ABSTRACT. We derive general, yet simple, sharp bounds on the size of the omitted variable bias for a broad class of causal parameters that can be identified as linear functionals of the conditional expectation function of the outcome. Such functionals encompass many of the traditional targets of investigation in causal inference studies, such as, for example, (weighted) average of potential outcomes, average treatment effects (including subgroup effects, such as the effect on the treated), (weighted) average derivatives, and policy effects from shifts in covariate distribution—all for general, nonparametric causal models. Our construction relies on the Riesz-Fréchet representation of the target functional. Specifically, we show how the bound on the bias depends only on the additional variation that the latent variables create both in the outcome and in the Riesz representer for the parameter of interest. Moreover, in many important cases (e.g., average treatment effects and average derivatives) the bound is shown to depend on easily interpretable quantities that measure the explanatory power of the omitted variables. Therefore, simple plausibility judgments on the maximum explanatory power of omitted variables (in explaining treatment and outcome variation) are sufficient to place overall bounds on the size of the bias. Furthermore, we use debiased machine learning to provide flexible and efficient statistical inference on learnable components of the bounds. Finally, empirical examples demonstrate the usefulness of the approach.

Keywords: sensitivity analysis, short regression, long regression, omitted variable bias, omitted confounders, causal models, machine learning, confidence bounds.
1. Introduction

Causal inference with observational data usually relies on the assumption that the treatment assignment mechanism is “exogenous” or “ignorable” (i.e., independent of potential outcomes) conditional on a set of observed variables; or, equivalently, that the set of observed covariates satisfy the “backdoor” (or, more generally, adjustment) criterion (Rosenbaum and Rubin, 1983a; Pearl, 1995; 2009a; Angrist and Pischke, 2009; Shpitser et al., 2012; Imbens and Rubin, 2015). Investigators who rely on the conditional ignorability assumption for drawing causal inferences from non-experimental studies must, therefore, also be able to cogently argue that there are no unobserved confounders of the treatment-outcome relationship. Yet, claiming the absence of unmeasured confounders is not only fundamentally unverifiable from the data, but often an assumption that is very hard to defend in practice.

When the assumption of no unobserved confounders is called into question, researchers are advised to perform sensitivity analyses, consisting of a formal and systematic assessment of the robustness of their findings against plausible violations of unconfoundedness. The problem of sensitivity analysis has been studied across several disciplines, dating back to, at least, the classical work of Cornfield et al. (1959), and with more recent works from Rosenbaum and Rubin (1983b); Rosenbaum (1987); Robins (1999); Frank (2000); Rosenbaum (2002); Imbens (2003); Brumback et al. (2004); Altonji et al. (2005b); Hosman et al. (2010); Imai et al. (2010); Vanderweele and Aron (2011); Blackwell (2013); Frank et al. (2013); Carnegie et al. (2016); Dorie et al. (2016); Middleton et al. (2016); Oster (2017); VanderWeele and Ding (2017); Yadlowsky et al. (2018); Masten and Poirier (2018); Kallus and Zhou (2018); Kallus et al. (2019); Cinelli et al. (2019); Franks et al. (2020); Cinelli and Hazlett (2020a, b); Bonvini and Kennedy (2021); Scharfstein et al. (2021); Jesson et al. (2021), among others. Most of this work, however, either focus exclusively on binary treatments, target a specific estimand of interest (e.g., a causal risk-ratio, or a causal risk difference), or impose parametric assumptions on the observed data, or on the nature of unobserved confounding (see Section 3.4 for further discussion).

In this paper, we generalize the traditional “omitted variable bias” framework for a broad class of causal parameters that can be identified as linear functionals of the conditional expectation function of the outcome. Such functionals encompass many of the traditional targets of investigation in causal inference studies, such as, for example, (weighted) average of potential outcomes, average treatment effects (including subgroup effects, such as the effect on the treated), (weighted) average derivatives, policy effects from shifts in covariate distribution, and others—all for general, nonparametric causal models. Our construction relies on the Riesz-Frechet representation of the target functional. Specifically, we show how the bound on the bias has a simple characterization, depending only on the additional variation that the latent variables create both in the outcome and in the Riesz representer (RR) for the parameter of interest. We can thus perform sensitivity analysis with respect to violations of conditional ignorability in a broad class of causal models and target estimands.

Moreover, in many important cases (e.g., average treatment effects in partially linear models, or in nonseparable models with a binary treatment), we further show how the bias can be reparameterized
in terms of easily interpretable quantities that measure the additional gains in precision, both in predicting the outcome and the treatment, due to unobserved variables. Therefore, simple plausibility judgments on the maximum explanatory power of omitted variables (in explaining treatment and outcome variation) are sufficient to place overall bounds on the size of the bias. These results recover and generalize recent works on sensitivity analysis such as Cinelli and Hazlett (2020a) and Detommaso et al. (2021).

Finally, we provide flexible statistical inference for these bounds using debiased machine learning (DML) and auto-DML (Chernozhukov et al. 2018a, 2016a, 2018c, 2020, 2018d) as well as targeted MLE (Van der Laan and Rose 2011). DML methods can be seen as implementing the classical “one-step” semi-parametric correction (Levit 1975; Hasminskii and Ibragimov 1978; Pfanzagl and Wefelmeyer 1985; Bickel et al. 1993) based on regression scores (Newey 1994) combined with cross-fitting, an efficient form of data-splitting. The construction makes it possible to use modern machine learning methods for estimating the identifiable components of the bounds, including regression functions, Riesz represeners, the norm of regression residuals, and the norm of RR. Auto-DML further automates the process and estimates RR using their variational or adversarial characterization, without needing to know their analytical form. Auto-TML (Chernozhukov et al. 2018d) provides further refinements.

In what follows, Section 2 presents our method in the simpler context of partially linear models. The results in that section serve not only as an introduction to the main ideas of the more general, abstract framework, but are also important in their own right, since partially linear models are widely used in applied work. Section 3 then develops a general theory of omitted variable bias for continuous linear functionals of the conditional expectation of the outcome, based on their Riesz-Frechet representations. Section 4 expands on popular target functionals of interest more formally. In Section 5 we construct high-quality inference methods for the bounds on the target parameters by leveraging recent advances in debiased machine learning with Riesz represeners. In Section 6 we apply these tools to assess the robustness of causal claims in two empirical examples: (i) the impact of 401(k) eligibility on financial assets; and, (ii) the average price elasticity of gasoline demand. We conclude with Section 7 by offering some final remarks, and suggesting possible extensions.

Notation. All random vectors are defined on the probability space with probability measure \(P\). We consider a random vector \(Z = (Y, W)\) with distribution \(P\) taking values \(z\) in its support \(\mathcal{Z}\); we use \(P_V\) to denote the probability law of any subvector \(V\) and \(\mathcal{V}\) denote its support. We use \(\|f\|_{p,q} = \|f(Z)\|_{p,q}\) to denote the \(L^q(P)\) norm of a measurable function \(f : \mathcal{Z} \rightarrow \mathbb{R}\) and also the \(L^q(P)\) norm of random variable \(f(Z)\). For a differentiable map \(x \mapsto g(x)\), from \(\mathbb{R}^d\) to \(\mathbb{R}^k\), \(\partial_x g\) abbreviates the partial derivatives \(\partial / \partial x' g(x)\), and \(\partial_x g(x_0)\) means \(\partial_x g(x) \big|_{x=x_0}\). We use \(x'\) to denote the transpose of a column vector \(x\); we use \(R^2_{U \sim V}\) to denote the \(R^2\) from the orthogonal linear projection of a scalar random variable \(U\) on a random vector \(V\). We use the conventional notation \(dL/dP\) to denote the Radon-Nykodym derivative of measure \(L\) with respect to \(P\).
2. OMITTED VARIABLE BIAS IN PARTIALLY LINEAR MODELS

To fix ideas, we begin our discussion in the context of partially linear models (PLM), i.e., the case in which the conditional expectation functions (CEF) of the outcome are linearly separable in the treatment. These results not only provide the key intuitions and the building blocks for the general case of nonseparable, nonparametric models of Section 3 but they are also important in their own right, as these models are widely used in applied work.

2.1. Problem Set-Up. Consider the partially linear regression model of the form

\[ Y = \theta D + f(X,A) + \epsilon. \]  

(1)

Here \( Y \) denotes a real-valued outcome, \( D \) a real-valued treatment, \( X \) an observed vector of covariates, and \( A \) an unobserved vector of covariates. We refer to \( W := (D,X,A) \) as the “long” list of regressors, and to equation (1) as the “long” regression. For now, we assume the error term \( \epsilon \) obeys \( E[\epsilon|D,X,A] = 0 \) and thus \( E[Y|D,X,A] = \theta D + f(X,A) \).

Under the traditional assumption of conditional exogeneity (or ignorability), we have that

\[ E[Y(d+1) - Y(d)] = E[E[Y|D = d+1,X,A] - E[Y|D = d,X,A]] = \theta, \]

where \( Y(d) \) denotes the potential outcome of \( Y \) when the treatment \( D \) is experimentally set to \( d \). In other words, the assumptions of ignorability and a linearly separable CEF endow the regression coefficient \( \theta \) with a causal meaning: the average treatment effect of a unit increase of \( D \) on the outcome \( Y \). The problem, however, is that \( A \) is not observed, and thus both the long regression, and the regression coefficient \( \theta \) cannot be identified.

Since the latent variables \( A \) are not measured, an alternative route to obtain an approximate estimate of \( \theta \) is to consider the regression of \( Y \) on the “short” list of observed regressors \( W^\delta := (D,X) \subset W \), as in,

\[ Y = \theta_s D + f_s(X) + \epsilon_s. \]  

(2)

Following convention, we call equation (2) the “short” regression. Here, again, we assume the error term \( \epsilon_s \) obeys \( E[\epsilon_s | D,X] = 0 \) and we thus have \( E[Y|D,X] = \theta_s D + f_s(X) \).\(^1\) We can then use the “short” regression parameter \( \theta_s \) as a proxy for \( \theta \). Evidently, in general they are not equal, \( \theta_s \neq \theta \), and this naturally leads to the question of how far our “proxy” \( \theta_s \) can deviate from the true inferential target \( \theta \).

Our goal is, thus, to analyze the difference between the short and long parameters—the omitted variable bias (OVB):

\[ \theta_s - \theta, \]

\(^1\)We can also consider, more generally, the case where the error term \( \epsilon \) is centered and simply obeys \( E[\epsilon(D - E[D | X,A])] = 0 \). In this case, we lose the interpretation of \( \theta D + f(X,A) \) as the CEF of the outcome, and it can be interpreted as the projection of the CEF on the space of functions that are partially linear in \( D \).

\(^2\)As before, one can also consider the case where \( \epsilon_s \) is centered and simply obeys the orthogonality condition \( E[\epsilon_s(D - E[D | X])] = 0 \).
and perform inference on this bias under various hypotheses on the strength of the latent confounders $A$.

2.2. OVB as the Covariance of Approximation Errors. Recall that, using a Frisch-Waugh-Lovell partialling out argument, one can express the long and short regression parameters, $\theta$ and $\theta_s$, as the linear projection coefficients of $Y$ on the residuals $D - E[D \mid X,A]$ and $D - E[D \mid X]$, respectively. That is,

$$\theta = E\alpha(W), \quad \theta^s = E\alpha_s(W^s);$$

where here we define

$$\alpha(W) := \frac{D - E[D \mid X,A]}{E(D - E[D \mid X,A])^2}, \quad \alpha_s(W^s) := \frac{D - E[D \mid X]}{E(D - E[D \mid X])^2}.$$  

For reasons that will become clear in the next section, we can refer to $\alpha(W)$ and $\alpha_s(W^s)$ as the “long” and “short” Riesz representers (RR).

Now let $g(W) := E[Y \mid D,X,A]$ and $g_s(W^s) := E[Y \mid D,X]$ denote the long and short regression functions, respectively. Using the orthogonality conditions in (1) and (2), we can further express $\theta$ and $\theta_s$ as

$$E\alpha(W) = Eg(W)\alpha(W), \quad E\alpha_s(W^s) = Eg_s(W^s)\alpha_s(W^s).$$

Our first characterization of the OVB is thus as follows, where we use the shorthand notation: $\alpha = \alpha(W)$, $\alpha_s = \alpha_s(W^s)$.

**Theorem 1 (OVB in PLM).** Assume that $Y$ and $D$ are square integrable with:

$$E(D - E[D \mid X,A])^2 > 0.$$  

Then the OVB for the partially linear model of equations (1) - (2) is given by

$$\theta_s - \theta = E(g_s - g)(\alpha_s - \alpha),$$

that is, it is the covariance between the regression error and the RR error. Furthermore, the squared bias can be bounded as

$$|\theta_s - \theta|^2 := \rho^2 B^2 \leq B^2,$$

where

$$B^2 := E(g - g_s)^2E(\alpha - \alpha_s)^2, \quad \rho^2 := \text{Cor}^2(g - g_s, \alpha - \alpha_s).$$

The bound $B^2$ is the product of additional variations that omitted confounders generate in the regression function and in the RR. This bound is sharp for the adversarial confounding that maximizes $\rho^2$ to 1 over choices of $\alpha$ and $g$, holding $E(\alpha - \alpha_s)^2$ and $E(g - g_s)^2 \leq E(Y - g_s)^2$ fixed, provided that the observed distribution of $(Y,D,X)$ places no further constraints on the problem.

