Abstract

Semiclassical Hamiltonian field theory is investigated from the axiomatic point of view. A notion of a semiclassical state is introduced. An "elementary" semiclassical state is specified by a set of classical field configuration and quantum state in this external field. "Composed" semiclassical states viewed as formal superpositions of "elementary" states are nontrivial only if the Maslov isotropic condition is satisfied; the inner product of "composed" semiclassical states is degenerate. The mathematical proof of Poincare invariance of semiclassical field theory is obtained for "elementary" and "composed" semiclassical states. The notion of semiclassical field is introduced; its Poincare invariance is also mathematically proved.

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1 Introduction

Different approaches to semiclassical field theory have been developed. Most of them were based on the functional integral technique: physical quantities were expressed via functional integrals which were evaluated with the help of saddle-point or stationary-phase technique. Since energy spectrum and S-matrix elements can be found from the functional integral \( \int [D\phi] \exp \left( i S[\phi] - i \frac{1}{2} \int d^4x \int d^4x' \partial_\mu \phi(x) \Phi^\dagger(x') \lambda V(x') \phi(x') \right) \), this approach appeared to be useful for the soliton quantization theory \( [1, 2, 3, 4, 5] \).

Another important partial case of the semiclassical field theory is the theory of quantization in a strong external background classical field \( [4] \) or in curved space-time \( [6] \): one decomposes the field as a sum of a classical c-number component and a quantum component. Then the theory is quantized.

The one-loop approximation \( [8, 9, 10, 11] \), the time-dependent Hartree-Fock approximation \( [12, 13] \) and the Gaussian approximation developed in \( [14, 15, 16, 17] \) may be also viewed as examples of applications of semiclassical conceptions.

On the other hand, the axiomatic field theory \( [18, 19, 20] \) tells us that main objects of QFT are states and observables. The Poincare group is represented in the Hilbert state space, so that evolution, boosts and other Poincare transformations are viewed as unitary operators.

The purpose of this paper is to introduce the semiclassical analogs of such QFT notions as states, fields and Poincare transformations. The analogs of Wightman Poincare invariance and field axioms for the semiclassical field theory are to be formulated and checked.

Unfortunately, ”exact” QFT is mathematically constructed for a restricted class of models only (see, for example, \( [21, 22, 23, 24] \)). Therefore, formal approximate methods such as perturbation theory seem to be ways to quantize the field theory rather than to construct approximations for the exact solutions of QFT equations. The conception of field quantization within the perturbation framework is popular \( [25, 26] \). One can expect that the semiclassical approximation plays an analogous role.

To construct the semiclassical formalism based on the notion of a state, one should use the equation-of-motion formulation of QFT rather than the usual S-matrix formulation. It is well-known that additional difficulties such as Stueckelberg divergences \( [27] \) and problems associated with the Haag theorem \( [28, 19, 20] \) arise in the equation-of-motion approach. There are some ways to overcome them. The vacuum divergences can be eliminated in the perturbation theory with the help of the Faddeev transformation \( [29] \). Stueckelberg divergences can be treated analogously \( [30] \) (exactly solvable models with Stueckelberg divergences have been suggested recently \( [31, 32] \)). These investigations are important for the semiclassical Hamiltonian field theory \( [33] \).

The semiclassical approaches are formally applicable to the quantum field theory models if the Lagrangian depends on the fields \( \phi \) and the small parameter \( \lambda \) as follows (see, for example, \( [4] \)):

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{1}{\lambda} V(\sqrt{\lambda} \phi),
\]

where \( V \) is an interaction potential. To illustrate the formal semiclassical ansatz for the state vector, use the functional Schrodinger representation (see, for example, \( [12, 13, 16, 17] \)). States at fixed moment of time are represented as functionals \( \psi[\phi(\cdot)] \) depending on fields \( \phi(x) \), \( x \in \mathbb{R}^d \), the field operator \( \hat{\phi}(x) \) is the operator of multiplication by \( \phi(x) \), while the canonically conjugated momentum \( \hat{\pi}(x) \) is represented as a differentiation operator \( -i\delta/\delta \phi(x) \). The functional Schrodinger equation reads

\[
i \frac{d\psi(t)}{dt} = \mathcal{H} \psi(t),
\]

where

\[
\mathcal{H} = \int dx \left[ -\frac{1}{2} \frac{\delta^2}{\delta \phi(x) \delta \phi^*(x)} + \frac{1}{2} (\nabla \phi(x))^2 + m^2 \phi^2(x) + \frac{1}{\lambda} V(\sqrt{\lambda} \phi(x)) \right]
\]

The simplest semiclassical state corresponds to the Maslov theory of complex germ in a point \( [34, 35, 36] \). It depends on the small parameter \( \lambda \) as

\[
\psi^t[\phi(\cdot)] = e^{\frac{i}{\lambda} S^t} e^{\frac{i}{\lambda} \int d\pi^t(x)[\phi(x)\sqrt{\lambda} - \Phi^t(x)]} f^t \left( \phi(\cdot) - \frac{\Phi^t(\cdot)}{\sqrt{\lambda}} \right) \equiv (K_{S^t,\Pi^t,\Phi^t} f^t)[\phi(\cdot)],
\]
where \( S', \Pi'(x), \Phi'(x), t \in \mathbb{R}, x \in \mathbb{R}^d \) are smooth real functions which rapidly damp with all their derivatives as \( x \to \infty \), \( f^t[\phi(\cdot)] \) is a \( t \)-dependent functional.

As \( \lambda \to 0 \), the substitution \((\ref{1.3})\) satisfies eq.\((\ref{1.2})\) in the leading order in \( \lambda \) if the following relations are obeyed. First, for the "action" \( S' \) one finds,

\[
\frac{dS'}{dt} = \int dx [\Pi'(x)\dot{\Phi}'(x) - \frac{1}{2}(\Pi'(x))^2 - \frac{1}{2}(\nabla \Phi'(x))^2 - m^2(\Phi'(x))^2 - V(\Phi'(x))],
\]

Second, \( \Pi', \Phi' \) obeys the classical Hamiltonian system

\[
\dot{\Phi}' = \Pi', -\dot{\Pi}' = (-\Delta + m^2)\Phi' + V'(\Phi'),
\]

Finally, the functional \( f^t \) satisfies the functional Schrödinger equation with the quadratic Hamiltonian

\[
if^t[\phi(\cdot)] = \int dx \left[ -\frac{1}{2}\delta^2_{\phi(x)}\delta^2_{\phi(x)} + \frac{1}{2}(\nabla \phi(x))^2 + \frac{m^2}{2}\phi^2(x) + \frac{1}{2}V''(\Phi'(x))\phi^2(x) \right] f^t[\phi(\cdot)].
\]

There are more complicated semiclassical states that also approximately satisfy the functional Schrödinger equation \((\ref{1.2})\). These ansätze correspond to the Maslov theory of Lagrangian manifolds with complex germs \([34, 35, 36]\). They are discussed in section 5.

However, the QFT divergences lead to the following difficulties.

It is not evident how one should specify the class of possible functionals \( f \) and introduce the inner product on such a space via functional integral. This class was constructed in \([32]\). In particular, it was found when the Gaussian functional

\[
f[\phi(\cdot)] = \text{const}\exp\left(\frac{i}{2}\int dx dy \phi(x)\phi(y)\mathcal{R}(x, y)\right)
\]

belongs to this class. The condition on the quadratic form \( \mathcal{R} \) which was obtained in \([32]\) depends on \( \Phi, \Pi \) and differs from the analogous condition in the free theory. This is in agreement with the statement of \([37, 38]\) that nonequivalent representations of the canonical commutation relations at different moments of time should be considered if QFT in the strong external field is investigated in the leading order in \( \lambda \). However, this does not lead to non-unitarity of the exact theory: the simple example has been presented in \([32]\).

Another problem is to formulate the semiclassical theory in terms of the axiomatic field theory. Section 2 deals with formulation of axioms of relativistic invariance and field for the semiclassical theory. Section 3 is devoted to construction of Poincaré transformations. In section 4 the notion of semiclassical field is investigated. More complicated semiclassical states are constructed in section 5. Section 6 contains concluding remarks.

2 Axioms of semiclassical field theory

In the Wightman axiomatic approach the main object of QFT is a notion of a state space \([18, 19, 20]\). Formula \((\ref{1.3})\) shows us that in the semiclassical field theory a state at fixed moment of time should be viewed as a set \((S, \Pi(\cdot), \Phi(\cdot), f[\phi(\cdot)])\) of a real number \( S \), real functions \( \Pi(x), \Phi(x), x \in \mathbb{R}^d \) and a functional \( f[\phi(\cdot)] \) from some class. This class depends on \( \Pi \) and \( \Phi \). Superposition of semiclassical states \((S_1, \Pi_1, \Phi_1, f_1)\) and \((S_2, \Pi_2, \Phi_2, f_2)\) is of the semiclassical type \((\ref{1.3})\) if and only if \( S_1 = S_2, \Phi_1 = \Phi_2, \Pi_1 = \Pi_2 \).

Thus, one introduces \([33, 34]\) the structure of a vector bundle (called as a "semiclassical bundle" in \([10]\)) on the set of semiclassical states of the type \((\ref{1.3})\). The base of the bundle being a space of sets \((S, \Pi, \Phi)\) ("extended phase space" \([33]\) will be denoted as \( \mathcal{X} \). The fibers are classes of functionals which depend on \( \Phi \) and \( \Pi \). Making use of the result concerning the class of functionals \([33]\), one makes
the bundle trivial as follows. Consider the $\Phi$, $\Pi$- dependent mapping $V$ which defines a correspondence between functionals $f$ and elements of the Fock space $\mathcal{F}$:

$$
V : \Psi \mapsto f, \quad \Psi \in \mathcal{F}, \quad f = f[\phi(\cdot)].
$$

as follows (see, for example, [11]). Let $\hat{R}(x,y)$ be an $\Phi$, $\Pi$ - dependent symmetric function such that its imaginary part is a kernel of a positively definite operator and the condition of ref. [3] (see eq. (3.41) of subsection 3.6) is satisfied. By $\hat{R}$ we denote the operator with kernel $\hat{R}$, while $\Gamma$ has a kernel $i^{-1}(\hat{R} - \hat{R}^*)$. The vacuum vector of the Fock space corresponds to the Gaussian functional (1.7). The operator $V$ is uniquely defined from the relations

$$
V^{-1} \phi(x)V = i(\hat{\Gamma}^{-1/2}(A^+ - A^-))(x),
$$

$$
V^{-1} \frac{i}{\delta \phi(x)} V = i(\hat{R}\hat{\Gamma}^{-1/2}A^+ - \hat{R}^*\hat{\Gamma}^{-1/2}A^-)(x).
$$

(2.1)

Here $A^\pm(x)$ are creation and annihilation operators in the Fock space.

**Definition 2.1.** A semiclassical state is a point on the trivial bundle $X \times \mathcal{F} \to X$.

An important postulate of QFT is Poincare invariance. This means that a representation of the Poincare group in the state space should be specified. For each Poincare transformation of the form

$$
x'^\mu = \Lambda^\mu_{\nu}x^\nu + a^\mu, \quad \mu, \nu = 0, d
$$

which is denoted as $(a, \Lambda)$, the unitary operator $U_{a,\Lambda}$ should be specified. The group property

$$
U_{(a_1,\Lambda_1)}U_{(a_2,\Lambda_2)} = U_{(a_1,\Lambda_1)(a_2,\Lambda_2)}
$$

with

$$(a_1,\Lambda_1)(a_2,\Lambda_2) = (a_1 + \Lambda_1a_2, \Lambda_1\Lambda_2).
$$

should be satisfied.

Formulate an analog of the Poincare invariance axiom for the semiclassical theory. Suppose that the Poincare transformation $U_{a,\Lambda}$ takes any semiclassical state $(X, f)$ to a semiclassical state $(\tilde{X}, \tilde{f})$ in the leading order in $\lambda^{1/2}$. Denote $\tilde{X} = u_{a,\Lambda}X$, $\tilde{f} = U(u_{a,\Lambda}X \leftarrow X)f$.

**Axiom 1 (Poincare invariance)**

1) the mappings $u_{a,\Lambda} : X \to X$ are specified, the group properties for them $u_{a_1,\Lambda_1}u_{a_2,\Lambda_2} = u_{(a_1,\Lambda_1)(a_2,\Lambda_2)}$ are satisfied;

2) for all $X \in X$ the unitary operators $U_{a,\Lambda}(u_{a,\Lambda}X \leftarrow X) : \mathcal{F} \to \mathcal{F}$, obeying the group property

$$
U_{a_1,\Lambda_1}(u_{(a_1,\Lambda_1)(a_2,\Lambda_2)}X \leftarrow u_{(a_2,\Lambda_2)}X)U_{a_2,\Lambda_2}(u_{(a_2,\Lambda_2)}X \leftarrow X) = U_{(a_1,\Lambda_1)(a_2,\Lambda_2)}(u_{(a_1,\Lambda_1)(a_2,\Lambda_2)}X \leftarrow X)
$$

(2.3)

are specified.

An important feature of QFT is the notion of a field: it is assumed that an operator distribution $\hat{\varphi}(x,t)$ is specified. Investigate it in the semiclassical theory. Applying the operator $\varphi(x)$ to the semiclassical state (1.3), we obtain an analogous state:

$$
e^{\frac{i}{\hbar}S^t}e^{\frac{i}{\hbar}\int d\pi(x)[\varphi(x)\sqrt{\lambda} - \Phi^i(x)]}f^i(\varphi(\cdot) - \frac{\Phi^i(\cdot)}{\sqrt{\lambda}}),
$$

where

$$
\hat{f}^i[\phi(\cdot)] = (\lambda^{-1/2}\Phi^i(x) + \phi(x))f^i[\phi(\cdot)]
$$

As $\lambda \to 0$, one has

$$
\hat{\varphi}(x, t) = \lambda^{-1/2}\Phi^i(x) + \hat{\phi}(x, t : X),
$$

where $\hat{\phi}(x, t : X)$ is a $\Pi, \Phi$-dependent operator in $\mathcal{F}$, $\Phi^i(x) \equiv \Phi(x : X)$ is a solution to the Cauchy problem for eq. (1.5). The field axiom can be reformulated as follows.
Axiom 2. For each $X \in \mathcal{X}$ the operator distribution $\phi(x, t; X) : \mathcal{F} \to \mathcal{F}$ is specified.

An important feature of the relativistic quantum field theory is the property of Poincare invariance of fields. The operator distribution $\hat{\phi}(x, t)$ should obey the following property

$$U_{a, \Lambda} \hat{\phi}(x) = \hat{\phi}(\Lambda x + a)U_{a, \Lambda}.$$  

Apply this identity to a semiclassical state $(X, f)$. In leading orders in $\lambda^{1/2}$, one obtains:

$$\lambda^{-1/2}\Phi(x : X)(u_{a, \Lambda}X, U_{a, \Lambda}(u_{a, \Lambda}X \leftarrow X)f) + \lambda^{-1/2}\Phi(\Lambda x + a : u_{a, \Lambda}X)(u_{a, \Lambda}X, U_{a, \Lambda}(u_{a, \Lambda}X \leftarrow X)f) + (u_{a, \Lambda}X, \hat{\phi}(\Lambda x + a : u_{a, \Lambda}X)U_{a, \Lambda}(u_{a, \Lambda}X \leftarrow X)f).$$

Therefore, we formulate the following axiom.

Axiom 3. (Poincare invariance of fields). The following properties are satisfied:

$$\Phi(x : X) = \Phi(\Lambda x + a : u_{a, \Lambda}X);$$

(2.4)

$$\hat{\phi}(\Lambda x + a : u_{a, \Lambda}X)U_{a, \Lambda}(u_{a, \Lambda}X \leftarrow X) = U_{a, \Lambda}(u_{a, \Lambda}X \leftarrow X)\hat{\phi}(x : X).$$

(2.5)

3 Semiclassical Poincare transformations

3.1 Construction of poincare transformations in the functional representation

1. Let us construct the mappings $u_{a, \Lambda}$ and unitary operators $U_{a, \Lambda}(u_{a, \Lambda}X \leftarrow X)$. Since any Poincare transformation is a composition of time and space translations, boost and spatial rotations,

$$(a, \Lambda) = (a^0, 0)(a, 0)(0, \exp(\alpha^k P_0^k))(0, \exp(\frac{1}{2}\theta_{sm} P^m))$$

with $\theta_{sm} = -\theta_{ms}$,

$$(\lambda^\alpha_\beta)^\nu = -g^{\lambda \alpha} \delta^\nu_\beta + g^{\mu \alpha} \delta^\nu_\beta,$$

it is sufficient to specify operators $U_{a, \Lambda}$ for these special cases and then apply a group property.

In the "exact" theory, the operator $U_{a, \Lambda}$ has the form

$$U_{a, \Lambda} = \exp[i P^0 a^0] \exp[-i P^j a^j] \exp[i \alpha^k M^{0k}] \exp[i \frac{1}{2} \lambda^{lm} \theta_{lm}].$$

(3.1)

The momentum and angular momentum operators entering to formula (3.1) have the well-known form (see, for example, [25])

$$\mathcal{P}^\mu = \int dx T^{\mu 0}(x), \quad M^{\mu \lambda} = \int dx [x^{\mu} \lambda^{0}(x) - x^\lambda T^{00}(x)],$$

(3.2)

where formally

$$T^{00} = \frac{1}{2} \tilde{\pi}^2 + \frac{1}{2} \partial_i \tilde{\phi} \partial_i \tilde{\phi} + \frac{m^2}{2} \tilde{\phi}^2 + \frac{1}{\lambda} V(\sqrt{\lambda} \tilde{\phi}), \quad T^{k0} = -\partial_k \tilde{\phi} \tilde{\pi}.$$  

We are going to apply the operator (3.1) to the semiclassical state (1.3). Note that the operators $\mathcal{P}^\mu$ and $M^{\mu \nu}$ (3.2) depend on field $\tilde{\phi}$ and momentum $\tilde{\pi}$ semiclassically,

$$\mathcal{P}^\mu = \frac{1}{\lambda} \lambda^\mu \left(\sqrt{\lambda} \tilde{\pi}(\cdot), \sqrt{\lambda} \tilde{\phi}(\cdot)\right),$$  

$$M^{\mu \nu} = \frac{1}{\lambda} \lambda^{\mu \nu} \left(\sqrt{\lambda} \tilde{\pi}(\cdot), \sqrt{\lambda} \tilde{\phi}(\cdot)\right),$$ 

5
It is convenient to consider the more general problem (cf. [35]). Let us find as \( \lambda \to 0 \) the state

\[
\exp(-iA)K_{s^0,\Pi^0,\phi^0}f^0,
\]

where \( K_{s,\Pi,\phi} \) has the form (1.3),

\[
A = \frac{1}{\lambda}A(\sqrt{\lambda}\pi(\cdot), \sqrt{\lambda}\phi(\cdot)).
\]

Note that the state functional (3.3) may be viewed as a solution to the Cauchy problem of the form

\[
i\frac{\partial\Psi^\tau}{\partial\tau} = \frac{1}{\lambda}A(\sqrt{\lambda}\pi, \sqrt{\lambda}\phi(\cdot))\Psi^\tau, \\
\Psi^0[\phi(\cdot)] = (K_{s^0,\Pi^0,\phi^0}f^0)[\phi(\cdot)]
\]

at \( \tau = 1 \). Let us look for the asymptotic solution to eq. (3.4) in the following form:

\[
\Psi^\tau[\phi(\cdot)] = (K_{s^\tau,\Pi^\tau,\phi^\tau}f^\tau)[\phi(\cdot)].
\]

Substitution of functional (3.5) to eq. (3.4) gives us the following relation:

\[
\left[-\frac{i}{\lambda}(\dot{S}^\tau - \int dx\Pi^\tau(x)\dot{\Phi}^\tau(x)) - \frac{1}{\sqrt{\lambda}}\int dx(\Pi^\tau(x)\phi(x) + \dot{\Phi}^\tau(x)i\frac{\delta}{\delta\phi(x)} + i\frac{\delta}{\delta\tau})\right]f^\tau[\phi(\cdot)] = \frac{1}{\lambda}A(\Pi^\tau(\cdot) - i\sqrt{\lambda}\frac{\delta}{\delta\phi(\cdot)}, \Phi^\tau(\cdot) + \sqrt{\lambda}\phi(\cdot))f^\tau[\phi(\cdot)].
\]

Considering the terms of the orders \( O(\lambda^{-1}) \), \( O(\lambda^{-1/2}) \) and \( O(1) \) in eq. (3.6), we obtain

\[
\dot{S}^\tau = \int dx\Pi^\tau(x)\Phi^\tau(x) - A(\Pi^\tau(\cdot), \Phi^\tau(\cdot)),
\]

\[
\dot{\Phi}^\tau(x) = \frac{\delta A(\Pi^\tau(\cdot), \Phi^\tau(\cdot))}{\delta\Pi(x)}, \quad \Pi^\tau(x) = -\frac{\delta A(\Pi^\tau(\cdot), \Phi^\tau(\cdot))}{\delta\Phi(x)}
\]

\[
\frac{i}{\sqrt{\lambda}}\frac{\partial f^\tau[\phi(\cdot)]}{\partial\tau} = \left(\int dx dy \left[\frac{1}{2}\frac{\delta}{\delta\phi(x)} - \frac{1}{2}\frac{\delta}{\delta\phi(y)}\Pi(x)\Phi(y) + \frac{1}{2}\frac{\delta}{\delta\phi(x)} - \frac{1}{2}\frac{\delta}{\delta\phi(y)}\Pi(y)\Phi(x)\right] + A_1\right)f^\tau[\phi(\cdot)].
\]

Here \( A_1 \) is a c-number quantity which depends on the ordering of the operators \( \dot{\phi} \) and \( \dot{\pi} \) and is relevant to the renormalization problem.

