MONOMIAL ALGEBRAS DEFINED BY LYNDON WORDS

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Abstract. Assume that \( X = \{x_1, \cdots, x_g\} \) is a finite alphabet and \( K \) is a field. We study monomial algebras \( A = K\langle X \rangle/(W) \), where \( W \) is an antichain of Lyndon words in \( X \) of arbitrary cardinality. We find a Poincaré-Birkhoff-Witt type basis of \( A \) in terms of its Lyndon atoms \( N \), but, in general, \( N \) may be infinite. We prove that if \( A \) has polynomial growth of degree \( d \) then \( A \) has global dimension \( d \) and is standard finitely presented, with \( d - 1 \leq |W| \leq d(d - 1)/2 \). Furthermore, \( A \) has polynomial growth iff the set of Lyndon atoms \( N \) is finite. In this case \( A \) has a \( K \)-basis \( N = \{l_{\alpha_1}l_{\alpha_2} \cdots l_{\alpha_d} | \alpha_i \geq 0, 1 \leq i \leq d\} \), where \( N = \{l_1, \cdots, l_d\} \). We give an extremal class of monomial algebras, the Fibonacci-Lyndon algebras, \( F_n \), with global dimension \( n \) and polynomial growth, and show that the algebra \( F_6 \) of global dimension 6 cannot be deformed, keeping the multigrading, to an Artin-Schelter regular algebra.

1. Introduction

Let \( X = \{x_1, x_2, \cdots, x_g\} \) be a finite alphabet. Denote by \( X^\ast \) the free monoid generated by \( X \), the empty word is denoted by 1. \( X^+ \) is the free semigroup generated by \( X \), \( X^+ = X^\ast - \{1\} \). Throughout the paper \( K\langle X \rangle \) stands for the free associative \( K \)-algebra generated by \( X \), where \( K \) is a field. As usual, the length of a word \( w \in X^+ \) is denoted by \( |w| \). We shall consider the canonical grading on \( K\langle X \rangle \), by length of words. We assume that each \( x \in X \) has degree 1.

Given an antichain of monomials \( W \subset X^+ \), the monomial algebra \( A = K\langle X \rangle/(W) \) is a particular case of a finitely generated augmented graded algebra with a set of obstructions \( W \), see [2] and [3]. Here and in the sequel \((W)\) denotes the two-sided ideal in \( K\langle X \rangle \) generated by \( W \). We shall study monomial algebras defined by Lyndon words.

This work together with [12] initiate the study of algebraic and homological properties of graded associative algebras for which the set of obstructions consists of Lyndon words. Lyndon words and Lyndon-Shirshov bases are widely used in the context of Lie algebras and their enveloping algebras, and also for PI algebras (see for example the celebrated Shirshov theorem of heights, [17]). It will be interesting to explore the remarkable combinatorial properties of Lyndon words in a more general context of associative algebras.

Anick studies the class of monomial algebras with finite global dimension \( d < \infty \), [2]. He proves that every such algebra either i) contains a free subalgebra generated by two monomials (and therefore has exponential growth); or ii) \( A \) is finitely
presented and has polynomial growth. In the second case he defines recursively a finite set $N$ of new generators for $A$, called atoms, with $|N| = d$, and uses the atoms to build a Poincaré-Birkhoff-Witt type $K$-basis of $A$ and to describe the structure of the monomial relations in $W$. Moreover, he proves that $A$ has the Hilbert series of a (usually nonstandard) graded polynomial ring:

$$H_A(t) = \prod_{i=1}^{n} \frac{1}{1 - t^{e_i}}$$

for some positive integers $e_1, \ldots, e_d$. It is amazing to see how Anick discovered that his atoms satisfy all good combinatorial properties of Lyndon words. (Possibly he did not know about Lyndon words or Lyndon’s theorem).

In this paper we study monomial algebras $A = K\langle X \rangle / (W)$, where $W$ is an antichain of Lyndon words in $X$. As a starting point we consider the most general case, when $W$ has arbitrary cardinality, and no assumptions for finiteness of growth, or global dimension are made. In this setting we introduce the set $N$ of Lyndon atoms, these are the Lyndon words which are normal modulo $(W)$. (In the context of Lie algebras these are often called standard Lyndon words, see [13]). The set $N$ contains $X$, and, in general, may be infinite, but exactly the atoms are involved in a constructive description of both the normal $K$-basis of $A$ and the set of relations $W$, so that it is easy to control the growth and the global dimension of $A$. Using the good combinatorial properties of Lyndon words we show that the set $W$ of monomial relations and the set $N$ of Lyndon atoms are very closely related. We prove that the monomial algebras $A$ defined by Lyndon words have a remarkable property:

If $A$ has polynomial growth of degree $d$ then $A$ has finite global dimension $d$ and is standard finitely presented with $d - 1 \leq |W| \leq d(d - 1)/2$.

In this case the normal $K$-basis of $A$ is $\mathfrak{N} = \{t_1^{k_1}t_2^{k_2} \cdots t_d^{k_d} \mid k_i \geq 0, 1 \leq i \leq d\}$, where $N = \{l_1, \ldots, l_d\}$ is the set of Lyndon atoms. Clearly, $\mathfrak{N}$ is a Poincaré-Birkhoff-Witt type $K$-basis of $A$, one can consider it as a particular case of Shirshov basis of height $d$.

Note that in the class of monomial algebras defined by Lyndon words our result complements a result by Anick which states that if a monomial algebra $A$ has a finite global dimension $d$ and does not contain two-generated free subalgebras, then $A$ has polynomial growth of degree $d$, [2], Theorem 6, but the proof of our results is independent of this theorem of Anick.

A natural question arises: whether our monomial algebras deform to Artin-Schelter regular algebras. We find a class of monomial algebras, Fibonacci-Lyndon algebras $F_n$, $n \geq 2$, which are extremal in the class of monomial algebras defined by Lyndon words. Each $F_n$ has global dimension $n$ and polynomial growth of degree $n$ and is uniquely determined up to isomorphism. The algebras $F_n$ are generated by two variables, hence they are $\mathbb{Z}^2$-graded. While $F_n$, with $n \leq 5$, has $\mathbb{Z}^2$-graded Artin-Schelter deformations, [9], we show that this is not the case for $F_6$. However we do not exclude that it may have singly graded such deformations.

One of our goals in the paper is to read off the properties of $A$ directly from its presentation, and, when this is possible, independently of Anick’s results. What we really use is his notion of $n$-chains and his purely combinatorial condition in terms of $n$-chains, necessary and sufficient for finite global dimension, see Fact 6.2 extracted from [2], Theorem 4. Note that all concrete monomial algebras $A$ with
i) finite global dimension and ii) polynomial growth given as examples in [2] are defined by antichains of Lyndon words. So it is natural to ask: is it true that if a monomial algebra $A$ satisfies i) and ii), then the set of defining relations $W$ consists of Lyndon words, w.r.t. appropriate enumeration of the generating set $X$. The answer is affirmative if $A$ is a quadratic algebra, i.e. $W$ consists of monomials of length 2, see [11], Theorem 1.1. In Section 3 we give an example of an algebra with three generators, satisfying i) and ii) and such that $W$ is not a set of Lyndon words, w.r.t. any ordering of $X$.

The paper is organized as follows. In Section 2 we give basic notions and state our main results, Theorems A and B. In Section 3 we give some background and motivation. In Section 4 we prove some results about Lyndon words, essential for the paper, we use these and Lyndon’s theorem to show that the normal $K$-basis of $A$ is built out of its Lyndon atoms $N$ and prove Theorem A. In Section 5 we investigate the close relations between the set $W$ of defining Lyndon words, and the set $N$ of Lyndon atoms. In Section 6 we find some combinatorial properties of $n$-chains, we show that the algebra has finite global dimension whenever the set $W$ is finite and prove Theorem B. In Section 7 we define and study the Fibonacci-Lyndon algebras $F_n$, and in Section 8 we show that $F_6$ has no deformation which is a bigraded Artin-Schelter algebra.

2. Definitions and results

As usual, $X^*$ and $X^+$ denote, respectively, the free monoid, and the free semigroup generated by $X$, ($X^+ = X^* - \{1\}$).

Consider the partial ordering on the set $X^+$ defined as: $a \preceq b$ iff $a$ is a proper subword (segment) of $b$, i.e. $b = uav$, $|b| > |a|$, but $u = 1$, or $v = 1$ is possible. In the case when $b = av$, $a, v \in X^+$, $a$ is called a proper left factor (segment) of $b$. Proper right factors are defined analogously.

Let $W \subseteq X^+$. If no two elements of $W$ are comparable for this partial order, $W$ is called an antichain of monomials. A monomial $a \in X^*$ is $W$-normal ($W$-standard) if $a$ does not contain as a subword any $u \in W$. Denote by $\mathcal{N}(W)$ the set of $W$-normal words.

$$\mathcal{N}(W) = \{ a \in X^* \mid a \text{ is } W\text{-normal} \}.$$ 

Note that the set $\mathcal{N}(W)$ is closed under taking subwords, Anick calls such a set an order ideal of monomials, [3, Sec. 1].

Order the alphabet by $x_1 < x_2 < \cdots < x_g$. The lexicographic order $<$ on $X^+$ is defined as follows: For any $u, v \in X^+$, $u < v$ iff either $u$ is a proper left factor of $v$, or $u = axb, v = ayc$ with $x < y, x, y \in X, a, b, c \in X^*$.

The following are well-known, see for example [13].

L1. For every $u \in X^*$ one has $a < b$ iff $ua < ub$.

L2. If $a$ is not a left segment of $b$, then for all $u, v \in X^+$ the inequality $a < b$ implies $au < bv$.

So “$<$” is a linear ordering on the set $X^+$ compatible with the left multiplication in $X^+$.

