ERROR ESTIMATES FOR APPROXIMATIONS OF DISTRIBUTED ORDER TIME FRACTIONAL DIFFUSION WITH NONSMOOTH DATA

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Abstract. In this work, we consider the numerical solution of an initial boundary value problem for the distributed order time fractional diffusion equation. The model arises in the mathematical modeling of ultra-slow diffusion processes observed in some physical problems, whose solution decays only logarithmically as the time $t$ tends to infinity. We develop a space semidiscrete scheme based on the standard Galerkin finite element method, and establish error estimates optimal with respect to data regularity in $L^2(\Omega)$ and $H^1(\Omega)$ norms for both smooth and nonsmooth initial data. Further, we propose two fully discrete schemes, based on the Laplace transform and convolution quadrature generated by the backward Euler method, respectively, and provide optimal convergence rates in the $L^2(\Omega)$ norm, which exhibits exponential convergence and first-order convergence in time, respectively. Extensive numerical experiments are provided to verify the error estimates for both smooth and nonsmooth initial data, and to examine the asymptotic behavior of the solution.

Keywords: distributed order, time fractional diffusion, Galerkin finite element method, fully discrete scheme, Laplace transform, error estimates

1. Introduction

We consider an initial-boundary value problem for the following distributed order time fractional diffusion equation for $u(x, t)$:

\begin{equation}
D_t^{[\mu]} u - \Delta u = f \quad \text{in } \Omega, \quad T > t > 0,
\end{equation}

(1.1)

\begin{equation}
u = 0 \quad \text{on } \partial \Omega, \quad T > t > 0,
\end{equation}

\begin{equation}
u(0) = v \quad \text{in } \Omega,
\end{equation}

where $\Omega$ is a bounded convex polygonal domain in $\mathbb{R}^d (d = 1, 2, 3)$ with a boundary $\partial \Omega$, $v$ is a given function on $\Omega$, and $T > 0$ is a fixed value. Here, $D_t^{[\mu]} u$ denotes the distributed order fractional derivative of $u$ in time $t$ (with respect to the weight function $\mu$) defined by

\begin{equation}D_t^{[\mu]} u(t) = \int_0^1 D_t^\alpha u(t) \mu(\alpha) \, d\alpha,
\end{equation}

where $D_t^\alpha u, \, 0 < \alpha < 1$, denotes the left-sided Caputo fractional derivative of order $\alpha$ with respect to $t$ and it is defined by (see, e.g. [20, p. 91])

\begin{equation}D_t^\alpha u(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \frac{d}{ds} u(s) \, ds,
\end{equation}

where $\Gamma(\cdot)$ denotes Euler’s Gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$, for all $x > 0$. In this paper we consider the case that $\mu \in C([0,1])$ is a nonzero nonnegative weight function with $0 \leq \mu < 1$, $\mu(0)\mu(1) > 0$.

In the last three decades, fractional calculus has been extensively studied and successfully employed to model anomalous diffusion, in which the mean squared variance grows faster (superdiffusion) or slower (subdiffusion) than that in a Gaussian process. The subdiffusion model, which is a diffusion equation involving a Caputo fractional derivative $D_t^{\alpha_0} u$ of order $\alpha_0 \in (0, 1)$ in time:

\begin{equation}D_t^{\alpha_0} u - \Delta u = f \quad \text{in } \Omega, \quad T > t > 0,
\end{equation}

\hrule
is often employed to model subdiffusion processes in which the mean squared variance grows at a sublinear (power type) rate, slower than the linear growth in a Gaussian process for normal diffusion. Formally, the subdiffusion model (1.4) can be recovered from the distributed order model (1.1) with a singular weight \( \mu(\alpha) = \delta(\alpha - \alpha_0) \), where \( \delta(\alpha - \alpha_0) \) is a Dirac delta function at \( \alpha_0 \). Physically, the subdiffusion process can be characterized by a unique diffusion exponent (commonly known as Hurst exponent) showing the time dependence of the characteristic displacement [5]. In practice, the physical process may not possess a unique Hurst exponent, and the distributed order model (1.1) provides a flexible framework for describing a host of continuous and nonstationary signals [5, 6, 41]. Problem (1.1) is frequently applied to describe ultraslow diffusion, where the mean squared variance grows only logarithmically with time, e.g., Sinai model [40]. The distributed-order fractional model arises often in disordered media, and has been successfully used in several applications. For example, Caputo [4] proposed the use of the distributed order derivative in generalizing the stress-strain relation in dielectrics, and Atanackovic et al. [1] suggested a distributed order wave equation as the constitutive relation for viscoelastic materials to describe stress relaxation in a rod.

In recent years, the theoretical study of problem (1.1) has attracted some attention [34, 21, 28, 33, 27, 11, 22, 13, 2]. Kochubei [21] made some early contributions to the rigorous analysis of the model (1.1), by constructing fundamental solutions to the problem and establishing their positivity and subordination property. Mainardi et al. [28] studied the existence of a solution, asymptotic behavior, and positivity etc. for the case \( \mu(0) \geq 0 \) and \( \int_0^1 \mu(\alpha)d\alpha = c > 0 \). Meerschaert and Scheffler [34] gave a stochastic model for ultraslow diffusion, based on random walks with a random waiting time between jumps whose probability tail falls off at a logarithmic rate. Meerschaert et al. [33] provided explicit strong solutions and their stochastic analogues. Luckho [27] showed a weak maximum principle for the problem. Li et al. [22] established sharp asymptotic behavior of the solution for \( t \to 0 \) and \( t \to \infty \), in the case of continuous density \( \mu \) with \( \mu(1) > 0 \). Jia et al. [13] studied the well-posedness of a Cauchy problem for an abstract distributed-order differential equation using a functional calculus approach. Very recently, Bazhlekova [2] analyzed problem (1.1) for \( \mu \in C[0,1], \mu \geq 0 \) and \( \mu(\alpha) \neq 0 \) on a set of positive measure.

The solution to the model (1.1) is rarely available in closed form, which necessitates the development of efficient numerical schemes, to enable the successful use of the model (1.1) in practice. Despite the extensive studies on the simpler subdiffusion model (1.4) (cf. [23, 30, 46, 7, 45, 36, 16, 17]) for an incomplete list of works on the numerical approximation of the Caputo fractional derivative \( D_{\alpha}^C u(t) \), there are only very few studies [8, 19, 35] on the distributed order model (1.1). Diethelm and Ford [8] developed a numerical scheme for distributed order fractional ODEs. It approximates the distributed order derivative \( D_{\alpha}^\mu u(t) \) by quadrature, leading to a multi-term time fractional ODE, which can then be solved by fractional multi-step methods. Error estimates of the approximation were discussed in [8]. Such a technique was also employed to solve nonlinear distributed-order fractional ODEs in [19], but without any analysis. Just recently, Morgado and Rebelo [35] developed an implicit finite difference method for the model (1.1) with a Lipschitz nonlinear source term in one space dimension. The scheme is based on a quadrature approximation of \( D_{\alpha}^\mu u(t) \) together with the backward finite difference approximation for the Caputo derivative \( D_{\alpha}^C u(t) \), and the second-order finite difference approximation in space. The stability of the scheme, and a convergence rate \( O(h^2 + \tau + (\delta \alpha)^2) \) (with \( h, \tau \) and \( \delta \alpha \) being the mesh size, time step size and step size for quadrature rule, respectively) were established under the assumption that the solution \( u \) is \( C^2 \) in time and \( C^4 \) in space and the weight function \( \mu(\alpha) \) is sufficiently regular. In view of the limited smoothing property of the solution operator, cf. Theorem 2.1 below, the regularity required by the convergence analysis is restrictive, especially for nonsmooth data. To the best of our knowledge, the development of robust numerical schemes for the model (1.1) with nonsmooth data and their rigorous analysis have not been carried out, despite its immense practical importance, e.g., in solving inverse and/or optimal control problems [18].

In this work, we develop a Galerkin finite element method (FEM) for problem (1.1) and establish optimal (with respect to data regularity) error estimates for both smooth and nonsmooth initial data \( v \). The approximation is based on the finite element space \( X_h \) of continuous piecewise linear functions over a family of shape regular quasi-uniform partitions \( \{ T_h \}_{0<h<1} \) of the domain \( \Omega \) into \( d \)-simplexes, where \( h \)
For initial data

\[ v \in H^1_0(\Omega) \text{ and } v_h \in X_h \]

where \((\cdot, \cdot)\) denotes the \(L^2(\Omega)\)-inner product, \(a(u, v) = (\nabla u, \nabla v)\) for \(u, v \in H^1_0(\Omega)\), and \(v_h \in X_h\) is an approximation of the initial data \(v\). Our default choices for \(v_h\) are the \(L^2(\Omega)\)-projection \(v_h = P_h v\), for \(v \in L^2(\Omega)\), and the Ritz projection \(v_h = R_h v\), for \(A \in L^2(\Omega)\), where \(A = -\Delta\) with a homogeneous Dirichlet boundary condition. Further, we develop two fully discrete schemes based on the Laplace transform and convolution quadrature generated by the backward Euler method, and provide optimal error estimates for both space semidiscrete and fully discrete schemes.

Our main contributions are as follows. First, in Theorem 2.1, we establish the sharp regularity estimates for the solution to problem (1.1). The proof relies essentially on various refined properties of the kernel function \(w(z)\) defined in (2.4) in Lemmas 2.1-2.3, which also enable one to apply the established techniques for analyzing the semidiscrete and fully discrete schemes. Second, in Theorems 3.1 and 3.2, we derive the following error estimates for the space semidiscrete Galerkin scheme (1.5) for \(t \in (0, T]\):

\[
\|(u(t) - u_h(t))\|_{L^2(\Omega)} + h\|\nabla(u(t) - u_h(t))\|_{L^2(\Omega)} \leq \begin{cases} 
 c_T h^2 \left( t \log \frac{2T}{t} \right)^{-1} \|v\|_{L^2(\Omega)} & \text{if } v \in L^2(\Omega), \\
 ch^2 \|Av\|_{L^2(\Omega)} & \text{if } Av \in L^2(\Omega).
\end{cases}
\]

For initial data \(v \in L^2(\Omega)\), the estimates deteriorate as the time \(t\) approaches 0, with an extra \(\frac{1}{\log t}\) factor in comparison with that for the standard diffusion case [42]. Third, we develop a fully discrete scheme based on the Laplace transform. It relies on a contour representation of the semidiscrete solution with a hyperbolic contour, and trapezoidal quadrature, cf. Theorem 4.1. Specifically, the fully discrete solution \(U_{N,h}(t)\) with \(N + 1\) quadrature points satisfies the following error bound for \(t \in (0, T]\):

\[
\|(u(t) - U_{N,h}(t))\|_{L^2(\Omega)} \leq \begin{cases} 
 c_T \left( e^{-c_1 N} + h^2 \left( t \log \frac{2T}{t} \right)^{-1} \right) \|v\|_{L^2(\Omega)} & \text{if } v \in L^2(\Omega), \\
 c \left( e^{-c_1 N} + h^2 \right) \|Av\|_{L^2(\Omega)} & \text{if } Av \in L^2(\Omega).
\end{cases}
\]

Last, we develop a second fully discrete scheme based on convolution quadrature, generated by the backward Euler method, and in Theorem 5.3, establish the first-order convergence of the scheme for both smooth and nonsmooth initial data. For example, for nonsmooth initial data \(v \in L^2(\Omega)\), the fully discrete solution \(U^*_h\) approximating the continuous solution \(u(t_n), t_n \in (0, T]\) (on a uniform grid in time with a step size \(\tau\)) satisfies the following bound:

\[
\|(u(t_n) - U^*_h)\|_{L^2(\Omega)} \leq c_T \left( \tau + h^2 \left( \log \frac{2T}{\tau} \right)^{-1} \right)^{-1} t_n^{-1} \|v\|_{L^2(\Omega)}.
\]

It is worth noting that all error estimates are nearly optimal and expressed in terms of the regularity of the initial data directly, and fully verified by extensive numerical experiments. Theoretically, these results extend our earlier studies [15, 16, 14] on the subdiffusion model (1.4), which contribute to the development and rigorous analysis of robust numerical schemes for the distributed order model (1.1).

The model (1.1) is closely related to parabolic equations with a positive type memory term, for which there are many important studies on numerical schemes based on convolution quadrature [25, 26, 7] and Laplace transform [10, 24, 38, 39, 31, 44, 32]. For example, Cuesta et al. [7] developed an abstract framework for analyzing convolution quadrature generated by the backward Euler method and second-order backward difference. It covers also inhomogeneous and nonlinear problems. Their analysis uses the generating function, and the Laplace transform involves \(w(z) = z^\alpha, 0 < \alpha < 1\). McLean and Thomée [32] studied the Laplace transform method for a fractional order model, whose Laplace transform involves \(w(z) = z^{\alpha}, -1 < \alpha < 1\). These interest works have inspired the current work on the model (1.1). However, the existing error analysis does not cover directly the model (1.1), due to the general kernel function involved, cf. (2.4). Instead, we shall opt for the general strategy outlined in [26], by deriving various refined estimates for the kernel function, especially identifying the suitable condition on the weight function \(\mu\). These estimates are also essential for analyzing the Laplace transform approach.

The rest of the paper is organized as follows. In Section 2, we recall the solution theory of the mathematical model (1.1) following [21, 22]. In Section 3, we develop a space semidiscrete Galerkin
scheme, and provide optimal error estimates. Two fully discrete schemes, based on the Laplace transform and convolution quadrature, are given in Sections 4 and 5, respectively. Finally, to test and to verify the convergence theory, we present in Section 6 extensive numerical experiments. Throughout, the notation $c$, with or without a subscript, denotes a generic constant, which may differ at different occurrences, but it is always independent of the mesh size $h$, the number $N$ of quadrature points, and time step size $\tau$.