We see that for the cases \( \mathcal{A} = -\mathcal{P}^0a^0, \mathcal{A} = \mathcal{P}^i a^i, \mathcal{A} = -\alpha^k \mathcal{M}^{0k}, \mathcal{A} = \frac{1}{2}\theta_{sm} \mathcal{M}^{sm} \) the mapping \( u_{a,\Lambda} \) takes the initial condition for the system (3.7), (3.8) to the solution of the Cauchy problem for this system at \( \tau = 1 \). The operators \( \tilde{U}_{a,\Lambda} \) transforms the initial condition for eq. (3.3) to the solution at \( \tau = 1 \).

2. The classical mappings \( u_{a,\Lambda} \) for our partial cases are presented in table 1.

One can write down the following general formula. Let \( (a,\Lambda) \) be an arbitrary Poincare transformation. It happens that the mapping \( \tilde{u}_{a,\Lambda} : (S,\Pi,\Phi) \mapsto (\tilde{S},\tilde{\Pi},\tilde{\Phi}) \) has the following form. Let \( \Phi(x,t) \equiv \Phi(x) \) be a solution of the Cauchy problem

\[
\partial_\mu\partial^\mu\Phi(x) + m^2\Phi(x) + V'(\Phi(x)) = 0, \\
\Phi(x,0) = \Phi(x), \quad \frac{\partial}{\partial t}\Phi(x,t)|_{t=0} = \Pi(x).
\]

Denote

\[
\tilde{\Phi}(x) = \Phi(\Lambda^{-1}(x-a)).
\]

It appears that

\[
\tilde{\Phi}(x) = \Phi(x,0), \quad \tilde{\Pi}(x) = \frac{\delta}{\delta t}\tilde{\Phi}(x,t)|_{t=0}, \\
\tilde{S} = S + \int dx\theta(x')\theta(-\Lambda x + a)\theta((\Lambda x + a)(x') - \theta(-x')\theta((\Lambda x + a)(x')) \\
\times \left[\frac{1}{2}\partial_\mu\Phi(x)\partial^\mu\Phi(x) - \frac{m^2}{2}\Phi^2(x) - V(\Phi(x))\right].
\]

\[
(3.10)
\]

### Table 1: Poincare transformations in classical theory

| Element of Poincare group \((a_\tau, \Lambda_\tau)\) | Classical Poincare transformation \(u_{a_\tau, \Lambda_\tau}: (S^0, \Pi^0, \Phi^0) \mapsto (S^\tau, \Pi^\tau, \Phi^\tau)\) | Classical Lie derivative \(\delta F(S, \Pi, \Phi) = \frac{d}{dT}|_{T=0} F(S^\tau, \Pi^\tau, \Phi^\tau)\) |
|---|---|---|
| \(a_\tau = 0, \Lambda_\tau = \exp(\frac{\tau}{2}I_{sm}\theta_{sm})\); spatial rotation | \(\Phi^\tau(x) = \Phi^0(e^{-\frac{\tau}{2}I_{sm}\theta_{sm}}x)\); \(\Pi^\tau(x) = \Pi^0(e^{-\frac{\tau}{2}I_{sm}\theta_{sm}}x)\); \(S^\tau = S^0\) | \(\frac{1}{2}\theta_{lm}\delta_{lm} = \frac{1}{2}\theta_{lm} \int dx((x^l \partial_m - x^m \partial_l)\Phi(x)\frac{\delta}{\delta \Phi(x)}) + (x^l \partial_m - x^m \partial_l)\Pi(x)\frac{\delta}{\delta \Pi(x)})\) |
| \(a_\tau = 0, \Lambda_\tau = 1, a_\tau = b\tau\); spatial translation | \(\Phi^\tau(x) = \Phi^0(x - b\tau)\); \(\Pi^\tau(x) = \Pi^0(x - b\tau)\); \(S^\tau = S^0\) | \(-b^k \delta^k = \int dx(\partial_k \Phi(x)\frac{\delta}{\delta \Phi(x)}) + \partial_k \Phi(x)\frac{\delta}{\delta \Phi(x)})\) |
| \(a^0 = -\tau, a = 0; \Lambda = 1;\) evolution | Resolving operator for the Cauchy problem: \(\dot{\Phi} = \Pi^\tau;\) \(-\Pi^\tau = (-\Delta + m^2)\Phi^\tau + V'(\Phi^\tau)\); \(\dot{S}^\tau = \int dx[\Pi^\tau \dot{\Phi} - \frac{1}{2}(\Pi^\tau)^2 - \frac{1}{2}(\nabla \Phi^\tau)^2 - \frac{m^2}{2}(\Phi^\tau)^2 - V(\Phi^\tau)]\). | \(-\delta_H = \int dx[\Pi(x)\frac{\delta}{\delta \Phi(x)} - (-\Delta \Phi(x)) + m^2\Phi(x) + V'(\Phi(x)))\frac{\delta}{\delta \Pi(x)} - \int dx[\frac{1}{2}\Pi^2(x) - \frac{1}{2}(\nabla \Phi(x))^2 - \frac{m^2}{2}(\Phi^2(x) - V(\Phi(x)])\frac{\partial}{\partial S}\) |
| \(a_\tau = 0; \Lambda_\tau = \exp(-\tau n^k I_{nk});\) boost | Resolving operator for the Cauchy problem: \(\dot{\Phi} = n^k x^k \Pi^\tau;\) \(-\Pi^\tau = -\nabla x^n n^k \nabla \Phi^\tau + x^n n^k (m^2 \Phi^\tau + V'(\Phi^\tau))\); \(\dot{S}^\tau = \int dx[\Pi^\tau \dot{\Phi} - x^n n^k \frac{1}{2}(\Pi^\tau)^2 + \frac{1}{2}(\nabla \Phi^\tau)^2 + \frac{m^2}{2}(\Phi^\tau)^2 + V(\Phi^\tau)]\). | \(-\frac{n^m \delta^m}{2} = \int dx[n^m \Pi(x)\frac{\delta}{\delta \Phi(x)} - (-\partial_i x^m \partial_i \Phi(x)) + x^n m^2 \Phi(x) + x^n V'(\Phi(x)))\frac{\delta}{\delta \Pi(x)}] + \int dx x^n m^2 \Phi^2(x) - \frac{1}{2}(\nabla \Phi(x))^2 - \frac{m^2}{2}(\Phi^2(x) - V(\Phi(x)])\frac{\partial}{\partial S}\) |
For spatial translations, rotations and evolution, agreement between (3.11) and table 1 is evident. Consider the $x\tau$ functions $\Phi(x\tau)$, the infinitesimal Poincare transformations and check the algebraic analog of (2.3).

Since the operator $\tilde{\Lambda}$, the subgroup of the Poincare group with the tangent vector $\tilde{\nu}$, is a space of sections $S$, it is a space of sets $S_{\nu}$. One can also notice that the group property for eq. (3.11) is satisfied. Let us make more precise the definition of the space $S$.

### Definition 3.1
$X$ is a space of sets $(S, \Pi, \Phi)$ of a number $S$ and functions $\Pi, \Phi \in S(\mathbb{R}^d)$ such that there exists a unique solution of the Cauchy problem (3.10) such that the functions $\Phi(\Lambda x + a)$ are of the class $S(\mathbb{R}^d)$ for all $a, \Lambda$.

We see that the transformation $u_{a,\Lambda} : X \rightarrow X$ is defined.

### 3
The operators $\tilde{U}_{a,\Lambda}(u_{a,\Lambda}X \leftarrow X)$ are presented in table 2.

### However, it is not easy to check the group property (2.3). It is much more convenient to investigate the infinitesimal Poincare transformations and check the algebraic analog of (2.3).

It happens that operators $\tilde{U}_{a,\Lambda}(u_{a,\Lambda}X \leftarrow X)$ induce a Poincare group representation in a specific space. It is a space of sections $f(x; \phi(\cdot))$ of the semiclassical bundle. The operators $\tilde{U}_{a,\Lambda}$ act as

$$ (\tilde{U}_{a,\Lambda}f)(X) = \tilde{U}_{a,\Lambda}(X \leftarrow u^{-1}_{a,\Lambda}X)f(u^{-1}_{a,\Lambda}X). $$

The group property for the operators $\tilde{U}$ is equivalent to relation (2.3). Let $(a_{a,\Lambda}, \Lambda)$ be a one-parametric subgroup of the Poincare group with the tangent vector $\Lambda$ being an element of the Poincare algebra. Since the operator $\tilde{U}_{a,\Lambda}(u_{a,\Lambda}X \leftarrow X)$ takes the initial condition for the cauchy problem for equation

$$ i\dot{\phi} = \tilde{H}(A : u_{a,\Lambda}X)f,$$
to the solution of this equation. Therefore, the generator of representation (3.12) is
\[
(\tilde{H}(A)f)(X) = \frac{i}{\hbar} \frac{d}{dt}|_{t=0}(\tilde{U}_{a_{\tau}, A_{\tau}} f)(X) = [\tilde{H}(A : X) - i\delta[A]]f(X),
\]
where
\[
\delta[A] = \frac{d}{dt}|_{t=0}f(u_{a_{\tau}, A_{\tau}} X)
\]
is a Lie derivative presented in table 1. Therefore, the infinitesimal analog of the group property (2.3) is
\[
[\tilde{H}(A_1 : X) - i\delta[A_1]; \tilde{H}(A_2 : X) - i\delta[A_2]] = i(\tilde{H}([A_1; A_2] : X) - i\delta[A_1; A_2]).
\]
It follows from notations of tables 1 and 2 that relation (3.13) can be rewritten for the Poincare algebra as
\[
[M^\lambda, \tilde{P}^\mu] = 0; \quad [\tilde{M}^\mu, \tilde{P}^\nu] = i(g^{\mu\sigma}\tilde{M}^\sigma - g^{\nu\rho}\tilde{M}^\rho);
\]
for operators
\[
\tilde{M}^m = \tilde{M}^{ms} + i\delta_M^m, \quad \tilde{P}^m = \tilde{P}^{m} + i\delta_P^m, \quad \tilde{P} = \tilde{H} + i\delta_H, \quad \tilde{M} = B^k + i\delta_B
\]
It is checked by direct calculations that eqs.(3.14) are formally satisfied. However, there is a problem of divergences and renormalization which requires more careful investigations.

### 3.2 Semiclassical Poincare Transformations in Fock Space

For renormalization, let us construct the semiclassical Poincare transformations in the Fock space. They are related with the constructed operators $\tilde{U}_{a_{\Lambda}}(u_{a_{\Lambda}} X \leftarrow X)$ by the relation:
\[
\tilde{U}_{a_{\Lambda}}(u_{a_{\Lambda}} X \leftarrow X) = V_{a_{\Lambda}}(u_{a_{\Lambda}} X \leftarrow X) V^{-1}_{X}.
\]
The operator $V$ taking the Fock space vector $\Psi \in \mathcal{F}$ to the functional $f[\phi(\cdot)]$ is defined from the relation
\[
V : |0 \rangle \mapsto c \exp[i \frac{1}{2} \int dx dy \tilde{R}(x, y) \phi(x) \phi(y)]
\]
and from formulas (2.1), which can be rewritten as
\[
VA^+(x)V^{-1} = A^+(x) \equiv (\hat{\Gamma}^{-1/2} \hat{R} \phi - \hat{\Gamma}^{-1/2} \frac{1}{i} \frac{\delta}{\delta \phi})(x),
\]
\[
VA^{-}(x)V^{-1} = A^-(x) \equiv (\hat{\Gamma}^{-1/2} \hat{R} \phi - \hat{\Gamma}^{-1/2} \frac{1}{i} \frac{\delta}{\delta \phi})(x).
\]
$|c|$ can be formally found from the normalization condition
\[
|c|^2 \int D\phi \exp[i \frac{1}{2} \int dx dy \phi(x) \tilde{R}(x, y) \phi(y)]^2 = 1
\]
The argument can be chosen to be arbitrary, for example,
\[
Argc = 0.
\]
Notice that the operator $V$ is defined from the relations (3.16) - (3.19) uniquely.
Namely, any element of the Fock space can be presented via its components, vacuum state and creation operators as
\[
\Psi = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int d \mathbf{x}_1 \ldots d \mathbf{x}_n \Psi_n(\mathbf{x}_1, \ldots, \mathbf{x}_n) A^+(\mathbf{x}_1) \ldots A^+(\mathbf{x}_n) |0 >
The problem of divergence of the series is related with the problem of correctness of the functional Schrodinger representation. It is not investigated here.

Since the operators $A^\pm(x)$ satisfy usual canonical commutation relations and $A^-(x)|0 >= 0$, we obtain $VA^\pm(x) = A^\pm(x)V$.

The operator $V$ depend on $R$. It is useful to find an explicit form of the operator $V^{-1}\delta V$.

It happens that the following property is satisfied:

$$V^{-1}\delta V = -\frac{i}{2}A^+\hat{\Gamma}^{-1/2}\delta \hat{\Gamma}\hat{\Gamma}^{-1/2}A^+ + \frac{i}{2}A^-\hat{\Gamma}^{-1/2}\delta \hat{\Gamma}\hat{\Gamma}^{-1/2}A^- + A^+\hat{\Gamma}^{-1/2}\delta \hat{\Gamma}\hat{\Gamma}^{-1/2}A^+ + \frac{i}{4}Tr[\delta(\hat{\Gamma} + \hat{\Gamma}^*)\hat{\Gamma}]^{-1}$$ (3.20)

The notations of the type $A^\pm\hat{B}A^-$ are used for the operators like $\int dxdy A^+(x)\hat{B}(x,y)A^-(y)$, where $\hat{B}(x,y)$ is a kernel of the operator $\hat{B}$.

To check formula (3.20), consider the variation of the formula (2.1) if $R$ is varied:

$$[A^+(x); V^{-1}\delta V] = (\hat{\Gamma}^{1/2}\delta \hat{\Gamma}^{1/2}A^+)(x) - i(\hat{\Gamma}^{-1/2}\delta \hat{\Gamma}\hat{\Gamma}^{-1/2}A^+)(x) + i(\hat{\Gamma}^{-1/2}\delta \hat{\Gamma}\hat{\Gamma}^{-1/2}A^-)(x).$$

Therefore, formula (3.20) is correct up to an additive constant. To find it, note that

$$\delta V|0 > = [\frac{i}{2}\int dxdy\phi(x)\delta \hat{\Gamma}(x,y)\phi(y) + \delta ln c]V|0 > .$$

This relation and formula (2.1) imply

$$< 0|V^{-1}\delta V|0 > = \frac{i}{2}Tr(\delta \hat{\Gamma}^{-1}) + \delta ln c.$$

It follows from the normalization conditions (3.18) and (3.19) that $c = (det\hat{\Gamma})^{1/4}$. Therefore, $\delta ln c = \frac{i}{2}Tr\delta\hat{\Gamma}\hat{\Gamma}^{-1}$. Thus, $< 0|V^{-1}\delta V|0 > = \frac{i}{2}Tr\delta(\hat{\Gamma} + \hat{\Gamma}^*)\hat{\Gamma}^{-1}$. Formula (3.20) is checked.

It follows from formula (3.13) that the generators $H(A : X)$ in the Fock representation are related with $\tilde{H}(A : X)$ by the following relation:

$$\tilde{H}(A : X) = H(A : X) - i\delta[A] = V^{-1}_x(\tilde{H}[A : X] - i\delta[A])V_x.$$

We see that commutation relations (3.13) are invariant under change of representation.

An explicit form of operators $H(A : X)$ will be simplified if we consider the case when the quadratic form $R$ is invariant under spatial translations and rotations:

$$\tilde{R}(x,y : u(a,L)X) = \tilde{R}(L^{-1}(x-a),L^{-1}(y-a) : X).$$ (3.21)

This property implies that

$$[\partial_k; \tilde{R}] = \delta_k^l\tilde{R}; \quad [\partial_k; \hat{\Gamma}^{1/2}] = \delta_k^l\hat{\Gamma}^{1/2};$$

$$[(x^k\partial_l - x^l\partial_k); \tilde{R}] = \delta_M^k\tilde{R}; \quad [(x^k\partial_l - x^l\partial_k); \hat{\Gamma}^{1/2}] = \delta_M^k(\hat{\Gamma}^{1/2}).$$ (3.22)

The generators $H(A : X)$ are presented in table 3.

We see that renormalization is necessary since the evolution and boost generators contain divergent terms $\frac{1}{4}Tr\hat{\Gamma}$ and $\frac{1}{4}Trx^k\hat{\Gamma}$ which are to be changed by finite renormalization terms $\frac{1}{4}TrR\hat{\Gamma}$ and $\frac{1}{4}TrRx^k\hat{\Gamma}$.

