UNRECTIFIABLE NORMAL CURRENTS IN EUCLIDEAN SPACES

ANDREA SCHIOPPA

Abstract. We construct in $\mathbb{R}^{k+2}$ a $k$-dimensional simple normal current whose support is purely 2-unrectifiable. The result is sharp because the support of a normal current cannot be purely 1-unrectifiable and a $(k+1)$-dimensional normal current can be represented as an integral of $(k+1)$-rectifiable currents. This gives a negative answer to the (revised version) of a question of Frank Morgan (1984).

Contents

1. Introduction 1
2. 2-current in Hilbert space 4
3. 2-current in $\mathbb{R}^4$ 11
4. $k$-current in $\mathbb{R}^{k+2}$ 21
References 26

1. Introduction

1.1. Results. This paper is a continuation of [Sch15] to which we refer for more background and notation. The main motivation behind [Sch15] was to provide new examples of Ambrosio-Kirchheim metric currents [AK00a] and to prove that higher-dimensional analogues of some results in [Sch16a, Sch14] do not hold. Specifically, in [Sch16a] it was shown that in metric measure spaces vector fields can be concretely described as a superposition of partial derivative operators associated with curve fragments. In particular, the background measure $\mu$ appearing in the definition of vector fields (see for example Subsec. 2.1 in [Sch15] about Weaver derivations) has to admit Alberti representations or, more precisely, has to be 1-rectifiably representable; this means that $\mu$ can be represented as an integral of 1-dimensional Hausdorff measures associated with curve fragments $\gamma$: $\mu = \int H^1(\gamma) dQ(\gamma)$. In the case in which a higher order representation exists, i.e. $\mu = \int H^k(\sigma) dQ(\sigma)$ where $\sigma$ is a $k$-rectifiable compact set (see [AK00b] for the theory of rectifiable sets in metric spaces) we will say that $\mu$ is $k$-rectifiably representable.

In [Sch14] it was later shown that 1-dimensional metric currents admitted an integral representation in terms of 1-rectifiable metric currents $T = \int [\gamma] dQ(\gamma)$ ($[\gamma]$ being the current associated to an oriented fragment) and that $k$-dimensional metric currents could be canonically associated to $k$-dimensional vector fields obtaining

2010 Mathematics Subject Classification. 49Q15, 28A75.
Key words and phrases. Normal current, Rectifiability.
a parallel between the metric theory of Ambrosio-Kirchheim [AK00a] and the classical theory of Federer and Fleming [Fed69, Ch. 4]. A natural question we had at the time was whether a 2-dimensional metric current $T$ could be represented as an integral of 2-rectifiable currents $T = \int [[\sigma]] dQ(\sigma)$. Some specific examples of non-simple (i.e. the associated vector fields are not simple) 2-dimensional currents with 2-purely unrectifiable supports had been obtained by Marshall Williams [Wil12] in Carnot groups. In [Sch15] we obtained a general negative answer constructing for each $k$ a simple $k$-dimensional normal metric current whose support is purely 2-unrectifiable. Unfortunately, those currents could not be constructed in Euclidean spaces. In this paper we complete the treatment by:

**Construction 1.1.** In $\mathbb{R}^{k+2}$ there is a $k$-dimensional normal current whose support is purely 2-unrectifiable.

Note that our normal currents are also classical normal currents, thus providing examples of normal currents which live on 2-unrectifiable subsets.

1.2. Relation to previous work. Even though I came across this problem while finishing my dissertation in 2014, I later found that other researchers had previously considered it. I learned from Giovanni Alberti that he had considered also this problem, and later found out the following question of Frank Morgan [mor86, Problem 3.8, pg. 446]:

(Q-Morgan): Question of Frank Morgan. “Can every normal current in $\mathbb{R}^s$ be decomposed as a convex integral combination of integral currents? In codimension one the answer is yes if $\partial T = 0$, see [Fed69, # 4.5.9(13)]”. (In my own words): Is a $k$-dimensional normal current $T$ in $\mathbb{R}^s$ representable as an integral $\int [[\sigma]] dQ(\sigma)$ of $k$-integral currents enforcing the **mass constraint** (here is what “convex” probably means) $\|T\| = \int \|[[\sigma]]\| dQ(\sigma)$?

Here $\|T\|$ denotes the mass measure of $T$.

The answer to (Q-Morgan) for $k = 1$ is positive by the beautiful work of Stanislav Smirnov [Smi93], and recently we have learned from Alberti and Massaci that this is also the case for $k = s - 1$ as a consequence of the coarea formula for BV functions [Mas14, AFP00]; essentially they find a “good filling” for the boundary of the normal current to reduce the problem to the case $\partial T = 0$ sketched by Morgan.

In general the answer to (Q-Morgan) is negative: Zworski [Zwo88] gives as counterexample $T = \xi \mathcal{H}^s$ where $\xi$ is a suitable non-involutive $k$-field (a small gap in his argument is pointed out and fixed in [Mas14, Chap. 2]). However, these examples are still representable as integrals of integral currents if one drops the mass constraint, and if one wants to keep the mass constraint, one can use a remarkable Theorem of Alberti [Alb91] to obtain an integral decomposition into rectifiable currents by finding rectifiable sets tangent to the non-involutive distribution. This suggests the following revised version of (Q-Morgan):

(Q-MorganRev): Is a $k$-dimensional normal current $T$ in $\mathbb{R}^s$ representable as an integral $\int [[\sigma]] dQ(\sigma)$ of $k$-rectifiable currents without necessarily satisfying the mass constraint?

Our result answers (Q-MorganRev) in the negative for all $k \leq s - 2$: the support of a $k$-current does not need even to intersect a 2-rectifiable set in positive area.

Our work has also applications to the recent structure theory for measures developed in [ACP10]. In particular, this answers the problem of whether measures that admit a $k$-tangent field (this essentially gives the directions along which
a Rademacher Theorem on the differentiability of Lipschitz functions holds) in the sense of [ACP10] are $k$-rectifiably representable. Following [Sch14, DR16] we rephrase the problem in the language of normal currents:

(Q-ACP): Question of Alberti, Csörnyei and Preiss [ACP10, Sec. 2]. If $\mu$ is a Radon measure on $\mathbb{R}^s$ and for $1 < k \leq s$ there are $k$ 1-dimensional normal currents $\{N_i\}_{i=1}^k$ with $\mu \ll \|N_i\|$ and such that at $\mu$-a.e. point the vector fields associated to the $N_i$ are independent, is then $\mu$ $k$-rectifiably representable?

For $k = s$ (Q-ACP) has a positive answer by the recent work of de Philippis and Rindler [DR16]. For $s = 3$ and $k = 2$ a negative result has been announced by Andras Mathe [Mát]. Our construction answers (Q-ACP) in the negative for all $k \in \{2, \ldots, s-2\}$. It is likely that modifications to our approach can also yield the negative answer for $k = s - 1$, but we do not pursue it further because it is likely to follow also from the announced results of [Mát].

1.3. Organization. In the paper we follow the same approach in which we discovered the result: there are the following 3-layers:

Layer 1: A 2-normal current in the Hilbert space $l^2$ whose support is purely 2-unrectifiable.

Layer 2: A 2-normal current in $\mathbb{R}^4$ whose support is purely 2-unrectifiable.

Layer 3: A $k$-normal current in $\mathbb{R}^{k+2}$ whose support is purely 2-unrectifiable.

Layer 1 (Sec. 2) is already non-trivial because the Hilbert space has the Radon-Nikodym property, i.e. Lipschitz Hilbert-valued functions are differentiable a.e. It is not hard to show that this implies that the examples in [Sch15] cannot be bi-Lipschitz embedded in Hilbert space. However, we are able to find a topological embedding of those examples which is Lipschitz; an examination of the construction allows to find a “rate of collapse” of the fibers of the double covers used in [Sch15] which allows to prove 2-unrectifiability. Unfortunately, the Radon-Nikodym property prevents the use of a simple blow-up argument as in [Sch15] and we must resort to a quantitative estimate based on holonomy.

In Layer 2 (Sec 3) we pass from Hilbert space to $\mathbb{R}^4$ by resorting to kernel methods (see for example [MRT12, Ch. 5], [GBV93]) which are well-known in the SVMs literature. Essentially the kernel trick allows to train an SVM on an $\infty$-dimensional implicit set of features even though the data set has (obviously) only features living in a finite dimensional space. For example, in $\mathbb{R}^4$ we can fabricate something like the Hilbert space $l^2$ (countable sequences) using kernel functions. Unfortunately, this approach destroys the approximate “self-similarity” of the construction in Hilbert space making the details more technical and lengthy. In particular, we must resort to curvilinear $(1 + \varepsilon)$-Lipschitz projections to resolve the fine structure of the support of the current at a given scale.

In Layer 3 (Sec 4) we obtain the general case using a simple idea from [Sch15] (I am indebted to Bruce Kleiner for it) which consists in destroying Lipschitz surfaces which are graphs on any pair of coordinate axes.

1.4. Notational conventions. For notational conventions, background and terminology we refer the reader to [Sch15, Sec. 2]. Here we use a more general notion of weak* convergence for Lipschitz functions.
Definition 1.2 (Weak* convergence for Lipschitz maps). Let \( \{f_n\}_n \) be a sequence of Lipschitz maps \( f_n : X \to Y \). We say that \( f_n \) converges to a Lipschitz map \( f : X \to Y \) in the \textbf{weak* sense} (and write \( f_n \xrightarrow{w*} f \)) if \( f_n \to f \) pointwise and \( \sup_n L(f_n) < \infty \), where \( L(f_n) \) denotes the Lipschitz constant of \( f_n \).

Assume that the sets \( X_n \subset Z \) converge to the set \( X \subset Z \) in the Hausdorff sense. For \( x \in X \) we say that \( \{x_n\}_n \subset Z \) with \( x_n \in X_n \) \textbf{represents} \( x \in X \) if \( x_n \to x \). Let \( \{f_n\}_n \) be a sequence of Lipschitz maps \( f_n : X_n \to Y \). We say that \( f_n \) converges to a Lipschitz map \( f : X \to Y \) in the \textbf{weak* sense} (and write \( f_n \xrightarrow{w*} f \)) if \( \sup_n L(f_n) < \infty \), and whenever \( \{x_n\}_n \) represents \( x \), \( f_n(x_n) \to f(x) \).

Note that in the previous definition one may check, for each \( x \), that \( f_n(x_n) \to f(x) \) just for one sequence \( \{x_n\}_n \) representing \( x \), thanks to the uniform bound on the Lipschitz constants of the functions \( f_n \).

Now assume also that the sets \( Y_n \subset W \) converge to the set \( Y \subset W \) in the Hausdorff sense. We say that a sequence \( f_n : X_n \to Y_n \) converges to a Lipschitz map \( f : X \to Y \) in the \textbf{weak* sense} (and write \( f_n \xrightarrow{w*} f \)) if \( \sup_n L(f_n) < \infty \), and whenever \( \{x_n\}_n \) represents \( x \), \( f_n(x_n) \) represents \( f(x) \).

In this paper there are only a couple of points where we use measured Gromov-Hausdorff convergence. For background and notational conventions we refer to [Sch16b, Subsec. 3.1]. However, here we always reduce to the classical case by assuming that convergence takes place in a \textbf{container} \( Z \): if \( (X_n, \mu_n) \) converges to \( (X, \mu) \) in the measured Gromov-Hausdorff sense, we assume that \( X_n \) and \( X \) are isometrically embedded in \( Z \), and then that \( X_n \to X \) in the Hausdorff sense and \( \mu_n \to \mu \) in the weak* sense for Radon measures (i.e. as functionals on continuous functions defined on \( Z \) which are bounded and have bounded support).

Finally, we use the convention \( a \simeq b \) (or \( a \approx b \)) to say that \( a/b, b/a \in [C^{-1}, C] \) where \( C \) is a universal constant; when we want to highlight \( C \) we write \( a \simeq_C b \). We similarly use notations like \( a \lesssim b \) and \( a \gtrsim_C b \).

Acknowledgements. This work has been partially supported by the “ETH Zurich Postdoctoral Fellowship Program and the Marie Curie Actions for People COFUND Program”.

2. 2-current in Hilbert space

Let \( \{X_i\}_i \) denote the inverse system of square complexes in [Sch15, Sec. 4], denote by \( X_{\infty} \) the corresponding inverse limit, and for \( m \leq n \) (\( n = \infty \) being allowed) let \( \pi_{n,m} : X_n \to X_m \) denote the corresponding 1-Lipschitz projection. We let \( \delta_n \) denote a sequence with \( \sum_n \delta_n = \infty \) and \( \sum_n \delta_n^2 < \infty \): the precise form of \( \delta_n \) will be determined later.

We briefly recall how \( X_{i+1} \) is obtained out of \( X_i \). Let \( \text{Sq}_i(X_i) \) denote the set of squares of generation \( i \) of \( X_i \), whose side length is \( l_i = 2^{-i} \). To get \( X_{i+1} \) one subdivides each square \( Q \in \text{Sq}_i(X_i) \) and applies the following operation. The square \( Q \) is subdivided into squares of generation \( i + 1 \); there are \( 5^2 \) such squares that, up to identifying \( Q \) with \( [0, 5]^2 \), can be indexed by the location of their south-west corner by pairs \( (j_1, j_2) \in \{0, \cdots, 4\} \). These squares are grouped into three pieces:

- The central square \( Q_c \) corresponding to \( (j_1, j_2) = (2, 2) \).
- The outer annulus \( Q_o \) corresponding to the squares where either \( j_1 \in \{0, 4\} \) or \( j_2 \in \{0, 4\} \).
The middle annulus $Q_\delta$ consisting of the squares neither in $Q_c$ nor in $Q_o$. We make the simple observation $\mathcal{H}^2(Q_\delta) \geq \frac{8}{\pi^2} \mathcal{H}^2(Q)$ and replace $Q_\delta$ by a double cover $\tilde{Q}_\delta$, split the Lebesgue measure on $Q_\delta$ in half and glue $\tilde{Q}_\delta$ back to $Q_c$ and $Q_o$ by collapsing the fibers of the cover on the boundary $\partial \tilde{Q}_\delta$ to match $\partial Q_c$ and the inner component of $\partial Q_o$. Let $Q$ denote the square-complex thus obtained.

