by a recent work of Ali, Sharma and Ravichandran [?] wherein similar classes were investigated. These classes contain non-univalent functions and this makes the study of such functions interesting. We compute \( G \)-radius when \( G \) is one of the following subclasses of starlike functions studied recently in the literature. A starlike univalent function \( f \) is characterized by the condition \( \text{Re}(zf'(z)/f(z)) > 0 \). If we define the class \( \mathcal{P} \) as the class of all functions \( p(z) = 1 + c_1z + \cdots \) defined on \( \mathbb{D} \) satisfying \( \text{Re}p(z) > 0 \), it follows that a function \( f \) is starlike and only if \( zf'(z)/f(z) \in \mathcal{P} \). Several subclasses of starlike functions were studied in the past by restricting the function \( zf'(z)/f(z) \) to take values in a region \( \Omega \) on the right-half plane, or, equivalently, by requiring the function \( zf'(z)/f(z) \) to be subordinate to the corresponding mapping \( \varphi : \mathbb{D} \to \Omega : zf'(z)/f(z) \prec \varphi(z) \). The functions \( \varphi \) defined by \( \varphi(z) := (1 + Az)/(1 + Bz) \), with \(-1 < B < A < 1, e^z, 1 + (4/3)z + (2/3)z^2, 1 + \sin z, z + \sqrt{1+z^2}, 1 + (zk + z^2)/(k^2 - kz) \) where \( k = \sqrt{2} + 1, 1 + (2\log((1 + \sqrt{z})/(1 - \sqrt{z}))^2\pi^2), 2/(1 + e^{-z}) \) and \( 1 + z - z^3/3 \), we denote the class of all functions \( f \in \mathcal{A} \) with \( zf'(z)/f(z) \prec \varphi(z) \) respectively by \( S^*_{[A,B]}, S^*_{\exp}, S^*_{\sin}, S^*_{\cos}, S^*_{R}, S^*_{p}, S^*_{SG} \) and \( S^*_{Ne} \).

2. The radius of univalence

2.1. Class \( \mathcal{T}_1 \). This class consists of the analytic functions \( f \) such that \( f(z)/g(z) \) is subordinate to \( e^z \) and \( p \) is subordinate to \( \sqrt{1+z} \) for some analytic function \( g \). This class is non-empty as the function \( f_1 : \mathbb{D} \to \mathbb{C} \), defined by

\[
f_1(z) = ze^{2z}\sqrt{1+z}
\]

belongs to the class \( \mathcal{T}_1 \). The function \( f_1 \) satisfies the subordination condition for this class along with the functions \( g_1, p_1 : \mathbb{D} \to \mathbb{C} \) by

\[
g_1(z) = ze^{2z}\sqrt{1+z} \quad \text{and} \quad p_1(z) = \sqrt{1+z}.
\]

The function \( f_1 \) plays the role of extremal for this class. Since

\[
f_1'(z) = \frac{e^{2z}(4z^2 + 7z + 2)}{2\sqrt{1+z}},
\]

it is clear that \( f'(-7 + \sqrt{17}/8) = 0 \), and so the radius of univalence is at most \((-7 + \sqrt{17})/8\). As the function \( f_1 \) is not univalent in \( \mathbb{D} \), the class \( \mathcal{T}_1 \) contains non-univalent functions. Here also we find that radius of univalence and the radius of starlikeness for \( \mathcal{T}_1 \) are same and is \((-7 + \sqrt{17})/8 \approx -0.359612 \).
To do this, we need to first find a disk into which the disk $D_r$ is mapped by the function $f \in \mathcal{T}_1$. Let $f \in \mathcal{T}_1$ and define the functions $p_1, p_2$ by $p_1(z) = f(z)/g(z)$ and $p_2(z) = g(z)/zp(z)$. Then $f(z) = zp(z)p_1(z)p_2(z)$ and
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{zp'(z)}{p(z)} \right| + \left| \frac{zp'_1(z)}{p_1(z)} \right| + \left| \frac{zp'_2(z)}{p_2(z)} \right|.
\]
Using the bounds for $p, p_1, p_2$, we obtain
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \begin{cases} \frac{r(5-4r)}{2(1-r)}; & r \leq \sqrt{2} - 1 \\ \frac{3r^3+7r^2+3r^4}{2(1-r^2)}; & r \geq \sqrt{2} - 1. \end{cases}
\]
for each function $f \in \mathcal{T}_1$. Clearly, for $|z| = r \leq (-7 + \sqrt{17})/8$, we have
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{r(5-4r)}{2(1-r)} \leq 1.
\]
This shows that the radius of starlikeness is at least $(-7 + \sqrt{17})/8$. Since the radius of univalence is atmost $(-7 + \sqrt{17})/8$, it follows that the radius univalence and radius of starlikeness are both equal to $(-7 + \sqrt{17})/8$.