Remark 2.1. Note that the right multiplication does not necessarily preserve inequalities, for example $a < ax_2$, but $ax_3 > ax_2x_3$. Furthermore, the decreasing
chain condition on monomials is not satisfied on \((X^+, <)\), for if \(x, y \in X, x < y\), one has \(xy > x^2y > x^3y > \cdots\).

**Definition 2.2.** A nonperiodic word \(u \in X^+\) is a Lyndon word if it is minimal (with respect to \(<\)) in its conjugate class. In other words, \(u = ab, a, b \in X^+\) implies \(u < ba\). The set of Lyndon words in \(X^+\) will be denoted by \(L\). By definition \(X \subset L\).

Given an antichain \(W\) of Lyndon words, the set of \(W\)-normal Lyndon words will be denoted by \(N = N(W)\), we shall refer to \(N\) as the set of Lyndon atoms corresponding to \(W\). By definition it satisfies \(N = N(W) = N(W) \cap L\).

We shall study finitely generated monomial algebras \(A = K\langle X \rangle / (W)\), where \(|X| \geq 2\) and \(W\) is a nonempty antichain of Lyndon words. By convention we shall consider only minimal presentations of \(A\), so \(W \cap X = \emptyset\) and therefore \(X \subset N \subset \mathfrak{N}\).

In this case, inspired by Anick, \(\cite{2}\), we call \(N = N(W)\) the set of Lyndon atoms for \(A\).

Recall that the graded associative algebra \(A = K\langle X \rangle / (W)\) has polynomial growth if there is a real number \(d\) and a positive constant \(C\) such that for all \(n \geq 0\)

\[
\dim K A_n \leq Cn^d.
\]

The infimum of the possible \(d\)'s is the Gelfand-Kirillov dimension of \(A\).

The main results of the paper are the following two theorems which are stated under the the same hypothesis:

Assume that \(A = K\langle X \rangle / (W)\) is a monomial algebra, where \(W\) is an antichain of Lyndon words of arbitrary cardinality, \(N = N(W)\) is the set of Lyndon atoms, and \(\mathfrak{N}\) is the set of normal words modulo \((W)\).

**Theorem A.**  
(1) The set

\[
\{1^{k_1}2^{k_2} \cdots s^{k_s} | s \geq 1, \{l_1 > l_2 > \cdots > l_s\} \subseteq N, k_i \geq 0, 1 \leq i \leq s\}.
\]

is a \(K\)-basis of \(A\). It coincides with the set of normal words \(\mathfrak{N}\).

(2) The following conditions are equivalent:

(i) \(N\) is a finite set;

(ii) \(A\) has polynomial growth;

(iii) \(A\) is a PI algebra;

(iv) \(A\) can be embedded in a matrix ring over \(K\).

In this case \(N = \{l_1 > l_2 > \cdots > l_d\}\), \(A\) has a Poincaré-Birkhoff-Witt type \(K\)-basis \(\mathfrak{N} = \{l_1^{\alpha_1}l_2^{\alpha_2} \cdots l_d^{\alpha_d} | \alpha_i \geq 0, 1 \leq i \leq d\}\), so \(GK \dim A = d\) and its Hilbert series is:

\[
H_A(t) = \prod_{1 \leq i \leq d} \frac{1}{(1 - t^{\beta_i})}.
\]

**Theorem B.**  
(1) \(A = K\langle X \rangle / (W)\) is a standard finite presentation iff \(W\) is finite.

(2) Suppose \(W\) is a finite set of order \(|W| = r\), and \(m\) is the maximal length of words in \(W\). Then:

(i) the global dimension of \(A\) is finite and equals at most \(r + 1\);
(ii) the algebra $A$ has either polynomial growth or it contains a free subalgebra generated by two monomials;
(iii) $A$ has polynomial growth iff every word $l$ in $N$ has length $|l| \leq m - 1$.

(3) Suppose $A$ has polynomial growth of degree $d$. Then $A$ has finite global dimension $d$, and $W$ is of finite order with
\[ d - 1 \leq |W| \leq d(d - 1)/2, \]
so $A$ is standard finitely presented. Furthermore, the following conditions are equivalent:
(i) $|W| = d(d - 1)/2$;
(ii) $W = \{ x_i x_j \mid 1 \leq i < j \leq d \}$;
(iii) $N = X$.

Corollary 2.3. A has polynomial growth of degree $d$ if and only if $A$ has global dimension $d$ and does not contain a free subalgebra generated by two monomials.

3. Background and Motivation

It was shown by the first author, [11], Theorem 1.1, that an arbitrary finitely presented monomial algebra $A^0 = K \langle x_1, \ldots, x_n \rangle / (W)$ with quadratic monomial relations $W$ has polynomial growth and finite global dimension if and only if there is a (possibly new) enumeration of the generating set $X = \{ x_1, \ldots, x_n \}$, so that $X = \{ y_1 > y_2 > \cdots > y_n \}$ and $W = \{ y_i y_j \mid 1 \leq i \leq j \leq n \}$, that is $W$ consists of all Lyndon words of length 2 (w.r.t the new ordering). We believe it is interesting to know that assuming only quadratic monomial relations $W$, but no restrictions of their shape, or number, together with certain algebraic properties (see the equivalent conditions (1) ··· (6) below) lead to exactly $\binom{n}{2}$ defining relations $W$, each of which is a Lyndon word of length 2. For convenience of the reader we give the precise result which, of course, agrees with the general results of this paper.

Theorem 3.1. [11] Let $A^0 = K \langle x_1, \ldots, x_n \rangle / (W)$ be a quadratic monomial algebra. The following conditions are equivalent:

(1) $A^0$ has finite global dimension and polynomial growth.

(2) $A^0$ has finite global dimension and $|W| = \binom{n}{2}$.

(3) $A^0$ has polynomial growth, $W$ contains no square $x_i^2$, and $|W| = \binom{n}{2}$.

(4) The Hilbert series of $A^0$ is
\[ H_{A^0}(z) = \frac{1}{(1 - z)^n}. \]

(5) There is a permutation $y_1, \ldots, y_n$ of $x_1, \ldots, x_n$ such that the set
\[ \mathcal{M} = \{ y_1^{a_1} \cdots y_n^{a_n} \mid a_i \geq 0, 1 \leq i \leq n \}. \]
is a $K$-basis of $A^0$.

(6) There is a permutation $y_1, \ldots, y_n$ of $x_1, \ldots, x_n$, such that
\[ W = \{ y_i y_j \mid 1 \leq i < j \leq n \}. \]

One can consider Corollary 2.3 as a generalization of the equivalence of conditions (2) and (3) for Lyndon-type relations of arbitrarily high degrees.

It is shown in [11], Theorem 1.2, that each of the monomial algebras $A^0$ as above share the same obstruction set $W$ with various noncommutative quadratic algebras with binomial relations which are Artin-Schelter regular, produce solutions of the
Yang-Baxter equation, and have all good properties of the ring of commutative polynomials, like being Koszul and Noetherian domains.

**Example 3.2.** Let $A = K(x, y, z)/(xz, zy, xxy, xyy, zxy)$. Then $\omega = xzxyy$ is a 3-chain, but there are no 4-chains on $W$ so $A$ has global dimension 4. One uses the Ufnarovski graph $\Gamma(A)$, [19], (see also Section 5) to verify that $A$ has polynomial growth of degree 4, the cycles $\Gamma(A)$ correspond to the atoms (in the sense of Anick). In this case the atoms are $y > xy > z > x$, and the normal $K$-basis of $A$ is the set

\[ \mathcal{B} = \{ y^{\alpha_1} (xy)^{\alpha_2} z^{\alpha_3} x^{\alpha_4} | \alpha_i \geq 0, 1 \leq i \leq 4 \} \]

An easy computation verifies that there is no ordering $<$ on the alphabet $X = \{x, y, z\}$ such that each of the monomial relations is a Lyndon word w.r.t. $<$.

One can extract from the proof of Theorem 6 in [2] the following.

**Remark 3.3.** Let $A$ be a monomial algebra with global dimension $d$ and polynomial growth. Suppose $Y = \{y_1, y_2, \cdots, y_d\}$ is the set of its atoms (in the sense of Anick) enumerated according Anick’s total order “$ightarrow$”: $y_d \rightarrow y_{d-1} \rightarrow \cdots \rightarrow y_2 \rightarrow y_1$, see [1]. Then $A$ has a $K$-basis

\[ \mathcal{B} = \{ y_1^{k_1} y_2^{k_2} \cdots y_d^{k_d} | k_j \geq 0, 1 \leq j \leq d \} \]

This induces a new presentation of $A$ in terms of a new generating set $Y$ and new relations $W_0$:

\[ A \simeq K\langle Y \rangle / (W_0), \]

where $W_0 = \{y_jy_i | 1 \leq i < j \leq d\}$. The ordering is ”reverse” to the enumeration, i.e. $y_j \rightarrow y_i$ iff $j > i$, so the new relations are Lyndon words in the alphabet $Y$, with total ordering $\rightarrow$. However, whenever the original set of relations $W$ contains a monomial of degree $> 2$, some of the new generators (atoms) have degree $> 1$.

### 4. The normal bases of algebras defined by Lyndon words

In this section we prove some results on Lyndon words which are essential for the paper. We then describe the normal basis of an algebra defined by Lyndon words, and prove Theorem A.

We start with some basic facts about Lyndon words, our main reference is [14], Section 5.1., and [15], Section 11.5. (for Lyndon’s theorem). As usual, $L$ denotes the set of all Lyndon words in the alphabet $X$.

