ON THE EXISTENCE OF A CREPANT RESOLUTION OF SOME MODULI SPACES OF SHEAVES ON AN ABELIAN SURFACE

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Abstract. Let $J$ be an abelian surface with a generic ample line bundle $\mathcal{O}_J(1)$. For $n \geq 1$, the moduli space $M_J(2,0,2n)$ of $\mathcal{O}(1)$-semistable sheaves $F$ of rank 2 with Chern classes $c_1(F) = 0$, $c_2(F) = 2n$ is a singular projective variety, endowed with a holomorphic symplectic structure on the smooth locus. In this paper, we show that there does not exist a crepant resolution of $M_J(2,0,2n)$ for $n \geq 2$. This certainly implies that there is no symplectic desingularization of $M_J(2,0,2n)$ for $n \geq 2$.

1. Introduction

An irreducible symplectic manifold $X$ is a compact simply connected complex manifold, endowed with a nondegenerate holomorphic 2-form $\omega$ which spans $H^0(\Omega^2_X)$. By the Bogomolov decomposition [2], irreducible symplectic manifolds are building blocks of Kähler manifolds in the sense that for any compact Kähler manifold with trivial first Chern class there is an étale cover from the product of tori, Calabi-Yau manifolds and irreducible symplectic manifolds, which we call the Bogomolov factors. Two standard series of examples were provided by Beauville: Hilbert schemes of points on K3 surfaces and the generalized Kummer varieties [2].

Recently O’Grady proposed a strategy for finding new examples of irreducible symplectic manifolds as follows [23, 24]:

1. Consider a singular moduli space $M(r,c_1,c_2)$ of semistable sheaves on a K3 or abelian surface $S$ of rank $r$ with Chern classes $c_1, c_2 \in H^*(S, \mathbb{Z})$. By Mukai’s theorem [20], there is a symplectic form, called the Mukai form, on the open subset of stable sheaves $M(r,c_1,c_2)^s$.

2. Find a desingularization $\tilde{M}(r,c_1,c_2)$ of $\tilde{M}(r,c_1,c_2)$ on which the Mukai form extends everywhere without degeneration.

3. Look at the Bogomolov factors for a new irreducible symplectic manifold.

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Actually, O’Grady successfully implemented his program in two cases and found new irreducible symplectic manifolds of (complex) dimension 10 and 6 respectively:

1. a symplectic desingularization of the moduli space $M_{K3}(2,0,4)$ of rank 2 semistable sheaves on a K3 surface with Chern classes $c_1 = 0, c_2 = 4$ (see [23]),
2. a Bogomolov factor of a symplectic desingularization of the moduli space $M_{Ab}(2,0,2)$ of rank 2 semistable sheaves on an abelian surface with Chern classes $c_1 = 0, c_2 = 2$ (see [24]).

A natural question raised by O’Grady asks whether one can do the same with $M_{K3}(2,0,2^n)$ with $n \geq 3$ or $M_{Ab}(2,0,2n)$ with $n \geq 2$, i.e.

**Question 1.1.** Does there exist a symplectic desingularization of $M_{K3}(2,0,2m)$ with $m \geq 3$ or $M_{Ab}(2,0,2n)$ with $n \geq 2$?

In [4, 14], it was proved that unfortunately the answer is NO for the K3 case: there is no symplectic desingularization of $M_{K3}(2,0,2m)$ for $m \geq 3$. However, the question remains open for $M_{Ab}(2,0,2n)$ with $n \geq 2$. The purpose of this paper is to show that the answer to the above question is also NO for the abelian case, i.e. there is no symplectic desingularization of $M_{Ab}(2,0,2n)$ with $n \geq 2$.

Fix any integer $n \geq 2$. Let $J$ be a complex projective abelian surface equipped with a generic ample divisor $\Theta$, which satisfies

**Assumption 1.2.** ([24, (1.3)]) There is no divisor $A$ orthogonal to $\Theta$ with $-2n \leq A^2 < 0$.

This condition is satisfied if for instance the Néron-Severi group is $NS(J) = \mathbb{Z}c_1(\Theta)$. Let

$$M = M_{2n} = M_J(2,0,2n)$$

denote the moduli space of $\Theta$-semistable sheaves $F$ on $J$ of rank 2 with $c_1(F) = 0, c_2(F) = 2n$ in $H^*(J, \mathbb{Z})$. This is an irreducible normal projective variety of dimension $8n + 2$ ([8, 20, 27]) with Gorenstein singularities.\(^1\) The main result of this paper is the following.

**Theorem 1.3.** If $n \geq 2$, there is no crepant resolution of $M_{2n}$.

By Mukai’s theorem [20], the canonical line bundle of $M_{2n}$ is trivial and hence any desingularization of $M_{2n}$ equipped with a holomorphic symplectic form is a crepant resolution. So we deduce from Theorem 1.3 the following corollary which answers Question 1.1.

**Corollary 1.4.** If $n \geq 2$, there is no symplectic desingularization of $M_{2n}$.

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\(^1\)A normal variety $X$ is Gorenstein if the canonical divisor $K_X$ is a Cartier divisor [1]. In our case, the stable part $M^s$ of $M$ is equipped with a symplectic form [20] and $\text{codim}(M - M^s) \geq 2$ (see §3). Therefore, the canonical divisor $K_M$ is zero and thus $K_M$ is Cartier.
As in [4], the idea of the proof of Theorem 1.3 is to use properties of the stringy E-function [1]. Recall that \( M \) is a normal irreducible variety with Gorenstein singularities. Also we will see in section 3 that the singularities are terminal. Hence, the stringy E-function \( E_{st}(M; u, v) \) of \( M \) is a well-defined rational function in formal variables \( u, v \). By Kontsevich’s Theorem (Theorem 2.2), if there is a crepant resolution \( \tilde{M} \) of \( M \), then the stringy E-function of \( M \) is equal to the Hodge-Deligne polynomial (E-polynomial) of \( \tilde{M} \). In particular, we deduce that the stringy E-function \( E_{st}(M; u, v) \) must be a polynomial. Therefore, Theorem 1.3 is a consequence of the following.

**Proposition 1.5.** The stringy E-function \( E_{st}(M; u, v) \) is not a polynomial for \( n \geq 2 \).

In [23, 21], O’Grady studies Kirwan’s desingularization \( \hat{M} \) of \( M \) which is obtained as the result of three blow-ups. We use O’Grady’s analysis of Kirwan’s desingularization of \( M \) to prove Proposition 1.5. In section 2 we recall some properties of stringy E-functions and we prove Proposition 1.5 in section 3. In section 3 we analyze Kirwan’s desingularization.

After completion of this paper, Kaledin, Lehn and Sorger proved nonexistence of symplectic desingularization for arbitrary rank by a different method. See [15]. We are grateful to C. Sorger for delightful conversations about singular symplectic moduli spaces at the Korea Institute of Advanced Study during a workshop on Vector Bundles on Algebraic Curves organized by S. Ramanan and J.-M. Hwang in April 2005.

### 2. Some properties of Poincaré polynomials, Hodge-Deligne polynomials and stringy E-functions

In this section we collect some facts that we shall use later.

For a topological space \( V \), the Poincaré polynomial of \( V \) is defined by

\[
P(V; z) = \sum_i (-1)^i b_i(V) z^i
\]

where \( b_i(V) \) is the \( i \)-th Betti number of \( V \).

Next we recall the definition and basic facts about stringy E-functions from [16]. Let \( W \) be a normal irreducible variety with at worst log-terminal singularities, i.e.

1. \( W \) is \( \mathbb{Q} \)-Gorenstein;
2. for a resolution of singularities \( \rho : V \to W \) such that the exceptional locus of \( \rho \) is a divisor \( D \) whose irreducible components \( D_1, \ldots, D_r \) are smooth divisors with only normal crossings, we have

\[
K_V = \rho^* K_W + \sum_{i=1}^r a_i D_i
\]

with \( a_i > -1 \) for all \( i \), where \( D_i \) runs over all irreducible components of \( D \). The divisor \( \sum_{i=1}^r a_i D_i \) is called the discrepancy divisor.
Definition 2.1. For each subset $J \subset I = \{1, 2, \ldots, r\}$, define $D_J = \cap_{j \in J} D_j$, $D_\emptyset = V$ and $D^0_J = D_J - \cup_{i \in I - J} D_i$. Then the stringy E-function of $W$ is defined by

\[ E_{st}(W; u, v) = \sum_{J \subset I} E(D^0_J; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1} \]

where

\[ E(Z; u, v) = \sum_{p,q} \sum_{k \geq 0} (-1)^k h^{p,q}(H^k_c(Z; \mathbb{C})) u^p v^q \]

is the Hodge-Deligne polynomial (= E-polynomial) for a variety $Z$.

Note that the Hodge-Deligne polynomials have

1. the additive property: $E(Z; u, v) = E(U; u, v) + E(Z - U; u, v)$ if $U$ is an open subvariety of $Z$;

2. the multiplicative property: $E(Z; u, v) = E(B; u, v)E(F; u, v)$ if $Z$ is a Zariski locally trivial $F$-bundle over $B$.

By \cite[Theorem 6.27]{1}, the function $E_{st}$ is independent of the choice of a resolution (Theorem 3.4 in \cite{1}) and the following holds.