2.2. Class $\mathcal{T}_2$. Functions of this class are analytic functions $f$ satisfying the subordinations
\[
\frac{f(z)}{g(z)} \prec \sqrt{1+z}, \quad \frac{g(z)}{zp(z)} \prec \sqrt{1+z} \quad \text{and} \quad p(z) \prec e^z,
\]
where the functions $g \in \mathcal{A}$ and $p \in \mathcal{P}$.

This class is non-empty as the function $f_2 : D \to \mathbb{C}$ defined by
\[
f_2(z) = z(1+z)e^z
\]
belongs to the class $\mathcal{T}_2$. The function $f_2$ satisfies the required subordinations when we define the functions $g_2, p_2 : D \to \mathbb{C}$ by
\[
g_2(z) = ze^z\sqrt{1+z} \quad \text{and} \quad p_2(z) = e^z.
\]
This function $f_2$ plays the role of extremal for this class. Since
\[
f_2'(z) = e^z(1 + 3z + z^2),
\]
it is clear that $f_2'(-3 + \sqrt{5}/2) = 0$, and so the radius of univalence is atmost $(-3 + \sqrt{5})/2$. We shall show that the radius of univalence and the radius of starlikeness for $\mathcal{T}_2$ are equal and common value of the radius is precisely $(-3 + \sqrt{5})/2 \approx -0.381966$.

To do this, we need to first find a disk into which the disk $D_r$ is mapped by the function $f \in \mathcal{T}_2$. Let $f \in \mathcal{T}_2$ and define the functions $p_1, p_2$ by $p_1(z) = f(z)/g(z)$ and $p_2(z) = g(z)/zp(z)$. Then $f(z) = zp(z)p_1(z)p_2(z)$ and
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{zp'(z)}{p(z)} \right| + \left| \frac{zp'_1(z)}{p_1(z)} \right| + \left| \frac{zp'_2(z)}{p_2(z)} \right|.
\]
Using the bounds for $p, p_1, p_2$, we obtain
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \begin{cases} \frac{r(-2+r)}{1-1+r}; & r \leq \sqrt{2} - 1 \\ \frac{4r^3+6r^2+3r^4}{4(1-r^2)}; & r \geq \sqrt{2} - 1. \end{cases}
\]
for each function $f \in \mathcal{T}_2$. Clearly, for $|z| = r \leq (-3 + \sqrt{5})/2$, we have
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{r(-2+r)}{1-1+r} \leq 1.
\]
The function \( f \) is clear that \( f \) \( \geq \). Class 2.3. are both equal to \( ( -3 + \sqrt{5} ) / 2 \). Since the radius of univalence is atmost \( ( -3 + \sqrt{5} ) / 2 \), it follows that the radius univalence and radius of starlikeness are both equal to \( ( -3 + \sqrt{5} ) / 2 \).

2.3. Class \( T_3 \). Recall that a function \( f \in A \) belongs the class \( T_3 \) if there are functions \( g \in A \) and \( p \in P \) satisfying the subordinations

\[
\frac{f(z)}{g(z)} < z + \sqrt{1 + z^2}, \quad \frac{g(z)}{zp(z)} < z + \sqrt{1 + z^2} \quad \text{and} \quad p(z) < z + \sqrt{1 + z^2}.
\]

This class is non-empty as the function \( f_3 : \mathbb{D} \to \mathbb{C} \) defined by

\[
f_3(z) = (z + \sqrt{1 + z^2})^3
\]

belongs to the class \( T_3 \). Indeed, the function \( f_3 \) satisfies the required subordinations when we define the functions \( g_3, p_3 : \mathbb{D} \to \mathbb{C} \) by

\[
g_3(z) = (z + \sqrt{1 + z^2})^2 \quad \text{and} \quad p_3(z) = z + \sqrt{1 + z^2}.
\]

The function \( f_3 \) plays the role of extremal for this class. Since

\[
f'_3(z) = \frac{(z + \sqrt{1 + z^2})^3 (3z + \sqrt{1 + z^2})}{\sqrt{1 + z^2}},
\]

it is clear that \( f'_3(-1/\sqrt{5}) = 0 \), and so the radius of univalence is at most \( 1/\sqrt{5} \). Also, since the function \( f_3 \) is not univalent in \( \mathbb{D} \), the class \( T_3 \) contains non-univalent functions. Indeed, we shall show that the radius of univalence and the radius of starlikeness for \( T_3 \) are equal and the common value of the radius is precisely \( 1/\sqrt{8} \approx 0.353553 \).

