Uniform Stability and Error Analysis
for Some Discontinuous Galerkin Methods

Qingguo Hong and Jinchao Xu

Abstract

In this paper, we provide a number of new estimates on the stability and convergence of both hybrid discontinuous Galerkin (HDG) and weak Galerkin (WG) methods. By using the standard Brezzi theory on mixed methods, we carefully define appropriate norms for the various discretization variables and then establish that the stability and error estimates hold uniformly with respect to stabilization and discretization parameters. As a result, by taking appropriate limit of the stabilization parameters, we show that the HDG method converges to a primal conforming method and the WG method converge to a mixed conforming method.

Keywords. Uniform Stability, Uniform Error Estimate, Hybrid Discontinuous Galerkin, Weak Galerkin

1 Introduction

In the last few decades, one variant of finite element method called the discontinuous Galerkin (DG) method [1, 2] has been developed to solve various differential equations due to their flexibility in constructing feasible local shape-function spaces and the advantage of effectively capturing non-smooth or oscillatory solutions. Since DG methods use discontinuous space as trial space, the number of degrees of freedom is usually much higher than the standard conforming method. To reduce the number of globally coupled degrees of freedom of DG methods, a hybrid DG (HDG) has been developed. The idea of hybrid methods can be tracked to the 1960s [3]. A new hybridization approach in [4] was put forward by Cockburn and Gopalakrishnan in 2004 and was successfully applied to a discontinuous Galerkin method in [5]. Using the local discontinuous Galerkin (LDG) method to define the local solvers, a super-convergent LDG-hybridizable Galerkin method for second-order elliptic problems was designed in [6]. In 2009, a unified analysis for the hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second-order elliptic problems was presented in [7] by Cockburn, Gopalakrishnan, and Lazarov. A projection-based error analysis of HDG
methods was presented in [8], where a projection was constructed to obtain the $L^2$ error estimate for the potential and flux. However, the error estimate was dependent on the stabilization parameter. A projection-based analysis of the hybridized discontinuous Galerkin methods for convection-diffusion equations for semi-matching nonconforming meshes was presented in [9]. An analysis for a hybridized discontinuous Galerkin method with reduced stabilization for second-order elliptic problem was given in [10].

Based on a new concept, namely the weak gradient, introduced in [11], Wang and Ye proposed a weak Galerkin (WG) method for elliptic equations. Similar to the concept introduced in [11], Wang and Ye [12] introduced a concept called weak divergence. Based on the newly introduced concept, Wang and Ye [12] proposed and analyzed a WG method for the second-order elliptic equation formulated as a system of two first-order linear equations. Then a similar idea was applied to Darcy-Stokes flow in [13]. A primal-dual WG finite element method for second-order elliptic equations in non-divergence form was presented in [14] and a further similar method was applied to Fokker-Planck type equations in [15]. A bridge building the connection between the WG method and HDG method was shown in [16]. A summary of the idea and applications of WG methods to various problem were provided in [17].

In this paper, in contrast to the projection-based error analysis in [8, 10], we use the Ladyzhenskaya-Babuška-Brezzi (LBB) theory to prove two types of uniform stability results under some carefully constructed parameter-dependent norms for HDG methods. Based on the uniform stability results, we prove uniform and optimal error estimates for HDG methods. In addition, by using properly defined parameter-dependent norms, we further prove two types of uniform stability results for WG methods. Similarly based on the uniform stability results, we provide uniform and optimal error estimates for WG methods. These uniform stability results and error estimates for WG methods are meaningful and interesting improvement for the results in [11, 12]. Following these uniform stability results for HDG methods and WG methods presented in this paper, an HDG method is shown to converge to a primal conforming method, whereas a WG method is shown to converge to a mixed conforming method by taking the limit of the stabilization parameters.

We illustrate the main idea and results by using the following elliptic boundary value problem:

\[
\begin{aligned}
-\text{div}(\alpha \nabla u) &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain and $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded and symmetric positive definite matrix, and its inverse is denoted by $c = \alpha^{-1}$. Setting $p = -\alpha \nabla u$, the above problem can be written as:

\[
\begin{aligned}
cp + \nabla u &= 0 & \text{in } \Omega, \\
-\text{div}p &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]

(1.2)
The rest of the paper is organized as follows. In Section 2, some preliminary materials are provided. In Section 3, we set up the HDG and WG methods and provide the main uniform well-posedness results. Based on the uniform well-posedness results, we present uniform and optimal error estimates for HDG and WG in Section 4, and show that an HDG method converges to a primal conforming method, whereas a WG method converges to a mixed conforming method by taking the limit of the stabilization parameters in Section 5. In Section 6, we provide proof of the uniform well-posedness of HDG and WG under the specific parameter-dependent norms. We provide a brief summary in the last section.

2 Preliminaries

In this section, we describe some basic notation. Throughout this paper, we use letter $C$ to denote a generic positive constant, which may stand for different values at different occurrences, but not depending the mesh size and the stability parameters. The notations $x \lesssim y$ and $x \gtrsim y$ mean $x \leq C y$ and $x \geq C y$, respectively.

2.1 Discontinuous Galerkin Notation

Given a bounded domain $D \subset \mathbb{R}^d$ and a positive integer $m$, $H^m(D)$ is the Sobolev space with the corresponding usual norm and semi-norm, which are denoted respectively by $\| \cdot \|_{m,D}$ and $| \cdot |_{m,D}$. We abbreviate them by $\| \cdot \|_m$ and $| \cdot |_m$, respectively, when $D$ is chosen as $\Omega$. The $L^2$-inner products on $D$ and $\partial D$ are denoted by $(\cdot, \cdot)_D$ and $(\cdot, \cdot)_{\partial D}$, respectively. Moreover, $\| \cdot \|_{0,D}$ and $\| \cdot \|_{0,\partial D}$ are the norms of Lebesgue spaces $L^2(D)$ and $L^2(\partial D)$, respectively, and $\| \cdot \|_{\Omega} = \| \cdot \|_{0,\Omega}$. We also set $H(\text{div}, \Omega) = \{ u \in L^2(\Omega) : \text{div} u \in L^2(\Omega) \}$ equipped with the norm $\| u \|^2_{\text{div}} = (u, u) + (\text{div} u, \text{div} u)$.

We assume $\Omega$ is a polygonal domain, and a family of triangulations of $\Omega$ is denoted by $\{ T_h \}_h$, with the minimal angle condition satisfied. Let $h_K = \text{diam}(K)$ and $h = \max\{ h_K : K \in T_h \}$. We denote $E^i_h$ the set of interior edges (or faces) of $T_h$ and $E^\partial_h$ the set of boundary edges (or faces), and let $E_h = E^\partial_h \cup E^i_h$. For $e \in E_h$, let $h_e = \text{diam}(e)$. For $e \in E^i_h$, we choose a fixed normal unit direction denoted by $n_e$, and for $e \in E^\partial_h$, we take the outward unit normal as $n_e$. Let $e$ be the common edge of two elements $K^+$ and $K^-$, and $n^i = n|_{\partial K^i}$ be the unit outward normal vector on $\partial K^i$ with $i = +, -$. For any scalar-valued function $v$ and vector-valued function $q$, let $v^\pm = v|_{\partial K^\pm}$ and $q^\pm = q|_{\partial K^\pm}$. Then, we define averages $\{ \cdot \}, \{ \cdot \}, \{ \cdot \}$ and jumps $[ \cdot ], [ \cdot ], [ \cdot ]$ as follows:

\[
\begin{align*}
\{ v \} &= \frac{1}{2} (v^+ + v^-), \\
\{ q \} &= \frac{1}{2} (q^+ + q^-), \\
\{ q \} &= \frac{1}{2} (q^+ \cdot n^+ - q^- \cdot n^-) \\
[v] &= v^+ n^+ + v^- n^-, \\
[q] &= q^+ \cdot n^+ + q^- \cdot n^- \\
[q] &= v n, \\
[q] &= v, \\
\{ q \} &= q, \\
\{ q \} &= q \cdot n
\end{align*}
\]

on $e \in E^i_h$

on $e \in E^\partial_h$

on $e \in E^\partial_h$

Here, we specify $n$ as the outward unit normal direction on $\partial \Omega$. 

We define some inner products as follows:

\[
\langle \cdot, \cdot \rangle_{T_h} = \sum_{K \in T_h} \langle \cdot, \cdot \rangle_K, \quad \langle \cdot, \cdot \rangle_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} \langle \cdot, \cdot \rangle_e, \quad \langle \cdot, \cdot \rangle_{\mathcal{E}^i_h} = \sum_{e \in \mathcal{E}^i_h} \langle \cdot, \cdot \rangle_e, \quad \langle \cdot, \cdot \rangle_{\partial T_h} = \sum_{K \in T_h} \langle \cdot, \cdot \rangle_{\partial K}. \tag{2.1}
\]

We now give more details about the last notation of the inner product. For any scalar-valued function \(v\) and vector-valued function \(q\),

\[
\langle v, q \cdot n \rangle_{\partial T_h} = \sum_{K \in T_h} \langle v, q \cdot n \rangle_{\partial K} = \sum_{K \in T_h} \langle v, q \cdot n_K \rangle_{\partial K}.
\]

Here, we specify the outward unit normal direction \(n\) corresponding to the element \(K\), namely \(n_K\).

For the piecewise smooth scalar-valued function \(v\) and vector-valued function \(q\), let \(\nabla_h\) and \(\text{div}_h\) be defined by the relation

\[
(\nabla_h v)|_K = \nabla(v|_K), \quad (\text{div}_h q)|_K = \text{div}(q|_K),
\]

on any element \(K \in T_h\), respectively.

With the definition of averages and jumps, we have the following identity:

\[
\langle v, q \cdot n \rangle_{\partial T_h} = \langle \{ q \}, \{ v \} \rangle_{\mathcal{E}_h} + \langle \{ q \}, \{ v \} \rangle_{\mathcal{E}^i_h},
\]

and

\[
\{ q \} : \{ v \} = \{ q \} \cdot \{ v \}. \tag{2.3}
\]

Before discussing various Galerkin methods, we need to introduce the finite element spaces associated with the triangulation \(T_h\). First, \(V_h\) and \(Q_h\) are the piecewise scalar and vector-valued discrete spaces on the triangulation \(T_h\), respectively and for \(k \geq 0\), we define the spaces as follows:

\[
V_h^k = \{ v_h \in L^2(\Omega) : v_h|_K \in \mathcal{P}_k(K), \forall K \in T_h \},
\]

\[
Q_h^k = \{ p_h \in L^2(\Omega) : p_h|_K \in \mathcal{P}_k(K), \forall K \in T_h \},
\]

\[
Q_h^{k,RT} = \{ p_h \in L^2(\Omega) : p_h|_K \in \mathcal{P}_k(K) + x \mathcal{P}_k(K), \forall K \in T_h \},
\]

where \(\mathcal{P}_k(K)\) is the space of polynomial functions of degree at most \(k\) on \(K\). We also use the following spaces associated with \(\mathcal{E}_h\):

\[
\hat{Q}_h = \{ \hat{p}_h : \hat{p}_h|_e \in \hat{Q}(e)n_e, \forall e \in \mathcal{E}_h \},
\]

\[
\hat{Q}_h = \{ \hat{p}_h : \hat{p}_h|_e \in \hat{Q}(e), \forall e \in \mathcal{E}_h \},
\]

\[
\hat{V}_h = \{ \hat{v}_h : \hat{v}_h|_e \in \hat{V}(e), \forall e \in \mathcal{E}_h \},
\]

\[
\hat{V}_h = \{ \hat{v}_h : \hat{v}_h|_e \in \hat{V}(e), \forall e \in \mathcal{E}_h \},
\]

\[
\hat{Q}_h^k = \{ \hat{p}_h \in L^2(\mathcal{E}_h) : \hat{p}_h|_e \in \mathcal{P}_k(e), \forall e \in \mathcal{E}_h \},
\]

\[
\hat{Q}_h^k = \{ \hat{p}_h \in L^2(\mathcal{E}_h) : \hat{p}_h|_e \in \mathcal{P}_k(e), \forall e \in \mathcal{E}_h \},
\]

where \(\hat{Q}(e)\) and \(\hat{V}(e)\) are some local spaces on \(e\) and \(\mathcal{P}_k(e)\) is the space of polynomial functions of degree at most \(k\) on \(e\). For convenience, we denote \(\hat{Q}_h = Q_h \times \hat{Q}_h\) and \(\hat{V}_h = V_h \times \hat{V}_h\).
3 Uniform Stability for HDG and WG Methods

In this section, we set up the HDG and WG methods first and then provide the uniform well-posedness for both HDG and WG methods under proper parameter-dependent defined norms.