**Construction 2.1** (A map $\Psi : \tilde{Q} \to \mathbb{R}^2$ depending on a parameter $\delta$). Fix $\delta > 0$ small. Let $\tilde{Q}_\delta \subset Q$ be the central annulus of the first subdivision of $Q_\delta$ consisting of those squares in $\mathrm{Sq}_{i+2}(Q_\delta)$ which are at distance $\geq 5^{-i-2}$ from $\partial Q_\delta$. We observe that: $\mathcal{H}^2(Q_\delta) \geq \frac{3}{\pi^2} \mathcal{H}^2(Q_\delta)$.

Choose a 1-cell $\sigma$ in the 1-skeleton of $\mathrm{Sq}_{i+1}(Q_\delta)$ which joins the two components of $\partial Q_\delta$. Note that $\sigma$ can be used to choose an “origin” of the angles for a polar coordinate system $(r, \theta)$ on $Q_\delta$. Formally, we identify $Q_\delta \setminus \sigma$ with polar coordinates $(r, \theta) : Q_\delta \setminus \sigma \to [0, \pi) \times (0, 2\pi)$. Moreover, the set $\tilde{Q}_\delta \setminus \sigma$ is determined by the condition $r \in [5^{-i-2}, 5^{-i-1} - 5^{-i-2}]$.

Let $\tilde{\pi} : \tilde{Q} \to Q$ denote the double cover and note that on $\Sigma = \tilde{\pi}^{-1}(Q_\delta \setminus \sigma)$ we get a polar coordinate system $(r, \theta) : \Sigma \to [0, 5^{-i-1}] \times [(0, 4\pi) \setminus \{2\pi\}]$, and that the map $\tilde{\pi}$, in polar coordinates, assumes the form $\tilde{\pi}(r, \theta) = (r, \theta \bmod 2\pi)$. In particular, $\tilde{\pi}^{-1}(\sigma)$ divides $\Sigma$ in two sheets: $\Sigma_+$ where $\theta \in (2\pi, 4\pi)$, and $\Sigma_-$ where $\theta \in (0, 2\pi)$. We let $\chi$ denote the characteristic function of $\Sigma_+$; the following observation is crucial in the following:

**ShSep:** If $p, q \in \Sigma$, $d_{\tilde{Q}}(p, q) \leq 5^{-i-3}$ and $\tilde{\pi}(p)$ and $\tilde{\pi}(q)$ are on opposite sides of $\sigma$ (i.e. $|\theta(\tilde{\pi}(p)) - \theta(\tilde{\pi}(q))| \geq \pi$), then $\chi(p) \neq \chi(q)$.

We now define two helper functions $h_1, h_2 : [0, 4\pi) \to \mathbb{R}$:

\begin{align}
\label{eq:2.2}
h_1(\theta) &= \frac{\delta}{2\pi} (2\pi - |\theta - 2\pi|), \\
\label{eq:2.3}\begin{cases}
-\frac{\theta}{\delta} & \text{if } \theta \in [0, \pi], \\
-\delta + \frac{\pi}{\delta} (\theta - \pi) & \text{if } \theta \in [\pi, 3\pi], \\
\delta - \frac{\pi}{\delta} (\theta - 3\pi) & \text{if } \theta \in [3\pi, 4\pi].
\end{cases}
\end{align}

Note that the global Lipschitz constants of $h_1$ and $h_2$ are: $L(h_1) = \delta/(2\pi)$ and $L(h_2) = \delta/\pi$. One also has the lower bound:

\begin{equation}
\label{eq:2.4}
\inf_{\theta \in [0, 2\pi]} \left[ (h_1(\theta) - h_1(\theta + 2\pi))^2 + (h_2(\theta) - h_2(\theta + 2\pi))^2 \right]^{1/2} \geq \frac{\delta}{2},
\end{equation}

which is proven in three cases; case $\theta \in [0, \pi/2]$: then $h_1(\theta) \leq \delta/4$ and $h_1(\theta + 2\pi) \geq 3\delta/4$; case $\theta \in [\pi/2, 3\pi/2]$; then $h_2(\theta) \in [-\delta, -\delta/2]$ and $h_2(\theta + \pi) \in [\delta/2, \delta]$; case $\theta \in [3\pi/2, 2\pi]$: then $h_1(\theta) \geq 3\delta/4$ and $h_1(\theta + 2\pi) \leq \delta/4$.

We now define the 5-Lipschitz cut-off function $\phi : [0, 5^{-i-1}] \to \mathbb{R}$:

\begin{equation}
\label{eq:2.5}
\phi(r) = \begin{cases}
5r & \text{if } r \in [0, 5^{-i-2}], \\
5^{-i-1} & \text{if } r \in [5^{-i-2}, 5^{-i-1} - 5^{-i-2}], \\
5^{-i-1} [1 - 5^{-i+2}(r - 5^{-i-1} + 5^{-i-2})] & \text{if } r \in [5^{-i-1} - 5^{-i-2}, 5^{-i-1}],
\end{cases}
\end{equation}

and note that $\|\phi\|_\infty \leq 5^{-i-1}$.
We now define $\Psi$ using polar coordinates:

$$
\Psi : \Sigma \to \mathbb{R}^2
$$

$$
(r, \theta) \mapsto (\phi(r)h_1(\theta), \phi(r)h_2(\theta)),
$$

and find the unique continuous extension $\Psi : \tilde{Q} \to \mathbb{R}^2$ with $\Psi = 0$ on $Q_c \cup Q_0$. We now collect the important properties of $\Psi$. First, if $p_1, p_2 \in \tilde{\pi}^{-1}(q)$ for $q \in Q_a \setminus \sigma$ and $|\theta(p_1) - \theta(p_2)| = \pi$, then (2.4) implies:

$$
(2.7) \quad \|\Psi(p_1) - \Psi(p_2)\|_{\mathbb{R}^2} \geq \frac{\delta}{2} \phi(r(p_1)).
$$

Second from the upper bound on $\phi$ we get:

$$
(2.8) \quad \|\Psi\|_{\mathbb{R}^2} \leq \delta \text{ diam } Q,
$$

and third, from computing $d\Psi$ and using the standard Riemannian metric $r^2d\theta^2 + dr^2$ on $\Sigma$, we estimate the global Lipschitz constant of $\Psi$:

$$
(2.9) \quad L(\Psi) \in [\delta, 7\delta].
$$

In the following we let $\{e_i\}_{i=1}^\infty$ denote the standard orthonormal basis of $l^2$.

**Construction 2.10** (Construction of maps $F_i : X_i \to l^2$). The map $F_0 : X_0 \to l^2$ is just an isometric embedding of the square $X_0$ in the plane $e_1 \oplus e_2$. To get $F_1 : X_1 \to l^2$ we modify $F_0 \circ \pi_{1,0}$ by adding to it $\Psi_{\delta_1} \otimes (e_3 \oplus e_4)$: this notation means that we take the map $\Psi$ from Construction 2.1 with $\delta = \delta_1$ and with $\tilde{Q}$ the unique square $\{Q\} = Sq_0(X_0)$, and then we identify the codomain of $\Psi$ with the plane $e_3 \oplus e_4$. In particular note that:

$$
(2.11) \quad \|F_0 \circ \pi_{1,0} - F_1\|_\infty \lesssim \delta_1 5^{-i-1}
$$

$$
(2.12) \quad L(F_1) \lesssim (1 + \delta_1^2)^{1/2}.
$$

For $i \geq 1$, the map $F_{i+1}$ is defined by induction. We first have that $\text{Im } F_i$ is a subset of the hyperplane of $l^2$ spanned by the vectors $\{e_\alpha\}_{1 \leq \alpha \leq 2i+2}$; then for each $Q \in Sq_i(X_i)$ we choose $\Psi_{\delta_i, Q} : \tilde{Q} \to \mathbb{R}^2$ as in Construction 2.1 setting $\delta = \delta_{i+1}$, and we then let:

$$
(2.13) \quad F_{i+1} = F_i \circ \pi_{i+1,i} + \sum_{Q \in Sq_i(X_i)} \Psi_{\delta_i, Q} \otimes (e_{2i+3} \oplus e_{2i+4}).
$$

As we have inserted the new contributions in a plane orthogonal to $\text{Im } F_i$ we conclude that:

$$
(2.14) \quad L(F_{i+1}) \lesssim (1 + \delta_1^2 + \cdots + \delta_i^2)^{1/2},
$$

and moreover:

$$
(2.15) \quad \|F_i \circ \pi_{i+1,i} - F_{i+1}\|_\infty \lesssim \delta_i 5^{-i-1}.
$$

**Lemma 2.16** (Convergence of the maps $F_i \circ \pi_{\infty,i}$). The pull-backs $F_i \circ \pi_{\infty,i}$ converge uniformly to a map $F_\infty : X_\infty \to l^2$ whose Lipschitz constant satisfies:

$$
(2.17) \quad L(F_\infty) \lesssim \left(1 + \sum_i \delta_i^2\right)^{1/2}.
$$
Let $P_i : l^2 \to l^2$ denote the orthogonal projection of $l^2$ onto the hyperplane spanned by $\{e_1, e_2, \cdots, e_{2i+1}, e_{2i+2}\}$ and let $i \leq j$ where $j = \infty$ is admissible. Defining $Y_j = F_j(X_j)$ we have a commutative diagram:

\[
\begin{array}{ccc}
X_j & \xrightarrow{F_j} & Y_j \\
\Downarrow{\pi_{j,i}} & & \Downarrow{P_i} \\
X_i & \xrightarrow{F_i} & Y_i
\end{array}
\]

(2.18)

Proof. By (2.15) the $F_i \circ \pi_{i+1,i}$ converge uniformly and the limit map $F_\infty$ satisfies the Lipschitz bound (2.17) as (2.14) implies a uniform bound on the Lipschitz constants of the $\{F_i\}_i$. When $j < \infty$ the commutativity of the diagram (2.18) follows from the definition of the maps $\{F_i\}_i$; for $j = \infty$ one passes the commutativity to the limit. \qed

In the following we let $N_\infty$ be the 2-normal current canonically associated to $X_\infty$: details and the precise definition of $N_\infty$ are in [Sch15, Sec. 3]. Recall also that, even though $N_\infty$ is a metric current, the calculus on $X_\infty$ is similar to the classical one in $\mathbb{R}^2$, and $N_\infty$ admits a “classical” 2-vector-field representation: $N_\infty = \partial_x \wedge \partial_y \ d\mu_{X_\infty}$.

**Lemma 2.19** (Existence and nontriviality of the 2-current). The push-forward $F_\infty\#N_\infty$ is a nontrivial 2-normal current in $l^2$ supported on $Y_\infty$.

Proof. As $F_\infty$ is Lipschitz (actually it is a Lipschitz embedding, but not biLipschitz as the biLipschitz constants of the $F_i$ degrade as $i \to \infty$), we only have to show that $F_\infty\#N_\infty$ is nontrivial. Let $x, y$ denote the standard “coordinate” functions on $e_1 \oplus e_2$, and assume that $Y_0$ is normalized to be a unit square in that plane. Using the commutativity of the diagram (2.18) for $j = \infty$ and $i = 0$ we get:

\[
(2.20) \quad P_0\#F_\infty\#N_\infty(dx \wedge dy) = (F_0 \circ \pi_\infty,0)\#N_\infty(dx \wedge dy) = F_0\#N_0(dx \wedge dy) = 1,
\]

where $N_0$ denotes the current associated to $X_0$, i.e. the anticlockwise-oriented unit square with the Lebesgue measure. \qed

**Theorem 2.21** (2-rectifiability of $Y_\infty$). $Y_\infty$ is purely 2-rectifiable in the sense that whenever $K \subset \mathbb{R}^2$ is compact and $\Phi : K \to l^2$ is Lipschitz, $\mathcal{H}^2(\Phi^{-1}(Y_\infty) \cap K) = 0$.

Proof. We will argue by contradiction assuming that $K \subset \Phi^{-1}(Y_\infty)$ and that $\mathcal{H}^2(K) > 0$.

Step 1: Reduction to the case in which $\Phi$ is a graph over $Y_0$.

Let $\Phi_n = P_n \circ \Phi$ and, using the Radon-Nikodym property of $l^2$, note that at each point $p \in K$ of differentiability of $\Phi$ one has that each $\Phi_n$ is also differentiable at $p$ and that:

\[
(2.22) \quad \lim_{n \to \infty} d\Phi_n(p) = d\Phi(p),
\]

where the limit is in the norm-topology of linear maps $\mathbb{R}^2 \to l^2$. Following the notation of [AK00b, Sec. 4&5], we let $J_2$ denote the Jacobian appearing in the area formula; by dominated convergence we then have:

\[
(2.23) \quad \lim_{n \to \infty} \int_K \chi_E J_2(d\Phi_n) \, d\mathcal{H}^2 = \int_K \chi_E J_2(d\Phi) \, d\mathcal{H}^2
\]
whenever \( E \subset K \) is a Borel set.