This result for partially linear regression models is new, and generalizes results for classical linear regression models. Moreover, the theorem naturally generalizes for completely nonseparable regression models, as we show in Section 3.
Note that the bound $B^2$ is the maximum amount of squared bias generated by confounding; the actual bias is amortized by the correlation $\rho$. For a given value of $B$, adversarial confounding would select this correlation to maximize the bias, by setting $\rho^2 = 1$, while amicable confounding would minimize the bias, and set $\rho^2 = 0$. In principle $\rho^2$ could be set to various values less than 1 (say, $\rho^2 = .5$) when confounding is assumed to be “natural” rather than adversarial. Here we focus on the case in which the researcher has no knowledge of the functional form of the CEFs in order to limit $\rho$, and accordingly, interpret the bound $B^2$ as resulting from the presence of adversarial confounding.

2.3. **Further Characterization of the Bias.** Sensitivity analysis requires making plausibility judgments on the values of the sensitivity parameters. Therefore, it is important that such parameters be well-understood, and easily interpretable in applied settings. Here we show how the bias of Theorem 1 can be further interpreted in terms of conventional $R^2$’s. This interpretation is inspired by Imbens (2003) and, specifically, by the partial $R^2$ characterizations of the OVB in linear models in Cinelli and Hazlett (2020a).

Let us use $R^2_{U \rightarrow V} = \text{Cor}^2(U, V)$ to denote the $R^2$ from the orthogonal linear projection of random variable $U$ on random vector $V$. Also, recall that, when the CEF is not linear, a natural measure of the strength of relationship between some variable $W$ and $V \in \{Y, D\}$ is the nonparametric $R^2$ (also known as Pearson’s correlation ratio (Pearson, 1905; Doksum and Samarov, 1995)):

$$\eta^2_{V \sim W} := \frac{\text{Var}(E[V|W])}{\text{Var}(V)}.$$  

Further, the nonparametric partial $R^2$ of variable $V \in \{Y, D\}$ with $A$ given $(D, X)$ measures the additional gain in the explanatory power that $A$ provides:

$$\eta^2_{V \sim A|D, X} := \frac{\text{Var}(E[V|A, D, X]) - \text{Var}(E[V|D, X])}{\text{Var}(V) - \text{Var}(E[V|D, X])} = \frac{\eta^2_{V \sim ADX} - \eta^2_{V \sim DX}}{1 - \eta^2_{V \sim DX}}.$$  

We are now ready to rewrite the bound of Theorem 1.

**Corollary 1 (Interpreting OVB Bounds in Terms of $R^2$).** Under the conditions of Theorem 1 we can further express the bound $B^2$ as

$$B^2 = S^2 C^2_Y C^2_D, \quad S^2 := \frac{\text{E}(Y - g_s)^2}{\text{E} \alpha_s^2},$$  

where

$$C^2_Y = R^2_{Y \rightarrow g_s \sim g_s} = \eta^2_{Y \sim A | D, X}, \quad C^2_D := \frac{1 - R^2_{\alpha \sim \alpha_s}}{R^2_{\alpha \sim \alpha_s}} = \frac{\eta^2_{D \sim A | X}}{1 - \eta^2_{D \sim A | X}}.$$  

Here we see that the bound is the product of the term $S^2$, which is directly identifiable from the observed distribution of $(Y, D, X)$, and the term $C^2_Y C^2_D$, which is not identifiable, and needs to be

3For instance, suppose that nature chooses $\rho^2 \sim U(0, 1)$. This yields an expected value for $\rho^2$ of .5. Remark 8 shows, however, that there is no natural way to set the level of natural confounding.
restricted through hypotheses that limit strength of confounding. The factors $C_Y$ and $C_D$ measure the strength of confounding that the omitted variables generate in the outcome and treatment regressions:

- $\eta^2_{Y \sim A|DX}$ in the first factor measures the proportion of residual variation of the outcome explained by latent confounders; and,
- $\eta^2_{D \sim A|X}$ in the second factor measures the proportion of residual variation of the treatment explained by latent confounders.

In addition, as we will see in Section 3, the corollary emphasizes a universal OVB formula that holds for general models, and any target parameter that can be expressed as a linear functional of the CEF. In particular, $C_D$ is determined by $1 - R^2_{\alpha \sim \alpha_s}$—the proportion of residual variance of the long RR generated by latent confounders. In the partially linear model, it turns out that $1 - R^2_{\alpha \sim \alpha_s}$ is simply given by $\eta^2_{D \sim A|X}$.

Finally, the above results hold for population data. In practice, both $\theta_s$ and $S^2$ need to be estimated from finite samples. This can be readily done using debiased machine learning, as we discuss in Section 5. This enables efficient statistical inference on the bounds for $\theta$ under any hypothetical strength of the sensitivity parameters $C_\alpha$ and $C_g$. These results allow researchers to perform sharp sensitivity analyses in a flexible class of machine-learned causal models using very simple, and interpretable, tools.

2.4. Sensitivity Analysis: A Brief Overview. The importance of the previous corollaries stems from the fact that it greatly reduces the complexity of plausibility judgments—no matter how complicated the nonlinear term $f(X,A)$ of $E[Y|D,X,A]$ actually is, or no matter how complicated $E[D|X,A]$ actually is, to place bounds on the size of the bias, researchers need only to reason about the maximum explanatory power that unobserved confounders $A$ have in explaining treatment and outcome variation.

As an overview of how these bounds can be used for sensitivity analysis in practice, let us briefly consider a real example: the robustness of the estimated impact of 401(k) eligibility on financial assets against the presence of unobserved confounders, such as firm characteristics (Poterba and Venti 1994; Poterba et al. 1995a; Chernozhukov et al. 2018b). This example is discussed in detail in Section 6. Here the short regression coefficient is estimated to be $\hat{\theta}_s = 9189$, suggesting that 401(k) availability leads to an extra $9,189 in financial assets. But how robust is this result to presence of omitted confounders?

Confounding Scenarios. Suppose that latent variables $A$ can explain at most 4% of the residual variation of the outcome, and 3% of the residual variation of the treatment; that is, we have that $\eta^2_{Y \sim A|DX} = .04$ and $\eta^2_{D \sim A|X} = .03$. In Section 6, we explain why this may actually be a conservative scenario. Given the estimate for $\hat{S} = 119,837$, these values for the partial $R^2$ of $A$ with $Y$ and $D$
Nonparametric partial $R^2$ of confounder(s) with the outcome

Nonparametric partial $R^2$ of confounder(s) with the treatment

2000
4000
6000
8000
10000
12000
14000
16000
18000
20000

0.00
0.05
0.10
0.15

0.00
0.05
0.10
0.15

0

Max match (3 years)
(4215)

(A) Contours for $|B|$.

Nonparametric partial $R^2$ of confounder(s) with the outcome

Nonparametric partial $R^2$ of confounder(s) with the treatment

Unadjusted
(9200)
Max match (3 years)
(4974)

(B) Contours for $\theta_- = \theta_s - |B|$.

FIGURE 1. Sensitivity contour plots in the 401(k) example.

Note: The vertical axis shows the nonparametric partial $R^2$ of latent variables with the outcome. The horizontal axis shows the nonparametric partial $R^2$ of latent variables with the treatment. Fig. 1a shows the contours for the absolute value of the bound on the bias, $|B|$. Fig. 1b shows the contours for the lower bound of the target parameter itself, i.e., $\theta_- = \theta_s - |B|$, which could be brought to the critical value of zero (dashed red contour), or beyond zero.

translate into an estimated bound on the absolute value of the bias of:

$$|\hat{B}| = \sqrt{\hat{S}^2 \left( \frac{\eta_Y^2_{D,A|X} \eta_D^2-A|X}}{1-\eta_Y^2_{D,A|X}} \right)} = 119837 \times \sqrt{\frac{0.04 \times 0.03}{1-0.03}} \approx 4215$$

In other words, such confounding would lead us to consider a bias of at most $4,215 in our original estimate of $9,189. This implies the following estimated bounds for the target parameter $\theta$,

$$\hat{\theta}_\pm := \hat{\theta}_s \pm |\hat{B}| \approx 9189 \pm 4215 = [4974, 13404]$$

That is, even under such violation of conditional ignorability, our estimate of the effect of 401(k) availability on financial assets is still large, and it could be between $4,975 to $13,405.

Sensitivity Contours. Notice there is a trade-off between the parameters that bound the bias: in order to maintain the same bound, a higher partial $R^2$ with the treatment can be offset by a lower partial $R^2$ with the outcome. Therefore, a useful tool for visualizing the whole sensitivity range of the target parameter, under different assumptions regarding the strength of confounding, is a bivariate contour plot (Imbens, 2003; Cinelli and Hazlett, 2020a) showing the collection of curves in the space of nonparametric partial $R^2$ values ($\eta_Y^2_{D,A|X}, \eta_D^2-A|X)$ along which the bounds are constant. Figure 1 illustrates such curves for the 401(k) example, both for the estimated bound on the absolute value of the bias $|B|$ (Fig 1a), and for the estimated lower bound of the target parameter
itself, i.e, $\theta_{-} = \theta_{s} - |B|$ (Fig 1b). In Fig 1b the black triangle in the lower corner shows the original estimate of $9,189. The red diamond shows the lower bound implied by the particular confounding scenario described above, $4,784. But now notice that, with the contour plot, we can readily assess the sensitivity of our estimate to any confounding scenario. In this particular example, for instance, even substantially stronger confounders that explain, say, 10% percent of the residual variation of the treatment and 5% of the residual variation of the outcome (or vice-versa) would not be sufficiently strong to bring down the estimate of the lower bound to the critical threshold of zero (although, the estimate would be substantially reduced).

All values here were flexibly estimated using Random Forests with debiased machine learning, and we can also account for sampling uncertainty by constructing valid asymptotic confidence intervals for the bounds. Details are provided in Sections 5 and 6.

3. Omitted Variable Bias in Nonparametric Causal Models

In this section we derive the main partial identification theorems of the paper, and construct sharp bounds on the size of the omitted variable bias for a broad class of causal parameters that can be identified as linear functionals of the conditional expectation function of the outcome, all for general nonparametric causal models. Although more abstract, the presentation of this section largely parallels the special case of partially linear models given in Section 2.

3.1. Problem Set-Up. Consider the following modern acyclical structural equations model (SEM) as an example:

$$
Y \leftarrow g_Y(D, X, A, \varepsilon_Y),
$$

$$
D \leftarrow g_D(X, A, \varepsilon_D),
$$

$$
A \leftarrow g_A(X, \varepsilon_A),
$$

$$
X \leftarrow \varepsilon_X,
$$

where $Y$ is an outcome variable, $D$ is a treatment variable, $X$ is a vector-valued confounder variable, $A$ is a vector-valued latent confounder variable, $\varepsilon_Y, \varepsilon_D, \varepsilon_A$ are vector-valued structural disturbances that are mutually independent, and $\leftarrow$ denotes assignment. This model has an associated Directed Acyclic Graph (DAG) (Pearl, 1995, 2009a) as shown in Figure 2.
The SEM above induces the potential outcome \( Y(d) \) under the intervention that replaces \( D \) in the first equation by the fixed value \( d \). That is,

\[
Y(d) := g_Y(d, X, A, \varepsilon_Y).
\]

Additionally, the independence of the structural disturbances implies the following conditional exogeneity (or, ignorability) condition:

\[
Y(d) \perp\!\!\!\!\perp D \mid \{X, A\},
\]

which states that the realized treatment \( D \) is independent of the potential outcomes, conditional on \( X \) and \( A \).

More generally, we can work with any structural equation model that implies the existence of the potential outcomes \( Y(d) \), and such that the conditional exogeneity \([7]\) holds. In fact there are many structural causal models that satisfy such assumptions (see e.g. Pearl (2009b) and Figure 3 for concrete examples). The causal interpretation of our results rely only on conditional exogeneity. Under this set-up and when \( d \) is in the support of \( D \) given \( X, A \), we then have the following (well-known) identification result

\[
\mathbb{E}[Y(d) \mid X, A] = \mathbb{E}[Y \mid D = d, X, A] =: g(d, X, A),
\]

that is, the conditional average potential outcome coincides with the “long” regression function of \( Y \) on \( D, X, \) and \( A \). Therefore, we can identify various causal parameters—functionals of the average potential outcome—from the regression function. Important examples include: (i) the average causal effect (ACE)

\[
\theta = \mathbb{E}[Y(1) - Y(0)] = \mathbb{E}[g(1, X, A) - g(0, X, A)],
\]

for the case of a binary treatment \( D \); and, (ii) the average causal derivative (ACD)

\[
\theta = \mathbb{E} [\partial_d Y(d) \mid d=D] = \mathbb{E} [\partial_d g(D, X, A)],
\]

for the case of a continuous treatment \( D \).
In fact, our framework is considerably more general, in that it covers any target parameter of the following general form.

**Assumption 1 (Target “Long” Parameter).** The target parameter $\theta$ is a continuous linear functional of the long regression:

$$\theta := E m(W, g);$$

where the mapping $f \mapsto m(w; f)$ is linear in $f \in L^2(P_W)$, and the mapping $f \mapsto E m(W, f)$ is continuous in $f$ with respect to the $L^2(P_W)$ norm.

This formulation covers the two working examples above with $m(W, g) = g(1, X, A) - g(0, X, A)$ for the ACE and $m(W, g) = \partial_d g(D, X, A)$ for the ACD, and the continuity condition holds under the regularity condition provided in the remark below. In addition to these examples, we show that many other examples in Section 4 are of this form; and further examples of this form (e.g., consumer surplus, decomposition functionals) can be found in Chernozhukov et al. (2018d).

