TRIANGULAR FEJÉR SUMMABILITY OF TWO-DIMENSIONAL WALSH-FOURIER SERIES

GYÖRGY GÁT AND USHANGI GOGINAVA

ABSTRACT. It is proved that the operators $\sigma_n^\square$ of the triangular-Fejér-means of a two-dimensional Walsh–Fourier series are uniformly bounded from the dyadic Hardy space $H_p$ to $L_p$ for all $4/5 < p \leq \infty$.

1. Introduction

Lebesgue’s [13] theorem is well known for trigonometric Fourier series: the Fejér means $\sigma_n f$ of $f$ converge to $f$ almost everywhere if $f \in L_1(\mathbb{T}), \mathbb{T} := [-\pi, \pi)$ (see also Zygmund [27]).

An analogous result for Walsh–Fourier series is due to Fine [1]. Later, Schipp [16] showed that the maximal operator $\sigma^*$ of the Fejér means of the one-dimensional Walsh–Fourier series is of weak type $(1,1)$, from which the a.e. convergence follows by standard arguments. Schipp’s result implies by interpolation also the boundedness of $\sigma^*: L_p(G) \to L_p(G)$, where $1 < p \leq \infty$. This fails to hold for $p = 1$, but Fujii [3] proved that $\sigma^*$ is bounded from the dyadic Hardy space $H_1(G)$ to the space $L_1(G)$ (see also Simon [18]). Fujii’s theorem was extended by Weisz [20]. Namely, he proved that $\sigma^*$ is bounded from the martingale Hardy space $H_p(G)$ to the space $L_p(G)$ for $p > 1/2$. Simon [19] gave a counterexample, which shows that this boundedness does not hold for $0 < p < 1/2$. In the endpoint case $p = 1/2$, Weisz [22] proved that $\sigma^*$ is bounded from the Hardy space $H_{1/2}(G)$ to the space weak-$L_{1/2}(G)$. Goginava proved in [8] that the maximal operator of the Fejér means of the one dimensional Walsh–Fourier series is not bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$.

Marcinkiewicz [14] verified for two-dimensional trigonometric Fourier series that the Marcinkiewicz-Fejér means

$$\sigma_n^\square f = \frac{1}{n} \sum_{j=0}^{n-1} S_j^\square (f)$$

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of a function \( f \in L \log L(T \times T) \) converge a.e. to \( f \) as \( n \to \infty \), where \( S_j^\square (f) \) denotes the qubical partial sums of the Fourier series of \( f \). Later Zhizhiashvili \cite{25,26} extended this result to all \( f \in L_1(T \times T) \).

An analogous result for two-dimensional Walsh–Fourier series is due to Weisz \cite{21}. Moreover, he proved that the maximal operator \( \sigma_n^\triangle f = \sup_{n \geq 1} |\sigma_n^\triangle f| \) is bounded from the dyadic martingale Hardy space \( H_p(G \times G) \) to the space \( L_p(G \times G) \) for \( p > 2/3 \). The second author \cite{11,8} proved that the maximal operator \( \sigma_n^\square f \) is bounded from \( H_{2/3}(G \times G) \) to \( L_{2/3}(G \times G) \) and is not bounded from \( H_{2/3}(G \times G) \) to \( L_{2/3}(G \times G) \).

Weisz \cite{23,24} studied the triangular partial sums and the Fejér means

\[
\sigma_n^\triangle f = \frac{1}{n} \sum_{j=0}^{n-1} S_j^\triangle f
\]

of the two-dimensional trigonometric Fourier series. This summability method is rarely investigated in the literature (see the references in \cite{24}). In \cite{12} it is proved that the maximal operator \( \sigma_n^\triangle := \sup_n |\sigma_n^\triangle f| \) of the Fejér means of the triangular partial sums of the double Walsh–Fourier series is bounded from the dyadic Hardy space \( H_p(G \times G) \) to the \( L_p(G \times G) \) if \( p > 1/2 \), is bounded from \( H_{1/3}(G \times G) \) to the space weak- \( L_{1/2}(G \times G) \) and it is not bounded from \( H_{1/2}(G \times G) \) to \( L_{1/2}(G \times G) \).

For triangular partial sums it is well-known \cite{17} the operators \( S_{2n}^\triangle \) are not uniformly bounded on \( L_p \) for \( 1 \leq p \neq 2 \).

It is proved that the operators \( \sigma_n^\triangle \) of the triangular-Fejér-means of a two-dimensional Walsh–Fourier series are uniformly bounded from the dyadic Hardy space \( H_p \) to \( L_p \) for all \( 4/5 < p \leq \infty \).

The results for summability of quadratical partial sums of two-dimensional Walsh-Fourier series can be found in \cite{7,10,4,5}.

2. Definitions and the notation

Let \( \mathbb{P} \) denote the set of positive integers, \( \mathbb{N} := \mathbb{P} \cup \{0\} \). Denote by \( Z_2 \) the discrete cyclic group of order 2, that is \( Z_2 = \{0,1\} \), where the group operation is the modulo 2 addition and every subset is open. A Haar measure on \( Z_2 \) is given such that the measure of a singleton is 1/2. Let \( G \) be the complete direct product of the countable infinite copies of the compact groups \( Z_2 \). The elements of \( G \) are of the form \( x = (x_0, x_1, \ldots, x_k, \ldots) \) with \( x_k \in \{0,1\} \) \((k \in \mathbb{N})\). The group operation on \( G \) is the coordinate-wise addition, the measure (denoted by \( \mu \)) and the topology are the product measure and topology. The compact Abelian group \( G \) is called the Walsh group. A base for the neighborhoods of \( G \) can be given by

\[
I_n(x) := I_n(x_0, \ldots, x_{n-1}) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\},
\]
where $I_0(x) := G$ and $x \in G, n \in \mathbb{N}$. These sets are called the dyadic intervals. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of $G$, $I_n := I_n(0)$ ($n \in \mathbb{N}$), $\overline{T}_n := G \setminus I_n$. Denote $2^N x := (x_N, x_{N+1}, ...), x \in G$.

