Classical signal model reproducing quantum probabilities for single and coincidence detections

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Abstract. We present a simple classical (random) signal model reproducing Born’s rule. The crucial point of our approach is that the presence of detector’s threshold and calibration procedure have to be treated not as simply experimental technicalities, but as the basic counterparts of the theoretical model. We call this approach threshold signal detection model (TSD). The experiment on coincidence detection which was done by Grangier in 1986 [22] played a crucial role in rejection of (semi-)classical field models in favour of quantum mechanics (QM): impossibility to resolve the wave-particle duality in favour of a purely wave model. QM predicts that the relative probability of coincidence detection, the coefficient \( g^{(2)}(0) \), is zero (for one photon states), but in (semi-)classical models \( g^{(2)}(0) \geq 1 \). In TSD the coefficient \( g^{(2)}(0) \) decreases as \( 1/E_d^2 \), where \( E_d > 0 \) is the detection threshold. Hence, by increasing this threshold an experimenter can make the coefficient \( g^{(2)}(0) \) essentially less than 1. The TSD-prediction can be tested experimentally in new Grangier type experiments presenting a detailed monitoring of dependence of the coefficient \( g^{(2)}(0) \) on the detection threshold. Structurally our model has some similarity with the prequantum model of Grössing et al. Subquantum stochasticity is composed of the two counterparts: a stationary process in the space of internal degrees of freedom and the random walk type motion describing the temporal dynamics.

1. Introduction
We study the old problem of a possibility to construct a classical field model reproducing probabilistic predictions of quantum mechanics. The common opinion, see Bell [1], is that this is impossible to do, but see [2]–[16] for numerous attempts to proceed with classical wave (oscillatory) prequantum models. The main argument is that for composite quantum systems (e.g., a pair of entangled photons) the correlation predicted by QM (and confirmed by experiment [17]–[19]) cannot be reproduced by models with (local) hidden variables. This is also the common opinion that “Bell’s theorem” [1] is formulated in so general abstract framework that it rejects all (local) models with hidden variables, including variables of classical field (e.g., electromagnetic) type. Although nowadays Bell’s inequality and entangled systems are the hot topics [20], [21], the problem of a possibility of the classical field description of a single quantum system is also of large importance for quantum theory, including quantum information theory. One of the most important tests of a possibility to represent a photon simply as a pulse of classical electromagnetic field is the experiment on coincidence detection in two output channels of the polarization beam splitter (PBS) [22], [23]. QM predicts that if the source can be considered as one-photon source, then the probability of coincidence detection is equal to zero: one photon cannot be split between
two channels, it is either in one or another channel. Any pulse of the classical electromagnetic field is split between two output channels of PBS. Hence, the probability of coincidence detection is nonzero. Of course, the real experimental situation is more complicated: for any "one-photon source" the probability of emission of e.g. two photons is nonzero (although it can be made very small), there is also the contribution of noise, including so called duck counts (i.e., counts in detectors in the absence of the source). Therefore, instead of the absolute probability of coincidence detection $P_{12}$, experimenters use the relative probability [22], [23]:

$$g^{(2)}(0) = \frac{P_{12}}{P_1 P_2},$$

where $P_{12}$ is the probability of coincidence detection in channels $i = 1, 2$ and $P_i$ are probabilities of detection in corresponding channels. Known (semi-) classical models of QM predict that $g^{(2)}(0) \geq 1$. If photon is not a classical pulse, one can expect that $g^{(2)}(0) < 1$ (even by taking into account noise and emission of double photons). The first experiment of this type was performed by [22], see also [23] and [24] for review. It was shown that the number of double clicks is relatively small. This experiment played an important role in quantum foundations. As is commonly accepted, this experiment justified rejection of (semi-)classical field theories as “prequantum theories”; in particular, photons cannot be interpreted as pulses of classical electromagnetic field.

The aim of this paper is to show that the main reason of dis-matching of predictions of (semi-)classical field models with QM and experimental data is that such models do not take into account the impact of detectors to creation of the quantum statistics.

