Non proper HNN extensions and uniform uniform exponential growth

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Abstract

If a finitely generated torsion free group $K$ has the property that all finitely generated subgroups $S$ of $K$ are either small or have growth constant bounded uniformly away from 1 then a non proper HNN extension $G = K \rtimes_{\alpha} \mathbb{Z}$ of $K$ has the same property. Here small means cyclic or, if $\alpha$ has no periodic conjugacy classes, free abelian of bounded rank.

1 Introduction

If $A$ is a finite generating set for the group $G$ then the growth function $\gamma_A : \mathbb{N} \mapsto \mathbb{N}$ of $G$ with respect to $A$ is the number of elements in $G$ with word length at most $n$ when written as a product of the elements of $A$ and their inverses. The exponential growth rate $\omega(G, A)$ of $G$ with respect to $A$ is defined to be the limit as $n$ tends to infinity of $\gamma_A(n)^{1/n}$: this limit always exists as $\gamma_A$ is a submultiplicative function. If $\omega(G, A) > 1$ for some $A$ then this holds for all finite generating sets and so we have the division of finitely generated groups into those with exponential and non-exponential word growth.

However we also have the newer concept of uniform exponential growth. Here we define the growth constant $\omega(G)$ to be the infimum over all finite
generating sets \( A \) of \( \omega(G, A) \) and say that \( G \) has uniform exponential growth if this infimum is strictly bigger than 1. (However the supremum is always infinity for groups of exponential growth: for instance \cite{31} Proposition 12.10(b) shows that for any \( c > 1 \) there is a finite set \( S \subseteq G \) where the group \( \langle S \rangle \) has \( \gamma_S(n) \geq c^n \) for arbitrarily large \( n \). But now we can add a finite generating set of \( G \) to \( S \) without reducing the growth function.) The existence of a finitely generated group having exponential but non uniform exponential growth was eventually established in \cite{34}. Nevertheless there are many classes where any group with exponential growth is also known to have uniform exponential growth: here we just mention word hyperbolic groups \cite{25} and indeed groups which are hyperbolic relative to a collection of proper subgroups \cite{35}, groups which are linear over a field of characteristic 0 \cite{16}, elementary amenable groups \cite{30} and 1-relator groups \cite{20}.

Very recently there has been interest in the concept of what is sometimes called uniform uniform growth: we say that a class of groups has uniform uniform growth if there is \( k > 1 \) such that every group in the class has growth constant at least \( k \). In particular one consequence of the work of Breuillard and Breuillard-Gelander culminating in \cite{9} Corollary 1.2 is that if \( d \) is an integer then the class of non virtually soluble linear groups of dimension \( d \) over any field has uniform uniform growth. We also have \cite{17} which establishes uniform uniform growth for the fundamental groups of closed 3-manifolds which have exponential growth. Moreover whenever we have a group \( G \) with uniform exponential growth there is always an opportunity to ask about uniform uniform growth: namely is there \( k > 1 \) such that any finitely generated subgroup of \( G \) which is not “small” (for instance virtually cyclic, virtually nilpotent, virtually soluble) has growth constant at least \( k \)? Note that if \( G \) fails this property when small means virtually soluble then \( G \) cannot be a linear group by the above. For examples which do possess this property we have in \cite{29} the existence of \( k_S > 1 \) such that any finitely generated subgroup of the mapping class group of a compact orientable surface \( S \) is virtually abelian or has growth constant at least \( k_S \).

Another example is that in the paper \cite{20} on 1-relator groups, we have uniform uniform growth over all 1-relator groups with \( k = 2^{1/4} \) apart from cyclic groups, \( \mathbb{Z} \times \mathbb{Z} \) or the fundamental group of the Klein bottle. This is established using the results of \cite{13} which obtained uniform uniform growth with the same \( k \) for most amalgamated free products. Also this \( k \) applies for HNN extensions \( G \) except where both associated subgroups are equal to the base \( H \), in which case \( G \) is a semidirect product of the form \( H \rtimes \mathbb{Z} \).
In this paper our aim is to give results on when such a group $G$ has uniform growth. In particular we consider the case when the class of finitely generated subgroups of $H$ which are not “small” has uniform uniform growth and we give conditions that will imply $G$ has uniform growth. Moreover the methods establish that the class of finitely generated subgroups of the new group $G$ which are not “small” also has uniform uniform growth. Sometimes the definition of being small in $G$ is slightly wider than being small in $H$ but we investigate when we can take both to be the same, which allows us to apply our results to iterated HNN extensions.

In particular we show in Section 2 that the lower bound $2^{1/4}$ for the growth constant of 1-relator groups also holds for finitely presented groups of deficiency 1 (with exactly the same exceptions) and finitely generated groups possessing a homomorphism onto $\mathbb{Z}$ with infinitely generated kernel. We then consider in Section 3 groups $G$ of the form $K \rtimes_{\alpha} \mathbb{Z}$ where $K$ is finitely generated and torsion free. We show that if the growth constants of the non-cyclic finitely generated subgroups of $K$ are uniformly bounded away from 1 then the same is true for the finitely generated subgroups of $G$ which are not cyclic, $\mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group. This has applications to the growth constant of non proper HNN extensions of word hyperbolic groups, including free-by-$\mathbb{Z}$ and surface-by-$\mathbb{Z}$ groups, as well as extensions of Baumslag-Solitar groups. In Section 5 we prove the same result with the non-cyclic finitely generated subgroups of $K$ replaced by those finitely generated subgroups which are not isomorphic to a free abelian group of bounded rank. However we need to impose the condition that the automorphism $\alpha$ forming the semidirect product has no periodic conjugacy classes. If this condition holds then the finitely generated subgroups of $G$ without uniform exponential growth will be the same as those of $K$, so the result can be iterated.

In the last section we show how to adapt this theorem to allow the Klein bottle group as an exceptional subgroup, both of $K$ and $G$ so again we can form iterated HNN extensions with uniform exponential growth. Here we require that $K$ is locally indicable. This has applications to the growth of non proper HNN extensions of finitely generated groups of cohomological dimension (at most) 2, because the only virtually nilpotent groups in this class are $(\{e\}, \mathbb{Z}), \mathbb{Z} \times \mathbb{Z}$ and the Klein bottle group. The methods of proof are for the most part standard exponential growth and group theoretical arguments, although the Alexander polynomial also plays a role.
2 Groups of deficiency 1

For background in uniform exponential growth, see [21] Chapter VI and related references. Two easily proved but invaluable results for a finitely generated group $G$ are:

1. If there exists a surjective homomorphism from $G$ to $Q$ then $\omega(G) \geq \omega(Q)$.
2. (Shalen-Wagreich Lemma) If $H$ is an index $i$ subgroup of $G$ then $\omega(H) \leq \omega(G)^{2i-1}$.

However it is not true that if $H$ is a finitely generated subgroup of $G$ then $\omega(G) \geq \omega(H)$. For although it is true that if the finite set $A$ generates $G$ and $S$ is a subset of $A$ then $\omega(G, A) \geq \omega(\langle S \rangle, S)$, so a finitely generated group with a finitely generated subgroup of exponential growth also has exponential growth, the example [34] due to J. S. Wilson of the group with non-uniform exponential growth is shown to have exponential growth because it contains a non-abelian free group.

We also quote here a few standard facts about HNN extensions. We can form an HNN extension $G$ with stable letter $t$, base $H$ and associated subgroups $A$, $B$ whenever $H$ is a group possessing subgroups $A$ and $B$ where there exists an isomorphism $\theta$ from $A$ to $B$. Note that if $H = A = B$ then we obtain a semidirect product $G = H \rtimes_\theta \mathbb{Z}$. We call this a non-proper HNN extension whereas a proper HNN extension is where at least one of $A$ or $B$ is not equal to $H$. Other possibilities are if one of $A$ or $B$ is equal to $H$ in which case we say $G$ is an ascending HNN extension (and if exactly one is equal we call the HNN extension strictly ascending). Any HNN extension gives rise to a homomorphism $\chi$ onto $\mathbb{Z}$ (called the associated homomorphism of the HNN extension), which is defined by sending $t$ to $1$ and $H$ to $0$. It is clear that if $H = A = B$ then the kernel of $\chi$ is equal to $H$ because in this case every element of $G$ can be written in the form $ht^i$ for $h \in H$.

Conversely a homomorphism $\chi$ from a group $G$ onto $\mathbb{Z}$ allows us to express $G$ as an HNN extension. However this is ambiguous if further restrictions are not imposed: for instance we always have $G = \ker(\chi) \rtimes \mathbb{Z}$. If we insist that the base is finitely generated then a better picture emerges: by [30] Section 4 if we express $G$ as two HNN extensions of this more restricted form and the corresponding associated homomorphisms are the same then they must both be non-proper, strictly ascending or non-ascending together. In fact we are in the first case if and only if $\ker(\chi)$ is finitely generated whereupon it is equal to the base. In the second case there is a small amount of ambiguity in choosing the base but a great deal for non-ascending HNN extensions. As an example
the Baumslag Solitar group $BS(2, 3) = \langle a, t|tat^{-1} = a^3 \rangle$ can be formed with base $\langle a \rangle$ equal to $\mathbb{Z}$, but we can also write it as $\langle a, b, t|b^2 = a^3, tat^{-1} = b \rangle$ which is an HNN extension with base the fundamental group of the trefoil knot.

A finitely presented group $G$ is said to have deficiency $d$ if there exists a presentation for $G$ consisting of $n$ generators and $n - d$ relators. It is well known that $\omega(F_n) = 2n - 1$ for $F_n$ the free group of rank $n$ and Gromov conjectured that a group $G$ with deficiency $d$ has $\omega(G) \geq 2d - 1$. It is readily seen using (1) and (2) that if $d \geq 2$ then $G$ has uniform exponential growth by quoting a famous result of Baumslag and Pride [2] that $G$ has a finite index subgroup surjecting onto $F_2$ but we have no uniform control over this index. In [33] J. S. Wilson established Gromov’s conjecture by use of pro-$p$ presentations. If $d = 1$ then we certainly have groups $G$ for which $\omega(G) = 1$, for instance $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ and the Klein bottle group $\langle a, t|tat^{-1} = a^{-1} \rangle$. But in another paper [32] of the same author, it was shown that any soluble group of deficiency 1 is isomorphic to a Baumslag-Solitar group $G_k = \langle t, a|tat^{-1} = a^k \rangle$ with $k \in \mathbb{Z}$ (so the above 3 groups correspond to $k = 0, 1, -1$ respectively). However it is known that $G_k$ has uniform exponential growth for all other values of $k$, so it is worth asking if non-soluble groups of deficiency 1 have uniform growth or even uniform uniform growth.