For $k \in \mathbb{N}$ and $x \in G$ denote $r_k(x) := (-1)^{x_k}$ ($x \in G, \ k \in \mathbb{N}$) the $k$-th Rademacher function. If $n \in \mathbb{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}$ ($i \in \mathbb{N}$), i.e. $n$ is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} \left(r_k(x)\right)^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{\left|n\right|-1} n_k x_k} \quad (x \in G, \ n \in \mathbb{P}) .$$

The Walsh–Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x), D_0(x) = 0 .$$

Recall that (17)

$$D_{2n}(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{if } x \in \overline{T}_n \end{cases},$$

(1)

$$D_n(x) = -w_{n(i+1)}(x) \left( \sum_{r=0}^{i-1} n_r 2^r - n_i 2^i \right) \text{ for } x \in I_i \setminus I_{i+1} .$$

In this paper we consider the double system $\{w_i(x) w_j(y) : i, j \in \mathbb{N}\}$ on $G \times G$.

The rectangular partial sums of the 2-dimensional Walsh–Fourier series are defined as

$$S_{M,N} f(x,y) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \hat{f}(i,j) w_i(x) w_j(y) ,$$

where the number

$$\hat{f}(i,j) = \int_{G \times G} f(x,y) w_i(x) w_j(y) d\mu(x,y)$$

is said to be the $(i,j)$th Walsh–Fourier coefficient of the function $f$. Denote

$$S_{M}^{\square} f(x,y) := S_{M,M} f(x,y) .$$
The triangular partial sums of the 2-dimensional Walsh–Fourier series are defined as
\[
S^\triangle_k f(x, y) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \hat{f}(i, j) w_i(x) w_j(y).
\]
Denote
\[
D^\square_k (x, y) := D_k(x) D_k(y)
\]
and
\[
D^\triangle_k (x, y) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} w_i(x) w_j(y).
\]
The norm (or the quasinorm) of the space \(L^p(G \times G)\) is defined by
\[
\|f\|_p := \left( \int_{G \times G} |f(x, y)|^p \, d\mu(x, y) \right)^{1/p} \quad (0 < p \leq \infty).
\]
The space weak-\(L^p(G \times G)\) consists of all measurable functions \(f\) for which
\[
\|f\|_{\text{weak}-L^p(G \times G)} := \sup_{\lambda > 0} \lambda \mu(|f| > \lambda)^{1/p} < +\infty.
\]
The \(\sigma\)-algebra generated by the dyadic 2-dimensional \(I_k(x) \times I_k(y)\) cubes of measure \(2^{-k} \times 2^{-k}\) will be denoted by \(F_k(k \in \mathbb{N})\). Denote by \(f = (f^{(n)}, n \in \mathbb{N})\) one-parameter martingales with respect to \((F_n, n \in \mathbb{N})\) (for details see e.g. [22]). The maximal function of a martingale \(f\) is defined by
\[
f^* := \sup_{n \in \mathbb{N}} |f^{(n)}|.
\]
In case \(f \in L^1(G \times G)\), the maximal function can also be given by
\[
f^*(x, y) = \sup_{n \geq 1} \frac{1}{\mu(I_n(x) \times I_n(y))} \left| \int_{I_n(x) \times I_n(y)} f(s, t) \, d\mu(s, t) \right|,
\]
\((x, y) \in G \times G\).

For \(0 < p < \infty\) the martingale Hardy space \(H^p(G \times G)\) consists of all martingales for which
\[
\|f\|_{H^p} := \|f^*\|_p < \infty.
\]
If \(f \in L^1(G \times G)\), then it is easy to show that the sequence \((S^\triangle_{2^n} f : n \in \mathbb{N})\) is a martingale. If \(f\) is a martingale, that is \(f = (f^{(0)}, f^{(1)}, \ldots)\), then the Walsh–Fourier coefficients must be defined in a little bit different way:
\[
\hat{f}(i, j) = \lim_{k \to \infty} \int_{G \times G} f^{(k)}(x, y) w_i(x) w_j(y) \, d\mu(x, y).
\]
The Walsh–Fourier coefficients of \(f \in L^1(G \times G)\) are the same as the ones of the martingale \((S^\triangle_{2^n} f : n \in \mathbb{N})\) obtained from \(f\).
For $n \in \mathbb{P}$ and a martingale $f$ the Marcinkiewicz-Fejér means and triangular Fejér means of order $n$ of the 2-dimensional Walsh–Fourier series of a function $f$ is given by

$$
\sigma_n^\Box f(x, y) = \frac{1}{n} \sum_{j=0}^{n-1} S_j^\Box f(x, y)
$$

and

$$
\sigma_n^\triangle f(x, y) = \frac{1}{n} \sum_{j=0}^{n-1} S_j^\triangle f(x, y),
$$

respectively. It is easy to show that

$$
\sigma_n^\Box f(x, y) = \int_{G \times G} f(s, t) K_n^\Box (x + s, y + t) \, d\mu(s, t)
$$

and

$$
(2) \quad \sigma_n^\triangle f(x, y) = \int_{G \times G} f(s, t) K_n^\triangle (x + s, y + t) \, d\mu(s, t),
$$

where

$$
K_n^\Box (x, y) := \frac{1}{n} \sum_{j=0}^{n-1} D_j^\Box (x, y)
$$

and

$$
K_n^\triangle (x, y) := \frac{1}{n} \sum_{j=0}^{n-1} D_j^\triangle (x, y).
$$

We can write

$$
(3) \quad K_n^\triangle (x, y) = \frac{1}{n} \sum_{k=0}^{n-1} D_k^\triangle (x, y)
$$

$$
= \frac{1}{n} \sum_{k=1}^{n-1} \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} w_i (x) w_j (y)
$$

$$
= \frac{1}{n} \sum_{k=1}^{n-1} \sum_{i=0}^{k-1} w_i (x) D_{k-i} (y)
$$

$$
= \frac{1}{n} \sum_{k=1}^{n-1} \sum_{i=1}^{k} w_{k-i} (x) D_i (y)
$$

$$
= \frac{1}{n} \sum_{i=1}^{n-1} \sum_{k=i}^{n-1} w_{k-i} (x) D_i (y)
$$

$$
= \frac{1}{n} \sum_{i=1}^{n-1} D_{n-i} (x) D_i (y).
$$
A bounded measurable function $a$ is a $p$-atom if there exists a dyadic 2-dimensional cube $I$ such that

a) $\int_I ad\mu = 0$;

b) $\|a\|_\infty \leq \mu(I)^{-1/p}$;

c) $\text{supp } a \subset I$.

