On 2-partitionable clutters and the MFMC property

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Abstract

We introduce 2-partitionable clutters as the simplest case of the class of \(k\)-partitionable clutters and study some of their combinatorial properties. In particular, we study properties of the rank of the incidence matrix of these clutters and properties of their minors.

A well known conjecture of Conforti and Cornuéjols [1, 2] states: That all the clutters with the packing property have the max-flow min-cut property, i.e. are mengerian. Among the general classes of clutters known to verify the conjecture are: balanced clutters (Fulkerson, Hoffman and Oppenheim [5]), binary clutters (Seymour [11]) and dyadic clutters (Cornuéjols, Guenin and Margot [3]). We find a new infinite family of 2-partitionable clutters, that verifies the conjecture.

On the other hand we are interested in studying the normality of the Rees algebra associated to a clutter and possible relations with the Conforti and Cornuéjols conjecture. In fact this conjecture is equivalent to an algebraic statement about the normality of the Rees algebra [6].

1 Introduction

We briefly describe the main results in this paper. Theorem 5.2 characterizes when an ideal clutter is mengerian in terms of the existence of an edge \(e\) of \(H\) for which \(\tau^\omega(H) = \tau^{\omega-e}(H) + 1\), for all \(\tau^\omega(H) > 0\). Hence the Conforti-Cornuéjols conjecture

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\(\text{2000 Mathematics Subject Classification.}\) Primary 05C75; Secondary 05C85, 05C20, 13H10.

\(\text{Key words and phrases.}\) max-flow min-cut, mengerian, clutters, hypergraphs, normality, Rees algebras

1LIDETEA Universidad La Salle
2This work was partially supported by CONACyT grants 49251-F, 49835-F, and SNI.
3Partially supported by COFAA-IPN.
reduces to proving that for every hypergraph $H$ with the packing property if $\tau^\omega (H) > 0$, for some $\omega$, then there exists $e \in E(H)$, for which $\tau^\omega (H) = \tau^\omega - e (H) + 1$.

After introducing and showing that the family of hypergraphs $Q_{pq}$ has the packing property, we use Theorem 5.2 to prove that it is also mengerian. We give an explicit algorithm in pseudo code to obtain inductively the edge $e$ required in Theorem 5.2. This algorithm can be generalized to prove that other ideal hypergraphs are mengerian.

We introduce 2-partitionable clutters. We then prove Propositions 5.1 and 5.3 that give information on the rank of the incidence matrix $A$ of the clutter $H$ and the minors of $H$ (when $H$ is 2-partitionable). We propose the Conforti and Cornuejos conjecture for 2-partitionable hypergraphs since we believe that in the class of $k$-partitionable hypergraphs a counterexample can in principle be possible.

In proposition 4.3 we prove the following conjecture for the case of the hypergraphs $Q_{pq}$, thus giving support to this conjecture.

Conjecture 2.14: If $\tau(C') = \nu(C')$ for all minors $C'$ of $C$ and $x^{v_1}, \ldots, x^{v_q}$ have degree $d \geq 2$, then the group $\mathbb{Z}^{n+1}/((v_1,1), \ldots, (v_q,1))$ is free, or equivalently $\Delta_r(B) = 1$ where $r = \text{rank}(B)$.

2 Preliminaries

We now give several definitions that help to clarify the algebraic translations of the combinatorial optimization problems studied in this paper.

A hypergraph $H$ is defined by a pair $(V, E)$ where $V$ represents a finite set called the vertices of $H$, and $E$ represents a collection of subsets from $V$ called the edges of $H$. In some cases, the notation $V(H), E(H)$ will be used to refer to the vertices and the edges of $H$, respectively.

A clutter $C$ is a particular type of hypergraph, with the property that $S_1 \not\subseteq S_2$ for all distinct $S_1, S_2 \in E(C)$.

For every hypergraph $H = (V, E)$ there is an associated clutter $H^\min$, defined by:

$$H^\min = \{ e \in E : e \not\supsetneq f \in E \}$$

The contraction $H/i$ and deletion $H\setminus i$ are hypergraphs with vertex set $V(H) \setminus \{i\}$ where: $E(H/i) = \{ S \setminus \{i\} : S \in E(H) \}$ (for clutters we take the set of inclusionwise minimal members of this set) and $E(H\setminus i) = \{ S \in E(H) : i \notin S \}$. Contractions and deletions of distinct vertices can be performed sequentially and the result does not depend on the order. An hypergraph obtained from $H$ by a sequence of deletions $I_d$ and contractions $I_c$ ($I_d \cap I_c = \emptyset$) is called a minor of $H$ and is denoted by $H\setminus I_d/I_c$. If $I_d \neq \emptyset$ or $I_c \neq \emptyset$, the minor is proper. For general properties of hypergraphs, clutters and their blockers we refer the reader to [10].

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field $K$ and let $I$ be an ideal of $R$ of height $g \geq 2$, minimally generated by a finite set $F = \{x^{v_1}, \ldots, x^{v_q}\}$ of square-free
monomials of degree at least two. Where a monomial \( f \) in \( R \) is called square-free if \( f = x_{i_1} \ldots x_{i_r} \) for some \( 1 \leq i_1 < \cdots < i_r \leq n \). For technical reasons we shall assume that each variable \( x_i \) occurs in at least one monomial of \( F \).

There is a natural one to one correspondence between the family of square-free monomial ideals and the family of clutters: We associate to the ideal \( I \) a clutter \( C \) by taking the set of indeterminates \( V = \{x_1, \ldots, x_n\} \) as vertex set and \( E = \{S_1, \ldots, S_q\} \) as edge set, where

\[
S_k = \{x_i \mid \langle e_i, v_k \rangle = 1, e_i \text{ denotes the } i\text{'th unit vector} \} = \text{supp}(x^{v_k}),
\]

and the support of a monomial \( x^a = x_1^{a_1} \cdots x_n^{a_n} \) in \( R \) is given by \( \text{supp}(x^a) = \{x_i \mid a_i > 0\} \).

The ideal \( I \) is called the edge ideal of \( C \). To stress the relationship between \( I \) and \( C \) we will use the notation \( I = I(C) \). The \( \{0,1\} \)-vector \( v_k \) is called the characteristic vector of \( S_k \). We denote by \( 1 \) the vector whose entries are all ones.

Let \( A \) be the incidence matrix whose column vectors are \( v_1, \ldots, v_q \). The set covering polyhedron associated with \( A \) is defined as:

\[
Q(A) = \{x \in \mathbb{R}^n \mid x \geq 0; xA \geq 1\}.
\]

We say that a hypergraph (clutter) is ideal if the polyhedron \( Q(A) \) is integral.

A set \( C \subseteq V \) is a vertex cover or transversal of the hypergraph \( H \) if every edge of \( H \) contains at least one vertex in \( C \). We call \( C \) a minimal vertex cover or minimal transversal if \( C \) is minimal with respect to this property. A set of edges of the hypergraph \( H \) is called independent or a matching if no two of them have a vertex in common.

Let \( B(H) \) be the collection of all the transversals of \( H \), then the blocker of \( H \), \( b(H) \), is defined by:

\[
b(H) := B(H)^{\text{min}}
\]

It is well known that the following dual relationship holds for the blocker.

**Theorem 2.1** For every hypergraph \( H \), \( b(b(H)) = H^{\text{min}} \). In particular, if \( H \) is a clutter, then \( H = b(b(H)) \).

Observe that for minors we have: \( b(H \setminus i) = (b(H)/i)^{\text{min}} \) and \( b(H/i) = b(H \setminus i) \).

Let us denote by \( f^c \) the complement of \( f \). Then as an immediate consequence of the former theorem we have that.