2. Solution theory

In this part, we discuss the solution theory of problem (1.1) using the Laplace transform. The stability estimates will play an essential role in developing optimal error estimates. We denote by $\hat{\cdot}$ the Laplace transform. First we recall the following well known relation [20, Lemma 2.24, p. 98]

\begin{equation}
\partial_t^\alpha u = z^\alpha \hat{u} - z^{\alpha-1}u(0).
\end{equation}

Next we denote by $A$ the operator $-\Delta$ with a homogeneous Dirichlet boundary condition with a domain $D(A) = H^3_0(\Omega) \cap H^2(\Omega)$. The $H^2(\Omega)$ regularity of the elliptic problem is essential for the error analysis below and it follows from the convexity assumption on the domain $\Omega$. It is well known that the operator $A$ generates a bounded analytic semigroup of angle $\pi/2$, i.e., for any $\theta \in (\pi/2, \pi)$ [12, p. 321, Proposition C.4.2]

\begin{equation}
\|(zI + A)^{-1}\| \leq \frac{1}{|\Im(z)|} \leq \frac{1}{|\sin(\theta)|}, \quad \forall z \in \Sigma'_\theta,
\end{equation}

where $\Sigma'_\theta$ is a sector with the origin excluded, i.e.,

\[ \Sigma_\theta = \{ z \in \mathbb{C} : |\arg(z)| < \theta \}, \quad \Sigma'_\theta = \Sigma_\theta \setminus \{0\}. \]

Now it follows from (1.1) and (2.1) that

\begin{equation}
zw(z)\hat{u}(z) + A\hat{u}(z) = w(z)v,
\end{equation}

where the function $w(z)$ is defined by

\begin{equation}
w(z) = \int_0^1 z^\alpha \mu(\alpha) \, d\alpha.
\end{equation}

By means of the inverse Laplace transform, the solution $u(t)$ can be represented by

\begin{equation}
u(t) = S(t)v := \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt} H(z)v \, dz,
\end{equation}

where the kernel $H(z)$ is defined by

\[ H(z) = (zw(z)I + A)^{-1}w(z), \]

and the contour $\Gamma_{\theta,\delta}$ by

\begin{equation}
\Gamma_{\theta,\delta} = \{ z \in \mathbb{C} : |z| = \delta, |\arg z| \leq \theta \} \cup \{ z \in \mathbb{C} : z = \rho e^{\pm i\theta}, \rho \geq \delta \}.
\end{equation}

We begin by discussing the regularity estimates of the solution. To this end, we first give a few elementary properties of the function $w(z)$. The first is the sector-preserving property, which enables applying the resolvent estimate (2.2) in the error analysis to be developed in Sections 3-5.

**Lemma 2.1.** Let $\theta \in (\pi/2, \pi)$ and assume that $\mu(0)\mu(1) > 0$. Then $zw(z) \in \Sigma_{\theta'}$ with $\theta' \in (\pi/2, \pi)$ for all $z \in \Sigma_\theta$ and $\theta'$ depends only on $\mu$ and $\theta$.

**Proof.** Let $z = re^{i\varphi}$ with $\varphi \in [-\theta, \theta]$. If $\varphi \in (-\pi/2, \pi/2)$, we have

\[ \Re(zw(z)) = \int_0^1 r^\alpha \cos(\alpha \varphi) \mu(\alpha) \, d\alpha > 0. \]
It suffices to consider the case $\varphi \in [\pi/2, \theta]$. First we claim that there exists an $r_0 \in (0, 1)$ only dependent on $\mu$ such that $\Re(zw(z)) > 0$ for all $r < r_0$. By the assumption $\mu(0) > 0$, we can find a small $\epsilon_0 > 0$ such that $\min_{\alpha \in [0, \epsilon_0]} \cos(\alpha \pi) \mu(\alpha) = 0 > 0$. Hence

$$\Re(zw(z)) \geq \int_0^{r_0} r^\alpha \cos(\alpha \varphi) \mu(\alpha) \, d\alpha - \int_{r_0}^1 r^\alpha |\cos(\alpha \varphi)| \mu(\alpha) \, d\alpha$$

(2.7)

$$\geq \delta_0 \int_0^{r_0} r^\alpha \, d\alpha - ||\mu||_{C[0, 1]} \int_{r_0}^1 r^\alpha \, d\alpha$$

$$\geq -(\ln r)^{-1} \left[ \delta_0 - r^{\epsilon_0}(\delta_0 + ||\mu||_{C[0, 1]}) \right].$$

Then direct calculation yields

$$\delta_0 - r^{\epsilon_0}(\delta_0 + ||\mu||_{C[0, 1]}) > 0 \quad \forall r < r_0 =: \left( \frac{\delta_0}{\delta_0 + ||\mu||_{C[0, 1]}} \right)^{1/\epsilon_0} \in (0, 1),$$

and the desired claim $\Re(zw(z)) > 0$ follows. Now we consider the case $r \geq r_0$ and $\varphi \in [\pi/2, \theta]$. In fact,

$$|\tan(\arg(zw(z)))| = \frac{\int_0^1 r^\alpha \sin(\alpha \varphi) \mu(\alpha) \, d\alpha}{\int_0^1 r^\alpha \cos(\alpha \varphi) \mu(\alpha) \, d\alpha} \geq \frac{\int_0^{r_0} r^\alpha \sin(\alpha \varphi) \mu(\alpha) \, d\alpha}{||\mu||_{C[0, 1]} \int_0^1 r^\alpha \, d\alpha}.$$

In view of the assumption $\mu(1) > 0$ and $\mu \in C[0, 1]$, we may find a small $\epsilon_1 > 0$ such that $\min_{\alpha \in [1-\epsilon_1, 1]} \mu(\alpha) \geq \delta_1 > 0$ and

$$\int_0^1 r^\alpha \sin(\alpha \varphi) \mu(\alpha) \, d\alpha \geq \int_{1-\epsilon_1}^1 r^\alpha \sin(\alpha \varphi) \mu(\alpha) \, d\alpha$$

$$\geq \delta_1 \sin(\theta) \int_{1-\epsilon_1}^1 r^\alpha \, d\alpha.$$

For $r \geq r_0$, clearly there holds

$$\int_{1-\epsilon_1}^1 r^\alpha \, d\alpha = \int_{1-\epsilon_1}^1 r_0^\alpha \left( \frac{r}{r_0} \right)^\alpha \, d\alpha \geq r_0 \int_{1-\epsilon_1}^1 \left( \frac{r}{r_0} \right)^\alpha \, d\alpha$$

(2.8)

$$\geq r_0 \epsilon_1 \int_0^1 \left( \frac{r}{r_0} \right)^\alpha \, d\alpha \geq r_0 \epsilon_1 \int_0^1 r^\alpha \, d\alpha.$$

Then we have

(2.9) $$|\tan(\arg(zw(z)))| \geq \delta_1 \sin(\theta) r_0 \epsilon_1 ||\mu||_{C[0, 1]} =: c'.$$

Hence $zw(z) \in \Sigma_{\theta'}$ with $\theta' = \pi - \arctan(c')$. \hfill \qed

**Remark 2.1.** We note that the constants $\delta_1$, $r_0$ and $\epsilon_1$ in (2.9) are all independent of the choice $\theta$. Hence in case of $\theta = \pi - \epsilon$ for a small $\epsilon > 0$, $zw(z) \in \Sigma_{\theta'}$ with $\theta' = \pi - \arctan(\tan(\theta)) = \pi - \arctan(\tan(\epsilon)) \approx \pi - \epsilon \epsilon$, where $c = \delta_1 r_0 \epsilon_1 ||\mu||_{C[0, 1]}$.

The second result is an upper bound on the kernel $w(z)$, which can be obtained by an elementary calculation.

**Lemma 2.2.** Let $\mu \in C[0, 1]$ be a nonnegative function. Then there holds

$$|w(z)| \leq \frac{||\mu||_{C[0, 1]} |z| - 1}{|z| \log |z|}.$$  

The third result gives a lower bound on the function $zw(z)$.

**Lemma 2.3.** Let $\theta \in (\pi/2, \pi)$ and assume that $\mu(0)\mu(1) > 0$. Then there exists a constant $c > 0$ dependent only on $\theta$ and $\mu$ such that for any $z \in \Sigma_{\theta}$

(2.10) $$|zw(z)| \geq c \int_0^1 r^\alpha \, d\alpha = \frac{|z| - 1}{\log |z|},$$

(2.11) $$|z|w(|z|) \geq |zw(z)| \geq c|zw(|z|)|.$$
Proof. Let \( z = re^{i\theta} \). Using \( \mu(1) > 0 \) and \( \mu \in C[0,1] \), we can find a small \( \epsilon_1 > 0 \) such that \( \min_{\alpha \in [1-\epsilon_1,1]} \mu(\alpha) \geq \delta_1 > 0 \). Then we have for all \( r \geq 1 \)

\[
\int_0^1 r^\alpha \mu(\alpha) \, d\alpha \geq \int_{1-\epsilon_1}^1 r^\alpha \mu(\alpha) \, d\alpha \geq \delta_1 \int_{1-\epsilon_1}^1 r^\alpha \, d\alpha \geq \epsilon_1 \delta_1 \int_0^1 r^\alpha \, d\alpha.
\]

Similarly, we may find a small \( \epsilon_2 > 0 \) such that \( \min_{\alpha \in [0,\epsilon_2]} \mu(\alpha) \geq \delta_2 > 0 \) and then for all \( r < 1 \)

\[
\int_0^1 r^\alpha \mu(\alpha) \, d\alpha \geq \epsilon_2 \delta_2 \int_0^1 r^\alpha \, d\alpha.
\]

Hence for \( \varphi \in (\theta - \pi, \pi - \theta) \), we get for \( c_1 = \min(\epsilon_1 \delta_1, \epsilon_2 \delta_2) \)

\[
|zw(z)| \geq \Re(zw(z)) \geq \cos(\pi - \theta) \int_0^1 r^\alpha \mu(\alpha) \, d\alpha \geq c_1 \cos(\pi - \theta) \int_0^1 r^\alpha \, d\alpha.
\]

Now it suffices to consider the case \( \varphi \in [\pi - \theta, \theta] \), and the case \( \varphi \in [-\pi, \theta - \pi] \) follows analogously. From (2.7), we deduce

\[
|zw(z)| \geq \Re(zw(z)) \geq \frac{\delta_0}{2} \int_0^1 r^\alpha \, d\alpha \quad \forall r \leq r_0 = \left( \frac{\delta_0}{2(\delta_0 + \|\mu\|_{C[0,1]})} \right)^{1/\delta_0}.
\]

Then a similar argument for deriving (2.8) shows that the inequality (2.10) holds for \( r \geq r_0 \) and \( \varphi \in [\pi - \theta, \theta] \), thereby showing (2.10). The inequality (2.11) follows from

\[
\|\mu\|_{C[0,1]} \int_0^1 r^\alpha \, d\alpha \geq \int_0^1 r^\alpha \mu(\alpha) \, d\alpha = |zw(z)|
\]

and the trivial inequality \( |zw(z)| \leq |z|w(|z|) \). \( \square \)

Now we give the main result of this section, namely, stability of problem (1.1) with \( f \equiv 0 \).

Theorem 2.1. Let \( \mu \in C[0,1] \) be a non-negative function with \( \mu(0)\mu(1) > 0 \). Then the solution \( u \) to problem (1.1) with \( f \equiv 0 \) satisfies the following stability estimates for \( t \in (0,T] \) and \( \nu = 0,1 \):

\[
(A^\nu S^{(m)}(t)v)_{L^2(\Omega)} \leq ct^{-m-\nu} \ell_1(t)^\nu \|v\|_{L^2(\Omega)}, \quad v \in L^2(\Omega), m \geq 0,
\]

\[
(A^\nu S^{(m)}(t)v)_{L^2(\Omega)} \leq ct^{-m+1-\nu} \ell_2(t)^{1-\nu} \|Av\|_{L^2(\Omega)}, \quad v \in D(A), \nu + m \geq 1,
\]

where \( \ell_1(t) = (\log(2T/t))^{-1} \), \( \ell_2(t) = \log(\max(t^{-1},2)) \) and \( c_T > 0 \) is a constant that may depend on \( d \), \( \Omega \), \( \mu \), \( M \), \( m \) and \( T \).

Proof. The existence and uniqueness of a weak solution was already shown in [22], and it suffices to show the stability estimates (2.12) and (2.13). First, by the resolvent estimate (2.2), we obtain the following basic estimate on the kernel \( H(z) \)

\[
\|H(z)\| = \|(zw(z)I + A)^{-1}\|w(z)\| \leq M/|z| \quad \forall z \in \Sigma_d^\nu.
\]

Let \( t > 0 \), \( \theta \in (\pi/2, \pi) \), \( \delta > 0 \). We choose \( \delta = 1/t \) and denote for short \( \Gamma = \Gamma_{\theta,\delta} \). First we derive the estimate (2.12) for \( \nu = 0 \) and \( m \geq 0 \). By the solution representation (2.5), we deduce

\[
\|S^{(m)}(t)\| = \left\| \frac{1}{2\pi i} \int_{\Gamma} z^m e^{zt} H(z) \, dz \right\| \leq c \int_{\Gamma} |z|^m e^{R(z)t} \|H(z)\| |dz|
\]

\[
\leq c \left( \int_{1/t}^{\infty} r^{-m-1} e^{rt\cos\theta} \, dr + \int_{-\theta}^{\theta} e^{r\cos\psi} t^{-m} \, d\psi \right) \leq ct^{-m}.
\]

Next we prove estimate (2.12) for \( \nu = 1 \) and \( m \geq 0 \). To this end, we take \( \delta = 2T/t \) in the contour \( \Gamma \). By applying the operator \( A \) to both sides of (2.5) and differentiating with respect to time \( t \) we arrive at

\[
AS^{(m)}(t) = \frac{1}{2\pi i} \int_{\Gamma} z^m e^{zt} AH(z) \, dz.
\]

Owing to the identity

\[
AH(z) = A(zw(z)I + A)^{-1}w(z) = (I - zH(z))w(z),
\]
it follows from Lemma 2.2 that
\[(2.15) \quad \|AH(z)\| \leq c|w(z)| \leq c\frac{|z| - 1}{|z| \log |z|} \quad \forall z \in \Sigma_f.
\]

Hence we obtain from (2.14)
\[
\|AS^{(m)}(t)\| \leq c \int \frac{|z|^m |z| - 1}{|z| \log |z|} e^{\Re(z)t} \, dz \\
\leq c \int_0^{2T/t} r^{m-1} \frac{r - 1}{\log r} e^rt \cos \theta \, dr + cT^{1-m} \frac{2T/t - 1}{\log(2T/t)} \int_0^\theta e^{2T \cos \psi} \, d\psi =: I + II.
\]

Since $2T/t \geq 2$, we can bound the first term $I$ by
\[
I \leq c \int_0^{2T/t} r^{m} (\log r)^{-1} e^{rt \cos \theta} \, dr \leq c \ell_1(t) \int_0^{\infty} r^{m} e^{rt \cos \theta} \, dr \leq cT^{1-m-1} \ell_1(t).
\]