An operator $T$ which maps the set of martingale into the collection of measurable functions will be called $p$-quasi-local if there exists a constant $c_p > 0$ such that for every $p$-atom $a$

$$\int_{G \times G \setminus I} |Ta|^p \leq c_p < \infty,$$

where $I$ is the support of the atom.

3. Formulation of main results

**Theorem 1.** If $4/5 < p \leq \infty$, then the operators $\sigma_n^\Delta$ are uniformly bounded from the Hardy space $H_p(G \times G)$ to the space $L_p(G \times G)$. In particular, if $f \in H_p(G \times G)$, then

$$\sup_n \left\| \sigma_n^\Delta f \right\|_p \leq c_p \|f\|_{H_p}.$$

4. Auxiliary proposition

We shall need the following lemma (see [22]).

**Lemma 1.** Suppose that the operator $T$ is $\sigma$-sublinear and $p$-quasi-local for each $0 < p_0 < p \leq 1$. If $T$ is bounded from $L_\infty(G \times G)$ to $L_\infty(G \times G)$, then

$$\|Tf\|_p \leq c(p) \|f\|_{H_p} \quad (f \in H_p(G \times G))$$

for every $0 < p_0 < p < \infty$.

**Lemma 2.** Let $p \in (1/2, 1]$. Then

$$\int_G \left( \sup_{1 \leq n \leq 2^N} \left| \sum_{j=1}^n D_j(x) \right|^p \right) d\mu(x) \leq c_p 2^{N(2p-1)}.$$

The proof can be found in [9].

**Lemma 3.** Let $p \in (1/2, 1]$. Then

$$\int_G \sup_{1 \leq n \leq 2^N} \left| \sum_{k=1}^n D_k(x) (n - k + 1) \right|^p d\mu(x) \leq c_p 2^{N(3p-1)}.$$

**Proof of Lemma 3.** Applying Lemma 2 and Abel’s transformation we obtain

$$\int_G \sup_{1 \leq n \leq 2^N} \left| \sum_{k=1}^n D_k(x) (n - k + 1) \right|^p d\mu(x)$$
Lemma 4. Let $p \in (1/2, 1]$. Then

$$
\int_G \left( \sup_{0 \leq q < 2^N} \left| \sum_{k=q}^{2^N-1} D_k(x)(k-q+1) \right| \right)^p d\mu(x) \leq c_p 2^{Np}.
$$

Proof of Lemma 4. From Lemma 2 we have

$$
\int_G \left( \sup_{0 \leq q < 2^N} \left| \sum_{k=q}^{2^N-1} D_k(x)(k-q+1) \right| \right)^p d\mu(x) \leq c_p 2^{Np}.
$$

Lemma 4 is proved.

Lemma 5. Let $p \in (1/2, 1]$. Then

$$
\sup_{1 \leq n \leq 2^N} \int_{T^n \times T^n} \left| \sum_{k=0}^{n-1} D_{\alpha_1(n,k)}(x) D_{\alpha_2(n,k)}(y) \right|^p d\mu(x,y) \leq c_p 2^{N(3p-2)}.
$$

Let $\alpha := (\alpha_1, \alpha_2) : \mathbb{N}^2 \to \mathbb{N}^2$ be a function. Define the following Marcinkiewicz-like kernels

$$
\frac{1}{n} \sum_{k=0}^{n-1} D_{\alpha_1(n,k)}(x) D_{\alpha_2(n,k)}(y), n \in \mathbb{P}.
$$

Denote by $\#B$ the cardinality of set $B$. Suppose that

(4) $\# \{ l \in \mathbb{N} : \alpha_j(n,l) = \alpha_j(n,k), l < n \} \leq C$ $(k < n, n \in \mathbb{P}, j = 1, 2)$.

Lemma 5. Let $p \in (4/5, 1]$. Then

$$
\sup_{1 \leq n \leq 2^N} \int_{T^n \times T^n} \left| \sum_{k=0}^{n-1} D_{\alpha_1(n,k)}(x) D_{\alpha_2(n,k)}(y) \right|^p d\mu(x,y) \leq c_p 2^{N(3p-2)}.
$$
Proof of Lemma 5. In the sequel we use terminology and methods of paper [6]. Set

\[ n^{(s)} := \sum_{k=s}^{\infty} n_k 2^s, \quad J_k := \mathcal{I}_k \setminus \mathcal{I}_{k+1}. \]  

Thus \( n^{(0)} = n, n^{(N)} = 0, n, k \in \mathbb{N} \).

Then by

\[
\sum_{k=0}^{n-1} D_{\alpha_1(n,k)}(x) D_{\alpha_2(n,k)}(y) = \sum_{s=0}^{N-1} n_s \sum_{k=0}^{2^s - 1} D_{\alpha_1(n,k+n(s+1))}(x) D_{\alpha_1(n,k+n(s+1))}(y)
\]

and \( \mathcal{T}_N = \bigcup_{i=0}^{N-1} J_i = \bigcup_{i=0}^{N-1} (\mathcal{I}_i \setminus \mathcal{I}_{i+1}) \) we have

\[
\int_{\mathcal{T}_N \times \mathcal{T}_N} \left( \sum_{k=0}^{n-1} D_{\alpha_1(n,k)}(x) D_{\alpha_2(n,k)}(y) \right)^p d\mu(x,y)
\]

\[
\leq \sum_{s=0}^{N-1} \int_{\mathcal{T}_N \times \mathcal{T}_N} \left( \sum_{k=0}^{2^s - 1} D_{\alpha_1(n,k+n(s+1))} \right)^p d\mu(x,y)
\]

\[
= \sum_{i=0}^{N-1} \sum_{j=i}^{N-1} \int_{J_i \times J_j} \left( \sum_{k=0}^{2^s - 1} D_{\alpha_1(n,k+n(s+1))} \right)^p d\mu(x,y)
\]

where \( n \leq 2^N \). We can write

\[
A = \sum_{i=0}^{N-1} \sum_{j=i}^{N-1} \int_{J_i \times J_j} \left( \sum_{k=0}^{2^s - 1} D_{\alpha_1(n,k+n(s+1))} \right)^p d\mu(x,y)
\]

\[
+ \sum_{i=0}^{N-1} \sum_{j=i}^{N-1} \int_{J_i \times J_j} \left( \sum_{k=0}^{2^s - 1} D_{\alpha_1(n,k+n(s+1))} \right)^p d\mu(x,y)
\]

\[
= A_1 + A_2.
\]
Since for every \( n \in \mathbb{N} \) and \( x \in J_i = I_i \setminus I_{i+1} \) we have \(|D_n(x)| \leq c2^i\) then we also have

\[
|D_{\alpha_1(n,k)}(x)D_{\alpha_2(n,k)}(y)| \leq c2^{i+j}, \quad (x,y) \in J_i \times J_j.
\]

