Model error compensator design for continuous- and discrete-time non-minimum phase systems with polytopic-type uncertainties

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ABSTRACT
This paper presents a design for a model error compensator with a parallel feedforward compensator for continuous- and discrete-time non-minimum phase multiple input multiple output (MIMO) plants. The model error compensator can easily achieve robustness for several types of control systems. By appending the compensator to the actual plant, the output trajectory of the plant can be made close to that of the control system with the intended nominal model. Our previous study proposed a design for the model error compensator using particle swarm optimization and linear matrix inequalities based on the common Lyapunov function. The compensator design for the plants addresses polytopic-type uncertainties. However, it is challenging to design the appropriate gain for the model error compensator if the plant is a non-minimum phase MIMO system. In this study, a parallel feedforward compensator is attached to the model error compensator to achieve minimum phase characteristics. An evaluation system, including a parallel feedforward compensator, can be derived as a system with polytopic uncertainties via the addition of some assumptions. Thus, it is easy to design the gain of the model error compensator in the proposed method and achieve robust performance. The effectiveness of the proposed design is evaluated using numerical examples.

1. Introduction

Control systems are often designed according to model-based control methods in control engineering fields. Figure 1 shows the flow of a standard model-based control approach. Herein, we first obtain the dynamic model $P_m$ of an actual plant $P$ using system identification. Then, we design the controller $C$ for the obtained mathematical model. Finally, we obtain the desired control performance by applying the controller to the actual plant. When the model is constructed well, the control system for the actual plant also works well. In contrast, the controller designed for a mathematical model is inadequate if there is a difference between the actual plant dynamics and the model dynamics, such as modelling errors and degradation over time.

To overcome this problem, a model error compensator (MEC) was proposed for continuous-time linear time-invariant single input single output (SISO) systems in a previous study [1]. By attaching the MEC to the actual plant, the output trajectory can be made close to that of the model. Then, the difference in the dynamics between the controller and plant output can be minimized, and the performance degradation caused by uncertainties can be reduced drastically. The MEC is pre-designed offline for the plant and nominal model. The systems with MEC can be together with various existing designed controllers. Thus, the control system with MEC is expected to be more robust without compromising on nominal control performance. In addition, many studies on MECs have been proposed, with focus on discussion of continuous-time systems [2–13]. For example, MECs can be applied to non-linear systems [2] without complicated extensions. Robust feedback linearization could be achieved and verified via numerical examples. A robust path-following method based on feedback linearization [2] was presented in [3]. In addition, the MEC was applied to a benchmark problem of the SICE committee using a three-inertia system [4]. In [5–7], application examples were presented for welfare vehicles with MECs to overcome the weight variabilities of the occupants. In [14], application example was presented for quadcopter.

In [8], an analysis method of the MEC for multiple input multiple output (MIMO) plants with polytopic-type uncertainties [15–18] was proposed; here, the plant was given by continuous-time state-space representation. An evaluation system is therefore introduced and reduced to a system with polytopic-type uncertainties, such that it is easy to analyse the entire system using well-known standard linear matrix inequality (LMI) methods. In [9], the analysis method given in [8] was expanded as a design for the MEC. The
design of the MEC in [9] is based on meta-heuristics and LMIs using the common Lyapunov function. The effectiveness of the design for systems with polytopic-type uncertainties and disturbances was evaluated by numerical simulations.

It is difficult to design an appropriate MEC when the plant is given as a non-minimum phase system. It is widely known that non-minimum phase systems are difficult to control compared with minimum phase systems. To apply the MEC design in [9] to non-minimum phase systems, a large amount of time is required because finding the initial parameters for a stable generalized plant is a complicated process. In addition, the design results tend to be not good. If the plant is not a complex system, we can achieve satisfactory performance by setting the MEC with high gain. However, the feedback gain must be designed appropriately for many types of complex systems, such as MIMO and non-minimum phase systems.

The disturbance observer [19] is one of the effective methods to add robustness for systems, and the effectiveness is addressed in many studies [20, 21]. Its conceptual function is similar to that of MEC, and the method to apply disturbance observer to non-minimum phase systems is addressed in [22, 23]. The disturbance observer uses an inverse model in the system structure and it is difficult to apply the disturbance observer for unstable systems because unstable pole-zero cancellation occurs. Therefore, the disturbance observer has a more limited scope of application than MEC.

In this study, we address a design method of MEC for MIMO non-minimum phase systems with polytopic-type uncertainties. This work expands the achievements in [9] for non-minimum phase MIMO systems using a parallel feedforward compensator (PFC). The PFC is known to be an effective method for controlling non-minimum phase systems [24]. A control system with the PFC can improve the apparent internal feedback dynamics from the design perspective. We propose a design of the MEC with PFC for MIMO non-minimum phase systems and particle swarm optimization (PSO) with an analysis of the $H_{\infty}$ performance. Herein, the generalized plant is reduced to a matrix polytope form as well as the plant. Then, it is easy to derive the performance analysis for continuous- and discrete-time settings. The design of the MEC with PFC is proposed according to the derived performance analysis.

Note that the MEC structure with PFC is already presented in [10, 11] for SISO continuous-time settings. The effectiveness of the MEC with PFC has also been discussed in the frequency domain. Consequently, this work involves the design of the MEC with PFC for MIMO non-minimum phase systems. The proposed design is based on state–space representation; therefore, there is an advantage that we can address the design for the MIMO plant in an easy manner. Moreover, this work handles discrete- as well as continuous-time systems.

