ON THE MODULI SPACES OF COMMUTING ELEMENTS IN THE PROJECTIVE UNITARY GROUPS

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ABSTRACT. We provide descriptions for the moduli spaces $\text{Rep}(\Gamma, PU(m))$, where $\Gamma$ is any finitely generated abelian group and $PU(m)$ is the group of $m \times m$ projective unitary matrices. As an application we show that for any connected CW–complex $X$ with $\pi_1(X) \cong \mathbb{Z}^n$, the natural map

$$\pi_0(\text{Rep}(\pi_1(X), PU(m))) \to [X, BP\text{U}(m)]$$

is injective, hence providing a complete enumeration of the isomorphism classes of flat principal $PU(m)$–bundles over $X$.

1. Introduction

The space of ordered commuting $n$–tuples in a compact, connected Lie group $G$ is by definition the subspace $\text{Hom}(\mathbb{Z}^n, G) \subset G^n$ (see [4] for background and basic properties). Its orbit space under conjugation, denoted $\text{Rep}(\mathbb{Z}^n, G)$, can be identified with the moduli space of isomorphism classes of flat connections on principal $G$–bundles over the $n$–dimensional torus $\mathbb{T}^n$. In the case when all of the maximal abelian subgroups of $G$ are path connected, it can be shown that this moduli space has a single connected component, corresponding to the identity element $(1, \ldots, 1) \in G^n$ (see [4], Proposition 2.3). For example, $\text{Rep}(\mathbb{Z}^n, U(m)) \cong SP^m(\mathbb{T}^n)$, the $m$-fold symmetric product of the $n$–torus. However by a result due to Borel (see [6], page 216), for any prime number $p$ the fundamental group of $G$ has $p$–torsion if and only if there exists a rank two elementary abelian $p$–subgroup of $G$ which is not a subgroup of any torus. In this case $\text{Rep}(\mathbb{Z}^n, G)$ fails to be path–connected for all $n \geq 2$ and determining the number and exact structure of the components can be fairly complicated. Borel also shows that $H^*(G, \mathbb{Z})$ has $p$–torsion if and only if there exists a subgroup of the form $(\mathbb{Z}/p\mathbb{Z})^3 \subset G$ which is not contained in any torus. This can be used to show for example that $\text{Rep}(\mathbb{Z}^3, \text{Spin}(7))$ is not path connected, even though $\text{Spin}(7)$ is simply connected.

In this note we consider the case when $G = PU(m)$, the group of $m \times m$ projective unitary matrices, which has fundamental group isomorphic to $\mathbb{Z}/m\mathbb{Z}$ and

$$H^2(BPU(m), \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$$

has a canonical generator $\nu$ of order $m$, corresponding to the central extension

$$1 \to \mathbb{Z}/m\mathbb{Z} \to SU(m) \to PU(m) \to 1.$$ 

Given a homomorphism $h : \mathbb{Z}^n \to PU(m)$, one can associate to it the cohomology class $\alpha = h^*(\nu) \in H^2(\mathbb{Z}^n, \mathbb{Z}/m\mathbb{Z})$. For our purposes it’s convenient to identify $H^2(\mathbb{Z}^n, \mathbb{Z}/m\mathbb{Z})$
with the set $T(n, \mathbb{Z}/m\mathbb{Z})$ of $n \times n$ skew-symmetric matrices over $\mathbb{Z}/m\mathbb{Z}$: given a basis $z_1, \ldots, z_n$ for $\mathbb{Z}^n$, then $D = (d_{ij}) \in T(n, \mathbb{Z}/m\mathbb{Z})$ corresponds to $\sum_{i<j} d_{ij} z_i^* z_j^*$.

Now given a skew-symmetric $n \times n$ matrix $D$ over $\mathbb{Z}/m\mathbb{Z}$ representing $\alpha$, we define $\sigma(\alpha) = \sigma(D) = \sqrt{|R(D)|}$, where $R(D) \subset (\mathbb{Z}/m\mathbb{Z})^n$ is the row space of $D$ (see \cite{1}, Definition 2). Alternatively, for $\alpha \in H^2(\mathbb{Z}^n, \mathbb{Z}/m\mathbb{Z})$, by \cite{8}, Proposition 4.1 we can find a basis $g_1, \ldots, g_n$ of $\mathbb{Z}^n$ such that $\alpha = c_1 g_1^* g_2^* + \cdots + c_r g_{2r-1}^* g_{2r}^*$, where $2r \leq n$, $c_1, \ldots, c_r \in \mathbb{Z}/m\mathbb{Z}$ and $|c_r| \geq |c_{r-1}| \geq \cdots \geq |c_1|$. Using this basis it follows that $\sigma(\alpha) = \prod_{i=1}^r |c_i|$.

