RESEARCH ARTICLE

Approximation on Durrmeyer modification of generalized Szász–Mirakjan operators

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This paper deals with the approximations of the functions by generalized Durrmeyer operators of Szász–Mirakjan, which are linear positive operators. Several approximation results are presented well, and we estimate the approximation properties along with the order of approximation and the convergence theorem of the proposed operators. For an explicit explanation of the operators, we determine the properties using the weight function. A quantitative approach is discussed for the operators; quantitative Voronovskaya type and Grüsst type theorems are established, showing the operators’ more efficient work. We investigate the A-statistical convergence properties for the said operators, including the rate of approximation in a statistical sense. An important property for the rate of convergence of the operators is obtained in terms of the function with a derivative of the bounded variation. At last, the graphical representations and numerical analysis are discussed and shown to support our theoretical findings.

KEYWORDS
Lipschitz function, modulus of continuity, statistical convergence, function of bounded variation, Szász–Mirakjan operators

MSC CLASSIFICATION
41A25, 41A35, 41A36

1 INTRODUCTION

In 1941, Mirakjan [1] and in 1950, Szász [2] independently introduced as well as studied the approximation properties of a generalization of Bernstein’s operators on an infinite interval, which are known as Szász–Mirakjan operators. After two decades, in 1977, Jain and Pethe [3] introduced new type of Szász–Mirakjan operators which are as follows:

\[ \mathcal{L}C_n^{[\alpha]}(g; x) = \sum_{i=0}^{\infty} (1 + n\alpha)^{\frac{-\alpha}{n}} \left( a + \frac{1}{n} \right)^{-\frac{i}{n}} g \left( \frac{i}{n} \right). \]  

(1.1)

where \( x^{(i-\alpha)} = x(x + \alpha) \cdots (x + (i-1)\alpha) \), \( x^{(0,-\alpha)} = 1 \) and the function \( g \) is considered to be of exponential type such that \( |g(x)| \leq Me^{Ax} \), \( (x \geq 0, A > 0) \), with positive constant \( M \), denoted by \( E \) and here \( \alpha = \{a_n\}, n \in \mathbb{N} \) is as \( 0 \leq a_n \leq \frac{1}{n} \). They determined the approximation properties of the said operators. For \( \alpha = \frac{1}{n^c} \), the above operators (1.1) reduce to the operators which Agratini has defined [4]. In 2007, Abel and Ivan [5] replaced \( \alpha \) by \( \frac{1}{nc} \) in the above operators (1.1) and obtained the generalized version operators, which are as follows:
\[ A_{O_n^c} = \sum_{i=0}^{\infty} \left( \frac{c+1}{c} \right)^{-i} n^i \left( \frac{n}{c} + i - 1 \right) g \left( \frac{i}{n} \right), \]  
(1.2)

where \( c = c_n, n \in \mathbb{N} \) is restricted with a certain constant \( \beta > 0 \) such that \( c_n \geq \beta \) for \( n \in \mathbb{N} \). The main purpose of defining the above operators (1.2) was to investigate the local approximation properties and check the asymptotic behavior. Very recently, Dhamiza et al. [6] defined a Kantorovich variant of the operators (1.1) for studying the local approximations properties and rate of convergence. For bounded and integrable function on \([0, \infty)\), the operators are given by

\[ MO_n^c(g; x) = \sum_{i=0}^{\infty} (1 + na)^{\frac{a}{n}} \left( a + \frac{1}{n} \right)^{-i} \frac{x^{i-\alpha}}{i!} \int_{\frac{i}{n}}^{\frac{i+1}{n}} g(t) \, dt. \]  
(1.3)

In a particular case, when \( \alpha \to 0 \), the above operators reduce to Szász–Mirakjan–Kantorovich operators defined by Totik [7]. But for the Durrmeyer point of view of the Szász–Mirakjan operators, in 1985, Mazhar and Totik [8] modified the Szász–Mirakjan operators into summation integral type operators, which are defined by

\[ I_n(g; x) = n \sum_{i=0}^{\infty} u_{n,i}(x) \int_{0}^{\infty} u_{n,i}(t) g(t) \, dt, \]  
(1.4)

where \( u_{n,i}(x) = e^{-nx(u_i/a)} \), and independently, the related approximations properties have been discussed by Kasana et al. [9]. In this regard, Gupta and Pant [10] determined the convergence rate and other approximations properties. Some approximation properties can be seen in Gupta et al. [11], and other modifications regarding Durremnyer of the various operators can be obtained from earlier studies [12, 13]. In 2016, Mishra et al. [14] modified Szász–Mirakjan–Durrmeyer operators defined by (1.4) for studying the properties like simultaneous approximation, rate of convergence, and so on. The modified operators are as follows:

\[ D_n(g; x) = d_n \sum_{i=0}^{\infty} u_{d_n,i}(x) \int_{0}^{\infty} u_{d_n,i}(t) g(t) \, dt, \]  
(1.5)

where \( d_n \to \infty \) as \( n \to \infty \) be a positive sequence of real numbers which is increasing as well as \( d_1 \geq 1 \) and \( u_{d_n,i}(x) = e^{-d_n x (u_i/a)} \). Moreover, the operators (1.4) can be obtained when \( d_n = n \) in the operators (1.5). On the other hand, some modifications and generalizations in terms of the Durrmeyer variant can be seen in previous studies [15, 16]. Motivated by the above works, we define the Durrmeyer modification of the above operators (1.1) by considering the function \( g : [0, \infty) \to \mathbb{R} \) is integrable and bounded on the interval \([0, \infty)\) as follows:

\[ L_n^{(1)}(g; x) = a \sum_{i=0}^{\infty} (1 + na)^{-i} \frac{x^{i-\alpha}}{i!} \int_{0}^{\infty} e^{-nt} \frac{(nt)^i}{i!} g(t) \, dt, \]  
(1.6)

where \( a = a_n, n \in \mathbb{N} \) is as \( 0 \leq a_n \leq \frac{1}{n} \). For \( \alpha \to 0 \), the above operators are reduced into Szász–Mirakjan–Durrmeyer operators, which are defined by Equation (1.4). Also, if we replace \( a \) by \( \frac{1}{nc} \), we can reach to Case 1 on Govil et al. (page 196, [15]).

The main motive of this article is to investigate the approximation properties of the defined operators (1.6). To express the novelty of the present manuscript, the different perspectives of the research can be seen. Our proposed operators have not been addressed yet, and our goal is to work on this being considered approximation of the functions. The operators also exhibit some new developments, like statistical convergence, which is a new property for the operators. If we move towards the other development, we can find the convergence rate in terms of the function with the derivative of bounded variation. Overall, the properties that are yet to be discussed can be written within some statements. Here, we divide it into sections. The defined operators' convergence rate is obtained in terms of modulus of continuity, second-order modulus of continuity with the relations of Peetre's \( K \)-functional in Section 3. Section 4 consists of weighted approximations properties. To
determine the properties of the operators (1.6) via quantitatively, quantitative Voronovskaya type theorem and Grüss Voronovskaya type theorem are studied in Section 5. In Section 6, graphical and numerical representations are presented to support approximation results. Section 7 offers A-statistical properties of the operators, and in Section 8, an important property is studied for the rate of convergence by means of the derivative of bounded variations. Finally, the conclusion, result discussion, and applications are discussed.

## 2 Preliminaries

This section contains some important results, lemmas, and theorem to study the approximations properties of the proposed operators.

### Some important results:

For all \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \cup \{0\} \), we have

1. 
\[
\int_0^\infty e^{-nt} \frac{(nt)^i}{i!} u^m \, dt = \frac{1}{n^{m+1}} \frac{(i+m)!}{i!}.
\]

in particular, if \( m = 0 \) then

\[
\int_0^\infty e^{-nt} \frac{t^i}{i!} \, dt = \frac{1}{n},
\]

if \( m = 1 \) then

\[
\int_0^\infty e^{-nt} \frac{t^{i+1} \, dt}{i!} = \frac{(i+1)}{n^2},
\]

and so on …

2. 
\[
(1 + n\alpha)^z = \sum_{i=0}^{\infty} \left( \alpha + \frac{1}{n} \right)^{i} \frac{x^{i-i\alpha}}{i!}.
\]

**Lemma 2.1.** The following results hold for all \( n \in \mathbb{N} \):

\[
U_n^{[a]}(1;x) = 1,
\]

\[
U_n^{[a]}(i;x) = \frac{1 + nx}{n},
\]

\[
U_n^{[a]}(i^2;x) = \frac{2 + 4nx + n^2x^2 + n^2x\alpha}{n^2},
\]

\[
U_n^{[a]}(i^3;x) = \frac{6 + 18nx + 9n^2x(x + \alpha) + n^3x(x^2 + 3x\alpha + 2\alpha^2)}{n^3},
\]

\[
U_n^{[a]}(i^4;x) = \frac{24 + 96nx + 72n^2x(x + \alpha) + 16n^3x(x^2 + 3x\alpha + 2\alpha^2) + n^4x(x^3 + 6x^2\alpha + 11x\alpha^2 + 6\alpha^3)}{n^4},
\]

\[
U_n^{[a]}(i^5;x) = \frac{1}{n^{\frac{1}{2}}} \left( n^5x \left( 24\alpha^4 + x^4 + 10x\alpha^3 + 35\alpha^2x^2 + 50\alpha^3x \right) + 25n^4x \left( 6\alpha^3 + x^3 + 6x\alpha^2 + 11x\alpha^2 \right) 
\right.
\]
\[
\left. + 200n^3x \left( 2\alpha^2 + x^2 + 3ax \right) + 600n^2x(x + \alpha) + 600nx + 120),
\]

\[
U_n^{[a]}(i^6;x) = \frac{1}{n^{\frac{1}{3}}} \left( n^6x \left( 120\alpha^5 + x^5 + 15ax^4 + 85\alpha^2x^3 + 225\alpha^3x^2 + 274\alpha^4x \right) + 36n^5x \left( 24\alpha^4 + x^4 + 10x\alpha^3 
\right.
\right.
\]
\[
\left. + 35\alpha^2x^2 + 50\alpha^3x) + 450n^4x \left( 6\alpha^3 + x^3 + 6x\alpha^2 + 11x\alpha^2 \right) + 2400n^3x \left( 2\alpha^2 + x^2 + 3ax \right) 
\right.
\]
\[
\left. + 5400n^2x(x + \alpha) + 4320nx + 720)\right)\]
Proof. Using the above results (1) and (2), we get

\[ U_n^{[\alpha]}(1; x) = \sum_{i=0}^{\infty} (1 + na)^{-i} \left( \alpha + \frac{1}{n} \right)^{-i} x^{(i-\alpha)} \frac{1}{i!} \]

Similarly, we can prove the other equalities.

