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Static and Dynamic Response of FG-CNT-Reinforced Rhombic Laminates

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Abstract: The present study focuses on the static and dynamic response of functionally graded carbon nanotube (FG-CNT)-reinforced rhombic laminates. The cubic variation of thickness coordinate in the displacement field is considered in terms of Taylor’s series expansion, which represents the higher-order transverse cross-sectional deformation modes. The condition of zero-transverse shear strain at upper and lower surface of FG-CNT-reinforced rhombic laminates is imposed in the present formulation. The present two-dimensional model is formulated in a finite element, with the \( C_0 \) element consisting of seven nodal unknowns per node. The final material properties of FG-CNT-reinforced rhombic laminates are estimated using the rule of mixture. The obtained numerical are compared with the results available in the literature to verify the reliability of the present model. The present study investigates the effect of CNT distribution, loading pattern, volume fraction, and various combinations of boundary constraints by developing a finite element code in FORTRAN.

Keywords: plates; carbon nanotubes; structure; skew; finite element method; deflection; frequency

1. Introduction

Recently, composite plates reinforced by carbon nanotubes (CNTs) have gained significant attention in civil, aeronautical, mechanical, and marine engineering due to their high strength/weight ratio and low density. The CNTs discovered by Iijima [1] are made up of the molecular-scale tube-like structure of carbon allotropes having fine materialistic properties. The CNTs are generally used in composites to improve their elastomechanical and thermal properties by dispersing in the matrix [2,3]. Various plate theories have been developed by the researchers to analyze the plates. The classical plate theory (CPT) model is based on the Kirchhoff–Love hypothesis that straight lines remain straight and perpendicular to the midplane after deformation. The transverse shear stress components are neglected in the CPT, where it is included in the shear deformation theory by Reissner [4]. In Reissner’s shear deformation theory, a shear correction factor is required for strain equations. Zhu et al. [5] discussed the effect of single-walled CNTs on the bending and vibration analysis of a CNT-reinforced...
composite (CNTRC) plate using first-order shear deformation theory (FSDT). They used the rule of the mixture to calculate the final material properties of CNTRC plate. The free vibration analysis of uniaxially aligned single-walled carbon nanotubes (SWCNTs)-reinforced composite was conducted by Lei et al. [6] by incorporating FSDT along with element-free kp-Ritz method. The static analysis of CNTRC cylinders using the mesh-free method was studied by Dastjerdi et al. [7]. Yas et al. [8] developed a three-dimensional solution for free vibration analysis of SWCNTs-reinforced composite cylindrical panel. The differential quadrature method and FSDT-based governing equation were used by Malekzadeh and Zarei [9] to examine the free vibration behavior of FG-CNT-reinforced laminated plate. The free vibration analysis of FG-CNTRC plate was presented by Nami and Janghorban [10] using the differential quadrature method. Shahrbabaki and Alibeigloo [11] used the Ritz method for the three-dimensional vibration analysis of CNTRC plate. They have calculated the effective material properties of the reinforced composite using the modified rule of mixture. Sankar et al. [12] used HSDT to study the static and free vibrations of FG-CNTRC plates and sandwich plates. The effect of uniform thermal environment on FG-CNTRC plate was studied by Mehar et al. [13] while the effect of non-uniform thermal load on FG-CNTRC beam was analyzed by Mayandi and Jeyaraj [14] using the finite element method. Zhang et al. [15] incorporated FSDT along with the element-free Ritz method to analyze a CNTRC plate with elastically restrained edges. Huang et al. [16] discussed the vibration and bending behavior of antisymmetric laminated functionally graded CNT-reinforced plate using FSDT containing four variables. Macias et al. [17] used HSDT for the static and free vibration analysis of FG-CNTRC skew plate using a four-noded shell element. Mirzaei and Kiani [18] used FSDT and Song et al. [19] considered HSDT for the vibration analysis of functionally graded CNTRC cylindrical panels and the ultimate properties of the composite were calculated by the refined rule of mixture. Thomas and Roy [20] studied the influence of UD-CNT and FG-CNT on the vibration of different type of shell structures reinforced by CNTs. The advanced fabrication and properties of aligned carbon nanotube composites were explained by Duong et al. [21]. Both Selim et al. [22] and Zhang and Selim [23] used a Reddy’s higher order shear deformable model for the vibration analysis of FG-CNTRC plate using the mesh-free method. Manevitch et al. [24] explored the nonlinear optical vibration behavior of SWCNTs. Fantuzzi et al. [25] studied the free vibration analysis of an arbitrarily shaped FG-CNT-reinforced plate using NURBS curves. The effect of agglomerated CNT on the linear static analysis of nanocomposite plate and shell was studied by Tornabene et al. [26]. Size phenomena in elastic nanobeams can be conveniently carried out by stress-driven nonlocal continuum mechanics proposed by Romano and Barretta [27] to overcome all difficulties of the strain-driven nonlocal theory of elasticity. A general treatment on nonlocal integral formulations of elasticity for nanomaterials is presented in [28] and successfully applied by [29–32] to size-dependent static and dynamic structural problems of current technical interest. Ansari et al. [33] provided a numerical solution for the vibration analysis of FG-CNTRC elliptical plates and the extended rule of the mixture was used for calculating the effective material properties. The free vibration analysis of nanocomposite plate and shell using FSDT was explored by Tornabene et al. [34]. Ardestani et al. [35] discussed the effect of orientation of CNT on the static and vibration behavior of FG-CNTRC skew plates.

The literature review indicates that few works for the static and dynamic analysis of FG-CNT-reinforced rhombic laminates are available. Therefore, in this paper, an effort was made to conduct a behavioral study of the FG-CNT rhombic laminates using HSDT, which removes the need for a shear correction factor. The finite element coding was developed by authors of the present mathematical model using a $C^0$ nine-noded finite element. Since results for FG-CNT-reinforced rhombic laminates subjected to trigonometrical loading are not available in the literature, the present analysis results may serve as a benchmark for the researchers working in this field.

2. Carbon Nanotube-Reinforced Laminates

Figure 1 depicts the geometry of the FG-CNT-reinforced plate used in the present study. The length and width of the plate are taken as $a$ and $b$ with the total thickness $h$. The middle section was taken
as a reference for the material coordinates (x, y, z) of the rhombic plate. Four types of distribution were considered, as adopted by Zhu et al. [5], namely UD, FG-O, FG-X and FG-V of CNTs inside a polymer matrix within CNT-reinforced composite plate in the thickness direction. The ultimate material properties of the FG-CNT-reinforced rhombic plate were determined in accordance with the rule of mixture [36,37].

![Figure 1](image_url)

**Figure 1.** The geometry and configuration of carbon nanotube reinforced plate.

The volume fraction of uniform distribution and functionally graded distributions of the CNTs along the thickness direction of the CNT-reinforced rhombic plates shown in Figure 1 was assumed to be as follows:

\[
V_{\text{CNT}}(z) = \begin{cases} 
V_{\text{CNT}}^* & (\text{UD}) \\
2 \left(1 - \frac{2|z|}{h}\right) V_{\text{CNT}}^* & (\text{FG-O}) \\
2 \left(\frac{2|z|}{h}\right) V_{\text{CNT}}^* & (\text{FG-X}) \\
(1 + \frac{2|z|}{h}) V_{\text{CNT}}^* & (\text{FG-V}) 
\end{cases} 
\]

(1)

where \( V_{\text{CNT}}^* = \frac{w_{\text{CNT}}}{\rho_{\text{CNT}}/(\rho_{\text{m}} - (\rho_{\text{CNT}}/\rho_{\text{m}})w_{\text{CNT}})} \),

where \( w_{\text{CNT}} \) is the mass fraction of the CNTs in the CNT-reinforced rhombic plates, whereas \( \rho_{\text{m}} \) and \( \rho_{\text{CNT}} \) are the densities of the polymer matrix and carbon nanotubes, respectively.
In line with the rule of mixture, the effective material properties of FG-CNT-reinforced plates were employed by introducing the CNT efficiency parameters; thus, the final properties can be written as follows [38]:

\[ E_{11} = \eta_1 V_{\text{CNT}} E_{11}^{\text{CNT}} + V_m E^m \]  
\[ \frac{\eta_2}{E_{22}} = \frac{V_{\text{CNT}}}{E_{22}^{\text{CNT}}} + \frac{V_m}{E^m} \]  
\[ \frac{\eta_3}{G_{12}} = \frac{V_{\text{CNT}}}{G_{12}^{\text{CNT}}} + \frac{V_m}{G^m} \]  
\[ v'_{12} = V_{\text{CNT}}^* v_{12}^{\text{CNT}} + V_m v^m \]  
\[ \rho_{12} = V_{\text{CNT}}^* \rho_{12}^{\text{CNT}} + V_m \rho^m, \]  

where \( E_{11}^{\text{CNT}} \) and \( E_{22}^{\text{CNT}} \) are Young’s moduli and \( G_{12}^{\text{CNT}} \) is the shear modulus of singly walled CNTs, respectively. \( E^m \) and \( G^m \) are known as Young’s modulus and shear modulus of the isotropic matrix. \( V_{12}^{\text{CNT}} \) and \( v^m \) represent the Poisson’s ratio of CNTs and matrix respectively. \( V_m \) and \( V_{\text{CNT}} \) are the volume fractions of the matrix and carbon nanotubes, respectively, and the sum of both volume fractions equals to unity. \( \eta_1, \eta_2, \eta_3 \) are the scale-dependent material properties and they can be calculated by matching the effective properties of CNT-reinforced composite obtained from the MD simulations with those from the rule of mixture.