Meanwhile the second term $II$ can be bounded by
\[
II = cT^{m-1} (2T/t - 1)/\log(2T/t) \leq cT^{m-1} (2T/t)/\log(2T/t) = cT^{m-1} \ell_1(t).
\]

This shows the first estimate (2.12). To prove the second estimate (2.13) with $\nu = 0$, we choose $\delta = 1/t$ and denote again $\Gamma = \Gamma_{\theta, \delta}$. Then
\[
S^{(m)}(t)v = \frac{1}{2\pi i} \int_{\Gamma} z^{m} e^{zt} H(z)v \, dz = \frac{1}{2\pi i} \int_{\Gamma} z^{m-1} e^{zt} zA^{-1} H(z)Av \, dz.
\]

Upon noting the identity
\[
zA^{-1} H(z) = zw(z)A^{-1}(zw(z)I + A)^{-1} = A^{-1} - (zw(z)I + A)^{-1}
\]
and the fact that $\int_{\Gamma} z^{m-1} e^{zt} \, dz = 0$ for $m \geq 1$, we have
\[
S^{(m)}(t)v = \frac{1}{2\pi i} \int_{\Gamma} z^{m-1} e^{zt} vz \, dz + \frac{1}{2\pi i} \int_{\Gamma} z^{m-1} e^{zt} (zw(z)I + A)^{-1} \, dz Av \\
= -\frac{1}{2\pi i} \int_{\Gamma} z^{m-1} e^{zt} (zw(z)I + A)^{-1} \, dz Av.
\]

By (2.2) and Lemma 2.3 we obtain
\[
\|(zw(z)I + A)^{-1}\| \leq M|zw(z)|^{-1} \leq c \frac{\log |z|}{|z| - 1},
\]

and thus using this estimate and the the monotone decreasing property of the function $f(x) = -\frac{\log(x)}{1-x}$ on the positive real axis $\mathbb{R}^+$, we get
\[
\|S^{(m)}(t)v\|_{L^2(\Omega)} \leq c \left( \int_{\Gamma} \left| \int_{\Omega} z^{m-1} e^{\Re(z)t} \|(zw(z)I + A)^{-1}\| \, \, |dz| \right| \right) \|Av\|_{L^2(\Omega)} \\
\leq c \left( \int_{1/t}^{\infty} e^{rt \cos \theta} r^{m-1} \frac{\log r}{r - 1} \, dr + t^{-m} \frac{\log(1/t)}{1/t - 1} \int_0^\theta e^{\cos \psi} \, d\psi \right) \|Av\|_{L^2(\Omega)} \\
\leq cT^{m+1} \frac{\log(t^{(-1)})}{1-t} \|Av\|_{L^2(\Omega)}.
\]

We observe that if $t_n^{-1} \geq 2$, i.e. $t_n \leq 1/2$, then $\frac{\log(t_n)}{1-t_n} \leq 2 \log(t_n^{-1})$. Otherwise if $t_n^{-1} < 2$, i.e. $t_n \geq 1/2$, then by the monotonicity of the function $f(x) = -\frac{\log(x)}{1-x}$ on $\mathbb{R}^+$, $\frac{\log(t_n^{-1})}{1-t_n} = \frac{\log(t_n)}{t_n-1} \leq 2 \log(2)$. Then we deduce
\[
\|S^{(m)}(t)v\|_{L^2(\Omega)} \leq cT^{m+1} \ell_2(t) \|Av\|_{L^2(\Omega)}.
\]

Lastly, note that (2.13) with $\nu = 1$ is equivalent to (2.12) with $\nu = 0$ and $v$ replaced by $Av$. This completes the proof of the theorem. \qed
Remark 2.2. The a priori estimate of the solution at short time is given in Theorem 2.1, in which the constant $C_T$ depends on the final time $T$ (see also [22, Theorem 2.2] for the special case $v \in L^2(\Omega)$, $\nu = 1$ and $m = 0$). The long time asymptotic behavior of the solution in case of $v \in D(A)$ was given in [22, Theorem 2.1], i.e., it decays like $(\log t)^{-1}$ as $t \to \infty$; see also [43, example 6.5] for related discussions on asymptotic decay.

3. Semidiscrete discretization by Galerkin FEM

Now we discuss the space semidiscrete scheme (1.5) based on the Galerkin FEM. On the finite element space $X_h$, we define the $L^2(\Omega)$-orthogonal projection $P_h : L^2(\Omega) \to X_h$ and the Ritz projection $R_h : H^1_0(\Omega) \to X_h$, respectively, by

$$
(P_h \varphi, \psi) = (\varphi, \psi) \quad \forall \psi \in X_h,
$$

$$
(\nabla R_h \varphi, \nabla \chi) = (\nabla \varphi, \nabla \chi) \quad \forall \chi \in X_h.
$$

The Ritz projection $R_h$ and the $L^2(\Omega)$-projection $P_h$ have the following properties [42].

Lemma 3.1. Let the mesh $T_h$ be quasi-uniform. Then the operators $R_h$ and $P_h$ satisfy:

$$
\|\varphi - R_h \varphi\|_{L^2(\Omega)} + h\|\nabla (\varphi - R_h \varphi)\|_{L^2(\Omega)} \leq ch^q\|\varphi\|_{H^q(\Omega)} \quad \forall \varphi \in H^1_0(\Omega) \cap H^q(\Omega), \quad q = 1, 2,
$$

$$
\|\varphi - P_h \varphi\|_{L^2(\Omega)} + h\|\nabla (\varphi - P_h \varphi)\|_{L^2(\Omega)} \leq ch^q\|\varphi\|_{H^q(\Omega)} \quad \forall \varphi \in H^1_0(\Omega) \cap H^q(\Omega), \quad q = 1, 2.
$$

In addition, $P_h$ is stable on $H^2_0(\Omega)$ for $0 \leq q \leq 1$.

The space semidiscrete Galerkin scheme for problem (1.1) reads: find $u_h(t) \in X_h$ such that

$$
(\mathcal{D}[u] u_h, \chi) + (\nabla u_h, \nabla \chi) = (f, \chi) \quad \forall \chi \in X_h,
$$

with $u_h(0) = u_h \in X_h$. Upon introducing the discrete Laplacian $\Delta_h : X_h \to X_h$ defined by

$$
-(\Delta_h \varphi, \chi) = (\nabla \varphi, \nabla \chi) \quad \forall \varphi, \chi \in X_h,
$$

and $A_h = -\Delta_h$, the space semidiscrete Galerkin scheme (3.1) can be rewritten as

$$
\mathcal{D}[u] u_h(t) + A_h u_h(t) = 0, \quad t > 0
$$

(3.2)

with $u_h(0) = u_h \in X_h$ and $A_h = -\Delta_h$.

For the error analysis of the semidiscrete scheme (3.2), we employ an operator trick due to Fujita and Suzuki [9]. To this end, we first represent the semidiscrete solution $u_h$ to (3.2) by

$$
uh(t) = S_h(t)v_h := \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt}(zw(z)I + A_h)^{-1}w(z)v_h \, dz.
$$

Lemma 3.2. For any $\psi \in H^1_0(\Omega)$ and $z \in \Sigma_{\theta,\delta}$ for $\theta \in (\pi/2, \pi)$, there holds

$$
|zw(z)||\psi||_{L^2(\Omega)} + ||\nabla \psi||_{L^2(\Omega)} \leq c \left|zw(z)||\psi||_{L^2(\Omega)} + ||\nabla \psi||_{L^2(\Omega)} \right|.
$$

Proof. With Lemma 2.1, the proof is identical to that of [3, Lemma 3.3], and hence omitted. □

Now we introduce the error function $e(t) := u(t) - u_h(t)$ which, in view of (2.5) and (3.3), can be represented by

$$
e(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt}w(z)(\varphi_h(z) - \varphi(z)) \, dz,
$$

(3.5)

with $\varphi(z) = (zw(z)I + A_h)^{-1}v$ and $\varphi_h(z) = (zw(z)I + A_h)^{-1}P_h v$. The following lemma shows a bound on the error $\varphi_h - \varphi$. It follows directly from Lemma 3.2, similar to [3, Lemma 3.4], and hence the proof is omitted.

Lemma 3.3. Let $v \in L^2(\Omega)$, $z \in \Sigma_\theta$ with $\theta \in (\pi/2, \pi)$, $\varphi(z) = (zw(z)I + A_h)^{-1}v$ and $\varphi_h(z) = (zw(z)I + A_h)^{-1}P_h v$. Then there holds

$$
||\varphi(z) - \varphi_h(z)||_{L^2(\Omega)} + h||\nabla (\varphi(z) - \varphi_h(z))||_{L^2(\Omega)} \leq ch^2\|v\|_{L^2(\Omega)}.
$$

(3.6)
Now we can state an error estimate for nonsmooth initial data $v \in L^2(\Omega)$.

**Theorem 3.1.** Let $u$ and $u_h$ be the solutions of problem (1.1) and (3.2) with $v \in L^2(\Omega)$ and $v_h = P_h v$, respectively. Then for $t > 0$ and $\ell(t) = \log(2T/t)^{-1}$, there holds

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} + h \|\nabla (u(t) - u_h(t))\|_{L^2(\Omega)} \leq cT h^2 t^{-1}\ell_1(t)\|v\|_{L^2(\Omega)}.$$  

**Proof.** In the error representation (3.5), by choosing $\delta = 2T/t$ in the contour $\Gamma_{\theta,\delta}$ and appealing to Lemmas 3.3 and 2.2, we deduce

$$\|\nabla e(t)\|_{L^2(\Omega)} \leq c \int_{2T/t}^{\infty} e^{r\cos \theta} \frac{r-1}{r \log r} dr \|v\|_{L^2(\Omega)} + c \int_{-\theta}^{0} e^{2T\cos \psi} \frac{2T/t - 1}{\log(2T/t)} d\psi \|v\|_{L^2(\Omega)} := I + II.$$  

Now the first term $I$ can be bounded by

$$I \leq c h \int_{2T/t}^{\infty} e^{r\cos \theta} \frac{1}{r \log r} dr \|v\|_{L^2(\Omega)} \leq \frac{c h}{\log(2T/t)} \int_{2T/t}^{\infty} e^{r\cos \theta} dr \leq cT h t^{-1}\ell_1(t)\|v\|_{L^2(\Omega)},$$

and the second term $II$ is bounded by

$$II \leq \frac{cT h}{t \log(2T/t)} \int_{-\theta}^{0} e^{2T\cos \psi} d\psi \|v\|_{L^2(\Omega)} \leq cT h t^{-1}\ell_1(t)\|v\|_{L^2(\Omega)}.$$  

The bound on $\|\nabla e(t)\|_{L^2(\Omega)}$ now follows by the triangle inequality. A similar argument yields the desired $L^2(\Omega)$ error estimate. \qed

Next we turn to the case of smooth initial data, i.e., $Av \in L^2(\Omega)$, and derive the following error estimate.

**Theorem 3.2.** Let $u$ and $u_h$ be the solutions of problem (1.1) and (3.2) with $v \in H^2(\Omega)$ and $v_h = P_h v$, respectively. Then for $t > 0$, there holds:

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} + h \|\nabla (u(t) - u_h(t))\|_{L^2(\Omega)} \leq c h^2 \|Av\|_{L^2(\Omega)}.$$  

**Proof.** Like before, we take $\theta \in (\pi/2, \pi)$ and $\delta = 1/t$ in the contour $\Gamma_{\theta,\delta}$. Then the error $e_h(t) = u(t) - u_h(t)$ can be represented by

$$e_h(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt} w(z) ((zw(z)I + A)^{-1} - (zw(z)I + A_h)^{-1}R_h) v dz.$$  

Using the identity

$$w(z)(zw(z)I + A)^{-1} = z^{-1}I - z^{-1}(zw(z)I + A)^{-1}A,$$

we deduce

$$e_h(t) = \frac{1}{2\pi i} \left( \int_{\Gamma_{\theta,\delta}} e^{zt} z^{-1}(\widehat{\varphi}_h(z) - \widehat{\varphi}(z)) dz + \int_{\Gamma_{\theta,\delta}} e^{zt} z^{-1} (v - R_h v) dz \right),$$

where $\widehat{\varphi}(z) = (zw(z)I + A)^{-1}Av$ and $\widehat{\varphi}_h(z) = (zw(z)I + A_h)^{-1}A_h R_h v$. Then Lemmas 3.1 and 3.3, and the identity $A_h R_h = P_h A$ give

$$\|\widehat{\varphi}(z) - \widehat{\varphi}_h(z)\|_{L^2(\Omega)} + h \|\nabla (\widehat{\varphi}(z) - \widehat{\varphi}_h(z))\|_{L^2(\Omega)} \leq c h^2 \|Av\|_{L^2(\Omega)}.$$  

Now it follows from this and the representation (3.8) that

$$\|e_h(t)\| \leq c h^2 \|Av\|_{L^2(\Omega)} \left( \int_{1/t}^{\infty} e^{r\cos \theta} r^{-1} dr + \int_{-\theta}^{0} e^{\cos \psi} d\psi \right) \leq c h^2 \|Av\|_{L^2(\Omega)},$$

which gives the $L^2(\Omega)$-error estimate. The $H^1(\Omega)$ estimate follows analogously. \qed

**Remark 3.1.** The error estimate for nonsmooth initial data $v \in L^2(\Omega)$ deteriorates like $t^{-1}\ell_1(t)$ as $t \to 0^+$. The behavior agrees with the solution singularity in Theorem 2.1. The factor $t^{-1}\ell_1(t)$ is different from that for subdiffusion [15] and multi-term time fractional diffusion [14]. In contrast, for smooth initial data $Av \in L^2(\Omega)$, the error estimate is uniform in $t$. 


