Irregular isomonodromic deformations for Garnier systems and Okamoto’s canonical transformations

Marta Mazzocco

Abstract

In this paper we describe the Garnier systems as isomonodromic deformation equations of a linear system with a simple pole at 0 and a Poincaré rank two singularity at infinity. We discuss the extension of Okamoto’s birational canonical transformations to the Garnier systems in more than one variable and to the Schlesinger systems.

1 Introduction

The n-variables Garnier systems $G_n$ were introduced in [6,7] as isomonodromic deformation equations of scalar differential equations of the form

$$\frac{d^2 y}{dx^2} + p_1(x) \frac{dy}{dx} + p_2(x) y = 0$$ (1)

with n apparent singularities $\lambda_1, \ldots, \lambda_n$ of indices (0,2) and $n + 3$ pairwise distinct Fuchsian singularities $t_1, \ldots, t_{n+3}$ (in particular $t_{n+1} = 0$, $t_{n+2} = 1$, $t_{n+3} = \infty$) with indices $(0, \theta_1), \ldots, (0, \theta_{n+2})$, \(\left(-\frac{\sum_{k=1}^{n+2} \theta_k + \theta_\infty}{2}, -\frac{\sum_{k=1}^{n+2} \theta_k - \theta_\infty}{2} + 1\right)\) respectively. The constraints on the indices are enough to determine $p_1(x)$, $p_2(x)$ in terms of $\lambda_1, \ldots, \lambda_n$ and $n$ other quantities $\mu_1, \ldots, \mu_n$ (see [11])

$$p_1(x) = \sum_{k=1}^{n+2} \frac{1-\theta_k}{x-t_k} - \sum_{i=1}^{n} \frac{1}{x(t_i-1)K_i},$$

$$p_2(x) = \frac{\kappa}{x(x-1)} - \sum_{i=1}^{n} \frac{\Lambda(t_i) K_i}{x(x-1)(x-t_i)} + \sum_{i=1}^{n} \frac{\lambda_i(\lambda_i-1) \mu_i}{x(x-1)(x-\lambda_i)},$$ (2)

where $\kappa = \frac{1}{4} \left\{ (\sum_{k=1}^{n+2} \theta_k - 1)^2 - (\theta_\infty + 1)^2 \right\}$ and

$$K_i = -\frac{\Lambda(t_i)}{T'(t_i)} \left[ \sum_{k=1}^{n} \frac{T(\lambda_k)}{(\lambda_k-t_i)N'(\lambda_k)} \left( \mu_k^2 - \sum_{m=1}^{n+2} \frac{\theta_m - \delta_{im}}{\lambda_k-t_m} \mu_k + \frac{\kappa}{\lambda_k(\lambda_k-1)} \right) \right]$$ (3)

*DPMMS, Cambridge University, Cambridge CB3 0WB, UK.
with $\Lambda(u) := \Pi_{k=1}^{n}(u - \lambda_k)$ and $T(u) := \Pi_{k=1}^{n+2}(u - t_k)$.

The isomonodromic deformations equations of the above equation are described by the following completely integrable Hamiltonian system \([6, 7, 17, 11]\), the Garnier systems $\mathcal{G}_n$:

$$\begin{cases}
\frac{\partial \lambda_i}{\partial t_j} = \frac{\partial K_{ij}}{\partial \tilde{\mu}_j}, & i, j = 1, \ldots, n, \\
\frac{\partial \tilde{\mu}_j}{\partial t_i} = -\frac{\partial K_{ij}}{\partial \lambda_i}, & i, j = 1, \ldots, n.
\end{cases}$$ (4)

In this paper we show that the Garnier systems $\mathcal{G}_n$ can be interpreted as isomonodromic deformations equations of the following $(n+2) \times (n+2)$ linear system with a simple pole at 0 and a Poincaré rank two singularity at $\infty$

$$\frac{d}{dz} Y = \left( \begin{pmatrix} t_1 & \ldots & t_{n+2} \end{pmatrix} + V - \frac{1}{z} \right) Y,$$

where $V$ is a $(n+2) \times (n+2)$ diagonalizable matrix with $n$ null eigenvalues

$$G^{-1}VG = \rho := \begin{pmatrix} \rho_1 & 0 & \ldots & 0 \\ 0 & \rho_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$ (6)

with $\rho_1 = \frac{1}{2}( - \sum \theta_k + \vartheta_\infty )$, $\rho_2 = \frac{1}{2}( - \sum \theta_k - \vartheta_\infty )$. We show that if $\lambda_1, \ldots, \lambda_n$, $\mu_1, \ldots, \mu_n$ are solutions of the Garnier system (4), then they uniquely determine the matrix $V(t_1, \ldots, t_n)$ up to diagonal conjugation in such a way that the monodromy data of the system

$$\frac{d}{dz} Y = \left( \begin{pmatrix} t_1 & \ldots & t_{n+2} \end{pmatrix} + V(t_1, \ldots, t_n) - \frac{1}{z} \right) Y,$$

are constant. In particular $V$ is given by the following formulae

$$V_{ii} = -\theta_i, \quad i = 1, \ldots, n+2,$$ (7)

$$V_{ij} = -\theta_j + M_j W_j - M_j W_i, \quad i, j = 1, \ldots, n+2,$$ (8)

where $M_i, W_i$ are given in terms of the variables $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ (here we use the same notations of [11]) by:

$$M_i := -\frac{\Lambda(t_i)}{T(t_i)}, \quad i = 1, \ldots, n+2,$$

$$W_j := \sum_{k=1}^{n} \frac{T(\lambda_k)}{(\lambda_h - t_j) M(\lambda_h)} \left( \mu_k - \frac{\sum_{k=1}^{n+2} \theta_k + \theta_\infty}{2\lambda_k} \right), \quad j = 1, \ldots, n,$$

$$W_{n+1} = \frac{1}{M_{n+1}} \left( \sum_{j=1}^{n} (t_j - 1) M_j W_j + \frac{\sum_{k=1}^{n+2} \theta_k + \theta_\infty}{2} \right),$$

$$W_{n+2} = -\frac{1}{M_{n+2}} \sum_{j=1}^{n} t_j M_j W_j.$$ (9)
In the case of \( n = 1 \), corresponding to the Painlevé sixth equation, such result was already known [8, 16]. Nevertheless the generalization to the Garnier systems \( \mathcal{G}_n \) with \( n > 1 \) variables is non-trivial.