3.1 Setting up the HDG and WG Methods

Now we start with the second-order elliptic equation and set \( p = -\alpha \nabla u \) to obtain the following form:

\[
\begin{aligned}
\alpha p + \nabla u &= 0 & \text{in } \Omega, \\
\text{div} p &= f & \text{in } \Omega.
\end{aligned}
\]  

Multiplying the first and second equations by \( q \) and \( K \), respectively, then integrating on an element \( K \in \mathcal{T}_h \), we obtain:

\[
\begin{aligned}
\int (c \alpha p, q)_K &= -\int (u, \text{div} q)_K + \int (u, q \cdot n_K)_{\partial K} = 0 & \forall q_h \in Q_h, \\
\int (p \cdot \nabla v_h)_K - \int (p \cdot n_K, v_h)_{\partial K} &= -(f, v_h)_K & \forall v_h \in V_h.
\end{aligned}
\]  

Summing on all \( K \in \mathcal{T}_h \), we have:

\[
\begin{aligned}
\int (c \alpha p, q_h)_{T_h} &= -\int (u, \text{div} q_h)_{T_h} + \int (u, q_h \cdot n)_{\partial T_h} = 0 & \forall q_h \in Q_h, \\
\int (p \cdot \nabla v_h)_{T_h} - \int (p \cdot n, v_h)_{\partial T_h} &= -(f, v_h)_{T_h} & \forall v_h \in V_h.
\end{aligned}
\]

Now we approximate \( u, p \) by \( u_h \in V_h \), and \( p_h \in Q_h \), respectively, and the trace of \( u \) and the flux \( p \cdot n \) on \( \partial K \) by \( \hat{u}_h, \hat{p}_h \cdot n \). Hence, we have:

\[
\begin{aligned}
\int (c \alpha p_h, q_h)_{T_h} &= -\int (u_h, \text{div} q_h)_{T_h} + \int (u_h, q_h \cdot n)_{\partial T_h} = 0, \forall q_h \in Q_h, \\
\int (p_h \cdot \nabla v_h)_{T_h} - \int (\hat{p}_h \cdot n, v_h)_{\partial T_h} &= -(f, v_h)_{T_h} & \forall v_h \in V_h.
\end{aligned}
\]

Next, we need to derive appropriate equations for the variables of \( \hat{u}_h \) and \( \hat{p}_h \). The starting point is the following relationship:

\[
\hat{p}_h \cdot n_K + \tau \hat{u}_h = p_h \cdot n_K + \tau u_h, \quad \hat{p}_h = \hat{p}_h n_e.
\]  

The idea is that we only use either \( \hat{p}_h \) or \( \hat{u}_h \) as an unknown and then use (3.5) to determine the other variable. There are two different approaches; one approach is for deriving HDG methods, and the other one is for deriving WG methods.

**First approach: (Hybridized Discontinuous Galerkin)** Set \( \hat{u}_h = \hat{u}_h \in \hat{V}_h \) as an unknown that is single-valued. The “continuity” of \( \hat{p}_h \) is then enforced weakly as follows:

\[
\int (\hat{p}_h \cdot n, \hat{v}_h)_{\partial T_h} = 0, \forall \hat{v}_h \in \hat{V}_h.
\]  

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where $\tilde{p}_h$ is given by (3.5). From the identity (2.2) and the fact that $[\tilde{v}_h] = 0$, a straightforward calculation shows that (3.6) can be rewritten as:

\[
\langle [\tilde{p}_h], \tilde{v}_h \rangle_{E_h} := \sum_{e \in E_h} \langle [\tilde{p}_h], \tilde{v}_h \rangle_{e} = 0, \quad \forall \tilde{v}_h \in \tilde{V}_h. \tag{3.7}
\]

Collecting (3.4), (3.5), and (3.6), the HDG methods read: Find $(p_h, \tilde{u}_h) \in Q_h \times \tilde{V}_h$ such that for any $(q_h, \tilde{v}_h) \in Q_h \times \tilde{V}_h$,

\[
\begin{aligned}
& a_h(p_h, q_h) + b_h(q_h, \tilde{u}_h) = 0, \\
& b_h(p_h, \tilde{v}_h) + c_h(\tilde{u}_h, \tilde{v}_h) = -(f, v_h)_{T_h}.
\end{aligned} \tag{3.8}
\]

Here

\[
\begin{aligned}
& a_h(p_h, q_h) = (cp_h, q_h)_{T_h}, \\
& b_h(q_h, \tilde{u}_h) = -(u_h, \text{div} q_h)_{T_h} + \langle \tilde{u}_h, q_h \cdot n_K \rangle_{\partial T_h}, \\
& c_h(\tilde{u}_h, \tilde{v}_h) = -\tau (u_h - \tilde{u}_h, v_h - \tilde{v}_h)_{\partial T_h},
\end{aligned} \tag{3.9}
\]

where $\tau$ is the stabilization parameter.

The HDG method can be written in a compact form: Find $(p_h, \tilde{u}_h) \in Q_h \times \tilde{V}_h$ such that for any $(q_h, \tilde{v}_h) \in Q_h \times \tilde{V}_h$, \n
\[
A_h((p_h, \tilde{u}_h), (q_h, \tilde{v}_h)) = -(f, v_h)_{T_h}, \tag{3.10}
\]

where

\[
A_h((p_h, \tilde{u}_h), (q_h, \tilde{v}_h)) = a_h(p_h, q_h) + b_h(q_h, \tilde{u}_h) + b_h(p_h, \tilde{v}_h) + c_h(\tilde{u}_h, \tilde{v}_h). \tag{3.11}
\]

In the first case, we choose $\tau = \rho h_K$ in (3.9) and for any $\tilde{v} \in \tilde{V}_h$ and $q_h \in Q_h$, we define

\[
\begin{aligned}
& \|\tilde{v}_h\|^2_{\tilde{V},\rho,h} = (v_h, v_h)_{T_h} + \rho \sum_{e \in E_h} h_e \langle \tilde{v}_h, \tilde{v}_h \rangle_{e}, \\
& \|q_h\|^2_{\tilde{V},\rho,h} = (cq_h, q_h)_{T_h} + (\text{div} q_h, \text{div} q_h)_{T_h} + \rho^{-1} \sum_{e \in E_h} h_e^{-1} \langle \hat{P}_e([q_h]), \hat{P}_e([q_h]) \rangle_{e},
\end{aligned} \tag{3.12}
\]

where $\hat{P}_e : L^2(e) \rightarrow V(e)$ is the $L^2$ projection.

In the second case, we choose $\tau = \rho^{-1} h_K^{-1}$ in (3.9) and for any $\tilde{v} \in \tilde{V}_h$ and $q_h \in Q_h$, we define

\[
\begin{aligned}
& \|	ilde{v}_h\|^2_{\tilde{V},\rho,h} = (\nabla_h v_h, \nabla_h v_h)_{T_h} + \rho^{-1} \sum_{K \in T_h} h_K^{-1} \langle v_h - \tilde{v}_h, v_h - \tilde{v}_h \rangle_{\partial K}, \\
& \|q_h\|^2 = (cq_h, q_h)_{T_h}. \tag{3.13}
\end{aligned}
\]

By noting that

\[
\langle v_h - \tilde{v}_h, v_h - \tilde{v}_h \rangle_{\partial T_h} = 2 \langle \{v_h - \tilde{v}_h\}, \{v_h - \tilde{v}_h\} \rangle_{E_h} + \frac{1}{2} \langle \|v_h - \tilde{v}_h\|_{E_h}, \|v_h - \tilde{v}_h\|_{E_h} \rangle_{E_h},
\]

$\|	ilde{v}_h\|^2_{\tilde{V},\rho,h}$ is indeed a norm on $\tilde{V}_h$.
Second approach: (Weak Galerkin) We set $\tilde{p}_h := \tilde{p}_h = \tilde{p}_h n_e \in \tilde{Q}_h$ as an unknown that is single-valued. The “continuity” of $\tilde{u}_h$ is then enforced weakly as follows:

$$\langle [\tilde{u}_h], \tilde{q}_h \cdot n \rangle_{\partial T_h} = 0 \quad \forall \tilde{q}_h \in \tilde{Q}_h,$$

where $\tilde{a}_h$ is again given by (3.5). From the identity (2.2) and the fact that $[\tilde{q}_h] = 0$, a straightforward calculation shows that (3.14) can be rewritten as:

$$\langle [\tilde{u}_h], \tilde{q}_h \rangle_{E_h} := \sum_{e \in E_h} \langle [\tilde{u}_h], \tilde{q}_h \rangle_e = 0 \quad \forall \tilde{q}_h \in \tilde{Q}_h.$$ 

Collecting (3.4), (3.5), and (3.14), the WG methods read: Find $(\tilde{p}_h, u_h) \in \tilde{Q}_h \times V_h$ such that for any $(\tilde{q}_h, v_h) \in \tilde{Q}_h \times V_h$,

$$\begin{align*}
aw(\tilde{p}_h, \tilde{q}_h) + bw(u_h, \tilde{q}_h) &= 0, \\
bw(\tilde{p}_h, v_h) &= -(f, v_h).
\end{align*}$$

Here

$$\begin{align*}
aw(\tilde{p}_h, \tilde{q}_h) &= (c p_h, q_h)_{T_h} + \eta(\langle p_h - \tilde{p}_h \rangle \cdot n, (q_h - \tilde{q}_h) \cdot n)_{\partial T_h}, \\
bw(\tilde{p}_h, v_h) &= (p_h, \nabla_h v_h)_{T_h} - (\tilde{p}_h \cdot n_K, v_h)_{\partial T_h},
\end{align*}$$

(3.17)

where $\eta$ is the stabilized parameter.

The WG method can be rewritten in a compact form: Find $(\tilde{p}_h, u_h) \in \tilde{Q}_h \times V_h$ such that for any $(\tilde{q}_h, v_h) \in \tilde{Q}_h \times V_h$:

$$A_w((\tilde{p}_h, u_h), (\tilde{q}_h, v_h)) = -(f, v_h),$$

(3.18)

where

$$A_w((\tilde{p}_h, u_h), (\tilde{q}_h, v_h)) = aw(\tilde{p}_h, \tilde{q}_h) + bw(\tilde{q}_h, u_h) + bw(\tilde{p}_h, v_h).$$

(3.19)

In the first case, we choose parameter $\eta$ as $\eta = \rho h_{K}$ in (3.17) and for any $v_h \in V_h$ and $\tilde{q}_h \in \tilde{Q}_h$, we define the norms as follows:

$$\begin{align*}
\|v_h\|_{1, h, \rho}^2 &= \|\nabla_h v_h\|^2 + \rho^{-1} \sum_{e \in E_h} h_e^{-1} \|\tilde{Q}_e([v_h])\|_{0, e}^2, \\
\|\tilde{q}_h\|_{0, h, \rho}^2 &= (c q_h, q_h)_{T_h} + \rho \sum_{K \in T_h} h_K \langle (q_h - \tilde{q}_h) \cdot n_K, (q_h - \tilde{q}_h) \cdot n_K \rangle_{\partial K}.
\end{align*}$$

(3.20)

where $\tilde{Q}_e$ is the $L^2$ projection from $L^2(e)$ to $\tilde{Q}(e)$.

In the second case, we choose parameter $\eta$ as $\eta = \rho^{-1} h_{K}^{-1}$ in (3.17) and for any $v_h \in V_h$ and $\tilde{q}_h \in \tilde{Q}_h$, we define the norms as follows:

$$\|u_h\|^2 = (u_h, u_h)_{T_h}; \quad \|\tilde{q}_h\|_{\text{div}, h, \rho}^2 = \sum_{K \in T_h} \|\tilde{q}_h\|_{\text{div}, \rho, K}^2,$$

(3.21)

where $\|\tilde{q}_h\|_{\text{div}, \rho, K}^2 = (c q_h, q_h)_K + (\text{div} q_h, \text{div} q_h)_K + \rho^{-1} h_{K}^{-1} \langle (q_h - \tilde{q}_h) \cdot n_K, (q_h - \tilde{q}_h) \cdot n_K \rangle_{\partial K}$. 

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3.2 Uniform Well-posedness of HDG and WG

For the elliptic problem (1.2), we set a discretization: Find \( U_h \in U_h \), such that:

\[
A_{h,\theta}(U_h, V_h) = F(V_h) \quad \forall \ V_h \in U_h,
\]

(3.22)

where \( U_h \) is a finite dimensional space according to partition \( T_h \) and \( A_{h,\theta}(U_h, V_h) \) is a general symmetric \( \theta \)-parameter-dependent bilinear form and \( F(V_h) = -(f, v_h)_{T_h} \).

Let \( U = (p, u) \) be the true solution of (1.2).

1. We say that the discretization (3.22) is consistent if

\[
A_{h,\theta}(U_h, V_h) = F(V_h) \quad \forall \ V_h \in U_h.
\]

(3.23)

2. We say that the bilinear form \( A_{h,\theta}(U_h, V_h) \) is uniformly continuous with respect to the norm \( \| \cdot \|_{U_{h,\theta}} \) if

\[
|A_{h,\theta}(U_h, V_h)| \leq M_0 \|U_h\|_{U_{h,\theta}} \|V_h\|_{U_{h,\theta}},
\]

(3.24)

where \( M_0 \) is independent of the parameter \( \theta \) and the mesh size \( h \).

3. We say that the bilinear form \( A_{h,\theta}(U_h, V_h) \) satisfies the inf-sup condition uniformly with respect to the norm \( \| \cdot \|_{U_{h,\theta}} \) if there exists a constant \( \beta_1 > 0 \) that does not depend on the parameter \( \theta \) and the mesh size \( h \) such that:

\[
\inf_{V_h \in U_h} \sup_{U_h \in U_h} \frac{A_{h,\theta}(U_h, V_h)}{\|U_h\|_{U_{h,\theta}} \|V_h\|_{U_{h,\theta}}} \geq \beta_1.
\]

(3.25)

By the Céa’s lemma, see [18], we have

**Theorem 3.1** If a discretization (3.22) satisfies

1. consistency, namely (3.23);

2. continuity uniformly, namely (3.24);

3. inf-sup condition uniformly with respect to the norm \( \| \cdot \|_{U_{h,\theta}} \), namely (3.25),

then we have

\[
\|U - U_h\|_{U_{h,\theta}} \leq C_1 \inf_{V_h \in U_h} \|U - V_h\|_{U_{h,\theta}},
\]

(3.26)

where \( C_1 \) is independent of the parameter \( \theta \) and the mesh size \( h \). Further, we say the discretization (3.22) is uniformly stable.
Now for the HDG method, the parameter $\theta = \tau$ in (3.22) and the bilinear form is given by (3.11), the space $U_h = Q_h \times V_h$, $U_h = (p_h, \tilde{u}_h)$. In the first case, the parameter $\tau = \rho h \kappa$, and the norm $\|U_h\|_{U_{h, \theta}}^2 = \|p_h\|_{\text{div}, p, h}^2 + \|\tilde{u}_h\|_{p, h}^2$. In the second case, the parameter $\tau = \rho^{-1} h_{K}^{-1}$ and the norm $\|U_h\|_{U_{h, \theta}}^2 = \|p_h\|^2 + \|\tilde{u}_h\|_{p, h}^2$.

