Fokker–Planck–Rosenbluth-type equations for self-gravitating systems in the 1PN approximation

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Received 5 September 2007, in final form 13 December 2007
Published 5 February 2008
Online at stacks.iop.org/CQG/25/045011

Abstract
We present two formulations of Fokker–Planck–Rosenbluth-type (FPR) equations for many-particle self-gravitating systems, with first-order relativistic corrections in the post-Newtonian approach (1PN). The first starts from a covariant Fokker–Planck equation for a simple gas, introduced recently by Chacón-Acosta and Kremer (2007 Phys. Rev. E 76 021201). The second derivation is based on the establishment of an 1PN-BBGKY hierarchy, developed systematically from the 1PN microscopic law of force and using the Klimontovich–Dupree (KD) method. We close the hierarchy by the introduction of a two-point correlation function that describes adequately the relaxation process. This picture reveals an aspect that is not considered in the first formulation: the contribution of ternary correlation patterns to the diffusion coefficients, as a consequence of the nature of 1PN interaction. Both formulations can be considered as a generalization of the equation derived by Rezania and Sobouti (2000 Astron. Astrophys. 354 1110), to stellar systems where the relativistic effects of gravitation play a significant role.

PACS numbers: 05.20.Dd, 05.10.Gg, 04.40.-b, 04.25.Nx

1. Introduction

The collisionless Boltzmann equation in the 1PN approximation for a self-gravitating system, imbedded in an otherwise flat spacetime, was derived by Rezania and Sobouti in [1], finding some relevant solutions. The aim of the present paper is to go beyond the collisionless case and incorporate situations where encounters between particles play a significant role, and derive tractable kinetic equations describing the evolution of this class of self-gravitating systems. As pointed out by Kandrup, the astrophysical objects where the relativistic effects could play a significant role in the relaxation process are the galactic nucleus and, perhaps, the relativistic star clusters with age shorter than its relaxation time [11]. There is a solid
observational evidence that there exists nuclei containing massive black holes formed due to the dynamical instability of relativistic systems of stars. In particular, the relativistic effects might be important in the case of a galactic nucleus decoupled from the remainder of the galaxy, which becomes even more dense (and relativistic) at a time scale of the order of its relaxation time [2, 3]. Since these systems are composed of stars with velocity much less than the speed of light, the post-Newtonian approach supplies the adequate tool to investigate their behavior.

It is well known that the evolution of typical globular clusters, where it is assumed that relativistic corrections are unimportant, is described satisfactorily by the Fokker–Planck kinetic equation. In the local approximation, it is possible to obtain explicit relations for the diffusion coefficients, in terms of Rosenbluth potentials, and such an equation takes a very tractable form [4] (in this paper we call it the FPR equation):

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{g} \cdot \frac{\partial f}{\partial \mathbf{v}} = -\frac{\partial}{\partial v_i} \left(A_i^N f\right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(B_{ij}^N f\right),$$

(1)

where we have used the sum convention \((i, j = 1, 2, 3)\). Here \(A_i^N\) and \(B_{ij}^N\) are the diffusion coefficients or Rosenbluth's coefficients (the subscript \(N\) remarks the fact that we are dealing with the classical Newtonian theory). Such equation describes the evolution of the distribution function \(f(x, v, t)\) of a test star, moving through a homogenous sea of field stars. They are described by a static space-independent distribution function, \(\Psi_1(v)\), which determines \(D_i^N\) and \(D_{ij}^N\). If each field star has the same mass \(m\) than the test star, the Rosenbluth coefficients take the form [4, 5]

$$A_i^N = 8\pi G^2 m^2 \ln \Lambda \frac{\partial}{\partial v_i} \int d^3v' \frac{\Psi(v')}{|v - v'|},$$

(2)

$$B_{ij}^N = 4\pi G^2 m^2 \ln \Lambda \frac{\partial^2}{\partial v_i \partial v_j} \int d^3v' \Psi(v')|v - v'|,$$

(3)

where \(\ln \Lambda\) is the Coulomb logarithm and \(G\) is the gravitation constant. As it is well known, derivation of (2) and (3) requires three fundamental facts: (i) to assume that the binary weak encounters dominate the system’s relaxation process [6, 7]; (ii) to assume that all of these encounters are local (i.e. with impact parameter much less than the system size); (iii) a detailed knowledge of the two-body microscopic dynamics.

Perhaps, a natural way to obtain an 1PN version of (1) is to start from a covariant Fokker–Planck equation and then perform its post-Newtonian approximation. At present, different versions of this kind of kinetic relation can be found, such as that derived by Kandrup [9–11] based on a stochastic standpoint (i.e. from a covariant master equation). A more fundamental approach was introduced recently by Chacón-Acosta and Kremer [8], who derived a covariant Fokker–Planck equation for a simple gas in the presence of gravitational field (see equation (4)), starting from the general-relativistic Boltzmann equation. There are several reasons impelling us to adopt this scheme. Since such a derivation is supported on the Boltzmann equation, the corresponding Fokker–Planck relation is consistent with situations where the system evolves toward an equilibrium state, characterized by the Maxwell–Jüttner distribution function [12]. On the other hand, such an equation is valid for systems whose relaxation process is dominated by grazing collisions, which is the usual picture for modelling the behavior of self-gravitating systems (in this context it is more convenient to use the term encounters instead of collisions). Moreover, the diffusion coefficients characterizing (4) are presented in a form that facilitates the implementation of the post-Newtonian approach. We shall show this in section 2, which is dedicated to obtaining the 1PN approximation of (4).
The result is an equation that is not manifestly covariant (this is a characteristic of the 1PN scheme), whose convective (l.h.s.) and collision (r.h.s) terms can be split on a Newtonian and post-Newtonian contributions.

The limitations of the above scheme can be viewed through the consideration of a more fundamental standpoint, i.e. a BBGKY formalism [16–19], which will be the subject of later sections. By examining the system’s dynamics at a microscopic level, one can note that a fundamental aspect of the 1PN interaction has not been contemplated in the formulation of (4). As we shall show in section 3, the 1PN force exerted on each particle has a ternary nature (see (35), (38)). This feature leads to the apparition of three-order correlation patterns in the first BBGKY equation and not all of those vanish after the implementation of binary encounters assumption (see (49), (52)), which is the basic statement employed in the formulation of collision terms of the Landau [13–15] or Fokker–Planck type.

Despite the apparition of third-order correlation patterns a collision term originates that does not resemble a standard fashion (equation (52)), it can be put in a FPR form by choosing adequately the two-point correlation function \( g_2 \) and setting \( g_3 = 0 \). Here, two facts will play an important role: (i) The classical FPR equation can be derived from the first BBGKY equation by choosing a correlation function corresponding to a weakly coupled gas (WCG) in the hydrodynamical regime (it is not a surprising fact, since in the usual derivation of the FPR equation it is assumed that the weak encounters play a dominant role in the relaxation process); (ii) in the 1PN approach the momentum conservation law for a system of point particles is the same as in Newtonian theory, and it holds if and only if each particle obeys the Newtonian equation of motion [22]. This means that the scattering process and, in consequence, the explicit form of \( g_2 \) in the 1PN approximation are the same as in the Newtonian case. These considerations permit us to incorporate the hydrodynamical WCG correlation function in the post-Newtonian contribution of the collision term for its subsequent simplification.