Our result seems to be of fundamental importance if one wants to investigate the possibility to extend Okamoto’s birational canonical transformations defined for the Painlevé sixth equation to the case of Garnier systems. In [18] it is shown that the group of all birational canonical transformations of the Painlevé sixth equation is isomorphic to the affine extension of \( W(F_4) \). The generators are given by the generators \( w_1, w_2, w_3, w_4 \) of \( W(D_4) \), the parallel transformations \( l_1, l_2, l_3 \), and the symmetries \( x_1, x_2, x_3 \) (we shall remind details of these in Section 3). The symmetries can be generalized to the case of Garnier systems [11], and indeed to the Schlesinger systems in any dimension [5]. Recently Tsuda [21] defined the analogues of \( w_1, w_3, w_4, l_1, l_2, l_3 \) for the Garnier system and together with Sakai conjectured that the analogue of \( w_2 \) does not exist [19]. Here we give further strong evidence that Sakai-Tsuda conjecture is valid. In fact, in the case of the Painlevé sixth equation, \( n = 1 \), we generate a birational canonical transformation \( \tilde{w} \) equivalent to \( w_2 \), \( \tilde{w} = x_2 w_2 x_2 \), as a simple scalar gauge transformation of the irregular system \( (5) \)

\[
\tilde{Y}(z) = z^{-\rho} Y(z), \quad V \rightarrow V - \rho \mathbb{1},
\]

where \( \rho \) is one of the eigenvalues of \( V \). In this way the condition that one of the eigenvalues of \( V \) is zero is preserved. The idea is that for \( n > 1 \), \( V \) has \( n > 1 \) null eigenvalues, and this condition is not preserved by the above gauge transformation. Conversely it is possible to extend \( \tilde{w} \) to a special case of the Schlesinger systems.

A second application of our result is in the theory of Frobenius manifolds [2]. A \( \tilde{n} \)-dimensional Frobenius manifolds is locally parameterized by the monodromy data of a \( \tilde{n} \times \tilde{n} \) linear system similar to \( (5) \):

\[
\frac{d}{dz} \tilde{Y} = \left( \begin{array}{ccc}
  t_1 & \cdots & \tilde{t}_1 \\
  \vdots & \ddots & \vdots \\
  \tilde{t}_{\tilde{n}} & \cdots & t_{\tilde{n}}
\end{array} \right) + \frac{W}{z} \tilde{Y},
\]

where \( W \) is a \( \tilde{n} \times \tilde{n} \) diagonalizable antisymmetric matrix. The eigenvalues \( \tilde{\rho}_1, \ldots, \tilde{\rho}_{\tilde{n}} \) of \( W \) are related to the degrees \( d_1, \ldots, d_{\tilde{n}} \) of the Frobenius manifold by \( d_i = 1 - \tilde{\rho}_i - \frac{d}{2} \). In the special case

\[
d_1 = 1, \quad d_{\tilde{n}} = 1 - d, \\
d_i = 1 - \frac{d}{2}, \quad i \neq 1, \tilde{n},
\]

the system \((10)\) is equivalent to the system \((5)\) with \( n = \tilde{n} - 2 \), \( \theta_1 = \theta_2 = \cdots = \theta_{n+2} = 0 \) and \( \theta_\infty = d \).\(^1\) In other words we show that the Garnier

\(^1\)Such equivalence is realized by \( \tilde{Y} = z Y \). The fact that \( V \) becomes antisymmetric follows immediately from the formula \((5)\) after a suitable diagonal conjugation.
system (4) describes the analytic structure of Frobenius manifolds of degrees 

\[ 1, 1 - d, 1 - \frac{d}{2}, \ldots, 1 - \frac{d}{2} \]

for any \( d \).

This paper is organized as follows: in Section 2 we show that the Garnier systems \( \mathcal{G}_n \) can be interpreted as isomonodromic deformations equations of the \((n+2) \times (n+2)\) linear system (5). In Section 3 we give a brief resume of Okamoto’s results on birational canonical transformations for the Painlevé sixth equation. In Section 4 we discuss the isomonodromic meaning of Okamoto’s birational canonical transformation \( w_2 \) and its extension to the Schlesinger systems.

Acknowledgments The author is grateful to H. Sakai and T. Tsuda for helpful discussions and to Prof. Okamoto for kindly inviting her to Tokyo graduate school of Mathematics where this paper was started.

2 Isomonodromic interpretation of the Garnier systems

In this section we prove that the Garnier systems are isomonodromic deformations equations of the system (5). These are described in the following theorem proved in [10, 3]:

**Theorem 1.** Let \( V(t), t = (t_1, \ldots, t_n) \), be a diagonalizable matrix function with eigenvalues \( \rho_1, \rho_2, 0, \ldots, 0 \). Suppose that for every \( i = 1, \ldots, n \) the matrix function \( V(t) \) satisfies

\[
\frac{\partial}{\partial t_i} V = [V_i, V], \tag{11}
\]

where

\[
V_i = \text{ad}_{E_i} \text{ad}_T^{-1}(V), \quad T = \begin{pmatrix} t_1 & \cdots & \cdots & t_{n+2} \end{pmatrix},
\]

and \( E_{ikl} = \delta_{ik} \delta_{il} \). Then the monodromy data\(^2\) of the system (5)

\[
\frac{d}{dz} Y = \left( T + \frac{V(t) - \mathbb{1}}{z} \right) Y,
\]

are constant.

