Measuring communication complexity using instance complexity with oracles

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Abstract—We establish a connection between non-deterministic communication complexity and instance complexity, a measure of information based on algorithmic entropy. Let $\pi$, $\gamma$ and $Y_1(\pi)$ be respectively the input known by Alice, the input known by Bob, and the set of all values of $y$ such that $f(\pi, y) = 1$; a string is a witness of the non-deterministic communication protocol iff it is a program $p$ that “corresponds exactly” to the instance complexity $ic_{f,t}(\gamma : Y_1(\pi))$.

I. INTRODUCTION

In a general scenario of communication complexity there are two parties, Alice and Bob, and the goal is to find the minimal quantity of information, measured in number of bits, that they must exchange in order to compute the value of a given function of their inputs, $f : X \times Y \rightarrow \{0, 1\}$. The instance complexity $ic_{f,t}(x : A)$, is a rigorous measure of information, based on algorithmic entropy, is the length of the shortest program with access to the oracle $O$ that, in time $t$,

1) answers correctly the question “$x \in A$?";
2) does not “lie” about the set $A$ (the program may however answer “I don’t know” by outputting ⊥).

Thus, the communication complexity measures the communication costs while instance complexity is related with computational complexity. The objective of this paper is to establish a relationship between these two apparently unrelated measures of complexity.

Let $\pi$ and $\gamma$ be the inputs of size $n$ of Alice and Bob respectively. Consider $Y_1(\pi)$, the set of all possible inputs $y$ given to Bob such that $f(\pi, y) = 1$. We prove that, apart from a constant, $\max_{|x|=|y|=n} \{|c_{f,t}^{ic} (\gamma : Y_1(\pi))\}$, where $ic_{f,t}$ is a “onesided” version of instance complexity is equal to the non-deterministic communication complexity $N^{f}(f)$; as a consequence of this result the maximum value of $ic_{f,t}^{ic} (\gamma : Y_1(\pi))$ over all inputs $(\pi, \gamma)$ equals the non-deterministic communication complexity $N(f)$. The main ingredient for the proof of this result is a protocol in which Alice uses the non-deterministic word $p$ as a program that eventually corresponds to $ic_{f,t}^{ic} (\gamma : Y_1(\pi))$. It is important to notice that neither Alice nor Bob alone (i.e., without communication and without the help of the oracle $f$) can compute $ic_{f,t}^{ic} (\gamma : Y_1(\pi))$; the reason is that Alice only knows $\pi$ and Bob only knows $\gamma$.

We mention two previous works where the communication complexity has been analyzed in a non-standard way: the paper [2] on individual communication complexity in which Kolmogorov complexity is used as the main analysis tool and [5] where “distinguishers” are used to obtain bounds on communication complexity.

Our results use a bounded resource version of instance complexity with access to an oracle. Notice that, in the communication complexity scenario, the time of the computations performed by each party is irrelevant. The program $p$ used as a guess must have access to the description of $f$; however, the description of non-uniform functions $f$, which in general is infinite, can not be incorporated into a, necessarily finite, program $p$. Our solution to this problem is based on an oracle which, for each size $n$, gives to $p$ a description of $f$ restricted to inputs $x$ and $y$ of length $n$ (which is of course finite). We will show that the program $p$ used as a guess must have access to the description of $f$ and so, if $f$ is not uniform, and $p$ does not have access to its description for free, then $p$ would have to build in the description of $f$, which is only possible if its length is unlimited.

The rest of the paper is organized as follows. The next section contains some background and notation on communication complexity and instance complexity. In Section III we study the one-sided protocols and in Section IV we focus on two sided protocols. These two sections contain the main results of this paper, namely Theorems III.1 and IV.1. Section V contains some comments on the relationship between individual communication complexity and instance complexity.

II. PRELIMINARIES

In the rest of this work, $\mathbb{N}$ denotes the set of natural numbers (including 0). The alphabet that we will be using is $\{0, 1\}$. A word is this alphabet is a sequence (possibly empty) of 0’s and 1’s and will be denoted by $x$, and 1’s and will be denoted by $x$ and $y$, possibly overlined. The length and the $i$-th bit of $x$ are denoted by $|x|$ and $x_i$ respectively.