**Corollary 2.2** For every hypergraph \( H = (V, E) \), and \( f \subseteq V \), either \( f \supseteq e \in E \), or \( f^c \supseteq t \in b(H) \), but not both.

**Proposition 2.3** \([6]\) The following are equivalent:

(a) \( p = (x_1, \ldots, x_r) \) is a minimal prime of \( I = I(C) \).

(b) \( C = \{x_1, \ldots, x_r\} \in b(C) \).

(c) \( \alpha = e_1 + \cdots + e_r \) is a vertex of \( Q(A) \).
2.1 The Conforti and Cornuèjols conjecture

**Definition 2.4** The clutter $C$ satisfies the max-flow min-cut (MFMC) property if both sides of the LP-duality equation

$$\tau_\omega = \min \{ \langle \omega, x \rangle \mid x \geq 0; xA \geq 1 \} = \max \{ \langle y, 1 \rangle \mid y \geq 0; Ay \leq 1 \} = \nu_\omega \quad (1)$$

have integral optimum solutions $x$ and $y$ for each non-negative integral vector $\omega$. Clutters that satisfy the MFMC property are called mengerian.

Recall that a monomial subring $K[F] \subset R$ is normal if $K[F] = \overline{K[F]}$, where the integral closure $\overline{K[F]}$ is given by:

$$\overline{K[F]} = K[\{ x^a \mid a \in \mathbb{Z}A \cap R_+A \}], \quad (2)$$

where $\mathbb{Z}A$ is the subgroup spanned by $A$ and $R_+A$ is the polyhedral cone

$$R_+A = \left\{ \sum_{i=1}^q a_i v_i \mid a_i \in R_+ \text{ for all } i \right\}$$

generated by $A = \{ v_1, \ldots, v_q \}$. Here $R_+$ denotes the set of non negative real numbers.

**Theorem 2.5** [4, 7, 8] The following are equivalent

(i) $C$ has the max-flow min-cut property.

(ii) $R[I^t]$ is normal and $Q(A)$ is an integral polyhedron, where $R[I^t]$ is the subring of $R[t]$ generated by $x_1, \ldots, x_n, x^a t, \ldots, x^a t_q$, over the field $K$.

(iii) $x \geq 0; xA \geq 1$ is totally dual integral (a TDI system).

**Proposition 2.6** [11] If a clutter $C$ has the MFMC property, then so do all its minors.

Let us denote by $\tau(C)$ the minimum size of a vertex cover in $C$ and by $\nu(C)$ the maximum size of a matching in $C$, then:

$$\tau(C) \geq \min \{ \langle 1, x \rangle \mid x \geq 0; xA \geq 1 \} = \max \{ \langle y, 1 \rangle \mid y \geq 0; Ay \leq 1 \} \geq \nu(C).$$

**Definition 2.7** A clutter $C$ has the König property or packs if $\tau(C) = \nu(C)$.

The simplest example of a clutter with the König property is a bipartite graph.
Definition 2.8 A clutter $\mathcal{C}$ satisfies the packing property (PP) if all its minors satisfy the König property, that is, $\tau(C') = \nu(C')$ for every minor $C'$ of $\mathcal{C}$.

Theorem 2.9 [9] If $C$ has the packing property, then $Q(A)$ is integral.

It is well known that:

Proposition 2.10 If $C$ has the max-flow min-cut property, then $C$ has the packing property.

Conjecture 2.11 (Conforti-Cornuéjols [1, 2]) If the clutter $\mathcal{C}$ has the packing property, then $\mathcal{C}$ has the max-flow min-cut property.

Next we state an algebraic version of Conjecture 2.11.

Conjecture 2.12 [6] If $\tau(C') = \nu(C')$ for all minors $C'$ of $\mathcal{C}$, then $R[It]$ is normal.

Notation For an integral matrix $B \neq (0)$, the greatest common divisor of all the nonzero $r \times r$ subdeterminants of $B$ will be denoted by $\Delta_r(B)$.

Theorem 2.13 [6] If $x^{v_1}, \ldots, x^{v_q}$ are monomials of degree $d \geq 2$, i.e., all the edges of the clutter have $d$ vertices, such that $\mathcal{C}$ satisfies MFMC and the matrix

$$B = \begin{pmatrix} v_1 & \cdots & v_q \\ 1 & \cdots & 1 \end{pmatrix}$$

has rank $r$, then $\Delta_r(B) = 1$.

By using the previous result we obtain that a positive answer to Conjecture 2.12 implies the following:

Conjecture 2.14 [6] If $\tau(C') = \nu(C')$ for all minors $C'$ of $\mathcal{C}$ and $x^{v_1}, \ldots, x^{v_q}$ have degree $d \geq 2$, then the group

$$\mathbb{Z}^{n+1}/((v_1, 1), \ldots, (v_q, 1))$$

is free, or equivalently $\Delta_r(B) = 1$ where $r = \text{rank}(B)$.
3 On 2-partitionable clutters

Let $H = (V, E)$ be a hypergraph with $|V| = 2d$ and $E = \{S_1, \ldots, S_k\}$. Let

$$V = X_1 \cup X_2 \cup \cdots \cup X_d$$

be a partition of $V = \{x_1, \ldots, x_n\}$ into $d$ subsets of size two. We set $X_i = \{x_{2i-1}, x_{2i}\}$ for $i = 1, \ldots, d$ and $d \geq 2$. If

$$|S_i \cap X_j| = 1 \quad \forall \ i, j. \quad (3)$$

we say that $H$ is 2-partitionable. This definition could be generalized to $k$-partitionable hypergraphs, where $|X_j| = k \geq 2$.

Note that for $I = I(C)$ minimally generated by $F = \{x_{v_1}, \ldots, x_{v_q}\}$ we have that equation (3) becomes:

$$|\text{supp}(x_{v_i}) \cap X_j| = 1 \quad \forall \ i, j. \quad (4)$$

Observation: In our situation, by the pigeon hole principle, any minimal vertex cover $C$ of the clutter $C$ satisfies $2 \leq |C| \leq d$. Notice that for each odd integer $k$ the sum of rows $k$ and $k+1$ of the matrix $A$ is equal to $1 = (1, \ldots, 1)$. Thus the rank of $A$ is bounded by $d+1$.

The next result shows that $A$ has “maximal rank” if $C$ has a cover of maximum possible size.

**Proposition 3.1** Let $C$ be a 2-partitionable clutter. If there exists a minimal vertex cover $C$ such that $|C| = d \geq 3$ and $C$ satisfies the König property, then $\text{rank}(A) = d + 1$.

**Proof.** First notice that $C$ contains exactly one element of each $X_j$ because $X_j \not\subset C$. Thus we may assume

$$C = \{x_1, x_3, \ldots, x_{2d-1}\}.$$

Consider the monomial $x^\alpha = x_2 x_4 \cdots x_{2d}$ and notice that $x_k x^\alpha \in I$ for each $x_k \in C$ because the monomial $x_k x^\alpha$ is clearly in every minimal prime of $I$. Writing $x_k = x_{2i-1}$ with $1 \leq i \leq d$ we conclude that the monomial

$$x^\alpha i = x_2 x_4 \cdots x_{2(i-1)} x_{2i-1} x_{2(i+1)} x_2 \cdots x_{2d}$$

is a minimal generator of $I$. Thus we may assume $x^\alpha i = x^{v_i}$ for $i = 1, \ldots, d$. The vector $1$ belongs to the linear space generated by $v_1, \ldots, v_q$ because $C$ has the König property. It follows readily that the matrix with rows $v_1, \ldots, v_q, 1$ has rank $d + 1$. \hfill \Box

**Remark 3.2** If $C$ is 2-partitionable then $\text{rank}(A) = \text{rank}(B) = d + 1$, where

$$B = \begin{pmatrix} v_1 & \cdots & v_q \\ 1 & \cdots & 1 \end{pmatrix}$$
Proposition 3.3 Let $H$ be a 2-partitionable hypergraph with $W, Z \subseteq V(H)$ such that $W \cap Z = \emptyset$ and $Z \cap X_i = \emptyset$ for some $X_i$. If $H' \setminus W/Z$ is a proper minor of $H$, then $H'$ has the König property or $W \cap X_i = \emptyset$.