Theorem 2.2. \cite[Theorem 3.12]{1} Suppose $W$ is a $\mathbb{Q}$-Gorenstein algebraic variety with at worst log-terminal singularities. If $\rho: V \to W$ is a crepant desingularization (i.e. $\rho^* K_W = K_V$) then $E_{st}(W; u, v) = E(V; u, v)$. In particular, $E_{st}(W; u, v)$ is a polynomial.

3. Kirwan’s desingularization of $M$

In this section, we analyze Kirwan’s desingularization $\rho: \hat{M} \to M$ constructed in \cite[\S 2.1]{24}. Only Propositions 3.6 and 3.7 will be used in section 4.

Let $C$ be a smooth irreducible projective curve of genus 2 and $J = \text{Pic}^0(C)$. Fix a Weierstrass point $p_0$ of $C$ and let $\Theta$ be the image of the Abel-Jacobi map $C \to J$ defined by $p \mapsto p - p_0$. In this paper, we always suppose Assumption 1.2 is satisfied as in \cite[(1.3)]{24}. This is obviously satisfied if for instance the Néron-Severi group is $\text{NS}(J) = \mathbb{Z} c_1(\Theta)$. From now on, (semi)stability of a torsion-free sheaf on $J$ means (semi)stability with respect to the ample divisor $\Theta := \mathcal{O}_J(1)$. Let $J[n]$ denote the Hilbert scheme of $n$ points in $J$ and $\hat{J} = \text{Pic}^0(J)$.

Assumption 1.2 is necessary for the following (\cite[Lemma 2.1.2]{24}):

Lemma 3.1. A torsion-free sheaf $F$ of rank 2 with $c_1(F) = 0$ on $J$ is strictly semistable if and only if $F$ fits into a short exact sequence

\[ 0 \to I_{Z_1} \otimes \xi_1 \to F \to I_{Z_2} \otimes \xi_2 \to 0 \]

where $I_{Z_i}$ is the ideal sheaf of a zero dimensional subscheme $Z_i \in J[n]$ of length $n$ and $\xi_i \in \text{Pic}^0(J) = \hat{J}$. 


Now consider Simpson’s construction of the moduli space $M = M_{2n}$ [21 §1.1]. Let $Q$ be the closure of the set of semistable points $Q^{ss}$ in the Quot-scheme whose quotient by the natural $PGL(N)$ action is $M$ for some even integer $N$. Then $Q^{ss}$ parameterizes semistable sheaves $F$ together with surjective homomorphisms $h : \mathcal{O}^\oplus\mathcal{N} \to F(k)$ which induces an isomorphism $\mathbb{C}^N \cong H^0(F(k))$. Let $\Omega_Q$ denote the subset of $Q^{ss}$ which parameterizes sheaves of the form $(I_Z \otimes \xi)^{\oplus 2}$ for some $Z \in J^{[n]}$ and $\xi \in \text{Pic}^0(J) = \hat{J}$. Then $\Omega_Q$ is precisely the locus of closed orbits with maximal dimensional stabilizers, isomorphic to $PGL(2)$ and the quotient of $\Omega_Q$ by $PGL(N)$ is

$$\Omega := \Omega_Q/PGL(N) \cong J^{[n]} \times \hat{J}.$$ 

Let $\Sigma_Q$ be the subscheme of $Q^{ss}$ which parameterizes sheaves of the form $(I_{Z_1} \otimes \xi_1)^{\oplus} (I_{Z_2} \otimes \xi_2)$ for some $Z_1, Z_2 \in J^{[n]}$ and $\xi_1, \xi_2 \in \text{Pic}^0(J) = \hat{J}$. Then $\Sigma_Q - \Omega_Q$ is precisely the locus of closed orbits with 1-dimensional stabilizers isomorphic to $\mathbb{C}^*$. The quotient of $\Sigma_Q$ by $PGL(N)$ is

$$\Sigma := \Sigma_Q/PGL(N) \cong (J^{[n]} \times \hat{J}) \times (J^{[n]} \times \hat{J})/\mathbb{Z}_2$$

where the $\mathbb{Z}_2$-action is the involution which interchanges the two components while $\Omega$ sits in $\Sigma$ as the diagonal. So we have a stratification of $M$:

$$M = M^s \sqcup (\Sigma - \Omega) \sqcup \Omega$$

where $M^s$ is the locus of stable sheaves which is smooth by [20]. To obtain a desingularization of $M$ we blow up $M$ along $\Omega$ and then along the proper transform of $\Sigma$. The result of these two blow-ups is an orbifold and by blowing up once more along the singular locus we get a smooth model of $M$, which we call Kirwan’s desingularization [24 §2.1], [23 Proposition 1.8.3].

For a detailed analysis of $\Omega_Q$ and $\Sigma_Q$, we need to make some observations. To begin with, note that at each $(Z, \xi) \in J^{[n]} \times \hat{J}$ the tangent space $T_{J^{[n]} \times \hat{J}, (Z, \xi)}$ of $J^{[n]} \times \hat{J} \cong M_J(1, 0, n)$ is canonically isomorphic to $\text{Ext}^1(I_Z, I_Z)$ where $I_Z$ is the ideal sheaf of $Z$. By the Yoneda pairing map and the Serre duality, we have a skew-symmetric pairing

$$\omega : \text{Ext}^1(I_Z, I_Z) \otimes \text{Ext}^1(I_Z, I_Z) \to \text{Ext}^2(I_Z, I_Z) \cong \mathbb{C}$$

which gives us a symplectic form $\omega$ on the tangent bundle $T_{J^{[n]} \times \hat{J}}$ by [20 Theorem 0.1].

Let $W = sl(2)^{\vee} \cong sl(2) \cong \mathbb{C}^3$. The adjoint action of $PGL(2)$ on $W$ gives us an identification $SO(W) \cong PGL(2)$ ([21 §1.5]). For a symplectic vector space $(V, \omega)$, let $\text{Hom}^\omega(W, V)$ be the space of homomorphisms from $W$ to $V$ whose image is isotropic, i.e. the restriction of $\omega$ to the image is trivial. Let $\text{Hom}^\omega(W, T_{J^{[n]} \times \hat{J}})$ be the bundle over $J^{[n]} \times \hat{J}$ whose fiber over $(Z, \xi) \in J^{[n]} \times \hat{J}$ is $\text{Hom}^\omega(W, T_{J^{[n]} \times \hat{J}, (Z, \xi)})$. As an algebraic vector bundle, $T_{J^{[n]} \times \hat{J}}$ is a Zariski locally trivial bundle. By elementary linear algebra, we can furthermore find local trivializations so that the symplectic form $\omega$ is given by a constant skew-symmetric matrix on each open
set. Therefore, the bundle $\text{Hom}^{\omega}(W, T_{J[n] \times J})$ is Zariski locally trivial. Let $\text{Hom}^{\omega}_k(W, T_{J[n] \times J})$ be the subbundle of $\text{Hom}^{\omega}(W, T_{J[n] \times J})$ of rank $\leq k$ elements in $\text{Hom}^{\omega}(W, T_{J[n] \times J})$. Also let $\text{Gr}^\omega(3, T_{J[n] \times J})$ be the relative Grassmannian of isotropic 3-dimensional subspaces in $T_{J[n] \times J}$ and let $\mathcal{B}$ denote the tautological rank 3 bundle on $\text{Gr}^\omega(3, T_{J[n] \times J})$. Obviously these bundles are all Zariski locally trivial as well.

Let $\mathbb{P}\text{Hom}^{\omega}(W, T_{J[n] \times J})$ (resp. $\mathbb{P}\text{Hom}^{\omega}_k(W, T_{J[n] \times J})$) be the projectivization of $\text{Hom}^{\omega}(W, T_{J[n] \times J})$ (resp. $\text{Hom}^{\omega}_k(W, T_{J[n] \times J})$). Likewise, let $\text{Hom}(W, \mathcal{B})$ and $\text{Hom}_k(W, \mathcal{B})$ denote the projectivizations of the bundles $\text{Hom}(W, \mathcal{B})$ and $\text{Hom}_k(W, \mathcal{B})$. Note that there are obvious forgetful maps

$$f : \mathbb{P}\text{Hom}(W, \mathcal{B}) \to \mathbb{P}\text{Hom}^{\omega}(W, T_{J[n] \times J})$$

$$f_k : \mathbb{P}\text{Hom}_k(W, \mathcal{B}) \to \mathbb{P}\text{Hom}^{\omega}_k(W, T_{J[n] \times J})$$

Since the pull-back of the defining ideal of $\mathbb{P}\text{Hom}^{\omega}_k(W, T_{J[n] \times J})$ is the ideal of $\mathbb{P}\text{Hom}_1(W, \mathcal{B})$ (both are actually given by the determinants of $2 \times 2$ minor matrices), $f$ gives rise to a map between blow-ups

$$\overline{f} : Bl_{\mathbb{P}\text{Hom}_1(W, \mathcal{B})}\mathbb{P}\text{Hom}(W, \mathcal{B}) \to Bl_{\mathbb{P}\text{Hom}^{\omega}_k(W, T_{J[n] \times J})}\mathbb{P}\text{Hom}^{\omega}(W, T_{J[n] \times J}).$$

Let us denote $Bl_{\mathbb{P}\text{Hom}_1(W, \mathcal{B})}\mathbb{P}\text{Hom}(W, \mathcal{B})$ by $Bl^\mathcal{B}$ and $Bl_{\mathbb{P}\text{Hom}^{\omega}_k(W, T_{J[n] \times J})}\mathbb{P}\text{Hom}^{\omega}(W, T_{J[n] \times J})$ by $Bl^T$. We denote the proper transform of $\mathbb{P}\text{Hom}_2(W, \mathcal{B})$ in $Bl^\mathcal{B}$ by $Bl^\mathcal{B}_2$ and the proper transform of $\text{Hom}_2(W, T_{J[n] \times J})$ by $Bl^T_2$. Since $Bl^T_2$ is a smooth divisor which is mapped onto $Bl^T$ and the pull-back of the defining ideal of $Bl^T_2$ is the ideal sheaf of $Bl^\mathcal{B}_2$, $\overline{f}$ lifts to

$$(3.1) \quad \hat{f} : Bl^\mathcal{B} \to Bl^T, \overline{f}$$

By [21], §3.1 IV, $\hat{f}$ is an isomorphism on each fiber over $J^{[n]}$, so in particular $\hat{f}$ is bijective. Therefore, $\hat{f}$ is an isomorphism.