A. Communication complexity

We introduce the basic concepts of communication complexity. For more detailed information see, for example, [4]. Let $f : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}$ be a boolean function. Alice and Bob want to determine the value $f(\pi, \gamma)$ where $\pi$ is only known by Alice and $\gamma$ is only known by Bob. To achieve the goal is imperative that Alice and Bob communicate. Non deterministic protocols $P$ for $f$ involve the usage of a “guess”

1This fact improves the results proved on a previous version of this paper presented at CiE 2007.
which is given to Alice and Bob. These protocols $P$ satisfy, for $z = 0$ or 1 the following conditions:

$$\begin{align*}
[f(x, y) = z] &\Rightarrow [\exists w : P(w, x, y) = z] \quad (I.1) \\
[f(x, y) \neq z] &\Rightarrow [\forall w : P(w, x, y) \neq z] \quad (I.2)
\end{align*}$$

For $z \in \{0, 1\}$ a “one-sided” protocol $P^z$ has output either $z$ or ⊥ and satisfies

$$\begin{align*}
[f(x, y) = z] &\Rightarrow [\exists w : P^z(w, x, y) = z] \quad (I.3) \\
[f(x, y) \neq z] &\Rightarrow [\forall w : P^z(w, x, y) = ⊥] \quad (I.4)
\end{align*}$$

It is easy to build a non-deterministic protocol for $f$ using the one-sided protocols $P^0$ and $P^1$.

It is important to notice that at the end of any protocol, Alice and Bob must be convinced about the veracity of the value produced, in the sense that “false guesses” must be detected and rejected (output ⊥). This requirement corresponds to the “∨ . . .” predicates above. In other words, this means that Alice and Bob do not trust the oracle. Notice that if both Alice and Bob trusted the oracle the problem would be trivially solved by sending the bit corresponding to the value of the function on their input.

**Definition II.1** (non-deterministic communication complexities). Standard and individual (non-deterministic) communication complexities are denoted by $N$ and $N$ respectively.

- Individual communication complexity of protocol $P$ with output set $\{1, ⊥\}$: $N^1_P(f, x, y) = \min_w |c(w)| : P(w, x, y) = 1$ where $w$ is the guess and $c(w)$ (“conversation”) is the sequence of bits exchanged between Alice and Bob when the guess is $w$. Notice that $N^1_P(f, x, y)$ is only defined if $f(x, y) = 1$. Notice also that the behavior of the protocol $P$ for the other inputs $(x, y)$ is irrelevant.

- Individual communication complexity with output set $\{1, ⊥\}$: $N^1_P(f, x, y) = \min_P \{N^1_P(f, x, y)\}$ where the protocols $P$ considered for minimization are one-sided protocols with output set $\{1, ⊥\}$ for the function $f$.

- Communication complexity of protocol $P$ with output set $\{1, ⊥\}$: $N^1_P(f) = \max_{f, x, y} \{N^1_P(f, x, y)\}$.

- Communication complexity of function $f$ with output set $\{1, ⊥\}$: $N^1(f) = \min_P \{N^1_P(f)\}$.

The complexities $N^0_P(f, x, y)$, $N^1_P(f)$, and $N^0(f)$, are defined in a similar way.

Define also $N_P(f, x, y) = N^1_P(f, x, y)$ if $f(x, y) = 0$ and $N_P(f, x, y) = N^1_P(f, x, y)$ if $f(x, y) = 1$; $N_P(f) = \log(2N^0_P(f) + 2N^1_P(f))$; $N(f) = \min_P \{N_P(f)\}$.

A witness is a guess that causes the protocol to output a value different from ⊥.

The following result from [4] proves that for every function there is a simple optimal non-deterministic protocol.