4. Fully discrete scheme I: Laplace transform

The first fully discrete scheme is based on the Laplace transform. To this end, we select a proper contour $\Gamma_{\theta,\delta}$ in the integral representation (3.3) of the semidiscrete solution $u_h$, and then apply a quadrature rule. We follow the works [10, 24, 32, 38, 39, 44] and deform the contour $\Gamma_{\theta,\delta}$ to be a curve with the following parametric representation

$$z(\xi) := \lambda(1 + \sin(i\xi - \psi)),$$

with $\lambda > 0$, $\psi \in (0, \pi/2)$ and $\xi \in \mathbb{R}$. The optimal choices of $\lambda$ and $\psi$ will be given below, in the proof of Lemma 4.4. This deformation is valid since it does not transverse the poles of the kernel function $H(z)v = (zw(z) + A_h)^{-1}w(z)v$, cf., Lemma 2.1 and Lemma 4.3 below. Upon letting $z = x + iy$, we deduce that the contour (4.1) is the left branch of the hyperbola

$$\left(\frac{x - \lambda}{\lambda \sin \psi}\right)^2 - \left(\frac{y}{\lambda \cos \psi}\right)^2 = 1,$$

which intersects the real axis at $x = \lambda(1 \pm \sin \psi)$ and has asymptotes $y = \pm(\lambda - x)\cot \psi$. Now we can represent the semidiscrete solution $u_h(t)$ by

$$u_h(t) = \int_{-\infty}^{\infty} \hat{g}(\xi, t) \, d\xi,$$

with the integrand $\hat{g}(\xi, t)$ being defined by

$$\hat{g}(\xi, t) = \frac{1}{2\pi i} e^{z(\xi)t} (z(\xi)w(z(\xi))I + A_h)^{-1}w(z(\xi))z'(\xi) v_h.$$

**Remark 4.1.** The integrand $\hat{g}(\xi, t)$ exhibits a double exponential decay as $|\xi| \to \infty$ for $t > 0$.

Now we describe the quadrature rule for approximating (4.3). By setting $z_j = z(\xi_j)$ and $z'_j := z'(\xi_j)$ with $\xi_j = jk$ and $k$ being the step size, we have the following quadrature approximation

$$U_h(t) = \frac{k}{2\pi i} \sum_{j=-\infty}^{\infty} e^{z_j t} \hat{\phi}_j z'_j v_h,$$

and the truncated quadrature approximation

$$U_{N,h}(t) = \frac{k}{2\pi i} \sum_{j=-N}^{N} e^{z_j t} \hat{\phi}_j z'_j,$$

with $\hat{\phi}_j := (z_j w(z_j)I + A_h)^{-1}w(z_j)v_h$. To compute $U_{N,h}(t)$, we need to solve only $N + 1$ elliptic problems, instead of $2N + 1$ elliptic problems, by exploiting the conjugacy relations: $z_{-j} = \overline{z}_j$, $w(z_{-j}) = \overline{w(z_j)}$, $\hat{\phi}_{-j} = \overline{\hat{\phi}_j}$, $j = 1, \cdots, N$. Indeed, since $z'_j = z'(\xi_j) = i\lambda \cos(i\xi_j - \psi)$, denoting by $\zeta_j = \lambda \cos(i\xi_j - \psi)$, (4.6) is reduced to

$$U_{N,h}(t) = \frac{k}{\pi} \left[ \frac{1}{2} e^{z_0 t} \hat{\phi}_0 \zeta_0 + \sum_{j=1}^{N} \Re\{e^{z_j t} \hat{\phi}_j \zeta_j\} \right].$$

Hence we solve the following complex–valued elliptic problems

$$(z_j w(z_j)I + A_h) \hat{\phi}_j = w(z_j)v_h, \quad j = 0, \ldots, N.$$  

These problems are independent of each other and can be solved in parallel, if desired. 

Next, we define a strip $S_{a,b} \subset \mathbb{C}$ by

$$S_{a,b} = \{p = \xi + i\eta : \text{for all } \xi \in \mathbb{R} \text{ and } \eta \in (-b, a)\}.$$  

The following lemma recalls a known error estimate for the quadrature [29] [44, Theorem 2.1]. The quadrature is exponentially convergent, provided that the integrand $g$ is analytic on a strip $S_{a,b}$ with some additional conditions.
Lemma 4.1. Let $g$ be an analytic function in a strip $S_{a,b}$ for some $a, b > 0$, and $I$ and $I_k$, for $k > 0$, be defined by

$$I = \int_{-\infty}^{\infty} g(x) \, dx \quad \text{and} \quad I_k = k \sum_{j=-\infty}^{\infty} g(jk),$$

respectively. Furthermore, assume that $g(z) \to 0$ uniformly as $|z| \to \infty$ in the strip $S_{a,b}$, and that there exist $M_+ > 0$ and $M_- > 0$, which may depend on $a$ and $b$ such that

$$\lim_{r \to a^-} \int_{-\infty}^{\infty} |g(x + ir)| \, dx \leq M_+, \quad \lim_{s \to b^+} \int_{-\infty}^{\infty} |g(x - is)| \, dx \leq M_-. $$

Then the approximation error can be bounded by

$$|I - I_k| \leq E^+ + E^-,$$

where

$$E^+ = \frac{M_+}{e^{2a\pi/k} - 1} \quad \text{and} \quad E^- = \frac{M_-}{e^{2b\pi/k} - 1}.$$ 

The next lemma gives one crucial estimate on the map $z(p)$ over the strip $S_{a,b}$. Even though the hyperbolic contour (4.1) has been extensively used, the estimate on the map $z(p)$ below seems to be new and it is of independent interest.

Lemma 4.2. Let $p = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}$. Then with $a = \pi/2 - \psi - \epsilon$ and $b = \psi - \epsilon$, for small $\epsilon > 0$, there holds

$$z(p) \in \Sigma_{\pi - \psi} \quad \text{and} \quad \left| \frac{z'(p)}{z(p)} \right| \leq \frac{c}{\epsilon} \quad \forall p \in \mathcal{S}_{a,0},$$

(4.9)

$$z(p) \in \Sigma_{\pi - \epsilon} \quad \text{and} \quad \left| \frac{z'(p)}{z(p)} \right| \leq c \quad \forall p \in \mathcal{S}_{0,b}.$$ 

(4.10)

Proof. For $p = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}$, then the image $z(p)$ in the parameterization (4.1) is given by

$$z(p) = \lambda(1 - \sin(\psi + \eta) \cosh(\xi)) + i\lambda\cos(\psi + \eta)\sinh(\xi),$$

and its derivative $z'(p)$ is given by

$$z'(p) = \lambda \cosh \xi \cos(\psi + \eta) - i \sin \xi \sin(\psi + \eta).$$

By writing $z = x + iy$, it can be expressed as the left branch of the hyperbola

$$\left( \frac{x - \lambda}{\lambda \sin(\psi + \eta)} \right)^2 - \left( \frac{y}{\cos(\psi + \eta)} \right)^2 = 1.$$ 

It intersects the real axis at $x = \lambda(1 - \sin(\psi + \eta))$ and has the asymptotes $y = \pm(x - \lambda) \cot(\psi + \eta)$. Next we show the estimates (4.9) and (4.10). First, for $p \in \mathcal{S}_{a,0}$, i.e., $\eta \in [0, a]$, $z(p)$ lies in the sector $\Sigma_{\pi - \psi}$. Using the elementary identity $\sin^2 x = \cosh^2 x - 1$, the fact $\varphi := \eta + \psi \in (\psi, \pi/2 - \epsilon)$, and the estimate $\sin(\pi/2 - \epsilon) \sim 1 - \epsilon^2/2 \leq 1 - \epsilon^2/3$ for small $\epsilon$, we have for all $\xi \in \mathbb{R}$

$$\left| \frac{z'(p)}{z(p)} \right|^2 = \frac{\cos(\varphi) \cosh(\xi) - i \sin(\varphi) \sinh(\xi)}{\cosh(\xi) \sin(\varphi) + i \sinh(\xi) \cosh(\varphi)} \leq \frac{\cos^2(\varphi) \cosh^2(\xi) + \sin^2(\varphi) \sinh^2(\xi)}{1 - 2 \cosh(\xi) \sin(\varphi) + \cosh^2(\xi) \sin^2(\varphi) + \sinh^2(\xi) \cos^2(\varphi)} \leq \frac{\cosh^2(\xi) - \sin^2(\varphi)}{(\cosh(\xi) - \sin(\varphi))^2} \leq \frac{2}{\left( 1 - \sin(\varphi) \right)^2} \leq \frac{2}{1 - \epsilon^2/3} \leq \frac{6}{\epsilon^2}.$$ 

Hence the estimate (4.9) holds true. Now we turn to the case $p \in \mathcal{S}_{0,b}$, i.e., $\eta \in [-b, 0]$. Then $z(p)$ lies in the sector $\Sigma_{\pi-(\psi+\eta)} \subset \Sigma_{\pi-\epsilon}$. Further, by noting $\varphi := \eta + \psi \in (\epsilon, \psi)$, we have for all $\xi \in \mathbb{R}$

$$\left| \frac{z'(p)}{z(p)} \right|^2 \leq \frac{1 + \sin(\varphi)}{1 - \sin(\varphi)} \leq \frac{1 + \sin(\psi)}{1 - \sin(\psi)}.$$ 

Then the desired result (4.10) follows directly. \hfill \Box
The next result gives the analyticity of and an estimate on the integrand $g(\xi, \xi)$ on the strip $S_{a,b}$.

**Lemma 4.3.** Let $p = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}$ and $\tilde{g}(p)$ be defined by (4.4). Then $\tilde{g}(p)$ is analytic on the strip $S_{a,b}$, and the following estimate holds:

$$\|\tilde{g}(p, t)\| \leq \frac{C}{\epsilon} e^{\lambda(1 - \sin(\psi + \eta) \cosh(\xi))} \|v_h\|_{L^2(\Omega)} \quad \forall p \in S_{a,b}.$$ 

**Proof.** For $p = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}$, the image $z(p)$ in (4.1) is given by

$$z(p) = \lambda(1 - \sin(\psi + \eta) \cosh(\xi)) + i\lambda \cos(\psi + \eta) \sinh(\xi).$$

By Lemmas 4.2 and 2.1, and Remark 2.1, $z(p)w(z(p)) \in \Sigma_{\pi - \epsilon'}$, with $\epsilon' > 0$. Hence the function

$$\tilde{g}(p, t) = \frac{1}{2\pi i} e^{z(p)t} (z(p)w(z(p))I + A_h)^{-1} w(z(p))z'(p)v_h$$

is analytic in the strip $S_{a,b}$. It remains to show the estimate. First, we consider the case $p \in \mathfrak{S}_{0,b}$. By (4.10), $z(p) \in \Sigma_{-\epsilon'}$. Then, by Lemma 2.1 and Remark 2.1, $z(p)w(z(p)) \in \Sigma_{\pi - \epsilon'}$, with $\epsilon' = \epsilon c$. By the resolvent estimate (2.2), we deduce that for small $\epsilon > 0$, there holds

$$\|z(I + A_h)^{-1}\| \leq c/|3(z)| \leq c/|z\sin(\pi - \epsilon)| \leq c/|z| \epsilon' \quad \forall z \in \Sigma_{\pi - \epsilon'}.$$ 

Meanwhile, for any $p \in \mathfrak{S}_{a,b}$, there holds

$$\Re(z(p)) = \lambda(1 - \sin(\psi + \eta) \cosh(\xi)),$$

which together with the resolvent estimate (4.11) and Lemma 2.1 yields

$$\|\tilde{g}(p, t)\| \leq c e^{\Re(z(p))t} |z'(p)w(z(p))| \|z(p)w(z(p)) + A\|^{-1} \|v_h\|_{L^2(\Omega)} \leq \frac{c}{\epsilon} e^{\lambda(1 - \sin(\psi + \eta) \cosh(\xi))} \|v_h\|_{L^2(\Omega)}.$$ 

This together with (4.10) yields the desired assertion. The case $p \in \mathfrak{S}_{a,0}$ is more direct. Then (4.10) and Lemma 2.1 imply that $z(p)w(z(p)) \in \Sigma_{\theta'}$ with $\theta' \in (\pi/2, \pi)$ depending only on $\psi$. Then the desired assertion follows from (4.9) and the resolvent estimate (2.2). \(\square\)

Now we can give an error estimate for the quadrature approximation $U_{N,h}$.

**Lemma 4.4.** Let $u_h(t)$ and $U_{N,h}(t)$ be defined in (4.3) and (4.6), respectively, and the contour be parametrically represented by (4.1). Then with the choice $k = c_0/N$ and $\lambda = c_1 N/t$, there holds

$$\|u_h(t) - U_{N,h}(t)\|_{L^2(\Omega)} \leq c e^{-c' N} \|v\|_{L^2(\Omega)},$$

where the constant $c$ and $c'$ depend on the choice of $\psi$ in (4.1).

**Proof.** We use the following splitting

$$u_h - U_{N,h} = (u_h - U_h) + (U_h - U_{N,h}) =: E_q + E_t,$$

where $E_q$ and $E_t$ denote the quadrature and truncation error, respectively. We apply Lemma 4.1 to bound $\|E_q\|_{L^2(\Omega)}$. To this end, we set $a = \pi/2 - \psi - \epsilon$ and $b = \psi - \epsilon$. For $p = \xi + ia$, $z(p)$ lies on the sector $\Sigma_\theta$ for some $\theta \in (\pi/2, \pi)$. Note the elementary inequalities $\cosh(\xi) \geq 1 + \epsilon^2/2$ and $1 - \sin(\pi/2 - \epsilon) \leq \epsilon$ for small $\epsilon > 0$. These together with the choice $\lambda = c_1 N/t$ and Lemma 4.3 yield

$$\left\| \int_{-\infty}^{\infty} |\tilde{g}(\xi + ia)| \, d\xi \right\|_{L^2(\Omega)} \leq \frac{c}{\epsilon} \int_0^\infty e^{cN(1 - \sin(\pi/2 - \epsilon) \cosh(\xi))} d\xi \|v_h\|_{L^2(\Omega)} \leq \frac{c e^{c_1 Nt}}{\epsilon} \int_0^\infty e^{-c_1 N \sin(\pi/2 - \epsilon)} \xi^2/2 d\xi \|v_h\|_{L^2(\Omega)} \leq \frac{c}{\epsilon} N^{-\frac{1}{2}} e^{c_1 Nt} \|v_h\|_{L^2(\Omega)}.$$ 