The remainder of this paper is organized as follows: In Section 2, we describe the relevant plant and model. In Section 3, the research outline and characteristics of the MEC are presented. In Section 4, the analysis for the $H_{\infty}$ performance of the MEC with PFC based on LMIs [25] is given. In Section 5, we describe the design of the PFC and MEC combined with PSO and proposed analysis. Finally, in Section 6, we evaluate the effectiveness through a numerical example.

Note that the $H_{\infty}$ norms of $\Phi(s)$ for continuous-time systems and $\Phi(z)$ for discrete-time systems are given by the following equations.

$$\|\Phi(s)\|_{\infty} := \sup_{\omega \in \mathbb{R}} \sigma_{\text{max}}(\Phi(j\omega)) \quad (1)$$

$$\|\Phi(z)\|_{\infty} := \sup_{\theta \in [0, 2\pi]} \sigma_{\text{max}}(\Phi(e^{j\theta})) \quad (2)$$

where $\sigma(\cdot)$ is the maximum singular value.

2. Plant and model

2.1. State–space representation

In this work, we assume control of the continuous- and discrete-time linear time-invariant MIMO non-minimum phase systems. The state equation, which represents the dynamics of the plant $P$, is given as follows:

$$P : \left\{ \begin{array}{l}
\delta x(t) = Ax(t) + B \delta u(t) + B_w w_u(t) \\
y(t) = C x(t) + D_w w_y(t)
\end{array} \right. \quad (3)$$

where $\delta$ denotes the derivative operator $\delta x(t) = \dot{x}(t)$ for continuous-time systems and forward-shift operator $\delta x(k) = x(k + 1)$ for discrete-time systems, with $k \in \mathbb{Z}$. Further, $A$ is square matrix that represents the plant dynamics; $B$, $B_w$, $C$, and $D_w$ are the appropriate matrices depending on the number of inputs/outputs. The degree of the plant in (3) is $m_s$, and $x(t) \in \mathbb{R}^{m_s}$, $\delta u(t) \in \mathbb{R}^{m_u}$, $y(t) \in \mathbb{R}^{m_y}$, $w_u(t) \in \mathbb{R}^{m_w}$, and $w_y(t) \in \mathbb{R}^{m_y}$ are the state, control input, plant output, disturbance input, and observation noise, respectively. Here, $A$, $B$, $C$, $D_w$, and $D_w$ are the appropriate matrices depending on the number of inputs/outputs.
and $C$ are assumed to have polytopic-type uncertainties and are given as follows using $\lambda = [\lambda_i]$ and vertex matrices $A_i$, $B_i$, and $C_i$ of appropriate dimensions.

$$A = \sum_{i=1}^{N} \lambda_i A_i, \quad B = \sum_{i=1}^{N} \lambda_i B_i, \quad C = \sum_{i=1}^{N} \lambda_i C_i$$

(4)

where $\lambda$ is a time-invariant parameter called the standard simplex and belongs to the following set $\varepsilon$:

$$\varepsilon := \left\{ \lambda \in \mathbb{R}^N : \lambda_i \geq 0, \sum_{i=0}^{N} \lambda_i = 1 \right\}.$$  

(5)

The matrix polytope representation is a system uncertainty where true parameters cannot be obtained, but we know that they are included in the set represented by the vertex matrices. An arbitrary point in the set can be represented by (4). This type of plant can be handled using an LMI easily. Additionally, using polytopic-type uncertainties is practical and valuable for systems whose models are obtained from the physical or nominal models. In this study, it is assumed that the plant is controllable and observable about the arbitrary $\lambda \in \varepsilon$.

The dynamics of the nominal model $P_m$ used for MEC is given as a linear time-invariant system by the following equation:

$$P_m : \begin{cases} \delta x_m(t) = A_m x_m(t) + B_m u_m(t) \\ y_m(t) = C_m x_m(t) \end{cases}$$

(6)

where $A_m$ is a square matrix representing the plant dynamics, and $B_m$ and $C_m$ are appropriate matrices that depend on the number of inputs/outputs. $x_m(t) \in \mathbb{R}^{m_i}$, $u_m(t) \in \mathbb{R}^{m_i}$, and $y_m(t) \in \mathbb{R}^{m_o}$ are the state of the nominal model, control input, and nominal model output, respectively. Now, we consider the modelling errors $\Delta A = A - A_m$, $\Delta B = B - B_m$, and $\Delta C = C - C_m$. $\Delta A$, $\Delta B$, and $\Delta C$ are given as follows using (4):

$$\Delta A = \sum_{i=1}^{N} \lambda_i \Delta A_i, \quad \Delta B = \sum_{i=1}^{N} \lambda_i \Delta B_i,$$

$$\Delta C = \sum_{i=1}^{N} \lambda_i \Delta C_i.$$  

(7)

where $\Delta A_i = A_i - A_m$, $\Delta B_i = B_i - B_m$, and $\Delta C_i = C_i - C_m$.

One of the methods of deciding the model parameters $A_m$, $B_m$, and $C_m$ is by setting them to value $\lambda_i = 1/N$ in (4) (where $N$ is the number of vertices). This means that $A_m$, $B_m$, and $C_m$ are the centres of the polytope (4), and the model with the centre parameters is the maximum likelihood estimate of the plant. Hence, the model parameters are set to values that represent the plant dynamics accurately. Note that there is theoretically no problem in setting the model parameters to values outside (4). The relationships between the model settings and performances are discussed in [8].