A special class of deficiency 1 groups are those with a 2-generator 1-relator presentation which were dealt with in [20] (following partial results in [15]) where it was established that such a group $G$ has $\omega(G) \geq 2^{1/4}$ with the above three exceptions. This used the following theorem by de la Harpe and Bucher in [13].

Theorem 2.1 A finitely generated group $G$ which is an HNN extension with base $H$ and associated subgroups $A$ and $B$ has $\omega(G) \geq 2^{1/4}$ provided that $[H : A] + [H : B] \geq 3$.

The inequality allows infinite index subgroups, and so the condition in this Theorem is equivalent to saying that $G$ can be expressed as a proper HNN extension. We can apply this in a similar way as for 1-relator group presentations.

Theorem 2.2 If $G$ has a deficiency 1 presentation then $\omega(G) \geq 2^{1/4}$, with the exception of $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ and the Klein bottle group where $\omega(G) = 1$.

Proof. By abelianising this presentation we see that $G$ must have a homomorphism onto $\mathbb{Z}$. A result [7] of Bieri and Strebel states that any finitely
presented group with a homomorphism $\chi$ onto $\mathbb{Z}$ can be written as an HNN extension with $\chi$ as the associated homomorphism where the base $H$ and the associated subgroups $A, B$ are all finitely generated. If $A < H$ or $B < H$ then the above result applies, but if $A = B = H$ then we know that $H$ is equal to $\ker(\chi)$. Now a recent result [24] of Kochloukova is that a finitely generated kernel of any homomorphism from a deficiency 1 group onto $\mathbb{Z}$ is free. If it is free of rank 0 or 1 then we obtain the three exceptions but any free by cyclic group of the form $G = F_n \rtimes_\alpha \mathbb{Z}$ where $n \geq 2$ has $\omega(G) \geq 3^{1/6} > 2^{1/4}$. This result is Lemma 2.3 in [15] and we will give more details in the next section where our aim is to generalise this proof for other groups.

In [20] an immediate corollary of their result on 1-relator groups is that if $G$ is finitely presented and has a finite generating set $S$ such that $\omega(G, S) < 2^{1/4}$ then $G$ has deficiency at most 1. Consequently by Theorem 2.2 we can strengthen this conclusion by saying that $G$ has deficiency at most 0, or $G$ is one of our three exceptional groups in which case $\omega(G, S) = 1$.

It might be wondered if we can have uniform exponential growth not just for deficiency 1 groups but for any subgroup of a deficiency 1 group, or at least for any subgroup which is not virtually nilpotent. This cannot hold because any group $G$ having a finite presentation of deficiency at most zero can appear as a subgroup of deficiency 1: merely add new generators to the presentation for $G$ to form the free product $G * F_n$ until the new presentation has deficiency 1 and then $G$ will be a free factor. However the chief ingredients in the above proof are that our group has a surjection to $\mathbb{Z}$ and that if the kernel of this surjection is finitely generated then this kernel belongs to a well behaved class of groups. The first condition holds for all non-trivial subgroups whenever $G$ is a torsion free 1-relator group (the torsion free condition is equivalent here to the relator not being a proper power in the free group providing the generators for the 1-relator presentation for $G$) because it was proved by Brodskii [12] and Howie [23] that $G$ is locally indicable: that is any (non-trivial) finitely generated subgroup of $G$ has a homomorphism onto $\mathbb{Z}$. As for the second condition, we will need to invoke a longstanding conjecture on 1-relator groups (required here only for the torsion free ones) which is that they are coherent, namely every finitely generated subgroup is finitely presented.

**Proposition 2.3** Suppose that $G$ is a torsion free 1-relator group which is coherent. Then any finitely generated subgroup $H$ of $G$ has $\omega(H) \geq 2^{1/4}$
unless $H$ is trivial, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group in which case $\omega(H) = 1$.

**Proof.** We know that there exists a homomorphism $\chi$ from $H$ onto $\mathbb{Z}$ if $H$ is non-trivial. By the above $\omega(H) \geq 2^{1/4}$ if $K = \ker(\chi)$ fails to be finitely generated. But otherwise $H = K \rtimes \mathbb{Z}$. An old result of Lyndon [28] is that $G$ has cohomological dimension 2, so $H$ does as well. The coherence assumption means that $H$ and $K$ are finitely presented. We then apply [5] Corollary 8.6 which states that a normal finitely presented subgroup of infinite index in a finitely presented group of cohomological dimension 2 must be free. Thus $H$ is of the form $F_n \rtimes \mathbb{Z}$ and hence we obtain $\omega(H) \geq 3^{1/6}$ as before unless $n = 0$ or 1, in which case $H$ is again one of our three exceptional groups.

We have seen how the de la Harpe-Bucher result immediately gives us uniform exponential growth for finitely presented groups having a homomorphism to $\mathbb{Z}$ where the kernel is infinitely generated. We finish this section with a few words on how to extend this to finitely generated groups with the same property. This follows as part of a wide ranging result by Osin which is Proposition 3.2 in [30], stating that a finitely generated group $G$ having an infinitely generated normal subgroup where the quotient is nilpotent of class (also called degree) $d$ has $\omega(G) \geq 2^{1/\alpha}$, where $\alpha = 3.4^{d+1}$. For our situation where $d = 1$, this gives us $\omega(G) \geq 2^{1/48}$. However we can recover the previous $2^{1/4}$ bound.

**Proposition 2.4** If $G$ is a finitely generated group having a homomorphism $\chi$ onto $\mathbb{Z}$ with infinitely generated kernel then $\omega(G) \geq 2^{1/4}$.

This follows immediately from Theorem 2.1 (which does allow the base and associated subgroups to be infinitely generated) and the following theorem, which can be seen as a variation on the Bieri-Strebel result, replacing the finitely presented hypothesis for $G$ with that of being finitely generated. Note that the Bieri-Strebel result itself does not hold for finitely generated groups as shown in [6] Section 7 by the example of $F/F''$ for $F$ a non-abelian free group.

**Theorem 2.5** If $G$ is a finitely generated group with a homomorphism onto $\mathbb{Z}$ then either the kernel is finitely generated or $G$ can be written as an HNN extension (with base and associated subgroups possibly infinitely generated) where at least one of the associated subgroups $A, B$ is not equal to the base $H$. 
Proof. We use the fact that $G$ will have a presentation (with finitely many generators but possibly infinitely many relators) where one generator $t$ appears with zero exponent sum in each of the relators (and the homomorphism $\chi$ merely sends an element $g$ of $G$ to the exponent sum of $t$ in any word representing $g$). We will assume that there are two other generators $x, y$ for $G$ in order to reduce numbers of subscripts but the idea behind the proof is exactly the same in the more general case.

Given our presentation

$$G = \langle t, x, y | r_j(t, x, y) : j \in \mathbb{N} \rangle$$

where $t$ has 0 exponent sum in all of the $r_j$, we can use Reidemeister-Schreier rewriting to get a presentation for the kernel $K$ of $\chi$ as follows: the generators are $x_i = t^i x t^{-i}$ and $y_i = t^i y t^{-i}$ for $i \in \mathbb{Z}$ and the relators $\tau_{j,k}$ for $j \in \mathbb{N}$ and $k \in \mathbb{Z}$ are formed by taking the relators $r_j = t^k r_j t^{-k}$ (which are words in $t, x, y$ with zero exponent sum in $t$) and rewriting them in terms of $x_i$ and $y_i$ where $i \in \mathbb{Z}$. We then have a (rather long) alternative presentation for $G$ of the form

$$\langle t, x_i, y_i : i \in \mathbb{Z} | r_{j,k}(x_i, y_i) : j \in \mathbb{N}, k \in \mathbb{Z}; t x_i t^{-1} = x_{i+1}, t y_i t^{-1} = y_{i+1} : i \in \mathbb{Z} \rangle$$

Consequently $G$ is a non-proper HNN extension with base $K$. However we attempt to express $G$ as a strictly ascending (and hence proper) extension by setting $H = A = \langle x_i, y_i \rangle$ where now $i$ ranges over $0, 1, 2, \ldots$, not $\mathbb{Z}$, and $B = \langle x_i, y_i \rangle$ for $i = 1, 2, 3, \ldots$. We can certainly form the HNN extension $\Gamma = \langle H, s \rangle$ with base $H$, stable letter $s$ and associated subgroups $A, B$. This is allowed as $A$ is isomorphic to $B$ inside $H$ because we have $t \in G$ with $t A t^{-1} = B$. The generators for $\Gamma$ are then $s, x_i, y_i$ for $i \geq 0$ and the relations for $\Gamma$ are the relations that hold in $H$ along with

$$s x_i s^{-1} = x_{i+1}, s y_i s^{-1} = y_{i+1} \text{ for } i = 0, 1, 2, \ldots$$

We now alter the presentation for $\Gamma$ somewhat. We add generators $x_i, y_i$ and relators $s x_i s^{-1} = x_{i+1}$ and $s y_i s^{-1} = y_{i+1}$ for negative values of $i$. Also, rather than including all relations that hold in $H$, we take for each original relator $r_j$ a high enough value $l(j)$ of $k$ such that the resulting relator $\tau_{j,l(j)}$ is written in terms of only those $x_i, y_i$ where $i \geq 0$. Then any relation that holds amongst the generators of $H$ also holds in $K$ and so is a consequence of the $\tau_{j,k}$ over all $j$ and $k$. But for each $j$ we included $\tau_{j,l(j)}$ in our presentation.
for $\Gamma$, and so as conjugating by $s^{\pm 1}$ sends $\overline{r}_{j,k}$ to $\overline{r}_{j,k\pm 1}$, all $\overline{r}_{j,k}$ are equal to the identity in $\Gamma$. Consequently in the presentation for $\Gamma$ we can now throw in all remaining $\overline{r}_{j,k}$. Finally, on renaming $s$ as $t$, we see that this presentation for $\Gamma$ is identical to the long one for $G$.