Using Lemma 4.1, for $k = c_0/N$ we have

$$\|E_q^+\|_{L^2(\Omega)} \leq \frac{c}{\epsilon} N^{-\frac{1}{2}} e^{-c_0 c_1 N}.$$
Next we bound the error due to the lower half. For the choice $p = \xi - ib$, $\lambda = c_1 N/t$ and appealing again to the inequality $\cosh \xi \geq 1 + \xi^2/2$, we deduce
\[
\left\| \int_{-\infty}^{\infty} |\hat{g}(\xi - ib)| d\xi \right\|_{L^2(\Omega)} \leq \frac{C}{\epsilon} \int_{0}^{\infty} e^{c_1 N(1 - \sin(\epsilon) \cosh(\xi))} d\xi \left\| v_h \right\|_{L^2(\Omega)}
\leq \frac{C}{\epsilon} e^{c_1 N(1 - \sin(\epsilon))} \int_{0}^{\infty} e^{-c_1 N \sin(\epsilon) \xi^2/2} d\xi \left\| v_h \right\|_{L^2(\Omega)}
\leq \frac{c}{\epsilon^{3/2}} N^{-\frac{3}{2}} e^{c_1 N(1 - \epsilon)} \left\| v_h \right\|_{L^2(\Omega)}.
\]
Then for the choice $k = c_0/N$, Lemma 4.1 yields the following estimate
\[
\left\| E_q^- \right\|_{L^2(\Omega)} \leq \frac{c}{\epsilon^{3/2}} N^{-\frac{3}{2}} e^{-2(\pi - \epsilon)/c_0 - c_1 (1 - \epsilon) N}.
\]

Further, by using $\cosh(\xi) \geq \cosh(c_0) + \sinh(c_0)(\xi - c_0)$ for $\xi \geq c_0$, the truncation error $\left\| E_i \right\|_{L^2(\Omega)}$ can be simply estimated by
\[
\left\| E_i \right\|_{L^2(\Omega)} \leq \frac{c}{\epsilon} \int_{0}^{\infty} e^{c_1 N(1 - \sin(\psi) \cosh(\xi))} d\xi \left\| v_h \right\|_{L^2(\Omega)}
\leq \frac{c}{\epsilon} e^{c_1 N(1 - \sin(\psi) \cosh(c_0))} \int_{0}^{\infty} e^{-c_1 N \sin(\psi) \sinh(c_0) (\xi - c_0)} d\xi \left\| v_h \right\|_{L^2(\Omega)}
= \frac{c}{c_1 \sin(\psi) \sinh(c_0)} e^{N^{-1} c_1 N(1 - \sin(\psi) \cosh(c_0))} \left\| v_h \right\|_{L^2(\Omega)}.
\]

Finally, by disregarding $\epsilon$ terms, balancing asymptotically the exponential parts in $\left\| E_q^+ \right\|_{L^2(\Omega)}$, $\left\| E_q^- \right\|_{L^2(\Omega)}$ and $\left\| E_i^- \right\|_{L^2(\Omega)}$, we arrive at
\[
2\pi(\pi/2 - \psi)/c_0 = 2\pi\psi/c_0 - c_1 = c_1(1 - \sin(\psi) \cosh(c_0)).
\]

We may express the parameters $c_0$ and $c_1$ in terms of $\psi$:
\[
c_0 = \cosh^{-1}\left(\frac{2\pi\psi}{(4\pi\psi - \pi^2) \sin(\psi)}\right) \quad \text{and} \quad c_1 = (4\pi\psi - \pi^2)/\cosh^{-1}\left(\frac{2\pi\psi}{(4\pi\psi - \pi^2) \sin(\psi)}\right).
\]

Finally we minimize the ratio
\[
B(\psi) = c_1 - 2\pi\psi/c_0
\]
with respect to the parameter $\psi$, which achieves the minimum at $\psi = 1.1721$ and hence,
\[
c_0 = 1.0818, \quad c_1 = 4.4920 \quad \text{and} \quad B(\psi) = -2.32,
\]
which are identical to those values given in [44]. Then collecting the balanced asymptotic bound and the rest from $\left\| E_q^+ \right\|_{L^2(\Omega)}$, $\left\| E_q^- \right\|_{L^2(\Omega)}$ and $\left\| E_i^- \right\|_{L^2(\Omega)}$ yields
\[
\left\| u_h(t) - U_{N,h}(t) \right\|_{L^2(\Omega)} \leq c \left( e^{-1} N^{-1/2} + e^{-3/2} N^{-1/2} + e^{-1} N \right) e^{-2.32 - (2\pi/c_0 + c_1) \epsilon} N \left\| v_h \right\|_{L^2(\Omega)}.
\]

Now by choosing $\epsilon = 1/N$, we get
\[
\left\| u_h(t) - U_{N,h}(t) \right\|_{L^2(\Omega)} \leq c e^{-2.32 + \frac{4\pi\psi}{(4\pi\psi - \pi^2) \sin(\psi)}} N \left\| v_h \right\|_{L^2(\Omega)}
\]
which together with the fact $(\log x)/x \leq 1/e$ for $x \geq 1$ and the $L^2$-stability of the projection $P_h$ yields the desired result.

Last, we give the main result of this section, i.e., error estimates for the fully discrete scheme (4.6). It follows from Theorems 3.1 and 3.2, and Lemma 4.4 and the triangle inequality.

**Theorem 4.1.** Let $u(t)$ be the solution of problem (1.1), and $U_{N,h}(t)$ be the quadrature approximation defined in (4.6), with the parameters chosen as in Lemma 4.4. Then with $t_1(t) = (\log 2T/t)^{-1}$, the following estimates hold.

(a) If $Av \in L^2(\Omega)$ and $v_h = R_h v$, then
\[
\left\| u(t) - U_{N,h}(t) \right\|_{L^2(\Omega)} \leq c \left( e^{-c_1 N} + h^2 \right) \left\| Av \right\|_{L^2(\Omega)}.
\]
(b) If \( v \in L^2(\Omega) \) and \( v_h = P_h v \), then
\[
\|u(t) - U_{N,h}\|_{L^2(\Omega)} \leq c_T \left( e^{-c N} + h^2 t^{-1} \ell_1(t) \right) \|v\|_{L^2(\Omega)}.
\]

5. Fully Discrete Scheme II: Convolution Quadrature

Now we develop a second fully discrete scheme based on convolution quadrature generated by the backward Euler method, and show that the scheme is first order convergent.

5.1. Time stepping based on convolution quadratures. To describe the fully discrete scheme, we divide the interval \([0, T]\) into a uniform grid with a time step size \( \tau = T/N \), \( N \in \mathbb{N} \), with \( 0 = t_0 < t_1 < \ldots < t_N = T \), and \( t_n = n\tau \), \( n = 0, \ldots, N \). The general construction of convolution quadrature is as follows [25, 7]. Let \((\sigma, \rho)\) be a stable and consistent implicit linear multistep method, with \((\sigma, \rho)\) being its characteristic polynomials. Then we define convolution quadrature weights \(\{b_j\}_{j=0}^\infty\) by the expansion coefficients of
\[
\tilde{\omega}(\xi) = \sum_{j=0}^\infty b_j \xi^j = \int_0^1 \left( \frac{\sigma(1/\xi)}{\rho(1/\xi)} \right)^\alpha \mu(\alpha) \, d\alpha.
\]

We consider only the simplest case, i.e., the backward Euler method, for which the convolution quadrature weights \(\{b_j\}_{j=0}^\infty\) are defined by
\[
\tilde{\omega}(\xi) = \sum_{j=0}^\infty b_j \xi^j = \int_0^1 \left( \frac{1 - \xi}{\tau} \right)^\alpha \mu(\alpha) \, d\alpha = \left( \frac{1 - \xi}{\tau} \right) w \left( \frac{1 - \xi}{\tau} \right).
\]

The convolution quadrature weights \(\{b_j\}\) can be computed efficiently using the fast Fourier transform [37], in view of Cauchy’s theorem. Then the convolution quadrature \(Q_{\tau} \varphi\) for a Riemann-Liouville fractional derivative \(R^\alpha D_t^\alpha \varphi(t)\) generated by the backward Euler method is given by
\[
(Q_{\tau} \varphi)(t_n) = \sum_{j=0}^n b_{n-j} \varphi(j\tau).
\]

Following this general construction, we now derive the time stepping scheme. The approximation \(Q_n(\varphi)\) to the Riemann-Liouville fractional derivative \(R^\alpha D_t^\alpha \varphi(t_n)\) is given by [7, 16]: for any \( n = 1, 2, \ldots, N \):
\[
Q_n(\varphi) = \sum_{j=1}^n b_{n-j} \varphi(t_j),
\]

where the weights \(\{b_j\}\) are generated by (5.1). Recall also the defining relation of the Caputo derivative using the Riemann-Liouville derivative [20, p. 91, equation (2.4.4)] \(D_t^\alpha u = R^\alpha D_t^\alpha (u - u(0))\). Upon applying the convolution quadrature to the term on the right hand side and using it for the semidiscrete problem (3.2), we arrive at the following fully discrete scheme for the model (1.1): for \( n = 1, 2, \ldots, N \)
\[
Q_n(U_h) + A_h U_n^h = Q_n(1)v_h,
\]
with \(U_0^h = v_h\). Throughout, we denote the generating function \(\tilde{\beta}\) of a sequence \(\{\beta_j\}_{j=0}^\infty\) by \(\tilde{\beta}(\xi) = \sum_{j=0}^\infty \beta_j \xi^j\).

Remark 5.1. Compared with the general construction (5.2), the term corresponding to \(j = 0\) is omitted in our fully discrete scheme (5.4). This choice was taken earlier in [26, 3].

5.2. Error analysis. Now we carry out the error analysis of the fully discrete scheme (5.4), following the strategy outlined in the pioneering work [26]. To derive \(L^2(\Omega)\)-error estimates, we split the error into
\[
e^n = u(t_n) - U^n_h = (u(t_n) - u_h(t_n)) + (u_h(t_n) - U^n_h).
\]

In view of Theorems 3.1 and 3.2, it suffices to establish a bound on \(\|u_h(t_n) - U^n_h\|_{L^2(\Omega)}\). The proof relies on the following splitting
\[
u_h(t_n) - U^n_h = y_h(t) - Y^n_h.
\]
where

\[ y_h(t) = u_h(t) - v_h \quad \text{and} \quad Y^n_h = U^n_r - v_h. \]

First, we derive representations of the semidiscrete solution \( y_h \) and fully discrete solution \( Y_h \).

**Lemma 5.1.** Let the kernel \( K(z) \) be defined by

\[ K(z) = -z^{-1}(zw(z)I + A_h)^{-1}A_h \]

and \( \chi(z) = \frac{1 - e^{-z}}{\tau} \). Then \( y_h \) and \( Y^n_h \) can be represented by

\[ y_h(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta, \delta}} e^{zt}K(z)\,v_h\,dz \quad \text{and} \quad Y^n_h = \frac{1}{2\pi i} \int_{\Gamma_r} e^{zt\,n-1}K(\chi(z))v_h\,dz, \]

respectively, with the contour \( \Gamma_r = \{ z \in \Gamma_{\theta, \delta} : |\Im(z)| \leq \pi/\tau \} \).

**Proof.** By its definition, \( y_h \) satisfies the problem:

\[ D^\mu_t y_h + Ah y_h = -Ah v_h, \]

with \( y_h(0) = 0 \). The Laplace transform gives

\[ zw(z)\hat{y}_h(z) + Ah \hat{y}_h(z) = -z^{-1}Ah v_h. \]

Hence, \( \hat{y}_h(z) = K(z)v_h \), with \( K(z) = -z^{-1}(zw(z)I + A_h)^{-1}A_h \), and the desired representation for \( y_h(t) \) follows from the inverse Laplace transform. Next, the fully discrete solution \( Y^n_h \) satisfies the following time stepping scheme

\[ Q_n(Y_h) + AY^n_h = -Ah v_h, \]

with \( Y^0_h = 0 \). Now multiplying both sides by \( \xi^n \), summing from \( 1 \) to \( \infty \) and noting \( Y^0_h = 0 \) yield

\[ \sum_{n=1}^{\infty} Q_n(Y_h) \xi^n + Ah \tilde{Y}_h(\xi) = -\xi/(1 - \xi)Ah v_h. \]

Using the condition \( Y^0_h = 0 \), we have

\[ \sum_{n=1}^{\infty} Q_n(Y_h) \xi^n = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (b_{n-j}\xi^{n-j}) \left( Y^j_h \xi^j \right) = ((1 - \xi)/\tau)w((1 - \xi)/\tau)\tilde{Y}_h(\xi). \]

Thus, by simple calculation, we deduce

\[ \tilde{Y}_h(\xi) = (\xi/\tau)K((1 - \xi)/\tau)v_h, \]

and it is analytic at \( \xi = 0 \). Then Cauchy theorem implies that for \( \theta \) small enough, there holds

\[ Y^n_h = \frac{1}{2\pi i} \int_{|\xi| = \theta} \xi^{-n}K((1 - \xi)/\tau)v_h\,d\xi. \]

Now, by changing variable \( \xi = e^{-zt} \), we obtain

\[ Y^n_h = \frac{1}{2\pi i} \int_{\Gamma_0} e^{zt\,n-1}K((1 - e^{-zt})/\tau)v_h\,dz, \]

where the contour \( \Gamma_0 = \{ z = -\ln(\theta)/\tau + iy : |y| \leq \pi/\tau \} \) is oriented counterclockwise. We obtain the desired representation by deforming the contour \( \Gamma_0 \) to \( \Gamma_r = \{ z \in \Gamma_{\theta, \delta} : |\Im(z)| \leq \pi/\tau \} \) and using the periodicity of the exponential function. \( \square \)

By Lemma 5.1, we can write the difference between \( Y^n_h \) and \( y_h(t_n) \) as

\[ y_h(t_n) - Y^n_h = I + II, \]

where the terms \( I \) and \( II \) are given by

\[ I = \frac{1}{2\pi i} \int_{\Gamma_{\theta, \delta} \setminus \Gamma_r} e^{zt}\,K(z)v_h\,dz \]
and
\[ II = \frac{1}{2\pi i} \int_{\Gamma} e^{z\tau} \left( K(z) - e^{-z\tau} K(\chi(z)) \right) v_k dz. \]

This splitting is essential for the error analysis below. Since the function \( |e^{-z\tau}| \) is uniformly bounded on the contour \( \Gamma_\tau \), we have
\[
\| K(z) - e^{-z\tau} K(\chi(z)) \| \leq |e^{-z\tau}|||K(z)|-K(\chi(z))| + |1-e^{-z\tau}|||K(z)|
\]
\[
\leq c \|K(z)-K(\chi(z))\| + c\tau ||K(z)||
\]
(5.8)
where the last line, using the resolvent estimate (2.2), follows from the inequality
\[ \|K(z)\| = |z|^{-1} - I + zw(z)(zw(z) + A_k)^{-1} \leq c|z|^{-1}. \]

Hence, it remains to bound the term \( \|K(z) - K(\chi(z))\| \), which will be carried out in several steps. First we recall a bound on the function \( \chi(z) = \tau^{-1}(1 - e^{-z\tau}) [17, \text{Lemma 3.1}] \).

