TRAVELING WAVE SOLUTIONS TO DIFFUSIVE HOLLING-TANNER PREDATOR-PREY MODELS

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Dedicated to Professor Sze-Bi Hsu

ABSTRACT. In this paper, we first establish the existence of semi-traveling wave solutions to a diffusive generalized Holling-Tanner predator-prey model in which the functional response may depend on both the predator and prey populations. Then, by constructing the Lyapunov function, we apply the obtained result to show the existence of traveling wave solutions to the diffusive Holling-Tanner predator-prey models with various functional responses, including the Lotka-Volterra type functional response, the Holling type II functional response and the Beddington-DeAngelis functional response.

1. Introduction. Traveling wave solutions to diffusive predator-prey systems have been studied extensively. In recent years, some researchers focused on the study of the diffusive Holling-Tanner predator-prey model. For example, Chen, Guo and Yao in [3] studied the diffusive Holling-Tanner model of Lotka-Volterra type functional response

\begin{align}
  u_t &= u_{xx} + ru(1-u) - rkuv, \\
  v_t &= dv_{xx} + sv \left(1 - \frac{v}{u}\right),
\end{align}

where \( d, r, s, \) and \( k \) are positive constants. Under the condition \( 0 < k < 1, \) they showed that system (1) admits a traveling wave solution with speed \( c \) iff \( c \geq c^* := 2\sqrt{ds}. \) Ai, Du and Peng in [1] first established the existence of semi-traveling wave solutions of a generalized Holling-Tanner model

\begin{align}
  u_t &= u_{xx} + B(u) - f(u)v, \\
  v_t &= dv_{xx} + sv \left(1 - \frac{v}{u}\right),
\end{align}

and then applied it to show the existence of traveling wave solutions to the following diffusive Holling-Tanner model

\begin{align}
  u_t &= u_{xx} + u(1-u) - \frac{ku^m}{1+bu^m}v, \\
  v_t &= dv_{xx} + sv \left(1 - \frac{v}{u}\right).
\end{align}

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They showed that system (2) admits a traveling wave solution with speed \( c \geq c^* \) for the two cases: (i) \( m = 1, \ 0 < k < 1, \) and \( b \geq 0; \) (ii) \( m = 2, \) and either \( k > 0, \ \ 0 \leq b < 3 \) or \( 0 < k < b^{2/3}(3 - b^{1/3}), \ 0 < b < 27. \) To the authors’ knowledge, there are no results in the literature for traveling wave solutions of the diffusive Holling-Tanner system with the Beddington-DeAngelis functional response

\[
\begin{align*}
  u_t &= u_{xx} + ru(1 - u) - \frac{rkuv}{1 + bu + ev}, \\
  v_t &= dv_{xx} + sv \left(1 - \frac{v}{u}\right),
\end{align*}
\]

where \( k, \ r, \ s, \ b, \) and \( e \) are positive constants. Here \( u(x, t) \) and \( v(x, t) \) represent the density of prey and predators at position \( x \) and time \( t, \) respectively; \( d \) denotes the ratio of the diffusivity of the predator to that of the prey. Besides, let us consider the case when \( b = e = 0. \) In this case, (3) is reduced to system (1) or system (2) with \( m = 1 \) and \( b = 0. \) For the case \( b > 0 \) and \( e = 0, \) (3) is reduced to system (2) with \( m = 1 \) and \( b > 0. \) So the existence of traveling wave solutions to these two cases has been investigated in [3] and [1]. However, the results are under the restriction \( 0 < k < 1. \) Based on the above reasons, in this paper, we will first establish the existence of semi-traveling wave solutions to the following generalized diffusive Holling-Tanner model

\[
\begin{align*}
  u_t &= u_{xx} + B(u) - F(u, v)v, \\
  v_t &= dv_{xx} + sv \left(1 - \frac{v}{u}\right),
\end{align*}
\]

and then use the obtained result to show the existence of traveling wave solutions of system (3) with \( b \geq 0 \) and \( e \geq 0. \) In particular, for \( b = e = 0, \) we will relax the restriction on \( k. \)

For convenience, we rewrite (4) in the following form

\[
\begin{align*}
  u_t &= u_{xx} + h(u)|f(u) - g(u, v)v|, \\
  v_t &= dv_{xx} + sv \left(1 - \frac{v}{u}\right).
\end{align*}
\]

For functions \( h, \ f, \) and \( g, \) we impose the following hypotheses:

\begin{itemize}
  \item[(H1)] \( h(u) \) and \( f(u) \) are twice continuously differentiable functions on \( [0, \infty) \) and \( g(u, v) \) is a twice continuously differentiable function on \( [0, \infty) \times [0, \infty). \)
  \item[(H2)] \( h(0) = 0 \) and \( h(u) > 0 \) for all \( u \in (0, 1). \)
  \item[(H3)] \( f(0) > 0, \ f(1) = 0 \) and \( f(u) \geq 0 \) for all \( u \in (0, 1). \)
  \item[(H4)] \( g(1, v) > 0 \) and \( g(u, v) \geq 0 \) for all \( u \in [0, 1) \) and \( v \in [0, 1]. \)
  \item[(H5)] There exists a unique positive number \( \eta^* \in (0, 1) \) such that \( f(\eta^*) - g(\eta^*, \eta^*)\eta^* = 0. \)
\end{itemize}

Under the hypotheses (H3) and (H5), system (5) has a boundary equilibrium point \((1, 0)\) and a unique interior equilibrium point (i.e., the coexistence equilibrium point) \((\eta^*, \eta^*)\). A solution \((u, v)\) of system (5) is called a traveling wave solution if it is of the form

\[
(u(x, t), v(x, t)) = (U(z), V(z)), \quad z = x + ct,
\]

where \( c \) denotes the wave speed and \( (U, V) \in C^2(\mathbb{R}) \times C^2(\mathbb{R}) \) is a pair of positive functions satisfying the boundary conditions \((U, V)(-\infty) = (1, 0)\) and \((U, V)(+\infty) = (\eta^*, \eta^*).\) Upon substituting (6) into systems (5), we are led to the governing
system for \((U, V)\) as follows:
\[
\begin{align*}
U'' - cU' + h(U)[f(U) - g(U, V)V] &= 0, \\
dV'' - cV' + sV \left(1 - \frac{V}{U}\right) &= 0
\end{align*}
\] (7)
on \mathbb{R}, together with the boundary conditions
\[
(U, V)(-\infty) = (1, 0) \quad \text{and} \quad (U, V)(+\infty) = (\eta^*, \eta^*).
\] (8)

Therefore, to show the existence of traveling wave solutions of system (5) is equivalent to show the existence of positive solutions of system (7) satisfying (8). Our main results are stated in the following.

**Theorem 1.1.** Suppose (H1)–(H5) hold. Then system (5) admits a semi-traveling wave solution \((u, v)\) with speed \(c\) iff \(c \geq c^*\). In addition, \(\delta < u < 1\) and \(0 < v < 1\) for some positive constant \(\delta\).

