Compactified Little String Theories and Compact Moduli Spaces of Vacua

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It is emphasized that compactified little string theories have compact moduli spaces of vacua, which globally probe compact string geometry. Compactifying various little string theories on $T^3$ leads to 3d theories with exact, quantum Coulomb branch given by: an arbitrary $T^4$ of volume $M^2_s$, an arbitrary K3 of volume $M^2_s$, and moduli spaces of $G = SU(N)$, $SO(2N)$, or $E_{6,7,8}$ instantons on an arbitrary $T^4$ or K3 of fixed volume. Compactifying instead on a $T^2$ leads to 4d theories with a compact Coulomb branch base which, when combined with the exact photon gauge coupling fiber, is a compact, elliptically-fibered space related to the above spaces.

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1. Introduction

Over the past few years, there have been a variety of connections between the moduli spaces of supersymmetric gauge theories and stringy geometry. For example, singular background geometry or gauge bundles can lead to enhanced, non-perturbative, gauge theories, whose moduli spaces reproduce the local singularity \[1,2\]. Another connection is via branes, whose world-volume supersymmetric gauge theory has a moduli space which “probes” \[3,4\] the geometry in which the branes live. In an extreme form of this connection \[5\], we perhaps actually live in the moduli space of a supersymmetric theory. It is thus interesting to consider, generally, what types of geometry can be reproduced via moduli spaces of vacua.

A basic issue is whether moduli spaces of vacua can be compact. Moduli spaces of vacua of standard gauge theories are generally non-compact cones: if a given set of scalar expectation values \( \langle \phi_i \rangle \) is a D-flat vacuum, so is \( \lambda \langle \phi_i \rangle \) for arbitrary scaling factor \( \lambda \to \infty \). (This is slightly modified by Fayet-Iliopoulos terms for \( U(1) \) factors.) An exception is the Coulomb branch moduli, associated with the Wilson lines, of gauge theories which are compactified on tori; these moduli live on dual tori, modded out by the Weyl group.

The present note is devoted to emphasizing that toroidally compactified “little string theories” \[6,7\] can have a variety of interesting, compact, moduli spaces of vacua. The present discussion is an elaboration of a footnote which appeared in \[11\]. The basic message is that, while world-volume gauge theories only \( \text{locally} \) probe the geometry transverse to the brane, little string extensions can \( \text{globally} \) probe \( \text{compact} \) geometry. While this fact is perhaps well-known to some experts, it is hoped that some readers will find it of interest.

For example, the basic \( \mathcal{N} = (1,0) \) heterotic little string theory, when compactified on \( T^3 \), is argued to have a Coulomb branch moduli space of vacua which is a K3 of volume \( M_s^2 \). (Since a 3d scalar has mass dimension \( \frac{1}{2} \), this has the correct dimensions.) The Coulomb branch is a non-linear sigma model with exact, quantum metric equal to the Ricci-flat metric of K3. The K3 is an arbitrary one of fixed volume, whose parameter space coincides with that of the \( T^3 \) compactified heterotic little string theory; the map between these parameter spaces is the same as enters in the duality between the 10d heterotic string on \( T^3 \) and M theory on K3. Geometric symmetries of the K3 Coulomb branch map to non-trivial T-dualities of the \( T^3 \) compactified little string theory.

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\[1\] The extent to which these 6d theories decouple from the 10d bulk, for energies above some gap value, is subtle \[8,9,10\]; we will ignore these issues and only discuss the vacuum manifold.
More generally, it will be argued that the little string theories obtained in [11] from $K$ heterotic (or type II) NS branes at a transverse $\mathbb{C}^2/\Gamma_G$ singularity, when compactified on $T^3$, have a compact Coulomb branch moduli space of vacua given by the moduli space of $K G$-instantons on $K3$ (or $T^4$). Here $G$ is an arbitrary $A, D, E$ group and $\Gamma_G$ is the corresponding $SU(2)$ subgroup. The $K3$ or $T^4$ appearing here is precisely that of $M$ theory duality, which the compactified little string theory globally probes. The volume of the compact Coulomb branch is again set by $M_s$. In each case, the Coulomb branch sigma model metric must be the unique one which is Ricci flat.

Similarly, it will be argued that little string theories, when compactified to 4d on a $T^2$, have Coulomb branches which globally probe $F$ theory. The Coulomb branch is the base space of $F$ theory, and the photon kinetic terms are the elliptic fibration. For example, compactifying the basic $\mathcal{N} = (1, 0)$ little string theory on $T^2$ leads to a 4d theory whose total space of Coulomb branch base and Seiberg-Witten curve is an elliptically fibered $K3$ of volume $M_s^2$. The map between the $T^2$ compactification data and the parameter space of fixed volume, elliptically fibered, $K3$ spaces is the same as in the duality between the 10d heterotic theory on $T^2$ and $F$ theory on an elliptically fibered $K3$ [12].

