A SIMPLE METHOD TO EXTRACT THE ZEROS OF SOME EISENSTEIN SERIES OF LEVEL N

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Abstract. This paper provides a simple method to extract the zeros of some weight two Eisenstein series of level N where N = 2, 3, 5 and 7. The method is based on the observation that these Eisenstein series are integral over the graded algebra of modular forms on SL(2, Z) and their zeros are ‘controlled’ by those of E4 and E6 in the fundamental domain of Γ0(N).

1. Introduction

In the work of Rankin and Swinnerton Dyer [RSD] the location of zeros of all Eisenstein series E_k (of weight k ≥ 4, even) of full modular group SL_2(Z) had been determined. In the fundamental domain this was found to be always on the arc |τ| = 1, with 2π/3 ≥ arg(τ) ≥ π/2. The method has been generalized to Fricke groups in recent works of [SJ] and to the subgroups of SL_2(Z) in [GS]. The zeros of weight two Eisenstein series E_2(q) were studied by [RW], [BS]. In this work we shall find the zeros of Eisenstein series \( \tilde{E}_N \) which are holomorphic modular forms of weight 2 of \( \Gamma_0(N) \) defined as

\[
\tilde{E}_N(\tau) := \frac{1}{N-1}(NE_2(N\tau) - E_2(\tau)),
\]

(1.1)

where \( E_2(\tau) \) is the quasimodular Eisenstein series defined by,

\[
E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n, \quad q = e^{2\pi i \tau},
\]

(1.2)

with \( \sigma_1(n) \) being the sum over all the divisors of \( n \).

The method which we present is quite different from that of [RSD] but however can only be applied for \( N = 2, 3, 5, 7 \). Our main observation is that \( \tilde{E}_N \) is integral over the graded algebra

\[
M(SL_2(\mathbb{Z})) = \bigoplus_{k \geq 0} M_k(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6].
\]

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\(^1\)In general \( \tilde{E}_N \) is defined as negative of what it is defined here in (1.1).
with $E_4$, $E_6$ being the Eisenstein series of weight 4 and 6 defined as,

$$E_4(\tau) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n,$$

(1.3)

$$E_6(\tau) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n,$$

(1.4)

where $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$.

Let us denote the fundamental domain for $\Gamma_0(N)$ by $F_N$. For $N = 2, 3, 5, 7$ the zeros of $\tilde{E}_N$ are ‘controlled’ by those of $E_4$ and $E_6$ in $F_N$.

We now state the main result of our paper which would be proved in section 3.

**Theorem 1.** All the zeros of $\tilde{E}_N(\tau)$ in the fundamental domain of $\Gamma_0(N)$ lie as follows:

| $N$ | 2   | 3   | 5   | 7   |
|-----|-----|-----|-----|-----|
| $\tau$ | $-\frac{1}{1+i}$ | $-\frac{1}{e^{2\pi i/3}+2}$ | $-\frac{1}{e^{2\pi i/3}+2}$ | $-\frac{1}{e^{2\pi i/3}+2}$ |

Table 1. Zeros of $\tilde{E}_N(\tau)$ for $N = 2, 3, 5, 7$.

### 2. Notations and Preliminaries

For $k \geq 2$, even we denote the space of all modular forms on $SL(2, \mathbb{Z})$ by $M_k$ and the space of all cusp forms on $SL(2, \mathbb{Z})$ by $S_k$ and the space of all modular forms on $\Gamma_0(N)$ by $M_k(\Gamma_0(N))$ and the space of all cusp forms on $\Gamma_0(N)$ by $S_k(\Gamma_0(N))$.

Let $f(z)$ be a holomorphic function in the upper half plane and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$ then

$$f|_k \gamma = \text{det}(\gamma)^{k/2} (cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right).$$

With $\omega_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ denoting the usual Fricke involution on $M_k$, one can readily verify the fact that $\tilde{E}_N|_2 \omega_N = -\tilde{E}_N$.

Let $F_\Gamma$ be the fundamental domain (see [TMA] for more details and definition) of $\Gamma := SL_2(\mathbb{Z})$ given by the region in the upper half plane such as $\tau = x + iy$ where
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$-1/2 < x < 1/2$ and $x^2 + y^2 \geq 1$, then $F_p$, the fundamental domain for $\Gamma_0(p)$ for a prime $p$ can be given as

$$F_p = F_\Gamma \cup \bigcup_{k=0}^{p-1} ST^k(F_\Gamma),$$

(2.1)

where $\{\{ST^k\}_{k=0}^{p-1} \cup \text{Id}\}$ is a set of coset representatives of $\Gamma_0(p)$ in $\Gamma$.