Our hope is that $\Gamma$ is a proper HNN extension with, say, $x_0 \notin H$. However it could be that $x_0$ and $y_0$ are both in $H$, so can be expressed as words $v(x_i, y_i)$ and $w(x_i, y_i)$ where all $i$ appearing in both words are at least 1. As only finitely many $i$ can appear in these two words, let $M$ be the maximum value occurring. Note that $x_i^{-1}$, $y_i^{-1}$ are in $H$ too as on conjugating by $t$ they can be expressed in terms of $x_0, y_0, \ldots, x_{M-1}, y_{M-1}$ and then $x_0, y_0$ can be replaced. Continuing this process, we see that $x_i, y_i \in H$ for all negative values of $i$.

Now what we do is to run the whole proof over again but the other way, meaning that our new $H$ and $A$ are now $\langle x_0, y_0, x_{-1}, y_{-1}, \ldots \rangle$ and $B$ is set equal to $\langle x_{-1}, y_{-1}, x_{-2}, y_{-2}, \ldots \rangle$. If we find that both $x_0$ and $y_0$ are words in $x_i, y_i$ for strictly negative $i$ then let $m$ be the lowest value of $i$ appearing. Just as before we see that $x_1, y_1, x_2, y_2, \ldots$ can be written in terms of $x_m, y_m, \ldots, x_0, y_0$. But now $K$ is finitely generated by $x_m, y_m, \ldots, x_M, y_M$. 

\section{Non-proper HNN extensions}

It was noted in \cite{14} that a free-by-cyclic group $G$ of the form $F_n \rtimes \mathbb{Z}$ for $F_n$ the free group of rank $n \geq 2$ has uniform exponential growth by utilising the fact that the group is either large or word hyperbolic. However a simpler and more direct argument in \cite{15} Lemma 2.3 which we now examine tells us that $\omega(G) \geq 3^{1/6}$. The previous Lemma in that paper notes that if a group $G$ is generated by two finite sets $X$ and $Y$ then $\omega(G, X) \geq (\omega(G, Y))^{1/L}$ where $L$ is an upper bound for the length of each element of $Y$ when written as a word in $X^{\pm 1}$. Consequently on being given any finite set of generators $A = \{a_1, \ldots, a_l\}$ for $G = F_n \rtimes \mathbb{Z}$, we consider the set $C$ of commutators of length 1 words, that is $C = \{[a_j^{\pm 1}, a_k^{\pm 1}] : 1 \leq j, k \leq l\}$, and further the set of conjugates by length 0 or 1 words of elements in $C$, obtaining $B = \{a_i^{\pm 1}c a_i^{\mp 1}\} \cup C$ for $1 \leq i \leq l$ and $c \in C$. Then as we have that any element of $A \cup B$ has length at most 6 in the elements of $A$, we obtain $\omega(G, A)^6 \geq \omega(G, A \cup B) \geq \omega((B), B)$. Now $B$ has been chosen to lie in $F_n$. 

\end{proof}
Thus we are done if $\omega(\langle B \rangle) \geq 3$ which means we just have to show that the free group $\langle B \rangle$ is non-abelian. This is straightforward but requires using standard properties of free groups.

If we abstract this approach with the aim of applying it to other groups of the form $G = K \rtimes \mathbb{Z}$ where $K$ is finitely generated, we see that we require of $K$ that there is a uniform lower bound $u > 1$ for $\omega(H)$ over finitely generated subgroups $H$ of $K$ which are not “small”, which here means non-cyclic (in the above case $u = 3$). Then on being given any finite generating set $A$ for $G$ we can take words which are of bounded length $b$ in the elements of $A$ (such as length 6 for free by cyclic groups but this quantity has to be independent of $A$) and such that these words lie in $K$. If these words happen to generate a subgroup of $K$ that is not “small” then we conclude as before that $\omega(G, A) \geq u^{1/b}$.

**Theorem 3.1** Let $K$ (not equal to $\{e\}$ or $\mathbb{Z}$) be a finitely generated torsion-free group where there exists $u > 1$ such that if $H$ is a finitely generated subgroup of $K$ then either $\omega(H) \geq u$ or $H$ is cyclic. Then for any non-proper HNN extension $G = K \rtimes \mathbb{Z}$ we have that $G$ has uniform exponential growth, and further $\omega(G) \geq \min (u^{1/6}, 2^{1/16})$.

**Proof.** Given any finite generating set $A = \{a_1, \ldots, a_l\}$ for $G$, we consider for each $1 \leq i, j, k \leq l$ the 2-generator subgroup $H_{i,j,k} = \langle a_i, [a_j, a_k] \rangle$. Note that if $\omega(H_{i,j,k}) = c$ then $\omega(G, A) \geq c^{1/4}$. If $a_i \in K$ then $H_{i,j,k} \leq K$ so the inequality would be true with $c = u$ unless $H_{i,j,k}$ is cyclic in which case $a_i$ commutes with $[a_j, a_k]$. But if $a_i \notin K$ then the associated homomorphism from $G$ to $\mathbb{Z}$ with kernel $K$ restricts to a non-trivial (so without loss of generality surjective) homomorphism $\chi : H_{i,j,k} \to \mathbb{Z}$. If the kernel $L = H_{i,j,k} \cap K$ of this restriction is infinitely generated then we immediately get by Proposition 2.4 that $\omega(H_{i,j,k}) \geq 2^{1/4}$ and so $\omega(G) \geq 2^{1/16}$. But on setting $t = a_i$ and $x = [a_j, a_k]$ we have that $L$ is the normal closure of $x$ in $H_{i,j,k}$ and so is generated by $x_n = t^n xt^{-n}$ for $n \in \mathbb{Z}$.

However we can now assume that $L$ is finitely generated. Let us define $L_0$ to be the cyclic group $\langle x_0 \rangle$ and $L_{\pm 1}$ to be the 2-generator group $\langle x_0, x_{\pm 1} \rangle$ for each choice of sign. If one of $L_{\pm 1}$ is not cyclic then we have $\omega(L_{\pm 1}) \geq u$ and so as the length of $x_0$ and $x_{\pm 1}$ is at most 6 in $A$, we obtain $\omega(G, A) \geq u^{1/6}$.

Thus we are done unless we find for all $i, j, k$ that $H_{i,j,k}$ is cyclic whenever $a_i$ is in $K$ and both of $L_{\pm 1}$ are cyclic whenever $a_i$ is not in $K$. Let us assume for now that this always implies that $L_{\pm 1} = L_0$ and explain how we
Having $L_1 = L_0$ means that $t$ conjugates $x$ into a power of itself, and $L_{-1} = L_0$ gives us the same for $t^{-1}$. This means that $t(x)t^{-1} = \langle x \rangle$ and furthermore that $txt^{-1} = x^{\pm 1}$. If we now fix $j$ and $k$ but let $i$ vary, we conclude that $\langle [a_j, a_k] \rangle$ is conjugated to itself by every element in the generating set $A$ and so $\langle [a_j, a_k] \rangle$ is a normal cyclic subgroup of $G$. Let $P$ be the product over $j$ and $k$ of these subgroups. By Fitting’s Theorem, a (finite) product of normal nilpotent subgroups of $G$ is also nilpotent, and as $P$ lies in $K$ we have that $P$ is cyclic. But $G/P$ is abelian as now all generators commute. Moreover if $p$ is a generator for $P$ then we must have $gpg^{-1} = p^{\pm 1}$ for all $g \in G$ because $P$ is normal in $G$. This means that the centraliser $C(p)$ of $p$ in $G$ contains all squares, so contains the subgroup $G^2$ generated by all squares. Now $G/G^2$ is a group of exponent 2, hence abelian, and is also finitely generated so must be finite. But $P$ is contained in $G^2$ and hence is in the centre of $G^2$. Quotienting $G^2$ by $P$, we are left with the abelian group $G^2/P$, telling us that $G^2$ is nilpotent and $G$ is virtually nilpotent. This means that $K$ is too, so $\omega(K) = 1$ which leaves only $K = \{e\}$ or $\mathbb{Z}$.

However we must now address the fact that $L_1$ being cyclic does not mean that $L_0 = L_1$, only that $L_0$ is a cyclic subgroup of $L_1$ and so is of finite index. Consequently there is a power of $x_1$ which lies in $L_0$ and so we must have a relation holding in $L$ of the form $x_0^\alpha x_1^\beta$. We can assume that $\alpha$ and $\beta$ are coprime because $x_0$ and $x_1$ commute and $L$ is torsion free. Also if $\beta = \pm 1$ then we have $L_1 = L_0$ anyway.

We now set $L_2 = \langle x_0, x_1, x_2 \rangle$ and so on. We have that either $L_2$ is non-cyclic, so has uniformly exponential growth, or it remains cyclic and contains $L_1$ with finite index. But as $L$ is finitely generated, this sequence of subgroups must eventually stabilise when $L_n = L_{n+1} = L$. We need to deal with two possibilities. The first is that $L_n$ is still cyclic. The other is that at some point $m$ (possibly for a large $m$ but which could be a lot smaller than $n$) we have that $L_m$ stops being cyclic and so has uniformly exponential growth. We will show that in fact neither of these cases can occur. The first uses the Alexander polynomial of $H_{i,j,k}$ and is based on the fact that the degree of the polynomial is $\beta_1(L; \mathbb{Q})$ but this contradicts $L$ being finitely generated. The second case is again based on $L$ being finitely generated to get that $L_n$ is still abelian.