Consider the \( \Theta_{n,m}^{[\alpha]}(x) = U_n^{[\alpha]}((t-x)^m; x) \), \( m = 1, 2, 3, 4, 6 \) are the central moments and here we obtain the following the results.

**Lemma 2.2.** For each \( x \geq 0 \) and \( n \in \mathbb{N} \), it holds the following:

\[ \Theta_{n,1}^{[\alpha]}(x) = \frac{1}{n}, \]

\[ \Theta_{n,2}^{[\alpha]}(x) = \frac{an^2x + 2nx + 2}{n^2}, \]

\[ \Theta_{n,3}^{[\alpha]}(x) = \frac{2a^2n^3x + 9an^2x + 12nx + 6}{n^3}, \]

\[ \Theta_{n,4}^{[\alpha]}(x) = \frac{3a^2n^4(2a + x) + 4an^3(x(8a + 3x) + 12n^2x(6a + x) + 72nx + 24)}{n^4}, \]

\[ \Theta_{n,6}^{[\alpha]}(x) = \frac{1}{n^6} \left( 5a^3n^6x \left( 24a^2 + 3x^2 + 26ax \right) + 18a^2n^5x \left( 48a^2 + 5x^2 + 50ax \right) \\ + 30an^4x \left( 90a^2 + 6x^2 + 85ax \right) + 120n^3x \left( 40a^2 + x^2 + 30ax \right) + 1080n^2x(5a + 2x) + 3600nx + 720 \right). \]

Proof. We can easily prove the above lemma by using Lemma 2.1, so we omit the proof.

**Remark 2.1.** For \( x \in [0, \infty) \) and for \( n \in \mathbb{N} \), we obtain

\[ \Theta_{n,1}^{[\alpha]}(x) = \frac{an^2x + 2nx + 2}{n^2} = ax + \frac{2x}{n^2} \leq \frac{3x}{n^2} + \frac{2}{n^2} = \frac{3}{n} \left( x + \frac{1}{n} \right) = \frac{3}{n} \eta^{[\alpha]}_n(x). \]

**Remark 2.2.** The above operators 1.6 can be written as

\[ U_n^{[\alpha]}(g; x) = \int_0^\infty u_n^{[\alpha]}(x, t)g(t)dt, \]

where \( u_n^{[\alpha]}(x, t) = n \sum_{i=0}^{\infty} r_{n,i}^{[\alpha]}(x)p_n(t), r_{n,i}^{[\alpha]}(x) = (1 + na)^{-i} \left( \alpha + \frac{1}{n} \right)^{-i} \frac{x^{(i-\alpha)}}{i!} \) and \( p_n(x) = e^{-\alpha x (mx)^{\alpha}} \).
Lemma 2.3. For every $x \geq 0$ and $\max \alpha = \frac{1}{n}$ for all $n \in \mathbb{N}$, then it holds the following:

$$
\begin{align*}
\lim_{n \to \infty} \{n \Theta_{n,1}^a(x)\} &= 1, \\
\lim_{n \to \infty} \{n \Theta_{n,2}^a(x)\} &= 3x, \\
\lim_{n \to \infty} \{n^2 \Theta_{n,3}^a(x)\} &= 27x^2, \\
\lim_{n \to \infty} \{n^3 \Theta_{n,4}^a(x)\} &= 405x^3.
\end{align*}
$$

Proof. For all $x \geq 0$ and $n \in \mathbb{N}$. Using Lemma 2.2 and the first part of the lemma is $\Theta_{n,1}^a(x) = \frac{1}{n}$ and then it is obvious that $\lim_{n \to \infty} \{n \Theta_{n,1}^a(x)\} = 1$. Now, we come to the other parts.

$$
\lim_{n \to \infty} \{n \Theta_{n,2}^a(x)\} = \lim_{n \to \infty} \frac{an^2x + 2nx + 2}{n} = \lim_{n \to \infty} \frac{nx + 2nx + 2}{n} = 3x,
$$

$$
\lim_{n \to \infty} \{n^2 \Theta_{n,3}^a(x)\} = \lim_{n \to \infty} \frac{3a^2n^4x(2\alpha + x) + 4an^7x(8\alpha + 3x) + 12n^2x(6\alpha + x) + 72nx + 24}{n^2} = \lim_{n \to \infty} \frac{6nx + 3n^2x^2 + 32nx + 12n^2x^2 + 72nx + 24}{n^2} = 27x^2.
$$

Similarly, the last part can also be proved, and the proof is completed. \(\square\)

Lemma 2.4. If a function $g$ defined on $[0, \infty)$ and bounded with supremum norm $||g|| = \sup_{x \geq 0} |g(x)|$, then there is an inequality that holds as follows:

$$
|L_n^{|a|}(g;x)| \leq ||g||.
$$

Proof. For a bounded function $g$, defined on $[0, \infty)$ endowed with the norm $||g|| = \sup_{x \geq 0} |g(x)|$, the operators (1.6) are

$$
|L_n^{|a|}(g;x)| = \left| \sum_{l=0}^{\infty} (1 + na)^{-\alpha} \left( a + \frac{1}{n} \right)^{-i x^{(l,-a)}} \frac{1}{l!} \int_0^{\infty} e^{-nt}(nt)^j g(t) dt \right|
$$

$$
\leq \sum_{l=0}^{\infty} (1 + na)^{-\alpha} \left( a + \frac{1}{n} \right)^{-i x^{(l,-a)}} \frac{1}{l!} \int_0^{\infty} e^{-nt}(nt)^j g(t) dt
$$

$$
\leq \sum_{l=0}^{\infty} (1 + na)^{-\alpha} \left( a + \frac{1}{n} \right)^{-i x^{(l,-a)}} |g(t)| \int_0^{\infty} e^{-nt}(nt)^j dt
$$

$$
\leq ||g|| \sum_{l=0}^{\infty} (1 + na)^{-\alpha} \left( a + \frac{1}{n} \right)^{-i x^{(l,-a)}} \int_0^{\infty} e^{-nt}(nt)^j dt = ||g||.
$$

Hence, it is proved. \(\square\)

Theorem 2.1. Consider $g \in C[0, \infty) \cap W$, with $\alpha \in [0, \frac{1}{n}]$ and $\alpha \to 0$ as $n \to \infty$ then we get

$$
\lim_{n \to \infty} L_n^{|a|}(g;x) = g(x)
$$
uniformly on each finite interval of \([0, \infty)\). Here, \(C[0, \infty)\) is denoted as the set of all continuous functions defined on \([0, \infty)\) and \(W = \{g : g(t) = O(t^\gamma), \ast t \to \infty, \gamma > 0\}\).

3 | DIRECT RESULTS

This segment consists of the uniform convergence theorem, the convergence rate of the proposed operators using second-order modulus of smoothness and continuity, and a relation exists with Peetre’s \(K\)-functional.

To study the approximation properties, we suppose \(C_B[0, \infty)\) be the set of all continuous and bounded function \(g\) defined on \([0, \infty)\) with supremum norm \(\|g\| = \sup_{x \in [0, \infty)} \{g(x)\}\). And here for any \(\xi > 0\), Peetre’s \(K\)-functional is defined by the following:

\[
K_2(g; \xi) = \inf_{g_1 \in C_B[0, \infty)} \{\|g - g_1\| + \xi\|g''\|\}, \quad \text{where} \quad C^2_B[0, \infty) = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}.
\]

In 1993, an important relation was introduced by De Vore [17] by considering Peetre’s \(K\)-functional and second-order modulus of smoothness, which is given below:

\[
K_2(g; \xi) \leq M \omega_2(g; \sqrt{\xi}), \tag{3.1}
\]

where \(\omega_2(g; \sqrt{\xi})\) is the second-order modulus of smoothness and is defined by

\[
\omega_2(g; \sqrt{\xi}) = \sup_{h \in [0, \sqrt{\xi}], x \in [0, \infty)} \{|g(x + 2h) - 2g(x + h) + g(x)|\}, \ g \in C_B[0, \infty).
\]

Also, a further relation introduced in De Vore and Lorentz [17], for which there exists a positive constant \(M\) such that

\[
K_2(f; \delta) \leq M \{\omega_2(f; \sqrt{\delta}) + \min(1, \delta)\|f\|_{C_B[0, \infty)}\},
\]

where \(\omega_2(f; \sqrt{\delta})\) is the second-order modulus of smoothness.