Note that $\mathbb{P}\text{Hom}(W, \mathcal{B})/\mathbb{P}(SO(W))$ (resp. $\mathbb{P}\text{Hom}_k(W, \mathcal{B})/\mathbb{P}(SO(W))$) is isomorphic to the space of conics $\mathbb{P}(S^2 \mathcal{B})$ (resp. rank $\leq k$ conics $\mathbb{P}(S^2 \mathcal{B})$) where the $SO(W)$-quotient map is given by $\alpha \mapsto \alpha \circ \alpha^t$ ([21], §3.3). Let $\mathbb{P}(S^2 \mathcal{B}) \cong Bl_{\mathbb{P}(SO(W))}\mathbb{P}(S^2 \mathcal{B})$ denote the blow-up along the locus of rank 1 conics. Then $Bl^T/\mathbb{P}(SO(W))$ is canonically isomorphic to $\mathbb{P}(S^2 \mathcal{B})$ by [18] Lemma 3.11. Since $\mathcal{B}$ is Zariski locally trivial, so is $\mathbb{P}(S^2 \mathcal{B})$ over $\text{Gr}^\omega(3, T_{J[n] \times J})$.

Now we can give a more precise description of $\Omega_Q$ as follows. Let $\mathcal{L}$ be a universal rank 1 sheaf over $(J^{[n]} \times \hat{J}) \times J = M_J(1, 0, n) \times J$ such that $\mathcal{L}|(Z, \xi) \times J$ is isomorphic to $I_Z \otimes \xi$. By [13] Theorem 10.2.1, the tangent bundle $T_{J[n] \times J}$ is in fact isomorphic to $\mathcal{E}_{\mathcal{L}}T_J^1(J^{[n]} \times J, \mathcal{L}, \mathcal{L})$. Let $p : (J^{[n]} \times \hat{J}) \times J \to J^{[n]} \times \hat{J}$ be the projection onto $J^{[n]} \times \hat{J}$ and $p_J$ be the projection onto $J$. By tensoring with the pull-back of $\mathcal{O}_J(k)$ for suitable $k$, $p_*\mathcal{L}(k)$ is a vector bundle of rank $N/2$ where $\mathcal{L}(k) := \mathcal{L} \otimes p_J^*\mathcal{O}_J(k)$. Let

$$(3.2) \quad q : \text{Pisom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k)) \to J^{[n]} \times \hat{J}$$
be the $PGL(N)$-bundle over $J^{[n]} \times \hat{J}$ whose fiber over $(Z, \xi)$ is
$$\mathbb{P} \text{Isom}(\mathbb{C}^N, H^0((I_Z \otimes \xi)^{\oplus 2} \otimes \mathcal{O}_J(k))).$$
Note that the standard action of $PGL(N)$ on $\mathbb{C}^N$ commutes with the obvious action of $PGL(2) \cong SO(W)$ on $p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k)$.

**Lemma 3.2.** (1) $\Omega_Q \cong \mathbb{P} \text{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k))/SO(W)$.
(2) Via the above isomorphism, the normal cone of $\Omega_Q$ in $Q^{ss}$ is
$$q^*\text{Hom}^\omega(W, T_{J^{[n]} \times \hat{J}})/SO(W) \rightarrow \mathbb{P} \text{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k))/SO(W)$$
whose fiber is $\text{Hom}^\omega(W, T_{J^{[n]} \times \hat{J}}(Z, \xi))$.

**Proof.** (1) This is standard and we omit the proof.
(2) Let $\mathcal{E}$ denote the universal quotient sheaf on $Q^{ss} \times J$. The Kodaira-Spencer map associated to $\mathcal{E}$ restricted to $\Omega_Q$ gives us a map from the tangent sheaf $T_{Q^{ss}}|_{\Omega_Q}$ to the sheaf $\text{Ext}^1_{\Omega_Q}(\mathcal{E}, \mathcal{E})$ whose kernel is the tangent sheaf of the orbits. Via the isomorphism of (1), we have a map
$$\delta : \mathbb{P} \text{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k)) \rightarrow \mathbb{P} \text{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k))/SO(W) \cong \Omega_Q.$$
From the proof of (1) above, the pull-back of $\mathcal{E}$ by $\delta$ is isomorphic to $(q \times 1)^*(\mathcal{L}(k) \oplus \mathcal{L}(k)) \otimes H$ and thus the vector bundle $\delta^*\text{Ext}^1_{\Omega_Q}(\mathcal{E}, \mathcal{E})$ is isomorphic to
$$q^*\text{Ext}^1_{J^{[n]} \times \hat{J}}(\mathcal{E}, \mathcal{L}) \otimes gl(2) \cong q^*T_{J^{[n]} \times \hat{J}} \otimes gl(2).$$
The pull-back of the tangent sheaf of $J^{[n]} \times \hat{J}$ sits in it as $q^*T_{J^{[n]} \times \hat{J}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
and thus the pull-back by $\delta$ of the normal sheaf to $\Omega_Q$ is isomorphic to
$$q^*T_{J^{[n]} \times \hat{J}} \otimes sl(2) \cong q^*\text{Hom}(W, T_{J^{[n]} \times \hat{J}}).$$
By $[21]$ (1.4.10), the normal cone is the same as the Hessian cone fiber-wisely. Since the normal cone is contained in the Hessian cone, the normal cone is equal to the Hessian cone which is the inverse image of zero by the Yoneda square map $\mathcal{Y} : \text{Ext}^1_{\Omega_Q}(\mathcal{E}, \mathcal{E}) \rightarrow \text{Ext}^2_{\Omega_Q}(\mathcal{E}, \mathcal{E})$. It is elementary to see that $\delta^*\mathcal{Y}^{-1}(0)$ is precisely $q^*\text{Hom}(W, T_{J^{[n]} \times \hat{J}})$. Since $SO(W)$ acts freely we obtain (2). □

Let $\pi_R : R \rightarrow Q^{ss}$ be the blow-up of $Q^{ss}$ along $\Omega_Q$. Let $\Omega_R$ be the exceptional divisor of $\pi_R$ and $\Sigma_R$ be the proper transform of $\Sigma_Q$. By the above lemma, we have
$$\Omega_R \cong q^*\mathbb{P} \text{Hom}^\omega(W, T_{J^{[n]} \times \hat{J}})/SO(W).$$
The following lemma is an easy exercise.

**Lemma 3.3.** (1) The locus of points in $\mathbb{P} \text{Hom}^\omega(W, T_{J^{[n]} \times \hat{J}}(Z, \xi))^{ss}$ whose stabilizer is 1-dimensional by the action of $SO(W)$ is precisely $\mathbb{P} \text{Hom}^\omega(W, T_{J^{[n]} \times \hat{J}}(Z, \xi))^{ss}$. 
(2) The locus of nontrivial stabilizers is $\mathbb{P}\text{Hom}^\omega_2(W, T_Jn) \times J, (Z, \xi) )^{ss}$ and the stabilizers are isomorphic to $\mathbb{Z}_2$ or $\mathbb{C}^*$.

Let
\begin{equation}
\Delta_R = q^*\mathbb{P}\text{Hom}^\omega_2(W, T_Jn) // SO(W).
\end{equation}

Note that $\Sigma_Q - \Omega_Q$ is precisely the locus of points in $Q^{ss}$ whose stabilizer is isomorphic to $\mathbb{C}^*$ and hence $\Sigma_R^{ss}$ is precisely the locus of points in $R^{ss}$ with 1-dimensional stabilizers by [13]. Therefore we have the following from Lemma 3.3.