To do this, we need to first find a disk into which the disk \( \mathbb{D} \), is mapped by the function \( f \in T_3 \). Let \( f \in T_3 \) and define the functions \( p_1, p_2 \) by \( p_1(z) = f(z)/g(z) \) and \( p_2(z) = g(z)/zp(z) \). Then \( f(z) = zp(z)p_1(z)p_2(z) \) and

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{zp'(z)}{p(z)} \right| + \left| \frac{z p'_1(z)}{p_1(z)} \right| + \left| \frac{z p'_2(z)}{p_2(z)} \right|. \tag{2.5}
\]

For \( p \in P \) with \( p(z) < z + \sqrt{1 + z^2} \), Afis and Noor [14] have shown that

\[
|p(z)| \leq r + \sqrt{1 + r^2}, \quad \left| \frac{zp'(z)}{p(z)} \right| \leq \frac{r}{\sqrt{1 + r^2}} \quad (|z| \leq r).
\]

Using these inequalities for \( p, p_1, p_2 < z + \sqrt{1 + z^2} \), we see that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{3r}{\sqrt{1 + r^2}}, \quad (|z| \leq r, \tag{2.6}
\]

for each function \( f \in T_3 \). Clearly, for \( |z| = r \leq 1/\sqrt{8} \), we have

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{3r}{\sqrt{1 + r^2}} \leq 1.
\]
This shows that the radius of starlikeness is at least $1/\sqrt{8}$. Since the radius of univalence is at most $1/\sqrt{8}$, it follows that the radius of univalence and the radius of starlikeness are both equal to $1/\sqrt{8}$.

3. Radius of starlikeness

Our first theorem gives the sharp radius of starlikeness of order $\alpha$ of the classes $T_1$, $T_2$ and $T_3$. We shall show that this radius is the same for the subclass $S_1^\ast$ consisting of all functions $f \in S^\ast(\alpha)$ satisfying $|zf'(z)/f(z) - 1| < 1 - \alpha$.

**Theorem 3.1.** The following results hold for the classes $S^\ast(\alpha)$ and $S_1^\ast$.

(i) $R_{S(\alpha)}(T_1) = R_{S_1^\ast}(T_1) = (7 - 2\alpha - \sqrt{17 + 4\alpha + 4\alpha^2})/8$

(ii) $R_{S(\alpha)}(T_2) = R_{S_1^\ast}(T_2) = (3 - \alpha - \sqrt{5 - 2\alpha + \alpha^2})/2$

(iii) $R_{S(\alpha)}(T_3) = R_{S_1^\ast}(T_3) = (1 - \alpha)/(\sqrt{8 + 2\alpha - \alpha^2})$

**Proof.** (i) The function defined by $m(r) = (4r^2 - 7r + 2)/2(1 - r), 0 \leq r < 1$ is a decreasing function. Let $\rho = R_{S_1^\ast}(T_1)$ is the root of the equation $m(r) = \alpha$. From (2.2), it follows that

$$\Re \frac{zf'(z)}{f(z)} \geq \frac{4r^2 - 7r + 2}{2(1 - r)} = m(r) \geq m(\rho) = \alpha.$$ 

This shows that $R_{S_1^\ast}(T_1)$ is at least $\rho$. At $z = -R_{S_1^\ast}(T_1) = -\rho$, the function $f_1$ satisfies

$$\frac{zf_1'(z)}{f_1(z)} = \frac{4\rho^2 - 7\rho + 2}{2(1 - \rho)} = \alpha.$$

Thus the result is sharp.

Also,

$$\left|\frac{zf_1'(z)}{f_1(z)} - 1\right| = 1 - \alpha.$$

This proves that the radii $R_{S(\alpha)}(T_1)$ and $R_{S_1^\ast}(T_1)$ are same.

(ii) The function defined by $n(r) = (r^2 - 3r + 1)/(1 - r), 0 \leq r < 1$ is a decreasing function. Let $\rho = R_{S_1^\ast}(T_2)$ is the root of the equation $n(r) = \alpha$. From (2.4), it follows that

$$\Re \frac{zf'(z)}{f(z)} \geq \frac{1 - 3r + r^2}{1 - r} = n(r) \geq n(\rho) = \alpha.$$ 

This shows that $R_{S_1^\ast}(T_2)$ is at least $\rho$. At $z = -R_{S_1^\ast}(T_2) = -\rho$, the function $f_2$ satisfies

$$\frac{zf_2'(z)}{f_2(z)} = \frac{1 - 3\rho + \rho^2}{1 - \rho} = \alpha.$$

Also, for the function $f_2$ we have

$$\left|\frac{zf_2'(z)}{f_2(z)} - 1\right| = 1 - \alpha.$$

Hence, the radii $R_{S(\alpha)}$ and $R_{S_1^\ast}$ are same for the class $T_2$.

(iii) The function $s(r) = 1 - 3r/\sqrt{1 + r^2}, 0 \leq r < 1$ is a decreasing function. Let $\rho = R_{S_1^\ast}(T_3)$ is the root of the equation $s(r) = \alpha$. From (2.6), it follows

$$\left|\frac{zf_3'(z)}{f_3(z)} - 1\right| = 1 - \alpha.$$
Corollary 3.2. The following results holds for the class $S_p^*$. 