Proof. If $\emptyset \in E(H')$ or $E(H') = \emptyset$, then $H'$ has the König property. We assume that $\emptyset \notin E(H')$ and $E(H') \neq \emptyset$. If $\tau (H') = 1$ then $H'$ has the König property. Let us consider then that $\tau (H') \geq 2$, and $X_i = X_1 = \{x_1, x_2\}$. As $Z \cap X_1 = \emptyset$ then $\{x_1, x_2\} \in b(H/Z) = b(H) \setminus Z$. Hence, $|\{x_1, x_2\} \setminus W| \geq \tau (H') \geq 2$ and therefore $W \cap X_1 = \emptyset$. \qed

Conjecture 3.4 Let $\mathcal{C}$ be a 2-partitionable clutter. Then $\mathcal{C}$ has the packing property if and only if it is mengerian.

4 The $Q_6$ property class of hypergraphs

A hypergraph $H$ is minimally non packing (MNP) if it does not pack, but every minor of it does. Cornuéjols, Guenin and Margot [3], give an infinite class of ideal MNP clutters, which they call the $Q_6$ property class (before their work, only two MNP clutters were known).

A clutter has the $Q_6$ property, when $V(\mathcal{C})$ can be partitioned into nonempty sets $I_1, \ldots, I_6$ such that there are edges $S_1, \ldots, S_4$ in $\mathcal{C}$ of the form:

$$S_1 = I_1 \cup I_3 \cup I_5, \quad S_2 = I_1 \cup I_4 \cup I_6, \quad S_3 = I_2 \cup I_4 \cup I_5, \quad S_4 = I_2 \cup I_3 \cup I_6.$$ 

The Cornuéjols, Guenin and Margot MNP family of $Q_6$-property clutters is described as follows. Given $p, q \in \mathbb{N}$, we construct the incidence matrix of the clutter $Q_{pq}$ by partitioning the set $V(Q_{pq})$ in 6 blocks which we will call $P, P^*, Q, Q^*, r, r^*$, with elements $P = \{p_1, \ldots, p_p\}$, $P^* = \{p_1^*, \ldots, p_p^*\}$, $Q = \{q_1, \ldots, q_q\}$, $Q^* = \{q_1^*, \ldots, q_q^*\}$. Furthermore, denote by $M_{m \times n}(\mathbb{B})$ the set of 0,1 matrices and let $H_p \in M_{((2p-1)\times p)}(\mathbb{B})$ be a matrix whose rows represent the characteristic vectors of the non empty subsets of a set with $p$ elements. Let $H_p^*$ be its complement, i.e. $H_p + H_p^* = J$, where $J$ denotes the matrix whose entries are all one. Then the transpose $A^t$ of the incidence matrix $A$ of the clutter $Q_{pq}$ is given by:

$$A^t (Q_{pq}) = \begin{bmatrix} p_1 \ldots p_p & p_1^* \ldots p_p^* & q_1 \ldots q_q & q_1^* \ldots q_q^* & r & r^* \\
H_p & H_p^* & J & 0 & 1 & 0 \\
H_p^* & H_p & 0 & J & 1 & 0 \\
J & 0 & H_q^* & H_q & 0 & 1 \\
0 & J & H_q & H_q^* & 0 & 1 \end{bmatrix}$$

The hypergraph $Q_6$ giving name to the class, corresponds to $Q_{1,1}$.
As an example of an hypergraph in this class, we show the incidence matrix of $Q_{2,1}$:

$A^t(Q_{2,1}) = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
\end{bmatrix}$

### 4.1 The family $Q^F_{pq}$ of 2-partitionable hypergraphs

For homogeneity reasons, from here on we will assume that $p, q > 1$. From the construction of $Q_{pq}$, it follows that every pair of vertices of the form $vv^*$ with $v \in PQR$ is contained in $b(Q_{pq})$ ($XYz$ will be used as a shorthand for the union of sets $X \cup Y \cup \{z\}$). Moreover, it is not hard to show that the elements of $b(Q_{pq})$ correspond to one of the following types:

- $vv^*$, where $v \in PQR$ (5a)
- $p_ip_jq_kq_l^*$, where $i \neq j, k \neq l$ (5b)
- $p_ip_j^*r$, where $i \neq j$ (5c)
- $q_iq_j^*r^*$, where $i \neq j$ (5d)
- $Pq_i^*r^*$ (5e)
- $P_i^*q_i^*r^*$ (5f)
- $p_iQr$ (5g)
- $P_i^*Q^*r$ (5h)

For simplicity, we will denote by $V_{pq}, E_{pq}$ the set of vertices $V(Q_{pq})$ and the set of edges $E(Q_{pq})$, respectively. If $A \in \{P,Q\}$ and $A_1 \subseteq A$ we define $A_1^* = A \setminus A_1$. We will also use the following definitions:

$$F_{pq} := \{PQR, P^*Q^*r, PQ^*r^*, P^*Qr^*\}$$

$$Q^F_{pq} := (V_{pq}, E_{pq} \cup F), \text{ where } F \subseteq E_{pq}^*$$

$$F^* := \{f^* : f \in F\} \text{ note that } f = (f^*)^*$$

It is easy to verify that by construction the hypergraphs $Q^F_{pq}$ are 2-partitionable. We next show how for some suitable sets $F$, the hypergraphs $Q^F_{pq}$ are packing. To do so, we start with the following Lemma.
Lemma 4.1 Let \( t \) be an element of the blocker \( b(Q^F_{pq}) \).

I. If \( F \subseteq (F_{pq})^* \) then \( t \) is of the form (5.a-d) or \( t \) is of type (5.e-h) whenever \( t \not\subseteq f^* \), \( f \in F \).

II. If \( F = (E_{pq} \setminus F')^* \) where \( F' \subseteq F_{pq}, F' \neq \emptyset \) then \( t \) is of the form (5.a,b) or \( t \in F^* \) and \( t \not\subseteq f^* \), \( f \in F \).

Proof. Since \( E(Q^F_{pq}) \supseteq E_{pq} \), we have that for every \( t' \in b(Q^F_{pq}) \), \( \exists t \in b(Q_{pq}) \) such that \( t' = t \cup x' \).

Case I. If \( t \in b(Q_{pq}) \) of type (5.a-d) then \( t \in b(Q^F_{pq}) \) and \( t' = t \). Now, for \( t = Abc \) a minimal transversal of type (5.e-h), if \( f = A^*B^*c^* \in F \), as \( t' \cap f \neq \emptyset \) and \( t \cap f = \emptyset \) then \( x' \cap f = \emptyset \). If \( x' \cap A^*c^* \neq \emptyset \) then \( t' \) contains \( t \in b(Q_{pq}) \) of the form (5.a). Thus \( x' \subseteq B^* \), but this implies that \( t' \) contains at least one \( t \in b(Q_{pq}) \) of type (5.a,b). Therefore \( f \not\subseteq F \), and \( t \in b(Q^F_{pq}) \) implying \( t' = t \).