**Corollary 3.4.** $\Sigma_R^{ss} \cap \Omega_R = q^*\mathbb{P}\text{Hom}^\omega_2(W, T_Jn) // SO(W)$.

We have an explicit description of $\Sigma_R^{ss}$ which is parallel to [21] §1.7 II] as follows. Let
\begin{equation}
\beta : \mathfrak{g}^m \rightarrow (J^n \times \hat{J}) \times (J^n \times \hat{J})
\end{equation}
be the blow-up along the diagonal and let $\mathfrak{g}^m_0 = (J^n \times \hat{J}) \times (J^n \times \hat{J}) - \Delta$ where $\Delta$ is the diagonal. Let $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) be the pull-back to $\mathfrak{g}^m_0$ of the universal sheaf $\mathcal{L} \rightarrow (J^n \times \hat{J}) \times J$ by $p_{13} \circ (\beta \times 1)$ (resp. $p_{23} \circ (\beta \times 1)$) where $p_{ij}$ is the projection onto the first (resp. second) and third components. Let $p : \mathfrak{g}^m_0 \times J \rightarrow \mathfrak{g}^m_0$ be the projection onto the first component. Then $p_* \mathcal{L}_1(k) \oplus p_* \mathcal{L}_2(k)$ is a vector bundle of rank $N$. Let
\begin{equation}
q : \mathbb{P}\text{Isom}(\mathbb{C}^N, p_* \mathcal{L}_1(k) \oplus p_* \mathcal{L}_2(k)) \rightarrow \mathfrak{g}^m_0
\end{equation}
be the $\mathbb{P}GL(N)$-bundle. There is an action of $O(2)$ on $\mathbb{P}\text{Isom}(\mathbb{C}^N, p_* \mathcal{L}_1(k) \oplus p_* \mathcal{L}_2(k))$. The following lemma is obtained by (a proof parallel to) [21] (1.7.10) and (1.7.1).

**Lemma 3.5.** (1) $\Sigma_R^{ss} \cong \mathbb{P}\text{Isom}(\mathbb{C}^N, p_* \mathcal{L}_1(k) \oplus p_* \mathcal{L}_2(k)) // O(2)$
(2) The normal cone of $\Sigma_R^{ss}$ in $R^{ss}$ is a locally trivial bundle over $\Sigma_R^{ss}$ with fiber the cone over a smooth quadric in $\mathbb{P}^{4n-1}$.

In fact we can give a more explicit description of the normal cone when restricted to $\Sigma^0_R := \Sigma_R^{ss} - \Omega_R$. Similarly as in the proof of Lemma 3.2 the normal sheaf to $\Sigma^0_R$ is isomorphic to the vector bundle (of rank 4n)
\begin{equation}
q[\mathbb{E}xt^1_{\delta_0[n]}(\mathcal{L}_1, \mathcal{L}_2) \oplus \mathbb{E}xt^1_{\delta_0[n]}(\mathcal{L}_2, \mathcal{L}_1)] // O(2)
\end{equation}
over $\mathbb{P}\text{Isom}(\mathbb{C}^N, p_* \mathcal{L}_1(k) \oplus p_* \mathcal{L}_2(k)) // O(2)$ where $O(2)$ acts as follows: if we realize $O(2)$ as the subgroup of $\mathbb{P}GL(2)$ generated by $SO(2) = \{ \theta_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \} / \{ \pm Id \}$, $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\theta_\alpha$ multiplies $\alpha$ (resp. $\alpha^{-1}$) to $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) and $\tau$ interchanges $\mathcal{L}_1$ and $\mathcal{L}_2$. The normal cone is the inverse image $q^* \Upsilon^{-1}(0)$ of zero in terms of the Yoneda pairing
\begin{equation}
\Upsilon : \mathbb{E}xt^1_{\delta_0[n]}(\mathcal{L}_1, \mathcal{L}_2) \oplus \mathbb{E}xt^1_{\delta_0[n]}(\mathcal{L}_2, \mathcal{L}_1) \rightarrow \mathbb{E}xt^2_{\delta_0[n]}(\mathcal{L}_1, \mathcal{L}_1).
\end{equation}
Let $\pi_S : S \to R^{ss}$ be the blow-up of $R^{ss}$ along $\Sigma_R^{ss}$. Let $\Sigma_S$ be the exceptional divisor of $\pi_S$ and $\Omega_S$ (resp. $\Delta_S$) be the proper transform of $\Omega_R$ (resp. $\Delta_R$). By (3.3), we have

$\Sigma_S^{\pi_S^{-1}(\tau_R)} \cong q^* \mathbb{P}Y^{-1}(0)/O(2) \subset q^* \mathbb{P}[\mathbb{E}xt^1_{\mathbb{A}^0}(\mathcal{L}_2, \mathcal{L}_2) \oplus \mathbb{E}xt^1_{\mathbb{A}^0}(\mathcal{L}_2, \mathcal{L}_1)]/O(2)$.

By (a proof parallel to) [21, (1.8.10)], $S^a = S^{ss}$ and $S^a$ is smooth. The quotient $S/PGL(N)$ has only $\mathbb{Z}_2$-quotient singularities along $\Delta_S/PGL(N)$.

Finally let $\pi_T : T \to S^a$ be the blow-up of $S^a$ along $\Delta_S^a$. Let $\Delta_T$ be the exceptional divisor of $\pi_T$ and $\Omega_T$ (resp. $\Sigma_T$) be the proper transform of $\Omega_S$ (resp. $\Sigma_S$). Since $\Omega_T^a$, $\Sigma_T^a$ and $\Delta_T^a$ are smooth divisors with finite stabilizers $T/PGL(N)$ is nonsingular and this is Kirwan’s desingularization $\rho : \check{M} \to \mathcal{M}$.

The quotients $\Omega_T/PGL(N)$, $\Sigma_T/PGL(N)$ and $\Delta_T/PGL(N)$ are denoted by $D_1 = \check{\Omega}$, $D_2 = \check{\Sigma}$ and $D_3 = \check{\Delta}$ respectively.

We are ready to describe all the intersections of the smooth divisors $D_1$, $D_2$ and $D_3$. Let $\check{\mathbb{P}}^5$ be the blow-up of $\mathbb{P}^5$ (projectivization of the space of 3 × 3 symmetric matrices) along $\mathbb{P}^2$ (the locus of rank 1 matrices). For a symplectic vector space $(\mathbb{C}^{2n}, \omega)$, $\text{Gr}^\omega(k, 2n)$ denotes the Grassmannian of $k$-dimensional subspaces of $\mathbb{C}^{2n}$, isotropic with respect to the symplectic form $\omega$ (i.e. the restriction of $\omega$ to the subspace is zero).

**Proposition 3.6.** Let $n \geq 2$.

1. $D_1$ is a $\check{\mathbb{P}}^5$-bundle over a $\text{Gr}^\omega(3, 2n + 2)$-bundle over $\check{J}^{|n|} \times \check{J}$.
2. $D_2^0$ is a free $\mathbb{Z}_2$-quotient of a Zariski locally trivial $\check{I}_{2n-1}$-bundle over $\check{J}^{|n|} = (\check{J}^{|n|} \times \check{J}) \times (\check{J}^{|n|} \times \check{J}) - \Delta$ where $\Delta$ is the diagonal and $\check{I}_{2n-1}$ is the incidence variety given by

$$I_{2n-1} = \{(p, H) \in \mathbb{P}^{2n-1} \times \mathbb{P}^{2n-1} | p \in H\}.$$

3. $D_3$ is a $\mathbb{P}^{2n-2}$-bundle over a Zariski locally trivial $\mathbb{P}^2$-bundle over a Zariski locally trivial $\text{Gr}^\omega(2, 2n + 2)$-bundle over $\check{J}^{|n|} \times \check{J}$.
4. $D_1 \cap D_2$ is a $\mathbb{P}^2 \times \mathbb{P}^2$-bundle over a $\text{Gr}^\omega(3, 2n + 2)$-bundle over $\check{J}^{|n|} \times \check{J}$.
5. $D_2 \cap D_3$ is a $\mathbb{P}^{2n-2}$-bundle over a Zariski locally trivial $\mathbb{P}^1$-bundle over a Zariski locally trivial $\text{Gr}^\omega(2, 2n + 2)$-bundle over $\check{J}^{|n|} \times \check{J}$.
6. $D_1 \cap D_3$ is a $\mathbb{P}^2 \times \mathbb{P}^2$-bundle over a $\text{Gr}^\omega(3, 2n + 2)$-bundle over $\check{J}^{|n|} \times \check{J}$.
7. $D_1 \cap D_2 \cap D_3$ is a $\mathbb{P}^1 \times \mathbb{P}^2$-bundle over a $\text{Gr}^\omega(3, 2n + 2)$-bundle over $\check{J}^{|n|} \times \check{J}$.

All the above bundles except in (2), (3) and (5) are Zariski locally trivial. Moreover, $D_i$ ($i = 1, 2, 3$) are smooth divisors such that $D_1 \cup D_2 \cup D_3$ is normal crossing.

**Proof.** (1) By (3.3) and Corollary 3.4, $\Omega_S$ is the blow-up of

$q^* \mathbb{P}Hom^\omega(W, T_{J^{|n|} \times \check{J}})/SO(W)$ along $q^* \mathbb{P}Hom^\omega(W, T_{J^{|n|} \times \check{J}})/SO(W)$. 


By (3.1), \( \Omega_T \) is the blow-up of \( \Omega_S \) along the proper transform of
\[ q^*\text{Hom}^\omega_2(W, T_{j[n] \times j})//SO(W) \]
and \( D_1 = \hat{\Omega} \) is the quotient of \( \Omega_T \) by the action of \( PGL(N) \). Since the action
of \( PGL(N) \) commutes with the action of \( SO(W) \), \( D_1 \) is in fact the quotient
by \( SO(W) \times PGL(N) \) of the variety obtained from \( q^*\text{Hom}^\omega_2(W, T_{j[n] \times j}) \)
by two blow-ups. So \( D_1 \) is also the consequence of taking the quotient by
\( PGL(N) \) first and then the quotient by \( SO(W) \) second. Since \( q \) \((3.2)\) is a
principal \( PGL(N) \)-bundle, the result of the first quotient is just \( Bl_{Bl^2} Bl^T \)
in \((3.3)\) which is isomorphic to \( Bl^B \). If we take further the quotient by
\( SO(W) \), then as discussed above the result is \( D_1 = \hat{\mathbb{P}}(S^2B) \).