**Theorem 1.2.** Suppose that \(0 \leq b \leq 1\), \(e \geq 0\), and
\[
0 < k < \frac{1}{2} \left[1 + b + 2e + \sqrt{(1 + b + 2e)^2 + 16(1 + b + e)^2}/[2(1 + b + e)]\right].
\]
Then system (3) admits a traveling wave solution \((u, v)\) with speed \(c\) iff \(c \geq c^*\). In addition, \(\delta < u < 1\) and \(0 < v < 1\) for some positive constant \(\delta\).

In particular, when \(b = e = 0\), the restriction on \(k\) can be further relaxed as follows.

**Theorem 1.3.** Suppose that \(b = e = 0\). There exists a constant \(k_0 \in (4, 5)\) such that if \(0 < k < k_0\), then system (3) admits a traveling wave solution \((u, v)\) with speed \(c\) iff \(c \geq c^*\). In addition, \(\delta < u < 1\) and \(0 < v < 1\) for some positive constant \(\delta\).

Now we present some numerical results for system (3). In Fig.1 and Fig.2, we see that the large time behaviors of the solutions of the initial value problem of system (3) are two traveling waves propagating outwards in opposite directions, where the initial data \((u_0, v_0)\) is chosen so that \(u_0 = 1\) and \(v_0 = 0.05(1 + \text{sign}(51 - x))(1 + \text{sign}(x - 49))/4\). From Fig.3 and Fig.4, we conjecture that the restrictions \(b \leq 1\) in Theorem 1.2 and on \(k\) in Theorem 1.2 and Theorem 1.3 should be technical assumptions (the values of the parameters \(b\) and \(k\) in Fig.3 and Fig.4 do not satisfy the restrictions).

There are two parts in the proof of the existence of traveling wave solutions. In the first part, we show that system (5) admits a positive solution \((u, v)\) of the form (6) with \((U, V)(-\infty) = (1, 0)\). Such a solution is so-called a semi-traveling wave solution. In the second part, we apply the result in the first part to system (3). By proving \((U, V)(+\infty) = (\eta^*, \eta^*)\), we show that the obtained semi-traveling wave solutions are actually traveling wave solutions under certain conditions.

Our proof is outlined as follows. First, since system (7) has a singularity at \(U = 0\), we follow the idea of [1] to consider a modified system in which the reaction term \(sV(1 - V/U)\) is replace by the function \(sV(1 - V/\sigma_\epsilon(U))\). Here \(\sigma_\epsilon : [0, 1] \rightarrow (0, 1]\) is a continuous function defined by
\[
\sigma_\epsilon(U) := \begin{cases} 
U, & \text{if } U \geq \epsilon, \\
U + \epsilon e U - \epsilon, & \text{if } 0 \leq U < \epsilon,
\end{cases}
\] (9)
where \(\epsilon\) is a sufficiently small constant. Then we get a semi-traveling wave solution \((U_\epsilon, V_\epsilon)\) to the modification system. Next, since \(U_\epsilon\) has a positive lower bound \(\delta\),
Figure 1. The solution as a function of the spatial variable $x$ is plotted at $t=0$, $t=10$, $t=20$ and $t=30$. The initial data $(u_0, v_0)$ is chosen so that $u_0 = 1$ and $v_0 = 0.05 \times (1 + \text{sign}(51 - x)) \times (1 + \text{sign}(x - 49))/4$. The parameter values are $k = 1.4$, $b = e = 1$, $d = 1$, $r = 4$ and $s = 0.6$.

Figure 2. The solution as a function of the spatial variable $x$ is plotted at $t=0$, $t=5$, $t=10$ and $t=20$. The initial data $(u_0, v_0)$ is chosen so that $u_0 = 1$ and $v_0 = 0.05 \times (1 + \text{sign}(51 - x)) \times (1 + \text{sign}(x - 49))/4$. The parameter values are $k = 4$, $b = e = 0$, $d = 1$, $r = 2$ and $s = 0.5$. 
Figure 3. The solution as a function of the spatial variable $x$ is plotted at $t=0$, $t=10$, $t=20$ and $t=30$. The initial data $(u_0, v_0)$ is chosen so that $u_0 = 1$ and $v_0 = 0.05 \times (1 + \text{sign}(51 - x)) \times (1 + \text{sign}(x - 49))/4$. The parameter values are $k = 10$, $b = 5$, $e = 1$, $d = 1$, $r = 4$ and $s = 0.6$.

Figure 4. The solution as a function of the spatial variable $x$ is plotted at $t=0$, $t=5$, $t=10$ and $t=20$. The initial data $(u_0, v_0)$ is chosen so that $u_0 = 1$ and $v_0 = 0.05 \times (1 + \text{sign}(51 - x)) \times (1 + \text{sign}(x - 49))/4$. The parameter values are $k = 10$, $b = e = 0$, $d = 1$, $r = 2$ and $s = 0.5$. 
which is independent of $\epsilon$, it follows that $(U_\epsilon, V_\epsilon)$ with $\epsilon \leq \delta$ is actually a semi-traveling wave solution of (5). The argument of the proof for the existence of semi-traveling wave solutions of (5) is followed from that of [1] with a modification. Finally, we apply the obtained result to system (3) and show that, under certain conditions, the semi-traveling wave solutions are indeed traveling wave solutions by constructing the Lyapunov function and applying the LaSalle’s invariance principle. Though this argument is standard, the main difficulty is the construction of the Lyapunov function. Motivated by [4], we construct a Lyapunov function different from that in [1] to improve the known result.

The rest of the paper is organized as follows. In Section 2, we analyze the trajectory of system (7) near the equilibrium point $(1, 0)$ to show that there exist no semi-traveling wave solutions with speed $c < c^*$. For the existence of traveling wave solutions, since system (7) has a singularity, we follow the idea of [1] to first show that the modified system mentioned in the preceding paragraph has a positive solution $(U_\epsilon, V_\epsilon)$ and $U_\epsilon$ has a positive lower bound $\delta$, which is independent of $\epsilon$. Then $(U_\epsilon, V_\epsilon)$ with $\epsilon \leq \delta$ is actually a semi-traveling wave solution of (5). In Section 3, by constructing the Lyapunov function and using the LaSalle’s invariance principle, we prove that the obtained semi-traveling wave solutions to system (3) are actually traveling wave solutions under certain assumptions.

2. Semi-traveling wave solutions. In this section, we prove Theorem 1.1.

2.1. Non-existence of semi-traveling wave solutions. We show the nonexistence of semi-traveling wave solutions of system (5) with speed $c < c^*$ in the following.