Case II. Notice that for any \( t \in b(Q_{pq}) \) of types (5.a,b) we have that \( t \in b(Q^F_{pq}) \). Let \( t = a_i^*a_j^*c \) be a minimal transversal of types (5.c,d), and choose \( A_1 \subseteq A \) such that \( a_i \in A_1, a_j \not\in A_1 \). Since, \( A_i^*A_j^*Bc^* \) and \( A_i^*A_j^*B^*c^* \) are both edges of \( Q^F_{pq} \), we have that, \( t' \) could only be a minimal transversal of \( Q^F_{pq} \) if \( a_i^*, a_j, b_kb_l^* \) or \( c^* \) are contained in \( x' \), which implies that \( t' \) contains at least one \( t \in b(Q_{pq}) \) of type (5.a,b).

Now, let \( t = Ab_{1}c \) be a minimal transversal of type (5.e-h). If \( \emptyset \neq x' \cap A^*B^*c^* \), this implies that \( t' \) contains a \( t \in b(Q_{pq}) \) of type (5.a), or we are in the case \( t \) is a minimal transversal of type (5.c,d). Hence \( x' \subseteq B \), and \( f = A^*B^*c^* \not\subseteq F \).

On the other hand, if \( B_1 \) is a proper subset of \( B \) then \( A^*B_1(B_1^c)^*c^* \) is an edge of \( Q^F_{pq} \), but this implies that \( Abc \subseteq t' \), therefore \( t' = Abc \), which is a minimal transversal of \( Q^F_{pq} \).

We construct a new class of clutters with the packing property obtained by adding some new hyperedges to the hypergraphs \( Q_{pq} \). \( \square \)
Theorem 4.2 The hypergraphs $Q_{pq}^F$ have the packing property for:

I. $F \subseteq (F_{pq})^*$, $F \neq \emptyset$, or

II. $F = (E_{pq} \setminus F')^*$ with $F' \subseteq F_{pq}$, $F' \neq \emptyset$.

Proof. From the construction of $Q_{pq}^F$ we have that $\tau(Q_{pq}^F) = \nu(Q_{pq}^F) = 2$. Now let $q_{pq}^F := Q_{pq}^F \setminus W/Z$ be a proper minor of $Q_{pq}^F$. If $\emptyset \in E(q_{pq}^F)$ or $\emptyset = E(q_{pq}^F)$ then by convention it packs. Therefore, for both cases we must show that for every proper minor $q_{pq}^F$ such that $\emptyset \notin E(q_{pq}^F)$ and $\emptyset \neq E(q_{pq}^F)$ the following equality holds:

$$1 \leq \tau(q_{pq}^F) = \nu(q_{pq}^F)$$ (6)

For the proof, we will use the fact that $b(H \setminus v) = (b(H) / v)^{\min}$, $b(H / v) = b(H) \setminus v$ and that deletion and contraction are associative and commute. We will do first the contractions, and then deletions. If $\tau(q_{pq}^F) = 1$, as $\emptyset \notin E(q_{pq}^F)$ and $\emptyset \neq E(q_{pq}^F)$ then $q_{pq}^F$ packs. Thus we can assume $\tau(q_{pq}^F) \geq 2$.

We have that if $t \in b(Q_{pq}^F)$, then there exists $t' \in b(Q_{pq})$ such that $t' \subseteq t$, therefore, $\tau(Q_{pq}^F) \leq \tau(q_{pq}^F)$.

If $\tau(q_{pq}^F) = \tau(Q_{pq}^F \setminus W/Z)$ then $\nu(q_{pq}^F) = \nu(Q_{pq}^F)$ since $E(q_{pq}^F) \subseteq E(Q_{pq}^F \setminus W/Z)$ and $Q_{pq}^F$ packs. Hence, we must prove that $\tau(q_{pq}^F) = \nu(q_{pq}^F)$ when $\tau(q_{pq}^F) > \tau(Q_{pq}^F \setminus W/Z)$.

[Case I] $(F \subseteq (F_{pq})^* \text{ with } F \neq \emptyset)$: Notice that $\tau(q_{pq}^F) > \tau(Q_{pq}^F \setminus W/Z)$ could only happen if $\tau(Q_{pq}^F \setminus W/Z) = |A_{bc} \setminus W|$ and $Z \cap A_{bc} = \emptyset$, where $A_{bc}$ is a minimal transversal of type (e-h), and $A_{bc} \notin b(q_{pq}^F)$. But this implies that $A^* B^* c^* \in F$.

[1.1] If $Z \cap vv^* \neq \emptyset$ for all $v \in PQr$, then $A_1 (A_1')^* B_1 (B_1')^* d \subseteq Z$ where $A_1 \subseteq A$, $B_1 \subseteq B$ and $d \in rr^*$. As $Z \cap A_{bc} = \emptyset$, then $d = c^*$, $A_1 = \emptyset$ and $b \notin B_1$. Therefore $Z = A^* B_2^* c^*$, where $B_1 \subseteq B_2 \subseteq B$, $(B_1')^* \subseteq B_3' \subseteq B^*$ and $b \notin B_2$. If $B^* = B_3'$ then $\emptyset \in E(q_{pq}^F)$, but we assumed that $\emptyset \notin E(q_{pq}^F)$. Thus, there exists $b_1^* \in B^*$ such that $b_1^* \notin B_3'$, and consequently $B_2 \neq \emptyset$.

By the form of the elements of $b(Q_{pq})$ and of $b(Q_{pq}^F)$ we have that:

$b(Q_{pq}/Z) = \{b, b_1^* c, A_{bc} \} \setminus B_2 B_3$, $b(Q_{pq}^F/Z) = \{b, b_1^* c\} \setminus B_2 B_3$ and $b b_1^* c \in b(Q_{pq}/Z) \cap b(Q_{pq}^F/Z)$. Therefore $|b b_1^* c \setminus W| > |A_{bc} \setminus W|$ for all $b \notin B_3^*$. This is only possible if $A \subseteq W$, and $b_2 \notin W$ for all $b_2 \notin B_3^*$; i.e. $W \cap (B_3^*)^c = \emptyset$. Consequently, $3 \geq \tau(q_{pq}^F) = \tau(Q_{pq}^F \setminus W/Z) + 1 \geq 1$. Thus, either $W = AB_4$ or $W = AB_4 c$, where $B_4 \subseteq B \setminus B_3$. Since $B_2 \neq \emptyset$ then $B_4 \neq B$. Even more, $\emptyset \neq f = (B_3^*)^* = A^* B^* c \setminus A^* B_2 B_3 c^* \in E(q_{pq}^F \setminus E(Q_{pq}^F \setminus W/Z))$. Now, by the form of the elements of $E(Q_{pq}^F)$ we have that $E(Q_{pq}^F \setminus W/Z) = \{V (V')^c : V \subseteq B \} \cup B / B_2 B_3 \setminus W'$ where $W' = W \setminus A$ and $E(q_{pq}^F) \supseteq E(Q_{pq}^F \setminus W/Z) \cup (B_3^*)^*$. 

[1.1.1] If $c \in W'$ then $E(Q_{pq}^F \setminus W/Z) = \{B\} / B_2 B_4$. Now, $(B_3^*)^*$ is independent with the edges of $Q_{pq} \setminus W/Z$, which implies that $q_{pq}^F$ packs.
[I.1.2] Now if \( c \notin W \) then \( E(Q_{pq}|W/Z) \supseteq ((V(V^c)^* \ c : V \subseteq B_3^\ast)) / B_2B_3^* \); since \( B_2 \cap B_4 = \emptyset \), we have \( B_2 (B_2^\ast)^c \setminus B_2B_3^\ast \in E(Q_{pq}|W/Z) \), but \( B_1 \subseteq B_2 \), therefore \( B_2^\ast \subseteq B_1^\ast \subseteq B_3 \). Thus \( c = B_2 (B_2^\ast)^c \setminus B_2B_3^\ast \in E(Q_{pq}|W/Z) \). We have the following two remaining cases.