(2) We use Lemma \(3.5 \) and \((3.7)\). Note that \( \Sigma^0_n \) does not intersect
with \( \Omega_R \) and \( \Delta_R \). Hence \( D^0_2 \) is the quotient of \( q^*\mathbb{P}^T^{-1}(0)//O(2) \) which
is a subset of \( q^*\mathbb{P}[^\mathbb{P}E_{xt1}^1_{\mathbb{E}_0}(\mathcal{L}_1, \mathcal{L}_2) \oplus E_{xt1}^1_{\mathbb{E}_0}(\mathcal{L}_2, \mathcal{L}_1)]]//O(2) \), by the action
of \( PGL(N) \). The above are bundles over the restriction of
\[ \mathbb{P}\text{Isom}(\mathbb{C}^N, p_\mathcal{L}_1(k) \oplus p_\mathcal{L}_2(k))/O(2) \]
to \( \mathbb{E}_0[n] \). As in the proof of (1), observe that \( D^0_2 \) is in fact the quotient of
\( q^*\mathbb{P}^{-1}(0) \) by the action of \( PGL(N) \times O(2) \) since the actions commute.
So we can first take the quotient by the action of \( PGL(N) \), then by the
action of \( SO(2) \), and finally by the action of \( Z_2 = O(2)/SO(2) \). Since
\[ \mathbb{P}\text{Isom}(\mathbb{C}^N, p_\mathcal{L}_1(k) \oplus p_\mathcal{L}_2(k)) \] is a principal \( PGL(N) \)-bundle, the quotient
by \( PGL(N) \) gives us
\[ \mathbb{P}^T^{-1}(0) \subset \mathbb{P}[E_{xt1}^1_{\mathbb{E}_0}(\mathcal{L}_1, \mathcal{L}_2) \oplus E_{xt1}^1_{\mathbb{E}_0}(\mathcal{L}_2, \mathcal{L}_1)] \]
over \( \mathbb{E}_0[n] \). The algebraic vector bundles \( E_{xt1}^1_{\mathbb{E}_0}(\mathcal{L}_1, \mathcal{L}_2) \) and \( E_{xt1}^1_{\mathbb{E}_0}(\mathcal{L}_2, \mathcal{L}_1) \)
are certainly Zariski locally trivial and in fact these bundles are dual to each
other by the Yoneda pairing \( \Upsilon \) which is non-degenerate. In particular,
\( \Upsilon^{-1}(0) \) is Zariski locally trivial.

Next we take the quotient of \( \mathbb{P}^T(0)//SO(2) \) which is non-degenerate. This action is
trivial on the base \( \mathbb{E}_0[n] \) and \( SO(2) \) acts on the fibers. Hence \( \mathbb{P}T^{-1}(0)/SO(2) \) is a
Zariski locally trivial subbundle of
\[ \mathbb{P}[E_{xt1}^1_{\mathbb{E}_0}(\mathcal{L}_1, \mathcal{L}_2) \oplus E_{xt1}^1_{\mathbb{E}_0}(\mathcal{L}_2, \mathcal{L}_1)]/\mathbb{C}^* \cong \mathbb{P}E_{xt1}^1_{\mathbb{E}_0}(\mathcal{L}_1, \mathcal{L}_2) \times E_{xt1}^1_{\mathbb{E}_0}(\mathcal{L}_2, \mathcal{L}_1) \]
over \( \mathbb{E}_0[n] \) given by the incidence relations in terms of the identification
\[ \mathbb{P}E_{xt1}^1_{\mathbb{E}_0}(\mathcal{L}_1, \mathcal{L}_2) \cong \mathbb{P}E_{xt1}^1_{\mathbb{E}_0}(\mathcal{L}_2, \mathcal{L}_1)^\vee. \]
Finally, \( D^0_2 \) is the \( Z_2 \)-quotient of \( \mathbb{P}T^{-1}(0)/SO(2) \).

(3) By (a proof parallel to \(21 \) (1.7.10), the intersection of \( \Sigma^r_{\mathbb{R}} \) and \( \Omega_R \) is
smooth. By Corollary \(3.4 \) and \((3.2)\), \( \delta_S \) is the blow-up of \( q^*\text{Hom}^\omega_2(\mathbb{W}, T_{j[n] \times j})//SO(W) \)
along $q^*\mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J})/\mathbb{P}(W)$. Hence $\Delta_S/\mathbb{P}GL(N)$ is the quotient of

$$Bl_q^*\mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J}),$$

by the action of $SO(W) \times PGL(N)$. By taking the quotient by the action of $PGL(N)$ we get

$$Bl_{\mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J})}^*\mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J})$$

since $q$ is a principal $PGL(N)$-bundle. Next we take the quotient by the action of $SO(W)$. Let $Gr^\omega(2,T_{J[n]}|_{X,J})$ be the relative Grassmannian of isotropic 2-dimensional subspaces in $T_{J[n]}|_{X,J}$ and let $A$ be the tautological rank 2 bundle on $Gr^\omega(2,T_{J[n]}|_{X,J})$. We claim

$$Bl_{\mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J})}^*\mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J})/\mathbb{P}(SO(W) \cong \mathbb{P}(S^2A)$$

which is a $\mathbb{P}^2$-bundle over a $Gr^\omega(2,2n)$-bundle over $J[n]$. It is obvious that the bundles are Zariski locally trivial.

There are forgetful maps

$$f : \mathbb{P}\mathrm{Hom}(W,A) \to \mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J})$$

and

$$f_1 : \mathbb{P}\mathrm{Hom}_{1}(W,A) \to \mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J})$$

where the subscript 1 denotes the locus of rank $\leq 1$ homomorphisms. Because the ideal of $\mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J})$ pulls back to the ideal of $\mathbb{P}\mathrm{Hom}_{1}(W,A)$, $f$ lifts to

$$\hat{f} : Bl_{\mathbb{P}\mathrm{Hom}(W,A)}^*\mathbb{P}\mathrm{Hom}(W,A) \to Bl_{\mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J})}^*\mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J}).$$

This map is bijective (21.3.5.1)] and hence $\hat{f}$ is an isomorphism. Now observe that the quotient $Bl_{\mathbb{P}\mathrm{Hom}_{1}(W,A)}^*\mathbb{P}\mathrm{Hom}(W,A)/\mathbb{P}(SO(W)$ is isomorphic to $\mathbb{P}(S^2A)$ where the quotient map is given by $\alpha \mapsto \alpha \circ \alpha^t$. So we proved that

$$\Delta_S/\mathbb{P}GL(N) \cong \mathbb{P}(S^2A).$$

Finally $S/\mathbb{P}GL(N)$ is singular only along $\Delta_S/\mathbb{P}GL(N)$ and the singularities are $\mathbb{C}^{2n-1}/\{\pm 1\}$ by Luna’s slice theorem [21.1.2,1]. Since $D_3$ is the exceptional divisor of the blow-up of $S/\mathbb{P}GL(N)$ along $\Delta_S/\mathbb{P}GL(N)$, we conclude that $D_3$ is a $\mathbb{P}^{2n-2}$-bundle over $\mathbb{P}(S^2A)$.

(4) By Corollary 3.34, $\Sigma_S \cap \Omega_S$ is the exceptional divisor of the blow-up $Bl_{q^*\mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J})}^*q^*\mathbb{P}\mathrm{Hom}(W,T_{J[n]}|_{X,J})/\mathbb{P}(SO(W)$ and $\Sigma_T \cap \Omega_T$ is now the blow-up of the exceptional divisor along the proper transform of

$$q^*\mathbb{P}\mathrm{Hom}_{2}'(W,T_{J[n]}|_{X,J})/\mathbb{P}(SO(W).$$

Using the isomorphism 3.34, this is the exceptional divisor of

$$q^*Bl_{\mathbb{P}(S^2A)}\mathbb{P}(S^2B) \to q^*\mathbb{P}(S^2B)$$
over $\text{Gr}^w(3, T_{J[n \times j]})$. Since $q$ is a principal $\text{PGL}(N)$-bundle, $D_1 \cap D_2 = \Sigma_{T} \cap \Omega_T // \text{PGL}(N)$ is the exceptional divisor of the blow-up $\text{Bl}_{P(S^2 B)} P(S^2 B)$. As $P(S^2 B)$ is a $P^2$-bundle over $\text{Gr}^w(3, T_{J[n \times j]})$, the exceptional divisor is a $P^2 \times P^2$-bundle over $\text{Gr}^w(3, T_{J[n \times j]})$. This is obviously Zariski locally trivial.

(5) From the above proof of (3) it follows immediately that $\Sigma_S \cap \Delta_S // \text{SO}(W)$ is $P(S^2 A)$ and $D_2 \cap D_3$ is a $P^{2n-2}$ bundle over $P(S^2 A)$.

(6) As in the above proof of (4), we start with (3.4) and use the isomorphism (3.1) to see that $D_1 \cap D_3$ is the proper transform of $P(S^2 A)$ in the blow-up $\text{Bl}_{P(S^2 B)} P(S^2 B)$. This is a Zariski locally trivial $P^2 \times P^2$-bundle over $\text{Gr}^w(3, T_{J[n \times j]})$.