The next section will review little string theories and their compactification, with several new minor comments included. Sect. 3 outlines classical $T^3$ dimensional reduction of ordinary 6d $U(1)$ and $SU(2)$ gauge fields. This already leads to compact Coulomb branches, of fixed volume $g_6^{-2}$; the Coulomb branch for $U(1)$ is $T^4$, while that of $SU(2)$ is $K3$. Sect. 4 extends the probe argument of [13] to argue for our main message: that compactifying little string theories on $T^3$ leads to theories whose exact Coulomb branch is compact and globally probes the $T^4$ or $K3$ of $M$ theory duality. Sect. 4 discusses $T^2$ compactification of little string theories, which similarly have compact Coulomb branches that globally probe compactification of $F$ theory, e.g. on elliptically fibered $K3$s.

The proposed relation of [11] between compactified type II little string theories and moduli spaces of instantons on $T^4$ also entered in [14,15,16], where it was extended to moduli spaces of instantons on a non-commutative $T^4$ by introducing R-symmetry twists in the compactification. A relation between the twisted, compactified $(2, 0)$ theory and $K3$ was proposed in [13]; this appears to be unrelated to the presently discussed appearance of $K3$ in the context of the untwisted, compactified, heterotic little string theories. Much as in [15,16], it should also be possible to introduce R-symmetry twists for the compactified heterotic little string theories, perhaps leading to moduli spaces of instantons on a non-commutative $K3$, though this will not be done here.
2. Review of little string theories and their compactification

Four classes of 6d little string theories were obtained in [7] via the world-volume of 5-branes in the limit $g_s \rightarrow 0$ with $M_s$ held fixed:

(iia) $\mathcal{N} = (1, 1)$ supersymmetric, via $K$ IIB NS five-branes [7] or via IIA or M theory with a $\mathbb{C}^2/\Gamma_G$ ALE singularity [17][18].

(iib) $\mathcal{N} = (2, 0)$ supersymmetric, via $K$ type IIA five branes [7] or type IIB with a $\mathbb{C}^2/\Gamma_G$ singularity [17].

(o) $\mathcal{N} = (1, 0)$ supersymmetric, via $K$ $SO(32)$ heterotic small instantons.

(e) $\mathcal{N} = (1, 0)$ supersymmetric, via $K$ heterotic $E_8 \times E_8$ small instantons.

Cases (iia) and (o) contain gauge fields with coupling $g_6^{-2} = M_s^2$ and are IR free. Instantons in the 6d gauge theories are fundamental strings, with tension $g_6^{-2} = M_s^2$. Cases (iib) and (e) instead contain tensor multiplet two-form gauge fields, with self-dual field strength, and lead to interacting RG fixed point field theories in the IR.

$\mathcal{N} = (1, 0)$ tensor multiplet theories (of which $\mathcal{N} = (2, 0)$ is a special case) always have an associated group $G$. For cases (iib) it is $SU(K)$ or the $ADE$ singularity group $G$, while for (e) it is $Sp(K)$. There is a $r =$rank($G$) dimensional, compact Coulomb branch moduli space, with the real scalars in the $\mathcal{N} = (1, 0)$ tensor multiplets taking values $\langle \vec{\Phi} \rangle$ in the “$G$-Coxeter box” $(S^1)^{\otimes r}/W_G$, where the $S_1$ is of radius $M_s^2$ and $W_G$ is the Weyl group of $G$. The theory is interacting at the boundaries of the Coxeter box but, in the bulk, behaves in the IR as $r$ free self-dual tensor multiplets. Strings are charged under the $r$ 2-form gauge fields of these tensor multiplets, with charge vectors $\vec{\alpha}$ in the $G$ root lattice. Via a BPS formula, a string with charges $\vec{\alpha}$ has tension $Z = \vec{\alpha} \cdot \vec{\Phi}$, becoming tensionless at the origin of the Coulomb branch. Reducing to 5d leads to a gauge theory with non-Abelian gauge group $G$ at the origin, so the 6d theory can be regarded as a non-Abelian self-dual two-form gauge theory with group $G$ (whatever that means).