**Remark 1 (Regularity Conditions for ACE and ACD).** As regularity conditions for the ACE we assume $E Y^2 < \infty$ and the weak overlap condition:

$$E [P(D = 1 \mid X, A)^{-1} P(D = 0 \mid X, A)^{-1}] < \infty.$$

As regularity conditions for the ACD we assume $E Y^2 < \infty$, that the conditional density $d \mapsto f(d \mid x, a)$ is continuously differentiable on its support $\mathcal{D}_{x,a}$, the regression function $d \mapsto g(d, x, a)$ is continuously differentiable on $\mathcal{D}_{x,a}$, and we have that $f(d \mid x, a)$ vanishes whenever $d$ is on the boundary of $\mathcal{D}_{x,a}$. The above needs to hold for all values $x$ and $a$ in the support of $(X, A)$. We also impose the bounded information assumption:

$$E (\partial_d \log f(D \mid X, A))^2 < \infty.$$

These conditions imply that Assumption 1 holds, by Lemma 3 given in Section 4.

The key problem is that we do not observe $A$, and therefore we can only identify the “short” conditional expectation of $Y$ given $D$ and $X$, i.e.

$$g_s(D, X) := E[Y \mid D, X] = E[g(D, X, A) \mid D, X],$$

which is the conditional expectation of the long regression $g(D, X, A)$ given the observed $D$ and $X$. Given the short regression, we can compute proxies (or approximations) $\theta_s$ for $\theta$. In particular, for the ACE, the short parameter consists of

$$\theta_s = E[g_s(1, X) - g_s(0, X)],$$

and for the ACD,

$$\theta_s = E[\partial_d g_s(D, X)].$$

In this general framework, the proxy parameters can also be expressed as the same linear functionals applied to the short regression, $g_s(W^s)$. 


Assumption 2 (Proxy “Short” Parameter). The proxy parameter $\theta_s$ is defined by replacing the long regression $g$ with the short regression $g_s$ in the definition of the target parameter:

$$\theta_s := Em(W, g_s).$$

We require $m(W, g_s) = m(W^s, g_s)$, i.e., the score depends only on $W^s$ when evaluated at $g_s$.

Indeed, in the two working examples this assumption is satisfied, since $m(W, g_s) = m(W^s, g_s) = g_s(1, X) - g_s(0, X)$ for the ACE and $m(W, g_s) = m(W^s, g_s) = \partial_d g_s(D, X)$ for the ACD. Section 4 verifies this assumption for other examples.

Our goal is to provide bounds on the omitted variable bias (OVB), i.e., the difference between the “short” and “long” functionals,

$$\theta_s - \theta,$$

under assumptions that limit the strength of confounding, and perform statistical inference on its size.

3.2. Omitted Variable Bias for Linear Functionals of the CEF. The key to bounding the bias is the following lemma that characterizes the target parameters and their proxies as inner products of regressions with terms called Riesz representers (RR).

Lemma 1 (Riesz Representation). There exist unique square integrable random variables $\alpha(W)$ and $\alpha_s(W^s)$, the long and short Riesz representers, such that

$$\theta = Em(W, g) = Eg(W)\alpha(W), \quad \theta_s = Em(W^s, g_s) = Eg_s(W^s)\alpha_s(W^s),$$

for all square-integrable $g$’s and $g_s$. Furthermore, $\alpha_s(W^s)$ is the projection of $\alpha_s$ in the sense that $\alpha_s(W^s) = E[\alpha(W) \mid W^s]$.

In the case of the ACE with a binary treatment, we have that

$$\alpha(W) = \frac{1(D = 1)}{P(D = 1 \mid X, A)} - \frac{1(D = 0)}{P(D = 0 \mid X, A)}, \quad \alpha_s(W) = \frac{1(D = 1)}{P(D = 1 \mid X)} - \frac{1(D = 0)}{P(D = 0 \mid X)},$$

and in the case of the ACD with a continuous treatment, we have that

$$\alpha(W) = -\partial_d \log f(D \mid X, A), \quad \alpha_s(W^s) = -\partial_d \log f(D \mid X).$$

Sometimes it is useful to impose restrictions on the regression functions, such as partial linearity or additivity. The next lemma describes the RR property for the long and short target parameters in this case.

Lemma 2 (Riesz Representation for Restricted Regression Classes). Furthermore, if $g$ is known to belong to a closed linear subspace $\Gamma$ of $L^2(P_W)$, and $g_s$ is known to belong to a closed linear subspace $\Gamma_s = \Gamma \cap L^2(P_{W^s})$, then there exist unique long RR $\bar{\alpha}$ in $\Gamma$ and unique short RR $\bar{\alpha}_s$ in $\Gamma_s$ that continue to have the representation property

$$\theta = Em(W, g) = Eg(W)\bar{\alpha}(W), \quad \theta_s = Em(W^s, g_s) = Eg_s(W^s)\bar{\alpha}_s(W^s),$$
for all $g \in \Gamma$ and $g_s \in \Gamma_s$. Moreover, they are given by the orthogonal projections of $\alpha$ and $\alpha_s$ on $\Gamma$ and $\Gamma_s$, respectively. Since projections reduce the norm, we have $E\bar{\alpha}^2 \leq E\alpha^2$ and $E\bar{\alpha}_s^2 \leq E\alpha_s^2$. Furthermore, the best linear projection of $\bar{\alpha}$ on $\bar{\alpha}_s$ is given by $\bar{\alpha}_s$, namely,

$$\min_{b \in \mathbb{R}} E((\bar{\alpha} - b\bar{\alpha}_s)^2) = E((\bar{\alpha} - \bar{\alpha}_s)^2) = E\bar{\alpha}^2 - E\bar{\alpha}_s^2.$$ 

To illustrate, suppose that the regression functions are partially linear, as in Section 2

$$g(W) = \beta D + f(X,A), \quad g_s(W^s) = \beta_s D + f_s(X),$$

then for either the ACE or the ACD we have that the RR are given by

$$\alpha(W) = \frac{D - E[D \mid X,A]}{E(D - E[D \mid X,A])^2}, \quad \alpha_s(W^s) = \frac{D - E[D \mid X]}{E(D - E[D \mid X])^2}.$$

In what follows we use the notation $\alpha$ and $\alpha_s$ without bars, with the understanding that if such restrictions have been made, then we work with $\bar{\alpha}$ and $\bar{\alpha}_s$.

Using these lemmas, we immediately obtain the following characterization of the OVB.

**Theorem 2 (OVB and Sharp Bounds).** Consider the long and short parameters $\theta$ and $\theta_s$ as given by Assumptions 1 and 2. We then have that the OVB is

$$\theta_s - \theta = E(g_s - g)(\alpha_s - \alpha),$$

that is, it is the covariance between the regression error and the RR error. Therefore, the squared bias can be bounded as

$$|\theta_s - \theta|^2 \leq \rho^2 B^2 \leq B^2,$$

where

$$B^2 := E(g - g_s)^2E(\alpha - \alpha_s)^2, \quad \rho^2 := \text{Cor}^2(g - g_s, \alpha - \alpha_s).$$

The bound $B^2$ is the product of additional variations that omitted confounders generate in the regression function and in the RR. This bound is sharp for the adversarial confounding that maximizes $\rho^2$ to 1 over choices of $\alpha$ and $g$, holding $E(\alpha - \alpha_s)^2$ and $E(g - g_s)^2 \leq E(Y - g_s)^2$ fixed, provided that the observed distribution of $(Y,D,X)$ places no further constraints on the problem.

This is a general OVB formula that covers a wide variety of causal estimands of interest, as long as they can be written as linear functionals of the long regression. In particular, it recovers classical OVB formulas for linear regression; for the case of average causal derivatives, it recovers the OVB formula in Detommaso et al. (2021) (which was derived via a different method using a flow representation of a DAG). It applies to rich classes of examples analyzed in Section 4 and many other examples (e.g., consumer surplus, decomposition of total effect into direct and indirect and others) discussed in Chernozhukov et al. (2018d).

Finally, we note the following interesting fact.
Remark 2 (Tighter Bounds under Restrictions). When we work with restricted parameter spaces, the restricted RRs obey
\[ E(\bar{\alpha} - \bar{\alpha})^2 \leq E(\alpha - \alpha_s)^2, \]
since the orthogonal projection on a closed subspace reduces the \( L^2(P) \) norm. This means that the bounds become tighter in this case. Therefore, by default, when restrictions have been made, we work with restricted RRs. □

3.3. Characterization of the OVB Bounds. In the same spirit of Section 2, we can further derive useful characterizations of the bounds.

Corollary 2 (Interpreting Bounds). The bound of Theorem 2 can be re-expressed as
\[ B^2 = S^2C_Y^2C_D^2, \quad S^2 := E(Y - g_s)^2E\alpha_s^2, \]
where
\[ C_Y^2 := \frac{E(g - g_s)^2}{E(Y - g_s)^2} = R_{Y-g_s\sim g-g_s}^2, \quad C_D^2 := \frac{E\alpha^2 - E\alpha_s^2}{E\alpha_s^2} = \frac{1 - R_{\alpha\sim\alpha_s}^2}{R_{\alpha\sim\alpha_s}^2}. \]

This generalizes the result of Corollary 1 to fully nonlinear models, and general target parameters defined as linear functionals of the long regression. As before, the bound is the product of the term \( S^2 \), which is directly identifiable from the observed distribution of \((Y,D,X)\), and the term \( C_Y^2C_D^2 \), which is not identifiable, and needs to be restricted through hypotheses that limit strength of confounding.

Thus, as before, the terms \( C_Y^2 \) and \( C_D^2 \) generally measure the strength of confounding that the omitted variables generate in the outcome regression and in the treatment:

- \( R_{Y-g_s\sim g-g_s}^2 \) in the first factor measures the proportion of residual variance in the outcome explained by confounders;
- \( 1 - R_{\alpha\sim\alpha_s}^2 \) in the second factor measures the proportion of residual variance of the long RR generated by latent confounders.

The case of zero adversarial confounding arises whenever one of these two parameters is zero. Here, again, there is a trade-off: higher \( R_{Y-g_s\sim g-g_s}^2 \) can be compensated by lower \( 1 - R_{\alpha\sim\alpha_s}^2 \), and vice-versa.

As before we again have the following useful interpretation of \( C_Y^2 \) as the nonparametric partial \( R^2 \) of \( A \) with \( Y \), given \( D \) and \( X \), namely, \( C_Y^2 = \eta_{Y\sim A|D,X}^2 \). The interpretation of \( C_D^2 \) can be further specialized for different cases, as follows.

Remark 3 (Interpretation of \( C_D^2 \) for ACE with a Binary Treatment). For the ACE example, we have that
\[ C_D^2 = \frac{E[1/(\pi(X,A)(1 - \pi(X,A)))]}{E[1/(\pi(X)(1 - \pi(X)))]} - 1, \]
where $\pi(X) = P(D = 1 \mid X)$ and $\pi(X,A) = P(D = 1 \mid X, A)$, which is the average gain in the conditional precision with which we predict $D$ by using $A$ in addition to $X$. Therefore, the interpretation of $C^2_D$ for the ACE with a binary treatment is similar in spirit to the interpretation for the case of the partially linear model.

And a similar interpretation applies for average causal derivatives.

**Remark 4 (Interpretation of $C_D$ for Average Causal Derivatives).** For the ACE example,

$$C^2_D = \frac{E[(\partial_d \log f(D \mid X,A))^2]}{E[(\partial_d \log f(D \mid X))^2]} - 1,$$

(11)

which can be interpreted as the gain in the information that the confounder $A$ provides about the location of $D$. If $D$ is homoscedastic Gaussian, conditional on both $X$ and $(X,A)$, we have

$$\partial_d \log f(D \mid X,A) = -\frac{D - E[D \mid X,A]}{E(D - E[D \mid X,A])^2}, \quad \partial_d \log f(D \mid X,A) = -\frac{D - E[D \mid X]}{E(D - E[D \mid X])^2},$$

so that $C^2_D$ simplifies to the term $C^2_D$ found for the partially linear model.

### 3.4. Connections to Related Literature.

Sensitivity analysis to unobserved confounders can be traced back to, at least, the classical work of [Cornfield et al. (1959)](Cornfield1959). In that paper, Cornfield and colleagues derived the minimum strength that a binary unobserved confounder must have to nullify the observed risk-ratio of a binary treatment with a binary outcome. Since then, a plethora of sensitivity analysis methods have been proposed. In this section, we provide a brief overview of the main approaches to sensitivity analysis, with a focus on recent methods, and specifically on how they differ from our proposal. We refer readers to [Liu et al. (2013)](Liu2013), [Richardson et al. (2014)](Richardson2014), [Cinelli and Hazlett (2020a)](Cinelli2020), and, more recently, [Scharfstein et al. (2021)](Scharfstein2021) for further discussion.

It is useful to summarize the results we derive in this paper. Here we provide bounds for a broad class of causal parameters that can be expressed as linear functionals of the conditional expectation function of the outcome—all for general, non-parametric, models. In particular, we allow for arbitrary (e.g., binary or continuous) treatment and outcome variables, and the bounds depends only on the additional gains in variation that latent variables create both in the outcome regression, $C_Y$, and in the Riesz representer of the target functional, $C_D$. Moreover, since $C_Y$ is itself bounded by unity, bounds on $C_D$ alone are sufficient to bound the parameter of interest.