(7) The description of $D_1 \cap D_2 \cap D_3$ follows immediately from the proof of (4) and (6).

From the above descriptions, it is clear that $D_i$ ($i = 1, 2, 3$) are normal crossing smooth divisors. □

In order to compute the stringy E-function of $M$ by using Kirwan’s desingularization $\widehat{M}$ and Definition 2.1, we also need the discrepancy divisor $K_{\widehat{M}} - \rho^* K_M$.

Proposition 3.7. The discrepancy divisor of $\rho : \widehat{M} \to M$ is

$$(6n - 1)D_1 + (2n - 2)D_2 + (4n - 2)D_3$$

Proof. The proof is identical to that of [21, (3.4.1)] and so we omit the details. □

In particular, the singularities of $M = M_{2n}$ are terminal for $n \geq 2$.

Remark 3.8. Another way to prove Proposition 3.7 is as follows. First observe as in [21] that $\widehat{M}$ can be blown-down twice:

$$\widehat{M} \to \overline{\widehat{M}} \to \widetilde{M}$$

The first map is the contraction of $D_3$ along the $P^2$-fiber and the second map is the contraction of $D_1$ along the $P^5$-fiber (after the first contraction $\overline{\tilde{P}^5}$ becomes $\tilde{P}^5$). The result of the two contractions is also a desingularization $\nu : \tilde{M} \to M$. Since the singularities along $\Sigma$ are toric, it is easy to compute the discrepancy along $D_2$ of $\nu$ which is precisely $2n - 2$ by toric geometry. It is not hard to check that the pull-back of the closure of $D_2$ in $\tilde{M}$ to $\tilde{M}$ is $3D_1 + D_2 + 2D_3$. From the well-known formula [12, II Ex. 8.5], we deduce that the discrepancy divisor for $\rho$ is

$$(2n - 2)(3D_1 + D_2 + 2D_3) + 5D_1 + 2D_3 = (6n - 1)D_1 + (2n - 2)D_2 + (4n - 2)D_3.$$
4. Nonexistence of a crepant resolution

In this section we first find an expression for the stringy E-function of the moduli space $M = M_{2n}$ with $n \geq 2$ by using the detailed analysis of Kirwan’s desingularization in [§]. Then we show that it cannot be a polynomial, which proves Proposition 1.5.

By (2.2) and Proposition 3.7, the stringy E-function of $M$ is given by

$$
E(M; u, v) + E(D^0_1; u, v) \frac{1 - uv}{1 - (uv)^{2n}} + E(D^0_2; u, v) \frac{1 - uv}{1 - (uv)^{2n-1}}
$$

$$
+ E(D^0_3; u, v) \frac{1 - uv}{1 - (uv)^n} + E(D^0_{12}; u, v) \frac{1 - uv}{1 - (uv)^{2n-1}}
$$

$$
+ E(D^0_{23}; u, v) \frac{1 - uv}{1 - (uv)^{2n-1}} + E(D^0_{13}; u, v) \frac{1 - uv}{1 - (uv)^n}
$$

$$
+ E(D^0_{123}; u, v) \frac{1 - uv}{1 - (uv)^n} + E(D^0; u, v) \frac{1 - uv}{1 - (uv)^n}.
$$

(4.1)

We need to compute the Hodge-Deligne polynomials of $D^0_j$ for $J \subset \{1, 2, 3\}$. Recall that for a symplectic vector space $(\mathbb{C}^{2n}, \omega)$, $\text{Gr}^\omega(k, 2n)$ denotes the Grassmannian of $k$-dimensional subspaces of $\mathbb{C}^{2n}$, isotropic with respect to the symplectic form $\omega$ (i.e. the restriction of $\omega$ to the subspace is zero).

**Lemma 4.1.** [1 Lemma 3.1] For $k \leq n$, the Hodge-Deligne polynomial of $\text{Gr}^\omega(k, 2n)$ is

$$
\prod_{1 \leq i \leq k} \frac{1 - (uv)^{2n-2k+2i}}{1 - (uv)^i}.
$$

From Lemma 4.1 and Proposition 3.6, we have the following corollary by the additive and multiplicative properties of the Hodge-Deligne polynomial.

**Corollary 4.2.**

$$
E(D_1; u, v) = \left(\frac{1 - (uv)^6}{1 - uv} - \frac{1 - (uv)^3}{1 - uv}\right) \frac{1 - (uv)^3}{1 - uv} \times E(J[n] \times \hat{J}; u, v),
$$

$$
E(D_3; u, v) = \frac{1 - (uv)^{2n-1}}{1 - uv} \times \prod_{1 \leq i \leq 2} \frac{1 - (uv)^{2n-2+2i}}{1 - (uv)^i} \times E(J[n] \times \hat{J}; u, v),
$$

$$
E(D_{12}; u, v) = \left(\frac{1 - (uv)^3}{1 - uv}\right)^2 \times \prod_{1 \leq i \leq 3} \frac{1 - (uv)^{2n-4+2i}}{1 - (uv)^i} \times E(J[n] \times \hat{J}; u, v),
$$

$$
E(D_{23}; u, v) = \frac{1 - (uv)^{2n-1}}{1 - uv} \times \prod_{1 \leq i \leq 2} \frac{1 - (uv)^{2n-2+2i}}{1 - (uv)^i} \times E(J[n] \times \hat{J}; u, v),
$$

$$
E(D_{13}; u, v) = \frac{1 - (uv)^3}{1 - uv} \times \prod_{1 \leq i \leq 2} \frac{1 - (uv)^{2n-2+2i}}{1 - (uv)^i} \times E(J[n] \times \hat{J}; u, v),
$$

$$
E(D_{123}; u, v) = \frac{1 - (uv)^2}{1 - uv} \times \prod_{1 \leq i \leq 2} \frac{1 - (uv)^{2n-2+2i}}{1 - (uv)^i} \times E(J[n] \times \hat{J}; u, v).
$$
Proof. The only thing that doesn’t follow from the multiplicative property of Hodge-Deligne polynomial is the equations for $D_3$ and $D_{23}$ but this is a direct consequence of the Leray-Hirsch theorem [26, p.195].

For the E-polynomial of $D_2^0$ we have the following lemma.

**Lemma 4.3.** $E(D_2^0; z, z)$ is divisible by $\frac{1-(z^2)^{2n-1}}{1-z^2}$.

**Proof.** Note that

$$I_{2n-1} = \{ ((x_i), (y_j)) \in \mathbb{P}^{2n-1} \times \mathbb{P}^{2n-1} | \sum_{i=0}^{2n-1} x_i y_i = 0 \}$$

and that it admits a $\mathbb{Z}_2$-action interchanging $x_i$ and $y_i$. It is elementary ([10, p. 606]) to see that $H^*(I_{2n-1}; \mathbb{Q}) \cong \mathbb{Q}[a, b]/(a^{2n}, b^{2n}, a^{2n-1} + a^{2n-2}b + a^{2n-3}b^2 + \ldots + b^{2n-1})$ where $a$ (resp. $b$) is the pull-back of the first Chern class of the tautological line bundle of the first (resp. second) $\mathbb{P}^{2n-1}$. The $\mathbb{Z}_2$-action interchanges $a$ and $b$. Let $H^*(I_{2n-1}; \mathbb{Q})^\pm$ be the $\pm 1$-eigenspace of the $\mathbb{Z}_2$-action in $H^*(I_{2n-1}; \mathbb{Q})$. The invariant subspace $H^*(I_{2n-1}; \mathbb{Q})^+$ of $H^*(I_{2n-1}; \mathbb{Q})$ is generated by classes of the form $a^ib^j + a^jb^i$. As a vector space $H^*(I_{2n-1}; \mathbb{Q})$ is

\[(4.2) \quad \mathbb{Q}\text{-span}\{a^ib^j | 0 \leq i \leq 2n-1, 0 \leq j \leq 2n-2\}\]

while the invariant subspace is

\[\mathbb{Q}\text{-span}\{a^ib^j + a^jb^i | 0 \leq i \leq j \leq 2n-2\} \]

The index set $\{(i, j) | 0 \leq i \leq j \leq 2n-2\}$ is mapped to its complement in $\{(i, j) | 0 \leq i \leq 2n-1, 0 \leq j \leq 2n-2\}$ by the map $(i, j) \mapsto (j+1, i)$. This immediately implies that the Poincaré polynomial satisfies

\[(4.3) \quad P(I_{2n-1}; z) = (1 + z^2)P^+(I_{2n-1}; z) \]

where $P^\pm(I_{2n-1}; z) = \sum (-1)^r z^r \dim H^r(I_{2n-1})^\pm$. By (4.2), we have

\[P(I_{2n-1}; z) = \frac{1 - (z^2)^{2n}}{1 - z^2} - \frac{1 - (z^2)^{2n-1}}{1 - z^2}.\]