**Fact 4.1.**

1. A word $l$ is a Lyndon word if and only if $l < b$ for any proper right segment $b$ of $l$.
2. For all $w \in L$, the equality $w = ab$, with $a, b \in X^+$, implies $a < w < b$.
3. If $a < b$ are Lyndon words, then $ab$ is a Lyndon word, so $a < ab < b$.
4. If $b$ is the longest proper right segment of $l$ which is a Lyndon word, then $l = ab$, where $a$ is a Lyndon word. This is called the standard factorization of $l$ and denoted as $(a, b)$.
5. If $(a, b)$ is a standard factorization of a Lyndon word and $c$ is a Lyndon word with $ab < c \leq b$, then $abc$ is a Lyndon word with standard factorization $(ab, c)$.
6. (Lyndon’s Theorem) Any word $w \in X^+$ can be written uniquely as a non-increasing product $w = l_1 l_2 \cdots l_s$ of Lyndon words.
4.1. More results on Lyndon words. Next we prove some technical results on Lyndon words (in general context).

Notation 4.2. For monomials \(a, b \in X^+\) we shall write \(\widehat{a, b}\) if \(a = uv, b = vw\), where \(u, w \in X^*, v, uw \in X^+\) \((b = aw, or a = ub\) is possible). In the case when \(u, v, w \in X^+\) we say that \(a\) and \(b\) overlap.

We shall write \(a, \omega, b\) if \(\omega \subseteq ab\), and \(\omega\) overlaps with both \(a\) and \(b\), so that \(\widehat{a, \omega}\) and \(\widehat{\omega, b}\).

Lemma 4.3. (1) If \(v\) is a proper right segment of \(l \in L\), then \(v\) is not a left segment of \(l\). In other words a monomial of the shape \(l = va = bv\), \(a, b, v \in X^+\) can not be a Lyndon word.

(2) If \(v\) is a proper right segment of \(l = uv \in L\), then
\[
(4.1) lw = uvw < vw \quad \text{for all } w \in X^*.
\]

(3) Let \(uv\) and \(vw\) be Lyndon words. Then \(uvw\) is a Lyndon word.

(4) If \(a, b \in L\) and \(\widehat{a, b}\), then \(a < b\).

(5) Suppose that \(a, b, w \in L\), with \(a, w, b\). Then \(a < w < b\).

Proof. 1. Let \(w \in L\), and suppose \(v\) is a proper right segment of \(w\). Fact 4.1 implies \(w < v\). If we assume that \(v\) is also a proper left segment of \(w\), then one has \(v < w\), and therefore \(v < w < v\), which is impossible.

2. Let \(v\) be a proper right segment of \(l \in L\), then \(l = uv < v\), by Fact 4.1. Moreover, \(l\) is not in \(vX^+\), by part 1 so L2 implies (4.1).

3. Assume that \(uv, vw \in L\), and note first that since \(v\) is a proper right segment of the Lyndon word \(l = uv\), (4.1) is in force for all \(w \in X^+\).

We have to show that \(uvw\) is a Lyndon word, so by Fact 4.1 (1) it will be enough to verify that \(uvw < b\) holds whenever \(b\) is a proper right segment of \(uvw\). Three cases are possible: i) \(b\) is a proper right segment of \(vw\); ii) \(b = vw\); iii) \(b = cvw\), where \(c \in X^+\) is a proper right segment of \(u\). Assume (i) holds. Fact 4.1 implies \(vw < b\) which together with (4.1) implies \(uvw < vw < b\). In case ii) the relation (4.1) gives straightforwardly \(uvw < b = vw\). Assume iii) holds. The monomial \(cv\) is a proper right segment of the Lyndon word \(uv\), therefore part 2 implies
\[
(4.2) uvw < cvw = b, \quad \forall w \in X^*.
\]

We have verified part (3).

4. By assumption \(a = uv \in L\), and \(b = vw \in L\), where \(v, uv \in X^+\). If \(u = 1\), then \(w \neq 1\), so \(a = v\) is a proper left segment of \(b\) and therefore \(a < b\). Similarly, if \(w = 1, b = v\), then \(u \neq 1, a = uv = ub \in L\), hence \(a < b\). If \(u, w \in X^+\), then by part 3 uvw is a Lyndon word with a proper left segment \(a\) and a proper right segment \(b\), hence \(a < uvw < b\).

5. By assumption \(a, b, w \in L\) and \(\widehat{a, w, b}\), hence (by definition) \(\widehat{a, w}\) and \(\widehat{w, b}\) which, by part 4, implies \(a < w\), and \(w < b\). \(\square\)

Lemma 4.4. (1) Let \(a < b\) be Lyndon words. Then \(a^k b^l\) are Lyndon words for all \(k, l \geq 1\).

(2) If \(l = ab\) is the standard factorization of the Lyndon word \(l\), then the standard factorization of \(ab^k\) is \((ab^{k-1}, b)\).
Proof. Let $1$. We use induction on $k$ and $l$. By Fact 4.1(3), $ab$ is a Lyndon word. Suppose some $a^kb^l$ is a Lyndon word with $k,l \geq 1$. By Fact 4.1 part (2), the product $a(a^kb^l) = a^{k+1}b^l$ of the Lyndon words $a < a^kb^l$ is a Lyndon word. Similarly, $a^kb^l < b$ are Lyndon words, so $a^kb^l+1$ is also a Lyndon word.

2. This follows by induction and by Fact 4.1(9). □

Lemma 4.5. Let $l_1 \geq l_2 \geq \cdots \geq l_s$ be Lyndon words, $s \geq 2$. If a Lyndon word $u$ is a subword of $l_1l_2 \cdots l_s$, then $u$ is a subword of $l_i$, for some $i$, $1 \leq i \leq s$.

Proof. Let $u \in L$ be a subword of $l_1l_2 \cdots l_s$ and assume that $u$ is not a subword of $l_i$ for $1 \leq i \leq s$. Then there are overlaps $l_i, u, l_j$ for some $1 \leq i < j \leq s$, hence, by Lemma 4.3(1) one has $l_i < u < l_j$, which contradicts the hypothesis. □

Corollary 4.6. Every Lyndon word $a$ of length $\geq 2$ contains a subword of the form $x_i x_j$, $1 \leq i < j \leq g$.

Proof. Let $a \in L$ with $|a| \geq 2$, and assume, on the contrary, that $a$ does not contain any subword $x_i x_j$, where $1 \leq i < j \leq g$. Then $a = y_1y_2 \cdots y_s$, where $y_1 \geq y_2 \geq \cdots \geq y_s$ are in $X$. By definition $X \subseteq L$, so the Lyndon word $a$ is a (non-proper) subword of the product $y_1y_2 \cdots y_s$ of non-increasing Lyndon words $y_i \in L$. Lemma 4.5 implies that $a$ is a subword of some $y_i$, which is impossible. □

4.2. The normal basis of $A$. In this subsection, as usual, each of the sets $W$ and $N = N(W)$ may have arbitrary cardinality. Lemma 4.5 implies straightforwardly the following.

Lemma 4.7. Let $W$ be an antichain of Lyndon words and let $N = N(W)$ be the corresponding set of $W$-normal Lyndon atoms in $X^+$. Let $\mathfrak{N}$ be the set of all words in $X^+$ which are normal modulo the ideal $(W)$. Assume that $N$ contains the Lyndon words $l_1 > l_2 > \cdots > l_s$. Then $\mathfrak{N}$ contains the set

$$(4.3) \quad T(l_1, \cdots, l_s) = \{ w \in X^* \mid w = t_{i_1} t_{i_2} \cdots t_{i_s}, k_i \geq 0 \}.$$ 

Proposition 4.8. Suppose $A = K(X)/(W)$ is a monomial algebra, where $W$ is an antichain of Lyndon words of arbitrary cardinality. Let $N = N(W)$ be the corresponding set of $W$-normal Lyndon atoms in $X^+$, and let $\mathfrak{N}$ be the set of normal words modulo $(W)$. Then $\mathfrak{N}$ coincides with the set given in (4.3), and is a $K$-basis of $A$.

Proof. It is well-known in the theory of non-commutative Groebner bases that the set of normal monomials $\mathfrak{N}$ is a $K$-basis of $A$. Clearly, the set given in (4.3) is the union

$$T = \bigcup_{s \geq 1} \{ l_1 > l_2 > \cdots > l_s \} \subseteq N$$

We shall show that $\mathfrak{N} = T$. Let $u \in \mathfrak{N} - \{ 1 \}$. Clearly $u \in X^+$, hence by Lyndon’s Theorem (see Fact 1.1) it can be written uniquely as a product $u = u_1^{k_1} u_2^{k_2} \cdots u_s^{k_s}$ of Lyndon words, where $s \geq 1$, $u_1 > u_2 > \cdots > u_s$, $k_i \geq 1, 1 \leq i \leq s$. As a subword of a normal word, each $u_i$ in this product is also normal, so $u_1, u_2, \cdots, u_s \in N$. Therefore $u \in T(u_1, \cdots, u_s)$, see (4.3). We have shown the inclusion $\mathfrak{N} \subseteq T$. The reverse inclusion follows from Lemma 4.7 □
Proof of Theorem A. Part (1) follows by Proposition \[4.8\]

(2) First we show the implication $|N| < \infty \implies GK \dim A = |N|$. Assume that $N$ has finite order $d$, so $N = \{l_1 > l_2 > \cdots > l_d\} \subseteq L$, and by part (1) the $k$ normal basis of $A$ has the desired form. The vector spaces isomorphism $A \cong \text{Span}_K \mathfrak{M}$ implies that $A$ has polynomial growth of degree $d$ and the Hilbert series of $A$ is the same as the Hilbert series of a polynomial ring where the generators have degrees $|l_i|$ for $i = 1, \ldots, d$. This gives the stated form of the Hilbert series.