For a topological group $G$, let $\hat{SP}^n(G)$ denote the reduced $n$-fold symmetric product of $G$, defined as the quotient $SP^n(G)/G$, where $G$ acts by translation on each unordered coordinate. Then our main result can be stated as follows:

**Theorem A.** For all $m, n \geq 1$ there are homeomorphisms

$$Rep(\mathbb{Z}^n, PU(m)) \cong \coprod_{D \in T(n, \mathbb{Z}/m\mathbb{Z})} SP^{\pi(D)}(\mathbb{T}^n/R(D))$$

Note that $\mathbb{T}^n/R(D) \cong \mathbb{T}^n$, so our formula expresses the moduli space as a disjoint union of reduced symmetric products of the $n$-torus. The number of path-connected components of $Rep(\mathbb{Z}^n, PU(m))$ is equal to $N(n, m) = |\{ D \in T(n, \mathbb{Z}/m\mathbb{Z}) \mid \sigma(D) \text{ divides } m \}|$, a rather intricate number that has been computed in Adem-Cheng (see \cite{1}, Corollary 3.9). We also show how to apply our methods to provide a description of $Rep(\Gamma, PU(m))$ for any finitely generated abelian group $\Gamma$ (see Theorem 2.11). Note that by the results in \cite{3}, for these groups there is a homotopy equivalence $Rep(\Gamma, PGL(m, \mathbb{C})) \simeq Rep(\Gamma, PU(m))$.

Recall that for $G$ a topological group and $X$ a CW–complex, the homotopy classes of maps from $X$ to $BG$, denoted $[X, BG]$, classify isomorphism classes of principal $G$ bundles over $X$. Representations play a key in this through the theory of flat bundles; in our setting the key connection is via the induced map on components

$$\Psi^{PU(m)}_{\mathbb{T}^n} : \pi_0(Rep(\mathbb{Z}^n, PU(m))) \to [\mathbb{T}^n, BPU(m)].$$

Taking composition with the classifying map $c_X : X \to B\pi_1(X)$, we obtain the following general result

**Theorem B.** Let $X$ denote a connected CW–complex with $\pi_1(X) = \mathbb{Z}^n$; then the map

$$\Psi^{PU(m)}_X : \pi_0(Rep(\pi_1(X), PU(m))) \to [X, BPU(m)]$$

is injective for all $n, m \geq 1$ and so there are $N(n, m)$ distinct isomorphism classes of flat $PU(m)$–bundles on $X$.

Regarding surjectivity, we obtain that for all $m \geq 2$,

$$\Psi^{PU(m)}_{\mathbb{T}^n} : \pi_0(Rep(\mathbb{Z}^n, PU(m))) \to [\mathbb{T}^n, BPU(m)]$$

is surjective if and only if $n \leq 3$ (Proposition 3.2). It follows that there exists a principal $PU(m)$-bundle on the $n$–torus $\mathbb{T}^n$ which does not admit a flat structure if and only if $n \geq 4$. Here we apply the results in \cite{9}, which provide a classification of principal $PU(m)$–bundles over low–dimensional complexes.
The results in this paper can be viewed as an application and refinement of the analysis carried out in our previous work [1], where we described the space \( B_n(U(m)) \) of almost commuting \( n \)-tuples in \( U(m) \). We are grateful to M. Bergeron, J.M. Gómez, Z. Reichstein and B. Williams for their helpful comments.

2. Projective Representations and Almost Commuting Elements

In this section we will apply the methods from [1] to give a description of \( \text{Rep}(\mathbb{Z}^n, PU(m)) \). The almost commuting elements will play a crucial role.

**Definition 2.1.** We define \( B_n(G) \), the almost commuting elements in a Lie group \( G \), as the set of all ordered \( n \)-tuples \((A_1, A_2, \ldots, A_n) \) in \( G^n \) such that the commutators \([A_i, A_j] \in Z(G)\), the centre of \( G \), for all \( 1 \leq i, j \leq n \).

Let \( F_n \) denote the free group on \( n \) generators \( a_1, \ldots, a_n \). We will identify a map from \( F_n \) to \( G \) with the \( n \)-tuple of images of these generators in \( G \). Suppose that \( f : F_n \rightarrow U(m) \) is in \( B_n(U(m)) \). For any \( u, v \in F_n \), \([f(u), f(v)] = \gamma I_m \) for some \( m \)-th root of unity \( \gamma \), as the determinant of a commutator of invertible matrices is equal to one. The exponential function \( z \mapsto e^{2\pi i z} \) establishes a group isomorphism between \( \mathbb{R}/\mathbb{Z} \) and \( S^1 \subset \mathbb{C} \) with inverse \( w \mapsto \frac{1}{2\pi i} \log w \). The multiplicative groups of \( m \)-th roots of unity and all roots of unity correspond to the subgroup \( \mathbb{Z}[\frac{1}{m}]/\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \) and \( \mathbb{Q}/\mathbb{Z} \) of \( \mathbb{R}/\mathbb{Z} \) respectively under this isomorphism. We will be using this identification from now on. Hence there is a map \( F_n \times F_n \rightarrow \mathbb{Z}/m\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z} \) defined by \((u, v) \mapsto \frac{1}{2\pi i} \log \gamma \). Since \( f([F_n, F_n]) \subset Z(U(m)) \), the map factors through the abelianization of \( F_n \times F_n \) and thus gives rise to a \( \mathbb{Z}/m\mathbb{Z} \)-valued skew-symmetric bilinear form \( \omega_f : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}/m\mathbb{Z} \).