**Theorem II.2.** For every boolean function $f$ there is an optimal one-sided non-deterministic protocol $P$ for $f$, that is, a protocol $P$ such that $N^1_P(f) = N^1(f)$, with the following form where the witness $w$, $1 \leq w \leq m$, is the index of the first rectangle $R_w = A \times B$ containing $(x, y)$ in the first minimum $1$-cover:

1. Alice guesses $w$ and checks if $\exists x \in A$.
2. Alice sends $w$ to Bob.
3. Bob checks if $\exists y \in B$.

Define the sets:

$$\begin{align*}
X_0(\overline{x}) &= \{x : f(x, \overline{y}) = 0\}, \quad X_1(\overline{x}) = \{x : f(x, \overline{y}) = 1\}, \\
Y_0(\overline{y}) &= \{y : f(\overline{x}, y) = 0\}, \quad Y_1(\overline{y}) = \{y : f(\overline{x}, y) = 1\}.
\end{align*}$$

Notice that Alice knows $Y_0(\overline{x})$ and $Y_1(\overline{x})$ while Bob knows $X_0(\overline{y})$ and $X_1(\overline{y})$. The set $Y_1$ is often mentioned in this paper.

**Definition II.3.** A function is uniform if it is computed by a fixed (independent of the length of the input) algorithm.

Every function that can be described by an algorithm is uniform; for instance equality and parity are uniform functions. An example of a function which with almost certainty is not uniform is the random function defined as $f(x, y) = 0$ or $f(x, y) = 1$ with probability 1/2. Notice that in the case that $f$ is uniform we can build in the program $p$ a description of $f$ with a small cost (a constant number of bits) in the length of program. On the other hand, if $f$ is not uniform, then the description of $f$ is no longer a constant. To avoid programs of high length for non-uniform functions we allow the program to have oracle access to the description of $f$.

**B. Instance complexity**

Instance complexity is a rigorous measure of information of a string relatively to the belonging to a set $A$. It is based on algorithmic entropy, which is, up to a constant term, equal to the expected value of Shannon entropy. We define several forms of instance complexity; for a more complete presentation see [7]. It is assumed that programs always terminate, and output either 0, 1 or ⊥ (“don’t know”). In the communication complexity the ‘cost’ is the number of bits exchanged between Alice and Bob who have unlimited computational power. In order to establish a relationship with instance complexity we use a time bounded version of instance complexity where it is assumed that the time is sufficiently large.

**Definition II.4.** A program $p$ is consistent with a set $A$ if $x \in A$ whenever $p(x) = 1$ and $x \notin A$ whenever $p(x) = 0$.

**Definition II.5** (time bounded instance complexity). Let $t$ be a constructible time bound, $A$ be a set, $x$ an element and $p$ a total program with access to an oracle $O$. Consider the following conditions: (C1) for all $y$, $p(y)$ runs in time not exceeding $t(|y|)$; (C2) for all $y$, $p(y)$ outputs 0, 1 or ⊥, (C3) $p$ is consistent with $A$, and (C4) $p(x) \neq ⊥$. The $t$-bounded instance complexity with oracle access to $O$ of $x$ relative to the set $A$ is

$$ic^{O,t}(x : A) = \min_{p : \text{a total program with oracle access to } O \text{ that satisfies (C1), (C2), (C3) and (C4) \quad (\exists y : p = ic^{O,t}(x : A)) \text{ we say that } p \text{ corresponds exactly to } ic^{O,t}(x : A) \quad \square}$$

We say that a program $p$ corresponds to $ic^{O,t}(x : A)$ if it satisfies conditions (C1), (C2), (C3) and (C4); if moreover $|p| = ic^{O,t}(x : A)$ we say that $p$ corresponds exactly to $ic^{O,t}(x : A)$. \square
Notice that in the communication complexity the time is not an important issue since Alice and Bob have unlimited power of computation and the communication complexity is measured in number of bits exchanged and not by the time required to transmit the information. The reason why we consider a time bound version of instance complexity is because Alice must have a reference for the time that she can expect for the program, that is given to her as a guess, to stop. This is a technical detail. Notice that if the possible guess $p$ is not a total program then there are data for which the $p$ will not stop and then Alice cannot compute the set of $y$ such that $f(x, y) \neq \perp$, unless she can compute the Halting problem.