Our main result is the following:

**Theorem 2.** Consider the \((n+2) \times (n+2)\) diagonalizable matrix \( V(t), t = (t_1, \ldots, t_n) \), satisfying the conditions (6) and (7). Then its off diagonal entries can be expressed in terms of the variables \( \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n \) by (8), where \( M_i \) and \( W_i \) are given in (9). Moreover \( V(t) \) satisfies (11) if and only if \( \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n \) satisfy the Garnier system \( \mathcal{G}_n \) given in (4).

\(^2\)See appendix for a description of the monodromy data of the system (5).
Proof. The idea of the proof is to show that the systems of the form (11) satisfying the conditions (6) and (7), are equivalent to 2 × 2 Fuchsian systems of the form

\[ \frac{d}{dx} \Phi = \left( \sum_{k=1}^{n+2} \frac{A_k}{x-t_k} \right) \Phi, \]

(12)

where the residue matrices \( A_k \) satisfy the following conditions:

\[
\text{eigen } (A_k) = (0, \theta_k) \quad \text{and} \quad -\sum_{k=1}^{n+2} A_k = A_\infty := \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix},
\]

(13)

with \( \rho_1 = \frac{1}{2}(-\sum \theta_k + \vartheta_\infty), \rho_2 = \frac{1}{2}(-\sum \theta_k - \vartheta_\infty). \) We then shall use the fact the Garnier systems are isomonodromic deformation equations for such 2 × 2 Fuchsian systems.

More precisely, we introduce the following

**Definition 3.** Two \((n+2)\times(n+2)\) systems of the form (11) are equivalent up to diagonal conjugation if they have the same matrix \(T\) and the same matrix \(V\) up to \(V \rightarrow \rho = \text{diagonal}(\rho_1, \rho_2, 0, \ldots, 0)\), \( \rho \) being any \((n+2)\times(n+2)\) diagonal matrix. Analogously, two 2 × 2 Fuchsian systems of the form (12) are equivalent up to diagonal conjugation if they have the same matrices \(A_k, k = 1, 2, 3\), up to \(A_k \rightarrow \rho^{-1}A_k\rho\), \( \rho \) being any diagonal matrix.

**Lemma 4.** There is a one to one correspondence between classes of equivalence of \((n+2)\times(n+2)\) systems of the form (11) and classes of equivalence of 2 × 2 systems of the form (12) where \(A_k\) satisfy (13).

**Proof.** Consider the following \((n+2)\times(n+2)\) Fuchsian system

\[ \frac{d}{dz} Y = \left( T + \frac{V - \mathbb{I}}{z} \right) Y, \]

where \( V \) is a \((n+2)\times(n+2)\) diagonalizable matrix satisfying the conditions (12) and (13) and classes of equivalence of 2 × 2 systems of the form (12), where \( A_k \) satisfy (13).
where $\tilde{B}_k = -G^{-1}E_k G \rho$, i.e. all matrices $\tilde{B}_k$ have all last $n$ columns equal to zero. The system (15) therefore reduces to a $2 \times 2$ Fuchsian system of the form (12) for the first two rows of $\tilde{X}$, $\Phi = \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}$, with $2 \times 2$ residue matrices

$$A_k = \begin{pmatrix} (\tilde{B}_k)_{11} & (\tilde{B}_k)_{12} \\ (\tilde{B}_k)_{21} & (\tilde{B}_k)_{22} \end{pmatrix}. \quad (16)$$

By construction, the residue matrices $A_k$ satisfy (13), where $\rho_1 = \frac{1}{2}(-\sum \theta_k + \vartheta_{\infty})$, $\rho_2 = \frac{1}{2}(-\sum \theta_k - \vartheta_{\infty})$ are the two non-null eigenvalues of $V$.

It is very easy to verify that the off–diagonal elements of $V$ satisfy the following relation

$$V_{kl}V_{lk} = \text{Tr}(B_k B_l) = \text{Tr}(\tilde{B}_k \tilde{B}_l) = \text{Tr}(A_k A_l),$$

therefore given $A_1, \ldots, A_{n+2}$, we determine $V$ up to diagonal conjugation. Vice versa given $V$, it uniquely determines $B_k = -E_k V$ and $G$ up to $G \to G S$, where $S$ is a $(n+2) \times (n+2)$ invertible matrix such that $S_{12} = S_{21} = 0$ and $(S^{-1})_{11} = \frac{1}{S_{11}}$, $(S^{-1})_{22} = \frac{1}{S_{22}}$, $(S^{-1})_{12} = 0$, $(S^{-1})_{21} = 0$. This determines $\tilde{B}_k = G^{-1}B_k G$ up to $\tilde{B}_k \to S^{-1}\tilde{B}_k S$, therefore $V$ uniquely determines $A_k$ up to diagonal conjugation. \(\triangle\)

The isomonodromic deformations equations of the Fuchsian system (12) are described in the following theorem proved in [20].

**Theorem 5.** Let $A_1(t), \ldots, A_{n+2}(t)$, $t = (t_1, \ldots, t_n)$, be diagonalizable matrices satisfying (13). Suppose that for every $i = 1, \ldots, n$ the matrix functions $A_1(t), \ldots, A_{n+2}(t)$ satisfy

$$\begin{align*}
\frac{\partial}{\partial t_i} A_i &= \frac{[A_i, A_j]}{t_i - t_j}, & i \neq j, \\
\frac{\partial}{\partial t_i} A_i &= -\sum_{j \neq i} \frac{[A_i, A_j]}{t_i - t_j}.
\end{align*} \quad (17)$$

Then the monodromy data of the system

$$\frac{d}{dx} \Phi = \sum_{k=1}^{n+2} \frac{A_k}{x - t_k} \Phi$$

are constant.