(i) $R_{S_p^*}(T_1) = (3 - \sqrt{5})/4 \approx 0.190983$

(ii) $R_{S_p^*}(T_2) = (5 - \sqrt{17})/4 \approx 0.219224$

(iii) $R_{S_p^*}(T_3) = 1/\sqrt{35} \approx 0.169031$

Proof. In equation (3.1), putting $a = 1$ gives $S_{1/2}^* \subset S_p^*$. Every parabolic starlike function is also starlike function of order $1/2$, whence the inclusion $S_{1/2}^* \subset S_p^* \subset S^*(1/2)$. Therefore, for any class $F$, readily $R_{S_{1/2}^*}(F) \leq R_{S_p^*}(F) \leq R_{S^*(1/2)}(F)$.

When $F = T_i, i = 1, 2, 3$, Theorem 3.1 gives $R_{S^*(\alpha)}(T_i) = R_{S_{1/2}^*}(T_i)$. This shows that $R_{S_{1/2}^*}(T_i) = R_{S_p^*}(T_i) = R_{S^*(1/2)}(T_i)$. So for $\alpha = 1/2$, from Theorem 3.1 it follows that $R_{S_p^*}(T_i) = (3 - \sqrt{5})/4, R_{S_p^*}(T_2) = (5 - \sqrt{17})/4$ and $R_{S_p^*}(T_3) = 1/\sqrt{35}$. 

The function $\varphi_{PAR} : \mathbb{D} \rightarrow \mathbb{C}$ given by

$$\varphi_{PAR} := 1 + \frac{2}{\pi} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2, \quad \operatorname{Im} \sqrt{z} \geq 0$$

maps $\mathbb{D}$ on the parabolic region given by $\varphi_{PAR}(\mathbb{D}) = \{w = u + iv : v^2 < 2u - 1 = \{w : \operatorname{Re} w > |w - 1|\}$. The class $S_p^* := S^*(\varphi_{PAR}) = \{f \in \mathcal{A} : zf'(z)/f(z) < \varphi_{PAR}(z)\}$ was introduced by Rønning [14], and is known as the class of parabolic starlike functions. The class $S_p^*$ consists of the functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right|, z \in \mathbb{D}.$$

Evidently, every parabolic starlike function is also starlike of order 1/2.

Shanmugam and Ravichandran [15] had proved that for $1/2 < a < 3/2$, then

$$\{w : |w - a| < a - 1/2\} \subset \{w : \operatorname{Re} w > |w - 1|\}. \tag{3.1}$$

The following result gives the radius of parabolic starlikeness for the classes $T_1, T_2$ and $T_3$.

Corollary 3.2. The following results holds for the class $S_p^*$.

\[\begin{align*}
(i) \quad & R_{S_p^*}(T_1) = (3 - \sqrt{5})/4 \approx 0.190983 \\
(ii) \quad & R_{S_p^*}(T_2) = (5 - \sqrt{17})/4 \approx 0.219224 \\
(iii) \quad & R_{S_p^*}(T_3) = 1/\sqrt{35} \approx 0.169031
\end{align*}\]

Proof. In equation (3.1), putting $a = 1$ gives $S_{1/2}^* \subset S_p^*$. Every parabolic starlike function is also starlike function of order $1/2$, whence the inclusion $S_{1/2}^* \subset S_p^* \subset S^*(1/2)$. Therefore, for any class $F$, readily $R_{S_{1/2}^*}(F) \leq R_{S_p^*}(F) \leq R_{S^*(1/2)}(F)$.

When $F = T_i, i = 1, 2, 3$, Theorem 3.1 gives $R_{S^*(\alpha)}(T_i) = R_{S_{1/2}^*}(T_i)$. This shows that $R_{S_{1/2}^*}(T_i) = R_{S_p^*}(T_i) = R_{S^*(1/2)}(T_i)$. So for $\alpha = 1/2$, from Theorem 3.1 it follows that $R_{S_p^*}(T_i) = (3 - \sqrt{5})/4, R_{S_p^*}(T_2) = (5 - \sqrt{17})/4$ and $R_{S_p^*}(T_3) = 1/\sqrt{35}$.
(ii) For the function \( f_2(z) = z(1 + z)e^z \), at \( z = R_{S^*_e}(T_2) = -\rho \), we obtain
\[
\Re \frac{zf'_2(z)}{f_2(z)} = \frac{\rho^2 + 3\rho + 1}{1 + \rho} = \frac{1}{2} = \left| \frac{zf'_2(z)}{f_2(z)} - 1 \right|.
\]
Thus, \( R_{S^*_e}(T_2) \leq (5 - \sqrt{11})/4 \).