3. Mathematical Formulation

3.1. Displacement Fields and Strains

The displacement field for the FG-CNT-reinforced rhombic plate is considered to derive the mathematical model based on the third-order shear deformation theory [39]:

\[ u(x, y, z) = U_0(x, y) + z \xi_x(x, y) + z^2 \xi_x(x, y) + z^3 \xi_x(x, y) \]  
\[ v(x, y, z) = V_0(x, y) + z \xi_y(x, y) + z^2 \xi_y(x, y) + z^3 \xi_y(x, y) \]  
\[ w(x, y, z) = W_0(x, y) \]  
\[ (7) \]

where \( u, v \) and \( w \) are the displacements of any generic point in the plate geometry, \((U_0, V_0)\) and \((W_0)\) are displacements at the mid-plane and \( \xi_x, \xi_y \) are the bending rotations defined at the midplane about the \( y \) and \( x \)-axes respectively. \( \xi_x, \xi_y, \bar{\xi}_x \) and \( \bar{\xi}_y \) are higher order terms of Taylor’s series expansion. The function \( \bar{\xi}_x, \bar{\xi}_y \) will be calculated by vanishing shear stress at top and bottom of the plate. By applying the boundary conditions \( \gamma_{xy}(x, y, \pm h/2) = \gamma_{yz}(x, y, \pm h/2) = 0 \) at the upper layer and lower layer of the plate in Equation (7) and rearranging the terms that appear in the displacement field \((u)\) and \((v)\), we obtained

\[ \bar{\xi}_x = \bar{\xi}_y = 0 \]  
\[ (8) \]
\[ \xi_x = \frac{4}{3h^2} \left( \frac{\partial w}{\partial x} \right), \xi_y = \frac{4}{3h^2} \left( \frac{\partial w}{\partial y} \right). \]  
\[ (9) \]

By substituting Equations (8) and (9) into Equation (7), we obtain

\[ u(x, y, z) = u_0 + \left( z - \frac{4z^3}{3h^2} \right) \xi_x - \frac{4z^3}{3h^2} \left( \frac{\partial w}{\partial x} \right) \]  
\[ v(x, y, z) = v_0 + \left( z - \frac{4z^3}{3h^2} \right) \xi_y - \frac{4z^3}{3h^2} \left( \frac{\partial w}{\partial y} \right) \]  
\[ w(x, y, z) = w_0 \]  
\[ (10) \]

If the displacement field represented in Equation (10) is implemented in the strain part, the problem of \( C^1 \) continuity requirement in the higher-order theory may arise due to the existence of
first-order derivatives of transverse displacement. By applying $C^0$ continuity to the present problem, the out of plane derivatives are exchanged by the following relations in Equation (10):

$$
\psi_x = \frac{\partial w}{\partial x}, \quad \psi_y = \frac{\partial w}{\partial y}
$$

(11)

The final form of higher order theory possessing $C^0$ continuity may be presented in the following manner:

$$
\begin{align*}
u(x, y, z) &= u_0 + \left(z - \frac{4z^3}{3h^2}\right) \theta_x - \frac{4z^3}{3h^2} \psi_x, \\
v(x, y, z) &= v_0 + \left(z - \frac{4z^3}{3h^2}\right) \theta_y - \frac{4z^3}{3h^2} \psi_y, \\
w(x, y, z) &= w_0.
\end{align*}
$$

(12)

Hence, the basic field variables interpreted in the present investigation with the assumption of constant transverse displacement component are $u_0$, $v_0$, $w_0$, $\theta_x$, $\theta_y$, $\psi_x$, and $\psi_y$ for each node. Mathematically, the nodal displacement vector $\{\delta\}$ corresponding to displacement field in Equation (12) may be represented as

$$
\{\delta\} = [u_0 \ v_0 \ w_0 \ \theta_x \ \theta_y \ \psi_x \ \psi_y]^T
$$

(13)

From the displacement field presented above in Equation (12), the strain can be written as

$$
\{\varepsilon\} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}\}^T.
$$

(14)

Furthermore, the expression of strain vector $\{\varepsilon\}$ can be correlated with the displacement vector $\{\delta\}$ by means of the following relationship:

$$
\{\varepsilon\} = [B]\{\delta\},
$$

(15)

where $[B]$ is known as a strain-displacement matrix and involves the derivatives of shape function terms. Since the plane stress problem is considered in the analysis, the components of strain vector may be represented as

$$
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x}, \\
\varepsilon_y &= \frac{\partial v}{\partial y}, \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.
\end{align*}
$$

(16)

The strain relationships can be written as

$$
\begin{align*}
\varepsilon_x &= \varepsilon_{x0} + z \left(1 - \frac{4z^3}{3h^2}\right) k_x - \frac{4z^3}{3h^2} k_x^s, \\
\varepsilon_y &= \varepsilon_{y0} + z \left(1 - \frac{4z^3}{3h^2}\right) k_y - \frac{4z^3}{3h^2} k_y^s, \\
\gamma_{xy} &= \gamma_{xy0} + z \left(1 - \frac{4z^3}{3h^2}\right) k_{xy} + \frac{4z^3}{3h^2} k_{xy}^s, \\
\gamma_{xz} &= \phi_x - \frac{4z^3}{3h^2} k_{xz} - \frac{4z^3}{3h^2} k_{xz}^s, \\
\gamma_{yz} &= \phi_y - \frac{4z^3}{3h^2} k_{yz} - \frac{4z^3}{3h^2} k_{yz}^s.
\end{align*}
$$

(17)
where
\[ \varepsilon_{x0} = \frac{\partial u_0}{\partial x}, \quad \varepsilon_{y0} = \frac{\partial v_0}{\partial y}, \quad \gamma_{xy0} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}, \]
\[ \phi_x = \frac{\partial w_0}{\partial x} + \theta_x, \quad \phi_y = \frac{\partial w_0}{\partial y} + \theta_y, \]
\[ k_x = \frac{\partial \phi_x}{\partial x}, \quad k_y = \frac{\partial \phi_y}{\partial y}, \quad k_{xy} = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}, \quad k_{xz} = \theta_x, \quad k_{yz} = \theta_y, \]
\[ k^*_x = \frac{\partial \psi_x}{\partial x}, \quad k^*_y = \frac{\partial \psi_y}{\partial y}, \quad k^*_{xy} = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}, \quad k^*_{xz} = \psi_x, \quad k^*_{yz} = \psi_y. \]

3.2. Constitutive Relationship

The stress-strain relationship for the CNTRC rhombic plate can be written as
\[ \{\sigma\} = [Q] \{\varepsilon\}, \]
where the constitutive matrix is
\[ \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix}, \]
where \( Q_{11} = \frac{E_{11}}{1 - \nu_{12} \nu_{21}}, \) \( Q_{22} = \frac{E_{22}}{1 - \nu_{12} \nu_{21}}, \) \( Q_{12} = \frac{v_{12}E_{11}}{1 - \nu_{12} \nu_{21}}, \)
\( Q_{33} = Q_{44} = Q_{55} = G_{12}. \)

4. Finite Element Formulation

4.1. Element Description

A nine-node \( C^0 \) isoparametric Lagrangian element was utilized in the present investigation. The element has a total of 63 degrees of freedom and each node has seven degrees of freedom. The element has inconsistent rectangular geometry in the \( x-y \) coordinate system. In order to ensure a consistent rectangular geometry, the element was plotted to the \( \xi-\eta \) plane. For the assumed nine-node element, the expressions for shape functions \( N_i \) are described below.

For corner nodes:
\[ N_1 = \frac{1}{4} (\xi - 1)(\eta - 1) \xi \eta, \quad N_3 = \frac{1}{4} (\xi + 1)(\eta - 1) \xi \eta, \quad N_7 = \frac{1}{4} (\xi - 1)(\eta + 1) \xi \eta, \quad N_9 = \frac{1}{4} (\xi + 1)(\eta + 1) \xi \eta. \]  

For middle nodes:
\[ N_2 = \frac{1}{2} (1 - \xi^2)(\eta^2 - \eta), \quad N_4 = \frac{1}{2} (\xi^2 - \xi)(1 - \eta^2), \quad N_6 = \frac{1}{2} (\xi^2 + \xi)(1 - \eta^2), \quad N_8 = \frac{1}{2} (1 - \xi^2)(\eta^2 + \eta). \]

For the center node:
\[ N_5 = (1 - \xi^2)(1 - \eta^2). \]

4.2. Governing Equation for Bending Analysis

The expression of strain energy may be given as
\[ \delta U = \iiint \{\delta \varepsilon\}^T \{\sigma\} \, dx \, dy \, dz. \]
By utilizing the relationship of Equation (18), the above equation can be written as
\[
\delta U = \iint \{ \delta \varepsilon \}^T [D] \{ \varepsilon \} \, dx \, dy
\]
where 
\[
[D] = \int [H]^T [Q] [H] \, dz,
\]  
(25)
where \([H]\) is the matrix that contains the terms involving \(z\) and \(h\).

The change in strain vector may be written as 
\[
\{ \delta \varepsilon \} = [B] \{ \delta X \}.
\]

By using Equations (15) and (25), the stiffness matrix \([K]\) can be written in the following form:
\[
[K] = \iint [B]^T [D][B] \, dx \, dy.
\]  
(26)

### 4.3. Governing Equation for Free Vibration Analysis

The time derivative of velocity at any given point within the element may be expressed in terms of the mid-surface displacement parameters \((u_0, v_0\) and \(w_0)\) as
\[
\frac{\partial^2}{\partial t^2} \{ f \} = -\omega^2 \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = -\omega^2 [F]\{ f \},
\]  
(27)
where the vector \(\{ f \}\) represents the nodal unknowns, which is of the \(7 \times 1\) order and contains the terms of Equation (7).