Lemma 2.1. Suppose (H1)–(H5) hold. For $c < c^*$, there exist no positive solutions of system (7) satisfying

$$(U, V)(-\infty) = (1, 0).$$

Proof. First, we consider the linearized system of (7) around $(1, 0)$

\begin{align}
U'' - cU' + h(U)[f'(U)-g(U,V)V] &= 0, \\
V'' - cV' + sV &= 0.
\end{align}

Note that (10b) has two eigenvalues $\lambda_1$ and $\lambda_2$, where

$$\lambda_1 := \frac{c - \sqrt{c^2 - 4ds}}{2}, \quad \lambda_2 := \frac{c + \sqrt{c^2 - 4ds}}{2}.$$  

For contradiction, we assume $(U,V)$ is a positive solution of system (7) with $c < 2\sqrt{ds}$ such that $(U,V)(-\infty) = (1,0)$. Suppose that $c \leq -2\sqrt{ds}$. Then we have $\lambda_i < 0, \ i = 1, 2$, and so $V(z)$ is unbounded as $z \to -\infty$, a contradiction. Suppose $|c| < 2\sqrt{ds}$, then $\lambda_1$ and $\lambda_2$ form a complex conjugate pair. This would imply that $V(z)$ cannot be of the same sign for $z$ near negative infinity, a contradiction again. Hence we complete the proof of this lemma.

2.2. Existence of semi-traveling wave solutions. We will establish the existence of semi-traveling wave solutions of (5) with speed $c \geq c^*$. Since system (7) has a singularity at $U = 0$, we first consider the following modified system

\begin{align}
U'' - cU' + h(U)[f(U) - g(U,V)V] &= 0, \\
V'' - cV' + sV \left( 1 - \frac{V}{\sigma(U)} \right) &= 0
\end{align}

(11)
on $\mathbb{R}$, where $\sigma_{\epsilon}$ is the function defined by (9).

Following the arguments of [1, Lemma 2.3], we establish the following lemma for the existence of semi-traveling wave solutions to the modified system.

**Lemma 2.2.** Suppose (H1)–(H5) hold. For $c \geq c^*$, system (11) admits a positive solution $(U_{\epsilon}, V_{\epsilon})$ on $\mathbb{R}$ satisfying

$$0 < U_{\epsilon}(z) < 1, \quad 0 < V_{\epsilon}(z) < 1, \quad \forall z \in \mathbb{R},$$

and

$$(U_{\epsilon}, U'_{\epsilon}, V_{\epsilon}, V'_{\epsilon})(-\infty) = (1, 0, 0, 0).$$

Furthermore, $U'_{\epsilon}$ and $V'_{\epsilon}$ are bounded on $\mathbb{R}$, and there exist sufficiently small positive constants $\delta$ and $\epsilon_0$ such that, for $0 < \epsilon < \epsilon_0$,

$$U_{\epsilon}(z) > \delta, \quad \forall z \in \mathbb{R}. \quad (14)$$

**Proof.** By applying [1, Theorem 2.1], system (11) admits at least one nonnegative solution $(U_{\epsilon}, V_{\epsilon})$ satisfying (13) if $c \geq c^*$. In addition, $(U_{\epsilon}, V_{\epsilon})$ satisfies

$$0 < U_{\epsilon}(z) < 1 \quad \text{and} \quad 0 \leq V_{\epsilon}(z) \leq 1, \quad \forall z \in \mathbb{R}, \quad (15)$$

and $U'_{\epsilon}, V'_{\epsilon}$ are bounded on $\mathbb{R}$. Furthermore, we claim that $0 < V_{\epsilon}(z) < 1$ for all $z \in \mathbb{R}$. For contradiction, we assume that there exists $z_1 > 0$ such that $V_{\epsilon}(z_1) = 0$. Then $V'_{\epsilon}(z_1) = 0$ and so the existence and unique theorem gives that $V_{\epsilon} = 0$ for all $z \in \mathbb{R}$, which contradicts (15). Assume that there exists $\hat{z}_1 \in \mathbb{R}$ such that $V_{\epsilon}(\hat{z}_1) = 1$. Since $V_{\epsilon}$ attains the maximum at the point $z = \hat{z}_1$, it follows that $V'_{\epsilon}(\hat{z}_1) = 0$ and $V''_{\epsilon}(\hat{z}_1) \leq 0$. On the other hand, by (11), we have

$$dV''_{\epsilon}(\hat{z}_1) = cV'_{\epsilon}(\hat{z}_1) - sV_{\epsilon}(\hat{z}_1) \left(1 - \frac{V_{\epsilon}(\hat{z}_1)}{\sigma_{\epsilon}(U_{\epsilon}(\hat{z}_1))}\right) > 0,$$

a contradiction. Hence $V_{\epsilon}(z) < 1$ for all $z \in \mathbb{R}$.

Now it suffices to claim that there exist sufficiently small positive constants $\delta$ and $\epsilon_0$ such that $U_{\epsilon}(z) > \delta$ for all $z \in \mathbb{R}$ and $0 < \epsilon < \epsilon_0$. We divide the proof of this claim into several steps. For convenience, we denote $U := U_{\epsilon}, \ V := V_{\epsilon}$, and $\sigma(U) := \sigma_{\epsilon}(U)$ in the remaining proof.

**Step 1.** Show that

$$|U'(z)|/U(z) \leq \gamma_+, \quad \forall z \in \mathbb{R}. \quad (16)$$

Here $\gamma_+ := (c + \sqrt{c^2 + 4M_1})/2$ is a positive solution of $c\gamma + M_1 - \gamma^2 = 0$, where

$$M_1 := \sup_{(U,V) \in (0,1] \times (0,1]} \frac{h(U)[g(U,V)V - f(U)]}{U} > 0.$$  

By (11), the function $\phi_1 := U'/U$ satisfies

$$\phi_1' = \frac{U''}{U} - \phi_1^2 = c\phi_1 - \frac{h(U)[f(U) - g(U,V)V]}{U} - \phi_1^2 \leq c\phi_1 + M_1 - \phi_1^2.$$

Since $\psi(z) := \gamma_+ \geq 0$ satisfies $\psi' = c\psi + M_1 - \psi^2$ and $\phi_1(-\infty) = 0 < \psi(-\infty)$, one can easily verify that $\phi_1(z) \leq \psi(z)$ for all $z \in \mathbb{R}$. Hence $\phi_1 \leq \gamma_+$ on $\mathbb{R}$.