For \( A_1 \) this implies in the case of \( 1 > (p > 2/3) \)

\[
A_1 \leq c_p \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{s=0}^{i-1} \frac{2^{p(s+i+j)}}{2^{i+j}} \leq c_p 2^{N(3p-2)}.
\]

If \( p = 1 \), then for \( A_1 \) we have

\[
A_1 \leq \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{s=0}^{i-1} \frac{2^{p(s+i+j)}}{2^{i+j}} \leq c_1 \sum_{j=0}^{N} \sum_{s=0}^{N-i} 2^i \leq c_1 2^N \leq c_p 2^{N(3p-2)}.
\]

That is, for every \( (p > 2/3) \)

\[
(7) \quad A_1 \leq c_p \sum_{i=0}^{N} \sum_{j=1}^{N} \sum_{s=0}^{i-1} \frac{2^{p(s+i+j)}}{2^{i+j}} \leq c_p 2^{N(3p-2)}.
\]

Turn our attention to \( A_2 \). By (11) for \( x \in J_i = I_i \setminus I_{i+1} \) we can write

\[
D_{\alpha_1(n,k+n(s+1))}(x) = \alpha_1(n,k+n(s+1))^{(i+1)}(x)
\]

\[
\times \left( \sum_{r=0}^{i-1} \left( \alpha_1(n,k+n(s+1)) \right)_r 2^r - \left( \alpha_1(n,k+n(s+1)) \right)_i 2^i \right).
\]

Set

\[
\theta(s,k,i) := \sum_{r=0}^{i-1} \left( \alpha_1(n,k+n(s+1)) \right)_r 2^r - \left( \alpha_1(n,k+n(s+1)) \right)_i 2^i.
\]

Apply the Cauchy-Schwarz inequality and follow the method of [6]:

\[
\int_{J_i \times J_j} \left| \sum_{k=0}^{2^i-1} D_{\alpha_1(n,k+n(s+1))}(x) D_{\alpha_2(n,k+n(s+1))}(y) \right|^p d\mu(x,y)
\]

\[
\leq \int_{J_j} 2^{-i(1-p/2)} \left( \int_{J_i} \left| \sum_{k=0}^{2^i-1} D_{\alpha_1(n,k+n(s+1))}(x) \right|^2 d\mu(x) \right)^{p/2}
\]

\[
\times D_{\alpha_2(n,k+n(s+1))}(y)^2 d\mu(y)
\]
\[
= \int_{J_i} 2^{-i(1-p/2)} \left( \int_{J_i} \sum_{k,l=0}^{2^s-1} D_{\alpha_1(n, n, n^{(s+1)})} \right) \left( x \right) d\mu(y)
\]
\[
\times D_{\alpha_2(n, n, n^{(s+1)})} \left( x \right) D_{\alpha_2(n, n, n^{(s+1)})} \left( y \right) \frac{d\mu(x)}{d\mu(y)}
\]
\[
= \int_{J_i} 2^{-i(1-p/2)} \left( \int_{J_i} \sum_{k,l=0}^{2^s-1} w^{(\alpha_1(n, n, n^{(s+1)}))} \right) \left( x \right) \theta(s, k, i) \theta(s, l, i)
\]
\[
\times D_{\alpha_2(n, n, n^{(s+1)})} \left( y \right) \frac{d\mu(x)}{d\mu(y)}
\]
\[
= \int_{J_i} 2^{-i(1-p/2)} \left( \sum_{k,l=0}^{2^s-1} \theta(s, k, i) \theta(s, l, i)
\]
\[
\times D_{\alpha_2(n, n, n^{(s+1)})} \left( y \right) \frac{d\mu(x)}{d\mu(y)}
\]
\[
\times w^{(\alpha_1(n, n, n^{(s+1)}))} \left( x \right) d\mu(x)
\]

Discuss the integral on the set \(J_i = I_i \setminus I_{i+1}\)

\[
(8) \quad \int_{J_i} w^{(\alpha_1(n, n, n^{(s+1)}))} \left( x \right) w^{(\alpha_1(n, n, n^{(s+1)}))} \left( x \right) d\mu(x).
\]

If it differs from zero, then the \(i+1\)-th, \(i+2\)-th, ... coordinates of \(\alpha_1(n, n, n^{(s+1)} + k)\) and \(\alpha_1(n, n, n^{(s+1)} + l)\) should be equal.

We have that for every \(k\) there exists only a bounded numbers of \(l\)'s for which \(\alpha_1(n, n^{(s+1)} + k) = \alpha_1(n, n^{(s+1)} + l)\). These facts give that for every \(k\) there exists - at most - \(2^s\) number of \(l\)'s for which this integral is not zero. This will be very important in the estimation of \(A_2\) because - at first sight - \(k\) and \(l\) are elements of \(\{0, 1, \ldots, 2^n - 1\}\) and consequently this would mean \(2^{2s}\) addends. But this is not the case, because for every \(k \in \{0, 1, \ldots, 2^n - 1\}\) the number of \(l\)'s to be taken is \(2^i\) because we need to take the integrals only when they are not zero. Thus, the number of \((k, l)\) pairs we have to take account is only \(2^{s+i}\) and not \(2^{2s}\).