Next we show that, conversely, $GK \dim A = d < \infty \implies |N| = d$. Suppose that $GK \dim A = d$. If we assume that for some $s > d$, $N$ contains the set of Lyndon atoms $\{l_1 > l_2 > \cdots > l_s\}$, then by Lemma \[4.7\] the normal $k$-basis $\mathfrak{M}$ of $A$ contains the set $T$ given in \[4.3\]. This implies that $GK \dim A \geq s > d$, a contradiction. Therefore $N$ is a finite set with $|N| \leq d$. It follows from the first implication that $GK \dim A = |N|$. This gives the equivalence of (i) and (ii). It is proven in a more general context (no restriction on the shape of $W$) that for a finitely presented monomial algebra $A$ conditions (ii), (iii), and (iv) are equivalent, see \[8\]. Part (2) has been proved.

5. Determining polynomial growth

5.1. Relations between $W$ and $N(W)$. We have seen that each antichain $W$ of Lyndon monomials determines uniquely a set $N = N(W) \subseteq L$, we refer to it as the set of Lyndon atoms corresponding to $W$. It satisfies the following conditions:

- **C1.** $X \subseteq N$.
- **C2.** $\forall v \in L, \forall u \in N, v \sqsubseteq u \implies v \in N$.
- **C3.** $u \in N \iff u \in L$ and $u \notin (W)$.

Conversely, each set $N$ of Lyndon words satisfying conditions **C1** and **C2** determines uniquely an antichain of Lyndon monomials $W = W(N)$, such that condition **C3** holds, and $N$ is exactly the set of Lyndon atoms corresponding to $W$. Indeed, let $C = L - N$ be the complement of $N$ in $L$, and let $W = W(N)$ be the antichain of all minimal w.r.t. $\sqsubseteq$ elements in $C$. Then one has $N = N(W(N))$. We shall refer to $W = W(N)$ as the antichain of Lyndon words corresponding to $N$. Proposition \[5.4\] below gives some of the close relations between the sets $W$ and $N(W)$ on set-theoretic level. Our previous discussion implies straightforwardly parts (1), (2) and the first statement in part (3). The remaining parts are extracted from Theorems A and B and are given only for completeness.

**Proposition 5.1.**

1. There exists a one-to-one correspondence between the set $\mathcal{W}$ of all antichains $W$ of Lyndon words with $X \cap W = \emptyset$ and the set $\mathcal{N}$ consisting of all sets $N$ of Lyndon words satisfying conditions **C1** and **C2**. In notation as above this correspondence is defined as

$$\phi : \mathcal{W} \rightarrow \mathcal{N}, \quad W \mapsto N(W)$$

and each pair $(N = N(W), W)$ (respectively $(N, W = W(N))$ obtained via this correspondence satisfies condition **C3**.

2. If $N \in \mathcal{N}$ is a finite set of order $d$, then the corresponding antichain $W = W(N)$ is also finite with $|W| \leq d(d-1)/2$. 

(3) If $N \in \mathcal{N}$ is a finite set of order $d$, then the corresponding antichain $W = W(N)$ is also finite with $|W| \leq d(d-1)/2$. 

\[\square\]
(4) Each finite antichain $W \in \mathcal{W}$ determines a monomial algebra $A = K\langle X \rangle / (W)$ of finite global dimension, $\text{gl} \dim A \leq |W| + 1$.

(5) Each $N \in \mathbb{N}$ determines uniquely a monomial algebra $A = K\langle X \rangle / (W)$, with a set of defining relation $W = W(N)$ and a set of Lyndon atoms precisely $N$. The algebra $A$ has polynomial growth of degree $d$ iff $|N| = d$.

We shall need the following Lemma, extracted from a more general result in [12].

**Lemma 5.2.** Let $W$ be an antichain of Lyndon words, and assume $N = \{l_1 < l_2 < \cdots < l_d\}$ has finite order $d$. Then there is an inclusion of sets:

$$\{l_1 l_{i+1} \mid 1 \leq i < j \leq d\} \subseteq W.$$ 

In particular, $d - 1 \leq |W|$.

The following theorem gives some of the intimate relations between $W$ and $N(W)$ on the level of words.

**Theorem 5.3.** Let $W$ be an antichain of Lyndon words, let $N = N(W)$ be the corresponding set of Lyndon atoms.

1. If $u$ is a proper Lyndon subword of some $w \in W$, then $u$ is a Lyndon atom, so $u \in N$.

2. Every word $w \in W$ factors as $uv$, where $u < v \in N$.

3. If $N = \{l_1 < l_2 < \cdots < l_d\}$ has finite order $d$, then $W$ is also finite with $d - 1 \leq |W| \leq d(d - 1)/2$, and there are inclusions of sets:

$$\{l_i l_{i+1} \mid 1 \leq i < j \leq d\} \subseteq W \subseteq \{l_i l_j \mid 1 \leq i < j \leq d\}.$$ 

Moreover, if $s$ is the maximal length of words in $N$, then each $w \in W$ has length $|w| \leq 2s$.

4. Assume $W$ is finite and let $m$ be the maximal length of words in $W$. Let $N^{(m-1)}$ be the set of all Lyndon atoms $u$ of length $\leq m - 1$. The following conditions are equivalent:

   (a) $N$ is finite;

   (b) every word $l \in N$ has length $|l| \leq m - 1$, that is $N = N^{(m-1)}$.

   (c) Every word $ab$, where $a, b \in N^{(m-1)}$ with $a < b$ and $|ab| \geq m$ contains as a subword some $w \in W$.

**Proof.** Part (1) follows straightforwardly from the definition of an antichain.

2. Let $w \in W$. As a Lyndon word $w$ has a standard factorization $w = uv$, where $u, v \in L$ and $v$ is the longest proper right Lyndon segment of $w$. By (1) $u$ and $v$ are Lyndon atoms.

3. Assume now that $N = \{l_1 < l_2 < \cdots < l_d\}$. Lemma 5.2 implies the left-hand side inclusion in (5.1) and the inequality $d - 1 \leq |W|$. Part (2) implies that $W \subseteq \{l_i l_j \mid 1 \leq i < j \leq d\}$, hence $|W| \leq d(d - 1)/2$. This also implies that the length of each $w \in W$ is at most $2s$, where $s$ is the maximal length of a Lyndon atom.

4. Suppose $m$ is the maximal length of words in $W$.

   The implications (4b) $\Rightarrow$ (4a) and (1a) $\Rightarrow$ (4c) are clear.

   (4c) $\Rightarrow$ (1a). Suppose every monomial $u = ab$, where $a, b \in N^{(m-1)}$, $a < b$, and $m \leq |a| + |b| \leq 2(m - 1)$, contains as a subword some $w \in W$. We claim that every Lyndon word $l$ of length $|l| \geq m$ is in the ideal $(W)$. Assume the contrary. Let $l \in L$ be of minimal length, such that $|l| \geq m$, and $l \notin (W)$. Clearly, $l \in N$ so the
Lyndon words $a, b$ in its standard factorization $l = ab, a < b$ are also Lyndon atoms. The lengths of $a$ and $b$ satisfy either i) $|a| \leq m - 1$ and $|b| \leq m - 1$; or ii) at least one of the monomials $a$ and $b$ has length $\geq m$. Note that (i) is impossible, since it contradicts condition (1). Suppose (ii) holds. Without loss of generality, we may assume $|a| \geq m$. We have found a Lyndon word $a \in W$, such that $m \leq |a| < |l|$ which contradicts the choice of $l$.

Suppose $N$ is a finite set of order $d$, and let $N = \{i_1 < i_2 < \cdots < i_d\}$. It is proven in [12], that $i_{i+1} \in W$ for all $1 \leq i \leq d$. Therefore $|i_{i+1}| \leq m$, which gives

$$|i_1| \leq m - 1, \quad 1 \leq i \leq d.$$ 

Remark 5.4. As we have seen, when $N$ is finite of order $d$ the lower and the upper bounds for the order $|W|$ are exact. Theorem B implies that in this case the equality $|W| = d(d - 1)/2$ determines the set $W$ uniquely and explicitly. In contrast, when $|W| = d - 1$, there may be various $W$’s reaching this bound. One example is the antichain $W$ defining the Fibonacci algebra $F_d$ of Section 1. Another example is defined via its set of Lyndon atoms:

$$N = \{x < x^{d-2}y < x^{d-3}y < \cdots < xy < y\}$$

5.2. An algorithm to determine polynomial growth. In the general case of an s.f.p. monomial algebra $A = K\langle X \rangle/(W)$, where $W \subset X^+$ is a finite antichain of monomials, one can use Ufnarovski’s graph $\Gamma(A)$ to decide whether the algebra has polynomial or exponential growth. For convenience of the reader we recall the definition and an important result.

The Ufnarovski graph $\Gamma = \Gamma(A)$ of normal words is a directed graph defined as follows. The vertices of $\Gamma(A)$ are the non-zero words $u$ of $A$ of length $m - 1$, (that is $u \in \mathcal{R}(W)$), where $m$ is the maximal length of a word in $W$. There is an arrow $u \rightarrow v$ iff $ux = yv \in \mathcal{R}(W)$ for some $x, y \in X$. A cyclic route is called a cycle, this is a path beginning and ending at a vertex $u$.

Fact 5.5. [18]

1. For every $k \geq m$ there is a one-to-one correspondence between the set of normal words of length $k$ and the set of paths of length $k - m + 1$ in the graph $\Gamma$. The path $y_1 \cdots y_{m-1} \rightarrow y_2 \cdots y_m \cdots \rightarrow y_k \cdots y_{k+m+1} \cdots y_k$ (these are not necessarily distinct vertices) corresponds to the word $y_1y_2 \cdots y_k \in \mathcal{R}$ ($y_1, \cdots, y_k \in X$).

2. $A$ has exponential growth iff the graph $\Gamma$ has two intersecting cycles.

3. $A$ has polynomial growth of degree $d$ iff $\Gamma$ has no intersecting cyclic routes (cycles) and $d$ is the largest number of (oriented) cycles occurring in a path of $\Gamma$.