Because $1 + z^2$ divides $\frac{1 - (z^2)^{2n}}{1 - z^2}$, $\frac{1 - (z^2)^{2n-1}}{1 - z^2}$ also divides $P^+(I_{2n-1}; z)$. By (4.3), $P^-(I_{2n-1}; z) = z^2P^+(I_{2n-1}; z)$ and hence $\frac{1 - (z^2)^{2n-1}}{1 - z^2}$ also divides $P^-(I_{2n-1}; z)$. Let

$$\psi : \mathcal{D} := \mathbb{P}Y^{-1}(0)/SO(2) \to \mathcal{D}_0^{[n]} = (J^{[n]} \times \hat{J}) \times (J^{[n]} \times \hat{J}) - \Delta$$

be the Zariski locally trivial $I_{2n-1}$-bundle in the proof of Proposition 3.6 (2). Recall that $D^0_2 = \mathcal{D}/\mathbb{Z}_2$. We have seen in the proof of Proposition 3.6 (2) that there is a $\mathbb{Z}_2$-equivariant embedding

$$\nu : \mathcal{D} \hookrightarrow \mathbb{P}Ext_{\mathcal{D}_0^{[n]}}^1(\mathcal{L}_1, \mathcal{L}_2) \times_{\mathcal{D}_0^{[n]}} \mathbb{P}Ext_{\mathcal{D}_0^{[n]}}^1(\mathcal{L}_2, \mathcal{L}_1)$$

where the $\mathbb{Z}_2$-action interchanges $\mathcal{L}_1$ and $\mathcal{L}_2$.
Let $\lambda$ (resp. $\eta$) be the pull-back to $\mathcal{D}$ of the first Chern class of the tautological line bundle over $\mathbb{P} \mathcal{E}xt^1_{\mathcal{D}_0}(\mathcal{L}_1, \mathcal{L}_2)$ (resp. $\mathbb{P} \mathcal{E}xt^1_{\mathcal{D}_0}(\mathcal{L}_2, \mathcal{L}_1)$). By definition, $\lambda$ and $\eta$ restrict to $a$ and $b$ respectively. The $\mathbb{Z}_2$-action interchanges $\lambda$ and $\eta$. By the Leray-Hirsch theorem ([26] p.195), we have an isomorphism

\[(4.4) \quad H^*_c(\mathcal{D}) \cong H^*_c(\mathcal{D}_0^{[n]}) \otimes H^*(I_{2n-1}).\]

As the pull-back and the cup product preserve mixed Hodge structure, (4.4) determines the mixed Hodge structure of $H^*_c(\mathcal{D})$. The $\mathbb{Z}_2$-invariant part is

\[H^*_c(\mathcal{D})^+ \cong \left( H^*_c(\mathcal{D}_0^{[n]})^+ \otimes H^*(I_{2n-1})^+ \right) \oplus \left( H^*_c(\mathcal{D}_0^{[n]})^- \otimes H^*(I_{2n-1})^- \right)\]

where the superscript $\pm$ denotes the $\pm 1$-eigenspace of the $\mathbb{Z}_2$-action. Because $H^*_c(\mathcal{D}_0^{[n]}) \cong H^*_c(\mathcal{D}/\mathbb{Z}_2) \cong H^*_c(\mathcal{D})^+$ ([11] Theorem 5.3.1 and Proposition 5.2.3), $E(D^0_2; u, v)$ is equal to

\[E^+(D; u, v) = E^+(\mathcal{D}_0^{[n]}; u, v)E^+(I_{2n-1}; u, v) + E^-(\mathcal{D}_0^{[0]}; u, v)E^-(I_{2n-1}; u, v),\]

where $E^+(Y; u, v) = \sum_{p,q} \sum_{k \geq 0} (1)^k h^{p,q}(H^k_c(Y))u^p v^q$. Since the smooth projective variety $I_{2n-1}$ has pure Hodge structure,

\[E^+(I_{2n-1}; z, z) = P^+(I_{2n-1}; z) \quad \text{and} \quad E^-(I_{2n-1}; z, z) = P^-(I_{2n-1}; z).\]

As $\frac{1 - (z^2)^{2n-1}}{1 - z^2}$ divides $P^\pm(I_{2n-1}; z)$, it divides $E(D^0_2; u, v)$ as well. \hfill \Box

**Proof of Proposition 4.2**

Let us prove that (4.1) cannot be a polynomial. Let

\[S(z) = E_{\text{at}}(M; z, z) - E(M^0; z, z) - \frac{1 - z^2}{1 - (z^2)^{2n-1}}E(D^0_2; z, z).\]

It suffices to show that $S(z)$ is not a polynomial for all $n \geq 2$ because $E(M^0; z, z)$ and $\frac{1 - z^2}{1 - (z^2)^{2n-1}}E(D^0_2; z, z)$ are polynomials by Lemma 4.3.

Express the rational function $S(z)$ as

\[\frac{N(z)}{(1 - (z^2)^{2n-1})(1 - (z^2)^{4n-1})(1 - (z^2)^{6n})}.\]

By direct computation using (4.1) and Corollary 4.2 $N(z)$ modulo $1 - (z^2)^{2n-1}$ is congruent to

\[(4.5) \quad (1 - z^2)^2(1 - (z^2)^{4n-1}) \times \left( \frac{1 - (z^2)^3}{1 - z^2} \right)^2 \times \prod_{1 \leq i \leq 3} \left( \frac{1 - (z^2)^{2n-4+2i}}{1 - (z^2)^2} \right) \times P(\mathfrak{J}; z)\]

\[-(1 - z^2)^2(1 - (z^2)^{4n-1}) \times \frac{1 - (z^2)^2}{1 - z^2} \times \prod_{1 \leq i \leq 2} \left( \frac{1 - (z^2)^{2n-2+2i}}{1 - (z^2)^2} \right) \times P(\mathfrak{J}; z)\]

\[-(1 - z^2)^2(1 - (z^2)^{6n}) \times \frac{1 - (z^2)^2}{1 - z^2} \times \prod_{1 \leq i \leq 2} \left( \frac{1 - (z^2)^{2n-2+2i}}{1 - (z^2)^2} \right) \times P(\mathfrak{J}; z)\]

\[+(1 - z^2)^3 \times \frac{1 - (z^2)^2}{1 - z^2} \times \prod_{1 \leq i \leq 2} \left( \frac{1 - (z^2)^{2n-2+2i}}{1 - (z^2)^2} \right) \times P(\mathfrak{J}; z)\]

where $\mathfrak{J} := J^{[n]} \times \mathfrak{J}$. 
All we need to show is that the numerator $N(z)$ is not divisible by the denominator $(1 - (z^2)^{2n-1}) (1 - (z^2)^{4n-1})(1 - (z^2)^{6n})$. We write Eq. (4.5) as a product $s(t) \cdot P(\bar{z}; z)$ for some polynomial $s(t)$ with $t = z^2$. For the proof of Proposition 4.3 for $n \geq 3$ (the $n = 2$ case will be treated separately), it suffices to prove the following:

1. If $n + 1$ is not divisible by 3, then $1 - z^2$ is the GCD of $1 - (z^2)^{2n-1}$ and $s(z^2)$, and $\frac{1 - (z^2)^{2n-1}}{1 - z^2}$ does not divide $P(\bar{z}; z)$;
2. If $n + 1$ is divisible by 3, then $1 - (z^2)^3$ is the GCD of $1 - (z^2)^{2n-1}$ and $s(z^2)$, and $\frac{1 - (z^2)^{2n-1}}{1 - (z^2)^3}$ does not divide $P(\bar{z}; z)$.

For (1), suppose $n + 1$ is not divisible by 3. From Eq. (4.5), $s(t)$ is divisible by $1 - t$. We claim that $s(t)$ is not divisible by any irreducible factor of $\frac{1 - (z^2)^{2n-1}}{1 - z^2}$, i.e. for any root $\alpha$ of $1 - t^{2n-1}$ which is not 1, $s(\alpha) \neq 0$. Using the relation $\alpha^{2n-1} = 1$, we compute directly that

$$s(\alpha) = \frac{\alpha(1 - \alpha^{-1})(1 - \alpha^3)^2}{1 + \alpha},$$

which is not 0 because 3 does not divide $2n - 1$.

Next we check that $\frac{1 - (z^2)^{2n-1}}{1 - z^2}$ does not divide $P(\bar{z}; z)$. Note that

$$P(\bar{z}; z) = P(J[n]; z)P(\bar{J}; z) = (1 - z)^4 P(J[n]; z)$$

and hence it suffices to show that $\frac{1 - (z^2)^{2n-1}}{1 - z^2}$ does not divide $P(J[n]; z)$. We put $P(J[n]; z) = \sum_{0 \leq i \leq 4n} a_{i} z^{i}$ and write

$$\sum_{0 \leq i \leq 4n} a_{i} z^{i} = a_{0} + a_{1} z + a_{2} z^{2} + ... + a_{4n-4} z^{4n-4}$$

$$+ a_{4n-3} z \left( \frac{1 - (z^2)^{2n-1}}{1 - z^2} - \sum_{i=0}^{2n-3} (z^2)^i \right) + a_{4n-2} (z^{4n-2} - 1) + a_{4n-2}$$

$$+ a_{4n-1} z (z^{4n-2} - 1) + a_{4n-1} z + a_{4n} z^2 (z^{4n-2} - 1) + a_{4n} z^2.$$

We see from this that $P(J[n]; z)$ is divisible by $\frac{1 - (z^2)^{2n-1}}{1 - z^2}$ if and only if

$$a_{0} + a_{1} z + a_{2} z^{2} + ... + a_{4n-4} z^{4n-4}$$

$$+ a_{4n-3} z \left( - \sum_{i=0}^{2n-3} (z^2)^i \right) + a_{4n-2} + a_{4n-1} z + a_{4n} z^2$$

is divisible by $\frac{1 - (z^2)^{2n-1}}{1 - z^2}$. Since Eq. (4.8) is of degree $\leq 4n - 4$, it is divisible by $\frac{1 - (z^2)^{2n-1}}{1 - z^2}$ if and only if (4.8) is a constant multiple of $\frac{1 - (z^2)^{2n-1}}{1 - z^2}$. If this were true then the coefficient of $z$ must be zero, i.e. $a_{1} - a_{4n-3} + a_{4n-1} = 0$.