The nodal unknowns \(\{ f \}\) are decoupled into a matrix \([C]\) that involves the shape functions \(N_i\) and global displacement vector \(\{ X \}\):
\[
\{ f \} = [C]\{ X \},
\]  
(28)
where the matrix \(\{ X \}\) contains the nodal unknowns of the nine nodes.

By utilizing the Equations (27) and (28), the mass matrix of an element can be written as
\[
[m] = \iint [C]^T [L][C] \, dA,
\]  
(29)
where the expression of the matrix \([L]\) can be expressed as
\[
[L] = \int z \rho [F]^T [F] \, dz,
\]  
(30)
where \(\rho\) is termed as the density of the CNT-reinforced rhombic plate. The derivation of element stiffness matrix and the mass matrix is given in the Appendix A. Hence, the governing equation for free vibration analysis of rhombic plate becomes
\[
([K] - \omega^2 [M]) \{ X \} = \{ 0 \},
\]  
(31)
where \([K]\) and \([M]\) are the linear stiffness matrix and mass matrix, respectively.

### 4.4. Skew Boundary Transformation

For the rhombic plate shown in Figure 2, it is important to alter the element matrices from global axes \((x, y)\) to local axes \((x', y')\) because the skew boundary of the laminate is not parallel to the global
axes of the rhombic laminate. Hence, the transformation matrix \([T]\) is required at the element level to transform the element matrices from global to local axes.

\[
\text{Transformation matrix } [T] = \begin{pmatrix}
c & -s & 0 & 0 & 0 & 0 \\
s & c & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & c & -s & 0 \\
0 & 0 & 0 & s & c & 0 \\
0 & 0 & 0 & 0 & c & -s \\
0 & 0 & 0 & 0 & s & c
\end{pmatrix},
\]

where \(c = \cos \alpha, s = \sin \alpha\) and \(\alpha\) is the skew angle of the plate.

5. Numerical Results and Discussion

The static and free vibration analyses were performed for FG-CNT-reinforced rhombic plate under different combination of end support, volume fraction, and several geometric parameters. The above-discussed formulation has been incorporated into a computer code. The nine-noded isoparametric elements with seven degrees of freedom per node were chosen for the present model for discretizing the FG-CNT-reinforced rhombic plate. Poly[(m-phenylenevinylene)-co-(2,5-dioctoxy-p-phenylene)vinylene]], basically known as PmPV [40], was chosen as the matrix and the armchair \((10, 10)\) SWCNTs were considered as the reinforcing material. The material properties of the matrix were taken as \(E^m = 2.1\) GPa, \(\rho^m = 1150\) kg/m\(^3\) and \(v^m = 0.34\) at room temperature \(300\) K. The material properties of \((10,10)\) SWCNTs at 300K are tabulated in Table 1. Three types volume fraction were used in present study. In the case of \(V^{\ast}_{\text{CNT}} = 0.11, \eta_1 = 0.934\) and \(\eta_2 = 0.149\), in the case of \(V^{\ast}_{\text{CNT}} = 0.14, \eta_1 = 0.150\) and \(\eta_2 = 0.941\), and for \(V^{\ast}_{\text{CNT}} = 0.17, \eta_1 = 0.149\) and \(\eta_2 = 1.381\). We assume that \(\eta_2 = \eta_3\) and \(G_{12} = G_{13} = G_{23}\). The abovementioned values are used for the following numerical results.

| Temperature (K) | \(E_{11}^{\text{CNT}}\) (TPa) | \(E_{22}^{\text{CNT}}\) (TPa) | \(G_{12}^{\text{CNT}}\) (TPa) | \(f_{11}^{\text{CNT}} (10^{-6}/\text{K})\) | \(f_{12}^{\text{CNT}} (10^{-6}/\text{K})\) |
|----------------|-------------------------------|-------------------------------|-------------------------------|----------------------------------|----------------------------------|
| 300            | 5.6466                        | 7.0800                        | 1.9445                        | 3.4584                           | 5.1682                           |
| 500            | 5.5308                        | 6.9348                        | 1.9643                        | 4.5361                           | 5.0189                           |
| 700            | 5.4744                        | 6.8641                        | 1.9644                        | 4.6677                           | 4.8943                           |

Figure 2. The coordinate system of the rhombic plate.
The non-dimensional quantities used are:

- For the bending analysis
  \[ \bar{w} = \frac{w}{h}, \quad \bar{\sigma}_x = \sigma_x \left( \frac{a}{2}, \frac{b}{2}, z \right) \frac{h^2}{q_0 a^2}. \]  
  \( \text{(33)} \)

- For the free vibration analysis
  \[ \bar{\omega} = \omega \left( \frac{a^2}{h} \right) \sqrt{\frac{\rho m}{E m}}. \]  
  \( \text{(34)} \)

The loading patterns used for the bending analysis are:

- For uniform loading \( q = q_0 \);
- For sin-sin loading \( q = q_0 \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \);
- For cos-cos loading \( q = q_0 \cos \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) \).

\( \text{(35)} \)

The boundary conditions taken in the present analysis are as mentioned below:

1. Simply supported (SSSS):
   \[ v = w = \theta_y = \psi_y = 0 \text{ at } x = 0, a \]
   \[ u = w = \theta_x = \psi_x = 0 \text{ at } y = 0, b. \]

2. Clamped (CCCC):
   \[ u = v = w = \theta_x = \theta_y = \psi_x = \psi_y = 0 \text{ at } x = 0, a \text{ and } y = 0, b. \]

3. Clamped and simply supported (CCSS):
   \[ u = v = w = \theta_x = \theta_y = \psi_x = \psi_y = 0 \text{ at } x = 0, a \]
   \[ u = w = \theta_x = \psi_x = 0 \text{ at } y = 0, b. \]

5.1. Convergence and Validation of Present Formulation

5.1.1. Free Vibration Analysis

Example 1. The convergence study for the non-dimensional frequency parameter was carried out for UD and FG-CNT-reinforced rhombic plate shown in Table 2. The dimensionless frequency parameter of the UD, FG-V, FG-O, and FG-X type distributed CNTRC rhombic plate was computed for different mesh sizes and clamped boundary conditions. The results were computed for \( V^{*}_{\text{CNT}} = 0.11 \) and a skew angle of 15°. The convergence study indicated that 16 × 16 mesh is satisfactory for the free vibration analysis of functionally graded CNTRC rhombic plate using current nine-noded isoparametric elements. Hence, 16 × 16 mesh size was adopted for all the cases of free vibration analysis of a functionally graded CNT-reinforced rhombic plate.

Table 2. Convergence of non-dimensional frequency of FG-CNT-reinforced clamped rhombic plate.

| Mesh | UD   | FG-V | FG-O | FG-X |
|------|------|------|------|------|
| 8 x 8| 18.5932 | 18.0731 | 16.6424 | 19.2637 |
| 10 x 10 | 18.5915 | 18.0719 | 16.4112 | 19.2623 |
| 12 x 12 | 18.5906 | 18.0709 | 16.4093 | 19.2612 |
| 14 x 14 | 18.5894 | 18.0697 | 16.4085 | 19.2606 |
| 16 x 16 | 18.5892 | 18.0694 | 16.4084 | 19.2604 |
Example 2. Tables 3 and 4 show the results of the bending and free vibration analyses for an isotropic square plate ($\nu = 0.3$), respectively. The maximum deflection and axial stress were compared with the results provided by Reddy [41] and the frequency parameter of the isotropic plate was compared with an exact solution [42] and HSDT results for a moderately thick plate [43].

| $a/h$ | Source | $w$  | $\sigma_x$ |
|------|--------|------|-----------|
| 10   | Present| 4.666| 0.289     |
|      | Reddy [41]| 4.770| 0.289     |
| 20   | Present| 4.491| 0.287     |
|      | Reddy [41]| 4.570| 0.268     |
| 50   | Present| 4.408| 0.284     |
|      | Reddy [41]| 4.496| 0.266     |

Table 4. Comparison of non-dimensional frequency parameter of the simply supported square isotropic plate.

| Mode | Source Mode | (1, 1) | (1, 2) | (1, 3) |
|------|-------------|--------|--------|--------|
| Present | 0.093 | 0.221 | 0.415 |
| Mantari et al. [43] | 0.093 | 0.222 | 0.415 |
| Srinivas et al. [42] | 0.093 | 0.223 | 0.417 |

Example 3. Free vibration analyses of the FG-V CNT-reinforced composite plate for various $a/h$ ratios were presented in Tables 5 and 6. The first six non-dimensional frequencies were compared with Zhu et al. [5]. Three volume fractions ($V^*_{\text{CNT}} = 0.11$ and 0.14) and three side-to-thickness ratios ($a/h = 10, 20$ and 50) were taken for comparison. The frequency parameter of simply supported and clamped boundary condition was found to be closer to Zhu et al. [5].

