ATTRACTORS FOR FIRST ORDER LATTICE SYSTEMS WITH ALMOST PERIODIC NONLINEAR PART

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Abstract. We study the existence of the uniform global attractor for a family of infinite dimensional first order non-autonomous lattice dynamical systems of the following form:

\[ u + Au + \alpha u + f(u, t) = g(t), \quad (g, f) \in \mathcal{H}((g_0, f_0)), \quad t > \tau, \tau \in \mathbb{R}, \]

with initial data

\[ u(\tau) = u_{\tau}. \]

The nonlinear part of the system \( f(u, t) \) presents the main difficulty of this work. To overcome this difficulty we introduce a suitable Banach space \( W \) of functions satisfying (3)-(7) with norm (8) such that \( f_0(\cdot, t) \) is an almost periodic function of \( t \) with values in \( W \) and \( (g, f) \in \mathcal{H}((g_0, f_0)) \).

1. Introduction. Lattice dynamical systems (LDSs) occur in a wide variety of applications in science and engineering, for instance, in propagation of nerve pulses in myelinated axons [10, 11, 28, 29], electrical engineering [14], pattern recognition [18, 20, 33], image processing [21, 22, 23], chemical reaction theory [24, 27], etc. In each case, they have their own form, but in some cases, they appear as spatial discretizations of corresponding continuous partial differential equations (PDEs) on unbounded domains [15, 17]. The traveling solutions for LDSs have been examined by [7, 8, 19, 24, 45] and the chaotic properties or pattern formation properties of solutions for LDSs have been investigated by [18, 20, 33].

The global attractor is a significant tool to investigate the asymptotic behavior of a dissipative dynamical system since it is the smallest compact set, with respect to inclusion, that is invariant, attracts all the trajectories originated from the whole phase space and sometimes it has finite dimension. On unbounded domains, it is not easy to introduce the global attractors for continuous PDEs because some difficulties appear such as well-posedness and lack of compactness of Sobolev embeddings. Therefore it is important to study the existence of global attractors for LDSs because of the importance of such systems and they can be regarded as spatial discretizations of such PDEs.

Recently, The existence of global attractors, uniform attractors, pullback attractors, and random attractors for different types of autonomous, non-autonomous,
and stochastic LDSs in standard and weighted spaces have been carefully investigated \cite{1, 2, 3, 4, 5, 6, 9, 12, 13, 25, 26, 31, 34, 36, 37, 38, 39, 40, 41, 43, 44}. For first order LDSs, the existence of global attractors for autonomous systems \cite{9, 42, 43} and the existence of uniform global attractors for non-autonomous systems \cite{4, 36} have been studied. We will investigate the existence of the uniform global attractor for a new family of first order non-autonomous LDSs of the following form,

\[
\dot{u}_i + (Au)_i + \alpha u_i + f_i (u_i, t) = g_i (t), \quad i \in \mathbb{Z}^n, t > \tau, \tau \in \mathbb{R}
\]

with initial data

\[
u_i (\tau) = u_{i, \tau}, \forall i \in \mathbb{Z}^n.
\]

2. Preliminaries. For \( n \in \mathbb{N} \), consider the Hilbert space

\[
l^2 = \left\{ u = (u_i)_{i \in \mathbb{Z}^n} : i = (i_1, i_2, ..., i_n) \in \mathbb{Z}^n, u_i \in \mathbb{R}, \sum_{i \in \mathbb{Z}^n} u_i^2 < \infty \right\},
\]

whose inner product and norm are given by:

\[
\langle u, v \rangle_{l^2} = \sum_{i \in \mathbb{Z}^n} u_i v_i, \|u\|_{l^2} = \langle u, u \rangle_{l^2}^{1/2}, \forall u = (u_i)_{i \in \mathbb{Z}^n}, v = (v_i)_{i \in \mathbb{Z}^n} \in l^2.
\]

Let \( W \) be the set of functions \( \varphi \) such that

\[
\varphi (u) = (\varphi_i (u_i))_{i \in \mathbb{Z}^n}, \quad \forall u = (u_i)_{i \in \mathbb{Z}^n}, u_i \in \mathbb{R},
\]

\[
\varphi_i \in C^1 (\mathbb{R}, \mathbb{R}), \quad \forall i \in \mathbb{Z}^n,
\]

\[
\varphi_i (0) = 0, \quad \forall i \in \mathbb{Z}^n,
\]

and

\[
\sup_{i \in \mathbb{Z}^n} \sup_{s \in \mathbb{R}} \left( \frac{\varphi_i (s)}{|s|^{p+1} + a_i} + \frac{\varphi'_i (s)}{|s|^p + a_i} \right) < \infty,
\]

where

\[
p \geq 0, \quad 0 < a_i < 1, \quad a = (a_i)_{i \in \mathbb{Z}^n} \in l^2,
\]

and \( C^1 (\mathbb{R}, \mathbb{R}) \) is the space of continuously differentiable functions from \( \mathbb{R} \) into \( \mathbb{R} \).

In section 5, we show that \( W \) is a real Banach space whose norm is defined by

\[
\| \varphi \|_W = \sup_{i \in \mathbb{Z}^n} \sup_{s \in \mathbb{R}} \left( \frac{\varphi_i (s)}{|s|^{p+1} + a_i} + \frac{\varphi'_i (s)}{|s|^p + a_i} \right), \quad \forall \varphi \in W.
\]

Note that (6) is acceptable. In fact, it is enough to assume that there exists a positive constant \( \beta (\varphi) \) depending on \( \varphi \) such that

\[
|\varphi'_i (s)| \leq \beta (\varphi) (|s|^p + a_i), \quad i \in \mathbb{Z}^n, s \in \mathbb{R},
\]

then taking into account (5) and (7), it follows that

\[
|\varphi_i (s)| \leq \beta (\varphi) \left( \frac{|s|^{p+1}}{p+1} + a_i |s| \right) \leq 2 \beta (\varphi) \left( |s|^{p+1} + a_i \right), \quad i \in \mathbb{Z}^n, s \in \mathbb{R}.
\]

Lemma 2.1. If \( \varphi \in W \), then \( \varphi \) maps \( l^2 \) into \( l^2 \) and it is locally Lipschitz continuous from \( l^2 \) into \( l^2 \).
Proof. Let $u = (u_i)_{i \in \mathbb{Z}^n} \in l^2$ and $\varphi \in W$ with $\varphi(u) = (\varphi_i(u_i))_{i \in \mathbb{Z}^n}$, from (8) there exists a positive constant $\beta(\varphi)$ such that for $i \in \mathbb{Z}^n$ we have
\[
|\varphi_i(u_i)| \leq \beta(\varphi) \left(|u_i|^{p+1} + a_i\right),
\]
and
\[
|\varphi'_i(u_i)| \leq \beta(\varphi) (|u_i|^p + a_i).
\]
From (11) it is clear that $\varphi$ maps $l^2$ into $l^2$, and from the mean value theorem and (12) it is clear that $\varphi$ is locally Lipschitz continuous from $l^2$ into $l^2$.