**Theorem 3.2** We have two uniform stability results for the HDG method as follows:

1. For any $0 < \rho \leq 1$, and for $k \geq 0$, if $Q_h = Q_h^{k+1}, V_h = V_h^k$ and $\hat{V}_h = \hat{V}_h^r$ where $0 \leq r \leq k + 1$, or $Q_h = Q_h^{k,RT}, V_h = V_h^k$ and $\hat{V}_h = \hat{V}_h^r$ where $0 \leq r \leq k$, then the bilinear form $A_h((\cdot, \cdot), (\cdot, \cdot))$ with $\tau = \rho h \kappa$ is uniformly stable with respect to the norms defined by (3.12);

2. Assume that $\nabla_h V_h \subset Q_h$, then there exists a positive constant $\rho_0$ such that for any $0 < \rho \leq \rho_0$ the bilinear form $A_h((\cdot, \cdot), (\cdot, \cdot))$ with $\tau = \rho^{-1} h_{K}^{-1}$ is uniformly stable with respect to the norms defined by (3.13).

From part 2 of the above theorem, we have the following corollary:

**Corollary 3.3** Assume $\nabla_h V_h \subset Q_h$, then there exists a unique solution $(p_h, \tilde{u}_h) \in Q_h \times \hat{V}_h$ that satisfies (3.8) with $\tau = \rho^{-1} h_{K}^{-1}$, and there exists a positive constant $\rho_0$ such that for any $0 < \rho \leq \rho_0$ the following estimate holds:

$$\|p_h\| + \|\tilde{u}_h\|_{1, p, h} \leq C_2 \|f\|_{*, p},$$  \hspace{1cm} (3.27)

where $C_2$ is a constant independent of $\rho$ and $h$ and $\|f\|_{*, p} = \sup_{v_h \in V_h} \frac{(f, v_h)}{\|v_h\|_{1, p, h}}$.

**Remark 3.4** From the above corollary and the discrete Poincaré–Friedrichs inequalities for piecewise $H^1$ functions [19], that is $\|v_h\| \lesssim \|\nabla_h v_h\| + \sum_{e \in \hat{E}_h} h^{-1}_e \|v_h\|_{0, e}$, we further have $\|p_h\| + \|\tilde{u}_h\|_{1, p, h} \leq C_2 \|f\|$.

**Remark 3.5** By the uniform stability results of the HDG method, namely Corollary 3.3, we can prove that the solution of the HDG method converges to the solution of the primal conforming method when the parameter $\rho$ approaches to zero, see Section 5.

Next, for the WG method, the parameter $\theta = \eta$ in (3.22) and the bilinear form is given by (3.19), and the space $U_h = \hat{Q}_h \times V_h$, $U_h = (\tilde{p}_h, u_h)$. In first case, the parameter $\eta = \rho h \kappa$ and the norm $\|U_h\|_{U_{h, \theta}}^2 = \|\tilde{p}_h\|_{0, h, \rho}^2 + \|u_h\|_{1, h, \rho}^2$. In the second case, the parameter $\eta = \rho^{-1} h_{K}^{-1}$, and the norm $\|U_h\|_{U_{h, \theta}}^2 = \|\tilde{p}_h\|_{\text{div}, p, h}^2 + \|u_h\|_{h, \rho}^2$.

**Theorem 3.6** We have two uniform stability results for the WG method as follows:

1. Assume $\nabla_h V_h \subset Q_h$, then for any $0 < \rho \leq 1$ the bilinear form $A_w((\cdot, \cdot), (\cdot, \cdot))$ with $\eta = \rho h \kappa$ is uniformly stable with respect to the norms defined by (3.20);
2. Let \( R_h \subset H(\text{div}, \Omega) \cap Q_h \) be the Raviart-Thomas finite element space. Assume that \( \{ R_h \} \subset \tilde{Q}_h \) and \( V_h = \text{div}_h Q_h \), then for any \( 0 < \rho \leq 1 \) the bilinear form \( A_w((\cdot,\cdot),(\cdot,\cdot)) \) with \( \eta = \rho^{-1}h^{-1}_K \) is uniformly stable with respect to the norms defined by (3.21).

From part 1 of the above theorem, we have the following corollary:

**Corollary 3.7** Assume \( \nabla_h V_h \subset Q_h \), then there exists a unique solution \( (\tilde{p}_h, u_h) \in \tilde{Q}_h \times V_h \) that satisfies (3.16) with \( \eta = \rho h_K \), and for any \( 0 < \rho \leq 1 \) the following estimates holds:

\[
\| \tilde{p}_h \|_{0,h,\rho} + \| u_h \|_{1,h,\rho} \leq C_3 \| f \|_{*,\rho},
\]

where \( C_3 \) is a constant uniform with respect to \( \rho \) and \( h \) and \( \| f \|_{*,\rho} = \sup_{v_h \in \tilde{V}_h} \frac{(f,v_h)_{\tilde{h}}}{\| v_h \|_{1,h,\rho}} \).

**Remark 3.8** From the above theorem, we improved the result in [12] by proving the well-posedness of the WG method for any \( 0 < \rho \leq 1 \), while in [12] the inf-sup condition for some constant \( \rho \) (for example \( \rho = 1 \)) was proved.

From part 2 of Theorem 3.6, we have the following corollary:

**Corollary 3.9** Assume the spaces \( \tilde{Q}_h \times V_h \) satisfy the conditions in part 2 of Theorem 3.6, then there exists a unique solution \( (\tilde{p}_h, u_h) \in \tilde{Q}_h \times V_h \) that satisfies (3.16) with \( \eta = \rho^{-1}h^{-1}_K \), and for any \( 0 < \rho \leq 1 \) the following estimates holds:

\[
\| \tilde{p}_h \|_{\text{div},h,\rho} + \| u_h \| \leq C_4 \| f \|,
\]

where \( C_4 \) is a uniform constant with respect to \( \rho \) and \( h \).

**Remark 3.10** By the above uniform stability result of the WG method, namely Corollary 3.9, we can prove that the solution of the WG method converges to the solution of the mixed conforming method when the parameter \( \rho \) approaches to zero, see Section 5.

## 4 Uniform Error Estimates of HDG and WG

In this section, based on the uniform stability results shown in Section 3, we provide the error analysis for HDG and WG methods and obtain uniformly optimal error estimates for HDG and WG methods.

### 4.1 Error Estimate of HDG Method

**Theorem 4.1** Let \( (p, u) \in H(\text{div}, \Omega) \times L^2(\Omega) \) be the solution of (1.2) and \( p \in H^{k+1}(\Omega), \text{div} p \in H^{k+1}(\Omega), u \in H^{k+1}(\Omega)(k \geq 0) \), and \( (p_h, \tilde{u}_h) \in Q_h \times \tilde{V}_h \) be the solution of (3.8) with \( \tau = \rho h_K \). If we choose the spaces
\( V_h \times Q_h \times \hat{V}_h = V_h^k \times Q_h^{k,RT} \times V_h^k \), then for any \( 0 < \rho \leq 1 \) the following estimate holds:

\[
\| p - p_h \|_{\text{div}, \rho, h} + \| u - \hat{u}_h \|_{0, \rho, h} \leq C_{r,1} h^{k+1} (\| p \|_{k+1} + |\text{div} p|_{k+1} + |u|_{k+1}),
\]

where \( C_{r,1} \) is a constant independent of \( h \) and \( \rho \).

**Proof.** From part 1 of Theorem 3.2 and Theorem 3.1, we have

\[
\| p - p_h \|_{\text{div}, \rho, h} + \| u - \hat{u}_h \|_{0, \rho, h} \leq \inf_{q_h \in Q_h, \hat{v}_h \in \hat{V}_h} \left( \| p - q_h \|_{\text{div}, \rho, h} + \| u - \hat{v}_h \|_{0, \rho, h} \right).
\]

Hence we need to estimate:

\[
\inf_{q_h \in Q_h, \hat{v}_h \in \hat{V}_h} \left( \| p - q_h \|_{\text{div}, \rho, h} + \| u - \hat{v}_h \|_{0, \rho, h} \right).
\]

Now we choose \( q_h = \pi_h^{\text{div}} p \in Q_h \), where \( \pi_h^{\text{div}} \) be the interpolation of \( p \) into the \( H(\text{div}) \)-conforming Raviart-Thomas (RT) finite element space, namely \( q_h \in Q_h \cap H(\text{div}, \Omega) \). Since \( q_h \cdot n \) is single-valued, then by the approximation property of the RT finite element space, we obtain:

\[
\| p - q_h \|_{\text{div}, \rho, h}^2 = (p - q_h, p - q_h)_{\mathcal{T}_h} + (\text{div}(p - q_h), \text{div}(p - q_h))_{\mathcal{E}_h} + \rho^{-1} \sum_{e \in \mathcal{E}_h} h_e^{-1} (\hat{p}_e((p - q_h)_e), \hat{p}_e((p - q_h)_e))_e \leq h^{2k+2} (\| p \|_{k+1}^2 + |\text{div} p|_{k+1}^2).
\]

Further, we choose \( v_h = Q_h(u), \hat{v}_h = \{Q_h(u)\} \), where \( Q_h \) is \( L^2 \) projection from \( L^2(\Omega) \) to \( V_h \). Then, by using the approximation of \( L^2 \) projection, trace inequality and noting that \( 0 < \rho \leq 1 \), we have:

\[
\| u - \hat{v}_h \|_{0, \rho, h}^2 = (u - v_h, u - v_h)_{\mathcal{T}_h} + \rho \sum_{e \in \mathcal{E}_h} h_e (u - \hat{v}_h, u - \hat{v}_h)_e \\
= (u - Q_h(u), u - Q_h(u))_{\mathcal{T}_h} + \rho \sum_{e \in \mathcal{E}_h} h_e (u - Q_h(u), u - Q_h(u))_e \\
\lesssim \| u - Q_h(u) \|_{0, \rho, h}^2 + \rho \sum_{e \in \mathcal{E}_h} h_e (h^{-1}_e \| u - Q_h(u) \|_{K_{e,1} \cup K_{e,2}}^2 + h_e \| \nabla_h (u - Q_h(u)) \|_{K_{e,1} \cup K_{e,2}}^2) \\
\lesssim \| u - Q_h(u) \|_{0, \rho, h}^2 + h^2_e \| \nabla_h (u - Q_h(u)) \|_2^2 \lesssim h^{2k+2} |u|_{k+1}^2,
\]

where \( K_{e,1}, K_{e,2} \) are the elements sharing the edge \( e \).

Combining (4.4) and (4.5), we get the desired result. 

**Theorem 4.2** Let \( (p, u) \in L^2(\Omega) \times H^1(\Omega) \) be the solution of (1.2) and \( p \in H^{k+1}(\Omega), u \in H^{k+2}(\Omega) \) \( (k \geq 0) \), and \( (p_h, \hat{u}_h) \in Q_h \times \hat{V}_h \) be the solution of (3.8) with \( \tau = \rho^{-1} h^{-1}_K \). If we choose the spaces
\[ V_h \times Q_h \times \hat{V}_h = V_h^{k+1} \times Q_h^k \times V_h^{k+1}, \] then there exists \( \rho_0 > 0 \) such that for any \( 0 < \rho \leq \rho_0 \) the following estimate holds:

\[ \|p - p_h\| + \|u - \tilde{u}_h\|_{1, \rho, h} \leq C_{r.2} h^{k+1}(p|_{k+1} + |u|_{k+2}), \tag{4.6} \]

where \( C_{r.2} \) is independent of \( h \) and \( \rho \).