2.2. Characteristics of non-minimum phase systems

We assume that the system represented by (3) is non-minimum phase system. Non-minimum phase systems generally have unstable zeros. The zero of the system (3) is the complex number $s$, which reduces the rank of the following matrix (that is, zero is the root of $\det |P(s)| = 0$):

$$P(s) = \begin{bmatrix} s I - A & B \\ -C & 0 \end{bmatrix}$$

(8)

The unstable zeros are those with $\Re(s) > 0$ for continuous-time systems or $|s| > 1$ for discrete-time systems. Non-minimum phase systems are known to be difficult to control, where undershoots occur in the step responses. For example, consider the transfer function $G(s)$ of a SISO system as follows:

$$G(s) = \frac{-s + 1}{s^2 + 2s + 3}.$$  

(9)

$G(s)$ has an unstable zero at $s = 1$, and the step response of $G(s)$ is shown in Figure 2. We can see that an undershoot occurs from $t = 0$ to $t = 1$. Unstable zeros are often included in systems with non-collocated sensors and actuators.

The difficulties in controlling non-minimum phase systems are due to the inverse model that cannot be formulated strictly because it becomes unstable if the plant is non-minimum phase; the non-minimum phase system thus has a control performance limitation, as shown in [26]. Furthermore, when applying MEC to systems, high gain feedback of the difference between the outputs of the plant and model are supposed for minimum phase systems, whereas there are undershoots for non-minimum phase systems, with design difficulties.
3. Model error compensator

3.1. Method for achieving robust control systems

In model-based control, the designer obtains the nominal model of the actual plant by system identification or using physics formulas. Then, the designer formulates a controller for the nominal model, which achieves the desired control performance. However, if there are modelling errors, the desired control performance cannot be achieved because the design is for the nominal model. To overcome this problem, an MEC was proposed in a previous study [1]. The MEC \( H \) is attached to plant \( P \), as shown in Figure 3. By designing the MEC \( H \) appropriately, the influences of modelling errors and disturbances are reduced, and the dynamics of \( P' \) are close to those of the nominal model. Therefore, the dynamics of the controller are close to those of the nominal model, and the controller achieves good performance according to design. The controller is designed using various existing design methods. Hence, the system to which that MEC is applied is expected to achieve greater robustness without compromising the control performance.

Figure 4 shows the structure of the MEC, where \( P, P_m, \) and \( D \) are the actual plant, nominal model, and differential compensator, respectively. The MEC includes the nominal model \( P_m \) and uses the output difference between the existing plant \( P \) and nominal model \( P_m \) as the feedback signal. Moreover, we can ensure that the dynamics of \( P' \) are close to those of \( P_m \) given an appropriate differential compensator \( D \).

3.2. Design of MEC for polytopic-type uncertainties

In [9], we proposed a design for the MEC with polytopic-type uncertainties using LMIs and PSO. Specifically, we derive the generalized plant \( \eta \) shown in Figure 5, apply the generalized plant to the LMI analysis problem, and design the MEC by PSO using the analysis results. If the plant is given as a matrix polytope, the generalized plant \( \eta \) can be represented by a matrix polytope as well when at least one of the assumptions, namely the uncertainty of input matrix of the plant is zero, the uncertainty of the output matrix of the plant is zero, or feed-through matrix of the differential compensator \( D \) is zero, holds; the generalized plant with the polytopes can be analysed with LMI easily. Further, using meta-heuristics in the design allows flexible evaluations or constraints. In [9], we show some numerical examples and the effects on minimum phase systems. However, if we apply the method in [9] to non-minimum phase systems, although the modelling errors and constant disturbances are reduced than the case without the MEC, the results tend to be not good. Additionally, for complex non-minimum phase systems, such as plants with multiple unstable zeros, it is difficult to find the parameters of the differential compensator \( D \) such that the generalized plant is stable in PSO. Thus, excess amounts of time are required, or the design is infeasible.

4. Main results

To overcome the problem of non-minimum phase systems shown in Section 3.2, we propose a method using the PFC. In [9], the initial parameters are not set in the PSO because the differential compensator \( D \) cannot find the parameters for the stable generalized plant. Therefore, by changing the apparent dynamics of the plant with the PFC, the design is expected to work well.

This section presents the system representation of the PFC and differential compensator \( D \) shown in Figure 6, and the apparent dynamics of the plant with polytopic-type uncertainties and model from the viewpoint of the differential compensator \( D \) are designed. Then, we show the generalized plant for analysing the \( H_\infty \) performance. The generalized plant is based on [8], but we extend this scheme to non-minimum phase systems to include the PFC. Finally, we describe the analysis for \( H_\infty \) performance of the generalized continuous- or discrete-time plant. This analysis method is used for the PSO as an evaluation function.

4.1. MEC with PFC

In simple adaptive control (SAC), the PFC is used to simplify the system design by changing the apparent dynamics of the plant [24]. As the same manner, we proposed a design method for \( D \) for the non-minimum phase system based on transfer function representation using a PFC to change the plant dynamics from the point of view of \( D \) in previous studies [10, 11], as shown in Figure 6. By designing the MEC under the constraints of \( D \) to have the poles at 0 and \( F \) and to have the zero at 0, the effects of constant disturbances and steady-state errors can be removed [10].