Table 5. Comparison study of first six non-dimensional frequency parameter of UD CNT-reinforced composite plate.

| BC     | $V^*_{\text{CNT}}$ | Mode | $a/h = 10$ | $a/h = 20$ | $a/h = 50$ |
|--------|--------------------|------|------------|------------|------------|
| CCCC   | 0.11               | 1    | 17.625     | 18.284     | 28.400     |
|        |                    | 2    | 23.041     | 23.793     | 33.114     |
|        |                    | 3    | 33.729     | 35.188     | 44.559     |
|        |                    | 4    | 37.011     | 38.536     | 59.198     |
|        |                    | 5    | 37.317     | 38.738     | 61.851     |
|        |                    | 6    | 37.921     | 39.384     | 64.199     |
|        | 0.14               | 1    | 18.127     | 18.854     | 29.911     |
|        |                    | 2    | 23.572     | 24.374     | 34.516     |
|        |                    | 3    | 34.252     | 34.874     | 45.898     |
|        |                    | 4    | 34.650     | 36.267     | 45.898     |
|        |                    | 5    | 37.921     | 39.384     | 50.199     |
|        |                    | 6    | 37.972     | 39.592     | 50.199     |
| SSSS   | 0.11               | 1    | 13.332     | 13.885     | 17.355     |
|        |                    | 2    | 17.700     | 18.199     | 21.511     |
|        |                    | 3    | 19.449     | 19.422     | 23.299     |
|        |                    | 4    | 19.449     | 19.427     | 23.299     |
|        |                    | 5    | 27.569     | 28.121     | 33.898     |
|        |                    | 6    | 32.563     | 33.291     | 40.524     |
|        | 0.14               | 1    | 14.306     | 14.668     | 18.921     |
|        |                    | 2    | 18.362     | 18.870     | 22.867     |
|        |                    | 3    | 19.791     | 19.769     | 23.570     |
|        |                    | 4    | 19.791     | 19.774     | 23.570     |
|        |                    | 5    | 28.230     | 28.784     | 33.898     |
|        |                    | 6    | 33.646     | 34.492     | 51.422     |
Table 6. Comparison study of first six non-dimensional frequency parameter of FG-V CNT-reinforced composite plate.

| BC   | V*^CNT | Mode | a/h = 10 | a/h = 20 | a/h = 50 |
|------|--------|------|----------|----------|----------|
|      |        |      | Zhu et al. [5] | Present | Zhu et al. [5] | Present | Zhu et al. [5] | Present |
| CCCC | 0.11   | 1    | 17.211   | 17.753   | 26.304   | 26.693   | 34.165   | 34.480   |
|      |        | 2    | 22.812   | 23.462   | 31.496   | 32.099   | 39.043   | 39.584   |
|      |        | 3    | 33.070   | 34.035   | 43.589   | 44.133   | 51.204   | 51.815   |
|      |        | 4    | 33.552   | 34.355   | 56.249   | 57.061   | 72.202   | 71.954   |
|      |        | 5    | 36.528   | 37.889   | 59.249   | 60.253   | 86.291   | 86.133   |
|      |        | 6    | 37.437   | 38.841   | 62.608   | 62.218   | 89.054   | 89.105   |
|      | 0.14   | 1    | 17.791   | 18.405   | 27.926   | 28.371   | 37.568   | 37.909   |
|      |        | 2    | 23.413   | 24.113   | 32.976   | 33.629   | 42.175   | 42.733   |
|      |        | 3    | 34.101   | 34.792   | 44.989   | 45.573   | 53.963   | 54.590   |
|      |        | 4    | 34.275   | 35.553   | 58.951   | 59.698   | 74.785   | 74.546   |
|      |        | 5    | 37.538   | 39.053   | 61.816   | 63.051   | 94.022   | 93.911   |
|      |        | 6    | 38.159   | 39.574   | 64.135   | 63.758   | 96.573   | 96.680   |
| SSSS | 0.11   | 1    | 12.452   | 12.601   | 15.110   | 15.291   | 16.252   | 16.465   |
|      |        | 2    | 17.060   | 17.409   | 19.903   | 20.297   | 21.142   | 21.573   |
|      |        | 3    | 19.499   | 19.479   | 31.561   | 32.106   | 33.350   | 33.993   |
|      |        | 4    | 19.499   | 19.484   | 38.998   | 38.959   | 53.430   | 53.670   |
|      |        | 5    | 27.340   | 27.762   | 38.998   | 38.969   | 60.188   | 60.337   |
|      |        | 6    | 31.417   | 31.903   | 47.739   | 47.899   | 62.198   | 62.042   |
|      | 0.14   | 1    | 13.256   | 13.415   | 16.510   | 16.701   | 17.995   | 18.228   |
|      |        | 2    | 17.734   | 18.090   | 21.087   | 21.483   | 22.643   | 23.082   |
|      |        | 3    | 19.879   | 19.871   | 32.617   | 33.163   | 34.660   | 35.306   |
|      |        | 4    | 19.879   | 19.876   | 39.759   | 39.742   | 54.833   | 55.062   |
|      |        | 5    | 28.021   | 28.449   | 39.759   | 39.752   | 66.552   | 66.712   |
|      |        | 6    | 32.678   | 33.284   | 51.078   | 51.122   | 68.940   | 69.212   |

5.1.2. Bending Analysis

**Example 4.** Table 7 depicts the convergence study for dimensionless maximum deflection for UD and FG-CNT-reinforced functionally graded rhombic plate. The dimensionless maximum deflection of the UD, FG-V, FG-O and FG-X CNTRC rhombic plate was computed for different mesh size and clamped boundary condition. The results were computed for $V^*$^CNT = 0.11. The convergence study showed that $16 \times 16$ mesh size is acceptable for the present model using the discussed nine-noded isoparametric elements. Hence, $16 \times 16$ mesh size was chosen for all the parametric studies of the bending analysis of functionally graded CNTRC rhombic plate.

Table 7. Convergence of non-dimensional maximum deflection of CNT-reinforced functionally graded clamped rhombic plate.

| Mesh | UD   | FG-V  | FG-O  | FG-X  |
|------|------|-------|-------|-------|
| 8 ×  8 | 0.00372 | 0.00422 | 0.00588 | 0.00309 |
| 10 × 10 | 0.00361 | 0.00411 | 0.00576 | 0.00297 |
| 12 × 12 | 0.00355 | 0.00406 | 0.00568 | 0.00292 |
| 14 × 14 | 0.00349 | 0.00402 | 0.00561 | 0.00289 |
| 16 × 16 | 0.00345 | 0.00401 | 0.00557 | 0.00287 |

**Example 5.** The dimensionless central deflection ($w/h$) of the uniformly distributed CNT-reinforced composite plate with different side-to-thickness ratios and end supports were presented in Tables 8 and 9. The non-dimensional central deflection was compared with Zhu et al. [5]; $V^*$^CNT = 0.11, 0.14 and $a/h = 10, 20, 50$ were used for the comparison study. The dimensionless central deflection of different types of boundary condition was found to be in decent agreement with Zhu et al. [5].
Table 8. Comparison study of non-dimensional maximum deflection of various volume fraction of UD CNT-reinforced composite plate.

| BC  | V<sup>*</sup>CNT | a/h = 10 | | a/h = 20 | | a/h = 50 |
|-----|-----------------|---------|---------|---------|---------|---------|
|     | Zhu et al. [5]  | Present | Zhu et al. [5] | Present | Zhu et al. [5] | Present |
| CCCC | 0.11 | 0.00222 | 0.00207 | 0.01339 | 0.01257 | 0.2618 | 0.24056 |
|     | 0.14 | 0.00208 | 0.00192 | 0.01188 | 0.01115 | 0.2131 | 0.19644 |
| SSSS | 0.11 | 0.00373 | 0.00354 | 0.03628 | 0.03352 | 1.1550 | 1.04729 |
|     | 0.14 | 0.00330 | 0.00314 | 0.03001 | 0.02779 | 0.9175 | 0.83205 |
| SCSC | 0.11 | 0.00332 | 0.00313 | 0.03393 | 0.03127 | 1.0990 | 0.99624 |
|     | 0.14 | 0.00297 | 0.00281 | 0.02652 | 0.02634 | 0.8890 | 0.80555 |
| SFSF | 0.11 | 0.00344 | 0.00339 | 0.03341 | 0.03223 | 1.0680 | 1.01428 |
|     | 0.14 | 0.00302 | 0.00297 | 0.02760 | 0.02654 | 0.8505 | 0.80295 |

Table 9. Comparison study of non-dimensional maximum deflection of various volume fraction of UD CNT-reinforced composite plate.

| BC  | V<sup>*</sup>CNT | a/h = 10 | | a/h = 20 | | a/h = 50 |
|-----|-----------------|---------|---------|---------|---------|---------|
|     | Zhu et al. [5]  | Present | Zhu et al. [5] | Present | Zhu et al. [5] | Present |
| CCCC | 0.11 | 0.00211 | 0.00191 | 0.01150 | 0.01052 | 0.18940 | 0.16721 |
|     | 0.14 | 0.00198 | 0.00179 | 0.01036 | 0.00954 | 0.15600 | 0.13941 |
| SSSS | 0.11 | 0.00318 | 0.00294 | 0.02701 | 0.02398 | 0.79000 | 0.67655 |
|     | 0.14 | 0.00284 | 0.00266 | 0.02256 | 0.02021 | 0.62710 | 0.53777 |
| SCSC | 0.11 | 0.00286 | 0.00264 | 0.02587 | 0.02297 | 0.77280 | 0.66351 |
|     | 0.14 | 0.00258 | 0.00240 | 0.02184 | 0.01955 | 0.62060 | 0.53313 |
| SFSF | 0.11 | 0.00290 | 0.00276 | 0.02484 | 0.02281 | 0.73380 | 0.70308 |
|     | 0.14 | 0.00259 | 0.00248 | 0.02078 | 0.01916 | 0.58540 | 0.54605 |

5.2. Results and Discussion

The comparison study indicates that the present mathematical model and its finite element implementation results are in agreement with the previously published results. The present study has been conducted to investigate the effect of loading pattern, side-to-thickness ratios (a/h), aspect ratio (a/b), skew angle (α), volume fraction of CNT (V<sup>*</sup>CNT), and different boundary conditions (SSSS, CCCC, CCSS, CSCS, CCFF, and CFCF) on the bending and free vibration behavior of functionally graded CNT-reinforced composite rhombic plate.