In contrast, many sensitivity analysis proposals demand from users a rather extensive specification of the unobserved confounders. This could range from positing the marginal (or conditional) distribution of latent variables, along with specifying how such confounders would enter the outcome/treatment equations (e.g., entering linearly). Among such proposals, with varying degrees of requirements and parametric assumptions, we can find [Rosenbaum and Rubin (1983b)](Rosenbaum1983), [Imbens (2003)](Imbens2003), [Vanderweele and Arah (2011)](Vanderweele2011), [Carnegie et al. (2016)](Carnegie2016), [Dorie et al. (2016)](Dorie2016), [Altonji et al. (2005a)](Altonji2005), [Oster (2017)](Oster2017), and [Veitch and Zaveri (2020)](Veitch2020).
Another branch of literature, such as Robins (1999), Brumback et al. (2004), Blackwell (2013), relies instead on the specification of a “tilting,” “selection,” or “bias” function, directly relating the difference between the conditional distribution of the outcome under treatment (control) between treated and control units (or just parameterizing the difference in conditional means). Recent work in this area includes Franks et al. (2020) and Scharfstein et al. (2021), with a special focus on binary treatments, and flexible semi-parametric estimation procedures.

Continuing with binary treatments, many sensitivity analysis proposals focus on this special case, differing mainly on how to parameterize departures from random assignment. For instance, Masten and Poirier (2018) places bounds on the difference between the treatment assignment distribution, conditioning and not conditioning on potential outcomes, whereas Rosenbaum (1987, 2002) and more recently Tan (2006); Yadlowsky et al. (2018); Kallus and Zhou (2018); Kallus et al. (2019); Jesson et al. (2021) place bounds on the odds of such distributions. Bonvini and Kennedy (2021), on the other hand, propose a contamination model approach, placing restriction on the proportion of confounded units.

Other approaches allow for general confounders, treatments and outcomes, but target specific estimands. For example, following the tradition of Cornfield et al. (1959), Ding and VanderWeele (2016) have recently derived general “Cornfield conditions” for the risk-ratio, relaxing the assumption of binary variables. Finally, and closely related to our approach, Cinelli and Hazlett (2020a) derive general bounds for partial regression coefficients in ordinary least squares, that requires no assumptions beyond the maximum explanatory power of omitted variables with the treatment and the outcome.

4. Details of Leading Examples

We were using the ACE and ACD as working examples. Here we provide more general example classes, covering a wide variety of interesting and important causal estimands. The presentation of examples draws on Chernozhukov et al. (2018c).

4.1. Examples. We first present some examples for the binary treatment case, with the understanding that finitely discrete treatments can be analyzed similarly. Recall that we use $W = (D, X, A)$ to denote the “long” set of regressors and $W^s = (D, X)$ to denote the “short” list of regressors.

Example 1 (Weighted Average Potential Outcome). Let $D \in \{0, 1\}$ be the indicator of the receipt of the treatment. Define the long parameter as

$$\theta = E[g(\bar{d}, X, A)\ell(W^s)],$$

where $w^s \mapsto \ell(w^s)$ is a bounded nonnegative weighting function and $\bar{d}$ is a fixed value in $\{0, 1\}$. We define the short parameter as

$$\theta_s = E[g_s(\bar{d}, X)\ell(W^s)].$$
We assume \( \text{E}Y^2 < \infty \) and the weak overlap condition
\[
\text{E}[\ell^2(W^s)/P(D = \tilde{d} \mid X, A)] < \infty.
\]

The long parameter is a weighted average potential outcome (PO) when we set the treatment to \( \tilde{d} \), under the standard conditional exogeneity assumption \( [7] \). The short parameter is a statistical approximation based on the short regression. In this example, setting

- \( \ell(w^s) = 1 \) gives the average PO in the entire population;
- \( \ell(w^s) = 1(x \in \mathcal{N})/P(X \in \mathcal{N}) \) the average PO for group \( \mathcal{N} \);
- \( \ell(w^s) = 1(d = 1)/P(D = 1) \) the average PO for the treated.

Above we can consider \( \mathcal{N} \) as small regions shrinking in volume with the sample size, to make the averages local, as in Chernozhukov et al. (2018c), but for simplicity we take them as fixed in this paper.

**Example 2 (Weighted Average Treatment Effects).** In the setting of the previous example, define the long parameter
\[
\theta = \text{E}[(g(1,X,A) - g(0,X,A))\ell(W^s)],
\]
and the short parameter as
\[
\theta_s = \text{E}[(g_s(1,X) - g_s(0,X))\ell(W^s)].
\]
We further assume \( \text{E}Y^2 < \infty \) and the weak overlap condition
\[
\text{E}[\ell^2(W^s)/\{P(D = 0 \mid X, A)P(D = 1 \mid X, A)\}] < \infty.
\]

The long parameter is a weighted average treatment effect under the standard conditional exogeneity assumption. In this example, setting

- \( \ell(w^s) = 1 \) gives ACE in the entire population;
- \( \ell(w^s) = 1(x \in \mathcal{N})/P(X \in \mathcal{N}) \) the ACE for group \( \mathcal{N} \);
- \( \ell(w^s) = 1(d = 1)/P(D = 1) \) the ACE for the treated;
- \( \ell(x) = \pi(x) \) the average value of policy (APV) \( \pi \),

where the policy \( \pi \) assigns a fraction \( 0 \leq \pi(x) \leq 1 \) of the subpopulation with observed covariate value \( x \) to receive the treatment.

In what follows \( D \) does not need to be binary. We next consider a weighted average effect of changing observed covariates \( W^s \) according to a transport map \( w^s \mapsto T(w^s) \), where \( T \) is deterministic measurable map from \( \tilde{\mathcal{W}}^s \) to \( \mathcal{W}^s \). For example, the policy
\[
(D,X,A) \mapsto (D + 1,X,A)
\]
adds a unit to the treatment \( D \), that is \( T(W^s) = (D + 1,X) \). This has a causal interpretation if the policy induces the equivariant change in the regression function, namely the counterfactual outcome
As regularity conditions, we suppose that the support of $P$ under the policy obeys $\mathbb{E}[\tilde{Y} | X, A] = g(T(W^s), A)$, and the counterfactual covariates are given by $\tilde{W} = (T(W^s), A)$.

**Example 3 (Average Policy Effect from Transporting $W^s$).** For a bounded weighting function $w^s \mapsto \ell(w^s)$, the long parameter is given by

$$\theta = \mathbb{E}[\{g(T(W^s), A) - g(W^s, A)\} \ell(W^s)].$$

The short form of this parameter is

$$\theta_s = \mathbb{E}[\{g_s(T(W^s)) - g_s(W^s)\} \ell(W^s)].$$

As the regularity conditions we require that the support of $P_W = \text{Law}(T(W^s), A)$ is included in the support of $P$, and require the weak overlap condition

$$\mathbb{E}[(\ell(dP_W - dP_W)/dP_W)^2] < \infty.$$

We now turn to examples with continuous treatments $D$ taking values in $\mathbb{R}^k$. Consider the average causal effect of the policy that shifts the distribution of covariates via the map $W = (D, X, A) \mapsto (T(W^s), A) = (D + rt(W^s), X, A)$ weighted by $\ell(W^s)$, keeping the long regression function invariant. The following long parameter $\theta$ is an approximation to $1/r$ times this average causal effect for small values of $r$. This example is a differential version of the previous example.

**Example 4 (Weighted Average Incremental Effects).** Consider the long parameter taking the form of the average directional derivative:

$$\theta = \mathbb{E}[\ell(W^s)t(W^s)' \partial_d g(D, X, A)],$$

where $\ell$ is a bounded weighting function and $t$ is a bounded direction function. The short form of this parameter is

$$\theta_s = \mathbb{E}[\ell(W^s)t(W^s)' \partial_d g_s(D, X)].$$

As regularity conditions, we suppose that $\mathbb{E}Y^2 < \infty$. Further for each $(x, a)$ in the support of $(X, A)$, and each $d$ in $\mathcal{D}_{x,a}$, the support of $D$ given $(X, A) = (x, a)$, the derivative maps $d \mapsto \partial_d g(d, x, a)$ and $d \mapsto g(w)\omega(w)$, for $\omega(w) := \ell(d, x)t(d, x)f(d|x, a)$, are continuously differentiable; the set $\mathcal{D}_{x,a}$ is bounded, and its boundary is piecewise-smooth; and $\omega(w)$ vanishes for each $d$ in this boundary. Moreover, we assume the weak overlap:

$$\mathbb{E}[(\text{div}_d \omega(W)/f(D|X, A))^2] < \infty.$$

Another example is that of a policy that shifts the entire distribution of observed covariates, independently of $A$. The following long parameter corresponds to the average causal contrast of two policies that set the distribution of observed covariates $W^s$ to $F_0$ and $F_1$, independently of $A$. Note that this example is different from the transport example, since here the dependence between $A$ and $W^s$ is eliminated under the interventions.

**Example 5 (Policy Effect from Changing Distribution of $W^s$).** Define the long parameter as

$$\theta = \int \left[ \int g(w^s, a)dP_A(a) \right] \ell(w^s)d\mu(w^s); \quad \mu(w^s) = F_1(w^s) - F_0(w^s),$$
where \( \ell \) is a bounded weight function, and the short parameter as

\[
\theta_s = \int g_s(w^s)\ell(w^s)d\mu(w^s); \quad \mu(w^s) = F_1(w^s) - F_0(w^s).
\]

As the regularity conditions we require that the supports of \( F_0 \) and \( F_1 \) are contained in the support of \( W \), and that the measure \( P_A \times dF_k \) is absolutely continuous with respect to the measure \( dP_W \) on \( \mathcal{A} \times \text{support}(\ell) \). We further assume that \( \mathbb{E}Y^2 < \infty \) and the weak overlap:

\[
\mathbb{E}[(\ell[dP_A \times d(F_1 - F_0)]/dP)^2] < \infty.
\]

The following main result for this section establishes that the OVB formulas and bounds are valid.

**Theorem 3 (OVB Validity in Examples 1-5).** Under the conditions stated in Examples 1,2,3,5, Assumptions 1 and 2 are satisfied. Under conditions stated in Example 4, Assumptions 1 and 2 are satisfied for the Hahn-Banach extension of the mapping \( g \mapsto \mathbb{E}_m(W,g) \) to the entire \( L^2(P_W) \), given by \( g \mapsto \mathbb{E}_g(W)\alpha(W) \). The m-scores and the corresponding short m-scores in Examples 1-5 are given by:

1. \( m(w,g) = (g(d,x,a))\ell(w^s); \)
2. \( m(w,g) = (g(1,x,a) - g(0,x,a))\ell(w^s); \)
3. \( m(w,g) = (g(T(w^s),a) - g(w^s,a))\ell(w^s); \)
4. \( m(w,g) = \ell(w^s)t(w^s)\partial g(w); \)
5. \( m(w,g) = \int [\int g(w^s,a)dP_A(a)]\ell(w^s)d\mu(w^s); \)

The long RR and corresponding short RR are given by:

1. \( \alpha(w) = \frac{1}{\prod(d(x,a))} \bar{\ell}(x,a); \)
2. \( \alpha(w) = \frac{1}{\prod(d(x,a))} \bar{\ell}(x,a); \)
3. \( \alpha(w) = \frac{dP_A(w)}{dP_A(w) - dP_A(w)} \ell(w^s); \)
4. \( \alpha(w) = -\frac{\text{div}(\ell(w^s)t(w^s)f(x,a))}{\int(d(x,a))}; \)
5. \( \alpha(w) = \frac{dP_A(a) \times d(F_1(w^s) - F_0(w^s))}{dP_A(w)} \ell(w^s); \)

where above we used the notations: \( \ell(X,A) := \mathbb{E}[\ell(W^s)|X,A], \bar{\ell}(X) := \mathbb{E}[\ell(W^s)|X], p(d|x,a) := P(D = d|X = x, A = a), p(d|x) := P(D = d|X = x). \) In Examples 1-2, when the weight function only depends on \( X \), namely \( \ell(W^s) = \ell(X) \), we have the simplifications \( \bar{\ell}(X,A) = \bar{\ell}(X) = \ell(X) \).

5. **Statistical Inference on the Bounds**

The bounds for the target parameter \( \theta \) take the form

\[
\theta_{\text{m}} = \theta_s \pm |\rho|S_C g C_\alpha, \quad S^2 = \mathbb{E}(Y - g_s)^2 \mathbb{E}_\alpha^2.
\]
The components $C_g$, $C_\alpha$ are set through hypotheses on the explanatory power of omitted variables. The correlation $|\rho|$ can be set to 1 under adversarial confounding. The unknown components of the bounds are $S$ and $\theta_s$. We can estimate these components via debiased machine learning (DML), which is a form of the classical “one-step” semi-parametric correction (Levit, 1975; Hasminskii and Ibragimov, 1979; Pfanzagl and Wefelmeyer, 1985; Bickel et al., 1993; Chernozhukov et al., 2016b) based on regression (Newey, 1994) and other Neyman orthogonal scores, including a novel score for the second moment of the RR. These are combined with cross-fitting, an efficient form of data-splitting.

For debiased machine learning of $\theta_s$, we exploit the representation

$$\theta_s = E[m(W^s, g_s) + (Y - g_s)\alpha_s],$$

as in Chernozhukov et al. (2018d, 2021a). This representation is Neyman orthogonal with respect to perturbations of $(g_s, \alpha_s)$, which is a key property required for DML. Another component to be estimated is

$$E(Y - g_s)^2 = \sigma_s^2,$$

which is also Neyman-orthogonal with respect to $g_s$. The final component to be estimated is $E\alpha_s^2$. For this we explore the following formulation:

$$E\alpha_s^2 = 2E m(W^s, \alpha_s) - E\alpha_s^2 =: \nu_s^2,$$

where the latter parameterization is Neyman-orthogonal. Specifically Neyman orthogonality refers to the property:

$$\partial_{g,\alpha}E[m(W^s, g) + (Y - g)\alpha]\bigg|_{\alpha=\alpha_s, g=g_s} = 0;$$

$$\partial_gE(Y - g)^2\bigg|_{g=g_s} = 0;$$

$$\partial_\alphaE[2m(W^s, \alpha) - \alpha^2]\bigg|_{\alpha=\alpha_s} = 0;$$

where $\partial$ is the Gateaux (pathwise derivative) operator over directions $h \in L^2(P_{W^s})$.