2. Derivation from a covariant Fokker–Planck equation

We start by considering that the distribution function \( f(x^\mu, p^i) \) of a self-gravitating gas of particles with identical rest mass \( m \), satisfies the covariant Fokker–Planck equation [8],

\[
p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^\mu_{\nu\sigma} p^\nu \frac{\partial f}{\partial p^\sigma} = -\frac{\partial}{\partial p^\mu} (f D^\mu) + \frac{1}{2} \frac{\partial^2}{\partial p^\mu \partial p^\nu} (f D^{\mu\nu}),
\]

(4)

where \( p^\mu = m U^\mu \) is the 4-momentum, \( \Gamma^\mu_{\nu\sigma} \) are the Christoffel’s symbols, \( x^\mu = (ct, \mathbf{x}) \) is the set of configuration coordinates and \( D^\mu, D^{\mu\nu} \) are the diffusion coefficients given by

\[
D^\mu = \int f_s \Delta p_s^\mu F \sigma \, d\Omega \sqrt{-g} \frac{d^3 p_s}{p_{s0}},
\]

(5)

\[
D^{\mu\nu} = \int f_s \Delta p_s^\mu \Delta p_s^\nu F \sigma \, d\Omega \sqrt{-g} \frac{d^3 p_s}{p_{s0}},
\]

(6)

where \( f_s = f(x^\mu, p^i_s), \Delta p_s^\mu = p_s^\mu - p_i^\mu, g = \det(g_{\mu\nu}), F = \sqrt{(p_s^\mu p_s^\nu)^2 - m^4 c^4} \) is the invariant flux, \( \sigma \) is the differential cross-section and \( d\Omega \) is an element of solid angle which characterizes the scattering process. The quantities \( p^\mu \) and \( p_s^\mu \) denote the 4-momentum of the test and field stars before the encounter, respectively, while \( p_i^\mu \) and \( p_i^\mu \) represent their 4-momentum after the encounter.

In order to perform the 1PN approximation of (4), we have to take into account the following facts. We need an expansion of (4)–(6) up to order \((\bar{v}/c)^4\), where \( \bar{v} \) is a typical Newtonian speed in the system and \( c \) is the speed of light. In this approximation \( g_{\mu\nu} \) is given
in terms of the Newtonian field $\Phi$ and the post-Newtonian fields $\psi, \xi_i$ (see appendix A),

$$g_{00} = -1 - 2\Phi/c^2 - 2(\Phi^2 + \psi)/c^4,$$

$$g_{0k} = \xi_k / c^3,$$

$$g_{ij} = (1 - 2\Phi/c^2)\delta_{ij}. \hspace{1cm} (9)$$

(Latin indices run from 1 to 3). On the other hand, the 4-velocity $U^\mu$ is related to the classical velocity $v^i = dx^i/dt$ through the equation

$$U^\mu = U_0^\nu V^\mu, \hspace{1cm} V^\mu = (1, v^i / c), \hspace{1cm} (10)$$

and it is restricted by the relation

$$g_{\mu\nu}U^\mu U^\nu = -c^2. \hspace{1cm} (11)$$

Another important fact is that in the 1PN approximation the momentum conservation for a system of point particles is satisfied if and only if each particle obeys the Newtonian equation of motion [22]. In other words, they obey the classical momentum conservation law. Immediately we note two implications that will play a fundamental role in the calculation of the diffusion coefficients. They were obtained in [8] by choosing the center-of-mass system corresponding to two colliding particles. Since in the 1PN approximation they obey the classical momentum conservation law, it implies that $\Delta U^0 = 0$ and, in consequence, $D^0 = D^{00} = 0$. On the other hand, also in virtue of the classical momentum conservation law, we must consider $\sigma d\Omega$ as the usual Newtonian scattering cross-section. The above considerations enable us to rewrite (4)–(6) as

$$L_U f = -\frac{\partial}{\partial t} f(A^t) + \frac{1}{2} \frac{\partial^2}{\partial t^2} f(B^{ij}), \hspace{1cm} (12)$$

where $L_U$ is the Liouville’s operator defined as

$$L_U = U^\mu \frac{\partial}{\partial x^\mu} - \Gamma^i_{\mu\nu} U^\mu U^\nu \frac{\partial}{\partial U^i} \hspace{1cm} (13)$$

and

$$A^t = \int f_s U^0_s \left( \frac{-c^2}{U^0_s} \right)^{\frac{3}{2}} \frac{1}{g} \frac{d^3 U_s}{U^0_s}, \hspace{1cm} (14)$$

$$B^{ij} = \int f_s U^0_s \Delta U^i_s \left( \frac{-c^2}{U^0_s} \right)^{\frac{3}{2}} \frac{1}{g} \frac{d^3 U_s}{U^0_s}. \hspace{1cm} (15)$$

We have decided to express the Fokker–Planck equation in terms of the 4-velocity instead of the 4-momentum in order to facilitate the implementation of some results used by Rezania and Sobouti in the collisionless case [1]. Now $f(x^\mu, U^i)$ is a phase density in the six-dimensional $(x^i, U^i)$-space and $\sigma$ is the Newtonian differential cross-section. The l.h.s. of (12) in the 1PN approximation can be written as [1]

$$L_U f = \frac{U^0}{c} \left\{ \frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} - \frac{\partial \Phi}{\partial x^i} \frac{\partial f}{\partial v^i} - \frac{1}{c^2} \left[ (4\Phi + v^2) \frac{\partial \Phi}{\partial x^i} - v^i v^j \frac{\partial \Phi}{\partial x^j} + \frac{\partial \psi}{\partial x^i} \right] v^j + \frac{\partial \xi_j}{\partial x^i} \right\}, \hspace{1cm} (16)$$