5.2.1. Free Vibration Analysis

Tables 10 and 11 represent the first six dimensionless frequency parameter of UD-CNT- and FG-CNT-reinforced rhombic plate for the three different types of CNT volume fraction (V<sup>*</sup>CNT = 0.11, 0.14 and 0.17) and four different skew angles (α = 15°, 30°, 45° and 60°). The results were tabulated for simply supported and clamped boundary condition, respectively. It was noticed that an increase in the skew angle results in an increase in dimensionless frequency parameter for all types of CNTs distribution and all considered CNT volume fractions. An approximately 6% increase was noticed in the dimensionless fundamental frequency parameter of functionally graded CNTRC rhombic plate when skew angle changes from 15° to 30°, 22% and 60% increase was noticed for 15° to 45° and for 15° to 60°. Table 12 shows the dimensionless frequency parameter of FG-CNT-reinforced rhombic plate for CCSS-, CSCS-, CCFF-, and CFCF-type boundary support. For the all considered boundary conditions and skew angles, the FG-O distribution retains the minimum dimensionless frequency parameter while the FG-X distribution shows maximum values of dimensionless frequency parameter among other kinds of distribution. Additional distribution of CNTs should be provided at the top and the bottom section rather than at the mid-section for attaining maximum stiffness.
| Types | $V^\text{CNT}$ | Skew Angle | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 |
|-------|------------|------------|--------|--------|--------|--------|--------|--------|
| UD    | 0.11       | 15°        | 14.0884| 18.3200| 18.8946| 22.6183| 29.3328| 33.6414|
|       | 0.14       | 30°        | 14.8977| 18.8299| 21.3717| 28.4322| 33.1078| 35.1178|
|       | 0.17       | 45°        | 17.1862| 20.5115| 26.9900| 37.0144| 39.6322| 39.6568|
|       | 0.17       | 60°        | 22.7998| 24.0366| 38.2235| 46.2487| 47.8936| 50.9246|
|       | 0.11       | 15°        | 14.8687| 18.6487| 19.5642| 23.0233| 30.0198| 34.8422|
|       | 0.14       | 30°        | 15.6755| 19.1758| 22.0498| 28.9455| 33.9035| 36.3247|
|       | 0.17       | 45°        | 17.9727| 20.9102| 27.7332| 37.7063| 39.6322| 40.8949|
|       | 0.17       | 60°        | 23.2808| 24.8769| 38.2235| 46.2487| 47.8936| 50.9246|
| FG-V  | 0.11       | 15°        | 12.8212| 18.1269| 18.3759| 22.6786| 28.9595| 32.7212|
|       | 0.14       | 30°        | 13.6868| 18.8299| 21.3717| 28.4322| 33.1078| 35.1178|
|       | 0.17       | 45°        | 14.4859| 19.2484| 21.3970| 29.0713| 33.4956| 35.1799|
|       | 0.17       | 60°        | 22.7998| 24.0366| 38.2235| 46.2487| 47.8936| 50.9246|
| FG-O  | 0.11       | 15°        | 11.2246| 16.8568| 18.3759| 22.6786| 28.9595| 32.7212|
|       | 0.14       | 30°        | 12.1133| 18.8528| 20.6993| 28.4857| 32.6175| 33.8100|
|       | 0.17       | 45°        | 14.5197| 20.4388| 24.7329| 36.9725| 38.5706| 38.8945|
|       | 0.17       | 60°        | 22.7998| 24.0366| 38.2235| 46.2487| 47.8936| 50.9246|
| FG-X  | 0.11       | 15°        | 15.3579| 18.4618| 19.9956| 22.7929| 30.4082| 35.0962|
|       | 0.14       | 30°        | 16.1745| 18.9803| 22.4702| 28.6545| 34.2453| 36.6028|
|       | 0.17       | 45°        | 18.4949| 20.6851| 28.1124| 37.3147| 40.9182| 41.2241|
|       | 0.17       | 60°        | 23.0073| 25.4117| 39.5510| 46.7919| 48.3756| 52.4841|
|       | 0.11       | 15°        | 16.0673| 18.8614| 20.7220| 23.8551| 31.2995| 36.1507|
|       | 0.14       | 30°        | 16.9039| 19.3980| 23.2367| 29.2774| 35.2156| 37.6006|
|       | 0.17       | 45°        | 19.2847| 21.1602| 28.9762| 38.1470| 42.0428| 42.4396|
|       | 0.17       | 60°        | 23.5712| 26.3605| 40.6662| 48.1906| 49.6636| 53.9309|
|       | 0.11       | 15°        | 19.0583| 23.2864| 25.1259| 28.7507| 38.4760| 43.6835|
|       | 0.14       | 30°        | 20.1180| 23.9280| 28.2892| 36.1352| 43.2193| 45.9398|
|       | 0.17       | 45°        | 23.1055| 26.0614| 35.2483| 47.0376| 51.4225| 51.5545|
|       | 0.17       | 60°        | 28.9750| 31.9346| 49.7188| 58.8146| 60.8862| 65.8386|
Table 11. Variation of first six natural non-dimensional frequency parameters of clamped FG-CNT-reinforced rhombic plate.

| Types  | $V^\circ_{\text{CNT}}$ | Skew Angle | Mode     |
|--------|------------------------|------------|----------|
|        |                        |            | 1        | 2        | 3        | 4        | 5        | 6        |
| UD     | 0.11                   | 15°        | 18.5892  | 24.6235  | 35.3293  | 35.6482  | 39.8059  | 40.1566  |
|        |                        | 30°        | 19.7907  | 27.5192  | 37.5758  | 38.7334  | 44.7699  | 44.8668  |
|        |                        | 45°        | 23.1240  | 33.7718  | 43.5280  | 45.2188  | 54.8007  | 56.3342  |
|        |                        | 60°        | 32.8768  | 46.6645  | 58.8579  | 62.5042  | 70.8589  | 76.4394  |
|        | 0.14                   | 15°        | 19.1594  | 25.2162  | 36.0641  | 36.7261  | 40.8256  | 40.8465  |
|        |                        | 30°        | 20.3677  | 28.1613  | 38.6503  | 39.6187  | 45.6147  | 45.7882  |
|        |                        | 45°        | 23.7315  | 34.5566  | 44.6059  | 46.2941  | 55.7146  | 57.5412  |
|        |                        | 60°        | 33.6101  | 47.7344  | 60.1998  | 63.7591  | 72.4446  | 77.7145  |
| FG-V   | 0.11                   | 15°        | 18.1069  | 24.2993  | 34.8395  | 35.1140  | 39.2444  | 40.2623  |
|        |                        | 30°        | 19.3073  | 27.1997  | 36.8576  | 38.3514  | 44.4246  | 44.9823  |
|        |                        | 45°        | 22.7117  | 33.9303  | 43.0375  | 44.7306  | 54.9298  | 55.8988  |
|        |                        | 60°        | 32.5888  | 46.2233  | 58.3954  | 62.3472  | 70.4436  | 76.6148  |
|        | 0.14                   | 15°        | 18.7213  | 25.2162  | 36.0641  | 36.7261  | 40.8256  | 40.8465  |
|        |                        | 30°        | 20.3677  | 28.1613  | 38.6503  | 39.6187  | 45.6147  | 45.7882  |
|        |                        | 45°        | 23.7315  | 34.5566  | 44.6059  | 46.2941  | 55.7146  | 57.5412  |
|        |                        | 60°        | 33.6101  | 47.7344  | 60.1998  | 63.7591  | 72.4446  | 77.7145  |
| FG-O   | 0.11                   | 15°        | 16.4084  | 22.9046  | 31.6159  | 33.6029  | 36.6562  | 40.0537  |
|        |                        | 30°        | 17.7077  | 25.7058  | 33.8501  | 36.3280  | 42.3527  | 44.7306  |
|        |                        | 45°        | 21.2099  | 31.6586  | 40.5674  | 42.2311  | 52.8152  | 53.9899  |
|        |                        | 60°        | 31.2046  | 44.1065  | 55.5461  | 60.6186  | 66.8835  | 76.2352  |
|        | 0.14                   | 15°        | 17.0452  | 23.4601  | 32.8241  | 34.3062  | 37.6525  | 40.7285  |
|        |                        | 30°        | 18.3275  | 27.9173  | 38.0427  | 39.3412  | 45.5331  | 45.8315  |
|        |                        | 45°        | 23.3936  | 34.2698  | 44.2154  | 45.9247  | 55.9691  | 57.3257  |
|        |                        | 60°        | 33.3992  | 47.4078  | 59.8831  | 63.7136  | 72.2043  | 78.0564  |
| FG-X   | 0.11                   | 15°        | 19.2604  | 25.3608  | 36.1914  | 36.8063  | 40.4675  | 40.9340  |
|        |                        | 30°        | 20.4754  | 28.2998  | 38.7220  | 39.7120  | 45.2151  | 45.8357  |
|        |                        | 45°        | 23.8434  | 34.6543  | 44.6261  | 46.3103  | 55.2276  | 57.4279  |
|        |                        | 60°        | 33.6486  | 47.6758  | 59.9923  | 63.4780  | 72.0686  | 77.0374  |
|        | 0.14                   | 15°        | 19.8084  | 26.0430  | 37.1106  | 37.8532  | 41.2910  | 42.0442  |
|        |                        | 30°        | 21.0506  | 29.0493  | 39.8008  | 40.7351  | 46.1353  | 47.0096  |
|        |                        | 45°        | 24.9290  | 35.5521  | 45.7961  | 47.9199  | 56.3516  | 58.8017  |
|        |                        | 60°        | 34.4934  | 48.8548  | 61.4386  | 64.9465  | 73.7658  | 76.0494  |
|        | 0.17                   | 15°        | 24.0671  | 31.9388  | 45.6821  | 45.9560  | 50.9173  | 51.3375  |
|        |                        | 30°        | 25.6288  | 35.6476  | 48.4261  | 49.9805  | 56.8856  | 57.6593  |
|        |                        | 45°        | 29.9300  | 43.6056  | 55.9981  | 58.1580  | 69.4279  | 72.2037  |
|        |                        | 60°        | 42.3389  | 59.8760  | 75.2616  | 79.8475  | 90.3762  | 96.8913  |
Table 12. Variation of fundamental natural non-dimensional frequency parameters of FG-CNT-reinforced rhombic plate.