We briefly review the facts we will need about the Alexander polynomial. The Alexander polynomial $\Delta(t)$ is defined for a homomorphism $\chi$ from a finitely generated group $G$ onto $\mathbb{Z}$. Usually it is supposed that $G$ is finitely presented but we will only ever need it for 2-generator groups $G = \langle t, x \rangle$. The Alexander polynomial $\Delta(t)$ is defined for a homomorphism $\chi$ from a finitely generated group $G$ onto $\mathbb{Z}$. Usually it is supposed that $G$ is finitely presented but we will only ever need it for 2-generator groups $G = \langle t, x \rangle$. The Alexander polynomial $\Delta(t)$ is defined for a homomorphism $\chi$ from a finitely generated group $G$ onto $\mathbb{Z}$. Usually it is supposed that $G$ is finitely presented but we will only ever need it for 2-generator groups $G = \langle t, x \rangle$. The Alexander polynomial $\Delta(t)$ is defined for a homomorphism $\chi$ from a finitely generated group $G$ onto $\mathbb{Z}$. Usually it is supposed that $G$ is finitely presented but we will only ever need it for 2-generator groups $G = \langle t, x \rangle$. The Alexander polynomial $\Delta(t)$ is defined for a homomorphism $\chi$ from a finitely generated group $G$ onto $\mathbb{Z}$. Usually it is supposed that $G$ is finitely presented but we will only ever need it for 2-generator groups $G = \langle t, x \rangle$. The Alexander polynomial $\Delta(t)$ is defined for a homomorphism $\chi$ from a finitely generated group $G$ onto $\mathbb{Z}$. Usually it is supposed that $G$ is finitely presented but we will only ever need it for 2-generator groups $G = \langle t, x \rangle$. The Alexander polynomial $\Delta(t)$ is defined for a homomorphism $\chi$ from a finitely generated group $G$ onto $\mathbb{Z}$. Usually it is supposed that $G$ is finitely presented but we will only ever need it for 2-generator groups $G = \langle t, x \rangle$. The Alexander polynomial $\Delta(t)$ is defined for a homomorphism $\chi$ from a finitely generated group $G$ onto $\mathbb{Z}$. Usually it is supposed that $G$ is finitely presented but we will only ever need it for 2-generator groups $G = \langle t, x \rangle$. The Alexander polynomial $\Delta(t)$ is defined for a homomorphism $\chi$ from a finitely generated group $G$ onto $\mathbb{Z}$. Usually it is supposed that $G$ is finitely presented but we will only ever need it for 2-generator groups $G = \langle t, x \rangle$.
where we can suppose that $\chi(t) = 1$ and $\chi(x) = 0$. In this restricted setting, the results we require will continue to apply even if $G$ is not finitely presented. Let us set $K = \ker(\chi)$ and note that $K$ is generated by $x_i = t^i xt^{-i}$ for $i \in \mathbb{Z}$. On abelianising we have that $K/K'$ is a finitely generated module $M$ over the unique factorisation domain $R = \mathbb{Z}[t^{\pm 1}]$ where the polynomial variable $t$ acts on elements of $M$ via conjugation by the group element $t$. In our 2-generator case we have that $M$ is a cyclic module $R/I$ generated by $x_0$ and here we define the Alexander polynomial $\Delta(t)$ to be the highest common factor of the elements in $I$, or equivalently the generator of the smallest principal ideal containing $I$. It is defined up to multiplication by units which are $\pm t^k$ for $k \in \mathbb{Z}$.

The three facts that we require here are:

1. The degree of $\Delta(t)$ (meaning the degree of the highest power of $t$ minus the lowest) is equal to the first Betti number $\beta_1(K; \mathbb{Q})$ which is defined to be the dimension of the vector space $H_1(K; \mathbb{Q}) = H_1(K; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. This is clear here because $H_1(K; \mathbb{Z}) = K/K'$ so we just work over the ring $\mathbb{Q}[t^{\pm 1}]$ instead.

2. If $K$ is finitely generated as a group then $\Delta(t)$ is monic at both ends: that is the highest and lowest coefficients must both be $\pm 1$. This follows because the sequence of subgroups $S_0 = \langle x_0 \rangle$, $S_1 = \langle x_0, x_1 \rangle$, $S_2 = \langle x_0, x_1, x_2 \rangle, \ldots$ must stabilise at $n$ say, so that $x_n$ is a word in $x_0, \ldots, x_{n-1}$. On abelianising, this gives us that $t^n + a_{n-1}t^{n-1} + \ldots + a_0$ must hold in $M$, thus $\Delta(t)$ divides a monic polynomial so is itself monic. As for the lowest coefficient, we argue the other way with $x_0, x_{-1}, x_{-2}, \ldots$.

3. If we consider the sequence of subgroups $S_i$ above then we must clearly have $\beta_1(S_i) \leq i + 1$ as this is bounded by the number of generators of a group. But on taking the first $i$ where $\beta_1(S_i) \leq i$ (if one exists) we have $\beta_1(S_j)$ is non increasing for $j \geq i$. This is because once we have a relationship of the form $\alpha_0x_0 + \ldots + \alpha_ix_i = 0$ with $\alpha_i \neq 0$ in the abelianisation of $S_i$, this relation will also hold in the abelianisation of $S_{i+1}$ along with the same relation with the subscripts shifted up by 1. Thus in $S_{i+1}/S_{i+1}'$ we see that $x_{i+1}$ will lie in the span of $x_0, \ldots, x_i$ which means we cannot increase $\beta_1$. We can now repeat this argument.

We apply the above to the homomorphism $\chi : H_{i,j,k} \rightarrow \mathbb{Z}$ with kernel $L$. As the relation $x_i^\alpha x_j^\beta$ holds in $L$, we must have that the Alexander polynomial $\Delta(t)$ of $\chi$ divides $\beta t + \alpha$. However $L$ being finitely generated means that $\Delta(t)$ must be monic, so equal to $\pm 1$ because $\beta \neq \pm 1$ and $\alpha$ and $\beta$ are coprime.
But this implies that $\beta_1(L; \mathbb{Q}) = 0$ which contradicts the fact that $L \cong \mathbb{Z}$.

As for when $L$ is non-cyclic, we have $L = \langle x_0, x_1, \ldots, x_{n-1}, x_n \rangle$. By using the conjugation action of $t$ on $L$, we have that $x_i^\alpha x_{i+1}^\beta = e$ for all $i \in \mathbb{Z}$ and that $x_i$ commutes with $x_{i \pm 1}$. We will show that a very high power of $x_0$ is in the centre of $L$. Note that $x_0^\alpha$ is a power of $x_1$ and so commutes with $x_2$. Moreover we have $x_0^{\alpha^2} = (x_1^\alpha)^{-\beta}$ which commutes with $x_3$ because $x_1^\alpha$ does. Continuing in this way, we see that $x_0^{\alpha^{n-1}}$ commutes with $x_n$ and all $x_i$ from 0 to $n$, thus is in the centre $Z(L)$ of $L$. But $Z(L)$ is characteristic in $L$, hence invariant under conjugation by $t$ so $x_0^{\alpha^{n-1}}$ is in $Z(L)$ for all $i$.

To complete this argument we now start with $x_n$ and work backwards, giving us that $x_i^{\alpha^{n-1}}$ is in $Z(L)$ too. But $\alpha^{n-1}$ and $\beta^{n-1}$ are coprime, so $x_i \in Z(L)$ for all $i$ thus $L$ is abelian.

Thus in both cases we have shown that $L_1$ being cyclic implies that $L_0 = L_1$. We now repeat the argument with $L_{-1}$ in place of $L_1$.

Our hypothesis required that every finitely generated subgroup of $K$ was either “small” or had uniform exponential growth bounded below away from 1. However a straightforward adaptation of our argument obtains the same conclusion for $G$, although with a slight widening of the concept of “small”.

**Corollary 3.2** Let $K$ be a finitely generated torsion-free group where there exists $u > 1$ such that if $H$ is a finitely generated subgroup of $K$ then either $\omega(H) \geq u$ or $H$ is cyclic. Let $G = K \rtimes \mathbb{Z}$ be any non-proper HNN extension of $K$. Then for any finitely generated subgroup $S$ of $G$ we have that $S$ has uniform exponential growth with $\omega(S) \geq \min (u^{1/6}, 2^{1/16})$ or $S$ is $\{e\}, \mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group.

**Proof.** We follow the proof of Theorem 3.1 with $G$ replaced by $S$. If $S$ lies in $K$ anyway then we have our bound for $\omega(S)$ unless $S = \{e\}$ or $\mathbb{Z}$. If $S$ has elements outside $K$ then the natural homomorphism from $G$ to $\mathbb{Z}$ with kernel $K$ restricts to a non-trivial homomorphism from $S$. If the kernel $R = S \cap K$ of the restriction is infinitely generated then we have $\omega(S) \geq 2^{1/4}$ by Proposition 2.4. Otherwise we can regard $S$ as a semidirect product $R \rtimes \mathbb{Z}$ and so can replace $G$ with $S$ and $K$ with $R$ in Theorem 3.1. The hypotheses are satisfied unless $R = \{e\}$ or $\mathbb{Z}$, in which case $S$ can only be $\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group.

$\square$
4 Applications and examples

The first examples of groups we can think of where every subgroup is either cyclic or "big" are of course free groups. This is not surprising in light of the fact that Theorem 3.1 takes as its starting point the Ceccherini-Silberstein and Grigorchuk result that $G = F_n \rtimes \mathbb{Z}$ has $\omega(G) \geq 3^{1/6}$ for $F_n$ the finitely generated free group of rank $n \geq 2$. Here we can obtain a slight generalisation.

**Proposition 4.1** If $G = F \rtimes \mathbb{Z}$ where $F$ is a free group of any cardinality then all finitely generated subgroups $H$ of $G$ have $\omega(H) \geq 3^{1/6}$ unless $H$ is $\{e\}$, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group.