Define a map
\[
\rho : B_n(U(m)) \rightarrow T(n, \mathbb{Z}/m\mathbb{Z})
\]
by \( \rho(f) = (d_{ij}) \), where \([f(a_i), f(a_j)] = e^{2\pi i d_{ij}} I_m \). For \( D \in T(n, \mathbb{Z}/m\mathbb{Z}) \), let \( B_n(U(m))_D = \rho^{-1}(D) \). For \( f \in B_n(U(m))_D \), the ordered \( n \)-tuple \((f(a_1), \ldots, f(a_n)) \) is said to be \( D \)-commuting. Note that \( \rho(f) \) is the skew-symmetric matrix associated to the bilinear form \( \omega_f \).

**Definition 2.2.** For any \( n \times n \) matrix \( D \in M_{n \times n}(\mathbb{Z}/m\mathbb{Z}) \), the row space \( R(D) \) is the submodule of \((\mathbb{Z}/m\mathbb{Z})^n \) generated by the rows of \( D \) over \( \mathbb{Z} \). Let \( R_i(D) \subset \mathbb{Z}/m\mathbb{Z} \) be the image of \( R(D) \) under the projection onto the \( i \)-th factor. Let \( r_i(D) = |R_i(D)| \) for \( i = 1, \ldots, n \) and \( \sigma(D) = \sqrt{|R(D)|} \).

We recall the structure of the almost commuting \( n \)-tuples in \( U(m) \), established in [1], Corollary 3.6.

**Proposition 2.3.** For \( D \in T(n, \mathbb{Z}/m\mathbb{Z}) \), the space \( B_n(U(m))_D \) is non-empty and path connected if \( \sigma(D) \) divides \( m \), and is empty otherwise. The space \( B_n(U(m)) \) can be expressed as a disjoint union of path connected components
\[
B_n(U(m)) = \bigcup_{D \in T(n, \mathbb{Z}/m\mathbb{Z})} B_n(U(m))_D.
\]

We begin our analysis by focusing on the basic case when \( m = \sigma(D) \). Let \( 0 \leq t \leq n/2 \) and \( d_1, d_2, \ldots, d_t \neq 0 \in \mathbb{Z}/m\mathbb{Z} \). Define \( D_n(d_1, d_2, \ldots, d_t) = (d_{ij}) \in T(n, \mathbb{Z}/m\mathbb{Z}) \) be the skew-symmetric matrix with

\[
    d_{ij} = \begin{cases} 
        d_k & \text{if } (i, j) = (k + t, k), 1 \leq k \leq t; \\
        -d_k & \text{if } (i, j) = (k, k + t), 1 \leq k \leq t; \\
        0 & \text{otherwise}.
    \end{cases}
\]

**Lemma 2.4.** Suppose \( D \in T(n, \mathbb{Z}/m\mathbb{Z}) \), \( m = \sigma(D) \). Let \((A_1, \ldots, A_n) \in B_n(U(m))_D \). If \( B \in U(m) \) commutes with \( A_i \) for any \( i = 1, \ldots, n \), then \( B \) is a scalar matrix.

**Proof.** By [8, Proposition 4.1], there exists \( Q = (q_{ij}) \in GL(n, \mathbb{Z}) \) such that \( Q^T D Q = D' = D_n(d_1, \ldots, d_t) \). Define \( D'' = D_{n+1}(d_1, \ldots, d_t) \) and \( A'_j = A_1^{q_{1j}} A_2^{q_{2j}} \cdots A_n^{q_{nj}} \) for \( j = 1, \ldots, n \). Then \( \sigma(D'') = m \) and \((A'_1, \ldots, A'_n, B) \in B_{n+1}(U(m))_{D''} \). By [1, theorem 3.3], there exists an orthonormal basis of \( \mathbb{C}^m \) consisting of eigenvectors of \( B \) corresponding to a common eigenvalue. Hence, \( B \) is a scalar matrix. \( \square \)

**Lemma 2.5.** Suppose \( D \in T(n, \mathbb{Z}/m\mathbb{Z}) \) and \( m = \sigma(D) \). Let \((A_1, \ldots, A_n) \in B_n(U(m))_D \) and \( \omega = e^{2\pi i / r_j(D)} \). If \( \lambda \) is an eigenvalue of \( A_j \), then \( \omega^q \lambda \), where \( q = 0, \ldots, r_j(D) - 1 \), are all the distinct eigenvalues of \( A_j \).