Remark 5.6. Given $X$ and $W$, formally one can decide effectively whether $A = K\langle X \rangle/(W)$ has polynomial growth of degree $d$.

Note that this method does not give a sharp upper bound for the length of normal monomials (or equivalently routes in $\Gamma$) that have to be checked in order to find the growth. Clearly, the length of a cyclic route in $\Gamma$ is bounded by the number of its vertices (that is by the number of normal words of length $m - 1$). Translated to words, this method involves the words in $\mathcal{R}$ of length $\leq m - 1 + |\mathcal{R}_{m-1}|$, where $\mathcal{R}_{m-1}$ is the set of all normal words of length $m - 1$. In contrast with the general
case, when $W$ is an antichain of Lyndon words instead of working with general
normal words one works only with Lyndon atoms. Furthermore, there exists a
sharp upper bound for the admissible length of normal Lyndon atoms in order to
have polynomial growth. This bound is $m$, it is common for all monomial algebras
with sets of defining relations $W \subset L$ such that $m = \max\{|w| \mid w \in W\}$. Here we
have to study whether or not there exists an atom $u$ of length $m \leq u \leq 2m - 2$.
More precisely, knowing all atoms of length $\leq m - 1$, whose number is say $d$, one
has only to check all possible products $ab$, where $a < b$, are atoms of length $\leq m - 1$, and $|ab| \geq m$. The number of such products is bounded by $d(d - 1)/2$.

Condition (4c) of Theorem 5.3 implies a simple method to decide whether $A$
has polynomial growth. Consider the following problem.

Problem. Given

\begin{align*}
X &= \{x_1, \ldots, x_g\} \quad \text{a finite alphabet} \\
W &= \{w_1, \ldots, w_r\} \subset L \quad \text{a finite antichain of Lyndon monomials} \\
m &= \max_{1 \leq i \leq r} |w_i| \\
k &= \min_{1 \leq i \leq r} |w_i|.
\end{align*}

(1) For each $s = 1, \ldots, m - 1$, find the set $N_s$ of Lyndon atoms of length $s$.
Find $N^{(m-1)} := \bigcup_{1 \leq s \leq m-1} N_s$.
(2) Decide whether the monomial algebra $A = K\langle X \rangle / (W)$ has polynomial
growth, and if "yes",
(3) Find $\text{GK} \dim A$, the Gelfand-Kirillov dimension of $A$, and $\text{gl} \dim A$, the
global dimension of $A$.

In the settings of this problem the question of finding the global dimension
of $A$ (only in case of polynomial growth) is answered straightforwardly. In the
general case of finitely presented monomial algebras, or s.f.p. associative algebras,
there exist various algorithms, which (implementing Anick’s results) find the global
dimension directly using the reduced Groebner basis, and Anick’s resolution, see
[10], [20], et al.

Method.

(1) For $1 \leq i \leq k - 1$, we set $N_i = \{u \in L \mid |u| = i\}$.
For $k \leq s \leq m - 1$ we find $N_s$ recursively.
Suppose $N_j$ is found for all $1 \leq j \leq s - 1$, denote $N^{(s-1)} = \bigcup_{1 \leq i \leq s-1} N_i$.
Then
\begin{align*}
N_s &= \{ab \mid a, b \in N^{(s-1)}, a < b, |a| + |b| = s, \\
&\quad \text{no } w \in W \text{ is a subword of } ab\} \\
N^{(m-1)} &= \bigcup_{1 \leq i \leq m-1} N_i \\
d &= |N^{(m-1)}|.
\end{align*}

(2) For each pair $a, b \in N^{(m-1)}$ with $a < b$ and $|ab| \geq m$ check whether $ab$
has some $w \in W$ as a subword.
If "YES", then $A$ has polynomial growth, proceed to 3.
If, “NO” (i. e. there exist $a, b \in N^{(m-1)}$ with $a < b$ and $|ab| \geq m$, such
that no $w \in W$ is a segment of $ab$), then $A$ has exponential growth. The
process halts.
(3) Set 

\[ \text{GK dim} := d; \quad \text{gl dim} A := d. \]

Remark 5.7. Note that in contrast with Anick’s proof, see [2], the special shape of the elements of \( W \) makes it possible to straightforwardly determine the so called “atoms” of \( A \), which are difficult to find explicitly using Anick’s result. In our case these are the Lyndon atoms. To find the atoms in the general case of a finitely presented monomial algebra with polynomial growth, one can use the graph \( \Gamma \): the atoms correspond to the cycles in \( \Gamma \).

6. The Global Dimension

Given an antichain of monomials \( W \subset X^+ \) the monomial algebra \( A = K \langle X \rangle/(W) \) is a particular case of finitely generated augmented graded algebras with a set of obstructions \( W \), see [2] and [3]. Anick constructs a resolution, of the field \( K \) considered as an \( A \)-module, and obtains important results on algebras with polynomial growth and finite global dimension, see [3] and [2]. The ”bricks” of Anick’s resolution are the so called \( n \)-chains on \( W \). Anick’s resolution is minimal whenever \( A \) is a monomial algebra. We recall first the definition of an \( n \)-chain and a result from [2 Sec. 3].

Definition 6.1. The set of \( n \)-chains on \( W \) is defined recursively. A \((-1)\)-chain is the monomial 1, a 0-chain is any element of \( X \), an 1-chain is a word in \( W \). An \((n + 1)\)-prechain is a word \( w \in X^+ \), which can be factored in two different ways \( w = uwq = ust \) such that \( t \in W \), \( u \) is an \((n - 1)\)-chain, \( uv \) is an \( n \)-chain, and \( s \) is a proper left segment of \( v \). An \((n + 1)\)-prechain is an \((n + 1)\)-chain if no proper left segment of it is an \( n \)-chain. In this case the monomial \( q \) is called the tail of the \((n + 1)\)-chain \( w \).

The following fact can be extracted from [2 Theorem 4].

Fact 6.2. Let \( A = K \langle X \rangle/(W) \) be a monomial algebra, where \( X \) is a nonempty set of arbitrary cardinality, and \( W \) is an antichain of monomials in \( X^+ \). The global dimension of \( A \) is \( n \) iff there exists an \((n - 1)\)-chain but there are no \( n \)-chains on \( W \).

One can read off Anick’s definition that every \( n \)-chain is “built” out of a string of \( n \)-monomials from \( W \) which overlap successively in a special way. The lemma below is straightforward.

Lemma 6.3. Let \( W \) be a nonempty antichain of monomials in \( X^+ \), \( n \geq 2 \). The word \( \omega \in X^+ \) is an \( n \)-prechain if and only if it has a presentation

\[ \omega = v_0t_1v_1t_2v_2\cdots t_{n-2}v_{n-2}t_{n-1}v_{n-1}t_nv_n, \]

such that

1. \( v_0 \in X \), and \( v_i \in X^+, t_i \in X^*, 1 \leq i \leq n; \)
2. Each \( u_i = v_it_iv_{i+1} \) is a word in \( W \), \( 0 \leq i \leq n - 1 \), and one has \( \overrightarrow{u_i}, \overrightarrow{u_{i+1}} \) for all \( 0 \leq i \leq n - 2 \).
3. For \( 1 \leq m \leq n - 1 \), the left proper segment

\[ w_m = v_0t_1v_1t_2v_2\cdots v_{m-1}t_mv_m \]

of \( \omega \) is an \( m \)-chain on \( W \) with a tail \( t_mv_m \). Furthermore,

\[ v_0t_1v_1t_2v_2\cdots t_{n-2}v_{n-2}t_{n-1}v_{n-1} \]
is the unique \((n - 1)\)-chain contained as a left segment of \(\omega\). (The initial letter, \(w_0 = v_0 \in X\) is a 0-chain).

An \(n\)-prechain is an \(n\)-chain if no proper left segment of it is an \(n\)-prechain.

**Proposition 6.4.** Let \(W\) be a nonempty antichain of Lyndon words (of arbitrary cardinality). Then

1. Every 2-prechain is a Lyndon word.
2. Every \(n\)-chain \(\omega, n \geq 1\), is a Lyndon word.
3. Suppose \(\omega\) is an \(n\)-chain, \(n \geq 1\). Let \(u_i = v_{i-1}t_iv_i \in W\), \(1 \leq i \leq n\) be the Lyndon monomials involved in its presentation (6.1). Then \(u_i u_{i+1}\), for all \(1 \leq i \leq n - 1\), so there are strict inequalities

\[u_1 < u_2 < \cdots < u_n.\]

4. If \(W\) is a set of order \(r\), then there are no \((r + 1)\)-chains on \(W\).

**Proof.**

1. Let \(\omega\) be a 2-prechain, then it factors as \(\omega = uvw\), where \(u, v, w \in X^+\), and \(uvw, vw \in W \subseteq L\). Hence \(uv\) and \(vw\) are Lyndon words, and by Lemma 4.3 the product \(uvw = \omega\) is also a Lyndon word.

2. We prove that every \(n\)-chain \(\omega\) is a Lyndon word by induction on \(n\). By definition each 1-chain is an element of \(W\), and therefore it is a Lyndon word. We just proved that the 2-chains are also Lyndon words. Suppose now that for \(2 \leq m \leq n\) every \(m\)-chain is a Lyndon word, and let \(\omega\) be an \((n + 1)\)-chain. Then in notation as in Lemma 6.3, \(\omega = (w_n t_n) v_n (t_{n+1} v_{n+1}) = uvw\), where \(w_{n-1}\) is an \((n - 1)\)-chain, \(u = w_{n-1} t_n, v = v_n, uv = w_{n-1} t_n v_n\) is the unique \(n\)-chain contained as a left segment of \(\omega\), and \(vw = v_n t_{n+1} v_{n+1} \in W\). By the inductive assumption the \(n\)-chain \(uv\) is a Lyndon word, clearly \(vw \in W\) is also a Lyndon word. It follows then from Lemma 4.3 that \(\omega = uvw\) is a Lyndon word, which proves part (2) of the proposition.