Because of the self-duality, these strings can be regarded as either “electrically” or “magnetically” charged. The Dirac quantization condition thus implies that the lattice $\Lambda$ must be an integer lattice, i.e. the dot product of any two lattice vectors is an integer, so $\Lambda \subset \tilde{\Lambda}$, where $\tilde{\Lambda}$ is the dual lattice. This is, of course, a weaker condition than self-duality of the lattice. For example, the root lattice of a simple group $G$ is generally not self-dual but, rather, a subgroup, of degree given by the center of $G$, in the dual lattice, which is the weight lattice.
Each of the four above classes has either vector or tensor multiplets, but not both. Theories containing both vector and tensor multiplets were discussed in [11] by combining 5-branes with \( \mathbb{C}^2 / \Gamma_G \) orbifold singularities in the transverse dimensions, using results obtained in [19,22]. In this way, new theories can be obtained for each of the four classes of branes, type IIA, IIB, \( SO(32) \) heterotic, and \( E_8 \times E_8 \) heterotic, at \( \mathbb{C}^2 / \Gamma_G \) singularities. All of these theories generally have \( \mathcal{N} = (1,0) \) supersymmetry.

E.g. K type IIB NS 5-branes at a \( \mathbb{C}^2 / \Gamma_G \) singularity [21] has a quiver gauge theory, based on the extended Dynkin diagram of the \( ADE \) singularity group \( G \), with gauge group \( U(1)_D \times \prod_{\mu=0}^r SU(Kn_\mu) \) and bi-fundamental matter. \( n_\mu \) are the \( G \) Dynkin indices and \( r = \text{rank}(G) \). There are \( r \mathcal{N} = (1,0) \) tensor multiplets, which are associated, as described above, with the singularity group \( G \). Via an anomaly cancellation mechanism, \( SU(Kn_\mu) \) has gauge coupling \( g_{\mu,\text{eff}}^2 = M_s^2 \delta_{\mu,0} + \vec{\alpha}_\mu \cdot \vec{\Phi} \) and an \( SU(Kn_\mu) \) instanton, which is a string in 6d, has tensor-multiplet charges \( \vec{\alpha}_\mu \) and BPS tension \( Z_\mu = g_{\mu,\text{eff}}^2 \). Here the \( \vec{\alpha}_\mu \) are the \( G \) root vectors (\( \vec{\alpha}_0 \) is the extending root) and the condition that all \( g_{\mu,\text{eff}}^2 \geq 0 \) is precisely that the Coulomb branch \( \langle \vec{\Phi} \rangle \) is the \( G \) Coxeter box, of side length \( M_s^2 \). The \( SU(Kn_\mu) \) instanton string charges span the \( G \) root lattice.

The instanton string for a diagonal \( SU(K)_D \subset \prod_{\mu=0}^r SU(Kn_\mu) \), with index of embedding \( n_\mu \) in \( SU(Kn_\mu) \), has tension \( n^\mu Z_\mu = M_s^2 \), and is identified with the fundamental IIB string. The other \( r \) independent instanton strings in \( \prod_{\mu=0}^r SU(Kn_\mu) \) are to be identified with the strings obtained [23] by wrapping the type IIB 3-brane on the \( r \) independent, fully collapsed, two-cycles of the \( \mathbb{C}^2 / \Gamma_G \) singularity; \( m = 1 \ldots r \) of these strings become tensionless for \( \langle \vec{\Phi} \rangle \) at a codimension \( m \) boundary of the Coulomb branch Coxeter box.

The simplest heterotic case is \( K SO(32) \) 5-branes at a \( \mathbb{C}^2 / \Gamma_G \) singularity [19,22]. The theories are associated [11] with a subgroup \( H \) of the singularity group \( G \), with \( G \rightarrow H \) as \( SU(2P) \rightarrow Sp(P) \), \( SO(4P+2) \rightarrow SO(4P+1) \), \( SO(4P) \rightarrow SO(4P) \), \( E_6 \rightarrow F_4 \), \( E_7 \rightarrow E_7 \), \( E_8 \rightarrow E_8 \). The gauge group and matter content is given by a quiver diagram, which is the extended \( H \) Dynkin diagram, with \( SO \), \( Sp \), and \( SU \) groups at various nodes, e.g. the group at the \( \mu = 0 \) node is \( Sp(K) \). There are \( r = \text{rank}(H) \) \( \mathcal{N} = (1,0) \) tensor multiplets, which are associated with the group \( H \). Via an anomaly cancellation mechanism, the gauge group at node \( \mu = 0 \ldots r \) of the quiver diagram has coupling \( g_{\mu,\text{eff}}^{-2} = M_s^2 \delta_{\mu,0} + \vec{\alpha}_\mu \cdot \vec{\Phi} \), and an instantons string in this group has tensor multiplet charges \( \vec{\alpha}_\mu \) and BPS tension \( Z_\mu = g_{\mu,\text{eff}}^{-2} \). Here \( \vec{\alpha}_\mu \) are the simple and extending roots of \( H \), so the instanton strings span the \( H \) root lattice. Instantons in a diagonal \( Sp(K)_D \) are identified with the fundamental heterotic string, of tension \( M_s^2 \). The other \( r \) independent instanton strings can again be
identified with 3-branes wrapped on collapsed two-cycles; \( m = 1 \ldots r \) of these become massless at a codimension \( m \) boundary of the Coulomb branch (the \( H \) Coxeter box).