Let us check the commutation relations between $\tilde{H}(A : X)$. Since the divergences arise in terms $\tilde{B}$ and $\tilde{H}$ only, so that we suppose them to be arbitrary and then find the conditions that provide Poincare invariance.
Table 3: Semiclassical Poincare transformations in Fock representation

| Element of Poincare group \((a_\tau, \Lambda_\tau)\) | Semiclassical operator \(U_{a_\tau, \Lambda_\tau}(u_{a_\tau, \Lambda_\tau} X \leftrightarrow X) : \Psi_0 \mapsto \Psi_t\) in the Fock representation takes the initial condition for the Cauchy problem to the solution of the Cauchy problem for the equation: |
|---------------------------------------------------|--------------------------------------------------------------------------------------------------|
| \(a_\tau = 0, \) \(\Lambda_\tau = \exp(\frac{\tau}{2}i \text{sm} \theta_{sm});\) spatial rotation | \(i \dot{\Psi}_\tau = -\frac{1}{2} \theta_{sm} M^{sm} \Psi_\tau;\) \(\Lambda_\lambda = 1; \) \(\tau \theta = 0, \) \(\Lambda \tau = 0;\) \(a_\tau = b \tau;\) spatial translation |
| \(a_\tau = -\tau, \) \(a = 0;\) \(\Lambda = 1;\) evolution | \(i \dot{\Psi}_\tau = H(X_\tau) \Psi_\tau;\) \(H(X) = \frac{1}{2} A^-(\hat{\mathcal{H}}^- \hat{\mathcal{H}}^0 + \hat{\mathcal{H}}^0 \hat{\mathcal{H}}^- A^+ + \frac{1}{2} A^+ \hat{\mathcal{H}}^{++}(X) A^+ + \hat{\mathcal{P}};\) \(\hat{\mathcal{H}}^{++}(X) = \hat{\mathcal{H}}^{-1/2}[\delta H \hat{\mathcal{R}} - \hat{\mathcal{R}} \hat{\mathcal{H}} - (-\Delta + m^2 + V''(\Phi(x))) \hat{\mathcal{I}}^{-1/2};\) \(\hat{\mathcal{H}}^- = (\hat{\mathcal{H}}^{++})^*;\) \(\hat{\mathcal{H}}(X) = \hat{\mathcal{H}}^{-1/2}(\hat{\mathcal{R}} \hat{\mathcal{R}}^* + (-\Delta + m^2 + V''(\Phi(x))) - \frac{1}{2} \delta H(\hat{\mathcal{R}} + \hat{\mathcal{R}}^*) + \frac{i}{2}[\delta H(\hat{\mathcal{I}}^{1/2}, \hat{\mathcal{I}}^{1/2})] \hat{\mathcal{I}}^{-1/2} - \hat{\omega};\) \(\hat{\omega} = \sqrt{-\Delta + m^2};\) \(\hat{\mathcal{P}} = \hat{\mathcal{P}}_{\text{reg}} + \frac{1}{4} Tr \hat{\mathcal{I}};\) \(\hat{\mathcal{P}}_{\text{reg}} = -\frac{1}{4} Tr[\hat{\mathcal{H}}^{++} + \hat{\mathcal{H}}^-].\) |
| \(a_\tau = 0;\) \(\Lambda_\tau = \exp(-\tau n^k \partial^k);\) boost | \(i \dot{\Psi}_\tau = n^m B^{m}(X_\tau) \Psi_\tau;\) \(B^k(X) = \frac{1}{2} A^- B^{k^-}(X) A^+ + \frac{1}{2} A^+ B^{k^+}(X) A^+ + \frac{1}{2} A^+ B^{k^+}(X) A^+ + B^k;\) \(B^{k^+}(X) = \hat{\mathcal{H}}^{-1/2}[\delta^B \hat{\mathcal{R}} - \hat{\mathcal{R}} x^k \hat{\mathcal{R}} - (-\partial_i x^k \partial_i + x^k m^2 + x^k V''(\Phi(x))) \hat{\mathcal{I}}^{-1/2};\) \(B^{k^-} = (B^{k^+})^*;\) \(B^k = \hat{\mathcal{H}}^{-1/2}[\hat{\mathcal{R}} x^k \hat{\mathcal{R}}^* + (-\partial_i x^k \partial_i + x^k m^2 + x^k V''(\Phi(x))) - \frac{1}{2} \delta^B (\hat{\mathcal{R}} + \hat{\mathcal{R}}^*) + \frac{i}{2}[\delta^B(\hat{\mathcal{I}}^{1/2}, \hat{\mathcal{I}}^{1/2})] \hat{\mathcal{I}}^{-1/2} - L_k;\) \(L_k = \frac{1}{2} \Delta^{-1/2}[\hat{\omega} x^k \hat{\omega} + (-\partial_i x^k \partial_i + x^k m^2)] \hat{\omega}^{-1/2};\) \(\hat{\mathcal{P}}_{\text{reg}} = -\frac{1}{4} Tr[B^{k^+} + B^{k^-}].\) |
be arbitrary quadratic Hamiltonians. Then the property \([\hat{H}_1, \hat{H}_2] = i\hat{H}_3\) under condition \([i\delta_1, i\delta_2] = i^2\delta_3\) means that

\[
\mathcal{H}_3^{++} = -i[\mathcal{H}_1^{++}, \mathcal{H}_2^{++}] + \mathcal{H}_2^{++}(\mathcal{H}_1^{++})^* - \mathcal{H}_1^{++}(\mathcal{H}_2^{++})^* - \mathcal{H}_1^{++}\mathcal{H}_2^{++} + \delta_1\mathcal{H}_2^{++} - \delta_2\mathcal{H}_1^{++}. \\
\mathcal{H}_3^{+-} = -i[\mathcal{H}_2^{+-}, \mathcal{H}_1^{+-}] - \mathcal{H}_1^{+-}(\mathcal{H}_2^{+-})^* + \mathcal{H}_1^{+-}(\mathcal{H}_2^{+-})^* + \delta_1\mathcal{H}_2^{+-} - \delta_2\mathcal{H}_1^{+-},
\]

(3.23)

(3.24)

\[
\mathcal{P}_3 = -\frac{i}{2}Tr[\mathcal{H}_2^{++}(\mathcal{H}_1^{++})^* - \mathcal{H}_1^{++}(\mathcal{H}_2^{++})^*] + \delta_1\mathcal{P}_2 - \delta_2\mathcal{P}_1.
\]

(3.25)

Relations (3.23), (3.24), (3.25) are treated in sense of bilinear forms on \(D(T)\).

Consider now the commutation relations.

1. The relations

\[
[\hat{P}^k, \hat{P}^l] = 0, \quad [\hat{M}^{lm}, \hat{P}^s] = i(g^{ms}\hat{P}^l - g^{ls}\hat{P}^m)
\]

are satisfied automatically since

\[
[\partial_k, \partial_l] = 0, \quad -[x^l\partial_m - x^m\partial_l, \partial_s] = g^{ms}\partial_l - g^{ls}\partial_m.
\]

2. The relation

\[
[\hat{M}^{lm}, \hat{M}^{rs}] = -i(g^{lr}\hat{M}^{ms} - g^{mr}\hat{M}^{ls} + g^{ms}\hat{M}^{lr} - g^{ls}\hat{M}^{mr})
\]

is also satisfied.

3. For the relation

\[
[\hat{P}^k, \hat{P}^0] = 0
\]

eqs (3.23) - (3.24) take the form

\[
\delta_k^{\hat{P}}\mathcal{H}^{++} - [\partial_k; \mathcal{H}^{++}] = 0, \quad \delta_k^{\hat{P}}\mathcal{H}^{+-} - [\partial_k; \mathcal{H}^{+-}] = 0,
\]

(3.26)

\[
\delta_k^{\hat{P}}\mathcal{P} = 0.
\]

(3.27)

4. For the relation

\[
[\hat{M}^{kl}, \hat{P}^0] = 0,
\]

eqs. (3.23) - (3.25) are written as

\[
\delta_k^{\hat{M}}\mathcal{H}^{++} - [x^k\partial_l - x^l\partial_k; \mathcal{H}^{++}] = 0; \quad \delta_k^{\hat{M}}\mathcal{H}^{+-} - [x^k\partial_l - x^l\partial_k; \mathcal{H}^{+-}] = 0;
\]

(3.28)

\[
\delta_k^{\hat{M}}\mathcal{P} = 0.
\]

(3.29)

5. Consider the relation

\[
[\hat{M}^{kl}, \hat{P}^s] = -ig^{ks}\hat{P}^0.
\]

We write eqs. (3.23) - (3.25) as follows:

\[
[\partial_s, \mathcal{B}^{++}] - \delta_p^{\hat{B}}\mathcal{B}^{++} = -g^{ks}\mathcal{H}^{++}, \quad [\partial_s, \mathcal{B}^{+-}] - \delta_p^{\hat{B}}\mathcal{B}^{+-} = -g^{ks}\mathcal{H}^{+-},
\]

(3.30)

\[
\delta_p^{\hat{B}}\mathcal{B} = g^{ks}\mathcal{P}.
\]

(3.31)

6. The commutation relation

\[
[\hat{M}^{lm}, \hat{M}^{k0}] = -i(g^{lk}\hat{M}^{m0} - g^{mk}\hat{M}^{l0})
\]

is equivalent to

\[
[x^l\partial_m - x^m\partial_l; \mathcal{B}^{++}] - \delta_k^{\hat{B}}\mathcal{B}^{++} = g^{lk}\mathcal{B}^{m++} - g^{mk}\mathcal{B}^{l++},
\]
They can be rewritten as follows:

\[ 0 = -i \{ B^{k+} H^{++} + H^{++}(B^{k+})^* - B^{k+}(H^{++})^* - H^{++}(B^{k+}) \} + \delta_B^k H^{++} - \delta_H B^{k+}; \]
\[ -i \partial_k = -i \{ B^{k+} H^{++} - B^{k+}(H^{++})^* + [B^{k+}; H^{++}] \} + \delta_B^k H^{++} - \delta_H B^{k+}; \]  \hspace{1cm} (3.34)

\[ 0 = -\frac{i}{2} Tr[H^{++}(B^{k+})^* - B^{k+}(H^{++})^*] + \delta_B^k \overline{B^k} - \delta_H \overline{B^k} \]  \hspace{1cm} (3.35)

and

\[ 0 = -i \{ B^{l+} H^{++} + H^{++}(B^{l+})^* - B^{l+}(H^{++})^* - H^{++}(B^{l+}) \} + \delta_B^l H^{++} - \delta_B B^{l+}; \]
\[ i(x^k \partial_l - x^l \partial_k) = -i \{ B^{l+} B^{k+} - B^{k+}(B^{l+})^* + [B^{l+}; B^{k+}] \} + \delta_B^l B^{l+} - \delta_B B^{k+}; \]  \hspace{1cm} (3.36)

\[ 0 = -\frac{i}{2} Tr[B^{l+}(B^{k+})^* - B^{k+}(B^{l+})^*] + \delta_B^l B^l - \delta_B^k B^k. \]  \hspace{1cm} (3.37)

3. Properties (3.26), (3.28), (3.30), (3.32) are obvious corollaries of relations (3.22). Properties (3.34) and (3.36) are checked by nontrivial but also direct computations.

Properties (3.27), (3.29), (3.31), (3.33), (3.35), (3.37) will be satisfied if the renormalized trace satisfies the following properties:

\[ \delta^k_{TR} \hat{T} = 0; \quad \delta^M_{TR} \hat{T} = 0; \]
\[ \delta^P_{TR} x^k \hat{\Gamma} = -\delta^k_{TR} \hat{T}; \quad \delta^M_{TR} x^k \hat{\Gamma} = \delta^k_{TR} \hat{T}; \]
\[ Tr[x^l(\delta^P_{TR} x^k \hat{\Gamma} - \hat{\Gamma} x^k \hat{\Lambda}) - x^k(\delta^P_{TR} x^l \hat{\Gamma} - \hat{\Gamma} x^l \hat{\Lambda})] + \delta^P_{TR} x^k \hat{\Gamma} - \delta^P_{TR} x^l \hat{\Gamma} = 0; \]
\[ Tr[x^l(\delta^M_{TR} x^k \hat{\Gamma} - \hat{\Gamma} x^k \hat{\Lambda}) - (\delta^M_{TR} x^l \hat{\Gamma} - \hat{\Gamma} x^l \hat{\Lambda})] + \delta^M_{TR} x^k \hat{\Gamma} - \delta^M_{TR} x^l \hat{\Gamma} = 0, \]

where \( \hat{\Lambda} = \frac{1}{2} (\hat{\mathcal{R}} + \hat{\mathcal{R}}^*) \).

Thus, algebraic commutation relations are checked.

### 3.3 Conditions of integrability

The problem of reconstructing a representation of a local Lie group from a representation of a Lie algebra ("integrability problem") is mathematically nontrivial. Different conditions of integrability were presented in [13, 14, 15, 16, 17].

The problem of reconstructing the operators \( U_g(u_g X \leftarrow X) \) and checking the group property was discussed in details in [18]. It has been shown that the operators \( U_g(u_g X \leftarrow X) \) are correctly defined under the following sufficient conditions.

Let \( h(\alpha) \) be an arbitrary smooth curve on the Poincare group.

**P1.** For self-adjoint operators

\[ A_k = L_k, \quad A_{d+k} = -i \partial_k, \quad A_{2d+k+d+1} = -i(x^k \partial_l - x^l \partial_k), \quad A_{2d+d^2+1} = \hat{\omega} \]

there exists such a positively definite operator \( T \) that

1. \( ||T^{-1/2} A_j T^{-1/2}|| < \infty, ||A_j T^{-1/2}|| < \infty. \)
2. for all \( t_1 \) there exists such a constant \( C \) that \( ||T^{1/2} e^{i\lambda_j^T} T^{-1/2}|| \leq C, ||T e^{-i\lambda_j^T} T^{-1}|| \leq C, t \in [-t_1, t_1]. \)

**P2.** The \( \alpha \)-dependent operator functions \( T \mathcal{B}^{k+}(u_{h(\alpha)} X) \) and \( T \mathcal{H}^{++}(u_{h(\alpha)} X) \) are continuous in the Hilbert-Schmidt topology \( || \cdot ||_2 \).
It is necessary to investigate the Poincare transformation properties of the operators \( \hat{b} \) are bounded if

\[ \text{(b) The functions } B_\alpha \text{ are differentiable with respect to } \omega. \]

To prove the second part of P1, represent it in the following form:

\[ \text{Notice that the following relations are satisfied:} \]

\[ \text{The } P_3. \text{ The } \alpha\text{-dependent operator functions } B^{\alpha+}(u_{\alpha}(X)) \text{ and } \mathcal{H}^{\alpha+}(u_{\alpha}(X)) \text{ are continuously differentiable with respect to } \alpha \text{ in the Hilbert-Schmidt topology.} \]

\[ \text{P4. The } \alpha\text{-dependent operator functions } B^{\alpha}(u_{\alpha}(X)), \mathcal{H}(u_{\alpha}(X)), TB^{\alpha}(u_{\alpha}(X))T^{-1}, T^{1/2}B^{\alpha}(u_{\alpha}(X))T^{-1/2}, T\mathcal{H}(u_{\alpha}(X))T^{-1}, T^{1/2}\mathcal{H}(u_{\alpha}(X))T^{-1/2} \text{ are strongly continuous.} \]

\[ \text{P5. The } \alpha\text{-dependent operator functions } T^{-1/2}\mathcal{H}(u_{\alpha}(X))T^{-1/2}, T^{-1/2}B^{\alpha}(u_{\alpha}(X))T^{-1/2}, \mathcal{H}(u_{\alpha}(X))T^{-1}, B^{\alpha}(u_{\alpha}(X))T^{-1} \text{ are continuously differentiable with respect to } \alpha \text{ in the operator norm } || \cdot || \text{ topology.} \]

\[ \text{P6. The functions } \overline{\mathcal{H}}(u_{\alpha}(X)) \text{ and } \overline{B}^{\alpha}(u_{\alpha}(X)) \text{ are continuous.} \]

The property P6 can be substituted by the following property.

\[ \text{P6'. (a) The operators } B^{\alpha+} \text{ and } \mathcal{H}^{\alpha+} \text{ are of the trace class and } TrB^{\alpha+}(u_{\alpha}(X)) \text{ and } Tr\mathcal{H}^{\alpha+}(u_{\alpha}(X)) \text{ are continuous functions of } \alpha. \]

\[ \text{(b) The functions } TrR\Gamma(X) \text{ and } TrRx^{s}\Gamma(X_{\alpha}(X)) \text{ are continuous.} \]

Let us first justify property P1.

\[ \text{Let } \hat{K} = \hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}. \]

This is a bounded self-adjoint positively definite operator without zero eigenvalues. Therefore, \( \hat{K}^{-1} \equiv T^{1/2} \) is a (non-bounded) self-adjoint operator and

\[ T = \hat{\omega}^{1/4}(x^2 + 1)\hat{\omega}^{1/2}(x^2 + 1)\hat{\omega}^{1/4}; \]

\[ T \geq c > 0 \text{ for some } c. \]

The first part of property P1 is justified as follows. One should check that the following norms are finite:

\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]

\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]

\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}(k^s x^2 - k^s x^2)\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]

\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]

\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]

\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]

\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]

\[ ||(k^s x^2 - k^s x^2)\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]

\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||, \]

where \( \hat{k} = -i\partial/\partial x^j. \)

To check this statement, it is sufficient to notice that lemma A.29 of Appendix A implies that the operators

\[ [\hat{\omega}^{\alpha}; (x^2 + 1)^{-1}]; \quad [\hat{\omega}^{\alpha}; x^s (x^2 + 1)^{-1}]; \quad [\hat{\omega}^{\alpha}; x^s x^s (x^2 + 1)^{-1}] \quad (3.39) \]

are bounded if \( \alpha \leq 1. \)

To prove the second part of P1, represent it in the following form:

\[ ||e^{iA_x T^{1/2}} e^{-iA_x T^{-1/2}}|| = ||T^{1/2} \hat{t} T^{-1/2}|| \leq C; \quad ||T^{-1/2} \hat{t}|| \leq C. \quad (3.40) \]

It is necessary to investigate the Poincare transformation properties of the operators \( \hat{k}^s \) and \( \hat{k}^j. \)

Notice that the following relations are satisfied:

\[ e^{i\theta t} \hat{x}^l e^{-i\theta t} = \hat{x}^l + \hat{k}^l \theta^{-1} t, \quad e^{i\theta t} \hat{k}^l e^{-i\theta t} = \hat{k}^l; \]

\[ e^{i\theta t} \hat{x}^l e^{-i\theta t} = \hat{x}^l + \hat{k}^l \theta^{-1} t, \quad e^{i\theta t} \hat{k}^l e^{-i\theta t} = \hat{k}^l; \]

\[ e^{i\theta t} \hat{x}^l e^{-i\theta t} = \hat{x}^l + \hat{k}^l \theta^{-1} t, \quad e^{i\theta t} \hat{k}^l e^{-i\theta t} = \hat{k}^l; \]

\[ e^{i\theta t} \hat{x}^l e^{-i\theta t} = \hat{x}^l + \hat{k}^l \theta^{-1} t, \quad e^{i\theta t} \hat{k}^l e^{-i\theta t} = \hat{k}^l; \]

\[ e^{i\theta t} \hat{x}^l e^{-i\theta t} = \hat{x}^l + \hat{k}^l \theta^{-1} t, \quad e^{i\theta t} \hat{k}^l e^{-i\theta t} = \hat{k}^l; \]

\[ e^{i\theta t} \hat{x}^l e^{-i\theta t} = \hat{x}^l + \hat{k}^l \theta^{-1} t, \quad e^{i\theta t} \hat{k}^l e^{-i\theta t} = \hat{k}^l. \]
The operators $\hat{X}^l(\tau) = e^{iL^l\tau}x^l e^{-iL^l\tau}$ have the following Weyl symbols:

$$X^1 = \frac{\omega_k}{\omega_k \cosh \tau - k^1 \sinh \tau} x^1; \quad X^\alpha = x^\alpha + \frac{k^\alpha \sinh \tau x^1}{\omega_k \cosh \tau - k^1 \sinh \tau}$$

To check the properties, it is sufficient to show that they are satisfied at $\tau = 0$ and show that the derivatives of left-hand and right-hand sides of these relations coincide.

Making use of commutation relations $[x^s, f(\hat{k})] = i \frac{\partial f}{\partial k^s}(\hat{k})$ and boundedness of the operators (3.39), we find that operators (3.40) are bounded uniformly with respect to $t \in [0, t_1]$. Property P1 is checked.

### 3.4 Choice of the operator $R$

Let us choose operator $R$ in order to satisfy properties P1-P5, P7. We will use the notions of Appendix A (subsection A.5). First, we construct such an asymptotic expansion of a Weyl symbol $R_N$ that for $R = R_N$

$$\text{deg}[\delta_B R - R \ast x^l \ast R - x^l(\omega_k^2 + V''(\Phi(x)))] > \max\{d/2, d - 1\}; \quad \text{deg}[\delta_H R - R \ast R - (\omega_k^2 + V''(\Phi(x)))] > \max\{d/2, d - 1\}. \quad (3.41)$$

Next, we will construct another asymptotic expansion of a Weyl symbol $R$ which obeys the condition $\text{Im} R > 0$ and approximately equals to $R_N$ at large $|k|$, so that eqs. (3.41) are satisfied.

This will imply that properties P2-P5, P6' are satisfied.