Let $C_b(\mathbb{R}, X)$ be the Banach space of bounded continuous functions on $\mathbb{R}$ with values in a Banach space $X$ whose norm is given by
\[
\|\sigma\|_{C_b(\mathbb{R}, X)} = \sup_{t \in \mathbb{R}} \|\sigma(t)\|_X, \quad \sigma \in C_b(\mathbb{R}, X),
\]
and let $\sigma_0 : \mathbb{R} \to X$ be an almost periodic function in time $t$ with values in $X$. Along the lines of the Bochner’s criterion [30], the set of translations $\{\sigma_0(\cdot + h) : h \in \mathbb{R}\}$ is precompact in $C_b(\mathbb{R}, X)$. The closure of this set in $C_b(\mathbb{R}, X)$ is said to be the Hull $\mathcal{H}(\sigma_0)$ of the function $\sigma_0(t)$:
\[
\mathcal{H}(\sigma_0) = \{\sigma_0(\cdot + h) : h \in \mathbb{R}\} \subset C_b(\mathbb{R}, X).
\]
Moreover, for any $\sigma(t) \in \mathcal{H}(\sigma_0)$, $\sigma$ is almost periodic and $\mathcal{H}(\sigma) = \mathcal{H}(\sigma_0)$.

In this work we assume that:

(A1) $g_0 : \mathbb{R} \to l^2$ with $g_0(t) = (g_{0i}(t))_{i \in \mathbb{Z}^n}$ is an almost periodic function of $t$ with values in $l^2$.

(A2) $f_0(u, t) = (f_{0i}(u_i, t))_{i \in \mathbb{Z}^n}$ is a nonlinear function of $t \in \mathbb{R}$ and $u = (u_i)_{i \in \mathbb{Z}^n}$, $u_i \in \mathbb{R}$, with $f_{0i} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f_0(\cdot, t)$ is an almost periodic function of $t$ with values in $W$ and
\[
f_{0i}(s, t), s \geq 0, \quad s, t \in \mathbb{R}, i \in \mathbb{Z}^n.
\]

(A3) $A : l^2 \to l^2$ is a bounded linear operator of the following form
\[
A = A_1 + A_2 + \ldots + A_n,
\]
and for $j = 1, 2, \ldots, n$, there exist bounded linear operators $D_j : l^2 \to l^2$ given by
\[
(D_j u)_i = \sum_{l=-m_0}^{m_0} d_{j,l} u_{i+l}, \quad u = (u_i)_{i \in \mathbb{Z}^n} \in l^2,
\]
with
\[
\|D_j\|_O \leq c_0,
\]
where $d_{j,l} \in \mathbb{R}$, $m_0$ is a positive integer, $i_{j,l} = (i_1, \ldots, i_{j-1}, i_j + l, i_{j+1}, \ldots, i_n) \in \mathbb{Z}^n$, and $\|\cdot\|_O$ is the norm of a bounded linear operator from $l^2$ into $l^2$, such that
\[
A_j = D_j^* D_j = D_j D_j^*,
\]
where $D_j^*$ is the adjoint operator of $D_j$, that is,
\[
(D_j^* u)_i = \sum_{l=-m_0}^{m_0} d_{j,-l} u_{i+l}, \quad \langle D_j u, v \rangle_{l^2} = \langle u, D_j^* v \rangle_{l^2}, \quad u = (u_i)_{i \in \mathbb{Z}^n}, v = (v_i)_{i \in \mathbb{Z}^n} \in l^2.
\]
From (17)-(18), it follows that
\[
|d_{j,l}| \leq c_0, \quad j = 1, \ldots, n, \quad l = -m_0, \ldots, m_0.
\]
In this work we consider the Banach space $\mathcal{M} = l^2 \times W$ and the time symbol $\sigma_0(t) = (g_0(t), f_0(u(t)))$, where $g_0$ and $f_0$ are given by assumptions (A1) and (A2), respectively, then $\sigma_0(t)$ is almost periodic with values in $\mathcal{M}$, and we consider the symbol space $\mathcal{H}(\sigma_0) = \mathcal{H}((g_0, f_0))$. In such a case, any $\sigma(t) = (g(t), f(u(t))) \in \mathcal{H}((g_0, f_0))$ is almost periodic in $\mathcal{M}$.

Taking into account the LDS (1)-(2), we shall study the existence of the uniform global attractor with respect to $\sigma(t) = (g(t), f(u(t))) \in \mathcal{H}((g_0, f_0))$ for the family of first order non-autonomous LDSs of the following form:

$$\dot{u} + Au + \alpha u + f(u(t)) = g(t) , (g, f) \in \mathcal{H}((g_0, f_0))$$

with initial data

$$u(\tau) = (u_{i, \tau})_{i \in \mathbb{Z}^n} = u_{\tau},$$

where $u = (u_i)_{i \in \mathbb{Z}^n}$, $Au = ((Au_i)_{i \in \mathbb{Z}^n}$, $f(u(t)) = (f_i(u_i(t)))_{i \in \mathbb{Z}^n}$, and $g(t) = (g_i(t))_{i \in \mathbb{Z}^n}$.

3. Global solutions and absorbing sets.

**Lemma 3.1.** Considering the almost periodic functions $g_0$ and $f_0$ given by assumptions (A1) and (A2), respectively, there exist constants $\delta(g_0) \geq 0$ and $\beta(f_0) \geq 0$ depending on $g_0$ and $f_0$, respectively, such that for $\sigma(t) = (g(t), f(u(t))) \in \mathcal{H}(\sigma_0)$ with $\sigma_0(t) = (g_0(t), f_0(u(t)))$, we have

$$\|g\|_{C_b(\mathbb{R}, l^2)} = \sup_{t \in \mathbb{R}} \|g(t)\|_{l^2} = \delta(g_0),$$

and for $s, t \in \mathbb{R}$, and $i \in \mathbb{Z}^n$,

$$f_i(s, t) \geq 0,$$

$$|f_i(s, t)| \leq \beta(f_0) \left(|s|^{p+1} + a_i\right),$$

and

$$|\partial_s f_i(s, t)| \leq \beta(f_0) \left(|s|^p + a_i\right),$$

where $\partial_s f_i(s, t)$ is the partial derivative of $f_i(s, t)$ with respect to $s$.