Application of DML theory in Chernozhukov et al. (2018a) and the delta-method gives the statistical properties of the estimated bounds under the condition that machine learning of $g_s$ and $\alpha_s$ is of sufficiently high quality, with learning rate faster than $n^{-1/4}$.

The estimation relies on the following generic algorithm.

**Definition 1 (DML($\psi$)).** Input the Neyman-orthogonal score $\psi(Z; \beta, \eta)$, where $\eta = (g, \alpha)$. Then (1), given a sample $(Z_i := (Y_i, D_i, X_i))_{i=1}^n$, randomly partition the sample into folds $(I_\ell)_{\ell=1}^L$ of approximately equal size. Denote by $I^c_\ell$ the complement of $I_\ell$. (2) For each $\ell$, estimate $\tilde{\eta}_\ell = (\tilde{g}_\ell, \tilde{\alpha}_\ell)$ from observations in $I^c_\ell$. (3) Estimate $\beta$ as a root of: $0 = n^{-1} \sum_{\ell=1}^L \sum_{i \in I_\ell} \psi(\beta, Z_i; \tilde{\eta}_\ell)$. Output $\beta$ and the estimated scores $\tilde{\psi}_i^0(Z_i) = \psi(\hat{\beta}, Z_i; \tilde{\eta}_\ell)$ for each $i \in I_\ell$ and each $\ell$.

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4Or other values less than 1, if one is willing to entertain non-adversarial confounding.
Therefore the estimators are defined as
\[ \hat{\theta}_s := \text{DML}(\psi_0); \quad \hat{\sigma}_s^2 := \text{DML}(\psi_{\sigma^2}); \quad \hat{v}_s^2 := \text{DML}(\psi_{v^2}); \]
for the scores
\[ \psi_0(Z; \theta, g, \alpha) := m(W^s, g) + (Y - g(W^s))\alpha(W^s) - \theta; \]
\[ \psi_{\sigma^2}(Z; \sigma^2, g) := (Y - g(W^s))^2 - \sigma^2; \]
\[ \psi_{v^2}(Z; v^2, \alpha) := (2m(W^s, \alpha) - \alpha^2) - v^2. \]

We say that an estimator \( \hat{B} \) of \( B \) is asymptotically linear and Gaussian with the centered influence function \( \psi^o(Z) \) if
\[ \sqrt{n}(\hat{B} - B) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi^o(Z_i) + o_P(1) \sim N(0, \psi^o(Z)). \]

The application of the results in Chernozhukov et al. (2018a) for linear score functions yields the following result.

**Lemma 3 (DML for Bound Components).** Suppose that each of \( \psi \)'s listed above and the machine learners \( \hat{\eta}_\ell = (\alpha_\ell, g_\ell) \) of \( \eta_0 = (g_s, \alpha_s) \) in \( L^2(P_{\psi^s}) \) obey Assumptions 3.1 and 3.2 in Chernozhukov et al. (2018a), in particular the rate of learning \( \eta_0 \) in the \( L^2(P_{\psi^s}) \) norm needs to be \( o_P(n^{-1/4}) \). Then the estimators are asymptotically linear and Gaussian with influence functions:
\[ \psi^o_\theta(Z) := \psi_0(Z; \theta_s, g_s, \alpha_s); \quad \psi^o_{\sigma^2}(Z) := \psi_{\sigma^2}(Z; \sigma^2_s, g_s); \quad \psi^o_{v^2}(Z) := \psi_{v^2}(Z; v^2_s, \alpha_s). \]

The covariance of the scores can be estimated by the empirical analogues using the covariance of the estimated scores.

The resulting plug-in estimator for the bounds is then:
\[ \hat{\theta}_\pm = \hat{\theta}_s \pm \hat{S}_\pm |C_g C_\alpha|, \quad \hat{S}\hat{\sigma}_s^2 = \hat{\sigma}_s^2 \hat{v}_s^2. \]

**Theorem 4 (DML Confidence Bounds for Bounds).** Under the conditions of Lemma 3 the plug-in estimator \( \hat{\theta}_\pm \) is also asymptotically linear and Gaussian with the influence function:
\[ \varphi^o_\pm(Z) = \psi^o_\theta(Z) \pm \frac{|C_g C_\alpha|}{2} \frac{C_g C_\alpha}{S} \left( \sigma^2_s \psi^o_{v^2}(Z) + v^2_s \psi^o_{\sigma^2}(Z) \right). \]

Therefore, the confidence bound
\[ [\ell, u] = \left[ \hat{\theta}_- - \Phi^{-1}(1 - a) \sqrt{\frac{E\varphi^o_{\sigma^2}}{n}}, \hat{\theta}_+ + \Phi^{-1}(1 - a) \sqrt{\frac{E\varphi^o_{+\sigma^2}}{n}} \right] \]
has the one-sided covering property, namely
\[ P(\theta_\leq \ell) \to 1 - a \text{ and } P(\theta_+ \leq u) \to 1 - a. \]
The same results continue to hold if $E\phi_{\pm}^2(Z)^2$ are replaced by the empirical analogue

$$\frac{1}{n}\sum_{\ell=1}^L \sum_{i\in I_\ell} \phi_{\pm}^2(Z_i).$$

Remark 5 (Confidence Bounds). The above interval has the following set-wise covering property for the region $[\theta_-, \theta_+]$: the region is covered with probability no less than $1 - 2a - o(1)$, by the union bound. However, as argued by [Imbens and Manski (2004)], the goal is often to cover a true value $\theta \in [\theta_-, \theta_+]$ with a prescribed probability, which is a two-sided point-wise covering property. The above confidence interval does cover any such $\theta$ with probability no less than $1 - a - o(1)$, when the width of the bound $B$ is bounded away from zero. This follows by the argument of [Imbens and Manski (2004)]. If we want to have the two-sided pointwise covering property to be uniform in $B \approx 0$, then we can use the further adjustment of [Stoye (2009)] to guarantee that. For simplicity though, we focus on the one-sided covering property stated in the theorem, because in applications the relevant hypotheses are often one-sided.

The following remark discusses learning the regression function $g_s$ and the Riesz representer $\alpha_s$.

Remark 6 (Machine Learning of $\alpha_s$ and $g_s$). Estimation of the short regression $g_s$ is standard and a variety of modern methods can be used (neural networks, random forests, penalized regressions). Estimation of the short RR $\alpha_s$ can proceed in one of the following ways. First, we can use analytical formulas for $\alpha_s$ (see e.g., [Chernozhukov et al. (2018a); Semenova and Chernozhukov (2021), and references therein, for practical details]). Second, we can use a variational characterization of $\alpha_s$:

$$\alpha_s = \arg\min_{\alpha \in \mathcal{A}} E[\alpha^2(W^s) - 2m(W^s, \alpha)],$$

where $\mathcal{A}$ is the parameter space for $\alpha_s$, as proposed in [Chernozhukov et al. (2021a, 2018d)]. This avoids inverting propensity scores or conditional densities, as usually required when using analytical formulas. This approach is motivated by the first-order-conditions of the variational characterization:

$$E\alpha_s g = Em(W^s, g) \quad \text{for all } g \in \mathcal{G},$$

which is the definition of the RR. Neural network (RieszNet) and random forest (ForestRiesz) implementations of this approach are given in [Chernozhukov et al. (2021b)], and the Lasso implementation in [Chernozhukov et al. (2018d)]. Third, we may use a minimax (adversarial) characterization of $\alpha_s$, as in [Chernozhukov et al. (2018c, 2020)]:

$$\alpha_s = \arg\min_{\alpha \in \mathcal{A}} \max_{g \in \mathcal{G}} |Em(W^s, g) - E\alpha g|,$$

where $\mathcal{A}$ is the parameter space for $\alpha_s$. The Dantzig selector implementation of this approach is given in [Chernozhukov et al. (2018c)]. The neural network implementation of this approach is given in [Chernozhukov et al. (2020)].
6. EMPIRICAL EXAMPLES

We now re-analyze two empirical examples: (i) the estimation of the causal effect of 401(k) eligibility on net financial assets (Poterba et al., 1994, 1995b; Chernozhukov et al., 2018a); and, (ii) the estimation of the price elasticity of gasoline demand (Blundell et al., 2012, 2017; Chetverikov and Wilhelm, 2017). Our goal is to complement previous studies with a sensitivity analysis, utilizing the methods developed in the present paper. More specifically we want to determine whether prior conclusions, reached under the assumption of conditional ignorability, are robust to potential uncontrolled confounding.

6.1. **The effect of 401(k) Eligibility on Financial Assets.** A 401(k) plan is an employed sponsored tax-deferred savings option that allows individuals to deduct contributions from their taxable income, and accrue tax-free interest on investments within the plan. Introduced in the early 1980s as an incentive to increase individual savings for retirement, an important question in the savings literature is precisely to quantify the *causal* impact of 401(k) eligibility on net financial assets. Indeed, a naive comparison of net financial assets between those individuals with and without 401(k) eligibility suggests a positive and large impact: using data from the 1991 *Survey of Income and Program Participation* (SIPP), this difference amounts to $19,559.

The problem of this naive comparison, however, is that 401(k) plans can be obtained only by those individuals that work for a firm that offers such savings option—and employment decisions are far from randomized. As an attempt to overcome this lack of random assignment, Poterba et al. (1994), Poterba et al. (1995b), and more recently DML, leveraged the 1991 SIPP data to adjust for potential confounding factors between 401(k) eligibility and the financial assets of an individual. Their main argument is that eligibility for enrolling in a 401(k) plan can be taken as exogenous after conditioning on a few observed variables, most importantly, income. As explained in Poterba et al. (1994), at least around the time 401(k) plans initially became available, people were unlikely to make employment decisions based on whether an employer offered a 401(k) plan; instead, their main focus were on salary and other aspects of the job. Thus, whether one is eligible for a 401(k) plan could be taken as ignorable once we condition on income and other covariates related to job choice.

It is useful to think about causal diagrams (Pearl, 2009a) that represent this identification strategy. One possible model is shown Figure 4a. Here our outcome variable, $Y$, consists of net financial assets, the treatment variable, $D$, is an indicator for being eligible to enroll in a 401(k) plan; finally, the vector of observed covariates, $X$, consists of: (i) age; (ii) income; (iii) family size; (iv) years of education; (iv) a binary variable indicating marital status; (v) a “two-earner” status indicator; (vi) a defined benefit pension status indicator; (vii) an IRA participation indicator; and, (viii) a home ownership indicator. We consider that the decision to work for a firm that offers a 401(k) plan could be taken as ignorable once we condition on income and other covariates related to job choice.

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5Defined as the sum of IRA balances, 401(k) balances, checking accounts, U.S. saving bonds, other interest-earning accounts in banks and other financial institutions, other interest-earning assets (such as bonds held personally), stocks, and mutual funds less non-mortgage debt.
plan depends both on the observed covariates $X$, but also on latent firm characteristics, denoted by $F$; moreover, $X$, $F$, and $D$ are jointly affected by a set of latent factors $U$. Most importantly, note the assumption of absence of direct arrows, both from $F$ and $U$, to $Y$. Under such assumption, conditional ignorability holds adjusting for $X$ only. The story represented by the DAG of Figure 4a is one way of rationalizing the identification strategy used in earlier papers.

The first three columns of Table 1 shows the estimates for the effect of 401(k) eligibility on net financial assets under this conditional ignorability assumption. For these estimates, we follow the same strategy used in Chernozhukov et al. (2018a), and we estimate the causal effect using DML with Random Forests, considering both a partially linear model, and a fully non-parametric model. As we can see, after flexibly taking into account observed confounding factors, although the estimates of the effect of 401(k) eligibility on net financial assets are substantially attenuated, they are still large, positive and statistically significant (more precisely, $9,189$ for the PLM and $8,247$ for the nonparametric model).

With the nonparametric model, we further explore heterogeneous treatment effects, by analyzing the ATE within income quartile groups. The results are shown in Figure 5a. We see that the ATE varies substantially across groups, with effects ranging from approximately $5,000$ (first quartile) to almost $20,000$ (last quartile).

6.1.1. Sensitivity Analysis. Although an improvement over a naive comparison, the previous estimates may still be threatened by unobserved confounders. It is thus advisable to consider

\[
\eta^2_{Y \sim F | D, X} \approx 4\% , \quad 1 - R^2_{a \sim a_i} \approx 3\%
\]

| Model               | Estimates Assuming Conditional Ignorability | Robustness Values |
|---------------------|---------------------------------------------|-------------------|
|                     | Short Estimate | Std. Error | t-value | RV   | RV$_{a=0.05}$|
| Partially Linear    | 9,189         | 1,314      | 6.99    | 7.4% | 5.5%        |
| Fully Nonparametric | 8,247         | 1,227      | 6.72    | 6.5% | 4.9%        |

**Table 1.** Minimal sensitivity reporting: original estimates and robustness values.
scenarios in which conditional ignorability fails. Figure 5b presents one such scenario, where a violation of conditional ignorability is credible. Employers often offer a benefit in which they “match” a proportion of an employee’s contribution to their 401(k) up to 5% of the employee’s salaries. The model in Figure 5b allows this “matched amount,” denoted by \( M \), to be determined by unobserved firm characteristics \( F \), observed worker characteristics \( X \), and by 401(k) eligibility \( D \). Note that, in this model, adjustment for \( X \) alone is not sufficient for control of confounding, as there are open confounding paths due to the latent variables \( F \) and \( U \) passing through \( M \). Instead, we now need to condition both on observed covariates \( X \) and latent confounders \( F \) for ignorability to hold.