Relaxing the condition “$p(x) \neq \perp$” we get two weaker forms of instance complexity:

**Definition II.6** (inside instance complexity). Let $t$ be a constructible time bound, $A$ be a set, $x$ an element and $p$ a total program with access to an oracle $O$. Consider the following conditions: (C1) for all $y$, $p(y)$ runs in time not exceeding $t(|y|)$, (C2) for all $y$, $p(y)$ outputs either 0 or $\perp$, (C3) $p$ is consistent with $A$ and (C4) $x \notin A \implies p(x) = 0$.

The $t$-bounded inside instance complexity with oracle access to $O$ of $x$ relative to the set $A$ is

$$ic_{yes}^{O,t}(x : A) = \min \left\{ |p| : p \text{ is a total program with oracle access to } O \text{ that satisfies (C1), (C2), (C3) and (C4)} \right\}$$

A program $p$ corresponds to $ic_{yes}^{O,t}(x : A)$ if it satisfies conditions (C1), (C2), (C3) and (C4); if moreover $|p| = ic_{yes}^{O,t}(x : A)$ we say that $p$ corresponds exactly to $ic_{yes}^{O,t}(x : A)$. □

**Definition II.7** (outside instance complexity). Let $t$ be a constructible time bound, $A$ be a set, $x$ an element and $p$ a total program with access to an oracle $O$. Consider the following conditions: (C1) for all $y$, $p(y)$ runs in time not exceeding $t(|y|)$, (C2) for all $y$, $p(y)$ outputs either 0 or $\perp$, (C3) $p$ is consistent with $A$ and (C4) $x \notin A \implies p(x) = 0$.

The $t$-bounded outside instance complexity with oracle access to $O$ of $x$ relative to the set $A$ is

$$ic_{no}^{O,t}(x : A) = \min \left\{ |p| : p \text{ is a total program with oracle access to } O \text{ that satisfies (C1), (C2), (C3) and (C4)} \right\}$$

A program $p$ corresponds to $ic_{no}^{O,t}(x : A)$ if it satisfies conditions (C1), (C2), (C3) and (C4); if moreover $|p| = ic_{no}^{O,t}(x : A)$ we say that $p$ corresponds exactly to $ic_{no}^{O,t}(x : A)$. □

Notice that if $x \notin A$ then $ic_{yes}^{O,t}(x : A)$ is a constant (independent of $x$) for a time bound $t$ (namely a constant), because the program $p(x) = \perp$ has fixed length and is consistent with every set; similarly if $x \in A$ then $ic_{no}^{O,t}(x : A)$ is a constant. Notice also that for every element $x$ and set $A$ we have $ic_{yes}^{O,t}(x : A) \leq ic_{no}^{O,t}(x : A)$ and $ic_{no}^{O,t}(x : A) \leq ic_{yes}^{O,t}(x : A)$. On the other hand, from a program $p$ corresponding to $ic_{yes}^{O,t}(x : A)$ and a program $p'$ corresponding to $ic_{no}^{O,t}(x : A)$ we can define a program $r$ as follows: $r(x) = 1$ if $p(x) = 1$, $r(x) = 0$ if $p'(x) = 0$ and $r(x) = \perp$ otherwise, concluding that

$$ic_{yes}^{O,t}(x : A) \leq ic_{yes}^{O,t}(x : A) + ic_{no}^{O,t}(x : A) + O(\log(\min(\{ic_{yes}^{O,t}(x : A), ic_{no}^{O,t}(x : A)\})))$$

where the function $f$ represents the time overhead needed for the simulation of $p(x)$ for $t_1$ steps followed by simulation of $p'(x)$ for $t_2$ steps; the logarithmic term comes from the need to delimit $p$ from $p'$ in the concatenation $pp'$.