Note. The first-order modulus of continuity is \(\omega(g; \xi) = \sup_{h \in [0, \xi], x \in [0, \infty)} \{|g(x + h) - g(x)|\}, \ g \in C_B[0, \infty)\).

Theorem 3.1. Consider the function \(g \in C_B[0, \infty)\) and every \(x \in [0, \infty)\), then it holds as follows:

\[
\left|U_n^{[\alpha]}(g; x) - g(x)\right| \leq M \omega_2(g; \sqrt{\rho_n(x)}) + \omega(g, \nu_n(x)),
\]

where \(\nu_n(x) = \Theta_{n,1}^{[\alpha]}(x)\) and \(\rho_n(x) = \Theta_{n,2}^{[\alpha]}(x) + \frac{1}{n}\).

Proof. For \(g \in C_B[0, \infty)\), consider

\[
U_n^{[\alpha]}(g; x) = U_n^{[\alpha]}(g; x) + g(x) - g\left(1 + \frac{n\alpha}{n}\right).
\]

Here, \(U_n^{[\alpha]}(1; x) = 1\) and \(U_n^{[\alpha]}(t; x) = x\), that is, the auxiliary operators preserve the linear function and constant term. Now, using Taylor’s remainder formula for integral for the function \(u \in C^2_B[0, \infty)\), we have

\[
u(t) - u(x) = (t - x)u'(x) + \int_x^t (t - v)u''(v)dv.
\]
Applying the operators $U_n^{[\alpha]}$ to both sides, we obtain

$$U_n^{[\alpha]}(u(t) - u(x); x) = U_n^{[\alpha]}\left(\int_x^t (t - v)u''(v) \, dv\right) = U_n^{[\alpha]}\left(\int_x^t (t - v)u''(v) \, dv\right) - \left(\frac{1 + nx}{n} - v\right)u''(v) \, dv.$$}

Here,

$$\left|\int_x^t (t - v)u''(v) \, dv\right| \leq \int_x^t |t - v| |u''(v)| \, dv \leq |u''|(t - x)^2.$$

Similarly,

$$\left|\int_x^t \left(\frac{1 + nx}{n} - v\right)u''(v) \, dv\right| \leq \frac{|u''|}{n^2}.$$

Therefore, one has

$$|U_n^{[\alpha]}(g; x) - g(x)| \leq |u''|\Theta_n^{[\alpha]}(x) + \frac{|u''|}{n^2} = \rho_n(x)|u''|.$$ 

Also,

$$|U_n^{[\alpha]}(g; x)| = |U_n^{[\alpha]}(g; x)| + 2||g(x)|| \leq 3||g(x)||.$$ 

By considering all the above inequalities, we obtain

$$\left|U_n^{[\alpha]}(g; x) - g(x)\right| \leq |U_n^{[\alpha]}(g - u; x)| + |U_n^{[\alpha]}(u(t) - u(x); x)| + |u(x) - g(x)| + |g\left(\frac{1 + nx}{n}\right) - g(x)| \leq 4||g - u|| + \rho_n(x)||u''|| + \left|g\left(\frac{1 + nx}{n}\right) - g(x)\right| + \rho_n(x)||u''|| + \omega(g; \nu_n(x)).$$

Now, taking minimum overall $u \in C^n_2[0, \infty)$ on the right-hand side of the above inequality and using Peetre $K$-functional, we obtain

$$\left|U_n^{[\alpha]}(g; x) - g(x)\right| \leq MK_2\rho_n(x) + \omega(g; \nu_n(x)) \leq M\omega(g; \sqrt[2]{\rho_n(x)}) + \omega(g; \nu_n(x)).$$

\[\square\]

Now, we estimate the approximation of the defined operators (1.6), by a new type of Lipschitz maximal function with order $r \in (0, 1]$, defined by Lenze [18] as

$$\kappa_r(g, x) = \sup_{x, y \geq 0} \frac{|g(y) - g(x)|}{|y - x|^r}, \quad x \neq y. \quad (3.2)$$

Using Lipschitz maximal function, we have an upper bound with the function given by a theorem.

**Theorem 3.2.** Consider $g \in C_2[0, \infty)$ with $r \in (0, 1]$, then we obtain

$$\left|U_n^{[\alpha]}(g; x) - g(x)\right| \leq \kappa_r(g, x)(\nu_n(x))^\frac{1}{r}.$$
Proof. By Equation (3.2), we can write
\[ |U_n^{[a]}(g; x) - g(x)| \leq \kappa_j(g, x)U_n^{[a]}(|t - x|^z; x). \]

Using Hölder’s inequality with \( j = \frac{2}{s}, l = \frac{2}{s}, \) one can get
\[ |U_n^{[a]}(g; x) - g(x)| \leq \kappa_j(g, x)(U_n^{[a]}((t - x)^2; x))^{\frac{1}{2}} = \kappa_j(g, x)(\varphi_n(x))^{\frac{1}{2}}. \]

Hence, it is proved. \( \square \)

The next theorem is based on modified Lipschitz type space [19], and this space is defined by
\[ Lip_M^{\lambda_1, \lambda_2}(s) = \left \{ g \in C_\mathbb{R}[0, \infty) : |g(j) - g(k)| \leq M \frac{|j - k|^s}{(j + k^2 \lambda_1 + k \lambda_2)^{\frac{s}{2}}} \right \}, \]
where \( \lambda_1, \lambda_2 \) are the fixed numbers.

**Theorem 3.3.** For \( g \in Lip_M^{\lambda_1, \lambda_2}(s) \) and \( 0 < s \leq 1 \), we have an inequality holds:
\[ |U_n^{[a]}(g; x) - g(x)| \leq M \left ( \frac{\Theta_{n,2}^{[a]}(x)}{x(x\lambda_1 + \lambda_2)} \right )^{\frac{1}{2}}. \]

Proof. To prove the above theorem, we can distribute its proof into two parts by considering the case discussion. So here:

**Case 1.** If \( s = 1 \), proceed ahead, we can observe that \( \frac{1}{(y + x^2 \lambda_1 + x \lambda_2)} \leq \frac{1}{x(x\lambda_1 + \lambda_2)} \), then one has
\[ |U_n^{[a]}(g; x) - g(x)| \leq U_n^{[a]}(|g(t) - g(x)|; x) \]
\[ \leq MU_n^{[a]} \left ( \frac{|t - x|}{(t + x^2 \lambda_1 + x \lambda_2)^{\frac{s}{2}}} ; x \right ) \]
\[ \leq \frac{M}{(x(x\lambda_1 + \lambda_2))^{\frac{s}{2}}} \left ( \frac{\Theta_{n,2}^{[a]}(x)}{x(x\lambda_1 + \lambda_2)} \right )^{\frac{1}{2}} \]
\[ \leq M \left ( \frac{\Theta_{n,2}^{[a]}(x)}{x(x\lambda_1 + \lambda_2)} \right )^{\frac{1}{2}}. \]

**Case 2.** If \( s \in (0, 1) \), then with the help of Hölder inequality by considering \( l = \frac{2}{s}, m = \frac{2}{2-s} \), we get
\[ |U_n^{[a]}(g; x) - g(x)| \leq \left ( U_n^{[a]}(|g(t) - g(x)|^{\frac{1}{2}}; x) \right )^{\frac{1}{2}} \leq MU_n^{[a]} \left ( \frac{|t - x|^2}{(t + x^2 \lambda_1 + x \lambda_2)^{\frac{s}{2}}} ; x \right )^{\frac{1}{2}} \]
\[ \leq MU_n^{[a]} \left ( \frac{|t - x|^2}{x(x\lambda_1 + \lambda_2)} ; x \right )^{\frac{1}{2}} \]
\[ \leq M \left ( \frac{\Theta_{n,2}^{[a]}(x)}{x(x\lambda_1 + \lambda_2)} \right )^{\frac{1}{2}}. \]

Thus, the proof is completed. \( \square \)
Let \( g \in C_b[0, \infty) \) and we define Steklov mean function, which is as follows:

\[
G_h(x) = \frac{1}{h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} (2(g(x + \kappa + \lambda)) - g(x + 2(\kappa + \lambda))) d\kappa d\lambda, \quad \kappa, \lambda \geq 0 \text{ and } h > 0.
\]

To approximate continuous functions by smoother functions, the Steklov function is used, and we appeal to investigate the approximation properties. So, we collect some properties in the following lemma, which are used to prove the main theorem.

**Lemma 3.1.** Let \( g \in C_b[0, \infty) \), it holds following inequalities:

1. \( \|G_h - g\|_{C_b[0, \infty)} \leq \omega_2(g, h) \),
2. \( \|G_h'\|_{C_b[0, \infty)} \leq \frac{5\omega(g, h)}{h^2} \) for \( G_h' \in C_b[0, \infty) \),
3. \( \|G_h''\|_{C_b[0, \infty)} \leq \frac{9\omega(g, h)}{h^4} \) for \( G_h'' \in C_b[0, \infty) \),

where \( \omega(g, h) \) and \( \omega_2(g, h) \) are the modulus of continuity and second order modulus of continuity, respectively, and can be defined as another way, given by

\[
\omega(g, h) = \sup_{x, x, \kappa, \lambda \in [0, \infty] } |g(x + \kappa) - g(x + \lambda)|,
\]

\[
\omega_2(g, h) = \sup_{x, x, \kappa, \lambda \in [0, \infty] } |g(x + 2\kappa) - 2g(x + \kappa + \lambda) + g(x + 2\lambda)|.
\]

**Theorem 3.4.** Consider \( g \in C_b[0, \infty) \), for every \( x \in [0, \infty) \), we get

\[
|U_n^{[a]}(g; x) - g(x)| \leq 5 \left( \omega(g, \sqrt{\Theta_n^{[a]}}) + \frac{13}{10} \omega_2(g, \sqrt{\Theta_n^{[a]}}) \right),
\]

where \( \Theta_n^{[a]} \) is defined by Lemma 2.2.