We present a classical field model which in combination with the procedure of detection of random signals by threshold type detectors (which are properly calibrated) reproduces quantum probabilities and, in particular, the coefficient $g^{(2)}(0) << 1$ for a sufficiently high detection threshold $E_d$. Hence, the prediction of our model differs crucially from the prediction of the known classical field models which do not take into account the evident fact that measurement is not simply detection (of the monitoring-type) of continuous classical signals, but creation of discrete counts with the aid of threshold detectors. Detectors also have to be properly calibrated.

Our model, threshold signal detection model (TSD), predicts that the quantum prediction, $g^{(2)}(0) << 1$, matches better measurements for high value of the detection threshold. The basic prediction of TSD is that the coefficient

$$g^{(2)}(0) \leq \frac{K}{E_d^2},$$

where $K > 0$ is a constant depending on the signal. This $K$ depends on the brightness of the source: higher brightness implies larger $K$. It also depends on elements of the density matrix $\rho = (\rho_{ij})$ corresponding to the prequantum signal. (In TSD $\rho$ corresponds to the normalization (by the trace) of the covariance operator $B$ of the prequantum random signal, i.e., $\rho = B/\text{Tr}B$.) Larger $\rho_{12}$ implies larger $K$.

It is interesting to compare this prediction with the real experiment. Unfortunately, it seems that Grangier’s type experiments with detailed monitoring of dependence of the coincidence probability on the value of the threshold have never been done; more specifically, that is: In Grangier’s experiment, did the calibration of detectors play a crucial role to eliminate coincidences?

This is clearly a crucial question. It is hard to give a definite answer as to how exactly it was influencing the result at a fundamental level because Grangier et al [22], [23] didn’t study the influence of threshold on the $g^{(2)}(0)$ parameter he was measuring. It is nevertheless clear that with a lower threshold, he would have gotten a less good result. Let me translate the part of
Grangier’s thesis [23] where he explains for the first time the role of the threshold and how its level was chosen:

“[...] In this configuration, the threshold has a double role of acquisition of timing information and of selection of the pulses (the too weak pulses are not taken into account). The problems connected to the choice of the discriminator threshold and of the high voltage of the photomultipliers are discussed in detail in reference [17]. We have in the present experiment chosen a rather high threshold, which amounts to give the priority of the stability of the counting rates and the reproducibility of the results, rather than to the global detection efficiencies.” (I stressed with bold the important fact that Grangier proceeded with rather high threshold.)

TSD provides a strong motivation to perform Grangier’s type experiment with monitoring of dependence of the coincidence probability on the detection threshold and source’s brightness.

TSD can be considered as measurement theory for recently developed prequantum classical statistical field theory, PCSFT, [32],[33]. The latter reproduced all quantum averages and correlations including correlations for entangled quantum states. In particular, PCSFT correlations violated Bell’s inequality. The main problem for matching of PCSFT and conventional QM was that PCSFT (nor other classical field models) was not able to describe probabilities of discrete clicks of detectors. In particular, PCSFT is theory of correlations of continuous signals. “Prequantum observables” are given by quadratic forms of signals. These forms are unbounded and this is not surprising that correlations of such observables can violate Bell’s type inequalities, see [34] for discussion and an elementary example. The condition of coincidence of ranges of values of quantum observables and corresponding “prequantum variables” plays a crucial role in Bell’s argument. TSD solved the measurement problem of PCSFT. In the same way as in Bell’s consideration, TSD operates with discrete observables. In particular, in the case of photon polarization (its projection to a fixed axis) TSD operates with dichotomous variables taking values \( \pm 1 \).

TSD/PCSFT for composite quantum systems was presented in [35]. However, random signals considered in [35] have a complex structure of temporal correlations, even in the case of a single system. In the present paper, we use simply a combination of the Wiener process (as the career of temporal correlations) and a Gaussian stationary process valued in the space of internal degrees of freedom of a quantum system (e.g., its polarization). We do not consider spatial degrees of freedom.