How strong would the omitted firm characteristics \( F \) have to be (in terms of partial \( R^2 \)s) in order to overturn our previous conclusions? How would our estimates have changed under certain posited strengths of the explanatory power of firm characteristics? And how plausible are the strengths revealed to be problematic? In what follows, we use our sensitivity analysis results to address these questions.

**Minimal Sensitivity Reporting.** Table 1 adapts the suggestion of Cinelli and Hazlett (2020a,b) for a minimal sensitivity analysis reporting. Beyond the original results, it shows the robustness value (RV) of the short estimate, both for the PLM and the fully nonparametric model. The RV measures the minimal equal strength of both sensitivity parameters such that the (confidence) bound...
for the ATE would include zero[7] Starting with the PLM, the RV of 7.4% means that unobserved confounders that explain less than 7.4% of the residual variation, both of the treatment, and of the outcome, are not sufficiently strong to explain away the observed effect. If we further account for sampling uncertainty (at the 5% significance level), we obtain an RV_\alpha=0.05 of 5.6%, meaning that if latent firm characteristics explain less than 5.6% of the residual variation, both of 401(k) eligibility and net financial assets, this would not be sufficient to bring down the lower limit of the confidence bound for the ATE to zero. Moving to the fully nonparametric model, we obtain similar, but somewhat lower values of RV = 6.5% and RV_\alpha=0.05 = 4.9%. The RV thus provides a quick and meaningful reference point that summarizes the robustness of the short estimate against unobserved confounding—any postulated confounding that does not meet this minimal criterion of strength cannot overwrite the results of the original study.

The right corner of Table[1] shows plausible bounds on the strength of confounding. We start with the assumption that F explains as much variation in net financial assets as the total variation of the maximal matched amount of income (5%) over the period of three years. This would lead to an additional 3% of total variation explained, resulting in a partial R^2 with the outcome of η_{Y\sim F|DX} = \frac{\eta^2_{FDX} - \eta^2_{DX}}{1 - \eta^2_{DX}} = 0.28+0.03-0.28 = \frac{0.28}{1-0.28} \approx 4\%, which amounts to a relative increase of approximately 10% of in corresponding the baseline R^2 of 28%. Following similar reasoning, and more conservatively, we further posit that omitted firm characteristics can explain an additional 2.5% of the variation in 401(k) eligibility, a 20% relative increase in the corresponding baseline R^2 with the treatment of 13%. This results in 1 - R^2_{\alpha \sim \alpha} = η^2_{D\sim F|X} = \frac{\eta^2_{D,FX} - \eta^2_{D,X}}{1 - \eta^2_{DX}} = 0.13+0.25-0.13 = \frac{0.13}{1-0.13} \approx 3\%. Since both η_{Y\sim F|DX} \approx 4\% and 1 - R^2_{\alpha \sim \alpha} \approx 3\% are below the robustness value of 5.6%, we immediately conclude that such confounding scenario is not capable of bringing the lower limit of the confidence bound of the ATE to zero.

| Model               | Short Estimate | Bias Bound | ATE Bounds   | Confidence Bounds |
|--------------------|----------------|------------|--------------|-------------------|
| Partially Linear   | 9,189 (1,314)  | 4,215 (302)| [4,974; 13,404]| [2,798; 15,659]    |
| Fully Nonparametric| 8,247 (1,227)  | 4,289 (399)| [3,958; 12,536]| [1,878; 14,699]    |

*Note: ρ^2 = 1; C_\hat{\rho}_Y = 0.04; C_D \approx 0.031. Significance level of 5%. Standard errors in parenthesis.*

**TABLE 2.** Estimate, bias, and bounds for the ATE.

Next we determine the exact bias, bounds, and confidence bounds on the ATE implied by the the posited scenario, as shown in Table[2] Starting with the partially linear model, we see that the confounding scenario would create an estimated absolute bias of at most $4,215. Accounting for statistical uncertainty, we obtain a lower limit for the confidence bound of $2,796. The results for the fully nonparametric model are qualitatively similar, with point estimates and bounds for the ATE

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7 More precisely, RV stands for the minimum bound on both sensitivity parameters, η_{Y\sim F|DX} ≤ RV and 1 - R^2_{\alpha \sim \alpha} ≤ RV, such that the interval [\hat{\theta}_-, \hat{\theta}_+] of Theorem 4 includes zero (ρ is left unconstrained). RV_\alpha stands for the minimum bound such that the interval [l, u] of Theorem 4 includes zero, at the significance level α. We can also compute RVs for thresholds other than zero, as well as extreme robustness values, by also leaving the sensitivity parameter η_{Y\sim F|DX} unconstrained. See Cinelli and Hazlett [2020a,b] for further discussion.
shifted down by roughly one thousand dollars. Confidence bounds for group-wise ATEs can also be computed, and are shown in Figure 5b. Note how the bounds are still largely positive, with only a small excursion into the negative side in the case of the second quartile group. These results suggest that the main qualitative findings reported in earlier studies are rather robust to such violation of unconfoundedness.

**Sensitivity Contour Plots.** Thus far we have: (i) characterized the overall robustness of the short estimate against unobserved confounding; and, (ii) obtained (confidence) bounds for the ATE under a specific confounding scenario. We now turn our attention to sensitivity contour plots (Imbens, 2003; Cinelli and Hazlett, 2020a,b), allowing investigators to quickly and easily assess the robustness of their findings against any postulated degree of confounding. Sensitivity contour plots for the absolute value of the bias, as well as for the estimated lower bound of the ATE, were already given in Figure 1 of Section 2.4. Here we provide analogous plots for the lower limit and upper limit of the confidence bounds for the ATE, considering the partially linear model (results are qualitatively similar for the nonparametric model).

The results are shown in Figure 6. As before, the horizontal axis describes the fraction of residual variation of the treatment explained by unobserved confounders, whereas the vertical axis describes the share of residual variation of the outcome explained by unobserved confounders. The contour lines show the lower limit (Figure 6a) and upper limit (Figure 6b) of the confidence bounds \([l, u]\) for the ATE (see Theorem 4), with a given pair of hypothesized values of partial \(R^2\) (and the conservative assumption of adversarial confounding, \(\rho = 1\)). Note \(RV_{a=0.05}\) of Table 1 is the point

**Figure 6.** Sensitivity contour plots for the lower (a) and upper (b) limits of the confidence bounds, using the partially linear model. Significance level \(\alpha = 0.05\).
where the 45-degree line crosses the critical contour of zero (red dashed line), offering a convenient and interpretable summary of the critical contour.

We can further place reference points on the contour plots, indicating plausible bounds on the strength of confounding, under alternative assumptions about the maximum explanatory power of omitted variables. The first reference point on the plot—Max match (3 years)—shows the bounds on the partial $R^2$ as previously discussed, resulting in confidence bounds for the ATE of $2,796$ to $15,664$, in accordance with Table 2. The second reference point—$l \times income$—shows bounds on the strength of confounding under the assumption that adjusting for latent firm characteristics brings about similar additive gains in explanatory power, both with the treatment and with the outcome, as the observed gains in $R^2$ due to adjusting for income. This implies a partial $R^2$ of the unobserved confounder $F$ with the outcome of $\eta_{Y\sim F|DX}^2 \approx 15\%$, and a partial $R^2$ of the confounder with the treatment of $\eta_{D\sim F|X}^2 \approx 1.5\%$, resulting in confidence bounds for the ATE of $1,251$ to $17,236$. Note here we consider adversarial confounding, by setting $\rho = 1$. Setting $\rho$ to a value similar to what is observed for income results in a much weaker scenario (see Appendix C for details).

**Other Observed Confounders.** We also consider other observed covariates as reference points. As shown in Appendix C, apart from income, all other covariates have an estimated explanatory power, either with the outcome or with the treatment, of zero. This would imply a trivial bias of zero, and thus their results are omitted. We conclude that, for latent confounders to completely eliminate the observed effect, they would need to generate much higher gains in explanatory power than the observed gains of key observed covariates.

### 6.2. Average Price Elasticity of Gasoline Demand.
An important part of estimating the welfare consequences of price changes is to identify the price elasticity of demand. Here we re-analyze the data on gasoline demand from the 2001 *National Household Travel Survey* (NHTS) (Blundell et al., 2012; 2017; Chetverikov and Wilhelm, 2017). This is a household level survey conducted by telephone and complemented by travel diaries and odometer readings (see Blundell et al. (2012) and ORNL (2004) for details). Important variables in the survey include household income, gasoline price, and annual gasoline consumption (as inferred by odometer readings and fuel efficiency of vehicles). Income data corresponds to the median of the income bracket of the household, with 15 income brackets equally spaced apart in the logarithmic scale. The survey also contains 24 covariates related to population density, urbanization, demographics and US Census region indicators.

Under the assumption of conditional ignorability, we estimate the average causal derivative of log price on log demand, adjusting for the 24 observed covariates.\(^8\) We consider both a partially linear

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\(^8\)The data is available on the npiv STATA package (Chetverikov et al., 2018). The full data contains 3,640 observations. After applying the same filters suggested by Blundell et al. (2017) and Chetverikov et al. (2018), the final data used in the analysis contains 3,466 observations.

\(^9\)This can be interpreted as the average price elasticity of demand. We approximate the derivative numerically using a finite difference (e.g., $f'(x) \approx (f(x+0.01) - f(x-0.01))/0.01$).
model, and a fully non-parametric model. The results are shown in the first column of Table 3. In both models, we obtain estimates similar to the ones obtained in prior literature, with an estimated price elasticity of approximately $-0.7$.

| Model                  | Short Estimate (Std. Error) | $RV_v = -1.5$ | $RV_v = -1.5, \alpha = 0.05$ | $RV_v = 0$ | $RV_v = 0, \alpha = 0.05$ |
|------------------------|----------------------------|---------------|-------------------------------|------------|---------------------------|
| Partially linear       | -0.701 (0.257)             | 0.054         | 0.026                         | 0.047      | 0.019                     |
| Non-parametric         | -0.761 (0.360)             | 0.047         | 0.010                         | 0.049      | 0.011                     |

**Note:** $\rho^2 = 1$; Significance level of 5%. Standard errors in parenthesis.

**Table 3.** Estimate and robustness values for the average marginal effect.

6.2.1. **Sensitivity Analysis.** Despite having a large number of control variables, there are several reasons why one should worry about the assumption of no unobserved confounders in this setting. For instance, as was argued in Blundell et al. (2017), prices vary at the local market level, and unobserved factors that affect consumer preferences could act as unobserved confounders. Another potential source of endogeneity is the fact that we only observe the median of the income bracket of each household, and not the actual income. Since these brackets correspond to large income intervals, the remnant variation in the true income could be another major source of unobserved confounding. This is exacerbated in the larger income brackets, which correspond to larger intervals (and explains the reason why these larger income brackets were not included in prior work). We thus applied our sensitivity analysis tools to assess the sensitivity of the previous estimates to unobserved confounding.

The second part of Table 3 reports the robustness values for price elasticity, such that the sensitivity bounds would contain a target value $v$. Here we consider $v = -1.5$ (very elastic) and $v = -0.7$. For the partially linear specification we use DML with a cross-validated generic machine learning regression to residualize the outcome and the treatment. For the fully non-parametric specification, we use a generic machine learning approach to estimate both the regression function and the Riesz Representer. In both cases, the regression estimator uses 5-fold cross-validation to select the best among: (i) lasso models with feature expansions; (ii) random forests; and, (iii) local polynomial forests. The Riesz representer is estimated based on the loss outlined in Remark 6. We again use 5-fold cross-validation to choose the best model among a penalized linear Riesz representation with expanded features and a combination of $\ell_1$ and $\ell_2$ penalty (Chernozhukov et al., 2021a, 2018d), and a random forest representation (ForestRiesz) (Chernozhukov et al., 2021b). In both analyses, in order to reduce the variance that stems from sample splitting for cross-validation and for cross-fitting, we repeat the experiment for 5 random partitions of the data and average the final estimate, incorporating variation across experiments into the standard error, as described in Chernozhukov et al. (2018b). Moreover, since samples are highly correlated within states, we perform grouped cross-validation, where samples of the same state are always in the same fold and we stratify the folds by the census region variable.

Prior work has also analyzed this data via instrumental variable (IV) approaches (Blundell et al., 2017; Chetverikov and Wilhelm, 2017), using the distance to the closest major oil platform as an instrument. They find that IV estimates are close to the ones based on unconfoundedness (Chetverikov and Wilhelm, 2017). Further, note that the above threats to conditional ignorability are also credible threats to the validity of this proposed instrument. Extensions of our sensitivity results to IV is left to future work.
\( \nu = 0 \) (perfectly inelastic). We find that, at the 5% confidence level, these robustness values are at around 1% (or 2%, for the partially linear model). Therefore, these results show that, unless researchers are able to rule out confounding that explains at about 1-2% of the residual variation of gasoline price and gasoline consumption, the evidence provided by the data is not strong enough to distinguish between extremes such as a “very elastic,” or a “perfectly inelastic” demand function.

Finally, we explore how price elasticity varies with income under a specific confounding scenario. We consider three overlapping income groups defined as observations with income within ±0.5 in log-scale around the income points $42,500, $57,500 and $72,500, as well as a fourth high income group of all units with income above 11.6 on the log scale (≈ $110,000). To illustrate, we consider a confounding scenario of approximately 3% for both sensitivity parameters, and repeat our non-parametric and partially linear estimation and sensitivity analysis for each sub-group. Point-estimates, confidence intervals, bounds and confidence bounds are reported in Figure 7. Note that, under this scenario, the evidence for effect heterogeneity is substantially weakened, especially when using a fully non-parametric model.

