A Novel Differential System Construction Method for Complex Surface Based on Aerodynamic Design

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Abstract. In this paper, a novel complex surface construction method for aerodynamic design is proposed based on differential system. In order to simplify the process of calculating and analyzing the flow field above the surface, we introduce a high-dimensional truncation method of Navier-Stokes equations to transform the complex partial differential system into an ordinary one. Concretely, we execute the Fourier expansion along some selected directions which are called wave vector sets, preserving the local properties of the solutions of Navier-Stokes equations. Further, we use the truncated ordinary differential system to describe the shape of complex surface. Experiments show that our differential system construction method for complex surface dedicated to aerodynamic design has better fitting results than the traditional linear fitting method.

Introduction

Complex surface modeling technology is one of the most critical branches of computer aided design (CAD). With the development of CAD/CAM technology, researchers have developed a lot of modeling technologies. It mainly focuses on the representation, design, display and analysis of curved surface under the environment of computer graphics system. After decades of development, it has now formed a geometric theory system with parameterized feature design represented by Bezier and b-spline method and implicit algebraic surface representation method as the main body and interpolation, fitting and approximation as the skeleton.

At present, the popular methods to generate parametric curves and surfaces by computational geometry include Bezier method, Coons method, non-uniform rational B-spline (NURBS) method, etc. Among them, NURBS becomes the most important foundation in the development trend of surface modeling technology. However, the calculation of NURBS method is complex, and improper selection of weighting factors will lead to bad parameterization and damage the surface structure. By contrast, PDE surfaces have recently emerged as a powerful modeling technique and started to gain popularity and strength for surface modeling and design [3-7]. Compared with traditional control point technique, PDE surface has many advantages. Firstly, natural physical processes are often expressed by partial differential equations, which are more useful for design and analysis. Secondly, a smooth surface with high order continuity can be defined easily by using complex partial differential equations. Moreover, each parameter in the partial differential equation generally has its physical significance, and researchers can more easily adjust and control the surface by adjusting the physical parameters. Finally, the partial differential equation unifies the geometric and physical properties, which is of great significance for modeling.
When modeling a complex surface based on aerodynamic design, we can use the surface flow field above the part to represent its shape. From a physical point of view, streamlines are the motion trajectories of air molecules, determined by a series of equations that control the motion of molecules, such as Navier-Stokes (NS) equations. These equations describe certain dynamic processes, which are rather complicated and difficult to solve. To avoid solving the NS equation directly, we simplify the original equation by using a truncation method, which is first induced in \[2\]. It transforms the NS equation into an ordinary differential equation (ODE), maintaining the primary properties \[1,8-15\].

**High-dimensional Truncation of Navier-Stokes Equations**

NS equation is a set of equations describing the motion of fluid substances such as liquid and air, which is a nonlinear partial differential equation. Consider the equations:

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u &= -\nabla p + f + \nu \Delta u \\
\text{div} u &= 0 \\
\int_{\Omega} u \, dx &= 0 \\
u \bigg|_{\Omega} &= u_{\infty}
\end{aligned}
\]  

where \(u = (u_1, u_2)\) is the velocity field on the torus \(\Omega = [0,2\pi] \times [0,2\pi]\), \(p\) is the pressure, \(\nu\) is the viscosity coefficient, and \(f\) is a periodic volume force. The equation \(\text{div} u = 0\) means that the fluid is incompressible.

We use the Fourier expansion on \(u, f, p\) in the original Navier-Stokes equation:

\[
\begin{aligned}
u(x) &= \sum_{k \neq 0} e^{ik \cdot x} \hat{u}_k = k_{\perp} \\
p(x) &= \sum_{k \neq 0} e^{ik \cdot x} \hat{p}_k \\
f(x) &= \sum_{k \neq 0} e^{ik \cdot x} \left( \frac{\hat{f}_k}{k_{\perp}} + \frac{\hat{f}_k}{k} \right)
\end{aligned}
\]

