On the string equation at $c = 1$

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ABSTRACT

The analogue of the string equation which specifies the partition function of $c = 1$ string with a compactification radius $\beta \in \mathbb{Z}_{\geq 1}$ is described in the framework of the Toda lattice hierarchy.
Recently much attention has been paid to the understanding of $c = 1$ string theory from the view of integrable hierarchy. In particular, the tachyon dynamics of the theory has been described in the framework of the (dispersionless) Toda lattice hierarchy [1], [2], [3]. In spite of these developments, our understanding of the nonperturbative aspects for $c = 1$ string still has a gap from that of non-critical string theory. In the non-critical string theory the full partition function is the $\tau$ function of Kadomtsev-Petriashvili (KP) hierarchy specified by the solution of string equation [4]:

$$[P, Q] = 1$$

where $P$, $Q$ are differential operators. These pairs can be given in terms of the Lax and Orlov operators of KP hierarchy [5]. On the other hand, even in the framework of the Toda lattice (TL) hierarchy, the analogue of the string equation at $c = 1$ has not been clarified yet. In this letter, by utilizing the concepts of these integrable hierarchies, we try to obtain this nonperturbative counterpart which characterizes the generating function for the tachyon correlation functions of $c = 1$ string with a compactification radius $\beta \in \mathbb{Z}_{\geq 1}$.

\section{$c = 1$ string}

The generating function $F$ for the tachyon correlation functions of $c = 1$ string with a compactification radius $\beta \in \mathbb{Z}_{\geq 1}$ is described in terms of free fermion system: $\psi(z) = \sum \psi_l z^{-l}$, $\psi^*_l(z) = \sum \psi^*_l z^{-l-1}$. The partition function is given by

$$\exp\{\frac{F(t, \bar{t}, \bar{h})}{\beta}\} g_0 e^{-\overline{H}(\bar{t})} |0\rangle,$$

where $t = (t_1, t_2, \cdots)$ and $\bar{t} = (\bar{t}_1, \bar{t}_2, \cdots)$. $t_k$ and $\bar{t}_k$ ($k \in \mathbb{Z}_{\geq 1}$) are parameters coupled to the tachyons with momentum $\frac{k}{\beta}$ and $\frac{-k}{\beta}$ respectively. "$1/\hbar"$ will play the role of cosmological constant of this string theory, that is,

$$\mathcal{F}(t, \bar{t}, \bar{h}) = \sum_{g \geq 0} \hbar^{2g-2} \mathcal{F}_g(t, \bar{t})$$

where $\mathcal{F}_g$ is the free energy on the 2 surface of genus $g$. $g_0$ is an element of $GL(\infty)$ [1], [3]:

$$g_0 = \exp\{\sum \alpha_l : \psi_l \psi^*_l :\}$$

$$e^{\alpha_l} = \hbar^{-\frac{l \cdot \bar{l}}{\beta}} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{\beta} - \frac{l \cdot \bar{l}}{\beta}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{\beta}\right)}.$$

$H(t)$ and $\overline{H}(\bar{t})$ are respectively the positive and negative modes of a $U(1)$ current $J(z) = : \psi \psi^* : (z)$, that is,

$$H(t) = \sum_{k \geq 1} t_k J_k \quad \overline{H}(\bar{t}) = \sum_{k \geq 1} \bar{t}_k J_{-k}.$$ 

The fermion vacuum $|n\rangle$ is introduced by the conditions:

$$\psi^*_k |n\rangle = 0 \quad \text{for} \quad k \geq n,$$

$$\psi_k |n\rangle = 0 \quad \text{for} \quad k \geq 1 - n.$$
In order to study the partition function $e^{F}$ in the framework of the TL hierarchy we shall introduce the quantities:

$$\tau(n : t, \bar{t}) = <n|e^{H(t)}g_0e^{-\bar{H}(\bar{t})}|n>.$$  \hspace{1cm} (3)

These quantities are precisely $\tau$-function of the TL hierarchy \cite{7, 8}. Since, by fixing $t = (t_1, t_2, \cdots) (\bar{t} = (\bar{t}_1, \bar{t}_2, \cdots))$, $\tau(n; t, \bar{t})$ \hspace{1cm} (3) can be considered as a special solution of the KP hierarchy which time variables are \hspace{1cm} $\bar{t}$ (t), we shall begin by studying $\tau(n; t, \bar{t})$ \hspace{1cm} (3) from the view of the KP hierarchy \cite{9, 10}.

2 \hspace{1cm} $t$ as KP times

By introducing the Miwa variables $\lambda = (\lambda_1, \cdots, \lambda_N)$ through the relation:

$$\frac{t_k}{\hbar} = \frac{1}{k} \sum_{i=1}^{N} \lambda_i^{-k} \quad \text{for} \quad \forall k \in \mathbb{Z}_{\geq 1},$$

and then utilizing the bozonization rule of free fermion $\psi(z)$ and $\psi^*(z)$, one can rewrite $\tau(n)(3)$ into

$$\tau(n; t, \bar{t}) = \frac{\text{const.}}{\Delta(\lambda)} \det \left| \lambda_j^n < n|\psi_{i-n}\psi^*(\lambda_j)g_0e^{-\bar{H}(\bar{t})}|n> \right|_{1 \leq i, j \leq N},$$

$$= \text{const.} \left( \prod_{j=1}^{N} \frac{\lambda_i}{\Delta(\lambda)} \right) \det \left| \phi_{i-n}(\lambda_j; \frac{\bar{t}}{\hbar}) \right|_{1 \leq i, j \leq N}.$$