The relation between Schlesinger equations and Garnier systems is given in the following

**Lemma 6.** [11] Let $\lambda_1, \ldots, \lambda_n$ be the roots of the equation

$$\sum_{k=1}^{n+2} \frac{A_{k,12}}{x - t_k} = 0$$
and define $\mu_1, \ldots, \mu_n$ as

$$
\mu_i := \sum_{k=1}^{n+2} \frac{A_{k11}}{\lambda_i - t_k}, \quad i = 1, \ldots, n.
$$

Then $A_1, \ldots, A_{n+2}$ are uniquely determined up to diagonal conjugation by $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$

$$
\begin{align*}
A_{k11} &= M_k(W_k - W) \\
A_{k12} &= -M_i \\
A_{k21} &= -(W_k - W)[M_k(W - W_k) + \theta_k] \\
A_{k22} &= \theta_k - M_k(W_k - W),
\end{align*}
$$

where $M_k$ and $W_k$ are given in terms of the variables $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ by (19) and $\theta_\infty W = \sum_{j=1}^{n+2} W_j(M_j W_j - \theta_j)$. Moreover $A_1, \ldots, A_{n+2}$ satisfy the Schlesinger equations if and only if $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ satisfy the Garnier system $G_n$ given in (4).

In particular, using the formulae (18) for the matrices $A_1, \ldots, A_{n+2}$ and we obtain that the off-diagonal elements of $V$ are given by the formulae (8) up to diagonal conjugation.

In order to complete the proof of Theorem 2 it is enough to prove the following

Lemma 7. The matrix $V(t)$, $t = (t_1, \ldots, t_n)$ satisfies (17) if and only if the matrices $A_1, \ldots, A_{n+2}$ defined by (18) satisfy the Schlesinger equations.

Proof. The matrix $V(t)$, $t = (t_1, \ldots, t_n)$ satisfies (11) if and only if the diagonalizing matrix $G$ such that $G^{-1}V_i G = \rho$, satisfies

$$
\left[ G^{-1} \frac{\partial G}{\partial t_i} \rho \right] = \left[ G^{-1} V_i G, \rho \right], \quad \forall i.
$$

This equation is valid if and only if $G^{-1} \frac{\partial G}{\partial t_i} = G^{-1} V_i G + T_i$, where $T_i$ is a $(n + 2) \times (n + 2)$ matrix such that $T_{i1} = T_{i21} = 0$. Let us define $A_k := \Pi B_k$, where $\Pi = \text{diagonal}(1, 1, 0, \ldots, 0)$. It is clear that $A_1, \ldots, A_{n+2}$ satisfy the Schlesinger equations if and only if $A_1, \ldots, A_{n+2}$ do. This happens if and only if

$$
\Pi \left[ G^{-1} \frac{\partial G}{\partial t_i} G^{-1} E_k G \right] \rho = \Pi \left[ G^{-1} V_i G, G^{-1} E_k G \right] \rho, \quad \forall k \neq i,
$$

and

$$
\Pi \left[ G^{-1} \frac{\partial G}{\partial t_i} G^{-1} E_i G \right] \rho = -\sum_{k \neq i} \Pi \left[ G^{-1} V_i G, G^{-1} E_k G \right] \rho.
$$

These equations are valid if and only if $G^{-1} \frac{\partial G}{\partial t_i} = G^{-1} V_i G + R_i$, where $R_i$ is a $(n + 2) \times (n + 2)$ matrix such that $\Pi[R_i, G^{-1} E_k G] \rho = 0$ for every $k \neq i$. Since
G is determined by $V$ up to $G \rightarrow GS$, where $S$ is a $(n + 2) \times (n + 2)$ matrix such that $S_{12} = S_{21} = 0$, and $(S^{-1})_{11} = \frac{1}{S_{11}}$, $(S^{-1})_{22} = \frac{1}{S_{22}}$, $(S^{-1})_{12} = 0$, $(S^{-1})_{21} = 0$, we can choose $G$ in such a way that $\Pi[R_i, G^{-1}E_kG] \rho$ is zero if and only if $R_i = T_i$. This shows that (19) is valid if and only if (20) is. △

3 Birational canonical transformations for the Painlevé sixth equation

For $n = 1$, the Garnier system becomes and ODE in the variable $t_1 = t$, the Painlevé sixth equation

$$\frac{d^2 \lambda}{dt^2} = \frac{1}{2} \left( \frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t} \right) \left( \frac{d \lambda}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t} \right) \frac{d \lambda}{dt} + \frac{\lambda(\lambda - 1)(\lambda - t)}{t^2(t - 1)^2} \left[ \alpha + \beta \frac{\lambda}{\lambda^2} + \gamma \frac{t - 1}{(\lambda - 1)^2} + \delta \frac{t(t - 1)}{(\lambda - t)^2} \right],$$

where

$$\alpha = \left( \theta_\infty - 1 \right)^2, \quad \beta = -\frac{\theta_\infty^2}{2}, \quad \gamma = \frac{\theta_\infty^2}{2}, \quad \delta = \frac{1 - \theta_\infty^2}{2}. \tag{21}$$