**Proof.** From part 2 of Theorem 3.2 and Theorem 3.1, we have

\[ \|p - p_h\| + \|u - \tilde{u}_h\|_{1, \rho, h} \lesssim \inf_{q_h \in Q_h, v_h \in \hat{V}_h} \left( \|p - q_h\| + \|u - v_h\|_{1, \rho, h} \right). \tag{4.7} \]

Hence we need to estimate:

\[ \inf_{q_h \in Q_h, v_h \in \hat{V}_h} \left( \|p - q_h\| + \|u - v_h\|_{1, \rho, h} \right). \tag{4.8} \]

Now we choose \( q_h = Q_h(p) \), where \( Q_h \) is \( L^2 \) projection from \( L^2(\Omega) \) to \( Q_h \). Then, by using the approximation of \( L^2 \) projection, we obtain:

\[ \|p - q_h\| \leq h^{k+1} |p|_{k+1}. \tag{4.9} \]

Further, we choose \( v_h = \pi_h u \), where \( \pi_h \) is the interpolation of \( u \) to the continuous finite element space, namely \( v_h \in V_h \cap H^1(\Omega) \). Since \( v_h \) is in \( H^1(\Omega) \), we can choose \( \tilde{v}_h \) such that \( \tilde{v}_h|_{\partial K} = v_h|_{\partial K} \), for any \( K \in T_h \), then we get the following convergence rate result:

\[ \|u - \tilde{v}_h\|_{1, \rho, h}^2 = \langle \nabla_h (u - v_h), \nabla_h (u - v_h) \rangle_{T_h} \]
\[ + \rho^{-1} \sum_{K \in T_h} h_K^{-1} ((u - v_h) - (u - \tilde{v}_h), (u - v_h) - (u - \tilde{v}_h)|_{\partial K}) \tag{4.10} \]
\[ = \langle \nabla (u - \pi_h u), \nabla (u - \pi_h u) \rangle_{T_h} \lesssim h^{2k+2} |u|_{k+2}^2. \]

Combining (4.9) and (4.10), we get the desired result. \qed

### 4.2 Error Estimate of the WG Method

**Theorem 4.3** Let \((p, u) \in L^2(\Omega) \times H^1(\Omega)\) be the solution of (1.2) and \( p \in H^{k+1}(\Omega), u \in H^{k+2}(\Omega) \), and \((\tilde{p}_h, u_h) \in \hat{Q}_h \times \hat{V}_h\) be the solution of (3.16) with \( \eta = ph_K \). If we choose the spaces \( V_h \times Q_h \times \hat{Q}_h = V_h^{k+1} \times Q_h^k \times \hat{Q}_h^k \), then for any \( 0 < \rho \leq 1 \) the following estimate holds:

\[ \|p - \tilde{p}_h\|_{0, \rho, h} + \|u - u_h\|_{1, \rho, h} \leq C_{r.3} h^{k+1}(p|_{k+1} + |u|_{k+2}), \tag{4.11} \]

where \( C_{r.3} \) is independent of \( h \) and \( \rho \).

**Proof.** From part 1 of Theorem 3.6 and Theorem 3.1, we have

\[ \|p - \tilde{p}_h\|_{0, \rho, h} + \|u - u_h\|_{1, \rho, h} \lesssim \inf_{\tilde{q}_h \in Q_h, v_h \in V_h} \left( \|p - \tilde{q}_h\|_{0, \rho, h} + \|u - v_h\|_{1, \rho, h} \right). \tag{4.12} \]
Hence we need to estimate:

\[
\inf_{\tilde{q}_h \in Q_h, v_h \in V_h} \left( \| p - \tilde{q}_h \|_{0,h,\rho} + \| u - v_h \|_{1,h,\rho} \right).
\] (4.13)

Now for any \( K \in T_h \), we choose \( \tilde{q}_h = (q_h, \tilde{q}_h) = (Q_h(p), \{Q_h(p)\}) \), where \( Q_h \) is the local \( L^2 \) projection from \( L^2(\Omega) \) to \( Q_h \). By the approximation property of the \( L^2 \) projection, trace inequality and noting that \( 0 < \rho \leq 1 \), we obtain:

\[
\| p - \tilde{q}_h \|_{0,h,\rho}^2 = (c(p - Q_h(p)), p - Q_h(p))_{T_h} + \rho \sum_{K \in T_h} h_K \| (p - Q_h(p)) \cdot n_K \|_{0,\partial K}^2 
\leq \| p - Q_h(p) \|_{T_h}^2 + \sum_{K \in T_h} h_K \| (p - Q_h(p)) \cdot n_K \|_{0,\partial K}^2 + \| p - \{Q_h(p)\} \cdot n_K \|_{0,\partial K}^2 
= \| p - Q_h(p) \|_{T_h}^2 + \sum_{K \in T_h} \| p - Q_h(p) \|_{0,K}^2 + h_K^2 \| \nabla (p - Q_h(p)) \|_{0,K}^2 
\leq h^{2k+2} |p|_{k+1}^2.
\] (4.14)

Next we choose \( v_h = \pi_h u \), where \( \pi_h \) is the interpolation of \( u \) to the continuous finite element space, namely, \( v_h \in V_h \cap H^1_0(\Omega) \), we immediately have:

\[
\| u - v_h \|_{1,h,\rho}^2 = \| \nabla_h (u - v_h) \|^2 + \rho^{-1} \sum_{e \in \mathcal{E}_h} h_e^{-1} \| \hat{Q}_e([u - v_h]) \|_{0,\mathcal{E}}^2
\]

\[
= \| \nabla_h (u - v_h) \|^2 = \| \nabla_h (u - \pi_h u) \|^2 \leq h^{2k+2} |u|_{k+2}^2.
\] (4.15)

Combining (4.14) and (4.15), we get the desired result. \( \blacksquare \)

**Theorem 4.4** Let \( (p, u) \in H(\text{div}, \Omega) \times L^2(\Omega) \) be the solution of (1.2) and \( p \in H^{k+1}(\Omega), \text{div} p \in H^{k+1}(\Omega), u \in H^{k+1}(\Omega) \), and \( (\tilde{p}_h, u_h) \in \tilde{Q}_h \times V_h \) be the solution of (3.16) with \( \eta = \rho^{-1} h_K^{-1} \). If we choose the spaces \( V_h \times Q_h \times \tilde{Q}_h = V_h \times Q_h^{k,RT} \times \tilde{Q}_h \), then for any \( 0 < \rho \leq 1 \) the following estimate holds:

\[
\| p - \tilde{p}_h \|_{\text{div},h,\rho} + \| u - u_h \| \leq C_{r,4} h^{k+1} |p|_{k+1} + |\text{div} p|_{k+1} + |u|_{k+1}.
\] (4.16)

where \( C_{r,4} \) is independent of \( h \) and \( \rho \).

**Proof.** From part 2 of Theorem 3.6 and Theorem 3.1, we have

\[
\| p - \tilde{p}_h \|_{\text{div},h,\rho} + \| u - u_h \| \leq \inf_{\tilde{q}_h \in \tilde{Q}_h \cap Q_h} \left( \| p - \tilde{q}_h \|_{\text{div},h,\rho} + \| u - v_h \| \right).
\] (4.17)

We need to estimate:

\[
\inf_{\tilde{q}_h \in \tilde{Q}_h \cap Q_h} \left( \| p - \tilde{q}_h \|_{\text{div},h,\rho} + \| u - v_h \| \right).
\] (4.18)

Now we choose \( q_h = \pi_h \text{div} p \in Q_h \), where \( \pi_h \text{div} \) is the interpolation of \( p \) into the \( H(\text{div}) \)-conforming \( RT \) finite element space, namely \( q_h \in Q_h \cap H(\text{div}, \Omega) \). Since \( q_h \cdot n \) is single-valued, we can choose \( \tilde{q}_h = (q_h \cdot n)n \),
then by the approximation property of the $RT$ finite element space, we obtain
\[
\|p - \tilde{q}_h\|_{\text{div},h,\rho} = (c(p - q_h), p - q_h)_{\mathcal{T}_h} + (\text{div}(p - q_h), \text{div}(p - q_h))_{\mathcal{T}_h}
\]
\[
+ \sum_{K \in \mathcal{T}_h} \rho^{-1} h^{-1}_K \| (p - p_h - (p - \hat{p}_h) \cdot n_K\|_{0,\partial K}
\]
\[
= (c(p - q_h), p - q_h)_{\mathcal{T}_h} + (\text{div}(p - q_h), \text{div}(p - q_h))_{\mathcal{T}_h}
\]
\[
\lesssim h^{2k+2}(|p|_{k+1}^2 + |\text{div}p|_{k+1}^2).
\]
(4.19)

Next, we choose $v_h = Q_h(u)$, where $Q_h$ is the $L^2$ projection from $L^2(\Omega)$ to $V_h$, and we immediately have:
\[
\|u - v_h\| \lesssim h^{k+1}|u|_{k+1}.
\]
(4.20)

Combining (4.19) and (4.20), we get the desired result. 

**Remark 4.5** We must point out that the error estimates obtained here are uniform with respect to the parameter $\rho$. Namely, all the constants $C_{r,1}, C_{r,2}, C_{r,3},$ and $C_{r,4}$ are independent of $\rho$. We figure out the following Table 1.

| FEM | norms | $U_h$ | parameter $\theta$ | order |
|-----|-------|------|--------------------|-------|
| HDG | $\|p_h\|_{\text{div},\rho, h}$ | $V_h^k \times Q_h^{k, RT} \times \hat{V}_h^k$ | $\theta = \tau = \rho h_K$ | $k + 1$ |
| HDG | $\|\tilde{u}_h\|_{0,\rho, h}$ | $V_h^{k+1} \times Q_h^{k} \times \hat{V}_h^k$ | $\theta = \tau = \rho^{-1} h^{-1}_K$ | $k + 1$ |
| WG  | $\|\tilde{p}_h\|_{0,\rho, h}$ | $V_h^{k+1} \times Q_h^{k} \times \hat{Q}_h^k$ | $\theta = \eta = \rho h_K$ | $k + 1$ |
| WG  | $\|\tilde{\pi}_h\|_{1,\rho, h}$ | $V_h^{k+1} \times Q_h^{k, RT} \times \hat{Q}_h^k$ | $\theta = \eta = \rho^{-1} h^{-1}_K$ | $k + 1$ |

Table 1: Convergence of HDG and WG

5 Relationships between HDG, primal conforming methods and between WG, mixed conforming method

In this section, as an application of the uniform stability results, we shall discuss the relationships between HDG and primal conforming methods and the relationship between WG and mixed conforming method. The proof for the results of this section are also shown in [20]. For the convenience of reading and the self-consistency of the paper, we show the proof here again. Further, the numerical results verifying the results of this section can be found in [20].
5.1 Primal conforming methods as the limiting case of HDG methods

For a given mesh, consider the $H^1$-conforming subspace $V_h^c = V_h \cap H^1_0(\Omega) \subset V_h$, then the primal conforming methods in the variational form are written as: Find $(u_h^c, p_h^c) \in V_h^c \times Q_h$ such that

$$
\begin{cases}
(cp_h^c, q_h)_{\kappa_h} + \langle \nabla u_h^c, q_h \rangle_{\kappa_h} = \langle g_1, q_h \rangle_{\kappa_h} + \langle g_2, q_h \cdot n \rangle_{\partial \kappa_h} & \forall q_h \in Q_h, \\
-(p_h^c, \nabla v_h^c)_{\kappa_h} = \langle f, v_h^c \rangle_{\kappa_h} & \forall v_h^c \in V_h^c,
\end{cases}
$$

(5.1)

where $g_1 = 0$ and $g_2 = 0$ when applied to the Poisson equation (1.2).

We try to prove that the HDG methods (3.8) with the stabilization parameter $\tau = \rho^{-1} h^{-1}$ converge to primal conforming methods (5.1) when $\rho \to 0$.

First, by $\nabla V_h^c \subset \nabla_h V_h \subset Q_h$, the well-posedness of the primal conforming methods (cf. [21]) implies that

$$
\|p_h^c\| + \|u_h^c\| \leq C_P \left( \|f\|_{-1,h} + \sup_{q_h \in Q_h} \frac{(g_1, q_h)_{\kappa_h} + \langle g_2, q_h \cdot n \rangle_{\partial \kappa_h}}{\|q_h\|} \right),
$$

(5.2)

where $\|f\|_{-1,h} = \sup_{v_h^c \neq 0, v_h^c \in V_h^c} \frac{(f, v_h^c)_{\kappa_h}}{\|v_h^c\|_1}$.

Recall that the space define on $E_h$ (see (2.5)) of HDG methods is given by

$$
\hat{V}_h = \{ \hat{v}_h : \hat{v}_h|_e \in \hat{V}(e), \forall e \in E_h^i, \hat{v}_h|_{E^o} = 0 \}.
$$

We make the following assumption on the finite element spaces of stabilized hybrid mixed methods.

**Assumption 5.1** Assume that the spaces $Q_h$, $V_h$ and $\hat{V}_h$ satisfy

1. $\nabla_h V_h \subset Q_h$;
2. $\{V_h\}_e \subset \hat{V}(e), \forall e \in E_h^i$;
3. There exists a constant $C_P^I$ independent of $h$, such that for any $u_h \in V_h$,

$$
\inf_{u_h^c \in V_h^c} (\|(u_h^c - u_h\| + \|\nabla_h(u_h^c - u_h)\|) \leq C_P^I \sum_{e \in E_h} h_e^{-1/2}\|u_h\|_{0,e},
$$

(5.3)

where $V_h^c = V_h \cap H^1(\Omega)$.

We note that the first assumption in Assumption 5.1 ensures the well-posedness of the primal conforming methods (5.1). The following example satisfies Assumption 5.1 (see the conforming relatives in [22, 21]).

**Example 5.2** $Q_h = Q_h^k$, $V_h = V_h^{k+1}$, $\hat{V}(e) = P_{k+1}(e)$, for $k \geq 0$. 

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For any given $\tau = \rho^{-1}h_K^{-1}$, we rewrite the HDG methods (3.8) in the variational form as: Find $(p_h^\tau, u_h^\tau, \tilde{u}_h^\tau) \in Q_h \times V_h \times \hat{V}_h$ such that for any $(q_h, v_h, \hat{v}_h) \in Q_h \times V_h \times \hat{V}_h$

\[
\begin{align*}
\langle c(p_h^\tau, q_h)_{\tau_h} - (u_h^\tau, \text{div} q_h)_{\tau_h} + \langle \tilde{u}_h^\tau, q_h \cdot n \rangle_{\partial \tau_h} = 0, \\
\langle \text{div} p_h^\tau, v_h \rangle_{\tau_h} - \langle p_h^\tau \cdot n, \hat{v}_h \rangle_{\partial \tau_h} + \rho^{-1}h_K^{-1}(u_h^\tau - \tilde{u}_h^\tau), v_h - \hat{v}_h \rangle_{\partial \tau_h} = (f, v_h)_{\tau_h}.
\end{align*}
\]

(5.4)

**Theorem 5.3** Under the Assumption 5.1, the HDG methods (3.8) with $\tau = \rho^{-1}h_K^{-1}$ converge to the primal conforming methods (5.1) as $\rho \to 0$. More precisely, we have

\[
\|p_h^\tau - p_h^C\| + \|u_h^\tau - u_h^C\|_{1,h} \leq C_{d,3}\rho^{1/2}\|f\|_{-1,\rho,h},
\]

(5.5)

where $C_{d,3}$ is independent of both mesh size $h$ and $\rho$, and $\|f\|_{-1,\rho,h} = \sup_{\hat{v}_h \in \hat{V}_h}(f, \hat{v}_h)_{\tau_h}$.