Consequently, apply the facts that \(|\theta(s, k, i)| \leq c2^i\), \(|D_{\alpha_2(n, n, n^{(s+1)})}(y)| \leq c2^i\) we obtain
\[
\int_{j} 2^{-i(1-p/2)} \left( \sum_{k,l=0}^{\infty} \theta (s, k; i) \theta (s, l; i) \right) \\
\times D_{\alpha_2(n,k+n(s+1))}(y) D_{\alpha_2(n,l+n(s+1))}(y) \int_{J_i} w(\alpha_1(n,k+n(s+1))^{(i+1)})(x) \\
\times w(\alpha_1(n,l+n(s+1))^{(i+1)})(x) d\mu(x) \right)^{p/2} d\mu(y) \\
\leq c \int_{j} 2^{-i(1-p/2)} \left(2^{2i+2i^2+2j^2}\right)^{p/2} d\mu(y).
\]

This gives that for \( A_2 \) we obtain

\[
A_2 \leq c_p \sum_{i=0}^{N} \sum_{j=i}^{N} \sum_{s=i}^{N} \int_{j} 2^{-i(1-p/2)} \left(2^{2i+2i^2+2j^2}\right)^{p/2} d\mu(y)
\]

\[
\leq c_p 2^{Np/2} \sum_{i=0}^{N} 2^{i(3p/2-1)} \sum_{j=i}^{N} 2^{j(p-1)}
\]

\[
\leq c_p 2^{Np/2} \sum_{i=0}^{N} 2^{i(5p/2-2)}
\]

\[
\leq c_p 2^{N(3p-2)},
\]

for \( p \in (4/5, 1) \). Let \( p = 1 \). Then

\[
A_2 \leq 2^{N/2} \sum_{i=0}^{N} 2^{i/2} (N - i + 1)
\]

\[
\leq 2^{N} \sum_{i=0}^{N} \frac{(N - i + 1)}{2^{(N-i)/2}} \leq c_2 N.
\]

Combining (5)-(10) we complete the proof of Lemma 5.

\[\square\]

**Corollary 1.** Let \( \alpha_1(n,k) = k, \alpha_2(n,k) = n - k \), \( k = 0, \ldots, n - 1 \), \( n \in \mathbb{N} \), \( p \in (4/5, 1) \). Then

\[
\sup_{1 \leq n \leq 2^N} \int_{T_N \times T_N} \left| \sum_{k=0}^{n-1} D_k(x) D_{n-k}(y) \right|^p d\mu(x,y) \leq c_p 2^{N(3p-2)}.
\]
Corollary 2. Let \( p \in (4/5, 1] \). Then
\[
\sup_{0 \leq q < 2^N} \int_{T_N \times T_N} \left| \sum_{k=q}^{2^N-1} D_k (x) D_{k-q} (y) \right|^p d\mu (x, y) \leq c_p 2^{N(3p-2)}.
\]

Proof of Corollary 2 Since
\[ w_{2N-1} D_j = D_{2N} - D_{2N-j}, \quad j = 0, 1, ..., 2^N - 1 \]
from Lemma 2 and Corollary 1 we can write
\[
\int_{T_N \times T_N} \left| \sum_{k=q}^{2^N-1} D_k (x) D_{k-q} (y) \right|^p d\mu (x, y)
\]
\[
\leq \int_{T_N \times T_N} \left| \sum_{k=0}^{2^N-q} \left( D_{2N} (x) - D_{2N-(k+q)} (x) \right) D_k (y) \right|^p d\mu (x, y)
\]
\[
\leq c_p \int_{T_N \times T_N} D_{2N} (x) \left| \sum_{k=0}^{2^N-q} D_k (y) \right|^p d\mu (x, y)
\]
\[
+ c_p \int_{T_N \times T_N} \left| \sum_{k=0}^{2^N-q} D_{2N-q-k} (x) D_k (y) \right|^p d\mu (x, y) \leq c_p 2^{N(3p-2)}.
\]
Corollary 2 is proved. \( \square \)

It is well known that \( D_0 = 0 \) but to have symmetry, that is for some technical reason, at same places we have to indicate the 0th Dirichlet kernel in sums below.

Lemma 6. Let \( n = 2^N q_1 + q_2, 0 \leq q_2 < 2^N \). Then
\[
K_n^\Delta (x, y) = \sum_{l=0}^{q_1-1} w_l (2^N x) w_{q_1-l} (2^N y) \sum_{k=1}^{q_2-1} D_k (x) D_{q_2-k} (y)
\]
\[
+ D_{2N} (y) \sum_{l=0}^{q_1-1} w_l (2^N x) D_{q_1-l} (2^N y) \sum_{k=1}^{2^N} D_k (x)
\]
\[
+ D_{2N} (x) \sum_{l=0}^{q_1-1} D_l (2^N x) w_{q_1-l} (2^N y) q_2 K_{q_2} (y)
\]
\[
+ D_{2N} (x) D_{2N} (y) \sum_{l=0}^{q_1-1} D_l (2^N x) D_{q_1-l} (2^N y) 2^N
\]
\[
- w_{2^N-1} (y) \sum_{l=0}^{q_1-1} w_l (2^N x) w_{q_1-l-1} (2^N y) \sum_{k=q_2+1}^{2^N} D_k (x) D_{k-q_2-1} (y)
\]
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\[-w_{2N-1} (y) D_{2N} (x) \sum_{l=0}^{q_1 - 1} D_l (2^N x) w_{q_1 - l - 1} (2^N y) \sum_{k=q_2 + 1}^{2^N} D_{k-q_2} (y)\]

\[+w_{q_1} (2^N x) \sum_{k=1}^{q_2 - 1} D_k (x) D_{q_2 - k} (y) + D_{2N} (x) D_{q_1} (2^N x) \sum_{v=1}^{q_2 - 1} D_v (y) .\]

Proof of Lemma \[\text{We can write}\]

\[(11) \quad K_n^\triangle (x,y) = \sum_{k=1}^{2^N q_1 + q_2 - 1} D_k (x) D_{n-k} (y)\]

\[= \sum_{k=1}^{2^N q_1} D_k (x) D_{n-k} (y) + \sum_{k=1}^{q_2 - 1} D_{k+2^N q_1} (x) D_{q_2-k} (y)\]

\[= I_1 + I_2.\]

Since (see [2 (6.1)]))

\[(12) \quad D_{k+2^N} (x) = w_l (2^N x) D_k (x) + D_{2N} (x) D_l (2^N x)\]

for \(I_2\) we have

\[(13) \quad I_2 = w_{q_1} (2^N x) \sum_{k=1}^{q_2 - 1} D_k (x) D_{q_2-k} (y)\]

\[+D_{2N} (x) D_{q_1} (2^N x) \sum_{k=1}^{q_2 - 1} D_k (y)\]