In Okamoto’s papers the parameters

$$\kappa_0 = \theta_2, \quad \kappa_1 = \theta_3, \quad \kappa_\infty = \theta_\infty - 1, \quad \theta = \theta_1, \tag{22}$$

are used. To fix notations, and for convenience of the reader, let us remind the results of [18]. In the canonical variables $(\lambda, \mu)$, the Hamiltonian function $K$ reads:

$$K = \frac{1}{t(t - 1)} \left\{ \lambda(\lambda - 1)(\lambda - t) \mu^2 + b_3 b_4 (\lambda - t) - \left[ (b_1 + b_2)(\lambda - 1)(\lambda - t) + (b_1 - b_2) \lambda(\lambda - t) + (b_3 + b_4) \lambda(\lambda - 1) \right] \mu \right\}, \tag{23}$$

where the parameters $(b_1, b_2, b_3, b_4)$ are given by

$$b_1 = \frac{\kappa_0 + \kappa_1}{2}, \quad b_2 = \frac{\kappa_0 - \kappa_1}{2}, \quad b_3 = \frac{\theta - 1 + \kappa_\infty}{2}, \quad b_4 = \frac{\theta - 1 - \kappa_\infty}{2}. \tag{24}$$

Okamoto’s birational canonical transformations are given in terms of transformations of a certain auxiliary Hamiltonian function

$$h(t) = t(t - 1)K(t) + (b_1 b_3 + b_1 b_4 + b_3 b_4) t - \frac{b_1 b_2 + b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4}{2},$$

that satisfies the following nonlinear ordinary differential equation

$$\frac{dh}{dt} \left[ t(t - 1) \frac{d^2 h}{dt^2} \right]^2 + \left\{ \frac{dh}{dt} \left[ 2h - (2t - 1) \frac{d h}{dt} \right] + b_1 b_2 b_3 b_4 \right\}^2 = \prod_{i=1}^{4} \left( \frac{dh}{dt} + b_i^2 \right). \tag{25}$$
A singular solution $h$ to (23) is a solution linear in $t$:

$$h(t) = at + b,$$

for some constants $a$ and $b$. For the case $\theta_{1,2,3} = 0$, these singular solutions coincide with $\lambda = t_i$ and $\lambda = \infty$. Okamoto (see [18]) proves the following:

**Lemma 8.** There is a one-to-one correspondence between non singular solutions $h$ to (23) and solutions $(\mu, \lambda)$ to the Hamiltonian system with Hamiltonian function $K$ given by (23), or equivalently solutions $\lambda$ of the Painlevé VI equation with parameters $\alpha, \beta, \gamma, \delta$ given by (21).

Such a correspondence is realized by the formulae at page 354 of [18]. The canonical transformations of the general PVI are all listed by Okamoto and consist of three families:

1. **Generators** $w_1, w_2, w_3, w_4$ of $W(D_4)$, which leave the equation (23) invariant

   $$w_1(b_1, b_2, b_3, b_4) = (b_2, b_1, b_3, b_4),$$

   $$w_2(b_1, b_2, b_3, b_4) = (b_1, b_3, b_2, b_4),$$

   $$w_3(b_1, b_2, b_3, b_4) = (b_1, b_2, b_4, b_3),$$

   $$w_4(b_1, b_2, b_3, b_4) = (b_1, b_2, -b_3, -b_4).$$

   The action of the transformations $w_1, \ldots, w_4$ on $\mu, \lambda$ is given by formula (2.10) in [18].

2. **Parallel transformations** $l_i$, which change the auxiliary Hamiltonian $h$. They act on the parameters as follows:

   $$l_i(b_j) = b_j \quad \text{for } j \neq i, \quad l_i(b_i) = b_i + 1$$

   and on the auxiliary Hamiltonian $h$ as:

   $$l_i(h(b_1, b_2, b_3, b_4)) = h(l_i(b_1, b_2, b_3, b_4)).$$

3. The symmetries $x_i$, which change also the variable $t$:

   $$x_1(\mu, \lambda, t, b_1, \ldots, b_4) = (-\mu, 1 - \lambda, 1 - t, b_1, -b_2, b_3, b_4),$$

   $$x_2(\mu, \lambda, t, b_1, \ldots, b_4) = \left((b_1 + b_3)\lambda - \lambda^2 \mu, \frac{1}{\lambda}, \frac{1}{\lambda}, \frac{b_1 - b_2 + b_3 - b_4}{2}, \frac{b_2 - b_1 + b_3 - b_4}{2}, \frac{b_1 - b_2 + b_3 + b_4}{2}, \frac{b_2 - b_1 + b_3 + b_4}{2}\right),$$

   $$x_3(\mu, \lambda, t, b_1, \ldots, b_4) = \left(-\frac{t}{2} - \frac{\mu}{\lambda}, \frac{1 - \mu}{2}, \frac{1}{\lambda}, \frac{b_1 - b_2 + b_3 + b_4 + 1}{2}, \frac{b_2 - b_1 + b_3 + b_4 + 1}{2}, \frac{b_1 + b_2 - b_3 - b_4 - 1}{2}, \frac{b_1 + b_2 + b_3 + b_4 - 1}{2}\right).$$

Okamoto proves that all these transformations are realized as birational canonical transformations of $(\mu, \lambda)$, provided that the correspondent auxiliary Hamiltonian is non singular.
4 Okamoto’s $w_2$ transformation

As we mentioned in the introduction, the symmetries $x_1, x_2, x_3$ can all be obtained as conformal transformations on the variable $x$ of the linear differential equation (1). This fact can be extended to Garnier systems as explained in [11] and to Schlesinger systems in any dimensions [5]. All transformations $w_i$ apart from $w_2$ can be obtained as constant gauge transformations on the Fuchsian system (12), that can be generalized to Garnier systems [21]. The affine transformations can be obtained as Schlesinger transformations on the Fuchsian system and can also be generalized to Garnier systems [21]. The isomonodromic meaning of $w_2$ was until now unknown, we explain it here below using the irregular system (5). We actually explain the meaning of an equivalent transformation $\tilde{w} = x_2 w_2 x_2$ such that

$$\tilde{w}(\mu, \lambda, t, b_1, b_2, b_3, b_4) = (\tilde{\mu}, \tilde{\lambda}, t, -b_4, b_2, b_3, -b_1),$$

where $(\tilde{\mu}, \tilde{\lambda})$ can be obtained by formula (2.10) in [18]. Our transformation $\tilde{w}$ is obtained by the following simple gauge transformation of the irregular system (5)

$$\tilde{Y}(z) = z^\gamma Y(z), \quad \gamma = \frac{\theta_1 + \theta_2 + \theta_3 - \theta_\infty}{2} = b_1 + b_4 = -\rho_1.$$