**Proof.** Let \( D = (d_{ij}) \). There exists \( a_1, \ldots, a_n \in \mathbb{Z} \) such that \( a_1 d_{ij} + \ldots + a_n d_{nj} = \left[-\frac{1}{r_j(D)}\right] \).

Let \( B = A_1^{a_1} \cdots A_n^{a_n} \). Then \( \omega B A_j = A_j^{-1} B A_j \). If \( v \) is an eigenvector of \( A_j \) corresponding to eigenvalue \( \lambda \), then \( A_j B v = \omega B A_j v = \omega B (\lambda v) = \omega \lambda B v \) and so \( \omega \lambda \) is also an eigenvalue of \( A_j \).

Inductively \( \omega^q \lambda \) is also an eigenvalue of \( A_j \) for any \( q \). On the other hand, \( A_j^{r_j(D)} \) commutes with \( A_1, \ldots, A_n \). By lemma 2.4, \( A_j^{r_j(D)} \) is a scalar matrix. Since \( \lambda \) is an eigenvalue of \( A_j \), \( A_j^{r_j(D)} = \lambda^{r_j(D)} I_m \) and so any eigenvalue of \( A_j \) is of the form \( \omega^q \lambda \). \( \square \)

In the situation of lemma 2.5, take an eigenvalue \( \lambda_j \) of \( A_j \). Define

\[
    c_j = \frac{1}{2\pi \sqrt{-1}} \log \lambda_j \in \mathbb{R}/\langle \frac{1}{r_j(D)} \rangle \cong S^1/R_j(D).
\]

This element \( c_j \) is independent of the choice of \( \lambda_j \).

**Theorem 2.6.** Let \( m = \sigma(D) \). There exists a \( \mathbb{T}^n \)-equivariant homeomorphism

\[
    B_n(U(m))_D/U(m) \cong \mathbb{T}^n/R(D)
\]

such that its composition with the quotient map

\[
    B_n(U(m))_D \to B_n(U(m))_D/U(m) \cong \mathbb{T}^n/R(D) \to \prod_{j=1}^n S^1/R_j(D)
\]

sends \((A_1, \ldots, A_n)\) to \((c_1, \ldots, c_n)\).

**Proof.** If \( D \) is of the special form \( D = D_n(d_1, \ldots, d_t) \), the theorem follows easily from the proof of theorem 3.4 in [1]. In the general case, let \((C_1, \ldots, C_n) \in B_n(U(m))_D \). By multiplying scalars if necessary, we assume that \( 1 \) is an eigenvalue of each \( C_i \). The map \( f : \mathbb{T}^n \to B_n(U(m))_D \) with \( f(\theta_1, \ldots, \theta_n) = (\theta_1 C_1, \ldots, \theta_n C_n) \) is \( \mathbb{T}^n \)-equivariant. Also, there exists \( Q = (q_{ij}) \in GL(n, \mathbb{Z}) \) such that \( Q^T D Q = D' = D_n(d_1, \ldots, d_t) \). Define
\( \phi : \mathbb{T}^n \to \mathbb{T}^n \) by \( \phi(\theta_1, \ldots, \theta_n) = (\theta'_1, \ldots, \theta'_n) \) and \( g : B_n(U(m))_D \to B_n(U(m))_{D'} \) by \( g(A_1, \ldots, A_n) = (A'_1, \ldots, A'_n) \), where \( \theta'_j = \theta_{q(j)}^1 \cdot \cdots \cdot \theta_{q(j)}^n \) and \( A'_j = A_{q(j)}^1 A_{q(j)}^2 \cdots A_{q(j)}^n \) for \( j = 1, \ldots, n \). Note that \( g \) is a \( \phi \)-equivariant and \( U(m) \)-equivariant homeomorphism. Using the result for the special case \( D' \), it can be deduced that the composition

\[
\mathbb{T}^n/R(D') \cong \mathbb{T}^n/R(D) \xrightarrow{\tilde{f}} B_n(U(m))_D/U(m) \cong B_n(U(m))_{D'/U(m)}
\]

is a homeomorphism and so is \( \tilde{f} \). It is clear that \( \tilde{f}^{-1} \) has the desired properties. \( \square \)

Note that \( R(D) \) is a finite subgroup of \( \mathbb{T}^n \), acting by translation; thus \( \mathbb{T}^n \to \mathbb{T}^n/R(D) \) is a covering space and \( \mathbb{T}^n/R(D) \) is homeomorphic to \( \mathbb{T}^n \).