The other heterotic case, \( K E_8 \times E_8 \) 5-branes at a \( \mathbb{C}^2/\Gamma_G \) singularity, leads to little string theories with a more involved spectrum of tensor multiplets, gauge groups, and matter content \[ [22,11]. \]

Compactifying on a circle, 6d vector and tensor multiplets both lead to 5d vector multiplets. A 6d \( \mathcal{N} = (1,0) \) theory with a gauge group of rank \( r_V \) and \( n_T \) tensor multiplets, when compactified, leads to a 5d theory with a Coulomb branch moduli space of vacua of dimension \( d_C = r_V + n_T \). Compactifying to 4d, the Coulomb branch has real dimension \( 2(r_V + n_T) \) and in 3d, upon dualizing the \( d_C \) photons, there is a Coulomb branch of real dimension \( 4(r_V + n_T) \).

Little string theories exhibit T-duality when compactified on a circle \[ [7], with the (iia) theory on a circle of radius \( R \) identical to the (iib) theory on a circle of radius \( 1/M_s^2R \). Similarly, the (o) heterotic theory, on a circle of radius \( R \), and with a Wilson line around the circle breaking \( SO(32) \) to \( SO(16) \times SO(16) \), is identical to the (e) heterotic theory on a circle of radius \( 1/M_s^2R \), again with a Wilson line breaking \( E_8 \times E_8 \) to \( SO(16) \times SO(16) \). (See \[ 24 \] for the heterotic T-duality with general Wilson lines.) In these cases, T-duality exchanges 6d tensor and vector multiplets, \( r_V \leftrightarrow n_T \). This is nice because the 5d classical kinetic terms for the scalars coming from 6d tensor multiplets, \( M_s^4R(d\Phi)^2 \), is indeed exchanged with the kinetic term, \( g_6^{-2}R(R^{-1}d\Phi)^2 \), of a vector multiplet on a circle of radius \( R \). In both cases \( \Phi \) is a compact scalar, normalized so \( \Phi \in [0,1] \), and the two kinetic terms are exchanged by \( R \leftrightarrow (M_s^2R)^{-1} \) upon setting \( g_6^{-2} = M_s^2 \).

More generally, there is an expected T-duality, with \( R \leftrightarrow (M_s^2R)^{-1} \) exchanging the theories coming from IIA and IIB or \( SO(32) \) and \( E_8 \times E_8 \) heterotic branes at \( \mathbb{C}^2/\Gamma_G \) singularities. T-dual theories must have \( r_V + n_T = \tilde{r}_V + \tilde{n}_T \). As was noted in \[ 24,11 \], this is the case for the \( SO(32) \) and \( E_8 \times E_8 \) branes at singularities: both cases have \( r_V + n_T = C_2(G)K - |G| \), where \( C_2(G) \) is the dual Coxeter number of the singularity group \( G \) and \( |G| \) is its dimension. This formula will be important in what follows. A point of concern mentioned in \[ 11 \] is that a stronger condition, \( r_V = \tilde{n}_T \) and \( n_T = \tilde{r}_V \), needed for T-duality to exchange the classical kinetic terms as above, is not satisfied. The present situation is, in fact, closely connected to that of \[ 23 \], where it was argued that T-duality can fail. Here, however, there is a simpler resolution: the Coulomb branch metric can get quantum corrections and, while the quantum corrected metrics are expected to agree, the classical metrics need not. The stronger condition is thus unnecessary.
All little string theories, when compactified on $T^D$, have the parameter space

$$O(D + y, D; \mathbb{Z}) \backslash O(D + y, D)/O(D + y) \times O(D),$$

(2.1)

where $y = 0$ for the type II cases and $y = 16$ for the heterotic cases. These are the $T^D$ metric and $B_{NS}$ fields ($D^2$ real parameters), and also the $SO(32)$ or $E_8 \times E_8$ Wilson lines in the heterotic cases ($16D$ real parameters). $O(D + y, D; \mathbb{Z})$ is the full $T$ duality group.