$\Delta(\lambda)$ is the Vandermonde’s determinant: $\Delta(\lambda) = \prod_{i>j}^{N} (\lambda_i - \lambda_j)$. $\phi_i(\lambda)$, which will play an important role in the subsequent discussion, has the form:

$$\phi_i(\lambda; \frac{\bar{t}}{\hbar}) = \sum_{l \geq 0} \lambda^{i-l} e^{\eta(\lambda, \bar{t})} P_l(\frac{-\bar{t}}{\hbar}),$$

where $P_l(\bar{t})$ is a Schur polynomial which generating function is

$$\sum_{l \geq 0} P_l(\bar{t}) \lambda^l = e^{\eta(\bar{t}, \lambda)} \quad \eta(\bar{t}, \lambda) = \sum_{i \geq 1} \bar{t}_i \lambda^i.$$

One can consider $\tau(n; t, \bar{t})$ \hspace{1cm} (3) as a $\tau$-function of the KP hierarchy by fixing $\bar{t} = (\bar{t}_1, \bar{t}_2, \cdots)$. The corresponding point in the Universal Grassmann manifold (UGM) \cite{4, 11} will be described in terms of these $\phi_i(x; \frac{\bar{t}}{\hbar})$ or the Laplace transforms of $\phi_i(\lambda; \frac{\bar{t}}{\hbar})$:

$$\phi_i(x; \frac{\bar{t}}{\hbar}) = \int d\lambda e^{\frac{\lambda x}{\hbar}} \phi_i(\lambda; \frac{\bar{t}}{\hbar}).$$ \hspace{1cm} (4)

Let us define a vector space $V(n; \bar{t})$ (a point of the UGM) as

$$V(n; \bar{t}) = \bigoplus_{i \geq 1} C \phi_{i-n}(x; \frac{\bar{t}}{\hbar}).$$ \hspace{1cm} (5)
Notice that the set \( \{ V(n; \tilde{t}) \}_{n \in \mathbb{Z}} \) defines a flag of the UGM. Namely it satisfies the conditions:

\[ \cdots \subset V(n; \tilde{t}) \subset V(n + 1; \tilde{t}) \subset \cdots. \]

\[ \dim_{\mathbb{C}} \frac{V(n + 1; \tilde{t})}{V(n; \tilde{t})} = 1. \]

We can also introduce \( W(n; \hbar \partial_x) \), the wave operator of KP hierarchy which corresponds to \( \tau(n) \) \([8]\). This wave operator has the form:

\[ W(n; \hbar \partial_x) = \sum_{j \geq 0} u_{j,\alpha=\frac{1}{2}}^{(\infty)}(n)(\hbar \partial_x)^{-j+n}, \quad (6) \]

where \( u_{j,\alpha=\frac{1}{2}}^{(\infty)}(n) = u_{j,\alpha=\frac{1}{2}}^{(\infty)}(n; t, \tilde{t}) \). \( t = (t_1, t_2, \cdots) \) is playing the role of KP times and especially we now identify \( t_1 = x \). The parameter \( \alpha \) will play a gauge parameter in our description of the TL hierarchy. The relation between the vector space \( V(n; \tilde{t}) \) \([8]\) and the wave operator \( W(n; \hbar \partial_x) \) \([8]\) is given by the theorem:

**Theorem 1** \([7, 8]\)

\[ V(n; \tilde{t}) = W_0(n; \hbar \partial_x)^{-1} V_\emptyset, \quad (7) \]

where

\[ W_0(n; \hbar \partial_x) = W(n; \hbar \partial_x)|_{t=(x,0,0,...)}. \]

\( V_\emptyset \) is the point of the UGM which corresponds to the fermionic vacuum \( |0 > \). \([8]\)

Owing to the above theorem we can characterize the wave operator \( W(n) \equiv W(n; \hbar \partial_x) \) \([8]\) from the study of the point \( V(n; \tilde{t}) \) \([8]\). In particular one can see \( \phi_i(x; \frac{\tilde{t}}{\hbar}) \) \([11]\) satisfies the following equations: For \( \forall k \in \mathbb{Z}_{\geq 1} \),

(i) \[ D_\beta(x; \hbar \partial_x)^k \phi_i(x; \frac{\tilde{t}}{\hbar}) = \hbar^k \frac{\Gamma(k + \frac{1}{2} + \frac{1}{\hbar} - \frac{i \beta}{2})}{\Gamma(\frac{1}{2} + \frac{1}{\hbar} - \frac{i \beta}{2})} \phi_{i-k \beta}(x; \frac{\tilde{t}}{\hbar}). \]

(ii) \[ -\hbar \frac{\partial}{\partial t_\beta} \phi_i(x; \frac{\tilde{t}}{\hbar}) = (-\hbar \frac{\partial}{\partial t_\beta})^k \phi_i(x; \frac{\tilde{t}}{\hbar}) \]

\[ = \hbar^k \frac{\Gamma(k + \frac{1}{2} + \frac{1}{\hbar} - \frac{i \beta}{2})}{\Gamma(\frac{1}{2} + \frac{1}{\hbar} - \frac{i \beta}{2})} \phi_{i-k \beta}(x; \frac{\tilde{t}}{\hbar}). \quad (8) \]