### III. One-sided protocols

To give an idea of the relationship between instance complexity and communication complexity we first analyze, in subsection III-A, the special case of function inequality defined by $NEQ(\overline{x}, \overline{y}) = 1$ if and only if $\overline{x} \neq \overline{y}$. We show how to use programs corresponding to instance complexity as guesses of (optimal) non-deterministic protocols. This usage is later generalized to any function in subsection III-B.

#### A. Inequality: an optimal “$ic_{yes}$-protocol”

Consider the predicate NEQ and suppose that $\overline{x} \neq \overline{y}$; then for some $i$, $1 \leq i \leq n$, we have $x_i \neq y_i$. A possible program $p_i$ corresponding to $ic_{yes}^{NEQ,t}(\overline{y} : Y_i(\overline{f})) = 1$ if $y_i = x_i$, $p_i(y) = \perp$ if $y_i \neq x_i$. If the reader computes the set $Y_i = \{ y : p_i(y) = 1 \}$ it is easy to see that $Y_i \subset Y_i(\overline{f})$. So, if $p(\overline{f}) = 1$ and if $|p|$ is minimum, this program corresponds exactly to $ic_{yes}^{NEQ,t}(\overline{y} : Y_i(\overline{f}))$ for some function $t$.

Consider now the following protocol $P$ for NEQ where $t$ is a time bound sufficiently large (see more details in subsection III-B). Alice receives a word $p$ as a guess; $p$ may eventually be the program $p_i$ above. Then she runs $p(y)$ for every $y \in Y$ until the program halts or until $t(|y|)$ steps have elapsed. If $p(y)$ does not halt in time $t(|y|)$, the word $p$ is not a valid witness and the protocol halts. Otherwise Alice defines the set $Y_i = \{ y : p(y) = 1 \}$. If $Y_i \subset Y_i(\overline{f})$, i.e., if $p$ is consistent with $Y_i(\overline{f})$, she sends $p$ to Bob, otherwise outputs $\perp$ and halts. Bob tests if $p(\overline{f}) = 1$; if yes, outputs 1, otherwise outputs $\perp$.

**Correctness conditions:**

1) If $\overline{x} \neq \overline{y}$, there is a witness $p$ that corresponds to $ic_{yes}^{NEQ,t}(\overline{y} : Y_i(\overline{f}))$.

We have $Y_i \subset Y_i(\overline{f})$ for some $i$, $0 \leq i \leq n$. Then, if $p$ happens to be the program $p_i$ above, the protocol $P$ outputs 1 so $p$ corresponds to $ic_{yes}^{NEQ,t}(\overline{y} : Y_i(\overline{f}))$, that is, we have $Y_i \subset Y_i(\overline{f})$ consistent with $Y_i(\overline{f})$ (verified by Alice) and $p(\overline{f}) = 1$ (verified by Bob).

2) If a guess is wrong, the output is $\perp$.

If the guess is wrong, then either some $p(y)$ does not run in time $t(|y|)$ or $p$ is not consistent with $Y_i(\overline{f})$ or $p(\overline{f}) = \perp$; all these cases are possible to detect by Alice and Bob.

3) If $\overline{x} = \overline{y}$, no guess $p$ can cause output 1.

This follows directly from the definition of the protocol.

**Complexity:**

The length of $p_i$ need not to exceed $\log n + O(1)$ and $\max_{0 \leq i \leq n} \{ |p_i| \}$ is $\log n + O(1)$. Thus the complexity of the protocol $P$ is $\log(n) + O(1)$. But the non-deterministic
communication complexity of \( NEQ \) is also \( \log n + O(1) \) (see [4]), thus the protocol is optimal.

B. “ic\(_{\text{yes}}\) protocols” are optimal

In this section we prove the main theorem of this paper by showing how to use a program corresponding to \( t \)-bounded inside instance complexity as a guess in a nondeterministic protocol. In the general case, the function \( f \), which is known by Alice and Bob, is arbitrarily complex; therefore the description of \( f \) can not be included into an “instance complexity program” \( p \) unless \( \lim_{n \to \infty} |p| = \infty \). But the scenario is different if we give the program \( p \) free access to the description of the function \( f \).