We design the MEC with meta-heuristics using the structure shown in Figure 6; however, the evaluation and constraints can be flexible, which is one of the advantages of using meta-heuristics. Additionally, in this work, we propose a design based on state-space representation with polytopic uncertainties. Therefore, it is expected that the versatility of the MEC can be improved for easy application to MIMO plants.
4.2. System representations

Herein, we present the system representations of the PFC for the plant and model. The state equation represents the dynamics of the PFC for the plant and model, which are denoted as $F$ and $F_m$, respectively, in Figure 6 and given by the following equations:

\[
F : \begin{cases}
\delta x_f(t) = A_f x_f(t) + B_f \tilde{u} \\
y_f(t) = C_f x_f(t)
\end{cases}
\quad (10)
\]

\[
F_m : \begin{cases}
\delta x_{fm}(t) = A_{f} x_{fm}(t) + B_{f} u_m(t) \\
y_{fm}(t) = C_{f} x_{fm}(t)
\end{cases}
\quad (11)
\]

where $A_f$ is a square matrix that represents the PFC dynamics; $B_f$ and $C_f$ are appropriate matrices that are given depending on the number of inputs/outputs. The degree of the PFCs in (10) and (11) is $m_f$, and $x_f(t) \in \mathbb{R}^{m_f}$, $x_{fm}(t) \in \mathbb{R}^{m_f}$, $y_f(t) \in \mathbb{R}^{m_o}$, and $y_{fm}(t) \in \mathbb{R}^{m_o}$ are the state of $F$, state of $F_m$, output of $F$, and output of $F_m$, respectively.
Summarizing (3), (6), (10), and (11), from the MEC, the apparent dynamics of the plant $P_F$ and model $P_{Fm}$ are obtained as follows:

\begin{equation}
P_F: \begin{cases}
\delta x_F(t) = A_F x_F(t) + B_F \tilde{u}(t) + B_{Fw} w_u(t) \\
y_F(t) = C_F x_F(t) + D_{Fw} w_y(t)
\end{cases}
\end{equation}

\begin{equation}
P_{Fm}: \begin{cases}
\delta x_{Fm}(t) = A_{Fm} x_{Fm}(t) + B_{Fm} u_m(t) \\
y_{Fm}(t) = C_{Fm} x_{Fm}(t)
\end{cases}
\end{equation}

where

\begin{equation}
x_F(t) = \begin{bmatrix} x(t) \\
x_F(t) \end{bmatrix}, \quad x_{Fm}(t) = \begin{bmatrix} x_m(t) \\
x_{Fm}(t) \end{bmatrix}
\end{equation}

\begin{equation}
A_F = \begin{bmatrix} A & 0 \\
0 & A_f \end{bmatrix}, \quad B_F = \begin{bmatrix} B \\
B_f \end{bmatrix}, \quad B_{Fw} = \begin{bmatrix} B_w \\
0 \end{bmatrix},
\end{equation}

\begin{equation}
C_F = \begin{bmatrix} C & C_f \end{bmatrix}, \quad D_{Fw} = D_w
\end{equation}

\begin{equation}
A_{Fm} = \begin{bmatrix} A_m & 0 \\
0 & A_f \end{bmatrix}, \quad B_{Fm} = \begin{bmatrix} B_m \\
B_f \end{bmatrix},
\end{equation}

\begin{equation}
C_{Fm} = \begin{bmatrix} C_m & C_f \end{bmatrix}
\end{equation}

where $y_F(t)$ is the output of $P + F$, and $y_{Fm}(t)$ is the output of $P_m + F_m$. The point to be noted here is that the system represented by (12) can be represented by the matrix polytope as follows:

\begin{equation}
A_F = \sum_{i=1}^{N} \lambda_i A_{Fi}, \quad B_F = \sum_{i=1}^{N} \lambda_i B_{Fi}, \quad C_F = \sum_{i=1}^{N} \lambda_i C_{Fi}.
\end{equation}

Now, we consider $\Delta A_F = A_F - A_{Fm}$, $\Delta B_F = B_F - B_{Fm}$, and $\Delta C_F = C_F - C_{Fm}$, which are the apparent modelling errors. $\Delta A_F$, $\Delta B_F$, and $\Delta C_F$ are given as follows using (18):

\begin{equation}
\Delta A_F = \sum_{i=1}^{N} \lambda_i \Delta A_{Fi}, \quad \Delta B_F = \sum_{i=1}^{N} \lambda_i \Delta B_{Fi},
\end{equation}

\begin{equation}
\Delta C_F = \sum_{i=1}^{N} \lambda_i \Delta C_{Fi}.
\end{equation}

where $\Delta A_{Fi} = A_{Fi} - A_{Fm}$, $\Delta B_{Fi} = B_{Fi} - B_{Fm}$ and $\Delta C_{Fi} = C_{Fi} - C_{Fm}$.

Considering that the MEC is applied to the above plant $P$ and nominal model $P_m$ with the PFCs, the differential compensator $D$ is given by the following equation:

\begin{equation}
D: \begin{cases}
\delta x_d(t) = A_d x_d(t) + B_d (y_F(t) - y_{Fm}(t)) \\
y_d(t) = C_d x_d(t) + D_d (y_F(t) - y_{Fm}(t))
\end{cases}
\end{equation}

where the degree of the differential compensator (20) is $m_d$, and $x_d(t) \in \mathbb{R}^{m_d}$ is the state of the differential compensator $D$. Applying the MEC to the plant, the input to $P_m$ and $F_m$ is given as $u_m(t) = u(t)$, and the input to $P$ and $F$ is given as $\tilde{u}(t) = u(t) - y_d(t)$, where $u(t)$ is the output of the controller $C$, and $y_d(t) \in \mathbb{R}^{m_d}$ is the compensation input.