**Proof.** For every \( x \geq 0 \), using Steklov function, we can write as

\[
|U_n^{[a]}(g; x) - g(x)| \leq U_n^{[a]}(|g - G_h|; x) + |U_n^{[a]}(G_h - G_h(x); x)| + |G_h(x) - g(x)|.
\]

Since \( |U_n^{[a]}(f; x)| \leq \|g(x)\|_{C_b[0, \infty)} \) as \( g \in C_b[0, \infty) \) and \( x \geq 0 \). Then using Steklov mean property, we can have

\[
U_n^{[a]}(|g - G_h|; x) \leq \|U_n^{[a]}(g - G_h; x)\|_{C_b[0, \infty)} \leq \|g - G_h\|_{C_b[0, \infty)} \leq \omega_2(g, h).
\]

Using Taylor’s formula and applying the operators (1.6), we can write

\[
|U_n^{[a]}(G_h - G_h(x); x)| \leq \|G_h'\|_{C_b[0, \infty)} \sqrt{\Theta_n^{[a]}} + \frac{\|G_h''\|_{C_b[0, \infty)}}{2!} \Theta_n^{[a]}.
\]

Using the property of the Steklov mean, we can write as

\[
|U_n^{[a]}(G_h - G_h(x); x)| \leq \frac{5\omega(g, h)}{h^2} \sqrt{\Theta_n^{[a]}} + \frac{9\omega_2(g, h)}{2h^4} \Theta_n^{[a]}.
\]

Choosing \( h = \sqrt{\Theta_n^{[a]}} \), we obtain our required result. \( \square \)

**Remark 3.1.** If \( \Theta_n^{[a]} \to 0 \) as \( n \to \infty \) and then \( \omega(g, \sqrt{\Theta_n^{[a]}}) \to 0, \omega_2(g, \sqrt{\Theta_n^{[a]}}) \to 0 \), this implies that \( U_n^{[a]}(g; x) \) converge to the function \( g(x) \), while function should be continuous.
4 | WEIGHTED APPROXIMATION

For describing the approximation properties of any sequence of linear positive operators, Gadzhiev [20, 21] developed some fundamental results using the weighted spaces for linear positive operators. Recall from there, and we consider the some functions classes, which are as follows:

\[ B_w[0, \infty) = \{ g : [0, \infty) \rightarrow \mathbb{R} | |g(x)| \leq Mw(x) \}. \]

Also, define the spaces

\[ C_w[0, \infty) = \{ g \in B_w[0, \infty), g \text{ is continuous function} \}. \]

\[ C^k_w[0, \infty) = \{ g \in C_w[0, \infty), \lim_{x \to \infty} \frac{|g(x)|}{w(x)} = k_w < +\infty \}. \]

where \( k_w \) is a constant depending on \( g \) and \( w(x) = 1 + x^2 \) is the weight function.

**Lemma 4.1** ([22]). Let \( L_n : C_w[0, \infty) \rightarrow B_w[0, \infty) \) with the conditions \( \lim_{n \to \infty} ||L_n(t; x) - x^r||_w = 0, r = 0, 1, 2, \) then for \( g \in C^k_w[0, \infty) \), we have

\[ \lim_{n \to \infty} ||L_n(g; x) - g(x)||_w = 0. \]

**Theorem 4.1.** Let \( \{ L_n^{[\alpha]} \} \) be a sequence defined by (1.6) and \( \max \alpha = \frac{1}{n} \) for \( n \in \mathbb{N} \) then it holds as follows:

\[ \lim_{n \to \infty} ||L_n^{[\alpha]}(g; x) - g(x)||_w = 0, \text{ for } g \in C^k_w[0, \infty). \]

**Proof.** If we show that \( \lim_{n \to \infty} ||L_n^{[\alpha]}(t^r; x) - x^r||_w = 0 \) holds for \( r = 0, 1, 2 \) then the above theorem will be proved. Here, it is obvious

\[ \lim_{n \to \infty} ||L_n^{[\alpha]}(1; x) - 1||_w = 0. \] (4.1)

Using Lemma 2.1, we have

\[ ||L_n^{[\alpha]}(t; x) - x||_w = \frac{1}{n} \sup_{n \geq 0} \frac{1}{1 + x^2} \leq \frac{1}{n} \Rightarrow ||L_n^{[\alpha]}(t; x) - x||_w \to 0, \text{ as } n \to \infty. \]

Also,

\[ ||L_n^{[\alpha]}(t^2; x) - x^2||_w = \sup_{n \geq 0} \left[ \frac{2 + 4ax + ax^2 + x^3a}{1 + x^2} - x^2 \right] \]

\[ \leq \frac{1}{n} \left( \frac{2}{n} \sup_{n \geq 0} \frac{1}{1 + x^2} + 5 \sup_{n \geq 0} \frac{x}{1 + x^2} \right) \]

\[ \leq \frac{2}{n^2} + \frac{5}{2n}. \]

\[ \Rightarrow ||L_n^{[\alpha]}(t^2; x) - x^2||_w \to 0 \text{ as } n \to \infty. \]

And hence, the proof is completed. \( \square \)

**Theorem 4.2.** For every \( x \geq 0 \) and \( \alpha = \alpha(n) \to 0 \) as \( n \to \infty \), let \( g \in C^k_w[0, \infty) \) and \( l > 0 \) then we obtain

\[ \lim_{n \to \infty} \sup_{n \geq 0} \frac{|L_n^{[\alpha]}(g; x) - g(x)|}{(1 + x^2)^{1+l}} = 0. \] (4.2)
Proof. Consider $x_0$ be a fixed point, then we can write as

\[
\sup_{x \geq 0} \frac{|L_n^{[a]}(g; x) - g(x)|}{(1 + x^2)^{1+l}} \leq \sup_{x \leq x_0} \frac{|L_n^{[a]}(g; x) - g(x)|}{(1 + x^2)^{1+l}} + \sup_{x > x_0} \frac{|L_n^{[a]}(g; x) - g(x)|}{(1 + x^2)^{1+l}}
\]

\[
\leq \|L_n^{[a]}(g; x) - g(x)\| + \|g\| \sup_{x > x_0} \frac{|L_n^{[a]}((1 + t^2); x)|}{(1 + x^2)^{1+l}} + \frac{|g|}{(1 + x_0^2)^{1+l}}
\]

\[
= L_1 + L_2 + L_3 \text{ (say)}.
\]

Here,

\[
L_3 = \sup_{x > x_0} \frac{|g|}{(1 + x^2)^{1+l}} \leq \frac{\|g\|_w}{(1 + x_0^2)^{l}} \quad \text{(as } |g(x)| \leq M(1 + x^2))
\]

so, for large value of $x_0$, we can consider an arbitrary $c > 0$ such that

\[
L_3 = \frac{\|g\|_w}{(1 + x_0^2)^l} \leq \frac{c}{3}.
\] (4.4)

Since, $\lim_{n \to \infty} \sup_{x > x_0} \frac{|L_n^{[a]}((1 + t^2); x)|}{(1 + x^2)^l} = 1$, so let us consider for any arbitrary $c > 0$, there exists $n_1 \in \mathbb{N}$, such that

\[
L_2 = \|g\|_w \sup_{x > x_0} \frac{|L_n^{[a]}((1 + t^2); x)|}{(1 + x^2)^{1+l}} \leq \frac{\|g\|_w}{(1 + x_0^2)^l} \leq \frac{\|g\|_w}{(1 + x_0^2)^{l}} < \frac{c}{3}.
\] (4.5)

Applying Theorem 2.1, we can have

\[
L_1 = \|L_n^{[a]}(g; x) - g(x)\|_{C[0, x_0]} \leq \frac{c}{3}.
\] (4.6)

Combining (4.4)–(4.6) and using (4.3), we obtain our required result.

**Theorem 4.3.** For $g \in C_w[0, \infty)$, one can obtain

\[
|L_n^{[a]}(g; x) - g(x)| \leq 4N_g(1 + x^2)\Theta_{n,2}^{[a]}(x) + 2\omega_{l+1} \left( g; \sqrt{\Theta_{n,2}^{[a]}} \right),
\]

where $N_g$ is a positive constant depending on $g$.