| Types | $V_{\text{CNT}}$ | Skew Angle $\theta$ | Boundary Condition |
|-------|------------------|---------------------|------------------|
|       |                  |                     | CCSS             | CSCS             | CCFF             | CFCF             |
| UD    | 0.11             | $15^\circ$          | 17.5656          | 16.1325          | 17.1211          | 5.8025           |
|       |                  | $30^\circ$          | 18.3524          | 17.1106          | 17.4540          | 5.8490           |
|       |                  | $45^\circ$          | 20.7037          | 19.8447          | 18.4603          | 5.9919           |
|       |                  | $60^\circ$          | 28.2486          | 27.7948          | 21.7009          | 6.3069           |
|       | 0.14             | $15^\circ$          | 18.1342          | 16.7688          | 17.7012          | 6.2424           |
|       |                  | $30^\circ$          | 18.9229          | 17.7494          | 18.0355          | 6.2905           |
|       |                  | $45^\circ$          | 21.2864          | 20.5072          | 19.0480          | 6.4353           |
|       |                  | $60^\circ$          | 28.9146          | 28.5757          | 22.3161          | 6.7554           |
|       | 0.17             | $15^\circ$          | 21.8982          | 20.0801          | 21.3412          | 6.2424           |
|       |                  | $30^\circ$          | 22.8820          | 21.3025          | 21.7584          | 6.2905           |
|       |                  | $45^\circ$          | 25.8166          | 24.7178          | 23.0172          | 7.1146           |
|       |                  | $60^\circ$          | 35.2478          | 34.6467          | 27.0680          | 7.8039           |
| FG-V  | 0.11             | $15^\circ$          | 17.0086          | 15.3184          | 16.5114          | 5.0487           |
|       |                  | $30^\circ$          | 17.8273          | 16.3356          | 16.8584          | 5.0936           |
|       |                  | $45^\circ$          | 20.4411          | 19.1336          | 17.8998          | 5.2408           |
|       |                  | $60^\circ$          | 27.9303          | 27.1387          | 21.2303          | 5.5617           |
|       | 0.14             | $15^\circ$          | 17.6618          | 16.0166          | 17.1825          | 5.4678           |
|       |                  | $30^\circ$          | 18.4787          | 17.0329          | 17.5285          | 5.5141           |
|       |                  | $45^\circ$          | 20.9045          | 19.8505          | 18.5709          | 5.6624           |
|       |                  | $60^\circ$          | 28.6673          | 27.9750          | 21.9207          | 5.9884           |
|       | 0.17             | $15^\circ$          | 21.2161          | 19.0620          | 20.5857          | 6.2207           |
|       |                  | $30^\circ$          | 22.2464          | 20.3420          | 21.0234          | 6.2774           |
|       |                  | $45^\circ$          | 25.2838          | 23.8570          | 22.3339          | 6.4629           |
|       |                  | $60^\circ$          | 34.9349          | 33.9038          | 26.5208          | 6.8657           |
| FG-O  | 0.11             | $15^\circ$          | 15.2966          | 13.7187          | 14.7110          | 4.3016           |
|       |                  | $30^\circ$          | 16.1749          | 14.7745          | 15.0980          | 4.3467           |
|       |                  | $45^\circ$          | 18.6942          | 17.6072          | 16.2095          | 4.4929           |
|       |                  | $60^\circ$          | 26.5093          | 25.5699          | 19.6311          | 4.7975           |
|       | 0.14             | $15^\circ$          | 15.9573          | 14.3811          | 15.4087          | 4.6680           |
|       |                  | $30^\circ$          | 16.8163          | 15.4187          | 15.7844          | 4.7147           |
|       |                  | $45^\circ$          | 19.3078          | 18.2378          | 16.8762          | 4.8622           |
|       |                  | $60^\circ$          | 27.1313          | 26.2562          | 20.2771          | 5.1739           |
|       | 0.17             | $15^\circ$          | 19.1484          | 17.0768          | 18.4438          | 5.2956           |
|       |                  | $30^\circ$          | 20.2204          | 18.3681          | 18.9177          | 5.3519           |
|       |                  | $45^\circ$          | 23.3092          | 21.8497          | 20.2813          | 5.5337           |
|       |                  | $60^\circ$          | 32.9609          | 31.7016          | 24.4972          | 5.9132           |
| FG-X  | 0.11             | $15^\circ$          | 18.2140          | 17.0254          | 17.7792          | 6.5860           |
|       |                  | $30^\circ$          | 19.0078          | 18.0146          | 18.1123          | 6.6352           |
|       |                  | $45^\circ$          | 21.3832          | 20.7913          | 19.1261          | 6.7811           |
|       |                  | $60^\circ$          | 29.0061          | 28.8746          | 22.3997          | 7.1022           |
|       | 0.14             | $15^\circ$          | 18.7355          | 17.6246          | 18.2942          | 7.0150           |
|       |                  | $30^\circ$          | 19.5470          | 18.6364          | 18.6340          | 7.0670           |
|       |                  | $45^\circ$          | 21.9758          | 21.4797          | 19.6696          | 7.2180           |
|       |                  | $60^\circ$          | 29.7641          | 29.7602          | 23.0163          | 7.5489           |
|       | 0.17             | $15^\circ$          | 22.7038          | 21.2134          | 22.1242          | 8.1485           |
|       |                  | $30^\circ$          | 23.7285          | 22.4908          | 22.5506          | 8.2111           |
|       |                  | $45^\circ$          | 26.7784          | 26.0495          | 23.8494          | 8.3974           |
|       |                  | $60^\circ$          | 36.4902          | 36.3121          | 28.0370          | 8.8035           |

Therefore, the FG-X and FG-O distributions yield maximum and minimum stiffness, respectively. Apart from this, the CCCC end support yields the highest dimensionless frequency parameter while the CFCF end support shows the lowest value among all considered boundary condition resulting from the fact that the higher constraints at support impart higher stiffness to the FG-CNTRC rhombic plate. The effect of the side-to-thickness ratio for various types of skew angles was presented in Table 13. The dimensionless fundamental frequency parameter for all type of CNT distribution was
increased along with the \( a/h \) ratio. The dimensionless frequency parameter also increases with the aspect ratio of FG-CNTR-reinforced rhombic plate, as depicted in Figure 3. The results were calculated for skew angles of 15° and 30°. The first four mode shapes for FG-V CNT-reinforced rhombic square plate having simply supported boundary condition and 30° skew angle are presented in Figure 4.