where $U^0/c$, determined from (11), is given by

$$\frac{U^0}{c} = 1 + \frac{v^2}{2c^2} - \frac{\Phi}{c^2}. \hspace{1cm} (17)$$
In consequence, equation (12) can be put in the form
\[
(\mathcal{L}_N + \mathcal{L}_{PN}) f = \lambda \left[ -\frac{\partial}{\partial U^i} (f A^i) + \frac{1}{2} \frac{\partial^2}{\partial U^i \partial U^j} (f B^{ij}) \right],
\]
where \( \mathcal{L}_N \) and \( \mathcal{L}_{PN} \) are the classical and post-Newtonian Liouville’s operators, given by
\[
\mathcal{L}_N = \frac{\partial}{\partial t} + v^j \frac{\partial}{\partial x^j} - \frac{\partial}{\partial \phi_1} \frac{\partial}{\partial x^j} \quad \text{and} \quad \mathcal{L}_{PN} = -\frac{1}{c^2} \left[ (4\Phi + v^2) \frac{\partial}{\partial x^j} - v^j v^l \frac{\partial}{\partial x^j} - v^j \frac{\partial}{\partial x^j} + \left( \frac{\partial \xi_i}{\partial x^j} - \frac{\partial \xi_j}{\partial x^i} \right) v^j + \frac{\partial \xi_i}{\partial t} \right] \frac{\partial}{\partial v^j}.
\]
and \( \lambda = c / U^0, \) up to order \( c^{-2}, \) equals to
\[
\lambda = 1 - \frac{v^2}{2c^2} + \frac{\Phi}{c^2}.
\]
According to (18), we need an expansion of the Fokker–Planck operator (the term in parenthesis in the r.h.s.) up to order \( c^{-2}. \) Here we have to take into account that \( \partial v^i / \partial U^j \) is given by [1]
\[
\frac{\partial v^i}{\partial U^j} = \begin{cases} Q^{-1} v^i (g_{00} + g_{0i} v^j / c) & \text{for } i \neq k \\ -Q^{-1} \left[ c^3 U^0 v^i + \sum_{k \neq i} v^k (g_{0k} + g_{ki} v^j / c) \right] & \text{for } i = k, \end{cases}
\]
where
\[
Q = U^0 (g_{00} + g_{0i} v^j / c).
\]
Then we obtain
\[
\frac{\partial}{\partial U^i} = \frac{\partial v^i}{\partial U^j} \frac{\partial}{\partial v^j} = \beta \frac{\partial}{\partial v^j} + \frac{1}{c^2} \left( \sum_{k \neq j} v^k v^j \frac{\partial}{\partial v^j} - \sum_{j \neq i} v^i v^j \frac{\partial}{\partial v^i} \right)
\]
with
\[
\beta = 1 - \frac{3v^2}{2c^2} + \frac{\Phi}{c^2}.
\]
Now we shall obtain \( A^i \) and \( B^{ij} \) up to order \( c^{-2}. \) It is useful to consider that in the center-of-mass system \( U^0 = U^0, U^i = -U^i \) and, as a consequence of condition (11), we have that \( g_{00} = g_{00}, g_{0i} = -g_{0i}, g_{ij} = g_{ij}. \) This will help us to simplify the calculation of quantities like \( \Delta U^i / U^0, d^3 v^* \) and \( U^* U^0. \) In fact, after some calculations, we find
\[
\frac{\Delta U^i}{U^0} = \left( -1 + \frac{2\Phi}{c^2} \right) \frac{\Delta v^i}{c}, \quad \frac{\Delta v^i}{c} = 1 - \frac{2\Phi}{c^2}, \quad \frac{d^3 U_*}{\Omega} = \left( 1 + \frac{5v^2}{2c^2} - \frac{3\Phi}{c^2} \right) \frac{d^3 v_*},
\]
and
\[
\sqrt{(U_* U^0)^2 - c^4} = c \left( 1 + \frac{v^2}{2c^2} - \frac{4\Phi}{c^2} \right) |v - v_*|.
\]
By introducing the above relations in (14) and (15), we obtain up to order \( c^{-2} \)
\[
A^i = \int \gamma |v - v_*| f_* \Delta v^i \tilde{\sigma} \ d\Omega \frac{d^3 v_*}{d^3 v},
\]
\[
B^{ij} = \int \eta |v - v_*| f_* \Delta v^i \Delta v^j \tilde{\sigma} \ d\Omega \frac{d^3 v_*}{d^3 v}.\]
with
\[
\gamma = -1 - \frac{7v^2}{2c^2} + \frac{11\Phi}{c^2}, \quad \eta = -1 - \frac{4v^2}{c^2} + \frac{12\Phi}{c^2}.
\] (27)

From equations (25) and (26) one can see the relation between \(A^i, B^{ij}\) and the classical Rosenbluth coefficients. In order to do this one has to assume that \(f_*\) can be replaced by a distribution function of a homogeneous field stars population in equilibrium and to consider the differential cross-section corresponding to the gravitational inverse-square law of force. That is
\[
f(x, v_*, t) \to \Psi(v_*), \quad \tilde{\sigma} = G^2m^2|v - v_*|^{-4} \sin^{-4}(\theta/2),
\] (28)

where \(\Psi(v_*) = \Psi_*\) is the field star DF and \(\theta\) is the scattering angle as measured in the center-of-mass frame. By implementing (28) in (25) and (26) one can see that the \(c_0\)-factors of \(A^i\) and \(B^{ij}\) are equivalent to the Rosenbluth’s coefficients \(A^i_N\) and \(B^{ij}_N\), given by (2) and (3) (details of this calculation can be seen in [5]). Therefore, we can write
\[
A^i = -A^i_N - A^i_{PN}, \quad B^{ij} = -B^{ij}_N - B^{ij}_{PN},
\] (29)

where \(A^i_N\) and \(B^{ij}_N\) are given by (2), (3) and
\[
A^i_{PN} = \frac{G^2m^2}{c^2} \int \left( \frac{7v^2}{2c^2} - \frac{11\Phi}{c^2} \right) \Psi(v_*) \Delta v^i \, d\Omega \, d^3v_*,
\] (30)
\[
B^{ij}_{PN} = \frac{G^2m^2}{c^2} \int \left( \frac{4v^2}{c^2} - \frac{12\Phi}{c^2} \right) \Psi(v_*) \Delta v^i \Delta v^j \, d\Omega \, d^3v_*.
\] (31)

Finally, introducing (29) and (23) in (18), we find that the 1PN approximation of (12) can be written as
\[
(L_N + L_{PN}) f = -\frac{\partial}{\partial v_i} (A^i_N f) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} (B^{ij}_N f) - \frac{\partial}{\partial v_i} (A^i_{PN} f) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} (B^{ij}_{PN} f)
\] + \[\frac{1}{c^2} \left[ J_i (A^i_N f) + K_{ij} (B^{ij}_N f) \right],
\] (32)

where we have defined the operators \(J_i\) and \(K_{ij}\) as
\[
J_i = (v^2 - 2\Phi) \frac{\partial}{\partial v^i} + \delta_{ij} v^j \left( 2 + v^i \frac{\partial}{\partial v^i} \right),
\] (33)
\[
K_{ij} = \left( v^2 - \frac{11\Phi}{4} \right) \frac{\partial^2}{\partial v_i \partial v_j} + \frac{3v^i v^j}{4} \frac{\partial^2}{\partial v_i \partial v_j} + \delta_{ij} \left( 2 + v^i \frac{\partial}{\partial v^i} \right) + \frac{11v^i}{2} \frac{\partial}{\partial v^i}.
\] (34)

The relation (32), in contrast with (4), is not a manifestly covariant kinetic equation that can be interpreted as an 1PN extension of the classical FPR equation (1). There, the Newtonian and post-Newtonian contributions appear separated both in the l.h.s. and in the r.h.s. We note that the post-Newtonian fields \(\psi\) and \(\xi_i\) contribute only through \(L_{PN}\) and do not appear in the collision term. In the following sections, we will see that a derivation of the kinetic equation from microscopic dynamics reveals additional contributions to the diffusion coefficients. The reason is that the 1PN interaction has ternary contributions, coming from \(\psi\) and \(\xi_i\), that do not disappear when the BBGKY sequence is closed.
3. Derivation from the microscopic dynamics: the first 1PN-BBGKY equation

In this section, we will derive the statistical picture of the self-gravitating system starting from its microscopic dynamics. In particular, we shall deal with the BBGKY hierarchy that formally equals to the Liouville’s equation. In particular, we shall consider only the first equation of the sequence, from which we will obtain the kinetic equation. Since we start from the non-covariant 1PN law of force, the corresponding evolution equation will also be expressed in a non-manifestly covariant fashion. As pointed out in the previous section, this is a feature characterizing the 1PN approximation.