**Proof.** From the assumption $\{V_h\}_e \subset \hat{V}(e)$, by taking $v_h = v_h^e$ and $\hat{v}_h|_e = v_h^e|_e$ in (5.4) and integrating by parts, we see that

\[
-(p_h^\tau, \nabla v_h^e)_{\tau_h} = (f, v_h^e)_{\tau_h} \quad \forall v_h^e \in V_h^e.
\]

(5.6)

Subtracting (5.1) from the first equation of (5.4) and (5.6), we have

\[
\begin{align*}
\langle c(p_h^\tau - p_h^C), q_h \rangle_{\tau_h} + (\nabla u_h^\tau - \nabla u_h^C, q_h)_{\tau_h} &= \langle u_h^\tau - \tilde{u}_h^\tau, q_h \cdot n \rangle_{\partial \tau_h} \quad \forall q_h \in Q_h, \\
-(p_h^\tau - p_h^C, \nabla v_h^C)_{\tau_h} &= 0 \quad \forall v_h^C \in V_h^C.
\end{align*}
\]

(5.7)

Again, for any $u_h^\tau \in V_h^e$, we have

\[
\begin{align*}
\langle c(p_h^\tau - p_h^C), q_h \rangle_{\tau_h} + (\nabla u_h^\tau - \nabla u_h^C, q_h)_{\tau_h} &= \langle u_h^\tau - \tilde{u}_h^\tau, q_h \cdot n \rangle_{\partial \tau_h} + (\nabla u_h^\tau - \nabla u_h^C, q_h)_{\tau_h} \quad \forall q_h \in Q_h, \\
-(p_h^\tau - p_h^C, \nabla v_h^C)_{\tau_h} &= 0 \quad \forall v_h^C \in V_h^C.
\end{align*}
\]

(5.8)

Because $p_h^\tau - p_h^C \in Q_h$ and $v_h^C - u_h^C \in V_h^C$, using (5.2), trace inequality, inverse inequality and Cauchy inequality, we obtain

\[
\|p_h^\tau - p_h^C\| + \|u_h^\tau - u_h^C\|_1 \leq C_p \sup_{q_h \in Q_h} \frac{\langle u_h^\tau - \tilde{u}_h^\tau, q_h \cdot n \rangle_{\partial \tau_h} + (\nabla u_h^\tau - \nabla u_h^C, q_h)_{\tau_h}}{\|q_h\|} 
\]

\[
\lesssim |u_h^\tau - u_h^C|_{1,h} + \langle h^{-1}(u_h^\tau - \tilde{u}_h^\tau), u_h^\tau - \tilde{u}_h^\tau \rangle_{\partial \tau_h}^{1/2}.
\]

(5.9)

Noting that $\{V_h\}_e \subset \hat{V}(e)$, and

\[
\langle u_h^\tau - \tilde{u}_h^\tau, u_h^\tau - \tilde{u}_h^\tau \rangle_{\partial \tau_h} = 2(\{u_h^\tau - \tilde{u}_h^\tau\}, \{u_h^\tau - \tilde{u}_h^\tau\})_{\tau_h} + \frac{1}{2}(\|u_h^\tau\|, \|u_h^\tau\|)_{\tau_h}.
\]

(5.10)
Therefore, Assumption 5.1, (5.9), and (5.10) imply that
\[
\|p_h^r - p_h^c + u_h^r - u_h^c\|_{1,h} \leq \inf_{u_h^r \in V_h^c} (\|p_h^r - p_h^c\| + \|u_h^r - u_h^c\| + \|u_h^r - u_h^c\|_{1,h}) \\
\lesssim (h^{-1}(u_h^r - \hat{u}_h^r), u_h^r - \hat{u}_h^r)_{\partial T_h} + \inf_{u_h^r \in V_h^c} \|u_h^r - u_h^c\|_{1,h} \\
\lesssim (h^{-1}(u_h^r - \hat{u}_h^r), u_h^r - \hat{u}_h^r)_{\partial T_h} + \sum_{e \in E_h} h_e^{-1/2}\|u_h\|_{0,e} \\
\lesssim (h^{-1}(u_h^r - \hat{u}_h^r), u_h^r - \hat{u}_h^r)_{\partial T_h} \\
\lesssim \rho^{1/2}\|f\|_{-1,\rho,h},
\]
where Corollary 3.3 was used in the last step. 

**Remark 5.4** From the definition of \(\|\cdot\|_{1,\rho,h}\), when \(\rho \lesssim 1\), we have
\[
\inf_{\hat{v}_h \in V_h} \|\hat{v}_h\|_{1,\rho,h}^2 = \inf_{\hat{v}_h \in V_h} (\nabla v_h, \nabla v_h)_{T_h} + \sum_{K \in T_h} \rho^{-1} h_K^{-1}(v_h - \hat{v}_h, v_h - \hat{v}_h)_{\partial K}
\approx (\nabla v_h, \nabla v_h)_{T_h} + \rho^{-1} \sum_{e \in E_h} h_e^{-1}\|v_h\|_{0,e} \gtrsim \|v_h\|_{1,h}.
\]
Hence, when \(\rho \lesssim 1\),
\[
\|f\|_{-1,\rho,h} = \sup_{\hat{v}_h \in V_h} \frac{(f, v_h)_{T_h}}{\|\hat{v}_h\|_{1,\rho,h}} = \sup_{v_h \in V_h} \frac{(f, v_h)_{T_h}}{\inf_{\hat{v}_h \in V_h} \|\hat{v}_h\|_{1,\rho,h}} \lesssim \sup_{v_h \in V_h} \frac{(f, v_h)_{T_h}}{\|v_h\|_{1,h}} \lesssim \|f\|,
\]
which means that the solutions of HDG methods converge to those of primal conforming methods with order \(\rho^{1/2}\) at least.

### 5.2 Mixed conforming methods as the limiting case of WG methods

For a given mesh, consider the \(H(\text{div})\)-conforming subspace \(Q_h^c := Q_h \cap H(\text{div}, \Omega) \subset Q_h\), the mixed conforming methods in variational form are written as: Find \((p_h^c, u_h^c) \in Q_h^c \times V_h\) such that
\[
\begin{cases}
(c p_h^c, q_h^c)_{T_h} - (u_h^c, \text{div} q_h^c)_{T_h} = (g_1, q_h^c)_{T_h} & \forall q_h^c \in Q_h^c, \\
(\text{div} p_h^c, v_h)_{T_h} = (f, v_h)_{T_h} + (g_2, v_h)_{\partial T_h} & \forall v_h \in V_h,
\end{cases}
\tag{5.11}
\]
where \(g_1 = 0\) and \(g_2 = 0\) when applied to the Poisson equation (1.1).

We will now try to prove that WG methods (3.16) with \(\eta = \rho^{-1} h_K^{-1}\) converge to mixed conforming methods (5.11) when \(\rho \to 0\).

First, by \(V_h \subset \text{div} Q_h^c \subset \text{div} Q_h \subset V_h\), the well-posedness of the mixed conforming methods (cf. [23, 24]) implies that
\[
\|p_h^c\|_{H(\text{div})} + \|v_h^c\| \leq C_M \left(\|f\| + \sup_{q_h^c \in Q_h^c} \frac{(g_1, q_h^c)_{T_h}}{\|q_h^c\|_{H(\text{div})}} + \sup_{v_h \in V_h} \frac{(g_2, v_h)_{\partial T_h}}{\|v_h\|} \right),
\tag{5.12}
\]
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Recall that the spaces defined on $\mathcal{E}_h$ (see (2.5)) of WG methods are given by

$$
\hat{Q}_h = \{\hat{p}_h : \hat{p}_h|_e \in \hat{Q}(e) n_e, \forall e \in \mathcal{E}_h\}, \quad \hat{Q}_h = \{\hat{p}_h : \hat{p}_h|_e \in \hat{Q}(e), \forall e \in \mathcal{E}_h\}.
$$

We make the following assumption on the finite element spaces of WG methods.

**Assumption 5.5** Assume that the spaces $Q_h, \hat{Q}_h$ and $V_h$ satisfy

1. $\text{div}_h Q_h = V_h$;
2. $\|Q_h\|_e \subset \hat{Q}(e), \forall e \in \mathcal{E}_h$;
3. There exists a constant $C_M$ independent of $h$, such that for any $p_h \in Q_h$,

$$
\inf_{p_h^i \in Q^1_h}(\|p^I_h - p_h\| + \|\text{div}_h(p^I_h - p_h)\|) \leq C_M \sum_{e \in \mathcal{E}_h} h_e^{-1/2}\|p_h\|_{0,e},
$$

where $Q^1_h = Q_h \cap H(\text{div}; \Omega)$.

We note that the first assumption in Assumption 5.5 ensures well-posedness of the mixed conforming methods (5.11). Several examples are given below.

**Example 5.6** Raviart-Thomas type: $Q_h = Q_h^{k,RT}, \hat{Q}(e) = P_k(e), V_h = V_h^k$, for $k \geq 0$.

**Example 5.7** Brezzi-Douglas-Marini type: $Q_h = Q_h^{k+1,}, \hat{Q}(e) = P_{k+1}(e), V_h = V_h^k$, for $k \geq 0$.

**Lemma 5.8** If we choose the spaces as in Example 5.7 or Example 5.6, then Assumption 5.5 holds.

**Proof.** We only sketch the proof of (5.13) in Assumption 5.5. Denote the set of degrees of freedom of RT or BDM element by $D$, see [23, 24]. We then define $p^I_h$ as

$$
d(p^I_h) = \frac{1}{|T_d|} \sum_{K \in T_d} d(p_h|_T) \quad \forall d \in D,
$$

where $T_d$ denotes the set of elements that share the degrees of freedom $d$ and $|T_d|$ denotes the cardinality of this set. By the standard scaling argument,

$$
\sum_{K \in T_h} \|p^I_h - p_h\| \lesssim \sum_{e \in \mathcal{E}_h} h_e^{1/2}\|p_h\|_{0,e}.
$$

Then (5.13) follows from the inverse inequality. 

For any given $\eta = \rho^{-1} h_K^{-1}$, we rewrite the WG methods (3.16) in the variational form as: Find $(p^0_h, u^0_h, \hat{p}^0_h) \in Q_h \times V_h \times \hat{Q}_h$ such that for any $(q_h, v_h, \hat{q}_h) \in Q_h \times V_h \times \hat{Q}_h$

$$
\begin{align*}
\langle \rho^{-1}(h_K^{-1}(p^0_h - \hat{p}_h)) \cdot n, (q_h - \hat{q}_h) \cdot n \rangle_{\partial T_h} + \langle \nabla u_h, q_h \rangle_{\partial T_h} - \langle u_h, q_h \cdot n \rangle_{\partial T_h} &= 0, \\
-\langle p^0_h \cdot \nabla v_h \rangle_{\partial T_h} + \langle \hat{p}^0_h \cdot n, v_h \rangle_{\partial T_h} &= (f, v_h).
\end{align*}
$$

(5.14)
Theorem 5.9 Under the Assumption 5.5, WG WG methods (3.16) converge to the mixed conforming methods (5.11) as \( \rho \to 0 \) with \( \eta = \rho^{-1} h^{-1} \). More precisely, we have
\[
\|p_h^\eta - p_h^\xi\| \cdot H(\text{div}) + \|u_h^\eta - u_h^\xi\| \leq C_{\text{w,3}} \rho^{1/2} \|f\|,
\]
where \( C_{\text{w,3}} \) is independent of both mesh size \( h \) and \( \rho \).