Now we claim that $\phi_1 \geq -\gamma_+$. For contradiction, suppose that $\phi_1(z_2) < -\gamma_+$ at some point $z_2 \in \mathbb{R}$. Let $\phi(z)$ be the unique solution of the equation

$$\phi' = c\phi + M_1 - \phi^2$$

satisfying $\phi(0) = 0$. Since $\phi_1(z) \leq \phi(z)$, it follows that $\phi(z) > 0$ for all $z \in \mathbb{R}$. Therefore, $\phi_1(z) < -\gamma_+$ for all $z \in \mathbb{R}$, which contradicts the above assumption. Hence $\phi_1 \geq -\gamma_+$ and $\phi(z)$ is the unique solution of the equation

$$\phi' = c\phi + M_1 - \phi^2.$$
such that \( \phi(z_2) = \phi_1(z_2) \). By the comparison principle, we have

\[
\phi_1(z) \leq \phi(z), \quad \forall z \geq z_2.
\]  

(18)

Let \( \gamma_- := (c - \sqrt{c^2 + 4M_1})/2 \) be another root of \( c\gamma + M_1 - \gamma^2 = 0 \). By solving (17), we have

\[
\phi(z) = \frac{\gamma_+ - \gamma_- e^{-\sqrt{c^2 + 4M_1}(z - z_3)}}{1 - e^{-\sqrt{c^2 + 4M_1}(z - z_3)}},
\]

where

\[
z_3 := z_2 + \frac{1}{\sqrt{c^2 + 4M_1}} \ln \left( \frac{\gamma_+ - \phi(z_2)}{\gamma_- - \phi(z_2)} \right) > z_2.
\]

This yields \( \phi \rightarrow -\infty \) as \( z \rightarrow z_3^- \). Together with (18), we find that \( \phi_1 \rightarrow -\infty \) as \( z \rightarrow z_4^- \) for some point \( z_4 \in (z_2, z_3) \), which contradicts the fact that \( \phi_1 \) is defined for all \( z \in \mathbb{R} \). Hence (16) holds.

**Step 2.** Show that

\[
V'(z)/V(z) \leq \lambda, \quad \forall z \in \mathbb{R},
\]

(19)

where \( \lambda := (c - \sqrt{c^2 - 4d^2s})/(2d) \).

For contradiction, we assume that \( (V'/V)(\hat{z}_1) > \lambda \) at some point \( \hat{z}_1 \). Then \( (V'/V)(z) > \lambda \) for all \( z \geq \hat{z}_1 \). To see this, we also use a contradictory argument and assume that there exists \( \hat{z}_2 > \hat{z}_1 \) such that \( (V'/V)(\hat{z}_2) = \lambda \) and \( (V'/V)'(\hat{z}_2) \leq 0 \). Then, by using \( d\lambda^2 - c\lambda + s = 0 \), we have

\[
\left( \frac{V''}{V'} \right) (\hat{z}_2) = \frac{cV'(\hat{z}_2) - sV(\hat{z}_2)(1 - V(\hat{z}_2)/\sigma(U(\hat{z}_2)))}{d\lambda V(\hat{z}_2)} > \frac{c\lambda V(\hat{z}_2) - sV(\hat{z}_2)}{d\lambda V(\hat{z}_2)} = \frac{c\lambda - s}{d\lambda} = \lambda,
\]

which, together with \( (V'/V)(\hat{z}_2) = \lambda \), yields that

\[
\left( \frac{V''}{V'} \right)'(\hat{z}_2) = \left( \frac{V''}{V'} \right)(\hat{z}_2) - \left( \frac{V''}{V} \right)^2(\hat{z}_2) > 0,
\]

a contradiction. Hence \( V'(z)/V(z) > \lambda \) for all \( z \geq \hat{z}_1 \) and so \( V(z) \geq V(\hat{z}_1)e^{\lambda(z - \hat{z}_1)} \rightarrow \infty \) as \( z \rightarrow \infty \), which contradicts the fact that \( V \leq 1 \). Therefore, (19) holds.

**Step 3.** Show that there exist a positive constant \( \epsilon_1 \) such that, for \( 0 < \epsilon < \epsilon_1 \),

\[
V(z)/\sigma(U(z)) < dM_2/s, \quad \forall z \in \mathbb{R},
\]

(20)

where \( M_2 := s/d + c|1/d - 1|\gamma_+ + M_1 + \gamma_2^2 \).

Let \( \rho := V/\sigma(U) \). By computation, we obtain that

\[
\rho' = \left( \frac{V'}{V} - \frac{(\sigma(U))'}{\sigma(U)} \right) \rho
\]

(21)
and
\[
\rho'' = \left( \frac{V'}{V} - \frac{(\sigma(U))'}{\sigma(U)} \right) \rho' + \left[ \frac{V''}{V} - \left( \frac{V'}{V} \right)^2 - \frac{\sigma''(U)(U')^2 + \sigma'(U)U''}{\sigma(U)} \right] \rho \\
= \left( \frac{V'}{V} - \frac{(\sigma(U))'}{\sigma(U)} \right) \rho' + \left\{ \frac{c V'}{d} - \frac{s}{d} (1 - \rho) - \left( \frac{V'}{V} \right)^2 - \frac{\sigma''(U)(U')^2}{\sigma(U)} \right\} \rho \\
= \left( \frac{V'}{V} - \frac{(\sigma(U))'}{\sigma(U)} \right) \rho' + \left\{ \frac{c V'}{d} - \frac{s}{d} (1 - \rho) - \frac{\sigma'(U)U'}{\sigma(U)} \cdot h(U)[f(U) - g(U,V)V] \right\} \rho \\
= \left( \frac{V'}{V} - \frac{(\sigma(U))'}{\sigma(U)} \right) \rho' + \left\{ \frac{c V'}{d} - \frac{s}{d} (1 - \rho) - \frac{\sigma'(U)U'}{\sigma(U)} \right\} \rho \\
= \left( \frac{c}{d} - \frac{2\sigma(U)}{\sigma(U)} \right) \rho' + \left\{ \frac{s}{d} \rho - \frac{s}{d} \frac{1}{d} (1 - 1) \frac{\sigma'(U)U'}{\sigma(U)} \right\} \rho.
\]

From step 3 in [1, Lemma 2.3], we can find \( \epsilon_1 > 0 \) such that, for \( 0 < \epsilon < \epsilon_1 \) and \( U > 0 \),
\[
\max\{U, \epsilon e^{-1/\epsilon}\} \leq \sigma(U) \leq \min\{1, U + \epsilon\}, \quad 0 < \sigma'(0) \leq \sigma'(U) \leq 1, \quad 0 \leq \sigma''(U) \leq \sigma''(0) < 1,
\]
so that
\[
\frac{\sigma'(U)}{\sigma(U)} \leq \frac{1}{U} \quad \text{and} \quad \frac{\sigma''(U)}{\sigma(U)} \leq \frac{1}{U^2}.
\]

Using (24), (16), (19), and the definition of \( M_1 \), we get from (22) that
\[
\rho'' > \left( \frac{c}{d} - \frac{2\sigma(U)}{\sigma(U)} \right) \rho' + \left( \frac{s}{d} \rho - M_2 \right) \rho.
\]

Multiplying (25) by the integrating factor \( Q(z) := \sigma^2(U)e^{-cz/d} \), we obtain that
\[
[Q(z)\rho'(z)]' > Q(z) \left[ \frac{s}{d} \rho(z) - M_2 \right] \rho(z), \quad \forall z \in \mathbb{R}.
\]