3. Compactification preliminaries

We first consider the classical dimensional reduction of a 6d $U(1)$ gauge field,

$$\int d^6x(-\frac{1}{4g_6^2}F_{\mu\nu}F^{\mu\nu} + B_{NS} \wedge F \wedge F),$$

(3.1)

on a $T^3$ to three dimensions. $B_{NS}$ is an external, background, two-form gauge field. We take space to be $\mathbb{R}^3 \times T^3$, with $\mathbb{R}^3$ coordinates $x^i$, $i = 1, 2, 3$, and periodic coordinates $\rho^a \in [0, 1], a = 1, 2, 3$, for the $T^3$; the metric is $ds^2 = \delta_{ij}dx^i dx^j + h_{ab}\rho^a d\rho^b$. Taking all fields to be independent of the $T^3$ coordinates $\rho^a$, (3.1) becomes

$$S = \int d^3x \left[ \frac{\sqrt{\det h}}{g_6^2}(-\frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}(h^{-1})^{ab}\partial_i \phi_a \partial^i \phi_b) + \theta^a \epsilon^{ijk} F_{ij} \partial_k \phi_a \right],$$

(3.2)

where $B_{NS} = \epsilon_{abc} \theta^a d\rho^b \wedge d\rho^c$ for some constants $\theta^a$, $a = 1, 2, 3$. The three real scalars $\phi_a$ are associated with the Wilson lines of the gauge field around the cycles $d\rho^a$ of the $T^3$ and are periodic, normalized so that $\phi_a \in [0, 1]$.

The 3d $U(1)$ gauge field can be dualized to another real scalar, which also lives on a circle. This is done as in [27]: we replace $F_{ij} \rightarrow F_{ij} - H_{ij}$ in (3.2) and introduce an additional term $\epsilon^{ijk} H_{ij} \partial_k \phi_4$, with the scalar $\phi_4$ periodic, normalized so that $\phi_4 \in [0, 1]$. First integrating out $\phi_4$ leads back to the original theory. First integrating out $H$ sets $F_{ij} = 0$ and leads to $\phi_4$ kinetic terms. Combining with the $\phi_a$ kinetic terms in (3.2), the upshot is a $T^4$ Coulomb branch moduli space of vacua $\langle \phi_A \rangle$, $A = 1, \ldots, 4$, with metric

$$ds^2 = \frac{\sqrt{\det h}}{g_6^2}(h^{-1})^{ab}d\phi_ad\phi_b + \frac{g_6^2}{\sqrt{\det h}}(d\phi_4 - \theta^a d\phi_a)^2 \equiv G^{AB}d\phi_A d\phi_B,$$

(3.3)

where $a$ runs over 1, 2, 3 and $A = 1, \ldots , 4$. This metric $G^{AB}$ has 10 real components, which depend on the 9 real parameters $h^{ab}$ and $\theta^a$, and thus satisfies one constraint. The relation is that the $T^4$ of (3.3) has fixed volume, independent of the $h_{ab}$ and the $\theta^a$:

$$\text{Volume}(T^4) = \sqrt{\det(G^{AB})} = \frac{1}{g_6^2} = M_s^2.$$

(3.4)
Although the above discussion was purely classical, the map (3.3) between the $T^3$ metric $h_{ab}$ and $B$ fields $\theta^a$ is exactly the relevant one for relating type IIB string theory on $T^3$ to M theory on $T^4$, to be discussed in the next section. Indeed, the map (3.3) was also obtained in [16] in the context of the compactified $(2,0)$ theory via a chain of string duality gymnastics.

We pause to note that the metric (3.3) nicely exhibits properties to be expected based on its connection to M theory. In particular, the obvious, geometric $SL(4; \mathbb{Z})$ discrete symmetries of the $T^4$ correspond to non-trivial T-dualities, in a subgroup of the T-duality group appearing in (2.1). For example, consider the obvious requirement that the $T^4$ be invariant under the relabeling exchange $\phi_3 \leftrightarrow -\phi_4$. Taking, for simplicity, $T^3$ with $h_{ab} = L_2^2 \delta_{ab}$ and $\theta_a = 0$, it follows from (3.3) that this operation corresponds to the operation

$$L_1 \rightarrow (M_s^2 L_2)^{-1}, \quad L_2 \rightarrow (M_s^2 L_1)^{-1}, \quad L_3 \rightarrow L_3,$$

where we set $g_{6^{-2}} = M_s^2$. This is a T duality in two circles, which is non-trivial but, nevertheless, a symmetry taking the IIA or IIB theory back to itself. The generalization of the T-duality (3.3) for general $h_{ab}$ and $\theta_a$ is quite complicated, see e.g. [28]; remarkably, it is indeed reproduced from (3.3) by simply requiring the $\phi_3 \leftrightarrow -\phi_4$ symmetry.