In fact this gauge transformation maps the system (5) to

$$\frac{d}{dz} \tilde{Y} = \left( \begin{pmatrix} t_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & t_{n+2} \end{pmatrix} + \frac{\tilde{V} - 1}{z} \right) \tilde{Y},$$

where $\tilde{V} = V - \rho_1$. This gauge transformation is therefore compatible with the condition that $V$ has one null eigenvalue. In fact

$$\theta_1 \rightarrow \tilde{\theta}_1 = \theta_1 + \rho_1,$$

$$\theta_2 \rightarrow \tilde{\theta}_2 = \theta_2 + \rho_1,$$

$$\theta_3 \rightarrow \tilde{\theta}_3 = \theta_3 + \rho_1,$$

$$\theta_\infty \rightarrow \tilde{\theta}_\infty = \theta_\infty - \rho_1,$$

$$\rho_1 \rightarrow 0,$$

$$\rho_2 \rightarrow \tilde{\rho}_2 = \rho_2 - \rho_1,$$

$$0 \rightarrow \tilde{\rho}_1 = -\rho_1.$$

Using (22) and (24), we immediately obtain that such gauge transformation maps $(b_1, b_2, b_3, b_4)$ to $(-b_4, b_2, b_3, -b_1)$. The transformation law for $(\tilde{\mu}, \tilde{\lambda})$ follows immediately from the fact that $\tilde{V}$ satisfies (14) if and only if $\tilde{V}$ does. Then it is possible to apply the same procedure as in Section 2 to $\tilde{V}$, i.e. express $\tilde{V}$ in terms of $\lambda, \tilde{\mu}$, solutions of the Garnier system (Painlevé sixth equation)
in one variable \( t \) with parameters \( \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_\infty \). Then \((\mu, \lambda)\) must be related to \((\mu, \lambda)\) by formula (2.10) in [18]. It is clear that in the case of Garnier systems with \( n > 1 \) variables, the condition that \( V \) has \( n > 1 \) null eigenvalues is not preserved by the above gauge transformation. Therefore we expect that the analogous of the transformation \( \tilde{w} \) or equivalently of \( w_2 \) does not exist for Garnier systems with \( n > 1 \) variables [19]. Conversely, we expect such a transformation to extend to the special cases of Schlesinger systems (17) as explained here below.

**Theorem 9.** Let \( m \) be any integer \( 0 < m < n + 2 \), and let \( \rho_1, \ldots, \rho_m \) pairwise distinct non-zero complex numbers. There is a one-to-one correspondence between equivalence classes of \((n + 2) \times (n + 2)\) systems of the form (5) where \( V \) is a \((n + 2) \times (n + 2)\) diagonalizable matrix satisfying the conditions (6) and

\[
G^{-1}VG = \rho := \begin{pmatrix} \rho_1 & \cdots & \rho_m \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix},
\]

(26)

and classes of equivalence of \( m \times m \) systems of the form (12)

\[
\frac{d}{dx} \Phi = \sum_{k=1}^{n+2} \frac{A_k}{x-t_k} \Phi
\]

where \( A_1, \ldots, A_{n+2} \) satisfy

\[
eigen(A_k) = (0, \theta_k) \quad \text{and} \quad - \sum_{k=1}^{n+2} A_k = A_\infty := \begin{pmatrix} \rho_1 & \cdots \\ \cdots & \cdots \\ \rho_m & \cdots \end{pmatrix}.
\]

(27)

Moreover \( V(t) \) satisfies (17) if and only if \( A_1, \ldots, A_{n+2} \) satisfy (17).

**Proof.** The proof of this theorem is very similar to the proof of Lemmata 4 and 7. The only difference is that instead of dealing with the first \( 2 \times 2 \) block with deal with the first \( m \times m \) block. For example \( \Pi \) will be replaced by a diagonal matrix with the first \( m \) diagonal elements equal to 1 and the remaining ones equal to 0.

\( \triangle \)

If in the above theorem we take \( m = n + 1 \), the matrix \( V \) has only one null eigenvalue and the gauge transformation

\[
\tilde{Y}(z) = z^{-\rho}Y(z),
\]

where \( \rho \) is one of the eigenvalues of \( V \), is compatible with the condition that \( V \) has one null eigenvalue. Therefore the above gauge transformation induces a canonical trasformation of the Schlesinger systems specified by (27) for \( m = n + 1 \).

11
Remark 10. It is worth observing that our birational canonical transformation \( \tilde{\omega} \) is precisely the one linking the special case PVI\( \mu \) of the Painlevé sixth equation studied in [4, 15] and the one studied in [9, 10]. The case of PVI\( \mu \) is specified by the following choice of the parameters
\[
\alpha = \frac{(2\mu - 1)^2}{2}, \quad \beta = 0, \quad \gamma = 0, \quad \delta = \frac{1}{2},
\]
that is \( \theta_1 = \theta_2 = \theta_3 = 0 \quad \theta_\infty = 2\mu \). Hitchin’s case, say PVI\( k \), is specified by
\[
\alpha = \frac{(\sqrt{2k} - 1)^2}{2}, \quad \beta = -k, \quad \gamma = k, \quad \delta = \frac{1 - 2k}{2},
\]
that is \( \theta_1, \theta_2, \theta_3, \theta_\infty = \pm \sqrt{2k} \). Therefore the transformation \( \tilde{\omega} \) permits to map the classification results of [4, 15] to Hitchin’s case PVI\( k \).