We now consider the Borel set \( E \subset K \) consisting of those points which are Lebesgue density points of the set of points where \( \Phi \) is differentiable and where \( d\Phi_0 \) has rank \( < 2 \), and our goal is to show that:

\[
(2.24) \quad \mathcal{H}^2(\text{Im} \, \Phi \cap \mathcal{P}_0^{-1}(\Phi_0(E))) = 0.
\]

Note that the area formula \([AK00b, \text{Thm. 5.1}]\) gives \( \mathcal{H}^2(\Phi_0(E) \cap Y_0) = 0 \). For each \( n \geq 1 \), using the square complex structure of \( \{X_i\}_{i \leq n} \), the set \( Y_n \) can be partitioned into finitely many closed sets \( \{S_\alpha\}_\alpha \) such that each restriction \( P_0|_{S_\alpha} : S_\alpha \to P_0(S_\alpha) \) is biLipschitz, thus giving:

\[
(2.25) \quad \mathcal{H}^2(\Phi_n(E) \cap Y_n) = 0.
\]

In particular, the area formula implies that:

\[
(2.26) \quad \int_K \chi_{EJ_2(d\Phi_n)} d\mathcal{H}^2 = 0.
\]

Therefore, by (2.24) we conclude that:

\[
(2.27) \quad \int_K \chi_{EJ_2(d\Phi)} d\mathcal{H}^2 = 0
\]

and then (2.24) follows from the area formula.

Therefore by (2.24) we can assume that \( d\Phi_0 \) has full rank 2 on the set of Lebesgue density points of the set of differentiability points of \( \Phi \). Using \([Kir94, \text{Thm. 9}]\), which is essentially a Lipschitz version of the Inverse Function Theorem, up to further partitioning \( K \) and throwing away a set of null measure, we can assume that \( \Phi \) is \( C \)-biLipschitz and that \( \Phi_0 \circ \Phi = \text{Id}_{\mathcal{P}_0(K)} \). In particular, we can assume that \( K \subset Y_0 \) and that \( \Phi_0 \) is just the identity map.

**Step 2: Existence of square holes at scale \( 5^{-n} \).**

Note that the square-complex structure of \( X_n \) induces a square-complex structure on \( Y_n \) via the homeomorphism \( F_n \); in the following for \( i \geq n \) we will implicitly identify \( \text{Sq}_i(Y_n) \) with \( \text{Sq}_i(X_n) \).

Fix now \( n \) and a square \( Q \in \text{Sq}_{n-1}(Y_0) \). Let \( \hat{Q}_\alpha \) and \( \sigma \) be as in Construction 2.1 and recall that \( \hat{Q}_\alpha \) consists of squares of \( \text{Sq}_{n+2}(Y_0) \).

We now fix a small parameter \( c \) to be determined later in function of the bi-Lipschitz constant \( C \) of \( \Phi \) and the Lipschitz constant of \( F_\infty \). Let

\[
i_n = \lceil -\log_5(5^{n-2}c\delta_n) \rceil
\]

and partition \( \hat{Q}_\alpha \) into \( 5^{n-n} \) annuli consisting of squares of \( \text{Sq}_{j_{n-1}}(Y_0) \). We consider one such an annulus \( A \). Our goal is to show that \( K \) has to miss the interior of one of the squares in \( A \).

We first order the squares \( \{R_\alpha\}_{1 \leq \alpha \leq t} \) of \( A \) anticlockwise so that \( R_{\alpha+1} \) follows \( R_\alpha \), and \( R_1 \) follows \( R_t \), and \( R_1 \) and \( R_t \) meet along a subsegment of \( \sigma \). Assume that \( K \) intersects each \( \text{Int}(R_\alpha) \) and let \( p_\alpha \in K \cap \text{Int}(R_\alpha) \).

We first show that for \( 0 \leq j \leq n \) the points \( \Phi_j(p_\alpha) \) and \( \Phi_j(p_{(\alpha+1) \mod t}) \) belong to the same square of \( \text{Sq}_j(Y_j) \). In the following we use \( \beta \) to denote \( \alpha \) or \( (\alpha + 1) \mod t \) and we will just write \( \alpha + 1 \) for \( (\alpha + 1) \mod t \).

For \( j = 0 \) by construction \( \Phi_0(p_\alpha) \) and \( \Phi_0(p_{\alpha+1}) \) belong to the same square of \( \text{Sq}_0(Y_0) \), and for \( j \geq 1 \) we assume by induction that \( \Phi_{j-1}(p_\alpha), \Phi_{j-1}(p_{\alpha+1}) \) belong to the same \( Q_{j-1}^{(j-1)} \in \text{Sq}_{j-1}(Y_{j-1}) \). Let \( Q_{j,\beta}^{(j)} \in \text{Sq}_j(Y_j) \) denote the square containing
Φ_j(p_β) and assume by contradiction that \(Q(j)_{\alpha, \beta} \neq Q(j)_{\alpha, \alpha + 1}\). In the following we will use the decorators \(\alpha, \alpha, \beta, \ldots\) as in Construction 2.1: for example \(Q(j-1)_{\alpha, \alpha + 1}\) is obtained as \(\hat{Q}\) if we let \(Q = Q(j-1)_{\alpha, \alpha + 1}\). In particular, as \(Q(j)_{\alpha, \beta} \neq Q(j)_{\alpha, \alpha + 1}\) we must have \(P(j-1)_{\alpha, \beta} \subset Q(j-1)_{\alpha, \alpha + 1}\). Let now \(Q(j)_{\alpha, \beta} \in S_{Q(j)_{\alpha, \beta}}(0)\) denote the square containing \(p_\beta\), let \(q_\beta\) be its center, set \(q(j)_{\alpha, \beta} = P^{-1}(Q(j)_{\alpha, \beta}) \cap Q(j)_{\alpha, \beta}\) and let \(q(j)_{\alpha, \beta}\) denote its center.

As \(\Phi\) is \(C\)-Lipschitz,
\[
d(\Phi_j(p_\alpha), \Phi_j(p_{\alpha + 1})) \leq 4C \times c5^{-n}\delta_n;
\]
as \(F_j\) is \(L(F_\infty)\)-Lipschitz,
\[
d(q(j)_{\alpha, \beta}, \Phi_j(p_\beta)) \leq 2L(F_\infty) \times c5^{-n}\delta_n,
\]
so that:
\[
d(q(j)_{\alpha}, q(j)_{\alpha + 1}) \leq 4(C + L(F_\infty)) \times c5^{-n}\delta_n.
\]
Let \(S(j-1)_{\alpha, \beta} = F^{-1}(Q(j-1)_{\alpha, \beta})\) and \(S(j)_{\alpha, \beta} = F^{-1}(Q(j)_{\alpha, \beta})\); we must have \(S(j)_{\alpha, \beta} \neq S(j)_{\alpha, \alpha + 1}\) and \(\pi_{j-1}(F^{-1}(q(j)_{\alpha})) \in S(j-1)_{\alpha, \alpha + 1}\). Note that \(F^{-1}(q(j)_{\alpha})\) must be at distance \(\geq 5^{-n}\) from \(\partial S(j-1)_{\alpha, \alpha + 1}\) if \(j \leq n - 1\) and at distance \(\geq 5^{-n-3}\) if \(j = n\) (in this case we use that \(p_\beta \in Q(j)_{\alpha, \alpha + 1}\), so that:
\[
\phi(r(F^{-1}(q(j)_{\alpha}))) \geq 5^{-n-3}.
\]
As \(F^{-1}(q(j)_{\alpha}) \neq F^{-1}(q(j)_{\alpha + 1})\), they belong to different sheets of the double cover, and as \(\pi_{j-1}(S(j)_{\alpha})\) and \(\pi_{j-1}(S(j)_{\alpha + 1})\) are adjacent, we let \(q(j)_{\alpha}\) be the center of the square of \(S_{Q(j)_{\alpha}}(Y_j)\) adjacent to \(Q(j)_{\alpha, \alpha + 1}\) and such that \(\pi_{j-1}(F^{-1}(q(j)_{\alpha})) = \pi_{j-1}(F^{-1}(q(j)_{\alpha + 1}))\).

We now have:
\[
r(F^{-1}(q(j)_{\alpha})) = r(F^{-1}(q(j)_{\alpha})),
\]
\[
|\theta(F^{-1}(q(j)_{\alpha})) - \theta(F^{-1}(q(j)_{\alpha}))| = \pi,
\]
and invoking (2.7) we get:
\[
d(q(j)_{\alpha}, q(j)_{\alpha + 1}) \geq \frac{5}{2} \delta_j \geq \frac{5}{2} \delta_n.
\]
But as \(q(j)_{\alpha}\) is the center of the square of \(S_{Q(j)_{\alpha}}(Y_j)\) adjacent to \(Q(j)_{\alpha, \alpha + 1}\), from (2.31) we get:
\[
d(q(j)_{\alpha}, q(j)_{\alpha}) \leq 8(C + L(F_\infty)) \times c5^{-n}\delta_n.
\]
Now, combining (2.36) and (2.35) and choosing \(c \leq 10^{-6}/(C + L(F_\infty))\) we get a contradiction and conclude that \(Q(j)_{\alpha, \beta} = Q(j)_{\alpha, \alpha + 1}\).

A consequence of the previous discussion, specialized to \(j = n\), is that \(\Phi_n(p_1)\) and \(\Phi_n(p_n)\) belong to the same sheet of the double cover \(P^{-1}(Q(n-1)_{\alpha, \alpha + 1}) \cap Y_n \rightarrow Q(n-1)_{\alpha, \alpha + 1}\), while the choice of \(c\) gives:
\[
d(F^{-1}(\Phi_n(p_1)), F^{-1}(\Phi_n(p_n))) \leq 5^{-n-3},
\]
which contradicts (ShSep).
Let now \( R_A \) denote the interior of the square of \( A \) that \( K \) misses. For each of the \( \approx 5^{n-n} \) annuli we can find such a square and group them in a set \( \text{Holes}(Q_{n-1}^{(0)}) \), and we have that:

\[
\mathcal{H}^2 \left( \text{Holes}(Q_{n-1}^{(0)}) \right) \geq \gamma \delta_n \mathcal{H}^2(Q_{n-1}^{(0)})
\]

for a constant \( \gamma > 0 \) which does not depend on \( n \) or \( Q_{n-1}^{(0)} \). We thus conclude that

\[
\mathcal{H}^2(K) \leq \mathcal{H}^2 \left( Y_0 \setminus \bigcup_{Q_{n-1}^{(0)} \in \text{Sq}_{n-1}(Y_0)} \text{Holes}(Q_{n-1}^{(0)}) \right) \leq (1 - \gamma \delta_n)\mathcal{H}^2(Y_0).
\]

**Step 3:** **Cumulating the effects of holes and the choice of \( \{\delta_n\} \).**

Let \( Q_0 \) denote the unique square of \( \text{Sq}_0(Y_0) \). By Step 2 we have:

\[
\mathcal{H}^2(K) \leq \mathcal{H}^2 \left( Y_0 \setminus \bigcup \text{Holes}(Q_0) \right) \leq (1 - \gamma \delta_1)\mathcal{H}^2(Y_0).
\]

Now \( \text{Holes}(Q_0) \) consists of squares of generation \( \leq k_2 = 1 + \lceil G \log(1/\delta_1) \rceil \) where \( G \) is an appropriate constant which depends on \( c \) and \( C \). As squares are nested, if we apply Step 2 on each of the squares of \( \text{Sq}_{k_2}(Y_0) \) which do not intersect the interior of \( \bigcup \text{Holes}(Q_0) \) we get:

\[
\mathcal{H}^2(K) \leq (1 - \gamma \delta_1)(1 - \gamma \delta_{k_2}).
\]

In general, we can reiterate, and get:

\[
\mathcal{H}^2(K) \leq \prod_j (1 - \gamma \delta_{k_j}),
\]

where \( k_1 = 1 \) and \( k_{j+1} = k_j + \lceil G \log(1/\delta_j) \rceil \). If we had

\[
\sum_j \delta_{k_j} = \infty
\]

we would finish obtaining the contradiction \( \mathcal{H}^2(K) = 0 \).

We show that (2.43) holds if \( \delta_n = \frac{1}{10^{n-n}} \). For simplicity we assume that logarithms are in base 10. We use the estimate:

\[
\sum_{j=10^t}^{10^{t+1}} \frac{1}{j} \geq \frac{\log 10^{t+1} - \log 10^t}{16} = \frac{1}{16}
\]

If \( k_t \in (10^t, 10^{t+1}) \) then \( k_t \) and \( k_{t+1} \) are separated by a distance \( \leq 23(t+1) \). Hence we have:

\[
\sum_{10^t \leq k_t < 10^{t+1}} \delta_{k_t} \geq \frac{1}{42(t+1)}.
\]

We thus have:

\[
\sum_j \delta_{k_j} \geq \lim_{T \to \infty} \sum_{t=2}^{T} \frac{1}{42(t+1)} = \infty.
\]
3. 2-CURRENT IN $\mathbb{R}^4$

In $\mathbb{R}^4$ we have both to construct the metric spaces $X_n$ and the embeddings as the construction cannot be self-similar.