Suppose that \( \rho(z) < dM_2/s \) for all \( z \in \mathbb{R} \) is false. Due to \( \rho(-\infty) = 0 \), there exists a smallest \( z_5 \) such that \( \rho(z_5) = dM_2/s \) and \( \rho'(z_5) \geq 0 \). Together with (26), we get \( [Q(z_5)\rho'(z_5)]' > 0 \). So there exists a positive constant \( \eta_2 \) such that \( Q(z)\rho'(z) > Q(z_5)\rho'(z_5) \geq 0 \) for all \( z \in (z_5, z_5 + \eta_2) \). Since \( Q(z) > 0 \), it implies that \( \rho'(z) > 0 \) and \( \rho(z) > dM_2/s \) for all \( z \in (z_5, z_5 + \eta_2) \). Further, we claim that \( \rho'(z) > 0 \) for all \( z > z_5 \). For contradiction, we assume that there exists a smallest \( z_6 > z_5 \) such that \( \rho'(z_6) \leq 0 \). By integrating (26) from \( z_5 \) to \( z_6 \) and using the fact that \( \rho'(z_5) \geq 0, \rho'(z_6) \leq 0, \) and \( \rho' > 0 \) in \( (z_5, z_6) \), we have
\[
0 \geq Q(z_6)\rho'(z_6) - Q(z_5)\rho'(z_5) \geq \int_{z_5}^{z_6} Q(z) \left[ \frac{s}{d} \rho(z) - M_2 \right] \rho(z)dz > 0,
\]
a contradiction. So \( \rho'(z) > 0 \) and \( \rho(z) > dM_2/s \) for all \( z > z_5 \), which gives
\[
[Q(z)\rho'(z)]' > 0, \quad \forall z > z_5 \quad \text{and} \quad Q(z)\rho'(z) > Q(z_5 + \eta_2)\rho'(z_5 + \eta_2), \quad \forall z > z_5 + \eta_2,
\]
and therefore
\[ \rho'(z) > \frac{Q(z + \eta_2)}{Q(z)} \cdot \rho'(z + \eta_2) = \frac{\sigma^2(U(z + \eta_2))}{\sigma^2(U(z))} \cdot e^{c(z-(z+\eta_2))/d} \rho'(z + \eta_2) \geq \frac{\sigma^2(U(z + \eta_2)) \cdot e^{c(z-(z+\eta_2))/d} \rho'(z + \eta_2)}{\rho(z + \eta_2)} \cdot z > z \]
for all \( z > z \).

Combining (19) and (29), we get
\[ \rho(z) \to \infty \text{ as } z \to \infty. \] (27)

By (24) and (16), we have
\[ \left| \frac{(\sigma(U(z)))'}{\sigma(U(z))} \right| = \left| \frac{\sigma'(U(z))U'(z)}{\sigma(U(z))} \right| < \gamma_+. \] (28)

For all \( z > z \), we use (21) and \( \rho'(z) > 0 \) to deduce that
\[ \frac{V'(z)}{V(z)} = \frac{(\sigma(U(z)))'}{\sigma(U(z))} > 0, \]
which, together with (28), yields
\[ \frac{V'(z)}{V(z)} > -\gamma_. \] (29)

Combining (19) and (29), we get
\[ \frac{|V'(z)|}{V(z)} \leq M_3 \] (30)
for all \( z > z \), where \( M_3 := \max\{c/(2d), \gamma_+\} \). By (21), (30), and (28), we deduce that
\[ \left| \left( \frac{c}{d} - \frac{2(\sigma(U))'}{\sigma(U)} \right) \rho' \right| = \left| \left( \frac{c}{d} - \frac{2(\sigma(U))'}{\sigma(U)} \right) \left( \frac{V'}{V} - \frac{(\sigma(U))'}{\sigma(U)} \right) \rho \right| \leq M_4 \rho, \]
where \( M_4 := (c/d + 2\gamma_+)(M_3 + \gamma_+) \). Then, by (25), we get
\[ \rho''(z) > \left( \frac{s}{d} \rho(z) - M_2 - M_4 \right) \rho(z) \]
for all \( z > z \). Due to \( \rho(\infty) = \infty \), there exists \( z_6 > z \) such that, for \( z > z_6 \),
\[ \rho(z) > 2d(M_2 + M_4)/s, \]
which follows that
\[ \rho''(z) > \frac{s}{2d} \rho(z)^2, \]
and so, by multiplying the above inequality by \( \rho' \) and integrating the resulting inequality, we obtain
\[ (\rho')^2(z) > (\rho')^2(z_6) + \frac{s}{3d} [\rho(z)^3 - \rho(z_6)^3]. \]

Now we take \( z_7 > z_6 \) such that, for \( z > z_7 \),
\[ \rho(z)^3 > 2\rho(z_6)^3, \]
which yields that
\[ (\rho')^2(z) > \frac{s}{6d} \rho(z)^3, \]
and therefore,
\[ \rho'(z) > \sqrt{\frac{s}{6d} \rho(z)^3/2}. \]
This implies that \( \rho \) blows up in finite time, a contradiction. Thus we finish the proof of step 3.

**Step 4.** Show that there exist positive constants \( \epsilon_0 \) and \( \delta \) such that, for \( 0 < \epsilon < \epsilon_0 \),

\[
U(z) > \delta, \forall z \in \mathbb{R}.
\]  

Set

\[
M_0 := \max_{(u,v) \in [0,1] \times [0,1]} q(u,v).
\]

We consider the function \( \Theta(\tau) := f(\tau) - (dM_0M_2/s)(\tau + \alpha) \), where \( \alpha := f(0)s/(2dM_0M_2) \). Due to \( \Theta(0) > 0 \), there exist a constant \( \delta \in (0,1) \) such that

\[
\Theta_0(\tau) > 0, \forall \tau \in [0,\delta].
\]

Recall that \( U(-\infty) = 1 \). For contradiction, we assume that there exists a smallest \( z_8 \) such that \( U(z_8) = \delta \) and \( U'(z_8) \leq 0 \). For \( 0 < \epsilon < \epsilon_0 := \min\{\epsilon, \alpha\} \), it follows from (11), (20), (23), and the definitions of \( M_0 \) that, as long as \( U \leq \delta \) and \( U' \leq 0 \),

\[
U'' = cU'' - h(U)[f(U) - g(U,V)V] \\
\leq cU'' - h(U)[f(U) - \frac{dM_0M_2}{s}(U + \alpha)] < 0.
\]

Hence one can easily verify that \( U''(z) < 0 \) and \( U'(z) < 0 \) for all \( z > z_8 \) and therefore \( U(z) \to -\infty \) as \( z \to \infty \), which contradicts the boundedness of \( U \). Thus we finish the proof of (31) and the theorem. \( \square \)

With the help of Lemma 2.2, we establish the following result for the existence of semi-traveling wave solutions of system (7), which, together with Lemma 2.1, gives Theorem 1.1.

**Lemma 2.3.** Suppose (H1)–(H5) hold. For any \( c \geq c^* \), there is a sufficiently small constant \( \delta > 0 \) such that the system (7) has at least one positive solution \( (U,V) \) on \( \mathbb{R} \) satisfying

\[
\delta < U(z) < 1, \quad 0 < V(z) < 1, \quad \forall z \in \mathbb{R}; \quad V(z) > \delta, \quad \forall z \geq z_0
\]

for some \( z_0 \in \mathbb{R} \), and

\[
(U,U',V,V')(\infty) = (1,0,0,0).
\]

Furthermore, \( U',V' \) are bounded on \( \mathbb{R} \).