For \(I_1\) we have

\[(14) \quad I_1 = \sum_{l=0}^{q_1 - 1} \sum_{k=1}^{2^N} D_{k+l2^N} (x) D_{2^N q_1 + q_2 - l2^N - k} (y)\]

\[= \sum_{l=0}^{q_1 - 1} \sum_{k=1}^{q_2} D_{k+l2^N} (x) D_{2^N (q_1-l) + q_2-k} (y)\]

\[+ \sum_{l=0}^{q_1 - 1} \sum_{k=q_2 + 1}^{2^N} D_{k+l2^N} (x) D_{2^N (q_1-l) + q_2-k} (y)\]

\[= I_{11} + I_{12}.\]
From (12) we get

\begin{align*}
I_{11} &= \sum_{l=0}^{q_1-1} w_l (2^N x) w_{q_1-l} (2^N y) \sum_{k=1}^{q_2} D_k (x) D_{q_2-k} (y) \\
&\quad + D_{2N} (y) \sum_{l=0}^{q_1-1} w_l (2^N x) D_{q_1-l} (2^N y) \sum_{k=1}^{q_2} D_k (x) \\
&\quad + D_{2N} (x) \sum_{l=0}^{q_1-1} D_l (2^N x) w_{q_1-l} (2^N y) q_2 K_{q_2} (y) \\
&\quad + D_{2N} (x) D_{2N} (y) \sum_{l=0}^{q_1-1} D_l (2^N x) D_{q_1-l} (2^N y) q_2.
\end{align*}

Turn our attention to $I_{12}$. From the simple calculation we can write $(q_2 < k \leq 2^N, 0 \leq l < q_1)$

\begin{align*}
D_{2N (q_1-l)-(k-q_2)} (y) &= D_{2N (q_1-l-1)+2N-(k-q_2)} (y) \\
&= D_{2N (q_1-l-1)} (y) + w_{2N (q_1-l-1)} (y) D_{2N-(k-q_2)} (y).
\end{align*}

Since

\begin{align*}
D_{2N-(k-q_2)} (y) &= D_{2N} (y) - w_{2N-1} (y) D_{k-q_2} (y),
\end{align*}

we obtain

\begin{align*}
D_{2N (q_1-l)+q_2-k} (y) &= D_{2N} (y) D_{q_1-l-1} (2^N y) + w_{q_1-l-1} (2^N y) D_{2N} (y) \\
&\quad - w_{q_1-l-1} (2^N y) w_{2N-1} (y) D_{k-q_2} (y).
\end{align*}

Hence from (12), we have

\begin{align*}
I_{12} &= D_{2N} (y) \sum_{l=0}^{q_1-1} w_l (2^N x) D_{q_1-l-1} (2^N y) \sum_{k=q_2+1}^{2^N} D_k (x) \\
&\quad + D_{2N} (y) D_{2N} (x) \sum_{l=0}^{q_1-1} D_l (2^N x) D_{q_1-l-1} (2^N y) (2^N - q_2) \\
&\quad + D_{2N} (y) \sum_{l=0}^{q_1-1} w_l (2^N x) w_{q_1-l-1} (2^N y) \sum_{k=q_2+1}^{2^N} D_k (x) \\
&\quad + D_{2N} (y) D_{2N} (x) \sum_{l=0}^{q_1-1} D_l (2^N x) w_{q_1-l-1} (2^N y) (2^N - q_2) \\
&\quad - w_{2N-1} (y) \sum_{l=0}^{q_1-1} w_l (2^N x) w_{q_1-l-1} (2^N y) \sum_{k=q_2+1}^{2^N} D_k (x) D_{k-q_2} (y).
\end{align*}
Consequently, Proof of Lemma 7.

Since

\begin{align*}
&-w_{2N-1}(y)D_{2N}(x)\sum_{l=0}^{q_1-1}D_l(2^N x)w_{q_1-l-1}(2^N y)\sum_{k=q_2+1}^{2^N}D_{k-q_2}(y) \\
&= D_{2N}(y)\sum_{l=0}^{q_1-1}D_l(2^N x)D_{q_1-l}(2^N y)\sum_{k=q_2+1}^{2^N}D_k(x) \\
&+ D_{2N}(y)D_{2N}(x)\sum_{l=0}^{q_1-1}D_l(2^N x)D_{q_1-l}(2^N y)(2^N - q_2) \\
&- w_{2N-1}(y)\sum_{l=0}^{q_1-1}w_l(2^N x)w_{q_1-l-1}(2^N y)\sum_{k=q_2+1}^{2^N}D_k(x)D_{k-q_2}(y) \\
&- w_{2N-1}(y)D_{2N}(x)\sum_{l=0}^{q_1-1}D_l(2^N x)w_{q_1-l-1}(2^N y)\sum_{k=q_2+1}^{2^N}D_{k-q_2}(y).
\end{align*}

Combining (11), (14), (15) and (16) we complete the proof of Lemma 7. \qed

Lemma 7. Let \((x, y) \in T_N \times T_N\) and \(n = 2^Nq_1 + q_2, 0 \leq q_2 < 2^N\). Then the following inequality holds

\[
\left| K_n^\triangle (x + s, y + t) \right| d\mu(s, t) \leq \frac{c}{23^N} \left\{ \left| \sum_{k=0}^{q_2-1} D_k(x) D_{q_2-k}(y) \right| + \left| \sum_{k=q_2+1}^{2^N} D_k(x) D_{k-q_2}(y) \right| \right\}.
\]

Proof of Lemma 7. Since \(x + s, y + t \notin I_N\), by lemma 6 we have

\[
nK_n^\triangle (x, y) = \sum_{l=0}^{q_1-1}w_l(2^N x)w_{q_1-l}(2^N y)\sum_{k=0}^{q_2-1}D_k(x)D_{q_2-k}(y) \\
- w_{2N-1}(y)\sum_{l=0}^{q_1-1}w_l(2^N x)w_{q_1-l-1}(2^N y)\sum_{k=q_2+1}^{2^N}D_k(x)D_{k-q_2}(y) \\
+ w_{q_1}(2^N x)\sum_{k=0}^{q_2-1}D_k(x)D_{q_2-k}(y).
\]