**Proof.** From the assumption \( \{Q_h\}_e \subset \hat{Q}(e) \), by taking \( q_h = q_h^\xi \) and \( \hat{q}_h|_e = (q_h^\xi \cdot n_e)n_e \) in (5.14) and integrating by parts, we see that \((p_h^\eta, u_h^\eta)\) satisfies
\[
(c(p_h^\eta, q_h^\eta))_{\mathcal{T}h} - (u_h^\eta - u_h^\xi, \text{div} q_h^\eta)_{\mathcal{T}h} = 0 \quad \forall q_h^\eta \in Q_h^\xi.
\]
Subtracting (5.11) from (5.16) and the second equation of (5.14), we have
\[
\begin{cases}
(c(p_h^\eta - p_h^\xi, q_h^\eta))_{\mathcal{T}h} - (u_h^\eta - u_h^\xi, \text{div} q_h^\eta)_{\mathcal{T}h} = 0 & \forall q_h^\eta \in Q_h^\xi, \\
(\text{div}(p_h^\eta - p_h^\xi), v_h)_{\mathcal{T}h} = ((p_h^\eta - \hat{p}_h^\eta) \cdot n, v_h)_{\partial \mathcal{T}h} & \forall v_h \in V_h.
\end{cases}
\]
Noting that \( p_h^\eta \notin Q_h^\xi \), we have that, for any \( p_h^I \in Q_h^\xi \),
\[
\begin{cases}
(c(p_h^I - p_h^\xi, q_h^\eta))_{\mathcal{T}h} - (u_h^I - u_h^\xi, \text{div} q_h^\eta)_{\mathcal{T}h} = (c(p_h^I - p_h^\eta, q_h^\eta))_{\mathcal{T}h} & \forall q_h^\eta \in Q_h^\xi, \\
(\text{div}(p_h^I - p_h^\eta), v_h)_{\mathcal{T}h} = (\text{div}(p_h^I - p_h^\eta), v_h)_{\partial \mathcal{T}h} & \forall v_h \in V_h.
\end{cases}
\]
Because \((p_h^I - p_h^\eta) \in Q_h^\xi, (u_h^I - u_h^\xi) \in V_h\), by the well-posedness of the mixed conforming methods (5.12), trace inequality, inverse inequality and Cauchy inequality, we have
\[
\|p_h^I - p_h^\xi\| \cdot H(\text{div}) + \|u_h^I - u_h^\xi\|
\leq C_M \left( \sup_{q_h^\eta \in Q_h^\xi} \frac{(c(p_h^I - p_h^\eta, q_h^\eta))_{\mathcal{T}h}}{\|q_h^\eta\| \cdot H(\text{div})} + \sup_{v_h \in V_h} \frac{(\text{div}(p_h^I - p_h^\eta), v_h)_{\partial \mathcal{T}h}}{\|v_h\|} \right)
\leq \|p_h^I - p_h^\eta\| + \|\text{div}_h(p_h^I - p_h^\eta)\| + \langle h^{-1}(p_h^I - \hat{p}_h^\eta) \cdot n, (p_h^I - \hat{p}_h^\eta) \cdot n \rangle_{\partial \mathcal{T}h}^{1/2}.
\]
Hence, by Assumption 5.5 and inverse inequality, we have
\[
\|p_h^\eta - p_h^\xi\| \cdot H(\text{div}) + \|u_h^\eta - u_h^\xi\| \lesssim \langle h^{-1}(p_h^\eta - \hat{p}_h^\eta) \cdot n, (p_h^\eta - \hat{p}_h^\eta) \cdot n \rangle_{\partial \mathcal{T}h}^{1/2} + \inf_{p_h^I \in Q_h^\xi} \left( \|p_h^I - p_h^\eta\| + \|\text{div}_h(p_h^I - p_h^\eta)\| \right)
\lesssim \langle h^{-1}(p_h^\eta - \hat{p}_h^\eta) \cdot n, (p_h^\eta - \hat{p}_h^\eta) \cdot n \rangle_{\partial \mathcal{T}h}^{1/2} + \sum_{e \in \mathcal{E}_h^h} h_e^{-1/2} \|p_h^\eta\|_{0,e}.
\]
From the fact that
\[
\langle (p_h^\eta - \hat{p}_h^\eta) \cdot n, (p_h^\eta - \hat{p}_h^\eta) \cdot n \rangle_{\partial \mathcal{T}h} = 2\langle (p_h^\eta - \hat{p}_h^\eta), (p_h^\eta - \hat{p}_h^\eta) \rangle_{\mathcal{E}_h^h} + \frac{1}{2} \langle (p_h^\eta, (p_h^\eta)_{\mathcal{E}_h^h}\rangle_{\mathcal{E}_h^h},
\]
we obtain
\[
\|p_h^\eta - p_h^\xi\| \cdot H(\text{div}) + \|u_h^\eta - u_h^\xi\| \lesssim \langle h^{-1}(p_h^\eta - \hat{p}_h^\eta) \cdot n, (p_h^\eta - \hat{p}_h^\eta) \cdot n \rangle_{\partial \mathcal{T}h}^{1/2} + \sum_{e \in \mathcal{E}_h^h} h_e^{-1/2} \|p_h^\eta\|_{0,e}
\lesssim \langle h^{-1}(p_h^\eta - \hat{p}_h^\eta) \cdot n, (p_h^\eta - \hat{p}_h^\eta) \cdot n \rangle_{\partial \mathcal{T}h}^{1/2} \lesssim \rho^{1/2} \|f\|,
\]
where we used Corollary 3.9 in the last step. This completes the proof. ■
6 Analysis of HDG and WG

In this section, we present the analysis of HDG and WG methods and hence prove the uniformly well-posed results of the HDG and WG methods provided in Section 3. Namely, we prove the Theorem 3.2 for HDG methods and the Theorem 3.6 for WG methods. That means we need to prove that the HDG and WG methods satisfy the consistency, the uniform continuity and the inf-sup condition uniformly with respect to the corresponding norms.

Lemma 6.1 Both the HDG methods and WG methods are consistent.

Proof. By the verification of (3.23), the proof is obvious.

6.1 Proof for Part 1 of Theorem 3.2

Theorem 6.2 For any $0 < \rho \leq 1$, the bilinear form $A_h((p_h, \tilde{u}_h), (q_h, \tilde{v}_h))$ is uniformly continuous.

Proof. The boundedness of $a_h(p_h, q_h)$ is obvious. Before we discuss the boundedness of $b_h(q_h, \tilde{u}_h)$, by (2.2) and noting $[\hat{u}_h] = 0$, we rewrite $b_h(q_h, \tilde{u}_h)$ as:

$$b_h(q_h, \tilde{u}_h) = -(u_h, \text{div}_h q_h)_{\tau_h} + \sum_{K \in T_h} \langle \hat{u}_h, q_h \cdot n_K \rangle_{\partial K} = -(u_h, \text{div}_h q_h)_{\tau_h} + \sum_{e \in \mathcal{E}_h^i} \langle \hat{u}_h, [q_h] \rangle_e. \quad (6.1)$$

Now we show the boundedness of $b_h(q_h, \tilde{u}_h)$ here. By (6.1) and the definition of $\hat{P}_e$, we have:

$$b_h(q_h, \tilde{u}_h) = -(u_h, \text{div}_h q_h)_{\tau_h} + \sum_{K \in T_h} \langle \hat{u}_h, q_h \cdot n_K \rangle_{\partial K} = -(u_h, \text{div}_h q_h)_{\tau_h} + \sum_{e \in \mathcal{E}_h^i} \langle \hat{u}_h, [q_h] \rangle_e. \quad (6.2)$$

$$\leq \|\text{div}_h q_h\|_{\infty} \|u_h\| + \left(\rho^{-1} \sum_{e \in \mathcal{E}_h^i} h_e^{-1} \langle \hat{P}_e([q_h]), \hat{P}_e([q_h]) \rangle_e \right)^{\frac{1}{2}} \left(\rho \sum_{e \in \mathcal{E}_h^i} h_e \langle \hat{u}_h, \hat{u}_h \rangle_e \right)^{\frac{1}{2}} \quad (6.3)$$

$$\leq \|q_h\|_{\text{div}, \rho, h} \|\tilde{u}_h\|_{0, \rho, h}. \quad (6.4)$$

which proves the boundedness of $b_h(q_h, \tilde{u}_h)$.

Next we prove the boundedness of $c_h(\tilde{u}_h, \tilde{v}_h)$.

By the Cauchy inequality, we have:

$$|c_h(\tilde{u}_h, \tilde{v}_h)| = \rho \sum_{K \in T_h} h_K \langle u_h - \tilde{u}_h, v_h - \tilde{v}_h \rangle_{\partial K} \leq \left( \rho \sum_{K \in T_h} h_K \langle u_h - \tilde{u}_h, u_h - \tilde{u}_h \rangle_{\partial K} \right)^{\frac{1}{2}} \left( \rho \sum_{K \in T_h} h_K \langle v_h - \tilde{v}_h, v_h - \tilde{v}_h \rangle_{\partial K} \right)^{\frac{1}{2}}. \quad (6.5)$$
By the trace inequality, inverse inequality, and noting that $0 < \rho \leq 1$, we have:

\[
\rho \sum_{K \in \mathcal{T}_h} h_K \langle u_h - \hat{u}_h, u_h - \hat{u}_h \rangle_{\partial K} \quad (6.6)
\]

\[
= \rho \sum_{K \in \mathcal{T}_h} h_K \left( \langle u_h, u_h \rangle_{\partial K} - 2 \langle u_h, \hat{u}_h \rangle_{\partial K} + \langle \hat{u}_h, \hat{u}_h \rangle_{\partial K} \right) \quad (6.7)
\]

\[
\leq 2 \rho \sum_{K \in \mathcal{T}_h} h_K \left( \langle u_h, u_h \rangle_{\partial K} + \langle \hat{u}_h, \hat{u}_h \rangle_{\partial K} \right) \quad (6.8)
\]

\[
\leq 2 \rho \sum_{K \in \mathcal{T}_h} h_K \langle u_h, u_h \rangle_{\partial K} + 4 \rho \sum_{K \in E_h^+} h_e \langle \hat{u}_h, \hat{u}_h \rangle_e \quad (6.9)
\]

\[
\leq 2 \sum_{K \in \mathcal{T}_h} (u_h, u_h)_K + 4 \rho \sum_{K \in E_h^+} h_e \langle \hat{u}_h, \hat{u}_h \rangle_e \leq 4 \|\hat{u}_h\|_{0, \rho, h}^2. \quad (6.10)
\]

Similarly, we have:

\[
\rho \sum_{K \in \mathcal{T}_h} h_K \langle v_h - \hat{v}_h, v_h - \hat{v}_h \rangle_{\partial K} \lesssim \|\hat{v}_h\|_{0, \rho, h}^2. \quad (6.11)
\]

Hence, we obtain $|c_h(\hat{u}_h, \hat{v}_h)| \lesssim \|\hat{u}_h\|_{0, \rho, h} \|\hat{v}_h\|_{0, \rho, h}$.  

We denote

\[
\text{Ker}(B) = \{q_h \in \mathcal{Q}_h : b_h(q_h, \hat{u}_h) = 0, \forall \hat{u}_h \in \hat{V}_h\}. \quad (6.12)
\]

Then, we have the coercivity of $a_h(\cdot, \cdot)$ on the Ker($B$) as follows:

**Theorem 6.3** Assume that $\text{div}_h \mathcal{Q}_h \subset \mathcal{V}_h$, then

\[
a_h(p_h, p_h) \geq \|p_h\|_{\text{div}, \rho, h}^2, \forall p_h \in \text{Ker}(B). \quad (6.13)
\]

**Proof.** Since

\[
\text{Ker}(B) = \{q_h \in \mathcal{Q}_h : b_h(q_h, \hat{u}_h) = 0, \forall \hat{u}_h \in \hat{V}_h\}, \quad (6.14)
\]

then by (6.1) and under the assumption that $\text{div}_h \mathcal{Q}_h \subset \mathcal{V}_h$, we have:

\[
\text{Ker}(B) = \{q_h \in \mathcal{Q}_h : - (u_h, \text{div} q_h)_{\mathcal{T}_h} + \langle \hat{u}_h, [q_h] \rangle_{E_h^+} = 0, \forall \hat{u}_h \in \hat{V}_h\} \quad (6.15)
\]

\[
= \{q_h \in \mathcal{Q}_h : - (u_h, \text{div} q_h)_{\mathcal{T}_h} + \langle \hat{u}_h, \hat{P}_h([q_h]) \rangle_{E_h^+} = 0, \forall \hat{u}_h \in \hat{V}_h\} \quad (6.16)
\]

\[
= \{q_h \in \mathcal{Q}_h : \text{div}_h q_h = 0, \hat{P}_h([q_h]) = 0\}. \quad (6.17)
\]

Hence, by the definition of $\|q_h\|_{\text{div}, \rho, h}$, we obtain $a_h(p_h, p_h) \geq \|p_h\|_{\text{div}, \rho, h}^2, \forall p_h \in \text{Ker}(B)$.  

**Lemma 6.4** Given the edges (faces) $e_1, e_2, \cdots, e_{d+1}$ of the simplex $K$ and functions $q \in L^2(K)$ and $\hat{\zeta}_i \in L^2(e_i), i = 1, \cdots, d+1$, there is a unique function $z \in \mathcal{P}_r(K) \oplus x \mathcal{P}_{r-1}(K), r \geq 0$ such that,

\[
(z - q, p)_K = 0, \forall p \in \mathcal{P}_{r-1}(K), \quad (6.18)
\]

\[
(z \cdot n_i - \hat{\zeta}_i, \hat{v})_{e_i} = 0, \forall \hat{v} \in \mathcal{P}_r(e_i), i = 1, \cdots, d+1, \quad (6.19)
\]

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where \( n_i \) is the outward normal unit vector of \( e_i \). Moreover:

\[
\|z\|_{0,K} \leq C_{d,r} \left( \|q\|_{0,K} + h^{1/2} \sum_{i=1}^{d+1} \|\hat{\zeta}_i\|_{0,e_i} \right),
\]

(6.20)

where \( C_{d,r} \) depends only on \( d, r \), and the shape regular constant.