On the other hand, T-duality in an odd number of cycles, such as the $O(3,3; \mathbb{Z})$ element taking all $L_i \rightarrow (M_s^2 L_i)^{-1}$, for $i = 1, 2, 3$, is not a geometric $SL(4; \mathbb{Z})$ symmetry of (3.3). This is sensible, since such operations are not symmetries of IIA or IIB string compactifications but, rather, exchange IIA and IIB.

In particular, starting instead from a 6d tensor multiplet, dimensional reduction on a $T^3$ leads to a $T^4$ Coulomb branch moduli space, with metric related to (3.3) by T-duality in an odd number of the $T^3$ cycles, corresponding to the exchange of IIA and IIB.

Now consider $T^3$ reduction of a 6d $SU(2)$ gauge theory. The above discussion for $U(1)$ carries over to this case with almost no changes. The only difference is that the real scalars $\phi_A$ must be modded out by the Weyl group action $\phi_A \sim -\phi_A$. Modding out the $T^4$ by this $Z_2$ action leads to a $K3$. Thus the Coulomb branch of a 6d $SU(2)$ gauge theory reduced to 3d on a $T^3$ is given by $\langle \phi_A \rangle$ in a compact K3. The volume of the K3 is again set by $g_{6^{-2}}$, and equal to $M_s^2$. The full parameter space of K3 metrics of fixed volume is 57 dimensional and given by (2.1) with $D = 3$ and $y = 16$, while that obtained here only depends on the 9 dimensional subspace given by (2.1) with $D = 3$ and $y = 0$. The remaining parameters will come from 3 real masses for each of 16 $SU(2)$ fundamental matter flavors; these enter as the Wilson loop parameters in (2.1).
4. The probe argument, checks, and comments

The parameter space (2.1) for $T^3$ compactified heterotic (or type II) little string theories coincides with the geometric parameter space of a K3 (or $T^4$) of fixed volume. These are, of course, the standard miracles which enter in the duality between the 10d heterotic (or type II) string on $T^3$ and M theory on $Y=K3$ (or $T^4$). The fundamental string arises as the M5 brane wrapped on $Y$, so $M_p^6 Vol(Y) = M_s^2$. We will here extend the probe argument of [13] to argue that these M theory dualities provide the solution for the exact, quantum, Coulomb branch metric of the $T^3$ compactified little string theories.

Recall that the argument of [13] started with 3d $\mathcal{N} = 4$ supersymmetric (8 supercharges) $SU(2)$ gauge theory with fundamental matter, which is the world-volume field theory in a D2 brane in type I' string theory on $T^3$. This maps to a M2 brane in M theory on K3, which can be at an arbitrary point in the transverse $\mathbb{R}^4 \times K3$. The $\mathbb{R}^4$ corresponds to a decoupled hypermultiplet in the world-volume theory. The K3 factor is more interesting: it was thus argued in [13] that the full, quantum-corrected metric on the Coulomb branch of the D2 brane world-volume field theory must be a local piece of the corresponding K3; this was confirmed in [27] purely in the context of 3d field theory.

The D2 brane world-volume field theory only locally probes the K3 because of the particular limit taken to decouple the bulk dynamics: $g_s \to 0$ and $M_s \to \infty$. On the other hand, we can take $g_s \to 0$, but with $M_s$ held fixed. This theory is precisely the 6d heterotic little string theory (o), compactified to 3d on the same $T^3$ as the 10d heterotic or type I' bulk theory. The $T^3$ compactified little string theory (o) globally probes the fixed volume K3 of M theory, and must thus have a Coulomb branch moduli space of vacua which is the same K3. The geometric K3 has volume $M_s^2 M_p^{-6}$ and, taking into account how the properly normalized Coulomb moduli scalars probe geometry, the volume of the Coulomb branch K3 is $M_s^2$. This matches with the result of the previous section. This compact Coulomb branch properly becomes non-compact in the field theory limit $M_s^2 \to \infty$.

The K3 Coulomb branch can have singularities, depending on the choice of parameters in (2.1). As in [13], these singularities mark the intersection of the Coulomb branch with a Higgs branch, with an interacting 3d infra-red conformal field theory at the intersection.