**Theorem III.1** (ic\(_{\text{yes}}\) protocols are optimal). Let \( f \) be an arbitrary function. There is a computable function \( t(n) \) such that

\[
N^1(f) = \max_{|x| = |y| = n} \left( ic_{\text{yes}}^f(Y; Y_i(x)) \right) + O(1)
\]  

(III.5)

**Proof.** Let \( p \) be the non-deterministic word given to Alice by the third entity; the protocol \( P \) is described in Figure 1 Notice that the protocol specifies that Alice should interpret \( p \) as a program and execute \( p \) for all \( y \) for \( t(|y|) \) steps.

- **Alice:**
  - Receive program \( p(y) \) (as a possible witness)
  - Test if, for every \( y \in Y_i \), \( p \) halts and produces \( \perp \) or \( \top \) in time \( t(n) \)
  - If not, output \( \perp \) and halt
  - Compute the set \( B = \{ y : p(y) = 1 \} \)
  - Using the oracle access to the description of \( f \)
  - Select the first (in lexicographic order) such cover \( \{ R_1, R_2, ..., R_m \} \)
  - Select a rectangle \( R_i = A \times B \) from that cover
  - Where \( B \subseteq Y \) is the set computed above
  - As the cover is minimum, there can be at most one such rectangle. If there is none, output \( \perp \) and halt
  - Test if \( \tau \in A \)
  - If not, output \( \perp \) and halt
  - Send \( p \) to Bob

- **Bob:**
  - Verify if \( p(\tau) = 1 \)
  - If yes, output \( \top \) and halt
  - Output \( \perp \) and halt

Fig. 1. A family of one-sided non-deterministic protocols \( P \). The guess is based on a program \( p \) that corresponds to \( ic_{\text{yes}}^f(\tau; Y_i(\tau)) \).

The program \( p \), being an arbitrary guess, may behave in many different ways; in particular, if \( f(\tau, \tau) = 1 \), the behavior can be described as follows:

If \( i \) is chosen so that \( (\tau, \tau) \in R_i \) (if \( f(\tau, \tau) = 1 \) there is at least one such \( i \), otherwise there is none) then \( p \) is consistent with \( Y_i(\tau) \) and \( p(\tau) = 1 \). Then \( |p| \geq ic_{\text{yes}}^f(\tau; Y_i(\tau)) \).

Moreover, if \( p \) is not "correct", that fact can be detected by Alice or by Bob; thus, conditions (II.3) and (II.4) (see page 2) are verified.

How much time \( t(n) \) must Alice run \( p(y) \) (for each \( y \)) so that, there is at least a witness for every pair \( (\tau, \tau) \) with \( f(\tau, \tau) = 1 \)? It is possible to obtain an upper bound \( t(n) \) in a constructive way by detailing and analyzing the algorithm that the witness \( p \) should implement, see Figure 2. In fact, \( t(n) \) is a computable function that Alice can determine\(^2\).

Suppose now that \( f(\tau, \tau) = 1 \). If the protocol accepts \( (\tau, \tau) \) with guess \( p \), we have \( |p| \leq \log m + O(1) \) and \( \max_{|\tau| = |\tau| = n} \{ |p| \} \leq \log m + O(1) \). Thus

\[
N^1(f) = \log C^1(f) + O(1) \quad (\text{III.6})
\]

\[
\geq \log m + O(1) \quad (\text{III.7})
\]

\[
\geq \max_{|\tau| = |\tau| = n} \{ |p| \} + O(1) \quad (\text{III.8})
\]

\[
\geq \max_{|\tau| = |\tau| = n} \{ ic_{\text{yes}}^f(\tau; Y_i(\tau)) \} + O(1) \quad (\text{III.9})
\]