**Proof.** Pick \( \epsilon \) sufficiently small such that \( 0 < \epsilon < \min\{\epsilon_0,\delta\} \), where \( \epsilon_0 \) and \( \delta \) are constants defined in Lemma 2.2. By Lemma 2.2, system (11) admits a solution \((U_\epsilon,V_\epsilon)\) satisfying (12), (13), and (14). Since \( U_\epsilon > \delta > \epsilon \), it follows from definition of \( \sigma_\epsilon \) that \( \sigma_\epsilon(U_\epsilon) = U_\epsilon \) and so \((U,V) := (U_\epsilon,V_\epsilon)\) is a solution of (7) satisfying

\[
\delta < U(z) < 1, \quad 0 < V(z) < 1, \quad \forall z \in \mathbb{R},
\]

and

\[
(U,U',V,V')(\infty) = (1,0,0,0).
\]

Now we claim that there is a \( z_0 \in \mathbb{R} \) such that \( V(z_0) > \delta \). For contradiction, we assume that

\[
V(z) \leq \delta, \quad \forall z \in \mathbb{R}.
\]  

Since \( V(-\infty) = 0 \) and \( V(z) > 0 \) for all \( z \in \mathbb{R} \), there are two possibilities: (i) there is a smallest \( \hat{z} \) such that \( V'(\hat{z}) \leq 0 \); (ii) \( V'(z) > 0 \) for all \( z \in \mathbb{R} \). For the case (i), we note that, as long as \( V \leq \delta \) and \( V' \leq 0 \), it follows from (31) that

\[
dV'' = cV' - sV \left(1 - \frac{V}{U}\right) < 0,
\]
which implies that \( V''(z) < 0 \) and \( V'(z) < 0 \) for all \( z > \hat{z} \). This leads to \( V(z) \to -\infty \) as \( z \to \infty \), that contradicts the boundedness of \( V(z) \). For the case (ii), \( V(\infty) \) exists and is positive. Since \( V \) and \( sV(1 - V/U) \) are bounded on \( \mathbb{R} \), it follows from Lemma A.3 that \( V'' \) and \( V''' \) are bounded on \( \mathbb{R} \). Then, differentiating the second equation of (7) and using Lemma A.3 again, we also have \( V''' \) is bounded on \( \mathbb{R} \). Thus, by Lemma A.1, we have \( V'(\infty) = V''(\infty) = 0 \). This, together with (7) and the fact that \( V(\infty) > 0 \), yields that \( U(\infty) \) also exists and \( U(\infty) = V(\infty) \). From (31) and (35), we get \( U(\infty) = V(\infty) = \delta \). Arguing as above, we have \( U'(\infty) = 0 \) and \( U''(\infty) = 0 \). Therefore, using the first equation of (7), we get that \( f(\delta) - g(\delta, \delta) \delta = 0 \). On the other hand, by the definitions of \( M_2 \) in step 3, we know that \( dM_2/s > 1 \). Together with definition of \( M_0 \) and (32), we get that

\[
f(\delta) - g(\delta, \delta) \delta > f(\delta) - dM_2/s \cdot \delta > 0,
\]

a contradiction. Hence we conclude that there is a \( z_0 \in \mathbb{R} \) such that \( V(z_0) > \delta \).

Furthermore, we claim that \( V(z) > \delta \) for all \( z \geq z_0 \). For contradiction, we assume that there exists \( \hat{z} > z_0 \) such that \( V(\hat{z}) = \delta \) and \( V'(\hat{z}) \leq 0 \). Recall we have shown in the case (i) that \( V'' < 0 \) as long as \( V \leq \delta \) and \( V' \leq 0 \). It follows that \( V''(z) < 0 \) and \( V'(z) < 0 \) for all \( z > \hat{z} \) and therefore \( V(z) \to -\infty \) as \( z \to \infty \), which contradicts the boundedness of \( V \). So we finish the proof of this lemma.

3. Existence of traveling wave solutions. In this section, we prove Theorem 1.2 and Theorem 1.3 for the existence of traveling wave solutions of system (3). To be specific, we apply the results obtained in Lemma 2.3 to system (3) and show that, under certain conditions, the semi-traveling wave solutions of (3) are actually traveling wave solutions by constructing Lyapunov function and using the Lasalle’s invariance principle.

To begin with, we set \( h(u) := ru, f(u) := 1 - u, \) and \( g(u, v) := k/(1 + bu + ev) \), where \( r > 0, k > 0, b \geq 0 \) and \( e \geq 0 \). Then (5) and (7) become (3) and

\[
\begin{align*}
U'' - dU' + ru(1 - U) - \frac{rkUV}{1 + bu + ev} &= 0, \\
dV'' - cV' + sV(1 - \frac{V}{U}) &= 0,
\end{align*}
\]

(36)

respectively. Besides, it is easy to check that the functions \( f, g, \) and \( h \) satisfy the assumptions (H1)–(H5) with

\[
\eta' = \begin{cases} 
\frac{1}{k + 1}, & \text{if } b = e = 0, \\
\frac{b + e - 1 - k + \sqrt{(b + e - 1 - k)^2 + 4(b + e)}}{2(b + e)}, & \text{otherwise}.
\end{cases}
\]

So it follows from Lemma 2.3 that for any \( c \geq c^* \), there is a sufficiently small constant \( \delta > 0 \) such that system (36) has at least one positive solution \((U, V)\) on \( \mathbb{R} \) satisfying

\[
\delta < U(z) < 1, \quad 0 < V(z) < 1, \quad \forall z \in \mathbb{R}; \quad V(z) > \delta, \quad \forall z \geq z_0
\]

(38)

for some \( z_0 \in \mathbb{R} \), and

\[
(U, U', V, V')(\infty) = (1, 0, 0, 0).
\]

(39)

In addition, there exists a positive constant \( M \) such that

\[
|U'| < M, \quad |V'| < M, \quad \forall z \in \mathbb{R}.
\]

(40)
Moreover, we will use the LaSalle’s invariance principle to prove that
\[(U, U', V, V')(+\infty) = (\eta^*, 0, \eta^*, 0).\] (41)

For this, we first rewrite system (36) as a first-order ODEs system:

\[
U' = Y,
Y' = cY - rU \left[ 1 - U - \frac{kV}{1 + bU + eV} \right],
V' = Z,
dZ' = cZ - sV \left( 1 - \frac{V}{U} \right),
\] (42)

and set
\[\Sigma := (\delta, 1) \times (-M, M) \times (\delta, 1) \times (-M, M).\]

Next, motivated by [4], we define the Lyapunov function
\[L(U, Y, V, Z) := \frac{cU}{r} - \frac{V}{r} + \left( \frac{\eta^*}{r} \right)^2 + \frac{c(\eta^*)^2}{rU} + \kappa \left( cV - dZ + \frac{d\eta^* Z}{V} - c\eta^* \ln V \right),\]
where \(\kappa\) is a positive constant to be determined later. To proceed, we need the following lemma.