Let a system composed by \( N \) identical point particles, with mass \( m \), interacting gravitationally and moving with velocity \( \ll c \). According to the 1PN approximation, each particle experiences an acceleration given by

\[
\dot{\mathbf{v}} = \mathbf{g}_K(\mathbf{x}, \mathbf{v}, t) + \mathbf{\Gamma}_K(\mathbf{x}, \mathbf{v}, t),
\]

where \( \mathbf{g}_K \) and \( \mathbf{\Gamma}_K \) are the Newtonian and post-Newtonian gravitational forces (per unit mass), respectively. They are given by \[22\]

\[
\mathbf{g}_K(\mathbf{x}, \mathbf{v}, t) = \frac{Gm}{r_{\mathbf{x}i}^3} \left[ 4mG \mathbf{r}_{\mathbf{x}i} - \mathbf{v}_i^2 - 2\mathbf{v} \cdot \mathbf{v}_i + \frac{3}{2} \left( \mathbf{v}_i \cdot \mathbf{r}_{\mathbf{x}i} \right)^2 \right],
\]

\[35\]

\[
\mathbf{\Gamma}_K(\mathbf{x}, \mathbf{v}, t) = \frac{c^2}{\partial \mathbf{x}} \left( 2\Phi^2 + \psi \right) - \partial^2 \Phi \partial t - \mathbf{v} \times (\partial^2 \Phi \partial \mathbf{x}) + \frac{3}{r_{\mathbf{x}i}^2} \mathbf{r}_{\mathbf{x}i} \left( \mathbf{v} \cdot \mathbf{v}_i \right)^2 - \frac{3}{2} \left( \mathbf{v}_i \cdot \mathbf{r}_{\mathbf{x}i} \right)^2.
\]

The last equation represents the relativistic correction to the force in the 1PN approximation, which includes the contribution of the post-Newtonian potentials \( \xi \) and \( \psi \). In this case (identical point-like masses), we can obtain an explicit form for \( \mathbf{\Gamma}_K \), in terms of positions and velocities. We have found that this post-Newtonian contribution can be written in a very suggestive form (see appendix A),

\[
\mathbf{\Gamma}_K = \sum_{i=1}^{N} \mathbf{A}(\mathbf{w}, \mathbf{w}_i) + \sum_{i=1}^{N} \sum_{j \neq i}^{N} \mathbf{\Upsilon}(\mathbf{x}, \mathbf{x}_i, \mathbf{x}_j),
\]

where \( \mathbf{w} \equiv (\mathbf{x}, \mathbf{v}) \) (this notation will be used henceforward). Here, the total relativistic force is shown as the result of two contributions: a velocity-dependent binary interaction term \( \mathbf{A} \), and a velocity-independent ternary interaction term \( \mathbf{\Upsilon} \). They are defined as follows:

\[
\mathbf{A}(\mathbf{w}, \mathbf{w}_i) = \frac{Gm}{c^2} \left[ \frac{\mathbf{r}_{\mathbf{x}i}}{r_{\mathbf{x}i}^3} \left( \left( \frac{4mG}{r_{\mathbf{x}i}} - \mathbf{v}_i^2 - 2\mathbf{v} \cdot \mathbf{v}_i + \frac{3}{2} \left( \mathbf{v}_i \cdot \mathbf{r}_{\mathbf{x}i} \right)^2 \right) \right) \right],
\]

\[39\]

\[
\mathbf{\Upsilon}(\mathbf{x}, \mathbf{x}_i, \mathbf{x}_j) = \frac{G^2m^2}{2c^3} \left[ \frac{\mathbf{r}_{\mathbf{x}j}}{r_{\mathbf{x}j}^3} \left( \frac{r_{\mathbf{x}i}}{r_{\mathbf{x}j}} + \frac{r_{\mathbf{x}j}}{r_{\mathbf{x}i}} + \frac{2r_{\mathbf{x}i}^2 - r_{\mathbf{x}i}^2 \cdot \mathbf{r}_{\mathbf{x}j}}{r_{\mathbf{x}i}^3 \cdot r_{\mathbf{x}j}^3} \right) \right].
\]

As we shall show, \( \mathbf{A} \) leads to the incorporation of the two-point correlation function in the statistical description. On the other hand, \( \mathbf{\Upsilon} \) causes the apparition of third-order correlation patterns, which is an essential difference with the purely Newtonian case.
In order to introduce the statistical description of the evolution for this system, we use the KD approach [20, 21]. Such a formulation of non-equilibrium statistical mechanics had been widely used in the probabilistic treatment of systems dominated by interactions of kind (35) (see [23–25], as examples). The method starts by introducing the microscopic one-particle phase space density (KD function)

\[ f_K(x, v, t) = \sum_{i=1}^{N} \delta[x - x_i(t)] \delta[v - v_i(t)]. \]  

(42)

where \( \delta \) is the three-dimensional Dirac delta function. Clearly, this function satisfies the following evolution equation (KD equation):

\[ \frac{\partial f_K}{\partial t} + v \cdot \frac{\partial f_K}{\partial x} + g_K \cdot \frac{\partial f_K}{\partial v} + \Gamma_K \cdot \frac{\partial f_K}{\partial v} = 0, \]

(43)

where \( g_K \) and \( \Gamma_K \) are given by (36) and (38), respectively, or by the equivalent relations

\[ g_K(x, t) = -Gm \int d^6w f_K(w', t) \frac{r_{xx'} f}{r_{xx'}}, \]

(44)

\[ \Gamma_K(x, t) = \int d^6w f_K(w', t) \Lambda(w, w') + \int d^6w' d^6w'' f_K(w', t) f_K(w'', t) \Upsilon(x, x', x''). \]

(45)

The one-particle distribution function (representing the system state), as much as the first BBGKY equation (describing its temporal evolution), is obtained averaging the KD function and the KD equation, respectively. Indeed, it is easy to see that the average of \( f_K \) is

\[ \langle f_K(w, t) \rangle = f(w, t), \]

(46)

where \( f \) is the usual one-particle distribution function, which represents the behavior of an average particle in the system. Also, there exist relations connecting correlation functions, \( f \) and \( f_K \) [14, 24],

\[ \langle f_K f'_K \rangle = ff' + g_2(w, w', t) + \delta(w' - w)f, \]

(47)

\[ \langle f_K f'_K f''_K \rangle = ff' f'' + g_3(w, w', w'', t) + fg_2(w', w'', t) + f'g_2(w, w'', t) + f''g_2(w, w', t) \]

\[ + \delta(w - w')[ff'' + g_2(w, w'', t)] + \delta(w - w')\delta(w - w')f \]

\[ + \delta(w' - w'')ff' + g_2(w, w', t)]. \]

(48)

Here \( g_2 \) and \( g_3 \) are the two- and three-point correlation functions, respectively, and \( f = \langle f(w, t) \rangle, f' = \langle f(w', t) \rangle, \) etc.