**Proof.** From Ibikli and Gadjieva [23], for $0 \leq x \leq l$ and $u \geq 0$, it holds

\[
|g(u) - g(x)| \leq 4N_g(1 + x^2)(u - x)^2 + \left( 1 + \frac{|u - x|}{\theta} \right) \omega_{l+1}(g; \theta), \quad \theta > 0.
\]

Now, applying the operators defined by (1.6) and applying the Cauchy–Schwarz inequality, we can obtain

\[
|L_n^{[a]}(g; x) - g(x)| \leq 4N_g(1 + x^2)\Theta_{n,2}^{[a]}(x) + \left( 1 + \frac{\Theta_{n,2}^{[a]}|u - x|}{\theta} \right) \omega_{l+1}(g; \theta)
\]

\[
\leq 4N_g(1 + x^2)\Theta_{n,2}^{[a]}(x) + \left( 1 + \frac{\sqrt{\Theta_{n,2}^{[a]}}}{\theta} \right) \omega_{l+1}(g; \theta)
\]

\[
\leq 4N_g(1 + x^2)\Theta_{n,2}^{[a]}(x) + (1 + 1)\omega_{l+1} \left( g; \sqrt{\Theta_{n,2}^{[a]}} \right)
\]

\[
= 4N_g(1 + x^2)\Theta_{n,2}^{[a]}(x) + 2\omega_{l+1} \left( g; \sqrt{\Theta_{n,2}^{[a]}} \right).
\]

Hence, the proof is completed.
5 | QUANTITATIVE APPROXIMATION

To determine the rate of convergence of any sequence of linear positive operators in the weighted space $C_k^0[0, \infty)$, Ispir [24] proposed the weighted modulus of continuity $\Delta(g; \xi)$ for any $\xi > 0$, as follows:

$$
\Delta(g; \xi) = \sup_{0 \leq h \leq \xi, 0 \leq x \leq \infty} \frac{|g(x + h) - g(x)|}{(1 + h^2)(1 + x^2)}, \quad g \in C_k^0[0, \infty).
$$

(5.1)

**Remark 5.1.** For $g \in C_k^0[0, \infty)$

$$
\lim_{\xi \to 0} \Delta(g; \xi) = 0.
$$

One can obtain as follows: $\Delta(g; \lambda \xi) \leq 2(1 + \xi^2)(1 + \lambda)\Delta(g; \xi)$, $\lambda > 0$. Using the weighted modulus of continuity and defined inequality, we can show the following:

$$
|g(t) - g(x)| \leq (1 + x^2)(1 + (t - x)^2)\Delta(g; |t - x|)
\leq 2 \left(1 + \frac{|t - x|}{\xi}\right)(1 + \xi^2)(1 + (t - x)^2)(1 + x^2)\Delta(g; |t - x|), \quad \text{for every } g \in C_k^0[0, \infty).
$$

(5.2)

As the consequence of the weighted modulus of continuity, we determine the estimation of the proposed operators (1.6) by means of quantitative approximation in the weighted space $C_k^0[0, \infty)$.

5.1 | Quantitative Voronovskaya type theorem

**Theorem 5.1.** Let $g', g'' \in C_k^0[0, \infty)$ and for sufficiently large value of $n \in \mathbb{N}$. Then for each $x \geq 0$, we get

$$
n \left| U_n^{[a]}(g; x) - g(x) - g'(x)\Theta_n^{[a]} - \frac{g''(x)}{2!}\Theta_n^{[a]} \right| = O(1)\Delta \left(g; \sqrt{\frac{1}{n}}\right), \quad \text{as } n \to \infty.
$$

Proof. Through Taylor’s expansion, one can obtain

$$
g(t) - g(x) = g'(x)(t - x) + \frac{g''(x)}{2}(t - x)^2 + \zeta(t, x),
$$

(5.3)

where $\zeta(t, x) = \frac{g''(\theta) - g''(x)}{2!}(\theta - x)^2$ and $\theta \in (t, x)$. Applying operators (1.6) on both sides to the above expansion, then one can obtain

$$
n \left| U_n^{[a]}(g; x) - g(x) - g'(x)\Theta_n^{[a]} - \frac{g''(x)}{2}\Theta_n^{[a]} \right| \leq nU_n^{[a]}(|\eta(t, x)|; x).
$$

(5.4)

Now using the property of weighted modulus of continuity, we get

$$
\frac{g''(\theta) - g''(x)}{2} \leq \left(1 + \frac{|t - x|}{\xi}\right)(1 + \xi^2)(1 + (t - x)^2)(1 + x^2)\Delta(g'', \xi)
$$

and also

$$
\left| \frac{g''(\theta) - g''(x)}{2} \right| \leq \begin{cases} 
2(1 + \xi^2)(1 + x^2)\Delta(g'', \xi), & |t - x| < \xi, \\
2(1 + \xi^2)(1 + x^2)\frac{|t - x|}{\xi^4}\Delta(g'', \xi), & |t - x| \geq \xi.
\end{cases}
$$

(5.5)

Now for $\xi \in (0, 1)$, we get

$$
\left| \frac{g''(\theta) - g''(x)}{2} \right| \leq 8(1 + x^2) \left(1 + \frac{(t - x)^4}{\xi^4}\right)\Delta(g'', \xi).
$$

(5.6)
Hence,
\[
(|\zeta(t,x);x|) \leq 8(1+x^2) \left( (t-x)^2 + \frac{(t-x)^6}{\xi^4} \right) \Delta(g'',\xi).
\]

Thus, applying Lemma 2.3, it holds
\[
U_n^{[a]}(|\zeta(t,x);x|) \leq 8(1+x^2)\Delta(g'',\xi) \left( U_n^{[a]}((t-x)^2;x) + \frac{U_n^{[a]((t-x)^6;x)}}{\xi^4} \right)
\]
\[
\leq 8(1+x^2)\Delta(g'',\xi) \left( O\left(\frac{1}{n}\right) + \frac{1}{\xi^4}O\left(\frac{1}{n^3}\right) \right), \text{ as } n \to \infty.
\]

Choose, \(\xi = \sqrt{\frac{1}{n}}\), then
\[
U_n^{[a]}(|\zeta(t,x);x|) \leq 8O\left(\frac{1}{n}\right) \Delta \left( g'', \sqrt{\frac{1}{n}} \right) (1 + x^2).
\]

So, we have
\[
nU_n^{[a]}(|\zeta(t,x);x|) = O(1)\Delta \left( g'', \sqrt{\frac{1}{n}} \right).
\]

By (5.4) and (5.8), we obtain the required result. \(\square\)

5.2 | Grüss Voronovskaya type theorem

In 1935, Grüss [25] defined an inequality, known as Grüss type inequality after his name. This inequality has a crucial impact on the theory of approximation. The application of Grüss inequality is seen in Acu et al. [26] and some good results can be seen in Gonska and Tachev [27] regarding the Grüss inequality. Moreover, using the Grüss inequality, Gal and Gonska [28] estimated a Voronovskaya type theorem for the Bernstein operators, known as Grüss-Voronovskaya type theorem. So, motivated by the above works, we study the Grüss-Voronovskaya type theorem in the next theorem.

**Theorem 5.2.** Let \(f, g \in C_0^2[0,\infty)\) then for \(f', f'', g', g'' \in C_0^2[0,\infty)\), it holds
\[
\lim_{n \to \infty} n \left( U_n^{[a]}(fg;x) - U_n^{[a]}(f;x)U_n^{[a]}(g;x) \right) = 2xf'(x)g'(x).
\]

**Proof.** By making suitable arrangements and using well-known properties of the derivative of the multiplication of two functions, we get
\[
n \left( U_n^{[a]}(fg;x) - U_n^{[a]}(f;x)U_n^{[a]}(g;x) \right) = n \left\{ \left( U_n^{[a]}(fg;x) - f(x)g(x) - (fg)\Theta_{n,1}^{[a]} \right) \right.
\]
\[
- \frac{(fg)'(x)\Theta_{n,2}^{[a]}}{2!} - f(x) \left( U_n^{[a]}(f;x) - f(x) \right)
\]
\[
- f'(x)g^{[a]}_{n,1} - \frac{f''(x)}{2!}g^{[a]}_{n,2} \right) \right.
\]
\[
- U_n^{[a]}(f;x) \left( U_n^{[a]}(g;x) - g(x) - g'(x)\Theta_{n,1}^{[a]} \right)
\]
\[
- \frac{g''(x)}{2!}g^{[a]}_{n,2} \right) \right.
\]
\[
\times \left( f - U_n^{[a]}(f;x) + f'(x)g'(x)\Theta_{n,2}^{[a]} \right)
\]
\[
+ g'(x)\Theta_{n,1}^{[a]} \left( f - U_n^{[a]}(f;x) \right) \right\}.
\]

For a sufficiently large value of \(n\), i.e., for \(n \to \infty, a \to 0\), with the help of Theorems 2.1 and 5.2 as well as talking the limit on both sides to the above equation, we obtain the required result. \(\square\)
FIGURE 1  The convergence of the operators $L_n^{\alpha}(g; x)$ to the function $g(x)$ (red). [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 2  The convergence of the operators $L_n^{\alpha}(g; x)$ to the function $g(x)$ (red). [Colour figure can be viewed at wileyonlinelibrary.com]

6  |  GRAPHICAL AND NUMERICAL REPRESENTATION

This section consists of the graphical approach and numerical analysis for the convergence of the operators to the function.

Example 6.1. Consider the function is $g(x) = x^2 \sin \pi x$ with $x \in [0, 2]$ and choosing $\alpha = \frac{1}{60}$. Then the corresponding operators for $n = 15, 35$ are $L_{15}^{\alpha}(g; x)$ (green), $L_{35}^{\alpha}(g; x)$ (blue), respectively. One can observe that the better rate of convergence can be obtained by graphical representation 1, which is given below.

Example 6.2. Let us consider the function $g(x) = xe^{-7x}$ (red) for which the rate of convergence of the defined operators (1.6) is discussed by taking different values of $n \in \mathbb{N}$. Choosing $n = 5, 10, 20, 25, 35, 40, 45$, the corresponding operators are represented by blue, green, cyan, brown, yellow, magenta, and purple colors in the given Figure 2. Here we take $\alpha = \frac{1}{45}$.