On the other hand, there exists a non-deterministic protocol with complexity \( \max_{|\tau| = |\tau| = n} \{ ic_{\text{yes}}^f(\tau; Y_i(\tau)) \} + O(1) \); this is the protocol of Figure 3. Notice that protocol \( p \) can be any total program running in time \( t \) which is consistent with \( Y_i(\tau) \) and such that \( p(\tau) = 1 \) (and, if \( f(\tau, \tau) = 1 \), there is at least one such program, as we have seen above); thus it can be the shortest such program, \( |p| = ic_{\text{yes}}^f(\tau; Y_i(\tau)) \). Taking the maximum over all \( \tau \) and \( \tau \) with \( |\tau| = |\tau| = n \) (see Definition II.1) we get

\[
N^1(f) \leq \max_{|\tau| = |\tau| = n} \{ ic_{\text{yes}}^f(\tau; Y_i(\tau)) \} + O(1) \quad (\text{III.10})
\]

because \( N^1(f) \) is the smallest complexity among all the protocols for \( f \). Combining this result with inequality (III.9) we get

\[
N^1(f) = \max_{|\tau| = |\tau| = n} \{ ic_{\text{yes}}^f(\tau; Y_i(\tau)) \} + O(1) \quad \square
\]

\(^2\)Notice that the time required must be, at least, exponential since the determination of the minimal cover can be determine in exponential time.

| Program \( p \), input \( y \): |
|---------------------------------|
| From the description of \( f \) (which is given by the oracle) and \( i \):
| Find the set \( S_i \) of smallest 1-covers
| Select the first (in lexicographic order) cover \( \{ R_1, R_2, ..., R_m \} \in S_i \)
| Select rectangle \( R_i = A \times B \) in that cover
| With input \( y \), output |
| \( p(y) = 1 \) if \( y \in B \)
| \( p(y) = \perp \) otherwise

Fig. 2. A possible behavior of the program \( p \) which may cause the protocol \( P \) (see Figure 1) to output 1. A string \( p \) with this behavior can be specified in length \( |p| \). The existence of this program, which has length \( \log m \) where \( m \) is the size of the minimum covers, justifies the step between equation (III.7) and inequality (III.3).
do not allow access to an oracle the result is valid for uniform functions since the description of \( f \) in this cases requires a constant number of bits and hence can be built in the program that is used as a guess with a cost of a constant in the number of bits. The idea to prove that the result is false without oracle access is to use the Kolmogorov complexity as a tool. Denote by \( C(x) \) the (plain) Kolmogorov complexity of \( x \) which is defined as \( C(x) = \min \{|p| : U(p) = x\} \) where \( U \) is some fixed universal Turing machine, see [6].

Consider a monochromatic cover of a non uniform function such that (i) the number \( m \) of rectangles in the cover is very small and (ii) the horizontal side \( B \) of the first rectangle in the cover has a Kolmogorov random length, \( C(|B|) \approx n \). The length \( B \) can be obtained from \( p \), thus \( C(|B|) \leq C(p) + O(1) \) which implies \( C(p) \geq n + O(1) \gg \log m \); thus the step (III.7) → (III.8) in the proof is not valid.

IV. TWO-SIDED PROTOCOLS

Now we consider the two-sided protocols for non-deterministic communication complexity. Similarly to the result of the previous section we show that there are optimum protocols whose guesses correspond exactly to \( ic(f,y) \) of the first rectangle in the cover has a Kolmogorov random length, \( C(|B|) \approx n \). The length \( B \) can be obtained from \( p \), thus \( C(|B|) \leq C(p) + O(1) \) which implies \( C(p) \geq n + O(1) \gg \log m \); thus the step (III.7) → (III.8) in the proof is not valid.