3. Suppose \(\omega\) is an \(n\)-chain. Then each of the monomials \(u_m = v_{m-1} t_m v_m\), \(1 \leq m \leq n\), involved in its presentation (6.1) is an element of \(W\), so it is a Lyndon word. Clearly, for \(1 \leq i \leq n - 1\) one has \(u_i u_{i+1}\), so by Lemma 4.3 \(u_i < u_{i+1}\). This yields \(u_1 < u_2 < \cdots < u_n\), which proves (3).

Part (1) follows straightforwardly from (3).

**Proof of Theorem B.**

1. It is well-known that any antichain of monomials \(W\) is a minimal Gröbner basis of the ideal \((W)\), thus \(A = K\langle X \rangle / (W)\) is a standard finite presentation of \(A\) iff \(W\) is finite.

2. Suppose \(|W| = r\). By Proposition 6.3 part (1) there are no \((r + 1)\)-chains on \(W\), so Fact 6.2 implies that \(gdim A \leq r + 1\) which proves part (i). Part (ii) follows straightforwardly from the results of Ufnarovski. 19. For convenience of the reader, only, we shall give a sketch of a proof. We shall use the results of Ufnarovski recalled in Section 3.

It follows from Fact 5.6 that an s.f.p. algebra \(A\) has either polynomial or exponential growth. Assume that \(A\) has exponential growth, so by Fact 5.5 the graph \(\Gamma\) has two intersecting cycles \(C_1\) and \(C_2\). Let \(a_1, a_2\), respectively, be the corresponding normal words. Then every word \(u\) in the alphabet \(a_1, a_2\) corresponds to a route in \(\Gamma\) and therefore \(u \in \mathfrak{N}\). This implies that \(A\) contains the free algebra generated by \(a_1\) and \(a_2\).

Part (iii) follows from Theorem 5.3 [1].
Assume \( \text{GK} \dim A = d \), then, as we have already shown, \(|N| = d\) and therefore, by Theorem 5.3 (3), one has \(|W| \leq d(d - 1)/2\).

A resent result of [12] verifies (independently of Anick’s results) that if \( W \) is an antichain of Lyndon words, and \(|N| = d\), then there exists a \((d - 1)\)-chain, but there is no \(d\)-chain on \( W \), and therefore by Fact 5.2, \( A \) has global dimension \( d \). By Theorem A the order \(|N| = d\), is also the GK-dimension of \( A \).

We give now a second (indirect) argument for the equality between the global dimension and GK-dimension of \( A \).

By assumption \( A \) has polynomial growth, so \( W \) is a finite set, and by part (2) \( A \) has finite global dimension. Clearly \( A \) does not contain a free subalgebra generated by two monomials, and therefore by [2, Theorem 6] there is an equality \( \text{GK} \dim A = \text{gl} \dim A \).

Suppose \( N = \{l_1 < l_2 < \cdots < l_d\} \). By Theorem 5.3
\[
W \subseteq \{l_i l_j \mid 1 \leq i < j \leq d\},
\]
so the equality \(|W| = d(d - 1)/2\) implies an equality of the sets above.

By convention \( X \subseteq N \), and therefore
\[
W_0 = \{x_i x_j \mid 1 \leq i < j \leq g\} \subseteq W.
\]

Note that \( W_0 \), consists of all Lyndon words of length 2. Corollary 4.6 implies that every Lyndon word \( a \) of length \( \geq 2 \) contains a subword of the form \( x_i x_j \), \( 1 \leq i < j \leq g \), hence \( a \in (W_0) \). Note that \( W \) is an antichain of Lyndon monomials of length \( \geq 2 \), it follows then that \( W = W_0 \). This implies the equalities \( N = X \), \( g = d \), so \( A \) is presented as
\[
(6.2) \quad A = k\langle x_1, \cdots, x_d \rangle/(W_0), \quad W_0 = \{x_i x_j \mid 1 \leq i < j \leq d\}.
\]

Conversely, if the monomial algebra \( A \) is defined by (6.2) where \( g = d \), then \( A \) satisfies the hypothesis of the theorem, the set of its Lyndon atoms is \( N = X \), and \(|W| = d(d - 1)/2\).

\( \Box \)

Corollary 2.3 is straightforward from Theorem B (3), and Anick’s result [2, Theorem 6].

Remark 6.5. The fact that every standard finitely presented graded algebra \( A \) has either polynomial or exponential growth is already classical. It follows from Ufnarovski’s results, [19], see also Fact 5.3. As we have seen for finitely presented monomial algebras exponential growth is equivalent to the existence of a free subalgebra generated by two monomials, a condition which is, in general, stronger than having exponential growth.

Corollary 6.6. Let \( A^0 = K\langle x_1, \cdots, x_n \rangle/(W_0) \) be a monomial algebra, where \( W_0 \subseteq X^+ \) is an arbitrary antichain of monomials. The following conditions are equivalent:

1. The set of monomial relations \( W_0 \) consists of Lyndon words, \( A^0 \) has polynomial growth of degree \( d \), and \(|W_0| = d(d - 1)/2\).
2. \( A^0 \) is a quadratic algebra, i.e. \( W_0 \) consists of monomials of length 2, \( n = d \), and at least one of conditions (1) through (6) of Theorem 3.1 is satisfied.

In this case all conditions (1), \( \cdots, (6) \) of Theorem 3.1 hold.
7. An extremal algebra: The Fibonacci algebra

7.1. Fibonacci-Lyndon words. Consider the alphabet \( X = \{x, y\} \). Define the sequence of Fibonacci-Lyndon words \( \{f_n(x, y)\} \) by the initial conditions \( f_0 = x, f_1 = y \) and, then for \( n \geq 1 \)

\[
(7.1) \quad f_{2n} = f_{2n-2}f_{2n-1}, \quad f_{2n+1} = f_{2n}f_{2n-1}.
\]

This gives the sequence

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
x & y & xy & xyy & xyxyy & xyxyyxyy & xyxyyxyyxyy
\end{array}
\]

Note that if we let \( a \) be \( f_2(x, y) = xy \) and, then for Fibonacci-Lyndon words \( a, b \) be \( f_3(x, y) = xyy \), then the Fibonacci-Lyndon word \( f_m(a, b) = f_m+2(x, y) \).

**Lemma 7.1.** The following holds:

a. The word \( f_n(x, y) \) is a Lyndon word and its length is the \( n \)'th Fibonacci number.

b. For the lexicographic order we have

\[
f_0 < f_2 < \cdots < f_{2n} < \cdots < f_{2n+1} < \cdots < f_3 < f_1.
\]

**Proof.** By induction we see that \( f_{2n} \) and \( f_{2n+1} \) are Lyndon words and their lengths are as stated, which gives part a.

Now the recursive definition \( (7.1) \) and Fact 4.1 (2), imply that for each \( n \geq 1 \) the Fibonacci-Lyndon words satisfy

\[
(7.2) \quad f_{2n-2} < f_{2n} = f_{2n-2}f_{2n-1} < f_{2n-1}
\]

\[
f_{2n} < f_{2n+1} = f_{2n}f_{2n-1} < f_{2n-1}.
\]

This straightforwardly proves part (b). \( \square \)

Let \( U \) consist of all Lyndon words \( f_{2n-2}f_{2n} \) and \( f_{2n+1}f_{2n-1} \), where \( n \geq 1 \).

**Proposition 7.2.** A Lyndon word \( w \) in \( x \) and \( y \) is not in the ideal \( (U) \) if and only if it is a Fibonacci-Lyndon word.

**Proof.** We argue by induction on the length of \( w \), the statement clearly holds when the length is one. Suppose the length of \( w \) is \( \geq 2 \). Note that \( f_0f_2 = xxy \) and \( f_3f_1 = xyy \), so, if a Lyndon word \( w \) is not in the ideal \( (U) \) we see that \( w \) must be a word in \( a = xy \) and \( b = xyy \). Since \( f_m(a, b) = f_{m+2}(x, y) \) we see that \( w \) is not divisible by any of

\[
f_{2p-2}(a, b)f_{2p}(a, b), \quad f_{2p+1}(a, b)f_{2p-1}(a, b).
\]

Considering the length of \( w \) written in terms of \( a \) and \( b \) and using induction, we prove that \( w = f_m(a, b) = f_{m+2}(x, y) \) for some \( m \). \( \square \)

7.2. The extremal algebra. Let \( W_n \) be the antichain of all minimal elements in \( U \cup \{f_n(x, y)\} \), with respect to the divisibility order \( \sqsubset \). This is a finite set, since \( f_n \) is a factor of the Fibonacci-Lyndon words later in the sequence.

The set of Lyndon atoms with respect to the ideal \( (W_n) \) is \( \mathcal{N}_n = \{f_0, \ldots, f_{n-1}\} \), and so we obtain a monomial algebra, the Fibonacci algebra

\[
F_n = k\langle X \rangle/(W_n),
\]
whose Hilbert series is
\[
\prod_{i=0}^{n-1} \frac{1}{1-u_i u_i}.
\]
where \( f_i \) is the \( i \)'th Fibonacci number. Clearly, the global dimension and the Gelfand-Kirillov dimension of \( A \) are both \( n \).

This algebra is extremal in the following sense.

**Proposition 7.3.** Let \( W \) be a finite set of Lyndon words such that the corresponding set of Lyndon atoms, \( N(W) = \{w_0, \ldots, w_{n-1}\} \), is finite and enumerated according to increasing lengths of \( w_p \). Then the lengths satisfy \( |w_p| \leq |f_p| \). If we have an equality for each \( p, 0 \leq p \leq n - 1 \), then the algebra \( A = K\langle X \rangle / (W) \) is isomorphic to the Fibonacci algebra \( F_n \).