### 4.3. State equation and evaluation output of the generalized plant

This section presents the derivations of the generalized plant, nominal model, and PFC described in the previous section. Figure 7 shows the generalized plant with the MEC.

By defining $e_F(t) = x_F(t) - x_{Fm}(t)$, $\xi(t) = [e_F(t)^T, x_d(t)^T, x_{Fm}(t)^T]^T$ as the state and $v(t) = [w_u(t)^T, w_y(t)^T, u(t)^T]^T$ as the input, we obtain the following equations:

\begin{equation}
\delta \xi(t) = \bar{A} \xi(t) + \bar{B} v(t)
\end{equation}

\begin{equation}
\bar{A} = \begin{bmatrix} A_F - B_F D_d C_F & -B_F C_d \\
B_d C_F & A_d \end{bmatrix}
\end{equation}

\begin{equation}
\bar{B} = \begin{bmatrix} B_{Fw} & -B_F D_d D_{Fm} & \Delta B_F \\
0 & B_d D_{Fw} & 0 \\
0 & 0 & B_{Fm} \end{bmatrix}
\end{equation}

where the objective of the MEC is to reduce the gap between $y$ and $y_m$, hence, we consider following evaluation output:

\begin{equation}
e_c(t) = C x(t) - C_m x_m(t) = \bar{E} \xi(t)
\end{equation}

where $\bar{E} = [C, 0^{m_m \times (m_f + m_d)}, C, 0^{m_m \times m_f}]$. Note that in previous studies [10, 11], the MEC is evaluated using the errors of the summation output of the plant and PFC with the summation outputs of the model and PFC (i.e. evaluating $y_F(t) - y_{Fm}(t)$), but the proposed method can evaluate the errors of outputs of the plant and model directly.
To analyze the MEC based on polytopic-type uncertainties, we define the following matrices:

\[
\begin{align*}
\tilde{A}_i &= \begin{bmatrix}
A_{Fi} - B_{Fi}D_dC_{Fi} & -B_{Fi}C_d \\
B_dC_{Fi} & A_d \\
\Delta A_{Fi} - B_{Fi}D_d\Delta C_{Fi} & B_d\Delta C_{Fi} \\
A_{Fi} & 
\end{bmatrix} \\
\tilde{B}_i &= \begin{bmatrix}
B_{Fw} & -B_{Fi}D_dD_{Fw} & \Delta B_{Fi} \\
0 & B_dD_{Fw} & 0 \\
0 & 0 & B_{Fm}
\end{bmatrix} \\
\tilde{E}_i &= \begin{bmatrix}
C_i & 0 & \Delta C_{Fi} & 0 & 0
\end{bmatrix}
\end{align*}
\]

where the 1,1 element of \( \tilde{A}_i \) includes the bilinear term of \( B_{Fi} \) and \( C_{Fi} \), and the 1,3 element of \( \tilde{A}_i \) includes the bilinear term of \( B_{Fi} \) and \( \Delta C_{Fi} \). Therefore, it cannot become a matrix polytope; however, if \( \Delta B_{Fi} = 0 \), \( \Delta C_{Fi} = 0 \), or \( D_d = 0 \) is satisfied, then we obtain the following equation representing the dynamics of the generalized plant as a matrix polytope using \( \tilde{A}_i, \tilde{B}_i, \tilde{E}_i \), and \( \lambda \in \mathcal{E} \):

\[
\hat{A} = \sum_{i=1}^{N} \lambda_i \tilde{A}_i, \quad \hat{B} = \sum_{i=1}^{N} \lambda_i \tilde{B}_i, \quad \hat{C} = \sum_{i=1}^{N} \lambda_i \tilde{E}_i. \quad (25)
\]

The output of a system often represents its state, and such systems satisfy \( \Delta C_{Fi} = 0 \). Moreover, \( D_d = 0 \) implies that there is no direct term in the differential compensator and has often been used in control systems design theories previously. Therefore, satisfying the assumption is not difficult. We denote the input–output system from \( v(t) \) to \( e_{r}(t) \) as \( \Phi \). The system \( \Phi \) is affected by not only disturbance inputs and observation noise but also input \( u \).

4.4. \( H_\infty \) performance analysis for continuous-time systems

This section describes the analysis of the \( H_\infty \) performance for system \( \Phi \) for a continuous-time plant shown in Section 4.3. Here, we assume that \( \Delta B_{Fi} = 0 \), \( \Delta C_{Fi} = 0 \), or \( D_d = 0 \) holds for the system \( \Phi \) given by (21). As described before, under this assumption, the input and output of system \( \Phi \) can be represented by polytopic matrices, and we can easily analyze the system.