in which \(k = (k_1, k_2)\) is a wave vector and it is the basis for the Fourier expansion, which can be thought as the frequency of the unfolded wave. \(k_1, k_2\) are integer components. \(k_{\perp} = -k_2, k_1\) is the vertical direction of \(k\). \(\gamma_k, \hat{p}_k, \hat{f}_k\) are coefficients of the expansion. \(k \in L\) and \(L\) is the set of the whole wave vectors and their inverse vectors. The expansion of the vector field is equal to expand it in the direction tangent to the flow of the fluid. After a series of simplification, we can get the following form:

\[
\begin{aligned}
\gamma_k' &= f_k - \nu|k|^2 \gamma_k - i \sum_{k_1, k_2 \neq 0} \frac{(k_1 \cdot k_2)(k_2^2 - k_1^2)}{2|k_1||k_2||k|} \gamma_{k_1} \gamma_{k_2} \\
p_k &= -i \sum_{k_1, k_2 \neq 0, k_1 + k_2 = k} \frac{(k_1 \cdot k_2)(k_1^2 - k_2^2)}{|k_1||k_2||k|} \gamma_{k_1} \gamma_{k_2} + \hat{f}_k
\end{aligned}
\]

Considering that each component of the wave vector is an integer, we consider selecting a wave vector in a square \([-a, a] \times [-a, a]\), where \(a\) is a positive integer.
The system after truncation is a nonlinear system of ordinary differential equations with respect to $\gamma_k$. We select the classic Runge-Kutta approach. The entire process of the algorithm is as follows:

1. Given the length $a$ of the sides of the square, find all the pairs of integer points in it. Eliminate redundant pairs of integers with the same slope. The rest of the set consists of the selected set $L$ of wave vectors.
2. Search for the pairs of integer points which satisfy $k_1 + k_2 = k$ in $L$ and mark their positions in the set of wave vectors.
3. Initialize the variables $f_k, \nu, u_0$, the iteration step length $h$ and the maximum number of iterations $n$.
4. In each iteration, calculate the four coefficients of the classical Runge-Kutta method.
5. Plug in the value of $\gamma_k$ for each step into the expansion of $u$ to get the curve $u(x)$ at different times.

Fig.1 shows the change of flow field after truncation as the truncation radius gradually increases.

![Figure 1](image_url)

Figure 1. The change of flow field diagram with different $a$.

It can be seen from Fig.1 that the truncated system basically maintains the flow field characteristics of the original system. However, as the truncation radius increases, some singularities (such as saddle points, as can be seen from the figure) gradually appear in the truncated system. It can be concluded that the flow field of the original system is well restored after truncation, but simply increasing the truncation radius cannot improve the approximate accuracy. By selecting appropriate wave vectors, the truncated equation can guarantee the properties of the original NS equation to a large extent. That is to say, the linear term and the quadratic cross term such as $xy$ can preserve the complexity of NS equation. Therefore, we take the ordinary differential equation in the following form $\dot{\gamma}_k = a\gamma_k + \sum_{h, k_2} \gamma_{h k} \gamma_{k_2}$ as the basis to fit the shape of the surface based on hydromechanics design.

**Differential System Construction Methods for Complex Surface**

The number of velocity field directions is the order of the truncated ordinary differential system. The NS equation controls the motion of fluid on the blade surface. If separation is not considered, the solution of NS equation should stick to the curved surface. The following ordinary differential system is adopted:
\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36}
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}.
\]

(4)

where \( x, y, z \) are the coordinates of a surface in three-dimensional Euclidean space. We use the simplest first-order difference scheme, and the original system becomes as follows:

\[
\begin{pmatrix}
x_{n+1} - x_n \\
y_{n+1} - y_n \\
z_{n+1} - z_n
\end{pmatrix} = \Delta t
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36}
\end{pmatrix}
\begin{pmatrix}
x_n \\
y_n \\
z_n
\end{pmatrix}.
\]

(5)