If \( B_4 \neq \emptyset \), \( E(Q_{pq}|W/Z) = ((V(V^c)^* : V \subseteq B_3^\ast)) / B_2B_3^* \), then \( \tau (Q_{pq}|W/Z) = 1 \) and \( c \) is independent with \((B_3^\ast)^\ast\) hence, \( q_{pq}^F \) packs.

If \( B_4 = \emptyset \), \( B_3^\ast = \{ B \} / B_2B_3^\ast \in E(Q_{pq}|W/Z) \) then \( B_2^\ast, c, (B_3^\ast)^\ast \) are independent in \( q_{pq}^F \).

As \( \tau (q_{pq}^F) \leq 3 \) then \( q_{pq}^F \) packs.

[I.2] If \( Z \cap vv^c = \emptyset \) for some \( v \in PQr \) then, \( 2 \geq |vv^c\setminus W| \geq \tau (q_{pq}^F) > \tau (Q_{pq}|W/Z) = |\text{Abc\setminus W}| \) for all \( v \) such that \( Z \cap vv^c = \emptyset \). If \( \text{Abc} \subseteq W \), as \( \text{Abc} \in b(Q_{pq}) \) then \( E(q_{pq}^F) = \{ A^\ast B^c^\ast \} / Z \setminus W \), and \( q_{pq}^F \) packs. Assume \( \text{Abc} \notin W \), then \( \tau (Q_{pq}|W/Z) = 1 \) and \( \tau (q_{pq}^F) = 2 \). Notice that if \( u \in W \), then \( u^* \in Z \); since otherwise, knowing that \( W \cap Z = \emptyset \) then \( uu^* \cap Z = \emptyset \). Thus \( 1 \geq |uu^\ast \setminus W| > |\text{Abc}\setminus W| = \tau (Q_{pq}|W/Z) \) which is a contradiction.

[I.2.1] If \( bc \subseteq W \), then \( b^\ast \in B^c \subseteq Z \). In this case \( \text{Abc} \setminus W = \{ a \} \subseteq A \), and \( c^\ast \in Z \); implying \((A\setminus \{ a \})^\ast B^c \subseteq Z \) and \((A\setminus \{ a \})bc \subseteq W \). Now, by construction of \( E(Q_{pq}) \), we have that \( E(q_{pq}^F) = \{ a, a^\ast \} \setminus (W \cap Ba^\ast) / (Z \cap Ba), \) but \( \tau (q_{pq}^F) = 2 \), and consequently \( q_{pq}^F \) packs.

[I.2.2] Now, if \( bc \notin W \), then \( A \subseteq W \) and \( A^\ast \subseteq Z \). Even more, \( b \in W \) or \( c \in W \).

If \( c \in W \), then \( c^\ast \in Z \). Therefore \( Ac \subseteq W \) and \( A^c \subseteq Z \). By the form of the elements of \( E(Q_{pq}) \) and of \( F \) we have that \( E(q_{pq}^F) = \{ B, B^\ast \} \setminus (W \cap BB^\ast) / (Z \cap BB^c) \), hence \( q_{pq}^F \) packs.

If \( b \in W \), then \( Ab \subseteq W \) and \( A^c \subseteq Z \). So the edge set of \( q_{pq}^F \) is given by \( E(q_{pq}^F) = ((V(V^c)^* \ c : V \subseteq B^\ast) \cup (B^\ast)^c \setminus (W \cap BB^c) / (Z \cap BB^c)) \). Since \( E(q_{pq}^F) \neq E(Q_{pq}|W/Z) \) we have that \( W \cap (B^\ast)^c \setminus (W \cap BB^c) = \emptyset \). Therefore \( W = AB_5 \), \( Z \subseteq A^cB_6^c \) with \( b \in B_5 \subseteq B_6 \subseteq B \).

As \( \tau (Q_{pq}|W/Z) = 1 \), we must have \( B_5^c \setminus Z = (B_5^c) B_5^c \setminus Z \in E(Q_{pq}|W/Z) \). Then \( B_5^c \setminus Z \) and \((B^\ast)^c \setminus Z \) are independent in \( E(q_{pq}^F) \) and hence, \( q_{pq}^F \) packs.

[Case II] ( \( F = (E_{pq} \setminus F')^\ast \) with \( F' \subseteq F_{pq} \), \( F' \neq \emptyset \)):

[II.1] If \( \tau (Q_{pq}|Z) = 4 \), then \( vv^\ast \cap Z \neq \emptyset \) for all \( v \in PQr \). Thus \( Z = P_1P_2^\ast Q_1Q_2^\ast D \), where \( P_1 \cup P_2 = P, Q_1 \cup Q_2 = Q \) and \( \emptyset \neq D \subseteq rr^\ast \). Since \( 4 = \tau (Q_{pq}|Z) = |p_1p_2^\ast q_1q_2^\ast|, p_1p_2^\ast q_1q_2^\ast \cap Z = \emptyset \) then \( P_1, P_2 \neq P \) and \( Q_1, Q_2 \neq Q \). Moreover, \( P_1, P_2 \neq \emptyset, Q_1, Q_2 \neq \emptyset \). By the form of \( E(Q_{pq}^c) \) we have that \( E(Q_{pq}^c) = \{ P_1^c, (P_2^c)^\ast, Q_1^c, (Q_2^c)^\ast \} \). Therefore \( q_{pq}^F \) packs.

[II.2] If \( \tau (Q_{pq}|Z) = p + q + 1 \), then \( \tau (Q_{pq}|Z) = |ABC| \) and \( ABC \cap Z = \emptyset \), implying \( A^\ast B^c \in F \). Moreover, since \( vv^\ast \cap Z \neq \emptyset \) for all \( v \in PQr \), then \( Z = A^\ast B_5^c \). By the form of \( E(Q_{pq}^c) \) we have that \( E(Q_{pq}^c) = \{ v : v \in ABC \} \). Therefore \( q_{pq}^F \) packs.
[II.3] If \( \tau(Q^F_{pq}/Z) = 2 \), then \( \tau(Q^F_{pq}/Z) = |vv^*| \) for some \( v \in PQr ; \) i.e. \( vv^* \cap Z = \emptyset \). Let us consider the case where

\[
2 = \tau(q^F_{pq}) = |vv^* \setminus W| > \tau(Q_{pq}/W/Z)
\]

By Proposition 3.3 we have that \( \forall u \in u \cup uu^* = \emptyset \) for all \( u \) such that \( uu^* \cap Z = \emptyset \). So, if \( u \in W \), then \( uu^* \in Z \) since otherwise \( \tau(q^F_{pq}) < 2 \), a contradiction.

For the proof, let \( W = A_1A_2^*B_1^*B_2^*D_1 \) and \( Z = A_3A_4^*B_3^*B_4^*D_2 \), with \( A_1 \subseteq A_2 \subseteq A_3 \), \( B_1 \subseteq B_2 \subseteq B_3 \subseteq B_4 \subseteq D_2 \subseteq \text{rr}^* \). Furthermore, since \( W \cap Z = \emptyset \) we have that \( A_1 \cap A_3 = \emptyset \), \( A_2 \cap A_4 = \emptyset \), \( B_1 \cap B_3 = \emptyset \), \( B_2 \cap B_4 = \emptyset \) and \( D_1 \cap D_2 = \emptyset \). Moreover, \( A_1 \cap A_2 = \emptyset \), \( B_1 \cap B_2 = \emptyset \) and \( |D_1| \leq 1 \).

On the other hand, some of the sets \( A_1, A_2, B_1 \) or \( B_2 \) is empty. This is necessary, since otherwise this would imply that \( \emptyset \in b(q^F_{pq}) \), which is a contradiction. Therefore, assume without loss of generality that \( B_2 = \emptyset \). For the rest of the proof assume that \( f_1, f_2 \in E(q^F_{pq}) \).