As it is usual, we assume that \( f \) must satisfy a kinetic equation describing its evolution. In order to derive such an equation, we start from the BBGKY first equation, which is obtained by taking the average of (43), using relations (46) and (48),

\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + g \cdot \frac{\partial f}{\partial v} + \Gamma \cdot \frac{\partial f}{\partial v} = -\frac{\partial}{\partial v} \cdot F, \]

(49)

where \( g, \Gamma \) (Newtonian and post-Newtonian average gravitational fields) are given by

\[ g(x, t) = -Gm \int d^6w' f' \frac{F_{xx'}}{r_{xx'}}, \]

(50)

\[ \Gamma(w, t) = \int d^6w f' \Lambda(w, w') + \int d^6w' d^6w'' f' f'' \Upsilon(x, x', x''), \]

(51)
while in the r.h.s. (collision term), $F$ has the form
\[
F \equiv \int d^6w g_2(w, w', t) \left[ \Lambda(w, w') - G m_{\text{r}x'x} \right] + \int d^6w' d^6w'' \left[ g_2(w', w'', t) f + g_2(w, w', t) f'' + g_3(w, w', w'', t) \right] \Upsilon(x, x', x'').
\] (52)

Equation (49) is the first equation of the 1PN-BBGKY hierarchy. The l.h.s. (convective term) depends on the mean fields $g, \Gamma$, and has the same structure of a mean-field kinetic term (the l.h.s. of a Vlasov equation [26, 27]). Moreover, it is easy to see that this convective term is equivalent to that characterizing equation (18). However, the collision term is given in a non-standard form. We can see it as a sum of two contributions: (i) a two-point term, characterized by $g_2$ and binary interactions $g, \Lambda$; (ii) a three-point term, characterized by the ternary interaction $\Upsilon$ and ternary correlation patterns of the form $fg_2$ and $g_3$. This last contribution makes the task of closing the hierarchy non-trivial, in order to derive a tractable kinetic equation. However, as it will be shown later, we find that is possible to do this, establishing some reasonable assumptions about the correlation functions behavior. In particular, we will assume that the star cluster relaxation process follows a sort of WCG correlation dynamics. Such an assumption comes from the fact that the classical FPR equation can be derived directly from the first BBGKY equation by introducing a two-point correlation function corresponding to a WCG in hydrodynamical regime. The validity of such a statement will be the subject of the following section.

3.1. An alternative derivation of the classical FPR equation: the hydrodynamical-WCG approximation

As pointed out in section 1, weak encounters play a dominant role in the star cluster relaxation process [6, 7]. This fact suggests that we can expect a close analogy between the star cluster relaxation process (dominated by weak encounters) and the relaxation process of a WCG (based on weak interactions). In fact, we can show that the introduction of a WCG correlation function (in the hydrodynamic regime) in the collision term of the classical first BBGKY relation, leads to the usual FPR equation. Before doing this, we expose briefly some relevant aspects of the WCG correlation dynamics.

A classical WCG is commonly described as a set of particles interacting via Gaussian potentials of the form [14]
\[
V(r) = V_0 e^{-\left(\frac{r}{\alpha}\right)^2},
\] (53)
where $r$ is the inter-particle separation, $V_0$ represents the potential’s maximum value and $\alpha$ is an interaction length (for $r$ greater than $\alpha$ the interaction practically vanishes). The evolution in time, for this class of systems, is described by the proper kinetic equation to order 2 (see [14, p 574]) and there is an aspect we are especially interested in, around such an equation: the explicit form of correlation functions. Here we note that (to order 2) $g_3, g_4$, etc vanish and that $g_2$ is given by the relation [14],
\[
g_2(w, w', t) = \int_0^\infty dt V(w, w', \tau) \left( \nabla_{w'} + \tau \nabla_{x'} \right) [ff'],
\] (54)
where we have used the notation
\[
\nabla_{x'} \equiv \frac{\partial}{\partial x} - \frac{\partial}{\partial x'}, \quad \nabla_{w'} \equiv \frac{\partial}{\partial w} - \frac{\partial}{\partial w'}, \quad u_{w'} \equiv v - v'.
\] (55)
and
\[
V(w, w', \tau) = V_0 e^{-\left(\tau u_{w'}/\alpha\right)^2}.
\] (56)
Moreover, if we consider that the WCG is in the hydrodynamic regime, then the spatial variation of $f$ is characterized by a hydrodynamical length $L_h$, defined as \[ L_h = \max_f \frac{f}{|\partial f/\partial x|} \gg \alpha. \] (57)

Since $L_h \gg \alpha$, the system can be considered practically homogeneous ($f$ is spatially independent), over a distance of the order of $\alpha$. We may take advantage of this fact in order to simplify (54). In the Taylor expansion of $f'$ at $x'$ around $x$,

\[ f(x', v', t) = f(x, v', t) - r_{xx'} \cdot \frac{\partial f(x, v', t)}{\partial x} + \cdots, \] (58)

the second term is of the order $\alpha/L_h$ compared to the first (the next terms are smaller). So we can take $f(x', v', t) \approx f(x, v', t)$ and then

\[ \frac{\partial}{\partial x'} [f f'] \approx 0. \]

Performing a similar procedure, starting from the expansion of $f$ at $x$ around $x'$, we also obtain

\[ \frac{\partial}{\partial x} [f f'] \approx 0. \]

This means that, in the hydrodynamical limit, we can set $\nabla_{xx'} [f f'] \approx 0$ and replace $f'$ by $f(x, v', t)$ in the remaining factor $\nabla_{vv'} [f f']$ of equation (54).