The pseudo-differential operator \( D_\beta \equiv D_\beta(x; \hbar \partial_x) \) in (i) of \([8]\) is:

\[ D_\beta(x; \hbar \partial_x) = \frac{1}{\beta} (\hbar \partial_x)^{1-\beta} x + \{ 1 + \frac{\hbar}{2} (1 - \frac{1}{\beta}) \} (\hbar \partial_x)^{-\beta}. \quad (9) \]

\(^1\)By introducing the bases \( \{ e_i(x) \} \), \( V_\emptyset = \bigoplus_{i \geq 1} C e_{-i}(x) \) holds. Notice that, in our notation, \( \hbar \partial_x \) and \( x \) act on these bases as \( \hbar \partial_x e_i(x) = e_{i-1}(x), \; xe_i(x) = \hbar(i + 1) e_{i+1}(x). \)

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Therefore, from theorem 1, the evolution of the relation:

\[-\hbar \partial_{t_{k\beta}} \xi = \mathcal{P}_{k\beta}(x; \hbar \partial_x) \xi,\]

for \(\forall \xi \in V = \bigcup_{\nu} V(n; \tilde{t}) (\equiv \bigoplus_i C \phi_i(x; \frac{\tilde{t}}{\hbar}))\), the first line of (ii) in (8) tells us the relation:

\[\mathcal{P}_{k\beta} = \mathcal{T}_\beta \cdots \mathcal{T}_k.\]

Let us look into the content of the equation (ii) in (8). Introducing pseudo-differential operators \(\mathcal{P}_{k\beta} \equiv \mathcal{P}_{k\beta}(x; \hbar \partial_x) (k \in \mathbb{Z}_{\geq 1})\) which satisfy

\[-\hbar \partial_{t_{k\beta}} \xi = \mathcal{P}_{k\beta}(x; \hbar \partial_x) \xi,\]

for \(\forall \xi \in V = \bigcup_{\nu} V(n; \tilde{t}) (\equiv \bigoplus_i C \phi_i(x; \frac{\tilde{t}}{\hbar}))\), the first line of (ii) in (8) tells us the relation:

\[\mathcal{P}_{k\beta} = \mathcal{T}_\beta \cdots \mathcal{T}_k.\]

Therefore, from theorem 1, the evolution of \(W_0(n) \equiv W_0(x; \hbar \partial_x)\) with respect to \(\tilde{t}_{\beta k}\) has the form \[\text{Eqn. 3}\]:

\[\hbar \partial_{t_{k\beta}} W_0(n) = (W_0(n)\mathcal{P}_{k\beta} W_0(n)^{-1})_0 W_0(n).\tag{10}\]

At this stage we shall turn to (i) in (8). By combining (i) with (ii), we also obtain the relation:

\[\mathcal{P}_{k\beta} = D_{\beta} \tag{11}\]

This can be rephrased by using the Orlov formulation of KP hierarchy \[\text{Ref. 12}\]. Let us introduce the Lax operator \(L_{KP}(n) (\equiv L_{KP}(n; \hbar \partial_x))\) and the Orlov operator \(M_{KP}(n) (\equiv M_{KP}(n; \hbar \partial_x))\) as

\[L_{KP}(n; \hbar \partial_x) = W(n)h \partial_x W(n)^{-1},\]
\[M_{KP}(n; \hbar \partial_x) = W(n) \left(\sum_{l \geq 1} \tilde{t}_l (h \partial_x)^{l-1}\right) W(n)^{-1}.\]

In terms of these operators the relation (11) becomes:

\[W_0(n)\mathcal{P}_{\beta} W(n)^{-1}_0 = F(L_{0KP}(n), M_{0KP}(n)),\tag{12}\]

where

\[F(x, y) = \frac{1}{\beta} x^{1-\beta} y + \left\{1 + \frac{\hbar}{2}(1 - \frac{1}{\beta})\right\} x^{-\beta}.\tag{13}\]

\(L_{0KP}(n)\) and \(M_{0KP}(n)\) are those evaluated at the initial time \(t = (x, 0, 0, \cdots)\), that is,

\[L_{0KP}(n) = L_{KP}(n)|_{t=(x,0,0,\cdots)}, \quad M_{0KP}(n) = M_{KP}(n)|_{t=(x,0,0,\cdots)}.\]

\(^2(\ )_\pm\) are introduced as projection operators with respect to \(\partial_x;\)

\[\begin{align*}
(\partial_x^m)_- &= \partial_x^m - (\partial_x^m)_+, & (\partial_x^m)_+ &= \left\{ \begin{array}{ll} \partial_x^m & m \geq 0 \\ 0 & m < -1 \end{array} \right. \\
\end{align*}\]
Finally we will return to the equation (10). Though \( \bar{t} = (\bar{t}_1, \bar{t}_2, \cdots) \) can not be considered as KP times in our present approach, it is possible to define their flows as

\[
\bar{h}\partial_{\bar{t}_m}W(n) = B_{m}^{KP}(n)W(n),
\]

where \( B_{m}^{KP}(n) = \sum_{l \geq 1} P_l(-\bar{h}\partial_{\bar{t}_m})\ln \tau(n)L^{KP}(n)^{-l}, \)

(14)

where \( \bar{h} = (\partial_1, \frac{1}{2}\partial_2, \cdots) \). Hence the equation (10) means

\[
(W_0(n)P_{\beta}^kW_0(n)^{-1})_\tau = B_{k\beta}^{KP}(n)|_{t=(x,0,\cdots)} \quad \text{for } \forall k \in \mathbb{Z}_{\geq 1}.
\]

(15)

3 \( \bar{t} \) as KP times

Nextly we will consider \( \tau(n; t, \bar{t}) \) \( \mathbf{[3]} \) as a \( \tau \) function of the KP hierarchy with respect to time variables \( \bar{t} = (\bar{t}_1, \bar{t}_2, \cdots) \).