[II.3.1] Assume that \( A_1 \neq \emptyset \) and \( A_2 \neq \emptyset \), then:

[II.3.1.1] If \( B_1 \neq \emptyset \), we have \( B_1^* = B^* \), hence:

If \( D_1 = \emptyset \); by the form of \( E(Q^F_{pq}) \), there exists \( f_1 \subseteq (A_i^c \setminus A_2)^* c \) and \( f_2 \subseteq (A_3^c \setminus A_1)^* c^* \) then \( E(q^F_{pq}) \) packs.

If \( D_1 \neq \emptyset \); assume \( D_1 = c \). If \( A_1 \cup A_2 = A \) then \( \emptyset \in E(q^F_{pq}) \). Thus, suppose that \( A_1 \cup A_2 \neq A \). By the form of the elements of \( E(Q^F_{pq}) \), there exists \( f_1 \subseteq (A_i^c \setminus A_2)^* c \) and \( f_2 \subseteq (A_3^c \setminus A_1)^* c^* \), therefore \( E(q^F_{pq}) \) packs.

[II.3.1.2] If \( B_1 = \emptyset \). Consider the case where \( D_1 = \emptyset \), then there are \( f_1, f_2 \) such that \( f_1 \subseteq (A_i^c \setminus A_1) Bc \) and \( f_2 \subseteq (A_i^c \setminus A_2)^* B^* c^* \). Then \( q^F_{pq} \) packs.

On the other hand, if \( D_1 \neq \emptyset \), assume \( D_1 = c \). This implies that \( c^* \in Z \). For this case there are \( f_1, f_2 \) such that \( f_1 \subseteq (A_i^c \setminus A_1) B \) and \( f_2 \subseteq (A_i^c \setminus A_2)^* B^* \).

[II.3.2] Assume that \( A_1 = \emptyset \) or \( A_2 = \emptyset \). Without loss of generality let \( A_2 = \emptyset \), and let \( A_1, B_1 \neq \emptyset \).

[II.3.2.1] So if \( A_1 \neq A \) or \( B_1 \neq B \) then:

If \( D = \emptyset \), then there exists \( f_1 \subseteq A_i (B_1^c)^* c \) and \( f_2 \subseteq (A_i^c)^* B_1^c c^* \), then \( E^F_{pq} \) packs.

If \( D \neq \emptyset \), let \( D = c \), implying \( c^* \in Z \). Then there are \( f_1, f_2 \in E(q^F_{pq}) \) such that \( f_1 \subseteq (A_i^c) (B_1^c)^* c \) and \( f_2 \subseteq (A_i^c)^* B_1^c c^* \).

[II.3.2.2] So if \( A_1 = A \) and \( B_1 = B \), then \( D = \emptyset \) and \( f_1 = c \) and \( f_2 = c^* \).

[II.3.3] Assume that \( B_1 = \emptyset \), \( A_2 = \emptyset \) and \( A_1 \neq \emptyset \), then:

[II.3.3.1] If \( D_1 = \emptyset \), then \( f_1 \subseteq A_i^c B^* c \) and \( f_2 \subseteq (A_i^c)^* B^* c^* \).

[II.3.3.2] If \( D_1 \neq \emptyset \), then assume that \( D_1 = c \) and \( c^* \in Z \). If \( A_1 \neq A \), there are \( f_1, f_2 \) such that \( f_1 \subseteq (B')^c (B')^* c \) and \( f_2 \subseteq (B') ((B')^c)^* \) for some \( \emptyset \neq B' \subset B \), therefore \( q^F_{pq} \) packs.
Assume that $B_1 = \emptyset$, $A_2 = \emptyset$ and $A_1 \neq \emptyset$. Now let $W \supseteq rr^*$ then notice that for this case there are $f_1, f_2 \in E(Q_{pq} \setminus W)$ two independent edges, therefore, $f_1 \setminus Z$ and $f_2 \setminus Z$ should contain independent edges in $q_F^F$, therefore $q_F^F$ packs.

Concluding with this the proof that $Q_{pq}^F$ has the packing property. \qed

As mentioned before, by construction all the clutters in the class $Q_{pq}^F$ are 2-partitionable. In particular for the clutters given in Theorem 4.2 we have:

**Proposition 4.3** Consider $Q_{pq}^F$ for $F$ as in cases I and II of Theorem 4.2. Then, $\Delta_r(B(Q_{pq}^F)) = 1$.

**Proof.** We have that $Q_{pq}^F$ is 2-partitionable and has the König property. By the observation before Proposition 3.1 and Remark 3.2 we have that $\text{rank}(A) = \text{rank}(B(Q_{pq}^F)) \leq d + 1$. In this case $d = p + q + 1$, so $\text{rank}(B(Q_{pq}^F)) \leq p + q + 2$.

On the other hand,

$$L = \begin{bmatrix}
P^* & Q^* & r^* & r \\
0 & I & 1 & 0 \\
0 & 0 & 0 & 1 \\
I & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}$$

is a submatrix of $B(Q_{pq}^F)$, and $L$ is reducible by elementary matrix transformations to $I_{p+q+2}$ (the identity matrix of order $p + q + 2$). Hence $\det(L) = 1$ and it follows that $\Delta_r(B(Q_{pq}^F)) = 1$. \qed

Note that the previous result gives support to Conjecture 2.14.

## 5 A new infinite family of mengerian clutters

In this section we will prove that if $F = F^*$ then $Q_{pq}^F$ is a mengerian hypergraph.

Among the general classes of clutters known to verify the Conjecture of Conforti and Cornuőjols are: *binary*, *balanced* and *dyadic* clutters. We now prove that the family $Q_{pq}^F$ of Theorem 4.2 does not belong to any of these classes.

Let us denote by $\triangle$ the symmetric difference operator. A hypergraph $H = (V, E)$ is:

a) *Binary* if for every $e_1, e_2, e_3 \in E$ there is an $e \in E$ such that $e \subseteq e_1 \triangle e_2 \triangle e_3$.

b) *Dyadic* if for every pair $(e, t)$ with $e \in E, t \in b(H)$ the inequality $|t \cap e| \leq 2$ holds.

c) *Balanced* if no square submatrix of odd order contains exactly two 1’s per row and per column.
**Lemma 5.1** The hypergraphs $Q_{pq}^F$ of Theorem 4.2 are not dyadic, nor binary, nor balanced.

**Proof.** For the proof let $e, f \in E_{pq}$ such that $e^c \in F, f^c \notin F$ (the existence of both elements is a consequence of the construction given in Theorem 4.2). Now, to prove that $Q_{pq}^F$ is not dyadic, notice that there is an edge $e \in E_{pq}$ and a minimal transversal $t$ of type (5.b) such that $|t \cap f| \geq 3$. Moreover, notice that $e \triangle e^c \triangle f = f^c \notin E(Q_{pq}^F)$ so $Q_{pq}^F$ is not binary. Now, to prove that $Q_{pq}^F$ is not balanced, notice that the edges $PQrr^*$ and $P^*Qr^*$ intersected with $p \in P, q \in Q$ and $r^*$ constitute a counterexample. \(\square\)

Observe that if we consider for $Q_{pq}^F$ that $F = E_{pq}^c$ (which was not taken into account for the second case of Theorem 4.2), then $Q_{pq}^F$ is a binary hypergraph.

**Theorem 5.2** Let $H$ be an ideal hypergraph. Then the following statements are equivalent.

1. $H$ is mengerian
2. If $\tau^w(H) > 0$, then there exists $e \in E(H)$ such that $\tau^w(H) = \tau^{w-e}(H) + 1$.