Let us define the expansions $R_N$ with the help of the following recursive relations. Set

$$\mathcal{R}_0 = i\omega_k; \quad \mathcal{S}_n = -\delta_H \mathcal{R}_n + \mathcal{R}_n \ast \mathcal{R}_n + \omega_k^2 + V''(\Phi(x)); \quad \mathcal{R}_{n+1} = \mathcal{R}_n + \frac{i}{2\omega_k} \mathcal{S}_n. \quad (3.42)$$

**Lemma 3.1.** The following relation is satisfied:

$$\text{deg} \mathcal{S}_n = n.$$  

**Proof.** For $n = 0$, $\mathcal{S}_0 = V''(\Phi(x))$, so that statement of lemma is satisfied. Suppose that statement of lemma is justified for $n < N$. Check it for $n = N$. One has

$$\mathcal{S}_N = \mathcal{S}_{N-1} + \mathcal{R}_N \ast \left( \frac{i}{2\omega_k} \mathcal{S}_{N-1} \right) + \left( \frac{i}{2\omega_k} \mathcal{S}_{N-1} \right) \ast \mathcal{R}_N + \left( \frac{i}{2\omega_k} \mathcal{S}_{N-1} \right) \ast \left( \frac{i}{2\omega_k} \mathcal{S}_{N-1} \right) - \frac{i}{2\omega_k} \delta_H \mathcal{S}_{N-1}.$$  

Since

$$\text{deg} \left[ \frac{i}{2\omega_k} \mathcal{S}_{N-1} \right] \ast \left( \frac{i}{2\omega_k} \mathcal{S}_{N-1} \right) - \frac{i}{2\omega_k} \delta_H \mathcal{S}_{N-1} \right] \geq \text{deg} \mathcal{S}_{N-1} + 1 = N$$

and

$$\mathcal{S}_N = \mathcal{S}_{N-1} + \mathcal{R}_N \ast \left( \frac{i}{2\omega_k} \mathcal{S}_{N-1} \right) + \left( \frac{i}{2\omega_k} \mathcal{S}_{N-1} \right) \ast \mathcal{R}_N \simeq$$

$$\mathcal{S}_{N-1} + \mathcal{R}_N \left( \frac{i}{2\omega_k} \mathcal{S}_{N-1} \right) + \left( \frac{i}{2\omega_k} \mathcal{S}_{N-1} \right) \mathcal{R}_N = 0$$

up to terms of the degree $N$, one finds

$$\text{deg} \mathcal{S}_N = N.$$  

Lemma 3.1 is proved.

Denote

$$X^l_n = -\delta_B \mathcal{R}_n + \mathcal{R}_n \ast x^l \ast \mathcal{R}_n + x^l(\omega_k^2 + V''(\Phi(x))).$$

**Lemma 3.2.** The following property is obeyed:

$$\delta_B \mathcal{S}_n - \delta_H X^l_n = -X^l_n \ast \mathcal{R}_n - \mathcal{R}_n \ast X^l_n + \mathcal{S}_n \ast x^l \ast \mathcal{R}_n + \mathcal{R}_n \ast x^l \ast \mathcal{S}_n. \quad (3.43)$$
It follows from the definition of the Weyl symbol that

\[ F_n^t = \delta B_n S_n - \delta H X_n^t + \delta H X_n^t * R_n + R_n * X_n^t - S_n * x^t * R_n - R_n * x^t * S_n. \]

One has

\[ F_n^t = (\delta B - x^t \delta H) V''(\Phi(x)) + [\delta H; \delta B] R_n - [x^t (\omega_k^2 + V''(\Phi(x))) - (\omega_k^2 + V''(\Phi(x))) - x^t * (\omega_k^2 + V''(\Phi(x)))] \]

It follows from the definition of the Weyl symbol that

\[ x^t * f(x, k) = (x^t + \frac{i \partial}{2 \partial k}) f(x, k) \]

One also has

\[ (\delta B - x^t \delta H) V''(\Phi(x)) = 0. \]

Thus,

\[ F_n^t = [\delta H; \delta B] R_n + i k^t * R_n - R_n * ik^t = \frac{\partial R_n}{\partial x^t} - \delta^t R_n. \]

However, the property

\[ \frac{\partial R_n}{\partial x^t} = \delta^t R_n \]

which means that eq. (3.21) is satisfied is checked by induction. Lemma 3.2 is proved.

**Lemma 3.3.** The following properties are satisfied:

1. \( \deg X_n^t = n. \)
2. \( \deg (X_n^t - x^t S_n) \geq n + 1. \)

**Proof.** It follows from the results of Appendix A that \( X_n^t \) is an asymptotic expansion of a Weyl symbol. Let \( \deg X_n^t = \alpha. \)

Suppose that \( \alpha < n. \) Then the left-hand side of eqs. (3.43) is of the degree \( \alpha, \) the degree of the right-hand side of eq. (3.43) is greater than or equal to \( \alpha - 1. \) In the leading order in \( 1/|k| \) the right-hand side has the form one has \((-2i\omega_k X_n^t)\) and its degree should be greater than or equal to \( \alpha. \) Therefore, \( \deg X_n^t \geq \alpha + 1. \) We obtain a contradiction.

Suppose \( \alpha > n. \) Then the left-hand side of eq. (3.43) is of the degree \( n, \) the right-hand side in the leading order in \( 1/|k| \) has the form \( 2i\omega_k x^t S_n, \) so that \( \deg S_n \) should obey the inequality \( \deg S_n \geq n + 1. \) We also obtain a contradiction.

Thus, \( \alpha = n. \) In the leading order in \( 1/|k| \) one has

\[ 0 \simeq -2i\omega_k (X_n^t - x^t S_n) \]

up to terms of the degree \( n, \) so that \( \deg (X_n^t - x^t S_n) \geq n + 1. \) Lemma 3.3 is proved.

We see that for \( N \geq \max\{d/2, d - 1\} \) the properties (3.41) are satisfied.

**Lemma 3.4.** Let \( R^{(1)} \) and \( R^{(2)} \) be asymptotic expansions of Weyl symbols, \( \deg R^{(1)} = \deg R^{(2)} = -1 \) and \( \deg (R^{(1)} - R^{(2)}) = N + 1. \) Then

\[ \deg (X^{(1)l} - X^{(2)l}) = N \]

and

\[ \deg (S^{(1)} - S^{(2)}) = N. \]

**Proof.** Denote \( R^{(1)} - R^{(2)} = D. \) Then

\[ X^{(1)l} - X^{(2)l} = -\delta B D + R^{(1)} * x^l * D + D * x^l * R^{(1)} + D * D * x^l * D. \]
We see that $\text{deg}(A^{(1)} - A^{(2)}) = 1$. The second statement is checked analogously. Lemma 3.4 is proved.

Let us construct such an asymptotic expansion $R$ that $\text{deg}(R - R_N) = N + 1$ and $\text{Im} R > 0$. We will look for $R$ as follows (cf. [34]),

$$ R = A + i \omega_k^{1/4} \exp B \star \omega_k^{1/4} \exp B \star \omega_k^{1/4}, $$

where $A$ and $B$ are real asymptotic expansions. Then

$$ \Gamma^{1/2} = \omega_k^{1/4} \exp B \star \omega_k^{1/4}; $$

$$ \Gamma^{-1/2} = \omega_k^{-1/4} \exp(-B) \star \omega_k^{-1/4} $$

are also asymptotic expansions of Weyl symbols. Choose $A$ and $B$ to be polynomials,

$$ A = \sum_{s=1}^{S_1} A_s(x, k/\omega_k), \quad B = \sum_{s=1}^{S_2} B_s(x, k/\omega_k), $$

where $S_1 = [N/2], S_2 = [N+1]/2$.

**Lemma 3.5.** There exists unique functions $A_1, \ldots, A_{S_1}, B_1, \ldots, B_{S_2}$ such that $\text{deg}(R - R_N) = N + 1$.

**Proof.** It follows from recursive relations (3.42) that

$$ \text{Re} R_N = \sum_{s=1}^{\infty} \frac{A_{N,s}(x, k/\omega_k)}{\omega_k^{2s}}, $$

$$ \text{Im} R_N = \omega_k + \sum_{s=1}^{\infty} \frac{C_{N,s}(x, k/\omega_k)}{\omega_k^{2s}}. $$

Therefore, $A_s = A_{N,s}$, so that $A$ is uniquely defined. Denote

$$ B_s = \frac{B_s(x, k/\omega_k)}{\omega_k^{2s}}. $$

Show that $B_s$ is uniquely defined. In the leading order in $1/\|k\|$, one has

$$ \text{Im} R \simeq \omega_k + 2B_1\omega_k, $$

so that $B_1 = C_{N,1}/2$. Suppose that one can choose $B_1, \ldots, B_{s-1}$ in such a way that the degree of the asymptotic expansion of a Weyl symbol

$$ E_{N,s} = \text{Im} R_N - \omega_k^{1/4} \exp(B_1 + \ldots + B_{s-1}) \star \omega_k^{1/2} \exp(B_1 + \ldots + B_{s-1}) \star \omega_k^{1/4} $$

satisfies the inequality

$$ \text{deg} E_{N,s} \geq 2s - 1. $$

Choose $B_s$ in such a way that $\text{deg} E_{N,s} \geq 2s - 1$. One has

$$ E_{N,s+1} = \text{Im} R_N - \omega_k^{1/4} \sum_{l_1=0}^{\infty} \frac{(B_1 + \ldots + B_{s-1} + B_s)^{l_1}}{l_1!} \star \omega_k^{1/2} \sum_{l_2=0}^{\infty} \frac{(B_1 + \ldots + B_{s-1} + B_s)^{l_2}}{l_2!} \star \omega_k^{1/4}. $$

Up to terms of the degree $2s + 1$, one has

$$ E_{N,s+1} \simeq \text{Im} R_N - \omega_k^{1/4} \left( \sum_{l_1=0}^{\infty} \frac{(B_1 + \ldots + B_{s-1} + B_s)^{l_1}}{l_1!} + B_s \right) \star \omega_k^{1/2} \left( \sum_{l_2=0}^{\infty} \frac{(B_1 + \ldots + B_{s-1} + B_s)^{l_2}}{l_2!} + B_s \right) \star \omega_k^{1/4} \simeq E_{N,s} - 2B_s \omega_k. $$

Since

$$ E_{N,s} = \frac{1}{\omega_k^{2s-1}} \sum_{l=0}^{\infty} F_{N,s,l}(x, k/\omega_k), $$

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one finds that

$$B_s = \frac{1}{2\omega_k^2} F_{N,s,0}(x, k/\omega_k)$$

is uniquely defined. Lemma 3.5 is proved.

Thus, we have constructed the operator $\mathcal{R}$ such that properties (3.41) are satisfied. We obtain the following theorem.

**Theorem 3.6.** Properties (3.24), P2-P5, P6′(a) are satisfied.

This theorem is a direct corollary of the results of Appendix A. Property P1 is satisfied because of construction of the operator $\mathcal{R}$. Properties P2-P5, P6′(a) are corollaries of Theorems A.31, A.32, A.33, properties (3.41) and lemmas A.8, A.9, A.19.

### 3.5 Regularization and renormalization of a trace

The purpose of this subsection is to specify functionals $\text{Tr}_R \Gamma$ and $\text{Tr}_R x^k \Gamma$ of arguments $\Phi, \Pi$ in order to satisfy properties P6′(b), (3.38). We want the renormalized trace to satisfy properties like these:

(i) $\text{Tr}_R [\tilde{\mathcal{A}}] = \text{Tr} \mathcal{A}$ if $\mathcal{A}$ is of the trace class;

(ii) $\text{Tr}_R (\mathcal{A} + \lambda \mathcal{B}) = \text{Tr}_R \mathcal{A} + \lambda \text{Tr}_R \mathcal{B}$;

(iii) $\text{Tr}_R [\mathcal{A}; \mathcal{B}] = 0$;

(iv) $\text{Tr}_R \mathcal{A} \rightarrow 0$ if $\mathcal{A} \rightarrow 0$

for such class of operators that is as wide as possible. Under these conditions, properties P6′(b) and (3.38) are satisfied. However, one cannot specify such a renormalized trace. Namely, one should have

$$\text{Tr}_R [\hat{x}_j; \mathcal{W}(\frac{k_j}{\omega_k} f(x))] = 0,$$  \hspace{1cm} (3.44)

where $f \in S(\mathbb{R}^d)$. $\mathcal{W}(A)$ is a Weyl quantization of the function $A$ (see appendix A). Property (3.44) means that

$$\text{Tr}_R (i \frac{\partial}{\partial k_i} \frac{k_j}{\omega_k} f(x)) = 0.$$  \hspace{1cm} (3.45)

Therefore,

$$\delta_{ij} \text{Tr}_R \mathcal{W}(\frac{f(x)}{\omega_k}) - l \text{Tr}_R \mathcal{W}(\frac{k_i k_j}{\omega_k^{l+2}} f(x)) = 0. \hspace{1cm} (3.45)$$

Choose $l = d$. Consider $i = j$ in eq.(3.45) and perform the summation over $i$. Making use of the relation

$$\omega_k^2 - k_i k_i = m^2,$$

we find

$$\text{Tr}_R \mathcal{W}(m^2 \omega_k^{-d-2} f(x)) = 0.$$

However, the operator with Weyl symbol $m^2 f(x) \omega_k^{-d-2}$ is of the trace class. Its trace is nonzero, provided that $\int dxf(x) \neq 0$.

However, we can introduce a notion of a trace for *asymptotic expansions of Weyl symbols*. The trace will be specified not only by operator but also by its asymptotic expansion which is not unique (see remark after definition A.6).

Let $\underline{A} = (A, \mathcal{A})$ be asymptotic expansion of a Weyl symbol. Suppose that the coefficients $A_l$ of the formal asymptotic expansion

$$\mathcal{A} \equiv \sum_{l=0}^{\infty} \omega_k^{-a-l} A_l(x, k/\omega_k)$$

are polynomial in $k/\omega_k$. One formally has

$$\text{Tr}_R \underline{A} = \sum_{l=0}^{l_0} \int \frac{dkdx}{(2\pi)^d} \frac{1}{\omega_k^{a+l}} A_l(x, k/\omega_k) + \int \frac{dkdx}{(2\pi)^d} (A(x, k) - \sum_{l=0}^{l_0} \frac{1}{\omega_k^{a+l}} A_l(x, k/\omega_k)). \hspace{1cm} (3.46)$$
For \( \alpha + l_0 + 1 > d \), the last integral in the right-hand side of eq.(3.46) converges. To specify trace, it is sufficient then to specify values of integrals

\[
I_{i_1 \ldots i_n}^{s,n} = \int \frac{dk}{\omega_k} \frac{k_{i_1}}{\omega_k} \ldots \frac{k_{i_n}}{\omega_k}
\]  
(3.47)

for \( s \leq d \) which are divergent. We will define the quantities (3.47), making use of the following argumentation.

1. We are going to specify trace in such a way that

\[
Tr_R \frac{\partial}{\partial k_i} A = 0.
\]  
(3.48)

Let

\[
A = \frac{1}{\omega_k^{s-d}} \frac{k_{j_1}}{\omega_k} \ldots \frac{k_{j_{n+1}}}{\omega_k}
\]

property (3.48) implies the following recursive relations

\[
\sum_{s=1}^{n+1} \delta_{ij} I_{j_1 \ldots j_{s-1} j_{s+1} \ldots j_{n+1}}^{s,n} = (s + n) I_{j_1 \ldots j_{n+1}}^{s,n+2}.
\]  
(3.49)

Therefore, \( I_{s,n} = 0 \) for odd \( n \), while for even \( n \) \( I_{s,n} \) is defined from eqs.(3.49), for example, \( I_{ij}^{s,2} = \frac{1}{s} \delta_{ij} I_{i}^{s,0} \).

Therefore, it is sufficient to define integrals

\[
I_{s,n}^{s,0} = \int d\omega_k \omega_k^{-s}.
\]  
(3.50)

Let us use the approach based on the dimensional regularization [49, 50]. It is based on considering integrals (3.50) at arbitrary dimensionality of space-time. Expression (3.50) appears to be a meromorphic function of \( d \). Subtracting the poles corresponding to sufficiently small positive integer values of \( d \), we obtain a finite expression.

Formally, one has

\[
I_{s,0}^{s,0} = \frac{1}{\Gamma(s/2)} \int_0^\infty d\alpha \alpha^{s/2-1} \int \frac{dk}{\omega_k} e^{-\alpha(k^2 + m^2)} = \frac{\pi^{d/2}}{\Gamma(s/2)} \frac{\Gamma(\frac{s-d}{2})}{m^{s-d}}.
\]

If \( \frac{s-d}{2} = -N \) is a nonpositive integer number, one should modify the definition of \( I_{s,0}^{s,0} \). Change \( d \to d-2\varepsilon \). One finds:

\[
I_{s,0}^{s,0} = \frac{\pi^{d/2}}{\Gamma(s/2)m^{s-d}} \frac{\Gamma(1+\varepsilon)(\pi m^2)^{-\varepsilon}}{(-N+\varepsilon)\ldots(-1+\varepsilon)} \simeq \frac{\pi^{d/2}(1-\varepsilon)^N}{\Gamma(s/2)m^{s-d}N!\varepsilon} (1 + \varepsilon(-ln(\pi m^2) + \Gamma'(1) + 1 + \ldots + N^{-1})) + O(\varepsilon).
\]

In the \( MS \) renormalization scheme [50], one should omit the term \( O(\varepsilon^{-1}) \). There is also an \( \overline{MS} \) renormalization scheme in which one omits also a fixed term of order \( O(1) \). Let us omit the term \(-ln(\pi m^2) + \Gamma'(1)\). We obtain the following renormalized value of the integral:

\[
I_{s,0}^{s,0,\text{ren}} = \frac{\pi^{d/2}}{\Gamma(s/2)m^{s-d}} \frac{(-1)^N}{N!} (1 + \ldots + 1/N),
\]

provided that \( N = \frac{d-s}{2} \) is a nonnegative integer number.

Therefore, we have defined the renormalized trace of an asymptotic expansion of a Weyl symbol by formula (3.46), provided that the coefficient functions are polynomials in \( k/\omega_k \).

Let us investigate properties of the renormalized trace. Some properties are direct corollaries of definition (3.46).

**Lemma 3.7.** The following properties are satisfied:

(i) \( Tr_R(A + \lambda B) = 0; \)
(ii) \( \text{Tr}_{R_{a,b}} \frac{\partial A}{\partial X_{a,b}} = 0; \text{Tr}_{R_{a,b}} \frac{\partial A}{\partial X_{a,b}} = 0; \)

(III) Let \( E = -\lim_{n \to \infty} A_n = A \). Then \( \lim_{n \to \infty} \text{Tr}_{R} A_n = \text{Tr}_{R} A \).

(iv) Let \( \text{deg} A > 0 \). Then \( \text{Tr}_{R} A = \text{Tr} A. \)

**Corollary.** The property AP9 is satisfied.

Let us check that \( \text{Tr}_{R_{a,b}} (A \ast B - B \ast A) = 0 \). First of all, prove the following statement.

**Lemma 3.8.** \( \text{Tr}_{R_{a,b}} A \ast B = \text{Tr}_{R} AB \).

**Proof.** Making use of eq. (A.4), we find

\[
(A \ast B)(x, k) - (AB)(x, k) = \int \frac{dp_1 dp_2 dy_1 dy_2}{(2\pi)^2} \int_0^1 da \frac{1}{\alpha} \left[ A(x + y_1; k + \alpha p_2) B(x + y_2, k_2 - \alpha p_1) \right] e^{-ip_1 y_1 - ip_2 y_2} =
\]

\[
= -i \int \frac{dp_1 dp_2 dy_1 dy_2}{(2\pi)^2} \int_0^1 da \left[ \frac{\partial}{\partial y_1} A(x + y_1; k + \alpha p_2) \frac{\partial}{\partial y_2} B(x + y_2, k_2 - \alpha p_1) - \frac{\partial}{\partial y_2} A(x + y_1; k + \alpha p_2) \frac{\partial}{\partial y_1} B(x + y_2, k_2 - \alpha p_1) \right] e^{-ip_1 y_1 - ip_2 y_2}.
\]

with

\[
C_i(x, k) = \int \frac{dp_1 dp_2 dy_1 dy_2}{(2\pi)^2} \int_0^1 da \left[ A(x + y_1; k + \alpha p_2) \frac{\partial}{\partial y_2} B(x + y_2, k_2 - \alpha p_1) - B(x + y_2, k_2 - \alpha p_1) \frac{\partial}{\partial y_1} A(x + y_1; k + \alpha p_2) \right] e^{-ip_1 y_1 - ip_2 y_2}.
\]

One also has

\[
\hat{A} \ast \hat{B} - \hat{A} \hat{B} = \frac{i}{2} \frac{\partial C^j}{\partial k^j}
\]

with

\[
\hat{C}^j(x, k) = \sum_{s=0}^{\infty} \sum_{t_1, t_2 \geq 0} \frac{(-i)^1 \gamma_2}{2 \gamma_1 + 1} \frac{\partial^{1+t_2} A}{\partial x_{1}^{\gamma_1} \gamma_1 \partial x_{1}^{t_2 + 1}} \frac{\partial^{1+t_2} B}{\partial x_{1}^{\gamma_1} \gamma_2 \partial x_{1}^{t_2 + 1}}
\]

Analogously to Appendix C, one finds that \( (C^j, \hat{C}^j) \equiv \hat{C}^j \) is an asymptotic expansion of a Weyl symbol.