Introducing the Miwa variables \( \lambda = (\lambda_1, \cdots, \lambda_N) \) though the relation:

\[
\frac{\bar{t}_k}{\bar{h}} = \frac{1}{k} \sum_{i=1}^{N} \lambda_i^{-k} \quad \text{for } \forall k \in \mathbb{Z}_{\geq 1},
\]

the \( \tau \) function \( \tau(n + 1) \) \( \mathbf{[3]} \) can be written into

\[
\tau(n + 1; t, \bar{t}) = \frac{\text{const.}}{(\Pi_{i=1}^{N} \lambda_i)^{n+1}\Delta(\lambda)} \det \left| \frac{\partial}{\partial \lambda} \phi_{i+n+1}(\lambda; \frac{t}{\bar{h}}) \right|_{1 \leq i, j \leq N},
\]

where

\[
\frac{\partial}{\partial \lambda} \phi_{i}(\lambda; \frac{t}{\bar{h}}) = \sum_{l \geq 0} \lambda^{i-l-1} e^{\alpha_{1-i} \lambda} P_l(-\frac{t}{\bar{h}}).
\]

One can consider \( \tau(n + 1) \) \( \mathbf{[3]} \) as a \( \tau \) function of the KP hierarchy with KP times \( \bar{t} \). The corresponding point in the UGM which we will denote as \( \nabla(n; t) \) can be described in terms of \( \phi_{i}(x; \frac{t}{\bar{h}}) = \int d\lambda e^{\frac{\lambda}{\bar{h}} \phi_{i}(\lambda; \frac{t}{\bar{h}})} \):

\[
\nabla(n; t) = \bigoplus_{i \geq 1} C_{\phi_{i+n+1}}(x; \frac{t}{\bar{h}}).
\]

(16)

Note that, as we see in the previous section, the set \( \{ \nabla(n; t) \}_{n \in \mathbb{Z}} \) defines a flag in the UGM.

\( \nabla(n; h\partial_x) \), the wave operator of KP hierarchy which corresponds to \( \nabla(n; t) \) \( \mathbf{[10]} \), will be realized as

\[
\nabla(n; h\partial_x) = \sum_{j \geq 0} u_{j, a=-\frac{1}{2}}^{(0)}(n)(h\partial_x)^{-j-n-1},
\]

(17)
where \( u_{j,\alpha}^{(0)}(n) = u_{j,\alpha}^{(0)}(n; t, \bar{t}) \) and we specify \( x = \bar{t}_1 \). The relation between \( \nabla(n; t) \) and \( \mathcal{W}(n; \hbar \partial_x) \) is

\[
\nabla(n; t) = \mathcal{W}_0(n; \partial_x)^{-1} V_0, 
\]

where

\[
\mathcal{W}_0(n; \hbar \partial_x) = \mathcal{W}(n; \hbar \partial_x)|_{t=(x,0,\ldots)}. 
\]

Since \( \bar{\phi}_i(x, \frac{t}{n}) \) satisfies the similar equations as those presented in (8) we can follow the same steps as in the previous section. Especially the analogue of the relation (11) is

\[
\text{where} \\
P_{\beta} = -\mathcal{D}_{\beta}. 
\]

\( \mathcal{D}_{\beta} = \mathcal{D}_\beta(x; h\partial_x) \) is the pseudo-differential operator with the following form

\[
\mathcal{D}_\beta(x; h\partial_x) = \frac{1}{\beta}(h\partial_x)^{1-\beta}x - \{1 - \frac{h}{2}(1 - \frac{1}{\beta})\}(h\partial_x)^{-\beta}. 
\]

On the other hand the pseudo-differential operator \( P_{\beta} = P_{\beta}(x; h\partial_x) \) is given by the conditions :

\[
-h\partial_{x,\beta}\xi = P_{\beta}(x, h\partial_x)\xi, 
\]

for \( \xi \in \nabla = \bigcup_n \nabla(n; t) \).

Introducing the Lax operator \( \mathcal{L}^{KP}(n) \left( \equiv \mathcal{L}^{KP}(n; h\partial_x) \right) \) and the Orlov operator \( \mathcal{M}^{KP}(n) \left( \equiv \mathcal{M}^{KP}(n; h\partial_x) \right) \) as

\[
\mathcal{L}^{KP}(n; h\partial_x) = \mathcal{W}(n)(h\partial_x)\mathcal{W}(n)^{-1}, \\
\mathcal{M}^{KP}(n; h\partial_x) = \mathcal{W}(n)\left(\sum_{l \geq 1} l\tilde{t}_l(h\partial_x)^{l-1}\right)\mathcal{W}(n)^{-1},
\]

then we can rewrite the relation (13) into :

\[
\mathcal{W}_0(n)P_{\beta}^k\mathcal{W}_0(n)^{-1} = -G(\mathcal{L}_0^{KP}(n), \mathcal{M}_0^{KP}(n)), 
\]

where

\[
G(x, y) = \frac{1}{\beta} x^{1-\beta}x + \{-1 + \frac{h}{2}(1 - \frac{1}{\beta})\}x^{-\beta}. 
\]

This is the analogue of the relation (12).