**Proof.** (2 $\Rightarrow$ 1) As $H$ is ideal, $\tau^w(H) = wx$, where $x \in b(H)$. We only need to prove that $\tau^w(H) = \nu^w(H) = y1$, where $Ay \leq w$, $y \geq 0$ and $y$ integer. We will prove this by induction over $\tau^w(H)$. If $\tau^w(H) = 0$, then $y = 0$ satisfies the proposition. Now suppose this true for $\tau^w(H) < n$. If $\tau^w(H) = n$ by 2) we know that $\tau^w(H) = \tau^{w-e}(H) + 1$, where $e \in E(H)$. Suppose that $e = v_k$, then by induction hypothesis there exists $y' = (y_1', \ldots, y_k')$ such that $\tau^{w-e}(H) = y'1$ and $Ay' \leq w - e, y' \geq 0, y_1'v_1 + \cdots + y_k'v_k \leq w - v_k$. Let $y = (y_1', \ldots, y_k' + 1)$ then $Ay \leq w, y \geq 0$ and integer and $y1 = y'1 + 1 = \tau^{w-e}(H) + 1 = \tau^w(H)$. But this implies $H$ is mengerian.

(1 $\Rightarrow$ 2) As $H$ is mengerian, $\tau^w(H) = wx = \nu^w(H) = y1$ where $x \geq 0, xA \geq 1$, $y \geq 0, Ay \leq w$ and $x, y$ integer vectors. Suppose that $\tau^w(H) > 0$. If $y = (y_1, \ldots, y_k)$, we have that $y_1v_1 + \cdots + y_kv_k \leq w$ and as $0 < \tau^w(H) = \nu^w(H) = y_1 + \cdots + y_k$, then $y_i > 0$ for some $i$. Suppose $y_k$ is precisely that element. Then define $w' = w - v_k > 0$, and $y' = (y_1, \ldots, y_k - 1)$. Then $y' \geq 0$, $Ay' \leq w - v_k = w'$, and $\nu^{w'}(H) \geq y'1 = y1 - 1$. Thus $\nu^{w'}(H) \geq \nu^w(H) - 1$ and $\tau^{w'}(H) \geq \tau^w(H) - 1$.

Now, let $p_{\text{min}} := \min_{p \in P} \{w(p)\}$ and define $p_{\text{min}}^*, q_{\text{min}}$ and $q_{\text{min}}^*$ in an analogous way. We construct the sets $P_\geq := \{p \in P : w(p) > p_{\text{min}}\}$ and $P_\geq^* := \{p^* \in P^* : w(p^*) > p_{\text{min}}^*\}$. In an analogous manner we construct the sets $Q_\geq$ and $Q_\geq^*$. Now take $\mathbb{P} \subseteq P_\geq \cup P_\geq^*$ and $Q_\geq \subseteq Q_\geq$ with maximum cardinality such that either $p_i$ or $p_i^*$ ($q_i$ or $q_i^*$) but not both is an element of $\mathbb{P}$ ($Q$). These subsets fulfill the following conditions.
Lemma 5.3 If \( p_{\text{min}} = p_{\text{min}}^* = 0 \) (\( q_{\text{min}} = q_{\text{min}}^* = 0 \)), and \( \tau^w > 0 \) then \( p = |P| \) (\( q = |Q| \)) and \( q_{\text{min}} + q_{\text{min}}^* \), \( w(r) \geq \tau^w \) (\( p_{\text{min}} + p_{\text{min}}^*, w(r^*) \geq \tau^w \)).

Proof. If \( p_{\text{min}} = p_{\text{min}}^* = 0 \) then \( p = |P| \) (if not, there would be \( p_i, p_i^* \) such that \( w(p_i, p_i^*) = 0 \), which is a contradiction). Now, that \( q_{\text{min}} + q_{\text{min}}^* \) and \( w(r) \) are greater or equal to \( \tau^w \) is a consequence of (5.b) and (5.c).

In the sequel we denote by \( \bar{t} \) a minimum weight \( t \in b(Q_{pq}^F) \).

Corollary 5.4 For the case stated in the former Lemma, we can pick \( e \) from:

\[
\begin{align*}
    e &= P Q r & \text{if } |P| = p \text{ and } q_{\text{min}} > 0 \\
    e &= P Q^* r & \text{if } |P| = p \text{ and } q_{\text{min}}^* > 0 \\
    e &= P Q r^* & \text{if } |Q| = q \text{ and } p_{\text{min}} > 0 \\
    e &= P^* Q r^* & \text{if } |Q| = q \text{ and } p_{\text{min}}^* > 0
\end{align*}
\]

such that \( \tau^w = \tau^{w'} + 1 \) and \( w' \geq 0 \).

Proof. That \( w' \geq 0 \) follows directly from the selection of \( e \). Thus, to prove that \( \tau^w = \tau^{w'} + 1 \) let \( p_{\text{min}} = p_{\text{min}}^* = 0 \). Therefore \( e \) will be either \( P Q r \) or \( P Q^* r \). Moreover, notice that every \( \bar{t} \) of type (5.a-d) intersects \( e \) in a single vertex. Consequently \( \tau^w = \tau^{w'} + 1 \).

Lemma 5.5 If \( \tau^w > 0 \) and \( \bar{T} \) is of type (5.b) or both (5.c,d), then:

\[
p = |P| \text{ or } q = |Q|
\]

Proof. If (9) is false then there must be \( p_i, q_k \) such that \( w(p_i) = p_{\text{min}}, w(p_i^*) = p_{\text{min}}^*, w(q_k) = q_{\text{min}}, w(q_k^*) = q_{\text{min}}^* \). Now, assume \( \bar{t} \) is of type (5.b), then since \( w(v v^*) \geq \tau^w \) we have that:

\[
\tau^w = w(p_i p_i^* q_k q_k^*) = p_{\text{min}} + p_{\text{min}}^* + q_{\text{min}} + q_{\text{min}}^* = w(p_i p_i^*) + w(q_k q_k^*) \geq 2 \tau^w
\]

which is a contradiction.

Now, consider that \( \bar{t} \) is of both types (5.c,d), and notice that either \( w(r) \) or \( w(r^*) \) are greater than zero (as a consequence of (5.a)). Assume that \( w(r) > 0 \), then it follows that:

\[
\tau^w = w(p_i p_i^* r) = p_{\text{min}} + p_{\text{min}}^* + w(r) = w(p_i p_i^*) + w(r) \geq \tau^w + 1
\]

once again, a contradiction.

Corollary 5.6 If \( \tau^w > 0 \), max \{p_{\text{min}}, p_{\text{min}}^* \} > 0, \max \{q_{\text{min}}, q_{\text{min}}^* \} > 0 \text{ and } \bar{T} \text{ is of type (5.b) or both types (5.c,d), then there is an edge } e \text{ of } Q_{pq}^F \text{ such that } w' \geq 0 \text{ and } \tau^w = \tau^{w'} + 1.\]
Proof. From Lemma 5.5 we know that either \( p = |\mathbb{P}| \) or \( q = |\mathbb{Q}| \). Therefore, consider that \( \tau^w = w(p_ip_jq_{k}q_{l}) \), then we have that \( w(p_ip_jq_{k}q_{l}) \leq w(p_ip_jr) \) and \( w(p_ip_jq_{k}q_{l}) \leq w(q_{k}q_{l}r^*) \) but this implies that:

\[
w(r) \geq w(q_{k}q_{l}^*) > 0, \quad w(r^*) \geq w(p_ip_j^*) > 0 \tag{10}
\]

On the other hand, if \( \tau^w = w(p_ip_jr) = w(q_{k}q_{l}r^*) = \tau^w \) we have that \( w(rr^*) \geq w(p_ip_jr) \), \( w(q_{k}q_{l}r^*) \) but this also implies (10). Therefore, we must pick \( e \) from (\ref{eq:5.9}). The selection of \( e \) guarantees that \( w' \geq 0 \), while the occurrence of \( \mathbb{P} \) or \( \mathbb{Q} \) in \( e \) guarantees that \( \tau^w = \tau^w' + 1 \).

\[\square\]

Corollary 5.7 If \( \tau^w > 0 \), \( \max \{p_{\text{min}}, p_{\text{min}}^*\} > 0 \), \( \max \{q_{\text{min}}, q_{\text{min}}^*\} > 0 \) and \( \mathcal{I} \) is not of type (\ref{eq:5.1} b) or both (\ref{eq:5.1} c) or (\ref{eq:5.1} d) then there is an \( e \in \mathbb{F}_{pq} \cup \mathbb{F}_{pq}^* \) such that \( \tau^w = \tau^w' + 1 \) and \( w' \geq 0 \).