**Proof.** Since $g_0$ and $f_0$ are almost periodic functions of $t$ with values in $l^2$ and $W$, respectively, then $g_0 \in C_b(\mathbb{R}, l^2)$ and $f_0 \in C_b(\mathbb{R}, W)$, that is, there exist constants $\delta(g_0) \geq 0$ and $\beta(f_0) \geq 0$ depending on $g_0$ and $f_0$, respectively, such that

$$\|g_0\|_{C_b(\mathbb{R}, l^2)} = \sup_{t \in \mathbb{R}} \|g_0(t)\|_{l^2} = \delta(g_0),$$

and

$$\|f_0\|_{C_b(\mathbb{R}, l^2)} = \sup_{t \in \mathbb{R}} \sup_{i \in \mathbb{Z}^n} \sup_{s \in \mathbb{R}} \left(\frac{|f_{0i}(s, t)|}{|s|^{p+1} + a_i} + \frac{|\partial_s f_{0i}(s, t)|}{|s|^p + a_i}\right) = \beta(f_0),$$

that is, for $s, t \in \mathbb{R}$, and $i \in \mathbb{Z}^n$,

$$|f_{0i}(s, t)| \leq \beta(f_0) \left(|s|^{p+1} + a_i\right),$$

$$|\partial_s f_{0i}(s, t)| \leq \beta(f_0) \left(|s|^p + a_i\right).$$

Since $\sigma(t) = (g(t), f(u(t))) \in \mathcal{H}((g_0, f_0))$ and $\mathcal{H}((g_0, f_0))$ is given by (14) with $\sigma_0(t) = (g_0(t), f_0(u(t)))$, there exists a sequence of functions $\{\sigma_0(\cdot + h_m)\}_{m \in \mathbb{N}}$ such that as $m \to \infty$,

$$\|\sigma(\cdot) - \sigma_0(\cdot + h_m)\|_{C_b(\mathbb{R}, l^2)} = \sup_{t \in \mathbb{R}} \|\sigma(t) - \sigma_0(t + h_m)\|_{l^2} \to 0,$$
which implies that
\[ \sup_{t \in \mathbb{R}} \| g(t) - g_0(t + h_m) \|_{L^2} \to 0, \quad (32) \]
and
\[ \sup_{t \in \mathbb{R}} \| f(\cdot, t) - f_0(\cdot, t + h_m) \|_{W} \to 0, \]
that is,
\[ \sup_{t \in \mathbb{R}} \sup_{v \in \mathbb{R}} \sup_{s \in \mathbb{R}} \left( \left| f_i(s, t) - f_0(i, s + t + h_m) \right| + \left| \partial_s f_i(s, t) - \partial_s f_0(i, s + t + h_m) \right| \right) \to 0. \quad (33) \]

From (28) and (32), it is clear that (24) is satisfied. Following (33), we find that (25), (26), and (27) are satisfied since (15), (30), and (31) are satisfied.

**Lemma 3.2.** For \( \sigma(t) = (g(t), f(u, t)) \in \mathcal{H}((g_0, f_0)) \), the following are satisfied:
\( (a) \) \( f(u, t) \) maps \( L^2 \times \mathbb{R} \) into \( L^2 \) and for \( R > 0 \), \( u, v \in L^2 \), with \( \| u \|_{L^2} \leq R \), \( \| v \|_{L^2} \leq R \), and \( t \in \mathbb{R} \),
\[ \| f(u, t) - f(v, t) \|_{L^2} \leq \beta(f_0)((2R)^p + \| a \|_{L^2}) \| u - v \|_{L^2}, \quad (34) \]
that is, \( f : L^2 \times \mathbb{R} \to L^2 \) is a locally Lipschitz function of \( u \) uniformly on \( t \).

\( (b) \) \( g : R \to L^2 \) and \( f : L^2 \times \mathbb{R} \to L^2 \) are continuous functions of \( t \).

**Proof.** (a) From (26) and (27) the proof is completed.

(b) Since \( (g(t), f(u, t)) \in \mathcal{H}((g_0, f_0)) \subset C_b(R, \mathcal{M}) \) and \( \mathcal{M} = L^2 \times W \), it is clear that \( g : R \to L^2 \) and \( f(\cdot, t) : R \to W \) are continuous functions. We need to show that \( f : L^2 \times \mathbb{R} \to L^2 \) is a continuous function of \( t \). Indeed, since \( f(\cdot, t) : R \to W \) is continuous, then for \( t \in \mathbb{R} \), given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for \( x \in \mathbb{R} \) with \( |x - t| < \delta \) we have
\[ \| f(\cdot, x) - f_0(\cdot, t) \|_{W} < \frac{\varepsilon}{c}, \]
where \( c \) is any positive constant. In such a case we have
\[ \sup_{i \in \mathbb{Z}^n} \sup_{s \in \mathbb{R}} \left( \left| f_i(s, x) - f_i(s, t) \right| \right) \leq \frac{\varepsilon}{c}, \]
and for \( u = (u_i)_{i \in \mathbb{Z}^n} \in L^2 \),
\[ |f_i(u, x) - f_i(u, t)| \leq \frac{\varepsilon}{c} \left( |w_i|^{p+1} + a_i \right), \quad i \in \mathbb{Z}^n. \quad (35) \]
For any fixed \( w = (w_i)_{i \in \mathbb{Z}^n} \in L^2 \), we can assume that \( c \) is sufficiently large such that
\[ \sum_{i \in \mathbb{Z}^n} \left( |w_i|^{p+1} + a_i \right)^2 \leq 2 \sum_{i \in \mathbb{Z}^n} \left( |w_i|^{2p+2} + a_i^2 \right), \quad i \in \mathbb{Z}^n. \quad (36) \]
Using (35)-(36), it follows that
\[ \| f(w, x) - f(w, t) \|_{L^2} < \varepsilon. \]
That is \( f : L^2 \times \mathbb{R} \to L^2 \) is a continuous function of \( t \).

**Lemma 3.3.** Given \( \tau \in \mathbb{R} \), \( u_\tau \in L^2 \), and \( (g, f) \in \mathcal{H}(g_0, f_0) \), there exists a unique local maximal classical solution \( u \) satisfying (22)-(23) in \( L^2 \) such that \( u \in C^1([\tau, T), L^2) \), for some \( T > \tau \). Moreover, if \( T < +\infty \), then
\[ \lim_{t \to T^-} \| u(t) \|_{L^2} = +\infty. \quad (37) \]
Proof. From Lemma 3.2, it is clear that \( g(t) : \mathbb{R} \to l^2 \) and \( f(u, t) : l^2 \times \mathbb{R} \to l^2 \) are continuous functions of \( t \) and \( f : l^2 \times \mathbb{R} \to l^2 \) is a locally Lipschitz function of \( u \) uniformly on \( t \). Since \( A : l^2 \to l^2 \) is a bounded linear operator, \(-A\) is the infinitesimal generator of a \( C_0 \) semigroup on \( l^2 \). Moreover, the domain of \( A \) is \( l^2 \).