**Proof.** Either $H$ is contained in $F$, in which case $H$ is free and we have $\omega(H) \geq 3$ or $H = \{e\}$ or $\mathbb{Z}$, or $H$ is a semidirect product $(H \cap F) \rtimes \mathbb{Z}$. If $H \cap F$ is finitely generated then we have $\omega(H) \geq 3^{1/6}$ or $H = \mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group. If $H \cap F$ is infinitely generated then, although we know that $\omega(H) \geq 2^{1/4}$, we can do better by using an observation of [14] which is based on the results of [18]. The latter proves that $H$ is finitely presented and it is pointed out in Section 5 of the former that $H$ must have a presentation with deficiency at least 2, so we have $\omega(H) \geq 3$ by J. S. Wilson’s result.

Our next examples are the surface groups $\Gamma_g$; these are the fundamental groups of the closed orientable surface of genus $g \geq 2$. It is known that $\omega(\Gamma_g) \geq 4g - 3$ (see [21] VII.B.15 although it is not known if this is best possible). Using Theorem 3.1 means we can treat HNN extensions of free groups and surface groups in a unified fashion.

**Corollary 4.2** If $G = \Gamma_g \rtimes \mathbb{Z}$ then we have $\omega(S) \geq 3^{1/6}$ for any finitely generated subgroup $S$ of $G$, apart from $S$ equal to $\{e\}$, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group.

**Proof.** It is well known that the subgroups of $\Gamma_g$ are isomorphic to $\Gamma_h$ in the finite index case and are free if they are of infinite index. Therefore the conditions of Theorem 3.1 and Corollary 3.2 apply with $u = 3$. By the proof of Theorem 3.1, we either have that one of the subgroups $H_{i,j,k}$ of $S$ lies in $\Gamma_g$ and is non-cyclic, in which case we have $\omega(S) \geq u^{1/4}$, or for some $i, j, k$ we have that the kernel $L$ is finitely generated and one of $L_{i+1}$ is non-cyclic.
so $\omega(S) \geq u^{1/6}$, or $L$ is infinitely generated. If that occurs then $H_{i,j,k}$ is (free of infinite rank)-by-$\mathbb{Z}$ and so $\omega(H_{i,j,k}) \geq 3$ by Proposition 4.1, giving $\omega(S) \geq 3^{1/4}$.

The groups $G$ in Corollary 4.2 are all fundamental groups of closed 3-manifolds, where the 3-manifold is fibred over the circle. In [17] it is shown (using Geometrisation) that we have uniform uniform growth for the class of fundamental groups of closed 3-manifolds, with the exception of the virtually nilpotent groups.

A much wider class of groups (including free and surface groups) where every subgroup is either cyclic or contains a non-abelian free group is the class of torsion free word hyperbolic groups (note that a torsion free virtually cyclic group must be $\mathbb{Z}$ or $\{e\}$, for instance by [22] Lemma 11.4). Therefore Theorem 3.1 and Corollary 3.2 will apply to such a group $G$ whenever we have a lower bound away from 1 of the growth constant of all non-cyclic finitely generated subgroups. We present two cases where this would be so.

**Corollary 4.3** Let $K$ be a torsion free word hyperbolic linear group. Then there exists a constant $k > 1$ such that if $G = K \rtimes \mathbb{Z}$ is any non-proper HNN extension of $K$ then we have $\omega(S) \geq k$ for any finitely generated subgroup $S$ of $G$ that is not isomorphic to $\{e\}$, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group.

**Proof.** By the results [9], [10] of Breuillard and Breuillard-Gelander, we have that for any integer $d > 1$ there is a constant $c(d) > 1$ such that if $K$ is a finitely generated non-virtually solvable subgroup of $GL_d(\mathbb{F})$, where $\mathbb{F}$ is any field, then $\omega(K) \geq c(d)$. But if $K$ is also torsion free and word hyperbolic then any non-cyclic finitely generated subgroup $H$ (though it may not be word hyperbolic) will also have $\omega(H) \geq c(d)$. Therefore Corollary 3.2 applies with $u = c(d)$.

Note that although $G$ above will be torsion free, it will not in general be word hyperbolic because it could easily contain $\mathbb{Z} \times \mathbb{Z}$, as discussed in the next section. The question of whether $G$ is linear seems interesting; it is open even in the case where $K$ is the free group $F_r$ for $r \geq 3$.

Another result that could be of use here is that of Arzhantseva and Lyksenok in [1]. This states that if $K$ is a word hyperbolic group then there is an $\alpha > 0$, effectively computable from $K$, such that for any finitely generated subgroup $H$ of $K$ which is not virtually cyclic and any finite generating set
$C$ for $H$, we have $\omega(H, C) \geq \alpha |C|$. This implies that if $K$ is torsion free word hyperbolic and the value of $\alpha$ above is greater than $1/2$ then Corollary 3.2 applies to $K$ because when $H$ is non-cyclic we have $|C| \geq 2$ and so $\omega(H, C) \geq u = 2\alpha > 1$.

We should say that if a torsion free 1-ended word hyperbolic group $H$ has trivial JSJ decomposition then $\text{Out}(H)$ is finite and therefore any group of the form $H \rtimes_\alpha \mathbb{Z}$ has uniform exponential growth. This is because if $\alpha$ has order $a$ in $\text{Out}(H)$ then the degree $a$ cyclic cover of $H \rtimes_\alpha \mathbb{Z}$ is isomorphic to $H \times \mathbb{Z}$ and so $\omega(H \rtimes \mathbb{Z})^{2a-1} \geq \omega(H \times \mathbb{Z}) \geq \omega(H) > 1$. However $a$ could be arbitrarily high and moreover if $\text{Out}(H)$ is infinite then this argument will only apply to the automorphisms of finite order.

Finally we wish to display some examples which are not word hyperbolic; indeed which are as far away from being word hyperbolic as possible. For non-zero integers $m, n$ let $BS(m, n)$ be the Baumslag-Solitar group with presentation $\langle a, t | ta^m t^{-1} = a^n \rangle$. These are “poison” subgroups in the sense that any group containing one of these cannot be word hyperbolic. However $BS(1, n)$ does not contain $\mathbb{Z} \times \mathbb{Z}$ if $n \neq \pm 1$. Thus we obtain:

**Corollary 4.4** Let $G$ be any non-proper HNN extension of $B = BS(1, n)$ for $n \neq \pm 1$ then we have $\omega(S) \geq 2^{1/4}$ for any finitely generated subgroup $S$ of $G$ that is not isomorphic to $\{e\}, \mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group.

**Proof.** The group $B$ is a strictly ascending HNN extension with base $\mathbb{Z}$ so the associated homomorphism has a kernel which is not finitely generated but which is an ascending union of copies of $\mathbb{Z}$, so is locally $\mathbb{Z}$. If $H$ is a finitely generated subgroup of $B$ which is not in this kernel, then if it intersects the kernel in an infinitely generated subgroup we have $\omega(H) \geq 2^{1/4}$. If however the intersection is finitely generated then it must be $\{e\}$ or $\mathbb{Z}$, so $H$ is $\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group but we have ruled out the last two. Therefore Corollary 3.2 applies with $u = 2^{1/4}$.

The above discussion also applies to generalised Baumslag-Solitar groups. These are finitely generated groups which act on a tree with all edge and vertex stabilisers infinite cyclic. If such a group does not contain $\mathbb{Z} \times \mathbb{Z}$ then Corollary 4.4 will apply. To see this we note that any generalised Baumslag-Solitar group has a surjection to $\mathbb{Z}$ and any finitely generated subgroup is either free or also a generalised Baumslag-Solitar group. If the kernel of this surjection is infinitely generated then we have the $2^{1/4}$ bound but if the kernel
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is finitely generated then we use the fact that generalised Baumslag-Solitar groups have cohomological dimension 2 and are coherent. By the result of Bieri mentioned before in Proposition 2.3, we have that this kernel must be free and hence we get in this case a lower bound of $3^{1/6}$ once we have ruled out $\mathbb{Z} \times \mathbb{Z}$ and the Klein bottle.

The paper [27] describes the automorphism group of a generalised Baumslag Solitar group, which in some cases can be a big group. It also quotes the results used above and provides references.

5 Periodic conjugacy classes

We know that a group containing $\mathbb{Z} \times \mathbb{Z}$ cannot be word hyperbolic and so in general $G = K \rtimes_{\alpha} \mathbb{Z}$ will not be word hyperbolic even if $K$ is. We can see this because if the automorphism $\alpha$ has a periodic conjugacy class, that is there exists a non-identity element $k$ of $K$ and a non-zero integer $n$ such that $\alpha^n(k)$ is conjugate by $c \in K$ to $k$, then $t^nkt^{-n} = ckc^{-1}$ where conjugation by $t$ acts as $\alpha$ on $K$. Thus $c^{-1}t^n$ commutes with $k$ and if $K$ is torsion free then this must generate $\mathbb{Z} \times \mathbb{Z}$ because the image in $\mathbb{Z}$ of any power of $c^{-1}t^n$ is never trivial so no power can lie in $K$. In fact the converse is true too.

Proposition 5.1 Let $K$ be a group that does not contain $\mathbb{Z} \times \mathbb{Z}$ and let $\alpha$ be an automorphism of $K$. Then if $G = K \rtimes_{\alpha} \mathbb{Z}$ contains $\mathbb{Z} \times \mathbb{Z}$, we have that $\alpha$ has a periodic conjugacy class.

Proof. We take two generators of $\mathbb{Z} \times \mathbb{Z}$ in $G$ which we can write as $x = kt^i$ and $y = lt^j$ for $k, l \in K$. Without loss of generality $i \neq 0$ because if both are then $x, y \in K$. We then set $z = x^iy^{-i}$ which (on taking the exponent sum of $t$ in $z$) will lie in $K$ and will not equal the identity. But then $xzx^{-1} = z$ implies that $\alpha^i(z) = k^{-1}zk$.