**Construction 3.1 (2-Normal current in $\mathbb{R}^4$).** Step 1: Affine approximation of $\Psi_\delta$.

Let $Q$, $\tilde{Q}$, $\Psi_\delta$, etc... be as in Construction 2.1. Thee maps $h_1$, $h_2$ and $\phi$ are piecewise-affine, while $\theta$ and $r$, which are defined on $\Sigma$, are not so. However, by taking iterated subdivisions of $Q$ and $\tilde{Q}$, we can approximate $\theta$ and $r$ by maps which are affine on each square of $\Sigma^{(N)}$; letting $N \to \infty$ one can take the approximations as close as one wants in the uniform topology while keeping the Lipschitz constants bounded. Thus, there are an $N \in \mathbb{N}$, independent of $\delta$, and a piecewise-affine map

$$\Phi_\delta : \tilde{Q}^{(N)} \to \mathbb{R}^2$$

such that the corresponding of (2.7), (2.8) and (2.9) hold:

$$L(\Phi_\delta) \in \left[\frac{\delta}{16}, 23\delta\right]$$

$$\|\Phi_\delta(p_1) - \Phi_\delta(p_2)\|_{\mathbb{R}^2} \geq \frac{\delta}{3}\phi(r(p_1))$$

$$\|\Phi_\delta\|_{\mathbb{R}^2} \leq 2\delta \text{diam } Q.$$  

Step 2: Construction of $F_1$.

Let $X_0 = [0, 1]^2$ and $F_0 : X_0 \to e_1 \oplus e_2 \subset \mathbb{R}^4$ be the standard isometric embedding; $X_1$ is obtained by applying to $X_0$ Construction 2.1 as in the $l^2$-case and then we let:

$$F_1 = F_0 \circ \pi_{1,0} + \Phi_\delta \otimes (e_3 \oplus e_4).$$

Note that we have bounds on the Lipschitz constant of $F_1$:

$$L(F_1) \in \left[\frac{(1 + \delta_1^2)^{1/2}}{16}, 23(1 + \delta_1^2)^{1/2}\right]$$

and that because of (3.4) $F_1$ is a topological embedding, being injective. Let $\text{Sq}(X_1)$ denote the set of squares of $X_1$ and let $Y_1 = F_1(X_1)$. As $F_1$ is piecewise affine, each $Q \in \text{Sq}(X_1)$ determines a unique affine 2-plane $\tau(Q) \subset \mathbb{R}^4$ which contains $F_1(Q)$; the corresponding unique 2-plane parallel to $\tau(Q)$ and passing through the origin will be denoted by $\tau_0(Q)$; we finally let:

$$\text{Th}(1) = \bigcup_{Q \in \text{Sq}(X_1)} \tau(Q)$$

$$\text{Th}_0(1) = \bigcup_{Q \in \text{Sq}(X_1)} \tau_0(Q),$$

and note that both sets are finite.

Step 3: The Radial Basis Neighbourhood.

For $Q \in \text{Sq}(X_1)$ we let $\pi_{\tau(Q)}$ denote the orthogonal projection onto $\tau(Q)$ and define the radial-basis function:

$$\varphi_Q(x) = \begin{cases} 
\exp\left(-\frac{\varphi_0}{\text{dist}(\pi_{\tau(Q)}(x), F_1(Q))}\right) \times 46 \text{diam}(F_1(Q)) & \text{if } \pi_{\tau(Q)} \in \text{Int}(F_1(Q)) \\
0 & \text{otherwise},
\end{cases}$$
where $\sigma_1 > 0$ is a parameter to be chosen later. We then define the radial basis neighbourhood:

\[(3.11) \quad \text{RN}(1) = \left\{ p \in \mathbb{R}^4 : \text{there is a } Q \in \text{Sq}(X_1) : p = x + y, x \in F_1(Q), \right. \]

\[ \left. y \perp \tau(Q), \text{and } \|y\| \leq \varphi(Q(x)) \right\}. \]

RN(1) is not a neighbourhood of $Y_1$ as about each point of $F_1(\partial Q)$ it has empty interior; however, it is close to being a neighbourhood of $Y_1$ as it contains a neighbourhood of:

\[(3.12) \quad \bigcup_{Q \in \text{Sq}(X_1)} \text{Int}(F_1(Q)). \]

We define $P_1 : \text{RN}(1) \to Y_1$ by $p = x + y \mapsto x$. Note that if $\sigma_1$ is sufficiently large $P_1$ is well-defined (see Lemma 3.37), and that:

(\textbf{Claim1}): For each $\varepsilon_1 > 0$ there is a $\sigma_1 > 0$ such that $P_1$ is $(1 + \varepsilon_1)$-Lipschitz.

\textbf{Step 4: The adaptative subdivision of $X_1$ and the construction of $X_2$}.

Let $\text{Sk}_1(X_1)$ denote the 1-skeleton of $X_1$ (i.e. the union of 1-and-0-dimensional cells) and $\text{Sq}_k(X_1)$ the set of squares obtained by subdividing the squares of $\text{Sq}(X_1)$ $k$-times (i.e. we get $5^{2k}$-isometric subsquares from each $Q \in \text{Sq}(X_1)$). Let

\[(3.13) \quad \text{Sq}_\infty(X_1) = \bigcup_{k \geq 1} \text{Sq}_k(X_1); \]

we say that $Q \in \text{Sq}_\infty(X_1)$ is \textbf{adapted} to $\text{RN}(1)$ if the $(23\delta_1 \text{ diam } F_1(Q))$-neighborhood of $F_1(Q)$ is contained in $\text{RN}(1)$ and if, denoting by PAR($Q$) $\in \text{Sq}(X_1)$ the unique square containing $Q$, one has:

\[(3.14) \quad \max_{x \in Q} \text{dist}(x, \partial Q) \leq \delta_1 \max_{x \in Q} \text{dist}(x, \partial \text{PAR}(Q)). \]

Now the set of adapted squares is partially ordered by inclusion and we let $\text{Sq}_{\text{ad}}(X_1)$ denote the set of its maximal elements. Note that the elements of $\text{Sq}_{\text{ad}}(X_1)$ must have pairwise disjoint interiors and:

\[(3.15) \quad X_1 \setminus \text{Sk}_1(X_1) = \bigcup_{Q \in \text{Sq}_{\text{ad}}(X_1)} Q. \]

We obtain $X_2$ from $X_1$ by applying Construction 2.1 to each $Q \in \text{Sq}_{\text{ad}}(X_1)$, and subdividing the resulting squares $N$-times as in \textbf{Step 1}. Now $X_2$ is not a square complex, but it is almost so. First, $X_2$ is the limit on an admissible inverse system in the sense of Definition 3.1 in [Sch15]. As on $X_0$ and $X_1$ we considered the canonical measures constructed in Section 2, we obtain a canonical measure $\mu_2$ on $X_2$ so that $(X_2, \mu_2)$ is a $(1,1)$-PI space (see Theorem 3.8 in [Sch15]). As the metric on $X_2$ we will consider the length metric and we observe that $X_2$ is doubling with doubling constant $\leq 15$. We also obtain a 1-Lipschitz map $\pi_{2,1} : X_2 \to X_1$ as the inverse limit system associated to $X_2$ is built on top of $X_1$. By Theorem 3.20 in [Sch15] we obtain a 2-dimensional simple normal current $N_2$ with $\|N_2\| = \mu_2$ and $\pi_{2,1} \# N_2 = N_1$, $N_1$ being the canonical normal current associated to $X_1$.

Second $\text{Sk}_1(X_1)$ embeds isometrically in $X_2$ and, away from $\text{Sk}_1(X_1)$, $X_2$ has a square complex structure. In fact, each $Q \in \text{Sq}_{\text{ad}}(X_1)$ gives rise to at most
10 \times 5^{N+7} \text{ squares in } X_2; \text{ we thus denote the set of such squares by } \text{Sq}(X_2) \text{ and let:}

\begin{equation}
\text{Sk}_1(X_2) = \text{Sk}_1(X_1) \cup \bigcup_{Q \in \text{Sq}(X_2)} \text{Sk}_1(Q).
\end{equation}

**Step 5: The construction of } F_2.\)**

To get } X_2 \text{ we have applied to each } Q \in \text{Sq}_{\text{ad}}(X_1) \text{ Construction 2.1 and we have further subdivided } N \text{-times the squares of the branched cover } \tilde{Q} \to Q \text{ so that we can define } \Phi_{Q,\delta_2} : \tilde{Q} \to \mathbb{R}^2 \text{ as in Step 1. However, we need a bit extra care to get finitely many possibilities for the tangent space of } Y_2: \text{ this will be useful in the proof of Lemma 3.37. First, for } Q_1 \neq Q_2 \in \text{Sq}_{\text{ad}}(X_1) \text{ the maps } \Phi_{Q_1,\delta_2} \text{ and } \Phi_{Q_2,\delta_2} \text{ can be taken to be the same up to composition with translations and dilations. Second, each } Q \in \text{Sq}_{\text{ad}}(X_1) \text{ belongs to a unique parent } \text{PAR}(Q) \in \text{Sq}(X_1). \text{ As } \text{Th}_0(1) \text{ is finite, we can choose a finite set of pairs } \{(e_1,Q, e_2,Q)\}_{Q \in \text{Sq}_{\text{ad}}(X_1)} \text{ such that each pair } (e_1,Q, e_2,Q) \text{ is an orthonormal basis of the 2-plane orthogonal to } \tau_0(\text{PAR}(Q)).\text{ We let:}

\begin{equation}
F_2(x) = F_1 \circ \pi_{2,1}(x) + \sum_{Q \in \text{Sq}_{\text{ad}}(X_1)} \Phi_{Q,\delta_2}(x) \otimes (e_1,Q, e_2,Q),
\end{equation}

and observe that by (3.4) } F_2 \text{ is a topological embedding. As } X_2 \text{ is a length space and as } \Phi_{Q,\delta_2} \text{ adds a contribution to the gradient of } F_1 \text{ orthogonally to } \tau_0(\text{PAR}(Q)), \text{ we get:}

\begin{equation}
16^{-1}(1 + \delta_1^2 + \delta_2^2)^{1/2} \leq L(F_2) \leq 23(1 + \delta_1^2 + \delta_2^2)^{1/2},
\end{equation}

and we also have:

\begin{equation}
\|F_1 \circ \pi_{2,1} - F_2\|_{\infty} \leq 56 \times 5^{-2}\delta_2.
\end{equation}

Let } Y_2 = F_2(X_2) \text{ and note that } F_2 \text{ is affine when restricted to each } Q \in \text{Sq}(X_2). \text{ We let } \tau(F_2(Q)) \text{ denote the affine 2-plane containing } F_2(Q) \text{ and } \tau_0(F_2(Q)) \text{ the corresponding 2-plane passing through the origin. We finally let}

\begin{equation}
\text{Th}(2) = \bigcup_{Q \in \text{Sq}(X_2)} \tau(Q)
\end{equation}

\begin{equation}
\text{Th}_0(2) = \bigcup_{Q \in \text{Sq}(X_2)} \tau_0(Q),
\end{equation}

and note that } \text{Th}_0(2) \text{ is finite by the choice of } \{(e_1,Q, e_2,Q)\}_{Q \in \text{Sq}_{\text{ad}}(X_1)} \text{ (while } \text{Th}(2) \text{ is not finite). By construction we also have the commutative diagram:}

\begin{equation}
\begin{array}{ccc}
X_2 & \xrightarrow{F_2} & Y_2 \\
\downarrow{\pi_{2,1}} & & \downarrow{P_1} \\
X_1 & \xrightarrow{F_1} & Y_1 \\
\downarrow{\pi_{1,0}} & & \downarrow{P_0} \\
X_0 & \xrightarrow{P_0} & Y_0
\end{array}
\end{equation}

**Step 6: The general iteration.**
Assume we have constructed \( \{X_k\}_{k \leq j}, \{\text{RN}(k)\}_{k \leq j-1} \) and \( \{F_k\}_{k \leq j} \); for \( Q \in \text{Sq}(X_j) \) we define the \textbf{radial basis function}:

\[
\varphi_Q(x) = \begin{cases} 
\exp \left( -\frac{\sigma_j}{\text{dist}(\pi_{\tau(Q)}(x), F_j(Q))} \right) \times 46 \text{diam}(F_j(Q)) & \text{if } \pi_{\tau(Q)} \in F_j(\text{Int}(Q)) \\
0 & \text{otherwise},
\end{cases}
\]

where \( \sigma_j > 0 \) is a parameter to be chosen later. We then define the \textbf{radial basis neighbourhood}:

\[
\text{RN}(j) = \left\{ p \in \mathbb{R}^d : \text{there is a } Q \in \text{Sq}(X_j) : p = x + y, x \in F_j(Q), \\ y \perp \tau(Q), \text{ and } \|y\| \leq \varphi_Q(x) \right\}.
\]

As for \( \text{RN}(1) \), \( \text{RN}(j) \) is not a neighbourhood of \( Y_j \) but it is a neighbourhood of

\[
\bigcup_{Q \in \text{Sq}(X_j)} \text{Int}(F_j(Q)).
\]

We define \( P_j : \text{RN}(j) \to Y_j \) by \( p = x + y \mapsto x \) and will later show that if \( \sigma_j \) is sufficiently large, \( P_j \) is well-defined (see Lemma 3.37), and that:

**(Claim j):** For each \( \varepsilon_j > 0 \) there is a \( \sigma_j > 0 \) such that \( P_j \) is \((1+\varepsilon_j)\)-Lipschitz.