The valence formula: Let $f \neq 0$ be a modular form of weight $k$ in $SL(2,\mathbb{Z})$, $v_p(f)$ is the order of the zero of $f$ at point $p \in F_\Gamma$ and $\rho = e^{2\pi i/3}$ then the valence formula gives,

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_p(f) + \sum_{p \in \Gamma, p \neq i, \rho} v_p = \frac{k}{12}$$

(2.2)

From the above valence formula one obtains that in the fundamental domain of the full modular group the only zeros of $E_4$ and $E_6$ lie at $\tau = e^{2\pi i/3}$ and $\tau = i$ respectively.

$$E_6(i) = 0, \quad E_4(e^{2\pi i/3}) = 0.$$  

(2.3)

3. Proof of Theorem

We shall begin this section with the following lemmas regarding the zeros of Eisenstein series which are essential to prove the Theorem

**Lemma 1.** If $\tau \in F_N$ and $E_6(\tau) = 0$ then either $\tau = i$ or $\tau = \frac{1}{1+i}$. Similarly if $\tau \in F_N$ and $E_4(\tau) = 0$ then either $\tau = e^{2\pi i/3}$ or $\tau = \frac{1}{e^{2\pi i/3}+i}$ where $0 \leq k \leq N - 1$ and these are the only possible zeros of $E_4$ and $E_6$ in $F_N$.

**Proof.** The lemma follows from the expression of fundamental domain $F_p$ (2.1) and the zeros of $E_4$ and $E_6$ in $F_\Gamma$.

**Lemma 2.** There exists a polynomial

$$(\tilde{E}_N(\tau))^{N+1} = \sum_{i=0}^{N-1} a_i(\tilde{E}_N(\tau))^i m_{N+1-i}(\tau),$$
where \( a_i \) is a constant and \( m_{N+1-i} \in M_{2(N+1-i)}(SL_2(\mathbb{Z})) \). These polynomials are given by,

\[
\begin{align*}
\tilde{E}_2^\prime(\tau)^3 &= \frac{1}{4}E_6(\tau) + \frac{3}{4}E_4\tilde{E}_2(\tau), \\
\tilde{E}_3^\prime(\tau)^4 &= \frac{1}{27}E_4^3(\tau) + \frac{8}{27}E_6(\tau)\tilde{E}_3(\tau) + \frac{2}{3}\tilde{E}_3(\tau)^2E_4(\tau), \\
\tilde{E}_5^\prime(\tau)^6 &= \frac{1}{3125}E_6^5(\tau) + \frac{24}{3125}E_6E_4\tilde{E}_5(\tau) + \frac{9}{125}E_4^2\tilde{E}_5(\tau)^2 + \frac{8}{25}E_6
\end{align*}
\]

\[
\tilde{E}_5(\tau)^3 + \frac{3}{5}\tilde{E}_5(\tau)^4E_4(\tau), \\
\tilde{E}_7^\prime(\tau)^8 &= \frac{1}{77}E_4^7 + \frac{48}{77}E_5^3E_4\tilde{E}_7 + \left( \frac{64}{64827}E_6^2 + \frac{92}{453789}E_4^3 \right)\tilde{E}_7^2 + \frac{32}{2401}E_4E_6\tilde{E}_7^3
\]

\[
+ \frac{30}{2401}E_4^2\tilde{E}_7^4 + \frac{16}{49}E_6\tilde{E}_7^5 + \frac{4}{7}\tilde{E}_7^6E_4. \\
\]

\[
(3.1) \quad (3.2) \quad (3.3) \quad (3.4)
\]

**Proof.** Considering the \( q \) expansion of both sides of the above equations upto Strum’s bound the lemma follows. The program zeros.nb is given with this paper for reference.

**Remark:** The equation (3.1) was used in finding the twisted elliptic genus and new supersymmetric index of the non-standard embedding of heterotic string compactified on \( K3 \times T^2 \) orbifolded with \( \mathbb{Z}_2 \) group corresponding to the 2A conjugacy class of Mathieu group \( M_{24} \). [CD]

**Proof of Theorem [1]:** From Lemma [1] we know the location of all zeros of \( E_4 \) and \( E_6 \) in \( F_N \). Let us denote these sets of zeros in \( F_N \) as \( \mathbb{L}_{4,N} \) and \( \mathbb{L}_{6,N} \) respectively. Also let us denote the set of all zeros of \( \tilde{E}_N \) in \( F_N \) as \( \mathbb{L}_N \). From Lemma [2] we observe that

\[
\mathbb{L}_2 \subseteq \mathbb{L}_{6,2}, \quad \mathbb{L}_3 \subseteq \mathbb{L}_{4,3}, \quad \mathbb{L}_5 \subseteq \mathbb{L}_{6,5}, \quad \mathbb{L}_7 \subseteq \mathbb{L}_{4,7}.
\]