To escape misunderstanding, we stress that the presented detection model is purely classical; in particular, detectors are classical detectors which are sensitive to the energy density of a signal in the domain of detection \( V \). Such a detector clicks at the first instance of time \( \tau = \tau_d \) when the total energy of signal in \( V \) approaches the detection threshold \( \mathcal{E}_d \). Since we consider random signals, this instant of detection is a random variable \( \tau = \tau(\omega) \). Our main aim is to find its average \( \bar{\tau} \). The quantity \( 1/\bar{\tau} \) determines the probability of detection. At the moment we proceed with such an operational description of the classical detectors. We plan to consider a more detailed scheme of the classical threshold detection in another paper in which we shall study classical signals with spatial degrees of freedom, i.e., signals in physical space-time. (In the present paper we restrict the model to “internal degrees of freedom” such as polarization. The model with spatial degrees of freedom is essentially more complicated, since it involves processes with infinite-dimensional state space.)

2. Threshold detection

2.1. The class of random signals

We consider a special class of classical random signals. The model is phenomenological: we cannot present physical motivations for selection of this class of signals, besides the fact that detection of such signals with the aid of threshold type detectors reproduces the correct quantum
probabilities. Another class of classical random signals serving for the same aim was introduced in [35]. Signals considered in the present paper have essentially simpler temporal structure, simply the Wiener process.

Structurally our model has some similarity with the prequantum model of Grössing et al [15]. Subquantum stochasticity is combined of the two counterparts: a stationary process in the space of internal degrees of freedom and the random walk type motion describing the temporal dynamics.

We start with stationary signals. We proceed in the finite dimensional state space corresponding to internal degrees of freedom such as polarization. Generalization to spatial degrees of freedom is evident, but it has essentially more complicated mathematical structure.

Let $H$ be the $m$ dimensional complex Hilbert space. Let $\phi(\omega)$ be the $H$-valued Gaussian random variable with zero average and the covariance operator $B$. This operator is Hermitian and positively defined. Take in $H$ an orthonormal basis $\{e_j\}$ and consider corresponding signal’s components $\phi_j(\omega) = \langle \phi(\omega), e_j \rangle$. The mathematical expansion of the random vector $\phi(\omega)$ with respect to the basis $\phi(\omega) = \sum_j \phi_j(\omega)e_j$ physically corresponds to splitting of the signal into disjoint channels. We remark that correlations of signal’s components are given by

$$E\phi_i\overline{\phi_j} = \langle Be_i, e_j \rangle = b_{ij}.$$ 

Now we introduce the temporal stochastics by simply using the one dimensional Wiener process $w(t)$ which is independent from the stationary process $\phi(\omega)$. Consider the random (non-stationary) signal $\phi(t, \omega) = w(t)\phi(\omega)$ and its components corresponding to internal degrees of freedom: $\phi_j(t, \omega) = w(t)\phi_j(\omega)$. We remark that correlations of signal’s components are given by

$$E\phi_i(t)\overline{\phi_j(s)} = \min(t, s)b_{ij}.$$ 

The energy of the $i$th component of the complex signal $\phi(t)$ (at the instance of time $t$) is given by the square of its absolute value

$$E_i(t, \omega) = |\phi_i(t, \omega)|^2.$$ 

The total energy of the signal is

$$E_i(t, \omega) = \sum_i |\phi_i(t, \omega)|^2 = \|\phi(t, \omega)\|^2.$$ 

2.2. The scheme of threshold measurement

We consider the measurement scheme in which each channel, $i = 1, 2, ..., m$, goes to a threshold type detector. We assume that all detectors have the same threshold $E_d > 0$. The detection procedure under consideration is reduced to the condition of the energy level approaching the detection threshold. The instant of time $\tau$ corresponding to the signal’s detection (“click”) by $j$th detector is determined by the condition:

$$E_j(\tau, \omega) = E_d.$$ 

We remark that the instant of the signal detection is a random variable:

$$\tau = \tau(\omega).$$ 

Mathematically our aim is to find average of the instance of detection, $\bar{\tau} = E\tau$. The quantity $1/\bar{\tau}$ will be used to find the probability of detection, “how often the detector produces clicks,” see section 3.
We apply the mathematical expectation (average) operator to both sides of the detection condition (2) and we obtain

\[ E \mathcal{E}_j(\tau(\omega), \omega) = \mathcal{E}_d, \] (3)

or

\[ E\omega^2(\tau(\omega), \omega)E|\phi_j(\omega)|^2 = \mathcal{E}_d. \] (4)

To find the first average, we use the formula of total probability:

\[ E\omega^2(\tau(\omega), \omega) = \int_0^\infty d\tau P(\tau(\omega) = \tau)E\omega^2(\tau, \omega). \]

We know that, for the fixed \( \tau \), \( E\omega^2(\tau, \omega) = \tau \). Hence,

\[ E\omega^2(\tau(\omega), \omega) = \int_0^\infty d\tau P(\tau(\omega) = \tau)\tau = E\tau \equiv \bar{\tau}. \]

Thus, the detection condition (4) has the form:

\[ \bar{\tau} E|\phi_j(\omega)|^2 = \mathcal{E}_d, \] (5)

or

\[ \bar{\tau} b_{ii} = \mathcal{E}_d. \] (6)

We remark that \( \tau = \tau_i, i = 1, 2, ..., m \). Thus

\[ \frac{1}{\bar{\tau}_i} = \frac{b_{ii}}{\mathcal{E}_d}. \] (7)

3. Probabilities of clicks in detection channels

Hence, during a long period of time \( T \) such a detector clicks \( N_{\text{click}} \)-times, where

\[ N_i \approx \frac{T}{\bar{\tau}_i} = \frac{b_{ii}T}{\mathcal{E}_d}. \] (8)

To find the probability of detection and match the real detection scheme which is used in quantum experiments we have to use a proper normalization of \( N_i \). This is an important point of our considerations. (The normalization problem is typically ignored in standard books on quantum foundations, cf., however, [34].) In QM-experiments probabilities are obtained through normalization corresponding to the sum of clicks in all detectors involved in the experiment, e.g., spin up and spin down detectors.

Hence, the total number of clicks:

\[ N = \sum_i N_i = \frac{T\sum_i b_{ii}}{\mathcal{E}_d} = \frac{T\mathbf{Tr} B}{\mathcal{E}_d}. \] (9)

We remark that the total number of clicks does not depend on the split of the signal into disjoint channels, i.e., on the selection of an orthonormal basis \( \{e_j\} \) in \( H \). In fact,

\[ E\|\phi(\omega)\|^2 = \mathbf{Tr} B. \]

The probability of detection for the \( j \)th detector is given by

\[ P_j = N_j/N = \frac{b_{ii}}{\mathbf{Tr} B}. \] (10)
In fact, this is the Born’s rule of QM. Consider the operator

\[ \rho = B/\text{Tr}B. \]  

This is the Hermitian positive trace one operator; so it has all properties of the density operator used in QM to describe the state of a quantum system.

Set \( \hat{C}_i = |e_i\rangle\langle e_i| \), the orthogonal projection onto the vector \( e_i \). Then the equality for the probability of detection (10) can be written as

\[ P_i = \text{Tr}\rho\hat{C}_i. \]  

This is the QM-rule for calculation of probabilities of detection.

4. Coincidence detection

Coincidence of clicks corresponds to matching of two conditions of threshold approaching corresponds to two constraints:

\[ \mathcal{E}_1(\tau_1(\omega), \omega) = \mathcal{E}_d, \mathcal{E}_2(\tau_2(\omega), \omega) = \mathcal{E}_d = \mathcal{E}_d, \]  

where matching has the form

\[ \tau_1(\omega) = \tau_2(\omega) = \tau(\omega). \]  

Our aim is to estimate the probability of coincidence \( P_{12} \). To shorter notation, we set \( \mathcal{E}_i(\omega) \equiv \mathcal{E}_i(\tau(\omega); \omega) \) or even simply \( \Gamma_i, i = 1, 2 \).