We then define as above:

\[
\text{Sq}_\infty(X_j) = \bigcup_{k \geq 1} \text{Sq}_k(X_j);
\]

we say that \( Q \in \text{Sq}_\infty(X_j) \) is \textit{adapted} to \( \text{RN}(j) \) if the \((23\delta_j \text{diam } F_j(Q))\)-neighborhood of \( F_j(Q) \) is contained in \( \text{RN}(j) \) and if, denoting by \( \text{PAR}(Q) \in \text{Sq}(X_1) \) the unique square containing \( Q \), one has:

\[
\max_{x \in Q} \text{dist}(x, \partial Q) \leq \delta_j \max_{x \in Q} \text{dist}(x, \partial \text{PAR}(Q)).
\]

As above we let \( \text{Sq}_{\text{ad}}(X_j) \) be the set of maximal adapted squares, which must then have pairwise disjoint interiors and satisfy:

\[
X_j \setminus \text{Sk}_1(X_j) = \bigcup_{Q \in \text{Sq}_{\text{ad}}(X_j)} Q.
\]

We obtain \( X_{j+1} \) from \( X_j \) by applying Construction 2.1 to each \( Q \in \text{Sq}_{\text{ad}}(X_j) \) and subdividing the obtained squares other \( N \)-times. As discussed above, \( X_{j+1} \) is not a square complex, but it is almost so. In fact, \( X_{j+1} \) is the limit of an admissible inverse system in the sense of Definition 3.1 of [Sch15]. We get a 1-Lipschitz map \( \pi_{j+1,j} : (X_{j+1}, \mu_{j+1}) \to (X_j, \mu_j) \) and \( X_{j+1} \) is a doubling length space with doubling constant \( \leq 50 \) (the projection of a square of \( \text{Sq}_{\text{ad}}(X_j) \) contains at most 50 squares of 1/5-the side length). As in **Step 4** we find that to \( X_{j+1} \) is canonically associated a normal metric current \( N_{j+1} \) with \( \pi_{j+1,j} \# N_{j+1} = N_j \) and \( \|N_{j+1}\| = \|N_j\| \). We let \( \text{Sq}(X_{j+1}) \) be the corresponding set of squares of \( X_{j+1} \), which has a square-complex structure away from:

\[
\text{Sk}_1(X_{j+1}) = \bigcup_{k \leq j} \text{Sk}_1(X_j) \cup \bigcup_{Q \in \text{Sq}(X_{j+1})} \text{Sk}_1(Q);
\]
note also that:

\[
X_{j+1} \setminus \bigcup_{k \leq j} \text{Sk}_1(X_j) = \bigcup_{Q \in \text{Sq}(X_{j+1})} Q.
\]

To construct \(F_{j+1}\) we proceed as for \(F_2\): for \(Q \in \text{Sq}_{\text{td}}(X_j)\) we choose \(\Phi_{Q,\delta_{j+1}} : \tilde{Q} \to \mathbb{R}^2\) such that for \(Q_1 \neq Q_2\) the maps \(\Phi_{Q_1,\delta_{j+1}}\) and \(\Phi_{Q_2,\delta_{j+1}}\) can be taken to differ up to composition with translations and dilations. Secondly, each \(Q \in \text{Sq}_{\text{td}}(X_j)\) belongs to a unique parent \(\text{PAR}(Q) \in \text{Sq}(X_j)\) and \(\text{Th}_0(j)\) is finite. Thus we can choose a finite set of pairs \(\{(e_{1,Q}, e_{2,Q})\}_{Q \in \text{Sq}_{\text{td}}(X_j)}\) such that each \((e_{1,Q}, e_{2,Q})\) is an orthonormal basis of the orthogonal complement of \(\tau_0(\text{PAR}(Q))\). We define:

\[
F_{j+1}(x) = F_j \circ \pi_{j+1,j}(x) + \sum_{Q \in \text{Sq}_{\text{td}}(X_j)} \Phi_{Q,\delta_{j+1}}(x) \otimes (e_{1,Q} \oplus e_{2,Q}),
\]

and observe that by (3.4) \(F_{j+1}\) is a topological embedding. As \(X_{j+1}\) is a length space and as \(\Phi_{Q,\delta_{j+1}}\) adds a contribution to the gradient of \(F_j\) orthogonally to \(\tau_0(\text{PAR}(Q))\), we get:

\[
16^{-1}(1 + \sum_{l=1}^{j+1} \delta_l^2)^{1/2} \leq L(F_{j+1}) \leq 23(1 + \sum_{l=1}^{j+1} \delta_l^2)^{1/2},
\]

and we also have:

\[
\|F_j \circ \pi_{j+1,j} - F_{j+1}\|_{\infty} \leq 56 \times 5^{-j} \delta_{j+1}.
\]

Let \(Y_{j+1} = F_{j+1}(X_{j+1})\) and note that \(F_{j+1}\) is affine when restricted to each \(Q \in \text{Sq}(X_{j+1})\); as in Step 5 we define \(\tau(F_{j+1}(Q))\), \(\tau_0(F_{j+1}(Q))\), \(\text{Th}(j+1)\) and \(\text{Th}_0(j+1)\), and observe that \(\text{Th}_0(j+1)\) is finite. Finally for \(j \leq k\) one has the following commutative diagrams:

\[
\begin{array}{ccc}
X_{j+1} & \xrightarrow{F_{j+1}} & Y_{j+1} \\
\pi_{j+1,k} & & \phi_{k} \circ \phi_{k-1} \circ \cdots \circ \phi_{j} \\
X_{k} & \xrightarrow{F_{k}} & Y_{k}
\end{array}
\]

**Lemma 3.35** (Convergence of the spaces and currents). The metric measure spaces \((X_n, \mu_n)\) converge in the mGH-sense to \((X_\infty, \mu_\infty)\); having arranged convergence in a container, the normal currents \(N_n\) converge weakly to a normal current \(N_\infty\) supported in \(X_\infty\) with \(\|N_\infty\| = \mu_\infty\); the maps \(\pi_{n,i} : X_n \to X_i\) also converge to 1-Lipschitz maps \(\pi_{\infty,i} : X_\infty \to X_i\) as \(n \to \infty\) and, for each pair \(l < i\), one has commutative diagrams:

\[
\begin{array}{ccc}
(X_\infty, \mu_\infty, N_\infty) & \xrightarrow{\pi_{\infty,i}} & (X_i, \mu_i, N_i) \\
\pi_{\infty,l} & & \pi_{i,l} \\
(X_i, \mu_i, N_l) & \xrightarrow{\pi_{i,l}} & (X_l, \mu_l, N_l)
\end{array}
\]

**Proof.** The proof is routine as \((X_\infty, \mu_\infty)\) is an inverse limit of the metric measure spaces \((X_k, \mu_k)\). Even though here we work with a slightly more general cube
Choosing (note that for the case in which $j > 3.39$) $x$

**Proof.**

**Step 1:** The case $j = 1$.

As Th(1) is finite and $F_1$ is an isometric embedding plus a small Lipschitz perturbation, we can find an $\alpha > 0$ such that if $\{Q_1, Q_2\} \subseteq Sq(X_1)$ are distinct and $x_t \in F_t(Q_t)$ ($t = 1, 2$) then:

$$\|x_t - x_2\| \geq \alpha \max_{t=1,2} \text{dist}(x_t, F_1(\partial Q_t)).$$

Let $x_1 + y_1, x_2 + y_2 \in \text{RN}(1)$; then

$$\|y_t\| \leq c(\sigma_1) \text{dist}(x_t, F_1(\partial Q_t)),$$

where $\lim_{\sigma_1 \to \infty} c(\sigma_1) = 0$. Therefore,

$$\|(x_1 + y_1) - (x_2 + y_2)\|_2 \geq \|x_1 - x_2\| - c(\sigma_1)(\|y_1\| + \|y_2\|),$$

from which we get:

$$\left(1 + \frac{2}{\alpha} c(\sigma_1)\right) \|(x_1 + y_1) - (x_2 + y_2)\| \geq \|x_1 - x_2\| = \|P_1(x_1 + y_1) - P_1(x_2 + y_2)\|.$$ 

Choosing $\sigma_1$ sufficiently small we obtain that $P_1$ is well-defined and $(1+\varepsilon_1)$-Lipschitz (note that for the case in which $Q_1 = Q_2$ we have $\alpha = 1$ in (3.42)).

**Step 2:** The case $j > 1$.

By induction we assume the existence of $\eta > 0$ such that if $k \leq j - 1$, $x_t \in F_k(Q_t)$ ($t = 1, 2$ and $Q_t \in \text{Sq}(X_k)$) where $Q_1 \neq Q_2$, then:

$$\|x_t - x_2\| \geq \eta \max_{t=1,2} \text{dist}(x_t, F_k(\partial Q_t)).$$

We want to establish an analogue of (3.42), but we will need to consider 3 possibilities; we define:

$$P_t, k = P_k \circ \cdots \circ P_{t-1} \circ P_t \quad \text{(compare (3.58))},$$

and we let $Q_{k,t}$ denote the square of $\text{Sq}(X_k)$ containing $F_k^{-1}(P_{j-1,k}(x_t))$.

First assume that for some $k \leq j - 1$ $Q_{k,1} \neq Q_{k,2}$ and let $k_0$ be the minimal value of $k$ such that this happens. Then:

$$\|P_{j-1,k_0}(x_t) - P_{j-1,k_0}(x_2)\| \geq \eta \max_{t=1,2} \text{dist}(P_{j-1,k_0}(x_t), F_{k_0}(\partial Q_{k_0,t})).$$

By induction we will assume that $P_{j-1,k_0}$ is well-defined with $L(P_{j-1,k_0}) < \infty$. Let $q_t \in F_{k_0}(\partial Q_{k_0,t})$ be a closest point to $x_t$. As $F_{k_0}|Q_{k_0,t}$ is affine satisfying (3.32), we conclude that:

$$\frac{\|P_{j-1,k_0}(x_t) - q_t\|}{d(F_{k_0}^{-1}(P_{j-1,k_0}(x_t)), F_{k_0}^{-1}(q_t))} \in \left[\left(1 + \sum_{k=k_0} \delta^2_k\right)^{1/2}, \frac{23(1 + \sum_{k=k_0} \delta^2_k)^{1/2}}{16}\right] \left(1 + \sum_{k=k_0} \delta^2_k\right)^{1/2}.$$
For $k_0 < k \leq j - 1$ let $Q_{j,k,t}^{(\text{par})}$ denote the square of $\text{S} \text{q} \text{d}_a(X_{k-1})$ containing $\pi_{k,k-1}(Q_{k,t})$. From the definition of $F_k$ we get:

$$(3.47) \quad x_t - P_{j-1,k_0}(x_t) = \sum_{k_0 + 1 \leq k \leq j} \Phi_{Q_{j,k,t}}^{(\text{par})} \cdot \delta_k (\pi_{j,k} \circ F_j^{-1}(x_t)) \otimes (e_{1,Q_{j,k,t}}^{(\text{par})} + e_{2,Q_{j,k,t}}^{(\text{par})}).$$

From the bound on the Lipschitz constant of $\Phi_{Q_{j,k,t}}^{(\text{par})} \cdot \delta_k$ we get:

$$(3.48) \quad ||\Phi_{Q_{j,k,t}}^{(\text{par})} \cdot \delta_k (\pi_{j,k} \circ F_j^{-1}(x_t))||_{\mathbb{R}^2} \leq 28\delta_k \text{dist}(\pi_{j,k} \circ F_j^{-1}(x_t), \partial Q_{j,k,t}^{(\text{par})});$$

recall from Step 6 in 3.1 that $\partial Q_{k_0,t}$ is isometrically embedded in $X_k$ for $k \geq k_0$, as geodesic paths joining a point $p \in X_k$ to a point $q \in \text{S} \text{q} \text{d}_1(X_k)$ can be taken not to pass through different sheets of the double covers and, minding (3.27), we have for $k_0 < k \leq j - 1$:

$$(3.49) \quad \text{dist}(\pi_{j,k} \circ F_j^{-1}(x_t), \partial Q_{j,k,t}^{(\text{par})}) \leq \delta_k \text{dist}(\pi_{j,k_0} \circ F_j^{-1}(x_t), \partial Q_{k_0,t}).$$

Combining (3.47), (3.48) and (3.49) we get:

$$(3.50) \quad ||x_t - P_{j-1,k_0}(x_t)||_{\mathbb{R}^2} \leq 28 \left( \sum_{k_0 < k \leq j} \delta_k^2 \right) \text{dist}(\pi_{j,k_0} \circ F_j^{-1}(x_t), \partial Q_{k_0,t}).$$

Recalling (3.47)

$$(3.51) \quad ||x_t - q_t|| \leq ||P_{j-1,k_0}(x_t) - q_t|| + 28 \times 16(1 + \sum_{k \leq k_0} \delta_k^2)^{1/2}||P_{j-1,k_0}(x_t) - q_t||,$$

and the choice (3.38) of the sequence $\delta_k$, we get:

$$(3.52) \quad ||x_t - q_t|| \leq \frac{9}{8}||P_{j-1,k_0}(x_t) - q_t||.$$**

Now:

$$(3.53) \quad ||x_1 - x_2|| \geq \frac{1}{L(P_{j-1,k_0})}||P_{j-1,k_0}(x_1) - P_{j-1,k_0}(x_2)|| \geq \frac{\eta}{L(P_{j-1,k_0})} \max_{t = 1,2} \text{dist}(\pi_{j,k_0} \circ F_j^{-1}(x_t), \partial Q_{k_0,t}) \geq \frac{8\eta}{9L(P_{j-1,k_0})L(F_{k_0})} \text{dist}(x_t, F_{k_0}(\partial Q_{k_0,t})).$$

If $x_t + y_t \in \text{RN}(j)$ then

$$(3.54) \quad ||(x_1 + y_1) - (x_2 + y_2)|| \geq ||x_1 - x_2|| - e(\sigma_2)(||y_1|| + ||y_2||),$$

where $\lim_{\sigma_2 \to 0} e(\sigma_2) = 0$, and we can conclude as in Step 1.

In the second case assume that $Q_{j-1,1} = Q_{j-1,2}$ but $Q_{j-1,1}^{(\text{par})} \neq Q_{j-1,2}^{(\text{par})}$ Then $P_{j-1}(x_1)$ and $P_{j-1}(x_2)$ lie on the same affine plane of $\text{Th}(j - 1)$ and thus:

$$(3.55) \quad ||P_{j-1}(x_1) - P_{j-1}(x_2)|| \geq \max_{t = 1,2} \text{dist}(P_{j-1}(x_1), F_{j-1}(\partial Q_{k_0,t})), $$

and we can then argue as in the first case.