Example 6.3. For the same function, which has been taken in the above example, we observe by changing the different values of $a$, that is, choosing $\alpha = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{60}, \frac{1}{80}$, for which the corresponding operators for the fixed value of $n = 10$ are represented by black, orange, pink, blue, and green colors in given Figure 3.

Concluding remark:
1. By the above Figures 1 and 2, one can observe that as the value of $n$ is increased, the approximation will improve while taking the particular value of $\alpha$. that is, by the suitable choice of $\alpha$, we can show a better approximation by taking the large value of $n$. But in Figure 3, convergence can be seen when the value of $n$ is fixed, and the value of $\alpha$ is decreased.
2. On choosing appropriate values of $\alpha$ and $n$, we can find the better approximation.
The convergence of the operators $L_n^{[\alpha]}(g; x)$ to the function $g(x)$ (red). [Colour figure can be viewed at wileyonlinelibrary.com]

**Example 6.4.** Let the function $g = (x^2 + 1)e^x$, take $n = 5, 10, 20, 25, 30, 40, 50, 70, 90, 130, 150, 190, 240, 250, 400, 500$

and $\alpha = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \frac{1}{50}, \frac{1}{100}, \frac{1}{150}, \frac{1}{200}, \frac{1}{250}, \frac{1}{500}$ then we obtain the approximation by given table.

**Observation:** By the above Table 1, we can observe that as the value of $\alpha$ decreases, the error is minimal for a particular value of $n$. And the same time, if we see the table, we can observe that on increasing the values of $n$, the errors are decreased for a particular value of $\alpha$ (excluding the “dash-”). This process is running on a specific point of $x$.

**7 A-STATISTICAL CONVERGENCE OF THE DEFINED OPERATORS**

This section contains the statistical convergence theorem. We establish some approximation properties to study statistical convergence. Fast [29] first introduced the basic idea of statistical convergence even though the first publication related to statistical convergence was in 1935 (published in Warsaw), and credit goes to Zygmund in his monograph; also, independently, work is seen in the research of Steinhaus [30] in 1951. It can be seen that Schoenberg [31] reintroduced statistical convergence, and nowadays, it has become an area of active research in approximation theory.

In 2002, the application of statistical convergence in approximation theory was seen by Gadjiev [32]. We recall the symbol from Gadjiev [21], as $A_{m} = \{a_{ni}\}$ is an infinite nonnegative infinite summability matrix. Here we denote $A$-transform.
of matrix $A_{n}$ for a given sequence $\{x_i\}$ by

$$A_{ni}x_i = \sum_{i=0}^{\infty} a_{ni}x_i,$$

provided $(A_{ni}x_i)$ converges for each $n \in \mathbb{N}$. Now if $\lim_{n \to \infty}(A_{ni}x_i) = \sigma$ whenever $\lim x_i = \sigma$ [33] then $A_{ni}$ is said to be regular and also, $\lim a_{ni} = 0, \forall i \in \mathbb{N}$. In this case, $(x_i)$ is said to be $A$-statistically convergent, that is, for every $\epsilon > 0$, $\lim \sum_{i:||x_i - \sigma|| > \epsilon ||a_{ni}|| = 0$, and it is written as $st_A - \lim x_i = 0$. For more information, we refer the reader to see earlier studies [34, 35].

**Theorem 7.1.** Consider $A_{ni} = \{a_{ni}\}$ be nonnegative regular summability matrix and for each $g \in C_0^\infty([0, \infty))$ then for every $x \in [0, \infty)$, we have

$$st_A - \lim_{n \to \infty} \|L_n^{[\sigma]}(g; x) - g(x)\|_{w(x)} = 0.$$

**Proof.** If we show

$$st_A - \lim_{n \to \infty} \|L_n^{[\sigma]}(t', x) - x\|_{w(x)} = 0 \quad \text{for } r = 0, 1, 2, \quad (7.1)$$

then we obtain the required result.

It is clear that

$$st_A - \lim_{n \to \infty} \|L_n^{[\sigma]}(1; x) - 1\|_{w(x)} = 0.$$

Also,

$$\|L_n^{[\sigma]}(t; x) - x\|_{w(x)} = \sup_{x \geq 0} \frac{1}{n} \leq \frac{1}{n}.$$

So we define the following sets for given $\epsilon > 0$, as

$$\mathcal{V}_1 = \{ n : \|L_n^{[\sigma]}(t; x) - x\| \geq \epsilon \} \mathcal{V}_2 = \left\{ n : \frac{1}{n} \geq \epsilon \right\}.$$

Obviously, $\mathcal{V}_1 \subset \mathcal{V}_2$ and hence, $\sum_{i \in \mathcal{V}_1} a_{ni} \leq \sum_{i \in \mathcal{V}_2} a_{ni}$. Therefore,

$$st_A - \lim_{n \to \infty} \|L_n^{[\sigma]}(t; x) - x\|_{w(x)} = 0.$$

Further,

$$\|L_n^{[\sigma]}(t^2; x) - x^2\|_{w(x)} = \sup_{x \geq 0} \frac{1}{1 + x^2} \left( \frac{2 + 4nx + n^2x^2 + n^2x\sigma}{n^2} - x^2 \right) \leq \left( \frac{2}{n^2} + \frac{5}{2n} \right).$$

Since the R.H.S. of the above inequality tends to zero as $n \to \infty$. So for given $\epsilon > 0$, we can consider the following sets, which is shown as follows:

$$\mathcal{W}_1 = \{ n : \|L_n^{[\sigma]}(t^2; x) - x^2\| \geq \epsilon \}$$

$$\mathcal{W}_2 = \left\{ n : \frac{2}{n^2} \geq \frac{\epsilon}{2} \right\}$$

$$\mathcal{W}_3 = \left\{ n : \frac{5}{2n} \geq \frac{\epsilon}{2} \right\},$$

which implies that $\mathcal{W}_1 \subset \mathcal{W}_2 \cup \mathcal{W}_3$, and hence, $\sum_{i \in \mathcal{W}_1} a_{ni} \leq \sum_{i \in \mathcal{W}_2} a_{ni} + \sum_{i \in \mathcal{W}_3} a_{ni}$. Therefore,

$$st_A - \lim_{n \to \infty} \|L_n^{[\sigma]}(t^2; x) - x^2\|_{w(x)} = 0.$$

Thus, the proof is completed. □
Corollary 1. Let \( w_\gamma(x) \geq 1 \) be a continuous function such that \( \lim_{|x| \to \infty} \frac{w_\gamma(x)}{|x|} = 0 \) then for each \( g \in C^0_\beta(0, \infty) \) with \( \{a_{ni}\} \) be nonnegative regular summability matrix, we have

\[
st_A \left\| U_\gamma^{[\alpha]}(t^2; x) - x^2 \right\|_{w_\gamma(x)} = 0.
\]

By means of Peetre’s \( K \)-functional, \( A \)-Statistical convergence is introduced for the operators \( U_\gamma^{[\alpha]} \) in the next theorem.

Theorem 7.2. Let \( g \in C^0_B(0, \infty) \) and for every \( x \in [0, \infty) \), we have

\[
st_A \lim_{n \to \infty} \left\| U_\gamma^{[\alpha]}(g; x) - g \right\|_{C_\beta(0, \infty)} = 0, \forall n \in \mathbb{N}.
\]

Proof. Using the expansion of Taylor, we have

\[
g(t) = g(x) + g'(x)(t - x) + \frac{1}{2} g''(\tau)(t - x)^2,
\]

where \( \tau \in [t, x] \). Now applying \( U_\gamma^{[\alpha]} \), we obtain

\[
U_\gamma^{[\alpha]}(g(t) - g(x); x) = g'(x)\Theta^{[\alpha]}_{n,1}(x) + \frac{g''(\tau)}{2} \Theta^{[\alpha]}_{n,2}(x).
\]

In this way,

\[
\left\| U_\gamma^{[\alpha]}(g(t) - g(x); x) \right\|_{C_\beta(0, \infty)} \leq \left\| g'(x) \right\|_{C_\beta(0, \infty)} \left\| \Theta^{[\alpha]}_{n,1}(x) \right\|_{C_\beta(0, \infty)} + \left\| g''(\tau) \right\|_{C_\beta(0, \infty)} \left\| \Theta^{[\alpha]}_{n,2}(x) \right\|_{C_\beta(0, \infty)} = E_1 + E_2, \text{ (say)},
\]

that is,

\[
\left\| U_\gamma^{[\alpha]}(g(t) - g(x); x) \right\|_{C_\beta(0, \infty)} \leq E_1 + E_2.
\]

From (7.1), we can write

\[
\lim_{n \to \infty} \sum_{\{i \in \mathbb{N} : \varepsilon_i \geq \varepsilon_2 \}} a_{ni} = 0,
\]

\[
\lim_{n \to \infty} \sum_{\{i \in \mathbb{N} : \varepsilon_i \geq \varepsilon_2 \}} a_{ni} = 0.
\]

So by Equation (7.2), we get

\[
\lim_{n \to \infty} \sum_{\{i \in \mathbb{N} : \| U_\gamma^{[\alpha]}(f(t) - f(x); x) \|_{C_\beta(0, \infty)} \geq \varepsilon_1 \}} a_{ni} \leq \lim_{n \to \infty} \sum_{\{i \in \mathbb{N} : \varepsilon_i \geq \varepsilon_2 \}} a_{ni} + \lim_{n \to \infty} \sum_{\{i \in \mathbb{N} : \varepsilon_i \geq \varepsilon_2 \}} a_{ni}.
\]

Thus,

\[
st_A \lim_{n \to \infty} \left\| U_\gamma^{[\alpha]}(g; x) - g \right\| = 0.
\]

Hence, it is proved.