7. Extensions and Conclusions

In this paper, we have derived simple, practical, yet sharp bounds for a rich class of estimands in causal models with unobserved confounders. We further provided high-quality statistical inference for such bounds using debiased machine learning methods. Finally, we demonstrated how to use these tools for sensitivity analysis in practice by examining the robustness of the estimated effect of
401(k) eligibility on net financial assets against the presence of omitted firm characteristics. Many interesting extensions are possible.

The causal estimands we address in this paper are given by linear functionals of the long regression that one would have run had they observed the latent confounders. These results can potentially be extended to nonlinear functionals. For example, consider a variant of the IV problem (Imbens and Angrist, 1994), where the instrumental variable $Z$ is confounded by observed covariates $X$, and latent variables $A$. In this case, the IV estimand is given by the ratio of two average causal effects,

$$\text{IV} = \frac{ACE(Z \rightarrow Y)}{ACE(Z \rightarrow D)}$$

The numerator and denominator can be bounded using the methods for bounding the ACE proposed in this paper.

Another potentially interesting direction of investigation is to consider causal estimands that are functionals of the long quantile regression, or causal estimands that are values of a policy in dynamic stochastic programming. When the degree of confounding is small, it seems possible to use the results in Chernozhukov et al. (2016a) to derive approximate bounds on the bias that can be estimated using debiased ML approaches. Another interesting direction for further explorations is the use of shape restrictions on the long regression $g$ that can potentially sharpen the bounds.

**APPENDIX A. PRELIMINARIES**

**A.1. Few Preliminaries.** To prove supporting lemmas we recall the following definitions and results. Given two normed vector spaces $V$ and $W$ over the field of real numbers $\mathbb{R}$, a linear map $A : V \rightarrow W$ is continuous if and only if it has a bounded operator norm:

$$\|A\|_{\text{op}} := \inf \{ c \geq 0 : \|Av\| \leq c\|v\| \text{ for all } v \in V \} < \infty,$$

where $\|\cdot\|_{\text{op}}$ is the operator norm. The operator norm depends on the choice of norms for the normed vector spaces $V$ and $W$. A Hilbert space is a complete linear space equipped with an inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|_{1/2}$. The space $L^2(P)$ is the Hilbert space with the inner product $\langle f, g \rangle = \int f g dP$ and norm $\|f\|_{P,2}$. The closed linear subspaces of $L^2(P)$ equipped with the same inner product and norm are Hilbert spaces.

**Hahn-Banach Extension for Normed Vector Spaces.** If $V$ is a normed vector space with linear subspace $U$ (not necessarily closed) and if $\phi : U \rightarrow K$ is continuous and linear, then there exists an extension $\psi : V \rightarrow K$ of $\phi$ which is also continuous and linear and which has the same operator norm as $\phi$.

**Riesz-Frechet Representation Theorem.** Let $H$ be a Hilbert space over $\mathbb{R}$ with an inner product $\langle \cdot, \cdot \rangle$, and $T$ a bounded linear functional mapping $H$ to $\mathbb{R}$. If $T$ is bounded then there exists a unique $g \in H$ such that for every $f \in H$ we have $T(f) = \langle f, g \rangle$. It is given by $g = z(Tz)$, where $z$ is unit-norm element of the orthogonal complement of the kernel subspace $K = \{a \in H : Ta = 0\}$. 
Moreover, \( \|T\|_{op} = \|g\| \), where \( \|T\|_{op} \) denotes the operator norm of \( T \), while \( \|g\| \) denotes the Hilbert space norm of \( g \).

**Radon-Nykodym Derivative.** Consider a measure space \((\mathcal{X}, \mathcal{A})\) on which two \( \sigma \)-finite measure are defined, \( \mu \) and \( \nu \). If \( \nu \ll \mu \) (i.e. \( \nu \) is absolutely continuous with respect to \( \mu \)), then there is a measurable function \( f : \mathcal{X} \to [0, \infty) \), such that for any measurable set \( A \subseteq \mathcal{X} \), \( \nu(A) = \int_A f \, d\mu \). The function \( f \) is conventionally denoted by \( d\nu/d\mu \).

**Integration by Parts.** Consider a closed measurable subset \( \mathcal{X} \) of \( \mathbb{R}^k \) equipped with Lebesgue measure \( V \) and piecewise smooth boundary \( \partial \mathcal{X} \), and suppose that \( \nu : \mathcal{X} \to \mathbb{R}^k \) and \( \phi : \mathcal{X} \to \mathbb{R} \) are both \( C^1(\mathcal{X}) \), then

\[
\int_{\mathcal{X}} \phi \operatorname{div} \nu \, dV = \int_{\partial \mathcal{X}} \phi \nu^1 \, dS - \int_{\mathcal{X}} \nu^1 \, \operatorname{grad} \phi \, dV,
\]

where \( S \) is the surface measure over the surface \( \partial \mathcal{X} \) induced by \( V \), and \( n \) is the outward normal vector.

**APPENDIX B. DEFERRED PROOFS**

**B.1. Proof of Theorem 1 and Corollary 1** The result follows from

\[
\begin{align*}
Eg\alpha - Eg_s\alpha_s & = E(g_s + g - g_s)(\alpha_s + \alpha - \alpha_s) - Eg_s\alpha_s \\
& = Eg_s(\alpha - \alpha_s) + E\alpha_s(g - g_s) + E(g - g_s)(\alpha - \alpha_s) \\
& = E(g - g_s)(\alpha - \alpha_s),
\end{align*}
\]

using the fact that \( \alpha_s \) is orthogonal to \( g - g_s \) and \( g_s \) is orthogonal to \( \alpha - \alpha_s \) by definition of \( \alpha, \alpha_s \) and \( g_s \).

To show the bound \( |E(g - g_s)(\alpha - \alpha_s)|^2 \leq (g - g_s)^2 E(\alpha - \alpha_s)^2 \) is sharp, we need to show that

\[
1 = \max \{ \rho^2 \mid (\alpha, g) : E(\alpha - \alpha_s)^2 = B^2_{\alpha}, \; E(g - g_s)^2 = B^2_{g} \},
\]

where \( B_{\alpha} \) and \( B_g \) are nonnegative constants such that \( B^2_{g} \leq E(Y - g_s)^2 \), and \( \rho^2 = \operatorname{Cor}^2(g - g_s, \alpha - \alpha_s) \). To do so, choose \( g = g_s + A \), where \( A \) is square-integrable and independent of \( X \) such that

\[
E(g - g_s)^2 = B^2_{g},
\]

then set

\[
\alpha - \alpha_s = B_{\alpha}(g - g_s)/B_g.
\]

This yields an admissible RR, and sets \( \rho^2 = 1 \).

Corollary 1 follows from observing that the bound factorizes as

\[
B^2 = S^2 C_F^2 C_D^2,
\]

where \( S^2 := E(Y - g_s)^2 E\alpha_s^2 \), and

\[
C_F^2 = \frac{E(g - g_s)^2}{E(Y - g_s)^2} = R^2_{Y - g_s} g - g_s.
\]
and
\[ C^2_D = \frac{E(\alpha - \alpha_s)^2}{E\alpha^2} = \frac{E\alpha^2 - E\alpha_s^2}{E\alpha^2} = \frac{1/E\tilde{D}^2 - 1/E\tilde{D}_s^2}{1/E\tilde{D}_s^2} = \frac{E\tilde{D}_s^2 - E\tilde{D}^2}{E\tilde{D}_s^2} = \frac{R^2_{D_s,\tilde{A}}}{1 - R^2_{D,\tilde{A}}}, \]
where \( \tilde{D} := D - E[D \mid X,A], \tilde{D}_s := D - E[D \mid X], \) and \( \tilde{A} = E[D \mid X,A] - E[D \mid X]. \)

Here we used the observation that
\[ E(\alpha - \alpha_s)^2 = E\alpha^2 + E\alpha_s^2 - 2E\alpha\alpha_s = E\alpha^2 - E\alpha_s^2, \]
holds because
\[ E\alpha\alpha_s = \frac{E\tilde{D}\tilde{D}_s}{E\tilde{D}^2E\tilde{D}_s^2} = \frac{E\tilde{D}_s^2}{E\tilde{D}^2E\tilde{D}_s^2} = \frac{1}{E\tilde{D}_s^2} = E\alpha_s. \]
The corollary now follows immediately from the definitions of \( \eta^2 \), since
\[ R^2_{Y \sim g \sim g - g_s} = \eta^2_{Y \sim A \mid D,X} \] and \( R^2_{D_s,\tilde{A}} = \eta^2_{D \sim A \mid X}. \)

In addition, we note
\[ \frac{E\alpha^2 - E\alpha_s^2}{E\alpha^2} = \frac{E\alpha^2 - E\alpha_s^2}{E\alpha_s^2} = \frac{1 - R^2_{\alpha \sim \alpha_s}}{R^2_{\alpha \sim \alpha_s}}. \]

**Remark 7** (Rationalization of any \( \rho \) and \( B^2_\alpha \), and \( B^2_\alpha \)). The above argument can be modified to show that we can achieve any \(-1 \leq \rho \leq 1, 0 \leq B^2_\alpha \), and \( 0 \leq B^2_\alpha \leq \text{Var}(Y - g_s) \) by a suitable confounding model as follows. Choose
\[ g - g_s = \mu^I_1 A \] and \( \alpha - \alpha_s = \mu^I_2 A \),
where \( A \sim N(0,I_2) \), independently of \( X \). Then set
\[ \left( \begin{array}{c} \mu^I_1 \\ \mu^I_2 \end{array} \right) = \left( \begin{array}{cc} B^2_2 & \rho B_2 B_\alpha \\ \rho B_2 B_\alpha & B^2_\alpha \end{array} \right)^{1/2}. \]

**Remark 8** (On “Natural” Confounding). The preceding remark suggests a way to generate models of “natural confounding” that are not strictly adversarial and then calculate the expected R-squared \( \rho^2 \) given some “natural priors” on the model’s hyper-parameters. Consider the confounding model: \( g - g_s = \mu^I_1 A \) and \( \alpha - \alpha_s = \mu^I_2 A \), where \( A \sim N(0,I_K) \) is \( K \)-dimensional vector of confounders. Suppose the hyperparameters \( \mu \) are drawn from \( N(0,I) \), then \( E\rho^2 = 1/K \), which attains maximal value of 1 when \( K = 1 \) and decreases to 0 as \( K \to \infty \). Therefore, there does not appear to exist a good formal way to set the level of “natural” confounding.

**B.2. Proof of Lemma [1]** The existence of the unique long RR \( \alpha \in L^2(W) \) follows from the Riesz-Frechet representation theory. To show that we can take \( \alpha_s(W^s) := E[\alpha(W) \mid W^s] \) to be the short RR, we first observe that the long RR obeys
\[ \text{Em}(W, g_s) = E g_s(W^s) \alpha(W) \]
for all \( g_s \in L^2(P_{W_s}) \). That is, the long RR \( \alpha \) can represent the linear functionals over the smaller space \( L^2(P_{W_s}) \subset L^2(P_W) \), but \( \alpha \) itself is not in \( L^2(P_{W_s}) \). Then, we decompose the long RR into the orthogonal projection \( \alpha_s \) and the residual \( e \):

\[
\alpha(W) = \alpha_s(W^s) + e(W), \quad Ee(W)g_s(W) = 0, \text{ for all } g_s \text{ in } L^2(P_{W_s}).
\]

Then

\[
Eg_s(W)\alpha(W) = E[g_s(W^s)(\alpha_s(W^s) + e(W^s))] = E[g_s(W^s)\alpha_s(W^s)].
\]

Therefore \( E[\alpha(W) \mid W^s] \) is a short RR, and it is unique in \( L^2(P_{W_s}) \) by the RF theory. We also have that \( E\alpha^2 = E\alpha_s^2 + Ee^2 \), establishing that \( E\alpha^2 \geq E\alpha_s^2 \). \( \square \)

B.3. **Proof of Lemma** \( \square \) We have from the Riesz-Frechet theory that

\[
Em(W,g_r) = Eg_r(W)\alpha(W),
\]

for all \( g_r \in \Gamma \), that is the RR \( \alpha \) continues to represent the functional over the restricted linear subspace \( \Gamma \subset L^2(P_W) \). Decompose \( \alpha \) in the orthogonal projection \( \bar{\alpha} \) and the residual \( e \):

\[
\alpha(W) = \bar{\alpha}(W) + e(W), \quad Ee(W)g_r(W) = 0, \text{ for all } g_r \in \Gamma.
\]

Then we have that

\[
Eg_r(W)\alpha(W) = Eg_r(W)\bar{\alpha}(W) + Eg_r(W)e(W) = Eg_r(W)\bar{\alpha}(W).
\]

That is, \( \bar{\alpha} \) is a RR, and it is unique in \( \Gamma \) by the RF theory. We also have that \( E\alpha^2 = E\bar{\alpha}^2 + Ee^2 \), establishing that \( E\alpha^2 \geq E\bar{\alpha}^2 \).

Analogous argument yields the result for the closed linear subsets \( \Gamma_s \) of \( L^2(P_{W_s}) \).