**Lemma 3.1.** The equation \(-k^3 - k^2 + 16k + 32 = 0\) has a unique solution, saying \(k_0\). In addition, \(k_0 \in (4, 5)\) and

\[-k^3 - k^2 + 16k + 32 > 0, \forall k < k_0.\]

**Proof.** Note that \(P_0(k) := -k^3 - k^2 + 16k + 32\) has a positive local minimum at \(k = -8/3\) and a positive local maximum at \(k = 2\). In addition, the graph of \(P_0\) is concave up on \((-\infty, -1/3)\) and concave down on \((-1/3, \infty)\). Since \(P_0(4) > 0\) and \(P_0(5) < 0\), we can find a number \(k_0 \in (4, 5)\) such that \(P_0(k_0) = 0\), and \(P_0 > 0\) in \((-\infty, k_0)\) and \(P_0 < 0\) in \((k_0, \infty)\). Hence we complete the proof of this lemma. \(\square\)

**Lemma 3.2.** Suppose one of the following conditions holds:

(i) \(b = e = 0\) and \(0 < k < k_0\), where \(k_0\) is defined in Lemma 3.1;
(ii) \(0 \leq b \leq 1, e \geq 0\) and \(0 < k < \left[ -(1+b+2e) + \sqrt{(1+b+2e)^2 + 16(1+b+e)^2} \right] / [2(1+b+e)] < 2.\)

If \(c \geq c^*\), then the orbital derivative of \(L\) along any trajectory \(X(z) := (U(z), Y(z), V(z), Z(z))\) of (42) lying in \(\Sigma\) is non-positive; that is,

\[\frac{d}{dz} L(X(z)) \leq 0.\]

**Proof.** By (42), we see that

\[1 - \eta^* = \frac{k\eta^*}{1 + (b + e)\eta^*}.\] (43)
Together with (42), we deduce that
\[
\frac{d}{dz} L(X(z)) = \nabla L(X(z)) \cdot X'(z)
\]
\[
= \frac{eY}{r} - \left[ \frac{eY}{r} - U \left( 1 - U - \frac{kV}{1 + bU + eV} \right) \right]
\]
\[
+ \frac{(\eta^*)^2}{U^2} \cdot \left[ \frac{eY}{r} - U \left( 1 - U - \frac{kV}{1 + bU + eV} \right) \right]
\]
\[
- \frac{2(\eta^*)^2 Y^2}{r} - \frac{e(\eta^*)^2}{r U^2} Y + c k Z - \kappa \left[ c Z - s V(1 - V) \right]
\]
\[
+ \frac{\kappa \eta^*}{V} \cdot \left[ c Z - s V(1 - V) \right] - \frac{k c \eta^* Z^2}{V^2} - \frac{\kappa c \eta^* Z}{V}.
\]
\[
\leq \frac{U^2 - (\eta^*)^2}{U} \cdot \left[ 1 - U - \frac{k V}{1 + bU + eV} \right] + \frac{\kappa s}{U} (V - \eta^*)(U - V)
\]
\[
= \frac{1}{U} (U + \eta^*)(U - \eta^*) \left[ - (U - \eta^*) + \frac{k(\eta^* - V) + k b \eta^*(U - V)}{(1 + (b + e) \eta^*)(1 + bU + eV)} \right]
\]
\[
= \frac{1}{U} \left\{ - A(U - \eta^*)^2 + B(U - \eta^*)(V - \eta^*) - C(V - \eta^*)^2 \right\},
\]
where
\[
A := \left[ 1 - \frac{k b \eta^*}{(1 + (b + e) \eta^*)(1 + bU + eV)} \right] (U + \eta^*),
\]
\[
B := \kappa s - \frac{k(1 + b \eta^*)(U + \eta^*)}{(1 + (b + e) \eta^*)(1 + bU + eV)},
\]
and
\[
C := \kappa s.
\]

By \( U, V > 0 \) and using (43), and (i) and (ii), we get
\[
\frac{k b \eta^*}{1 + (b + e) \eta^*}(1 + bU + eV) < \frac{k b \eta^*}{1 + (b + e) \eta^*} = b(1 - \eta^*) < 1,
\]
and so \( A > 0 \). Thus, if \( B^2 - 4 A C < 0 \), that is,
\[
\left[ \frac{\kappa s}{k} - \frac{k(1 + b \eta^*)(U + \eta^*)}{(1 + (b + e) \eta^*)(1 + bU + eV)} \right]^2
\]
\[
- 4 \kappa s \left[ 1 - \frac{k b \eta^*}{(1 + (b + e) \eta^*)(1 + bU + eV)} (U + \eta^*) \right] < 0,
\]
then the quadratic form
\[
- A(U - \eta^*)^2 + B(U - \eta^*)(V - \eta^*) - C(V - \eta^*)^2
\]
is negative unless \( U = \eta^* \) and \( V = \eta^* \). So we get \( dL(X(z))/dz \leq 0 \).

Now we must find \( \kappa > 0 \) such that (45) holds. For this, we rewrite (45) as the form
\[
(k \kappa)^2 - 2(U + \eta^*) \left[ \frac{k(1 - b \eta^*)}{1 + (b + e) \eta^*(1 + bU + eV)} + 2 \right] (k \kappa)
\]
\[
+ \frac{k^2(1 + b \eta^*)^2(U + \eta^*)^2}{(1 + (b + e) \eta^*)^2(1 + bU + eV)^2} < 0.
\]
We claim that
\[
D := \left[ \frac{k(1 - b\eta^*)}{(1 + (b + e)\eta^*)(1 + bU + eV)} + 2 \right]^2 - \frac{k^2(1 + b\eta^*)^2}{(1 + (b + e)\eta^*)^2(1 + bU + eV)^2} > 0. 
\]

After computation, we use \(1 + bU + eV > 1\) and (43) to get
\[
D = 4 \left[ 1 + \frac{k(1 - b\eta^*)}{(1 + (b + e)\eta^*)(1 + bU + eV)} - \frac{k^2 b\eta^*}{k(1 - b\eta^*)} \right] > 4 \left[ 1 + \frac{1 - b}{1 + b + e} \left( \frac{1}{\eta^*} - 1 \right) \right]
\]

(47)

Since \(b \leq 1\), (47) implies \(D > 0\). So (46) holds if and only if
\[
(U + \eta^*)(H - \sqrt{D}) < \kappa s < (U + \eta^*)(H + \sqrt{D}),
\]

(48)

where
\[
H := 2 + \frac{k(1 - b\eta^*)}{(1 + (b + e)\eta^*)(1 + bU + eV)}. 
\]