Let \( A \) be the coefficient matrix and \( \{x_n, y_n, z_n\} \) is the set of data points, \( X = \)

\[
\begin{pmatrix}
x_1 & \cdots & x_{n-1} \\
y_1 & \cdots & y_{n-1} \\
z_1 & \cdots & z_{n-1} \\
x_1 y_1 & \cdots & x_{n-1} y_{n-1} \\
x_1 z_1 & \cdots & x_{n-1} z_{n-1} \\
y_1 z_1 & \cdots & y_{n-1} z_{n-1}
\end{pmatrix}
\]

and multiply both sides of this equation by \( X^T \). \( XX^T \) is positive definite when the set of data points satisfies certain conditions. By multiplying both sides of this equation by the inverse \( XX^T \) and we obtain the following equation:

\[
\Delta tA = \begin{pmatrix}
x_2 - x_1 & \cdots & x_{n+1} - x_n \\
y_2 - y_1 & \cdots & y_{n+1} - y_n \\
z_2 - z_1 & \cdots & z_{n+1} - z_n
\end{pmatrix} X^T (XX^T)^{-1}.
\]

(6)

In this form, \( \Delta tA \) can be solved. By plugging in the first point in the curve, the whole curve can be reconstructed through Eq. 5. The process of the differential system fitting method is as following:

Step 1. Get the surface points. Denoise and filter the data points.
Step 2. Use the presented differential system construction method to fit \( N \) primitive curve.
Step 3. Construct the overall description of the surface. Each primitive curve is represented by a corresponding parameter matrix, and the curves between the two primitive curves are obtained by the nonlinear homotopy method.

Experiment

Comparison of the Two Construction Methods

We sample data points of a rotor blade of a certain type of compressor as the original data for comparison. The data set includes 15 groups, and 2 of each group form a closed curve.

Firstly, we use the presented differential system construction method to fit 15 groups of primitive curves.
The fitting result pairs are shown in Fig.2. The red line shows the experimental data. The blue line and pink line respectively represent the fitting effect of the differential system under the first-order difference scheme and the second-order difference scheme. The green line represents the fitting effect of our method of differential system construction based on truncation. Obviously, the difference scheme basically does not affect the accuracy of fitting.

We fitted each set of leaves in the data. Matching the results, we find that the fitting method proposed in this paper performs obviously better than the general linear fitting method. To some extent, the truncation system represents the original NS system well. On the other hand, we can also see that the blade shape does follow the NS equation.

An Improved Method

The velocity distribution for each point in the flow field is inconsistent. In order to fit more closely to the flow field, we require that the velocity field of the two systems at the scatter position also fit well. In the previous experiment of rotor blade fitting, the time from the data point to the adjacent data points is taken as a unit step, and the step length between every two adjacent points is considered to be certain. By observing the original data points, we find that there are fewer sampling points in the places where the curve curvature is large, and more sampling points in the places where the curve curvature is small. Therefore, the method of selecting the step size mentioned above is to consider that the fluid has a high velocity at the place with a smooth curve and a low velocity at the place with a steep curve. If the step size in the fitting process is modified, the velocity distribution mode of the fitted system is changed. In the constant velocity field, the velocity is identical at every point, and the direction is not the same. The qualified formula is modified as follows:

\[
\frac{X_{i+1} - X_i}{\|X_{i+1} - X_i\|} = A \frac{X_{i+1} + X_i}{2}. \tag{7}
\]

where \(s\) is the entire length of arc length. We select the scatter point on the stator blade for fitting and compare it with the original linear fitting result.

Figure 3. Comparison of the fitting effects of two step size selection methods on the suction surface of the stator blade.
In Fig.3, the red line represents the original data, the black line is the fitted curve before the step size improvement, and the blue line is the fitted curve after the improvement. Obviously, the improved method is better than the original method in curve fitting. It also shows that the streamline velocity is almost constant on the stator blade.

Conclusion and Future Work

We have proposed a construction method of differential system for complex surface based on aerodynamic design. Through introducing a high-dimensional truncation of Navier-Stokes equations, we transform the complex partial differential system into an ordinary one. On the premise of preserving the local nature of the solutions, we propose a new method for selecting wave vector sets to simplify the expression of the equations. Finally, we use the truncated ordinary differential system to describe the shape of complex surface. Furthermore, we improved the step selection in the fitting method. Experimental results show that our method is superior to the traditional linear fitting method. Our future work may focus on the optimization and manufacture of compressor blades integrated with this model.

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