Theorem 7.3. Consider \( g \in C_\beta(0, \infty) \) and for each \( n \in \mathbb{N} \), an inequality holds

\[
\left\| U_\gamma^{[\alpha]}(g(t) - g(x); x) \right\|_{C_\beta(0, \infty)} \leq C \omega_2(g; \sqrt{\varepsilon}).
\]
Proof. Consider a function $h \in C^2_b(0, \infty)$. We can write

$$\|U_n^{[\alpha]}(h(t) - h(x); x)\|_{C^1[0, \infty)} \leq \|h'\|_{C^1[0, \infty)} \|U_n^{[\alpha]}((t-x); x)\|_{C^0[0, \infty)} + \frac{1}{2} \|h''\|_{C^0[0, \infty)} \|\Theta_n^{[\alpha]}\|_{C^0[0, \infty)} \leq \eta \|h\|_{C^2_b(0, \infty)}.$$

Since, $g \in C^1_b[0, \infty)$ and $h \in C^2_b[0, \infty)$, therefore, by above inequality, one can write

$$\|U_n^{[\alpha]}(g(t) - g(x); x)\|_{C^1[0, \infty)} \leq \|U_n^{[\alpha]}(g; x) - U_n^{[\alpha]}(h; x)\|_{C^1[0, \infty)} + \|U_n^{[\alpha]}(h(t) - h(x); x)\|_{C^0[0, \infty)} \|h - g\|_{C^1[0, \infty)}$$

$$\leq 2\|h - g\|_{C^1[0, \infty)} + \|U_n^{[\alpha]}(h(t) - h(x); x)\|_{C^0[0, \infty)} \leq 2\|h - g\|_{C^0[0, \infty)} + \eta \|h\|_{C^2_b[0, \infty)}.$$

Using the property of Peetre’s $K$-functional (3.1), one has

$$\|U_n^{[\alpha]}(g(t) - g(x); x)\|_{C^1[0, \infty)} \leq C \{\omega_2(g; \sqrt{\eta}) + \min(1, \eta) \|g\|_{C^1[0, \infty)} \}.$$

Using (7.1), we obtain

$$\eta \to 0$$

statistically as $n \to \infty$.

In this way,

$$\omega_2(g; \sqrt{\eta}) \to 0$$

statistically as $n \to \infty$.

Hence, the rate of convergence $\eta$-statistically is obtained of the sequence of linear positive operators defined by 1.6 to the function $g(x)$.

Remark 7.1. If $A_n = I$, then the ordinary rate of convergence is obtained.

8 | RATE OF CONVERGENCE BY MEANS OF THE FUNCTION OF BOUNDED VARIATION

In this section, the said operators’ convergence rate is determined in the space of the functions with the derivative of bounded variation. Here, we consider $DBV[0, \infty)$, the set of all continuous functions having derivative of bounded variation on every finite subinterval of the $[0, \infty)$. On observing that for each $g \in DBV[0, \infty)$, it is yielded as

$$g(x) = \int_0^x h(s) \, ds + g(0),$$

where $h$ is a function of bounded variation on each finite subinterval of $[0, \infty)$. Here, we use an auxiliary operator $g_\varepsilon$ for every $g \in DBV[0, \infty)$ to get the rate of convergence of the proposed operators, which is defined as follows:

$$g_\varepsilon(t) = \begin{cases} g(t) - g(x_\varepsilon), & 0 \leq t < x, \\ 0, & t = x, \\ g(t) - g(x_\varepsilon), & x < t < \infty. \end{cases}$$

Generally, we denote $V^b_{a,b}g$ is the total variation of a real valued function $g$ defined on $[a, b] \subset [0, \infty)$ with the expression

$$V^b_{a,b}g = \sup_P \left( \sum_{k=0}^{n-1} |g(x_{k+1}) - g(x_k)| \right),$$

where $P$ is the set of all partition $P = \{a = x_0, \ldots, x_{n_p} = b\}$ of the interval $[a, b]$. 

Lemma 8.1. For sufficiently large value of $n$, for every $x \geq 0$ then there exists a positive constant $M > 0$ such that

1. $h_n^{[a]}(x, y) = \int_0^y u_n^{[a]}(x, t) \, dt \leq \frac{3\eta_n^2(x)}{n(x-y)^2}$, $0 \leq y < x$,

2. $1 - h_n^{[a]}(x, z) = \int_z^\infty u_n^{[a]}(x, t) \, dt \leq \frac{3\eta_n^2(x)}{n(z-x)^2}$, $x < z < \infty$.

Proof. For $y \in [0, x)$, it holds

$$\int_0^y u_n^{[a]}(x, t) \, dt \leq \frac{1}{(x-y)^2} L_n^{[a]}((t-x)^2; x) \leq \frac{3\eta_n^2(x)}{n(x-y)^2}.$$ 

Similarly, other result can be proved.

Theorem 8.1. Let $g \in DBV(0, \infty)$, for sufficiently large value of $n$ and $\max a = \frac{1}{n}$. If $g(t) = O(t^3)$ as $t \to \infty$. Then for $x \in [0, \infty)$, we obtain

$$|L_n^{[a]}(g; x) - g(x)| \leq \frac{3\eta_n^2(x)}{nx^2} \left| \frac{g(2x) - g(x) - x g'(x+))}{\sqrt{n}} \right| + \frac{x V_n^{[a]}(g')}{\sqrt{n}} + \frac{3\eta_n^2(x)}{n} \sum_{l=1}^{\lfloor \sqrt{n} \rfloor} V_n^{[a]}(g'_l) + M_{n,x},$$

$$\quad + \frac{3}{n} \eta_n(x) |g'(x)| + \sqrt{n} \eta_n(x) |g'(x)| + \frac{3}{n} \eta_n(x) |g'(x) - g'(x-x)| \eta_n(x) + \frac{|g'(x) - g'(x-x)|}{2n},$$

where

$$M_{n,x} = M 2^\gamma \left( \int_0^\infty (t-x)^{2a} u_n^{[a]}(x, t) \, dt \right)^{\frac{1}{2}}.$$

Proof. We have

$$|L_n^{[a]}(g; x) - g(x)| = \int_0^\infty u_n^{[a]}(x, t) (g(t) - g(x)) \, dt = \int_0^\infty u_n^{[a]}(x, t) \left( \int_0^t g'(u) \, du \right).$$

Since $g \in DBV[0, \infty)$, one can write an identity

$$g'(v) = \frac{1}{2} \left[ g'(x+ +) - g'(x-) \right] + g'_x(v) + \frac{1}{2} \left[ g'(x+ +) - g'(x- -) \right] \text{sgn}(v-x)$$

$$\quad + \xi_x(v) \left( g'(v) - \frac{1}{2} \left( g'(x+) + g'(x-) \right) \right),$$

where

$$\xi_x(v) = \begin{cases} 1 & v = x \\ 0 & v \neq x. \end{cases}$$

(8.4)

(8.5)

Now, one can easily obtain as

$$\int_0^\infty u_n^{[a]}(x, t) \left( \int_0^t \left( \xi_x(v) \left( g'(v) - \frac{1}{2} \left( g'(x+) + g'(x-) \right) \right) dv \right) \right) \, dt = 0;$$

also,

$$\int_0^\infty u_n^{[a]}(x, t) \left( \int_0^t \frac{1}{2} \left( g'(x+) - g'(x-) \right) \text{sgn}(v-x) \, dv \right) \, dt \leq \frac{1}{2} \left[ g'(x+) - g'(x-) \right] L_n^{[a]}[(t-x)^2; x]$$

$$\quad \leq \frac{1}{2} \left[ g'(x+) - g'(x-) \right] \left( L_n^{[a]}((t-x)^2; x) \right)^{\frac{1}{2}},$$

(8.6)
and
\[
\int_0^\infty u_n^{[a]}(x, t) \left( \int_0^t \frac{1}{2} \{ g'(x^+) - g'(x^-) \} dv \right) \, dt = \frac{1}{2} \{ g'(x^+) - g'(x^-) \} U_n^{[a]}((t - x); x). 
\] (8.7)

So, we have
\[
| U_n^{[a]}(g; x) - g(x) | \leq \left| \int_0^x \left( \int_0^t g'_v(v) \, dv \right) u_n^{[a]}(x, t) \, dt + \int_x^\infty \left( \int_0^t g'_v(v) \, dv \right) u_n^{[a]}(x, t) \, dt \right|
\]
\[+ \frac{1}{2} \{ g'(x^+) - g'(x^-) \} \left( \frac{1}{2} \{ U_n^{[a]}((t - x)^2); x \} \right)
\]
\[+ \frac{1}{2} \{ g'(x^+) - g'(x^-) \} U_n^{[a]}((t - x); x)
\]
\[\leq E_{nx} + F_{nx} + \frac{1}{2} \sqrt{n} \{ g'(x^+) - g'(x^-) \} \eta_n(x) + \frac{1}{2n} \{ g'(x^+) - g'(x^-) \}, \]

where
\[
E_{nx} = \left| \int_0^x \left( \int_0^t g'_v(v) \, dv \right) u_n^{[a]}(x, t) \, dt \right|
\]
and
\[
F_{nx} = \left| \int_x^\infty \left( \int_0^t g'_v(v) \, dv \right) u_n^{[a]}(x, t) \, dt \right|. \] (8.9)