In such a case, following [35], specifically Theorems 1.4 and 1.7 of chapter 6, the proof is completed. \( \square \)

**Lemma 3.4.** Given \((g, f) \in \mathcal{H}(g_0, f_0), \tau \in \mathbb{R}, \) and \( u_\tau \in l^2 \), the solution \( u \) of (22)-(23) in \( l^2 \) satisfies

\[
\|u(t)\|_{l^2}^2 \leq e^{\alpha(\tau-t)} \|u_\tau\|_{l^2}^2 + \left(\frac{\delta(g_0)}{\alpha}\right)^2 \left(1-e^{\alpha(\tau-t)}\right), \quad t \geq \tau. \tag{38}
\]

**Proof.** Considering the inner product of (22) with \( u(t) \) in \( l^2 \), we obtain that

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{l^2}^2 + \alpha \|u(t)\|_{l^2}^2 + \sum_{j=1}^{n} \|D_j u(t)\|_{l^2}^2 + (f(u(t), t), u(t))_{l^2} = (g(t), u(t))_{l^2} , \quad t \geq \tau. \tag{39}
\]

From (24), we get

\[
\langle g(t), u(t) \rangle_{l^2} \leq \frac{\alpha}{2} \|u(t)\|_{l^2}^2 + \frac{1}{2\alpha} \|g(t)\|_{l^2}^2 \leq \frac{\alpha}{2} \|u(t)\|_{l^2}^2 + \frac{1}{2\alpha} \left(\delta(g_0)\right)^2, \quad t \geq \tau, \tag{40}
\]

and from (25), we find

\[
(f(u(t), t), u(t))_{l^2} = \sum_{i \in \mathbb{Z}_n} f_i(u_i(t), t) u_i(t) \geq 0, \quad t \geq \tau. \tag{41}
\]

Putting (40)-(41) into (39), it follows that

\[
\frac{d}{dt} \left(e^{\alpha t} \|u(t)\|_{l^2}^2\right) \leq \frac{1}{\alpha} \left(\delta(g_0)\right)^2 e^{\alpha t}, \quad t \geq \tau.
\]

Integrating the last inequality from \( \tau \) into \( t \), we get (38). \( \square \)

Following (37) and (38), we find that the solution \( u \) of (22)-(23) is defined globally for \( t \geq \tau \). In such a case we can introduce a family of processes \( \{U^{g,f}(t, \tau) : t \geq \tau, \tau \in \mathbb{R}\}_{(g,f) \in \mathcal{H}(g_0, f_0)} \) on \( l^2 \) such that for \( \tau \in \mathbb{R}, t \geq \tau, \) and \( u_\tau \in l^2 \), we have \( U^{g,f}(t, \tau) u_\tau = u(t) \), where \( u \) is the solution of (22)-(23). By the unique solvability of (22)-(23), the family of processes \( \{U^{g,f}(t, \tau)\}_{g,f \in \mathcal{H}(g_0, f_0)} \) satisfies the multiplicative properties:

\[
U^{g,f}(t, s) U^{g,f}(s, \tau) = U^{g,f}(t, \tau), \quad t \geq s \geq \tau, \tau \in \mathbb{R},
\]

\[
U^{g,f}(\tau, \tau) = I, \quad \tau \in \mathbb{R},
\]

where \( I \) is the identity operator. Moreover the following translation identity holds for the processes and the translation group \( \{T(h)\}_{h \in \mathbb{R}} : \)

\[
U^{g,f}(t + h, \tau + h) = U^{T(h)(g,f)}(t, \tau), \quad h, \tau \in \mathbb{R}, t \geq \tau,
\]

where

\[
T(h)(g, f) = (f(\cdot + h), g(\cdot + h)), \quad (g, f) \in \mathcal{H}(g_0, f_0). \tag{43}
\]

Along the lines of (38), we get the following lemma.
Lemma 3.5. In $l^2$, the closed bounded ball $B = B(0, R_0)$ with center 0 and radius $R_0 > \frac{d(\gamma_0)\alpha}{a}$ is a uniform absorbing set for the family of processes $\{U^{\gamma,f}(t, \tau) : t \geq \tau, \tau \in \mathbb{R}\}$ corresponding to the LDS (22)-(23). That is, for $\tau \in \mathbb{R}$ and $R_1 \geq 0$, if $\|u_0\|_{l^2} \leq R_1$, then there exists a constant $T_0 = T_0(R_1) \geq 0$ such that

$$
\|U^{\gamma,f}(t, \tau) u_\tau\|_{l^2} \leq R_0, \quad (g, f) \in \mathcal{H}(g_0, f_0), \ t - \tau \geq T_0.
$$

Moreover, there exists a constant $R_2 = R_2(R_1)$ such that

$$
\|U^{\gamma,f}(t, \tau) u_\tau\|_{l^2} \leq R_2, \quad (g, f) \in \mathcal{H}(g_0, f_0), \ t - \tau \in [0, T_0).
$$

Here we present the continuity of the family of processes $\{U^{\gamma,f}(t, \tau)\}_{(\gamma,f) \in \mathcal{H}(g_0, f_0)}$ with respect to $(g, f) \in \mathcal{H}(g_0, f_0)$ and initial data which is necessary to investigate the uniform asymptotic behavior of the family of processes.