(iii) For the function \( f_3(z) = z(z + \sqrt{1 + z^2})^3 \), at \( z = R_{S^*_e}(T_3) = -\rho \), we have
\[
\Re \frac{zf'_3(z)}{f_3(z)} = 1 - \frac{3\rho}{\sqrt{1 + \rho^2}} = \frac{1}{2} = \left| \frac{zf'_3(z)}{f_3(z)} - 1 \right|.
\]
This proves that \( R_{S^*_e}(T_3) \leq 1/\sqrt{35} \).

In 2015, Mendiratta et al. [12] introduced the class of starlike functions associated with the exponential function as \( S^*_e = S^*(e^z) \) and it satisfies the condition \( |\log zf''(z)/f(z)| < 1 \). They had also proved that, for \( e^{-1} \leq a \leq (e + e^{-1})/2 \),
\[
\{ w \in \mathbb{C} : |w - a| < a - e^{-1} \} \subseteq \{ w \in \mathbb{C} : |\log w| < 1 \}. \tag{3.2}
\]

**Corollary 3.3.** The following results hold for the class \( S^*_e \).

(i) \( R_{S^*_e}(T_1) = (-2 + 7e - \sqrt{4 + 4e + 17e^2})/8e \approx 0.237983 \)

(ii) \( R_{S^*_e}(T_2) = (-1 + 3e - \sqrt{1 - 2e + 5e^2})/2e \approx 0.267302 \)

(iii) \( R_{S^*_e}(T_3) = (1 - 2e + e^2)/(-1 + 2e + 8e^2) \approx 0.215546 \)

**Proof.** Mendiratta et al. provided the inclusion (3.2), which gives \( S^*_{1/e} \subseteq S^*_e \). It was also shown in [12, Theorem 2.1(i)] that \( S^*_e \subseteq S^*_{1/e} \). Therefore, \( S^*_{1/e} \subseteq S^*_e \subseteq S^* \), which provides the required radii as a consequence of Theorem 3.1

(i) For the function \( f_1(z) = ze^{2z}\sqrt{1 + z} \), at \( z = -R_{S^*_e}(T_1) = -\rho \), we have
\[
\left| \log \frac{zf'_1(z)}{f_1(z)} \right| = \left| \log \frac{4\rho^2 - 7\rho + 2}{2(1 - \rho)} \right| = 1.
\]
Thus, \( R_{S^*_e}(T_1) \leq (-2 + 7e - \sqrt{4 + 4e + 17e^2})/8e \).

(ii) For the function \( f_2(z) = z(1 + z)e^z \), at \( z = -R_{S^*_e}(T_2) = -\rho \), we get
\[
\left| \log \frac{zf'_2(z)}{f_2(z)} \right| = \left| \log \frac{1 - 3\rho + \rho^2}{(1 - \rho)} \right| = 1.
\]
Thus, \( R_{S^*_e}(T_2) \leq (-1 + 3e - \sqrt{1 - 2e + 5e^2})/2e \).

(iii) For the function \( f_3(z) = z(z + \sqrt{1 + z^2})^3 \), at \( z = -R_{S^*_e}(T_3) = -\rho \), we obtain
\[
\left| \log \frac{zf'_3(z)}{f_3(z)} \right| = \left| \log \left( 1 - \frac{3\rho}{\sqrt{1 + \rho^2}} \right) \right| = 1.
\]
This proves that \( R_{S^*_e}(T_3) \leq (1 - 2e + e^2)/(1 - 2e + 8e^2) \).
Corollary 3.4. The following result holds for the class $S^*_c$.

(i) $R_{S^*_c}(T_1) = 1/4 = 0.25$
(ii) $R_{S^*_c}(T_2) = (4 - \sqrt{10})/3 \approx 0.279241$
(iii) $R_{S^*_c}(T_3) = 2/\sqrt{77} \approx 0.227921$

Proof. Equation (3.3) provides the inclusion $S^*_{1/3} \subset S^*_c$ for $a = 1$. Thus $R_{S^*_{1/3}}(T_i) \leq R_{S^*_c}(T_i)$ for $i = 1, 2, 3$. The proof is completed by demonstrating $R_{S^*_c}(T_i) \leq R_{S^*_{1/3}}(T_i)$ for $i = 1, 2, 3$.

(i) For the function $f_1(z) = ze^{2z}\sqrt{1 + z}$, at $z = -R_{S^*_c}(T_1) = -\rho$, we obtain

$$\left| \frac{zf_1'(z)}{f_1(z)} \right| = \frac{1}{3}.$$ 

Thus, $R_{S^*_c}(T_1) \leq 1/4$.