Because the analogue of the equation (13) is

\[
h\partial_{t,\beta}\mathcal{W}_0(n) = (\mathcal{W}_0(n)P_{\beta}^k\mathcal{W}_0(n)^{-1})_{-}\mathcal{W}_0(n), 
\]

we can conclude that \( (\mathcal{W}_0(n)P_{\beta}^k\mathcal{W}_0(n)^{-1})_{-} \) \( (k \in \mathbb{Z}_{\geq 1}) \) has the expression :

\[
(\mathcal{W}_0(n)P_{\beta}^k\mathcal{W}_0(n)^{-1})_{-} = B_{k,\beta}^{KP}(n)|_{t=(x,0,\ldots)}, 
\]

where

\[
B_{k,\beta}^{KP}(n) = \sum_{j \geq 1} P_{j}(-h\tilde{\partial}_l)(h\partial_{t,\beta})\ln(n + 1)\mathcal{L}^{KP}(n)^{-j}. 
\]

This corresponds to the equation (13).
4 TL hierarchy

In this section we will give equivalent expressions for the relations \((\text{12})\) and \((\text{19})\) in the terminology of the TL hierarchy. These conditions specify \(\tau(n)\) \((\text{3})\). For this purpose let us begin by reviewing the TL hierarchy along the line of those in \([7, 13]\).

The wave operators of TL hierarchy in \(\alpha\)-gauge which we will denote as \(W^{(\infty)}_{\alpha}(n; e^{\partial_n})\) are given by

\[
W^{(\infty)}_{\alpha}(n; e^{\partial_n}) = \sum_{j \geq 0} u_{j, \alpha}^{(\infty)}(n) e^{\mp j \partial_n},\]  

(23)

where \(u_{j, \alpha}^{(\infty)}(n) = u_{j, \alpha}^{(\infty)}(n; t, \tilde{t})\), and \(e^{\partial_n}\) is a difference operator with respect to \(n\). The coefficients \(u_{j, \alpha}^{(\infty)}(n)\) can be described in terms of a \(\tau\) function of the TL hierarchy :

\[
u_{j, \alpha}^{(\infty)}(n) = \frac{P_{\tau}(-\hbar \bar{\partial}_\alpha) \tau(n)}{\tau(n + 1)^{\frac{1}{2} - \alpha} \tau(n)^{\frac{1}{2} + \alpha}}, \quad u_{j, \alpha}^{(0)}(n) = \frac{P_{\tau}(-\hbar \bar{\partial}_\alpha) \tau(n + 1)}{\tau(n + 1)^{\frac{1}{2} - \alpha} \tau(n)^{\frac{1}{2} + \alpha}}.\]  

(24)

The time evolutions of these wave operators are

\[
\hbar \partial_{t_m} W^{(\infty)}_{\alpha}(n) = B^{\text{TL}}_{m, \alpha}(n) W^{(\infty)}_{\alpha}(n) - W^{(\infty)}_{\alpha}(n) e^{m \partial_n},
\]

\[
\hbar \partial_{t_m} W^{(0)}_{\alpha}(n) = \overline{B}^{\text{TL}}_{m, \alpha}(n) W^{(0)}_{\alpha}(n),
\]

\[
\hbar \partial_{\bar{t}_m} W^{(0)}_{\alpha}(n) = \overline{B}^{\text{TL}}_{m, \alpha}(n) W^{(0)}_{\alpha}(n) - W^{(0)}_{\alpha}(n) e^{-m \partial_n},
\]

where the difference operators \(B^{\text{TL}}_{m, \alpha}(n)\) and \(\overline{B}^{\text{TL}}_{m, \alpha}(n)\) are \(\bar{1}\)

\[
B^{\text{TL}}_{m, \alpha}(n) = (I^{\text{TL}}_{\alpha}(n)^m)_+ + \left(\frac{1}{2} + \alpha\right)(L^{\text{TL}}_{\alpha}(n)^m)_0,
\]

\[
\overline{B}^{\text{TL}}_{m, \alpha}(n) = (I^{\text{TL}}_{\alpha}(n)^m)_- + \left(\frac{1}{2} - \alpha\right)(L^{\text{TL}}_{\alpha}(n)^m)_0,
\]

with the following Lax operators :

\[
I^{\text{TL}}_{\alpha}(n) = W^{(\infty)}_{\alpha}(n) e^{\partial_n} W^{(\infty)}_{\alpha}(n)^{-1},
\]

\[
\overline{I}^{\text{TL}}_{\alpha}(n) = W^{(0)}_{\alpha}(n) e^{-\partial_n} W^{(0)}_{\alpha}(n)^{-1}.
\]