### Table 13. Variation of fundamental natural non-dimensional frequency parameters of FG-CNT-reinforced rhombic plate.

| Types | \( V^* \) CNT | Skew Angle | \( a/h \) | 5  | 20  | 50  | 100 |
|-------|----------------|-----------|---------|----|-----|-----|-----|
| UD    | 0.11           | 15°       | 9.0743  | 18.2012 | 20.3073 | 20.6787 |
|       |                | 30°       | 9.4150  | 18.9711 | 21.0669 | 21.4408 |
|       |                | 45°       | 10.2558 | 21.3334 | 23.4743 | 23.8687 |
|       |                | 60°       | 11.3999 | 29.4063 | 32.2438 | 32.8085 |
|       | 0.14           | 15°       | 9.3243  | 19.7973 | 22.5305 | 23.0303 |
|       |                | 30°       | 9.5879  | 20.5395 | 23.2468 | 23.7472 |
|       |                | 45°       | 10.4551 | 22.8478 | 25.5594 | 26.0749 |
|       |                | 60°       | 11.6404 | 30.8885 | 34.2574 | 34.9408 |
|       | 0.17           | 15°       | 11.3132 | 22.4497 | 24.9546 | 25.3933 |
|       |                | 30°       | 11.7725 | 23.4173 | 25.9121 | 26.3544 |
|       |                | 45°       | 12.8172 | 26.3808 | 28.9389 | 29.4075 |
|       |                | 60°       | 14.2430 | 36.4827 | 39.9125 | 40.5928 |
| FG-V  | 0.11           | 15°       | 8.7986  | 15.5123 | 16.6902 | 16.8853 |
|       |                | 30°       | 9.4269  | 16.4011 | 17.5995 | 17.8014 |
|       |                | 45°       | 10.2475 | 19.0147 | 20.3313 | 20.5637 |
|       |                | 60°       | 11.3736 | 27.4772 | 29.5441 | 29.9513 |
|       | 0.14           | 15°       | 9.1228  | 16.9116 | 18.4397 | 18.6988 |
|       |                | 30°       | 9.6238  | 17.7666 | 19.3058 | 19.5706 |
|       |                | 45°       | 10.4761 | 20.3260 | 21.9636 | 22.2570 |
|       |                | 60°       | 11.6465 | 28.7997 | 31.1984 | 31.6713 |
|       | 0.17           | 15°       | 10.9766 | 19.0705 | 20.4470 | 20.6737 |
|       |                | 30°       | 11.8498 | 20.1959 | 21.5996 | 21.8348 |
|       |                | 45°       | 12.8769 | 23.4938 | 25.0480 | 25.3214 |
|       |                | 60°       | 14.2886 | 34.1237 | 36.6062 | 37.0957 |
| FG-O  | 0.11           | 15°       | 8.0040  | 13.1635 | 13.9508 | 14.0776 |
|       |                | 30°       | 8.7975  | 14.0945 | 14.9077 | 15.0418 |
|       |                | 45°       | 10.2194 | 16.7496 | 17.6897 | 17.8561 |
|       |                | 60°       | 11.3501 | 25.0679 | 26.6826 | 27.0149 |
|       | 0.14           | 15°       | 8.3081  | 14.3293 | 15.3437 | 15.5098 |
|       |                | 30°       | 9.1008  | 15.2139 | 16.2464 | 16.4188 |
|       |                | 45°       | 10.4216 | 17.7910 | 18.9359 | 19.1387 |
|       |                | 60°       | 11.5944 | 26.0648 | 27.8938 | 28.2650 |
|       | 0.17           | 15°       | 9.9887  | 16.1395 | 17.0377 | 17.1815 |
|       |                | 30°       | 10.9641 | 17.2789 | 18.2081 | 18.3607 |
|       |                | 45°       | 12.7826 | 20.5334 | 21.6118 | 21.8026 |
|       |                | 60°       | 14.1922 | 30.7514 | 32.6180 | 33.0048 |
| FG-X  | 0.11           | 15°       | 9.2309  | 21.2410 | 24.8589 | 25.5557 |
|       |                | 30°       | 9.4901  | 21.9714 | 25.5444 | 26.2392 |
|       |                | 45°       | 10.3425 | 24.2577 | 27.7886 | 28.4923 |
|       |                | 60°       | 11.5036 | 32.2825 | 36.4515 | 37.3272 |
|       | 0.14           | 15°       | 9.4307  | 22.9391 | 27.5913 | 28.3538 |
|       |                | 30°       | 9.6990  | 23.6674 | 28.2521 | 29.1913 |
|       |                | 45°       | 10.5801 | 25.9622 | 30.4437 | 31.3820 |
|       |                | 60°       | 11.7856 | 34.0782 | 39.1212 | 40.2237 |
|       | 0.17           | 15°       | 11.6432 | 26.2338 | 30.6098 | 31.4489 |
|       |                | 30°       | 11.9640 | 27.1943 | 31.5188 | 32.3564 |
|       |                | 45°       | 13.0307 | 30.1785 | 34.4733 | 35.3262 |
|       |                | 60°       | 14.4875 | 40.5269 | 45.7146 | 46.8021 |
5.2.2. Static Analysis

The dimensionless maximum deflection of a FG-CNT-reinforced rhombic plate under uniform loading for simply supported and clamped boundary conditions is shown in Tables 14 and 15, respectively. The volume fraction of CNT was taken as 0.11, 0.14 and 0.17. The results were tabulated for UD and FG-CNT-reinforced rhombic plate with $a/h = 1$ and $a/h = 10$. It can be observed that an increase in the volume fraction of CNTs results in a decrease in the deflection of CNTRC rhombic plate because of the fact that the higher value of volume fraction has higher stiffness; thus, the deflection is reduced. It is anticipated that there is a nearly 36% decrease shown in maximum deflection for both clamped and simply supported boundary conditions as the value of $V^\text{**}_\text{CNT}$ increases from 0.11 to 0.17 and approximately 6% decreases are noticed when $V^\text{**}_\text{CNT}$ changes from 0.11 to 0.14. Maximum dimensionless deflection decreases with an increase in the skew angle because it reduces the length of the shorter diagonal leading to an enhancement in the stiffness of the rhombic plate. Thus, the deflection is reduced.

Figure 3. Variation of non-dimensional frequency parameter of FG-CNT-reinforced rhombic plate with aspect ratio; (A) $\alpha = 15^\circ$ and (B) $\alpha = 30^\circ$. 
Figure 4. The free vibration mode shapes of a SSSS square FG-V CNT-reinforced rhombic plate for skew angle 30° (A) 1st Mode; (B) 2nd Mode; (C) 3rd Mode and (D) 4th Mode.

Table 14. Variation of non-dimensional maximum deflection of FG-CNT-reinforced simply supported rhombic plate under uniform loading.

| Types  | $V^*_{CNT}$ | 15°    | 30°    | 45°    | 60°    |
|--------|-------------|--------|--------|--------|--------|
| UD     |             |        |        |        |        |
| 0.11   | 0.00345     | 0.00311| 0.00234| 0.00119|        |
| 0.14   | 0.00306     | 0.00278| 0.00212| 0.00110|        |
| 0.17   | 0.00222     | 0.00199| 0.00150| 0.00076|        |
| FG-V   |             |        |        |        |        |
| 0.11   | 0.00401     | 0.00365| 0.00267| 0.00128|        |
| 0.14   | 0.00354     | 0.00323| 0.00240| 0.00118|        |
| 0.17   | 0.00257     | 0.00234| 0.00170| 0.00081|        |
| FG-O   |             |        |        |        |        |
| 0.11   | 0.00557     | 0.00479| 0.00333| 0.00149|        |
| 0.14   | 0.00486     | 0.00424| 0.00302| 0.00140|        |
| 0.17   | 0.00360     | 0.00310| 0.00216| 0.00097|        |
| FG-X   |             |        |        |        |        |
| 0.11   | 0.00287     | 0.00261| 0.00201| 0.00106|        |
| 0.14   | 0.00260     | 0.00237| 0.00183| 0.00098|        |
| 0.17   | 0.00185     | 0.00167| 0.00127| 0.00066|        |
Table 15. Variation of non-dimensional maximum deflection of FG-CNT-reinforced clamped rhombic plate under uniform loading.

| Types  | $V^*_{CNT}$ | 15°  | 30°  | 45°  | 60°  |
|--------|-------------|------|------|------|------|
| UD     | 0.11        | 0.00201 | 0.00178 | 0.00131 | 0.00065 |
|        | 0.14        | 0.00187 | 0.00167 | 0.00124 | 0.00061 |
|        | 0.17        | 0.00127 | 0.00113 | 0.00083 | 0.00041 |
| FG-V   | 0.11        | 0.00214 | 0.00189 | 0.00137 | 0.00066 |
|        | 0.14        | 0.00197 | 0.00175 | 0.00128 | 0.00062 |
|        | 0.17        | 0.00136 | 0.00120 | 0.00087 | 0.00042 |
| FG-O   | 0.11        | 0.00263 | 0.00226 | 0.00158 | 0.00072 |
|        | 0.14        | 0.00241 | 0.00210 | 0.00148 | 0.00069 |
|        | 0.17        | 0.00166 | 0.00144 | 0.00101 | 0.00046 |
| FG-X   | 0.11        | 0.00186 | 0.00166 | 0.00123 | 0.00061 |
|        | 0.14        | 0.00174 | 0.00155 | 0.00116 | 0.00058 |
|        | 0.17        | 0.00117 | 0.00104 | 0.00077 | 0.00038 |

Tables 16 and 17 represent the dimensionless maximum deflection of simply supported and clamped FG-CNT-reinforced rhombic plate under sin-sin loading, respectively. Here, an approximately 25% decrease in the maximum dimensionless deflection is noticed when the skew angle changes from 15° to 30°; 40% decreases when the skew angle changes from 30° to 45° and 55% decreases when the skew angle changes from 45° to 60° for both uniform loading and sin-sin loading. The lowest and highest dimensionless deflection was found for FG-O- and FG-X-type CNT distribution, respectively.

Table 16. Variation of non-dimensional maximum deflection of FG-CNT-reinforced simply supported rhombic plate under sin-sin loading.

| Types  | $V^*_{CNT}$ | 15°  | 30°  | 45°  | 60°  |
|--------|-------------|------|------|------|------|
| UD     | 0.11        | 0.00215 | 0.00158 | 0.00092 | 0.00038 |
|        | 0.14        | 0.00192 | 0.00142 | 0.00084 | 0.00036 |
|        | 0.17        | 0.00138 | 0.00101 | 0.00059 | 0.00024 |
| FG-V   | 0.11        | 0.00248 | 0.00183 | 0.00102 | 0.00040 |
|        | 0.14        | 0.00220 | 0.00163 | 0.00093 | 0.00037 |
|        | 0.17        | 0.00159 | 0.00117 | 0.00065 | 0.00026 |
| FG-O   | 0.11        | 0.00339 | 0.00237 | 0.00126 | 0.00047 |
|        | 0.14        | 0.00297 | 0.00211 | 0.00115 | 0.00044 |
|        | 0.17        | 0.00219 | 0.00153 | 0.00081 | 0.00030 |
| FG-X   | 0.11        | 0.00181 | 0.00135 | 0.00081 | 0.00035 |
|        | 0.14        | 0.00165 | 0.00123 | 0.00074 | 0.00032 |
|        | 0.17        | 0.00116 | 0.00086 | 0.00051 | 0.00022 |

Figure 5 shows the variation of dimensionless deflection of FG-V CNT-reinforced rhombic plate along the length ($x/a$) at $y/b = 0.50$ for four skew angles under sin-sin loading. It can be seen that all values of CNT volume fraction have the same nature of deflection along the length and for the skew angle 60°, negative deflection is noticed for the farther end subjected to sin-sin loading.
Table 17. Variation of non-dimensional maximum deflection of FG-CNT-reinforced clamped rhombic plate under sin-sin loading.

| Types | $V^*_\text{CNT}$ | Skew Angle | 15°  | 30°  | 45°  | 60°  |
|-------|------------------|------------|------|------|------|------|
| UD    | 0.11             | 0.00136    | 0.00101 | 0.00059 | 0.00025 |
|       | 0.14             | 0.00127    | 0.00095 | 0.00056 | 0.00023 |
|       | 0.17             | 0.00086    | 0.00064 | 0.00038 | 0.00016 |
| FG-V  | 0.11             | 0.00145    | 0.00106 | 0.00062 | 0.00025 |
|       | 0.14             | 0.00134    | 0.00099 | 0.00058 | 0.00024 |
|       | 0.17             | 0.00092    | 0.00067 | 0.00039 | 0.00016 |
| FG-O  | 0.11             | 0.00177    | 0.00128 | 0.00071 | 0.00027 |
|       | 0.14             | 0.00163    | 0.00118 | 0.00067 | 0.00026 |
|       | 0.17             | 0.00112    | 0.00081 | 0.00045 | 0.00017 |
| FG-X  | 0.11             | 0.00126    | 0.00094 | 0.00056 | 0.00023 |
|       | 0.14             | 0.00118    | 0.00088 | 0.00052 | 0.00022 |
|       | 0.17             | 0.00080    | 0.00059 | 0.00035 | 0.00015 |

Figure 5. Non-dimensional deflection of FG-V CNT-reinforced rhombic plate along the central line for (A) 15°; (B) 30°; (C) 45° and (D) 60° skew angle subjected to sin-sin loading.