For the general case \( m = l \cdot \sigma(D) \), we recall that from \([1]\), Corollary 3.10 that

\[
B_n(U(m))_D/U(m) \cong (B_n(\sigma(D))_D/U(\sigma(D)))^l/\Sigma_l = SP^l(B_n(\sigma(D))_D/U(\sigma(D))).
\]

Hence, we obtain that

**Theorem 2.7.** The moduli space of ordered almost commuting \( n \)-tuples in \( U(m) \) can be expressed as a disjoint union of symmetric products of the \( n \)-torus \( \mathbb{T}^n \):

\[
B_n(U(m))/U(m) \cong \coprod_{D \in T(n, \mathbb{Z}/m\mathbb{Z})} SP^\frac{m}{\sigma(D)}(\mathbb{T}^n/R(D))
\]

Now we apply the following result.

**Lemma 2.8 (\([2]\), Lemma 2.3).** The projection map \( U(m) \to PU(m) \) induces a \( U(m) \)-equivariant, \( \mathbb{T}^n \)-principal bundle

\[
B_n(U(m)) \to Hom(\mathbb{Z}^n, PU(m))
\]

which gives rise to a homeomorphism

\[
\mathbb{T}^n \backslash B_n(U(m))/U(m) \cong Rep(\mathbb{Z}^n, PU(m)).
\]

In particular it induces a bijection between \( \pi_0(B_n(U(m))/U(m)) \) and \( \pi_0(Rep(\mathbb{Z}^n, PU(m))) \).

The components \( B_n(U(m))_D \) each give rise to a principal \( \mathbb{T}^n \)-bundle, and after dividing out by conjugation we see that the action of \( \mathbb{T}^n \) is given by the simultaneous action on each unordered coordinate in the symmetric product through the homomorphism \( \mathbb{T}^n \to \mathbb{T}^n/R(D) \);

we denote these reduced symmetric products by \( \overset{\text{m}}{SP} (\mathbb{T}^n/R(D)) \). Note that they are all in fact homeomorphic to the corresponding reduced symmetric product of \( \mathbb{T}^n \). Combining the two results above we obtain

**Theorem A.**

\[
Rep(\mathbb{Z}^n, PU(m)) \cong \coprod_{D \in T(n, \mathbb{Z}/m\mathbb{Z})} SP^\frac{m}{\sigma(D)}(\mathbb{T}^n/R(D))
\]
The labelling of components can be understood using cohomology. As before we can identify $T(n,\mathbb{Z}/m\mathbb{Z})$ with $H^2(\mathbb{Z}^n,\mathbb{Z}/m\mathbb{Z})$ using a basis. For a projective representation $h : \mathbb{Z}^n \to PU(m)$, there is a cohomology class in $H^2(\mathbb{Z}^n,\mathbb{Z}/m\mathbb{Z})$ associated to it, defined as the pullback $\alpha = h^* (\nu)$, where $\nu \in H^2(BPU(m),\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$ is the canonical generator associated to $SU(m)$. The component corresponding to $h$ is precisely the one labelled by a skew symmetric matrix $D$ which represents $\alpha$. As this component is non-empty we must have $\sigma (\alpha) \mid m$.

**Example 2.9.** In the case when $n = 2$, $H^2(\mathbb{Z}^2,\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$ and $\sigma (D) = |D|$, the order of $D$, which always divides $m$ and so $N(2, m) = m$. Our decomposition can be written as

$$
\text{Rep}(\mathbb{Z}^2, PU(m)) \cong \bigcup_{D \in \mathbb{Z}/m\mathbb{Z}} \bigcup_{1 < |D| < m} \bigcup \{x_1, \ldots, x_{\Phi(m)}\}
$$

where $\Phi$ is Euler’s function (see [3], Proposition 9).

**Example 2.10.** We now consider the case when $m = p^2$ and $n \geq 4$. From the analysis in [1], section 3, we see that $\text{Rep}(\mathbb{Z}^n, PU(p^2))$ has

$$
N(n, p^2) = 1 + \frac{(p^{n-1} - 1)(p^n - 1)(p^{2n+1} - p^n - p^{n-1} + p^4 + p^2 - 1)}{(p^2 - 1)(p^4 - 1)}
$$

components, of which

$$
u(p) = \frac{(p^{n-1} - 1)(p^n - 1)}{p^2 - 1}
$$

correspond to $\sigma (D) = p$ and so we have

$$
\text{Rep}(\mathbb{Z}^n, PU(p^2)) \cong \bigcup \bigcup \{x_1, \ldots, x_{N(n,p^2) - u(p) - 1}\}
$$

Our analysis can be applied to describe the projective representations of any finitely generated abelian group. Let

$$
\Gamma = \mathbb{Z}/k_1 \oplus \ldots \oplus \mathbb{Z}/k_s \oplus \mathbb{Z}^n \cong \text{Tor}(\Gamma) \oplus \mathbb{Z}^n.
$$