(ii) For the function $f_2(z) = z(1 + z)e^z$, at $z = -R_{S^*_c}(T_2) = -\rho$, we have

$$\left| \frac{zf_2'(z)}{f_2(z)} \right| = \frac{1}{3}.$$ 

Thus, $R_{S^*_c}(T_2) \leq (4 - \sqrt{10})/3$.

(iii) For the function $f_3(z) = z(z + \sqrt{1 + z^2})^3$, at $z = -R_{S^*_c}(T_3) = -\rho$,

$$\left| \frac{zf_3'(z)}{f_3(z)} \right| = \frac{1}{3}.$$ 

This proves that $R_{S^*_c}(T_3) \leq 2/\sqrt{77}$.

In 2019, Cho et al. [5] considered the class of starlike functions associated with sine function. The class $S^*_{sin}$ is defined as $S^*_{sin} = \{ f \in A : zf'(z)/f(z) < 1 + \sin z := q_0(z) \}$ for $z \in \mathbb{D}$. For $|a - 1| \leq \sin 1$, the following inclusion holds

$$\{ w \in \mathbb{C} : |w - a| < \sin 1 - |a - 1| \} \subseteq \Omega_s.$$ 

Here $\Omega_s := q_0(\mathbb{D})$ is the image of the unit disk $\mathbb{D}$ under the mapping $q_0(z) = 1 + \sin z$.

We find the $S^*_{sin}$-radius for the classes $T_1, T_2$ and $T_3$.

Corollary 3.5. The following results hold for the class $S^*_{sin}$.

(i) $R_{S^*_{sin}}(T_1) = (5 + 2\sin 1 - \sqrt{25 - 12\sin 1 + 4\sin 1^2})/8 \approx 0.308961$
(ii) $R_{S^*_{sin}}(T_2) = (2 + \sin 1 - \sqrt{4 + 1\sin 1^2})/2 \approx 0.335831$
(iii) $R_{S^*_{sin}}(T_3) = \sin 1/\sqrt{9 - \sin 1^2} \approx 0.292221$
Proof. By putting $a = 1$ in equation (3.1), we obtain the inclusion $S_{1 - \sin 1}^* \subset S_{\sin 1}^*$. Thus $R S_{1 - \sin 1}^* \leq R S_{\sin 1}^*$ for $i = 1, 2, 3$. The proof is completed by demonstrating $R S_{1 - \sin 1}^* \leq R S_{\sin 1}^*$ for $i = 1, 2, 3$. (i) For the function $f_1(z) = z e^{2z} \sqrt{1 + z}$, at $z = -R S_{\sin 1}^* (T_1) = -\rho$, we get
\[
\left| \frac{z f_1'(z)}{f_1(z)} \right| = \left| \frac{4 \rho^2 - 7 \rho + 2}{2(1 - \rho)} \right| = 1 + \sin 1.
\]
Thus, $R S_{\sin 1}^* (T_1) \leq (5 + 2 \sin 1 - \sqrt{25 - 12 \sin 1 + 4 \sin 1^2})/8$.

(ii) For the function $f_2(z) = z(1 + z) e^z$, at $z = -R S_{\sin 1}^* (T_2) = -\rho$, we have
\[
\left| \frac{z f_2'(z)}{f_2(z)} \right| = \left| \frac{1 - 3 \rho + \rho^2}{(1 - \rho)} \right| = 1 + \sin 1.
\]
Thus, $R S_{\sin 1}^* (T_2) \leq (2 + \sin 1 - \sqrt{4 + \sin 1^2})/2$.

(iii) For the function $f_3(z) = z(z + (1 + z)^2)^3$, at $z = -R S_{\sin 1}^* (T_3) = -\rho$, we get
\[
\left| \frac{z f_3'(z)}{f_3(z)} \right| = \left| 1 - \frac{3 \rho}{\sqrt{1 + \rho^2}} \right| = 1 + \sin 1.
\]
This proves that $R S_{\sin 1}^* (T_3) \leq \sin 1/\sqrt{9 - \sin 1^2}$.

In the next result, we find the radius for starlike functions associated with a rational function. Kumar and Ravichandran [3] introduced the class of starlike functions associated with a rational function, $\psi(z) = 1 + (z^2 k + z^2)/(k^2 - k z)$ where $k = \sqrt{2} + 1$, defined by $S_R = S^*(\psi(z))$. For $2(\sqrt{2} - 1) < a \leq \sqrt{2}$, they had proved that
\[
\{ w \in \mathbb{C} : |w - a| < a - 2(\sqrt{2} - 1) \} \subseteq \psi(\mathbb{D}). \tag{3.5}
\]

**Corollary 3.6.** The following results holds for the class $S_R^*$.

(i) $R S_R^* (T_1) = (11 - 4 \sqrt{2} - \sqrt{57 - 24 \sqrt{2}})/8 \approx 0.0676475$

(ii) $R S_R^* (T_2) = (5 - 2 \sqrt{2} - \sqrt{21 - 12 \sqrt{2}})/2 \approx 0.0821135$

(iii) $R S_R^* (T_3) = (\sqrt{-38 + 27 \sqrt{2}})/2 \sqrt{14} \approx 0.0572847$

Proof. For $a = 1$ equation (3.5) gives the inclusion $S_{2(\sqrt{2} - 1)}^* \subset S_R^*$. Thus $R S_{2(\sqrt{2} - 1)}^* (T_i) \leq R S_R^* (T_i)$ for $i = 1, 2, 3$. We next show that $R S_R^* (T_i) \leq R S_{2(\sqrt{2} - 1)}^* (T_i)$ for $i = 1, 2, 3$.