The effect of loading type on the non-dimensional deflection of simply supported and clamped FG-V type CNT-reinforced rhombic plate with skew angle was shown in Figure 6.
The effect of loading type on the non-dimensional deflection of simply supported and clamped FG-V type CNT-reinforced rhombic plate with skew angle was shown in Figure 6. (A) 

Figure 6. Variation of non-dimensional deflection of FG-V CNT-reinforced rhombic plate with the skew angle for (A) SSSS and (B) CCCC boundary condition.

The non-dimensional maximum deflection decreased with an increase in the skew angle for uniform and sin-sin loading, while under the cos-cos loading, the value of \( \tilde{w} \) increases first and then decreases as the skew angle grows. The effect of the skew angle on the maximum dimensionless deflection of CNT-reinforced rhombic plate subjected to sin-sin loading having various types of boundary condition was shown in Figure 7. For all considered boundary conditions except CFCF, the pattern of dimensionless deflection along the skew angle is linear. Figure 8 shows the variation in dimensionless deflection of FG-CNT-reinforced rhombic plate along the length of the central line for four types of side-to-thickness ratios subjected to sin-sin load. The same nature of deflection along the length was noticed for all values of \( a/h \). The results were calculated for the skew angle of 30° and simply supported boundary condition.
dimensionless deflection of FG-CNT-reinforced rhombic plate along the length of the central line for four types of side-to-thickness ratios subjected to sin-sin load. The same nature of deflection along the length was noticed for all values of a/h. The results were calculated for the skew angle of 30° and simply supported boundary condition.

Figure 7. Variation of non-dimensional deflection of FG-CNT-reinforced rhombic plate with the skew angle subjected to sin-sin loading for (A) CCSS, (B) CSCS, (C) CCFF and (D) CFCF boundary condition.

Figure 8. Cont.
The dimensionless value of axial stress decreases with an increase in the skew angle and same nature subjected to sin-sin loading for simply supported and clamped boundary condition, respectively. Appliance of variation in thickness coordinate was noticed for all values of skew angles.

Figures 8 and 10 show the variation of non-dimensional axial stress for FG-CNTRC rhombic plate subjected to sin-sin loading for simply supported and clamped boundary condition, respectively. The dimensionless value of axial stress decreases with an increase in the skew angle and same nature of variation in thickness coordinate was noticed for all values of skew angles.

![Figure 8](image1.png)

**Figure 8.** Non-dimensional deflection of FG-CNTRC skew plate along the length (y/b = 0.5) subjected to sin-sin loading for (A) a/h = 10; (B) a/h = 20; (C) a/h = 50 and (D) a/h = 100.

Figures 9 and 10 show the variation of non-dimensional axial stress for FG-CNTRC rhombic plate subjected to sin-sin loading for simply supported and clamped boundary condition, respectively. The dimensionless value of axial stress decreases with an increase in the skew angle and same nature of variation in thickness coordinate was noticed for all values of skew angles.

![Figure 9](image2.png)

**Figure 9.** Variation of non-dimensional axial stress for FG-CNT-reinforced rhombic plate subjected to sin-sin loading for (A) α = 15°, (B) α = 30°, (C) α = 45° and (D) α = 60°.
6. Conclusions

The static and free vibration analyses of FG-CNT-reinforced rhombic plate under various types of load considering various combinations of end support using an efficient $C^0$ finite element model based on TSDT were presented. The actual material properties at any given section are calculated using the rule of mixture. The following conclusions written below were drawn from the obtained results for numerous values of side-to-thickness ratio, skew angle, and aspect ratio, and different types of end support.

- The FG-O and FG-X type distributions inside the CNT rhombic plates have lower and higher non-dimensional frequency parameter as well as higher and lower dimensionless deflection, respectively.
- The rise in the CNTs volume fraction results in a decrease in the deflection and an increase in the frequency parameter of the CNT-reinforced rhombic plate.
- The dimensionless frequency parameter increases along with the skew angle, irrespective of the CNT distribution and boundary condition.
- Maximum dimensionless deflection and dimensionless normal stresses decrease along with the skew angle.

Figure 10. Variation of non-dimensional axial stress for FG-CNT-reinforced rhombic plate subjected to sin-sin loading (A) $\alpha = 15^\circ$, (B) $\alpha = 30^\circ$, (C) $\alpha = 45^\circ$ and (D) $\alpha = 60^\circ$. 

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Higher values of non-dimensional fundamental frequencies and lower values of dimensionless deflection are found for greater constraints on boundaries.

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Appendix A

Appendix A.1. Element Stiffness Matrix

The derivation of element stiffness matrix for the static and free vibration analysis was presented in this section. By employing the principle of minimum potential energy, the element stiffness matrix can be written as follows:

\[
[k] = \sum_{i=1}^{nu+n1} \int \left[ \left[ B \right]^T \left[ H \right] \left[ Q \right] \left[ H \right]^T \left[ B \right] \right] \text{dxdydz} + [p0],
\]

where \([D] = \sum_{k=1}^{n} \int [H]^T[H]dz\), in which \([B]\) is the strain matrix, \([Q]\) is the transformed material constant matrix and \([H]\) is the matrix consisting of the terms containing ‘\(z\)’ and some term related to material properties.

In the abovementioned expression, \([p0]\) is the penalty matrix added to the stiffness part to compensate for the replacement of the derivatives of transverse displacement \((\partial w/\partial x \text{ and } \partial w/\partial y)\) demanding \(C_1\) continuity by new \(C_0\) continuous variables \((w_1 \text{ and } w_2)\) following penalty approach, which is a well-known procedure in the finite element analysis. The penalty \([p0]\) matrix, is expressed as

\[
[p0] = \gamma \int \left( \left\{ \frac{\partial w}{\partial x} - w_1 \right\}^T \left\{ \frac{\partial w}{\partial x} - w_1 \right\} + \left\{ \frac{\partial w}{\partial y} - w_2 \right\}^T \left\{ \frac{\partial w}{\partial y} - w_2 \right\} \right) \text{dxdy},
\]

where \(\gamma\) is the penalty parameter; the value of \(\gamma\) in the present study was assumed as \(10^5\).

Appendix A.2. Element Mass Matrix

The consistent mass matrix can be derived in a similar manner to that of the stiffness matrix and was used in the vibration analysis. For the free vibration problem, the acceleration at any point within the plate may be expressed in terms of reference plane parameters:

\[
\{ \ddot{f} \} = \frac{\partial^2}{\partial t^2} \{ f \} = \left\{ \ddot{u}_0 \right\} = -\omega^2 \left\{ \begin{array}{c} u_0 \\ v_0 \\ w_0 \\ \theta_x \\ \theta_y \\ \psi_x \\ \psi_y \end{array} \right\} = -\omega^2 [F] \{ f \},
\]

where the matrix \([F]\) of order 3 \(\times\) 7 contains \(z\) and some constant quantities like that of \([H]\) and

\[
\{ f \} = \left[ u_0 \ v_0 \ w_0 \ \theta_x \ \theta_y \ \psi_x \ \psi_y \right]^T.
\]
It can finally be expressed in terms of nodal displacement vector \( \{ \delta \} \), as presented below:

\[
\{ f \} = [C] \{ X \},
\]

where \([C]\) is the matrix having an order of \( 7 \times 63 \) containing shape functions and its derivatives and \([X]\) is the nodal displacement vector containing nodal unknowns for all nine nodes and thus forming a matrix of order \( 63 \times 1 \).

Using the abovementioned equations, the consistent mass matrix of an element can be derived by applying Hamilton’s principle and it may be expressed as

\[
[m_e] = \sum_{i=1}^{n_u+n_l} \int \rho_i [C]^T[F]^T[F][C] \, dx\, dy\, dz = \int [C]^T[L][C] \, dx\, dy,
\]

where \( \rho_i \) is the mass density of the \( i \)-th layer and the matrix \([L]\) is

\[
[L] = \sum_{i=1}^{n_u+n_l} \int \rho_i[F]^T[F] \, dz.
\]

The stiffness matrix \([K_e]\) (which is \( 63 \times 63 \) in the present formulation) and mass matrix \([m_e]\) are computed for all the elements and assembled to form the overall stiffness matrix, \([K]\), and mass matrix, \([M]\), for the total structure. The skyline storage technique is used to keep these large size matrices \([K]\) and \([M]\) in a single array; thus, a considerable amount of storage space in core memory is saved in an efficient manner. This has been implemented systematically in the computer code developed in the present study.

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