However it is far from immediate that an absence of $\mathbb{Z} \times \mathbb{Z}$ in $G$ along with $K$ being word hyperbolic is enough to imply that $G$ is too. Even when $K$ is the free group $F_n$ one requires the combined results of [3], [4] and [11] to prove this. Even so, as word hyperbolicity is not explicitly mentioned in Theorem 3.1 or Corollary 3.2, we can use the absence of periodic conjugacy classes to obtain iterated versions of these results.
Corollary 5.2 Let $K$ be a finitely generated torsion-free group where there exists $u > 1$ such that if $H$ is a finitely generated subgroup of $K$ then either $\omega(H) \geq u$ or $H$ is cyclic. Let $\alpha$ be an automorphism of $K$ without periodic conjugacy classes and let $G = K \rtimes_\alpha \mathbb{Z}$. Now let $\beta$ be any automorphism of $G$ and let us form the repeated non-proper HNN extension $D = (K \rtimes_\alpha \mathbb{Z}) \rtimes_\beta \mathbb{Z}$. Then there exists $w > 1$ such that for any finitely generated subgroup $S$ of $D$ we have that $S$ has uniform exponential growth with $\omega(S) \geq w$ or $S$ is $\{e\}, \mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group.

Proof. On applying Corollary 3.2 to $K$, we obtain that $G$ is finitely generated and torsion free, with $v > 1$ such that any finitely generated subgroup $H$ of $G$ having $\omega(H) < v$ must be cyclic or isomorphic to $\mathbb{Z} \times \mathbb{Z}$ or the Klein bottle. But the last two cases are ruled out by Proposition 5.1 if $\alpha$ has no periodic conjugacy classes. Now we can apply Corollary 3.2 again but this time to $G$.

In particular, by the above and Proposition 4.1, if $G = F_n \rtimes \mathbb{Z}$ is a word hyperbolic free-by-cyclic group then any finitely generated subgroup $D$ of a group of the form $G \rtimes \mathbb{Z}$ has $\omega(D) \geq 3^{1/36}$. The same is true by Corollary 4.2 if $G$ is the fundamental group of a closed hyperbolic 3-manifold which is fibred over the circle. Also if we have any number of repeated non-proper HNN extensions of a group $K$ satisfying the conditions of Theorem 3.1 where all automorphisms (except possibly the last) have no periodic conjugacy classes then the resulting group has uniform exponential growth with constant depending only on $K$ and the number of HNN extensions.

We might hope to develop versions of Theorem 3.1 and Corollary 3.2 where the allowable “small groups” are more than just cyclic, for instance we could permit free abelian groups of bounded rank. The main problem here appears to be finding an “exit strategy” in the sense that we may always find that $L$ in the proof turns out to be abelian but it is hard to see what this implies for the whole group $G$. Nevertheless the absence of periodic conjugacy classes allows us to get round this.

Theorem 5.3 Let $K$ (not equal to $\{e\}$) be a finitely generated torsion-free group where there exists $u > 1$ and $d \in \mathbb{N}$ such that if $H$ is a finitely generated subgroup of $K$ then either $\omega(H) \geq u$ or $H \cong \mathbb{Z}^n$ for $n \leq d$. Then there exists $k > 1$ depending only on $u$ and $d$ such that for any non-proper HNN extension $G = K \rtimes_\alpha \mathbb{Z}$ where $\alpha$ has no periodic conjugacy classes, we have $\omega(G) \geq k$. 
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**Proof.** We follow the proof of Theorem 3.1 and indicate where we need to strengthen it. Suppose we have \( H_{i,j,k} \) not contained in \( K \) with finitely generated kernel \( L \neq \{e\} \). (Note that there will be some generator \( a_i \) of \( G \) which is not in \( K \), and some pair of generators \( a_j, a_k \) which do not commute or else \( G \) is abelian and so has periodic conjugacy classes. Thus this case will occur for any generating set.) One considers as before the ascending sequence of subgroups \( L_0 \cong \mathbb{Z}, L_1, L_2, \ldots \). Suppose that \( L_{s-1} \) is (free) abelian then we must have on moving from \( L_{s-1} \) to \( L_s \) that either \( L_s \) is free abelian of one rank higher, or \( L_s \) and all further \( L_{s+1}, L_{s+2}, \ldots \) are no longer abelian, or we “stick”, which is where \( L_s \) remains abelian but has the same rank as \( L_{s-1} \).

Suppose we never stick but drop out of the abelian category then we must have \( L_s \) non-abelian for \( s \leq d \), in which case we have \( \omega(L_s) \geq u \) for \( L_s = \langle x, txt^{-1}, \ldots, t^sxt^{-s} \rangle \) with each element of this generating set having length at most \( 2d + 4 \) in \( A \), so \( \omega(G, A) \geq u^{1/(2d+4)} \).

Now let us see what happens if we do stick, supposing that \( s \) is the first time this has occurred so that \( L_{s-1} \cong L_s \cong \mathbb{Z}^s \) with \( s \leq d \). If \( L_{s-1} = L_s \) then we have \( L_{s+1} = L_{s+2} = \ldots \). We now repeat our argument in the other direction, starting with \( M_0 = \langle x_{s-1}, x_{s-2}, \ldots, x_0 \rangle \) and \( M_1 = \langle M_0, x_{-1} \rangle \), \( M_2 = \langle M_1, x_{-2} \rangle \) and so on. Once again we either increase the rank by 1 but can only do this a maximum of \( d-s \) times, or we are no longer abelian but then the above bound again applies for \( \omega(G, A) \), or we again stick. If further we have \( M_{t-1} = M_t \) in our new sequence of subgroups then we have found that \( L \) is isomorphic to \( \mathbb{Z}^n \) for \( n \leq d \) and consequently the subgroup \( H_{i,j,k} = \langle t, x \rangle \) is \( \mathbb{Z}^n \rtimes_\beta \mathbb{Z} \) where conjugation by \( t \) induces the automorphism \( \beta \) of \( \mathbb{Z}^n \).

Now a lot is known about the word growth of groups of this form. Let \( M \) be the corresponding element of \( GL(n, \mathbb{Z}) \) obtained from \( \beta \). Set \( m \) to be the modulus of the largest eigenvalue of \( M \); by a famous result of Kronecker we have that \( m > 1 \) unless all solutions of the characteristic equation are roots of unity. In the former case our group \( H_{i,j,k} \) has uniform exponential growth with growth constant that depends only on \( m \) and this again translates into a lower bound for \( \omega(G, A) \). Moreover as \( m \) varies, this growth constant is bounded away from 1 if \( m \) is; see for instance the example in [13]. Whether or not our growth constants are bounded away from 1 over all \( m \) would be implied by a positive answer to a longstanding question of Lehmer on Mahler measure. However we are in a position where \( M \) is an \( n \) by \( n \) matrix for \( n \leq d \). When the degree of the polynomial is bounded we can use [8] which states that if \( p \) is a monic polynomial in \( \mathbb{Z}[t] \) of degree at most \( d \)
with \( m \leq 1 + 1/(30d^2 \log 6d) \) then the solutions of \( p \) are all roots of unity, where \( m \) is the modulus of the largest root. Consequently we have uniform exponential growth for \( H_{i,j,k} \) with growth constant that depends only on \( d \).

In the case where all eigenvalues are roots of unity our group is virtually nilpotent. Let one of these eigenvalues have order \( r \) then the subgroup \( \langle t^r, x \rangle \) of \( H_{i,j,k} \) corresponds to the matrix \( M^r \) which will have 1 as an eigenvalue. Consequently there will exist a non-zero \( v \in \mathbb{Q}^n \), and also in \( \mathbb{Z}^n \) by clearing out denominators, with \( M^r v = v \). We can think of this \( v \in \mathbb{Z}^n \) as an element of \( H_{i,j,k} \cap K \), for \( K \) the kernel of the natural homomorphism from \( G \) to \( \mathbb{Z} \).

If \( s \in G \) induces the automorphism \( \alpha \) of \( K \) by conjugation then we have \( t = ks^i \) for \( k \in K \) and \( i \neq 0 \), obtaining \( t^r = ls^{ir} \) for \( l \) also in \( K \). Thus we have \( v = t^{ir} v^{-r} = ls^{ir} vs^{-ir}l^{-1} \) so \( \alpha^{ir}(v) = l^{-1} vl \), giving us a periodic conjugacy class for \( \alpha \).

We now have to consider what happens when we stick for the first time at \( s \), so that \( L_{s-1} \cong L_s \cong \mathbb{Z}^s \), but where \( L_{s-1} \) is not equal to \( L_s \). We will have \( L_{s-1} \) of finite index in \( L_s \) and a relation holding of the form \( x_0^{\alpha_0} \ldots x_s^{\alpha_s} = e \) where (as \( L_s \) is free abelian) there is no common factor of the integers \( \alpha_0, \ldots, \alpha_s \) and (as \( L_{s-1} \neq L_s \)) \( \alpha_i \neq \pm 1 \). As this relation continues to hold by conjugation amongst \( x_1, \ldots, x_{s+1} \) and so on, we must have that either \( L_{s+1} \) is no longer abelian or it is abelian with \( L_s \leq f L_{s+1} \).

In the case where we never reach a non-abelian subgroup, the previous proof generalises in a straightforward manner. We will have an ascending chain of finite index subgroups until we terminate at \( L_t = L \cong \mathbb{Z}^s \). Now we invoke the Alexander polynomial argument again which is that it must be a monic polynomial (as \( L \) is finitely generated) of degree \( \beta_1(L; \mathbb{Q}) = s \). But it must also divide \( \alpha_0 + \alpha_1 t + \ldots + \alpha_s t^s \) which is a contradiction.

However the case where along the way we stop being abelian is more awkward. We suppose that \( L = \langle x_0, x_1, \ldots, x_M \rangle \) for some large \( M \) and is non-abelian, with

\[
x_0^{\alpha_0} \ldots x_s^{\alpha_s} = e
\]

holding as before. Our previous argument immediately generalises to showing that \( x_i^b \) is in the centre \( Z(L) \) for \( b \) a high power of \( \alpha_0 \), and similarly for a high power of \( \alpha_s \). If \( \alpha_0 \) and \( \alpha_s \) are coprime then we are immediately done but this need not be so, thus we have to proceed more carefully.