It follows from lemma 3.7 that \( \text{Tr}_{R_{a,b}} \frac{\partial C^j}{\partial k^j} = 0 \). We obtain statement of lemma 3.8.

**Lemma 3.9.** For \( \text{deg} B \geq 2 \), \( \text{Tr}_{R_{a,b}} x^k \omega_k * B = \text{Tr}_{R} x^k \omega_k B \) and \( \text{Tr}_{R_{a,b}} \omega_k * B = \text{Tr}_{R} \omega_k B \).

The proof is analogous.

**Corollary 1.** The following relations are satisfied:

1. \( \text{Tr}_{R_{a,b}} (A \ast B - B \ast A) = 0; \)
2. \( \text{Tr}_{R_{a,b}} (x^k \omega_k \ast B - B \ast x^k \omega_k) = 0; \)
3. \( \text{Tr}_{R_{a,b}} (\omega_k \ast B - B \ast \omega_k) = 0. \)

**Corollary 2.** Property \( (3.38) \) is satisfied.

Thus, we have constructed functionals \( \text{Tr}_{R} x^k \Gamma = \text{Tr}_{R} x^k \Gamma \) and \( \text{Tr}_{R} \hat{\Gamma} = \text{Tr}_{R} \hat{\Gamma} \) such that properties \( (3.38) \) and Property 6(b) are satisfied.

Note that the "finite renormalization" \( [23] \) can be also be made. One can add quantities \( \Delta \text{Tr}_{R} x^k \Gamma \) and \( \Delta \text{Tr}_{R} \hat{\Gamma} \) to renormalized traces in such a way that

\[
\delta^P_k \Delta \text{Tr}_{R} x^k \Gamma = 0; \quad \delta^M_{kl} \Delta \text{Tr}_{R} \hat{\Gamma} = 0;
\]

\[
\delta^P_l \Delta \text{Tr}_{R} x^k \hat{\Gamma} = -\delta^{kl} \Delta \text{Tr}_{R} x^k \Gamma; \quad \delta^M_{kl} \Delta \text{Tr}_{R} x^k \Gamma \Gamma = \delta^{kl} \Delta \text{Tr}_{R} x^m \hat{\Gamma} - \delta^{mk} \Delta \text{Tr}_{R} x^l \hat{\Gamma};
\]

\[
\delta^P_l \Delta \text{Tr}_{R} x^k \hat{\Gamma} = \delta^F \Delta \text{Tr}_{R} x^k \Gamma - \delta^P_l \Delta \text{Tr}_{R} x^k \Gamma = 0; \quad \delta^F \Delta \text{Tr}_{R} x^k \hat{\Gamma} = 0.
\]

This corresponds to the possibility of adding the finite one-loop counterterm to the Lagrangian.

## 4 Semiclassical field

An important feature of QFT is a notion of field. In this section we introduce the notion of a semiclassical field and check its Poincare invariance.
4.1 Definition of semiclassical field

First of all, introduce the notion of a semiclassical field $\tilde{\phi}(x, t : X)$ in the functional Schrödinger representation. At $t = 0$, this is the operator of multiplication by $\phi(x)$. For arbitrary $t$, one has

$$\tilde{\phi}(x, t : X) = \tilde{U}_{-t}(X \leftarrow u_t X) \phi(x) \tilde{U}_t(u_t X \leftarrow X),$$

where $\tilde{U}_t(u_t X \leftarrow X)$ is the operator transforming the initial condition for the Cauchy problem for eq. (1.6) to the solution to the Cauchy problem.

The field operator in the Fock representation is related with $\tilde{\phi}$ by the transformation (2.1),

$$\hat{\phi}(x, t : X) = V_X^{-1} \tilde{\phi}(x, t : X) V_X.$$  \hspace{1cm} (4.1)

Making use of eq. (3.15), one finds

$$\hat{\phi}(x, t : X) = (U_H^t(X))^{-1} \hat{\phi}(x, u_t X) U_H^t(X)$$

Here $\hat{\phi}(x : X) = i(\Gamma^{-1/2}(A^+ - A^-))(x)$, while

$$U_H^t(X) \equiv U_{a, \Lambda}(u_{a, \Lambda} X \leftarrow X), \quad \Lambda = 1, a = 0, a^0 = -t.$$

Let us define $\hat{\phi}$ mathematically as an operator distribution.

Let $S(\mathbb{R}^d)$ be a space of complex smooth functions $u : \mathbb{R}^d \to \mathbb{C}$ such that

$$||u||_{l,m} = \max_{\alpha_1 + \ldots + \alpha_d \leq 1} \sup_{x \in \mathbb{R}^d} (1 + |x|)^m \left| \frac{\partial^{\alpha_1 + \ldots + \alpha_d}}{\partial x^{\alpha_1} \ldots \partial x^{\alpha_d}} u(x) \right| \to_{k \to \infty} 0.$$  \hspace{1cm} (4.1)

We say that the sequence $\{u_k\} \in S(\mathbb{R}^d)$, $k \to \infty$ tends to zero if $||u_k||_{l,m} \to_{k \to \infty} 0$ for all $l, m$.

Denote $\mathcal{D} = \{ \Psi \in \mathcal{F} |||A^+ T A^- \Psi|| < \infty \}$ (cf. [18]).

**Definition 4.1.** (cf. [20]). 1. An operator distribution $\phi$ defined on $\mathcal{D} \in \mathcal{F}$ is a linear mapping taking functions $f \in S(\mathbb{R}^d)$ to the linear operator $\phi[f] : \mathcal{D} \to \mathcal{F},$

$$\phi : f \in S(\mathbb{R}^d) \mapsto \phi[f] : \mathcal{D} \to \mathcal{F},$$

such that $||\phi[f_n]|| \to_{n \to \infty} 0$ if $f_n \to_{n \to \infty} 0$.

2. A sequence of operator distributions $\phi_n$ is called convergent to the operator distribution $\phi$ if

$$||\phi_n[f] \Phi - \phi[f] \Phi|| \to_{n \to \infty} 0$$

for all $\Phi \in \mathcal{D}, f \in S(\mathbb{R}^d)$.

We will write

$$\phi[f] \equiv \int d x \phi(x) f(x), \quad x \in \mathbb{R}^d.$$  \hspace{1cm} (4.2)

Consider the mapping $f \mapsto \phi_t \{ f \}, f \in S(\mathbb{R}^d)$ of the form

$$\phi_t \{ f : X \} = \int d x \tilde{\phi}(x, t : X) f(x).$$

It follows from the results of [18] that $\phi_t$ is an operator distribution being continuous with respect to $t$.

Consider the mapping $f \mapsto \phi[f], f \in S(\mathbb{R}^{d+1})$ of the form

$$\phi[f : X] = \int d t \phi_t \{ f(\cdot, t) : X \}.$$  \hspace{1cm} (4.2)

Analogously, note also that $\phi$ is an operator distribution.
4.2 Poincare invariance of the semiclassical field

4.2.1 Algebraic properties

To check the property of Poincare invariance, notice that it is sufficient to check it for partial cases: spatial translations, rotations, evolution, boost, since any Poincare transformation can be presented as a composition of these transformations. Let $g_B(\tau) = (a(\tau), \Lambda(\tau))$ be a one-parametric subgroup of Poincare group corresponding to the element $B$ of the Poincare algebra. The Poincare invariance property can be rewritten as

$$\hat{\phi}[f : X] = (U_B^\tau(X))\phi[v_{g_B(\tau)}f : u_{g_B(\tau)}X]U_B^\tau[X],$$

(4.2)

where

$$(v_{g_B(\tau)}f)(x) = f(\Lambda^{-1}(\tau)(x - a(\tau))).$$

Obviously, $v_{g_1}v_{g_2} = v_{g_1g_2}$.

Let us check relation (4.2). It is convenient to reduce the group property to an algebraic property. The formal derivative with respect to $\tau$ of the right-hand side of eq.(4.2) is

$$(U_B^\tau(X))^\dagger \{i[H(\tau) : u_{g_B(\tau)}X]; \phi[v_{g_B(\tau)}f : u_{g_B(\tau)}X]] + \partial\partial\phi[v_{g_B(\tau)}f : u_{g_B(\tau)}X]U_B^\tau(X)$$

(4.3)

If the quantity (4.3) vanishes, the property (4.2) will be satisfied since it is obeyed at $\tau = 0$. Making use of the group property $g_B(\tau + \delta\tau) = g_B(\delta\tau)g_B(\tau)$, we find that vanishing of expression (4.3) is equivalent to the property:

$$\partial\partial\{\tau = 0\phi[v_{g_B(\tau)}f : u_{g_B(\tau)}X] - i[\phi[f : X]; H(\tau : X)] = 0.$$

(4.4)

We obtain the following lemma.

**Lemma 4.1.** Let the bilinear form (4.4) vanish on $D$. Then the property (4.2) is satisfied on $D$.

**Proof.** Consider the matrix element

$$\chi^\tau = (U_B^\tau(X)\Psi_1, \phi[v_{g_B(\tau)}f : u_{g_B(\tau)}X]U_B^\tau(X)\Psi_2) - (\Psi_1, \hat{\phi}[f : X]\Psi_2),$$

where $\Psi_1, \Psi_2 \in D$. Show it to be differentiable with respect to $\tau$. Let us check that for $\Psi \in D$, the vector $\phi[v_{g_B(\tau)}f : u_{g_B(\tau)}X]\Psi$ is strongly continuously differentiable with respect to $\tau$.

One has:

$$\phi[v_{g_B(\tau + \delta\tau)}f : u_{g_B(\tau + \delta\tau)}X]\Psi = \phi[v_{g_B(\tau + \delta\tau)}f : u_{g_B(\tau)}X] \Psi + \phi[v_{g_B(\tau + \delta\tau)}f : u_{g_B(\tau)}X] \Psi.$$ 

It follows from (4.8) that the first term tends to $\phi[\partial_\tau|_t v_{g_B(t)}v_{g_B(\tau)}f : u_{g_B(\tau)}X]\Psi$, while the second term tends to $\delta[B] \phi[v_{g_B(\tau)}f : u_{g_B(\tau)}X]\Psi$.

Notice that

$$\sum_{\delta\tau}^\tau \phi[v_{g_B(\tau + \delta\tau)}f : u_{g_B(\tau + \delta\tau)}X]U_B^{\tau + \delta\tau}(X)\Psi_1; \frac{U_B^{\tau + \delta\tau}(X) - U_B^{\tau}}{\delta\tau}\Psi_2) = (U_B^{\tau + \delta\tau}(X)\Psi_1; (\phi[v_{g_B(\tau + \delta\tau)}f : u_{g_B(\tau + \delta\tau)}X] - \phi[v_{g_B(\tau)}f : u_{g_B(\tau)}X])U_B^\tau(X)\Psi_2) + ((U_B^{\tau + \delta\tau}(X) - U_B^\tau(X))\Psi_1; \phi[v_{g_B(\tau)}f : u_{g_B(\tau)}X]U_B^\tau(X)\Psi_2).$$

This quantity tends as $\delta\tau \to 0$ to the matrix element of the bilinear form (4.3) and vanishes under condition (4.4). Lemma 4.1 is proved.
4.2.2 Check of invariance

One should check property (4.2) for spatial translations and rotations, evolution and boost transformations.

For spatial translations and rotations, property (4.2) reads:

\[ \hat{\phi}(x, t : X) = U_{0,a,L}^{-1} \phi(Lx + a, t : u_{0,a,L}X) U_{0,a,L} \]  

(4.5)

It follows from commutativity of \( U_{0,a,L} \) and \( U_t \) and table 3 that property (4.3) is satisfied.

For evolution operator, property (4.2) is rewritten as:

\[ \hat{\phi}(x, t : X) = (U_H^t(X))^{-1} \phi(x, t - \tau : u_tX) U_H^t(X) \]  

(4.6)

Relation (1.4) is a direct corollary of definition (1.1) and group property for evolution operators.

Consider now the \( n \)-boost transformation. Check property (4.4). It can be presented as

\[ [\hat{B}_k; \hat{\phi}(x, t; X)] = -i(x^k \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_k})\hat{\phi}(x, t : X) \]

or

\[ [B_k(X); (U_H^t(X))^{-1} \phi(x : u_tX) U_H^t(X)] + i\delta_B \{(U_H^t(X))^{-1} \phi(x : u_tX) U_H^t(X)\} = -i(x^k \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_k})(U_H^t(X))^{-1} \phi(x : u_tX) U_H^t(X) \]  

(4.7)

Let us make use of the property

\[ U_H^t(X)B_k(X) = i(\delta_B U_H^t(X)) + [B_k(u_tX) - tP_k(u_tX)]U_H^t(X) \]  

(4.8)

which can be checked by multiplication by \((U_H^t(X))^{-1}\) and differentiation with respect to \( t \) in a weak sense (cf. [3]). We take relation (4.7) to the form

\[ [\hat{B}_k(Y) - \hat{P}_k(Y)t; \phi(x : Y)] = x^k [\hat{H}(Y); \phi(x : Y)] - it \frac{\partial \phi(x : Y)}{\partial x_k}, \]

where \( Y = u_tX \). The property

\[ i \frac{\partial \phi(x : Y)}{\partial x_k} = [\hat{P}_k(Y); \phi(x : Y)] \]

is a corollary of relation (4.22). The relation

\[ [\hat{B}_k(Y) - x^k \hat{H}(Y); \phi(x : Y)] = 0 \]

is also checked by direct calculation.

Thus, we have obtained that the invariance property (2.5) is satisfied.

5 Remarks on composed semiclassical states

In the soliton quantization theory and in gauge field theories, the zero-mode problem arises [2]. To resolve it, one can consider the superposition of the "elementary" quantum field semiclassical states (1.4) of the form (cf. [30])

\[ \psi[\varphi(\cdot)] = \int \frac{d\alpha}{\lambda^{d/2}} e^{\frac{1}{\lambda} S(\alpha)} e^{\frac{1}{\lambda} \Pi(\alpha ; X)(\varphi(\cdot) - \Phi(\alpha, x))} g(\alpha, \varphi(\cdot) - \frac{\Phi(\alpha, \cdot)}{\sqrt{\lambda}}) \]  

(5.1)

where \( \alpha \in \mathbb{R}^k, S(\alpha), \Pi(\alpha ; x), \Phi(\alpha, x) \) are smooth functions. Calculate (formally) the functional integral for \( \psi, \phi \):

\[ \int \frac{d\alpha}{\lambda^{d/2}} \int D\varphi e^{-\frac{1}{\lambda} S(\alpha)} e^{\frac{1}{\lambda} \Pi(\alpha ; x)(\varphi(\cdot) - \Phi(\alpha, x))} g(\alpha, \varphi(\cdot) - \frac{\Phi(\alpha, \cdot)}{\sqrt{\lambda}}) \]

(5.2)
After substitution $\gamma = \alpha + \sqrt{\lambda} \beta$, $\varphi(\cdot) = \frac{\Phi(\alpha)}{\sqrt{\lambda}} + \phi(\cdot)$ we obtain as $\lambda \to 0$:  

$$
(\psi, \psi) \simeq \int d\alpha d\beta e^{\frac{1}{\sqrt{\lambda}} \left( \partial S_{\alpha} \right) - \int d\alpha \Pi(\alpha, x) \frac{\partial \Phi(\alpha, x)}{\partial x} \right)} \frac{1}{2} \partial \beta_{\alpha} \frac{\partial S_{\alpha}}{\partial x} - \int d\alpha \Pi(\alpha, x) \frac{\partial \Phi(\alpha, x)}{\partial x} \right) \beta_{\lambda} 
$$

$$
\int D\phi^*(\alpha, \phi(\cdot)) e^{i\beta \cdot \int d\alpha \left( \frac{\partial \Pi(\alpha, x)}{\partial x} \phi(\cdot \right) - \frac{\partial \Phi(\alpha, x)}{\partial x} \frac{1}{i \delta \phi(\cdot \right)} g(\alpha, \phi(\cdot))} \tag{5.2}
$$

The condition

$$
\frac{\partial S}{\partial \alpha} = \int d\alpha \Pi(\alpha, x) \frac{\partial \Phi(\alpha, x)}{\partial \alpha} \tag{5.3}
$$

should be satisfied. Otherwise, the integral (5.2) will be exponentially small as $\lambda \to 0$, so that state (5.1) will be trivial. Under condition (5.3), one has

$$
(\psi, \psi) \to \lambda \to 0 \int d\alpha d\beta \int D\phi^*(\alpha, \phi(\cdot)) e^{i\beta \cdot \int d\alpha \left( \frac{\partial \Pi(\alpha, x)}{\partial x} \phi(\cdot \right) - \frac{\partial \Phi(\alpha, x)}{\partial x} \frac{1}{i \delta \phi(\cdot \right)} g(\alpha, \phi(\cdot))} = 0 \tag{5.4}
$$

To specify the composed semiclassical state in the functional representation, one should:

- (i) specify the smooth functions $(S(\alpha), \Pi(\alpha, x), \Phi(\alpha, x)) \equiv X(\alpha)$ obeying eq. (5.3) (determine the $k$-dimensional isotropic manifold in the extended phase space $\mathcal{X}$);

- (ii) specify the $\alpha$-dependent functional $g(\alpha, \phi(\cdot))$.

The inner product of composed semiclassical states is given by expression (5.4).

Since the inner product (5.4) may vanish for nonzero $g$, one should factorize the space of composed semiclassical states. Such functionals $g$ that obey the property

$$
\int d\alpha (g^*(\alpha, \cdot) \prod_i \delta \left[ \int d\alpha \left( \frac{\partial \Pi(\alpha, x)}{\partial x} \phi(\cdot \right) - \frac{\partial \Phi(\alpha, x)}{\partial x} \frac{1}{i \delta \phi(\cdot \right) \right] g(\alpha, \cdot) = 0 \tag{5.5}
$$

should be set to be equal to zero, $g \sim 0$.

One can determine the Poincare transformation of the composed semiclassical state as follows. The transformation of $(S(\alpha), \Pi(\alpha, \cdot), \Phi(\alpha, \cdot))$ is $u_{a, \Lambda}(S(\alpha), \Pi(\alpha, \cdot), \Phi(\alpha, \cdot))$. The transformation of $g(\alpha, \phi(\cdot))$ is

$$
\tilde{U}_{a, \Lambda}(u_{a, \Lambda}(S, \Pi, \Phi)) \leftarrow (S, \Pi, \Phi)) g(\alpha, \phi(\cdot)).
$$

One should check that the inner product entering to eq. (5.3) is invariant under Poincare transformations. This will also imply that equivalent states are taken to equivalent.

Since the functional Schrodinger representation is not well-defined, let us consider the Fock representation. One should then specify the $\alpha$-dependent Fock vector $Y(\alpha) = V^{-1} g(\alpha, \cdot)$ instead of the $\alpha$-dependent functional $g(\alpha, \phi(\cdot))$. Making use of formulas (2.1), we find that the inner product (5.4) takes the form

$$
\left( \begin{pmatrix} \Lambda^k \\ Y(\cdot) \end{pmatrix}, \begin{pmatrix} \Lambda^k \\ Y(\cdot) \end{pmatrix} \right) = \int d\alpha d\beta (Y(\alpha), e^{\beta \cdot \int d\alpha (B_s(\alpha, \cdot) A^+(\cdot)) - B_s^*(\alpha, \cdot) A^+(\cdot)) Y(\alpha)) \tag{5.6}
$$

where

$$
B_s(\alpha, \cdot) = \hat{\Gamma}^{-1/2} \hat{\mathcal{R}} \frac{\partial \Phi(\alpha, \cdot)}{\partial \alpha} - \frac{\partial \Pi(\alpha, \cdot)}{\partial \alpha}, \tag{5.7}
$$

$$
\hat{\Gamma} = \hat{\Gamma}(\Phi(\alpha, \cdot), \Pi(\alpha, \cdot)), \hat{\mathcal{R}} = \hat{\mathcal{R}}(\Phi(\alpha, \cdot), \Pi(\alpha, \cdot)). \tag{5.8}
$$

If the isotropic manifold $(\Phi(\alpha, \cdot), \Pi(\alpha, \cdot))$ is non-degenerate, the functions $B_s(\alpha, x)$ are linearly independent.