**Proof.** By convention, \( X \) has at least two elements, and \( X \subseteq N \). Clearly, then \( |w_i| \leq |f_i| \) for \( i = 0, 1 \). Let \( n - 1 \leq p \geq 2 \), and let \( w_p = l m \) be the standard factorization of \( w_p \). Then \( l, m \) must be in \( N \) so we may write \( w_p = w_i w_j \) where \( i \) and \( j \) are distinct integers \( < p \). If \( i < j < p \), then \( i \leq p - 2, j \leq p - 1 \) so by the inductive assumption, \( |w_i| \leq |f_{p-2}| \) and \( |w_j| \leq |f_{p-1}| \), and therefore \( |w_p| = |w_i| + |w_j| \leq |f_p| \).

The case \( i > j \) is analogous.

Assume now \( |w_p| = |f_p| \), \( 0 \leq p \leq n - 1 \). Clearly, \( X \) must consist of two elements \( x < y \), and there are equalities \( w_0 = x \) and \( w_1 = y \), or the other way around. In any case we must have \( w_2 = xy \). There are now two possibilities for \( w_3 \). It is either \( xyy \) or \( xxy \). Assume \( w_3 = xyy \). We then prove by induction that \( w_p = f_p \) for \( p = 0, \ldots, n - 1 \). Clearly, the standard factorization of \( w_{2r} \) is either \( w_{2r-1} w_{2r-2} \) or \( w_{2r-2} w_{2r-1} \). Since \( f_p = w_p \) for \( p < 2r \) and given the ordering of the Fibonacci-Lyndon words, the latter must hold, and so \( w_{2r} = f_{2r} \). Similarly, we may argue that \( w_{2r+1} = f_{2r+1} \).

In the case \( w_3 = xxy \) we take \( w_0 = y \) and \( w_1 = x \). There is an involution \( \tau \) on \( k\langle x, y \rangle \) which takes a word \( a_1 a_2 \cdots a_r \) and arranges it in the opposite order \( a_r \cdots a_1 \). There is also an involution \( \iota \) which replaces each \( x \) with \( y \) and each \( y \) with \( x \). It is not difficult to argue that \( w_p = \iota \circ \tau (f_p) \), so \( A \) and \( F_n \) become isomorphic via the map \( \iota \circ \tau \).

\[
8. \text{Fibonacci algebras not deforming to Artin-Schelter regular algebras}
\]

It is known that to each Lyndon word \( l \) one may associate a Lie monomial (called bracketing of \( l \), and denoted by \( [l] \)), [13], Chapter 5, or [16], Chapter 4. The Lie monomials corresponding to Lyndon words form a basis for the free Lie algebra, \( \text{Lie}(X) \), generated by \( X \).

To each monomial algebra \( A = K\langle X \rangle / (W) \) defined by an antichain of Lyndon words \( W \) we associate canonically the (associative) algebra \( \tilde{A} = K\langle X \rangle / ([W]) \) with the same generating set \( X \), and the set \([W]\) of Lie monomials associated with \( W \) as defining relations. (As usual, a Lie element \([a, b] \in \text{Lie}(X)\) is considered also as an "associative" element \([a, b] = ab - ba \in K\langle X \rangle\)). In this case the algebra \( \tilde{A} \) is an enveloping algebra of the Lie algebra \( \mathcal{L} \) generated by \( X \) and with the same set of defining relations (considered as elements in \( \text{Lie}(X) \)). The first question to ask is whether the monomial algebra \( A \) and the corresponding enveloping algebra \( \tilde{A} \) share the same \( K\)-basis, or, equivalently, the same Hilbert series. This is not so, in general, but when this holds is further investigated in [12]. Enveloping algebras
of finite dimensional graded Lie algebras are special cases of Artin-Schelter regular algebras. It is then natural to ask if our monomial algebras may deform to algebras in this more general class.

An algebra \( A = k \oplus A_1 \oplus A_2 \oplus \cdots \) is an Artin-Schelter regular algebra of dimension \( d \) if:

- \( A \) has finite global dimension \( d \).
- \( A \) has finite Gelfand-Kirillov dimension.
- \( A \) is Gorenstein, i.e.

\[
\operatorname{Ext}_A^i(K, A) = \begin{cases} 0 & i \neq d \\ K(l) & i = d \end{cases}
\]

for some shift \( l \).

The monomial algebras defined by Lyndon words with a finite set \( N \) of Lyndon atoms have the two first properties. It is therefore natural to ask if they can be deformed to Artin-Schelter regular algebras. If \( B \) is such a monomial algebra, its Hilbert series is

\[
H_B(t) = \prod_{l \in N} \frac{1}{1 - t^{|l|}}.
\]

If the resolution of the residue field \( K \) of \( B \) is

\[ B \leftarrow \cdots \leftarrow \oplus_{j \in \mathbb{Z}} B(-j)^{\beta_{ij}} \leftarrow \cdots \]

then

\[
H_B(t) \left( \sum_{i,j \in \mathbb{Z}} (-1)^i \beta_{ij} t^j \right) = 1
\]

so

\[
\sum_{i,j \in \mathbb{Z}} (-1)^i \beta_{ij} t^j = \prod_{l \in N} (1 - t^{|l|}).
\]

Thus, since this polynomial is symmetric up to sign, it is numerically possible that the monomial algebra \( B \) deforms to an algebra with the Gorenstein property.

Fløystad and J.E.Vatne show that the \( \mathbb{Z}_2 \)-graded Fibonacci-Lyndon monomial algebras \( F_n \) for \( n \leq 5 \) all deform to Artin-Schelter regular algebras which are also \( \mathbb{Z}_2 \)-graded, \([9]\). For \( n \leq 4 \) these deformations are enveloping algebras of Lie algebras but it is not so for \( n = 5 \). However we have the following.

**Proposition 8.1.** The Fibonacci-Lyndon monomial algebra \( F_6 \) does not deform to a bigraded Artin-Schelter regular algebra.

**Remark.** There still remains the possibility though that it might deform to a singly graded Artin-Schelter regular algebra.

**Proof.** Since the complete argument involves a lot of computation, we will give only a sketch for the last parts of the proof.

Part 1. Let \( B \) be the monomial algebra \( F_6 \). The resolution of its residue field may be worked out to be (we write the multidegrees of the generators of the free modules below):

\[
\begin{array}{cccccccc}
B & \leftarrow & B^2 & \leftarrow & B^5 & \leftarrow & B^{10} & \leftarrow & B^9 & \leftarrow & B^4 & \leftarrow & B \\
(0, 0) & \leftarrow & (1, 0) & \leftarrow & (2, 1) & \leftarrow & (3, 2) & \leftarrow & (4, 8) & \leftarrow & (5, 8) & \leftarrow & (8, 12) \\
(0, 1) & \leftarrow & (1, 3) & \leftarrow & (2, 4) & \leftarrow & (3, 5) & \leftarrow & (4, 11) & \leftarrow & (5, 9) & \leftarrow & (7, 11) \\
(5, 8) & \leftarrow & (5, 7) & \leftarrow & (6, 9) & \leftarrow & (6, 10) & \leftarrow & (7, 10) & \leftarrow & (8, 11) \\
(4, 7) & \leftarrow & (6, 8) & \leftarrow & (6, 10) & \leftarrow & (6, 9) & \leftarrow & (7, 10) & \leftarrow & (8, 11) \\
\end{array}
\]
If $B$ deforms to a bigraded Artin-Schelter regular algebra $A$, its resolution is obtained by canceling adjacent terms of the same multidegrees in the above resolution, and it must have a selfdual form, since $\text{Tor}_i^A(k, k)$ can be computed by taking a resolution of $K$ either as a left or as a right module. The only possibility for the minimal resolution of $A$ is then

\begin{align*}
A &\twoheadrightarrow A^2 & \xrightarrow{d_1} & A^4 & \xrightarrow{d_2} & A^6 & \xrightarrow{d_3} & A^4 & \xrightarrow{d_4} & A^2 & \twoheadrightarrow A
\end{align*}

where $d_0 = [x, y]$.

The differentials here are represented by matrices whose entries are in $A$. These entries are then of the form $\pi(p)$, where $\pi : K < x, y > \rightarrow A$ is the natural quotient map, and $p \in K < x, y >$. By abuse of notation we shall simply write $p$ for such an entry. Since the composition of successive differentials is zero, the product of any two successive matrices will then have entries which are relations for $A$, i.e. they are in the kernel of $\pi$.

In particular, the defining relations of $A$ are given by the elements of the product matrix $d_0 \cdot d_1$, which have bidegrees $(2, 1), (1, 3), (3, 4)$ and $(4, 7)$.