By (i), (ii), \(0 < U \leq 1\) and \(0 < V \leq 1\), we have
\[
2 + \frac{1 - b\eta^*}{1 + b + e} \left( \frac{1}{\eta^*} - 1 \right) < H < 2 + \frac{k(1 - b\eta^*)}{1 + (b + e)\eta^*}. 
\]

This, together with (43), yields
\[
2 + \frac{1 - b\eta^*}{1 + b + e} \left( \frac{1}{\eta^*} - 1 \right) < H < 2 + (1 - b\eta^*) \left( \frac{1}{\eta^*} - 1 \right). 
\]

(49)

Note that \(H - \sqrt{D}\) is positive. By (47), (49), and \(0 < U \leq 1\), we can find a suitable \(\kappa > 0\) such that (48) holds if
\[
(1 + \eta^* \left[ 2 + (1 - b\eta^*) \left( \frac{1}{\eta^*} - 1 \right) - 2 \sqrt{1 + \frac{1 - b}{1 + b + e} \left( \frac{1}{\eta^*} - 1 \right)} \right] < \eta^* \left[ 2 + \frac{1 - b\eta^*}{1 + b + e} \left( \frac{1}{\eta^*} - 1 \right) + 2 \sqrt{1 + \frac{1 - b}{1 + b + e} \left( \frac{1}{\eta^*} - 1 \right)} \right]. 
\]

(50)

Rearranging (50), we get
\[
(1 + \eta^* \left[ 2 + (1 - b\eta^*) \left( \frac{1}{\eta^*} - 1 \right) \right] - \eta^* \left[ 2 + \frac{1 - b\eta^*}{1 + b + e} \left( \frac{1}{\eta^*} - 1 \right) \right] < (2 + 4\eta^*) \left[ 1 + \frac{1 - b}{1 + b + e} \left( \frac{1}{\eta^*} - 1 \right) \right],
\]

which is equivalent to
\[
1 - b + \frac{b + e}{1 + b + e} + \frac{1}{\eta^*} - \frac{\eta^*}{1 + b + e} + \frac{b(b + e)(\eta^*)^2}{1 + b + e} < (2 + 4\eta^*) \left[ \frac{2b + e}{1 + b + e} + \left( \frac{1 - b}{1 + b + e} \right) \frac{1}{\eta^*} \right]. 
\]

(51)
For (i), we use \( b = e = 0 \) and \( \eta^* = 1/(k + 1) \) to simplify (51) in the form

\[
-k^3 - k^2 + 16k + 32 > 0.
\] (52)

Since \( 0 < k < k_0 \), it follows from Lemma 3.1 that (52) holds, and so does (50). For (ii), we first claim that \( \eta^* > 1/(1 + k) \). Using (43) and the fact \( 1 + (b + e)\eta^* > 1 \), we have \( 1 - \eta^* < k\eta^* \), which implies \( \eta^* > 1/(1 + k) \). By \( 1/(1 + k) < \eta^* < 1 \), (51) holds if

\[
1 - b + \frac{b + e}{1 + b + e} + 1 + k - \frac{e}{(1 + b + e)(1 + k)} + \frac{b(b + e)}{1 + b + e} < 2 + \frac{2k + 6}{1 + k} \sqrt{2b + e} + \frac{1 - b}{1 + b + e}.
\] (53)

Simplifying (53), we get

\[
2 + k + \frac{e}{1 + b + e} - \frac{e}{(1 + b + e)(1 + k)} < 2 + \frac{4}{1 + k}.
\] (54)

After some simple calculations, we get (54) holds if

\[
(1 + b + e)k^2 + (1 + b + e)k - 4(1 + b + e) < 0.
\] (55)

From the assumption on \( k \), (55) holds, and so does (50). Hence the proof is complete.

Finally, since \( L \) is continuous and bounded below in \( \Sigma \) and \( X(z) := (U(z), Y(z), V(z), Z(z)) \) is a solution of (42) obtained in Lemma 2.3, is positively invariant in \( \Sigma \) for all \( z \geq z_0 \), it follows from Lemma 3.2 and the Lasalle’s invariance principle that \( (U, Y, V, Z)(\infty) = (\eta^*, 0, \eta^*, 0) \). So we establish the existence of traveling wave solutions of system (3) in the following lemma, which, together with Lemma 2.1, gives Theorem 1.2 and Theorem 1.3.

**Lemma 3.3.** Suppose one of the following conditions holds:

(i) \( b = e = 0 \) and \( 0 < k < k_0 \), where \( k_0 \) is defined in Lemma 3.1;

(ii) \( 0 \leq b \leq 1, e \geq 0 \) and \( 0 < k < \left[-(1 + b + 2e) + \sqrt{(1 + b + 2e)^2 + 16(1 + b + e)^2}\right]/[2(1 + b + e)] \).

For \( c \geq c^* \), system (36) has a positive solution \( (U, V) \) satisfying (8). In addition, there exists a positive constant \( \delta \) such that

\[
\delta < U(z) < 1, \quad 0 < V(z) < 1, \quad \forall z \in \mathbb{R}.
\]

**Appendix.** In this section, we collect some useful lemmas which are used in the proof.

**Lemma A.1.** (Barbălat’s Lemma [2]) Suppose \( w \in C^1(b, \infty) \) and \( \lim_{t \to \infty} w(t) \) exists. If \( w' \) is uniformly continuous, then \( \lim_{t \to \infty} w'(t) = 0 \).

**Lemma A.2.** (LaSalle’s Invariance Principle [6].) Consider the following initial value problem:

\[
X' = f(X), \quad X \in \mathbb{R}^n. \tag{A.1}
\]

Let \( \Sigma \subset \mathbb{R}^n \) be an open set in \( \mathbb{R}^n \). Suppose \( X(z) \) is a solution of (A.1) which is positive invariant in \( \Sigma \). If there is a continuous and bounded below function \( V : \Sigma \to \mathbb{R} \) such that the orbital derivative of \( V \) along \( X(z) \) is non-positive, i.e.,

\[
\frac{d}{dz} L(X(z)) = \nabla L(X(z)) \cdot X'(z) \leq 0,
\]
then the $\omega$-limit set of $X(z)$ is contained in $\mathcal{I}$, where $\mathcal{I}$ is the largest invariant set in $\{X \in \Sigma : dV/dz = 0\}$.

The following a priori estimates for the second-order differential equations can be found in [5].

**Lemma A.3.** Let $B$ be a positive number and $G \in C(\mathbb{R})$. Suppose that $w \in C^2(\mathbb{R})$ is a solution of

$$w'' - Bw' = G(z)$$

in $\mathbb{R}$. If $w$ and $G$ are bounded in $\mathbb{R}$, then so are $w'$ and $w''$. Moreover,

$$\|w'\|_{L^\infty(\mathbb{R})} \leq \frac{\|G\|_{L^\infty(\mathbb{R})}}{B}$$

and

$$\|w''\|_{L^\infty(\mathbb{R})} \leq 2\|G\|_{L^\infty(\mathbb{R})}.$$

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