Let us chose an element \( \omega_{4,N} \) from \( \mathbb{L}_{4,N} \) for \( N = 3, 7 \) and \( \omega_{6,N} \) from \( \mathbb{L}_{6,N} \) for \( N = 2, 5 \). Now we re-write the resulting polynomial equations (3.1) – (3.4) as a product of two factors as follows:

\[
\begin{align*}
\tilde{E}_2(\omega_{6,2}) \left( \tilde{E}_2(\omega_{6,2})^2 - \frac{3}{4}E_4(\omega_{6,2}) \right) &= 0, (3.5) \\
\tilde{E}_3(\omega_{4,3}) \left( \tilde{E}_3(\omega_{4,3})^3 - \frac{8}{27}E_6(\omega_{4,3}) \right) &= 0, (3.6) \\
\tilde{E}_5(\omega_{6,5})^2 \left( \tilde{E}_5(\omega_{6,5})^4 - \frac{3}{5}E_5(\omega_{6,5})^2E_4(\omega_{6,5}) - \frac{9}{125}E_4(\omega_{6,5})^2 \right) &= 0, (3.7) \\
\tilde{E}_7^2(\omega_{4,7}) \left( \tilde{E}_7(\omega_{4,7})^6 - \frac{16}{49}E_6(\omega_{4,7})\tilde{E}_7(\omega_{4,7})^3 - \frac{64}{64827}E_6(\omega_{4,7})^2 \right) &= 0, (3.8)
\end{align*}
\]
Since the right hand side of the above set of equations are zero so at least one of these factors must be zero. Now we need to check the numerical values of these factors to determine the location of zeros of $\tilde{E}_N$ in $F_N$.

So we estimate the bounds of $\tilde{E}_N(\omega)$, where $\omega = \omega_{6,N}$ for $N = 2, 5$ and $\omega = \omega_{4,N}$ for $N = 3, 7$. We write the $q$ expansion of $\tilde{E}_N$ as follows:

$$\tilde{E}_N(\tau) = \sum_{n=0}^{m} a_n q^n + \sum_{n=m+1}^{\infty} a_n q^n = \sum_{n=0}^{m} a_n q^n + R(m, q).$$

Now we have,

$$|R(m, q)| = | \sum_{n=m+1}^{\infty} a_n (x + iy)^n | \leq \sum_{n=m+1}^{\infty} |a_n| (|x| + |y|)^n \leq \sum_{n=m+1}^{\infty} b_n (|x| + |y|)^n < \sum_{n=m+1}^{\infty} b_n (|x| + |y| + \epsilon)^n$$

where, $b_n = (N + 1)n(n + 1)/2 > (N + 1)\sigma_1(n) > |a_n|, \; q = x + iy, \; \epsilon > 0$.\footnote{\textsuperscript{2} $\epsilon$ is needed to approximate $|x|$ and $|y|$ in the computer program.}

Note that our choice of $b_n$ is such that $\sum_{n=m+1}^{\infty} b_n (|x| + |y| + \epsilon)^n$ can be estimated exactly in terms of $|x| + |y| + \epsilon$ and $m$. Also note that $\sum_{n=m+1}^{\infty} b_n (|x| + |y| + \epsilon)^n$ is convergent only if $|x| + |y| + \epsilon < 1$.

In the region $|x| + |y| + \epsilon < 1$ we estimate the numerical value of $\tilde{E}_N(\omega)$ up to first $m + 1$ terms in the $q$ expansion using equation (3.9) we estimate the upper bound of $|R(m, q)|$ in $F_N$. From this we have estimated an upper and lower bound of $|\tilde{E}_N(\omega)|$.\footnote{\textsuperscript{3} Numerical values for these estimates can be found in the mathematica notebook with this paper.}