Let \( c \) be the highest common factor of \( \alpha_0 \) and \( \alpha_s \). Using the old argument, we know that there is some integer \( a \) such that for all \( 0 \leq i \leq M \), we have \( x_i \) raised to the power of \( c^a \) is in the centre \( Z(L) \). Let us now work modulo
the centre and take a prime $p$ dividing $c$, so that there exists $r$ coprime to $p$ and a potentially high integer $k$ with $x_i^{rp^k} = e$. We then raise the equation (1) to the power of $rp^{k-1}$ so that if $j_1, \ldots, j_l$ are the values of $i$ where $p$ does not divide $\alpha_i$, we have

$$x_{j_1}^{\alpha_{j_1} rp^{k-1}} \cdots x_{j_l}^{\alpha_{j_l} rp^{k-1}} = e.$$ 

Consequently $x_{j_i}^{\alpha_{j_i} rp^{k-1}}$ is in the subgroup generated by $x_i^{rp^k-1}$ for values of $i$ lower than $j_i$. But $\alpha_{j_i}$ is coprime to $p$ so in fact $x_{j_i}^{rp^{k-1}}$ is in this subgroup too. But now on conjugating this relation so that $j_i = M$ and working down, we obtain that $x_M^{rp^{k-1}}$ is in the centre of $L/Z(L)$. We can now quotient out by the new centre, obtaining $x_i^{rp^{k-1}} = e$ and then repeat the argument with $k$ lowered by 1. Continuing in this way we reduce $k$ until after successive quotients we have $x_i = e$. We can now replace $c$ with $r$, pull out another prime from $r$ and repeat the argument, always quotienting out by the centre. Eventually we have $x_i = e$ so that $L$ is actually nilpotent. As it is a subgroup of $K$, we conclude that $L$ was abelian all along.

\[ \square \]

Again we have our usual corollary.

**Corollary 5.4** Let $K$ be a finitely generated torsion-free group where there exists $u > 1$ and $d \in \mathbb{N}$ such that if $H$ is a finitely generated subgroup of $K$ then either $\omega(H) \geq u$ or $H \cong \mathbb{Z}^n$ for $n \leq d$. Then there exists $k > 1$ depending only on $u$ and $d$ such that for any non-proper HNN extension $G = K \rtimes_\alpha \mathbb{Z}$ where $\alpha$ has no periodic conjugacy classes, and any finitely generated subgroup $S$ of $G$, we have $\omega(S) \geq k$ or $S \cong \mathbb{Z}^n$.

**Proof.** We are done if $S \leq K$ or if $S \cap K$ is infinitely generated. If $S \cap K = R$ is finitely generated then we have $S = R \rtimes_\beta \mathbb{Z}$ where the action of $\beta$ is conjugation by $ks^l$ for $k \in K$ and where conjugation by $s$ on $K$ induces $\alpha$. We can then apply Theorem 5.3 to $S$ unless $\beta$ has periodic conjugacy classes, say $\beta^j(r) = lrl^{-1}$ for $r, l \in R$. But as $(ks^l)^j = k_0 s^{ij}$ for $k_0 \in K$, we have $s^{ij} r s^{-ij} = k_0^{-1} l r l^{-1} k_0$ so $\alpha$ has periodic conjugacy classes too.

\[ \square \]
6 The Klein bottle group

It would be good to extend Theorem 5.3 to cases where other small groups are allowed. In this section we will show how to alter the argument to incorporate the Klein bottle group given by the presentation \(<\alpha, \beta | \beta\alpha\beta^{-1} = \alpha^{-1}>\) because we have often seen results where the groups left over are cyclic, \(\mathbb{Z} \times \mathbb{Z}\) or the Klein bottle. However the ad hoc nature of this case means that we need to impose an extra condition on our group \(K\), which is that it is locally indicable.

**Theorem 6.1** Let \(K\) (not equal to \(\{e\}\)) be a finitely generated locally indicable group where there exists \(u > 1\) and \(d \in \mathbb{N}\) such that if \(H\) is a finitely generated subgroup of \(K\) then either \(\omega(H) \geq u\) or \(H \cong \mathbb{Z}^n\) for \(n \leq d\) or \(H\) is isomorphic to the Klein bottle group. Then there exists \(k > 1\) depending only on \(u\) and \(d\) such that for any non-proper HNN extension \(G = K \rtimes_{\alpha} \mathbb{Z}\) where \(\alpha\) has no periodic conjugacy classes, we have \(\omega(G) \geq k\).

**Proof.** We use the proof of Theorem 5.3 and indicate where we branch off to deal with the Klein bottle group. Let us take a subgroup \(H_{i,j,k} = \langle t, x \rangle\) with \(t \notin K\) but \(x \in K - \{e\}\) as before and again let \(L\) (which we can assume is finitely generated) be \(H_{i,j,k} \cap K\). We still consider our ascending sequence of subgroups \(L_0 = \langle x_0 \rangle \cong \mathbb{Z}, L_1 = \langle x_0, x_1 \rangle, \ldots\) and everything will work as before unless we come across a group \(L_i\) on the way which is isomorphic to the Klein bottle group. As this is our only small subgroup which is non-abelian, we will have \(L\) is itself isomorphic to the Klein bottle group or we will drop out of the small subgroups during the sequence.

As the first Betti number of the Klein bottle group \(L_i\) is 1, we have \(\beta_1(L) \leq 1\) by point (3) on the Alexander polynomial. Now we use the local indicability of \(K\) so we have \(\beta_1(L) = 1\) (as \(L\) is not trivial). This means that the Alexander polynomial of the homomorphism from \(H_{i,j,k}\) onto \(\mathbb{Z}\) with kernel \(L\) can only be \(t \pm 1\). Let us assume it is \(t - 1\) as the other case just needs changes of signs. Therefore by point (1) there is (up to sign) only one homomorphism \(\theta\) from \(L\) onto \(\mathbb{Z}\) and the Alexander polynomial above tells us that \(x_i = x_{i+1}\) holds in \(L/L'\), so \(\theta(x_i) = 1\) for all \(i\). Now for any \(i\) we have that \(\theta\) restricts to a surjective homomorphism on \(L_i\), so whenever \(L_i\) is isomorphic to the Klein bottle group we must have that \(\theta\) agrees with the unique homomorphism (up to sign) from the Klein bottle group onto \(\mathbb{Z}\). Any element in the Klein bottle group can be expressed as \(\alpha^i\beta^j\) for \(i, j \in \mathbb{Z}\) which will have image \(j\) under this homomorphism (changing \(\beta\) to \(\beta^{-1}\) if necessary).
We now show that if a Klein bottle group appears at all amongst the $L_i$, it must first happen at $L_1$. Otherwise $L_1$ would be abelian and if it is cyclic with $x^a_0x^b_1 = e$ for coprime $a$ and $b$ then, although this might not imply that all subsequent $L_i$ are cyclic or abelian, it does mean that the abelianisation of all subsequent $L_i$ will be $\mathbb{Z}$ but the Klein bottle group has abelianisation $C_2 \times \mathbb{Z}$. Thus $L_1$ is $\mathbb{Z} \times \mathbb{Z}$. Suppose that everything from $L_1$ to $L_{k-1}$ is $\mathbb{Z} \times \mathbb{Z}$ (so $tL_{k-1}t^{-1}$ is too) but $L_k$ is the Klein bottle group. Then $x_1, \ldots, x_{k-1}$ will all be in the centre of $L_k$ because these elements commute with $x_0$ and $x_k$, as well as with each other. But the centre of the Klein bottle group is generated by $\beta^2$, thus $\theta(x_1)$ would be even, not 1, which is a contradiction if $k \geq 2$. Consequently our generators $x_0, x_1$ of the Klein bottle group $L_1$ must be $\alpha^i\beta$ and $\alpha^j\beta$ when written in this form. But this implies that the relation $x^2_0 = x^2_1 = (\beta^2)$ holds in $L_1$ and hence in $L$.

Let us now look at the case where $L_i$ is always a Klein bottle group for $i \geq 1$ but where $L_1 \neq L_2$. Note that if a Klein bottle group is contained in another then this has to be of finite index. We can take our homomorphism $\theta$ from $L$ to $\mathbb{Z}$ and then to the cyclic group $C_2$, with the composition called $\lambda$. Note that the kernel $M$ of this homomorphism will be isomorphic to $\mathbb{Z} \times \mathbb{Z}$ as it contains the elements in $L$ of the form $\alpha^i\beta^j$ for even $j$. Also we have $t^{\pm 1}Mt^{\pm 1} = M$ because $M$ is the set of elements whose total exponent sum is even in the $x_i$s. Consequently we can obtain an increasing sequence of subgroups $M_i = M \cap L_i$ of $L$, each of which is of index 2 in the corresponding subgroup $L_i$ so that $L_1 < L_2$ implies $M_1 < M_2$, and also that $M_1$ and $M_2$ are isomorphic to $\mathbb{Z} \times \mathbb{Z}$. We have that $M_1 = \langle z, y_1 \rangle$ and $M_2 = \langle z, y_1, y_2 \rangle$, where we have set $z = x^2_0 = x^2_1 = x^2_2 = \ldots$ and $y_i = x_{i-1}x_i$. Both $y_1$ and $y_2$ go to 2 under $\theta$, so on writing the elements of $L_2$ in $\alpha, \beta$ form we have that $y_1 = \alpha^i\beta^2$ and $y_2 = \alpha^m\beta^2$, with $l, m \neq 0$ as $z = \beta^2$ but $\mathbb{Z} \not\cong M_1 < M_2$ implies that $z, y_1, y_2$ are three distinct elements. As $\alpha$ and $\beta^2$ sit inside the torsion free abelian group $M_2$, we cannot have a relation of the form $y_1^l = y_2^m$ because applying $\theta$ tells us that $a = b$ which would imply that $y_1 = y_2$. Thus $\langle y_1, y_2 \rangle$ is $\mathbb{Z} \times \mathbb{Z}$ and we will have a relation holding of the form $y_2^a = y_1^{b+c}$ with $a \neq \pm 1$. Moreover we can assume that $a$ and $b$ are coprime, because applying $\theta$ tells us that $a = b + c$, so we could pull out a common factor from all three indices.