Now applying Lemma 8.1, and let \( x = \frac{r \sqrt{n}}{\sqrt{n-1}} \), then integrating by part of
\[
E_{nx} = \left| \int_0^x \left( \int_0^t g'_v(v) \, dv \right) u_n^{[a]}(x, t) \, dt \right|
\]
\[= \left| \int_0^x h_n^{[a]}(x, t) g'_v(t) \, dt \right|
\]
\[\leq \int_0^x | h_n^{[a]}(x, t) | | g'_v(t) | \, dt + \int_x^\infty | h_n^{[a]}(x, t) | | g'_v(t) | \, dt
\]
\[= \int_0^{x - \frac{r}{\sqrt{n}}} h_n^{[a]}(x, t) g'_v(t) \, dt + \int_x^{x - \frac{r}{\sqrt{n}}} h_n^{[a]}(x, t) g'_v(t) \, dt.
\]

With the help of (8.2) and using Lemma 8.1, we get
\[
\int_{x - \frac{r}{\sqrt{n}}}^x h_n^{[a]}(x, t) | g'_v(t) | \, dt \leq \int_{x - \frac{r}{\sqrt{n}}}^x | g'_v(t) - g'_v(x) | \, dt \leq \frac{x}{\sqrt{n}} V_{x - \frac{r}{\sqrt{n}}}, \] (8.10)

Also, let \( v = 1 + \frac{1}{x - t} \) and using Lemma 8.1, we obtain
\[
\int_0^{x - \frac{r}{\sqrt{n}}} h_n^{[a]}(x, t) | g'_v(t) | \, dt = \frac{3n^2 \eta_2(x)}{n} \int_0^{x - \frac{r}{\sqrt{n}}} \frac{| g'_v(t) |}{(x - t)^2} \, dt = \frac{3n^2 \eta_2(x)}{n} \int_1^{\sqrt{n}} V_{x - \frac{r}{\sqrt{n}}} (g'_v) \, dv \leq \frac{3n^2 \eta_2(x)}{n} \sum_{i=1}^{\sqrt{n}} V_{x - \frac{r}{\sqrt{n}}} (g'_v). \] (8.11)

Hence,
\[
E_{nx} \leq \frac{x}{\sqrt{n}} V_{x - \frac{r}{\sqrt{n}}} + \frac{3n^2 \eta_2(x)}{n} \sum_{i=1}^{\sqrt{n}} V_{x - \frac{r}{\sqrt{n}}} (f'_v). \] (8.12)
Now, another part can be written as

\[
F_{nx} = \left| \int_{x}^{\infty} \left( \int_{0}^{t} g_{n}(v) \, dv \right) u_{n}^{[a]}(x, t) \, dt \right|
\]

\[
\leq \int_{x}^{2x} \left( \int_{0}^{t} g_{n}(v) \, dv \right) dt \left( 1 - h_{n}^{[a]}(x, t) \right) + \int_{x}^{\infty} \left( \int_{0}^{1} g_{n}(v) \, dv \right) u_{n}^{[a]}(x, t) \, dt \right|
\]

\[
= \int_{x}^{2x} g_{n}(x) \, dv \left( 1 - h_{n}^{[a]}(x, 2x) \right) - \int_{x}^{2x} g_{n}(t) \left( 1 - h_{n}^{[a]}(x, t) \right) \, dt \right|
\]

\[
+ \int_{2x}^{\infty} \left( \int_{0}^{t} (g_{n}(v) - g_{n}(x+)) \, dv \right) u_{n}^{[a]}(x, t) \, dt \right|
\]

\[
\leq \int_{x}^{2x} (g_{n}(v) - g_{n}(x+)) \, dv \left( 1 - h_{n}^{[a]}(x, 2x) \right) + \int_{x}^{2x} g_{n}(t) \left( 1 - h_{n}^{[a]}(x, t) \right) \, dt \right|
\]

\[
+ \int_{2x}^{\infty} \left( |g(t) - g(x)| \right) u_{n}^{[a]}(x, t) \, dt + \int_{2x}^{\infty} \left( |g(x+)| \right) u_{n}^{[a]}(x, t) \, dt \right|
\]

\[
\leq \frac{3\eta_{n}^{2}(x)}{nx^{2}} \left| (g(2x) - g(x) - xg_{n}(x+)) \right| + \int_{x}^{\frac{2x}{n}} |g(t)| \left| (1 - h_{n}^{[a]}(x, t)) \right| \, dt
\]

\[
+ M \int_{x}^{2x} t^{\frac{1}{2}} u_{n}^{[a]}(x, t) \, dt + |g(x)| \int_{x}^{2x} u_{n}^{[a]}(x, t) \, dt + |g(x+)| \left( \int_{0}^{x} u_{n}^{[a]}(x, t)(t-x)^{2} \, dt \right)^{\frac{1}{2}} \times \left( \int_{0}^{u_{n}^{[a]}(x, t)} \, dt \right)^{\frac{1}{2}}
\]

\[
\leq \frac{3\eta_{n}^{2}(x)}{nx^{2}} \left| (g(2x) - g(x) - xg_{n}(x+)) \right| + \frac{x}{\sqrt{n}} V_{x}^{f_{n}}(g_{x}) + \frac{3\eta_{n}^{2}(x)}{n} \int_{x}^{2x} \frac{V_{x}^{f_{n}}(g_{x})}{(x-t)^{2}} \, dt
\]

\[
+ M \int_{x}^{2x} t^{\frac{1}{2}} u_{n}^{[a]}(x, t) \, dt + |f(x)| \int_{x}^{2x} u_{n}^{[a]}(x, t) \, dt + |f(x+)| \sqrt{\frac{3}{n}} \eta_{n}(x)
\]

\[
\leq \frac{3\eta_{n}^{2}(x)}{nx^{2}} \left| (g(2x) - g(x) - xg_{n}(x+)) \right| + \frac{x}{\sqrt{n}} V_{x}^{f_{n}}(g_{x}) + \frac{3\eta_{n}^{2}(x)}{n} \sum_{i=1}^{\frac{\sqrt{n}}{x}} V_{x}^{x^{2}}(g_{x})
\]

\[
+ M \int_{x}^{2x} t^{\frac{1}{2}} u_{n}^{[a]}(x, t) \, dt + |g(x)| \int_{x}^{2x} u_{n}^{[a]}(x, t) \, dt + |g(x+)| \sqrt{\frac{3}{n}} \eta_{n}(x)
\]

Here, it is arising a case \( t \leq 2(t-x) \) and \( x \leq t-x \), when \( t \geq 2x \), then by using Hölder inequality, we can obtain

\[
F_{nx} \leq \frac{3\eta_{n}^{2}(x)}{nx^{2}} \left| (g(2x) - g(x) - xg_{n}(x+)) \right| + \frac{x}{\sqrt{n}} V_{x}^{f_{n}}(g_{x}) + \frac{3\eta_{n}^{2}(x)}{n} \sum_{i=1}^{\frac{\sqrt{n}}{x}} V_{x}^{x^{2}}(g_{x})
\]

\[
+ M 2^{\left( \int_{0}^{x} (t-x)^{2} u_{n}^{[a]}(x, t) \, dt \right)^{\frac{1}{2}}} + \left| \frac{3|g(x)| \eta_{n}(x)}{nx^{2}} \right| + |g(x+)| \sqrt{\frac{3}{n}} \eta_{n}(x)
\]

\[
= \frac{3\eta_{n}^{2}(x)}{nx^{2}} \left| (g(2x) - g(x) - xg_{n}(x+)) \right| + \frac{x}{\sqrt{n}} V_{x}^{f_{n}}(g_{x}) + \frac{3\eta_{n}^{2}(x)}{n} \sum_{i=1}^{\frac{\sqrt{n}}{x}} V_{x}^{x^{2}}(g_{x}) + M \int_{x}^{2x} t^{\frac{1}{2}} u_{n}^{[a]}(x, t) \, dt + |g(x+)| \sqrt{\frac{3}{n}} \eta_{n}(x)
\]

Using the values of \( E_{nx} \) and \( F_{nx} \) in Equation (8.8), we get our required result.
9 | CONCLUSIONS AND RESULT DISCUSSION

We have discussed properties that define the order of approximation in terms of modulus of continuity using modified Lipschitz-type space. The Steklov function is one of the most helpful functions the present article deals with. Some approximation results that show the rate of the convergence of the defined operators have been discussed in the weighted spaces, which enrich the quality of our works. The quantitative approximation has been studied, and the asymptotic behavior of the operators, Grüss Voronovskaya type theorem, has also been discussed quantitatively. We gave examples in support of our theoretical findings. An important property, the statistical convergence of the operators, has been established, and the statistical rate of convergence obtained, in addition to the rate of convergence, is determined in terms of the function with a derivative of bounded variation. This research article is beneficial for researchers working in the field of mathematical analysis. The literature review discusses various topics such as Mathematical Analysis, Applied Mathematics, and Quantum Calculus. Moreover, these operators can be studied in Fractal calculus using \( q \)-analogue and \((p, q)\)-analogue of the proposed operators, and thus this theory can be implemented into Chaos theory. It can be pretty practical for the convergence rate using \( q \)-integer in quantum calculus. Significant generalization is possible with complex domains. The induced operators can be generalized by considering the hypergeometric function.

AUTHOR CONTRIBUTIONS

Rishikesh Yadav: Conceptualization, investigation, writing—original draft, methodology, visualization, writing—review and editing, formal analysis, validation. Vishnu Narayan Mishra: Supervision. Ramakanta Meher: Supervision.

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