\(^{3}\) Notice that \((\ )\pm,0\) are projection operators with respect to \(e^{\partial_n}\):

\[
(e^{m \partial_n})_+ = \begin{cases} e^{m \partial_n} & m \geq 1, \\ 0 & m \leq 0 \end{cases},
\]

\[
(e^{m \partial_n})_0 = \delta_{m, 0}, \quad (e^{m \partial_n})_- = e^{m \partial_n} - (e^{m \partial_n})_+ - (e^{m \partial_n})_0.
\]
The wave functions of TL hierarchy are defined by
\[
\psi^{(\infty)}(\alpha; n; \lambda) = W^{(\infty)}(\alpha; n) e^{\frac{1}{\hbar} \eta(t; \lambda)} \lambda^n,
\]
\[
\psi^{(0)}(\alpha; n; \lambda) = W^{(0)}(\alpha; n) e^{\frac{1}{\hbar} \eta(t; \frac{1}{\hbar})} \lambda^n,
\]
on which the Lax operators act as
\[
L^{TL}_\alpha(n)\psi^{(\infty)}(\alpha; n; \lambda) = \lambda \psi^{(\infty)}(\alpha; n; \lambda),
\]
\[
M^{TL}_\alpha(n)\psi^{(\infty)}(\alpha; n; \lambda) = \hbar \partial_\lambda \psi^{(\infty)}(\alpha; n; \lambda),
\]
\[
\overline{L}^{TL}_\alpha(n)\psi^{(0)}(\alpha; n; \lambda) = \frac{1}{\lambda} \psi^{(0)}(\alpha; n; \lambda),
\]
\[
\overline{M}^{TL}_\alpha(n)\psi^{(0)}(\alpha; n; \lambda) = -\hbar \lambda^2 \partial_\lambda \psi^{(0)}(\alpha; n; \lambda),
\]
where we also introduce the Orlov operators \( M^{TL}_\alpha(n) \) (\( \equiv M^{TL}_\alpha(n; e^{\bar{\lambda} \hbar}) \)), \( \overline{M}^{TL}_\alpha(n) \) (\( \equiv \overline{M}^{TL}_\alpha(n; e^{\bar{\lambda} \hbar}) \)) by
\[
M^{TL}_\alpha(n) = W^{(\infty)}(\alpha; n) \left( \sum_{l \geq 1} h_l e^{-(l-1)\partial_\alpha + \hbar n e^{-\partial_\alpha}} \right) W^{(\infty)}(\alpha; n)^{-1},
\]
\[
\overline{M}^{TL}_\alpha(n) = W^{(0)}(\alpha; n) \left( \sum_{l \geq 1} \bar{h}_l e^{-(l-1)\partial_\alpha - \hbar n e^{\partial_\alpha}} \right) W^{(0)}(\alpha; n)^{-1}.
\]

With the above brief review of the TL hierarchy we shall return to our specific solution of the TL hierarchy \((\ref{eq:specific_solution})\). We first notice that, in \( \alpha = \frac{1}{2} \) gauge, one can expand \( \overline{B}^{TL}_{k; \beta, \alpha=\frac{1}{2}}(n) \) by negative powers of \( L^{TL}_{\alpha=\frac{1}{2}} \):
\[
\overline{B}^{TL}_{k; \beta, \alpha=\frac{1}{2}}(n) \equiv \left( \overline{L}^{TL}_{\alpha=\frac{1}{2}}(n; k\beta) \right) = \sum_{l \geq 1} P_l(\hbar \tilde{\partial}_l)(\hbar \partial_{\tilde{\partial}_l}) \ln \tau(n) L^{TL}_{\alpha=\frac{1}{2}}(n)^{-1}.
\]

By comparing the equation \((\ref{eq:specific_solution})\) with the equation \((\ref{eq:specific_solution1})\) one can see \( W(n) \overline{P}_{\beta} W(n)^{-1} \) plays the same role as \( \overline{L}^{TL}_{\alpha=\frac{1}{2}}(n; \beta) \). In particular the actions of these two operators on the wave function \( \psi^{(\infty)}(\alpha; n; \lambda) \) should be same:
\[
W(n) \overline{P}_{\beta} W(n)^{-1} \psi^{(\infty)}(\alpha=\frac{1}{2}; n; \lambda) = \overline{L}^{TL}_{\alpha=\frac{1}{2}}(n)^{\beta} \psi^{(\infty)}(\alpha=\frac{1}{2}; n; \lambda).
\]

Note that the Lax operators \( L^{KP}_{\alpha=\frac{1}{2}}(n) \), \( L^{TL}_{\alpha=\frac{1}{2}}(n) \) and the Orlov operators \( M^{KP}_{\alpha=\frac{1}{2}}(n), M^{TL}_{\alpha=\frac{1}{2}}(n) \) act on the wave function \( \psi^{(\infty)}(\alpha=\frac{1}{2}; n; \lambda) \) as
\[
L^{KP}_{\alpha=\frac{1}{2}}(n) \psi^{(\infty)}(\alpha=\frac{1}{2}; n; \lambda) = \lambda \psi^{(\infty)}(\alpha=\frac{1}{2}; n; \lambda),
\]
\[
M^{KP}_{\alpha=\frac{1}{2}}(n) \psi^{(\infty)}(\alpha=\frac{1}{2}; n; \lambda) = \hbar \partial_\lambda \psi^{(\infty)}(\alpha=\frac{1}{2}; n; \lambda).
\]
Hence from the above equalities (28) and (27) we can express the relation (12) in the terminology of the TL hierarchy:

$$T^{TL}_{\alpha=-\frac{1}{2}}(n)^\beta = F(T^{TL}_{\alpha=\frac{1}{2}}(n), M^{TL}_{\alpha=-\frac{1}{2}}(n)).$$