Unfortunately, both sides in the present equivalence, between the quantum Coulomb branch of the $T^3$ compactified little string theory on the one hand, and the metric of K3 on the other, are presently not well understood. Perhaps the present equivalence will eventually be useful for using one of the two sides to learn about the other.
A direct generalization of the above is to consider a $T^3$ compactification of the little string theory (o) associated with $K \text{SO}(32)$ heterotic small instanton. This maps to $K\text{M2}$ branes at points on $\mathbb{R}^4 \times K3$. The Coulomb branch is, correspondingly, the symmetric product $(K3)^{\otimes K}/S_K$, where each $K3$ is again of fixed volume $M^2_s$.

The geometric symmetries (see, e.g. [29]) of the Coulomb branch $K3$ correspond to non-trivial T-dualities in (2.1) though, as in (3.5), only the subgroup which takes the $\text{SO}(32)$ heterotic theory back to itself. An additional $\mathbb{Z}_2$ component of T-dualities in $O(19,3;\mathbb{Z})$ reflects the fact that, instead compactifying the $E_3 \times E_8$ heterotic little string (e), with T-dual $T^3$ compactification data, also yields the same 3d theory, with the same $K3$ compact Coulomb branch as described above.

Each of the little string theories reviewed in sect. 2 can be compactified to 3d on a $T^3$, and each has an exact quantum Coulomb branch which globally probes the dual M theory compactification. In each case, there is a compact Coulomb branch component, with unit volume in units of $M_s$. The 3d field theory limit is recovered by taking $M_s \rightarrow \infty$.

The $\mathcal{N} = (1,1)$ little string theories with group $U(K)$, when compactified on $T^3$, have a Coulomb branch which is $(\mathbb{R}^4 \times T^4)^{\otimes K}/S_K$ (more generally, $(\mathbb{R}^4 \times T^d)^{\text{rank}(G)}/\text{Weyl}(G)$), which probes the duality between type II strings on $T^3$ and M theory on $T^4$. The Coulomb branch $T^4$ has metric $G^{AB}$ which is given exactly in terms of the $T^3$ compactification date by (3.3), with volume $M^2_s$. There is a similar statement for the $\mathcal{N} = (2,0)$ little string theory on $T^3$, differing from the $\mathcal{N} = (1,1)$ case by a T-duality in one of the $T^3$ cycles; the fixed volume $T^4$ in this context was also discussed in [15,16].

The $\mathcal{N} = (1,0)$ little string theories associated with $K$ type II or heterotic 5-branes at an $X_G \equiv \mathbb{C}^2/\Gamma_G$ singularity, when compactified on $T^3$, similarly probe M theory geometry. In the heterotic (or type II) cases, the M theory dual is given by $K\text{M2}$ branes with a transverse space $X_G \times K3$ (or $X_G \times T^4$). In both cases, M theory with a $X_G$ singularity has an enhanced $G$ gauge symmetry and $\text{M2}$ branes, when sitting directly on top of the $G$ singularity of $X_G$, can be interpreted as small $G$ instantons. In the heterotic (or type II) cases, these $K G$-instantons have the fixed volume $K3$ (or $T^4$) as their four spatial coordinates. There is a moduli space for these instantons given by their positions in these four spatial coordinates, as well as their moduli for fattening up and rotating in $G$.

Thus, by the probe argument, the little string theory associated with $K$ heterotic (or type II) branes at a $\mathbb{C}^2/\Gamma_G$ singularity, when compactified on a $T^3$, has a compact Coulomb branch moduli space of vacua which is exactly given by the moduli space of $K G$-instantons on a $K3$ (or $T^4$) of volume $M^2_s$. A quick check is that the dimension of the Coulomb branch
of the $T^3$ compactified little string theories indeed agrees with the dimension of the moduli space of $K G$-instantons\(^3\) on $T^4$ or K3: the type II cases indeed have $4(r_V + n_T) = 4KC_2(G)$ and the heterotic cases indeed have $4(r_V + n_T) = 4(KC_2(G) - |G|)$. This latter fact also played a role in the mirror symmetry of [25].

Another check is to consider the limit $M_s^2 \to \infty$, where $T^4 \to \mathbb{R}^4$ or $K^3$ becomes a non-compact piece of K3, and where the compactified little string theory goes over to its 3d field theory limit. In the type II cases, the resulting 3d field theory has the quiver gauge group $\prod_{\mu=0}^r U(K n_\mu)$, based on the extended $G$-Dynkin diagram, which was indeed argued in [30,31] to have a quantum Coulomb branch which is the moduli space of $K G$-instantons on $\mathbb{R}^4$. This theory was argued in [30] to have a hidden, global $G$ symmetry. Because $M$ theory has $G$ gauge symmetry even for finite $M_s$, the full compactified little string theory is expected to also have this hidden global symmetry. Similar statements should hold in the heterotic cases.