However if $|x| + |y| \geq 1$ we use the following results

$$\tilde{E}_N\left(\frac{\tau + k}{N}\right) = \tilde{E}_N\left(\frac{\tau + k - N}{N}\right),$$

$$\tilde{E}_N\left(-\frac{1}{\tau + k}\right) = -\left(\frac{\tau + k}{N}\right)^2 \tilde{E}_N\left(\frac{\tau + k}{N}\right),$$

which is true for any $k \in \mathbb{Z}$. This implies that

$$\tilde{E}_N\left(-\frac{1}{\tau + k - N}\right) = \left(\frac{\tau + k - N}{(\tau + k)^2}\right)^2 \tilde{E}_N\left(-\frac{1}{\tau + k}\right),$$

where $0 \leq k \leq N - 1$. Putting $\omega = -\frac{1}{\tau + k}$ in (3.11) we have,

$$\tilde{E}_N\left(\frac{1}{1/\omega + N}\right) = (1/\omega + N)^2 \omega^2 \tilde{E}_N(\omega).$$

Note that if $\tau \in F_N$, then for $0 \leq k \leq N - 1$, we have $\frac{-1}{\tau + k} \in F_N$ and $\frac{-1}{\tau + k - N} \notin F_N$. So if $\omega = \omega_{4,N}$ for $N = 3, 7$ and $\omega = \omega_{6,N}$ for $N = 2, 5$ then the equation (3.12)
implies that there is a relation between the value of $\tilde{E}_N$ at a point outside $L_{4,N}$ (respectively $L_{6,N}$) and a point inside $L_{4,N}$ (respectively $L_{6,N}$). So at the point $\tau = \frac{1}{1 + \omega + N}$ where $|\Re(e^{2\pi i \tau})| + |\Im(e^{2\pi i \tau})| < 1$ we repeat the process as before. Thus we can estimate the upper and lower bound of $|\tilde{E}_N(\tau)|$ which in turn puts an upper and lower bound on $|\tilde{E}_N(\omega)|$ (see the mathematica file with this paper).

Thus we obtain the points where $\tilde{E}_N$ can be zero. Now to prove that these are the only zeros of $\tilde{E}_N$ in $F_N$ as given in table (1) we must check that at these points the second factor will not become zero. We shall do this as follows:

From the definition of $E_4$ and $E_6$ [1.30 and 1.31] we see that $E_4(i) > 1$ and $E_6(e^{2\pi i/3}) > 1$. Hence, from the modular transformation properties of $E_4$ and $E_6$ $|E_4(\omega_{0,2})| > 1, |E_4(\omega_{0,5})| > 1$ and $|E_6(\omega_{4,2})| > 1, |E_6(\omega_{4,7})| > 1$. So one can easily see that the second factor in equations (3.5) to (3.8) can only become zero if $|\tilde{E}_N(\tau)| > 10^{-2}$. So when the second factor would be zero that would never give a zero of $\tilde{E}_N(\omega)$ and vice versa. Now estimating the bounds on $|\tilde{E}_N|$ we find all the zeros of $\tilde{E}_N(\omega)$ which are given as:

$$\tilde{E}_2 \left( \frac{-1}{i+1} \right) = 0, \quad (3.13)$$

$$\tilde{E}_3 \left( \frac{-1}{e^{2\pi i/3} + 2} \right) = 0,$$

$$\tilde{E}_5 \left( \frac{-1}{i+2} \right) = \tilde{E}_5 \left( \frac{-1}{i+3} \right) = 0,$$

$$\tilde{E}_7 \left( \frac{-1}{e^{2\pi i/3} + 3} \right) = \tilde{E}_7 \left( \frac{-1}{e^{2\pi i/3} + 5} \right) = 0.$$

So these are the only possible zeros of $\tilde{E}_N$ in $F_N$. This completes the proof of Theorem 1.

Remarks

(1) The cusps of $\Gamma_0(N)$ are $0$ and $i\infty$ for prime $N$. However from the $q$ expansions it is obvious that $E_N(i\infty) = 1$. Also using Fricke involutions one can see that

$$\lim_{\epsilon \to 0} \tilde{E}_N \left( \frac{-1}{ie} \right) = \lim_{\epsilon \to 0} \left( \frac{-(ie)^2}{N} \right) \tilde{E}_N \left( \frac{i\epsilon}{N} \right),$$

so $\tilde{E}_N$ is non-zero at the cusps. So the only possible zeros of $\tilde{E}_N$ are at equations (3.13) in the fundamental domain $F_N$.

(2) This method does not easily generalize to $\tilde{E}_{11}$ (or higher prime numbers) because of the presence of cusp forms of weight 2 in $\Gamma_0(11)$. Also it doesn’t easily generalize to composite numbers.

Numerical estimates can be found in the mathematica notebook attached with this paper.
(3) One may try to generalize the method to different types of modular functions where there is a possibility of finding a polynomial relation in terms of modular functions whose zeros are known.

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