APPENDIX

We describe the monodromy data of the system (5). A general description of monodromy data of linear systems of ODE can be found in [12, 13, 14]. The treatment of systems of the form (5) can be found in [2] where the case of antisymmetric \( V \) is dealt with. Here we essentially adapt (omitting all proofs) the description of [2] to any \( V \) satisfying our conditions (6).

A.1. Local theory at zero

The system (5) has a simple pole at \( z = 0 \). We assume that a branch-cut between zero and infinity has been chosen along a fixed line \( l \) and a branch of \( \log z \) has been selected.

Proposition 11. There exists a gauge transformation \( Y = G(z)\tilde{Y} \) where \( G(z) = \sum_{k=0}^{\infty} G_k z^k \), convergent near 0, with principal term \( G_0 := G \) defined by \( V = G \rho G^{-1} \), that maps the system (5) into the Birkhoff Normal form:
\[
\frac{d}{dz} \tilde{Y} = \left( \frac{\rho - \mathbb{I}}{z} + \sum_{k \geq 1} R_k z^{k-1} \right) \tilde{Y}
\]  
(A-1)

where \( R = \sum_{k \geq 1} R_k \), with
\[
R_{k_{ij}} \neq 0, \quad \iff \quad \rho_i - \rho_j = k.
\] As a consequence there exists a fundamental matrix of the system (5) of the form
\[
Y_0(z) = G(z)z^{\rho - \mathbb{I}} z^R, \quad \text{as} \quad z \to 0.
\]  
(A-2)
The monodromy $\mathcal{M}_0$ of the system (A-2) with respect to the normalized fundamental matrix generated by a simple closed loop around the origin is

$$\mathcal{M}_0 = \exp(2\pi i \rho) \exp(2\pi i R).$$

A.2. Local theory at infinity

Proposition 12. For the system (A-3) there exists a unique formal power series

$$P(z) = \sum_{k=0}^{\infty} P_k z^{-k}$$

with $P_0 = 1$, such that the formal gauge transformation $Y = P(z) \tilde{Y}$ changes the system (A-4) into the system in normal form:

$$\frac{d}{dz} \tilde{Y} = \left( T + \frac{\Theta}{z} \right) \tilde{Y},$$

where $\Theta$ is a diagonal matrix of entries $-\theta_1 - 1, \ldots, -\theta_{n+2} - 1$. As a consequence, there is a unique formal fundamental solution $Y_f^\infty$ to the system (A-5) at $z = \infty$:

$$Y_f^\infty = P(z) z^{\Theta} e^{zT}. \quad (A-5)$$

The above result establishes only the existence of formal solutions. Regarding the true solutions, we need to set up some more machinery.

Definition 13. The half-line

$$R_{ij} = \{ z | \text{Re} [z(t_i - t_j)] = 0, \text{Im} [z(t_i - t_j)] < 0 \}$$

oriented from zero to infinity is called Stokes-ray.

Definition 14. An oriented line $l$ in the complex plane is called admissible with respect to the points $(t_1, \ldots, t_{n+2})$ if it is such that all the Stokes rays $R_{ij}$ with $i < j$ lie on the left of $l$. Vice versa, fixed any oriented line $l$ in the complex plane, the set of points $(t_1, \ldots, t_{n+2})$ is called admissible with respect to the line $l$, if all the Stokes rays $R_{ij}$ with $i < j$ lie on the left of $l$.

Definition 15. (Valid for the particular case we are working with). Let $\Sigma$ be some sector near $\infty$ and $f(z)$ analytic for $z \in \Sigma \cap \{|z| > N\}$, $N \in \mathbb{R}$. We say that $\sum_{k=0}^{\infty} \frac{a_k}{z^k}$ is the asymptotic series expansion of $f(z)$ in $\Sigma \cap \{|z| > N\}$,

$$f(z) \sim \sum_{k=0}^{\infty} \frac{a_k}{z^k},$$

We fix the branch of the logarithm required in the definition of $z^{\Theta}$ as in the previous section.
if for all $m \in \mathbb{Z}_+$, and for $z \to \infty$ inside $\Sigma$

$$\lim_{z \to \infty} z^m \left( f(z) - \sum_{k=0}^{m} \frac{a_k}{z^k} \right) = 0.$$ 

**Lemma 16.** Fix an admissible oriented line $l$ and consider the system (5) and its unique formal fundamental solution (A-5). Then there exists $\varepsilon > 0$ small enough, two sectors $\Pi_L$ and $\Pi_R$ defined as

$$\Pi_R = \{ z : \arg(l) - \pi - \varepsilon < \arg(z) < \arg(l) + \varepsilon \}$$

$$\Pi_L = \{ z : \arg(l) - \varepsilon < \arg(z) < \arg(l) + \pi + \varepsilon \}$$  \hspace{1cm} (A-7)

and two fundamental solutions $Y_L(z)$ in $\Pi_L$ and $Y_R(z)$ in $\Pi_R$ such that

$$Y_{L,R} \sim \left( 1 + O \left( \frac{1}{z} \right) \right) z^{\Theta} e^{zT}, \quad \text{as} \quad z \to \infty, \quad z \in \Pi_{L,R}. \quad \text{(A-8)}$$

The fundamental solutions $Y_{L,R}(z)$ are uniquely determined by (A-8).

The proof can be found in [1].