Moduli spaces of instantons on $T^4$ or K3 have made a variety of appearances in physics and mathematics, though usually with $G = U(N)$ as the gauge group. In that case, the moduli space also depend on $v_a = \int_{\Sigma_a} \text{Tr} F$, where $\Sigma_a$ is a basis for the two cycles of $T^4$ or K3. In the present case, $G$ is a simple $A, D, E$ group so $\text{Tr} F = 0$. ($B$ fields can possibly still contribute to $v_a \neq 0$, e.g. as in [21].)

The moduli spaces of instantons obtained above have many interesting singularities. At these Coulomb branch singularities, there is an attached Higgs branch, with an interacting 3d IR CFT at the intersection.

All of the above compact Coulomb branches are hyper-Kahler, with $c_1 = 0$, and the sigma model metric is the unique one which is Ricci-flat.

5. $T^2$ compactification: probing $F$ theory

Compactifying the heterotic (or type II) little string theories to 4d on a $T^2$ leads to quantum Coulomb branches which globally probe $F$ theory compactifications on a fixed volume, elliptically fibered K3 (or $T^4$). For example, consider the $K = 1$ case of the heterotic little string theory (o), whose low-energy field theory content is that of the world-volume of D3 branes in type I’ on $T^2$. This latter theory has a non-compact, quantum

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\(^3\) For a general 4 manifold with Euler character $\chi$ and signature $\sigma$, the dimension is $4KC_2(G) - \frac{1}{2}|G|(\chi + \sigma)$. For $T^4$, $\chi = \sigma = 0$ and, for K3, $\chi = 24$ and $\sigma = -16$. 

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Coulomb branch which was argued \cite{32,4} to locally probe the duality to F-theory on an elliptically fibered K3. The Coulomb branch in the 4d field theory is the non-compact complex u plane, over which the photon coupling $\tau_{\text{eff}}(u)$ is fibered according to the Seiberg-Witten curve \cite{33}; the total space of u-plane base and $\tau_{\text{eff}}(u)$ fiber is a local, non-compact piece of K3. This is the $M_s \to \infty$ limit of the $T^2$ compactified little string theory.

Considering now the $T^2$ compactified little string theory for finite $M_s^2$, the u-plane base is a compact box of volume $M_s^2$ (this is the correct mass dimension for 4d scalars). As in sect. 3, reducing a 6d $U(1)$ gauge field on a $T^2$ with metric $h_{ab}d\rho^a d\rho^b$ leads to scalars living on a dual $T^2$, with metric $g_6^{-2} \sqrt{\det h} (h^{-1})^{ab} d\phi_a d\phi_b$, which has volume $g_6^{-2} = M_s^2$ for all $h_{ab}$. For $SU(2)$ rather than $U(1)$, we mod out by the Weyl group $\phi_a \sim -\phi_a$, yielding a 2d box of volume $M_s^2$. Considering the elliptic fiber $\tau(u)$ over the compact base as a dimensionless coordinate, the total space of base and fiber is an elliptically fibered K3 of volume $M_s^2$. This elliptically fibered K3 of fixed volume is that of the F-theory dual to the 10d heterotic string on $T^2$. As was the case there, the parameter space (2.1) of data in the $T^2$ compactification of the heterotic string matches that of the fixed volume, elliptically fibered K3s. This can be regarded as a special case of the $T^3$ compactification considered in the previous sections, where one of the radii is taken to infinity. It is thus good that we again get a K3 of volume $M_s^2$, since that was the case in the previous sections for all radii.

More generally, $T^2$ compactifying the little string theories associated with $K$ type II 5-branes at a $\mathbb{C}^2/\Gamma_G$ singularity leads to a compact Coulomb branch which is a $2(r_V + n_T) = 2KC_2(G)$ dimensional torus of unit volume in units of $M_s$. Including the $KC_2(G)$ complex dimensional elliptic fiber, associated with the kinetic terms of the $KC_2(G)$ photons, the total space is the moduli space of $K G$-instantons on a $T^4$ of volume $M_s^2$, where both the $T^4$ and the resulting instanton moduli space are regarded as an elliptic fibration.

Compactifying on a $T^2$ the little string theories associated with $K$ heterotic 5-branes at a $\mathbb{C}^2/\Gamma_G$ singularity leads to a compact Coulomb branch which is a $2(r_V + n_T) = 2(KC_2(G) - |G|)$ dimensional box, of unit volume in units of $M_s$. Including the fiber associated with the photons, the total space is an elliptically fibered space which is exactly the moduli space of $K G$-instantons on an elliptically fibered K3 of volume $M_s^2$.

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