**Lemma 5.2.** Let \( \chi(z) = \tau^{-1}(1 - e^{-z\tau}) \). Then for all \( z \in \Gamma_\tau \), there hold
\[ |\chi(z) - z| \leq c|z|^2 \tau \quad \text{and} \quad c_1|z| \leq |\chi(z)| \leq c_2|z|, \]
and \( \chi(z) \) lies in a sector \( \Sigma_{\theta'} \) for some \( \theta' \in (\pi/2, \pi) \).

Next we give one crucial error estimate on the approximation \( \chi(z)w(\chi(z)) \) to the kernel \( zw(z) \).

**Lemma 5.3.** For \( z \in \Gamma_\tau \), the following bound holds:
\[ |\chi(z)w(\chi(z)) - zw(z)| \leq c\tau|z|^2w(|z|). \]

**Proof.** By the intermediate value theorem, for \( z \in \Gamma_\tau \), we have
\[ |\chi(z)^\alpha - z^\alpha| = a \left| \int_z^{\chi(z)} s^{\alpha-1} ds \right| \leq a|\chi(z) - z| \max_{\eta \in [0,1]} |z_\eta|^\alpha - 1, \]
where \( z_\eta = \eta \chi(z) + (1 - \eta)z \) with \( \eta \in [0,1] \). Next we claim \( |z_\eta|^{-1} \leq c|z|^{-1} \) for \( \eta \in [0,1] \). To this end, we split \( \Gamma_\tau \) into \( \Gamma_\tau = \Gamma_\tau^+ \cup \Gamma_\tau^- \cup \Gamma_\tau^0 \), with \( \Gamma_\tau^\pm \) being the rays in the upper and lower half plane, respectively, and \( \Gamma_\tau^0 \) is the circular arc. For \( z \in \Gamma_\tau^0 \), by the Taylor expansion of \( e^{-z\tau} \), we have
\[ z_\eta = z \left( 1 + \eta \sum_{j=1}^{\infty} (-1)^j \frac{z^j \tau^j}{(j+1)!} \right). \]

In view of the trivial inequality \( |z\tau| \leq 1 \) for \( z \in \Gamma_\tau^0 \), we deduce \( |z_\eta|^{-1} \leq c|z|^{-1} \) for \( z \in \Gamma_\tau^0 \). It remains to show the assertion for \( z \in \Gamma_\tau^\pm \), and the case \( z \in \Gamma_\tau^0 \) follows analogously. First we show \( \Im(\chi(z)) > 0 \) for \( z \in \Gamma_\tau^+ \). For \( z = re^{(\pi - \theta)} \) with \( r\tau \in (\delta, \pi/\sin \theta) \) we have
\[ \chi(z) = \frac{1}{r} \left( 1 - e^{r\tau \cos \theta} e^{-r\tau \sin \theta} \right), \]
and therefore using \( r\tau \sin \theta \leq \pi \), we get \( \Im(\chi(z)) \geq 0 \). Then Lemma 5.2 yields
\[ |z_\eta| > \min(|z|, |\chi(z)|) \frac{\cos \theta}{2} \geq c|z|. \]

This shows the desired claim. Hence, appealing to Lemma 5.2 again implies that for \( z \in \Gamma_\tau \) there holds
\[
\int_0^1 (\chi(z)^\alpha - z^\alpha) \mu(\alpha) d\alpha \leq \int_0^1 |\chi(z)^\alpha - z^\alpha| \mu(\alpha) d\alpha \leq c\tau|z| \int_0^1 |z^\alpha| \mu(\alpha) d\alpha = c\tau |z|^2w(|z|),
\]
which concludes the proof of the lemma. \( \square \)

Next we give a crucial error estimate on the approximation \( K(\chi(z)) \) to the kernel function \( K(z) \).

**Lemma 5.4.** Let \( \chi(z) = (1 - e^{-z\tau})/\tau \). Then for the kernel \( K(z) \) in (5.5), there holds
\[ \|K(z) - K(\chi(z))\| \leq c\tau \quad \forall z \in \Gamma_\tau. \]
Proof. Let $B(z) = zK(z)$. Simple computation shows
\[
B(z) - B(\chi(z)) = zw(z)(zw(z)I + A_h)^{-1} - \chi(z)w(\chi(z))(\chi(z)w(z))I + A_h)^{-1}
\]
\[
= zw(z)\left((zw(z)I + A_h)^{-1} - (\chi(z)w(\chi(z))I + A_h)^{-1}\right)
\]
\[
+ (zw(z) - (\chi(z)w(\chi(z))) (\chi(z)w(z))I + A_h)^{-1} := I + II.
\]
First, by Lemmas 2.3 and 5.2, there holds
\[
|\chi(z)w(\chi(z))| \geq c|\chi(z)|w(|\chi(z)|) \geq c|z|w(|z|).
\]
Further, by Lemma 2.1 and (2.2) and Lemma 2.3, we have
\[
(5.9) \quad \|(zw(z)I + A_h)^{-1}\| \leq c|zw(z)|^{-1} \leq c(|z|w(|z|))^{-1}.
\]
Likewise, in view of Lemmas 5.2 and 2.1 and (2.2), we have
\[
(5.10) \quad \|(\chi(z)w(\chi(z))I + A_h)^{-1}\| \leq c|\chi(z)w(\chi(z))|^{-1} \leq c(|z|w(|z|))^{-1}.
\]
Now, by the identity
\[
(zw(z)I + A_h)^{-1} - (\chi(z)w(\chi(z))I + A_h)^{-1}
\]
\[
= (zw(z) - (\chi(z)w(\chi(z))) (zw(z)I + A_h)^{-1} (\chi(z)w(\chi(z))I + A_h)^{-1},
\]
Lemma 2.3, (5.9) and (5.10), the first term $I$ can be bounded by
\[
\|I\| \leq c\tau|z|^3w(|z|)^2\|(zw(z)I + A_h)^{-1}\|\|(\chi(z)w(\chi(z))I + A_h)^{-1}\| \leq c\tau|z|.
\]
Likewise, with Lemma 5.3 and (5.10), the second term $II$ can be bounded by
\[
\|II\| \leq |zw(z) - (\chi(z)w(\chi(z))|\|(\chi(z)w(\chi(z))I + A_h)^{-1}\|
\]
\[
\leq c\tau|z|^2w(|z|)|zw(|z|))^{-1} \leq c\tau|z|.
\]
Hence we bound $\|B(z) - B(\chi(z))\|$ by
\[
\|B(z) - B(\chi(z))\| \leq c\tau|z|.
\]
Last, by Lemma 5.2 and $\|B(z)\| \leq c$, we bound $\|K(z) - K(\chi(z))\|$ by
\[
\|K(z) - K(\chi(z))\| \leq |z|^{-1} - (\chi(z))^{-1}\|B(z)\| + |z|^{-1}\|B(z) - B(\chi(z))\|
\]
\[
\leq c|z - \chi(z)||z|^{-2} + c\tau \leq c\tau,
\]
which completes the proof of the lemma. \qed

Now we can state an error estimate on the time discretization error for nonsmooth initial data, i.e.,
v \in L^2(\Omega).

**Theorem 5.1.** Let $u_h$ and $U^n_h$ be the solutions of problems (3.2) and (5.4) with $v \in L^2(\Omega)$, $U^n_0 = v_h = P_h v$
and $f \equiv 0$, respectively. Then there holds
\[
\|u_h(t_n) - U^n_h\|_{L^2(\Omega)} \leq c\tau t_n^{-1}\|v\|_{L^2(\Omega)}.
\]

Proof. It suffices to bound the terms $I$ and $II$ defined in (5.6) and (5.7), respectively. We choose $\delta = t_n^{-1}$ in the contour $\Gamma_{\delta, \theta}$. By (2.2) and direct calculation, we bound the first term $I$ by
\[
\|I\|_{L^2(\Omega)} \leq c\int_{\pi/(\tau \sin \theta)}^{\pi/(\tau \sin \theta)} e^{r_n \cos \theta} r_n^{-1} \, dr \|v_h\|_{L^2(\Omega)}
\]
\[
\leq c\tau \|v_h\|_{L^2(\Omega)} \int_0^\infty e^{r_n \cos \theta} \, dr \leq c\tau t_n^{-1}\|v_h\|_{L^2(\Omega)}.
\]
Using Lemma 5.4, we arrive at the following bound for the second term $II$ :
\[
\|II\|_{L^2(\Omega)} \leq c\tau \|v_h\|_{L^2(\Omega)} \left(\int_{1/t_n}^{\pi/(\tau \sin \theta)} e^{r_n \cos \theta} \, dr + \int_0^\theta e^{\cos \psi t_n^{-1}} d\psi\right) \leq c\tau t_n^{-1}\|v_h\|_{L^2(\Omega)}.
\]
Combining estimates (5.11) and (5.12) yields
\[
\|y_h(t_n) - Y_h^n\|_{L^2(\Omega)} \leq crt_n^{-1}\|v_h\|_{L^2(\Omega)},
\]
and the desired result follows directly from the identity \(U_h^n - u_h(t_n) = Y_h^n - y_h(t_n)\) and the stability of the projection \(P_h\) in \(L^2(\Omega)\).

**Remark 5.2.** The \(L^2(\Omega)\) stability of the time stepping scheme (5.4) follows directly from Theorem 5.1.

Next we turn to smooth initial data, i.e., \(Av \in L^2(\Omega)\). To this end, we first state an alternative estimate on the solution kernel \(K(z)\).

**Lemma 5.5.** Let \(K^s(z) = -z^{-1}(zw(z)I + A_h)^{-1}\). Then for any \(z \in \Gamma_r\), there holds
\[
\|K^s(z) - K^s(\chi(z))\| \leq cr\frac{\log|z|}{|z| - 1}.
\]

**Proof.** Let \(B^s(z) = -(zw(z)I + A_h)^{-1}\). Then by the trivial inequality
\[
B^s(z) - B^s(\chi(z)) = \chi(z)w(\chi(z)) - zw(z) (zw(z)I + A_h)^{-1} (\chi(z)w(\chi(z))I + A_h)^{-1}
\]
Lemma 5.3, and (5.9) and (5.10), we deduce immediately
\[
\|B^s(z) - B^s(\chi(z))\| \leq c|zw(z)|^{-1}.
\]

Appealing to 5.9 again, we have \(\|B^s(z)\| \leq c|zw(z)|^{-1}\), and thus
\[
\|K^s(z) - K^s(\chi(z))\| \leq |z|^{-1} - \chi(z)^{-1}\|B^s(z)\| + |\chi(z)|^{-1}\|B^s(z) - B^s(\chi(z))\|
\]
\[
\leq c|z - \chi(z)||z|^{-3}|w(z)|^{-1} + cr|zw(z)|^{-1} \leq cr|zw(z)|^{-1}.
\]

Then the desired result follows from Lemma 2.3.

Now we can state an error estimate for smooth initial data \(Av \in L^2(\Omega)\).

**Theorem 5.2.** Let \(u_h\) and \(U_h^n\) be the solutions of problems (3.2) and (5.4) with \(Av \in L^2(\Omega), U_h^0 = v_h = R_hv\) and \(f \equiv 0\), respectively. Then for \(\ell_2(t) = \log(\max(t^{-1}, 2))\), there holds
\[
\|u_h(t_n) - U_h^n\|_{L^2(\Omega)} \leq c\ell_2(t)\|Av\|_{L^2(\Omega)}.
\]

**Proof.** Let \(K^s(z) = -z^{-1}(zw(z)I + A_h)^{-1}\). Then we can rewrite the error as
\[
y_h(t_n) - Y_h^n = \frac{1}{2\pi i} \int_{\Gamma_r} e^{zt} K^s(z) A_h v_h dz
\]
(5.13)
\[
+ \frac{1}{2\pi i} \int_{\Gamma_r} e^{zt} \left(K^s(z) - e^{-z\tau} K^s(\chi(z))\right) A_h v_h dz := I + II.
\]

By Lemma 5.5 we have for \(z \in \Gamma_r\)
\[
\|K^s(z) - e^{-z\tau} K^s(\chi(z))\| \leq cr\frac{\log|z|}{|z| - 1}.
\]

By setting \(\delta = 1/t_n\) and by the monotonicity of the function \(f(x) = \frac{\log(x)}{1-x}\) on \(\mathbb{R}^+\), we derive the following bound for the term \(II\)
\[
\|II\|_{L^2(\Omega)} \leq c\tau\|A_h v_h\|_{L^2(\Omega)} \left(\int_{1/t_n}^{\pi/(t \sin \theta)} e^{\pi t_n \cos \theta} \frac{\log r}{r-1} dr + \int_{-\theta}^{\theta} e^{\cos \psi} \frac{\log(t_n^{-1})}{1-t_n^{-1}} d\psi\right) \leq c\frac{\log(t_n^{-1})}{1-t_n^{-1}} \tau\|A_h v_h\|_{L^2(\Omega)}.
\]

Now (2.2) implies that for all \(z \in \Gamma_{\theta, \delta}\), \(\|K^s(z)\| \leq c|z|^{-1}|zw(z)|^{-1}\). Therefore, using Lemma 2.3, we deduce
\[
\|I\|_{L^2(\Omega)} \leq c\|A_h v_h\|_{L^2(\Omega)} \int_{1/t_n}^{\infty} e^{\pi t_n \cos \theta} r^{-2}|w(r)|^{-1} dr
\]
(5.14)
\[
\leq c\tau\|A_h v_h\|_{L^2(\Omega)} \int_{1/t_n}^{\infty} e^{\pi t_n \cos \theta} \frac{\log r}{r-1} dr \leq c\frac{\log(t_n^{-1})}{1-t_n^{-1}} \tau\|A_h v_h\|_{L^2(\Omega)}.
\]
Finally, we observe that if \( t_{n}^{-1} \geq 2 \), i.e. \( t_{n} \leq 1/2 \), then \( \frac{\log(t_{n}^{-1})}{t_{n}} \leq 2 \log(t_{n}^{-1}) \). Otherwise if \( t_{n}^{-1} < 2 \), i.e. \( t_{n} \geq 1/2 \), then by the monotonicity of the function \( f(x) = \frac{\log(x)}{x} \) on \( \mathbb{R}^+ \), we deduce \( \frac{\log(t_{n}^{-1})}{t_{n}} = \frac{\log(t_{n}^{-1})}{t_{n}} \leq 2 \log(2) \). Then the desired result follows from (5.2), (5.14) and the identities \( U_{h}^{n} - u_{h}(t_{n}) = Y_{h}^{n} - y_{h}(t_{n}) \) and \( A_{h}R_{h} = P_{h}A \).