**Proof.** Similar to the definition of the local Raviart-Thomas finite element, the well-posedness of \( z \) is obvious. Then, from a simple scaling argument, the estimate is desired. \( \square \)

Let \( \bar{\mathbf{P}}_{r-1}(K) = (\bar{\mathbf{P}}_{r-1}(K))^d \) be the vector homogeneous polynomials of degree \( r - 1 \). Similar to Lemma 6.4, we also have:

**Lemma 6.5** Given the edges (faces) \( e_1, e_2, \cdots, e_{d+1} \) of the simplex \( K \) and functions \( q \in L^2(K) \) and \( \hat{\zeta}_i \in L^2(e_i), i = 1, \cdots, d + 1 \), there is a unique function \( z \in \mathcal{P}_r(K), r \geq 1 \) such that,

\[
(z - q, p)_K = 0, \quad \forall \ p \in \mathcal{P}_{r-2}(K) \oplus \mathcal{S}_{r-1}(K),
\]

(6.21)

\[
(z \cdot n_i - \hat{\zeta}_i, v)_{e_i} = 0, \quad \forall \ v \in \mathcal{P}_r(e_i), i = 1, \cdots, d + 1,
\]

(6.22)

where \( n_i \) is the outward normal unit vector of \( e_i \) and \( \mathcal{S}_{r-1}(K) = \{ v \in \bar{\mathbf{P}}_{r-1}(K) : x \cdot v = 0 \} \). Moreover:

\[
\|z\|_{0,K} \leq C_{d,r} \left( \|q\|_{0,K} + h^{1/2} \sum_{i=1}^{d+1} \|\hat{\zeta}_i\|_{0,e_i} \right),
\]

(6.23)

where \( C_{d,r} \) depends only on \( d, r \), and the shape regular constant.

**Proof.** Similar to the definition of the local Brezzi-Douglas-Marini (BDM) finite element, the well-posedness of \( z \) is obvious. From a simple scaling argument, the estimate is desired. \( \square \)

Now we consider the inf-sup condition of \( b_h(q_h, \bar{u}_h) \).

**Theorem 6.6** For \( k \geq 1 \), assume that \( Q_h = Q_k^h, V_h = V_k^{h-1} \) and \( \hat{V}_h = \hat{V}_h^r \), where \( 0 \leq r \leq k \), or \( Q_h = Q_k^{h,RT}, V_h = V_k^{h-1} \) and \( \hat{V}_h = \hat{V}_h^r \), where \( 0 \leq r \leq k - 1 \), then we have:

\[
\inf_{\bar{u}_h \in \hat{V}_h} \sup_{q_h \in Q_h} \frac{b_h(q_h, \bar{u}_h)}{\|q_h\|_{\text{div},\rho,h} \|\bar{u}_h\|_{0,\rho,h}} \geq \beta_2,
\]

(6.24)

where \( \beta_2 > 0 \) is a constant independent of \( \rho \) and mesh size \( h \).

**Proof.** Here we only give the proof under the assumption \( Q_h = Q_k^h, V_h = V_k^{h-1} \) and \( \hat{V}_h = \hat{V}_h^r \). The other case is similar.

For any \( \bar{u}_h \in \hat{V}_h \), namely for any \( u_h \in V_h, \bar{u}_h \in \hat{V}_h \), we need to construct a \( q_h \in Q_h \), such that:

\[
b_h(q_h, \bar{u}_h) = \|\bar{u}_h\|_{0,\rho,h}^2 \quad \text{and} \quad \|q_h\|_{\text{div},\rho,h} \lesssim \|\bar{u}_h\|_{0,\rho,h}.
\]

(6.25)
We define $z_h$ piecewisely on any $K$, namely $z_h \in Q_h, z_h|_K = z_K$ and $z_K \in \mathcal{P}_r(K)$ is defined as follows

\[
(z_K, p)_K = 0, \quad \forall \ p \in \mathcal{P}_{r-2}(K) \oplus S_{r-1}(K),
\]

\[
(z_K \cdot n_i - \rho h e_i \hat{u}_h, \hat{v})_{e_i} = 0, \quad \forall \ \hat{v} \in \mathcal{P}_r(e_i), i = 1, \cdots, d + 1.
\]  

(6.26) (6.27)

Then, by Lemma 6.5, we have:

\[
\|z_h\|^2 \leq \rho^2 |K| h^2 \sum_{e \in E_h^i} \|\hat{u}_h\|_{0, e}^2.
\]  

(6.28)

In fact, we also have that for any $e \in E_h^i$,

\[
[z_h]_e = \rho h e_i \hat{u}_h|_e.
\]  

(6.29)

Next, noting that $\text{div} z_K \in \mathcal{P}_{r-1}(K)$, then for $-\text{div} z_h - u_h$, there exists $r_h \in H(\text{div}, \Omega) \cap Q_h$ such that:

\[
\text{div} r_h = -\text{div} z_h - u_h,
\]  

(6.30)

\[
\|r_h\| + \|\text{div} r_h\| \leq \|\text{div} z_h - u_h\|.
\]  

(6.31)

Now we define $q_h = z_h + r_h$, noting that $r_h \in H(\text{div}, \Omega) \cap Q_h$, namely for any $e \in E_h^i, |r_h|_e = 0$; hence, for any $e \in E_h^i$:

\[
[q_h]_e = [r_h]_e + [z_h]_e = [z_h]_e = \rho h e_i \hat{u}_h|_e
\]  

(6.32)

and

\[
\text{div} q_h = \text{div} z_h + r_h = -u_h.
\]  

(6.33)

Substituting (6.32) and (6.33) into $b_h(q_h, \hat{u}_h)$, we immediately obtain:

\[
b_h(q_h, \hat{u}_h) = -(u_h, \text{div} q_h)_{\Omega} + \sum_{e \in E_h^i} \langle \hat{u}_h, [q_h]_e \rangle = \|\hat{u}_h\|_{0, p, h}^2.
\]

Finally, by (6.28), (6.30), inverse inequality, and (6.29), noting that for any $e \in E_h^i, |r_h|_e = 0$, we have:

\[
\|q_h\|_{\text{div}, p, h} \leq \|r_h + z_h\|_{\text{div}} \leq \|r_h\|_{\text{div}} + \|z_h\|_{\text{div}} \leq \|r_h\| + \|\text{div} r_h\| + \|z_h\|_{\text{div}}
\]

\[
\lesssim \|z_h\| + h^{-1} \|z_h\| + \left( \rho^{-1} \sum_{e \in E_h^i} h_e^{-1} \langle\hat{P}_e([z_h]), \hat{P}_e([z_h])\rangle \right)^{1/2}
\]

(6.34)

\[
\lesssim \|z_h\| + (1 + h^{-1}) \|z_h\| + \left( \rho^{-1} \sum_{e \in E_h^i} h_e^{-1} \langle[z_h], [z_h]\rangle \right)^{1/2}
\]

(6.35)

\[
= \|z_h\| + (1 + h^{-1}) \left( h \sum_{e \in E_h^i} \rho^2 h_e^2 \|\hat{u}_h\|_{0, e}^2 \right)^{1/2} + \left( \rho^{-1} \sum_{e \in E_h^i} h_e^{-1} \langle \rho h e_i \hat{u}_h, \rho h e_i \hat{u}_h \rangle \right)^{1/2}
\]

(6.36)

\[
\lesssim \|z_h\| + \left( \rho \sum_{e \in E_h^i} h_e \langle \hat{u}_h, \hat{u}_h \rangle \right)^{1/2} \lesssim \|\hat{u}_h\|_{0, p, h}.
\]

(6.37)

Therefore, we obtain the proof. ■
Remark 6.7 In fact, we also can choose $\tau = 0$ in (3.9) and for any $\hat{v} \in \hat{V}_h$, we define:

$$\|\hat{v}_h\|^2_{0,\rho,h} = (v_h, v_h)_{\tau_h} + \sum_{e \in E_h} h_e \langle \hat{v}_h, \hat{v}_h \rangle_e.$$

Define norms for $p_h \in Q_h$ as follows:

$$\|p_h\|^2_{\text{div},\rho,h} = \left(\frac{c}{p_h}, p_h\right)_{\tau_h} + \left(\frac{\text{div} p_h, \text{div} p_h}{\tau_h} + \sum_{e \in E_h} h_e \langle \hat{P}_e([p_h]), \hat{P}_e([p_h]) \rangle_e, \right.$$

where $\hat{P}_e : L^2(e) \to \hat{V}(e)$ still is the $L^2$ projection. Then, we can get the stability result by a similar proof.

6.2 Proof for Part 2 of Theorem 3.2

Next, we prove part 2 of Theorem 3.2. The uniform boundedness of $A_h((p_h, \tilde{u}_h), (q_h, \tilde{v}_h))$ is obvious. The uniform inf-sup condition for $A_h((p_h, \tilde{u}_h), (q_h, \tilde{v}_h))$ is as follows:

Theorem 6.8 Assume $\nabla h V_h \subset Q_h$, then there exists a positive constant $\rho_0$ that only depends on the shape regularity of the mesh, such that for any $0 < \rho \leq \rho_0$, we have:

$$\inf_{(p_h, \tilde{u}_h) \in Q_h \times \hat{V}_h} \sup_{q_h, \tilde{v}_h \in Q_h \times \hat{V}_h} \frac{A_h((p_h, \tilde{u}_h), (q_h, \tilde{v}_h))}{(\|\tilde{u}_h\|_{\text{1},\rho,h} + \|p_h\|)(\|\tilde{v}_h\|_{\text{1},\rho,h} + \|q_h\|)} \geq \beta_3,$$  \hspace{1cm} (6.38)

where $\beta_3 > 0$ is a constant independent of $\rho$ and mesh size $h$.

Proof. For any given $(p_h, \tilde{u}_h) \in Q_h \times \hat{V}_h$, since $\nabla h V_h \subset Q_h$, we choose $q_h = p_h + \nabla h u_h$ and $\tilde{v}_h = -\tilde{u}_h$, and then we have the following boundedness of $q_h$ and $\tilde{v}_h$ by $q_h$ and $\tilde{v}_h$

$$\|q_h\|^2 = (c(p_h + \nabla h u_h), p_h + \nabla h u_h)_{\tau_h} \leq 2(\|p_h\|^2 + \|\nabla h u_h\|^2), \hspace{1cm} (6.39)$$

and

$$\|\tilde{v}_h\|_{\text{1},\rho,h} = \|\tilde{u}_h\|_{\text{1},\rho,h} = \|\tilde{u}_h\|_{\text{1},\rho,h}.$$

(6.40)
On the other hand,
\[
A_h((p_h, \tilde{v}_h), (q_h, \tilde{v}_h)) \\
= (p_h, q_h)_{\mathcal{T}_h} + (\nabla_h u_h, q_h)_{\mathcal{T}_h} - (u_h - \tilde{u}_h, q_h \cdot n)_{\partial \mathcal{T}_h} \\
+ (\nabla_h v_h, p_h)_{\mathcal{T}_h} - (v_h - \tilde{v}_h, p_h \cdot n)_{\partial \mathcal{T}_h} - \rho^{-1} \sum_{K \in \mathcal{T}_h} h_K^{-1} \langle u_h - \tilde{u}_h, v_h - \tilde{v}_h \rangle_{\partial K} \\
= (p_h, p_h + \nabla_h u_h)_{\mathcal{T}_h} + (\nabla_h u_h, p_h + \nabla_h u_h)_{\mathcal{T}_h} - (u_h - \tilde{u}_h, q_h \cdot n)_{\partial \mathcal{T}_h} \\
+ (-\nabla_h u_h, p_h)_{\mathcal{T}_h} - ((u_h - \tilde{u}_h), (p_h \cdot n))_{\partial \mathcal{T}_h} + \rho^{-1} \sum_{K \in \mathcal{T}_h} h_K^{-1} \langle u_h - \tilde{u}_h, u_h - \tilde{u}_h \rangle_{\partial K} \\
= (p_h, p_h)_{\mathcal{T}_h} + (c p_h, \nabla_h u_h)_{\mathcal{T}_h} + (\nabla_h u_h, \nabla_h u_h)_{\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} (\langle u_h - \tilde{u}_h \rangle, \langle p_h - q_h \rangle \cdot n_K)_{\partial K} \\
+ \rho^{-1} \sum_{K \in \mathcal{T}_h} h_K^{-1} \langle u_h - \tilde{u}_h, u_h - \tilde{u}_h \rangle_{\partial K}
\]
where \(C_5\) is a constant independent of \(\rho\) and \(h\).