Lemma 3.6. Let $(g, f), (g_k, f_k) \in \mathcal{H}(g_0, f_0)$ and $\phi, \phi_k \in l^2$, where $k = 1, 2, \ldots$. If $(g_k, f_k) \rightarrow (g, f)$ in $C_b([0, R \times W])$ and $\phi_k \rightarrow \phi$ in $l^2$ as $k \rightarrow +\infty$, then for $\tau \in \mathbb{R}$ and $t \geq \tau$,

$$
\|U^{g_k,f_k}(t, \tau) \phi_k - U^{\gamma,f}(t, \tau) \phi\|_{l^2} \rightarrow 0, \quad \text{as} \quad k \rightarrow +\infty.
$$

Proof. Let $u_k(t, \tau) = U^{g_k,f_k}(t, \tau) \phi_k, u(t, \tau) = U^{\gamma,f}(t, \tau) \phi, \phi_k \rightarrow \phi$ in $l^2$ as $k \rightarrow +\infty$, then for $\tau \in \mathbb{R}$ and $t \geq \tau$,

$$
\|w_k + Aw_k + \alpha w_k + f_k(u_k, t) - f(u, t) - g_k - g\|_{l^2} < \tau.
$$

Considering the inner product of (47) with $w_k$ in $l^2$, we get

$$
\left\| \frac{1}{2} \partial^2 \alpha \| w_k \|_{l^2}^2 + \sum_{j=1}^n \| D_j w_k \|_{l^2}^2 + \alpha \| w_k \|_{l^2}^2 + \| f_k(u_k, t) - f(u, t), w_k \|_{l^2} = \| g_k - g, w_k \|_{l^2}, \ t > \tau.
$$

By Lemma 3.5, since $\{\phi_k\}_{k=1}^{+\infty}$ is bounded in $l^2$, there exists a constant $R_3 > 0$ such that

$$
\|u\|_{l^2} \leq R_3, \|u_k\|_{l^2} \leq R_3, \|w_k\|_{l^2} \leq R_3, \quad k \geq 1, t \geq \tau.
$$

Using (49), we obtain

$$
\|g_k - g, w_k\|_{l^2} \leq \|g_k - g\|_{l^2} \|w_k\|_{l^2} \leq R_3 \|g_k - g\|_{l^2}, \ t \geq \tau.
$$

and following (34) and (49), we get

$$
\|f_k(u_k, t) - f(u, t), w_k\|_{l^2} \leq \|f_k(u_k, t) - f(u, t), w_k\|_{l^2} \leq \|g_k - g\|_{l^2} \|w_k\|_{l^2}
$$

By (48), (50), and (51), it follows that for $k \geq 1, t > \tau$,

$$
\frac{d}{dt} \|w_k\|_{l^2}^2 - K_1 \|w_k\|_{l^2}^2 \leq 2R_3 \|g_k - g\|_{l^2} \|w_k\|_{l^2} + \|f_k(u, t) - f(u, t), w_k\|_{l^2},
$$

where

$$
K_1 = 1 + 2\beta (\gamma_0)((2R_3)^p + \|a\|_{l^2}).
$$

That is, for $k \geq 1, t > \tau$,

$$
\frac{d}{dt} \left( e^{-K_1 t} \|w_k\|_{l^2}^2 \right) \leq e^{-K_1 t} \left( 2R_3 \|g_k - g\|_{l^2} \|w_k\|_{l^2} + \|f_k(u, t) - f(u, t), w_k\|_{l^2} \right).
$$
Integrating both sides of the last inequality from \( \tau \) into \( t \), we get for \( k \geq 1, t \geq \tau \),
\[
+ \frac{e^{K_1(t-\tau)}}{K_1} \sup_{x \in \mathbb{R}} \|w_k(t,\tau)\|_{L^2}^2 \leq e^{K_1(t-\tau)} \|w_k(\tau,\tau)\|_{L^2}^2
\]
\[
+ \frac{e^{K_1(t-\tau)}}{K_1} \sup_{x \in \mathbb{R}} \left( 2R_3 \|g_k(x) - g(x)\|_{L^2} + \|f_k(u(x,\tau),x) - f(u(x,\tau),x)\|_{L^2} \right).
\]

But
\[
\sup_{x \in \mathbb{R}} \|g_k(x) - g(x)\|_{L^2} \leq \sup_{x \in \mathbb{R}} \|g_k(x) - g(x)\|_{L^2} = \|g_k - g\|_{C_{\infty}(\mathbb{R},L^2)},
\]
and
\[
\sup_{x \in \mathbb{R}} \|f_k(u(x,\tau),x) - f(u(x,\tau),x)\|_{L^2} \leq \left( \sup_{x \in \mathbb{R}} \sup_{\|a\|_{L^2} \leq 2} \left( \left( u_i(x,\tau) \right)^{p+1} + a_i \right)^2 \right) \sum_{x \in \mathbb{R}, n \in \mathbb{Z}} \left( \frac{\|f_k(u(x,\tau),x) - f(u(x,\tau),x)\|_{L^2}}{(u_i(x,\tau))^{p+1} + a_i} \right)^2
\]
\[
\leq \|f_k - f\|_{C_{\infty}(\mathbb{R},W)} \sup_{x \in \mathbb{R}} \sum_{\|a\|_{L^2} \leq 2} \left( \left( u_i(x,\tau) \right)^{p+1} + a_i \right)^2.
\]

Recalling (49), it follows that
\[
\sum_{i \in \mathbb{Z}} \left( u_i(x,\tau) \right)^{p+1} + a_i \leq 2 \sum_{i \in \mathbb{Z}} \left( u_i(x,\tau) \right)^{2p+2} + a_i \leq K_2, \quad x \geq \tau,
\]
where
\[
K_2 = 2 \left( R_3^{2p+2} + \|a\|_{L^2}^2 \right).
\]

From (54)-(55), we get
\[
\sup_{x \in \mathbb{R}} \|f_k(u(x,\tau),x) - f(u(x,\tau),x)\|_{L^2} \leq K_2 \|f_k - f\|_{C_{\infty}(\mathbb{R},W)}.
\]

Putting (53) and (56) into (52), we obtain that for \( k \geq 1, t \geq \tau \),
\[
+ \frac{e^{K_1(t-\tau)}}{K_1} \sup_{x \in \mathbb{R}} \left( 2R_3 \|g_k - g\|_{C_{\infty}(\mathbb{R},L^2)} + K_2 \|f_k - f\|_{C_{\infty}(\mathbb{R},W)} \right).
\]

From the last inequality, it is clear that If \( (g_k, f_k) \to (g, f) \) in \( C_b(\mathbb{R}, L^2 \times W) \) and \( \phi_k \to \phi \) in \( L^2 \) as \( k \to +\infty \), then for \( \tau \in \mathbb{R} \) and \( t \geq \tau \), (46) is satisfied.