We consider the set of random parameters corresponding to coincidence detection: \( A_{12} = \{ \omega : \mathcal{E}_i = \mathcal{E}_d, i = 1, 2 \} \). \( A_{12} \) is the event of coincidence detection. We have to estimate its probability, \( P_{12} = P(A_{12}) \). We shall get a rather rough estimate which, nevertheless, will be sufficient for our purpose. However, we shall see that better estimates of this probability will clarify essentially inter-relation between our “prequantum classical field theory”, QM, and experiment. In principle, one may hope to derive an approximative expression for \( P_{12} \) as we did for probabilities \( P_j, j = 1, 2 \). However, this is a complicated probabilistic problem.

We remark that \( A_{12} \) is a subset of the set

\[ A_{1 \times 2} = \{ \omega : \mathcal{E}_1\mathcal{E}_2 = \mathcal{E}_d^2 \}. \]

Hence, \( P(A_{12}) \leq P(A_{1 \times 2}) \). And the set \( A_{1 \times 2} \) is a subset of the set

\[ A_{1 \times 2 \geq \mathcal{E}_d^2} = \{ \omega : \mathcal{E}_1\mathcal{E}_2 \geq \mathcal{E}_d^2 \} \]

and hence \( P(A_{12}) \leq P(A_{1 \times 2 \geq \mathcal{E}_d^2}) \). The latter probability we can (roughly) estimate by using Chebyshov inequality (which usage is standard for such estimates, cf. [36]). In the simplest form, for a random variable \( u = u(\omega) \) and a constant \( k > 0 \), this inequality has the form:

\[ P(\omega : u \geq k) \leq \frac{E[u(\omega)]}{k}. \]

In our case \( u = \mathcal{E}_1\mathcal{E}_2, k = \mathcal{E}_d^2 \). We have

\[ P(A_{12}) \leq \frac{E\mathcal{E}_1\mathcal{E}_2}{\mathcal{E}_d^2}. \]  

We find this average by using the formula of total probability [36]:

\[ EE_1(\omega)E_2(\omega) = \int_0^\infty EE_1(\tau, \omega)E_2(\tau, \omega)P(\tau(\omega) = \tau)d\tau. \]  

(15)
Thus our main problem is to find the correlation of two energies for each instant of time \( \tau \). We have

\[
E \mathcal{E}_1(\tau, \omega) \mathcal{E}_2(\tau, \omega) = E w^4(\tau) E |\phi_1(\omega)|^2 |\phi_2(\omega)|^2.
\]

The first factor is known, \( E w^4(\tau) = 3 \tau^2 \).

To find the second factor, we shall use general theory of Gaussian integrals on complex Hilbert space \([37]\), see appendix. Consider in \( H \) (\( m \)-dimensional complex space) projection operators \( \hat{A}_k = |e_k\rangle \langle e_k|, k = 1, \ldots, m \). Set \( f_{\hat{A}_k}(\phi) = \langle \hat{A}_k \phi, \phi \rangle, \phi \in H \), the quadratic form corresponding to the operator \( \hat{A}_k \). By (27), appendix, we obtain

\[
E f_{\hat{A}_1} f_{\hat{A}_2} = \text{Tr} B \hat{A}_1 \text{Tr} B \hat{A}_2 + \text{Tr} B \hat{A}_2 B \hat{A}_1,
\]
where \( B \) is the covariance operator. We remark that

\[
b_{ij} = \rho_{ij} \text{Tr} B,
\]

where \( \rho = (\rho_{ij}) \) is a density operator. So, we consider the prequantum random signal corresponding to the quantum state \( \rho \) and we study the problem of detection coincidence for such a signal \( \phi(t) \equiv \phi_\rho(t, \omega) \).