Now setting \(\epsilon = \frac{1}{4C_5}, \rho_0 = \frac{3}{16C_5}\), then for any \(\rho \leq \rho_0\), we have:
\[
A_h((p_h, \tilde{u}_h), (q_h, \tilde{v}_h)) \geq \frac{1}{2} \| p_h \|^2 + \frac{1}{2} - \epsilon C_5 \| \nabla_h u_h \|^2 + \rho^{-1} (1 - \rho \epsilon^{-1}) \sum_{K \in \mathcal{T}_h} h_K^{-1} \langle u_h - \tilde{u}_h, u_h - \tilde{u}_h \rangle_{\partial K}
\]
\[
\geq \frac{1}{2} \| p_h \|^2 + \| \nabla_h u_h \|^2 + \frac{1}{4} \rho^{-1} \sum_{K \in \mathcal{T}_h} h_K^{-1} \langle u_h - \tilde{u}_h, u_h - \tilde{u}_h \rangle_{\partial K}
\]
\[
\geq \frac{1}{4} (\| p_h \|^2 + \| \tilde{u}_h \|_{W^{1,\infty}}^2).
\]
Hereby, we complete the proof. \(\blacksquare\)

6.3 Proof for Part 1 of Theorem 3.6

By the definition of the norms, the continuity and coercivity of \(a_w(\cdot, \cdot)\) is obvious, namely,

**Theorem 6.9** For any \(0 < \rho \leq 1\), we have:
\[
|a_w(\tilde{p}_h, \tilde{q}_h)| \leq \| \tilde{p}_h \|_{W^{1,\infty}} \| \tilde{q}_h \|_{W^{1,\infty}} \forall \tilde{p}_h \in \tilde{Q}_h, \tilde{q}_h \in \tilde{Q}_h.
\]

\[
a_w(\tilde{p}_h, \tilde{p}_h) \geq \| \tilde{p}_h \|_{W^{1,\infty}}^2 \forall \tilde{p}_h \in \tilde{Q}_h.
\]

Before we prove the boundedness and inf-sup condition of \(b_w(\tilde{p}_h, \tilde{v}_h)\), by identity (2.2) and noting that \([\tilde{p}_h] = 0\), we rewrite \(b_w(\tilde{p}_h, \tilde{v}_h)\) as:
\[
b_w(\tilde{p}_h, \tilde{v}_h) = (p_h, \nabla_h v_h)_{\mathcal{T}_h} - (\tilde{p}_h \cdot n_K, v_h)_{\partial \mathcal{T}_h} = (p_h, \nabla v_h)_{\mathcal{T}_h} - (\tilde{p}_h, [v_h])_{\mathcal{E}_h}.
\]
Then, the boundedness of $b_w(\hat{p}_h, v_h)$ is as follows:

**Theorem 6.10** For any $0 < \rho \leq 1$, and for any $\hat{p}_h \in \mathcal{Q}_h, v_h \in V_h$,

$$b_w(\hat{p}_h, v_h) \leq C_w \|\hat{p}_h\|_{0,h,p} \|v_h\|_{1,h,\rho}. \quad (6.42)$$

**Proof.** Using the Cauchy inequality for (6.41), we obtain:

$$|b_w(\hat{p}_h, v_h)| \leq \|p_h\| \|\nabla_h v_h\| + \sum_{e \in \mathcal{E}_h} \|\hat{p}_h \cdot n_e\|_{0,e} \|\hat{Q}_e([v_h])\|_{0,e}$$

$$\leq \|p_h\| \|\nabla_h v_h\| + \left(\rho \sum_{e \in \mathcal{E}_h} h_e \|\hat{p}_h \cdot n_e\|_{0,e}^2\right)^{1/2} \left(\rho^{-1} \sum_{e \in \mathcal{E}_h} h_e^{-1} \|\hat{Q}_e([v_h])\|_{0,e}^2\right)^{1/2}$$

$$\leq \left(\|p_h\| + \left(\rho \sum_{e \in \mathcal{E}_h} h_e \|\hat{p}_h \cdot n_e\|_{0,e}^2\right)^{1/2}\right) \|v_h\|_{1,h,\rho}. \quad (6.43)$$

Let $K$ be an element that takes $e$ as an edge or flat face. Then, using the trace inequality and the inverse inequality we obtain:

$$h_e \|\hat{p}_h \cdot n_e\|_{0,e}^2 \leq 2h_e \|(\hat{p}_h - p_h) \cdot n_e\|_{0,e}^2 + 2h_e \|p_h \cdot n_e\|_{0,e}^2 \leq C_1 h_e \|(\hat{p}_h - p_h) \cdot n_e\|_{0,e}^2 + \|p_h\|_{0,K}^2. \quad (6.44)$$

Substituting the above inequality (6.44) into (6.43) yields:

$$b_w(\hat{p}_h, v_h) \leq C_w \|\hat{p}_h\|_{0,h,p} \|v_h\|_{1,h,\rho}. \quad (6.45)$$

Hence, the lemma is proved. □

We also have the following uniform inf-sup condition for $b_w(\hat{p}_h, v_h)$:

**Theorem 6.11** Assume $\nabla_h V_h \subset \mathcal{Q}_h$, then for any $0 < \rho \leq 1$, we have:

$$\inf_{v_h \in V_h} \sup_{\hat{p}_h \in \mathcal{Q}_h} \frac{b_w(\hat{p}_h, v_h)}{\|v_h\|_{1,h,\rho} \|\hat{p}_h\|_{0,h,p}} \geq \beta_4, \quad (6.46)$$

where $\beta_4 > 0$ is independent of mesh size $h$ and $\rho$.

**Proof.** Since $\nabla_h V_h \subset \mathcal{Q}_h$, taking $p_h = \nabla_h v_h, \hat{p}_h = -\rho^{-1} h_e^{-1} \hat{Q}_e([v_h]) n_e$ in (6.41), we have

$$b_w(\hat{p}_h, v_h) = (\nabla_h v_h, \nabla_h v_h)_\mathcal{T}_h + \rho^{-1} \sum_{e \in \mathcal{E}_h} h_e^{-1} \|\hat{Q}_e([v_h])\|_{0,e} \|v_h\|_{1,h,\rho}^2.$$}

Noting that $\rho \leq 1$, we obtain:

$$\|\hat{p}_h\|_{0,h,p}^2 = (c \nabla_h v_h, \nabla_h v_h)_{\mathcal{T}_h} + \rho \sum_{K \in \mathcal{T}_h} h_K \|\nabla_h v_h \cdot n_K + \rho^{-1} h_e^{-1} \hat{Q}_e([v_h]) n_e \cdot n_K\|_{0,K}^2$$

$$\leq \beta_4 \left( (c \nabla_h v_h, \nabla_h v_h)_{\mathcal{T}_h} + \rho^{-1} \sum_{e \in \mathcal{E}_h} h_e^{-1} \|\hat{Q}_e([v_h])\|_{0,e}^2 \right) \quad (6.46)$$

$$\leq \beta_4 \|v_h\|_{1,h,\rho}^2.$$ Here, we obtain the desired result. □
6.4 Proof for Part 2 of Theorem 3.6

Next, we prove part 2 of Theorem 2. The uniform boundedness of $A_w((\cdot,\cdot),(\cdot,\cdot))$ is obvious. The uniform inf-sup of $A_w((\cdot,\cdot),(\cdot,\cdot))$ is as follows:

**Theorem 6.12** Let $R_h \subset H(\text{div}, \Omega) \cap Q_h$ be the Raviart-Thomas finite element space. Assume that $\{R_h\} \subset \hat{Q}_h$ and $V_h = \text{div}_h Q_h$. Then, for $0 < \rho \leq 1$ the bilinear form $A_w((\cdot,\cdot),(\cdot,\cdot))$ with $\eta = \rho^{-1} h^{-1}_K$ satisfies:

$$\inf_{(p_h,u_h)\in Q_h \times V_h} \sup_{(q_h,v_h)\in Q_h \times V_h} \frac{A_w((\hat{p}_h,u_h),(\hat{q}_h,v_h))}{\|u_h\| + \|p_h\|_{\text{div},\rho,h}(\|v_h\| + \|q_h\|_{\text{div},\rho,h})} \geq \beta_5, \quad (6.47)$$

where $\beta_5 > 0$ is a constant independent of both $\rho$ and mesh size $h$.

**Proof.** For any given $(\hat{p}_h,u_h) \in \hat{Q}_h \times V_h$, namely $(p_h,\hat{p}_h,u_h) \in Q_h \times \hat{Q}_h \times V_h$. Since $V_h \subset \text{div} R_h$ and $R_h \times V_h$ such that the mixed conforming method is well-defined, there exists $r_h \in R_h$ such that:

$$-\text{div}r_h = u_h \quad \text{and} \quad \|r_h\| + \|\text{div}r_h\| \leq C\|u_h\|. \quad (6.48)$$

Now we choose $q_h = r_h + \alpha p_h, \tilde{q}_h = \alpha \hat{p}_h + (r_h \cdot n_e)n_e, v_h = -\text{div}p_h - \alpha u_h$, where $\alpha$ is a constant that will be indicated later.

We then first verify the boundedness of $(\tilde{q}_h,v_h)$ by $(\hat{p}_h,u_h)$.

Noting that $(q_h - \tilde{q}_h) \cdot n_K|_{\partial K} = (r_h + \alpha p_h - \alpha \hat{p}_h - (r_h \cdot n_e)n_e) \cdot n_K|_{\partial K} = \alpha(p_h - \hat{p}_h) \cdot n_K|_{\partial K}$, we have:

$$\|\tilde{q}_h\|_{\text{div},\rho,K}^2 = (c q_h, q_h)_K + (\text{div} q_h, \text{div} q_h)_K + \rho^{-1}_h h^{-1}_K((q_h - \hat{q}_h) \cdot n_K, (q_h - \hat{q}_h) \cdot n_K)|_{\partial K} = (c r_h + \alpha p_h, r_h + \alpha p_h)_K + (\text{div} r_h + \alpha \text{div} p_h, \text{div} r_h + \alpha \text{div} p_h)_K$$

$$+ \alpha^2 \rho^{-1}_h h^{-1}_K((p_h - \hat{p}_h) \cdot n_K, (p_h - \hat{p}_h) \cdot n_K)|_{\partial K} \leq 2\|r_h\|^2 + 2\alpha^2\|p_h\|^2 + 2\|\text{div} r_h\|^2 + (\text{div} p_h, \text{div} p_h)_K$$

$$+ \alpha^2 \rho^{-1}_h h^{-1}_K((p_h - \hat{p}_h) \cdot n_K, (p_h - \hat{p}_h) \cdot n_K)|_{\partial K} \leq 2(C^2\|u_h\|_{0,K}^2 + \alpha^2\|\tilde{p}_h\|_{\text{div},\rho,K}^2). \quad (6.49)$$

Hence, $\|\tilde{q}_h\|_{\text{div},\rho,h}^2 \leq 2(C^2\|u_h\|^2 + \alpha^2\|\tilde{p}_h\|_{\text{div},h,\rho}^2)$. Further,

$$\|v_h\| = \| - \text{div}_h p_h - \alpha u_h \| = \|\text{div}_h p_h\| + \alpha\|u_h\| \leq \|\tilde{p}_h\|_{\text{div},h,\rho} + \alpha\|u_h\|. \quad (6.50)$$

Then, we prove the boundedness of $(\tilde{q}_h,v_h)$ by $(\hat{p}_h,u_h)$.
Now through integration by parts, we have the following:

\[
A_w((\tilde{p}_h, u_h), (\bar{q}_h, v_h)) = (cp_h, q_h)_{T_h} + \rho^{-1} \sum_{K \in T_h} h_K^{-1} \langle (p_h - \tilde{p}_h) \cdot n_K, (q_h - \bar{q}_h) \cdot n_K \rangle_{\partial K} \\
+ \langle q_h, \nabla u_h \rangle_{T_h} - \langle \tilde{q}_h \cdot n, u_h \rangle_{\partial T_h} + \langle p_h, \nabla v_h \rangle_{T_h} - \langle \tilde{p}_h \cdot n, v_h \rangle_{\partial T_h}
\]

By the Cauchy inequality and inverse inequality, we have:

\[
A_{w,K}((\tilde{p}_h, u_h), (\bar{q}_h, v_h)) = (cp_h, r_h + \alpha p_h)_{K} + \alpha \rho^{-1} h_K^{-1} \langle (p_h - \tilde{p}_h) \cdot n_K, (p_h - \tilde{p}_h) \cdot n_K \rangle_{\partial K} \\
- (\text{div} r_h + \alpha \text{div} p_h, u_h)_{K} + \alpha \langle (p_h - \tilde{p}_h) \cdot n_K, u_h \rangle_{\partial K} \\
- (\text{div} p_h, -\text{div} p_h - \alpha u_h)_{K} + \langle (p_h - \tilde{p}_h) \cdot n_K, -\text{div} p_h - \alpha u_h \rangle_{\partial K}
\]

Noting (6.48), we have the following:

\[
A_w((\tilde{p}_h, u_h), (\bar{q}_h, v_h)) \geq (1 - \epsilon_1 C) \|u_h\|^2 + (\alpha - \epsilon_1^{-1}) \|p_h\|^2 + (1 - \epsilon_2 C) \|\text{div}_h p_h, \text{div}_h p_h\| \\
+ (\alpha \rho^{-1} - \epsilon_2^{-1}) \sum_{K \in T_h} h_K^{-1} \langle (p_h - \tilde{p}_h) \cdot n_K, (p_h - \tilde{p}_h) \cdot n_K \rangle_{\partial K}.
\]

Now choosing \(\epsilon_1 = \frac{1}{2c}, \epsilon_2 = \frac{1}{2c_0}, \alpha = \max\{2C + \frac{1}{2}, 2C_0 + \frac{1}{2}\}, 0 < \rho \leq 1\), we have:

\[
A_w((\tilde{p}_h, u_h), (\bar{q}_h, v_h)) \geq \frac{1}{2} \left( \|p_h\|^2 + \sum_{K \in T_h} h_K^{-1} \|p_h - \tilde{p}_h\|_{0, K} \cdot \|n_K\|^2_{\partial K} + \|u_h\|^2 + \|\text{div}_h p_h\|^2 \right) \\
\geq \frac{1}{2} \left( \|u_h\|^2 + \|\tilde{p}_h\|_{\text{div}, h, \rho} \right).
\]

Thus, we prove the theorem. \(\square\)
7 Summary

In this paper we use the classic LBB theory to prove two types of uniform stability results under some proper parameter-dependent norms for HDG methods, which are uniformly stable with respect to the stabilization parameters and mesh size \( h \). Based on the uniform stability results, we further prove uniform and optimal error estimates for HDG methods, which are independent of the stabilization parameters. In addition, we also prove two types of uniform stability results for WG methods. Similarly based on the uniform stability results, we further prove uniform and optimal error estimates for WG methods. These uniform stability results and optimal error estimates for WG methods are meaningful. Following these uniform stability results for HDG methods and WG methods presented in this paper, an HDG method is shown to converge to a primal conforming method, whereas a WG method is shown to converge to a mixed conforming method by taking the limit of the stabilization parameters.

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