(i) For the function $f_1(z) = z e^{2z} \sqrt{1 + z}$, at $z = -R S_R^* (T_1) = -\rho$, we get
\[
\left| \frac{z f_1'(z)}{f_1(z)} \right| = \left| \frac{4 \rho^2 - 7 \rho + 2}{2(1 - \rho)} \right| = 2(\sqrt{2} - 1).
\]
Thus, $R S_R^* (T_1) \leq (11 - 4 \sqrt{2} - \sqrt{57 - 24 \sqrt{2}})/8$.

(ii) For the function $f_2(z) = z(1 + z) e^z$, at $z = -R S_R^* (T_2) = -\rho$, we have
\[
\left| \frac{z f_2'(z)}{f_2(z)} \right| = \left| \frac{1 - 3 \rho + \rho^2}{(1 - \rho)} \right| = 2(\sqrt{2} - 1).
\]
Thus, $R S_R^* (T_2) \leq (5 - 2 \sqrt{2} - \sqrt{21 - 12 \sqrt{2}})/2$.
(iii) For the function $f_3(z) = z(z + \sqrt{1 + z^2})^3$, at $z = -R_{S_r^*}(T_3) = -\rho$, we obtain
\[
|zf'_3(z)| = \left|1 - \frac{3\rho}{\sqrt{1 + \rho^2}}\right| = 2(\sqrt{2} - 1).
\]
Thus, $R_{S_r^*}(T_3) \leq (\sqrt{-38 + 27\sqrt{2}})/2\sqrt{14}$. 

In 2020, Wani and Swaminathan [20] introduced the class $S_{Ne}^* = S^*(1 + z - (z^3/3))$ that maps open disc $\mathbb{D}$ onto the interior of a two cusped kidney shaped curve $\Omega_{Ne} := \{u + iv : ((u - 1)^2 + v^2 - 4/9)^3 - 4v^2/3 < 0\}$. For $1/3 < a \leq 1$, they had proved that
\[
\{w \in \mathbb{C} : |w - a| < a - 1/3\} \subseteq \Omega_{Ne}.
\] (3.6)

Our next theorem determines the $S_{Ne}^*$ radius results for the classes $T_1$, $T_2$, and $T_3$.

**Corollary 3.7.** The following results hold for the class $S_{Ne}^*$.

(i) $R_{S_{Ne}^*}(T_1) = 1/4 = 0.25$

(ii) $R_{S_{Ne}^*}(T_2) = (4 - \sqrt{10})/3 \approx 0.279241$

(iii) $R_{S_{Ne}^*}(T_3) = 2/\sqrt{17} \approx 0.227921$

**Proof.** From equation (3.6), we obtain the inclusion $S_{1/3}^* \subset S_{Ne}^*$ for $a = 1$. This shows that $R_{S_{1/3}^*}(T_i) \leq R_{S_{Ne}^*}(T_i)$ for $i = 1, 2, 3$.

(i) For the function $f_1(z) = ze^{2z}\sqrt{1 + z}$, at $z = -R_{S_{Ne}^*}(T_1) = -\rho$, $|zf'_1(z)| = \left|\frac{4\rho^2 - 7\rho + 2}{2(1 - \rho)}\right| = \frac{1}{3}$

Thus, $R_{S_{Ne}^*}(T_1) \leq 1/4$.

(ii) For the function $f_2(z) = z(1 + z)e^z$, at $z = -R_{S_r^*}(T_2) = -\rho$, $|zf'_2(z)| = \left|\frac{1 - 3\rho + \rho^2}{(1 - \rho)}\right| = \frac{1}{3}$

Thus, $R_{S_{Ne}^*}(T_2) \leq (4 - \sqrt{10})/3$.

(iii) For the function $f_3(z) = z(z + \sqrt{1 + z^2})^3$, at $z = -R_{S_{Ne}^*}(T_3) = -\rho$, $|zf'_3(z)| = \left|1 - \frac{3\rho}{\sqrt{1 + \rho^2}}\right| = \frac{1}{3}$

This proves that $R_{S_{Ne}^*}(T_3) \leq 2/\sqrt{17}$.