Define

$$
B(\Gamma, U(m)) = \{(A_1, \ldots, A_{s+n}) \in B_{s+n}(U(m)) : A_{i}^{k_i} = I_m \text{ for } 1 \leq i \leq s\}.
$$

It is a subspace of $B_{s+n}(U(m))$ and is invariant under the action of the subgroup

$$
\{(\theta_1, \ldots, \theta_{s+n}) : \theta_i^{k_i} = 1 \text{ for } 1 \leq i \leq s\} \cong \text{Tor}(\Gamma) \times \mathbb{T}^n \subset \mathbb{T}^{s+n}.
$$

This group action commutes with the conjugation action of $U(m)$. Hence, $B(\Gamma, U(m))$ is a $U(m)$–equivariant, $(\text{Tor}(\Gamma) \times \mathbb{T}^n)$–principal bundle over $\text{Hom}(\Gamma, PU(m))$. Also, there is a decomposition of

$$
B(\Gamma, U(m)) = \bigsqcup B(\Gamma, U(m))_D
$$

into a disjoint union of subspaces indexed by $D \in T(s + n, \mathbb{Z}/m\mathbb{Z})$. The subspaces

$$
\text{Hom}(\Gamma, PU(m))_D \subset \text{Hom}(\Gamma, PU(m)) \text{ and } \text{Rep}(\Gamma, PU(m))_D \subset \text{Rep}(\Gamma, PU(m))
$$
can be similarly defined.

Note that for \( A_i \in U(m) \), \( A_i^{k_i} = I_m \) if and only if all the eigenvalues of \( A_i \) are \( k_i \)-th roots of unity. By Theorem 2.6 we obtain the following results similar to Theorem 2.7 and Theorem A.

**Theorem 2.11.** Let \( \Gamma = \mathbb{Z}/k_1 \oplus \ldots \oplus \mathbb{Z}/k_s \oplus \mathbb{Z}^n \cong \text{Tor}(\Gamma) \oplus \mathbb{Z}^n \). Then

\[
B(\Gamma, U(m))/U(m) \cong \prod_{D \in T(s+n, \mathbb{Z}/m\mathbb{Z})} \left( SP^{m}_{\pi(\hat{D})}((\text{Tor}(\Gamma) \times \mathbb{T}^n)/R(D)) \right)
\]

and

\[
\text{Rep}(\Gamma, PU(m)) \cong \prod_{D \in T(s+n, \mathbb{Z}/m\mathbb{Z})} \left( SP^{m}_{\pi(\hat{D})}((\text{Tor}(\Gamma) \times \mathbb{T}^n)/R(D)) \right)
\]

**Remark 2.12.** Suppose \( \sigma(D)|m \) and \( r_i(D)|k_i \). Let \( H \) be the cokernel of the composition \( R(D) \hookrightarrow \text{Tor}(\Gamma) \times \mathbb{T}^n \overset{\text{proj}_2}{\rightarrow} \text{Tor}(\Gamma) \) of inclusion followed by projection. Then, if we focus on the images of the components associated to \( D \) under the quotient

\[
B(\Gamma, U(m)) \rightarrow \text{Hom}(\Gamma, PU(m)),
\]

we have

\[
\pi_0(\text{Hom}(\Gamma, PU(m))_D) \cong \pi_0(\text{Rep}(\Gamma, PU(m))_D) \cong SP^{m}_{\pi(\hat{D})}(H).
\]

The number of orbits on the right hand side can be computed using Burnside’s lemma:

\[
\left| SP^{m}_{\pi(\hat{D})}(H) \right| = \frac{1}{|H|} \sum_{g \in H} \left| SP^{m}_{\pi(\hat{D})}(H)^g \right|
\]

where

\[
\left| SP^{m}_{\pi(\hat{D})}(H)^g \right| = \begin{cases} 
\left( \frac{m + |H|\sigma(D)}{|g|\sigma(D)} - 1 \right) & |g|\sigma(D) | m \\
0 & |g|\sigma(D) \nmid m
\end{cases}
\]

**Example 2.13** (Projective representations of finite abelian groups). Our results allow us to recover results about projective representations of finite abelian groups. Consider a finite abelian group \( \Gamma = \mathbb{Z}/k_1 \oplus \ldots \oplus \mathbb{Z}/k_s \), where \( k_i \) divides \( k_{i+1} \) for \( i = 1, \ldots, s - 1 \). By Theorem 2.11 the space of degree \( m \) projective unitary representation modulo projective equivalence is given by

\[
\text{Rep}(\Gamma, PU(m)) \cong \prod_{D \in T(s+n, m\mathbb{Z})} \left( SP^{m}_{\pi(\hat{D})}(\Gamma/R(D)) \right).
\]

If \( \sigma(D) = m \), then \( SP^{m}_{\pi(\hat{D})}(\Gamma/R(D)) \) is a single point and corresponds to an irreducible projective representation.
Removing the restriction on the dimension, we see that projective equivalence classes of irreducible projective representations of $\Gamma$ are in one-to-one correspondence with the $D \in T(n, \mathbb{Q}/\mathbb{Z})$ such that $r_i(D)|k_i$ for any $i = 1, \ldots, s$. Note that since $D$ is skew-symmetric, the condition $r_i(D)|k_i$ is equivalent to $|d_{ij}| = |d_{ji}|$ divides $k_i$ for $i < j$.