Consequently \((q_1 \leq n2^{-N})\)

\[
\left| K_n^\triangle (x + s, y + t) \right| d\mu(s, t)
\]
Lemma 8. Let \((x, y) \in \mathcal{T}_N \times I_N\) and \(n = 2^N q_1 + q_2, 0 \leq q_2 < 2^N\). Then the following inequality holds

\[
\int_{I_N \times I_N} \left| K_n^\Delta (x + s, y + t) \right| \, d\mu(s, t) \leq \frac{c}{2^{3N}} \left\{ \sum_{k=0}^{q_2-1} |D_k(x) D_{q_2-k}(y)| + \sum_{k=q_2+1}^{2^N} |D_k(x) D_{k-q_2}(y)| \right\} + D_{2^N}(y) \sum_{k=1}^{2^N} |D_k(x)|, \quad n \geq 2^N.
\]

Lemma 7 is proved. \(\square\)
Proof of Lemma 8. From (1) and Lemma 5 we get
\[ nK_n^\triangle (x, y) \]
\[ = \sum_{l=0}^{q_1-1} w_l (2^N x) w_{q_1-l} (2^N y) \sum_{k=0}^{q_2-1} D_k (x) D_{q_2-k} (y) \]
\[ + \left( \sum_{k=1}^{2N-1} D_k (x) \right) D_{2N} (y) \sum_{l=0}^{q_1-1} w_l (2^N x) D_{q_1-l} (2^N y) \]
\[ - w_{2N-1} (y) \sum_{l=0}^{q_1-1} w_l (2^N x) w_{q_1-l-1} (2^N y) \sum_{k=q_2+1}^{2N} D_k (x) D_{k-q_2} (y) \]
\[ + w_{q_1} (2^N x) \sum_{k=0}^{q_2-1} D_k (x) D_{q_2-k} (y). \]

Consequently,
\[ \int_{I_N \times I_N} \left| K_n^\triangle (x + s, y + t) \right| d\mu (s, t) \]
\[ \leq \frac{1}{n} \sum_{k=0}^{q_2-1} D_k (x) D_{q_2-k} (y) \]
\[ \times \int_{I_N \times I_N} \left| \sum_{l=0}^{q_1-1} w_l (2^N (x + s)) w_{q_1-l} (2^N (y + t)) \right| d\mu (s, t) \]
\[ + \frac{D_{2N} (y)}{n} \left| \sum_{k=1}^{2N-1} D_k (x) \right| \]
\[ \times \int_{I_N \times I_N} \left| \sum_{l=0}^{q_1-1} w_l (2^N (x + s)) D_{q_1-l} (2^N (y + t)) \right| d\mu (s, t) \]
\[ + \frac{1}{n} \left| \sum_{k=q_2}^{2N-1} D_k (x) D_{k-q_2} (y) \right| \]
\[ \times \int_{I_N \times I_N} \left| \sum_{l=0}^{q_1-1} w_l (2^N (x + s)) w_{q_1-l-1} (2^N (y + t)) \right| d\mu (s, t) \]
\[ + \frac{1}{2n^{2N}} \sum_{k=0}^{q_2-1} D_k (x) D_{q_2-k} (y) \]
\[ \leq \frac{q_1}{n^{2N}} \sum_{k=0}^{q_2-1} D_k (x) D_{q_2-k} (y) \]
\begin{align*}
&+ \frac{D_{2N}(y)}{n^{2N}} \left| \sum_{k=1}^{2N-1} D_k(x) \right| \left| \int_{G \times G} \left| \sum_{l=0}^{q_1-1} w_l(u) D_{q_1-l}(v) \right| d\mu(u,v) \\
&+ \frac{q_1}{n^{2N}} \left| \sum_{k=q_2+1}^{2N} D_k(x) D_{k-q_2}(y) \right| \\
&+ \frac{1}{2N} \left| \sum_{k=0}^{q_2-1} D_k(x) D_{q_2-k}(y) \right|.
\end{align*}

Since
\begin{align*}
\int_{G \times G} & \left| \sum_{l=0}^{q_1-1} w_l(u) D_{q_1-l}(v) \right| d\mu(u,v) \\
\leq & \left( \int_{G \times G} \left| \sum_{l=0}^{q_1-1} w_l(u) D_{q_1-l}(v) \right|^2 d\mu(u,v) \right)^{1/2} \\
= & \left( \int_G \sum_{l=0}^{q_1-1} D_{q_1-l}^2(v) d\mu(v) \right)^{1/2} \\
= & \sum_{l=0}^{q_1-1} (q_1 - l) \leq q_1,
\end{align*}
by (17) we complete the proof of Lemma 8.

5. Proofs of main results

Proof of Theorem 1. By Lemma 1, the proof of theorem will be complete if we show that the operator \( \sigma_n^\Delta \) is \( p \)-quasi-local for each \( 4/5 < p \leq 1 \) and bounded from \( L_\infty(G \times G) \) to \( L_\infty(G \times G) \).

First, we prove the boundedness from \( L_\infty(G \times G) \) to \( L_\infty(G \times G) \). We can write
\begin{align*}
&\int_{G \times G} \frac{1}{n} \sum_{k=1}^{n-1} D_k(x) D_{n-k}(y) d\mu(x,y) \\
= & \int_{T|n| \times T|n|} \frac{1}{n} \sum_{k=1}^{n-1} D_k(x) D_{n-k}(y) d\mu(x,y) \\
&+ \int_{T|n| \times I|n|} \frac{1}{n} \sum_{k=1}^{n-1} D_k(x) D_{n-k}(y) d\mu(x,y)
\end{align*}
+ \int_{I_n \times I_n} \frac{1}{n} \sum_{k=1}^{n-1} D_k(x) D_{n-k}(y) \, d\mu(x, y)

+ \int_{I_n \times I_n} \frac{1}{n} \sum_{k=1}^{n-1} D_k(x) D_{n-k}(y) \, d\mu(x, y)

From Corollary 1 we have

$$\sup_n \int_{T_N \times T_N} \frac{1}{n} \sum_{k=0}^{n-1} D_k(x) D_{n-k}(y) \, d\mu(x, y) < \infty.$$  (19)