Thus, by carrying out the integral (54), $g_2$ can be written as

\[ g_2(w, w', t) = G(w, w') \cdot \nabla_{vv'} [f(x, v', t) f(x, v, t)], \] (59)

with

\[ G(w, w') \equiv \frac{\sqrt{\pi} V_o}{u_{vv'} \alpha^2} e^{-r_{xx'}/\alpha^2} \left\{ \frac{u_{vv'}}{\sqrt{\pi}} + (Q_{ww'} u_{vv'} - u_{vv'} r_{xx'}/\alpha) [1 + \text{Erf}(Q_{ww'})] e^{Q_{ww'}^2} \right\}, \] (60)

where Erf$(Q_{ww'}$) is the error function and

\[ Q_{ww'} = \frac{u_{vv'} r_{xx'}}{u_{vv' \alpha}}. \] (61)

Now let us contemplate the implications concerning the implementation of (59) in the collision term of the classical first BBGKY equation, which we write as

\[ -\frac{\partial}{\partial v} \cdot F_N = -Gm \int d^6 w' g_2(w, w', t) \frac{r_{xx'}}{r_{xx'}.} \] (62)

Again the subscript $N$ denotes that we are dealing with the Newtonian case. In appendix B, we show that after the introduction of (59) in the above equation, it reduces to

\[ -\frac{\partial}{\partial v} \cdot F_N = -\frac{\partial}{\partial v_i} (\tilde{A}_N f) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} (\tilde{B}_N^{ij} f) \] (63)

where we have introduced the definitions

\[ \tilde{A}_N^i = -4\pi^{3/2} V_o Gm \frac{\partial}{\partial v_j} \int d^3 v' \frac{f(x, v', t)}{|v - v'|}, \] (64)

\[ \tilde{B}_N^{ij} = -2\pi^{3/2} V_o Gm \frac{\partial^2}{\partial v_i \partial v_j} \int d^3 v' f(x, v', t)|v - v'|. \] (65)

From the above relations, we immediately note that (63) equals to the classical FPR collision term (1) by choosing $V_o$ as

\[ V_o = -\frac{2Gm}{\alpha \sqrt{\pi}} \ln \Lambda, \] (66)

and assume that

\[ f(x, v', t) \approx \Psi(v'), \] (67)
where \( \psi(v') \), in the context of the FPR equation derived in [4, section 8.1], is a field star distribution function. With (66) and (67), \( A_i^j \) and \( B_{ij}^N \) are exactly the Rosenbluth diffusion coefficients (2) and (3), describing the drift and diffusivity generated by a homogeneous sea of background particles of mass \( m \) and distribution function \( \psi(v') \), over a test star with the same mass and distribution function \( f(w, t) \).

The assumption (67) is provided by the hydrodynamical approximation, which establishes that the system is practically space independent over the correlation range \( \alpha \). Moreover, as usual, we demand that the homogeneous distribution \( \psi(v') \) must satisfy the relation

\[
\int d^3v' \psi(v') = \frac{N}{V}, \tag{68}
\]

where \( V \) is the system’s volume.

All the above considerations lead us to establish that, in order to obtain the FPR equation starting from BBGKY hierarchy, we can choose \( g_3 = g_4 = \cdots = 0 \) and a two-point correlation function of the form

\[
g_2(w, w', t) = G(w, w') \cdot \nabla_{vv'} [\psi(v') f(w, t)], \tag{69}
\]

where \( G \) is given by equation (60), defining previously the constant \( V_0 \) through (66). We have to point out that this model of correlation function, in the context of a BBGKY scheme, is in close concordance with models based on scattering cross-section considerations as well as approaches stated on a master equation (stochastic pictures).

3.2. The Fokker–Planck–Rosenbluth equation in the 1PN approximation

At this stage we can derive an alternative closed kinetic equation for the self-gravitating system in the 1PN approximation. Here, an important fact, proceeding from our experience of section 2, will play a crucial role: since in the 1PN approach the momentum conservation for a system of point particles is satisfied if and only if each particle obeys the Newtonian equation of motion, the scattering process is characterized by a differential cross-section associated with the Newtonian law of force. In other words: the statistical correlation between two colliding particles, at 1PN approximation level, must be described by a two-point correlation function corresponding to the Newtonian scheme. This means that, as a consequence of the statements established in the previous section, we can model satisfactorily the relaxation process of the self-gravitating system by introducing (69) in the collision term (52), where we also have to set \( g_3 = 0 \).

Then, by introducing (69) in (49) and setting \( f' \approx \psi(v') \), \( f'' \approx \psi(v') \) (assumption (67)) in the collision term (52), we obtain (see appendix C)

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + g \cdot \frac{\partial f}{\partial v} + \Gamma \cdot \frac{\partial f}{\partial v} = - \frac{\partial}{\partial v_i} (A_i^j f) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} (B_{ij}^N f), \tag{70}
\]

where \( A_i^j \) and \( B_{ij}^N \) are the 1PN-drift vector and the 1PN-diffusion tensor, respectively. They are given by the relations

\[
A_i^j = 8 \pi G^2 m^2 \ln \Lambda \frac{\partial}{\partial v_i} \int d^3v' \frac{\psi(v')}{|v - v'|} + \int d^3v' \psi(v') \left[ \frac{\partial}{\partial v_j} - \frac{\partial}{\partial v_j} \right] \omega_{PN}^{ij}(w, v'),
\]

\[
B_{ij}^N = 4 \pi G^2 m^2 \ln \Lambda \frac{\partial^2}{\partial v_i \partial v_j} \int d^3v' \psi(v') |v - v'| - 2 \int d^3v' \psi(v') \omega_{PN}^{ij}(w, v'), \tag{71}
\]

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where

\[
\Omega_{\mu}^{ij}(w, w') = \int d^3x' \Lambda^i(w, x') G^j(w, w') + 2 \sum_{V} \int d^3x' d^3x'' \Gamma_3^i(x, x', x'') G^j(w, w'),
\]

(73)

\[
\Phi_{\mu}^{ij}(x, x', x'') = \int d^3x' d^3x'' \Gamma_3^i(x, x', x'') G^j(w', w'').
\]

(74)

The first term on the r.h.s. of (71) and (72) corresponds to the Newtonian contribution (classical Rosenbluth coefficients) and the remainder is the post-Newtonian contribution that can be calculated explicitly once we choose a particular expression for the field star distribution \(\Psi\).

In this picture, the post-Newtonian diffusion coefficients look much more involved than in the formulation shown in section 2. In this subject, we have to point out that they do not necessarily coincide, even vanishing ternary contributions. The form in which \(A_{\mu N}^i\) and \(B_{\mu N}^{ij}\) determine the collision term in the first formulation, differs from the present one.

4. Concluding remarks

We have obtained two formulations of FPR-type equations that can be used to model the evolution, in a diffusion approximation, of stellar systems where the relativistic effects play a significant role. In the first formulation the contribution of third-order correlation patterns to the diffusion coefficients is vanished, while in the second one this fact is taken into account. Since the l.h.s. of (32) and (70) are equivalent, one can expect that they be consistent with the post-Newtonian equations of hydrodynamics in general relativity [1]. However the generalization of the Eulerian equations of Newtonian hydrodynamics, derived from each approach, will differ in the term associated with a drift acceleration, since it will be determined by the diffusion coefficients.

Since both (32) and (70) are expressed in a non-manifestly covariant fashion (as a consequence of the 1PN scheme) its physical interpretation, as well as their differences with the usual FPR equation, can be elucidated from a classical Newtonian point of view. We have to point out that they have a common point: once we choose an adequate field star distribution function, the diffusion coefficients (30), (31) or (71), (72) can be calculated explicitly, making it possible to find numerical solutions of (32) and (70) (for example, using the formalism showed in [28]).