By the Poincaré duality $a_{1} - a_{4n-3} + a_{4n-1} = a_{1} - a_{3} + a_{1}$. This value is not 0 because $a_{1} = -b_{1}(J[n]) = -4$ and $a_{3} = -b_{3}(J[n]) = -40$ for $n \geq 3$ by Göttzsche’s formula [9]:

$$\sum_{n \geq 0} P(J[n]; z) t^n = \prod_{k \geq 1} \prod_{i=0}^{4} (1 - z^{2k-2+i} t^k (-1)^{i+1} b_i(J)).$$
For (2), suppose 3 divides $n + 1$ and $n \neq 2$. Then from (1.6), $(1 - t^3)$ divides $s(t)$. More precisely, for a third root of unity $\alpha$, $s(\alpha) = 0$. On the other hand, if $\alpha$ is a root of $1 - t^{2n-1}$ but not a third root of unity then we can observe that $s(\alpha) \neq 0$ by (1.6). Since every root of $1 - t^{2n-1}$ is a simple root, any irreducible factor of $\frac{1 - t^{2n-1}}{1 - t}$ does not divide $s(t)$.

We next check that the polynomial $\frac{1 - (z^2)^{2n-1}}{1 - (z^2)^3}$ does not divide $P(J; z)$. Again by (4.7), it suffices to show that $\frac{1 - (z^2)^{2n-1}}{1 - (z^2)^3}$ does not divide $P(J[n]; z)$. Let $P(J[n]; z) = \sum_{0 \leq i \leq 4n} a_i z^i$ and write $z^{4n-8}$ as $D(z) = \frac{1 - (z^2)^{2n-1}}{1 - (z^2)^3} - \sum_{i=0}^{2n-7} (z^2)^{3i}$. Then we have

$$\sum_{0 \leq i \leq 4n} a_i z^i = a_0 + a_1 z + \ldots + a_{4n-8} z^{4n-8}$$

$$+ a_{4n-7} z D(z) + a_{4n-6} z^2 D(z) + \ldots + a_{4n-3} z^5 D(z)$$

$$+ a_{4n-2}(z^{4n-2} - 1) + a_{4n-1} z (z^{4n-2} - 1) + a_{4n-1} z$$

$$+ a_{4n} z^2 (z^{4n-2} - 1) + a_{4n-2} z^2.$$ 

Therefore, $P(J[n]; z)$ is divisible by $\frac{1 - (z^2)^{2n-1}}{1 - (z^2)^3}$ only if

$$a_0 + a_1 z + \ldots + a_{4n-8} z^{4n-8}$$

$$+ (a_{4n-7} + a_{4n-6} z + \ldots + a_{4n-3} z^5)(- \sum_{i=0}^{2n-7} (z^2)^{3i})$$

$$+ a_{4n-2} + a_{4n-1} z + a_{4n} z^2$$

is divisible by $\frac{1 - (z^2)^{2n-1}}{1 - (z^2)^3}$. Since (4.10) is of degree $\leq 4n - 8$, it is divisible by $\frac{1 - (z^2)^{2n-1}}{1 - (z^2)^3}$ if and only if it is a constant multiple of $\frac{1 - (z^2)^{2n-1}}{1 - (z^2)^3}$. If this were true the coefficient of $z$ must be zero, i.e. $a_1 - a_{4n-7} + a_{4n-1} = 0$. By the Poincaré duality $a_1 - a_{4n-7} + a_{4n-1} = a_1 - a_7 + a_1$. This value is not zero because $a_1 = -4$ and $a_7 = -b_7(J[n]) \leq -196$ for $n \geq 3$ by direct computation using Göttsche’s formula again.

The case of $n = 2$ remains to be proved. We show that $N(z)$ is not divisible by $1 - (z^2)^6 n = 1 - (z^2)^{12}$. By direct computation using (4.11) and Corollary 4.2, we have

$$N(z) = (1 - z^2)^2 (1 + z^2)^2 ((z^2)^2 - z^2 + 1)(1 + z^2 + (z^2)^2)$$

$$\times ((z^2)^{12} + 3(z^2)^{11} + 3(z^2)^{10} + 2(z^2)^9 + 2(z^2)^8 + (z^2)^7 + 3(z^2)^6$$

$$+ (z^2)^5 + (z^2)^3 + (z^2)^2 + 1) \times P(J; z).$$

By Göttsche’s formula, we also have

$$P(J; z) = (1 - 4z + 13z^2 - 32z^3 + 44z^4 - 32z^5 + 13z^6 - 4z^7 + z^8)(1 - z)^4.$$ 

By plugging in a primitive root of $z^{24} = 1$, it is easy to check that $N(z)$ is not divisible by $1 - (z^2)^{12}$.

Therefore, $E_{st}(M; z, z)$ is not a polynomial for any $n \geq 2$. □
Remark 4.4. The sum of second Chern class together with the determinant map give us a morphism

\[ a : M \to J \times \Pic^0(J). \]

Let \( M = a^{-1}(0, 0) \). Like \( M \), \( M \) is a singular projective variety equipped with a holomorphic symplectic form on the smooth part. One may ask if there is a crepant resolution of \( M \). It is easy to modify our proof to show that there is no crepant resolution of \( M \) (and therefore no symplectic desingularization) either. We leave the details to the reader.

References

1. V. Batyrev. Stringy Hodge numbers of varieties with Gorenstein canonical singularities. Integrable systems and algebraic geometry (Kobe/Kyoto, 1997) (1998), 1–32.
2. A. Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. J. Diff. Geom. 18 (1983), 755–782.
3. F. A. Bogomolov. The decomposition of Kähler manifolds with a trivial canonical class. Mat. Sb. 93 (1974), 573–575, 630.
4. J. Choy and Y.-H. Kiem. Nonexistence of a crepant resolution of some moduli spaces of sheaves on a K3 surface. math.AG/0407100
5. V. Danilov and G. Khovanski. Newton polyhedra and an algorithm for computing Hodge-Deligne numbers. Math. USSR Izvestiya 29 no. 2 (1987), 279–298.
6. J. Denef and F. Loeser. Germs of arcs on singular varieties and motivic integration. Invent. Math. 135 no. 1 (1999), 201–232.
7. D. Eisenbud. Commutative algebra with a view toward algebraic geometry. Springer-Verlag, (1995)
8. D. Gieseker. On the moduli of vector bundles on an algebraic surface. Ann. of Math. 106 (1977), 45–60.
9. L. Göttsche. The Betti numbers of the Hilbert scheme of points on a smooth projective surface. Math. Ann. 286 (1990), 193–207.
10. P. Griffiths and J. Harris. Principles of algebraic geometry. A Wiley-Interscience Publication, John Wiley & Sons (1978).
11. A. Grothendieck. Sur quelques points d’algèbre homologique. Tôhoku Math. 9 (1957), 119–221.
12. R. Hartshorne. Algebraic Geometry. Graduate Texts in Mathematics 52. Springer-Verlag, 1977
13. D. Huybrechts and M. Lehn. The Geometry of moduli spaces of sheaves. A Publication of the Max-Planck-Institut für Mathematik, Bonn (1997)
14. D. Kaledin and M. Lehn. Local structure of hyperkähler singularities in O’Grady’s examples. math.AG/0405575
15. D. Kaledin, M. Lehn and C. Sorger. Singular symplectic moduli spaces. math.AG/0504202
16. Y.-H. Kiem. On the existence of a symplectic desingularization of some moduli spaces of sheaves on a K3 surface. To appear in Compositio Mathematica. math.AG/0404453
17. Y.-H. Kiem and J. Li. Desingularizations of the moduli space of rank 2 bundles over a curve. Math. Ann. 330 (2004), 491–518.
18. F. Kirwan. Partial desingularisations of quotients of nonsingular varieties and their Betti numbers. Ann. of Math. 122 (1985), 41–85.
19. I. G. Macdonald. The Poincare polynomial of a symmetric product. Proc. Cambridge Philos. Soc. 58 (1962), 563–568.
20. S. Mukai. Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. Invent. Math. 77 (1984), 101–116.
21. K.G. O’Grady. *Desingularized moduli sheaves on a K3*. [math.AG/9708009]
22. K.G. O’Grady. *Desingularized moduli sheaves on a K3, II*. [math.AG/9805099]
23. K.G. O’Grady. *Desingularized moduli spaces of sheaves on a K3*. J. Reine Angew. Math. 512 (1999), 49–117.
24. K.G. O’Grady. *A new six dimensional irreducible symplectic variety*. J. Alg. Geom. 12 (2003), 435–505.
25. C. Vafa and E. Witten. *A strong coupling test of S-duality*. Nuclear Phys. B 431 no. 1-2 (1994), 3–77.
26. C. Voisin. *Hodge theory and complex algebraic geometry, I*. Cambridge studies in advanced mathematics 76 Cambridge University Press (2002).
27. K. Yoshioka. *Twisted stability and Fourier-Mukai transform*. Compositio Math. 138, no. 3 (2003), 261–288; [math.AG/0106118]

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