Here we show that \( \bar{\alpha}_s \) is given by a projection of \( \bar{\alpha} \) onto \( \Gamma_s \). Indeed, \( \bar{\alpha} \) represents the functionals over \( \Gamma_s \) but it is not itself in \( \Gamma_s \). However, its projection onto \( \Gamma_s \) therefore can also represent the functionals, using the same arguments as above. By uniqueness of the RR over \( \Gamma_s \), we must have that the projected \( \bar{\alpha} \) coincides with \( \alpha_s \). Further,

\[
E(\bar{\alpha} - \alpha_s)^2 \geq \min_{b \in \mathbb{R}} E(\bar{\alpha} - b\alpha_s)^2 \geq \min_{a \in \Gamma_s} E(\bar{\alpha} - a)^2 \Rightarrow E(\bar{\alpha} - \alpha_s)^2.
\]

This shows that the linear orthogonal projection of \( \bar{\alpha} \) on \( \alpha_s \) is given by \( \bar{\alpha}_s \). The latter means that we can decompose:

\[
E(\alpha - \alpha_s)^2 = E\alpha^2 - E\alpha_s^2. \quad \square
\]
B.4. **Proof of Theorem 2 and Corollary 2** We decompose \( L^2(P_W) \) into \( L^2(P_{W^s}) \) and its orthocomplement \( L^2(P_{W^s})^\perp \),

\[
L^2(P_W) = L^2(P_{W^s}) + L^2(P_{W^s})^\perp.
\]

So that any element \( m_s \in L^2(P_{W^s}) \) is orthogonal to any \( e \in L^2(P_{W^s})^\perp \) in the sense that

\[
E m_s(W^s)e(W) = 0.
\]

The claim of the theorem follows from

\[
Eg_1\alpha - Eg_s\alpha_s = E(g_s + g - g_s)(\alpha_s + \alpha - \alpha_s) - Eg_s\alpha_s = Eg_s(\alpha - \alpha_s) + E\alpha_s(g - g_s) + E(g - g_s)(\alpha - \alpha_s) = E(g - g_s)(\alpha - \alpha_s),
\]

using the fact that \( \alpha_s \in L^2(P_{W^s}) \) is orthogonal to \( g - g_s \in L^2(P_{W^s})^\perp \) and \( g_s \in L^2(P_{W^s}) \) is orthogonal to \( \alpha - \alpha_s \in L^2(P_{W^s})^\perp \).

**Corollary 2** follows from observing that

\[
\frac{E(g - g_s)^2}{E(Y - g_s)^2} = R_{Y - g_s}^2,
\]

as before, and from

\[
\frac{E(\alpha - \alpha_s)^2}{E\alpha_s^2} = \frac{E\alpha^2 - E\alpha_s^2}{E\alpha_s^2} = \frac{E\alpha^2 - E\alpha_s^2}{E\alpha_s^2} \frac{E\alpha^2}{E\alpha_s^2} = \frac{1 - R_{\alpha - \alpha_s}^2}{R_{\alpha - \alpha_s}^2}.
\]

The proof for the case where \( g \)'s and \( \alpha \)'s are restricted follows similarly, replacing \( L^2(P_W) \) with \( \Gamma \subset L^2(P_W) \) and \( L^2(P_{W^s}) \) with \( \Gamma_s = \Gamma \cap L^2(P_{W_s}) \), and decomposing \( \Gamma = \Gamma_s + \Gamma_s^\perp \), where \( \Gamma_s^\perp \) is the orthogonal complement of \( \Gamma_s \) relative to \( \Gamma \). The remaining arguments are the same, utilizing Lemma 2.

To show the bound is sharp we need to show that

\[
1 = \max\{\rho^2 \mid (\alpha, g) : E(\alpha - \alpha_s)^2 = B_\alpha^2, \ E(g - g_s)^2 = B_g^2\},
\]

where \( B_\alpha \) and \( B_g \) are nonnegative constants such that \( B_g^2 \leq E(Y - g_s)^2 \). To do so, choose any \( \alpha \) such such that \( E(\alpha - \alpha_s)^2 = B_\alpha^2 \), then set

\[
g - g_s = B_g(\alpha - \alpha_s)/B_\alpha.
\]

This yields an admissible long regression function, and sets \( \rho^2 = 1 \). □

**Remark 9.** We note here that distribution of observed data \( P \) can place other restrictions on the problem, restricting admissible values of \( B_\alpha^2 \) or \( B_g^2 \) or \( \rho^2 < 1 \). For example, we have \( 0 \leq g, g_s \leq 1 \) when \( 0 \leq Y \leq 1 \). This implies \( \|g - g_s\|_\infty \leq 1 \), which can potentially result in the adversarial \( \rho^2 < 1 \). □

We also note that Remark 7 applies here as well.
B.5. Proof of Theorem 3. Here the argument is similar to Chernozhukov et al. (2018c), but we provide details for completeness.

The assumptions directly imply that the candidate long RR obey \( \alpha \in L^2(P) \) with \( \| \alpha \|_{L^2} \leq C \) in each of the examples, for some constant \( C \) that depends on \( P \). By \( EY^2 < \infty \), we have \( g \in L^2(P) \). Therefore, \( |E \alpha g| < \| \alpha \|_{L^2} \| g \|_{L^2} < \infty \) in any of the calculations below.

We first verify that long RR \( \alpha \)'s can indeed represent the functionals \( g \mapsto \theta(g) := \text{Em}(W,g) \) in Examples 1, 2, 3, 5 over \( g \in L^2(P) \). In Example 4, the long RR represents the Hanh-Banach extension of the mapping \( g \mapsto \theta(g) \) to \( L^2(P) \) over \( L^2(P) \).

In Example 1, recall that \( \tilde{\ell}(X,A) := \mathbb{E}[\ell(W^s)|X,A] \). Then since \( dP(d,x,a) = \sum_{j=0}^1 1(j=d)P[D = j|X = x,A = a]dP(x,a) \) by the Bayes rule, we have

\[
\begin{align*}
\mathbb{E}g(W)\alpha(W) &= \int g(d,x,a) \frac{1(d = \tilde{d})\tilde{\ell}(x,a)}{P[D = \tilde{d}|X = x,A = a]}dP(d,x,a) \\
&= \int g(\tilde{d},x,a)d\alpha(x,a) \\
&= \mathbb{E}g(\tilde{d},X,A)\tilde{\ell}(X,A) = \mathbb{E}g(\tilde{d},X,A)\ell(W^s) = \theta(g),
\end{align*}
\]

where the penultimate equality follows by the law of iterated expectations. The claim for Example 2 follows from the claim for Example 1.

Example 3 follows by the change of measure of \( dP_{\tilde{W}} \) to \( dP_W \), given the assumed absolute continuity of the former with respect to the latter. Then we have

\[
\mathbb{E}g(W)\alpha(W) = \int g(\ell(W^s),a) - g(w^s,a)dP_{\tilde{W}}(w) = \theta(g).
\]

In Example 4, we can write for any \( g \)'s that have the properties stated in this example:

\[
\begin{align*}
\mathbb{E}g(W)\alpha(W) &= -\int \int g(w) \frac{\text{div}_d(\ell(w^s) t(w^s) f(d|x,a))}{f(d|x,a)}f(d|x,a)d\alpha(x,a) \\
&= -\int \int g(w)\text{div}_d(\ell(w^s) t(w^s) f(d|x,a))d\alpha(x,a) \\
&= -\int \int g(w)\ell(w^s) t(w^s) f(d|x,a)n_{a,x}(d) dS(d)d\alpha(x,a) \\
&\quad + \int \int \partial_d g(w)\ell(w^s) t(w^s) f(d|x,a)d\alpha(x,a) \\
&= \int \int \partial_d g(w)\ell(w^s) t(w^s) f(d|x,a)d\alpha(x,a) = \theta(g),
\end{align*}
\]

where we used the integration by parts and that \( \ell(w^s) t(w^s) f(d|x,a) \) vanishes for any \( d \) in the boundary of \( \mathcal{D}_{x,a} \).
Example 5 follows by the change of measure $dP_A \times dF_k$ to $dP_W$, given the assumed absolutely continuity of the former with respect to the latter on $\mathcal{A} \times \text{support}(\ell)$. Then we have

$$
\text{Eg}(W)\alpha(W) = \int g(\ell\left(\frac{dP_A \times d(F_1 - F_0)}{dP_W}\right)) dP_W = \int g(w, a)\ell(w) dP_A(a) d(F_1 - F_0)(w) = \theta(g).
$$

In all examples, the continuity of $g \mapsto \theta(g)$ required in Assumption 1 now follows from the representation property and from $|E\alpha| \leq ||\alpha||_{P,2}||g||_{P,2} \leq C||g||_{P,2}$.

Verification of Assumption 2 follows directly from the inspection of m-scores given in Section 5.

Note that we do not need the analytical form of the short RRs to verify Assumptions 1 or 2. However, their analytical form can be found by exactly the same steps as above, or by taking the conditional expectation.

B.6. Proof of Lemma 3 and Theorem 4. The Lemma follows from the application of Theorem 3.1 and Theorem 3.2 in Chernozhukov et al. (2018a). Valid estimation of covariance follows similarly to the proof of Theorem 3.2 in Chernozhukov et al. (2018a). The first result of Theorem 4 follows from the delta method in van der Vaart and Wellner (1996). The validity of the confidence intervals follows from using the standard arguments for confidence intervals based on asymptotic normality.

Appendix C. Benchmarks

The comparison we make here is inspired by the ideas in Imbens (2003), Altonji et al. (2005b) and Cinelli and Hazlett (2020a). We consider the observed covariates: (i) worker’s income (“income”), (ii) whether a defined pension benefits program is offered by the employer (“defined benefits”); (iii) whether a worker has an individual retirement account (“IRA”); and (iv) whether the worker’s family has a two-earner status (“two-earners”). These observed covariates were chosen because they are the most likely to be acting as proxies of the latent firm characteristics—the “matching generosity”—which, together with income, would determine the matched amount.

For a given observed covariate $X_j$, let $\Delta \eta^2_{Y \sim X_j}$ and $\Delta \eta^2_{D \sim X_j}$ denote the observed additive gains in explanatory power, with the outcome and with the treatment, when including that covariate in the regression equation; that is,

$$
\Delta \eta^2_{Y \sim X_j} := \eta^2_{Y \sim DX} - \eta^2_{Y \sim DX_{-j}}, \quad \Delta \eta^2_{D \sim X_j} := \eta^2_{D \sim X} - \eta^2_{D \sim X_{-j}},
$$

where $X_{-j}$ denotes the set of observed covariates $X$ except $X_j$. Further, let $\rho_j$ denote the observed correlation of the “effective confounders” due to the omission of $X_j$; i.e., $\rho_j := \text{Cor}(g_s - g_{s-j}, \alpha_s - \alpha_{s-j})$, where $g_{s-j}$ and $\alpha_{s-j}$ denote the CEF and the RR excluding covariate $X_j$. Finally, define the “observed bias” due to the exclusion of $X_j$ as $\Delta \theta_{s,j} := E[g_s \alpha_s] - E[g_{s-j} \alpha_{s-j}]$. Table 4 shows these
quantities for the chosen covariates “income,” “defined benefits,” “IRA” and “two-earners.” These estimates were obtained via 5-fold cross-validation, thus estimated negative gains in explanatory power, as well the respective \( \rho_j \)'s for such cases, can be effectively interpreted as zeros.

| Observed covariate | Additive gain in \( R^2 \)'s | Corr. eff. conf. | Change in estimate |
|-------------------|-----------------------------|-----------------|--------------------|
| Income            | \( \Delta \eta^2_{Y \sim X_j} \) | \( \Delta \eta^2_{D \sim X_j} \) | \( \rho_j \) | \( \Delta \hat{\theta}_{s,j} \) |
| Income            | 0.11                        | 0.01            | 0.21               | 2957               |
| Defined benefits  | -0.01                       | 0.01            | -0.08              | -287               |
| IRA               | 0.03                        | -0.02           | 0.08               | 336                |
| Two-earners       | 0.01                        | -0.02           | -0.05              | -115               |

**Table 4.** Explanatory power of observed covariates.

Note how, apart from income, the other covariates have weak explanatory power either with the outcome or with the treatment; in this sense, they are all weak “observed confounders,” and do not meaningfully change the estimated short parameter\(^{12}\). Thus here we focus on income, which is also, ex-ante, suspected to be a strong confounder. The comparison we make is the following. First we assume that the gain in explanatory power due to omitted firm characteristics \( F \) is of the same magnitude of the observed gain in explanatory power due to observing income, that is, \( \eta^2_{Y \sim FXD} - \eta^2_{Y \sim XD} = \Delta \eta^2_{Y \sim X_j} \) and \( \eta^2_{D \sim FX} - \eta^2_{D \sim X} = \Delta \eta^2_{D \sim X_j} \). Next we consider that \( \rho \), the correlation of the “effective confounders” due to \( F \), remains similar to what was observed for correlation of the “effective confounders” due to \( X_j \), namely, \( \rho_j \).

These assumptions imply a higher partial \( R^2 \) of the unobserved confounder \( F \) with the outcome, \( \eta^2_{Y \sim F|D,X} \approx 15\% \), but a much lower partial \( R^2 \) of the confounder with the treatment, \( \eta^2_{D \sim F|X} \approx 1.5\% \). Combined with the non-adversarial \( \rho \) of 0.21, we obtain a smaller overall bias than before—\( \$2,314 \), or about half of the bias implied by the previous scenario. Moreover, even allowing for adversarial confounding (setting \( \rho = 1 \)) results in bias and confidence bounds of comparable magnitude to the previous scenario, as shown in the main text.

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\(^{12}\)Readers should keep in mind that this does not mean \( X_j \) is a weak confounder in an absolute sense. For instance, \( X_j \) could interact with the latent variables \( F \), and only reveal their full explanatory power in the presence of such variables. Therefore, plausibility judgments about the gains in explanatory power due to the inclusion of \( F \), or the relative importance of \( X_j \), should take that into account.
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