The next theorem gives error estimates for the fully discrete scheme (5.4), which follow from Theorems 3.1, 3.2, 5.1 and 5.2 and the triangle inequality.

**Theorem 5.3.** Let \( u \) and \( U_{h}^{n} \) be the solutions of problems (1.1) and (5.4) with \( U_{h}^{0} = v_{h} \) and \( f \equiv 0 \), respectively. Then for \( \ell_{1}(t) = \log(2T/t)^{-1} \) and \( \ell_{2}(t) = \log(\max(t^{-1}, 2)) \) and \( t_{n} = n\tau \), the following error estimates hold.

(a) If \( Av \in L^{2}(\Omega) \) and \( v_{h} = R_{h}v \), then for \( n \geq 1 \)

\[
\|u(t_{n}) - U_{h}^{n}\|_{L^{2}(\Omega)} \leq c(\tau \ell_{2}(t_{n}) + h^{2})\|Av\|_{L^{2}(\Omega)}.
\]

(b) If \( v \in L^{2}(\Omega) \) and \( v_{h} = P_{h}v \), then for \( n \geq 1 \)

\[
\|u(t_{n}) - U_{h}^{n}\|_{L^{2}(\Omega)} \leq c_{\tau} (\tau + h^{2}\ell_{1}(t_{n})) t_{n}^{-1}\|v\|_{L^{2}(\Omega)}.
\]

**Remark 5.3.** For distributed order time fractional diffusion, the error estimate involves a log factor in time for smooth initial data, which is reminiscent of the asymptotic behavior of the solution at small time, cf. Theorem 2.1. This factor is not present for the single term and multi-term time fractional diffusion [14, 15].

### 6. Numerical experiments and discussions

Now we present numerical results to verify the convergence theory. To this end, we let the domain \( \Omega \) be the unit interval \( \Omega = (0, 1) \) and consider the following three examples with smooth, discontinuous, and singular initial data:

(a) \( v(x) = \sin(2\pi x) \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega) \);

(b) \( v = \chi_{(0,1/2)} \in H^{1/2-\epsilon}(\Omega) \) with \( \epsilon \in (0, 1/2) \), and \( \chi_{S} \) the characteristic function of a set \( S \);

(c) \( v(x) = x^{-1/4} \in H^{1/4-\epsilon}(\Omega) \) with \( \epsilon \in (0, 1/4) \).

We measure the temporal discretization error by the normalized \( L^{2}(\Omega) \) errors \( \|u(t_{n}) - U_{h}(t_{n})\|_{L^{2}(\Omega)}/\|v\|_{L^{2}(\Omega)} \) or \( \|u(t_{n}) - U_{h}^{n}\|_{L^{2}(\Omega)}/\|v\|_{L^{2}(\Omega)} \), and the spatial discretization error by the normalized \( L^{2}(\Omega) \) and \( H^{1}(\Omega) \) errors, i.e., \( \|u(t) - u_{h}(t)\|_{L^{2}(\Omega)}/\|v\|_{L^{2}(\Omega)} \) and \( \|\nabla(u(t) - u_{h}(t))\|_{L^{2}(\Omega)}/\|v\|_{L^{2}(\Omega)} \). In the computations, we divide the domain \( \Omega \) into \( M \) equally spaced subintervals with a mesh size \( h = 1/M \). Since the exact solution \( u(t) \) is not available in closed form, we compute the reference solution using a much finer mesh.

#### 6.1. Numerical results for the semidiscrete scheme

First we examine the convergence behavior of the space semidiscrete scheme. To this end, we fix \( N = 10 \) in the Laplace transform approach such that the error due to time discretization is negligible. The numerical results are given in Tables 1–3. In the table, rate denotes the empirical convergence rates when the mesh size \( h \) halves, and the numbers in the bracket denote the theoretical rates. For all three initial data, the \( L^{2}(\Omega) \) and \( H^{1}(\Omega) \) norms of the error exhibit second and first order convergence rates, respectively, which agrees well with the theoretical prediction, cf. Theorems 3.1 and 3.2. The convergence of the semidiscrete scheme is robust in that the convergence rates hold for both smooth and nonsmooth initial data. The error increases as \( t \to 0 \), which is attributed to the weak singularity of the solution as \( t \to 0 \), cf. Theorem 2.1.

#### 6.2. Numerical results for the fully discrete scheme

Next we illustrate the convergence of the first fully discrete scheme based on the Laplace transform. To make the spatial discretization error negligible, we fix the spatial mesh size \( h = 10^{-5} \). In all numerical simulations, the optimal contour parameters \( \lambda \) and \( \psi \) in the parameterization (4.1) and \( k \) in (4.7) are chosen as suggested in the proof of Lemma 4.4 (see also [44]). Moreover, \( \lambda \) is fixed, independent of \( t \), with which the elliptic problems (4.8) are solved for each time \( t \). The numerical results are summarized in Tables 4 and 5 for the weight functions \( \mu(\alpha) = (\alpha - 1/2)^{2} \) and \( \mu(\alpha) = \chi_{[1/2,1]}(\alpha) \), respectively. The results indicate an exponential convergence with respect to the number \( N \) of quadrature points on hyperbolic contour, decaying at
an exponential decay for normal diffusion, as predicted by 

\[ v(t) \approx e^{-\mu t} \quad \text{for large } t \],

\[ v(t) \approx e^{-(\mu + 1) t} \quad \text{for subdiffusion} \]

agree well with the theoretical predictions from Theorem 1.2; see also Fig. 1.

Further, the convergence rate is independent of the time step size \( \tau \). This allows one to examine the asymptotic behavior of the solution as the time \( t \to \infty \); see Table 7 for the ultraslow decay asymptotics for distributed order diffusion processes, in comparison with sublinear decay for subdiffusion and exponential decay for normal diffusion.

6.3. Numerical results for the fully discrete scheme II. Last we verify the convergence of the fully discrete scheme II, i.e., convolution quadrature based on the backward Euler method. By Theorem 5.3, it exhibits a first order convergence with respect to the time step size \( \tau \). This is fully confirmed by the numerical results in Tables 8 and 9 for the weight functions \( \mu(\alpha) = (\alpha - 1/2)^2 \) and \( \mu(\alpha) = \chi_{[1,2]}(\alpha) \), respectively, which agree well with the theoretical predictions from Theorem 4.1. Note that even though the weight function \( \mu(\alpha) = \chi_{[1,2]}(\alpha) \) does not satisfy the assumption \( \mu(0)\mu(1) > 0 \), the empirical convergence rates still agree well with the theoretical prediction, which calls for further theoretical study. Further, the convergence rate is independent of the time \( t \), and thus the scheme is robust. Interestingly, the smoothness of the initial data \( v \) does not affect much the time discretization errors, even for small time instances, cf. Table 6.

One salient feature of the fully discrete scheme I is that it allows computing the solution at any arbitrarily large time directly. This allows one to examine the asymptotic behavior of the solution as the time \( t \to \infty \); see Table 7 and Fig. 1. In particular, one clearly observes the logarithmic decay of the solution, as predicted by [22, Theorem 2.1]; see also Fig. 1. This numerically verifies the ultraslow decay asymptotics for distributed order diffusion process, in comparison with sublinear decay for subdiffusion and exponential decay for normal diffusion.
Table 4. The $L^2$ errors for initial data (a)-(c) with $h = 10^{-5}$ and $\mu(\alpha) = (\alpha - 1/2)^2$, by the Laplace transform method. The notation $r$ denotes the exponential convergence rate in the error $\| u_{N,h}^{\alpha} - u(t_n) \|_{L^2(\Omega)} \leq C e^{-r N}$.

| case | $t \setminus N$ | 3 | 5 | 7 | 9 | 11 | 13 | $r$ |
|------|-----------------|---|---|---|---|----|----|-----|
| (a)  | $10^{-2}$       | 1.33e-6 | 1.49e-8 | 1.26e-10 | 2.20e-12 | 3.54e-14 | 8.24e-17 | 2.35 |
|      | $10^{-3}$       | 4.78e-6 | 7.36e-7 | 2.77e-9 | 5.45e-11 | 4.88e-13 | 2.23e-14 | 1.92 |
| (b)  | $10^{-2}$       | 3.34e-6 | 3.56e-8 | 2.86e-10 | 5.76e-12 | 8.66e-14 | 1.25e-15 | 2.17 |
|      | $10^{-3}$       | 1.24e-5 | 8.29e-7 | 2.31e-9 | 6.09e-11 | 4.78e-13 | 2.18e-14 | 2.02 |
| (c)  | $10^{-2}$       | 6.99e-5 | 1.73e-6 | 1.09e-8 | 5.38e-11 | 1.17e-12 | 1.59e-14 | 2.22 |
|      | $10^{-3}$       | 0.04e-6 | 9.05e-8 | 6.80e-10 | 1.39e-11 | 2.08e-13 | 3.02e-15 | 2.17 |

Table 5. The $L^2$ errors for initial data (a)-(c) with $h = 10^{-5}$ and $\mu(\alpha) = \chi_{[1/2,1]}(\alpha)$, by the Laplace transform method. The notation $r$ denotes the exponential convergence rate in the error $\| u_{N,h}^{\alpha} - u(t_n) \|_{L^2(\Omega)} \leq C e^{-r N}$.

| case | $t \setminus N$ | 3 | 5 | 7 | 9 | 11 | 13 | $r$ |
|------|-----------------|---|---|---|---|----|----|-----|
| (a)  | $10^{-2}$       | 4.54e-6 | 2.30e-7 | 1.63e-9 | 1.69e-11 | 2.36e-13 | 8.46e-15 | 2.02 |
|      | $10^{-3}$       | 6.21e-5 | 1.65e-6 | 3.71e-9 | 1.07e-10 | 7.00e-13 | 2.58e-14 | 2.16 |
| (b)  | $10^{-2}$       | 4.78e-6 | 4.74e-7 | 2.43e-9 | 3.44e-11 | 3.49e-13 | 1.87e-14 | 1.94 |
|      | $10^{-3}$       | 1.03e-4 | 1.13e-6 | 3.58e-9 | 8.78e-11 | 5.04e-13 | 1.93e-14 | 2.24 |
| (c)  | $10^{-2}$       | 4.79e-6 | 5.61e-7 | 2.75e-9 | 4.07e-11 | 3.94e-13 | 2.23e-14 | 1.92 |
|      | $10^{-3}$       | 1.18e-4 | 6.08e-7 | 3.37e-9 | 7.22e-11 | 2.84e-13 | 8.94e-14 | 2.10 |

Table 6. The $L^2$ errors for initial data (b) and (c) with $h = 10^{-5}$, $\mu(\alpha) = (\alpha - 1/2)^2$ and $N = 5$ at small time instances $t = 10^{-k}$, $k = 4, 5, \cdots, 9$, by the Laplace transform method.

| case \ $t$ | $10^{-4}$ | $10^{-5}$ | $10^{-6}$ | $10^{-7}$ | $10^{-8}$ | $10^{-9}$ |
|------------|-----------|-----------|-----------|-----------|-----------|-----------|
| (b)        | 7.05e-6   | 9.39e-6   | 1.58e-5   | 1.75e-5   | 1.81e-5   | 1.82e-5   |
| (c)        | 6.39e-6   | 1.17e-5   | 1.53e-5   | 1.68e-5   | 1.75e-5   | 1.79e-5   |

Table 7. The $L^2$ norm of the solution for initial data (a) and (c) with $h = 10^{-5}$, $\mu(\alpha) = (\alpha - 1/2)^2$ and $N = 10$ at large time instances $t = 10^k$, $k = 6, 8, \cdots, 18$, computed by the Laplace transform method.

| case \ $k$ | 6 | 8 | 10 | 12 | 14 | 16 | 18 | rate |
|------------|---|---|----|----|----|----|-----|------|
| (a)        | 3.33e-4 | 2.70e-4 | 2.26e-4 | 1.95e-4 | 1.71e-4 | 1.52e-4 | 1.37e-4 | $1/k$ |
| (c)        | 1.06e-3 | 8.54e-4 | 7.17e-4 | 6.17e-4 | 5.41e-4 | 4.82e-4 | 4.34e-4 | $1/k$ |

respectively. A first order convergence is observed for all three examples and at all time instances, showing the robustness of the scheme.

To examine more closely the convergence behavior of the scheme, we consider $t = 10^{-k}$, $k = 4, 5, \cdots, 9$, and for each time instance $t$, divide the interval $[0, t]$ into $N = 10$ subintervals. The scheme works well for the smooth initial data in example (a), however, it works poorly for the singular initial data in example (c), cf. Table 10. This behavior is predicted by Theorems 5.1 and 5.3: the error is dominated by the factor $\frac{1}{\tau}$ for $L^2(\Omega)$ initial data. In Fig. 2, we plot the error ratio $\| U_{k}^\alpha - u(\tau) \|/\tau$ against log $\tau$ for smooth
\[ || u(\cdot, 10^k) ||_{L^2(\Omega)} \]

**Figure 1.** The \( L^2 \) norm of the solution for initial data (a) and (c) at \( t = 10^k \), \( k = 6, 8, \cdots, 18 \), by the Laplace transform method.