The KD formalism allows us to derive, in a relatively simple way, the first 1PN-BBGKY relation, equation (49). It constitutes an interesting example of the importance of this method, when we deal with the statistical description of dynamical systems described by velocity-dependent interactions (see [24]). Perhaps the most relevant fact, that can be noted through (49), is the apparition of three-order correlation patterns in the collision term. This feature, absent in the purely Newtonian case, is a consequence of the ternary interaction \(\Upsilon\) (see equation (40)). One could go far away and think that more accurate post-Newtonian approximations (2PN, 3PN and so on) would lead to descriptions based on correlations of greater order. It would not be a surprising fact that Einstein’s gravity theory increases the correlation level, in a non-equilibrium statistical mechanics description.

Assumptions (69) and (67), reduce (49) (in the purely Newtonian case) to the usual FPR equation (1). This fact suggests that the local approximation, employed in the stochastic approaches of kinetic theory (see for example of [4, section 8.3]), is equivalent to the hydrodynamic-WCG approximation, used here. Through such approximation we model the star cluster relaxation process by the correlation dynamics corresponding to a gas dominated by attractive short-range interactions, \(V(r) = -(2Gm \ln M/\alpha \sqrt{\pi}) \exp(-r^2/\alpha^2)\). The interaction
range $\alpha$, introduced here, defines a characteristic scale of distances over which (i) encounters play a dominant role, and (ii) the distribution function can be considered homogeneous. One can estimate the magnitude of such a length, taking into account the considerations used in the local approximation, and say that $\alpha$ must be of the order of the minimum impact parameter for which such approximation holds. That is, $\alpha \sim R/N$, where $R$ is the system’s characteristic radius [4]. According to these statements, equation (70), shown in section 3.2, describes the evolution of the distribution function corresponding to a typical test particle interacting with a homogeneous sea of test stars in thermodynamic equilibrium, taking into account 1PN corrections.

Appendix A. Post-Newtonian potentials for $N$ point-like particles

In the 1PN approximation, $\Phi$, $\xi$ and $\psi$ are given by [22]

$$\Phi(x, t) = -G \int \frac{T^{00}(x', t)}{|x - x'|} \, d^3x', \quad \text{(A.1)}$$

$$\psi(x, t) = -\int \frac{d^3x'}{|x - x'|} \left[ \frac{l^2 \Phi(x', t)}{4\pi} \frac{\partial^2}{\partial t^2} + G T^{00}(x', t) + G T^{aa}(x', t) \right], \quad \text{(A.2)}$$

$$\xi^a(x, t) = -4G \int \frac{d^3x'}{|x - x'|} T^{a0}(x', t) \, \delta(x' - x_i(t)). \quad \text{(A.3)}$$

(We denote $a, b = 1, 2, 3$). For a system composed of $N$ identical point-like particles, the energy–momentum tensor components, at this order, are

$$T^{00}(x', t) = m \sum_{i=1}^{N} \delta(x' - x_i(t)), \quad \text{(A.4)}$$

$$T^{01}(x', t) = m \sum_{i=1}^{N} \left[ \Phi(x', t) + \frac{v_i^2}{2} \right] \delta(x' - x_i(t)), \quad \text{(A.5)}$$

$$T^{0a}(x', t) = m \sum_{i=1}^{N} v_i^a \delta(x' - x_i(t)), \quad \text{(A.6)}$$

$$T^{ab}(x', t) = m \sum_{i=1}^{N} v_i^a v_i^b \delta(x' - x_i(t)). \quad \text{(A.7)}$$

Introducing (A.4) in (A.1) and (A.6) in (A.3), we obtain

$$\Phi(x, t) = -\sum_{i=1}^{N} \frac{Gm}{|x - x_i(t)|}, \quad \text{(A.8)}$$

$$\xi(x, t) = -\sum_{i=1}^{N} \frac{4Gmv_i}{|x - x_i(t)|}. \quad \text{(A.9)}$$
while, introducing (A.5) and (A.7) in (A.2) (expression $T^{\mu\nu}$ indicates sum over $a$), $\psi$ takes the form

$$
\psi(x, t) = \sum_{i=1}^{N} \left( \frac{3}{2} GMm_i^2 \right) \frac{x_i^2}{|x - x_i|} + \sum_{i=1}^{N} \sum_{j \neq i} \frac{G^2m^2}{|x - x_i||x - x_j|} \\
+ GM \frac{\partial^2}{\partial t^2} \sum_{i=1}^{N} \int \frac{d^3x'}{|x - x'||x' - x_i(t)|}.
$$

(A.10)

In order to solve the integral on the right-hand side we introduce the identity [29]

$$
\frac{1}{|x - x'|} = \frac{1}{2\pi^2} \int \frac{d^3k}{k^2} e^{ik(x-x')}.
$$

that permits us to write

$$
\frac{1}{|x - x'||x' - x_i|} = \frac{1}{4\pi^4} \int \frac{d^3k}{k^2} \frac{d^3k'}{k'^2} e^{i(kx-k'x_i)} e^{i(k'-k)x'}.
$$

The integral over $x'$ of this last expression, in virtue of the identity

$$
\delta(k' - k) = \frac{1}{(2\pi)^3} \int d^3x' e^{i(k-k')x'},
$$

equals to

$$
\int \frac{d^3x'}{|x - x'||x' - x_i|} = \frac{2}{\pi} \int \frac{d^3k}{k^2} e^{i(kx-x_i)}.
$$

(A.11)

Defining Cartesian coordinates, such that $k_i$ is in the direction of

$$
r = x - x_i,
$$

(A.12)

and then, changing to spherical polar coordinates and introducing the transformation $u = kr = k|x - x_i|$, the right-hand side integral of (A.11) takes the form

$$
\int \frac{d^3k}{k^2} e^{i(kx-x_i)} = 4\pi |x - x_i| \int_{0}^{\infty} du \frac{\sin u}{u^3}.
$$

(A.13)

Since

$$
\int_{0}^{\infty} du \frac{\sin u}{u^3} = -\frac{\pi}{4},
$$

we find that (A.11) reduces to

$$
\int \frac{d^3x'}{|x - x'||x' - x_i|} = -2\pi |x - x_i|.
$$

(A.14)

Then, we have to take into account the term

$$
\frac{\partial^2}{\partial t^2} |x - x_i(t)| = \frac{v_i^2}{|x - x_i|} - \frac{|v_i \cdot (x - x_i)|^2}{|x - x_i|^3} = \frac{x - x_i}{|x - x_i|^3} \cdot \frac{dv_i}{dt}
$$

in (A.10). Here, in agreement with the order of approximation, we must take

$$
\frac{dv_i}{dt} = -GM \sum_{j \neq i} \frac{x_i - x_j}{|x - x_i||x - x_j|^3}.
$$

Finally, we can write (A.10) as

$$
\psi(x, t) = -G^2m^2 \sum_{i=1}^{N} \sum_{j \neq i} \left\{ \frac{(x - 3x_i + 2x_j) \cdot (x_i - x_j)}{2|x - x_i||x_i - x_j|^3} \right\}
\right.
\left. - GM \sum_{i=1}^{N} \left\{ \frac{2v_i^2}{|x - x_i|} - \frac{|v_i \cdot (x - x_i)|^2}{2|x - x_i|^3} \right\}.
$$

(A.15)

By introducing relations (A.8), (A.9) and (A.15) in (37), we can write it as (51).
Appendix B. Derivation of relations (63)–(65)

Introducing (59) in (62), it can be cast as
\[
- \frac{\partial}{\partial v} \cdot F_N = \int d^3v' \nabla_{v'} \cdot \Omega_N \cdot \nabla_{v'} \{ f(x, v', t) f(x, v, t) \},
\]
where the subscript \(N\) indicates the Newtonian case and \(\Omega_N\) is a second-rank tensor, defined as
\[
\Omega_N = \int_0^\infty d\tau \int d^3r \left[ \frac{\partial \Phi(r)}{\partial x} \right] \left[ \frac{\partial V(r - \tau u_{vv})}{\partial x} \right],
\]
where we have changed the integration domain \(x'\) by \(r = x - x'\), and called \(\Phi(r) = -Gm/r\).