Part 2. Now look at the subcomplex

\begin{align*}
A &\twoheadrightarrow A^2 & \xrightarrow{d_0} & A^2 & \xrightarrow{d_1'} & A^6 & \xrightarrow{d_2'} & A^4 & \xrightarrow{d_3'} & A
\end{align*}

After suitable base changes we may assume that

\[ d_1' = \begin{bmatrix} xy + \alpha_0yx \\ -\alpha_1x^2 & \beta_0xy^2 + \beta_1yxy + \beta_2y^2x \end{bmatrix}, \quad d_2' = \begin{bmatrix} y^3 \\ \gamma xy \end{bmatrix}. \]

Multiplying $d_0 = [x, y]$ with $d_1'$ we get the two first of the four defining relations for $A$:

\begin{align*}
\text{(8.2)} & \quad x^2y + \alpha_0xyx - \alpha_1yx^2 \\
\text{(8.3)} & \quad xy^3 + \beta_0xy^2 + \beta_1yxy + \beta_2y^3x.
\end{align*}

The product of $d_1'$ and $d_2'$ induces the following relations of $A$.

\begin{align*}
\text{(8.4)} & \quad xy^3 + \alpha_0xyy^2 + \gamma y^3x \\
\text{(8.5)} & \quad -\alpha_1x^2y^2 + \gamma \beta_0xy^2x + \gamma \beta_1yxyx + \gamma \beta_2y^2x^2.
\end{align*}

So (8.4) must be a consequence of (8.3) which gives

\[ \beta_1 = 0, \quad \beta_0 = \alpha_0, \quad \beta_2 = \gamma. \]

Then the relation (8.3) becomes

\begin{align*}
\text{(8.6)} & \quad -\alpha_1x^2y^2 + \gamma \alpha_0xy^2x + \gamma^2y^2x^2.
\end{align*}

The relation (8.6) must be a linear combination of the following expressions obtained by multiplying (8.2) with $y$ on the left and on the right.
\[x^2y^2 + \alpha_0 xyxy - \alpha_1 y^2x \]

If (8.6) is nonzero, then this linear combination is also nonzero. This gives \(\alpha_0 = 0\) and a dependence between

\[-\alpha_1 x^2y^2 + \gamma^2 y^2x^2 \quad \text{and} \quad x^2y^2 - \alpha_1^2 y^2 x^2,\]

which implies \(\alpha_1^3 = \gamma^2\).

Setting \(\alpha_1 = \gamma\) we obtain

\[d_1' = \begin{bmatrix} xy & y^3 \\ -\alpha x^2 & \gamma y^2 x \end{bmatrix}, d_2'' = \begin{bmatrix} y^2 \\ \gamma x \end{bmatrix},\]

where \(\gamma = \alpha_3\). Then the relations (8.2) and (8.3) are reduced straightforwardly to (8.7)

\[x^2y - \alpha xy^2, xy^3 + \gamma y^3x.\]

If \(\alpha,\) equivalently \(\gamma,\) is nonzero then each of the monomials \(x^2\) and \(y^3\) commutes with any word up to adjusting with constants.

In the next part we assume that \(\alpha\) and \(\gamma\) are nonzero.

Part 3. Consider the subcomplex of (8.1) given by

(8.8)

\[
\begin{array}{cccc}
A & \xleftarrow{d_0} & A^2 & \xleftarrow{d_1''} & A^3 \\
(0,0) & (1,0) & (2,1) & (2,3) & (3,4)
\end{array}
\]

where

\[d_1'' = \begin{bmatrix} xy & y^3 \\ -\alpha x^2 & \gamma y^2 x \end{bmatrix}, d_2'' = \begin{bmatrix} y^2 \\ \gamma x \\ \mu_1 x \\ \mu_2 y \end{bmatrix}.
\]

We make the following adjustments to \(d_1''\), noting that \(P\) has bidegree (2, 4). By i) subtracting from \(P\) right multiplicities of the first two columns of \(d_1''\) and ii) using the relations (8.7) we may assume

\[P = a_0 y^2 xy^2 x + a_1 y^2 xyxy + a_2 y^2 xy^2 xy + a_4 xyxyxy^2.\]

Also \(R_2\) has bidegree (1, 4). By subtracting from \(R_2\) right multiplicities of the first column in \(d_2''\) and using the relations (8.7) we may assume \(R_2 = 0\). Furthermore \(S_1\) has bidegree (3, 1) and again by subtracting right multiplicities of the first column and using the relations (8.7) we may assume \(S_1 = 0\). The matrices are now

\[d_1'' = \begin{bmatrix} xy & y^3 \\ -\alpha x^2 & \gamma y^2 x \end{bmatrix}, d_2'' = \begin{bmatrix} y^2 \\ \gamma x \\ 0 \\ \mu_1 x \\ \mu_2 y \end{bmatrix}.
\]

If \(\mu_1 = 0\) we get a relation \(x^2 R_1\) of bidegree (4, 3). This must be a consequence of the relations (8.7) which easily gives \(R_1 = 0\). Similarly if \(\mu_2 = 0\) we get a relation \(y^3 S_2 = 0\) of bidegree (2, 5) which again easily gives \(S_2 = 0\). Hence both \(\mu_1\) and \(\mu_2\)
must be nonzero, and by a base change of the generators we may assume they are both \(-1\). Multiplying the matrices above we then get relations

\[
\begin{align*}
xyR_1 &= Px, & \text{of bidegree (3, 4)} \\
\alpha x^2 R_1 &= Qx, & \text{of bidegree (4, 3)} \\
y^3 S_2 &= Py, & \text{of bidegree (2, 5)} \\
-\gamma y^2 x S_2 &= Qy, & \text{of bidegree (3, 4)}.
\end{align*}
\]

(8.9)

The relations of bidegree (2, 5) and (4, 3) must be a consequence of the relations (8.7). The only way this is possible for the (2, 5) relation is if \(a_0 = a_1 = a_2 = 0\) in \(P\), since the corresponding terms in \(Py\) cannot be rearranged. Hence we may assume \(P = axxyxy^2\) where \(a = a_3\). We then easily see that \(S_2 = axxyx\).

For the relation of bidegree (4, 3) to hold, \(Q\) must have the form

\[
ax^2 T_0 + \alpha^4 T'_1 x,
\]

where \(T_0\) and \(T'_1\) are bihomogeneous elements of \(K\langle x, y\rangle\).

Since \(x^2\) commutes up to coefficient change, we may as well assume that \(T'_1\) ends with \(y\) and also does not contain \(x^2\) as a subword, so we may write

\[
Q = ax^2 T_0 + \alpha^4 T_1 yx.
\]

Considering the relation above of bidegree (4, 3), shifting \(x^2\) to the left in \(Qx\) and adjusting the coefficients, we obtain

\[
R_1 = T_0 x + T_1 y,
\]

since we may cancel \(x^2\).

The added defining relation of degree (3, 4) of \(A\) coming from the product \(d_0.d_1\) is the relation

(8.10)

\[
xP - yQ = axxyxy^2 - yQ.
\]

The last relation of bidegree (3, 4) in (8.9) is

\[
\alpha x^2 T_0 y + \alpha^4 T_1 yxy + a\gamma y^2 xyxyx.
\]

If nonzero, the term \(axxyxy^2\) of the defining relation (8.10) of bidegree (3, 4) cannot be rearranged using the first relations (8.7). Moreover, it does not occur in the relation above and therefore, it must be a consequence of the first defining relations (8.7). This is also impossible, since last term cannot cancel. It follows then that \(a = 0\) and \(P\) and \(Q\) are zero.

The three relations of bidegree (3, 4) listed with the defining relation first are:

\[
\begin{align*}
yx^2 T_0 + \alpha^3 y T_1 yx &= 0 \\
x^2 T_0 y + \alpha^3 T_1 y yx &= 0 \\
xy T_0 x + xy T_1 y &= 0.
\end{align*}
\]

Note that \(T_1\) has bidegree (2, 2) and does not contain \(x^2\) as subword, so \(T_1\) can only contain the terms \(m = xyxy, xy^2 x,\) or \(yxyx\). But then the term \(xyxy\) in the last relation above cannot be rearranged by (8.7). Hence it should occur in the first equation which it does not. Therefore it must be that \(T_1 = 0\) and then we easily see that \(T_0 = 0\) and so \(Q = 0\). But it is impossible that both \(P\) and \(Q\) are zero.
Part 4. Now we consider the case when (8.6) is identically zero, that is \( \alpha_1 = \gamma = 0 \).

Then we obtain (letting \( \alpha_0 = -\alpha \))

\[
d'_1 = \begin{bmatrix} xy - \alpha yx & y^3 \\ 0 & -\alpha xy^2 \end{bmatrix}, \quad d'_2 = \begin{bmatrix} y^2 \\ 0 \end{bmatrix}.
\]

We again consider the subcomplex (8.8), where now

\[
d''_1 = \begin{bmatrix} xy - \alpha yx & y^3 & P \\ 0 & -\alpha xy^2 & -Q \end{bmatrix}, \quad d''_2 = \begin{bmatrix} y^2 & R_1 & R_2 \\ 0 & S_1 & S_2 \\ 0 & \mu_1 x & \mu_2 y \end{bmatrix}.
\]

By almost the same type of arguments as in Part 3, we may assume that \( R_2 = 0 \).

We work out that \( S_2 = 0 \) and we may assume \( \mu_2 y = -y \). Then it follows quickly that \( P \) is a multiple of \( yxyxy^2 - \alpha yxy^2 xy \).

Next we show that \( Q, S_1 \) and \( \mu_1 \) are all zero and \( R_1 = xxyy^2 - \alpha xxy^2 xy \). This gives the matrices

\[
d''_1 = \begin{bmatrix} xy - \alpha yx & y^3 & yxyy^2 - \alpha xxy^2 xy \\ 0 & -\alpha xy^2 & 0 \end{bmatrix},
\]

\[
d''_2 = \begin{bmatrix} y^2 & xyxy^2 - \alpha xxy^2 y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -y \end{bmatrix}.
\]

Part 5. Further computation shows that we must have

\[
d_1 = \begin{bmatrix} xy - \alpha yx & y^3 & yxyy^2 - \alpha xxy^2 xy & yxy^2 x^2 y^2 - \alpha y^2 xxy^2 xy^2 \\ 0 & -\alpha xy^2 & 0 \end{bmatrix}.
\]

But now computing the resolution of the algebra with the relations we get from \( d_0 \cdot d_1 \), we see that it is not the desired resolution (8.1). In particular the kernel of \( d_1 \) has a syzygy of degree 13, which is not the case in (8.1). \( \square \)

Acknowledgments. The first author worked on this paper during her visit to Max Planck Institute for Mathematics, Bonn, in 2011–2012. It is her pleasant duty to thank MPIM both for the support and for the inspiring and creative atmosphere during her visit.

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