Using Lemma 3 for \(p = 1\) we have

$$\int_{I_n \times I_n} \frac{1}{n} \sum_{k=1}^{n-1} D_k(x) D_{n-k}(y) \, d\mu(x, y) = \frac{1}{n^{2|n|}} \int_G \left| \sum_{k=1}^{n-1} (n-k) D_k(x) \right| d\mu(x, y) \leq c < \infty, n \in \mathbb{P}. $$  (20)

Analogously, we can prove that

$$\sup_n \int_{I_n \times T_n} \frac{1}{n} \sum_{k=1}^{n-1} D_k(x) D_{n-k}(y) \, d\mu(x, y) < \infty.$$  (21)

From a simple calculation we have

$$\int_{I_n \times I_n} \frac{1}{n} \sum_{k=1}^{n-1} D_k(x) D_{n-k}(y) \, d\mu(x, y) \leq \frac{1}{2^{3N}} \sum_{k=1}^{n-1} k(n-k) \leq c < \infty.$$  (22)

Combining (18)-(22) we conclude that

$$\sup_n \int_{G \times G} \frac{1}{n} \sum_{k=1}^{n-1} D_k(x) D_{n-k}(y) \, d\mu(x, y) < \infty.$$  (23)

Hence, from (2) and (3) we conclude that

$$\left\| \sigma_n^\triangle f \right\|_\infty \leq c \|f\|_\infty.$$  (24)

Now, we prove that \(\sigma_n^\triangle\) is \(p\)-quasi-local. We consider three cases. Let \(a\) be an arbitrary atom with support \(I_N(u) \times I_N(v)\). It is easy to see that \(\sigma_n^\triangle(a) = 0\) if \(n < 2^N\). Therefore we can suppose that \(n \geq 2^N\). Since the
dyadic addition is a measure preserving group operation, we may assume that \( u = v = 0 \).

**Step 1. Integrating over** \( \mathcal{T}_N \times \mathcal{T}_N \). Since \( \|a\|_{\infty} \leq c^{2N/p} \) from Lemma 7 we obtain

\[
(23) \quad |\sigma_n^a (x, y)| \leq \|a\|_{\infty} \int_{\mathcal{T}_N \times \mathcal{T}_N} |K_n^a (x + s, y + t)| \, d\mu (s, t)
\]

\[
\leq c^{2N/p} \frac{2^{2N}}{3N} \left| \sum_{k=1}^{q_2-1} D_k (x) D_{q_2-k} (y) \right|
\]

\[
+ c^{2N/p} \frac{2^{2N}}{3N} \left| \sum_{k=q_2+1}^{2N-1} D_k (x) D_{q_2-k} (y) \right|, n = 2^N q_1 + q_2, 0 \leq q_2 < 2^N.
\]

Applying the inequality

\[
\left( \sum_k a_k \right)^p \leq \sum_k a_k^p, \quad (0 < p \leq 1)
\]

by (23) we have

\[
|\sigma_n^a (x, y)|^p \leq \sum_k a_k^p, \quad (0 < p \leq 1)
\]

Consequently, by Corollary 1 and Corollary 2 we can write

\[
(24) \quad \int_{\mathcal{T}_N \times \mathcal{T}_N} |\sigma_n^a (x, y)|^p \, d\mu (x, y)
\]

\[
\leq c_p 2^{2N} \frac{2^{2N}}{3Np} \left\{ \sup_{1 \leq q_2 \leq 2^N} \int_{\mathcal{T}_N \times \mathcal{T}_N} \left| \sum_{k=1}^{q_2-1} D_k (x) D_{q_2-k} (y) \right|^p \, d\mu (x, y)
\]

\[
+ \sup_{1 \leq q_2 \leq 2^N} \int_{\mathcal{T}_N \times \mathcal{T}_N} \left| \sum_{k=q_2+1}^{2N-1} D_k (x) D_{q_2-k} (y) \right|^p \, d\mu (x, y) \right\}
\]

\[
\leq c_p < \infty, \quad 4/5 < p \leq 1.
\]
Step 2. Integrating over $T_N \times I_N$. From Lemma \ref{lem8} we obtain

$$
|\sigma_n \triangle a(x, y)| 
\leq \|a\|_\infty \int_{I_N \times I_N} \left| K_n \triangle (x + s, y + t) \right| \, d\mu(s, t)
$$

$$
\leq \frac{c^2 2^{Np}}{2^{3N}} \left\{ \sum_{k=1}^{2^N-1} D_k(x) D_{q_2-k}(y) + \sum_{k=q_2+1}^{2^N-1} D_k(x) D_{k-q_2}(y) \right\}
$$

Consequently, from Lemma \ref{lem2}, Lemma \ref{lem3} and Lemma \ref{lem4}, we get

(25) \hspace{1cm} \int_{T_N \times I_N} |\sigma_n \triangle a(x, y)|^p \, d\mu(x, y)

\hspace{1cm} \leq \frac{c_p 2^N}{2^{3NP}} \left\{ \sup_{T_N} \int_{1 \leq q_2 \leq 2^N} \left| \sum_{k=1}^{q_2-1} D_k(x) (q_2 - k) \right|^p \, d\mu(x) \right. 

\hspace{2cm} + \left. \sup_{T_N} \int_{1 \leq q_2 \leq 2^N} \left| \sum_{k=q_2}^{2^N-1} D_k(x) (k - q_2) \right|^p \, d\mu(x) \right\}

\hspace{1cm} + \frac{2^{Np}}{2^{2NP}} \left\{ \sum_{k=1}^{2^N-1} D_k(x) \right|^p \, d\mu(x) \left\}

\leq \frac{c_p 2^N}{2^{3NP}} 2^{(3p-1)N} = c_p < \infty \quad (4/5 < p \leq 1).$

Step 3. Integrating over $I_N \times T_N$. The case is analogous to step 2 and we have

(26) \hspace{1cm} \int_{I_N \times T_N} |\sigma_n \triangle a(x, y)|^p \, d\mu(x, y) \leq c_p < \infty \quad (4/5 < p \leq 1).$

Combining (24), (25) and (26) we complete the proof of Theorem \ref{thm1} \hfill \Box

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G. Gát, Institute of Mathematics and Computer Science, College of Nyíregyháza, P.O. Box 166, Nyiregyháza, H-4400 Hungary
E-mail address: gatgy@nyf.hu

U. Goginava, Department of Mathematics, Faculty of Exact and Natural Sciences, Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia
E-mail address: zazagoginava@gmail.com