Proof. Let us assume that \( \mathcal{I} = a_i a_j^* c \) is of type (\ref{eq:5.1} c or d). Then \( w(cc^*) \geq w(a_i a_j^* c) \), but this implies that \( w(c^*) \geq w(a_i a_j^*) \geq 0 \). Therefore \( e \) could be picked as:

\[
\begin{align*}
PQc^* & \quad \text{if } p_{\text{min}}, q_{\text{min}} > 0 \\
PQ^*c^* & \quad \text{if } p_{\text{min}}, q_{\text{min}} > 0 \\
P^*Qc^* & \quad \text{if } p_{\text{min}}, q_{\text{min}} > 0 \\
P^*Q^*c^* & \quad \text{if } p_{\text{min}}, q_{\text{min}} > 0
\end{align*}
\tag{11}
\]

Moreover, the selection of \( e \) guarantees that \( w' \geq 0 \) and since \( w(a_i a_j^* c) = w'(a_i a_j^* c) + 1 \) and \( w(a_i a_j^* c) < w(b_k b_l^* e^*) \) it is also true that \( w'(a_i a_j^* c) \leq w'(b_k b_l^* e^*) + 2 \). On the other hand, if \( \tau^w = w(vv) \) for some \( v \in \mathbb{P} \mathbb{Q} \) then we only need to pick \( e \) in such a way that \( w' \geq 0 \) since \( w(vv^*) = w'(vv^*) + 1 \), and considering that \( c^* \) is either \( r \) or \( r^* \) we can again pick \( e \) from (11).

As an example, consider \( \mathcal{I} = q_{k}q_{l}^* r^* \), and \( p_{\text{min}}, q_{\text{min}}^* > 0 \) then from (11) we know that \( e = PQ^*r \).

By the above results we obtain:

**Theorem 5.8** The hypergraph \( Q_{pq}^F \) with \( F = F_{pq}^* \) is mengerian.

Proof. By Theorem 4.2 \( Q_{pq}^F \) has the packing property. Thus \( Q_{pq}^F \) is an ideal hypergraph (Theorem 2.9). If \( \tau^w(Q_{pq}^F) > 0 \) then by the previous Corollaries 5.4, 5.6 and 5.7 there exists \( e \in E(Q_{pq}^F) \) such that \( \tau^w(Q_{pq}^F) = \tau^{w-e}(Q_{pq}^F) + 1 \). Therefore by Theorem 5.2 \( Q_{pq}^F \) is mengerian.

\[\square\]

Let us denote by \( I(Q_{pq}^F) = \mathcal{I} \) the ideal generated by \( F(Q_{pq}^F) = \{x_1, \ldots, x_k\} \), where \( v_i \) is the \( i \)th column of the matrix \( A(Q_{pq}^F) \), then:

**Corollary 5.9** \( \mathcal{R}[\mathcal{I}] \) is normal and the set covering polyhedron \( Q(A(Q_{pq}^F)) \) is integral.
Proof. It follows by applying Theorem 2.5.

Finally, we give an algorithm that constructs a list \( m \) with \( \tau^w \) edges from \( Q_{F}^E \) such that \( \sum_{e \in m} e \leq w \).

1. Set \( i = 0, m_0 = \emptyset \) and \( w_0 = w \)

2. while \( \tau^{w_i} \neq 0 \)
   
   (a) Obtain the values \( p_{\min}, p_{\min}^*, q_{\min}, q_{\min}^* \) for \( w_i \)
   
   (b) if \( \max \{p_{\min}, p_{\min}^*\} > 0, \max \{q_{\min}, q_{\min}^*\} > 0 \), then:
   
   i. if \( w_i(p_ip^*_jr) = w_i(q_kq^*_lr) = \tau^{w_i} \) or \( w_i(p_ip^*_jq_kq^*_l) = \tau^{w_i} \) then pick \( e \) from (8)
   
   ii. else pick \( e \) from (11)

   (c) else pick \( e \) from (8)

   (d) Set \( m_{i+1} = (e, m_i) \) and \( w_{i+1} = w_i - e \)

   (e) \( i = i + 1 \)

3. return \( m_i \)

Note that the former algorithm repeats the while statement in line 2, \( \tau^w \) times. This is so since for every iteration, \( e \) is picked in such a way that \( \tau^{w_i} = \tau^{w_{i-1}} + 1 \). Moreover, if \( m \) represents the returned value, we have that:

\[
\sum_{e \in m} e_v \leq w_v \text{ for every } v \in V_{pq}
\]

since otherwise, the vector \( w_{\tau^w} \) obtained at the end of the while cycle in line 2 would contain at least one negative entry. Therefore verifying if \( Q_{F}^E \) is mengerian. This algorithm can be generalized to other hypergraphs.

References

[1] M. Conforti, G. Cornuéjols, Clutters that Pack and the Max-Flow Min-Cut Property: A Conjecture, The fourth Bellairs Workshop on Combinatorial Optimization, W. R. Pulleyblank, F. B. Shepherd, eds. (1993).

[2] G. Cornuéjols, Combinatorial optimization: Packing and covering, CBMS-NSF Regional Conference Series in Applied Mathematics 74, SIAM (2001).
[3] G. Cornuéjols, F. Margot and B. Guenin, *The packing property*, Mathematical Programming, Ser. A, 89 (2000), 113-126.

[4] C. Escobar, R. Villarreal and Y. Yoshino, *Torsion freeness and normality of associated graded rings and Rees algebras of monomial ideals*, Commutative Algebra, Lect. Notes Pure Appl. Math. 244, Chapman and Hall/CRC, Boca Raton, FL, (2006), pp. 69-84.

[5] D. R. Fulkerson, A. J. Hoffman and R. Oppenheim, *On balanced matrices*, Mathematical Programming Study 1 (1974), 120-132.

[6] I. Gitler, E. Reyes and R.H. Villarreal, *Blowup algebras of square–free monomial ideals and some links to combinatorial optimization problems*, Rocky Mountain Journal of Mathematics, to appear, 2008.

[7] I. Gitler, C. Valencia and R. Villarreal, *A note on Rees algebras and the MFMC property*, Beiträge Algebra Geom. 48, No. 1 (2007), 141-150.

[8] C. Huneke, A. Simis and W. V. Vasconcelos, *Reduced normal cones are domains*, Contemp. Math. 88 (1989), 95-101.

[9] A. Lehman, *On the width-length inequality and degenerate projective planes*, (W. Cook and P.D. Seymour eds.), Polyhedral Combinatorics, DIMACS Series in Discrete Mathematics and Theoretical Computer Science 1, (1990), 101-105.

[10] A. Schrijver, *Combinatorial Optimization: Polyhedra and efficiency*, Algorithms and Combinatorics 24 (2003), Springer-Verlag, Berlin.

[11] P. D. Seymour, *The matroids with the max-flow min-cut property*, Journal of Combinatorial Theory, Series B 23 (1977), 189-222.