(28)

This condition should be independent of the gauge. From the explicit forms (24) of $W^{(\gamma)}_{\alpha}(n)$ their gauge transforms are simple: $W^{(\gamma)}_{\alpha}(n) = f(n; \gamma)W^{(\gamma)}_{\alpha}(n)$. The equation (28) is preserved under the gauge transformation. Therefore the relation (12) is equivalent to

$$L^{TL}(n)^\beta = \frac{1}{\beta} \left( L^{TL}(n) \right)^{\beta} \left\{ M^{TL}(n) + \left( \beta + \frac{\hbar}{2}(1 + \beta) \right) L^{TL}(n)^{-1} \right\} L^{TL}(n).$$

(29)

This is one of two relations which will specify $e^F$ (4) in the framework of TL hierarchy.

We shall try to obtain the another equation which is characteristic of $c = 1$ string with a compactification radius $\beta \in \mathbb{Z}_{\geq 1}$. For this purpose we will examine $\alpha = -\frac{1}{2}$ gauge of TL hierarchy. In $\alpha = -\frac{1}{2}$ gauge, $B^{TL}_{k\beta, \alpha=-\frac{1}{2}}(n)$ can be expanded by the negative powers of $T^{TL}_{\alpha=-\frac{1}{2}}(n)$:

$$B^{TL}_{k\beta, \alpha=-\frac{1}{2}}(n) = \left( L^{TL}_{\alpha=-\frac{1}{2}}(n)^{k\beta} \right)_+ = \sum_{l \geq 1} P_l(-h\partial_l)(h\partial_{l\beta}) \ln r(n + 1)\overline{T}^{TL}_{\alpha=-\frac{1}{2}}(n)^{-l}. \quad (30)$$

By comparing the equation (30) with the equation (22) one can see that $\overline{W}(n)P\overline{W}(n)^{-1}$ corresponds to $L^{TL}_{\alpha=-\frac{1}{2}}(n)^\beta$:

$$\overline{W}(n)P\overline{W}(n)^{-1}\psi^{(0)}_{\alpha=-\frac{1}{2}}(n; \lambda) = L^{TL}_{\alpha=-\frac{1}{2}}(n)^\beta\psi^{(0)}_{\alpha=-\frac{1}{2}}(n; \lambda).$$

Since the Lax operators $L^{KP}(n)$, $L^{TL}_{\alpha=-\frac{1}{2}}(n)$ and the Orlov operators $M^{KP}(n)$, $M^{TL}_{\alpha=-\frac{1}{2}}(n)$ act on the wave function $\psi^{(0)}_{\alpha=-\frac{1}{2}}(n; \lambda)$ as

$$L^{KP}(n)\psi^{(0)}_{\alpha=-\frac{1}{2}}(n; \lambda) = L^{TL}_{\alpha=-\frac{1}{2}}(n)\psi^{(0)}_{\alpha=-\frac{1}{2}}(n; \lambda) = \frac{1}{\lambda}\psi^{(0)}_{\alpha=-\frac{1}{2}}(n; \lambda),$$

$$\left( M^{KP}(n) + hL^{KP}(n)^{-1} \right)\psi^{(0)}_{\alpha=-\frac{1}{2}}(n; \lambda) = M^{TL}_{\alpha=-\frac{1}{2}}(n)\psi^{(0)}_{\alpha=-\frac{1}{2}}(n; \lambda) = -h\lambda^2\partial_\lambda\psi^{(0)}_{\alpha=-\frac{1}{2}}(n; \lambda),$$

the relation (14) can be rephrased into

$$L^{TL}_{\alpha=-\frac{1}{2}}(n)^\beta = -G(T^{TL}_{\alpha=-\frac{1}{2}}(n), M^{TL}_{\alpha=-\frac{1}{2}} - hL^{TL}_{\alpha=-\frac{1}{2}}(n)^{-1}).$$

\(^4\)Here we abbreviate the gauge parameter $\alpha$.\[\]
which turns out the following gauge invariant form:

\[ L^{TL}(n) = -\frac{1}{\beta} \left( L^{TL}(n) \right)^{-\beta} \{ M^{TL}(n) - \left( \beta + \frac{\hbar}{2}(1 - \beta) \right) \bar{L}^{TL}(n)^{-1} \} \bar{L}^{TL}(n). \]  

(31)

It is important to remark that the above obtained relations (29) and (31) are consistent with the commutation relations among the Lax and Orlov operators in the TL hierarchy:

\[ [L^{TL}(n), M^{TL}(n)] = \hbar, \quad [\bar{L}^{TL}(n), M^{TL}(n)] = \hbar. \]

This consistency tells us that one can regard the pair of these relations as the "twistor data" of TL hierarchy [14] which is associated with the solution (3). Thus the relations (29) and (31) characterize the generating function for the tachyon amplitudes of \( c = 1 \) string with a compactification radius \( \beta \in \mathbb{Z}_{\geq 1} \).