4. Uniform global attractors.

**Lemma 4.1.** Given \( \eta > 0 \), there exist constants \( I = I(\eta) \) and \( T = T(\eta) \) such that for \( (g, f) \in \mathcal{H}(g_0, f_0), \tau \in \mathbb{R}, t \geq \tau \), and \( u_\tau \in \mathcal{B} \), the solution \( u \) of (22)-(23) satisfies
\[
\sum_{\|u\|_{L^2} \geq I} (u_i(t))^2 \leq \eta, \forall t - \tau \geq T.
\]

**Proof.** Consider a smooth increasing function \( \theta \in C^1(\mathbb{R}^+, \mathbb{R}) \) such that
\[
\begin{cases}
\theta(s) = 0, & 0 \leq s < 1, \\
0 \leq \theta(s) \leq 1, & 1 \leq s < 2, \\
\theta(s) = 1, & 2 \leq s,
\end{cases}
\]
and there exists a constant \( M_0 \) such that
\[
|\theta'(s)| \leq M_0, \forall s \geq 0.
\]
Since $g$ is almost periodic with values in $l^2$, the set $\{(g_i(t))_{i \in \mathbb{Z}^n} : t \in \mathbb{R}\}$ is precompact in $l^2$, and hence for $\eta > 0$, there exists a constant $K_3(g, \eta)$ depending on $g$ and $\eta$ such that
\[
\sum_{\|i\|_m \geq K_3} g_i^2(t) \leq \frac{\eta \alpha^2}{4}, \quad \forall t \in \mathbb{R},
\] (59)
where $\alpha$ is given by (22). But $g \in \mathcal{H}(g_0)$ and $\mathcal{H}(g_0)$ is compact in $C_b(\mathbb{R}, l^2)$. In such a case, taking into account (59), it follows that there exists a constant $K_4(\eta)$ depends on $\eta$ but it is independent of $g$, such that
\[
\sum_{\|i\|_m \geq K_4} g_i^2(t) \leq \frac{\eta \alpha^2}{4}, \quad \forall t \in \mathbb{R}.
\] (60)

Let us choose $K_5 = K_5(\eta)$ to be a positive integer such that
\[
K_5 = \max \left\{ K_4, \frac{32nR_0^2M_0m_0^2\varepsilon_0^2}{\eta \alpha} \right\},
\] (61)
and for each $i \in \mathbb{Z}^n$, let $z_i = \theta \left( \frac{\|i\|_m}{K_5} \right) u_i$ and $z = (z_i)_{i \in \mathbb{Z}^n}$. Taking the inner product of (22) with $z$ in $l^2$, we obtain that
\[
\begin{align*}
\frac{1}{2} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) \frac{d}{dt} u_i^2(t) + \sum_{j=1}^{n} \langle D_j u(t), D_j z(t) \rangle + \alpha \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) u_i^2(t) \\
+ \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) f_i(u_i(t), t) u_i(t) = \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) g_i(t) u_i(t), \forall t - \tau \geq 0.
\end{align*}
\] (62)

Using (25), we get
\[
\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) f_i(u_i(t), t) u_i(t) \geq 0, \forall t - \tau \geq 0,
\] (63)
and by (57), (60), and (61), we find
\[
\begin{align*}
\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) g_i(t) u_i(t) \\
\leq \frac{1}{2n} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) u_i^2(t) + \frac{\alpha}{2} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) u_i^2(t) \\
\leq \frac{\eta \alpha}{8} + \frac{\alpha}{2} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) u_i^2(t), \forall t - \tau \geq 0.
\end{align*}
\] (64)

Taking into account (21), (44), (58), and following (31) of [42], there exists $T_1 = T_1(R_0)$ such that
\[
\begin{align*}
= \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) (D_j u(t))^2_i + \sum_{i \in \mathbb{Z}^n} (D_j u(t))^2_i \left( (D_j z(t))_i - \theta \left( \frac{\|i\|_m}{K_5} \right) (D_j u(t))_i \right) \\
\geq \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) (D_j u(t))^2_i - \frac{4R_0^2M_0m_0^2\varepsilon_0^2}{K_5}, \forall j = 1, 2, ..., n, t - \tau \geq T_1,
\end{align*}
\]
and from (61), we find
\[
\langle D_j u(t), D_j z(t) \rangle \geq \frac{\eta \alpha}{8n}, \forall j = 1, 2, ..., n, t - \tau \geq T_1.
\] (65)

Recalling (62)-(65), it follows that
\[
\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) \frac{d}{dt} u_i^2(t) + \alpha \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_m}{K_5} \right) u_i^2(t) \leq \frac{\eta \alpha}{2}, \forall t - \tau \geq T_1.
\]
That is,
\[
\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|\| m}{K_5} \right) \frac{d}{dt} (e^{\alpha t} u_i^2 (t)) \leq \frac{\eta \alpha}{2} e^{\alpha t}, \forall t - \tau \geq T_1.
\]
Integrating the last inequality from \(\tau\) into \(t\), we get
\[
\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|\| m}{K_5} \right) u_i^2 (t) \leq e^{\alpha (\tau - t)} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|\| m}{K_5} \right) u_i^2 (\tau) + 2 \frac{\eta}{\alpha}, \forall t - \tau \geq T_1.
\]
Thus for \(I = I (\eta) = 2K_5 (\eta)\) and \(T = T (\eta) = \max \left\{ T_1, \frac{1}{\alpha} \ln \left( \frac{2R_0^2}{\eta} \right) \right\}\), we have
\[
\sum_{\|\| m \geq I} (u_i (t))^2 \leq \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|\| m}{K_5} \right) u_i^2 (t) \leq \eta, \forall t - \tau \geq T.
\]

In \(L^2\), let us connect the family of processes \(\{ U^{g,f} (t, \tau) : t \geq \tau, \tau \in \mathbb{R} \}\) corresponding to the LDS (22)-(23) with an associated semigroup of nonlinear operators, where we use the semigroup theory to investigate the existence of the uniform global attractor of the processes. Along the lines of [16], we define the nonlinear semigroup \(\{ S (t) \} \) associated with the LDS (22)-(23) acting on the extended phase space \(L^2 \times H (g_0, f_0)\) by
\[
S (t) (u, (g, f)) = (U^{g,f} (t, 0) u, T (t) (g, f)) , t \geq 0, u \in L^2, (g, f) \in H (g_0, f_0).
\]
In such a case, we find that \(\{ S (t) \} \) satisfies the semigroup identities:
\[
S (s) S (t) = S (t + s), S (0) = I, \forall t, s \geq 0.
\]

**Lemma 4.2.** The solution semigroup \(\{ S (t) \} \) associated with the LDS (22)-(23) is asymptotically compact, that is, if \(\{ (u_n, (f_n, g_n)) \} \) is bounded in \(L^2 \times H (g_0, f_0)\), and \(t_n \to \infty\), then \(\{ S (t_n) (u_n, (f_n, g_n)) \} \) is precompact in \(L^2 \times H (g_0, f_0)\).