First, for the vertex matrices \( (\tilde{A}_i, \tilde{B}_i, \tilde{E}_i) \), the \( H_\infty \) performance analysis for a continuous-time system can be obtained by the following LMI using \( \gamma_\infty > 0 \):

\[
X > 0, \quad \begin{bmatrix}
\hat{A}X + X\hat{A}^T & X\hat{E}^T & \hat{B} \\
\hat{E}X & -\gamma_\infty^2 I & 0 \\
\hat{B}^T & 0 & -I
\end{bmatrix} < 0. \quad (26)
\]

where the LMI constraints of (26) are denoted as \( \Psi_i < 0 \). If we find \( X > 0 \) that satisfies \( \Psi_i < 0 \) for all \( i \), noting that \( \lambda_i \geq 0 \), the following inequality holds:

\[
\left( \sum_{i=1}^{N} \lambda_i \Psi_i \right) < 0 \quad (27)
\]

From \( \sum_{i=1}^{N} \lambda_i = 1 \), (25), (26), and (27), the following LMIs hold:

\[
X > 0, \quad \begin{bmatrix}
\hat{A}X + X\hat{A}^T & X\hat{E}^T & \hat{B} \\
\hat{E}X & -\gamma_{\infty}^2 I & 0 \\
\hat{B}^T & 0 & -I
\end{bmatrix} < 0 \quad (28)
\]

If \( X \) and \( \gamma_{\infty} \) satisfying (28) are found, the system \( \Phi \) holds for the following inequality:

\[
\| \Phi \|_{\infty} \leq \gamma_{\infty}. \quad (29)
\]

Therefore, by finding the minimum \( \gamma_{\infty} \) satisfying (28), we can analyze the system \( \Phi \) for the \( H_\infty \) performance.

4.5. \( H_\infty \) analysis for discrete-time systems

Herein, we describe the analysis for the system obtained in Section 4.3 for a discrete-time plant. Similar to
the continuous-time case, it is assumed that $\Delta B_F = 0$, $\Delta C_F = 0$, or $D_d = 0$ holds for the same reason.

For the given vertex matrices, the $H_\infty$ analysis is as follows given $\gamma$:

$$ X > 0, \quad \begin{bmatrix} -X + \bar{A}_iX\bar{A}_i^T + \bar{B}_i\bar{B}_i^T & \bar{A}_iX\bar{E}_i^T \\ \bar{E}_i\bar{X}^T & \bar{E}_i\bar{X}\bar{E}_i^T - \gamma^2_\infty I \end{bmatrix} < 0. $$

(30)

where the LMI constraints of (30) are denoted as $\Omega_i < 0$. If we find $X > 0$ that satisfies $\Omega_i < 0$ for all $i$, noting that $\lambda_i \geq 0$, the following inequality holds:

$$ \left( \sum_{i=1}^{N} \lambda_i \Omega_i \right) < 0 \quad (31) $$

From $\sum_{i=1}^{N} \lambda_i = 1$, (25), (30), and (31), the following LMIs hold:

$$ X > 0, \quad \begin{bmatrix} -X + \bar{X}\bar{A}_i^T + \bar{B}_i\bar{B}_i^T & \bar{A}_i\bar{X}\bar{E}_i^T \\ \bar{E}_i\bar{X}\bar{E}_i^T & \bar{E}_i\bar{X}\bar{E}_i^T - \gamma^2_\infty I \end{bmatrix} < 0. $$

(32)

If $X$ and $\gamma_\infty$ satisfying (32) are found, the system $\Phi$ holds for the following inequality:

$$ \| \Phi\|_{\infty} \leq \gamma_\infty. \quad (33) $$

Therefore, by finding the minimum $\gamma_\infty$ satisfying (32), we can analyse the discrete-time system $\Phi$ for the $H_\infty$ performance, similar to the continuous-time case.

Summarizing Sections 4.4 and 4.5, the analysis of $H_\infty$ reduces to the problem of finding the minimum $\gamma_\infty$ that satisfies (28) for continuous-time systems or (32) for discrete-time systems. This problem can be solved easily with a numerical calculation software, such as MATLAB.

In this work, we describe the analysis of only the $H_\infty$ performance. However, the LMI constraints of $H_2$ performance, the peak value of the impulse response, and pole placement can also be found, as described in [25]. Therefore, we can obtain these evaluation indexes by setting the necessary constraints and designing the MEC using these indexes.

5. Design of controller for MEC with PFC

As described in the previous section, if the parameters of the differential compensator $D$ and PFC $F$ are given, we can analyse the $H_\infty$ performance of the generalized plant $\Phi$. Hence, we provide the parameters using meta-heuristics, such as PSO [27], and the results of the analysis are used to evaluate the method.

PSO is one of the combination optimization methods that uses many particles and shares the solution information among them; it is also a multipoint search algorithm that simulates the search behaviours of creatures such as fish and birds. It is well known that the PSO algorithm is useful for control system designs. In PSO, each particle has the properties of position and velocity. The position is a candidate solution and is an actual parameter, whereas the position is updated at each step based on the velocity and solutions of the other particles.

PSO has the following features: the concept is easy to understand, it has few parameters that need to be set by the user, and it is suitable for searching real-number variables with continuous values. Further, the meta-heuristics can change the evaluation function flexibly. That is, the meta-heuristics can evaluate various indexes; thus, the design can be modified according to the purpose.

This section describes the design of the PFC first, and the design of the MEC combining PSO with the analysis method is described subsequently.

5.1. Design of PFC

One of the methods of the PFC $F$ and $F_m$ in Figure 7 is iteration-based design. For example, given the fixed appropriate differential compensator $D$, the PFC $F$ and $F_m$ are obtained to minimize $\gamma_\infty$ in (28) for continuous-time systems or (32) for discrete-time systems using meta-heuristics, such as PSO, as described in Section 5.2. Next, $F$ and $F_m$ are fixed, and the differential compensator $D$ is designed to minimize $\gamma_\infty$. By repeating these operations, we can obtain the MEC with minimized evaluation value. Additionally, if the design variables of $D$ and PFC are not many, we can design the PFC and $D$ simultaneously using PSO.