5 Comment

Let us first comment on the nature of \( F_0 \), the genus 0 contribution to \( F \). This quantity may be described in terms of the \( \tau \) function of the dispersionless Toda hierarchy [14] which is the \( \hbar \rightarrow 0 \) limit of the TL hierarchy with fixing \( s = \frac{n}{\hbar} \). In this limit the Lax and Orlov operators of the TL hierarchy turn to the counterparts of this dispersionless hierarchy:

\[ L(n) (\equiv L^{TL}(n)) \rightarrow \mathcal{L}(s), \quad M(n) (\equiv M^{TL}(n)L^{TL}(n)) \rightarrow \mathcal{M}(s), \]

\[ \hat{L}(n) (\equiv \bar{L}^{TL}(n)^{-1}) \rightarrow \hat{\mathcal{L}}(s), \quad \hat{M}(n) (\equiv -\bar{M}^{TL}(n)\bar{L}^{TL}(n)) \rightarrow \hat{\mathcal{M}}(s). \]

In this limit the string equation (29) and (31) becomes as follows:

\[ \hat{\mathcal{L}}(s)^{-\beta} = \frac{1}{\beta} \mathcal{L}(s)^{-\beta}(\mathcal{M}(s) + \beta), \quad \mathcal{L}(s)^{\beta} = \frac{1}{\beta} \hat{\mathcal{L}}(s)^{\beta}(\hat{\mathcal{M}}(s) + \beta). \]

Notice that, without any effect on the physical quantities, we can shift \( s \) to \( s - \beta \) in the definition of the Orlov operators \( \mathcal{M}(s) \) and \( \hat{\mathcal{M}}(s) \) [14]. With this harmless shift we obtain

\[ \hat{\mathcal{L}}(s)^{-\beta} = \frac{1}{\beta} \mathcal{L}(s)^{-\beta} \mathcal{M}(s), \quad \mathcal{L}(s)^{\beta} = \frac{1}{\beta} \hat{\mathcal{L}}(s)^{\beta} \hat{\mathcal{M}}(s). \]  

(32)

In the case of \( \beta = 1 \) this is precisely the proposed twistor data appropriate to describe the classical (genus 0) tachyon dynamics at the self dual radius \( \beta \).

\footnote{We follow the notation of the reference [14].}
In this classical limit the $\hbar$–dependent parts in (29) and (31) vanish. Nevertheless this vanishing parts can be expected to play an important role in the nonperturbative definition of the theory. For an example, in the case of $A_{N-1}$ topological string, the partition function is the $\tau$ function of the $N$–reduced KP hierarchy specified by the following form of the string equation [5]:

$$P = L^N, \quad Q = \frac{1}{N} \left( M - NL^N - \frac{N-1}{N} \hbar L^{-1} \right) L^{1-N}.$$  

The $\hbar$–dependent term in (33) is crucial for the matrix integral realization of this topological string [5]. And also, from the analysis of the above equation (33) some geometrical nature of the topological string has been revealed through the ”genus expansion” [15]. Thus we can also expect that some geometry of $c = 1$ string theory appears from the study of the characteristic relations (29) and (31). This study will be reported elsewhere.

The author would like to thank Prof.T.Eguchi for several discussions and hospitality during his stay at Hongo, Univ. of Tokyo. He would also like to thank Prof.M.Noumi, Dr.T.Takebe for useful discussions and comments, and Prof.K.Takasaki for the comments on the first version of this note.

References

[1] R.Dijkgraaf, G.Moore and R.Plesser, Nucl.Phys.B394 (1993)356.
[2] A.Hanany, Y.Oz and R.Plesser, Nucl.Phys.B425(1994)150.
L.Bonora and C.S.Xiong, preprint SISSA-ISAS54/94/EP (May,1994).
[3] K.Takasaki, Kyoto preprint KUCP0067/47 (March,1994).
T.Eguchi and H.Kanno, Phys.Lett.B331(1994)330.
[4] M.Douglas, Phys.Lett. B238(1990)176.
[5] M.Adler and P.Van Moerbeke,Commun.Math.Phys.147(1992)25.
A.S.Schwarz, Mod.Phys.Lett.A6(1991)2713.
[6] G.Moore,R.Plesser and S.Ramgoolam,Nucl.Phys.B377(1992)143.
I.R.Klebanov and D.Lowe,Nucl.Phys.B363(1991)543.
[7] K.Ueno and K.Takasaki, Advanced Studies in Pure Math.4 (1984)1.
[8] T.Takebe, Publ.RIMS.27 (1991)491.
[9] M.Sato and Y.Sato, Lect.Notes in Num.Anal.5(1982)259.
[10] E.Date, M.Kashiwara, M.Jimbo and T.Miwa, in *Nonlinear Integrable Systems* (WorldScientific, 1983)39.

[11] M.Sato and M.Noumi, Sophia Univ.Koukyuroku in Math.18(1984), in Japanese.

[12] P.G.Grinevich and A.Yu.Orlov, in *Problems of Modern Quantum Field theory* (Springer-Verlag, 1989)

[13] T.Takebe, Commun.Math.Phys.129(1990)281.

[14] K.Takasaki and T.Takebe, preprint UTMS 94-35 (May,1994).

[15] T.Nakatsu, A.Kato, M.Noumi and T.Takebe, Phys.Lett.*B*322 (1994)192. A.Kato, T.Nakatsu, M.Noumi and T.Takebe, in preparation.