**Proof.** Using Lemmas 3.5 and 4.1, above, the proof is similar to that of Lemma 5.4 [36].

**Definition 4.3.** In \(L^2\), a closed set \(A\) is called the uniform attractor for the family of processes \(\{ U^{g,f} (t, \tau) \} \) with respect to \((g, f) \in H (g_0, f_0)\) if:

(a) For any bounded set \(G \subseteq L^2\),
\[
\lim_{t \to \infty} \sup_{(g, f) \in H (g_0, f_0)} \text{dist} (U^{g,f} (t, \tau) G, A) = 0, \forall \tau \in \mathbb{R}.
\]

(b) (Minimal property) If \(\overline{A}\) is any closed subset of \(L^2\) satisfying property (a), then \(A \subseteq \overline{A}\).

**Definition 4.4.** Given \((g, f) \in H (g_0, f_0)\), a curve \(t \to u (t) \in L^2\) is said to be a complete solution for the process \(U^{g,f} (t, \tau)\), if it satisfies
\[
U^{g,f} (t, \tau) u (\tau) = u (t), \forall \tau \in \mathbb{R}, t \geq \tau.
\]
The kernel of the process \(U^{g,f} (t, \tau)\) is the collection \(K_{g,f}\) of all its bounded complete solutions, that is,
\[
K_{g,f} = \left\{ u (\cdot) \in C_b (\mathbb{R}, L^2) : u (\cdot) \text{ satisfies (4.11)} \right\}.
\]
The kernel section of the process \(U^{g,f} (t, \tau)\) at time \(s \in \mathbb{R}\) is the set
\[
K_{g,f} (s) = \{ u (s) : u (\cdot) \in K_{g,f} \}.
\]
Let $F_1$ and $F_2$ be the projectors of $l^2 \times H(g_0, f_0)$ onto $l^2$ and $H(g_0, f_0)$, respectively. Following the uniform attractor theory [16], we have the following proposition.

**Proposition 1.** In $l^2 \times H(g_0, f_0)$, if the semigroup $\{S(t)\}_{t \geq 0}$ is continuous, point dissipative, and asymptotically compact, then it has a compact global attractor $A$. Furthermore, in $l^2$, $A = F_1A_S$ is the compact uniform attractor for the family of processes $\{U^{g,f}(t, \tau)\}_{(g,f) \in H(g_0, f_0)}$. In addition,

(a) $A_S = \bigcup_{(g,f) \in H(g_0, f_0)} K_{g,f}(0) \times \{(g,f)\}$,
(b) $A = \bigcup_{(g,f) \in H(g_0, f_0)} K_{g,f}(0)$,
(c) $F_2A_S = H(g_0, f_0)$.

**Theorem 4.5.** In $l^2$, the family of processes $\{U^{g,f}(t, \tau)\}_{(g,f) \in H(g_0, f_0)}$ associated with the LDS (22)-(23) has a compact uniform attractor $A$ with respect to $(g, f) \in H(g_0, f_0)$.

**Proof.** Following Lemma 3.6, we find that the family of processes are continuous from $l^2 \times H(g_0, f_0)$ into $l^2$. In such a case, using the continuity of translation group $\{T(t)\}_{t \in \mathbb{R}}$, it follows that the solution semigroup $\{S(t)\}_{t \geq 0}$ associated with the LDS (22)-(23) is continuous in $l^2 \times H(g_0, f_0)$. Let $B_S = B \times H(g_0, f_0)$, where $B$ is the uniform absorbing set of the processes $\{U^{g,f}(t, \tau)\}_{(g,f) \in H(g_0, f_0)}$ given by Lemma 3.5, then $B_S$ is a bounded absorbing set for the solution semigroup $\{S(t)\}_{t \geq 0}$ in $l^2 \times H(g_0, f_0)$. From Proposition 1, taking into account Lemma 4.2, there exists a compact global attractor $A_S$ for the solution semigroup $\{S(t)\}_{t \geq 0}$ in $l^2 \times H(g_0, f_0)$ and $A = F_1A_S$ is the compact uniform global attractor for the family of processes $\{U^{g,f}(t, \tau)\}_{(g,f) \in H(g_0, f_0)}$ in $l^2$ with respect to $(g, f) \in H(g_0, f_0)$.

5. **$W$ is a real Banach space.** It is known that

$$X = \left\{ f : f \in C(\mathbb{R}, \mathbb{R}), \sup_{s \in \mathbb{R}} |f(s)| < \infty \right\}$$

is a real Banach space whose norm is given by

$$\|f\|_X = \sup_{s \in \mathbb{R}} |f(s)|, \forall f \in X,$$

where $C(\mathbb{R}, \mathbb{R})$ is the real vector space of continuous functions from $\mathbb{R}$ into $\mathbb{R}$.

**Lemma 5.1.** Let

$$Y = \left\{ f : f \in C^1(\mathbb{R}, \mathbb{R}), f(0) = 0, \sup_{s \in \mathbb{R}} \left( \frac{|f(s)|}{b(s)} + \frac{|f'(s)|}{c(s)} \right) < \infty \right\},$$

where $b, c : \mathbb{R} \to [a, \infty)$ are continuous functions with $a > 0$ and $C^1(\mathbb{R}, \mathbb{R})$ is the real vector space of continuously differentiable functions from $\mathbb{R}$ into $\mathbb{R}$. Then $Y$ is a real Banach space whose norm is given by

$$\|f\|_Y = \sup_{s \in \mathbb{R}} \left( \frac{|f(s)|}{b(s)} + \frac{|f'(s)|}{c(s)} \right), \forall f \in Y.$$

**Proof.** It is clear that $Y$ is a real normed space and it is enough to show that it is complete. That is every Cauchy sequence in $Y$ converges. Indeed, let $(f_m)_{m \in \mathbb{N}}$ be a Cauchy sequence in $Y$. Then for $\varepsilon > 0$ there exists a positive integer $N$ such that

$$\|f_m - f_n\|_Y = \sup_{s \in \mathbb{R}} \left( \frac{|f_m(s) - f_n(s)|}{b(s)} + \frac{|f'_m(s) - f'_n(s)|}{c(s)} \right) < \varepsilon, \ m, n > N.$$
In such a case, it is clear that \((f_m)_m\in\mathbb{N}\) and \((f'_m)_m\in\mathbb{N}\) are Cauchy sequences in the real Banach space \(X\), given by (68), and therefore have continuous limits which we denote by \(f\) and \(g\), respectively. That is,

\[
\left\| \frac{f_m}{b} - \frac{f}{b} \right\|_X = \sup_{s\in\mathbb{R}} \left( \frac{|f_m(s) - f(s)|}{b(s)} \right) \to 0, \tag{72}
\]

and

\[
\left\| \frac{f_m}{c} - \frac{f}{c} \right\|_X = \sup_{s\in\mathbb{R}} \left( \frac{|f'_m(s) - f(s)|}{c(s)} \right) \to 0. \tag{73}
\]