**Table 8.** The \( L^2 \) errors for initial data (a)-(c) with \( h = 10^{-4} \) and \( \mu(\alpha) = (\alpha - 1/2)^2 \), by the backward Euler convolution quadrature.

| case | \( t \) | \( N \) | 10 | 20 | 40 | 80 | 160 | 320 | rate |
|------|-------|-----|----|----|----|----|-----|-----|------|
| (a)  | 1     |     | 1.82e-5 | 8.78e-6 | 4.31e-6 | 2.12e-6 | 1.01e-6 | 4.74e-7 | 1.05 (1.00) |
|      | \( 10^{-2} \) |     | 8.64e-4 | 3.91e-4 | 1.88e-4 | 9.20e-5 | 4.55e-5 | 2.26e-5 | 1.05 (1.00) |
|      | \( 10^{-3} \) |     | 2.17e-2 | 1.10e-2 | 5.51e-3 | 2.76e-3 | 1.38e-3 | 6.92e-4 | 0.99 (1.00) |
| (b)  | 1     |     | 4.81e-5 | 2.32e-5 | 1.14e-5 | 5.60e-6 | 2.67e-6 | 1.26e-6 | 1.05 (1.00) |
|      | \( 10^{-2} \) |     | 8.11e-3 | 3.87e-3 | 1.88e-3 | 9.29e-4 | 4.61e-4 | 2.30e-4 | 1.03 (1.00) |
|      | \( 10^{-3} \) |     | 1.48e-2 | 7.46e-3 | 3.74e-3 | 1.88e-3 | 9.39e-4 | 4.70e-4 | 1.00 (1.00) |
| (c)  | 1     |     | 5.81e-5 | 2.81e-5 | 1.38e-5 | 7.67e-6 | 3.23e-6 | 1.52e-6 | 1.05 (1.00) |
|      | \( 10^{-2} \) |     | 1.01e-2 | 4.80e-3 | 2.34e-3 | 1.15e-3 | 5.72e-4 | 2.85e-4 | 1.03 (1.00) |
|      | \( 10^{-3} \) |     | 7.35e-3 | 3.66e-3 | 1.82e-3 | 9.11e-4 | 4.55e-4 | 2.27e-4 | 1.00 (1.00) |

**Table 9.** The \( L^2 \) errors for initial data (a)-(c) with \( h = 10^{-4} \) and \( \mu(\alpha) = \chi[1/2,1](\alpha) \), by the backward Euler convolution quadrature.

| case | \( t \) | \( N \) | 10 | 20 | 40 | 80 | 160 | 320 | rate |
|------|-------|-----|----|----|----|----|-----|-----|------|
| (a)  | 1     |     | 2.20e-4 | 1.06e-4 | 5.20e-5 | 2.58e-5 | 1.28e-5 | 6.40e-6 | 1.02 (1.00) |
|      | \( 10^{-2} \) |     | 1.76e-2 | 8.81e-3 | 4.40e-3 | 2.20e-3 | 1.10e-3 | 5.49e-4 | 1.00 (1.00) |
|      | \( 10^{-3} \) |     | 3.92e-3 | 1.98e-3 | 9.95e-4 | 4.99e-4 | 2.50e-4 | 1.25e-4 | 0.99 (1.00) |
| (b)  | 1     |     | 6.52e-4 | 3.11e-4 | 1.52e-4 | 7.55e-5 | 3.74e-5 | 1.87e-5 | 1.03 (1.00) |
|      | \( 10^{-2} \) |     | 1.25e-2 | 6.20e-3 | 3.13e-3 | 1.56e-3 | 7.82e-4 | 3.91e-4 | 1.00 (1.00) |
|      | \( 10^{-3} \) |     | 5.76e-3 | 2.88e-3 | 1.44e-3 | 7.18e-4 | 3.59e-4 | 1.79e-4 | 1.00 (1.00) |
| (c)  | 1     |     | 7.92e-4 | 3.78e-4 | 1.85e-4 | 9.14e-5 | 4.54e-5 | 2.27e-5 | 1.03 (1.00) |
|      | \( 10^{-2} \) |     | 7.40e-3 | 3.71e-3 | 1.86e-3 | 9.28e-4 | 4.64e-4 | 2.32e-4 | 1.00 (1.00) |
|      | \( 10^{-3} \) |     | 6.10e-3 | 3.06e-3 | 1.53e-3 | 7.65e-4 | 3.83e-4 | 1.91e-4 | 1.00 (1.00) |

initial data in example (a). Theorem 5.2 predicts an error estimate \( || U_h^1 - u(\tau) ||_{L^2(\Omega)} \leq c\tau \log \tau^{-1} \). The log factor \( \ell_2(t) \) in Theorem 5.2 is fully confirmed by Fig. 2, and thus the corresponding error estimate is sharp.
Table 10. The $L^2$ errors for initial data (a) and (c) with $h = 10^{-5}$ and $N = 10$, at time instances $t = 10^{-k}$, $k = 4, 5, \cdots, 9$, by backward Euler convolution quadrature.

| case | $t$       | $10^{-4}$ | $10^{-5}$ | $10^{-6}$ | $10^{-7}$ | $10^{-8}$ | $10^{-9}$ | rate  |
|------|-----------|-----------|-----------|-----------|-----------|-----------|----------|-------|
| (a)  |           | 2.42e-3   | 1.03e-4   | 7.87e-6   | 7.59e-7   | 7.58e-8   | 7.44e-9  | 1.01 (1.00) |
| (c)  |           | 7.44e-3   | 5.67e-3   | 4.30e-3   | 3.27e-3   | 2.49e-3   | 1.88e-3  | 0.12 (0.12) |

Figure 2. The $L^2$ errors for the backward Euler method for initial data (a) at small time instances $t_1 = \tau = 10^{-k}$, $k = 5, 6, \ldots, 11, 12$.

7. Concluding remarks

In this work, we have presented a first rigorous numerical analysis of two fully discrete schemes (one based on the Laplace transform and another based on convolution quadratures) for the distributed-order time fractional diffusion equation with nonsmooth initial data. We have provided regularity estimates for the solution and developed one space semidiscrete Galerkin method and two fully discrete schemes. Optimal error estimates for the semidiscrete scheme were shown using an operator trick due to Fujita and Suzuki. The first fully discrete scheme is based on quadrature approximation of the inverse Laplace transform with a deformed contour of hyperbolic type, and exhibits an exponential convergence. It is especially suited to computing the solution at many and large time instances. The second fully discrete scheme is based on convolution quadrature generated by the backward Euler method, and exhibits a first order convergence. The sharpness of the error estimates were fully verified by extensive numerical experiments for both smooth and nonsmooth initial data.

This work represents only a first step towards rigorous numerical analysis of distributed order subdiffusion, and there are a number of avenues for further research. First, the semidiscrete and fully discrete schemes may be extended to the distributed order diffusion wave equation, with a nonnegative weight $\mu(\alpha) \in C[0, 2]$. Second, the error estimates for the semidiscrete Galerkin scheme in the case of nonsmooth initial data $v \in L^2(\Omega)$ depend on the final time $T$. It remains unknown how to get rid of this factor. This is especially important if the solution is sought for large $T$. Third, the assumption $\mu(\alpha) \in C[0, 1]$ might be too restrictive and its is of much interest to relax it to $\mu(\alpha) \in L^\infty(0, 1)$.

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References

[1] T. M. Atanackovic, S. Pilipovic, and D. Zorica. Distributed-order fractional wave equation on a finite domain. Stress relaxation in a rod. Internat. J. Engrg. Sci., 49(2):175–190, 2011.
E. Bazhlekova. Completely monotone functions and some classes of fractional evolution equations. preprint, arXiv:1502.04647, 2015.

E. Bazhlekova, B. Jin, R. Lazarov, and Z. Zhou. An analysis of the Rayleigh-Stokes problem for the generalized second grade fluid. *Numer. Math.*, 2014. DOI:10.1007/s00211-014-0685-2 (arXiv:1401.8049).

M. Caputo. Distributed order differential equations modelling dielectric induction and diffusion. *Fract. Calc. Appl. Anal.*, 4(4):421–442, 2001.

A. V. Chechkin, R. Gorenflo, and I. M. Sokolov. Retarding subdiffusion and accelerating superdiffusion governed by distributed-order fractional diffusion equations. *Phys. Rev. E.*, 66:046129, 2002.

A. V. Chechkin, R. Gorenflo, I. M. Sokolov, and V. Y. Gonchar. Distributed order time fractional diffusion equation. *Fract. Calc. Appl. Anal.*, 6(3):259–279, 2003.

E. Cuesta, C. Lubich, and C. Palencia. Convolution quadrature time discretization of fractional diffusion-wave equations. *Math. Comp.*, 75(254):673–696, 2006.

K. Diethelm and N. J. Ford. Numerical analysis for distributed-order differential equations. *J. Comput. Appl. Math.*, 225(1):96–104, 2009.

H. Fujita and T. Suzuki. Evolution problems. In *Handbook of Numerical Analysis*, Vol. II, pages 789–928. North-Holland, Amsterdam, 1991.

I. P. Gavrilyuk, W. Hackbusch, and B. N. Khoromskij. $H$-matrix approximation for the operator exponential with applications. *Numer. Math.*, 92(1):83–111, 2002.

R. Gorenflo, Y. Luchko, and M. Stojanović. Fundamental solution of a distributed order time-fractional diffusion-wave equation as probability density. *Fract. Calc. Appl. Anal.*, 16(2):297–316, 2013.

M. Hasse. *The Functional Calculus for Sectorial Operators*. Birkhäuser-Verlag, Basel, 2006.

J. Jia, J. Peng, and K. Li. Well-posedness of abstract distributed-order fractional diffusion equations. *Commun. Pure Appl. Anal.*, 13(2):605–621, 2014.

B. Jin, R. Lazarov, Y. Liu, and Z. Zhou. The Galerkin finite element method for a multi-term time-fractional diffusion equation. *J. Comput. Phys.*, 281:825–843, 2015.

B. Jin, R. Lazarov, and Z. Zhou. Error estimates for a semidiscrete finite element method for fractional order parabolic equations. *SIAM J. Numer. Anal.*, 51(1):445–466, 2013.

B. Jin, R. Lazarov, and Z. Zhou. On two schemes for fractional diffusion and diffusion-wave equations. preprint, arXiv:1404.3800, 2014.

B. Jin, R. Lazarov, and Z. Zhou. An analysis of the L1 scheme for the subdiffusion equation with nonsmooth data. *IMA Numer. Anal.*, page in press, 2015.

B. Jin and W. Rundell. A tutorial on inverse problems for anomalous diffusion processes. *Inverse Problems*, 31(3):035003, 40p., 2015.

J. T. Katsikadelis. Numerical solution of distributed order fractional differential equations. *J. Comput. Phys.*, 259:11–22, 2014.

A. Kilbas, H. Srivastava, and J. Trujillo. *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.

A. N. Kochubei. Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl.*, 340(1):252–281, 2008.

Z. Li, Y. Luchko, and M. Yamamoto. Asymptotic estimates of solutions to initial-boundary-value problems for distributed order time-fractional diffusion equations. *Fract. Calc. Appl. Anal.*, 17(4):1114–1136, 2014.

Y. Lin and C. Xu. Finite difference/spectral approximations for the time-fractional diffusion equation. *J. Comput. Phys.*, 225(2):1533–1552, 2007.

M. López-Fernández, C. Palencia, and A. Schädle. A spectral order method for inverting sectorial Laplace transforms. *SIAM J. Numer. Anal.*, 44(3):1332–1350, 2006.

C. Lubich. Convolution quadrature and discretized operational calculus. I. *Numer. Math.*, 52(2):129–145, 1988.

C. Lubich, I. H. Sloan, and V. Thomée. Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term. *Math. Comp.*, 65(213):1–17, 1996.

Y. Luchko. Boundary value problems for the generalized time-fractional diffusion equation of distributed order. *Fract. Calc. Appl. Anal.*, 12(4):409–422, 2009.

F. Mainardi, A. Mura, G. Pagnini, and R. Gorenflo. Time-fractional diffusion of distributed order. *J. Vib. Control*, 14(9–10):1267–1290, 2008.

E. Martensen. Zur numerischen Auswertung eineigenlicher Integrale. *Z. Angew. Math. Mech.*, 48:T83–T85, 1968.

W. McLean and K. Mustapha. Time-stepping error bounds for fractional diffusion problems with non-smooth initial data. *J. Comput. Phys.*, in press. arXiv:1405.2140, 2014.

W. McLean, I. H. Sloan, and V. Thomée. Time discretization via Laplace transformation of an integro-differential equation of parabolic type. *Numer. Math.*, 102(3):497–522, 2006.

W. McLean and V. Thomée. Numerical solution via Laplace transforms of a fractional order evolution equation. *J. Integral Equations Appl.*, 22(1):57–94, 2010.

M. M. Meerschaert, E. Nane, and P. Vellaisamy. Distributed-order fractional diffusions on bounded domains. *J. Math. Anal. Appl.*, 379(1):216–228, 2011.
[34] M. M. Meerschaert and H.-P. Scheffler. Stochastic model for ultraslow diffusion. *Stochastic Process. Appl.*, 116(9):1215–1235, 2006.

[35] M. L. Morgado and M. Rebelo. Numerical approximation of distributed order reaction diffusion equations. *J. Comput. Appl. Math.*, 275:216–227, 2015.

[36] K. Mustapha, B. Abdallah, and K. M. Furati. A discontinuous Petrov–Galerkin method for time-fractional diffusion equations. *SIAM J. Numer. Anal.*, 52(2):2512–2529, 2014.

[37] I. Podlubny. *Fractional Differential Equations*. Academic Press, San Diego, CA, 1999.

[38] D. Sheen, I. Sloan, and V. Thomée. A parallel method for time-discretization of parabolic problems based on contour integral representation and quadrature. *Math. Comp.*, 69(229):177–195, 2000.

[39] D. Sheen, I. Sloan, and V. Thomée. A parallel method for time discretization of parabolic equations based on Laplace transformation and quadrature. *IMA J. Numer. Anal.*, 23(2):269–299, 2003.

[40] Y. G. Sinai. The limit behavior of a one-dimensional random walk in a random environment. *Teor. Veroyatnost. i Primenen.*, 27(2):247–258, 1982.

[41] I. M. Sokolov, A. V. Chechkin, and J. Klafter. Distributed-order fractional kinetics. *Acta Phys. Polon. B*, 35(4):1323–1341, 2004.

[42] V. Thomée. *Galerkin Finite Element Methods for Parabolic Problems*, volume 25 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2006.

[43] V. Vergara and R. Zacher. Optimal decay estimates for time-fractional and other nonlocal subdiffusion equations via energy methods. *SIAM J. Math. Anal.*, 47(1):210–239, 2015.

[44] J. A. C. Weideman and L. N. Trefethen. Parabolic and hyperbolic contours for computing the Bromwich integral. *Math. Comp.*, 76(259):1341–1356, 2007.

[45] F. Zeng, C. Li, F. Liu, and I. Turner. The use of finite difference/element approaches for solving the time-fractional subdiffusion equation. *SIAM J. Sci. Comput.*, 35(6):A2976–A3000, 2013.

[46] Y.-N. Zhang, Z.-Z. Sun, and H.-L. Liao. Finite difference methods for the time fractional diffusion equation on non-uniform meshes. *J. Comput. Phys.*, 265:195–210, 2014.

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