Third, if $Q_{j-1,1}^{(\text{par})} = Q_{j-1,2}^{(\text{par})}$ we can argue as in Step 1. In fact, by Step 6 in Construction 3.1 the set $\text{Th}_0(j - 1)$ is finite and, up to translations and dilations, there are only finitely many possibilities for the subcomplexes of $Y_j$ which project via $P_{j-1}$ onto some $F_{j-1}(Q)$ for $Q \in \text{S} \text{q} \text{d}_a(X_{j-1})$. Thus we can find an $\alpha > 0$ such
that if \( \{Q_1, Q_2\} \subset Sq(X_j) \) are distinct, \( x_t \in F(Q_t) \) \((t = 1, 2)\) and \( \pi_{j-1}(Q_1) \) and \( \pi_{j-1}(Q_2) \) belong to the same square of \( Sq_d(X_{j-1}) \) then

\[
\|x_1 - x_2\| \geq \alpha \max_{t=1,2} \text{dist}(x_t, F_j(\partial Q_t)),
\]

and then argue as in Step 1. \( \square \)

**Lemma 3.57** (Compositions of \( P_i \) are uniformly Lipschitz). Assume that \( \sum_t \epsilon_t < \infty \); then the Lipschitz maps \( P_i : \text{RN}(i) \to Y_i \) can be composed to give uniformly Lipschitz maps; specifically, for \( k < i \) let:

\[
P_{i,k} = P_k \circ \cdots \circ P_{i-1} \circ P_i;
\]

then:

\[
L(P_{i,k}) \leq \prod_t (1 + \epsilon_t).
\]

Let \( F_i(X_i) = Y_i \) \((i = \infty \) is admissible\); then the maps

\[
P_{i,k} : \text{RN}(i) \to Y_k
\]

as \( i \nearrow \infty \) converge weak* to a map:

\[
P_{\infty,k} : Y_\infty \to Y_k
\]

which satisfies \( L(P_{\infty,k}) \leq \prod_t (1 + \epsilon_t) \). For \( k < l \leq i \) \((i \text{ or } l \text{ can be } \infty \text{ with } P_{\infty,\infty} \text{ being taken to be the identity of } Y_\infty) \) one has:

\[
P_{l,k} \circ P_{i,l} = P_{i,k};
\]

as \( k \nearrow \infty \) \( P_{\infty,k} \) converges weak* to \( P_{\infty,\infty} \).

Proof. Assuming that \( \sum_t \epsilon_t < \infty \) we have a uniform bound on the Lipschitz constants of the maps \( P_{i,k} \):

\[
\sup_{i,k} L(P_{i,k}) \leq \prod_t (1 + \epsilon_t) < \infty.
\]

From the definition of \( \text{RN}(i) \) we get that if \( \sigma_i > 1 \) \( \text{(note that the } \sigma_i \text{'s are chosen very large in Lemma 3.37)} \) we have:

\[
\sup_{x \in \text{RN}(i)} \|P_i(x) - x\| \leq 100 \cdot 5^{-i}.
\]

In particular, for a universal constant \( C > 0 \) we have:

\[
\sup_{x \in \text{RN}(i)} \sup_l \|P_{i+l,k}(x) - P_{i,k}(x)\| \leq C5^{-i}.
\]

Therefore, on \( Y_\infty \) the maps \( P_{i,k} \) converge, uniformly as \( i \nearrow \infty \) to a map \( P_{\infty,k} \) which must be Lipschitz because of (3.63); the uniform bound (3.63) also ensures that convergence is in the weak* sense.

From the definition of \( P_{i,k} \) we have that (3.62) holds when all of \( \{i, l, k\} \) are finite. For \( l = \infty \) or \( i = \infty \) we establish the result by a limiting argument setting \( P_{\infty,\infty} \) equal to the identity of \( Y_\infty \). We are thus only left to show that \( P_{\infty,k} \) converges on \( Y_k \) uniformly to the identity. But this is immediate observing that (3.64) gives:

\[
\sup_{x \in Y_\infty = \bigcap_k \text{RN}(k)} \|P_{k,\infty}(x) - x\| \leq 10^3 \times 5^{-k}.
\]

\( \square \)
**Lemma 3.67** (Convergence of the Embeddings). The topological embeddings $F_i : X_i \hookrightarrow \mathbb{R}^4$ converge, as $i \not\to \infty$, to a topological embedding $F_\infty : X_\infty \hookrightarrow \mathbb{R}^4$ such that:

\[
16^{-1}(1 + \sum_i \delta_i^2)^{1/2} \leq L(F_\infty) \leq 23(1 + \sum_i \delta_i^2)^{1/2}.
\]

For each $k < i$ ($i = \infty$ being admissible) one has a commutative diagram:

\[
\begin{array}{ccc}
X_i & \xrightarrow{F_i} & Y_i \\
\downarrow{\pi_{i,k}} & & \downarrow{P_{i,k}} \\
X_k & \xrightarrow{F_k} & Y_k
\end{array}
\]

(3.69)

**Proof.** Note that from (3.33) we have:

\[
\sup_{x \in X_{i+k}} \|F_{i+k}(x) - F_i(\pi_{i+k,i}(x))\| \leq 200 \cdot 5^{-i},
\]

and so the embeddings $F_i : X_i \hookrightarrow \mathbb{R}^4$ converge uniformly to a map $F_\infty : X_\infty \to \mathbb{R}^4$ which must satisfy (3.68) because of (3.32).

The diagram (3.69) commutes because of (3.34) (the case $i = \infty$ is handled by a limiting argument).

Finally, as $X_\infty$ is compact, in order to conclude that $F_\infty$ is an embedding it suffices to establish that it is injective. Let $x, y$ be distinct points of $X_\infty$; then for some $k$: $\pi_{\infty,k}(x) \neq \pi_{\infty,k}(y)$ and, as $F_k$ is an embedding:

\[
F_k(\pi_{\infty,k}(x)) \neq F_k(\pi_{\infty,k}(y));
\]

but as the diagrams (3.69) commute:

\[
P_{\infty,k}(F_\infty(x)) \neq P_{\infty,k}(F_\infty(y)).
\]

\[\square\]

**Lemma 3.73** (Existence and nontriviality of the 2-current). The pushforward $F_{\infty\#} N_\infty$ is a nontrivial normal current in $\mathbb{R}^4$ supported on $Y_\infty$; in fact:

\[
P_{\infty,0\#} F_{\infty\#} N_\infty = F_{0\#} N_0.
\]

**Proof.** One just needs to prove (3.74) and might argue from the commutative diagram (3.69) for $(i, k) = (\infty, 0)$. But some sleight of hand is concealed in this approach and for the Apprehensive Analyst we provide a direct computation which uses weak* continuity of normal currents:

\[
P_{\infty,0\#} F_{\infty\#} N_\infty (f dg_1 \wedge dg_2) = N_\infty(f \circ P_{\infty,0} \circ F_{\infty} d(g_1 \circ P_{\infty,0} \circ F_{\infty}) \wedge d(g_2 \circ P_{\infty,0} \circ F_{\infty}));
\]

but $P_{i,0} \circ F_i \circ \pi_{\infty,i} \xrightarrow{w*} P_{\infty,0} \circ F_{\infty}$ as $i \not\to \infty$ and thus:

\[
P_{\infty,0\#} \circ F_{\infty\#} N_\infty = \lim_{i \to \infty} N_\infty[(P_{i,0} \circ F_i \circ \pi_{i,\infty})^* f dg_1 \wedge dg_2]
\]

(3.76)

\[
= \lim_{i \to \infty} N_\infty[(P_{i,0} \circ F_i)^* f dg_1 \wedge dg_2)
\]

\[
= \lim_{i \to \infty} P_{i,0\#} F_{i\#} N_\infty(f dg_1 \wedge dg_2) = F_{0\#} N_0(f dg_1 \wedge dg_2).
\]

\[\square\]
**Theorem 3.77** (2-unrectifiability of $Y_\infty$). $Y_\infty$ is purely 2-unrectifiable in the sense that whenever $K \subset \mathbb{R}^2$ is compact and $\Phi : K \to \mathbb{R}^4$ is Lipschitz, $\mathcal{H}^2(\Phi^{-1}(Y_\infty) \cap K) = 0$.

**Proof.** We will argue by contradiction assuming that $K \subset \Phi^{-1}(Y_\infty)$ and that $\mathcal{H}^2(K) > 0$. The main difference from the proof of Theorem 2.21 is Step 1 where we resort to a weak* (approximate) lower-semicontinuity argument.

**Step 1:** Reduction to the case in which $\Phi$ is a graph over $Y_0$.

Let $\Phi_n = P_{\infty,n} \circ \Phi$, which are well-defined and uniformly Lipschitz. By Lemma 3.57 we also have that $\Phi_n$ converges weak* to $\Phi$.

We now consider the Borel set $E \subset K$ consisting of those points which are Lebesgue density points of the set of points where $\Phi$ and each $\Phi_n$ is differentiable and where $d\Phi_0$ has rank $< 2$; our goal is to show that

$$\mathcal{H}^2(\text{Im } \Phi \cap P_{\infty,0}^{-1}(\Phi_0(E))) = 0. \tag{3.78}$$

First, the area formula [AK00b, Thm. 5.1] gives $\mathcal{H}^2(Y_0 \cap \Phi_0(E)) = 0$. Secondly, for each $n$, using the square complex structure of $\{X_i\}_{i \in \mathbb{N}}$, the set $Y_n \setminus F_n$ can be partitioned into countably many closed sets $\{S_\alpha\}_\alpha$ (e.g. taking each $F_n(Q)$ for $Q \in \text{S}_{\text{ad}}(X_n)$) such that each restriction $P_{n,0}|S_\alpha : S_\alpha \to P_{n,0}(S_\alpha)$ is biLipschitz, thus giving:

$$\mathcal{H}^2(Y_n \cap P_{n,0}^{-1}(\Phi_0(E))) = 0. \tag{3.79}$$

In particular, the area formula implies that:

$$\int_K \chi_E J_2(d\Phi_n) \, d\mathcal{H}^2 = 0. \tag{3.80}$$

We want to use the lower-semicontinuity of the area functional (see for example [AFP00, Subsec. 2.6]), but we need the domain of the maps $\Phi_n, \Phi_\infty$ to be open. Fix $\varepsilon > 0$ and choose $U \subset E$ open with $\mathcal{H}^2(U \setminus E) < \varepsilon$. By McShane’s Lemma we can extend each $\Phi_n$ to a $7\mathcal{C}$-Lipschitz map $\tilde{\Phi}_n : U \to \mathbb{R}^4$ which coincides on $E$ with $\Phi_n$. Up to passing to a subsequence we can assume $\tilde{\Phi}_n \xrightarrow{w^*} \tilde{\Phi}_\infty$ were $\tilde{\Phi}_\infty|E = \tilde{\Phi}_\infty$. We can now invoke lower-semicontinuity of area:

$$\int_K \chi_E J_2(d\Phi_\infty) \, d\mathcal{H}^2 \leq \liminf_{n \to \infty} \int_U J_2(d\tilde{\Phi}_n) \, d\mathcal{H}^2 \leq \limsup_{n \to \infty} \int_U J_2(d\tilde{\Phi}_n) \, d\mathcal{H}^2 \leq 49C^2\varepsilon,$$

and (3.78) follows letting $\varepsilon \searrow 0$ and applying the area formula.

**Step 2:** Existence of square holes.

The same argument as in Step 2 of Theorem 2.21 goes through with minor modifications.

First, the (generalized) square-complex structure of $X_n \setminus \text{Sk}_1(X_n)$ induces a generalized square-complex structure on $Y_n \setminus F_n(\text{Sk}_1(X_n))$ via the homeomorphism $F_n$; thus, in the following, we will implicitly identify $\text{S}_{k}(X_n)$ (resp. $\text{S}_{\text{ad}}(X_n)$) with $\text{S}_{k}(Y_n)$ (resp. $\text{S}_{\text{ad}}(Y_n)$).

Second, compared to the $l^2$-case there are differences in indexing the $\text{S}_{k}(Y_n), \text{S}_{k}(X_n)$. In fact, as the construction is no longer self-similar, $\text{S}_{k}(X_n)$ does not
represent the set of squares of \( X_n \) of generation \( k \) (and side length \( 5^{-k} \)), but the set of squares obtained by subdividing each square of \( \text{Sq}(X_n) \) \( k \)-times (and so the side length is \( 5^{-k} \)-times the side length of the parent square in \( \text{Sq}(X_n) \)). Moreover, we need a notation for the set of squares obtained by subdividing each square of \( \text{Sq}_{\text{ad}}(X_n) \) \( k \)-times: we will use \( \text{Sq}_{\text{ad},k}(X_n) \).

Third, in Step 1 of Construction 3.1 we took a piecewise-affine approximation of \( \Psi_\beta \) which involved subdividing squares \( N \)-extra times. We must thus modify the definition of \( i_n \) (2.28) letting:

\[
i_n = \lceil -\log_5(5^{-n}-N^{-5}c\delta_n) \rceil.
\]

Fourth, we have to consider a square \( Q \in P_{n-1,0}(\text{Sq}_{\text{ad},i_n-n}(Y_{n-1})) \) and partition \( \tilde{Q}_n \) into \( \simeq 5^{n-n} \) annuli consisting of squares of \( P_{n-1,0}(\text{Sq}_{\text{ad},i_n-n}(Y_{n-1})) \). Having fixed such an annulus \( A \), the goal is again to show that \( K = \text{dom} \Phi \subset Y_0 \) (we have reduced to the case in which \( \Phi \) is a graph over a subset of \( Y_0 \) in the previous Step 1) has to miss one of the squares of \( A \).