From (72), it is clear that \(f(0) = 0\) since \(f_m(0) = 0\) for \(m\in\mathbb{N}\). It remains to show that \(f\) is differentiable and \(f' = g\). By the Fundamental Theorem of Calculus, for every \(m\in\mathbb{N}\), since \((f'_m)_m\in C(\mathbb{R},\mathbb{R})\),

\[
f_m(s) = f_m(s) - f_m(0) = \int_0^s f'_m(t) \, dt, \quad s \in \mathbb{R}. \tag{74}
\]

In \(X\), the sequence \((f'_m)_m\in\mathbb{N}\) is bounded since it is a Cauchy sequence. Hence there exists a constant \(\beta > 0\) such that

\[
\left\| \frac{f'_m}{c} \right\|_X = \sup_{s\in\mathbb{R}} \left( \frac{|f'_m(s)|}{c(s)} \right) \leq \beta, \quad m \in \mathbb{N},
\]

and

\[
|f'_m(s)| \leq \beta c(s), \quad s \in \mathbb{R}, \quad m \in \mathbb{N}. \tag{75}
\]

Following (72)-(73), it is clear that

\[
\lim_{m \to \infty} f_m(s) = f(s), \quad s \in \mathbb{R}, \tag{76}
\]

and

\[
\lim_{m \to \infty} f'_m(s) = g(s), \quad s \in \mathbb{R}. \tag{77}
\]

Recalling (75) and (77), by the Dominated Convergence Theorem

\[
\lim_{m \to \infty} \int_0^s f'_m(t) \, dt = \int_0^s \lim_{m \to \infty} f'_m(t) \, dt = \int_0^s g(t) \, dt. \tag{78}
\]

Taking into account (74), (76), and (78) it is clear that

\[
f(s) = f(s) - f(0) = \int_0^s g(t) \, dt
\]

and so \(f' = g\). \(\square\)

**Lemma 5.2.** Let \(W\) be the set of functions satisfying (3)-(7). Then \(W\) is a real Banach space whose norm is given by (8).

**Proof.** It is clear that \(W\) is a real normed space and it is enough to show that it is complete. That is, every Cauchy sequence in \(W\) converges. Indeed, let \((\varphi_m)_m\in\mathbb{N}\) be a Cauchy sequence in \(W\) with

\[
\varphi_m(u) = (\varphi_{mi}(u_i))_{i\in\mathbb{Z}^n}, \quad \forall u = (u_i)_{i\in\mathbb{Z}^n}, \, u_i \in \mathbb{R}.
\]

Then for \(\varepsilon > 0\) there exists a positive integer \(N\) such that

\[
\|\varphi_m - \varphi_n\|_W = \sup_{i\in\mathbb{Z}^n} \sup_{s\in\mathbb{R}} \left( \frac{|\varphi_{mi}(s) - \varphi_{ni}(s)|}{|s|^{p+1} + a_i} + \frac{|\varphi'_{mi}(s) - \varphi'_{ni}(s)|}{|s|^p + a_i} \right) < \varepsilon, \quad m, n > N.
\]
For every fixed $i \in \mathbb{Z}^n$, it is clear that $(\varphi_{mi})_{m \in \mathbb{N}}$ is a Cauchy sequence in the real Banach space $Y$, given by Lemma 5.1 with $b(s) = |s|^{p+1} + a_i$ and $c(s) = |s|^p + a_i$, that is,

$$
\|\varphi_{mi} - \varphi_{ni}\|_Y = \sup_{s \in \mathbb{R}} \left( \frac{|\varphi_{mi}(s) - \varphi_{ni}(s)|}{|s|^{p+1} + a_i} + \frac{|\varphi'_{mi}(s) - \varphi'_{ni}(s)|}{|s|^p + a_i} \right) < \varepsilon, \ m, n > N.
$$

(79)

Hence there exists $\psi_i \in Y$ such that $\varphi_{mi} \to \psi_i$ in $Y$. Let us define $\psi$ such that

$$
\psi(u) = (\psi_i(u_i))_{i \in \mathbb{Z}^n}, \ \forall u = (u_i)_{i \in \mathbb{Z}^n}, u_i \in \mathbb{R}.
$$

(80)

Since $\psi_i \in Y$ it is clear that

$$
\psi_i \in C^1(\mathbb{R}, \mathbb{R}), \ \forall i \in \mathbb{Z}^n,
$$

(81)

and

$$
\psi_i(0) = 0, \ \forall i \in \mathbb{Z}^n.
$$

(82)

We need to show that $\psi \in W$ and $\varphi_m \to \psi$ in $W$. Using (79) with $n \to \infty$ we get

$$
\|\varphi_{mi} - \psi_i\|_Y = \sup_{s \in \mathbb{R}} \left( \frac{|\varphi_{mi}(s) - \psi_i(s)|}{|s|^{p+1} + a_i} + \frac{|\varphi'_{mi}(s) - \psi'_i(s)|}{|s|^p + a_i} \right) \leq \varepsilon, \ \forall i \in \mathbb{Z}^n, m > N.
$$

(83)

In $W$ the sequence $(\varphi_m)_{m \in \mathbb{N}}$ is bounded since it is a Cauchy sequence, therefore there exists a constant $\beta > 0$ such that

$$
\|\varphi_m\|_W = \sup_{i \in \mathbb{Z}^n} \|\varphi_{mi}\|_Y \leq \beta, \ m \in \mathbb{N}.
$$

(84)

Using (83)-(84), it follows that

$$
\|\psi\|_Y \leq \|\psi_i - \varphi_{mi}\|_Y + \|\varphi_{mi}\|_Y \leq \varepsilon + \beta, \ i \in \mathbb{Z}^n, m > N,
$$

and

$$
\|\psi\|_W = \sup_{i \in \mathbb{Z}^n} \|\psi_i\|_Y \leq \varepsilon + \beta, \ m > N.
$$

(85)

From (80)-(82) and (85), it is clear that $\psi \in W$. Recalling (83) it is clear that

$$
\|\varphi_m - \psi\|_W = \sup_{i \in \mathbb{Z}^n} \|\varphi_{mi} - \psi_i\|_Y \leq \varepsilon, \ m > N,
$$

that is, $\varphi_m \to \psi$ in $W$. 

\[ \Box \]

**Acknowledgments.** We would like to thank Professor Roshdi Khalil for his helpful comments. This work was supported by the “Scientific Research Deanship” at The University of Jordan.

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Received February 2019; revised June 2019.

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