We are now going to define one of the main objects in the monodromy theory of our system at infinity: the Stokes matrices. In both of the narrow sectors

$$\Pi_+ := \{ z : \varphi - \varepsilon < \arg(z) < \varphi + \varepsilon \}$$

$$\Pi_- := \{ z : \varphi - \pi - \varepsilon < \arg(z) < \varphi - \pi + \varepsilon \}$$

obtained by the intersection of $\Pi_l$ and $\Pi_R$, we have two fundamental matrices. They must be related by multiplication by a constant invertible matrix

$$Y_L(z) = Y_R(z) S_+, \quad z \in \Pi_+.$$ 

$$Y_L(z) = Y_R(z) S_-, \quad z \in \Pi_-.$$ 

**Definition 17.** The matrices $S_+, S_-$ are called Stokes matrices of the system (5) with respect to the admissible line $l$.

**A.3. Monodromy data**

Resuming, for our system (5)

$$\frac{d}{dz} Y = \left( T + \frac{V - \mathbb{I}}{z} \right) Y,$$

where $T$ is a diagonal matrix with pairwise distinct entries $t_1, \ldots, t_{n+2}$ and $V$ is diagonalizable with $n$ null eigenvalues, we have built three distinguished bases
in the space of solutions, i.e. \( Y_0(z) \) near 0, and \( Y_{L,R}(z) \) near \( \infty \) depending on the choice of the admissible line \( \ell \).

To complete the list of the monodromy data we define the central connection matrix between 0 and \( \infty \)

\[
Y_0(z) = Y_{R,L}(z)C_{R,L}, \quad z \in \Pi_{R,L}.
\]

Recall that to define \( Y_0 \) we had to fix the branch cut of \( \log(z) \). We can choose it along the negative part \( l_- \) of the same line \( l \).

We can reduce the list of the monodromy data \((\rho, R, S_+, S_- , C_R, C_L)\), by noticing the following cyclic relation

\[
C_R^{-1} S_T S_+^{-1} C_R = C_L^{-1} S_T S_-^{-1} C_L M_0 = \exp(2\pi i \rho) \exp(2\pi R).
\]

This expresses a simple topological fact: on the punctured plane \( \mathbb{C} \setminus \{0\} \) a loop around infinity is homotopic to a loop around the origin. For a similar reason \( C_L = S_+^{-1} C_R \).

**Definition 18.** The **Monodromy data** of the system (5) are a collection of constant matrices

\((\rho, R, S_+, C_R)\)

**Theorem 19.** Two systems of the same form (5) coincide if and only of they have the same monodromy data.

The proof can be found in [2].

**References**

[1] W. Balser, W.B. Jurkat, and D.A. Lutz. Birkhoff invariants and stokes multipliers for meromorphic linear differential equations. *J. Math. Anal. Appl.*, 71:48–94, 1979.

[2] B. Dubrovin. *Geometry of 2D topological field theories*, volume 1620 of *Springer Lecture Notes in Math.* Integrable Systems and Quantum Groups, M. Francaviglia, S. Greco editors, 1996.

[3] B. Dubrovin. Painlevé transcendents in Two-Dimensional Topological Field Theory. *The Painlevé property One Century Later*, pages 287–412, 1999.

[4] B. Dubrovin and M. Mazzocco. Monodromy of certain Painlevé-VI transcendents and reflection groups. *Invent. Math.*, 141:55–147, 2000.
[5] B. Dubrovin and M. Mazzocco. Canonical structures for the Schlesinger systems. *preprint*, 2003.

[6] R. Garnier. Sur des équations différentielles du troisième ordre dont l’intégrale générale est uniforme et sur une classe d’équations nouvelles d’ordre supérieur dont l’intégrale générale a ses points critiques fixes. *Ann. Sci. École Norm. Sup.*, 29:no. 3, 1–126, 1912.

[7] R. Garnier. Solution du problème de Riemann pour les systemes différentielles linéaires du second ordre. *Ann. Sci. École Norm.. Sup.*, 43:239–352, 1926.

[8] J. Harnad. Dual isomonodromic deformations and moment maps to loop algebras. *Commun. Math. Phys.*, 166:337–365, 1994.

[9] N. Hitchin. Algebraic solutions of the Painlevé VI equation. *talk at Edinburgh, work in progress*, 1998.

[10] N. Hitchin. A lecture on the octahedron. *preprint*, 2002.

[11] K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida. *From Gauss to Painlevé, a Modern Theory of Special Functions*, volume E 16. Aspects of Mathematics, 1991.

[12] M. Jimbo and T. Miwa. Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I. *Physica 2D*, 2 no. 2:306–352, 1981.

[13] M. Jimbo and T. Miwa. Monodromy preserving deformations of linear ordinary differential equations with rational coefficients II. *Physica 2D*, 4 no.1:407–448, 1981.

[14] M. Jimbo, T. Miwa, and K. Ueno. Monodromy preserving deformations of linear ordinary differential equations with rational coefficients iii. *Physica 2D*, 4 no.1:26–46, 1982.

[15] M. Mazzocco. Picard and Chazy solutions to the PVI equation. *Math. Ann.*, 321:no. 1:131–169, 2001.

[16] M. Mazzocco. Painlevé sixth equation as isomonodromic deformations equation of an irregular system. *The Kowalevski property (Leeds, 2000), CRM Proc. Lecture Notes.*, 32:219–238, 2002.

[17] K. Okamoto. Isomonodromic deformations, Painlevé equations, and the Garnier system. *J. Fac. Sci. Univ. Tokyo, Sect. 1A, Math.*, 33:576–618, 1986.
[18] K. Okamoto. Studies on the Painlevé equations I, sixth Painlevé equation. *Ann. Mat. Pura Appl.*, 146:337–381, 1987.

[19] H. Sakai and T. Tsuda. *private communication*, 2002.

[20] L. Schlesinger. Über eine Klasse von Differentsial System Beliebliger Ordnung mit Festen Kritischer Punkten. *J. fur Math.*, 141:96–145, 1912.

[21] T. Tsuda. Universal characters and integrable systems. *PhD thesis, Tokyo Graduate School of Mathematics*, 2003.