In terms of cohomology, this indexing can be seen to arise from pulling back using the projection $p: \mathbb{Z}^s \to \Gamma$, which yields a factorization

$$H^2(BPU(m), \mathbb{Z}/m\mathbb{Z}) \to H^2(\Gamma, \mathbb{Z}/m\mathbb{Z}) \to H^2(\mathbb{Z}^s, \mathbb{Z}/m\mathbb{Z}).$$

Note that $p^*: H^2(\Gamma, \mathbb{Q}/\mathbb{Z}) \to H^2(\mathbb{Z}^s, \mathbb{Q}/\mathbb{Z})$ is injective, therefore if we consider all possible dimensions $m$ we see that the total indexing is in one-to-one correspondence with elements in $H^2(\Gamma, \mathbb{Q}/\mathbb{Z}) \cong H^3(\Gamma, \mathbb{Z})$. Moreover, for the projective equivalence class corresponding to $D$, its set of representatives, up to linear equivalence, is indexed by $\Gamma/R(D)$, and each such representation has degree $\sigma(D)$.

3. Projective Representations and Flat Bundles

We now reformulate the computation of path components using homotopy theory, following the approach in [4], Lemma 2.5. Recall that given a group homomorphism $h: \Gamma \to G$, it induces a continuous (pointed) map on classifying spaces $Bh: B\Gamma \to BG$. For $G$ a compact connected Lie group, the correspondence

$$Hom(\Gamma, G) \to Map_*(BG, BG)$$

is continuous and so induces a map on path components. As conjugation by $G$ is homotopically trivial on $BG$, it gives rise to a map

$$\pi_0(Rep(\Gamma, G)) \to [BG, BG].$$

If $X$ is a path-connected CW–complex with $\pi_1(X) = \Gamma$, then composing with the classifying map $c: X \to B\pi_1(X)$ of the universal cover $\tilde{X} \to X$ we obtain a map

$$\Psi^G_X: \pi_0(Rep(\pi_1(X), G)) \to [X, BG]$$

which measures the flat principal $G$–bundles on $X$.

In the case when $G = PU(m)$ we have a canonical homotopy class

$$\Omega: BPU(m) \to K(\mathbb{Z}/m\mathbb{Z}, 2)$$

associated to the central extension

$$1 \to \mathbb{Z}/m \to SU(m) \to PU(m) \to 1.$$ 

For a CW–complex $X$, this gives rise to the composition

$$\pi_0(Rep(\pi_1(X), PU(m))) \xrightarrow{\Psi^PU(m)} [X, BPU(m)] \xrightarrow{\Omega} [X, K(\mathbb{Z}/m\mathbb{Z}, 2)] = H^2(X, \mathbb{Z}/m\mathbb{Z}).$$

We have

**Theorem B.** Let $X$ denote a connected CW–complex with $\pi_1(X) = \mathbb{Z}^n$; then the map $\Psi^PU(m)_X: \pi_0(Rep(\pi_1(X), PU(m)) \to [X, BPU(m)]$ is injective for all $n, m \geq 1$ and so there are $N(n, m)$ distinct isomorphism classes of flat principal $PU(m)$–bundles on $X$. 
Proof. First we consider the basic case $X = BZ^n = T^n$: the composition
\[
\pi_0(\text{Rep}(Z^n, PU(m))) \xrightarrow{\Psi_T^{PU(m)}} [T^n, BPU(m)] \xrightarrow{\Omega} [T^n, K(Z/mZ, 2)] = H^2(BZ^n, Z/mZ)
\]
is the map on path components described in Section 2. We know that its image has precisely $N(n, m)$ elements, corresponding to the skew symmetric matrices $D$ with $\sigma(D)$ dividing $m$. In other words, this map distinguishes components, and is injective. If
\[
c^* : H^2(BZ^n, Z/mZ) \to H^2(X, Z/mZ)
\]
is induced by the classifying map, by naturality we have
\[
c^* \circ \Omega \circ \Psi_T^{PU(m)} = \Omega \circ \Psi_X^{PU(m)}.
\]
From the five-term exact sequence in cohomology for the fibration $\tilde{X} \to X \to B\pi_1 X$ we infer that $c^* : H^2(B\pi_1(X), Z/mZ) \to H^2(X, Z/mZ)$ is in fact injective. Therefore the composition $\Omega \circ \Psi_X^{PU(m)}$ is injective and so is $\Psi_X^{PU(m)}$. □