The Poincare transformation of the composed semiclassical state

$$
\left( \begin{pmatrix} \{X(\alpha)\} \\ Y(\alpha) \end{pmatrix} \right)
$$

is

$$
\left( \begin{pmatrix} u_{a, \Lambda} X(\alpha) \\ U_{a, \Lambda} (u_{a, \Lambda} X(\alpha) \leftarrow X(\alpha)) Y(\alpha) \end{pmatrix} \right)
$$
Consider the quantity

\[
(B_s, B_l) - (B_l, B_s) = \left( \frac{\partial \Phi}{\partial \alpha_s}, (\hat{R}^* - \hat{R}) \hat{\Gamma}^{-1} \frac{\partial \Pi}{\partial \alpha_l} \right) - \left( \frac{\partial \Phi}{\partial \alpha_l}, (\hat{R}^* - \hat{R}) \hat{\Gamma}^{-1} \frac{\partial \Pi}{\partial \alpha_s} \right) = i \int d\mathbf{x} \left( \frac{\partial \Phi(\alpha, \mathbf{x})}{\partial \alpha_s} \frac{\partial \Pi(\alpha, \mathbf{x})}{\partial \alpha_l} - \frac{\partial \Phi(\alpha, \mathbf{x})}{\partial \alpha_l} \frac{\partial \Pi(\alpha, \mathbf{x})}{\partial \alpha_s} \right).
\]

Differentiating (5.3) with respect to \(\alpha_l\), we obtain that quantity (5.8) vanishes. Thus, operators \(\beta_s \int d\mathbf{x} (B_s(\alpha, \mathbf{x}) A^+(\mathbf{x}) - B_l^*(\alpha, \mathbf{x}) A^+(\mathbf{x}))\) commute each other.

It follows from the results of [48] that the inner product (5.7) is correctly defined, while Poincare transformations of composed semiclassical states satisfy the group property and conserve the inner product (5.7).

6 Conclusions

In this paper a notion of a semiclassical state is introduced. "Elementary" semiclassical state are specified by a set \((X, \Psi)\) of classical field configuration \(X\) (point on the infinite-dimensional manifold \(\mathcal{X}\), see section 2 and subsection 3.2) and element \(\Psi\) of the space \(\mathcal{F}\). Set of all "elementary" semiclassical states may be viewed as a semiclassical bundle.

The physical meaning of classical field \(X\) is evident. Discuss the role of \(\Psi\). In the soliton quantization language [1, 2] \(\Psi\) specifies whether the quantum soliton is in the ground or excited state. For the Gaussian approach [14, 15, 16, 17], \(\Psi\) specifies the form of the Gaussian functional, while for QFT in the strong external classical field [6, 7] \(\Psi\) is a state of a quantum field in the classical background.

The "composed" semiclassical states have been also introduced (section 5). They can be viewed as superpositions of "elementary" semiclassical states and are specified by the functions \((X(\tau), \Psi(\tau))\) defined on some domain of \(\mathbb{R}^k\) with values on the semiclassical bundle.

Not arbitrary superposition of elementary semiclassical states is nontrivial. The isotropic condition (5.3) should be satisfied. Moreover, the inner product of the "composed" semiclassical states (eq.(5.6)) is degenerate, so that there is a "gauge freedom" (5.3) in specifying composed semiclassical states.

The composed semiclassical states are used [36] in soliton quantization, since there are translation zero modes and solitons can be shifted. They are useful if there are conserved integrals of motion like charges. The correspondence between composed and elementary semiclassical states in QFT resembles the relationship between WKB and wave packet approximations in quantum mechanics.

An important feature of QFT is the property of Poincare invariance. In this paper an explicit check of this property is presented for semiclassical QFT. The Poincare transformations of elementary and composed semiclassical states have been constructed as follows. First, the simplest Poincare transformations like spatial translations and rotations, evolution and boost are considered. The infinitesimal transformations are investigated, the Lie algebraic commutation relations have been checked and the group properties have been justified.

For the "composed" states, conservation of the degenerate inner product and isotropic condition under Poincare transformation have been checked.

An important feature of QFT is a notion of field. In this paper this notion is introduced for semiclassical QFT. The property of Poincare invariance of semiclassical field is checked.

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A Weyl calculus

The purpose of this appendix is to investigate some properties of Weyl symbols of operators which are useful in justification of properties P1-P6.
\section*{A.1 Definition of Weyl symbol}

First of all, remind the definition of Weyl symbol of operator (see, for example, [34, 51]). Let $A(x, k), \ x, k \in \mathbb{R}^d$ be a classical observable depending on coordinates $x = (x_1, \ldots, x_d)$ and momenta $k = (k_1, \ldots, k_d)$. To specify the corresponding quantum observable $\hat{A}$ (to “quantize” the observable $A$), one should present it as a superposition of exponents,

$$A(x, k) = \int d\alpha d\beta \tilde{A}(\alpha, \beta) e^{i\alpha k + i\beta x}$$

and set

$$\hat{A} = \int d\alpha d\beta \tilde{A}(\alpha, \beta) e^{i\alpha \hat{k} + i\beta \hat{x}}$$

Applying the formula for inverse Fourier transformation, we find

$$(\hat{A} f)(x) = \int \frac{d\alpha dp}{(2\pi)^d} A(x + \frac{\alpha}{2}; p) e^{-i\alpha p} f(x + \alpha). \quad (A.1)$$

We denote the operator $\hat{A}$ of the form (A.1) as $\hat{A} = \mathcal{W}(A)$. We will also write $A = \mathcal{W}(\hat{A})$ if $\hat{A} = \mathcal{W}(A)$.

**Definition A.1.** The operator $\mathcal{W}(A)$ is called a Weyl quantization of the function $A$. The function $\mathcal{W}(A)$ is called as a Weyl symbol of the operator $\hat{A}$.

\section*{A.2 Some classes of Weyl symbols}

\subsection*{A.2.1 Classes $A_N$ and $B_N$}

For investigations of QFT ultraviolet divergences, we are interested in behavior of Weyl symbols of operators at large values of momenta. Let us introduce some important spaces. Let $\omega_k = \sqrt{k^2 + m^2}$ for some $m$.

**Definition A.2.** 1. We say that a smooth function $A(x, k)$ is of the class $B_N$ if and only if the functions

$$\omega_k^{N+s} \frac{\partial^s A}{\partial k^{i_1} \cdots \partial k^{i_s}}$$

are bounded for all $s, i_1, \ldots, i_s$.

2. Let $A_n \in B_N, n = 1, \infty, A \in B_N$. We say that $B_N - \lim_{n \to \infty} A_n = A$ if and only if

$$\lim_{n \to \infty} \max_{k, x} \omega_k^{N+s} \frac{\partial^s (A_n - A)}{\partial k^{i_1} \cdots \partial k^{i_s}} = 0$$

for all $s, i_1, \ldots, i_s$.

3. We say that a function $A \in B_N$ is of the class $A_N$ if and only if

$$x_{j_1} \cdots x_{j_R} \frac{\partial}{\partial x_{s_1}} \cdots \frac{\partial}{\partial x_{s_P}} A \in B_N$$

for all $R, P, j_1, \ldots, j_R, s_1, \ldots, s_P$.

4. Let $A_n \in A_N, A \in A_N$. We say that $A_N - \lim_{n \to \infty} A_n = A$ if and only if

$$B_N - \lim_{n \to \infty} x_{j_1} \cdots x_{j_R} \frac{\partial}{\partial x_{s_1}} \cdots \frac{\partial}{\partial x_{s_P}} (A_n - A) = 0$$

for all $R, P, j_1, \ldots, j_R, s_1, \ldots, s_P$.

Let us investigate some properties of introduced classes $A_N$ and $B_N$.

**Lemma A.1.** 1. $A_{N+R} \subseteq A, B_{N+R} \subseteq B$ for $R \geq 0$.

2. Let $A_{N+R} - \lim_{n \to \infty} A_n = A$ and $R \geq 0$. Then $A_N - \lim_{n \to \infty} A_n = A$.

3. Let $B_{N+R} - \lim_{n \to \infty} A_n = A$ and $R \geq 0$. Then $B_N - \lim_{n \to \infty} A_n = A$. 


The proof is obvious: it is sufficient to notice that $\omega_k^R$ is a bounded function.

**Lemma A.2.** Let $A \in B_N$. Then $\frac{\partial A}{\partial k_i} \in B_{N+1}$.

2. Let $A \in A_N$. Then $x_i A \in A_N$, $\frac{\partial A}{\partial x_i} \in A_N$, $\frac{\partial A}{\partial k_i} \in A_{N+1}$, $f(x)A \in A_N$ for smooth bounded function $f(x)$.

3. Let $B_N = \lim_{n \to \infty} A_n = A$. Then $B_{N+1} = \lim_{n \to \infty} \frac{\partial A}{\partial k_i} A_n = \frac{\partial A}{\partial k_i} A$.

4. Let $A_n \in B_N$, $A_n = x_i A$. Then $A_N - \lim_{n \to \infty} \frac{\partial A_n}{\partial x_i} = \frac{\partial A}{\partial x_i}$, $A_N - \lim_{n \to \infty} f(x)A_n = f(x)A$ for smooth bounded function $f(x)$.

The proof is also obvious.

**Lemma A.3.** Let $A_1 \in B_{N_1}$, $A_2 \in B_{N_2}$. Then $A_1 A_2 \in B_{N_1+2}$.

**Proof.** It is sufficient to check that the expression

$$\omega_k^{N_1} \omega_k^{N_2} \omega_k^{N_3} \frac{\partial^s}{\partial k_{i_1} \cdots \partial k_{i_s}} (A_1 A_2)$$

is bounded. This statement is a corollary of properties $A_1 \in B_{N_1}$, $A_2 \in B_{N_2}$ and formula

$$\frac{\partial}{\partial k_i} (f g) = \frac{\partial}{\partial k_i} f \cdot g + f \cdot \frac{\partial}{\partial k_i} g.$$  

Lemma A.3 is proved.

**Lemma A.4.** The following properties are satisfied: $k_i \in B_{-1}$, $\omega_k^\alpha \in B_{-\alpha}$.

**Proof.** Since $|k_i/\omega_k| < 1$, we obtain the property $k_i \in B_{-1}$. For the function $\omega_k^\alpha$, one has

$$\frac{\partial}{\partial k_{i_1}} \cdots \frac{\partial}{\partial k_{i_s}} \omega_k^\alpha = \omega_k^\alpha P(k_i/\omega_k)$$

(A.3)

where $P$ is a polynomial in $k_i/\omega_k$. Property (A.3) is checked by induction. Therefore, functions (A.2) are bounded for $N = 1$. Lemma A.4 is proved.

**Lemma A.5.** Let $A \in B_N$. Then

$$k_i A \in B_{N-1}, \quad \omega_k^{-\alpha} A \in B_{N+\alpha}, \quad \frac{\partial A}{\partial k_i} \in B_{N+1}.$$  

2. Let $A \in A_N$. Then

$$k_i A \in A_{N-1}, \quad \omega_k^{-\alpha} A \in A_{N+\alpha}, \quad \frac{\partial A}{\partial k_i} \in A_{N+1}.$$  

**Proof.** Property 1 is a corollary of lemmas A.2 and A.4. Property 1 implies property 2. Lemma is proved.

**Lemma A.6.** Let $B_N = \lim_{n \to \infty} A_n = A$. Then

$$B_{N-1} = \lim_{n \to \infty} k_i A_n = k_i A, \quad B_{N+\alpha} = \lim_{n \to \infty} \omega_k^{-\alpha} A_n = \omega_k^{-\alpha} A, \quad B_{N+1} = \lim_{n \to \infty} \frac{\partial A_n}{\partial k_i} = \frac{\partial A}{\partial k_i}.$$  

2. Let $A_N = \lim_{n \to \infty} A_n = A$. Then

$$A_{N-1} = \lim_{n \to \infty} k_i A_n = k_i A, \quad A_{N+\alpha} = \lim_{n \to \infty} \omega_k^{-\alpha} A_n = \omega_k^{-\alpha} A, \quad A_{N+1} = \lim_{n \to \infty} \frac{\partial A_n}{\partial k_i} = \frac{\partial A}{\partial k_i}.$$  

The proof is analogous to the proof of lemma A.3.

**Lemma A.7.** Let $A_1 \in A_{N_1}$, $A_2 \in A_{N_2}$. Then $A_1 A_2 \in A_{N_1+N_2}$.

2. Let $A_{N_1} = \lim_{n \to \infty} A_{1,n} = A_1$, $A_{N_2} = \lim_{n \to \infty} A_{2,n} = A_2$. Then $A_{N_1+N_2} = \lim_{n \to \infty} A_{1,n} A_{2,n} = A_1 A_2$.

The proof is analogous to lemma A.3.
A.2.2 Properties of operators and symbols

Lemma A.8. 1. Let $A \in A_0$. Then the operator $W(A)$ (A.1) is bounded.
2. Let $A_0 = \lim_{n \to \infty} A_n = 0$. Then $\lim_{n \to \infty} ||W(A_n)|| = 0$.

Proof. (cf. [63]). Let us obtain an estimation for the norm $||\hat{A}||$. One has

$$\hat{A} = \int d\beta e^{i\beta \hat{k}} \int d\alpha e^{i\alpha (k+\beta/2)} \hat{A}(\alpha, \beta).$$

The estimation $||\int d\beta \hat{F}(\beta)|| \leq \int d\beta ||\hat{F}(\beta)||$ implies

$$||\hat{A}|| \leq \int d\beta || \int d\alpha e^{i\alpha (k+\beta/2)} \hat{A}(\alpha, \beta).$$

However, for operator $F(\hat{k})$ one has $||F(\hat{k})|| = \sup_k ||F(k)||$, since in the momentum representation $F(\hat{k})$ is the operator of multiplication by $F(k)$. Therefore,

$$|| \int d\alpha e^{i\alpha (k+\beta/2)} \hat{A}(\alpha, \beta) || = \max_k | \int d\alpha e^{i\alpha (k+\beta/2)} \hat{A}(\alpha, \beta) | = \max_k | \int d\alpha e^{i\alpha k} \hat{A}(\alpha, \beta) | =$$

$$\max_k | \int (\hat{\alpha} + \beta/2) \hat{A}(\alpha, \beta) | = \max_k | \int (\hat{\alpha} + \beta/2) A(x, k) e^{-i\beta x} | =$$

$$= \frac{1}{(2\pi)^2} \int \frac{d\beta d\alpha}{(\beta + 1)^N (x^2 + 1)^N} \max_k |(x^2 + 1)^N (-\Delta x + 1)^N A(x, k)|.$$

Here $N$ is an arbitrary number such that $N > d/2$. Thus,

$$||\hat{A}|| \leq \frac{1}{(2\pi)^2} \int \frac{d\beta d\alpha}{(\beta + 1)^N (x^2 + 1)^N} \max_k |(x^2 + 1)^N (-\Delta x + 1)^N A(x, k)|.$$

The first statement is justified. Proof of the second statement is analogous. Lemma A.8 is proved.

Lemma A.9.1. Let $A \in A_N$, $N > d/2$. Then $W(A)$ is a Hilbert-Schmidt operator.
2. Let $A_N = \lim_{n \to \infty} A_n = 0$, $N > d/2$. Then $\lim_{n \to \infty} ||W(A_n)||_2 = 0$.

Proof. Let us use the property [34, 71]

$$||\hat{A}||_2^2 = \int \frac{dx dk}{(2\pi)^2} |A(x, k)|^2$$

which can be obtained from definition (A.1). One has

$$||\hat{A}||_2^2 \leq \int \frac{dx}{(2\pi)^d (x^2 + 1)^N} \frac{dk}{\omega_k^{2N}} \max_k |(x^2 + 1)^{N/2} \omega_k^N A(x, k)|^2.$$

The first statement is justified. Proof of the second statement is analogous. Lemma A.9 is proved.

A.3 Properties of *-product

Remind that the Weyl symbol of the product of operators

$$A \ast B = \overline{W}(W(A)W(B))$$

can be presented as [34, 51]

$$(A \ast B)(x, k) = \int \frac{d\beta_1 d\beta_2 d\xi_1 d\xi_2}{(2\pi)^{2d}} A(x + \xi_1, k + \frac{\beta_1}{2}) B(x + \xi_2, k - \frac{\beta_2}{2}) e^{-i\beta_1 \xi_1 - i\beta_2 \xi_2} \quad (A.4)$$

Formula (A.4) can be obtained from definition (A.1).

Let us investigate some properties of formula (A.4). Let us find an expansion of formula (A.4) as $|k| \to \infty$. Formally, one has

$$A(x + \xi_1, k + \frac{\beta_1}{2}) = \sum_{n_2=0}^{\infty} \frac{1}{2^n n_2!} \frac{\partial^n A(x, \xi_1, k + \beta_1)}{\partial k_1^{n_2} \partial \xi_1^{n_2}} \beta_1^{n_2} \cdots$$

$$B(x + \xi_2, k - \frac{\beta_2}{2}) = \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{2^n n_1!} \frac{\partial^n B(x, \xi_2, k - \beta_2)}{\partial k_1^{n_1} \partial \xi_2^{n_1}} \beta_2^{n_1} \cdots.$$
Therefore,
\[
(A \ast B)(x, k) = \sum_{n_1 n_2 = 0}^{\infty} \frac{(-1)^n}{2^n n_1! n_2!} \int \frac{d\beta_1 d\beta_2 d\xi_1 d\xi_2}{(2\pi i)^d} e^{-i\beta_1 \xi_1 - i\beta_2 \xi_2} \frac{\partial^{n_1+n_2} A(x+\xi_1, k)}{\partial k^{n_1} \partial x^{n_2}} \frac{\partial^{n_1+n_2} B(x, k)}{\partial k^{n_2} \partial x^{n_1}} \frac{\partial^{n_1+n_2} A(x, k)}{\partial x^1 \cdots \partial x^{n_2}} \frac{\partial^{n_1+n_2} B(x, k)}{\partial x^1 \cdots \partial x^{n_1}}
\]
(A.5)

Denote
\[
(A^K \ast B)(x, k) = \sum_{n_1 n_2 \geq 0, n_1 + n_2 < K} \frac{i^{n_1-n_2}}{n_1! n_2!} \frac{\partial^{n_1+n_2} A(x, k)}{\partial k^{n_1} \partial x^{n_2}} \frac{\partial^{n_1+n_2} B(x, k)}{\partial k^{n_2} \partial x^{n_1}}
\]

This is an asymptotic expansion in $1/|k|$ as $|k| \to \infty$. Let us estimate an accuracy of the asymptotic series.