We have

\[
\text{Tr} B \hat{A}_i = b_{ii}, i = 1, 2.
\]

In the same way

\[
\text{Tr} B \hat{A}_2 B \hat{A}_1 = |\langle e_1 | B | e_2 \rangle|^2 = |b_{12}|^2.
\]

Finally, we obtain (for the fixed instant of time \( \tau \))

\[
E \mathcal{E}_1(\tau, \omega) \mathcal{E}_2(\tau, \omega) = 3 \tau^2 (b_{11} b_{22} + |b_{12}|^2).
\]

The formula of total probability (16) implies

\[
E \mathcal{E}_1(\tau(\omega); \omega) \mathcal{E}_2(\tau(\omega); \omega) = 3 \tau^2 (b_{11} b_{22} + |b_{12}|^2),
\]

where \( \tau^2 = \overline{\tau^2} \). By (11) we can rewrite this answer in terms of quantum mechanical density matrix

\[
E \mathcal{E}_1(\tau(\omega); \omega) \mathcal{E}_2(\tau(\omega); \omega) = 3 \tau^2 (\text{Tr} B)^2 (\rho_{ii} \rho_{jj} + |\rho_{ij}|^2),
\]

Thus by the Chebyshov inequality

\[
P(A_{12}) \leq \frac{3 (\text{Tr} B)^2 \overline{\tau^2}}{\mathcal{E}_d^2} (\rho_{ii} \rho_{jj} + |\rho_{ij}|^2).
\]

The quantity

\[
\Delta \equiv \sqrt{\overline{\tau^2}}
\]

can be interpreted as the time parameter scaling the time intervals between coincidence clicks. Therefore \( \mathcal{E}_\Delta = \Delta \text{Tr} B \) is average of signal’s energy distributed between clicks. Suppose that

\[
\epsilon = \frac{\overline{\tau^2}}{\mathcal{E}_d} << 1.
\]

In this case

\[
P(A_{12}) \leq 3 \epsilon^2 (\rho_{ii} \rho_{jj} + |\rho_{ij}|^2).
\]
As usual [], set \( g^{(2)}(0) = \frac{P_{12}}{P_1 P_2} \), where \( P_1 \) and \( P_2 \) are probabilities of detection in channels \( i \) and \( j \), respectively (it is assumed that the later probabilities are positive). We proved, see section 3, that in TSD model, these probabilities are equal to quantum probabilities: \( P_1 = \rho_{11} \) and \( P_2 = \rho_{22} \). Hence,
\[
g^{(2)}(0) = \frac{P_{12}}{P_1 P_2} \leq 3\epsilon^2 \left( 1 + \frac{|\rho_{ij}|^2}{\rho_{ii} \rho_{jj}} \right),
\]
(23)
Since \( \epsilon \sim \frac{1}{\mathcal{E}_d} \), by increasing the parameter \( \mathcal{E}_d \) the coefficient \( g_{12}(0) \) can be done less than 1. Hence, our prequantum classical field model can violate (under the selection of a proper threshold) the inequality \( g^{(2)}(0) \geq 1 \) which is valid for “standard” classical signal theory – its standard version which does not take into account the evident fact that measurement is not simply detection (of the monitoring-type) of continuous classical signals, but creation of discrete counts with the aid of threshold detectors. Detectors also have to be properly calibrated.

5. Appendix: Gaussian integrals
Details of theory of integration with respect to Gaussian measures on complex Hilbert spaces can be found in [37].