Next we study the surjectivity of the map
\[
\Psi_T^{PU(m)} : \pi_0(\text{Rep}(Z^n, PU(m))) \to [T^n, BPU(m)].
\]
We begin with the following lemma

Lemma 3.1. Let $m \geq 2$. If $n \leq 3$, $[T^n, BPU(m)]$ is finite, of cardinality equal to that of $T(n, Z/mZ)$. If $n \geq 4$, the set $[T^n, BPU(m)]$ has infinitely many elements.

Proof. This follows from Woodward’s classification of principal $PU(m)$-bundles for low dimensional complexes (see [9], page 514). For $T^n$, $n \leq 3$ he shows that the map
\[
[T^n, BPU(m)] \to H^2(T^n, Z/mZ) \cong T(n, Z/mZ)
\]
is a bijection. We outline a direct proof that $[T^n, BPU(m)]$ must be infinite for $n \geq 4$. For $n = 4$, there is a map $\Phi : T^4 \to S^4$ which induces an isomorphism on $H_4$. This arises from using the 4-dimensional cell in a CW-complex decomposition for $T^4$ from its structure as a product of circles, each having a single 0-cell and a single 1-cell. The map $BSU(m) \to BPU(m)$ induces an isomorphism $Z = \pi_4(BSU(m)) \cong \pi_4(BPU(m))$ and so it is possible to choose a map $\rho : S^4 \to BPU(m)$ realizing this isomorphism. The Hurewicz map $\pi_4(BPU(m)) \to H_4(BPU(m), Z)$ can be identified with the monomorphism $H_4(BSU(m), Z) \to H_4(BPU(m), Z)$, hence the composition $\rho \circ \Phi : T^4 \to BPU(m)$ is a map inducing an injection on the fundamental class in $T^4$. It follows that $[T^4, BPU(m)]$ cannot be finite. For $n \geq 4$, we can use the split surjection $[T^n, BPU(m)] \to [T^4, BPU(m)]$ to verify the claim. □

Proposition 3.2. Let $m \geq 2$, then $\Psi_T^{PU(m)} : \pi_0(\text{Rep}(Z^n, PU(m))) \to [T^n, BPU(m)]$ is surjective if and only if $n \leq 3$.

Proof. From the definition in cohomology (and using an appropriate basis when $n = 3$) it is easy to see that for $n = 1, 2, 3$, we have that $\{D \in T(n, Z/mZ) : \sigma(D) \mid m\} = T(n, Z/mZ)$. Thus we conclude that $\Omega \circ \Psi_T^{PU(m)}$ is a bijection and therefore by a cardinality argument so is $\Psi_T^{PU(m)}$ for $n = 1, 2, 3$. For $n \geq 4$ we have verified that $[T^n, BPU(m)]$ is infinite, whence the result follows. □
Corollary 3.3. There exists a principal $PU(m)$-bundle on the $n$–torus $\mathbb{T}^n$ which does not admit a flat structure if and only if $n \geq 4$.

References

[1] Alejandro Adem and Man Chuen Cheng. Representation spaces for central extensions and almost commuting unitary matrices. J. London Math. Soc. 94 (2): pp 503-524 (2016).

[2] Alejandro Adem, Frederick R. Cohen, and José Manuel Gómez. Stable splittings, spaces of representations and almost commuting elements in Lie groups Math. Proc. Camb. Phil. Soc., 149, 455-490, 2010.

[3] Alejandro Adem, Frederick R. Cohen, and José Manuel Gómez. Commuting elements in central products of special unitary groups Proc. Edinburgh Math. Soc. (Series 2) 56(1):1–12, 2013.

[4] Alejandro Adem and Frederick R. Cohen. Commuting elements and spaces of homomorphisms. Math. Ann., 338(3):587–626, 2007.

[5] Maxime Bergeron. The topology of nilpotent representations in reductive groups and their maximal compact subgroups. Geom. Topol., 19(3):1383–1407, 2015.

[6] Armand Borel. Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes. Tohoku Math. J., (2) Vol. 13, Number 2 (1961), 216-240.

[7] Armand Borel, Robert Friedman, and John W. Morgan. Almost commuting elements in compact Lie groups. Mem. Amer. Math. Soc., 157(747):x+136, 2002.

[8] Zinovy Reichstein and Nikolaus Vonessen. Rational central simple algebras. Israel J. Math., 95:253–280, 1996.

[9] L. M. Woodward The Classification of principal $PU(n)$–bundles over a 4–complex J. London Math. Soc., (2), 25 (1982), 513-524.

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