Goel and Kumar [7] introduced the class $S_{SG}^* := S^*(2/1 + e^{-z})$, where $2/(1 + e^{-z})$ is a modified sigmoid function that maps $\mathbb{D}$ onto the region $\Omega_{SG} := \{w = u + iv : |\log(w/(2 - w))| < 1\}$. Precisely, $f \in S_{SG}^*$ provided the function $zf'(z)/f(z)$ maps $\mathbb{D}$ onto the region lying inside the domain $\Omega_{SG}$. For $2/(e + 1) < a < 2e/(1 + e)$, Goel and Kumar [7] had proved the following inclusion
\[
\{w \in \mathbb{C} : |w - a| < r_{SG}\} \subset \Omega_{SG},
\] (3.7) provided $r_{SG} = ((e - 1)/(e + 1)) - |a - 1|$. Next result is about the $S_{SG}^*$ radius for the defined classes.
Theorem 3.8. The following results hold for the class $S_{SG}^*$.

(i) $R_{S_{SG}^*}(T_1) = (3 + 7e - \sqrt{41 + 42e + 17e^2})/(8 + 8e) \approx 0.177213$

(ii) $R_{S_{SG}^*}(T_2) = (1 + 3e - \sqrt{5 + 6e + 5e^2})(2 + 2e) \approx 0.204712$

(iii) $R_{S_{SG}^*}(T_3) = (\sqrt{1 - 2e + e^2})/(8 + 20e + 8e^2) \approx 0.1559$

Proof. (i) The function defined by $m(r) = (4r^2 - 7r + 2)/2(1-r), 0 \leq r < 1$ is a decreasing function. Let $\rho = R_{S_{SG}^*}(T_1)$ is the root of the equation $m(r) = 2/(1 + e)$. For $0 < r \leq R_{S_{SG}^*}(T_1)$, we have $m(r) \geq 2/(1 + e)$. That is

$$\frac{r(5 - 4r)}{2(1 - r)} \leq \frac{e - 1}{e + 1}.$$

For the class $T_1$, the centre of the disk is 1, therefore the disk obtained in (2.2) is contained in the region bounded by modified sigmoid, by equation (3.7). For the function $f_1(z) = ze^{\psi} \sqrt{1 + z}$, at $z = -R_{S_{SG}^*}(T_1) = -\rho$, we have

$$\left| \log \frac{zf_1(z)/f_1(z)}{2 - (zf_1(z)/f_1(z))} \right| = \left| \frac{(4\rho^2 - 7\rho + 2)/2(1 - \rho)}{2 - ((4\rho^2 - 7\rho + 2)/2(1 - \rho))} \right| = 1.$$

(ii) The function defined $n(r) = (r^2 - 3r + 1)/(1-r), 0 \leq r < 1$ is a decreasing function. Let $\rho = R_{S_{SG}^*}(T_2)$ is the root of the equation $n(r) = 1/3$. For $0 < r \leq R_{S_{SG}^*}(T_2)$, we have $n(r) \geq 2/1 + e$. That is,

$$\frac{r(-2 + r)}{(-1 + r)} \leq \frac{e - 1}{e + 1}.$$

For the class $T_2$, the centre of the disk is 1, therefore the disk obtained in (2.4) is contained in the region bounded by the modified sigmoid, using equation (3.7). For the function $f_2(z) = z(1 + e)^{\psi}, z = f_2(z) = -\rho$, we have

$$\left| \log \frac{zf_2(z)/f_2(z)}{2 - (zf_2(z)/f_2(z))} \right| = \left| \frac{(1 - 3\rho + \rho^2)/(1 - \rho)}{2 - ((1 - 3\rho + \rho^2)/(1 - \rho))} \right| = 1.$$

(iii) The function defined $s(r) = 1 - 3r/\sqrt{1 + r^2}, 0 \leq r < 1$ is a decreasing function. Let $\rho = R_{S_{SG}^*}(T_3)$ is the root of the equation $s(r) = 2/(1 + e)$. For $0 < r \leq R_{S_{SG}^*}(T_3)$, we have $s(r) \geq 2/1 + e$. That is

$$\frac{3r}{\sqrt{1 + r^2}} \leq \frac{e - 1}{e + 1}.$$

For the class $T_3$, the centre of the disk is 1, therefore the disk obtained in (2.6) is contained in the region bounded by the modified sigmoid, by equation (3.7). For the function $f_2(z) = z(z + \sqrt{1 + z^2})^3, z = -R_{S_{SG}^*}(T_3) = -\rho$, we have

$$\left| \log \frac{zf_3(z)/f_3(z)}{2 - (zf_3(z)/f_3(z))} \right| = \left| \frac{1 - (3\rho)/\sqrt{1 + \rho^2}}{2 - ((1 - (3\rho)/(\sqrt{1 + \rho^2}))} \right| = 1.$$
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