Let \( W \) be a real Hilbert space. Consider a \( \sigma \)-additive Gaussian measure \( p \) on the \( \sigma \)-field of Borel subsets of \( W \). This measure is determined by its covariance operator \( B : W \rightarrow W \) and mean value \( m \in W \). For example, \( B \) and \( m \) determine the Fourier transform of \( p \):
\[
\hat{p}(y) = \int_W e^{i(y, \phi)} dp(\phi) = e^{\frac{i}{2}(B y, y) + i(m, y)} , y \in W.
\]
(In probability theory it is called the characteristic functional of the probability distribution \( p \).) In what follows we restrict our considerations to Gaussian measures with zero mean value: \((m, y) = \int_W (y, \psi) dp(\psi) = 0 \) for any \( y \in W \). Sometimes there will be used the symbol \( p_B \) to denote the Gaussian measure with the covariance operator \( B \) and \( m = 0 \). We recall that the covariance operator \( B \) is defined by its bilinear form \((B y_1, y_2) = \int (y_1, \phi)(y_2, \phi) dp(\phi) \), \( y_1, y_2 \in W \).

Let \( Q \) and \( P \) be two copies of a real Hilbert space. Let us consider their Cartesian product \( H = Q \times P \), “phase space,” endowed with the symplectic operator \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Consider the class of Gaussian measures (with zero mean value) which are invariant with respect to the action of the operator \( J \); denote this class \( \mathcal{S}(H) \). It is easy to show that \( p \in \mathcal{S}(H) \) if and only if its covariance operator commutes with the symplectic operator, [32].

As always, we consider complexification of \( H \) (which will be denoted by the same symbol), \( H = Q \oplus i P \). The complex scalar product is denoted by the symbol \((\cdot, \cdot)\). The space of bounded Hermitian operators acting in \( H \) is denoted by the symbol \( \mathcal{L}_\sigma(H) \).

We introduce the complex covariance operator of a measure \( p \) on the complex Hilbert space \( H : \langle D y_1, y_2 \rangle = \int_H (y_1, \phi)(y_2, \phi) dp(\phi) \). Let \( p \) be a measure on the Cartesian product \( H_1 \times H_2 \) of two complex Hilbert spaces. Then its covariance operator has the block structure
\[
D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},
\]
(24)
where \( D_{ii} : H_i \rightarrow H_i \) and \( D_{ij} : H_j \rightarrow H_i \). The operator is Hermitian. Hence \( D_{ii}^* = D_{ii} \), and \( D_{12}^* = D_{21} \).

Let \( H \) be a complex Hilbert space and let \( \hat{A} \in \mathcal{L}_\sigma(H) \). We consider its quadratic form (which will play an important role in our further considerations) \( \phi \rightarrow f_A(\phi) = \langle \hat{A} \phi, \phi \rangle \). We make a trivial, but ideologically important remark: \( f_A : H \rightarrow \mathbb{R} \), is a “usual function” which is defined point wise. We use the equality, see, e.g., [32]:
\[
\int_H f_A(\phi) dp_D(\phi) = \operatorname{Tr} D \hat{A}.
\]
(25)
Let $p$ be a Gaussian measure of the class $\mathcal{S}(H_1 \times H_2)$ with the (complex) covariance operator $D$ and let operators $\hat{A}_i$ belong to the class $\mathcal{L}_s(H_i), i = 1, 2$. Then

$$\int_{H_1 \times H_2} f_{A_1}(\phi_1)f_{A_2}(\phi_2)d\phi = \text{Tr}D_{11}\hat{A}_1\text{Tr}D_{22}\hat{A}_2 + \text{Tr}D_{12}\hat{A}_2D_{21}\hat{A}_1$$

(26)

This equality is a consequence of the following general result [32]:

Let $p \in \mathcal{S}(H)$ with the (complex) covariance operator $D$ and let $\hat{A}_i \in \mathcal{L}_s(H)$. Then

$$\int_H f_{A_1}(\phi)f_{A_2}(\phi)d\phi = \text{Tr}\hat{D}_1\text{Tr}\hat{D}_2 + \text{Tr}\hat{D}_2\hat{D}_1$$

(27)

**Summary on coincidence probability:** Prequantum classical field model with threshold and properly calibrated detectors violates predictions of standard classical and semiclassical models, cf. [24], and gives the prediction compatible with known experimental data. More detailed experiments of Grangier’s type with monitoring of the dependence of $g^{(2)}(0)$ on the detection threshold and source’s brightness are demanded.

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