In previous studies [10, 11], the PFC was designed by considering the following evaluation function $J$ using the weighting transfer function $W(s)$:

$$ J = \min_{F(s)} \left\| W(s) \frac{F(s)}{P_m(s) + F(s)} \right\|_\infty, \quad (34) $$

whereas the design method using (28) or (32) evaluates the difference between the outputs of the plant $P$ and model $P_m$. This approach is an indirect method for designing $F(s)$. On the other hand, one of the advantages of our approach is that the PFC can be designed while evaluating the MEC performance by $\Phi$, directly.

If we apply the PFC to MEC, the PFC and differential compensator have to include a differentiator and an integrator, respectively [10], to remove the constant disturbances and steady-state errors. If we design the PFC with the transfer function representation, we can achieve the requirement for the PFC by setting the zeroth-order term of the numerator to 0. If we design the PFC by state-space representation, we can construct the controllable canonical form and constrain the terms of $C_f$. The purpose of attaching the PFC is to ensure the apparent dynamics of the plant minimum phase system. Therefore, in such a case, remember that adding that constraints in PSO.
5.2. Design of differential compensator

We proposed the design of a differential compensator $D$ for the minimum phase system with PSO in [9]. The parameters of $D$ are regarded as the design parameters, and each particle has a solution that is a candidate for the actual parameters of $D$. The general flow of the design is as follows:

(i) **Initialization**

Initialize the positions and velocities of all particles randomly. Depending on the initialized parameter, the generalized plant becomes unstable. At that time, redo initialization until the parameters that stabilize the system are obtained.

(ii) **Evaluation**

Obtain evaluation values for all particles by the analysis method proposed in Section 4.4 for continuous-time system or Section 4.5 for discrete-time system. Sometimes, the generalized plant becomes unstable depending on the parameter update results. At that time, a large value is added to the evaluation value as a penalty.

(iii) **Decision of termination**

Suppose the number of updates reaches a specified count set by the designer; then, output the best solution found by the particles and complete execution. If not, move to the next step, i.e. parameter update.

(iv) **Parameter update**

Update parameters of particles according to the following parameter update law and go back to the **Evaluation** step:

\[
\begin{align*}
\mathbf{z}_i^x(k+1) &= \mathbf{z}_i^x(k) + \mathbf{z}_i^v(k), \\
\mathbf{z}_i^v(k+1) &= \rho \mathbf{z}_i^v(k) + r_1 \mathbf{c}_1 (p - \mathbf{z}_i^x(k)) + r_2 \mathbf{c}_2 (g - \mathbf{z}_i^x(k)),
\end{align*}
\]

where, $k$ is the number of updates, $\mathbf{z}_i^x(k)$ and $\mathbf{z}_i^v(k)$ ($i = 1, \ldots, n$, where $n$ is the number of particles) are the position vector and velocity vector of the $i$th particle, respectively. The position vector $\mathbf{z}_i^x$ is the candidate of the solution that is actual parameters of $D$. $g$ is the global best, which is the solution with the best evaluation value found by all particles, and $p$ is the $i$th particle’s personal best, which is the best solution with the best evaluation value found by the $i$th particle. $r_1$ and $r_2$ are the random numbers in $[0, 1]$ and they are generated for each particle and number of updates. $\rho$ is the inertia weight that determines how much the current velocity $\mathbf{z}_i^v(k)$ affect the next step velocity $\mathbf{z}_i^v(k+1)$. $c_1$ and $c_2$ are the acceleration coefficients that direct the particles to the personal best and global best, respectively.

In the PSO algorithm, the parameters to be set by the designer are the number of particles, maximum update count, and weight coefficients used in the update.

As described in Section 5.1, to remove the effects of constant disturbances and steady-state errors caused by modelling errors, the differential compensator $D$ must include an integrator. If we design $D$ with the transfer function representation, this can be achieved by setting the zeroth-order term of the denominator to 0. If we design $D$ with the state-space representation, this can be achieved using the controllable canonical form with constraints on some terms of $A_f$. For example, if $D$ is designed with 4 states, 2 inputs, and 2 outputs, $D$ includes an integrator by the following constraint:

\[
A_d = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & z_{x1} & z_{x2} & 1 \\ 0 & 0 & 0 & 1 \\ 0 & z_{x3} & 0 & z_{x4} \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 & 0 \\ 1 & z_{x5} \\ 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
C_d = \begin{bmatrix} z_{x6} & z_{x7} & z_{x8} & z_{x9} \\ z_{x10} & z_{x11} & z_{x12} & z_{x13} \end{bmatrix},
\]

\[
D_d = \begin{bmatrix} z_{x14} & z_{x15} \\ z_{x16} & z_{x17} \end{bmatrix}.
\]

where $z_{xi}$ is the parameter to be designed by the PSO. In the above case, we set the degree of $D$ to 4, but a larger degree allows greater potential in general. However, if the degree increases, the number of parameters to be designed also increase, which requires additional time for initialization and evaluation. Thus, the degree should be decided by considering the balance between time and performance.