Making use of the relation
\[
A(x + \xi_1, k + \frac{\beta_2}{2}) - A(x + \xi_1, k) = \int_0^1 d(\alpha_2 - 1) \frac{\partial}{\partial \alpha_2} A(x + \xi_1, k + \alpha_2 \frac{\beta_2}{2})
\]
and integrating by parts $N_2$ times, we find
\[
A(x + \xi_1, k + \frac{\beta_2}{2}) = \sum_{n_2 = 0}^{N_2} \frac{1}{2^n n_2!} \frac{\partial^{n_2} A(x+\xi_1, k)}{\partial k^{n_2}} \frac{\partial^{n_2+1} A(x+\xi_1, k+\alpha_2 \frac{\beta_2}{2})}{\partial \alpha_2 \partial k^{n_2}} + \int_0^1 d\alpha_2 \frac{(1-\alpha_2) N_2}{2^n n_2!} \frac{\partial^{n_2+1} A(x+\xi_1, k+\alpha_2 \frac{\beta_2}{2})}{\partial \alpha_2 \partial k^{n_2}} \frac{\partial^{n_2+1} A(x+\xi_1, k+\alpha_2 \frac{\beta_2}{2})}{\partial \alpha_2 \partial k^{n_2}}.
\]

Analogously,
\[
B(x + \xi_2, k - \frac{\beta_1}{2}) = \sum_{n_1 = 0}^{N_1} \frac{(-1)^n}{2^n n_1!} \frac{\partial^{n_1} B(x+\xi_2, k)}{\partial k^{n_1}} \frac{\partial^{n_1+1} B(x+\xi_2, k-\alpha_1 \frac{\beta_1}{2})}{\partial \alpha_1 \partial k^{n_1}} \frac{\partial^{n_1+1} B(x+\xi_2, k-\alpha_1 \frac{\beta_1}{2})}{\partial \alpha_1 \partial k^{n_1}}.
\]

Therefore,
\[
(A \ast B)(x, k) = \sum_{n_1 n_2 = 0}^{N_1 N_2} \frac{i^{n_1-n_2}}{n_1! n_2!} \frac{\partial^{n_1+n_2} A(x, k)}{\partial k^{n_1} \partial x^{n_2}} \frac{\partial^{n_1+n_2} B(x, k)}{\partial k^{n_2} \partial x^{n_1}} + \sum_{n_1 = 0}^{N_1} r^{(1)}_{n_1 N_2} + \sum_{n_2 = 0}^{N_2} r^{(2)}_{N_1 n_2} + R_{N_1 N_2}
\]

with the following remaining terms,
\[
r^{(1)}_{n_1 N_2} = \int \frac{d\beta_1 d\beta_2 d\xi_1 d\xi_2}{(2\pi i)^d} e^{-i\beta_1 \xi_1 - i\beta_2 \xi_2} \int_0^1 d\alpha_2 \frac{(1-\alpha_2) N_2}{2^n n_2!} \frac{\partial^{n_2+1} A(x+\xi_1, k+\alpha_2 \frac{\beta_2}{2})}{\partial \alpha_2 \partial k^{n_2}} \frac{\partial^{n_2+1} A(x+\xi_1, k+\alpha_2 \frac{\beta_2}{2})}{\partial \alpha_2 \partial k^{n_2}} \frac{\partial^{n_2+1} A(x+\xi_1, k+\alpha_2 \frac{\beta_2}{2})}{\partial \alpha_2 \partial k^{n_2}} \frac{\partial^{n_2+1} A(x+\xi_1, k+\alpha_2 \frac{\beta_2}{2})}{\partial \alpha_2 \partial k^{n_2}}
\]
\[
r^{(2)}_{N_1 n_2} = \int \frac{d\beta_1 d\beta_2 d\xi_1 d\xi_2}{(2\pi i)^d} e^{-i\beta_1 \xi_1 - i\beta_2 \xi_2} \int_0^1 d\alpha_1 \frac{(1-\alpha_1) N_1}{2^n n_1!} \frac{\partial^{n_1+1} B(x+\xi_2, k-\alpha_1 \frac{\beta_1}{2})}{\partial \alpha_1 \partial k^{n_1}} \frac{\partial^{n_1+1} B(x+\xi_2, k-\alpha_1 \frac{\beta_1}{2})}{\partial \alpha_1 \partial k^{n_1}} \frac{\partial^{n_1+1} B(x+\xi_2, k-\alpha_1 \frac{\beta_1}{2})}{\partial \alpha_1 \partial k^{n_1}} \frac{\partial^{n_1+1} B(x+\xi_2, k-\alpha_1 \frac{\beta_1}{2})}{\partial \alpha_1 \partial k^{n_1}}
\]
\[
R_{N_1 N_2} = \int \frac{d\beta_1 d\beta_2 d\xi_1 d\xi_2}{(2\pi i)^d} e^{-i\beta_1 \xi_1 - i\beta_2 \xi_2} \int_0^1 d\alpha_1 \frac{(1-\alpha_2) N_2}{2^n n_2!} \frac{\partial^{n_2+1} A(x+\xi_1, k+\alpha_2 \frac{\beta_2}{2})}{\partial \alpha_2 \partial k^{n_2}} \frac{\partial^{n_2+1} A(x+\xi_1, k+\alpha_2 \frac{\beta_2}{2})}{\partial \alpha_2 \partial k^{n_2}} \frac{\partial^{n_2+1} A(x+\xi_1, k+\alpha_2 \frac{\beta_2}{2})}{\partial \alpha_2 \partial k^{n_2}} \frac{\partial^{n_2+1} A(x+\xi_1, k+\alpha_2 \frac{\beta_2}{2})}{\partial \alpha_2 \partial k^{n_2}}
\]

Let us investigate the remaining terms.

### A.3.1 The $k$-independent case

**Definition A.3.** We say that the function $f(x)$, $x \in \mathbb{R}^d$ is of the class $C$ if $f$ is a smooth function such that for each set $(i_1, \ldots, i_l)$ there exists $m > 0$ such that the function
\[
(x^2 + 1)^{-m} \frac{\partial^l}{\partial x^{i_1} \cdots \partial x^{i_l}} f
\]
is bounded.
Let $A = f(x), f \in \mathcal{C}$. Then the only nontrivial term is $r_{N_10}^{(2)}$ which is taken by integrating by parts to the form

$$r_{N_10}^{(2)} = \int \frac{d\beta_1 d\xi}{(2\pi)^d} e^{-i\beta_1 \xi} \frac{\partial^{N_1+1}}{\partial x^j_1 \cdots \partial x^j_{N_1+1}} f(x + \xi) \int_0^1 \alpha_1 \left( \frac{1 - \alpha_1}{N_1!} \frac{\partial^{N_1+1} B(x, k - \alpha_1 \frac{\beta_1}{2})}{\partial k^j_1 \cdots \partial k^j_{N_1+1}} \right). \tag{A.6}$$

Let us prove some auxiliary statements.

**Lemma A.10.** For some constant $A_1$ the estimation

$$\omega_k \leq A_1 \omega_p \omega_{k-p} \tag{A.7}$$

is satisfied.

**Proof.** Let $p = (\frac{1}{2} + \alpha)k + p_\perp, \alpha \in \mathbb{R}, p_\perp \perp k$. Then

$$\frac{\omega_k}{\omega_p \omega_{k-p}} = \frac{\omega_k}{\omega_1/2+\alpha_k \omega_1/2-\alpha_k} \equiv f(\alpha, k),$$

so that it is sufficient to check estimation (A.7) for $p = \alpha k$. For the function $1/f^2$, one has

$$\frac{1}{f^2(\alpha, k)} = \frac{1}{k^2 + m^2} \left[ \left( \frac{1}{2} + \alpha \right)^2 k^2 + m^2 \right] \left[ \left( \frac{1}{2} - \alpha \right)^2 k^2 + m^2 \right].$$

It has the following minimal value

$$\min_{\alpha} \frac{1}{f^2(\alpha, k)} = \begin{cases} \frac{(k^2/4+m^2)^2}{k^2+m^2}, & k^2 < 4m^2, \\ \frac{k^2 m^2}{k^2+m^2}, & k^2 > 4m^2, \end{cases} \alpha = \sqrt{1 - \frac{m^2}{k^2}}. \tag{A.8}$$

The quantity (A.8) is bounded below. Thus, lemma is proved.

**Corollary.** For $0 < \gamma < 1$,

$$\frac{\omega_k}{\omega_p \omega_{k-p}} \leq A_1.$$ 

**Lemma A.11.** Let $C \in \mathcal{A}_N$, $\chi \in \mathcal{C}$, $\varphi \in C[0, 1]$. Then for

$$F(x, k) = \int_0^1 d\alpha \varphi(\alpha) \int \frac{d\beta d\xi}{(2\pi)^d} e^{-i\beta \xi} \chi(x + \xi) C(x, k - \frac{\alpha \beta}{2}) \tag{A.9}$$

the function $\omega_k^N F$ is bounded.

**Proof.** Inserting the identity

$$e^{-i\beta \xi} = (\xi^2 + 1)^{-L_1} (\frac{\partial^2}{\partial \beta^2} + 1)^{L_1} e^{-i\beta \xi} \tag{A.10}$$

and integrating by parts, we obtain that

$$F(x, k) = \int_0^1 d\alpha \varphi(\alpha) \int \frac{d\beta d\xi}{(2\pi)^d} \left( \frac{1}{(\xi^2 + 1)^{L_1}} \right) e^{-i\beta \xi} \chi(x + \xi) \left( 1 - \frac{\alpha^2}{4} \frac{\partial^2}{\partial k^2} \right)^{L_1} C(x; k - \frac{\alpha \beta}{2}).$$

For the function $\omega_k^N F$, one has

$$\omega_k^N F = \int_0^1 d\alpha \varphi(\alpha) \int \frac{d\beta d\xi}{(2\pi)^d} \left( \frac{1}{(\xi^2 + 1)^{L_1}} \right) \chi(x + \xi) \left( -\frac{1}{4} \frac{\partial^2}{\partial k^2} + m^2 \right)^{L_1+1} e^{-i\beta \xi} \omega_k^N \frac{\omega_k^N \omega_k^N}{\omega_{k/2}^N} \times \left( 1 - \frac{\alpha^2}{4} \frac{\partial^2}{\partial k^2} \right)^{L_1} C(x; k - \frac{\alpha \beta}{2}). \tag{A.11}$$
Choose $L_2$ to be such a number that $\frac{L_2 + N}{2}$ is integer, $L_2 > d$. The property $\chi \in C$ implies that there exists such $K$ that

$$\frac{\partial^m}{\partial \xi_{i_1} \ldots \partial \xi_{i_m}} \chi(x + \xi) = ((x + \xi)^2 + 1)^K f_{m,i_1 \ldots i_m}(x + \xi), \quad m = 0, \frac{L_2 + N}{2},$$

where $f_{m,i_1 \ldots i_m}$ are bounded functions. Choose $L_1$ to be integer and $L_1 > \frac{K + d}{2}$. Integrating expression (A.11) by parts, making use of corollary of lemma A.10 and property $C \in \mathcal{A}_N$, we obtain that $\omega_k^N F$ is a bounded function. Lemma A.11 is proved.

**Lemma A.12.** Under conditions of lemma A.11 $F \in \mathcal{A}_N$.

**Proof.** It is sufficient to consider the functions

$$\omega_k^{N+1} \frac{\partial^j}{\partial k_{i_1} \ldots \partial k_{i_j}} x_{i_1} \ldots x_{i_j} \partial \partial x_{i_1} \ldots \partial x_{i_p} F$$

(A.12)

which are expressed via linear combinations of integrals of the type (A.9). Lemma A.12 is a corollary of lemma A.11.

**Lemma A.13.** Let $\mathcal{A}_N - \lim_{n \to \infty} C_n = C, \chi \in C, \varphi \in C[0,1]$. Then $\mathcal{A}_N - \lim_{n \to \infty} F_n = F$.

The proof is analogous to lemmas A.11 and A.12.

We obtain therefore the following theorem.

**Theorem A.14.**

1. Let $f \in C, B \in \mathcal{A}_N$. Then

$$f \ast B = f \ast B + R_K$$

with $R_K \in \mathcal{A}_{N+K+1}$.

2. Let $f \in C, \mathcal{A}_N - \lim_{n \to \infty} B_n = 0$. Then $\mathcal{A}_{N+K+1} - \lim_{n \to \infty} (f \ast B_n - f \ast B_n) = 0$.

### A.3.2 The $x$-independent case

Let $A = A(k), A \in \mathcal{B}_{M_1}, B \in \mathcal{A}_{M_2}$. The only nontrivial term is taken to the form:

$$r_{0N_2}^{(1)}(x,k) = \int_0^1 d\alpha_2 \left( -\frac{i}{2} \right)^{N_2+1} \frac{1 - \alpha_2}{N_2!} \int \frac{d\beta_2 d\xi_2}{(2\pi)^d} e^{-i\beta_2 \xi_2} \frac{\partial^{N_2+1} A(k + \frac{\alpha \beta_2}{2}) \partial^{N_2+1} B(x + \xi_2; k)}{\partial x_{i_1} \ldots \partial x_{i_{N_2+1}}}. $$

**Lemma A.15.** Let $C = C(k), C \in \mathcal{B}_{K_1}, K_1 > 0, D \in \mathcal{A}_{K_2}, \varphi \in C[0,1]$. Then for

$$F(x,k) = \int_0^1 d\alpha \varphi(\alpha) \int \frac{d\beta d\xi}{(2\pi)^d} e^{-i\beta \xi} C(k + \frac{\alpha \beta}{2}) \partial x_{i_1} \ldots \partial x_{i_m}$$

the function $\omega_k^{K_1+K_2} F$ is bounded.

**Proof.** Inserting the identity (A.10) and integrating by parts, we obtain that

$$F(x,k) = \int_0^1 d\alpha \varphi(\alpha) \int \frac{d\beta d\xi}{(2\pi)^d} \frac{1}{(\xi_{i+1}^2 + 1)^{L_1}} e^{-i\beta \xi} D(x + \xi, k) \left( 1 - \frac{\alpha^2}{4} \frac{\partial^2}{\partial k^2} \right)^{L_1} \times (-\frac{i}{2})^m \frac{\partial}{\partial k_{i_1}} \ldots \frac{\partial}{\partial k_{i_m}} C(k + \frac{\alpha \beta}{2}).$$

For the function $\omega_k^{K_1+K_2} F$, one has

$$\omega_k^{K_1+K_2} F(x,k) = \int_0^1 d\alpha \varphi(\alpha) \int \frac{d\beta d\xi}{(2\pi)^d} \frac{1}{(\xi_{i+1}^2 + 1)^{L_1}} \omega_k^{K_2} D(x + \xi, k) \left( -\frac{1}{4} \frac{\partial^2}{\partial k^2} + m^2 \right)^{\frac{L_1 + K_1}{2}}$$

$$e^{-i\beta \xi} \frac{\omega_k^{K_1}}{\omega^{K_2}_{\beta/2}} \left( 1 - \frac{\alpha^2}{4} \frac{\partial^2}{\partial k^2} \right)^{L_1} (-i\frac{\alpha}{2})^m \frac{\partial}{\partial k_{i_1}} \ldots \frac{\partial}{\partial k_{i_m}} C(k + \frac{\alpha \beta}{2}).$$

Integrating by parts for sufficiently large $L_1, L_2$, making use of lemmas A.10, we check proposition of lemma A.15.
Lemma A.16. Under conditions of lemma A.15 \( F \in \mathcal{A}_{K_1+K_2} \).

Lemma A.17. Let \( A_{K_2} - \lim_{n \to \infty} D_n = D \), \( C = C(k) \), \( C \in \mathcal{B}_{K_1} \), \( K_1 > 0 \), \( \varphi \in C[0,1] \). Then \( A_{K_1+K_2} - \lim_{n \to \infty} F_n = F \).

The proof is analogous to lemmas A.12 and A.13. We obtain then the following theorem.

Theorem A.18. 1. Let \( A = A(k) \), \( A \in \mathcal{B}_{M_1} \), \( B \in \mathcal{A}_{M_2} \). Then

\[
A \ast B = A \ast B + R_K
\]

with \( R_K \in \mathcal{A}_{M_1+M_2+K+1} \), provided that \( K + M_1 + 1 > 0 \).

2. Let \( A = A(k) \), \( A \in \mathcal{B}_{M_1} \), \( A_{M_2} - \lim_{n \to \infty} B_n = B \). Then

\[
A_{M_1+M_2+K+1} - \lim_{n \to \infty} (A \ast B_n - A \ast B_n) = 0,
\]

provided that \( K + M_1 + 1 > 0 \).

Remark. If the proposition of theorem A.18 is satisfied for \( K = K_0 \), it is satisfied for all \( K \leq K_0 \). Therefore, the condition \( K + M_1 + 1 > 0 \) can be omitted.

The following lemma is a corollary of theorem A.18.

Lemma A.19. 1. Let \( A \in \mathcal{A}_N \), \( N > d \). Then \( \mathcal{W}(A) \) is of the trace class.

2. Let \( \mathcal{A}_N - \lim_{n \to \infty} A_n = 0 \), \( N > d \). Then \( \lim_{n \to \infty} \text{Tr}\mathcal{W}(A_n) = 0 \).

Proof. Consider the operator

\[
\hat{B} = \mathcal{W}(B) = \tilde{\omega}^{N/2}(x^2 + 1)^{N/2}\mathcal{W}(A)
\]

with

\[
B = \omega_k^{N/2} \ast (x^2 + 1)^{N/2} \ast A
\]

Since \( B \in \mathcal{A}_N \), \( \mathcal{W}(B) \) is a Hilbert-Schmidt operator according to lemma A.9. Therefore, \( \mathcal{W}(A) \) is a product of two Hilbert-Schmidt operators \( (x^2 + 1)^{-N/2}\tilde{\omega}^{-N/2} \) and \( \mathcal{W}(B) \). Thus, \( \mathcal{W}(A) \) is of the trace class.

One also has:

\[
|\text{Tr}\mathcal{W}(A_n)| = |\text{Tr}(x^2 + 1)^{-N/2}\tilde{\omega}^{-N/2}\mathcal{W}(B_n)| \leq ||(x^2 + 1)^{-N/2}\tilde{\omega}^{-N/2}||_2||\mathcal{W}(B_n)||_2.
\]

Making use of lemma A.9, we prove lemma A.19.

A.3.3 The \( \mathcal{A}_N \)-case

Let \( A \in \mathcal{A}_{M_1}, B \in \mathcal{A}_{M_2} \). The \( r \)-terms can be investigated as follows.

1. We substitute \( \beta_{1,2} \beta_{1,2} e^{-i\xi_1 \xi_2} \equiv i\frac{\partial}{\partial \xi_1} e^{-i\xi_1 \xi_2} \) and integrate the expressions for \( r^{(1)}, r^{(2)}, R \) by parts with respect to \( \xi_1, \xi_2 \).

2. We consider the quantities like

\[
\omega_k^{N_1+N_2+M_1+M_2+1+L} \frac{\partial^L}{\partial k^1 \cdots \partial k^L x_1 \cdots x_j \partial x_{a_1} \cdots \partial x_{s_p}} r
\]

for \( r = r^{(1)}, r^{(2)}, R \) and show them to be bounded.

Lemma A.20. Let \( F \in \mathcal{A}_{K_1}, G \in \mathcal{A}_{K_2}, K_1, K_2 > 0 \). Then the function

\[
\int \frac{d\beta_1 d\beta_2 d\xi_1 d\xi_2}{(2\pi)^{2d}} e^{-i\beta_1 \xi_1 - i\beta_2 \xi_2} \omega_k^{K_1+K_2} F(x + \xi_1, k + \alpha_2 \frac{\beta_2}{2}) G(x + \xi_2, k - \alpha_1 \frac{\beta_1}{2}) \xi_1^j \cdots \xi_1^m
\]

is uniformly bounded with respect to \( \alpha_1, \alpha_2 \in [0,1] \).

This lemma is proved analogously to lemmas A.11 and A.15.

3. Analogously to previous subsubsections, we prove the following theorem.
Theorem A.21. 1. Let \( A \in A_{M_1}, B \in A_{M_2} \). Then
\[
A \ast B = A^K \ast B + R_K
\]
with \( R_K \in A_{M_1 + M_2 + K + 1} \).

2. Let \( A_n \in A_{M_1}, B_n \in A_{M_2} \). Then
\[
A_{M_1 + M_2 + K + 1} - \lim_{n \to \infty} (A_n \ast B_n - A_n^K \ast B_n) = A \ast B - A^K \ast B.
\]

A.4 Properties of the exponent

Let us investigate now the properties of the exponent of the operator \( \exp W(A) \equiv \mathcal{W}(\ast \exp A) \). It is convenient to consider the Fourier transformations of Weyl symbols,
\[
\tilde{A}(\gamma, k) = \int \frac{dx}{(2\pi)^d} e^{-i\gamma x} A(x,k).
\]

Introduce the following norms for Weyl symbols,
\[
||A||_{I,K} = \max_{J+M+N \leq K} \max_{\gamma,K} \left| \omega^J \frac{\partial^J}{\partial k_1 \ldots \partial k_{J+J}} \gamma_{m_1} \ldots \gamma_{m_M} \frac{\partial^N \tilde{A}}{\partial \gamma_{n_1} \ldots \partial \gamma_{n_N}} \right|. \quad (A.13)
\]

Lemma A.22. \( A \in A_I \) if and only if \( ||A||_{I,K} < \infty \) for all \( k = 0, \infty \).
The proof is obvious.

Let \( C = A \ast B \). Then the Fourier transformation \( \tilde{C} \) can be expressed via \( \tilde{A} \) and \( \tilde{B} \) as follows,
\[
\tilde{C}(\gamma, k) = \int d\alpha \tilde{A}(\alpha, k + \frac{\gamma - \alpha}{2}) \tilde{B}(\gamma - \alpha, k - \frac{\alpha}{2}). \quad (A.14)
\]

The following estimation is satisfied.