Theorem IV.1. Let \( f \) be any function. There exists a computable function \( t \) such that

\[
N(f) = \max_{|\pi| = |\tilde{y}| = n} \{ic(f,y) : Y_i(\pi) \} + O(1)
\]

The proof of this Theorem is similar to the proof of Theorem [III.7] we make only a few observations. The reader should compare Figures [II.8] and [II.9] with Figures [II.4] and [II.5] respectively. The main difference in the proof is that we have to consider a minimum cover of 0-rectangles and a minimum cover of 1-rectangles. Denote by \( m = C_0(f) \) and \( m' = C_1(f) \) the size of those covers; the witness (program) \( p \) has a description with length \( \log (m + m') + O(1) \). It is not difficult to verify the correctness of conditions (II.1) to (II.2), see page [II].

| Program \( p \), input \( y \): |
|----------------------------------|
| From the description of \( f \) given by the oracle and \( i \): |
| Find the set \( S_0 \) of smallest 0-covers and the set \( S_1 \) of smallest 1-covers |
| Select the first (in lexicographic order) sequence \( s = (R_1, \ldots, R_m, R_{m+1}, \ldots, R_{m+m'}) \) |
| where \( (R_1, \ldots, R_m) \in S_0 \) and \( (R_{m+1}, \ldots, R_{m+m'}) \in S_1 \) |
| Select the ith rectangle \( R_i = A \times B \) from \( s \) |
| With input \( y \), output: |
| \( p(y) = 1 \) if \( y \in B \) and rectangle \( A \times B \) has color \( z \in \{0,1\} \) |
| \( p(y) = ⊥ \) otherwise |

Fig. 5. A possible behavior of the program \( p \) which may cause the protocol \( P \) of Figure [II.4] to output a value different from ⊥. A string \( p \) with this behavior can be specified in length \( \log(m + m') \).

V. ABOUT INDIVIDUAL COMMUNICATION COMPLEXITY

The one sided individual communication complexity satisfies

\[
N^1(f, \pi, \tilde{y}) \geq \min \{ic(f,y) : Y_i(\pi) \} + O(1)
\]

for some constructible time \( t \). The complexity \( N^1(f, \pi, \tilde{y}) \) is obtained from a minimization over all protocols which must of course “work correctly” for every pair \( (x, y) \) and not only for \( (\pi, \tilde{y}) \) while no such restriction exists in the definition of instance complexity. The individual communication complexity may in a few rare cases (if \( i \) has a very short description), be much smaller than \( \log m \).

Finally we present a result relating the individual non-deterministic communication complexity with the instance complexity.

Theorem V.1. (Individual upper bound) For every function \( f \) and values \( x \) and \( y \) the individual non-deterministic communication complexity \( N(f, x, y) \) satisfies for some constructible time \( t \)

\[
N(f, x, y) = ic(f,y) + O(1) \leq N(f) + O(1)
\]

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REFERENCES

[1] S. Arora, B. Barak, Computational Complexity: A Modern Approach, Princeton University, 2006.
[2] H. Buhrman, H. Klauck, N. Vereshchagin and P. Vitányi, Individual communication complexity, Proc. of STACS 2004.
[3] I. Kremer, N. Nisan, D. Ron, On randomized one-round communication complexity, Proc. of STOC, pp 596-605, 1995.
[4] Eyal Kushilevitz, Noam Nisan, Communication Complexity, Cambridge University Press, New York, Springer-Verlag, 1996.
[5] S. Laplante, J. Rogers, Indistinguishability, TR-96-26, 1996.
[6] M. Li e P. Vitányi, An Introduction to Kolmogorov Complexity and its Applications, Springer, second edition, 1997.
[7] P. Orponen, K. Ko, U. Schönig, O. Watanabe, Instance Complexity, Journal of the ACM, 41:1, pp 96-121, 1994.
[8] A. Yao, Some complexity questions related to distributive computing, Proceedings of Symposium on Theory of Computing, pp 209-213, 1979.
[9] A. Yao. The entropic limitations on VLSI computations, Proceedings of Symposium on Theory of Computing, pp 308-311, 1981.

Fig. 4. A family of two-sided non-deterministic protocols \( P \). The guess is based on a program \( p \) that corresponds to \( ic(f,y) \). Compare with Figure [II.4] For simplicity we assume that whenever a test fails, the protocol outputs ⊥ and halts.