We now argue in a similar fashion as before by using the Alexander polynomial but this time applied to the group $\langle t, y_i : i \in \mathbb{Z} \rangle$ with homomorphism $t \mapsto 1, y_i \mapsto 0$, which works because $ty_it^{-1} = y_{i+1}$. Certainly the kernel $\langle y_i : i \in \mathbb{Z} \rangle$ is finitely generated and has first Betti number equal to 2 as
it is not cyclic but sits inside $M \cong \mathbb{Z} \times \mathbb{Z}$. But on combining the relations $y_2^2 = y_1^b z^c$ and $y_3^2 = y_2^d z^c$ which hold in $M$, we obtain $y_3^2 = y_2^{a+b} y_1^{-b}$. Thus we require our monic quadratic Alexander polynomial to divide $at^2 - (a+b)t + b$ which is a contradiction.

As for the case where $L_t$ stops being a Klein bottle group along the way, we again try for an argument involving the centre of $L$. We have that $L_0 = \mathbb{Z}$ with $L_1$ a Klein bottle group properly contained with finite index in $L_2$ which we can assume is also a Klein bottle group or else we have reached a growth constant bigger than 1. We still have the homomorphism $\lambda$ from $L$ to $C_2$ but with a different kernel $M$. However if we stabilise at $L_t = L_{t+1}$ then such a kernel will always be generated by $x_0^2, x_1^2, \ldots, x_t^2$ (which here are all the same element $z$) and $x_0x_1, x_1x_2, \ldots, x_{t-1}x_t$, which here are called $y_1, y_2, \ldots, y_t$. Now if we restrict $\lambda$ to $\langle x_i, x_{i+1} \rangle$ which being a conjugate of $L_1$ is also a Klein bottle group, we have as before that the intersection of this subgroup with $M$ is $\mathbb{Z} \times \mathbb{Z}$ and generated by $x_i^2 = x_{i+1}^2 = z$ and $x_i x_{i+1} = y_{i+1}$, so for any $i$ the two elements $z$ and $y_i$ commute. This means that $z$ is in the centre of $M$. As for $y_{i+1}$, we have that $L_1 \cap M = \langle z, y_1 \rangle$ has finite index in $L_2 \cap M = \langle z, y_1, y_2 \rangle$ which is also isomorphic to $\mathbb{Z} \times \mathbb{Z}$, because there is only one homomorphism from a Klein bottle group to $C_2$ which factors through $\mathbb{Z}$. Thus we must have a relation of the form $y_2^a = y_1^b z^c$ as before, where we can again assume that $a$ and $b$ are coprime. But now we argue as previously that $y_1^{b^2} = (y_2^b)^a z^{-cb} = (y_3^b z^{-c})^a z^{-cb}$ thus $y_1$ to the power $b^2$ commutes with $y_3$ and so on. We then get that $y_i$ to a high power of both $b$ and (by working back down again) $a$ is in the centre, thus $y_i$ is too and so $M$ is abelian after all. This tells us that $L$ has an abelian subgroup of index 2, so it can only be abelian or the Klein bottle group.

Thus we are done unless $L_1 = L_2$ is the Klein bottle group. We now argue the other way with $L'_0 = \langle x_1 \rangle$, $L'_1 = \langle L'_0, x_0 \rangle = L_1$, $L'_2 = \langle L'_1, x_1 \rangle$ and we are done unless $L'_2 = L'_1$, in which case $L = L_1$ with $H_{i,j,k}$ equal to $L \rtimes_\beta \mathbb{Z}$ for some automorphism $\beta$. But a Klein bottle group has a characteristic subgroup $C \cong \mathbb{Z} \times \mathbb{Z}$ with quotient $C_2 \times C_2$ so $H_{i,j,k}$ has a subgroup $(\mathbb{Z} \times \mathbb{Z}) \rtimes_\beta \mathbb{Z}$ of index 4. This gives us a lower bound for the growth constant of $H_{i,j,k}$ unless the subgroup is virtually nilpotent, in which case as before $\beta$ and hence $\alpha$ has a periodic conjugacy class.

$\Box$

As before we have a Corollary to this result which is proved similarly.
Corollary 6.2 Let $K$ be a finitely generated locally indicable group where there exists $u > 1$ and $d \in \mathbb{N}$ such that if $H$ is a finitely generated subgroup of $K$ then either $\omega(H) \geq u$ or $H \cong \mathbb{Z}^n$ for $n \leq d$ or $H$ is isomorphic to the Klein bottle group. Then there exists $k > 1$ depending only on $u$ and $d$ such that for any non-proper HNN extension $G = K \rtimes \alpha \mathbb{Z}$ where $\alpha$ has no periodic conjugacy classes and for any finitely generated subgroup $S$ of $G$, we have that either $\omega(S) \geq k$ or $S$ is $\mathbb{Z}^n$ for $n \leq d$ or is the Klein bottle group.

Proof. Once again, Theorem 6.1 applies to $S$ unless $R = S \cap K$ is $\mathbb{Z}^n$ or the Klein bottle group. But then $S$ is a non-proper extension of one of these groups, which will either mean that $S$ has uniform exponential growth depending only on $d$, or we have a periodic conjugacy class as at the end of the proof of Corollary 5.4.

As for applications, we have some when $d \leq 2$.

Corollary 6.3 Let $n \in \mathbb{N}$. Then there exists $k > 1$ depending only on $n$ with the following property. Let $K$ be any finitely generated coherent locally indicable group of cohomological dimension 2 and suppose we form an iterated sequence of $n$ non-proper HNN extensions to obtain the group $G$ where every automorphism has no periodic conjugacy classes. Then for any finitely generated subgroup $S$ of $G$, we have that either $\omega(G) \geq k$ or $S$ is $\mathbb{Z}^n$ for $n \leq 2$ or is the Klein bottle group.

Proof. First let us consider $G_1 = K \rtimes \alpha_1 \mathbb{Z}$ where $\alpha_1$ has no periodic conjugacy classes. The cohomological dimension means $K$ is torsion free. On taking a finitely generated (hence presented) subgroup $H$ of $K$ and using local indicability, we have a kernel of a homomorphism from $H$ to $\mathbb{Z}$ which if infinitely generated gives us $\omega(H) \geq 2^{1/4}$, and which if finitely generated must be free, as mentioned in Proposition 2.3, giving us a similar bound unless $H$ is $\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$ or the Klein bottle. Thus we can apply Corollary 6.2 with $u = 2^{1/4}$ and $d = 2$ to obtain our conclusion for $G_1$. Now we form $G_2 = G_1 \rtimes \alpha_2 \mathbb{Z}$. Although we may have lost coherence and have probably increased the cohomological dimension by 1, $G_1$ is still locally indicable because a finitely generated subgroup $S$ of $G_1$ will either lie in $K$ or will have a non-trivial image in $\mathbb{Z}$ on restriction of the associated homomorphism of the HNN extension. Thus we are now able to apply the Corollary repeatedly.
The conditions on $K$ in Corollary 6.3 may look restrictive but we give some cases where they are satisfied.

(1) If $K$ is a free-by-cyclic group of the form $F_n \times_{\alpha} \mathbb{Z}$, with coherence established in [18]. Thus we get uniform uniform growth for iterated sequences of non-proper HNN extensions of finitely generated free groups, as long as all automorphisms except for the first have no periodic conjugacy classes. We also have the same result if all automorphisms except for the last have no periodic conjugacy classes, by Proposition 5.1 and Corollary 5.2.

(2) If $K$ is a torsion free 1-relator group which is coherent, as mentioned in Proposition 2.3. Thus we can take such a group and apply repeated non-proper HNN extensions, retaining uniform uniform exponential growth if the automorphisms are non-periodic.

(3) Generalised Baumslag-Solitar groups, as mentioned at the end of Section 4. There we got uniform uniform growth for non-proper HNN extensions of generalised Baumslag-Solitar groups which do not contain $\mathbb{Z} \times \mathbb{Z}$ formed by any automorphism. Now we get uniform uniform growth for non-proper HNN extensions of any generalised Baumslag-Solitar group, but using non-periodic automorphisms and repeated extensions thereof.

(4) We can also apply this result to finitely generated linear locally indicable groups $K$ of cohomological dimension 2, in which case we can lose the coherence condition. This is because by [2] we have $u > 1$ depending only on the degree $r$ of $K$ such that any finitely generated subgroup $H$ of $K$ has $\omega(H) \geq u$ or is virtually soluble. But the finitely generated soluble groups of cohomological dimension at most 2 are known by [19] to be $\{e\}, \mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$, the Klein bottle or the Baumslag Solitar groups $BS(1,m)$ for $m \neq 0, \pm 1$ with the latter family having growth constant at least $2^{1/4}$. Moreover this result also applies to virtually soluble groups by Corollaries 3(ii) and 2 of [26] so we can apply Corollary 6.2 $n$ times with $d = 2$ to obtain $k$, which depends only on the degree $r$ and $n$, such that any iterated sequence of non-proper HNN extensions of $K$ using automorphisms with no periodic conjugacy classes results in a group $G$ where all finitely generated subgroups either have growth constant at least $k$ or must be $e, \mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group.

Finally we finish with a comment on amalgamated free products. In [13] (non-trivial) amalgamated free products are also considered, where it is again shown that such a group has growth constant at least $2^{1/4}$ unless the amalgamated subgroup has index 2 in both factors. It is remarked in the latter case that the resulting group may or may not have uniform exponential growth but it may be fruitful to regard it as an HNN extension. It is well
known that a group $F$ decomposes as an amalgamated free product $G*A=B\, H$ with $A$ of index 2 in $G$ and the isomorphic subgroup $B$ of index 2 in $H$ if and only if there is a surjection from $F$ to $C_2*C_2$. Consequently $F$ has an index 2 subgroup $S$ surjecting onto $\mathbb{Z}$, giving $\omega(S) \geq 2^{1/4}$ and thus $\omega(F) \geq 2^{1/12}$ if $S$ is a proper HNN extension. If $S$ is a non-proper HNN extension it may not have uniform exponential growth anyway, but we can at least see if any of the results here apply to $S$.

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