Lemma A.23. For arbitrary integer numbers \( K, L > d/2 \) there exists such a constant \( b_K \) that
\[
||A \ast B||_{0,K} \leq b_K ||A||_{0,K+2L} ||B||_{0,K}. \quad (A.15)
\]

To prove estimation \((A.15)\), one should use definition \((A.13)\) and formula \((A.14)\):
(i) the derivatives \( \partial/\partial \gamma_n \) are applied as
\[
\frac{\partial}{\partial \gamma_n}(\tilde{A}(\alpha, k + \frac{\gamma - \alpha}{2}) \tilde{B}(\gamma - \alpha, k - \frac{\alpha}{2})) = \frac{1}{2} \frac{\partial \tilde{A}}{\partial \gamma_n}(\alpha, k + \frac{\gamma - \alpha}{2}) \tilde{B}(\gamma - \alpha, k - \frac{\alpha}{2}) + \tilde{A}(\alpha, k + \frac{\gamma - \alpha}{2}) \frac{\partial \tilde{B}}{\partial \gamma_n}(\gamma - \alpha, k - \frac{\alpha}{2});
\]
(ii) the derivatives \( \partial/\partial k_j \) are applied analogously;
(iii) the multiplicators \( \gamma_m \) are written as \( \alpha_m + (\gamma_m - \alpha_m) \);
(iv) the estimations
\[
\omega_k \leq C\omega^{\alpha/2}w_{k-\alpha/2}, \quad \omega_k \leq C\omega^{\alpha/2}w_{k+\alpha/2}
\]
(lemma A.10) are taken into account.
(v) the integrating measure is written as
\[
d\alpha = \frac{d\alpha}{(\alpha^2 + 1)^L} d\alpha^2 L.
\]
We obtain the estimation \((A.15)\).
Consider the Weyl symbol of the exponent
\[ \ast \exp At - 1 = \sum_{n=1}^{\infty} \frac{A^n t^n}{n!} \]  
(A.16)
with \( A^n = A \ast \ldots \ast A \).

**Lemma A.24.** Let \( A \in A_M, M > 0 \). Then the estimation (A.16) is convergent in the \( \| \cdot \|_{0,K} \)-norm. The estimation \( \| \ast \exp At - 1 \|_{0,K} \leq C_K \) is satisfied for \( t \in [0,T] \).

**Proof.** One has
\[ ||A^n||_{0,K} \leq b_K^{-1}||A||_{0,K+2L}^{n-1}||A||_{0,K} \leq b_K^{-1}||A||_{0,K+2L}^{n} \]
Therefore,
\[ \| \ast \exp At - 1 \|_{0,K} \leq \sum_{n=1}^{\infty} \frac{1}{b_K} \left( t||A||_{0,K+2L} b_K \right)^n \leq \frac{e^{t||A||_{0,K+2L} b_K}}{b_K} - 1 \leq C_K \]
on \( t \in [0,T] \). Lemma A.24 is proved.

**Lemma A.25.** Let \( A \in A_M, M > 0 \). Then
\[ \sum_{m=N}^{\infty} \frac{A^m}{m!} \in A_{MN}. \]

**Proof.** One has
\[ \sum_{m=N}^{\infty} \frac{A^m}{m!} = A^N \left( \frac{1}{N!} + \int_0^1 d\tau \frac{(1 - \tau)^{N-1}}{(N-1)!} (\ast \exp A \tau - 1) \right) \]  
(A.17)
Lemma A.24 implies that
\[ \int_0^1 d\tau \frac{(1 - \tau)^{N-1}}{(N-1)!} (\ast \exp A \tau - 1) \in A_0. \]
It follows from theorem A.21 that the symbol (A.17) is of the \( A_{NM} \)-class. Lemma A.25 is proved.

**Lemma A.26.** Let \( A_n \in A_M, M > 0 \) and \( A_M - \lim_{n \to \infty} A_n = A \). Then
\[ A_{MN} - \lim_{n \to \infty} \sum_{m=N}^{\infty} \frac{A^m_n}{m!} = \sum_{m=N}^{\infty} \frac{A^m}{m!}. \]

**Proof.** Because relation (A.17) and theorem A.21 it is sufficient to prove that
\[ A_0 - \lim_{n \to \infty} \int_0^1 d\tau \frac{(1 - \tau)^{N-1}}{(N-1)!} (\ast \exp A_n \tau - \ast \exp At) = 0. \]  
(A.18)
One has
\[ \ast \exp A_n \tau - \ast \exp At = \int_0^\tau d\tau' \ast \exp A(t - \tau') (A_n - A) \ast \exp A_n \tau'. \]
Making use of lemma A.23, we obtain then estimation (A.18).

### A.5 Estimations for the commutator

Let \( \hat{A} = f(\hat{x}), \hat{B} = g(\hat{k}) \). To investigate the properties of the commutator \( \hat{K} = [\hat{A}; \hat{B}] \), it is convenient to introduce the notion of \( \hat{x}\hat{k} \)-symbol of the operator instead of Weyl symbol. For the \( \hat{x}\hat{k} \)-quantization, the operator \( e^{i\beta \hat{x}} e^{i\alpha \hat{k}} \) corresponds to the function \( e^{i\beta \hat{x}} e^{i\alpha \hat{k}} \). Therefore, the function
\[ A(x, k) = \int d\alpha d\beta \hat{A}(\alpha, \beta) e^{i\alpha \hat{k} + i\beta \hat{x}} \]
corresponds to the operator
\[ \hat{A} = \int d\alpha d\beta \tilde{A}(\alpha, \beta)e^{i\beta \hat{x}}e^{ik} \]

For \( \hat{x}k \)-quantization, the *-product defined from the relations \( \hat{C} = \hat{A}\hat{B}, C = A*B \) has the form \[ (A*B)(x, k) = A(x, k - i\frac{\partial}{\partial y})B(y, k)|_{y=x}. \]

**Lemma A.27.** 1. Let \( A(x, k) = \varphi_1(x)\varphi_2(k) \) with bounded functions \( \varphi_1, \varphi_2 \). Then \( ||\hat{A}|| < \infty \).
2. Let \( A \in L^2(\mathbb{R}^{2d}) \). Then \( \hat{A} \) is a Hilbert-Schmidt operator.

**Proof.** 1. One has \( \hat{A} = \varphi_1(\hat{x})\varphi_2(\hat{k}), ||\hat{A}|| \leq ||\varphi_1(\hat{x})||||\varphi_2(\hat{k})|| = \max |\varphi_1| \max |\varphi_2| < \infty. \)
2. One has \( TrA^+A = \frac{1}{(2\pi)^d} \int dk dx|A(x, k)|^2 < \infty. \)

The commutator \( \hat{K} = [f(\hat{x}), g(\hat{k})] \) has the following \( \hat{x}k \)-symbol:
\[ K(x, k) = [g(k) - g(k - i\frac{\partial}{\partial x})]f(x) = \sum_{n=0}^{L} \frac{\partial^n g}{\partial k^n} (-i)^n \frac{\partial^n f}{\partial x^n} \]
\[ - \int_0^1 d\alpha \frac{(1-\alpha)^L}{L!} (-i)^{L+1} \frac{\partial^{L+1} g(k-\alpha \hat{\pi})}{\partial x^{L+1}} \frac{\partial^{L+1} f}{\partial x^{L+1}}. \]

**Lemma A.28.** Let \( C(x, k) = A(k - i\alpha \partial/\partial x)B(x) \). Then \( ||C||_{L^2} = ||A||_{L^2}||B||_{L^2}. \)

**Proof.** Consider the Fourier transformation of the function \( A \):
\[ A(k) = \int d\gamma \tilde{A}(\gamma)e^{i\gamma k}. \]

One has \( ||A||_{L^2} = (2\pi)^{d/2}||\tilde{A}||_{L^2} \) and
\[ C(x, k) = \int d\gamma \tilde{A}(\gamma)e^{i\gamma k}e^{i\alpha \frac{\partial}{\partial x}} B(x). \]

Since \( e^{i\alpha \frac{\partial}{\partial x}} B(x) = B(x + \gamma \alpha) \), one has
\[ ||C||^2_{L^2} = \int dk d\gamma d\gamma_1 d\gamma_2 \tilde{A}^\ast(\gamma_1)e^{-i\gamma_1 k}B^\ast(x + \gamma_1 \alpha)\tilde{A}(\gamma_2)e^{i\gamma_2 k}B(x + \gamma_2 \alpha) = \]
\[ (2\pi)^d \int d\gamma ||\tilde{A}(\gamma)||^2 \int dx |B(x + \gamma \alpha)|^2 = ||A||^2_{L^2}||B||^2_{L^2}. \]

Lemma A.28 is proved.

We have obtained the following important statement.

**Lemma A.29.** Let \( \frac{\partial^m f}{\partial x^{i_1}}...\frac{\partial^m g}{\partial k^{i_1}} \) be bounded functions, \( m, n = 1, L, \) while
\[ \frac{\partial^{L+1} f}{\partial x^{i_1}...\partial x^{i_{L+1}}} \in L^2, \quad \frac{\partial^{L+1} g}{\partial k^{i_1}...\partial k^{i_{L+1}}} \in L^2. \]

Then \([f(\hat{x}), g(\hat{k})]\) is a bounded operator.

### A.6 Asymptotic expansions of Weyl symbol

To check the property of Poincare invariance, it is important to investigate the large-\( k \) expansion of the Weyl symbols. Introduce the corresponding definitions.

**Definition A.4.** 1. We say that a smooth function \( A(x, n), x, n \in \mathbb{R}^d, |n| < 1, \) is of the class \( L \) if the functions
\[ \frac{\partial^l}{\partial n_{i_1}...\partial n_{i_l}x_{j_1}...x_{j_l}} \frac{\partial^m}{\partial x_{m_1}...\partial x_{m_M}} A \] (A.19)
are bounded.
2. Let $A_s \in \mathcal{L}$, $s = 1, \infty$. We say that $\mathcal{L} - \lim_{s \to \infty} A_s = 0$ if
\[
\sup_{|n| \leq 1} \lim_{s \to \infty} \left| \frac{\partial^I}{\partial m_{i_1} \ldots \partial m_{i_j}} x_{j_1} \ldots x_{j_I} \frac{\partial^M}{\partial x_{m_1} \ldots \partial x_{m_M}} A \right| = 0.
\]

Definitions A.2 and A.4 imply the following statement.

Lemma A.30. 1. Let $A \in \mathcal{L}$. Then the function $B(x, k) = A(x, k/\omega_k)$ is of the class $\mathcal{B}_0$.

2. Let $\mathcal{L} - \lim_{s \to \infty} A_s = 0$. Then $\mathcal{B}_0 - \lim_{s \to \infty} A_s (x, k/\omega_k) = 0$.

Making use of definition A.2 and lemma A.25, we obtain the following corollary.

Corollary. 1. Let $A \in \mathcal{L}$. Then the function $\omega^{-\alpha} A(x, k/\omega_k)$ is of the class $\mathcal{A}_\alpha$.

2. Let $\mathcal{L} - \lim_{s \to \infty} A_s = 0$. Then $\mathcal{A}_\alpha - \lim_{s \to \infty} \omega^{-\alpha} A_s (x, k/\omega_k) = 0$.

Definition A.5. 1. A formal asymptotic expansion is a set $\alpha \in \mathbb{R}$ and functions $A_0, A_1, \ldots \in \mathcal{L}$. We say that the formal asymptotic expansions $\hat{A} = (\alpha, A_0, A_1, \ldots)$ and $\hat{B} = (\beta, B_0, B_1, \ldots)$ are equivalent if $\alpha - \beta$ is an integer number and $A_{l-\alpha+\beta} = B_l$ for all $l = -\infty, +\infty$ (we assume $A_l = 0$ and $B_l = 0$ for $l < 0$). We denote formal asymptotic expansions of Weyl symbols as
\[
\hat{A} \equiv \sum_{n=0}^{\infty} \omega^{-\alpha} A_n (x, k/\omega_k).
\]

If $A_0 = 0, \ldots, A_{l-1} = 0, A_l \neq 0$, the quantity $\deg \hat{A} \equiv \alpha + n$ is called as a degree of a formal asymptotic expansion $\hat{A}$.

2. Let $\hat{A}_s, s = 1, \infty$ and $\hat{A}$ be formal asymptotic expansions of Weyl symbols. We say that $F.E - \lim_{s \to \infty} \hat{A}_s = A$ if $\alpha_s = \alpha$ and $\mathcal{L} - \lim_{s \to \infty} (A_{s,n} - A_n) = 0$.

The summation and multiplication by numbers are obviously defined:
\[
\hat{A} + \lambda \hat{B} = \sum_{n=0}^{\infty} \omega^{-\alpha} (A_n (x, k/\omega_k) + \lambda B_n (x, k/\omega_k)).
\]

The product of formal asymptotic expansions of Weyl symbols
\[
\hat{A} \equiv \sum_{n=0}^{\infty} \omega^{-\alpha} A_n (x, k/\omega_k), \quad \hat{B} \equiv \sum_{n=0}^{\infty} \omega^{-\beta} B_n (x, k/\omega_k)
\]
is defined as
\[
\hat{A} \hat{B} \equiv \sum_{n=0}^{\infty} \omega^{-\alpha-\beta} \sum_{s,l \geq 0, s+l=n} A_s (x, k/\omega_k) B_l (x, k/\omega_k).
\]

Let $f = f(x), f \in \mathcal{C}$. Then
\[
f(x) \hat{A} \equiv \sum_{n=0}^{\infty} \omega^{-\alpha} f(x) A_n (x, k/\omega_k).
\]

One also defines
\[
\omega^{-\beta} \hat{A} \equiv \sum_{n=0}^{\infty} \omega^{-\alpha-\beta} A_n (x, k/\omega_k)
\]
and
\[
\frac{\partial \hat{A}}{\partial k_s} = \sum_{l=0}^{\infty} \omega^{-l+\alpha-1} \left[ -(l+\alpha) A_l (x, n) + \frac{\partial A_l}{\partial n_p} (x, n) \delta_{ps} - n_p \delta_{ns} \right] |_{n=k/\omega_k}
\]
The *-product of formal asymptotic expansions is introduced as
\[
\hat{A} * \hat{B} \equiv \sum_{K=0}^{\infty} \sum_{n_1 n_2 \geq 0, n_1 + n_2 = K} \sum_{\frac{n_1 n_2}{1 + n_1 + n_2} \frac{n_1 + n_2}{1 + n_1 + n_2} \frac{n_1 n_2}{1 + n_1 + n_2}} \sum_{l_1 = 0}^{\infty} \omega^{-l_1-\alpha_1} A_{l_1} (x, k/\omega_k) \times \sigma_{x_{n_1} \ldots x_{n_2} k_{n_1} \ldots k_{n_2}} \sum_{l_2 = 0}^{\infty} \omega^{-l_2-\alpha_2} A_{l_2} (x, k/\omega_k)
\]
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The formal asymptotic expansions $\tilde{A} \ast \omega_k^\alpha$, $\tilde{A} \ast f(x)$ are defined analogously. The *-exponent of a formal asymptotic expansion $\tilde{A}$ is defined as

$$\ast \exp \tilde{A} - 1 = \sum_{n=1}^{\infty} \frac{\tilde{A}^n}{n!}$$

provided that $\text{deg}A$ is a positive integer number.

**Definition A.6.** 1. An asymptotic expansion of the Weyl symbol is a set $\Delta \equiv (A, \tilde{A})$ of the Weyl symbol $A$ and a formal asymptotic expansion $\tilde{A}$ such that

$$A(x, k) - \sum_{i=0}^{n-1} A_i (x, k/\omega_k) \omega_k^{l+\alpha} \in \mathcal{A}_{n+\alpha}$$

for all $n = 0, \infty$.

2. We say that $E - \lim_{s \to \infty} A_s = \Delta$ if $F.E - \lim_{s \to \infty} \tilde{A}_s = \tilde{A}$ and

$$\mathcal{A}_{n+\alpha} - \lim_{s \to \infty} (A_s(x,k) - \sum_{i=0}^{n-1} A_i (x, k/\omega_k) \omega_k^{l+\alpha}) = A(x, k) - \sum_{i=0}^{n-1} A_i (x, k/\omega_k) \omega_k^{l+\alpha}$$

for all $n = 0, \infty$.

**Remark.** For given Weyl symbol $A$, the asymptotic expansion is not unique. For example, let

$$A(x, k) = m^2 f(x)/\omega_k.$$ 

One can choose $\alpha = 2$, $A_0(x, n) = m^2 f(x)$ and find $A(x, k) = \omega_k^{-2} A_0(x, k/\omega_k)$. On the other hand, one can set $\alpha = 0$, $A_0(x, n) = f(x)(1 - n, n)$ and obtain $A(x, k) = A_0(x, k/\omega_k)$ since $\omega_k^2 - k; k_i = m^2$. We see that a degree is a characteristic feature of an expansion rather than of a symbol.

Let $A = (A, \tilde{A})$, $B = (B, \tilde{B})$. Denote $A \ast B = (A \ast B, \tilde{A} \ast \tilde{B})$,

$\omega_k^\alpha \ast A \equiv (\omega_k^\alpha \ast A, \omega_k^\alpha \ast \tilde{A})$,

$f(x) \ast A \equiv (f(x) \ast A, f(x) \ast \tilde{A})$,

$\ast \exp A - 1 \equiv (\ast \exp A - 1, \ast \exp \tilde{A} - 1)$.

Theorems A.14, A.18, A.21 and lemmas A.25 and A.26 imply the following statements.

**Theorem A.31.** 1. Let $\Delta$ be an asymptotic expansion of a Weyl symbol. Then $\omega_k^\alpha \ast A$ and $f(x) \ast A$ are asymptotic expansions of Weyl symbols under conditions of theorem A.14, while $\ast \exp A - 1$ is an asymptotic expansion of a Weyl symbol, provided that $\text{deg} \tilde{A}$ is a positive integer number.

2. Let $\Delta$ and $B$ be asymptotic expansions of Weyl symbols. Then $\Delta \ast B$ is an asymptotic expansion of a Weyl symbol.

**Theorem A.32.** 1. Let $E - \lim_{n \to \infty} A_n = \Delta$. Then:

(a) $E - \lim_{n \to \infty} \omega_k^\alpha \ast A_n \ast \tilde{A}$;
(b) $E - \lim_{n \to \infty} f(x) \ast A_n \ast \tilde{A}$ under conditions of theorem A.14;
(c) $E - \lim_{n \to \infty} (\ast \exp A_n - 1) = \ast \exp \tilde{A} - 1$ if $\text{deg} \tilde{A}_n, \text{deg} \tilde{A}$ are positive integer numbers.

2. Let $E - \lim_{n \to \infty} A_n = \Delta$ and $E - \lim_{n \to \infty} B_n = B$. Then $E - \lim_{n \to \infty} A_n \ast B_n = \Delta \ast B$.

The time derivative of the asymptotic expansion $A(t)$ with respect to $t$ is defined in a standard way

$$E - \lim_{\delta t \to 0} \frac{A(t + \delta t) - A(t)}{\delta t} = \frac{dA(t)}{dt}.$$

The integral $\int_{t_1}^{t_2} A(t) dt$ is also defined in a standard way.

**Theorem A.33.** 1. Let $A(t)$ be a continuously differentiable asymptotic expansion of a Weyl symbol.

Then:

(a) $\frac{d}{dt}(\omega_k^\alpha \ast A) = \omega_k^\alpha \ast \frac{dA}{dt}$;
(b) $\frac{d}{dt}(f(x) \ast A) = f(x) \ast \frac{dA}{dt}$ under conditions of theorem A.14.
(c) \( \frac{d}{dt} (\exp A - 1) = \int_0^1 d\tau e^{A(t-\tau)} \frac{dA}{dt} e^{\tau} \);  

(d) \( \frac{d}{dt} (A \star B) = \frac{d}{dt} A \star B + A \star \frac{d}{dt} B \).

The only nontrivial statement is (c). It is proved by using the identity 51

\[ \star \exp A_1 - \exp A_2 = \int_0^1 d\tau \star \exp(A_1(1-\tau)) \star (A_1 - A_2) \star \exp(A_2\tau). \]
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