In addition to the above design method, $D$ can be designed by the iteration-based method with a common Lyapunov matrix and differential compensator. For example, an appropriate differential compensator is given first, and the performance is analysed using the common Lyapunov matrix; then, the common Lyapunov matrix is fixed and the LMI for the differential compensator is solved using design variables. By repeating these two operations, a differential compensator whose evaluation value for $\gamma_{\infty}$ is small can be obtained.

Moreover, it is possible to combine the PSO with the iteration-based method. Specifically, obtain the parameters of $D$ by the PSO method first, then execute the iteration-based method with the obtained $D$. In this combined method, we can save time and effort to search the initial values to give the iteration-based method. Furthermore, it is expected that we can obtain more robust parameters of $D$. In the next section, we design $D$ by combining the PSO and the iteration-based method.

6. Numerical examples

This section presents a numerical example of the design of the MEC for a 2-input, 2-output continuous-time
non-minimum phase plant. First, we show the conditions of the simulation and then present the simulation results. Finally, we compose the system shown in Figure 6 using the design results and evaluate the effects.

### 6.1. Conditions

The transfer function of the nominal model of the plant is given as follows:

\[ P_m(s) = \begin{bmatrix} -s + 2 & -2s^2 + 1 \\ s^2 + 2s + 1 & 2s^2 + 3s + 1 \\ -2s + 1 & -s + 2 \\ s^2 + 4s + 3 & s^2 + 3s + 2 \end{bmatrix}. \quad (38) \]

\( A_m, B_m, \) and \( C_m \) in (6) are obtained by converting (38) using the MATLAB command “ss.” The number of vertices is 4, and the vertex matrices \( A_i, B_i, \) and \( C_i \) are obtained by adding a random number to \( A_m, B_m, \) and \( C_m. \) The variabilities of the elements of the vertex matrices to the model were observed to be up to 16.5%. Note that in this simulation, \( \Delta C_F = 0 \) is assumed.

The transfer function of the PFC is also given as follows to ensure a minimum-phase plant:

\[ F_m(s) = \begin{bmatrix} 1.2s \\ s^2 + 2s + 1 \\ 2.4s \\ s^2 + 4s + 3 \end{bmatrix} \quad (39) \]

Similar to the model, \( A_f, B_f, \) and \( C_f \) in (10) and (11) are obtained by converting (39) using the MATLAB command “ss.”

Figure 8 shows the step responses of systems \( P_m, F, \) and \( P_m + F \) from the inputs to outputs. From Figure 8, the nominal model \( P_m \) is the non-minimum phase system, and \( P_m + F \) is the apparent dynamics from the viewpoint of the MEC that becomes a minimum phase system. Moreover, the given \( F \) is designed to ensure that all vertex matrices are minimum phase systems. The differential compensator \( D \) is then designed with (37).

Also, we use the method that PSO combined with the iteration-based method to design \( D, \) described in Section 5.2. As the configuration of the PSO method, the number of particles is 50, the number of maximum update count is 100, and inertia weight \( \rho, \) acceleration coefficients \( c_1, c_2 \) for personal best and for global best, these are used for updating particles, are set to 1, 0.8, and 0.8, respectively. In the iteration-based method, it is terminated when the decrease of the value of \( \gamma_\infty \) in iteration becomes less than \( 10^6. \)

### 6.2. Design results of MEC

The following are the design results of the differential compensator \( D \) for the condition described above.

\[ A_d = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & -5.6773 & 0 & -49.4331 \\ 0 & 0 & 0 & 1.0000 \\ 0 & 49.6089 & 0 & 0.0076 \end{bmatrix} \]
where the above differential compensator $D$ is applied to analyse the generalized plant $\Phi$. Then, we obtain the evaluation value $\gamma_\infty = 0.2404$, indicating that the formula holds:

$$\|\Phi\|_\infty \leq 0.2404.$$  

### 6.3. Verification of design results

We compose some systems shown in Figure 6 using the differential compensator $D$ designed in the previous section. Here, the values of the plant matrices are randomly determined in the polytope. The control input $u$ is given as 1 from $t = 0$ to 20; the disturbance input and observation noise are given as 0.5 from $t = 10$ to 20. Figure 9 shows the responses of the systems without MEC, with MEC [9], and the proposed method via red lines. Comparing Figure 9(a,c), we can see that the designed MEC reduces the influences of modelling errors and disturbances. The dashed lines in Figure 9 are the ideal outputs. Further, comparing Figure 9(b,c), the MEC with the PFC, which is the proposed method, is observed to have superior performance for reducing modelling errors and disturbances compared to the other approaches.

Note that we describe the results only for continuous-time systems in this work, but good results
can be obtained with the MEC design for discrete-time systems as well.

7. Conclusions

In this study, we present a design for the MECs of MIMO non-minimum phase systems using the PFC and PSO, along with the $H_\infty$ performance analysis of the MEC. First, a system representation of the components of the MEC is given, and the generalized plant $\Phi$ including the PFC is derived. The generalized plant can also be represented as a matrix polytope. Then, we describe the $H_\infty$ performance analysis of the generalized plant based on LMIs. The analysis with LMIs is used as an evaluation function for the design. By altering the evaluation function, we can also design the MEC according to the required purpose. Finally, a MEC design example for a non-minimum phase system is shown via numerical simulations. We compose the MEC using the design results and show the response waveforms for step disturbance. The results show that the MEC designed by the proposed method can reduce the influences of disturbances compared with the design in [9] for the non-minimum phase plant.

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