It is convenient to express \(\Phi_1(r)\) and \(V(r)\) in the Fourier expansion
\[
\Phi_1(r) = \int d^3k \Phi_k e^{ik \cdot r},
\]
\[
V(r) = \int d^3k V_k e^{ik \cdot r},
\]
with
\[
\Phi_k = -\frac{Gm}{2\pi^2 k^2},
\]
\[
V_k = \frac{\alpha^3 V_0}{8\pi^{3/2}} e^{-(k/2)^2}.
\]

Introducing (B.3) in (B.2), we find
\[
\Omega_N = 8\pi^3 \int_0^\infty d\tau \int d^3k e^{ik \cdot u_{vv}} \Phi_k V_k.
\]
The integral with respect to \(\tau\) can be evaluated using the representation (see appendix 2 of [30])
\[
\int_0^\infty d\tau e^{\pm ik \cdot \tau} = \pi \delta(x) \pm i\mathcal{P} \left( \frac{1}{x} \right),
\]
where \(\mathcal{P}(1/x)\) denotes the principal part. Taking into account that \(\delta(x)\) is an even function, \(\mathcal{P}(1/x)\) is odd and \(\Phi_k V_k \delta(k)\) is even in the vector \(k\), we obtain
\[
\Omega_N = 8\pi^4 \int d^3k \delta(k \cdot u_{vv}) \Phi_k V_k.
\]
The calculation of the above integral is simplified using spherical coordinates and choosing the \(Z\)-axis in the \(u_{vv}\) direction (details of this transformation are shown in [14], section 11.6). We find that
\[
\Omega_N = C \frac{\mu_{vv}^2 (I - u_{vv} u_{vv})}{\mu_{vv}^2},
\]
where \(I\) is the identity second-rank tensor and
\[
C = 8\pi^5 \int_0^\infty dk k^3 \Phi_k V_k = -\pi^{3/2} Gm\alpha V_0.
\]
The relation (B.8) allows us to write the velocity divergence of \(F_N\) in the Fokker–Planck form
\[
- \frac{\partial}{\partial v} \cdot F_N = - \frac{\partial}{\partial v_i} \left( \lambda_{N,i} f \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left( \tilde{B}^{ij}_{N} f \right),
\]
where we have used the sum convention \((i, j = 1, 2, 3)\) and
\[
\lambda_{N,i} = \int d^3v' f(x, v') \left[ \frac{\partial}{\partial v_j} - \frac{\partial}{\partial v'_j} \right] \Omega^{ij}_{N},
\]
\[
\tilde{B}^{ij}_{N} = 2 \int d^3v' f(x, v') \Omega^{ij}_{N}.
\]
Finally, introducing (B.8) and (B.9) in the last relations, we obtain (64) and (65).
Appendix C. Derivation of the post-Newtonian diffusion coefficients

It is possible to write the remaining part of the collision term in a Fokker–Planck form, similar to relation (63). We first write $\mathbf{F} = \mathbf{F}_N + \mathbf{F}_{PN}$, considering only $\mathbf{F}_{PN}$ (the post-Newtonian part of $\mathbf{F}$). From (52) we note that a simplification is introduced if we define symmetric and antisymmetric (with respect to $\mathbf{x}', \mathbf{x}''$) fields

$$\Upsilon_S(\mathbf{x}, \mathbf{x}', \mathbf{x}'') \equiv \frac{1}{2} [\Upsilon(\mathbf{x}, \mathbf{x}', \mathbf{x}'') + \Upsilon(\mathbf{x}, \mathbf{x}'', \mathbf{x}')]$$

(C.1)

$$\Upsilon_A(\mathbf{x}, \mathbf{x}', \mathbf{x}'') \equiv \frac{1}{2} [\Upsilon(\mathbf{x}, \mathbf{x}', \mathbf{x}'') - \Upsilon(\mathbf{x}, \mathbf{x}'', \mathbf{x}')]$$

(C.2)

With the help of these definitions, remembering that $g_2$ is symmetric and neglecting $g_3$, $\mathbf{F}_{PN}$ can be cast as

$$\mathbf{F}_{PN} = \int d^6w \Lambda(\mathbf{w}, \mathbf{w}')g_2(\mathbf{w}, \mathbf{w}') + \int d^6w d^6w'' \Upsilon_S(\mathbf{x}, \mathbf{x}', \mathbf{x}'') [g_2(\mathbf{w}', \mathbf{w}'') f + 2g_2(\mathbf{w}, \mathbf{w}') f'']$$

(C.3)

Introducing (69) in this relation, taking into account the assumption (67), by which one can set $f' \approx \Psi'(v')$, $f'' \approx \Psi''(v'')$ in the collision term, we find that the velocity divergence of the resulting expression can be written as

$$-\frac{\partial}{\partial v_j} \cdot \mathbf{F}_{PN} = -\frac{\partial}{\partial v_j} \left[A^j_{PN} f + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(B^{ij}_{PN} f \right) \right].$$

(C.4)

Here we have introduced the post-Newtonian diffusion coefficients

$$A^j_{PN} = \int d^3v' \Psi(v') \left[ \frac{\partial}{\partial v'_j} - \frac{\partial}{\partial v'_i} \right] \Omega^j_{PN} + \int d^3v' d^3v'' \Psi(v') \Psi(v'') \left[ \frac{\partial}{\partial v''_j} - \frac{\partial}{\partial v''_i} \right] \Phi^{ij}_{PN},$$

(C.5)

$$B^{ij}_{PN} = -2 \int d^3v' \Psi(v') \Omega^{ij}_{PN},$$

(C.6)

where

$$\Omega^j_{PN} = \int d^3x' \Lambda^j(\mathbf{x}, \mathbf{w}', \mathbf{w}) G^j(\mathbf{w}, \mathbf{w}') + 2 \frac{N}{V} \int d^3x' d^3x'' \Upsilon^j(x, \mathbf{x}', \mathbf{x}'') G^j(\mathbf{w}, \mathbf{w}''),$$

(C.7)

$$\Phi^{ij}_{PN} = \int d^3x' d^3x'' \Upsilon^j(x, \mathbf{x}', \mathbf{x}'') G^j(\mathbf{w}', \mathbf{w}'').$$

(C.8)

We obtain (70) adding (C.4) and (B.10) (choosing $V_o$ according to (66)).

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