Then the proof follows the same logic as in Step 2 of Theorem 2.21 with some minor notational modifications:

- \( \text{Sq}_l(Y_j) \) becomes \( \text{Sq}(Y_j) \), compare the previous discussion about indexing.
- \( \text{Sq}_{\text{ad}}(\text{Sq}_l^{-1}(Q_{n-1})) \) becomes \( \text{Sq}_{\text{ad},i_n-n}(Q_{n-1}) \), where \( \text{Sq}_{\text{ad}}(Q) \) denotes the set of sub-squares of \( Q \) obtained by taking \( k \)-iterated subdivisions.
- We cannot simply use the projection \( P_0 \), but must use \( P_{l,0} \) when projecting points from \( Y_j \) to \( Y_0 \). In particular, instead of writing \( Q_{i_{n,\beta}}^{(j)} = P_0^{-1}(Q_{i_{n,\beta}}^{(0)}) \cap Q_{j,\alpha}^{(j)} \), we need to consider \( Q_{i_{n,\beta}}^{(j)} = P_{l,0}^{-1}(Q_{i_{n,\beta}}^{(0)}) \cap Q_{j,\alpha}^{(j)} \).

**Step 3: The choice of the \( \delta_k \)'s.**
Here we have to guarantee that (3.38) holds; this can be achieved by shifting the sequence we used in Theorem 2.21 to the right:

\[
\delta_k = \frac{1}{10^9 + k}.
\]

\( \square \)

4. \( k \)-CURRENT IN \( \mathbb{R}^{k+2} \)

The \( k \)-current in \( \mathbb{R}^{k+2} \) is constructed resorting to a trick that was already employed in [Sch15, Sec. 4]: once one is able to construct a 2-current which meets all Lipschitz surfaces which are graphs over a coordinate plane in a \( \mathcal{H}^2 \)-null set, one can iterate over all planes parallel to a pair of coordinate axes. In the following we let \( \{\epsilon_\xi\}_{1 \leq \xi \leq t} \) denote the standard orthonormal basis of \( \mathbb{R}^l \) (where \( l = k \) or \( l = k+2 \)) and for \( \xi < \zeta \) we let \( \epsilon_\xi + \epsilon_\zeta \) denote the plane spanned by \( \epsilon_\xi \) and \( \epsilon_\zeta \). Finally, we will identify the set of planes \( \{\epsilon_\xi + \epsilon_\zeta\}_{1 \leq \xi < \zeta \leq k} \) with \( \mathbb{Z}_{(k)}^{(k)} \) and we will write equations like \( s = \epsilon_\xi + \epsilon_\zeta \mod \binom{k}{2} \) or \( \epsilon_\xi + \epsilon_\zeta = 2 \mod \binom{k}{2} \).

**Construction 4.1** (Modifications to Construction 2.1). Now Construction 2.1 is generalized adding an additional parameter: a 2-plane \( \epsilon_\xi + \epsilon_\zeta \). Let \( k \) be a \( k \)-cube isometric to \([0, 5^{-l}] \) and let \( p_{\xi,\zeta} \) denote the projection onto \( \epsilon_\xi + \epsilon_\zeta \) and set
Q = pj_{\xi,\zeta}(K). Let Q_a, Q_c, Q_o, \hat{Q}, etc. as in Construction 2.1 and set:

\begin{align*}
K_a &= p_{j_{\xi,\zeta}}^{-1}(Q_a) \\
K_o &= p_{j_{\xi,\zeta}}^{-1}(Q_o) \\
K_c &= p_{j_{\xi,\zeta}}^{-1}(Q_c).
\end{align*}

(4.2)

We use standard covering theory to find a double cover \( \tilde{\pi} : \tilde{K}_a \to K_a \), and a lift \( \tilde{p}_{j_{\xi,\zeta}} : \tilde{Q}_a \to Q_a \) such that the following diagram commutes:

\begin{align*}
\begin{array}{ccc}
\tilde{K}_a & \xrightarrow{\tilde{\pi}} & K_a \\
\downarrow \tilde{p}_{j_{\xi,\zeta}} & & \downarrow p_{j_{\xi,\zeta}} \\
\tilde{Q}_a & \xrightarrow{\tilde{\pi}_Q} & Q_a
\end{array}
\end{align*}

(4.3)

where \( \tilde{\pi}_Q : \tilde{Q}_a \to Q_a \) is the double cover from Construction 2.1. We then glue \( \tilde{K}_a \) back to \( K_o \cup K_c \) by gluing together the pair of points of \( \partial \tilde{K}_a \) that are mapped to the same point by \( \tilde{\pi} \). If \( \tilde{K} \) denotes the resulting cube-complex, then \( \tilde{\pi} \) extends to a branched covering \( \tilde{\pi} : \tilde{K} \to K \) and we also obtain an extension \( \tilde{p}_{j_{\xi,\zeta}} : \tilde{K} \to Q \) of \( \tilde{p}_{j_{\xi,\zeta}}|_{\text{Int}(\tilde{K}_a)} \) which makes the following diagram commute:

\begin{align*}
\begin{array}{ccc}
\tilde{K} & \xrightarrow{\tilde{\pi}} & K \\
\downarrow \tilde{p}_{j_{\xi,\zeta}} & & \downarrow p_{j_{\xi,\zeta}} \\
\tilde{Q} & \xrightarrow{\tilde{\pi}_Q} & Q
\end{array}
\end{align*}

(4.4)

we then obtain \( \Psi : \tilde{K} \to \mathbb{R}^2 \) as the composition \( \Psi = \Psi_Q \circ \tilde{p}_{j_{\xi,\zeta}} \) where \( \Psi_Q : \tilde{Q} \to \mathbb{R}^2 \) is the map we built in Construction 2.1.

**Construction 4.5** (Modification to Construction 3.1).

**Step 1:** Piecewise affine approximation.

For fixed \( \delta, \xi, \zeta \), let \( \Psi_\delta : \tilde{K} \to \mathbb{R}^2 \) be as in Construction 4.5 using the parameters \( \delta, e_\xi \oplus e_\zeta \). If \( K^{(m)} \) denotes the \( m \)-th iterated subdivision of \( K \), we can find \( N \in \mathbb{N} \) and a piecewise affine approximation \( \Phi_\delta : K^{(N)} \to \mathbb{R}^2 \) of \( \Psi_\delta \) such that the following analogs of (3.3), (3.4) and (3.5) hold:

\begin{align*}
\text{L(} \Phi_\delta \text{)} &\in \left[ \frac{\delta}{16}, 23\delta \right] \\
\| \Phi_\delta(p_1) - \Phi_\delta(p_2) \| &\geq \frac{\delta}{3} \phi(r(p_{j_{\xi,\zeta}}(p_1))) \\
\| \Phi_\delta \| &\leq 2\delta \text{ diam } K.
\end{align*}

(4.6) (4.7) (4.8)

We let \( X_0 = [0,1]^k \) and \( F_0 : X_0 \to \bigoplus_{1 \leq \xi \leq k} e_\xi \subset \mathbb{R}^{k+2} \) denote the standard isometric embedding. We obtain \( X_1 \) from \( X_0 \) by applying Construction 4.1 with \( e_\xi \oplus e_\zeta = 0 \mod \left( \begin{smallmatrix} k \\ 2 \end{smallmatrix} \right) \) and then let

\begin{align*}
F_1 &= F_0 \circ \pi_{1,0} + \Phi_\delta_1 \otimes (e_{k+1} \oplus e_{k+2}).
\end{align*}

(4.9)

**Step 2:** Construction of \( X_{j+1} \) and \( F_{j+1} \).
We need first to generalize the notation. We let Cell($X_j$) denote the set of k-dimensional cells of $X_j$; while $X_1$ is a k-cube complex, as in Construction 3.1, $X_j$ does not have a k-cube complex structure, but it is a union of its k-cells Cell($X_j$) away from the $(k - 1)$-skeleton $Sk_{k-1}(X_{j-1})$ of $X_{j-1}$, where $Sk_{k-1}(X_{j-1})$ embeds isometrically in $X_j$. Moreover, we let $Sk_{k-1}(X_j) = Sk_{k-1}(X_{j-1}) \cup \bigcup_{K \in Cell(X_j)} \partial K$; in particular:

\[(4.10) \quad X_j \setminus Sk_{k-1}(X_j) = \bigcup_{K \in Cell(X_j)} Int(K).\]

For $K \in Cell(X_j)$ we define the radial basis function

\[(4.11) \quad \varphi_K(x) = \begin{cases} 
\exp \left( -\frac{\sigma_j}{\text{dist}(\pi_{\tau(K)}(x), \partial K)} \right) \times 46 \text{diam}(F_j(K)) & \text{if } \pi_{\tau(K)} \in F_j(\text{Int}(K)) \\
0 & \text{otherwise},
\end{cases}\]

where $\pi_{\tau(K)}$ denotes the orthogonal projection onto the affine k-plane $\tau(K)$ containing $F_j(K)$. We then define the radial basis neighbourhood $RN(j)$ as:

\[(4.12) \quad RN(j) = \left\{ p \in \mathbb{R}^{k+2} : \text{there is a } K \in Cell(X_j) : p = x + y, x \in F_j(K), y \perp \tau(K), \text{ and } \|y\| \leq \varphi_K(x) \right\}.\]

We then define $P_j : RN(j) \to Y_j$ by $p = x + y \mapsto X$ and, as in Section 3, it follows that:

**Claim:** For each $\varepsilon_j > 0$ there is a $\sigma_j > 0$ such that $P_j$ is $(1 + \varepsilon_j)$-Lipschitz.

Let Cell$_m(X_j)$ denote the set of cells obtained by subdividing each cell of Cell($X_j$) $m$-times, and let:

\[(4.13) \quad \text{Cell}_\infty(X_j) = \bigcup_{m \geq 1} \text{Cell}_m(X_j).\]

Now a cell $K \in \text{Cell}_\infty(X_j)$ is adapted to $RN(j)$ if the $(23\delta_j \text{diam } F_j(K))$-neighborhood of $F_j(K)$ is contained in $RN(j)$ and if, denoting by $\text{PAR}(K) \in Cell(X_j)$ the unique cell containing $K$, one has:

\[(4.14) \quad \max_{x \in K} \text{dist}(x, \partial K) \leq \delta_j \max_{x \in K} \text{dist}(x, \partial \text{PAR}(K)).\]

We let Cell$_\text{ad}(X_j)$ denote the set of maximal adapted k-cubes of Cell$_\infty(X_j)$; the elements of Cell$_\text{ad}(X_j)$ have pairwise disjoint interia and satisfy:

\[(4.15) \quad X_j \setminus Sk_{k-1}(X_j) = \bigcup_{K \in Cell_\text{ad}(X_j)} \text{Int}(K).\]

Fix $e_\xi \oplus e_\eta = j \mod \binom{k}{2}$ and apply Construction 4.1 to each $K \in Cell_\text{ad}(X_j)$ to get $\Phi_{K,\delta_{j+1}} : K \to \mathbb{R}^2$. As in Construction 3.1 we can ensure that if $K_1 \neq K_2$ $\Phi_{K_1,\delta_{j+1}}$ and $\Phi_{K_2,\delta_{j+1}}$ can be taken to differ up to composition with translations and dilations. Let $Th_0(j) = \bigcup_{K \in Cell(X_j)} \tau_0(K)$ where $\tau_0(K)$ denotes the $k$-plane parallel to $\tau(K)$ and passing through the origin. By induction we assume $Th_0(j)$ to be finite and choose a finite set of pairs $\{(e_{1,K}, e_{2,K})\}_{K \in Cell_\text{ad}(X_j)}$ such that each
(e_{1,K}, e_{2,K}) is an orthonormal basis of the orthogonal complement of τ_0(PAR(K)) where PAR(K) ∈ Cell(X_j) is the k-cell containing K. We can then define:

(4.16) \quad F_{j+1}(x) = F_j \circ \pi_{j+1}(x) + \sum_{K \in Cell_{ad}(X_j)} \Phi_{K,\delta_{j+1}}(x) \otimes (e_{1,K} \oplus e_{2,K}),

and get

(4.17) \quad L(F_{j+1}) \in \left[16^{-1}(1 + \sum_{l \leq j+1} \delta_l^2)^{1/2}, 23(1 + \sum_{l \leq j+1} \delta_l^2)^{1/2}\right].

As in the \( \mathbb{R}^4 \)-case we let \( Y_i = F_i(X_i) \) and \( Y_\infty = F_\infty(X_\infty) \).

**Theorem 4.18** (2-rectifiability of \( Y_\infty \subset \mathbb{R}^{k+2} \)). \( Y_\infty \) is purely 2-rectifiable in the sense that whenever \( K \subset \mathbb{R}^2 \) is compact and \( \Phi : K \rightarrow \mathbb{R}^{k+2} \) is Lipschitz, \( \mathcal{H}^2(\Phi^{-1}(Y_\infty) \cap K) = 0 \).

**Proof.** We will focus on the differences with the proof of Theorem 3.77.

**Step 1:** Reduction to the case in which \( \Phi \) is a graph over \( Y_0 \).

Let \( \Phi : K \subset [0,1]^2 \rightarrow Y_\infty \) be Lipschitz with \( \mathcal{H}^2(K) > 0 \). Let \( \Phi_n = P_{\infty,n} \circ \Phi \) and \( E \subset K \) be the set of differentiability points \( p \) of \( \{\Phi_n\}_n \). \( \Phi \) such that for each pair \( (\xi, \zeta) \) with \( 1 \leq \xi < \zeta \leq k \) (note the \( k \), not \( k + 2 \)). Our construction has already screwed-up the behavior in the last two coordinates:

(4.19) \quad \det \begin{pmatrix} \langle e_\xi, d\Phi_0(\partial_\xi) \rangle & \langle e_\xi, d\Phi_0(\partial_\eta) \rangle \\ \langle e_\zeta, d\Phi_0(\partial_\xi) \rangle & \langle e_\zeta, d\Phi_0(\partial_\eta) \rangle \end{pmatrix}(p) = 0.

As \( Y_0 \) lies in \( \bigoplus_{\xi \leq k} e_\xi \) the area formula gives:

(4.20) \quad \mathcal{H}^2(Y_0 \cap \Phi_0(E)) = 0.

Now, using that \( \Phi_n \xrightarrow{w^*} \Phi \) and the weak* lower-semicontinuity of the area functional as in **Step 1** of Theorem 3.77 we conclude that:

(4.21) \quad \mathcal{H}^2(Y_\infty \cap \Phi(E)) = 0.

Thus, up to passing to a countable partition of \( K \) and throwing away an \( \mathcal{H}^2 \)-null set we can assume that there are \( 1 \leq \xi_0 < \zeta_0 \leq k \) such that for each \( p \in K \):

(4.22) \quad \det \begin{pmatrix} \langle e_{\xi_0}, d\Phi_0(\partial_\xi) \rangle & \langle e_{\xi_0}, d\Phi_0(\partial_\eta) \rangle \\ \langle e_{\zeta_0}, d\Phi_0(\partial_\xi) \rangle & \langle e_{\zeta_0}, d\Phi_0(\partial_\eta) \rangle \end{pmatrix}(p) \neq 0.

Using [Kir94, Thm. 9] in, which is essentially a measurable and Lipschitz version of the Inverse Function Theorem, up to further partioning and throwing away an \( \mathcal{H}^2 \)-null set we are reduced to the case \( K \subset \pi_{e_{\xi_0} \oplus e_{\zeta_0}}(Y_0) \) where \( \pi_{e_{\xi_0} \oplus e_{\zeta_0}} \) denotes the orthogonal projection onto \( e_{\xi_0} \oplus e_{\zeta_0} \).

**Step 2:** Existence of square holes.

The proof now proceeds as in **Step 2** of Theorems 3.77, 2.21 but we spell out more details because we deal both with squares and \( k \)-dimensional cells.

Let \( n-1 = e_{\xi_0} \oplus e_{\zeta_0} \mod \langle \delta \rangle \) and let \( Q \in \pi_{e_{\xi_0} \oplus e_{\zeta_0}}(P_{n-1,0}(Cell_{ad}(Y_{n-1}))) \) where

(4.23) \quad i_n = \left[ -\log_5(5^{n-N-5}e_{\delta_n}) \right];

let \( Q_n \) be as in Construction 2.1 and partition \( Q_n \) into \( \approx 5^{n-n} \) annuli consisting of squares of \( Sq_{i_n-n}(Q) \) (i.e. subdivide \( Q \) into 25 subquarces \( (i_n - n) \)-times). We consider one such an annulus \( A \). Our goal is to show that \( K \) has to miss the interior of one of the squares in \( A \). Let \( p_\alpha, p_{\alpha+1} \) be as in **Step 2** of the proof...
of Theorem 3.77, and we will show that $\Phi_j(p_\alpha)$ and $\Phi_j(p_{\alpha+1})$ belong to the same cell of Cell_{ad}(Y_j). This is true by construction when $j = 0$ and for $j \geq 1$ we assume by induction that $\Phi_{j-1}(p_\alpha)$, $\Phi_{j-1}(p_{\alpha+1})$ belong to the same $K^{(j-1)}_{j-1}$ in Cell_{ad}(Y_{j-1}). Let $K^{(j)}_{j,\beta} \in \text{Cell}_{ad}(Y_j)$ denote the cell containing $\Phi_j(p_\beta)$ and assume by contradiction that $K^{(j)}_{j,\alpha} \neq K^{(j)}_{j,\alpha+1}$. In the following we will use the decorators $_{n}, \alpha, \epsilon$ and $\gamma$ as in Constructions 2.1 and 4.1: for example $K^{(j-1)}_{j-1,n}$ is obtained as $K_n$ if we let $K = K^{(j-1)}_{j-1}$. In particular, as $K^{(j)}_{j,\alpha} \neq K^{(j)}_{j,\alpha+1}$ we must have $P_{j-1}(K^{(j)}_{j,\beta}) \subset K^{(j)}_{j-1,\alpha}$. Let now $Q^{(0)}_{i,n,\beta} \in \pi_{e_{\alpha}\oplus e_{\alpha}}(P_{j-1,0}(\text{Cell}_{n,j-1}(Y_{j-1})))$ be the square containing $p_\beta$. Note that $Q^{(0)}_{i,n,\beta} \subset Q$ can be identified with a square of an iterated subdivision of $Q$, more precisely, $Q^{(0)}_{i,n,\beta} \in \text{Sqd}_{n-n}(Q)$. Let $K^{(j)}_{i,n,\beta} = P_{j,0}^{-1}(Q^{(0)}_{i,n,\beta}) \cap K^{(j)}_{j,\beta}$, and let $q^{(j)}_{i,\beta}$ denote the center of the cell $K^{(j)}_{i,n,\beta}$. As $\Phi$ is $C$-Lipschitz:

$$d(\Phi_j(p_\alpha), \Phi_j(p_{\alpha+1})) \leq 4\sqrt{k}C \times c_\alpha \delta_n \text{ diam } Q.$$  

As $F_j$ is $L(F_\infty)$-Lipschitz and as $\text{diam } F_j^{-1}(K^{(j)}_{j,\beta}) \leq 2c_\alpha \sqrt{k} \text{ diam } Q$,

$$d(q^{(j)}_{\alpha}, \Phi_j(p_\beta)) \leq 2\sqrt{k}L(F_\infty) \times c_\beta \delta_n \text{ diam } Q$$

$$d(q^{(j)}_{\alpha}, q^{(j)}_{\alpha+1}) \leq 4(\sqrt{k}C + L(F_\infty)) \times c_\beta \delta_n \text{ diam } Q.$$  

Let $S^{(j)}_{j-1} = F_j^{-1}(K^{(j-1)}_{j-1})$ and $S^{(j)}_{j,\beta} = F_j^{-1}(K^{(j)}_{j,\beta})$; we must have $S^{(j)}_{j,\alpha} \neq S^{(j)}_{j,\alpha+1}$ and $\pi_{j-1}^{-1}(F_j^{-1}(q^{(j)}_{\beta})) \in S^{(j-1)}_{j-1,a}$. Note that $F_j^{-1}(q^{(j)}_{\beta})$ must be at distance $\geq 5^{-3} \text{ diam } (\partial S^{(j-1)}_{j-1,a})$ from $\partial S^{(j-1)}_{j-1,a}$ so that

$$\phi(r(p_{j,\xi,\zeta}(F^{-1}_{j}(q^{(j)}_{\beta})))) \geq 5^{-3} \text{ diam } (\partial S^{(j-1)}_{j-1,a}),$$

where $e_\xi \oplus e_\zeta = j - 1 \mod (k^n_2)$. As $F_j^{-1}(q^{(j)}_{\alpha}) \neq F_j^{-1}(q^{(j)}_{\alpha+1})$, they belong to different sheets of the double cover, and as $\pi_{j-1}(S^{(j)}_{j,\alpha})$ and $\pi_{j-1}(S^{(j)}_{j,\alpha+1})$ are adjacent, we let $q^{(j)}_{\alpha}$ be the center of the cell of Cell_{n,j}(X_j) adjacent to $K^{(j)}_{j,\alpha}$ and such that $\pi_{j-1}(F j^{-1}(q^{(j)}_{\alpha})) = \pi_{j-1}(F j^{-1}(q^{(j)}_{\alpha+1}))$. We now have:

$$r(p_{j,\xi,\zeta}(F^{-1}_{j}(q^{(j)}_{\alpha}))) = r(p_{j,\xi,\zeta}(F^{-1}_{j}(q^{(j)}_{\alpha+1})))$$

$$\theta(p_{j,\xi,\zeta}(F^{-1}_{j}(q^{(j)}_{\alpha}))) - \theta(p_{j,\xi,\zeta}(F^{-1}_{j}(q^{(j)}_{\alpha+1}))) = \pi.$$  

Invoking (2.7) we get:

$$d(q^{(j)}_{\alpha}, q^{(j)}_{\alpha+1}) \geq \frac{5^{-3}}{2} \delta_j \text{ diam } (\partial S^{(j-1)}_{j-1,a}) \geq \frac{5^{-3}}{2L(F_0)} \delta_n \text{ diam } Q,$$

where we used that $F_\infty$ and the maps $F_{n,n}$ are Lipschitz and that $Q$ lies in the $F_0$-image of $S^{(j-1)}_{j-1,a}$. But as $q^{(j)}_{\alpha}$ is the center of the cell of Cell_{n,j}(X_j) adjacent to $K^{(j)}_{j,\alpha}$ we get:

$$d(q^{(j)}_{\alpha}, q^{(j)}_{\alpha+1}) \leq 16(\sqrt{k}C + L(F_\infty)) \times c_\beta \delta_n \text{ diam } Q.$$  

Thus, if $c$ is chosen sufficiently small in function of $\sqrt{k}, C, L(F_\infty)$ we obtain a contradiction and conclude that $K^{(j)}_{j,\alpha} = K^{(j)}_{j,\alpha+1}$. A consequence of this discussion,
specialized to \( j = n \), is that \( \Phi_n(p_1) \) and \( \Phi_n(p_t) \) belong to the same sheet of the double cover \( P_{n-1} \left( \hat{K}_{n-1,a} \right) \cap Y_n \to \hat{K}_{n-1,a} \) while the choice of \( \epsilon \) gives:

\[
(4.32) \quad d \left( F_{n-1}^{-1}(\Phi_n(p_1)), F_{n-1}^{-1}(\Phi_n(p_t)) \right) \leq 5^{-3} \text{diam} Q.
\]

Note, however, that as \( n - 1 = e_{\xi_0} \oplus e_{\zeta_0} \mod (k_2) \), from the definition of \( \Psi \) in Construction 4.1 and \((\text{ShSep})\) in Construction 2.1 we get a contradiction. Thus \( K \) misses one of the squares of the annulus \( A \).

\[\square\]

References

- [ACP10] Giovanni Alberti, Marianna Cs"{o}rnyei, and David Preiss. Differentiability of Lipschitz functions, structure of null sets, and other problems. In *Proceedings of the International Congress of Mathematicians. Volume III*, pages 1379–1394, New Delhi, 2010. Hindustan Book Agency.
- [AFP00] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
- [AK00a] Luigi Ambrosio and Bernd Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1):1–80, 2000.
- [AK00b] Luigi Ambrosio and Bernd Kirchheim. Rectifiable sets in metric and Banach spaces. *Math. Ann.*, 318(3):527–555, 2000.
- [Alb91] Giovanni Alberti. A Luzin type theorem for gradients. *J. Funct. Anal.*, 100(1):110–118, 1991.
- [DR16] G. De Philippis and F. Rindler. On the structure of \( A \)-free measures and applications. *ArXiv e-prints*, January 2016.
- [Fed69] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
- [GBV93] I. Guyon, B. Boser, and V. Vapnik. Automatic capacity tuning of very large \( vc \)-dimension classifiers. In *Advances in Neural Information Processing Systems*, pages 147–155. Morgan Kaufmann, 1993.
- [Kir94] Bernd Kirchheim. Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. *Proc. Amer. Math. Soc.*, 121(1):113–123, 1994.
- [Mas14] Annalisa Massaccesi. *Currents with coefficients in groups, applications and other problems in Geometric Measure Theory*. PhD thesis, SNS, Pisa, 2014.
- [Mát] András Máté. Nonplanar measure in \( r^3 \) invariant under two independent flows. In preparation.
- [mor86] Some open problems in geometric measure theory and its applications suggested by participants of the 1984 AMS summer institute. In J. E. Brothers, editor, *Geometric measure theory and the calculus of variations (Arcata, Calif., 1984)*, volume 44 of *Proc. Sympos. Pure Math.*, pages 441–464. Amer. Math. Soc., Providence, RI, 1986.
- [MRT12] Mehryar Mohri, Afshin Rostamizadeh, and Ameet Talwalkar. *Foundations of machine learning*. Adaptive Computation and Machine Learning. MIT Press, Cambridge, MA, 2012.
- [Sch14] A. Schioppa. Metric Currents and Alberti representations. *ArXiv e-prints*, March 2014.
- [Sch15] A. Schioppa. Examples of 2-unrectifiable normal currents. *ArXiv e-prints*, August 2015.
- [Sch16a] Andrea Schioppa. Derivations and Alberti representations. *Adv. Math.*, 293:436–528, 2016.
- [Sch16b] Andrea Schioppa. The Lip-lip equality is stable under blow-up. *Calc. Var. Partial Differential Equations*, 55(1):Art. 22, 30, 2016.
- [Smirn93] S. K. Smirnov. Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows. *Algebra i Analiz*, 5(4):206–238, 1993.
- [W12] Marshall Williams. Metric currents, differentiable structures, and Carnot groups. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 11(2):259–302, 2012.
- [Zwo88] Maciej Zworski. Decomposition of normal currents. *Proc. Amer. Math. Soc.*, 102(4):831–839, 1988.

E-mail address: andrea.schioppa@math.ethz.ch